UNIFORM EXISTENCE OF THE INTEGRATED DENSITY OF STATES 
FOR MODELS ON $\mathbb{Z}^d$

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Abstract. We provide an ergodic theorem for certain Banach-space valued functions on structures over $\mathbb{Z}^d$, which allow for existence of frequencies of finite patterns. As an application we obtain existence of the integrated density of states for associated finite-range operators in the sense of convergence of the distributions with respect to the supremum norm. These results apply to various examples including periodic operators, percolation models and nearest-neighbour hopping on the set of visible points. Our method gives explicit bounds on the speed of convergence in terms of the speed of convergence of the underlying frequencies. It uses neither von Neumann algebras nor a framework of random operators on a probability space.

1. Introduction

This paper deals with existence of averages of Banach-space valued functions on subsets of $\mathbb{Z}^d$ and applications to finite-range operators. Existence of averages of Banach-space valued functions plays a role in both the study of thermodynamic formalism and the investigation of equivariant operators. For Banach-space valued functions on tiling-type structures exhibiting a special form of (dis)order known as aperiodic order, this has been investigated in [5, 9, 10]. Aperiodic order has attracted a lot of attention in recent years (see [2, 6, 13] for background and recent surveys). In the mentioned works the assumption of aperiodic order implies uniform existence of frequencies of finite patterns, i.e. existence of frequencies along arbitrary van Hove sequences. This is equivalent to unique ergodicity of the underlying dynamical system. The key idea introduced in [5] is then to decompose the underlying structure into nice pieces and use existence of frequencies of these pieces in order to conclude the desired existence of the averages. This type of program has been worked out in specific contexts in the above-cited literature.

The assumption of uniform existence of frequencies is well adjusted to aperiodic order. However, it will not be met in a more random context. Thus, the question arises to what extent similar ergodic-type theorems can be proven in a more general framework. It is quite clear that an answer to this problem is useful in various random and geometric contexts, see for instance [3]. Our first result, Theorem 1, answers this questions for models on $\mathbb{Z}^d$. It turns out that an analogue of the previously given results is valid including explicit bounds on convergence under the sole assumption of existence of frequencies along a fixed van Hove sequence. In particular, we do not even need a context of dynamical systems. The proof uses ideas from [9], where one-dimensional uniquely ergodic subshifts are considered.

While Theorem 1 may be of independent interest, in the present paper we use it to study uniform existence of the integrated density of states (IDS) for certain random operators. To do so, we follow the strategy of [10] and reduce the proof of uniform convergence of the IDS
to the validity of a Banach-space valued ergodic theorem. This leads to our second result, Theorem 2 which ensures convergence of the normalised finite-volume eigenvalue counting functions to the IDS with respect to the supremum norm. Moreover, the explicit bounds on convergence in Theorem 1 yield explicit bounds for the speed of convergence for the IDS.

As existence of frequencies is our only assumption, our approach is rather flexible and simple. In particular, we do not need von Neumann algebras or traces. We illustrate this flexibility by applying our abstract results to three particular situations: (i) periodic operators, (ii) percolation models and (iii) discrete Laplacians on the set of visible points. We emphasise that there is no good dynamical system available in the third situation.

Note added. When we were finishing up this work we learned about the recent preprint [4] of Gabor Elek entitled “Aperiodic order, integrated density of states and the continuous algebras of John von Neumann” containing related results. Using von Neumann algebras and a construction of Godearl, Elek can prove uniform convergence of the IDS for various models including percolation models and operators on self similar graphs.

2. Basic notions

This paper is about functions and operators on colourings over \( \mathbb{Z}^d \). We start by introducing the necessary background and notation.

The set of all finite subsets of \( \mathbb{Z}^d \) is denoted by \( \mathcal{F}(\mathbb{Z}^d) \). We write \( \sharp A \) for the cardinality of a general set \( A \). For the special case of \( Q \in \mathcal{F}(\mathbb{Z}^d) \), however, we use the notation \( |Q| \) instead of \( \sharp Q \). We will be particularly interested in boxes. A subset \( Q = [a_1, b_1] \times \cdots \times [a_d, b_d] \in \mathcal{F}(\mathbb{Z}^d) \) with \( a_j, b_j \in \mathbb{Z}, a_j \leq b_j, j = 1, \ldots, d \) is called a box. The set of all boxes is denoted by \( \mathcal{B}(\mathbb{Z}^d) \).

A box \( Q = [a_1, b_1] \times \cdots \times [a_d, b_d] \) with \( (a_1, \ldots, a_d) = (0, \ldots, 0) \) is said to lie at the origin. The set of all boxes at the origin is denoted by \( \mathcal{B}_0(\mathbb{Z}^d) \). Given \( M \in \mathbb{N} \), we write

\[
C_M := \{ x \in \mathbb{Z}^d : 0 \leq x_j \leq M - 1, j = 1, \ldots, d \}
\]

for the box with edges of length \( M - 1 \) that is centred at the origin and \( C_M(a) := a + C_M \) for \( a \in \mathbb{Z}^d \).

Let \( A \) be a finite set. A map \( A : \mathbb{Z}^d \to A \) is called an \( A \)-colouring of \( \mathbb{Z}^d \). A map \( P : Q(P) \to A \) with \( Q(P) \in \mathcal{F}(\mathbb{Z}^d) \) is called an \( A \)-pattern. The set \( Q(P) \) is called the domain of \( P \). The set of all \( A \)-patterns is denoted by \( \mathcal{P}(\mathbb{Z}^d) \). When the set \( A \) is understood from the context we will just speak about colourings and patterns. For a colouring \( \Lambda \) and a \( Q \in \mathcal{F}(\mathbb{Z}^d) \) we define \( \Lambda \cap Q \) by \( \Lambda \cap Q : Q \to A, y \mapsto \Lambda(y) \). Similarly, for a pattern \( P \) and a box \( Q \) with \( Q \subset Q(P) \), we define \( P \cap Q : Q \to A, y \mapsto P(y) \). For a pattern \( P \) and \( x \in \mathbb{Z}^d \) we define the shifted map \( x + P \) by \( x + P : x + Q(P) \to A, x + y \mapsto P(y) \). A pattern \( P \) with \( Q(P) \in \mathcal{B}(\mathbb{Z}^d) \) is called a box pattern. If, in addition, \( Q(P) \) lies at the origin, then \( P \) is said to be a box pattern at the origin. The set of all box patterns at the origin will be denoted by \( \mathcal{P}_0^{B}(M) \), and the set of all box patterns at the origin with domain \( C_M \) will be denoted by \( \mathcal{P}_0^{B}(M) \).

