Quantization of Friedmann cosmological models with two fluids: 
dust plus radiation

N. Pinto-Neto,¹ E. Sergio Santini,¹,²† and F. T. Falciano¹‡

¹Centro Brasileiro de Pesquisas Físicas, 
Coordenação de Cosmologia, Relatividade e Astrofísica: ICRA-BR, 
Rua Dr. Xavier Sigaud 150, Urca 22290-180 – Rio de Janeiro, RJ – Brasil
²Comissão Nacional de Energia Nuclear 
Rua General Severiano 90, Botafogo 22290-901 – Rio de Janeiro, RJ – Brasil

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Abstract

The causal interpretation of quantum mechanics is applied to a homogeneous and isotropic quantum universe, whose matter content is composed by non interacting dust and radiation. For wave functions which are eigenstates of the total dust mass operator, we find some bouncing quantum universes which reaches the classical limit for scale factors much larger than its minimum size. However these wave functions do not have unitary evolution. For wave functions which are not eigenstates of the dust total mass operator but do have unitary evolution, we show that, for flat spatial sections, matter can be created as a quantum effect in such a way that the universe can undergo a transition from an exotic matter dominated era to a matter dominated one.

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*Electronic address: nelsonpn@cbpf.br
†Electronic address: santini@cbpf.br
‡Electronic address: ftovar@cbpf.br
I. INTRODUCTION

The Bohm-de Broglie (BdB) interpretation\cite{1,2,3} has been successfully applied to quantum minisuperspace models\cite{4,5,6,7,8,9,10}, and to full superspace\cite{11,12,13}. In the first case, it was discussed the classical limit, the singularity problem, the cosmological constant problem, and the time issue. It was shown in scalar field and radiation models for the matter content of the early universe that quantum effects driven by the quantum potential can avoid the formation of a singularity through a repulsive quantum force that counteract the gravitational attraction. The quantum universe usually reach the classical limit for large scale factors. However, it is possible to have small classical universes and large quantum ones: it depends on the state vector and on initial conditions\cite{9}. It was also shown that the quantum evolution of homogeneous hypersurfaces form the same four-geometry independently on the choice of the lapse function\cite{5}.

In the present work we study the minisuperspace model given by a quantum Friedmann-Lemaître-Robertson-Walker (FLRW) universe filled with dust and radiation decoupled from each other. We write down the hamiltonian that comes from the velocity potential Schutz formalism\cite{14}. After implementing a canonical transformation, the momentum associated to the radiation fluid $p_T$ and to the dust fluid $p_\phi$ appear linearly in the superhamiltonian constraint. Both can be associated to time parameters, but physical reasons and mathematical simplicity led us to choose the coordinate $T$ associated with $p_T$ as the time parameter. This is equivalent to choose the (reversed) conformal time. We quantize this system obtaining a Schrödinger-like equation. We analyze its time dependent solutions applying the BdB interpretation in order to study the scale factor quantum dynamics.

We first consider an initial quantum state given by a gaussian superposition of the scale factor which is an eigenstate of the total dust mass operator (matter is not being created nor destroyed in such states), and we compute the solution at a general subsequent time by means of the propagator approach. We calculate the bohmian trajectories for the scale factor. For flat and negative curvature spatial sections, we find that the quantum solutions for the scale factor reach the classical behaviour for long times, but do not present any initial singularities due to quantum effects. In the same way, in the case of positive curvature spatial sections, the classical initial and final singularities are removed due to quantum effects, and the scale factor oscillates between a minimum and a maximum size. For large scale factor,
the classical behaviour is recovered. However, such eigenfunctions of the total dust mass operator do not have unitary evolution. This led us to consider an initial state given by gaussian superpositions of the total dust matter content. In this situation, dust and radiation can be created and destroyed. We calculate general solutions for flat, negative and positive curvature spatial sections. In particular, for flat spatial sections, we construct a wave packet whose quantum trajectories represent universes which begin classically in an epoch where the dust matter has negative energy density (exotic dust matter), evolving unitarily to a configuration where quantum effects avoid the subsequent classical big crunch singularity, performing a graceful exit to an expanding classical model filled with conventional matter and radiation. There is thus a transition from an exotic matter era to a conventional matter one due to quantum effects.

This paper is organized as follows. In section II we synthesize the basic features of the Bohm-de Broglie interpretation of quantum mechanics, which will be necessary to interpret our quantum model studied in other sections. In section III we briefly summarize the velocity potential Schutz formalism, and we apply it to construct the hamiltonian of the FLRW universe filled with two perfect fluids, which are dust and radiation. We then review and analyze the classical features of the two perfect fluids FLRW model in order to have the results to be contrasted with the quantum models of the following sections. In section IV we present some new results concerning the existence of singularities in the quantization of the one fluid case. We show that, when the fluid is radiation, all quantum solutions do not present singularities. In section V we quantize the model with two fluids, and we compute the solutions of the Schrödinger like equation for two different initial conditions: the first being an eigenstate of the total dust matter operator, and the second a gaussian superposition of total dust matter eigenstates. We interpret the solutions according to the BdB view and we develop the main results of the paper. Section VI is for discussion and conclusions.

II. THE BOHM-DE BROGLIE INTERPRETATION OF QUANTUM MECHANICS

In this section, we briefly review the basic principles of the Bohm-de Broglie (BdB) interpretation of quantum mechanics. According to this causal interpretation, an individual
physical system comprises a wave $\Psi(x, t)$, which is a solution of the Schrödinger equation, and a point particle that follows a trajectory $x(t)$, independent of observations, which is solution of the Bohm guidance equation

$$p = m\dot{x} = \nabla S(x, t)|_{x=x(t)},$$

where $S(x, t)$ is the phase of $\Psi$. In order to solve Eq. (1), we have to specify the initial condition $x(0) = x_0$. The uncertainty in the initial conditions define an ensemble of possible motions, [1, 2, 3].

It is sufficient for our purposes to analyze the Schrödinger equation for a non relativistic particle in a potential $V(x)$, which, in coordinate representation, is

$$i\frac{\partial \Psi(x, t)}{\partial t} = \left[ -\frac{1}{2m}\nabla^2 + V(x) \right] \Psi(x, t).$$

Substituting in (2) the wave function in polar form, $\Psi = A \exp(iS)$, and separating into real and imaginary parts, we obtain the following two equations for the fields $A$ and $S$

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{1}{2m} \frac{\nabla^2 A}{A} = 0,$$

$$\frac{\partial A^2}{\partial t} + \nabla \cdot \left( A^2 \frac{\nabla S}{m} \right) = 0.$$  

Equation (3) can be interpreted as a Hamilton-Jacobi type equation for a particle submitted to a potential, which is given by the classical potential $V(x)$ plus a quantum potential defined as

$$Q = -\frac{1}{2m} \frac{\nabla^2 A}{A}.$$  

It is possible to verify that the particle trajectory $x(t)$ satisfies the equation of motion

$$m \frac{d^2 x}{dt^2} = -\nabla V - \nabla Q.$$  

The classical limit is obtained when $Q = 0$. The BdB interpretation does not need a classical domain outside the quantized system to generate the physical facts out of potentialities. In a real measurement, we do not see superpositions of the pointer apparatus because the measurement interaction causes the wave function to split up into a set of non overlapping packets. The particle will enter in one of them, the rest being empty, and it will
be influenced by the unique quantum potential coming from the sole non zero wave function of this region. The particle cannot pass to another branch of the superposition because they are separated by regions where $\Psi = 0$, nodal regions.

