A time-discretized version of the Calogero-Moser model

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Abstract

We introduce an integrable time-discretized version of the classical Calogero-Moser model, which goes to the original model in a continuum limit. This discrete model is obtained from pole solutions of a discretized version of the Kadomtsev-Petviashvili equation, leading to a finite-dimensional symplectic mapping. Lax pair, symplectic structure and sufficient set of invariants of the discrete Calogero-Moser model are constructed. The classical $r$-matrix is the same as for the continuum model.
1. Introduction. The original Calogero-Sutherland-Moser model, \[1, 2, 3\], is a one-dimensional many-body system with pairwise inverse square interactions, and it is an integrable model at both classical and quantum levels, cf. the two review papers, \[4\], for a comprehensive account of both cases. (We will refer to the classical model as the Calogero-Moser (CM) model). This long-range interaction model has aroused renewed interest most recently in view of its close relations to the Haldane-Shastry SU(N) chain that has been studied in \[5\]. Intriguingly, there is strong evidence that this integrable model plays an important role in understanding the universal behaviour in quantum chaos and mesoscopic physics, \[6\]. Furthermore, rich algebraic structures, such as a dynamical $r$-matrix, have been found in this model in \[7\]; cf. also \[8\].

In this letter we introduce a time-discretized version of the classical CM model. Discrete-time systems are a subject of intensive research recently, (cf. e.g. \[9, 10\] for reviews), and they are, in our belief, important in understanding the true nature of integrability. Most known integrable systems, such as integrable partial differential equations (Korteweg-de-Vries equation, Kadomtsev-Petviashvili equation, etc.) or ordinary differential equations (the Painlevé transcendents and elliptic function equations), possess, as it has turned out to be the case, one or more integrable discrete counterparts which lead to the original model in a well-defined continuum limit. Finding such integrable discrete versions, however, is in general a highly non-trivial undertaking. In the present paper, we propose a new discrete integrable system, namely an exact discrete-time version of the Calogero-Moser system.

The method used here to obtain the discrete-time CM model is based on the observation of \[11\]; cf. also \[12\], that the dynamics of the poles of special solutions of integrable nonlinear evolution equations is connected with integrable systems of particles on the line. The connection between the pole solutions of the (continuum) Kadomtsev-Petviashvili (KP) equation and the CM system was found by Krichever in \[13\], cf. also \[14\]. Here we will perform a similar construction for the discrete
case, and we will show that the pole solutions of a semi-discretized version of the KP equation is connected with a time-discretized version of the CM model.

2. Discrete-time CM model. The semi-discretized version of the KP equation that we will use reads

\[(p - q + \hat{u} - \overline{\pi})y = (p - q + \hat{u} - \overline{\pi})(u + \hat{\pi} - \hat{u} - \overline{\pi}),\]

where \(p\) and \(q\) are two (lattice) parameters, \(u\) is the (classical) field, the \(\overline{\cdot}\) denotes the discrete time-shift corresponding to a translation in the “time” direction while the \(\hat{\cdot}\) denotes a shift or translation in the “spatial” direction. In (1) the time and spatial variables are discrete while the third variable \(y\) is continuous. Eq. (1) is obtained from the fully-discretized version of the KP equation of [15] by letting one of three lattice parameters tend to zero.

It can be easily proved that (1) is the compatibility condition of the following two equations

\[\overline{\phi} = \phi_y + (p + u - \overline{\pi})\phi,\]  
\[\hat{\phi} = \phi_y + (q + u - \hat{u})\phi,\]

which is the Lax representation of (1), and the starting point for the investigation of the integrability characteristics of the semi-discrete KP.

Direct calculation shows that

\[u = \frac{1}{x}\] with \(x = \frac{n}{p} + \frac{m}{q} + y\)

is a (pole) solution to the semi-discretized version of KP equation (1). Here \(n\) and \(m\) are two integers, and \((n, m)\) are considered as the coordinates of a two-dimensional lattice and

\[\overline{x} = \frac{n + 1}{p} + \frac{m}{q} + y, \quad \hat{x} = \frac{n}{p} + \frac{m + 1}{q} + y.\]

If we substitute this special solution (3) into the \(u\)’s in (2), then we find that

\[\phi = (1 - \frac{1}{k x})(p + k)^n(q + k)^m e^{x(ky)}\]
satisfies (2), where $k$ is a spectral parameter.

