Indecomposability of branched coverings of even degree on the projective plane

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Abstract

In this work we characterize branch data of branched coverings of even degree over the projective plane which are realizable by indecomposable branched coverings.

Key words: branched coverings, primitive groups, imprimitive groups, permutation groups, projective plane.

1 Introduction

It is a natural and standard question to know whether a map in a given class of maps is a composition or not of maps in the same class. In 1957 Borsuk and Molski [9] asked about the existence of a continuous map of finite order\(^1\), which is not a composition of simple maps (maps of order \(\leq 2\)). For the historical development of this question see [5] and [4]. More recently, in 2002, Krzempek [16] constructed covering maps on locally arcwise connected continua that are not factorizable into covering maps of order \(\leq n - 1\), for all \(n\). Also in 2002, Bogataya, Bogatyí and Zieschang [5] gave an example of a 4-fold covering of a surface of genus 2 by a surface of genus 5 that cannot be represented as a composition of two non-trivial open maps. The relevant and general problem of classifying the branched coverings from the viewpoint of decomposability is related with the Inverse Galois problem (see for example the references [18] and [13]) and with a construction of primitive

\[^1\]A continuous map \(\phi\) defined on a space \(X\) is said to be of order \(\leq k \in \mathbb{Z}^+\) if for any \(y \in \phi(X), \phi^{-1}(y)\) contains at most \(k\) points.

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and imprimitive monodromy groups as treated in [17]. Besides the facts mentioned above, the problem is interesting in its own right. The present work is a contribution to the study of this problem. Recently some contribution to this problem was given in [3] and [4]. More specifically in [4] this problem has been considered for the class of branched coverings \( \phi : M \to N \) between connected closed surfaces. It consists in classifying which branch data (see definition below) can be realized by indecomposable and which ones can be realized by decomposable. In the work [4] this problem is completely solved in the case where \( N \) is different from the sphere \( S^2 \) and the projective plane \( \mathbb{R}P^2 \). The main results are:

**Theorem 1.1.** ([4]). Every non-trivial admissible data are realized on any \( N \) with \( \chi(N) \leq 0 \), by an indecomposable primitive branched covering.

**Proposition 1.2.** ([4]). Admissible data \( D \) are decomposable on \( N \), with \( \chi(N) \leq 0 \), if and only if there exists a factorization of \( D \) such that its first factor is non-trivial admissible data.

The purpose of this work is to study the same question for branched coverings over \( N = \mathbb{R}P^2 \) where the degree of the covering map \( \phi : M \to \mathbb{R}P^2 \) is even.

A branched covering \( \phi : M \to N \) of degree \( d \) between closed connected surfaces determines a finite collection of partitions \( \mathcal{D} \) of \( d \), the branch data, in correspondence with the branch point set \( B_\phi \subset N \). The total defect of \( \mathcal{D} \) is defined by \( \nu(\mathcal{D}) = \sum_{x \in B_\phi} d - \#\phi^{-1}(x) \). Conversely, given a collection of partitions \( \mathcal{D} \) of \( d \) and \( N \neq S^2 \), the necessary conditions for \( \mathcal{D} \) to be a branch datum are sufficient to realize it. Let us point out that the realization problem for \( \mathbb{R}P^2 \) has been solved in [11], in this case we will call \( \mathcal{D} \) admissible datum.

From now on let \( N = \mathbb{R}P^2 \) and \( \phi : M \to \mathbb{R}P^2 \) a primitive branched covering, i.e. the induced homomorphism \( \pi_1(M) \to \pi_1(\mathbb{R}P^2) \) is surjective. Since \( \phi \) is an orientation-true map then \( M \) is nonorientable, see [6]. If the homomorphism is not surjective the map admits an obvious decomposition.

The main result is:

**Theorem 3.7**. Let \( d \) be even and \( \mathcal{D} = \{D_1, \ldots, D_s\} \) an admissible datum. Then, \( \mathcal{D} \) is realizable by an indecomposable branched covering if, and only if, either:

1. \( d = 2 \), or
2. There is \( i \in \{1, \ldots, s\} \) such that \( D_i \neq [2, \ldots, 2] \), or
3. \( d > 4 \) and \( s > 2 \).

