ON SOLUTIONS WITH COMPACT SPECTRUM TO NONLINEAR KLIE\-N–GORDON AND SCHR\-ÖDINGER EQUATIONS∗

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Abstract. We consider finite energy solutions to the nonlinear Schrödinger equation and nonlinear Klein–Gordon equation and find the condition on the nonlinearity so that the standard, one-frequency solitary waves are the only solutions with compact spectrum. We also construct an example of a four-frequency solitary wave solution to the nonlinear Dirac equation in three dimensions.

Key words. Multifrequency solitary waves, compact spectrum, nonlinear Klein–Gordon equation, nonlinear Schrödinger equation, soliton resolution conjecture, Titchmarsh convolution theorem

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1. Introduction. This article contains an improvement of the result obtained in [Com19] on non-existence of nontrivial solutions of compact spectrum to nonlinear Schrödinger and Klein–Gordon equations. We consider nonlinear Schrödinger and nonlinear Klein–Gordon equations,

\[ \begin{align*}
    i\partial_t u &= -\Delta u + \alpha(|u|^2)u, \\
    -\partial_t^2 u &= -\Delta u + m^2 u + \alpha(|u|^2)u,
\end{align*} \tag{1.1} \]

with \( u(x, t) \in \mathbb{C}, x \in \mathbb{R}^n, n \in \mathbb{N}. \) The nonlinearity in (1.1) is represented by a function \( \alpha \in C^1(\mathbb{R}, \mathbb{R}), \alpha(0) = 0. \) These \( U(1) \)-invariant equations are known to admit solitary wave solutions of the form

\[ u(x, t) = \phi(x)e^{-i\omega t}, \quad \omega \in \mathbb{R}, \tag{1.2} \]

with \( \phi(x) \) decaying at infinity [Str77, BL83]. Our aim is to prove under which conditions the one-frequency solitary waves are the only finite energy solutions with compact spectrum, defined as follows.

Definition 1.1. Let \( u \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}), \) and let \( \hat{u}(x, \omega) = \int_{\mathbb{R}} e^{i\omega t} u(x, t) dt \) be its partial Fourier transform in time. We say that the spectrum of \( u \) is compact if there is a bounded interval \( I \subset \mathbb{R} \) such that

\[ \text{supp } \hat{u} \subset \mathbb{R}^n \times I. \]

In [Com19], we proved that if the nonlinearity in (1.1) is represented by a function \( \alpha(\tau) \) which is either a polynomial or an algebraic function satisfying certain restrictions, and moreover satisfying the growth estimate

\[ |\alpha(\tau)| \leq C(1 + |\tau|^\kappa), \quad \forall \tau \geq 0, \text{ with } \kappa \text{ satisfying } \begin{cases} \kappa > 0, & n \leq 2, \\ 0 < \kappa \leq 2/(n-2), & n \geq 3, \end{cases} \]

then the only finite energy solutions with compact spectrum are solitary waves (1.2). The result was based on the Titchmarsh theorem for partial convolutions (see [Com19, Theorem 2]). In the present article, we show that under a slightly stronger restriction

\[ 0 < \kappa < 2/(n-2), \quad n \geq 3, \]

the only finite energy solutions with compact spectrum are solitary waves (1.2).
the proof can be simplified, and extend the result to a larger class of algebraic functions
\( \alpha(\tau) \); see Theorem 3.7 below. In the essence, we prove that if 
\( 0 < \kappa < 2/(n - 2) \), then solutions 
\( u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n)) \) with compact spectrum have improved regularity, 
\( u \in C^\infty_b(\mathbb{R}, L^q(\mathbb{R}^n) \cap C^{1,\alpha}(\mathbb{R}^n)) \), with any 
\( 2 \leq q \leq \infty \) and any \( \alpha \in (0, 1) \) (see Theorem 3.5 below). This allows one to consider a wider class of algebraic nonlinearities and to base the argument on a simple version of the Titchmarsh theorem for partial convolution in the case of continuous functions (see Theorem 2.6 below).

We mention that there could be multifrequency solitary wave solutions of the form 
\( \sum_{j=1}^N \phi_j(x)e^{-i\omega_j t} \), which are known to exist in similar models. In particular, there are multifrequency solitary waves in the Klein–Gordon equation with the mean-field self-interaction [KK09] and with several nonlinear oscillators [KK10b]. Bi-frequency solitary waves can exist in systems of nonlinear Schrödinger equations [BSS+12] and in the Soler model and Dirac–Klein–Gordon model with Yukawa self-interaction [BC18]. There are one-, two-, and four-frequency solitary wave solutions to the Klein–Gordon equation in discrete time-space coupled with a nonlinear oscillator [Com13]. In Appendix A, we give an example of a four-frequency solitary wave solution to the nonlinear Dirac equation.

The question of existence of multifrequency solitary waves and more generally the solutions of compact spectrum is related to the soliton resolution conjecture, which proposes that the long-time asymptotics of any finite energy solution to a nonlinear dispersive system is given by a superposition of outgoing solitary waves and an outgoing dispersive wave; see [Kom03, Sof06, Tao07, KK07] The related results for the nonlinear wave equation with the critical nonlinearity are obtained in [DKM16, DJKM17]. One strategy to attack this problem was proposed in [Kom03]:

1. Prove that any “omega-limit” radiationless solution of finite energy has a compact spectrum;
2. Prove that any solution with compact spectrum has a spectrum consisting of a single point, and hence is a solitary wave.

Both steps of the program were accomplished for several models without translation invariance, such as the Klein–Gordon and Dirac equations with several nonlinear oscillators and with the mean field self-interaction [KK09, KK10a, KK10b, Com12, Com13]. See also the review [Kom16]. While presently we can not prove that any radiationless solution of a sufficiently general system has a compact spectrum (this seems to be a hard task), in this article we prove that, under certain assumptions on the nonlinearity, any solution with a compact spectrum is a single-frequency solitary wave, completing the second, easier step of the program proposed in [Kom03]. In particular, our result excludes the existence of multifrequency solitary waves under rather general assumptions.

Remark 1.2. There are solutions to the sine-Gordon equation known as breathers, exponentially localized in space and are periodic in time, whose spectrum is not compact; see e.g. [AKNS73]. We point out that the nonlinearity in this equation is non-algebraic; our results on non-existence of solutions with compact spectrum do not apply to such systems. Similarly, the cubic nonlinear Schrödinger equation admits breather-type solutions [AEK87] with the noncompact spectrum; their charge and energy are infinite, so again our results do not cover this case.

We give the necessary results on the Titchmarsh theorem for partial convolution in Section 2. In Section 3, we derive the regularity results for the solutions with
compact spectrum (Theorem 3.5). Then in Theorem 3.7 we prove that the nonlinear
Schrödinger and Klein–Gordon equations with a certain class of nonlinearities do not
admit multifrequency solitary wave solutions. An example of a four-frequency solitary
wave solution to the nonlinear Dirac equation is presented in Appendix A.