In section \ref{sec:quantization} the FLRW minisuperspace model containing dust and radiation as two perfect decoupled fluids will be quantized. A preferred time variable can be chosen as one of the velocity potentials associated to the fluids (radiation), yielding a Schrödinger like equation. Then the BdB interpretation of our quantum model runs like in Ref. \cite{8}, in close analogy to the non relativistic particle model described above. In the present case, however, the scale factor of the universe will not be the only degree of freedom: the velocity potential associated with the dust field and its canonical momentum, interpreted as the dust total mass, are also present. They satisfy a Hamilton-Jacobi equation modified by an extra term, the quantum potential, so that their time evolution will be different from the classical one. The main features of this classical model we describe in the following section.

III. CLASSICAL DUST PLUS RADIATION MODEL IN THE VELOCITY POTENTIAL SCHUTZ FORMALISM

We start by considering a perfect fluid in a FLRW universe model. The line element is given by

$$ds^2 = -N^2 dt^2 + a^2(t) \gamma_{ij} dx^i dx^j \quad (7)$$

where $N$ is the lapse function, $a$ is the scale factor, and $\gamma_{ij}$ is the metric of the three-dimensional homogeneous isotropic static spatial section of constant curvature $\kappa = 1, 0, \text{or} -1$.

Following the Schutz's canonical formalism to describe the relativistic dynamics of a perfect fluid in interaction with the gravitational field \cite{14}, we introduce the five velocity potentials, $\alpha, \beta, \theta, \varphi$ and $s$. The potentials $\alpha$ and $\beta$, which describe vortex motion, vanish in the FLRW model because of its symmetry. The potential $s$ is the specific entropy and $\theta$ can be related with the temperature of the fluid. By now $\varphi$ works only as a mathematical tool.

The four-velocity of the fluid is obtained from the velocity potentials as
\[ U_\nu = \frac{1}{\mu} (\varphi_\nu + \theta s_\nu), \quad (8) \]

where \( \mu \) stands for the specific enthalpy. The four velocity is normalized as

\[ g_{\alpha \beta} U^\alpha U^\beta = -1. \quad (9) \]

Using this equation, it is possible to write the specific enthalpy \( \mu \) as a function of the velocity potentials.

The action for a relativistic perfect fluid and the gravitational field in the natural units \( c = \hbar = 1 \) is given by

\[ I = -\frac{1}{6l_p^2} \int_M d^4x \sqrt{-g} \mathcal{R} + \int_M d^4x \sqrt{-g} p + \frac{1}{3l_p^2} \int_{\partial M} d^3x \sqrt{h} h_{ij} K^{ij}, \quad (10) \]

where \( l_p \equiv (8\pi G/3)^{-1/2} \), \( G \) is the Newton’s constant (hence \( l_p \) is the Planck length in the natural units), \( \mathcal{R} \) is the scalar curvature of the spacetime, \( p \) is the pressure of the fluid, \( h_{ij} \) is the three metric on the boundary \( \partial M \) of the 4-dimensional manifold \( M \), and \( K^{ij} \) its extrinsic curvature. The velocity potentials are supposed to be functions of \( t \) only, in accordance with the homogeneity of spacetime. The perfect fluid follows the equation of state \( p = \lambda \rho \).

Substituting the metric \( (7) \) into the action \( (10) \), using the formalism of Schutz \[14\] to write the pressure of the fluid as

\[ p = p_{0r} \left[ \varphi + \frac{\theta \dot{s}}{N(\lambda + 1)} \right]^{\lambda+1 \over \lambda} \exp \left( -{s \over s_{0r} \lambda} \right), \quad (11) \]

with \( p_{0r} \) and \( s_{0r} \) constants, computing the canonical momenta \( p_\varphi, p_s, p_\theta \) for the fluid and \( p_a \) for the gravitational field, using the two constraints equations \( p_\theta = 0 \), \( \theta p_\varphi = p_s \), and performing the canonical transformation

\[ T = -{p_s \over 6^{1-3\lambda}} \exp \left( -{s \over s_{0r}} \right) p_\varphi^{-\lambda+1} \rho_0^{\lambda} s_{0r}, \quad (12) \]

and

\[ \varphi_N = \varphi + (\lambda + 1) s_{0r} p_s \rho_0 \rho_\varphi, \quad (13) \]

leading to the momenta

\[ p_r = 6^{1-3\lambda} p_\varphi^{\lambda+1} \rho_0^{\lambda} s_{0r} \exp \left( {s \over s_{0r}} \right), \quad (14) \]
and
\[ p_{\phi N} = p_{\phi}, \] (15)
we obtain for the final Hamiltonian (see Ref. [16] for details),
\[ H \equiv N\mathcal{H} = N \left( -\frac{p_a^2}{24a} - 6\kappa a + \frac{p_T}{a^{3\lambda}} \right), \] (16)
where \( N \) plays the role of a Lagrange multiplier whose variation yields the constraint equation
\[ \mathcal{H} \approx 0, \] (17)
where \( \approx \) means ‘weakly zero’ (this phase space function is constrained to be zero, but its Poisson bracket to other quantities is not). We have redefined \( \tilde{a} = \sqrt{V/(16\pi l_p^2)} \, a \) in order for \( \tilde{a} \) be dimensionless, and \( \tilde{N} = \sqrt{6}N \), where \( V \) is the total comoving volume of the spatial sections. The tilda have been omitted. Considering now two decoupled fluids, one being radiation \( (\lambda_r = 1/3) \), and the other dust matter \( (\lambda_d = 0) \), the Hamiltonian reads:
\[ H \equiv N\mathcal{H} = N \left( -\frac{p_a^2}{24a} - 6\kappa a + \frac{p_T}{a} + p_{\phi} \right) \] (18)

The classical Hamilton equations are:
\[ \dot{a} = \{a, H\} = -\frac{N}{12a}p_a \Rightarrow p_a = -\frac{12a\dot{a}}{N}, \] (19)
\[ \dot{p}_a = \{p_a, H\} = N \left( -\frac{p_a^2}{24a^2} + 6\kappa + \frac{p_T}{a^2} \right), \] (20)
\[ \dot{T} = \frac{N}{a}, \] (21)
\[ \dot{\phi} = \{\phi, H\} = N, \] (22)
\[ \dot{p}_T = \dot{p}_{\phi} = 0 \Rightarrow p_T, p_{\phi} \text{ are constants.} \] (23)

The superhamiltonian is constrained to vanish due to variation of the Hamiltonian with respect to the lapse function \( N \), \( \mathcal{H} \approx 0, \)
\[ -\frac{p_a^2}{24a} - 6\kappa a + \frac{p_T}{a} + p_{\phi} = 0. \] (24)
The constraint (24) combined with Eqs. (19) and (23) yield the Friedmann’s equation

