Non-local meta-conformal invariance in diffusion-limited erosion

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Received 20 June 2016, revised 17 October 2016
Accepted for publication 27 October 2016
Published 16 November 2016

Abstract
The non-stationary relaxation and physical ageing in the diffusion-limited erosion process (DLE) is studied through the exact solution of its Langevin equation, in $d$ spatial dimensions. The dynamical exponent $z = 1$, the growth exponent $\beta = \max(0, (1 - d)/2)$ and the ageing exponents $a = b = d - 1$ and $\lambda_C = \lambda_R = d$ are found. In $d = 1$ spatial dimension, a new representation of the meta-conformal Lie algebra, isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, acts as a dynamical symmetry of the noise-averaged DLE Langevin equation. Its infinitesimal generators are non-local in space. The exact form of the full time-space dependence of the two-time response function of DLE is reproduced for $d = 1$ from this symmetry. The relationship to the terrace-step-kink model of vicinal surfaces is discussed.

Keywords: diffusion-limited erosion, physical ageing, conformal invariance, local scale-invariance, terrace-step-kink model, non-locality, laplacian growth
on a hyper-cubic lattice \( \mathcal{L} \subset \mathbb{Z}^d \) of \(|\mathcal{L}| = L^d \) sites, where \( \langle \cdot \rangle \) denotes an average over many independent samples and \( \overline{h}(t) := L^{-d}\sum_{r \in \mathcal{L}} h(t, r) \) is the spatially averaged height. Herein, \( \alpha \) is the roughness exponent, \( \beta \) the growth exponent and \( z = \alpha / \beta > 0 \) the dynamical exponent. The interface is called rough if \( \beta > 0 \) and smooth if \( \lim_{t \to \infty} w(t) \) is finite. While theoretical studies abound, reliable experimental results are quite recent. Examples for the KPZ class are turbulent liquid crystals, cell colony growth, colloids, paper combustion, auto-catalytic reaction fronts, thin semiconductor films and sedimentation-electrodispersion, see [14, 42] for recent reviews and [20] for a list of measured values of these exponents. More subtle aspects can be studied through the non-equilibrium relaxation, analogous to physical ageing e.g. in glasses or simple magnets [17]. Analysis proceeds via the two-time correlator \( C(t; s; r) \) and the two-time response \( R(t; s; r) \). For sufficiently large lattices (where effectively \( L \to \infty \)) one expects, in the long-time scaling limit \( t, s \to \infty \) with \( y := t / s > 1 \) fixed, the scaling behaviour

\[
C(t; s; r) := \langle (h(t, r) - \overline{h}(t))(h(s, 0) - \overline{h}(s)) \rangle = s^{-\beta} F_C \left( \frac{t}{s}; \frac{r}{s^{1/z}} \right),
\]

\[
R(t; s; r) = \frac{\delta \langle h(t, r) - \overline{h}(t) \rangle}{\delta j(s, 0)} \bigg|_{j=0} = \langle h(t, r) \overline{h}(s, 0) \rangle = s^{-1-\beta} F_R \left( \frac{t}{s}; \frac{r}{s^{1/z}} \right),
\]

where spatial translation-invariance has been implicitly admitted and \( j \) is an external field conjugate to \( h \). The autocorrelation exponent \( \lambda_C \) and the autoresponse exponent \( \lambda_R \) are defined from the asymptotics \( F_C, R(y, 0) \sim y^{-\lambda_C z / 2} \) as \( y \to \infty \). For these non-equilibrium exponents, one has \( b = -2 \beta \) and the bound \( \lambda_C \geq (d + 2 \beta) / 2 \). For the EW, KPZ, WV and Arcetri classes, where the dynamical exponent \( z > \frac{3}{2} \), the values of \( a, b, \lambda_C, \lambda_R \) have been determined either analytically or in simulations [7, 9, 10, 13, 19, 20, 28, 31, 35] or else experimentally [41].

Here, we shall be interested in a different universality class, namely diffusion-limited erosion (DLE) [25], often also referred to as Laplacian growth. We shall first derive the width \( w(t) \), the correlator \( C(t; s; r) \) and the response \( R(t; s; r) \) from the exact solution of the defining Langevin equation. Then, for \( d = 1 \) spatial dimension, we shall construct a new representation of the conformal Lie algebra, in terms of spatially non-local operators. We shall show that (i) this representation acts as a dynamical symmetry of the equation of motion of DLE and (ii) that for \( d = 1 \), this dynamical symmetry (which has \( z = 1 \)), predicts the form of the response \( R(t; s; r) \).

