Hamiltonian Analysis of Minimal Massive Gravity Coupled to Galileon Tadpole Term

J. Klusoň

*Department of Theoretical Physics and Astrophysics*
*Faculty of Science, Masaryk University*
*Kotlářská 2, 611 37, Brno*
*Czech Republic*
*E-mail: klu@physics.muni.cz*

**Abstract:** We perform the Hamiltonian analysis of minimal massive gravity coupled to the Galileon tadpole term. We determine all constraints and we argue that the physical degrees of freedom correspond to ten modes of the massive gravity together with $2(D - 3)$ Galileons so that given model is ghost free.

**Keywords:** Massive Gravity, Hamiltonian Formalism
1. Introduction and Summary

The non-linear massive gravity is very nice and intriguing proposal of the ghost-free massive gravity (dRGT) \[1, 2\]. dRGT massive gravity is 3−parameter of potentials whose structure is based on the square root of the matrix $\hat{g}^{\mu\nu} \hat{f}_{\nu\sigma}$ where $\hat{g}^{\mu\nu}$ is dynamical four dimensional metric while $\hat{f}_{\mu\nu}$ is fixed four dimensional metric. The structure of given potential ensures an existence of the additional constraint that is sufficient for removing the ghost degree of freedom. Further important extension of given work was performed in \[3, 2, 4, 5\] while the general proof of the absence of the ghosts using the Hamiltonian formalism was presented in \[6\], for further works, see \[7, 8, 9, 10, 11, 25, 26, 27\].

Generally the presence of the mass term with fixed metric $\hat{f}_{\mu\nu}$ breaks the diffeomorphism invariance completely, however this can be restored by introduction of four the St"uckelberg fields so that $\hat{f}_{\mu\nu} \Rightarrow \partial_\mu \phi^A \partial_\nu \phi^B \eta_{AB}$ \[12\]. By construction St"uckelberg fields are pure gauge and original dRGT theory can be restored by gauge fixing of the diffeomorphism invariance by imposing the conditions $\phi^A = \delta^A_\mu x^\mu$.  

The generalization of given construction was presented recently in \[15, 14\] with following basic idea: Let us interpret the St"uckelberg fields as the embedding mapping of a sigma model $\Sigma \rightarrow M$ where both $\Sigma$ and $M$ are four dimensional Minkowski space-time \[2\]. In this picture the dynamical metric $\hat{g}_{\mu\nu}(x)$ is a world-volume metric that lives on $\Sigma$. Then there is a natural generalization when we consider the target space to be higher-dimensional and also it can be curved. In other words we can consider the metric $\hat{f}_{\mu\nu}$ in the form

$$\hat{f}_{\mu\nu} = \partial_\mu \phi^A \partial_\nu \phi^B \mathcal{G}_{AB}(\phi) , \quad (1.1)$$

where now $A, B, C... = 0, \ldots, D$. Due to this generalization we now have $D − 3$ scalar fields that cannot be gauged away so that they are physical scalars that couple to the physical

\[1\]For Hamiltonian analysis of the dRGT gravity formulated with St"uckelberg fields, see \[13\] and also \[8, 9, 10, 11\].

\[2\]For related works, see \[24\].
metric through the dRGT potential [15]. Due to the fact that given fields are dynamical we can consider more general form of the action that contain invariants constructed solely from $\hat{f}_{\mu\nu}$. The leading term in this construction is the DBI action (known as the tadpole term) $\sim \int d^4 x \sqrt{-\det \hat{f}_{\mu\nu}}$ and the higher Lovelock invariants give Galileons [16, 17]. It turns out that this theory possesses a Galileon-like symmetry for each isometry.

The construction presented in [15] is very interesting from different points of views. It is a generalization of the dRGT theory but also provides a way how to couple the Galileons to massive gravity while preserving the Galileon invariance. This is very important fact since it is well known that when we try to couple the Galileon to massless gravity we have to introduce non-minimal coupling in order to ensure the second order equations of motion and the Galileon symmetry is broken [18, 19]. These results suggest that Galileon could couple more naturally to the massive gravity than to the ordinary massless gravity.

It was argued in [15] that given theory is ghost-free for the flat target space metric in the decoupling limit and for simplifying choice of parameters. Then it was argued in [14], using the methods similar to those [23] that the full theory, for any target space metric $G_{AB}$ has the primary constraints that is necessary to eliminate the Boulware-Deser ghost.

Despite of this remarkable conclusion we fell that the coupled system of massive gravity and Galileon deserves further investigation from different point of view. Explicitly, we are not quit sure whether the analysis presented in [14] is sufficient to show that the ghost mode is eliminated since even if they identified the constraints that could eliminate the ghost mode it was not shown whether they are the first or the second class constraints. Further, it is not clear how to find the momenta conjugate to the scalar modes since the coupled action between massive gravity and Galileon is rather complicated and it is not immediately clear how to perform the Legendre transform from Lagrangian to Hamiltonian formulation.

Our goal is not to perform the analysis of the dRGT theory coupled to the Galileon actions in full generality. Rather we will be more modest and perform the Hamiltonian analysis of the particular model of the minimal dRGT gravity coupled with Galileon tadpole term that allows an explicit analysis. It turns out that it is convenient to perform this analysis when we consider dRGT theory with redefined shift functions [3, 2, 4, 5]. Then we will be able to perform the Hamiltonian analysis and find corresponding primary constraints. The analysis is similar to the analysis performed in case of pure massive gravity in [11] however now the presence of the Galileon tadpole term makes it more complicated. Despite of this fact we find the constraint structure of given theory and determine the character of given constraints. We show that there are really two additional constraints whose presence allow to eliminate the ghost mode. In other words our result confirms the results presented in [14] using the metric formulation of massive gravity at least for some particular case of minimal dRGT theory coupled to tadpole Galileon term.

