Structured exploration in the finite horizon linear quadratic dual control problem

Andrea Iannelli¹, Mohammad Khosravi¹ and Roy S. Smith¹

Abstract—This paper presents a novel approach to synthesize dual controllers for unknown linear time-invariant systems with the tasks of optimizing a quadratic cost while reducing the uncertainty. The work here builds on ideas from experiment design and robust control, and defines a synthesis problem where the feedback law has to simultaneously gain knowledge of the system and robustly optimize the given quadratic cost. Different from recent results, the problem is framed in a finite horizon setting, which provides more insights on the trade-offs arising when the tasks include both identification and control of an unknown plant. One of the main findings of the work is that efficient exploration strategies are achieved when the structure of the problem (e.g. properties of the system) is exploited.

I. INTRODUCTION

One of the most central problems in automatic control is the Linear Quadratic (LQ) regulator, whereby the goal is to design a control law which minimizes deviations of the states of a linear system from a given reference trajectory (e.g. the origin) while keeping as small as possible the necessary action. In the full state information case (standard LQR), when the dynamics is exactly known the problem has a well known optimal solution (see [1]). In the infinite horizon case, that is when transient features are negligible and so the problem can be approximated as an infinitely long time window, this consists of a static feedback law obtained through the solution of an Algebraic Riccati Equation (ARE). The main advantage is that by framing the problem in a finite horizon setting, which provides more insights on the trade-offs arising when the tasks include both identification and control of an unknown plant. One of the main findings of the work is that efficient exploration strategies are achieved when the structure of the problem (e.g. properties of the system) is exploited.

Despite important control theoretic works on robust $H_2$ analysis and filtering problems [2], [3], the solution of the LQ control problem when the dynamics is unknown is far less understood and constitutes an active area of research. Notably, this problem has been used in the last few years as principal case study to show possible complementarities of Reinforcement Learning (RL) and control theory-based approaches for the fundamental problem of optimally manipulating an unknown system by using the information carried in collected data [4]. Bridging these two communities has been the effort of many recent works, see e.g. [5], [6], [7], but despite the variety of techniques considered, no strategies which allow for an easy implementation on one hand, and provide optimal cost guarantees on the other, have been found [4]. Moreover, a fundamental unsolved problem is what is the best strategy to extract information about the system such that the performance can be improved while preserving at the same time safety. In other words, borrowing terminology from the reinforcement learning community (e.g. multi-armed bandits, [8]), specifying an optimal policy (control law) that robustly balances between exploration (acquiring knowledge of the system by testing and identification) and exploitation (operating the system to maximize the reward, or performance).

The approach considered here owes to the long and rich tradition of dual control from the 60s (see e.g. [9]), where the problem of simultaneously identifying and controlling a system was first formalized. An important research topic, also related to this work, that emerged from those early studies is experiment design, whereby one attempts to determine the most suitable inputs in order to extract information from the unknown plant (see [10] for a seminal paper and the recent survey in [11]). The material presented here is also inspired by a recent publication [12] where the unknown LQ problem is framed in a dual control setting. Specifically, given an initial estimate of the dynamics in the form of nominal state matrices and an ellipsoidal uncertainty set, the joint optimization of two robust feedback laws $G_{K1}$ and $G_{K2}$ is proposed therein (considering two distinct infinite horizon problems). The policy $G_{K1}$ is responsible for reducing the uncertainty such that when $G_{K2}$ is designed (based on the new uncertainty set) the cost is minimized.

While retaining the same application-oriented philosophy, basically consisting of promoting a reduction of the uncertainty which is beneficial for the purpose of minimizing the cost, the work here substantially changes the synthesis approach by framing the problem in a finite horizon setting. This is motivated by the fact that the dual control problem is more realistically described in a finite time window, due to the importance of transient features. The design of two robust static feedback laws tasked with different goals is thus shifted to the design of a single time-varying law $G_K$, which is responsible for dealing simultaneously with the two tasks. From an optimal control perspective, but in an uncertain setting now, the problem is formulated as the solution of an RDE rather than an ARE. The main advantage is that by framing the problem in a finite horizon setting the different trade-offs between exploration and exploitation are better captured. New insights into these trade-offs are in turn believed to help gaining a deeper understanding of the

¹ The authors are with the Department of Information Technology and Electrical Engineering, Automatic Control Lab, ETH, Zürich 8092, Switzerland iannelli/khosravm/rsmith@control.ee.ethz.ch
unknown LQ problem.

The main technical contribution of the paper is the formulation of a Semidefinite program (SDP) to solve the robust dual LQR control problem in the finite horizon setting optimally balancing exploration and exploitation. This is presented in Section III where also the corresponding programs for the nominal and robust (but with fixed uncertainty, i.e. without exploration) problems are derived. The other important contribution is gathered in Section IV, where features of the synthesised policies are shown through numerical examples. Crucially, the application-oriented nature of the exploration, here termed for this reason structured in order to distinguish it from standard approaches where it is essentially random, is emphasised and original related aspects are shown.