2 The DLE process [25] can be defined as a lattice model by considering the diffusive motion of a corrosive particle, which starts initially far away from the interface. When the particle finally reaches the interface, it erodes a particle from that interface. Repeating this process many times, an eroding interface forms which is described in terms of a fluctuating height \( h(t, r) \). This leads to the Langevin equation for \( h(t, r) \) in DLE. In Fourier space [25, 26]

\[
\partial_t \hat{h}(t, \mathbf{q}) = -v |\mathbf{q}| \hat{h}(t, \mathbf{q}) + \hat{j}(t, \mathbf{q}) + \hat{\eta}(t, \mathbf{q})
\]

In the context of Janssen–de Dominicis theory, \( \hat{h} \) is the conjugate response field to \( h \), see [40].
including the gaussian white noise \( \tilde{\eta} \), with the variance \( \langle \tilde{\eta}(t, q) \tilde{\eta}(t', q') \rangle = 2\nu T \delta(t - t') \delta(q + q') \) and the constants \( \nu, T \) and an external perturbation \( \tilde{f} \). Several lattice formulations of the model exist [1, 25, 43, 45]. Flat and radial geometries are compared in [15]. Potential applications of DLE may include contact lines of a liquid meniscus and crack propagation [29]. Remarkably, for \( d = 1 \) space dimension, the Langevin equation (4) has been argued [37] to be related to a system of non-interacting fermions, conditioned to an a-typically large flux. Consider the terrace-step-kink model of a vicinal surface, and interpret the steps as the world lines of fermions. Its transfer matrix is the matrix exponential of the quantum hamiltonian \( H \) of the asymmetric XXZ chain [37]. Use Pauli matrices \( \sigma_n^{\pm, z} \), attached to each site \( n \), such that the particle number at each site is \( \rho_n = \frac{1}{2}(1 + \sigma_n^z) = 0, 1 \). On a chain of \( N \) sites [24, 34, 37]

\[
H = -\frac{w}{2} \sum_{n=1}^{N} [2\nu \sigma_n^\alpha \sigma_{n+1}^\beta + 2\nu^{-1} \sigma_n^\beta \sigma_{n+1}^\alpha] + \Delta (\sigma_n^z \sigma_{n+1}^z - 1),
\]

(5)

where \( w = \sqrt{pq} \), \( v = \sqrt{p/q} \), \( \Delta = 2(\sqrt{p/q} + \sqrt{q/p}) e^{-\mu} \). Herein, \( p, q \) describe the left/right bias of single-particle hopping and \( \lambda, \mu \) are the grand-canonical parameters conjugate to the current and the mean particle number. In the continuum limit, the particle density \( \rho_n(t) \rightarrow \rho(t, r) \) is related to the height \( h \) which in turn obeys (4), with a gaussian white noise \( \eta \) [37]. This follows from the application of the theory of fluctuating hydrodynamics, see [6, 38] for recent reviews. The low-energy behaviour of \( H \) yields the dynamical exponent \( z = 1 \) [24, 34, 37]. If one conditions the system to an a-typically large current, the large-time, large-distance behaviour of (5) has very recently been shown [24] (i) to be described by a conformal field-theory with central charge \( c = 1 \) and (ii) the time-space scaling behaviour of the stationary structure function has been worked out explicitly, for \( \lambda \rightarrow \infty \). Therefore, one may conjecture that the so simple-looking equation (4) should furnish an effective continuum description of the large-time, long-range properties of quite non-trivial systems, such as (5).

The solution of (4) reads in momentum space

\[
\hat{h}(t, q) = e^{-\nu|q|t} \hat{h}(0, q) + \int_0^t d\tau e^{-\nu|q|\tau} \hat{\varphi}(\tau, q) + \hat{\eta}(\tau, q).
\]

(6)

In this letter, we focus on the non-equilibrium relaxation of DLE, starting from an an initially flat interface \( h(0, r) = 0 \). If \( \hat{\varphi}(t, q) = 0 \), the average interface position remains fixed, thus \( \langle \hat{h}(t, q) \rangle = 0 \) and \( \langle h(t, r) \rangle \) = 0. The two-time correlator and response are

\[
\hat{C}(t; s, q, q') := \langle \hat{h}(t, q) \hat{h}(s, q') \rangle = \frac{T}{|q|} [e^{-\nu|q|(t-s)} - e^{-\nu|q|(t+s)}] \delta(q + q'),
\]

(7a)

\[
\hat{R}(t; s, q, q') := \frac{\delta \hat{C}(t; s, q, q')}{\delta \hat{\varphi}(s, q')} \bigg|_{\hat{\varphi} = 0} = \Theta(t-s) e^{-\nu|q|(t-s)} \delta(q + q'),
\]

(7b)

which becomes in direct space, with \( C_0 = \frac{\nu}{d+1}((d+1)/2)^{d/2} \), and for \( d = 1 \)

\[
C(t; s, r) = \frac{T C_0}{d-1} [(\nu^2(t-s)^2 + r^2)^{-(d+1)/2} - (\nu^2(t+s)^2 + r^2)^{-(d+1)/2}],
\]

(8a)

\[
R(t; s, r) = C_0 \Theta(t-s) \nu(t-s)(\nu^2(t-s)^2 + r^2)^{-(d+1)/2}.
\]

(8b)

3 Below, we shall refer to (4) with \( \tilde{\eta} = 0 \) as the deterministic part of (4).
where the Heaviside function $\Theta$ expresses the causality condition $t > s$. In particular, the interface width $w^2(t) = C(t; t, 0)$ is (apply a high-momentum cut-off $\Lambda$ for $L \to \infty$, if $d > 1$)

$$w^2(t) = \frac{T\mathcal{C}_0}{1 - d} [t(2\nu t)^{1-d} - \mathcal{C}_0(\lambda)] t \to \infty \begin{cases} T\mathcal{C}_0\Theta(\lambda)/(d - 1); & \text{if } d > 1 \\ T\mathcal{C}_0\ln(2\nu t); & \text{if } d = 1 \\ [T\mathcal{C}_0(2\nu)^{1-d}/(1 - d)] t^{1-d}; & \text{if } d < 1. \end{cases}$$