In Proposition 2.6[4] the authors classify admissible data realizable by decomposable primitive branched coverings, for \( \chi(N) \leq 0 \). Notice that the same proof applies
when \( \chi(N) = 1 \), i.e. \( N = \mathbb{R}P^2 \). Then, except for the case \( d = 2 \), where the branched covering clearly is never decomposable, we can assume in Theorem 3.7 that \( \mathcal{R} \) is realizable by decomposable primitive branched coverings. This is, if we join the main result in this paper with Proposition 2.6[4] we characterize admissible data realizable by decomposable primitive branched coverings. This is, if we join the main result in this paper with Proposition 2.6[4] we characterize admissible data realizable by decomposable primitive branched coverings.

The case where \( N = \mathbb{R}P^2 \) and \( d \) odd looks more subtle and it is a work in progress.

The paper is divided into two sections apart from the introduction. In Section 1, we quote the main definitions and the results about realization of branched coverings over \( \mathbb{R}P^2 \). In Section 2, we characterize branch data realizable by indecomposable branched coverings with even degree.

## 2 Preliminaries, terminology and notation

### 2.1 Permutation groups

We denote by \( \Sigma_d \) the symmetric group on a set \( \Omega \) with \( d \) elements and by \( 1_d \) its identity element. If \( \alpha \in \Sigma_d \) and \( x \in \Omega \), \( x^\alpha \) is the image of \( x \) by \( \alpha \). An explicit permutation \( \alpha \) will be written either as a product of disjoint cycles, i.e. its cyclic decomposition, or in the following way:

\[
\alpha = \begin{pmatrix}
1 & 2 & \cdots & 2k + 1 \\
1^\alpha & 2^\alpha & \cdots & (2k + 1)^\alpha
\end{pmatrix},
\]

depending on what is more convenient. The set of lengths of the cycles in the cyclic decomposition of \( \alpha \), including the trivial ones, defines a partition of \( d \), say \( D_\alpha = [d_1, \ldots, d_t] \), called the cyclic structure of \( \alpha \). Define \( \nu(\alpha) := \sum_{i=1}^t (d_i - 1) \), then \( \nu(\alpha) \equiv 0 \) (mod 2). Given a partition \( D \) of \( d \), we say \( \alpha \in D \) if the cyclic structure of \( \alpha \) is \( D \) and we put \( \nu(D) := \nu(\alpha) \).

For \( 1 < r \leq d \), a permutation \( \alpha \in \Sigma_d \) is called a \( r \)-cycle if in its cyclic decomposition its unique non-trivial cycle has length \( r \). Permutations \( \alpha, \beta \in \Sigma_d \) are conjugate if there is \( \lambda \in \Sigma_d \) such that \( \alpha^\lambda := \lambda \alpha \lambda^{-1} = \beta \). It is a known fact that conjugate permutations have the same cyclic structure.

Given a permutation group \( G \) on \( \Omega \) and \( x \in \Omega \), one defines the isotropy subgroup of \( x \), \( G_x := \{ g \in G : x^g = x \} \), and the orbit of \( x \) by \( G \), \( x^G := \{ x^g : g \in G \} \). For \( H \subset G \), the subsets \( \text{Supp}(H) := \{ x \in \Omega : x^h \neq x \text{ for some } h \in H \} \) and \( \text{Fix}(H) := \{ x \in \Omega : x^h = x \text{ for all } h \in H \} \) are defined. For \( \Lambda \subset \Omega \) and \( g \in G \), \( \Lambda^g := \{ y^g : y \in \Lambda \} \).

The permutation group \( G \) is transitive if for all \( x, y \in \Omega \) there is \( g \in G \) such that \( x^g = y \). A nonempty subset \( \Lambda \subset \Omega \) is a block of a transitive group \( G \) if for each \( g \in G \) either \( \Lambda^g = \Lambda \) or \( \Lambda^g \cap \Lambda = \emptyset \). A block \( \Lambda \) is trivial if either \( \Lambda = \Omega \) or \( \Lambda = \{ x \} \).
for some \( x \in \Omega \). Given a block \( \Lambda \) of \( G \), the set \( \Gamma := \{ \Lambda^\alpha : \alpha \in G \} \) defines a partition of \( \Omega \) in blocks. This set is called a system of blocks containing \( \Lambda \) and the cardinality of \( \Lambda \) divides the cardinality of \( \Omega \). \( G \) acts naturally on \( \Gamma \). A transitive permutation group is primitive if it admits only trivial blocks. Otherwise it is imprimitive.

