ON A SOLVABLE $p-$DIMENSIONAL SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS

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Abstract. In this paper, we investigate the solutions of the following system of $p-$nonlinear difference equations

\[ x_{n+1}^{(i)} = \frac{a^{(i)}x_n^{(i+1 \mod p)}x_{n+1}^{(i+1 \mod p)}}{b^{(i)}x_{n-1}^{(i)} + c^{(i)}x_n^{(i+1 \mod p)}}, \quad n \in \mathbb{N}_0, p \in \mathbb{N}, i \in \{1, \ldots, p\}, \]

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the sequences $(a^{(i)})$, $(b^{(i)})$, $(c^{(i)})$, are non-zero real numbers and initial values $x_{-j}^{(i)}$, $j \in \{0, 1, 2\}$, $i \in \{1, \ldots, p\}$. Finally, we give some applications concerning aforementioned system of difference equations.

Keywords: Riccati difference equation; periodicity; general solution; system of difference equations; Pell sequence.

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1. INTRODUCTION

In the recent years, there has been a lot of interest in studying nonlinear difference equations and systems. Not surprisingly therefore, several studies have been published on this topic (see, e.g., [1]-[28], and the related references therein). Besides their theoretical value, most of the recent applications have appeared in many scientific areas such as biology (population dynamics in
particular), ecology, physics, engineering and economics (see, e.g. [8],[9], [12], [22]). It is very worthy to find systems belonging to solvable nonlinear difference equations systems in closed-form.

Since the paper by Brand [5], the following one-dimensional nonlinear difference equation of Riccati type,

\[ x_{n+1} = \frac{ax_n + d}{cx_n + b}, n \in \mathbb{N}_0, \]

where the initial value \( x_0 \) is a real number or complex number and the parameters \( a, b, c \) and \( d \) are the real numbers with the restrictions \( c \neq 0 \) and \( ab \neq cd \), have the most diverse and interesting properties, especially as regards the distribution of their cluster points. This finding, led Stević [24] to study the solutions of the Eq. (1.1).

In Abo-Zeid et al. [3] the authors presented the solutions of the one-dimensional system of nonlinear difference equations which reduced to the Riccati difference equation under appropriate transformations,

\[ x_{n+1} = \frac{x_nx_{n-2}}{\pm x_{n-1} \mp x_{n-2}}, n \in \mathbb{N}_0. \]

Then, in [10] and [11], equations in (1.3) were generalized to the following equations

\[ x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} \pm cx_{n-q}}, n \in \mathbb{N}_0, \]

where \( \max\{l,k,p,q\} \) is nonnegative integer and \( a, b, c \) are positive constants. Moreover, in [4] the authors presented the solutions of the following two-dimensional four systems of nonlinear difference equation generalization of Eq. (1.2):

\[ x_{n+1} = \frac{y_ny_{n-2}}{x_{n-1} + y_{n-2}}, y_{n+1} = \frac{x_nx_{n-2}}{\pm y_{n-1} \pm x_{n-2}}, n \in \mathbb{N}_0. \]

But, two (resp. three)—dimensional system of difference equations in (1.4) was extended to the following two (resp. three)—dimensional system of difference equations with constant coefficients

\[ x_{n+1} = \frac{y_ny_{n-2}}{bx_{n-1} + ay_{n-2}}, y_{n+1} = \frac{x_nx_{n-2}}{dy_{n-1} + cx_{n-2}}, n \in \mathbb{N}_0, \]
The remainder of the paper is organized as follows: In section 2, we study the solutions of the given system of the $p$–dimensional nonlinear rational difference equations by using convenient transformation. In the next section, we obtain well-known Fibonacci numbers and Pell numbers in the solutions of aforementioned system for some cases. Section 4 concludes.

