Geometrically non-linear forced vibrations of fully clamped functionally graded beams with multi-cracks resting on intermediate simple supports

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Abstract. The objective of this work is to investigate the non-linear forced dynamic response at large vibration amplitudes of fully clamped functionally graded beams containing a multiple open edge cracks and resting on intermediate simple supports. The theoretical model is based on the Euler-Bernoulli beam theory and the crack rotational spring model. The functionally graded beam properties are supposed to vary continuously through the beam thickness. A homogenization procedure based on the neutral surface approach taking into account the presence of the crack is developed to reduce the problem examined to that of an equivalent isotropic homogeneous multi-cracked beam. In the non-linear analysis, harmonic motion is assumed and the discretized expressions for the beam total strain and kinetic energies are derived by applying Hamilton’s Principle, so as to reduce the problem to a non-linear algebraic system solved using an approximate explicit method previously applied to various non-linear structural vibration problems. Considering the forced vibration case, the non-linear frequency response functions are obtained numerically in the neighbourhood of the predominant non-linear mode shape using a single mode approach. The effects of crack, the beam material gradient and the applied harmonic force are presented and discussed.

1. Introduction
As one of the advanced inhomogeneous sophisticated new materials, functionally graded materials (FGMs) are considered as an alternative solution to the problems and limitations of conventional composites owing to their outstanding properties characterized by the smooth and continuous transition in both material properties and compositional profile. FGMs have obtained a wide range of industrial applications, including thermal barrier coatings, aircraft parts, automotive components, and more particularly in harsh environments under high temperatures subjected to high levels of workloads. Furthermore, damage in the form of fatigue cracks can easily occur in this type of structure during their operation life or during the manufacturing process. A crack in a structural element decreases locally its rigidity and increases its flexibility and may induces a significant change in the dynamic characteristics of the whole structure. It is therefore necessary for designers, in order to take into account the effect of the cracks presence in structural elements, to understand their mechanical effects, both qualitatively and
quantitatively. Over the last few decades, several studies addressed the problem of linear and non-linear free and forced vibrations of functionally graded cracked beam-like structural members, using analytical or numerical techniques or both. Yang and Chen [1] studied the effect of an open edge crack on the free vibration and elastic buckling of functionally graded beams with different boundary conditions, using the Euler Bernoulli beam theory and the rotational spring model. Yu and Chu [2] developed a methodology based on the p-version of the finite element method to study the transverse vibration characteristics of functionally graded cracked beams. Wei et al [3] presented an analytical method combining the transfer matrix method with rotational spring model to study the free vibration of functionally graded cracked beams with axial loading, taking into account the effect of the rotary inertia and shear deformation. Lien et al [4] presented an approach to study the free vibrations of functionally graded beam with multiple cracks based on Timoshenko beam theory, dynamic stiffness method and the neutral surface approach. Kou et al [5] studied the free vibration of the functionally graded beams with edged cracks using a meshfree boundary-domain integral equation method. Panigrahi and Pohit [6] investigated the dynamic behaviour of a functionally graded cracked Timoshenko beam under an excitation force using the harmonic balance method in conjunction with an iterative technique. It may be noticed that most of the mentioned studies are concerned with single-span beams, while the investigations on multi-span cracked beams are very limited. To the best knowledge of the authors, there are a very few previous works treating the problem of cracked multi-span beams. Lien and Hao [7] determined the mode shapes of multi-span cracked beam using the dynamic stiffness method. Liu et al [8] and Nam et al [9] investigated the free vibrations of a multi-span continuous beam with an arbitrary number of cracks, using the transfer matrix method. Recently, Tan et al. [10] studied both the direct and inverse problems of free vibration analysis of a uniform continuous beam with an arbitrary number of cracks and spring-mass systems. All of the aforementioned investigations are even in the relatively simple cases of vibrations at small deflections and isotropic materials. Also, no work is reported in the available literature on the problem of the geometrically non-linear vibrations due to large transverse displacement amplitudes of multi-span functionally graded beams with multi-cracks. Therefore, the purpose of the present work was also to investigate the geometrically non-linear forced vibrations of fully clamped functionally graded beams with multi-cracks resting on intermediate simple supports, based on the Euler-Bernoulli beam theory and the rotational spring crack model followed by the Von Karman geometrical non-linearity assumptions. The material properties of the functionally graded beam examined are assumed to vary according to a continuous exponential function along the beam thickness. A homogenization method based on the neutral surface approach, used previously in [11–13], was employed to reduce the problem under consideration to that of an equivalent isotropic homogeneous multi-cracked beam. The closed-form solutions are employed and solved iteratively using the Newton Raphson method. Afterwards, the non-linear case was examined by expanding the non-linear multi-cracked beam transverse displacement function as a series of the linear modes calculated before. Using the model developed in [14], the modal functions obtained have been used as trial functions in the development of the so-called second formulation, leading to a multimode approach of the non-linear free response problem. The non-linear forced case has also been examined using a single mode approach as in [15], to obtain the non-linear frequency response curves in the neighbourhood of the predominant non-linear mode shape. A quite comprehensive parametric study was performed by varying the vibration amplitude, the crack depth, the beam material gradient and the applied distributed harmonic uniform force to investigate in each case the effect on the non-linear dynamic behaviour of a multi-span functionally graded cracked beam.
2. Problem formulation
As may be seen in figure 1, a clamped-clamped functionally graded (FG) beam with multi-cracks located at the positions \( x_{c_j} \), resting on a number of intermediate simple supports located at \( x_{ss_j} \) accounted from the left beam end and subjected to a uniformly distributed harmonic force \( F_d \). The beam, characterized by: length \( L \), width \( b \) and thickness \( h \). The material properties are assumed to vary according to a continuous exponential function along the beam thickness as follows:

\[
E(z) = E_1 \sqrt{k} e^{k \ln(k)}, \quad \rho(z) = \rho_1 \sqrt{k} e^{k \ln(k)} \quad (1a,b)
\]

Where \( E_1 \) and \( \rho_1 \) are the Young’s modulus and mass density of beam at top surface, respectively and \( z \) is ordinate of the point from the middle surface. The material index \( k = E_2 / E_1 \) is introduced to characterise the material properties. it should be pointed out that the Poisson’s ratio \( \nu \) is taken to be constant owing to its weak influence on the stress intensity factors (SIF) [3]. 

The present formulation is based on the neutral surface approach distinguishing between the beam middle surface and its neutral surface. the distance separating the two surfaces can be expressed as follows:

\[
\delta = \int_{-h/2}^{h/2} zE(z) \, dz / \int_{-h/2}^{h/2} E(z) \, dz \quad (2)
\]

Since the present study deals with the pure bending vibration, then only the mode I (The opening fracture mode) describing the local deformation of the crack tip and stress fields is considered. For that reason, the crack is assumed to be perpendicular to the beam surface and to remain always open. Based on the rotational spring crack model, the bending stiffness \( K_\tau \) of the cracked section is related to the flexibility \( C \) by:

\[
C = \int_0^\pi \frac{72\pi(1-\nu^2)}{E(\alpha)h^2} f^2(k, \frac{\alpha}{h}) d\alpha \quad (3)
\]

Where \( \alpha/h \) is the crack depth, \( E(\alpha) \) is the effective elastic modulus at the crack tip expressed as follows:

\[
E(\alpha) = E_1 \sqrt{k} e^{\left(-\frac{\alpha}{h} - \frac{\delta}{\pi + \frac{\alpha}{h}}\right) \ln(k)} \quad (4)
\]
4

and \( f(k, \frac{\alpha}{h}) \) is called the crack correction function given as [2]:

\[
f(k, \frac{\alpha}{h}) = \frac{1.1732 - 0.3539\ln(k) + 0.0289\ln(k)^2 - 0.0061\ln(k)^6 + 0.6625\left(\frac{\alpha}{h}\right)^2}{1 - 0.014\ln(k) - 0.0017\ln(k)^2 + 1.9917\left(\frac{\alpha}{h}\right)^2 - 3.0496\left(\frac{\alpha}{h}\right)^3}
\]

(5)

