Magnetic black holes in Weitzenböck geometry

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We derive magnetic black hole solutions using a general gauge potential in the framework of teleparallel equivalent general relativity. One of the solutions gives a non-trivial value of the scalar torsion. This non-triviality of the torsion scalar depends on some values of the magnetic field. The metric of those solutions behave asymptotically as Anti-de-Sitter/ de-Sitter (AdS/dS) spacetimes. The energy conditions are discussed in details. Also, we calculate the torsion and curvature invariants to discuss singularities. Additionally, we calculate the conserved quantities using the Einstein-Cartan geometry to understand the physics of the constants appearing into the solutions.

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I. Introduction

Several issues related to the gravitational field, ranging from quantum gravity up to cosmological dark energy and dark matter, encourage scientists to search for modifications of general relativity (GR) capable of addressing under a new standard the phenomenology \cite{1-7}. Clearly, a viable modification must be consistent with experimental tests and solve problems in quantum gravity and/or cosmology, that is at UV and IR scales. From a theoretical point of view, a first requirement is avoiding ghosts or other severe defects to achieve self-consistent theories \cite{8}.

Therefore, it is logic to start using different approaches with respect to GR. Among these constructions, there is the one used by Einstein himself \cite{9-11}. Assuming absolute parallelism, we can formulate gravitational field theory equivalent to GR using tetrad fields as building blocks instead of the metric \cite{12-31}. In this theory the gravitational field is due to torsion tensor, which is acting as a force \cite{32}, instead of the curvature tensor of GR. This theory is known in the literature as teleparallel equivalent of general relativity (TEGR) which is equivalent to GR and allows new conjectures into gravitation \cite{33}. The main advantage of this theory is that one can define a true gravitational energy-momentum tensor which is locally Lorentz invariant \cite{34-36}. Generally, we can consider TEGR as a gauge theory of translation \cite{37-41}. In this framework, the tetrad field has the role of the gauge translational potential of gravitational field \cite{42}.

It is worth stressing that Einstein himself used the Weitzenböck geometry with the aim to unify the electromagnetic and gravitational fields \cite{11}. The issue is connected to the fact that the tetrad field has 16-components while the metric field has 10-components. The 16-components of the tetrad can be regarded as 10-components encoded for the metric and 6-components as degrees of freedom responsible for the electromagnetic field. However, the possible
unification pointed out by Einstein failed but the concept to adopt the Weitzenböck geometry and TEGR to describe the gravitational field through the torsion field remained. In particular, considering tetrads as fields in the context of TEGR, many authors investigated the notion of gravitational energy in different physical cases \[43-45\]. Specifically, the notion of gravitational energy has been adopted to construct a non-local gravitational theory \[46\].

A crucial issue in both GR and TEGR is to derive exact black hole solutions in order to understand the main features of the theory. An important topic is to derive asymptotically flat black hole solutions in the context of Einstein-Maxwell equations of motions, either rotating or non-rotating. The applications of such solutions to stellar systems is straightforward: in particular, if these systems are endowed with a magnetic field they can represent realistic astrophysical objects. In fact many charged black hole solutions have been derived using Einstein-Maxwell field equations and not all of them, in static situations, coincide with the Schwarzschild spacetime \[47-57\].

Furthermore, many solutions have been derived in TEGR theory \[58, 59\], however, till now, no magnetic black holes with flat horizons have been derived. A flat horizon is a spacetime with cylindrical symmetry. This spacetime plays a main role in the discussion of internal consistency of a given solution as in the case of Levi-Civita \[60, 61\] and Chazy-Curzon \[62, 63\] static solutions, and the Lewis solutions \[64\]. In astrophysical context, cylindrical symmetry has been used to study cosmic strings \[65\]. In GR, for example, cylindrically symmetric rotating black hole solutions have been derived with a negative cosmological constant \[66, 67\]. The aim of the present study is to derive magnetic black holes using general gauge potential in the framework of TEGR and analyze their physical properties by discussing their energy conditions, studying their singularities and calculating the related conserved charges.

The layout of the paper is the following: In §II, a summary TEGR geometry is provided. In §III, a tetrad field with cylindrical symmetry is adopted for the TEGR charged field equations. By a general gauge potential, analytic solutions are derived. The black hole singularities are investigated in §IV. In §V, the energy conditions are discussed. In §VI, the conserved charges are obtained. It is shown that they are divergent in the time direction. In §VII, the method of “regularization through relocalization” is used and finite conserved charges are obtained. Summary of the results are discussed in final section §VIII.

