Weak field limit in vierbein-Einstein-Palatini formalism and
Fierz-Pauli Equation

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Abstract

We consider the weak field limit of gravity in the vierbein-Einstein-Palatini formalism, find the
action and the equations for perturbations around an arbitrary background, and compare them
with the usual metric perturbation equations. We also write the Fierz-Pauli equations for massive
gravitons on an arbitrary curved background in this formalism.

PACS numbers: 04.50.Kd, 04.25.Nx

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I. INTRODUCTION

General relativity is a non-linear theory. A linearized approximation to the theory can be obtained in by considering the gravitational field to be so weak that the space-time metric can be considered as a background metric plus a perturbation \[1\]. Einstein’s equation turns out to be a linear second order equation in the perturbation.

Another formulation of general relativity uses tetrads and spin connections instead of metric and Christoffel symbols. In this formulation, an internal Minkowski space is attached to each point of space-time, isomorphic to the tangent space of the space-time manifold at that point. Linear isomorphisms between the tangent space and the internal space are given by tetrads, also known as vierbeins. Any connection on this bundle, called a spin connection, gives the parallel transport of the sections of the internal space. The curvature of this connection is related to the space-time curvature, which allows us to write the Einstein-Hilbert action of general relativity in terms of the tetrads and the spin connection. This formulation is known as vierbein-Einstein-Palatani (VEP) formalism \[2 – 4\]. In this paper we find the weak field limit of the VEP action and equations.

The idea of perturbing the vierbein is not a new one (a representative list is \[5–12\]), but our paper differs from earlier work in an important manner. In all papers dealing with vierbein perturbations that we have been able to find, the perturbations were considered around a flat background spacetime. The background spacetime and the internal Lorentz space are then identical, and as a result the background tetrad can be written as a Kronecker delta, \(e^I_\mu = \delta^I_\mu\). Furthermore, the background connection is then trivially flat and torsion-free, which means no attention is paid to the independent nature of the spin connection. This paper is different from earlier work in both ways – we assume a general curved background and an independent spin connection, so tetrad fields are not constant, while the absence of torsion is implemented by the equations of motion for the spin connection only in the absence of fermionic matter. When fermions are coupled to the spin connection, torsion remains as a part of the spin connection, and the latter cannot be completely eliminated from the perturbation equations. As far as we are aware, this is the first work to write down tetrad perturbation equations on an arbitrary curved background, and in presence of torsion. While gravity with torsion has a long history (see \[13–17\] for some recent work which are close to our work in spirit), we find perturbation equations for tetrads and spin.
connection, and write the torsion in terms of these variables.

Another motivation for looking at vierbein perturbations is to investigate massive gravity \([18–24]\) in this formalism. Supernova data \([25, 26]\) reveal that the universe is in a phase of accelerated expansion, which can be explained by positing a dark energy component along with matter and dark matter in cosmological models based on general relativity. Dark energy is most simply modeled by the addition of a constant term, called the cosmological constant, in general relativity. However, there is no convincing model for the cosmological constant itself, which looks like a vacuum energy density, but the value of the cosmological constant differs from the value obtained from the vacuum energy in quantum field theory \([27]\) by a factor of \(\sim 10^{65}\). An alternative suggestion is that general relativity itself is modified in the infrared which may remove the need for dark energy to explain the acceleration. One particular infrared modification of general relativity is one with massive gravitons, i.e. one where metric perturbations obey a massive wave equation.

The story of massive gravity goes back a long way. Fierz and Pauli \([28, 29]\) proposed an action that describes a free massive graviton on a flat background. The Fierz-Pauli equations are linear in the metric perturbation, but it was shown later that the massless equation cannot be reached by a continuous limiting procedure \([30, 31]\). The problem seemed to lie in the linearization procedure \([32]\), but it was soon shown that any non-linear version introduces ghost instability \([33]\). Since then there have been several attempts to find a consistent theory of massive gravity and significant progress has been made recently, using a bimetric formulation \([34–44]\). An important building block of these constructions is a matrix of the form \(\sqrt{g^{-1}f}\), where \(g\) is the metric and \(f\) is the auxiliary metric, and the square root is defined so that \(\sqrt{g^{-1}f} \sqrt{g^{-1}f} = g^{-1}f\). It is this ‘square root of metric’ that hints at a possible reformulation in terms of tetrads, since the metric is related to the tetrads by \(g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}\). The tetrad formulations of bimetric gravity that are available in the literature has two sets of tetrads, one of which acts as an auxiliary tetrad and the other is the dynamical one. As in the case of tetrad perturbations, the auxiliary tetrad is usually taken to be the Kronecker delta, and the torsion is ignored. One may expect that in the bi-tetrad formulation of the full theory of massive gravity the auxiliary tetrad will correspond to an arbitrary background, including torsion.

This paper is, however, somewhat less ambitious. Here we consider classical perturbations of gravity in the VEP formalism, and then construct the Fierz-Pauli equation, and
the corresponding action, in terms of these perturbations. It should be mentioned that while the vierbein formulation of bimetric massive gravity has also received some attention recently \cite{6, 11, 12}, our approach cannot be directly compared to these theories, again because the mass term in this paper is written for perturbations around an arbitrary curved background. We note that bimetric gravity for an arbitrary background metric has been investigated recently \cite{22, 23}, although not in the vierbein formalism, and perturbations were not considered. It may be expected that linearization of that formulation, in terms of tetrads, can be related to what we find below. We leave a careful comparison, as well as an investigation into ghost degrees of freedom in this formalism, for future work.