2.1. Linear vibration analysis

Based on Euler-Bernoulli beam theory displacement fields such as longitudinal displacement \( U \) and transverse displacement \( W \) of an arbitrary point along the \( x \) and \( z \) axes are assumed as:

\[
U(x, z, t) = u(x, t) - z\frac{\partial w}{\partial x}, \quad W(x, z, t) = w(x, t)
\]

(6)

where \( u \) and \( w \) are displacement components of a point in the mid-plane. the strain-displacement relation is given as:

\[
\varepsilon_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}
\]

(7)

The stress-strain relation given by the generalised Hooke’s law:

\[
\sigma_x = \frac{E(z)}{(1 - \nu^2)}\varepsilon_x
\]

(8)

The axial force and bending moment resultants expressions are then obtained as:

\[
N_x = \int_S \sigma_x dA = A_{11}\frac{\partial u}{\partial x} - B_{11}\frac{\partial^2 w}{\partial x^2}
\]

(9)

\[
M_x = \int_S \sigma_x z dA = B_{11}\frac{\partial u}{\partial x} - D_{11}\frac{\partial^2 w}{\partial x^2}
\]

(10)

where \( A_{11}, B_{11} \) and \( D_{11} \) are extensional, coupling and bending rigidities coefficients respectively defined as follows:

\[
(A_{11}, B_{11}, D_{11}) = \int_{\frac{h}{2}+\delta}^{\frac{h}{2}+\delta} E(z) (1, z, z^2) \, dz
\]

(11)

Taking into account what is mentioned in equation (2), integrating the coefficients in equation (11). so, \( B_{11} \) vanishes. Consequently the equations of motion for the \( j \)th sub-beam neglecting the axial inertia can be derived as:

\[
\frac{\partial N_x}{\partial x} = 0
\]

(12)

\[
\frac{\partial^2 M_x}{\partial x^2} - I_1 \frac{\partial^2 w}{\partial t^2} = (EI)^{eff} \frac{\partial^4 w}{\partial x^4} + (\rho S)^{eff} \frac{\partial^4 w}{\partial t^4} = 0
\]

(13)

Equation (13) is effective for replacing the FG beam problem with an equivalent isotropic beam, where \( (EI)^{eff} = (D_{11} - B_{11}^2/A_{11}) \) and \( (\rho S)^{eff} = I_1 \) are the effective bending stiffness and distributed mass respectively. with \( I_1 \) given by:

\[
I_1 = \int_{-\frac{h}{2}+\delta}^{\frac{h}{2}+\delta} \rho(z) \, dz
\]

(14)
Applying the end conditions considered in this study (Clamped-Clamped) where \((u = 0 \text{ at } x = 0 \text{ and } L)\) and integrating equation (12) with respect to \(x:\)

\[
N_x = \frac{A_{11}}{2L} \int_0^L \left( \frac{\partial w}{\partial x} \right)^2 \, dx
\]

The expression of equation (15) is effective to equivalent isotropic beam with the effective expression of axial stiffness where \(A_{11} = (ES)_{\text{eff}}\). In the linear vibration analysis, the transverse displacements in the \(j^{th}\) beam span with \(x^* = x/L\) can be expressed as [16]:

\[
w_{ij}(x^*) = A_{j}cosh \left( \beta_iL \left( x^* - x_{j-1}^* \right) \right) + B_{j}sinh \left( \beta_iL \left( x^* - x_{j-1}^* \right) \right) + C_{j}cos \left( \beta_iL \left( x^* - x_{j-1}^* \right) \right)
\]

\[
+ D_{j}sin \left( \beta_iL \left( x^* - x_{j-1}^* \right) \right)
\]

\[
x_{j-1}^* = x^* \leq x_j^* \text{ for } j = 1, 2, ..., N + 1
\]

in which \(\beta_i = \sqrt{\omega_i^2 \frac{(ES)_{\text{eff}}}{(EI)_{\text{eff}}}}\) for \(i = 1 \text{ to } m\), are the eigenvalue parameters of the beam. The constants \((A_j, B_j, C_j, D_j)\) are determined by the imposed beam end conditions:

\[
w_{(j-1)i}(x^*) \bigg|_{x^* = 0} = w_{ji}(x^*) \bigg|_{x^* = 1} = 0
\]

\[
\frac{dw_{(j-1)i}(x^*)}{dx^*} \bigg|_{x^* = 0} = \frac{dw_{(j)j}(x^*)}{dx^*} \bigg|_{x^* = 1} = 0
\]

and the continuity and compatibility conditions at the crack and support locations:

- The continuity and compatibility conditions at the \(j^{th}\) crack location \(\xi_j = x_{ssj}^*\) [17]:

\[
w_{(j-1)i}(x^*) \bigg|_{x^* = \xi_j} = w_{ji}(x^*) \bigg|_{x^* = \xi_j}
\]

\[
\frac{d^2 w_{(j-1)i}(x^*)}{dx^*} \bigg|_{x^* = \xi_j} = \frac{d^2 w_{(ji)(x^*)}}{dx^*} \bigg|_{x^* = \xi_j}
\]

\[
\frac{d^3 w_{(j-1)i}(x^*)}{dx^*} \bigg|_{x^* = \xi_j} = \frac{d^3 w_{(ji)(x^*)}}{dx^*} \bigg|_{x^* = \xi_j}
\]

\[
\frac{d^4 w_{(j-1)i}(x^*)}{dx^*} \bigg|_{x^* = \xi_j} = \frac{d^4 w_{(ji)(x^*)}}{dx^*} \bigg|_{x^* = \xi_j} + \frac{L}{K^*} \frac{d^2 w_{(j)j}(x^*)}{dx^*} \bigg|_{x^* = \xi_j}
\]

Where \(K^*\) is the non-dimensional local rigidity due to the crack, related to the local flexibility coefficient \(C\) of the rotational spring by:

\[
K^* = \frac{L}{C(EI)_{\text{eff}}} \frac{K_{s}L}{(EI)_{\text{eff}}}
\]

The continuity conditions for the intermediate support at \(\eta_j = x_{ssj}^*\) [18]:

\[
w_{(j-1)i}(x^*) \bigg|_{x^* = \eta_j} = w_{ji}(x^*) \bigg|_{x^* = \eta_j} = 0
\]
\[
\frac{d w_{(j-1)}(x^*)}{dx^*} \bigg|_{x^* = \eta_j} = \frac{dw_j(x^*)}{dx^*} \bigg|_{x^* = \eta_j} \tag{25}
\]
\[
\frac{d^2 w_{(j-1)}(x^*)}{dx^{*2}} \bigg|_{x^* = \eta_j} = \frac{d^2 w_j(x^*)}{dx^{*2}} \bigg|_{x^* = \eta_j} \tag{26}
\]

The satisfaction of predefined conditions, leads to a homogeneous linear system, which has a non-trivial solution if its determinant is equal to zero. It has been solved iteratively using the Newton Raphson algorithm to find the natural frequencies. The corresponding mode shapes have been then calculated by the usual algebraic procedure.

2.2. Non-linear free vibrations

By adding the non-linear components into equation (7), The Von Kàrmàn type non-linear strain-displacement relationship expressed now as:

\[
\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \frac{1}{2} \frac{\partial^2 w}{\partial x^2} \tag{27}
\]

The elastic strain energy \( V \) of the beam is given as:

\[
V = \frac{1}{2} \int_0^L \left[ N_x \varepsilon_x - M_x \frac{\partial^2 w}{\partial x^2} \right] dx \tag{28}
\]

With equation (2) always in mind and considering that is mentioned in equation (15). The total strain energy \( V \) can be written now only in terms of the transverse displacement \( w \) as follows:

\[
V = \frac{(ES)_{\text{eff}}}{8L} \left[ \int_0^L \left( \frac{\partial w}{\partial x} \right)^2 dx \right] + \frac{(EI)_{\text{eff}}}{2} \int_0^L \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx \tag{29}
\]

The kinetic energy \( T \) and the strain energy of the crack \( V_c \) are given by:

\[
T = \frac{1}{2} \int_0^L \int_{-\frac{b+\delta}{2}}^{\frac{b+\delta}{2}} \rho(z) \left( \frac{\partial w}{\partial t} \right)^2 dx \tag{30}
\]
\[
V_c = \frac{(EI)^2_{\text{eff}}}{2K_{\tau}} \left( \frac{\partial^2 w}{\partial x^2} \right)^2_{x = x_c} \tag{31}
\]

