Asymptotics of Wigner $3nj$-symbols with small and large angular momenta: an elementary method

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Abstract

Yu and Littlejohn recently studied in (2011 Phys. Rev. A 83 052114 (arXiv:1104.1499)) some asymptotics of Wigner symbols with some small and large angular momenta. They found that in this regime the essential information is captured by the geometry of a tetrahedron, and gave new formulae for $9j$, $12j$- and $15j$-symbols. We present here an alternative derivation which leads to a simpler formula, based on the use of the Ponzano–Regge formula for the relevant tetrahedron. The approach is generalized to Wigner $3nj$-symbols with some large and small angular momenta, where more than one tetrahedron are needed, leading to new asymptotics for Wigner $3nj$-symbols. As an illustration, we present $15j$-symbols with one, two and four small angular momenta, and give an alternative formula to Yu’s recent $15j$-symbol with three small spins.

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(Some figures may appear in colour only in the online journal)

Introduction

Wigner symbols are re-coupling coefficients of $SU(2)$ representation theory. As such they naturally arise when dealing with sums of more than four spins and/or angular momenta in quantum mechanics, and are notoriously important in spectroscopy and atomic/molecular physics. Moreover, they also enter different parts of physics [1], like quantum computing [2] (in the presence of topological order, and mainly with a quantum group deformation [3]), and are expected to contain relevant aspects of quantum geometry (in loop quantum gravity and spin foam models [4–7]).

Although those objects are easily defined, using sums of Clebsch–Gordan coefficients, or inner products of wavefunctions, it is quite hard to extract their semi-classical behavior, for large angular momenta (not even to mention getting a rigorous proof). Moreover, in the
typical case of spin–orbit couplings, one may be interested in large angular momenta coupled to some intrinsic spins which cannot be scaled. Obviously, different behaviors are expected in those different regimes of a given symbol.

The most studied Wigner symbol is the $6j$-symbol, whose asymptotics is known when all spins are large (the Ponzano–Regge formula, [11–14]), or with one being small (Edmonds’ formula [15, 16]). Over the years, the $6j$-symbol has turned out to be an interesting topic in modern physics, see [17] which contains additional references. Very recent progress include subleading corrections to its Ponzano–Regge asymptotics [18–20], a new recursion relation on the square of the $6j$-symbol [21] and a derivation of its standard recursion relation as a Wheeler–DeWitt equation for three-dimensional Riemannian gravity [22].

When all angular momenta are large, a nice feature of the Ponzano–Regge asymptotics is that all the information is captured into a tetrahedron whose edge lengths are basically the six quantum angular momenta. Such a simple geometric interpretation is usually not available for larger symbols, which makes their analysis more difficult. While a new approach was recently proposed in [23] to classify the various asymptotic regimes of Wigner $3nj$-symbols, it remains an open issue. In contrast with the generic case, it is interesting to note that the $15j$-symbol admits a natural four-dimensional interpretation in the coherent state basis, in terms of the geometry of a 4-simplex [24, 25]. (The same method has been applied to the three-dimensional Ponzano–Regge model, see [26].) Just like for the $6j$-symbol, that geometric property can be understood via the recursions satisfied by the $15j$-symbol which naturally arise as Hamiltonian equations for the quantum 4-simplex [27].

Yu and Littlejohn [8] have recently obtained a new formula for the asymptotics of the $9j$-symbol with eight large angular momenta and one small spin (and also for larger symbols, but the method there focuses on this example, and we will do the same here). Their method is quite generic and powerful since it relies on previous works of the authors which extend the Born–Oppenheimer approximation in the case the fast degrees of freedom are coupled through a matrix of non-commuting operators. We refer the reader to [8] for further references to the method and its applications, as we will be more interested in the asymptotics of the $9j$-symbol itself.

The asymptotic formula itself is indeed quite interesting. The information is encoded into the geometry of a tetrahedron, and can be re-formulated more conveniently with three tetrahedra. The formula also displays ingredients which are familiar to the asymptotics of the $6j$-symbol, as noted by the authors of [8], more precisely the amplitude involving the inverse of the square root of the volume of the tetrahedron, and oscillations with part of the frequency given by the Regge action of the same tetrahedron.

This suggests that the asymptotic formula for the $9j$-symbol with a small spin may really be derived using the Ponzano–Regge asymptotic formula for some $6j$-symbol associated with the relevant tetrahedron. This is exactly what we show in this paper, ending up with a quite straightforward derivation. Remarkably, not only the derivation but also the final formula turns out to be simpler than that presented in [8]. Section 1 is devoted to showing this.

Our analysis further reveals the conditions so that an arbitrary Wigner symbol can be semi-classically described by a number of tetrahedra, when some spins remain small. This way we obtain generic asymptotic formulae for Wigner symbols, presented in section 2.

As an illustration, we show the formulae for $15j$-symbols in section 3 since they may have some interesting applications in four-dimensional models for gravity. The cases with one, two and four small angular momenta are new. The case with three small angular momenta is an alternative to the recent result of [10] and our formula is a bit simpler since all quantities can be evaluated using only two tetrahedra. It should be noted that the method used in [8–10] is surely quite powerful since it has given access to some regimes which cannot be probed with
our method. So at the end of the day, we think that both methods yield complementary results, with some overlap, and open an interesting window towards new asymptotics of re-coupling coefficients.

1. Asymptotics of the $9j$-symbol with one small angular momentum

1.1. Notations

We have tried to use as often as possible the same notations as [8]. The large angular momenta, or spins in the sense of irreducible $SU(2)$ representations, are denoted like $j \in \mathbb{N}/2$ and the small spins like $s \in \mathbb{N}/2$.

However, differences appear regarding angles. We systematically call $\phi_{a,b}$ the (internal) angle between two edges $a$ and $b$ of a triangle. Its value is given in terms of the lengths of the triangle,

$$\cos \phi_{a,b} = \frac{\ell_{a}^2 + \ell_{b}^2 - \ell_{c}^2}{2 \ell_{a} \ell_{b}},$$

where the third length $\ell_{c}$ will be mentioned when necessary.

Lengths are simple functions of spins which will be attached to the corresponding edges,

$$\ell_{a} \overset{\text{def}}{=} j_{a} + \frac{1}{2} \overset{\text{def}}{=} d_{j_{a}}/2.$$  

This is the relation which is necessary for the Ponzano–Regge formula to work. But note that when using Edmonds’ formula (given explicitly later), those dihedral angles are more naturally given in terms of different lengths $\sqrt{j_{a}(j_{a} + 1)}$. The latter are asymptotically equivalent to $\ell_{a}$ and the difference in Edmonds’ formula only appears at subleading orders. Note also that $d_{j}$ is the dimension of the representation of spin $j$.

We denote, respectively, $\Theta_{e}$ and $\theta_{e}$ as the external and internal dihedral angles between two triangles adjacent to the edge $e$ in a tetrahedron, with $\Theta_{e} = \pi - \theta_{e}$. They are obviously determined by the lengths and satisfy the following relation:

$$\cos \theta_{e} = \frac{\cos \phi_{b,c} - \cos \phi_{a,b} \cos \phi_{a,c}}{\sin \phi_{a,b} \sin \phi_{a,c}},$$

when the edges $a$, $b$ and $c$ meet at a node in a tetrahedron.

1.2. The formula

The $9j$-symbol comes as a re-coupling coefficient, i.e. a change of basis, when describing in different ways the rotational invariant subspace of a tensor product of five spins $j_{1}, j_{2}, s, j_{4}, j_{5}$. The invariant subspace is characterized by the fact that the sum of the angular momenta vanishes,

$$J_{1} + J_{2} + S + J_{4} + J_{5} = 0.$$  

(4)

The first basis is obtained by choosing a spin $j_{13}$ in the tensor product $j_{1} \otimes s$, a spin $j_{24}$ in $j_{2} \otimes j_{4}$ and then consider the unique (normalized) vector satisfying $J_{13} + J_{24} + J_{5} = 0$ in $j_{13} \otimes j_{24} \otimes j_{5}$. Denote this vector $| (j_{1}, s, j_{13}), (j_{2}, j_{4}, j_{24}) \rangle$. An equivalent basis is obtained by tensoring first $j_{1}$ with $j_{2}$ and choosing a spin $j_{12}$ in the decomposition, and similarly choosing a spin $j_{34}$ in $s \otimes j_{4}$. A basis vector is then formed by the unique vector satisfying $J_{12} + J_{34} + J_{5} = 0,$

1 It is actually equivalent to look at the projection of $j_{1} \otimes j_{2} \otimes s \otimes j_{4}$ onto the spin $j_{5}$. 

