Constructing elliptic curve isogenies in quantum subexponential time

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Abstract. Given two elliptic curves over a finite field having the same cardinality and endomorphism ring, it is known that the curves admit an isogeny between them, but finding such an isogeny is believed to be computationally difficult. The fastest known classical algorithm takes exponential time, and prior to our work no faster quantum algorithm was known. Recently, public-key cryptosystems based on the presumed hardness of this problem have been proposed as candidates for post-quantum cryptography. In this paper, we give a subexponential-time quantum algorithm for constructing isogenies, assuming the Generalized Riemann Hypothesis (but with no other assumptions). Our algorithm is based on a reduction to a hidden shift problem, together with a new subexponential-time algorithm for evaluating isogenies from kernel ideals (under only GRH), and represents the first nontrivial application of Kuperberg’s quantum algorithm for the hidden shift problem. This result suggests that isogeny-based cryptosystems may be uncompetitive with more mainstream quantum-resistant cryptosystems such as lattice-based cryptosystems.

1 Introduction

We consider the problem of constructing an isogeny between two given isogenous ordinary elliptic curves defined over a finite field $\mathbb{F}_q$ and having the same endomorphism ring. (Two such curves are called \textit{horizontally isogenous}.) This problem has led to several applications in elliptic curve cryptography, both constructive and destructive. The fastest known probabilistic algorithm for solving this problem is the algorithm of Galbraith and Stolbunov \cite{18}, based on the work of Galbraith, Hess, and Smart \cite{17}. Their algorithm is exponential, with a worst-case (and average-case) running time roughly proportional to $q^{3/2}$.

Although quantum attacks are known against several cryptographic protocols of an algebraic nature \cite{12,19,34}, until now there has been no nontrivial quantum algorithm for constructing isogenies. The difficulty of this problem has led to various constructions of public-key cryptosystems based on finding isogenies, beginning with a proposal of Couveignes \cite{10}. More recently, Rostovtsev and Stolbunov \cite{29} and Stolbunov \cite{36} proposed refined versions of these cryptosystems with the specific aim of obtaining cryptographic protocols that resist attacks by quantum computers.

In this work, we give a subexponential-time quantum algorithm for constructing an isogeny between two given horizontally isogenous elliptic curves, and show that the running time of our algorithm is bounded above by $L_q(\frac{1}{2}, \sqrt{q})$ under (only) the Generalized Riemann Hypothesis (GRH). This result raises serious questions about the viability of isogeny-based cryptosystems in the context...
of quantum computers. At present, isogeny-based cryptosystems are not especially attractive since their performance is poor compared to other quantum-resistant cryptosystems, such as lattice-based cryptography [20]. Nevertheless, they represent a distinct family of cryptosystems worthy of analysis (for reasons of diversity if nothing else, given the small number of quantum-resistant public-key cryptosystem families available [27]). Since isogeny-based cryptosystems already perform poorly at moderate security levels [36, Table 1], any improved attacks such as ours would seem to disqualify such systems from consideration in a post-quantum world.

1.1 Contributions

Our first main contribution, described in Section 5, is a reduction from the problem of isogeny construction to the abelian hidden shift problem. While a connection between isogenies and hidden shifts was noted previously by Stolbunov [36], we observe that the reduction gives an injective hidden shift problem. This allows us to apply an algorithm of Kuperberg [25] to solve the hidden shift problem using a subexponential number of queries to certain functions. This reduction constitutes the first nontrivial application of Kuperberg’s algorithm outside of the black-box setting.

The reduction to the hidden shift problem alone does not immediately give a subexponential-time algorithm for computing isogenies, because one must consider the time required to compute the hiding functions. Indeed, prior to our work there was no known subexponential-time algorithm to evaluate these functions. Our second main contribution, described in Section 4, is a subexponential-time (classical) algorithm to compute the isogeny star operator, which is defined as a certain action of an ideal class group on a set of elliptic curves. In this way we can compute the hiding functions in subexponential time and thus obtain a subexponential reduction to the hidden shift problem. Unlike previous algorithms for isogeny computation [16, 17, 23], our runtime analysis assumes only GRH, whereas all previous subexponential-time algorithms for isogeny problems have required additional heuristic assumptions. We achieve this improvement using expansion properties of a certain Cayley graph [22]. The same idea can also be used to obtain subexponential algorithms (under only GRH) for evaluating isogenies (see Remark 4.9). In addition, Bisson [4] has shown that our method yields a subexponential algorithm for computing endomorphism rings of ordinary elliptic curves under GRH.

Kuperberg’s algorithm for the abelian hidden shift problem uses superpolynomial space (i.e., a quantum computer with superpolynomially many qubits), so the same is true of the most straightforward version of our algorithm. Since it is difficult to build quantum computers with many qubits, this feature could limit the applicability of our result. However, we also obtain an algorithm using polynomial space by taking advantage of an alternative approach to the abelian hidden shift problem due to Regev [28]. Regev only explicitly considered the case of the hidden shift problem in a cyclic group whose order is a power of 2, and even in that case did not compute the constant in the exponent of the running time. We fill both of these gaps in our work, showing that the hidden shift problem in any finite abelian group $A$ can be solved in time $L_{|A|}(\frac{1}{2}, \sqrt{2})$ by a quantum computer using only polynomial space. Consequently, we give a polynomial-space quantum algorithm for isogeny construction using time $L_q(\frac{1}{2}, \frac{\sqrt{2}}{1} + \sqrt{2})$. The group relevant to isogeny construction is not always cyclic, so the extension to general abelian groups is necessary for our application.

1.2 Related work

Our algorithm for evaluating the isogeny star operator is based on reducing an ideal modulo principal ideals to obtain a smooth ideal. This idea is originally due to Galbraith, Hess, and Smart [17].
Bröker, Charles, and Lauter [7] and Jao and Soukharev [23] also use this idea to give algorithms for evaluating isogenies. Bisson and Sutherland [5] use a similar smoothing technique to compute endomorphism rings in subexponential time. We stress that, with the exception of [7], which is restricted in scope to small discriminants, all the results mentioned above make heuristic assumptions of varying severity [5, §4][17, p. 37][23, p. 224] in addition to the Generalized Riemann Hypothesis in the course of proving their respective runtime claims. Our work is the first to achieve provably subexponential running time with no heuristic assumptions other than GRH. In practice, the heuristic algorithms in [5] and [23] run slightly faster than our algorithms in Section 4, because they make use of an optimized exponent distribution (originating from [5]) that minimizes the number of large degree isogenies appearing in the smooth factorization. Our work does not use this optimization, because doing so would reintroduce the need for additional heuristic assumptions.

An alternative approach to computing isogenies, given in Couveignes [10, p. 11] and Stolbunov [36, p. 227], is to treat the class group as a $\mathbb{Z}$-module and use lattice basis reduction to compute the isogeny star operator. In practice, the lattice-based approach works well for moderate parameter sizes. However, since it amounts to solving the closest vector problem, the method asymptotically requires exponential time (even with known quantum algorithms), and thus is slower than our approach.

## 2 Isogenies

For general background on elliptic curves, we refer the reader to Silverman [35].

Let $E$ and $E'$ be elliptic curves defined over a field $F$. An isogeny $\phi: E \to E'$ is an algebraic morphism satisfying $\phi(\infty) = \infty$. The degree of an isogeny is its degree as an algebraic map. The endomorphism ring $\text{End}(E)$ is the set of isogenies from $E(\bar{F})$ to itself. This set forms a ring under pointwise addition and composition.

When $F$ is a finite field, the rank of $\text{End}(E)$ as a $\mathbb{Z}$-module is either 2 or 4. We say $E$ is supersingular if the rank is 4, and ordinary otherwise. A supersingular curve cannot be isogenous to an ordinary curve. Most elliptic curves are ordinary (in particular, supersingular curves have density zero [32]), and most current proposals for isogeny-based cryptography (including all published isogeny-based public-key cryptosystems) use ordinary curves. Thus, in this paper we restrict our attention to ordinary elliptic curves. It remains an interesting open problem to study cryptographic applications of isogenies between supersingular curves and to better understand the computational difficulty of computing such isogenies, but we do not address this issue.

Over a finite field $\mathbb{F}_q$, two elliptic curves $E$ and $E'$ are isogenous if and only if $\#E(\mathbb{F}_q) = \#E'(\mathbb{F}_q)$ [37]. The endomorphism ring of an ordinary elliptic curve over a finite field is an imaginary quadratic order $\mathcal{O}_\Delta$ of discriminant $\Delta < 0$. The set of all isomorphism classes (over $\bar{\mathbb{F}}_q$) of isogenous curves with endomorphism ring $\mathcal{O}_\Delta$ is denoted $\text{Ell}_{q,n}(\mathcal{O}_\Delta)$, where $n$ is the cardinality of any such curve. We represent elements of $\text{Ell}_{q,n}(\mathcal{O}_\Delta)$ by taking the $j$-invariant of any representative curve in the isomorphism class.

An isogeny between two curves having the same endomorphism ring is called a horizontal isogeny [15]. Likewise, we say that two isogenous curves are horizontally isogenous if their endomorphism rings are equal. Any separable horizontal isogeny $\phi: E \to E'$ between curves in $\text{Ell}_{q,n}(\mathcal{O}_\Delta)$ can be specified, up to isomorphism, by giving $E$ and $\ker \phi$ [35, III.4.12]. The kernel of an isogeny, in turn, can be represented as an ideal in $\mathcal{O}_\Delta$ [39, Thm. 4.5]. Denote by $\phi_b: E \to E_b$ the isogeny corresponding to an ideal $b$ (keeping in mind that $\phi_b$ is only defined up to isomorphism of $E_b$). Principal ideals correspond to isomorphisms, so any other ideal equivalent to $b$ in the ideal class
group $\text{Cl}(\mathcal{O}_\Delta)$ of $\mathcal{O}_\Delta$ induces the same isogeny, up to isomorphism [39, Thm. 3.11]. Hence one obtains a well-defined group action

$$*: \text{Cl}(\mathcal{O}_\Delta) \times \text{Ell}_{q,n}(\mathcal{O}_\Delta) \to \text{Ell}_{q,n}(\mathcal{O}_\Delta)$$

$$[b] * j(E) = j(E_b)$$

where $[b]$ denotes the ideal class of $b$. This group action, which we call the isogeny star operator, is free and transitive [39, Thm. 4.5], and thus $\text{Ell}_{q,n}(\mathcal{O}_\Delta)$ forms a principal homogeneous space over $\text{Cl}(\mathcal{O}_\Delta)$.

