CONGRUENCES BETWEEN HILBERT MODULAR FORMS:
CONSTRUCTING ORDINARY LIFTS, II

THOMAS BARNET-LAMB, TOBY GEE AND DAVID GERAGHTY

Abstract. In this paper, we improve on the results of our earlier paper [BLGG12], proving a near-optimal theorem on the existence of ordinary lifts of a mod $l$ Hilbert modular form for any odd prime $l$.

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1. Introduction

Let $F$ be a totally real field with absolute Galois group $G_F$, and let $l$ be an odd prime number. In our earlier paper [BLGG12], we proved a general result on the existence of ordinary modular lifts of a given modular representation $\rho : G_F \to \text{GL}_2(F_l)$; we refer the reader to the introduction of op. cit. for a detailed discussion of the problem of constructing such a lift, and of our techniques for doing so.

The purpose of this paper is to improve on the hypotheses imposed on $\rho$, removing some awkward assumptions on its image; in particular, if $l = 3$ then the results of [BLGG12] were limited to some cases where $\rho$ was induced from a quadratic character, whereas our main theorem is the following.

Theorem A. Suppose that $l > 2$ is prime, that $F$ is a totally real field, and that $\rho : G_F \to \text{GL}_2(F_l)$ is irreducible and modular. Assume that $\rho|_{G_{F_v}}$ is reducible at all places $v|l$ of $F$.

If $l = 5$ and the projective image of $\rho|_{G_{F,(5)}}$ is isomorphic to $\text{PSL}_2(F_5)$, assume further that there is a finite solvable totally real extension $F'/F$ such that $\rho|_{G_{F'}}$ is conjugate to a representation valued in $\text{GL}_2(F_5)$.

Then $\rho$ has a modular lift $\bar{\rho} : G_F \to \text{GL}_2(\mathbb{Q}_l)$, which is ordinary at all places $v|l$.

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Our methods are based on those of [BLGG12]. The reason that we are now able to prove a stronger result is that the automorphy lifting results that we employed in [BLGG12] have since been optimized in [BLGGT10] and [Tho12]; in particular, we make extensive use of the results of the appendix to [BLGG13], which improves on a lifting result of [BLGGT10], and classifies the subgroups of $\text{GL}_2(\mathbb{F}_l)$, which are adequate in the sense of [Tho12]. In Section 2, we use these results to prove Theorem A, except in the case that $l = 3$ or $5$ and the projective image of $\rho(G_F)\zeta_l$ is isomorphic to $\text{PSL}_2(\mathbb{F}_3)$, and certain cases where $\rho$ is dihedral. In the dihedral cases, the result is proved in [All12]. In the remaining cases, the adequacy hypothesis we require fails, but in Section 3 we handle this case completely when $l = 3$ by making use of the Langlands–Tunnell theorem, and we prove a partial result when $l = 5$ using the results of [SBT97].

1.1. Notation. If $M$ is a field, we let $G_M$ denote its absolute Galois group. We write $\overline{\varepsilon}$ for the mod $l$ cyclotomic character. We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and regard all algebraic extensions of $\mathbb{Q}$ as subfields of $\overline{\mathbb{Q}}$. For each prime $p$ we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and we fix an embedding $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_p$. In this way, if $v$ is a finite place of a number field $F$, we have a homomorphism $G_{F_v} \hookrightarrow G_F$. We also fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

We normalize the definition of Hodge–Tate weights so that all the Hodge–Tate weights of the $l$-adic cyclotomic character $\varepsilon$ are $-1$. We refer to a two-dimensional potentially crystalline representation with all pairs of labelled Hodge–Tate weights equal to $\{0, 1\}$ as a weight 0 representation. (The reason for this terminology is that the Galois representations associated to an automorphic representation, which is cohomological of weight 0 have these Hodge–Tate weights.)

If $F$ is a totally real field, then a continuous representation $\overline{\rho} : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_l)$ is said to be modular if there exists a regular algebraic automorphic representation $\pi$ of $\text{GL}_2(\mathbb{A}_F)$, such that $\overline{\rho}(\pi) \cong \overline{\rho}$, where $\overline{\rho}(\pi)$ is the $l$-adic Galois representation associated with $\pi$.