**Example 1.** A transitive permutation group \( G \leq \Sigma_d \) containing a \((d - 1)\)-cycle is primitive. Without loss of generality let us suppose that \( g = (1 \ldots d - 1)(d) \in G \). Then any proper subset \( \Lambda \) of \( \{1, \ldots, d\} \) containing \( d \) and at least one more element satisfies \( \Lambda^g \neq \Lambda \) and \( \Lambda^g \cap \Lambda \neq \emptyset \). Thus the blocks of \( G \) are trivial and \( G \) is primitive.

**Proposition 2.1** ([10], Cor. 1.5A). Let \( G \) be a transitive permutation group on a set \( \Omega \) with at least two points. Then \( G \) is primitive if and only if each isotropy subgroup \( G_x \), for \( x \in \Omega \), is a maximal subgroup of \( G \).

### 2.2 Branched coverings on the projective plane

A surjective continuous open map \( \phi : M \rightarrow N \) between closed surfaces such that:

- for \( x \in N \), \( \phi^{-1}(x) \) is a totally disconnected set, and

- there is a non-empty discrete set \( B_\phi \subset N \) such that the restriction \( \hat{\phi} := \phi|_{M - \phi^{-1}(B_\phi)} \) is an ordinary unbranched covering of degree \( d \),

is called a branched covering of degree \( d \) over \( N \) and it is denoted by \((M, \phi, N, B_\phi, d)\). \( N \) is the base surface, \( M \) is the covering surface and \( B_\phi \) is the branch point set. Its associated unbranched covering is denoted by \((\hat{M}, \hat{\phi}, \hat{N}, d)\), where \( \hat{N} := N - B_\phi \) and \( \hat{M} := M - \phi^{-1}(B_\phi) \). It is known that \( \chi(\hat{M}) = d\chi(\hat{N}) \), equivalently

\[
\chi(M) - \#\phi^{-1}(B_\phi) = d(\chi(N) - \#B_\phi) \tag{1}
\]

The set \( B_\phi \) is just the image of the points in \( M \) in which \( \phi \) fails to be a local homeomorphism. Then each \( x \in B_\phi \) determines a non-trivial partition \( D_x \) of \( d \), defined by the local degrees of \( \phi \) on each component in the preimage of a small disk \( U_x \) around \( x \), with \( U_x \cap B_\phi = \{x\} \). The collection \( \mathcal{D} := \{D_x\}_{x \in B_\phi} \) is called the branch data and its total defect is the positive integer defined by \( \nu(\mathcal{D}) := \sum_{x \in B_\phi} \nu(D_x) \). The total defect satisfies the Riemann-Hurwitz formula (see [11]):

\[
\nu(\mathcal{D}) = d\chi(N) - \chi(M) \tag{2}
\]

Associated to \((M, \phi, N, B_\phi, d)\) we have a permutation group, the monodromy group of \( \phi \), given by the image of the Hurwitz’s representation

\[
\rho_\phi : \pi_1(N - B_\phi, z) \rightarrow \Sigma_d, \tag{3}
\]
which sends each class $\alpha \in \pi_1(N - B_\phi, z)$ to a permutation of $\phi^{-1}(z) = \{z_1, \ldots, z_d\}$, which indicates the terminal point of the lifting of a loop in $\alpha$ after fixing the initial point. In particular, for $x \in B_\phi$, let $c_x$ be a path from $z$ to a small circle $a_x$ about $x$ and define the loop class $u_x := [c_xa_xc_x^{-1}]$. Then the cyclic structure of the permutation $\alpha_x := \rho_\phi(u_x)$ is given by $D_x$ and $\nu(\prod_{x \in B_\phi} \alpha_x) \equiv \nu(\mathcal{D}) \pmod{2}$. The problem of realization of a branch data is equivalent to an algebraic problem in term of representation on the symmetric group. More precisely:

**Theorem 2.2** (See [15]). Let $N$ be a surface, $\mathcal{D}$ a finite collection of partitions of $d$ and $F \subset N$ such that $\#F = \#\mathcal{D}$. If it is possible to define a representation $\pi_1(N - F, z) \longrightarrow \Sigma_d$ like $\rho_\phi$, then $\mathcal{D}$ is realizable as branch datum of a branched covering on $N$.

