COMBINATORIAL ASPECTS OF AN ODD LINKAGE PROPERTY FOR GENERAL LINEAR SUPERGROUPS

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Abstract. Let $G = GL(m|n)$ be a general linear supergroup and $G_{ev}$ be its even subsupergroup isomorphic to $GL(m) \times GL(n)$. In this paper we use the explicit description of $G_{ev}$-primitive vectors in the costandard supermodule $\nabla(\lambda)$, the largest polynomial $G$-subsupermodule of the induced supermodule $H^0_G(\lambda)$, for $(m|n)$-hook partition $\lambda$, and properties of certain morphisms $\psi_k$ to derive results related to odd linkage for $G$ over a field $F$ of characteristic different from 2.

Introduction

Throughout the paper, with the exception of its last Section 6 we assume that the ground field $F$ is algebraically closed and of characteristic zero.

For the definition of the general linear supergroup $G = GL(m|n)$, distribution superalgebra $\text{Dist}(G)$ and properties of induced supermodules $H^0_G(\lambda)$, the reader is asked to consult [1].

The linkage principle for reductive algebraic groups states that a weight $\mu$, of the simple factor $L(\mu)$ appearing in the composition series of the induced module $H^0(\lambda)$, is obtained via a repeated application of the dot action of the corresponding affine Weyl group on the weight $\lambda$. This linkage applied to the maximal even subgroup $G_{ev} \cong GL(m) \times GL(n)$ of $G$ gives rise to the even linkage of weights. However, in the general linear supergroup $G$ there is another type of linkage - the odd linkage - that appears due to the presence of odd roots of $G$.

The focus of this paper is the odd linkage of weights of $G$ and its combinatorial aspects. If a weight $\mu$ is such that the simple supermodule $L_G(\mu)$ is a composition factor of $H^0_G(\lambda)$ and $\mu$ is odd-linked to $\lambda$, then it is of the form $\mu = \lambda_I|J$, where the pair $(I|J)$ of multiindices $I = (i_1 \cdots i_k)$ and $J = (j_1 \cdots j_k)$ with $1 \leq i_1, \ldots, i_k \leq m$ and $1 \leq j_1, \ldots, j_k \leq n$ is admissible. Therefore, from the very beginning we concentrate our attention to weights of type $\lambda_I|J$ as above.

The combinatorial techniques we apply work for the largest polynomial subsupermodule $\nabla(\lambda)$ of $H^0_G(\lambda)$. Our main result states that if both $\lambda$ and $\lambda_I|J$ are dominant polynomial weights, and $L_G(\lambda_I|J)$ is a composition factor of $\nabla(\lambda)$, then $\lambda$ and $\lambda_I|J$ are odd-linked through a sequence of polynomial weights.

We work mostly with modules over the (even) subgroup $G_{ev}$ of $GL(m|n)$ and one of our main tools is the explicit description of $G_{ev}$-primitive vectors in $\nabla(\lambda)$ established in [8], using the terminology of marked tableaux that can be regarded as a realization of the concept of pictures in the sense of Zelevinsky ([11]) to the general linear supergroups setup.

We give a description of simple composition factors of costandard modules $\nabla(\lambda)$ that is related to the surjectivity of certain maps $\psi_k$. 

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In Section 1 we show that these maps $\psi_k$ are $G_{ev}$-morphisms, and in Section 2 we show that there is a plethora of explicit $G_{ev}$-primitive vectors $\pi_{I|J}$ in the domain of the maps $\psi_k$ and that all $G_{ev}$-primitive vectors in the codomain of $\psi_k$ are linear combinations of explicit vectors $\pi_{I|J}$.

In Section 3 we compute images $\psi_k(\pi_{I|J})$ and establish preliminary results relating surjectivity of $\psi_k$ to the strong linkage for $G$. In Section 4 we establish results related to the strong linkage principle in the case when the weight $\lambda$ is $(I|J)$-robust. This case is easier to handle, since there is a basis of $G_{ev}$-primitive vectors consisting of vectors $\pi_{I|J}$, but the main idea of the strong linkage is already visible in this case.

In Section 5 we consider only polynomial weights $\lambda$ and $\lambda_{I|J}$ and apply combinatorial techniques (using tableaux, Clausen order, pictures) to prove statements related to the linkage principle for $G$.

In Section 6 we formulate a few results connecting our previous investigations to the linkage principle for general linear supergroups over ground fields $F$ of characteristic $p > 2$.

1. Maps $\psi_k$

From now on assume that the characteristic of the ground field $F$ is zero. We will consider the case of the ground field of odd characteristic at the end of the paper.

Write a generic $(m+n) \times (m+n)$-matrix $C = (c_{ij})$ in a $(m|n)$-block form

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}. $$

Let $A(m|n)$ be the superalgebra freely generated by elements $c_{ij}$ for $1 \leq i, j \leq m+n$ subject to the supercommutativity relation

$$c_{ij}c_{kl} = (-1)^{|c_{ij}||c_{kl}|}c_{kl}c_{ij},$$

where the parity $|c_{ij}| = 0$ for $1 \leq i, j \leq m$ or $m+1 \leq i, j \leq m+n$ and $|c_{ij}| = 1$ otherwise. The coordinate superalgebra $F[G]$ of the supergroup $G = GL(m|n)$ is a localization of $A(m|n)$ by the element $det(C_{11})det(C_{22})$. Denote by $A_{ev}(m|n)$ the subsuperalgebra of $A(m|n)$ spanned by the elements $c_{ij}$ for $1 \leq i, j \leq m$ or $m+1 \leq i, j \leq m+n$, and by $K(m|n)$ the localization $A(m|n)(A_{ev}(m|n) \setminus 0)^{-1}$.

The maximal even subsupergroup $G_{ev}$ of $G$ satisfies $G_{ev} \simeq GL(m) \times GL(n)$. We will write a weight $\lambda$ of $G$ as $\lambda = (\lambda_1^+, \ldots, \lambda_m^+|\lambda_1^- \ldots, \lambda_n^-)$ and identify it with a pair of weights $\lambda^+ = (\lambda_1^+, \ldots, \lambda_m^+)$ of $GL(m)$ and $\lambda^- = (\lambda_1^-, \ldots, \lambda_n^-)$ of $GL(n)$.

We will be working inside the induced supermodule $H^0_G(\lambda)$ considered using the super space isomorphism $\tilde{\phi}$ given in Lemma 5.1 of [12] as $H^0_{C_{ev}}(\lambda) \otimes S(C_{12}) \rightarrow H^0_G(\lambda)$, where $S(C_{12})$ is the supersymmetric superalgebra of the superspace $C_{12}$. The map $\tilde{\phi}$ is a restriction of the multiplicative morphism $\phi : K[G] \rightarrow K[G]$ given on generators as follows:

$$C_{11} \mapsto C_{11}, C_{21} \mapsto C_{21}, C_{12} \mapsto C_{11}^{-1}C_{12}, C_{22} \mapsto C_{22} - C_{21}C_{11}^{-1}C_{12}.$$ 

for $1 \leq i \leq m$ and $m+1 \leq j \leq m+n$.

Let $D$ be the determinant of $C_{11}$ and $A = (A_{ij})$ be the adjoint matrix of $C_{11}$. Then

$$y_{ij} = \frac{\phi(c_{ij}) = A_{i1}c_{1j} + A_{i2}c_{2j} + \ldots + A_{im}c_{mj}}{D}$$
and \(H^0_G(\lambda)\) is identified with \(\oplus_{k=0}^{mn} V \otimes \wedge^k Y\), where \(V = H^0_{G_{ev}}(\lambda)\) is the even-induced supermodule and \(Y = F[y_{ij} | 1 \leq i \leq m, 1 \leq j \leq n]\).

The structure of induced modules over general linear groups is described using biternants. Since we consider \(H^0_{G_{ev}}(\lambda)\) embedded inside \(H^0_G(\lambda)\) using the map \(\phi\), we need to adjust the notation for bideterminants in order to accomodate the effect of the map \(\phi\).

If \(1 \leq i_1, \ldots, i_s \leq m\), then denote by \(D^+(i_1, \ldots, i_s)\) the determinant

\[
\begin{vmatrix}
  c_{1,i_1} & \cdots & c_{1,i_s} \\
  c_{2,i_1} & \cdots & c_{2,i_s} \\
  \vdots & \ddots & \vdots \\
  c_{s,i_1} & \cdots & c_{s,i_s}
\end{vmatrix}
\]

Clearly, if some of the numbers \(i_1, \ldots, i_s\) coincide, then \(D^+(i_1, \ldots, i_s) = 0\).

If \(m + 1 \leq j_1, \ldots, j_s \leq m + n\), then denote by \(D^-(j_1, \ldots, j_s)\) the determinant

\[
\begin{vmatrix}
  \phi(c_{m+1,j_1}) & \cdots & \phi(c_{m+1,j_s}) \\
  \phi(c_{m+2,j_1}) & \cdots & \phi(c_{m+2,j_s}) \\
  \vdots & \ddots & \vdots \\
  \phi(c_{m+s,j_1}) & \cdots & \phi(c_{m+s,j_s})
\end{vmatrix}
\]

Clearly, if some of the numbers \(j_1, \ldots, j_s\) coincide, then \(D^-(j_1, \ldots, j_s) = 0\).

The images of the highest vector \(v_+\) of \(H^0_{\text{Gl}(m)}(\lambda^+)\) and the highest vector \(v_\cdot\) of \(H^0_{\text{Gl}(m)}(\lambda^-)\) under the map \(\phi\) are identified as the following elements of \(H^0_{G_{ev}}(\lambda)\):

\[
v_+ = \prod_{a=1}^m D^+(1, \ldots, a)^{\lambda^+_{kl} - \lambda^+_{kl+1}}, \quad v_- = \prod_{b=1}^n D^-(m + 1, \ldots, m + b)^{\lambda^+_{kl} - \lambda^+_{kl+1}},
\]

where \(\lambda^+_{m+1} = 0 = \lambda^-_{m+1}\). Their product \(v = v_+v_-\) is the highest vector of \(H^0_G(\lambda)\).

The superderivation \(i_j D\) of parity \(|i_j D| = |c_{ij}|\) is given by \((c_{kl})i_j D = \delta_{i_l}c_{kj}\), where \(\delta_{il}\) stands for the Kronecker delta, and it satisfies the property

\[
(ab)_i j D = (-1)^{|b||i_j D|} (a)_i j D b + (a)_i j D b
\]

for \(a, b \in A(m|n)\). The action of \(i_j D\) extends to \(K[G]\) using the quotient rule

\[
(b)_{ij} D = \frac{(a)_i j D b - a(b)_{ij} D}{b^2}
\]

for \(a, b \in A(m|n)\) and \(b\) even.

The action of \(i_j D\) on elements of \(H^0_{G_{ev}}(\lambda)\) were computed in Section 2 of [7].

For \(k = 0, \ldots, mn\) denote by \(T_k\) the supermodule \(V \otimes Y \otimes^k Y\) and by \(F_k\) the supermodule \(V \otimes \wedge^k Y\) of \(H^0_G(\lambda)\). The supermodule \(F_k\) will be called the \(k\)-floor of \(H^0_{G_{ev}}(\lambda)\). It is important to note that both \(T_k\) and \(F_k\) are \(G_{ev}\)-modules.

Let us denote \(\ell(\mu^+) = \sum_{i=1}^m \mu^+_i\), \(\ell(\mu^-) = \sum_{j=1}^n \mu^-_j\) and \(\ell(\mu) = \ell(\mu^+) + \ell(\mu^-)\). If \(M\) is an indecomposable \(G\)-supermodule, the value of \(\ell(\mu)\) for all nonzero weight spaces \(M_\mu\) remains constant. If \(M\) is an indecomposable \(G_{ev}\)-module, then the values of \(\ell(\mu^+)\) and \(\ell(\mu^-)\) on all nonzero weight spaces \(M_\mu\) remain constant. Therefore, if study the \(G_{ev}\)-structure of \(H^0_{G_{ev}}(\lambda)\), this leads naturally to the grading by \(H^0_{G_{ev}}(\lambda)\) by the floors \(F_k\) as above.

In earlier papers [8, 2] we have considered \(G_{ev}\)-morphisms \(\phi_k : V \otimes \wedge^k Y \rightarrow V \otimes \wedge^k Y\) defined by \(\phi_k(v \otimes (y_{i_1 j_1} \wedge \cdots \wedge y_{i_s j_s})) = (v)_{i_1 j_1} D \cdots \dot{i_s j_s} D\) that proved useful when investigating \(G_{ev}\)-module structure of \(F_k\).
Let \( \text{Dist} \) be the superalgebra of distributions of \( G \) and \( \text{Lie}(G) \subseteq \text{Dist}(G) \) be the Lie superalgebra of \( G \). One can define (left) actions of \( \text{Dist}(G) \) on \( A \) by

\[
\phi \cdot a = \sum a_1 \phi(a_2)
\]

and by

\[
\phi \ast a = \sum (-1)^{|\phi||a_1|} a_1 \phi(a_2),
\]

respectively, where \( \Delta(a) = \sum a_1 \otimes a_2 \).

If \( \phi \in \text{Lie}(G) \), then \( \phi \) acts on \( A \) as a right superderivation with respect to the action \( \cdot \), and as a left superderivation with respect to the action \( \ast \). The relationship between both actions is \( \phi \cdot a = (-1)^{|\phi||a_1|+1} \phi \ast a \) for \( a \in A \). Therefore we can interchange the actions \( \ast \) and \( \cdot \); and \( \text{Dist}(G) \)-supermodules, with respect to these two actions, can be identified.

The proof of the next lemma uses the relationship of superderivation \( ijD \) to the action of the distribution algebra \( \text{Dist}(G) \).

**Lemma 1.1.** Every map \( \psi_k \) as above is a \( G_{ev} \)-morphism.

**Proof.** Since the action of a superderivation \( ijD \) on \( A(m|n) \) corresponds to the \( \cdot \) action of \( e_{ij} \) on \( A(m|n) \) - see [3], we obtain

\[
\psi_k(v \otimes y_{i_1 j_1} \otimes \cdots \otimes y_{i_{k-1} j_{k-1}} \otimes y_{i_k j_k}) = (v \wedge y_{i_1 j_1} \wedge \cdots \wedge y_{i_{k-1} j_{k-1}})_{i_k j_k} D = e_{i_k j_k} \cdot (v \wedge y_{i_1 j_1} \wedge \cdots \wedge y_{i_{k-1} j_{k-1}}),
\]

Using the comments preceding Lemma [4] and the corresponding modification of Lemma 1.2 of [10] we derive that for every \( g \in G_{ev} \) there is

\[
g(\psi_k(v \otimes y_{i_1 j_1} \otimes \cdots \otimes y_{i_{k-1} j_{k-1}} \otimes y_{i_k j_k})) = g(e_{i_k j_k} \cdot (v \wedge y_{i_1 j_1} \wedge \cdots \wedge y_{i_{k-1} j_{k-1}})) = \text{Ad}(g)(e_{i_k j_k}) \cdot (v \wedge y_{i_1 j_1} \wedge \cdots \wedge y_{i_{k-1} j_{k-1}}).
\]

Since the adjoint action of \( h \in G_{ev} \) on \( e_{ji} \), where \( 1 \leq i \leq m < j \leq m + n \), is given as

\[
\text{Ad}(h)e_{ji} = h e_{ji} h^{-1} = \sum_{l=m+1}^{m+n} \sum_{k=1}^{m} h_{lj} e_{lk} h_{ik}^{-1}
\]
and its action on $y_{ij}$ is given as

$$hy_{ij} = \sum_{k=1}^{m} \sum_{l=m+1}^{m+n} h_{ik}^{-1} y_{kl} h_{lj},$$

we conclude that the map $y_{ij} \mapsto e_{ji}$ induces an isomorphism of $G_{ce}$-supermodules $Y = \sum_{1 \leq i < j \leq m+n} K y_{ij}$ and $\sum_{1 \leq i < j \leq m+n} K e_{ji}$ and the claim follows. \hfill \Box

2. EVEN-PRIMITIVE VECTORS $\pi_{I,J}$

2.1. Notation. Before we start, we review the definition of $\pi_{I,J}$, $v_{I,J}$, $\rho_{I,J}$ from [7].

Assume that $(I|J) = (i_1 \ldots i_k | j_1 \ldots j_k)$ is a multi-index such that $1 \leq i_1, \ldots, i_k \leq m$ and $1 \leq j_1, \ldots, j_k \leq n$. Define the content $\text{cont}(I|J)$ of $(I|J)$ to be the $(m+n)$-tuple $\text{cont}(I|J) = (x_1^{+}, \ldots, x_n^{+}|x_1^{-}, \ldots, x_n^{-})$, where $x_s^{+}$ is the number of occurrences of the symbol $s$ in $i_1 \ldots i_k$ and $x_s^{-}$ is the number of occurrences of the symbol $t$ in $j_1 \ldots j_k$. Further, denote

$$\lambda_{I,J} = \lambda - \sum_{s=1}^{k} \delta_{i_s}^{+} + \sum_{s=1}^{k} \delta_{j_s}^{-},$$

where $\delta_{i_s}^{+}$ is the weight of $G$ that has all the components equal to zero except the component corresponding to the index $i_s$ in its $GL(m)$-part which equals to one. Analogously, $\delta_{j_s}^{-}$ is the weight of $G$ that has all the components equal to zero except the component corresponding to the index $j_s$ in its $GL(n)$-part which equals to one. In particular, if $I = \{i\}$ and $J = \{j\}$, then

$$\lambda_{ij} = \lambda_{I,J} = (\lambda_{i}^{+}, \ldots, \lambda_{i}^{+} - 1, \ldots, \lambda_{n}^{+} | \lambda_{1}^{-}, \ldots, \lambda_{n}^{-} + 1, \ldots, \lambda_{n}^{-}).$$

It is important to note that if the weight space of $H_{G}^{\nu}(\lambda)$ corresponding to the weight $\lambda_{I,J}$ is nonzero, then $\lambda_{I,J}$ is a weight of the $k$-th floor $F_k$, and $\ell(\lambda_{I,J}^{-}) = \ell(\lambda^{-}) + k$.

