A Two-Level Block Preconditioned Jacobi-Davidson Method for Computing Multiple and Clustered Eigenvalues

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Abstract: In this paper, we propose a two-level block preconditioned Jacobi-Davidson (BPJD) method for solving discrete eigenvalue problems resulting from finite element approximations of \(2m\)th \((m = 1, 2)\) order elliptic eigenvalue problems. A new and efficient preconditioner is constructed by an overlapping domain decomposition (DD). This method may compute the first several eigenpairs, including simple, multiple and clustered cases. In each iteration, we only need to solve a couple of parallel subproblems and one small scale eigenvalue problem. The rigorous theoretical analysis reveals that the convergence rate of the two-level BPJD method for the first several eigenvalues is bounded by \(c(H)(1 - C\frac{H^{2m-1}}{\delta^2})^2\), where \(H\) is the diameter of subdomains and \(\delta\) is the overlapping size among subdomains. The constant \(C\) is independent of the mesh size \(h\) and internal gaps among target eigenvalues, which means that our method is optimal and cluster robust. Meanwhile, the \(H\)-dependent constant \(c(H)\) decreases monotonically to 1, as \(H \to 0\), which means that more subdomains lead to the better convergence rate. Numerical results supporting our theory are given.

Keywords: Multiple and clustered eigenvalues, finite element, Jacobi-Davidson method, overlapping domain decomposition.

1 Introduction

Solving large scale eigenvalue problems is one of the fundamental problems in modern science and engineering (see, e.g., \[2, 4, 6, 10, 13, 27, 31, 34, 35\]). Compared with boundary value problems, there are fewer parallel solvers for solving PDE eigenvalue problems, among which computing multiple and clustered eigenvalues is more challenging. Therefore, we propose a two-level block preconditioned Jacobi-Davidson (BPJD) method for computing multiple and clustered eigenvalues, where the parallel preconditioner is constructed by an overlapping domain decomposition (DD). Then we give a rigorous theoretical analysis for our method, which shows that it is optimal and scalable, namely, the convergence rate does not deteriorate as the fine mesh size \(h \to 0\), or the number of subdomains increases. In particular, our method is cluster robust, namely, the convergence rate is not adversely affected by the gap among the clustered eigenvalues.

For the elliptic eigenvalue problem, Babuška and Osborn (see \[1\]) take advantage of the finite element method to compute eigenpairs. Two-grid methods have been widely used to solve elliptic eigenvalue problems (see, e.g., \[8, 32, 33\]). Under the conditions \(h = O(H^i)\) \((i = 2, 4, \text{see } 32, 33)\), asymptotically optimal accuracy may be obtained, respectively. For the discrete eigenvalue problem, many classical iterative algorithms are introduced (see \[21, 22\]). Inverse, shifted inverse, and Rayleigh quotient iteration may be regarded as the Newton’s method in \[25\], among which one of the most famous methods for
solving eigenvalue problems is the Jacobi-Davidson method proposed in [24]. At present, the Jacobi-Davidson method has been successfully applied to a variety of practical computations, such as Maxwell eigenvalue problems (see [10]), magnetohydrodynamics (MHD) eigenvalue problems (see [18]), polynomial PDE eigenvalue problems in quantum dot simulation (see [27, 28]), computations of large singular value decomposition (SVD) (see [9]) and so on.

For the large scale discrete eigenvalue problem, some preconditioning techniques are needed (see [12]). Cai (see [3]) and McCormick (see [17]) propose some multigrid methods to compute the eigenpairs. Recently, a class of multilevel correction methods for solving elliptic eigenvalue problems are studied in [6, 29–31]. Yang (see [34]) propose another multilevel correction method based on the shift and inverse technique for solving elliptic eigenvalue problems.

It is known that the DD method performs better than the MG method from the parallel point of view. For DD methods, Lui [15] proposes some two-subdomain DD methods to compute the principal eigenpair through solving an interface problem. For many subdomains case, Maliassov [16] constructs a Schwarz alternating method for solving the eigenvalue problem and it can be shown to be convergent if the initial guess satisfies a suitable assumption. By dealing with an interface condition, Genseberger (see [6]) presents some eigensolvers by combining the Jacobi-Davidson method with some non-overlapping domain decomposition methods. Zhao, Huang and Cai (see [35, 36]) propose a two-level preconditioned Jacobi-Davidson (PJD) method for eigenvalue problems. To obtain the quadratic convergence for the outer iteration, the Jacobi-Davidson correction equation needs to be solved accurately enough, which is the most expensive part. In order to reduce the cost of solving the Jacobi-Davidson correction equation, Wang and Xu (see [28]) construct a domain decomposition method to precondition the Jacobi-Davidson correction equation at one step in every outer iteration. A rigorous theoretical analysis of the two-level PJD method for the principal eigenvalue of 2mth ($m = 1, 2$) order elliptic operators is presented in [28]. Recently, we also present a two-level preconditioned Helmholtz-Jacobi-Davidson (PHJD) method for the Maxwell eigenvalue problem (see [14]). The two-level PHJD method works well in practical computations and it is proved to be optimal and scalable. However, for the two-level PJD method in [27, 28] or the two-level PHJD method in [14], the theoretical analysis is only valid for the simple principal eigenvalue. In this paper, we try to give a rigorous analysis for the non-principal eigenvalues, including multiple and clustered eigenvalues.

We must emphasize that the theoretical analysis from simple eigenvalue to multiple and clustered eigenvalues is nontrivial for the two-level BPJD method. Firstly, it is necessary to ensure that all of constants appearing in the convergence rate do not depend on internal gaps of target eigenvalues; Secondly, since the dimension of eigenspace corresponding to the multiple eigenvalue is greater than 1, many techniques developed for simple eigenvalues in [14, 27, 28] are not applicable; Thirdly, to measure the distance, the Hausdorff distance or the gap needs to be used in the theoretical analysis rather than the vector norm, which brings more difficulties. In this paper, we overcome these difficulties, and try to prove that the convergence result is optimal, scalable and cluster robust, i.e., (1.1) is true.

Combining an additive Schwarz domain decomposition method with a spectral decomposition on the coarse space, we propose an efficient and parallel eigensolver for solving the first several eigenvalues (including multiple and clustered cases) and the corresponding eigenvectors simultaneously. And then we give a rigorous analysis of multiple eigenvalues and clustered eigenvalues for 2mth ($m = 1, 2$) order elliptic operators. By constructing a stable decomposition for error space of target eigenvalues, technical estimates for the gap and other tricks, we finally get that the convergence rate of the first s eigenvalues of the two-level BPJD method is bounded by $\gamma \leq c(H)(1 - C \frac{s^{m-1}}{\Pi_{i=1}^{s}})^2$, i.e.,

$$\sum_{i=1}^{s} (\lambda_{i+1}^h - \lambda_i^h) \leq \gamma \sum_{i=1}^{s} (\lambda_i^h - \lambda_i^h),$$

where $\lambda_i^h$ is the current iterative approximation of the $i$th discrete eigenvalue $\lambda_i$. The constant $C$ is independent of $h, H, \delta$ and internal gaps among the first $s$ eigenvalues. Meanwhile, the $H$-dependent constant $c(H)$ decreases monotonically to 1, as $H \rightarrow 0$. Namely, the two-level BPJD method is optimal, cluster robust and scalable. Moreover, we have not any assumption on the relationship between $H$ and
h, and internal gaps among the first s eigenvalues. Numerical results presented in this paper verify our theoretical findings.

The rest of this paper is organized as follows: Some preliminaries are introduced in Section 2. In Section 3, the two-level BPJD method for 2mth order elliptic eigenvalue problems is proposed. Some properties of subspace method are presented in Section 4 and the main convergence analysis is given in Section 5. Finally, we present our numerical results in Section 6 and the conclusion in Section 7.

2 Model problems and preliminaries

In this section, we first introduce some notations and model problems in subsection 2.1, the corresponding discrete counterpart in subsection 2.2, and then present some results on domain decomposition methods in subsection 2.3.

2.1 Model problems

Throughout this paper, we use the standard notations for the Sobolev spaces \( W^{m,q}(\Omega) \) and \( W^{m,q}_0(\Omega) \) with their associated norms and seminorms. We denote by \( H^m(\Omega) := W^{m,q}(\Omega) \) and \( H^m_0(\Omega) := W^{m,q}_0(\Omega) \) for \( q = 2 \), and denote by \( L^2(\Omega) := H^0(\Omega) \) for \( m = 0 \). Consider the Laplacian and biharmonic eigenvalue problems as follows:

\[
\begin{align*}
-\Delta u &= \lambda u, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

and

\[
\begin{align*}
\Delta^2 u &= \lambda u, \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]

For simplicity, we assume that \( \Omega \) is a convex polygonal domain in \( \mathbb{R}^2 \) and \( \partial \Omega \) is the boundary of \( \Omega \). We denote by \( n \) the unit outward normal vector of \( \partial \Omega \).

The variational form of 2mth order elliptic eigenvalue problems may be written as:

\[
\begin{align*}
\text{Find } (\lambda, u) &\in \mathcal{R} \times V \text{ such that } b(u, u) = 1, \\
a(u, v) &= \lambda b(u, v) \quad \forall \ v \in V,
\end{align*}
\]

where \( V := H^m_0(\Omega) \), the bilinear forms \( a(\cdot, \cdot) : V \times V \to \mathcal{R} \), \( b(\cdot, \cdot) : L^2(\Omega) \times L^2(\Omega) \to \mathcal{R} \) are symmetric and positive. Define \( b(u, v) := \int_\Omega uv \ dx \) for all \( u, v \in L^2(\Omega) \) and \( ||v||_b^2 := b(v, v) \) for all \( v \in L^2(\Omega) \). Specifically, for (2.1),

\[
a(u, v) = \int_\Omega \nabla u \cdot \nabla v \ dx
\]

for all \( u, v \in V = H^1_0(\Omega) \), and for (2.2),

\[
a(u, v) = \frac{1}{\Omega} \int_\Omega \Delta u \Delta v \ dx
\]

for all \( u, v \in V = H^2_0(\Omega) \). It is easy to see that \( a(\cdot, \cdot) \) constructs an inner product on \( V \) and we define \( ||v||_a^2 := a(v, v) \) for all \( v \in V \). For convenience, we denote by \( R_b(v) := \frac{a(v, u)}{b(u, u)} > 0 \) for all \( v \neq 0 \) \( \in V \) the Rayleigh quotient functional. We also define \( R_a(v) := \frac{1}{R_b(v)} < 0 \) for all \( v \neq 0 \) \( \in V \).