For \( P \in \mathcal{P}_0^{B}(M) \) and \( P' \in \mathcal{P}(\mathbb{Z}^d) \) arbitrary we define the number of occurrences of the box pattern \( P \) in \( P' \) by

\[
\sharp_P P' := \sharp \left\{ x \in Q(P') : x + Q(P) \subset Q(P'), P' \cap (x + Q(P)) = x + P \right\}.
\]
For an arbitrary $Q \in \mathcal{F}(\mathbb{Z}^d)$ and $S \in \mathbb{N}$ we introduce the $S$-boundary of $Q$ as

$$\partial^S Q := \{ x \in \mathbb{Z}^d \setminus Q : \text{dist}(x, Q) \leq S \} \cup \{ x \in Q : \text{dist}(x, \mathbb{Z}^d \setminus Q) \leq S \}.$$ 

A sequence $(Q_j)_{j \in \mathbb{N}} \subset \mathcal{F}(\mathbb{Z}^d)$ is called a van Hove sequence in $\mathbb{Z}^d$, if

$$\lim_{j \to \infty} \frac{\| \partial^S Q_j \|}{|Q_j|} = 0$$

for every $S \in \mathbb{N}$. Finally, let $A : \mathbb{Z}^d \to \mathcal{A}$ be a colouring, $P \in \mathcal{P}_0^B$ a box pattern at the origin and $(Q_j)_{j \in \mathbb{N}}$ a van Hove sequence in $\mathbb{Z}^d$. If the limit

$$\nu_P := \lim_{j \to \infty} \frac{1}{|Q_j|} \# P(A \cap Q_j)$$

exists, it is called the frequency of $P$ in $A$ along $(Q_j)_{j \in \mathbb{N}}$. Existence of frequencies of box patterns is the only assumption we will pose on the colourings we consider. This will be sufficient to derive an ergodic-type theorem for certain Banach-space valued functions, which are introduced in Definition 2 below.

**Definition 1.** A map $b : \mathcal{F}(\mathbb{Z}^d) \to [0, \infty)$ is called a boundary term if $b(Q) = b(t + Q)$ for all $t \in \mathbb{Z}^d$ and $Q \in \mathcal{F}(\mathbb{Z}^d)$, $\lim_{j \to \infty} |Q_j|^{-1} b(Q_j) = 0$ for any van Hove sequence $(Q_j)$, and there exists $D > 0$ with $b(Q) \leq D |Q|$ for all $Q \in \mathcal{F}(\mathbb{Z}^d)$.

**Definition 2.** Let $(X, \| \cdot \|)$ be a Banach space and $F : \mathcal{F}(\mathbb{Z}^d) \to X$ be given.

(a) The function $F$ is said to be almost-additive if there exists a boundary term $b$ such that

$$\| F(\cup_{k=1}^m Q_k) - \sum_{k=1}^m F(Q_k) \| \leq \sum_{k=1}^m b(Q_k),$$

for all $m \in \mathbb{N}$ and all pairwise disjoint sets $Q_k \in \mathcal{F}(\mathbb{Z}^d)$, $k = 1, \ldots, m$.

(b) Let $A : \mathbb{Z}^d \to \mathcal{A}$ be a colouring. The function $F$ is said to be $A$-invariant if

$$F(Q) = F(t + Q)$$

for all $t \in \mathbb{Z}^d$ and all $Q \in \mathcal{F}(\mathbb{Z}^d)$ obeying $t + (A \cap Q) = A \cap (t + Q)$. In this case there exists a function $\tilde{F} : \mathcal{P}_0^B \to X$ on the set of box patterns at the origin such that

$$F(t + Q) = \tilde{F}(-t + (A \cap (t + Q)))$$

for all $t \in \mathbb{Z}^d$ and all $Q \in \mathcal{B}_0(\mathbb{Z}^d)$.

(c) The function $F$ is said to be bounded if there exists a finite constant $C > 0$ such that

$$\| F(Q) \| \leq C |Q|$$

for all $Q \in \mathcal{F}(\mathbb{Z}^d)$.

We will use the Banach-space valued ergodic theorem in Sect. 3 below to study certain types of operators. These operators will be introduced now.

Let $\mathcal{H}$ be a fixed Hilbert space with dimension $\dim(\mathcal{H}) < \infty$ and norm $\| \cdot \|$. Then,

$$\ell^2(\mathbb{Z}^d, \mathcal{H}) := \{ u : \mathbb{Z}^d \to \mathcal{H} : \sum_{x \in \mathbb{Z}^d} \| u(x) \|^2 < \infty \}$$

is a Hilbert space. The support of $u \in \ell^2(\mathbb{Z}^d, \mathcal{H})$ is the set of $x \in \mathbb{Z}^d$ with $u(x) \neq 0$. 

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For \( x \in \mathbb{Z}^d \), we define the natural projection \( p_x : \ell^2(\mathbb{Z}^d, \mathcal{H}) \to \mathcal{H}, u \mapsto p_x(u) := u(x) \). Let \( i_x : \mathcal{H} \to \ell^2(\mathbb{Z}^d, \mathcal{H}) \) be the adjoint of \( p_x \). Similarly, for a subset \( Q \subset \mathbb{Z}^d \) we define \( \ell^2(Q, \mathcal{H}) \) to be the subspace of \( \ell^2(\mathbb{Z}^d, \mathcal{H}) \) consisting of elements supported in \( Q \). The projection of \( \ell^2(\mathbb{Z}^d, \mathcal{H}) \) on \( \ell^2(Q, \mathcal{H}) \) is denoted by \( p_Q \) and its adjoint by \( i_Q \).

The operators and functions we are interested in are specified in the next two definitions.

**Definition 3.** Let \( A \) be a finite set, \( A : \mathbb{Z}^d \to A \) a colouring and \( H : \ell^2(\mathbb{Z}^d, \mathcal{H}) \to \ell^2(\mathbb{Z}^d, \mathcal{H}) \) a selfadjoint operator.

(a) The operator \( H \) is said to be of **finite range**, if there exists a length \( R_{fr} > 0 \) such that \( p_y Hi_x = 0 \), whenever \( x, y \in \mathbb{Z}^d \) have distance bigger than \( R_{fr} \).