Let us briefly recapitulate the formalism of metric perturbation theory. In this, the space-time metric is written as a background metric plus a perturbation. Often we are interested in a flat background, in which case

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \]  

with \(|h_{\mu\nu}| \ll 1\) for each component. The inverse is given by

\[ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} , \]

where all indices are raised and lowered with \(\eta\).

The linearized Riemann and Ricci tensors, and the Ricci scalar, are thus

\[ R_{\mu\nu\rho\sigma} = \partial_\rho \partial_\sigma h_{\mu\nu} + \partial_\sigma \partial_\mu h_{\nu\rho} - \partial_\sigma \partial_\nu h_{\rho\mu} - \partial_\rho \partial_\mu h_{\nu\sigma} , \]

\[ R_{\mu\nu} = \frac{1}{2} \left[ \partial_\sigma \partial_\nu h^\sigma_\mu + \partial_\sigma \partial_\mu h^\sigma_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} \right] , \]

\[ R = \partial_\rho \partial_\sigma h^{\rho\sigma} - \Box h , \]

where \(h\) is the trace of the perturbation, \(h = \eta^{\mu\nu} h_{\mu\nu}\). The linearized Einstein equation in vacuum follows from these,

\[ \frac{1}{2} \left[ \partial_\sigma \partial_\nu h^\sigma_\mu + \partial_\sigma \partial_\mu h^\sigma_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu} \Box h \right] = 8 \pi GT_{\mu\nu} , \]

which is the linearized form of \(G_{\mu\nu} = 8 \pi GT_{\mu\nu}\). Usually we are concerned with the vacuum equation, \(G_{\mu\nu} = 0\). The linearized version written above is for a special situation where we have taken the background to be flat. We will consider perturbations around a general curved background. Then let us denote the background metric by \(\bar{g}_{\mu\nu}\) so that the total
metric is written as \( g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \). Then the Christoffel symbol is

\[
\Gamma^\alpha_{\mu\nu} = \bar{\Gamma}^\alpha_{\mu\nu} - \frac{1}{2} \bar{g}^{\alpha\lambda} \left[ \partial_\mu \bar{g}_{\lambda\nu} + \partial_\nu \bar{g}_{\mu\lambda} - \partial_\lambda \bar{g}_{\mu\nu} \right] + \frac{1}{2} \bar{g}^{\alpha\lambda} \left[ \partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu} \right] + \mathcal{O}(h^2),
\]

(1.7)

where now the indices are lowered and raised by \( \bar{g}_{\mu\nu} \) and its inverse \( \bar{g}^{\mu\nu} \), respectively, and quantities pertaining to the background are denoted by a bar.

By calculating the Einstein tensor using these Christoffel symbols, we can write the Einstein equations for an arbitrary background,

\[
\bar{R}_{\mu\nu} - \frac{1}{2} \left( \bar{g}_{\mu\nu} \bar{g}^{\alpha\beta} + h_{\mu\nu} \bar{g}^{\alpha\beta} - \bar{g}_{\mu\nu} h^{\alpha\beta} \right) \bar{R}_{\alpha\beta} + R^1_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{g}^{\alpha\beta} R^1_{\alpha\beta} = 8\pi G \bar{T}_{\mu\nu},
\]

(1.8)

where the Ricci tensor \( \bar{R}_{\mu\nu} \) is derived from the background metric \( \bar{g}_{\mu\nu} \), and the quantity \( R^1_{\mu\nu} \) is the part of the Ricci tensor linear in \( h \). This equation, however, contains the background equation \( \bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = 8\pi G \bar{T}_{\mu\nu} \) which needs to be eliminated. Note that if the stress-energy tensor is calculated from a matter action by varying the metric, \( T_{\mu\nu} \) and \( \bar{T}_{\mu\nu} \) are not equal in general. Thus we can finally write the equation for gravitational perturbations as

\[
\frac{1}{2} \left( \bar{g}_{\mu\nu} h^{\alpha\beta} - h_{\mu\nu} \bar{g}^{\alpha\beta} \right) \bar{R}_{\alpha\beta} + R^1_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{g}^{\alpha\beta} R^1_{\alpha\beta} = 8\pi G \left( T_{\mu\nu} - \bar{T}_{\mu\nu} \right).
\]

(1.9)

In this paper we will consider the VEP formulation of Einstein gravity, and find the tetrad equivalent of the linearized Einstein’s equation. We will consider perturbations on a general curved background, and treat the perturbations of tetrads and spin connection independently. When the background spacetime is flat, we recover the standard tetrad perturbation equations. The paper is organized in the following way. In Sec. [II] we briefly recall the VEP formulation of gravity, to fix the notation and conventions. In Sec. [III] we find the equations for small perturbations of the vierbein around an arbitrary background spacetime with matter. In Sec. [IV] we consider different fields as matter source. In particular, we find that for the real scalar field and the electromagnetic field the perturbation equations can be written purely in terms of the vierbein perturbations. For a fermionic field on the other hand, there is a torsion component to the spin connection, so the latter cannot be eliminated from the perturbation equations. Finally in Sec. [V] we write a mass term for the vierbein perturbations, leading to the Fierz-Pauli equation and its generalization for a curved background.
II. VIERBEIN-EINSTEIN-PALATINI FORMALISM

In the vierbein-Einstein-Palatini formalism, the variables for gravity are the vierbein or tetrads $e^I_\mu$, and the spin connection $A^{IJ}_\mu$. We will denote space-time indices by lowercase Greek letters and internal indices by uppercase Roman letters. The internal space is a 4-dimensional flat space with metric $\eta_{IJ} = (-1, 1, 1, 1)$ attached to each point of space-time. Raising and lowering of the internal indices are done by $\eta$, while space-time indices are raised and lowered by the space-time metric $g$, which is also of signature $(-+++)$.

