On Spin II

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Abstract. Having previously identified the photon field with a (special) linear Complex, we give a brief account on identifications and reasoning so far. Then, in order to include spinorial degrees of freedom into the Lagrangean description, we discuss the mapping of lines to spins based on an old transfer principle by Lie. This introduces quaternionic reps and relates to our original group-based approach by SU(4) and SU*(4) \cong SL(2,H), respectively. Finally, we discuss some related geometrical aspects in terms of (spatial) projective geometry which point to a projective construction scheme and algebraic geometry.

1. Introduction
So far, we’ve transformed our original, group-based view (see e.g. [1] or [2]) from using classical point(like) reps\(^1\) in Lagrangean approaches towards a Lagrangean description which includes higher order objects as well as ’extended’ geometrical objects like lines and Complexe in order to describe physical observations.

In this context, the major building block has been the identification of certain geometrical objects, properties and symmetries which – when associated with ’point’ reps in \(P^5\) – yields line geometry in \(P^3\), i.e. in real 3-dim space represented by homogeneous line coordinates. We’ll summarize few aspects and references in section 2. Based on this identification, in [3] we have used Plücker’s four line coordinates \((r, \rho, s, \sigma)\) via the (Euclidean) line rep \(x = rz + \rho, y = sz + \sigma\) as well as Lie’s reasoning and transfer principle [4] between lines and spheres. By rearranging line coordinates, we have shown that we thus obtain a matrix rep in terms of Pauli matrices, i.e.

\[
\begin{pmatrix}
 r \\
 \rho \\
 s \\
 \sigma 
\end{pmatrix}
\sim
\begin{pmatrix}
 -Z \\
 X - iY \\
 X + iY \\
 +Z 
\end{pmatrix}
\sim
X\sigma_1 - Y\sigma_2 - Z\sigma_3
\tag{1}
\]

relating Lie’s two 3-dim spaces \(r\) and \(R\) ([3], section III.B). The lhs of eqn. (1) is based on the space \(r\) and describes a projective transformation in real 3-space with the usual projective geometry, whereas the rhs relates to point reps \((X,Y,Z)\) in the space \(R\) and Lie’s sphere geometry. We have mapped this sphere geometry to a Pauli rep ([3], section III.B, especially eq. (7), and ibd. section III.E), and discussed this ’Lie transfer’ and some related parallels between the respective geometries of \(r\) and \(R\) ([3], section III.G ff.). Thus we have related lines and Complexe with typical SL(2,C) spinor, or quaternionic, calculus. In [3], we have also shown that Cartan’s spinor calculus has its foundation (and we think its origin) in Study’s and Beck’s work [3], and that these topics have to be treated as a subset of rational curves and advanced (projective) geometry (PG).

\(^1\) As before, we use this shorthand notation for ’representation(s)’.
On the other hand, in [5] – using the same original line rep \( x = rz + \rho, \ y = sz + \sigma \) – we have discussed that Minkowski’s paper [6] on special relativity (SR) encapsulates certain aspects of line and projective geometry in the contemporarily emerging 4-vector description, and we have shown that this treatment and invariant theory can be simplified by switching to the linear Complex \( \mathbb{C} \) and their geometry. Especially, the two ‘invariants’ nowadays derived by the SU(2) \( \times \mathbb{C} \) interpretation of SR are directly related to the parameters of a linear Complex \( \mathbb{C} \).

So here, we want to use those prerequisites to approach Dirac theory and various spinor representations commonly used throughout Lagrangean descriptions of quantum field theory (QFT). Based on this reasoning, here we are going to discuss two possible approaches and related aspects. In section 2, we summarize few necessary aspects of [3] and [5]. In section 3, we ‘re-organize’ the spinorial rep in a suitable manner for use in sections 4 and 5. In section 4, we discuss a real interpretation of the \( C \) we ‘re-organize’ the spinorial rep in a suitable manner for use in sections 4 and 5. In section 4, we discuss a real interpretation of the C4x2 matrix in order to visualize the action of the Dirac algebra which can be related to the lhs of eq. (1) and the space \( r \). We close in section 5 with an interpretation related to \( R \) and a brief outlook.

2. Group and Rep Theory

In [7], appendix 1, we have discussed the situation of (compact) SU(4) reps where, departing from SO(6), we can connect SU(4) to various related real forms, and to the Lie algebras so(m,n) or groups SO(m,n) with \( m + n = 6 \). This gave a further hint to work in \( P^5 \) right from the beginning and discuss Complex (and line) geometry. So with respect to (non-compact) SU*(4) \( \cong \text{SL}(2,\mathbb{H}) \) (covering SO(5,1) twice), or ‘Dirac theory’ and appropriate reps, we want to discuss once more Dirac’s original problem to find linear reps here: Given the (quadratic) energy-momentum relation \( p_\mu p^\mu = E^2 - p^2 = m^2 \), and considering a rep of \( p_\mu \) by differential operators acting on a rep \( \psi \), how do we construct and identify the (linear) rep spaces and their geometry?

