The Application of the Functional Variable Method for Solving the Loaded Non-linear Evaluation Equations

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In this article, we construct exact traveling wave solutions of the loaded Korteweg-de Vries, the loaded modified Korteweg-de Vries, and the loaded Gardner equation by the functional variable method. The performance of this method is reliable and effective and gives the exact solitary and periodic wave solutions. All solutions to these equations have been examined and 3D graphics of the obtained solutions have been drawn by using the Matlab program. We get some traveling wave solutions, which are expressed by the hyperbolic functions and trigonometric functions. The graphical representations of some obtained solutions are demonstrated to better understand their physical features, including bell-shaped solitary wave solutions, singular soliton solutions, and solitary wave solutions of kink type. Our results reveal that the method is a very effective and straightforward way of formulating the exact traveling wave solutions of non-linear wave equations arising in mathematical physics and engineering.

Keywords: the loaded Korteweg-de Vries equation, the loaded modified Korteweg-de Vries equation, periodic wave solutions, soliton wave solutions, the loaded Gardner equation, functional variable method

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1. INTRODUCTION

The investigation of exact traveling wave solutions to non-linear evolutions equations plays an important role in the study of non-linear physical phenomena. These equations arise in several fields of science, such as fluid dynamics, physics of plasmas, biological models, non-linear optics, chemical kinetics, quantum mechanics, ecological systems, electricity, ocean, and sea. One of the most important non-linear evolution equations is Korteweg De Vries (KdV) equation.

The KdV equation was first observed by John Scott Russell in experiments, and then Lord Rayleigh and Joseph Boussinesq studied it theoretically. Finally, in 1895, Korteweg and De Vries formulated a model equation to describe the aforementioned water wave, which helped to prove the existence of solitary waves. In the mid-1960s, Zabusky and Kruskal discovered the remarkably stable particle-like behavior of solitary waves. According to their study, solitary waves described by the KdV equation can pass through each other keeping their speed and shape unchanged. As a result, the name “soliton” is defined. In the wake of these discoveries, solitary wave theory boosted the development of many areas of science and technology. After 100 years, integrable systems
developed deeply and soliton theory was widely applied in many areas. The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0,$$  \hspace{1cm} (1)

has many connections to several branches of physics. The Equation (1) is especially important due to the potential application of different properties of electrostatic waves in the development of new theories of chemical physics, space environments, plasma physics, fluid dynamics, astrophysics, optical physics, nuclear physics, geophysics, dusty plasma, fluid mechanics, and different other fields of applied physics [1–11].

In recent years, studying electrostatic waves specifically to discuss different properties of solitary waves in the field of soliton dynamics has played a significant role for many researchers and has received considerable attention from them. The ion-acoustic solitary wave is one of the fundamental non-linear wave phenomena appearing in plasma physics. In 1973, Hans Schamel studies a modified Korteweg-de Vries equation for ion-acoustic waves which is expressed in the following basic form

$$u_t + 6u^2 u_x + u_{xxx} = 0.$$  \hspace{1cm} (2)

This equation has been applied widely, e.g., in the molecular chain model, the generalized elastic solid, and so on [12–14]. Non-linear interactions between low-hybrid waves and plasmas can be described well by using the mKdV equation [15].

In 1968, the Gardner equation is an integrable non-linear partial differential equation introduced by the mathematician Clifford Gardner to generalize the KdV equation and modified the KdV equation. This equation can be written in a normalized form as follows:

$$u_t + 2auu_x + 3b^2 u_x + u_{xxx} = 0.$$  \hspace{1cm} (3)

If the coefficient $b > 0$, Equation (3) admits two families of solitons and oscillating wave packets (called breathers), whereas if $b < 0$, only one category of solitons exists [16]. The Gardner equation plays an important role in various branches of physics, such as plasma physics, fluid physics, and quantum field theory [17, 18]. It also describes a variety of wave phenomena in plasma and solid state physics [19, 20].

In arterial mechanics, a model is widely used in which the artery is considered as a thin-walled prestressed elastic tube with a variable radius (or with stenosis) and blood as an ideal fluid [21]. The governing equation that models weakly non-linear waves in such fluid-filled elastic tubes is the modified Korteweg–de Vries equation

$$u_t - 6u^2 u_x + u_{xxx} - h(t) u_x = 0,$$

where $t$ - is a scaled coordinate along the axis of the vessel after static deformation characterizing axisymmetric stenosis on the surface of the arterial wall. $x$ - is a variable that depends on time and coordinates along the axis of the vessel. $h(t)$ - is a form of stenosis and $u(x, t)$ characterizes the average axial velocity of the fluid.

