Optimal Local and Remote Controls of Multiple Systems with Multiplicative Noises and Unreliable Uplink Channels
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Abstract—In this paper, the optimal local and remote linear quadratic (LQ) control problem is studied for a networked control system (NCS) which consists of multiple subsystems and each of which is described by a general multiplicative noise stochastic system with one local controller and one remote controller. Due to the unreliable uplink channels, the remote controller can only access unreliable state information of all subsystems, while the downlink channels from the remote controller to the local controllers are perfect. The difficulties of the LQ control problem for such a system arise from the different information structures of the local controllers and the remote controller. By developing the Pontryagin maximum principle, the necessary and sufficient solvability conditions are derived, which are based on the solution to a group of forward and backward difference equations (G-FBSDEs). Furthermore, by proposing a new method to decouple the G-FBSDEs and introducing new coupled Riccati equations (CREs), the optimal control strategies are derived where we verify that the separation principle holds for the multiplicative noise NCSs with packet dropouts. This paper can be seen as an important contribution to the optimal control problem with asymmetric information structures.

Index Terms—Multiplicative noise system, multiple subsystems, optimal local and remote controls, Pontryagin maximum principle.

I. INTRODUCTION

As is well known, NCSs are systems in which actuators, sensors, and controllers exchange information through a shared bandwidth limited digital communication network. The research on NCSs has attracted significant interest in recent years, due to the advantages of NCSs such as low cost and simple installation, see [1]–[8] and the cited references therein. On the other hand, unreliable wireless communication channels and limited bandwidth make NCSs less reliable [3]–[5], which may cause performance loss and destabilize the NCSs. Thus, it is necessary to investigate control problems for NCSs with unreliable communication channels.

In this paper, we investigate multiplicative noise NCSs (MN-NCSs) with local and remote controls and unreliable uplink channels. As shown in Figure 1, the NCS is composed of $L$ subsystems, and each subsystem is regulated by one local controller and one remote controller. Due to the limited communication capability of each subsystem, the uplink channel from a local controller to the remote controller is affected by packet dropouts, while the downlink channel from the remote controller to each local controller is perfect. The LQ optimal control problem is considered in this paper. Our aim is to design $L$ local controllers and the remote controller such that a given cost function is minimized. Due to the uncertainty of the uplink channels, the information sets available to the local controllers and the remote controller are different. Hence, we are concerned with the optimal control problem with an asymmetric information structure which pose major challenges.

The pioneering study of asymmetric information control can be traced back to the 1960s, and it is well-known that finding optimal controls for an asymmetric-information-control problem is difficult, see [9]–[12]. For example, [9] gives the celebrated Witsenhausen’s Counterexample for which the associated explicit optimal control with asymmetric information structures still remains to be solved.

The optimal local and remote control problems were first studied in [13], [16]. By using the common information approach, the optimal local and remote controls are derived. Following [16], an elementary proof of the common information approach was given in [13]. Furthermore, [17] studies the optimal local and remote control problem with only one subsystem by using maximum principle approach. It should be pointed out that previous works [13]–[15] focused on NCSs with additive noise. Different from the previous works, this paper investigates the local control and remote control problem for networked systems with multiplicative noise and multiple subsystems. The motivations of this paper are: On one hand, multiplicative noise systems exist in many applications. The
existence of multiplicative noise results in the non-Gaussian property of the NCSs, see [19]–[24]. On the other hand, the unreliable uplink channels result in that the state estimation errors are involved with the control inputs, which may lead to the failure of the separation principle. In other words, it remains difficult to design the optimal output feedback control for MN-NCSs with unreliable uplink channels, see [16]–[19]. Furthermore, the optimal local and remote control problem for multiple subsystems can be regarded as a special case of optimal control for multi-agent systems, for which the optimal decentralized/distributed control design remains challenging, see [25], [26].

Based on the above discussions, we can conclude that the study of the local and remote control problem for MN-NCSs with multiple subsystems has the following difficulties and challenges: 1) Due to the possible failure of the separation principle for MN-NCSs with unreliable uplink channels, the derivation of the optimal “output feedback” controllers remains challenging. 2) When the Pontryagin maximum principle is adopted, how to decouple a group of forward and backward difference equations (G-FBSDEs) is difficult and unsolved.

In this paper, the optimal local and remote control problem for MN-NCSs with multiple subsystems is solved. Firstly, by developing the Pontryagin maximum principle, we show that the optimal control problem under consideration is uniquely solved if and only if a group of FBSDEs (G-FBSDEs) is uniquely solved; Consequently, a method is proposed to decouple the G-FBSDEs, it is shown that the solution to the original G-FBSDEs can be given by decoupling new G-FBSDEs and introducing new information filtrations. Furthermore, the optimal local and remote controllers are derived based on the solution to coupled CREs, which are asymmetric. As special cases, the additive noise NCSs case, the single subsystem case, and the indefinite weighting matrices case are also investigated.

As far as we know, the obtained results are new and innovative in the following aspects: 1) Compared with the common information approach adopted in previous works, the structure of the optimal local controllers and remote controller is not assumed in advance, see Lemma 3 in [13]; 2) In this paper, the multiple subsystems case is solved by using the Pontraygin Maximum principle, while previous works focused on the single subsystem case [17]; 3) The existence of multiplicative noise results in that the state estimation error and the error covariance are involved with the controls, which may cause the separation principle to fail. To overcome this, new asymmetric CREs are introduced; It is verified that the separation principle holds for the considered optimization problem, i.e., the optimal local controllers and optimal remote controller can be designed, and the control gain matrices and the estimation gain matrix can be calculated separately; 4) It is noted that the obtained results include the results in [13], [16], [17] as special cases.

The rest of the paper is organized as follows. Section II investigates the existence of the optimal control strategies. The main results are presented in Section III, where the optimal local controllers and optimal remote controller are derived by decoupling the G-FBSDEs. Consequently, some discussions are given in Section IV. In Section V, numerical examples are presented to illustrate the main results. We conclude this paper in Section VI. Finally, the proof of the main results is given in the Appendix.

The following notations will be used throughout this paper: \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space, and \( A^T \) means the transpose of matrix \( A \); The subscript of \( A_i, i = 1, \cdots, L \) means the \( i \)-th subsystem, and subscript of \( A_k \) denotes \( A \) to the power of \( k \). Symmetric matrix \( M > 0 \) (positive definite); \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space; \( I_n \) denotes the identity matrix of dimension \( n \); \( E[\cdot] \) means the mathematical expectation and \( E[\cdot|Y] \) signifies the conditional expectation with respect to \( Y \). \( \mathcal{N}(\mu, \Sigma) \) is the normal distribution with mean \( \mu \) and covariance \( \Sigma \), and \( Pr(A) \) denotes the probability of the occurrence of event \( A \). \( \mathcal{F}(X) \) means the filtration generated by random variable/vector \( X \), \( Tr(\cdot) \) means the trace of a matrix, \( vec(x), vec(x^2), vec(x^3), \cdots \) means the vector \( (x^1)^T, (x^2)^T, (x^3)^T, \cdots \), and \( \sigma(X) \) denotes the \( \sigma \)-algebra generated by random vector \( X \).

II. EXISTENCE OF OPTIMAL CONTROL STRATEGY

A. Problem Formulation

The system dynamics of the \( i \)-th subsystem \( (i = 1, \cdots, L) \) is given by

\[
x_{k+1}^i = [A^i + w_k^i A^i]x_k^i + [B^i + u_k^i B^i]u_k^i + [B_0^0 + u_k^i B_0^0]u_0^0 + v_k^i.
\]

(1)

where \( x_k^i \in \mathbb{R}^{n_i} \) is the system state of the \( i \)-th subsystem at time \( k \), \( A^i, A^i \in \mathbb{R}^{n_i \times n_i}, B^i, B^i \in \mathbb{R}^{n_i \times m_i}, B_0^0, B_0^0 \in \mathbb{R}^{n_i \times m_0} \) are the given system matrices, both \( w_k^i \in \mathbb{R}^{n_i} \) and \( v_k^i \in \mathbb{R}^{n_i} \) are Gaussian white noises satisfying \( w_k^i \sim \mathcal{N}(0, \Sigma_{w_k^i}), v_k^i \sim \mathcal{N}(0, \Sigma_{v_k^i}) \). The initial state \( x_0^i \sim \mathcal{N}(\mu^i, \Sigma_{x_0^i}) \), \( v_k^i \in \mathbb{R}^{m_i} \) is the local controller of the \( i \)-th subsystem, and \( u_k^i \in \mathbb{R}^{m_0} \) is the remote controller. Since the uplink channels are unreliable, let binary random variables \( \gamma_k^i, i = 1, 2, \ldots \) with probability distribution \( Pr(\gamma_k^i = 1) = p_i \) denote if a packet is successfully transmitted, i.e., \( \gamma_k^i = 1 \) means that the packet is successfully transmitted from the \( i \)-th local controller to the remote controller, and fails otherwise. \( p_i, i = 1, \cdots, L \) is called the packet dropout rate. Moreover, we assume \( x_0^i, \{\gamma_k^i\}^N_{k=0}, \{u_k^i\}^N_{k=0}, \{v_k^i\}^N_{k=0} \) are independent of each other for all \( i = 1, \cdots, L \).