The work presented here can be extended in different way. In particular, we could consider the general form of dRGT theory coupled to the Galileon or the minimal dRGT theory coupled to the general Galileon Lagrangian or finally the most general case of the general dRGT massive gravity coupled to the general Galileon action. However in all these cases the analysis is very complicated. In particular, the simplest analysis could be in case
of general dRGT gravity coupled with the Galileon tadpole term where when we can follow [11] and determine all constraints of the theory. On the other hand it is very difficult to determine the character of these constraints in case of the general dRGT massive gravity due to their complicated form as was shown in [11]. The situation could be even worse in case of more general Galileon term since it seems to be very difficult to express momenta as function of the time derivatives of the scalar fields and hence to perform the Legendre transformation from the Lagrangian to the Hamiltonian formalism.

This note is organized as follows. In the next section (2) we introduce the minimal version of dRGT gravity coupled to the Galileon tadpole term formulated with the transformed shift function. Then in section (3) we perform the Hamiltonian analysis of given action and argue that the constraint structure of given theory allows to eliminate the ghost mode. Finally in Appendix we briefly review the Hamiltonian analysis of the Galileon tadpole term.

2. Minimal dRGT Massive Gravity Coupled with Galileon Tadpole Term

Let us consider minimal dRGT theory coupled with the tadpole Galileon term. The action of the system has the form

\[
S = S_{m.g.} + S_{gal},
\]

\[
S_{m.g.} = M_p^2 \int d^4x \sqrt{-\hat{g}} \left[ (4) R[\hat{g}] + 2m^2(3 - \sqrt{\hat{g}}^{-1}\hat{f}) \right],
\]

(2.1)

where

\[
S_{gal} = -T \int d^4x \sqrt{-\det \hat{f}}
\]

(2.2)

and where \( \hat{g}_{\mu\nu} \) is four dimensional metric with signature \((-+++)\), \((4) R[\hat{g}]\) is scalar curvature calculated with \( \hat{g}_{\mu\nu} \) and finally \( \hat{f}_{\mu\nu} \) is induced metric on the world-volume of brane defined as \( \hat{f}_{\mu\nu} = \partial_{\mu}\phi^A \partial_{\nu} \phi^B \). For simplicity of further analysis we consider the case when \( \mathcal{G}_{AB} \) is the constant tensor keeping in mind that the generalization to the case when \( \mathcal{G}_{AB} \) depends on \( \phi \) is straightforward. Finally, \( M_p \) is four dimensional Planck mass and \( T \) is the tension of four-dimensional brane.

The coupling between gravity and Galileon is described through the massive term \( \sqrt{g^{\mu\nu}\hat{f}_{\nu\rho}} \) where the square root is defined as \( \sqrt{\hat{g}^{\mu\nu}\hat{f}_{\nu\rho}} \sqrt{\hat{g}^{\rho\sigma}\hat{f}_{\sigma\omega}} = \hat{g}^{\mu\nu}\hat{f}_{\nu\omega} \). To proceed further we use 3 + 1 decomposition of the four dimensional metric \( g_{\mu\nu} \) [20, 21]

\[
\hat{g}_{00} = -N^2 + N_i g^{ij} N_j, \quad \hat{g}_{0i} = N_i, \quad \hat{g}_{ij} = g_{ij},
\]

\[
\hat{g}^{00} = -\frac{1}{N^2}, \quad \hat{g}^{0i} = \frac{N^i}{N^2}, \quad \hat{g}^{ij} = g^{ij} - \frac{N^i N^j}{N^2}
\]

(2.3)

so that we find

\[
N^2 \hat{g}^{-1} f = \left( \begin{array}{cc}
-f_{00} + N^i f_{i0} & -f_{0j} + N^i f_{ij} \\
N^2 g^{il} f_{l0} - N^i(-f_{00} + N^l f_{l0}) & N^2 g^{il} f_{lj} - N^i(-f_{0j} + N^l f_{lj})
\end{array} \right),
\]

(2.4)
Following [3, 2, 4, 5] we perform the redefinition of the shift function $N^i$

$$N^i = M\tilde{n}^i + f^{ik}f_{0k} + N\tilde{D}^i j\tilde{n}^j, \quad (2.5)$$

where

$$\tilde{x} = 1 - \tilde{n}^i f_{ij}\tilde{n}^j, \quad M^2 = -f_{00} + f_{0k}f^{kl}f_{l0} \quad (2.6)$$

and where we defined $f_{ij}$ as the inverse to $f_{ij}$ in the sense

$$f_{ik}f^{kj} = \delta^j_i. \quad (2.7)$$

Finally note that the matrix $\tilde{D}^i j$ obeys the equation [3, 2, 4, 5]

$$\sqrt{\tilde{x}}\tilde{D}^i j = \sqrt{(g^{ik} - \tilde{D}^i m\tilde{n}^m\tilde{D}^k n\tilde{n}^n)f_{kj}} \quad (2.8)$$

and also following important identity

$$f_{ik}\tilde{D}^k j = f_{jk}\tilde{D}^k i. \quad (2.9)$$

Now we proceed to the case of the tadpole Galileon action. Using the property of the determinant we obtain

$$S_{gal} = -T \int d^4x \sqrt{-\det f_{\mu\nu}} = -T \int d^4x \sqrt{-(f_{00} - f_{0i}f_{ij}f_{j0}) \sqrt{\det f_{ij}}} - T \int d^4x M\sqrt{f}, \quad (2.10)$$

where

$$f \equiv \det f_{ij}. \quad (2.11)$$

Then using the results derived in [3, 2, 4, 5] and (2.10) we find the action in the form