2. Explicit Formulas for the Solutions of System (1.8)

Let $\left\{ x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)} \right\}_{n \geq -2}$ be a solution of system (1.8). If at least one of the initial values $x_{-j}^{(i)}$, $j \in \{0, 1, 2\}$, $i \in \{1, \ldots, p\}$, is equal to zero, then the solutions of system (1.8) is not defined. For example, if $x_n^{(i_0)} = 0$ for some $n_0 \geq -2$, $i_0 \in \{1, \ldots, p\}$. Then from the system (1.7) it follows that $x_{n_0+1}^{(i_0)} = 0$, and consequently $b^{(i_0)} x_{n_0+1}^{(i_0)} + c^{(i_0)} x_{n_0}^{(i_0+1)\mathrm{mod}(p)} = 0$, from which it follows that $x_{n_0+3}$
is not defined. Thus, for every well-defined solution of system (1.8), we get that \( \prod_{i=1}^{p} x_{n}^{(i)} \neq 0 \), \( n \geq -2 \), if and only if \( \prod_{i=1}^{p} x_{-j}^{(i)} \neq 0 \), \( j \in \{0, 1, 2\} \). Note that the system (1.8) can be written in the form

\[
(2.1) \quad b^{(i)} \frac{x_{n-1}^{(i)}}{x_{n-2}^{(i)}} + c^{(i)} = \frac{x_{n}^{(i)}}{x_{n+1}^{(i)}}^{(i+1) \mod (p)}, \quad n \in \mathbb{N}_0, \quad p \in \mathbb{N}, \quad i \in \{1, \ldots, p\}.
\]

Next, by employing the change of variables

\[
y_n^{(i)} = \frac{x_{n-1}^{(i)}}{x_{n}^{(i)}}^{(i+1) \mod (p)}, \quad n \geq -2, \quad p \in \mathbb{N}, \quad i \in \{1, \ldots, p\}.
\]

Then system (2.1) can be written as

\[
(2.2) \quad y_{n+1}^{(i)} = \frac{b^{(i)}}{y_{n-1}^{(i)}} + c^{(i)}, \quad n \in \mathbb{N}_0, \quad p \in \mathbb{N}, \quad i \in \{1, \ldots, p\}.
\]

Let \( z_{m,k}^{(i)} = y_{2m+k}^{(i)} \) for \( m \geq -1, \quad k \in \{1, 2\}, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N} \). Then, from (2.2) we see that \( \left( z_{m,k}^{(i)} \right)_{m \geq -1}^{k = 1, 2, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N} } \) are \( 2p \)-solutions to the following system of difference equations

\[
(2.3) \quad t_{m}^{(i)} = \frac{b^{(i)}}{t_{m-1}^{(i)}} + c^{(i)}, \quad m \in \mathbb{N}_0, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N}.
\]

System (2.3) is solvable. Let

\[
(2.4) \quad t_{m}^{(i)} = u_{m+1}^{(i)} \left( u_{m}^{(i)} \right)^{-1}, \quad m \geq -1, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N},
\]

where \( u_{-1}^{(i)} = 1, \quad u_{0}^{(i)} = t_{-1}^{(i)}, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N} \). From now on, we assume that the sequence \( \left( t_{m}^{(i)} \right)_{m \geq -1, \quad i \in \{1, \ldots, p\}} \) is well defined. Then system (2.3) becomes

\[
(2.5) \quad u_{m+1}^{(i)} = c^{(i)} u_{m}^{(i)} + b^{(i)} u_{m-1}^{(i)}, \quad m \in \mathbb{N}_0, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N}.
\]

To solve system (2.5) we need to use the following lemma.

**Lemma 2.1.** For \( a, b \in \mathbb{R} \), consider the homogeneous linear second-order difference equation with constant coefficients

\[
(2.6) \quad x_{n+1} = ax_{n} + bx_{n-1}, \quad n \in \mathbb{N}_0,
\]
where \( b \neq 0 \) and \( a^2 + 4b \neq 0 \), the general solution for equation (2.6) as follows:

\[
x_n = bx_{n-1}s_{n-1} + x_0s_n, \quad n \geq -1,
\]

\( s_{-2} \) is calculated by using the following relations \( bs_{n-1} = s_{n+1} - as_n \) for \( n = -1 \).

**Proof.** The proof follows essentially the same arguments as in Stević [24]. □

**Remark 2.1.** By taking \( a = 1 \) (resp. \( a = 2 \)) and \( b = 1 \) in equation (2.6), with \( s_{-1} = 0, s_0 = 1 \), then the sequence \( (s_n)_{n \geq -1} \) reduce to the well-known Fibonacci (resp. Pell) sequence.