The tetrads are defined to be orthonormal,

$$g^{\mu\nu}e^I_\mu e^J_\nu = \eta^{IJ},$$  \hspace{1cm} \text{(2.1)}

which we can rewrite as

$$e^I_\mu e^I_\mu = \delta^I_I, \quad e^J_\mu e^J_\nu = \delta^J_\nu,$$  \hspace{1cm} \text{(2.2)}

identifying $e^I_\mu \equiv \eta_{IJ} g^{\mu\nu}e^J_\nu$ as the co-tetrad. It is easy to see that the determinants are related by $|e| = \sqrt{-g}$.

A connection $D$ on this bundle is defined by its action on any smooth section $S$,

$$D_\mu S^I = \partial_\mu S^I + A^{IJ}_\mu S^J,$$  \hspace{1cm} \text{(2.3)}

where $A^{IJ}_\mu$ are the components of what is called the spin connection. It follows from definition that $A^{IJ}_\mu$ is antisymmetric,

$$0 = D_\mu \eta^{IJ} = \partial_\mu \eta^{IJ} - A^{IJ}_K \eta^{KJ} - A^{IJ}_K \eta^{IK}$$

$$\Rightarrow A^{IJ}_\mu = -A^{JI}_\mu.$$  \hspace{1cm} \text{(2.4)}

The curvature of $D$ can be written as

$$F^{IJ}_{\mu\nu} = [D_\mu, D_\nu]^{IJ}$$

$$= \partial_\mu A^{IJ}_\nu - \partial_\nu A^{IJ}_\mu + A^{IJ}_K A^{KJ}_\nu - A^{IJ}_K A^{KJ}_\mu$$

$$= \partial_\mu A^{IJ}_\nu - \partial_\nu A^{IJ}_\mu + [A_\mu, A_\nu]^{IJ}. $$  \hspace{1cm} \text{(2.5)}

In order to write the connection on space-time, we define a set of Christoffel symbols

$$\Gamma^\alpha_{\mu\nu} = e^\alpha_I \partial_\mu e^I_\nu + A^{IJ}_\mu e^J_\nu e^\alpha_I.$$  \hspace{1cm} \text{(2.6)}
This leads to a metric-compatible connection, as we see from the following calculation,

$$
\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\beta_{\alpha\mu} g_{\beta\nu} - \Gamma^\beta_{\alpha\nu} g_{\mu\beta}
$$

$$
= \eta_{IJ} \partial_\alpha (e^J_\mu e^I_\nu) - \eta_{IJ} e^J_\mu \partial_\alpha e^I_\nu - \eta_{IJ} e^I_\mu \partial_\alpha e^J_\nu - A^I_{\alpha J} \eta_{IL} e^L_\mu e^J_\nu - A^J_{\alpha I} \eta_{KL} e^I_\mu e^K_\nu
$$

$$
= -A^I_{\alpha J} e_J_\mu e_I_\nu - A^J_{\alpha I} e_J_\nu e_I_\mu = 0,
$$

(2.7)

where we have used the antisymmetry of the spin-connection $A$ in the last step.

We can now calculate the Riemann tensor in terms of the tetrads and the spin connection,

$$
R^\rho_{\sigma\mu\nu} = F^I_{\mu\nu J} e_\rho^I e_J^J,
$$

(2.8)

thereby getting the Ricci tensor and Ricci scalar respectively as

$$
R_{\sigma\nu} = F^I_{\mu\nu J} e_\nu^I e_J^J,
$$

(2.9)

$$
R = F^I_{\mu\nu} e_\mu^I e_\nu^I.
$$

(2.10)

In the Einstein-Hilbert action for gravity, we replace the Ricci scalar by Eq. (2.10), and the metric determinant by that of tetrads, to write the action as

$$
S[e \, A] = \int_M |e| d^4x F^I_{\mu\nu} e^\mu_I e^\nu_J.
$$

(2.11)

This action is extremized under variations of the vierbein $e^\mu_I$, keeping $A^I_{\mu J}$ fixed. Variation of the determinant gives

$$
\delta |e| = -|e| e^\mu_I \delta e_I^\mu.
$$

(2.12)

Using the antisymmetry of $F^I_{\mu\nu}$, we can then derive the field equations quite easily,

$$
2 F^J_{\lambda\nu} e^\lambda_I - e^\mu_I F^K_{\mu\sigma} e^K_L e^\sigma_J = 0.
$$

(2.13)

Contracting with $e_{\mu J}$, and using Eq. (2.9), we get the familiar form

$$
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0.
$$

(2.14)

This equation would be the vacuum Einstein equation if we could show that $\nabla$ is torsion free, i.e., if $\Gamma^\alpha_{\mu\nu}$ is symmetric in $\mu, \nu$. For this purpose, we vary the action of Eq. (2.11) again, but this time with respect to the spin connection $A^I_{\mu J}$, keeping the vierbein fixed. To do this we first simplify the action using the antisymmetry of the spin connection,

$$
S[e \, A] = \int |e| d^4x \left( 2 \partial_\mu A^J_{\nu } e^\mu_J e_\nu^J + [A_\mu, A_\nu]_{IJ} e^\mu_I e^\nu_J \right).
$$

(2.15)
Variation with respect to $A^I_{\mu J}$ produces the equation
\[-e^K_{\mu} (\partial_{\mu} e^K_{\alpha}) e^\alpha_I e^\nu_J - e^K_{\nu} (\partial_{\nu} e^K_{\alpha}) e^\alpha_I e^\mu_J + A^K_{\mu} e^K_{\nu} e^\nu_I e^\mu_J - A^K_{\nu} e^K_{\mu} e^\mu_I e^\nu_J = 0. \tag{2.16}\]