2. Titchmarsh theorem for partial convolution. The Titchmarsh convolution
theorem [Tit26] states that \( \sup \text{supp } \phi \ast \psi = \sup \text{supp } \phi + \sup \text{supp } \psi \), for any
\( \phi, \psi \in \mathcal{D}'(\mathbb{R}) \), where \( \mathcal{D}'(\mathbb{R}) \) is the space of distributions with compact support (dual
to the space \( \mathcal{D}(\mathbb{R}) \) which is \( C^\infty(\mathbb{R}) \) with the seminorms \( \sup_\omega |f^{(k)}(\omega)| \)). We need
a version of this theorem for a partial convolution with respect to only a subset of
variables; for the proofs, we refer to [Com19].

**Lemma 2.1.** For any function \( \mu : \mathbb{R}^n \to \mathbb{R} \), \( n \geq 1 \), there is a maximal lower
semicontinuous function on \( \mathbb{R}^n \) which does not exceed \( \mu \); we denote this function by
\( \mu^L(x) \). Similarly, there is a minimal upper semicontinuous function on \( \mathbb{R}^n \) which is
not exceeded by \( \mu \); we denote this function by \( \mu^U(x) \). For any \( \mu, \nu : \mathbb{R}^n \to \mathbb{R} \),
\begin{align}
&\mu^L \leq \mu \leq \mu^U, \\
&\mu^L + \nu^L \leq \mu^U + \nu^U.
\end{align}

We recall that the space of distributions \( \mathcal{D}'(\mathbb{R}^n) \) is defined as the dual to \( \mathcal{D}(\mathbb{R}^n) = 
C^\infty_{\text{comp}}(\mathbb{R}^n) \) (with the standard seminorms), while \( \mathcal{D}'(\mathbb{R}^n) \) is the space of distributions
with compact support (the dual to \( C^\infty(\mathbb{R}^n) \)).

**Definition 2.2.** Let \( f \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) \). We define the functions \( A_f \) and \( B_f \) by
\begin{align*}
A_f : \mathbb{R}^n \to \mathbb{R} \sqcup \{ \pm \infty \}, & \quad x \mapsto \inf \{ \omega \in \mathbb{R} ; (x,\omega) \in \text{supp } f \}; \\
B_f : \mathbb{R}^n \to \mathbb{R} \sqcup \{ \pm \infty \}, & \quad x \mapsto \sup \{ \omega \in \mathbb{R} ; (x,\omega) \in \text{supp } f \}.
\end{align*}

It follows that \( A_f \) is lower semicontinuous, while \( B_f \) is upper semicontinuous:
\[ A_f(x) = A_f^L(x), \quad B_f(x) = B_f^U(x), \quad \forall x \in \mathbb{R}^n. \]

**Definition 2.3.** Let \( f \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) \). We denote by \( \Sigma_f \subset \mathbb{R}^n \) the projection of
\( \text{supp } f \subset \mathbb{R}^n \times \mathbb{R} \) onto the first factor:
\[ \Sigma_f = \{ x \in \mathbb{R}^n ; \{ x \times \mathbb{R} \} \cap \text{supp } f \neq \emptyset \} \subset \mathbb{R}^n. \]

Thus, the following three statements are equivalent:
\[ x \notin \Sigma_f \quad \iff \quad A_f(x) = +\infty \quad \iff \quad B_f(x) = -\infty. \]

**Lemma 2.4.** For \( f \in \mathcal{D}(\mathbb{R}^n, \mathcal{D}')(\mathbb{R}) \), the set \( \Sigma_f \subset \mathbb{R}^n \) is closed.

**Lemma 2.5.** For any distribution \( f \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) \), one has:
\begin{align}
&A_f(x) \leq A_f^U(x) \leq B_f(x), \quad A_f(x) \leq B_f^U(x) \leq B_f(x), \quad \forall x \in \Sigma_f; \\
&(A_f^U)^L \geq A_f, \quad (B_f^U)^U \leq B_f.
\end{align}

For \( f, g \in C^\infty_{\text{comp}}(\mathbb{R}^n \times \mathbb{R}) \), the partial convolution
\[ * : C^\infty_{\text{comp}}(\mathbb{R}^n \times \mathbb{R}) \times C^\infty_{\text{comp}}(\mathbb{R}^n \times \mathbb{R}) \to C^\infty_{\text{comp}}(\mathbb{R}^n \times \mathbb{R}) \]
is defined by

\[(2.5) \quad (f \ast g)(x, \omega) = \int_{\mathbb{R}} f(x, \omega - \tau)g(x, \tau) d\tau, \quad (x, \omega) \in \mathbb{R}^n \times \mathbb{R}.
\]

This operation is continuously extended to \(f, g \in \mathcal{E}'(\mathbb{R}, C(\mathbb{R}^n))\):

\[
\ast : \mathcal{E}'(\mathbb{R}, C(\mathbb{R}^n)) \times \mathcal{E}'(\mathbb{R}, C(\mathbb{R}^n)) \to \mathcal{E}'(\mathbb{R}, C(\mathbb{R}^n)).
\]

**Theorem 2.6.** Let \(f, g \in \mathcal{E}'(\mathbb{R}, C(\mathbb{R}^n))\). One has:

\[
A^U_{f \ast g} = A_f^U + A_g^U, \quad B^U_{f \ast g} = B_f^U + B_g^U.
\]

**Proof.** Since \(f\) and \(g\) depend continuously on \(x\), the Titchmarsh convolution theorem can be applied pointwise in \(x\), yielding

\[(2.6) \quad \inf \text{supp}(f \ast g)(x, \cdot) = \inf \text{supp} f(x, \cdot) + \inf \text{supp} g(x, \cdot), \quad \forall x \in \mathbb{R}^n,
\]

and similarly for sup. Let \(f \in \mathcal{E}'(\mathbb{R}, C(\mathbb{R}^n))\) and let \(\rho \in \mathcal{D}(\mathbb{R})\). If \(\mathcal{O} \subset \mathbb{R}^n\) is an open set such that \(\langle \rho, f(x, \cdot) \rangle = 0\) for all \(x \in \mathcal{O}\), then, by continuity of \(f\) in \(x\), one also has \(\langle \rho, f(x, \cdot) \rangle = 0\) for all \(x\) from the closure of \(\mathcal{O}\). Therefore, given an open set \(\Omega \subset \mathbb{R}\), if \(\Omega \cap \text{supp}\ f(x, \cdot) = \emptyset\) for \(x \in \mathcal{O} \subset \mathbb{R}^n\), then \(\Omega \cap \text{supp}\ f(x, \cdot) = \emptyset\) for \(x\) from the closure of \(\mathcal{O}\); it follows that

\[
A^U_f(x) = \inf \text{supp} f(x, \cdot), \quad B^U_f(x) = \sup \text{supp} f(x, \cdot), \quad \forall x \text{ such that } f(x, \cdot) \neq 0.
\]