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{\dot{\kappa}}{a^2} + \frac{1}{6} \left( \frac{p_T}{a^4} + \frac{p_\varphi}{a^3} \right)
\]

(25)

Note that the conjugate momenta \( p_T \) and \( p_\varphi \), classical constants of motion, can be identified to the total content of dust and radiation in the universe:

\[
p_\varphi = 16\pi G a^3 \rho_m,
\]

(26)

\[
p_T = 16\pi G a^4 \rho_r.
\]

(27)

Note also that Eq. (22) implies that \( d\varphi = N dt \), hence \( \varphi \) is cosmic time, while Eq. (21) yields \( adT = N dt \) so \( T \) is conformal time. Consequently, choosing \( N = 1 \) means taking coordinate time \( t \) as cosmic time \( \varphi \), while choosing \( N = a \) imposes coordinate time to be conformal time \( T \). Explicit analytic solutions of Eqs. (19, 20, 23, 25) can be obtained only in the gauge \( N = a \). In this gauge, besides the constraint (25) with \( N = a \), we obtain the simple second order equation,

\[
a'' + \kappa a = \frac{p_\varphi}{12},
\]

(28)

where a prime means differentiation with respect to conformal time, which we denote \( \eta \) from now on. The solutions read:

\[
a = \left\{ \begin{array}{ll}
\left( \frac{2a_{eq}}{\eta_{eq}} \right) \left[ 1 - \cos(\eta) + \eta_{eq} \sin(\eta) \right] & ; \kappa = 1, \\
\eta_{eq} \left( 2 - \left( \frac{\eta}{\eta_{eq}} \right)^2 \right) & ; \kappa = 0, \\
\left( \frac{2a_{eq}}{\eta_{eq}} \right) \left[ \cosh(\eta) + \eta_{eq} \sinh(\eta) - 1 \right] & ; \kappa = -1.
\end{array} \right.
\]

(29)

The quantity \( a_{eq} \) is defined to be the value of the scale factor at the equilibrium time where \( \rho_m = \rho_r \), and \( \eta_{eq}^2 = 3/(2\pi G \rho_r a^4) = 24 a_{eq}/|p_\varphi| \).

As we will see in section (V), the presence of quantum effects can create exotic dust matter content. Hence, for comparison, we analyze a classical universe filled with exotic dust, which means \( \rho_m < 0 \), i.e \( p_\varphi < 0 \). For simplicity, let us focus on the flat spatial case.

In the presence of exotic dust, the behaviour of the scale factor is drastically different. From the Friedmann Eq. (25), since \( p_\varphi < 0 \), the radiation density must always be equal or
greater than the dust density, otherwise the Friedmann equation

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{6} \left( \frac{p_\eta}{a^2} - \frac{|p_\phi|}{a} \right)
\]

has no solution. For small values of the scale factor, the radiation term dominates. As the scale factor grows, the exotic dust term begins to be comparable to the radiation term up to the critical point where both are equal and \( \dot{a} = 0 \). From this point, the scale factor decreases until the universe recollapses. Note that Eq. (29) for \( \kappa = 0 \) and \( p_\phi < 0 \) implies that \( a'' < 0 \) at all times. Hence, contrary to the normal dust matter case where after the big bang the universe expands forever [see Eq. (29) for \( \kappa = 0 \)], in the exotic case the universe recollapses in a big crunch. The qualitative evolution of the scale factor is plotted in figure 1: The deceleration parameter in conformal time is given by

\[
q = \frac{-a''a}{a'^2} + 1.
\]

It diverges when the scale factor reaches its maximum value (\( a' = 0 \) and \( a'' < 0 \)).

**IV. FLRW QUANTUM MODEL WITH RADIATION**

In this section, we present a general result concerning the presence of singularities in the quantization of a FLRW model with radiation. The hamiltonian constraint in this case is

\[
\mathcal{H} = -\frac{p_\eta^2}{24a} - 6\kappa a + \frac{p_\eta}{a} \approx 0,
\]
and \( \eta \) is conformal time, as discussed above.

Using the Dirac quantization procedure, the Hamiltonian constraint phase space function \( \mathcal{H} \) becomes an operator which must annihilate the quantum wave function: \( \hat{\mathcal{H}} \Psi = 0 \). One then obtains in natural units the Wheeler-De Witt equation for the minisuperspace FLRW metric with radiation:

\[
\frac{i}{\partial \eta} \Psi (a, \eta) = \left( -\frac{1}{24} \frac{\partial^2}{\partial a^2} + 6 \kappa a^2 \right) \Psi (a, \eta).
\]

(32)

Note that a particular factor ordering has been chosen and, because \( p_\eta \) appears linearly in Eq. (31), \( \eta \rightarrow -\eta \) is chosen to be the time label in which the wave function evolves (the sign reversing was done in order to express this quantum equation in a familiar Schröedinger form \[16\]).

The scale factor is defined only in the half line \([0, \infty)\), which means that the superhamiltonian \(31\) is not in general hermitian. Hence, if one requires unitary evolution, the Hilbert space is restricted to functions in \( L^2(0, \infty) \) satisfying the condition

\[
\frac{\partial \Psi}{\partial a} (0, \eta) = \alpha \Psi (0, \eta),
\]

(33)

where \( \alpha \) is a real parameter \[25\].

We will now show that condition (33), together with the assumption that \( \Psi (a, \eta) \) is analytic in \( \eta \) at \( a = 0 \), implies that general quantum solutions of Eq. (32), when interpreted using the BdB interpretation, yield quantum cosmological models without any singularity.

We can rearrange Eq. (32) in order to isolate the second spatial derivative:

\[
\frac{\partial^2}{\partial a^2} \Psi (a, \eta) = 24 \left[ -i \frac{\partial}{\partial \eta} \Psi (a, \eta) + 6 \kappa a^2 \Psi (a, \eta) \right].
\]

(34)

Using the BdB interpretation, the scale factor equation of motion is given by the gradient of the phase \( S (a, \eta) \) of the wave function

\[
a' = \frac{1}{12} S_a (a, \eta) = -\frac{i}{24} \left( \frac{\Psi \Psi^* - \Psi_a \Psi^*}{\Psi \Psi^*} \right) \equiv f (a, \eta),
\]

where the index \( a \) means derivative with respect to \( a \). Taking the boundary condition (33) at \( a = 0 \), the velocity function \( f (0, \eta) \) vanishes. Hence, if there is a time \( \eta_0 \) where \( a(\eta_0) = 0 \), then \( a'(\eta_0) = 0 \). For \( a'' \) one has:

\[
a'' = \frac{\partial f}{\partial a} a' + \frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial a} + \frac{\partial f}{\partial \eta}.
\]
This is also zero at $a = 0$ unless $\partial f / \partial a$ diverges there. However,

$$
\frac{\partial f}{\partial a} = -\frac{i}{24} \left( \frac{\Psi a - \Psi a^\prime}{\Psi^*} \right) - \frac{i}{24} \frac{\left( (\Psi a^*)^2 - (\Psi a^* a^*) \Psi \Psi^* \right)}{\Psi^*^2} - \frac{i}{24} \frac{\left( (\Psi a^*)^2 - (\Psi a^* a^*) \Psi \Psi^* \right)}{\Psi^*^2}
$$

is obviously finite if condition (33) and analyticity of $\Psi$ in $\eta$ is satisfied at $a = 0$. The case when $(\Psi^*^2) = 0$ does not need to be analyzed because bohmian trajectories cannot pass through nodal regions of the wave function.