(b) The operator \( H \) is said to be **\( A \)-invariant** if there exists a length \( R_{inv} \in \mathbb{N} \) such that \( p_y Hi_x = p_{y+t} Hi_{x+t} \) for all \( x, y, t \in \mathbb{Z}^d \) obeying

\[
t + \left( A \cap (C_{R_{inv}}(x) \cup C_{R_{inv}}(y)) \right) = A \cap (C_{R_{inv}}(x+t) \cup C_{R_{inv}}(y+t)).
\]

(c) If \( H \) is both of finite range and \( A \)-invariant, the number \( R(H) := \max\{R_{fr}, R_{inv}\} \) is called the **overall range** of \( H \).

A given finite-range, \( A \)-invariant operator \( H \) is fully determined by specifying finitely many \( \dim(\mathcal{H}) \times \dim(\mathcal{H}) \) matrices \( p_y Hi_x \). In particular, such operators \( H \) are bounded.

**Definition 4.** Let \( \mathcal{L}(\mathbb{R}) \) be the Banach space of right-continuous, bounded functions equipped with the supremum norm. For a selfadjoint operator \( A \) on a finite-dimensional Hilbert space we define its **cumulative eigenvalue counting function** \( n(A) \in \mathcal{L}(\mathbb{R}) \) by setting

\[
n(A)(\lambda) := \sharp\{\text{eigenvalues of } A \text{ not exceeding } \lambda\}
\]

for all \( \lambda \in \mathbb{R} \), where each eigenvalue is counted according to its multiplicity.

We will be particularly concerned with functions of the form \( n(p_Q Hi_Q) \) for finite-range operators \( H \) and \( Q \in \mathcal{F}(\mathbb{Z}^d) \).

### 3. An ergodic theorem

In this section, we present an ergodic theorem for certain Banach-space valued functions. It is the main abstract input upon which we base our results on uniform convergence of the integrated density of states in Sect. 4. We will consider the following situation:

(E) Let \( A \) be a finite set, \( A : \mathbb{Z}^d \to A \) an \( A \)-colouring and \((X, \| \cdot \|)\) a Banach-space. Let \( (Q_j)_{j \in \mathbb{N}} \) be a van Hove sequence such that for every \( P \in \mathcal{P}_0^B \) the frequency 

\[
\nu_P = \lim_{j \to \infty} |Q_j|^{-1} \sharp_P (A \cap Q_j)
\]

exists.

**Theorem 1.** Assume (E). Let \( F : \mathcal{F}(\mathbb{Z}^d) \to X \) be a \( \Lambda \)-invariant, almost-additive bounded function. Then, the limits

\[
\overline{F} := \lim_{j \to \infty} \frac{F(Q_j)}{|Q_j|} = \lim_{M \to \infty} \sum_{P \in \mathcal{P}_0^B(M)} \nu_P \frac{\overline{F}(P)}{|C_M|}
\]
exist and are equal. Moreover, for \( j, M \in \mathbb{N} \) the difference

\[
\Delta(j, M) := \left\| \frac{F(Q_j)}{|Q_j|} - \sum_{P \in \mathcal{P}_0^B(M)} \nu_P \frac{\tilde{F}(P)}{|C_M|} \right\|
\]

of the finite-volume approximants of \( \tilde{F} \) satisfies the estimate

\[
\Delta(j, M) \leq \frac{b(C_M)}{|C_M|} + (C + D) \frac{\partial^M Q_j}{|Q_j|} + C \sum_{P \in \mathcal{P}_0^B(M)} \left\| \# P \cap Q_j - \nu_P \right\| \frac{|\partial M Q_j|}{|Q_j|},
\]

for all \( j, M \in \mathbb{N} \) and

\[
\left\| F - \sum_{P \in \mathcal{P}_0^B(M)} \nu_P \frac{\tilde{F}(P)}{|C_M|} \right\| \leq \frac{b(C_M)}{|C_M|}
\]

for all \( M \in \mathbb{N} \).

**Proof of Theorem** We will derive the explicit bound (1) on \( \Delta(j, M) \). This bound gives immediately that

\[
\lim_{M \to \infty} \lim_{j \to \infty} \Delta(j, M) = 0.
\]

Due to

\[
\left\| \frac{F(Q_j)}{|Q_j|} - \frac{F(Q_m)}{|Q_m|} \right\| \leq \Delta(j, M) + \Delta(m, M)
\]

for all \( M \in \mathbb{N} \), this in turn shows that \( (|Q_j|^{-1} F(Q_j))_{j \in \mathbb{N}} \) is a Cauchy sequence and hence convergent. This convergence then implies convergence of \( \sum_{P \in \mathcal{P}_0^B(M)} \nu_P \frac{\tilde{F}(P)}{|C_M|} \) to the same limit.

We are now going to prove (1). By the triangle inequality we have for arbitrary \( j, M \in \mathbb{N} \) that

\[
\Delta(j, M) \leq D_1(j, M) + D_2(j, M)
\]

with

\[
D_1(j, M) := \left\| \frac{F(Q_j)}{|Q_j|} - \sum_{P \in \mathcal{P}_0^B(M)} \frac{\# P \cap Q_j}{|Q_j|} \frac{\tilde{F}(P)}{|C_M|} \right\|
\]

and

\[
D_2(j, M) := \sum_{P \in \mathcal{P}_0^B(M)} \left\| \frac{\# P \cap Q_j}{|Q_j|} - \nu_P \right\| \frac{\| \tilde{F}(P) \|}{|C_M|}.
\]
By the boundedness assumption on $F$ we have $\|F(P)\|_{C_M} \leq C$ for any $P \in P_0^R(M)$. Thus

$$D_2(j, M) \leq C \sum_{P \in P_0^R(M)} \left| \frac{\sharp_P(A \cap Q_j)}{|Q_j|} - \nu_P \right|$$

(5)

provides the last term in the error estimate (1).

