Deutsch-Jozsa Algorithm Revisited in the Domain of Cryptographically Significant Boolean Functions

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Abstract

Boolean functions are important building blocks in cryptography for their wide application in both stream and block cipher systems. For cryptanalysis of such systems one tries to find out linear functions that are correlated to the Boolean functions used in the crypto system. Let \( f \) be an \( n \)-variable Boolean function and its Walsh spectra is denoted by \( W_f(\omega) \) at the point \( \omega \in \{0,1\}^n \). The Boolean function is available in the form of an oracle. We like to find an \( \omega \) such that \( W_f(\omega) \neq 0 \) as this will provide one of the linear functions which are correlated to \( f \). We show that the quantum algorithm proposed by Deutsch and Jozsa (1992) solves the above mentioned problem in constant time. However, the best known classical algorithm to solve this problem requires exponential time in \( n \). We also analyse certain classes of cryptographically significant Boolean functions and highlight how the basic Deutsch-Jozsa algorithm performs on them.

Keywords: Boolean Functions, Query Complexity, Quantum Algorithms, Walsh Spectra.

1 Introduction

Many of the symmetric (private) key crypto systems use nonlinear Boolean functions in the design process. Nonlinearity is an important property of Boolean functions to resist the linear cryptanalysis [15] on block cipher systems like DES. Apart from nonlinearity, the Boolean functions should also possess other cryptographic properties. In the nonlinear combiner model of stream cipher systems, correlation immunity is an important cryptographic property for a Boolean function to be used in the schemes [23, 24]. Both the nonlinearity and the correlation immunity can be described in terms of Walsh spectra of the Boolean function (see Subsection 1.1 for exact details). Construction of highly nonlinear and correlation immune Boolean functions are available in literature (see [23, 24, 22, 16, 5].
and the references in these papers). Even if a Boolean function is highly nonlinear and correlation immune of certain order, due to the Parseval’s relation [8], there always exist linear functions which are correlated to the Boolean function in use. In the design, it is always attempted to reduce the correlation, which is the job of the cryptographer. On the other hand, the cryptanalyst tries to exploit the correlation to mount the attack (see [4, 14] and the references in these papers for more details). To device such an attack, one needs a linear function which is correlated to the Boolean function. Given an $n$-variable Boolean function $f$, this requires the Walsh spectra of the Boolean function and the Fast Walsh Transform algorithm requires $O(n2^n)$ time when the truth table of the Boolean function is available. If the Boolean function is available in the form of an oracle (black box), then $2^n$ steps are required to get the truth table and then only the Fast Walsh Transform can be applied. This is the best known classical algorithm known in this area. On the other hand we identify that the well known Deutsch-Jozsa algorithm [7] can solve this problem in constant time under the quantum computational framework. It has been commented [17, Page 36] that the Deutsch-Jozsa algorithm has not much application in practical sense. This is the first time we show how this algorithm can be used to solve a problem which naturally comes from cryptographic domain.

Now we like to point out the importance of the problem from the quantum complexity theoretic viewpoint. For detailed discussion on complexity classes and their hierarchies see [17, 9]. The Deutsch-Jozsa problem [7] (distinguishing between balanced and constant Boolean functions) presents relativized separation of P and EQP, but not of BPP and BQP. In [2], Bernstein and Vazirani presented relativized separation between BPP and BQP using recursive Fourier sampling. Though the problem is important from complexity theoretic point of view, it has been commented to be artificial [11]. Bernstein and Vazirani [3] have further shown the relativized separation of NP and even MA from BQP and conjectured that recursive Fourier sampling is not in PH (related discussion is also available in [11]). Green and Pruim [10] presented relativized separation between BQP and P$^{NP}$ using a nice technique based on Grover’s algorithm [11]. Aaronson has commented in [11] that it may need a completely different problem than recursive Fourier sampling to provide a relativized separation between BQP and PH. The problems we mention here (specifically see Problem 5 in Section 2) may be a good candidate in this direction.

1.1 Preliminaries: Boolean Functions

A Boolean function on $n$ variables may be viewed as a mapping from $\{0, 1\}^n$ into $\{0, 1\}$. The set of all $n$-variable Boolean functions is denoted by $\Omega_n$.

