Fractal Structure with a Typical Scale

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Abstract

In order to understand characteristics common to distributions which have both fractal and non-fractal scale regions in a unified framework, we introduce a concept of typical scale. We employ a model of 2d gravity modified by the $R^2$ term as a tool to understand such distributions through the typical scale. This model is obtained by adding an interaction term with a typical scale to a scale invariant system. A distribution derived in the model provides power law one in the large scale region, but Weibull-like one in the small scale region. As examples of distributions which have both fractal and non-fractal regions, we take those of personal income and citation number of scientific papers. We show that these distributions are fitted fairly well by the distribution curves derived analytically in the $R^2$ 2d gravity model. As a result, we consider that the typical scale is a useful concept to understand various distributions observed in the real world in a unified way. We also point out that the $R^2$ 2d gravity model provides us with an effective tool to read the typical scales of various distributions in a systematic way.

1 Introduction

A self-similar system is called fractal [1] and it is one of the subjects which attract attention broadly not only in natural science but also in social science, in recent years. In many cases, fractal structure appears in some restricted scale, does not do in all scale of the system concerned. For example, in the case of distribution of personal income interested in econophysics [2], the distribution of top several percent income earners follows fractal power law [3], while that of the rest earners does not do [4, 5, 6]. In other words, in such a system, self-similarity is not maintained over all scale but is broken in small scale region.
In usual, the fractal property and the deviation from it in each system are discussed by using individual models for the system. For example, as for the personal income distribution, the fractal power law in high income region is studied using a stochastic evolution equation [7]. In addition, whole profile of the distribution is investigated based on a concept of small world network [8], and in [9], the distribution is explained using q-Gaussian distribution emerged from nonextensive statistical mechanics [10]. In this paper, however, we will discuss characteristics common to systems which show both fractal and non-fractal properties independent of the detail of the individual models.

A scale invariant model does not have any scale, so we expect that distributions derived from the model follow power law. This is thought to be a universal property that does not depend on the detailed structure of the model. One of the simplest methods to realize both fractal and non-fractal scale regions analytically is to introduce a typical scale into a scale invariant model to break the original scale invariance. If the interaction term with a typical scale is added to the scale invariant model, the model obtains a typical scale and we expect that the distributions derived form the model become non-fractal in the scale region where the typical scale is meaningful, while keeping fractal property in the large scale region. In this paper, we will discuss fractal and deviation from it in the above framework.

There are various ways to construct a scale invariant model. As a tool to discuss fractal property and deviation from it concretely, we take a model of 2-dimensional quantum gravity coupled with conformal matter fields. Because of special nature of 2-dimension and the conformal property of the matter fields, the model has scale invariance and fractal property. Some distributions derived from the model follow power law[11, 12]. One of the reasons to take this model is that the model is decided by the action only and it is suitable to treat the whole system analytically. In addition, the model has a simple geometrical meaning, so that it is easy to understand the fractal property and the deviation from it intuitively.  

The above model, the standard 2d gravity model coupled with conformal matter fields, is scale invariant. To introduce a typical scale and break the original scale invariance, we add the $R^2$ interaction term to the action. Here, $R^2$ is the square of scalar curvature. The obtained model is called $R^2$ 2d gravity theory. Because of the typical scale introduced, we expect the deviation from fractal in the scale region where the typical scale is meaningful[13, 14], keeping the original fractal property in the large scale region.

In this paper, we would like to point out that the typical scale is a useful new concept to understand various distributions which have both fractal and non-fractal regions, employing the $R^2$ 2d gravity model as a tool to understand the features of such distributions in a unified way. As examples of distributions which have fractal and non-fractal regions, we take those of personal income and citation number of scientific papers. We show that these distributions are well understood by the typical scales and theoretical curves derived from the framework of the $R^2$ 2d gravity model. We also point out that the $R^2$ 2d gravity model also provides us with an effective tool to read the typical scales of various distributions in a systematic way.