$$S = M_p^2 \int d^3x dt [N\sqrt{g}\tilde{K}^{ijkl}\tilde{K}_{kl} + N\sqrt{g}R - \sqrt{g}MU' - 2m^2(N\sqrt{g}\sqrt{x}D^i i - 3N\sqrt{g})], \quad (2.12)$$

where

$$U' = 2m^2\sqrt{\tilde{x}} + \frac{T}{M_p^2\sqrt{g}}, \quad (2.13)$$

and where we used the 3 + 1 decomposition of the four dimensional scalar curvature

$$(^4R) = \tilde{K}^{ijkl}\tilde{K}_{kl} + R, \quad (2.14)$$

\footnote{Note that in our convention $f^{ik}$ coincides with $(^3f^{-1})^{ik}$ presented in [3, 2, 4, 5].}
where $R$ is three dimensional scalar curvature and where

\[ G^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - g^{ij}g^{kl} \]  

(2.15)

with inverse

\[ G_{ijkl} = \frac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk}) - \frac{1}{2}g_{ij}g_{kl} \]  

(2.16)

Note that in (2.14) we ignored the total derivative terms. Finally note that $\tilde{K}_{ij}$ is defined as

\[ \tilde{K}_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i N_j(\tilde{n}, g) - \nabla_j N_i(\tilde{n}, g)) \]  

(2.17)

where $N_i$ depends on $\tilde{n}^i$ and $g$ through the relation (2.5).

3. Hamiltonian Formalism

Now we are ready to proceed to the Hamiltonian formalism, following [11]. From (2.12) we find the momenta conjugate to $N$, $\tilde{n}^i$ and $g_{ij}$

\[ \pi_N \approx 0 \, , \quad \pi_i \approx 0 \, , \quad \pi^{ij} = M_p^2 \sqrt{g} G^{ijkl} \tilde{K}_{kl} \]  

(3.1)

and the momentum conjugate to $\phi^A$

\[ p_A = -\left( \frac{\delta M}{\delta \partial_t \phi^A} \tilde{n}^i + f^{ij} \partial_j \phi_A \right) \mathcal{R}_i - M_p^2 \sqrt{g} \frac{\delta M}{\partial_t \phi^A} U' , \]  

(3.2)

where

\[ \mathcal{R}_i = -2 \eta_{ik} \nabla_j \pi^{kj} . \]  

(3.3)

It turns out that it is useful to write $M^2$ in the form

\[ M^2 = -\partial_t \phi^A \mathcal{M}_{AB} \partial_t \phi^B \, , \quad \mathcal{M}_{AB} = \eta_{AB} - \partial_t \phi_A f^{ij} \partial_j \phi_B \]  

(3.4)

where by definition the matrix $\mathcal{M}_{AB}$ obeys following relations

\[ \mathcal{M}_{AB} \eta^{BC} \mathcal{M}_{CD} = \mathcal{M}_{AD} \, , \quad \det \mathcal{M}_B^A = 1 \]  

(3.5)

together with

\[ \partial_t \phi^A \mathcal{M}_{AB} = \partial_t \phi_B - \partial_t \phi_A \mathcal{M}_{kl} \partial_k \phi_B = 0 . \]  

(3.6)

With the help of these results we find

\[ p_A + \mathcal{R}_i f^{ij} \partial_j \phi_A = \left( \tilde{n}^i \mathcal{R}_i + M_p^2 \sqrt{g} U' \right) \frac{1}{M} \mathcal{M}_{AB} \partial_t \phi^B \]  

(3.7)
and then following primary constraint
\[ \Sigma_p = (\tilde{n}^i \mathcal{R}_i + M_p^2 \sqrt{g} U')^2 + (p_A + \mathcal{R}_i f^{ij} \partial_j \phi_A)(p^A + \mathcal{R}_i f^{ij} \partial_j \phi^A) \approx 0 . \]

Note that using (3.6) we obtain another set of the primary constraints
\[ \partial_i \phi^A \Pi_A = \partial_i \phi^A p_A + \mathcal{R}_i = \Sigma_i \approx 0 . \]  

(3.8)

Observe that using (3.8) we can write
\[ p_A + \mathcal{R}_i f^{ij} \partial_j \phi_A = \mathcal{M}_{AC} \eta^{CB} p_B + \Sigma_i f^{ij} \partial_j \phi_A \]
so that we can rewrite \( \Sigma_p \) into the form
\[ \Sigma_p = (\tilde{n}^i \mathcal{R}_i + M_p^2 \sqrt{g} U')^2 + p_A \mathcal{M}^{AB} p_B + H^i \Sigma_i , \]
where \( H^i \) are functions of the phase space variables. As a result we see that it is natural to consider following independent constraint \( \Sigma_p \)
\[ \Sigma_p = (\tilde{n}^i \mathcal{R}_i + M_p^2 \sqrt{g} U')^2 + p_A \mathcal{M}^{AB} p_B \approx 0 . \]  

(3.11)

We return to the analysis of the constraint \( \Sigma_p \) below.

Now we are ready to write the extended Hamiltonian which includes all the primary constraints
\[ H_E = \int d^3x (N \mathcal{C}_0 + v_N \pi_N + v^i \pi_i + \Omega_p \Sigma_p + \Omega^i \tilde{\Sigma}_i) , \]
(3.12)

where
\[ \mathcal{C}_0 = \frac{1}{\sqrt{g} M_p^2} \pi^{ij} g_{ijkl} \pi^{kl} - M_p^2 \sqrt{g} \mathcal{R} + 2m^2 M_p^2 \sqrt{g} \sqrt{\tilde{\mathcal{D}}} \mathcal{D}_i - 6m^2 M_p^2 \sqrt{g} + \tilde{\mathcal{D}}_j \tilde{n}^j \mathcal{R}_i \]
(3.13)

and where we introduced the constraints \( \tilde{\Sigma}_i \) defined as
\[ \tilde{\Sigma}_i = \Sigma_i + \partial_i \tilde{n}^i \pi_i + \partial_j (\tilde{n}^j \pi_i) . \]  

(3.14)

Note that \( \tilde{\Sigma}_i \) is defined as linear combination of the constraints \( \Sigma_i \approx 0 \) together with the constraints \( \pi_i \approx 0 \).

