Conversion Measure of Faraday Rotation–Conversion with Application to Fast Radio Bursts

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Abstract

Faraday rotation–conversion is the simultaneous rotation of all three Stokes polarization parameters, $Q$, $U$, and $V$, as an electromagnetic wave propagates through a magnetized plasma. In this regime, the Faraday plasma screen is characterized by more than just a rotation measure. We define the conversion measure that characterizes the wavelength-dependent conversion between the linear and circular polarization. In a cold plasma, the conversion occurs at the localized regions along the wave’s path, where the large-scale magnetic field is perpendicular to the propagation direction. We show that the number of these regions along the line of sight through the screen, and their individual contributions to the conversion measure, can be inferred from the polarization measurements. We argue that the simultaneous measurement of wavelength-dependent linear and circular polarization might give an important insight into the magnetic-field geometry of the Faraday screen in FRB 121102 and other repeating fast radio bursts.

Key words: magnetic fields – plasmas – polarization

1. Introduction

Faraday conversion (FC) between the linear and circular polarizations is thought to be responsible for producing the measured circular polarization of some radio sources in galactic nuclei (e.g., Jones & O’Dell 1977; Bower et al. 1999; Ruzskowski & Begelman 2002; Homan & Lister 2006). More recently, Vedantham & Ravi (2019) have pointed out that FC might be relevant for fast radio bursts (FRBs), especially for the repeating FRB 121102. This source, discovered by Spitler et al. (2014, 2016) and studied extensively since then, provides an important test bed for the models of the origin of FRBs (e.g., Beloborodov 2017; Thompson 2017; Waxman 2017; Margalit & Metzger 2018). FC reflects the birefringence of the plasma surrounding the FRB, and it is potentially a sensitive probe of the magnetico-ionic environment of this enigmatic source, since (i) the measured linear polarization is close to 100%, with extremely high rotation measure (RM) of $1.46 \times 10^7$ rad/m$^2$ (Michilli et al. 2018); (ii) while the currently measured circular polarization is consistent with zero at frequencies higher than 5 GHz, this may not be so at lower frequencies and is potentially measurable.

Vedantham & Ravi (2019) stated that in the cold plasma, FC is suppressed, because the normal-mode waves are nearly circular, and that instead, the conversion would happen in a relativistic plasma. However, in this paper we show that FC can occur efficiently in cold plasma near the points where the Faraday rotation frequency is close to zero, i.e., near the “field reversals” where the magnetic field component along the line of sight is zero. The importance of reversals in FC has been studied in the past starting from Cohen (1960) and Zheleznyakov & Zlotnik (1964). Ruzskowski & Begelman (2002) explored the role of reversals in the context of circular polarization from galactic nuclei. Melrose et al. (1995), Broderick & Blandford (2010), and Melrose (2010) give a comprehensive discussion of the passage through a reversal in terms of rotation of the Poincare sphere.

In this paper, we identify several novel diagnostics for the polarization state of the pulse traveling through a series of reversals. We present a simple theory of the effect in Section 2, and we use the theory in Section 3 to predict what might be seen in FRB 121102 if the polarization is measured at sufficiently low frequency $\nu$.

In this work, we focus on cold plasma. Our theoretical results can be summarized as follows. A 100% linear polarized wave, after passing through several field reversals in a magnetized plasma (called the Faraday screen in this context) becomes partially circularly polarized. After a passage through the reversal, the circular polarization $V$ oscillates quasiperiodically as a function of $\lambda$, where $\lambda = c/\nu$ is the wavelength. For the screen parameters and wave frequencies that are expected to be relevant for FRB 121102.