Define a linear operator \( T : L^2(\Omega) \to V \) such that for any \( f \in L^2(\Omega) \),

\[
a(T f, v) = b(f, v) \quad \forall \ v \in V.
\]
Since \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are symmetric and \( V \) is embedded compactly in \( L^2(\Omega) \), we know that \( T : L^2(\Omega) \to L^2(\Omega) \) is compact and symmetric. Moreover, \( T : V \to V \) is also compact and symmetric. By the Hilbert-Schmidt Theorem, we get that \( Tu_i = \mu_i u_i \) (\( \mu_i = (\lambda_i)^{-1} \)), and the eigenvalues of (2.5) are
\[
\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to +\infty,
\]
and the corresponding eigenvectors are \( u_1, u_2, \ldots, u_n, \ldots \), which satisfy \( a(u_i, u_j) = \lambda_i b(u_i, u_j) = \lambda_i \delta_{ij} \) (\( \delta_{ij} \) represents the Kronecker delta). In the sequence \( \{\lambda_i\}_{i=1}^{\infty} \), \( \lambda_i \) is repeated according to its geometric multiplicity. For convenience, we call \( d_{i-1}^{-} := (\lambda_i - \lambda_{i-1}) \) and \( d_{i+1}^{+} := (\lambda_{i+1} - \lambda_i) \) as the left gap and the right gap of the eigenvalue \( \lambda_i \), respectively. In particular, \( d_{i-1}^{-} := \lambda_1 \).

We are interested in the first \( s \) eigenvalues \( \{\lambda_i\}_{i=1}^{s} \) and the corresponding eigenvectors \( \{u_i\}_{i=1}^{s} \). For our theoretical analysis, we first introduce a reasonable assumption.

**Assumption 1** Assume that there is an ‘obvious’ gap between the \( s \)th eigenvalue \( \lambda_s \) and the \( (s+1) \)th eigenvalue \( \lambda_{s+1} \).

**Remark 2.1** Assumption \([1]\) excludes two cases: (i) \( \lambda_s = \lambda_{s+1} \), (ii) \( \lambda_s \approx \lambda_{s+1} \), but there are not any assumptions about the left gaps of the eigenvalues \( \{\lambda_i\}_{i=2}^{s} \). In practical computation, either for case (i) or for case (ii), we may consider the first \( s + s_1 \) eigenvalues so that \( \lambda_{s+s_1} \) and \( \lambda_{s+s_1+1} \) satisfy the Assumption 1, where \( s_1 ( \geq 1 \) is a positive integer. So actually this assumption is not a limitation for our practical computation.

It is known that the following spacial decomposition property holds
\[
V = U_s \oplus U_{s+1},
\]
where \( U_s = \text{span}\{u_1, u_2, \ldots, u_s\} \), \( \oplus \) denotes the orthogonal direct sum with respect to \( b(\cdot, \cdot) \) (also \( a(\cdot, \cdot) \)) and \( U_{s+1} \) is the orthogonal complement of \( U_s \).

In order to measure the ‘distance’ between two closed subspaces included in a Hilbert space, we introduce the following definition. For more details, please see [11], Section 2 in [13] and references therein.

**Definition 2.1** For any Hilbert space \((X, (\cdot, \cdot))\), define \( \Sigma_X := \{ W \mid W \) is a closed subspace of \( X \}\). A binary mapping \( \theta \) (called as the gap) : \( \Sigma_X \times \Sigma_X \to [0, 1] \) is defined by
\[
\theta(W_1, W_2) = \max\{\sin(W_1; W_2), \sin(W_2; W_1)\} \quad \text{for all } W_1, W_2 \in \Sigma_X,
\]
where
\[
\sin(W_1; W_2) = \sup_{u \in W_1, ||u||=1} \inf_{v \in W_2} ||u - v|| \quad \text{for all } W_1, W_2 \in \Sigma_X.
\]
If \( W_1 = 0 \), set \( \sin(W_1, W_2) = 0 \) for all \( W_2 \in \Sigma_X \). If \( W_2 = 0 \), set \( \sin(W_1, W_2) = 1 \) for all \( W_1 (\neq 0) \in \Sigma_X \).

**Remark 2.2** For any Hilbert space \((X, (\cdot, \cdot))\), if \( W_1, W_2 \in \Sigma_X \) and \( \dim(W_1) = \dim(W_2) < +\infty \), it is easy to know that
\[
\theta(W_1, W_2) = \sin(W_1; W_2) = \sin(W_2; W_1).
\]
For any \( W_1, W_2, W_3 \in \Sigma_X \) and \( \dim(W_1) = \dim(W_2) = \dim(W_3) < +\infty \), it is easy to check that
\[
\sin(W_1; W_2) \leq \sin(W_1; W_3) + \sin(W_3; W_2).
\]
If \( W_1 = \text{span}\{u\} \), then \( \sin(W_1; W_2) \) is denoted through \( \sin\{u; W_2\} \). Similarly, if \( W_2 = \text{span}\{v\} \), then \( \sin(W_1; W_2) \) is denoted through \( \sin(W_1; v) \).

In the rest of this paper, we shall use the notations \( \sin_b(\cdot, \cdot) \) and \( \sin_a(\cdot, \cdot) \) with respect to \( b(\cdot, \cdot) \) and \( a(\cdot, \cdot) \), respectively. We also denote by \( \theta_b(\cdot, \cdot) \) and \( \theta_a(\cdot, \cdot) \) the gaps with respect to \( b(\cdot, \cdot) \) and \( a(\cdot, \cdot) \), respectively.
2.2 Finite element discretization

Let $V^h$ be a conforming finite element space based on a shape regular and quasi-uniform triangular or rectangular partition $\mathcal{G}_h$ with the mesh size $h$. We consider the discrete variational form of (2.3) as:

\[
\begin{align*}
\text{Find } (\lambda^h, u^h) \in \mathcal{R} \times V^h \text{ such that } &\|u^h\|_0 = 1, \\
& a(u^h, v) = \lambda^h b(u^h, v) \quad \forall v \in V^h.
\end{align*}
\]

(2.6)

Define a discrete linear operator $T^h : L^2(\Omega) \to V^h$ such that for any $f \in L^2(\Omega)$,

\[
a(T^h f, v) = b(f, v) \quad \forall v \in V^h.
\]

(2.7)

It is easy to see that the operator $T^h$ is compact and symmetric (For convenience of notations, $T^h|_{Y^h}$ is also denoted through $T^h$ in the following). Hence, we get that $T^h u^h_i = \mu^h_i u^h_i$ ($\mu^h_i = (\lambda^h_i)^{-1}$). Meanwhile, the eigenvalues of (2.6) are $\lambda^h_1 \leq \lambda^h_2 \leq \ldots \leq \lambda^h_n$, and the corresponding eigenvectors are $u^h_1, u^h_2, \ldots, u^h_n$, which satisfy $a(u^h_i, u^h_j) = \lambda^h_i b(u^h_i, u^h_j) = \lambda^h_i \delta_{ij}$ and $ad = \dim(V^h)$. We also define $A^h : V^h \to V^h$ such that $b(A^h u, v) = a(u, v)$ for all $u, v \in V^h$, and it is obvious to see that $A^h u^h_i = \lambda^h_i u^h_i$.

The finite element space $V^h$ may be decomposed as:

\[
V^h = U^h_s \oplus U^h_{s+1} = V^h_1 \oplus V^h_2 \oplus \ldots \oplus V^h_s \oplus U^h_{s+1},
\]

(2.8)

where $U^h_s = V^h_1 \oplus V^h_2 \oplus \ldots \oplus V^h_i$, $V^h_s = \text{span}\{u^h_i\}$, $i = 1, 2, \ldots, s$ and $U^h_{s+1}$ denotes the $b(\cdot, \cdot)$-orthogonal (also $a(\cdot, \cdot)$-orthogonal) complement of $U^h_s$. Let $Q^h_s, Q^h_{s+1}$ and $Q^h_{i+1}$, $i = 1, 2, \ldots, s$ be the $b(\cdot, \cdot)$-orthogonal (also $a(\cdot, \cdot)$-orthogonal) projectors from $V^h$ onto $U^h_s, U^h_{s+1}$ and $V^h_i$ ($i = 1, 2, \ldots, s$), respectively. For any subspace $U \subset V^h$, $U^\perp$ represents the orthogonal complement of $U$ with respect to $b(\cdot, \cdot)$, and let $Q_U$ and $P_U$ be the $b(\cdot, \cdot)$-orthogonal and the $a(\cdot, \cdot)$-orthogonal projectors from $V^h$ onto $U$, respectively. If $U = \text{span}\{u\}$, then we denote $Q_u = Q_U$ and $P_u = P_U$. Unless otherwise stated, the letters $C$ (with or without subscripts) in this paper denote generic positive constants independent of $h$, $H$, $\delta$ and the left gaps of the eigenvalues $\{\lambda_i\}_{i=2}^n$, which may be different at different occurrences.

2.2.1 The Laplacian eigenvalue problem

In order to make the ideas clearer, we use $V^h$, the continuous piecewise and linear finite element space with vanishing trace, to approximate the Sobolev space $H^1(\Omega)$ for the Laplacian eigenvalue problem. The following a priori error estimates are useful in this paper. For the first conclusion in Theorem 2.2, please see [11] and [13] for more details. For the proof of (2.9), please see Theorem 3.1 and Theorem 3.3 in [13]. In order to focus on the algorithm in Section 3 and the corresponding theoretical analysis in Section 4 and 5, we give proofs of (2.10) and (2.11) in Appendix.

