On Information Asymmetry in Competitive Multi-Agent Reinforcement Learning: Convergence and Optimaly

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Abstract

In this work, we study the system of interacting non-cooperative two Q-learning agents, where one agent has the privilege of observing the other’s actions. We show that this information asymmetry can lead to a stable outcome of population learning, which does not occur in an environment of general independent learners. Furthermore, we discuss the resulted post-learning policies, show that they are almost optimal in the underlying game sense, and provide numerical hints of almost welfare-optimal of the resulted policies.

Index Terms

Information Asymmetry, Q-learning, Markov Game, Reinforcement Learning, Online Optimization

I. INTRODUCTION

Information asymmetry in applications: In widespread multi-agent systems, information distribution is often asymmetrical, meaning that some agents have more or better information than the other. This property has been a subject of extensive study emphasizing the characterization of undesirable consequences such as market failure, moral hazards, monopoly of information, and adverse selection [1]. Likewise, information asymmetry arises in technical applications usually as an effect of hierarchical structures and cross-layer perspectives, which grows in importance with the systems’ increasing complexity and growing interlinkage enabled by groundbreaking infrastructures, such as 5G and IoT. A specific example of an asymmetrical information relationship is that between the base stations (BSs) and the (mobile) users (USs) in a wireless communication system, where BSs often has (implicit) information about USs’ service request. Among USs themselves, information asymmetry might also occur, due to the decision-making order, such as in the setting of primary user (PU) and secondary users (SUs) [2], [3]. Another possible occurrence of information asymmetry is in the relation between defender and attacker in security systems [4]–[8].

Reinforcement learning: In recent years, machine learning (ML) techniques have gained significant importance in academia and industry. Reinforcement learning (RL) [9], [10] is a ML paradigm suited for dynamical applications. It allows a single agent to learn a reward maximizing policy in an unknown Markovian environment. One fundamental technique in RL is the so-called Q-learning. Q-learning explores and exploits the state-action space and generate the so-called optimal Q-function, giving rise to the optimal strategy for optimizing the accumulated discounted reward. Q-learning has been successfully adapted in several applications, reaching from single-device systems [11]–[13] to networked multi-device systems, found, e.g., in wireless communication [14]–[18], wireless sensor networks [19], and edge computing [20].

Problem formulation: Since learning methods have gained importance in widespread practical applications, and the tendency of increasing interconnection between heterogeneous autonomous systems is observable, it is crucial to study the interaction of (egoistic) learning systems. In this work, we are specifically interested in the setting of two Q-learning agents interacting non-cooperatively, where one of them has the privilege of observing the other’s actions. At first, we aim to investigate the impact of this information asymmetry on the learning outcome’s stability. Generally, for non-cooperative Q-learners, the learning outcome is not stable [21], [22], although some exceptions [23], [24] exist. Furthermore, we aim to describe the corresponding outcome in case the informational asymmetric Q-learning converges. In this direction, our attention lies in the question of whether the informational advantage (resp. disadvantage) of the globalized (resp. localized) agent has a positive (resp. negative) impact on her.

Our contributions: In this work, we show that informational asymmetry can foster the stability of the two agent Q-learning in the above asymmetrical information setting. Furthermore, we provide theoretical and numerical analysis of the policies that result from the convergent multi-agent Q-learning. In this respect, our main result is the insight that the information asymmetry is reflected in the outcome of the learning: The localized agent generates via Q-learning greedy policy, optimal given that the GA applies the long-term training policy, which is a model mismatch since the globalized agent likely apply the corresponding greedy policy in the post-learning phase; In contrast, the GA generates greedy policy optimal given that the localized agent generates the greedy post-learning policy. However, despite of the model mismatch of the localized agent, we are able to show
that the greedy post-learning policy is optimal in the game sense, meaning that no agent have incentives to deviate from this strategy.

**Relation to prior works:** Our work is a contribution to the large field of multi-agent reinforcement learning (for excellent overview see [25]). Relevant to our work are the works concerning value-based reinforcement learning agents. In contrast to our work, most of the works in this topic derive and investigate algorithms for joint learner, where the iterate of one agent requires the knowledge of other agents’ action or reward proposing several minimax Q-learning [26]–[28]. While the focus of the mentioned works is on designing algorithms converging to the Nash equilibrium of the underlying Markov game, our focus is to investigate the population dynamic of non-cooperative Q-learning agents with information asymmetry. Nevertheless, we are able to show that in some sense the outcome of the learning Q-learning is an almost solution of the underlying game. Lastly, another type of information asymmetry has also been considered in the literature [29], such as the case where an agent in contrast to others has local information about the underlying system. However, such approach, in contrast to ours, usually requires additional assumption on the system dynamic such as the existence of post-decision state where some system characteristics are revealed.

**II. Model Description**

**A. Markov Decision Process and Q-learning**

To begin with, we first recall the setting of a Markov decision process (MDP) [30] of a single agent, defined by the tuple $(S, A, r, P)$, where $S$ and $A$ are finite sets of system states and the agent’s action space, $r : S \times A \rightarrow \mathbb{R}$ describing the reward of the agent, and $P \in \Delta(S \times A)(S)$ describing the probabilistic state evolution given the present state and agent action. The notation $\Delta_X(Y)$ stands for the set of Markov kernel with source $X$ and target $Y$, and $\Delta(Y)$ for the set of probability distribution on $Y$. In the repeated setting, the aim of the agent in MDP is to determine a policy $\pi \in \Delta_S(A)$ optimizing the obtained reward, where $\pi(a_t | s_t)$ is the probability that the agent chooses the action $a_t \in A$ given that the system is in the state $s_t \in S$. One may consider a more general class of policy such as time-varying policies depending on the state-action history. However, it is usually enough to consider the class of (stationary Markov) policy $\Delta_S(A)$ (see, e.g., [30]). One of the important subclass of the policy is the class of deterministic policy. A deterministic policy $\pi \in \Delta_S(A)$ is a policy satisfying for every $s \in S \forall a \in A$. Therefore, a deterministic policy $\pi$ can be seen with slight abuse of notation as a function $\pi : S \rightarrow A$. A type of deterministic policy of particular interest in MDP is the greedy policy $\pi$ w.r.t. $Q \in \mathbb{R}^{S \times A}$ defined as $\pi(s) = \arg \max_{a \in A} Q(s, a)$.

The performance of a policy $\pi \in \Delta_S(A)$ in a discounted MDP $(S, A, r, P, \beta)$ is measured by the so-called value function $V_\pi : S \rightarrow \mathbb{R}$ given by $V_\pi(s) = \mathbb{E} \left[ \sum_{t=0}^\infty \beta^t r_t(S_t, A_t) \right]$, where $\beta \in (0, 1)$ is a chosen discount factor, and where the expectation is with respect to $S_0 = s, S_{t+1} \sim P(\cdot|S_t, A_t)$, and $A_t \sim \pi(\cdot|S_t)$, for all $t \in \mathbb{N}_0$. We refer the MDP with discount factor $\mathcal{DM}_0 := (S, A, r, P, \beta)$ to as discounted MDP, and $V_\pi$ as to the value function of $\pi$ in $\mathcal{DM}_0$. Related to the value function, is the following quantity called the Q-function of the policy $\pi$ in $\mathcal{DM}_0$, defined as $Q_\pi(s, a) = \mathbb{E} \left[ \sum_{t=0}^\infty \beta^t r_t(S_t, A_t) \right]$, where the expectation is w.r.t. $S_0 = s, A_0 = a, A_t \sim \pi(\cdot|S_t)$, and $S_{t+1} \sim P(\cdot|S_t, A_t)$, for all $t \in \mathbb{N}_0$. Clearly, it holds $V_\pi(s) = \mathbb{E}_{A \sim \pi(\cdot|s)}[Q_\pi(s, A)]$.

The goal of the discounted MDP is to optimize the value function w.r.t. the policy. In this direction, it is convenient to consider the so called optimal value function of $\mathcal{DM}_0$ given by $V_\ast(s) := \max_{\pi \in \Delta_S(A)} V_\pi(s)$. It is sometimes convenient to consider Q-function corresponds to $V_\ast$, i.e., the function $Q_\ast : \mathbb{R}^{S \times A} \rightarrow \mathbb{R}$ given by $Q_\ast(s, a) := \max_{\pi \in \Delta_S(A)} Q_\pi(s, a)$. We refer $Q_\ast$ as the optimal Q-function for $\mathcal{DM}_0$. If we know $Q_\ast$, then the optimal policy $\pi_\ast$ for $\mathcal{DM}_0$, i.e. the policy satisfying $Q_\ast = Q_{\pi_\ast}$ and $V_\ast = V_{\pi_\ast}$ is the greedy policy w.r.t. $Q_\ast$.

In many practical applications, the agent in an MDP $(S, A, r, P)$ has no knowledge about the reward $r$ and the transition probability $P$. Consequently, she cannot simply solve the Bellman equations (4) and (5), or (6). One way to do this is by the so-called Q-learning algorithm. This algorithm maintains at each time step $t \in \mathbb{N}_0$, the so-called Q-table $Q_t \in \mathbb{R}^{S \times A}$ serving as an approximation of the optimal Q-function. After executing the the action $A_t \sim \eta_t(Q_t, S_t)$, experiencing the reward $R_t := r(S_t, A_t)$, where $\eta_t \in \Delta(S \times A) \rightarrow (S \times A)$ is a prespecified learning (history dependent) policy, and querying the system state update $S_{t+1} \sim P(\cdot|S_t, A_t)$, the agent update the Q-table at time $t \in \mathbb{N}$ as follows:

$$Q_{t+1}(S_t, A_t) = (1 - \gamma_t)Q_t(S_t, A_t) + \gamma_t[R_t + \beta \max_{a \in A} Q_t(S_{t+1}, a)],$$

and leaves the remaining entries unupdated, i.e., $Q_{t+1}(s, a) = Q_t(s, a)$, for all $(s, a) \neq (S_t, A_t)$.

The choice of the learning policy $\eta_t$ follows the famous trade-off principle of exploration and exploitation. An instance of a learning policy satisfying this principle is the Boltzmann strategy with the temperature $\tau$, given by $\eta_t(a | Q_t, S_t) \propto \exp(Q_t(S_t, a)/\tau)$. For $\tau \rightarrow 0$, the Boltzmann strategy tends to be the greedy w.r.t. $Q_t$ (exploitation), and for $\tau \rightarrow \infty$, it tends to choose the action with equal probability (exploration).

