Ad \((G)\) is of type \(R\) implies \(G\) is of type \(R\) for certain \(p\)-adic Lie groups

C. R. E. Raja

Abstract

We provide a sufficient condition for a \(p\)-adic Lie group to be of type \(R\) when its adjoint image is of type \(R\).

2000 Mathematics Subject Classification: 22E15, 22E20.

Key words: \(p\)-adic Lie group, type \(R\), adjoint representation.

Let \(G\) be a \(p\)-adic Lie group and \( \mathfrak{g} \) be the Lie algebra of \(G\). Then there is an analytic homomorphism \(\text{Ad}: G \rightarrow \text{GL}( \mathfrak{g})\), called the adjoint representation, such that \(\text{Ad}(x)\) is the differential of the inner-automorphism \( g \mapsto xgx^{-1} \) on \(G\) defined by \(x\). We refer to \[Bo-89\] and \[Se-06\] for basics and results concerning \(p\)-adic Lie groups.

We say that a \(p\)-adic Lie group \(G\) is of type \(R\) if the eigenvalues of \(\text{Ad}(x)\) are of \(p\)-adic absolute value one for any \(x \in G\).

Example 1 We now give some examples of \(p\)-adic Lie groups.

(1) Abelian groups: \(p\)-adic vector spaces such as \(\mathbb{Q}_p^k\).

(2) \(p\)-adic Heisenberg group: \(\{(a, x, z) \mid a, x \in \mathbb{Q}_p, \ z \in \mathbb{Q}_p\}\) with multiplication given by
\[
(a, x, z)(a', x', z') = (a + a', x + x', z + z' + \langle a, x' \rangle)
\]
where \(\langle u, v \rangle = \sum u_i v_i\) for any two \(u, v \in \mathbb{Q}_p^k\).

(3) The solvable group \((\mathbb{Q}_p \setminus \{0\}) \ltimes \mathbb{Q}_p\): \(\{(a, x) \mid a \in \mathbb{Q}_p \setminus \{0\}, \ x \in \mathbb{Q}_p\}\) with multiplication given by
\[
(a, x)(a', x') = (aa', x + ax').
\]

(4) In general any closed subgroup of \(GL_n(\mathbb{Q}_p)\) (see \[Se-06\] and \[Bo-89\]).
Among the above examples, (1) and (2) are of type R but (3) is not of type R.

For an automorphism $\alpha$ of $G$, we define the subgroups $U_\pm(\alpha)$ by $U_\pm(\alpha) = \{ x \in G \mid \lim_{n \to \pm\infty} \alpha^n(x) = e \}$: when only one automorphism is under consideration, we write $U_\pm$ instead of $U_\pm(\alpha)$. The study of these subgroups for innerautomorphisms plays a crucial role in proving type R. For instance, the following observation which is essentially contained in [Wa-84] and Theorem 1 of [Ra-99].

Proposition 1 A $p$-adic Lie group $G$ is of type R if and only if $U_\pm(\alpha)$ is trivial for any innerautomorphism $\alpha$.

Proof If $G$ is of type R and $\alpha$ is any innerautomorphism, then by Theorem 3.5 and Corollary 1 of [Wa-84] we get that $G$ has arbitrarily small compact open subgroups stable under $\alpha$. Since $U_\pm(\alpha)$ is contained in any $\alpha$-stable open subgroup, $U_\pm(\alpha)$ is trivial.

Conversely, suppose $U_\pm(\alpha)$ is trivial for any innerautomorphism $\alpha$ of $G$. Then by Theorem 3.5 (iii) of [Wa-84] we see that condition (2) of Theorem 1 in [Ra-99] is satisfied and hence $G$ is of type R.

In this note we would like to explore the question: $\text{Ad}(G)$ is of type R implies $G$ is of type R.

The answer to this question is positive if the kernel of Ad is the center of $G$ but for a $p$-adic Lie group kernel of Ad need not be the center of $G$ (see Example 2). It may be noted that for Zariski-connected $p$-adic algebraic groups, kernel of Ad is the center of $G$.

The following example shows that $\text{Ad}(G)$ is of type R need not imply $G$ is of type R, that is, answer to our question is not always positive.