### Isogeny graphs under GRH

Our runtime analysis in Section 4 relies on the following result of [22] which states, roughly, that random short products of small primes in $\text{Cl}(\mathcal{O}_\Delta)$ yield nearly uniformly random elements of $\text{Cl}(\mathcal{O}_\Delta)$, under GRH.

**Theorem 2.1.** Let $\mathcal{O}_\Delta$ be an imaginary quadratic order of discriminant $\Delta < 0$ and conductor $c$. Set $G = \text{Cl}(\mathcal{O}_\Delta)$. Let $B$ and $x$ be real numbers satisfying $B > 2$ and $x \geq (\ln |\Delta|)^B$. Let $S_x$ be the multiset $A \cup A^{-1}$ where

$$A = \{[p] \in G : \gcd(c, p) = 1 \text{ and } N(p) \leq x \text{ is prime}\}$$

with $N(p)$ denoting the norm of $p$. Then, assuming GRH, there exists a positive absolute constant $C > 1$, depending only on $B$, such that for all $\Delta$, a random walk of length

$$t \geq C \frac{\ln |G|}{\ln \ln |\Delta|}$$

in the Cayley graph $\text{Cay}(G, S_x)$ from any starting vertex lands in any fixed subset $S \subset G$ with probability at least $\frac{1}{2} |S|/|G|$. 

**Proof.** Apply Corollary 1.3 of [22] with the parameters

- $K = \text{the field of fractions of } \mathcal{O}_\Delta$
- $G = \text{Cl}(\mathcal{O}_\Delta)$
- $q = |\Delta|$. 

Following [11], we refer to $G = \text{Cl}(\mathcal{O}_\Delta)$ as the ring class group of $\Delta$. Observe that by Remark 1.2(a) of [22], Corollary 1.3 of [22] applies to the ring class group $G$, since ring class groups are quotients of narrow ray class groups [11, p. 160]. By Corollary 1.3 of [22], Theorem 2.1 holds for all sufficiently large values of $|\Delta|$, i.e., for all but finitely many $|\Delta|$. To prove the theorem for all $|\Delta|$, simply take a larger (but still finite) value of $C$.

**Corollary 2.2** Theorem 2.1 still holds with the set $A$ redefined as

$$A = \{[p] \in G : \gcd(m\Delta, p) = 1 \text{ and } N(p) \leq x \text{ is prime}\}$$

where $m$ is any integer having at most $O(x^{1/2-\varepsilon} \log |\Delta|)$ prime divisors.

**Proof.** The alternative definition of the set $A$ differs from the original definition by no more than $O(x^{1/2-\varepsilon} \log |\Delta|)$ primes. As stated in [22, p. 1497], the contribution of these primes can be absorbed into the error term $O(x^{1/2} \log(x) \log(xq))$, and hence does not affect the conclusion of the theorem.
3 The group action inverse problem

For a fixed discriminant $\Delta$, the vectorization [10, §2] or group action inverse [36, §2.4] problem is the problem of finding an ideal class $[b] \in \text{Cl}(O_\Delta)$ such that $[b] \ast j(E) = j(E')$, given $j(E)$ and $j(E')$. We refer to $[b]$ as the quotient of $j(E)$ and $j(E')$. The computational infeasibility of finding quotients in $\text{Ell}_{q,n}(O_\Delta)$ is a necessary condition for the security of isogeny-based cryptosystems [10, §3][36, §7]. In the remainder of this paper, we present our subexponential algorithm for evaluating quotients in $\text{Ell}_{q,n}(O_\Delta)$ on a quantum computer.

A notable property of isogeny-based cryptosystems is that they do not require the ability to evaluate the isogeny star operator efficiently on arbitrary inputs. It is enough to sample from random smooth ideals (for which $\ast$ can be evaluated efficiently) when performing operations such as key generation [10, §5.4][36, §6.2]. However, to attack these cryptosystems using our approach, we do require the ability to evaluate the isogeny star operator on arbitrary inputs. We turn to this problem in the next section.

4 Computing the isogeny star operator

In this section, we describe a new classical (i.e., non-quantum) algorithm to evaluate the isogeny star operator. All notation is as in Section 2. Given an ideal class $[b] \in \text{Cl}(O_\Delta)$, and a $j$-invariant $E$ of an ordinary elliptic curve of endomorphism ring $O_\Delta$ over $\mathbb{F}_q$, we wish to evaluate $[b] \ast j(E)$. We define

$$L_N(\frac{1}{2}, c) := \exp[(c + o(1))\sqrt{\ln N \ln \ln N}].$$

For convenience, we denote $L_{\max(|\Delta|, q)}(\frac{1}{2}, c)$ by $L(c)$.

In Section 4.1 we show that, under GRH, our algorithm has a running time of $L_q(\frac{1}{2}, \frac{\sqrt{\ln \Delta}}{\min(\sqrt{\ln \Delta}, \sqrt{\ln q})})$, which is subexponential in the input size. For clarity, we present our algorithms and analysis in full instead of as “patches” to existing work. We emphasize that the basic structure of these algorithms appeared in prior work; our main contribution is to the analysis, which is facilitated by small changes to the algorithms. Specifically, Algorithm 1 is based on [23, Algorithm 3], which is in turn based on Seyeen’s algorithm [33]; Algorithm 2 is based on [7, Algorithm 4.1]. Our bounds on $t$ in Algorithm 1 are new, and allow us to prove the crucial runtime bound (Proposition 4.3).

**Computing a relation.** Given an ideal class $[b] \in \text{Cl}(O_\Delta)$, Algorithm 1 produces a relation vector $z = (z_1, \ldots, z_f) \in \mathbb{Z}^f$ for $[b]$, with respect to a factor base $\mathcal{F} = \{p_1, \ldots, p_f\}$, satisfying $[b] = \mathcal{F}^z := p_1^{z_1} \cdots p_f^{z_f}$, with the additional property (cf. Proposition 4.5) that the $L^1$-norm $|z|_1$ of $z$ is less than $O(\ln |\Delta|)$ for some absolute implied constant (here the $L^1$ norm of a vector denotes the sum of the absolute values of its coordinates). Algorithm 1 is similar to Algorithm 11.2 in [8], except that we impose a constraint on $|v|_1$ in Step 5 in order to keep $|z|_1$ small, and (for performance reasons) we use Bernstein’s algorithm instead of trial division to find smooth elements. We remark that Corollary 9.3.12 of [8] together with the restriction $C > 1$ in Theorem 2.1 implies that there exists a value of $t$ satisfying the inequality in Algorithm 1.

**Computing $j(E')$.** Algorithm 2 is the main algorithm for evaluating the isogeny star operator. It takes as input a discriminant $\Delta < 0$, an ideal class $[b] \in \text{Cl}(O_\Delta)$, and a $j$-invariant $j(E) \in \text{Ell}_{q,n}(O_\Delta)$, and produces the element $j(E') \in \text{Ell}_{q,n}(O_\Delta)$ such that $[b] \ast j(E) = j(E')$. Eliminating
Algorithm 1 Computing a relation

Input: $\Delta$, $q$, $n$, $z$, $[b]$, and an integer $t$ satisfying $C^{|\ln(\ln |\Delta|)|} \leq t \leq C \ln |\Delta|$ where $C$ is the constant of Theorem 2.1/Corollary 2.2.

Output: A relation vector $z \in \mathbb{Z}^j$ such that $[b] = [F^z]$, or nil.

1: Using Algorithm 1 with any valid choice of $\phi$, compute a sequence of isogenies $(\phi_1, \ldots, \phi_s)$ such that the composition $\phi_c : E \to E_c$ of the sequence has kernel $E[p_1^{s_1} p_2^{s_2} \cdots p_f^{s_f}]$, using the method of [7, §3]

Algorithm 2 Computing $j(E')$

Input: $\Delta$, $q$, $b$, and a $j$-invariant $j(E) \in \text{Ell}_{q,n}(\mathcal{O}_\Delta)$

Output: The element $j(E') \in \text{Ell}_{q,n}(\mathcal{O}_\Delta)$ such that $[b] * j(E) = j(E')$

1: Using Algorithm 1 with any valid choice of $t$, compute a relation $z \in \mathbb{Z}^j$ such that $[b] = [F^z] = [p_1^{s_1} p_2^{s_2} \cdots p_f^{s_f}]$

2: Compute a sequence of isogenies $(\phi_1, \ldots, \phi_s)$ such that the composition $\phi_c : E \to E_c$ of the sequence has kernel $E[p_1^{s_1} p_2^{s_2} \cdots p_f^{s_f}]$.

3: Return $j(E_c)$

the primes dividing $qn$ is necessary for the computation of the isogenies in the final step of the algorithm.

Algorithm 2 is correct since the ideals $b$ and $F^z$ belong to the same ideal class, and thus act identically on $\text{Ell}_{q,n}(\mathcal{O}_\Delta)$.

4.1 Runtime analysis

Here we determine the theoretical running time of Algorithm 2, as well as the optimal value of the parameter $z$ in Algorithm 1. As is typical for subexponential-time factorization algorithms involving a factor base, these two quantities depend on each other, and hence both are calculated simultaneously.