Let \( (s_{m}^{(i)})_{m \geq -1, i \in \{1, \ldots, p\}} \) be the solution to system (2.5) such that \( s_{-1}^{(i)} = 0 \) and \( s_0^{(i)} = 1 \), for \( i \in \{1, \ldots, p\} \). Then, from Lemma 2.1, the general solutions to system (2.5) can be written in the following form

\[
(2.7) \quad u_m^{(i)} = b^{(i)}u_{m-1}^{(i)} + u_0^{(i)}s_{m-1}^{(i)}, \quad m \geq -1, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N}.
\]

\( s_{-2}^{(i)}, i \in \{1, \ldots, p\} \) are calculated by using the following relations \( b^{(i)}s_{m-1}^{(i)} = s_{m+1}^{(i)} - c^{(i)}s_m^{(i)}, \quad i \in \{1, \ldots, p\} \) for \( m = -1 \). From the system in (2.4) and the system (2.7), it follows that

\[
t_m^{(i)} = \frac{b^{(i)}u_{m-1}^{(i)} + u_0^{(i)}s_{m-1}^{(i)}}{b^{(i)}u_{m-1}^{(i)} + u_0^{(i)}s_{m}^{(i)}}, \quad m \geq -1, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N},
\]

Hence,

\[
z_{m,k}^{(i)} = \frac{b^{(i)}s_{m}^{(i)} + c^{(i)}s_{m+1}^{(i)}}{b^{(i)}s_{m-1}^{(i)} + c^{(i)}s_{m}^{(i)}}, \quad m \geq -1, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N},
\]

for \( k \in \{1, 2\} \), that is,

\[
y_{2m+k}^{(i)} = \frac{b^{(i)}s_{m+2}^{(i)} + c^{(i)}s_{m+1}^{(i)}}{b^{(i)}s_{m-1}^{(i)} + c^{(i)}s_{m}^{(i)}}, \quad m \geq -1, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N}, \quad \text{for} \quad k \in \{1, 2\}.
\]

Using the change of variables

\[
(2.8) \quad y_n^{(i)} = \frac{x_{n-1}^{(i+1 \mod(p))}}{x_n^{(i)}}, \quad n \geq -2, \quad p \in \mathbb{N}, \quad i \in \{1, \ldots, p\},
\]

thus we have

\[
x_n^{(i)} = x_{n-1}^{(i+1 \mod(p))}y_n^{(i)} = x_{n-2}^{(i+2 \mod(p))}y_n^{(i+1 \mod(p))} = \ldots = x_{n-p+1}^{(i)}y_{n-j}^{(i+j \mod(p))} = x_{n-p}^{(i)}, \quad n \geq p, \quad p \in \mathbb{N}, \quad i \in \{1, \ldots, p\},
\]
where $\prod_{j=i}^{l} y_j = 1$ if $l < i$. From all above mentioned we see that the following corollary holds.

**Corollary 2.1.** Let $\left\{ x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)} \right\}_{n \geq 2}$ be a solution of system (1.8). Then for $n \geq p$,

$$x_{pn+k}^{(i)} = \frac{x_k^{(i)}}{\prod_{l=0}^{n-1-p-1} y_{p(n-l)+k-j}^{(i+j) \mod (p)}}, \quad k \in \{0, 1, \ldots, p-1\}, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N}.$$

**Theorem 2.1.** Let $\left\{ x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)} \right\}_{n \geq 2}$ be a solution of system (1.8). Then for $n \geq p$, if $p$ is even,

$$x_{pn+k}^{(i)} = \frac{\left\{ \prod_{j=1}^{\left[ \frac{k}{2} \right]} y_{2j}^{(i+k-2j) \mod (p)} \right\}^{-1} \left\{ \prod_{j=\left[ \frac{k}{2} \right]+1}^{k} y_{2(k-j)+1}^{(i-k+2j-1) \mod (p)} \right\}^{-1} x_{0}^{(i+k) \mod (p)}}{\prod_{l=0}^{n-1-2} y_{2(2l+1)}^{(i+2j+t) \mod (p)}},$$