Example 2 Let $G = \{(n, a, x, z+\mathbb{Z}_p) \mid n \in \mathbb{Z}, a, x, z \in \mathbb{Q}_p \}$ with multiplication given by

$$(n, a, x, z+\mathbb{Z}_p)(m, b, y, z'+\mathbb{Z}_p) = (n+m, a+p^n b, x+p^{-n} y, z+z'+p^{-n} < a, y > +\mathbb{Z}_p).$$

Let $\alpha: G \to G$ be the innerautomorphism defined by $(n, 0, 0, 0)$. Then $U_+ = \{(0, a, 0, 0) \mid a \in \mathbb{Q}_p \}$ and $U_- = \{(0, 0, x, 0) \mid x \in \mathbb{Q}_p \}$. But $\text{Ad}(G) \cong \mathbb{Z}$, hence $\text{Ad}(G)$ is of type R but $G$ is not of type R as $p$ and $p^{-1}$ are eigenvalues of innerautomorphism given by $(n, 0, 0, 0)$.

Following provides a sufficient condition for affirmative answer.

Theorem 1 Let $G$ be a $p$-adic Lie group such that $\text{Ad}(G)$ is of type R and open subgroups of $G$ and their quotients are unimodular. Suppose any topologically finitely generated subgroup of the kernel of the adjoint representation is a discrete extension of a $k$-step solvable group for some $k > 0$. Then $G$ is also of type R.
Remark 1 The condition that open subgroups of $G$ and their quotients are unimodular is necessary. If $G$ is of type $R$, then any open subgroup and its quotients are also of type $R$. Now it follows from Theorem 1 and Corollary 1 of [Ra-99] that open subgroups of $G$ and their quotients are unimodular.

We first introduce a few notations. For any two closed subgroups $A$ and $B$ of a Hausdorff topological group $X$. Let $C_0(A, B) = \langle A, B \rangle$ denote the closed subgroup generated by $A$ and $B$ in $X$, and $C_1(A, B) = [A, B], C_k(A, B) = [C_{k-1}(A, B), C_{k-1}(A, B)]$ for all $k > 1$. If $A = B$, let $C_k(A) = C_k(A, A)$ for all $k \geq 1$. $X$ is called a $k$-step solvable group if $C_k(X) = \{e\}$.

We next develop a few general results.

Lemma 1 Let $G$ be a $p$-adic Lie group such that Ad $(G)$ is of type $R$. For $g \in G$, if $\alpha$ is the inner-automorphism of $G$ defined by $g$, then there is a $\alpha$-invariant compact subgroup $O$ such that $O$ is centralized by $U_+ (\alpha)$ and $U_+ (\alpha) O U_-(\alpha)$ is an open subgroup of $G$.

Proof Let $\alpha$ be the inner-automorphism of $G$ defined by $g \in G$ and $U_\pm = \{x \in G \mid \lim_{n \to \pm \infty} \alpha^n(x) = e\}$. Since Ad $(G)$ is of type $R$, $U_\pm \subset \text{Ker (Ad)}$. It follows from [Wa-84] that $U_\pm$ are unipotent closed subgroups of $G$. Since Lie algebra of $\text{Ker (Ad)}$ is the center of the Lie algebra of $G$, $U_\pm$ is abelian. Since Lie algebra of $\text{Ker (Ad)}$ is abelian, $\text{Ker (Ad)}$ contains a compact open abelian subgroup (cf. Corollary 3, Section 4.1, Chapter III of [Bo-89]). Thus, there exists compact open subgroups $K_\pm \subset U_\pm$ such that $K_+ K_-$ is an abelian group. Since $\alpha^\pm |_{U_\pm}$ is a contraction, we may assume that $\alpha^{-i}(K_\pm)$ is increasing and $U_\pm = \cup_{i \geq 0} \alpha^{-i}(K_\pm)$. Since $K_\pm$ are compact subgroups of $U_\pm$ which are $p$-adic vector spaces, $K_\pm$ have a dense finitely generated subgroup. Hence $K_+ K_-$ has a dense finitely generated subgroup. Since $K_+ K_-$ is contained in the kernel of the adjoint representation, each element of $K_+ K_-$ centralize an open subgroup of $G$ (cf. Theorem 3, Section 7, Chapter III of [Bo-89]). Since $K_+ K_-$ has a dense finitely generated subgroup, $K_+ K_-$ centralize an open subgroup of $G$.

Let $M = \{x \in G \mid \overline{\{\alpha^n(x)\}} \text{ is compact}\}$. Then $M$ is a closed $\alpha$-stable subgroup of $G$ and $M$ contains arbitrarily small $\alpha$-stable compact open subgroups. Since $K_+ K_-$ centralize an open subgroup of $G$, there exists a $\alpha$-stable compact open subgroup $O$ of $M$ such that $O$ is centralized by $K_+ K_-$. Since $\alpha(O) = O$ and $\cup \alpha^n(K_\pm) = U_\pm$, we get that $O$ is centralized by $U_\pm$. It also follows from Theorem 3.5 (iii) of [Wa-84] that $K_+ O K_-$ is an open (subgroup) in $G$.

