CHARACTERISATION OF THE PRESSURE TERM IN THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS ON THE WHOLE SPACE

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Abstract. We characterise the pressure term in the incompressible 2D and 3D Navier–Stokes equations for solutions defined on the whole space.

Introduction. In the context of the Cauchy initial value problem for Navier–Stokes equations on $\mathbb{R}^d$ (with $d = 2$ or $d = 3$)

$$\begin{cases}
\partial_t u = \Delta u - (u \cdot \nabla)u - \nabla p + \nabla \cdot F \\
\nabla \cdot u = 0, \\
\quad u(0, \cdot) = u_0
\end{cases}$$

an important problem is to propose a formula for the gradient of the pressure, which is an auxiliary unknown (usually interpreted as a Lagrange multiplier for the constraint of incompressibility).

As we shall not assume differentiability of $u$ in our computations, it is better to write the equations as

$$\begin{cases}
\partial_t u = \Delta u - \nabla \cdot (u \otimes u) - \nabla p + \nabla \cdot F \\
\nabla \cdot u = 0, \\
\quad u(0, \cdot) = u_0
\end{cases}$$

Taking the Laplacian of equations (NS), since we have for a vector field $w$ the identity

$$-\Delta w = \nabla \wedge (\nabla \wedge w) - \nabla (\nabla \cdot w)$$

we get the equations

$$\partial_t \Delta u = \Delta^2 u + \nabla \wedge (\nabla \wedge (\nabla \cdot (u \otimes u - F)))$$

and

$$0 = -\Delta \nabla p - \nabla (\nabla \cdot (u \otimes u - F)) = -\Delta \nabla p - \nabla (\sum_{1 \leq i,j \leq d} \partial_i \partial_j (u_i u_j - F_{ij})).$$

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Thus, the rotational-free unknown $\nabla p$ obeys a Poisson equation. If $G_d$ is the fundamental solution of the operator $-\Delta$
\[ G_2 = \frac{1}{2\pi} \ln\left(\frac{1}{|x|}\right), \quad G_3 = \frac{1}{4\pi|x|} \]
(which satisfies $-\Delta G_d = \delta$), we formally have
\[ \nabla p = G_d \ast \nabla \left( \sum_{1 \leq i,j \leq d} \partial_i \partial_j (u_i u_j - F_{i,j}) \right) + H \tag{1} \]
with $\Delta H = 0$. In the literature, one usually finds the assumption that $\nabla p$ vanishes at infinity and this is read as $H = 0$. Equivalently, this is read as
\[ \partial_i u = G_d \ast \nabla \wedge (\nabla \wedge \partial_i u); \]
the operator
\[ \mathcal{P} = G_d \ast \nabla \wedge (\nabla \wedge \cdot) \]
is called the Leray projection operator and the decomposition (when justified)
\[ w = \mathcal{P} w - G_d \ast (\nabla \cdot w) \]
the Hodge decomposition of the vector field $w$.

Hence, an important issue when dealing with the Navier–Stokes equations is to study whether in formula (1) the first half of the right-hand term is well-defined, and if so which values the second half (the harmonic part $H$) may have.

In order to give some meaning to the formal convolution $G_d \ast \nabla \partial_i \partial_j (u_i u_j)$ or to $(\nabla \partial_i \partial_j G_d) \ast (u_i u_j)$, we should require $u_i$ to be locally $L^2 L^2_x$ (in order to define $u_i u_j$ as a distribution) and to have small increase at infinity, since the distribution $\nabla \partial_i \partial_j G_d$ has small decay at infinity (it belongs to $L^1 \cap L^\infty$ far from the origin and is $O(|x|^{-(d+1)})$). Thus, we will focus on solutions $u$ that belong to $L^2(0, T), L^2(\mathbb{R}^d, w_{d+1} \, dx)$ where
\[ w_{\gamma}(x) = (1 + |x|)^{-\gamma}. \]

We shall recall various examples (from recent or older literature) of solutions belonging to the space $L^2((0, T), L^2(\mathbb{R}^d, w_{\gamma} \, dx))$ (with $\gamma \in \{d, d+1\}$). As $F_{i,j}$ plays a role similar to $u_i u_j$, we shall assume that $\mathcal{F} \in L^1((0, T), L^1(\mathbb{R}^d, w_{\gamma} \, dx))$.

1. Main results. Before stating the results, we precise the meaning of $\nabla p$ in equations (NS):

**Lemma 1.1.** Consider the dimension $d \in \{2, 3\}$ and $\gamma \geq 0$. Let $0 < T < +\infty$. Let $u$ be a vector field $u(t, x) = (u_i(t, x))_{1 \leq i \leq d}$ such that $\nabla \cdot u = 0$ and $u$ belongs to $L^2((0, T), L^2(\mathbb{R}^d, w_{\gamma} \, dx))$, and let $\mathcal{F}$ be a tensor $\mathcal{F}(t, x) = (F_{i,j}(t, x))_{1 \leq i,j \leq d}$ such that $\mathcal{F}$ belongs to $L^1((0, T), L^1(\mathbb{R}^d, w_{\gamma} \, dx))$. Define the distribution $S$ by
\[ S = \Delta u - \nabla \cdot (u \otimes u - \mathcal{F}) - \partial_t u. \]

Then the following assertions are equivalent:
(A) $S$ is curl-free : $\nabla \wedge S = 0$.
(B) There exists a distribution $p \in \mathcal{D}'((0, T) \times \mathbb{R}^d)$ such that $S = \nabla p$.

Remark that no regularity (neither in space nor in time) is assumed on $u$, so that the derivatives in the definition of $S$ are understood as derivatives in the sense of distributions. Our main result is then the following one (where again all derivatives are to be understood as derivatives in the sense of distributions):
Theorem 1.2. Consider the dimension $d \in \{2, 3\}$. Let $0 < T < +\infty$. Let $\mathbb{F}$ be a tensor $\mathbb{F}(t, x) = (F_{i,j}(t, x))^1_{i,j \leq d}$ such that $\mathbb{F}$ belongs to $L^1((0, T), L^2(\mathbb{R}^d, w_{d+1} \, dx))$. Let $u$ be a solution of the following problem

$$
\begin{align*}
\begin{cases}
\partial_t u &= \Delta u - \nabla \cdot (u \otimes u) - S + \nabla \cdot \mathbb{F} \\
\nabla \cdot u &= 0, \quad \nabla \wedge S = 0, \quad u(0, x) = u_0(x)
\end{cases}
\end{align*}
$$

(2)

such that : $u$ belongs to $L^2((0, T), L^2(\mathbb{R}^d, w_{d+1}))$, and $S$ belongs to $\mathcal{D}’((0, T) \times \mathbb{R}^d)$.