In order to bound the contribution $D_1$, we first introduce some notation. For $M \in \mathbb{N}$, a box $Q \in \mathcal{B}(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$ we consider a (disjoint) covering of $Q$ with boxes $C_M(a)$ shifted from the origin by vectors $a$ in the scaled lattice $\mathbb{Z}_x^M := x + (M \mathbb{Z})^d$. We write

$$S(Q, x, M) := \{ a \in \mathbb{Z}_x^M : C_M(a) \cap Q \neq \emptyset \}$$

for the minimal set of shift vectors needed to generate such a covering of $Q$. The set $S(Q, x, M)$ decomposes into two disjoint parts,

$$S(Q, x, M) = \partial(Q, x, M) \cup I(Q, x, M),$$

(6)

where

$$I(Q, x, M) := \{ a \in \mathbb{Z}_x^M : C_M(a) \subset Q \},$$

$$\partial(Q, x, M) := \{ a \in \mathbb{Z}_x^M : a \notin I(Q, x, M) \text{ and } C_M(a) \cap Q \neq \emptyset \}.$$ 

The interpretation of the sets $I(Q, x, M)$ and $\partial(Q, x, M)$ is as follows: the boxes of the covering that are shifted by $a \in I(Q, x, M)$ are fully contained in $Q$, while those shifted by $a \in \partial(Q, x, M)$ are only partly contained in $Q$. For later purpose we remark that

$$|\partial(Q, x, M)| \leq \frac{\partial M Q}{|C_M|}.$$ 

(7)

Given $M \in \mathbb{N}$, $Q \in \mathcal{B}(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$ arbitrary, we then estimate

$$T(Q, x, M) := \left\| F(Q) - \sum_{a \in I(Q, x, M)} F(C_M(a)) \right\|$$

$$\leq \left\| F(Q) - \sum_{a \in I(Q, x, M)} F(C_M(a) \cap Q) \right\| + \left\| \sum_{a \in \partial(Q, x, M)} F(C_M(a) \cap Q) \right\|$$

$$\leq \left( \sum_{a \in I(Q, x, M)} b(C_M(a)) + \sum_{a \in \partial(Q, x, M)} b(C_M(a) \cap Q) \right) + C \sum_{a \in \partial(Q, x, M)} |C_M(a)|$$

$$\leq |Q| \frac{b(C_M)}{|C_M|} + |\partial(Q, x, M)| (C + D)|C_M|.$$ 

(8)

The first inequality in (8) follows from (6), the second from the almost-additivity of $F$ and the third from the boundedness of $F$ and of the boundary term.

Now, we come back to $D_1$ defined in (3). From

$$\sum_{P \in P_0^R(M)} \sharp_P(A \cap Q_j) \bar{F}(P) = \sum_{z \in \mathbb{Z}^d : C_M(z) \subset Q_j} F(C_M(z)) = \sum_{x \in C_M} \sum_{a \in I(Q_j, x, M)} F(C_M(a))$$
we deduce
\[ |Q_j| D_1(j, M) = \left\| F(Q_j) - \frac{1}{|C_M|} \sum_{x \in C_M} \sum_{a \in \Gamma(Q_j, x, M)} F(C_M(a)) \right\| \leq \frac{1}{|C_M|} \sum_{x \in C_M} T(Q_j, x, M). \]

Therefore, (7) and (8) give
\[ D_1(j, M) \leq \frac{b(C_M)}{|C_M|} + \frac{C + D}{|Q_j|} |\partial^M Q_j|. \]

Together with (2) and (5), this finishes the proof of (11). \qed

Proof of Corollary 11. For every \( j, M \in \mathbb{N} \) we have
\[ \left\| F - \frac{F(Q_j)}{|Q_j|} \right\| = \lim_{m \to \infty} \left\| F(Q_m) - \frac{F(Q_j)}{|Q_j|} \right\| \leq \limsup_{m \to \infty} [\Delta(m, M) + \Delta(j, M)], \]
and the first estimate follows from (11). Similarly, we have
\[ \left\| F - \sum_{P \in \mathcal{P}_0(M)} \nu_P \frac{\tilde{F}(P)}{|C_M|} \right\| = \lim_{j \to \infty} \left\| F(Q_j) - \sum_{P \in \mathcal{P}_0^0(M)} \nu_P \frac{\tilde{F}(P)}{|C_M|} \right\| = \lim_{j \to \infty} \Delta(j, M) \]
for all \( M \in \mathbb{N} \), and (11) implies the second estimate. \qed

4. Uniform existence of the integrated density of states – abstract results

In this section we assume the following situation:

(S) Let \( \Lambda : \mathbb{Z}^d \to \mathcal{A} \) be a colouring and \((Q_j)_{j \in \mathbb{N}}\) a van Hove sequence along which the frequencies \( \nu_P \) of all patterns \( P \in \mathcal{P}_0^0 \) exist. Let \( H : \ell^2(\mathbb{Z}^d, \mathcal{H}) \to \ell^2(\mathbb{Z}^d, \mathcal{H}) \) be a selfadjoint, \( \Lambda \)-invariant finite-range operator. Let \( R = R(H) \) denote the overall range of \( H \) as given in Definition 3.

For certain statements it will be necessary to assume as well Condition (+) The frequencies \( \nu_P \) are strictly positive for all patterns \( P \) which occur in \( \Lambda \), i.e. for which there exists \( x \in \mathbb{Z}^d \) with \( \Lambda \cap (x + Q(P)) = x + P \).

Theorem 2. Assume (S). Then, there exists a unique probability measure \( \mu_H \) on \( \mathbb{R} \) with distribution function \( N_H \) such that \( \frac{1}{|Q_j|} n(p_{Q_j} H_{iQ_j}) \) converges to \( N_H \) with respect to the supremum norm as \( j \to \infty \). The distribution function \( N_H \) is called the integrated density of states (IDS) of \( H \). In fact, the estimate
\[ \left\| \frac{n(p_{Q_j} H_{iQ_j})}{|Q_j|} - N_H \right\|_\infty \leq 8 |\partial^R C_M| \dim(\mathcal{H}) + \sum_{P \in \mathcal{P}_0^0(M)} \left| \nu_P(A \cap Q_j) \right| - \nu_P \]
\[ + 5 |C_R| \frac{|\partial^M Q_j|}{|Q_j|} \dim(\mathcal{H}) \]
holds for all \( j, M \in \mathbb{N} \). If (+) holds as well, then the spectrum of \( H \) is the topological support of \( \mu_H \).
Remark 1. Assumption (+) is necessary to obtain equality of the spectrum and of the topological support of $\mu_H$. This can easily be seen from examples. Take, e.g., the identity operator $id$ on $\ell^2(\mathbb{Z})$ and perform a rank one perturbation $B = (\delta_0, \cdot)\delta_0$ at the origin. Then, the IDS of $id$ and of $id + B$ coincide, but their spectra do not.

We also obtain an analogue of Corollary 1 for the IDS. For this purpose, and for later use as well, we introduce for a given $S \in \mathbb{N}$ the interior core $Q_S := \{x \in Q : \text{dist}(x, \mathbb{Z}^d \setminus Q) > S\} = Q \setminus \partial S Q$ of a bounded set $Q \in F(\mathbb{Z}^d)$.