There is (at least) twofold interest in this discussion:

On the one hand, \( x_\mu x^\mu = 0 \) in point space is related to a quadratic Complex in line space, \( x_\alpha \) denoting homogeneous point coordinates. So given \( \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 + \alpha_0 x_0^2 = 0 \) to describe a second order surface \( S \), the same surface in line coordinates reads as

\[
\alpha_1 \alpha_2 p_{12}^2 + \alpha_1 \alpha_3 p_{13}^2 + \alpha_2 \alpha_3 p_{23}^2 + \alpha_0 \alpha_1 p_{01}^2 + \alpha_0 \alpha_2 p_{02}^2 + \alpha_0 \alpha_3 p_{03}^2 = 0
\]  

(2)

In the case of the ‘Minkowski metric’, \( x_1^2 + x_2^2 + x_3^2 - x_0^2 = 0 \), eq. (2) yields the quadratic Complex \( p_{12}^2 + p_{13}^2 + p_{23}^2 - p_{01}^2 - p_{02}^2 - p_{03}^2 = 0 \), or formally an SO(3,3) symmetry \( \cong \text{PGL}(4,\mathbb{C}) \).

Note for later use, that this description of the surface in terms of line coordinates is defined only up to a multiple of the Plücker condition \( F = p_{01} p_{23} + p_{02} p_{31} + p_{03} p_{12} = 0 \), i.e. we are allowed to add \( m \, P \) to the lhs of eq. (2), \( m \) being a number.

In [3], section III.I, we have drawn attention already to Clebsch’s paper [8] on Complex symbolism to treat powers of Complexes and associate linear Complexes which may serve to obtain linear reps. This, however, shifts the focus to the (linear) reps themselves \( \cong x_\mu \) or \( p_\mu \) are of dimension 4 while Complexes are of dimension 6. Whereas we have already associated the photon to a special (linear) Complex [9], [5] which has to be treated carefully due to degeneracies and thus appears in different roles and contexts, a regular linear Complex \( \mathfrak{H} \) is associated to a (non-degenerate) correlation, the null system, mapping point to plane coordinates, \( u_\beta = (\mathfrak{H})_{\beta \alpha} x_\alpha \).

2 We restrict the discussion to the historical question focusing on the tangent rep and differential geometry. More general, there are additional possibilities from within PG. One approach based on Plücker’s line variable \( \eta \) will be given in section 5.

3 We suppress a detailed discussion of the associated polar theory here!

4 We have discussed already possible complexifications, or ‘transfers’, to SO(n,m) with \( n + m = 6 \) [7]. The covering groups (have to) change appropriately, too.

5 See also [7], appendix, with respect to a discussion of reps and rep dimensions in SU(4) context and a geometrical identification.
and vice versa. This is automatically related to an incidence relation $x \cdot u = 0$, because $x_{\beta}u_{\beta} = \mathcal{R}_{\beta\alpha}x_{\alpha} = (\mathcal{R})_{\beta\alpha}x_{\alpha} = 0$ due to the antisymmetry of the $\mathfrak{R}$ rep\(^6\). While the calculus of SR treats '4-dim points' (and tensorial reps thereof), and requires additional assumptions and rules with respect to a certain ('Minkowski') metric, quadratic Complexes are embedded in the well-defined and well-known framework of (advanced) projective geometry, so e.g. the metric can be simply derived by the Cayley-Klein mechanism. We know from the early days from mechanics (e.g. in Möbius’ work or see Plücker [11]), that the central notion of 'a force' has to be represented by 6-dim line reps and null systems, and we know from Minkowski and Poincaré that one should map the 6-dim line rep (or more general a linear Complex) to an antisymmetric twofold tensor $F^{\mu\nu}$ to incorporate forces into the 4-dim rep theory of special relativity [6] [5]. So why not just go back to the original force definition\(^7\) and see how to extract correct reps or even irreps?

On the other hand, the notion of tangents (and as such of lines!) has been transported to metric (tangent) spaces, differential geometry and 4-dim (point) reps of a 'momentum' $p^\mu$, and used throughout literature, whereas some problematic parts of this identification like mass (see e.g. [12], ch. 4.1) or the separation of spin (and thus, of course, orbital angular momentum) from the common 6-dim rep are treated separately in terms of additional linear reps or (in some approaches) as 'perturbations'. In order to keep both pictures (lines and 4-vector calculus) alive, we focus on the 6-dim rep, either in terms of linear Complexes using line coordinates, or in terms of the twofold antisymmetric tensor rep $F^{\mu\nu}$ (as in the case of a special linear Complex in electromagnetism). Even from the viewpoint of 'classical' gauge theories, this can be pursued using a usual gauge boson vertex on a (fundamental) spinor, which we might rearrange into a gauge boson with momentum $k$ splitting into two conjugate or even adjoint spinor reps like sketched in Fig. 1. Formally, this rises once more the question to find two reps (square roots or conjugate objects) which combine to an appropriate boson (or vectorial) rep. The major

\[ r^{-2} = u^2 + v^2 + w^2 \]