The main reason to consider the weights $\lambda_{I,J}$ is that they are the only weights of $H_{G}^{\nu}(\lambda)$ that can be odd-linked to $\lambda$. For the definition of odd linkage see Section 4.

For $1 \leq i \leq m$ and $m+1 \leq j \leq m+n$ we set

$$\rho_{ij} = \sum_{r=i}^{m} D^{+}(1, \ldots, i-1, r) \sum_{s=1}^{j} (-1)^{s+j} D^{-}(m+1, \ldots, \hat{m+s}, \ldots, m+j) y_{r,m+s}.$$

For each $(I|J) = (i_1 \ldots i_k | j_1 \ldots j_k)$ such that $1 \leq i_1, \ldots, i_k \leq m$ and $1 \leq j_1, \ldots, j_k \leq n$ denote the element $y_{i_1,j_1} \otimes \cdots \otimes y_{i_k,j_k}$ by $y_{I,J}$, the element $y_{i_1,j_1} \wedge \cdots \wedge y_{i_k,j_k}$ by $y_{I,J}$ and $\otimes_{s=1}^{k} \rho_{i_s,j_s}$ by $\rho_{I,J}$. For such $(I|J)$ denote the elements

$$v_{I,J} = \frac{v}{\prod_{s=1}^{k} D^{+}(1, \ldots, i_s) \prod_{s=1}^{k} D^{-}(m+1, \ldots, m+j_s - 1)},$$

where we set $D^{-}(m+1, \ldots, m+j_s - 1) = 1$ for $j_s = 1$.

Then the elements

$$\pi_{I,J} = v_{I,J} \rho_{I,J}$$

are $G_{ce}$-primitive vectors of $Q_k = K(m|n) \otimes Y^{\otimes k}$.
Denote by $\overline{\Omega}_{I\mid J}$ the images of $\rho_{I\mid J}$ under the natural map $Q_k = K(m\mid n) \otimes Y^\otimes k \to \overline{Q}_k = K(m\mid n) \otimes \wedge^k Y$, and $\overline{\Omega}_{I\mid J} = v_{I\mid J}\overline{\Omega}_{I\mid J}$.

Every $w \in T_k$ of weight $\mu$ can be written in the form $w = \sum_{\kappa \leq \lambda} w_\kappa$, where

$$w_\kappa = \sum_{\text{cont}(K|L) = \mu - \kappa} w_{K|L} \otimes y_{K|L}$$

and each $w_{K|L}$ is a vector from $H^0_{G_{ev}}(\lambda)$ of weight $\kappa$.

We will say that a weight $\kappa \leq \lambda$ is a leading weight in $w$ if $w_\kappa \neq 0$ and $w_\mu = 0$ for every $\mu$ such that $\kappa \leq \mu \leq \lambda$. Note that there can be more than one leading weight for $w$.

We call $(I|J)$ admissible if $i_1 \leq \ldots \leq i_k$, and if indices $i_1 < i_2$ are such that $i_{t_1} = i_{t_2}$, then $j_{t_1} < j_{t_2}$.

For $w \in F_k$, we have $w = \sum_{\kappa \leq \lambda} w_\kappa$, where

$$w_\kappa = \sum_{\text{cont}(K|L) \text{ admissible}} w_{K|L} \otimes \overline{\Omega}_{K|L}$$

and $w_{K|L} \in H^0_{G_{ev}}(\lambda)$ is of weight $\kappa$, and the definition of a leading weight is analogous to the one above.

We will apply analogous descriptions to elements from $Q_k$ and $\overline{Q}_k$, in particular to vectors $\pi_{I\mid J}$ and $\overline{\Omega}_{I\mid J}$. Then $\lambda$ is the unique leading weight of both $\pi_{I\mid J}$ and $\overline{\Omega}_{I\mid J}$, while $(\pi_{I\mid J})_\lambda = vy_{I\mid J}$ and $(\overline{\Omega}_{I\mid J})_\lambda = \nu\overline{\Omega}_{I\mid J}$.

### 2.2. Even-primitive vectors of weights $\lambda_{I\mid J}$ in $H^0_{G_{ev}}(\lambda)$

Let $(I|J)$ be an admissible multiindex. Following [7], we say that the weight $\lambda$ is $(I|J)$-robust provided the symbol $i_s < m$ appears at most $\lambda_s^+ - \lambda_{s+1}^+$ times in $I$, symbol $m$ appears at most $\lambda_m^+$ times in $I$; and symbol $j_t > 1$ appears at most $\lambda_{t-1}^+ - \lambda_t^+$ times in $J$.

In this paper, we will consider robust weights first since the even-primitive vectors of weight $\lambda_{I\mid J}$ such that $\lambda$ is $(I|J)$-robust have an easy description; many results will have more transparent formulations and proofs for robust weights. Later we will handle the general case.

Let us first describe a basis of even-primitive vectors in $F_k$, of weight $\mu$ under the assumption that the weight $\lambda$ is $(I|J)$-robust and $\mu = \lambda + \text{cont}(I|J)$.

**Proposition 2.1.** Assume the weight $\lambda$ is $(I|J)$-robust. Then the vectors $\pi_{K|L}$, where $\text{cont}(K|L) = \text{cont}(I|J)$, form a basis of even-primitive vectors of weight $\lambda_{I\mid J}$ in $T_k$. Additionally, the vectors $\pi_{K|L}$, where $\text{cont}(K|L) = \text{cont}(I|J)$ and $(K|L)$ admissible, form a basis of even-primitive vectors of weight $\lambda_{I\mid J}$ in $F_k$.

**Proof.** The statements that vectors $\pi_{K|L}$ are even-primitive vectors in $T_k$ and $\pi_{K|L}$ are even-primitive vectors in $F_k$ follows from Lemma 4.1 of [7]. It is clear that vectors $\pi_{K|L}$, where $\text{cont}(K|L) = \text{cont}(I|J)$, are linearly independent vectors of weight $\lambda_{I\mid J}$. By Lemmas 4.2 and 4.3 of [7], the vectors $\pi_{K|L}$, where $\text{cont}(K|L) = \text{cont}(I|J)$ and $(K|L)$ is admissible, are linearly independent of weight $\lambda_{I\mid J}$.

Let $w \neq 0$ be an even-primitive vector of weight $\mu = \lambda_{I\mid J}$ from $F_k$ and write $w = \sum_{\kappa \leq \lambda} w_\kappa$ as above. We will show that $\lambda$ is the leading weight of $w$. Assume this not the case and that $\nu \neq \lambda$ is a leading weight of $w$. Then

$$w_{\kappa'} = \sum_{\text{cont}(K|L) = \mu - \nu} w_{\kappa'|L} \otimes y_{\kappa'|L}$$

for every $\kappa' \leq \lambda$ and $\nu$, where $w_{\kappa'|L} \in H^0_{G_{ev}}(\nu)$ is of weight $\kappa'$. Note that

$$w_{\kappa'} = \sum_{\text{cont}(K|L) = \mu - \nu} w_{\kappa'|L} \otimes \overline{\Omega}_{\kappa'|L}$$

for every $\kappa' \leq \lambda$ and $\nu$, where $w_{\kappa'|L} \in H^0_{G_{ev}}(\nu)$ is of weight $\kappa'$.
Let $ijD$ be an even superderivation. Then $(w)_{ij}D = 0$ implies $(w_\nu)_{ij}D = 0$. Since $w_\nu = \sum_{(K|L) \text{ admissible} \atop \cont(K|L) = \mu - \nu} w_{K|L} \otimes \varpi_{K|L}$ and $\nu$ is a leading weight of $w$, from the action of $ijD$ on elements of $H^0_G(\lambda)$ and on $\varpi_{K|L}$, we conclude that $(w_{K|L})_{ij}D = 0$ for each admissible $(K|L)$ with $\cont(K|L) = \mu - \nu$. Since this is true for every even superderivation $ijD$, we obtain that each such $w_{K|L}$ is an even-primitive vector of $H^0_G(\nu)$ of weight $\nu$. Since the characteristic of the field $F$ is zero, the $G_{ev}$-module $H^0_G(\nu)$ is irreducible and its only nonzero primitive vectors are of the weight $\lambda$.

Therefore, $w_{K|L} = 0$ for every each admissible $(K|L)$ with $\cont(K|L) = \mu - \nu$, which implies $w_\nu = 0$ and contradicts the assumption that $\nu \neq \lambda$ is a leading weight of $w$.

Finally, let $w$ be an-even primitive vector of weight $\mu$ in $F_k$ and write $w_\lambda = \sum_{(K|L) \text{ admissible} \atop \cont(K|L) \neq \mu} c_{K|L} \varpi_{K|L}$. Then $w - \sum_{(K|L) \text{ admissible} \atop \cont(K|L) \neq \mu} c_{K|L} \varpi_{K|L}$ is an even-primitive vector in $F_k$, which does not have $\lambda$ as its leading weight. By the above argument, we obtain $w = \sum_{(K|L) \text{ admissible} \atop \cont(K|L) \neq \mu} c_{K|L} \varpi_{K|L}$ showing that the vectors $\varpi_{K|L}$ for $(K|L)$ admissible and weight $\mu$ form a basis of all even-primitive vectors of weight $\mu$ in $F_k$.

The proof that every even-primitive vector in $T_k$ of weight $\lambda_{I|J}$ is a linear combination of vectors $\pi_{K|L}$, where $\cont(K|L) = \cont(I|J)$, is analogous.

Note that if all entries in $I$ are distinct, all entries in $J$ are distinct and $\lambda$ is $(I|J)$-robust, then the dimension of even-primitive vectors of weight $\lambda_{I|J}$ is $k!$.

In the general case, even-primitive vectors of weight $\lambda_{I|J}$ in $H^0_G(\lambda)$ are certain linear combinations of elements $\varpi_{K|L}$ with $\cont(K|L) = \cont(I|J)$.

**Proposition 2.2.** Every even-primitive vector $w$ in $T_k$ of weight $\lambda_{I|J}$ is a linear combination of vectors $\pi_{K|L}$, where each $\cont(K|L) = \cont(I|J)$. Additionally, every even-primitive vector $w$ in $F_k$ of weight $\lambda_{I|J}$ is a linear combination of vectors $\varpi_{K|L}$, where each $(K|L)$ is admissible and $\cont(K|L) = \cont(I|J)$.

**Proof.** Denote by $M$ the $G_{ev}$-module generated by $F_k$ and vectors $\varpi_{K|L}$, where $\cont(K|L) = \cont(I|J)$. Assume that $w \in M$ is an even-primitive vector of weight $\lambda_{I|J}$ and write $w = \sum_{\kappa \leq \lambda} w_\kappa$ as before. The same argument as in Proposition 2.1 gives that $\lambda$ is the leading weight of $w$. If the leading term of $w$ is $w_\lambda = \sum_{(K|L) \text{ admissible} \atop \cont(K|L) \neq \mu} c_{K|L} \varpi_{K|L}$, then $w - \sum_{(K|L) \text{ admissible} \atop \cont(K|L) \neq \mu} c_{K|L} \varpi_{K|L}$ is an even-primitive vector in $M$ that does not have $\lambda$ as its leading weight. Therefore we conclude that $w = \sum_{(K|L) \text{ admissible} \atop \cont(K|L) \neq \mu} c_{K|L} \varpi_{K|L}$.

The proof for even-primitive vectors in $T_k$ is analogous.

**Theorem 7.1** of [8] gives an explicit description of a basis of all even-primitive vectors in $\nabla(\lambda)$ for a hook partition $\lambda$, which will be used later.

3. **Image of vectors $\pi_{I|J}$ under the map $\psi_k$**

3.1. **Image $\psi_k(\pi_{I|J})$.** It is important to note that the vectors $\pi_{I|J}$ do not necessarily belong to $V \otimes Y^\otimes k$. Therefore, we extend the previously defined map $\psi_k$ in a natural way to a map from $Q_k$ to $\bar{Q}_k$. By abuse of notation we will denote this new map by the same symbol $\psi_k : Q_k \to \bar{Q}_k$.

Let $E$ be a linear combination of elements $\varpi_{K|L}$. The vectors $\varpi_{K|L}$ are linearly independent by Lemma 4.3 of [7]. We define $\text{lead}(E)$ to be the leading term of
$E$, that is the linear combination of all terms in $E$ which are scalar multiples of expressions of type $v_{L|M}$.

Since the leading terms $v_{K|L}$ of elements $\pi_{K|L}$ are by itself linearly independent, in order to determine the coefficients in the above linear combination of $\pi_{K|L}$ for $\psi_k(\pi_{l|m})$ it is enough to determine the coefficients at their leading terms $v_{L|M}$.

We will write $\rho_{ij}$ as

$$D^+(1,\ldots,i-1,i)D^-(m+1,\ldots,m+j,\ldots,m+j)y_{r,m+j}$$

$$+ \sum_{r=i+1}^{m} D^+(1,\ldots,i-1,r)D^-(m+1,\ldots,m+j,\ldots,m+j)y_{r,m+j}$$

$$+ \sum_{s=1}^{j-1} D^+(1,\ldots,i-1,i)(-1)^{s+j}D^-(m+1,\ldots,m+s,\ldots,m+j)y_{r,m+s}$$

$$+ F_{ij},$$

where

$$F_{ij} = \sum_{r=i+1}^{m} D^+(1,\ldots,i-1,r) \sum_{s=1}^{j-1} (-1)^{s+j}D^-(m+1,\ldots,m+s,\ldots,m+j)y_{r,m+s}$$

is a sum of multiples of $y_{r,m+s}$, where $r > i$ and $s < j$.