**Theorem 2.2** Let $\Omega$ be a bounded convex polygonal domain, then the eigenvalues of discrete problem (2.6) $\lambda^h_1, \lambda^h_2, \ldots, \lambda^h_n$ converge to the eigenvalues of problem (2.3) $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, as $h \to 0$. Moreover, there exists $h_0 > 0$ such that for $0 < h < h_0$, the following inequalities hold:

\[
0 \leq \lambda^h_i - \lambda_i \leq Ch^2, \quad i = 1, 2, \ldots, s,
\]

(2.9)

and

\[
\theta_a(U_s, U^h_s) \leq Ch,
\]

(2.10)

\[
\theta_b(U_s, U^h_s) \leq Ch^2,
\]

(2.11)

where the constant $C$ is independent of the left gaps of the eigenvalues $\{\lambda_i\}_{i=2}^n$, but depends on $d^+$, $\theta_a(U_s, U^h_s)$ and $\theta_b(U_s, U^h_s)$ denote the gaps between $U_s$ and $U^h_s$ with respect to $\|\cdot\|_a$ and $\|\cdot\|_b$, respectively.
2.2.2 The biharmonic eigenvalue problem

For problem (2.2), we shall use $V_h$, the Böcher-Fox-Schmit (BFS) finite element space with vanishing trace and vanishing trace of outer normal derivative, to approximate the Sobolev space $H_0^2(\Omega)$. For more details about BFS finite element, please see [23] and references therein. Under the regularity assumption that the eigenfunction $u_i \in H^3(\Omega) \cap H_0^2(\Omega)$ (i = 1, 2, ..., s) for the biharmonic eigenvalue problem, we also have the same theoretical results as Theorem 2.2.

2.3 Domain decomposition

In this subsection, we introduce some results on overlapping domain decomposition.

Let $\{\Omega_l\}_{l=1}^N$ be a coarse shape regular and quasi-uniform partition of $\Omega$, and we denote it by $J_H$. We define $H := \max\{H_l \mid l = 1, 2, ..., N\}$, where $H_l = \text{diam}(\Omega_l)$. The fine shape regular and quasi-uniform partition $J_h$ is obtained by subdividing $J_H$. We may construct the finite element spaces $V^h \subset V^h$ on $J_H$ and $J_h$, respectively. To get the overlapping subdomains $\{\Omega_l\}$, 1 ≤ $l$ ≤ $N$, we enlarge the subdomains $\Omega_l$ by adding fine elements inside $\Omega$ layer by layer such that $\partial \Omega_l$ does not cut through any fine element. To measure the overlapping width between neighboring subdomains, we define $\delta := \min\{\delta_l \mid l = 1, 2, ..., N\}$, where $\delta_l = \text{dist}(\partial \Omega_l \setminus \partial \Omega, \partial \Omega_l \setminus \partial \Omega)$. We also assume that $H_l$ is the diameter of $\Omega_l$. Let $\Omega_{l, \delta_l} (\subset \Omega_l)$ be the set of the points that are within a distance $\delta_l$ of $\partial \Omega_l \setminus \partial \Omega$, l = 1, 2, ..., $N$. The local subspaces may be defined by $V^{(l)} := V^h \cap H_0^2(\Omega_l)$ (for the Laplacian operator) or $V^{(l)} := V^h \cap H_0^3(\Omega_l)$ (for the biharmonic operator). It is obvious to see $V^{(l)} \subset V^h$ by a trivial extension.

Assumption 2 The partition $\{\Omega_l\}_{l=1}^N$ may be colored using at most $N_0$ colors, in such a way that subdomains with the same color are disjoint. The integer $N_0$ is independent of $N$.

There exists a family of continuous piecewise and linear functions $\{\theta_l\}_{l=1}^N$ which satisfy the following properties (see [23] or the Chapter 3 in [26]):

$$\text{supp}(\theta_l) \subset \overline{\Omega_l}, \quad 0 \leq \theta_l \leq 1, \quad \sum_{l=1}^N \theta_l(x) = 1, \quad x \in \Omega, \quad ||\nabla \theta_l||_{0,\infty,\Omega_{l,\delta_l}} \leq \frac{C}{\delta_l}. \quad (2.12)$$

We also note that $\nabla \theta_l$ differs from zero only in a strip $\Omega_{l,\delta_l}$. The strengthened Cauchy-Schwarz inequality holds over the local subspaces $V^{(l)}$, i.e., there exists $\eta_{pl}$ (0 ≤ $\eta_{pl} \leq 1, \ 1 \leq p, l \leq N$) such that

$$|b(u^{(p)}, v^{(l)})| \leq \eta_{pl}||u^{(p)}||_b||v^{(l)}||_b \quad \forall \ v^{(p)} \in V^{(p)}, \ v^{(l)} \in V^{(l)}, \ 1 \leq p, l \leq N.$$

Let $\rho(\Lambda)$ be the spectral radius of the matrix $(\eta_{pl})_{1 \leq p, l \leq N}$, then the following result holds (see [26]).

**Lemma 2.3** If Assumption 2 holds, then $\rho(\Lambda) \leq N_0$. Moreover, for any $v^{(p)} \in V^{(p)}, v^{(l)} \in V^{(l)}$ (p, l = 1, 2, ..., N),

$$\sum_{p,l=1}^N b(u^{(p)}, v^{(l)}) \leq N_0 \sum_{l=1}^N b(u^{(l)}, v^{(l)}), \quad \sum_{p,l=1}^N a(u^{(p)}, v^{(l)}) \leq N_0 \sum_{l=1}^N a(u^{(l)}, v^{(l)}).$$

The following result holds in $H^1(\Omega_l)$ (see [26]).

**Lemma 2.4** It holds that

$$||u||_{H^2(\Omega_{l,\delta_l})}^2 \leq C\delta_l^2 \left\{ (1 + \frac{H_l}{\delta_l}) ||u||_{H^2(\Omega_l)}^2 + \frac{1}{H_l\delta_l} ||u||_{L^2(\Omega_{l,\delta_l})}^2 \right\} \quad \forall \ u \in H^1(\Omega_l), \ l = 1, 2, ..., N.$$


3 The two-level BPJD method

In this section, we present our two-level BPJD method and some remarks about our algorithm.

In order to present our new preconditioner, we denote by \( Q^H : V^h \rightarrow V^H \), \( Q^{(l)} : V^h \rightarrow V^{(l)} \) \((l = 1, 2, ..., N)\) \( b(\cdot, \cdot)\)-orthogonal projectors. We also define \( A^{(l)} : V^{(l)} \rightarrow V^{(l)} \) such that \( b(A^{(l)} v, w) = a(v, w) \) for all \( v, w \in V^{(l)} \), and \( A^H : V^h \rightarrow V^H \) such that \( b(A^H v, w) = a(v, w) \) for all \( v, w \in V^H \). For convenience, denote by \( B_{(0,1)}^k := A^H - \lambda_i^k \) and \( B_{(0,i)}^k := A^{(l)} - \lambda_i^k \) \((i = 1, 2, ..., s, l = 1, 2, ..., N)\), where \( \lambda_i^k \) denotes the \(i\)th iterative approximation of the \(i\)th discrete eigenvalue \( \lambda_i^k \) in Algorithm 3.1. By using a scaling argument, it is easy to check that

\[
\lambda_{\min}(B_{(l,i)}^k) = O(H^{-2m}), \quad \lambda_{\max}(B_{(l,i)}^k) = O(h^{-2m}), \quad m = 1, 2, i = 1, 2, ..., s, l = 1, 2, ..., N. \tag{3.1}
\]

Corresponding to (2.8), there is a spectral decomposition on the coarse space \( V^H \) \((s < \dim(V^H))\):

\[
V^H = U_s^H \oplus U_{s+1}^H = V_1^H \oplus V_2^H \oplus ... \oplus V_s^H \oplus U_{s+1}^H,
\]

where \( V_i^H = \text{span}\{u_i^H\}, u_i^H \) is the \(i\)th discrete eigenvector of \( A^H \), \( U_s^H = v_1^H \oplus v_2^H \oplus ... \oplus v_s^H \), \( U_{s+1}^H \) denotes the orthogonal direct sum with respect to \( b(\cdot, \cdot) \) (also \( a(\cdot, \cdot) \)), and \( U_{s+1}^H \) denotes the orthogonal complement of \( U_s^H \). Furthermore,

\[
\lambda_{\min}(B_{(0,1)}^k|_{U_{s+1}^H}) = \lambda_{s+1}^k - \lambda_i^k, \quad \lambda_{\max}(B_{(0,i)}^k) = O(H^{-2m}), \quad m = 1, 2, i = 1, 2, ..., s. \tag{3.2}
\]

We also denote by \( Q^{(l)} : V^H \rightarrow U_{s+1}^H \), \( Q_{s+1}^{(l)} : V^H \rightarrow U_{s+1}^H \) and \( Q^H : V^H \rightarrow V^H \) \( b(\cdot, \cdot)\)-orthogonal (also \( a(\cdot, \cdot)\)-orthogonal) \(P\) projectors. The core of our two-level BPJD method is to design parallel preconditioners defined as

\[
(B_{(l,i)}^k)^{-1} = (B_{(0,1)}^k)^{-1} Q_{s+1}^H Q^H + \sum_{l=1}^N (B_{(l,i)}^k)^{-1} Q^{(l)}, \tag{3.3}
\]

to solve the block-version Jacobi-Davidson correction equations:

\[
\begin{cases}
\text{Find } t_{(l,i)}^{k+1} \in (U_i^H)^{\perp}, \ i = 1, 2, ..., s, \text{ such that } \\
\quad b((A^H - \lambda_i^k)t_{(l,i)}^{k+1}, v) = b(r_i^k, v), \quad v \in (U_i^H)^{\perp},
\end{cases} \tag{3.4}
\]

where \( U^k = \text{span}\{u_i^k\}_{i=1}^s \), \( u_i^k \) is the iterative approximation of \( u_i^H \), and \( r_i^k = \lambda_i^k u_i^k - A^H u_i^k, \ i = 1, 2, ..., s. \)

**Algorithm 3.1: Two-Level BPJD Algorithm**

**Step 1** Solve the following coarse eigenvalue problem:

\[
A_i^H u_i^H = \lambda_i^H u_i^H, \quad b(u_i^H, u_j^H) = \delta_{ij}, \quad i, j = 1, 2, ..., s, \quad s < \dim(V^H),
\]

such that \( \lambda_i^H < \lambda_i^k \). Set \( u_0^0 = u_i^H, \lambda_0^0 = Rg(u_0^0), W^0 = U^0 = \text{span}\{u_0^0\}_{i=1}^s, \Lambda^0 = \{\lambda_0^0\}_{i=1}^s. \)

**Step 2** For \( k = 0, 1, 2, ... \), solve (3.4) inexact through solving some parallel preconditioned systems:

\[
t_{(l,i)}^{k+1} = (I - Q_{s+1}^H)(B_{(l,i)}^k)^{-1} r_i^k = (B_{(l,i)}^k)^{-1} r_i^k - \sum_{i=1}^s b((B_{(l,i)}^k)^{-1} r_i^k, u_i^k) u_i^k, \ i = 1, 2, ..., s, \tag{3.5}
\]

where \( (B_{(l,i)}^k)^{-1} \) is defined in (3.3).