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and is denoted \(((j_1, j_2, j_{12}), j_3, (s, j_4, j_{34}))\). The 9\(j\)-symbol is just
\[
\langle (j_1, j_2, j_{12}), j_3, (s, j_4, j_{34}) | (j_1, s, j_{33}), j_5, (j_2, j_4, j_{24}) \rangle
\]
\[
= \frac{1}{d_{j_12} d_{j_14} d_{j_13} d_{j_24}} \sum_{e \in \text{tet}_1} \left( j_1 + j_2 + j_{12} + j_3 + s + j_{34} + j_4 + j_{24} \right) \cos \left( \frac{1}{2} \Theta^{(1)}_e \right) d_{j_1}^0 \left( \pi - \varphi_{1,34} \right).
\]

In the above expression:

- the sum in the cosine runs over the six large angular momenta \(j_e\) with \(e = 1, 2, 3, 4, 5, 12, 24\);
- \(\mu \triangleq j_{12} - j_1 \) and \(\nu \triangleq j_{34} - j_4\), those two differences have the same order of magnitude as \(s\), due to the triangle inequalities (or Clebsch–Gordan conditions) between the spins \((j_1, s, j_{33})\) and also between \((s, j_4, j_{34})\);
- \(V_1\) and \(\{\Theta_e^{(1)}\}_{e=1,2,3,4,5,12,24}\) are geometric quantities associated with the tetrahedron \(\text{tet}_1\) in figure 1, \(V_1\) is the volume of the tetrahedron and \(\Theta_e^{(1)}\) is the angle between two external normals to the faces adjacent to the edge \(e\) (external dihedral angle);
- \(\varphi_{1,34}\) is associated with the tetrahedron \(\text{tet}_2\) given in figure 1; it is the angle between the edges 1 and 34; \(\theta_{1}^{(2)}\) and \(\theta_{4}^{(2)}\) are, respectively, the internal dihedral angles between the faces adjacent to the edges 1 and 34 in \(\text{tet}_2\);
- \(d^{(s)}\) is the Wigner \(d\)-matrix with spin \(s\), with the convention \(d_{\mu}^{s}(\phi) = \langle s, \mu | e^{-\frac{i}{2} \phi} | s, \nu \rangle\).
The formula holds in the classically allowed region away from the caustic, i.e. where the volume $V_1$ of the tetrahedron $\text{tet}_1$ is not close to zero.

The tetrahedron $\text{tet}_2$ is built by gluing the triangles $(2, 34, 24)$ and $(1, 5, 24)$ along 24, similar to the tetrahedron $\text{tet}_1$, but after flipping one of the triangles. This means that 2 and 5 meet at one end of 24, and 1 and 34 meet at the other node. To completely determine $\text{tet}_2$, one has to set the dihedral angle between the two triangles. It is $\theta_{24}^{(2)} = \Theta_{24}^{(1)}$. This can be deduced, like in [8], from the vectors representing the classical angular momenta, which are drawn in figure 1. This is indeed equivalent to adding to $\text{tet}_1$ the parallelogram spanned by $J_2$ and $J_{34}$, and observe that $\text{tet}_2$ is the tetrahedron between the parallelogram and $\text{tet}_1$. That observation is summarized in figure 2 which offers a full view on the two relevant tetrahedra.

Our tetrahedra are closely related to those used in [8]. Their first tetrahedron is like ours but with $J_4$ instead of $J_{34}$ (note indeed that they differ only by a vector of order $s$). Their second tetrahedron is built just like $\text{tet}_2$ from $\text{tet}_1$. Because we are using a different first tetrahedron, the Regge action $\sum_{e \subset \text{tet}_1} (j_e + \frac{1}{2}) \Theta_e^{(1)}$ in the cosine is not the same. The difference, which can be observed in [8], is a term $\nu/\Theta_{34}^{(1)}$. This adds to the explicit $\nu$ contribution in (6), which becomes $\nu(\Theta_{34}^{(1)} - \theta_{24}^{(2)})$. That new angle can be interpreted with the introduction of a third tetrahedron, which is the additional tetrahedron of [8].

1.3. Direct derivation of the formula

Our derivation starts from a decomposition\(^2\) of the $9j$-symbol in terms of $6j$-symbols [28]

$$\left\{ \begin{array}{c} j_1 \ j_2 \ j_{12} \\ s \ j_4 \ j_{34} \\ j_{13} \ j_{24} \ j_5 \end{array} \right\} = \sum_x (-1)^{2x} \left\{ \begin{array}{c} j_1 \ j_2 \ j_{12} \\ j_{34} \ j_4 \ x \\ j_{13} \ j_{24} \ j_5 \end{array} \right\} \left\{ \begin{array}{c} s \ j_4 \ j_{34} \\ j_2 \ x \ j_{24} \end{array} \right\} \left\{ \begin{array}{c} j_{13} \ j_{24} \ j_5 \\ x \ j_1 \ s \end{array} \right\}. \tag{7}$$

The range of summation of $x$ is finite due to triangle inequalities. $x$ is bounded from below by $\max(\{|J_{24} - s|, |j_1 - j_5|, |J_{34}|\}$ and from above by $\min(j_{34} + s, j_1 + j_5, j_2 + j_{34})$. Now the regime we are looking at, away from the caustic where the volume of $\text{tet}_1$ becomes close to zero, ensures that neither $\max(\{|j_1 - j_5|, |J_{24} - s|\}$ nor $\min(j_1 + j_5, j_2 + j_{34})$ is close to $J_{34}$ up to terms of order $s$. Hence, from the non-degeneracy of $\text{tet}_1$, we know that $x$ 

\(^2\) There are several such decompositions. Those we are interested in are such that the summed variable, here $x$, has to satisfy Clebsch–Gordan conditions with the small spin $s$. There are four such possibilities. In our choice, $j_{24}$ will play a special role in the following. But we could have equally well-chosen $j_2, j_{12}$ or $j_{34}$ instead.
runs from $j_{24} - s$ to $j_{24} + s$. Introduce $\xi \equiv x - j_{24}$ which lives in $[-s, \ldots, s]$, and write

\[
\begin{align*}
\left\{ j_1, j_2, j_{12} \atop s, j_{34}, j_3, j_4, j_{34} \right\} &= \sum_{\xi = -s}^{s} d_{j_{24} + \xi} \left( -1 \right)^{j_{24} + 2s} \left\{ s, j_{34} - v, j_{34} \atop j_2, j_{24} + \xi, j_{24} \right\} \\
&\times \left\{ j_1 + \mu, j_{24}, j_5 \atop j_{24} + \xi, j_1, j_{12} \atop s, j_{34}, j_5, j_2, j_{24} + \xi \right\} .
\end{align*}
\]  

(8)

We observe that there are two different situations for the above $6j$-symbols. The last one does not contain $s$, so that its large spin behavior is given by the Ponzano–Regge formula. The other two $6j$-symbols only have five large spins, and their asymptotics are given by Edmonds’ formula.