Let us choose $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\varphi(x) = 1$ on a neighborhood of $0$ and define

$$A_{i,j,\varphi} = (1 - \varphi)\partial_i \partial_j G_d.$$ 

Then, there exists $g(t) \in L^1((0, T))$ such that

$$S = \nabla p_\varphi + \partial_t g$$

with

$$p_\varphi = \sum_{i,j} (\varphi \partial_i \partial_j G_d) * (u_i u_j - F_{i,j})$$

$$+ \sum_{i,j} \int (A_{i,j,\varphi}(x - y) - A_{i,j,\varphi}(-y))(u_i(t, y)u_j(t, y) - F_{i,j}(t, y)) \, dy.$$ 

Moreover,

- $\nabla p_\varphi$ does not depend on the choice of $\varphi$ : if we change $\varphi$ in $\psi$, then

$$p_\varphi(t, x) - p_\psi(t, x) = \sum_{i,j} \int (A_{i,j,\psi}(-y) - A_{i,j,\psi}(-y))(u_i(t, y)u_j(t, y) - F_{i,j}(t, y)) \, dy.$$ 

- $\nabla p_\varphi$ is the unique solution of the Poisson problem

$$\Delta w = -\nabla \cdot (\nabla \cdot (u \otimes u - \mathbb{F}))$$

with

$$\lim_{\tau \to +\infty} e^{\tau \Delta} w = 0 \text{ in } \mathcal{D}'.$$

- if we assume more precisely that $\mathbb{F}$ belongs to $L^1((0, T), L^1(\mathbb{R}^d))$ and $u$ belongs to $L^2((0, T), L^2(\mathbb{R}^d))$, then $g = 0$ and $\nabla p_\varphi = \nabla p_0$ where

$$p_0 = \sum_{i,j} (\varphi \partial_i \partial_j G_d) * (u_i u_j - F_{i,j}) + \sum_{i,j} ((1 - \varphi)\partial_i \partial_j G_d) * (u_i u_j - F_{i,j}).$$

($p_0$ does not actually depend on $\varphi$ and could have been defined as $p_0 = \sum_{i,j} (\partial_i \partial_j G_d) * (u_i u_j - F_{i,j}).$)

When $\mathbb{F} = 0$, the case $g \neq 0$ can easily be reduced to a change of referential, due to the extended Galilean invariance of the Navier–Stokes equations:

Theorem 1.3. Consider the dimension $d \in \{2, 3\}$. Let $0 < T < +\infty$. Let $u$ be a solution of the following problem

$$
\begin{align*}
\begin{cases}
\partial_t u &= \Delta u - \nabla \cdot (u \otimes u) - S \\
\nabla \cdot u &= 0, \quad \nabla \wedge S = 0, \quad u(0, x) = u_0(x)
\end{cases}
\end{align*}
$$

(3)

such that : $u$ belongs to $L^2((0, T), L^2(\mathbb{R}^d, w_{d+1}))$, $\lim_{t \to 0} u(t, \cdot) = u_0 \in L^2_{d+1}$ in $\mathcal{D}$, and $S$ belongs to $\mathcal{D}’((0, T) \times \mathbb{R}^d)$. 

Let us choose \( \varphi \in \mathcal{D}(\mathbb{R}^d) \) such that \( \varphi(x) = 1 \) on a neighborhood of 0 and define 
\[
A_{i,j,\varphi} = (1 - \varphi) \partial_i \partial_j G_d.
\]

We decompose \( S \) into 
\[
S = \nabla p_\varphi + \partial_t g
\]
with 
\[
p_\varphi = \sum_{i,j} (\varphi \partial_i \partial_j G_d) * (u_i u_j)
+ \sum_{i,j} \int (A_{i,j,\varphi}(x - y) - A_{i,j,\varphi}(-y))(u_i(t,y)u_j(t,y)) dy
\]
and 
\[
g(t) \in L^1((0,T)).
\]

Let us define 
\[
E(t) = \int_0^t g(\lambda) d\lambda
\]
and 
\[
w(t,x) = u(t,x - E(t)) + g(t).
\]
Then, \( w \) is a solution of the Navier–Stokes problem
\[
\begin{align*}
\partial_t w &= \Delta w - \nabla \cdot (w \otimes w) - \nabla q_\varphi
\quad \nabla \cdot w = 0, \quad w(0,x) = u_0(x)
\quad q_\varphi = \sum_{i,j} (\varphi \partial_i \partial_j G_d) * (w_i w_j) + \sum_{i,j} \int (A_{i,j,\varphi}(x - y) - A_{i,j,\varphi}(-y))(w_i(t,y)w_j(t,y)) dy
\end{align*}
\]

2. Curl-free vector fields. In this section we prove Lemma 1.1 with simple arguments:

Proof. We take a partition of unity on \((0,T)\)
\[
\sum_{j \in \mathbb{Z}} \omega_j = 1
\]
with \( \omega_j \) supported in \((2^{j-2}T,2^j T)\) for \( j < 0 \), in \((T/4,3T/4)\) for \( j = 0 \) and in \((T - 2^{-j}T,T - 2^{-(j+2)}T)\) for \( j > 1 \). We define 
\[
V_j = -\omega_j u + \int_0^t \omega_j \Delta u - \omega_j \nabla \cdot (u \otimes u - F) + (\partial_t \omega_j) u \ ds.
\]
Then \( V_j \) is a sum of the form \( A + \Delta B + \nabla \cdot C + D \) with \( A, B, C \) and \( D \) in \( L^1((0,T), L^1(\mathbb{R}^d, w_{d+1} dx)) \); thus, by Fubini’s theorem, we may see it as a time-dependent tempered distribution. Moreover, \( \partial_t V_j = \omega_j S, V_j \) is equal to 0 for \( t \) in a neighbourhood of 0, and \( \nabla \wedge V_j = 0. \) Moreover, \( S = \sum_{j \in \mathbb{Z}} \partial_t V_j. \)

We choose \( \Phi \in \mathcal{S}(\mathbb{R}^d) \) such that the Fourier transform of \( \Phi \) is compactly supported and is equal to 1 in the neighbourhood of 0. Then \( \Phi * V_j \) is well-defined and \( \nabla \wedge (\Phi * V_j) = 0. \) We define 
\[
X_j = \Phi * V_j \text{ and } Y_j = V_j - X_j.
\]
We have
\[ Y_j = \nabla \left( \frac{1}{\Delta} \nabla \cdot Y_j \right) \]

and (due to Poincaré’s lemma)
\[ X_j = \nabla \left( \int_0^1 x \cdot X_j(t, \lambda x) d\lambda \right). \]

We find \( S = \nabla p \) with
\[ p = \partial_t \sum_{j \in \mathbb{Z}} \left( \int_0^1 x \cdot X_j(t, \lambda x) d\lambda + \frac{1}{\Delta} \nabla \cdot Y_j \right). \]

3. The Poisson problem \(-\Delta U = \partial_k \partial_i \partial_j h\). We first consider a simple Poisson problem:

**Proposition 1.** Let \( h \in L^1(\mathbb{R}^d, (1 + |x|)^{-(d+1)} dx) \) then
\[ U = U_1 + U_2 = (\partial_k (\varphi \partial_i \partial_j G_d)) * h + \partial_k ((1 - \varphi) \partial_i \partial_j G_d) * h. \]

is a distribution such that \( U_2 \) belongs to \( L^1(\mathbb{R}^d, (1 + |x|)^{-(d+1)} dx) \) and \( U \) is a solution of the problem
\[ -\Delta U = \partial_k \partial_i \partial_j h. \quad (5) \]

More precisely, \( U \) is the unique solution in \( S' \) such that \( \lim_{\tau \to 0} e^{\tau \Delta} U = 0 \) in \( S' \).