A Boolean function $f(x_1, \ldots, x_n)$ is also interpreted as the output column of its truth table $f$, i.e., a binary string of length $2^n$,

$$f = [f(0, 0, \cdots, 0), f(1, 0, \cdots, 0), f(0, 1, \cdots, 0), \ldots, f(1, 1, \cdots, 1)].$$

If a Boolean function is presented as an oracle (a black box), then one can only present an $n$-bit input and get the 1-bit output corresponding to that. Thus, to get the truth table, one needs to query the oracle $2^n$ times in a classical computational model.
The Hamming distance between \(S_1, S_2\) is denoted by \(d(S_1, S_2)\), i.e., \(d(S_1, S_2) = |S_1 \neq S_2|\). Also the Hamming weight or simply the weight of a binary string \(S\) is the number of ones in \(S\). This is denoted by \(wt(S)\). An \(n\)-variable function \(f\) is said to be balanced if its output column in the truth table contains equal number of 0’s and 1’s (i.e., \(wt(f) = 2^{n-1}\)).

Let us denote addition operator over \(GF(2)\) by \(\oplus\). An \(n\)-variable Boolean function \(f(x_1, \ldots, x_n)\) can be considered to be a multivariate polynomial over \(GF(2)\). This polynomial can be expressed as a sum of products representation of all distinct \(k\)-th order products \((0 \leq k \leq n)\) of the variables. More precisely, \(f(x_1, \ldots, x_n)\) can be written as

\[
a_0 \oplus \bigoplus_{1 \leq i \leq n} a_i x_i \oplus \bigoplus_{1 \leq i < j \leq n} a_{ij} x_i x_j \oplus \cdots \oplus a_{12 \ldots n} x_1 x_2 \ldots x_n,
\]

where the coefficients \(a_0, a_{ij}, \ldots, a_{12 \ldots n} \in \{0, 1\}\). This representation of \(f\) is called the algebraic normal form (ANF) of \(f\). The number of variables in the highest order product term with nonzero coefficient is called the algebraic degree, or simply the degree of \(f\) and denoted by \(deg(f)\).

Functions of degree at most one are called affine functions. An affine function with constant term equal to zero is called a linear function. The set of all \(n\)-variable affine (respectively linear) functions is denoted by \(A(n)\) (respectively \(L(n)\)). The nonlinearity of an \(n\)-variable function \(f\) is

\[
nl(f) = \min_{g \in A(n)} (d(f, g)),
\]

i.e., the distance from the set of all \(n\)-variable affine functions.

Let \(x = (x_1, \ldots, x_n)\) and \(\omega = (\omega_1, \ldots, \omega_n)\) both belong to \(\{0, 1\}^n\) and the inner product

\[
x \cdot \omega = x_1 \omega_1 \oplus \cdots \oplus x_n \omega_n.
\]

Let \(f(x)\) be a Boolean function on \(n\) variables. Then the Walsh transform of \(f(x)\) is a real valued function over \(\{0, 1\}^n\) which is defined as

\[
W_f(\omega) = \sum_{x \in \{0,1\}^n} (-1)^{f(x) \cdot x \cdot \omega}.
\]

Given a Boolean function \(f\), \(W_f(\omega) = \#(f = l) - \#(f \neq l)\), where \(l = \omega \cdot x\) is a linear function. If \(W_f(\omega) = 0\), then there is no correlation between \(f\) and \(l\). However, if \(W_f(\omega) > 0\), then there is correlation between \(f, l\) as \(\#(f = l) > \#(f \neq l)\). Similarly, if \(W_f(\omega) < 0\), then there is correlation between \(f, 1 \oplus l\) as \(\#(f = l) < \#(f \neq l)\), which gives \(\#(f = 1 \oplus l) > \#(f \neq 1 \oplus l)\). This correlation between the Boolean function \(f\) and the linear function \(l\) (or the affine function \(1 \oplus l\)) is exploited for cryptanalytic attacks \(\square\). Thus, given a Boolean function \(f\), it is important to find out some \(\omega\) such that \(W_f(\omega) \neq 0\).

It should be noted that getting the Walsh spectra is not an easy problem in general. See Algorithm\(\square\) in this Section and Proposition\(\square\) in Section\(\square\) later for further discussion.