Using these relations for each of the terms in (2.6) (and similarly for sup supp) leads to the desired relations. \(\square\)

**Remark 2.7.** In a more general case \(f, g \in \mathcal{E}'(\mathbb{R}, L^2_{\text{loc}}(\mathbb{R}^n))\), there is a result similar to Theorem 2.6 (see [Com19, Theorem 2]):

\[
A_{f \ast g} = (A_f^U + A_g)^L = (A_f + A_g^U)^L, \quad B_{f \ast g} = (B_f^L + B_g)^U = (B_f + B_g^L)^U.
\]

### 3. Compact spectrum solutions to nonlinear Klein–Gordon equation.

**Assumption 3.1.** In (1.1), the nonlinearity is represented by \(f(u) = \alpha(|u|^2)u\), with \(\alpha \in C(\mathbb{R}_+, \mathbb{R})\), \(\alpha(0) = 0\), and there is \(C < \infty\) such that

\[(3.1) \quad |\alpha(\tau)| \leq C(1 + |\tau|^\kappa), \quad \forall \tau \geq 0, \quad \text{where} \quad \begin{cases} \kappa > 0, & n \leq 2; \\ 0 < \kappa < 2/(n - 2), & n \geq 3. \end{cases}
\]

**Assumption 3.2.** \(\alpha \in C(\mathbb{R}_+, \mathbb{R})\) is a non-constant function such that there is \(J \in \mathbb{N}\) and polynomials \(M_j(\tau), 0 \leq j \leq J\), with \(M_j(\tau) \neq 0\) and with

\[(3.2) \quad \deg M_0 > \deg M_j + j, \quad \forall j, \quad 1 \leq j \leq J,
\]

such that \(w(\tau) := \tau \alpha(\tau)\) satisfies the relation

\[
\mathcal{M}(\tau, w(\tau)) = 0, \quad \forall \tau \geq 0, \quad \text{where} \quad \mathcal{M}(\tau, w) := \sum_{j=0}^{J} M_j(\tau)w^j.
\]

We follow the convention that the degree of the zero polynomial equals \(-\infty\), so that (3.2) is formally satisfied when \(M_j(\tau) \equiv 0\) for some \(j \geq 1\).
Example 3.3. Let
\[ \alpha(\tau) = \pm A(\tau)^{1/N}, \]
with \( N \in \mathbb{N}, N \geq 2 \), and with \( A(\tau) = \sum_{j=0}^{a} A_j \tau^j \), a polynomial with real coefficients of degree \( a = \deg A \geq 1 \); if \( N \) is even, we additionally assume that \( A(\tau) \geq 0 \) for \( \tau \geq 0 \). Let \( M_0(\tau) = -(\pm \tau)^N A(\tau) \) and \( M_N(\tau) = 1 \), with \( \deg M_0 = a + N \) and \( \deg M_N = 0 \). Then, for all \( \tau \geq 0 \),
\[ \mathcal{M}(\tau, \tau \alpha(\tau)) = M_0(\tau) + M_N(\tau)(\tau \alpha(\tau))^N = -(\pm \tau)^N A(\tau) \cdot 1 + 1 \cdot (\tau \alpha(\tau))^N = 0. \]
If \( n \leq 2 \), then both Assumptions 3.1 and 3.2 are satisfied for any \( N \in \mathbb{N}, N \geq 2 \), and \( a \in \mathbb{N} \). If \( n \geq 3 \), we additionally need \( \kappa = a/N \) to satisfy \( \kappa < 2/(n - 2) \).

We note that in [Com19], one could only consider the case \( a = 1, N = 2 \) (thus \( \alpha(\tau) = \pm \sqrt{A_0 + A_1 \tau} \), with \( A_0 \geq 0 \) and \( A_1 > 0 \), \( 3 \leq n \leq 6 \)). Since in the present article we do not cover the case \( \kappa = \frac{a}{n} \), our results do not apply to this nonlinearity in dimension \( n = 6 \), the case which is covered by [Com19].

Example 3.4. Let
\[ \alpha(\tau) = \pm \left( \frac{A(\tau)}{B(\tau)} \right)^{1/N}, \]
with \( N \in \mathbb{N} \), and \( A(\tau), B(\tau) \) polynomials with real coefficients of degrees \( a = \deg A \in \mathbb{N}_0 \) and \( b = \deg B \in \mathbb{N}_0 \); \( B(\tau) \neq 0 \) for \( \tau \geq 0 \). We assume that either \( A(\tau)/B(\tau) \geq 0 \) for all \( \tau \geq 0 \) or that \( N \) is odd. Let \( M_0(\tau) = -(\pm \tau)^N A(\tau), M_j = 0 \) for \( 1 \leq j < N \), \( M_N(\tau) = B(\tau) \); \( \deg M_0 = a + N \), \( \deg M_N = b \). Then, for all \( \tau \geq 0 \),
\[ \mathcal{M}(\tau, \tau \alpha(\tau)) = M_0(\tau) + M_N(\tau)(\tau \alpha(\tau))^N = -(\pm \tau)^N A(\tau) \cdot 1 + B(\tau) \cdot (\tau \alpha(\tau))^N = 0. \]
If \( n \leq 2 \), then both Assumption 3.1 and Assumption 3.2 are satisfied if \( a > b \) (so that (3.2) holds). If \( n \geq 3 \), then we additionally need \( N \in \mathbb{N} \) to be large enough so that \( \kappa = (a - b)/N \) satisfies \( \kappa < 2/(n - 2) \).

Let us mention that in [Com19], besides the case \( n \leq 2 \) (with any \( a > b \geq 0 \) and any \( N \in \mathbb{N} \)), we could only consider the case \( a = 2, b = 1, N = 1, n = 3 \) and the case \( a = 1, b = 0, N = 2, 3 \leq n \leq 6 \). Since in this article we exclude the case \( \kappa = 2/(n - 2) \), our present results do not apply to the case \( a = 1, b = 0, N = 2 \) (which is again \( \alpha(\tau) = \pm \sqrt{A_0 + A_1 \tau} \), with \( A_0 \geq 0 \) and \( A_1 > 0 \)) in dimension \( n = 6 \).