For the sake of discussion, the following notions are denoted:

\[
X_k = vec(x_1^i, \cdots, x_L^i), U_k = vec(u_k^i, u_k^i, \cdots, u_k^L),
\]

\[
V_k = vec(v_1^i, \cdots, v_L^i), N_L = \sum_{i=1}^{L} n_i, M_L = \sum_{i=0}^{L} m_i,
\]

\[
A = \begin{bmatrix}
A^1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A^L
\end{bmatrix},
B = \begin{bmatrix}
B_0^0 & B_1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
B^L_0 & 0 & \cdots & B^L
\end{bmatrix},
\]

\[
L_{1:r} = \begin{bmatrix}
L_{1:0} & L_{2:0} & \cdots & L_{r:0}
\end{bmatrix},
\]

\[
\mathcal{F}(X) = \mathcal{F}(x_0), \mathcal{F}(U_k), \mathcal{F}(V_k), \mathcal{F}(A), \mathcal{F}(B), \mathcal{F}(L_{1:r}), \mathcal{F}(N_L), \mathcal{F}(M_L).
\]
Using the notations in (2), we can rewrite system (1) as
\[ X_{k+1} = AX_k + BU_k + \sum_{i=1}^{L} w_{ik}^i (\hat{A}^i X_k + \hat{B}^i U_k) + V_k \] (3)
with initial \( X_0 = \text{vec}(x_0^0, \ldots, x_0^L) \).

Associated with system (1), the following cost function is introduced
\[ J_N = E \left[ \sum_{k=0}^{N} (X_k^T Q X_k + U_k^T R U_k) + X_{N+1}^T P_{N+1} X_{N+1} \right] \] (4)
where \( Q, R, P_{N+1} \) are symmetric weighting matrices of appropriate dimensions with
\[ Q = \begin{bmatrix} Q^{11} & \cdots & Q^{1L} \\ \vdots & \ddots & \vdots \\ Q^{L1} & \cdots & Q^{LL} \end{bmatrix}, \quad R = \begin{bmatrix} R^{00} & \cdots & R^{0L} \\ \vdots & \ddots & \vdots \\ R^{L0} & \cdots & R^{LL} \end{bmatrix}, \]
\[ P_{N+1} = \begin{bmatrix} P^{01}_{N+1} & \cdots & P^{0L}_{N+1} \\ \vdots & \ddots & \vdots \\ P^{L0}_{N+1} & \cdots & P^{LL}_{N+1} \end{bmatrix}, \]
and block matrices \( Q^{ij}, P^{ij}_{N+1} \in \mathbb{R}^{n_i \times n_j}, i, j = 1, \ldots, L, R^{ij} \in \mathbb{R}^{m_i \times m_j}, i, j = 0, \ldots, L \).

Corresponding with the system model described in Figure 1, the feasibility of controllers \( u_k^i, i = 0, \ldots, L \) is given in the following assumption.

**Assumption 1.** The remote controller \( u_k^0 \) is \( \mathcal{F}_k^0 \)-measurable, and the local controller \( u_k^i \) is measurable with respect to \( \mathcal{F}_k^i \), \( i = 1, \ldots, L \), where
\[ \mathcal{F}_k^0 = \sigma \left\{ \{\Gamma_m \}_{m=0}^k, \{\Gamma_m X_m \}_{m=0}^k, \{u_m^0 \}_{m=0}^{k-1} \right\}, \]
\[ \mathcal{F}_k^i = \sigma \left\{ \{\Gamma_m \}_{m=0}^k, \{\Gamma_m X_m \}_{m=0}^k, \{u_m^0 \}_{m=0}^{k-1}, \{x_m^i \}_{m=0}^{k-1} \right\}, \quad i = 1, \ldots, L. \]

It can be easily judged from (6)-(7) that \( \mathcal{F}_k^0 \subset \mathcal{F}_k^i, i = 1, \ldots, L \).

Next, we will introduce the LQ control problem to be solved in this paper.

**Problem 1.** For system (1)-(3), find \( \mathcal{F}_k^i \)-measurable control \( u_k^i \) to minimize cost function (4), where \( k = 0, \ldots, N, i = 0, \ldots, L \).

Throughout this paper, the assumption on the weighting matrices of (4) is given as follows.

**Assumption 2.** \( Q \geq 0, R > 0, \) and \( P_{N+1} \geq 0 \).

**Remark 1.** It is stressed that Problem 1 has not been solved in the existing literatures. The previous works [13–17] mainly focused on additive noise systems, and their multiplicative noise counterpart remains less investigated. The existence of unreliable uplink channels for multiplicative noise systems may result in the failure of “separation principle”, making the design of optimal control in Problem 1 difficult. Furthermore, the maximum principle was adopted to solve only the single subsystem case in [17], while the multiple subsystems case hasn’t been solved in the framework of maximum principle. The main challenge for the multiple subsystems case is that the solution for the G-FBSDEs is difficult.

B. Necessary and Sufficient Solvability Conditions

In this section, we will derive the necessary and sufficient solvability conditions for Problem 1.

**Lemma 1.** Given system (1)-(3) and cost function (4), for \( i = 1, \ldots, L, k = 0, \ldots, N, \) let \( u_k^i = u_k^i + \varepsilon u_k^i \), where \( \varepsilon \in \mathbb{R}^1 \) and \( \delta u_k^i \) is \( \mathcal{F}_k^i \)-measurable satisfying \( \sum_{k=0}^{N} E[ (\delta u_k^i)^T \delta u_k^i ] < +\infty \). Denote \( x_k^i, x_k^e \) and \( J_N \) the corresponding state and cost function associated with \( u_k^i, u_k^e = \text{vec}(u_k^0, \ldots, u_k^L) \), \( X_k = \text{vec}(x_k^0, \ldots, x_k^L) \), and \( \delta U_k = \text{vec}(\delta u_k^0, \ldots, \delta u_k^L) \). Then we have
\[ J_N - J_N = \varepsilon^2 \left\{ \sum_{k=0}^{N} E[ Y_k^T Q Y_k + \delta U_k R \delta U_k] + E[ Y_k^T P_{N+1} Y_{N+1}] \right\} + 2\varepsilon \sum_{k=0}^{N} E[ (\Theta_k^T (B + \sum_{i=1}^{L} w_i^i B^i) + RU_k)^T \delta U_k], \]
where \( Y_{k+1} = (A + \sum_{i=1}^{L} w_i^i \hat{A}^i) Y_k + (B + \sum_{i=1}^{L} w_i^i \hat{B}^i) \delta U_k \) (9)
with initial condition \( Y_0 = 0 \), and \( \Theta_k = \text{vec}(\Theta_k^0, \ldots, \Theta_k^L) \) satisfies the iteration
\[ \Theta_{k-1} = E \left[ (A + \sum_{i=1}^{L} w_i^i \hat{A}^i)^T \Theta_k + Q X_k \right| \mathcal{G}_k \] (10)
with terminal condition \( \Theta_N = P_{N+1} X_{N+1} \) and the information filtration \( \mathcal{G}_k \) given by
\[ \mathcal{G}_k = \sigma \left\{ \{\Gamma_m \}_{m=0}^k, \{U_m \}_{m=0}^{k-1}, \{X_m \}_{m=0}^{k-1} \right\}. \] (11)

**Proof.** By setting \( Y_k = \frac{X_k^i - X_k^e}{\varepsilon} \), we know that \( Y_0 = 0 \) and (9) holds.
Subsequently, it can be calculated that
\[
J_N^* - J_N = \sum_{k=0}^{N} E\left[ \left( X_k + \varepsilon Y_k \right)^T Q \left( X_k + \varepsilon Y_k \right) \right] \\
+ \left[ U_k + \varepsilon \delta U_k \right]^T R \left[ U_k + \varepsilon \delta U_k \right] \\
+ E\left[ X_{N+1} + \varepsilon Y_{N+1} \right]^T \left( P_{N+1} \left[ X_{N+1} + \varepsilon Y_{N+1} \right] + \varepsilon \left( J_N^* - J_N \right) \right) \\
- \sum_{k=0}^{N} E \left[ X_k^T Q X_k + U_k^T RU_k \right] \\
= 2 \varepsilon E \left[ \sum_{k=0}^{N} \left( X_k^T Q Y_k + \delta U_k^T RU_k \right) \right] \\
+ \varepsilon^2 E \left[ \sum_{k=0}^{N} \left( Y_k^T Q Y_k + \delta U_k^T RU_k \right) \right] \\
+ \varepsilon E \left[ \sum_{k=0}^{N} \left( Y_k^T Q Y_k + \delta U_k^T \delta U_k \right) \right] \\
+ \varepsilon^2 E \left[ \sum_{k=0}^{N} \left( Y_k^T Q Y_k + \delta U_k^T \delta U_k \right) \right].
\]

