Puzzle in the Charmed $D_s$ meson decays into pions: Could the light quarks be not so light?

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Abstract

The inclusive decay rate into pions of the charmed $D_s$ meson is surprisingly larger than estimates expected from the $W$ annihilation, adopting commonly used values of current-algebra up and down quark masses. We then go beyond this tree diagram and consider possible QCD effects that might cause such a large rate. There are two; the first one is related to the spectator decay $c\bar{s} \rightarrow s\bar{s} + u\bar{d}$ followed by $s\bar{s} \rightarrow d\bar{d} + u\bar{u}$ via two-gluon exchange box diagram. The second one is a gluon emission in weak annihilation for which the usual helicity suppression is vitiated: $D_s \rightarrow W + g$ followed by $W \rightarrow u\bar{d}$, $g \rightarrow d\bar{d}$, $u\bar{u}$.

These two contributions, however, turn out to be insufficient to explain data, implying that the puzzle could be understood if the up, down quarks have higher mass values.

Furthermore, on the basis of experimental informations on the spectral function $\rho_{3\pi}(Q^2)$ deduced from the exclusive $D_s \rightarrow 3\pi$ mode, the QCD sum rules also point to a higher mass for light quarks.
The inclusive decay rate of the charmed $D_s$ meson into non-strange ordinary hadrons (mainly pions) is surprisingly large [1]. Its amplitude, being governed by the $W$ annihilation mechanism at the electroweak tree diagram and directly related to the divergence of the axial current, is expected to be strongly suppressed by the familiar helicity argument (similar to the suppression of $\pi \to e\nu$ compared to $\pi \to \mu\nu$), and/or by the partial conservation of the axial current (PCAC). Experiments do not confirm this expectation, however.

We observe first that the $D_s$ decay into pions cannot be described either by the dominant spectator mechanism (both color-favoured and color-suppressed) or by the small penguin diagram, because the spectator constituent $\bar{s}$ of the $D_s^+$ is absent in the decay products. Therefore only the $W$ annihilation mechanism can give rise - at the tree level - to the decays of $D_s$ into pions. The gluonic effects will be also considered.

Experimentally, the inclusive branching ratio for $D_s$ decays into non-strange hadrons ($X_{ud}$) can be estimated to be at least $(1.65 \pm 0.35)\%$ from the Particle Data Group (PDG) [1]. For this lower bound, we only retain the three and five charged pion modes. All other modes with four, six, seven pions are disregarded because they come mainly from the quasi two-body modes $D_s \to \eta + \pi(\rho)$ and $D_s \to \eta' + \pi(\rho)$ followed by the subsequent decays of $\eta$ and $\eta'$ into pions. Since $\eta$ and $\eta'$ have a large $s\bar{s}$ component, these modes must be attributed to the dominant spectator diagram and not to the $W$ annihilation in which we are interested here.

Let us remark that with the $W$ annihilation, the decays of $D_s$ into $\eta + \pi$ and $\eta' + \pi$ vanish by the conserved vector current (CVC) in the standard factorization approach. Another remarkable feature of the experimental data is the important fraction of $D_s$ decays into non-resonant $\pi^+\pi^-\pi^+$ state. Its rate, which is already around 1/3 of the dominant spectator $\phi\pi^+$ one, is really intriguing.

In the rest of the paper, we use two different methods - appropriated to inclusive and exclusive decays - to estimate the up, down quark masses. Both approaches converge to mass values higher than the ones estimated in literature.

### 1 Inclusive Decay of $D_s$ into pions:

Compared to the pure leptonic rate $D_s \to \mu + \nu$, the inclusive decay of $D_s$ into non-strange hadrons $X_{ud}$ is given, at the effective electroweak tree level, by:

$$R_{D_s} \equiv \frac{\Gamma(D_s^+ \to X_{ud})}{\Gamma(D_s^+ \to \mu^+\nu)} = \frac{\Gamma(D_s^+ \to ud)}{\Gamma(D_s^+ \to \mu^+\nu)} = 3a_1^2|V_{ud}|^2 J(x_u, x_d) \frac{m_u^2 + m_d^2}{m_{\mu}^2 (1 - x_\mu)^2}. \quad (1)$$

By considering the ratio $R_{D_s}$ in Eq.(1), the large uncertainty in the decay constant $f_{D_s}$ can be avoided. The coefficient 3 comes from color. Here $a_1$ is the Bauer-Stech-Wirbel (BSW) phenomenological parameter [2] taken from the QCD corrected effective Lagrangian first calculated by Gaillard-Lee [3], Altarelli-Maiani [3] following the Wilson operator product expansion method [4].

$$\mathcal{L}_{eff} = \frac{G}{\sqrt{2}} V_{cs}^* V_{ud} \left[ c_1 (\bar{s}\gamma_{\mu}Lc)(\bar{u}\gamma_{\mu}Ld) + c_2 (\bar{s}\gamma_{\mu}Ld)(\bar{u}\gamma_{\mu}Lc) \right] \quad (2)$$
In the vacuum insertion approximation, known as the factorization method à la BSW
[2], the relevant effective Lagrangian appropriated to our case can be written as [2]
\[
\mathcal{L}_{\text{eff}} = \frac{G}{\sqrt{2}} V^* c V a_1 (\bar{s} \gamma_\mu L c) H (\bar{u} \gamma_\mu L d) H
\]
with \(a_1 = c_1 + \frac{1}{N_c} c_2\) and the subscript \(H\) stands for color singlet hadronic currents.