1It is believed that a typical 2d surface has self-similar structure (Fig. 1).
2 2-dimensional gravity with $R^2$ term

In this section, we review $R^2$ 2d gravity model. First let us consider standard 2-dimensional quantum gravity coupled with conformal matter fields. To make the argument concrete, as conformal matter fields we take scalar fields $X^i$ ($i = 1, 2, \cdots, c$). The action of the matter part takes the form

$$S_M(X^i; g) = \frac{1}{8\pi} \int d^2x \sqrt{g} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^i ,$$

(1)

where $g_{\mu\nu}(\mu, \nu = 0, 1)$ is the metric of 2d surface. In 2-dimension, the standard Einstein action $\frac{1}{4\pi} \int d^2x \sqrt{g} R$, where $R$ is the scalar curvature, merely yields a constant which characterizes the topology of the 2d surface, so that we can neglect the Einstein term. The total action is given by $S_{\text{total}}(X^i; g) = S_M(X^i; g)$, and it is invariant under the scale transformation of the metric $g_{\mu\nu}$.

The partition function for fixed area $A$ of 2d surface is given by

$$Z(A) = \int \frac{Dg DX}{\text{vol(Diff)}} e^{-S_{\text{total}}(X^i; g)} \delta(\int d^2x \sqrt{g} - A) .$$

(2)

The action and the integration measure $Dg DX$ are invariant under 2d diffeomorphisms, so that the measure should be divided by the volume of the diffeomorphisms, which is denoted by $\text{vol(Diff)}$. The partition function is evaluated to be $Z(A) \propto A^{\tilde{\gamma}_0 - 3}$ [11], where $\tilde{\gamma}_0$ is a constant determined by the central charge $c$ and the number of handles of the 2d surface $h$,

$$\tilde{\gamma}_0(c, h) = \frac{c - 25 - \sqrt{(25 - c)(1 - c)}}{12} (1 - h) + 2 .$$

(3)

Note that this model has no scale parameter, so that the partition function $Z(A)$ follows power law. It is expected that a typical 2d surface has self-similar structure (Fig. 1) [11].

Next let us turn to 2d gravity with $R^2$ term. In order to introduce a typical scale into the standard 2d gravity, we add the scale variant $R^2$ term $1/(32\pi m^2) \int d^2x \sqrt{g} R^2$ to the action (1). The total action is given by

$$S_{\text{total}}(X^i; g) = \frac{1}{32\pi m^2} \int d^2x \sqrt{g} R^2 + S_M(X^i; g) ,$$

(4)

where $m$ is a coupling constant of length dimension $-1$. The first term in the action (4) is not scale invariant, and it provides the typical scale $2\pi/m^2$ to the theory. As a result, the fractal structure of 2d surface collapses in the region where the typical scale $2\pi/m^2$ becomes meaningful, while 2d surface maintains fractal at area scale much larger than $2\pi/m^2$. In fact, the asymptotic forms of the partition function are evaluated as [13]

$$Z(A) \sim C_0 A^{\gamma_0 - 3} \exp \left[ - \frac{2\pi}{m^2 A} (1 - h)^2 \right] \quad \text{for} \quad A \ll \frac{2\pi}{m^2} ,$$

(5)

$$\sim C_\infty A^{\gamma_\infty - 3} \quad \text{for} \quad A \gg \frac{2\pi}{m^2} ,$$

(6)

where $C_0$ and $C_\infty$ are the proportional constants, and

$$\gamma_0(c, h) = \frac{(c - 12)}{6} (1 - h) + 2 .$$

(7)
Here we can observe that fractal power law is broken in the region $A \ll 2\pi/m^2$.

In order to investigate the breaking of fractal structure concretely, it is appropriate to treat 2d surface discretely. In study of 2d gravity, one of the useful methods of discretizing 2d surface is known as Dynamical Triangulation (DT) [12]. In usual, 2d surface is discretized using small equilateral triangles, where each triangle has the same size. From various evidences, DT is believed to be equivalent to the continuum theory of 2d gravity in the continuum limit [15].