**Proof.** We may write \( \partial_j G_d \) as \( \partial_j G_d = \int_0^{\infty} \partial_j W_t dt \)

where \( W_t(x) \) is the heat kernel \( W_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}, \) so that on \( \mathbb{R}^d \setminus \{0\} \), we have
\[ \partial_j G_d = c_d \frac{x_j}{|x|^d} \text{ with } c_d = -\frac{1}{2(4\pi)^{d/2}} \int_0^{+\infty} e^{-\frac{u}{4t}} \frac{du}{u^{\frac{d+2}{2}}}. \]

The first part defining \( U, U_1 = (\partial_k (\varphi \partial_i \partial_j G_d)) * h, \) is well defined, since \( \partial_k (\varphi \partial_i \partial_j G_d) \) is a compactly supported distribution. To control \( U_2, \) we write
\[ \int \int \frac{1}{(1 + |x|)^{d+1}} |\partial_k ((1 - \varphi) \partial_i \partial_j G_d)(x - y)||h(y)||dy dx \]
\[ \leq \int \int \frac{C}{(1 + |x|)^{d+1}} (1 + |x - y|)^{d+1} |h(y)||dy dx \]
\[ \leq C' \int \frac{1}{(1 + |y|)^{d+1}} |h(y)| dy \]
Let \( h \in L^1(\mathbb{R}^d) \) and consider the Poisson problem 
\[
\Delta U = h \quad \text{in} \quad \mathbb{R}^d.
\]
By the dominated convergence theorem, we get that \( \lim_{\tau \to 0} \int_{|x| > \tau} h(x) \, dx \) exists. If \( \Delta U_1 \) is well defined, we may compute 
\[
\Delta U_1 = \int V \, dV = \int \sum_{i=1}^d \int \phi \, dV.
\]
In order to compute \( \Delta U_2 \), we see that we can differentiate under the integration sign and find 
\[
\Delta U_2 = \int \sum_{i=1}^d \int \phi \, dV.
\]
Thus, \( U \) is a solution of the Poisson problem.

\[ e^{\tau \Delta} U = (e^{\tau \Delta} \partial_k \partial_l G_d) * h \]
and hence 
\[
|e^{\tau \Delta} U(x)| \leq C \int \frac{1}{(\sqrt{\tau + |x|})^{d+1}} |h(y)| \, dy.
\]
By the dominated convergence theorem, we get that \( \lim_{\tau \to 0} e^{\tau \Delta} U = 0 \) in \( L^1(\mathbb{R}^d, (1 + |x|)^{-d} \, dx) \). If \( V \) is another solution of the same Poisson problem with \( V \in \mathcal{S}' \) and \( \Delta(U - V) = 0 \) in \( \mathcal{S}' \), then \( \Delta(U - V) = 0 \) and \( U - V \in \mathcal{S}' \), so that \( U - V \) is a polynomial; with the assumption that \( \lim_{\tau \to 0} e^{\tau \Delta} (U - V) = 0 \), we find that this polynomial is equal to 0.

If we have better integrability of \( h \), then of course we have better integrability of \( U_2 \). For instance, we have:

**Proposition 2.** Let \( h \in L^1(\mathbb{R}^d, (1 + |x|)^{-d} \, dx) \) then 
\[ U_2 = \partial_k((1 - \varphi)\partial_l G_d) * h \]
belongs to \( L^1(\mathbb{R}^d, (1 + |x|)^{-d} \, dx) \).

**Proof.** We write 
\[
\int \int \frac{1}{(1 + |x|)^d} \frac{1}{(1 + |x-y|)^{d+1}} |h(y)| \, dy \, dx 
\leq C \int \int \frac{1}{(1 + |x|)^d} \frac{1}{(1 + |x-y|)^{d+1}} |h(y)| \, dy \, dx.
\]
For \(|y| < 1\), we have 
\[
\int \frac{1}{(1 + |x|)^d} \frac{1}{(1 + |x-y|)^{d+1}} \, dx \leq \int \frac{1}{(1 + |x|)^{d+1}} \, dx \leq C
\]
and for $|y| > 1$, as the real number \( \int_{|x| < \frac{1}{2}} \frac{1}{|x|^{d-1} |x - \frac{1}{y}|} dx \) is finite and does not depend on \( y \), we can write
\[
\int \frac{1}{(1 + |x|)^d} \frac{1}{(1 + |x - y|)^{d+1}} dx \leq \int_{|x| > \frac{1}{2}} \frac{1}{(1 + |x|)^d} \frac{1}{(1 + |x - y|)^{d+1}} dx + \int_{|x| < \frac{1}{2}} \frac{1}{(1 + |x|)^d} \frac{1}{(1 + |x - y|)^{d+1}} dx
\]
\[
\leq \frac{2^d}{(1 + |y|)^d} \int (1 + |x|)^{d+1} dx + \frac{2^{d-1}}{(1 + |y|)^{d-1}} \int_{|x| < \frac{1}{2}} \frac{1}{|x|^{d-1} |x - y|^2} dx
\]
\[
\leq C \frac{1}{(1 + |y|)^d} + C \frac{1}{(1 + |y|)^{d-1}} \frac{1}{|y|} \int |x|^{d-1} |x - y|^2 dx
\]
\[
\leq C' \frac{1}{(1 + |y|)^d}.
\]
This concludes the proof. \( \square \)

4. The Poisson problem \(-\Delta V = \partial_i \partial_j h.\)

**Proposition 3.** Let \( h \in L^1((1 + |x|)^{-d-1} dx) \) and \( A_\varphi = (1 - \varphi) \partial_i \partial_j G d \) then
\[
V = V_1 + V_2 = (\varphi \partial_i \partial_j G d) * h + \int (A_\varphi(x - y) - A_\varphi(-y)) h(y) dy
\]
is a distribution such that \( V_2 \) belongs to \( L^1((1 + |x|)^{-\gamma}) \), for \( \gamma > d + 1 \), and \( V \) is a solution of the problem
\[-\Delta V = \partial_i \partial_j h.\] (6)

**Proof.** We know that \( V_1 \) is well defined since \( \varphi \partial_i \partial_j G_d \) is a supported compactly distribution, and we will verify that \( V_2 \) is well defined.