3.1. Regularity.

Theorem 3.5. Let \( u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n)) \), \( n \in \mathbb{N} \). If there is a bounded interval \( I \subset \mathbb{R} \) such that \( \text{supp} \hat{u} \subset \mathbb{R}^n \times I \), with \( \hat{u}(x, \omega) \) the Fourier transform of \( u \) with respect to time, then
\[ u \in C_b^\infty(\mathbb{R}, H^1(\mathbb{R}^n)). \]
Further, assume that \( u \) is a solution to the nonlinear Schrödinger or Klein–Gordon equation (1.1) with some \( m > 0 \) and with \( \alpha(\cdot) \) satisfying Assumption 3.1. Then
\[ u \in C_b^\infty(\mathbb{R}, L^Q(\mathbb{R}^n) \cap C^{1,a}(\mathbb{R}^n)), \quad \forall Q \in [2, \infty], \quad \forall a \in (0, 1). \]
Above,
\[ C_b^\infty(\mathbb{R}) = \{ f \in C_b^\infty(\mathbb{R}) ; \sup_{t \in \mathbb{R}} |\partial_t f(t)| < \infty \quad \forall j \in \mathbb{N}_0 \} \]
and \( C^{1,a}(\mathbb{R}^n) \) is a Banach space with the standard norm
\[ \| u \|_{C^{1,a}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |u(x)| + \sup_{x \in \mathbb{R}^n} |\nabla u(x)| + \sup_{x, y \in \mathbb{R}^n; x \neq y} \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^a}. \]
Proof. We recall that the quasimeasures $\mathcal{D}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$ are defined as distributions whose Fourier transform belongs to $L^\infty(\mathbb{R})$ [Kom03, KK07]. Since $u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n))$, by definition, its Fourier transform is a quasimasure:

$$
\hat{u}(x, \omega) \in \mathcal{D}(\mathbb{R}, H^1(\mathbb{R}^n)).
$$

Let $j \in \mathbb{N}$. Pick $\rho \in C^\infty_{\text{comp}}(\mathbb{R})$ such that $\rho(\omega) = (-\omega)^j$ for all $\omega \in I$. The inverse Fourier transform of $\rho$ satisfies $\hat{\rho} = \mathcal{F}^{-1}[\rho] \in \mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$, hence

$$
(3.5) \quad \partial_t^j u(x, t) = \mathcal{F}^{-1}[\rho(\cdot)|\hat{u}(x, \cdot)|(t) = \hat{\rho} * u(x, \cdot) \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n)).
$$

In other words, $\rho$ is a multiplier in the space of quasimeasures (for more details, see [KK07]). The inclusion (3.5) with any $j \in \mathbb{N}$ shows that $u \in C^\infty(\mathbb{R}, H^1(\mathbb{R}^n))$, with $||\partial_t^j u(\cdot, t)||_{H^1}$, for each $j \in \mathbb{N}$, bounded uniformly in $t \in \mathbb{R}$. This proves (3.3).

Let us prove (3.4). There is nothing to do in the case $n = 1$ since the inclusion

$$
\partial_x^2 u = \partial_t^2 u + m^2 u + \alpha(|u|^2) u \in C^\infty(\mathbb{R}, H^1(\mathbb{R}))
$$

implies that $u \in C^\infty(\mathbb{R}, H^3(\mathbb{R})) \subset C^\infty_b(\mathbb{R}, C^{2,\alpha}(\mathbb{R}))$, with $\alpha = 1/2$.

For $n \geq 2$, the proof of (3.4) is by induction. We start with (3.3) and use the Sobolev embedding:

$$
(3.6) \quad u \in C^\infty_b(\mathbb{R}, H^1(\mathbb{R}^n)) \subset C^\infty(\mathbb{R}, L^q(\mathbb{R}^n)), \quad \begin{cases} q \in [2, \infty), & n = 2; \\ q \in [2, 2n/(n - 2)], & n \geq 3. \end{cases}
$$

Fix $R > 0$. By (3.6), we may assume that for each $x_0 \in \mathbb{R}^n$ one has

$$
(3.7) \quad u \in C^\infty_b(\mathbb{R}, L^q(\mathbb{B}_R(x_0))), \quad \text{with some } q \geq q_0 = \begin{cases} 2 + 4\kappa, & n = 2, \\ 2n/(n - 2), & n \geq 3, \end{cases}
$$

with the norm independent of $x_0 \in \mathbb{R}^n$. Then, in the case when $u$ is a solution to the nonlinear Klein–Gordon equation from (1.1) (for definiteness), one has

$$
(3.8) \quad \Delta u = \partial_t^2 u + m^2 u + \alpha(|u|^2) u \in C^\infty(\mathbb{R}, L^p(\mathbb{B}_R(x_0))), \quad P := \frac{q}{1 + 2\kappa}, \quad n \geq 2,
$$

with the norm independent of $x_0$. Note that our assumptions on $q$ in (3.7) are such that

$$
(3.9) \quad P = \frac{q}{1 + 2\kappa} \geq \begin{cases} (2 + 4\kappa)/(1 + 2\kappa) = 2, & n = 2; \\ 2n/(n - 2) \frac{2n/(n - 2)}{1 + 2\kappa} > \frac{2n/(n - 2)}{1 + 4/(n - 2)} = \frac{2n}{n + 2} > 1, & n \geq 3. \end{cases}
$$

Above, for $n \geq 3$, we used Assumption 3.1 on $\kappa$. To arrive at (3.8), we took into account the uniform bound on the time derivative (3.3) and the inclusions $u \in C^\infty_b(\mathbb{R}, L^q(\mathbb{B}_R(x_0))) \subset C^\infty(\mathbb{R}, L^p(\mathbb{B}_R(x_0)))$ and $\alpha(|u|^2) u \in C^\infty(\mathbb{R}, L^\infty(\mathbb{B}_R(x_0)))$ (by Assumption 3.1), with the norm independent of $x_0$.

Remark 3.6. If $n \geq 3$ and $0 < \kappa \leq 1/(n - 2)$, then $P = \frac{q}{1 + 2\kappa} \geq \frac{2n/(n - 2)}{1 + 4/(n - 2)} = 2$ (similarly, if $n = 2$, then $q = 2 + 4\kappa$, $P = 2$), and we do not need to restrict the functions to $\mathbb{B}_R(x_0)$: indeed, since $2 \leq P < q$, one has $u \in C^\infty(\mathbb{R}, L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)) \subset C^\infty(\mathbb{R}, L^p(\mathbb{R}^n))$, hence $|\alpha(|u|^2) u| \leq C(|u| + |u|^{1 + 2\kappa}) \in C^\infty(\mathbb{R}, L^p(\mathbb{R}^n))$, and then $\Delta u = \partial_t^2 u + m^2 u + \alpha(|u|^2) u \in C^\infty(\mathbb{R}, L^p(\mathbb{R}^n))$. 


