INTERPRETATION OF THE ARITHMETIC IN CERTAIN GROUPS OF PIECEWISE AFFINE PERMUTATIONS OF AN INTERVAL

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Abstract. The Arithmetic is interpreted in all the groups of Richard Thompson and Graham Higman, as well as in other groups of piecewise affine permutations of an interval which generalize the groups of Thompson and Higman. In particular, the elementary theories of all these groups are undecidable. Moreover, Thompson’s group \( F \) and some of its generalizations interpret the Arithmetic without parameters.

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1. INTRODUCTION

Valery Bardakov and Vladimir Tolstykh [2] showed recently that Richard Thompson’s group \( F \) interprets the Arithmetic. In other words, \( F \) interprets the structure \((\mathbb{N},+,{\times})\) by first-order formulae with parameters. In this work we generalize their result in two directions. On the one hand, in Sections 3 and 4, we generalize the approach of Bardakov and Tolstykh to the construction of the Arithmetic from \( F \), and show that it works for all the

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groups defined by Melanie Stein in [34], which include all the three groups of Thompson and all the groups of Graham Higman [14]. On the other hand, in Section 5 we show that the group $F$ and some of its generalizations interpret the Arithmetic without parameters. (The difference between with and without parameters shall be explained in Section 2.3.)

The elementary theory of the Arithmetic $(\mathbb{N}, +, \times)$ is famous for its complexity since Gödel’s incompleteness theorems [13]. If a structure of finite signature interprets the Arithmetic, then the elementary theory of this structure is hereditarily undecidable. (A theory of finite signature is called hereditarily undecidable if every its subtheory of the same signature is undecidable, see [35, §3].) Indeed, it is well known from the work Andrzej Mostowski, Raphael Robinson, and Alfred Tarski [19, 20, 35] that the elementary theory of the Arithmetic is hereditarily undecidable. It is also well known to specialists that if a structure $N$ of finite signature interprets with parameters another structure $M$ of finite signature whose elementary theory is hereditarily undecidable, then the elementary theory of $N$ is hereditarily undecidable as well.¹ Thus Bardakov and Tolstykh showed that the elementary theory of $F$ is hereditarily undecidable, and hence Question 4.16 by Mark Sapir in [37] is partially resolved. The same argument shows that the elementary theories of all the groups that we study in this paper are hereditarily undecidable.

For the reader’s convenience, we present in Section 6 our version of a proof, also based on a result of Mostowski, Robinson, and Tarski [19, Theorem 9], that if a structure $S$ of finite signature interprets the Arithmetic with parameters, then the elementary theory of $S$ is hereditarily undecidable.

The groups that are the object of our study appear naturally as generalizations of the tree groups defined by Thompson in 1965 and customarily denoted $F$, $T$, and $V$.² Thompson’s groups are presented in detail in [3, 8]. All the three are infinite finitely presented. The group $V$ was the first known example of a finitely presented infinite simple group. The group $T$ is also simple. The group $F$ embeds in $T$, and $T$ embeds in $V$. The group $V$ was generalized by Higman [14] into a family of finitely presented groups $G_{n,r}$, $n = 2, 3, 4, \ldots$, $r = 1, 2, 3, \ldots$, where $G_{2,1} \cong V$. Higman’s group $G_{n,r}$ is simple if $n$ is even; if $n$ is odd, the derived subgroup $[G_{n,r}, G_{n,r}]$ is simple of index 2 in $G_{n,r}$. Kenneth Brown [7, Section 4] similarly generalized the groups $F$ and $T$.

Thompson’s groups have representations by piecewise affine permutations of an interval, where the group $F$ is represented by homeomorphisms with respect to the usual topology, and $T$ is represented by homeomorphisms with respect to the topology of circle. Stein [34] studied three families of groups of such permutations which generalize respectively the three groups of Thompson. In order to state our main results, we shall review here the definitions of these families.

¹In [2] the authors state this fact with a reference to [12]. We show this fact as Lemma 6.2.

²Other letters were also used to denote these groups (see [8]). It is rather common, for example, to denote the group $V$ by $G$. 
Let $r$ be a positive real number and $\Lambda$ a subgroup of the multiplicative group $\mathbb{R}^*_+$ of positive reals. Let $A$ be an additive subgroup of $\mathbb{R}$ containing $r$ and invariant under the action of $\Lambda$ by multiplications. Then define $\mathcal{V}(r, \Lambda, A)$ to be the group of all the bijections $x: [0, r) \to [0, r)$ that satisfy the following conditions:

1. $x$ is piecewise affine with finitely many cuts and singularities;
2. $x$ is right-continuous at every point (in the usual sense);
3. the slope of each affine part of $x$ is in $\Lambda$;
4. all cut and singular points of $x$, as well as their images, are in $A$.

The family of groups $\mathcal{V}(r, \Lambda, A)$ contains all the groups of Higman: for every $n = 2, 3, \ldots$ and every $r = 1, 2, \ldots$,

$$G_{n,r} \cong \mathcal{V}(r, \langle n \rangle, \mathbb{Z}[1/n]).$$

Define subgroups $\mathcal{F}(r, \Lambda, A)$ and $\mathcal{T}(r, \Lambda, A)$ of $\mathcal{V}(r, \Lambda, A)$ as follows:

- $\mathcal{F}(r, \Lambda, A)$ is the subgroup of all the elements of $\mathcal{V}(r, \Lambda, A)$ continuous with respect to the usual topology of $[0, r)$,
- $\mathcal{T}(r, \Lambda, A)$ is the subgroup of all the elements of $\mathcal{V}(r, \Lambda, A)$ continuous with respect to the topology of circle on $[0, r)$.

The groups $\mathcal{F}$, $\mathcal{T}$, and $\mathcal{V}$ of Thompson are isomorphic to $\mathcal{F}(1, \langle 2 \rangle, \mathbb{Z}[1/2])$, $\mathcal{T}(1, \langle 2 \rangle, \mathbb{Z}[1/2])$, and $\mathcal{V}(1, \langle 2 \rangle, \mathbb{Z}[1/2])$, respectively. Groups of the form $\mathcal{F}(r, \Lambda, A)$ were studied already by Robert Bieri and Ralph Strebel in [4] (unpublished).

For the rest we shall always assume that $\Lambda \neq \{1\}$.

**Theorem A.** If $G$ is a subgroup of $\mathcal{V}(r, \mathbb{R}^*_+, \mathbb{R})$ such that

$$G \cap \mathcal{F}(r, \mathbb{R}^*_+, \mathbb{R}) = \mathcal{F}(r, \Lambda, A),$$

then $G$ interprets the Arithmetic $(\mathbb{N}, +, \times)$ with parameters.

**Theorem B.** If $\Lambda$ is cyclic, then $\mathcal{F}(r, \Lambda, A)$ interprets the Arithmetic without parameters.

**Theorem C.** If $G$ is a group as in Theorem A, then the elementary theory of $G$ is hereditarily undecidable.

In particular, all the groups of Thompson and Higman interpret the Arithmetic with parameters, while Thompson’s group $F$ also interprets it without parameters, and the elementary theories of all these groups are hereditarily undecidable.

Theorems A and B are proved in Section 5. To the best of our knowledge, the interpretation constructed in the proof of Theorem B is entirely original. Theorem C is proved in Section 6 as a corollary of Theorem A. In the Appendix we show that every element of the derived subgroup of $\mathcal{F}(r, \Lambda, A)$ is the product of two commutators, and hence the derived subgroup is definable in $\mathcal{F}(r, \Lambda, A)$.

The main idea of the proof of Theorem A is, as in [2], to find in $G$ a definable subgroup isomorphic to the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$, because it is known that the latter group interprets the Arithmetic. Note that, as opposed to $\mathbb{Z} \wr \mathbb{Z}$ and to the groups that we study here, neither abelian...
groups, nor virtually abelian, nor free groups, nor torsion-free hyperbolic ones can interpret the Arithmetic because their elementary theories are all stable, while the elementary theory of every structure that interprets the Arithmetic is "strongly" unstable. *Stability* is a fundamental concept of Model Theory. The textbooks [24, 25, 27] are all excellent introductions to the subject. The best sources for learning about *stable groups*, i.e. the groups whose elementary theories are stable, are, in our opinion, [26, 28, 36]. A proof of the stability of abelian groups can be found in [29, Theorem 3.1]. Every non-elementary torsion-free hyperbolic group is stable according to a recent result of Zlil Sela [33].

A definable subgroup of $F$ isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ was chosen by Bardakov and Tolstykh [2] as follows. Let $x_0$ and $x_1$ be the “standard” generators of $F$, and let $a = x_0^2$, $b = x_1 x_0^{-1} x_1^{-1} x_0$ (see Figure 1). One can verify without major difficulty that $\langle a, b \rangle = \langle b \rangle \langle a \rangle \cong \mathbb{Z} \wr \mathbb{Z}$. The centralizer of $x_0$ in $F$ coincides with the subgroup generated by $x_0$. Consequently, the subgroup $\langle a \rangle$ is definable in $F$ by a formula with the parameter $x_0$. Then it is shown that the centralizer of the subset $\{ a^{-k} b a^k \mid k \in \mathbb{Z} \}$ coincides with the subgroup $\langle a^{-k} b a^k \mid k \in \mathbb{Z} \rangle$. As the subset $\{ a^{-k} b a^k \mid k \in \mathbb{Z} \}$ is clearly definable with the parameters $x_0$ and $x_1$, so is the subgroup $\langle a^{-k} b a^k \mid k \in \mathbb{Z} \rangle$. Thus the subgroup $\langle a, b \rangle \cong \mathbb{Z} \wr \mathbb{Z}$ is definable in $F$ with parameters.
In the proof of Theorem A, we follow a similar approach for the group $F(r, \Lambda, A)$.

2. Generalities

In this section we present some basic definitions and facts.

2.1. Permutations and piecewise affine maps.

**Definition.** A bijection of a set onto itself shall be called a permutation of this set. A map $f$ shall be said to permute a set $S$ if the restriction $f|_S$ is a permutation of $S$.

**Definition.** Let $S$ be a set and $f$ be a bijection of $S$ onto itself. We shall call the support of $f$, denoted $\text{Supp}(f)$, the complement in $S$ of the set of fixed points of $f$, denoted $\text{Fix}(f)$.

Customarily, in the context of study of Thompson and Higman’s groups, all maps act on the right. We adopt the same convention in this article; for example: $(\alpha)(xy) = ((\alpha)x)y$ if $x$ and $y$ are permutations of a set $S$, and $\alpha \in S$.

We are going to write $Xf$, or sometimes $(X)f$, to denote the image of the set $X$ under the map $f$.

**Lemma 2.1.** Two permutations of the same set commute if their supports are disjoint.

**Lemma 2.2.** Let $f$ and $g$ be two permutations of the same set. Then

$$\text{Fix}(g^{-1}fg) = (\text{Fix}(f))g,$$

$$\text{Supp}(g^{-1}fg) = (\text{Supp}(f))g.$$ 

**Lemma 2.3.** Let $f$ and $g$ be two commuting permutations of the same set. Then $g$ permutes each of the sets $\text{Fix}(f)$ and $\text{Supp}(f)$.

**Lemma 2.4.** Let $I$ be an interval in $\mathbb{R}$, $f$ an increasing permutation of $I$, and $n \in \mathbb{Z} \setminus \{0\}$. Then $\text{Supp}(f^n) = \text{Supp}(f)$.

