Superconformal index on $\mathbb{RP}^2 \times S^1$ and mirror symmetry

Akinori Tanaka$^a$, Hironori Mori$^a$, and Takeshi Morita$^b$

$^a$Department of Physics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

$^b$Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan

Abstract

We study $\mathcal{N} = 2$ supersymmetric gauge theories on $\mathbb{RP}^2 \times S^1$ and compute the superconformal index by using the localization technique. We consider not only the round real projective plane $\mathbb{RP}^2$ but also the squashed real projective plane $\mathbb{RP}_b^2$, which turns back to $\mathbb{RP}^2$ by taking a squashing parameter $b$ as 1. In addition, we found that the result is independent of the squashing parameter $b$. We apply our new superconformal index to the check of the simplest 3d mirror symmetry, i.e. the equivalence between the $\mathcal{N} = 2$ SQED and the XYZ model on $\mathbb{RP}^2 \times S^1$. We prove it by using a mathematical formula called the $q$-binomial theorem. We comment on the $\mathcal{N} = 4$ version of mirror symmetry, mirror symmetry via generalized indices, and possibilities of generalizations from mathematical viewpoints.

*akinori@het.phys.sci.osaka-u.ac.jp
†hiromori@het.phys.sci.osaka-u.ac.jp
‡t-morita@cr.math.sci.osaka-u.ac.jp
1 Introduction

The remarkable recent progress in 3d supersymmetric gauge theories is that we can exactly investigate theories with interactions on various curved geometries by making use of the localization [1, 2, 3, 4, 5, 6, 7, 8, 9]. One of the interesting quantities to which we can apply this exact calculation is the superconformal index (SCI) [10, 11] defined as a refinement of
the Witten index. The SCI of $\mathcal{N} = 2$ superconformal theories is defined by \[ [12] \]
\[
\mathcal{I}(x, e^{i\mu_a}) = \text{Tr}_\mathcal{H} \left[ (-1)^F e^{(Q, Q^\dagger)_{x} \hat{H} + \hat{j}_{3}} \prod_{a} e^{i\mu_a \hat{f}_{a}} \right],
\]
(1.1)
where $\mathcal{H}$ is the Hilbert space of the theory, and the trace is taken over this Hilbert space (see Section 3 for details). Basically, it counts the number of BPS states with eigenvalues of certain operators commuting with both the Hamiltonian $\{Q, Q^\dagger\}$ and the Fermion number operator $\hat{F}$. The SCI on $S^2 \times S^1$ has been computed by the localization in [13, 14].

An application of the SCI is to study 3d mirror symmetry [15, 16, 17, 18] of which the duality between the $\mathcal{N} = 2$ SQED and the XYZ model is the simplest example. Mirror symmetry on $S^2 \times S^1$ based on SCIs has been studied numerically in [14] and has been manifested in [19] by using the $q$-binomial theorem and the Ramanujan’s summation (the special case for SCIs with gauging flavor symmetries, called generalized indices, also has been proven in the same way [20]). An advantage to utilize the SCI is that we can establish mirror symmetry rigorously in the mathematical sense thanks to the localization.

On the other hand, one can construct 2d theories on the real projective plane $\mathbb{RP}^2$ by taking precise boundary conditions of fields on the two-sphere $S^2$ under the antipodal identification

\[
(\pi - \vartheta, \pi + \varphi) \sim (\vartheta, \varphi),
\]
(1.2)
where $\vartheta, \varphi$ are coordinates of $S^2$. The partition function on $\mathbb{RP}^2$ has been computed exactly in [22]. The authors also showed how to define 2d SUSY theories on the squashed real projective plane $\mathbb{RP}^2_b$ by turning on an appropriate background $U(1)_{R^\prime}$-gauge field. This method was developed in [23] in the context of localization calculus on the squashed two-sphere $S^2_b$.

In this paper, we show that their constructions can be lifted naturally to these on $\mathbb{RP}^2_b \times S^1$ by adding the 3rd coordinate $y$. We can get this curved space from $S^2_b \times S^1$ by identifying

\[
(\pi - \vartheta, \pi + \varphi, y) \sim (\vartheta, \varphi, y),
\]
(1.3)
where $\vartheta \in [0, \pi], \varphi \in [0, 2\pi]$, and $y \in [0, 2\pi]$. Note that there is no difference between $S^2_b \times S^1$ and $\mathbb{RP}^2_b \times S^1$ if we only discuss the local quantities. The difference between them comes from the global distinction of topology and the boundary conditions of fields under the antipodal identification (1.3). With these setups, we calculate the SCI of $\mathcal{N} = 2$ supersymmetric gauge theories on $\mathbb{RP}^2_b \times S^1$ by the localization. First of all, we take the
Kaluza-Klein expansion for all fields along the $S^1$ direction, which reduces Lagrangians on $\mathbb{R}P^2_b \times S^1$ to the sum of Lagrangians on $\mathbb{R}P^2$ over KK modes. Then one-loop determinant of the vector multiplet and the matter multiplet can be obtained as a product of one-loop determinants on $\mathbb{R}P^2$ computed in [22]. Furthermore, we specify parity conditions, named the B-parity condition, in order to make all fields consistent with the antipodal identification. The B-parity condition is concluded by plausible requirements from physical consideration. The one-loop determinant is expressed by the contribution of the $Z_2$-holonomy plus or minus sector due to the B-parity condition. As a result, the SCI is written as the sum of each contribution when the vector multiplet is considered. This is different from the case where the SCI on $S^2_b \times S^1$ receives the contribution of the monopole as the infinite sum over integers. In addition, the one-loop determinants and the SCI on $\mathbb{R}P^2_b \times S^1$ are independent of the squashing parameter $b$.

With our exact results, we check $\mathcal{N} = 2$ Abelian mirror symmetry on $\mathbb{R}P^2_b \times S^1$. Again, the B-parity condition carry a crucial role to establish this duality. We verify it exactly as an equality of SCIs involving the $q$-shifted factorial and the basic hypergeometric series.

The rest of this paper is organized as follows: In Section 2, we construct $\mathcal{N} = 2$ supersymmetric gauge theories with $U(1)$ gauge group on $\mathbb{R}P^2_b \times S^1$. Also, we indicate the B-parity condition for a single flavor and its generalization to $N_f$ flavors. In Section 3 we show the main idea of the localization computation on $\mathbb{R}P^2_b \times S^1$ and one-loop determinants for the vector multiplet and the matter multiplet. In Section 4 mirror symmetry on $\mathbb{R}P^2_b \times S^1$ is established as the relation of the SCI for the SQED and the XYZ model with an appropriate identification of variables. We must take account of the B-parity condition correctly to get these SCIs. We justify it mathematically by employing the $q$-binomial theorem. In Section 5 we summarize our results and comment on some open questions. In Appendix A and B, we explain calculation details of the one-loop determinants. In Appendix C we discuss mathematical generalizations of our relation obtained as mirror symmetry.

\footnote{We follow these names used in [23, 19]. The authors of [20] use the $q$-product instead of the $q$-shifted factorial.}
2 Supersymmetry on $\mathbb{R}P^2_b \times S^1$

In this section, we show how to define 3d SUSY theories with U(1) gauge group on $\mathbb{R}P^2_b \times S^1$. Of course, we can also define the theories on $S^2_b \times S^1$. In fact, the arguments in Section 2.1, 2.2, and 2.3 can be applied to the theories on $S^2_b \times S^1$. However, we omit explanations for the calculations of the index on $S^2_b \times S^1$ because the final results are free from the squashing parameter and just reproduce the known results on $S^2 \times S^1$ [13, 3, 14]. On the other hand, discussions on $\mathbb{R}P^2_b \times S^1$ produce truly new results even though they are free from the squashing parameter. Therefore, we concentrate on the explanations of the theories on $\mathbb{R}P^2_b \times S^1$.

2.1 Our background and conventions

We extend the construction of 2d Killing spinors and the background U(1)$_R$ gauge field in [23, 22] to the 3d case.

Our background We consider the following dreibein and background U(1)$_R$-gauge field:

$$e^1 = f(\vartheta)d\vartheta, \quad e^2 = l \sin \vartheta d\varphi, \quad e^3 = dy,$$

(2.1)

$$V^R = \frac{1}{2}(1 - \frac{l}{f})d\varphi + \frac{-i}{2l}(1 - \frac{l}{f})dy,$$

(2.2)

where $f(\vartheta)^2 = \tilde{l}^2 \sin^2 \vartheta + l^2 \cos^2 \vartheta$. We use alphabets $a, b, c$ for the local Lorentz indices.

Covariant derivative The 3d covariant derivative is defined by

$$\mathcal{D}_\mu = \partial_\mu + \frac{1}{4} \omega^{ab}_\mu \mathcal{J}_{ab} - i\hat{R}V^R_\mu,$$

(2.3)

where $\omega^{ab}_\mu$ is the spin connection computed from the dreibein (2.1), $\mathcal{J}_{ab}$ are Lorentz generators of the fields characterized by its spin:

spin 0 $\Rightarrow \mathcal{J}_{ab} = 0$,

spin 1/2 $\Rightarrow \mathcal{J}_{ab} = \gamma_{ab}$,

(2.4)

where $\gamma_{ab}$ are antisymmetrized gamma matrices defined in (2.7), and $\hat{R}$ is a R-charge. See Table 4 for assignments of R-charges to each field.

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2We focus on U(1) gauge theories in this note in order to check 3d Abelian mirror symmetry. It is straightforward to extend our results to non-Abelian SUSY gauge theories.
Table 1: Charge assignments for each field

| Field | $A_\mu$ | $A$ | $\lambda$ | $D$ | $\phi$ | $\bar{\phi}$ | $\psi$ | $\bar{\psi}$ | $F$ | $\bar{F}$ |
|-------|--------|-----|----------|-----|--------|-------------|-------|-------------|-----|--------|
| spin  | 1      | 0   | 1/2      | 0   | 0      | 0           | 1/2   | 0           | 0   | 0      |
| $\hat{R}$ | 0     | 0   | +1       | -1  | 0      | -\Delta     | $\Delta$ | $-(\Delta - 1)$ | $\Delta - 1$ | $-(\Delta - 2)$ | $\Delta - 2$ |

**Killing spinors**  Then the spinors

$$
\epsilon(\vartheta, \varphi, y) = e^{\frac{1}{2}(\vartheta + i\varphi)} \left( \begin{array}{c} \cos \frac{\vartheta}{2} \\ \sin \frac{\vartheta}{2} \end{array} \right), \quad \bar{\epsilon}(\vartheta, \varphi, y) = e^{-\frac{1}{2}(\vartheta - i\varphi)} \left( \begin{array}{c} \sin \frac{\vartheta}{2} \\ \cos \frac{\vartheta}{2} \end{array} \right)
$$

satisfy the Killing spinor equations,

$$
D_\mu \epsilon = \frac{1}{2} f_{\mu
u} \gamma^\nu \epsilon, \quad D_\mu \bar{\epsilon} = -\frac{1}{2} f_{\mu
u} \gamma^\nu \bar{\epsilon}.
$$

**Gamma matrices**  The gamma matrices $\gamma_a$ are defined by the Pauli matrices

$$
\gamma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \gamma_2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \gamma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \gamma_{ab} = \frac{1}{2}(\gamma_a \gamma_b - \gamma_b \gamma_a).
$$

**Spinor bilinear**  Let us denote generic spinors by $\epsilon, \bar{\epsilon},$ and $\lambda.$ We take spinor bilinears as

$$
\epsilon \lambda = \left( \begin{array}{cc} \epsilon_1 & \epsilon_2 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right), \quad \epsilon \gamma_a \lambda = \left( \begin{array}{cc} \epsilon_1 & \epsilon_2 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \gamma_a \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right).
$$

Using this convention, one can prove the following formulas:

$$
\epsilon \lambda = (-1)^{1+|\epsilon|} |\lambda| \epsilon \lambda, \quad \epsilon \gamma_a \lambda = (-1)^{|\epsilon|} |\lambda| \gamma_a \epsilon, \quad (\gamma_a \epsilon) \lambda = -\epsilon \gamma_a \lambda, \quad \bar{\epsilon}(\epsilon \lambda) + (-1)^{1+|\epsilon|} |\lambda| \epsilon(\bar{\epsilon} \lambda) + (\bar{\epsilon} \epsilon) \lambda = 0, \quad (\gamma_a \epsilon) \lambda + (-1)^{|\epsilon|} |\lambda| (\epsilon \gamma_a \lambda) \gamma^a \epsilon = 0,
$$

where $|\epsilon|$ means the spinor $\epsilon$’s statistics such that $|\epsilon| = 0$ for a bosonic $\epsilon$ and $|\epsilon| = 1$ for a fermionic $\epsilon.$

### 2.2 Supersymmetry

We show our definition of supersymmetry in this subsection. There are two kinds of multiplets in the 3d $\mathcal{N} = 2$ theory. The first one is the vector multiplet composed of gauge
field $A_\mu$, scalar $\sigma$, gauginos $\lambda, \bar{\lambda}$, and an auxiliary field $D$. The supersymmetry for the vector multiplet is given by

$$
\delta_\epsilon A_\mu = -\frac{i}{2} \bar{\lambda} \gamma_\mu \epsilon, \quad \delta_\epsilon A_\mu = -\frac{i}{2} \bar{\tau} \gamma_\mu \lambda, \quad (2.8)
$$

$$
\delta_\epsilon \sigma = +\frac{1}{2} \bar{\epsilon} \epsilon, \quad \delta_\epsilon \sigma = +\frac{1}{2} \bar{\tau} \lambda, \quad (2.9)
$$

$$
\delta_\epsilon \lambda = \frac{1}{2} \gamma^{\mu \nu} \epsilon F_{\mu \nu} - D\epsilon + i \gamma^\mu \epsilon D_\mu \sigma + \frac{i}{f} \sigma \gamma_3 \epsilon, \quad \delta_\epsilon \lambda = 0, \quad (2.10)
$$

$$
\delta_\epsilon \bar{\lambda} = 0, \quad \delta_\epsilon \bar{\lambda} = \frac{1}{2} \gamma^{\mu \nu} \bar{\epsilon} F_{\mu \nu} + D\bar{\epsilon} - i \gamma^\mu \bar{\epsilon} D_\mu \sigma + \frac{i}{f} \sigma \gamma_3 \bar{\epsilon}, \quad (2.11)
$$

$$
\delta_\epsilon D = i \frac{1}{2} D_\mu \bar{\lambda} \gamma^\mu \epsilon + \frac{i}{4 f} \bar{\lambda} \gamma_3 \epsilon, \quad \delta_\epsilon D = -i \frac{1}{2} \bar{\tau} \gamma^\mu D_\mu \lambda + \frac{i}{4 f} \bar{\tau} \gamma_3 \lambda, \quad (2.12)
$$

where we use the same supersymmetry as in [25] where $\delta_\epsilon$ and $\delta_\tau$ are purely fermionic. We use the Killing spinors in (2.5) as supersymmetry parameters.