Proposition 4.1 The running time of Algorithm 1 is at most $L(z) + L(\frac{1}{4z})$, assuming GRH.

Proof. Step 1 of Algorithm 1 takes time $L(z)$ [8, Lemmas 11.3.1 and 11.3.2]. Step 2 of the algorithm requires $L(z)$ norm computations. Step 3 is negligible. Step 6 requires $C \ln |\Delta|$ multiplications in the class group, each of which requires $O((\ln |\Delta|)^{1+x})$ bit operations [30]. Hence the for loop in Steps 4–8 has running time $L(\frac{1}{4z}) \cdot O((\ln |\Delta|)^{2+x}).$ Bernstein’s algorithm [3] in Step 9 has a running
time of \( b(\log_2 b)^{2+\varepsilon} \) where \( b = L(z) + L\left(\frac{1}{4z}\right) \) is the combined size of \( \mathcal{S} \) and \( \mathcal{P} \). Finding the prime factorization in Step 10 costs \( L(z) \) using trial division, and Seysen’s algorithm [33, Thm. 3.1] in Step 11 has negligible cost under ERH (and hence GRH). Accordingly, we find that the running time is
\[
L(z) + O((\ln |\Delta|)^{2+\varepsilon}) \cdot L\left(\frac{1}{4z}\right) + b(\log_2 b)^{2+\varepsilon} + L(z) = L(z) + L\left(\frac{1}{4z}\right),
\]
as desired.

Remark 4.2. If we use quantum algorithms, then the performance boost obtained from Bernstein’s algorithm is not necessary, since quantum computers can factor integers in polynomial time [34]. This allows for some simplification in Algorithm 1 in the quantum setting: there is no need to store elements of \( \mathcal{S} \) (since one can test directly for smooth integers via factoring), and the algorithm no longer requires superpolynomial space.

Proposition 4.3 Under GRH, the probability that a single iteration of the for loop of Algorithm 1 produces an \( \mathcal{F} \)-smooth ideal \( \mathfrak{a}_v \) is at least \( L\left(-\frac{1}{4z}\right) \).

Proof. We adopt the notation used in Theorem 2.1 and Corollary 2.2. Apply Corollary 2.2 with the values \( m = qn, B = 3, \) and \( x = f = L(z) \gg (\ln |\Delta|)^B \). Observe that \( m \) has at most \( O(\log q) \) prime divisors, and \( O(\log q) \ll L\left(\frac{1}{4z}\right) \leq \frac{1}{4z} \). Therefore Corollary 2.2 applies. The ideal class \( [\mathfrak{a}] \cdot [\mathcal{F}^v] \) is equal to the ideal class obtained by taking the walk of length \( t \) in the Cayley graph \( \text{Cay}(G, S_x) \), having initial vertex \( [\mathfrak{a}] \), and whose edges correspond to the nonzero coordinates of the vector \( v \). Hence a random choice of vector \( v \) under the constraints of Algorithm 1 yields the same probability distribution as a random walk in \( \text{Cay}(G, S_x) \) starting from \( [\mathfrak{a}] \).

Let \( S \) be the set of reduced ideals in \( G \) with \( L(z) \)-smooth norm. By [8, Lemma 11.4.4], \( |S| \geq \sqrt{|\Delta|} L(z) \gg \frac{1}{4z} \). Hence, by Corollary 2.2, the probability that \( \mathfrak{a}_v \) lies in \( S \) is at least
\[
\frac{1}{2} \cdot \frac{|S|}{|G|} \geq \frac{1}{2} \cdot \frac{\sqrt{|\Delta|}}{|G|} \cdot L\left(-\frac{1}{4z}\right).
\]
Finally, Theorem 9.3.11 of [8] states that \( \frac{\sqrt{|\Delta|}}{|G|} \geq \frac{1}{\ln |\Delta|} \). Hence the probability that \( \mathfrak{a}_v \) is \( \mathcal{F} \)-smooth is at least
\[
\frac{1}{2} \cdot \frac{1}{\ln |\Delta|} \cdot L\left(-\frac{1}{4z}\right) = L\left(-\frac{1}{4z}\right),
\]
as desired.

Corollary 4.4 Under GRH, Algorithm 1 succeeds with probability at least \( 1 - \frac{1}{e} \).

Proof. Algorithm 1 loops through \( t = L\left(\frac{1}{4z}\right) \) vectors \( v \), and by Proposition 4.3, each such choice of \( v \) has an independent \( 1/t \) chance of producing a smooth ideal \( \mathfrak{a}_v \). Therefore the probability of success is at least \( 1 - \left(1 - \frac{1}{t}\right)^t > 1 - \frac{1}{e} \) as claimed.

The following proposition shows that the relation vector \( \mathbf{z} \) produced by Algorithm 1 is guaranteed to have small coefficients.
Proposition 4.5 Any vector $z$ output by Algorithm 1 satisfies $|z|_1 < (C + 1) \ln |\Delta|$.

Proof. Since $z = a - v$, we have $|z|_1 \leq |a|_1 + |v|_1$. But $|v|_1 \leq C \ln |\Delta|$ by construction, and the norm of $a_\nu$ is less than $\sqrt{\Delta/3}$ [8, Prop. 9.1.7], so

$$|a|_1 < \log_2 \sqrt{|\Delta|/3} < \log_2 \sqrt{|\Delta|} < \ln |\Delta|.$$  

This completes the proof.

Finally, we analyze the running time of Algorithm 2.

Theorem 4.6. Under GRH, Algorithm 2 succeeds with probability at least $1 - \frac{1}{q}$ and runs in time at most

$$L(\frac{1}{v}) + \max\{L(3z), L(z)(\ln q)^{3+\varepsilon}\}.$$  

Proof. We have shown that Algorithm 1 has running time $L(z) + L(\frac{1}{v})$ and success probability at least $1 - \frac{1}{q}$. Assuming that it succeeds, the computation of the individual isogenies $\phi_i$ in Step 2 of Algorithm 2 proceeds in one of two ways, depending on whether the characteristic of $F_q$ is large [7, §3.1] or small [7, §3.2]. The large characteristic algorithm fails when the characteristic is small, whereas the small characteristic algorithm succeeds in all situations, but is slightly slower in large characteristic. For simplicity, we consider only the latter, and more general, algorithm.

The general algorithm proceeds in two steps. In the first step, we compute the kernel polynomial of the isogeny. The time to perform one such calculation is $O((\ell(\ln q) \max(\ell, \ln q)^2)^{1+\varepsilon})$ in all cases ([26, Thm. 1] for characteristic $\geq 5$ and [14, Thm. 1] for characteristic 2 or 3). In the second step, we compute the equation of the isogenous curve using Vélu’s formulae [38]. This second step has a running time of $O(\ell^2 \varepsilon(\ln q)^{1+\varepsilon})$ [21, p. 214]. Hence the running time of Step 2 is at most

$$|z|_1 (O((\ell(\ln q) \max(\ell, \ln q)^2)^{1+\varepsilon}) + O(\ell^{2+\varepsilon}(\ln q)^{1+\varepsilon})).$$  

By Proposition 4.5, this expression is at most

$$(C + 1)(\ln |\Delta|)(\max\{L(3z), L(z)(\ln q)^{3+\varepsilon}\} + L(2z)(\ln q)^{1+\varepsilon}) = \max\{L(3z), L(z)(\ln q)^{3+\varepsilon}\}.$$  

The theorem follows.

Corollary 4.7 Under GRH, Algorithm 2 has a worst-case running time of at most $L_q(\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Proof. Using the inequality $|\Delta| \leq 4q$, we may rewrite Theorem 4.6 in terms of $q$. We obtain the following upper bound for the running time:

$$L(\frac{1}{v}) + \max\{L(3z), L(z)(\ln q)^{3+\varepsilon}\} \leq L_q(\frac{1}{2}, \frac{1}{4z} + 3z).$$  

The optimal choice of $z = \frac{1}{2\sqrt{3}}$ yields the running time bound of $L_q(\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Remark 4.8 Using our technique for eliminating heuristics, Bisson [4] has recently developed a subexponential-time algorithm for determining endomorphism rings of elliptic curves, assuming only GRH. As part of that work, Bisson presents a faster algorithm [4, Prop. 4.4] for determining the curves appearing in the sequence of isogenies in Step 2 of Algorithm 2, with running time quadratic in the isogeny degrees, improving upon the cubic time required in prior algorithms. Using this algorithm, the running time of Algorithm 2 improves to $L_q(\frac{1}{4}, \frac{\sqrt{3}}{2})$. 
Remark 4.9. Our algorithm for computing the isogeny star operator readily extends to an algorithm for evaluating isogenies in subexponential time. As in [7, 23], we specify an isogeny \( \phi: E \to E' \) by providing the ideal \( b \subset \text{End}(E) = \mathcal{O}_\Delta \) corresponding to the kernel of \( \phi \). To distinguish between isogenies that are identical up to isomorphism, we define a *normalized isogeny* [6, 7] to be one where \( \phi^*(w_{E'}) = w_E \). Algorithm 2 applied to the input \( b \) yields an (unnormalized) isogeny \( \phi_c: E \to E_c \) isomorphic to the desired isogeny \( \phi \). To find the normalized isogeny, we must evaluate the necessary isomorphism explicitly. This can be easily done by using [23, Algorithm 3, Steps 20–23] in conjunction with [7, Algorithm 4.1, Steps 4–6] on the relation produced by Algorithm 1. These additional steps are not rate-limiting, so the running time of the algorithm is unchanged. Bisson’s improvement (Remark 4.8) does not apply here, since we need to evaluate the actual isogeny, rather than just find the isogenous curve.