if $p$ is odd,

$$x_{pn+k}^{(i)} = \frac{\left\{ \prod_{j=1}^{\left[ \frac{k}{2} \right]} y_{2j}^{(i+k-2j) \mod (p)} \right\}^{-1} \left\{ \prod_{j=\left[ \frac{k}{2} \right]+1}^{k} y_{2(k-j)+1}^{(i-k+2j-1) \mod (p)} \right\}^{-1} x_{0}^{(i+k) \mod (p)}}{\prod_{l=0}^{n-1-2} y_{2(2l+1)}^{(i+2j+t) \mod (p)}},$$

$$\times \left\{ \prod_{j=\left[ \frac{k}{2} \right]}^{\left[ \frac{p+2n-2t}{2} \right]} y_{2(h_3(l,k,p,n)-j-1+h)}^{(i+2j+1-h) \mod (p)} \right\}^{-1},$$

$k \in \{0, 1, \ldots, p-1\}, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N}$, where $t = k - 2 \left[ \frac{k}{2} \right] \in \{0, 1\}, \quad t_2 = n - 2 \left[ \frac{n}{2} \right] \in \{0, 1\}$, $h = t \lor t_2 - t \land t_2, \quad h_1 (l,k,p,n) = \left[ \frac{n}{2} \right] (n-l) + \left[ \frac{k}{2} \right], \quad h_2 (l,k,p,n) = h_1 (2l,k,p,n) - l + \left[ \frac{n}{2} \right], \quad h_3 (l,k,p,n) = \left[ \frac{n}{2} \right] (2l+t_2-1) + \left[ \frac{k}{2} \right] + l$, $[x]$ is integral part of $x$ and

$$y_{2m+1}^{(i)} = \frac{b_{s_m}^{(i)} + y_{-1}^{(i)} s_{m+1}^{(i)} + y_{1}^{(i)} s_{m}^{(i)}}{b_{s_m}^{(i)} + y_{0}^{(i)} s_{m+1}^{(i)} + y_{0}^{(i)} s_{m}^{(i)}}$$

$m \geq -1, \quad i \in \{1, \ldots, p\}, \quad p \in \mathbb{N},$
with \((s^{(i)}_m)_{m \geq -1, i \in \{1, \ldots, p\}}\) be the solution to system (2.5) such that \(s^{(i)}_{-1} = 0\) and \(s^{(i)}_0 = 1\), for \(i \in \{1, \ldots, p\}\).

**Proof.** By Corollary 2.1, we obtain

\[
x^{(i)}_{pn+k} = \frac{x^{(i)}_k}{\prod_{l=0}^{n-1} \prod_{j=0}^{p(n-l)+k-j} (i+j) \mod (p)}, k \in \{0, 1, \ldots, p-1\}, i \in \{1, \ldots, p\}, p \in \mathbb{N}.
\]

Using (2.8), we get

\[
x^{(i)}_{pn+k} = \left\{ \prod_{l=0}^{n-1} \prod_{j=0}^{p(n-l)+k-j} (i+j) \mod (p) \right\} \left\{ \prod_{j=0}^{k-1} (i+j) \mod (p) \right\} \left( \prod_{j=0}^{k-j} \right), k \in \{0, 1, \ldots, p-1\}, i \in \{1, \ldots, p\}, p \in \mathbb{N},
\]

The rest of assertions are immediate. \(\square\)

# 3. SOME APPLICATIONS

In this section, we will give some applications for some special cases of the coefficients of the system (1.7).

**Corollary 3.1.** Let \(\{x^{(1)}_n, \ldots, x^{(2p)}_n\}_{n \geq -2}\) be a well-defined solution to the following system,

\[
x^{(i)}_{n+1} = \frac{x^{(i+1) \mod (2p)}_n}{x^{(i) \mod (2p)}_{n-1} + x^{(i+1) \mod (2p)}_{n-2}}, n \in \mathbb{N}_0, i \in \{1, \ldots, 2p\}, p \in \mathbb{N}.
\]