**B. Game Setting**

In this work, we consider the setting of Markov game (see, e.g., [25] with two players: The localized agent (LA) and the globalized agent (GA). The (finite) state space of system containing those agents is denoted by $S$. $A_0$ stands for the (finite)
action space of the LA, and \( A_g \) for the action space of the GA. The function \( r^b : S \times A_{lo} \times A_g \rightarrow \mathbb{R} \) specifying the LA’s reward depends on the state of the system, the action of the GA, and her own action. Similarly, GA’s reward function is given by \( r^g : S \times A_{lo} \times A_g \rightarrow \mathbb{R} \). Throughout, we assume that both agents are unaware of the reward functions. Assuming that the system is in the state \( s \in S \), and that the agents apply the joint action \((a_{lo}, a_g) \in A_{lo} \times A_g\), the system state changes in Markovian manner as in a MDP described by a probability distribution \( P(\cdot | s, a_{lo}, a_g) \in \Delta(S) \), unknown to both agents.

We consider this game in a repeated setting with the information asymmetrical feature, that LA only knows about the instantaneous system evolution, while GA knows the latter and LA action. Specifically, we have, starting with the initial state \( S_0 = s \in S \), the following procedure for each time \( t \in \mathbb{N}_0 \): First, LA chooses the (randomized) action \( A_{lo}^t \) possibly utilizing the historical and actual system dynamic \((S_t)_{t \in [t]}\), and some implicit information about the historical GA actions \((A_g^t)_{t \in [t-1]}\). Afterwards GA observes LA’s action \( A_{lo}^t \), and by utilizing this information, the historical and actual system dynamic \((S_t)_{t \in [t]}\), GA chooses the action \( A_g^t \). Finally LA (resp. GA) obtain the reward \( r^b(S_t, A_{lo}^t, A_g) \) (resp. \( r^g(S_t, A_{lo}^t) \)) and subsequently the system transits to the state \( S_{t+1} \sim P(\cdot | S_t, A_{lo}^t, A_g^t) \).

For analysis purposes, we can represent the scope of LA information by means of the filtration \( (\mathcal{F}_t)_{t \in \mathbb{N}_0} \), where \( \mathcal{F}_t \) is the sigma-algebra generated by \((S_t)_{t \in [t-1]}, (A_{lo}^t)_{t \in [t-1]}, (A_g^t)_{t \in [t-1]}\), and \( S_t \), representing the implicit and explicit information LA can utilize at time \( t \) for choosing the action \( A_{lo}^t \). As modeled in the previous paragraph, the scope of GA information is different than LA. Thus, we represent this by a different filtration \( (\hat{\mathcal{F}}_t)_{t \in \mathbb{N}_0} \), where \( \hat{\mathcal{F}}_t \) denotes the \( \sigma \)-algebra generated by \( \mathcal{F}_t \) and \( A_{lo}^t \).

III. LA-GA Q-LEARNING – CONVERGENCE RESULT

Algorithm 1 LA Q-learning with Globalized Interference (LAQGI)

- **Extrinsic Parameter:** LA reward \( r^b \), system dynamic \( P \), GA policy \( (\eta_t^g)_{t \in \mathbb{N}_0} \)
- **LA parameter:** Policy \( \eta_t^b \in \Delta_{|S| \times |A_{lo}|}(A_{lo}) \), learning rate \( \gamma^b_t > 0 \), discount factor \( \beta^b \)

for \( t = 1, 2, \ldots \) do
  execute the action \( A_{lo}^t \sim \eta_t^b(\cdot | \hat{Q}_t^b) \)
  experience \( Q_{lo}^b(S_t, A_{lo}^t, A_g^t) \), where \( A_g^t \sim \eta_t^g(\cdot | \mathcal{F}_t) \)
  query system state information \( S_{t+1} \sim P(\cdot | S_t, A_{lo}^t, A_g^t) \)
  update: \( Q_{t+1}^b(S_t, A_{lo}^t) = (1 - \gamma^b_t)Q_{lo}^b(S_t, A_{lo}^t) + \gamma^b_t H_{t+1}^b \)
  for all \((s, a_{lo}) \neq (S_t, A_{lo}^t)\) do
    update \( Q_{t+1}^b(s, a_{lo}) = Q_{t+1}^b(s, a_{lo}) \)
  end for
end for

In this section, we extend the single agent Q-learning paradigm to the informational asymmetrical Markov game setting given in Subsection II-B. Our particular interest is on the convergence behaviour of the given Q-learning extension.

A. LA Q-learning

First, we model the Q-learning iterate for LA by straightforwardly extend the single agent Q-learning to the LA reward structure having additional dependency on the GA action. In our LA Q-learning model, we assume firstly that the GA chooses her action according to a general time-varying policy \( \Omega \times S \ni (\omega, s) \rightarrow \eta_t^g(\cdot | \mathcal{F}_t)(\omega) \), which might depend on the (implicit information of the) state-action history according to the GA scope information (see Subsection II-B). Formally, we assume that for every \( t \in \mathbb{N}_0, \eta_t^g(s | \mathcal{F}_t) \) is \( \mathcal{F}_t \)-measurable. Later in the next subsection, we will specify this policy to Q-learning-based algorithm. Our Q-learning model for the LA is specifically given in Algorithm I.

In the following, we specify some conditions leading to the convergence of LAQGI and determine the corresponding limit:

**Theorem 1:** Suppose that a.s. \( \lim_{t \to \infty} \eta_t^g(\cdot | \hat{\mathcal{F}}_t) = \eta^g(\cdot) \in \Delta(A_g) \). Moreover, suppose that \( \sum_{t=0}^{\infty} \psi_t^{lo} = \infty \) and \( \sum_{t=0}^{\infty} \psi_t^{lo, 2} < \infty \), where \( \psi_t^{lo}(s, a) = 1_{\{S_t = s, A_{lo} = a\}} \gamma^b_t \). Then the limit \( \hat{Q}_{LAQGI}^b \) of LAQGI’s iterate is the optimal Q-function of the discounted MDP \((S, A_{lo}, P_{lo}, \hat{P}_b, \beta_{lo})\), where for \( b = (s, a_{lo}) \in S \times A_{lo}, \tilde{r}^b(b) := \mathbb{E}_{A_g \sim \eta^g \sigma} [r^b(b, A_g)] \) and \( \tilde{P}^b(\cdot | b) := \mathbb{E}_{A_g \sim \eta^g \sigma} [P(\cdot | b, A_g)] \).

The proof of above Theorem is given in Subsubsection VII-C1. Above theorem gives hint that LA learns via Q-learning how to act optimally in expectation given GA’s stationary strategy (see Lemma 6), remarkably without knowing the latter. This property is interesting for e.g., security applications, where LA is a defender and GA is an attacker, since it implies that Q-learning helps the defender to learn optimal defend policy. However, it is not yet clear, whether, by applying the greedy policy resulted from learning phase, LA has indeed an optimal discounted cumulative reward. The discounted yields of the LA. We will clarify this aspect in the next section.
B. GA Q-Learning

Suppose that the LA applies Algorithm [1]. Our actual interest is on the behaviour of the Q-learning applying GA. As GA has informational advantage over LA, we assume that she utilizes this information in the learning phase and executes Q-table update for each LA action. Our proposal of GA Q-learning is specifically given in Algorithm [2].

Algorithm 2 GA Q-Learning (GAQL)

**Extrinsic Parameter:** GA reward $r^g$, system dynamic $P$, LA policy $(\pi_L^{lo})_{t\in\mathbb{N}_0}$

**GA parameter:** Policy $\eta^gl$, learning rate $\gamma^lo$, discount factor $\beta^lo$

for $t = 0, 2, \ldots$ do

- Execute the action $A^gl_t \sim \eta^gl(\cdot | S^t, A^lo_t)$
- Experience the reward $R^gl_t := r^g(S^t, A^gl_t, A^lo_t)$
- Query system state $S^t$ and LAs’ action $A^lo_t$
- Update $Q^gl_{A^{lo}_{t+1}}(S^t, A^gl_t) = (1 - \gamma^lo)Q^gl_{A^{lo}_{t+1}}(S^t, A^gl_t) + \gamma^lo H^gl_{t+1}$,
  where $H^gl_{t+1} = R^gl_t + \beta^gl \max_{a^l \in A^l} Q^gl_{A^{lo}_{t+1}}(S^t+1, a^l)$.

for all $(s, a^gl, a^lo) \neq (S^t, A^gl, A^lo)$ do

- Update $Q^gl_{A^{lo}_{t+1}}(s, a^gl) = Q^lo_{A^{lo}_{t+1}}(s, a^gl)$

end for

end for

The following Theorem gives sufficient conditions for the convergence of GAQL:

**Theorem 2:** Let $\eta^gl$ be policy generated in LAQGI. Suppose that $\sum_{t=0}^{\infty} \psi^gl_t = 1$ and $\sum_{t=0}^{\infty} \psi^gl_t^2 < \infty$, where $\psi^gl_t(s, a^gl, a^lo)$ := $1 \{S^t = s, A^gl_t = a^gl, A^lo_t = a^lo\} \gamma^lo$. Then $\forall a \in A^lo$, the limit of GAQL’s iterates $(Q^gl_{A^{lo}_{t}})_{t \in \mathbb{N}_0}$ is the optimal Q-function of $(S, A^g, \pi^lo_{A^g}, P^lo_{A^g}, \beta^lo)$, where $\pi^lo_{A^g}(s, a^g) := \pi^gl(s, a^gl, a^lo)$ and $\tilde{P}^gl_{A^g}(s'|s, a^g) := P(s'|s, a^g, a^lo)$.

The proof of above Theorem is given in Subsubsection VII-C2. Above theorem gives the hint that GA learns via GAQL the optimal strategies given that LA executes a constant action, it learns the optimal Q-function of the relevant MDP. At the first sight, this might affect adversely GA performance since LA's action rather changes over the time. However, we will see later in the next section (Lemma 3) that this is not true: GA learns via GAQL greedy policy given that LA applies a strategy from the class of deterministic strategies including LA optimal policy according to MDP theory.