We suppose that a form of stenosis $h(t)$ is proportional to $u(0, t)$, and we consider the loaded KdV, the loaded modified KdV and the loaded Gardner equation

$$u_t - 6au^2 u_x + u_{xxx} + \gamma(t)u(0, t)u_x = 0,$$  \hspace{1cm} (4)

$$u_t - 12a^2 u_x + u_{xxx} + \gamma(t)u(0, t)u_x = 0,$$  \hspace{1cm} (5)

$$u_t + 2au^2 u_x + 3b^2 u_x + u_{xxx} + \gamma(t)u(0, t)u_x = 0,$$  \hspace{1cm} (6)

where $u(x, t)$ is an unknown function, $x \in \mathbb{R}$, $t \geq 0$, $a$, and $b$ are any constants, $\gamma(t)$ is the given real continuous function.

Many powerful and direct methods have been developed to find special solutions of non-linear evolution equations such as, Weierstrass elliptic function method [22], Jacobi elliptic function expansion method [23], tanh-function method [24], inverse scattering transform method [25], Hirota method [26], Backlund transform method [27], exp-function method [28], truncated Painleve expansion method [29], extended tanh-method [30], and the homogeneous balance method [31] are used for searching the exact solutions.

We establish exact traveling wave solutions of the loaded KdV, the loaded modified KdV, and the loaded Gardner equation by the functional variable method. The performance of this method is reliable and effective and gives the exact solitary wave solutions and periodic wave solutions. The traveling wave solutions obtained via this method are expressed by hyperbolic functions and trigonometric functions. The graphical representations of some obtained solutions are demonstrated to better understand their physical features, including bell-shaped solitary wave solutions, singular soliton solutions, and solitary wave solutions of kink type. This method presents wider applicability for handling non-linear wave equations.

In the recent years, the study of the stability of traveling waves of periodic and soliton types associated with non-linear dispersive equations has increased significantly. A rich variety of new mathematical problems have emerged, as well as the physical importance related to them. This subject is often studied in relation to the natural symmetries associated with the model and by perturbations of symmetric classes, e.g., the class of periodic functions with the same minimal period as the underlying wave. In the case of shallow-water wave models, a formal stability theory of periodic and soliton traveling waves has started.

It is known that the loaded differential equations contain some traces of an unknown function. In [32–38], the term “loaded equation” was used for the first time, the most general definitions of the loaded differential equation were given, and also detailed classifications of the differential loaded equations, as well as their numerous applications, were presented. A complete description of solutions of the non-linear loaded equations and their applications can be found in articles [39–45].

2. DESCRIPTION OF THE FUNCTIONAL VARIABLE METHOD

Consider non-linear evolution equations with independent variables $x, y$, and $t$ is of the form

$$F(u, u_x, u_y, u_t, u_{xx}, u_{tt}, u_{yy}, u_{xy}, u_{xt}, u_{yt}, \ldots) = 0,$$  \hspace{1cm} (7)
where \( F \) is a polynomial in \( u = u(x,y,t) \) and its partial derivatives. In [46, 47], Zerarka and others have summarized the functional variable method in the following:

**Step 1.** We use the wave transformation

\[
\xi = px + qy - kt, \tag{8}
\]

where \( p \) and \( q \) are constants, and \( k \) is the speed of the traveling wave.

Next, we can introduce the following transformation for a traveling wave solution of Equation (7)

\[
u(x,y,t) = v(\xi), \tag{9}
\]

and the chain

\[
\frac{\partial u}{\partial x} = p \frac{du}{d\xi}, \frac{\partial u}{\partial y} = q \frac{du}{d\xi}, \frac{\partial u}{\partial t} = -k \frac{du}{d\xi}, \ldots \tag{10}
\]

Using Equations (9) and (10), the non-linear partial differential Equation (7) can be transformed into an ordinary differential equation of the form

\[
P(u, u', u'', u'''\ldots) = 0, \tag{11}
\]

where \( P \) is a polynomial in \( u(\xi) \) and its total derivatives, \( u' = \frac{du}{d\xi} \).