In the non-linear analysis, by assuming the harmonic motion, the transverse displacement is expressed as time dependent function \( a(t) \) and the linear fundamental vibration mode \( w(x) \):

\[
w(x,t) = a(t) w_i(x); \text{ where } a(t) = a_i \sin(\omega t) \tag{32}
\]

in which \( a_i \)'s are the unknown basic function contribution coefficients. Inserting equation (33) into equations (29-31) and introducing the following dimensionless parameters:

\[
w(x) = hw_i^*(\frac{x}{L}) = hw_i^*(x^*) \tag{33}
\]

By applying Hamilton’s principle considering the dynamic behaviour of a system is conservative:

\[
\frac{\partial}{\partial t} \int_0^{2\pi} (V + V_c - T) \, dt = 0 \tag{34}
\]
the following set of non-linear algebraic equations is then obtained:

$$2a_1K_{ir}^* + 3a_1a_ja_kB_{ijkr}^* - 2\omega^2a_iM_{ir}^* = 0 \quad r=1,2,...m$$ (35)

Where $K_{ir}^*$, $B_{ijkr}^*$ and $M_{ir}^*$ denote the dimensionless classical rigidity tensor due to the bending strain energy, the non-linearity tensor due to the axial strain energy induced by large deflections and the mass tensor attributable to the kinetic energy respectively, defined by [17]:

$$K_{ij}^* = \int_0^1 \frac{\partial^2 w_i^*}{\partial x^*} \frac{\partial^2 w_j^*}{\partial x^*} dx^* + \sum_{c=1}^N \frac{(EI)_{eff}}{K_c^*} \left| \frac{\partial^2 w_i^*}{\partial x^*} \right|_{x^*=\xi_j}$$ (36)

$$B_{ijkl}^* = \frac{h^2(ES)_{eff}}{4(EI)_{eff}} \left( \int_0^1 \frac{\partial w_i^*}{\partial x^*} \frac{\partial w_j^*}{\partial x^*} dx^* \right) \left( \int_0^1 \frac{\partial w_k^*}{\partial x^*} \frac{\partial w_l^*}{\partial x^*} dx^* \right)$$ (37)

$$M_{ij}^* = \int_0^1 w_i^* w_j^* dx^*$$ (38)

2.3. Non-linear forced vibrations

The generalized forces $F_i(t)$ associated to a distributed force $F(x,t)$ are given by:

$$F_i(t) = \int_S F(x,t) w_i(x) dx$$ (39)

The harmonic force is uniformly distributed and equal to $F^d$, it excites the beam $i^{th}$ mode by:

$$F_{i}^{d}(t) = F_{i}^{d} \sin(\omega_{e} t) \int_0^L w_i(x) dx = f_{i}^{d} \sin(\omega_{e} t)$$ (40)

where $\omega_{e}$ is the excitation frequency. As will be shown below, the predominant mode present in the beam response to such an excitation is the third mode. This allows us to explore an approximate solution based on the single mode approach, as has been shown in [15]. First, a right-hand side corresponding to the forcing term is added to the non-linear algebraic system (35), leading to the so-called multidimensional duffing equation. Then, by keeping only the equation corresponding to the predominant mode, the non-linear system reduces, after applying the harmonic balance method, to:

$$\left( \frac{\omega_{e}^*}{\omega_{l}^*} \right)^2 = 1 + \frac{3}{2} \left( \frac{B_{3333}^*}{K_{33}^*} \right) a_2^a - \left( \frac{f_{i}^{d}}{K_{33}^*} \right) a_3^a \quad \text{with} \quad f_{3}^{d} = \frac{F_{i}^{d}}{h(ET)_{eff}} \int_0^1 w_{3}^*(x^*) dx^*$$ (41)

where $\omega_{l}^* = \frac{K_{33}^*}{B_{3333}^*}$. Equation (41) can also be written as:

$$a_3^a = \frac{2}{3} \left( \frac{K_{33}^*}{B_{3333}^*} \right) \left[ 1 - \left( \frac{\omega_{e}^*}{\omega_{l}^*} \right)^2 \right] a_3^a - \frac{2}{3} \left( \frac{f_{3}^{d}}{B_{3333}^*} \right) = 0$$ (42)