From the general fit of charm non-leptonic decays, the value of \(a_1 = 1.26\) is commonly used [5].

Finally \(J(x_u, x_d)\) in Eq.(1) is the phase-space correction factor [4] due to the quark masses
\[
J(x_u, x_d) = [1 - \frac{(x_d - x_u)^2}{x_d + x_u}] \lambda(1, x_u, x_d)
\]
with \(\lambda(1, x, y) = \sqrt{(1 - x - y)^2 - 4xy}\). \(x_{u,d,\mu} = \frac{m_{u,d,\mu}^2}{M^2}\) where \(m_u, m_d, m_\mu\) and \(M\) are respectively masses of up, down quarks, muon and charmed \(D_s\) meson.

We remark the similarity of the ratio \(R_{D_s}\) with the hadrons/lepton ratio \(R_\tau\) in the tau’s lepton decay
\[
R_\tau \equiv \frac{\Gamma(\tau^+ \to \bar{\nu}_\tau X_{ud})}{\Gamma(\tau^+ \to \bar{\nu}_\tau e^+ \nu)} = 3 |V_{ud}|^2 F(x_u, x_d)
\]
with the phase space factor [3] \(F(x_u, x_d)\) given by
\[
F(x, y) = \lambda(1, x, y) [1 - 7(x + y) - (x^2 + y^2) - 6(x - y)^2 + (x + y)(x - y)^2 - 6xy(x + y)] + 12[x^2(1 - y^2) \ln \frac{1 + x - y + \lambda(1, x, y)}{1 + x - y - \lambda(1, x, y)} + (x \leftrightarrow y)]
\]
\(J(0, 0) = F(0, 0) = 1\).

Eqs.(1) and (5) are only exact at the tree level, i.e. when QCD effects - at the final up, down vertex- are not taken into account. When gluonic effects ( up to three loops) at the final quarks vertex are kept [4], then both \(R_{D_s}\) and \(R_\tau\) in Eqs.(1) and (5) respectively have to be multiplied by a common correction factor [5] \(G\) given by [7]
\[
G = 1 + \frac{\alpha_s}{\pi} \delta_1(x_u, x_d) + \left(\frac{\alpha_s}{\pi}\right)^2 \delta_2(x_u, x_d) + \left(\frac{\alpha_s}{\pi}\right)^3 \delta_3(x_u, x_d)
\]
\(*\) Let us emphasize that there is no confusion ( or double counting) possible between the QCD correction in the effective Lagrangian Eq.(2) symbolized by the \(c_{1,2}\) coefficients on the one hand, and the QCD corrected coefficient \(G\) at the \(u, d\) vertex on the other hand.

The first one \(c_{1,2}\) are quantities that issue from the renormalization group equation that sums up large logarithmic enhancement due to gluons crossing the \(W\) line, i.e. gluons connecting the initial \(sc\) to the final \(ud\). This is why \(\tau\) decay is not concerned with this effective Lagrangian ( gluons ignore \(\tau\) lepton).

\(†\) The QCD corrected coefficient \(G\) in Eq.(7), on the other hand, operates only at the final state \(u, d\) vertex, therefore \(G\) is common to both \(R_\tau\) and \(R_{D_s}\).
with \[ \delta_1(0,0) = 1, \; \delta_2(0,0) = 5.2, \; \delta_3(0,0) = 26.36 \] (8)

In Eq.(7) non-perturbative QCD contributions turn out to be tiny and are consequently neglected.

For non-zero arguments \( x_u, x_d \), the one-loop function \( \delta_1(x,y) \) has also been computed with a rather surprising feature \( \delta_1(x,y) > \delta_1(0,0) \) recently confirmed. To our knowledge, \( \delta_2(x,y) \) and \( \delta_3(x,y) \) for non-zero \( x, y \) are not yet computed.

Now the first question that arises is which mass-current algebra mass or constituent mass must be used for the up, down quarks?

We argue for the first one (current mass) due both to theoretical and phenomenological reasons. Theoretically, in QCD, the divergence of the observable axial weak current is given in terms of current mass, as extensively discussed in Ref.[11]. The constituent mass is only appropriate for the bound state problem not considered here.

Let us also remark that the hadronization of \( u, d \) quarks into pions is conceptually different from the boundstate problem.

At the phenomenological level, the use of constituent mass \( \simeq 300 \) MeV would firstly spoil the excellent arguments in favour of QCD tests in \( \tau \) decays: indeed, the enhancement of about 20\% through the QCD factor \( G \) in Eq.(7) would be diminished by a similar amount if constituent mass is used because of the phase space factor \( F(x,y) \) of Eq.(6). Secondly, for constituent mass, the ratio \( R_{D_s} \) of Eq.(1): would be in strong disagreement with data by two orders of magnitude.

Once the principle of current algebra masses is retained, let us return to Eq.(1) and ask ourselves which numerical value must be used for \( m_u \) and \( m_d \).

This is because, contrary to \( R_\tau \) in Eq.(5) which is insensitive to exact values for current masses, the ratio \( R_{D_s} \) as given by Eq.(1) is very sensitive to them.