Enlightened by the above simple exercise, we now suppose \( u = \sum_{i=1}^{N} \frac{1}{y - x_i} \), \( \phi = (1 - \frac{1}{k} \sum_{i=1}^{N} \frac{b_i}{y - x_i}) (p + k)^n (q + k)^m \exp(ky) \), \( \) \hspace{1cm} (6a) \hspace{1cm} (6b) \]

where $x_i$ and $b_i$ are independent of $y$ (but they depend on the time variable), and we are to find the conditions these $x_i$ and $b_i$ should satisfy such that equation (2a) is valid. Substituting (6) into (2a) and equating to zero the coefficients of $(y - x_i)^{-1}$ and $(y - x_i)^{-1}$, we obtain the following equations:

\[ (p + k)b_i = k + \sum_{j=1}^{N} \frac{b_j}{x_i - x_j} - \sum_{j=1, j \neq i}^{N} \frac{b_i + b_j}{x_i - x_j}, \] \hspace{1cm} (7a) \hspace{1cm} \( (p + k)b_i = k - \sum_{j=1}^{N} \frac{b_j}{x_i - x_j}, \) \hspace{1cm} (7b) \]

for all $i = 1, 2, ..., N$, where $\overline{b_i}$ and $\overline{x_i}$ denotes the discrete time-shift as stated before. If these conditions (7) are satisfied, then the $u$ and $\phi$ given by (6) satisfy (2a).

Now, by introducing the vectors $B = (b_1, b_2, ..., b_N)^T$ and $E = (1, 1, 1, ..., 1)^T$ and the matrices

\[ L_{ij} = \left( \sum_{l=1}^{N} \frac{1}{x_i - x_l} - \sum_{l=1, l \neq i}^{N} \frac{1}{x_i - x_l} \right) \delta_{ij} - \frac{1}{x_i - x_j} (1 - \delta_{ij}) \], \hspace{1cm} (8a) \hspace{1cm} \[ M_{ij} = \frac{1}{x_i - x_l} \delta_{ij} - \frac{1}{x_i - x_j} (1 - \delta_{ij}) \], \hspace{1cm} (8b) \]

we can rewrite (7) in the following form

\[ (p + k)B = kE + LB, \] \hspace{1cm} (9a) \hspace{1cm} \[ (p + k)\overline{B} = kE + MB, \] \hspace{1cm} (9b) \]

We note here that if we take the more restrictive Ansatz $\phi = \frac{1}{k} (\sum_{i=1}^{N} \frac{b_i}{y - x_i}) \exp(ky)$ instead of that given in (6) we still get the same result, i.e., eqs. (11) and (13) below.
and the compatibility of (9) leads to the equation
\[(\bar{L}M - ML)B + k(\bar{L} - M)E = 0.\]  (10)

(10) is a discrete non-homogeneous Lax’s equation. It can be readily checked that the resulting equations of (10), i.e.
\[\bar{L}M = ML\]  (11)
and
\[(\bar{L} - M)E = 0,\]  (12)
are consistent and give the same discrete equations of motion of a \(N\)-particle system:
\[\frac{1}{x_i - \bar{x}_i} + \frac{1}{x_i - \bar{x}_i} + \sum_{j=1}^{N} \left( \frac{1}{x_i - \bar{x}_j} + \frac{1}{x_i - x_j} - 2 \frac{1}{x_i - x_j} \right) = 0,\]
\[i = 1, 2, ..., N,\]  (13)
where \(x_i\) denotes the discrete time-shift in the opposite direction to the one of \(\bar{x}_i\).

We will call the model, for which eq. (13) are the equations of motion, the discrete-time CM model. We will show below that in a continuum limit these equations go to that of the original CM model.

The Lax pair for this discrete-time CM model are given by \(L\) and \(M\) of (8). From eq. (11), it can be readily seen that
\[\bar{I}_k = I_k\]  (14)
for any \(k = 1, 2, ...,\), where
\[I_k \equiv Tr(L^k),\]  (15)
leading to a sufficient number of invariants (or conservation laws) of the discrete-time flow given by (13).
3. **Symplectic Structure.** In order to establish the exact integrability of the discrete CM model (13), we need first to establish an appropriate symplectic structure for the N-particle system. The discrete-time flow will then have the interpretation of the iterate of a canonical transformation with respect to that symplectic structure.