We let $\zeta_l$ be a primitive $l$th root of unity.

2. The adequate case

2.1. The notion of an adequate subgroup of $\text{GL}_n(\overline{\mathbb{F}}_l)$ is defined in [Tho12]. We will not need to make use of the actual definition; instead, we will use the following classification result. Note that by definition an adequate subgroup of $\text{GL}_n(\overline{\mathbb{F}}_l)$ necessarily acts irreducibly on $\overline{\mathbb{F}}_l^n$.

Proposition 2.1.1. Suppose that $l > 2$ is a prime, and that $G$ is a finite subgroup of $\text{GL}_2(\overline{\mathbb{F}}_l)$, which acts irreducibly on $\overline{\mathbb{F}}_l^2$. Then precisely one of the following is true:

- We have $l = 3$, and the image of $G$ in $\text{PGL}_2(\overline{\mathbb{F}}_3)$ is conjugate to $\text{PSL}_2(\overline{\mathbb{F}}_3)$.
- We have $l = 5$, and the image of $G$ in $\text{PGL}_2(\overline{\mathbb{F}}_5)$ is conjugate to $\text{PSL}_2(\overline{\mathbb{F}}_5)$.
- $G$ is adequate.

Proof. This is Proposition A.2.1 of [BLGG13].
In the case that \( \bar{\rho}(G_{F(\zeta)}) \) is adequate, our main result follows exactly as in section 6 of [BLGG12], using the results of Appendix A of [BLGG13] (which in turn build on the results of [BLGGT10]). We obtain the following theorem.

**Theorem 2.1.2.** Suppose that \( l > 2 \) is prime, that \( F \) is a totally real field, and that \( \bar{\rho} : G_F \to \text{GL}_2(\mathbb{F}_l) \) is irreducible and modular. Suppose also that \( \bar{\rho}(G_{F(\zeta)}) \) is adequate. Then:

1. There is a finite solvable extension of totally real fields \( L/F \) which is linearly disjoint from \( F^{\text{ker} \bar{\rho}} \) over \( F \), such that \( \bar{\rho}|_{G_L} \) has a modular lift \( \rho_L : G_L \to \text{GL}_2(\overline{\mathbb{Q}}_l) \) of weight 0, which is ordinary at all places \( v \mid l \).
2. If furthermore \( \bar{\rho}|_{G_{F_v}} \) is reducible at all places \( v \mid l \), then \( \bar{\rho} \) itself has a modular lift \( \rho : G_F \to \text{GL}_2(\overline{\mathbb{Q}}_l) \) of weight 0, which is ordinary at all places \( v \mid l \).

**Proof.** First, note that (2) is easily deduced from (1) using the results of Section 3 of [Gee11] (which build on Kisin’s reinterpretation of the Khare–Wintenberger method). Indeed, the proofs of Theorems 6.1.5 and 6.1.7 of [BLGG12] go through unchanged to show that there is a finite solvable extension of totally real fields \( L/F \) such that \( \bar{\rho}(G_{F(\zeta)}) \) is adequate. Our main result follows exactly as in section 6 of [BLGG10] for all places \( w \mid l \).

Similarly, (1) is easily proved in the same way as Proposition 6.1.3 of [BLGG12] (and in fact the proof is much shorter). First, note that the proof of Lemma 6.1.1 of [BLGG12] goes through unchanged to show that there is a finite solvable extension of totally real fields \( L/F \) which is linearly disjoint from \( F^{\text{ker} \bar{\rho}} \) over \( F \), such that \( \bar{\rho}|_{G_L} \) has a modular lift \( \rho' : G_L \to \text{GL}_2(\overline{\mathbb{Q}}_l) \) of weight 0 which is potentially crystalline at all places dividing \( l \), and in addition both \( \bar{\rho}|_{G_{L_w}} \) and \( \bar{\rho}|_{G_{L_w}} \) are trivial for each place \( w \mid l \) (and in particular, \( \bar{\rho}|_{G_{L_w}} \) admits an ordinary lift of weight 0), and \( \bar{\rho} \) is unramified at all finite places. By Lemma 4.4.1 of [GK12], \( \rho'|_{G_{L_w}} \) is potentially diagonalizable in the sense of [BLGG10] for all places \( w \mid l \) of \( L \).