1. The rms value of $V$ oscillations is given by

$$\langle \Pi_0 \rangle = \text{CM} \lambda^2,$$

where $\Pi_0 \equiv |V|/I$ is the degree of circular polarization. Total intensity $I = 1$ is assumed throughout the paper, and we have defined a conversion measure (CM) measured in units of $1/m^2$. The result is valid over a finite wavelength interval, where conversion is small and rotation is large:

$$\text{CM} \lambda^2 \ll 1, \ \text{RM} \lambda^2 \gg 1.$$

Here, RM is the usual Faraday RM. We also show that for a passage through a single smooth reversal, an asymptotically exact expression is

$$\langle \Pi_0 \rangle = \sqrt{2(e^{-c^2/2} - e^{-c^2})}, \ c \equiv \text{CM} \lambda^2.$$

This equation is valid with a high degree of precision for \text{RM} \lambda^2 \gg 1 and arbitrary \text{CM} \lambda^2. We note that this
solution was already found in Zheleznyakov & Zlotnik (1964) in a slightly different form.\footnote{We thank Harish Vedantham for pointing this out after the first version of this paper appeared on the arXiv. While the $\lambda$-dependence for the conversion angle follows directly Zheleznyakov and Zlotnik’s results, it was not emphasized in that paper, and CM is defined here for the first time.}

2. For FRB 121102 the expected CM is

\[ \text{CM} \sim 1 \text{m}^{-2}, \]  

(4)

giving the rms degree of circular polarization as

\[ (\Pi_0) \sim 10\% \]  

(5)

at $\nu \sim 1 \text{GHz}$, but with very high uncertainty, as explained in Section 3.

3. The number of different quasiperiods, i.e., the number of peaks of the Fourier transform of $V$ as a function of $\lambda^2$, is equal to the number of reversals in the large-scale magnetic field. The locations of the peaks represent the RMs of the reversal points. This feature survives if the field has small-scale fluctuations due to a turbulent cascade at short wavelengths. Even though, in this case, there is a multitude of reversal points, they are strongly clustered around the reversals of the large-scale field, and each of the clusters produces a potentially measurable quasiperiod.

2. Faraday Rotation–Conversion

Propagation of an electromagnetic wave in a Faraday screen changes all three Stokes polarization parameters, $Q$, $U$, and $V$. We will assume that the wave propagates along $z$ and takes $Q^2 + U^2 + V^2 = 1$. Then $Q = \pm 1$ is 100% linear polarization along $x, y$; $U = \pm 1$ is 100% linear along $x \pm y$; $V = \pm 1$ is 100% right/left circular polarization.

Assuming the plasma is cold, we have a simple equation for the polarization evolution in the screen (e.g., Sazonov 1969, an ab initio derivation is given in the Appendix):

\[ P' = \Omega \times P, \quad \Omega \equiv (Q, U, V). \]  

(6)

The polarization vector $P$ rotates with angular velocity $\Omega$ (measured in units of $\text{m}^{-1}$) as the wave propagates along $z$, and $' \equiv d/dz$. The components of the polarization rotation–conversion rate

\[ \Omega \equiv (g, h, f) \]  

(7)

are the Faraday rotation rate

\[ f = -\frac{1}{c} \frac{\omega_p^2 \omega_B}{\omega^2} \hat{B}_z \]  

(8)

and the FC rate

\[ h + ig = -\frac{1}{2c} \frac{\omega_p^2 \omega_B^2}{\omega^3} (\hat{B}_x + i\hat{B}_y)^2, \]  

(9)

where $\hat{B} \equiv B/B$ is the unit vector along the magnetic field of the screen $B$, $\omega$ is the angular frequency of the wave, and

\[ \omega_p^2 = \frac{4\pi n e^2}{m}, \quad \omega_B = \frac{eB}{mc} \]  

(10)

are the plasma and Larmor frequencies in the screen.

It is important to note the hierarchy of the components of the angular velocity $\Omega$ responsible for rotation–conversion:

\[ \frac{g}{f} \sim \frac{h}{f} \sim \frac{\omega_B}{\omega} = \frac{\nu_B}{\nu} = \frac{2.8B_G}{\nu}, \]  

(11)

where $B_G \equiv B/1\text{G}$. It is important to note, however, that these estimates are only valid if $B_x \sim B_y \sim B_z$ and may not hold in all regions along the path. If, say, $B \sim 1\text{mG}$ and $\nu \sim 1\text{GHz}$, we have $g/f \sim 3 \times 10^{-6}$, and the angular velocity $\Omega$ is nearly aligned with the $V$-axis.