**Step 2.** Then we make a transformation in which the unknown function \( u \) is considered a functional variable in the form

\[
u' = F(u), \tag{12}
\]

then, the solution can be found by the relation

\[
\int \frac{du}{F(u)} = \xi + \xi_0,
\]

here, \( \xi_0 \) is a constant of integration which is set equal to zero for convenience. Some successive differentiations of \( u \) in terms of \( F \) are given as

\[
u'' = \frac{d^2F(u)}{du^2} \frac{du}{dF(u)} F(u) = \frac{1}{2} \frac{d(F^2(u))}{du},
\]

\[
u''' = \frac{d^3F(u)}{du^3} \frac{du}{dF(u)} \frac{2}{F^2(u)},
\]

\[
u'''' = \frac{1}{2} \left[ \frac{d(F^2(u))}{du^2} F^2(u) + \frac{d^2(F^2(u))}{du^2} \frac{d(F^2(u))}{du} \right].
\]

**Step 3.** The ordinary differential Equation (11) can be reduced in terms of \( u, F, \) and its derivatives upon using the expressions of Equation (13) into Equation (7) gives

\[
H(u, \frac{dF(u)}{du}, \frac{d^2F(u)}{du^2}, \frac{d^3F(u)}{du^3}, \ldots) = 0. \tag{14}
\]

The key idea of this particular form Equation (14) is of special interest because it admits analytical solutions for a large class of non-linear wave type equations. After integration, Equation (14) provides the expression of \( F \) and this, together with Equation (12), give appropriate solutions to the original problem.

### 3. SOLUTIONS OF THE LOADED KDV EQUATION

We will show how to find the exact solution of the loaded KdV by the functional variable method. Using the wave variable

\[
u(x,t) = u(\xi), \xi = px - kt,
\]

that will convert Equation (4) to an ordinary differential equation

\[- ku'' - 6\alpha pu' + p^3 u''' + \gamma(t)pu(0,t)u' = 0. \tag{15}\]

Integrating once Equation (15) with respect to \( \xi \), and put the constant of integration zero, we have

\[
u'' = \frac{1}{p^3} \left( 3\alpha pu^2 + (k - \gamma(t)pu(0,t))u \right). \tag{16}\]

Following Equation (13), it is easy to deduce from Equation (16) an expression for the function \( F(u) \)

\[
\frac{1}{2} \frac{d(F^2(u))}{du} = \frac{1}{p^3} \left( 3\alpha pu^2 + (k - \gamma(t)pu(0,t))u \right). \tag{17}\]

Integrating Equation (17) and setting the constant of integration to zero yields

\[
F^2(u) = \frac{1}{p^2} \left( 2\alpha pu^3 + (k - \gamma(t)pu(0,t))u^2 \right)
\]

\[
F(u) = u \sqrt{\frac{2\alpha}{p^2} u - \eta(t)), \tag{18}\]

where \( \eta(t) = \frac{\gamma(t)pu(0,t) - k}{2\alpha} \). From Equation (12) and Equation (18), we deduce that

\[
\frac{du}{u\sqrt{u - \eta(t)}} = \sqrt{\frac{2\alpha}{p^2}} d\xi. \tag{19}\]

After integrating Equation (19), with zero constant of integration, we have the following exact solution

\[
u(x,t) = \frac{\gamma(t)pu(0,t) - k}{2\alpha p} \sqrt{\frac{1}{\cos^2 \sqrt{\frac{\gamma(t)pu(0,t) - k}{4\alpha}}}} \left( px - kt \right). \tag{20}\]

It is obvious that the function \( u(0, t) \) can be easily found based on expression (20).

We have several types of traveling wave solutions of the loaded KdV equation as follows:

1) When \( \sqrt{\frac{\gamma(t)pu(0,t) - k}{4\alpha}} > 0 \), we have the periodic wave solution

\[
v(x,t) = \frac{\gamma(t)pu(0,t) - k}{2\alpha p} \cos \sqrt{\frac{\gamma(t)pu(0,t) - k}{4\alpha}} \left( px - kt \right). \tag{21}\]
2) When $\sqrt{\frac{\gamma(t)pu(0,t) - k}{4p^2}} < 0$, we have the solitary wave solution

$$u(x,t) = \frac{\gamma(t)pu(0,t) - k}{2\alpha p} \frac{1}{\cosh^2 \sqrt{\frac{\gamma(t)pu(0,t) - k}{4p^2}} (px - kt)}.$$  

Now, by choosing free parameters, we will write the traveling wave solutions of the loaded KdV equation in the simple form which can be used for the graphical illustrations.