We have
\[
\int \frac{1}{(1 + |x|)^\gamma} |A_\varphi(x - y) - A_\varphi(-y)| dx \leq C \|A_\varphi\|_{L^\infty} \int \frac{1}{(1 + |x|)^\gamma} dx.
\]
For \( |y| > 1 \), we have by the mean value inequality
\[
\int_{|x| < \frac{1}{2}} \frac{1}{(1 + |x|)^\gamma} |A_\varphi(x - y) - A_\varphi(-y)| dx \leq C \frac{1}{|y|^{d+1}} \int_{|x| < \frac{1}{2}} \frac{|x|}{(1 + |x|)^\gamma} dx
\]
and we can control the other part as follows
\[
\int_{|x| > \frac{1}{2}} \frac{1}{(1 + |x|)^\gamma} |A_\varphi(-y)| dx \leq C \frac{1}{|y|^{d+1}} \int_{|x| > \frac{1}{2}} \frac{1}{(1 + |x|)^\gamma} dx \leq C \frac{1}{|y|^{\gamma}}
\]
and for \( \epsilon > 0 \) such that \( \gamma - \epsilon \geq d + 1 \), we have
\[
\int_{|x| > \frac{1}{2}} \frac{1}{(1 + |x|)^\gamma} |A_\varphi(x - y)| dx \leq C \int_{|x| > \frac{1}{2}} \frac{1}{|x|^{\gamma}} (1 + \frac{1}{|x - y|})^d dx
\]
\[
\leq C \int_{|x| > \frac{1}{2}} \frac{1}{|x|^{\gamma}} \frac{1}{|x - y|^{d+\epsilon}} dx
\]
\[
= C \frac{1}{|y|^{\gamma+\epsilon}} \int_{|x| > \frac{1}{2}} \frac{1}{|x|^{\gamma}} \frac{1}{|x - y|^{d+\epsilon}} dx
\]
\[
\leq C' \frac{1}{|y|^{d+1}}.
\]
\( \square \)
5. **Proof of Theorems 1.2 and 1.3.** We may now prove Theorem 1.2:

*Proof.* Taking the divergence of 
\[ \partial_t u = \Delta u - \nabla \cdot (u \otimes u) - S + \nabla \cdot F, \]
we obtain 
\[ -\sum_{i,j} \partial_i \partial_j (u_i u_j) + \sum_{i,j} \partial_i \partial_j F_{i,j} - \nabla \cdot S = 0 \]
and 
\[ -\Delta S = \nabla \left( \sum_{i,j} \partial_i \partial_j (u_i u_j - F_{i,j}) \right). \]

We write \( h_{i,j} = u_i u_j - F_{i,j} \), and \( A_{i,j,\varphi} = (1 - \varphi) \partial_i \partial_j G_d \). By Proposition 3, we can define 
\[ p_\varphi = \sum_{i,j} (\varphi \partial_i \partial_j G_d) * h_{i,j} + \sum_{i,j} \int (A_{i,j,\varphi}(x - y) - A_{i,j,\varphi}(-y)) h_{i,j}(y) dy \]
and 
\[ U = U_1 + U_2 = \nabla \sum_{i,j} (\varphi \partial_i \partial_j G_d) * h_{i,j} + \nabla \sum_{i,j} ((1 - \varphi) \partial_i \partial_j G_d) * h_{i,j} = \nabla p_\varphi. \]

Let \( \tilde{U} = S - U \). First, we remark that \( \Delta U = \Delta S \) so that \( \Delta \tilde{U} = 0 \), hence \( \tilde{U} \) is harmonic in the space variable.

On the other hand, for a test function \( \alpha \in D(\mathbb{R}) \) such that \( \alpha(t) = 0 \) for all \( |t| \geq \varepsilon \), and a test function \( \beta \in D(\mathbb{R}^3) \), and for \( t \in (\varepsilon, T - \varepsilon) \), we have 
\[ \tilde{U}(t) *_{t,x} (\alpha \otimes \beta) = (u * (-\partial_t \alpha \otimes \beta + \alpha \otimes \Delta \beta) + (u \otimes u + F) \cdot *(\alpha \otimes \nabla \beta))(t,\cdot) + \sum_{i,j} (h_{i,j} * (\nabla (\varphi \partial_i \partial_j G_d) * (\alpha \otimes \beta)))(t,\cdot) - (U_2 * (\alpha \otimes \beta))(t, \cdot). \]

By Proposition 1, we may conclude that \( \tilde{U} * (\alpha \otimes \beta)(t,\cdot) \) belongs to the space \( L^1(\mathbb{R}^d, (1 + |x|^{-d-1}) dx) \). Thus, it is a tempered distribution; as it is harmonic, it must be polynomial. The integrability in \( L^1(\mathbb{R}^d, (1 + |x|^{-d-1}) dx) \) implies that this polynomial is constant.

If \( F \) belongs to \( L^1((0, T), L^1_{w_{d}}(\mathbb{R}^d)) \) and \( u \) belongs to \( L^2((0, T), L^2_{w_{d}}(\mathbb{R}^d)) \), we find more precisely that this polynomial belongs to \( L^1(\mathbb{R}^d, w_{d} dx) \), hence is equal to 0.

Then, using the identity approximation \( \Phi_\varepsilon = \frac{1}{4\pi} \alpha(\frac{\varepsilon}{2}) \beta(\frac{x}{2}) \) and letting \( \varepsilon \) go to 0, we obtain a similar result for \( \tilde{U} \). Thus \( S = \nabla p_\varphi + f(t) \), with \( f(t) = 0 \) if \( F \) belongs to \( L^1((0, T), L^1_{w_{d}}(\mathbb{R}^d)) \) and \( u \) belongs to \( L^2((0, T), L^2_{w_{d}}(\mathbb{R}^d)) \).

As \( f \) does not depend on \( x \), we may take a function \( \beta \in D(\mathbb{R}^d) \) with \( \int \beta dx = 1 \) and write \( f = f *_x \beta \); we find that 
\[ f(t) = \partial_t (u_0 * \beta - u * \beta) + \int_0^t u * \Delta \beta - (u \otimes u + F) \cdot * \nabla \beta - p_\varphi * \nabla \beta ds = \partial_t g. \]

As \( \partial_t \partial_t f = 0 \) and \( \partial_t g(0,\cdot) = 0 \), we find that \( g \) depends only on \( t \); moreover, the formula giving \( g \) proves that \( g \in L^1((0, T)) \).

