Abstract: From the paper [JHEP 01 (2019) 021], it is known that the effective action of a massless $U(1)$ gauge vector field on a codimension-2 brane is gauge invariant due to the coupling between the vector Kaluza-Klein (KK) modes with two types of scalar KK modes. In a brane model with three extra dimensions, there are three types of scalar KK modes. The couplings between different KK modes would be more complex. The vector KK mode would couple with the three types of scalar ones, and the three types of scalar ones also couple to each other. We demonstrate that all the KK modes also make up a gauge invariant effective action. The relationships between the coupling constants of the KK modes finally lead to the gauge invariant effective action. This gauge invariant effective action implies that the masses of the vector KK mode contain three parts, which come from the three extra dimensions. However, the scalar ones only can obtain masses from two of the three extra dimensions.

Keywords: braneworld, extra dimensions

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1 Introduction

Early in the beginning of last century Kaluza and Klein considered the possibility that our four-dimensional observable universe might have extra spatial dimensions. This idea has been extensively explored and applied in modern string theory. At the end of last century, some TeV scale brane world models were proposed [1–3], and after that it became realistic to look for the signs of extra dimensions in experiments [4–8]. In the famous Randall-Sundrum (RS) model, the introduction of a warp factor $e^{A(z)}$ ($z$ denotes the extra dimension) in the geometry of the space-time gives an alternative way for solving the hierarchy problem of the standard model of particle physics. That is why this theory has been received so much attention [9–29]. This warp factor also determines whether the high dimensional fields can be localized on the brane [30–46].

In our recent works we considered the localization of a $U(1)$ gauge field in RS-type of brane world models [42–44]. There have been many works on this issue [47–56], where before performed a KK decomposition for the bulk $U(1)$ gauge field, the authors usually choose a gauge condition to simplify the calculation. As a cost, they failed to observe the scalar KK modes in the effective action. However, we found that these scalar KK modes, by coupling with the vector modes, can lead to a gauge invariant effective action in a brane with one or two extra dimensions [42, 44], which is extremely important for the massive vector KK modes. This gauge invariant effective action suggests that the vector KK modes can obtain masses from the extra dimensions. We know that with the Higgs mechanism the vector field also can obtain masses by coupling with the Higgs field. The remarkable difference is that here the scalar modes are from a massless bulk $U(1)$ gauge vector field, and the geometry of the extra dimensions determines whether all these KK modes could obtain masses and how many masses they will have.

Interestingly, in a previous work [43], we found that in a brane with two extra dimensions, the gauge invariant effective action got for the bulk $U(1)$ gauge field also suggests
that scalar KK modes could obtain masses from one of the two extra dimensions [44]. However, in a brane only with one extra dimension all the scalar KK modes are massless [42]. Then it is wondered that whether this gauge invariant effective action can be derived in the brane with more extra dimensions, and then what new properties of all KK modes will present. In this work, we will investigate this question in a brane with three extra dimensions, because when there is one more extra dimension, there will be one more type of scalar KK mode, which will make the couplings between the KK modes more complex, and it is difficult to generalize the work to models with an arbitrary number of extra dimensions.

For a codimension-3 brane, there will be three types of scalar KK modes. In addition to coupling with the vector modes, the scalar modes also couple mutually. Thus many different coupling constants will appear in the effective action. It is not easy to discern the gauge invariance of the action. To check the gauge invariance of the action, we will use the method in Ref. [44], i.e., compare the equations of motion for the KK modes got by two ways to find the relationships between these coupling constants, and then use these relationships to simplify the effective action and check the gauge invariance of it. Meanwhile we are expected to get some Schrödinger-like equations which constrain the KK modes.

