No-signaling, entanglement-breaking, and localizability in bipartite channels

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A bipartite quantum channel represents the interaction between systems, generally allowing for exchange of information. A special class of bipartite channels are the no-signaling ones, which do not allow for communication. In Ref. [1] it has been conjectured that all no-signaling channels are mixtures of entanglement-breaking and localizable channels, which require only local operations and entanglement. Here we provide the general realization scheme, giving a counterexample to the conjecture.

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Causality is the basic assumption of science, the building block of any mechanism and any prediction scheme [2]. It is the grey eminence of physical theories, taking apparently different forms, such as retarded potentials in classical physics, Minkowskian causality in Relativity, (anti)commutation relations in quantum field theory. The modern paradigm of causality is communication, where we identify the causal relation with information exchange. Causality should not be confused with determinism: indeed, any communication scheme from Alice to Bob can be regarded as a dependence of the outcome probability distributions at Bob’s location on Alice’s choice. It is easy to recognize that such scheme contains all customary definitions of causality, including determinism as a very special case. In synthesis, we define causality as the dependence of a probability distribution on a choice.

In the past quantum entanglement has been claimed as a resource for communication [3], regarding Alice’s choice of local measurement as a way of changing Bob’s probabilities—the spooky action at a distance of Einstein [4]. The impossibility of communicating by local operations—today commonly referred to as no-signaling—is instead an immediate consequence of causality of the theory, as proved in Ref. [5].

In order to have a causal relation between two systems one needs an interaction between the two systems A (Alice) and B (Bob). In Quantum Theory such interaction is represented by a bipartite channel for A and B, with communication from A to B corresponding to the dependence of the local output state of system B on the choice of the input state of system A. Indeed one can generalize the scheme to the case of A and B at the input being different from A’ and B’ at the output, considering the causal relation e. g. from A to A’. More generally we can include the case of one-dimensional systems, thus recovering also the situation of mono-partite channels (the case of both inputs and/or both outputs one-dimensional is uninteresting, since there is no input and/or no output). Notice that causality by definition is a pairwise relation, whence the bipartite channel is the most general interaction scenario. For simplicity we will restrict to finite dimensions, and use the same capital Roman letter to denote the system and the corresponding Hilbert space, writing \( \mathcal{L}(A) \) for the space of operators on A. The graphical representation of the bipartite quantum channel \( \mathcal{C} : \mathcal{L}(A) \otimes \mathcal{L}(B) \rightarrow \mathcal{L}(A') \otimes \mathcal{L}(B') \) is given by the following circuit:

\[
\begin{array}{c}
\begin{array}{c}
A \\
B \\
C \\
A' \\
B' \\
\end{array}
\end{array}
\]

The natural question is now which interactions allow for communication between input and output. In Ref. [6] it has been shown that not every no-signaling channel is localizable, i.e. it can be implemented with local operations using entangled ancillas (see Definition [1]). In the same reference it has been conjectured that all semi-causal channels (namely no-signaling from B to A’, but not necessarily from A to B’) are also semi-localizable, namely they are of the form

\[
\begin{array}{c}
\begin{array}{c}
A \\
B \\
V_1 \\
V_2 \\
A' \\
B' \\
\end{array}
\end{array}
\]

for some system \( E' \) and suitable quantum channels \( V_1 \) and \( V_2 \). Such conjecture has later been proved in Ref. [8]. An alternative proof was given in Ref. [1], where the authors also proposed the following

Conjecture 1 All no-signaling channels are mixtures of entanglement-breaking and localizable channels.

We will show that Conjecture [1] is false. We will also provide the general realization scheme for the no-signaling bipartite channel, along with a concrete counterexample to Conjecture [1].