**Remark.** If $N = \mathbb{R}P^2$ and $\mathcal{D} = \{D_1, \ldots, D_s\}$, to define $\rho_\phi$, it is necessary and sufficient to have permutations $\alpha_i \in D_i$, for $i = 1, \ldots, s$, such that $\prod_{i=1}^s \alpha_i$ is a square. This follows from the presentation $\pi_1(\mathbb{R}P^2 - \{x_1, \ldots, x_s\}) = \langle a, u_1, \ldots, u_s | \prod_{i=1}^s u_i = a^{-2} \rangle$.

**Example 2.** If $r > 0$ is an odd natural number then every $r$-cycle is the square of a permutation:

if $\alpha = (a_1 a_2 \ldots a_r)$ then $\alpha = \beta^2$ where $\beta = (a_1 a_{(r+1)/2+1} a_2 a_{(r+1)/2+2} \ldots a_r a_{(r+1)/2})$.

The brach data which can be realized by branched coverings over $\mathbb{R}P^2$ are given by:

**Theorem 2.3** (See [11]). Let $\mathcal{D}$ be a collection of partitions of $d$. Then there is a branched covering $\phi : M \rightarrow \mathbb{R}P^2$ of degree $d$, with $M$ connected and with branch data $\mathcal{D}$ if and only if

$$d - 1 \leq \nu(\mathcal{D}) \equiv 0 \pmod{2}. \quad (4)$$

Moreover, $M$ can be chosen to be nonorientable.

**Remark.** The realization result above does not tell which branch data can be realized by an orientable covering. In fact it is not hard to show that there is a bijection between the set of branched coverings over $\mathbb{R}P^2$ where the covering surface is orientable and the set of branched coverings over the sphere $S^2$ which have an even number of branched points. It is certainly an interesting problem to classify such realizable brached data over the sphere $S^2$ from the viewpoint of decomposibility.

**Definition 2.4.** A collection of partitions $\mathcal{D}$ of $d$ satisfying (4) will be called admissible datum.
3 Decomposability

Given a covering, it is decomposable if it can be written as a composition of two non-trivial coverings (i.e., both with degree bigger than 1), otherwise it is called indecomposable. In a decomposition of a branched covering at least one of its components is a branched covering having proper branching. Moreover, since the degree of a decomposable covering is the product of the degrees of its components (see [5], theorem 2.3), we are interested in branched coverings with non-prime degree.

Proposition 3.1 (See [4]). A primitive branched covering is decomposable if and only if its monodromy group is imprimitive. □

Let \((M, \phi, \mathbb{R}P^2, B_\phi, d)\) be primitive with branch data \(\mathcal{D} = \{D_1, \ldots, D_s\}\), where \(s := \#B_\phi\). If \(d\) is even, by (2) and (4), \(\chi(M)\) is even and since \(M\) is non-orientable \(\chi(M) \leq 0\). Then by (1), \(s > 1\).

Proposition 3.2. Let \(d > 2\) be an even number and \(\mathcal{D} = \{D_1, \ldots, D_s\}\) an admissible datum. If \(\mathcal{D}\) contains a partition different of \([2, \ldots, 2]\), then \(\mathcal{D}\) can be realized by an indecomposable branched covering.

Proof. Without loss of generality, let us suppose \(D_s \neq [2, \ldots, 2]\). Since \(d \leq \nu(\mathcal{D}) \equiv 0 \pmod{2}\), there is \(q \geq 0\) such that \(\nu(\mathcal{D}) = d + 2q\) and \(\nu(D_1) + \nu(D_2) = d + 2q - \sum_{i=3}^{s} \nu(D_i)\). If \(t := \sum_{i=3}^{s} \nu(D_i) - 2q\) is bigger than zero, applying Lemma 4.2 [11], there are permutations \(\gamma_1 \in D_1, \gamma_2 \in D_2\) such that \(\langle \gamma_1, \gamma_2 \rangle\) acts in \(\{1, \ldots, d\}\) with \(t\) orbits and

\[
\nu(\gamma_1\gamma_2) = d - t = d - \sum_{i=3}^{s} \nu(D_i) + 2q.
\] (5)

If \(t \leq 0\), define \(r := 1 - t\) and applying Lemma 4.3 [11], for \(k := -(1 - \sum_{i=3}^{s} \nu(D_i)) \equiv r \pmod{2}\), there exist \(\gamma_1 \in D_1, \gamma_2 \in D_2\) such that \(\langle \gamma_1, \gamma_2 \rangle\) acts transitively on \(\{1, \ldots, d\}\) and

\[
\nu(\gamma_1\gamma_2) = (d - 1) - k = d - \sum_{i=3}^{s} \nu(D_i).
\] (6)