The line-element of our model is assumed as

$$ds^2 = e^{2A(z,y,w)}(\hat{g}_{\mu\nu}dx^\mu dx^\nu + dz^2 + dy^2 + dw^2),$$

(1.1)

where the warp factor $e^{A(z,y,w)}$ is a function of the three extra dimensional coordinates $z, y$ and $w$, and $\hat{g}_{\mu\nu}$ is the induced metric on the brane. The Greek letters $\mu, \nu = 0, 1, 2, 3$ are used for the brane indexes. It has been pointed out the significance of the form of the ansatz like (1.1) for the final result of our work in brane with one or two extra dimensions [42–44]. Although the solution of this brane world model with (1.1) has not been found, we can also investigate how the warp factor generally impacts on the KK modes. It will be discovered that there will be a series of effective potentials, which are generated by the warp factor, influence the properties of the KK modes.

## 2 The effective action of the KK modes

The action of a bulk massless $U(1)$ gauge vector field $X_M(x_\mu, z, y, w)$ in a seven-dimensional space is

$$S = -\frac{1}{4} \int d^7x \sqrt{-g} \ Y^{MN}Y_{MN},$$

(2.1)

where $Y_{MN} = \frac{1}{2}(\partial_MX_N - \partial_NX_M)$ is the field strength of the $U(1)$ gauge field and the metric is given by the line-element (1.1). On our four-dimensional brane, the bulk field acts as a tower of KK modes. The bulk $U(1)$ gauge field has two types of KK modes, i.e., the vector and the scalar ones. The vector KK modes come from the components of the bulk field $X_\mu(x_\mu, z, y, w)$ and the scalar types of KK modes come from the components $X_z(x_\mu, z, y, w), X_y(x_\mu, z, y, w)$ and $X_w(x_\mu, z, y, w)$.
In order to investigate the properties of the KK modes, we prefer to do a gauge free KK decomposition for the field components:

\[
X_{\mu_1}(x_\mu, y, z) = \sum_n \hat{X}^{(n)}_{\mu_1}(x_\mu) W_1^{(n)}(z, y, w) e^{a A(z, y, w)}, 
\]

(2.2a)

\[
X_\varepsilon(x_\mu, y, z) = \sum_n \phi^{(n)}(x_\mu) W_2^{(n)}(z, y, w) e^{a A(z, y, w)}, 
\]

(2.2b)

\[
X_y(x_\mu, y, z) = \sum_n \varphi^{(n)}(x_\mu) W_3^{(n)}(z, y, w) e^{a A(z, y, w)}, 
\]

(2.2c)

\[
X_w(x_\mu, y, z) = \sum_n \chi^{(n)}(x_\mu) W_4^{(n)}(z, y, w) e^{a A(z, y, w)}, 
\]

(2.2d)

where \((n)\) denotes the \(n\)-level KK modes. \(\hat{X}^{(n)}_{\mu_1}(x_\mu)\) is the vector KK mode, and \(\phi^{(n)}(x_\mu), \varphi^{(n)}(x_\mu), \chi^{(n)}(x_\mu)\) are three different types of scalar modes. The \(W_1^{(n)}(z, y, w), W_2^{(n)}(z, y, w), W_3^{(n)}(z, y, w), W_4^{(n)}(z, y, w)\) are functions of the extra dimensions only, and \(a\) is a constant, which is introduced for convenience.

By substituting the KK decomposition (2.2) into the bulk action (2.1) we can get the effective action:

\[
S = -\frac{1}{4} \sum_n \sum_{n'} \int d^4 x \sqrt{-g} \left[ I_1^{(nn')} \hat{Y}_{\mu_1\mu_2} Y_{\mu_1\mu_2}^{(nn')} + \left( I_2^{(nn')} + I_4^{(nn')} + I_{11}^{(nn')} \right) \hat{X}^{(n)}_{\mu_1} \hat{X}^{(n)}_{\mu_1} \right.