Suppose we are interested in the evolution of an initially linearly polarized pulse, as is the case with the repeating radio bursts from FRB 121102. The polarization vector $P$ is initially in the $U$–$Q$ plane and therefore we may consider the rotation of the whole plane with $\Omega$ as the angular velocity. Clearly, this is equivalent to considering rotation of the vector normal to the $U$–$Q$ plane, i.e., rotation of the unit vector $V$ that represents initially purely circular polarization. We can immediately see that as $V$ and the $U$–$Q$ plane rotate around $\Omega$, and they turn at the most by the angle $\theta_{\text{max}} \approx 2\sqrt{g^2 + h^2}/f \ll 1$, and thus the circular polarization $V$ of any initially linearly polarized pulse does not exceed $\theta_{\text{max}}$. This is the Faraday rotation regime—common case in astrophysics, with negligible conversion between linear $(Q, U)$ and circular $V$ polarizations. In this regime, the Faraday screen is fully characterized by a single parameter—the RM:

\[ \text{RM} \equiv \frac{1}{2\lambda^2} \int dz f = 8.1 \times 10^5 \text{rad m}^2 / \text{pc cm}^{-3} \text{G}, \]  

(12)

relating to initial and final linear polarizations:

\[ (Q + iU)|_f = e^{2\text{RM} \lambda^2} (Q + iU)|_i. \]  

(13)

2.1. Adiabatic Invariant

It is clear that FC will remain small, so long as the angle between the angular velocity vector $\Omega$ and the $V$-axis remains small. This statement can be made with greater rigor by noting that the polarization transfer Equation (6) has an adiabatic invariant of

\[ P_\parallel \equiv \hat{\Omega} \cdot P = \text{inv}, \quad \text{RM} \lambda^2 \gg 1, \]  

(14)

because

\[ P_\parallel' = \hat{\Omega}' \cdot P \rightarrow \hat{\Omega}' \cdot \langle P \rangle = \hat{\Omega}' \cdot \langle P_\parallel \hat{\Omega} \rangle = 0. \]  

(15)

This invariant has been expensively discussed in the literature (e.g., Melrose 2010). One can show that the invariant is conserved so long as

\[ |\hat{\Omega}'| \ll \Omega, \]  

(16)

i.e., when the Faraday rotation is faster than the rotation of direction of $\Omega$. In that case, the vectors that were initially in the $U$–$Q$ plane, remained nearly perpendicular to $\Omega$, and, if the latter remains close to the $V$-axis, the circular polarization remained very small.

At first glance, for $\nu_B/\nu \ll 1$ the $V - \Omega$, alignment and adiabaticity are always satisfied in cold plasma. However, this argument is flawed. As the pulse travels along the line of sight, it is likely to encounter field reversals, i.e., the locations where
Figure 1. Evolution of the conversion angle during a passage through a reversal. The coordinate $z$ is plotted on the horizontal in terms of the characteristic length $h/f$. With increasing $z$, the passage through a reversal becomes more gradual, reaching the adiabatic regime where the conversion angle tracks the angle by which $\Omega$ rotates. The $\xi = 15$ curve is indistinguishable from $\alpha(z)$, i.e., the angle by which the angular velocity $\Omega$ turns during the passage. The conversion is large for intermediate values of $\xi \sim 1$. The curves are qualitatively similar to those in Figures (2)-(4) of Melrose et al. (1995).

the field is perpendicular to the line of sight and both $B_z$ and $f$ are zero. At or near these locations, $\Omega$ is strongly misaligned with the $V$-axis, and its direction changes rapidly as the pulse travels through the reversal. Therefore, the adiabaticity can be broken, in which case the pulse can develop a substantial circular polarization. In the next section, we discuss FC in more detail as the pulse crosses the field reversal.