Choose a CM quadratic extension \( M/L \) that is linearly disjoint from \( L(\zeta) \) over \( L \), in which all places of \( L \) dividing \( l \) split. We can now apply Theorem A.4.1 of [BLGG13] (with \( F' = F = M \), \( S \) the set of places of \( L \) dividing \( l \), and \( \rho_v \) an ordinary lift of \( \bar{\rho}|_{G_{L_w}} \) for each \( w \mid l \)) to see that \( \bar{\rho}|_{G_M} \) has an ordinary automorphic lift \( \rho_M : G_M \to \text{GL}_2(\overline{\mathbb{Q}}_l) \) of weight 0.

The argument of the last paragraph of the proof of Proposition 6.1.3 of [BLGG12] (which uses the Khare–Wintenberger method to compare deformation rings for \( \bar{\rho}|_{G_L} \) and \( \bar{\rho}|_{G_M} \)) now goes over unchanged to complete the proof. \( \square \)

3. Inadequate cases

3.1. The first inadequate case. We now consider the case that \( l = 3 \) and \( \bar{\rho}|_{G_{F(\zeta)}} \) is irreducible, but \( \bar{\rho}(G_{F(\zeta)}) \) is not adequate. By Proposition 2.1.1, this means that the projective image of \( \bar{\rho}(G_{F(\zeta)}) \) is isomorphic to \( \text{PSL}_2(\mathbb{F}_3) \), and is in particular solvable. We now use the Langlands–Tunnell theorem to prove our main theorem in this case.

**Theorem 3.1.1.** Suppose that \( F \) is a totally real field, and that \( \bar{\rho} : G_F \to \text{GL}_2(\mathbb{F}_3) \) is irreducible and modular. Assume that \( \bar{\rho}|_{G_{F_v}} \) is reducible at all places \( v \mid 3 \) of \( F \), and that the projective image of \( \bar{\rho}(G_{F(\zeta)}) \) is isomorphic to \( \text{PSL}_2(\mathbb{F}_3) \).

Then \( \bar{\rho} \) has a modular lift \( \rho : G_F \to \text{GL}_2(\overline{\mathbb{Q}}_3) \) which is ordinary at all places \( v \mid 3 \).
3.2. The second inadequate case. We now suppose that $l = 5$, that $\bar{\rho}(G_{F(3)})$ is irreducible but its image is not adequate. Then $\bar{\rho}(G_{F(3)})$ has projective image conjugate to $\text{PSL}_2(\mathbb{F}_5)$, which is isomorphic to $\text{PGL}_2(\mathbb{F}_5)$. We see that $\bar{\rho}(G_F)$ has projective image conjugate to either $\text{PGL}_2(\mathbb{F}_5)$ or $\text{PSL}_2(\mathbb{F}_5)$. (This follows from [DDT97, Prop. 2.47].) Thus, after conjugating, we may assume that $\bar{\rho} : G_{F(3)} \to \text{GL}_2(\mathbb{F}_5)$ takes values in $\text{PSL}_2(\mathbb{F}_5)$.

In order to apply the results of [SBT97], we need to assume further that there is a finite solvable totally real extension $F'/F$ such that $\bar{\rho}(G_{F'})$ is valued in $\text{GL}_2(\mathbb{F}_5)$. (This condition is not automatic, but it holds if the projective image of $\bar{\rho}(G_F)$ is isomorphic to $\text{PSL}_2(\mathbb{F}_5)$.)

**Theorem 3.2.1.** Suppose that $F$ is a totally real field, and that $\bar{\rho} : G_F \to \text{GL}_2(\mathbb{F}_5)$ is irreducible and modular. Assume that $\bar{\rho}(G_{F(3)})$ is reducible at all places $v|5$ of $F$, and that the projective image of $\bar{\rho}(G_{F(3)})$ is isomorphic to $\text{PSL}_2(\mathbb{F}_3)$. Assume further that there is a finite solvable totally real extension $F'/F$ so that $\bar{\rho}(G_{F'})$ is conjugate to a representation valued in $\text{GL}_2(\mathbb{F}_5)$.