In order to identify $\Gamma$ in this equation, we contract with $e^I_{\rho} e^J_{\lambda}$,
\[-e^K_{\rho} (\partial_{\rho} e^K_{\alpha}) \delta^\nu_{\lambda} - (e^K_{\rho} \partial_{\rho} e^K_{\alpha}) \delta^\nu_{\lambda} - (e^K_{\mu} \partial_{\mu} e^K_{\nu}) \delta^\rho_{\lambda} - A^K_{\mu} e^K_{\nu} e^\nu_{\rho} e^\mu_{\rho} = 0. \tag{2.17}\]

Antisymmetrising in $\rho, \lambda$ and using the expression Eq. (2.6) for $\Gamma$, we get
\[\delta^\nu_{\lambda} (\Gamma^\alpha_{\rho \lambda} - \Gamma^\alpha_{\rho \alpha}) + \delta^\nu_{\rho} (\Gamma^\alpha_{\lambda \alpha} - \Gamma^\alpha_{\rho \lambda}) + \Gamma^\nu_{\rho \lambda} - \Gamma^\nu_{\lambda \rho} = 0, \tag{2.18}\]
whose trace produces $\Gamma^\alpha_{\rho \alpha} - \Gamma^\alpha_{\rho \rho} = 0$. Putting this back into Eq. (2.18) gives the desired equation,
\[\Gamma^\nu_{\rho \lambda} = \Gamma^\nu_{\lambda \rho}. \tag{2.19}\]

Thus we can identify $\nabla$ as the unique metric-compatible torsion-free connection on the space-time. Although in usual General Relativity the torsion-free condition is imposed a priori, in the VEP formulation only metric-compatibility follows from the definition of $\Gamma$; the torsion-free condition comes from the equations of motion. Thus we can now identify Eq. (2.14) with Einstein’s equations in vacuum.

Once the connection has been identified as torsion-free, the spin connection becomes expressible as a function of tetrads,
\[A^J_{\mu} = \frac{1}{2} e^{\alpha I} e^{\beta J} e^K_{[\alpha} \partial_{\mu} e^K_{\beta]}, \tag{2.20}\]
using the definition of $\Gamma$.

So far we have considered the vacuum equations. In presence of matter, the total VEP action reads
\[S_{Total} = \frac{1}{16\pi G} \int |e| d^4x F^{IJ}_{\mu \nu} e^\mu_I e^\nu_J + S_M, \tag{2.21}\]
where $S_M = \int |e| d^4x \mathcal{L}_M$ is the action for the matter. The total VEP equation, obtained by variation with respect to the tetrad, is then
\[F^{IJ}_{\alpha \mu} e^\alpha_I - \frac{1}{2} e^\beta_J F^{KL}_{\alpha \beta} e^K_{\alpha} e^\beta_L = 8\pi G T^\alpha_{\mu \alpha} e^\alpha_J, \tag{2.22}\]
where $T^\alpha_{\mu \alpha}$ is the usual energy-momentum tensor for the matter. As before, we can contract this equation with the tetrad to obtain the familiar form, $G_{\mu \nu} = 8\pi G T_{\mu \nu}$. However, the
inclusion of matter dictates whether the spin connection can be expressed entirely in terms of tetrads. More precisely, if the Lagrangian of the matter field contains the spin connection, we get a nonvanishing quantity on the right-hand side of Eq. (2.16) and thus the torsion-free condition is not obtained. This happens in particular in case of fermionic matter, for which torsion remains an independent entity, as does the spin connection.

III. VEP PERTURBATIONS

The spin connection is expressible in terms of the tetrad in the absence of matter, in particular fermionic matter, as Eq. (2.20) relies crucially on the connection being torsion-free, which in this formalism follows from the matter action being independent of the spin connection $A^I_I$. Thus we can relate the spin connection to the tetrad when the spin connection does not couple to matter.

However, perturbations of tetrad and spin connection around some background solution of the equations of motion should be considered as independent objects. This is because perturbations are off-shell objects a priori and need not satisfy the same equations as the background solutions, and even the background tetrad and spin connection cannot be related when there is a fermionic energy-momentum density on the background space-time.

We write the tetrad $e^\mu_I$ as a sum of the background and perturbation,

$$e^\mu_I = \bar{e}^\mu_I + f^\mu_I; \quad (3.1)$$

where $f^\mu_I$ is much smaller than $\bar{e}^\mu_I$ (more precisely, $\text{Tr}(\bar{e}^\mu_I f^\mu_I) \ll 1$). As before, we will denote background quantities by a bar on top. In order to calculate the co-tetrads, we use their definition $e^\mu_I e^\nu_I = \delta^\mu_\nu = \bar{e}^\mu_I \bar{e}^\nu_I$, where the internal index is raised and lowered with $\eta_{IJ}$ as before. Thus we find

$$e^I_\mu = \bar{e}_I^\mu - \bar{e}^\alpha_J f^\alpha_J. \quad (3.2)$$

We will often denote $-\bar{e}^I_\mu \bar{e}^\alpha_J f^\alpha_J$ as $\tilde{f}^I_\mu$. The space-time indices will be raised and lowered by the total space-time metric $g_{\mu\nu} = e^l_I e_{l
u}$ when needed. By writing the background metric as $\bar{g}_{\mu\nu} = \bar{e}_I^l \bar{e}_{l\nu}$, we can identify the metric perturbation $h_{\mu\nu}$ in terms of the background tetrad $\bar{e}_I^l$ and the tetrad perturbation $\tilde{f}^I_\mu$ as

$$h_{\mu\nu} = \bar{e}_I^l \bar{e}_{l\nu} + \bar{e}_{l\nu} \tilde{f}^I_\mu. \quad (3.3)$$
The background now is any general space-time, not necessarily flat. Let us also write the spin connection as a sum of its value in the background space-time and a perturbation,

\[ A^{IJ}_{\mu} = \bar{A}^{IJ}_{\mu} + a^{IJ}_{\mu}. \]  