To get the generating function of this canonical transformation, we will follow the point of view of ref. [16]. We start by noting that eq. (13) can actually be obtained from the variation of a discrete action, given by

\[
S = \sum_n \mathcal{L}(x, \bar{x}) = \sum_n \left( \sum_{i,j=1}^{N} \log |x_i - x_j| - \sum_{i,j=1}^{N} \log |x_i - x_j| \right), \tag{16}
\]

in which the sum over \( n \) denotes the sum over all discrete-time iterates. The discrete Euler-Lagrange equations

\[
\frac{\partial L}{\partial x_i} + \frac{\partial L}{\partial \dot{x}_i} = 0, \quad i = 1, 2, \ldots, N, \tag{17}
\]

yield the equations of motion (13).

The form of the ‘kinetic’ term in our action (16) invites to perform a Legendre transformation in the following form

\[
\mathcal{H}(\mathbf{p}, x) = \sum_{i,j=1}^{N} \mathbf{p}_{ij}(x_i - x_j) - \mathcal{L}(x, \bar{x}), \tag{18}
\]

from which we obtain

\[
\sum_{i=1}^{N} \mathbf{p}_{ij} = -\frac{\partial L}{\partial \dot{x}_j}, \tag{19}
\]

and

\[
\frac{\partial \mathcal{H}}{\partial x_i} = \sum_{j=1}^{N} (\mathbf{p}_{ij} - \mathbf{p}_{ji}), \tag{20a}
\]

\[
\frac{\partial \mathcal{H}}{\partial \mathbf{p}_{ij}} = x_i - x_j. \tag{20b}
\]

Eqs. (20a) and (20b) can be interpreted as discrete Hamilton equations, but, of course, \( \mathcal{H} \) is not a Hamiltonian in the usual sense of the word. \( \mathcal{H} \) is the generating
function of the canonical transformation \( x \mapsto \mathbf{x}, p \mapsto \mathbf{p} \). This transformation, as a consequence of eqs. \((20)\), will leave the following symplectic form invariant:

\[
\Omega = \sum_{i,j=1}^{N} dp_{ij} \wedge dx_{j},
\]

for which we have \( \Omega = \Omega \).

Eqs. \((18)-(21)\) are still general. Coming back to the special case of the action \((16)\) we will find from \((19)\) that

\[
\frac{1}{x_{i} - x_{j}}
\]

and from \((18)\) we find \( \mathcal{H} \):

\[
\mathcal{H}(\mathbf{p}, x) = \sum_{i,j=1}^{N} \log |p_{ij}| + \sum_{i,j=1}^{N} \log |x_{i} - x_{j}|.
\]

The symplectic structure \((21)\) leads to the following Poisson brackets

\[
\sum_{i=1}^{N} \{p_{ij}, x_{k}\} = \delta_{jk}.
\]

4. Classical r-matrix. In order to prove the complete integrability of the discrete-time CM model, we now construct its r-matrix structure. From the above discussion it follows that we can make the following choice of canonical variables \((p_{i}, x_{i})\) of the \(i\)th particle

\[
p_{i} = \sum_{j=1}^{N} p_{ji} + \sum_{j \neq i}^{N} \frac{1}{x_{i} - x_{j}},
\]

leading to the standard Poisson brackets

\[
\{p_{i}, x_{j}\} = \delta_{ij}, \quad \{p_{i}, p_{j}\} = \{x_{i}, x_{j}\} = 0.
\]

In terms of these canonical variables, the Lax matrix \(L\) can be written as:

\[
L = \sum_{i=1}^{N} p_{i}e_{ii} - \sum_{i,j=1}^{N} e_{ij} \frac{x_{i} - x_{j}}{x_{i} - x_{j}},
\]
where the elements of the matrix \( e_{ij} \) are defined as \((e_{ij})_{kl} = \delta_{ik}\delta_{jl}\). Eq. (27) is the usual \( L \)-matrix for the CM model, and thus we can immediately use the result of \([4]\) in order to establish the involutivity of the invariants \([13]\). In fact, using (26) and the expression for \( L \), we can calculate the fundamental Poisson bracket structure in terms of the matrices \( L \), leading to the well-known result

\[
\{L \otimes L\} = [r_{12}, L \otimes 1] - [r_{21}, 1 \otimes L],
\]

where \( \otimes \) is the tensor product, \([\cdot, \cdot] \) denotes the usual commutator and

\[
\{L \otimes L\} \equiv \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} \{L_{ij}, L_{kl}\} e_{ij} \otimes e_{kl}.
\]