To proceed further we have to check the stability of all constraints. To do this we have to calculate the Poisson brackets between all constraints and the Hamiltonian \( H_E \). Note that we have following set of the canonical variables \( g_{ij}, \pi^{ij}, \phi^A, p_A, \tilde{n}^i, \pi_i \) and \( N, \pi_N \) with non-zero Poisson brackets
\[
\{ g_{ij}(x), \pi^{kl}(y) \} = \frac{1}{2} (\delta^k_i \delta^l_j + \delta^k_j \delta^l_i) \delta(x - y) , \quad \{ \phi^A(x), p_B(y) \} = \delta^A_B \delta(x - y) , \\
\{ N(x), \pi_N(y) \} = \delta(x - y) , \quad \{ \tilde{n}^i(x), \pi_j(y) \} = \delta^i_j \delta(x - y) . 
\]
(3.15)
The constraint $\tilde{\Sigma}_i$ has the same form as in \cite{11} where it was shown that the smeared form of this constraint
\[ T_S(N^i) = \int d^3x N^i \tilde{\Sigma}_i \] (3.16)
is the generator of the spatial diffeomorphism so that
\[
\begin{align*}
\{ T_S(N^i), \tilde{n}^k \} &= -N^i \partial_i \tilde{n}^k + \tilde{n}^j \partial_j N^k , \\
\{ T_S(N^i), R_j \} &= -\partial_i N^i R_j - N^i \partial_i R_j - R_i \partial_j N^i , \\
\{ T_S(N^i), p_A \} &= -N^i \partial_i p_A - \partial_i N^i p_A , \\
\{ T_S(N^i), \phi^A \} &= -N^i \partial_i \phi^A , \\
\{ T_S(N^i), g_{ij} \} &= -N^k \partial_k g_{ij} - \partial_i N^k g_{kj} - g_{ik} \partial_j N^k , \\
\{ T_S(N^i), \pi^j \} &= -\partial_k (N^k \pi^j) + \partial_k N^i \pi_k^j + \pi_k \partial_k N^j , \\
\{ T_S(N^i), f_{ij} \} &= -N^k \partial_k f_{ij} - \partial_i N^k f_{kj} - f_{ik} \partial_j N^k , \\
\{ T_S(N^i), \pi^j \} &= -\partial_i N^i \pi^j - N^i \partial_i \pi^j + \partial_j N^i \pi^j 
\end{align*}
\] (3.17)
and also
\[
\begin{align*}
\{ T_S(N^i), C_0 \} &= -N^m \partial_m C_0 - \partial_m N^m C_0 , \\
\{ T_S(N^i), \Sigma_p \} &= -N^m \partial_m \Sigma_p - \partial_m N^m \Sigma_p .
\end{align*}
\] (3.18)

Finally it is easy to show that following Poisson bracket holds
\[ \{ T_S(N^i), T_S(M^j) \} = T_S(N^j \partial_j M^i - M^j \partial_j N^i) . \] (3.19)

Now we are ready to analyze the stability of all primary constraints. As usual the requirement of the preservation of the constraint $\pi_N \approx 0$ implies an existence of the secondary constraint $C_0 \approx 0$. However the fact that $C_0$ is the constraint immediately implies that the constraint $\tilde{\Sigma}_i \approx 0$ is preserved during the time evolution of the system, using (3.18) and (3.19). As the next step we analyze the requirement of the preservation of the constraints $\pi_i \approx 0$ during the time evolution of the system
\[ \partial_t \pi_i = \{ \pi_i, H_E \} = - \left( \Omega_p \delta_i^k + \frac{\partial (\tilde{D}_j \tilde{n}^j)}{\partial \tilde{n}^i} \right) \left( \mathcal{R}_k - 2m^2 M_p^2 \sqrt{g} f_{km} \tilde{n}^m \right) = 0 . \] (3.20)

It turns out that the following matrix
\[ \Omega_p \delta_i^k + \frac{\partial (\tilde{D}_j \tilde{n}^j)}{\partial \tilde{n}^i} = 0 \] (3.21)
cannot be solved for $\Omega_p$ and hence we have to demand an existence of following secondary constraints \cite{3, 4, 5, 6}
\[ \mathcal{C}_i \equiv \mathcal{R}_i - \frac{2m^2 M_p^2 \sqrt{g}}{\sqrt{x}} f_{ij} \tilde{n}^j \approx 0 . \] (3.22)
Finally we have to proceed to the analysis of the time development of the constraint \( \Sigma_p \approx 0 \). Following [11] we simplify this constraint as follows. Using \( C_i \) and \( \Sigma_i \) we express \( \tilde{n}^i \) as a function of the phase space variables \( p_A, \phi^A \) and \( g_{ij}, \pi^{ij} \) [11]

\[
\tilde{n}^i = -\frac{\partial^j\phi^A p_A f^{ji}}{\sqrt{p_A \partial^k \phi^A \partial^j \phi^B p_B + 4m^4 M_p^4 g}} + \tilde{F}^{ij} \Sigma_j + \tilde{G}^{ij} C_j , \tag{3.23}
\]

where \( \tilde{F}^{ij}, \tilde{G}^{ij} \) are phase space functions whose explicit form is not needed for us.