We will stick on the graphical representation of a bipartite quantum channel in Eq. (1). By “quantum channel” we mean a completely positive, trace-preserving map between the density-matrix space of the input systems and that of the output systems.
The preparation of a state $\rho$ and measurement of a POVM $\{P_x\}$ on some system $A$ are special classes of channels, graphically represented as

$$\rho \rightarrow A, \quad A \rightarrow P_x. \quad (3)$$

We will use the bijection between states and operators

$$A = \sum_{m} A_{mn} |m\rangle \langle n| \leftrightarrow |A\rangle = \sum_{mn} A_{mn} |m\rangle \langle n| \quad (4)$$

summarized by the identity

$$|A\rangle = (A \otimes I) |I\rangle, \quad (5)$$

where $|I\rangle = \sum_n |n\rangle |n\rangle$ is the (unnormalized) maximally entangled state. It will also be useful to introduce the Choi-Jamiolkowski isomorphism between channels $C : \mathcal{L}(A) \rightarrow \mathcal{L}(B)$ and positive operators on $B \otimes A$

$$R_C := C \otimes I_B(|I\rangle \langle I|)$$

$$C(\rho) = \text{Tr}[(I \otimes \rho^T) R_C], \quad (6)$$

where $\rho^T$ denotes the transposition of the operator $\rho$ with respect to the orthonormal basis in Eq. (4).

We are now in position to make the above mentioned concepts more precise:

**Definition 1** The channel $C : \mathcal{L}(A) \otimes \mathcal{L}(B) \rightarrow \mathcal{L}(A') \otimes \mathcal{L}(B')$ is “localizable” if it can be realized by local operations on $A$ and $B$ with a shared entangled ancilla $\frac{1}{\sqrt{d}} |I\rangle$ on a couple of $d$-dimensional systems $E_A, E_B$ but without communication:

$$\begin{array}{ccc}
A & \rightarrow & A' \\
B & \rightarrow & B' \\
C & \rightarrow & C' \\
\end{array} \quad \begin{array}{ccc}
A & \rightarrow & A' \\
B & \rightarrow & B' \\
E_A & \rightarrow & E_A' \\
E_B & \rightarrow & E_B' \\
C & \rightarrow & C' \\
\end{array} \quad . \quad (7)$$

**Definition 2** A bipartite quantum channel $C : \mathcal{L}(A) \otimes \mathcal{L}(B) \rightarrow \mathcal{L}(A') \otimes \mathcal{L}(B')$ is “$A \rightarrow B'$ no-signaling” if $\text{Tr}_{A'}[R_C] = I_A \otimes S_{BB'}$ where $S_{BB'}$ is the Choi operator of some channel $S : \mathcal{L}(B) \rightarrow \mathcal{L}(B')$. We say that $C$ is “no-signaling” if it is both $A \rightarrow B'$ no-signaling and $B \rightarrow A'$ no-signaling.

The following theorem holds

**Theorem 1** The following are equivalent:

1. The channel $C : \mathcal{L}(A) \otimes \mathcal{L}(B) \rightarrow \mathcal{L}(A') \otimes \mathcal{L}(B')$ is no-signaling

2. There are equivalent $d$-dimensional quantum systems $E_A, E_B$, instruments $\{C_A^{(x)}\}_{x \in X}$ and $\{D_B^{(y)}\}_{y \in Y}$ with outcome space $X$, and channels $C_A^{(x)} : D_B^{(x)}$ for each $x \in X$ with

$$\begin{align*}
C_A^{(x)} : & \mathcal{L}(A) \otimes \mathcal{L}(E_A) \rightarrow \mathcal{L}(A') \\
C_B^{(x)} : & \mathcal{L}(B) \otimes \mathcal{L}(E_B) \rightarrow \mathcal{L}(B') \\
D_B^{(x)} : & \mathcal{L}(B) \otimes \mathcal{L}(E_B) \rightarrow \mathcal{L}(B') \\
D_A^{(x)} : & \mathcal{L}(A) \otimes \mathcal{L}(E_A) \rightarrow \mathcal{L}(A')
\end{align*} \quad (8)$$