Then $\pi_{l|m}$ can be written as a sum

$$v \otimes y_{l|m}$$

$$+ \sum_{r_1=i+1}^{m} \frac{D^+(1,\ldots,i_1-1,r_1)}{D^+(1,\ldots,i_1-1,i_1)} \otimes y_{r_1,m+j_1} \otimes y_{r_2,m+j_2} \otimes \cdots \otimes y_{r_k,m+j_k}$$

$$+ \ldots$$

$$+ \sum_{r_k=i_k+1}^{m} \frac{D^+(1,\ldots,i_k-1,r_k)}{D^+(1,\ldots,i_k-1,i_k)} \otimes y_{i_1,m+j_1} \otimes y_{i_2,m+j_2} \otimes \cdots \otimes y_{i_k,m+j_k}$$

$$+ \sum_{s_1=1}^{j_1-1} (-1)^{s_1+j_1} v \frac{D^-(m+1,\ldots,m+s_1,\ldots,m+j_1)}{D^-(m+1,\ldots,m+j_1-1)}$$

$$\otimes y_{i_1,m+s_1} \otimes y_{i_2,m+j_2} \otimes \cdots \otimes y_{i_k,m+j_k}$$

$$+ \ldots$$

$$+ \sum_{s_k=1}^{j_k-1} (-1)^{s_k+j_k} v \frac{D^-(m+1,\ldots,m+s_k,\ldots,m+j_k)}{D^-(m+1,\ldots,m+j_k-1)}$$

$$\otimes y_{i_1,m+j_1} \otimes \cdots \otimes y_{i_{k-1},m+j_{k-1}} \otimes y_{i_k,m+s_k}$$

$$+ F_{l|m},$$

where $F_{l|m}$ is a sum of multiples of various $y_{k|l}$, where at least two entries in $(K|L)$ differ from the corresponding entries in $(l|m)$.

Next, we compute the leading parts of images under $\psi_k$ of various summands appearing in the equation (4). Since the coefficients at $y_{k|l}$ appearing in $F_{l|m}$ contain a product of at least two different expressions $D^-(m+1,\ldots,m+l-1)$ in their denominators, we infer that

$$\text{lead}(\psi_k(F_{l|m})) = 0.$$
Since

\[ \psi_k(v \otimes y_{i,J}) = (-1)^{k-1}(v)_{i_k,j_k} D \otimes y_{i_1,m+j_1} \wedge \cdots \wedge y_{i_{k-1},m+j_{k-1}} + \cdots + v \otimes y_{i_1,m+j_1} \wedge \cdots \wedge (y_{i_{k-1},m+j_{k-1}})_{i_k,j_k} D, \]

using Corollary 2.20 and Lemma 2.10 of [7], we compute

\[ \text{lead}(\psi_k(v \otimes y_{i,J})) = (\lambda^+_{i_k} + \lambda^-_{j_k}) v \otimes y_{i_1,m+j_1} \wedge \cdots \wedge y_{i_{k-1},m+j_{k-1}} \wedge y_{i_k,m+j_k} + \cdots + v \otimes y_{i_1,m+j_1} \wedge \cdots \wedge (y_{i_{k-1},m+j_{k-1}})_{i_k,j_k} D \otimes \]

\[ \text{lead}(\psi_k(v \otimes y_{i,J})) = (\lambda^+_{i_k} + \lambda^-_{j_k}) v \otimes \mathcal{F}_{i,J} + v \otimes \mathcal{F}_{i_{j_2} \cdots j_{k-1}} + \cdots + v \otimes \mathcal{F}_{i_{j_1} \cdots j_{k-2} j_k j_{k-1}}. \]

We have

\[ \text{lead}(\psi_k(\sum_{r_1=1}^{m} D^+(1,\ldots,i_1-1,1+r_1) \wedge y_{r_1,m+j_1} \otimes y_{i_2,m+j_2} \otimes \cdots \otimes y_{i_k,m+j_k})) = \text{lead}((-1)^{k-1} \sum_{r_1=1}^{m} (v \otimes D^+(1,\ldots,i_1-1,r_1))_{i_k,j_k} D \otimes \]

\[ y_{r_1,m+j_1} \wedge y_{i_2,m+j_2} \wedge \cdots \wedge y_{i_{k-1},m+j_{k-1}}). \]

Using Corollary 2.20 and Lemma 2.10 of [7], we argue that the terms contributing to the leading part of the last expression correspond to

\[ D^+(1,\ldots,i_1-1,r_1)_{i_k,j_k} D = \sum_{a=1}^{m} D^+(1,\ldots,i_1-1,a) y_{a,j_k}, \]

provided \( i_k = r_1 \) (which implies \( i_k > i_1 \)) and the only summand contributing to the leading part corresponds to \( a = i_1 \). Therefore the last expression equals

\[ (-1)^{k-1} \delta_{i_k > i_1} v \otimes y_{i_1,j_k} \wedge y_{i_2,j_1} \wedge \cdots \wedge y_{i_{k-1},j_{k-1}} \]

\[ = -\delta_{i_k > i_1} v \otimes \mathcal{F}_{i_{j_2} \cdots j_{k-1}}, \]

where \( \delta_{i_k > i_1} = 1 \) if \( i_k > i_1 \) and equals zero otherwise.

Analogous formulae are valid for the summands corresponding to sums involving \( r_2 \) up to \( r_{k-1} \), and the last one is

\[ \text{lead}(\psi_k(\sum_{r_{k-1} = 1}^{m} D^+(1,\ldots,i_{k-1}-1,r_{k-1}) \wedge y_{i_1,m+j_1} \otimes \cdots \otimes y_{i_{k-1},m+j_{k-1}} \otimes y_{i_k,m+j_k}) = -\delta_{i_k > i_{k-1}} v \otimes \mathcal{F}_{i_{j_2} \cdots j_{k-2} j_{k-1}}. \]
The next sum corresponding to $r_k$ behaves differently. Using Lemma 2.10 of [7] again we get

\[
\text{lead}(\psi_k(\sum_{r_k=i_k+1}^{m} D^+(1, \ldots, i_k - 1, r_k) \\
\quad \times y_{i_1, m+j_1} \otimes \cdots \otimes y_{i_{k-1}, m+j_{k-1}} \otimes y_{r_k, m+j_k}))
\]

\[
= \text{lead}((-1)^{k-1} \sum_{r_k=i_k+1}^{m} (v D^+(1, \ldots, i_k - 1, r_k) )_{r_k,j_k} D \otimes \\
\quad y_{i_1, m+j_1} \wedge y_{i_2, m+j_2} \wedge \cdots \wedge y_{i_{k-1}, m+j_{k-1}})
\]

\[
= (-1)^{k-1}(m - i_k) v \otimes y_{i_k,j_k} \wedge y_{i_2,j_2} \wedge \cdots \wedge y_{i_{k-1},j_{k-1}} = (m - i_k) v \otimes \mathcal{F}_{I,J}.
\]

We have

\[
\text{lead}(\psi_k(\sum_{s_1=1}^{j_1-i_k} (-1)^{s_1+j_1} D^-(m + 1, \ldots, m + s_1, \ldots, m + j_1) \\
\quad \times y_{i_1, m+s_1} \otimes y_{i_2, m+j_2} \otimes \cdots \otimes y_{i_k, m+j_k}))
\]

\[
= \text{lead}(\sum_{s_1=1}^{j_1-i_k} (-1)^{s_1+j_1-k+1} (v D^-(m + 1, \ldots, m + s_1, \ldots, m + j_1) )_{i_k,j_k} D \\
\quad \otimes y_{i_1, m+s_1} \wedge y_{i_2, m+j_2} \wedge \cdots \wedge y_{i_{k-1}, m+j_{k-1}})
\]

Using Corollary 2.20 and Lemma 2.13 of [7] we argue that the terms contributing to the leading part of the last expression correspond to

\[
D^-(m + 1, \ldots, m + j_1)_{i_k,j_k} D = \\
D^-(m + j_k, m + 2, \ldots, m + s_1, \ldots, m + j_1) y_{i_k, m+1} \\
+ D^-(m + 1, m + j_k, \ldots, m + s_1, \ldots, m + j_1) y_{i_k, m+2} \\
+ \ldots \\
+ D^-(m + 1, \ldots, m + s_1, \ldots, m + j_1 - 1, m + j_k) y_{i_k, m+j_1}
\]

provided $j_k = s_1$ (which implies $j_k < j_1$) and the only term contributing to the leading part corresponds to the last summand

\[
D^-(m + 1, \ldots, m + j_k) = (-1)^{j_1+j_k-1} D^-(m + 1, \ldots, m + j_1 - 1)
\]

Since

\[
D^-(m + 1, \ldots, m + j_k, \ldots, m + j_1 - 1, m + j_k) = (-1)^{j_1+j_k-1} D^-(m + 1, \ldots, m + j_1 - 1)
\]

we infer that

\[
\text{lead}(\psi_k(\sum_{s_1=1}^{j_1-i_k} (-1)^{s_1+j_1} D^-(m + 1, \ldots, m + s_1, \ldots, m + j_1) \\
\quad \otimes y_{i_1, m+s_1} \otimes y_{i_2, m+j_2} \otimes \cdots \otimes y_{i_k, m+j_k}))
\]

\[
= (-1)^{k} \delta_{j_k < j_1} v \otimes y_{i_1, m+j_1} \wedge y_{i_2, m+j_2} \wedge \cdots \wedge y_{i_{k-1}, m+j_{k-1}} \\
\quad = -\delta_{j_k < j_1} v \otimes \mathcal{F}_{I|\{j_1,j_2,\ldots,j_{k-1},j_1\}}.
\]
Analogous formulae are valid for the summands corresponding to sums involving $s_2$ up to $s_{k-1}$, and the last one is

$$lead(\psi_k \left( \sum_{s_{k-1}=1}^{j_{k-1}-1} (-1)^{s_{k-1}+j_{k-1}-1} \sum_{s_{k-1}=1}^{j_{k-1}-1} (-1)^{s_{k-1}+j_{k-1}-1} D^-(m+1, \ldots, m+s_{k-1}, m+j_{k-1}) \right)$$

$$\otimes y_{i_1, m+j_1} \otimes \cdots \otimes y_{i_{k-1}, m+s_{k-1}} \otimes y_{i_k, m+j_k})$$

$$= -\delta_{j_k < j_{k-1}} v \otimes \mathcal{F}_{I_{j_k-1} I_{j_k-2} I_{j_k-1}}'.

The last sum corresponding to $s_k$ behaves differently. We have

$$lead(\psi_k \left( \sum_{s_{k-1}=1}^{j_{k-1}-1} (-1)^{s_{k-1}+j_{k-1}-1} D^-(m+1, \ldots, m+s_{k-1}, m+j_{k-1}) \right)$$

$$\otimes y_{i_1, m+j_1} \otimes \cdots \otimes y_{i_{k-1}, m+j_{k-1}} \otimes y_{i_k, m+s_k})$$

$$= lead(\left( \sum_{s_{k-1}=1}^{j_{k-1}-1} (-1)^{s_k+j_k+k-1} \left( D^-(m+1, \ldots, m+s_k), \ldots, m+j_k) y_{i_k, m+s_k} D$$

$$\otimes y_{i_1, m+j_1} \wedge y_{i_2, m+j_2} \wedge \cdots \wedge y_{i_{k-1}, m+j_{k-1}} \right)$$

Using Corollary 2.20 and Lemma 2.13 of [7], we derive that the terms contributing to the leading part of the last expression correspond to

$$D^-(m+1, \ldots, m+s_k, \ldots, m+j_k) y_{i_k, m+j_k} D =$$

$$D^-(m+s_k, m+2, \ldots, m+s_k, \ldots, m+j_k) y_{i_k, m+1}$$

$$+ D^-(m+1, m+s_k, \ldots, m+s_k, \ldots, m+j_k) y_{i_k, m+2}$$

$$+ \ldots$$

$$+ D^-(m+1, \ldots, m+s_k, \ldots, m+j_k-1, m+s_k) y_{i_k, m+j_k}$$

for each $s_k$, and the only term contributing to the leading part corresponds to the last summand

$$D^-(m+1, \ldots, m+s_k, \ldots, m+j_k-1, m+s_k) y_{i_k, m+j_k}.$$ 

Since

$$D^-(m+1, \ldots, m+s_k, \ldots, m+j_k-1, m+s_k)$$

$$= (-1)^{s_k+j_k-1} D^-(m+1, \ldots, m+j_k-1),$$

we infer that

$$lead(\psi_k \left( \sum_{s_{k-1}=1}^{j_{k-1}-1} (-1)^{s_k+j_k-1} D^-(m+1, \ldots, m+j_k-1) \right)$$

$$\otimes y_{i_1, m+j_1} \otimes \cdots \otimes y_{i_{k-1}, m+j_{k-1}} \otimes y_{i_k, m+s_k})$$

$$= (-1)^{j_k-1} v \otimes y_{i_k, m+j_k} \wedge y_{i_1, m+j_1} \wedge y_{i_2, m+j_2} \wedge \cdots \wedge y_{i_{k-1}, m+j_{k-1}}$$

$$= -(j_k-1)v \otimes \mathcal{F}_{I_{j_k-1} J}.$$

Recall that Definition 1.1 of [7] defines the expression $\omega_{ij}$ as

$$\omega_{ij} = \omega_{ij}(\lambda) = \lambda_i^+ + \lambda_j^- + m + 1 - i - j.$$ 

Recall the definition of $\pi_{I_{j_k-1} J}$ given in [7]. We are ready to state the following proposition.
Proposition 3.1. Let \((I,J)\) be admissible of length \(k\). Then
\[
\psi_k(\pi_{I,J}) = \omega_{I,J} + (1 - \delta_{i_k > j_1} - \delta_{j_k < j_1}) M_{I,J} + \cdots
\]

Proof. We have computed above that
\[
\text{lead}(\psi_k(\pi_{I,J})) = (\lambda^+_{i_k} + \lambda^-_{j_k} + m - i_k - j_k + 1)v \otimes \overline{M}_{I,J}
\]

Since \(\pi_{I,J}\) is a \(G_{ev}\)-primitive vector and \(\psi_k\) is a \(G_{ev}\)-morphism, we infer that \(\psi_k(\pi_{I,J})\) is a primitive vector. By Proposition \(2.2\) it is a linear combination of vectors \(\pi_{M,N}\) for admissible \((M,N)\) such that \(\text{cont}(M,N) = \text{cont}(I,J)\). Since the leading part of \(\psi_k(\pi_{I,J})\) is the same linear combination of leading parts \(v \otimes \overline{M}_{M,N}\) of \(\pi_{M,N}\), the statement follows from the above description of \(\text{lead}(\psi_k(\pi_{I,J}))\).

As a consequence of the previous Proposition, the map \(\psi_k : Q_k \rightarrow \overline{S}_k\) restricts to \(\psi_k : S_k \rightarrow \overline{S}_k\), where \(S_k\) is a span of all vectors \(\pi_{I,J}\) for admissible \(I\) and \(J\) of length \(k\) and \(\overline{S}_k\) is a span of all vectors \(\pi_{I,J}\) for admissible \(I\) and \(J\) of length \(k\).

We illustrate this Proposition on the following example that will be used later.

Example 3.2. Let \(m = n = 3\) and a weight \(\lambda\) is such that \(\lambda^+_1 > \lambda^+_2 > \lambda^+_3\) and \(\lambda^-_1 > \lambda^-_2 > \lambda^-_3\). Assume that \(I = J = (123)\), hence \(\lambda\) is \((I,J)\)-robust. Then
\[
\mathcal{B} = \left\{ \overline{b}_1 = \pi_{123,123}, \overline{b}_2 = \pi_{123,213}, \overline{b}_3 = \pi_{123,312}, \overline{b}_4 = \pi_{123,321}, \overline{b}_5 = \pi_{123,231}, \overline{b}_6 = \pi_{123,321} \right\}
\]
is a basis of all even-primitive vectors of weight \(\lambda_{I,J}\) in \(F_k\). The set \(\mathcal{A} = \{a_1, a_2, \ldots, a_{36}\}\) is linearly independent and spans the space \(A\) of even-primitive vectors of weight \(\lambda_{I,J}\) in \(T_k\). For an explanation how was the set \(\mathcal{A}\) ordered see Lemma \(4.7\) and Example \(4.2\) later.