By (3.6) and (3.8), both \( u \) and \( \Delta u \) belong to \( C^\infty_b(\mathbb{R}, L^P(B_R^n(x_0))) \), with \( P = q/(1 + 2\kappa) \), with the seminorms \( \sup_{t \in \mathbb{R}} \| \partial_t^j u \|_{L^P(B_R^n(x_0))} \) and \( \sup_{t \in \mathbb{R}} \| \partial_t^j \Delta u \|_{L^P(B_R^n(x_0))} \), \( j \in \mathbb{N}_0 \), dependent on \( \sup_{t \in \mathbb{R}} \| u(t) \|_{H^j(\mathbb{R}^n)} \) (and fixed \( R > 0 \)), but not on \( x_0 \in \mathbb{R}^n \), hence

\[
\begin{align*}
 u &\in C^\infty_b(\mathbb{R}, H^{2,P}(B_R^n(x_0))) \cong C^\infty_b(\mathbb{R}, W^{2,P}(B_R^n(x_0))), && P = q/(1 + 2\kappa) \in [1, +\infty).
\end{align*}
\]

The Sobolev embedding gives

\[
(3.10) \quad u \in C^\infty_{\text{loc}}(\mathbb{R}, W^{2,P}(B_R^n(x_0))) \subset C^\infty_b(\mathbb{R}, L^Q(B_R^n(x_0))),
\]

where we can choose any \( Q \in [1, +\infty] \) if \( \frac{1}{P} < \frac{2}{n} \). If, on the contrary, \( \frac{1}{P} \geq \frac{2}{n} \), then we can take any \( Q \in [1, +\infty] \) satisfying \( \frac{1}{P} - \frac{2}{Q} \leq \frac{2}{n} \), we choose \( Q \) such that

\[
(3.11) \quad \frac{1}{Q} = \frac{1}{P} - \frac{2}{n}, \quad \text{hence} \quad \frac{1}{Q} = \frac{1}{q} + \frac{2\kappa}{q} - \frac{2}{n},
\]

One can see from (3.11) that the inclusion \( u \in C^\infty_b(\mathbb{R}, L^Q(B_R^n(x_0))) \) is a strict improvement over (3.7):

\[
(3.12) \quad \frac{1}{q} - \frac{1}{Q} = \frac{2}{n} - \frac{2\kappa}{q} \geq \frac{2}{n} - \frac{2\kappa}{q_0} = \begin{cases} \frac{1}{2} - \frac{2\kappa}{2+4\kappa} \geq \frac{1}{2}, & n = 2, \\ \frac{2}{n} - \frac{n-2\kappa}{n} > 0, & n \geq 3, \end{cases}
\]

with \( q_0 \) defined in (3.7). We note that the right-hand side of (3.12) in the case \( n \geq 3 \) is strictly positive due to the condition on \( \kappa \) from Assumption 3.1. Now we can return to (3.7) with \( q_{\text{new}} = Q > q \) instead of \( q \) and proceed by induction. Since the right-hand side of (3.12) does not depend on \( q \), in finitely many steps of the induction argument we arrive at \( 1/P = (1 + 2\kappa)/q < 2/n \), and then in (3.10) we can choose an arbitrary value \( Q \in [1, +\infty] \). Thus, \( u \in C^\infty_{\text{loc}}(\mathbb{R}, L^\infty(B_R^n(x_0))) \), uniformly in \( x_0 \in \mathbb{R}^n \), and hence \( u \in C^\infty_b(\mathbb{R}, L^\infty(\mathbb{R}^n)) \), which we now interpolate with (3.3). The inclusion \( u \in C^\infty_{\text{loc}}(\mathbb{R}, L^Q(\mathbb{R}^n)) \), \( Q \in [2, +\infty] \), leads to

\[
(3.13) \quad \Delta u = \partial_t^2 u + m^2 u + \alpha(|u|^2) u \in C^\infty_b(\mathbb{R}, L^Q(\mathbb{R}^n)), \quad \forall Q \in [2, +\infty],
\]

hence \( u \in C^\infty_b(\mathbb{R}, W^{2,Q}(\mathbb{R}^n)) \), \( \forall Q \in [2, +\infty] \), and by the Sobolev embedding theorem this leads to \( u \in C^\infty_b(\mathbb{R}, C^{1,a}(\mathbb{R}^n)) \), with any \( a \in (0, 1) \).

### 3.2. Reduction to one frequency.

Now we can prove the main result about the absence of nontrivial solutions with compact spectrum.

**Theorem 3.7.** Let \( n \in \mathbb{N}, m \geq 0, \) and assume that \( \alpha(\cdot) \) satisfies both Assumption 3.1 and Assumption 3.2. Assume that \( u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n)) \) is a solution to the nonlinear Schrödinger or Klein–Gordon equation (1.1). If there is a bounded interval \( I \subset \mathbb{R} \) such that \( \text{supp} \hat{u} \subset \mathbb{R}^n \times I \), with \( \hat{u}(x, \omega) \) the Fourier transform of \( u \) with respect to time, then

\[
u(x, t) = \phi_0(x) e^{-i\omega_0 t}, \quad \text{with some} \; \phi_0 \in H^1(\mathbb{R}^n) \; \text{and} \; \omega_0 \in \mathbb{R}.
\]

Note that, in particular, the above theorem applies to finite energy solutions of the nonlinear Klein–Gordon equation from [Kat86, Proposition 2.1].
Proof. The proof closely follows that of [Com19, Theorem 6], simplified in view of Theorem 3.5; we provide the proof, shortening the repeating parts. Assume that \( u \in L^\infty((\mathbb{R}, H^1(\mathbb{R}^n))) \) is a solution to the nonlinear Klein–Gordon equation from (1.1) with compact spectrum, so that the Fourier transform of \( u \) in time,

\[
\hat{u}(x, \omega) = \int_{\mathbb{R}} u(x, t) e^{i \omega t} dt, \quad \hat{u} \in \mathcal{S}'(\mathbb{R}, H^1(\mathbb{R}^n)),
\]
satisfies \( \text{supp} \hat{u} \subset \mathbb{R}^n \times [a, b] \), with some \( a, b \in \mathbb{R}, a < b \). We denote

\[
\Sigma := \Sigma_\delta = \{ x \in \mathbb{R}^n ; (\{ x \} \times \mathbb{R}) \cap \text{supp} u \neq \emptyset \}
\]

(3.14) to be the projection of the support of \( u \) onto \( \mathbb{R}^n \). Then, since \( \text{supp} \hat{u} \subset \mathbb{R}^n \times [a, b] \),

\[
\mathbb{B}_a \supseteq \mathbb{E}_a \quad \Rightarrow \quad \mathbb{B}_a^L \supseteq \mathbb{E}_a \quad \Rightarrow \quad \mathbb{A}_a^U \supseteq \mathbb{E}_a \quad \forall x \in \Sigma.
\]