In terms of Walsh spectra, the nonlinearity of \(f\) is given by

\[
nl(f) = 2^n - \frac{1}{2} \max_{\omega \in \{0,1\}^n} |W_f(\omega)|.
\]
One important identity related to the Walsh spectra of any \( n \)-variable Boolean function \( f \) is the Parseval’s identity which gives

\[
\sum_{\omega \in \{0,1\}^n} W_f^2(\omega) = 2^{2n}.
\]

It is clear that the maximum nonlinearity is achieved when the maximum absolute value of the Walsh spectra is minimized. For \( n \) even, this happens when \( W_f(\omega) = \pm 2^{n/2} \), for each \( \omega \in \{0,1\}^n \). These functions, having nonlinearity \( 2^{n-1} - 2^{n/2-1} \), are well known as bent functions in literature. For \( n \) odd, \( \frac{n}{2} \) is not an integer and hence the situation becomes more complicated. For \( n \leq 7 \), it is known that the maximum possible nonlinearity can be \( 2^{n-1} - 2^{n/2-1} \). It has been shown in [18] that one can achieve nonlinearity strictly greater than \( 2^{n-1} - 2^{n/2-1} \) for \( n \geq 15 \).

In [12], an important characterization of resilient (balanced and correlation immune) functions has been presented, which we use as the definition here. A function \( f(x_1, \ldots, x_n) \) is \( m \)-resilient iff its Walsh transform satisfies

\[
W_f(\omega) = 0, \text{ for } 0 \leq wt(\omega) \leq m.
\]

As the notation used in [21, 22], by an \((n, m, d, \sigma)\) function we denote an \( n \)-variable, \( m \)-resilient function with degree \( d \) and nonlinearity \( \sigma \). For recent results on such functions see [21, 22, 5] and the references in these papers.

Now let us present the best known classical algorithm for calculating the Walsh spectra of a Boolean function. If the function is given as a black box, then one needs to get the truth table first, which requires \( 2^n \) many query to the oracle.

**Algorithm 1**

**Input:**
(i) A Boolean function \( f \) on \( n \) variables is available in the form of an oracle (black box);

1. Oracle \( f \) is queried \( 2^n \) many times to get the truth table as an integer array \( f[0, \ldots, 2^n - 1] \) of \( 0,1 \);
2. for \( (i = 0; i < 2^n; i = i + 1) \) \( f[i] = (-1)^{f[i]} \);
3. for \( (i = 0; i < n; i = i + 1) \) \{ 
   \( 3a. \) for \( (k = 0; k < 2^n; k = k + 2^{i+1}) \) \{ 
      \( 3a(i). \) for \( (j = k; j < k + 2^i; j = j + 1) \) \{ 
         \( 3a(i)A. \) \( a = f[j] + f[j + 2^i]; \)
         \( 3a(i)B. \) \( b = f[j] - f[j + 2^i]; \)
         \( 3a(i)C. \) \( f[j] = a; \)
         \( 3a(i)D. \) \( f[j + 2^i] = b; \)
      \}
   \}
   \( 3b. \) \}
4. \}
In the following we present an example how the Algorithm 1 runs. Note that the function used is a 3-variable one, and \( i \) varies from 0 to 2, i.e., \( n = 3 \) steps. The inner steps (using \( k, j \)) runs \( 2^3 = 8 \) many times.

| \( x_3 \) | \( x_2 \) | \( x_1 \) | \( f \) | \((-1)^i\) | \( i = 0 \) | \( i = 1 \) | \( i = 2 \) |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | -1 | -2 | -2 | 0 |
| 0 | 0 | 1 | 1 | -1 | 0 | 2 | 0 |
| 0 | 1 | 0 | 0 | 1 | 0 | -2 | -4 |
| 0 | 1 | 1 | 1 | -1 | 2 | -2 | 4 |
| 1 | 0 | 0 | 1 | -1 | 0 | 2 | -4 |
| 1 | 0 | 1 | 0 | 1 | -2 | -2 | 4 |
| 1 | 1 | 0 | 0 | 1 | 2 | -2 | 0 |
| 1 | 1 | 1 | 0 | 1 | 0 | -2 | 0 |

1.2 Preliminaries: The Deutsch-Jozsa Algorithm

Given \( f \) is either constant or balanced, one may ask for an algorithm, that can answer what exactly it is. In this case the Boolean function \( f \) is available in the form of an oracle (black box), where one can apply an input to the black box to get the output. A classical algorithm needs to check the function for \( 2^{n-1} + 1 \) many inputs in worst case to decide whether the function is constant or balanced.