II. PROBLEM DESCRIPTION

The notation is standard. Given a finite sequence \( \{x_t\}_{t=1}^N \), \( x \) denotes the stacked vector gathering its terms. In a symmetric matrix \( M \in \mathbb{S}^n \), \( (\cdot) \) is used to concisely denote the part below the main diagonal, while \( M \geq 0 \) indicates positive semidefiniteness.

A. Background

Consider the discrete linear time-invariant system:

\[
x_{t+1} = Ax_t + Bu_t + w_t, \quad w_t \sim \mathcal{N}(0, \sigma^2_{w}I_{n_w}), \quad x_0 = 0
\]

where \( x_t \in \mathbb{R}^{n_x} \) is the (measured) state, \( u_t \in \mathbb{R}^{n_u} \) is the control input, and \( w_t \in \mathbb{R}^{n_w} \) is the normally distributed process noise with zero mean and covariance \( \sigma^2_{w}I_{n_w} \). Given cost matrices \( Q \geq 0 \) and \( R \geq 0 \), the objective is to design a feedback law minimizing the expected finite horizon quadratic cost \( J \) in \([1, T]\) (with \( 1 < T < \infty \)):

\[
J = \mathbb{E} \left[ \sum_{t=1}^{T} \left( x_t^\top Q x_t + u_t^\top R u_t + x_t^\top Q x_t \right) \right]
\]

where the expectation is with respect to \( w \). When \( A \) and \( B \) are known, the optimal input is given by the time-varying state-feedback law \( u_t = K_t^{DRD} x_t \), where \( K_t^{DRD} \) is associated with the stabilizing solution of the discrete time Riccati difference equation (DRDE) for \( \mathbb{I} \).

The case of unknown \( A \) and \( B \), where the only access to information on \( \mathbb{I} \) is through measurements of \( x \) and \( u \), is considered here. Similarly to [12], an estimation of the unknown dynamics is obtained through the so-called Coarse-ID approach [7]. Given a dataset made of \( N \) samples \( S = \{ (x_t, u_t) : 1 \leq t \leq N \} \), the nominal dynamics is estimated through least squares problem:

\[
(\hat{A}, \hat{B}) = \arg \min_{A,B} \sum_{t=1}^{N-1} \| x_{t+1} - Ax_t + Bu_t \|_2^2
\]

and the true dynamics is assumed to belong to the ellipsoidal set \( \Omega(X,D) \):

\[
\Omega(X,D) = \{ X : X^\top DX \preceq I \}, \quad D \in \mathbb{S}^{n_x+n_u}, \quad X = \left( \begin{array}{c} \hat{A} - A \\ \hat{B} - B \end{array} \right), \quad X \in \mathbb{R}^{(n_x+n_u) \times n_x}.
\]
minimizing $\mathbf{H}_2$. Since the dynamics are not known, a simple strategy consists of choosing an intermediate time $T_{sw}$, and dividing the horizon into two phases. In the first (exploration, or $ID$-phase), the system is excited with random input $u_t \sim \mathcal{N}(0, \sigma_u^2 I)$ and the measured response (e.g. in the form of $S$) is used to identify the nominal matrices through $\mathbf{H}_2$. In the second (exploitation, or $K$-phase), a controller which optimizes $\mathbf{H}_2$ for the identified nominal matrices is synthesised. One possible option is the use of time-varying feedback $K_t^{\text{DRDE}}$ on the remaining horizon $[T_{sw}, T]$. That is, a pure exploration phase is followed by a pure exploitation phase. This is sometimes referred to as $\epsilon$-first strategy in the RL literature. This naturally leads to a trade-off between the duration of these two phases: while prolonging the first phase has the benefit of more accurately estimating the model, it also means that the system is optimally controlled for a shorter time. Conversely, stopping the $ID$-phase too early has the disadvantage that the controller might be designed with respect to a plant which is far from the true one.

In order to exemplify this aspect, an experiment is performed on a horizon of length $T=100$ with two randomly generated stable plants having $n_x=3$, $n_u=2$, and using $\sigma_u=0.3$, $\sigma_w=0.5$. Figure 1 shows the total expected cost $J_{\text{tot}}$ (obtained by averaging over 100 realizations of noise and random excitation) as a function of the switching time $T_{sw} < T$ when the $ID$-phase is stopped and the $K$-phase is started. $J_{ID}$ is the expected cost associated with the identification part when this is stopped at $T_{sw}$, while $J_K$ is the expected cost associated with the deployment of the controller in the horizon $[T_{sw}, T]$.

![Fig. 1. Expected costs for the two unknown plants as a function of the switching time $T_{sw}$.](image)

It can be observed that there exists an optimal switching time where the benefit of further exploring the unknown dynamics is overcome by the cost of exploration, and thus it is more advantageous to operate the system in closed-loop. This trade-off depends on the unknown true system. Moreover, it is apparent that it can only be captured in a finite horizon setting, where distinctive transient features of the $ID$-phase and $K$-phase are retained.