(3.4)

However, since all components \( \bar{A}^{IJ}_{\mu} \) of the background spin connection may vanish, it is not sensible to treat \( a^{IJ}_{\mu} \) as small perturbation. In particular, we will not neglect terms quadratic in \( a^{IJ}_{\mu} \) when calculating the action. Thus we calculate the Christoffel symbols up to first order in the perturbation \( f \) as

\[ \Gamma^{\alpha}_{\mu\nu} = \bar{\Gamma}^{\alpha}_{\mu\nu} + \bar{e}^{\alpha}_{I} \partial_{\mu} \bar{e}^{J}_{\nu} + f^{I}_{\mu} \partial_{\mu} \bar{e}^{J}_{\nu} + \bar{A}^{IJ}_{\mu} \bar{e}^{J}_{\nu} f^{I}_{\alpha} + a^{IJ}_{\mu} \bar{e}^{J}_{\nu} f^{I}_{\alpha} + a^{IJ}_{\mu} \bar{e}^{J}_I \bar{e}^{J}_J f^{I}_{\alpha}. \]  

(3.5)

Here \( \bar{\Gamma} \) corresponds to the background space-time,

\[ \bar{\Gamma}^{\alpha}_{\mu\nu} = \bar{e}^{\alpha}_{I} \partial_{\mu} \bar{e}^{J}_{\nu} + \bar{A}^{IJ}_{\mu} \bar{e}^{J}_{\nu} \bar{e}^{\alpha}_I. \]  

(3.6)

Let us also write the curvature in terms of \( \bar{A} \) and \( a \),

\[ F^{IJ}_{\mu\nu} = \bar{F}^{IJ}_{\mu\nu} + \mathcal{F}^{IJ}_{\mu\nu}, \]  

(3.7)

where \( \bar{F} \) is the background curvature and \( \mathcal{F} \), the extra part due to perturbation, can be written in the form

\[ \mathcal{F}^{IJ}_{\mu\nu} = \bar{D}_{\mu} a^{IJ}_{\nu} - \bar{D}_{\nu} a^{IJ}_{\mu} + [a_{\mu}, a_{\nu}]^{IJ}, \]  

(3.8)

with \( \bar{D} \) being the covariant derivative corresponding to the background spin connection \( \bar{A} \).

The perturbed VEP action is thus

\[ S = \frac{1}{16\pi G} \int |e| d^4x \left[ \bar{F}^{IJ}_{\mu\nu} \bar{e}^{I}_{\alpha} \bar{e}^{J}_{\beta} + 2\mathcal{F}^{IJ}_{\mu\nu} \bar{e}^{I}_{\alpha} \bar{e}^{J}_{\beta} + \mathcal{F}^{IJ}_{\mu\nu} f^{I}_{\alpha} f^{J}_{\beta} \right] + S_{M}. \]  

(3.9)

The determinant \( |e| \equiv |\bar{e}^{I}_{\mu} + \bar{f}^{I}_{\mu}| \) is now a polynomial in \( f \). The lowest order field equations will be of first order in \( f \). The variation of the determinant produces

\[ \delta |e| = -|e|(\bar{e}^{K}_{\alpha} + \bar{f}^{K}_{\alpha}) \delta f^{\alpha}_{K}, \]  

(3.10)

using which we obtain field equations by varying the VEP action of Eq. (3.9),

\[ (\bar{F}^{IJ}_{\alpha\mu} + \mathcal{F}^{IJ}_{\alpha\mu})(\bar{e}^{I}_{\mu} + f^{I}_{\mu}) - \frac{1}{2}(\bar{e}^{I}_{\mu} + \bar{f}^{I}_{\mu})(\mathcal{F}^{IJ}_{\alpha\beta} + \mathcal{F}^{IJ}_{\alpha\beta})(\bar{e}^{K}_{\alpha} + f^{K}_{\alpha})(\bar{e}^{\beta}_{L} + f^{\beta}_{L}) = 8\pi G T^{\nu}_{\mu} e^{\nu}_{J}. \]  

(3.11)

Of course, we could have obtained these directly from Eq. (2.22) by replacing \( e \to \bar{e} + \bar{f} \) and \( A \to \bar{A} + a \). However, it is useful to construct the action for the perturbations, as we will later consider matter couplings and a tetrad version of Fierz-Pauli equations.
Subtracting the VEP equation for the background, we get the equation of motion for the VEP perturbations,

\[ F^J_{\alpha I} f^\alpha_I - \frac{1}{2} F^K_{\alpha \beta} \left[ \tilde{f}^J_{\mu} e^K_{\alpha} e^\beta_L + 2 \tilde{f}^J_{\mu} f^K_{\alpha} e^\beta_L \right] + F^K_{\mu \alpha} (\tilde{e}^\alpha_I + f^{\alpha I}) \]

\[ - \frac{1}{2} F^K_{\alpha \beta} \left[ \tilde{e}^J_{\mu} e^K_{\alpha} e^\beta_L + f^K_{\mu} e^K_{\alpha} e^\beta_L + 2 \tilde{e}^J_{\mu} f^K_{\alpha} e^\beta_L \right] = 8 \pi G \left( T_{\mu \nu} e^{\nu J} - T_{\mu \nu} e^\nu J \right). \] (3.12)

This is a generic equation in the sense that we have not considered any particular background, or required that the background be flat, so this is the equation of perturbations around a general background space-time. In order to compare this equation with the metric perturbation equations of Eq. (1.9), let us consider some specific types of matter field theories. Whether or not the perturbation of the spin connection depends on tetrads relies entirely on the type of field considered. For non-fermionic matter, we will be able to use Eq. (2.20) for the spin connection and thus write Eq. (3.12) in terms of the tetrad and its perturbation. Let us investigate the VEP equation with scalar, electromagnetic and fermionic fields as examples of background matter.