**Proof.** Clearly $\text{Fix}(f) \subset \text{Fix}(f^n)$ and $\text{Supp}(f) \supset \text{Supp}(f^n)$. Consider an arbitrarily chosen $\alpha \in \text{Supp}(f)$. Without loss of generality, suppose that $(\alpha)f > \alpha$. Then

$$\alpha < (\alpha)f < (\alpha)f^2 < \cdots < (\alpha)f^n,$$

and hence $\alpha \in \text{Supp}(f^n)$. \hfill \Box

**Lemma 2.5.** Let $I$ be a compact interval in $\mathbb{R}$, $f$ be an increasing permutation of $I$, and $\alpha \in I$. Then

$$\lim_{n \to +\infty} (\alpha)f^n \in \text{Fix}(f).$$

**Proof.** As $f$ preserves the order, the sequence $((\alpha)f^n)_{n=0,1,\ldots}$ is monotone, and hence the limit exists and belongs to $I$. If $\beta = \lim_{n \to +\infty}(\alpha)f^n$, then $(\beta)f = \beta$ by continuity. \hfill \Box
Definition. Let $f$ be a map and $\alpha$ be a real number. The map $f$ shall be called affine to the right of $\alpha$ if there exists $\beta > \alpha$ such that the restriction $f|_{(\alpha, \beta)}$ is an affine map $(\alpha, \beta) \to \mathbb{R}$. The map $f$ shall be called affine to the left of $\alpha$ if there exists $\beta < \alpha$ such that $f|_{(\beta, \alpha)}$ is an affine map $(\beta, \alpha) \to \mathbb{R}$.

Remark. For all $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$, and for every map $f$ such that $f|_{(\alpha, \beta)}$ is an affine map $(\alpha, \beta) \to \mathbb{R}$, we shall denote the slope of $f|_{(\alpha, \beta)}$ by $(\alpha)f^+$ and also by $(\beta)f^-$. The number $(\alpha)f^+$ shall be called the slope of $f$ to the right of $\alpha$, and $(\beta)f^-$ shall be called the slope of $f$ to the left of $\beta$.

The following lemma can be proved the same way in which one determines the derivative of a composite function.

Lemma 2.6. Let $f$ and $g$ be two maps between subsets of $\mathbb{R}$, and let $\alpha \in \mathbb{R}$.

1. If $f$ is affine to the right of $\alpha$ with $(\alpha)f^+ > 0$, $f$ is right-continuous at $\alpha$, and $g$ is affine to the right of $(\alpha)f$, then

$$(\alpha)(fg)^+ = (\alpha)f^+ \cdot ((\alpha)f)g^+.$$

2. If $f$ is affine to the left of $\alpha$ with $(\alpha)f^- > 0$, $f$ is left-continuous at $\alpha$, and $g$ is affine to the left of $(\alpha)f$, then

$$(\alpha)(fg)^- = (\alpha)f^- \cdot ((\alpha)f)g^-.$$
For the rest of this article, we fix \((r, \Lambda, A)\), and moreover, we assume that \(\Lambda\) is nontrivial: \(\Lambda \neq \{1\}\). To simplify the notation, we abbreviate:

\[
F = F(r, \Lambda, A), \quad T = T(r, \Lambda, A), \quad V = V(r, \Lambda, A).
\]

We are also going to write \(F^\downarrow\) instead of \(F^\downarrow(r, \Lambda, A)\), etc.

### 2.3. Theories and models.

In this article we talk about structures in the sense of Model Theory (or in the sense of Universal Algebra, up to a few linguistic distinctions). When the terms “formula,” “sentence,” and “theory” are used in the formal sense, they always denote first-order formulae, sentences, and theories. The terms “theory,” “elementary theory,” and “first-order theory” are to be used as synonyms. “A model” and “a structure” will usually be synonyms as well. Except when indicated otherwise, the formulae are without parameters.

A structure \(M\) of signature \(\Sigma\), also called \(\Sigma\)-structure, is a model of a set of sentences \(S\), this fact being denoted \(M \models S\), if \(M\) satisfies every sentence \(\alpha\) of \(S\), which be denoted \(M \models \alpha\). A sentence \(\alpha\) is called a consequence of a set of sentences \(S\) in a signature \(\Sigma\), which be denoted \(S \models_\Sigma \alpha\), or \(S \models \alpha\) in the case when \(\Sigma\) is well understood, if every \(\Sigma\)-model of \(S\) is also a model of \(\alpha\). A set of sentences is consistent in a signature \(\Sigma\) if it has a \(\Sigma\)-model.\(^3\)

A set of sentences is deductively closed in a signature \(\Sigma\) if it contains all its consequences in \(\Sigma\). A theory of signature \(\Sigma\), also called \(\Sigma\)-theory, is a consistent and deductively closed in \(\Sigma\) set of sentences. If \(T\) is a \(\Sigma\)-theory, then \(T \models_\Sigma \alpha\) is equivalent to \(\alpha \in T\). The theory of a structure \(M\), denoted \(\text{Th}(M)\), is the set of all the sentences in the signature of \(M\) satisfied by \(M\). A theory is called complete if it is the theory of a structure. The class of all the \(\Sigma\)-models of a set of \(\Sigma\)-sentences \(S\) is denoted \(\text{Mod}_\Sigma(S)\).

We are going to use implicitly the Compactness Theorem, which guarantees that a consequence of a set of sentences is always a consequence, in the same signature, of its finite subset. We recommend any of the textbooks \([15, 16, 25, 27, 31]\) for references on general results of Model Theory.

If \(M\) is a structure, a set definable in \(M\) in general is a subset of \(M^n\), where \(n \in \mathbb{N}\), definable by a first-order formula in the signature of \(M\) and possibly with parameters from \(M\). The parameters are new constant symbols added to the language and interpreted by elements of \(M\). (Usually, to name an element \(a \in M\), one uses \(a\) itself as a parameter.) For example, in a group, the centralizer of every element \(g\) is definable by the formula \(φ(x) = ^rgx = xg^{-1}\) with \(g\) as a parameter, but in general there is no reason for that centralizer to be definable by a formula without parameters in the pure-group signature, which only contains a single binary function symbol \(\cdot^{-1}\) to denote the group operation \((x, y) \mapsto x \cdot y\). To be explicit, a set is said to be “definable with” or “without” parameters” according to whether or not parameters are allowed in its definition. However, we are not going to specify in which structure a given set is definable, except if there are several equally natural choices in the context. Of course it is possible as well to talk about definability of relations and operations.

\(^3\)The signature \(\Sigma\) in this definition is of minor importance except in the case when \(S\) has an empty model, which would imply in particular that no element of \(S\) contain any constant symbols.
If $f$ is a map $A \to B$, and $n$ is a positive integer, we denote by $f^n$ the map $A^n \to B^n$ induced by $f$. We are going to slightly abuse the notation by assuming that if the domain of $f$ is a subset of $A^m$, then the domain of $f^n$ is naturally identified with a subset of $A^{mn}$. We shall call the $f$-preimage of a given set its preimage under $f^n$ when the choice of $n$ is clear (so not necessarily under $f$ itself).

Consider two structures, $M$ of signature $\Sigma$ and $N$ of signature $\Gamma$.

**Definition.** We call an interpretation of $M$ in $N$ with parameters a pair $(n,f)$ where $n \in N$ and $f$ is a surjective map of a subset of $N^n$ onto $M$ such that for every set $X$ definable in $M$ without parameters, the $f$-preimage of $X$ is definable (in $N$) with (possibly) parameters. An interpretation $(n,f)$ with parameters is called an interpretation without parameters if the $f$-preimage of every set definable without parameters is definable without parameters as well.

See [15, Chapter 5] for detailed explanation of interpretability and related notions.

In what follows, in accordance with conventions of Model Theory, the terms “definable,” “0-definable,” “interpretation,” “0-interpretation” shall be used to denote, respectively, “definable with parameters,” “definable without parameters,” “interpretation with parameters,” and “interpretation without parameters.” Furthermore, as in our case the value of $n$ for an interpretation $(n,f)$ under consideration will be often either well understood, or hardly important, we shall simplify the notation and call $f$ itself an interpretation. To indicate that $(n,f)$ is an interpretation of $M$ in $N$, we shall write either $(n,f): M \hookrightarrow N$, or $f: M \to^N$. To indicate that $f$ is a 0-interpretation of $M$ in $N$, we shall write $f: M \to^0 N$. To indicate that $M$ is interpretable or 0-interpretable in $N$, we write $M \hookrightarrow N$ or $M \to^0 N$, respectively.

**Remark 2.9.** If $f: M \to^N$, then the $f$-preimage of every set definable in $M$ is also definable (in $N$).

**Remark 2.10.** Let $n \in N$ and $B \subset N^n$. Then a surjective map $f$ of $B$ onto $M$ is an interpretation of $M$ in $N$ if and only if

1. the domain $B$ is definable,
2. the equivalence relation on $B$ induced by $f$ (the kernel of $f$) is definable, and
3. for every relation, operation, and constant of the structure $M$ (named by a symbol of $\Sigma$), the $f$-preimage of its graph is definable.

The map $f$ is a 0-interpretation if and only if all these sets are 0-definable.

**Remark 2.11.** If $L$, $M$, and $N$ are three structures, and $(m,f): L \to^M$ and $(n,g): M \to^N$, then $(mn,g^{2n}f): L \to^N$. If moreover $f$ and $g$ are 0-interpretations, then so is $g^{2n}f$.

**Definition** (see [1] et [15, Section 5.4(c)]). Two structures $M$ and $N$ are said to be bi-interpretable if there exist two interpretations $(m,f): M \to^N$ and $(n,g): N \to^M$ such that the map $g^{2n}f$ is definable in $M$, and $f^{2m}g$ is 0-definable.

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4The arrow here is pointing in the opposite direction than that in the notation of [1].
definable in \( N \). The interpretations \((m, f)\) and \((n, g)\) in this case are called bi-interpretations.

2.4. Decidability. Let \( A \) be a finite set viewed as an alphabet, and denote \( A^* \) the set of all finite words in \( A \). We say that a set \( X \subset A^* \) is recursive or decidable if there exists an algorithm which for every input \( w \in A^* \) answers the question whether \( w \in X \). A map \( f : X \to A^* \) is said to be computable if there exists an algorithm which computes \( f(w) \) for every input \( w \in X \), and which never stops for any input \( w \notin X \).

Usually a set is said to be recursive or non-recursive, while a theory is said to be decidable or undecidable.

In the rest of this section, let \( \Sigma \) be an arbitrary finite signature.

Definition. A \( \Sigma \)-theory \( T \) is said to be essentially undecidable if every \( \Sigma \)-theory containing \( T \) is undecidable.

Remark 2.12. If \( T \) is a theory of finite signature, and a subset of \( T \) is an essentially undecidable theory (possibly of smaller signature), then \( T \) is undecidable, and even essentially undecidable.

Definition. A \( \Sigma \)-theory \( T \) is said to be hereditarily undecidable if every \( \Sigma \)-subtheory of \( T \) is undecidable.

Lemma 2.13 ([35, Theorem 6]). If \( T \) is a theory of finite signature, and \( T \) has a finitely axiomatized essentially undecidable subtheory (possibly of smaller signature), then \( T \) is hereditarily undecidable.

Proof. Denote by \( \Sigma \) the signature of \( T \). Let \( S \) be a finitely axiomatized essentially undecidable subtheory of \( T \). Choose a sentence \( \theta \in S \) which axiomatizes \( S \).

Arguing by contradiction, let \( U \) be a decidable \( \Sigma \)-subtheory of \( T \). Let \( R \) be the \( \Sigma \)-theory axiomatized (generated) by \( U \cup S \). Then

\[
R = \{ \alpha \mid U, \theta \vdash_{\Sigma} \alpha \} = \{ \alpha \mid \forall \theta \to \alpha \in U \},
\]

and hence \( R \) is decidable since so is \( U \). This contradicts the essential undecidability of \( S \) (see Remark 2.12).

3. Two lemmas

In this section we will prove two technical lemmas about piecewise affine homeomorphisms of an interval. These lemmas will be essential for the proof of Theorem A, namely for showing that certain centralizers are preserved when passing from \( \mathcal{F} \) to \( \mathcal{V} \), and thus being able to pass from an interpretation of the Arithmetic in \( \mathcal{F} \) to its interpretations in \( \mathcal{V} \) and \( \mathcal{T} \).

Lemma 3.1. Let \( r \in \mathbb{R}_+^* \). Let \( z \) be a homeomorphism \([0, r) \to [0, r)\) such that \((\alpha)z > \alpha\) for all \( \alpha \in (0, r) \). Let \( f \) be a permutation of \([0, r)\) such that:

1. \( f \) commutes with \( z \),
2. \( f \) is (right-)continuous at 0,
3. \( f \) has a finite number of points of discontinuity.