For example, the $r^{-2}$ dependence of forces with respect to the (Euclidean) radius can be mapped to quadratic plane coordinates using the (dual) class picture instead of the description by orders, i.e. we use tangent planes to envelop the sphere. The radius $r$ may then be written as $r = (ua + vb + wc + 1)\sqrt{u^2 + v^2 + w^2}^{-1}$ ([10], p. 26/27), and the sphere with origin in its center reads as $r^{-2} = u^2 + v^2 + w^2$, $u, v, w$ describing Euclidean/affine plane coordinates. So instead of writing differential expressions, we may use simple global expressions as well, of course being quadratic in both sets of coordinates, but with global (and not only infinitesimal) validity.

\[ \text{Figure 1. Figure caption} \]

\(^6\) We want to discuss details of this correlation in the comprehensive part VI of this series which we expect to be published soon. Right here, it is obvious that we may switch to class view instead of working with orders (and points), so that $u_{\alpha}u_{\beta} = (\mathcal{R})_{\alpha\gamma}x_{\gamma} = (\mathcal{R})_{\alpha\beta}x_{\beta} = x_{\gamma}(\mathcal{R})_{\gamma\alpha}(\mathcal{R})_{\beta\alpha}x_{\beta}$ which according to the antisymmetry of $\mathfrak{R}$ comprises not only the 'classical' null systems and forces, but Lie (algebra) theory as well. On the other hand, we have to relate this to second order surfaces and their polar theory, so that Clifford algebras and Dirac’s approach emerge as well whereas a representation in terms of line reps or Complexes has to include generating lines and appropriate involutions in the tangent plane(s) [10].

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clue, however, will be given in section 5 when invoking the mighty transfer principles from classical projective geometry, here mainly Lie transfer, which in the background may be related to Poncelet’s classification of second order surfaces and their relation to the absolute plane.

3. Dirac Theory, Spinors and the Point Picture

Instead of recalling the historical discussion on how to obtain a linear equation of motion and appropriate reps, we want to separate some hard structural requirements from a plenty of technical details which lead to algebraic and analytic ‘equations’ being nowadays put in the center of attention, although being formally nothing but an artefact of the chosen rep. A simple example is the discussion of the ‘linear’ Dirac equation, where the 4-dim momentum rep\(^9\) \(p^\mu\) is ‘obtained’ by acting typically with partial derivatives on ‘plane waves’, \(\partial_\mu \exp(-ip \cdot x) \sim \overleftarrow{\partial\mu} \exp(-ip \cdot x) \sim -ip_\mu \exp(-ip \cdot x)\) which formally yields the identification \(-i\partial_\mu \sim p_\mu\) (‘quantization\(^{10}\)). So the usual ingredients of this approach are a plane wave borrowed from physical and Fourier arguments, the requirement of linearity applied to the reps to preserve the formal apparatus, and a polar-like/tangential reasoning when acting with partial derivatives on forms in homogeneous coordinates\(^{11}\). The problems are shifted into the ‘god-given’ metric of SR which corresponds to the related polar form, thus serves to map \(-i\partial_\mu \sim p_\mu\) by (polar) incidence \(x^\mu p_\mu=0\), and allows for the (euclidean) identification of ‘mass’ in the ‘momentum’\(^{12}\).

The spinorial rep itself is usually treated in QFT as ‘a spinor’ in \(\mathbb{C}_{4\times 1}\) (see e.g. \([13]\), eq. (3.2), or ibd., eq. (3.16), or \([14]\), eq. 1(47)) to fulfill the equations of motion, although it is well-known that the Dirac matrices \((\vec{\alpha}, \vec{\beta})\) or \((\vec{\gamma}, \vec{\beta})\) exhibit their (also well-known) interesting 2\times2-block structure. As for our purposes (and especially from the viewpoint of 6-dim line reps or Complexe as the underlying rep of forces, i.e. comprising translatinal and rotational momenta as well), it is obvious to find unified descriptions of (3-dim) momentum and (3-dim) ‘spin’ \([3]\). So we use the ‘decomposition’ \(\mathbb{C}_{4\times 1} \leftrightarrow \mathbb{C}_{4\times 2} \mathbb{C}_{2\times 1}\) in order to respect the usual block structure of the Dirac matrices and separate the ‘spin’ interpretation as discussed in \([3]\), the more since in practical calculations and formalisms, spin often is averaged and ‘disappears’. So it is usually sufficient to respect related invariants. Moreover, a very classical but obviously long forgotten explanation of ‘spin’ will be given in section 5.

So the next step is to interpret the \(\mathbb{C}_{4\times 2}\) part above on which the Dirac matrices act by their 2\times2 block structure. The direct approach would be to grasp e.g. the Dirac spinor reps being indirectly visible in \([13]\), eq. (3.7), and the grouping in eq. (3.16), or directly the reps given in \([14]\), eq. 1(51a), 1(51b), in terms of objects

\[ u_{\bar{\sigma}} = \sqrt{\frac{E+m}{2m}} \left( \frac{1}{\sigma\bar{\sigma}} \right) \xi_{\sigma}, \quad v_{\bar{\sigma}} = \sqrt{\frac{E+m}{2m}} \left( \frac{\sigma\bar{\sigma}}{E+m} \right) \xi_{\sigma}, \quad (3) \]

\(^8\) For details, we refer to \([13]\) or \([14]\), which serve as our references in the text, too.

\(^9\) In the upcoming, comprehensive part VI, we are going to discuss some aspects to identify \(p\) in general with a planar rep related to polar theory and respecting duality, and not as derived from point rep generalizations.