5 A quantum algorithm for constructing isogenies

Our quantum algorithm for constructing isogenies uses a simple reduction to the abelian hidden shift problem. To define this problem, let \( A \) be a known finite abelian group (with the group operation written multiplicatively) and let \( f_0, f_1: A \to S \) be black-box functions, where \( S \) is a known finite set. We say that \( f_0, f_1 \) hide a shift \( s \in A \) if \( f_0 \) is injective and \( f_1(x) = f_0(xs) \) (i.e., \( f_1 \) is a shifted version of \( f_0 \)). The goal of the hidden shift problem is to determine \( s \) using queries to such black-box functions. Note that this problem is equivalent to the hidden subgroup problem in the \( A \)-dihedral group, the nonabelian group \( A \rtimes \mathbb{Z}_2 \) where \( \mathbb{Z}_2 \) acts on \( A \) by inversion.

Isogeny construction is easily reduced to the hidden shift problem using the group action defined in Section 2. Given horizontally isogenous curves \( E_0, E_1 \) with endomorphism ring \( \mathcal{O}_\Delta \), we define functions \( f_0, f_1: \text{Cl}(\mathcal{O}_\Delta) \to \text{Ell}_{g,n}(\mathcal{O}_\Delta) \) that hide \( [s] \in \text{Cl}(\mathcal{O}_\Delta) \), where \( [s] \) is the ideal class such that \( [s] * j(E_0) = j(E_1) \). Specifically, let \( f_*([b]) = [b] * j(E_c) \). Then it is immediate that \( f_0, f_1 \) hide \( [s] \):

**Lemma 5.1.** The function \( f_0 \) is injective and \( f_1([b]) = f_0([b][s]) \).

**Proof.** Since \( * \) is a group action,

\[
f_1([b]) = [b] * j(E_1) = [b] * ([s] * j(E_0)) = ([b][s]) * j(E_0) = f_0([b][s]).
\]

If there are distinct ideal classes \([b], [b']\) such that \( f_0([b]) = f_0([b']) \), then \([b] * j(E_0) = [b'] * j(E_0)\), which contradicts the fact that the action is free and transitive [39, Thm. 4.5]. Thus \( f_0 \) is injective.

Note that a similar connection between isogenies and hidden shift problems was described in [36, Section 7.2]. However, that paper did not recognize the significance of the reduction, and in particular did not appreciate the role played by injectivity. Without the assumption that \( f_0 \) is injective, the hidden shift problem can be as hard as the search problem, and hence requires exponentially many queries [2] (although for non-injective functions \( f_0 \) with appropriate structure, such as the Legendre symbol, the non-injective hidden shift problem can be solved by a quantum computer in polynomial time [12]). On the other hand, injectivity implies that the problem has polynomial quantum query complexity [13], allowing for the possibility of faster quantum algorithms.
This reduction allows us to apply quantum algorithms for the hidden shift problem to construct isogenies. The (injective) hidden shift problem can be solved in quantum subexponential time assuming we can evaluate the group action in subexponential time. The latter is possible due to Algorithm 2.

We consider two different approaches to solving the hidden shift problem in subexponential time on a quantum computer. The first, due to Kuperberg [25], has a faster running time but requires superpolynomial space. The second approach generalizes an algorithm of Regev [28]. It uses only polynomial space, but is slower than Kuperberg’s original algorithm.

Method 1: Kuperberg’s algorithm. Kuperberg’s approach to the abelian hidden shift problem is based on the idea of performing a Clebsch-Gordan sieve on coset states. The following appears as Theorem 7.1 of [25].

**Theorem 5.1.** The abelian hidden shift problem has a [quantum] algorithm with time and query complexity $2^{O(\sqrt{n})}$, where $n$ is the length of the output, uniformly for all finitely generated abelian groups.

In our context, $2^{O(\sqrt{n})} = 2^{O(\sqrt{\ln |\Delta|})}$ since $|\text{Cl}(\mathcal{O}_\Delta)| = O(\sqrt{\Delta \ln \Delta})$ [8, Theorem 9.3.11]. Furthermore, $2^{O(\sqrt{\ln |\Delta|})} = L(o(1)) = L(0)$ regardless of the value of the implied constant in the exponent, since the exponent on the left has no $\sqrt{\ln \ln |\Delta|}$ term, whereas $L(0)$ does. As mentioned above, Kuperberg’s algorithm also requires superpolynomial space (specifically, it uses $2^{O(\sqrt{\ln |\Delta|})}$ qubits).

Method 2: Regev’s algorithm. Regev [28] showed that a variant of Kuperberg’s sieve leads to a slightly slower algorithm using only polynomial space. In particular, he proved Theorem 5.2 below in the case where $A$ is a cyclic group whose order is a power of 2 (without giving an explicit value for the constant in the exponent). Theorem 5.2 generalizes Regev’s algorithm to arbitrary finite abelian groups. A detailed proof of Theorem 5.2 appears in the Appendix (see Theorem A.1).

**Theorem 5.2.** Let $A$ be a finite abelian group and let functions $f_0, f_1$ hide some unknown $s \in A$. Then there is a quantum algorithm that finds $s$ with time and query complexity $L(|A|)^{\frac{1}{2}, \sqrt{2}}$ using space $\text{poly}(\log |A|)$.

We now return to the original problem of constructing isogenies. Note that to use the hidden shift approach, the group structure of $\text{Cl}(\mathcal{O}_\Delta)$ must be known. Given $\Delta$, it is straightforward to compute $\text{Cl}(\mathcal{O}_\Delta)$ using existing quantum algorithms (see the proof of Theorem 5.4). Thus, we assume for simplicity that the discriminant $\Delta$ is given as part of the input. This requirement poses no difficulty, since all existing proposals for isogeny-based public-key cryptosystems [10, 29, 36] stipulate that $\mathcal{O}_\Delta$ is a maximal order, in which case its discriminant can be computed easily: simply calculate the trace $t(E)$ of the curve using Schoof’s algorithm [31], and factor $t(E)^2 - 4q$ to obtain the fundamental discriminant $\Delta$ (note of course that factoring is easy on a quantum computer [34]).

**Remark 5.3.** One can conceivably imagine a situation where one is asked to construct an isogeny between two given isogenous curves of unknown but identical endomorphism ring. Although we are not aware of any cryptographic applications of this scenario, it presents no essential difficulty. Bisson [4] has shown using Corollary 2.2 that the discriminant $\Delta$ of an elliptic curve can be computed in $L_q^{\frac{1}{2}, \sqrt{2}}$ time under only GRH (assuming that factoring is easy).
Algorithm 3 Isogeny construction

Input: A finite field \( \mathbb{F}_q \), a discriminant \( \Delta < 0 \), and Weierstrass equations of horizontally isogenous elliptic curves \( E_0, E_1 \)

Output: \([s] \in \text{Cl}(\mathcal{O}_\Delta)\) such that \([s] \ast j(E_0) = j(E_1)\)

1: Decompose \( \text{Cl}(\mathcal{O}_\Delta) = \langle [b_1] \rangle \oplus \cdots \oplus \langle [b_k] \rangle \) where \(|\langle [b_j] \rangle| = n_j\)
2: Solve the hidden shift problem defined by functions \( f_0, f_1: \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \rightarrow \text{Ell}_{q,n} (\mathcal{O}_\Delta) \) satisfying \( f_c(x_1, \ldots, x_k) = ([b_1]^{x_1} \cdots [b_k]^{x_k}) \ast j(E_c) \), giving some \((s_1, \ldots, s_k) \in \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}\)
3: Output \([s] = [b_1]^{s_1} \cdots [b_k]^{s_k}\)

Assuming \( \Delta \) is known, we decompose \( \text{Cl}(\mathcal{O}_\Delta) \) as a direct sum of cyclic groups, with a known generator for each, and then solve the hidden shift problem. The overall procedure is described in Algorithm 3.

Theorem 5.4. Assuming GRH, Algorithm 3 runs in time \( L_q(\frac{1}{2}, \frac{1}{\sqrt{2}}) \) (respectively, \( L_q(\frac{1}{2}, \frac{1}{\sqrt{2}} + \sqrt{2}) \)) using Theorem 5.1 (respectively, Theorem 5.2) to solve the hidden shift problem.

Proof. We perform Step 1 using [9, Algorithm 10], which determines the structure of an abelian group given a generating set and a unique representation for the group elements. We represent the elements uniquely using reduced quadratic forms, and we use the fact that, under ERH (and hence GRH), the set of ideal classes of norm at most \( 6 \ln^2 |\Delta| \) forms a generating set [1, p. 376]. By Theorem 5.1 (resp. Theorem 5.2), Step 2 uses \( L(o(1)) = L(0) \) (resp. \( L(\sqrt{2}) \)) evaluations of the functions \( f_i \). By Corollary 4.7, these functions can be evaluated in time \( L_q(\frac{1}{2}, \frac{1}{\sqrt{2}}) \) using Algorithm 2, assuming GRH. Overall, Step 2 takes time \( L_q(\frac{1}{2}, \frac{1}{\sqrt{2}} + o(1)) = L_q(\frac{1}{2}, \frac{1}{\sqrt{2}}) \) if Theorem 5.1 is used, or \( L_q(\frac{1}{2}, \frac{1}{\sqrt{2}} + \sqrt{2}) \) if Theorem 5.2 is used. The cost of Step 3 is negligible.

Remark 5.5. Using the improved algorithm for evaluating the isogeny star operator described in Remark 4.8, the running time of Algorithm 3 is improved to \( L_q(\frac{1}{2}, \frac{1}{\sqrt{2}} + \sqrt{2}) \) using Theorem 5.1 to solve the hidden shift problem (requiring superpolynomial space), and to \( L_q(\frac{1}{2}, \frac{1}{\sqrt{2}} + \sqrt{2}) = L_q(\frac{1}{2}, \frac{1}{\sqrt{2}}) \) using Theorem 5.2 (requiring only polynomial space).