Then \( f \) is continuous (with respect to the usual topology).
Proof. Suppose that \( f \) were not continuous. Then let \( \alpha \) be the least element of \([0, r)\) at which \( f \) is not continuous. Since \( f \) is continuous at 0, \( \alpha \in (0, r) \). Hence \((\alpha)z^{-1} < \alpha\) and \( f \) is continuous at \((\alpha)z^{-1}\). It follows that \( f \) is continuous at \( \alpha \) because \( f = z^{-1}fz \), where \( z^{-1} \) and \( z \) are continuous everywhere, and \( f \) is continuous at \((\alpha)z^{-1}\). This gives a contradiction. \( \square \)

Lemma 3.2. Let \( r \in \mathbb{R}_+^* \). Let \( Z \) be a set of homeomorphisms \([0, r) \to [0, r)\) such that:

1. \((\alpha)z \geq \alpha\) for all \( z \in Z \) and all \( \alpha \in [0, r)\),
2. \( \text{Supp}(z) \) is an interval for every \( z \in Z \),
3. \( \bigcup_{z \in Z} \text{Supp}(z) \) is dense in \((0, r)\).

Let \( f \) be a permutation of \([0, r)\) such that:

1. \( f \) commutes with every \( z \in Z \),
2. \( f \) is right-continuous at every point of \([0, r)\),
3. \( f \) has a finite number of points of discontinuity.

Then \( f \) is continuous.

Proof. We shall show first that \( f \) is increasing on each of its intervals of continuity.

We shall say that an interval \( I \) is right-closed if \( \sup I \in I \), and that it is left-closed if \( \inf I \in I \). If an interval is not right- or left-closed, we shall say that is right- or left-open, respectively.

As the number of points of discontinuity of \( f \) is finite, the number of its maximal intervals of continuity is finite as well. Since \( f \) is right-continuous everywhere in its domain \([0, r)\), every maximal interval of continuity is left-closed and right-open. Therefore the image under \( f \) of every maximal interval of continuity \( I \) is left-closed if \( f \) increases on \( I \) and right-closed if \( f \) decreases on \( I \).

Clearly \([0, r)\) is the disjoint union of the images of the maximal intervals of continuity of \( f \). Since every such image is either a right- or a left-open interval, the only possibility is that they all are left-closed and right-open. Hence \( f \) is increasing on each of its intervals of continuity.

Suppose now that \( f \) were not continuous.

By Lemma 2.3, for every \( z \in Z \), \( f \) permutes the set \( \text{Supp}(z) \). We shall show that \( f \) is continuous on each of these intervals. Arguing by contradiction, take \( z \in Z \) such that \( f \) is not continuous on \( \text{Supp}(z) \). Let \( \gamma \) be the least element of \( \text{Supp}(z) \) at which \( f \) is not continuous. Then \( f \) is continuous at \( \gamma \) because \( f = z^{-1}fz \), where \( z \) is a homeomorphism, and \( f \) is continuous at \((\gamma)z^{-1}\), since \((\gamma)z^{-1} < \gamma\). This gives a contradiction.

For every \( z \in Z \), \( f \) is increasing on \( \text{Supp}(z) \) since it is continuous there. Therefore for every \( z \in Z \),

\[
\lim_{\alpha \to (\text{sup \text{Supp}(z)})^-} (\alpha)f = \sup \text{Supp}(z).
\]

By right continuity, \((\text{inf \text{Supp}(z)})f = \text{inf \text{Supp}(z)}\) for all \( z \in Z \).

Let \( S = \bigcup_{z \in Z} \text{Supp}(z) \) and \( L = \{ \text{inf \text{Supp}(z)} | z \in Z \} \). Then \( S \) is open and dense in \([0, r)\), \( f|_S \) is continuous, and \( f|_L = \text{id}_L \).

Consider an arbitrary \( \alpha \in [0, r)\) \( \setminus S \). It is easy to see that for any \( \beta \in (\alpha, r) \), \( L \cap (\alpha, \beta) \) is not empty. Hence \((\alpha)f = \alpha\) by right continuity at \( \alpha \). We have shown that \( f|_{[0, r) \setminus S} = \text{id}_{[0, r) \setminus S} \).
The map \( f \) is increasing. Indeed, if \( f \) is a map from a linearly ordered set into itself, and this set is covered by intervals such that \( f \) sends each of them into itself in a strictly increasing way, then \( f \) is strictly increasing. In our case we have:

\[
[0, r) = \bigcup_{z \in Z} \text{Supp}(z) \cup \bigcup_{\alpha \in [0, r) \setminus S} [\alpha, \alpha].
\]

Therefore \( f \) is continuous as an increasing surjection of a subset of \( \mathbb{R} \) onto an interval of \( \mathbb{R} \).

These last two lemmas already suffice to show, using the result of Bar-dakov and Tolstykh, that the definable subgroup of \( F \) isomorphic to \( \mathbb{Z} \wr \mathbb{Z} \) used in [2] is definable in \( T \) and \( V \) as well.

**Proposition 3.3.** Let \( a \) and \( b \) be the elements of \( F \) shown on Figure 1. Then

\[
\langle a, b \rangle = \langle b \rangle \wr \langle a \rangle \cong \mathbb{Z} \wr \mathbb{Z},
\]

and the subgroup \( \langle a, b \rangle \) is definable in \( F \), in \( T \), and in \( V \) by the same first-order formula with parameters.

**Proof.** We consider the same elements \( x_0, x_1, a = x_0^2 \), and \( b = x_1 x_0^{-1} x_1^{-1} x_0 \) of \( F \) as in [2], see Figure 1. It is shown in [2] that:

1. \( \langle a, b \rangle = \langle b \rangle \wr \langle a \rangle \cong \mathbb{Z} \wr \mathbb{Z} \),
2. \( C_F(x_0) = \langle x_0 \rangle \),
3. \( C_F(\{ a^{-k}ba^k \mid k \in \mathbb{Z} \}) = \langle a^{-k}ba^k \mid k \in \mathbb{Z} \rangle \).

In particular, \( \langle a, b \rangle \) is the semi-direct product of \( C_F(\{ a^{-k}ba^k \mid k \in \mathbb{Z} \}) \) and \( \langle a \rangle \), where \( \langle a \rangle = \{ x^2 \mid x \in C_F(x_0) \} \).

Define \( \alpha_k = 2^{-1+2k} \) for \( k = -1, -2, -3, \ldots \), and \( \alpha_k = 1 - 2^{-1-2k} \) for \( k = 0, 1, 2, \ldots \). Then

\[
0 < \cdots < \alpha_{-2} < \alpha_{-1} < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < 1.
\]

A direct calculation facilitated by Lemma 2.2 shows that:

1. \( \text{Supp}(x_0) = (0, 1) \);
2. \( (\alpha)x_0 \geq \alpha \) for all \( \alpha \in [0, 1) \);
3. \( \text{Supp}(a^{-k}ba^k) = (\alpha_k, \alpha_{k+1}) \) for all \( k \in \mathbb{Z} \);
4. \( (\alpha)a^{-k}ba^k \geq \alpha \) for all \( \alpha \in [0, 1) \) and all \( k \in \mathbb{Z} \).

Choosing \( x_0 \) as \( z \) in Lemma 3.1, we conclude that every element of \( C_V(x_0) \) is continuous. Setting \( Z = \{ a^{-k}ba^k \mid k \in \mathbb{Z} \} \) in Lemma 3.2, we conclude that every element of \( C_V(\{ a^{-k}ba^k \mid k \in \mathbb{Z} \}) \) is continuous. Since an element of \( V \) belongs to \( F \) if and only if it is continuous, the centralizer of the element \( x_0 \) and the centralizer of the set \( \{ a^{-k}ba^k \mid k \in \mathbb{Z} \} \) are preserved when passing from \( F \) to \( V \). Hence \( \langle a, b \rangle \) is definable in \( F, T, \) and \( V \) by the same first-order formula with parameters.

\[ \square \]

4. **Definable copies of** \( \mathbb{Z} \wr \mathbb{Z} \)

In this section we show that all the groups \( F, T, \) and \( V \) have definable subgroups isomorphic to \( \mathbb{Z} \wr \mathbb{Z} \).
Lemma 4.1. Let $\alpha, \beta \in [0, r] \cap A$ be such that $\alpha < \beta$, and denote $I = (\alpha, \beta)$. Let $x \in F_I$ be such that $Fix(x) \cap I \cap A = \emptyset$. Let $\phi : F_I \rightarrow \Lambda$ be the map $y \mapsto (\alpha)y^+$. Let $C$ be the centralizer of $x$ in $F_I$. Then $\phi$ is a homomorphism, and its restriction to $C$ is injective. The same holds for $\phi' : F_I \rightarrow \Lambda$, $y \mapsto (\beta)y^-$. 

Proof. We shall only consider the case of $\phi : F_I \rightarrow \Lambda$, $y \mapsto (\alpha)y^+$, because the case of $y \mapsto (\beta)y^-$ is analogous. It is easy to verify that $\phi$ is a homomorphism (see Lemma 2.6). Suppose that $\phi$ is not injective on $C$.

Let $y \in C$ be such that $\phi(y) = 1$ but $y \neq id$. Let $\gamma \in (\alpha, \beta)$ be such that

$$y|[0, \gamma] = id_{[0, \gamma]} \quad \text{but} \quad (\gamma)y^+ \neq 1.$$

Then $\gamma \in A$, and hence $(\gamma)x \neq \gamma$. Without loss of generality, assume that $(\gamma)x > \gamma$, because otherwise $(\gamma)x^{-1} > \gamma$, and one can use $x^{-1}$ instead of $x$.

Then

$$[0, (\gamma)x] = [0, \gamma]^x \subset Fix(y),$$

see Lemma 2.3, and hence $(\gamma)y^+ = 1$. This gives a contradiction. \hfill \Box

Lemma 4.2. The centralizer $C$ in Lemma 4.1 is cyclic.

This lemma follows from the description of centralizers in $F(r, \mathbb{R}^*_+, \mathbb{R})$ obtained by Matthew Brin and Craig Squier [6], but for the reader’s convenience we prefer to provide a self-contained proof.

Proof of Lemma 4.2. First of all, note that if $\Lambda$ is cyclic itself, the conclusion of this lemma is an obvious corollary of Lemma 4.1.

Let $\alpha, \beta, I, x, \phi$, and $C$ be such as in Lemma 4.1. Without loss of generality, assume that $(\alpha)x^+ > 1$.

We are going to use the fact that a multiplicative subgroup of $\mathbb{R}^*_+$ is either cyclic (the trivial subgroup is considered cyclic), or dense in $\mathbb{R}^*_+$ with respect to the usual topology.

Let $\Gamma$ be the image of the group $C$ under the homomorphism $\phi : F_I \rightarrow \Lambda$. By Lemma 4.1, $\phi$ is injective, and hence $C \cong \Gamma$. It remains to show that $\Gamma$ is not dense in $\mathbb{R}^*_+$.

Observe that $Fix(x) \cap I$ is a finite set, and that

$$Fix(y) \cap I = Fix(x) \cap I \quad \text{for all} \quad y \in C \setminus \{id\}.$$

Indeed, it is clear that $Fix(x) \cap I$ is finite because $Fix(x) \cap I \cap A$ is empty and $A$ is dense in $\mathbb{R}$. Consider now an arbitrary $y \in C \setminus \{id\}$. By the injectivity of $\phi$, $(\alpha)y^+ \neq 1$. If $Fix(y) \cap I \cap A$ were nonempty, then it would have a least element $\gamma$, and this $\gamma$ would be fixed by $x$: $(\gamma)x = x$. This would contradict $Fix(x) \cap I \cap A = \emptyset$, hence $Fix(y) \cap I \cap A = \emptyset$ and $Fix(y) \cap I$ is finite. As $x$ and $y$ commute and each of them permutes $I$, it follows that $x$ permutes $Fix(y) \cap I$ and that $y$ permutes $Fix(x) \cap I$. Since these sets are finite, and because $x$ and $y$ preserve the order, it follows that $Fix(y) \cap I \subset Fix(x)$, $Fix(x) \cap I \subset Fix(y)$, an hence $Fix(y) \cap I = Fix(x) \cap I$.