2.2. FC at Field Reversals: Conversion Measure

Consider now the passage of a linearly polarized pulse near a field reversal at $z = 0$. It is convenient to choose $f'$ as the axis in the direction of the magnetic field at the reversal. Near the reversal, the angular frequency of the $P$ rotation is given by

$$\Omega = (0, h, f'z),$$

where we consider $f'$, $h$ as constants. Furthermore, we assume that this approximation is valid for a range of $-z_0 < z < z_0$ such that $f'z_0 \gg h$, i.e., we assume that the pulse is in the Faraday rotation limit as it both enters and exits the reversal. In this approximation, the evolution of the polarization vector is entirely characterized by a single dimensionless parameter,

$$\xi = h^2/f'.$$

In Figure 1, we show several examples of evolution of the conversion angle $\theta(z)$ as the pulse passes through the reversal. Here, $\theta$ is the angle by which the plane of linearly polarized Stokes vectors $Q-U$ (or, equivalently, a vector perpendicular to this plane) rotates as a result of FC. The circular polarization $|V| < \sin \theta$. The figure also shows the evolution of the angle $\alpha(z)$ by which the angular velocity $\Omega$ turns during the passage. It is clear that for $\xi \gg 1$, the orientation of the plane follows adiabatically that of $\Omega$. The circular polarization achieves its maximum at $z = 0$, but then the plane flips into alignment with the original $Q-U$ plane and the circular polarization becomes small again. For $\xi \ll 1$, the rate of Faraday rotation passes zero so quickly that FC does not have time to occur (see e.g., Melrose 2010 or Broderick & Blandford 2010).

It is possible to find the final value of the conversion angle $\theta_f = \theta(z = +\infty)$ analytically, by solving Equation (6) using a Laplace transform; it was also obtained by Zheleznyakov & Zlotnik (1964) using a different method. The answer is given by

$$\theta_f(\xi) = \arccos(2e^{-\pi\xi^2/2} - 1).$$

The greatest FC occurs at values of $\xi \sim 1$, with the angle $\theta_f$ reaching $\pi/2$ at $\xi = (2/\pi)\log 2$. At this value of $\xi$, a full conversion between the linear and circular polarizations is possible. The rms amplitude of $V$ is given by

$$\langle \Pi_0 \rangle = \frac{1}{\sqrt{2}} \sin \theta_f = \sqrt{2(e^{-\pi\xi^2/2} - e^{-\pi^2})}.$$

For small values of $\xi$,

$$\theta_f \sim \sqrt{2\pi\xi},$$

and this turns out to be a good approximation for $\theta_f < 1$ rad. It is easy to understand qualitatively where this scaling is coming from. Near the reversal, $P$ makes one Faraday rotation over the length scale $\Delta z \sim 1/\sqrt{f'\xi}$. During this interval, the conversion angle is $\theta \sim h\Delta z \sim \sqrt{\xi}$.

Assume now, for simplicity, that the coherence length of the magnetic field is comparable to the screen thickness $l$. If $B_z(z = 0) = 0$, then

$$f(z) \sim \frac{f_0}{l}, \quad |z| \lesssim l, \quad f_0 \sim \frac{\text{RM} \lambda^2}{l},$$

where $f_0$ is the characteristic value of $f$ far from the reversal. From Equation (23), we see that the produced circular polarization (starting from 100% linear) is about

$$V \sim \frac{\mu}{f_0} (\Pi_0)^{1/2} \sim \frac{\mu}{\nu} (\text{RM} \lambda^2)^{1/2}.$$

For small conversion angles, the rms value of $V$ is given by

$$\langle \Pi_0 \rangle = \text{CM} \lambda^2,$$

while for large conversion angles,

$$\langle \Pi_0 \rangle = \sqrt{2[1-e^{-(\text{CM} \lambda^2)^{1/2}} - e^{-(\text{CM} \lambda^2)^{1/2}}]}.$$
screen at \( z = -a \). The Faraday rotation angle is given by

\[
\int_{-a}^{z} dz f(z) = \frac{f_0}{2l} (z^2 - a^2).
\]