Corollary 2. Assume (S) and let $N_H$ be given by Theorem 2. For $M \in \mathbb{N}$ and a pattern $P \in \mathcal{P}^0_B(M)$ with $\nu_P > 0$ choose $x \in \mathbb{Z}^d$ such that $x + P = \Lambda \cap (x + Q(P))$ and define $n_P \in \mathcal{L}(\mathbb{R})$ by

$$n_P := \nu_P \mathcal{N}_{x+(Q(P))_R} H i_{x+(Q(P))_R}.$$

Then, $n_P$ does not depend on the choice of $x$ and the bound

$$\left\|N_H - \sum_{P \in \mathcal{P}^0_B(M)} \nu_P \frac{n_P}{|C_M|}\right\|_\infty \leq 8 \dim(\mathcal{H}) \frac{|\partial^R C_M|}{|C_M|}$$

holds for all $M \in \mathbb{N}$.

Remark 2. We emphasise that the dependence on $H$ of the error bounds (9) and (10) is very weak. In fact, it is only the overall range $R$ of $H$ which enters. Thus, our estimates work simultaneously for all $H$ with the same $R$.

Corollary 3. Assume (S) and (+), and let $\lambda \in \mathbb{R}$. Then the following assertions are equivalent:

(i) $\lambda$ is a point of discontinuity of $N_H$.
(ii) there exists a compactly supported eigenfunction of $H$ corresponding to $\lambda$.

The remainder of this section is devoted to a proof of these results. We begin with some preliminary considerations.

Proposition 1 (See Prop. 5.2 in [10]). Let $U$ be a subspace of a finite-dimensional Hilbert space $V$ with inclusion $i: U \to V$ and orthogonal projection $p: V \to U$. Then,

$$\|n(A) - n(pA)\|_\infty \leq 4 \cdot \text{rank}(1 - i \circ p)$$

for every selfadjoint operator $A$ on $V$.

Proposition 2. Assume (S). Then $F^H_R: \mathcal{F}(\mathbb{Z}^d) \to \mathcal{L}(\mathbb{R})$, $Q \mapsto F^H_R(Q) := n(p_{Q_R} Hi_{Q_R})$, is an almost-additive, bounded and $\Lambda$-invariant function with $C = 1$ and boundary term $b(Q) := 4|\partial^R Q| \dim(\mathcal{H})$.

Proof. This is a consequence of the previous proposition. Boundedness with constant $C = 1$ is clear from the definition of $n(\cdot)$. Similarly, $\Lambda$-invariance follows by construction of $F^H_R$ (namely, $F^H_R(Q)$ only depends on $Q \cap \Lambda$). Almost-additivity follows from a decoupling argument (see
More precisely, let \( Q \) be given such that \( Q \) is the disjoint union of \( Q_k \) for \( k = 1, \ldots, m \). By our choice of \( R \) we then have

\[
p_{\bigcup_{k=1}^m Q_k, R} H i_{\bigcup_{k=1}^m Q_k, R} = \bigoplus_{k=1}^m p_{Q_k, R} H i_{Q_k, R}
\]

and hence

\[
n(p_{\bigcup_{k=1}^m Q_k, R} H i_{\bigcup_{k=1}^m Q_k, R}) = \sum_{k=1}^m n(p_{Q_k, R} H i_{Q_k, R}).
\]

Moreover, Proposition \[\ref{prop:uniform-additivity}\] shows that

\[
\|n(p_{Q, R} H i_{Q, R}) - n(p_{\bigcup_{k=1}^m Q_k, R} H i_{\bigcup_{k=1}^m Q_k, R})\|_\infty \leq 4 \sum_{k=1}^m |\partial^R Q_k| \dim(\mathcal{H}).
\]

This gives almost-additivity of \( F^H_R \) with the boundary term

\[
b(Q) := 4|\partial^R Q| \dim(\mathcal{H}),
\]

and the proof is complete. \( \square \)

**Proposition 3.** Let \( A \) be a selfadjoint operator on a finite-dimensional Hilbert space \( V \). Let \( \lambda \in \mathbb{R} \) and \( \varepsilon > 0 \) be given and denote by \( U \) the subspace of \( V \) spanned by the eigenvectors of \( A \) belonging to eigenvalues in the open interval \( (\lambda - \varepsilon, \lambda + \varepsilon) \). If there exist \( k \) pairwise orthogonal and normalised vectors \( u_1, \ldots, u_k \in V \) such that \( (A - \lambda)u_j, j = 1, \ldots, k, \) are pairwise orthogonal and satisfy \( \|(A - \lambda)u_j\| < \varepsilon \), then \( \dim(U) \geq k \).

**Proof.** Denote the linear span of \( u_1, \ldots, u_k \), by \( S \), and let \( P: S \to U \) be the orthogonal projection from \( S \) to \( U \). It suffices to show that \( P \) is one-to-one. Assume the contrary, then there exists a unit vector \( u \in S \) which is orthogonal to \( U \). This orthogonality yields \( \|(A - \lambda)u\| \geq \varepsilon \). On the other hand, the assumptions on the \( u_j, j = 1, \ldots, k \) imply \( \|(A - \lambda)u\| < \varepsilon \). \( \square \)

Now we are prepared for the

**Proof of Theorem \[\ref{thm:uniform-existence}].** The statement on convergence is a direct consequence of Proposition \[\ref{prop:uniform-additivity}\] and Theorem \[\ref{thm:uniform-existence}\]. The explicit bound then follows from Corollary \[\ref{cor:uniform-bound} \] and Proposition \[\ref{prop:uniform-additivity}\].

The last statement on the topological support follows by Weyl-sequence-type arguments. Here are the details: first, note that

\[
\|(p_Q H i_Q - \lambda)u\| = \|(H - \lambda)u\| \quad (11)
\]

whenever \( Q \) is a finite subset of \( \mathbb{Z}^d \) and \( u \) is supported in \( Q_R \), as \( H \) is of range \( R \).