\(^{10}\) We have discussed in \([7]\), appendix, the dimensionality 4 of this rep already.

\(^{11}\) We’ll discuss some aspects of these requirements later as a special case of the identification of \(p\cdot x\) as well as their intrinsic assumptions. Instead of equations of motions and (linear) rep theory, it is better to start from conic sections (or more general a quadric) of energy and momentum, and discuss linear reps of appropriate square roots. Or in terms of PG, given a quadric in a plane or in 3-space, how do we generate the quadric linearly. This ‘problem’, however, is not exhaustive with respect to generation of only the quadric as well as to the restriction to possible linear reps and objects.

\(^{12}\) Note, that according to the identification by 6-vectors (or linear Complexe) \([6]\) \([5]\), one irrep comprises linear and angular momentum, i.e. the polar and the axial 3-vector, when decomposed according to affine and euclidean coordinates.
ξₐ being an appropriately chosen rep in \( \mathbb{C}_{2 \times 1} \). Now, if we act with Dirac/Clifford operators like \( \gamma^i \), \( \gamma^0 \) or \( \gamma^5 \) on the blocks of this structure, the Pauli/quaternion algebra acts within the blocks and as such on the respective individual block content. Whereas we want to discuss this block structure by a (real) toy model in section 4 and based on previous work [3] in section 5, it is the major aspect of section 5 to reconnect some geometrical aspects discussed in [3] to the reps given in eq. (3). Here, we want to emphasize once more that in eq. (3) the ‘mass’ obviously can be divided out, i.e. we are left with the 4-velocity\(^{13} \) \( u_\mu \), only, in the rep.

The interpretation of this rep so far is typically related to assumptions and interpretations of negative energy and 'anti'-particles [13], whereas practically the \( v \)-contributions of eq. (3) often are simply neglected in calculations. More 'extended versions' transport additional symmetries like chirality or helicity even to spontaneously broken symmetries and nonlinear reps [15], or mix them with compact group structures. Moreover, the usual treatment includes metric arguments and metric interpretations of the variables, combined with appropriate 'rules' and 'gauge equations', although the 'light cone' is known to be an absolute element.

Here, we do not want to extend this criticism, but instead, we depart from the reps in eq. (3), and try to gain some more insight (hopefully) independent of 'the known' physical motivation. So now the simple question is: Given objects like in eq. (3), what are the possibilities to identify those objects geometrically, and are these identifications unique? Both answers are simple: There is definitely more than one well-known possibility, and as such: NO, of course, it is NOT unique!

### 4. Real Approach: Line-Coordinates and Transformations

Another important fitting piece of the puzzle can be treated by a real toy model. So before proceeding to some applications of Lie transfer, this picture can be attached to a special interpretation of the Dirac 4\( \times \)2 'spinor' in terms of two 4-dim (point) 'vector' reps, i.e. we represent 3-dim (real) points in terms of their four homogeneous coordinates \( x_\alpha \) and \( y_\beta \), respectively, and use the 'Dirac spinor' notation to identify the \( (4 \times 2) = 6 \) Plücker line coordinates \( p_{\alpha\beta} = x_\alpha y_\beta - x_\beta y_\alpha \). This can be interpreted as well as identifying the six independent \( 2 \times 2 \) subdeterminants in this 'spinor' rep, and transformations by the Clifford/Dirac algebra transform the six underlying, basic line coordinates of a \( P^5 \). So the \( 4 \times 2 \) notation serves as a 'container' to denote either two (homogeneous) point reps as well as the related line rep.

#### 4.1. Some Algebra

Here, it is of course necessary to investigate the action of Dirac’s gamma matrices on this \( 4 \times 2 \) 'spinor' and its intrinsic line coordinates\(^{14} \) in detail! So defining

\[
\psi := \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix},
\]

---

\(^{13}\) Here, we use the spinorial notation \( u \) and \( v \) as well as the 4-velocity notation \( u_\mu \) in their respective contexts only to respect common notation, see e.g. [12], ch. 4.1. Our own notation uses \( u \) and \( v \) typically as plane coordinates.

\(^{14}\) We comment on the Dirac picture, including 'spin' and 'momentum', as well as on the (quaternionic) Weyl picture in section 5. Note already here, that using the null system as correlation, \( x_\alpha \) and \( y_\beta \) in eq. (4) can be transferred to planar coordinates \( u_\alpha \) and \( v_\beta \) as well while preserving the line interpretation. So with respect to \( \psi \), we find an equivalent rep \( \Psi \) in terms of \( u_\alpha \) and \( v_\beta \). In both cases, we are left with reps \( [\mathbb{H}]_{9,9} \) from section 2. The complete description, however, has to take care of symmetric (polar) transformations besides the null system because \( R^2 = 1 \) also allows for symmetric reps. So the diagonal (anti-commutator) part has to be considered besides the skew (commutator) part, i.e. we expect both aspects in the (Clifford) algebraic description when acting on \( u \) and \( v \).
and using the reps (see [13]) of $\gamma^\mu$, $\sigma^{\mu\nu}$ and $\gamma^5 = \gamma_5$, we can identify the following transformed line coordinates $p'_{\alpha\beta} \sim A p_{\alpha\beta}$, $p'_{\alpha\beta}$ being extracted from the transformed spinor $\psi'$ given in the top row of each table\textsuperscript{15},