If $k = -1, \alpha = 0.5, p = 1$ and $\gamma(t) = t$, then we have

$$u(x,t) = \frac{tu(0,t) + 1}{\cos^2 \sqrt{\frac{tu(0,t) + 1}{4}} (x + t)}.$$  

(21)

It is obvious that the function $u(0,t)$ can be easily found based on expression (21).

If $k = 1, \alpha = 0.5, p = -1$, and $\gamma(t) = -t$, then we have

$$u(x,t) = -\frac{tu(0,t) + 1}{\cosh^2 \sqrt{\frac{tu(0,t) + 1}{4}} (t - x)}.$$  

(22)

It is obvious that the function $u(0,t)$ can be easily found based on expression (22).

### 4. GRAPHICAL REPRESENTATION OF THE LOADED KDV EQUATION

We have presented some graphs of solitary and periodic waves constructed by taking suitable values of the involved unknown parameters to visualize the underlying mechanism of the original physical phenomena. Using mathematical software Matlab, 3D plots of the obtained solutions have been shown in Figures 1, 2. A soliton or solitary wave in the concept of mathematical physics is defined as a self-reinforcing wave package that retains its shape. It propagates at a constant amplitude and velocity. Solitons are solutions of a common class of non-linearly partially differential equations with weak linearity describing physical systems. The existence of periodic traveling waves usually depends on the parameter values in a mathematical equation. If there is a periodic traveling wave solution, then there is typically a family of such solutions, with different wave speeds. For partial
differential equations, periodic traveling waves typically occur for a continuous range of wave speeds. The physical description of the 3D loaded KdV equation of the installed exact moving wave solutions is discussed in this section. In the physical definition section, 3D surface drawings, contour maps, and 2D drawings of the developed moving wave solutions of the latest 3D loaded KdV equations are discussed. The 3D line plot emphasizes the amount of variability over time or compares multiple wave elements. The wave points were sequentially designed using equal interval breaks and connected by a line to emphasize the relationship of the wave points. Three-dimensional elegance is used to give visual attention to the diagram. Two-dimensional line drawings are used to represent very high and low frequencies and amplitudes.

5. SOLUTIONS OF THE LOADED MODIFIED KDV EQUATION

Assume that Equation (5) has an exact solution in the form of a traveling wave

\[ u(x, t) = u(\xi), \quad \xi = px - kt, \]

the Equation (5) can be converted to an ordinary differential equation

\[ -ku' - 12\alpha pu^3 u' + p^3 u'' + \gamma(t)pu(0,t)u' = 0. \quad (23) \]

Once integrating (23), setting the constant of integrating to zero, we obtain

\[ u'' = \frac{1}{p^2} (4\alpha pu^3 + (k - \gamma(t)pu(0,t)) u) . \quad (24) \]

Following Equation (13), it is easy to deduce from Equation (24) an expression for the function \( F(u) \)

\[ \frac{1}{2} \frac{d\left(F^2(u)\right)}{du} = \frac{1}{p^2} (4\alpha pu^3 + (k - \gamma(t)pu(0,t)) u) . \quad (25) \]

Integrating Equation (25) with respect to \( u \) and after the mathematical manipulations, we have

\[ F^2(u) = \frac{1}{p^3} \left(2\alpha pu^4 + (k - \gamma(t)pu(0,t)) u^2 \right) \]

\[ F(u) = \frac{u}{p} \sqrt{2\alpha} \left[ u^2 - \frac{\gamma(t)pu(0,t) - k}{2\alpha p} \right] \]

\[ F(u) = \frac{u}{p} \sqrt{2\alpha} \sqrt{u^2 - \varphi(t)}, \quad (26) \]

where \( \varphi(t) = \frac{\gamma(t)pu(0,t) - k}{2ap} \). From Equation (12) and Equation (26), we deduce that

\[ \frac{du}{u \sqrt{u^2 - \varphi(t)}} = \frac{\sqrt{2\alpha}}{p} d\xi . \quad (27) \]

After integrating Equation (27), with zero constant of integration, we have the following exact solution

\[ u(x, t) = \sqrt{\frac{\gamma(t)pu(0,t) - k}{2\alpha p}} \cos \frac{1}{p} \sqrt{\frac{\gamma(t)pu(0,t) - k}{p}} (px - kt) . \quad (28) \]

It is obvious that the function \( u(0, t) \) can be easily found based on expression (28).