**Step 3** Solve the first \( s \) eigenpairs in \( W^{k+1} \):

\[
a(u_i^{k+1}, v) = \lambda_i^{k+1} b(u_i^{k+1}, v) \quad \forall v \in W^{k+1}, \quad b(u_j^{k+1}, u_j^{k+1}) = \delta_{jj}, \tag{3.6}
\]

where \( i, j = 1, 2, ..., s, W^{k+1} = W^k + \text{span}\{t_{(l,i)}^{k+1}\}_{i=1}^s. \)

Set \( U^{k+1} = \text{span}\{u_i^{k+1}\}_{i=1}^s, \Lambda^{k+1} = \{\lambda_i^{k+1}\}_{i=1}^s. \)

**Step 4** If \( \sum_{i=1}^s |\lambda_i^{k+1} - \lambda_i^k| < \text{tol} \), return \((\Lambda^{k+1}, U^{k+1})\). Otherwise, goto **Step 2**.
Remark 3.1 Actually, the choice of $W^{k+1}$ may be different. We may choose

$$W^{k+1} = V^k + \text{span}\{W^{k+1}_i\}_{i=1}^s,$$

where $V^k$ is a smaller subspace satisfying $U^k \subset V^k$. For example, we can take $V^k := U^k$ or $V^k := U^{k-1} + U^k$. The advantage of these two is that $\dim W^{k+1}$ is independent of $k$, which can reduce the cost for solving the approximate eigenpairs in $W^{k+1}$.

Remark 3.2 The meaning of the ‘block’ in two-level BPJD method is understood as follows: Let

$$I \subset \mathbb{R}^{nd \times nd} = X^k(X^k)^tM,$$

where $M$ is the mass matrix corresponding to the finite element basis, the matrix $X^k$ is

$$\begin{pmatrix}
\sigma^1(u^k_1) & \sigma^2(u^k_1) & \ldots & \sigma^1(u^k_d) \\
\sigma^2(u^k_1) & \sigma^2(u^k_2) & \ldots & \sigma^2(u^k_d) \\
\vdots & \vdots & \ddots & \vdots \\
\sigma^{nd}(u^k_1) & \sigma^{nd}(u^k_2) & \ldots & \sigma^{nd}(u^k_d)
\end{pmatrix}_{nd \times s},$$

and $(X^k)^t$ is the transpose of $X^k$. In particular, for $s = 1$, the matrix version of the operator $I - Q_{u^k}$ is

$I_{nd \times nd} - \sigma(u^k_1)(\sigma(u^k_1))^tM$, where $\sigma(u^k_1)$ is the transpose of $(\sigma^1(u^k_1), \sigma^2(u^k_1), \ldots, \sigma^{nd}(u^k_1))$.

Remark 3.3 The purpose of Step 1 is to give an initial approximation for the proposed iteration algorithm. The condition $\lambda^H_i < \lambda^H$, which can be achieved through $H = \frac{H}{\tau}$ for any positive integer $\tau$, is to ensure that $(B_{0,1}^{1,1})^{-1}$ is well-defined in theoretical analysis. But we find that it is not necessary in practical computation.

Remark 3.4 For $s = 1$, our algorithm may be regarded as a parallel preconditioned solver which solves

$$u^h_1 = \arg \min_{v \neq 0, v \in V^h} \frac{a(v, v)}{b(v, v)} = \arg \min_{v \neq 0, \text{span}(v) \subset V^h} \text{Tr}(Q_vA^hQ_v).$$

For $s \geq 2$, our algorithm may be seen as a parallel preconditioned solver which solves

$$U^h_s = \arg \min_{U \subset V^h, \text{dim}(U) = s} \text{Tr}(Q_UA^hQ_U).$$

We may consider a functional $G : \Sigma_{V^h} \to \mathbb{R}$ such that $G(U) = \text{Tr}(Q_UA^hQ_U)$ for all $U \in \Sigma_{V^h}$, where $\Sigma_{V^h}$ includes all closed subspaces of $V^h$. Hence, we need to minimize the functional $G$ in $\Sigma_{V^h} := \{U \in \Sigma_{V^h} \mid \text{dim}(U) = s\}$ to obtain $\{\{U^h_i\}_{i=1}^s\}$.  

4 Some properties of subspace method

In this section, we present some useful properties in convergence analysis.

Since

$$U^k + \text{span}\{W^{k+1}_i\}_{i=1}^s \subset W^k + \text{span}\{W^{k+1}_i\}_{i=1}^s = W^{k+1} \subset V^h,$$

by the Courant-Fischer principle, we obtain

$$\lambda^H_i \leq \lambda^{k+1}_i \leq \lambda^H_i, \quad k = 0, 1, 2, ..., i = 1, 2, ..., s. \quad (4.1)$$

By Step 1 in Algorithm 3.1, we know that $\lambda^0_i = \lambda^H_i < \lambda^H_i, \quad i = 1, 2, ..., s$. Using (4.1) and Assumption 1, we have

$$\lambda_i \leq \lambda^H_i \leq \lambda^H_i \leq \lambda^H_i < \lambda^H_i \leq \lambda^{k+1}_i, \quad k = 0, 1, 2, ..., i = 1, 2, ..., s. \quad (4.2)$$
For our theoretical analysis, we define \((v, w)_{E^h} := a(v, w) - \lambda^h b(v, w)\), \((v, w)_{E^h_i} := a(v, w) - \lambda_i b(v, w)\) for all \(v, w, v \in V^h\). It is easy to check that the bilinear forms \((\cdot, \cdot)_{E^h}, (\cdot, \cdot)_{E^h_i}\) and \((\cdot, \cdot)_{E_i}\) construct inner products in \(U^h_{s+1}\). The norms \(||\cdot||_{E^h}, ||\cdot||_{E^h_i}\) and \(||\cdot||_{E_i}\) induced by \((\cdot, \cdot)_{E^h}, (\cdot, \cdot)_{E^h_i}\) and \((\cdot, \cdot)_{E_i}\), respectively, are equivalent to the norm \(||\cdot||_a\) over \(U^h_{s+1}\). In fact, on one hand, \((w, w)_{E^h_i} \leq (w, w)_{E^h} \leq (w, w)_{E_i} \leq a(w, w)\) for all \(w \in V^h\). On the other hand, by (4.2), it is easy to get that \((v, v)_{E^h_i} \geq (\lambda^h_{s+1} - \lambda^h_i) b(v, v)\) for all \(v \in U^h_{s+1}\). Moreover,

\[
(a(v, v) = (v, v)_{E^h_i} + \lambda^h_i b(v, v) \leq (\lambda^h_{s+1}) b(v, v) \quad \forall v \in V^h.
\]

where the real valued function \(\beta(\lambda) := 1 + \frac{\lambda^h}{\lambda - \lambda^h}\). Throughout the paper, we denote by \(\mu^k_i := (\lambda^k_i)^{-1}\), \(g^k_i := \mu^k_i u^k_i - T^h u^k_i, Q^k_i := I - Q^h U, P^k_i := I - P^h U\), and \(v^k_{s+1} := -Q^h_k u^k_i\), where \(Q^h: V^h \to U^k\) denotes a \(b(\cdot, \cdot)\)-orthogonal projector and \(P^h U: V^h \to U^k\) denotes an \(a(\cdot, \cdot)\)-orthogonal projector.

**Lemma 4.1** It holds that

\[
||Q^h v||^2_a \leq \frac{1}{\lambda^h} \sum_{i=1}^s \lambda^h_i ||v||^2_{a_i}, \quad ||Q^h v||^2_a \leq (2 + \frac{2}{\lambda^h} \sum_{i=1}^s \lambda^h_i) ||v||^2_a \quad \forall v \in V^h.
\]

**Proof.** By the fact that \(a(u^k_i, u^k_j) = \lambda^k_i b(u^k_i, u^k_j) = \lambda^k_i \delta_{ij}\), we know that \(\{\sqrt{\mu^k_i u^k_i}\}_{i=1}^s\) forms a group of normal and orthogonal basis with respect to \(a(\cdot, \cdot)\). For any \(v \in V^h\), by the Cauchy-Schwarz inequality, we obtain

\[
||Q^h v||^2_a = ||\sum_{i=1}^s b(v, u^k_i) u^k_i||^2_a = ||\sum_{i=1}^s b(v, \sqrt{\lambda^h_i} u^k_i) \sqrt{\mu^k_i} u^k_i||^2_a = \sum_{i=1}^s ||b(v, \sqrt{\lambda^h_i} u^k_i)||^2_a
\]

\[
\leq ||v||^2_a \sum_{i=1}^s ||\sqrt{\lambda^h_i} u^k_i||^2_a \leq \frac{1}{\lambda^h} \sum_{i=1}^s \lambda^h_i ||v||^2_a.
\]

Moreover,

\[
||Q^h_{k-1} v||^2_a = ||(I - Q^h U)v||^2_a \leq 2(||v||^2_a + ||Q^h v||^2_a) \leq 2(1 + \frac{1}{\lambda^h} \sum_{i=1}^s \lambda^h_i) ||v||^2_a,
\]

which completes the proof of this lemma. \(\square\)

**Lemma 4.2** For any \(v^H \in V^H\), it holds that

\[
||Q^h_{k+1} v^H||_b \leq CH^2 ||Q^h_{k+1} v^H||_b, \quad ||Q^h_{k+1} v^H||_a \leq CH ||Q^h_{k+1} v^H||_a.
\]

**Proof.** By Remark 2.2 and Theorem 2.2 and the fact \(\dim(U^h_s) = \dim(U^H_s) = \dim(U_s)\), we have

\[
\sin \{U^h_s, U^H_s\} \leq \sin \{U^h_s, U_s\} + \sin \{U_s, U^H_s\} \leq CH^2.
\]

If \(Q^h_{k+1} v^H = 0\), (4.3) holds. For \(Q^h_{k+1} v^H (\neq 0) \in U^h_s\), take \(\bar{w}^h_s = \frac{Q^h_{k+1} v^H}{||Q^h_{k+1} v^H||_b}\). By (4.4), there exists a \(v^H (\neq 0) \in U^H_s\) such that

\[
||\bar{w}^h_s - v^H_s||_b \leq CH^2.
\]

Then we have,

\[
||\bar{w}^h_s - \frac{v^H_s}{||v^H_s||_b}||_b = ||\bar{w}^h_s - \frac{v^H_s}{||v^H_s||_b}||_b \leq \frac{2}{||v^H_s||_b} ||\bar{w}^h_s - v^H_s||_b \leq CH^2.
\]

Taking \(w^H_s = \frac{Q^h_{k+1} v^H}{||v^H_s||_b} v^H_s \in U^H_s\), we get

\[
||Q^h_{k+1} v^H - w^H_s||_b \leq CH^2 ||Q^h_{k+1} v^H||_b.
\]
Moreover,
\[ ||Q_s^b Q_{s+1}^H v^H||_b^2 = b(Q_s^b Q_{s+1}^H v^H - w_s^H, Q_{s+1}^H v^H) \leq CH^2 ||Q_s^b Q_{s+1}^H v^H||_b ||Q_{s+1}^H v^H||_b, \]
which yields the first inequality in (4.3). Similarly, we may prove that \( ||Q_s^b Q_{s+1}^H v^H||_a \leq CH ||Q_{s+1}^H v^H||_a \), and then obtain the proof of this lemma.