Light quark masses have a reputation of being "not well measurable" and even their indirect experimental determinations have so far not even been attempted, while there exists a huge number of theoretical estimates ranging between 4 MeV and 12 MeV for the average mass \( \bar{m} \) of the \( u,d \) quarks.

\[ \bar{m} = \frac{1}{2}(m_u + m_d) \] (9)

Since consensus holds for the ratio \( r = \frac{m_d-m_u}{m_d+m_u} = 0.28 \pm 0.03 \), the problem resides only on \( \bar{m} \), more precisely on the running mass \( \bar{m}(s_0) \) at the scale \( s_0 \) of few GeV$^2$.

Returning to the left-hand side of Eq.(1), we first estimate the experimental value of \( R_{D_s} \) from the PDG to be 2.8±1.2 where the branching ratio \( Br(D_s^+ \to X_{ud}) = (1.65 \pm 0.35) \times 10^{-2} \) has been used together with \( Br(D_s^+ \to \mu^+\nu) = (0.59 \pm 0.22) \times 10^{-2} \).

Within the one standard deviation limit of data for \( R_{D_s} \), Eqs.(1), (7) can only be satisfied with \( \bar{m} \) around 38 MeV - a huge value much larger than those commonly expected. In principle, there is nothing wrong with such a large \( \bar{m} \) value, although this would imply a considerably lower value for the \( - < \bar{\psi}\psi > \) condensate than the standard chiral perturbation theory could support. Therefore before taking seriously this crude 38 MeV value, we must ask ourselves the next question, what could be the
other sources that may contribute to the substantially observed inclusive branching ratio $Br(D_s \to X_{ud})$?

There exists at least two possibilities. The first one is via the so-called Zweig forbidden rule as depicted in Fig.1: the dominant colorfavoured decay mode $c \to s + (u\bar{d})$ when combined with the spectator $\bar{s}$ could induce, through a two-gluon box diagram, the chain $c\bar{s} \to (s\bar{s}) + (u\bar{d}) \to (q\bar{q}) + (u\bar{d})$ where $q$ stands for $u$ and $d$ quarks. It is the final state interaction at the quark level (appropriated to inclusive processes), in which the final state $s\bar{s}$ turns into $q\bar{q}$.

The second one, as depicted in Fig.2, is a weak annihilation accompanied by a gluon emitted from the initial $\bar{s}$ and $c$ quarks bound inside $D_s^-$ mechanism, first proposed in Ref.[14] to vitiate the helicity suppression and recently being reexamined in details [15], could à priori yield a large $D_s \to X_{ud}$ rate (since the width is no longer suppressed by $\frac{m_s^5}{M}$).

Our task now is to compute these two contributions of Fig.1 and Fig.2, that we called respectively final state strong transition (FT) and gluonic weak annihilation (GA).

1) For the first one (FT), the decay rate can be written in the form

$$\Gamma_{FT}(c\bar{s} \to (q\bar{q}) + (u\bar{d})) = \Gamma_{SP}(c \to s + (u\bar{d})) P(s\bar{s} \to gg \to q\bar{q})$$

where $\Gamma_{SP}(c \to s + (u\bar{d}))$ is the familiar spectator inclusive rate and $P(s\bar{s} \to gg \to q\bar{q})$ is the transition probability for the $s\bar{s}$ pair in the final state transforming into a $q\bar{q}$ pair through a box diagram.

The first term in Eq. (10) $\Gamma_{SP}(c \to sud)$ is given by:

$$\Gamma_{SP}(c \to s + (u\bar{d})) = 3\alpha_s^3 \frac{G^2 m_c^5}{192\pi^3} V_{cs} V_{ud}^* I(x_s, x_u, x_d) \left[ 1 - \frac{2\alpha_s}{3\pi} \left( \pi^2 - \frac{31}{4} \right) \right]$$

In Eq.(11), the phase space factor $I(x, y, z)$ is known [6] with $I(x, 0, 0) = 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x$. The QCD correction factor $C = \frac{-2\alpha_s}{3\pi} (\pi^2 - \frac{31}{4})$ corresponds only to massless $s$, $u$, $d$ quarks and can be decomposed into two parts, the “upper vertex” associated to the $c\bar{s}$ part is $\frac{-2\alpha_s}{3\pi} (\pi^2 - \frac{25}{4})$ and the “lower vertex” associated to $ud$ part is $+\frac{\alpha_s}{\pi}$. For massive $s$, $u$, $d$ quarks, both “upper” and “lower” parts have already been computed [6], the explicit analytic expression for the “upper” part is also given [10, 16].