The second one is the matter multiplet composed of scalars $\phi, \bar{\phi}$, spinors $\psi, \bar{\psi}$, and auxiliary fields $F, \bar{F}$ which couple with the vector multiplet via the gauge symmetry with a charge $q$. Also, we have the following SUSY transformations for the matter multiplet:

$$
\delta_\epsilon \phi = 0, \quad \delta_\epsilon \bar{\phi} = \bar{\tau} \psi, \quad (2.13)
$$

$$
\delta_\epsilon \bar{\phi} = \epsilon \psi, \quad \delta_\epsilon \phi = 0, \quad (2.14)
$$

$$
\delta_\epsilon \psi = i \gamma^\mu D_\mu^A \phi + i q \phi \sigma \phi - \frac{i}{f} \Delta \epsilon \gamma_3 \phi, \quad \delta_\epsilon \bar{\psi} = \bar{\tau} F, \quad (2.15)
$$

$$
\delta_\epsilon \bar{\psi} = F \epsilon, \quad \delta_\epsilon \psi = i \gamma^\mu D_\mu^A \bar{\phi} + i q \bar{\phi} \sigma \bar{\epsilon} - \frac{i}{f} \bar{\phi} \gamma_3 \bar{\epsilon}, \quad (2.16)
$$

$$
\delta_\epsilon F = i \gamma^\mu D_\mu^A \psi - i q \sigma \epsilon \psi - i q \epsilon \lambda \phi + \frac{i (2 \Delta - 1)}{2 f} \epsilon \gamma_3 \psi, \quad \delta_\epsilon \bar{F} = 0, \quad (2.17)
$$

$$
\delta_\epsilon \bar{F} = 0, \quad \delta_\epsilon F = i \bar{\tau} \gamma^\mu D_\mu^A \bar{\psi} - i q \bar{\psi} \sigma \bar{\phi} + i q \bar{\phi} \bar{\sigma} \lambda - \frac{i (2 \Delta - 1)}{2 f} \bar{\tau} \gamma_3 \bar{\psi}. \quad (2.18)
$$

Of course, the SUSY algebra is closed within the translation, rotation, R-symmetry, and the gauge transformation. Here, we use the covariant derivative coupled with $A$

$$
D_\mu^A \Phi = D_\mu \Phi - i q A_\mu \Phi, \quad \bar{D}_\mu^A \bar{\Phi} = \bar{D}_\mu \bar{\Phi} + i q \bar{\Phi} A_\mu. \quad (2.19)
$$

If one wants a neutral matter, it is achieved by turning off $q$. In our convention, $A_\mu$ has the same dimension as $\partial_\mu$, thus, the charge $q$ must be dimensionless, or equivalently, $q$ is just a number. One of the most important features of the above SUSY is nilpotency

$$
\delta_\epsilon^2 = \delta_\tau^2 = 0. \quad (2.20)
$$
2.3 Lagrangians

SUSY-exact terms  In order to use the localization method, we need so-called SUSY-exact terms for the vector multiplet and the matter multiplet. For the vector multiplet, we can use the super Yang-Mills term as a SUSY-exact term. In fact, one can verify

\[ L_{YM} = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D^2 + D_\mu \sigma D^\mu \sigma + \epsilon^{\rho\sigma} \frac{\sigma}{f^2} F_{\rho\sigma} + \frac{\sigma^2}{f^2} + i \bar{\lambda} \gamma^\mu D_\mu \lambda - \frac{i}{2f} \bar{\lambda} \gamma_3 \lambda \]

(2.21)

up to a total derivative term. The notation \( \delta \bar{\epsilon} \rightarrow \epsilon^\dagger \) is defined in the same way in [25]. In addition, the following term for the matter multiplet is also SUSY-exact:

\[ L_{\text{mat}} = -i(\bar{\psi} \gamma^\mu D^\mu \psi) + i q(\bar{\psi} \sigma \psi) - i q\bar{\phi}(\bar{\lambda} \psi) - \frac{i \Delta}{2f} (\bar{\psi} \gamma_3 \psi) + T^F \]

\[ + i q(\bar{\psi} \lambda) \phi + g^{\mu\nu} D^\mu \phi D^\nu \phi + q^2 \bar{\phi} \sigma^2 \phi + i q \bar{\phi} D \phi - \frac{\Delta}{f^2} \phi D^3 \phi - \frac{\Delta}{f} \bar{\phi} \phi + \frac{\Delta}{4} R \bar{\phi} \phi \]

\[ - \frac{\Delta - 1}{f} \bar{v} \phi D^i \phi - \frac{\Delta - 1}{f} \omega \bar{\phi} \sigma \phi - i \frac{\Delta - 1}{2f} \bar{v}(\bar{\psi} \gamma_i \psi) - i \frac{\Delta - 1}{2f} \omega(\bar{\psi} \psi) \]

(2.22)

where \( i \) runs for 1, 2, or equivalently, \( \vartheta, \varphi \). Here, we define

\[ v^\mu = \bar{\epsilon} \gamma^\mu \epsilon, \quad \omega = \bar{\epsilon} \epsilon. \]

(2.23)

We use these actions as “regulators” for the localization. Thanks to the nilpotency (2.20), these terms are \( \delta_{\bar{\epsilon}} \)-invariant automatically.

Other terms  Of course, we can construct other SUSY-invariant terms. The famous one is the supersymmetric Chern-Simons term

\[ L_{CS} = \frac{1}{\sqrt{g}} \epsilon^{\mu\nu\lambda} (A_\mu \partial_\nu A_\lambda) - \bar{\lambda} \lambda + 2 D \sigma. \]

(2.24)

This term is, however, prohibited on \( \mathbb{RP}^2_6 \times S^1 \) as we explain later. We consider U(1) gauge group, therefore, the Fayet-Iliopoulos term

\[ L_{FI} = D - \frac{1}{f} A_3 \]

(2.25)

can be taken into account. If there is an additional dynamical vector multiplet, say \( (B_\mu, \bar{\sigma}, \bar{\lambda}, \tilde{\lambda}, \tilde{D}) \), which has the same SUSY transformations as (2.8) - (2.12), then we can write down the following supersymmetric BF term

\[ L_{BF} = \frac{1}{\sqrt{g}} \epsilon^{\mu\nu\lambda} (B_\mu F_{\nu\lambda}) - \bar{\lambda} \lambda - \bar{\lambda} \lambda + 2 D \bar{\sigma} + 2 \tilde{D} \sigma. \]

(2.26)
However, this term is also prohibited on $\mathbb{RP}^2 \times S^1$. In addition to them, the superpotential terms for the matter multiplet are also SUSY-invariant. It may be interesting to construct them explicitly on our curved space. For example, there is a known result how to write them explicitly on $S^3$ [26].

### 2.4 Parity conditions

As studied in [22], we have to find parity conditions compatible with the antipodal identification (1.3) for component fields. Our guiding principles are as follows.

- The squared parity transformation becomes $+1$ for bosons and $-1$ for fermions.
- Regulator Lagrangian (2.21) and (2.22) must be invariant under the parity.
- SUSY $\delta_{\epsilon}, \delta_{\zeta}$ must be consistent with the parity.

**Vector multiplet** After simple calculus, we find a unique set of parity conditions for the vector multiplet,

\[
A_1(\pi - \vartheta, \pi + \varphi, y) = -A_1(\vartheta, \varphi, y), \quad A_{2,3}(\pi - \vartheta, \pi + \varphi, y) = +A_{2,3}(\vartheta, \varphi, y), \\
\sigma(\pi - \vartheta, \pi + \varphi, y) = -\sigma(\vartheta, \varphi, y), \\
\lambda(\pi - \vartheta, \pi + \varphi, y) = +i\gamma_1\lambda(\vartheta, \varphi, y), \quad \overline{\lambda}(\pi - \vartheta, \pi + \varphi, y) = -i\gamma_1\overline{\lambda}(\vartheta, \varphi, y), \\
D(\pi - \vartheta, \pi + \varphi, y) = +D(\vartheta, \varphi, y).
\] (2.27)

These are similar to the ones in [22] called $B$-parity. Therefore, we would like to call the conditions in (2.27) the $B$-parity condition.

**Matter multiplet** The one flavor matter multiplet has two choices:

\[
\phi(\pi - \vartheta, \pi + \varphi, y) = \pm \phi(\vartheta, \varphi, y), \quad \overline{\phi}(\pi - \vartheta, \pi + \varphi, y) = \pm \overline{\phi}(\vartheta, \varphi, y), \\
\psi(\pi - \vartheta, \pi + \varphi, y) = \mp i\gamma_1\psi(\vartheta, \varphi, y), \quad \overline{\psi}(\pi - \vartheta, \pi + \varphi, y) = \pm i\gamma_1\overline{\psi}(\vartheta, \varphi, y), \\
F(\pi - \vartheta, \pi + \varphi, y) = \pm F(\vartheta, \varphi, y), \quad \overline{F}(\pi - \vartheta, \pi + \varphi, y) = \pm \overline{F}(\vartheta, \varphi, y).
\] (2.28)

The existence of compatible two choices can be regarded as the existence of a holonomy with respect to a background $U(1)_{\text{flavor}}$ flat gauge field $B_{\text{flat}}^{\text{flavor}}$. In other words, the parity conditions are characterized by the holonomy of $B_{\text{flat}}^{\text{flavor}}$,

\[
\pm 1 = e^{i \oint \vartheta B_{\text{flat}}^{\text{flavor}}}, \quad (2.29)
\]
where \( \gamma \) is a non-contractible cycle of \( \mathbb{R}P^2 \), and \( f \) is the corresponding U(1)_{\text{flavor}} charge defined by \( \hat{f}\Phi = f\Phi \). \( \hat{f} \) is a flavor charge operator used later in the definition of the superconformal index. This is an analogue of the background U(1)_{\text{flavor}} monopole gauge field on \( S^2 \times S^1 \) in [20]. If we have many flavors, we can generalize these conditions. Let us denote a multi-flavor field by

\[
\vec{\Phi} = \begin{pmatrix}
\Phi_1 \\
\Phi_2 \\
\vdots \\
\Phi_{N_f}
\end{pmatrix},
\]

then the generic B-parity condition is

\[
\begin{align*}
\tilde{\phi}(\pi - \vartheta, \pi + \varphi, y) &= M\tilde{\phi}(\vartheta, \varphi, y), \\
\tilde{\phi}(\pi - \vartheta, \pi + \varphi, y) &= N\tilde{\phi}(\vartheta, \varphi, y), \\
\tilde{\psi}(\pi - \vartheta, \pi + \varphi, y) &= -i\gamma_1 M\tilde{\psi}(\vartheta, \varphi, y), \\
\tilde{\psi}(\pi - \vartheta, \pi + \varphi, y) &= i\gamma_1 N\tilde{\psi}(\vartheta, \varphi, y),
\end{align*}
\]

where \( M \) and \( N \) are \( N_f \times N_f \) matrices constrained by

\[
N^T M = 1, \quad M^2 = N^2 = 1.
\]

**Comments on the B-parity condition** There are two comments. The first one is related to the vector multiplet. In order to use \( \mathcal{L}_{\text{YM}} \) (2.21) as a regulator when we perform the localization, we would like to maintain it to be invariant under the B-parity (2.27) as we noted in our guiding principles. In fact, \( \mathcal{L}_{\text{YM}} \) is invariant under (2.27). On the other hand, \( \mathcal{L}_{\text{CS}} \) (2.24) and \( \mathcal{L}_{\text{BF}} \) (2.26) become parity odd

\[
\mathcal{L}_{\text{CS/ BF}}(\pi - \vartheta, \pi + \varphi, y) = - \mathcal{L}_{\text{CS/ BF}}(\vartheta, \varphi, y).
\]

It means that we cannot take it on \( \mathbb{R}P^2_b \times S^1 \) as we commented just below (2.24) and (2.26).

The second comment concerns the matter multiplet. Suppose we have two flavors and the B-parity condition described by the \( 2 \times 2 \) matrices

\[
M = N = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\]

then we can lift its Lagrangian on \( \mathbb{R}P^2_b \times S^1 \) to the one on \( S^2_b \times S^1 \) by defining a new matter multiplet on \( S^2_b \times S^1 \) as

\[
\Phi(\vartheta, \varphi, y) = \begin{cases}
\Phi_1(\vartheta, \varphi, y), & \vartheta \in [0, \pi/2], \\
\Phi_2(\vartheta, \varphi, y), & \vartheta \in [\pi/2, \pi].
\end{cases}
\]
The authors of [22] also commented on this fact. This is quite similar to the doubling trick in string theory. In Section 4, we use such B-parity condition exactly in the context of 3d mirror symmetry.