Neglecting conversion, Equation (6) gives

\[
Q + iU = e^{i(\phi / 2l)(z^2 - a^2)}.
\]

From Equation (6), the circular polarization can be calculated perturbatively as

\[
V \sim h \int dz \cos \left[ \frac{f_0}{2l} (z^2 - a^2) \right]
\]

\[
\sim \sqrt{\xi} \cos \left[ \frac{f_0}{2l} a^2 - \frac{\pi}{4} \right].
\]

Putting in the \( \lambda \)-dependence and recalling the definition of the CM in Equation (24), we can rewrite the above equation as

\[
V(\lambda^2) = \sqrt{2} \lambda^2 \Sigma_i \cos \left[ \text{RM}_i \lambda^2 - \frac{\pi}{4} \right].
\]

where RM is the rotation measure determined at the point of reversal. For several reversals, the equation above generalizes to a sum:

\[
V(\lambda^2) = \sqrt{2} \lambda^2 \sum_i \cos [\text{RM}_i \lambda^2 + \phi_i].
\]

Here, RM is the rotation measure measured at the \( i \)th field reversal, and the phase \( \phi_i \) is determined by the orientation of the perpendicular component of the magnetic field at the reversal. The equation above is valid in the limit of small conversion and, in our opinion, is the most useful from a practical point of view. By fitting the observed \( V(\lambda^2) \) directly, or by taking a Fourier transform and analyzing the peaks, one can infer the information about the RM and CM of each of the reversal points. This is illustrated in Figure 2.

So far, we assumed that the reversals are smooth; however, if the field has a small-scale structure, it is not a priori clear that the peaks in the Fourier transform of \( V(\lambda^2) \) survive. This issue is studied in the next subsection.

2.3. Turbulent Magnetic Field

If the magnetic field is intrinsic to the plasma, it is most likely turbulent. We will assume the Kolmogorov spectrum:

\[
B_r \sim B \left( \frac{r}{l_c} \right)^{1/3}.
\]

Here, \( B_r \) is the characteristic random component of the magnetic field at length scale \( \sim r \), \( l_c \) is the macroscopic length scale that contains most of the magnetic energy, and \( B \) is the characteristic magnetic field at the scale of \( l_c \). For simplicity, we take \( l_c \sim l \), where \( l \) is the screen thickness.

It would seem that the screen with a turbulent magnetic field is not described by the analysis of Section 2.2. The magnetic field gradient \( \sim B_r / r \propto r^{-2/3} \) is now dominated by the magnetic field fluctuations at small scales. Each large-scale zero of \( B_z \) splits into infinitely many zeros with an infinite derivative, and the adiabatic invariant is not conserved. The small-scale cutoff of the turbulence might show up, complicating the picture.

We numerically integrated the polarization transfer of Equation (6) for dozens of realizations of a turbulent magnetic field with the Kolmogorov spectrum of Equation (32). Much to our surprise, we found that the concept of the CM survives, as well as the possibility to count the number of “pronounced zeros” of \( B_z \) by counting the number of quasiperiods of \( V \) as a function of \( \lambda^2 \). An example of such numerical integration is shown in Figure 3. We are able to explain this result analytically, as follows.
For \( \nu_B/\nu \ll 1 \), we solve Equation (6) perturbatively:

\[ Q + iU = e^{i\alpha} \int \frac{dz}{n_B}, \quad \alpha = \frac{e}{me^2} \frac{\omega_p^2}{\omega^2}, \quad (33) \]

\[ V = \beta \Re \int dz \, n(B_x + iB_y)^2 (Q + iU), \]

\[ \beta = \frac{e^2}{2mc^2} \frac{\omega_p^2}{\omega^2}. \quad (34) \]

We then calculate the expectation value of \( V^2 \), assuming a Gaussian isotropic parity-invariant magnetic field in the screen,

\[ \langle B_i(k)B_j(k') \rangle = (2\pi)^3 \delta (\mathbf{k} + \mathbf{k'}) M(k) (\delta_{ij} - \hat{k}_i \hat{k}_j). \quad (35) \]