We now compute the box diagrams (there are two, with crossed gluons). The dimensionless box diagram amplitude corresponding to $s(p_1) + \bar{s}(p_2) \to q(p_3) + \bar{q}(p_4)$ can be conveniently written as:

$$A = \left( \frac{4\alpha_s}{3} \right)^2 \frac{\bar{u}(p_2) \gamma^\mu u(p_1) \bar{u}(p_3) \gamma_\nu v(p_4)}{Q^2} B(Q^2, t, m_s^2)$$

with

$$Q^2 = (p_1 + p_2)^2, \; t = (p_1 - p_3)^2$$

(12)
The dimensionless quantity \( B(Q^2, t, m_s^2) \) coming from loop integration has the following representation (we neglect \( m_q^2 \))

\[
B(Q^2, t, m_s^2) = \int_0^1 dx \int_0^1 dy \int_0^1 dzz(1-z) \frac{Q^2[Q^2x(1-x)(1-z)^2 - t(1-z) - m_s^2 yz^2]}{D^2(x, y, z, Q^2, t, m_s^2)}
\]

where

\[
D(x, y, z, Q^2, t, m_s^2) = -Q^2x(1-x)(1-z)^2 - ty(1-y)z^2 + m_s^2 yz^2
\]

Its explicit expression is given by:

\[
B(Q^2, t, m_s^2) = \ln^2(\frac{\eta_1}{\eta_2}) + \frac{1}{\xi_1 - \xi_2} \mathcal{L}(Q^2, t, m_s^2)
\]

\[
+ \frac{Q^2}{Q^2 + t} \left\{ \frac{Q^2 + 2t + 2m_s^2}{2(Q^2 + t)} [L_{i2}(1 + \frac{t}{m_s^2}) - \frac{\pi^2}{6} - \ln^2(\frac{\eta_1}{\eta_2})] \right. \\
+ \frac{1}{2} L_{i2}(-\frac{t}{m_s^2}) + \frac{\eta_2 - \eta_1}{2} \ln(\frac{\eta_1}{\eta_2}) \\
+ \left. \frac{Q^2 + 2t - 4m_s^2 x^2 + 2m_s^4}{2(Q^2 + t)(\xi_2 - \xi_1)} \mathcal{L}(Q^2, t, m_s^2) \right\}
\]

with

\[
\mathcal{L}(Q^2, t, m_s^2) = L_{i2}(\frac{\xi_1}{\xi_1 - \xi_2}) + L_{i2}(\frac{\xi_1}{\xi_1 - \eta_2}) - L_{i2}(\frac{\xi_2}{\xi_2 - \eta_1}) - L_{i2}(\frac{\xi_2}{\xi_2 - \eta_2})
\]

\[
2\eta_{1,2} = 1 \pm \sqrt{1 - \frac{4m_s^2}{Q^2}}
\]

\[
2\xi_{1,2} = 1 \pm \sqrt{1 - \frac{4m_s^2}{Q^2}(1 + \frac{m_s^2}{t})}
\]

where \( L_{i2} \) is the Spence or dilogarithmic function.

Once the box amplitude \( \mathcal{A} \) for \( s\bar{s} \rightarrow g\bar{g} \rightarrow q\bar{q} \) is known, the transition probability \( Y \equiv P(s\bar{s} \rightarrow g\bar{g} \rightarrow q\bar{q}) \) in Eq.(10) is then obtained by:

\[
Y = \frac{1}{4\pi^2} \int \frac{d\vec{p}_3}{2E_3} \int \frac{d\vec{p}_4}{2E_4} \delta^4(Q - p_3 - p_4)|\mathcal{A}|^2
\]

Since \( \int \frac{d\vec{p}_3}{2E_3} \int \frac{d\vec{p}_4}{2E_4} \delta^4(Q - p_3 - p_4) = \frac{\pi}{4} \), the quantity \( Y \) can be conveniently rewritten as

\[
Y = \left( \frac{4\alpha_s}{3} \right)^4 \frac{1}{\pi} \mathcal{A}(Q^2, m_s^2)
\]

The range of variation for \( Q^2 \):

\[
4m_s^2 \leq Q^2 \leq \frac{m_s}{m_c}(m_s + m_c)^2
\]
is deduced from $0 \leq t^2 \leq (m_c - m_s)^2$ where $l$ is the invariant mass of the $u\bar{d}$ pair issuing from the spectator mechanism $c \rightarrow s + (u\bar{d})$.

The angular integration over the variable $t$ in Eq.(17) is done numerically, and $A(Q^2, m_s^2)$ turns out to be inside $0.36 \leq A(Q^2, m_s^2) \leq 0.95$ for $Q^2$ inside Eq.(18), using $m_s = 0.15$ GeV, $m_c = 1.45$ GeV.

With these values put into Eqs.(10), (11) and (18), the FT contribution yields a branching ratio $\text{Br}(D_s \rightarrow \text{pions})_{\text{FT}} \leq 0.68\%$ which is still far from the experimental data, which at least equals to $(1.65 \pm 0.35)\%$.

2) For the gluonic weak annihilation (GA) of Fig.2, the amplitude can be directly taken from Eq.(4) of Ref.[14] with the following substitution:

$$q \Rightarrow k_1 + k_2 \ , \ g_s \epsilon_j^\nu(q) \Rightarrow g_s^2 \bar{u}(k_1) \gamma^\nu \frac{\lambda_j}{2} v(k_2) \frac{1}{(k_1 + k_2)^2},$$

(20)

where $k_1$ and $k_2$ are the $q$ and $\bar{q}$ momentum, and $j$ denotes the color index.