II. A summary of teleparallel equivalent of general relativity

In the framework of TEGR theory, the basic field variables responsible for gravity are the tetrad fields \(b_i^\mu\) \[68\]. The teleparallel condition leads to:

\[
b^i_{\mu \nu} = b^i_{\mu \nu} - W^\lambda_{\mu \nu} b^i_{\lambda} = 0,
\]

where \(\mu\) and \(\nu\) are the ordinary derivative and the covariant derivative respectively. \(W^\lambda_{\mu \nu}\) is the non-symmetric Weitzenböck connection. \(W^\mu_{\lambda \nu}\) is a non-symmetric connection and it is defined as:

\[
W^\lambda_{\mu \nu} := b^i_{\lambda} b^i_{\mu \nu}.
\]

Using Eq. \((1)\) one can show that the curvature tensor vanishes identically. In TEGR theory, one can define the spacetime metric, \(g_{\mu \nu}\), as

\[
g_{\mu \nu} := \lambda_{ij} b^i_{\mu} b^j_{\nu},
\]

with \(\lambda_{ij} = (-1, +1, +1, +1)\) being the Minkowski spacetime. A main property of TEGR is that one can relate to any tetrad field \(b_i^\mu\) a unique metric while, for a given metric \(g_{\mu \nu}\), one can connect many tetrad fields due to the local Lorentz transformations.

The torsion and the contortion tensors are defined as:

\[
T^\alpha_{\mu \nu} := W^\alpha_{\nu \mu} - W^\alpha_{\mu \nu} = b^a_{\mu} (b^a_{\nu, \mu} - b^a_{\mu, \nu}),
\]

\[
K^\mu \nu \alpha := -\frac{1}{2} (T^\mu _\alpha - T^\nu _\mu \alpha - T^\nu _\alpha \mu),
\]

(4)
where the contortion tensor can be rewritten in terms of connections as $K^{\mu}_{\nu\rho} = W^\rho_{\nu\rho} - \{_{\nu\rho}\}^{\mu}$, with $\{_{\nu\rho}\}^{\mu}$ being the Levi-Civita connection. The super-potential tensor $S^{\alpha}_{\mu\nu}$ is defined as:

$$S^{\alpha}_{\mu\nu} := \frac{1}{2}(K^{\mu\nu}_{\alpha} + \delta^\mu_{\alpha}T^\beta_{\nu\beta} - \delta^\nu_{\alpha}T^\beta_{\mu\beta}),$$

and the torsion scalar takes the form

$$T := T^\alpha_{\mu\nu}S^{\alpha}_{\mu\nu}. \tag{6}$$

The gravitational action of TEGR, involving the cosmological constant, is defined as:

$$\mathcal{L}(b^i_{\mu}) = \int d^4x \ b \left[ \frac{1}{16\pi}(T - 2\Lambda) + \mathcal{L}_{em} \right], \tag{7}$$

where $b := \sqrt{-g} = det (b^i_{\mu})$, $\mathcal{L}_{em}$ is the Lagrangian of the electromagnetic field and $\Lambda$ is the cosmological constant. We assume natural units where the gravitational constant and the speed of light are $G = c = 1$. The Lagrangian of the electromagnetic field is defined as

$$\mathcal{L}_{em} := -\frac{1}{2}F \wedge^* F, \tag{8}$$

where $F$ is the electromagnetic strength field which is defined as

$$F := dA, \tag{9}$$

with $A = A_{\mu}dx^{\mu}$ being the electromagnetic gauge potential 1-form [69]. Carrying out the variation of Lagrangian (7) with respect to the tetrad field $b^i_{\mu}$, one obtains the following field equations of TEGR:

$$I^{\nu}_{\mu} \equiv e^{-1}e^i_{\mu}\partial_\rho (ee^i_\alpha S^{\rho\nu}_{\alpha}) - T^{\alpha}_{\lambda\mu}S^{\nu\lambda} - \frac{1}{4}g^{\alpha}_{\mu}(T - 2\Lambda) + 4\pi \Theta^{\nu}_{\mu} = 0, \tag{10}$$

$$\partial_\nu (\sqrt{-g}F^{\mu\nu}) = 0, \tag{11}$$

where

$$\Theta^{\nu}_{\mu} = g_{\rho\sigma}F^{\nu\rho}F_{\mu\sigma} - \frac{1}{4}\delta^{\nu}_{\mu}g^{\lambda\rho}g^{\sigma\epsilon}F_{\lambda\rho}F_{\mu\sigma},$$

is the energy momentum tensor of the electromagnetic field. In the following section we are going to apply the field equations (10) and (11) to a tetrad field with cylindrical symmetry.

### III. EXACT SOLUTIONS WITH ELECTROMAGNETIC FIELDS

Now we are going to apply the charged field equations of TEGR, Eqs. (10) and (11), to the flat horizon spacetime, which directly gives rise to the following vierbein, written in terms of cylindrical coordinates $(t, r, \phi, z)$ (see also [69]):

$$(b^i_{\mu}) = \left( \sqrt{A(r)}, \frac{1}{\sqrt{A_1(r)}}, r, r \right), \tag{12}$$

where $A(r)$ and $A_1(r)$ are two unknown functions of the radial coordinate $r$. Substituting Eq. (12) into Eq. (6), we evaluate the torsion scalar as\footnote{For the sake of simplicity, we will write $A(r) \equiv A$, $A_1(r) \equiv A_1$, $A' \equiv \frac{dA}{dr}$, $A'_1 \equiv \frac{dA_1}{dr}$, $A'' \equiv \frac{d^2A}{dr^2}$ and $A''_1 \equiv \frac{d^2A_1}{dr^2}$.}