Now we discuss the quantum computational model. It is known that given a classical circuit \( f \), there is a quantum circuit of comparable efficiency which computes the transformation \( U_f \) that takes input like \( |x, y\rangle \) and produces output like \( |x, y \oplus f(x)\rangle \). Given such an \( U_f \) is available, Deutsch-Jozsa [7] provided a quantum algorithm that can solve this problem in constant time. We first present how the quantum circuit looks like in Figure 1 and then explain the algorithm in Algorithm 2.

![Figure 1: Quantum circuit to implement Deutsch-Jozsa Algorithm](image-url)
Algorithm 2 Deutsch-Jozsa Algorithm \[7\]

1. $|\psi_0\rangle = |0\rangle^{\otimes n} |1\rangle$
2. $|\psi_1\rangle = \sum_{x \in \{0,1\}^n} \frac{|x\rangle}{\sqrt{2^n}} \left[ |0\rangle - |1\rangle \right]$ \[0\rangle - |1\rangle \right]$
3. $|\psi_2\rangle = \sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)}}{\sqrt{2^n}} \left[ |0\rangle - |1\rangle \right]\sqrt{2\frac{1}{2^n}}$ \[0\rangle - |1\rangle \right]$
4. $|\psi_3\rangle = \sum_{z \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot z + f(x) \cdot z} \frac{|x\rangle}{\sqrt{2^n}} \left[ |0\rangle - |1\rangle \right]$ \[0\rangle - |1\rangle \right]$
5. Measurement at $M$: all zero state implies that the function is constant, otherwise it is balanced.

In the next section we will keep the Algorithm 2 as it is and interpret the Step 5 of it according to our need.

## 2 Problems in EQP

Let us start with some technical results on hardness of calculating the Walsh spectra.

**Proposition 1** A Boolean function $f$ is available in the form of an oracle.

1. SAT is Turing reducible to computing Walsh transform at the point 0.
2. Finding $W_f(0)$ is outside $P^{NP}$.
3. Given a non zero $\omega$, finding $W_f(\omega)$ is outside $P^{NP}$.

**Proof:** The function $f$ is not satisfiable, iff $W_f(0) = 2^n$. This proves item 1.

Now we prove item 2. In [10], the following problem has been presented which is outside $P^{NP}$. A Boolean function $f$ with $wt(f)$ either $2^n - 2$ or $3 \cdot 2^{n-2}$ is given in the form of an oracle. One has to identify which one is this. Note that $wt(f) = 2^n - 2$ iff $W_f(0) = 2^{n-1}$ and $wt(f) = 3 \cdot 2^{n-2}$ iff $W_f(0) = -2^{n-1}$.

The proof of item 3 is as follows. $W_f(0) = W_{f \oplus \omega \cdot x}(\omega)$. If the oracle of $f$ is available, then it is easy to construct the oracle of $f \oplus \omega \cdot x$. Hence the proof.

We have already discussed in Algorithm 1 that the best known classical algorithm for calculating the Walsh spectra of an $n$-variable Boolean function requires the truth table of size $2^n$ as an input and then the algorithm requires $O(2^n)$ time. Let us now describe our interpretation of Deutsch-Jozsa Algorithm in terms of Walsh spectra. Note that $\sum_{z \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot z + f(x) \cdot z} \frac{W_f(z)}{2^n} |z\rangle$, i.e., the associated probability with a state $|z\rangle$ is $\frac{W_f^2(z)}{2^n}$. Hence we have the following result.

**Proposition 2** Given an $n$-variable Boolean function $f$, the Deutsch-Jozsa algorithm (Algorithm 2) produces a super position of all the states $z \in \{0,1\}^n$ at the measurement point $M$ with amplitude $\frac{W_f^2(z)}{2^n}$ corresponding to each state $z$.

Now let us describe the following problem which has been presented in [2] as parity problem.
Problem 1  \[2\] Let $f$ be an linear $n$-variable Boolean function, i.e., $f(x) = \omega \cdot x$, available in the form of an oracle. Find out the $\omega$.