The same reasoning can be used for all higher derivatives $a^\prime a / \eta$ at $a = 0$ to show that they are all zero: one just have to use equation (33) to substitute $\partial^2 \Psi / \partial a^2$ for $\partial \Psi / \partial \eta$ and condition (33) to substitute $\Psi / \partial a $ for $\alpha \Psi / \partial \eta$ at $a = 0$ whenever they appear, and then use analyticity of $\Psi$ in $\eta$ at $a = 0$.

With these results, if there is a time $\eta_0$ where $a(\eta_0) = 0$, expanding $a(\eta)$ in Taylor series around $\eta_0$ shows that $a(\eta) \equiv 0$. This means that the only singular bohmian trajectory is the trivial one of not having a universe at all! All non trivial quantum solutions have to be non singular.

V. QUANTUM BEHAVIOUR OF A FLRW MODEL WITH DUST AND RADIATION

As we have seen in section III, the superhamiltonian constraint for a FLRW model with non interacting dust and radiation is given by Eq. (24):

$$
\mathcal{H} \equiv -\frac{p^2_a}{24a} - 6\kappa a + \frac{p_n}{a} + p_\varphi \approx 0,
$$

We see that both $p_\eta$ and $p_\varphi$ appear linearly in $\mathcal{H}$, and their canonical coordinates $\eta$ and $\varphi$ are, respectively, conformal and cosmic time. As in the preceding section, from the Dirac quantization procedure one obtains the quantum equation $\hat{\mathcal{H}} \Psi = 0$, which reads

$$
\left( \frac{1}{24a} \frac{\partial^2}{\partial a^2} - \frac{i}{a} \frac{\partial}{\partial \eta} - i \frac{\partial}{\partial \varphi} \right) \Psi(a, \varphi, \eta) = 0,
$$

where we have used the usual coordinate representation $\hat{p} = -i \partial / \partial q$.  

11
Either $\eta$ or $\varphi$ can be chosen as time parameters on which $\Psi$ evolves. However, the classical solutions can be expressed explicitly only in conformal time $\eta$ [see Eq. (29)]. Furthermore, cosmic time $\varphi$ depends on the constants characterizing each particular solution through $\varphi = \int d\eta a(\eta)$, and it is not the same parameter for all classical solutions (see Ref. [17] for details). Hence, we will take $\eta$ (in fact $-\eta$, for the reasons mentioned in the previous section) as the time parameter of the quantum theory\(^1\). With this choice, and for a particular factor ordering, Eq. (36) can be written as:

$$i \frac{\partial}{\partial \eta} \Psi(a, \varphi, \eta) = \left( -\frac{1}{24} \frac{\partial^2}{\partial a^2} + 6\kappa a^2 + ia \frac{\partial}{\partial \varphi} \right) \Psi(a, \varphi, \eta).$$

(37)

A. Eigenstates of total matter content

In this subsection we only consider initial states $|\Psi(\eta_0)\rangle$ which are eigenstates of the total dust matter operator $\hat{p}_\varphi$. It follows that these states at time $\eta$, $|\Psi(\eta)\rangle$ will also be eigenstates of $\hat{p}_\varphi$ with the same eigenvalue because $[\hat{H}, \hat{p}_\varphi] = 0$. In other words, we consider that dust matter is not created nor destroyed. In such a way, we have $\hat{p}_\varphi |\Psi(\eta)\rangle = p_\varphi |\Psi(\eta)\rangle$ and the wave function in the $a, \varphi$ representation, $\langle a, \varphi | \Psi(\eta) \rangle = \Psi(a, \varphi, \eta)$, is given by

$$\Psi(a, \varphi, \eta) = \Psi(a, \eta) e^{ip_\varphi \varphi}.$$  

(38)

From the Schrödinger’s equation (37), we have for $\Psi(a, \eta)$

$$i \frac{\partial}{\partial \eta} \Psi(a, \eta) = \left( -\frac{1}{24} \frac{\partial^2}{\partial a^2} + 6\kappa a^2 - p_\varphi a \right) \Psi(a, \eta),$$

(39)

which is the Schrödinger equation for a particle of mass $m = 12$ in a one dimensional forced oscillator with frequency $w = \sqrt{\kappa}$ and constant force $p_\varphi$, which we write as

$$i \frac{\partial}{\partial \eta} \Psi(a, \eta) = \left( -\frac{1}{2m} \frac{\partial^2}{\partial a^2} + \frac{mw^2}{2} a^2 - p_\varphi a \right) \Psi(a, \eta).$$

(40)

The scale factor is defined only in the half line $[0, \infty)$, which means that the hamiltonian (35) is not in general hermitian. Hence, if one requires unitary evolution, the Hilbert subspace is restricted to functions on $L^2(0, \infty; -\infty, \infty)$ satisfying the condition:

$$\int_{-\infty}^{\infty} d\varphi \left[ \frac{\partial \Psi_2 (a, \varphi, \eta)}{\partial a} \Psi_1 (a, \varphi, \eta) \right]_{a=0} = \int_{-\infty}^{\infty} d\varphi \left[ \frac{\partial \Psi_1 (a, \varphi, \eta)}{\partial a} \Psi_2 (a, \varphi, \eta) \right]_{a=0}$$

(41)

\(^1\) The choice of $\varphi$ will probably yield a different theory, with a different Hilbert space. The kinetic term is more complicated and the measure is not the trivial one. We will not study this possibility here.
for any $\Psi_1(a, \varphi, \eta), \Psi_2(a, \varphi, \eta) \in L^2(0, \infty; -\infty, \infty)$. In the special case considered in this section, this condition is reduced to

$$\frac{\partial \Psi}{\partial a}(0, \eta) = \alpha \Psi(0, \eta), \quad (42)$$

where $\alpha$ is a real parameter \[25\]. We will analyze the two extreme cases: $\alpha = 0$ and $\alpha = \infty$, which are the simpler and usually studied in the literature on quantum cosmology \[8, 16, 18, 19, 21, 25\].

For the case $\alpha = 0$ we have that

$$\frac{\partial \Psi}{\partial a}(0, \eta) = 0, \quad (43)$$

which is satisfied for even functions of $a$. For the case $\alpha = \infty$, we have

$$\Psi(0, \eta) = 0, \quad (44)$$

which is satisfied for odd functions of $a$. Both of them address the boundary conditions of the wave packet near the singularity at $a = 0$.