$$T = 2\frac{A'A_1}{rA} + 2\frac{A'_1}{r^2}, \tag{13}$$
Applying Eq. (12) to the field equation (10) we get the following non-vanishing components:

\[
I_{tt} = \frac{A}{r^4} \left\{ r^2 A_1 (a_\phi [2b'_3 - a_\phi] + a_{1z} [2s'_1 - a_{1z}]) + s_\phi [2b_z - s_\phi] + b_z^2 - r^2 [A_1 (s'_1 + b'_1)] + A_1 \\
+ r A'_1 + r^2 A \right\} = 0,
\]

\[
I_{rr} = \frac{1}{r^4 A A_1} \left\{ r^2 A A_1 (a_\phi [2b'_3 - a_\phi] + a_{1z} [2s'_1 - a_{1z}]) + A s_\phi [s_\phi - 2b_z] + A s_{1z}^2 - r^2 [AA_1 (s'_1 + b'_1)] \\
- A_1 A' - A (A_1 + r^2 A) \right\} = 0,
\]

\[
I_{\phi r} = I_{\phi z} = \frac{2 (a_{1z} - s'_1) (s_\phi - b_z)}{r^2} = 0, \quad I_{z r} = I_{z r} = \frac{2 (a_\phi - b'_1) (s_\phi - b_z)}{r^2} = 0,
\]

\[
I_{\phi z} = \frac{1}{4 r^2 A^2} \left\{ 2 r^4 A A_1 A'' - r^4 A A'' + r^3 A A' [r A'_1 + 2 A_1] + 2 A^2 \left[ 2 r^2 A a_\phi [2b'_3 - a_\phi] - 2 s_\phi [s_\phi - 2b_z] \\
+ r^3 A'_1 - 2 r^2 A a_{1z} [2s'_1 - a_{1z}] - 2 b_z^2 - 2 r^2 (b'_1 A'_1 - r^2 A - A_1 s'_1) \right] \right\} = 0,
\]

\[
I_{zz} = \frac{1}{4 r^2 A^2} \left\{ 2 r^4 A A_1 A'' - r^4 A A'' + r^3 A A' [r A'_1 + 2 A_1] + 2 A^2 \left[ 2 r^2 A a_\phi [a_\phi - 2b'_3] - 2 s_\phi [s_\phi - 2a_{1z}] \\
+ r^3 A'_1 + 2 r^2 A a_{1z} (2s'_1 - a_{1z}) - 2 b_z^2 + 2 r^2 (b'_1 A'_1 + r^2 A - A_1 s'_1) \right] \right\} = 0,
\]

\]

\[
(14)
\]

where \( a_\phi = \frac{d a(\phi)}{d \phi} \), \( a_{1z} = \frac{d a_1(z)}{d z} \), \( b'_1 = \frac{d b_1(r)}{d r} \), \( b_z = \frac{d b(z)}{d z} \), \( s'_1 = \frac{d s_1(r)}{d r} \), \( s_\phi = \frac{d s(\phi)}{d \phi} \) and \( a(\phi), a_1(z), b(z), b_1(r), s_1(r) \) and \( s(\phi) \) are the magnetic field strengths given by the general gauge potential as

\[
v := [a(\phi) + a_1(z)] dr + [b(z) + b_1(r)] d\phi + [s(\phi) + s_1(r)] dz.
\]

The general solutions of the non-linear differential Eqs. (14) have the form:

\[
i) \quad A(r) = \frac{1}{A_1(r)} = \left( \frac{\Lambda r^3 - 3c_3}{3r} \right), \quad a(\phi) = c_2 \phi, \quad a_1(z) = c_3 z, \quad s(\phi) = c_4 \phi, \quad b(z) = c_5 z,
\]

\[
s_1(r) = c_6 r, \quad b_1(r) = c_7 r,
\]

\[
ii) \quad A(r) = \frac{1}{A_1(r)} = \left( \frac{\Lambda r^3 - 3c_1 r - 3c_8}{3r^2} \right), \quad a(\phi) = c_2 \phi, \quad a_1(z) = c_3 z, \quad s(\phi) = \pm \varphi \phi,
\]

\[
b(z) = \varphi^2 z, \quad s_1(r) = c_6 r, \quad b_1(r) = c_7 r, \quad \varphi = \frac{1 \pm \sqrt{1 + 4c_8}}{2},
\]

\]

\[
(15)
\]

where \( c_i, i = 1 \cdots 8 \) are constants of integration. Eqs. (16) shows that when the constant \( c_8 = 0 \) then the second set will be identical to the first set and the constant \( \varphi \), after some re-scaling, can be related to the constants \( c_4 \) and \( c_5 \) of the first set.
FIG. 1: The behavior of the torsion scalar for the second solution (16).