3 Localization calculus on $\mathbb{RP}^2_b \times S^1$

In this section, we calculate the superconformal index (SCI)

$$I(x, \alpha) = \text{Tr}_{\mathcal{H}_{\mathbb{RP}^2_b}} \left[ (-1)^F x^{j_3} R^{-j_3} x^{H + j_3} \alpha \hat{f} \right],$$

(3.1)

where $\hat{F}$ is the fermion number operator, $\hat{H}$ is the energy operator, $\hat{R}$ is the R-charge operator, $\hat{j}_3$ is the third component of the orbital angular momentum operator which acts on $\mathbb{RP}^2_b$, and $\hat{f}$ is the flavor charge operator. Note that we have opposite R-charge assignments compared with [14, 19, 20, 21]. $\mathcal{H}_{\mathbb{RP}^2_b}$ represents the Hilbert space of the theory on $\mathbb{RP}^2_b$. The squashing procedure is compatible with the definition (3.1) because this procedure preserves the isometry generated by $\hat{j}_3$. We take each fugacity as

$$x' = e^{-\beta_1}, \quad x = e^{-\beta_2}, \quad \alpha = e^{i\mu},$$

(3.2)

where $\mu$ is a chemical potential and define the relations

$$\beta_1 + \beta_2 = \frac{2\pi}{l}, \quad \Omega = \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2},$$

(3.3)

where we introduce the parameter $\Omega$ for later simplicity.

3.1 Vector multiplet contribution

First, we have to identify the locus of the Lagrangian $\mathcal{L}_{\text{YM}}$ (2.21) characterized by $\mathcal{L}_{\text{YM}} = 0$. In order to find it, it is useful to introduce the combination of the fields

$$F^\mu = \frac{1}{2} \epsilon^{\mu\rho\sigma} F_{\rho\sigma} + \partial^\mu \sigma + \frac{1}{f} \delta_3^\mu \sigma.$$ 

(3.4)

The Lagrangian $\mathcal{L}_{\text{YM}}$ can be rewritten as

$$\mathcal{L}_{\text{YM}} = F_\mu F^\mu + D^2 + i \bar{\lambda} \gamma^\mu D_\mu \lambda - \frac{i}{2f} \bar{\lambda} \gamma_3 \lambda,$$

(3.5)

up to total derivative. Now, the locus is obtained by

$$F_\mu = 0,$$

(3.6)

$$D = 0, \quad \lambda = 0, \quad \bar{\lambda} = 0.$$

(3.7)
locus on $\mathbb{RP}_b^2 \times S^1$ A nontrivial equation is (3.6). This is equivalent to the following equation expressed by differential forms

$$ *F + d\sigma + \frac{e^3}{f}\sigma = 0. $$

(3.8)

We have to know the configuration invariant under the B-parity condition (2.27) which satisfies (3.8). It can be characterized by

$$ F = 0, \quad \sigma = 0. $$

(3.9)

The first equation in (3.9) means, of course, the flat connection. The flat connection $A$ on $\mathbb{RP}_b^2 \times S^1$ is expressed by

$$ A = A_{\text{flat}} + \frac{\theta}{2\pi}dy, $$

(3.10)

where $A_{\text{flat}}$ is a flat connection of $\mathbb{RP}_b^2$ related to the holonomy along the noncontractible cycle $\gamma$ of $\mathbb{RP}_b^2$. There are two choices for $A_{\text{flat}} = A_{\text{flat}}^\pm$ characterized by

$$ \exp \left( i \oint_\gamma A_{\text{flat}}^\pm \right) = \pm 1, $$

(3.11)

Also, there is a constraint on the $\theta$ as

$$ \theta \sim \theta + 2\pi. $$

(3.12)

Therefore, we have to sum up these contributions weighted by the gaussian parts, or equivalently, the one-loop determinants $Z_{1-\text{loop}}^\pm$:

$$ \mathcal{I}(x, \alpha) = \int_0^{2\pi} \frac{d\theta}{2\pi} Z_{1-\text{loop}}^+ + \int_0^{2\pi} \frac{d\theta}{2\pi} Z_{1-\text{loop}}^- . $$

(3.13)

One important thing is that we can perform calculus even if we do not know the explicit form of $A_{\text{flat}}^\pm$. This is similar to the calculation of the partition function on $\mathbb{RP}_b^2$ in [22].

3d to 2d One might think that the U(1) vector multiplet contribution is trivial because the result on $S^2 \times S^1$ was so [13, 3, 14]. However, there is a nontrivial contribution once we put the theory on $\mathbb{RP}_b^2 \times S^1$. We can use results of 2d calculations [22] to compute our 3d SCI (3.13). Let us show how it works. First, we expand each component field around the locus (3.7), (3.9), and (3.10), then we get the following linearized Lagrangians:

$$ \mathcal{L}_{\text{boson}} = \frac{1}{2} \left[ \partial_\mu A_\nu - \partial_\nu A_\mu \right]^2 + (\partial_\mu \sigma)^2 + \epsilon^{\mu\nu\sigma} \frac{f}{j} [\partial_\mu A_\nu - \partial_\nu A_\mu] + \frac{\sigma^2}{f^2}, $$

(3.14)

$$ \mathcal{L}_{\text{fermion}} = i \bar{\lambda} \gamma^\mu D_\mu \lambda - \frac{i}{2f} \bar{\lambda} \gamma_3 \lambda. $$

(3.15)
Here, our starting Lagrangian has only a U(1) gauge symmetry. In other words, (2.21) is the one of a gaussian type theory. Therefore, the above Lagrangians have nothing but the same form as the original one (2.21).

Usually, in the context of localization calculus, one expand these fields with respect to the direct product of some harmonic functions on $\mathbb{R}P^2$ and Kaluza-Klein modes of $S^1$. Here, however, we take much quicker route. We expand each field with respect to the Kaluza-Klein modes only:

\[
A_i = \sum_n \frac{1}{\sqrt{2\pi}} e^{\left(\frac{i n - \beta_1 - \beta_2}{2\pi} j_3\right) y} A_i^{(n)}(\vartheta, \varphi) \quad (i = 1, 2),
\]

\[
A_3 = \sum_n \frac{1}{\sqrt{2\pi}} e^{\left(\frac{i n - \beta_1 - \beta_2}{2\pi} j_3\right) y} A_3^{(n)}(\vartheta, \varphi),
\]

\[
\sigma = \sum_n \frac{1}{\sqrt{2\pi}} e^{\left(\frac{i n - \beta_1 - \beta_2}{2\pi} j_3\right) y} \sigma^{(n)}(\vartheta, \varphi),
\]

\[
\lambda = \sum_n \frac{1}{\sqrt{2\pi}} e^{\left(\frac{i n + (1 - j_3)}{2\pi} \frac{\hat{j}_3}{2\pi} + \frac{\hat{j}_3}{2\pi}\right) y} \lambda^{(n)}(\vartheta, \varphi),
\]

\[
\overline{\lambda} = \sum_n \frac{1}{\sqrt{2\pi}} e^{\left(\frac{i n + (1 - j_3)}{2\pi} \frac{\hat{j}_3}{2\pi} + \frac{\hat{j}_3}{2\pi}\right) y} \overline{\lambda}^{(n)}(\vartheta, \varphi),
\]

where $\hat{j}_3$ is the orbital angular momentum operator:

\[
\hat{j}_3 = -i \partial_\varphi.
\]

Then one can get a sum of 2d Lagrangians $L^{2d \ (n)}$ of Kaluza-Klein fields labelled by $n$ after performing the integral along $S^1$,

\[
\int d^3x \sqrt{g_3} L = \int d^2x \sqrt{g_2} \sum_n L^{2d \ (n)}.
\]

The bosonic part and the fermionic part are as follows:

\[
\sqrt{g_2} L^{2d \ (n)}_{\text{boson}} = \begin{pmatrix}
A^{(-n)}_i \\
A^{(-n)}_3 \\
\sigma^{(-n)}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
-\ast d^2 + h_n^2 & -i h_n d & -\ast d \frac{1}{2} \\
-i h_n d & *2 & 0 \\
+\ast \ast d & 0 & -\ast d^2 + \frac{1}{2} + h_n^2
\end{pmatrix} & \begin{pmatrix}
A^{(n)}_i \\
A^{(n)}_3 \\
\sigma^{(n)}
\end{pmatrix}
\end{pmatrix}
\end{pmatrix}
\]

\[
L^{2d \ (n)}_{\text{fermion}} = \overline{\lambda}^{(-n)} \left(i \gamma^i D_i + \gamma_3 (h_n + \frac{i}{2l} \Omega)\right) \lambda^{(n)},
\]

\(^3\)When one generalizes it with non-Abelian gauge group, one should replace $\partial_\varphi$ by the covariant derivative defined by the precise flat connection $A_{\text{flat}}^n$ in (3.21) corresponding to the locus around which the fluctuation fields are expanded.
where \(*_2\) is the Hodge star of \(\mathbb{R}P^2_b\), and the exterior derivative \(d\) and the gauge field \(A^{(n)}\) are 1-forms on \(\mathbb{R}P^2_b\). The symbol \(h_n\) represents an operator defined by

\[
h_n = -(n + i \Omega j_3).
\]  

(3.25)

The Lagrangian (3.23) and (3.24) are quite similar to the ones on \(\mathbb{R}P^2_b\) in [22] by identifying \(h_n \sim \alpha \cdot \sigma\). Although, in the fermionic term (3.24), a slightly different contribution exists, we can do the same procedure performed in Appendix in [22]. See Appendix A for the details.

**One-loop determinant** The final result is

\[
Z^{\text{vector}+}_{1-\text{loop}} = Z^{\text{vector}−}_{1-\text{loop}} = Z^{\text{vector}}_{1-\text{loop}} = x^{+\frac{4}{2}} \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} f_{\text{vector}}(x^m) \right),
\]  

(3.26)

\[
f_{\text{vector}}(x_2) = \frac{x^2}{1 - x^4} - \frac{x^4}{1 - x^4},
\]  

(3.27)

where the prefactor preceding the exponent is the Casimir energy explained in detail in Appendix A and B.

### 3.2 Matter multiplet contribution

Second, we have to know the locus of the matter Lagrangian \(L_{\text{mat}}\) (2.22). However, it is somewhat trivial because the configuration is realized by turning off all fields in the matter multiplet. Therefore, by expanding around it, we get the following linearized Lagrangians:

\[
L_{\text{boson}} = g^{\mu \nu} D^A_{\mu} \phi D^A_{\nu} \phi - \frac{\Delta}{f} \phi \Gamma D_3 \phi - \frac{\Delta}{2f^2} \phi \phi + \frac{\Delta}{4} \overline{\phi} \Gamma \phi - \frac{\Delta - 1}{f} v \overline{\phi} D^A_i \phi,
\]  

(3.28)

\[
L_{\text{fermion}} = -i \overline{\psi} \gamma^\mu D^A_{\mu} \psi - \frac{i \Delta}{2f} \overline{\psi} \gamma_3 \psi - \frac{i \Delta - 1}{2f} v \overline{\psi} (\gamma_i \psi) - i \frac{\Delta - 1}{2f} \omega (\overline{\psi} \psi).
\]  

(3.29)

Here, the superscript \(A\) means the covariant derivative (2.19) defined with the locus value of the gauge field (3.10).

**3d to 2d** By expanding Kaluza-Klein modes first, we can get the 2d action as well as the case of the vector multiplet. In order to preserve SUSY, we have to read the precise
boundary conditions from the fugacities in the index (3.1):

\[
\phi(\vartheta, \varphi, y) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(in + (\Delta - \hat{j}_3) \frac{\vartheta}{2\pi} + \hat{j}_3 \frac{\varphi}{2\pi} - \frac{i\mu}{2\pi}\right)} y^{(n)}(\vartheta, \varphi),
\]

(3.30)

\[
\overline{\phi}(\vartheta, \varphi, y) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(i(n+\Delta - \hat{j}_3) \frac{\vartheta}{2\pi} + \hat{j}_3 \frac{\varphi}{2\pi} + \frac{i\mu}{2\pi}\right)} y^{(n)}(\vartheta, \varphi),
\]

(3.31)

\[
\psi(\vartheta, \varphi, y) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(in + (\Delta + 1 - \hat{j}_3) \frac{\vartheta}{2\pi} + \hat{j}_3 \frac{\varphi}{2\pi} - \frac{i\mu}{2\pi}\right)} y^{(n)}(\vartheta, \varphi),
\]

(3.32)

\[
\overline{\psi}(\vartheta, \varphi, y) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(in + (\Delta - 1 - \hat{j}_3) \frac{\vartheta}{2\pi} + \hat{j}_3 \frac{\varphi}{2\pi} + \frac{i\mu}{2\pi}\right)} y^{(n)}(\vartheta, \varphi),
\]

(3.33)

where \(\hat{j}_3\) is the orbital angular momentum operator

\[
\hat{j}_3 = -i\left(\partial_\varphi - i\hat{q}A_{\varphi}^{\text{flat}}\right).
\]

(3.34)

Note that there is a nontrivial contribution from the gauge field on the locus because the matter multiplet couples with the vector multiplet via the gauge symmetry. This effect is absent in the vector multiplet itself’s case because it is neutral when the gauge group is U(1).