+ \left( I_3^{(nn')} \partial_{\mu_1} \phi^{(n)} \partial^{\mu_1} \phi^{(n')} - I_6^{(nn')} \left( \partial_{\mu_1} \phi^{(n)} \hat{X}^{(n)}_{\mu_1} + \hat{X}^{(n)}_{\mu_1} \partial^{\mu_1} \phi^{(n')} \right) \right)

+ \left( I_5^{(nn')} \partial_{\mu_1} \varphi^{(n)} \partial^{\mu_1} \varphi^{(n')} - I_8^{(nn')} \left( \partial_{\mu_1} \varphi^{(n)} \hat{X}^{(n)}_{\mu_1} + \hat{X}^{(n)}_{\mu_1} \partial^{\mu_1} \varphi^{(n')} \right) \right)

+ \left( I_{13}^{(nn')} \partial_{\mu_1} \chi^{(n)} \partial^{\mu_1} \chi^{(n')} - I_{12}^{(nn')} \left( \partial_{\mu_1} \chi^{(n)} \hat{X}^{(n)}_{\mu_1} + \hat{X}^{(n)}_{\mu_1} \partial^{\mu_1} \chi^{(n')} \right) \right)

+ \left( I_7^{(nn')} + I_{18}^{(nn')} \right) \phi^{(n)} \phi^{(n')} + \left( I_9^{(nn')} + I_{19}^{(nn')} \right) \varphi^{(n)} \varphi^{(n')} + \left( I_{15}^{(nn')} + I_{17}^{(nn')} \right) \chi^{(n)} \chi^{(n')}

- I_{10}^{(nn')} \left( \phi^{(n)} \varphi^{(n')} + \varphi^{(n)} \phi^{(n')} \right) - I_{14}^{(nn')} \left( \phi^{(n)} \chi^{(n')} + \chi^{(n)} \phi^{(n')} \right)

- I_{16}^{(nn')} \left( \chi^{(n)} \varphi^{(n')} + \varphi^{(n)} \chi^{(n')} \right) \right].
\]

(2.3)

where we have set \(a = -3/2\), and assumed all the integrals of the extra dimensions \(I_i^{(nn')}\) are finite \(I_i^{(nn')} < \infty (i = 1, 2, 3...).\) Especially \(I_1^{(nn')}, I_3^{(nn')}, I_5^{(nn')}\) and \(I_{13}^{(nn')}\) are integrals defined as follows:

\[
I_1^{(nn')} = \int dy \, dz \, dw \, W_1^{(n)}(n') = \delta_{nn'}, 
\]

(2.4a)

\[
I_3^{(nn')} \equiv \frac{1}{2} \int dy \, dz \, dw \, W_2^{(n)}(n') = 2 \delta_{nn'}, 
\]

(2.4b)

\[
I_5^{(nn')} \equiv \frac{1}{2} \int dy \, dz \, dw \, W_3^{(n)}(n') = 2 \delta_{nn'}, 
\]

(2.4c)

\[
I_{13}^{(nn')} \equiv \frac{1}{2} \int dy \, dz \, dw \, W_4^{(n)}(n') = 2 \delta_{nn'}, 
\]

(2.4d)
which are the orthonormality conditions satisfied by $W_{1}^{(n)}(z, y, w)$, $W_{2}^{(n)}(z, y, w)$, $W_{3}^{(n)}(z, y, w)$ and $W_{4}^{(n)}(z, y, w)$. For this effective action there are two important points to be explained:

- In order to get a viable four-dimensional effective theory, we need to ensure that all the integrals $I_{i}^{(nn')}$ are finite. In our previous work [44], it was proved that if the orthonormality condition for $W_{1}^{(n)}$ can be satisfied, the other $I_{i}^{(nn')}$ must be finite. This is due to the relationships between $W_{1}^{(n)}$, $W_{2}^{(n)}$ and $W_{3}^{(n)}$, which are also the relationships between the coupling constants of the KK modes. By comparing two groups of equations of motion (EOM) for the KK modes got through two approaches, we can get these relationships along with some Schrödinger-like equations for the KK modes. The KK modes satisfying these Schrödinger-like equations with the orthonormality conditions are exactly those implied by the effective action. Therefore, in this work, we will use the same method to precisely calculate the relationships between $W_{1}^{(n)}$, $W_{2}^{(n)}$, $W_{3}^{(n)}$ and $W_{4}^{(n)}$ and to derive the Schrödinger-like equations for the KK modes.