such that

$$\begin{align*}
C & = \sum_{x \in X} C_A^{(x)} \circ C_B^{(x)} (d^{-1} |I\rangle \langle I|) \langle E_A E_B |)
= \sum_{x \in X} D_A^{(x)} \circ D_B^{(x)} (d^{-1} |I\rangle \langle I|) \langle E_A E_B |), \quad \quad \quad (9)
\end{align*}$$

namely, $C$ has the two equivalent circuit realizations

$$\begin{array}{cccc}
A & \rightarrow & A' \\
E_A & \rightarrow & E_A' \quad , \quad (10) \\
E_B & \rightarrow & E_B' \\
C & \rightarrow & C' \\
\end{array} \quad \quad \begin{array}{cccc}
A & \rightarrow & A' \\
E_A & \rightarrow & E_A' \quad , \quad (11) \\
E_B & \rightarrow & E_B' \\
D_A & \rightarrow & D_A' \\
\end{array}$$

**Proof.**

Proof of (1) $\Rightarrow$ (2).

$C$ is $B \rightarrow A'$ no-signaling, therefore it can be realized as in Eq. (2), where $E'$ is a $d'$-dimensional system. This system can be teleported using the entangled state $\frac{1}{\sqrt{d}} |I\rangle \langle I| \rangle$ of systems $E_A' E_B'$, the Bell measurement $|B_x\rangle$ on systems $E'$ and $E_A'$, and classical communication of the outcome $x$ followed by a controlled unitary $U_x$ on system $E_B'$, corresponding to the circuit

$$\begin{array}{cccc}
A & \rightarrow & A' \\
E_A & \rightarrow & E_A' \quad , \quad (12) \\
E_B & \rightarrow & E_B' \\
B & \rightarrow & B' \\
\end{array}$$

(the double wire represents the classical communication of the outcome $x$ of the measurement).

The quantum operation $C_A^{(x)}$ and the channel $C_B^{(x)}$ are the grouped circuital elements in Eq. (12), and are given by

$$\begin{align*}
C_A^{(x)}(\rho) & := \langle B_x | (\mathcal{I} \otimes E_A') \rho | B_x \rangle \\
C_B^{(x)}(\rho) & := \mathcal{O}_2(U(x) \otimes I_B) \rho (U(x) \otimes I_B)^{\dagger}.
\end{align*} \quad (13)$$
The final circuit is thus
\[ C = \begin{array}{cccc}
A & E_{A}^{\prime} & C_{A}^{(x)} & A' \\
B & W_{1} & C_{B}^{(x)} & B'
\end{array} \quad (14) \]

Since the channel \( C \) is also \( A \rightarrow B' \) no-signaling, the same argument gives:
\[ C = \begin{array}{cccc}
A & E_{A}^{\prime} & D_{A}^{(x)} & A' \\
B & W_{1} & D_{B}^{(x)} & B'
\end{array} \quad (15) \]

with \( D_{A}^{(x)} \) and \( D_{B}^{(x)} \) given by
\[
D_{B}^{(x)}(\rho) := \langle B_{x} | \{ W_{1} \otimes I_{E_{B}^{\prime}} \}(\rho) | B_{x} \rangle
\]  
\[
D_{A}^{(x)}(\rho) := W_{2}(U(\rho x) \otimes I_{A}) \rho (U(\rho x) \otimes I_{A})^\dagger.
\]  

We obtain the statement by defining \( E_{A} \) and \( E_{B} \) as \( d \)-dimensional systems, where \( d := \max\{d', d''\} \), and embedding \( E_{J}^{\prime} \) and \( E_{J}'' \) in \( E_{J} \), for \( J = A, B \).

Proof of (2) \( \Rightarrow \) (1).

Suppose that \( C \) admits the realization circuit given in Eq. (10). We can group \( E_{B} \) and \( X \) in the composite system \( E' \). Then \( C \) is also of the form of Eq. (2), thus being \( B \rightarrow A' \) no-signaling, as proved in Ref. [7, 8]. In the same way, exploiting the second realization circuit in Eq. (11), one can prove that \( C \) is also \( A \rightarrow B' \) no-signaling.