LEMMA 3.8. \( \alpha(|u(x,t)|^2) \) and \( |u(x,t)| \) do not depend on time, and moreover

\[
\mathbb{B}_a^L = \mathbb{A}_a, \quad \mathbb{A}_a = \mathbb{A}_a^U, \quad \forall x \in \Sigma.
\]

Proof. Theorem 3.5 and (3.1) show that the function

\[
v(x,t) := \alpha(|u(x,t)|^2)
\]
satisfies

(3.15) \( v \in C_{\mathcal{B}}^\infty (\mathbb{R}, C_b(\mathbb{R}^n, \mathbb{R})). \)

By (1.1) and (3.15),

(3.16) \( (\partial_t^2 - \Delta + m^2) u = -\alpha(|u|^2) u \in C_{\mathcal{B}}^\infty (\mathbb{R}, C_b(\mathbb{R}^n)). \)

We apply the Fourier transform to (3.16); denoting by \( \tilde{v}(x, \omega) \) the Fourier transform of \( v(x,t) := \alpha(|u(x,t)|^2) \) in time, one has

(3.17) \( (m^2 - \omega^2 - \Delta) \tilde{u} = -\tilde{v} \leq \tilde{u}. \)

Multiplying (3.16) by \( \tilde{u} \), we have:

(3.18) \( \tilde{u}(m^2 + \partial_t^2 - \Delta) u = -|u|^2 \alpha(|u|^2) \in C_{\mathcal{B}}^\infty (\mathbb{R}, C_b(\mathbb{R}^n)). \)

Let \( \mathcal{M} \) be as in Assumption 3.2. Applying \( \mathcal{M}(|u|^2, \cdot) \) to both sides of the relation (3.18) leads to

(3.19) \( 0 = \mathcal{M}(|u|^2, |u|^2 \alpha(|u|^2)) = \mathcal{M}(|u|^2, \tilde{u}(m^2 - \omega^2 - \Delta) u) \)

\[
= \sum_{j=0}^J \mathcal{M}_j(|u|^2)(-\tilde{u}(m^2 + \partial_t^2 - \Delta) u)^j.
\]

We note that \( \tilde{u} \leq \tilde{u} = |u|^2 \), where

(3.20) \( f^t(x, \omega) = f(x, -\omega). \)
Lemma 3.9. \( B_{\tilde{a}^t \tilde{u}}(m^2-\omega^2-\Delta)\tilde{u}(x) \leq B_{\tilde{a}^t \tilde{u}}(x) \quad \forall x \in \mathbb{R}^n. \)

Proof. Since \( \text{supp}(m^2-\omega^2-\Delta)\tilde{u} \subset \text{supp} \tilde{u}, \) there is the inequality

\[
(3.21) \quad B_{(m^2-\omega^2-\Delta)\tilde{u}}(x) \leq B\tilde{u}(x), \quad \forall x \in \mathbb{R}^n.
\]

Therefore, applying twice the Titchmarsh theorem for partial convolution (Theorem 2.6) and (3.21), we derive:

\[
\begin{align*}
B_{\tilde{a}^t \tilde{u}} &= (B_{\tilde{a}^t} + B\tilde{u})^U \geq (B_{\tilde{a}^t} + B_{(m^2-\omega^2-\Delta)\tilde{u}})^U \\
&= (B_{\tilde{a}^t} + B_{(m^2-\omega^2-\Delta)\tilde{u}})^U = B_{\tilde{a}^t \tilde{u}} + (m^2-\omega^2-\Delta)\tilde{u}.
\end{align*}
\]

For the last equality, we used Lemma 2.5.

Now we apply Theorem 2.6 to the Fourier transform (in time) of the relation (3.19) and use Assumption 3.2, getting \( B_{\tilde{a}^t \tilde{u}}(m^2-\omega^2-\Delta)\tilde{u} \leq 0; \) then \( B_{\tilde{a}^t \tilde{u}}(m^2-\omega^2-\Delta)\tilde{u} \leq 0, \) and similarly \( A_{\tilde{a}^t \tilde{u}}(m^2-\omega^2-\Delta)\tilde{u} \geq 0. \) It follows that

\[
(3.22) \quad \text{supp} \tilde{u}^2 \subseteq (m^2-\omega^2-\Delta)\tilde{u} \subset \mathbb{R}^n \times \{0\}.
\]

Lemma 3.10. \(|u|^2\alpha(|u|^2)\) is time-independent.

Proof. By (3.22), the function \( G(x,t) := |u|^2\alpha(|u|^2) = \tilde{u}(m^2+\partial_t^2-\Delta)u \) satisfies \( \text{supp} \tilde{G}(x,\omega) \subset \mathbb{R}^n \times \{0\}. \) We conclude that

\[
(3.23) \quad \tilde{G}(x,\omega) = \sum_{j \in \mathbb{N}_0} \delta^{(j)}(\omega)G_j(x), \quad x \in \mathbb{R}^n, \quad \omega \in \mathbb{R}.
\]

Above, in agreement with the general theory of distributions \([Hör83]\), the summation in \( j \in \mathbb{N}_0 \) is locally finite: for each compact subset \( K \subset \mathbb{R}^n, \) there are finitely many terms \( J(K) \in \mathbb{N} \) in the restriction of (3.23) onto \( K \) (cf. \([Hör83, \text{Theorem 2.3.5}]\)). Moreover, the terms with derivatives of \( \delta(\omega) \) should not appear in (3.23). Indeed, the Fourier transform of (3.23) coupled with a test function \( \varphi \in \mathcal{D}(K) \) is

\[
(3.24) \quad \langle \varphi, G(\cdot,t) \rangle = \frac{1}{2\pi} \sum_{j=0}^{J(K)} (it)^j \langle \varphi, G_j \rangle, \quad x \in K, \quad t \in \mathbb{R};
\]

if \( G_j \neq 0 \) for some \( 1 \leq j \leq J(K), \) then for some nonzero test function the right-hand side of (3.24) would be growing in time, in contradiction to (3.15). This implies that in (3.23) the only nonzero term is the one with \( j = 0. \) Thus, \( G(x,t) = |u(x,t)|^2\alpha(|u(x,t)|^2) = G_0(x) \) does not depend on time.