We have

$$A = \frac{g_s^2 G|V_{cs}^* V_{ud}|}{\sqrt{2}} \cdot \frac{R_{\mu\nu}(p, k_1 + k_2)}{(k_1 + k_2)^2} \cdot \bar{u}(q_1) \gamma^\mu (1 - \gamma_5) v(q_2) \bar{u}(k_1) \gamma^\nu \frac{\lambda}{2} v(k_2),$$

(21)

with

$$R_{\mu\nu}(p, q) = \frac{M}{(p.q)} \{(F_A(q_\mu p_\nu - (q.p) g_{\mu\nu}) + i F_V \epsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta)\},$$

(22)

where $F_{V,A}$ are the two dimensionless form factors somehow reflected the wave function at the origin of the $c\bar{s}$ bound state, which in turn is proportional to the decay constant $f_{D_s}$ via the Van Royen-Weisskopf formula. The tensor $R_{\mu\nu}$ is reminiscent of the standard fermionic triangular loop describing $W^+ \rightarrow \gamma \pi^+$ or $\pi^0 \rightarrow \gamma \gamma$, taken as an example.

Here $p = q_1 + q_2 + k_1 + k_2$ with $p^2 = M^2$. Neglecting $m_{u,d}^2$ compared to $M^2$, the four-body phase space integration of (spin and color summing up) $|A|^2$ can be simplified, and the decay rate is computed to be:

$$\Gamma_{GA}(D_s \rightarrow u\bar{d} + q\bar{q}) = a_1^2 G^2 |V_{cs}^* V_{ud}|^2 \frac{\alpha_s^2(F_V^2 + F_A^2)}{12\pi^3 M} \int_0^{M^2} dt_1 t_1 (M^2 - t_1)^2 \int_{t_1}^{M^2} dt_2 \frac{(t_2 - t_1) (M^2 - t_2)}{t_2^2 (M^2 - t_1 + t_2)^2},$$

(23)

with $t_1 = (q_1 + q_2)^2$, $t_2 = (k_1 + k_2)^2$.

The double integration over $t_1, t_2$ can be done analytically and the result is

$$\Gamma_{GA}(D_s \rightarrow u\bar{d} + q\bar{q}) = a_1^2 G^2 |V_{cs}^* V_{ud}|^2 \frac{\alpha_s^2(F_V^2 + F_A^2)}{12\pi^3 M^5} \left(\frac{\pi^2}{6} + \frac{10}{3} \ln 2 - \frac{71}{18}\right).$$

(24)

Following Ref.[14], we will take

$$F_V = \frac{f_{D_s}}{\sqrt{12m_s}} , \quad F_A = \frac{f_{D_s}(m_s - m_c)}{\sqrt{12m_sm_c}}.$$
Numerically, it turns out that the $\Gamma_{GA}$ rate is extremely small, the corresponding branching ratio $Br(D_s \to pions)$ via gluonic weak annihilation mechanism is at most (0.1)%. In Eqs.(12) and (24) we take $\alpha_s = 0.3$.

Putting altogether both $FT$ and $GA$ contributions, the inclusive branching ratio $Br(D_s \to pions)$ due to these gluonic processes cannot exceed 0.8% and is still far from the observed inclusive branching ratio larger than $(1.65 \pm 0.35)$% as estimated from PDG [1].

The difference between these two numbers must be attributed to the pure $W$ annihilation tree diagram ‡, from which at the one standard deviation lower bound of data, we get $\bar{m} \simeq 22$ MeV using again Eqs.(1) and (7).

This value, although smaller than the crude 38 MeV obtained above (when the gluonic backgrounds are neglected) is still at least twice as large as the common estimates.

2 Exclusive three pion mode: Mass determination from QCD sum rules

We now consider the exclusive decay mode $D_s^+ \to \pi^+\pi^+\pi^-$ which allows us to extract the spectral function $\rho(Q^2)$ that will be in turns exploited in the QCD sum rules to obtain $\bar{m}$.

The starting point is the QCD sum rule [17] for the two-point correlator of the divergence of the axial-current, put in the form [11]:

$$\bar{m}^2(s_0) = H^{-1}(w, s_0) \int_0^\infty dQ^2 w(Q^2, s_0) \rho(Q^2)$$  \hspace{1cm} (26)

We also consider its finite energy version closely followed Ref.[18]

$$\bar{m}^2(\mu, s_0) \equiv \left( \frac{lns_0/\Lambda^2}{ln\mu^2/\Lambda^2} \right)^{\frac{21}{3}} \frac{4\pi^2}{3s_0^2} [1 + R_2(s_0) + 2c_4 < O_4 > / s_0^2 ]^{-1} \{ f_\pi^2 m_\pi^4 + \int_0^{s_0} dQ^2 \rho(Q^2) \}$$  \hspace{1cm} (27)

In Eq.(26), $w(Q^2, s_0)$ denotes weight function and $H(w, s_0)$ is defined by the large $Q^2$ behavior of the two-point correlator, it has a perturbative QCD part and a non-perturbative part parametrized in terms of vacuum condensates. Also the two-loop expression for $R_2(s_0)$ as well as the dimension-4 condensate $c_4 < O_4 >$ in Eq.(27) can be found in Refs.[18, 22].

These types of sum rules have been extensively used in the literature to estimate quark mass and as explained in Ref.[14], the main problem in these estimates is not the sum-rule technique itself but rather the complete absence of experimental information on the magnitude of $\rho(Q^2)$ beyond the one-pion contribution. The later $f_\pi^2 m_\pi^4$ is

Note: The $W$ annihilation and the two gluonic contributions have different decay products, therefore they contribute incoherently to the $D_s \to pions$ rate, without interference.
singled out in Eq.\((24)\) with \(f_\pi \simeq 132\) MeV. Keeping only this pion contribution, one finds \([11, 18]\) a lower bound \(\bar{m}(1\) GeV\() > 4(5)\) MeV due to the positivity of \(\rho(Q^2)\).