Denote

$$\beta_0 = \min((Fix(x) \cap I) \cup \{\beta\});$$

it exists but does not belong to $A$ unless $\beta_0 = \beta$. Then $(\gamma)x > \gamma$ for all $\gamma \in (\alpha, \beta_0)$, since $(\alpha)x^+ > 1$. 

Choose $\alpha_1, \beta_1 \in (\alpha, \beta_0)$ such that $x^{-1}$ be affine on $[\alpha, \alpha_1]$ and $x$ be affine on $[\beta_1, \beta_0]$. Then $x$ is also affine on $[\alpha, (\alpha_1)x^{-1}]$, and $x^{-1}$ is affine on $[(\beta_1)x, \beta_0]$.

We shall show now that if $y \in C$ and $(\alpha)y^+ > 1$, then $y$ is affine on $[\alpha, (\alpha_1)y^{-1}]$ and on $[\beta_1, \beta_0]$. Consider one such $y$. Thus $(\gamma)y > \gamma$ for all $\gamma \in (\alpha, \beta_0)$. Then for every $\gamma \in (\alpha, \alpha_1)$,

$$x^{-1}|_{[\alpha,\alpha_1]} \cdot y^{-1}|_{[\alpha,\alpha_1]} \cdot x|_{[\alpha,\alpha_1]} = y^{-1}|_{[\alpha,\alpha_1]},$$

and for every $\gamma \in [\beta_1, \beta_0]$,

$$x|_{[\beta_1,\beta_0]} \cdot y|_{[\gamma,\beta_0]} \cdot x^{-1}|_{[\beta_1,\beta_0]} = y|_{[\gamma,\beta_0]}.$$

These obvious equalities imply that:

1. for every $\gamma \in (\alpha_1, \gamma)$, $y^{-1}$ is affine on $[\alpha, \gamma]$ if it is affine on $[\alpha, (\gamma)x^{-1}]$,

2. for every $\gamma \in [\beta_1, \beta_0)$, $y$ is affine on $[\gamma, \beta_0]$ if it is affine on $[(\gamma)x, \beta_0]$.

This is possible only if $y^{-1}$ is affine on $[\alpha, \alpha_1]$, and $y$ is affine on $[\beta_1, \beta_0]$.

Clearly $\lim_{n \to \infty} (\gamma)x^n = \beta_0$ for all $\gamma \in (\alpha, \beta_0)$ (see the proof of Lemma 2.5). Choose an integer $n$ such that

$$(\alpha_1)x^n > \beta_1,$$

and hence $$(\beta_1)x^{-n} < \alpha_1.$$

Denote $\rho = (\beta_1)x^{-n}$. Then $p^{-1} = (\beta_1)x^{-n}(x^n)^\rho$. Choose $\gamma$ in $((\beta_1)x^{-n}, \alpha_1)$ such that $x^n$ be affine on $[(\beta_1)x^{-n}, \gamma]$ (with the slope $p^{-1}$).

Suppose that $y$ is an element of $C$ such that

$$1 < (y)\phi \leq \frac{\gamma - \alpha}{(\beta_1)x^{-n} - \alpha}.$$

Then $y^{-1}$ is affine on $[\alpha, \alpha_1]$, $y$ is affine on $[\beta_1, \beta_0]$, and $(\beta_0)y^{-1} < 1$ (because $(\gamma)y > \gamma$ for all $\gamma \in (\alpha, \beta_0)$). Denote $q = (\alpha)y^+ = (y)\phi$. Then

$$(\beta_1)x^{-n} < \alpha + (\gamma - \alpha)q^{-1} = (\gamma)y^{-1} \leq \alpha + (\alpha_1 - \alpha)q^{-1} = (\alpha_1)y^{-1},$$

and therefore $(\beta_1)x^{-n} < (\beta_1)x^{-n}y < \gamma$. Since $y = x^{-n}yx^n$,

$$(\beta_1)y^+ = (\beta_1)(x^{-n})^\rho \cdot ((\beta_1)x^{-n})y^+ \cdot ((\beta_1)x^{-n}y)(x^n)^\rho
= pq^+q^{-1} = q > 1.$$

As $(\beta_1)y^+ = (\beta_0)y^+ < 1$, this gives a contradiction, which means that

$$\Gamma \cap \left(1, \frac{\gamma - \alpha}{(\beta_1)x^{-n} - \alpha}\right) = \emptyset.$$

\[ \square \]

Lemma 4.3. Let $\alpha, \beta \in [0, r] \cap A$ be such that $\alpha < \beta$. Let $p, q \in \Lambda$ be such that $p > 1 > q$. Then there exists $x \in \mathcal{F}^1$ such that $\text{Supp}(x) = (\alpha, \beta)$, $(\alpha)x^+ = p$, and $(\beta)x^- = q$.

Proof. Let $s$ be an element of $\Lambda$ such that $(2 + p + q)s \leq 1$. Denote $l = \beta - \alpha$. Consider two subdivisions of the interval $[\alpha, \beta]$: the first one—into subintervals of lengths $sl, qsl, (1 - (2 + p + q)s)l, psl$, and $sl$, in this order, and the second—into subintervals of lengths $psl, sl, (1 - (2 + p + q)s)l, sl$, and $qsl$, in this order. Let $x$ be the continuous map $[0, r] \to [0, r]$ which is the identity on $[0, \alpha] \cup [\beta, r]$, and which sends every interval of the first subdivision of $[\alpha, \beta]$ in the affine manner onto the corresponding interval of the second. It is easy to verify that $\text{Supp}(x) = (\alpha, \beta)$, $(\alpha)x^+ = p$, $(\beta)x^- = q$, and $x|_{[0, r]} \in \mathcal{F}^1$. \[ \square \]
Choose \( a \in \mathcal{F}^\uparrow \) such that \( \text{Supp}(a) = (0, r) \). Choose \( \alpha_0 \in (0, r) \cap A \) arbitrarily. For every \( k \in \mathbb{Z} \), define \( \alpha_k = (\alpha_0) a^k \). Observe that

\[
0 < \cdots < \alpha_{-2} < \alpha_{-1} < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < r,
\]

and

\[
\lim_{n \to -\infty} \alpha_n = 0, \quad \lim_{n \to +\infty} \alpha_n = r
\]

(see Lemma 2.5 and Remark 2.7). Choose \( b \in \mathcal{F}^\uparrow \) such that \( \text{Supp}(b) = (\alpha_0, \alpha_1) \) (see Figure 2). Such \( a \) and \( b \) exist by Lemma 4.3.

**Lemma 4.4.** The group generated by \( a \) and \( b \) is isomorphic to the restricted wreath product \( \mathbb{Z} \wr \mathbb{Z} \); more precisely,

\[
\langle a, b \rangle = \langle b \rangle \wr \langle a \rangle \cong \mathbb{Z}.
\]

**Proof.** First of all, observe that \( \text{Supp}(a^{-k}ba^k) = (\alpha_k, \alpha_{k+1}) \) for all \( k \in \mathbb{Z} \) (see Lemma 2.2). In particular, the supports of the maps \( a^{-k}ba^k, k \in \mathbb{Z} \), are pairwise disjoint, and \( \text{Supp}(a) = (0, r) \) is not equal to the support of any element of the group \( \langle a^{-k}ba^k \mid k \in \mathbb{Z} \rangle \). Thus

\[
\langle a^{-k}ba^k \mid k \in \mathbb{Z} \rangle = \bigoplus_{k \in \mathbb{Z}} \langle a^{-k}ba^k \rangle \quad \text{and} \quad \langle a^{-k}ba^k \mid k \in \mathbb{Z} \rangle \cap \langle a \rangle = \{\text{id}\},
\]

which implies \( \langle a, b \rangle = \langle a^{-k}ba^k \mid k \in \mathbb{Z} \rangle \times \langle a \rangle = \langle b \rangle \wr \langle a \rangle \). Since \( \mathcal{F} \) has no torsion, \( \langle a \rangle \cong \langle b \rangle \cong \mathbb{Z} \). \( \square \)

In the rest of this section, \( G \) is a subgroup of \( \mathcal{V}(r, \mathbb{R}_+^*, \mathbb{R}) \) such that

\[
G \cap \mathcal{F}(r, \mathbb{R}_+^*, \mathbb{R}) = \mathcal{F} = \mathcal{F}(r, \Lambda, A)
\]

(it is even possible to generalize some results of this section to the case when \( G \) satisfies a weaker condition than this one).
Proposition 4.5. Let $a$, $b$, and $G$ be the elements and the group defined above. Then there exists a first-order formula $\phi$ with $a$ and $b$ as parameters and with exactly one free variable such that $\langle a, b \rangle$ is defined in $G$ by $\phi$. Moreover, $\phi$ can be chosen based only on $a$, $b$, and $F$ (without knowing the whole of $G$).

In order to find such a $\phi$ and thus prove this proposition, choose first of all $c, d \in F$ and $s, t \in \mathbb{N}$ such that

1. $c$ is a generator of the centralizer of $a$ in $F$,
2. $d$ is a generator of the centralizer of $b$ in $F_{(\alpha_0, \alpha_1)}$,
3. $a = c^s$ and $b = d^t$.

The elements $c$ and $d$ and the numbers $s$ and $t$ exist by Lemma 4.2. By Lemma 2.4, $\text{Supp}(c) = (0, r)$ and $\text{Supp}(d) = (\alpha_0, \alpha_1)$. Clearly $c, d \in F^1$.

Observe that $a^{-k}da^k \in F^1$ and $\text{Supp}(a^{-k}da^k) = (\alpha_k, \alpha_{k+1})$ for all $k$. Since for every $k$ the conjugation by $a^k$ is an automorphism of $F$ which sends $F_{(\alpha_0, \alpha_1)}$ onto $F_{(\alpha_k, \alpha_{k+1})}$, $a^{-k}da^k$ is a generator of the centralizer of $a^{-k}ba^k$ in $F_{(\alpha_k, \alpha_{k+1})}$ for every $k$.

Lemma 4.6. The centralizer of $\{ a^{-k}ba^k \mid k \in \mathbb{Z} \}$ in $F$ is generated by $\{ a^{-k}da^k \mid k \in \mathbb{Z} \}$.

Proof. The inclusion

$$C_F(\{ a^{-k}ba^k \mid k \in \mathbb{Z} \}) \supset (a^{-k}da^k \mid k \in \mathbb{Z})$$

is obvious, it remains show the inverse one. So let $x$ be an arbitrary element of $F$ which commutes with every $a^{-k}ba^k$, $k \in \mathbb{Z}$.

By Lemma 2.2, $x$ permutes each of the intervals $(\alpha_k, \alpha_{k+1})$, $k \in \mathbb{Z}$. By continuity and monotonicity, $(\alpha_k)x = \alpha_k$ for all $k$. For every $k \in \mathbb{Z}$, let $y_k$ be the permutation of $[0, r)$ such that

$$y_k(\alpha_k, \alpha_{k+1}) = x(\alpha_k, \alpha_{k+1}) \quad \text{and} \quad y_k|_{[0, \alpha_k] \cup [\alpha_{k+1}, r)} = \text{id}_{[0, \alpha_k] \cup [\alpha_{k+1}, r)}.$$  

Then for every $k \in \mathbb{Z}$, $y_k \in F_{(\alpha_k, \alpha_{k+1})}$ and $y_k$ commutes with $a^{-k}ba^k$. Hence $y_k \in \langle a^{-k}da^k \rangle$ for all $k$.

Choose $\beta, \gamma \in (0, r)$ such that $x$ be affine on $[0, \beta]$ and on $[\gamma, r)$. Then $x|[0, \beta] \cup [\gamma, r) = \text{id}_{[0, \beta] \cup [\gamma, r)}$. Choose $n \in \mathbb{N}$ such that $\alpha_n \in (0, \beta]$ and $\alpha_{n+1} \in [\gamma, r)$. Then $\text{Supp}(x) \subseteq (\alpha_n, \alpha_{n+1})$, and hence

$$x = y_n y_{n+1} \cdots y_{n-1} y_n \in \langle a^{-k}da^k \mid k \in \mathbb{Z} \rangle.$$  

Lemma 4.7. The centralizer of the element $a$ and the centralizer of the set $\{ a^{-k}ba^k \mid k \in \mathbb{Z} \}$ do not change when passing from $F$ to $G$.