Let \(D_{12}\) be the partition determined by the cyclic structure of \(\gamma_1\gamma_2\). Then (5) implies \(\nu(D_{12}) + \nu(D_3) = d + 2q - \sum_{i=4}^{s} \nu(D_i)\) and we can repeat the analysis done before. On the other hand, the situation in (6) implies that \(\nu(D_{12}) + \nu(D_3) = d - \sum_{i=4}^{s} \nu(D_i)\) and since \(\sum_{i=4}^{s} \nu(D_i) > 0\), by Lemma 4.2 [11], there are \(\gamma_{12} \in D_{12}, \gamma_3 \in D_3\) such that
\( \langle \gamma_{12}, \gamma_3 \rangle \) acts with \( \sum_{i=4}^s \nu(D_i) \) orbits and \( \nu(\gamma_{12}\gamma_3) = d - \sum_{i=4}^s \nu(D_i) \). It is clear that repeating the analysis done before we will obtain one of the following conditions:

\[
\nu(D_{12...s-1}) + \nu(D_s) = \left\{ \begin{array}{ll}
3 + 2q & \text{applying Lemma 4.2}\|, \\
d & \text{applying Lemma 4.3}\|,
\end{array} \right.
\]

(7)

where \( D_{12...j} \) denotes the partition determined by \( \gamma_{1...j-1}; \gamma_j \), where \( \gamma_{1...j-1} \in D_{1...j-1} \) and \( \gamma_j \in D_j \) is obtained by successive applications of Lemmas 4.2\| and 4.3\|, for \( j = 2, \ldots, s - 1 \). Whichever the case, we are under the hypothesis of Lemma 4.5\| then, since \( D_s \neq [2, \ldots, 2] \), there are permutations \( \gamma_{12...s-1} \in D_{12...s-1}, \gamma_s \in D_s \) such that the group \( \langle \gamma_{12...s-1}, \gamma_s \rangle \) acts transitively on \( \{1, \ldots, d\} \) and the product \( \gamma_{12...s-1}; \gamma_s \) is a \( d - 1 \) cycle. Moreover by Example\| there is a permutation \( \alpha \in \Sigma_d \) such that \( \gamma_{12...s-1}; \gamma_s = \alpha^2 \) and the permutation group \( \langle \gamma_{12...s-1}, \gamma_s \rangle \) is primitive by Example\|. On the other hand, for \( j = 2, \ldots, s - 1 \), there are \( \lambda_1, \ldots, \lambda_j \in \Sigma_d \) such that \( \gamma_{12...j} = \gamma_1^{\lambda_2} \gamma_2^{\lambda_3} \cdots \gamma_j^{\lambda_j} \) (recall that \( \gamma_j^{\lambda_j} := \lambda_j^{\gamma_j} \gamma_j^{\lambda_j} \)). Thus, we define the representation

\[
\rho : \langle a, u_1, \ldots, u_s | a^{2\Pi_{i=1}^s u_i = 1} \rangle \rightarrow \Sigma_d \\
a \mapsto \alpha^{-1}, \\
u_i \mapsto \gamma_i^{\lambda_i}, \\
u_s \mapsto \gamma_s.
\]

Then, there exists a primitive branched covering \( (M, \phi, \mathbb{R}P^2, B_\phi, d) \) with \( M \) nonorientable realizing \( \mathcal{D} \) as branch data. Moreover, since \( \langle \gamma_{12...s-1}, \gamma_s \rangle < G := \text{Im}\rho \), then \( G \) is a primitive permutation group and by Proposition 3.1\| \((M, \phi, \mathbb{R}P^2, B_\phi, d)\) is indecomposable.

Proposition 3.3. Let \( d > 4 \) be even and \( \mathcal{D} = \{D_1, \ldots, D_s\} \) an admissible datum such that \( D_i = [2, \ldots, 2] \) for \( i = 1, \ldots, s \). If \( s > 2 \), there is an indecomposable branched covering \( (M, \phi, \mathbb{R}P^2, B_\phi, d) \) realizing \( \mathcal{D} \).