In order to develop the BdB interpretation, we substitute into the Schrödinger’s equation (40), the wave function in the polar form $\Psi = A e^{iS}$, obtaining for the real part

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial a} \right)^2 - a p_{\varphi} + \frac{mw^2}{2} a^2 + Q = 0, \quad (45)$$

where

$$Q \equiv -\frac{1}{2mA} \frac{\partial^2 A}{\partial a^2} \quad (46)$$

is the quantum potential. The Bohm guidance equation reads

$$ma' = \frac{\partial S}{\partial a}. \quad (47)$$

A solution $\Psi(a, \eta)$ of Eq. (40) can be obtained from an initial wave function $\Psi_0(a)$ using the propagator of a forced harmonic oscillator. Let us do it for the two boundary conditions just presented.
1. The case of boundary condition \( \alpha = 0 \)

We denote the propagator \( K^{\alpha=0}(2, 1) \equiv K^{\alpha=0}(\eta_2, a_2; \eta_1, a_1) \), where 1 stands for the initial time and initial scale factor \( \eta_1, a_1 \) respectively, and 2 stands for their final values.

The propagator when the Hilbert space is restricted to \( a > 0 \) can be obtained from the usual one (i.e with coordinate \(-\infty < a < \infty\)) which is associated to a particle in a forced oscillator \( K(2, 1) \equiv K(\eta_2, a_2; \eta_1, a_1) \) by means of

\[
K^{\alpha=0}(2, 1) = K(\eta_2, a_2; \eta_1, a_1) + K(\eta_2, a_2; \eta_1, -a_1)
\]  
(48)

This symmetry condition is necessary to consistently eliminate the contribution of the negative values of the scale factor \( \eta \). The usual propagator associated to a particle in a forced oscillator is \( K(2, 1) \):

\[
K(2, 1) = \sqrt{\frac{mw}{2\pi i \sin(w\eta)}} \exp(i S_{cl})
\]  
(49)

where \( \eta \equiv \eta_2 - \eta_1 \). The classical action \( S_{cl} \) is given by

\[
S_{cl} = \frac{mw}{2 \sin(w\eta)} \left\{ \cos(w\eta)(a_2^2 + a_1^2) - 2a_2a_1 + (a_2 + a_1) \frac{2p_{\varphi}}{mw^2}[1 - \cos(w\eta)] - \right.
\]

\[
\left. [1 - \cos(w\eta)] \frac{2p_{\varphi}^2}{m^2 w^4} + \frac{p_{\varphi}^2}{m^2 w^2} \sin(w\eta)w\eta \right\}.
\]  
(50)

We assume that for \( \eta_1 = 0 \), the initial wave function is given by

\[
\Psi_0(a) = \left( \frac{8\sigma}{\pi} \right)^{1/4} \exp(-\sigma a^2),
\]  
(51)

where \( \sigma > 0 \). The wave function in a future time \( \eta_2 \) is

\[
\Psi(a_2, \eta_2) = \int_0^\infty K^{\alpha=0}(2, 1)\Psi_0(a_1)da_1 = \int_{-\infty}^\infty K(2, 1)\Psi_0(a_1)da_1,
\]  
(52)

where the even character of \( \Psi(a, 0) \) has been taken into account to extend the integral. Integrating and renaming \( \eta \equiv \eta_2, a \equiv a_2 \) we have

\[
\Psi^{\alpha=0}(a, \eta) = \left( \frac{8\sigma}{\pi} \right)^{1/4} \frac{mw}{\cos^2(w\eta)[2\sigma \tan(w\eta) - imw]} \exp\left\{ \frac{imw}{2\tan(w\eta)} \left[ a^2 + \right. \right.
\]

\[
\left. \left. -a + \frac{p_{\varphi}}{mw^2}[1 - \cos(w\eta)] \right)^2 + \frac{2ap_{\varphi}}{m} \left[ \frac{1 - \cos(w\eta)}{w^2 \cos(w\eta)} + \frac{\tan(w\eta)}{w^3} \right] \right\}.
\]  
(53)
Flat spatial section ($\kappa = 0$)

We consider the case $\kappa = 0$, which is obtained by taking the limit of the wave function given by Eq. (53) for $w \to 0$. We compute its phase $S$ from $\Psi \equiv Ae^{iS}$ and calculate the derivative $\partial S/\partial a$. In this way we have, for the Bohm guidance equation Eq. (47),

$$a' - \frac{4\sigma^2 \eta}{4\sigma^2 \eta^2 + m^2} a = \frac{1}{m} \frac{(2\sigma^2 \eta^2 + m^2)}{4\sigma^2 \eta^2 + m^2} p_\varphi \eta. \quad (54)$$

Comparing with the radiation case studied in [8], we see that here it appears a term proportional to $p_\varphi$ in the RHS of the Bohm equation. The general solution is:

$$a(\eta) = C_0 \sqrt{4\sigma^2 \eta^2 + m^2 + \frac{p_\varphi}{2m} \eta^2}, \quad (55)$$

where $C_0$ is a positive integration constant. We can see that, contrary to the classical solution (29), there is no singularity at $\eta = 0$. The quantum effects avoid it. Furthermore, for long times $\eta \gg m/2\sigma$, Eq. (55) reproduces the classical behaviour (29) for the scale factor.

For the case in which the evolution starts from a shifted gaussian wave function

$$\Psi_0(a) = \left(\frac{8\sigma}{\pi}\right)^{1/4} \exp\left[-\sigma(a - a_0)^2\right], \quad (56)$$

the Bohm guidance relation contains an additional term yielding the general solution

$$a(\eta) = C_0 \sqrt{4\sigma^2 \eta^2 + m^2 + \frac{p_\varphi}{2m} \eta^2 + \frac{a_0}{2}}, \quad (57)$$

which has exactly the same behaviour, apart from the shift on the minimal value of the scale factor by the $a_0/2$ term.

Positive curvature spatial section ($\kappa = 1$)

Setting $w = \sqrt{\kappa} = 1$ in the wave function given by Eq. (53) and computing its phase $S$, we obtain for the bohmian trajectories

$$a(\eta) = C_0 \sqrt{4\sigma^2 \sin^2(\eta) + m^2 \cos^2(\eta) + \frac{p_\varphi}{2m} [1 - \cos(\eta)]}, \quad (58)$$

where $C_0$ is a positive integration constant. This is a non-singular cyclic universe, see figure 2, which presents classical behaviour for $\eta$ such that $|\tan(\eta)| \gg m/2$ [see Eq. (29)]. Quantum effects avoid the classical big bang and big crunch.

Negative curvature spatial section ($\kappa = -1$)
Setting now \( w = \sqrt{\kappa} = i \) in the wave function (53) yields the bohmian trajectories:

\[
a(\eta) = C_0 \sqrt{4 \sigma^2 \sinh^2(\eta) + m^2 \cosh^2(\eta) + \frac{p_\phi^2}{2m} [\cosh(\eta) - 1]}.
\]  

(59)

Again, \( C_0 \) is a positive integration constant. This is a non singular ever expanding universe which presents classical behaviour for \( \eta \) such that \( |\tanh(\eta)| \gg m/2 \) [see Eq. (29)]. Quantum effects avoid the classical big bang.

As in the \( \kappa = 0 \) case, a shift in the center of the initial gaussian will not modify these solutions qualitatively.