Theorem 1 shows that the most general no-signaling channel differs from a localizable channel because it also admits a single round of classical communication, with the constraint that it must be possible to implement the channel exploiting communication in either directions.

We now provide a counterexample to Conjecture 1 in terms of a no-signaling channel that is atomic, (i.e. it cannot be written as a convex combination of different channels whence also of no-signaling channels) and that is neither entanglement-breaking nor localizable. Let \( A, B, X_{A}, X_{B}, W_{A}, W_{B} \) be qubits. We define the channel \( \mathcal{R}_{\alpha} \) depending on \( \alpha, 0 \leq \alpha \leq 1 \):
\[
\mathcal{R}_{\alpha} = \begin{array}{cccc}
A & X_{A} & E & A' \\
B & X_{B} & E & B'
\end{array} \quad (17)
\]

where \( E \) is the swap operator, \( |\Psi_{\alpha}\rangle := \sqrt{\alpha} |00\rangle + \sqrt{1 - \alpha} |11\rangle \), the two-qubit gate in the dashed box is a controlled-\( \sigma_{x} \) given by \( \Sigma_{AW_{\alpha}} := |1\rangle_{W_{\alpha}} \otimes (\sigma_{x})_{A} + |0\rangle_{W_{\alpha}} \otimes I_{A} \) classically controlled by the outcomes of the measurements on the computational basis (represented by the circuitual element \( 0/1 \)). Notice that the classical control works as a logical AND, implying that the box \( \Sigma_{AW_{\alpha}} \) is performed if and only if both outcomes of the measurements \( 0/1 \) are equal to 1.

We notice that circuit \( \mathcal{R}_{\alpha} \) in Eq. (17) is implemented using local operations, entanglement, and one round of classical communication from Bob to Alice, thus being of the form of Eq. (11). One can verify that \( \mathcal{R}_{\alpha} \) can be equivalently realized applying the controlled-\( \sigma_{x} \) on systems \( B \) and \( W_{B} \) as follows
\[
\mathcal{R}_{\alpha} = \begin{array}{cccc}
A & X_{A} & E & A' \\
B & X_{B} & E & B'
\end{array} \quad (18)
\]

Consequently \( \mathcal{R}_{\alpha} \) also admits a realization of the form given in Eq. (11). By Theorem 1, we can conclude that this is a no-signaling channel. The Choi-Jamio\-kowski operator of \( \mathcal{R}_{\alpha} \) is:
\[
\mathcal{R}_{\alpha} = \sum_{m,n=0}^{1} |K_{mn}^{\alpha}\rangle \langle K_{mn}^{\alpha}| \quad (19)
\]

with
\[
|K_{mn}^{\alpha}\rangle = \langle (\Sigma_{AW_{\alpha}})^{mn} \otimes |m\rangle_{X_{A}} \langle n|_{X_{B}} |\Phi_{\alpha}\rangle | \quad (20)
\]

and
\[
|\Phi_{\alpha}\rangle = (E \otimes I_{AB}) \left( |I\rangle_{AB} \otimes \frac{1}{\sqrt{2}} |I\rangle_{X_{A}X_{B}} \otimes |\Psi_{\alpha}\rangle \right). \quad (21)
\]
where $E$ denotes the tensor product of the two controlled-swaps.

Using Mathematica, we prove that $R_{\alpha}$ with $\alpha := 1/6$ is a counterexample by showing that it satisfies the following properties: (1) It is not entanglement-breaking, (2) It is not localizable (3) It is atomic.