- From the effective action (2.3), it is clear that the vector KK modes couple with all the three types of scalar modes, and the scalar KK modes also couple to each other mutually. All the KK modes have mass term as they also couple with themselves. It is expected that the effective action (2.3) is gauge invariant despite it is not easy to see this directly from the original action. However, one can check that if the action (2.3) can be rearranged as the form

$$S = -\frac{1}{4} \sum_{n} \sum_{n'} \int d^{4}x \sqrt{-g} \left[ \hat{Y}_{\mu_{1}\mu_{2}}^{(n)} \hat{Y}_{\mu_{1}\mu_{2}}^{(n')} + (\partial_{\mu}\phi^{(n)} - G_{1}\chi^{(n)})^{2} + (\partial_{\mu}\varphi^{(n)})^{2} + (\partial_{\mu}\varphi^{(n)} - G_{3})^{2} + (G_{2}\chi^{(n)} - G_{3}\varphi^{(n)})^{2} + (\chi^{(n)} - G_{1}\varphi^{(n)})^{2} \right], \tag{2.5}$$

it would be gauge invariant under the following transformation

$$\hat{X}_{\mu}^{(n)} \rightarrow \hat{\phi}^{(n)} + \partial_{\mu} \rho^{(n)}, \quad \phi^{(n)} \rightarrow \phi^{(n)} + G_{1} \rho^{(n)}, \quad \varphi^{(n)} \rightarrow \varphi^{(n)} + G_{2} \rho^{(n)}, \quad \chi^{(n)} \rightarrow \chi^{(n)} + G_{3} \rho^{(n)} \tag{2.6}$$

where $G_{1}, G_{2}, G_{3}$ are constants and $\rho^{(n)}$ are scalar fields. If Eq. (2.5) is equivalent with Eq. (2.3), the coupling constants $I_{i}^{(nn')}$ in action (2.3) cannot be all independent, which suggests that there are must be some relationships between $W_{1}^{(n)}$, $W_{2}^{(n)}$, $W_{3}^{(n)}$ and $W_{4}^{(n)}$. Once these relationships are obtained, we can simplify the action (2.3) into the form presented in Eq. (2.5).

In the next section, we will derive these relationships.

3 The relationships between the coupling constants

Firstly, we will derive the EOM for the KK modes by inserting the KK decomposition (2.2a)-(2.2c) into the EOM for the bulk field, as we have done for the model with
where \(\lambda_i\) (\(i = 1, 2, \ldots, 21\)) are defined as:

\[
\begin{align*}
\lambda_1 &= \partial_z (\tilde{\partial}_z \tilde{W}_1^{(n)})/(2\tilde{W}_1^{(n)}), \\
\lambda_2 &= \tilde{\partial}_y (\tilde{\partial}_y \tilde{W}_1^{(n)})/(2\tilde{W}_1^{(n)}), \\
\lambda_3 &= \tilde{\partial}_w (\tilde{\partial}_w \tilde{W}_1^{(n)})/(2\tilde{W}_1^{(n)}), \\
\lambda_4 &= \partial_z \tilde{W}_2^{(n)}/(2\tilde{W}_2^{(n)}), \\
\lambda_5 &= \tilde{\partial}_y \tilde{W}_3^{(n)}/(2\tilde{W}_2^{(n)}), \\
\lambda_6 &= \tilde{\partial}_w \tilde{W}_4^{(n)}/(2\tilde{W}_2^{(n)}), \\
\lambda_7 &= \tilde{\partial}_z \tilde{W}_1^{(n)}/\tilde{W}_2, \\
\lambda_8 &= \tilde{\partial}_z (\tilde{\partial}_w \tilde{W}_2^{(n)})/\tilde{W}_2, \\
\lambda_9 &= \tilde{\partial}_w (\tilde{\partial}_w \tilde{W}_2^{(n)})/\tilde{W}_2, \\
\lambda_{10} &= \tilde{\partial}_y (\tilde{\partial}_z \tilde{W}_3^{(n)})/\tilde{W}_2, \\
\lambda_{11} &= \tilde{\partial}_w (\tilde{\partial}_z \tilde{W}_4^{(n)})/\tilde{W}_2, \\
\lambda_{12} &= \tilde{\partial}_y (\tilde{\partial}_w \tilde{W}_3^{(n)})/\tilde{W}_2, \\
\lambda_{13} &= \tilde{\partial}_z (\tilde{\partial}_w \tilde{W}_3^{(n)})/\tilde{W}_3, \\
\lambda_{14} &= \tilde{\partial}_w (\tilde{\partial}_w \tilde{W}_3^{(n)})/\tilde{W}_3, \\
\lambda_{15} &= \tilde{\partial}_z (\tilde{\partial}_w \tilde{W}_4^{(n)})/\tilde{W}_3, \\
\lambda_{16} &= \tilde{\partial}_w (\tilde{\partial}_w \tilde{W}_4^{(n)})/\tilde{W}_3, \\
\lambda_{17} &= \tilde{\partial}_w (\tilde{\partial}_w \tilde{W}_4^{(n)})/\tilde{W}_4, \\
\lambda_{18} &= \tilde{\partial}_z (\tilde{\partial}_w \tilde{W}_4^{(n)})/\tilde{W}_4, \\
\lambda_{19} &= \tilde{\partial}_y (\tilde{\partial}_w \tilde{W}_4^{(n)})/\tilde{W}_4, \\
\lambda_{20} &= \tilde{\partial}_w (\tilde{\partial}_w \tilde{W}_2^{(n)})/\tilde{W}_4, \\
\lambda_{21} &= \tilde{\partial}_w (\tilde{\partial}_w \tilde{W}_3^{(n)})/\tilde{W}_4.
\end{align*}
\]

with \(\tilde{W}_j^{(n)} \equiv e^{-A}W_j^{(n)}, \tilde{W}_j^{(n)} \equiv e^{A}W_j^{(n)}\) (\(j = 1, 2, 3, 4\)) and \(\tilde{\partial}_\alpha \equiv e^{-A}\partial_\alpha, \tilde{\partial}_\alpha \equiv e^{A}\partial_\alpha\) (\(\alpha = z, y, w\)).

On the other hand, the EOM for the KK modes also can be derived from the effective action (2.3), and the result should be consistent with (3.1). Then with this consistence we can find the relationships between \(\lambda_i\) and \(f^{(mn')}_j\).

- We first consider the mass term of the vector modes in action (2.3). This term contains three parts, i.e., \(f_2^{(mn)} \equiv \frac{1}{2}m_1^{(n)m_2^{(m)}}, f_4^{(mn)} \equiv \frac{1}{2}m_2^{(n)m_2^{(m)}}, f_{11}^{(mn)} \equiv \frac{1}{2}m_3^{(n)m_2^{(m)}}\). After comparing with the mass term in first line of (3.1a), we can get \(\lambda_1 = -f_2^{(mn)}, \lambda_2 = -f_4^{(mn)}\) and \(\lambda_3 = -f_{11}^{(mn)}\). By substituting in the definition of \(\lambda_i\), we obtain three Schrödinger-like equations