The presence of the symbol \( \{ j_1, j_2, j_{12}, j_{34}, j_3, j_4, j_{34} \} \) in (8) will make clear why the geometry of the tetrahedron $\text{tet}_1$ is important. It is indeed well known that the asymptotics of that symbol is described using a tetrahedron geometry, with oscillations following the Regge action for that tetrahedron. Precisely,

\[
\begin{align*}
\left\{ j_1, j_2, j_{12} \atop j_{34}, j_3, j_4, j_{34} + \xi \right\} &\approx \frac{\cos \left( S_R(\xi) + \frac{\pi}{4} \right)}{\sqrt{12\pi V(\xi)}} .
\end{align*}
\]  

(9)

$V(\xi)$ and $S_R(\xi)$ are the volume and the Regge action of the tetrahedron constructed with the lengths $j_1 + 1/2, j_2 + 1/2, j_{12} + 1/2, j_{34} + 1/2, j_3 + 1/2, j_4 + 1/2, j_{34} + \xi + 1/2$; in such a way that the triangles are formed by the spins which are coupled in the $6j$-symbol (through Clebsch–Gordan couplings). Since that tetrahedron differs from the tetrahedron $\text{tet}_1$ only due to a small $\xi$ of order $\ell$ along the vector $\mathbf{j}_{24}$, one can relate $S_R(\xi)$ to the Regge action $S_R(\text{tet}_1)$ of tetrahedron $\text{tet}_1$ via

\[
S_R(\xi) \approx S_R^{(1)} + \xi \Theta_\text{tet}_1^{(1)}, \quad \text{with} \quad S_R^{(1)} \equiv \sum_{e \in \text{tet}_1} \left( j_e + \frac{1}{2} \right) \Theta_e^{(1)} .
\]  

(10)

Here, the difference between the dihedral angles $\Theta_e(\xi)$ appearing in $S_R(\xi)$ and the dihedral angles $\Theta_e^{(1)}$ of $\text{tet}_1$ has been omitted thanks to Schläfli’s identity (it makes sure that those sum to zero on the six edges of the tetrahedron). Besides, the variations of $V$ with $\xi$ are irrelevant at the leading order, so that $V(\xi) \approx V_1$. The other two $6j$-symbols can be expressed using Edmonds’ formula\(^3\) [28]:

\[
\begin{align*}
\left\{ j_{34} - v \atop j_2, j_{24} + \xi, j_{24} \right\} &\approx \frac{(-1)^{j_{24} + j_{34} + s} d^{(s)}_{j_2, j_{34}} d^{(s)}_{j_{24} - v}(\varphi_{34,24})}{\sqrt{d_{j_2} d_{j_{34}}}} ,
\end{align*}
\]  

(11)

\[
\begin{align*}
\left\{ j_1 + \mu, j_{24}, j_5 \atop j_{24} + \xi, j_{12}, j_1 \right\} &\approx \frac{(-1)^{j_{12} + j_{24} + s} (-1)^{s+\xi} d^{(s)}_{\mu, j_{12}} d^{(s)}_{j_1, j_{24}}}{\sqrt{d_{j_1} d_{j_{24}}}} .
\end{align*}
\]  

(12)

We use symmetries of the Wigner matrices to write $d^{(s)}_{-v, \varphi_{34,24}} = (-1)^{s+\xi} d^{(s)}_{-\varphi_{34,24}, \varphi_{34,24}}$, inserting those asymptotics in (8) and relabeling the sum by $\xi \mapsto -\xi$, we obtain

\[
\begin{align*}
\{9j\} &\approx \frac{(-1)^{j_{12} + j_{24} + s} \sum_{\xi = -s}^{s} \cos \left( S_R^{(1)} + \frac{\pi}{4} - \xi \Theta_\text{tet}_1^{(1)} \right) d^{(s)}_{\mu, j_{12}} d^{(s)}_{j_1, j_{24}}}{\sqrt{d_{j_1} d_{j_{24}} (12\pi V_1)}} \times \\
&\sum_{\xi = -s}^{s} \cos \left( S_R^{(1)} + \frac{\pi}{4} - \xi \Theta_\text{tet}_1^{(1)} \right) d^{(s)}_{\mu, j_{12}} d^{(s)}_{j_1, j_{24}} .
\end{align*}
\]  

(13)

\(^3\) It gives the asymptotic behavior of a $6j$-symbol when one spin is much smaller than the five others,

\[
\begin{pmatrix} a & b & e \\ b + m & a + n & f \end{pmatrix}_{a,b,c,m,n,f} \approx \frac{(-1)^{b+c+m} d^{(m)}_{b+c}(\varphi_{a,b})}{\sqrt{2a + 1)(2b + 1)}}
\]
Our convention for the Euler angles is the form $\phi$ and $\sigma$. The sum over $J. Phys. A: Math. Theor.$ is almost a matrix product between the two Wigner $D$-matrices, but not exactly because the argument of the cosine does depend on $\xi$. Nevertheless, the sum over $\xi$ is a matrix product of Wigner $D$-matrices. To see that, we simply write the cosine as a sum of exponentials. Let us call $A$ the prefactor of the sum. We get the following expression:

$$[9J] \approx A e^{i \sigma_1 (\pi/4)} \sum_{\xi=\mu} d^{(2)}_{\mu}(\phi_{1,24}) e^{-i \rho_{\xi}(\phi_{24})} d^{(2)}_{\xi}(\pi - \phi_{34,24}) + \text{c.c.},$$

where 'c.c.' denotes the complex conjugate of the whole expression. By definition of the Wigner $D$-matrices, the sum can then be re-expressed as a matrix product,

$$\sum_{\xi} d^{(2)}_{\mu}(\phi_{1,24}) e^{-i \sigma_{\xi}(\phi_{24})} d^{(2)}_{\xi}(\pi - \phi_{34,24}) = D^{(2)}_{\mu} e^{-i \pi_{1,24}} e^{-i \pi_{24}} e^{-i \pi_{34,24}}.$$

We have used above the notation in terms of $SU(2)$ rotations, written with the Pauli matrices as generators. The last step of the calculation is to rewrite the relevant product of rotations as a single rotation parameterized by its Euler angles$^4$. Rewriting that, from the angles $\phi_{1,24}$, $\Theta_{24}^{(1)}$ and $\phi_{34,24}$ to Euler angles, will naturally be encoded into the geometry of the tetrahedron tet$_2$. Explicitly, we define three new angles $\theta_{1}^{(2)}$, $\phi_{1,34}$ and $\phi_{34,24}$ by

$$e^{-i \phi_{1,24}} e^{-i \phi_{24}} e^{-i \phi_{34,24}} = e^{-i \phi_{1,34}} e^{-i \phi_{34,24}} e^{-i \phi_{34,24}}.$$

The relation between both sets can be found in textbooks like [28]. Since all our angles lie in $[0, \pi]$, we can simply make use of the $SO(3)$ relations,

$$\cos \phi_{1,34} = \cos \phi_{1,24} \cos \phi_{34,24} + \sin \phi_{1,24} \sin \phi_{34,24} \cos \Theta_{24}^{(1)},$$

$$\cos \phi_{34,24} = \frac{\cos \phi_{1,34}}{\sin \phi_{34,24}},$$

$$\cos \phi_{1,34} = \frac{\cos \phi_{1,24} \cos \phi_{34,24} \cos \phi_{1,34}}{\sin \phi_{34,24}}.$$

As our notation suggests, those new angles have a nice geometric interpretation with the help of the tetrahedron tet$_2$. The latter appears in the following way. Those equations characterize the relationship between 2D and 3D angles for three triangles forming the 'top' of a tetrahedron, as displayed in figure 3. The initial angles $\phi_{1,24}$, $\Theta_{24}^{(1)}$ and $\phi_{34,24}$ enable to draw three links, 1, 34 and 24 meeting at a node, with two 2D angles being $\phi_{1,24}$ and $\phi_{34,24}$ and the (internal)

$^4$ Our convention for the Euler angles is the form $g = e^{-i \frac{\phi}{2}} e^{-i \frac{\phi}{2}} e^{-i \frac{\phi}{2}}$ for $g \in SU(2)$, together with $\sigma_z = \text{diag}(1, -1)$, $\sigma_y = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$.
angle between the planes (34, 24) and (1, 24) being $\theta_{34}^{(1)}$. Then the above formulae make it possible to evaluate the three remaining angles. The first equation states that $\psi_{1,34}$ is the angle between the edges 1 and 34. And $\theta_{34}^{(2)}$ and $\theta_{13}^{(2)}$ given by the above formulae are the two other internal dihedral angles. Those geometric considerations are summarized in Figure 3. This is exactly the geometry of the ‘top’ of tet$_2$, see Figure 1, where 1, 34 and 24 meet.