Proof of (1) $R_{\alpha}$ is not entanglement breaking. A channel is entanglement breaking if and only if the corresponding Choi-Jamiolkowski operator is separable. Thus, we can prove that $R_{\alpha}$ is not entanglement breaking by showing that $R_{\alpha}$ violates the Peres-Horodecki criterion for separability [6, 11]. According to the criterion, if a state is separable it has a positive definite partial transpose. Numerically one can check that $R_{\alpha}$ has a partial transpose with negative eigenvalues, whence we conclude that it is entangled and $R_{\alpha}$ is not entanglement-breaking.

Proof of (2) $R_{\alpha}$ is not localizable. If $R_{\alpha}$ were localizable (see Eq. (23)), the following observables $A_{n}, B_{m}$

\begin{align*}
(\sigma_{z}^{A} \otimes |n⟩) \rightarrow σ_{z}^{A} \otimes σ_{z}^{B} \otimes |m⟩ \rightarrow (\sigma_{A}^{B} \otimes |I⟩) \rightarrow (\sigma_{B}^{B} \otimes |O⟩) = 2 \sqrt{2}.
\end{align*}

We have that

\begin{align*}
(⟨A_{n}B_{m}⟩) = Tr[(σ_{z}^{A} \otimes |n⟩) ⟨n⟩_{A} \otimes σ_{z}^{B} \otimes |m⟩ ⟨m⟩_{B} \otimes I_{W_{A}W_{B}}] R_{α}
\end{align*}

whence (using expression in Eq. (19) for $R_{α}$) one finds $c_{α} = |4 - 6α|$. Since $c_{α} = 3 > 2\sqrt{2}$, the Cirel’son bound is violated and $R_{α}$ cannot be localizable.

Proof of (3) $R_{α}$ is extremal. One can check that the matrices \{$K_{m}^{α} K_{m'}^{α'}$\} are linearly independent. By Choi’s theorem on extremality [12] the channel $R_{α}$ is extremal.

For a multipartite channel satisfying two different no-signalling conditions, an analog of Theorem 1 holds. In fact, let us consider a channel $C$ with input systems labelled by a set of indices $I$ and output systems labelled by a set $O$. Suppose that $C$ satisfies the following no-signalling conditions

\begin{align*}
Tr_{O'}[R_{C}] = I_{O'} \otimes S_{O \Delta O'}
\end{align*}

for certain subsets $I', O'' \subseteq I$ and $O', O'' \subseteq O$, where $\overline{S}$ represents the set complement of $S$, and for suitable Choi-Jamiolkowski operators $S$ and $T$. Following the proof of Theorem 1 we can show that two circuits realizing $C$ are

\begin{align*}
\begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\
\mathcal{C}_{A} & \mathcal{C}_{B} & \mathcal{C}_{C} & \mathcal{C}_{D} \\
\mathcal{D}_{A} & \mathcal{D}_{B} & \mathcal{D}_{C} & \mathcal{D}_{D} \\
\end{array}
\end{align*}

In general the subsets $I', O''$ are not a partition of $I$. In this case we have that the circuits cannot be realized partitioning the systems between the two local parties $A$ and $B$. In particular the input systems in $I' \cap \overline{O'}$ are assigned to the party which sends the classical message, and input systems in $I' \cap \overline{O''}$ are assigned to the party which receives the classical message (and similarly for output systems). One can also consider more complex scenarios, i.e. channels with more than two no-signaling conditions of the kind in Eq. (25), or channels with nested conditions, for example when the Choi-Jamiolkowski operators $S$ and $T$ in Eq. (25) satisfy no-signaling conditions on their own. However the analysis of the classical communication required in these cases is complicated, and is left as an open problem.

In conclusion, we have provided the general realization scheme of no-signaling channels, giving a counterexample to the conjecture of Ref. [1], stating that such channels are mixtures of entanglement-breaking and localizable channels. The general realization scheme looks counter-intuitive, due to the presence of classical communication. However, the nontrivial constraint is the fact that an equivalent scheme must exist, with communication in the reverse direction, and it is remarkable that this constraint is sufficient to make the channel no-signaling.

The result has an intrinsic foundational relevance, involving the pivotal role of causality in theoretical physics and computer science.

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