By the analysis above, for any \( v^H \in V^H \), we could easily estimate \( ||Q_s^b Q_{s+1}^H v^H||_b \) and \( ||Q_s^h Q_{s+1}^H v^H||_a \), similarly. For any \( v^H \in V^h \), we have estimates which are
\[ ||Q_s^h Q_{s+1}^H v^H||_b \leq CH^2 ||Q_{s+1}^H v^H||_b. \]  
(4.5)
\[ ||Q_s^H Q_{s+1}^H v^H||_b \leq CH^2 ||Q_s^b v^H||_b. \]  
(4.6)

**Lemma 4.3** Let \( a(u_i^k, u_j^k) = \lambda_k^b b(u_i^k, u_j^k) = \lambda_k^b \delta_{ij} \). It holds that
\[ \sum_{i=1}^s a(g_i^k, g_i^k) \leq \sum_{i=1}^s (\mu_i^b - \mu_i^h). \]  
(4.7)

In particular, \( \sum_{i=1}^s ||g_i^k||_a^2 \leq CH^2. \)

**Proof.** As \( a(u_i^k, u_i^k) = \lambda_k^b b(u_i^k, u_i^k) = \lambda_k^h a(T_h u_i^k, u_i^k) = \lambda_k^h \), we have
\[ a(g_i^k, g_i^k) = a((T_h)^\frac{1}{2} u_i^k, (T_h)^\frac{1}{2} u_i^k) - \mu_i^h = a((T_h)^\frac{1}{2} u_i^k, (T_h)^\frac{1}{2} u_i^k) - \mu_i^h. \]  
(4.8)
Since \( a((T_h)^\frac{1}{2} u_i^k, (T_h)^\frac{1}{2} u_i^k) = \delta_{ij} \), we may consider the eigenvalue problem \( a(T_h w_i, v) = \nu_i^h a(w_i, v) \) for all \( v \in \text{span}\{(T_h)^\frac{1}{2} u_i^k\}_{i=1}^s \). Moreover,
\[ \sum_{i=1}^s a(g_i^k, g_i^k) = \sum_{i=1}^s a((T_h)^\frac{1}{2} u_i^k, (T_h)^\frac{1}{2} u_i^k) = \sum_{i=1}^s a((T_h)^\frac{1}{2} u_i^k, (T_h)^\frac{1}{2} u_i^k) - \mu_i^h. \]  
(4.9)
As \( \text{span}\{(T_h)^\frac{1}{2} u_i^k\}_{i=1}^s \subset V^h \), we know \( \nu_i^h \leq \mu_i^h \), which, together with (4.8), (4.9), yields
\[ \sum_{i=1}^s a(g_i^k, g_i^k) = \sum_{i=1}^s \{ a((T_h)^\frac{1}{2} u_i^k, (T_h)^\frac{1}{2} u_i^k) - \mu_i^h \} \leq \sum_{i=1}^s (\mu_i^h - \mu_i^h). \]
By (4.3) and Theorem 2.2, we get
\[ \sum_{i=1}^s a(g_i^k, g_i^k) \leq \sum_{i=1}^s (\mu_i^h - \mu_i^h) \leq C \sum_{i=1}^s (\lambda_i^h - \lambda_i^h) \leq C \sum_{i=1}^s (\lambda_i^h - \lambda_i^h) \leq CH^2, \]
which completes the proof of this lemma.

The following lemma illustrates that the gap between \( U_b^h \) and \( U^h \) with respect to \( ||\cdot||_b \) is bounded by the total error of eigenvalues. In particular, it is bounded by \( CH^2 \).

**Lemma 4.4** Let \( a(u_i^k, u_j^k) = \lambda_k^b b(u_i^k, u_j^k) = \lambda_k^b \delta_{ij} \) and Assumption 1 hold, then
\[ (\theta_b^k)^2 \leq \frac{1}{\lambda_{s+1}^h - \lambda_s^h} \sum_{i=1}^s (\lambda_i^h - \lambda_i^h), \]  
(4.10)
where \( \theta_b^k \) is the gap between \( U_b^h \) and \( U^h \) with respect to \( ||\cdot||_b \). In particular, \( (\theta_b^k)^2 \leq CH^2 \).
Proof. Since \( a(u_k^i, u_k^j) = \lambda_k^i b(u_k^i, u_k^j) = \lambda_k^i \delta_{ij} \), it is easy to know that \( \{u_k^i\}_{i=1}^s \) forms a group of normal and orthogonal basis for \( U^k \) with respect to \( b(\cdot, \cdot) \). By (2.8), we get

\[
\lambda_{s+1}^b - \lambda_s^b = b((\lambda_{s+1}^b - A^b)u_k^i, u_k^j) = b((\lambda_{s+1}^b - A^b)Q_s^{h_b}u_k^i, Q_s^{h_b}u_k^j)
+ b((\lambda_{s+1}^b - A^b)Q_{s+1}^{h_b}u_k^i, Q_{s+1}^{h_b}u_k^j) \leq b((\lambda_{s+1}^b - A^b)Q_s^{h_b}u_k^i, Q_s^{h_b}u_k^j).
\]

Combining (4.11) and the fact that \( Q_s^{h_b}v = \sum_{i=1}^s Q_s^{h_b}v \) for all \( v \in V^h \), we have

\[
\sum_{i=1}^s (\lambda_{s+1}^b - \lambda_s^b) = \sum_{i=1}^s \sum_{j=1}^s (\lambda_{s+1}^b - \lambda_j^b) |b(u_k^i, u_k^j)|^2
= \sum_{i=1}^s (\lambda_{s+1}^b - \lambda_j^b) \sum_{j=1}^s |b(u_k^i, u_k^j)|^2
= \sum_{i=1}^s (\lambda_{s+1}^b - \lambda_j^b) (1 - \|Q_{u_k^i}u_k^j\|^2_h)
= \sum_{i=1}^s (\lambda_{s+1}^b - \lambda_j^b) (1 - \sin^2 \{u_k^i; U^k\}).
\]

Hence, \( \sum_{i=1}^s (\lambda_{s+1}^b - \lambda_j^b) \sin^2 \{u_k^i; U^k\} \leq \sum_{i=1}^s (\lambda_j^b - \lambda_{s+1}^b) \). If Assumption 1 holds (also \( \lambda_j^b < \lambda_{s+1}^b \) as \( h \to 0 \)), we obtain

\[
\sum_{i=1}^s \sin^2 \{u_k^i; U^k\} \leq \frac{1}{\lambda_{s+1}^b - \lambda_s^b} \sum_{i=1}^s (\lambda_j^b - \lambda_{s+1}^b),
\]

which, together with the fact that \( \theta_k^b \leq \sum_{i=1}^s \sin^2 \{u_k^i; U^k\} \) (see Lemma 3.4 in [3] or Corollary 2.2 in [13]), yields (4.10). By (4.2) and Theorem 2.2 we get

\[
(\theta_k^b)^2 \leq \frac{1}{\lambda_{s+1}^b - \lambda_s^b} \sum_{i=1}^s (\lambda_j^b - \lambda_{s+1}^b) \leq C \sum_{i=1}^s (\lambda_j^b - \lambda_i) \leq CH^2,
\]

which completes the proof of this lemma.

We may use a similar argument as in the proof of Lemma 4.4 to obtain the following result.

Corollary 4.5 Let \( a(u_k^i, u_k^j) = \lambda_k^b b(u_k^i, u_k^j) = \lambda_k^b \delta_{ij} \) and Assumption 1 hold, then

\[
(\theta_k^b)^2 \leq \frac{1}{\mu_s^b - \mu_{s+1}^b} \sum_{i=1}^s (\mu_i^b - \mu_{s+1}^b),
\]

where \( \theta_k^b \) is the gap between \( U^h \) and \( U^k \) with respect to \( \| \cdot \|_a \). In particular, \( (\theta_k^b)^2 \leq CH^2 \).

The gap \( \theta_a^k \) and \( \theta_b^k \) can be characterized by the a-norm and the b-norm of the operators, respectively.

Lemma 4.6 It holds that

\[
\theta_a^k = \|P_1^{h_b}Q_s^{h_b}\|_a = \|Q_s^{h_b}P_1^{h_b}\|_a = \|Q_{s+1}^{h_b}P_{U^k}\|_a = \|P_{U^k}Q_{s+1}^{h_b}\|_a,
\]

and

\[
\theta_b^k = \|Q_1^{h_b}Q_s^{h_b}\|_b = \|Q_s^{h_b}Q_1^{h_b}\|_b = \|Q_{s+1}^{h_b}Q_{U^k}\|_b = \|Q_{U^k}Q_{s+1}^{h_b}\|_b.
\]

Proof. Combining Definition 2.1 and Remark 2.2, it is easy to prove this lemma. In order to focus on our main theoretical analysis, we ignore this proof here. \( \square \)
Lemma 4.7 It holds that
\[ ||e_{i,s+1}^k||_{E^2}^2 = (\lambda_i^k - Rq(Q_s^h u_i^k))||Q_s^h u_i^k||_{b}^2, \] (4.13)

\[ 0 \leq \lambda_i^k - Rq(Q_s^h u_i^k) \leq CH^2, \] (4.14)

and
\[ \lambda_i^k - Rq(Q_s^h u_i^k) \leq CH||g_i^k||_a. \] (4.15)

Moreover,
\[ ||(\lambda_i^k - A^h)Q_s^h u_i^k||_b \leq C||g_i^k||_a. \] (4.16)

**Proof.** We first prove (4.13) and (4.14). Using (3.6), we have
\[ ||e_{i,s+1}^k||_{E^2}^2 = b((\lambda_i^k - A^h)Q_s^h u_i^k, Q_s^h u_i^k) = (\lambda_i^k - Rq(Q_s^h u_i^k))||Q_s^h u_i^k||_{b}^2, \] (4.17)

which means that (4.13) holds and \( \lambda_i^k - Rq(Q_s^h u_i^k) \geq 0 \). By Lemma 4.4 and Lemma 4.6, we have
\[ (\lambda_i^k - Rq(Q_s^h u_i^k))||Q_s^h u_i^k||_{b}^2 = (\lambda_i^k - Rq(Q_s^h u_i^k))(1 - ||Q_s^h+1 P_{U^h} u_i^k||_{b}^2) \]
\[ \geq (\lambda_i^k - Rq(Q_s^h u_i^k))(1 - (\theta_h)^2||u_i^k||_{b}^2) \] (4.18)

By (4.17), Corollary 4.5 and Lemma 4.6 we obtain
\[ (\lambda_i^k - Rq(Q_s^h u_i^k))||Q_s^h u_i^k||_{b}^2 \leq ||e_{i,s+1}^k||_{E^2}^2 \leq ||Q_s^h+1 P_{U^h} u_i^k||_{a}^2 \leq CH^2. \] (4.19)

Using (4.18) and (4.19), we get (4.14).