Remark 5.6. The running time of the algorithm is ultimately limited by two factors: the best known quantum algorithm for the hidden shift problem runs in superpolynomial time, and the same holds for the best known (classical or quantum) algorithm for computing the isogeny star operator. Improving only one of these results to take polynomial time would still result in a superpolynomial-time algorithm.

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A Subexponential-time and polynomial-space quantum algorithm for the general abelian hidden shift problem

Following Kuperberg’s discovery of a subexponential-time quantum algorithm for the hidden shift problem in any finite abelian group \( A \) [25], Regev presented a modification of Kuperberg’s algorithm that requires only polynomial space, with a slight increase in the running time [28]. However, Regev only explicitly considered the case \( A = \mathbb{Z}_2^n \), and while he showed that the running time is \( L_{|A|}(\frac{1}{2}, c) \), he did not determine the value of the constant \( c \).

In this appendix we describe a polynomial-space quantum algorithm for the general abelian hidden shift problem using time \( L_{|A|}(\frac{1}{2}, \sqrt{2}) \). We use several of the same techniques employed by Kuperberg [25, Algorithm 5.1 and Theorem 7.1] to go beyond the case \( A = \mathbb{Z}_2^n \), adapted to work with a Regev-style sieve that only uses polynomial space.

Let \( A = \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_t} \) be a finite abelian group. Consider the hidden shift problem with hidden shift \( s = (s_1, \ldots, s_t) \in A \). By Fourier sampling, one (coherent) evaluation of the hiding functions \( f_0, f_1 \) can produce the state

\[
|\psi_x\rangle := \frac{1}{\sqrt{2}} \left( |0\rangle + \exp \left[ 2\pi i \left( \frac{s_1 x_1}{N_1} + \cdots + \frac{s_t x_t}{N_t} \right) \right] |1\rangle \right) \tag{\psi}
\]
with a known value \( x = (x_1, \ldots, x_i) \in \mathbb{R} A \) (see for example the proof of Theorem 7.1 in [25]), where \( x \in \mathbb{R} A \) denotes that \( x \) occurs uniformly at random from \( A \). For simplicity, we begin by considering the case where \( A = \mathbb{Z}_N \) is cyclic. Then Fourier sampling produces states

\[
|\psi_x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \omega^{sx}|1\rangle)
\]

where \( x \in \mathbb{R} \mathbb{Z}_N \) is known and \( \omega := e^{2\pi i/N} \).

If we could make states \(|\psi_x\rangle\) with chosen values of \( x \), then we could determine \( s \). In particular, the following observation is attributed to Peter Høyer in [25]:

**Lemma A.1.** Given one copy each of the states \(|\psi_1\rangle, |\psi_2\rangle, |\psi_4\rangle, \ldots, |\psi_{2^k-1}\rangle\), where \( 2^k = \Omega(N) \), one can reconstruct \( s \) in polynomial time with probability \( \Omega(1) \).

**Proof.** We have

\[
\bigotimes_{j=0}^{k-1} |\psi_{2^j}\rangle = \frac{1}{\sqrt{2^k}} \sum_{y=0}^{2^k-1} \omega^{sy}|y\rangle.
\]

Apply the inverse quantum Fourier transform over \( \mathbb{Z}_N \) (which runs in \( \text{poly}(\log N) \) time [24]) and measure in the computational basis. The Fourier transform of \(|s\rangle\), namely \( \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \omega^{sy}|s\rangle \), has overlap squared with this state of \( 2^k/N \), which implies the claim.

We aim to produce states of the form \(|\psi_{2^j}\rangle\) using a sieve that combines states to prepare new ones with more desirable labels. A basic building block is Algorithm 4, which can be used to produce states with smaller labels.

**Lemma A.2.** Algorithm 4 runs in time \( 2^k \text{poly}(\log N) \) and succeeds with probability \( \Omega(1) \) provided \( 4k \leq B/B' \leq 2^k/k \).

**Proof.** The running time is dominated by the brute force calculation in Step 6 and the projection in Step 10, both of which can be performed in time \( 2^k \text{poly}(\log N) \).

The probability of aborting in Step 2 for any one \( x_i \) is \( 1 - \frac{2B'}{B'} \left| \frac{B}{2B'} \right| \leq \frac{2B'}{B'} \), so by the union bound, the overall probability of aborting in this step is at most \( k \frac{2B'}{B'} \leq 1/2 \). Conditioned on not aborting in Step 2, \( x_i \in \mathbb{R} \{0, 1, \ldots, 2B'/B' - 1\} \).

Let \( x \cdot y^j = q(2B') + r^j \) where \( 0 \leq r^j < 2B' \) (\( q \) is the measurement outcome, which is independent of \( j \)). By the uniformity of the \( x_i \)'s, each \( r^j = x \cdot y^j \mod 2B' \) is uniformly distributed over \( \{0, 1, \ldots, 2B' - 1\} \). Thus the output label is \( x' = x \cdot (y^* - y^) = |r^* - r^*| \) where \( r^*, r^* \in \mathbb{R} \{0, 1, \ldots, 2B' - 1\} \). A simple calculation shows that the distribution of \(|r^* - r^*|\) is

\[
\Pr(|r^* - r^*| = \Delta) = \begin{cases} 
\frac{1}{2B'} & \text{for } \Delta = 0 \\
\frac{\Delta}{2B'^2} & \text{for } \Delta \in \{1, \ldots, 2B' - 1\}.
\end{cases}
\]

Thus the probability that we abort in Steps 12–16 is 1/2, and conditioned on not aborting in these steps, \( x' \in \mathbb{R} \{0, 1, \ldots, B' - 1\} \). Thus the algorithm is correct if it reaches Step 17.

It remains to show that the algorithm succeeds with constant probability. We have already bounded the probability that we abort in Step 2 and Steps 12–16. Since \( y = 0 \) occurs with probability
Algorithm 4 Combining states to give smaller labels

**Input:** Parameters $B, B'$ and states $|\psi_{x_1}\rangle, \ldots, |\psi_{x_k}\rangle$ with known $x_1, \ldots, x_k \in \mathbb{R} \{0,1,\ldots,B-1\}$

**Output:** State $|\psi'_{x'}\rangle$ with known $x' \in \mathbb{R} \{0,1,\ldots,B'-1\}$

1: if $\exists i: x_i \geq 2B'[B/2B']$ then
2: Abort
3: end if
4: Introduce an ancilla register and compute
   \[ \frac{1}{\sqrt{2^k}} \sum_{y \in \{0,1\}^k} \omega^{x \cdot y} |y\rangle |\langle x \cdot y|/2B'| \]
   where $x \cdot y := \sum_{i=0}^{k} x_i y_i$
5: Measure the ancilla register, giving an outcome $q$ and a state
   \[ \frac{1}{\sqrt{B'}} \sum_{j=1}^{\nu} \omega^{x \cdot y} |y\rangle \]
   where $y^1, \ldots, y^\nu \neq 0^k$ are the $k$-bit strings such that $|(x \cdot y^j)/2B'| = q$
6: Compute $y^1, \ldots, y^\nu$ by brute force
7: if $\nu = 1$ then
8: Abort
9: end if
10: Project onto $\text{span}\{|y^1\rangle, |y^2\rangle\}$ or $\text{span}\{|y^3\rangle, |y^\nu\rangle\}$ or \ldots or $\text{span}\{|y^{2^\nu\nu} \rangle, |y^{2^\nu\nu} \rangle\}$, giving an outcome
11: Let $x' = x \cdot (y^\nu - y^\nu)$ where $x \cdot y^\nu \geq x \cdot y^\nu$ WLOG
12: if $x' \in \{1, \ldots, B'-1\}$ then
13: Abort with probability $B'/(2B'-x')$
14: else if $x' \in \{B', \ldots, 2B'-1\}$ then
15: Abort
16: end if
17: Relabel $|y^\nu\rangle \mapsto |0\rangle$ and $|y^\nu\rangle \mapsto |1\rangle$, giving a state $|\psi'_{x'}\rangle$

$2^{-k}$ and at most one state $|y^\nu\rangle$ can be unpaired (and this only happens when $\nu$ is odd), the projection in Step 10 fails with probability at most $\nu^{-1} + 2^{-k} \leq 1/3 + o(1)$. We claim that the probability of aborting in Step 8 (i.e., the probability that $\nu = 1$) is also bounded away from 1. Call a value of $q$ bad if $\nu = 1$. Since $0 \leq x \cdot y \leq k(B-1)$, there are at most $kB/2B'$ possible values of $q$, and in particular, there can be at most $kB/2B'$ bad values of $q$. Since the probability of any particular bad $q$ is $1/2^k$, the probability that $q$ is bad is at most $kB/2B'2^{k+1} \leq 1/2$. This completes the proof.

We apply this combination procedure using the generalized sieve of Algorithm 5, which is equivalent to Regev’s “pipeline of routines” [28].

**Lemma A.3.** Suppose $m \cdot 2^{-k} = o(1)$. Then Algorithm 5 is correct, succeeds with probability $1 - o(1)$ using $k^{(1+o(1))m}$ state preparations and combination operations, and uses space $O(mk)$.

**Proof.** If Algorithm 5 outputs a state from $S_m$ then it is correct. Since the algorithm never stores more than $O(mk)$ states at a time, it uses space $O(mk)$. It remains to show that the algorithm is likely to succeed using only $k^{(1+o(1))m}$ state preparations and combination operations.
Algorithm 5: Sieving quantum states

**Input:** Procedures to prepare states from a set $S_0$ and to combine $k$ states from $S_i$ to make a state from $S_{i+1}$ with probability at least $p$ for each $i \in \{1, \ldots, m\}$

**Output:** State from $S_m$

1: repeat
2: while for all $i$ we have fewer than $k$ states from $S_i$ do
3: Make a state from $S_0$
4: end while
5: Combine $k$ states from some $S_i$ to make a state from $S_{i+1}$ with probability at least $p$
6: until there is a state from $S_m$

If we could perform combinations deterministically, we would need

- 1 state from $S_m$,
- $k$ states from $S_{m-1}$,
- $k^2$ states from $S_{m-2}$,
- $\vdots$
- $k^m$ states from $S_0$.