In DT, the evaluation of the partition function is performed by replacing the path integral over the metric with the sum over possible triangulations of 2d surface. Here, we represent the number of triangles sharing the vertex $i$ as $q_i$, which is called a coordination number. In DT, the $R^2$ term in the action (4) is expressed by

$$\int d^2x \sqrt{g} R^2 = \frac{4\pi^2}{3a^2} \sum_i \frac{(6-q_i)^2}{q_i},$$

(8)

from the correspondence $\int d^2x \sqrt{g} \approx a^2 \sum_i q_i/3$ and $R_i = 2\pi(6-q_i)/(a^2q_i)$. Here $a^2$ is the area of a triangle and $R_i$ is the discretized local scalar curvature at the $i$-th vertex. From Eq. (8), we can recognize that the $R^2$ term has the effect to make 2d surface flat ($q_i = 6$). This effect is parametrized by the coefficient of the $R^2$ term.

### 3 MINBU distribution

In DT, fractal structure (and non-fractal structure) of 2d surface can be discussed by considering so-called minimum-neck baby universe (MINBU) [16]. A MINBU is defined as a simply connected area region of 2d surface whose neck is composed of three links (three sides of triangles), where the neck is closed and non-self intersecting. In general, a lot of MINBUs of various sizes are formed on a 2d surface. A typical dynamically triangulated surface is shown in Fig. 1. Distribution of the area of MINBU is one of the important observable quantities in DT.

Now let us evaluate the distribution of MINBU. Consider a closed 2d surface of area $A$. There are many MINBUs on the surface, and each one is connected by a minimum neck one another. Paying attention to one of the minimum necks, the whole surface can be divided into two MINBUs (Fig. 2), where one has area $A - B$ and the other has area $B$. Representing the partition functions of the two MINBUs as $Z(A - B, 3)$ and $Z(B, 3)$ respectively, the statistical average number of finding a MINBU of area $B$ on a closed surface of area $A$, $n_A(B)$, can be expressed as

$$n_A(B) \sim \frac{Z(B, 3)Z(A - B, 3)}{Z(A)}.$$

(9)

Here we set $a^2 = 1$ for simplicity, and $Z(A)$ denotes the partition function of a closed surface of area $A$.

On the other hand, a MINBU of area $C$ can be constructed by removing one triangle from a closed 2d surface of area $C + 1$. We have $C + 1$ ways to choose the triangle to remove, so we have the relation

$$Z(C, 3) \sim (C + 1)Z(C + 1).$$

(10)
Using this relation, the partition functions of the two MINBUs in Eq. (9) are given by

\[ Z(B, 3) \sim (B + 1)Z(B + 1), \tag{11} \]
\[ Z(A - B, 3) \sim (A - B + 1)Z(A - B + 1). \tag{12} \]

Substituting these relations into Eq. (9), we can express \( n_A(B) \) in terms of the ordinary partition functions of closed surfaces. Using the asymptotic forms of the partition function (5) and (6), we obtain, in the end, the asymptotic expression of \( n_A(B) \)

\[ n_A(B) \sim C_0 A B^{\gamma_0 - 2} \exp\left[ -\frac{2\pi}{m^2 B} (1 - h)^2 \right] \quad \text{for} \ 1 \ll B \ll \frac{2\pi}{m^2} \ll A, \tag{13} \]
\[ \sim C_\infty A \left( \frac{1 - B}{A} \right)^{\gamma_\infty - 2} \quad \text{for} \ \frac{2\pi}{m^2} \ll B < A/2. \tag{14} \]

Here, the restriction \( B < A/2 \) comes from the strict definition of MINBU, where the area of a MINBU is less than half of the total area.

As for the case \( 2\pi/m^2 \ll B < A/2 \), the asymptotic form (14) follows power law, therefore, the surfaces are expected to be fractal \(^2\). In this range, even if the model contains the \( R^2 \) term, at an area scale much larger than \( 2\pi/m^2 \), the surfaces are fractal. On the other hand, as for the case \( 1 \ll B \ll 2\pi/m^2 \), the asymptotic form (13) is highly suppressed by the exponential factor \( \exp\left[ -\frac{2\pi}{m^2 B} (1 - h)^2 \right] \), hence, the fractal structure of 2d surface is broken. In this range, at an area scale much smaller than \( 2\pi/m^2 \), the surfaces are affected by the typical length scale, and are not fractal. In the case of \( \gamma_0 = 0 \), the distribution (13) is known as Weibull distribution. In this paper, we call the distribution (13) as Weibull-like distribution.