\begin{equation}
\begin{array}{c|c|c|c|c|c}
 p'_{\alpha\beta} & 1 \psi & \gamma^1 \psi & \gamma^2 \psi & \gamma^3 \psi & \gamma^0 \psi \\
p'_{01} & +p_{01} & -p_{23} & -p_{01} & -p_{01} & +p_{01} \\
p'_{23} & +p_{23} & -p_{01} & -p_{01} & +p_{23} & +p_{01} \\
p'_{02} & +p_{02} & +p_{13} & -p_{13} & +p_{02} & -p_{02} \\
p'_{13} & +p_{13} & +p_{02} & -p_{02} & +p_{13} & -p_{13} \\
p'_{03} & +p_{03} & +p_{03} & +p_{03} & -p_{12} & -p_{03} \\
p'_{12} & +p_{12} & +p_{12} & +p_{12} & -p_{03} & -p_{12} \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c|c|c|c|c|c}
 p'_{\alpha\beta} & \gamma^5 \psi & \gamma^5 \gamma^1 \psi & \gamma^5 \gamma^2 \psi & \gamma^5 \gamma^3 \psi & \gamma^5 \gamma^0 \psi \\
p'_{01} & +p_{23} & -p_{01} & -p_{01} & -p_{01} & +p_{23} \\
p'_{23} & +p_{01} & -p_{23} & -p_{23} & -p_{23} & +p_{01} \\
p'_{02} & -p_{02} & -p_{13} & +p_{13} & -p_{02} & +p_{02} \\
p'_{13} & -p_{13} & -p_{02} & +p_{02} & -p_{13} & +p_{13} \\
p'_{03} & -p_{12} & -p_{12} & +p_{03} & +p_{12} & +p_{12} \\
p'_{12} & -p_{03} & -p_{03} & +p_{12} & +p_{03} & -p_{12} \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c|c|c|c|c|c|c|c|c|c}
 p'_{\alpha\beta} & \sigma^{01} \psi & \sigma^{02} \psi & \sigma^{03} \psi & \sigma^{12} \psi & \sigma^{13} \psi & \sigma^{23} \psi \\
p'_{01} & +p_{23} & +p_{23} & +p_{23} & -p_{01} & -p_{01} & -p_{01} \\
p'_{23} & +p_{01} & +p_{01} & +p_{01} & -p_{23} & -p_{23} & -p_{23} \\
p'_{02} & +p_{13} & -p_{13} & -p_{13} & +p_{02} & +p_{02} & +p_{13} \\
p'_{13} & +p_{02} & -p_{02} & -p_{02} & +p_{13} & +p_{13} & +p_{02} \\
p'_{03} & +p_{03} & +p_{03} & +p_{03} & -p_{12} & -p_{12} & -p_{12} \\
p'_{12} & +p_{12} & +p_{12} & +p_{12} & +p_{03} & +p_{03} & +p_{03} \\
\end{array}
\end{equation}

This table shows some remarkable features when acting with $\gamma$-matrices (or 'the Dirac/Clifford algebra') on $4 \times 2$ 'spinors' $\psi$ of eq. (4):

- The six line coordinates (or $2 \times 2$ determinants) $p_{\alpha\beta}$ are defined by their respective row positions within the $4 \times 2$ 'spinor' only, i.e. by combinatorics and antisymmetry from the point (or plane) reps of the underlying coordinate system.

- Thus the action of the Dirac or Clifford algebra is independent of the (physical/mathematical) identification or content of the respective variables and/or coordinates at the respective position in the 'spinor', but it depends only on the (row) position and the antisymmetry of the line coordinate (or the determinant), and it preserves the line coordinate structure within the $4 \times 2$ 'spinor'.

- The transformations of the Dirac algebra rep given by [13] – although intrinsically complex – map real line coordinates $p_{\alpha\beta}$ to real line coordinates $p'_{\alpha\beta}$, independent of the respective content at the position in the $4 \times 2$ 'spinor'. As such, although transforming six (real) line coordinates into six (real) line coordinates, Dirac’s approach performs the transformations within a $4 \times 4$ matrix rep (NOT by means of a $6 \times 6$ matrix rep!) which increases the intricacy of the rep but (in some special reps) is suitable to represent 4-dim transformations.

- So in the background, using the 'two-point' interpretation of the $4 \times 2$ 'spinor' given above, we perform nothing but real PG in terms of Plücker coordinates, i.e. a projective geometry of 3-space using lines as basic geometric elements in $P^3$. And as such, the generalization leads to Complex geometry in $P^5$, or equivalently, to the Lie transferred sphere geometry.

\textsuperscript{15}In the tables, we’ve arranged the order of the line coordinates $p'_{\alpha\beta}$ in appropriate pairs already to simplify and shorten the subsequent discussion.
- In other words, the $4 \times 2$ 'spinor' may serve as a rep to perform line calculus when acting with linear combinations of individual base elements of the Dirac algebra on $\psi$, or as a rep of a linear (line) Complex when acting with linear combinations of individual base elements of the Dirac algebra on $\psi$ and equating this action to 0.