IV. Optimality Analysis

In this section we formally investigate the joint performance of both agents respective to the policies yielded from the training through the Q-learning. Specifically, we analyze the joint performance of the greedy policy $\pi^{lo}_{A^g}$ (w.r.t. the limit $Q^lo_{A^g}$ of LAQGI’s iterate) of LA and $\pi^gl$ (w.r.t. the limit $Q^gl_{A^g}$ of GAQL’s iterate) of GA. For this purpose, we extend the notion of value function for the single agent to our game setting by defining the value function $V^{lo}_{\pi^lo, \pi^gl}$ of LA policy $\pi^lo \in \Delta_S(A^g)$ and GA policy $\pi^gl \in \Delta_{S \times A^lo}(A^g)$ as $V^{lo}_{\pi^lo, \pi^gl}(s) := E \left[ \sum_{t=0}^{\infty} \beta^lo \psi^gl_t(S^t, A^lo_t, A^gl_t) \right]$, $s \in S$. The expectation in previous definition is w.r.t. $S^0 = s, A^lo_t \sim \pi^lo(\cdot | S^t)$, $A^gl_t \sim \pi^gl(\cdot | S^t, A^lo_t)$, and $S_{t+1} \sim P(\cdot | S^t, A^lo_t, A^gl_t)$, for all $t \in \mathbb{N}_0$.

Our first result is that the greedy policy of the LA Q-learning is indeed optimal for LA given that the GA applies the asymptotic training policy. The formal statement is as follows:

**Lemma 3:** Suppose that the assumptions of Theorem 2 holds, and let $\pi^{lo}_{A^g}$ be the greedy policy (w.r.t. $Q^lo_{A^g}$) of LA resulted from LAQGI (Algorithm 1) with a given sequence $(\eta^gl(\cdot | S^t))_{t \in \mathbb{N}_0}$ of GA’s policies. Then, we have $V^{lo}_{\pi^lo, \pi^gl} \geq V^{lo}_{\pi^lo, \pi^{gl}_{A^g}}$, $\forall \pi^{gl}_{A^g} \in \Delta(A^g)$, where $\pi^{gl}_{A^g}$ is the asymptotic policy of the GA.

The proof of this statement is given in Subsection VII-D. In contrast, the greedy policy of the GA Q-learning is optimal for GA given that LA applies deterministic policy, which we expect since the optimal policy in a discounted MDP is deterministic. Formally, we have:

**Lemma 4:** Let $\pi^lo \in \Delta_S(A^lo)$ be a deterministic LA policy, and $\pi^gl_{A^g}$ be the greedy strategy (w.r.t. $Q^gl_{A^g}$) resulted from GA Q-learning (Algorithm 2). Then it holds $V^{gl}_{\pi^lo, \pi^gl_{A^g}} \geq V^{gl}_{\pi^lo, \pi^{gl}_{A^g}}$, $\forall \pi^{gl}_{A^g} \in \Delta_S(A^g)$.

The proof of this result is given in Subsection VII-D.

To sum up we have from above that the GA learning anticipates LA’s post-learning strategy, while LA learning results in the best response strategy respective to long-term GA learning strategy. As a consequence, the tuple $(\pi^{lo}_{A^g}, \pi^{gl}_{A^g})$ of post-Q-learning policies can in general not be the solution concept of the underlying game, since LA might be better off by applying another strategy. However, if the GA’s long-term learning strategy is equal to GA’s post-learning greedy strategy, it is likely that the latter tuple is an (almost) solution concept. To ensure the former, GA can use the Boltzmann strategy with low temperature as the learning policy.
have V of the local-global Markov game, in the sense that for any deterministic \( \eta \), decreasing exponentially with the temperature of the in-training Boltzmann policy, no agent applying the post-learning \( \epsilon \) MDP is deterministic, one can expect that LA applies this kind of strategy. In the previous equations, erfc denotes the Gauss complementary error function. In the Q-learning phase, we choose the factors \( c \) probabilites by \( t \).

\[ \rho(s, a, a') = \log_2(\det(I_s - \rho(s, a, a')h(s)h(s)^T)) - c|p(s) - (a_0 + a_0)|, \]

composed by the capacity term (with \( h(s) \) denotes the state-dependent gain) and by the scaled (with factor \( c > 0 \) penalization of over and under-use of the power respective to the given a state-dependent power-capacity \( p(s) \). To construct the state-transition model, we calculate the Signal to Noise Ratio (SNR) at each state by \( \rho(s, a_0, a_0) = f(s, a_0, a_0)/N(s) \), where \( f \) specifies the signal power and \( N(s) \) the state-dependent noise power. We then model the transition probabilities by \( P(s'|s, a_1, a_2) = \text{erfc}(\sqrt{\rho(s, a_1, a_2)}/2) \) if \( s = s' \), and \( P(s'|s, a_1, a_2) = 1 - \text{erfc}(\sqrt{\rho(s, a_1, a_2)}/2)/3 \) otherwise. In the previous equations, erfc denotes the Gauss complementary error function. In the Q-learning phase, we choose the Boltzmann strategy as the training policy.

**Theorem 5:** Let be \( \tau > 0 \). Suppose that GA applies the Boltzmann strategy as the training policy, defined for any \( a_g \in A_g \) as \( \eta_g^G(a_g|S_t, A^*) \propto \exp(\tilde{Q}^G_{A^*, t}(S_t, a_g)/\tau) \). The tuple \((\pi^\text{LAQGI}_{gl}, \pi^\text{GAQL}_{gl}) \in \Delta_S(A_{lo}) \times \Delta_S \times A_g(A_{gl})\) is an almost Nash-equilibrium of the local-global Markov game, in the sense that for any deterministic \( \pi_{lo} \in \Delta_S(A_{lo}) \) and for any \( \pi_{gl} \in \Delta_S \times A_g(A_{gl}) \), we have \( V^\text{LAQGI}_{lo, \pi_{lo}} \geq V^\text{lo}_{\pi_{lo}, \pi_{gl}} - \epsilon \) and \( V^\text{GAQL}_{lo, \pi_{lo}} \geq V^\text{lo}_{\pi_{lo}, \pi_{gl}} \), where \( \epsilon \leq 2\|p\|_D \exp(-C/\tau) \), with \( C, D > 0 \) are constants.

The proof of this theorem can be found in Subsection VII-D. So from above Theorem, we have that, up to a deviation \( \epsilon \) decreasing exponentially with the temperature of the in-training Boltzmann policy, no agent applying the post-Q-learning greedy strategy has incentives to change her strategy. One thing which is unusual in above Theorem is that the statement is respective to deterministic LA strategies and not general strategies. However, since the optimal strategy in a (single-agent) MDP is deterministic, one can expect that LA applies this kind of strategy.

**V. Numerical Simulations**

We consider \( S = \{1, 2, 3\} \). Moreover, we set \( A_{lo} = \{1, 2, 3\}, A_{gl} = \{1, 2, 3, 4\} \), which one can interpret as possible power allocation. We set the reward functions of both agents equal to \( r(s, a_{lo}, a_{gl}) = \log_2(\det(I_s - \rho(s, a_{lo}, a_{gl})h(s)h(s)^T)) - c|p(s) - (a_0 + a_0)|, \) composed by the capacity term (with \( h(s) \) denotes the state-dependent gain) and by the scaled (with factor \( c > 0 \) penalization of over and under-use of the power respective to the given a state-dependent power-capacity \( p(s) \). To construct the state-transition model, we calculate the Signal to Noise Ratio (SNR) at each state by \( \rho(s, a_0, a_0) = f(s, a_0, a_0)/N(s) \), where \( f \) specifies the signal power and \( N(s) \) the state-dependent noise power. We then model the transition probabilities by \( P(s'|s, a_1, a_2) = \text{erfc}(\sqrt{\rho(s, a_1, a_2)}/2) \) if \( s = s' \), and \( P(s'|s, a_1, a_2) = 1 - \text{erfc}(\sqrt{\rho(s, a_1, a_2)}/2)/3 \) otherwise. In the previous equations, erfc denotes the Gauss complementary error function. In the Q-learning phase, we choose the Boltzmann strategy as the training policy.

![Fig. 1. Cum. disc. reward for different policies and temperatures.](image1)

(a) \( \tau = 1.3 \)

![Fig. 2. Joint. coop. vs. asymm. (this work) vs. fully non-cooperative](image2)

(b) \( \tau = 0.1 \)

For Boltzmann temperature \( \tau = 1.3 \), figure 1 (a) compares the cumulative discounted reward over time for both (local and global) agents different strategy choices, i.e., the post-learning greedy strategies \((\pi^\text{LAQGI}_{lo}, \pi^\text{GAQL}_{gl})\) and the long-term Boltzmann learning strategy \((\eta^\text{LAQGI}_{lo}, \eta^\text{GAQL}_{gl})\). We observe, that if GA applies the Boltzmann strategy, it is better for LA to apply...
the greedy strategy, and that if LA applies the greedy strategy, it is also better for LA to apply the greedy strategy. This observation supports in particular the claims in Lemmas [3] and [4]. Moreover, we see that best overall performance yields if both agents acts greedily. This observation is not surprising, since it follows from the fact that the agents’ rewards (and therefore the value function) are the same and from our analysis (Lemmas [3] and [4]).

In Figure [4] (b), we compare the same policy tuples, however with smaller $\tau = 1.3$. We observe that the cumulative discounted rewards are the approximately the same for any strategy choice, which is the effect of the fact that the Boltzmann strategy morphs into a greedy like strategy (c.f. the discussion above the Theorem 5). With increasing $\tau$, we observe in our simulation that the discrepancy between the strategy tuples’ performances becomes larger. These observations gives in particular insight into the Theorem 5. Moreover, we observe that too small $\tau$ leads to the lack of state-action exploration giving a sub-optimal solution. One can see the latter effect in Figure 1 which shows that the best possible value in case $\tau = 1.3$ is dominated by the best possible value in case $\tau = 10$.

At last, we compare in Figure 2 the performance of our asymmetrical Q-learning (Asymmetrical (AS)) with the jointly cooperative Q-learning (Jointly Cooperative (JC)), i.e., the single-agent Q-learning in the MDP $(\mathcal{S}, \mathcal{A}_{[i]}, \mathcal{A}_{[\overline{i}]}, r, \mathcal{P})$, and (fully) non-cooperative Q-learning (Non-Cooperative (NC)), i.e., the Q-learning where GA has no knowledge about LA action. There, we specifically compare the corresponding post-learning greedy policies. We observe, that the JC has the best performance, which is to be expected due to the knowledge of the agents. However, it is remarkable to see that AS greatly outperform the non-cooperative case, and its performance is only marginally worse than the jointly cooperative one. This leads to the belief, that even under asymmetry of information, the agents are able to approach an almost fully cooperative amount of reward, as well as outperform NC case.