Now using these results we find that the constraint \( \Sigma_p \) takes the form

\[
\Sigma_p = \tilde{\Sigma}_p + H^i \Sigma_i + G^i C_i , \tag{3.24}
\]

where we introduced new independent constraint \( \tilde{\Sigma}_p \)

\[
\tilde{\Sigma}_p = 4m^4 M_p^4 g + p_A G^{AB} p_B + 2T \sqrt{f} \sqrt{p_A \partial^i \phi^A f^{ij} \partial^j \phi^B p_B + 4m^4 M_p^4 g} + T^2 f = 0 \tag{3.25}
\]

which is more complicated than in pure massive case due to the term proportional to \( T \). On the other hand we observe that in case when \( m = 0 \) this constraint takes the form \( \tilde{\Sigma}_p = p_A G^{AB} p_B + 2T \sqrt{f} \sqrt{p_A \partial^i \phi^A f^{ij} \partial^j \phi^B p_B + 4m^4 M_p^4 g} + T^2 f \) which means that \( \tilde{\Sigma}_p \) is the linear combination of the Hamiltonian and diffeomorphism constraints of the pure Galileon action that is reviewed in appendix. In other words in the limit \( m \to 0 \) the theory possesses eight first class constraints \( C_0, R_i, H_i, H_T \) that reflects the fact that this theory is invariant under two independent diffeomorphism.

Returning to the case \( m \neq 0 \) we define the total Hamiltonian with all constraints included

\[
H_T = \int d^3 x (N C_0 + v_N \pi_N + v^i \pi_i + \Omega^i \tilde{\Sigma}_p + \Omega^i \tilde{\Sigma}_i + \Gamma^i C_i) . \tag{3.26}
\]

Now we are ready to analyze the stability of all constraints that appear in (3.26). First of all we find that \( \pi_N \approx 0 \) is automatically preserved while the preservation of the constraint \( \pi_i \approx 0 \) gives

\[
\partial_t \pi_i = \{ \pi_i, H_T \} \approx \int d^3 x \nabla^j(x) \{ \pi_i, C_j(x) \} = -2m^2 \Gamma^j \frac{1}{\sqrt{x}} (f_{ij} - f_{ik} \tilde{n}^k f_{jl} \tilde{n}^l) \equiv -\triangle_{\pi_i, C_j} \Gamma^j . \tag{3.27}
\]

By definition

\[
\det(f_{ij} - f_{ik} \tilde{n}^k f_{jl} \tilde{n}^l) = \tilde{x} \det f_{ij} \neq 0 \tag{3.28}
\]

and hence the matrix \( \triangle_{\pi_i, C_j} \) is non-singular. Then the only solution of the equation (3.27) is \( \Gamma^i = 0 \).
As the next step we perform the analysis of the stability of the constraint \( \tilde{\Sigma}_p \). We introduce the smeared form of this constraint

\[
\Sigma(N) = \int d^3 x N(x) \tilde{\Sigma}_p(x). \tag{3.29}
\]

To proceed further we need following Poisson brackets

\[
\{ p_A(x), f_{ij}(y) \} = -\partial_{y^i}\delta(x - y)G_{AB}\partial_{y^j}\phi^B(y) - \partial_{y^j}\phi^B(\partial_{y^i}\delta(x - y),
\{ p_A(x), f(y) \} = [ -\partial_{y^i}\delta(x - y)G_{AB}\partial_{y^j}\phi^B(y), f^{ij}(y) - \partial_{y^j}\phi^B(\partial_{y^i}\delta(x - y), f^{ij}(y)]f(y),
\{ p_A(x), f^{ij}(y) \} = -f^{im}(y)\{ p_A(x), f_{mn}(y) \} f^{nj}(y) =
\]

\[
= f^{im}(y)\{ \partial_{y^m}\delta(x - y)G_{AB}\partial_{y^n}\phi^B(y) + \partial_{y^n}\phi^B(\partial_{y^m}\delta(x - y), f^{nj}(y). \tag{3.30}
\]

With the help of these results and after some calculations we derive following Poisson bracket

\[
\{ \Sigma(N), \Sigma(M) \} = 4T \int d^3 x (N\partial_i M - M\partial_i N) f^{ij}(\partial_j \phi^A p_A) \sqrt{\frac{T}{A}} \times \]

\[
\times (p_A G^{AB} p_B + T^2 f + 4m^2 M_p^2 g + 2T \sqrt{f A}) =
\]

\[
= 4T \Sigma (N\partial_i M - M\partial_i N) f^{ij}(\partial_j \phi^A p_A) \sqrt{\frac{T}{A}}, \tag{3.31}
\]

where

\[
A = p_A \partial_i \phi^A f^{ij} \partial_j \phi^B p_B + 4m^4 M_p^4 g. \tag{3.32}
\]

This is very important result that shows that the Poisson bracket between \( \tilde{\Sigma}_p \) vanishes on the constraint surface.