Now, once we perform the integral over \(S^1\) as for the vector multiplet, we can get 2d Lagrangians,

\[
\mathcal{L}^{2d\ (n)}_{\text{boson}} = \overline{\phi}^{(-n)} \left( -g^{ij} D_i^{A_{\text{flat}}} D_j^{A_{\text{flat}}} + (p_n - \frac{\Delta - 1}{2l} \Omega)^2 + \frac{\Delta^2 - 2\Delta}{4f^2} + \frac{\Delta}{f} v_i D_i^{A_{\text{flat}}} \right) \phi^{(n)},
\]

(3.35)

\[
\mathcal{L}^{2d\ (n)}_{\text{fermion}} = \overline{\psi}^{(-n)} \left( -i\gamma^i D_i^{A_{\text{flat}}} - \gamma_3 (p_n - \frac{\Delta - 1}{2l} \Omega) - \frac{\Delta}{2f} \gamma_3 - i\frac{\Delta - 1}{2f} v_i \gamma_i - i\frac{\Delta - 1}{2f} \omega \right) \psi^{(n)},
\]

(3.36)

where the symbol \(p_n\) represents an operator defined by

\[
p_n = -(n + \frac{i}{l} \hat{j}_3 \Omega) + \frac{q\theta + f\mu}{2\pi}.
\]

(3.37)

The Lagrangian (3.35) and (3.36) are also similar to the ones on \(\mathbb{RP}^2_\theta\) in [22] by identifying \(p_n \sim \sigma\). As we can see in the fermionic part of the Lagrangian for the vector multiplet, there are also distinctions between (3.35), (3.36) and the corresponding ones in [22]. Even with these extra terms, we can preform exact calculations. See more details in Appendix A.
One-loop determinant  The final result is

\[ Z_{\text{matter}}^+ = x + \frac{\Delta - 1}{4} e^{\frac{iq \theta}{f}} + \frac{1}{4} f \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} f_{\text{matter}}^+ (e^{imq \theta}, x^m, \alpha^m f) \right), \]

\[ f_{\text{matter}}^+ (e^{iq \theta}, x, \alpha f) = e^{iq \theta} \alpha^{1/4} f \frac{x^\Delta}{1 - x^4} - e^{-iq \theta} \alpha^{-1/4} f \frac{x^{2-\Delta}}{1 - x^4} \]

for the even holonomy sector which gives \( e^{i \oint \gamma (q A_{\text{flat}} + f B_{\text{flavor}})} = +1 \). The other final form is

\[ Z_{\text{matter}}^- = x^{-\frac{\Delta - 1}{4}} e^{-\frac{iq \theta}{f}} + \frac{1}{4} f \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} f_{\text{matter}}^- (e^{imq \theta}, x^m, \alpha^m f) \right), \]

\[ f_{\text{matter}}^- (e^{iq \theta}, x, \alpha f) = e^{iq \theta} \alpha^{1/4} f \frac{x^{2+\Delta}}{1 - x^4} - e^{-iq \theta} \alpha^{-1/4} f \frac{x^{4-\Delta}}{1 - x^4} \]

when we have the odd holonomy sector \( e^{i \oint \gamma (q A_{\text{flat}} + f B_{\text{flavor}})} = -1 \).

4 Abelian Mirror Symmetry

We start with the review of Abelian mirror symmetry for 3d \( \mathcal{N} = 2 \) theories \([15, 16, 17, 18]\) with a single flavor. Then we explain how this duality can be realized in terms of SCIs for theories on \( \mathbb{R}^2 \times S^1 \) in the physical sense and provide the mathematically exact verification to it.

4.1 Review of 3d mirror symmetry

\( \mathcal{N} = 2 \) mirror symmetry states the duality between the SQED and the XYZ model which flow the same IR fixed point. The \( \mathcal{N} = 2 \) SQED has one vector multiplet \( V \) and one flavor consisting of two chiral fields \( Q, \bar{Q} \) with charges \( q = +1, -1 \) under U(1) gauge group, respectively. This theory possesses extra U(1) global symmetries: one is a topological U(1)\( _J \), and the other is a flavor symmetry U(1)\( _A \) with a charge \( f = +1 \) which rotates \( Q \) and \( \bar{Q} \) by the phase with the same weight as seen in Table 2. On the other hand, the XYZ model is the theory containing three chiral fields \( X, Y, Z \) interacting through the superpotential \( W = XYZ \). This theory has two U(1) global symmetries, named U(1)\( _V \) and U(1)\( _A \) in \([20]\), whose charges assigned on each field are shown in Table 3.

U(1)\( _J \) and U(1)\( _A \) in the SQED are identified with U(1)\( _V \) and U(1)\( _A \) in the XYZ model, respectively, and the currents \( J_A \) associated with each U(1)\( _A \) are mapped with flipping the sign (see Table 4). Furthermore, there exists the correspondence between the moduli

\(^4\)In the literature \([14, 19, 20]\), they are named \( q, \bar{q}, \) and \( S \), respectively.
Table 2: Charges in the SQED

|   | U(1) | U(1)_J | U(1)_A | \( \hat{R} \) |
|---|------|--------|--------|-----------|
| \( Q \)  | +1   | 0      | +1     | \(-\Delta\) |
| \( \tilde{Q} \) | -1   | 0      | +1     | \(-\Delta\) |

Table 3: Charges in the XYZ model

|   | U(1)_V | U(1)_A | \( \hat{R} \) |
|---|--------|--------|-----------|
| \( X \)  | +1     | +1     | \(-(1 - \Delta)\) |
| \( Y \)  | -1     | +1     | \(-(1 - \Delta)\) |
| \( Z \)  | 0      | -2     | \(-2\Delta\) |

Table 4: The mirror map

|   | SQED | XYZ |
|---|------|-----|
| U(1)_J | \( \leftrightarrow \) | U(1)_V |
| U(1)_A, \( J_A \) | \( \leftrightarrow \) | U(1)_A, \(-J_A\) |
| \( e^{(\sigma + i\rho)/g^2}, e^{-(\sigma + i\rho)/g^2} \) | \( \leftrightarrow \) | X, Y |
| \( Q\tilde{Q} \) | \( \leftrightarrow \) | Z |

spaces of those theories (at least on the flat space). The moduli parameters of the SQED are \( Q\tilde{Q} \) characterizing the Higgs branch and \((\sigma + i\rho)\) where \( \rho \) is the dual photon defined by

\[
\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho} = \partial_\mu \rho. \quad (4.1)
\]

The expectation values of two chiral superfields \( e^{(\sigma + i\rho)/g^2}, e^{-(\sigma + i\rho)/g^2} \) (\( g \) is a coupling constant) parametrize the corresponding regions of the Coulomb branch. In the context of mirror symmetry, we can identify \( e^{(\sigma + i\rho)/g^2}, e^{-(\sigma + i\rho)/g^2} \), and \( Q\tilde{Q} \) with \( X, Y, \) and \( Z \) on the moduli space of the XYZ model, respectively (Table 4).

We can also construct the \( \mathcal{N} = 4 \) version of mirror symmetry. In the SQED, we introduce an adjoint (uncharged) chiral field \( \tilde{S} \) coupling to \( Q\tilde{Q} \). Similarly for the XYZ model, \( \tilde{Z} \) is added via the superpotential \( ZZ\tilde{Z} \) making \( Z \) and \( \tilde{Z} \) massive. We can obtain the (twisted) free theory with chiral fields \( X \) and \( Y \) by integrating out \( Z \) and \( \tilde{Z} \). The duality between those theories is referred to as \( \mathcal{N} = 4 \) mirror symmetry.

Let’s now consider gauging a flavor symmetry and denote a corresponding background gauge field by \( B^{\text{flavor}} \). In addition to \( J_A \), there is a topological current\( J_T = *F \) associated with U(1)_J where \( * \) is the Hodge star defined by a 3d metric. The flavor symmetry can be gauged by coupling \( B^{\text{flavor}} \) with \( J_T \), which is the same thing to add a BF term to the original action \[27, 17\]. This fact can be employed to demonstrate mirror symmetry with\footnote{For non-Abelian theories, a topological current should be in the form \( J_T = *\text{Tr}F \).}
4.2 Physical derivation on $\mathbb{RP}^2_b \times S^1$

In this subsection, we construct Abelian mirror symmetry on $\mathbb{RP}^2_b \times S^1$ from physical point of view. Before proceeding to details of mirror symmetry, we will rewrite (3.26), (3.38), and (3.40) as more convenient forms. We now focus on the exponential part, called the plethystic exponential, of the one-loop determinant of the vector multiplet (3.26). It can be rewritten as follows. We use a geometric series for the one-particle index (3.27) and perform the sum over $m$, then the plethystic exponential becomes

$$
\exp \left( \sum_{m \geq 1} \frac{1}{m} f_{\text{vector}}(x^m) \right) = \exp \left( \sum_{k \geq 0} \left\{ \log(1 - x^4 x^{4k}) - \log(1 - x^2 x^{4k}) \right\} \right)
= \prod_{k \geq 0} \frac{(1 - x^4 x^{4k})}{(1 - x^2 x^{4k})}
= \frac{(x^4; x^4)_\infty}{(x^2; x^4)_\infty}, \quad (4.2)
$$

where we use the $q$-shifted factorial defined by [24]

$$(z; q)_n := \begin{cases} 
1 & \text{for } n = 0, \\
\prod_{k=0}^{n-1} (1 - zq^k) & \text{for } n \geq 1, \\
\prod_{k=1}^{-n} (1 - zq^{-k})^{-1} & \text{for } n \leq -1,
\end{cases} \quad (4.3)
$$

with $0 < |q| < 1$, and $(z; q)_\infty := \lim_{n \to \infty} (z; q)_n$. For simplicity, we will use the notation

$$(z_1, z_2, \cdots, z_r; q)_\infty := (z_1; q)_\infty (z_2; q)_\infty \cdots (z_r; q)_\infty. \quad (4.4)$$

The plethystic exponential of (3.38) and (3.40) can be written in the same manner with the $q$-shifted factorial,

$$
\exp \left( \sum_{m \geq 1} \frac{1}{m} f_{\text{matter}}^+(z^m, x^m, \alpha f^m) \right) = \frac{(z^{-q\alpha - f_x x^{2 - \Delta}}; x^4)_\infty}{(z^{+q\alpha + f_x x^2 + \Delta}; x^4)_\infty}, \quad (4.5)
$$
$$
\exp \left( \sum_{m \geq 1} \frac{1}{m} f_{\text{matter}}^-(z^m, x^m, \alpha f^m) \right) = \frac{(z^{-q\alpha - f_x x^{4 - \Delta}}; x^4)_\infty}{(z^{+q\alpha + f_x x^{2 + \Delta}}; x^4)_\infty}, \quad (4.6)
$$

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where \( z := e^{i\theta} \). Combining the Casimir energy (B.6), (B.8), and (B.10) together, we have the following one-loop determinants for each multiplet:

\[
Z_{\text{vector}}^{1-\text{loop}} = x^{+\frac{1}{4}} \frac{(x^4; x^4)_\infty}{(x^2; x^4)_\infty},
\]

\[
Z_{\text{matter}^+}^{1-\text{loop}} = x^{-\frac{\Delta-1}{4}} z^{+\frac{1}{4} q^2} \alpha^{+\frac{1}{4} f} \frac{(z^{-q_\alpha f} x^{(2-\Delta)}; x^4)_\infty}{(z^{q_\alpha f} x^{2-\Delta}; x^4)_\infty},
\]

\[
Z_{\text{matter}^-}^{1-\text{loop}} = x^{-\frac{\Delta-1}{4}} z^{-\frac{1}{4} q^2} \alpha^{-\frac{1}{4} f} \frac{(z^{-q_\alpha f} x^{(4-\Delta)}; x^4)_\infty}{(z^{q_\alpha f} x^{2+\Delta}; x^4)_\infty}.
\]

Moreover, to make the SCIs easy to deal with, we introduce new variables,

\[
q = x^2, \quad a = \alpha^{-2f} x^{2(1-\Delta)}.
\]

Here, we should note that \( U(1)_J \) in the SQED and \( U(1)_V \) in the XYZ model on \( \mathbb{RP}_b^2 \times S^1 \) cannot be turned on. This is because, for \( U(1)_J \), a BF term is parity odd under the B-parity condition as well as the Chern-Simons term (see (2.26) and (2.33)). On the dual side, since \( \sigma \) receives the change of the sign by the antipodal identification (2.27), \( X \) and \( Y \) seem to be interchanged each other from the mirror map (the third line of Table 4). However, this violates \( U(1)_V \) because \( X \) has its charge opposite to that of \( Y \). This is why there do not exist variables in the SCIs parametrizing \( U(1)_J \) and \( U(1)_V \) in the later argument.

**SQED**

As explained in the previous section, there is the contribution of the vector multiplet even for \( U(1) \) gauge group. For the matter multiplet, the SCI should be the sum of the even and odd holonomy sector as described in (3.13). Precisely, the SCI for the SQED \( \mathcal{I}_{\text{SQED}} \) is written by

\[
\mathcal{I}_{\text{SQED}}(x, \alpha) = Z_{\text{vector}}^{1-\text{loop}} \left\{ \oint \frac{dz}{2\pi i z} Z_{\text{vector}}^{Q^+} Z_{\text{vector}}^{\tilde{Q}^+} + \oint \frac{dz}{2\pi i z} Z_{\text{vector}}^{Q^-} Z_{\text{vector}}^{\tilde{Q}^-} \right\}
\]

\[
= q^{+\frac{1}{8}} \left( \frac{q^2; q^2)_\infty}{(q; q^2)_\infty} \right) \left\{ \oint \frac{dz}{2\pi i z} a^{+\frac{1}{4}} \frac{(z^{-1} a^{-\frac{1}{2}} q^2, z^{+1} a^{+\frac{1}{2}} q^2; q^2)_\infty}{(z^{+1} a^{-\frac{1}{2}} q^2, z^{-1} a^{+\frac{1}{2}} q^2; q^2)_\infty} \right\}
\]

\[
+ \oint \frac{dz}{2\pi i z} a^{+\frac{1}{4}} \frac{(z^{-1} a^{-\frac{1}{2}} q^2, z^{+1} a^{+\frac{1}{2}} q^2; q^2)_\infty}{(z^{+1} a^{-\frac{1}{2}} q^2, z^{-1} a^{+\frac{1}{2}} q^2; q^2)_\infty} \right\}.
\]

where letter superscripts of one-loop determinants represent corresponding field contents, and we used the precise values of charges of each field as shown in the previous subsection. We follow the way of [19, 20] to perform the above integrals. Firstly, we handle the first integral in (4.9). There are many single poles coming from the origin and the \( q \)-shifted
factorial. Those poles can be separated into the set inside and outside the unit circle. We set $|q| < 1$ for the convergence of the $q$-shifted factorial and assume $|a^{-\frac{1}{2}}q^{\frac{1}{2}}| < 1$. Then the poles we should take into account are the ones inside the unit circle,

$$z = a^{-\frac{1}{2}}q^{\frac{1}{2}+2j}, \quad j = 0, 1, 2, \cdots. \quad (4.10)$$