While the identification and control strategy considered in this section captures fundamental conflicting aspects arising in the dual control problem, it has important limitations, among which are robustness and optimality. Note indeed that the design of $K_t^{\text{DRDE}}$ is nominal, and thus it does not take into account any error in the least-square estimation process. Moreover, the interplay between identification and control is not exploited since exploration and exploitation are sequentially applied, to the detriment of performance. The next section addresses these fundamental aspects of the dual control problem by proposing a novel synthesis strategy.

III. Semidefinite Programs Formulations

The goal of this Section is to derive a convex formulation for synthesising a feedback law $G_K$ which optimizes $J_{\text{WC}}$. In order to clearly present the steps involved and highlight their meaning, the presentation is broken down in 3 parts. Section III-A deals with the nominal case (i.e. the estimated coincides with the true system), which thus is conceptually equivalent to solving the associated RDE. Section III-B instead considers a worst-case design where the set of uncertainty is fixed throughout the horizon, which thus is a robust version of the RDE. Finally, Section III-C establishes the dual control formulation, where exploration is promoted and thus the uncertainty of the system can be reduced while robustly controlling the plant.

The key idea is to use the application-oriented approach first introduced in [12], whereby the exploration is optimized such that the reduction in the uncertainty is driven by the performance objective. The differences with respect to [12] include the formulation of the problem in the finite horizon (whose importance was motivated earlier, and will be also emphasized in Section IV), and the synthesis of a single (time-varying) policy responsible for simultaneously exploring and controlling.

Exploration is used only to update the uncertainty matrix $D$ in (5), while the nominal matrices $\hat{A}$ and $\hat{B}$ are kept fixed. This is in line with related works [12], [13] that, in order to preserve convexity, consider a fixed nominal system while optimizing exploration.

The first step consists of deriving an expression for the cost $J$ which can be used for the robust optimization problem in (6). Let us begin by denoting by $P_t$ the covariance matrix of the state at timestep $t$:

$$P_t = \mathbb{E} [x_t x_t^\top] \in \mathbb{S}^{n_x}$$

Define also $Q_t := \begin{bmatrix} Q_t^\top & 0 \\ 0 & K_t \end{bmatrix} \in \mathbb{R}^{(n_x+n_u)\times n_x}$, $\bar{R}_t := \begin{bmatrix} 0 \\ R_t^\top \end{bmatrix} \in \mathbb{R}^{(n_x+n_u)\times n_x}$. Then the following result, proved in the Appendix, holds.

**Lemma 1:** The cost $J$ in (2) is equivalent to:

$$J = \text{tr} \left( \sum_{t=1}^{T-1} (\bar{Q}_t P_{t-1} \bar{Q}_t^\top + \bar{R}_t S_t \bar{R}_t^\top) + Q P_T Q^\top \right)$$

(9)

The key benefit of Lemma 1 is that it allows the finite horizon cost to be rewritten as a function of the timestep covariance $P_t$, rather than the horizon covariance $\mathcal{W} \in \mathbb{S}^{(T-1)n_x}$ (see the Appendix for more details). This gives an expression which has formal similarities to the infinite horizon case, whose minimization in turn is equivalent to the computation of the $\mathcal{H}_2$ norm of $[\bar{Q}_t]$.
A. Nominal design

The nominal case, which consists of problem (6) without inner maximization since $A = A$ and $B = B$ are known, is considered first. The solution, as with the SDP-based computation of the $H_2$ norm, can be obtained by minimizing (9) while constraining the covariance $P_t$ to satisfy the discrete time Lyapunov inequalities associated with the closed loop $P_t$ while constraining the covariance $S$ on the state (that is, (10c) comes from the assumed zero initial condition $\bar{Z}_t$). Schur complement as:

$$
\begin{align*}
&\min_{G_k} \text{tr} \left( \sum_{t=1}^{T-1} (\bar{Q}_t P_t \bar{Q}_t^T + \bar{R}_t S_t \bar{R}_t^T) + Q P_T Q^T \right) \tag{10a} \\
&P_{t+1} \succeq (A + BK_t) P_t (A + BK_t)^T + \sigma_w^2 I + BS_t B^T \\
&\forall t \in [1, T-1] \\
&P_t \succeq \sigma_w^2 I, \tag{10c}
\end{align*}
$$

where (10a) comes from the assumed zero initial condition on the state (that is, $P_0 = 0$). For generality, and to make more clear the changes among the 3 proposed formulations, the problem is formulated with the generic policy $G_k(K_t, S_t)$ (7). However, as intuitive and confirmed later by the results, the random excitation part $S_t$ will be zero in this case.