The element $D_{\mu \nu}^{(i)}$ of the Wigner D-matrix of the relevant SU(2) rotation can then be written in terms of the element $d_{\mu \nu}^{(i)}$ of a $d$-matrix, so that the 9j-symbol takes the following form:

$$[9j] \approx A \exp \left[ i \left( S_{R}^{(1)} + \frac{\pi}{4} - \mu \left( \pi - \theta_{34}^{(2)} \right) - v \theta_{13}^{(2)} \right) \right] d_{\mu \nu}^{(i)}(\pi - \psi_{1,34}) + \text{c.c.,} \quad (18)$$

that is to say

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ s & j_1 & j_{34} \\ j_{13} & j_24 & j_5 \end{bmatrix} \approx \frac{(-1)^{j_1 + j_2 + j_{34} + j_{12}}}{\sqrt{d_{\mu} d_{\nu} (12\pi V_1)}} \cos \left[ S_{R}^{(1)} + \frac{\pi}{4} - \mu \left( \pi - \theta_{34}^{(2)} \right) - v \theta_{13}^{(2)} \right] d_{\mu \nu}^{(i)}(\pi - \psi_{1,34}). \quad (19)$$

This is exactly formula (6).

Since our formula is different from that of Yu and Littlejohn, it is worth comparing it directly with numerics. From the derivation, it is clear that our formula holds as long as the approximation is still good when tet$_1$ is distorted with some angular momenta larger than other. For the smallest values of $j_1$ from 63 to 160. For the largest values of $j_1$, the approximation breaks down because the volume of tet$_1$ becomes negative. The same phenomenon is observed in Figure 4(a) and the error plot 4(b), where the error increases when we reach the low and high values of $j_{34}$.

Note that the corresponding symbol is $\left\{ \begin{array}{c} j_1 + \frac{i}{2} \\ j_2 + \frac{i}{2} \\ j_3 + \frac{i}{2} \end{array} \right\} \approx \left\{ \begin{array}{c} 430 \\ 431 \\ 430 \end{array} \right\}$, whose large spins differ by a factor of 10. It means that the approximation is still good when tet$_1$ is distorted with some angular momenta larger than other.

2. Asymptotics of 3nj-symbols with small and large angular momenta

2.1. Decomposition of a 3nj-symbol

For $n \geq 4$, there exist several ways to define a 3nj-symbol. For instance, in the case $n = 4$, one can define two different kinds of (irreducible) 12j-symbols. We restrict the discussion to the first kind, in the terminology of [29], whose decomposition in terms of 6j-symbols is

$$\begin{bmatrix} j_1 & j_2 & \cdots & j_n \\ l_1 & l_2 & \cdots & l_n \\ k_1 & k_2 & \cdots & k_n \end{bmatrix} = \sum_{x} d_{x}(-1)^{R_{x} + (n-1)x} \times \begin{bmatrix} j_1 & k_1 & x \\ j_2 & k_2 & x \\ \vdots & \vdots & \vdots \\ j_{n-1} & k_{n-1} & x \\ j_n & k_n & x \end{bmatrix} \begin{bmatrix} j_1 & k_1 & x \\ j_2 & k_2 & x \\ \vdots & \vdots & \vdots \\ j_{n-1} & k_{n-1} & x \\ j_n & k_n & x \end{bmatrix}. \quad (20)$$

Note that the edges 2 and 5 are somehow irrelevant here. This means that from the tetrahedron tet$_1$ only the apex where (1, 34, 24) meet is interesting, while the base triangle formed by 2, 14 and 5 does not provide interesting information.
5.10^7 

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\begin{align*} R_n \overset{\text{def}}{=} \sum_{i=1}^{n} j_i + k_i + l_i. \end{align*}

In equation (20), we wrote the 3nj-symbol so that the coupled spin triads appear easily, namely \((j_1, l_1, j_2), (j_2, l_2, j_3), \ldots, (j_{n-1}, l_{n-1}, j_n), (j_n, l_n, k_1), (k_1, l_1, k_2), \ldots, (k_{n-1}, l_{n-1}, k_n), (k_n, l_n, j_1).\) Those couplings are also fully represented in the involved 6j-symbols.

Symmetries of the 3nj-symbols can be deduced from the symmetries of the 6j-symbols in the above decomposition [28]. It is convenient to distinguish two groups of spins which have different roles, \((j_1, \ldots, j_n, k_1, \ldots, k_n)\) and \((l_1, \ldots, l_n)\). Then, the 3nj-symbol is invariant under simultaneous circular permutations within those groups. It is also invariant under the exchange of the \(j\)th row with the \(k\)th row.

2.2. Hypotheses—applicability of the method

Almost all the angular momenta are large, and a few of them can be chosen not to scale. Our method applies when the following conditions are satisfied:

1. **one and only one** spin among \(j_1, \ldots, j_n, k_1, \ldots, k_n\) is small, every other small spin must be an \(l_i\);
2. we are **away from the caustic**, i.e. the volumes of the tetrahedra associated with the 6j-symbols in (20) which do only have large spins are far from zero;
3. the small \(l_i\) must be chosen so that in each 6j-symbol of the decomposition (20), there is **at most one** small spin.
The first two conditions ensure that the summed variable $x$ in (20) is of the order of the large spins, while its range is controlled by the small spin chosen among $(j_1, \ldots, j_n, k_1, \ldots, k_n)$. Precisely, if $j_1$ is small, then $k_1 - j_1 \leq x \leq k_1 + j_1$. The last restriction enables us to use standard asymptotic expressions of the 6$j$-symbol, the Ponzano–Regge and Edmonds’ formulae.

2.3. Asymptotics of the summand

Without loss of generality (thanks to the symmetries of the symbol) we choose $j_1$ to be small. Let also $\{l_m\}_{m \in \mathcal{E}}$ be small, for a set of integers $\mathcal{E} \subset \{2, \ldots, n-1\}$. We introduce the following half-integers which are small due to the triangular inequalities:

$$
\begin{align*}
\xi & \equiv x - k_1 \\
\mu & \equiv j_2 - l_1 \\
\nu & \equiv k_n - l_n
\end{align*}
$$

and

$$
\forall m \in \mathcal{E} \quad \begin{align*}
\eta_m & \equiv j_{m+1} - j_m \\
\kappa_m & \equiv k_{m+1} - k_m
\end{align*}
$$

where $\xi, \mu, \nu \in \{-j_1, \ldots, j_1\}$ and $\forall m \in \mathcal{E} \quad \eta_m, \kappa_m \in \{-l_m, \ldots, l_m\}$. We are now ready to use asymptotic formulae for each 6$j$-symbol involved in (20).

2.3.1. 6$j$-symbols with one small spin. Edmonds’ formula applies to the 6$j$-symbols with one small spin,

$$
\begin{align*}
\begin{cases}
 j_1 & k_1 \\ k_2 & j_2 \\ l_1 & l_1
\end{cases} & \approx \frac{(-1)^{j_1+j_2+k_1+k_2+l_1}}{d_{\xi} d_{\mu} d_{\nu}} d^{(j_1)}_{\mu \nu} (\phi_1), \\
\begin{cases}
 j_n & k_n \\ j_1 & k_1 \\ l_n & l_n
\end{cases} & \approx \frac{(-1)^{j_1+j_2+k_1+k_2+l_1}}{d_{\xi} d_{\mu} d_{\nu}} d^{(j_1)}_{\mu \nu} (\phi_n), \\
\begin{cases}
 j_m & k_m \\ k_{m+1} & j_{m+1} \\ l_m & l_m
\end{cases} & \approx \frac{(-1)^{j_m+k_m+l_m}}{d_{\xi} d_{\mu} d_{\nu}} d^{(j_m)}_{\mu \nu \kappa_m} (\phi_m).
\end{align*}
$$

The angles $\phi_1, \phi_n$ and $(\phi_m)_{m \in \mathcal{E}}$ are defined geometrically in figure 5. As we perform all the calculations to the leading order, we can neglect the variations of $\phi_m$ with $\xi$, namely we take $k_1$ instead of $k_1 + \xi$ for the construction of the triangle $(j_m, k_m, k_1 + \xi)$ in figure 5. Also note that $l_1 \approx j_2$ and $l_n \approx k_n$ because $j_1$ is small.