- The 15-dim transformations using real parameters with Dirac or Clifford algebra elements thus describe nothing but projective transformations of lines onto lines in 3-dim space, however, one has to take care because lines are mapped to lines by duality, and thus correlations as well as collineations are described by 15-dim transformation groups each, whose algebraic/purely formal effects 'overlap'.

- It is necessary to understand and see that the apparent complexifications in the matrix definitions of the Dirac algebra elements – emerging in pairs or $2 \times 2$-blocks – as well as overall 'i's are absorbed in signs by the definition of Plücker coordinates in terms of $2 \times 2$-determinants of the $4 \times 2$ 'spinor'. This shifts the question on the meaning of such complexifications back to point reps and especially to properties of the respective transfer principle applied in general to the elements of projective 3-space; it cannot be answered from the viewpoint of (2nd order) line or polar geometry only.

- From above, it is obvious that Dirac’s approach in terms of $\gamma$-matrices acting on $\psi$ represents a symbolic scheme (or calculus, in the sense of the old German 'Kalkül'), and as such the trace mechanism and the various relations/formulae between traces and determinants are justified from above.

### 4.2. Transformed Views of the Dirac Algebra

Using the transformation results $p'_{\alpha \beta}$ of the tables given in eqns. (5) and (6), it is obvious that we can identify the individual coordinate 'positions' of the $p'_{\alpha \beta}$ in a $P^5$ rep. As such, if we 'regroup' the coordinates $p'_{\alpha \beta}$ into 'new' structures according to $(p_{01}, p_{23})$, $(p_{02}, p_{13})$, and $(p_{03}, p_{12})$, when acting with the Dirac algebra, we only find changes in the overall signs and, in some cases, an exchange of both line coordinates within these doublets. In order to quantify the analytic properties, it is thus self-evident to introduce the notation\(^{16}\)

\[
\mathcal{A}_\pm = (p_{01}, \pm p_{23}), \quad \mathcal{B}_\pm = (p_{02}, \pm p_{13}), \quad \mathcal{C}_\pm = (p_{03}, \pm p_{23})
\]  
\[\mathcal{A}'_\pm = (p_{23}, \pm p_{01}), \quad \mathcal{B}'_\pm = (p_{13}, \pm p_{02}), \quad \mathcal{C}'_\pm = (p_{23}, \pm p_{03}).
\]

Accordingly, the tables in eqns. (5) and (6) can be rewritten as

\[
\begin{array}{c|ccc|ccc}
\hline
& 1 \psi & \gamma^1 \psi & \gamma^2 \psi & \gamma^3 \psi & \gamma^0 \psi \\
\hline
\mathcal{A}'_\pm & +\mathcal{A}'_\pm & +\mathcal{A}'_\pm & +\mathcal{A}'_\pm & +\mathcal{A}'_\pm & +\mathcal{A}'_\pm \\
\mathcal{B}'_\pm & +\mathcal{B}'_\pm & +\mathcal{B}'_\pm & +\mathcal{B}'_\pm & +\mathcal{B}'_\pm & +\mathcal{B}'_\pm \\
\mathcal{C}'_\pm & +\mathcal{C}'_\pm & +\mathcal{C}'_\pm & +\mathcal{C}'_\pm & +\mathcal{C}'_\pm & +\mathcal{C}'_\pm \\
\hline
\mathcal{A}'_\pm & \pm \mathcal{A}'_\pm & -\mathcal{A}'_\pm & -\mathcal{A}'_\pm & -\mathcal{A}'_\pm & -\mathcal{A}'_\pm \\
\mathcal{B}'_\pm & -\mathcal{B}'_\pm & \mp \mathcal{B}'_\pm & \mp \mathcal{B}'_\pm & \mp \mathcal{B}'_\pm & \mp \mathcal{B}'_\pm \\
\mathcal{C}'_\pm & \mp \mathcal{C}'_\pm & \mp \mathcal{C}'_\pm & \mp \mathcal{C}'_\pm & \mp \mathcal{C}'_\pm & \mp \mathcal{C}'_\pm \\
\hline
\mathcal{A}'_\pm & \pm \mathcal{A}'_\pm & \pm \mathcal{A}'_\pm & \pm \mathcal{A}'_\pm & \pm \mathcal{A}'_\pm & \pm \mathcal{A}'_\pm \\
\mathcal{B}'_\pm & \pm \mathcal{B}'_\pm & \pm \mathcal{B}'_\pm & \pm \mathcal{B}'_\pm & \pm \mathcal{B}'_\pm & \pm \mathcal{B}'_\pm \\
\mathcal{C}'_\pm & \pm \mathcal{C}'_\pm & \pm \mathcal{C}'_\pm & \pm \mathcal{C}'_\pm & \pm \mathcal{C}'_\pm & \pm \mathcal{C}'_\pm \\
\end{array}
\]

\[\sigma_{01} \psi, \sigma_{02} \psi, \sigma_{03} \psi, \sigma_{12} \psi, \sigma_{13} \psi, \sigma_{23} \psi\]