The matrix of \(\psi_k\) with respect to the bases \(\mathcal{A}\) and \(\mathcal{B}\) is given as
\[
\begin{pmatrix}
\omega_{33} & -1 & 0 & -1 & -\omega_{33} & 0 & 1 & 0 & 0 & 1 \\
0 & \omega_{33} & 0 & -1 & -1 & 0 & 0 & -\omega_{33} & 0 & 1 & 0 \\
0 & 0 & \omega_{32} & 0 & -1 & 0 & 0 & 0 & -\omega_{32} & 0 & 1 & 0 \\
0 & 0 & 0 & \omega_{32} & 0 & -1 & 0 & 0 & 0 & -\omega_{32} & 0 & 1 \\
0 & 0 & 0 & 0 & \omega_{31} & 0 & 0 & 0 & 0 & -\omega_{31} & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & \omega_{31} & 0 & 0 & 0 & 0 & -\omega_{31} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \omega_{31} & 0 & 0 & 0 & 0 & -\omega_{31}
\end{pmatrix}
\]
\[
\begin{pmatrix}
-\omega_{22} & 1 & -1 & 0 & 0 & 0 & \omega_{22} & -1 & 1 & 0 & 0 & 0 \\
0 & -\omega_{21} & 0 & 0 & -1 & 0 & 0 & \omega_{21} & 0 & 0 & 1 & 0 \\
0 & 0 & -\omega_{23} & 1 & 0 & 0 & 0 & 0 & \omega_{23} & -1 & 0 & 0 \\
0 & 0 & 0 & -\omega_{22} & 0 & -1 & 0 & 0 & 0 & \omega_{22} & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -\omega_{23} & 1 & 0 & 0 & 0 & 0 & \omega_{23} & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{22}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\omega_{11} & 1 & 0 & 0 & 0 & 1 & -\omega_{11} & -1 & 0 & 0 & 0 & -1 \\
0 & \omega_{12} & 0 & 1 & 0 & 0 & 0 & -\omega_{12} & 0 & -1 & 0 & 0 \\
0 & 0 & \omega_{11} & 1 & 1 & 0 & 0 & 0 & -\omega_{11} & -1 & -1 & 0 \\
0 & 0 & 0 & \omega_{13} & 0 & 0 & 0 & 0 & -\omega_{13} & 0 & 0 \\
0 & 0 & 0 & 0 & \omega_{12} & 1 & 0 & 0 & 0 & -\omega_{12} & -1 \\
0 & 0 & 0 & 0 & 0 & \omega_{13} & 0 & 0 & 0 & 0 & -\omega_{13}
\end{pmatrix}
\]

Here \( \omega_{33} = \lambda_1^+ + \lambda_3^+ - 2 \), \( \omega_{32} = \lambda_1^+ + \lambda_2^+ - 1 \), \( \omega_{31} = \lambda_1^+ + \lambda_1^- \), \( \omega_{23} = \lambda_1^+ + \lambda_3^- - 1 \), \( \omega_{22} = \lambda_2^+ + \lambda_2^- \), \( \omega_{21} = \lambda_2^+ + \lambda_1^+ + 1 \), \( \omega_{13} = \lambda_1^- + \lambda_3^- \), \( \omega_{12} = \lambda_1^+ + \lambda_2^+ + 1 \) and \( \omega_{11} = \lambda_1^+ + \lambda_1^- + 2 \).

3.2. Map \( \psi_\mu \) and the strong linkage. Assume that admissible \( (I|J) \) has the length \( k \) and denote the weight \( \lambda_{I|J} \) by \( \mu \). Denote by \( S_{\mu} \) or \( S_{I|J} \) the span of all even-primitive vectors of weight \( \mu \) in \( T_k \) and by \( \overline{S}_\mu \) or \( \overline{S}_{I|J} \) the span of all even-primitive vectors of weight \( \mu \) in \( F_k \). Consider a restriction \( \psi_\mu \) of the map \( \psi_k \) on \( S_\mu \).

By Lemma 1.1, \( \psi_\mu \) maps \( S_\mu \) to \( \overline{S}_\mu \).

**Proposition 3.3.** Let \( \lambda \) be a dominant weight of \( G \), \( (I|J) \) be admissible of length \( k \) such that \( \mu = \lambda_{I|J} \) is dominant. Then \( L_G(\mu) \) is a composition factor of \( H^0_G(\lambda) \) if and only if the map \( \psi_\mu : S_\mu \to \overline{S}_\mu \) is not surjective. If both \( \lambda \) and \( \mu \) are polynomial weights, then \( L_G(\mu) \) is a composition factor of \( \nabla(\lambda) \) if and only if the map \( \psi_\mu : S_\mu \to \overline{S}_\mu \) is not surjective.

**Proof.** It is well-known that the category of \( G_{ev} \)-modules is semisimple, its simple objects \( L_{G_{ev}}(\mu) \) are indexed by weights \( \mu \) and are generated by \( G_{ev} \)-primitive vectors \( \psi_\mu \). On the other hand, the supermodule \( H^0_G(\lambda) \) need not be semisimple. Its simple composition factors have highest weights \( \mu \), where \( \mu = \lambda_{I|J} \) for admissible \( (I|J) \).

The supermodule \( L_G(\mu) \) is a composition factor of \( H^0_G(\lambda) \) if and only if there is a \( G_{ev} \)-primitive vector in \( F_k \) of weight \( \mu \) that does not belong to the \( G \)-subsupersubmodule \( M \) of \( H^0_G(\lambda) \) generated by all vectors on the floors \( F_s \) of index \( s \) smaller than \( k \).

The map \( \psi_k : T_k \to F_k \) factors through the natural projection

\[
\text{proj} : T_k = V \otimes Y^\otimes k \to V \otimes (\lambda^{k-1} Y) \otimes Y = F_{k-1} \otimes Y = T_k
\]

which is a surjective \( G_{ev} \)-morphism. Denote by \( \text{proj}_\mu \) the restriction of \( \text{proj} \) to the weight spaces corresponding to \( \mu \), by \( \tilde{S}_\mu \) the span of even-primitive vectors of weight \( \mu \) in \( T_k \) and by \( \psi_\mu \) the induced map satisfying \( \psi_\mu = \psi_k \circ \text{proj}_\mu \).

From the definitions of the supermodule \( M \) and the above factorisation of \( \psi_k \), it is clear that \( \psi_k(T_k) \subseteq M \cap F_k \). To see the converse inclusion, consider one of the generators \( w \in M \cap F_l \) of \( M \), where \( l < k \). Denote by \( W \) the \( G \)-supersubmodule generated by \( w \). Using Poincare-Birkhoff-Witt theorem, we order superderivations \( i_j D \) from the distribution algebra \( Dist(G) \) (universal enveloping algebra of the Lie
superalgebra \( g \) in such a way that odd superderivations \( j_a \cdot_i D \), where \( m + 1 \leq j_a \leq m + n \) and \( 1 \leq i_a \leq m \), are listed first, followed by even superderivations and odd superderivations \( i_a \cdot_j D \), where \( 1 \leq i_a \leq m \) and \( m + 1 \leq j_a \leq m + n \) come last. Then every \( w' \in W' \cap F_k \) is a sum of terms of type \((w'')_{ij} D\), where \( w'' \in F_{k-1} \) and odd superderivation \( ij D \) is such that \( 1 \leq i \leq m \) and \( m + 1 \leq j \leq m + n \). Since every \((w'')_{ij} D = \psi_k(w'' \otimes ij) D \in \psi_k(T_k)\), we conclude that \( W' \cap F_k \subseteq \psi_k(T_k) \) and hence \( M \cap F_k \subseteq \psi_k(T_k) \). Therefore \( M \cap F_k = \psi_k(T_k) \).

By Lemma 13 the map \( \psi_k \) sends \( G_{ev} \)-primitive vectors of weight \( \mu \in T_k \) to \( G_{ev} \)-primitive vectors in \( F_k \). Therefore, there is a \( G_{ev} \)-primitive vector of weight \( \mu \in F_k \) that does not belong to \( M \) if and only if the map \( \psi_\mu : S_\mu \to \overline{S}_\mu \) is not surjective.

The last statement now follows from Corollary 7.2 of [8].

Explicit elements of \( S_{I|J} \) and an explicit basis of \( \overline{S}_{I|J} \), in the case when \( \lambda \) is \((I|J)\)-robust, were given in [7] and in the general case in [8].

Using these and our description of the matrix of \( \psi_k \), we will determine necessary conditions when the map \( \psi_\mu : S_\mu \to \overline{S}_\mu \) is surjective. Using Proposition 5.3 we then obtain a description of simple composition factors of \( H^0_G(\lambda) \) and/or \( \nabla(\lambda) \). We start first with the case of a robust weight and deal with the general case later.

In what follows we will abuse the notation and denote various restrictions of the map \( \psi_k \) (and \( \psi_\mu \)) just by \( \psi_k \). We will indicate the domain and codomain of the map \( \psi_k \) every time it is required in order to avoid a confusion.

### 4. Odd linkage for robust weights

In this section we assume that the weight \( \lambda \) is \((I|J)\)-robust.

#### 4.1. \( \lambda \) is \((I|J)\)-robust, entries in \( I \) are distinct and entries in \( J \) are distinct.

Assume first that all entries in \( I \) are distinct and all entries in \( J \) are distinct. In this case, the condition that \( \lambda \) is \((I|J)\)-robust is equivalent to the condition that all entries \( \lambda^+_i \) for \( i \in I \) and all entries \( \lambda^-_j \) for \( j \in J \) are distinct. We will remove the assumption that all entries in \( I \) are distinct and all entries in \( J \) are distinct in the next subsection.

Let \( I_0 = (i_1 < i_2 < \ldots < i_k) \) be a multiindex of the same content as \( I \) and \( J_0 = (j_1 < j_2 < \ldots < j_k) \) be a multiindex of the same content as \( J \). Then \( \overline{S}_{I|J} \) is the span of vectors \( \overline{\pi}_{I_0|L} \) for all permutations \( L \) of \( J \).

Denote by \( S_J \) the set of all multiindices \( L \) of content \( \text{cont}(J) \). We impose the reverse Semitic lexicographic order \(<\) on the set \( S_J \). This means that we compare entries of two elements of \( S_J \) by reading from right to left and we impose the reverse order on the individual symbols. For example, if \( J_0 = \{3 < 2 \} \), then the order \(<\) on \( S_J \) is \( 123 < 213 < 132 < 312 < 231 < 321 \). The order \( L_1 < L_2 < \ldots < L_k! \) on \( S_J \) induces the corresponding order \(<\) on basis elements \( \overline{\pi}_{I_0|L} \) of \( \overline{S}_{I|J} \) given as \( \overline{\pi}_{I_0|L_1} < \overline{\pi}_{I_0|L_2} < \ldots < \overline{\pi}_{I_0|L_k!} \).

For a permutation \( \eta \in \Sigma_k \) of the set \( \{1, \ldots, k\} \) and \( L = (l_1, \ldots, l_k) \), we denote \( \eta L = (l_{\eta(1)}, \ldots, l_{\eta(k)}) \). The vector space \( S_{\eta I|J} \) contains a subspace that is a direct sum of spaces \( S_{\eta I_0|J} \) for all \( \eta \in \Sigma_k \), where each \( S_{\eta I_0|J} \) is the span of all vectors \( \pi_{\eta I_0|L} \) for all multiindices \( L \) of content \( \text{cont}(J) \). The dimension of \( S_{\eta I_0|J} \) is \( k! \) and the matrix of \( \psi_k : S_{\eta I_0|J} \to \overline{S}_{I|J} \) with respect to bases consisting of vectors \( \pi_{\eta I_0|L} \) and \( \overline{\pi}_{I_0|L} \), respectively, is a square matrix of dimension \( k! \).
Related to the order \(< \) on \( S_J \) and the permutation \( \eta \in \Sigma_k \), we define an order \( <^\eta \) on the basis elements \( \pi_{\eta I_0} \) of \( S_{\eta I_0} \) as follows. If \( L_1 < L_2 \cdots < L_k \) is the listing of the elements of \( S_J \) according to the order \(< \), then the order \( <^\eta \) on the elements \( \pi_{\eta I_0} \) is given as \( \pi_{\eta I_0} \) \( <^\eta \) \( \pi_{\eta I_0} \) \( <^\eta \) \( \cdots \) \( <^\eta \) \( \pi_{\eta I_0} \). This is compatible with the order \(< \) on basis elements \( \pi_{I_0} \) of \( S_{I_0} \) since \( \pi_{\eta I_0} = \pm \pi_{I_0} \).

Recall the definition of \( \omega_{i,j} \) given in \([3]\).

**Lemma 4.1.** Assume that all entries in \( I \) are distinct, all entries in \( J \) are distinct and \( \lambda \) is \( (|I|,|J|) \)-robust. Then the matrix of the map \( \psi_k : S_{\eta I_0,J} \rightarrow S_{I_0,J} \) with respect to the bases consisting of vectors \( \pi_{\eta I_0} \) ordered by \( <^\eta \), and \( \pi_{I_0} \) ordered by \(< \), is an upper-triangular matrix. Its diagonal entry corresponding to \( \pi_{\eta I_0} \) and \( \pi_{I_0} \) equals \((-1)^{\eta I_0,J_{\eta I_0}} \).

**Proof.** By Proposition \([3.1]\) we have

\[
\psi_k(\pi_{\eta I_0,J}) = \omega_{i_{\eta I_0},J_{\eta I_0}} \pi_{I_0,J} + (1 - \delta_{l_{\eta I_0},j_{\eta I_0}} - \delta_{l_{\eta I_0},l_{\eta I_0}}) \pi_{\eta I_0,J} \pi_{l_{\eta I_0},l_{\eta I_0}} \cdots \pi_{l_{\eta I_0},l_{\eta I_0}} \pi_{l_{\eta I_0},l_{\eta I_0}} + \cdots
\]

Therefore, the diagonal entries of our matrix are the same as described in the statement of the lemma and it only remains to show that our matrix is upper-triangular.

Consider the general term

\[
(1 - \delta_{l_{\eta I_0},j_{\eta I_0}} - \delta_{l_{\eta I_0},l_{\eta I_0}}) \pi_{\eta I_0,J} \pi_{l_{\eta I_0},l_{\eta I_0}} \cdots \pi_{l_{\eta I_0},l_{\eta I_0}} \pi_{l_{\eta I_0},l_{\eta I_0}}
\]

in the above formula for \( \psi_k(\pi_{\eta I_0,J}) \). Let \( \gamma = (\eta(t) \eta(k)) \) be the transposition switching entries in positions \( \eta(t) \) and \( \eta(k) \) and \( K = \gamma L \).

Then \( \eta K = (l_{\gamma t}) \cdots l_{(t-1)} l_{\eta(k)} l_{\eta(t-1)} \cdots l_{\eta(k)} l_{\eta(t)} \) and the above general term becomes

\[
(1 - \delta_{l_{\eta I_0},j_{\eta I_0}} - \delta_{l_{\eta I_0},l_{\eta I_0}}) \pi_{\eta I_0,K}.
\]

If \( i_{\eta(k)} > i_{\eta(t)} \) and \( l_{\eta(k)} > l_{\eta(t)} \), or \( i_{\eta(k)} < i_{\eta(t)} \) and \( l_{\eta(k)} < l_{\eta(t)} \), then this term vanishes.

If \( i_{\eta(k)} > i_{\eta(t)} \) and \( l_{\eta(k)} < l_{\eta(t)} \), then \( \eta(t) < \eta(k) \) and \( l_{\eta(t)} > l_{\eta(k)} \) imply that \( K = \gamma L \). If \( i_{\eta(k)} < i_{\eta(t)} \) and \( l_{\eta(k)} > l_{\eta(t)} \), then \( \eta(t) > \eta(k) \) and \( l_{\eta(t)} < l_{\eta(k)} \) imply again that \( K = \gamma L \). Therefore, in both cases \( \pi_{\eta I_0,K} \leq \eta \pi_{\eta I_0,J} \) and \( \pi_{I_0,K} \leq \pi_{I_0,J} \), it implies that our matrix is upper-triangular.

The above lemma implies immediately that the determinant of the matrix of the map \( \psi_k : S_{\eta I_0,J} \rightarrow S_{I_0,J} \) is \((\prod_{I=1}^{k} \omega_{i_{\eta I_0},J_{\eta I_0}}))^{(k-1)}\).

We illustrate this lemma on the following example.