This shows the upper critical dimension $d^* = 1$ of DLE, such that at late times the interface is smooth for $d > 1$ and rough for $d \leq 1$ [25]. In the long-time stationary limit, $s \to \infty$ with the time difference $\tau = t - s$ being kept fixed, one has the fluctuation–dissipation relationship $\partial \mathcal{C}(s + \tau, s; r)/\partial \tau = -\nu/T\mathcal{R}(s + \tau, s; r)$. This was to be expected, since there exist lattice model versions in the DLE class which can be formulated in terms of an equilibrium system [43]. Finally, in the long-time scaling limit, $s \to \infty$ with $y = t/s > 1$ being kept fixed, one may read off from (8) and (9) the exponents

$$\beta = \alpha = \begin{cases} 0; & \text{if } d > 1 \\ ((1 - d)/2); & \text{if } d < 1. \end{cases}$$

In contrast to the interface width $w(t)$, which shows a logarithmic growth at $d = d^* = 1$, logarithms cancel in the two-time correlator $C$ and response $R$, up to additive logarithmic corrections to scaling. This is well-established in the physical ageing of magnetic systems [17].

3. Can one explain the form of the two-time scaling functions of the DLE in terms of a dynamical symmetry? Such an approach, based on extensions of the dynamical scaling $t \mapsto b^z t$ and $r \mapsto br$ to a larger set of transformations where $b = b(t, r)$ becomes effectively time-space-dependent, has been applied and tested in the physical ageing of magnetic systems, quenched either to their critical temperature $T = T_\mathcal{C} > 0$ or else to $T < T_\mathcal{C}$ (where $z = 2$), see [17] for a detailed review. More recently, this was also done for the relaxation dynamics in interface growth, namely for the the EW class [35] where $z = 2$ and the $(1 + 1)D$ KPZ class [19], where $z = \frac{1}{2}$. These tests mainly involved the fitting of the auto-response $R(t, s; 0)$ to the exact solutions or the numerical data. Since in the DLE class, one has $z = 1$, a different set of local time-space transformations must be sought. It might look tempting to consider conformal invariance [5], well-known from equilibrium critical phenomena, by simply relabelling one of the spatial directions as ‘time’, since this would give $z = 1$. However, as we shall see, a more precise definition is needed. For notational simplicity, we now restrict to the case of $1 + 1$ time-space dimensions, labelled by a ‘time coordinate’ $t$ and a ‘space coordinate’ $r$.

**Definition.** (1) A set of *ortho-conformal transformations*\(^4\) (usually called ‘conformal transformation’) $\mathcal{C}$ is a set of maps $(t, r) \mapsto (t', r') = \mathcal{C}(t, r)$ of local coordinate transformations, depending analytically on several parameters, such that angles in the coordinate space of the points $(t, r)$ are kept invariant. The maximal finite-dimensional Lie sub-algebra of ortho-conformal transformations is isomorphic to $\text{conf}(2) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$.

\(^4\) From the greek prefix $\alpha\beta\theta\omega$: right, standard.
A physical system is ortho-conformally invariant if its n-point functions transform covariantly under ortho-conformal transformations.

(2) A set of meta-conformal transformations is a set of maps \((t, r) \mapsto (t', r') = \mathcal{H}(t, r)\), depending analytically on several parameters, whose maximal finite-dimensional Lie sub-algebra of meta-conformal transformations is isomorphic to \(\mathfrak{conf}(2)\). A physical system is meta-conformally invariant if its n-point functions transform covariantly under meta-conformal transformations.

Hence, ortho-conformal transformations are also meta-conformal transformations.

In \((1 + 1)D\), ortho-conformal transformations are all analytic or anti-analytic maps, \(z \mapsto f(z)\) or \(z \mapsto \bar{f}(\bar{z})\), of the complex variables \(z = t + ir, \bar{z} = t - ir\). For our purposes, we restrict here to the projective conformal transformations \(z \mapsto z + \frac{a z + b}{c z + d}\) with \(a d - b c = 1\) and similarly for \(\bar{z}\). Then the Lie algebra generators \(f_n = -z^{n+1}\partial_z\) and \(\bar{f}_n = -\bar{z}^{n+1}\partial_{\bar{z}}\) with \(n = \pm 1, 0\) span the Lie algebra \(\mathfrak{conf}(2) \cong sl(2, \mathbb{R}) \oplus \mathbb{R}^2\). We shall use below the basis \(X_a = f_n + \bar{f}_n\) and \(Y_a = f_n - \bar{f}_n\). In an ortho-conformally invariant physical system, these generators act on physical ‘quasi-primary’ scaling operators \(\phi = \phi(z, \bar{z}) = \varphi(t, r)\) and then contain also terms which describe how these quasi-primary operators should transform, namely

\[
\ell_a = -z^{n+1}\partial_z = \Delta(n + 1)z^n, \quad \bar{\ell}_a = -\bar{z}^{n+1}\partial_{\bar{z}} = \bar{\Delta}(n + 1)\bar{z}^n,
\]

where \(\Delta, \bar{\Delta}\) are the conformal weights of the scaling operator \(\phi\).