By Corollary 4.5 Lemma 4.6 and the facts \( P_{U^h} g_i^k = g_i^k \) and \( r_i^k = -\lambda_i^k A^h g_i^k \), we deduce
\[ (\lambda_i^k - Rq(Q_s^h u_i^k))||Q_s^h u_i^k||_{b}^2 = b((\lambda_i^k - A^h)Q_s^h u_i^k, Q_s^h u_i^k) = -\lambda_i^k a(g_i^k, Q_s^h u_i^k) \]
\[ = -\lambda_i^k a(Q_s^h P_{U^h} g_i^k, Q_s^h u_i^k) \leq C\theta_h^2||g_i^k||_a||Q_s^h u_i^k||_a \leq CH||g_i^k||_a, \]

which, together with (4.18), yields (4.15).

Since \( A^h|_{U^h} : U^h_s \to U^h_s \) is a linear isomorphism, we know that there exists an unique \( v_i^k,s \in U^h_s \) such that \( A^h v_i^k,s = (\lambda_i^k - A^h)Q_s^h u_i^k \). Accordingly,
\[ ||(\lambda_i^k - A^h)Q_s^h u_i^k||_{b}^2 = b(A^h(A^h)^\dagger v_i^k,s, (A^h)^\dagger v_i^k,s) \leq \lambda_i^k a(v_i^k,s, v_i^k,s) \]
\[ = \lambda_i^k((\mu_i^k - T^h)Q_s^h u_i^k||_{a}^2 = \lambda_i^k(\lambda_i^k)^2||Q_s^h u_i^k||_a^2 \leq \lambda_i^k(\lambda_i^k)^2||g_i^k||_a^2. \]

This leads to (4.16). \( \qed \)

5 Convergence analysis

In this section, we focus on giving a rigorous convergence analysis for the two-level BPJD method. We first present the main theoretical result in this paper. The rest of this section is organized as follows: In subsection 5.1, by choosing a suitable coarse component and using some overlapping DD techniques, we deduce the error reduction from \( e_{i,s+1}^k \) to \( e_{i,s+1}^{k+1} \) \( (e_{i,s+1}^{k+1} := -Q_s^h \tilde{u}_{i,s+1}^{k+1}, \) where \( \tilde{u}_{i,s+1}^{k+1} \) shall be defined in (5.4), \( i = 1, 2, ..., s \). In subsection 5.2, by constructing an auxiliary eigenvalue problem in \( \text{span}\{\tilde{u}_{i}^{k+1}\}_{i=1}^s \) which shall be presented in (5.6), we may establish the total error reduction of the first \( s \) eigenvalues.
**Theorem 5.1** Assume that Assumption 1 and Assumption 2 hold, then

\[ \sum_{i=1}^{s} (\lambda_i^{k+1} - \lambda_i^0) \leq \gamma \sum_{i=1}^{s} (\lambda_i^k - \lambda_i^0), \quad (5.1) \]

and

\[ (\theta_n^k)^2 \leq C \gamma^k, \quad (5.2) \]

\[ (\theta_b^k)^2 \leq C \gamma^k. \quad (5.3) \]

Here \( \gamma = c(H)(1 - C \frac{3^{2m-1}}{H^{2m-1}})^2 \), \( m = 1, 2 \). The constant \( C \) is independent of \( h \), \( H \), \( \delta \) and the left gaps of eigenvalues \( \{\lambda_i\}_{i=2}^{s} \), and the \( H \)-dependent constant \( c(H) = 1 + \frac{CH}{(1 - C \frac{3^{2m-1}}{H^{2m-1}})^2} \) decreases monotonically to 1, as \( H \to 0 \).

**Remark 5.1** In order to make the main idea of the proof of Theorem 5.1 clear, we only consider the model problem \((2.1)\). The proof for problem \((2.2)\) is similar.

For the convenience of the following convergence analysis, we first choose some special functions defined as

\[ \tilde{u}_i^{k+1} := u_i + \alpha_i^k (A_i - 1) \tilde{r}_i^k \in U_i + \text{span}\{\tilde{r}_i^{k+1}\}_{i=1}^{s} \subset W^{k+1}, \quad i = 1, 2, ..., s, \quad (5.4) \]

to analyze the error reduction, where \( \alpha_i^k (i = 1, 2, ..., s) \) are some undetermined parameters dependent on \( N_0 \). From \((3.5)\) and \((5.4)\), we know

\[ \tilde{u}_i^{k+1} = u_i^k + \alpha_i^k Q_i^k (B_i^k)^{-1} r_i^k, \quad i = 1, 2, ..., s, \quad (5.5) \]

which are linearly independent. So we may construct an auxiliary eigenvalue problem in \( \text{span}\{\tilde{u}_i^{k+1}\}_{i=1}^{s} \):

\[ a(u_i^{k+1}, v) = \lambda_i^{k+1} b(u_i^{k+1}, v) \quad \forall \ v \in \tilde{U}^{k+1} := \text{span}\{\tilde{u}_i^{k+1}\}_{i=1}^{s}. \quad (5.6) \]

Since \( \tilde{U}^{k+1} \subset W^{k+1} \), it is easy to see that \( \lambda_i^{k+1} \leq \tilde{\lambda}_i^{k+1} \).

The idea of the proof of Theorem 5.1 is to design an auxiliary eigenvalue problem \((5.6)\). By the ‘bridge’ term \( G(\tilde{U}^{k+1}) - G(U_i^h) \), we may obtain the error reduction from \( G(U_i^k) - G(U_i^h) \) to \( G(U_i^{k+1}) - G(U_i^h) \), i.e., by the ‘bridge’ term \( \sum_{i=1}^{s} (\tilde{\lambda}_i^{k+1} - \lambda_i^0) \), we may obtain the total error reduction from \( \sum_{i=1}^{s} (\lambda_i^k - \lambda_i^0) \) to \( \sum_{i=1}^{s} (\tilde{\lambda}_i^{k+1} - \lambda_i^0) \).

### 5.1 The error from the new DD preconditioner

The block-version Jacobi-Davidson correction equations \((3.4)\) are solved inexactly, i.e.,

\[ e_i^{k+1} = Q_i^k (B_i^k)^{-1} r_i^k. \]

From \((5.5)\), we first analyze the error reduction from \( e_{i,s+1}^k (:= -Q_{s+1}^k u_{i}^k) \) to \( e_{i,s+1}^{k+1} (:= -Q_{s+1}^k \tilde{u}_{i}^k) \). The orthogonal projection \( -Q_{s+1}^k \) is applied to both sides of \((5.5)\), we obtain

\[ \tilde{e}_{i,s+1}^{k+1} = e_{i,s+1}^k - \alpha_i^k Q_{s+1}^k Q_i^k (B_i^k)^{-1} r_i^k. \quad (5.7) \]

Moreover, by the splitting of the identity operator on \( V^h \) corresponding to \((2.8)\), we deduce

\[
\begin{align*}
\tilde{e}_{i,s+1}^{k+1} &= e_{i,s+1}^k - \alpha_i^k Q_{s+1}^k (B_i^k)^{-1} r_i^k + \alpha_i^k Q_{s+1}^k Q_{U^k} (B_i^k)^{-1} r_i^k \\
&= e_{i,s+1}^k - \alpha_i^k Q_{s+1}^k (B_i^k)^{-1} (\lambda_i^k - A_i h) (Q_i^k u_i^k - e_{i,s+1}^k) + \alpha_i^k Q_{s+1}^k Q_{U^k} (B_i^k)^{-1} r_i^k \\
&= \{ e_{i,s+1}^k + \alpha_i^k Q_{s+1}^k (B_i^k)^{-1} (\lambda_i^k - A_i h) e_{i,s+1}^k \} + \alpha_i^k Q_{s+1}^k (B_i^k)^{-1} (A_i h - \lambda_i^k) Q_i^k u_i^k \\
&\quad + Q_{s+1}^k Q_{U^k} (B_i^k)^{-1} r_i^k =: I_{1,i}^k + I_{2,i}^k.
\end{align*}
\]
For simplicity, we define $G^k_i := I + \alpha^k_i Q^{h_{s+1}}(B^k_i)^{-1}(\lambda^k_i - A^h)$. It is easy to see that $T^k_{1,i} = G^k_i \ell^k_{i,s+1}$. In this paper, we call $T^k_{1,i}$ the principal error term and $G^k_i : U_{s+1}^h \to U_{s+1}^h$ the principal error operator. Meanwhile, we call $T^k_{2,i}$ the additional error term.

### 5.1.1 Estimate of the principal error term $T^k_{1,i}$

In this subsection, we shall use the theory of the two-level domain decomposition method to analyze the principal error term $T^k_{1,i}$. Actually, we only need to estimate the spectral radius of the principal error operator $G^k_i$.