**Example 4.2.** Let \( m = n = 3 \) and a weight \( \lambda \) is such that \( \lambda_1^+ > \lambda_2^+ > \lambda_3^+ \) and \( \lambda_1^- > \lambda_2^- > \lambda_3^- \). Assume that \( I = J = (123) \), hence \( \lambda \) is \( (|I|,|J|) \)-robust. In the notation of Example \([3,3]\) we have \( \mathfrak{B} = \{b_1, b_2, b_3, b_4, b_5, b_6\} \) and

\[
\mathfrak{B}_{id} = \{a_1, a_2, a_3, a_4, a_5, a_6\}, \quad \mathfrak{A}_{(12)} = \{a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\},
\]

\[
\mathfrak{A}_{(23)} = \{a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, a_{18}\}, \quad \mathfrak{A}_{(31)} = \{a_{19}, a_{20}, a_{21}, a_{22}, a_{23}, a_{24}\},
\]

\[
\mathfrak{A}_{(21)} = \{a_{25}, a_{26}, a_{27}, a_{28}, a_{29}, a_{30}\} \text{ and } \mathfrak{A}_{(13)} = \{a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, a_{36}\}.
\]

The matrices considered in Lemma \([4,4]\) can be identified with blocks, formed by consecutive sextuples of columns, of the matrix in Example \([3,3]\).
To compare various even-primitive vectors in $\overline{Q}_k$ or in $\overline{S}_k$, it is useful to mention that $\overline{\pi}_{\mathfrak{I}_L} = (-1)^{\mathfrak{I}} \overline{\pi}_{\mathfrak{I}_L \eta_L}$.

Recall the definition of $\omega_{ij}(\lambda)$ given in [4]. We say that weights $\lambda$ and $\mu$ are simply-odd-linked, and write $\lambda \sim_{\text{odd}} \mu$, if $\lambda_{ij} = \mu$ or $\mu_{ij} = \lambda$ and $\omega_{ij}(\lambda) = 0$ (in this case also $\omega_{ij}(\mu) = 0$). We say that $\lambda$ and $\mu$ are odd-linked, and write $\lambda \sim_{\text{odd}} \mu$ if there is a chain of weights such that $\lambda = \lambda_1 \sim_{\text{odd}} \lambda_2 \sim_{\text{odd}} \cdots \sim_{\text{odd}} \lambda_t = \mu$.

Let us note that if a weight $\mu$ of $H_G^0(\lambda)$ is odd-linked to $\lambda$, then $\mu = \lambda_{I,J}$ for some admissible $(I|J)$.

If $\lambda \sim_{\text{odd}} \mu$, and $\lambda, \mu$ and all intermediate $\lambda_i$ in the above chain are polynomial weights, then we write $\lambda \sim_{\text{odd}} \mu$.

**Proposition 4.3.** Assume that all entries in $I$ are distinct, all entries in $J$ are distinct and $\lambda$ is $(I|J)$-robust. If the simple $G$-supermodule $L_G(\lambda_{I,J})$ is a composition factor of $H_G^0(\lambda)$, then $\lambda \sim_{\text{odd}} \lambda_{I,J}$.

If both $\lambda$ and $\lambda_{I,J}$ are polynomial weights and the simple $G$-supermodule $L_G(\lambda_{I,J})$ is a composition factor of $\nabla(\lambda)$, then $\lambda \sim_{\text{odd}} \lambda_{I,J}$.

**Proof.** We will use Proposition 3.3 and assume that the map $\psi_k: S_{I,J} \to \overline{S}_{I,J}$ is not surjective.

Create collections $C_N$, each indexed by a multiindex $N = (n_1, \ldots, n_k)$ of content $\text{cont}(J)$. The collection $C_N$ consists of vectors $\pi_{\eta_l|\eta_N}$ for all $\eta \in \Sigma_k$. The images $\psi_k(\pi_{\eta_l|\eta_N})$ of elements in the collection $C_N$ expressed as a linear combination of vectors $\overline{\pi}_{\mathfrak{I}_L}$ have the property that the coefficient at $\overline{\pi}_{\mathfrak{I}_L N}$ equals $\omega_{\eta(k)|\eta(k)}(\lambda)$, and all other nonzero coefficients appear only at $\overline{\pi}_{\mathfrak{I}_L N}$ for $L < N$. To each collection $C_N$ we assign the set $O_N = \{\omega_{\eta(k)|\eta(k)}| \eta \in \Sigma_k\} = \{\omega_{\eta|\eta}, t = 1, \ldots, k\}$. It is a crucial observation that if every $O_N$ contains a nonzero element, then the map $\psi_k : S_{I,J} \to \overline{S}_{I,J}$ is surjective. This is because by Lemma 11 we can find a set of $k!$ vectors $\pi_{K|L}$ of weight $\lambda_{I,J}$ such that the matrix of $\psi_k$ restricted on the span of these vectors is invertible.

Therefore, if $\psi_k : S_{I,J} \to \overline{S}_{I,J}$ is not surjective, then there is $N$ such that all elements of $O_N$ are equal to zero. In this case we derive that $\lambda$ and $\lambda_{I,J}$ are odd-linked via a sequence $\lambda \sim_{\text{odd}} \lambda_{I,J} \sim_{\text{odd}} \lambda_{I,J}$ because $\omega_{ij}(\lambda) = 0$ and so on.

The last statement follows from Corollary 7.2 of [8].

**Example 4.4.** Let $m = n = 3$ and a weight $\lambda$ is such that $\lambda_1^+ > \lambda_2^+ > \lambda_3^-$ and $\lambda_1^- > \lambda_2^- > \lambda_3^-$. Assume that $I = J = (123)$, hence $\lambda$ is $(I|J)$-robust. In the notation of Example 3.2 we have $\mathfrak{B} = \{b_1, b_2, b_3, b_4, b_5, b_6\}$. To illustrate the argument in the previous proposition, we determine that collections are

\[
\begin{align*}
C_{123} &= \{a_1, a_7, a_{13}, a_{19}, a_{25}, a_{31}\}, & C_{213} &= \{a_2, a_8, a_{14}, a_{20}, a_{26}, a_{32}\}, \\
C_{132} &= \{a_3, a_9, a_{15}, a_{21}, a_{27}, a_{33}\}, & C_{312} &= \{a_4, a_{10}, a_{16}, a_{22}, a_{28}, a_{34}\}, \\
C_{231} &= \{a_5, a_{11}, a_{17}, a_{23}, a_{29}, a_{35}\} & C_{321} &= \{a_6, a_{12}, a_{18}, a_{24}, a_{30}, a_{36}\},
\end{align*}
\]

while the corresponding sets are

\[
\begin{align*}
O_{123} &= \{\omega_{33}, \omega_{22}, \omega_{11}\}, & O_{213} &= \{\omega_{33}, \omega_{22}, \omega_{11}\}, & O_{132} &= \{\omega_{32}, \omega_{23}, \omega_{11}\}, \\
O_{312} &= \{\omega_{32}, \omega_{22}, \omega_{13}\}, & O_{231} &= \{\omega_{31}, \omega_{23}, \omega_{12}\} & O_{321} &= \{\omega_{31}, \omega_{22}, \omega_{13}\}.
\end{align*}
\]

**4.2. General case when the weight $\lambda$ is $(I|J)$-robust.** Let us start with the following three examples.
Example 4.5. Let \(m = n = 3\) and a weight \(\lambda\) is such that \(\lambda^+_1 - 1 > \lambda^+_2 > \lambda^+_3\) and \(\lambda^-_1 > \lambda^-_2 > \lambda^-_3\). Assume that \(I = (113)\) and \(J = (123)\), hence \(\lambda\) is \((I|J)\)-robust.

Then
\[
\mathcal{B} = \{ \vec{b}_1 = \pi_{113,123}, \vec{b}_2 = \pi_{113,132}, \vec{b}_3 = \pi_{113,231} \}
\]
is a basis of all even-primitive vectors of weight \(\lambda_{1|J}\) in \(F_k\). The set \(\mathcal{A} = \{ \vec{a}_1 = \pi_{113,123}, \vec{a}_2 = \pi_{113,213}, \vec{a}_3 = \pi_{113,132}, \vec{a}_4 = \pi_{113,312}, \vec{a}_5 = \pi_{113,231}, \vec{a}_6 = \pi_{113,321}, \vec{a}_7 = \pi_{131,132}, \vec{a}_8 = \pi_{131,231}, \vec{a}_9 = \pi_{131,123}, \vec{a}_{10} = \pi_{131,321}, \vec{a}_{11} = \pi_{131,213}, \vec{a}_{12} = \pi_{131,312}, \vec{a}_{13} = \pi_{311,312}, \vec{a}_{14} = \pi_{311,231}, \vec{a}_{15} = \pi_{311,213}, \vec{a}_{16} = \pi_{311,231}, \vec{a}_{17} = \pi_{311,123}, \vec{a}_{18} = \pi_{311,132} \}
is linearly independent and spans the space \(A\) of even-primitive vectors of weight \(\lambda_{1|J}\) in \(T_k\).

The matrix of \(\psi_k\) with respect to the bases \(\mathcal{A}\) and \(\mathcal{B}\) computed using Proposition 5.7 is given as
\[
\begin{pmatrix}
\omega_{33} & -\omega_{33} & -1 & 1 & 1 & -1 & -\omega_{12} + 1 & \omega_{11} & -1 & 0 \\
0 & 0 & \omega_{32} & -\omega_{32} & -1 & 1 & 0 & 0 & -\omega_{13} + 1 & \omega_{11} \\
0 & 0 & 0 & 0 & \omega_{31} & -\omega_{31} & 0 & 0 & 0 & 0 \\
1 & 0 & \omega_{12} & -1 & -\omega_{11} & 1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & \omega_{13} & -1 & -\omega_{11} & 0 & -1 & 0 \\
-\omega_{13} + 1 & \omega_{12} & 0 & 0 & 0 & 0 & \omega_{13} - 1 & -\omega_{12} \\
\end{pmatrix}
\]
The collections are
\[
C_1 = \{ \vec{a}_1, \vec{a}_2, \vec{a}_7, \vec{a}_8, \vec{a}_{13}, \vec{a}_{14} \}, \quad C_2 = \{ \vec{a}_3, \vec{a}_4, \vec{a}_9, \vec{a}_{10}, \vec{a}_{15}, \vec{a}_{16} \}, \quad C_3 = \{ \vec{a}_5, \vec{a}_6, \vec{a}_{11}, \vec{a}_{12}, \vec{a}_{17}, \vec{a}_{18} \},
\]
while the corresponding sets are
\[
O_1 = \{ \omega_{33}, \omega_{12} - 1, \omega_{11} \}, \quad O_2 = \{ \omega_{32}, \omega_{11}, \omega_{13} - 1 \}, \quad O_3 = \{ \omega_{31}, \omega_{12}, \omega_{13} - 1 \}.
\]

If all entries in \(O_1\) are zeroes, then \(\omega_{12}(\lambda_{11}) = 0\) and we get \(\lambda \sim_{sodd} \lambda_{11} \sim_{sodd} \lambda_{111|12} \sim_{sodd} \lambda_{111|123}\).

If all entries in \(O_2\) are zeroes, then \(\omega_{13}(\lambda_{11}) = 0\) and we get \(\lambda \sim_{sodd} \lambda_{11} \sim_{sodd} \lambda_{111|13} \sim_{sodd} \lambda_{111|132}\).

If all entries in \(O_3\) are zeroes, then \(\omega_{13}(\lambda_{12}) = 0\) and we get \(\lambda \sim_{sodd} \lambda_{12} \sim_{sodd} \lambda_{111|23} \sim_{sodd} \lambda_{111|231}\).

Example 4.6. Let \(m = n = 3\) and a weight \(\lambda\) is such that \(\lambda^+_1 - 1 > \lambda^+_2 > \lambda^+_3\) and \(\lambda^-_1 - 1 > \lambda^-_2 > \lambda^-_3\). Assume that \(I = (113)\) and \(J = (112)\), hence \(\lambda\) is \((I|J)\)-robust.

Then
\[
\mathcal{B} = \{ \vec{b}_1 = \pi_{113,121} \}
\]
is a basis of all even-primitive vectors of weight \(\lambda_{1|J}\) in \(F_k\). The set \(\mathcal{A} = \{ \vec{a}_1 = \pi_{113,121}, \vec{a}_2 = \pi_{113,211}, \vec{a}_3 = \pi_{113,122}, \vec{a}_4 = \pi_{131,211}, \vec{a}_5 = \pi_{311,112}, \vec{a}_6 = \pi_{311,121} \}
is linearly independent and spans the space \(A\) of even-primitive vectors of weight \(\lambda_{1|J}\) in \(T_k\).

The matrix of \(\psi_k\) with respect to the bases \(\mathcal{A}\) and \(\mathcal{B}\) computed using Proposition 5.7 is given as
\[
\begin{pmatrix}
\omega_{31} & -\omega_{31} & 1 - \omega_{12} & \omega_{11} + 1 & \omega_{12} - 1 & -\omega_{11} - 1 \\
\end{pmatrix}
\]
There is one collection
\[ C_1 = \{ \vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5, \vec{a}_6 \} \]
and the corresponding set is
\[ O_1 = \{ \omega_{31}, \omega_{12} - 1, \omega_{11} + 1 \}. \]

If all entries in \( O_1 \) are zeroes, then \( \omega_{11}(\lambda_{31}) = 0 \) and \( \omega_{12}(\lambda_{31|11}) = 0 \) we get \( \lambda \sim_{\text{sodd}} \lambda_{31} \sim_{\text{sodd}} \lambda_{31|11} \sim_{\text{sodd}} \lambda_{32|11|112}. \)

**Example 4.7.** Let \( m = n = 3 \) and a weight \( \lambda \) is such that \( \lambda_1^+ > \lambda_2^+ > \lambda_3^+ \) and \( \lambda_1^- - 2 > \lambda_2^- > \lambda_3^- \). Assume that \( I = (123) \) and \( J = (111) \), hence \( \lambda \) is \((I|J)\)-robust. Then
\[ \mathcal{B} = \{ \vec{b}_1 = \pi_{123,111} \} \]
is a basis of all even-primitive vectors of weight \( \lambda_{I|J} \) in \( F_k \). The set \( \mathfrak{A} = \{ \vec{a}_1 = \pi_{123,111}, \vec{a}_2 = \pi_{213,111}, \vec{a}_3 = \pi_{132,111}, \vec{a}_4 = \pi_{312,111}, \vec{a}_5 = \pi_{313,111}, \vec{a}_6 = \pi_{321,111} \} \]
is linearly independent and spans the space \( A \) of even-primitive vectors of weight \( \lambda_{I|J} \) in \( T_k \).

The matrix of \( \psi_k \) with respect to the bases \( \mathfrak{A} \) and \( \mathcal{B} \) computed using Proposition 3.1 is given as
\[
\begin{pmatrix}
\omega_{31} & -\omega_{31} & -\omega_{21} - 1 & \omega_{21} + 1 & \omega_{11} + 2 & -\omega_{11} - 2
\end{pmatrix}
\]

There is one collection
\[ C_1 = \{ \vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5, \vec{a}_6 \} \]
and the corresponding set is
\[ O_1 = \{ \omega_{31}, \omega_{21} + 1, \omega_{11} + 2 \}. \]

If all entries in \( O_1 \) are zeroes, then \( \omega_{21}(\lambda_{31}) = 0 \) and \( \omega_{11}(\lambda_{32|11}) = 0 \) we get \( \lambda \sim_{\text{sodd}} \lambda_{31} \sim_{\text{sodd}} \lambda_{32|11} \sim_{\text{sodd}} \lambda_{32|11|112}. \)

Since \( \lambda \) is \((I|J)\)-robust, by Proposition 2.1 the basis \( \mathcal{B} \) of even-primitive vectors of weight \( \lambda_{I|J} \) in \( H^0_G(\lambda) \) consists of elements \( \pi_{K|L} \), where \( |K|L \) is admissible and \( \text{cont}(K|L) = \text{cont}(I|J) \). Denote by \( S(\lambda, I, J) \) the set of all multi-indices \( L \) such that \((I_0|L)\) is admissible and \( \text{cont}(L) = \text{cont}(J) \). Then \( S(\lambda, I, J) \) serves as an index set for the basis \( \mathcal{B} \). The previously defined order \(<\) on \( S_J \) induces the order \(<\) on \( \mathcal{B} \) and the order \(<^\eta\) on \( S(\lambda, I, J) \).