The proof of Theorem 1.3 is classical and the result is known as the extended Galilean invariance of the Navier–Stokes equations:
Proof. Let us suppose that
\[ \partial_t u = \Delta u - (u \cdot \nabla)u - \nabla p_\varphi - \frac{d}{dt} g(t), \]
with \( g \in L^1((0,T)) \). We define
\[ E(t) = \int_0^t g(\lambda)d\lambda \text{ and } w = u(t, x - E(t)) + g(t). \]
We have
\[ \partial_t w = \partial_t u(t, x - E(t)) - g(t) \cdot \nabla u(t, x - E(t)) + \frac{d}{dt} g(t) \]
\[ = \Delta u(t, x - E(t)) - [(u \cdot \nabla)u](t, x - E(t)) - \nabla p_\varphi(t, x - E(t)) - \frac{d}{dt} g(t) \]
\[ = \Delta w - (w \cdot \nabla)w - \nabla p_\varphi(t, x - E(t)). \]
If we define \( q_\varphi(t, x) = p_\varphi(t, x - E(t)) \), we find that we have
\[ q_\varphi = \sum_{i,j} (\varphi \partial_i \partial_j G_d) * (w_i w_j) + \sum_{i,j} \int (A_{i,j,\varphi}(x - y) - A_{i,j,\varphi}(-y))(w_i(t, y)w_j(t, y)) dy. \]
The theorem is proved. \( \square \)

6. Applications. As a consequence of Proposition 1, we know that we may define the Leray projection operator on the divergence of tensors that belong to \( L^1((0,T), L^1(\mathbb{R}^d, w_{d+1} dx)) \):

**Definition 6.1.** Let \( \mathbb{H} \in L^1((0,T), L^1(\mathbb{R}^d, w_{d+1} dx)) \) and \( w = \nabla \cdot \mathbb{H} \). The Leray projection \( P(w) \) of \( w \) on solenoidal vector fields is defined by
\[ Pw = w - \nabla p_\varphi \]
where \( \nabla p_\varphi \) is the unique solution of
\[ -\Delta \nabla p = \nabla (\nabla \cdot w) \]
such that
\[ \lim_{\tau \to +\infty} e^{\tau \Delta} \nabla p = 0. \]

A special form of the Navier–Stokes equations is then given by
\[ (MNS) \quad \partial_t u = \Delta u - P \nabla \cdot (u \otimes u - \mathcal{F}), \quad u(0,.) = u_0. \]
This leads to the integro-differential equation
\[ u = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} P \nabla \cdot (u \otimes u - \mathcal{F}) ds. \]
The kernel of the convolution operator \( e^{(t-s)\Delta} P \nabla \cdot \) is called the Oseen kernel; its study is the core of the method of mild solutions of Kato and Fujita [13]. Thus, we will call equations (MNS) a mild formulation of the Navier–Stokes equations.

The mild formulation together with the local Leray energy inequality has been as well a key tool for extending Leray’s theory of weak solutions in \( L^2 \) to the setting of weak solutions with infinite energy. We may propose a general definition of suitable Leray-type weak solutions:
We consider the Navier–Stokes problem on $(0, T)$ better described by using the extended Galilean invariance of the equations (when the systems (NS) and (MNS) are no longer equivalent and general solutions are not specifications on the behaviour of $u$ at the boundary, the estimates on the pressure (and the Leray projection operator) are no longer available. However, Wolf described in 2017 [24] a local decomposition of the pressure into a term similar to the Leray projection of $\nabla \cdot (u \otimes u)$ and a harmonic term; he could generalize the notion of suitability to this new description of the pressure. On the equivalence of various notions of suitability, see the paper by Chamorro, Lemarié-Rieusset and Mayoufi [8].

c) The relationship between the system (NS) and its mild formulation (MNS) described in Theorem 1.2 has been described by Furioli, Lemarié-Rieusset and Terraneo in 2000 [14, 19] in the context of uniformly locally square integrable solutions. See the paper by Dubois [10], as well. Their results show the equivalence between (NS) and (MNS) in the case when $u$ and $F$ decay at infinity (more precisely, when $u$ belongs to the closure of test functions in $(L^1_t L^2_x)_{uloc}$ and when $F$ belongs to the closure of test functions in $(L^1_t L^1_x)_{uloc}$). Similar results were proved in 1972 for mild solutions in $L^1_t L^2_x$ with $\frac{d}{2} + \frac{d}{q} \leq 1$ and $d < q < +\infty$ by Fabes, Jones and Riviere [11], a much simpler case where the theory of singular integrals may be directly used.

d) The case of non-decaying solutions has been discussed by Kukavica in 2003 [16] for the study of the Cauchy problem with initial value in $L^\infty$ and by Kukavica and Vicol in 2010 [17] for the study of the Cauchy problem with initial value in $\text{BMO}^{-1}$. The systems (NS) and (MNS) are no longer equivalent and general solutions are better described by using the extended Galilean invariance of the equations (when

**Definition 6.2** (Suitable Leray-type solution).

Let $P \in L^2((0, T), L^2(\mathbb{R}^d, \frac{1}{(1+|x|)^{d+p}}))$ and $u_0 \in L^2(\mathbb{R}^d, \frac{1}{(1+|x|)^{d+p}})$ with $\nabla \cdot u_0 = 0$.

We consider the Navier–Stokes problem on $(0, T) \times \mathbb{R}^d$:

$$\partial_t u = \Delta u - P(u \otimes u - F),$$

$$\nabla \cdot u = 0, \quad u(0, .) = u_0.$$

A suitable Leray-type solution $u$ of the Navier–Stokes equations is a vector field $u$ defined on $(0, T) \times \mathbb{R}^d$ such that:

- $u$ is locally $L^2_t H^1_x$ on $(0, T) \times \mathbb{R}^d$.
- $\sup_{0 < t < T} \int |u(t, x)|^2 \frac{1}{(1+|x|)^{d+p}} dx < +\infty$.
- $\int_{(0,T) \times \mathbb{R}^d} |\nabla \otimes u(t, x)|^2 \frac{1}{(1+|x|)^{d+p}} dx dt < +\infty$.
- The application $t \mapsto \int u(t, x) \cdot w(x) dx$ is continuous for every smooth compactly supported vector field $w$.
- For every compact subset $K$ of $\mathbb{R}^d$, $\lim_{t \to 0} \int_K |u(t, x) - u_0(x)|^2 dx = 0$.
- Defining $p_\varphi$ as the solution of $-\Delta p_\varphi = \sum_{i,j} \partial_i \partial_j (u_i u_j - F_{i,j})$ given by Proposition 3, $u$ is suitable in the sense of Caffarelli, Kohn and Nirenberg: there exists a non-negative locally bounded Borel measure $\mu$ on $(0, T) \times \mathbb{R}^d$ such that

$$\partial_t \left( \frac{|u|^2}{2} \right) = \Delta \left( \frac{|u|^2}{2} \right) - |\nabla \otimes u|^2 - \nabla \cdot (\left( \frac{|u|^2}{2} + p_\varphi \right) u) + u \cdot (\nabla \cdot F) - \mu.$$