In fact, that explains only why we need at least one small spin among $(j_1, \ldots, j_n, k_1, \ldots, k_n)$. The reason why we must have at most one appears later. The main idea is to ensure that the sum over $x$ is a product of two Wigner matrices.
2.3.2. 6j-symbols with six large spins. Let \( P \) be the set of labels \( p \) corresponding to 6j-symbols with six large spins, namely \( \mathcal{P} \equiv \{ 2, \ldots, n-1 \} \setminus \varepsilon \). We apply the Ponzano–Regge formula to those symbols and use the same development of the Regge action as in equation (10) to get for all \( p \in \mathcal{P} \):

\[
\left\{ \begin{array}{ccc} j_p & k_p & x \\ k_{p+1} & f_{p+1} & l_p \end{array} \right\} \approx \frac{1}{\sqrt{12\pi V_p}} \cos \left( S_{R}^{(p)} + \xi \Theta^{(p)}_{k_1} + \frac{\pi}{4} \right),
\]

where \( V_p \) and \( S_{R}^{(p)} \) are, respectively, the volume and the Regge action of the tetrahedron \( \text{tet}_p \) depicted in figure 6. It is constructed in the usual way from the 6j-symbol on the left-hand side (25), for \( \xi = 0 \).

The Regge action \( S_{R}^{(p)} \) of that tetrahedron is explicitly given by

\[
S_{R}^{(p)} \equiv \sum_{e \subset \text{tet}_p} \left( j_e + \frac{1}{2} \right)/\Theta_{1}^{(p)}(e),
\]

where \( \Theta_{1}^{(p)}(e) \) is the external dihedral angle at the edge \( e \) in \( \text{tet}_p \).

2.4. Evaluation of the sum

Gathering the above pieces into the decomposition (20) of the 3nj-symbol, we obtain

\[
\{ 3nj \} \approx \frac{(-1)^{j(\varepsilon)}}{d_0 d_1} \prod_{\mu \in \varepsilon} \frac{d_{\text{me}(\phi_{\mu})}}{\sqrt{d_{\mu} d_{\kappa}}} \prod_{\mu \in \varepsilon} \frac{1}{\sqrt{12\pi V_p}} \cos \left( S_{R}^{(p)} + \xi \Theta^{(p)}_{k_1} + \frac{\pi}{4} \right),
\]

with \( M \equiv |\varepsilon| \) and \( r_{n}(\varepsilon) \equiv R_{n} + (n + M - 1) (k_{j_1} + j_1) + (\mu - j_1) + (k_{1} + k_{2} + l_{1}) + (k_{1} + j_{n} + l_{n}) + \sum_{m \in \varepsilon} j_{m} + l_{m} + k_{m+1} \). In practice, we expect that this asymptotic formula can be used numerically. Every angle which is involved has a precise definition in terms of the angular momenta \( \{ j_{i}, k_{i}, l_{i} \} \), and sums and products are finite.

But we can go one step further and recast the sum over \( \xi \) as a \( D \)-matrix product. That leads to a general asymptotic formula for 3nj-symbols, which we do not think is more powerful in terms of numerical computations, but which has a clearer geometric meaning.

As in the case of the 9j-symbol, we extract \( \xi \) from the argument of the cosines by writing each cosine as a sum of complex exponentials. Expanding the product of cosines and re-organizing its terms, we obtain the following combinatorial expression:
\[
\prod_{\mu,\nu} \cos \left( S_{R}^{(\mu)} + \xi \Theta_{k_{1}}^{(\mu)} + \frac{\pi}{4} \right)
\]
\[
= \frac{1}{2^{|P|}} \sum_{|\sigma|=\pm 1, \mu, \nu} \exp \left[ \sum_{\mu,\nu} \sigma_{\mu} \left( S_{R}^{(\mu)} + \frac{\pi}{4} \right) + \xi \sum_{\mu,\nu} \sigma_{\mu} \Theta_{k_{1}}^{(\mu)} \right] + c.c., \quad (27)
\]

where \( |P| \) is the number of Ponzano–Regge formulae which we have used. Note that in (27) we have used the symmetry \( (\sigma_{\mu} \rightarrow -\sigma_{\mu}) \) of the sum to make the complex conjugation explicit.

We perform the sum over \( \xi \) for each configuration \( |\sigma\rangle \) independently. Since \( j_{1} \) and \( \xi \) may not be integers, we write \( (-1)^{(n+M)(j_{1}-\xi)} = \exp i\pi(n+M)(j_{1} - \xi) \) and define the angle
\[
\omega_{k_{1}}^{(\sigma)} \overset{\text{def}}{=} (n+M)\pi - \sum_{\mu,\nu} \sigma_{\mu} \Theta_{k_{1}}^{(\mu)} \quad (\text{mod } 4\pi). \quad (28)
\]

The reason why it is defined only modulo \( 4\pi \) is that generically \( j_{1} \in \mathbb{N}/2 \). The sum over \( \xi \) reads
\[
\sum_{\xi=-j_{1}}^{j_{1}} d^{(j_{1})}_{\mu,\nu}(\phi_{1}) e^{-i\omega_{k_{1}}^{(\sigma)} \xi} d^{(j_{1})}_{\nu,\mu}(\phi_{n}) = D_{\mu,\nu}^{(j_{1})}(e^{-i\omega_{k_{1}}^{(\sigma)} \xi} e^{-i\omega_{k_{1}}^{(\sigma)} \xi} e^{-i\omega_{k_{1}}^{(\sigma)} \xi}). \quad (29)
\]

Like in the case of the \( 9j \)-symbol, the final step is to find the Euler angles of the \( SU(2) \) rotation on the right-hand side. They have a nice geometric picture as angles of a tetrahedron, but provided \( \omega \) ranges in an interval of size \( \pi \) (so that it can be a dihedral angle). Since \( \omega_{k_{1}}^{(\sigma)} \) is generically in \([-2\pi, 2\pi]\), we distinguish four cases.

1. **Case \( \omega_{k_{1}}^{(\sigma)} \in [0, \pi] \).** The formulae for the Euler angles show that \( \omega_{k_{1}}^{(\sigma)} \) has a natural interpretation as an *external* dihedral angle. So, we consider the tetrahedron \( tet_{\{\sigma\}} \) depicted in figure 7, defined by the gluing of the triangles \((k_{1}, k_{2}, l_{1})\) and \((j_{n}, l_{n}, k_{2})\) (which carry the angles \( \phi_{1} \) and \( \phi_{n} \), see figure 5) with the dihedral angle
\[
\theta_{k_{1}}^{(\sigma)} \overset{\text{def}}{=} \pi - \omega_{k_{1}}^{(\sigma)}. \quad (30)
\]

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tetrahedron.png}
\caption{Tetrahedron \( tet_{\{\sigma\}} \) associated with a sign configuration \( \{\sigma\} \). It is determined by the triangles \((k_{1}, k_{2}, l_{1})\) and \((j_{n}, l_{n}, k_{2})\), and by the angle \( \theta_{k_{1}}^{(\sigma)} \) between them, which in turn determine the length \( AB \).}
\end{figure}
\]
It is such that

$$D_{\mu\nu}^{(ij)}(e^{-i\phi_i\sigma_i} e^{-i(\tau-\delta_i^{(\mu)}\sigma_i}) e^{-i\phi_j\sigma_j}) = e^{-i\mu\delta_i^{(\mu)} d_{\mu\nu}^{(ij)}(\psi_{ji,\eta})} e^{-i\nu\delta_i^{(\nu)}}, \quad (31)$$

where the angles are shown in figure 7 and satisfy relations (3) and (17).

(2) Case $\alpha_{k1}^{(\sigma)} \in [\pi, -\pi]$. Then, we write

$$\theta_{k1}^{(\sigma)} \equiv -\pi - \alpha_{k1}^{(\sigma)}, \quad (32)$$

which is in $[0, \pi]$. Note that

$$e^{-i\pi\alpha_{k1}^{(\sigma)}} = e^{2\pi i \xi} e^{-i\pi\delta_i^{(\pi)} \xi} = e^{2\pi i \xi} e^{-i\pi\delta_i^{(\pi)} \xi}. \quad (33)$$

Hence, we are back to the case (1), and the final formula only differs by a phase $(-1)^{2\pi}$.