\[^{16}\text{Note, that with respect to the bi-linear invariant, we are free to introduce the additional notion of an adjoint (i.e. of transposed, conjugated or hermitean) elements as well, see e.g. Hamermesh ([16], p. 369/370), or } \mathcal{X}' = \mathcal{X}^{\gamma \gamma \gamma \gamma} \text{, etc.}\]
Geometrically, the line coordinate sets $\mathfrak{A}_{\pm}$, $\mathfrak{B}_{\pm}$ and $\mathfrak{C}_{\pm}$ describe opposite, non-intersecting edges of the fundamental tetrahedron in 3-space when we interpret the individual six line coordinates as base 'vectors', or basic lines. As such, in the non-degenerate case, each of the three sets comprises two non-intersecting lines, and – besides polar theory – we can apply the notion of Congruences\textsuperscript{17} \cite{17}. So even the interchange of the line coordinates $p_{\alpha\beta} \longleftrightarrow p_{\gamma\delta}$, $\alpha\beta \neq \gamma\delta$ (or $p_{\alpha\beta} \sim t_{\alpha\beta} \sim t_{\gamma\delta}$) within the sets according to $\mathfrak{A}_{\pm} \rightarrow \mathfrak{A}'_{\pm}$, $\mathfrak{B}_{\pm} \rightarrow \mathfrak{B}'_{\pm}$ and $\mathfrak{C}_{\pm} \rightarrow \mathfrak{C}'_{\pm}$ exchanges only the two lines within each of the Congruences but respects the Congruence itself and doesn’t mix it with the two other Congruences. This is valid for the full action of the Dirac/Clifford algebra. In this context, if we stress the notion of irreps, with respect to the Dirac algebra, we have thus found an irreducible structure in terms of three individual Congruences, and it depends on the (line) coordinates (or the linear Complex) to specify the Congruence in detail. It is noteworthy, that in the language of QFT, we have found threefold intrinsic symmetry depends on the (line) coordinates (or the linear Complex) to specify the Congruence in detail.

It is noteworthy, that in the language of QFT, we have found threefold intrinsic symmetry of the respective coefficient(s) $\lambda$, then $\mathfrak{A}_{\pm} \rightarrow \mathfrak{A}'_{\pm}$, $\mathfrak{B}_{\pm} \rightarrow \mathfrak{B}'_{\pm}$, and $\mathfrak{C}_{\pm} \rightarrow \mathfrak{C}'_{\pm}$. The action of the Dirac/Clifford algebra doesn’t change or mix $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, and moreover, it respects and doesn’t change the subscripts $\pm$. Thus we have found another justification of Klein’s six fundamental Complexes and $SU(2) \times SU(2)$ symmetry with respect to their 3$x$3 handedness. So dependent on the original rep $\psi$, the transformation of the Dirac algebra respects a threefold substructure induced by the geometrical properties of the fundamental (coordinate) tetrahedron when expressed in line coordinates.

With respect to applications expressed, there is however an additional and very interesting aspect\textsuperscript{18} which we may thus connect to Complexes and SU(4). Here, we discuss the thesis \cite{19} as a bridge to related topics and further applications of algebra theory in physics, and we cannot go into details. However, \cite{19}, chapter 4.1 yields a definition of a Lie algebra with (skew) generators $X_{\alpha\beta}$ denoted as 'tetrahedron algebra'. After having defined the 'three-point $sl_2$ Loop algebra' and stated a Lie algebra isomorphism, a related homomorphism $\psi$ has been given/cited \cite{19} by

$$
\begin{align*}
\psi(X_{12}) &= x \otimes 1 & \psi(X_{03}) &= y \otimes t + z \otimes (t - 1) \\
\psi(X_{23}) &= y \otimes 1 & \psi(X_{01}) &= z \otimes t' + x \otimes (t' - 1) \\
\psi(X_{31}) &= z \otimes 1 & \psi(X_{02}) &= x \otimes t'' + y \otimes (t'' - 1)
\end{align*}
$$

\textsuperscript{17} We do not want to blur the discussion, or be imprecise or vague, however, out of various possibilities to identify the geometrical setup, here, we discuss only two identifications: Two opposite edges of the fundamental tetrahedron, each can be seen as axis of a special Complex, so lines fulfilling both constraints satisfy a Congruence (see also \cite{9}). The second case, in section 4.3, is devoted to a linear Complex in the additive rep, i.e. forming one constraint in line coordinates. Related, there is deep historic background, see e.g. \cite{18}.

\textsuperscript{18} We want to thank B. Schmeikal for helpful private discussions and remembering Onsager theory.

\textsuperscript{19} We have adapted to our notation.
with $t' = 1 - t^{-1}$, $t'' = (1 - t)^{-1}$, and $x, y, z$ from $sl_2$ (‘equitable basis’). The important point from our discussion above, however, is the identification of their operators $u_0, u_1,$ and $u_2$ by $4u_0 = \psi(X_{02} + X_{34}), 4u_1 = \psi(X_{03} + X_{12}),$ and $4u_2 = \psi(X_{01} + X_{23})$ as generators of $sl_2 \otimes k[t, t^{-1}, (1 - t)^{-1}]$ being a Lie algebra over $k$. The labels of every two non-adjacent edges generate a subalgebra isomorphic to the Onsager algebra, i.e. ([19], Prop. 4.2.4) for mutually distinct $\alpha, \beta, \gamma, \delta \in \{0, 1, 2, 3\}$, the subalgebra of the ‘tetrahedron algebra’ generated by $X_{\alpha \beta}$ and $X_{\gamma \delta}$ is isomorphic to the Onsager algebra. The ‘tetrahedron algebra’ thus is a (direct) sum of three Onsager algebras.