Laplace’s equation \(\mathcal{S}\phi = 4\partial_z\partial_{\bar{z}}\phi = 0\) is a simple example of an ortho-conformally invariant system, since the commutator

\[
[S, \ell_a]\phi(z, \bar{z}) = -((n + 1)z^n\mathcal{S}\phi(z, \bar{z}) - 4\Delta n(n + 1)z^{n-1}\partial_z\phi(z, \bar{z})
\]

shows that for a scaling operator \(\phi\) with \(\Delta = \bar{\Delta} = 0\), the space of solutions of the Laplace equation \(\mathcal{S}\phi = 0\) is conformally invariant, since any solution is mapped onto another solution in the transformed coordinates. A two-point function of quasi-primary scaling operators \(\phi = \phi(z, \bar{z}) = \varphi(t, r)\) is covariance under ortho-conformal transformations is expressed by the ‘projective Ward identities’ \(X_n \phi = Y_n \phi = 0\) for \(n = \pm 1, 0\) [5]. For scalars, such that \(\Delta = \bar{\Delta} = x,\) this gives \([33]\)

\[
\mathcal{C}(t_1, t_2; r_1, r_2) = \mathcal{C}_0 \delta_{x_1, x_2} ((t_1 - t_2)^2 + (r_1 - r_2)^2)^{-x},
\]

where \(\mathcal{C}_0\) is a normalisation constant.

An example of meta-conformal transformations in \((1 + 1)\) dimensions is given by [16]

\[
X_a = -(t + 1)\partial_t - \mu \left[ (t + \mu r)^{n+1} - (t + 1)\partial_t + (n + 1)\gamma \right],
\]

\[
Y_a = -(t + \mu r)^{n+1}\partial_r + (n + 1)\gamma (t + \mu r)^n,
\]

where \(\gamma\) are the scaling dimension and the ‘rapidity’ of the scaling operator \(\varphi = \varphi(t, r)\) on which these generators act and the constant \(1/\mu\) has the dimensions of a velocity. The Lie

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5 From the greek prefix \(\mu \infty \gamma \delta \): of secondary rank.
6 Interpretation: \(X_\mu, Y_\mu\): generate time- and space-translations, \(X_0\) global dilatations \(t \mapsto bt, r \mapsto br\), \(Y_0\) rigid time-space rotations and \(X_\gamma, Y_\gamma\) generate the ‘special’ conformal transformations.
7 This concept of a dynamical symmetry, for the free diffusion equation, goes back to Jacobi (1842) and Lie (1881) and was re-introduced into physics by Niederer (1972) [30].
The covariant two-point function of the DLE in 1 + 1 dimensions is realized in terms of local second-order differential operators, suitable as a dynamical symmetry operator $S$ and the covariant two-point function $\langle \mathcal{C}(t; r) = \langle \varphi(t, r) \varphi(0, 0) \rangle$, up to normalisation. The physical nature of $\mathcal{C}$ is also indicated. For ortho-conformal invariance and meta-conformal invariance 1, one has the constraints $x_1 = x_2$ and $\gamma_1 = \gamma_2$. For the meta-conformal invariance 2, we list only case A from the text. One has $\mu^{-1} = i\nu$ with $\nu > 0$, and the constraints $\gamma_1 + \gamma_2 = \mu$ and $\gamma_1 - \gamma_2 = \mu (x_1 - x_2)$.

A comparison of the two-point functions for ortho- and meta-conformal invariance is shown in table 1. Comparing the two-point functions (13) and (15) shows that even for the same dynamical exponent $z = 1$, different forms of the scaling functions are possible for ortho- and meta-conformal invariance.

4. Are there examples of ortho- or meta-conformal invariance, which have $z = 1$ and are realised in terms of local first-order differential operators, suitable as a dynamical symmetry of the DLE in 1 + 1 dimensions? This must be answered in the negative, for the following reasons.

(1) The DLE response function (8b) is distinct from the predictions (13) and (15), see also table 1. For the meta-conformal two-point function (15), the functional form is clearly different for finite values of the scaling variable $v = (t_1 - t_2)/(t_1 - t_2)$). The ortho-conformal two-point function (13) looks to be much closer, with the choice $x_1 = 1/2$ and the scale factor fixed to $\nu = 1$, were it not for the extra factor $v(t - s)$. On the other hand, the two-time DLE correlator (8a) does not agree with (13) either, but might be similar to a two-point function computed in a semi-infinite space $t \geq 0$, $r \in \mathbb{R}$ with a boundary at $t = 0$.