4 Numerical analysis of DT

The analytic results (13) and (14) can be confirmed in the simulation of DT for the simple case that 2d surface is sphere \((h = 0)\) and there is no matter field on it \((c = 0)\) [14]. The simulation results are expressed in Fig. 3. Here, we plot MINBU distributions, \( n_A(B) \) versus \((1 - B/A)B\) with a log-log scale for \( \beta_L = 0, 50, 150, 200, 250, 300 \), which are coefficients of the discretized \( R^2 \) term (8) \(^3\). In this simulation, the total number of triangles is 100,000. These MINBU distributions can be well explained by the asymptotic formulae (13) and (14) with \( \gamma_0 = 0 \) and \( \gamma_\infty = -1/2 \), which are obtained from \( h = c = 0 \). We can read the typical scale \( 2\pi/m^2 \) for each case. For example, the data fittings for the cases of \( \beta_L = 50 \) and 100 are represented in Figs. 4 and 5, and we obtain 14.5 and 47.0 as the value of \( 2\pi/m^2 \) respectively. In each of these figures, several data points for small MINBUs are apart from the line of Weibull distribution (13). We consider that it is the finite lattice effect. In small \( B \) region, each of the corresponding MINBUs consists of a small number of triangles, so that it is not appropriate to treat the area of MINBU \( B \) as a continuous variable.

\(^2\)A similar phenomenon can be seen in 2d gravity without the \( R^2 \) term [11], where no typical length scale exists.

\(^3\)In Eq. (13), we can replace \( B \) with \((1 - B/A)B\) because of the range \( B \ll 2\pi/m^2 \ll A \).
5 Distributions of personal income and citation number of scientific papers

We apply the distribution of MINBU in 2d $R^2$ gravity to other distributions observed in the real world, and examine whether it can explain these distributions. Here, we investigate distributions of personal income and citation number of scientific papers [17]. These two kinds of distributions have fractal power law and non-fractal regions, so it is possible that the theoretical curves (13) and (14) can explain them.

First, let us consider the personal income distributions of Japan in the years 1997 and 1998 [6]. The distributions and data fittings are shown in Figs. 6 and 7. Here, we do not accumulate the data in this analysis. The horizontal axis indicates the income $x$ in units of thousand yen and the vertical axis indicates the number density of persons $N(x)$ per a period of 100 thousand yen.

In both distributions, from the data fittings for the power law regions, we obtain $\gamma - 2 \equiv -I_p = -3.0$. From Eq. (3), we see that this value is realized by choosing $c = -2$, $h = 0$. Substituting these values into Eq. (7), we obtain $\gamma_0 - 2 = -7/3$ for the Weibull-like distribution (13). The analytical functions employed to fit the personal income distributions are given by

$$N(x) \sim C_w x^{-7/3} \exp \left[ -\frac{2\pi}{m^2} \frac{1}{x} \right] \quad \text{for} \quad 1 \ll x \ll \frac{2\pi}{m^2}, \quad (15)$$

$$\sim C_p x^{-3} \quad \text{for} \quad \frac{2\pi}{m^2} \ll x. \quad (16)$$

In Figs. 6 and 7, we fit the data in the non-fractal regions by the Weibull-like distribution (15), and find that the typical scales $2\pi/m^2$ in 1997- and 1998-Japan are 4090 and 5210 thousand yen respectively. These values are almost the same as the averages of income, and we consider that these values are quite natural. We note that the scale transformation of $x$ and the adjustment of the normalization of $N(x)$ can always make the normalization constants $C_w$ and $C_p$ agree with the corresponding constants $C_0$ and $C_\infty$ in Eqs. (13) and (14) respectively. In the end, we consider that these two distributions of personal income are well explained by a concept of typical scale in distributions.