Which is a third-degree algebraic equation, solved classically using the Cardan’s method for each specified set of the parameters $k_{33}^*$, $b_{3333}^*$ and $f_{3}^{d}$. 7
3. Numerical results and discussions

In the present work, the FG beam under investigation is clamped at the two ends and in addition rests on two simple supports located at \( \eta_1 = 2/6, \eta_2 = 4/6 \) denoted in what follows as CCFGBRSS. It is supposed to contain three cracks, located at \( \xi_1 = 1/6, \xi_2 = 3/6, \xi_3 = 5/6 \), and to have the following material and physical parameters: the top surface of the FGM \( E_1 = 70 \text{GPa} \), \( \rho_1 = 2780 \text{Kg/m}^3 \) and \( h = 0.1 \text{m} \). First of all, the present linear numerical results need to be validated before being used in the non-linear analysis. So, the values of the first six modal frequencies of multi-cracked isotropic homogeneous beam with clamped end conditions and resting on one simple support at the middle with total beam length \( L = 2 \text{m} \) are calculated and compared with the available data in the literature and listed in table 1. The results show an excellent agreement of the present study with those given in [9].

Table 1: Comparison of the vibration frequencies of a multi-cracked homogeneous beam resting on one simple support at \( \eta = 1/2 \).

| Crack scenario | No crack | \( \xi_1 = 0.6, \xi_2 = 0.9 \) | \( \xi_1 = 0.25, \xi_2 = 0.6, \xi_3 = 0.9 \) | \( \xi_1 = 0.1, \xi_2 = 0.4, \xi_3 = 0.6, \xi_4 = 0.9 \) |
|----------------|----------|------------------|------------------|------------------|
| Modes         | Present  | [9] Present [9]  | Present [9]     | Present [9]     |
| \( \beta_1 \) | 3.9266   | 3.9163           | 3.9038           | 3.9057           |
| \( \beta_2 \) | 4.7300   | 4.7292           | 4.7135           | 4.7284           |
| \( \beta_3 \) | 7.0685   | 7.0065           | 6.9980           | 6.9953           |
| \( \beta_4 \) | 7.8532   | 7.8318           | 7.8329           | 7.8087           |
| \( \beta_5 \) | 10.2101  | 10.1974          | 10.0705          | 10.0800          |
| \( \beta_6 \) | 10.9956  | 10.9265          | 10.8678          | 10.8361          |

Figure 2: (a) The normalized first non-linear mode and (b) the associated curvature distribution of a C-C isotropic beam with three cracks resting on two intermediate supports corresponding to various values of the vibration amplitudes.

The normalized first non-linear modes and associated curvature distributions obtained via the present model for a clamped-clamped isotropic beam resting on two intermediate supports
are plotted in figure 2, for maximum non-dimensional amplitudes up to about once the beam thickness, corresponding to \( \alpha_1/h = 0.1 \), \( \alpha_2/h = 0.5 \) and \( \alpha_3/h = 0.3 \) and \( k = 1 \). The mode shapes and associated curvature distribution amplitude dependence, shown in the figure, indicate that the flexural stresses increase non-linearly near to the clamped ends. It can also be seen that the effect of the cracks and simple supports become clearly significant with the increase of the vibration amplitude. It may be concluded that the corrections obtained by the geometrically non-linear theory considered here have to be taken into account to get more accurate estimates for the frequency and mode data, compared with those based on the linear theory.

Figure 3: Non-linear frequency ratio versus the vibration amplitude curve of a CCFGCB, in the vicinity of the first mode, for various values of the crack depth.