**Remark 1.**

a) With those hypotheses, $p_\varphi$ belongs locally to $L^{3/2}_t L^{3/2}_x$ and $u$ belongs locally to $L^3_{t,x}$ so that the distribution $\left( \frac{|u|^2}{2} + p_\varphi \right) u$ is well-defined.

b) Suitability is a local assumption. It has been introduced by Caffarelli, Kohn and Nirenberg in 1982 [6] to get estimates on partial regularity for weak Leray solutions. If we consider a solution of the Navier–Stokes equations on a small domain with no specifications on the behaviour of $u$ at the boundary, the estimates on the pressure (and the Leray projection operator) are no longer available. However, Wolf described in 2017 [24] a local decomposition of the pressure into a term similar to the Leray projection of $\nabla \cdot (u \otimes u)$ and a harmonic term; he could generalize the notion of suitability to this new description of the pressure. On the equivalence of various notions of suitability, see the paper by Chamorro, Lemarié-Rieusset and Mayoufi [8].

c) The relationship between the system (NS) and its mild formulation (MNS) described in Theorem 1.2 has been described by Furioli, Lemarié-Rieusset and Terraneo in 2000 [14, 19] in the context of uniformly locally square integrable solutions. See the paper by Dubois [10], as well. Their results show the equivalence between (NS) and (MNS) in the case when $u$ and $F$ decay at infinity (more precisely, when $u$ belongs to the closure of test functions in $(L^1_t L^2_x)_{uloc}$ and when $F$ belongs to the closure of test functions in $(L^1_t L^1_x)_{uloc}$). Similar results were proved in 1972 for mild solutions in $L^1_t L^2_x$ with $\frac{d}{2} + \frac{d}{q} \leq 1$ and $d < q < +\infty$ by Fabes, Jones and Riviere [11], a much simpler case where the theory of singular integrals may be directly used.

d) The case of non-decaying solutions has been discussed by Kukavica in 2003 [16] for the study of the Cauchy problem with initial value in $L^\infty$ and by Kukavica and Vicol in 2010 [17] for the study of the Cauchy problem with initial value in $\text{BMO}^{-1}$. The systems (NS) and (MNS) are no longer equivalent and general solutions are better described by using the extended Galilean invariance of the equations (when
F = 0). In this paper, we find again such a description in the case of more general weak solutions, for which the integral formulation does not provide any existence or uniqueness results, in contrast to the case of solutions in $L^\infty$ or $\text{BMO}^{-1}$ data.

We list here a few examples to be found in the literature:

1. Solutions in $L^2$: in 1934, Leray [21] studied the Navier–Stokes problem (NS) with an initial data $u_0 \in L^2$ and a forcing tensor $F \in L^2_t L^2_x$. He then obtained a solution $u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1$. Remark that this solution is automatically a solution of the mild formulation of the Navier–Stokes equations (MNS). Leray’s construction by mollification provides suitable solutions.

2. Solutions in $L^2_{uloc}$: in 1999, Lemarié-Rieusset [18, 19] studied the Navier–Stokes problem (MNS) with an initial data $u_0 \in L^2_{uloc}$ (and, later in [20], a forcing tensor $F \in L^2_t L^2_x \otimes L^2_{uloc}$). He obtained (local in time) existence of a suitable solution $u$ on a small strip $(0, T_0) \times \mathbb{R}^d$ such that

$$\sup_{x_0 \in \mathbb{R}^d} \sup_{0 < t < T_0} \int_{B(x_0, 1)} |u(t, x)|^2 \, dx < +\infty$$

and

$$\sup_{x_0 \in \mathbb{R}^d} \int_{0}^{T_0} \int_{B(x_0, 1)} |\nabla \otimes u(t, x)|^2 \, dx < +\infty.$$  

Remark that we have $u \in L^2((0, T_0), L^2(\mathbb{R}^d, \frac{1}{(1+|x|)^{d+1}} \, dx))$ but $u$ does not belong to $L^2((0, T_0), L^2(\mathbb{R}^d, \frac{1}{(1+|x|)^d} \, dx))$; thus, in this setting, problems (NS) and (MNS) are not equivalent.

Various reformulations of local Leray solutions in $L^2_{uloc}$ have been provided, such as Kikuchi and Seregin in 2007 [15] or Bradshaw and Tsai in 2019 [4]. The formulas proposed for the pressure, however, are actually equivalent, as they all imply that $u$ is solution to the (MNS) problem.

In the case of dimension $d = 2$, Basson [1] proved in 2006 that the solution $u$ is indeed global (i.e. $T_0 = T$) and that, moreover, the solution is unique.

3. Solutions in a weighted Lebesgue space: in 2019, Fernández-Dalgo and Lemarié–Rieusset [12] considered (non-uniformly locally square integrable) data $u_0 \in L^2(\mathbb{R}^3, w_\gamma \, dx)$ and $F \in L^2((0, +\infty), L^2(\mathbb{R}^3, w_\gamma \, dx))$ with $0 < \gamma \leq 2$. They proved (global in time) existence of a suitable solution $u$ such that, for all $T_0 < +\infty$,

$$\sup_{0 < t < T_0} \int |u(t, x)|^2 w_\gamma(x) \, dx < +\infty$$

and

$$\int_{0}^{T_0} \int |\nabla \otimes u(t, x)|^2 \, dx < +\infty.$$  

[Of course, for such solutions, (NS) and (MNS) are equivalent.] They showed that, for $\frac{4}{3} < \gamma \leq 2$, this frame of work is well adapted to the study of discretely self-similar solutions with locally $L^2$ initial value, providing a new proof of the results of Chae and Wolf in 2018 [7] and of Bradshaw and Tsai in 2019 [3]. By Theorem 1.2, we can see that the formula proposed for the pressure by Bradshaw and Tsai can be derived from the weighted integrability of the solution which implies that $u$ is solution to the (MNS) problem.

4. Homogeneous Statistical Solutions: in 1977, Vishik and Fursikov [22] considered the (MNS) problem with a random initial value $u_0(\omega)$. The statistics of

THE PRESSURE TERM IN THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS 11
With a kind of local Morrey space, the space $B^p_{\gamma}$ appears naturally as substitutes to the weighted spaces $L^p_{w_\gamma}$.