IV. VEP EQUATION WITH MATTER FIELDS

A. Scalar field

The action for a massive real scalar field \( \phi \) is

\[ S_M = \int |e| d^4 x \left[ -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} m^2 \phi^2 \right]. \] (4.1)

The equation for the background vierbein with this matter field is given by

\[ \bar{F}^J_{\alpha I} e^\alpha_I = 8 \pi G \left[ \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} e^{\nu I} \left( e^J_{\nu} e^\beta J - \nabla_\alpha \phi \nabla_\beta \phi + m^2 \phi^2 \right) \right] e^{\nu J}. \] (4.2)

The right-hand side of this equation contains the background energy-momentum tensor of the field. Taking the trace of the equation by contraction with \( e^\mu J \), and then eliminating the trace, the equation can be written as

\[ \bar{F}^J_{\alpha I} e^\alpha_I = 8 \pi G \left[ \nabla_\mu \phi \nabla_\nu \phi e^{\alpha J} + \frac{1}{2} m^2 e^J_{\mu} \phi^2 \right]. \] (4.3)
For the massive scalar field, the VEP equation for vierbein perturbation, Eq. (3.12), can be written as

\[
\begin{align*}
\bar{F}^{IJ}_{\alpha I} f^\alpha_I - \frac{1}{2} F^{KL}_{\alpha \beta} \left[ \bar{f}^J J e^K e^\beta_L + 2 e^K f^K e^\beta_L + \bar{F}^{IJ}_{\alpha \beta} (e^K e^\beta_L + f^K e^\beta_L) \right] + F^{IJ}_{\alpha \beta} (e^K e^\beta_L + f^K e^\beta_L) \\
- \frac{1}{2} F^{KL}_{\alpha \beta} \left[ e^K e^\beta_L + \bar{f}^J J e^K e^\beta_L + 2 e^K f^K e^\beta_L \right] \\
= -4 \pi G \left( \bar{f}^J J e^K e^\beta K + 2 e^K f^K e^\beta K \right) \nabla_\alpha \phi \nabla_\beta \phi + \bar{f}^J J m^2 \phi^2.
\end{align*}
\] (4.4)

Interestingly, for the scalar field, the matter part can be completely removed from this equation. We use Eq. (4.3) to write the equation in the form

\[
\begin{align*}
F^{IJ}_{\alpha I} f^\alpha_I + F^{IJ}_{\alpha \beta} (e^K e^\beta_L + f^K e^\beta_L) - \frac{1}{2} F^{KL}_{\alpha \beta} \left[ e^K e^\beta_L + \bar{f}^J J e^K e^\beta_L + 2 e^K f^K e^\beta_L \right] = 0.
\end{align*}
\] (4.5)

Also, since the Lagrangian of scalar field does not contain the spin connection, variation of the action with respect to the connection yields the torsion-free condition as in vacuum. This enables us to express both the background spin connection and its perturbation in terms of \( e \) and \( f \). At the lowest order, the perturbation \( a^{IJ} \) turns out to be of first order in \( f \) and is given by

\[
a^{IJ}_\mu = \frac{1}{2} \left( e^{\alpha I} \bar{e}^{J J} - \bar{e}^{\beta I} e^{\alpha J} \right) \left[ \partial_\mu \left( \bar{e}^K \bar{f}^{\alpha K} \right) + \partial_\beta \left( \bar{e}^K \bar{f}^{\beta K} \right) - \partial_\alpha \left( \bar{e}^K \bar{f}^{\beta K} \right) \right].
\] (4.6)

Hence \( \mathcal{F} \) is also of first order in \( f \) at the lowest order. Thus neglecting higher order terms, Eq. (4.3) can be rewritten as

\[
\begin{align*}
F^{IJ}_{\alpha I} f^\alpha_I + F^{IJ}_{\alpha \beta} e^K e^\beta_L - \frac{1}{2} F^{KL}_{\alpha \beta} e^K e^\beta_L = 0.
\end{align*}
\] (4.7)

### B. Electromagnetic field

Next we consider the VEP equation with electromagnetic energy-momentum as source. Let us denote the electromagnetic field strength as \( F_{\mu \nu} \), then the Lagrangian of the field can be written as \(-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}\). The energy-momentum tensor for this is

\[
T_{\mu \nu} = F_{\mu \alpha} F^{\alpha \nu} - \frac{1}{4} e^\nu_I f^{I M} F_{\alpha \beta} F^\alpha \beta,
\] (4.8)

which is traceless and independent of the connection components. Thus the variation with respect to the connection again leads to the torsion-free condition as in the case of the scalar field. We can then write Eq. (3.12) with the electromagnetic field as matter source,

\[
\begin{align*}
\bar{F}^{IJ}_{\alpha I} f^\alpha_I + F^{IJ}_{\alpha \beta} e^K e^\beta_L = 8 \pi G \left[ F_{\mu \lambda} F_{\nu \rho} \left( e^\lambda e^{\mu I} f^{\rho I} + e^{\sigma I} e^{\nu J} f^{\lambda J} \right) - \frac{1}{4} f^{J J} F_{\alpha \beta} F^\alpha \beta - F_{\alpha \beta} e^K e^\beta_L f^{\sigma K} \right].
\end{align*}
\] (4.9)
Unlike for the scalar field, here we cannot eliminate the matter part. Thus the above equation is the final form of the VEP equation under perturbation with electromagnetic field as source.