Proof. Let \( s > 2 \) be a natural number. Since \( \nu(D_1) + \nu(D_2) = d \), by Lemma 4.5\| there are permutations \( \gamma_1 \in D_1, \gamma_2 \in D_2 \) such that \( \langle \gamma_1, \gamma_2 \rangle \) is transitive and \( \gamma_1 \gamma_2 \in D_{12} := \langle d/2, d/2 \rangle \). Then \( \nu(D_{12}) + \nu(D_3) = d - 2 + d/2 \). If \( d/2 \) is even, we apply again Lemma 4.5\| and we obtain permutations \( \gamma_1 \in D_{12}, \gamma_3 \in D_3 \) such that \( \langle \gamma_{12}, \gamma_3 \rangle \) is transitive and \( \gamma_{12}\gamma_3 \in D_{123} := \langle d - 1, 1 \rangle \) is a \( (d - 1) \)-cycle (because \( d/2 \neq 2 \)). Then by Example\| there exist \( \alpha \in \Sigma_d \) such that \( \gamma_{12}\gamma_3 = \alpha^2 \) and \( \langle \gamma_{12}, \gamma_3 \rangle \) is primitive. Since \( \gamma_{12} \) and \( \gamma_1 \gamma_2 \) are conjugates, there is \( \lambda \in \Sigma_d \) such that
\[ \gamma_{12} = \lambda \gamma_1 \gamma_2 \lambda^{-1} \]. If \( s \) is odd, we define the following representation:

\[
\rho : \langle a, \{u_j\}_{j=1}^s | a^2 \prod_{j=1}^s u_j = 1 \rangle \rightarrow \Sigma_d \\
a \mapsto \alpha^{-1}, \\
u \mapsto \lambda \gamma_1 \lambda^{-1}, \\
\{u_j\}_{i=2}^{s-1} \mapsto \lambda \gamma_2 \lambda^{-1}, \\
u_s \mapsto \gamma_3.
\]

If \( s \) is even, then \( \nu(D_{123}) + \nu(D_4) = d - 2 + d/2 \) and again, applying Lemma 4.5 \[11\] we obtain permutations \( \gamma_{123} \in D_{123} \) and \( \gamma_4 \in D_4 \) such that \( \langle \gamma_{123}, \gamma_4 \rangle \) is transitive and \( \gamma_{123} \gamma_4 \) is a \((d - 1)\)-cycle. Then by Example 2 there is \( \alpha \in \Sigma_d \) such that \( \gamma_{123} \gamma_4 = \alpha^2 \) and \( \langle \gamma_{123}, \gamma_4 \rangle \) is primitive by Example \[11\]. Notice that there exist \( \lambda_1, \lambda_2, \lambda_3 \in \Sigma_d \) such that \( \gamma_{123} = \gamma_1^{\lambda_1} \gamma_2^{\lambda_2} \gamma_3^{\lambda_3} \) and in this case we define the following representation:

\[
\rho : \langle a, \{u_j\}_{j=1}^s | a^2 \prod_{j=1}^s u_j = 1 \rangle \rightarrow \Sigma_d \\
a \mapsto \alpha^{-1}, \\
u \mapsto \gamma_1^{\lambda_1}, \\
\{u_j\}_{i=2}^{s-1} \mapsto \gamma_2^{\lambda_2}, \\
u_s \mapsto \gamma_3^{\lambda_3}, \\
u_{s-1} \mapsto \gamma_4.
\]

Whichever the case \( G := \text{Im}(\rho) \) is primitive by Proposition 3.1, the branched covering associated to \( G \) is indecomposable.

If \( d/2 \) is odd, since \( \nu(\emptyset) = sd/2 \), the hypothesis implies \( s \geq 4 \) even. Since \( \nu(D_{12}) + \nu(D_3) = (d - 1) + (d/2 - 1) \), by Lemma 4.3 \[11\] there are \( \gamma_{12} \in D_{12}, \gamma_3 \in D_3 \) such that \( \langle \gamma_{12}, \gamma_3 \rangle \) is transitive and \( \gamma_{12} \gamma_3 \) is a \( d \)-cycle. Let \( D_{123} := [d] \) thus \( \nu(D_{123}) + \nu(D_4) = d + (d/2 - 1) \) and by Lemma 4.5 in \[11\], we obtain permutations \( \gamma_{123} \in D_{123}, \gamma_4 \in D_4 \) such that \( \langle \gamma_{123}, \gamma_4 \rangle \) is transitive and \( \gamma_{123} \gamma_4 \) is a \((d - 1)\)-cycle. Then by Example 2 there is \( \alpha \in \Sigma_d \) such that \( \gamma_{123} \gamma_4 = \alpha^2 \) and, by Example \[11\], \( \langle \gamma_{123}, \gamma_4 \rangle \) is primitive. For this case we define a representation like the last in the case before.

\[ \square \]

**Lemma 3.4.** Let \( d \neq 2 \) be even and \( \alpha, \beta \in [2, \ldots, 2] \), such that \( G := \langle \alpha, \beta \rangle \) is transitive. Then \( G \) is imprimitive and unique up to conjugation.