By comparison, the definitions of $X_{\alpha \beta}$ and their figure 4.1 ([19], p. 77), the operators $X$ obviously represent classical line coordinates algebraically. Understanding the homomorphism as transfer, $u_0, u_1,$ and $u_2$ correspond to appropriate reps of our Congruences above, i.e. from our point of view [19] yields another transfer of projective geometry to a special notion.

5. Spin

Now with respect to general transformations parametrized by real numbers in the Dirac algebra, already here it is obvious that we have switched to a coordinate description using lines as base elements (respectively the appropriate Plücker coordinates). So by going back to the well-known Dirac algebra description, with respect to the real toy model of section 4 there are two open issues at this time: We have used two points to define our ‘spinor’ $\psi$ in terms of their homogeneous coordinates, so how does the rep given e.g. in [13] – as well as the various other spinorial reps floating around in literature – relate to this description or fit into this picture? Moreover, thinking in terms of (4-dim) quaternions (and considering their conjugates as well), we can also define $4 \times 2$ spinors while switching to their complex $2 \times 2$ rep, e.g. in terms of complexified Pauli matrices, and use SL$(2, \mathbb{H})$, or SU$(4)$ rep theory.

For now, however, we want to depart once more from$^{20}$ $p_\mu p^\mu = E^2 - \vec{p}^2 = m^2$, or $u_\mu u^\mu = 1$ using 4-velocities.

If we recall from [3], section III.E, our equations (16) and (17),

$$r = \frac{1}{2}(\pm H' - Z'), \rho = \frac{1}{2}(X' + iY'), \sigma = \frac{1}{2}(\pm H' + Z')$$

(14)

and

$$\frac{1}{2} \left( \begin{array}{cc} -Z' + H' & X' + iY' \\ X' - iY' & +Z' + H' \end{array} \right) \sim X'\sigma_1 - Y'\sigma_2 - Z'\sigma_3 \pm H'\sigma_0,$$

(15)

we have shown there, that Plücker’s fifth coordinate yields $\eta = r\sigma - s\rho = H'^2 - Z'^2 - X'^2 - Y'^2$. The variables $X', Y', Z'$ were point coordinates of Lie’s second, transferred 3-dim space $R, H'$ denoted a ‘sphere radius’ and plays a special rôle which we are going to discuss elsewhere in more detail. For analytic calculations and reps, we can use the geometry in $R$-space (or ‘the spin geometry’) which we began to develop in [3], III.G ff.

So if we now choose to define $\eta = p_\mu p^\mu$ (although being in $R$-space but relying on Lie’s transfer), and $\eta$ being the determinant of the line coordinates in $r$-space, then $\eta$ maps to a squared mass, i.e. $\eta = m^2$. If instead we use unimodular reps, $\eta = 1$, then $\eta = u_\mu u^\mu$ yields 4-velocities, and a quadratic equation as well.

In both approaches, $\eta$ is quadratic (which Plücker needed to preserve the grade of his coordinates during transformations), and we may (formally) introduce a ‘spinorial picture’ by two ‘doublets’ ($r, s$) and ($\rho, \sigma$) according to

$$\eta = \begin{pmatrix} r & s \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \rho \\ \sigma \end{pmatrix} \sim i, \quad i^2 = -1,$$

(16)

$^{20}$For us, to connect to PG above, it seems more natural to use the surface $\vec{p}^2 - E^2$, as can be seen from the minus sign of $\eta$ in the next footnote.
so that $\eta$ seems to behave as a ‘singlet’ (which corresponds to an interpretation of the complex structure $i$ above). However, care has to be taken in that $r, s, \rho$ and $\sigma$ are velocities in $r$-space, and the full formal treatment within PG$^{21}$ has to use (homogeneous) line coordinates $p_{\alpha\beta}$.

For now, in the context of finding linear reps with respect to expressions like $p_{\mu}p^{\mu}$ it should be noted that line geometry yields such a possibility by eqn. (16). The origin of such an approach, of course, is located in the theory of second order surfaces and their generation by lines.

Last not least, if we ‘translate back’, the ‘radius’ $H' = r + \sigma$ [3], II.E, reads as $H = (p_{01} + p_{23})p_{03}^{-1}$, and the value – being a ratio – is fixed and (usually) finite. So $H'$ is 0 denotes the Complex $p_{01} + p_{23} = 0$ in $r$-space (i.e. in real 3-dim space) which we already know from above. $\eta = -p_{12}p_{03}^{-1}$, so this also amounts to a finite and measurable value which we have to compare to mass calculations. Note the $-$-sign in $\eta$!

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$^{22}$In order to perform a direct comparison to Dirac and the spinor reps (3), recall that we want to resolve $E^2$ into linear reps.

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