The program in (10) is convex and can be recast as an SDP with well known Linear Matrix Inequalities (LMI) manipulations [16]. First, (10a) is upper bounded by replacing the argument of the summation in (10a) with $Y_t \in \mathbb{S}^{n_x+n_u}$, and the new objective function (10a) is written by using Schur complements as:

$$
\begin{align*}
&\min_{G_k} \text{tr} \left( \sum_{t=1}^{T-1} Y_t + Q P_T Q^T \right) \\
&\left[ Y_t - \bar{R} S_t \bar{R}_t^T \bar{Q}_t P_t \right] \succeq 0, \quad \forall t \in [1, T-1] \tag{11}
\end{align*}
$$

Note that $\bar{Q}_t$ and $P_t$ give rise to bilinear terms, thus the auxiliary variable $Z_t = P_t K_t^T$ is defined. As for the inequalities (10b), they can be recast as coupled LMI’s by application of Schur complement.

Program 1: Nominal design

$$
\begin{align*}
J = \min_{Y_t, P_t, Z_t, S_t} \text{tr} \left( \sum_{t=1}^{T-1} Y_t + Q P_T Q^T \right) \tag{12a} \\
&\left[ Y_t - \bar{R} S_t \bar{R}_t^T \begin{bmatrix} \bar{Q}_t P_t \\ \bar{R}_t Z_t \end{bmatrix} \right] \succeq 0, \tag{12b} \\
&\begin{bmatrix} P_t \\ \begin{bmatrix} P_t \\ P_{t+1} - \sigma_w^2 I - BS_t B^T \end{bmatrix} \end{bmatrix} \succeq 0 \tag{12c} \\
&Y_t \succeq 0, S_t \succeq 0, P_{t+1}, \forall t \in [1, T-1] \succeq 0 \\
&P_t \succeq \sigma_w^2 I, \tag{12d}
\end{align*}
$$

where $F_t := P_t A^T + Z_t B^T$. Solving Program 1 is conceptually equivalent to solving the DRDE and indeed, as shown in Section IV, leads to identical results. The advantage of this formulation is that it allows robustness constraints to be enforced and the effect of exploration to included, as shown in the following sections.

B. Robust control design

In the unknown dynamics case, the only knowledge is that $(A, B) \in \Omega(X, D_0)$, where $\dot{A}$, $\dot{B}$, and $D_0$ are assumed to be obtained from prior experiments. This compels us to modify the LMI’s such that they are guaranteed to hold for all possible $(A, B)$ inside the set. To this end, $(A, B)$ are written as a function of $X$, $\dot{A}$, and $\dot{B}$ by using definition (15b) and a Schur complement is applied to overcome the nonlinearity arising from $BS_t B^T$. Then, since $X$ has to lie inside an ellipsoidal set (16), an extension of the S-lemma to the matrix case, proposed in [17], is employed to solve the robust optimization problem.

Program 2: Robust control design

$$
\begin{align*}
J_{WC} = \min_{Y_t, P_t, Z_t, S_t, p_t} \text{tr} \left( \sum_{t=1}^{T-1} Y_t + Q P_T Q^T \right) \tag{13a} \\
&\left[ Y_t - \bar{R} S_t \bar{R}_t^T \begin{bmatrix} \bar{Q}_t P_t \\ \bar{R}_t Z_t \end{bmatrix} \right] \succeq 0, \tag{13b} \\
&\begin{bmatrix} P_t \\ \begin{bmatrix} P_t \\ P_{t+1} - \sigma_w^2 I - BS_t B^T \end{bmatrix} \end{bmatrix} \succeq 0 \tag{13c} \\
&Y_t \succeq 0, S_t \succeq 0, P_{t+1}, p_t \geq 0, \forall t \in [1, T-1] \succeq 0 \\
&P_t \succeq \sigma_w^2 I
\end{align*}
$$

where $G_t := -\begin{bmatrix} P_t \\ 0 \end{bmatrix} Z_t$, $H_t := G_t A^T B^T$, and $p_t$ is a multiplier from the S-lemma. The crucial feature of Program 2 is that the ellipsoid $\Omega(X, D_0)$ defining the uncertainty is fixed throughout the horizon. The consequence of this is that exploration is not encouraged, since it has an associated cost for which is not rewarded. In other words, the generation of control inputs with a different goal than just minimizing the performance objective will inevitably incur in a higher cost (or regret). This argument has a clear interpretation for $S_t$, where the LMI’s (15b)-(15c) show that a non-zero $S_t$ always determine an additional contribution to the cost via $Y_t$. In fact, the random excitation part $S_t$ will be zero here (as it was commented earlier for the nominal case). As for $K_t$, this will correspond to the stabilizing solution of the DRDE formed by taking at each time-step the worst-case matrices (which are in principle time-varying), i.e. it is the robust optimal policy.