Lemma 2 Let $G$ be a $p$-adic Lie group and $\alpha$ be an automorphism of $G$ such that the closed subgroup $P$ generated by $U_\pm (\alpha)$ is a discrete extension of a solvable group.
Suppose Haar measures on $P$ and any of its quotients is $\alpha$-invariant. Then $P$ is trivial.

**Proof** The group $Q := P/C_1(P)$ is abelian and $\alpha$ defines a factor automorphism on $Q$ which will be denoted by $\beta$. Since $Q$ is also a $p$-adic Lie group, by [Wa-84] we get that $U_\pm(\beta)$ are also closed and normal subgroups of $Q$.

Consider $Q/U_\pm(\beta)$. Let $\delta$ be the factor automorphisms of $\beta$ defined on $Q/U_\pm(\beta)$. It is easy to see that $U_\pm(\alpha)C_1(P) \subset U_\pm(\beta)$ and $U_+(\beta)U_-(\beta) \subset U_+(\delta)$: in fact, $U_\pm(\alpha)C_1(P) = U_\pm(\beta)$ and $U_+(\beta)U_-(\beta) = U_+(\delta)$ follows from [BaW-04]. Since $P$ is the closed subgroup generated by $U_-(\alpha)$ and $U_+(\alpha)$, $U_+(\delta) = Q/U_\pm(\beta)$. This shows that $Q/U_\pm(\beta)$ is contracted by $\delta$, that is, $\delta^n(x) \to e$ as $n \to \infty$ uniformly on compact sets (see [Wa-84]). But by assumption $\delta$ preserves the Haar measure on $Q/U_\pm(\beta)$, hence $Q/U_\pm(\beta)$ is trivial. This implies that $P/C_1(P) = U_\pm(\beta)$. This implies that $P/C_1(P)$ is contracted by $\beta^{-1}$. Since $\beta$ preserves the Haar measure on $P/C_1(P)$, we get that $P/C_1(P)$ is trivial. Thus, $P \subset C_1(P)$. Since $P$ is a discrete extension of a solvable group, $P$ is discrete. Since $P$ is generated by $U_\pm$, $P$ is trivial.

**Proof of Theorem** Let $\alpha$ be the inner-automorphism of $G$ defined by $g \in G$ and $U_\pm = \{x \in G | \lim_{n \to \pm \infty} \alpha^n(x) = e\}$. Since $\text{Ad} (G)$ is of type $R$, $U_\pm \subset \text{Ker} (\text{Ad})$. Since $\alpha^\pm |_{U_\pm}$ is a contraction, there are compact open subgroups $K_\pm$ in $U_\pm$ such that $\alpha^{-i}(K_\pm)$ is increasing and $U_\pm = \bigcup_{i \geq 0} \alpha^{-i}(K_\pm)$.

By assumption $C_k(\alpha^i(K_-), \alpha^{-i}(K_+))$ is discrete. Since $U_\pm = \bigcup_{i \geq 0} \alpha^{-i}(K_\pm)$, we get that $C_k(U_+, U_-) = \bigcup_{i \geq 0} C_k(\alpha^i(K_-), \alpha^{-i}(K_+))$ (because an increasing union of closed subgroups is closed in a $p$-adic Lie group). Thus, $C_k(U_+, U_-)$ is a countable group, hence discrete.

By Lemma there is a $\alpha$-invariant compact subgroup $O$ such that $O$ is centralized by $U_\pm$ and $U_+OU_-$ is an open subgroup of $G$. Let $N = \langle U_\pm, O \rangle$. Then $N$ is an $\alpha$-invariant open subgroup of $G$. Considering the group generated by $N$ and $g$, we get that Haar measures on $N$ and its quotient are $\alpha$-invariant. Since $N$ is generated by $U_\pm$ and $O$, $C_k(N/O)$ is discrete. This implies that $N/O$ is a discrete extension of a solvable group. It follows from Lemma that $U_\pm \subset O$. Since $O$ is compact and $U_\pm$ is a $p$-adic vector space, we get that $U_\pm$ is trivial. This proves that $G$ is of type $R$.

The following provides an example where the condition on the kernel of the adjoint representation is satisfied.

We say that a finitely generated group $A$ is virtually $\mathbb{Z}^k$ if $A$ contains a normal subgroup $B$ of finite index with $B \simeq \mathbb{Z}^k$.

**Corollary** Let $G$ be a $p$-adic Lie group such that $\text{Ad} (G)$ is of type $R$ and open subgroups of $G$ and their quotients are unimodular. Assume that finitely generated quotients of any open subgroup of $G$ is virtually $\mathbb{Z}^k$ for $k \leq 2$. Then $G$ is of type $R$. 