Proof. By Lemmas 3.1 and 3.2, all elements of $C_G(a)$ and of $C_G(\{ a^{-k}ba^k \mid k \in \mathbb{Z} \})$ are continuous. Since an element of $G$ belongs to $F$ if and only if it is continuous, the proof is complete.

The group $\langle c \rangle$ is definable in $G$ with the parameter $a$ because it is the centralizer of $a$ (see Lemma 4.7). The group $\langle a \rangle$ is definable in $G$ with the same parameter because

$$\langle a \rangle = \{ x^s \mid x \in \langle c \rangle \}.$$

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Thus the set \( \{ a^{-k}ba^k \mid k \in \mathbb{Z} \} \) is definable in \( G \) with the parameters \( a \) and \( b \), and hence so does its centralizer. By Lemmas 4.6 and 4.7, the centralizer of the set \( \{ a^{-k}ba^k \mid k \in \mathbb{Z} \} \) in \( G \) is the group \( \langle a^{-k}da^k \mid k \in \mathbb{Z} \rangle \). Since
\[
\langle a^{-k}ba^k \mid k \in \mathbb{Z} \rangle = \{ x^t \mid x \in \langle a^{-k}da^k \mid k \in \mathbb{Z} \rangle \},
\]
the group \( \langle a^{-k}ba^k \mid k \in \mathbb{Z} \rangle \) is definable with the same parameters. The group \( \langle a, b \rangle \) is definable with the parameters \( a \) and \( b \) since it is the semi-direct product of \( \langle a \rangle \) and \( \langle a^{-k}ba^k \mid k \in \mathbb{Z} \rangle \).

The following formula defines \( \langle a, b \rangle \) in \( G \) and depends only on \( a, b, s, \) and \( t \):
\[
\phi(x) = \exists y, z \left( x = y^s z^t \land ya = ay \land \left( \forall w \right) \left( wa = aw \rightarrow zw^{-s}bw^{-s} = w^{-s}bw^s z \right) \right).
\]

We have proved Proposition 4.5. We thus conclude:

**Corollary 4.8.** The group \( F \) has subgroups isomorphic to \( \mathbb{Z} \wr \mathbb{Z} \) and definable with parameters in \( F \), in \( T \), and in \( V \).

5. **Interpretations of the Arithmetic**

In this section we complete our proof of the interpretability of the Arithmetic in \( F \), \( T \), and \( V \) with parameters. Furthermore, in the case of the group \( F \), or more generally, of the group \( F \) with non-trivial cyclic \( \Lambda \), we exhibit an interpretation of the Arithmetic which does not require parameters.

Apparently it is well known to specialists that every finitely generated virtually solvable but not virtually abelian group interprets the Arithmetic (see [22, 23], and also [10]). For the reader’s convenience, we present here our self-contained proof for the group \( \mathbb{Z} \wr \mathbb{Z} \).

**Lemma 5.1.** The group \( \mathbb{Z} \wr \mathbb{Z} \) interprets the Arithmetic with parameters. More precisely, if \( \mathbb{Z} \wr \mathbb{Z} = \langle b \rangle \wr \langle a \rangle \), \( \langle b \rangle \cong \langle a \rangle \cong \mathbb{Z} \), then the bijection
\[
f : \{ a^n \mid n \in \mathbb{N} \} \to \mathbb{N}, \quad a^n \mapsto n
\]
is an interpretation of \( \langle \mathbb{N}, +, \times \rangle \) in \( \langle \mathbb{Z} \wr \mathbb{Z}, \times \rangle \) with parameters.

**Proof.** Denote
\[
G = \mathbb{Z} \wr \mathbb{Z} = \langle b \rangle \wr \langle a \rangle \quad \text{et} \quad H = \langle a^{-k}ba^k \mid k \in \mathbb{Z} \rangle.
\]
Recall that \( G = H \rtimes \langle a \rangle \), and that \( H \) is a free abelian group with the basis \( \langle a^{-k}ba^k \rangle_{k \in \mathbb{Z}} \). It is easy to verify that \( \langle a \rangle = C_G(a) \) and \( H = C_G(b) \).

Consider the bijection
\[
g : \langle a \rangle \to \mathbb{Z}, \quad a^n \mapsto n.
\]
It will suffice to show that \( g \) is an interpretation of \( \langle \mathbb{N}, +, \times \rangle \) in \( \langle G, \times \rangle \). Indeed, \( \langle \mathbb{N}, +, \times \rangle \) is a substructure of \( \langle \mathbb{Z} \wr \mathbb{Z}, \times \rangle \), and \( \mathbb{N} \) is 0-definable in \( \langle \mathbb{Z} \wr \mathbb{Z}, \times \rangle \) because, by Lagrange’s four-square theorem, every positive integer is the sum of four squares.

The domain of \( g \) is the centralizer of \( a \), hence definable. The operation induced on \( \langle a \rangle \) by the addition of \( \mathbb{Z} \) via \( g \) is simply the restriction of the multiplication of \( G \), hence 0-definable. It remains to show that the operation induced on \( \langle a \rangle \) by the multiplication of \( \mathbb{Z} \) via \( g \) is definable.
Observe the following facts:

1. For every \( x \in G \setminus H \), \( C_G(x) \) is cyclic,
2. For every \( n \in \mathbb{Z} \setminus \{0\} \), \( C_G(ba^n) = \langle ba^n \rangle \),
3. For every \( n \in \mathbb{Z} \), \( HC_G(ba^n) = H(a^n) \).

(The second fact is due to the homomorphism \( G \to \mathbb{Z}, a \mapsto 0, b \mapsto 1 \).)

Denote by \( | \) the relation of the divisibility in \( \mathbb{Z} \). Observe that for all \( m, n \in \mathbb{Z} \),

\[ m|n \iff HC_G(ba^m) \supset HC_G(ba^n). \]

The relation \( HC_G(bx) \supset HC_G(by) \) between \( x, y \in G \) can be expressed by a first-order formula with the parameter \( b \). Hence the relation induced on \( \langle a \rangle \) by \( | \) via \( g \) is definable.

The multiplication in \( \mathbb{Z} \) is definable in terms of the addition, the divisibility, and the constant 1, as can be seen from the following equivalences satisfied in \( \mathbb{Z} \):

\[ n = k(k + 1) \iff (\forall m) \left( n|m \leftrightarrow k|m \wedge (k + 1)|m \right) \wedge (2k + 1)|(2n - k), \]

\[ n = kl \iff (k + l)(k + l + 1) = k(k + 1) + l(l + 1) + 2n \]

(see [30, §5a] for details). Thus the operation induced on \( \langle a \rangle \) by the multiplication of \( \mathbb{Z} \) via \( g \) is definable.

Theorem A (see the Introduction) is a corollary of Proposition 4.5 and Lemma 5.1. In order to prove Theorem B, we shall construct new 0-interpretations of the Arithmetic in groups of \( \mathcal{F} \)-kind:

**Proposition 5.2.** If \( \Lambda \) is cyclic, \( \Lambda = \langle p \rangle \), then the map

\[ f : \{ x \in \mathcal{F} \mid (0)x^p = (r)x^r > 1 \} \to \mathbb{N}, \quad x \mapsto \log_p((0)x^p) \]

is an interpretation of \( (\mathbb{N}, +, \times) \) in \( (\mathcal{F}, \times) \) without parameters.

One of the main ideas of the proof of Proposition 5.2 is the use of the centralizers of pairs of elements. The following lemma is similar to Theorem 5.5 in [6]:

**Lemma 5.3.** Let \( H \) be a subgroup of \( \mathcal{F} \). Then \( H \) is the centralizer of an element if and only if \( H \) can be decomposed into a direct product of subgroups \( H_1, \ldots, H_n, n \in \mathbb{N}, \) such that there exist \( \alpha_0, \ldots, \alpha_n \in A \) such that:

1. \( 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = r; \)
2. for every \( i = 1, \ldots, n \), either \( H_i = \mathcal{F}(\alpha_{i-1}, \alpha_{i}) \), or there exists \( x \) in \( \mathcal{F}(\alpha_{i-1}, \alpha_{i}) \) such that \( H_i \) is the centralizer of \( x \) in \( \mathcal{F}(\alpha_{i-1}, \alpha_{i}) \), and \( H_i = \langle x \rangle \);
3. for every \( i = 1, \ldots, n-1 \), if \( H_i = \mathcal{F}(\alpha_{i-1}, \alpha_{i}) \), then \( H_{i+1} \neq \mathcal{F}(\alpha_i, \alpha_{i+1}) \).

**Proof.** Let \( x \in \mathcal{F} \) and \( H = C_\mathcal{F}(x) \). Choose \( \alpha_0, \ldots, \alpha_n \) such that \( 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = r \) and

\[ \{\alpha_1, \ldots, \alpha_{n-1}\} = \{ \alpha \in \langle 0, r \rangle \cap A \cap \text{Fix}(x) \mid (\alpha)x^\alpha \neq 1 \text{ or } (\alpha)x^\alpha \neq 1 \} .\]

---

5Centralizers in \( \mathcal{F}(r, \mathbb{R}_+, \mathbb{R}) \) have been described by Brin and Squier [6]. Collin Bleak and others [5] have recently announced classification of all centralizers in \( T(1, \langle n \rangle, \mathbb{Z}[\frac{1}{n}]) \) and \( V(1, \langle n \rangle, \mathbb{Z}[\frac{1}{n}]), n = 2, 3, \ldots \).
For every $i = 1, \ldots, n$, choose $x_i \in \mathcal{F}_{(a_i,1,a_i)}$ such that $x_i|_{(a_{i-1},a_i)} = x|_{(a_{i-1},a_i)}$, and let $H_i$ be the centralizer of $x_i$ in $\mathcal{F}_{(a_i,1,a_i)}$. Note that $x = x_1 \cdot \cdots \cdot x_n$. For every $i = 1, \ldots, n$, if $x_i \neq \text{id}$, then $H_i$ is cyclic (see Lemma 4.2).

Similarly to Lemma 2.3, it is easy to prove that every element $y$ of $H$ permutes the set $\{a_0, \ldots, a_{n-1}\}$, and hence, as this set is finite, $y$ fixes all its elements. Thus $H = H_1 \times \cdots \times H_n$.

Conversely, suppose that $H = H_1 \times \cdots \times H_n$, $n \in \mathbb{N}$, and that the subgroups $H_1, \ldots, H_n$ and the points $a_0, \ldots, a_n \in A$ are as in the statement of this lemma. For every $i = 1, \ldots, n$, choose $x_i \in \mathcal{F}_{(a_i,1,a_i)}$ such that $H_i$ be the centralizer of $x_i$ in $\mathcal{F}_{(a_i,1,a_i)}$. Then $H = C_F(x_1 \cdot \cdots \cdot x_n)$. □

Lemma 5.4. Let $H$ be a subgroup of $\mathcal{F}$. Then $H$ is the centralizer of a pair of elements (possibly equal) if and only if $H$ can be decomposed into a direct product of subgroups $H_1, \ldots, H_n$, $n \in \mathbb{N}$, such that there exist $a_0, \ldots, a_n \in A$ such that:

1. $0 = a_0 < a_1 < \cdots < a_n = r$;
2. for every $i = 1, \ldots, n$, either $H_i = \{\text{id}\}$, or $H_i = \mathcal{F}_{(a_i,1,a_i)}$, or there exists $x$ in $\mathcal{F}_{(a_i,1,a_i)}$ such that $H_i$ is the centralizer of $x$ in $\mathcal{F}_{(a_i,1,a_i)}$, and $H_i = \langle x \rangle$;
3. for every $i = 1, \ldots, n-1$, if $H_i = \mathcal{F}_{(a_i,1,a_i)}$, then $H_{i+1} \neq \mathcal{F}_{(a_i,1,a_i+1)}$.