VI. Conclusion and Future Work

We have studied the long-term outcome of multi-agent (independent) Q-learning with information asymmetry. We have shown that the latter can foster the stability of the learning method. Despite of the information asymmetry, we have shown that the post-learning joint strategy of the agents is an almost solution concept. A point worth for further discussion in the convergence aspect is the summability-condition $\psi^0$ and $\psi^1$ given in the corresponding theorems. The achievement of this depends not only on the model’s transition probability and the considered agent itself, but also on other extrinsic factor: One agent’s policy has to allow other’s to explore the MDP. We leave the detailed treatment of this aspect for the future.

VII. Appendix

A. Basic Notations

Let $\mathcal{X}$ and $\mathcal{Y}$ be a finite sets. We denote the set of probability density on $\mathcal{X}$ by $\Delta(\mathcal{X})$, i.e.:

$$\Delta(\mathcal{X}) := \left\{ p : \mathcal{X} \rightarrow [0, 1] : \sum_{x \in \mathcal{X}} p(x) = 1 \right\}.$$ 

We write the set of Markov kernel with source $\mathcal{X}$ and target $\mathcal{Y}$ by $\Delta_X(\mathcal{Y})$, i.e.:

$$\Delta_X(\mathcal{Y}) := \left\{ p : (\mathcal{X}, \mathcal{Y}) \rightarrow [0, 1], p(x,y) \rightarrow p(y|x) : \sum_{y \in \mathcal{Y}} p(y|x) = 1, \forall x \in \mathcal{X} \right\}$$

Given two vectors (or matrices) $x, y$ having the same dimensions, we denote the entrywise multiplication of $x$ and $y$ by $x \odot y$. For analysis of the value function $V_\pi$ of a policy $\pi \in \Delta_F(\mathcal{A})$ it is useful to describe it implicitly as a solution of an equation. Specifically, one can show (see Theorem 6.1.1 in [30]), that $V_\pi$ is the unique solution of the the Bellman equation:

$$V_\pi(s) = E_{A \sim \pi(\cdot|s)} \left[ r(s,A) + \beta E_{S' \sim P(\cdot|s,A)} [V_\pi(s')] \right].$$

Similarly, the $Q_\pi$-function of $\pi$ is the unique solution of equation:

$$Q_\pi(s,a) = r(s,a) + \beta E_{S' \sim P(\cdot|s,a)} \left[ E_{A' \sim \pi(\cdot|S')} [Q_\pi(S',A')] \right].$$

For optimal value function $V_*$ of $\mathcal{DRI}$ we have also implicit description similar to the previous one for the value function of a policy. Specifically, it holds that $V_*$ is the unique solution of the equation:

$$V_*(s) = \max_{a \in \mathcal{A}} \left[ r(s, a) + \gamma \sum_{s' \in \mathcal{S}} P(s'|s, a) V_*(s') \right].$$
Moreover for the corresponding Q-function $Q_\pi$, it holds that it is the unique solution of the equation:

$$Q_\pi(s, a) = r(s, a) + \gamma \mathbb{E}_{S' \sim P(S'|s, a)} \left[ \max_{a' \in A} Q_\pi(s', a') \right]. \quad (3)$$

Working with a discounted MDP $\mathcal{DM}$ and a policy $\pi \in \Delta_S(A)$, it is convenient to utilize the operator $T_\pi : \mathbb{R}^{S \times A} \to \mathbb{R}^{S \times A}$ given by:

$$(T_\pi Q)(s, a) := r(s, a) + \gamma \mathbb{E}_{S' \sim \rho}(Q(S', A')),$$

called the Bellman operator of $\pi$ in $\mathcal{DM}$. Furthermore, the operator $T_* : \mathbb{R}^{S \times A} \to \mathbb{R}^{S \times A}$,

$$(T_* Q)(s, a) := r(s, a) + \gamma \mathbb{E}_{s' \sim \rho}[\max_{a' \in A} Q(s', a')],$$

called the optimal Bellman operator of $\mathcal{DM}$, is also useful for MDP analysis. It directly follows from the discussion in the previous paragraph that the Q-function of the policy $\pi$ is the unique fixed point of the Bellman operator $T_\pi$ of the policy $\pi$ in $\mathcal{DM}$. The same relation holds also between the optimal Bellman operator and the optimal Q-function for $\mathcal{DM}$. Formally, $Q_\pi$ and $Q_*$ are the unique solution of:

$$T_\pi Q_\pi = Q_\pi \quad \text{and} \quad T_* Q_* = Q_* \quad (4)$$

One property of the Bellman operator useful for our later approach, is that both $T_\pi$ and $T_*$ are $\gamma$-contractions (w.r.t. $\| \cdot \|_\infty$), i.e.:

$$\|TQ - TQ'\|_\infty \leq \gamma \|Q - Q'\|_\infty, \quad \forall Q, Q' \in \mathbb{R}^{S \times A}, \quad (5)$$

where $T$ is either $T_\pi$ or $T_*$. 

C. Missing proofs in Section [II]

Our convergence proof is based on the following well-known statement [30]:

**Proposition 6:** Given a filtration $\mathcal{G} := (\mathcal{G}_t)_{t \in \mathbb{N}_0}$. Let $H : \mathbb{R}^D \to \mathbb{R}^D$, $(\gamma_t)_{t \in \mathbb{N}_0} \subset \mathbb{R}^D$, and $(U_t)_{t \in \mathbb{N}_0}$, $(W_t)_{t \in \mathbb{N}}$ are sequences of $\mathbb{R}^D$-valued RV. Let $(X_t)_{t \in \mathbb{N}_0} \subset \mathbb{R}^D$ be a sequence generated by the iteration:

$$X_{t+1}(i) = (1 - \gamma_t(i))X_t(i) + \gamma_t(i) \left[ (HX_t)(i) + U_t(i) + W_{t+1}(i) \right]. \quad (6)$$

Suppose that:

1) $(W_t)_{t \in \mathbb{N}}$ is $\mathcal{G}$-adapted and fulfills:

$$\mathbb{E}[W_{t+1} | \mathcal{F}_t] = 0 \quad \mathbb{E}[W_{t+1}^2 | \mathcal{F}_t] \leq A + B \|X_t\|_\infty,$$

for some $A, B > 0$.

2) $(\gamma_t)_{t \in \mathbb{N}}$ is sequence of non-negative $\mathcal{G}$-adapted RVs and fulfills:

$$\sum_{t=0}^\infty \gamma_t(i) = \infty \quad \text{and} \quad \sum_{t=0}^\infty \gamma_t(i)^2 < \infty \quad a.s.$$

3) there exists $x_0 \in \mathbb{R}^D$ and $\beta \in [0, 1)$ s.t.:

$$\|HX_t - x_0\|_\infty \leq \beta \|X_t - x_0\|_\infty$$

4) $(U_t)$ is $\mathcal{G}$-adapted, and there exists a sequence $(\theta_t)_{t \in \mathbb{N}_0}$ of $\mathbb{R}^D$-valued RV converging to 0 a.s. such that:

$$|U_t(i)| \leq \theta_t(\|X_t\|_\infty + 1),$$

Then:

$$X_t \xrightarrow{t \to \infty} x_0 \quad a.s.$$
1) Proof of LAQGI convergence (Theorem 1): Our strategy is to write the iterate (1) in the form (6). To achieve this, we first notice that the iterate of (1) can be written as:

\[ Q_{t+1}^{lo} = (1 - \psi_t^{lo}) \odot Q_t^{lo} + \psi_t^{lo} \odot \tilde{T}_t^{lo} Q_t^{lo}, \]

where \( \tilde{T}_t^{lo} \) is the optimal Bellman operator of the discounted MDP \((S, A_{lo}, r_t^{lo}, \delta_{s_{t+1}}, \beta_{lo})\), and where:

\[ \psi_t^{lo}(s, a) = 1 \{ S_t = s, A_t = a \} \gamma_t^{lo}. \]

Next, by means of the \( T_{lo,t} \) of the MDP \((S, A_{lo+t, lo}, \tilde{P}_t^{lo})\), where:

\[ \tilde{P}_t^{lo}(s, a) \] where:

\[ \tilde{P}_t^{lo}(s, a) := E_{A_t \sim \eta_t^{lo}(\cdot | \tilde{F}_t)}[P(s, a, A_t^{gl})], \]

and the optimal Bellman operator \( \tilde{T}_t^{lo} \) for the discounted MDP \((S, A_{lo}, r_t^{lo}, P_t^{lo}, \beta_{lo})\), we can rewrite (7) as:

\[ Q_{t+1}^{lo} = (1 - \psi_t^{lo}) \odot Q_t^{lo} + \psi_t^{lo} \odot \tilde{T}_t^{lo} Q_t^{lo} + U_t + W_{t+1}, \]

where:

\[ W_{t+1} = \tilde{T}_t^{lo} Q_t^{lo} - T_{lo,t} Q_t^{lo}, \]

\[ U_t = T_{lo,t} Q_t^{lo} - \tilde{T}_lo Q_t^{lo}, \]

Lemma 7: The random sequence \((W_t)_{t \in \mathbb{N}}\) defined in (8) satisfies:

\[ E[W_{t+1} | \tilde{F}_t] = 0, \]

\[ E[(W_{t+1}(s, a))^2 | \tilde{F}_t] \leq 2\|r\|_\infty + 2\gamma_{lo}^2 \|Q\|_t \]

Proof: We have:

\[ E[r_t^{lo}(s, a, A_t^{gl}) | \tilde{F}_t] = \sum_{a_{gl}} \eta_t^{gl}(a_{gl} | \tilde{F}_t) r_t^{lo}(s, a, a_{gl}) = E_{A_t^{gl} \sim \eta_t^{lo}(\cdot | \tilde{F}_t)}[r_t^{lo}(s, a, A_t^{gl})] = \tilde{r}_t(s, a) \]