Since the combinations only succeed with probability $p$, we lower bound the probability of eventually producing $(2k/p)^{m-i}$ states from $S_i$ for each $i \in \{1, \ldots, m\}$ (so in particular, we produce one state from $S_m$). Given $(2k/p)^{m-i+1}$ states from $S_{i-1}$, the expected number of successful combinations is $p(2k/p)^{m-i+1}/k = 2(2k/p)^{m-i}$, whereas only $(2k/p)^{m-i}$ successful combinations are needed. By the Chernoff bound, the probability of having fewer than $(2k/p)^{m-i}$ states from $S_i$ after each successful combination is at most $e^{-p(2k/p)^{m-i}}$. Thus, by the union bound, the probability that the algorithm fails is at most

$$\sum_{i=1}^{m-1} e^{-p(2k/p)^{m-i}} \leq me^{-2k},$$

so the probability of success is $1 - o(1)$.

Finally, the number of states from $S_0$ is $(2k/p)^m = k^{(1+o(1))m}$ and the total number of combinations is $\sum_{i=0}^{m-1} (2k/p)^{m-i}/k = k^{(1+o(1))m}$.

When using the sieve, we have the freedom to choose the relationship between $k$ and $m$ to optimize the running time. Suppose that $mk = (1 + o(1)) \log_2 N$ (intuitively, to cancel $\log_2 N$ bits of the label), and also suppose that the combination operation takes time $2^k \text{poly}(\log N)$ (as in Lemma A.2). Then if we take $k = c \sqrt{\log_2 N \log_2 \log_2 N}$, we find that the overall running time of Algorithm 5 is $2^k 2^{(1+o(1))m \log_2 k} \text{poly}(\log N) = L_N(\frac{c}{2}, \frac{c+1}{2})$. Choosing $c = \frac{1}{\sqrt{2}}$ gives the best running time, $L_N(\frac{1}{2}, \sqrt{2})$.

We now consider how to apply the sieve. To use Lemma A.1, our goal is to prepare states of the form $|\psi_j\rangle$ for each $j \in \{0, 1, \ldots, \lfloor \log_2 N \rfloor \}$. First we show how to prepare the state $|\psi_1\rangle$ in time $L_N(\frac{1}{2}, \sqrt{2})$ using Algorithm 4 as the combination procedure in Algorithm 5. For $i \in \{0, 1, \ldots, m\}$, the $i$th stage of the sieve produces states with labels from $S_i = \{0, 1, \ldots, B_{i-1} - 1\}$. Lemma A.4 below shows that there is a choice of the $B_i$ with $B_0 = N$, $B_m = 2$, and successive ratios of the
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B_i satisfying the conditions of Lemma A.2, such that $2^{k_i}k_i^{(1+o(1))m} = L_N(\frac{1}{2}, \sqrt{2})$. It then follows that Algorithm 5 produces a uniformly random label from $S_m = \{0, 1\}$ with constant probability in time $L_N(\frac{1}{2}, \sqrt{2})$, and in particular, can be used to produce a copy of $|\psi_1\rangle$ in time $L_N(\frac{1}{2}, \sqrt{2})$.

**Lemma A.4.** There is a constant $N_0$ such that for all $N > N_0$, letting $B_i = \lfloor N/\rho^i \rfloor$ where $\rho = (N/2)^1/m$ and

$$k = \left[ \sqrt{\frac{1}{2} \log_2 N \log_2 \log_2 N} \right], \quad m = \left\lfloor \frac{\log_2 N/2}{k - \log_2 2k} \right\rfloor = \Theta \left( \sqrt{\frac{\log_2 N}{\log_2 \log_2 N}} \right),$$

we have $B_0 = N$, $B_m = 2$, and $4k \leq B_{i-1}/B_i \leq 2^k/k$ for all $i \in \{1, \ldots, m\}$.

**Proof.** Clearly $B_0 = N$, and the value of $\rho$ is chosen so that $B_m = 2$.

For $i \in \{1, \ldots, m\}$, we have

$$\frac{B_{i-1}}{B_i} = \frac{\lfloor N/\rho^{i-1} \rfloor}{\lfloor N/\rho^i \rfloor} \leq \frac{N/\rho^{i-1}}{N/\rho^i - 1} = \frac{\rho}{1 - \rho^i/N}. $$

Since $\rho^i/N \leq \rho^m/N = 1/2$, we have $B_{i-1}/B_i \leq 2\rho$. Then using

$$\rho \leq (N/2)^{\frac{\log_2 2k}{\log_2 N/2}} = \frac{2^k}{2^k}$$

gives $B_{i-1}/B_i \leq 2^k/k$ as claimed.

Similarly, we have

$$\frac{B_{i-1}}{B_i} = \frac{\lfloor N/\rho^{i-1} \rfloor}{\lfloor N/\rho^i \rfloor} \geq \frac{N/\rho^{i-1} - 1}{N/\rho^i} = \rho - \rho^i/N \geq \rho - \frac{1}{2}. $$

Since

$$\rho = (N/2)^{\Theta(\sqrt{\log_2 \log_2 N/\log_2 N})} = 2^{\Theta(\sqrt{\log_2 N \log_2 \log_2 N})} = 2^{\Theta(k)},$$

we have $\rho - \frac{1}{2} \geq 4k$ for sufficiently large $N$. This completes the proof.

If $N$ is odd, then division by 2 is an automorphism of $\mathbb{Z}_N$. Thus we can prepare $|\psi_2\rangle$ by performing the above sieve under the automorphism $x \mapsto 2^{-j}x$. It follows that the abelian hidden shift problem in a cyclic group of odd order $N$ can be solved in time $L_N(\frac{1}{2}, \sqrt{2})$.

Now suppose that $N = 2^n$ is a power of 2. In this case, we first use a combination procedure that zeros out low-order bits, as described in Algorithm 6. We use the notation $xS := \{xz : z \in S\}$ for any $x \in \mathbb{Z}$ and $S \subset \mathbb{Z}$.

**Lemma A.5.** Algorithm 6 runs in time $2^k \text{poly}(\log N)$ and succeeds with probability $\Omega(1)$ provided $k \geq l^\ell - \ell + 1$.

**Proof.** The proof is similar to that of Lemma A.2. Again the running time is dominated by the brute force calculation in Step 3 and the projection in Step 7, both of which can be performed in time $2^k \text{poly}(\log N)$.
Algorithm 6 Combining states to cancel low-order bits

**Input:** Parameters $\ell, \ell'$ and states $|\psi_x\rangle, \ldots, |\psi_y\rangle$ with known $x_1, \ldots, x_k \in_R 2^\ell \{0, 1, \ldots, N/2^{\ell'} - 1\}$

**Output:** State $|\psi_{x'}\rangle$ with known $x' \in_R 2^{\ell'} \{0, 1, \ldots, N/2^{\ell'} - 1\}$

1. Introduce an ancilla register and compute
   \[
   \frac{1}{\sqrt{2^k}} \sum_{y \in \{0, 1\}^k} \omega^{x(y)} |y\rangle |x \cdot y \mod 2^{\ell'}\rangle
   \]

2. Measure the ancilla register, giving an outcome $r$ and a state
   \[
   \frac{1}{\sqrt{v}} \sum_{j=1}^{\ell} \omega^{x(j)} |y^j\rangle
   \]

   where $y^1, \ldots, y^{\nu} \neq 0^k$ are the $k$-bit strings such that $x \cdot y^j \mod 2^{\ell'} = r$

3. Compute $y^1, \ldots, y^{\nu}$ by brute force
4. if $\nu = 1$ then
5. Abort
6. end if
7. Project onto span$\{|y^1\rangle, |y^2\rangle\}$ or span$\{|y^3\rangle, |y^4\rangle\}$ or ... or span$\{|y^{2(\nu/2)-1}\rangle, |y^{2(\nu/2)}\rangle\}$, giving an outcome span$\{|y^*\rangle, |y'^*\rangle\}$
8. Relabel $|y^*\rangle \mapsto |0\rangle$ and $|y'^*\rangle \mapsto |1\rangle$, giving a state $|\psi_{x'}\rangle$ with $x' = x \cdot (y^* - y'^*) \mod N$

We claim that the algorithm is correct if it reaches Step 8. Observe that $x \cdot y^j \mod N = q^j 2^{\ell'} + r$ where $r$ is independent of $j$. Since $y^j \neq 0^k$, $x \cdot y^j \mod N \in_R 2^\ell \{0, 1, \ldots, N/2^{\ell'} - 1\}$, so $q^j \in_R \{0, 1, \ldots, N/2^{\ell'} - 1\}$, and hence $x' = (q^* - q'^*) 2^{\ell'} \mod N \in_R 2^{\ell'} \{0, 1, \ldots, N/2^{\ell'} - 1\}$ as required.

The projection in Step 7 fails with probability at most $1/3 + o(1)$. It remains to show that the algorithm reduces Step 7 with probability $\Omega(1)$, i.e., to upper bound the probability that $\nu = 1$. Call a value of $r$ bad if $\nu = 1$. There are $2^{\ell'-\ell}$ possible values of $r$, so in particular there are at most $2^{\ell'-\ell}$ bad values of $r$. Since the probability of any particular bad $r$ is $1/2^k$, the probability that $r$ is bad is at most $2^{\ell'-\ell-k} \leq 1/2$. This completes the proof.