We can relax the assumption by analytic continuation after obtaining the final result. At the moment, we ignore the contribution of the origin as we will mention later. The integral over $z$ with these assumptions gives the sum over residues from (4.10) as

$$a^{-\frac{1}{4}} \sum_{j \geq 0} \frac{(aq^{-2j}, q^{1+2j}; q^2)_\infty}{(a^{-1}q^{1+2j}, q^2; q^2)_\infty} \frac{1}{(q^{-2j}; q^2)_j}. \quad (4.11)$$

We also rewrite the sum over $j$ as follows. The dummy index $j$ in arguments of the $q$-shifted factorial can be subtracted outside such as

$$(aq^{-2j}; q^2)_\infty = (-1)^ja^j q^{-j(j+1)}(a^{-1}q^2; q^2)_j(a; q^2)_\infty, \quad (4.12)$$

$$(q^{1+2j}; q^2)_\infty = \frac{(q; q^2)_\infty}{(q; q^2)_j}. \quad (4.13)$$

With above expressions, (4.11) reduces to

$$a^{-\frac{1}{4}} \frac{(a, q; q^2)_\infty}{(a^{-1}q, q^2; q^2)_\infty} \sum_{j \geq 0} \frac{\frac{(a^{-1}q, a^{-1}q; q^2)_j}{(q; q^2)_j}}{\frac{(q; q^2)_j}{(q^2; q^2)_j}} a^j \quad \text{to} \quad \frac{(a, a^{-1}q^2; q^2)_\infty}{(a^{-1}q, q^2; q^2)_\infty} 2\varphi_1 \left( a^{-1}q^2, a^{-1}q; q^2, a \right), \quad (4.14)$$

where we use the basic hypergeometric series defined by [24]

$$r\varphi_s (\alpha_1, \alpha_2, \cdots, \alpha_r; \beta_1, \cdots, \beta_s; q, z) = \sum_{j \geq 0} \frac{(\alpha_1, \alpha_2, \cdots, \alpha_r; q)_j}{(\beta_1, \cdots, \beta_s; q)_j} \frac{z^j}{(q; q)_j} \left\{ (-1)^j q^{\frac{1}{2}j(j-1)} \right\}^{1+s-r}. \quad (4.15)$$

The convergence radius of the basic hypergeometric series is $\infty$, 1, or 0 for $r - s < 1$, $r - s = 1$, or $r - s > 1$, respectively. Secondly, we proceed the same way for the second integral in (4.9). We pick up the poles inside the unit circle

$$z = a^{-\frac{1}{2}}q^{\frac{1}{2}+2j}, \quad j = 0, 1, 2, \cdots, \quad (4.16)$$

and then the sum over residues in terms of the basic hypergeometric series becomes

$$a^{+\frac{1}{4}} \frac{(a, q^3; q^2)_\infty}{(a^{-1}q^3, q^2; q^2)_\infty} 2\varphi_1 \left( a^{-1}q^3, a^{-1}q^3; q^3; q^2, a \right). \quad (4.17)$$
Thus, (4.9) results in
\[
\mathcal{I}_{\text{SQED}}(x, \alpha) = q^\frac{1}{2} \frac{\langle q^2, q^2 \rangle_{\infty}}{\langle q, q^2 \rangle_{\infty}} \left\{ a - \frac{1}{4} \left( \frac{a, q; q^2}{a^{-1}, q, q^2} \right)_{\infty} \, 2 \varphi_1 \left( a^{-1}, q^2, a^{-1}, q; q^2, a \right) + a^\frac{1}{2} \frac{\langle a, q^3, q, q^2 \rangle_{\infty}}{\langle a^{-1}, q^3, q^2, q^2 \rangle_{\infty}} \, 2 \varphi_1 \left( a^{-1}, q^2, a^{-1}, q^3; q^2, a \right) \right\}. \quad (4.18)
\]

In terms of original variables, (4.18) is given by
\[
\mathcal{I}_{\text{SQED}}(x, \alpha) = x^\frac{1}{2} \frac{\langle x^4, x^4 \rangle_{\infty}}{\langle x^2, x^4 \rangle_{\infty}} \times \left\{ x^{-1} \frac{1}{2} \alpha + \frac{1}{2} \left( \frac{\alpha^{-2}, x^{2(1-\Delta)}}, x^2, x^4 \right)_{\infty} \, 2 \varphi_1 \left( \alpha^{+2}, x^{2(\Delta+1)}, x^{-2}, x^2, x^4, \alpha^{-2}, x^{2(1-\Delta)} \right) + x^{-1} \frac{1}{2} \alpha - \frac{1}{2} \left( \frac{\alpha^{-2}, x^{2(1-\Delta)}}, x^2, x^4 \right)_{\infty} \, 2 \varphi_1 \left( \alpha^{+2}, x^{2(\Delta+1)}, x^{-2}, x^2, x^4, \alpha^{-2}, x^{2(1-\Delta)} \right) \right\}. \quad (4.19)
\]

**XYZ model** We must determine the suitable B-parity condition for three chiral fields to obtain the correct result as well as for the SQED. As described above, $X$ turns into $Y$ under the antipodal identification, and vice versa. We assume that this observation is also hold for the quantum fluctuations of the XYZ model. Then we set the B-parity condition for these fields as
\[
\begin{align*}
X(\pi - \vartheta, \pi + \varphi, y) &= Y(\vartheta, \varphi, y), \\
Y(\pi - \vartheta, \pi + \varphi, y) &= X(\vartheta, \varphi, y), \\
Z(\pi - \vartheta, \pi + \varphi, y) &= Z(\vartheta, \varphi, y).
\end{align*}
\quad (4.20)
\]

This is the same situation as (2.35), namely, we should consider that $X$ and $Y$ join into a single field on $S^2_b \times S^1$. This means that the contribution of $X$ and $Y$ provide that of a field on $S^2_b \times S^1$ with the R-charge $-(1-\Delta)$ \[20\],
\[
\frac{\langle \tilde{a}^{-1}, x^{2(1-\Delta)}; x^2 \rangle_{\infty}}{\langle \tilde{a}^{+1}, x^{2(1-\Delta)}; x^2 \rangle_{\infty}} = \frac{\langle \tilde{a}^{-\frac{1}{2}}, q^2; q \rangle_{\infty}}{\langle \tilde{a}^{+\frac{1}{2}}, q^2; q \rangle_{\infty}}. \quad (4.21)
\]

where $\tilde{a} = \tilde{a}^{+2} x^{2(1-\Delta)}$. On the other hand, because $Z$ is a scalar invariant under the antipodal identification, the contribution of $Z$ corresponds to that of the even holonomy sector in the matter multiplet with the R-charge $-2\Delta$,
\[
q^{\frac{1}{2}} a^{-\frac{1}{2}} \frac{\langle \tilde{a}; q^2 \rangle_{\infty}}{\langle \tilde{a}^{-1}, q, q^2 \rangle_{\infty}}. \quad (4.22)
\]

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Again, we note that our R-charges of fields are opposite to these in [14, 19, 20, 21]. Bringing all together, the SCI for the XYZ model results in

$$I_{XYZ}(x, \tilde{\alpha}) = q^{\frac{1}{2} a} \prod_{\frac{1}{2}} \left( \frac{(a^{-1}; q)^\infty}{(a^{-1}; q^2)^\infty} \right).$$ (4.23)

Equivalently, (4.23) with original variables is written by

$$I_{XYZ}(x, \tilde{\alpha}) = x^{2\Delta} \prod_{\frac{1}{2}} \left( \frac{(\tilde{\alpha}^{-1} x^{(1+\Delta)}; x^2)^\infty}{(\tilde{\alpha}^{-1} x^{(1-\Delta)}; x^2)^\infty} \right).$$ (4.24)

In terms of the SCI, the usual mirror map for a flavor symmetry is realized by the identification $\alpha \sim \tilde{\alpha}^{-1}$, or equivalently, $a \sim \tilde{a}$ in our notation. Accordingly, we declare $N = 2$ Abelian mirror symmetry on $\mathbb{RP}^2_b \times S^1$ as the equality

$$I_{SQED}(x, \alpha) = I_{XYZ}(x, \tilde{\alpha}^{-1}).$$ (4.25)

Note that (4.25) should be true with an arbitrary $\Delta$, whereas the R-charge in theories without anomalous dimensions must take the canonical value as mentioned in [14]. In the next subsection, we will show the mathematically rigorous proof of (4.25).

$N = 4$ mirror symmetry As explained above, we can obtain $N = 4$ mirror symmetry by introducing an adjoint chiral field $\tilde{Z}$. In the XYZ model, the fact that the superpotential $Z\tilde{Z}$ must be uncharged for a flavor symmetry and have the R-charge 2 determines the $U(1)_A$ charge and the R-charge of $\tilde{Z}$ to be $+2$ and $2(1+\Delta)$, respectively. For the SCIs, the effect of $\tilde{Z}$ is identical with moving the contribution of $Z$ (4.22) in the RHS of (4.25) to the LHS. Concretely, we have

$$2\varphi_1 \left( a^{-1} q^2, a^{-1} q; q, q^2, a \right) + \frac{1}{2} \frac{(a^{-1} q, q^3; q^2)^\infty}{(a^{-1} q^3, q; q^2)^\infty} 2\varphi_1 \left( a^{-1} q^3, a^{-1} q^2, q^3, q^2, a \right) = \frac{(a^{-1} q; q)^\infty}{(a^{1/2}; q)^\infty}. \quad (4.26)$$

One can easily conform the correctness of (4.26) because this emerges on the way of the proof in the next subsection.

Generalized index The generalized index is defined as the SCI with gauging flavor symmetries [20]. In our context, we introduce a background flat gauge field $B^\text{flavor}_{\text{flat}}$ as gauging a $U(1)$ flavor symmetry, and the parity conditions must be classified in terms of both the holonomy of the dynamical gauge field $A^\text{flat}$ and the holonomy of $B^\text{flavor}_{\text{flat}}$, that is,

$$e^{i \oint C (a A^\text{flat} + f B^\text{flavor}_{\text{flat}})} = \pm 1, \quad \text{ (4.27)}$$
as explained in Appendix A.2. Since our argument for theories with a single flavor does not change, our generalized indices is still (4.9) and (4.23), and mirror symmetry with gauging a flavor symmetry can also be concluded as (4.25). The generalized index carries much more important roles when one discusses the multi-flavor case of mirror symmetry.

4.3 Mathematical proof on $\mathbb{RP}_b^2 \times S^1$

In this subsection, we give the proof of our new relation (4.25). At first, we review the $q$-binomial theorem [24] derived mainly by Cauchy [28] and Heine [29],

$$1 \varphi_0(a; -; q, x) = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |x| < 1. \quad (4.28)$$

This formula is the $q$-analogue of the binomial theorem

$$2F_1(a, c; c; z) = \sum_{n \geq 0} \frac{(a)_n}{n!} z^n = (1 - z)^{-a}, \quad (4.29)$$

where $|z| < 1$. $(a)_n$ is the classical shifted factorial $(a)_n = a(a + 1) \ldots (a + n - 1)$, and $(a)_0 = 1$. We prove our new relation (4.25) by utilizing the $q$-binomial theorem. The starting point is the SCI for the XYZ model (4.23),

$$\mathcal{I}_{\text{XYZ}}(x, \tilde{a}) = q^{+\frac{1}{8}}a^{-1} \frac{(\tilde{a}^{-\frac{1}{2}}q; q)_\infty}{(\tilde{a}^{+\frac{1}{2}}; q)_\infty} \frac{(\tilde{a}; q^2)_\infty}{(\tilde{a}^{-1}q; q^2)_\infty}$$

$$= q^{+\frac{1}{8}}a^{-1} \frac{(\tilde{a}; q^2)_\infty}{(\tilde{a}^{-1}q; q^2)_\infty} 1 \varphi_0(\tilde{a}^{-1}q; -; q, \tilde{a}^{+\frac{1}{2}})$$

$$= q^{+\frac{1}{8}}a^{-1} \frac{(\tilde{a}; q^2)_\infty}{(\tilde{a}^{-1}q; q^2)_\infty} \sum_{n \geq 0} (\tilde{a}^{-1}q; q^2)_n (\tilde{a}^{+\frac{1}{2}})^n$$

$$= q^{+\frac{1}{8}}a^{+\frac{1}{2}} \frac{(\tilde{a}^{-1}q; q^2)_\infty}{(\tilde{a}^{+\frac{1}{2}}; q^2)_\infty} \left\{ \sum_{m \geq 0} \frac{(\tilde{a}^{-1}q; q)_2m}{(q; q)_2m} (\tilde{a}^{+\frac{1}{2}})^{2m} + \sum_{m \geq 0} (\tilde{a}^{-1}q; q^2)_2m+1 (q; q^2)_2m+1 (\tilde{a}^{+\frac{1}{2}})^{2m+1} \right\}. \quad (4.30)$$

We remark that there are the following relations:

$$(a; q)_{2m} = (a, aq; q^2)_m, \quad (a; q)_{2m+1} = (1 - a)(aq, aq^2; q^2)_m. \quad (4.31)$$