**Definition 7.1.** For $1 \leq p < +\infty$, we denote $B^p_{\gamma}$ the Banach space of all functions $u \in L^p_{\text{loc}}$ such that:

$$
\|u\|_{B^p_{\gamma}} = \sup_{R \geq 1} \left( \frac{1}{R^\gamma} \int_{B(0,R)} |u|^p \, dx \right)^{1/p} < +\infty.
$$

Similarly, $B^p_{\gamma}L^p(0,T)$ is the Banach space of all functions $u \subset (L^p_{\text{loc}}L^p_x(0,T))$ such that

$$
\|u\|_{B^p_{\gamma}L^p(0,T)} = \sup_{R \geq 1} \left( \frac{1}{R^\gamma} \int_0^T \int_{B(0,R)} |u|^p \, dx \, ds \right)^{1/p}.
$$

As we shall see below, those spaces $B^p_{\gamma}$ appear naturally as substitutes to the spaces $L^p_{w_\gamma} = L^p(\mathbb{R}^d, w_\gamma(x) \, dx)$ (where $w_\gamma(x) = \frac{1}{(1 + |x|)^\gamma}$) for getting a more general existence result of weak solutions or for getting a more precise description of the homogeneous statistical solutions of Vishik and Fursikov.

We first remark that those spaces are very close to the weighted spaces $L^p_{w_\gamma}$:

**Lemma 7.2.** Let $\gamma \geq 0$ and $\gamma < \delta < +\infty$, we have the continuous embedding $L^p_{w_\gamma} \hookrightarrow B^p_{\gamma,0} \hookrightarrow B^p_{\gamma} \hookrightarrow L^p_{w_\gamma}$, where $B^p_{\gamma,0} \subset B^p_{\gamma}$ is the subspace of all functions $u \in B^p_{\gamma}$ such that $\lim_{R \to +\infty} \frac{1}{R^\gamma} \int_{B(0,R)} |u(x)|^p \, dx = 0$.

**Proof.** Let $u \in L^p_{w_\gamma}$. We verify easily that $\|u\|_{B^p_{\gamma}} \leq 2^{\gamma/p} \|u\|_{L^p_{w_\gamma}}$ and we see that

$$
\frac{1}{R^\gamma} \int_{|x| \leq R} |u|^p \, dx = \int_{|x| \leq R} \frac{|u|^p}{(1 + |x|)^\gamma} \frac{(1 + |x|)^\gamma}{R^\gamma} \, dx
$$

the initial distributions were supposed to be invariant though translation of the arguments of $u_0$ : for every Borel subset $B$ of $L^2_{\text{loc}}(\mathbb{R}^3)$ and every $x_0 \in \mathbb{R}^3$,

$$
Pr(u_0(\cdot - x_0) \in A) = Pr(u_0 \in A).
$$

Another assumption was that $u_0$ has a bounded mean energy density :

$$
e_0 = \mathbb{E} \left( \frac{\int_{|x| \leq 1} |u_0|^2 \, dx}{\int_{|x| \leq 1} \, dx} \right) < +\infty.
$$

Then

$$
Pr(u_0 \in L^2 \text{ and } u \neq 0) = 0
$$

while, for any $\epsilon > 0$,

$$
Pr\left( \int |u_0|^2 \frac{1}{(1 + |x|)^{3+\epsilon}} \, dx < +\infty \right) = 1.
$$

In [23], they constructed a solution $u(t, x, \omega)$ that solved the Navier–Stokes equation for almost every initial value $u_0(\omega)$, and the solution belonged almost surely to $L^1_t L^3_{x}(\frac{1}{1 + |x|^{3+\epsilon}} \, dx)$ with $\nabla \otimes u \in L^2_t L^2_{x}(\frac{1}{1 + |x|^{3+\epsilon}} \, dx)$.

In 2006, Basson [2] gave a precise description of the pressure in those equations (which is equivalent to our description through the Leray projection operator) and proved the suitability of the solutions.

### 7. The space $B^2_{\gamma}$

Instead of dealing with weighted Lebesgue spaces, one may deal with a kind of local Morrey space, the space $B^2_{\gamma}$.

**Definition 7.1.** For $1 \leq p < +\infty$, we denote $B^p_{\gamma}$ the Banach space of all functions $u \in L^p_{\text{loc}}$ such that :

$$
\|u\|_{B^p_{\gamma}} = \sup_{R \geq 1} \left( \frac{1}{R^\gamma} \int_{B(0,R)} |u|^p \, dx \right)^{1/p} < +\infty.
$$

Similarly, $B^p_{\gamma}L^p(0,T)$ is the Banach space of all functions $u \subset (L^p_{\text{loc}}L^p_x(0,T))$ such that

$$
\|u\|_{B^p_{\gamma}L^p(0,T)} = \sup_{R \geq 1} \left( \frac{1}{R^\gamma} \int_0^T \int_{B(0,R)} |u|^p \, dx \, ds \right)^{1/p}.
$$

As we shall see below, those spaces $B^p_{\gamma}$ appear naturally as substitutes to the spaces $L^p_{w_\gamma} = L^p(\mathbb{R}^d, w_\gamma(x) \, dx)$ (where $w_\gamma(x) = \frac{1}{(1 + |x|)^\gamma}$) for getting a more general existence result of weak solutions or for getting a more precise description of the homogeneous statistical solutions of Vishik and Fursikov.

We first remark that those spaces are very close to the weighted spaces $L^p_{w_\gamma}$:

**Lemma 7.2.** Let $\gamma \geq 0$ and $\gamma < \delta < +\infty$, we have the continuous embedding $L^p_{w_\gamma} \hookrightarrow B^p_{\gamma,0} \hookrightarrow B^p_{\gamma} \hookrightarrow L^p_{w_\gamma}$, where $B^p_{\gamma,0} \subset B^p_{\gamma}$ is the subspace of all functions $u \in B^p_{\gamma}$ such that $\lim_{R \to +\infty} \frac{1}{R^\gamma} \int_{B(0,R)} |u(x)|^p \, dx = 0$.

**Proof.** Let $u \in L^p_{w_\gamma}$. We verify easily that $\|u\|_{B^p_{\gamma}} \leq 2^{\gamma/p} \|u\|_{L^p_{w_\gamma}}$ and we see that

$$
\frac{1}{R^\gamma} \int_{|x| \leq R} |u|^p \, dx = \int_{|x| \leq R} \frac{|u|^p}{(1 + |x|)^\gamma} \frac{(1 + |x|)^\gamma}{R^\gamma} \, dx
$$
converges to zero when $R \to +\infty$ by dominated convergence, so $L_{w_0}^p \hookrightarrow B^p_{\gamma,0}$. To demonstrate the other part, we estimate

$$
\int \frac{|u|^p}{(1 + |x|)^\delta} \, dx = \int_{|x| \leq 1} \frac{|u|^p}{(1 + |x|)^\delta} \, dx + \sum_{n \in \mathbb{N}} \int_{2^{n-1} \leq |x| \leq 2^n} \frac{|u|^p}{(1 + |x|)^\delta} \, dx \\
\leq \int_{|x| \leq 1} |u|^p \, dx + \sum_{n \in \mathbb{N}} \frac{1}{(1 + 2^{n-1})^\delta} \int_{2^{n-1} \leq |x| \leq 2^n} |u|^p \, dx \\
\leq \int_{|x| \leq 1} |u|^p \, dx + c \sum_{n \in \mathbb{N}} \frac{1}{2^{n}} \int_{2^{n-1} \leq |x| \leq 2^n} |u|^p \, dx \\
\leq (1 + c \sum_{n \in \mathbb{N}} \frac{1}{2^{(\delta - \gamma)n}}) \sup_{R \geq 1} \frac{1}{R^\gamma} \int_{|x| \leq R} |u|^p \, dx,
$$

thus, $B^p_\gamma \subseteq L_{w_0}^p$.