IV. THE PHYSICAL PROPERTIES OF SOLUTIONS

The metric of solutions (16) take the form

\[ ds^2_1 = -\left( \frac{\Lambda r^3 - 3c_1}{3r} \right) dt^2 + \frac{dr^2}{\left( \frac{\Lambda r^3 - 3c_1}{3r} \right)} + r^2 (d\phi^2 + dz^2), \]

\[ ds^2_2 = -\left( \frac{\Lambda r^4 - 3c_1 r - 3c_8}{3r^2} \right) dt^2 + \frac{dr^2}{\left( \frac{\Lambda r^4 - 3c_1 r - 3c_8}{3r^2} \right)} + r^2 (d\phi^2 + dz^2). \]  

(17)

Eqs. (17) show that the metrics asymptotically behave as AdS/dS spacetime. Furthermore Eqs. (17) show that the first metric is static without any charge. The second metric has a charge which comes from the term of order \( O\left( \frac{1}{r^2} \right) \).

The second Eq. (17) can be rewritten as

\[ ds^2_1 = -\left( \frac{\Lambda r^2}{3} - \frac{m}{r} - \frac{q_m^2}{r^2} \right) dt^2 + \left( \frac{\Lambda r^2}{3} - \frac{m}{r} - \frac{q_m^2}{r^2} \right)^{-1} dr^2 + r^2 (d\phi^2 + dz^2), \]

(18)

where \( m = c_1 \) and \( q_m = \sqrt{c_m} \). The metric (18) is similar to the AdS/dS Reissner-Nordström solution [70]. Here we want to stress that the source of term \( O(r^{-2}) \) in metric (18) comes from the presence of magnetic field while, in the Reissner-Nordström case, such a term is related to the source of electric field.

Inserting Eqs. (16) into Eq. (6) we get

\[ T = 2\Lambda, \quad \Delta T = \frac{2(c_8 + \Lambda r^4)}{r^4}, \]

(19)

which shows that the scalar torsion is not constant in the second case. From Eqs. (19), it is easy to see that the torsion scalar of second case reduces to the first case as soon as the constant \( c_8 = 0 \). The behavior of the scalar torsion is given in Fig. 1.

Now we are going to calculate the singularities of solutions (16). The first step to discuss this issue is to find at which value of \( r \) the functions \( A(r) \) and \( A_1(r) \) become zero or infinity. The curvature and torsion invariants that arise from the first solution (16), using the Levi-Civita and Weitzenböck connections, take the form:

\[ R_{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} = \frac{4(2\Lambda^2 r^6 + 9c_1^2)}{3r^6}, \quad R_{\mu\nu\rho}R_{\mu\nu\rho} = 4\Lambda^2, \quad R = -4\Lambda, \]

\[ T_{\mu\nu}T_{\mu\nu} = \frac{4\Lambda^2 r^6 - 12\Lambda c_1 r^3 + 27c_1^2}{2r^3(3c_1 - \Lambda r^3)}, \quad \Delta T_{\mu} = \frac{3(3c_1 - 2\Lambda r^3)^2}{4r^3(3c_1 - \Lambda r^3)}, \quad T(r) = -2\Lambda, \]

\[ \nabla_\alpha T^\alpha = 3\Lambda, \quad \Rightarrow R = -T - 2\nabla_\alpha T^\alpha. \]  

(20)
and for the second solution, we get the invariants

\[ R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} = \frac{4(2\Lambda^2 r^8 + 6c_8[6c_1 + 7c_8r] + 9r^2 c_1^2)}{3r^8}, \quad R^{\mu\nu} R_{\mu\nu} = \frac{4(c_8^2 + \Lambda^2 r^8)}{r^8}, \quad R = -4\Lambda, \]

\[ T^{\mu\nu\lambda} T_{\mu\nu\lambda} = \frac{4\Lambda^2 r^8 - 8\Lambda c_8 r^4 - 12\Lambda c_1 r^5 + 27r^2 c_1^2 + 60c_1 c_8 r + 36c_8^2}{2r^4(3c_1 r - \Lambda r^4 + 3c_8)}, \]

\[ T^\mu T_\mu = \frac{3(3c_1 r - 2\Lambda r^4 + 2c_8)^2}{4r^4(3c_1 r - \Lambda r^4 + 3c_8)}, \quad T(r) = -\frac{2(\Lambda r^4 + c_8)}{r^4}, \quad \nabla_\alpha T^\alpha = \frac{(3\Lambda r^4 + c_8)}{r^4}, \]

(21)

The above calculations show that:

a)- Except for the scalars \( R^{\mu\nu} R_{\mu\nu}, R, \nabla_\alpha T^\alpha, T \) of the first solution and \( R \) of the second solution, all the above invariants show infinite behavior at \( r = 0 \) which represents a true singularity.

b)- For \( c_1 = \frac{r^3 \Lambda}{3} \) for the first solution, we get the horizon on the metric. For this value the curvature invariants are finite but torsion invariants diverge, i.e.

\[ T^{\mu\nu\lambda} T_{\mu\nu\lambda} \rightarrow \infty, \quad \text{and} \quad T^\mu T_\mu \rightarrow \infty. \]

This means that, on the horizon, the torsion invariants diverge. The reason that leads the curvature invariants to have finite value but torsion invariants diverge is the local Lorentz transformations. This can be seen clearly from the calculations of the scalar torsion \( T(r) \) which is finite on the horizon due to its invariant under local Lorentz transformations however, the scalars \( T^{\mu\nu\lambda} T_{\mu\nu\lambda} \) and \( T^\mu T_\mu \) are not finite because they are not invariant under local Lorentz transformations. The same discussion can be applied when \( c_8 = \frac{r^3 \Lambda - 3c_1}{3} \) for the second set of solution (16).

c)- The horizons of solutions (16) are respectively given for \( c_1 = \frac{r^3 \Lambda}{3} \) and \( c_8 = \frac{r^3 \Lambda - 3c_1}{3} \).