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We apply the relations (4.31) to (4.30),

\[
q^{+\frac{1}{2}}\tilde{a}^{-\frac{1}{2}} \frac{(\tilde{a}; q^2)_{\infty}}{(\tilde{a}^{-1}q; q^2)_{\infty}} \left\{ \sum_{m \geq 0} \frac{(\tilde{a}^{-1}q; q)_{2m}}{(q; q^2)_{2m}} (\tilde{a}^{+\frac{1}{2}})_{2m} + \sum_{m \geq 0} \frac{(\tilde{a}^{-1}q; q)_{2m+1}}{(q; q^2)_{2m+1}} (\tilde{a}^{+\frac{1}{2}})_{2m+1} \right\}
\]

\[= q^{+\frac{1}{2}}\tilde{a}^{-\frac{1}{2}} \frac{(\tilde{a}; q^2)_{\infty}}{(\tilde{a}^{-1}q; q^2)_{\infty}} \left\{ \sum_{m \geq 0} \frac{(\tilde{a}^{-1}q, \tilde{a}^{-1}q^2; q)_{m}}{(q; q^2)_m (q^2; q^2)_m} (\tilde{a})^m + \tilde{a}^{+\frac{1}{2}} \frac{1 - \tilde{a}^{-1}q}{1 - q} \sum_{m \geq 0} \frac{(\tilde{a}^{-1}q^2, \tilde{a}^{-1}q^3; q)_{m}}{(q^3; q^2)_m (q^2; q^2)_m} (\tilde{a})^m \right\}
\]

\[= q^{+\frac{1}{2}}\tilde{a}^{-\frac{1}{2}} \frac{(\tilde{a}; q^2)_{\infty}}{(\tilde{a}^{-1}q; q^2)_{\infty}} \left\{ \sum_{m \geq 0} \frac{(\tilde{a}^{-1}q, \tilde{a}^{-1}q^2; q^2; q^2, \tilde{a})}{(q; q^2)_m (q^2; q^2)_m} (\tilde{a})^m + \tilde{a}^{+\frac{1}{2}} \frac{1 - \tilde{a}^{-1}q}{1 - q} \sum_{m \geq 0} \frac{(\tilde{a}^{-1}q^2, \tilde{a}^{-1}q^3; q^2; q^2, \tilde{a})}{(q^3; q^2)_m (q^2; q^2)_m} (\tilde{a})^m \right\}.
\]

\[\text{(4.32)}
\]

The part \((1 - \tilde{a}^{-1}q)/(1 - q)\) can be rewritten as

\[
\frac{1 - \tilde{a}^{-1}q}{1 - q} = \frac{(\tilde{a}^{-1}q; q^2)_{\infty}}{(\tilde{a}^{-1}q^3; q^2)_{\infty}} (q^3; q^2)_{\infty}.
\]

\[\text{(4.33)}
\]

Combining the relation (4.32) and (4.33), we have

\[
q^{+\frac{1}{2}}\tilde{a}^{-\frac{1}{2}} \frac{(\tilde{a}; q^2)_{\infty}}{(\tilde{a}^{-1}q; q^2)_{\infty}} \left\{ 2 \varphi_1(\tilde{a}^{-1}q, \tilde{a}^{-1}q^2; q^2, \tilde{a})
\right.
\]

\[+ \tilde{a}^{+\frac{1}{2}} \frac{1 - \tilde{a}^{-1}q}{1 - q} 2 \varphi_1(\tilde{a}^{-1}q^2, \tilde{a}^{-1}q^3; q^2, \tilde{a}) \right\}
\]

\[= q^{+\frac{1}{2}} \frac{(\tilde{a}; q^2)_{\infty}}{(\tilde{a}^{-1}q; q^2)_{\infty}} \left\{ \tilde{a}^{+\frac{1}{2}} \varphi_1(\tilde{a}^{-1}q^2, \tilde{a}^{-1}q^3; q^2, \tilde{a})
\right.
\]

\[+ \tilde{a}^{+\frac{1}{2}} \frac{1 - \tilde{a}^{-1}q}{1 - q} 2 \varphi_1(\tilde{a}^{-1}q^2, \tilde{a}^{-1}q^3; q^2, \tilde{a}) \right\}
\]

\[= q^{+\frac{1}{2}} \frac{(q^3; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left\{ \tilde{a}^{+\frac{1}{2}} \frac{(\tilde{a}, q; q^2)_{\infty}}{(\tilde{a}^{-1}q, q^2; q^2)_{\infty}} 2 \varphi_1(\tilde{a}^{-1}q^2, \tilde{a}^{-1}q^3; q^2, \tilde{a})
\right.
\]

\[+ \tilde{a}^{+\frac{1}{2}} \frac{(\tilde{a}, q^3; q^2)_{\infty}}{(\tilde{a}^{-1}q^3, q^2; q^2)_{\infty}} 2 \varphi_1(\tilde{a}^{-1}q^2, \tilde{a}^{-1}q^3; q^2, \tilde{a}) \right\}.
\]

\[\text{(4.34)}
\]

Therefore, we obtain the conclusion

\[
q^{+\frac{1}{2}}\tilde{a}^{-\frac{1}{2}} \frac{(\tilde{a}^{-1}q; q)_{\infty}}{(\tilde{a}^{-1}q^2; q^2)_{\infty}} = q^{+\frac{1}{2}} \frac{(q^3; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left\{ \tilde{a}^{+\frac{1}{2}} \frac{(\tilde{a}, q; q^2)_{\infty}}{(\tilde{a}^{-1}q, q^2; q^2)_{\infty}} 2 \varphi_1(\tilde{a}^{-1}q^2, \tilde{a}^{-1}q^3; q^2, \tilde{a})
\right.
\]

\[+ \tilde{a}^{+\frac{1}{2}} \frac{(\tilde{a}, q^3; q^2)_{\infty}}{(\tilde{a}^{-1}q^3, q^2; q^2)_{\infty}} 2 \varphi_1(\tilde{a}^{-1}q^2, \tilde{a}^{-1}q^3; q^2, \tilde{a}) \right\}.
\]

\[\text{(4.35)}
\]

5 Discussions

We presented how to define \(\mathcal{N} = 2\) supersymmetric gauge theories on \(\mathbb{R}^2 \times S^1\) and got the exact form of the superconformal index with arbitrary number of vector multiplets.
and matter multiplets with U(1) gauge symmetry. As commented, the results are not dependent on \( l \) and \( \tilde{l} \), that is, the squashing parameter \( b \). This fact is expected because it is verified in 2d case \([23, 22]\). Also, we gave the exact check of \( \mathcal{N} = 2 \) and 4 Abelian mirror symmetry with the simplest case \( N_f = 1 \) on \( \mathbb{RP}^2_b \times S^1 \) by using the \( q \)-binomial theorem essentially. In this section, we would like to comment on some open questions and future directions briefly.

**Open questions and future directions** The first question is related to subtleties in our computation of the superconformal index. We used an ad-hoc way to regulate the Casimir energy presented in \([14, 30]\) (see Appendix A and B). They showed that the precise Chern-Simons level shift on \( S^2 \times S^1 \) emerges within this regularization scheme. However, as noted in Section 2, we cannot take the Chern-Simons term into account. Therefore, we cannot adopt the level shift as the guiding principle of the regularization. Furthermore, we ignored the pole coming from the origin in performing the integral (4.9) as mentioned in Section ???. Unlike the case on \( S^3 \) \([19, 20]\), we cannot simply take the path avoiding the origin. We do not know why our regularization of the Casimir energy works so well and why the contribution from the pole of the origin in the integral can be excluded to realize mirror symmetry on \( \mathbb{RP}^2 \times S^1 \) successfully. It is interesting to find more fundamental treatment to resolve them. As the second one, we would like to know the origin of our B-parity condition on the XYZ model side. We took a little bit ad-hoc way to determine it based on the correspondence between the moduli spaces. One straightforward way to solve this problem is using the brane construction of mirror symmetry \([31]\). We expect that the generalized mathematical formulas will emerge if this program is accomplished. The third question is related to the so-called “factorization” property of 3d exact results \([32, 33, 34, 35]\). The partition functions on \( S^3_b \) and the superconformal indices on \( S^2 \times S^1 \) can be decomposed into the product of more fundamental quantities called holomorphic blocks. This property of both cases naively comes from the fact that each curved space are characterized by solid-torus decomposition. However, \( \mathbb{RP}^2 \times S^1 \) cannot be expressed by using solid-torus decomposition. Instead of it, one can get \( \mathbb{RP}^2 \times S^1 \) by gluing the surface of solid-torus in an appropriate manner. One may find the unexpected description of our results in terms of holomorphic blocks via this method. The final comment is concerned with extension of our arguments. There are obvious open problems; we did not perform the check of Abelian mirror symmetry with general \( N_f \) flavors. Also, we did not present the generalization of the exact calculation with non-Abelian gauge symmetry in this note because there is a slight difficulty in the gauge fixing procedure. We hope to complete these
problems in the near future. Moreover, we found generalized mirror symmetry equalities in Appendix C. In Appendix C.1 we provided the generalized equality with the parameter $\lambda$ and its proof in terms of the $q$-binomial theorem. In Appendix C.2 we showed another relation derived by the properties of the theta function of Jacobi. The idea of the proof comes from connection problems on linear $q$-difference equations [36, 37]. The generalized relation also gives the connection formula for the $\varphi_0(\lambda; -; q, z)$-type equation between the solutions of the linear $q$-difference equations around the origin and around infinity. The important point is that we obtain the same relation (4.25) as the special case even though these relations in the subsections are essentially different each other. These formulas suggest the possibility to add one more parameter to our system, and its physical meaning may be found in the brane construction. If these are derived from string theory generally, our mathematical conclusion will give us new physical viewpoints.

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A Calculation details

In this appendix, we show the details of calculations for one-loop determinants. Our method discussed below is similar to the one discussed in [23, 22]. Their way did not respect the symmetry generated by $j_3$, whereas we derive (3.26) with preserving $j_3$ structure explicitly because it has an important meaning in our SCI (3.1). In the later discussions, we get the following type of infinite product in each final step:

$$\prod_{n \in \mathbb{Z}} \prod_{k \geq 0} \frac{2\pi i n + 2z_f(k)}{2\pi i n + 2z_b(k)},$$

(A.1)

where $z_{f/b}(k)$ represent certain $k$ dependent functions. By using the infinite product formula of sinh $z$, we can deform it to

$$\prod_{k \geq 0} \frac{2 \sinh z_f(k)}{2 \sinh z_b(k)} = \left( \prod_{k \geq 0} e^{z_f(k) - z_b(k)} \right) \times \exp \left( \sum_{m=1}^{\infty} \frac{-1}{m} \sum_{k \geq 0} (e^{-2mz_f(k)} - e^{-2mz_b(k)}) \right).$$

(A.2)
We call the first part in (A.2) the Casimir energy which must be regularized (see Appendix B for our regularization scheme) and the second part,

$$- \sum_{k \geq 0} \left( e^{-2z_f(k)} - e^{-2z_b(k)} \right) := f(x, \ldots),$$  

(A.3)

the one-particle index. As one can verify later, both the Casimir energy and the one-particle index do not depend on $x'$. In this appendix, we use 2d Killing spinors

$$\varepsilon(\vartheta, \varphi) = e^{\frac{i}{2} \varphi} \begin{pmatrix} \cos \frac{\varphi}{2} \\ \sin \frac{\varphi}{2} \end{pmatrix}, \quad \overline{\varepsilon}(\vartheta, \varphi) = e^{-\frac{i}{2} \varphi} \begin{pmatrix} \sin \frac{\varphi}{2} \\ \cos \frac{\varphi}{2} \end{pmatrix},$$  

(A.4)

which satisfy

$$D_i \varepsilon = \frac{1}{f} \gamma_i \gamma_3 \varepsilon, \quad D_i \overline{\varepsilon} = -\frac{1}{f} \gamma_i \gamma_3 \overline{\varepsilon},$$  

(A.5)

where $i$ runs for $\vartheta, \varphi$. We must consider the B-parity condition in order to get the index on $\mathbb{RP}^2_b \times S^1$. We can, of course, ignore the B-parity condition and then get the index on $S^2_b \times S^1$. However, as noted in the beginning of Section 2, the results do not depend on the squashing parameter. Consequently, the results without the B-parity condition reproduce the known results on $S^2 \times S^1$ [13, 3, 14].

**A.1 Vector multiplet**

**Gauge fixing** By repeating the same argument for the “shortcut” way of the gauge fixing [5, 23, 22], we can restrict the path integral onto the configuration satisfying

$$A^{(n)}_3 = 0, \quad *_{2d} *_{2} A^{(n)} = 0$$  

(A.6)

(A.7)

for all $n$ without any Fadeev-Popov determinant. Then we need to consider the operator’s determinants

$$\prod_{n \in \mathbb{Z}} \frac{\det \Delta_f^{(n)}}{\sqrt{\det \Delta_b^{(n)}}},$$  

(A.8)

where

$$\Delta_b^{(n)} = \begin{pmatrix} -(\ast_{2d})^2 + h_n^2 & -\ast_{2d} \frac{d}{f} \\ \frac{1}{f} \ast_{2d} \frac{d}{f} & -(\ast_{2d})^2 + \frac{1}{f^2} + h_n^2 \end{pmatrix},$$

$$\Delta_f^{(n)} = i \gamma^i D_i + \gamma_3 (h_n + \frac{i}{2l} \Omega),$$  

(A.9)

(A.10)
In addition, we can make this problem simpler by notifying
\[
\det \delta^{(n)}_b = \sqrt{\det \Delta^{(n)}_b}
\] (A.11)
up to the sign where
\[
\delta^{(n)}_b = \left( \begin{array}{cc}
h_n & -*2d \\
*2d & \frac{1}{2} + i h_n \end{array} \right).
\] (A.12)
The one-loop determinant which we should know is
\[
Z_{\text{vector}}^{1-\text{loop}} = \prod_{n \in \mathbb{Z}} \frac{\det \gamma_3 \Delta^{(n)}_f}{\det \delta^{(n)}_b}.
\] (A.13)
As one can see, the contribution of the U(1) vector multiplet already does not have the dependence on the holonomy. Therefore, we omit the superscript ± from now on.