For a linear function $f(x) = \omega \cdot x$, $W_f(\omega) = 2^n$ and $W_f(z) = 0$, for $z \neq \omega$. Thus the observed state of $n$ bits in the Step 5 of Algorithm 2 will clearly output $\omega$ itself (with probability $\frac{W_f(\omega)}{2^{2n}} = 1$). Thus the Deutsch-Jozsa algorithm solves this problem in constant time. In classical model this problem clearly needs $O(n)$ time. This difference has been exploited and it has been shown that BPP is not equal to BQP with respect to an oracle \[2\].

Now we present the problem we described.

Problem 2 A Boolean function $f$ is given in the form of an oracle. Find out an $\omega$, such that $W_f(\omega) \neq 0$.

The solution to this problem using the Deutsch-Jozsa algorithm works as follows. Let us consider that $S = \{\omega | W_f(\omega) \neq 0\}$. For any $\omega \in \{0,1\}^n \setminus S$, $W_f(\omega) = 0$. Note that for $\omega \in S$, $\sum_{x \in \{0,1\}^n} (-1)^{f(x) \cdot x \cdot \omega}$ is nonzero and for $\omega \in \{0,1\}^n \setminus S$, $\sum_{x \in \{0,1\}^n} (-1)^{f(x) \cdot x \cdot \omega}$ is zero.

We have already discussed that the associated probability with a state $|z\rangle$ is $\frac{W_f(z)}{2^n}$. Here the probability associated with $|z\rangle$ is nonzero when $z \in S$ and the probability associated with $|z\rangle$ is 0 when $z \in \{0,1\}^n \setminus S$. It is clear that the sum of probabilities associated with the states in $S$ is 1. Thus, the state, say $\omega$, observed after the measurement at Step 5 belongs to $S$ and for the observed $\omega$, $W_f(\omega) \neq 0$. Hence the Problem 2 can be solved in constant time using the Deutsch-Jozsa algorithm.

Based on the above discussion we have the following result.

Theorem 1 The Problem 2 belongs to EQP with respect to the oracle $f$.

Now we present a related problem where one needs to find out the maximally correlated linear or affine function with respect to $f$.

Problem 3 A Boolean function $f$ is given in the form of an oracle. Find out an $\hat{\omega}$, such that $W_f(\hat{\omega}) = \max_{\omega \in \{0,1\}^n} |W_f(\omega)|$.

Algorithm 2 does not guarantee the answer to Problem 3. Since Algorithm 2 is probabilistic in nature, it may very well happen that it outputs some $\omega'$, for which $W_f(\omega') \neq 0$, but $|W_f(\omega')| < |W_f(\omega)|$. That means we get a linear or affine function which is correlated to $f$, but not maximally correlated.

There exists a sub class of Boolean functions, the bent functions \[20\], for which one can solve Problems 2 3 in one step using classical computational model also. For a bent function $f$, $W_f(\omega) = \pm 2^n$, for any $\omega \in \{0,1\}^n$. Thus if it is known that the function is a bent function, then one can choose any $\omega$ and produce that as the output. However, it is very clear these problems are not easy in general.

One very interesting class of Boolean functions are the ones where the Walsh spectra become three valued $0, \pm 2^k$. These functions are referred as plateaued functions in literature \[25, 6\]. The class of plateaued functions contains cryptographically significant Boolean functions, including certain classes of resilient functions \[25, 22, 6\] and hence these functions are actually used in crypto systems. Now consider the following problem.
Problem 4 A plateaued Boolean function \( f \) (i.e., \( W_f(\omega) \) can take the values \( 0, \pm 2^k \)) is given in the form of an oracle. Find out an \( \hat{\omega} \), such that \(|W_f(\hat{\omega})| = \max_{\omega \in \{0,1\}^n} |W_f(\omega)|\), which is equivalent to find out an \( \hat{\omega} \), such that \( W_f(\hat{\omega}) \neq 0 \).

Clearly Algorithm 2 outputs proper solution in one step, but the best known classical algorithm till date which can deterministically solve this problem is the Fast Walsh transform which requires \( O(n2^n) \) time in worst case. The information that the Walsh spectra is three valued does not help in the calculation of Walsh spectra in a better way on the classical model.