The \( r \)-matrix in (28) is given by, \([7]\),

\[
r_{12} = \sum_{i,j=1}^{N} \frac{1}{x_j - x_i} e_{ij} \otimes e_{ji} + \frac{1}{2} \sum_{i,j=1}^{N} \frac{1}{x_j - x_i} e_{ii} \otimes (e_{ij} - e_{ji}),
\]

obeying \( r_{21} = Pr_{12}P \), where \( P \) is the permutation matrix: \( Px \otimes yP = y \otimes x \).

Thus, in going from the continuous to the discrete CM model, both the \( L \)-operator as well as the \( r \)-matrix remain the same. What changes is the \( M \)-matrix, which will now depend on the original momentum variables \( p_{ij} \) of eq. (22).

The involutivity of the invariants

\[
\{I_k, I_l\} = \{Tr(L^k), Tr(L^l)\} = 0 \quad \text{for all } k, l = 1, 2, \ldots ,
\]

that follows as a consequence of the \( r \)-matrix structure (28), will lead to the integrability of the discrete-time model by an argument presented elegantly in \([13]\), forming the discrete counterpart of the Arnol’d-Liouville theorem. Therefore, we have proved the complete integrability of the discrete-time CM model.

5. Continuum Limit. We now show that in a continuum limit \([13]\) goes to that of the original CM model. In fact, the considerations above show that the
invariants and the canonical variables remain the same on the continuous- as well as the discrete-time level. Thus, taking the invariants (13) as Hamiltonians, we get a hierarchy of continuous flows interpolating the discrete-time flow, the one of order \( k = 2 \) corresponding to the original model. In order to perform the continuum limit, we first set

\[ x_i = z_i + n\alpha, \quad i = 1, 2, ..., N, \]  

where \( \alpha \) is a small (constant) parameter and

\[ \overline{x_i} = \overline{z_i} + (n + 1)\alpha, \quad i = 1, 2, ..., N. \]  

Then, in the continuum limit, we write

\[ \overline{z_i} = z_i + \epsilon \dot{z}_i + 1/2\epsilon^2 \ddot{z}_i + ..., \]  

for all \( i = 1, 2, ..., N \), where \( \dot{z}_i \equiv \frac{d}{dt}z_i \) and \( \epsilon \) is the time-step parameter which, we suppose, is in the order of \( O(\alpha^2) \). Substituting (31)-(33) into (13), we get as the leading order term of (13), i.e.

\[ \ddot{z}_i = -\frac{2g}{N} \sum_{j=1}^{N} \frac{1}{(z_i - z_j)^3}, \quad i = 1, 2, ..., N, \]  

where \( g \equiv \alpha^4/\epsilon^2 \). It is clear that (34) are exactly the equations of motion of the (continuous) CM model. It is interesting to note that the coupling constant of the continuous model arises from the discrete-time step.

6. Discussion. In this letter, we have constructed a discrete-time CM model. This discrete model is also integrable and is the iterate of a canonical transformation generated by the discrete ‘Hamiltonian’ (23). The original CM model is a limit of this discrete-time CM model, and in this limit the coupling constant for the long-range interaction term is encoded in the discrete-time step parameter. The present result invites a number of interesting problems to be studied. First of all, one should
address the generalization of these results to the more generic elliptic potentials, cf. [13, 8]. Secondly, one may ask whether one can find time-discretizations of the relativistic version of the CM model, [17], cf. also [18]. In fact, Suris, in [19], has found an interesting connection between the discrete-time Toda model and its relativistic version. Furthermore, an intriguing connection exists between the sine-Gordon soliton solutions and the relativistic CM model, cf. [20]. This could lead to an alternative way to discretize the model. Thirdly, for the discrete-time model, following the similarities with the structure of the integrable quantum mappings studied in [21], one should investigate not only the $L$-part of the Lax pair, but also take the $M$-part under consideration in the $r$-matrix structure. This has been done for the integration of mappings of KdV type in [22]. Finally, these investigations should also be pursued on the quantum level. Although, the discrete-time model has an obvious quantum counterpart, in much the same way as the continuum model, the work of [21] on quantum mappings, as well as the work [23] on the quantum CM model, indicate that some important modification (e.g. with respect to the construction of exact quantum invariants of the discrete-time flow) might be expected. All these problems are being investigated and will be discussed in detail in future publications.