Using (9), (10), and noting that \( Y_k \) is \( \mathcal{G}_k \)-measurable, we have
\[
E \left[ \sum_{k=0}^{N} \left( X_k^T Q Y_k + \delta U_k^T RU_k \right) \right] \\
= E \left[ \sum_{k=0}^{N} \left( \Theta_k - 1 \right) \left( A + \sum_{i=1}^{L} w_i \hat{A}^i \right)^T \Theta_k \right] \\
+ \delta U_k^T RU_k + \Theta_N^T Y_{N+1} \\
= E \left[ \sum_{k=0}^{N} \left( \Theta_k^T \left( B + \sum_{i=1}^{L} w_i \hat{B}^i \right) + RU_k \right) \delta U_k \right],
\]
which ends the proof.

Accordingly, we have the following results.

**Theorem 1.** Under Assumptions 1 and 2, Problem 1 can be uniquely solved if and only if the equilibrium condition
\[
0 = RU_k + E \left[ \left( B + \sum_{i=1}^{L} w_i \hat{B}^i \right)^T \Theta_k \mathcal{G}_k \right] \tag{12}
\]
can be uniquely solved, where the costate \( \Theta_k \) satisfies (10).

**Proof.** ‘Necessity’: Suppose Problem 1 is uniquely solvable and \( U_k = vec(u_k^1, \ldots, u_k^L) \) are optimal controls for \( k = 0, \ldots, N \).

Using the symbols in Lemma 1 and from (8) we know that, for arbitrary \( \delta U_k \) and \( \varepsilon \in \mathbb{R} \), there holds
\[
J_N^* - J_N = \varepsilon^2 \delta J_N \\
+ 2 \varepsilon \sum_{k=0}^{N} E \left[ \left( \Theta_k^T \left( B + \sum_{i=1}^{L} w_i \hat{B}^i \right) + RU_k \right) \delta U_k \right] \\
\geq 0, \tag{13}
\]
where
\[
\delta J_N = \sum_{k=0}^{N} E \left[ Y_k^T Q Y_k + \delta U_k \delta U_k \right] \\
+ E \left[ Y_{N+1}^T P_{N+1} Y_{N+1} \right]. \tag{14}
\]
Observe from Assumption 2 that \( \delta J_N \geq 0 \). Next we will show (12) holds.

Suppose, by contradiction, that (12) is not satisfied. Let
\[
RU_k + E \left[ \left( B + \sum_{i=1}^{L} w_i \hat{B}^i \right)^T \Theta_k \mathcal{G}_k \right] = \tau_k \neq 0. \tag{15}
\]
In this case, if we choose \( \delta U_k = \tau_k \), then from (13) we have
\[
J_N^* - J_N = 2 \varepsilon \sum_{k=0}^{N} \tau_k^T \tau_k + \varepsilon^2 \delta J_N.
\]
Note that we can always find some \( \varepsilon < 0 \) such that \( J_N^* - J_N < 0 \), which contradicts with (13). Thus, \( \tau_k = 0 \). This ends the necessity proof.

‘Sufficiency’: If (12) is uniquely solvable, we shall show that Problem 1 is uniquely solvable under Assumptions 1, 2.

Actually, from (8) we know that for any \( \varepsilon \in \mathbb{R} \), \( J_N^* - J_N = \varepsilon^2 \delta J_N \geq 0 \), which means that Problem 1 is uniquely solvable.

It is noted that system dynamics (11) and (5) are forward, and the costate equation (10) is backward, then (11), (3), (10) and (12) constitute the G-FBSDEs. For the convenience of discussion, we denote the following G-FBSDEs composed of (1), (3), (10) and (12):
\[
\begin{align*}
\dot{x}_{k+1}^i &= \left( A^i + \sum_{j=1}^{L} w_i^j \hat{A}^j \right) x_k^i + \left( B^i + \sum_{j=1}^{L} w_i^j \hat{B}^j \right) u_k^j \\
&\quad + \sum_{j=1}^{L} w_i^j \hat{B}^j u_k^j + v_k^i, \quad x_0^i, i = 1, \ldots, L, \\
X_{k+1} &= \left( A + \sum_{i=1}^{L} w_i \hat{A} \right) X_k \\
&\quad + \sum_{i=1}^{L} w_i \hat{B} \mathcal{G}_k U_k + V_k, \\
\Theta_{k+1} &= E \left[ \left( A + \sum_{i=1}^{L} w_i \hat{A} \right)^T \Theta_k + \left( Q X_k \right) \mathcal{G}_k \right], \\
\Theta_N &= P_{N+1} X_{N+1}, \\
0 &= RU_k + E \left[ \left( B + \sum_{i=1}^{L} w_i \hat{B} \right)^T \Theta_k \mathcal{G}_k \right].
\end{align*}
\]

**Remark 2.** The necessary and sufficient solvability conditions of Problem 1 given in Theorem 1 are presented for the first time, which are based on the solution to G-FBSDEs (10). Consequently, to derive the optimal control strategies \( u_k^i, i = 0, \ldots, L, k = 0, \ldots, N \), we will find a method of decoupling G-FBSDEs (16).

Consequently, we will introduce some preliminary results.

**Lemma 2.** Denote \( \hat{u}_k^i = E\left[ u_k^i | \mathcal{G}_k^i \right] \), then the following relationship holds:
\[
E\left[ u_k^i | \mathcal{G}_k^i \right] = \left\{ \begin{array}{ll}
\hat{u}_k^i, & j = i, \\
u_k^i, & j \neq i.
\end{array} \right. \tag{17}
\]
where the information filtration \( \mathcal{G}_k^i \) is given by
\[
\mathcal{G}_k^i = \sigma \left\{ \{ \Gamma_m \}_{m=0}^{k}, \{ \Gamma_m X_k \}_{m=0}^{k}, \{ U_m \}_{m=0}^{k-1} \right\}. \tag{18}
\]

**Proof.** Due to the independence of \( \hat{u}_k^i \) and \( \{ \Gamma_m \}_{m=0}^{k}, \{ U_m \}_{m=0}^{k-1}, \{ u_m^j \}_{m=0}^{k}, j \neq i \), (17) can be obtained by using the properties of conditional expectation.

By using Lemma 2, the following result can be derived.
Lemma 3. Under Assumptions 2 and 2, the equilibrium condition (12) can be rewritten as:

\[
0 = R\hat{U}_k + E[(B + \sum_{l=1}^{L} w_k^l B^l)^T \Theta_k | F_k^0], \quad (19)
\]

\[
0 = R\hat{U}_k + E[(B + \sum_{l=1}^{L} w_k^l B^l)^T \Theta_k | G_k]
\]

\[
- E[(B + \sum_{l=1}^{L} w_k^l B^l)^T \Theta_k | F_k^0], \quad (20)
\]

\[
0 = R^{ii} \hat{u}_k^i + E[(B^i + w_k^i B_i^i)^T \Theta_k | G_k]
\]

\[
- E[(B^i + w_k^i B_i^i)^T \Theta_k | G_k^i], \quad (21)
\]

where \( \hat{U}_k = vec(u_k^1, u_k^2, \ldots, u_k^L) \), \( \hat{U}_k = U_k - \hat{U}_k \) and \( \Theta_k = vec(\Theta_k^1, \Theta_k^2, \ldots, \Theta_k^L) \) satisfies (10).

Proof. The results can be easily derived from Lemma 2.