For the boundary conditions \( \alpha = \infty \), or \( \Psi(0, t) = 0 \), the propagator \( K^{\alpha=\infty}(2, 1) \) can be obtained from the usual (i.e, with coordinate \( -\infty < a < \infty \)) propagator associated to a particle in a forced oscillator \( K(2, 1) \) by means of

\[
K^{\alpha=\infty}(2, 1) = K(\eta_2, a_2; \eta_1, a_1) - K(\eta_2, a_2; \eta_1, -a_1).
\]  

(60)

In order to satisfy the condition \( \Psi(0, \eta) = 0 \), we take as the initial wave function a wave packet given by

\[
\Psi_0(a) = \left( \frac{128 \sigma^3}{\pi} \right)^{1/4} a \exp(-\sigma a^2),
\]

(61)

where \( \sigma > 0 \). Following a similar procedure as in the case \( \alpha = 0 \), we calculate the wave function by propagating the initial wave function as
\[ \Psi(a_2, \eta_2) = \int_0^\infty K^{\alpha=\infty}(2, 1)\Psi_0(a_1)\,da_1 = \int_{-\infty}^\infty K(2, 1)\Psi_0(a_1)\,da_1, \quad (62) \]

where the odd character of \( \Psi \) has been used in order to extend the integral. Integrating this expression and renaming \( a \equiv a_2 \) and \( \eta \equiv \eta_2 \) with \( \eta_1 = 0 \), we have

\[ \Psi^{\alpha=\infty}(a, \eta) = \left( -\frac{C}{2D} \right) \Psi^{\alpha=0}(a, \eta) \quad (63) \]

where

\[ C \equiv \frac{imw}{\sin(w\eta)} \left[ -a + \frac{p_\phi}{mw^2}(1 - \cos(w\eta)) \right] \quad (64) \]

and

\[ D \equiv \frac{imw}{2\tan(w\eta)} - \sigma \quad (65) \]

The phase of \( \Psi^{\alpha=\infty}(a, \eta) \) can be expressed as the sum:

\[ \text{phase}[\Psi^{\alpha=\infty}(a, \eta)] = \text{phase}\left( -\frac{C}{2D} \right) + \text{phase}[\Psi^{\alpha=0}(a, \eta)], \quad (66) \]

and it is easy to see that the phase of \( (-C/2D) \) is independent of \( a \). Then, \( [\partial \text{phase}(\Psi^{\alpha=\infty}(a, \eta))]/\partial a = [\partial \text{phase}[\Psi^{\alpha=0}(a, \eta)]]/\partial a \), and the Bohm guidance relations are the same as in the previous cases. Therefore, the solutions are the same.

The quantum cosmological models obtained in this subsection have the nice properties of being non-singular and presenting classical behaviour for large \( a \). However, they suffer from a fundamental problem: the wave function \( (53) \) from which they are obtained does not have an unitary evolution. The reason is that propagators constructed from Eq.’s \( (48) \) and \( (60) \) do not in general preserve the hermiticity condition \( (42) \) imposed on the wave functions: it depends on the classical potential. In Ref. \( [26] \), there are obtained the potentials which allow propagators in the half line \( (a > 0) \) to preserve unitary evolution. The potentials of the previous section are some of them but the potentials of the present one are not. Hence, even though the initial wave function Eq. \( (51) \) satisfies the hermiticity condition, the wave function \( (53) \) does not. Let us then explore the more general case of initial superpositions of the total dust mass operator eigenstates.
B. Analysis of wave packets given by superpositions of total dust mass eigenstates

In this subsection we consider the case of a general solution of Eq.(37) which is not necessarily one of the eigenstates of \( \hat{p}_\varphi \), the total dust mass operator.

Following the BdB interpretation of quantum mechanics, we substitute in Eq.(37) the wave function in polar form: \( \Psi = A(a, \varphi, \eta) \exp\{iS(a, \varphi, \eta)\} \). The dynamical equation splits in two real coupled equation for the two real functions \( S \) and \( A \) (recall that \( w = \sqrt{\kappa} \) and \( m = 12 \)).

\[
\frac{\partial S}{\partial \eta} + \frac{1}{2m} \left( \frac{\partial S}{\partial a} \right)^2 - a \frac{\partial S}{\partial \varphi} + \frac{m w^2}{2} a^2 + Q = 0, \tag{67}
\]

\[
\frac{\partial A^2}{\partial \eta} + \frac{\partial}{\partial \varphi} (a A^2) + \frac{\partial}{\partial a} \left( A^2 \frac{1}{m} \frac{\partial S}{\partial a} \right) = 0, \tag{68}
\]

where

\[
Q \equiv -\frac{1}{2m A} \frac{\partial^2 A}{\partial a^2}. \tag{69}
\]

Equation (67) is the modified Hamilton-Jacobi equation where \( Q(a, \varphi, \eta) \) is the quantum potential which is responsible for all the peculiar non classical behaviours. When the quantum potential is zero, the equation is exactly the classical Hamilton-Jacobi equation. The momenta are given by the Bohm’s guidance equations

\[
p_a \equiv \frac{\partial S(a, \varphi, \eta)}{\partial a}, \tag{70}
\]

\[
p_\varphi \equiv \frac{\partial S(a, \varphi, \eta)}{\partial \varphi}. \tag{71}
\]

Note also that

\[
p_\eta = \frac{\partial S(a, \varphi, \eta)}{\partial \eta} \tag{72}
\]

is the total ‘energy’ of the system, which is interpreted, from its classical meaning, as the total amount of radiation in the universe model.

In the causal interpretation, equation (68) is a continuity equation where \( A^2 \) is a probability density. The generalised velocities can easily be identified as

\[
a' \equiv \frac{1}{m} \frac{\partial S(a, \varphi, \eta)}{\partial a}, \tag{73}
\]

\[
\varphi' \equiv \frac{1}{a}. \tag{74}
\]

Consider now the classical limit (\( Q = 0 \)). Then the solution of the principal Hamilton function \( S \) is just \( S = W(a) - E \eta + p_\varphi \varphi \), where \( E \) and \( p_\varphi \) are constants. Since \( p_\varphi \) is
proportional to the total amount of dust matter in the universe, and \( E \) to the total amount of radiation, there is no creation or annihilation of dust matter nor radiation. However, in the presence of a quantum potential, this solution is no longer valid, opening the possibility of non conservation of matter and radiation due to quantum effects.

1. Formal Solutions

We now turn to the problem of solving the Schrödinger’s equation (37).