The dependence of the non-linear frequency ratios on the vibration amplitudes (the backbone curve) of fully clamped beam with three cracks equitably distributed over the beam length and resting on two intermediate simple supports (as shown in figure 1) for various values of the material index \( k \) and showing the effect of the crack depth \( ac/h \) is depicted in figure 3. As may be expected the all curves of different cases of the beams show the hardening type behaviour i.e., the increase of the non-linear frequency ratio in accordance with the vibration amplitude. It can be clearly observed that the non-linear frequency ratio is always higher at the same vibration amplitude for a beam having a material index \( k = 0.2 \) which implies that the linear frequency in this case is minimum and lower. Consequently, a beam corresponding to “top surface rich ceramic” (b) is more sensitive to the presence of cracks especially when the crack depth increases above \( \alpha_c/h = 0.1 \). It is also noticed in figure 3 (a) and (c) that the beam corresponding to “isotropic metal” is slightly resistant to the presence of cracks compared to the functionally graded beam with \( k = 5 \), since the linear frequency is maximum when \( k = 1 \). As indicated before, it is necessary to determine the CCFGBRSS modes predominantly excited by a given physical excitation force, in order to justify use of the single mode approach on its neighbourhood. Therefore, the percentage of generalised forces exciting the first five symmetric modes, for the case of CCFGBRSS with three cracks equitably distributed over its length corresponding to \( \alpha_c/h = 0.1 \) and \( k = 5 \) excited by two types of uniformly distributed forces, (a) applied along the whole beam and (b) applied between the intermediate supports
are listed in tables 2 respectively. It can be concluded that the force excites in both cases predominantly the third mode.

Table 2: Percentage of generalised forces exciting the first five symmetric modes of a CCFGBRSS with three cracks excited by a uniformly distributed force.

| Modes | 1     | 3     | 5     | 7     | 9     |
|-------|-------|-------|-------|-------|-------|
| (a):  | $\int_0^1 w_i^* (x^*) / \sum_{i=1}^m | \int_0^1 w_i^* (x^*) |$ | 2.61E-09 | 69.34 | 3.52E-08 | 5.57E-08 | 30.65 |
| (b):  | $\int_{4/6}^{2/6} w_i^* (x^*) / \sum_{i=1}^m | \int_{4/6}^{2/6} w_i^* (x^*) |$ | 2.61E-09 | 69.34 | 3.52E-08 | 5.57E-08 | 30.65 |

Figure 4: The response curves based on the single mode approach of CCFGBRSS, for various value of crack depth.

The solution of equation (42) is presented in figure 4 in the neighbourhood of the third non-linear mode shape corresponding to $F^d = 200$, for the case of the CCFGBRSS shown in figure 1, having a material index $k = 5$, (a) uniformly excited along the whole beam span, and (b) uniformly excited between the intermediate supports. It can be seen that the applied distributed force, for an identical level of excitation has a different effect in the two cases. When the force is applied over the whole length of the beam, the curve has a larger opening, and the hardening effect is significantly accentuated each time the depth of the crack increases while the non-linear response curve remains narrow in the case of a beam partially excited (between the two intermediate simple supports).

Conclusion

The problem of geometrically non-linear forced vibrations of fully clamped multi-cracked functionally graded beams resting on intermediate simple supports are studied using an equivalent rotational spring model at each crack location. A homogenization procedure has been proposed, based on the neutral surface approach, leading to a simplified formulation of the
present problem which is reduced to that of an equivalent isotropic homogeneous multi-cracked beam. The linear free vibrations of an isotropic homogeneous multi-cracked beam resting on one simple supports is carried out and compared with data previously obtained. The theoretical model developed previously for non-linear vibrations of various thin elastic structures, based on Hamilton’s principle and spectral analysis, has then been used here to analyse the free non-linear vibrations of a CCFGBRSS with three cracks equitably distributed over the length beam for various values of the crack depths, and for vibration amplitudes up to about once the beam thickness are given, showing how crack depths amplified the deformation of the normalized first mode shape and the associated curvature distribution when the vibration amplitude increases. The forced vibration case of a CCFGBRSS with three cracks equitably distributed over the length beam for various values of the crack depths subjected to a uniform harmonic distributed force has been also investigated. The distributed force was applied in such a manner to ensure that the third mode is predominant in the beam response, justifying use of the single mode approach. The amplitude-frequency relationships have been obtained. The results show that the vibration amplitudes are significantly increases each time the crack depth increases and the response curve shift away the backbone curve (when Fd=0) and becomes more accentuated when the multi-cracked CCFGBRSS uniformly excited along the whole beam compared to the case of a beam excited between the intermediate supports. This results can be useful in the structural design or during the experimental analysis.

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