In view of Lemma 3, the G-FBSDEs (16) can be equivalently presented as:

\[
\begin{aligned}
\dot{x}_{k+1}^i &= [A^i + w_k^i A_i^i]x_k^i + [B^i + w_k^i B_i^i]u_k^i + [B_i^0 + w_k^i B_i^0]u_k^i + v_k^i, \\
X_{k+1} &= (A + \sum_{l=1}^{L} w_k^l A_i^l)x_k + V_k, \\
\Theta_{k-1} &= E[(A + \sum_{l=1}^{L} w_k^l A_i^l)\hat{u}_k + QX_k|G_k], \\
\Theta_N &= P_{N+1}X_{N+1}, \quad (22)
\end{aligned}
\]

\[
\begin{aligned}
0 &= R\hat{U}_k + E[(B + \sum_{l=1}^{L} w_k^l B_i^l)^T \Theta_k | F_k^0], \\
0 &= R\hat{U}_k + E[(B + \sum_{l=1}^{L} w_k^l B_i^l)^T \Theta_k | G_k], \\
0 &= E[(B + \sum_{l=1}^{L} w_k^l B_i^l)^T \Theta_k^0 | G_k^0], \\
0 &= E[(B^i + w_k^i B_i^i)^T \Theta_k^i | G_k^i].
\end{aligned}
\]

In the following lemma, we will introduce the preliminary results on the optimal estimation and the associated state estimation error.

Lemma 4. The optimal estimation \( \hat{x}_{k+1}^i \) \( \triangleq E[x_k^i | F_k^0] \), \( i = 1, \ldots, L \) and \( \bar{X}_k \triangleq E[X_k | F_k^0] \) can be calculated by:

\[
\begin{aligned}
\hat{x}_{k+1}^i &= (1 - \gamma_{k+1}^i)(A^i\hat{x}_k^i + B^i\hat{u}_k^i + B_i^0u_k^i) \\
+ \gamma_{k+1}^i \bar{x}_{k+1}, \\
\bar{X}_{k+1} &= (I_{N\times L} - \Gamma_{k+1})(A\bar{X}_k + B\bar{U}_k) + \Gamma_{k+1}X_{k+1}, \quad (23)
\end{aligned}
\]

with initial conditions \( \hat{x}_0^i = \gamma_0^i \mu_i + (1 - \gamma_0^i) x_0, \hat{X}_0 = (I_{N\times L} - \Gamma_0)x_0 \). In this case, the error covariance \( \tilde{x}_{k+1}^i = x_k^i - \hat{x}_{k+1}^i, i = 1, \ldots, L \) and \( \bar{X}_k = X_k - \bar{X}_k \) satisfy:

\[
\begin{aligned}
\tilde{x}_{k+1}^i &= (1 - \gamma_{k+1}^i)
[ A^i\tilde{x}_k^i + B\hat{u}_k^i + \bar{B}\tilde{u}_k^i + B_i^0u_k^i ] + \gamma_{k+1}^i \bar{x}_{k+1}, \\
\bar{X}_{k+1} &= (I_{N\times L} - \Gamma_{k+1})
[ A\bar{X}_k + B\hat{U}_k ] + \sum_{l=1}^{L} w_l^i (A_i^l\bar{X}_k + B_i^l(\hat{u}_k + \bar{u}_k)) + V_k, \quad (24)
\end{aligned}
\]

Proof. The detailed proof can be found in [3], [6], which is omitted here.

Remark 3. It can be observed from (25), (26) that the controls are involved with the state estimation error, which is the key difference from the additive noise case (i.e., \( w_k^i = 0 \)), see [13]–[17]. Moreover, since G-FBSDEs (16) and G-FBSDEs (22) are equivalent, we will derive the optimal controls by solving (22) instead.

III. Optimal Controls by Decoupling G-FBSDEs

In this section, the optimal controls \( u_k^i, i = 0, \ldots, L \) will be derived via decoupling the G-FBSDEs (16) (equivalently G-FBSDEs (24)).

Firstly, the following CREs are introduced:

\[
\begin{aligned}
P_k &= Q + A^T P_{k+1} A + \sum_{l=1}^{L} \Sigma_{w_l} (A_i^l)^T L_{k+1} \tilde{A}_i \\
H_k &= Q + A^T L_{k+1} A + \sum_{l=1}^{L} \Sigma_{w_l} (A_i^l)^T L_{k+1} \tilde{A}_i \\
\Psi_k &= (\Omega_k^T (\Pi_k - \Pi_k^i)^{-1} \Omega_k^i)^T \\
\Omega_k &= (\Omega_k^T (\Pi_k - \Pi_k^i)^{-1} \Omega_k^i)^{-1}. \quad (27)
\end{aligned}
\]

with terminal conditions \( L_{N+1} = H_{N+1} = P_{N+1}, L_i = P_i, i = 1, \ldots, N \) and the coefficients matrices \( \Lambda_k, \Psi_k, \Lambda_i, \Psi_i, \Pi_i^k, \Pi_i^i, \Pi_k, \Omega_k, \Lambda_k, \Lambda_i, \Pi_k, \Omega_k, \Pi_k, \Omega_k, \Pi_k, \Omega_k, \Pi_k, \Omega_k \) being given by:

\[
\begin{aligned}
\Lambda_k &= R + B^T P_{k+1} B + \sum_{l=1}^{L} \Sigma_{w_l} (B_i^l)^T L_{k+1} B_i^l, \\
\Psi_k &= B^T P_{k+1} + \sum_{l=1}^{L} \Sigma_{w_l} (B_i^l)^T L_{k+1} B_i^l, \\
\Pi_i^k &= R_i^0 + \sum_{l=1}^{L} \Sigma_{w_l} (B_i^l)^T L_{k+1} B_i^l, \\
\Pi_i^i &= R_i^0 + \sum_{l=1}^{L} \Sigma_{w_l} (B_i^l)^T L_{k+1} B_i^l, \\
\Theta_i^k &= (\Omega_i^0)^T (\Pi_i - \Pi_i^i)^{-1} \Omega_i^i \\
\Theta_i^i &= (\Omega_i^0)^T (\Pi_i - \Pi_i^i)^{-1} \Omega_i^i, \quad (28)
\end{aligned}
\]

Remark 4. For CREs (27), the following points should be noted:

- The CREs (27) are well defined (i.e., can be recursively calculated backwardly) if and only if \( \Lambda_k, \Lambda_i, \Pi_i^k, \Pi_i^i, \Pi_k, \Pi_i^k, \Pi_i^i, \Pi_k, \Omega_k, \Pi_k, \Omega_k, \Pi_k, \Omega_k, \Pi_k, \Omega_k, \Pi_k, \Omega_k, \Pi_k, \Omega_k \) are invertible.

- Different from traditional symmetric Riccati equation (27), \( P_k, H_k \) in (27) are asymmetric, while \( P_i, \Pi_i \) are symmetric.

Lemma 5. Under Assumptions 2 and 2, \( \Pi_i^k, \Pi_i^i \) are positive definite for \( k = 0, \ldots, N \).

Proof. To facilitate the discussions, we denote:

\[
\begin{aligned}
\tilde{g}_k &= -(\Pi_i^k)^{-1} \tilde{\Theta}_k, \\
\tilde{g}_k^i &= -(\Pi_i^i)^{-1} \tilde{\Theta}_k^i.
\end{aligned}
\]

With \( k = N \), from \( (27), (29) \) we know that:

\[
\begin{aligned}
\tilde{g}_N^i &= (\Omega_k^0)^T (\Pi_i^i)^{-1} \Omega_k^i \\
\tilde{g}_N &= (\Omega_k^0)^T (\Pi_i^i)^{-1} \Omega_k^i, \quad (30)
\end{aligned}
\]

Then we can rewrite \( P_i^k, P_i^i \) in CREs (27) as:

\[
P_i^k = Q^{ii} + (A_i^i)^T P_{i+k+1} A_i^i + \Sigma_{w_i} (A_i^i)^T L_{i+k+1} \tilde{A}_i
\]
Similarly, it is not hard to obtain that
\[ H^i_N = Q^{ii} + (g^{ii}_N)^T R^{ii} g^{ii}_N + (\hat{A}^i + B^i g^{ii}_N)^T P^i_{N+1} (\hat{A}^i + B^i g^{ii}_N) + \Sigma_w (\hat{A}^i + B^i g^{ii}_N)^T L^i_{N+1} (\hat{A}^i + B^i g^{ii}_N). \] (32)

From Assumption 2, we know that \( P^i_N \) and \( H^i_N \) are both positive semidefinite, then it can be observed from (28) that \( \Pi^i_N > 0 \) and \( \Pi^i_k > 0 \).

By repeating the above procedures backwardly, we can conclude that \( \Pi^i_k \) and \( \Pi^i_k \) are positive definite for \( k = 0, \ldots, N \). This ends the proof.

With the preliminaries introduced in Lemmas 115 we are in a position to present the solution to Problem 1.