\[ \square \]

**Remark 2.** Similarly, for all $\delta > \gamma$, $B^p_\delta L^p(0,T) \subseteq L^p((0,T), L^p_{w_{\delta}})$.

**Proposition 4.** The space $B^p_\gamma$ can be obtained by interpolation,

$$
B^p_\gamma = [L^p, L^p_{w_{\delta}}]_{\gamma, \infty}
$$

for all $0 < \gamma < \delta < \infty$, and the norms

$$
\| \cdot \|_{B^p_\gamma} \quad \text{and} \quad \| \cdot \|_{[L^p, L^p_{w_{\delta}}]_{\gamma, \infty}}
$$

are equivalent.

**Proof.** Let $f \in B^p_\gamma$. For $A < 1$, we write $f_0 = 0$ and $f_1 = f$, then we have $f = f_0 + f_1$ and

$$
\|f_1\|_{L^p_{w_0}} \leq CA^{\delta - 1}\|f\|_{B^p_\gamma}.
$$

For $A > 1$, we let $R = A^{\frac{\delta}{\gamma}} > 1$. We write $f_0 = f1_{|x| \leq R}$ and $f_1 = f1_{|x| > R}$, then

$$
\|f_0\|_p \leq C\|f\|_{B^p_\gamma} R^{\frac{\delta}{\gamma}} = CA^{\delta - 1}\|f\|_{B^p_\gamma}
$$

and

$$
\|f_1\|_{L^p_{w_0}}^p = \sum_{n \in \mathbb{N}} \int_{2^{n-1} \leq |x| \leq 2^n} \frac{|u|^p}{(1 + |x|)^\delta} \, dx \\
\leq CR^{1-\delta} \sum_{n \in \mathbb{N}} \frac{1}{2^{(\delta - \gamma)n}} \|f\|_{B^p_\gamma}^p \\
= CA^{(\delta - 1)p}\|f\|_{B^p_\gamma}^p.
$$

Thus, $B^p_\gamma \hookrightarrow [L^p, L^p_{w_{\delta}}]_{\gamma, \infty}$.

Let $f \in [L^p, L^p_{w_{\delta}}]_{\gamma, \infty}$, then there exist $c > 0$ such that for all $A > 0$, there exist $f_0 \in L^p$ and $f_1 \in L^p_{w_{\delta}}$ so that $f = f_0 + f_1$,

$$
\|f_0\|_p \leq cA^{\frac{\delta}{\gamma}} \quad \text{and} \quad \|f_1\|_{L^p_{w_\delta}} \leq cA^{\frac{\delta}{\gamma}-1}.
$$
For $j \in \mathbb{N}$ we take $A = 2^{2j}$, then
\[
\frac{1}{2^{2j}} \int_{|x| < 2^j} |f|^p \, dx 
\leq C \left( \frac{1}{2^{2j}} \int_{|x| < 2^j} |f_0|^p \, dx + \frac{1}{2^{2j}} \int_{|x| < 2^j} |f_1|^p \, dx \right) 
\leq C \left( \frac{1}{2^{2j}} \|f_0\|_p^p + \frac{C}{2^{2j}} \int_{|x| \leq 1} \frac{|f_1|^p}{(1 + |x|)^{q\delta}} \, dx + C \sum_{k=1}^{j} \frac{2^{k\delta}}{2^{2j}} \int_{2^{k-1} < |x| < 2^k} \frac{|f_1|^p}{(1 + |x|)^{q\delta}} \, dx \right) 
\leq C \left( \frac{1}{2^{2j}} \|f_0\|_p^p + C' 2^j(\delta - 1) \|f_1\|_{L^p_w}^p \right) 
\leq C''
\]

which implies $\sup_{j \in \mathbb{N}} \frac{1}{2^{2j}} \int_{|x| < 2^j} |f|^p \, dx < +\infty$, so $\sup_{R \geq 1} \frac{1}{R^2} \int_{|x| < R} |f|^p \, dx < +\infty$. $\square$

Thus, we can see that the local Morrey spaces $B^p_\gamma$ are very close to the weighted Lebesgue spaces $L^p_{w_\gamma}$. Indeed, the methods and results of Fernández-Dalgo and Lemarié–Rieusset [12] can be easily extended to the setting of local Morrey spaces in dimension $d = 2$ or $d = 3$: considering data $u_0 \in B^2_\gamma(\mathbb{R}^d)$ and $F \in (B^2_\gamma L^2)(0, T)(\mathbb{R}^d)$ with $0 < \gamma \leq 2$, one gets (local in time) existence of a suitable solution $u$ for the (MNS) system on a small strip $(0, T_0) \times \mathbb{R}^d$ such that $u \in L^\infty((0, T_0), B^2_\gamma)$ and $\nabla \otimes u \in (B^2_\gamma L^2)(0, T_0)$.

The case of $\gamma = 2$ deserves some comments. In the case $d = 3$, the results is slightly more general than the results in [12], as the class $B^2_\gamma$ is larger than the space $L^2_{w_2}$. Equations in $B^2_\gamma$ have been very recently discussed by Bradshaw, Kukavica and Tsai [5]. The case $d = 2$ is more intricate. Indeed, while the Leray projection operator is bounded on $B^2_\gamma(\mathbb{R}^3)$ (by interpolation with $L^2$ and $L^2_{w_2}$, with $2 < \delta < 3$, the Riesz transforms being bounded on $L^2_{w_2}$ by the theory of Muckenhoupt weights), this is no longer the case on $B^2_\gamma(\mathbb{R}^2)$. Thus, one must be careful in the handling of the pressure. This has been done by Basson in his Ph. D. thesis in 2006 [1].

Local Morrey spaces $B^2_\delta$ occur naturally in the setting of homogeneous statistical solutions. By using an ergodicity argument, Dostoglou [9] proved in 2001 that, under the assumptions of Vishik and Fursikov [22], we have
\[
Pr(u_0((,) \omega) \in B^2_\delta(\mathbb{R}^d)) = 1.
\]
Thus, the solutions of Vishik and Fursikov live in a smaller space than $L^2_{w_{d+1}}$.

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