Proof. This lemma is an easy corollary of Lemma 5.3 and the fact that for all $\alpha, \beta \in A$ such that $0 < \alpha < \beta < r$, there exist $x, y \in \mathcal{F}$ such that $\text{Supp}(x) = \text{Supp}(y) = (\alpha, \beta)$ and $xy \neq yx$, and hence the centralizer of $\{x, y\}$ in $\mathcal{F}_{(\alpha,\beta)}$ is trivial (see Lemmas 4.2 and 4.3). □

Proof of Proposition 5.2. Without loss of generality, suppose that $p > 1$.

Denote by $\mathcal{F}^0$ the subgroup of $\mathcal{F}$ formed by the elements that are identities in neighborhoods of 0 and $r$:

$$\mathcal{F}^0 = \{ x \in \mathcal{F} \mid (0)x^r = (r)x^r = 1 \}.$$ 

Denote by $B$ the domain of $f$:

$$B = \{ x \in \mathcal{F} \mid (0)x^r = (r)x^r > 1 \}.$$ 

Note that for any two elements $x$ and $y$ of $B$, $f(x) = f(y)$ if and only if $xy^{-1} \in \mathcal{F}^0$ (see Lemma 4.1).

Lemma 4.3 allows to conclude that $f$ is a surjection onto $\mathbb{N}$ (see Figure 3 for example). It will suffice to show that the set $B$, the group $\mathcal{F}^0$, and the relations induced on $B$ via $f$ by the addition and the divisibility of $\mathbb{N}$ are all 0-definable (see Remark 2.10). Indeed, the multiplication is 0-definable in $\mathbb{N}$ in terms of the addition and the divisibility—see [30, §4b] or the proof of Lemma 5.1.

Define

$$S = \{ \mathcal{F}_{(\alpha,\beta)} \mid \alpha, \beta \in A, \ 0 \leq \alpha < \beta \leq r \},$$
$$S_0 = \{ \mathcal{F}_{(\alpha,\beta)} \mid \alpha, \beta \in A, \ 0 < \alpha < \beta < r \},$$
$$S_1 = \{ \mathcal{F}_{(0,\beta)} \mid \beta \in A, \ 0 < \beta < r \} \cup \{ \mathcal{F}_{(\alpha,r)} \mid \alpha \in A, \ 0 < \alpha < r \}. $$
Figure 3. An element $a$ of $F$ such that $(0)a^+ = (1)a^- = 2$.

Then not only is the family $S$ uniformly definable, but also there exist two first-order formulae $\phi(x_1, x_2, x_3)$ and $\psi(x_1, x_2)$ (in the language of groups) without parameters such that:

1. for all $\alpha, \beta \in A$ such that $0 \leq \alpha < \beta \leq r$, there exist $x, y \in F$ such that $F \models \psi(x, y)$ and $F_{(\alpha, \beta)} = \{ z \in F \mid F \models \phi(x, y, z) \}$;
2. for all $x, y \in F$ such that $F \models \psi(x, y)$, there exist $\alpha, \beta \in A$ such that $0 \leq \alpha < \beta \leq r$ and $F_{(\alpha, \beta)} = \{ z \in F \mid F \models \phi(x, y, z) \}$.

Indeed, a subset of $F$ is of the form $F_{(\alpha, \beta)}$, where $\alpha, \beta \in A$ and $0 \leq \alpha < \beta \leq r$, if and only if it is the centralizer of a pair, is not abelian, and cannot be decomposed as the direct product of two other centralizers of pairs (see Lemma 5.4). All this can be expressed in the first-order language.

The families $S_0$ and $S_1$ of subsets of $F$ are “uniformly definable without parameters” in the same sense as $S$ because

\[
S_0 = \{ H_1 \cap H_2 \mid H_1, H_2 \in S, H_1 \not\subset H_2, \text{ and } H_2 \not\subset H_1 \},
\]

\[
S_1 = S \setminus (S_0 \cup \{ F \}).
\]

Since $F^0 = \bigcup S_0$, this subgroup is 0-definable.$^6$

Define

\[
E = \{ x \in F \mid (0)x^+ \neq 1 \text{ or } (r)x^- \neq 1 \} = F \setminus \bigcup S_0 = F \setminus F^0,
\]

\[
E_2 = \{ x \in F \mid (0)x^+ \neq 1 \text{ and } (r)x^- \neq 1 \} = F \setminus \bigcup S_1 = E \setminus \bigcup S_1.
\]

These sets are 0-definable.

Define

\[
P^+ = \{ x \in F \mid (0)x^+ > 1 \text{ and } (r)x^- > 1 \},
\]

\[
P^- = \{ x \in F \mid (0)x^+ < 1 \text{ and } (r)x^- < 1 \},
\]

and $P = P^+ \cup P^-$. These sets are 0-definable: for all $x \in F$,

\[
x \in P^+ \iff \left( \exists X \in S_0 \left( \forall Y \in S_1 \right) \left( Y \supset X \implies x^{-1}Yx \subseteq Y \right) \right),
\]

---

$^6$In the case when $F = F$, there exists a more natural proof that $F^0$ is definable in $F$ since in this case $F^0 = F^0 = [F, F] = [F, F]$ (see [8, Theorem 4.1]), and every element of $[F, F]$ is the product of two commutators (see the Appendix). In general $F^0$ and $[F, F]$ are not equal (see [7, Section 4D]).
and 

$$x \in P^- \iff \exists X \in S_0 \left( \forall Y \in S_1 \left( Y \supset X \rightarrow x^{-1}Yx \not\subset Y \right) \right).$$

It has been used here that for all $\alpha, \beta$ and for all $x \in \mathcal{F}$,

$$x^{-1}\mathcal{F}_{(\alpha, \beta)}x = \mathcal{F}_{((\alpha)x, (\beta)x)}.$$

Define 

$$U = \{ x \in \mathcal{F} \mid (0)x'^+ = ((r)x'^-)^{-1} \in \{ p^\pm 1 \} \}.$$ 

Then $U$ is $0$-definable: $U \subset E_2 \setminus P$, and for all $x \in E_2 \setminus P$,

$$x \in U \iff \left( \forall y \in P \right) \left( \forall z \in \mathcal{F}^0 \right) \left( \exists w_1, w_2 \in C_F(xz) \right) \left( (w_1w_2^{-1} \in E_2) \land (yw_1, yw_2 \notin E_2) \right).$$ 

To facilitate reading of the last formula, we remark that $xy^{-1} \notin E_2$ means exactly that either $(0)x'^+ = (0)y'^+$, or $(r)x'^- = (r)y'^-$. Then the implication “$\Rightarrow$” is easy to check: such $w_1, w_2$ can always be found even in $(xz)$. The direction “$\Leftarrow$” is less obvious, we shall rather prove its contrapositive.

Let $x \in (E_2 \setminus P) \setminus U$. Without loss of generality, suppose that $(0)x'^+ > p$, and hence $(r)x'^- < 1$. Choose $y \in P$ such that $(0)y'^+ = p$ (see Lemma 4.3).

Let $\gamma \in (0, r) \cap A$, and choose $x_1, x_2 \in \mathcal{F}$ such that:

1. $\text{Supp}(x_1) = (0, \gamma)$, $\text{Supp}(x_2) = (\gamma, r)$,
2. $(0)x'^+ = (0)x'^+$, $(\gamma)x'^- = p^{-1}$,
3. $(r)x'^- = (r)x'^-, \ (\gamma)x'^+ = p$.

Then the centralizer of $x_1x_2$ is the direct product $\langle x_1 \rangle \times \langle x_2 \rangle$ (see Lemmas 2.3 and 4.1). Choose $z \in \mathcal{F}^0$ such that $xz = x_1x_2$. Now suppose that

$$w_1, w_2 \in C_F(xz) = \langle x_1 \rangle \times \langle x_2 \rangle.$$

Then $(0)(yw_1)'^+ \neq 1$ and $(0)(yw_2)'^+ \neq 1$. Suppose that $yw_1, yw_2 \notin E_2$. Then $(r)(yw_1)'^- = (r)(yw_2)'^- = 1$, and hence $(r)(w_1w_2^{-1})'^- = 1$, or in other terms $w_1w_2^{-1} \notin E_2$. Thus $U$ is indeed $0$-definable.

For all $x \in P$,

$$\forall y \in U \iff \forall z \in C_F(y) \left( \exists w \in C_F(xz) \right) \left( zr, rz^{-1} \notin E_2 \right).$$

Thus, the set $B$ is $0$-definable.

The $f$-preimage of the graph of the addition of $\mathbb{N}$ is $0$-definable since it is the graph of the multiplication modulo $\mathcal{F}^0$: for all $x, y, z \in B$,

$$(x)f + (y)f = (z)f \iff xyz^{-1} \in \mathcal{F}^0.$$ 

It remains to show that the $f$-preimage of the graph of the divisibility of $\mathbb{N}$ is $0$-definable. This is indeed the case: for all $x, y \in B$,

$$\forall z \in \mathcal{F}^0 \left( \exists w \in C_F(xz) \right) \left( yw \in \mathcal{F}^0 \right).$$

The implication “$\iff$” here is the least evident. In order to establish its contrapositive, one can take $\gamma \in (0, r) \cap A$, choose $x_1, x_2 \in \mathcal{F}$ such that:

1. $\text{Supp}(x_1) = (0, \gamma)$, $\text{Supp}(x_2) = (\gamma, r)$,
2. $(0)x'^+ = (0)x'^+$, $(\gamma)x'^- = p^{-1}$,
3. $(r)x'^- = (r)x'^-, \ (\gamma)x'^+ = p^\pm 1$,
and choose $z \in F^\sigma$ such that $xz = x_1 x_2$ (and hence $C_F(xz) = \langle x_1 \rangle \times \langle x_2 \rangle$).

Theorem B is a corollary of Proposition 5.2.

6. Undecidability

In this section we deduce from [19, Theorem 9] (see Theorem 6.3 below) that the elementary theory of every structure of finite signature that interprets the Arithmetic with parameters is hereditarily undecidable.

In what follows, the constants are treated as functions of arity 0, and similarly constant symbols are viewed as a particular case of function symbols. If $\sigma$ is a relation symbol, function symbol, or a constant symbol, its arity shall be denoted by $\text{ar}(\sigma)$.

We shall say that an $n$-ary relation $R$ on a set $B$ is compatible with an equivalence relation $E$ on $B$ if $R$ is induced by a relation (of the same arity) on $B/E$, i.e. if belonging of an $n$-tuple $(b_1, \ldots, b_n)$ to $R$ is completely determined by the classes of $E$-equivalence of $b_1, \ldots, b_n$.

Let $\Sigma$ and $\Gamma$ be two signatures, and let $N$ be a $\Gamma$-structure. Let $n \in \mathbb{N}$, $B$ be a definable subset of $N^n$, and $E$ be an equivalence relation on $B$, also definable in $N$. Let $\phi, \psi,$ and $\xi_\sigma$ for all $\sigma \in \Sigma$ be $\Gamma$-formulae with parameters from $N$ such that:

1. $\phi$ defines $B$,
2. $\psi$ defines $E$,
3. for every relation symbol $\sigma \in \Sigma$, the formula $\xi_\sigma$ defines a relation on $B$ compatible with $E$ of arity $\text{ar}(\sigma)n$, and
4. for every function or constant symbol $\sigma \in \Sigma$, the formula $\xi_\sigma$ defines a relation on $B$ of arity $(\text{ar}(\sigma) + 1)n$ which is compatible with $E$ and which defines the graph of an operation on $B/E$ of arity $\text{ar}(\sigma)$.

Then denote by $\text{Ent}_\Sigma(N, \phi, \psi, (\xi_\sigma)_{\sigma \in \Sigma})$ the $\Sigma$-structure naturally defined on $B/E$ by the family $(\xi_\sigma)_{\sigma \in \Sigma}$.

Remark 6.1. In the same notation, the natural projection $p : B \rightarrow B/E$ is an interpretation of $\text{Ent}_\Sigma(N, \phi, \psi, (\xi_\sigma)_{\sigma \in \Sigma})$ in $N$.