Finally we have to determine the Poisson bracket between \( \Sigma(N) \) and \( C(M) \) where

\[
C(M) = \int d^3 x M(x) C_0(x). \tag{3.33}
\]

Using again (3.30) and after some calculations we find following result 4

\[
\{ \Sigma(N), C(M) \} = \int d^3 x \Sigma(N\Sigma^{IJ}) + \int d^3 x N\partial_i M \left[ 2T \sqrt{\frac{T}{A}} \partial_j \phi^A p_A \frac{\delta(\tilde{D}^k l)}{\delta f_{ij}} C_k + \right.
\]

\[
+ 2m^2 M_p^2 T \sqrt{\frac{T}{A}} \sqrt{g} \sqrt{x} \tilde{D}^i k f_{kj} \Sigma_j - 2m^2 M_p^2 T \sqrt{g} \sqrt{x} \tilde{D}^i k f_{kj} C_j +
\]

\[
+ 4\partial_j \phi^A p_A \frac{\delta(\tilde{D}^k l)}{\delta f_{ij}} C_k + 4m^2 M_p^2 \sqrt{g} \sqrt{x} \tilde{D}^i j f_{ik} \Sigma_k - 4m^2 M_p^2 \sqrt{g} \sqrt{x} \tilde{D}^i j f_{ik} C_k \right], \tag{3.35}
\]

\[^4\text{Note that during calculations we used the formula}
\]

\[
\frac{\delta(\sqrt{x} \tilde{D}^k)}{\delta f_{ij}} = \sqrt{\frac{T}{x}} \tilde{D}_l f_{ln} - \frac{1}{\sqrt{x}} f_{lm} \frac{\delta(\tilde{D}_m \tilde{D}^n)}{\delta f_{ij}} \tag{3.34}
\]

which follows from (2.9).
where

\[
\Sigma_{\Pi}^p = 4m^2 M^2 p_{\partial i} \left[ \delta(\tilde{D}^i)^{\Pi}_{\partial j} \phi^A \right] + 8m^4 M^4 p_{\partial i} \left[ \delta(\tilde{D}^i)^{\Pi}_{\partial j} \phi^A \right] + 
\]

\[
+ 2m^2 M^2 T \left( g^{ij} \tilde{g}_{ijkl} \pi^{kl} + 2m^4 M^4 p_{\partial i} \left[ \delta(\tilde{D}^i)^{\Pi}_{\partial j} \phi^A \right] + 
\]

so that we see that (3.35) vanishes on the constraint surface up to the expression that we denote as \( \tilde{\Sigma}^\Pi_p \)

\[
\tilde{\Sigma}^\Pi_p = 2m^2 M^2 \left( 2p_{\partial i} + T \tilde{T} \left( g^{ij} \tilde{g}_{ijkl} \pi^{kl} + 2m^4 M^4 p_{\partial i} \left[ \delta(\tilde{D}^i)^{\Pi}_{\partial j} \phi^A \right] + 
\]

Now we are ready to proceed to the analysis of the requirement of the preservation of the constraint \( \tilde{\Sigma}_p \)

\[
\partial_t \tilde{\Sigma}_p = \left\{ \tilde{\Sigma}_p, H_T \right\} \approx \int d^3x N(x) \left\{ \Sigma_p, C_0(x) \right\} \approx \int d^3x N(x) \tilde{\Sigma}^\Pi_p(x) .
\]

From this result we see that the constraint \( \tilde{\Sigma}_p \approx 0 \) is preserved during the time evolution of the system on condition when either \( N = 0 \) or when \( \tilde{\Sigma}^\Pi_p = 0 \). Note that we should interpreted \( N \) as the Lagrange multiplier so that it is possible to demand that \( N = 0 \) on condition when \( \tilde{\Sigma}^\Pi_p \neq 0 \) on the whole phase space. It seems to us that such a condition is too strong so that it is more natural to demand that \( \tilde{\Sigma}^\Pi_p \approx 0 \) and \( N \neq 0 \). In other words \( \tilde{\Sigma}^\Pi_p \approx 0 \) is the new secondary constraint.

It is convenient to have constraint \( \tilde{\Sigma}^\Pi_p \) independent on \( \tilde{n}^i \). This can be easily done when we use (3.23) and insert into the explicit form of \( \tilde{D}^i \)  

\[
\tilde{D}^i_j = \sqrt{g^{ik}f_{mn}Q^m_p(Q^{-1})^p} ,
\]

\[
Q^m_p = \tilde{x} \delta^m_p + \tilde{n}^m \tilde{n}^m f_{np} , \quad (Q^{-1})^p_j = \frac{1}{\tilde{x}}(\delta^m_p - \tilde{n}^p \tilde{n}^m f_{mj})
\]

(3.39)
so that we find

\[
Q^m_p = \frac{1}{A + 4M^4 p^4 g} (4m^4 M^4 p g \delta^m_p + \partial_j \phi^A p_A f^{jm} \partial_p \phi^B p_B),
\]

\[
(Q^{-1})^m_p = \frac{A + 4M^4 p^4 g}{4m^4 M^4 p g} \left( \delta^m_p - \frac{1}{A + 4M^4 p^4 g} \partial_j \phi^A p_A f^{jm} \partial_p \phi^B p_B \right)
\]

(3.40)

up to terms proportional to the constraints \( C_i, \Sigma_i \). With the help of these results it is easy to formulate \( \tilde{\Sigma}_p^{II} \) as a constraint that does not depend on \( n^i \) (Again up to the terms proportional to \( \Sigma_i, C_i \)). This fact simplifies further analysis considerably since now the Poisson brackets between \( \tilde{\Sigma}_p^{II} \) and \( \pi_i \) are zero.