Let now \( \lambda \) belong to the spectrum \( \sigma(H) \). Then, for each \( \varepsilon > 0 \) we can find a box \( Q \) and a normalised vector \( u \) with support in \( Q_R \) and \( \|(H - \lambda)u\| < \varepsilon \). As \( H \) has range \( R \) we infer that both \( u \) and \( (H - \lambda)u \) are supported in \( Q \) and \( (p_Q H i_Q - \lambda)u \) has norm strictly less than \( \varepsilon \) by \( (11) \). Assume now that there are \( k \) disjoint occurrences of translates of \( A \cap Q \) in a set \( Q_j \) for \( j \in \mathbb{N} \). These will provide mutually orthogonal normalised functions \( u_j, j = 1, \ldots, k, \) with \( (H - \lambda)u_j \) pairwise orthogonal and of norm strictly less than \( \varepsilon \). Proposition \[\ref{prop:uniform-additivity}\] shows

\[
n(p_{Q_j} H i_{Q_j})(\lambda + \varepsilon) - n(p_{Q_j} H i_{Q_j})(\lambda - \varepsilon) \geq k. \quad (12)
\]

as well).
By the assumption of positivity of the frequencies we see that the number of disjoint occurrences of a certain pattern grows linearly in the volume of $Q_j$ for large $n$, i.e. $k \geq c |Q_j|$ with $c > 0$ suitable. By uniform convergence of the $n(p_{Q_j} H_{iQ_j})$ this gives $\mu_H([\lambda - \varepsilon, \lambda + \varepsilon]) \geq c > 0$. As $\varepsilon > 0$ is arbitrary, we infer that $\lambda$ belongs to the support of $\mu_H$.

Conversely, let $\lambda$ belong to the support of $\mu_H$. Then there exists for each $\varepsilon > 0$ a $c > 0$ with $\mu_H([\lambda - \varepsilon, \lambda + \varepsilon]) \geq c > 0$. By uniform convergence this gives that

$$n(p_{Q_j} H_{iQ_j})(\lambda + \varepsilon) - n(p_{Q_j} H_{iQ_j})(\lambda - \varepsilon) \geq \frac{c}{2} |Q_j|$$

for sufficiently large $n$. Thus, by standard linear algebra (see [8] for similar reasoning), for sufficiently large $n$ we can find a normalised $u$ which is compactly supported in $Q_{n,R}$ and satisfies

$$\| (p_{Q_n} H_{iQ_n} - \lambda) u \| \leq \varepsilon.$$ 

Then, $\| (H - \lambda) u \| \leq \varepsilon$ by (11), and we see that $\sigma(H) \cap [\lambda - \varepsilon, \lambda + \varepsilon] \neq \emptyset$. As $\varepsilon > 0$ is arbitrary, we infer that $\lambda$ belongs to $\sigma(H)$. □

**Proof of Corollary 2.** This is a direct consequence of Proposition 2 and Corollary 1. □

**Proof of Corollary 3.** The corollary follows from Theorem 2 by mimicking the argument given in [8] to prove Theorem 2 there. □

5. Application to periodic operators

In this section we apply the above abstract results to periodic operators on graphs with a $\mathbb{Z}^d$-structure. It turns out that for a large class of such operators, it is sufficient to consider the case where the set $A$ has just one element. So we turn to this situation now.

If $|A| = 1$, the colouring $\Lambda$ is a trivial map, and the only information that is contained in an $A$-pattern $P$ is its domain $Q(P)$, which is by definition a finite subset of $\mathbb{Z}^d$. For $M$ twice as large as the diameter of $Q(P)$ and for any sequence van Hove sequence $(Q_j)_{j \in \mathbb{N}}$ we have

$$1 \geq \frac{4 p(A \cap Q_j)}{|Q_j|} \geq \frac{|Q_{j,M}|}{|Q_j|} \rightarrow 1 \quad \text{for } j \rightarrow \infty.$$ 

Thus the frequency of any pattern $P$ equals $\nu_P = 1$. In this situation, where the frequencies of patterns happen to exist along any van Hove sequence, it is particularly convenient to choose them as boxes, more precisely $Q_j = C_j$ for all $j \in \mathbb{N}$.

Now, for a bounded, selfadjoint operator $H$ as in Definition 3, Theorem 2 gives for all $j, M \in \mathbb{N}$ the estimate

$$\left\| \frac{n(p_{C_j} H_{iC_j})}{j^d} - N_H \right\|_\infty \leq 8 d \dim(\mathcal{H}) \left( \frac{4R}{M} + \frac{5R d M}{j} \right) + \frac{d M}{j}.$$ 

Similarly as in Corollary 2 we set $n_{C_M} := n(p_{C_{M,R}} H_{iC_{M,R}}) \in \mathcal{L}(\mathbb{R})$ and obtain thus

$$\| N_H - \frac{n_{C_M}}{|C_M|} \|_\infty \leq \dim(\mathcal{H}) \frac{16 d R}{M}$$

for all $M \in \mathbb{N}$.

Now we describe the geometry of the class of graphs with $\mathbb{Z}^d$-structure on which we will define our periodic operators later on. Let $G$ be a graph with a countable set of vertices
(which we again denote by $G$), and $T$ a representation of $\mathbb{Z}^d$ by isometric graph-isomorphisms $T_\gamma: G \to G$, $\gamma \in \mathbb{Z}^d$. We assume that the action $T$ of $\mathbb{Z}^d$ on $G$ is free and cocompact.

Let us denote by $\mathcal{D} \subset G$ a $\mathbb{Z}^d$-fundamental domain. Thus $\mathcal{D}$ contains exactly one element of each $\mathbb{Z}^d$-orbit in $G$. By the cocompactness assumption, $\mathcal{D}$ is finite. This implies in particular that the vertex degree of $G$ is uniformly bounded. From now on the fundamental domain $\mathcal{D}$ will be assumed fixed.

A simple example of such a graph is $\mathbb{Z}^d$ with the group $\mathbb{Z}^d$ acting on it by $T_\gamma(x) = x - N\gamma$, $N \geq 1$, for all $x \in \mathbb{Z}^d$ and all $\gamma \in \mathbb{Z}^d$. Another example would be the Cayley graph $G = G(\mathcal{G}, E)$ of a direct product group $G = \mathbb{Z}^d \otimes F$ where $F$ is any finite group and $E$ is a symmetric set of generators for $\mathcal{G}$. The action of $\mathbb{Z}^d$ on $G$ is given by

$$T_\gamma(x, f) = (x - \gamma, f), \quad (x, f) \in \mathcal{G}, \gamma \in \mathbb{Z}^d,$$

on the set of vertices $\mathcal{G}$ and analogously on the set of edges.