\[
\begin{align*}
(\partial_z z - V_{\text{eff1}})W_1^{(n)} &= m_1^{(n)} W_1^{(n)}, \\
(\partial_y y - V_{\text{eff2}})W_1^{(n)} &= m_2^{(n)} W_1^{(n)}, \\
(\partial_w w - V_{\text{eff3}})W_1^{(n)} &= m_3^{(n)} W_1^{(n)}.
\end{align*}
\]
We now turn to the mass term of the three types of scalar KK modes \( \phi^{(n)} \), \( \varphi^{(n)} \) and \( \chi^{(n)} \). From the effective action (2.3) we can see that each of these scalars has two parts of masses. We denote these mass terms as \( I_{\phi}^{(nn)} \equiv m_{\phi1}^{(n)} \), \( I_{\phi}^{(nn)} \equiv m_{\phi2}^{(n)} \), \( I_{\varphi}^{(nn)} = m_{\varphi1}^{(n)} \), \( I_{\varphi}^{(nn)} = m_{\varphi2}^{(n)} \) and \( I_{\chi}^{(nn)} = m_{\chi1}^{(n)} \), \( I_{\chi}^{(nn)} = m_{\chi2}^{(n)} \). Comparing these mass terms with those in (3.1b)–(3.1d), and using the definition of \( \lambda_i \), we get the following equations:

\[
(\partial_{y,y} - V_{\text{eff}2}) W_2^{(n)} = m_{\phi1}^{(n)} W_2^{(n)}, \quad (\partial_{w,w} - V_{\text{eff}3}) W_2^{(n)} = m_{\phi2}^{(n)} W_2^{(n)}, \quad (3.8)
\]

\[
(\partial_{y,y} - V_{\text{eff}2}) W_3^{(n)} = m_{\varphi1}^{(n)} W_3^{(n)}, \quad (\partial_{w,w} - V_{\text{eff}3}) W_3^{(n)} = m_{\varphi2}^{(n)} W_3^{(n)}, \quad (3.9)
\]

\[
(\partial_{z,z} - V_{\text{eff}1}) W_4^{(n)} = m_{\chi1}^{(n)} W_4^{(n)}, \quad (\partial_{y,y} - V_{\text{eff}2}) W_4^{(n)} = m_{\chi2}^{(n)} W_4^{(n)}. \quad (3.10)
\]

The definitions of the effective potentials can be found in Eqs. (3.5)-(3.7). If there are solutions for the equations (3.8) satisfying the orthonormality condition (2.4a), the scalar \( \phi^{(n)} \) can be localized on the brane, and can obtain masses from the extra dimensions \( y \) and \( w \). This can be easily understood, as the scalar \( \phi^{(n)} \) KK mode is from \( z \) component of the bulk field (see (2.2b)). For the other two scalars \( \varphi^{(n)} \) and \( \chi^{(n)} \) we also have similar results.

The coincidence between the Schrödinger-like equations of the scalar modes with those of the vector modes indicates that there must be some relationships between \( W_1^{(n)} \), \( W_2^{(n)} \), \( W_3^{(n)} \) and \( W_4^{(n)} \).

- These relationships can be found by studying the couplings between the vector and the scalar KK modes. For simplicity, we will rename the coupling constants \( I_{6}^{(nn)} \), \( I_{8}^{(nn)} \) and \( I_{12}^{(nn)} \) as \( C_{1}^{(n)} \), \( C_{2}^{(n)} \) and \( C_{3}^{(n)} \), respectively. By comparing the two groups of

\[\begin{align*}
V_{\text{eff}1} &= \frac{3}{2} \partial_{z,z} A + \frac{9}{4} \partial_{z} A \partial_{z} A, \\
V_{\text{eff}2} &= \frac{3}{2} \partial_{y,y} A + \frac{9}{4} \partial_{y} A \partial_{y} A, \\
V_{\text{eff}3} &= \frac{3}{2} \partial_{w,w} A + \frac{9}{4} \partial_{w} A \partial_{w} A.
\end{align*}\]
Finally, let us consider the couplings between the scalar KK modes. We replace the EOM we get the relations between the coupling constants with the consistence of the two groups of EOM we find:

\[ I_6^{(n)} = \lambda_4 \equiv C_1^{(n)} = -\bar{\partial}_z \tilde{W}_2^{(n)}/(2\tilde{W}_1^{(n)}), \]  
\[ I_6^{(n)} = \lambda_7 \equiv C_1^{(n)} = 2\bar{\partial}_z \tilde{W}_1^{(n)}/\tilde{W}_2, \]  
\[ I_8^{(n)} = \lambda_5 \equiv C_2^{(n)} = -\bar{\partial}_y \tilde{W}_3^{(n)}/(2\tilde{W}_1^{(n)}), \]  
\[ I_8^{(n)} = \lambda_12 \equiv C_2^{(n)} = 2\bar{\partial}_y \tilde{W}_1^{(n)}/\tilde{W}_3, \]  
\[ I_{12}^{(n)} = \lambda_6 \equiv C_3^{(n)} = -\bar{\partial}_w \tilde{W}_4^{(n)}/(2\tilde{W}_1^{(n)}), \]  
\[ I_{12}^{(n)} = \lambda_{17} \equiv C_3^{(n)} = 2\bar{\partial}_w \tilde{W}_1^{(n)}/\tilde{W}_4. \] 

With the above relations and the Schrödinger-like equations (3.2)–(3.4), one can easily find the following results:

\[ C_1^{(n)2} = m_1^{(n)2}, \]  
\[ C_2^{(n)2} = m_2^{(n)2}, \]  
\[ C_3^{(n)2} = m_3^{(n)2}, \]

which are similar to those obtained in models with codimension-1 and codimension-2, and are very important for the invariance of the effective action.

- Finally, let us consider the couplings between the scalar KK modes. We replace the coupling constants \( I_{10}^{(n)}, I_{14}^{(n)}, \) and \( I_{16}^{(n)} \) as \( C_4^{(n)}, C_5^{(n)}, \) and \( C_6^{(n)} \) for convenience, and with the consistence of the two groups of EOM we find:

\[ I_{10}^{(n)} = \lambda_{10} \equiv C_4^{(n)} = -2\bar{\partial}_y(\bar{\partial}_z \tilde{W}_3^{(n)})/\tilde{W}_2^{(n)}, \]  
\[ I_{10}^{(n)} = \lambda_{15} \equiv C_4^{(n)} = -2\bar{\partial}_z(\bar{\partial}_y \tilde{W}_2^{(n)})/\tilde{W}_3^{(n)}, \]  
\[ I_{14}^{(n)} = \lambda_{11} \equiv C_5^{(n)} = -2\bar{\partial}_w(\bar{\partial}_z \tilde{W}_4^{(n)})/\tilde{W}_2^{(n)}, \]  
\[ I_{14}^{(n)} = \lambda_{20} \equiv C_5^{(n)} = -2\bar{\partial}_w(\bar{\partial}_z \tilde{W}_2^{(n)})/\tilde{W}_4^{(n)}, \]  
\[ I_{16}^{(n)} = \lambda_{16} \equiv C_6^{(n)} = -2\bar{\partial}_w(\bar{\partial}_y \tilde{W}_4^{(n)})/\tilde{W}_3^{(n)}, \]  
\[ I_{16}^{(n)} = \lambda_{21} \equiv C_6^{(n)} = -2\bar{\partial}_w(\bar{\partial}_y \tilde{W}_3^{(n)})/\tilde{W}_4^{(n)}. \]

By using the relationships between \( W_1^{(n)} \) with \( W_2^{(n)}, W_3^{(n)}, W_4^{(n)} \), i.e., Eqs. (3.11) to the above equations, one would finally obtain

\[ C_4^{(n)2} = 4m_{1}^{(n)2}m_{1}^{(n)2}, \quad C_2^{(n)2} = m_{1}^{(n)2}, \quad C_1^{(n)2} = m_{1}^{(n)2}, \]  
\[ C_5^{(n)2} = 4m_{2}^{(n)2}m_{2}^{(n)2}, \quad C_3^{(n)2} = m_{2}^{(n)2}, \quad C_1^{(n)2} = m_{2}^{(n)2}, \]  
\[ C_6^{(n)2} = 4m_{3}^{(n)2}m_{3}^{(n)2}, \quad C_3^{(n)2} = m_{3}^{(n)2}, \quad C_3^{(n)2} = m_{3}^{(n)2}. \]