(2) The invariant equations $\mathcal{S} \varphi = 0$ are distinct from the deterministic part of the DLE Langevin equation (4). Recall the well-known fact [32] that for Langevin equations $\mathcal{S} \varphi = \eta$, where $\eta$ is a white noise, and where the noise-less equation $\mathcal{S} \varphi_{0} = 0$ has a local scale-invariance (including a generalised Galilei-invariance to derive Bargman super-selection rules [3]) all correlators and response functions can be reduced to responses found in the noise-less theory. In particular, the two-time response function of the full noisy equation $R(t, s; r) = R_0(t, s; r)$, is identical to the response $R_0$ found when the

8 See [39] for extensions as dynamical symmetries of the $(1 + 1)D$ Vlassov equation, isomorphic to $\text{conf}(2)$. 

| Table 1. Comparison of ortho- and two examples of meta-conformal invariance. |
|-----------------------------|-----------------------------|-----------------------------|
| Ortho | Meta-1 | Meta-2 |
| $[X_a, X_b] = (n - m)X_{a+b}$ | $[X_a, X_b] = (n - m)X_{a+b}$ | $[X_a, X_b] = (n - m)X_{a+b}$ |
| $[X_a, Y_b] = (n - m)Y_{a+b}$ | $[X_a, Y_b] = (n - m)Y_{a+b}$ | $[X_a, Y_b] = (n - m)Y_{a+b}$ |
| $\mathcal{S}$ | $\mathcal{S}$ | $\mathcal{S}$ |
| $\delta_{r}^{2} + \delta_{r}^{2}$ | $-\mu \delta_{r} + \delta_{r}$ | $-\mu \delta_{r} + \delta_{r}$ |
| $\mathcal{C}$ | $\mathcal{C}$ | $\mathcal{C}$ |
| $r^{-2\nu} \left( 1 + \frac{\mu}{\gamma_{1}} \left| \frac{r_{1} - r_{2}}{t_{1} - t_{2}} \right| \right)^{-2\gamma_{2}/\mu}$ | $r^{-2\nu} \left( 1 + \mu \left| \frac{r_{1} - r_{2}}{t_{1} - t_{2}} \right| \right)^{-2\gamma_{2}/\mu}$ | $r^{-2\nu} \left( 1 + \nu \left| \frac{r_{1} - r_{2}}{t_{1} - t_{2}} \right| \right)^{-2\gamma_{2}/\mu}$ |

Note. Listed are the commutators of the Lie algebra bases $(X_a, Y_a)_{a=\pm 1, 0}$ isomorphic to $\text{conf}(2)$ [21]. An invariant equation is $\mathcal{S} \varphi = (-\mu \delta_{r} + \delta_{r}) \varphi = 0$, provided only that $\gamma = \mu x$, since the only non-vanishing commutators of the Lie algebra with $\mathcal{S}$ are $[\mathcal{S}, X_0] \varphi = -\mathcal{S} \varphi$ and $[\mathcal{S}, X_1] \varphi = -2t \mathcal{S} \varphi + 2(\mu x - \gamma) \varphi$. The covariant two-point function is $\langle \mathcal{C}(t; r) \rangle \chi_{0} \delta_{\lambda_{1}} \delta_{\lambda_{2}} \delta_{\gamma_{1}} \delta_{\gamma_{2}} \chi_{0} \delta_{\lambda_{1}} \delta_{\lambda_{2}} \gamma_{1} \gamma_{2} \gamma_{1} \gamma_{2} \gamma_{1} \gamma_{2} \gamma_{1} \gamma_{2}$.

These well-known results are summarised in the first two columns of table 1. Comparing the two-point functions (13) and (15) shows that even for the same dynamical exponent $z = 1$, different forms of the scaling functions are possible for ortho- and meta-conformal invariance. 

4. Are there examples of ortho- or meta-conformal invariance, which have $z = 1$ and are realised in terms of local first-order differential operators, suitable as a dynamical symmetry of the DLE in 1 + 1 dimensions? This must be answered in the negative, for the following reasons.

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noise is turned off and computed from the dynamical symmetry [17, 32].
Indeed, in the example (7b) and (8b) of the DLE, one sees that the two-time response \( R \) is independent of \( T \), which characterises the white noise.

We shall look for dynamical symmetries of the equation \( S \varphi = (-\mu \partial_t + \nabla) \varphi = 0 \), which is the deterministic part of the DLE Langevin equation (4), in \( 1 + 1 \) dimensions. We shall seek to derive the form of the two-time response function \( R(t, s; r) \) from this dynamical symmetry. The two-time correlator \( C \) cannot be obtained in this way. Rather, we shall see that its ‘deterministic’ contribution \( C_0(t, s; r) = 0 \) simply vanishes. As shown in [32], the correlator must be obtained from an integral over three-point response functions. We leave this for future work.