One can show that the magnetic field components along a given line of sight, say along \( x = y = 0 \), are independent Gaussian random fields. This gives

\[ \langle V^2 \rangle \sim \beta^2 n^2 B^4 l \, \delta l, \quad (36) \]

where \( \delta l \) is the decorrelation length of \( Q + iU \). From

\[ \langle (Q - iU) |_{\text{lin}}(Q + iU) |_{\text{lin}} \rangle = \langle \exp i\alpha \int_{\text{lin}} dz \, nB_z \rangle \sim e^{-2\pi^2 n^2 B^2} |_{\text{lin}} |_{\text{lin}}, \quad (37) \]

we estimate

\[ \delta l \sim \frac{1}{\alpha nB} \quad (38) \]

and

\[ V \sim \frac{\nu_B}{\nu} (RM \lambda^2)^{1/2} \propto \lambda^2 \quad (39) \]

exactly as for the smooth field.

We also see numerically that \( V \) as a function of \( \lambda^2 \) oscillates quasiperiodically, although now the quasiperiodicity is to be understood not as a finite number of incommensurate frequencies as in Equation (31) but as a finite number of pronounced peaks in the Fourier transform of \( V \) as a function of \( \lambda^2 \). This happens because the infinite number of zeros of \( B_z \) (neglecting the small-scale cutoff) form well-defined clusters, with the number of clusters \( \sim l/l_c \). In Figure 3, we show a numerical example of this effect.

2.4. Non-perturbative Regime \( CM\lambda^2 \gtrsim 1 \)

Although \( CM \lambda^2 \lesssim 1 \) seems to be more relevant for FRBs, as discussed in Section 3, the non-perturbative regime \( CM \lambda^2 \gtrsim 1 \) cannot be excluded a priori. In this regime, the cases of smooth and turbulent magnetic field are very different.

For a turbulent field, we numerically find a full rotation–conversion regime with \( Q \sim U \sim V \) at all frequencies below \( CM \lambda^2 \sim 1 \). This would have been a natural expectation, as our perturbative result is \( V \sim CM \lambda^2 \) and \( V \) cannot be greater than 1, were it not at odds with the smooth magnetic field case, which we consider next.

In the smooth field case, one needs to break an adiabatic invariant to linearly convert into circular polarization, which occurs when

\[ |\Omega'| \gtrsim \Omega^2, \quad (40) \]

or, as can be seen by setting \( B_z \) zero at \( z = 0 \),

\[ \frac{f_0}{l} \gtrsim \left( \frac{f_0}{l} \right)^2 + g^2, \quad (41) \]

or

\[ 1 \gtrsim (RM \lambda^2)^{1/2} + (CM \lambda^2)^{1/2}. \quad (42) \]

In the perturbative regime, \( CM \lambda^2 \ll 1 \), this formula gives the thickness of the non-adiabatic region \( z \). But at \( CM \lambda^2 \gtrsim 1 \), the inequality of Equation (42) simply cannot be satisfied—the screen is adiabatic everywhere. Since adiabatic invariants are conserved to exponential accuracy, one gets an exponential cutoff of conversion at \( \lambda > \lambda_c \), where \( \lambda_c \) is model dependent but to the order of magnitude given by \( CM \lambda^2 \sim 1 \).

3. FRB 121102

What is the magnetic field inside the medium surrounding this source? The FRB is coincident with a radio nebula that generates synchrotron radiation with the luminosity of \( \sim 10^{39} \) erg s\(^{-1} \) (Chatterjee et al. 2017). Beloborodov (2017, Equation (4) of that paper) used the spectral shape of the observed radiation to derive the magnetic field of \( B \sim 0.1G \). This is consistent with the estimate from a one-zone model of Margalit & Metzger (2018, their Equation (17)), which is broadly based on Beloborodov’s scenario for powering the nebula and is designed to produce the observed RM.