C. Robust dual control design

In order to promote exploration, it is necessary to describe how the feedback law contributes to obtain knowledge of the system. More formally, a mapping between $G_k(K_t, S_t)$ and the uncertainty $D_t$ at a given timestep $t$, has to be formulated. By recalling its definition in (2), the following perturbation bound is proposed:

$$
D_t = \frac{1}{c_2^2} \sum_{l=1}^{T} \left[ P_l \begin{bmatrix} p_l^2 Z_l \\ P_l^{-1} Z_l \end{bmatrix} + S_l \right] \tag{14}
$$
Note the explicit influence of $S_t$ and $K_t$ (via $Z_t = P_tK_t^T$) on $D_t$, with the policy also having an indirect effect on $P_t$.

Due to the nonlinearity involving $Z_t$ and $P_t$ in the lower diagonal block, (14) cannot be readily used and thus a convex relaxation is sought. To this end, the matrix inequality in ([12], Lemma 1) is employed here to formulate, for a given matrix $\hat{D} \in \mathbb{R}^{n_u \times n_x}$, the following lower bound on $D_t$:

$$D_t \succeq \hat{D}_t = \frac{1}{c} \text{tr} \sum_{l=1}^{t} \left[ \begin{array}{c} P_l \\ Z_l \\ Z_l^T K_l^T + K_l P_l K_l^T + S_l \end{array} \right]$$

The bound is tight when $\hat{K} = K_t$. In this work $\hat{K}$ is chosen as the nominal controller, $K^{\text{Nom}}$, resulting from the optimization in Program 1.

The following dual control design problem is then proposed. It is basically obtained from Program 2 by adding to the fixed set $D_0$ in (13a), the (policy-dependent and time-varying) upper bound on the true uncertainty $\hat{D}_t$ (15).

Program 3: Robust dual control design

$$J_{\text{WC}} = \min \left\{ Y_t, P_t, Z_t, S_t, P_t \right\} \text{tr} \left( \sum_{t=1}^{T-1} Y_t + QP_tQ^T \right)$$

$$Y_t \succeq 0, S_t \succeq 0, P_t \geq 0, \forall t \in [1, T] \succeq 0$$

$$P_t \geq \sigma^2_w I$$

Program 3 clearly shows that the policy $G_K(K_t, S_t)$ can now perform application-oriented exploration. The key enabler of exploration is, with respect to (13a), the addition of the matrix $\hat{D}_t$ in (16c). Specifically, the feedback law is optimized so that, by obtaining a certain structure for $\hat{D}_t$, the worst-case matrices $A$ and $B$ are eliminated from the allowed set, to the benefit of the feasibility of the LMIs (16c) and in turn of the achievable cost. Exploration itself, however, has a cost and thus trade-offs will arise. The cost associated with $S_t$ is seen directly in (16b), while that related to $K_t$ can be interpreted as due to the deviation of $K_t$ from the robust optimal policy. The first trade-off is on which part of the policy $G_K(K_t, S_t)$ should be used for exploration, whether the state-feedback, the random excitation or both. Another trade-off is on which portion of the horizon exploration should be pursued (reminiscent of the scenario in Figure 1). It is important in this regard to note that a conceptually similar (convex) formulation for the mapping (15) between the policy and the uncertainty $\hat{D}_t$ was proposed in ([12]). However, while here the cost to pay to keep adding contributions to $\hat{D}_t$ is well captured in Program 3 it is not clear how this can be accounted for in an infinite horizon setting, where $J$ is effectively an averaged cost and thus does not depend on how many terms (i.e. samples) are featured in the summation leading to $\hat{D}_t$.

Finally, the bilinearity between $\hat{D}_t$ and $\hat{D}_t$ can be overcome by using a line search on $p_t$.

IV. RESULTS

Application of the proposed design framework is presented here. Numerical simulations are performed using YALMIP [18] in conjunction with the SDP solver SDPT3 [19].

A. Nominal and robust control designs

Considering the following true system:

$$A = \begin{bmatrix} 0.18 & 0.1 & 0 \\ 0 & 0.18 & 0.04 \\ 0 & -0.04 & 0.16 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0.6 \\ 0 & 0.6 \end{bmatrix}$$

with cost matrices $Q = I_n$ and $R = \text{blkdiag}(10, 1)$, and $\sigma_w^2 = 0.5$. First, an initial estimate of the dynamics in terms of $(A, B, D_0)$ is computed. Since the true system is known, the set membership condition (4a), which for given uncertainty matrix $D$ and nominal matrices boils down to a positive definiteness test, can be checked. The outcome of extensive tests is that the less conservative coefficient $c_1$ always provided an error estimate including the true system (for both roll-out and sampling strategies), hence it is used in the remainder for $D$ (5). Note that, for this problem, $c_2 \sigma_1 \approx 3$. It is also observed that, since the coefficient $c$ does not depend on the measurements and is simply a scaling of the ellipsoid, its choice will not influence the qualitative features of the exploration. The latter indeed will try to decrease the amount of uncertainty along the important directions by leveraging the parts of $D$ which depend on measurable quantities (i.e. the summation terms of Eq. 5).