5. In direct space, the invariant Schrödinger operator for DLE should be \( S = -\mu \partial_t + \nabla \), where \( \nabla \alpha \) denotes the Riesz–Feller fractional derivative [36] of order \( \alpha \). For functions \( f(r) \) of a single variable \( r \in \mathbb{R} \) (assuming that \( f(r) \) is such that the integral exists), we use the convention (for brevity, we often write \( \nabla = \nabla^1 \))

\[
\nabla^\alpha f(r) := \frac{i\alpha}{2\pi} \int_{\mathbb{R}^d} dk dx \left| k \right|^{\alpha} e^{i\alpha(r-x)} f(x).
\]

Then the following properties hold true, for formal manipulations [4], [17, appendix J.2]

\[
\begin{align*}
\nabla^\alpha \nabla^\beta f(r) &= \nabla^\alpha \nabla^\beta f(r), \\
\nabla^\alpha f(\nabla) f(r) &= \alpha \partial_r \nabla^{\alpha-2} f(r), \\
\nabla^\alpha f(\mu \partial_r) &= |\mu|^\alpha \nabla^\alpha f(\mu r), \\
\nabla^\alpha e^{iq\nabla} &= (i|q|)^\alpha e^{iq\nabla}, \\
\n\nabla^\alpha f(\nabla)(q) &= (i|q|^\alpha \hat{f}(q), \\
\n\nabla^2 f(r) &= \partial_r^2 f(r),
\end{align*}
\]

where \( \hat{f}(q) \) is the Fourier transform of \( f(r) \). In selecting the generators for the Lie algebra of dynamical symmetries, we follow [16] and require that time translations \( X_{-1} = -\partial_t \), dilatations \( X_0 = -t \partial_t - r \partial_r - x \), and space translations \( Y_{-1} \) are present. However, if one begins with the standard local generator \( -\partial_t \) of spatial translations, it turns out that the non-local generator \( \nabla_r \) is generated as well [4], [17, chapter 5.3]. The closure of this set of generators, for generic values of \( \alpha = 2 \), is still an open problem. In order to obtain a well-defined Lie algebra of dynamical symmetries of \( S \), we consider a non-local spatial translation operator \( Y_{-1} = -\nabla_r \). Consider the following set of single-particle generators

\[
\begin{align*}
X_{-1} &= -\partial_t, \\
X_0 &= -t \partial_t - r \partial_r - x, \\
X_1 &= -t^2 \partial_t - 2r \partial_r - \mu r^2 \nabla - 2xt - 2\gamma r \partial_r \nabla_r^{-1}, \\
Y_{-1} &= -\nabla, \\
Y_0 &= -t \nabla - \mu r \partial_r - \gamma, \\
Y_1 &= -t^2 \nabla - 2\mu r \partial_r - \mu^2 r^2 \nabla - 2\gamma t - 2\gamma \mu r \partial_r \nabla_r^{-1}.
\end{align*}
\]

As in the set (14) of meta-conformal transformations, the constants \( x \) and \( \gamma \), respectively, are the scaling dimension and rapidity of the scaling operator \( \varphi = \varphi(t, r) \) on which these generators act. It is now an afternoon’s exercise (before tea time) to check, with the help of (17)\(^9\), the following commutator relations, for \( n, m \in \{ \pm 1, 0 \} \)

\[
[X_n, X_m] = (n-m)X_{n+m}, \quad [X_n, Y_m] = (n-m)Y_{n+m}, \quad [Y_n, Y_m] = \mu(n-m)Y_{n+m}.
\]

This establishes the Lie algebra isomorphism \( \langle X_n, Y_n \rangle_{n=\pm 1,0} \cong \text{conf}(2) \). Furthermore, since \( [S, Y_n] \varphi = [S, X_{-1}] \varphi = 0, \quad [S, X_0] \varphi = -S \varphi, \quad [S, X_1] \varphi = -2tS \varphi + 2(\mu x - \gamma) \varphi \)

\[
\varphi = \varphi(t, r).
\]

\(^9\) Use the identities \([\nabla, r^2] = 2r \partial_r \nabla_r^{-1}, [r^2 \nabla, r \partial_r] = -r^2 \nabla_r \) and \([\nabla, \partial_t] = [\partial_t \nabla_r^{-1}, r] = 0 \).
the infinitesimal transformations (18) form a Lie algebra of meta-conformal dynamical symmetries (of the deterministic part) of the DLE equation (4), if \( \gamma = \chi \mu \). In contrast to the generators (14), the generators (18) are non-local and do not generate simple local changes of the coordinates \((t, r)\). In spite of an attempt to interpret non-local infinitesimal generators as the transformation of a distribution of coordinates [18], finding a clear geometrical interpretation of the generators (18) remains an open problem.

6. We look for the covariant \( n \)-point functions. We expect [17] that these will correspond physically to response functions, i.e. the two-time response \( R(t, s; r) = \langle \varphi(t, r)\varphi(s, 0) \rangle \), where \( \varphi \) is the response operator conjugate to the scaling operator \( \varphi \), in the context of Janssen–de Dominicis theory [40]. In order to write down the \( n \)-body operators analogous to (18), we must ascribe a ‘signature’ \( \varepsilon = \pm 1 \) to each scaling operator. We choose the convention that \( \varepsilon_1 = +1 \) for scaling operators \( \varphi_1 \) and \( \varepsilon_i = -1 \) for response operators \( \varphi_i \). Then

\[
Y_{-1} = Y^{[n]}_0 = \sum_i [-\varepsilon_i \nabla_i], \quad Y_0 = Y^{[n]}_0 = \sum_i [-\varepsilon_i t_i \nabla_i - \mu r_i D_i - \gamma_i]
\]