Fortunately, we show that the decay mode \(D_s \rightarrow 3\pi\) provides precious information on \(\rho(Q^2)\) at fixed \(Q^2 = M^2\). Indeed, the decay amplitude \(D_s^+ \rightarrow \pi^+\pi^+\pi^-\) is given by

\[
\frac{G}{\sqrt{2}} V_{cs} V_{ud} a_1 f_{D_s} Q_\mu < \pi^+(p_1)\pi^+(p_2)\pi^- (p_3) |A^\mu| 0 > \tag{28}
\]

where the most general matrix element \(< 3\pi | A^\mu | 0 >\) can be expressed in terms of three form factors \([19]\), the same occurred in \(\tau \rightarrow \nu + 3\pi\). However, for \(D_s \rightarrow 3\pi\), only one dimensionless form factor \([20]\) associated to the divergence of the axial current is involved and will be denoted by \(F(s_1, s_2)\):

\[
Q_\mu < \pi^+(p_1)\pi^+(p_2)\pi^- (p_3) |A^\mu| 0 >= M F(s_1, s_2) \tag{29}
\]

with \(s_i = (Q - p_i)^2, i = 1, 2\).

In terms of \(F(s_1, s_2)\), the \(D_s \rightarrow 3\pi\) rate is given by Ref.\([20]\):

\[
\Gamma(D_s \rightarrow 3\pi) = a_1^2 G^2 |V_{cs} V_{ud}|^2 M^3 f_{D_s}^2 K, \tag{30}
\]

where the dimensionless constant \(K\) is obtained by integrating the squared form factor \(|F(s_1, s_2)|^2\) over the whole Dalitz domain:

\[
K \equiv \frac{1}{4M^4} \int \int ds_1 ds_2 |F(s_1, s_2)|^2 \tag{31}
\]

\[
4m_\pi^2 \leq s_1 \leq (M - m_\pi)^2 \quad s_{min}(s_1) \leq s_2 \leq s_{max}(s_1) \tag{32}
\]

\[
s_{min, max}(s_1) = \frac{M^2 + s_1 - m_\pi^2}{2} \pm \sqrt{\frac{s_1 - 4m_\pi^2}{s_1}} \frac{\lambda(M^2, m_\pi^2, s_1)}{2}.
\]

The numerical value of \(K\) that we can extract from Eq.\((31)\) has two sources of errors, the first one is related to the experimental errors of the non-resonant branching ratio \(Br(D_s^+ \rightarrow \pi^+\pi^+\pi^-)_{NR} = (1.01 \pm 0.35) \times 10^{-2}\), the second one is the uncertainty of the decay constant \(f_{D_s}\). Fixing \(f_{D_s} = 280\) MeV\([1]\), we get \([20]\) \(K = 0.486 \pm 0.168\) where errors come from the ones of the experimental \(D_s \rightarrow \pi^+\pi^+\pi^-\) branching ratio. Let \([1]\) \(f_{D_s} = 280 \pm 70\) MeV, then we have:

\[
0.27 \pm 0.06 \leq K \leq 0.74 \pm 0.17 \tag{33}
\]

Now the crucial point is that the constant \(K\) can be directly related to the spectral function \(\rho_{3\pi}(Q^2)\) entering in the QCD sum rule Eqs.\((26),(27)\). At \(Q^2 = M^2\), we get:

\[
\rho_{3\pi}(Q^2 = M^2) = \frac{K}{64\pi^4 M^4}. \tag{34}
\]

\(^\ddagger\) Our dimensionless form factor \(F(s_1, s_2)\) is related to the \(F_4\) form factor of Ref.\([19]\) by \(F(s_1, s_2) = M F_4(s_1, s_2, Q^2 = M^2)\)
that at large $Q$, in terms of one unknown parameter ($\hat{\rho}_{\text{spectral function}}$ Eqs.(26) or (27) need the whole range of the low energy end of the $3\pi$ finite energy sum rule Eq.(27) with the lowest order chiral perturbation theory. Following Ref.[22], we will make use of the overall normalization $\lambda$ given in Ref.[22], and that is the crucial reason for getting high value of $\bar{m}$. Since the normalization at $Q^2 = M^2$ for $\rho_{3\pi}(Q^2 = M^2)$ is fixed by $\frac{5K}{256\pi^4} M^4$, let us follow the usual procedure [21] by adopting the simplest duality ansatz parametrization for $\rho(Q^2)$:

$$\rho(Q^2) = \frac{5K}{256\pi^4} M^2 Q^2 \left(\frac{\ln M^2/\Lambda^2}{2\sqrt{\pi}}\right)^{\frac{1}{2}} \frac{1 + \frac{17}{3} \alpha_s(Q^2)}{1 + \frac{17}{3} \alpha_s(M^2)}$$

which reduces to $\rho(M^2) = \frac{5K}{256\pi^4} M^4$ as it should be.