Furthermore:

\[ E \left[ \max_{a_{lo}} Q_t^{lo}(S_{t+1}, a_{lo}' | \tilde{F}_t) \right] \]

\[ = E \left[ E \left[ \max_{a_{lo}} Q_t^{lo}(S_{t+1}, a_{lo}') | \tilde{F}_t \right] \bigg| \tilde{F}_t \right] \]

\[ = E \left[ E_{S' \sim \mathcal{P}(s_{t+1}, a_{lo}')} \left[ \max_{a_{lo}} Q_t^{lo}(S', a_{lo}') \right] \bigg| \tilde{F}_t \right] \]

\[ = \sum_{a_{gl}} \eta_t^{gl}(a_{gl} | \tilde{F}_t) \sum_{s' \in S} \mathcal{P}(s' | S_t, A_t^{lo}, A_t^{gl}) \max_{a_{lo}} Q_t^{lo}(s', a_{lo}') \]

\[ = \sum_{s' \in S} \left[ \sum_{a_{gl}} \eta_t^{gl}(a_{gl} | \tilde{F}_t) \mathcal{P}(s' | S_t, A_t^{lo}, A_t^{gl}) \right] \max_{a_{lo}} Q_t^{lo}(s', a_{lo}') \]

\[ = \sum_{s' \in S} E_{A_t^{gl} \sim \eta_t^{lo}(\cdot | \tilde{F}_t)} \left[ \mathcal{P}(s' | S_t, A_t^{lo}, A_t^{gl}) \right] \max_{a_{lo}} Q_t^{lo}(s', a_{lo}') \]

\[ = \sum_{s' \in S} \tilde{P}_t^{lo}(s' | S_t, A_t^{lo}) \max_{a_{lo}} Q_t^{lo}(s', a_{lo}') \]

\[ = E_{S' \sim \mathcal{P}_t^{lo}(S_t, A_t^{lo})} \left[ \max_{a_{lo}} Q_t^{lo}(s', a_{lo}') \right] \]
Combining both previous computations, we have:

\[
E[(\tilde{T}^\eta_l Q)(S_t, A^\eta_l)|\tilde{F}_t]
\]

\[
= E \left[ r^{lo}(S_t, A_t^l, A^\eta_l) + \beta_{lo} \max_{a_{lo} \in A_{lo}} Q_{t+1}^{lo}(s_{t+1}, a_{lo}) \right] |\tilde{F}_t \]

\[
= E \left[ r^{lo}(S_t, A_t^l, A^\eta_l) |\tilde{F}_t \right] + \beta_{lo} E \left[ \max_{a_{lo} \in A_{lo}} Q_{t+1}^{lo}(s_{t+1}, a_{lo}) |\tilde{F}_t \right]
\]

where the last equality follows from the fact that \( \eta^l_{\tilde{T}}(\cdot)|\tilde{F}_t \) is \( \tilde{F}_t \)-measurable, and thus also \( P^l_{\tilde{T}}(\cdot|S_t, A^\eta_l) \). Above computation yields the first statement, since:

\[
E[W_{t+1}(s, a_{lo})|\tilde{F}_t]
\]

\[
= 1_{\{s = s', A^\eta_l = a_{lo}\}} E[ (\tilde{T}^\eta_l Q)(S_t, A^\eta_l) - (T_{lo, t} Q^lo)(S_t, A^lo)|\tilde{F}_t].
\]

For the second statement, we compute:

\[
E \left[ \left( (\tilde{T}^\eta_l Q)(s, a_{lo}) - (T_{lo, t} Q)(s, a_{lo}) \right)^2 |\tilde{F}_t \right]
\]

\[
\leq E \left[ \left( (\tilde{T}^\eta_l Q)(s, a_{lo}) \right)^2 |\tilde{F}_t \right] + E \left[ \left( (T_{lo, t} Q)(s, a_{lo}) \right)^2 |\tilde{F}_t \right]
\]

where the inequality follows from \((a - b)^2 \leq a^2 + b^2\) for any \(a, b \geq 0\). Now, we estimate each summand above. First, we have:

\[
E \left[ \left( r^{lo}(s, a_{lo}, A^\eta_l) \right)^2 |\tilde{F}_t \right] \leq \|r^{lo}\|_2^2,
\]

and:

\[
E \left[ \left( \max_{a_{lo} \in A_{lo}} Q_{t+1}^{lo}(s_{t+1}, a'_{lo}) \right)^2 |\tilde{F}_t \right] \leq \|Q_{t+1}^{lo}\|_2^2.
\]

Consequently:

\[
E \left[ \left( (\tilde{T}^\eta_l Q^lo)(s, a_{lo}) \right)^2 |\tilde{F}_t \right]
\]

\[
\leq E \left[ \left( r^{lo}(s, a_{lo}, A^\eta_l) \right)^2 |\tilde{F}_t \right] + \beta_{lo}^2 E \left[ \left( \max_{a_{lo} \in A_{lo}} Q_{t+1}^{lo}(s_{t+1}, a'_{lo}) \right)^2 |\tilde{F}_t \right]
\]

\[
\leq \|r^{lo}\|_2^2 + \beta_{lo}^2 \|Q_{t+1}^{lo}\|_2^2.
\]

Similar computation yields:

\[
E \left[ \left( (T_{lo, t} Q)(s, a_{lo}) \right)^2 |\tilde{F}_t \right] \leq \|r^{lo}\|_2^2 + \beta_{lo}^2 \|Q_{t}^{lo}\|_2^2.
\]

Combining both previous estimates, we obtain the desired statement.

**Lemma 8:** The random sequence \((U_t)_{t \geq 0}\) defined in (4) fulfills:

\[
|U_t(s, a)| \leq \theta_t (1 + \|Q^lo\|_2),
\]

where:

\[
\theta_t := \max \left\{ \|r^{lo}\|_2, \beta_{lo} \right\} \|\eta^gl(\cdot)|\tilde{F}_t\|_\infty - \|\eta^gl|\tilde{F}_t\|_\infty.
\]

**Proof:** Setting \( \pi^{(1)} = \eta^gl, \pi^{(2)} = \eta^gl(\cdot)|\tilde{F}_t \), \( \beta = \beta_{lo} \), and \( \pi \) equal to the greedy policy in Lemma 11 and by noticing that in this case we have \( r^{lo}_{(1)} = r^{lo}, P^{lo}_{(1)} = P^{lo}, r^{lo}_{(2)} = \tilde{r}^l, P^{lo}_{(2)} = P^{lo}_t \), we obtain as desired:

\[
|U_t(s, a)| = \left| (T_{lo, t} Q^lo)(s, a_{lo}) - (\tilde{T}^\eta_l Q^lo)(s, a_{lo}) \right|
\]

\[
\leq \left( \|r^{lo}\|_2 + \beta_{lo} \|Q^lo\|_2 \right) \|\eta^gl(\cdot)|\tilde{F}_t\|_\infty - \|\eta^gl(\cdot)|\tilde{F}_t\|_\infty.
\]

**Proof (Proof of Theorem 7):** The proof that \((W_t)\) (resp. \((U_t)\)) satisfies the first (resp. the fourth) condition of Proposition 6 is given in Lemma 7 (resp. Lemma 8). The third property follows from the fact that \( \tilde{T}^\eta_l \) as a Bellman operator is a contraction with respect to discount factor of the underlying MDP.
2) Proof of GAQL convergence (Theorem 2):

Proof (Proof of Theorem 2): For any \( a_{t_0} \), define:

\[
\tilde{\pi}^{gl}_{a_{t_0}}(s, a_{\text{gl}}) := r^{gl}(s, a_{\text{gl}}, a_{t_0}) \quad \text{and} \quad \tilde{P}^{gl}_{a_{t_0}}(s' | s, a_{\text{gl}}) := P(s' | s, a_{\text{gl}}, a_{t_0})
\]

(10)

Let \( \tilde{\Pi}^{gl}_{a_{t_0}, t} \) be the optimal Bellman operator of the discounted MDP \((S, A_{\text{gl}}, r^{\text{gl}}, \delta_{S_{t_1, t}}, \beta_{\text{gl}})\), and \( \tilde{T}^{gl}_{a_{t_0}} \) be the optimal Bellman operator of the discounted MDP \((S, A_{\text{gl}}, \tilde{\pi}^{gl}_{a_{t_0}}, \tilde{P}^{gl}_{a_{t_0}}, \beta_{\text{gl}})\). We can write the iterate of Algorithm 2 in the form:

\[
\psi^{gl}_{a_{t_0}, t}(s, a_{\text{gl}}) = 1_{\{S_t = s, A_t^v = a_{t_0}, A_t^{gl} = a_{t_0}^t\}} \tilde{T}^{gl}_{a_{t_0}, t}.
\]

We can write the iterate of Algorithm 2 in the form:

\[
Q^{gl}_{a_{t_0}, t+1} = (1 - \psi^{gl}_{a_{t_0}, t}) Q^{gl}_{a_{t_0}, t} + \psi^{gl}_{a_{t_0}, t} \left[ \tilde{T}^{gl}_{a_{t_0}, t} Q^{gl}_{a_{t_0}, t} + W^{gl}_{a_{t_0}, t+1} \right],
\]

where:

\[
W^{gl}_{a_{t_0}, t+1} := 1_{\{S_t = s, A_t^v = a_{t_0}, A_t^{gl} = a_{t_0}^t\}} \tilde{T}^{gl}_{a_{t_0}, t} Q^{gl}_{a_{t_0}, t} - \tilde{T}^{gl}_{a_{t_0}, t} Q^{gl}_{a_{t_0}, t+1},
\]

and where:

\[
\psi^{gl}_{a_{t_0}, t}(s, a_{\text{gl}}) = 1_{\{S_t = s, A_t^v = a_{t_0}, A_t^{gl} = a_{t_0}^t\}} \tilde{T}^{gl}_{a_{t_0}, t}.
\]

Notice that (11) has the form (6) with \( U_t = 0 \). So by checking the conditions (except the fourth condition) in Proposition 6 we can use the latter for showing the desired statement.

First, we check the third condition. Let be \( a_{t_0} \in A_{t_0} \) arbitrary. Let \( \tilde{Q}^{gl}_{a_{t_0}} \) be the optimal Q-function of the discounted MDP \((S, A_{\text{gl}}, \tilde{\pi}^{gl}_{a_{t_0}}, \tilde{P}^{gl}_{a_{t_0}}, \beta_{\text{gl}})\). We have for all \( Q \in \mathbb{R}^{S \times A_{t_0}} \):

\[
\| \tilde{T}^{gl}_{a_{t_0}} Q - \tilde{Q}^{gl}_{a_{t_0}} \|_\infty = \| \tilde{T}^{gl}_{a_{t_0}} Q - \tilde{T}^{gl}_{a_{t_0}} \tilde{Q}^{gl}_{a_{t_0}} \|_\infty \leq \beta_{\text{gl}} \| Q - \tilde{Q}^{gl}_{a_{t_0}} \|_\infty,
\]

where the equality follows from the fact that \( \tilde{Q}^{gl}_{a_{t_0}} \) is the fixed point of \( \tilde{\Pi}^{gl}_{a_{t_0}, t} \), and the inequality follows from the fact that the Bellman operator is a contraction w.r.t. the discount factor of the underlying discounted MDP. Thus, the third condition in Proposition 6 is shown.