**Theorem 2.** Under Assumptions 1 and 2, Problem 1 is uniquely solvable if and only if \( \Lambda_k \) and \( \hat{\Lambda}_k \) given by (28) are invertible. In this case, the optimal controls \( u^i_k, k = 0, \ldots, N, i = 0, \ldots, L \) minimizing the cost function (4) are given by
\[
\begin{align*}
\left\{ \begin{array}{l}
u^0_k = T^0 \hat{U}_k, \\
u^i_k = T^i \hat{U}_k + \hat{u}^i_k, i = 1, \ldots, L,
\end{array} \right.
\end{align*}
\] (33)
where
\[
\begin{align*}
\hat{U}_k &= -\Lambda_k^{-1} \Psi_k \hat{X}_k, \\
\hat{u}^i_k &= - (\hat{\Pi}^i_k)^{-1} \hat{\Omega}^i_k \hat{x}^i_k, \quad i = 1, \ldots, L,
\end{align*}
\] (34)
\[
T^i = [m^{0}, 0_{m^{0} \times m^{1}}, \ldots, 0_{m^{0} \times m^{L}}]_{m^{0} \times mL},
\] (36)
with \( \hat{X}_k, \hat{x}^i_k \) being calculated from Lemma 2 and the coefficient matrices \( \Lambda_k, \Psi_k, \hat{\Pi}_k, \hat{\Omega}_k \) being calculated via (27) backwardly.

Moreover, the optimal cost function is given by
\[
J^i_N = \sum_{i=0}^{L} E[(x^i_0)^T P^i_0 x^i_0] + \sum_{i=0}^{L} (1 - p^i) T^i R^i [\Sigma_{x^i} (P^i_0 + H^i_0)] + \sum_{i=0}^{L} \sum_{k=0}^{N} T^i (\Sigma_{x^i} L^i_{k+1}).
\] (37)

**Proof.** See Appendix.

**Remark 5.** In Theorem 2 we first derive the optimal control strategies by decoupling the G-FBSDEs (16) (equivalently G-FBSDEs (22)). The optimal control strategies are given in terms of new CREs (24), which can be calculated backwardly under Assumptions 1 and 2 and the conditions that \( \Lambda_k, \hat{\Lambda}_k \) are invertible. Moreover, it is noted that \( P_k \) and \( H_k \) in (27) are asymmetric, which is the essential difference from the additive noise case, see 13-17.

**IV. DISCUSSIONS**

In this section, we shall discuss some special cases of Problem 1 and demonstrate the novelty and significance of our results.

A. Additive Noise Case

In the case of \( w^i_k = 0 \), the MN-NCS (1) turns into the additive noise case. Using the results in Theorem 2, the solution to Problem 1 can be presented as follows.

**Corollary 1.** Under Assumptions 1 and 2, Problem 1 is uniquely solvable. Moreover, the control strategies \( u^i_k, k = 0, \ldots, N, i = 0, \ldots, L \) that minimize (4) can be given by
\[
\begin{align*}
u^0_k &= T^0 \hat{U}_k, \\
u^i_k &= T^i \hat{U}_k + \hat{u}^i_k, i = 1, \ldots, L,
\end{align*}
\] (38)
where
\[
\hat{U}_k = -\Lambda_k^{-1} \Psi_k \hat{X}_k, \\
\hat{u}^i_k = -(\hat{\Pi}^i_k)^{-1} \hat{\Omega}^i_k \hat{x}^i_k, \quad i = 1, \ldots, L.
\] (39)

In the above, \( \hat{X}_k, \hat{x}^i_k \) are given in Lemma 2 with \( w^i_k = 0 \), and the coefficient matrices \( \Lambda_k, \Psi_k, \hat{\Pi}_k, \hat{\Omega}_k \) can be calculated via the following Riccati equations:
\[
\begin{align*}
P_k &= Q + A^T P_{k+1} A - \Psi_k \Lambda_k^{-1} \Psi_k, \\
\hat{P}^i_k &= Q^{ii} + (A^i)^T P^i_{k+1} A^i - (\hat{\Omega}^i_k)^T (\hat{\Pi}^i_k)^{-1} \hat{\Omega}^i_k, \\
H^i_k &= Q^{ii} + (A^i)^T L^i_{k+1} A^i - (\hat{\Omega}^i_k)^T (\hat{\Pi}^i_k)^{-1} \hat{\Omega}^i_k, \\
L^i_{N+1} &= P^i_{N+1} + P^i_N, \text{ given in (28)}
\end{align*}
\] (41)
where \( \Lambda_k, \Psi_k, \hat{\Pi}_k, \hat{\Omega}_k \) being calculated from Lemma 2 and the coefficient matrices \( \Lambda_k, \Psi_k, \hat{\Pi}_k, \hat{\Omega}_k \) being calculated via (27) backwardly.

**Remark 6.** As shown in Corollary 1, the following points should be noted:

- The obtained results in Theorem 2 can be reduced to the additive noise case (i.e., \( w^i_k = 0 \)), which include the results of 13, 16, 17 as a special case.
- For the additive noise case, it is found that \( L_k = P_k = H_k \), hence CREs (27) can be reduced to (41), which are symmetric.
- For Riccati equations (41), by following the techniques of Lemma 2, it can be shown that \( \Lambda_k, \hat{\Pi}_k \) are positive definite under Assumptions 1 and 2. Thus the solvability of Problem 1 can be ensured. Furthermore, the control strategies (28) are unique in this case.

B. Single Subsystem Case

In this section, we will consider the single subsystem case, i.e., \( L = 1 \).

Following the results in Theorem 2, the solution to Problem 1 for the single subsystem case can be given as follows.
Corollary 2. Under Assumptions 7 and 2 Problem 7 is uniquely solvable, and the optimal controls $u^k_i, k = 0, \cdots, N, i = 0, 1$ can be given as

$$\begin{align*}
\mathbf{u}^0_k &= T\bar{U}_k, \\
\mathbf{u}^1_k &= T^1\bar{U}_k + \bar{u}^1, \\
\end{align*}$$

(43)

where

$$\begin{align*}
\bar{U}_k &= -\Lambda^{-1}_k \Psi_k \bar{x}_k, \\
\bar{u}_k &= -(\Lambda^{-1}_k)^T \Psi_k \bar{x}_k, \\
\end{align*}$$

(44)

$\Lambda_k, \Psi_k, \hat{\Lambda}_k, \hat{\Psi}_k$ given by

$$\begin{align*}
\Lambda_k &= R + B^T \Psi_{k+1} B + \Sigma_{w_k} B^T L_{k+1} B, \\
\Psi_k &= B^T P_{k+1} + A + \Sigma_{w_k} B^T L_{k+1} A, \\
\hat{\Lambda}_k &= R + B^T \hat{\Psi}_{k+1} B + \Sigma_{w_k} B^T \hat{L}_{k+1} B, \\
\hat{\Psi}_k &= B^T \hat{L}_{k+1} A + \Sigma_{w_k} B^T \hat{L}_{k+1} A, \\
\end{align*}$$

(46)

$P_k, H_k$ satisfying

$$\begin{align*}
P_k &= Q + A^T \Psi_{k+1} A + \Sigma_{w_k} A^T L_{k+1} A - \Psi_{k+1} A^T \Lambda_{k+1} \Psi_k, \\
H_k &= Q + A^T L_{k+1} A + \Sigma_{w_k} A^T L_{k+1} A - \Psi_{k+1} \Lambda_{k+1} \Psi_k, \\
\end{align*}$$

(47)

with $\Sigma_{w_k}$ is the covariance of system noise $\{w^k_i\}_{i=0}^N$ with $L = 1$ in 1 and 4).

As for the results given in Corollary 2, we have the following comments.

Remark 7. Firstly, different from the multiple subsystems case in Theorem 2, the optimal controls (43) are based on symmetric Riccati equations (47). Secondly, Assumptions 7 and 2 are sufficient to guarantee the solvability of Problem 1.

C. Solvability with Indefinite Weighting Matrices

In this section, we will investigate the case of indefinite weighting matrices in 4. In other words, we will just assume that weighting matrices $Q, R, P_{k+1}$ in 4 are symmetric.

Firstly, we will introduce the generalized Riccati equation:

$$\begin{align*}
\Delta_k &= \begin{cases}
Q + A^T \Delta_{k+1} A + \sum_{i=1}^L \Sigma_{w_i} (\Delta_i)^T \Delta_{k+1} \Delta_i - M_i \Gamma_i^T M_i, \\
- M_i \Gamma_i M_i, \\
\end{cases} \\
\Upsilon_k &= \begin{cases}
R + B^T \Delta_{k+1} B + \sum_{i=1}^L \Sigma_{w_i} (B_i)^T \Delta_{k+1} B_i, \\
M_k = B^T \Delta_{k+1} A + \sum_{i=1}^L \Sigma_{w_i} (B_i)^T \Delta_{k+1} A_i, \\
\end{cases} \\
\end{align*}$$

(48)

with terminal condition $\Gamma_{N+1} = P_{N+1}$, and $\dagger$ denotes the Moore-Penrose inverse.