For subsequent analyses, an estimate of (17) is obtained through 100 simulated roll outs, each of length $T_r = 5$. Figure 2 shows the time-varying gains of the feedback matrix $K_t$ optimized on the horizon $[1, 100]$ using different design schemes: $K^{\text{DRDE}}$ by solving the DRDE associated with $(A, B)$ (3); $K^{\text{Nom}}$ by solving Program 1 for $(A, B)$; $K^{\text{Rob}}$ by solving Program 2 for $(A, B, D_0)$.

![Fig. 2. Optimal controllers for the nominal and robust (fixed uncertainty) problem.](image-url)
was obtained from the known closed form solution \(J_{\text{DRDE}} = \sum_{t=1}^{T} \mathbb{E} \left( w_t^T X_{t+1} w_t \right)\) (where \(X_t\) is the stabilizing solution of the DRDE, \([1]\)), while the latter was directly provided by Program 3. As for the robust design, which achieved \(J_{\text{Rob}} \cong 240\), it is interesting the observe that \(K_{\text{Rob}}\) is generally far from the optimal controller for the nominal plant because of the constraint to guarantee robustness. This is the well known trade-off between performance and robustness, and it results in a controller which is optimized for the worst-case plant in the set. Finally, since the synthesis schemes used for the policies in Figure 2 do not take into account exploration, the optimizer always returns \(S_t = 0\) (not plotted here).

B. Structured exploration

Starting from the same initial estimate \((\hat{A}, \hat{B}, D_0)\) used to generate Figure 2, a new policy is designed using Program 3. The results are shown in Figure 3 by comparing the state-feedback dual controller \(K_{\text{Dc}}\) with the nominal \(K_{\text{Nom}}\) (already shown in Figure 2), and also by reporting the covariance \(S_t\) of the excitation input. Moreover, a plot with the timestep cost \(J_t^{\text{tot}}\), together with its two contributions \(J_t^{\text{tot}} = \mathbb{E} \left[ x_t^T (Q + K_t^T R K_t) x_t \right]\) and \(J_t = \mathbb{E} \left[ e_t^T R e_t \right]\), is shown in the bottom right plot.

There is a clear exploration action taking place in the first part of the finite horizon, essentially performed only by the state-feedback \(K_t\), while the covariance \(S_t\) is practically zero (note that the order of magnitude in the plot is \(10^{-9}\)). This can be also appreciated from the plot showing the cost added at each time step. While \(J_t^{\text{tot}} = 0\), it can be seen that \(J_t^{\text{tot}} = J_t^*\) has an initially increasing and later decreasing trend, before achieving a constant value. Indeed, since the cost would increase linearly in the optimal finite horizon problem, this can be read as a qualitative indication that, after approximately 20 timesteps, \(K_t\) has stopped exploring and is only devoted to (robust) exploitation. Note also that \(J_{\text{Dc}} \cong 180 < J_{\text{Rob}}\).

As in Figure 3, an exploration action can be detected in the first part of the finite horizon, however this time exploration is performed by both the state-feedback \(K_t\) and the covariance \(S_t\). This can be clearly seen from the plots, and in particular from the subplot with the costs, where it is also evident that the exploration performed by \(K_t\) lasts longer than that performed by \(S_t\). Two types of trade-offs arising in the dual control problem can be appreciated by looking at Figs. 3 and 4. The first is the one between exploration and exploitation, which is captured by the fact that the former only lasts for a certain fraction of the total mission. It is interesting to observe that, as already qualitatively seen in Figure 1, the extent to which exploration is carried out strongly depends on the type of plant to control. The second trade-off concerns the choice between \(K_t\) and \(S_t\) for the purpose of exploration. It is indeed not surprising the result obtained

The robust design strategy is not successful in this case. Indeed, when the number of roll-outs to determine the initial estimate \((\hat{A}, \hat{B}, D_0)\) is small (order of 100), the solver cannot find a feasible solution to Program 2. To interpret this, note that the system has now all its eigenvalues very close to the unit disk, and that the least damped mode is close to become uncontrollable. It follows that, depending on the tightness of the estimation error, the uncertainty set can contain unstable and uncontrollable plants. In this scenario, guaranteeing robustness can come with a high (worst-case) cost. Even though theoretically bounded because of the finite horizon, \(J_{\text{WC}}\) can become very high, and thus the solver might not able to find a feasible solution. The standard remedy would be to increase the number of samples used to compute an initial estimate, by either testing more roll-outs or sampling a longer trajectory. However, this has two important drawbacks. The first is the increased cost to pay to obtain the estimate. The second is that this blind exploration is inherently unsafe, because it is an open loop process with random inputs. Therefore, it would be desirable to keep it as short as possible. It is thus expected that the dual control problem proposed here can represent an efficient solution to safely and optimally control these type of plants.

Given an initial estimate \(D_0\), for which the robust control problem was unfeasible, a dual control policy is computed with Program 3 and the results are shown in Figure 4.