Algorithm 6 is similar to the combination procedure used in [28], but differs in that the latter requires $\nu = O(1)$, which is established in the analysis using a second moment argument. The modification of pairing as many values of $y$ as possible allows us to use a simpler analysis (with essentially the same performance).

To produce a state of the form $|\psi_{y^i}\rangle$, we first use Algorithm 6 to cancel low-order bits and then use Algorithm 4 to cancel high-order bits. Note that if all states $|\psi_x\rangle$ have labels $x$ with a common factor—say, $2^i |x\rangle$—then we can view the labels as elements of $\mathbb{Z}_{2n-1}$ and apply Algorithm 4 to affect the $n - j$ most significant bits. Specifically, to make the state $|\psi_{y^i}\rangle$, we apply Algorithm 5 using Algorithm 6 as the combination procedure that produces states from $S_i$ using states from $S_{i-1}$ for $i \in \{1, \ldots, m_1 + 1\}$, and Algorithm 4 (on the $n - j$ most significant bits) as the combination procedure for $i \in \{m_1 + 2, \ldots, m_1 + m_2 + 1\}$, taking

\[
S_i = \begin{cases} 
2^{(k-1)i} \{0, 1, \ldots, 2^{n-(k-1)i} - 1\} & \text{for } i \in \{0, 1, \ldots, m_1\} \\
2^i \{0, 1, \ldots, B_i - 1\} & \text{for } i \in \{m_1 + 1, \ldots, m_1 + m_2 + 1\}
\end{cases}
\]
where now

\[ B_i = \left\lfloor \frac{2^{n-j}}{\rho^{i-m_1-1}} \right\rfloor \quad m_1 = \left\lfloor \frac{j}{(k-1)} \right\rfloor \]

\[ \rho = 2^{\frac{(n-j-1)}{m_2}} \quad m_2 = \left\lfloor \frac{n-j}{k - \log_2 2k} \right\rfloor \]

and again \( k = \left\lfloor \frac{\sqrt{2} \log_2 N \log_2 \log_2 N}{\sqrt{2}} \right\rfloor \). When making states in \( S_1 \) from states in \( S_{i-1} \) for \( i \in \{1, \ldots, m_1\} \), we cancel \( k - 1 \) bits with \( k \) states, so the condition of Lemma A.5 is satisfied. For \( i = m_1 + 1 \), we cancel \( j - (k-1)m_1 = j - (k-1)\left\lfloor j/(k-1) \right\rfloor \leq j - (k-1)\left\lfloor j/(k-1) - 1 \right\rfloor = k-1 \) bits, so again the condition of Lemma A.5 is satisfied. For \( i \in \{m_1 + 2, \ldots, m_1 + m_2 + 1\} \), Lemma A.4 implies that the conditions of Lemma A.2 are satisfied provided \( 2^{n-j} \geq N_0 \). (If \( 2^{n-j} < N_0 \) then we only need to perform the first \( m_1 + 1 \) stages of the sieve, producing a state uniformly at random from \( S_{m_1+1} \), in this case \( |S_{m_1+1}| = O(1) \), so \( O(1) \) repetitions suffice to produce a copy of \( |\psi_2\rangle \).) Finally, since \((m_1 + m_2 + 1)k = (1 + o(1))n\), the discussion following Lemma A.3 shows that Algorithm 5 takes time \( L_N(\frac{x}{k}, \sqrt{2}) \).

So far we have covered the case where the group is \( A = \mathbb{Z}_N \) with \( N \) either odd or a power of 2. Now consider the case of a general finite abelian group \( A = \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_t} \). By the Chinese remainder theorem, we can assume without loss of generality that each \( N_i \) is either odd or a power of 2. Consider what happens if we apply Algorithm 4 or Algorithm 6 to one component of a product of cyclic groups. Suppose we combine \( k \) states of the form of Eqn. (ψ). For each \( i \in \{1, \ldots, k\} \), let \( x_i \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_t} \) denote the label of the \( i \)th state, with \( x_{i,j} \in \mathbb{Z}_{N_j} \) for \( j \in \{1, \ldots, t\} \). To address the \( \ell \)th component of \( A \), the combination procedure prepares a state

\[
\frac{1}{\sqrt{2^k}} \sum_{y \in \{0,1\}^k} \exp \left( 2\pi i \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{y_i x_{i,j} S_j}{N_j} \right) |y\rangle |h(\sum_{i=1}^{k} x_{i,\ell} y_i)\rangle
\]

for some function \( h \) (a quotient in Algorithm 4 or a remainder in Algorithm 6). For \( j \neq \ell \), if \( x_{i,j} = 0 \) for all \( i \in \{1, \ldots, k\} \) then \( x'_j = \sum_{i=1}^{k} x_{i,j} (y^*_i - y^*_j) = 0 \), so components that are initially zero remain zero. Thus, if we can prepare states \( |\psi_x\rangle \) with \( x_\ell \in \mathbb{Z}_{N_{\ell}} \) (for any desired \( \ell \in \{1, \ldots, t\} \)) and all other components zero, we effectively reduce the problem to the cyclic case.

To prepare such states, we use a new combination procedure, Algorithm 7. Without loss of generality, our goal is to zero out the first \( t - 1 \) components, leaving the last one uniformly random from \( \mathbb{Z}_{N_t} \). Algorithm 7 is similar to Algorithm 4, viewing the first \( t - 1 \) components of the label \( x_i \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_t} \) as the mixed-radix integer

\[
\mu(x_i) := \sum_{j=1}^{t-1} x_{i,j} \prod_{j' = 1}^{j-1} N_{j'}.
\]

Because we are merely trying to zero out certain components, we no longer require uniformity of the states output by the sieve, which simplifies the procedure and its analysis.

**Lemma A.6.** Algorithm 7 runs in time \( 2^k \text{poly}(\log N) \) and succeeds with probability \( \Omega(1) \) provided \( B/B' \leq 2^k/2k \).

**Proof.** As in Lemma A.2 and Lemma A.5, the running time is dominated by the brute force calculation in Step 3 and the projection in Step 7, both of which can be performed in time \( 2^k \text{poly}(\log N) \).
Algorithm 7 Combining non-cyclic states to reduce undesired components

Input: Parameters $B, B'$ and states $|\psi_{x_1}\rangle, \ldots, |\psi_{x_k}\rangle$ with known $x_1, \ldots, x_k \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_t}$ satisfying $\mu(x_i) \in \{0, \ldots, B - 1\}$ for each $i \in \{1, \ldots, k\}$, with $x_i \in \mathbb{Z}_{N_t}$

Output: State $|\psi'\rangle$ with known $x' \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_t}$ satisfying $\mu(x') \in \{0, 1, \ldots, B' - 1\}$, with $x'_i \in \mathbb{Z}_{N_t}$

1: Introduce an ancilla register and compute

$$\frac{1}{\sqrt{2^k}} \sum_{y \in \{0, 1\}^k} \exp \left( 2\pi i \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{y_i x_i y_j}{N_j} \right) |y\rangle \langle \sum_{i=1}^{k} \mu(x_i) y_i / B' \rangle$$

2: Measure the ancilla register, giving an outcome $q$ and a state

$$\frac{1}{\sqrt{2^\nu}} \sum_{y = 0}^{\nu - 1} \exp \left( 2\pi i \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{y_i x_i y_j}{N_j} \right) |y^\nu\rangle$$

where $y^1, \ldots, y^\nu \neq 0^k$ are the $k$-bit strings such that $\sum_{i=1}^{k} \mu(x_i) y^\nu_i / B' = q$

3: Compute $y^1, \ldots, y^\nu$ by brute force

4: if $\nu = 1$ then

5: Abort

6: end if

7: Project onto span$\{|y^1\rangle, |y^2\rangle\}$ or span$\{|y^3\rangle, |y^4\rangle\}$ or ... or span$\{|y^{2^\nu - 1}\rangle, |y^{2^\nu}\rangle\}$, giving an outcome span$\{|y^*\rangle, |y^*\rangle\}$

8: Relabel $|y^*\rangle \rightarrow |0\rangle$ and $|y^*\rangle \rightarrow |1\rangle$ where $\sum_{i=1}^{k} \mu(x_i) y^*_i \geq \sum_{i=1}^{k} \mu(x_i) y^*_i$ WLOG, giving a state $|\psi_{x'}\rangle$

with $x'_j = \sum_{i=1}^{k} x_i, j (y^*_i - y^*_i)$ for each $j \in \{1, \ldots, t\}$

We claim that the algorithm is correct if it reaches Step 8. Since $\sum_{i=1}^{k} \mu(x_i) y^*_{i} = q B' + r^j$ where $q$ is independent of $j$ and $0 \leq r^j < B'$, we have

$$\mu(x') = \sum_{j=1}^{t-1} x'_j \prod_{j'=1}^{j-1} N_{j'} = \sum_{j=1}^{t-1} \sum_{i=1}^{k} x'_i, j (y^*_i - y^*_i) \prod_{j'=1}^{j-1} N_{j'} = \sum_{i=1}^{k} \mu(x_i) (y^*_i - y^*_i) = r^* - r^* < B'$$

as required. Since $y^* \neq y^*$ and the $x'_i$ are uniformly random, $x'_i = \sum_{i=1}^{k} x_i, i (y^*_i - y^*_i)$ is uniformly random as required.

The projection in Step 7 fails with probability at most $1/3 + o(1)$. We claim the algorithm reaches Step 7 with probability $\Omega(1)$. To show this, we need to upper bound the probability that $\nu = 1$. Call a value of $q$ bad if $\nu = 1$. Since $0 \leq \sum_{i=1}^{k} \mu(x_i) y_i \leq k(B - 1)$, there are at most $kB / B'$ possible values of $q$, and in particular, there can be at most $kB / B'$ bad values of $q$. Since the probability of any particular bad $q$ is $1/2^k$, the probability that $q$ is bad is at most $kB / B'2^k \leq 1/2$. This completes the proof.