Putting now Eq.(38) into the sum rule Eq.(26), and taking for $K$ the one standard deviation most conservative lower limit in Eq.(33), we then obtain for $\bar{m}$ (1 GeV) a quite large value $20 \text{ MeV}$, consistent with our previous result from the inclusive decay $D_s \to X_{ud}$.

A different parametrization for $\rho(Q^2)$ can be obtained by making use of the shape of the low energy end of the $3\pi$-threshold $\rho_{3\pi}(t)$ calculated in a recent paper [22] using lowest order chiral perturbation theory. Following Ref.[22], we will make use of the finite energy sum rule Eq.(27) with $\rho(Q^2)$ given by

$$\rho(Q^2) = \lambda \frac{m_\pi^4}{64\pi^4 f_{\pi}^2} \frac{Q^2}{18} \rho_{\text{had}}(Q^2)$$

where $\rho_{\text{had}}(Q^2)$ is taken from Eq.(32) of Ref.[22] encoding the presence of two $\pi'$ resonances, the $1300 \text{ MeV}$ and $1770 \text{ MeV}$.

Here $\lambda$ is an overall normalization which, as emphasized by the authors of Ref.[11], can only be determined by experimental informations and not by theoretical calculations. Matching Eq.(34) to Eq.(39) at $Q^2 = M^2$, one can fix $\lambda$. It turns out that our overall normalization $\lambda$ provided by $D_s \to 3\pi$ decay rate is much larger than the one given in Ref.[22], and that is the crucial reason for getting high value of $\bar{m}$. 

$$\rho(Q^2) = \frac{1}{2\pi} \sum_n (2\pi)^4 \delta^4(Q-p_n) | < n | \partial_\mu A^\mu | 0 > |^2$$

and in particular for $\rho_{3\pi}(Q^2)$:

$$\rho_{3\pi} = \frac{1}{2\pi} \int \int \int \frac{d\bar{p}_1}{2E_1} \frac{d\bar{p}_2}{2E_2} \frac{d\bar{p}_3}{2E_3} (2\pi)^3 \delta^3(Q-p_1-p_2-p_3) | < 3\pi | A^\mu | 0 > |^2$$

In Eqs.(33) and (34), the constant $K$ and consequently the $\rho_{3\pi}(M^2)$ are only related to the charged $D_s^+ \to \pi^+\pi^+\pi^-$ mode. The neutral mode $D_s^+ \to \pi^0\pi^0\pi^+$ by isospin consideration [20] is presumably $1/4$ of the charged one. Hence the total charged and neutral must be $5/4$ of Eq.(34).

As shown in Eqs.(30) and (34), from $D_s \to 3\pi$ data, we get information for the spectral function $\rho(Q^2)$ at only one fixed value of $Q^2 = M^2$, while the QCD sum rules Eqs.(26) or (27) need the whole range of $Q^2$ in $\rho(Q^2)$. Fortunately, we also know that at large $Q^2$, perturbative QCD gives [18] explicit analytic expression for $\rho(Q^2)$ in terms of one unknown parameter $(\bar{m})^2$:

$$\rho(Q^2) \to \frac{3}{2\pi^2} \left(\frac{\bar{m}}{M^2}\right)^2 \left[ 1 + \frac{17}{3} \alpha_s(Q^2) \right]$$

Since the normalization at $Q^2 = M^2$ for $\rho_{3\pi}(Q^2 = M^2)$ is fixed by $\frac{5K}{256\pi^4} M^4$, let us follow the usual procedure [21] by adopting the simplest duality ansatz parametrization for $\rho(Q^2)$:

$$\rho(Q^2) = \frac{5K}{256\pi^4} M^2 Q^2 \left(\frac{\ln M^2/\Lambda^2}{2\sqrt{\pi}}\right)^{\frac{1}{2}} \frac{1 + \frac{17}{3} \alpha_s(Q^2)}{1 + \frac{17}{3} \alpha_s(M^2)}$$

where $\rho_{\text{had}}(Q^2)$ is taken from Eq.(32) of Ref.[22] encoding the presence of two $\pi'$ resonances, the $1300 \text{ MeV}$ and $1770 \text{ MeV}$.
3 Summary and Conclusion

The starting point is our observation that the decay rate of the charmed $D_s$ meson into pions is surprisingly larger than common estimates. The amplitude being governed by the $W$ annihilation is usually expected to be negligible by the partial conservation of the axial current (PCAC) at relatively high energy.

We then go beyond the tree $W$ annihilation diagram and consider two possible contributions. The first one, called finite state strong transition (FT) coming from the dominant spectator decay $c\bar{s} \to s\bar{s} + u\bar{d}$ followed by the Zweig violating rule $s\bar{s} \to q\bar{q}$ ($q = u, d$) via the two-gluon box diagram. The second one, called gluonic weak annihilation (GA), is a genuine weak annihilation accompanied by an emission of one gluon from the quarks $c, \bar{s}$ bound inside $D_s$, the mechanism that invalidates the helicity suppression.

We find out that these gluonic effects together cannot explain the large inclusive $D_s$ decay into pions, and consequently we suggest that $W$ annihilation (tree diagram) also contributes substantially to the rates. This would imply that $\bar{m}$, the averaged $u, d$ mass, could be around 22 MeV, few times as large as the usual estimates [13].