Let us now discuss some thermodynamical quantities related to the solution (18). To this aim, we calculate the horizons of the function

\[ \mathcal{N} = \frac{\Lambda r^2}{3} - \frac{m}{r} - \frac{q^2}{r^2}. \]

(22)

The above equation has 4 roots, 3 of them are imaginary while the fourth one is real and takes the form

\[ \frac{3^{2/3} 25^{1/6} \left(X^2/3 - 4q^2 \Lambda_1^{1/3}\right)^{3/4} + 27^{1/12} \sqrt{X^{2/3} \sqrt{2(X^{2/3} - 4q^2 \Lambda_1^{1/3})} + 25^{1/2} \Lambda_1^{1/3} q^2 (X^{2/3} - 4q^2 \Lambda_1^{1/3}) - 12m \sqrt{X}}}{12 \Lambda_1^{1/3} X^{1/6} [X^{2/3} - 4q^2 \Lambda_1^{1/3}]^{1/4}}. \]

(23)

where \( \Lambda_1 = 12\Lambda \) and \( X = 9m^2 + \sqrt{3(256q^2 \Lambda + 27m^4)} \). To ensure we have a real root, we must have \( \Lambda > -\frac{27m^4}{256q^6} \). The behavior of the horizon is drawn in Figure 2 which shows that we have only one horizon. The Hawking temperature is defined as [71]

\[ T_h = \frac{\mathcal{N}'(r_h)}{4\pi}. \]

(24)

where the event horizon is located at \( r = r_h \) which is the largest positive root of \( \mathcal{N}(r_h) = 0 \) that fulfills the condition \( \mathcal{N}'(r_h) \neq 0 \). The Hawking temperatures associated with the black hole solution (18) is calculated as

\[ T_h = \frac{3r_h^4 \Lambda + q^2}{4\pi r_h^3}, \]

(25)

where \( T_h \) is the Hawking temperature at the event horizon. We represent the Hawking temperature in Figure 3. This last figure shows that the temperature is always positive.
V. Energy conditions

An important issue is related to the possible violation of the energy conditions in cosmology or strong field regime. In GR, there are four types of energy conditions known as: The strong energy condition (SEC), the null energy condition (NEC), the dominant energy condition (DEC) and the weak energy condition (WEC) [72–75]. The SEC and the NEC arise from the structure of the gravitational field related to the dynamics of matter. It is related to the Raychaudhuri equation which leads the time expansion of the scalar \( \theta \) in terms of quantities like the Ricci tensor, the shear tensor \( \sigma_{\mu\nu} \) and the rotation \( \omega_{\mu\nu} \) for both time and light-like curves. These relations have the form:

\[
\frac{d\theta}{d\tau} = -\frac{1}{3} \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} + \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} u^\mu u^\nu = -\frac{1}{3} \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} + \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} u^\mu u^\nu, \tag{26}
\]

where \( u^\mu \) is an arbitrary time-like vector and \( k^\mu \) is an arbitrary null vector. As a consequence of the attraction, one can show

\[
R_{\mu\nu} u^\mu u^\nu \geq 0, \quad R_{\mu\nu} k^\mu k^\nu \geq 0. \tag{27}
\]

Eqs. (27) can be rewritten as

\[
R_{\mu\nu} u^\mu u^\nu = (T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}) u^\mu u^\nu \geq 0, \quad R_{\mu\nu} k^\mu k^\nu = (T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}) k^\mu k^\nu \geq 0. \tag{28}
\]
which are the SEC and the NEC, respectively for a given source of matter $T_{\mu\nu}$. In the case of perfect-fluid matter, the SEC and NEC given by (28) impose the following constraints $\rho + 3p \geq 0$ and $\rho + p \geq 0$ to be satisfied, while the WEC and DEC require the conditions $\rho \geq 0$ and $\rho \pm p \geq 0$, respectively for consistency.

The energy-momentum components of the first solution (16) are vanishing. This means that the first solution (16) is a vacuum solution. The non-vanishing components of the energy-momentum tensor of the second solution Eq. (16) have the form

$$T^0_0 = -T^3_3 = T^1_1 = -T^2_2 = -\frac{\epsilon_8}{2r^4}. \tag{29}$$

Eqs. (29) show that the WEC is violated unless $c_8 < 0$ and the DEC is satisfied for $\rho - p \geq 0$. However the DEC, NEC and SEC are all satisfied.