Then

\[
x^{(i)}_{2pn+k} = x^{(i+k) \mod (2p)}_0 \prod_{l=1}^{n} \left\{ \prod_{j=1}^{\left\lfloor \frac{k+2p-2}{2} \right\rfloor} x^{(i+k-2j) \mod (2p)}_{-l} F_{pl+j-1} + x^{(i+k-2j+1) \mod (2p)}_{-l} F_{pl+j} \right\}
\]

\[
\times \prod_{j=\left\lfloor \frac{k}{2} \right\rfloor}^{\left\lfloor \frac{k+2p-2}{2} \right\rfloor} x^{(i+k-2j) \mod (2p)}_{-l} F_{pl+j-1} + x^{(i+k-2j+1) \mod (2p)}_{-l} F_{pl+j}
\]

\[
\times \prod_{j=1}^{\left\lfloor \frac{k}{2} \right\rfloor} x^{(i+k-2j) \mod (2p)}_{-l} F_{j-1} + x^{(i+k-2j+1) \mod (2p)}_{-l} F_{j}
\]

\[
\times \prod_{j=\left\lfloor \frac{k}{2} \right\rfloor +1}^{\left\lfloor \frac{k+2p-2}{2} \right\rfloor} x^{(i+k-2j) \mod (2p)}_{-l} F_{j-1} + x^{(i+k-2j+1) \mod (2p)}_{-l} F_{j},
\]
for \( n \in \mathbb{N}_0, k \in \{0, 1, \ldots, 2p - 1\} \), \( i \in \{1, \ldots, 2p\} \), \( p \in \mathbb{N} \), where \((F_n)_{n \geq -1}\) is the solution to the following difference equation

\[ F_{n+1} = F_n + F_{n-1}, n \in \mathbb{N}_0, \]

satisfying the initial conditions \( F_{-1} = 0 \), \( F_0 = 1 \). The sequence \((F_n)_{n \geq -1}\) is called the well-known Fibonacci sequence in literature.

Proof. System (3.1) is obtained from system (1.7) with \( e^{(i)} = b^{(i)} = c^{(i)} = 1 \), \( i \in \{1, \ldots, 2p\} \), \( p \in \mathbb{N} \). Hence, the sequence \((s_n^{(i)})_{n \geq -1, i \in \{1, \ldots, 2p\}}\) satisfying conditions \( s_{-1}^{(i)} = 0 \) and \( s_0^{(i)} = 1 \), for \( i \in \{1, \ldots, 2p\} \) are the same and so we have \( s_n^{(i)} = F_n, n \geq -1 \). The rest of the proof is straightforward and hence omitted. \( \square \)

Corollary 3.2. Let \( \{x_n^{(1)}, \ldots, x_n^{(2p+1)}\}_{n \geq -2} \) be a well-defined solution to the following system,

\[
\begin{align*}
\begin{array}{c}
(i+1) \mod (2p+1) & (i+1) \mod (2p+1) \\
(i+2j-h) \mod (2p+1) & (i+2j-h+1) \mod (2p+1) \\
\end{array}
\end{align*}
\]

\[ x_{n+1}^{(i)} = \frac{x_n^{(i+1) \mod (2p+1)} (i+1) \mod (2p+1) x_{n-2}^{(i+1) \mod (2p+1)}}{x_n^{(i+1) \mod (2p+1)} + 2x_{n-2}^{(i+1) \mod (2p+1)}}, n \in \mathbb{N}_0, i \in \{1, \ldots, 2p+1\}, p \in \mathbb{N}.
\]

Then

\[ x_{(2p+1)n+k}^{(i)} = x_0^{(i+k) \mod (2p+1)} \prod_{l=0}^{n-\left\lfloor \frac{n-2}{2} \right\rfloor} \left\{ \prod_{j=h}^{p-1} \frac{x_0^{(i+2j-h) \mod (2p+1)} P_{m-1}^{(i+2j-h+1) \mod (2p+1)} P_m}{x_0^{(i+2j-h) \mod (2p+1)} P_{m-t} + x_0^{(i+2j-h+1) \mod (2p+1)} P_m} \right\} \]

\[ \times \prod_{j=0}^{p+1} x_0^{(i+2j) \mod (2p+1)} P_{m-p-t} + x_0^{(i+2j+1) \mod (2p+1)} P_{m-p} \]

\[ \times \prod_{j=0}^{p+2} x_0^{(i+2j) \mod (2p+1)} P_{m-p-t} + x_0^{(i+2j+1) \mod (2p+1)} P_{m-p-t-1} \]