One might argue that our quark parton type of analysis for inclusive decays of charm could only be accurate within a factor of two. However, by analyzing the exclusive $D_s \to 3\pi$ mode, we obtain the normalized spectral function $\rho_{3\pi}(Q^2)$, which is in turns exploited in the QCD sum rules. Again, by this completely different method, a higher value than common estimates is obtained for $\bar{m}$.

Our lower bound of 22 MeV for $\bar{m}$ depends, on the one hand, on the experimental data of both inclusive and exclusive $D_s$ decays into pions, and on the other hand, on the theoretical factorization method à la BSW. To obtain such a mass value, we carefully take into account experimental errors as well as the uncertainty of the decay constant $f_{D_s}$.

Independent of our proposition for a high $\bar{m}$ as a solution to the substantial rates of $D_s$ into pions, the understanding of the origin of such data constitutes a very interesting problem in its own right. The confirmation of experimental data is equally important, on the other side.

Finally we would like to emphasize the similarity as well as the complementarity between $\tau$ lepton and charmed $D_s$ meson decaying into (three and more) pions: While $\tau$ probes the dominant spin 1 and the small spin 0 axial spectral functions in the whole range of the momentum transfer $0 \leq Q^2 \leq M^2_\tau$ (to separate them is a challenging experimental problem [11]), the $D_s$ probes directly and easily the spin 0 spectral function at only one value $Q^2 = M^2_{D_s}$. This latter is the key for the determination of the light quark mass, a fundamental parameter "not well measured" in the standard model. Its importance in chiral symmetry breaking from both perturbative and nonperturbative aspects is well-known.
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References

[1] Particle Data Group PDG, Review of Particles Properties, Phys.Rev. D50, August 1994.

[2] M.Bauez, B.Stech and M.Wizbel, Z.Phys. C34 (1987), 103.

[3] M.K.Gaillard and B.W.Lee, Phys.Rev.Lett. 33 (1974), 108; G.Altarelli and L.Maiani, Phys.Lett. B32 (1974), 351.

[4] K.G.Wilson, Phys. Rev.179 (1969), 1499; See the update computation by A.J.Buras, M.Jamin, M.E.Lautenbacher and P.H.Weisz, Nucl.Phys. B370 (1992), 69.

[5] A.J.Buras, MPI-PhT/94-60 Max Planck Institut für Physik preprint, September 1994; A review can be found in M.Neubert, V.Rieckert, B.Stech and Q.P.Xu in Heavy Flavours. Ed.A.Buras (World Scientific 1992).

[6] J.L.Cortès, X.Y.Pham and A.Tounsi, Phys.Rev. D25, (1982), 188.

[7] E.Braaten, Phys.Rev. D39 (1989), 1458; The coefficient of the ($\alpha_s$)3 term taken from S.A.Gorishny, A.L.Kateev and S.A.Larin, Phys.Lett. B212 (1988), 238 is corrected by a factor of $\frac{1}{4}$ given here.

[8] E.Braaten, S.Narison and A.Pich, Nucl.Phys. 373 (1992), 581.

[9] Q.Ho-Kim and X.Y.Pham, Ann.Phys.155 (1984), 202; Phys.Lett. B122 (1983), 297.

[10] E.Bagan, P.Ball, V.M.Braun and P.Gosdzinsky Nucl.Phys. B432 (1994), 3; M.B.Voloshin, Phys. Rev. D51 (1995),3948.

[11] J.Stern, N.H.Fuchs and M.Knecht, Proc. of the 3rd Workshop on the Tau-Charm factory, Preprint IPN/TH 93-38.

[12] F.Le Diberder, Proc. of the 3rd Workshop on the Tau-Charm factory; See also I.Hinchliffe in Ref[1] p.1297.

[13] Some examples are S.Weinberg, Trans.N.Y.Acd.Sc. 38 (1977), 185; J.Gasser and H.Leutwyler Phys.Rep. 87 (1982), 77; C.Becchi, S.Narison, E.de Rafael and F.J.Yndurain, Z.Phys. C8 (1981), 335; A.L.Katev, N.Y.Krasnikov and A.A.Pivanov, Phys.Lettt. B123 (1983), 93; C.A.Dominguez and E.de Rafael, Ann.Phys. 174 (1987), 372; See also the review in Ref.[1] p.1435.
[14] M.Bander, D.Silverman and A.Soni, Phys.Rev.Lett. 44 (1980), 7; Errata 44 (1980), 962.

[15] I.I.Bigi and N.G.Uraltsev, Phys. Lett. B280 (1992), 271; Nucl.Phys. B423 (1994), 33.

[16] Y.Nir, Phys.Lett B221 (1989), 184.

[17] M.A.Shifman, A.I.Vainshtein and V.I.Zakharov, Nucl.Phys B147 (1979), 385, 448.

[18] See C.A.Dominguez and E.de Rafael, in Ref.[13].

[19] J.H.Kühn and E.Mirkes, Z.Phys.C56 (1992), 661.

[20] M.Gourdin, Y.Y.Keum and X.Y.Pham, Paris Preprint PAR/LPTHE/95-09 (March 1995) , hep-ph 9503326 (Revised version).

[21] S.Narison, QCD spectral sum rules University of Montpellier II preprint PM 95/05 hep-ph 9503234

[22] J.Bijnens, J.Prades and E.de Rafael, Phys.Lett B348 (1995), 226.