Let us argue that since by Lemma 3.10 the expression \(|u|^2\alpha(|u|^2)\) is time-independent, so is \(|u|^2.\) Since \( \tau\alpha(\tau) \equiv C \in \mathbb{R}, \) we substitute this value into \( \mathcal{M}, \) arriving at

\[
(3.25) \quad 0 \equiv \mathcal{M}(\tau, \tau\alpha(\tau)) = \mathcal{M}(\tau, C) = \sum_{j=0}^{J} M_j(\tau)C^j.
\]

Due to the conditions (3.2), the right-hand side of (3.25) is a polynomial of degree \( \deg M_0 > 0; \) thus, for each \( (x,t) \in \mathbb{R}^n \times \mathbb{R}, \) the value \( \tau = |u(x,t)|^2 \) has to be equal to one of the roots of \( \mathcal{M}(\tau, C); \) due to the continuous dependence of \(|u(x,t)|^2 \) of \( x \) and
By Lemma 3.8, one has $B_{\tilde{u}}^L = A_{\tilde{u}}$, and then it follows that $\text{supp } \tilde{u} \subset \mathbb{R}^n \times \{\omega_0\}$, with some $\omega_0 \in I$. Similarly to the proof of Lemma 3.10, this leads to $u(x, t) = e^{-i\omega_0t}\phi(x)$, concluding the proof of Theorem 3.7.

**Appendix A. Multifrequency solitary waves of nonlinear Dirac equation.**

Here we give an explicit construction of multifrequency solitary waves for the nonlinear Dirac equation with the scalar self-interaction, known as the Soler model [Iva83, Sol70]:

\begin{equation}
\label{eq:soler-model}
\begin{aligned}
i\partial_t \psi &= D_m \psi - f(\psi^* \beta \psi) \beta \psi, & \psi(x, t) \in \mathbb{C}^4, & x \in \mathbb{R}^3,
\end{aligned}
\end{equation}

where the free Dirac operator $D_m : L^2(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathcal{D}(D_m) = H^1(\mathbb{R}^3, \mathbb{C}^4)$ is given by

$$D_m = -i\alpha \cdot \nabla + \beta m,$$

with $\alpha^i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}$, $\sigma_i$, $1 \leq i \leq 3$ the Pauli matrices, and $\beta = \begin{bmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{bmatrix}$.

**Assumption A.1.** 1. There is $V \in C^\infty(\mathbb{R}^3)$ which is positive, spherically symmetric, and strictly monotonically decreasing, with $\lim_{|x| \to \infty} V(x) = 0$, such that there are eigenfunctions of the Dirac operator $D_m + \beta V$ corresponding to eigenvalues $\omega_0$ and $\omega_1$, with $0 < \omega_0 < \omega_1 < m$:

\begin{equation}
\label{eq:assumption-a-1}
\begin{aligned}
\omega_j \phi_j &= D_m \phi_j - \beta V \phi_j, & j = 0, 1.
\end{aligned}
\end{equation}
2. The corresponding eigenfunctions have the form

\[ \phi_j(x) = \begin{bmatrix} v_j(r) n_j \\ (u_j(r)) \sigma_r n_j \end{bmatrix}, \quad n_j \in \mathbb{C}^2, \quad |n_j| = 1, \quad j = 0, 1, \]

with \( r = |x|, \sigma_r = r^{-1} x \cdot \sigma \) (for \( x \neq 0 \)), and with \( v_j, u_j \) smooth (considered as functions of \( x \in \mathbb{R}^3 \)) and real-valued;

3. The function \( v_0 \) is strictly positive;

4. \( \rho(r) := |v_0(r)|^2 - |u_0(r)|^2 \) is monotonically decreasing in \( r \geq 0 \).

Thus, \( v_j \) and \( u_j \) satisfy the system

\[ \begin{align*}
\omega_j v_j &= \partial_r u_j + \frac{n_j}{r} u_j + (m - V)v_j, \\
\omega_j u_j &= -\partial_r v_j - (m - V)u_j,
\end{align*} \quad j = 0, 1. \tag{A.4} \]

**Remark A.2.** Let us sketch a construction of states \( \phi_j \) satisfying Assumption A.1. We assume that there is a spherically symmetric potential \( W(x) > 0 \) which we consider as a function of \( r = |x| \), monotonically decreasing at infinity with \( \lim_{r \to \infty} W(r) \to 0 \), such that \( H = -\frac{1}{2m} \Delta - W \) has a groundstate eigenvalue \( E_0 = -1 \) and an excited state with eigenvalue \( E_1 = -1/2 \), with the corresponding eigenfunctions \( \varphi_0 \) and \( \varphi_1 \):

\[ \begin{align*}
-\varphi_0 &= -\frac{1}{2m} \Delta \varphi_0 - W \varphi_0, \\
-\varphi_1 &= -\frac{1}{2m} \Delta \varphi_1 - W \varphi_1, \quad \varphi_0, \varphi_1 \in L^2(\mathbb{R}^3).
\end{align*} \tag{A.5} \]

Since \( \varphi_0 \) is a groundstate, we can assume that it is spherically symmetric, strictly positive, and monotonically decreasing to zero at infinity. For \( \omega \in (0, m) \), for \( r \geq 0 \), we define

\[ V(r) = (m - \omega)W((m - \omega)^{1/2}r), \]

\[ \hat{v}_j(r) = \varphi_j((m - \omega)^{1/2}r), \quad \hat{u}_j(r) = -\frac{\partial_r \hat{v}_j(r)}{2m} = -\frac{(m - \omega)^{1/2}}{2m} \varphi_j'(((m - \omega)^{1/2}r), \]

where \( j = 0, 1 \). Then \( (\hat{v}_0, \hat{u}_0) \) and \( (\hat{v}_1, \hat{u}_1) \) satisfy the relations

\[ -(m - \omega)\hat{v}_0 = -\frac{\Delta \hat{v}_0}{2m} - V \hat{v}_0 = \partial_r \hat{u}_0 + \frac{n - 1}{r} \hat{u}_0 - V \hat{v}_0, \]

\[ -\frac{m - \omega}{2} \hat{v}_1 = -\frac{\Delta \hat{v}_1}{2m} - V \hat{v}_1 = \partial_r \hat{u}_1 + \frac{n - 1}{r} \hat{u}_1 - V \hat{v}_1, \]

hence

\[ \omega \hat{v}_0 = \partial_r \hat{u}_0 + \frac{n - 1}{r} \hat{u}_0 + (m - V)\hat{v}_0, \quad \frac{m + \omega}{2} \hat{v}_1 = \partial_r \hat{u}_1 + \frac{n - 1}{r} \hat{u}_1 + (m - V)\hat{v}_1, \]

which coincide with the first equation in (A.4) (with \( \omega \) and \( (m + \omega)/2 \) in place of \( \omega_j \)). The function \( \varphi_0(r) \) is strictly monotonically decreasing, while \( \lim_{r \to \infty} |\varphi_0'(r)|/\varphi_0(r) | < \infty \) (one can show that \( f(r) = \varphi_0'(r)/\varphi_0(r) \) satisfies

\[ f'(r) = 1 - W(r) - (n - 1)f/r - f^2, \]