As in Figure 3, an exploration action can be detected in the first part of the finite horizon, however this time exploration is performed by both the state-feedback \(K_t\) and the covariance \(S_t\). This can be clearly seen from the plots, and in particular from the subplot with the costs, where it is also evident that the exploration performed by \(K_t\) lasts longer than that performed by \(S_t\). Two types of trade-offs arising in the dual control problem can be appreciated by looking at Figs. 3 and 4. The first is the one between exploration and exploitation, which is captured by the fact that the former only lasts for a certain fraction of the total mission. It is interesting to observe that, as already qualitatively seen in Figure 1, the extent to which exploration is carried out strongly depends on the type of plant to control. The second trade-off concerns the choice between \(K_t\) and \(S_t\) for the purpose of exploration. It is indeed not surprising the result obtained
here that whenever it is possible to explore in a controlled manner (i.e. without resorting to random excitation), this is the preferred way. This fact is captured by the LMI (16b) in Program 3 where it is apparent that there is always a cost associated with $S_t$. The best exploration strategy inevitably depends on the true (unknown) plant to control, for example its controllability and margin of stability. Thus, finding the optimal exploration is an unsolved problem (which bears similarities to a similarly circular problem in experiment design, [11]). This section however shows that the proposed design framework determines solutions which balance the different, and often conflicting, aspects concerning exploration and exploitation. In particular, it emerges a stark contrast with common approaches in RL where a randomized exploration is employed [8]. The results also corroborate the claim prompting this work that this type of dual control problem should be studied in a finite horizon setting.

Figs. 3-4 show that, unlike the nominal case where there is no sensible variation of the optimized controller within the horizon (except for the very last timesteps, recall Fig. 2), the presence of uncertainties on the one hand and the dual tasks on the other make the most important features of the problem (e.g. $K_t$, $S_t$, timestep costs) inherently time-varying.

V. Conclusions

The paper proposes a strategy to solve the finite horizon unknown Linear Quadratic problem with a dual control approach. A control law is designed with the twofold objective of minimizing the worst-case quadratic cost in the face of an ellipsoidal uncertainty set while reducing the uncertainty set based on the system response. This is achieved by formulating an exploration action which is application-oriented, since the effect of the policy on the ellipsoidal set is captured in the optimization problem, and safe, since the designed controller is robust. SDP programs to solve the nominal, robust (with fixed uncertainty) and dual control problems are proposed, and their application is shown. The resulting exploration encompasses different types of trade-offs and shows how the optimal actions depend on the features of the true plant. The finite horizon setting is the most appropriate for studying the dual control problem, since it enables the aforementioned trade-offs to be captured and the cost of exploring and exploiting to be cast in a more application relevant framework. Future steps will focus on alternative mappings between the control law and the uncertainty update which allow the effect of exploration on the nominal matrices to be included.

References

[1] D. P. Bertsekas, Dynamic Programming and Optimal Control, 3rd ed. Belmont, MA, USA: Athena Scientific, 2005, vol. I.
[2] M. Sznaier, T. Amishima, P. Parrilo, and J. Tienro, “A convex approach to robust $H_2$ performance analysis,” Automatica, vol. 38, pp. 957–966, 2002.
[3] K. Sun and A. Packard, “Robust $H_2$ and $H_\infty$ Filters for Uncertain LFT Systems,” IEEE Transactions on Automatic Control, vol. 50, pp. 715–720, 2005.
[4] B. Recht, “A tour of reinforcement learning: The view from continuous control,” Annual Review of Control, Robotics, and Autonomous Systems, vol. 2, no. 1, pp. 253–279, 2019.
[5] M. Fazel, R. Ge, S. Kakade, and M. Mesbahi, “Global convergence of policy gradient methods for the linear quadratic regulator,” in 35th International Conference on Machine Learning, 2018.
[6] A. Cohen, A. Hassidim, T. Koren, N. Lazic, Y. Mansour, and K. Talwar, “Online linear quadratic control,” in 35th International Conference on Machine Learning, 2018.
[7] S. Dean, H. Mania, N. Matni, B. Recht, and S. Tu, “On the sample complexity of the linear quadratic regulator,” Foundations of Computational Mathematics, Aug 2019.
[8] R. S. Sutton and A. G. Barto, Reinforcement Learning: An Introduction, 2nd ed. The MIT Press, 2018.
[9] K. Åström and B. Wittenmark, “Problems of identification and control,” J. Math. Anal. Appl., vol. 34, no. 1, pp. 90–113, 1971.
[10] M. Ferizbegovic, J. Umenberger, H. Hjalmarsson, and T. B. Schön, “Learning Robust $LQ$-Controllers Using Application Oriented Exploration,” IEEE Control Systems Letters, vol. 4, no. 1, pp. 19–24, Jan 2020.
[11] J. Umenberger, M. Ferizbegovic, T. B. Schön, and H. Hjalmarsson, “Robust exploration in linear quadratic reinforcement learning,” in (submitted to) 36th International Conference on Machine Learning, 2019.
[12] P. Auer, N. Cesa-Bianchi, and P. Fischer, “Finite-time analysis of the multiarmed bandit problem,” Machine Learning, vol. 47, no. 2, pp. 235–256, 2002.
[13] M. Lobo and S. Boyd, “Policies for simultaneous estimation and optimization,” in IEEE American Control Conference, 1999.
[14] C. Scherer and S. Weiland, Linear Matrix Inequalities in Control, Lecture Notes, Dutch Institute for Systems and Control, 2000.
[15] Z. Luo, J. Sturm, and S. Zhang, “Multivariate nonnegative quadratic mappings,” SIAM Journal on Optimization, vol. 14, no. 4, pp. 1140–1162, 2004.
[16] J. Löfberg, “YALMIP : A Toolbox for Modelizing and Optimization in MATLAB,” in IEEE International Symposium on Computer-Aided Control System Design, 2004.
[17] R. Tutuncu, K. Toh, and M. Todd, “Solving semidefinite-quadratic-linear programs using SDPT3,” Mathematical Programming, vol. 95, 2003.
**APPENDIX**