In summary we have following collection of constraints: \( \pi_N \approx 0, \pi_i \approx 0, C_0 \approx 0, C_i \approx 0, \Sigma_i \approx 0, \Sigma_p \approx 0, \tilde{\Sigma}_p^{II} \approx 0 \). The dynamics of these constraints is governed by the total Hamiltonian

\[
H_T = \int d^3x (N C_0 + v_N \pi_N + v^i \pi_i + \Omega_p \Sigma_p + \Omega^{II}_p \tilde{\Sigma}_p^{II} + \Omega^j \tilde{\Sigma}_i + \Gamma^i C_i).
\]

(3.41)

As the final step we have to analyze the preservation of all constraints, following [11]. The case of \( \pi_N \approx 0 \) is trivial. In case of \( \pi_i \approx 0 \) we obtain

\[
\partial_t \pi_i(x) = \{ \pi_i(x), H_T \} = \int d^3y \{ \pi_i(y), C_j(y) \} + \Omega^{II}_p(y) \left\{ \pi_i(x), \tilde{\Sigma}_p^{II}(y) \right\} = \Gamma^j \Delta_{pi} C_j(x) = 0
\]

(3.42)

due to the crucial fact that we used the formulation when \( \tilde{\Sigma}_p^{II} \) does not depend on \( n^i \). Then as we argued above the only solution of the equation is \( \Gamma^i = 0 \). Now the time development of \( C_i \) is given by the equation

\[
\partial_t C_i(x) = \{ C_i(x), H_T \} \approx \int d^3x (N(y) \{ C_i(x), C_0(y) \} + v^j(y) \{ C_i(x), \pi_j(y) \} + \Omega_p(y) \left\{ C_i(x), \tilde{\Sigma}_p(y) \right\} + \Omega^{II}_p(y) \left\{ C_i(x), \tilde{\Sigma}_p^{II}(y) \right\})
\]

(3.43)

and the time development of the constraint \( \tilde{\Sigma}_p \) is governed by the equation

\[
\partial_t \tilde{\Sigma}_p(x) = \left\{ \tilde{\Sigma}_p(x), H_T \right\} \approx \int d^3x \Omega^{II}_p(y) \left\{ \tilde{\Sigma}_p(x), \tilde{\Sigma}_p^{II}(y) \right\}.
\]

(3.44)

As follows from the explicit form of the constraint \( \tilde{\Sigma}_p^{II} \) we see that \( \left\{ \tilde{\Sigma}_p^{II}(x), \tilde{\Sigma}_p(y) \right\} \) is non-zero and proportional also to the higher order derivatives of the delta functions. As a
consequence we find that the only solution of the equation above is $\Omega^I_p = 0$. Further we analyze the time evolution of the constraint $\tilde{\Sigma}^{II}_p$

$$\partial_t \tilde{\Sigma}^{II}_p(x) = \left\{ \tilde{\Sigma}^{II}_p(x), H_T \right\} = \int d^3 x \left( N(y) \left\{ \tilde{\Sigma}^{II}_p(x), C_0(y) \right\} + \Omega_p(y) \left\{ \tilde{\Sigma}^{II}_p(x), \tilde{\Sigma}_p(y) \right\} \right) = 0 .$$

(3.45)

Now from the last equation we obtain $\Omega_p$ as a function of the phase space variables and $N$, at least in principle. Then inserting this result into the equation for the preservation of $C_i$ (3.43) we determine $v^j$ as functions of the phase space variables. Finally note also that the constraint $C_0$ is automatically preserved due to the fact that $\Gamma^i = \Omega^I_p = 0$ and also the fact that $\{C_0(x), C_0(y)\} \approx 0$ as was shown in [6].

In summary we obtain following picture. We have five first class constraints $\pi_N \approx 0$, $C_0 \approx 0$, $\tilde{\Sigma}_i \approx 0$ together with eight second class constraints $\pi_i \approx 0$, $C_i \approx 0$ and $\tilde{\Sigma}_p \approx 0$, $\tilde{\Sigma}^{II}_p \approx 0$. The constraints $\pi_i \approx 0$ together with $C_i \approx 0$ can be solved for $\pi_i$ and $\tilde{n}^i$. Then the constraint $\tilde{\Sigma}_p$ can be solved for one of the $D + 1$ momenta $p_A$ while the constraint $\tilde{\Sigma}^{II}_p$ can be solved for one of $D + 1 \phi^A$. As a result we have 12 gravitational degrees of freedom $g_{ij}$, $\pi^{ij}$, $2D$ scalars degrees of freedom together with 4 first class constraints $C_0 \approx 0$, $\tilde{\Sigma}_i \approx 0$. Then we find that the number of physical degrees of freedom is $12 + 2D - 8 = 10 + 2(D - 3)$ which corresponds to the number of physical degrees of freedom of the massive gravity coupled with $D - 3$ scalar Galileon fields. In other words we have shown that this specific model of the minimal dRGT gravity coupled with tadpole Galileon field is ghost free.

Acknowledgement: This work was supported by the Grant agency of the Czech republic under the grant P201/12/G028.

A. Hamiltonian Analysis of Galileon Tadpole Term

In this section we briefly review the Hamiltonian analysis of the Galileon tadpole term

$$S_{gal} = -T \int d^4 x \sqrt{-\det f_{\mu\nu}} .$$

(A.1)

The momentum conjugate to $\phi^A$ takes the form

$$p_A = -T G_{AB} \partial_\mu \phi^B (f^{-1})^{\mu0} \sqrt{-\det f_{\mu\nu}} .$$

(A.2)

Taking the square of given expression we find following primary constraint

$$H_T = p_A G^{AB} p_B + T^2 f \approx 0 .$$

(A.3)

On the other hand when we multiply (A.3) with $\partial_i \phi^A$ we obtain following 3 primary constraints

$$H_i = \partial_i \phi^A p_A \approx 0$$

(A.4)
Introducing the smeared forms of these constraints \( H_T(N) = \int d^3 x N^3 \mathcal{H}_T \) and \( H_S(N^i) = \int d^3 x N^i \mathcal{H}_i \) we easily find the Poisson brackets

\[
\{ H_T(N), H_T(M) \} = 2 T^2 H_S((\partial_i M N - \partial_i NM) f^{ij} \det f),
\]

\[
\{ H_S(N^i), H_T(M) \} = H_T((N^i \partial_i M - \partial_i N^i)),
\]

\[
\{ H_S(N^i), H_S(M^j) \} = H_S((N^i \partial_i M^j - M^i \partial_i N^j))
\]

(A.5)

that coincide with the Poisson brackets calculated for example in [22]. We see that \( \mathcal{H}_T, \mathcal{H}_i \) are the first class constraints which is a consequence of diffeomorphism invariance of given theory.