**Pairing structure** The calculation is based on the eigenvalues pairing structure as follows. Let \((A, \sigma)^T\) and \(\lambda\) be the eigenmodes:
\[
\delta^{(n)}_b \begin{pmatrix} A \\ \sigma \end{pmatrix} = -iM \begin{pmatrix} A \\ \sigma \end{pmatrix},
\] (A.14)
\[
\Delta^{(n)}_f \lambda = -M \lambda,
\] (A.15)
then we can map the one side to the other by defining
\[
\Lambda := \left( \begin{array}{c}
\gamma_3 \gamma_i A_i + i \sigma \gamma_3 \\
\end{array} \right) \epsilon,
\] (A.16)
\[
\begin{pmatrix} \Sigma \\ \Omega \end{pmatrix} := \left( \begin{array}{c}
-i(M + h_n) \epsilon \gamma_i \lambda e^{i} - d(\epsilon \gamma_3 \lambda) \\
(M + h_n) \epsilon \lambda
\end{array} \right).
\] (A.17)
The modes which have no pair only contribute to the one-loop determinant (A.13). In other words, we have to find the eigenvalues constrained by the following conditions:
\[
M = M_b \text{ which satisfy } (A.14) \text{ and } (A.16) = 0.
\] (A.18)
\[
M = M_f \text{ which satisfy } (A.15) \text{ and } (A.17) = 0.
\] (A.19)
The constraint (A.16) = 0 and (A.17) = 0 are solved by taking
\[
\begin{pmatrix} A \\ \sigma \end{pmatrix} = e^{i\frac{1}{2} \varphi} h_b(\vartheta) \left( e^i + i \cos \vartheta e^2 \right),
\] (A.20)
\[
\lambda = (M_f + h_n + \frac{i}{2\Omega}) e^{i\frac{1}{2} \varphi} h_f(\vartheta) \epsilon,
\] (A.21)
\footnote{The insertion of \(\gamma_3\) in the numerator does not spoil the validity and make the problem simple \cite{23, 22}.}
where $j_{3}^{b/f} \in \mathbb{Z}$. Substituting these representations into (A.14) and (A.15), we get the following sets of equations:

\[
\begin{align*}
\begin{cases}
\frac{1}{f(\vartheta)} \frac{\partial}{\partial \vartheta} h_b(\vartheta) + \frac{\cos \vartheta}{\sin \vartheta} \left( \frac{1}{f(\vartheta)} - \frac{j_{3}^{b}}{l} \right) h_b(\vartheta) = 0, \\
M_b = i \left( (\Omega - 1)j_{3}^{b} - iln \right) ,
\end{cases}
\end{align*}
\]

(A.22)

\[
\begin{align*}
\begin{cases}
\frac{1}{f(\vartheta)} \frac{\partial}{\partial \vartheta} h_f(\vartheta) + \frac{\cos \vartheta}{\sin \vartheta} \left( \frac{1}{f(\vartheta)} + \frac{j_{3}^{f}}{l} - \frac{1}{l} \right) h_f(\vartheta) = 0, \\
M_f = i \left( (\Omega - 1)(j_{3}^{f} - 1) - iln \right) .
\end{cases}
\end{align*}
\]

(A.23)

One can get the conditions of $j_{3}$ as $j_{3}^{b} \geq 1$ for bosons and $j_{3}^{f} \leq 0$ for fermions because, around $\vartheta \sim 0$, one can easily solve the equation for $h_b(\vartheta)$ and $h_f(\vartheta)$ in (A.22) and (A.23), respectively, as

\[
h_b(\vartheta) \sim \sin^{(j_{3}^{b} - 1)} \vartheta , \tag{A.24}
\]

\[
h_f(\vartheta) \sim \sin^{-j_{3}^{f}} \vartheta . \tag{A.25}
\]

Note that the coefficients of differential equations with respect to $\vartheta$ in (A.22) and (A.23) are invariant under the antipodal identification (1.2), that is,

\[
h_{b/f}(\pi - \vartheta) = h_{b/f}(\vartheta) . \tag{A.26}
\]

**B-parity condition** Usually, $j_{3}$ takes an arbitrary value in integers $\mathbb{Z}$. Therefore, one may think that $j_{3}^{b} = 1, 2, 3, \ldots$ and $j_{3}^{f} = 0, -1, -2, \ldots$, however it is not in our case because of the B-parity condition. We can determine the possible values for $j_{3}^{b/f}$ from the explicit forms of the eigenmode (A.20) and (A.21), the invariance of $h_{b/f}$ (A.26), and the B-parity condition (2.27). Combining these arguments, one can get the following condition

\[
e^{ij_{3}^{b/f} \pi} = -1 \tag{A.27}
\]

This means that we have

\[
\begin{align*}
\begin{cases}
M_b = \frac{i}{l} \left( (\Omega - 1)(2k + 1) - iln \right) , \\
M_f = \frac{i}{l} \left( - (\Omega - 1)(2k + 2) - iln \right) .
\end{cases}
\end{align*}
\]

(A.28)

where $k = 0, 1, 2, \ldots$, and $n \in \mathbb{Z}$. 28
One-loop determinant  We can get the explicit form of (A.13) just by substituting all relevant eigenvalues (A.28) into it:

\[
(A.13) = \prod_{M} \frac{M_f}{M_b} = \prod_{n \in \mathbb{Z}} \left( \prod_{k \geq 0} \frac{(1 - \Omega)(2k + 2) - iln}{(\Omega - 1)(2k + 1) - iln} \right)
\]

\[
\sim \prod_{n \in \mathbb{Z}} \left( \prod_{k \geq 0} \frac{(1 - \Omega)(2k + 2) + iln}{(1 - \Omega)(2k + 1) + iln} \right),
\]

(A.29)

where $\sim$ represents the equality up to the sign. This regularization is guaranteed in the 2d case [22]. From the above expression, substituting

\[
2z_f(k) = 2\beta_2(2k + 2), \quad 2z_b(k) = 2\beta_2(2k + 1)
\]

(A.30)

into (A.2), we can get (3.26) and (3.27). The Casimir energy can be regularized by using the zeta function regularization formula (B.4) explained in Appendix B.

A.2 Matter multiplet

There is, of course, no need of fixing gauge for the matter multiplet. Therefore, we start with the pairing structure of (3.35) and (3.36). To make our argument comprehensive, we define the differential operator $\Delta^{(n)}_{\phi}$ and $\Delta^{(n)}_{\psi}$ acting on $\phi^{(n)}$ and $\psi^{(n)}$, respectively, as

\[
\Delta^{(n)}_{\phi} = -g^{ij}D_i^{A_{flat}}D_j^{A_{flat}} + (p_n - i\frac{\Delta}{2\Omega})^2 + \frac{\Delta^2 - 2\Delta}{4f^2} + \frac{\Delta}{4} R - \frac{\Delta - 1}{f} v^i D_i^{A_{flat}},
\]

(A.31)

\[
\Delta^{(n)}_{\psi} = -i\gamma^i D_i^{A_{flat}} - \gamma_3 (p_n - i\frac{\Delta - 1}{2\Omega}) - i\frac{1}{2f} \gamma_3 - i\frac{1}{2f} v^i \gamma_i - i\frac{\Delta - 1}{2f} \omega,
\]

(A.32)

where $D_i^{A_{flat}}$ is defined with a flat connection $A_{flat}$.

Pairing structure  Let $\phi$ and $\psi$ be the eigenmode for $\Delta^{(n)}_{\phi}$ and $\Delta^{(n)}_{\psi}$, i.e.

\[
\Delta^{(n)}_{\phi} \phi = -M \left( M - 2(p_n - i\frac{\Delta}{2\Omega}) \right) \phi, \quad (A.33)
\]

\[
\gamma_3 \Delta^{(n)}_{\psi} \psi = M \psi, \quad (A.34)
\]

then

\[
\begin{pmatrix}
\Psi_1 \\
\Psi_2
\end{pmatrix} = \begin{pmatrix}
\gamma_3 \varepsilon \phi \\
(i\gamma^i D_i^{A_{flat}} \phi + \gamma_3 \varepsilon (p_n - i\frac{\Delta}{2\Omega}) \phi)
\end{pmatrix},
\]

\[
\Phi = \varepsilon \psi,
\]

(A.35)

(A.36)
satisfy the equations

\[
\gamma_3 \Delta^{(n)}_{\psi} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} -2(p_n - i\frac{\Delta - 1}{2} \Omega) & 1 \\ M(M - 2(p_n - i\frac{\Delta - 1}{2} \Omega)) & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{A.37}
\]

\[
\Delta^{(n)}_{\psi} \Phi = -M \left( M - 2(p_n - i\frac{\Delta - 1}{2} \Omega) \right) \Phi. \tag{A.38}
\]

As discussed in \cite{5, 23, 22}, one can find the relevant spectra characterized by

\[
M = M_\phi \text{ which satisfy (A.33) and } \Psi_2 = M \Psi_1, \tag{A.39}
\]

\[
M = M_\psi \text{ which satisfy (A.34) and (A.36) = 0.} \tag{A.40}
\]

Then we take each relevant mode as

\[
\phi = e^{i f} \Phi^{A_{nat}} e^{i j_3^b \vartheta} h_b(\vartheta), \tag{A.41}
\]

\[
\psi = e^{i f} \Phi^{A_{nat}} e^{i j_3^f \vartheta} h_f(\vartheta), \tag{A.42}
\]

where \( j_3^b/f \in \mathbb{Z} \). Substituting these forms into \( \Psi_2 = M_\phi \Psi_1 \) and (A.36) = 0, we get the following sets of equations:

\[
\begin{cases}
\frac{1}{f} \partial_\vartheta h_b(\vartheta) + \frac{\cos \vartheta}{\sin \vartheta} \left( \frac{\Delta}{2} h_b(\vartheta) - \frac{1}{l} \left( j_3^b + \frac{\Delta}{2} \right) h_b(\vartheta) \right) = 0, \\
M_\phi l = \frac{1}{l} \left( (1 - \Omega)(j_3^b + \frac{\Delta}{2}) + i \frac{l}{2\pi} [2\pi n - q \vartheta - f \mu] \right), \\
\frac{1}{f} \partial_\vartheta h_f(\vartheta) - \frac{\cos \vartheta}{\sin \vartheta} \left( \frac{\Delta - 2}{2} h_f(\vartheta) - \frac{1}{l} \left( j_3^f + \frac{\Delta - 2}{2} \right) h_f(\vartheta) \right) = 0, \\
M_\psi l = \frac{1}{l} \left( (\Omega - 1)(j_3^f + \frac{\Delta - 2}{2}) - i \frac{l}{2\pi} [2\pi n - q \vartheta - f \mu] \right),
\end{cases} \tag{A.43, A.44}
\]

One can get also the conditions of \( j_3 \) as \( j_3^b \geq 0 \) for bosons and \( j_3^f \leq 0 \) for fermions because the behaviors of \( h_b/f(\vartheta) \) around \( \vartheta \sim 0 \) become

\[
h_b(\vartheta) \sim \sin^{j_3^b} \vartheta, \tag{A.45}
\]

\[
h_f(\vartheta) \sim \sin^{-j_3^f} \vartheta. \tag{A.46}
\]

Note that these functions, the equation (A.43), and (A.44) have the symmetry (A.26).

**B-parity condition** We have to limit \( j_3 \) to preserve the B-parity condition (2.31) as we have done in the vector multiplet, but an additional issue occurs because the matter is charged through \( q \). The permitted region depends on the B-parity choice \( \pm \) and the value of the holonomy,

\[
e^{i f} \Phi^{A_{nat}} e^{i j_3 \pi} = \pm 1. \tag{A.47}
\]
Here, it is found that the consistent two choices of the B-parity condition correspond to the background $U(1)_{\text{flavor}}$ holonomies

$$\pm 1 = e^{i f_i B_{\text{flat}}^{\text{flavor}}},$$

(A.48)

where $f$ is the appropriate flavor charge. Therefore, we can get

$$e^{ij_3 \pi} = e^{i f_i (q A_{\text{flat}} + f B_{\text{flat}}^{\text{flavor}})}.$$  

(A.49)

It means that we have

$$e^{i f_i (q A_{\text{flat}} + f B_{\text{flat}}^{\text{flavor}}) = +1} \Rightarrow \begin{cases} M_\phi l = i \left( (1 - \Omega)(2k + \frac{\Delta}{2}) + i \frac{l}{2\pi} [2\pi n - q \theta - f \mu] \right), \\ M_\psi l = i \left( (\Omega - 1)(-2k - 1 + \frac{\Delta}{2}) - i \frac{l}{2\pi} [2\pi n - q \theta - f \mu] \right), \end{cases}$$

(A.50)

$$e^{i f_i (q A_{\text{flat}} + f B_{\text{flat}}^{\text{flavor}}) = -1} \Rightarrow \begin{cases} M_\phi l = i \left( (1 - \Omega)(2k + 1 + \frac{\Delta}{2}) + i \frac{l}{2\pi} [2\pi n - q \theta - f \mu] \right), \\ M_\psi l = i \left( (\Omega - 1)(-2k - 2 + \frac{\Delta}{2}) - i \frac{l}{2\pi} [2\pi n - q \theta - f \mu] \right), \end{cases}$$

(A.51)

Therefore, the one-loop determinant changes its form depending on the value of the total holonomy $e^{i f_i (q A_{\text{flat}} + f B_{\text{flat}}^{\text{flavor}})}$.

**One-loop determinant** We can get each one-loop determinant by calculating

$$\prod_{\text{all}} \frac{M_\phi}{M_\psi}.$$  

(A.52)

We read the eigenvalues of each holonomy sector from (A.50) and (A.51), and the corresponding infinite products (A.52) are written as

$$e^{i f_i (q A_{\text{flat}} + f B_{\text{flat}}^{\text{flavor}}) = +1} \Rightarrow \prod_{n \in \mathbb{Z}, k \geq 0} \left( 1 - \Omega)(2k + \frac{\Delta}{2}) + i \frac{l}{2\pi} [2\pi n - q \theta - f \mu] \right),$$

(A.53)

$$e^{i f_i (q A_{\text{flat}} + f B_{\text{flat}}^{\text{flavor}}) = -1} \Rightarrow \prod_{n \in \mathbb{Z}, k \geq 0} \left( 1 - \Omega)(2k + 2 + \frac{\Delta}{2}) - i \frac{l}{2\pi} [2\pi n - q \theta - f \mu] \right).$$

(A.54)
After substituting

\[ e^{i \oint \gamma (q A + f B)} = +1 \]  

\[ (A.55) \]

\[ e^{i \oint \gamma (q A + f B)} = -1 \]  

\[ (A.56) \]

into (A.2) and regularizing the Casimir energies by using (B.4), we can get the results (3.38) - (3.41).