**Theorem 5.2** For sufficiently small $\alpha^k_i$, it holds that

$$||G^k_i v||_{E^h_i} \leq (1 - C\frac{\delta}{H})||v||_{E^h_i} \quad \forall \ v \in U_{s+1}^h, \ i = 1, 2, \ldots, s. \quad (5.9)$$

First of all, we give two useful lemmas. The first lemma (Lemma 5.3) illustrates that the principal error operator $G^k_i : U_{s+1}^h \to U_{s+1}^h$ is symmetric and positive definite with respect to $(\cdot, \cdot)_{E^h_i}$. The second lemma (Lemma 5.4) gives a stable spacial decomposition for the error space $U_{s+1}^h$ instead of the whole space $V^h$. Hence, the constructions of both coarse component and local fine components in this paper are different from those in [26].

**Lemma 5.3** For any $i \ (i = 1, 2, \ldots, s)$, the operator $G^k_i : U_{s+1}^h \to U_{s+1}^h$ is symmetric with respect to $(\cdot, \cdot)_{E^h_i}$. Furthermore, if $\alpha^k_i$ is sufficiently small, the operator $G^k_i : U_{s+1}^h \to U_{s+1}^h$ is positive definite.

**Proof.** We first prove that the operator $G^k_i : U_{s+1}^h \to U_{s+1}^h$ is symmetric with respect to $(\cdot, \cdot)_{E^h_i}$. Since the operators $(B^k_{i,0})^{-1}$ and $(B^k_{i,N})^{-1}(i = 1, 2, \ldots, N)$ are symmetric with respect to $b(\cdot, \cdot)$, we have

$$(Q^{h_{s+1}}(B^k_i)^{-1}(\lambda^k_i - A^h)v, w)_{E^h_i} = ((B^k_i)^{-1}(\lambda^k_i - A^h)v, w)_{E^h_i}$$

$$= b((A^h - \lambda^k_i)v, (B^k_i)^{-1}(\lambda^k_i - A^h)w) = (v, Q^{h_{s+1}}(B^k_i)^{-1}(\lambda^k_i - A^h)w)_{E^h_i}, \quad \forall \ v, w \in U_{s+1}^h, \quad (5.10)$$

which means that the operator $G^k_i : U_{s+1}^h \to U_{s+1}^h$ is symmetric with respect to $(\cdot, \cdot)_{E^h_i}$.

Next, for any $i \ (i = 1, 2, \ldots, s)$, define an operator $T^k_{0,i} : U_{s+1}^h \to U_{s+1}^h$ such that for any $v \in U_{s+1}^h$,

$$(T^k_{0,i}v, w)_{E^h_i} = (v, w)_{E^h_i} \quad \forall \ w \in U_{s+1}^h. \quad (5.11)$$

By (4.2) and the Lax-Milgram Theorem, we know that the operator $T^k_{0,i}$ is well-defined. Similarly, we may define some operators $T^k_{l,i} : U_{s+1}^h \to V^l$ $(l = 1, 2, \ldots, N)$ such that for any $v \in U_{s+1}^h$,

$$(T^k_{l,i}v, w)_{E^h_i} = (v, w)_{E^h_i} \quad \forall \ w \in V^l. \quad (5.12)$$

By (3.1) and the Lax-Milgram Theorem, the operators $T^k_{l,i} (l = 1, 2, \ldots, N)$ are also well-defined. It is easy to check that $T^k_{0,i} = (B^k_{i,0})^{-1}Q^{h_{s+1}}Q^i(A^h - \lambda^k_i)$ and $T^k_{l,i} = (B^k_{i,l})^{-1}Q^l(A^h - \lambda^k_i)$. Moreover,

$$G^k_i = I + \alpha^k_i Q^{h_{s+1}}(B^k_i)^{-1}(\lambda^k_i - A^h) = I - \alpha^k_i Q^{h_{s+1}}T^k_{0,i} - \alpha^k_i \sum_{l=1}^{N} Q^{h_{s+1}}T^k_{l,i}.\quad (5.13)$$

For any $v \in U_{s+1}^h$, by (5.11) and (5.12), we have

$$(G^k_i v, v)_{E^h_i} = ||v||_{E^h_i}^2 - \alpha^k_i ||T^k_{0,i}v||_{E^h_i}^2 - \alpha^k_i \sum_{l=1}^{N} ||T^k_{l,i}v||_{E^h_i}^2. \quad (5.13)$$
For the second term of (5.13), by the Cauchy-Schwarz inequality, we get
\[
||T^h_{0,i}v||^2_{E^*_i} = (T^h_{0,i}v, v)_{E^*_i} = (Q^h_{s+1}T^k_{0,i}v, v)_{E^*_i} \leq ||Q^h_{s+1}T^k_{0,i}v||_{E^*_i}||v||_{E^*_i}
\]
\[
\leq ||Q^h_{s+1}T^k_{0,i}v||_{a}||v||_{E^*_i} \leq ||T^k_{0,i}v||_{a}||v||_{E^*_i} \leq \sqrt{\beta(H_{s+1})} ||T^k_{0,i}v||_{E^*_i},
\]
which yields
\[
||T^k_{0,i}v||^2_{E^*_i} \leq \beta(H_{s+1})||v||^2_{E^*_i}.
\]
(5.15)

For the third term of (5.13), by the Cauchy-Schwarz inequality, we deduce
\[
\sum_{l=1}^N||T^k_{l,i}v||^2_{E^*_i} = \sum_{l=1}^N(T^k_{l,i}v, v)_{E^*_i} = (Q^h_{s+1}\sum_{l=1}^NT^k_{l,i}v, v)_{E^*_i} \leq ||Q^h_{s+1}\sum_{l=1}^NT^k_{l,i}v||_{E^*_i}||v||_{E^*_i}.
\]
(5.16)

By Lemma 2.3, we get
\[
||Q^h_{s+1}\sum_{l=1}^NT^k_{l,i}v||^2_{E^*_i} \leq \sum_{l=1}^N||T^k_{l,i}v||^2_{a} \leq N_0 \sum_{l=1}^N||T^k_{l,i}v||^2_{a} \leq N_0 \max_{1 \leq \ell \leq N} (\lambda_{\ell,i}^h) \sum_{l=1}^N||T^k_{l,i}v||^2_{E^*_i},
\]
(5.17)

where $\lambda_{\ell,i}^h = \lambda_{\min}(A^{(i)}) = O(H_{s+1}^{-2})$. Using (5.16) and (5.17), we obtain
\[
\sum_{l=1}^N||T^k_{l,i}v||^2_{E^*_i} \leq N_0 \max_{1 \leq \ell \leq N} (\lambda_{\ell,i}^h)||v||^2_{E^*_i}.
\]
(5.18)

Combining (5.13), (5.15) and (5.18), we know that for any $v \in U^h_{s+1}$,
\[
(G^h_{s+1}v, v)_{E^*_i} = ||v||^2_{E^*_i} - \alpha_k^i||T^k_{0,i}v||^2_{E^*_i} - \alpha_k^i \sum_{l=1}^N||T^k_{l,i}v||^2_{E^*_i}
\]
\[
\geq (1 - \alpha_k^i(\beta(H_{s+1}^H) + N_0 \max_{1 \leq \ell \leq N} (\lambda_{\ell,i}^h)))||v||^2_{E^*_i}.
\]

Taking $0 < \alpha_k^i < \alpha_k^i_{\max} = \frac{1}{\beta(H_{s+1}) + N_0 \max_{1 \leq \ell \leq N} (\lambda_{\ell,i}^h)}$, we complete the proof of this lemma. \(\square\)

**Remark 5.2** By Lemma 2.3, (5.17) and (5.18), we have
\[
||\sum_{i=1}^sT^k_{l,i}v||^2_{a} \leq N_0 \sum_{l=1}^N||T^k_{l,i}v||^2_{a} \leq C \sum_{l=1}^N||T^k_{l,i}v||^2_{E^*_i} \leq C||v||^2_{E^*_i} \quad \forall v \in U^h_{s+1}.
\]
Moreover,
\[
||\sum_{i=1}^sT^k_{l,i}v||^2_{b} \leq N_0 \sum_{l=1}^N||T^k_{l,i}v||^2_{b} \leq CH^2 \sum_{l=1}^N||T^k_{l,i}v||^2_{a} \leq CH^2||v||^2_{E^*_i} \quad \forall v \in U^h_{s+1}.
\]