(3) Case $\alpha_{k1}^{(\sigma)} \in [-\pi, 0]$. We will have again the tetrahedron tet$_{[\sigma]}$ where the internal angle at $k_1$ is

$$\theta_{k1}^{(\sigma)} \equiv \pi + \alpha_{k1}^{(\sigma)}. \quad (34)$$

Using the symmetries of Wigner matrices, we find that the relevant matrix element of the SU(2) rotation is

$$D_{\mu\nu}^{(ij)}(e^{-i\phi_i\sigma_i} e^{-i(\tau-\delta_i^{(\mu)}\sigma_i}) e^{-i\phi_j\sigma_j}) = e^{i\pi \xi} D_{\nu\mu}^{(ij)}(e^{-i\phi_i\sigma_i} e^{-i(\tau-\delta_i^{(\nu)}\sigma_i}) e^{-i(\tau-\delta_i^{(\mu)}\sigma_i)}). \quad (35)$$

(4) Case $\alpha_{k1}^{(\sigma)} \in [\pi, 2\pi]$. The internal angle of tet$_{[\sigma]}$ is

$$\theta_{k1}^{(\sigma)} \equiv -\pi + \alpha_{k1}^{(\sigma)}. \quad (36)$$

Following the reasoning of (33), we conclude that it differs from the case (3) only by a factor $(-1)^{2\pi}$. 

2.5. Final asymptotics formula

The above cases fit into a not-so-complicated formula.

$$[3nj] \approx \frac{(-1)^{\varphi(\vec{\varphi})}}{2^p \sqrt{d_1 d_2}} \left[ \prod_{p \neq 0} \frac{1}{\sqrt{12\pi V_p}} \frac{d_{\mu\nu}^{(1n)}}{d_{\mu\nu}^{(1n)}} (\varphi_{ij}) \right] \times \sum_{\{\sigma_p = \pm 1\}} \cos \left[ \sum_{p \neq \varphi} \sigma_p \left( S_{\varphi}^{(p)} + \frac{\pi}{4} \right) + \pi (n + M) j_1 + f_{ij}^{(\sigma)} \right] \frac{d_{\mu\nu}^{(ij)}}{d_{\mu\nu}^{(ij)}} (\psi_{ji,\eta}). \quad (37)$$

- $\varphi$ is the set of tetrahedra on which the Ponzano–Regge formula has been applied, depicted in figure 6, and $P = |\varphi|$. $\varphi$ is the subset of $\{2, \ldots, n-1\}$ corresponding to the small spins $l_m$, with $M = |\varphi|$. The global sign is given by $r_{\sigma}(\vec{\varphi}) = R_n + (n + M - 1) (k_1 + j_1) + (\mu - j_1) + (k_1 + k_2 + l_1) + (k_1 + j_n + l_n) + \sum_{m \in \varphi} j_m + l_m + k_{m+1}$. 
- $\mu = j_2 - l_1$ and $\nu = k_n - l_n$ are small of the order of magnitude of $j_1$. Also, $k_m = k_{m+1} - k_m$, $l_m = j_m + 1 - j_m$ are of order $l_m$ for $m \in \varphi$. 
- $V_p$ and $S_{\varphi}^{(p)}$ are the volumes and Regge actions of the tetrahedra tet$_p$, depicted in figure 6 and which have only large spins.

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To each sign configuration $[\sigma]$, we have assigned a tetrahedron $\text{tet}_{[\sigma]}$, figure 7, defined by the gluing of the two triangles along $k_1$ with a dihedral angle $\theta_{k_1}^{[\sigma]}$, given in terms of $\omega_{k_1}^{[\sigma]} = (n + M)\pi - \sum_{\rho \in \sigma} \phi_{\rho, k_1}^{[\sigma]}$ in the above subsection. The function $f_{\mu \nu}^{[\sigma]}$ depends on the value of $\omega_{k_1}^{[\sigma]}$ as follows:

$$f_{\mu \nu}^{[\sigma]} = \begin{cases} 
-\left(\mu \theta_{k_1}^{[\sigma]} + \nu \phi_{k_1}^{[\sigma]}\right) + 2\pi j_1 & \text{if } \omega_{k_1}^{[\sigma]} \in [-2\pi, -\pi], \\
\left(\mu \theta_{k_1}^{[\sigma]} + \nu \phi_{k_1}^{[\sigma]}\right) & \text{if } \omega_{k_1}^{[\sigma]} \in [-\pi, 0], \\
-\left(\mu \theta_{k_1}^{[\sigma]} + \nu \phi_{k_1}^{[\sigma]}\right) & \text{if } \omega_{k_1}^{[\sigma]} \in [0, \pi], \\
\left(\mu \theta_{k_1}^{[\sigma]} + \nu \phi_{k_1}^{[\sigma]}\right) + 2\pi j_1 & \text{if } \omega_{k_1}^{[\sigma]} \in [\pi, 2\pi]. 
\end{cases}$$ (38)

The angles $\phi_{l_1, l_2}^{[\sigma]}$, $\theta_{l_1}^{[\sigma]}$ and $\theta_{l_2}^{[\sigma]}$ are evaluated from the angles $\phi_1$, $\phi_2$, and $\theta_{l_1}^{[\sigma]}$ following figure 7.

In our final result, there is a remaining sum over sign assignments $[\sigma]$. It is of combinatorial nature, and the initial sum over the intermediate spins $x$ has been fully performed. Note that the combinatorial sum contains a priori $2^p$ terms, but only $2^{p-1}$ are actually different. This sum assigns different frequencies to the oscillations, since in particular it sums over the Regge actions of individual tetrahedra with all possible signs. That phenomenon has also been observed in [26], where the authors looked at the asymptotics of the Ponzano–Regge model for 3d gravity on handlebodies and found a sum over ‘immersions’ with different frequencies like here.

3. Examples: asymptotics of 15j-symbols with small and large angular momenta

A simple example is obviously the 9j-symbol with a small spin, treated in the first section. It corresponds to the case where there is a single sum over $\sigma = \pm 1$ and $\omega_{k_1}^{(+1)} \in [-2\pi, -\pi]$.

We now derive an asymptotic formula for the 15j-symbol with three small angular momenta, producing an alternative to the formula of [10], and formulae with one, two, three and four small angular momenta. The latter are new to our knowledge. In particular, though the formulae derived in [10] look similar, they apply to different, non-equivalent choices of the small spins.

Using the notation of [29], we have

$$\begin{aligned}
\left\{ j_1, l_1, j_2, l_2, j_3, l_3, j_4, l_4, j_5, l_5 \right\} &= \sum_x d_x (-1)^{R_5} \left\{ j_1, k_2, k_1, j_2, l_1 \right\} \\
& \times \left\{ j_2, k_2, x, j_3, k_3, x, j_4, k_4, x, j_5, k_5, x, j_1, k_1, l_1 \right\},
\end{aligned}$$ (39)

where $R_5 = \sum_{i=1}^5 k_i + l_i$. In what follows, we always assume that $j_1$ is small. Thus, every other small spin must be chosen among $l_2$, $l_3$, and $l_4$ (see subsection 2.2).

3.1. Four small angular momenta

Assume that $j_1$, $l_2$, $l_3$, and $l_4$ are small. According to the notations of this section, we have

$$\mathcal{E} = \{2, 3, 4\} \quad \text{and} \quad \mathcal{F} = \emptyset.$$ (40)

It is easy to see from the derivation of (37) that if $\mathcal{F} = \emptyset$ then there is no sum over combinatorial sign assignments, $[\sigma] = \emptyset$. The corresponding tetrahedron $\text{tet}_{[\sigma]}$ is flattened to a triangle with $\phi_{l_1, l_2} = \phi_1 + \phi_5$. The reason is that in the decomposition (39), all 6j-symbols have
one small spin, so that no Ponzano–Regge formula gets involved. Consequently, there are no oscillations with some Regge action. Applying directly the general formula leads to

\[
15\, j = \frac{(-1)^{j_1 + j_2}}{\sqrt{d_{j_1} d_{j_2} d_{j_3} d_{j_4}}} d^{(l_{j_1})}_{\kappa \eta_1} (\phi_2) d^{(l_{j_2})}_{\kappa \eta_2} (\phi_3) d^{(l_{j_3})}_{\kappa \eta_3} (\phi_4) d^{(l_{j_4})}_{\kappa \eta_4} (\phi_5),
\]

with \(\kappa = k_{m+1} - k_m, \eta = j_m - j_m, \mu = j_2 - j_1\) and \(\nu = k_5 - k_5\).