\[ \times \prod_{j=1}^{k} x_0^{(i+k-2j) \mod (2p+1)} P_{j-2} + x_0^{(i+k-2j+1) \mod (2p+1)} P_{j-1} \]

\[ \times \prod_{j=\left\lceil \frac{k}{2} \right\rceil + 1}^{k} x_0^{(i+k-2j) \mod (2p+1)} P_{j-1} + x_0^{(i+k-2j) \mod (2p+1)} P_{j-1} \]
for \( n \in \mathbb{N}_0, k \in \{0,\ldots,2p\}, i \in \{1,\ldots,2p+1\}, p \in \mathbb{N}, \) where \( m = p(n-2l)-l + \left\lceil \frac{k}{2} \right\rceil + \left\lceil \frac{s}{2} \right\rceil + t \vee t_2 - j - 1 \) and \((P_n)_{n \geq -1}\) is the solution to the following difference equation

\[
P_{n+1} = 2P_n + P_{n-1}, n \in \mathbb{N}_0,
\]

satisfying the initial conditions \( P_{-1} = 0, P_0 = 1. \) The sequence \((P_n)_{n \geq -1}\) is called the Pell sequence in literature.

Proof. System \((3.2)\) is obtained from system \((1.7)\) with \( a^{(i)} = c^{(i)} = 1 \) and \( b^{(i)} = 2, i \in \{1,\ldots,2p+1\}, p \in \mathbb{N}. \) Hence, the sequence \((s^{(i)}_n)_{n \geq -1,i \in \{1,\ldots,2p+1\}}\) satisfying conditions \( s^{(i)}_{-1} = 0 \) and \( s^{(i)}_{0} = 1, \) for \( i \in \{1,\ldots,2p+1\} \) are the same and so we have \( s^{(i)}_n = P_n, n \geq -1. \) The rest of the proof is straightforward and hence omitted. \( \square \)

4. CONCLUSION

In this paper, we represented the general solutions of \( p-\)dimensional systems of nonlinear rational difference equations with constant coefficients using suitable transformation reducing to the equations in Riccati type. Secondly, the solutions of this system are related to both Fibonacci numbers and Pell numbers for some special cases. Finally, we will give the following important open problem for system of difference equations theory to researchers. The system \((1.7)\) can extend to equations more general than that in \((1.7)\). For example, the \( p-\)dimensional system of nonlinear rational difference equations of \((\max \{m,k,l,s\} + 1)\) -order,

\[
x^{(i)}_{n+1} = \frac{a^{(i)} x^{(i+1) \mod (p)}_{n-m} x^{(i+1) \mod (p)}_{n-l}}{b^{(i)} x^{(i)}_{n-k} + c^{(i)} x^{(i+1) \mod (p)}_{n-s}}, n \in \mathbb{N}_0, p \in \mathbb{N}, i \in \{1,\ldots,p\},
\]

where \( \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \) the sequences \((a^{(i)}), (b^{(i)}), (c^{(i)})\), are non-zero real numbers and initial values \( x^{(i)}_{-j}, j \in \{0,\ldots,\max \{m,k,l,s\}\}, i \in \{1,\ldots,p\}. \)