\[
Y_1 = Y^{[n]}_1 = \sum_i [-\varepsilon_i t_i^2 \nabla_i - 2\mu t_i r_i D_i - \mu^2 \varepsilon_i r_i^2 \nabla_i - 2\gamma_i t_i - 2\mu\gamma_i \varepsilon_i r_i D_i \nabla_i^{-1}]
\]

\[
X_{-1} = X^{[n]}_0 = \sum_i [-t_i \partial_i], \quad X_0 = X^{[n]}_0 = \sum_i [-t_i \partial_i - r_i D_i - x_i]
\]

\[
X_1 = X^{[n]}_1 = \sum_i [-t_i^2 \partial_i - 2t_i r_i D_i - \mu \varepsilon_i r_i^2 \nabla_i - 2x_i t_i - 2\gamma_i \varepsilon_i r_i D_i \nabla_i^{-1}]
\]

(21)

with the short-hands \( \partial_i = \frac{\partial}{\partial t_i}, D_i = \frac{\partial}{\partial x_i} \) and \( \nabla_i = \nabla_i \). It can be checked that the generators (21) obey the meta-conformal Lie algebra [19]. Now, for a \((n + m)\)-point function

\[
\tilde{g}_{n,m} = \tilde{g}_{n,m}(t_1, \ldots, t_{n+m}; r_1, \ldots, r_{n+m}),
\]

\[
= \langle \varphi(t_1, n) \cdots \varphi(n, r_1, \ldots, r_{n+m}) \varphi_{n+1}(t_{n+1}, \ldots, r_{n+m}) \rangle
\]

of quasi-primary scaling and response operators, the covariance is expressed through the projective Ward identities \( X^{[n+m]}_k \tilde{g}_{n,m} = X^{[n+m]}_k \tilde{g}_{n,m} = 0 \), for \( k = \pm 1, 0 \).

7. We apply this to the two-time response function \( \mathcal{R} = \mathcal{R}(t_1, t_2; r_1, r_2) = \tilde{g}_{1,1}(t_1, t_2; r_1, r_2) \). From \( X_{-1} \mathcal{R} = 0 \) it follows that \( \mathcal{R} = \mathcal{R}(t; r_1, r_2) \), with \( t = t_1 - t_2 \). On the other hand, the condition \( Y_{-1} \mathcal{R} = 0 \) would lead in Fourier space to \( \langle \varepsilon_1 q_1 + \varepsilon_2 q_2 \rangle \mathcal{R}(t; q_1, q_2) = 0 \). Because of the assigned signatures \( \varepsilon_1 = -\varepsilon_2 = 1 \), this equation can have a non-vanishing solution such that we can write \( \mathcal{R} = F(t, r) \), with \( r = r_1 - r_2 \). However, for a two-point correlator \( \tilde{g}_{2,0} \) with \( \varepsilon_1 = \varepsilon_2 = 1 \), the Ward identity \( Y_{-1} \tilde{g}_{2,0} = 0 \) would simply imply that \( \tilde{g}_{2,0} = 0 \), and in agreement with the fact that the DLE-correlator (7a) and (8a) vanishes as \( T \to 0 \). Standard calculations, see e.g. [17], lead to the following set of conditions for the function \( \mathcal{R} = F(t, r) \)

\[
[-t \partial_i - r \partial_i - x_i - x_2]F = 0,
\]

(22a)

\[
[-t \nabla_i - \mu r \partial_i - \gamma_i - \gamma_2]F = 0,
\]

(22b)

\[
[-t^2 \partial_i - 2r t \partial_i - \mu^2 \varepsilon_i \nabla_i - 2x_i t - 2\gamma_i \varepsilon_i r \partial_i \nabla_i^{-1}]F = 0,
\]

(22c)

\[
[-t^2 \varepsilon_i \nabla_i - 2\mu t \partial_i - \mu^2 \varepsilon_i r^2 \nabla_i - 2\gamma_i t - 2\mu \gamma_i \varepsilon_i r \partial_i \nabla_i^{-1}]F = 0.
\]

(22d)

Equations (22c) and (22d) can be further simplified by combining them with (22a), (22b) and reduce to
\[(x_1 - x_2)(t + \mu v_0 \partial_t \nabla_r^{-1})F = 0, \quad (\gamma_1 - \gamma_2) - \mu(x_1 - x_2)tF = 0. \quad (23)\]

If \( F \) does not contain a factor \( \sim \delta(t) \), the second equation (23) gives the constraint \( \gamma_1 - \gamma_2 = \mu(x_1 - x_2) \). Equation (22a) implies the scaling form \( F(t, r) = r^{-2\gamma}f(v) \), with \( v = r/t \) and \( x = 1/2(x_1 + x_2) \). From (22b) and (23), the scaling function \( f(v) \) must satisfy, with \( \gamma = \frac{1}{4}(\gamma_1 + \gamma_2) \)

\[
(\dot{\gamma}_1 \nabla_r + \mu v_0 \partial_r + 2\gamma)f(v) = 0 \quad \text{and} \quad \nabla_r^{-1}[(x_1 - x_2)(\dot{\gamma}_1 \nabla_r + \mu v_0 \partial_r + \mu)f(v) = 0. \quad (24)\]

The two conditions in equation (24) are compatible in two distinct cases:

**Case A:** \( 2\gamma = \mu \). Then \( (\dot{\gamma}_1 \nabla_r + \mu v_0 \partial_r + \mu)f(v) = 0 \) and \( x_1 = x_2 \) is still possible.