Now we introduce operators acting on $\ell^2(G)$ and $\ell^2(\mathbb{Z}, \mathcal{H})$. Let $A: \ell^2(G) \to \ell^2(G)$ be a selfadjoint linear operator satisfying the following equivariance condition

$$A(x, y) = A(T_\gamma x, T_\gamma y) \quad \text{for all } x, y \in G, \gamma \in \mathbb{Z}^d.$$

Furthermore we assume that $A(x, y) \neq 0$ implies that the graph-distance of $x$ and $y$ is smaller than $\rho \in \mathbb{R}$.

Set $\mathcal{H} := \ell^2(\mathcal{D})$, then $\dim(\mathcal{H}) = |\mathcal{D}|$, and define a unitary operator $U: \ell^2(\mathbb{Z}, \mathcal{H}) \to \ell^2(G)$ in the following way: for a $\psi \in \ell^2(\mathbb{Z}, \mathcal{H})$ and $\gamma \in \mathbb{Z}^d$ write $\psi(\gamma) = \sum_{i \in \mathcal{D}} \psi_i(\gamma) \delta_i$ where $(\delta_i)_{i \in \mathcal{D}}$ is the standard orthonormal basis of $\ell^2(\mathcal{D})$. Note that the coefficients $\psi_i(\gamma)$ are uniquely defined. Define now $U\psi \in \ell^2(G)$ by $(U\psi)(x) := \psi_i(\gamma)$ where $i \in \mathcal{D}$ and $\gamma \in \mathbb{Z}^d$ are the unique elements such that $x = T_\gamma i$. The inverse $U^*$ is given by $(U^*\phi)(\gamma) = \sum_{i \in \mathcal{D}} \phi(T_\gamma i) \delta_i$ for $\phi \in \ell^2(G)$.

Next we define the operator $H: \ell^2(\mathbb{Z}, \mathcal{H}) \to \ell^2(\mathbb{Z}, \mathcal{H})$ by $H := U^* AU$ and show that it is a $A$-invariant operator of finite range. Let $\alpha, \beta \in \mathbb{Z}^d$ be arbitrary. Then $U_{\alpha, \beta}$ maps $\mathcal{H}$ to $\ell^2(T_{\beta} \mathcal{D})$ and $p_{\alpha} U_{\beta}^*$ maps $\ell^2(T_{\alpha} \mathcal{D})$ to $\mathcal{H}$. Since $G$ and its (abelian) covering transformation group $\mathbb{Z}^d$ are quasi-isometric, it follows that if we choose $R_{fr} > 0$ sufficiently large, we have $|x - y| > \rho$ for all $x \in T_{\gamma} \mathcal{D}$, $y \in T_{\beta} \mathcal{D}$, $|\alpha - \beta| > R_{fr}$. Thus $p_{\alpha} U_{\beta}^* AU_{\beta} = 0$ for $|\alpha - \beta| > R_{fr}$. A straightforward calculation shows that $p_{\alpha + \gamma} H_{i_{\beta} + \gamma} = p_{\alpha} H_{i_{\beta}}$.

6. Application to Anderson-percolation models

In this section we discuss similar operators as in the previous one, however now some randomness enters the model. In particular the colouring $A$ is no longer trivial.

Let $G$ denote the graph $\mathbb{Z}^d \subset \mathbb{R}^d$ where two vertices are adjacent, if and only if their Euclidean distance is equal to one. For some $N \in \mathbb{N}$ fixed consider the action of the group $\Gamma = (N\mathbb{Z})^d$ on $G$ by translations, i.e. $T_\gamma x = x - \gamma$ for all $\gamma \in \Gamma$ and $x \in G$. Define a colouring by $\mathcal{M}: \mathbb{Z}^d \to C_N, \mathcal{M}(x) \equiv x \mod \Gamma$. Thus the colouring $\mathcal{M}$ is $\Gamma$-periodic, i.e. $\mathcal{M}(x) = \mathcal{M}(T_\gamma x)$ for all $\gamma \in \Gamma$ and $x \in G$. In particular $\mathcal{M}$ is uniquely defined by its values on the set $C_N \subset G$. Let $A: \ell^2(G) \to \ell^2(G)$ be a selfadjoint finite-range operator which is $\mathcal{M}$-invariant, in other words $\Gamma$-periodic. This will be the deterministic ingredient determining the hopping part of the random operators we want to consider in this section.
To define the random part, let $(Ω, ℙ)$ be a probability space and $τ_γ : Ω → Ω$ for $γ ∈ Γ$, an ergodic family of measure preserving transformations. Furthermore, let $A$ be an arbitrary finite subset of $ℝ ∪ \{+∞\}$ and $(ω, x) → V(ω, x) ∈ A$ a random field which is invariant under the transformations $τ_γ$, $γ ∈ Γ$. More precisely, for all $γ ∈ Γ$, $ω ∈ Ω$ and $x ∈ G$ we require $V(τ_γω, x) = V(ω, T_γx)$. Next we define random subsets of $G$ and $ℓ^2(G)$ induced by the random field $V$. For each $ω ∈ Ω$ define the subset of vertices $G_ω := \{x ∈ G : V(ω, x) < ∞\}$, the natural projection operator $p_ω : ℓ^2(G) → ℓ^2(G_ω)$ and its adjoint $i_ω : ℓ^2(G_ω) → ℓ^2(G)$.

The random Hamiltonian on $ℓ^2(G_ω)$ which we want to study is defined in the following way: the hopping part is given by

$$A_ω := p_ω A i_ω, \quad \text{dom}(A_ω) := ℓ^2(G_ω).$$

On the set $G_ω$, the mapping $x → V(ω, x)$ is a bounded, real-valued function and $V_ω := p_ω V(ω, ·) i_ω : ℓ^2(G_ω) → ℓ^2(G_ω)$ denotes the corresponding multiplication operator. The total Hamiltonian is then given by

$$H_ω := A_ω + V_ω, \quad \text{dom}(H_ω) := ℓ^2(G_ω)$$

It is $Γ$-stationary in the sense that $U_γH_ωU^*_γ = H_{τ_γω}$, where $U_γφ(x) := φ(x − γ)$ for all $x ∈ G$ and $γ ∈ Γ$. Since $A_ω$ is generated from the periodic operator $A$ by a site-percolation process on the graph $G$, it may be called a (site-) percolation Hamiltonian. On the other hand, $V_ω$ is a random potential as in the Anderson model on $ℓ^2(ℤ^d)$. For this reasons we call $H_ω$, which contains features of both models, an Anderson-percolation Hamiltonian. For such operators the existence of the IDS as a pointwise limit has been established in [15] and its continuity properties have been analysed in [16] and a contribution to [3].