It is unknown whether the Faraday screen is located inside or outside the nebula. However, the magnetic field of the Faraday screen is strongly constrained by the RM and the dispersion measure of the pulses. A dramatic reduction of 10% in the RM occurred after the initial measurement of the linear polarization of (Michilli et al. 2018; see Section 2 of Vedantham & Ravi 2019) for a summary). Since no measurable simultaneous change in the dispersion measure occurred (<1 pc cm\(^{-3} \)), one is able to derive a very conservative lower limit on the mean field:

\[ B > (0.02/\eta_B)G, \quad (43) \]

see Equation (3) of Vedantham & Ravi (2019). Here, \( \eta_B \) is the average value of \( B_z/B \) along the line of sight, which is expected to be considerably smaller than 1, especially if the medium has field reversals. We shall thus assume

\[ B \gtrsim 0.1G; \quad (44) \]

it is reassuring that this estimate is consistent with that of the magnetic field in the radio nebula.

We can now estimate the CM expected from the linearly polarized pulses of FRB 121102. Our Equation (26) gives

\[ CM \gtrsim 0.3 \frac{1}{m^2}. \quad (45) \]

This was reported in the introduction as \( CM \sim 1/m^2 \) because this is about the median value that we get numerically and is mostly due to the partial cancellation of positive and negative RM regions in the Faraday screens with zeros of \( B_z \).

We must stress that Faraday screens with a coherence length of the order of thickness, \( l_c \sim l \), have a very large scatter of the resulting CM. In particular, \( B_z \) may have no zeros and, as a
result, no conversion at all would occur, so

\[ \text{CM} \approx 0. \]  \hfill (46)

In other instances, the RM cancellation inside the screen is strong, and one gets

\[ \text{CM} \approx 10^{-1} \text{m}^2, \]  \hfill (47)

in which case Faraday rotation at \( \nu \sim 1 \text{ GHz} \) becomes a non-perturbative Faraday rotation–conversion, with observed \( Q \sim U \sim V. \)

4. Discussion

This paper provides qualitative predictions for the circular polarization of FRB pulses, if it is produced by FC of initially linearly polarized pulses, such as the ones in FRB 121102. If FC occurs in cold plasma, then it takes place near the field reversals along the line of sight. Each field reversal produces a quasiperiod in \( V(\lambda^2) \), with the frequency and amplitude containing information about the RM and CM of the reversal, respectively. Our work motivates narrow-band full polarization measurements at low frequencies; the pay-off is the measurement of the architecture of the magnetic environment of the FRB, as well as a confirmation of the beautiful physics of FC in cold plasmas.

The discussion of this paper is incomplete; we have concentrated only on cold plasma. FC in relativistic plasma with nontrivial magnetic geometry will be addressed in future work.

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Appendix

Rotation–Conversion in a Cold Magnetized Plasma

Rotation–conversion in a cold magnetized plasma is simple because there are no emission and absorption. To calculate the effect, one proceeds along standard lines—calculate the permittivity and then the radiation transfer. A common reference is Ginzburg (1964). We do it below in full and in somewhat different terms.

A.1. Permittivity

Let \( \mathbf{B} \) be the background constant uniform magnetic field, \( \mathbf{E} \) is the electric field of the wave, and \( \omega \) is the frequency of the wave. A cold electron moves according to the Lorentz equation \((c = 1 \text{ here and below}): \)

\[ m\mathbf{v}' = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \]  \hfill (48)
or

\[ -i\omega \mathbf{v} = \frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \]  \hfill (49)
or

\[ \mathbf{v} = \frac{i}{m} \frac{\omega}{\omega^2 - \omega_p^2} (\mathbf{E} - i\omega_p \mathbf{B} \times \mathbf{E} - \frac{\omega_p^2}{\omega^2} \mathbf{B} \cdot \mathbf{E}), \]

\[ \omega_p = \frac{e\mathbf{B}}{\mathbf{B}} \equiv \frac{\omega}{\mathbf{B}}. \]  \hfill (50)