Lemma 2: The cost $J$ in (2) is equivalent to:

$$J = \text{tr} \left( \sum_{t=1}^{T-1} (\bar{Q}_t P_t \bar{Q}_t^\top + \bar{R}_t S_t \bar{R}_t^\top) + Q P_T Q^\top \right)$$  \hspace{1cm} (19)

Proof: Recall from Section III the definitions: $P_t = \mathbb{E}[x_t x_t^\top]$, $\bar{Q}_t := \begin{bmatrix} Q_t^2 \\ R_t^2 \end{bmatrix}$, $\bar{R}_t := \begin{bmatrix} 0 \\ R_t^2 \end{bmatrix}$. Define $M_t = Q + K_t^\top R K_t$. By virtue of the chosen policy (7), $J$ can be rewritten as:

$$J = \mathbb{E} \left[ \sum_{t=1}^{T-1} (x_t^\top M_t x_t + e_t^\top R e_t) + x_T^\top Q x_T \right]$$  \hspace{1cm} (20)

Consider the first term in the summation (i.e. the one that depends on $x_t$). Simple matrix manipulations give it as:

$$\mathbb{E} \left[ \sum_{t=1}^{T-1} x_t^\top M_t x_t \right] = \mathbb{E} \left[ x^\top (I_{T-1} \otimes M_t) x \right]$$

$$= \mathbb{E} \left[ \text{tr} \left( x x^\top I_{T-1} \otimes M_t \right) \right] = \text{tr} \left( \mathbb{E} \left[ x x^\top \right] I_{T-1} \otimes M_t \right)$$  \hspace{1cm} (21)

where $\mathcal{W} = \mathbb{E} \left[ x x^\top \right] \in \mathbb{S}^{(T-1)n_x}$ denotes the covariance matrix of the state over the horizon and $\otimes$ is the Kronecker product. It follows that:

$$\text{tr} \left( \mathcal{W} I_{T-1} \otimes M_t \right) = \text{tr} \left( \left( I_{T-1} \otimes \bar{Q}_t \right) \mathcal{W} \left( I_{T-1} \otimes \bar{Q}_t \right)^\top \right)$$  \hspace{1cm} (22)

Due to the block diagonal structure of $\left( I_{T-1} \otimes \bar{Q}_t \right)$, it follows that:

$$\text{tr} \left( \left( I_{T-1} \otimes \bar{Q}_t \right) \mathcal{W} \left( I_{T-1} \otimes \bar{Q}_t \right)^\top \right) = \text{tr} \left( \sum_{t=1}^{T-1} (\bar{Q}_t P_t \bar{Q}_t^\top) \right)$$  \hspace{1cm} (23)

The contribution to the cost only depends on the diagonal terms of $\mathcal{W}$, which are the covariance matrices $P_t$ at the various timesteps:

$$\mathbb{E} \left[ \sum_{t=1}^{T-1} x_t^\top M_t x_t \right] = \text{tr} \left( \sum_{t=1}^{T-1} (\bar{Q}_t P_t \bar{Q}_t^\top) \right)$$  \hspace{1cm} (24)

The contribution of the state to the cost at $t = T$ directly follows from (24) specializing it to the case when $u$ is not penalized and thus $\bar{Q}_T \equiv Q$. Therefore:

$$\mathbb{E} \left[ Q P_T Q^\top \right] = \text{tr} \left( Q P_T Q^\top \right)$$  \hspace{1cm} (25)

Finally, the proof for the term depending on $e_t$ in the cost (20) follows along the same lines. This is further simplified by the fact that $e_t$ are uncorrelated for different times, and thus their covariance matrix in the horizon has the block diagonal structure $\left( I_{T-1} \otimes S_t \right)$. \qed