Now, we check the first condition in Proposition 6. We have:

\[
E[(\tilde{T}^{gl}_{A_t^v, t} Q^{gl}_{A_t^v, t})(S_t, A_t^v)|F_t] = r^{gl}_{A_t^v}(S_t, A_t^v) + \beta^{gl}_{A_t^v} E[\max_{a_{gl}} Q^{gl}_{A_t^v, t+1} (S_{t+1}, A_t^v, a_{gl}) | F_t]
\]

\[
= r^{gl}_{A_t^v}(S_t, A_t^v) + \beta^{gl}_{A_t^v} \sum_{s'} P(s' | S_t, A_t^v, A_t^{gl}) \max_{a_{gl}} Q^{gl}_{A_t^v, t+1} (s', A_t^v, a_{gl})
\]

\[
= r^{gl}_{A_t^v}(S_t, A_t^v) + \beta^{gl}_{A_t^v} E_{S_t \sim \mathcal{P}_{A_t^v}(|S_t, A_t^v, A_t^{gl})} [\max_{a_{gl}} Q^{gl}_{A_t^v, t+1} (S', A_t^v, a_{gl})]
\]

\[
= \tilde{T}^{gl}_{A_t^v} Q^{gl}_{A_t^v}(S_t, A_t^v) = E[(\tilde{T}^{gl}_{A_t^v} Q^{gl}_{A_t^v, t})(S_t, A_t^v)|F_t],
\]

where the first equality follows from the fact that \( r^{gl}_{A_t^v}(S_t, A_t^v) \) only depends on \((S_t, A_t^v, A_t^{gl})\) and thus \( F_t \)-measurable, and where the last equality follows from the fact that \( \tilde{T}^{gl}_{A_t^v} Q^{gl}_{A_t^v, t} \) only depends on \((S_t, A_t^v, A_t^{gl})\), \( t \in [0, t_0] \), and thus \( F_t \) measurable. Consequently, we have:

\[
E[W^{gl}_{t+1}(s, a)|F_t] = 1_{\{S_t = s, A_t = a\}} E[(\tilde{T}^{gl}_{A_t^v} Q^{gl}_{A_t^v, t})(S_t, A_t) - \tilde{T}^{gl}_{A_t^v} Q^{gl}_{A_t^v, t}(S_t, A_t)|F_t] = 0.
\]

To show that the first condition in Proposition 6 holds, it remains to derive the corresponding second moment bound. Similar argumentation as in the proof of Lemma 7 yields as desired:

\[
E[(W^{gl}_{t+1}(s, a_{\text{lo}}, a_{\text{gl}}))^2 | F_t] \leq 2 \left( \| r^{gl} \|_\infty + \beta^2_{\text{lo}} \| Q^{gl} \| \right)
\]
Clearly, \( \pi \in \Delta_S(A_\text{lo}) \). By above definition, it follows that \( V^\text{lo}_{\pi,\eta^\beta} \) is the value function of the policy \( \pi \) corresponds to the discounted MDP \( (S, A_\text{lo} \times A_{\text{gl}}, r^\text{lo}, P, \beta^\text{lo}) \). Denote \( A = (A_\text{lo}, A_{\text{gl}}) \) Therefore by means of (1), we can compute:

\[
V^\text{lo}_{\pi,\eta^\beta}(s) = \mathbb{E}_{A \sim \pi} \left[ r^\text{lo}(s, A) + \beta^\text{lo} \mathbb{E}_{S' \sim P(\cdot|s,A)} \left[ V^\text{lo}_{\pi,\eta^\beta}(S') \right] \right]
\]

\[
= \mathbb{E}_{A_\text{lo} \sim \eta^\text{lo}} \left[ r^\text{lo}(s, A_\text{lo}) \right] + \beta^\text{lo} \mathbb{E}_{A_\text{gl} \sim \eta^\text{gl}} \left[ \mathbb{E}_{S' \sim P(\cdot|s,A_\text{lo})} \left[ V^\text{lo}_{\pi,\eta^\beta}(S') \right] \right]
\]

where in the second equality follows by writing out the definition of \( \pi \) and the third inequality from the definition of \( r^\text{lo} \) and \( P^\text{lo} \). From above computation and (1) we have that \( V^\text{lo}_{\pi,\eta^\beta} \) is the value function of the policy \( \pi_{\text{lo}} \) in the discounted MDP \( (S, A_\text{lo}, r^\text{lo}, P^\text{lo}, \beta^\text{lo}) \). Now, Theorem 1 asserts that \( \pi^\text{QGL}_{\text{lo}} \) is the optimal policy of the MDP \( (S, A_\text{lo}, r^\text{lo}, P^\text{lo}, \beta^\text{lo}) \) and therefore, its value function in \( (S, A_\text{lo}, r^\text{lo}, P^\text{lo}, \beta^\text{lo}) \) dominates the value function of \( \pi_{\text{lo}} \) which is as shown before equal to \( V^\text{lo}_{\pi_{\text{lo}}, \eta^\beta} \). Finally, we obtain the desired statement by noticing that the value function of \( \pi^\text{QGL}_{\text{lo}} \) is equal to \( V^\text{lo}_{\pi^\text{QGL}_{\text{lo}}, \eta^\beta} \) by the similar argument as in (13).

**Proof (Proof of Lemma 2):** Let \( \pi_{\text{gl}} \in \Delta_S(A_{\text{gl}}) \) be arbitrary. By the similar argumentation as in (13), we have that:

\[
V^\text{gl}_{\pi_{\text{lo}}, \pi_{\text{gl}}}(s) = \mathbb{E}_{A_\text{lo} \sim \pi_{\text{lo}}(\cdot|s,a_{\text{lo}}(s))} \left[ r^\text{gl}_{\pi_{\text{lo}}}(s, A_{\text{gl}}) \right] + \beta^\text{gl} \mathbb{E}_{S' \sim P(\cdot|s,a_{\text{gl}}(s))} \left[ V^\text{gl}_{\pi_{\text{lo}}, \pi_{\text{gl}}}(S') \right]
\]

where \( r^\text{gl}_{\text{lo}}, \pi^\text{gl}_{\text{lo}} \), \( a_{\text{lo}} \in A_{\text{lo}} \) is defined in (10). Defining:

\[
r^\text{gl}_{\text{lo}}(s, a_{\text{gl}}) := r^\text{gl}_{\text{lo}}(s, a_{\text{gl}}) \text{ and } \pi^\text{gl}_{\text{lo}}(\cdot|s, a_{\text{gl}}) := \pi^\text{gl}_{\text{lo}}(\cdot|s, a_{\text{gl}}),
\]

we obtain from (14) that \( V^\text{gl}_{\pi_{\text{lo}}, \pi_{\text{gl}}} \) is the value function of the policy \( \pi_{\text{lo}} \in \Delta_S(A_{\text{lo}}) \), with \( \pi_{\text{gl}}(a_{\text{gl}}|s) := \pi_{\text{gl}}(a_{\text{gl}}|s, a_{\text{lo}}(s)) \), in the discounted MDP \( (S, A_{\text{gl}}, r^\text{gl}_{\text{lo}}, P^\text{gl}_{\text{lo}}, \beta^\text{gl}) \).

Now, the fact that for any \( a_{\text{lo}} \in A_{\text{lo}} \), \( \tilde{Q}^\text{gl}_{\text{lo}} \) is the optimal Q-function for the discounted MDP \( (S, A_{\text{gl}}, r^\text{gl}_{\text{lo}}, P^\text{gl}_{\text{lo}}, \beta^\text{gl}) \) and the relation (3) yields that the function \( \tilde{Q}^\text{gl}(s, a_{\text{gl}}) := \tilde{Q}^\text{gl}_{\text{lo}}(s, a_{\text{gl}}) \) satisfies:

\[
\tilde{Q}^\text{gl}(s, a_{\text{gl}}) = r^\text{gl}(s, a_{\text{gl}}) + \beta^\text{gl} \mathbb{E}_{S' \sim P^\text{gl}_{\text{lo}}(\cdot|s,a_{\text{gl}})} \left[ \max_{a_{\text{gl}}} Q^\text{gl}(S', a_{\text{gl}}) \right]
\]

and consequently \( \tilde{Q}^\text{gl} \) is the optimal Q-function of the discounted MDP \( (S, A_{\text{gl}}, r^\text{gl}_{\text{lo}}, P^\text{gl}_{\text{lo}}, \beta^\text{gl}) \). Now, let \( \nabla^\text{gl}(s) := \max_{a_{\text{gl}} \in A_{\text{gl}}} Q^\text{gl}(s, a_{\text{gl}}) \) be the corresponding value function. By the optimality \( \tilde{Q}^\text{gl} \) and the fact that \( V^\text{gl}_{\pi_{\text{lo}}, \pi_{\text{gl}}} \) is the value function of a policy in \( (S, A_{\text{gl}}, r^\text{gl}_{\text{lo}}, P^\text{gl}_{\text{lo}}, \beta^\text{gl}) \), we have that \( \nabla^\text{gl} \geq V^\text{gl}_{\pi_{\text{lo}}, \pi_{\text{gl}}} \).