4
We first prove the following lemmas.

**Lemma 3** If $A$ is a group and $B$ is subgroup of $A$ such that $B$ is in the center of $A$ and $A/B$ is finite, then $[A, A]$ is finite.

**Proof** Let $a_1, \ldots, a_k$ be such that $A = \cup a_iB$. Since $B$ is in the center of $A$, for $b_1, b_2 \in B$, $a_i b_1 a_j b_2 b_1^{-1} a_i^{-1} b_2^{-1} a_j^{-1} = a_i a_j a_i^{-1} a_j^{-1}$. Thus, $A$ has finitely many commutators, hence $[A, A]$ is finite.

**Lemma 4** If $A$ is virtually $\mathbb{Z}^k$ for $k \leq 2$, then $C_3(A)$ is finite.

**Proof** For any $x \in A$, let $i_x$ denote the inner automorphism defined by $x$ on $A$. If $B$ is a normal subgroup of finite index such that $B \simeq \mathbb{Z}^2$, define a homomorphism $f: A \to GL(2, \mathbb{Z})$ by $f(x) = i_x|_B$. Since $A/B$ is finite and $B$ is in the kernel of $f$, $f(A)$ is finite. Let $A_1 = \text{Ker}(f)$. Then $A_1$ is a normal subgroup of $A$ such that $A/A_1$ is a 2-step solvable group - any finite subgroup of $GL(2, \mathbb{Z})$ is subgroup of some orthogonal transformations $O(2, \mathbb{R})$ which is a 2-step solvable group. This shows that $C_2(A) \subset A_1$.

Since $f(x) = i_x|_B$, $B$ is in the center of $A_1$. Since $A/B$ is finite, $A_1/B$ is finite. By Lemma 3 $C_1(A_1)$ is finite. Since $C_2(A) \subset A_1$, we get that $C_3(A)$ is finite.

If $B \simeq \mathbb{Z}$, define a homomorphism $f': A \to \{\pm 1\}$ by $f'(x) = i_x|_B$. Let $A'_1 = \text{Ker}(f')$. Then $A'_1$ is a normal subgroup of $A$ such that $A/A'_1$ is an abelian group and $B$ is in the center of $A'_1$. This shows that $C_1(A) \subset A'_1$. Since $A'_1/B$ is finite and $B$ is in the center of $A'_1$, by Lemma 3 we conclude that $C_1(A'_1)$ is finite, hence $C_2(A)$ is finite.

**Proof of Corollary** Let $F$ be a finite subset of the kernel of the adjoint representation of $G$ and $H$ be the closed subgroup generated by $F$. We now claim that $C_4(H)$ is finite.

Since $F$ is in the kernel of the adjoint representation of $G$, each $x \in F$ centralizes a compact open subgroup $U_x$ of $G$ (see Theorem 3, Section 7, Chapter III of [Bo-89]). Let $U = \cap U_x$. Then $U$ is a compact open subgroup of $G$ centralized by all elements of $F$, hence $U$ is centralized by $H$.

Let $J = UH$. Then $J/U$ is a finitely generated group. So, by assumption $J/U$ is virtually $\mathbb{Z}^k$, for $k \leq 2$. By Lemma 4 $C_3(J/U)$ is finite. This implies that $C_3(J)$ is compact, hence $C_3(H)$ is compact. Since $U$ is open, $C_3(H)/C_3(H) \cap U$ is finite. Since $U$ is centralized by $H$, by Lemma 3 we get that $C_4(H)$ is finite. Now the result follows from Theorem 4.
References

[BaW-04] U. Baumgartner and G. A. Willis, Contraction groups and scales of automorphisms of totally disconnected locally compact groups, Israel J. Math. 142 (2004), 221248.

[Bo-89] N. Bourbaki, Nicolas Lie groups and Lie algebras. Chapters 13. Translated from the French. Reprint of the 1975 edition. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1989.

[Se-06] J-P. Serre, Lie algebras and Lie groups. 1964 lectures given at Harvard University. Corrected fifth printing of the second (1992) edition. Lecture Notes in Mathematics, 1500. Springer-Verlag, Berlin, 2006.

[Ra-99] C. R. E. Raja, On classes of p-adic Lie groups, New York J. Math. 5 (1999), 101105.

[Wa-84] J. S. P. Wang, The Mautner phenomenon for p-adic Lie groups, Math. Z. 185 (1984), 403412.

C. R. E. Raja
Stat-Math Unit
Indian Statistical Institute (ISI)
8th Mile Mysore Road
Bangalore 560 059, India.
creraja@isibang.ac.in