Moving electrons creates plasma current \( j = ne \mathbf{v} \), and ions contribute much less and are neglected. The dielectric permittivity tensor \( \epsilon \) is, by definition, given by \( 4\pi \mathbf{J} + \partial_t \mathbf{E} \equiv \partial_t (\epsilon \mathbf{E}) \) or \(-i\omega (\epsilon - 1) \mathbf{E} = 4\pi \mathbf{j} \), and we get

\[ \epsilon_{ij} = \epsilon_\perp \delta_{ij} + (\epsilon_\parallel - \epsilon_\perp) \mathbf{B}_i \mathbf{B}_j + i\epsilon_{g\ell} \mathbf{B}_i. \]  \hfill (51)

Here, \( \epsilon_{g\ell} \) is the antisymmetric unit tensor and

\[ \epsilon_\perp = 1 - \frac{\omega_p^2}{\omega^2 - \omega_p^2}, \quad \epsilon_\parallel = 1 - \frac{\omega_p^2}{\omega^2}, \]

\[ g = \frac{\omega_\perp^2}{(\omega^2 - \omega_p^2)\omega^2}, \quad \omega_\perp^2 \equiv \frac{4\pi ne^2}{m}. \]  \hfill (52)

As \( \omega \gg \omega_p \), \( \omega_p \) is assumed, we can replace it, to sufficient accuracy, as

\[ \epsilon_\perp \approx 1 - \frac{\omega_p^2}{\omega^2} + \frac{\omega_p^2}{\omega^2}, \quad g \approx -\frac{\omega_p^2}{\omega^2}. \]  \hfill (53)

A.2. Radiation Transfer

Maxwell equations give the eigenmode equation

\[ (k^2 - k_i k_j - \omega^2 \epsilon_{ij}) \mathbf{E}_j = 0, \]  \hfill (54)
or

\[ (k^2 - \omega^2 \epsilon_{ij}) \mathbf{E}_j = k_i k_j \mathbf{E}_j + \omega^2 (i\epsilon_{g\ell} \mathbf{B}_k + (\epsilon_\parallel - \epsilon_\perp) \mathbf{B}_i \mathbf{B}_j) \mathbf{E}_j, \]  \hfill (55)

where \( \mathbf{k} \) is the wavevector. Consider a wave propagating along \( \varphi, \mathbf{k} = (0, 0, k) \). To sufficient accuracy, one can neglect \( E_i \) in the first two equations of Equation (55) and also use \( \epsilon_{ij} \approx 1 - \frac{\omega_p^2}{\omega^2} \) in the left-hand side of Equation (55). Then

\[ (k^2 + \omega_\perp^2 - \omega^2) \mathbf{E}_a = -\omega^2 \left( \frac{\omega_\perp^2}{\omega^2} \mathbf{B}_a \epsilon_{ab} + \frac{\omega_p^2}{\omega^2} \mathbf{B}_a \mathbf{B}_b \right) \mathbf{E}_b, \]  \hfill (56)

where the indices \( a \) and \( b \) run from 1 to 2, and \( \epsilon_{ab} \) is the 2D antisymmetric unit tensor.

Replacement

\[ k = \sqrt{\omega^2 - \omega_p^2} - i\partial_\varphi \]  \hfill (57)
gives the polarization transfer equation for complex amplitudes \( \mathbf{E}_a^\omega \):

\[ \partial_\varphi \mathbf{E}_a = \frac{1}{2} \left( \frac{\omega_\perp^2}{\omega^2} \mathbf{B}_a \epsilon_{ab} - \frac{\omega_p^2}{\omega^2} \mathbf{B}_a \mathbf{B}_b \right) \mathbf{E}_b. \]  \hfill (58)
By defining Stokes parameters,

\[ E_aE_b^* = \frac{1}{2} \begin{pmatrix} I + Q & U + iV \\ U - V & I - Q \end{pmatrix} \]

and using Equation (58), we get Equation (6).

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