That expression can be simplified. Since \(l_2, l_3\) and \(l_4\) are small, we have \(j_2 \approx j_3 \approx j_4 \approx j_5\) and \(k_2 \approx k_3 \approx k_4 \approx k_5\). Thus, the triangles \((j_m, k_m, l_1, j_1, j_2, k_2)\) and \((k_1, j_2, k_2)\) defined in figure 5 are identical. Consequently, we have \(\phi_2 \approx \phi_3 \approx \phi_4 \approx \pi - \phi_1 - \phi_5\). The last (approximate) equality is simply due to the fact that the sum of the angles in a triangle is equal to \(\pi\). The situation is illustrated in figure 8. This finally leads to

\[
15\, j \approx \frac{1}{d_{j_1}^2 d_{j_2}^2} d^{(l_{j_1})}_{\kappa \eta_1} (\phi_2) d^{(l_{j_2})}_{\kappa \eta_2} (\phi_2) d^{(l_{j_3})}_{\kappa \eta_3} (\phi_2) d^{(l_{j_4})}_{\kappa \eta_4} (\phi_2).
\]

In [10], Yu gives an expression for the asymptotics of the 15j-symbol with four small spins chosen among the five \(l_i\). Although this configuration is different from ours, both formulæ look quite similar.

3.1.1. Three small angular momenta. We assume that \(j_1, l_2\) and \(l_3\) are small. The set of 6j-symbols on which we apply the Edmonds’ and Ponzano–Regge formulæ are

\(\varepsilon = \{2, 3\}\) and \(\beta = \{4\}\).

There are two sign configurations, \(\sigma_4 = \pm\), that we denote, respectively, \((+)\) and \((-)\). Their contributions in the combinatorial sum are equal, so we consider only the \((+\) situation. The formula, once simplified, is

\[
15\, j \approx \frac{(-1)^{j_1 + j_2 + j_3}}{d_{j_1} d_{j_2}} 12\pi V_4 \sqrt{d_{j_3} d_{j_4}} d^{(l_{j_3})}_{\kappa \eta_3} (\phi_2) d^{(l_{j_4})}_{\kappa \eta_4} (\phi_2) d^{(l_{j_5})}_{\kappa \eta_5} (\phi_2) \cos \left( S^{(4)}_R \frac{\pi}{4} - \mu \theta^{(+)}_{\mu} - \nu \theta^{(+)}_{\nu} + j_1 \pi \right),
\]

where \(\phi_2 \approx \phi_3\) is defined as usual; \(V_4\) and \(S^{(4)}_R\) are the volume and the Regge action of \(\text{tet}_4\), given in figure 9(a); and the angles \(\theta^{(+)}_{\mu}, \theta^{(+)}_{\nu}\) and \(\theta^{(+)}_{\kappa}\) belong to \(\text{tet}_{(+)}\), represented in figure 9(b). The latter is built by gluing the triangles \((k_1, l_1, k_2), (k_1, j_2, k_2)\) with the dihedral angle \(\theta^{(+)}_{\mu} \approx \pi - \theta^{(+)}_{k_1}\), i.e. the external dihedral angle of \(\text{tet}_4\).

The attentive reader may have noted that the tetrahedron \(\text{tet}_{(+)}\) includes the triangle \((j_4, k_4, k_1)\), whereas for the general case (\(\text{tet}_{(0)}\), in figure 7), we used \((l_1, k_2, k_1)\) instead. Both are actually equivalent since \(j_4 \approx j_3 \approx j_2\) and \(k_4 \approx k_3 \approx k_2\). Here, we have used that triangle to make contact with \(\text{tet}_4\).
The tetrahedron $\text{tet}_{(+) \downarrow}$ is built out of the tetrahedron $\text{tet}_4$ in exactly the same way the second tetrahedron (named $\text{tet}_2$) was built from the first tetrahedron in the case of the $9j$-symbol with one small spin. Indeed, one flips one of the two triangles which share $k_1$, and set the external angle $\theta_{k_1}$ as the new internal angle. The underlying reason is that for the $9j$-symbol with one small spin, just like for the $15j$-symbol with three small spins and for any $3nj$-symbol with $(n - 2)$ small spins (chosen according to the hypotheses of section 2.2), one makes use of a single Ponzano–Regge asymptotics formula. Therefore only one tetrahedron with large angular momenta is involved with the dihedral angle $\theta_{k_1}$ at $k_1$ and $\omega_{k_1} = 2\pi(n - 2) + \theta_{k_1}$. Studying the range of $\omega_{k_1}$, one finds that the internal angle at $k_1$ in $\text{tet}_{(+) \downarrow}$ is always $\theta_{k_1} + \pi - \theta_{k_1}$, i.e. the external angle of the reference tetrahedron. This means that the second tetrahedron is just built out of the first by the process we described.

In [10], Yu has obtained an asymptotic formula for the $15j$-symbol with $j_1, l_3$ and $l_4$ being small. In that case, the tetrahedron of reference on which we have to apply the Ponzano–Regge formula is $\text{tet}_{p=2}$ from figure 6. However, Yu used a slightly different tetrahedron, which makes the formula a bit more complicated in the sense that one cannot express all the needed angles as angles from two tetrahedra only. Consequently, the relevant Regge action is not the same as ours. Once the difference between the Regge actions is taken into account, it is possible to go from our formula to his and to recover the geometric interpretation of his variables.

### 3.1.2. Two small angular momenta

We assume that $j_1$ and $l_2$ are small. The set of $6j$-symbols on which we apply the Edmonds’ and Ponzano–Regge formulae are

$$\mathcal{E} = \{2\} \quad \text{and} \quad \mathcal{P} = \{3, 4\}. \quad (45)$$

There are $4^2 = 4$ sign configurations, $(\sigma_1, \sigma_4) = (++, (+-), (-+), (--)$. Thanks to the symmetry of the combinatorial sum, we can consider only the cases $++$.

We now have two tetrahedra, tet3 and tet4, with large angular momenta as in figure 6 for $p = 3$ and 4. To avoid distinguishing behaviors depending on their dihedral angles, we assume...
that they are nearly regular, so that their angles are close to $\theta_{\text{reg}} = \arccos 1/3$. Also, we assume without loss of generality that $\theta^{(3)}_{k_3} - \theta^{(4)}_{k_3} \geq 0$.

The asymptotic formula is then

$$[15\, j] \approx \frac{(-1)^{j_3 + j_4 + k_3 + k_4 + k_5 + j_5 + j_6 + 2\kappa_3}}{24\pi d_1 \sqrt{d_2 d_3 d_4 V_3 V_4}} \frac{d_{k_3 k_5}}{d_{k_4 k_5}}(\varphi_2) \left[ -d_{ij}^{(j_3)}(\varphi_j^{(\mp)}) \sin (S_{R}^{(3)} + S_{R}^{(4)} - \mu \theta^{(+)}_{j_3}) - \nu \theta^{(+)}_{k_3} + (-1)^{j_3}d_{ij}^{(j_3)}(\varphi_j^{(-)}) \cos (S_{R}^{(3)} - S_{R}^{(4)} - \mu \theta^{(-)}_{j_3} - \nu \theta^{(-)}_{k_3}) \right].$$

The volumes $V_3$ and $V_4$ and Regge actions $S_{R}^{(3)}$ and $S_{R}^{(4)}$ are associated with the tetrahedra $\text{tet}_3$ and $\text{tet}_4$, see figure 6. The angles $\varphi_j^{(+)}(\varphi_j^{(-)})$, $\theta_j^{(\pm)}$ are defined in figure 10, which illustrates $\text{tet}_{j}$.

Let us explain how to get the secondary tetrahedra $\text{tet}_{j}$. Note that $\text{tet}_{j}$ and $\text{tet}_{k}$ have a common triangle, $(k_1, j_3, k_4)$. Hence they can be glued together, in two different ways, either from the outside or one inside the other. We remove the common triangle $(k_1, j_3, k_4)$ so that the angle at $k_3$ between $(k_1, j_3, k_3)$ and $(k_3, j_5, k_5)$ is either $\theta^{(3)}_{k_3} + \theta^{(4)}_{k_3}$ or $\theta^{(3)}_{k_3} - \theta^{(4)}_{k_3}$.