Lemma 6.2. Let $M$ and $N$ be two structures of finite signatures such that $\text{Th}(M)$ is hereditarily undecidable, and $N$ interprets $M$ with parameters. Then $\text{Th}(N)$ is hereditarily undecidable as well.

Proof. Denote the signature of $M$ by $\Sigma$ and the signature of $N$ by $\Gamma$.

It suffices to consider only the case when $\Sigma$ contains no function or constant symbols. Indeed, let $\Sigma'$ be the signature obtained from $\Sigma$ by replacing every $n$-ary function symbol $f$ ($n \geq 0$) by an $(n+1)$-ary relation symbol $f'$. For every $\Sigma$-structure $M$, denote by $M'$ the $\Sigma'$-structure on the underlying set of $M$ in which every new relation symbol of $\Sigma'$ is interpreted by the graph of the function in $M$ named by the corresponding symbol of $\Sigma$, and all the other symbols of $\Sigma'$ are interpreted in $M'$ exactly like in $M$. For every $\Sigma$-theory $S$, denote by $S'$ the $\Sigma'$-theory of the class $\{ M' \mid M \models S \}$.

It is easy to see that:

1. the (class-)map $M \mapsto M'$, where $M$ is a $\Sigma$-model of $S$, is a (class-) bijection between $\text{Mod}_\Sigma S$ and $\text{Mod}_{\Sigma'} S'$ for every $\Sigma$-theory $S$,
(2) if $O_{\Sigma}$ denotes the minimal $\Sigma$-theory, then the map $S \mapsto S'$ of $\Sigma$-theories to $\Sigma'$-theories is a bijection between all the $\Sigma$-theories and all the $\Sigma'$-theories containing $O_{\Sigma}$ ($O_{\Sigma}'$ expresses simply that the new relation symbols of $\Sigma'$ are to be interpreted by graphs of functions),

(3) for every $\Sigma$-structure $M$, a set is 0-definable in $M$ if and only if it is 0-definable in $M'$, or in other words $\text{id}_M$ is a 0-interpretation of $M$ in $M'$ and of $M'$ in $M$.

It is easy to provide an algorithm which converts every $\Sigma$-sentence $\phi$ into a $\Sigma'$-sentence $\psi$ such that for every $\Sigma$-structure $M$, $M \models \phi$ if and only if $M' \models \psi$, and it is equally easy to provide and algorithm which converts every $\Sigma'$-sentence into a $\Sigma$-sentence equivalent in the same sense. Therefore, for every $\Sigma$-theory $S$, $S'$ is essentially undecidable if and only if such is $S$. Hence we suppose without loss of generality that $\Sigma$ contains only relation symbols.

Let $(n, f)$ be an interpretation of $M$ in $N$. Let $a_1, \ldots, a_m$ be a sequence of parameters from $N$ sufficient to define the domain and the kernel of $f$ and the $f$-preimage of the graph of every relation of $M$ ($\Sigma$ is finite): denote $\bar{a} = (a_1, \ldots, a_m)$. Let $x_1, \ldots, x_m, y_1, \ldots, y_n, y_1, \ldots, y_{2n}, \ldots$ be distinct variables, and denote $\bar{x} = (x_1, \ldots, x_m)$, $\bar{y} = (y_1, \ldots, y_n)$, $\bar{y}_1 = (y_{11}, \ldots, y_{1n})$, and so on. Let $\phi = \phi(\bar{x}, \bar{y})$, $\psi = \psi(\bar{x}, \bar{y}_1, \bar{y}_2)$, and $\xi_\sigma = \xi_\sigma(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k)$ for every $\sigma \in \Sigma$ of arity $k$ be $\Gamma$-formulae such that:

1. $\phi(\bar{a}, \bar{y})$ defines the domain of $f$ (which is a subset of $N^n$),
2. $\psi(\bar{a}, \bar{y}_1, \bar{y}_2)$ defines the kernel of $f$ (which is an equivalence relation on the domain),
3. for every symbol $\sigma \in \Sigma$, the formula $\xi_\sigma(\bar{a}, \bar{y}_1, \ldots, \bar{y}_{\ar(\sigma)})$ defines the $f$-preimage of the graph of the relation of $M$ named by $\sigma$.

Observe that the bijection induced by $f$ between the quotient of its domain by its kernel and its image is an isomorphism

$$\text{Int}_{\Sigma}(N, \phi(\bar{a}, \bar{y}), \psi(\bar{a}, \bar{y}_1, \bar{y}_2), (\xi_\sigma(\bar{a}, \bar{y}_1, \ldots, \bar{y}_{\ar(\sigma)}))_{\sigma \in \Sigma}) \cong M.$$ 

In what follows, let $\bar{c} = (c_1, \ldots, c_m)$ be a sequence of new constant symbols. Write $(\Gamma, \bar{c})$ to denote the signature obtained from $\Gamma$ by adding $c_1, \ldots, c_m$ (as constant symbols).

Let $\tau = \tau(\bar{x})$ be a $\Gamma$-formula such that the $(\Gamma, \bar{c})$-sentence $\tau(\bar{c})$ expresses that:

1. $\psi(\bar{c}, \bar{y}_1, \bar{y}_2)$ defines an equivalence relation on the set defined by $\phi(\bar{c}, \bar{y})$,
2. for every relation symbol $\sigma \in \Sigma$, the formula $\xi_\sigma(\bar{c}, \bar{y}_1, \ldots, \bar{y}_{\ar(\sigma)})$ defines a relation on the set defined by $\phi(\bar{c}, \bar{y})$ which is compatible with the equivalence relation defined by $\psi(\bar{c}, \bar{y}_1, \bar{y}_2)$.

Clearly all this can be expressed in the first-order language.\footnote{One can take as $\tau(\bar{c})$ the conjunction of the \textit{admissibility conditions} in the sense of [15, Section 5.3].} Note that

$$N \models \tau(\bar{a}).$$ 

Choose a recursive (i.e. computable by an algorithm) map $f$ from the set of $\Sigma$-sentences to the set of $\Gamma$-formulae all of whose free variables are among

$x_1, \ldots, x_m$ such that for every $\Sigma$-sentence $\alpha$ and every $(\Gamma, \vec{c})$-structure $L$ such that $L \models \tau(\vec{c})$,

$$L \models \alpha^I(\vec{c}) \iff (\exists \text{Int}(L, \phi(\vec{c}, \vec{y}), \psi(\vec{c}, \vec{y}_1, \vec{y}_2), (\xi_\sigma(\vec{c}, \vec{y}_1, \ldots, \vec{y}_{\text{ar}(\sigma)}))_{\sigma \in \Sigma}) \models \alpha).$$

It is easy to construct such a $t$ that uses the formula $\phi$ to relativize the quantifiers, the formula $\psi$ to replace $\neg =$, and the formula $\xi_\sigma$ to replace each $\sigma \in \Sigma$. Here is an example, where $\sigma$ is a binary relation symbol:

$$\alpha = \neg((\exists y_1, y_2, y_3) (\sigma(y_1, y_2) \land \sigma(y_1, y_3) \rightarrow y_2 = y_3)^7),$$

$$\alpha^I(x) = \neg((\exists y_1, y_2, y_3) (\phi(x, y_1) \land \phi(x, y_2) \land \phi(x, y_3)$$

$$\rightarrow (\xi_\sigma(x, y_1, y_2) \land \xi_\sigma(x, y_1, y_3) \rightarrow \psi(x, y_2, y_3))^7).$$

(As is customary, we do not show all the parentheses; they should be added according to the standard rules.)

Note the following properties of $t$:

(1) for every $\Sigma$-sentence $\alpha$,

$$\left(M \models \alpha \right) \iff \left(N \models \alpha^I(\vec{a}) \right);$$

(2) for every $\Sigma$-sentence $\alpha$,

$$\left(\models_{\Sigma} \alpha \right) \implies \left(\tau(\vec{c}) \models_{(\Gamma, \vec{c})} \alpha^I(c) \right);$$

(3) for every $\Sigma$-sentences $\alpha$ and $\beta$,

$$\tau(\vec{c}) \models_{(\Gamma, \vec{c})} (\alpha \land \beta)^I(c) \iff \alpha^I(c) \land \beta^I(c),$$

$$\tau(\vec{c}) \models_{(\Gamma, \vec{c})} (\neg \alpha)^I(c) \iff \neg \alpha^I(c),$$

and the same for the other boolean operations;

(4) for every $(\Gamma, \vec{c})$-theory $T$ such that $T \models_{(\Gamma, \vec{c})} \tau(\vec{c})$, the set

$$\{ \Sigma\text{-sentence } \alpha \mid T \models_{(\Gamma, \vec{c})} \alpha^I(\vec{c}) \}$$

is a $\Sigma$-theory.

Suppose now that $\text{Th}(N)$ were not hereditarily undecidable. Then let $T$ be a decidable $\Gamma$-subtheory of $\text{Th}(N)$. Let $S$ be the set of all the $\Gamma$-formulae $\alpha(\vec{x})$ such that

$$T \models_{\Gamma} (\forall \vec{x}) \left( \tau(\vec{x}) \rightarrow \alpha(\vec{x}) \right).$$

Then $\tau \in S$, $N \models \alpha(\vec{a})$ for all $\alpha \in S$, $S$ is a recursive (decidable) set, and

$$\{ \alpha(\vec{c}) \mid \alpha(\vec{x}) \in S \}$$

is a $(\Gamma, \vec{c})$-theory. Let $U$ be the preimage of $S$ under $t$. Then $U \subset \text{Th}(M)$, $U$ is a $\Sigma$-theory, and $U$ is decidable in contradiction with the hereditary undecidability of $\text{Th}(M)$. □

**Theorem 6.3** (Mostowski, Tarski, [20]). The elementary theory of the Arithmetic $(\mathbb{N}, +, \times)$ has an essentially undecidable finitely axiomatized subtheory.

For an improved proof of this fact, see [19, Theorem 9].

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8In [15, Section 5.3] such $t$ is called a reduction map.
**Proposition 6.4.** Let $M$ be a structure of finite signature which interprets the Arithmetic $(\mathbb{N}, +, \times)$ with parameters. Then $\text{Th}(M)$ is hereditarily undecidable.

**Proof.** This is a corollary of Theorem 6.3 and Lemmas 2.13 and 6.2. 

Theorem C (see the Introduction) follows now from Theorem A and Proposition 6.4.

7. Open questions

We conclude with two questions which, to our knowledge, are open.

Thompson’s group $F$ is definable in $T$. However, it is not known to the authors whether $F$ is definable in $V$.

**Question 1.** Is Thompson’s group $F$ definable in $V$ with parameters?

We have already shown that the Arithmetic is interpreted in $F$. In addition, $F$ is interpreted in the Arithmetic because the word problem for $F$ is decidable. (Matiyasevich’s theorem, see [18] or [9, Theorem 8.1], implies that every recursive or recursively enumerable subset of $\mathbb{N}^n$, $n \in \mathbb{N}$, is definable in the Arithmetic; thus every reasonable encoding of elements of $F$ by positive integers gives an interpretation of $F$ in the Arithmetic.) However, even if a structure interprets the Arithmetic and, reciprocally, is interpreted in the Arithmetic, they are not necessarily bi-interpretable (see [17, Theorem 6] or [21, Theorem 7.16]), hence the question:

**Question 2.** Is the group $F$ bi-interpretable with the Arithmetic with parameters?

We say that a structure $S$ is categorically finitely axiomatized in a class $C$ of structures of the same signature if $S \in C$ and there exists a first-order sentence $\phi$ such that $S \models \phi$ and every structure in $C$ that satisfies $\phi$ is isomorphic to $S$. According to Anatole Khélif [17], bi-interpretability with the Arithmetic can be used to demonstrate categoric finite axiomatization in classes of finitely generated structures of finite signature. Thomas Scanlon [32] has recently established bi-interpretability of the Arithmetic with all finitely generated fields and used it to show that all such fields are categorically finitely axiomatized within the class of finitely generated fields, and thus Pop’s conjecture holds true:

two finitely generated fields are elementary equivalent if and only if they are isomorphic.

André Nies raised the question whether there exists a finitely generated simple group categorically finitely axiomatized among all the finitely generated simple groups [21, Question 7.8].