### B  Zeta function regularization

In general, an infinite product is not well-defined and must be regulated with an appropriate method. Here, we adopt the zeta function regularization given as [30]

\[ \prod_{k \geq 0} f(k) = \exp \left( \frac{d}{ds} \sum_{k \geq 0} f(k)^s \right) \bigg|_{s=0} . \]  

\[ (B.1) \]

We make use of (B.1) to regularize the Casimir energy of the vector multiplet and the matter multiplet. Those forms shown explicitly in (A.2) are generally written as the infinite product

\[ \prod_{k \geq 0} \left( \frac{x^{2k+C_1}}{x^{2k+C_2}} \right)^r , \]  

\[ (B.2) \]

where \( C_1, C_2, \) and \( r \) are constants independent of \( k \). Applying (B.1) to the above expression, we expand the numerator and the denominator around \( s = 0 \) such that

\[ \frac{d}{ds} \sum_{k \geq 0} (x^{2k+C})^s = \frac{1}{2s^2 \log x} + \frac{-2 + 6C - 3C^2}{12 \log x + O(s)} . \]  

\[ (B.3) \]

Although this form is obliviously diverge at \( s = 0 \), unwanted terms can be canceled by taking a ratio of such infinite products. Consequently, (B.2) with \( s \to 0 \) results in

\[ \prod_{k \geq 0} \left( \frac{x^{2k+C_1}}{x^{2k+C_2}} \right)^r = \exp \left( r \left( \frac{-2 + 6C_1 - 3C_1^2}{12} - \frac{-2 + 6C_2 - 3C_2^2}{12} \right) \log x \right) \]

\[ = x^{-\frac{r}{2}(C_1-C_2)(C_1+C_2-2)} . \]  

\[ (B.4) \]
It is straightforward to apply this formula to each Casimir energy. Firstly, for the vector multiplet, its $k$-dependent functions (A.30) correspond to setting

$$r = 1, \quad C_1 = 1, \quad C_2 = 2,$$

then we can obtain the Casimir energy as

$$x^{+\frac{1}{4}}. \tag{B.5}$$

Secondly, for the even holonomy sector, its $k$-dependent functions (A.55) correspond to setting

$$r = 1, \quad C_1 = \frac{\Delta}{2} + \Theta, \quad C_2 = 1 - \frac{\Delta}{2} - \Theta, \quad x^\Theta := (e^{+iq\theta_0 + f})^{\frac{1}{2}}, \tag{B.6}$$

then we can obtain the Casimir energy as

$$x^{+\frac{\Delta-1}{4}}e^{+\frac{i}{4}q\theta_0 + \frac{1}{4}f}. \tag{B.7}$$

Lastly, for the odd holonomy sector, its $k$-dependent functions (A.56) correspond to setting

$$r = 1, \quad C_1 = 1 + \frac{\Delta}{2} + \Theta, \quad C_2 = 2 - \frac{\Delta}{2} - \Theta,$$

then we can obtain the Casimir energy as

$$x^{-\frac{\Delta-1}{4}}e^{-\frac{i}{4}q\theta_0 + \frac{1}{4}f}. \tag{B.8}$$

**C Mathematical generalizations of (4.25)**

In this section, we consider mathematical generalizations of the relation (4.25). In Appendix C.1 we give (4.25) as the special case of the generalization via the $q$-binomial theorem. In Appendix C.2 we also give (4.25) by using the connection formula of $1\varphi_0(\lambda; -; q, z)$. We remark that these formulae in each subsection are completely different, but we can derive the relation (4.25) as their special case.

**C.1 From the $q$-binomial theorem**

First, we derive more general form of (4.25) from the $q$-binomial theorem and its *alternative representation*. The $q$-binomial theorem is

$$\frac{(\lambda z; q)_\infty}{(z; q)_\infty} = 1\varphi_0(\lambda; -; q, z), \quad \forall |z| < 1, |q| < 1,$$
and \( _1\varphi_0(\lambda; -; q, z) \) can be deformed from its definition as
\[
1\varphi_0(\lambda; -; q, z) = \sum_{n \geq 0} (\lambda; q)_n z^n
\]
\[
= \sum_{m \geq 0} \frac{(\lambda; q)_{2m} z^{2m}}{(q; q)_m} + \sum_{m \geq 0} (\lambda; q)_{2m+1} z^{2m+1}
\]
\[
= \sum_{m \geq 0} \frac{(\lambda, \lambda q; q^2)_m (z^2)^m}{(q; q)_m (q^2; q^2)_m} + \frac{1 - \lambda}{1 - q} z \sum_{m \geq 0} \frac{(\lambda q, \lambda q^2; q^2)_m (z^2)^m}{(q^2; q^2)_m (q^2; q^2)_m}
\]
\[
= 2\varphi_1(\lambda, \lambda q; q; q^2, z^2) + \frac{1 - \lambda}{1 - q} z \varphi_1(\lambda q, \lambda q^2; q^2, z^2)
\]
\[
= 2\varphi_1(\lambda, \lambda q; q; q^2, z^2) + \frac{(\lambda q; q^2)_{\infty}}{(\lambda q^2, q^2)_{\infty}} (q^2; q^2)_{\infty} \varphi_1(\lambda, \lambda q; q; q^2, z^2) + \frac{(\lambda q^3; q^2)_{\infty}}{(\lambda q^2, q^2)_{\infty}} \varphi_1(\lambda, \lambda q^2; q^2, z^2)
\]
\[
= \left\{ \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \varphi_1(\lambda, \lambda q; q; q^2, z^2) + \frac{(\lambda q^3; q^2)_{\infty}}{(\lambda q^2, q^2)_{\infty}} \varphi_1(\lambda, \lambda q^2; q^2, z^2) \right\}
\]
Therefore, we acquire the alternative representation of the \( q \)-binomial theorem
\[
1\varphi_0(\lambda; -; q, z) = \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left\{ \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \varphi_1(\lambda, \lambda q; q; q^2, z^2) + \frac{(\lambda q^3; q^2)_{\infty}}{(\lambda q^2, q^2)_{\infty}} \varphi_1(\lambda, \lambda q^2; q^2, z^2) \right\}
\]
We now define the weight function
\[
w(z, \lambda; q) := q^{\frac{1}{2}} z^{-\frac{1}{2}} \frac{(z^2; q^2)_{\infty}}{(\lambda; q^2)_{\infty}}
\]
to make the generalization of the relation (4.25) clear. Multiplying the weight function \( w(z, \lambda; q) \) by the alternative representation (C.1), we obtain
\[
w(z, \lambda; q) \frac{(\lambda z; q)_{\infty}}{(z; q)_{\infty}} = q^{\frac{1}{2}} z^{-\frac{1}{2}} \frac{(z^2; q^2)_{\infty}}{(\lambda; q^2)_{\infty}} \frac{(\lambda z; q)_{\infty}}{(z; q)_{\infty}}
\]
\[
= q^{\frac{1}{2}} z^{-\frac{1}{2}} \frac{(z^2; q^2)_{\infty}}{(\lambda; q^2)_{\infty}} \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \left\{ \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \varphi_1(\lambda, \lambda q; q; q^2, z^2) + \frac{(\lambda q^3; q^2)_{\infty}}{(\lambda q^2, q^2)_{\infty}} \varphi_1(\lambda, \lambda q^2; q^2, z^2) \right\}
\]
\[
= q^{\frac{1}{2}} \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \left\{ z^{-\frac{1}{2}} \frac{(z^2; q^2)_{\infty}}{(\lambda, q^2; q^2)_{\infty}} \varphi_1(\lambda, \lambda q; q; q^2, z^2) + z^{\frac{1}{2}} \frac{(z^2; q^2)_{\infty}}{(\lambda q^2, q^2; q^2)_{\infty}} \varphi_1(\lambda, \lambda q^2; q^2, z^2) \right\},
\]
namevally,
\[
w(z, \lambda; q) \frac{(\lambda z; q)_{\infty}}{(z; q)_{\infty}} = q^{\frac{1}{2}} \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \left\{ z^{-\frac{1}{2}} \frac{(z^2; q^2)_{\infty}}{(\lambda, q^2; q^2)_{\infty}} \varphi_1(\lambda, \lambda q; q; q^2, z^2) + z^{\frac{1}{2}} \frac{(z^2; q^2)_{\infty}}{(\lambda q^2, q^2; q^2)_{\infty}} \varphi_1(\lambda, \lambda q^2; q^2, z^2) \right\}.
\]
When we put $z \mapsto \tilde{a}^{\frac{1}{2}}$ and $\lambda \mapsto \tilde{a}^{-\frac{1}{2}} q$, i.e. $\lambda z \mapsto \tilde{a}^{-\frac{1}{2}} q$ in (C.3), we obtain the relation (4.25).

C.2 From the triple product identity of the theta function of Jacobi

Next, we prove the relation (4.25) in terms of the theta function of Jacobi. The idea of the proof comes from the connection problems on linear $q$-difference equations [37]. The local theory and irregularity for $q$-difference equations are studied by J.-P. Ramis, J. Sauloy, and C. Zhang [36] by the using of the Newton polygon. Recently, C. Zhang and T. Morita provided some connection formulae with the irregular singular case. In connection problems, we study the elliptic functions associated with relations between the local solutions around the origin and around infinity. In this subsection, we deal with the first order $q$-difference equation (see Remark 1 for details) We begin with the review of the theta function [37]. The theta function is given by

$$\theta(x) = \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n, \quad \forall x \in \mathbb{C}. $$

The theta function has the triple product identity

$$\theta(x) = \left(q, -x, -\frac{q}{x}; q \right)_\infty. $$

(C.4)

For any $k \in \mathbb{Z}$, the theta function satisfies the $q$-difference equation

$$\theta(q^k x) = q^\frac{k(k-1)}{2} x^{-k} \theta(x).$$

(C.5)

The theta function also has the inversion formula

$$\theta(1/x) = \theta(x)/x. $$

(C.6)

The function $\phi_0(\lambda; -; q, z)$ can be rewritten by using the theta function as

$$\phi_0(\lambda; -; q, z) = \frac{(\lambda z; q)_\infty}{(z; q)_\infty} \frac{\theta(-\lambda z)}{\theta(-z)} \frac{(q/z; q)_\infty}{(q/\lambda z; q)_\infty} = \frac{\theta(-\lambda z)}{\theta(-z)} \phi_0 \left(\lambda; -; q, \frac{q}{\lambda z} \right),$$

(C.7)

provided that $|z| < 1$.

**Remark 1** The function $\phi_0(\lambda; -; q, z)$ satisfies the first order $q$-difference equation

$$(1 - \lambda z)u(q z) + (q - 1)u(z) = 0.$$  

(C.8)
We can check that the equation (C.8) has the solution around infinity

\[ u_\infty(z) := \frac{\theta(\lambda z)}{\theta(z)}_1 \varphi_0 \left( \lambda; -; q, \frac{q}{\lambda z} \right). \]  

(C.9)

With this solution, the relation (C.7) can be rewritten as

\[ _1 \varphi_0 (\lambda; -; q, z) = C_q(z)u_\infty(z), \]

where

\[ C_q(z) = \frac{\theta(-\lambda z)}{\theta(-z)} \frac{\theta(z)}{\theta(\lambda z)}. \]

Here, the function \( C_q(z) \) is the elliptic function, namely, \( q \)-periodic and unique valued:

\[ C_q(qz) = C_q(z), \quad C_q(e^{2\pi i}z) = C_q(z). \]

Therefore, the function \( C_q(z) \) gives the “true” connection coefficient \([37]\) between the function \( _1 \varphi_0(\lambda; -; q, z) \) and \( u_\infty(z) \).

The function \( _1 \varphi_0(\lambda; -; q, q/\lambda z) \) also has the alternative representation (C.1) as

\[ _1 \varphi_0 \left( \lambda; -; q, \frac{q}{\lambda z} \right) = 2 \varphi_1 \left( \lambda, \lambda q; q; q^2, \left( \frac{q}{\lambda z} \right)^2 \right) \]

\[ + \frac{(\lambda, q^3; q^2)_{\infty}}{(\lambda q^2, q; q^2)_{\infty}} \frac{q}{\lambda z}^2 \varphi_1 \left( \lambda q, \lambda q^2; q^3, q^2, \left( \frac{q}{\lambda z} \right)^2 \right). \]  

(C.10)

Combining the relation (C.7), (C.10), and the weight function \( w(z, \lambda; q) \) defined in Appendix C.1, we also obtain the following relation:

\[ w(z, \lambda; q) \frac{(\lambda z; q)_{\infty}}{(z; q)_{\infty}} = q^{\frac{1}{2}} (z^2; q^2)_{\infty} (\lambda z; q)_{\infty} \]

\[ = q^{\frac{1}{2}} (q^2; q^2)_{\infty} \frac{\theta(-q/\lambda z)}{\theta(-z)} \left\{ z^{-\frac{1}{2}} (z^2, q^2; q^2)_{\infty} \varphi_1 \left( \lambda, \lambda q; q^2, \left( \frac{q}{\lambda z} \right)^2 \right) \right. \]

\[ + \left. q \frac{(z^2, q^7, q^2; q^2)_{\infty}}{(\lambda q^2, q^2; q^2)_{\infty}} \varphi_1 \left( \lambda q, \lambda q^2; q^3, q^2, \left( \frac{q}{\lambda z} \right)^2 \right) \right\}. \]  

(C.11)

(C.11) gives the relation between the basic hypergeometric series \( _1 \varphi_0 \) around the origin and the basic hypergeometric series \( 2 \varphi_1 \) around infinity.

If we set \( z \mapsto \tilde{a} + \frac{z}{2} \) and \( \lambda \mapsto \tilde{a}^{-1}q, \) i.e. \( \lambda z \mapsto \tilde{a}^{-\frac{1}{2}}q, \) we again acquire the relation (4.25).
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