References

[1] C. de Rham, G. Gabadadze and A. J. Tolley, “Resummation of Massive Gravity,” Phys. Rev. Lett. 106 (2011) 231101 [arXiv:1011.1232 [hep-th]].

[2] S. F. Hassan and R. A. Rosen, “Resolving the Ghost Problem in non-Linear Massive Gravity,” Phys. Rev. Lett. 108 (2012) 041101 [arXiv:1106.3344 [hep-th]].

[3] S. F. Hassan and R. A. Rosen, “On Non-Linear Actions for Massive Gravity,” JHEP 1107 (2011) 009 [arXiv:1103.6055 [hep-th]].

[4] S. F. Hassan, R. A. Rosen and A. Schmidt-May, “Ghost-free Massive Gravity with a General Reference Metric,” JHEP 1202 (2012) 026 [arXiv:1109.3230 [hep-th]].

[5] S. F. Hassan and R. A. Rosen, “Bimetric Gravity from Ghost-free Massive Gravity,” JHEP 1202 (2012) 126 [arXiv:1109.3515 [hep-th]].

[6] S. F. Hassan and R. A. Rosen, “Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity,” JHEP 1204 (2012) 123 [arXiv:1111.2070 [hep-th]].

[7] A. Golovnev, “On the Hamiltonian analysis of non-linear massive gravity,” Phys. Lett. B 707 (2012) 404 [arXiv:1112.2134 [gr-qc]].

[8] J. Kluson, “Comments About Hamiltonian Formulation of Non-Linear Massive Gravity with Stuckelberg Fields,” JHEP 1206 (2012) 170 [arXiv:1112.5267 [hep-th]].

[9] J. Kluson, “Remark About Hamiltonian Formulation of Non-Linear Massive Gravity in Stuckelberg Formalism,” Phys. Rev. D 86 (2012) 124005 [arXiv:1202.5899 [hep-th]].

[10] J. Kluson, “Non-Linear Massive Gravity with Additional Primary Constraint and Absence of Ghosts,” Phys. Rev. D 86 (2012) 044024 [arXiv:1204.2957 [hep-th]].

[11] J. Kluson, “Note About Hamiltonian Formalism for General Non-Linear Massive Gravity Action in Stuckelberg Formalism,” arXiv:1209.3612 [hep-th].

[12] C. de Rham, G. Gabadadze and A. J. Tolley, “Ghost free Massive Gravity in the St¨uckelberg language,” Phys. Lett. B 711 (2012) 190 [arXiv:1107.3820 [hep-th]].

[13] S. F. Hassan, A. Schmidt-May and M. von Strauss, “Proof of Consistency of Nonlinear Massive Gravity in the St¨uckelberg Formulation,” Phys. Lett. B 715 (2012) 335 [arXiv:1203.5283 [hep-th]].
[14] M. Andrews, G. Goon, K. Hinterbichler, J. Stokes and M. Trodden, “Massive gravity coupled to DBI Galileons is ghost free,” arXiv:1303.1177 [hep-th].

[15] G. Gabadadze, K. Hinterbichler, J. Khoury, D. Pirtskhalava and M. Trodden, “A Covariant Master Theory for Novel Galilean Invariant Models and Massive Gravity,” Phys. Rev. D 86 (2012) 124004 [arXiv:1208.5773 [hep-th]].

[16] C. de Rham and A. J. Tolley, “DBI and the Galileon reunited,” JCAP 1005 (2010) 015 [arXiv:1003.5917 [hep-th]].

[17] K. Hinterbichler, M. Trodden and D. Wesley, “Multi-field galileons and higher co-dimension branes,” Phys. Rev. D 82 (2010) 124018 [arXiv:1008.1305 [hep-th]].

[18] C. Deffayet, G. Esposito-Farese and A. Vikman, “Covariant Galileon,” Phys. Rev. D 79 (2009) 084003 [arXiv:0901.1314 [hep-th]].

[19] C. Deffayet, S. Deser and G. Esposito-Farese, “Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress-tensors,” Phys. Rev. D 80 (2009) 064015 [arXiv:0906.1967 [gr-qc]].

[20] E. Gourgoulhon, “3+1 formalism and bases of numerical relativity,” gr-qc/0703035 [GR-QC].

[21] R. L. Arnowitt, S. Deser and C. W. Misner, “The Dynamics of general relativity,” Gen. Rel. Grav. 40 (2008) 1997 [gr-qc/0405109].

[22] I. Bengtsson, N. Barros e Sa and M. Zabzine, “A Note on topological brane theories,” Phys. Rev. D 62 (2000) 066005 [hep-th/0005092].

[23] K. Hinterbichler and R. A. Rosen, “Interacting Spin-2 Fields,” JHEP 1207 (2012) 047 [arXiv:1203.5783 [hep-th]].

[24] C. Lin, “SO(3) massive gravity,” arXiv:1305.2069 [hep-th].

[25] D. Comelli, M. Crisostomi, F. Nesti and L. Pilo, “Degrees of Freedom in Massive Gravity,” Phys. Rev. D 86 (2012) 101502 [arXiv:1204.1027 [hep-th]].

[26] D. Comelli, F. Nesti and L. Pilo, “Weak Massive Gravity,” arXiv:1302.4447 [hep-th].

[27] D. Comelli, F. Nesti and L. Pilo, “Massive gravity: a General Analysis,” arXiv:1305.0236 [hep-th].