C. Fermionic field

The action for a Dirac field $\psi$ on a curved background is given by 

$$ S_F = \int |e| d^4x \left[ i\bar{\psi} \gamma^K e^K_\mu \left( \partial_\mu + \frac{i}{4} A^{IJ}_\mu \sigma_{IJ} \right) \psi - m\bar{\psi} \psi \right], \quad (4.10) $$

where $\sigma_{IJ} = \frac{i}{2} [\gamma_I, \gamma_J]$. The perturbation equation Eq. (3.12) can be written as

$$ \tilde{F}^{IJ}_{\alpha \mu} f^a - \frac{1}{2} \tilde{F}^{KL}_{\alpha \beta} \left[ \tilde{f}^J_{\mu} e^K_\alpha e^\beta_L + 2 e^K_{\mu} f^a_\alpha e^\beta_L \right] + \mathcal{F}^{IJ}_{\alpha \mu} (e^a_\mu + f^a_\mu) 
- \frac{1}{2} \mathcal{F}^{KL}_{\alpha \beta} \left[ e^K_\mu e^K_\alpha e^\beta_L + \tilde{f}^J_{\mu} e^K_\alpha e^\beta_L + 2 e^K_{\mu} f^a_\alpha e^\beta_L \right] = 8\pi G (T_{\mu\nu} e^{\nu J} - \tilde{T}_{\mu\nu} e^{\nu J}) . \quad (4.11) $$

The right-hand side of this equation works out to be

$$ T_{\mu\nu} e^{\nu J} - \tilde{T}_{\mu\nu} e^{\nu J} = \frac{1}{4} \bar{\psi} \gamma^J a^K_{\alpha \mu} \sigma_{KL} \psi + \frac{1}{4} e^K_{\mu} e^K_\nu \bar{\psi} \gamma^K a^L_{\nu \mu} \sigma_{LM} \psi 
- e^K_{\mu} f^J_\mu \left( i\bar{\psi} \gamma^K \left( \partial_\nu + \frac{i}{4} \left( \bar{A}^{LM}_{\nu} + a^{LM}_{\nu} \right) \sigma_{LM} \right) \psi \right) 
- e^K_\nu f^J_\mu \left( i\bar{\psi} \gamma^K \left( \partial_\nu + \frac{i}{4} \left( \bar{A}^{LM}_{\nu} + a^{LM}_{\nu} \right) \sigma_{LM} \right) \psi \right) + m\tilde{f}^J_\mu \bar{\psi} \psi . \quad (4.12) $$

Next we vary the action with respect to $a^I_{\mu \nu}$. The resulting equation is

$$ -e^K_{\mu} (\partial_\nu e^K_\mu) e^K_\nu e^K_\nu - (\partial_\nu e^K_\mu) e^K_\nu - e^K_\mu (\partial_\nu e^K_\mu) + A^K_{\mu I} e^K_\nu + A^K_{\mu I} e^K_\nu + \pi G \bar{\psi} \gamma^K e^K_\nu \sigma_{IJ} \psi = 0 , \quad (4.13) $$

where we have written the equation in terms of the total vierbein and an overall factor of two has been eliminated. It is easy to show from this equation that the connection is not torsion free. If we contract with $e^K_\rho e^K_\lambda$ and then antisymmetrize in $\rho, \lambda$ as in Eq. (2.18), we get

$$ \delta_\nu^\lambda \left( \Gamma^\alpha_{\rho \alpha} - \Gamma^\alpha_{\alpha \rho} \right) + \delta_\rho^\nu \left( \Gamma^\alpha_{\alpha \lambda} - \Gamma^\alpha_{\lambda \alpha} \right) + (\Gamma^\nu_\lambda - \Gamma^\nu_\rho) = 2\pi G \bar{\psi} \gamma^K e^K_\nu \sigma_{IJ} e^K_\rho \psi . \quad (4.14) $$

Trace of this equation produces

$$ \Gamma^\alpha_{\rho \alpha} - \Gamma^\alpha_{\alpha \rho} = -3i\pi G \bar{\psi} \gamma_1 e^K_\rho \psi , \quad (4.15) $$

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where we have used $\gamma^K \sigma_{IK} = -3i \gamma_I$. Using this in Eq. (4.14) we find
\[
\Gamma^\nu_{\lambda\rho} - \Gamma^\nu_{\rho\lambda} = 2\pi G \bar{\psi} \gamma^K e^K_I e^I_\rho \psi + 3i\pi G \bar{\psi} \gamma_I (\delta^K_\lambda e^I_\rho - \delta^K_\rho e^I_\lambda) \psi.
\] (4.16)

This equation implies that $\Gamma$ is not torsion-free and the source of torsion is clearly the fermionic field as the equation indicates. It is easy to see that this expression for torsion can be written in the irreducible form as in [15, 16], with all the symmetries and trace identities.

If we have non-fermionic matter on a flat background, the VEP perturbation equations can be identified with the linearized Einstein equations, as we will see in the next section.