We have several types of traveling wave solutions of the loaded modified KdV equation as follows:

1) When \( \frac{\gamma(t)pu(0,t) - k}{p^3} > 0, \alpha > 0 \), we have the periodic wave solution

\[ u(x, t) = \sqrt{\frac{\gamma(t)pu(0,t) - k}{2\alpha p}} \cos \frac{1}{p} \sqrt{\frac{\gamma(t)pu(0,t) - k}{p}} (px - kt) . \]

2) When \( \frac{\gamma(t)pu(0,t) - k}{p^3} < 0, \alpha < 0 \), we have the solitary wave solution

\[ u(x, t) = \sqrt{\frac{\gamma(t)pu(0,t) - k}{2\alpha p}} \cosh \frac{1}{p} \sqrt{\frac{\gamma(t)pu(0,t) - k}{p}} (px - kt) . \]

Now, by choosing free parameters, we will write the traveling wave solutions of the loaded modified KdV equation in the simple form which can be used for the graphical illustrations.

If \( k = -1, \alpha = 0.5, p = 1 \), and \( \gamma(t) = t \), then we have

\[ u(x, t) = \frac{\sqrt{tu(0,t) + 1}}{\cos \sqrt{tu(0,t) + 1} (x + t)} . \quad (29) \]

It is obvious that the function \( u(0, t) \) can be easily found based on expression (29).

If \( k = 1, \alpha = -0.5, p = 1 \), and \( \gamma(t) = -t \), then we have

\[ u(x, t) = \frac{\sqrt{tu(0,t) + 1}}{\cosh \sqrt{tu(0,t) - 1} (x - t)} . \quad (30) \]

It is obvious that the function \( u(0, t) \) can be easily found based on expression (30).

6. PHYSICAL EXPLANATION OF THE LOADED MODIFIED KDV EQUATION

We have shown how to find the solutions of the loaded modified KdV equation in 3D plot formats to make it easier to imagine. Graphical representation is an effective tool for communication and it exemplifies evidently the solutions to the problems. The graphical illustrations of the solutions are depicted in Figures 3, 4. Solitary and periodic wave solutions represent an important type of solutions for non-linear partial differential equations as many non-linear partial differential equations have been found to have a variety of solitary wave solutions. The solitary wave solutions obtained in this article are encouraging, applicable, and could be helpful in analyzing long wave propagation on the surface of a fluid layer under the action of gravity, iron sound waves in plasma, and vibrations in a non-linear string.
7. SOLUTIONS OF THE LOADED GARDNER EQUATION

Assume that Equation (6) has an exact solution in the form of a traveling wave

$$u(x, t) = u(\xi), \quad \xi = px - kt,$$

that will convert Equation (6) to an ordinary differential equation

$$- ku' - 2\alpha pu' - 3\beta u^2 u' + p^3 u''' + \gamma(t)pu(0, t)u' = 0. \quad (31)$$

Integrating once Equation (31) with respect to $\xi$, and putting the constant of integration at zero, we have

$$u'' = \frac{1}{p^3} \left( \beta u^3 + \alpha pu^2 + (k - \gamma(t)pu(0, t)) u \right). \quad (32)$$

Following Equation (13), it is easy to deduce from Equation (32) an expression for the function $F(u)$

$$\frac{1}{2} \frac{d}{du} \left( F^2(u) \right) = \frac{1}{p^3} \left( \beta u^3 + \alpha pu^2 + (k - \gamma(t)pu(0, t)) u \right). \quad (33)$$

Integrating Equation (33) and setting the constant of integration to zero yields

$$F(u) = \frac{u}{p} \sqrt{\lambda u^2 + \tau u + \mu(t)}, \quad (34)$$

where $\lambda = \beta \frac{\rho}{p^2}, \tau = \frac{2\alpha}{p^3}, \mu(t) = \frac{k - \gamma(t)pu(0, t)}{p}$. From Equation (12) and Equation (34), we deduce that

$$\frac{du}{u \sqrt{\lambda u^2 + \tau u + \mu(t)}} = \frac{1}{p} d\xi. \quad (35)$$