For a permutation \( \eta \in \Sigma_k \) define \( \mathfrak{B}_\eta = \{ \pi_{\eta I_0,L} | L \in S(\lambda, I, J) \} \) and \( S_\eta = \text{span}(\mathfrak{B}_\eta) \).

For \( L \in S(\lambda, I, J) \) and \( \eta \in \Sigma_k \) define
\[ a_{L,\eta} = |\{ t = 1, \ldots, k | i_{\eta(t)} = i_{\eta(k)} \text{ and } l_{\eta(t)} < l_{\eta(k)} \}| \]
and
\[ b_{L,\eta} = |\{ t = 1, \ldots, k | l_{\eta(t)} = l_{\eta(k)} \text{ and } i_{\eta(t)} > i_{\eta(k)} \}|. \]

**Lemma 4.8.** Assume that \( \lambda \) is \((I|J)\)-robust. Let \( \eta \in \Sigma_k \) and \( L \in S(\lambda, I, J) \). Then the matrix of the map \( \psi_k : S_\eta \rightarrow \mathfrak{F}_{I|J} \) with respect to the bases \( \mathfrak{B}_\eta \) ordered by \(<^\eta\), and \( \mathcal{B} \) ordered by \(<\), is an upper-triangular matrix. Its diagonal entry corresponding to \( \pi_{\eta I_0|\eta L} \) and \( \pi_{I_0|L} \) is
\[ \alpha_{\eta(k),\eta(k)} = (-1)^{\eta}(\omega_{i_{\eta(k)},l_{\eta(k)}} - a_{L,\eta} + b_{L,\eta}). \]
Proof. By Proposition 3.1 we have

\[
\psi_k(\pi I_0|\eta_0 L) = \omega_{i_0(k)}(I_0(k)) \pi I_0|\eta_0 L \\
\quad + (1 - \delta_{l_0(k) > l_0(t)} - \delta_{l_0(k) < l_0(t)}) \pi I_0|\eta_0 L l_0(t)|l_0(t+1)|l_0(t+1)|l_0(t+1)|l_0(t+1) + \cdots \\
\quad + (1 - \delta_{l_0(k) > l_0(t)} - \delta_{l_0(k) < l_0(t)}) \pi I_0|\eta_0 L l_0(t)|l_0(t+1)|l_0(t+1)|l_0(t+1)|l_0(t+1) + \cdots.
\]

Consider the general term

\[(1 - \delta_{l_0(k) > l_0(t)} - \delta_{l_0(k) < l_0(t)}) \pi I_0|\eta_0 L l_0(t)|l_0(t+1)|l_0(t+1)|l_0(t+1)|l_0(t+1) + \cdots \]

in the above formula for \(\psi_k(\pi I_0|\eta_0 L)\).

If \(i_0(t) > i_0(k)\) and \(l_0(k) \geq l_0(t)\), or \(i_0(t) \leq i_0(k)\) and \(l_0(k) < l_0(t)\), then this term vanishes.

If \(i_0(t) = i_0(k)\) and \(l_0(k) < l_0(t)\), then

\[\pi I_0|\eta_0 L l_0(t)|l_0(t+1)|l_0(t+1)|l_0(t+1)|l_0(t+1) = -\pi I_0|\eta_0 L|L.\]

If \(i_0(t) > i_0(k)\) and \(l_0(t) = l_0(k)\), then

\[\pi I_0|\eta_0 L l_0(t)|l_0(t+1)|l_0(t+1)|l_0(t+1)|l_0(t+1) + \pi I_0|\eta_0 L|L.\]

Adding up \(\omega_{i_0(k)}(I_0(k)) \pi I_0|\eta_0 L\) and all other terms considered so far we obtain

\[(\omega_{i_0(k)}(I_0(k)) - a L, b_0, \eta) \pi I_0|\eta_0 L.\]

Since \(i_0(t) = i_0(k)\) and \(l_0(t) = l_0(k)\) is not possible, it remains to analyze two cases:

If \(i_0(k) > i_0(t)\) and \(l_0(k) < l_0(t)\), or \(i_0(t) < i_0(k)\) and \(l_0(t) > l_0(k)\).

Let \(\gamma = (\eta(t)\eta(k))\) be the transposition switching entries in positions \(\eta(t)\) and \(\eta(k)\) and \(K = \gamma L.\) Then the above general term becomes

\[(1 - \delta_{l_0(k) > l_0(t)} - \delta_{l_0(k) < l_0(t)}) \pi I_0|\eta_0 L|K.\]

If \(i_0(k) > i_0(t)\) and \(l_0(k) < l_0(t)\), then \(\eta(t) \leq l_0(k)\); applying \(\gamma\) brings higher value earlier and smaller value later, which imply \(K = \gamma L \leq L\).

Additionally, since \(\eta\) changes positions corresponding to different values of \(i\), we obtain that either \(\pi I_0|K = 0\), or otherwise it equals to an element from \(\mathcal{B}\) that is smaller than \(\pi I_0|L\). If \(i_0(k) < i_0(t)\) and \(l_0(k) > l_0(t)\), then \(\eta(t) > l_0(k)\) and \(l_0(t) < l_0(k)\) imply again that \(K = \gamma L \leq L\) and, analogously as above, we either get \(\pi I_0|K = 0\) or it equals to an element from \(\mathcal{B}\) that is smaller than \(\pi I_0|L\).

Therefore, nonzero coefficients in the expression for \(\psi_k(\pi I_0|\eta_0 L)\) as a linear combination of elements from \(\mathcal{B}\) occur only at \(\pi I_0|M\), where \(M \leq L\) and the statement of the Lemma follows.

\[\square\]

**Proposition 4.9.** Assume that \(\lambda\) is \((I,J)\)-robust. If the simple \(G\)-supermodule \(L_G(\lambda_{I,J})\) is a composition factor of \(H^0_G(\lambda)\), then \(\lambda \sim_{\text{odd}} \lambda_{I,J}\).

If \(\lambda\) and \(\lambda_{I,J}\) are both polynomial weights and the simple \(G\)-supermodule \(L_G(\lambda_{I,J})\) is a composition factor of \(\nabla(\lambda)\), then \(\lambda \sim_{\text{odd}} \lambda_{I,J}\).

**Proof.** We will use Proposition 3.3 and assume that the map \(\psi_k : S_{I,J} \rightarrow \nabla_{I,J}\) is not surjective.

Create collections \(C_N\); each collection indexed by a multi-index \(N = (n_1, \ldots, n_k)\) of content \(cont(J)\) such that \((I_0|N)\) is admissible. The collection \(C_N\) consists of vectors \(\pi I_0|\eta N\) for all \(\eta \in \Sigma_k\). According to Lemma 4.8, the images \(\psi_k(\pi I_0|\eta N)\) of elements in the collection \(C_N\) expressed as a linear combination of vectors \(\pi I_0|L\) have the property that the coefficient at \(\pi I_0|N\) equals \(\alpha_{\eta I_0|\eta N}\) and all other nonzero
coefficients appear only at $\pi_{I_0|L}$ for $L < N$. To each collection $C_N$ we assign a set $O_N = \{\alpha_{\eta_{I_0|N}} | \eta \in \Sigma_k \}$.

It is a crucial observation that if every $O_N$ contains a nonzero element, then the map $\psi_k : S_{I|J} \rightarrow \overline{S}_{I|J}$ is surjective. This is because by Lemma 4.8 we can find a set of vectors $\pi_{K|L}$ of weight $\lambda_{I|J}$ such that the matrix of $\psi_k$ restricted on the span of these vectors is invertible.

Therefore, if $\psi_k : S_{I|J} \rightarrow \overline{S}_{I|J}$ is not surjective, then there is $N$ as above such that all elements of $O_N$ are equal to zero.

For $t = k, \ldots, 1$ denote $\kappa_t = \lambda_{i_k \ldots i_{t-1}|n_k \ldots n_{t-1}}$. Assume $\eta(k) = t > 1$. Then

$$\kappa_t^+ = \lambda_{t_i}^+ - a_{N,\eta}$$ and $\kappa_t^- = \lambda_{t_i}^- + b_{N,\eta}$ imply

$$\omega_{i_{t-1},n_{t-1}}(\kappa_t) = \omega_{i_{t-1},n_{t-1}} - a_{N,\eta} + b_{N,\eta} = (-1)^t \alpha_{i_{t-1},n_{t-1}} = 0.$$

Therefore $\lambda$ and $\lambda_{I|J}$ are odd-linked via the sequence

$$\lambda \sim_{sodd} \lambda_{i_k|n_k} \sim_{sodd} \lambda_{i_k i_{k-1}|n_k n_{k-1}} \sim_{sodd} \cdots \sim_{sodd} \lambda_{i_k \ldots i_1|n_k \ldots n_1} = \lambda_{I|J}.$$

The last statement now follows from Corollary 7.2 of [8]. \qed

5. Odd linkage for polynomial weights

In this section we investigate simple composition factors of the costandard module $\nabla(\lambda)$. Since we are applying combinatorial techniques using tableaux, we will require that $\lambda$ is a dominant and polynomial weight, and $\lambda_{I|J}$ are also dominant and polynomial weights.

At this point the reader is asked to review the following material from the paper [8]: the general tableau setup from 4.1, the correspondence between tableaux $T^+$ and $T^-$ from 5.3, the definition and properties of Clausen preorders from 6.1; the definition and properties of marked tableaux and $\overline{\sigma}(T^+)$ from 6.2, and the connection to pictures from 7.3. Also, it is important to review the connection between operators $\sigma$ and $\tau$ from subsections 4.2 and 5.2.

In the general case we will still work with the vectors $\pi_{I|J}$ and rely on the formulas for $\psi_k(\pi_{I|J})$ derived in the previous sections. The approach we are going to explain works for arbitrary $\lambda_{I|J}$ and essentially reduces the general case to the case when $\lambda$ is $(I|J)$-robust.

Since $\pi_{I|J}$ do not necessarily belong to $T_k$, we will use Theorem 4.3 of [8] to obtain suitable even-primitive vectors of $T_k$ that are integral linear combinations of vectors $\pi_{K|L}$. These vectors belong to $S_{I|J}$ and are of the form $\nu_{I|J}\sigma_{\nu_{I|J}}$, where the operator $\sigma$ is defined in subsection 4.2 of [8]. In Section 6 of [8] there is a vector $\overline{\sigma}(T^+)$ assigned to each $T^+ \in M(\lambda^+/\mu', \nu/\omega)$, which is built using the operator $\tau$ acting on $T^+$ as defined in Section 5 of [8]. By Theorem 7.1 of [8] (see also Theorem 6.24 of [8]), the set $\mathcal{B} = \{\overline{\sigma}(T^+) | T^+ \in M(\lambda^+/\mu', \nu/\omega)\}$ form a basis of the space $B$ of all even-primitive vectors in $\nabla_k(\lambda) = \nabla(\lambda) \cap F_k$ of weight $(\mu|\nu)$.

Let us fix an arbitrary reading of all tableaux $T^+$ of the skew shape $\lambda^+/'\mu'$.

Corresponding to such a reading we assign two words. The word $J_{T^+}$ is obtained by listing the entries $j_i$ in the same order as the symbols $m + j_i$ are read. If the reading of the $t$-th symbol of $T^+$ appears at the location $(k_t, i_t) \in D^+$, then we define $I_0 = (i_1, \ldots, i_k)$. Note that $I_0$ is the same for all tableaux $T^+$ since it only depends on the fixed reading, but the entries of $I_0$ are no longer nondecreasing (as was the case earlier).
If we list the elements of $\mathfrak{B}$ as $\mathfrak{B} = \{\pi(T^+_1), \ldots, \pi(T^+_r)\}$ with respect to a certain order $\prec$, then we will write $J^* = J^+_T = (j_1^*, \ldots, j_r^*)$.

For a permutation $\eta \in \Sigma_k$ define $\mathfrak{B}_{\eta} = \{\sigma_{\pi_{\eta,1}|\eta,\eta'} \mid \eta' \in [1, \ldots, l]\}$ and $B_{\eta} = \text{span}(\mathfrak{B}_{\eta})$. The image of the element $\sigma_{\pi_{\eta,1}|\eta,\eta'}$ from $\mathfrak{B}_{\eta}$ under the map $\text{proj} : K(m|n) \otimes Y^{\otimes k} \to K(m|n) \otimes \Lambda^k Y$ is $\pi(T^+_\eta) = \sigma_{\pi_{\eta,1}|\eta,\eta'}$. Since $\psi_k$ is the $G_{ev}$-morphism by Lemma 5.1, the restriction of $\psi_k$ to $B_{\eta}$ gives a $G_{ev}$-morphism from $B_{\eta}$ to $B$. Since $\mathfrak{B}_{\eta}$ is a basis of $B_{\eta}$ and $\mathfrak{B}$ is a basis of $B$, we can consider the matrix $M_{\eta}$ of $\psi_k$ with respect to the bases $\mathfrak{B}_{\eta}$ and $\mathfrak{B}$.

For $T^+ \in M(\lambda^+/\mu', \nu/\omega)$ we define the set $\mathcal{C}_{T^+} = \{\sigma_{\pi_{\eta,1}|\eta,\eta'} \mid \eta \in \Sigma_k\}$ and denote by $C_{T^+}$ the $F$-span of elements in $\mathcal{C}_{T^+}$.

5.1. General case of $\lambda^+/\mu'$ with distinct entries in $I$ and distinct entries in $J$. Before dealing with the general case, we will describe the case when all entries in $I$ are distinct and all entries in $J$ are distinct. In this case the arguments are much simpler and we can avoid the full machinery of marked tableaux, pictures, and their properties.

Assume $I_0 = \{i_1 < \ldots < i_k\}$ and assume that the partition $\lambda^+$ satisfies
$$
\lambda^+_1 = \ldots = \lambda^+_{k_1} > \lambda^+_{k_1+1} > \ldots > \lambda^+_{k_1+k_2} > \ldots > \lambda^+_{k_1+\ldots+k_s-1} = \ldots = \lambda^+_{k_1+\ldots+k_s}
$$
and $k_1 + \ldots + k_s = k$. Then the diagram of the skew partition $\lambda^+/\mu'$ has $s$ rows consisting of positions
$$
\{[\lambda^+_{i_1}, i_1], \ldots, [\lambda^+_{i_k}, i_k]\},
$$
$$
\ldots
$$
$$
\{[\lambda^+_{i_k}, i_k], \ldots, [\lambda^+_{i_k+\ldots+k_s-1}, i_k]\}
$$

Analogously, assume $J_0 = \{j_1 < \ldots < j_k\}$ and assume that the partition $\lambda^-$ satisfies
$$
\lambda^-_1 = \ldots = \lambda^-_{j_1} > \lambda^-_{i_1+i_2} > \ldots > \lambda^-_{i_1+i_2+\ldots+i_t-1} = \ldots = \lambda^-_{i_1+i_2+\ldots+i_t}
$$
and $l_1 + \ldots + l_t = k$. Then the diagram of the skew partition $\nu/\omega$ has $t$ columns consisting of positions
$$
\{[j_1, \lambda^+_j], \ldots, [j_t, \lambda^+_j]\},
$$
$$
\ldots
$$
$$
\{[j_t, \lambda^+_j], \ldots, [j_1+i_1+i_2+\ldots+i_t-1, \lambda^+_j]\}
$$

Denote by $\overline{T^+}$ the row tabloid corresponding to $T^+$.

Marked tableaux are easy to understand in this case. A tableau $T^+$ belongs to $M(\lambda^+/\mu', \nu/\omega)$ if and only if entries in all rows of $T^+$ are strictly increasing and entries in all columns of $T^-$ are strictly decreasing.

**Lemma 5.1.** Every $\pi(T^+) \in \mathfrak{B}$, where $T^+ \in M(\lambda^+/\mu', \nu/\omega)$, is an integral linear combination of various $\pi_{I_0|J}$, where $\text{cont}(L) = \text{cont}(J)$. Each term $\pi_{I_0|J}$ can appear with non-zero coefficient in at most one $\pi(T^+) \in \mathfrak{B}$.