V. FIERZ-PAULI EQUATION

Massive spin-2 fields obey the Fierz-Pauli equation [29]. In this section we will write the Fierz-Pauli equation and the corresponding action in the VEP formalism. An action for massive gravitons, from which the Fierz-Pauli equation may be derived, can be written as
\[
S = \frac{1}{16\pi G} \int d^4x \left[ -\frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda} \partial^\nu h^{\mu\lambda} - \partial_\mu h_{\mu\nu} \partial_\nu h + \frac{1}{2} \partial_\lambda h \partial^\lambda h - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \right],
\] (5.1)

We can obtain the Fierz-Pauli equation for massive gravity by varying $h_{\mu\nu}$,
\[
\Box h_{\mu\nu} - \partial_\sigma \partial_\nu h^\sigma_\mu - \partial_\sigma \partial_\mu h^\sigma_\nu + \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h - m^2 (h_{\mu\nu} - \eta_{\mu\nu} h) = 0.
\] (5.2)

Usually this equation is written on a flat background space-time, and more specifically, with a background Minkowski metric, as we have done here. In order to get a theory of massive VEP perturbations, we add the Fierz-Pauli mass term to the action in terms of the VEP variables. Let us write the mass term for a background Minkowski metric as
\[
-\frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) = -m^2 (\bar{e}_I^f f^I_f f_{IJ}^f + \bar{e}_I^f f^I_f f_{IJ}^f - 2\bar{e}_I^f f^I_f f_{IJ}^f),
\] (5.3)

We note that this term depends only on the tetrad perturbations, not on the spin connection.

In the absence of matter, the connection is torsion-free, and thus $\mathcal{F}$ is of first order in $f$ at the lowest order. Therefore Eq. (3.12) reduces, at the lowest order in $f$, to
\[
\mathcal{F}^{IJ} e_{I}^\alpha e_{J}^\alpha = -\frac{1}{2} e_{I}^f e_{J}^f f_{IJ}^f = 0
\] (5.4)
in the absence of matter on a flat background, i.e. $\bar{F} = 0$. We can derive this equation from the action
\[
S_{\text{eff}} = \frac{1}{16\pi G} \int |e| d^4x \left( 2\mathcal{F}^{IJ} e_{I}^\alpha e_{J}^\alpha - e_{I}^f f_{IJ}^f \mathcal{F}^{KL}_{\alpha\beta} e_{I}^\alpha e_{L}^\beta \right).
\] (5.5)
The massive VEP action on a flat, matter-free background, is thus

\[
S = \frac{1}{16\pi G} \int |e| d^4x \left[ 2\mathcal{F}^{IJ}_{\alpha\mu} \bar{e}^I_\mu f^\mu_J - \bar{e}^I_\mu f^\mu_J \mathcal{F}^{KL}_{\alpha\beta} \bar{e}^K_\beta e^L_\alpha - m^2 \left( \bar{e}^I_\mu e^J_\nu f^\mu_J f^\nu_I + \bar{e}^I_\mu e^J_\nu f^\nu_J f^\mu_I - 2\bar{e}^I_\mu e^J_\nu f^\nu_J f^\mu_I \right) \right],
\]

which leads to the massive VEP equation by varying \( f \),

\[
\mathcal{F}^{IJ}_{\alpha\mu} \bar{e}^I_\mu - \frac{1}{2} e^J_\nu e^\alpha_K e^\beta_L - m^2 \left( \bar{e}^I_\mu e^J_\nu f^\mu_J f^\nu_I + \bar{e}^I_\mu e^J_\nu f^\nu_J f^\mu_I - 2\bar{e}^I_\mu e^J_\nu f^\nu_J f^\mu_I \right) = 0. \tag{5.7}
\]

As we have already mentioned, the Fierz-Pauli mass term is usually added to the linearized equations for perturbations — on a flat background, with the Minkowski metric — without any fermionic matter. In that case, the spin connection can be expressed as a function of the tetrads, as we have seen. Thus we could as well derive the corresponding equations for vierbein perturbations simply by replacing the metric and its perturbation in terms of those of the tetrads.

However, the way we have written the Fierz-Pauli equation and action allows an easy generalization to arbitrary curved backgrounds. We add the mass term on the right-hand side of Eq. (5.3) to the generalized VEP action Eq. (3.9) to get

\[
S = \frac{1}{16\pi G} \int |e| d^4x \left[ \mathcal{F}^{IJ}_{\mu\nu} \bar{e}^I_\mu \bar{e}^J_\nu + 2\mathcal{F}^{IJ}_{\mu\nu} \bar{e}^I_\mu f^\mu_J + \mathcal{F}^{IJ}_{\mu\nu} f^\mu_J f^\nu_I \\
- m^2 \left( \bar{e}^I_\mu \bar{e}^J_\nu f^\mu_J f^\nu_I + \bar{e}^I_\mu \bar{e}^J_\nu f^\nu_J f^\mu_I - 2\bar{e}^I_\mu \bar{e}^J_\nu f^\nu_J f^\mu_I \right) \right]. \tag{5.8}
\]

This is the VEP form of the Fierz-Pauli action in terms of a single set of tetrad fields, written as background plus perturbation. Written in this form, the action is not restricted to a flat background space-time, and we can add any type of matter source, including fermionic matter, to this action.

While this paper was in circulation as a preprint, a couple of papers have appeared regarding perturbations of bimetric gravity on an arbitrary background spacetime \[49, 50\]. Since these papers do not consider a vierbein formulation of the theory, their results cannot be easily compared with ours. However, one expects that their results and ours can be related by a careful comparison, which we leave for a future work.

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