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References

[1] F. Calogero, J. Math. Phys. 10 (1969) 2191, ibid. 12 (1971) 418.

[2] B. Sutherland, J. Math. Phys. 12 (1971), 246, 251, Phys. Rev. A4 (1971) 2019, ibid. A5 (1972) 1372.

[3] J. Moser, Adv. Math. 16 (1975) 197.

[4] M.A. Olshanetsky and A.M. Perelomov, Phys. Rep. 71 (1981) 313; ibid. 94 (1983) 313.

[5] F.D.M. Haldane, Phys. Rev. Lett. 60 (1988) 635, ibid. 66 (1991) 1529; B.S. Shastry, Phys. Rev. Lett. 60 (1988) 639; A.P. Polychronakos, Phys. Rev. Lett. 70 (1993) 2329.

[6] B.D. Simons, P.A. Lee and B.L. Altschuler, Phys. Rev. Lett. 72 (1994) 64; B.S. Shastry, The 1/r^2 Integrable system: The Universal Hamiltonian for Quantum Chaos, To appear in Correlation Effects in Low-Dimensional Electron Systems, eds. N. Kawakami and A. Okiji, (Springer Verlag, 1994).

[7] J. Avan, M. Talon, Phys. Lett. B303 (1993) 33; J. Avan, O. Babelon and M. Talon, Construction of the Classical R-matrices for the Toda and Calogero Models, Preprint LPTHE-93-31, hep-th/9306102.

[8] E. K. Sklyanin, Dynamical r-matrices for the Elliptic Calogero-Moser Model, preprint LPTHE-93-42, hep-th/9308060.

[9] A.P. Veselov, Russ. Math. Surv. 46 (1991) 1.

[10] F.W. Nijhoff, V.G. Papageorgiou and H.W. Capel, Integrable Time-Discrete Systems: Lattices and Mappings, in: Quantum Groups, ed. P.P. Kulish, Springer LNM 1510 (1992), p. 312.
[11] H. Airault, H. McKean, and J. Moser, Commun. Pure Appl. Math., 30 (1977) 95.

[12] D.V. Chudnovsky and G.V. Chudnovsky, Nuovo Cim. 40B (1977) 339.

[13] I. M. Krichever, Funct. Anal. Appl., 12 (1978) 59; ibid. 14 (1980) 282.

[14] A.P. Veselov, Russ. Math. Surv. 35 (1980) 239.

[15] F.W. Nijhoff, H.W. Capel, G.L. Wiersma and G.R.W. Quispel, Phys. Lett. 105A (1984) 267; F.W. Nijhoff, H. Capel and G.L. Wiersma, in Geometric Aspects of the Einstein Equations and Integrable Systems, ed. R. Martini, Springer Lect. Notes Phys. 239 (1985) 263.

[16] M. Bruschi, O. Ragnisco, P.M. Santini and G.-Z. Tu, Physica 49D (1991) 273.

[17] S.N.M. Ruijssenaars and H. Schneider, Ann. Phys. 170 (1986) 370; S.N.M. Ruijssenaars, Commun. Math. Phys. 110 (1987) 191; ibid. 133 (1990) 217.

[18] M. Bruschi and F. Calogero, Commun. Math. Phys. 109 (1987) 481; M. Bruschi and O. Ragnisco, Inv. Probl. 4 (1988) L15.

[19] Yu. B. Suris, Phys. Lett. 145A (1990) 113; ibid. 156A (1991) 467.

[20] O. Babelon and D. Bernard, The Sine-Gordon Solitons as a N-Body Problem, Preprint SPhT-93-072, LPTHE-93-40, [hep-th/9309154].

[21] F.W. Nijhoff, H.W. Capel and V.G. Papageorgiou, Phys. Rev. A46 (1992) 2155; F.W. Nijhoff and H.W. Capel, Phys. Lett. A163 (1992) 49; J. Phys. A26 (1993) 6385.

[22] F.W. Nijhoff and V.G. Papageorgiou, Hamiltonian Integration of Integrable Mappings of KdV Type, in preparation.

[23] H. Ujino, M. Wadati and K. Hikami, The quantum Calogero-Moser Model: Algebraic Structures, preprint, Tokyo (1993).