We will present the following corollary without proof.

Corollary 3. Under Assumption 7 Problem 7 is uniquely solvable if and only if $\Upsilon_k \geq 0$ in 45, and $\Lambda_k, \Psi_k, \bar{\Lambda}_k, \bar{\Psi}_k$ in 47, 48 are all invertible for $k = 0, \cdots, N, L = 1, \cdots, L$. In this case, the optimal controls are given by 34, in which the coefficients are based on the solution to the CREs 27.

Remark 8. The necessary and sufficient solvability conditions of Problem 7 are shown in Corollary 3 under the assumption that the weighting matrices of 4 are indefinite. The proposed results in Corollary 3 can be induced from Theorem 7 and its proof. It can be observed that the positive semi-definiteness of $\Upsilon_k$ is equivalent to the condition $\delta J_2 \geq 0$ in 114, which is the key of deriving Corollary 3. To avoid repetition, the detailed proof of Corollary 3 is omitted here.

V. Numerical Examples

In this section, some numerical simulation examples will be provided to show the effectiveness and feasibilities of the main results.

A. State Trajectory with Optimal Controls

Consider MN-NCS 11 and cost function 44 with

$$\begin{align*}
L = 3, p^1 = 0.8, p^2 = 0.6, p^3 = 0.8, N = 60, \\
A^1 = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}, A^2 = \begin{bmatrix} 0.8 & 2 \\ -1 & 0.6 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & -0.3 \\ 3 & -3 \end{bmatrix}, \\
B^1 = \begin{bmatrix} 1.1 & -1.9 \\ 0.6 & 2 \end{bmatrix}, B^2 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, B^3 = \begin{bmatrix} -1 & 1.5 \\ 2 & -1 \end{bmatrix}, \\
B^{10} = \begin{bmatrix} -1 & 1.5 \\ 2 & -1 \end{bmatrix}, B^{20} = \begin{bmatrix} 0.8 & -2 \\ 1 & -3 \end{bmatrix}, B^{30} = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}, \\
\bar{A}^1 = \begin{bmatrix} 0.9 & 2 \\ 2 & 3 \end{bmatrix}, \bar{A}^2 = \begin{bmatrix} 1 & -0.5 \\ 0.5 & 2 \end{bmatrix}, \bar{A}^3 = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}, \\
\bar{B}^1 = \begin{bmatrix} -1 & 0.2 \\ 3 & 2 \end{bmatrix}, \bar{B}^2 = \begin{bmatrix} 1.2 & 1.8 \\ -3 & 2 \end{bmatrix}, \bar{B}^3 = \begin{bmatrix} 1.1 & 2 \\ 3.1 & 2 \end{bmatrix}, \\
\bar{B}^{10} = \begin{bmatrix} -1 & 0.9 \\ 0.7 & 2 \end{bmatrix}, \bar{B}^{20} = \begin{bmatrix} 1.2 & 1 \\ -2 & 0 \end{bmatrix}, \bar{B}^{30} = \begin{bmatrix} 3 & -1 \\ 0.6 & 2 \end{bmatrix}, \\
\Sigma_{w_1} = \Sigma_{w_2} = \Sigma_{w_3} = 1, \Sigma_{u_1} = \Sigma_{u_2} = \Sigma_{u_3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\mu^1 = \begin{bmatrix} 1.2 \\ 2 \end{bmatrix}, \mu^2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mu^3 = \begin{bmatrix} -3 \\ 10 \end{bmatrix}, \\
\Sigma_{x_0} = \Sigma_{z_0} = \Sigma_{z_3} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \\
Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
R = \begin{bmatrix} -0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & -0.5 \end{bmatrix}, \\
P_{t_{01}} = \begin{bmatrix} 1.2 & -1.4 & -1.4 & -3.1 & -1.7 & -5.7 \\ -1.4 & 1.3 & 0.9 & 1.8 & -1.1 & 0.7 \\ -1.4 & 0.9 & 5.9 & 1.4 & -3.5 & -1.4 \\ -3.1 & 1.8 & 1.4 & 4.4 & -0.2 & 1.5 \\ -1.7 & -1.1 & -3.5 & -0.2 & 5.1 & 2.7 \\ -5.5 & 0.7 & -1.4 & 1.5 & 2.7 & 10.4 \end{bmatrix}. \\
\end{align*}$$

(49)

From Theorem 2 since Assumptions 11 and 2 hold for the coefficients given in 49, and $\Lambda_k$ and $\hat{\Lambda}_k$ in 28 are invertible, hence Problem 1 can be uniquely solved. In this case, by using the optimal controls $u^k_i, i = 1, 2, 3, k = 0, \cdots, 60$, the state trajectories of $x^k_1, x^k_2, x^k_3$ are presented as in Figures 2-4.

As shown, each subsystem state $x^k_j, i = 1, 2, 3$ is convergent with optimal controls $u^k_i, i = 0, \cdots, 3$ calculated via Theorem 2.

B. State Trajectory with Different Packet Dropout Rates

In this section, we will explore the effects on the state trajectories with different packet dropout rates $p^i, i = 1, 2, 3$. 
Without loss of generality, we choose the same coefficients with (49), and the state trajectory of subsystem 1 is given in Figures 5-6 with different packet dropout rates.

It can be observed that the convergence rate of $x_k^{1,1}$ decreases with the packet dropout rate $1 - p^i$, $i = 1, 2, 3$ becoming larger.

VI. CONCLUSION

In this paper, we have investigated the optimal local and remote control problem for MN-NCSs with unreliable uplink channels and multiple subsystems. By adopting the Pontryagin maximum principle, the necessary and sufficient solvability conditions have been derived. Moreover, we have proposed a novel approach to decouple G-BFSDEs associated with the Pontryagin maximum principle. Finally, by introducing the asymmetric CREs, the optimal local and remote control strategies were derived in terms of the solution to CREs, which can be calculated backwardly. The proposed methods and the obtained results in this paper provide some inspirations for studying general control problems with asymmetric information structures.

APPENDIX: PROOF OF THEOREM

Proof. We will show the main results by using an induction method.

Note that the terminal conditions are $P_{N+1} = H_{N+1} = L_{N+1}$, and $\Theta_N = P_{N+1}X_{N+1}$, it can be calculated from (19) that

$$0 = R\tilde{U}_N + E[(B + \sum_{i=1}^L w_i^N B^T)\Theta_N | \mathcal{F}_N]$$
Fig. 6. State trajectory of $x_t^1$, with $p^1 = p^2 = p^3 = 0.3$.

$$\begin{align*}
&= R\tilde{U}_N + E[(B + \sum_{i=1}^{L} w_i^N \tilde{B}^i)^T P_{N+1} X_{N+1} | \mathcal{F}_N^t] \\
&= [R + B^T P_{N+1} B + \sum_{i=1}^{L} (\tilde{B}^i)^T P_{N+1} \tilde{B}^i]\tilde{U}_N \\
&+ [B^T P_{N+1} A + \sum_{i=1}^{L} (\tilde{B}^i)^T P_{N+1} \tilde{A}^i] E[X_N | \mathcal{F}_N^0] \\
&= \Lambda_N \tilde{U}_N + \Psi_N E[X_N | \mathcal{F}_N^t].
\end{align*}$$

(50)

From Theorem 1 we know that Problem 1 is uniquely solvable if and only if $\text{G-FBSDEs (22)}$ is uniquely solvable under Assumptions 1 and 2. Moreover, (50) is uniquely solved if and only if $\Lambda_N$ is invertible. In this case, $\tilde{U}_N$ can be derived as in (53).