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Proof. Let us suppose \(d = 2k\), with \(1 < k \in \mathbb{N}\). To obtain \(\langle \alpha, \beta \rangle\) transitive, it is necessary that transpositions in \(\beta\) “link” \(k\) transpositions in \(\alpha\). For that, we need \(k - 1\) transpositions and thus \(\beta\) is automatically defined. Then, up to conjugation, we can consider \(\alpha = (1\ 2)(3\ 4) \ldots (d - 1\ d)\) and \(\beta = (2\ 3)(4\ 5) \ldots (d - 2\ d - 1)(d\ 1)\). Thus \(\alpha\beta = (1\ 3\ \ldots\ d - 1)(2\ 4\ \ldots\ d)\) and the set \(B = \{1, 3, \ldots, d - 1\}\) will be a nontrivial block of \(G = \langle \alpha, \beta \rangle\), which makes it imprimitive. \(\square\)

**Proposition 3.5.** A primitive branched covering realizing \([2, \ldots, 2], [2, \ldots, 2]\) is decomposable.

Proof. Let \((M, \phi, \mathbb{R}P^2, \{x, y\}, d)\) be a primitive branched covering with branch data \([2, \ldots, 2], [2, \ldots, 2]\). If

\[
\rho : \langle a, u_1, u_2 | a^2u_1u_2 = 1 \rangle \rightarrow \Sigma_d
\]

\[
a \mapsto \alpha,
\]

\[
u_1 \mapsto \gamma_1,
\]

\[
u_2 \mapsto \gamma_2,
\]

is its Hurwitz’s representation, \(G := \text{Im}\rho = \langle \alpha, \gamma_1, \gamma_2 | \gamma_1^2 = \gamma_2^2 = 1, \alpha^2\gamma_1\gamma_2 = 1 \rangle\) is a transitive permutation group with \(\gamma_1, \gamma_2 \in [2, \ldots, 2]\).

If \(\langle \gamma_1, \gamma_2 \rangle\) is transitive, by Lemma 3.4 it is imprimitive and the branched covering is decomposable. If not, using relations in \(G\), is easy to see that \([\text{Fix}(\gamma_1\gamma_2)]^n \subset \text{Fix}(\gamma_1\gamma_2)\), for \(i = 1, 2\), and \([\text{Fix}(\gamma_1\gamma_2)]^\alpha \subset \text{Fix}(\gamma_1\gamma_2)\).

Then

\[
\text{Fix}(\gamma_1\gamma_2) = [\text{Fix}(\gamma_1\gamma_2)]^n = [\text{Fix}(\gamma_1\gamma_2)]^\alpha,
\]