Then $\bar{\rho}$ has a modular lift $\rho : G_F \to \text{GL}_2(\overline{\mathbb{Q}_5})$ which is ordinary at all places $v|5$.

**Proof.** Since $\bar{\rho}$ is totally odd, we can replace $F'/F$ by a further finite solvable totally real extension and assume that $\bar{\rho}(G_{F'})$ takes values in $\text{GL}_2(\mathbb{F}_5)$ and has determinant equal to the cyclotomic character. Now, as in the proof of Theorem 2.1.2, to prove the current theorem, it suffices to show that $\bar{\rho}(G_{F'})$ has a modular lift of weight 0, which is ordinary at each $v|5$. (The only thing that needs to be checked is that Proposition 3.1.5 of [Gee11] applies to $\bar{\rho}(G_{F'})$. The only hypothesis which is not immediate is that if the projective image of $\bar{\rho}(G_{F'})$ is $\text{PGL}_2(\mathbb{F}_5)$, then $|F'(\zeta_5) : F'| = 4$. To see this, note that if $|F'(\zeta_5) : F'| = 2$, then since the determinant of $\bar{\rho}(G_{F'})$ is the mod 5 cyclotomic character, it has image $\{\pm 1\}$. This implies that the projective image is $\text{PSL}_2(\mathbb{F}_5)$, as required.)

By [SBT97, Theorem 1.2], there exists an elliptic curve $E/F'$ such that $E[5] \cong \bar{\rho}(G_{F'})$ and the image of $G_{F'}$ in $\text{Aut}(E[3])$ contains $\text{SL}_2(\mathbb{F}_3)$ (and hence its image is equal to $\text{Aut}(E[3])$ since the determinant is totally odd). We may further suppose that $E$ has good ordinary reduction at each prime of $F'$ dividing 5. (To see this, note that we may...
incorporate Ekedahl’s effective version of the Hilbert Irreducibility Theorem [Eke90] into the proof of [SBT97, Theorem 1.2] exactly as is done in [Tay03, Lemma 2.3]. By the Langlands–Tunnell theorem, $E[3]$ has a modular lift corresponding to a Hilbert modular form $f_0$ of parallel weight 1. Replacing $F'$ by a finite totally real solvable extension linearly disjoint from $\overline{F/k E[3]}$, we may assume that $f_0$ is ordinary at each prime dividing 3. By Hida theory, $E[3]$ then has a modular lift corresponding to a Hilbert modular form of parallel weight 2, which is ordinary at each prime dividing 3.

Note that the conditions of the modularity lifting theorem [Gee09, Theorem 1.1], applied to $\rho := T_3 E$, are satisfied. (For the third condition, note that $E[3]|_{G_{F'}}$ has non-dihedral image.) It follows that $T_3 E$ is modular and hence that $T_3 E$ is modular. Thus we have exhibited a modular lift of $\overline{\rho|_{G_{F'}}} \cong E[5]$ which has weight 0 and is ordinary at each prime above 5.

Finally, we deduce our main result from Theorems 2.1.2, 3.1.1 and 3.2.1.

Proof of Theorem A. If $\overline{\rho|_{G_{F'}}}$ is reducible, then $\rho$ is dihedral, and the result follows from Lemma 5.1.2 of [All12]. If $l = 3$ (respectively $l = 5$) and the projective image of $\overline{\rho|_{G_{F'}}}$ is isomorphic to $\text{PSL}_2(F_l)$, then the result follows from Theorem 3.1.1 (respectively, from Theorem 3.2.1). In all other cases, we see from Proposition 2.1.1 that $\overline{\rho|_{G_{F'}}}$ is adequate and the result follows from Theorem 2.1.2(2).

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DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY, 415 SOUTH ST, WALTHAM, MA 02453, USA
E-mail address: tbl@brandeis.edu

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, SOUTH KENSINGTON CAMPUS, EXHIBITION RD, LONDON SW7 2AZ, UK
E-mail address: toby.gee@imperial.ac.uk

PRINCETON UNIVERSITY AND INSTITUTE FOR ADVANCED STUDY, 1 EINSTEIN DR, PRINCETON, NJ 08540, USA
E-mail address: geraghty@math.princeton.edu