**Proof.** The first statement follows from Theorem 7.1 of [10]. For the second part, first observe that since every $T^+ \in M(\lambda^+/\mu', \nu/\omega)$ is semistandard, it is uniquely described by the content vectors of each row in the tableau $T^+$, or equivalently by its row tabloid $\overline{T^+}$. 


If \( T^+ \in M(\lambda^+ / \mu', \nu / \omega) \), then every tableau appearing in \( \tau T \) is obtained by permuting entries of \( T^+ \) that correspond to entries from the same columns of \( T^- \) (and same columns of the diagram \([\nu / \omega]\)).

Additionally, entries from the same column of \( T^- \) correspond to entries in \( T^+ \) that belong to different rows of \( T^+ \) (and different rows of the skew diagram \([\lambda^+ / \mu']\)).

Assume that \( T_1^+, T_2^+ \in M(\lambda^+ / \mu', \nu / \omega) \) are such that there is a common summand in \( \nu (T_1^+) \) and \( \nu (T_2^+) \). Then there is a common summand in \( \tau \nu (T_1^+) \) and \( \tau \nu (T_2^+) \), and, consequently, a common summand in \( \nu T_1^+ \) and \( \nu T_2^+ \). In this case there is a summand \( R_1 \) in \( \tau T_1^+ \) and a summand \( R_2 \) in \( \tau T_2^+ \) such that the row tabloids \( R_1 \) and \( R_2 \) coincide. Let \( R_1 = \sigma_{-1} T_1^+ \) and \( R_2 = \sigma_{-2} T_2^+ \), where \( \sigma_{-1}, \sigma_{-2} \) are column permutations of \([\nu / \omega]\). Then every entry \( m + j_i \) appears in the same row in \( R_1 \) as in \( R_2 \). Since entries in \( J \) are distinct and entries in columns of \( T_1^+, T_2^+ \) are strictly decreasing, we conclude that \( \sigma_{-1} = \sigma_{-2} \) and \( T_1^+ = T_2^+ \). Since both \( T_1^+, T_2^+ \) are semistandard, this implies \( T_1^+ = T_2^+ \). \( \square \)

To compute the entries in the matrix \( M_{T^+} \), we will use Proposition 3.1. The following example related to Example 3.2 illustrates this setup.

**Example 5.2.** Let \( m = n = 3 \) and a weight \( \lambda \) be such that \( \lambda_1^+ = \lambda_2^+ > \lambda_3^- \) and \( \lambda_1^- > \lambda_2^- > \lambda_3^- \). Assume that \( I = J = (123) \). Then in the notation of Example 3.2,

\[
\mathcal{B} = \{ \nu (T_1^+) = \vec{b}_1 + \vec{b}_2, \nu (T_2^+) = \vec{b}_3 + \vec{b}_4, \nu (T_3^+) = \vec{b}_5 + \vec{b}_6 \}
\]

is a basis of all even-primitive vectors of weight \( \lambda_{1|J} \) in \( \nabla_k (\lambda) \).

The sets \( \mathcal{E}^+_\nu \subset \mathcal{E}_s \) are given as follows.

\[
\mathcal{E}_1 = \{ \vec{a}_1 - \vec{a}_8, \vec{a}_7 - \vec{a}_2, \vec{a}_1 - \vec{a}_6, \vec{a}_2 - \vec{a}_5, \vec{a}_3, \vec{a}_4, \vec{a}_9, \vec{a}_1 \}
\]

\[
\mathcal{E}_2 = \{ \vec{a}_3 - \vec{a}_10, \vec{a}_9 - \vec{a}_4, \vec{a}_15 - \vec{a}_28, \vec{a}_21 - \vec{a}_34, \vec{a}_27 - \vec{a}_16, \vec{a}_33 - \vec{a}_22 \}
\]

\[
\mathcal{E}_3 = \{ \vec{a}_5 - \vec{a}_12, \vec{a}_11 - \vec{a}_6, \vec{a}_17 - \vec{a}_30, \vec{a}_23 - \vec{a}_36, \vec{a}_29 - \vec{a}_18, \vec{a}_35 - \vec{a}_24 \}
\]

The assumption \( \lambda_1^+ = \lambda_2^- \) implies \( \omega_{1j} = \omega_{2j} + 1 \). Using this, we evaluate the following matrices.

The matrix of \( \psi_k \) with respect to the bases \( \mathcal{E}_1 \) and \( \mathcal{B} \) is given as

\[
\begin{bmatrix}
\omega_{33} & -\omega_{33} & -\omega_{12} & \omega_{12} & \omega_{21} & -\omega_{21} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

the matrix of \( \psi_k \) with respect to the bases \( \mathcal{E}_2 \) and \( \mathcal{B} \) is given as

\[
\begin{bmatrix}
-1 & 1 & -1 & 1 & 0 & 0 \\
\omega_{32} & -\omega_{32} & -\omega_{13} & \omega_{13} & \omega_{21} & -\omega_{21} \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and the matrix of \( \psi_k \) with respect to the bases \( \mathcal{E}_3 \) and \( \mathcal{B} \) is given as

\[
\begin{bmatrix}
-1 & 1 & -1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & -1 \\
\omega_{31} & -\omega_{31} & -\omega_{13} & \omega_{13} & \omega_{22} & -\omega_{22}
\end{bmatrix}
\]

Therefore we have sets

\[
O_1 = \{ \omega_{33}, \omega_{12}, \omega_{21} \}, O_2 = \{ \omega_{32}, \omega_{13}, \omega_{21} \}, O_3 = \{ \omega_{31}, \omega_{13}, \omega_{22} \}
\]
and if \( \psi_k \) is not surjective, then at least one of the sets \( O_1, O_2, O_3 \) contains only zeros and therefore \( \lambda \sim_{\text{odd}} \lambda_{I,J} \).

It is helpful to keep in mind the above example in the following general consideration.

There is a unique reading of the tableau \( T^+ \), obtained by reading the entries in the tableau \( T^+ \) by rows from bottom to top, and in each row proceeding from left to right, such that \( I_0 = (i_1 \cdots i_k) \). Then the word \( J^* = (j_1^* \cdots j_k^*) \).

The reverse Semitic lexicographic order \( < \) on multiindices \( L \) of the same content as \( J \) extends to a linear order on tableaux \( T^+ \) and on vectors \( \pi_{I_0|J_{T^+}} \). List the elements of \( M(\lambda^+/\mu', \nu/\omega) \) with respect to this order \( < \) as \( T_1^+ < T_2^+ < \ldots < T_k^+ \).

Then we have
\[
\pi_{I_0|J_{T^+}} = \text{I}(T_1^+) < \pi_{I_0|J_{T^+}} = \text{I}(T_2^+) < \ldots < \pi_{I_0|J_{T^+}} = \text{I}(T_k^+).
\]
This induces the order \( < \) on elements of \( \mathfrak{B}_\eta \) such that \( \sigma \pi_{I_0|\eta|J_{T^+}} < \eta \sigma \pi_{I_0|\eta|J_{T^+}} \) if and only if \( a < b \).

If \( T^+ \in M(\lambda^+/\mu', \nu/\omega) \), then the summands in \( \pi(T^+) \) with respect to \( < \) are such that \( \pi(T^+) = \pi_{I_0|J_{T^+}} \) is the highest term in \( \pi(T^+) \). Therefore we will call \( \pi_{I_0|J_{T^+}} \) the leading element of \( \pi(T^+) \).

**Lemma 5.3.** Assume that all entries in \( I \) are distinct, all entries in \( J \) are distinct, \( \eta \in \Sigma_k \) and \( T^+ \in M(\lambda^+/\mu', \nu/\omega) \). The matrix of the map \( \psi_k : B_\eta \to B \) with respect to the bases \( \mathfrak{B}_\eta \) ordered by \( \prec \), and \( \mathfrak{B} \) ordered by \( < \), is an upper-triangular matrix. Its diagonal entry corresponding to \( \sigma \pi_{I_0|\eta|J_{T^+}} \) and \( \pi(T^+) \) is \((-1)^\eta \omega_{\pi_{I_0|\eta|J_{T^+}}} \).

**Proof.** If \( \pi_{K|M} \) is a summand of an element \( \sigma \pi_{I_0|\eta|J_{T^+}} \) from \( \mathfrak{B}_\eta \), then \( \pi_{K|M} \leq \pi_{I_0|J_{T^+}} = \text{I}(T^+) \).

Using Lemma 4.1, we derive that if the coefficient at \( \pi_{I_0|N} \) in the linear combination expressing \( \psi_k(\pi_{K|M}) \) is nonzero, then \( N \leq J_{T^+} \). Therefore, if the coefficient at \( \pi_{I_0|N} \) in the linear combination expressing \( \psi_k(\sigma \pi_{I_0|\eta|J_{T^+}}) \) is nonzero, then \( N \leq J_{T^+} \).

Moreover, the expression \( \pi_{I_0|\eta|J_{T^+}} \) is the only summand in \( \sigma \pi_{I_0|\eta|J_{T^+}} \) such that its image under \( \psi_k \) has a nonzero coefficient, namely \((-1)^\eta \), at \( \pi_{I_0|J_{T^+}} \).

Therefore, Lemmas 5.1 and 4.1 imply that if we express \( \psi_k(\sigma \pi_{I_0|\eta|J_{T^+}}) \) as a linear combination of elements \( \pi(T^+) \) for \( T^+ \in M(\lambda^+/\mu', \nu/\omega) \), then the coefficient at \( \pi(T^+) \) is \((-1)^\eta \omega_{\pi_{I_0|\eta|J_{T^+}}} \) and all coefficients at \( \pi(T^+) \) for \( t > s \) vanish. \( \square \)

**Theorem 5.4.** Assume that all entries in \( I \) are distinct and all entries in \( J \) are distinct. Moreover, assume that \( \lambda \) and \( \lambda_{I,J} \) are dominant and polynomial weights. If the simple \( G \)-supermodule \( L_G(\lambda_{I,J}) \) is a composition factor of \( \nabla(\lambda) \), then \( \lambda \sim_{\text{odd}} \lambda_{I,J} \).

**Proof.** We will use Corollary 7.2 of [8] and Proposition 3.3 and assume that the map \( \psi_k : S_{I,J} \to S_{I,J} \) is not surjective.

To each collection \( C_{T^+} \) we assign a set \( S_{T^+} = \{ \omega_{\pi_{I_0|\eta|J_{T^+}}(t) | \eta \in \Sigma_k \} = \{ \omega_{i, j^*} | t = 1, \ldots, k \} \).

It is a crucial observation that if every \( S_{T^+} \) contains a nonzero element, then the map \( \psi_k : S_{I,J} \to S_{I,J} \) is surjective. This is because using Lemma 5.1 and Theorem
7. of [8] we can find a set of primitive vectors $\sigma \pi_{\eta_i, I_0} |_{\eta_j J +}$ from $S_{I_1 J}$ such that the matrix of $\psi_k$ restricted on the span of these vectors is invertible.

Therefore, if $\psi_k : S_{I_1 J} \to S_{I_1 J}$ is not surjective, then there is $s$ such that all elements of $S_{I_1 J}$ are equal to zero. In this case we derive that $\lambda$ and $\lambda_{I_1 J}$ are odd-linked via a sequence $\lambda \sim \lambda_{I_1 J} \sim \lambda_{I_1 J_J} \sim \cdots \sim \lambda_{I_1 J_J}$ because $\omega_{\lambda_{I_1 J_J}}(\lambda) = 0$, $\omega_{\lambda_{I_1 J_J}}(\lambda_{I_1 J_J}) = \omega_{\lambda_{I_1 J_J}}(\lambda) = 0$ and so on until $\omega_{\lambda_{I_1 J_J}}(\lambda_{I_1 J_J}) = 0$. Thus, $\lambda \sim_{\text{odd}} \lambda_{I_1 J_J}$. Since $\lambda_{I_1 J_J}$ is a polynomial weight, all intermediate $\lambda_{I_1 J_J}$ are also polynomial weights, hence $\lambda \sim_{\text{odd}} \mu$. 

5.2. General case of $\lambda_{I_1 J_J}$. Before we make our final choices of the reading of the tableau $T^+$ and ordering $<$, let us comment on the previous special cases where other choices often seemed more natural.

If all entries in $I$ are distinct, then $T^+$ is a row strip; if all entries in $J$ are distinct, then $T^-$ is a column strip.

If the weight $\lambda$ is $(|I|, |J|)$-robust, then $T^+$ is a column strip, and $T^-$ is a row strip.

Assume $\lambda$ is $(|I|, |J|)$-robust. Then all entries in $I$ are distinct if and only if $T^+$ is an antichain with respect to the order $\prec$, defined in [8] (or [5]); and all entries in $J$ are distinct if and only if $T^-$ is an antichain with respect to the order $\prec$. The case of an antichain is the simplest, and the choice of the reading of tableaux does not matter.

Even the case when all entries in $I$ are distinct and all entries in $J$ are distinct - which we were considering above - is rather special from this point of view and it is not surprising that we could use various readings of the tableaux.

In the most general case we are going to consider now, the tableaux could assume complicated skew shapes and we really have a unique choice for our setup to work. Namely, from now on we assume that the reading of a tableau $T^+$ is by rows moving from right to left and moving from top to bottom.

Also, from now on we fix the order $<$ on tableaux $T^+$ and we will assume that it is the Clausen row order.

Previously, in the special cases described in Lemmas 4.1, 4.8 and 5.3 we have worked with the reverse Semitic order. Now we will show that the reverse Semitic order in Lemma 4.8 can be replaced by the Clausen row order that will work in general.

Lemma 5.5. If $\eta \in \Sigma_1$, then the matrix of the map $\psi_k : B_{\eta} \to B$ with respect to the bases $B_{\eta}$ ordered by $\prec$, and $B_{\eta}$ ordered by $\prec$, is an upper-triangular matrix with integral coefficients. Its diagonal entry corresponding to $\sigma \pi_{\eta L_0} |_{\eta L}$ and $\sigma \pi_{\eta L_0} |_{\eta L}$ is $\alpha_{\eta L_0} |_{\eta L}$. 

Proof. By Proposition 5.1 we have

$$
\psi_k(\pi_{\eta L_0} |_{\eta J^*}) = \omega_{\eta L_0} |_{\eta J^*} + (1 - \delta_{i_1, \eta L_0}) + (1 - \delta_{i_2, \eta L_0}) + \cdots
$$

Therefore, the diagonal entries of our matrix are the same as described in the statement of the lemma and it only remains to show that our matrix is upper-triangular.
Consider the general term
\[
(1 - \delta_{i_0^t(k)} > i_{\eta(t)} - \delta_{i_0^t(k)} < i_{\eta(t)}) \lambda_{\eta_0, T_s} \prod_{\eta_0} \xi_{\eta_0} \xi_{\eta_0} \cdots \xi_{\eta_0} \xi_{\eta_0} \xi_{\eta_0} \cdots \xi_{\eta_0} \xi_{\eta_0} \cdots \xi_{\eta_0} \xi_{\eta_0}
\]
in the above formula for \( \psi_k(\pi_{\eta_0, T_s}) \). Let \( \eta = (\eta(t), \eta(k)) \) be the transposition switching entries in positions \( \eta(t) \) and \( \eta(k) \) and \( K = \gamma \cdot J^s \).

Then \( \eta K = (j_{\eta(t)}^s \cdots j_{\eta(t-1)}^s j_{\eta(t)}^s \cdots j_{\eta(k-1)}^s j_{\eta(t)}^s) \) and the above general term becomes
\[
(1 - \delta_{i_0^t(k)} > i_{\eta(t)} - \delta_{i_0^t(k)} < i_{\eta(t)}) \lambda_{\eta_0, T_s} \eta K.
\]

If \( i_{\eta(t)} > i_{\eta(t)} \) and \( j_{\eta(t)}^s > i_{\eta(t)} \), or \( i_{\eta(t)} < i_{\eta(t)} \) and \( j_{\eta(t)}^s < i_{\eta(t)} \), then this term vanishes.