Next, from (21) we know

$$\Theta_i^1 = P_{N+1}^i x_{N+1}^1 + \cdots + P_{N+1}^{iL} x_{N+1}^L,$$

then there holds

$$0 = R^{ii} \tilde{u}_N^i + E[(B^i + w_i^N \tilde{B}^i)^T \sum_{j=1}^{L} P_{N+1}^{ij} x_{N+1}^j | \mathcal{G}_N]$$

$$- E[(B^i + w_i^N \tilde{B}^i)^T \sum_{j=1}^{L} P_{N+1}^{ij} x_{N+1}^j | \mathcal{G}_N]$$

$$= R^{ii} \tilde{u}_N + E[(B^i + w_i^N \tilde{B}^i)^T \sum_{j=1}^{L} P_{N+1}^{ij} [(A^j + w_j^N \tilde{A}^j)x_N]$$

$$+ (B^i + w_i^N \tilde{B}^i)u_N^i + (B^{i0} + w_i^N \tilde{B}^{i0})u_0^i] | \mathcal{G}_N]$$

$$- E[(B^i + w_i^N \tilde{B}^i)^T \sum_{j=1}^{L} P_{N+1}^{ij} [(A^j + w_j^N \tilde{A}^j)x_N]$$

$$+ (B^i + w_i^N \tilde{B}^i)u_N^i + (B^{i0} + w_i^N \tilde{B}^{i0})u_0^i] | \mathcal{G}_N]$$

$$= R^{ii} \tilde{u}_N + E[(B^i + w_i^N \tilde{B}^i)^T P_{N+1} [(A^i + w_i^N \tilde{A}^i)x_N]$$

$$+ (B^i + w_i^N \tilde{B}^i)u_N^i + (B^{i0} + w_i^N \tilde{B}^{i0})u_0^i] | \mathcal{G}_N]$$

$$- E[(B^i + w_i^N \tilde{B}^i)^T P_{N+1} [(A^i + w_i^N \tilde{A}^i)x_N]$$

$$+ (B^i + w_i^N \tilde{B}^i)u_N^i + (B^{i0} + w_i^N \tilde{B}^{i0})u_0^i] | \mathcal{G}_N]$$

$$+ (B^i + w_i^N \tilde{B}^i)u_N^i + (B^{i0} + w_i^N \tilde{B}^{i0})u_0^i] | \mathcal{G}_N]$$

$$= \tilde{\Pi}_N^i \tilde{u}_N^i + \tilde{\Omega}_N^i \tilde{x}_N^i.$$  (52)

Since $\tilde{\Pi}_N^i$ is positive definite as shown in Lemma 5, thus $\tilde{u}_N^i$ can be uniquely solved as in (55). Hence, the optimal remote controller $u_{0N}^i$ and the optimal local controllers $u_{N}^i, i = 1, \cdots, L$ can be derived as in (53).

Consequently, we will calculate $\Theta_{N-1}$. From (10) we have

$$\Theta_{N-1} = E \left[ (A + \sum_{i=1}^{L} w_i^N \tilde{A}^i)^T \Theta_N + Q X_N | \mathcal{G}_N \right]$$

$$= E \left[ (A + \sum_{i=1}^{L} w_i^N \tilde{A}^i)^T P_{N+1} X_{N+1} + Q X_N | \mathcal{G}_N \right]$$

$$= E \left[ (A + \sum_{i=1}^{L} w_i^N \tilde{A}^i)^T P_{N+1} (A + \sum_{i=1}^{L} w_i^N \tilde{A}^i) X_N$$

$$+ (B + \sum_{i=1}^{L} w_i^N \tilde{B}^i)(\tilde{U}_N + \tilde{U}_N) + V_N) | \mathcal{G}_N \right] + Q X_N$$

$$= [Q + A^T P_{N+1} A + \sum_{i=1}^{L} \Sigma w_i^N (\tilde{A}^i)^T P_{N+1} \tilde{A}^i] X_N$$

$$+ [A^T P_{N+1} B + \sum_{i=1}^{L} \Sigma w_i^N (\tilde{A}^i)^T P_{N+1} \tilde{B}^i]\tilde{U}_N$$

$$+ [A^T P_{N+1} B + \sum_{i=1}^{L} \Sigma w_i^N (\tilde{A}^i)^T P_{N+1} \tilde{B}^i]\tilde{U}_N$$

$$+ \Psi_N \Lambda_N \tilde{X}_N.$$  (53)

where $P_N, H_N$ satisfy (27) for $k = N$.

To complete the induction approach, we assume for $k = l + 1, \cdots, N$, there holds

- The optimal controls $u_k^i, i = 0, \cdots, L$ are given by (53)- (55);
- The relationship between the system state $X_k$ and costate $\Theta_k$ satisfies:

$$\Theta_{k-1} = P_k \tilde{X}_k + H_k \tilde{X}_k,$$  (54)

where $P_k, H_k$ can be calculated from (27) backwardly.

Next, we will calculate $u_i^i$. Note that $\Theta_i = P_{i+1} \tilde{X}_{i+1} + H_{i+1} \tilde{X}_{i+1}$, it can be calculated from (19) that

$$0 = R\tilde{U}_i + E[(B + \sum_{i=1}^{L} w_i^N \tilde{B}^i)^T \Theta_i | \mathcal{F}_i]$$

$$= R\tilde{U}_i + E[(B + \sum_{i=1}^{L} w_i^N \tilde{B}^i)^T (P_{i+1} \tilde{X}_{i+1}$$

$$+ H_{i+1} \tilde{X}_{i+1}) | \mathcal{F}_i]$$
\[ R\hat{U}_l + E\left[(B + \sum_{i=1}^L w_i^T \tilde{B}^i)^T \{ P_{l+1} [\Gamma_{l+1} X_{l+1} + (\Gamma_{l+1} - \Gamma_{l+1}) (A\bar{X}_l + B\bar{U}_l)]\} \right] \\
\quad + \{R + B^T P_{l+1} [pB + (I_{N_l} - p)B]\} \\
\quad + \{B^T P_{l+1} [pA + (I_{N_l} - p)A]\} \\
\quad + \{B^T [P_{l+1} + H_{l+1} (I_{N_l} - p)]\} \hat{X}_l \\
\quad + \{R + B^T [P_{l+1} + H_{l+1} (I_{N_l} - p)]\} \tilde{U}_l \\
\quad + \{B^T P_{l+1} pA + (I_{N_l} - p)A\} \\
\quad + \{B^T P_{l+1} pB + (I_{N_l} - p)B\} \\
\quad + \{\sum_{w_i} (\tilde{B}^i)^T [P_{l+1}p + H_{l+1} (I_{N_l} - p)]\} \tilde{U}_l \\
= \Lambda_l \hat{U}_l + \Psi_l \tilde{X}_l. \quad (55) \]

By following the discussions below \((50)\), we know that \((55)\) can be uniquely solved if and only if \(\Lambda_l\) is invertible. Then \(\hat{U}_l\) can be derived as in \((44)\).

Since \(\hat{U}_l = U_l - \tilde{U}_l\), we have

\[ 0 = R\hat{U}_l + E\left[(B + \sum_{i=1}^L w_i^T \tilde{B}^i)^T \Theta_l|G_l\right] \\
\quad - E\left[(B + \sum_{i=1}^L w_i^T \tilde{B}^i)^T \Theta_l|F_l^0\right] \\
= R\hat{U}_l + E\left[(B + \sum_{i=1}^L w_i^T \tilde{B}^i)^T \{ P_{l+1} [\Gamma_{l+1} X_{l+1} + (\Gamma_{l+1} - \Gamma_{l+1}) (A\bar{X}_l + B\bar{U}_l)]\} \right] \\
\quad + \{R + B^T P_{l+1} [pB + (I_{N_l} - p)B]\} \\
\quad + \{B^T P_{l+1} [pA + (I_{N_l} - p)A]\} \\
\quad + \{B^T [P_{l+1} + H_{l+1} (I_{N_l} - p)]\} \tilde{X}_l \\
\quad + \{R + B^T [P_{l+1} + H_{l+1} (I_{N_l} - p)]\} \tilde{U}_l \\
\quad + \{B^T P_{l+1} pA + (I_{N_l} - p)A\} \\
\quad + \{B^T P_{l+1} pB + (I_{N_l} - p)B\} \\
\quad + \{\sum_{w_i} (\tilde{B}^i)^T [P_{l+1}p + H_{l+1} (I_{N_l} - p)]\} \tilde{U}_l \]

Hence, the solvability of \((56)\) is equivalent to the invertibility of \(\Lambda_l\), and we have

\[ \hat{U}_l = -\Lambda_l^{-1} \Psi_l \tilde{X}_l, \quad (57) \]

i.e. \((55)\) can be verified for \(k = l\).