Flat spatial section

For the case of flat spatial section (\( \kappa = 0 \)), equation (37) simplify to

\[
i \frac{\partial \Psi (a, \varphi, \eta)}{\partial \eta} = -\frac{1}{2m} \frac{\partial^2 \Psi (a, \varphi, \eta)}{\partial a^2} + i a \frac{\partial \Psi (a, \varphi, \eta)}{\partial \varphi}
\]

(75)

To solve this equation we make the ansatz

\[
\Psi (a, \varphi, \eta) = \chi (a) \exp \left( -\frac{i}{2m} \beta \eta \right) \exp \left( \frac{i}{2m} \upsilon \varphi \right),
\]

(76)

where \( \chi (a) \) must satisfy the differential equation\(^2\)

\[
\frac{\partial^2 \chi (a)}{\partial a^2} + \upsilon a \chi (a) + \beta \chi (a) = 0.
\]

(77)

This is essentially an Airy equation with solution given by

\[
\chi (a) = \sqrt{a + \frac{\beta}{\upsilon}} \left\{ A Z_{-\frac{1}{3}} \left[ \frac{2\sqrt{\upsilon}}{3} \left( a + \frac{\beta}{\upsilon} \right)^{\frac{3}{2}} \right] + B Z_{-\frac{1}{3}} \left[ \frac{2\sqrt{\upsilon}}{3} \left( a + \frac{\beta}{\upsilon} \right)^{\frac{3}{2}} \right] \right\}
\]

(78)

The \( Z_{\pm \frac{1}{3}} \) function is the first kind Bessel function of degree \( \frac{1}{3} \), and the \( A \) and \( B \) can be any functions of \( \upsilon \) and \( \beta \).

The general solution is a superposition given by

\[
\Psi (a, \varphi, \eta) = \int d\beta d\upsilon \exp \left\{ -\frac{i}{2m} \beta \eta \right\} \exp \left\{ \frac{i}{2m} \upsilon \varphi \right\} \sqrt{a + \frac{\beta}{\upsilon}} \times \left\{ A (\beta, \upsilon) Z_{-\frac{1}{3}} \left[ \frac{2\sqrt{\upsilon}}{3} \left( a + \frac{\beta}{\upsilon} \right)^{\frac{3}{2}} \right] + B (\beta, \upsilon) Z_{-\frac{1}{3}} \left[ \frac{2\sqrt{\upsilon}}{3} \left( a + \frac{\beta}{\upsilon} \right)^{\frac{3}{2}} \right] \right\}
\]

\(^2\) As in the following we will make superpositions of eigenfunctions of the total dust matter operator, we will use from now on the letter \( \upsilon \) in order to not confuse it with the beable \( p_\varphi = \partial S/\partial \varphi \). We did not make this distinction before because they coincide for eigenfunctions of the total dust matter operator.
Positive curvature spatial section: Landau levels

In the positive curvature case ($\kappa = 1$), Eq. (37) reads

$$i\frac{\partial \Psi (a, \varphi, \eta)}{\partial \eta} = -\frac{1}{2m} \frac{\partial^2 \Psi (a, \varphi, \eta)}{\partial a^2} + \frac{m}{2} a^2 \Psi (a, \varphi, \eta) + \frac{ma}{2} \varphi \Psi (a, \varphi, \eta).$$

(79)

There is a canonical transformation which simplifies the problem. Let us define new variables given by

$$\xi \equiv \sqrt{m} a - \frac{p_\varphi}{\sqrt{m}} ; \quad \sigma \equiv -\sqrt{m} \varphi + \frac{p_a}{\sqrt{mw}},$$

$$p_\xi \equiv \frac{p_a}{\sqrt{m}} ; \quad p_\sigma \equiv -\frac{p_\varphi}{\sqrt{m}}.$$

Using these new variables, the Hamiltonian decouples in two parts, one describing a harmonic oscillator and the other a free particle:

$$\hat{H} = \frac{1}{2} \left( \hat{p}_\xi^2 + \hat{\xi}^2 \right) - \frac{1}{2} \hat{p}_\sigma^2.$$  \hspace{1cm} (80)

Decomposing the wave function as

$$\Psi (\xi, \sigma, \eta) = \chi (\xi) \exp \left\{-i \left( \epsilon \eta + \sqrt{2} k \sigma \right) \right\},$$

(81)

we immediately recognize that $\chi (\xi) = \exp \left\{-\frac{\xi^2}{2} \right\} h_n (\xi)$, where $h_n$ are the Hermite polynomial of degree $n$. Just as for the harmonic oscillator, the index $\epsilon$ is constrained to take the values

$$\epsilon_n = k + \left( n + \frac{1}{2} \right),$$

(82)

where $k$ can take any real positive value while $n$ is a positive integer. Eq. (82) determines a set of Landau levels for the cosmological model [22]. The most general solution is a superposition given by

$$\Psi (\xi, \sigma, \eta) = \sum_{n=0}^{\infty} \int dk \chi_n (\xi) \left[ D_n (k) \exp \left\{ i \sigma \sqrt{2} k \right\} + \right.$$  
$$G_n (k) \exp \left\{ -i \sigma \sqrt{2} k \right\} \times \exp \left\{-i \epsilon_n \eta \right\}.$$

(83)

The quantities $D_n (k)$ and $G_n (k)$ are arbitrary coefficients that can depend on the parameter $k$. Recall that we have performed a canonical transformation that mix coordinates...
and momenta, and these are not the proper variables to apply the causal interpretation. Instead, it is imperative to apply the inverse transformation to the coordinate basis before using the guidance relations. This is a necessary requirement to maintain the consistency of the causal interpretation of quantum mechanics [27]-[28].

For the negative curvature spatial section ($\kappa = -1$), the general solutions are hypergeometric functions whose asymptotic behaviours are rather complicated to study in order to obtain reasonable boundary conditions. Hence, we will not treat this case here. We proceed to the analysis of an interesting particular solution.

2. Transition from exotic dust to dust in the flat case

The quantum states of the matter and radiation FLRW universe studied in section V A 1 are eigenstates of the total dust matter operator $\hat{p}_\varphi$. The total wave function is given by $\Psi(a, \varphi, \eta) = \Psi(a, \eta) \exp(i \varphi p_\varphi)$ where $\Psi(a, \eta)$ is given by Eq. (53). Taking the limit $w \to 0$ in that equation, we obtain the wave function $\Psi(a, \eta)$ for the case of flat spatial section, $\kappa = 0$ which, after renaming the eigenvalues of total mass by $\nu \equiv p_\varphi$, is given by

$$
\Psi_{\nu}(a, \eta) = \left(\frac{8\sigma m^2}{\pi \mu}\right)^{\frac{1}{4}} \exp \left\{ -\frac{m^2 \sigma}{\mu} \left( a - \nu \frac{\eta^2}{2m} \right)^2 - i \frac{\nu^2 \eta^3}{6m} - i \frac{\theta}{2} + i \frac{m}{2 \eta} \left[ \left( a + \frac{\nu \eta^2}{2m} \right)^2 - \frac{m^2}{\mu} \left( a - \frac{\nu \eta^2}{2m} \right)^2 \right] \right\},
$$

where

$$
\mu = 4\sigma^2 \eta^2 + m^2,
$$
$$
\theta = \arctan \left( \frac{2\sigma \eta}{m} \right).
$$

Now we consider a more general situation than in section V A 1. We suppose an initial state at $\eta = 0$ which is given by a gaussian superposition of eigenstates of total matter

$$
\Psi(a, \varphi, 0) = \int_{-\infty}^{\infty} d\nu \exp{-\gamma(\nu-\nu_0)^2} \Psi_{\nu}(a, 0) \exp{-i \varphi \nu}.
$$