After integrating Equation (35), with zero constant of integration, we have the following exact solution

$$u(x, t) = \frac{2\mu(t)e^{-\sqrt{\mu(t)}(px-kt)}}{\left( e^{-\sqrt{\mu(t)}(px-kt)} - \frac{\tau}{2} \right)^2 - \frac{\tau^2}{2} - \lambda \mu(t)}. \quad (36)$$

It is obvious that the function $u(0, t)$ can be easily found based on expression (36).
We have several types of traveling wave solutions of the loaded Gardner equation as follows:

1) When \( \sqrt{\mu(t)} > 0 \), we have the periodic wave solution

\[
u(x, t) = \frac{2\mu(t)e^{-\sqrt{\mu(t)}(px-kt)}}{\left(e^{-\sqrt{\mu(t)}(px-kt)} - \frac{t}{2}\right)^2 - \frac{t^2}{4} - \lambda\mu(t)}.
\]

2) When \( \sqrt{\mu(t)} < 0 \), we have the solitary wave solution

\[
u(x, t) = \frac{2\mu(t)e^{-\sqrt{\mu(t)}(px-kt)i}}{\left(e^{-\sqrt{\mu(t)}(px-kt)i} - \frac{t}{2}\right)^2 - \frac{t^2}{4} - \lambda\mu(t)}.
\]

Now, by choosing free parameters, we will write the traveling wave solutions of the loaded Gardner equation in the simple form which can be used for the graphical illustrations.

If \( k = 1, \alpha = 3, p = -1, \beta = 2, \gamma(t) = -t \), then we have

\[
u(x, t) = \frac{2\left(t\nu(0, t) - 1\right)e^{\sqrt{t\nu(0, t) - 1}(x+t)}}{\left(e^{\sqrt{t\nu(0, t) - 1}(x+t)} - 1\right)^2 + t\nu(0, t) - 1}, \tag{37}
\]

It is obvious that the function \( \nu(0, t) \) can be easily found based on expression (37).

If \( k = 1, \alpha = 3, p = -1, \beta = 2, \gamma(t) = t \), then we have

\[
u(x, t) = \frac{-2\left(t\nu(0, t) + 1\right)e^{\sqrt{t\nu(0, t) + 1}(x+t)i}}{\left(e^{\sqrt{t\nu(0, t) + 1}(x+t)i} - 1\right)^2 - (t\nu(0, t) + 1)}, \tag{38}
\]

It is obvious that the function \( \nu(0, t) \) can be easily found based on expression (38).
8. GRAPHICAL REPRESENTATIONS OF TRAVELING WAVE SOLUTIONS OF THE LOADED GARDNER EQUATION

This section aims to present graphical illustrations of the obtained traveling wave solutions of the Gardner equation. Using mathematical software Matlab, 3D plots of the obtained solutions have been shown in Figures 5, 6. In the concept of mathematical physics, a soliton or solitary wave is defined as a self-reinforcing wave packet that upholds its shape. At the same time, it propagates at a constant amplitude and velocity. Solitary waves can be obtained from each traveling wave solution by setting particular values to its unknown parameters. By adjusting these parameters, one can get an internal localized mode. We have presented some graphs of solitary waves constructed by taking suitable values of the involved unknown parameters to visualize the underlying mechanism of the original physical phenomena.

9. CONCLUSION

The functional variable method has been successfully used to obtain several traveling wave solutions of the loaded KdV, the loaded modified KdV, and the loaded Gardner equation. The method does not require linearization of differential equations because it is a method of directly solving some non-linear physical models. A wide and general class of modern examples representing real physical problems from plasma physics, fluid dynamics, non-linear optics, and non-linear fields of gas dynamics can be solved easily and elegantly using this method. The exactness of the obtained results is studied by using the software Matlab. The received solutions with free parameters may be important to explain some physical phenomena. The advantage of the method is to give more solution functions such as periodic solutions and hyperbolic solutions than other popular analytical methods. We conclude that the functional variable method is significant and important for finding the exact traveling wave solutions of non-linear evolution equations. The proposed method can be applied to many other non-linear evolution equations in mathematical physics.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

BB and FA conceived of the presented idea. BB developed the theory and performed the computations. FA verified the analytical methods. Both authors discussed the results and contributed to the final manuscript. Both authors contributed to the article and approved the submitted version.

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