because \(\gamma_i\), for \(i = 1, 2\), and \(\alpha\) are permutations. Then for all \(g \in G\) we have \([\text{Fix}(\gamma_1\gamma_2)]^g = \text{Fix}(\gamma_1\gamma_2)\). If \(\text{Fix}(\gamma_1\gamma_2) \neq \emptyset\), since \(G\) is transitive, then \(\gamma_1\gamma_2 = 1\). Up to conjugation, \(\gamma_1 = \gamma_2 = (1\ 2)(3\ 4) \ldots (d - 3\ d - 2)(d - 1\ d)\). By the relation, the options for \(\alpha\) are either \(\alpha := (2\ 3)(4\ 5) \ldots (d - 2\ d - 1)(d\ 1)\) or \(\alpha := (1)(2\ 3)(4\ 5) \ldots (d - 2\ d - 1)(d)\). The first option implies \(G\) equal to the group in Lemma 3.4 then it is imprimitive. The second one implies \(\{1, d\}\) as a block. If \(\text{Fix}(\gamma_1\gamma_2) = \emptyset\) then every cycle of \(\alpha\) has length \(\geq 3\) and \(\gamma_1, \gamma_2\) have not common cycles. Let \(O_1, \ldots, O_k\) be the orbits of the action of \(\langle \gamma_1, \gamma_2 \rangle\) on \(\{1, \ldots, d\}\), with \(k > 1\). Notice that \(\#O_i \geq 4\) is even, because each transposition of \(\gamma_2\) linked transpositions of \(\gamma_1\) then, it connects an even number of elements. On the other hand, if \(\langle \gamma_1, \gamma_2 \rangle_i\) denotes the restriction of \(\langle \gamma_1, \gamma_2 \rangle\) on \(O_i\), we are in the situation of Lemma 3.4 therefore \(\langle \gamma_1, \gamma_2 \rangle_i\) is imprimitive. If \(\#O_i = 2n\) for \(n \in \mathbb{Z}^+\), then its elements appear in \(\gamma_1\gamma_2\) in the form \((a_{i_1} \ldots a_{i_n})(a_{i_{n+1}} \ldots a_{i_{2n}})\). Considering the
relation \( \gamma_1 \gamma_2 = \alpha^{-2} \), we conclude that \( \alpha \) will connect two orbits, \( O_i \) and \( O_j \), only if \( \#O_i = \#O_j \). But \( \alpha \) makes the group \( G \) transitive, and so all orbits have the same cardinality equal to \( 2n \). For example, if \( i \neq j \) and the elements of \( O_j \) are in the form \((b_{j1} \ldots b_{jn})(b_{j1+1} \ldots b_{jn})\), without loss of generality \((a_{i1} b_{j1} a_{i2} b_{j2} \ldots a_{in} b_{jn})\) is a cycle of \( \alpha \) and thus, the blocks of \( \langle \gamma_1, \gamma_2 \rangle_i \), for \( i = 1, \ldots, k \), become blocks for \( G \).

Then \( G \) is imprimitive and the branched covering is decomposable.

**Proposition 3.6.** A primitive branched covering of degree 4 realizing the finite collection \( \mathcal{D} = \{[2, 2], \ldots , [2, 2]\} \) is decomposable.

**Proof.** Let \((M, \phi, \mathbb{RP}^2, B_\phi, 4)\) be a primitive branched covering with branch data \( \mathcal{D} \). Suppose \( \nu(\mathcal{D}) = 2t \), \( 2 \leq t \in \mathbb{Z}^+ \). Let

\[
\rho : \langle a, u_1, \ldots, u_t | a^2 \prod_{i=1}^t u_i = 1 \rangle \rightarrow \Sigma_4
\]

\[
a \mapsto \alpha
\]

\[
u_i \mapsto \gamma_i.
\]

be its Hurwitz’s representation. Note that the possible images for \( \prod_{i=1}^t u_i \) are, without loss of generality, either \((1)(2)(3)(4)\) or \((12)(34)\). Define \( U := \langle \gamma_1, \ldots, \gamma_t \rangle \). If \( U \) is transitive, then \( U \cong \langle (12)(34), (13)(24) \rangle \) is imprimitive, because each pair of elements is a block. Thus, if \( \rho(\prod_{i=1}^t u_i) = 1 \), the group \( G := \text{Im} \rho = \langle U, \alpha | a^2 \prod_{i=1}^t u_i = 1 \gamma_i \rangle \) is imprimitive, for all \( \alpha \). On the other hand, if \( \rho(\prod_{i=1}^t u_i) = (12)(34) \) then, either \( \alpha = (1324) \) or \( \alpha = (1423) \). Whichever the case, \( \{1, 2\} \) is a block. If \( U \) is not transitive, then \( U \cong \langle (12)(34) \rangle \) and, for guarantee the transitivity of \( G \), we have \( \alpha = (1324) \). Thus \( \{1, 2\} \) is a block and \( G \) is imprimitive.

We summarize the case \( d \) even in the following theorem:

**Theorem 3.7.** Let \( d \) be even and \( \mathcal{D} = \{D_1, \ldots, D_s\} \) an admissible datum. Then, \( \mathcal{D} \) is realizable by an indecomposable branched covering if, and only if, either:

1. \( d = 2 \), or
2. There is \( i \in \{1, \ldots, s\} \) such that \( D_i \neq [2, \ldots, 2] \), or
3. \( d > 4 \) and \( s > 2 \).

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