Then, the state at time $\eta$ is given by

$$
\Psi(a, \varphi, \eta) = \int_{-\infty}^{\infty} d\nu \exp{-\gamma(\nu-\nu_0)^2} \Psi_{\nu}(a, \eta) \exp{-i \varphi \nu}.
$$
In this way, we have a square-integrable wave function. We find

\[ \Psi (a, \varphi, \eta) = \left( \frac{8\sigma \pi m^2}{\mu \nu} \right)^{\frac{1}{4}} \exp \left\{ \left( \frac{\Re (F)}{4\nu} - \frac{\sigma m^2}{\mu} \right) a^2 + \frac{\Re (G)}{4\nu} a \varphi + \frac{\Re (J)}{4\nu} \varphi^2 + \frac{\Re (L)}{4\nu} a + \frac{\Re (M)}{4\nu} \varphi + \frac{\Re (P)}{4\nu} \right\} \]

where we defined

\[ \nu = \left( \gamma + \frac{\sigma \eta^4}{4\mu} \right)^2 + \frac{\eta^6}{(24m\mu)^2} (\mu + 3m^2)^2 \]

\[ \tau = \arctan \left[ \frac{\eta^3 (\mu + 3m^2)}{24m(\gamma \mu + \sigma \eta^4)} \right] \]

\[ F = \left[ \frac{m \sigma \eta^2}{\mu} + i \frac{\eta}{2\mu} (\mu + m^2) \right]^2 \left[ \gamma + \frac{\sigma \eta^4}{4\mu} - i \frac{\eta^3}{24m\mu} (\mu + 3m^2) \right] \]

\[ G = -2i \left[ \frac{m \sigma \eta^2}{\mu} + i \frac{\eta}{2\mu} (\mu + m^2) \right] \left[ \gamma + \frac{\sigma \eta^4}{4\mu} - i \frac{\eta^3}{24m\mu} (\mu + 3m^2) \right] \]

\[ J = - \left[ \gamma + \frac{\sigma \eta^4}{4\mu} - i \frac{\eta^3}{24m\mu} (\mu + 3m^2) \right] \]

\[ L = -2i \gamma \nu_0 G \]

\[ M = 4i \gamma \nu_0 J \]

\[ P = -4\gamma^2 \nu_0^2 J \]

and \( \Re \) and \( \Im \) stands for the real and imaginary part, respectively.

If one calculates the squared norm of the wave function, one obtains

\[ \int_0^\infty \int_{-\infty}^\infty da d\varphi \| \Psi \|^2 = \sqrt{\frac{8\pi^3}{\gamma}} \left[ 1 + \frac{1}{\sqrt{\pi}} \text{erf} \left( \frac{\nu_0 \eta^2}{2m} \right) \right], \quad (87) \]

where \( \text{erf}(x) \) is the error function. The only dependence on time can be eliminated by choosing the gaussian to be centered at \( \nu_0 = 0 \). With this choice we guarantee unitary evolution of the total wave function. From equations (71)-(74), the trajectories can be computed by solving the given system of equations

\[ a' = \frac{2}{m} \left[ \frac{\Im (F)}{4\nu} + \frac{m}{2\mu \eta} (\mu - m^2) \right] a + \frac{\Im (G)}{4m\nu} \varphi \]

(88)
\[ \varphi' = a \quad (89) \]

\[ p_\varphi = \left[ \frac{2 \Im (J)}{4 \nu} \varphi + \frac{\Im (G)}{4 \nu} a \right] \quad (90) \]

Note that \( p_\varphi \) is no longer constant. We integrated numerically these equations with the renormalisation condition \( a(0) = 1 \). The quantum potential \( Q \equiv -\frac{1}{2m_A} \frac{\partial^2 A}{\partial a^2} \) is non zero only close to the origin as shown on figure 3. Hence, we expect that quantum effects be relevant only in this region. Far form the origin, the scale factor must behave classically.

The behaviour of \( p_\varphi \) is plotted on figure 4. From this plot we can see that far from the origin \( p_\varphi \) is constant. This is in accordance with classical behaviour as long as the quantum potential is zero in this region. The surprising feature is that in the far past the universe was filled with a classical exotic dust (\( p_\varphi < 0 \)).

From Eq.(72), one can also compute the amount of radiation. Figure 5 shows the result. Again, far from the origin, radiation is conserved while in the origin, due to quantum effects, it is not conserved.

For the evolution of the scale factor, numerical integration of equation (88) yields the plot of figure 6. In the far positive region, the scale factor behaves classically, as expected, and matter is conserved. On the other hand, in the far negative region the scale factor also behaves classically but with a universe filled with exotic dust, and here again matter is conserved (compare this region with figure 1). Both regions have a consistent classical behaviour. Hence, the universe begins classically from a big bang filled with exotic dust.
FIG. 4: Evolution of the total amount of matter in the universe ($p_\varphi$). In the far past, the universe was filled with exotic dust ($p_\varphi < 0$). Close to the origin, quantum effects transform it into conventional dust ($p_\varphi > 0$).

FIG. 5: Evolution of the total amount of radiation in the universe ($p_\eta$). Far from the origin the universe behaves classically and the total amount of radiation is conserved. It varies near the origin due to quantum effects.

and conventional radiation. It evolves until it reaches a configuration when quantum effects avoid the classical big crunch while transforming exotic dust into normal dust. From this point on, the universe expands classically filled with conventional dust and radiation.
VI. CONCLUSIONS

In the present work we studied some features of the minisuperspace quantization of FLRW universes with one and two fluids. For the one fluid case (radiation), we have generalized results in the literature by showing that all bohmian trajectories coming from reasonable general solutions of the wave equation obtained through the assumptions of unitarity and analyticity at the origin, do not present any singularity. Hence, this quantum minisuperspace theory is free of singularities.

For the two fluids case (non interacting radiation and dust), we first obtained bohmian quantum universes free of singularities reaching the classical limit for large scale factors. However, these trajectories arise from eigenfunctions of the total dust mass operator whose time evolution is not unitary. When considering the general case, we managed to obtain a wave solution presenting unitary evolution with some surprising effects. Now dust and radiation can be created but the new feature is the possibility of creation of exotic fluids. We have shown that dust matter can be created as a quantum effect in such a way that the universe can undergo a transition from an exotic dust matter era to a conventional dust matter one. In this transition, one can see from figure 5 that radiation also becomes exotic due to quantum effects, helping the formation of the bounce.

The fluid approach is not fundamental, but we expect that it can be quite accurate in
The description of quantum aspects of the Universe, in the same way the Landau description of superfluids in terms of fluid quantization was capable of showing many quantum features of this system [29]. After all, creation and annihilation of particles as well as quantum states with negative energy are usual in quantum field theory. The formalism developed in the present paper seems to be a simple and calculable way to grasp these features of quantum field theory. Their physical applications may be important: exotic fluids are relevant not only in causing cosmological bounces and avoiding cosmological singularities [30], but also for the formation of wormholes [31, 32] and for superluminal travels [33]. These are some developments of the present paper we want to explore in future works.

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