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.
REFERENCES

[1] M. Aloqeili, Dynamics of a $k$th order rational difference equation, Appl. Math. Comput. 181 (2) (2006), 1328 – 1335.
[2] M. Aloqeili, Dynamics of a rational difference equation, Appl. Math. Comput. 176 (2) (2006), 768 – 774.
[3] R. Abo-Zeid, H. Kamal, Global behavior of two rational third order difference equations, Univ. J. Math. Appl. 2(4) (2019), 212 – 217.
[4] A.M. Alotaibi, M.S.M. Noorani, M.A. El-Moneam, On the solutions of a system of third-order rational difference equations, Discr. Dyn. Nat. Soc. 2018 (2018), 1743540.
[5] L. Brand, A sequence defined by a difference equation, Amer. Math. Mon. 62 (7) (1955), 489 – 492.
[6] C. Çinar, On the positive solutions of the difference equation $x_{n+1} = x_{n-1} / (1 + ax_n x_{n-1})$, Appl. Math. Comput. 158(3) (2004), 809 – 812.
[7] D. Clark, M.R.S. Kulenovic, A coupled system of rational difference equations, Computers Math. Appl. 43 (6 – 7) (2002), 849 – 867.
[8] Q. Din, Global stability of a population model, Chaos Solitons Fractals. 59 (2014), 119 – 128.
[9] Q. Din, Global behavior of a plant-herbivore model. Adv. Differ. Equ. 2015(1) (2015), 119.
[10] E.M. Elabbasy, E.M. Elsayed, Dynamics of a rational difference equation, Chin. Ann. Math. Ser. B, 30B(2) (2009), 187 – 198.
[11] E.M. Elabbasy, H.A. El-Metwally, E.M. Elsayed, Global behavior of the solutions of some difference equations, Adv. Differ. Equ. 2011(1) (2011), 28.
[12] H. El-Metwally, Global behavior of an economic model, Chaos Solitons Fractals, 33(3) (2007), 994 – 1005.
[13] E.M., Elsayed, Solutions of rational difference system of order two. Math. Computer Model. 55 (3 – 4) (2012), 378 – 384.
[14] E.M. Elsayed, Solution for systems of difference equations of rational form of order two, Comput. Appl. Math. 33 (3) (2014), 751 – 765.
[15] E.M. Elsayed, H.S. Gafel, On some systems of three nonlinear difference equations, J. Comput. Anal. Appl. 29 (1) (2021), 86 – 108.
[16] M. Kara., Y. Yazlik, Solvability of a system of nonlinear difference equations of higher order, Turk. J. Math. 43(3) (2019), 1533 – 1565.
[17] M. Kara, Y. Yazlik, On the solutions of three-dimensional system of difference equations via recursive relations of order two and applications, J. Appl. Anal. Comput. 12 (2) (2022), 736 – 753.
[18] A.S. Kurbanli, C. Çinar, İ. Yalçinkaya, (2011a). On the behavior of positive solutions of the system of rational difference equations $x_{n+1} = x_{n-1} / (y_n x_{n-1} + 1), y_{n+1} = y_{n-1} / (x_n y_{n-1} + 1)$, Math. Computer Model. 53 (5 – 6), 1261 – 1267.
[19] A.S. Kurbanlı, On the behavior of solutions of the system of rational difference equations $x_{n+1} = x_{n-1}/(y_n x_{n-1} - 1)$, $y_{n+1} = y_{n-1}/(x_n y_{n-1} - 1)$, $z_{n+1} = z_{n-1}/(y_n z_{n-1} - 1)$, Discr. Dyn. Nat. Soc. 2011 (2011b), Article ID 932362.

[20] A.Y. Özban, On the positive solutions of the system of rational difference equations $x_{n+1} = 1/y_{n-k}$, $y_{n+1} = y_n/x_{n-m} y_{n-m-k}$, J. Math. Anal. Appl. 323(1) (2006), 26 – 32.

[21] S. Stević, More on a rational recurrence relation, Appl. Math. E-Notes, 4(1) (2004), 80 – 85.

[22] S. Stević, On a discrete epidemic model, Discr. Dyn. Nat. Soc. 2007 (2007), 087519.

[23] S. Stević, On some solvable systems of difference equations, Appl. Math. Comput. 218(9) (2012), 5010 – 5018.

[24] S. Stević, Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences, Electron. J. Qual. Theory Differ. Equ. 2014 (2014), 67.

[25] S. Stević, J. Diblik, B. Iričanin, Z. Šmarda, On a third-order system of difference equations with variable coefficients, Abstr. Appl. Anal. 2012 (2012), Article ID 508523.

[26] S. Stević, On a two-dimensional solvable system of difference equations, Electron. J. Qual. Theory Differ. Equ. 2018 (2018), 104.

[27] D.T. Tollu, Y. Yazlik, N. Taskara, On fourteen solvable systems of difference equations, Appl. Math. Comput. 233 (2014), 310 – 319.

[28] N. Touafek, E.M. Elsayed, On the solutions of systems of rational difference equations, Math. Computer Model. 55(7 – 8) (2012), 1987 – 1997.