Interestingly, we find that the masses of the scalar KK modes are related with the masses of the vector modes:

\[ m_{1}^{(n)2} = m_{2}^{(n)2} + m_{3}^{(n)2}, \quad m_{2}^{(n)2} = m_{1}^{(n)2} + m_{3}^{(n)2}, \quad m_{3}^{(n)2} = m_{1}^{(n)2} + m_{2}^{(n)2}. \] (3.24)

This is another crucial reason for the gauge invariance of the effective action.
4 The gauge-invariant effective action

Now with all above relationships between the coupling constants, we can simplified the effective action (2.3) as

\[
S = -\frac{1}{4} \sum_n \sum_{n'} \int d^4x \sqrt{-g} \left[ \mathcal{Y}(n)_{\mu_1\mu_2} \mathcal{Y}(n')_{\mu_1\mu_2} + 2 \left( (\partial_\mu \phi(n) - \frac{1}{2} m_1 \phi(n)) \right)^2 + \left( (\partial_\mu \varphi(n) - \frac{1}{2} m_2 \varphi(n)) \right)^2 + \left( (\partial_\mu \chi(n) - \frac{1}{2} m_3 \chi(n)) \right)^2 \right],
\]

which is invariant under the following gauge transform

\[
\begin{align*}
\mathcal{X}(n)_{\mu} &\rightarrow \mathcal{X}(n)_{\mu} + \partial_\mu \rho(n), \\
\phi(n) &\rightarrow \phi(n) + \frac{1}{2} m_1 \rho(n), \\
\varphi(n) &\rightarrow \varphi(n) + \frac{1}{2} m_2 \rho(n), \\
\chi(n) &\rightarrow \chi(n) + \frac{1}{2} m_3 \rho(n).
\end{align*}
\]

5 Summary and discussions

In this work, we investigated the KK modes of a bulk massless $U(1)$ gauge field for a seven-dimensional brane model, whose line-element was given in Eq. (1.1). Our results are summarized as follows:

- Using a general KK decomposition (2.2) without choosing any gauge for the bulk $U(1)$ gauge field, we found four types of KK modes, i.e., one vector and three types of scalar ones. In addition to their self-interactions, these modes also coupled mutually, as can be seen in (2.3), thus there are many coupling constants in the effective action, which should be not independent if the effective action is invariant;

- The EOM for the KK modes have been derived firstly from the bulk EOM, and then from the effective action. The EOM derived from these two approaches should be consistent. From this requirement we found that all the KK modes are constrained by some Schrödinger-like equations, for which the corresponding effective potentials depend on the warp factor of the background geometry. It was also found some precise relationships between the coupling constants in the effective action (2.3), which also are the relationships between $W_1(n), W_2(n), W_3(n)$ and $W_4(n)$;

- With the above relationships we simplified the effective action for the $U(1)$ gauge field and proved that it is gauge invariant. This action implies that the vector KK modes can obtain masses from the three extra dimensions, but the scalar ones only can obtain masses from two of the three extra dimensions. The masses of the scalars are related with the vectors.

In fact, it is not new that the extra dimension can lead to a gauge-invariant action for the massive vector field. In early works [57, 58], the authors had discussed this question in string theory, but the mechanism is different for models with even and odd number of
extra dimensions. However, as we have shown in Refs. [42, 44] and in the present work, for the Randall-Sundrum type brane model, the same mechanism applied even for different extra dimensions. In our mechanism, it is the vector-scalar couplings that to keep the gauge invariance. Although we have considered only the cases with one, two or three extra dimensions, one may also ask how to generalize this machenism to models with more extra dimensions, and what is the rule behind this.

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