The generic study tells us to consider the angle $\omega^{(\pm)}_{k_3} \equiv 6\pi - \Theta^{(3)}_{k_3} \mp \Theta^{(4)}_{k_3}$. One obtains

$$\omega^{(\pm)}_{k_3} = \theta^{(3)}_{k_3} \pm \theta^{(4)}_{k_3} - (1 \mp 1)\pi \quad (\text{mod } 4\pi),$$

which implies $\omega^{(+)}_{k_3} \in [0, \pi]$ and $\omega^{(-)}_{k_3} \in [-2\pi, -\pi]$. As a conclusion, the tetrahedra $\text{tet}_{j}^{(+\mp)}$ are built by flipping the triangle $(k_1, j_3, k_3)$ so that $k_5$ meet $j_3$, while $j_5$ meet $k_5$. Then, from those five lengths, we get a tetrahedron by setting as the new dihedral angle at $k_1$

$$\theta^{(\pm)}_{k_3} = \pi - (\theta^{(3)}_{k_3} \pm \theta^{(4)}_{k_3}),$$

like in figure 10.

In [10], a formula for the $15\, j$-symbol with two small angular momenta is given, which are $l_3$ and $l_4$. That configuration is not the same as here, but the formulae look quite similar. A notable difference is the denominator of the amplitude which is here the product of the volumes of the two relevant tetrahedra, while it is more complicated in [10].

Figure 10. Tetrahedra $\text{tet}_{j}^{(+\pm)}$ and $\text{tet}_{j}^{(-\pm)}$ which are defined by the angles $\phi_1$, $\phi_5$ and $\theta^{(\pm)}_{j_3}$, $\theta^{(\pm)}_{j_3} = \pi - (\theta^{(3)}_{k_3} \pm \theta^{(4)}_{k_3})$. They define in turn the angles $\varphi^{(3)}_{j_1 k_3}$, $\varphi^{(4)}_{j_3}$ and $\varphi^{(3)}_{j_5 k_5}$. They define in turn the angles $\varphi^{(\pm)}_{j_1 k_3}$, $\varphi^{(\pm)}_{j_3}$ and $\varphi^{(\pm)}_{j_5 k_5}$.
3.1.3. One small angular momentum. We assume that only $j_1$ is small. The set of 6 $j$-symbols on which we apply the Edmonds’ and Ponzano–Regge formulae are
\[ \varpi = \emptyset \quad \text{and} \quad \mathcal{P} = \{2, 3, 4\}. \] (49)

There are $2^3 = 8$ sign configurations ($\sigma_2, \sigma_3, \sigma_4$). Thanks to the symmetry of the combinatorial sum, we can consider only $(+++)$, $(+-+)$, $(+-+)$, and $(+-+)$. For simplicity, we assume that the three tetrahedra associated with $\mathcal{P}$ are almost regular. We obtain
\[ \{15j\} \approx \frac{(-1)^{j_1+j_2+j_3+j_4+k_1+k_2+k_3+k_4}}{48\pi \sqrt{12} d_{j_5} d_{j_6} V_5 V_4} \left[ d^{(jj)}_{\mu\nu} \left( \phi_{j_1 j_2 j_5}^{++} \right) \cos \left( \frac{S_{R}^{(2)} + S_{R}^{(3)} - S_{R}^{(4)}}{\pi} - \frac{\mu j_2}{4} - \mu \theta_{j_2}^{(++)} + \pi j_1 \right) \right. \]
\[ + d^{(jj)}_{\mu\nu} \left( \phi_{j_2 j_3 j_6}^{++} \right) \cos \left( \frac{S_{R}^{(2)} - S_{R}^{(3)} + S_{R}^{(4)}}{\pi} - \frac{\mu j_2}{4} - \mu \theta_{j_2}^{(++)} + \pi j_1 \right) \]
\[ + d^{(jj)}_{\mu\nu} \left( \phi_{j_3 j_4 j_5}^{++} \right) \cos \left( \frac{S_{R}^{(2)} + S_{R}^{(3)} + S_{R}^{(4)}}{\pi} + \frac{3\pi}{4} + \mu \theta_{j_2}^{(++)} + \pi j_1 \right) \]
\[ \left. + d^{(jj)}_{\mu\nu} \left( \phi_{j_4 j_5 j_6}^{++} \right) \right]. \] (50)

The volumes $V_5$, $V_4$ and $V_5$ and Regge actions $S_{R}^{(2)}$, $S_{R}^{(3)}$ and $S_{R}^{(4)}$ are associated with the tetrahedra $\text{tet}_1$, $\text{tet}_2$ and $\text{tet}_3$, respectively, see figure 6 for $p = 2, 3$ and $4$. The angles $\theta_{j_1 j_2 j_5}^{(++)}$ and $\theta_{j_4 j_5 j_6}^{(++)}$ belong to the tetrahedra $\text{tet}(4\pm\pm)$. They are defined by the five spins $k_1, j_2, j_3, j_4$ and $k_5$, which form two triangles glued along $k_1$ like in figure 11. To complete their definition, we set the dihedral angle at $k_1$ to be
\[ \theta_{k_1}^{(++)} = \theta_{k_1}^{(2)} + \theta_{k_1}^{(3)} + \theta_{k_1}^{(4)} - \pi, \]
\[ \theta_{k_1}^{(+-+)} = \pi - \theta_{k_1}^{(2)} - \theta_{k_1}^{(3)} + \theta_{k_1}^{(4)}, \]
\[ \theta_{k_1}^{(++)} = \pi - \theta_{k_1}^{(2)} + \theta_{k_1}^{(3)} - \theta_{k_1}^{(4)}, \]
\[ \theta_{k_1}^{(+-+)} = \pi - \theta_{k_1}^{(2)} - \theta_{k_1}^{(3)} - \theta_{k_1}^{(4)}. \] (51)
Note that since tet$_2$, tet$_3$ and tet$_4$ are close to being regular the above angle indeed lies in [0, π].

The intuitive way of building those tetrahedra is by first gluing tet$_2$ and tet$_3$ along their common triangle $(k_1, j_3, k_3)$. There are two different ways, either on the outside or one inside the other. Then, one adds tet$_4$ by gluing it along the triangle $(k_1, j_4, k_4)$, and there are here again two different ways to do so. Finally, we focus on the triangles $(k_1, j_2, k_2)$ and $(k_1, j_5, k_5)$ which can be completed to a tetrahedron, and we consider the tetrahedron which complements that one when drawing the parallelogram spanned by $(j_2, k_2)$. The final internal angle is given by (51), depending on the chosen way to glue the three initial tetrahedra. As an example, we give tet$_{1→→→}$ in figure 11.

4. Conclusion

We have shown that the result of [8] can be recovered by the direct application of the Ponzano–Regge formula (together with Edmonds’ formula), making clear the way the asymptotic information is encoded into the geometry of a tetrahedron. Moreover, the conditions of applicability of our method have been given and we derived new, explicit formulae for asymptotics of arbitrary Wigner symbols with some large and small angular momenta. Our method is simpler than that of [8], as far as Wigner symbols are concerned, and provides us with complementary results.

Some directions for future research are proposed in the conclusion of [8] and are definitely of interest.

We would like to mention in addition that beyond the asymptotic formulae for specific symbols, the regime where some spins remain small should deserve attention, because the geometric content is simply contained in tetrahedra. In particular, it may be investigated in terms of recurrence relations. As argued in [30], the latter provides a preferred way to encode the geometric properties of classical spin networks. It was further shown in [22], in the case of the 6j-symbol, that it is possible to derive those recurrences as quantum constraints (Wheeler–DeWitt equations) coming from three-dimensional gravity. The same result holds in the four-dimensional context [27] using a Hamiltonian from topological field theory. Hence, it would be interesting to look at the asymptotics of those relations with small and large spins, and to look for a similar regime where the asymptotic geometry is described in terms of 4-simplexes instead of tetrahedra.

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