If \( i_{\eta(t)} > i_{\eta(t)} \) and \( j_{\eta(t)}^s < i_{\eta(t)} \), then \( \eta(t) < \eta(k) \) and \( j_{\eta(t)}^s > j_{\eta(k)}^s \). This means that in the marked tableau \( T^s \) we have an entry \( m + j_{\eta(t)}^s \) appearing at the position \( [r_t, \eta(t)] \) and an entry \( m + j_{\eta(k)}^s \) appearing at the position \( [r_t, \eta(k)] \). If \( r_t \leq r_k \), then \( D^+ \) contains the position \( [r_k, \eta(k)] \) because \( \eta(t) < \eta(k) \). Since \( T^s \) is semistandard, this implies \( m + j_{\eta(t)}^s \leq m + j_{\eta(k)}^s \), which is a contradiction. Therefore \( r_t = r_k \), which means that \( K = \gamma \cdot J^s < J^s \) with respect to Clausen row order.

If \( i_{\eta(t)} < i_{\eta(t)} \) and \( j_{\eta(t)}^s > i_{\eta(t)} \), then \( \eta(t) > \eta(k) \) and \( j_{\eta(t)}^s < j_{\eta(k)}^s \). This implies again that \( K = \gamma \cdot J^s < J^s \). Therefore either \( \pi_{I_0} | K = 0 \) or otherwise \( \pi_{I_0} | K < \pi_{I_0} | J^s \).

Thus nonzero coefficients in the expression for \( \psi_k(\pi_{\eta_0, T_s}) \) as a linear combination of elements \( \pi_{I_0} | M \) occur only when \( M \leq J^s \) and the coefficient at \( \pi_{I_0} | J_s \) is \( \alpha_{\eta_0, T_s} \).

Next, if \( \pi_{K} | M \) is a summand of an element \( \sigma \pi_{\eta_0, T_s} \) from \( C \). Then \( \pi_{K} | M \leq \pi_{I_0} | T_s \). We infer that if the coefficient at \( \pi_{I_0} | N \) in the linear combination expressing \( \psi_k(\pi_{K} | M) \) is nonzero, then \( N \leq J^s \). Therefore, if the coefficient at \( \pi_{I_0} | N \) in the linear combination expressing \( \psi_k(\sigma \pi_{\eta_0, T_s}) \) is nonzero, then \( N \leq J^s \).

Moreover, the expression \( \pi_{T_s} \) is the only summand \( \pi_{U} \) in \( \sigma \pi_{\eta_0, T_s} \) such that \( \psi_k(\sigma \pi_{U}) \) has a nonzero coefficient, namely \( \alpha_{\sigma_{\eta_0}(t), s_{\eta_0}} \), at \( \pi_{I_0} | J^s \).

Since the leading terms \( \pi_{I_0} | J^s \) of the vectors \( \pi_{T_s} \) are linearly ordered with respect to \(<\), and all other terms in \( \pi(T^s) \) are lower than \( \pi_{I_0} | J^s \), if we express \( \psi_k(\sigma \pi_{\eta_0, T_s}) \) as a linear combination of elements \( \pi(T^s) \) for \( T^s = M(\lambda^s, s^s, s^{s^s}, \cdots) \), then the coefficient at \( \pi(T^s) \) is \( \alpha_{\eta_0, T_s} \) and all coefficients at \( \pi(T^s) \) for \( t > s \) vanish.

Before we proceed further, we need to adjust the sequence from Proposition 4.3 to make it adhere to our reading of the tableau \( T^s \). Corresponding to this reading, we define \( \kappa^s_0 = \lambda_i \cdots \lambda_j \) for \( t = 1, \ldots, k \) and \( \kappa_0 = \lambda \).

**Theorem 5.6.** Let \( \lambda \) and \( \lambda^s \) be dominant polynomial weights. If the simple \( G \)-supermodule \( L_G(\lambda^s) \) is a composition factor of \( \nabla(\lambda) \), then \( \lambda \simeq \lambda^s \).

**Proof.** We will use Corollary 7.2 of [8] and Proposition 8.3 and assume that the map \( \psi_k : S^1 | J \rightarrow S^1 | J \) is not surjective.

Consider the collections \( \zeta_s = \zeta^s_{T_s} \), consisting of vectors \( \sigma \pi_{\eta_0, T_s} \) for \( \eta \in \Sigma_s \), where \( J^s \) corresponds to \( T^s \), listed with respect to the order \( T^s_1 < \ldots < T^s_s \).

To each set \( \zeta_s \), we assign a set \( O_s = \{ \alpha_{\eta_0, T_s} \} \). It is a crucial observation that if every \( \alpha_{\eta_0, T_s} \) for \( s = 1, \ldots, l \) contains a nonzero element, then the map \( \psi_k : S^1 | J \rightarrow S^1 | J \) is surjective. This is because by Lemma 7.3 and Theorem 7.1 of [8]
we can find a set of vectors $\sigma \pi_{\eta,F_{0} \{t_{i}\}, J}$ of weight $\lambda_{i \mid J}$ such that the matrix of $\psi_{k}$ restricted on the span of these vectors is invertible.

Therefore, if $\psi_{k} : S_{i \mid J} \rightarrow \mathfrak{S}_{i \mid J}$ is not surjective, then there is an index $s$ such that all elements of $O_{s}$ vanish.

We will show that
\begin{equation}
\lambda = \kappa_{0}^{s} \sim \kappa_{1}^{s} \sim \kappa_{2}^{s} \cdots \sim \kappa_{k}^{s} = \lambda_{i \mid J}.
\end{equation}

Assume that $\eta$ is such that $\eta(k) = t$. Because of the order of the reading of the tableau $T_{s}^{+}$ and because the tableau $T_{s}^{+}$ is semi-standard, we infer that $a_{J^s, \eta}$ equals the number of entries in $D^{+}$ that appear in the $i_{s}$-th column that lie in rows with indexes less than $k_{s}^{i}$. Therefore, $(\kappa_{s-1}^{i})^{i}_{t} = \lambda_{s}^{i} - a_{J^s, \eta}$. We claim that $b_{J^s, \eta}$ equals the number of appearances of the symbol $m + j_{s}^{i}$ in the initial part of the reading of $T_{s}^{+}$ consisting of the first $t - 1$ elements. In order to see this, observe that since $T_{s}^{+}$ is semistandard, if the symbol $m + j_{s}^{i}$ appear in columns with index bigger than $i_{s}$, then it must lie in the rows with indexes not exceeding $k_{s}^{i}$. This means that these appearances of the symbol $m + j_{s}^{i}$ all lie in the initial part of the reading of $T_{s}^{+}$ consisting of the first $t - 1$ elements. On the other hand, since $T_{s}^{+}$ is semistandard, there cannot be any appearances of $m + j_{s}^{i}$ that lie in rows with index smaller than $k_{s}^{i}$ and in columns with indexes smaller or equal to $i_{s}$. This means that every appearance of $m + j_{s}^{i}$ in the initial part of the reading of $T_{s}^{+}$ consisting of the first $t - 1$ elements also count towards $b_{J^s, \eta}$. Therefore, $(\kappa_{s-1}^{i})^{i}_{t} = \lambda_{s}^{i} + b_{J^s, \eta}$. This implies $\omega_{i_{s}, j_{s}^{i}}(\kappa_{s-1}^{i}) = \alpha_{i_{s}, j_{s}^{i}} = 0$ and $\kappa_{s-1}^{i} \sim \lambda^{s}_{s}$ for each $t = 1, \ldots, k$, which proves $\lambda \sim \lambda_{i \mid J}$. Since all $\kappa_{s}^{i}$ are polynomial weights, the claim follows.

\section{Remarks for ground fields of odd characteristic}

In this section we assume that $F$ is a ground field of characteristic $p > 2$.

\begin{proposition}
Let $M$ be a submodule of $H_{G}^{0}(\lambda)$ generated by all elements from $F_{i}$ for $i < k$. Let $\psi_{k} : T_{k} \rightarrow F_{k}$ and $\tilde{\psi}_{k} : F_{k-1} \otimes Y \rightarrow F_{k}$ be the maps as before. Then $M \cap F_{k} = \psi_{k}(T_{k}) = \tilde{\psi}_{k}(F_{k-1} \otimes Y)$.

For a dominant weight $\mu$ corresponding to an element of $F_{k}$, the supermodule $L_{G}(\mu)$ is a composition factor of $H_{G}^{0}(\lambda)$ if and only if the module $L_{\text{ev}}(\mu)$ is a composition factor of $F_{k} / M \cap F_{k}$.

\end{proposition}

\begin{proof}
The equality $M \cap F_{k} = \psi_{k}(T_{k})$ follows by analogous arguments as in the proof of Proposition 5.3. Moreover, simple $G_{\text{ev}}$-composition factors $L_{\text{ev}}(\mu)$ of $F_{k} / M \cap F_{k}$ are in one-to-one correspondence to simple composition factors $L(\mu)$ of $H_{G}^{0}(\lambda)$ generated by elements from $F_{k}$.

Since $\text{Im}(\psi_{k}) = \text{Im}(\tilde{\psi}_{k})$, we can study the map $\tilde{\psi}_{k}$ instead of the map $\psi_{k}$. Note that whether $L(\mu)$ is a composition factor of $H_{G}^{0}(\lambda)$ depends only on the $F_{k-1}$ and on the map $\tilde{\psi}_{k}$, but not on other preceding floors $F_{i}$ for $i < k$. If the $G_{\text{ev}}$-structure of $F_{k-1} \otimes Y$ and $F_{k}$ is known, we can investigate $\tilde{\psi}_{k}$ as a $G_{\text{ev}}$-morphism. We will not, however, pursue this direction in this paper.

We would like to discuss the modular reduction from the ground field of rational numbers $\mathbb{Q}$ to a ground field $F$ of characteristic $p > 2$. From now on assume that $\lambda$ is dominant polynomial weight and $\mu = \lambda_{i \mid J}$ is a dominant polynomial weight belonging to the $k$-th floor $F_{k}$ of $H_{G}^{0}(\lambda)$. Denote by $S_{\mu,F}$ and $\mathfrak{S}_{\mu,F}$ the sets of even-primitive vectors of weight $\mu$ in $T_{k}$ and $F_{k}$, defined over the field $F$. 

Recall the definition of the sets $\mathcal{B}$ and $\mathcal{B}_\eta$ for $\eta \in \Sigma_k$ and their $F$-spans $B_{\eta,F}$ and $B_F$ from the beginning of Section 5.

Denote the $\mathbb{Z}$-span of elements from $\mathcal{B}_\eta$ by $Z_{\eta,\mathbb{Z}}$ and the $\mathbb{Z}$-span of elements from $\mathcal{B}$ by $Z_{\mathbb{Z}}$. Then $Z_{\eta,\mathbb{Z}} \otimes \mathbb{Q} \simeq B_{\eta,\mathbb{Q}}$ and $Z_{\mathbb{Z}} \otimes \mathbb{Q} \simeq B_{\mathbb{Q}}$.

Denote by $Z_\eta$ the image of $Z_{\eta,\mathbb{Z}}$ under the reduction modulo $p$, and by $Z$ the image of $Z_{\mathbb{Z}}$ under the reduction modulo $p$. Over a ground field $F$ of positive characteristic $p > 2$, the space $B_{\eta,F}$ contains $Z_{\eta,\mathbb{Z}} \otimes \mathbb{F}$ but is bigger in general, and the space $B_F$ contains $Z_{\mathbb{Z}} \otimes \mathbb{F}$ but is bigger in general.

Let us modify the definition of simple-odd-linkage of weights in the case when the ground field $F$ has characteristic $p > 2$ by replacing the requirement $\omega_{ij} = 0$ by $\omega_{ij} \equiv 0 \pmod{p}$.

It follows from Lemma 5.3 that $\psi_{k,Q}(Z_{\eta,\mathbb{Z}}) \subseteq Z_{\mathbb{Z}}$. Moreover, $\psi_{k,Q} : B_{\eta,Q} \rightarrow B_{\mathbb{Q}}$ is induced by $\psi_{k,Z} : Z_{\eta,\mathbb{Z}} \rightarrow Z_{\mathbb{Z}}$: When we reduce the map $\psi_{k,Z} : Z_{\eta,\mathbb{Z}} \rightarrow Z_{\mathbb{Z}}$ modulo $p$, we obtain a map $\psi_{\eta,Z} : Z_{\eta} \rightarrow Z$, which is a restriction and corestriction of the map $\psi_{k,F} : B_{\eta,F} \rightarrow B_F$. Combine different maps $\psi_{\eta,Z}$ to a map $\psi : \otimes_{\eta \in \Sigma_k} Z_\eta \rightarrow Z$ which is a restriction and corestriction of $\psi_k : S_{\mu,F} \rightarrow \overline{S}_{\mu,F}$.

The next statement gives a connection to the linkage principle for general linear supergroups over the field of characteristic $p > 2$ - see [10].

Proposition 6.2. Assume $\lambda$ and $\mu = \lambda_{|\lambda'}$ are dominant and polynomial weights and the characteristic of the ground field $F$ is $p > 2$. If $\psi^Z$ is not surjective, then $\lambda \sim \text{odd} \mu$.

Proof. We proceed as in the proof of Theorem 5.6 and consider the collections $\mathcal{E}_s = \mathcal{C}_{T_s^+, \mathbb{Z}}$, consisting of vectors $\sigma \pi_{\eta_{|\eta'}}$ for $\eta \in \Sigma_k$, where $J^s$ corresponds to $T_s^+$, listed with respect to the order $T_1^+ < \ldots < T_i^+$.

To each set $\mathcal{E}_s$ we assign a set $O_s = \{ \alpha_{\eta,\eta'_{|\eta'}} \eta \in \Sigma_k \}$. It is a crucial observation that if every $O_s$, for $s = 1, \ldots, l$, contains a nonzero element, then the map $\psi^Z$ is surjective. This is because by Lemma 5.3 and Theorem 7.1 of [8] we can find a set of vectors $\sigma \pi_{\eta_{|\eta'}}$ of weight $\lambda_{|\lambda'}$, such that the matrix of $\psi_k$ restricted on the span of these vectors is invertible.

Therefore, if $\psi^Z$ is not surjective, then there is an index $s$ such that all elements of $O_s$ vanish. The remainder of the proof is analogous to the second half of the proof of Theorem 5.6.

References

[1] Brundan, J. and Kujawa, J.: A new proof of the Mullineux conjecture. J. Algebraic Combin., 18 (2003), 13–39.
[2] Grishkov, A. N. and Marko, F.: Description of simple modules for Schur superalgebra $S(2,2)$, Glasg. Math. J. 55 (2013), no. 3, 695–719.
[3] La Scala R. and Zubkov, A. N.: Costandard modules over Schur superalgebras in characteristic $p$, J. Algebra and its Appl., 7 (2) (2008) 147–166.
[4] van Leeuwen, Marc A. A.: The Littlewood-Richardson rule, and related combinatorics. Interaction of combinatorics and representation theory, 95–145, MSJ Mem., 11, Math. Soc. Japan, Tokyo, 2001.
[5] van Leeuwen, Marc A. A.: Tableau algorithms defined naturally for pictures, Discrete Math. 157 (1996), 321–262.
[6] Marko, F.: Description of costandard modules of Schur superalgebra $S(31)$, Comm. Algebra 41 (2013), no. 7, 2665–2697.
[7] Marko, F.: Primitive vectors in induced supermodules for general linear supergroups, J. Pure Appl. Algebra 219 (2015), 978–1007.
[8] Marko, F.: Even-primitive vectors in induced supermodules for general linear supergroups and costandard supermodules for Schur superalgebras, submitted to J. Algebraic Combin., see also arXiv:1608.08989 [math.RT]

[9] Marko, F. and Zubkov, A. N.: Schur superalgebras in characteristic p, Algebras and Representation Theory 9 (1) (2006), 1–12.

[10] Marko, F. and Zubkov, A. N.: Blocks for general linear supergroup GL(m|n), Transformation Groups, DOI: 10.1007/s00031-017-9429-6, see also arXiv: 1507. 05027 [math.RT].

[11] Zelevinsky, A. V.: A generalization of the Littlewood-Richardson rule and the Richardson-Schensted-Knuth correspondence, J. Algebra 69 (1981), 82–94.

[12] Zubkov, A. N.: Some properties of general linear supergroups and of Schur superalgebras, Algebra Logic 45 (3) (2006), 147–171.

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