VI. EINSTEIN-CARTAN THEORY AND CONSERVED CURRENTS

The above considerations can be extended in the framework of the Einstein-Cartan theory. Let us define the Einstein-Cartan Lagrangian as [80]:

$$L(\theta^i, \Gamma^i_{jk}) = -\frac{1}{2\kappa} (R^{ij} \land \eta_{ij} - 2\Lambda \eta), \tag{30}$$

where $\theta^i$ is the co-frame, $\Gamma^i_{jk}$ is the connection one-form and $\kappa$ is the gravitational coupling constant that now we explicitly redefine. Lagrangian (30) is invariant under diffeomorphism and local Lorentz transformations [80]. The variation of Eq. (30) leads to the canonical energy-momentum and rotational gauge field momentum with the forms [80, 81]

$$E_i := -\frac{1}{2\kappa} (R^{jk} \land \eta_{ijk} - 2\Lambda \eta), \quad H_{ij} := \frac{1}{2\kappa} \eta_{ij}, \tag{31}$$

with $\eta_{ij}$ being a 2-form defined in Appendix A and $R^{jk}$ is the curvature 2-form. The conserved quantity of the gravitational field of (30) is [80]

$$j[\xi] = \frac{1}{2\kappa} d \left\{ \ast (dk + \xi \land (\theta^i \land T^i)) \right\}, \tag{32}$$

where $k = \xi_i \theta^i$, and $\xi^i = \xi \frac{\partial}{\partial \theta^i}$. Here $\ast$ denotes the Hodge duality, $\xi$ is an arbitrary vector field $\xi = \xi^i \partial_i$ and $\xi^i$ are four parameters $\xi^0, \xi^1, \xi^2$ and $\xi^3$. We are working in TEGR theory which is equivalent to GR, therefore the torsion is vanishing and the total charge, given by Eq. (32), takes the form

$$Q[\xi] = \frac{1}{2\kappa} \int_{@s} \ast dk. \tag{33}$$

This invariant conserved quantity $Q[\xi]$ was previously defined by Komar [76]-[79]. The quantity $Q[\xi]$ is conserved and invariant (for any given vector field $\xi$) under general coordinate transformations.

The coframe $\theta^i$ of solution (16), using tetrad (12), has the form:

$$\theta^0 = \sqrt{A(r)}dt, \quad \theta^1 = \frac{1}{\sqrt{A_1(r)}}dt, \quad \theta^2 = r d\phi, \quad \theta^3 = r dz. \tag{34}$$

Using Eqs. (34) into Eq. (32), we get

$$k = A(r) \xi_0 dt - \frac{\xi_1 dr}{A_1(r)} - r^2 \xi_2 d\phi - r^2 \xi_3 dz. \tag{35}$$
After some algebra, the total derivative of Eq. (35) has the form
\[ dk = A'(r)\xi_0(dr \wedge dt) + 2r\xi_2(d\phi \wedge dr) + 2r\xi_3(dz \wedge dr). \] (36)

Using the inverse of Eq. (35), (i.e. we write \( dt, dr, d\theta \) and \( d\phi \) in terms of \( \vartheta^0, \vartheta^1, \vartheta^2 \) and \( \vartheta^3 \)) and substituting Eq. (36) in Eq. (33) and applying the Hodge-dual to \( dk \), we finally get the total conserved charge in the form
\[ Q[\xi_t] = \xi_0(2r^3\Lambda + 3c_1), \quad Q[\xi_r] = Q[\xi_\theta] = Q[\xi_\phi] = 0, \] (37)

Using the same algorithm for the second solution (16), we get
\[ Q[\xi_t] = \xi_0(2r^4\Lambda + 3c_1 + 6c_7), \quad Q[\xi_r] = Q[\xi_\theta] = Q[\xi_\phi] = 0, \] (38)

Eqs. (37) and (38) show that the total conserved charges of solutions (16), using tetrad (12) and Eq. (33), are divergent when \( r \to \infty \). Therefore, Eq. (33) needs a regularization.

VII. REGULARIZATION VIA RELOCALIZATION

The conserved quantity given by Eq. (33) is invariant under diffeomorphism and local Lorentz transformations. Besides these transformations there is another issue in the definition of the conserved quantities which lies in the fact that the field equations allow for a relocalization of the gravitational field momenta [80]. Thus, the conserved currents can be altered through the relocalization of translational and rotational momenta. A relocalization generated by altering the Lagrangian of the gravitational field by a total derivative is given by
\[ L' = L + d\mathcal{N}, \quad \text{where} \quad \mathcal{N} = 2\mathcal{N}(\hat{\theta}, \Gamma_i^j, T_i^i, R_i^i). \] (39)

The second term exists in the Lagrangian, i.e., \( d\mathcal{N} \) modifies only the boundary part of the action, allowing the field equations to be invariant [80]. It is straightforward that the total conserved quantities can be regularized by means of a relocalization of the gravitational field momenta. It is shown that the most accurate method, that can solve the strange result derived in Eqs. (37) and (38), is to use relocalization which is originated by a boundary term in the Lagrangian. Here we use the relocalization
\[ H_{ij} \to H'_{ij} = H_{ij} - 2\alpha\eta_{ijkl}R^{kl}, \]
which is originated by altering the Lagrangian as [80]
\[ L \to L' = L + \alpha d\mathcal{N}, \]

where
\[ H'_{ij} = \left( \frac{1}{2\kappa} - \frac{4\alpha\Lambda}{3} \right) \eta_{ij} - 2\alpha\eta_{ijkl} \left( R^{kl} - \frac{\Lambda}{3} g^{kl} \right). \]