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9We shall not show this here.

10In the case when the class $C$ consists of all the finitely generated structures of a certain class, André Nies [21] and Anatole Khélif [17] used the term “quasi finitely axiomatized” in more or less the same sense as we use “categorically finitely axiomatized.” However, the definitions in [21] and [17] are not precise because the property of finite generation is not intrinsic and depends on the class in which a given structure is considered.
8. Appendix

Here we show that every element of the derived subgroup of $\mathcal{F}$ is the product of two commutators, and hence $[\mathcal{F}, \mathcal{F}]$ is 0-definable in $\mathcal{F}$. The proof of this fact, which, incidentally, has not been used in this paper, is known to specialists. However, this fact is closely related to the definable structure of the groups that we study here, and apparently it does not appear anywhere else in the literature. For Thompson’s group $F$, this result is probably part of folklore; we learned its proof from Matthew Brin, who had slightly modified the argument of Keith Dennis and Leonid Vaserstein [11, Proposition 1(c)]. Our argument is just a trivial generalization.

As in the proof of Proposition 5.2, define

$$\mathcal{F}^0 = \{ x \in \mathcal{F} \mid (0)x^{r+} = (r)x^{r-} = 1 \}. $$

**Proposition 8.1.** Every element of $[\mathcal{F}, \mathcal{F}]$ is the product of two commutators in $\mathcal{F}$, and even in $\mathcal{F}^0$:

$$[\mathcal{F}, \mathcal{F}] = \{ [x_1, x_2][x_3, x_4] \mid x_1, x_2, x_3, x_4 \in \mathcal{F}^0 \}. $$

**Proof.** First of all recall that $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}^0$ (see Lemma 4.1).

Consider any two elements $x, y \in \mathcal{F}$ and their commutator $c = [x, y]$. Choose $\alpha_1, \alpha_2, \beta_1, \beta_2 \in A$ such that $0 < \alpha_2 < \alpha_1 < \beta_1 < \beta_2 < r$ and $\text{Supp}(c) \subset (\alpha_1, \beta_1)$. Then there exists an endomorphism $h: \mathcal{F} \to \mathcal{F}(\alpha_2, \beta_2)$ such that $h$ is the identity on $\mathcal{F}(\alpha_1, \beta_1)$. Indeed, let $s: [0, r) \to [0, r)$ be a map which is the identity on $[\alpha_1, \beta_1]$ and which is affine on $[0, \alpha_1]$ and on $[\beta_1, r)$ with the slope $p \leq 1$, $p \in A$; then it is possible to take as $h$ the conjugation by $s$ composed with the natural embedding of permutations of the interval $[0, r)(s)(r)$ into permutation of $[0, r)$ (this $h$ is even injective). If $h$ is such an endomorphism, if $(x)h = x'$, and if $(y)h = y'$, then

$$c = (c)h = [x', y'] \quad \text{and} \quad \text{Supp}(x') \cup \text{Supp}(y') \subset (\alpha_2, \beta_2).$$

The above has two consequences of interest for us:

1. If $x, y \in \mathcal{F}$, then there exist $x', y' \in \mathcal{F}^0$ such that $[x, y] = [x', y']$;
2. If $c_1, c_2, \ldots, c_n$ are commutators in $\mathcal{F}$, and $I_1, I_2, \ldots, I_n$ are pairwise disjoint closed subintervals of $[0, r)$ such that $\text{Supp}(c_i) \subset I_i$ for all $i = 1, \ldots, n$, then the product $c_1c_2\cdots c_n$ is also a commutator.

Now let $c_1$, $c_2$, and $c_3$ be three arbitrary commutators in $\mathcal{F}$. Choose $\alpha, \beta \in A \cap (0, r)$ such that

$$\text{Supp}(c_1) \cup \text{Supp}(c_2) \cup \text{Supp}(c_3) \subset (\alpha, \beta).$$

Choose $b \in \mathcal{F}$ such that $(\alpha)b > \beta$. Then $0 < (\beta)b^{-1} < \alpha < \beta < (\alpha)b < r$. Since

$$\text{Supp}(c_1^b) \subset ((\alpha)b, r) \quad \text{and} \quad \text{Supp}(c_3^{b^{-1}}) \subset (0, (\beta)b^{-1}),$$

the product $c_1c_2c_3^{-b}$ is a commutator.\footnote{We are using the standard notation: $x^y = y^{-1}xy$, $[x, y] = x^{-1}y^{-1}xy$.} Hence

$$c_1c_2c_3 = c_1c_2c_3^{-b}
\left((c_3^{-1})b^{-1}c_2\right)
\left((c_2^{-1})bc_3\right)
= (c_1c_2c_3^{b^{-1}})[c_2^{-1}c_3^{-b^{-1}}, b]$$

is the product of two commutators.
References

1. Gisela Ahlbrandt and Martin Ziegler, Quasi finitely axiomatizable totally categorical theories, Ann. Pure Appl. Logic 30 (1986), no. 1, 63–82.

2. Valery G. Bardakov and Vladimir A. Tolstykh, Interpreting the arithmetic in Thompson’s group \( F \), J. Pure Appl. Algebra 211 (2007), no. 3, 633–637, Preprint: arXiv: math/0701748.

3. James M. Belk and Kenneth S. Brown, Forest diagrams for elements of Thompson’s group \( F \), Internat. J. Algebra Comput. 15 (2005), no. 5–6, 815–850.

4. Robert Bieri and Ralph Strebel, On groups of PL-homeomorphisms of the real line, Notes, Math. Sem. der Univ. Frankfurt, 1985.

5. Collin Bleak, Alison Gordon, Garrett Graham, Jacob Hughes, Francesco Matucci, Hannah Newfield-Plunkett, and Eugenia Sapir, Using dynamics to analyze centralizers in the generalized Higman-Thompson groups \( V_n \), Incomplete preprint.

6. Matthew G. Brin and Craig C. Squier, Presentations, conjugacy, roots, and centralizers in groups of piecewise linear homeomorphisms of the real line, Comm. Algebra 29 (2001), no. 10, 4557–4596.

7. Kenneth S. Brown, Finiteness properties of groups, J. Pure Appl. Algebra 44 (1987), no. 1–3, 45–75.

8. James W. Cannon, William J. Floyd, and Walter R. Parry, Introductory notes on Richard Thompson’s groups, Enseign. Math. (2) 42 (1996), no. 3–4, 215–256, Preprint: http://www.geom.uiuc.edu/docs/preprints/lib/GCG63/thompson.ps.

9. Martin Davis, Hilbert’s tenth problem is unsolvable, Amer. Math. Monthly 80 (1973), 233–269.

10. François Delon and Patrick Simonetta, Undecidable wreath products and skew power series fields, J. Symbolic Logic 63 (1998), no. 2, 671–676.

11. Yuri L. Ershov, Problemy razreshimosti i konstruktivnye modeli [Decision problems and constructive models], Mathematicheskaya Logika i Osnovaniya Matematiki [Mathematical Logic and Foundations of Mathematics], Nauka, Moscow, 1980 (Russian).

12. Yuri V. Matijasevich, Diofantovost’ perechislimykh mnozhestv [The Diophantiness of enumerable sets], Dokl. Akad. Nauk SSSR 191 (1970), 279–282 (Russian), English translation in Soviet Math. Dokl.

13. Wilfrid Hodges, Model theory, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, 1993.

14. Yuri L. Ershov, Problemy razreshimosti i konstruktivnye modeli [Decision problems and constructive models], Matematicheskaya Logika i Osnovaniya Matematiki [Mathematical Logic and Foundations of Mathematics], Nauka, Moscow, 1980 (Russian).

15. Kurt Gödel, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I [On formally undecidable propositions of Principia Mathematica and related systems, I], Monatsh. Math. 149 (2006), no. 1, 1–30 (German), Reprinted from Monatsh. Math. Phys. 38 (1931), 173–198 [MR1549910], With an introduction by Sy-David Friedman. MR MR2260656

16. Graham Higman, Finitely presented infinite simple groups, Notes on Pure Mathematics, vol. 8, Australian National University, Canberra, 1974.

17. Anatole Khelif, Bi-interprétabilité et structures QFA : étude de groupes résolubles et des anneaux commutatifs [bi-interpretable and qfa structures: study of some solvable groups and commutative rings], C. R. Math. Acad. Sci. Paris 345 (2007), no. 2, 59–61. MR MR2334552 (2008e:03058)

18. Yuri V. Matijasevich, Diofantovost’ perechislimykh mnozhestv [The Diophantiness of enumerable sets], Dokl. Akad. Nauk SSSR 191 (1970), 279–282 (Russian), English translation in Soviet Math. Dokl.

19. Andrzej Mostowski, Raphael M. Robinson, and Alfred Tarski, Undecidability and essential undecidability in arithmetic, Undecidable theories, Studies in logic and the foundations of mathematics, North-Holland Publishing Co., Amsterdam, 1971, pp. 36–74.

20. Andrzej Mostowski and Alfred Tarski, Undecidability in the arithmetic of integers and in the theory of rings, J. Symbolic Logic 14 (1949), 76.

21. André Nies, Describing groups, Bull. Symbolic Logic 13 (2007), no. 3, 305–339.

22. Gennady A. Noskov, Ob elementarnoj teorii konechno porozhdenykh pochti razreshimoj gruppy [On the elementary theory of a finitely generated almost solvable group],
23. On the elementary theory of a finitely generated almost solvable group, Math. USSR Izvestiya 22 (1984), no. 3, 465–482, Translated from Russian.

24. Anand Pillay, An introduction to stability theory, Oxford Logic Guides, vol. 8, The Clarendon Press, Oxford University Press, New York, 1983.

25. Bruno Poizat, Cours de théorie des modèles [Course on the theory of models]. Une introduction à la logique mathématique contemporaine [An introduction to contemporary mathematical logic]. Nur al-Mantiq wal-Ma’rifah, Bruno Poizat, Lyon, 1985 (French).

26. Groups stables [Stable groups]. Une tentative de conciliation entre la géométrie algébrique et la logique mathématique [An attempt at reconciling algebraic geometry and mathematical logic], Nur al-Mantiq wal-Ma’rifah [Light of Logic and Knowledge], vol. 2, Bruno Poizat, Lyon, 1987 (French).

27. A course in model theory. An introduction to contemporary mathematical logic, Universitext, Springer-Verlag, 2000, Translated from French by Moses Klein and revised by the author.

28. Stable groups, Mathematical Surveys and Monographs, vol. 87, American Mathematical Society, 2001, Translated from the 1987 French original by Moses Gabriel Klein.

29. Mike Y. Prest, Model theory and modules, London Mathematical Society Lecture Note Series, vol. 130, Cambridge University Press, 1988.

30. Raphael M. Robinson, Undecidable rings, Trans. Amer. Math. Soc. 70 (1951), 137–159.

31. Philipp Rothmaler, Introduction to model theory, Algebra, Logic and Applications, vol. 15, Gordon and Breach Science Publishers, Amsterdam, 2000, Translated and revised from the 1995 German original by the author.

32. Thomas Scanlon, Infinite finitely generated fields are biinterpretable with \( \mathbb{N} \), J. Amer. Math. Soc. 21 (2008), no. 3, 893–908, Preprint: http://math.berkeley.edu/~scanlon/papers/pc4sep07.pdf.

33. Zlil Sela, Diophantine geometry over groups VIII: stability, Preprint, arXiv:math/0609096, 2006.

34. Melanie Stein, Groups of piecewise linear homeomorphisms, Trans. Amer. Math. Soc. 332 (1992), no. 2, 477–514.

35. Alfred Tarski, A general method in proofs of undecidability, Undecidable theories, Studies in logic and the foundations of mathematics, North-Holland Publishing Co., Amsterdam, 1971, pp. 1–35.

36. Frank O. Wagner, Stable groups, Handbook of algebra, vol. 2, North-Holland Publishing Co., Amsterdam, 2000.

37. Thompson’s Group at 40 Years, 2004, Problem list of the workshop held 11–14 Jan., 2004, at American Institute of Mathematics, Palo Alto, California, http://www.aimath.org/WWN/thompsonsgroup/thompsonsgroup.pdf.