For each $ω ∈ Ω$ we define a colouring by

$$A_ω : ℤ^d → A × C_N, \quad A_ω(x) := (V(ω, x), M(x)).$$

For any pattern $P : Q(P) → A, Q(P) ∈ F(ℤ^d)$, the frequency $ν_P$ of $P$ in $A_ω$ along the van Hove sequence of boxes $C_j, j ∈ N$, exists by the ergodic theorem and is independent of $ω$ almost surely. If $ν_P$ is positive, let $Ω_P$ consist of all $ω ∈ Ω$ for which the frequency of $P$ exists and equals $ν_P$. If $ν_P$ is zero, then the set of $ω'$ in $Ω$ for which the pattern $P$ does occur at all has measure zero (as for each fixed position the set of $ω'$ for which $P$ occurs at this position has measure zero and there are only countably many positions). In this case denote the complement of this set by $Ω'_P$. In both cases, the set $Ω_P$ has full measure. Since there are only countably many finite subsets of $ℤ^d$, the intersection $Ω' := \bigcap_P Ω_P$ has again measure one. In the following we chose an $ω ∈ Ω'$ and keep it fixed.

To fit the operator $H_ω$ in the abstract setting we have been considering in Sections 2 to 4, we extend it to the whole of $ℓ^2(G)$ by setting it equal to zero on $ℓ^2(G \setminus G_ω)$. We denote this extension again by $H_ω$. Thus the operator $H_ω$ is $A_ω$-invariant with $R_{inv} = 1$ and is of finite range, since it is a sum of a finite-range and a diagonal operator. Therefore, Theorem 2 gives the existence of a unique probability measure $μ_{H_ω}$ on $ℝ$ with distribution function $N_{H_ω}$ such that $\frac{1}{|C_j|} n(p_{C_j} H_{C_j})$ converges to $N_{H_ω}$ with respect to the supremum norm: for every $j, M ∈ N$ we have the explicit estimate

$$\left\| \frac{n(p_{C_j} H_{C_j})}{|C_j|} - N_{H_ω} \right\|_∞ ≤ 8 |∂^R C_M| \sum_{P ∈ F_0(M)} |ζ_P(A_ω \cap C_j)| - ν_P + 5 |C_R| \frac{|∂^M C_j|}{|C_j|}.$$
The spectrum of $H_\omega$ and the topological support of $\mu_{H_\omega}$ coincide. Moreover, by Corollary 2 and the fact that the frequencies are independent of $\omega \in \Omega'$, we see that the distribution function $N_{H_\omega}$ is in fact the same for all $\omega \in \Omega'$ and that the spectrum of $\sigma(H_\omega)$ as a set does not depend on $\omega \in \Omega'$.

Let us close this section by pointing out to which situations the results presented here can be easily extended: (1) If the random part $V_\omega$ of the Hamiltonian $H_\omega$ is absent, we obtain a $\Gamma$-periodic, i.e. $\mathcal{M}$-invariant operator. This shows that in certain cases, one can use an alternative description of periodic operators by graph colourings to the one presented in Section 5.

(2) On the other hand, one can combine the constructions from Section 5 and the present one to define random Hamiltonians on more general $\mathbb{Z}^d$-equivariant graphs than $\mathbb{Z}^d$ itself.

(3) Analogous results to those derived here for site-percolation Hamiltonians can be derived for Hamiltonians on bond-percolation graphs by using the same arguments. For the construction of the IDS for such operators and for the asymptotics of the IDS at spectral edges, see [7, 12].

7. Application to visible points

The set of visible points in $\mathbb{Z}^d$ is a prominent example (and counterexample) in number theory and aperiodic order [11, 17]. In particular its diffraction theory has been well studied. Still, it seems that the corresponding nearest-neighbour hopping model has not received attention so far. In this section we provide a first modest step towards studying such a model by showing existence of the associated integrated density of states.

The set $V$ of visible points in $\mathbb{Z}^d$ consists of the origin and all $x \neq 0$ in $\mathbb{Z}^d$ with

$$\{tx : 0 < t < 1\} \cap \mathbb{Z}^d = \emptyset.$$ 

Put differently, $x \neq 0$ belongs to $V$, if and only if the greatest common divisor of its coordinates is 1. The obvious interpretation is that such an $x$ can be seen by an observer standing at the origin. This gives the name to this set. The characteristic function

$$\Lambda := \chi_V : \mathbb{Z}^d \to \mathcal{A} := \{0, 1\}$$

of $V$ provides a colouring. While $V$ is very regular in many respects, it has arbitrarily large holes. In particular, existence of the frequencies $\nu_p$ does not hold along arbitrary van Hove sequences. However, as was shown in [17] (see [11] for special cases as well), the frequencies exist and can be calculated explicitly for sequences of cubes containing the origin. Moreover, the frequencies of all patterns which occur are strictly positive.

Thus, all abstract results discussed in this paper are valid for $\chi_V$-invariant operators of finite range. One relevant such operator is the hopping Laplacian $\Delta_V$. We finish this section by defining this operator: Points $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $\mathbb{Z}^d$ are said to be neighbours, written as $x \sim y$, whenever

$$\sum_{j=1}^d |x_j - y_j| = 1.$$
Then, $\Delta_V: \ell^2(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)$ is defined by

$$(\Delta_V u)(x) := \chi_V(x) \sum_{y \sim x \in V} u(y)$$

for all $x \in \mathbb{Z}^d$ and all $u \in \ell^2(\mathbb{Z}^d)$. Obviously, assumptions (S) and (+) are fulfilled for this operator, and the results of Section 4 apply.

8. OPEN QUESTIONS

The considerations of the previous sections naturally raise various questions. Most prominent is the question to what extent similar ergodic theorems hold on more general groups than $\mathbb{Z}^d$. More specifically, one may consider finitely generated groups which are amenable and/or residually finite.

One may wonder about analogous statements for the IDS for suitable operators in continuous geometries. As an intermediate step between continuum and discrete models one may consider quantum graphs. In both cases the distributions in questions are no longer bounded.

Finally, it may be interesting to learn more about the spectral theory of the nearest-neighbour hopping Laplacian on the set of visible points.

Acknowledgments. Part of this work was done while one of the authors (D.L.) was visiting Göttingen. He would like to thank Thomas Schick for the invitation. He would also like to thank Michael Baake, Peter Pleasants and Bernd Sing for useful discussions concerning the set of visible points. Partial support from DFG is gratefully acknowledged.

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