We assume \( \alpha \), that appears in the above equation to have the form \( \frac{3}{8\Lambda\kappa} \) to insure the removal of the divergence that appear in Eqs. (37) and (38). Therefore, the conserved charge, using the relocalization method, takes the form
\[ J[\xi] = -\frac{3}{4\kappa\Lambda} \int_{\partial S} \eta_{ijkl} \Xi^{ij} W^{kl}, \] (40)
where $W^{ij}$ is the Weyl 2-form defined by

$$W^{ij} = \frac{1}{2} C_{kl}^{ij} \vartheta^k \wedge \vartheta^l,$$

(41)

with $C_{ij}^{kl} = b_i^{\mu} b_j^{\nu} b_k^{\alpha} b_l^{\beta} C_{\mu\nu}^{\alpha\beta}$ being the Weyl tensor and $\Xi^{ij}$ defined as

$$\Xi_{ij} := \frac{1}{2} e_j |e_i| dk.$$

(42)

The conserved currents $\mathcal{J}[\xi]$ are invariant under both coordinate and local Lorentz transformations. These currents $\mathcal{J}[\xi]$ are related to a given vector field $\xi$ on the spacetime of the manifold.

We calculate the necessary components needed for Eq. (40). The non-vanishing components of $\Xi^{ij}$ have the form

$$\Xi_{01} = -\frac{\xi_0(2\Lambda r^3 + 3c_1)}{6r^2}, \quad \Xi_{13} = \frac{\xi_3(3c_1 - \Lambda r^3)}{\sqrt{3}r}.$$

(43)

Using Eqs. (40), we get

$$\eta_{ijkl} \Xi^{ij} W^{kl} = \frac{2c_1 \xi_0(2\Lambda r^3 + 3c_1)(dz \wedge d\phi)}{3r^3}.$$

(44)

Substituting Eq. (44) in (40) we finally get

$$\mathcal{J}[\xi_t] = \frac{c_1}{2}, \quad \mathcal{J}[\xi_r] = \mathcal{J}[\xi_\theta] = \mathcal{J}[\xi_\phi] = 0.$$

(45)

Eqs. (45) show that the constant $c_1$ may take the value $c_1 = \frac{M^2}{2}$ such that the total mass of Eqs. (45) takes the form [83, 84]

$$E = M + \left( \frac{1}{r} \right).$$

(46)

By the same method, we can get the conserved charge of the second solution (16). It has the form

$$E = M + \left( \frac{c_8}{r} \right) + O \left( \frac{1}{r^2} \right),$$

(47)

which shows that the constant $c_8$ behaves as the electric charge.

### VIII. Discussion and conclusions

Including a magnetic field in the metric is a challenging issue to get exact solutions in theories of gravity. Despite of this difficulty, some analytic solutions have been derived, like [88, 89], where a magnetic "universe", including a magnetic field in the $z$ direction, is considered. Furthermore Gutsunaev and Man'ko found a solution where a magnetic dipole is present [48]. Of course it is always possible to study an arbitrary shape for the magnetic field and solve the

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2 The detailed derivation of Eq. (40) is found in references [88–82].

3 The non-vanishing components of Weyl tensor are given in Appendix B.
resulting Einstein equations. In this study, we have addressed the problem of deriving charged black hole solutions, in TTEGR theory, involving cosmological constant and using a general gauge potential including magnetic fields only. For this purpose, we have applied a tetrad field with two unknown functions and assumed a cylindrical symmetry for the charged field equations of TTEGR. We have used a gauge potential which contains 6 unknown functions. Finally, we obtained a system of nonlinear differential equations that has been solved exactly. The solution of this system has two cases: in the first one, the torsion scalar has a constant value and all the components of the energy momentum, which depends on the charge fields, are identically vanishing while all the components of the magnetic field have non-trivial values. The second case contains an integration constant which gives a nontrivial value to the torsion scalar which becomes trivial when this constant is equal zero. It is worth noticing that this constant is related to some component of the magnetic field.

We then discuss the energy conditions related to these solutions and show that the first set satisfies these conditions because it has a trivial value of the energy momentum tensor. However, for the second set, the energy conditions are satisfied under certain constraints. We have also discussed the singularities of the two sets and have discussed the horizons of each set. Finally, we have calculated the conserved quantities related to each set and have shown that the Komar formula gives a divergent quantity on the temporal components.

Therefore, we have applied the regularization through relocalization in order to calculate the conserved quantities. For the first set, we have shown that the only conserved quantity is the energy and have related the constant that appeared in the calculation of energy to the ADM mass. The conservation of the second set gives, besides the ADM mass, another term which is related to the constant that makes the torsion scalar a dynamical one. So we can explain the contribution of this constant as related to the magnetic field. The most interesting thing is that the sign of this term is different from the sign of the term of Reissner-Nordström spacetime whose source of charge comes from the electric charge [85]. In a forthcoming paper, we will extend these considerations to more general TTEGR models like those discussed in [6].