Secondly, we consider two distributions of citation number of scientific papers analyzed in Ref. [17]. One is the citation number distribution of the papers published in 1981 in journals which are cataloged by the Institute for Scientific Information (ISI). The second is that of the papers which were published in vols. 11-50 of Physical Review D (PRD), 1975-1994. As for the data of ISI, the distribution can be well explained by setting $c = -2$, $h = 0$ (Fig. 8). On the other hand, as for the data of PRD, the distribution can also be well explained by setting $c = 1/2$, $h = 0$ (Fig. 9). In the latter case, to fit the data, we employ the analytic functions

$$N(x) \sim C_w x^{-23/12} \exp \left[ -\frac{2\pi}{m^2} \frac{1}{x} \right] \quad \text{for} \quad 1 \ll x \ll \frac{2\pi}{m^2}, \quad (17)$$

$$\sim C_p x^{-7/3} \quad \text{for} \quad \frac{2\pi}{m^2} \ll x. \quad (18)$$

We find that the typical scales $2\pi/m^2$ of the citation number in ISI and PRD are 15.1 and 7.03 respectively. In the small $x$ regions in Figs. 8 and 9, several data points are apart from
the curves of the Weibull-like distributions (15) and (17). In the derivation of the Weibull-like function, the area of MINBU is treated as a continuous variable. However, it is not appropriate to treat $x$ continuously in small $x$ region. As a result, we consider that $x$ does not always follow the Weibull-like distribution in small $x$ region.

6 Summary and discussion

In this paper, we proposed the concept of a typical scale in order to understand distributions which have both fractal and non-fractal scale regions in a unified framework. The point was to introduce a typical scale into a scale invariant model to break the original scale invariance and to produce non-fractal feature in the small scale region. We employed the $R^2$ 2d gravity model as a tool to understand such distributions through the typical scale. MINBU distribution in this model followed the power law in the large scale region and provided Weibull-like one in the small scale region.

As examples of distributions where fractal and non-fractal regions coexist, we took those of personal income and citation number of scientific papers. We showed that these distributions were fitted fairly well by the theoretical curves of MINBU distribution, adjusting the values of $c$, $h$ and the typical scale. From these fittings, as for the personal income, we consider that there is no scale with respect to money for the top several percent high income earners, on the other hand, the rest earners are highly influenced by the typical scale of income. We can understand whole profile of the distribution merley by introducing the typical scale. As a result, we consider that the typical scale is a useful concept to understand various distributions where both fractal and non-fractal scale regions exist. We also consider that the $R^2$ 2d gravity model provides us with an effective tool to read the typical scales of various distributions in a systematic way.

In the distributions studied in this paper, the values of the typical scale are comparable with the average values of the distributions. We consider that the typical scale is a significant characteristic parameter similar to average in such distributions. The typical scale, however, can be read mainly from the data in the small scale region. It can thus be a more efficient characteristic parameter than average in some situations.

We use the $R^2$ 2d gravity model as a tool to discuss the significance of the typical scale concretely. Besides the coincidence of the distributions, is there any direct physical connection between $R^2$ 2d gravity and personal income or citation number? We can’t answer this question at present. However, 2d gravity can be also formulated by a stochastic evolution equation [18]. As we mentioned in Sect. 1, personal income distribution can be also described by a stochastic evolution equation [7]. It may be possible to find the physical relation by investigating these formulations in detail.

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Figure 1: A typical 2-dim surface.

Figure 2: Divided MINBs.

Figure 3: The simulation results of DT.
Figure 4: The fitting of $\beta_L = 50$ data.

Figure 5: The fitting of $\beta_L = 100$ data.
Income Distribution in 1997 Japan

Figure 6: The fitting of the personal income distribution in 1997 Japan.

Income Distribution in 1998 Japan

Figure 7: The fitting of the personal income distribution in 1998 Japan.
Figure 8: The fitting of the citation number distribution in ISI.

Figure 9: The fitting of the citation number distribution in PRD.