**Case B:** \( \dot{\gamma}_1 = x_2 \). Then \( \gamma_1 = \gamma_2 \) and \( (\dot{\gamma}_1 \nabla_r + \mu v_0 \partial_r + 2\gamma)f(v) = 0 \).

In Fourier space, equation (22b) gives \( (i\alpha|q| - \mu \partial_q + (2\gamma - \mu)) \hat{f}(q) = 0 \), which illustrates the difference between cases A and B. It follows that \( \hat{f}(q) = \hat{f}_0 q^{2\gamma/\mu - 1} \exp(i\alpha|q|/\mu), \) where \( \hat{f}_0 \) is a normalisation constant. Finally, comparison of the Schrödinger operator \( \hat{S} = -\mu \partial_q + \nabla_q \) with the DLE equation (4) shows that \( \mu^{-1} = iv \).

Transforming back into direct space, we find

\[
f(v) = \hat{f}_0 \times \begin{cases} \hat{f}_1(v^2 + \nu^2)^{-1}; & \text{case A,} \\ \Re(e^{-iv\phi}/(2(\hat{f}_1 v^2 + iv)^{-\nu - 1}); & \text{case B, with } \psi = 1; 2i\nu \gamma. \end{cases} \quad (25)\]

A linear combination of these two solutions is a solution of the linear system (24) as well.

**8.** In particular, for case A, the final form of the two-time response function \( \mathcal{A} \) becomes, with a normalisation constant \( F_0 \) and \( x = \frac{1}{2}(x_1 + x_2) \)

\[
\mathcal{A} = F(t, r) = F_0 r^{1-2\gamma} \frac{\hat{f}_1 v^2}{v_0^2 \nu^2 + r^2}; \quad \text{with } t = t_1 - t_2, \quad r = r_1 - r_2 \quad \text{case A}. \quad (26)\]

If one takes \( x = \frac{1}{2}, \) and \( \nu \in \mathbb{R}_+ \), this reproduces the exact solution (8b) of the response in \((1 + 1)D \) DLE. This is our main result: the non-local representation (21) of \( \text{conf}(2) \) is necessary to reproduce the correct scaling behaviour of the non-stationary response. The properties and predictions of this second example of a meta-conformal symmetry, for the special case A, are listed in the last column of table 1. An important difference is that ortho-conformal invariance and meta-conformal invariance 1 predict the form of a two-time correlator \( g = g_{2,0} \), whereas the meta-conformal invariance 2 predicts the form of a two-time response \( \mathcal{A} = g_{1,1} \).

**Summary:** we have proposed a meta-conformal dynamical symmetry for the DLE in \( 1 + 1 \) dimensions. This symmetry, isomorphic to the Lie algebra \( sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}) \), is realised in terms of non-local generators, see equations (18) and (21). It is distinct from other known representations of the conformal Lie algebra, both in the form of the invariant Schrödinger operator \( \hat{S} \) and in the predicted shape of the covariant two-point function, see table 1. In particular, the generator \( F_{1,1} \) which plays the role of ‘spatial translations’ is manifestly non-local. The full time-space form of the two-time response \( R(t, s; r) \) in (8b) can be derived from this dynamical symmetry. This is the first time that (i) the full response (and not only the auto-response \( R(t, s; 0) \)) can be confirmed and (ii) that the set of generators closes into a Lie algebra, for a system with \( z = 2 \).

In view of Spohn’s mapping [37], which relates the \((1 + 1)D \) DLE equation (4) with the quantum chain (5) and the terrace-step-kink model, a convenient linear combination of the prediction of \( R(t, s; r) \) from cases A and B might describe the non-stationary response of vicinal surfaces. Indeed, the explicit form of the connected stationary correlator \( \langle \varrho(t, r) \varrho(t, 0) \rangle \) of the particle density \( \varrho(t, r) = \partial_t h(t, r) \), obtained by Karevski and Schütz.
from (5) in the limit $\lambda \to \infty$ [24, equation (1)], contains two terms which look quite analogous to the responses (25) in cases A and B. While that is distinct from the non-stationary responses considered here, the qualitative analogy is encouraging. Certainly, a precise test is called for. This will require to work out higher $n$-point functions in order to be able to derive the form of non-equilibrium correlators. Stationary correlators might be included by considering an appropriate initial condition. An obvious further extension will be to dimensions $d > 1$. Conceptually, the consideration of manifestly non-local generators in local scale-invariance might lead to further insight for the construction of dynamical symmetries for different values of $z$.

Acknowledgments

I thank GM Schütz for fruitful discussions. This work was started at the workshop ‘Advanced Conformal Field Theory and Applications’ (ACFT) at the Institute Henri Poincaré Paris. It is a pleasure to thank the organisers for their warm hospitality. This work was also partly supported by the Collège Doctoral franco-allemand Nancy-Leipzig-Coventry (‘Statistical Physics of Complex Systems’) of UFA-DFH.

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