It is important to stress that magnetic teleparallel solutions can be obtained also in spherical symmetry adopting procedures similar to that considered in this paper. In fact, it is easy to see that the assumption (12) can be recast for spherical coordinates considering suitable vierbien fields. However, the magnetic field has to be adapted to the spherical symmetry and one obtains different forms for the functions $A(r)$ and $A_1(r)$.

A final remark concerns possible astrophysical applications of the present results. As we said, the value of the torsion scalar depends on the strength of the magnetic field and this fact could have observational consequences on magnetic astrophysical systems. As reported in [90], torsion plays a dynamical role on magnetic vortex line curves of magnetars. In particular, torsion contributes to the oscillations of the magnetar and to the equation of state of such systems. Furthermore, in [91], several observational evidences are reported for neutron star magnetospheres related to binary pulsars, Crab pulses and magnetars. In all these cases, the strict relation between torsion and magnetic field could contribute to figure out the dynamics. A detailed analysis in this direction will be developed in a forthcoming study.

Appendix A: Notation used in the calculations of conserved currents

The indices $i, j, \cdots$ are used for the (co)frame components while $\alpha, \beta, \cdots$ label the local holonomic spacetime coordinates. Exterior product is defined as $\wedge$, while the interior is denoted by $\xi \lrcorner \Psi$. The vector basis, dual to the 1-forms $\vartheta^i$, is denoted by $e_i$. They satisfy the condition $e_i \lrcorner \vartheta^j = \delta_i^j$. Using the local coordinates $x^\mu$, we have $\vartheta^i = b^i_\mu dx^\mu$ and $e_i = b_i^\mu \partial_\mu$ where $b_i^\mu$ and $b_i^\mu$ are the covariant and contravariant components of the tetrad field. The volume is defined as $\eta := \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3$ which is a 4-form. Moreover, by using the interior product one can define

$$\eta_i := e_i \lrcorner \eta = \frac{1}{3!} \epsilon_{ijkl} \vartheta^j \wedge \vartheta^k \wedge \vartheta^l.$$
where $\epsilon_{ijkl}$ is totally antisymmetric with $\epsilon_{0123} = 1$.

\[
\eta_{ij} := e_j|\eta_i = \frac{1}{2!}\epsilon_{ijkl} \vartheta^k \wedge \vartheta^l, \quad \eta_{ijk} := e_k|\eta_{ij} = \frac{1}{3!}\epsilon_{ijkl} \vartheta^l,
\]
that are the bases for 3-, 2- and 1-forms respectively. Finally,

\[
\eta_{ijkl} := e_l|\eta_{ijk} = e_l|e_k|e_j|\eta,
\]
is the Levi-Civita tensor density. The $\eta$-forms satisfy the useful identities:

\[
\vartheta^i \wedge \eta^j := \delta_i^j \eta, \quad \vartheta^i \wedge \eta^jk := \delta_i^j \eta_k - \delta_i^k \eta_j, \quad \vartheta^i \wedge \eta^lijk := \delta_i^j \eta^lijk + \delta_i^k \eta^lijk + \delta_i^j \eta^lijk,
\]

\[
\vartheta^i \wedge \eta^lijk := \delta_i^j \eta^lijk - \delta_i^j \eta^lijk + \delta_i^j \eta^lijk - \delta_i^j \eta^lijk.
\]

(1)

**Appendix B: Calculations of the Weyl and $W^{\mu\nu}$ tensors**

The non-vanishing components of Weyl tensor, using solutions (16), have the form:

\[
C_{0101} = -C_{0110} = C_{1010} = -C_{1001} = 2C_{0220} = -2C_{0202} = 2C_{0330} = -2C_{0303} = -2C_{2002} = 2C_{2020} = 2C_{2330} = -2C_{2303} = -2C_{3203} = 2C_{2112} = -2C_{2121} = 2C_{2131} = -2C_{1331} = -2C_{2112} = 2C_{2121} = 2C_{3113} = -2C_{3232} = 2C_{3233} = 2C_{3323} = 2C_{3332} = -c_1 \frac{r}{r^3},
\]

(1)

and the non-vanishing components of the tensor $W^{\mu\nu}$ take the form

\[
W^{01} = \frac{c_1}{r^3}(dt \wedge dr), \quad W^{02} = \frac{\sqrt{3c_1 - \Lambda r^3}}{2\sqrt{3r^3}}(d\phi \wedge dt),
\]

\[
W^{03} = \frac{\sqrt{3c_1 - \Lambda r^3}}{2\sqrt{3r^3}}(dz \wedge dt), \quad W^{12} = \frac{3c_1}{2\sqrt{3r^3(3c_1 - \Lambda r^3)}}(d\phi \wedge dr),
\]

\[
W^{13} = \frac{3c_1}{2\sqrt{3r^3(3c_1 - \Lambda r^3)}}(dr \wedge dz), \quad W^{23} = \frac{c_1(d\phi \wedge dz)}{r},
\]

(2)

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