There are different kinds of resilient, correlation immune and other cryptographically significant Boolean functions \([23, 21, 22, 5, 6]\) with three valued Walsh spectra. These functions are used for robust design of crypto systems. Getting a linear or affine function which is maximally correlated to the Boolean function in constant time directly helps in cryptanalysis of such crypto systems and presents an application to Algorithm 2, the Deutsch-Jozsa Algorithm \([7]\).

We further restrict the Problem 4 and present the following problem to highlight the exponential speed up of quantum algorithms over classical domain.

Problem 5 A plateaued \( n \)-variable (\( n \) odd) Boolean function \( f \) with three valued Walsh spectra \( 0, \pm 2^{n+1} \) is given in the form of an oracle. Find out an \( \omega \), such that \( W_f(\omega) \neq 0 \).

Algorithm 2 solves this problem in one step.

Theorem 2 Problem 5 belongs to EQP with respect to the oracle \( f \).

The best known classical algorithm, fast Walsh transform, needs \( O(n2^n) \) time and the structure of the problem does not reveal anything to present a better deterministic classical algorithm. To analyse the situation in more details, let us define restricted Walsh transform. The restricted Walsh transform of \( f(x) \) on a subset \( T \) of \( \{0,1\}^n \) is a real valued function over \( \{0,1\}^n \) which is defined as

\[
W_f(\omega)|_T = \sum_{x \in T} (-1)^{f(x) \oplus x \cdot \omega}.
\]

Any NP machine can guess an \( \omega \) but it is impossible to verify in polynomial time whether the value of Walsh spectra at chosen \( \omega \) is non zero. This is because \( f \) is presented as a black box and thus one needs to query the value of \( f \) in at least \( 2^{n-1} + 1 \) times at the best case to decide whether \( W_f(\omega) \) is non zero. Let \( T \subset \{0,1\}^n \) such that \(|T| = 2^{n-1} + 1 \). If one finds that \( W_f(\omega)|_T \) is \( 2^{n-1} + 1 \) or \(-2^{n-1} - 1 \), then it is clear that \( W_f(\omega) \) cannot be zero. However, it is not possible to decide whether \( W_f(\omega) \) is \( 0 \) or \( \pm 2^{n+1} \) from \( W_f(\omega)|_T \) when \(|T| \leq 2^{n-1} \). Thus the verification stage needs \( O(2^n) \) many queries to the oracle at the best case.

Though we can not present any formal proof, it seems that Problem 5 is outside BPP (may be even outside PH) with respect to the oracle \( f \) and once such a result can be proved, the Deutsch-Jozsa algorithm can be used to present a relativized separation between BPP (may be PH) and EQP. This we place as an important open problem in this direction.
3 Problems in BQP

Let us consider a subset of Boolean functions with the following property.

\[ \mathcal{L}_n = \{ f \in \Omega_n \, | \, d(f, l) \leq 2^{n-3}, l \in L(n) \}. \]

**Proposition 3** \(|\mathcal{L}_n| = 2^n \sum_{i=0}^{2n-3} \binom{2n}{i} \).

**Proof:** Let \( \mathcal{L}'_n = \{ f \in \Omega_n \, | \, d(f, l) \leq 2^{n-3} \} \). Since for distinct \( l_1, l_2 \in L(n) \), \( d(l_1, l_2) = 2^{n-1} \), we have \( \mathcal{L}'_n \cap \mathcal{L}''_n = \emptyset \). Also it is clear that \(|\mathcal{L}'_n| = |\mathcal{L}''_n| \). Since, \(|L(n)| = 2^n \), and \( \mathcal{L}_n = \bigcup_{l \in L(n)} \mathcal{L}'_n \), \(|\mathcal{L}_n| = 2^n |\mathcal{L}'_n| \) for some \( l \in L(n) \). Now \(|\mathcal{L}'_n| = \sum_{i=0}^{2n-3} \binom{2n}{i} \) as one can choose \( i \) \((0 \leq i \leq 2^{n-3})\) many positions in the truth table of the linear function \( l \) and complement them to get an \( f \). This gives the proof.

From [13, Page 165], \( \sum_{i=0}^{\lambda(n)} \binom{n}{i} \leq 2^n H(\lambda) \), where the binary entropy function \( H(\lambda) = -\lambda \log_2 \lambda - (1 - \lambda) \log_2 (1 - \lambda) \). Also it is clear that \( \sum_{i=0}^{2^{n-3}} \binom{2n-3}{i} < \sum_{i=0}^{2^{n-3}} \binom{2n}{i} \). Thus, \( 2^{2^{n-3}} < |\mathcal{L}_n'| = \sum_{i=0}^{2^{n-3}} \binom{2n}{i} \leq 2^{2^{n-3}} \).