\[ r > 1, \quad f(1) = \varphi_0'(1)/\varphi_0(1) < 0; \] solutions to this equation are either uniformly bounded from below or approach \(-\infty \) as \( r \to r_0 - 0 \), with some \( r_0 \in (1, \infty) \)). Therefore, the function

\[ \rho(r) := |\hat{v}_0(r)|^2 - |\hat{u}_0(r)|^2 = |\varphi_0((m - \omega)^{1/2}r)|^2 - \frac{m - \omega}{4m^2} |\varphi_0((m - \omega)^{1/2}r)|^2 \tag{A.6} \]
is also monotonically decreasing as long as $\omega$ is sufficiently close to $m$. The perturbation theory allows one to start with

$$\omega \left[ \begin{array}{c} \dot{v}_0 \\ \dot{u}_0 \end{array} \right] = \left[ \begin{array}{cc} m - V & \partial_r + \frac{n-1}{2} \\ -\partial_r & -2m + \omega \end{array} \right] \left[ \begin{array}{c} v_0 \\ u_0 \end{array} \right], \quad \frac{m + \omega}{2} \left[ \begin{array}{c} \dot{v}_1 \\ \dot{u}_1 \end{array} \right] = \left[ \begin{array}{cc} m - V & \partial_r + \frac{n-1}{2} \\ -\partial_r & -2m + \omega \end{array} \right] \left[ \begin{array}{c} v_1 \\ u_1 \end{array} \right]$$

and to construct eigenfunctions $(v_0, u_0) \in L^2(\mathbb{R}^n, \mathbb{C}^2)$ and $(v_1, u_1) \in L^2(\mathbb{R}^n, \mathbb{C}^2)$ to (A.4),

$$\omega_0 \left[ \begin{array}{c} v_0 \\ u_0 \end{array} \right] = \left[ \begin{array}{cc} m - V & \partial_r + \frac{n-1}{2} \\ -\partial_r & -m + V \end{array} \right] \left[ \begin{array}{c} v_0 \\ u_0 \end{array} \right], \quad \omega_1 \left[ \begin{array}{c} v_1 \\ u_1 \end{array} \right] = \left[ \begin{array}{cc} m - V & \partial_r + \frac{n-1}{2} \\ -\partial_r & -m + V \end{array} \right] \left[ \begin{array}{c} v_1 \\ u_1 \end{array} \right]$$

as bifurcations from $(\dot{v}_0, \dot{u}_0)$ and $(\dot{v}_1, \dot{u}_1)$, with corresponding eigenvalues $\omega_0 \approx \omega$ and $\omega_1 \approx (m + \omega)/2$, as long as $\omega \lesssim m$ is chosen sufficiently close to $m$. (We note that the operators in the right-hand sides are self-adjoint in $H^1_{\text{even, odd}}(\mathbb{R}, |r|^{n-1} \, dr; \mathbb{C}^2)$, the space consisting of $\mathbb{C}^2$-valued $H^1$-functions with respect to the measure $|r|^{n-1} \, dr$ on $\mathbb{R}$, with their first component being even and the second being odd as functions of $r \in \mathbb{R}$; see e.g. [BC17].) The analysis shows that $(v_j, u_j)$ can also be chosen real. Moreover, if $\omega$ is sufficiently close to $m$, then $\rho(r) := |v_0(r)|^2 - |u_0(r)|^2$ (cf. (A.6)) is monotonically decreasing.

Similarly to (A.2), we have

$$(A.7) \quad -\omega_j \chi_j = D_m \chi_j - \beta V \chi_j, \quad j = 0, 1,$$

with

$$(A.8) \quad \chi_j(x) = \left[ \begin{array}{c} \beta \phi_j(r) \\ \sigma_1 \partial_r \phi_j(r) \end{array} \right], \quad m_j \in \mathbb{C}^2, \quad |m_j| = 1, \quad j = 0, 1.$$  

(These expressions for $\chi_j$ can be obtained from $\phi_j$ by applying to (A.3) the charge conjugation operator $i\gamma^2 K$, with $\gamma^2 = \beta \alpha^2 = \left[ \begin{array}{cc} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{array} \right]$, with $K : \mathbb{C}^N \to \mathbb{C}^N$ the complex conjugation; for more details, see [BC18] or [BC19]). It follows that for any $a_j, b_j \in \mathbb{C}, j = 0, 1$, the function

$$(A.9) \quad \psi(x, t) = a_0 \phi_0(x) e^{-i\omega_0 t} + a_1 \phi_1(x) e^{-i\omega_1 t} + b_0 \chi_0(x) e^{i\omega_0 t} + b_1 \chi_1(x) e^{i\omega_1 t}$$

satisfies the linear Dirac equation $i\partial_t \psi = D_m \psi - \beta V \psi$.

We note that $\phi_j$ and $\chi_j, j = 0, 1$, defined in (A.3) and (A.8), satisfy

$$\chi^*_j \beta \phi_j = \phi_j^* \beta \chi_j = 0, \quad i, j = 0, 1,$$

for any choice of $n_j, m_j$. We choose $n_0, n_1 \in \mathbb{C}^2$ such that $|n_0| = |n_1| = 1, n_0^* n_1 = 0$; similarly, we choose $m_0, m_1 \in \mathbb{C}^2$ such that $|m_0| = |m_1| = 1, m_0^* m_1 = 0$; then

$$\phi_j^* \beta \phi_j = 0, \quad \chi_j^* \beta \chi_j = 0, \quad i, j = 0, 1, \quad i \neq j.$$  

Taking into account the relations

$$\phi_j^* \beta \phi_j = |v_j(r)|^2 - |u_j(r)|^2 \quad \text{and} \quad \chi_j^* \beta \chi_j = |u_j(r)|^2 - |v_j(r)|^2, \quad j = 0, 1,$$

we derive:

$$F(r) := \psi(x, t)^* \beta \psi(x, t)$$

$$= (|a_0|^2 - |b_0|^2)(v_0(r)^2 - u_0(r)^2) + (|a_1|^2 - |b_1|^2)(v_1(r)^2 - u_1(r)^2).$$
In view of Assumption A.1, the function $F(r)$ is positive, differentiable, and strictly monotonically decreasing to zero as $r \to \infty$ as long as $a_0, a_1, b_0, b_1 \in \mathbb{C}$ are chosen so that $|a_0|^2 > |b_0|^2$ and so that $|a_1|^2 - |b_1|^2$ is sufficiently small. Therefore, there is a differentiable monotonically increasing function $f$ with $f(0) = 0$ such that $f(F(r)) = V(r)$. Then the four-frequency solitary wave $\psi(x, t)$ from (A.9) satisfies the nonlinear Dirac equation (A.1).

Remark A.3. We do not know whether the potential $W$ in (A.5) could be chosen so that the function $f$ obtained in the above construction is polynomial or algebraic.

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