Next, from \((21)\) we know

\[ \Theta_l = P_{l+1}^1 \hat{x}_{l+1} + \cdots + P_{l+1}^L \hat{x}_{l+1} \]

Thus, using Lemmas \((23)\) there holds

\[ 0 = R^{ii} \hat{u}_l^i + E\left[(B^i + w_i^T \tilde{B}^i)^T \Theta_l|G_l\right] \\
\quad - E\left[(B^i + w_i^T \tilde{B}^i)^T \Theta_l|G_l\right] \]

\[ = R^{ii} \hat{u}_l^i + E\left[(B^i + w_i^T \tilde{B}^i)^T P_{l+1}^i \{ \gamma_{l+1}^i (A^i + w_i \tilde{A}) x_{l+1}^i + \gamma_{l+1}^i B^i x_{l+1}^i \} + (1 - \gamma_{l+1}^i) (A^i \hat{x}_l^i + B^i \hat{u}_l^i)\} \right] \\
\quad + \left[ \sum_{w_i} (\tilde{B}^i)^T (1 - \gamma_{l+1}^i) [(A^i + w_i \tilde{A}) x_{l+1}^i + (B^i + w_i \tilde{B}^i) x_{l+1}^i + (B^i + w_i \tilde{B}^i) u_{l+1}^i] \right] \right] \}

\[ \text{and } \]

\[ (58) \]
\[ + \sum_{i} (\overline{B}_i^i)^T (p^i P^i_{t+1} + (1 - p^i) H^i_{t+1}) \tilde{A}^i] \tilde{x}_t^i + A^T H_{t+1}(I_{N_L} - p) A \\
+ \sum_{i=1}^{L} \sum_{i} w_i^i (\overline{A}_i^i)^T H_{t+1}(I_{N_L} - p) \overline{A}_i^i] \tilde{x}_t + \overline{\Psi}_t \tilde{U}_t \\
= P_t \tilde{X}_t + H_t \tilde{X}_t. \]  

(60)

Thus, \( \Theta_{k} \) has been derived for \( k = l \). This ends the induction. Finally, we will calculate the optimal cost function. For simplicity, we denote

\[ V_k \triangleq E[X^T_k \Theta_{k-1}]. \]  

(61)

Hence, we have

\[ V_k - V_{k+1} = E[X^T_k \Theta_{k-1}] - E[X^T_{k+1} \Theta_{k}] \]

\[ = E[E[X^T_k (A + \sum_{i} w_i^i \overline{A}_i^i)^T \Theta_k + Q X_k | G_k] \]

\[ - E[(A + \sum_{i} w_i^i \overline{A}_i^i)X_k \]

\[ + (B + \sum_{i} w_i^i \overline{B}_i) U_k + V_k] \Theta_k] | G_k] \]

\[ = E[X^T_k Q X_k - V^T_k \Theta_k - U^T_k E[(B + \sum_{i} w_i^i \overline{B}_i)^T \Theta_k] | G_k] \]

\[ = E[X^T_k Q X_k + U^T_k R U_k - V^T_k \Theta_k]. \]  

(62)

Note \( \Theta_N = P_{N+1} X_{N+1} \), then taking summation of \( \Theta_N \) from 0 to \( N \), the optimal cost function can be given by

\[ J^*_N = \sum_{k=0}^{N} E[V^T_k \Theta_k] + E[X^T_0 \Theta_{-1}] \]

\[ = \sum_{k=0}^{N} E[V^T_k (P_{k+1} \tilde{X}_{k+1} + H_{k+1} \tilde{X}_{k+1})] \]

\[ + E[X^T_0 (P_0 \tilde{X}_0 + H_0 \tilde{X}_0)] \]

\[ = \sum_{k=0}^{N} E[V^T_k (P_{k+1} + H_{k+1}(I_{N_L} - p))X_k] \]

\[ + E[X^T_0 P_0 \tilde{X}_0 + \tilde{X}_0^T H_0 \tilde{X}_0] \]

\[ = \sum_{k=0}^{N} E[V^T_k L_{k+1} V_k] + E[X^T_0 P_0 \tilde{X}_0 + \sum_{i=0}^{L} (\tilde{x}_i^0)^T H_i \tilde{x}_i^0] \]

\[ + E[(v_k^i)^T L_{k+1} v_k^i] + \sum_{k=0}^{N} \sum_{i=0}^{L} E[(x_i^0)^T L_{k+1} x_i^0] \]

\[ + E[(x_i^0)^T P_0 x_i^0] + \sum_{i=0}^{L} (1 - p^i) Tr[\sum_{i=0}^{L} (P_0 + H_0)] \]

\[ + \sum_{i=0}^{L} \sum_{i=0}^{L} Tr(\Sigma w_i^i L_{k+1}). \]  

(63)

The proof is complete. \( \square \)
REFERENCES

[1] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, “A survey of recent results in networked control systems,” Proc. IEEE, vol. 95, no. 1, pp. 138-162, Jan. 2007.
[2] O. C. Imer, S. Yüksel, and T. Başar, “Optimal control of LTI systems over unreliable communication links,” Automatica, vol. 42, no. 9, pp. 1429-1439, 2006.
[3] G. C. Walsh, H. Ye, L. G. Bushnell, “Stability analysis of networked control systems”, IEEE Transactions on Control Systems Technology, vol. 10, no. 3, pp. 438-446, May 2002.
[4] W. Zhang, M. S. Branicky, S. M. Phillips, “Stability of networked control systems”, IEEE Control Systems Magazine, vol. 21, no. 1, pp. 84-99, Feb. 2001.
[5] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, S. S. Sastry, “Foundations of control and estimation over lossy networks.” Proceedings of IEEE, vol. 95, no. 1, pp. 163-187, Jan. 2007.
[6] Q. Qi and H. Zhang, “Output Feedback Control and Stabilization for Networked Control Systems With Packet Losses.” IEEE Transactions on Cybernetics, vol. 47, no. 8, pp. 2223-2234, Aug. 2017.
[7] Q. Qi and H. Zhang, “Output Feedback Control and Stabilization for Multiplicative Noise Systems With Intermittent Observations.” IEEE Transactions on Cybernetics, vol. 48, no. 7, pp. 2176-2189, Sep. 2012.
[8] T. Başar, “Variations on the theme of the Witsenhausen counterexample.” "47th IEEE Conference on Decision and Control, Cancun, 2008, pp. 1614-1619.
[9] G. M. Lipsa, N. C. Martins, “Optimal memoryless control in Gaussian noise: A simple counterexample.” Automatica, vol. 47, no. 3, pp. 552-558, 2011.
[10] M. Rabi, C. Ramesh, and K. H. Johansson, “Separated design of encoder and controller for networked linear quadratic optimal control.” SIAM Journal on Control and Optimization, vol. 54, no. 2, pp. 662-689, 2016.
[11] M. Afshari and A. Mahajan, “Optimal local and remote controllers with unreliable uplink channels: An elementary proof.” IEEE Transactions on Automatic Control. doi: 10.1109/TAC.2019.2951658.
[12] A. Nayyar, A. Mahajan, and D. Teneketzis, “Decentralized stochastic control with partial history sharing: A common information approach.” IEEE Trans. Autom. Control, vol. 58, no. 7, pp. 1644-1658, July 2013.
[13] Y. Ouyang, S. M. Asghari, A. Nayyar, “Optimal local and remote controllers with unreliable communication.” Proceedings of IEEE 55th Conference on Decision and Control, pp. 6024-6029, Dec. 2016.
[14] S. M. Asghari, Y. Ouyang, and A. Nayyar, “Optimal local and remote controllers with unreliable uplink channels.” IEEE Transactions on Automatic Control, vol. 64, no. 5, pp. 1816-1831, 2018.
[15] X. Liang, and J. Xu, “Control for networked control systems with remote and local controllers over unreliable communication channel.” Automatica, vol. 98, pp. 86-94, 2018.
[16] Y. Ouyang, S. M. Asghari and A. Nayyar, “Optimal infinite horizon decentralized networked controllers with unreliable communication.” arXiv preprint [arXiv:1906.06497] 2018.
[17] S. M. Joshi, “On optimal control of linear systems in the presence of multiplicative noise.” IEEE Transactions on Aerospace and Electronic Systems, vol. 12, no. 1, pp. 80-85, Jan. 1976.
[18] W. M. Wonham, “On a matrix Riccati equation of stochastic control.” SIAM Journal on Control and Optimization, vol. 6, no. 4, pp. 681-697, 1968.
[19] J. M. Bismut, “Linear quadratic optimal stochastic control with random coefficients.” SIAM Journal on Control and Optimization, vol. 14, no. 3, pp. 419-444, 1976.
[20] M. A. Rami and X. Zhou, “Linear matrix inequalities, Riccati equations, and indefinite stochastic linear quadratic controls.” IEEE Transactions on Automatic Control, vol. 45, no. 6, pp. 1131-1143, 2000.
[21] H. Zhang, L. Li, J. Xu and M. Fu, “Linear quadratic regulation and stabilization of discrete-time systems with delay and multiplicative noise.” IEEE Transactions on Automatic Control, vol. 60, no. 10, pp. 2599-2613, Oct. 2015.
[22] Q. Qi, H. Zhang and Z. Wu, “Stabilization control for linear continuous-time mean-field systems.” IEEE Transactions on Automatic Control, vol. 64, no.8, pp. 3461-3468, 2019.
[23] K. You and L. Xie, “Minimum Data Rate for Mean Square Stabilization of Discrete LTI Systems Over Lossy Channels.” IEEE Transactions on Automatic Control, vol. 55, no. 10, pp. 2373-2378, Oct. 2010.