To apply Algorithm 7 as the combination procedure for Algorithm 5, we require a straightforward variant of Lemma A.4, as follows.

Lemma A.7. There is a constant $N'_0$ such that for all $N > N_0$, letting $B_i = \lfloor N/\rho^i \rfloor$ where $\rho = N^{1/m}$ and

$$k = \left\lfloor \sqrt{\frac{1}{2} \log_2 N \log_2 \log_2 N} \right\rfloor \quad m = \left\lfloor \frac{\log_2 N}{k \log_2 4k} \right\rfloor = \Theta \left( \sqrt{\frac{\log_2 N}{\log_2 \log_2 N}} \right),$$
Algorithm 8 Abelian hidden shift problem

Input: Black box for the hidden shift problem in an abelian group \( A \)

Output: Hidden shift \( s \)

1: Write \( A = \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_t} \) where each \( N_i \) is either odd or a power of 2
2: for all \( i \in \{1, \ldots, t\} \) do
3: \hspace{1em} if \( N_i \) is odd then
4: \hspace{2em} for all \( j \in \{0, \ldots, \lfloor \log_2 N_i \rfloor \} \) do
5: \hspace{3em} Apply Algorithm 5, first using Algorithm 7 to zero out all components except the \( i \)th one
6: \hspace{3em} and then using Algorithm 4 under the \( \mathbb{Z}_{N_i} \)-automorphism \( x \mapsto 2^{-j}x \) to produce a copy of \( |\psi_{(0, \ldots, 0, 2^j a, \ldots, 0)} \rangle \) (see the proof of Theorem A.1 for detailed parameters)
7: \hspace{2em} end for
8: \hspace{1em} else
9: \hspace{2em} Let \( N_i = 2^n \)
10: \hspace{3em} for all \( j \in \{0, \ldots, n-1\} \) do
11: \hspace{4em} Apply Algorithm 5, first using Algorithm 7 to zero out all components except the \( i \)th one, then
12: \hspace{5em} using Algorithm 6 to make states \( |\psi_{(0, \ldots, 0, x, 0, \ldots, 0)} \rangle \) with \( 2^j |x \), and finally using Algorithm 4 to
13: \hspace{5em} produce a copy of \( |\psi_{(0, \ldots, 0, 2^j a, \ldots, 0)} \rangle \) (see the proof of Theorem A.1 for detailed parameters)
14: \hspace{2em} end for
15: \hspace{1em} end if
16: Apply Lemma A.1 with \( N = N_i \) to give \( s_i \)
17: end for
18: Output \( s = (s_1, \ldots, s_t) \)

we have \( B_0 = N \), \( B_m = 1 \), and \( B_{i-1}/B_i \leq 2^k/2k \) for all \( i \in \{1, \ldots, m\} \).

Proof. Clearly \( B_0 = N \), and the value of \( \rho \) is chosen so that \( B_m = 1 \).

We have \( B_{m-1} \leq N/\rho^{m-1} = \rho \), and since
\[
\rho \leq N^{\frac{\exp(3\frac{46}{\log_2 N})}{\log_2 N}} = \frac{2^k}{4k},
\]
the claimed inequality holds for \( i = m \).

For \( i \in \{1, \ldots, m-1\} \),
\[
\frac{B_{i-1}}{B_i} = \frac{[N/\rho^{i-1}]}{[N/\rho^i]} \leq \frac{N/\rho^{i-1}}{N/\rho^i - 1} = \frac{\rho}{1 - \rho^i/N}.
\]
Since \( \rho^i/N \leq \rho^{m-1}/N = 1/\rho \), we have \( B_{i-1}/B_i \leq \rho/(1 - 1/\rho) \). Then using
\[
\rho = N^{\Theta(\sqrt{\log_2 N/\log_2 N})} = 2^{\Theta(\sqrt{\log_2 N \log_2 \log_2 N})},
\]
we have \( \rho \geq 2 \) provided \( N > N'_0 \) for some constant \( N'_0 \), which implies \( B_{i-1}/B_i \leq 2^k/2k \). This completes the proof.

Combining these ideas, the overall procedure is presented in Algorithm 8.

Theorem A.1. Algorithm 8 runs in time \( L_{|A|}(\frac{1}{2}, \sqrt{2}) \).
Proof. In Step 1, if the structure of the group is not initially known, it can be determined in polynomial time using [9]. Given the structure of the group, for each term \( \mathbb{Z}_N \) we can easily factor \( N = 2^n M \) where \( M \) is odd; then \( \mathbb{Z}_N \cong \mathbb{Z}_{2^n} \times \mathbb{Z}_M \), and we obtain a decomposition of the desired form.

Now suppose without loss of generality that we are trying to determine \( s_i \) (i.e., \( i = t \) in Step 2). The main contribution to the running time comes from the sieves in Step 5 (for \( N_i \) odd) and Step 10 (for \( N_i \) a power of 2).

First suppose that \( N_i \) is odd. It suffices to handle the case where \( j = 0 \), so we are making the state \( |\psi_{(0,\ldots,0,1)}\rangle \). Then we apply Algorithm 5 with

\[
S_i = \begin{cases}
\{ x \in A : \mu(x) < B_i \} & \text{for } i \in \{0,1,\ldots,m_1\} \\
\{ x \in A : \mu(x) = 0 \text{ and } x_i < B_i \} & \text{for } i \in \{m_1 + 1,\ldots,m_2\}
\end{cases}
\]

where

\[
B_i = \begin{cases}
\lfloor (N/N_i)/\rho_1 \rfloor & \text{for } i \in \{0,1,\ldots,m_1\} \\
\lfloor N_i/\rho_2^{-m_1} \rfloor & \text{for } i \in \{m_1 + 1,\ldots,m_1 + m_2\}
\end{cases}
\]

with

\[
\rho_1 = (N/N_i)^{1/m_1} \quad \text{and} \quad m_1 = \frac{\log_2 N/N_i}{k - \log_2 4k} \\
\rho_2 = (N_i/2)^{1/m_2} \quad \text{and} \quad m_2 = \frac{\log_2 N_i/2}{k - \log_2 2k}
\]

and \( k = \lfloor \sqrt{\frac{1}{2}\log_2 N \log_2 \log_2 N} \rfloor \). We use Algorithm 7 as the combination procedure for the first \( m_1 \) stages of Algorithm 5. By Lemma A.7, the condition of Lemma A.6 is satisfied provided \( N/N_i > N_0 \); otherwise we can produce a state with a label from \( S_{m_1} \) in only \( O(1) \) trials. Then we proceed to apply Algorithm 4 as the combination procedure for the remaining \( m_2 \) stages of Algorithm 5. By Lemma A.4, the conditions of Lemma A.2 are satisfied provided \( N_i > N_0 \); otherwise, producing states with labels from \( S_{m_1} \) already suffices to produce the desired state with constant probability. Since \( (m_1 + m_2)k = (1+o(1)) \log_2 N \), Step 5 takes time \( L_{|A|}(\frac{4}{3}, \sqrt{2}) \) (see the discussion following the proof of Lemma A.3).

Now suppose that \( N_i = 2^n \) is a power of 2. Then we apply Algorithm 5 with

\[
S_i = \begin{cases}
\{ x \in A : \mu(x) < B_i \} & \text{for } i \in \{0,1,\ldots,m_1\} \\
\{ x \in A : \mu(x) = 0 \text{ and } x_i \in 2^{(k-1)i} \{0,1,\ldots,2^{n-(k-1)i}\} \} & \text{for } i \in \{m_1 + 1,\ldots,m_1 + m_2\} \\
\{ x \in A : \mu(x) = 0 \text{ and } x_i \in 2^i \{0,1,\ldots,B_i - 1\} \} & \text{for } i \in \{m_1 + m_2 + 1,\ldots,m_1 + m_2 + m_3 + 1\}
\end{cases}
\]

where

\[
B_i = \begin{cases}
\lfloor (N/N_i)/\rho_1 \rfloor & \text{for } i \in \{0,1,\ldots,m_1\} \\
\lfloor 2^{n-j}/\rho_2^{m_1-m_2-1} \rfloor & \text{for } i \in \{m_1 + m_2 + 1,\ldots,m_1 + m_2 + m_3 + 1\}
\end{cases}
\]
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with

\[ \rho_1 = \left( \frac{N}{N_t} \right)^{1/m_1} \]

\[ m_1 = \left\lfloor \frac{\log_2 N / N_t}{k - \log_2 4k} \right\rfloor \]

\[ m_2 = \left\lfloor \frac{j}{(k - 1)} \right\rfloor \]

\[ m_3 = \left\lceil \frac{n - j - 1}{k - \log_2 2k} \right\rceil \]

\[ \rho_3 = 2^{(n - j - 1)/m_3} \]

and again \( k = \lfloor \sqrt{\frac{1}{2} \log_2 N \log_2 \log_2 N} \rfloor \). We use Algorithm 7 as the combination procedure for the first \( m_1 \) stages, Algorithm 6 for the next \( m_2 + 1 \) stages, and Algorithm 4 (on the \( n - j \) most significant bits) for the final \( m_3 \) stages. By Lemma A.7 and Lemma A.4, the conditions of Lemma A.6 and Lemma A.2 are satisfied, respectively. Since we cancel at most \( k - 1 \) bits in each stage that uses Algorithm 6, the conditions of Lemma A.5 are satisfied for the intermediate stages. Finally, since \((m_1 + m_2 + m_3 + 1)k = (1 + o(1)) \log_2 N\), Step 10 takes time \( L_{|A|}(\frac{1}{2}, \sqrt{2}) \).

The loops in Step 2, Step 4, and Step 9 only introduce polynomial overhead. Step 13 takes polynomial time and Step 15 is negligible. Thus the overall running time is \( L_{|A|}(\frac{1}{2}, \sqrt{2}) \) as claimed.