Let us consider the following problem which is a restricted version of Problem 3.

**Problem 6** An \( n \)-variable (n odd) Boolean function \( f \in \mathcal{L}_n \) is given in the form of an oracle. Find out an \( \hat{\omega} \), such that \(|W_f(\hat{\omega})| = \max_{\omega \in \{0, 1\}^n} |W_f(\omega)|\).

**Lemma 1** Problem 6 belongs to BQP with respect to the oracle \( f \).

**Proof:** If \( f \in \mathcal{L}_n \), then \( d(f, \hat{\omega} \cdot x) \leq 2^{n-3} \), i.e., \( W_f(\hat{\omega}) \geq 2^n - 2d(f, \hat{\omega} \cdot x) = 2^n - 2^{n-2} \). Thus the success probability of Algorithm 2 is \( \geq (\frac{2^n - 2^{n-2}}{2^n})^2 = \frac{9}{16} \). The probability of getting a wrong answer is \( \leq \frac{7}{16} \).

Now we refine these results a little bit to extend the class \( \mathcal{L}_n \). Let

\[ \mathcal{L}_{n, \epsilon} = \{ f \in \Omega_n \, | \, d(f, l) \leq (1 + (3 - 2\sqrt{2} - 4\epsilon))2^{n-3}, l \in L(n), 0 < \epsilon < \frac{3 - 2\sqrt{2}}{4} \}. \]

It is clear that \(|\mathcal{L}_{n, \epsilon}| > |\mathcal{L}_n|\) as \( 3 - 2\sqrt{2} - 4\epsilon > 0 \), for the given range of \( \epsilon \).

If \( f \in \mathcal{L}_{n, \epsilon} \), then \( d(f, \hat{\omega} \cdot x) \leq (1 + (3 - 2\sqrt{2} - 4\epsilon))2^{n-3} \), i.e., \( W_f(\hat{\omega}) \geq 2^n - 2d(f, \hat{\omega} \cdot x) = 2^n - 2^{n-2}(4 - 2\sqrt{2} - 4\epsilon) \). Thus the success probability of Algorithm 2 is \( \geq \left( \frac{2^n - 2^{n-2}(4 - 2\sqrt{2} + 4\epsilon)}{2^n} \right)^2 = \left( \frac{1}{2} + \epsilon \right)^2 = \frac{1}{4} + \sqrt{2}\epsilon + \epsilon^2 > \frac{1}{2} + \epsilon \). The probability of getting a wrong answer is \( < \frac{1}{2} - \epsilon \).

Noting \( \sqrt{2} < 1.415 \), one can use a small constant \( \epsilon \) such that

\[ \mathcal{L}_{n, \epsilon} = \{ f \in \Omega_n \, | \, d(f, l) \leq 1.17 \cdot 2^{n-3}, l \in L(n) \}. \]

Based on the above results we present the following problem and corollary.

**Problem 7** An \( n \)-variable (n odd) Boolean function \( f \in \mathcal{L}_{n, \epsilon} \) is given in the form of an oracle. Find out an \( \hat{\omega} \), such that \(|W_f(\hat{\omega})| = \max_{\omega \in \{0, 1\}^n} |W_f(\omega)|\).

**Corollary 1** Problem 7 belongs to BQP with respect to the oracle \( f \).

To the best of our knowledge, there is no other way to solve Problem 6 and Problem 7 deterministically in classical domain without calculating the Walsh spectra.
4 Conclusion

In this note, we identify a large set of problems which are in EQP or BQP with respect to an oracle $f$, where $f$ is an $n$-variable Boolean function available in the form of a black box. We have used the basic Deutsch-Jozsa algorithm to prove our results and show further applications to this well known algorithm. The only known tool to solve these problems in classical computational model is calculation of Walsh spectra which requires $O(n2^n)$ time. It is left open whether these problems are indeed hard to solve from complexity theoretic viewpoint. If that can be shown then the problems mentioned here, along with the Deutsch-Jozsa algorithm can be used to prove important results related to relativized separation between BPP (may be PH) and EQP or BQP.

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