It remains now to show that \( \nabla^\text{gl} = V^\text{gl}_{\pi_{\text{lo}}, \pi^\text{QGL}_{\text{lo}}} \). Since:

\[
V^\text{gl}_{\pi_{\text{lo}}, \pi^\text{QGL}_{\text{lo}}}(s) = \mathbb{E}_{A_{\text{gl}} \sim \pi^\text{QGL}_{\text{lo}}(\cdot|s,a_{\text{lo}}(s))} \left[ \tilde{Q}^\text{gl}(s, A_{\text{gl}}) \right],
\]

and since:

\[
V^\text{gl}_{\pi_{\text{lo}}, \pi^\text{QGL}_{\text{lo}}}(s) = \mathbb{E}_{A_{\text{gl}} \sim \pi^\text{QGL}_{\text{lo}}(\cdot|s,a_{\text{lo}}(s))} \left[ Q_{\pi}(s, a_{\text{lo}}(s), A_{\text{gl}}) \right],
\]

where \( Q_{\pi} \) is the Q-function of the policy \( \pi \in \Delta_S(A_{\text{lo}} \times A_{\text{gl}}) \), with \( \pi(a_{\text{lo}}, a_{\text{gl}}|s) := \pi_{\text{lo}}(a_{\text{lo}}|s)\pi_{\text{gl}}(a_{\text{gl}}|s,a_{\text{lo}}(s)) \), it is sufficient to show that \( Q_{\pi}(s, a_{\text{lo}}(s), a_{\text{gl}}) = \tilde{Q}^\text{gl}(s, A_{\text{gl}}) \). Toward this end, it is straightforward to see that by (11-13) that \( Q_{\pi}(s, a_{\text{lo}}(s), a_{\text{gl}}) \) fulfills:

\[
Q_{\pi}(s, a_{\text{lo}}(s), a_{\text{gl}}) = r^\text{gl}(s, a_{\text{lo}}(s), a_{\text{gl}}) + \beta^\text{gl} \mathbb{E}_{S' \sim P(\cdot|s,a_{\text{lo}}(s),a_{\text{gl}})} \left[ Q_{\pi}(S', a_{\text{lo}}(S'), A_{\text{gl}}) \right]
\]

and

\[
Q_{\pi}(s, a_{\text{lo}}(s), a_{\text{gl}}) = r^\text{gl}(s, a_{\text{gl}}) + \beta^\text{gl} \mathbb{E}_{S' \sim P^\text{gl}_{\text{lo}}(\cdot|s,a_{\text{gl}})} \left[ Q_{\pi}(S', a_{\text{lo}}(S'), a_{\text{gl}}) \right].
\]
where $\tilde{\pi}_gl \in \Delta_S(A_{gl})$, with $\tilde{\pi}_gl(a_{gl}|s) := \pi^{GAQL}_gl(a_{gl}|s, \pi_{lo}(s))$. Moreover, we can write (13) by definition of $\pi^{GAQL}_gl$ as follows:

$$Q^{\pi}_gl(s, a_{gl}) = \pi^g(s, a_{gl}) + \beta_{gl}E_{S'\sim \pi_{lo}}[E_{A_{gl}\sim \pi_{lo}(|S')}[Q^{\pi}_{gl}(S', A_{gl})]$$

Consequently, $Q_{gl}(s, \pi_{lo}(s), a_{gl})$ and $Q^{\pi}_gl(s, a_{gl})$ are solutions for the Bellman equation for $Q$-function of $\pi_{lo}$ in $(S, A_{gl}, \pi_{lo}, \pi_{gl})$ (see (VII-B)). Uniqueness of the solution of a Bellman equation yields finally the desired statement. 

**Proof (Proof of Theorem 5):** Since $\pi^{LAQGI}_{lo} \in \Delta_S(A_{lo})$ is a deterministic policy, $V^{\pi_{lo}, \pi_{gl}}_{LAQGI} \geq V^{\pi_{lo}, \pi_{gl}}_{lo, \pi_{gl}}$ follows from Lemma 3.

So, it remains to show $V^{\pi_{lo}, \pi_{gl}}_{LAQGI} \geq V^{\pi_{lo}, \pi_{gl}}_{lo, \pi_{gl}}$. For this sake, notice first that by the similar argumentation as in (13), it holds that $\pi^{LAQGI}_{lo}$ is a deterministic policy, $\pi^{LAQGI}_{lo} \in \Delta_S(A_{lo})$. Therefore, we need only to check the desired inequality with $\pi^{LAQGI}_{lo}$ in place of $\pi_{gl}$.

First, notice that given a deterministic $\pi_{lo} \in \Delta_S(A_{lo})$, one can show that:

$$V^{\pi_{lo}, \pi_{gl}}_{LAQGI} = V^{\pi_{lo}, \pi_{gl}}_{lo, \pi_{gl}},$$

where $\pi^{LAQGI}_{lo} \in \Delta_S(A_{lo})$ is uniformly distributed in $\arg \max_{a_{gl}\in A_{gl}} \tilde{Q}_{\pi_{lo}}(s, a_{gl})$. Thus we need only to check the desired inequality with $\pi^{LAQGI}_{lo}$ in place of $\pi_{gl}$. For this sake, notice first that by the similar argumentation as in (13), it holds that $V^{\pi_{lo}, \pi_{gl}}_{LAQGI}$ is the value function of the policy $\tilde{\pi}_{gl}$ in $(S, A_{gl}, \pi_{lo}, \pi_{gl})$, where:

$$\tilde{\pi}_{lo}(s, a_{lo}) := \mathbb{E}_{A_{gl}\sim \pi^{LAQGI}_{lo}(-|s, a_{lo})}[\tilde{\pi}_{lo}(s, a_{lo}, A_{gl})]$$

$$\tilde{\pi}_{lo}(s, a_{lo}) := \mathbb{E}_{A_{gl}\sim \pi^{LAQGI}_{lo}(-|s, a_{lo})}[\tilde{\pi}_{lo}(s, a_{lo}, A_{gl})],$$

and $V^{\pi_{lo}, \pi_{gl}}_{LAQGI}$ is the value function of the policy $\tilde{\pi}_{gl}$ in $(S, A_{gl}, \pi_{lo}, \pi_{gl})$. Consequently by (21) in Lemma 11, we have:

$$\|V^{\pi_{lo}, \pi_{gl}}_{LAQGI} - V^{\pi_{lo}, \pi_{gl}}_{lo, \pi_{gl}}\|_\infty \leq \max_{s, a_{lo}} \|\tilde{\pi}_{gl}(-|s, a_{lo}) - \eta_{lo}^{gl}(-|s, a_{lo})\|_1 \leq D_{s, a_{lo}} \exp \left(-\frac{C_{s, a_{lo}}}{2\tau}\right),$$

where:

$$D_{s, a_{lo}} := \sqrt{\frac{|A_{gl}| - \arg \max_{a_{gl}\in A_{gl}} \tilde{Q}_{\pi_{lo}}(s, a_{gl})}{\arg \max_{a_{gl}\in A_{gl}} \tilde{Q}_{\pi_{lo}}(s, a_{gl})}}$$

and where:

$$C_{s, a_{lo}} := \min_{a_{gl}\in \arg \max_{a_{gl}\in A_{gl}} \tilde{Q}_{\pi_{lo}}(s, a_{gl})} \left(\max_{a_{gl}\in A_{gl}} \tilde{Q}_{\pi_{lo}}(s, a_{gl}) - \tilde{Q}_{\pi_{lo}}(s, a_{gl})\right).$$

Consequently, we have by combining both previous estimates:

$$\|V^{\pi_{lo}, \pi_{gl}}_{LAQGI} - V^{\pi_{lo}, \pi_{gl}}_{lo, \pi_{gl}}\|_\infty \leq \frac{\|\tilde{\pi}_{lo}\|_\infty D}{(1 - \beta_{lo})^2} \exp \left(-\frac{C}{\tau}\right)$$

By similar argumentation:

$$\|V^{\pi_{lo}, \pi_{gl}}_{LAQGI} - V^{\pi_{lo}, \pi_{gl}}_{lo, \pi_{gl}}\|_\infty \leq \frac{\|\tilde{\pi}_{lo}\|_\infty D}{(1 - \beta_{lo})^2} \exp \left(-\frac{C}{\tau}\right)$$

Consequently:

$$V^{\pi_{lo}, \pi_{gl}}_{LAQGI} + \epsilon \geq V^{\pi_{lo}, \pi_{gl}}_{lo, \pi_{gl}} \geq V^{\pi_{lo}, \pi_{gl}}_{lo, \pi_{gl}} \geq V^{\pi_{lo}, \pi_{gl}}_{lo, \pi_{gl}} - \frac{\epsilon}{2},$$

as desired.
E. Auxiliary Statements

Lemma 9: Let $V_i$ be the value function of the discounted MDP $(\mathcal{S}, \mathcal{A}, r, P, \beta)$ with stationary Markov policy $\pi$. It holds:

$$\|V_1 - V_2\|_\infty \leq \|r_1 - r_2\|_\infty + \frac{\beta |r_2|_\infty}{1 - \beta} \max_{s,a} |P_1(\cdot|s,a) - P_2(\cdot|s,a)|_1.$$ 

Proof: We denote:

$$f_i(s,a) := \mathbb{E}_{S', P(.|s,a)}[V_i(S')]$$

It holds:

$$|V_1(s) - V_2(s)| = \|\pi(.|s)r_1(s, \cdot) - r_2(s, \cdot) + \beta[f_1(s, \cdot) - f_2(s, \cdot)]\|_\infty \leq \|\pi(.|s)\|_1 \|r_1(s, \cdot) - r_2(s, \cdot) + \beta[f_1(s, \cdot) - f_2(s, \cdot)]\|_\infty \leq \|r_1(s, \cdot) - r_2(s, \cdot)\|_\infty + \beta|f_1(s, \cdot) - f_2(s, \cdot)|_\infty,$$

where the first inequality follows from Hölder’s inequality and the second inequality follows from the fact that $\pi(\cdot|s)$ is a probability distribution and from the triangle inequality. Taking the maximum over $s \in \mathcal{S}$ on the both sides of above inequality, it yields:

$$\|V_1 - V_2\|_\infty \leq \|r_1 - r_2\|_\infty + \beta|f_1 - f_2|_\infty.$$

Now, we compute:

$$|f_1(s,a) - f_2(s,a)| \leq \|P_1(.|s,a) - V_1 - V_2\| + \|P_1(.|s,a) - P_2(.|s,a), V_2\| \leq \|P_1(.|s,a)\|_1 \|V_1 - V_2\|_\infty + \|V_2\|_\infty \|P_1(.|s,a) - P_2(.|s,a)\|_1.$$ (16)

Clearly, we have that $\|P_1(.|s,a)\|_1 = 1$. Furthermore, it yields:

$$|V_2(s)| = \left| \mathbb{E}_{\pi} \left( \sum_{t=0}^{\infty} \beta^t r_2(S_t, A_t) \right) \right| \leq \mathbb{E}_{\pi} \left( \sum_{t=0}^{\infty} \beta^t |r_2(S_t, A_t)| \right) \leq \|r_2\|_\infty \sum_{t=0}^{\infty} \beta^t = \frac{\|r_2\|_\infty}{1 - \beta},$$

and thus $\|V_2\|_\infty \leq \|r_2\|_\infty/(1 - \beta)$. By previous observations, we can continue the estimate (17):

$$|f_1 - f_2|_\infty \leq \|V_1 - V_2\|_\infty + \frac{\|r_2\|_\infty}{1 - \beta} \max_{s,a} \|P_1(.|s,a) - P_2(.|s,a)\|_1,$$

and obtain:

Setting this into (16) and taking the maximum over $s$, we have:

$$\|V_1 - V_2\|_\infty \leq \|r_1 - r_2\|_\infty + \beta \left[ \|V_1 - V_2\|_\infty + \frac{\|r_2\|_\infty}{1 - \beta} \max_{s,a} \|P_1(.|s,a) - P_2(.|s,a)\|_1 \right].$$

Therefore:

$$\|V_1 - V_2\|_\infty \leq \frac{\|r_1 - r_2\|_\infty + \frac{\beta \|r_2\|_\infty}{1 - \beta} \max_{s,a} \|P_1(.|s,a) - P_2(.|s,a)\|_1}{1 - \beta}.$$ 

Lemma 10: Let $\mathcal{X}$ be a finite set, $f : \mathcal{X} \to \mathbb{R}$, and $\mathcal{X}^* := \arg \max f$. Consider the Boltzmann distribution with the inverse temperature parameter $\tau > 0$ and the potential $f$:

$$(\Phi_\tau(f))(x) = \frac{\exp \left( \frac{f(x)}{\tau} \right)}{\sum_{x'} \exp \left( \frac{f(x')}{\tau} \right)},$$

and the uniform distribution Uni($\mathcal{X}^*$) on the set of maximizer of $f$. It holds:

$$\|\Phi_\tau(f) - \text{Uni}(\mathcal{X}^*)\|_1 \leq \text{D} \exp \left( - \frac{\mathcal{C}}{\tau} \right).$$
where $C, D > 0$ given by:

$$C := \min_{x' \neq X^*_f} \left[ \max f - f(x') \right] \quad \text{and} \quad D := \left\lceil \frac{|X| - |X^*_f|}{|X^*_f|} \right\rceil$$

**Proof:** First we compute the Kullback-Leibler divergence from $\Phi_\tau(f)$ to $\text{Uni}(X^*_f)$:

$$\text{D}(\text{Uni}(X^*_f) | \Phi_\tau(f)) = E_{X \sim \text{Uni}(X^*_f)} \left[ \log \left( \frac{\text{Uni}(X)}{\Phi_\tau(f)(X)} \right) \right]$$

$$= \frac{1}{|X^*_f|} \sum_{x \in X^*_f} \log \left( \frac{\sum_{x' \neq X^*_f}^{x \neq X^*_f} \exp \left( \frac{f(x')}{\tau} \right)}{\sum_{x' \neq X^*_f}^{x \neq X^*_f} \exp \left( \frac{f(x)}{\tau} \right)} \right)$$

$$= \frac{1}{|X^*_f|} \sum_{x \in X^*_f} \log \left( \frac{A + B}{\sum_{x' \neq X^*_f}^{x \neq X^*_f} \exp \left( \frac{f(x)}{\tau} \right)} \right)$$

$$= - \log(|X^*_f|) - \frac{\sum_{x' \neq X^*_f}^{x \neq X^*_f} f(x)}{\tau} \log \left( \frac{A}{\sum_{x' \neq X^*_f}^{x \neq X^*_f} \exp \left( \frac{f(x')}{\tau} \right)} \right)$$

$$= - \log(|X^*_f|) - \frac{\max f}{\tau} \log(A) + \log \left( \frac{B}{A} \right) + \frac{B}{A}$$

where:

$$A := \sum_{x' \neq X^*_f} \exp \left( \frac{f(x')}{\tau} \right) \quad \text{and} \quad B := \sum_{x \neq X^*_f} \exp \left( \frac{f(x)}{\tau} \right).$$

The last inequality in above computation follows from the inequality $\log(x) \leq x - 1$, for all $x > 0$. Notice that $A = |X_f\setminus\exp(\max f/\tau)$. Thus we continue above estimation:

$$\text{D}(\text{Uni}(X^*_f) | \Phi_\tau(f))$$

$$\leq - \log(|X^*_f|) - \frac{\max f}{\tau} + \log(|X^*_f|) + \frac{B}{A}$$

$$= \frac{B}{A} \sum_{x' \neq X^*_f} \exp \left( \frac{f(x')}{\tau} \right) = \frac{|X| - |X^*_f|}{|X^*_f|} \exp \left( \frac{\max_{x' \neq X^*_f} \left[ f(x') - \max f \right]}{\tau} \right).$$

Now we apply the Pinsker’s inequality to obtain the desired statement:

$$\|\text{Uni}(X^*_f) - \Phi_\tau(f)\|_1 \leq \sqrt{2\text{D}(\text{Uni}(X^*_f) | \Phi_\tau(f))}$$

$$\leq \sqrt{\frac{2|X| - |X^*_f|}{|X^*_f|} \exp \left( \frac{\max_{x' \neq X^*_f} \left[ f(x') - \max f \right]}{2\tau} \right)}.$$ 

**Lemma 11:** Let be $\pi_{gl}^{(1)}, \pi_{gl}^{(2)} \in \Delta(A_{gl})$, and define for any $i \in \{1, 2\}$:

$$r_{lo}^{(i)}(s, a_{lo}) := E_{A_{gl} \sim \pi_{gl}^{(i)}} \left[ r_{lo}(s, a_{lo}, A_{gl}) \right]$$

$$P_{lo}^{(i)}(s'|s, a_{lo}) := E_{A_{gl} \sim \pi_{gl}^{(i)}} \left[ P(s'|s, a_{lo}, A_{gl}) \right]$$

Then it holds:

$$\|r_{lo}^{(1)} - r_{lo}^{(2)}\|_\infty \leq \|r_{lo}\|_\infty \|\pi_{gl}^{(1)} - \pi_{gl}^{(2)}\|_1 \leq \|r_{lo}^{(1)} - r_{lo}^{(2)}\|_1 \leq \|\pi_{gl}^{(1)} - \pi_{gl}^{(2)}\|_1$$

(18)

$$\|P_{lo}^{(1)}(\cdot|s, a_{lo}) - P_{lo}^{(2)}(\cdot|s, a_{lo})\|_1 \leq \|\pi_{gl}^{(1)} - \pi_{gl}^{(2)}\|_1$$

(19)
Moreover, let $T^{(1)}$ be the greedy Bellman operator of the discounted MDP $(\mathcal{S}, \mathcal{A}, r^{(1)}, P^{(1)}, \beta)$ with the policy $\pi \in \Delta_{\mathcal{S}}(\mathcal{A})$. Then it holds:

$$
\|T^{(1)}Q - T^{(2)}Q\|_\infty \leq (\|r^{(1)}\|_\infty + \beta \|Q\|_\infty) \|\pi^{(1)}_g - \pi^{(2)}_g\|_1.
$$

(20)

Let be $V^{(1)}_g$ be the value function of the policy $\pi_g \in \Delta_{\mathcal{S}}(\mathcal{A})$ in the discounted MDP $(\mathcal{S}, \mathcal{A}, r^{(1)}, P^{(1)})$. Then:

$$
\|V^{(1)}_g - V^{(2)}_g\|_\infty \leq \frac{\|r^{(1)}\|_\infty}{(1 - \beta)^2} \|\pi^{(1)}_g - \pi^{(2)}_g\|_1.
$$

(21)

**Proof:** The inequality (18) follows from Hölder’s inequality:

$$
\|r^{(1)}_g(s, a, \omega) - r^{(2)}_g(s, a, \omega)\| \leq \|r^{(1)}_g - r^{(2)}_g\|_X \|\pi^{(1)}_g - \pi^{(2)}_g\|_1 \leq \|r^{(1)}_g\|_X \|\pi^{(1)}_g - \pi^{(2)}_g\|_1
$$

Similarly, we obtain the inequality (19) by the following computation:

$$
\|P^{(1)}_g(s'|s, a, \omega) - P^{(2)}_g(s'|s, a, \omega)\| = \|\pi^{(1)}_g - \pi^{(2)}_g\|_1 \leq \|\pi^{(1)}_g - \pi^{(2)}_g\|_1.
$$

Now, we show (20). For any Q:

$$
\left| (T^{(1)}Q)(s, a, \omega) - (T^{(2)}Q)(s, a, \omega) \right|
\leq \|r^{(1)}_g(s, a, \omega) - r^{(2)}_g(s, a, \omega)\| + \beta \sum_{s'} \left( P^{(1)}_g(s'|s, a, \omega) - P^{(2)}_g(s'|s, a, \omega) \right) \mathbb{E}_{A' \sim \pi(\cdot|s')} [Q(s', A')]
$$

By means of Hölder’s inequality we can estimate the second summand in the right hand side of above inequality:

$$
\sum_{s'} \left( P^{(1)}_g(s'|s, a, \omega) - P^{(2)}_g(s'|s, a, \omega) \right) \mathbb{E}_{A' \sim \pi(\cdot|s')} [Q(s', A')]
\leq \|P^{(1)}_g(\cdot|s, a, \omega) - P^{(2)}_g(\cdot|s, a, \omega)\|_1 \max_{s, \omega} \mathbb{E}_{A' \sim \pi(\cdot|s')} [Q(s, A')]
\leq \|P^{(1)}_g(\cdot|s, a, \omega) - P^{(2)}_g(\cdot|s, a, \omega)\|_1 \|Q\|_\infty.
$$

Setting this estimate into the previous inequality and taking maximum over $(s, a, \omega)$, we obtain:

$$
\|T^{(1)}Q - T^{(2)}Q\|_\infty
\leq \|r^{(1)}_g - r^{(2)}_g\|_\infty + \beta \|Q\|_\infty \max_{s, \omega} \|P^{(1)}_g(\cdot|s, a, \omega) - P^{(2)}_g(\cdot|s, a, \omega)\|_1
$$

The desired statement yields by inserting the inequalities (18) and (19) into above estimate.

To show the last inequality (21), notice that by Lemma 9 we have:

$$
\|V^{(1)}_g - V^{(2)}_g\|_\infty
\leq \|r^{(1)}_g - r^{(2)}_g\|_\infty + \beta \|Q\|_\infty \max_{s, \omega} \|P^{(1)}_g(\cdot|s, a, \omega) - P^{(2)}_g(\cdot|s, a, \omega)\|_1.
$$

Clearly, $\|r^{(2)}_g\|_\infty \leq \|r^{(1)}\|$. Setting this estimate and the inequalities (18) and (19) into above inequality, we obtain the desired statement.

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