**Lemma 5.4** For any $v \in U^h_{s+1}$, there exist $w_0 \in U^H_{s+1}$ and $w^{(l)} \in V^{(l)}$ ($l = 1, 2, ..., N$), such that
\[
v = Q^h_{s+1}w_0 + \sum_{l=1}^NQ^h_{s+1}w^{(l)},
\]
and
\[
(w_0, w_0)_{E^*_i} + \sum_{l=1}^N(w^{(l)}, w^{(l)})_{E^*_i} \leq C(1 + \frac{H}{d})(v, v)_{E^*_i}, \quad i = 1, 2, ..., s.
\]
(5.19)
Proof. For any \( v \in U_{+1}^h \), set \( w_0 = Q_{+1}^H Q^H v \) and \( w^{(l)} = I^h(\theta_l(v - w_0)) \), where \( I^h : C^0(\bar{\Omega}) \to V^h \) is the usual nodal interpolation operator. It is easy to check that
\[
Q_{+1}^h w_0 + \sum_{l=1}^N Q_{+1}^h w^{(l)} = Q_{+1}^h w_0 + Q_{+1}^h \{ \sum_{l=1}^N \theta_l \} (v - w_0) = v.
\]
Next, we prove (5.19). For the coarse component, we deduce
\[
||w_0||_{E^h_1}^2 \leq ||Q_{+1}^h Q^H v||_{E^h_1}^2 \leq ||Q^H v||_{E^h_1}^2 \leq C||v||_{E^h_1}^2 \leq C||v||_{E^h_1}^2.
\]  (5.20)
For the local fine components, by the property of the operator \( I^h \) (see Lemma 3.9 in [20]), we have
\[
\sum_{l=1}^N (w^{(l)}, w^{(l)})_{E^h_1} \leq \sum_{l=1}^N a(w^{(l)}, w^{(l)}) = \sum_{l=1}^N |I^h(\theta_l(v - w_0)))|_{1, \alpha_l'}^2 \leq C \sum_{l=1}^N ||\nabla(v - w_0)||_{b, \alpha_l'}^2 + \frac{1}{\delta_l} ||v - w_0||_{b, \alpha_l, s_l}^2,
\]  (5.21)
where \( ||v||_{b, \alpha_l}^2 = \int_{\bar{\Omega}} v^2 dx, \ ||v||_{1, \alpha_l}^2 = \int_{\bar{\Omega}} \nabla v \cdot \nabla v dx \) for all \( \bar{\Omega} \subset \Omega \). On one hand, by (5.20), we get
\[
\sum_{l=1}^N ||\nabla(v - w_0)||_{b, \alpha_l'}^2 \leq C \sum_{l=1}^N ||\nabla(v - w_0)||_{b, \alpha_l'}^2 \leq C \sum_{l=1}^N ||v||_{b, \alpha_l'}^2 \leq C \sum_{l=1}^N ||w_0||_{b, \alpha_l'}^2 \leq C \sum_{l=1}^N ||v||_{E^h_1}^2.
\]  (5.22)
On the other hand, by Lemma 2.4 and (5.22), we obtain
\[
\sum_{l=1}^N \frac{1}{\delta_l} ||v - w_0||_{b, \alpha_l, s_l}^2 \leq C \sum_{l=1}^N \left( (1 + \frac{H_l}{\delta_l}) ||v - w_0||_{1, \alpha_l'} + \frac{1}{H_l \delta_l} ||v - w_0||_{b, \alpha_l'} \right) \leq C \sum_{l=1}^N \left( 1 + \frac{H_l}{\delta_l} ||v||_{E^h_1}^2 + \frac{1}{H_l \delta_l} ||w_0||_{b, \alpha_l'}^2 \right).
\]  (5.23)
Furthermore,
\[
\sum_{l=1}^N \frac{1}{H_l \delta_l} ||v - w_0||_{b, \alpha_l'}^2 \leq \frac{1}{\min_{1 \leq l \leq N} (H_l \delta_l)} \sum_{l=1}^N ||v - w_0||_{b, \alpha_l'}^2 \leq \frac{C}{\min_{1 \leq l \leq N} (H_l \delta_l)} ||v - w_0||_{b}^2.
\]  (5.24)
By (4.5) and the Poincaré inequality, we get
\[
||v - w_0||_b \leq ||v - Q^h v||_b + ||Q^h v - Q^h Q^h v||_b \leq CH ||v||_a + CH^2 ||v||_b \leq CH ||v||_{E^h_1},
\]
which, together with (5.23), (5.24) and the fact that \( J^H \) is quasi-uniform, yields
\[
\sum_{l=1}^N \frac{1}{\delta_l^2} ||v - w_0||_{b, \alpha_l, s_l}^2 \leq C(1 + \frac{H}{\delta}) ||v||_{E^h_1}^2.
\]  (5.25)
Combining (5.20), (5.21), (5.22) and (5.25) together, we complete the proof of (5.19). \( \square \)

Remark 5.3 After carefully checking our proof of Lemma 5.4, we know that the argument of the proof may be extended to the case of the fourth order elliptic operator. For the case of small overlap, we have the similar result as Lemma 5.4, and we only need to modify (5.19) to
\[
(w_0, w_0)_{E^h_1} + \sum_{l=1}^N (w^{(l)}, w^{(l)})_{E^h_1} \leq C(1 + \frac{H^3}{\delta^2}) \langle v, v \rangle_{E^h_1}, \quad i = 1, 2, \ldots, s.
\]
Proof of Theorem 5.2 For any \( v \in U_h^{i+1} \), by Lemma 5.4, there exist \( w_0 \in U_h^{i+1} \) and \( w_1 \in V^{(i)} \) such that
\[
v = Q_h^{i+1} w_0 + \sum_{l=1}^{N} Q_h^{i+1} w_l \quad \text{and} \quad \sum_{l=1}^{N} \| w_l \|_{E_i^h}^2 \leq C(1 + \frac{H}{\delta}) \| v \|_{E_i^h}^2. \tag{5.26}
\]
By (5.11), (5.12), (5.16), (5.26) and the Cauchy-Schwarz inequality, we may obtain
\[
(v, v)_{E_i^h} \leq C(1 + \frac{H}{\delta})(Q_h^{i+1} \sum_{l=1}^{N} T_{l,i}^k v, v)_{E_i^h}. \tag{5.27}
\]
Moreover,
\[
(G_k^i v, v)_{E_i^h} = (v, v)_{E_i^h} - \alpha_k^i (Q_h^{i+1} \sum_{l=1}^{N} T_{l,i}^k v, v)_{E_i^h} \leq \left( 1 - \frac{\alpha_k^i}{C(1 + \frac{H}{\delta})} \right) (v, v)_{E_i^h} \leq (1 - C\delta \frac{H}{\delta})(v, v)_{E_i^h}.
\]
By Lemma 5.3 we obtain
\[
\| G_k^i v \|_{E_i^h} \leq \| G_k^i \|_{E_i^h} \| v \|_{E_i^h} = \sup_{v \neq 0, v \in U_h^{i+1}} \frac{(G_k^i v, v)_{E_i^h}}{(v, v)_{E_i^h}} \| v \|_{E_i^h} \leq (1 - C\delta \frac{H}{\delta}) \| v \|_{E_i^h},
\]
which completes the proof of this theorem. \( \square \)

5.1.2 Estimate of the additional error term \( l_{2,i}^k \)

In this subsection, we give an estimate for the additional error term \( l_{2,i}^k \).

For convenience, denote by
\[
R_{i+1}^k := Q_h^{i+1}(B_{0,i}^k)^{-1}Q_h^{i+1}Q^H \quad \text{and} \quad \tilde{R}_{i+1}^k := (B_{0,i}^k)^{-1}Q_h^{i+1}Q^H.
\]
Similarly, denote by
\[
S_{i+1}^k := Q_h^{i+1} \sum_{l=1}^{N} (B_{l,i}^k)^{-1}Q^{(l)} \quad \text{and} \quad \tilde{S}_{i+1}^k := \sum_{l=1}^{N} (B_{l,i}^k)^{-1}Q^{(l)}.
\]
Hence, the additional error term \( l_{2,i}^k \), defined in (5.8) may be written as
\[
l_{2,i}^k = -\alpha_k^i (R_{i+1}^k + S_{i+1}^k) (\lambda_k^i - A^i)Q_{i+1}^k u_k^i + \alpha_k^i Q_{i+1}^k Q_{i+1}^k (B_{0,i}^k)^{-1} \lambda_k^i.
\] \tag{5.28}

Theorem 5.5 It holds that
\[
\| l_{2,i}^k \|_{E_i^h} \leq CH \| e_{i+1}^k \|_{E_i^h} + CH^2 \| g_k^i \|_a.
\]

Proof. Firstly, we estimate the first term of \( l_{2,i}^k \) in (5.28). For any \( w \in U_h^{i+1} \), by (4.6), (5.15) and the Cauchy-Schwarz inequality, we get
\[
\| R_{i+1}^k w \|_{E_i^h} = b(Q_h^{i+1}(B_{0,i}^k)^{-1}Q_h^{i+1}Q^H \cdot w, (A^i - \lambda_k^i)R_{i+1}^k w) = b((B_{0,i}^k)^{-1}Q_h^{i+1}Q^H \cdot w, (A^i - \lambda_k^i)R_{i+1}^k w) = b(Q_h^{i+1}Q^H w, T_{0,i}^k R_{i+1}^k w) \leq \| Q_h^{i+1}Q^H w \|_b \| T_{0,i}^k R_{i+1}^k w \|_{E_i^h} \leq CH^2 \| w \|_b \| R_{i+1}^k w \|_{E_i^h},
\]
which means that
\[
\| R_{i+1}^k w \|_{E_i^h} \leq CH^2 \| w \|_b. \quad \text{In particular, we take} \quad w = (\lambda_k^i - A^i)Q_{i+1}^k u_k^i \quad \text{and} \quad \| R_{i+1}^k (\lambda_k^i - A^i)Q_{i+1}^k u_k^i \|_{E_i^h} \leq CH^2 \| \lambda_k^i - A^i \|_{E_i^h} \| Q_{i+1}^k u_k^i \|_b. \tag{5.29}
\]
For any $w \in U^k_i$, by the Poincaré inequality in $V^{(l)}$ and (5.18), we obtain

$$
||S_{t,s+1}^k w||_{E_i^k}^2 = \sum_{l=1}^N b((B_{i,s})^{-1} Q^{(l)} w, (A^k - \lambda^k) S_{t,s+1}^k w) = \sum_{l=1}^N b(Q^{(l)} w, T_{i,s}^k S_{t,s+1}^k w) \leq \left\{ \sum_{l=1}^N ||Q^{(l)} w||_{b,\Omega_l}^2 \right\}^{\frac{1}{2}} \left\{ \sum_{l=1}^N ||T_{i,s}^k S_{t,s+1}^k w||_{b,\Omega_l}^2 \right\}^{\frac{1}{2}} \leq CH^2 ||w||_a ||S_{t,s+1}^k w||_{E_i^k},
$$

which means that $||S_{t,s+1}^k w||_{E_i^k} \leq CH^2 ||w||_a \leq C\sqrt{\lambda^k H^2} ||w||_b$. Specially, we take $w = (\lambda^k - A^k) Q_s^k u_i^k$.

By Lemma 4.7, we know

$$
||S_{t,s+1}^k (\lambda^k - A^k) Q_s^k u_i^k||_{E_i^k} \leq CH^2 (||\lambda^k - A^k|| Q_s^k u_i^k)||_b \leq CH^2 ||g_i^k||_a.
$$

which, together with (5.29), yields

$$
|| - \alpha^k (R_{i,s+1}^k + S_{i,s+1}^k) (\lambda^k - A^k) Q_s^k u_i^k||_{E_i^k} \leq CH^2 ||g_i^k||_a.
$$

Secondly, we estimate the second term of $I_{2,i}^k$ in (5.28). We divide it into three terms:

$$
Q_{s+1}^{h} U_s (B_{i,s})^{-1} r_{i,s}^k = Q_{s+1}^{h} U_s R_{i,s+1}^k (\lambda^k - A^k) Q_s^k u_i^k + Q_{s+1}^{h} U_s \tilde{S}_{i,s+1}^k (\lambda^k - A^k) Q_s^k u_i^k + Q_{s+1}^{h} U_s (B_{i,s})^{-1} (A^k - \lambda^k) e_{i,s+1} =: L_1 + L_2 + L_3.
$$

We estimate (5.32) one by one. Denote by $r_{i,s}^k := (\lambda^k - A^h) Q_s^k u_i^k$ and $r_{i,s+1}^k := (A^h - \lambda^k) e_{i,s+1}$, and we know $r_{i,s}^k = r_{i,s}^k + r_{i,s+1}^k$. For $L_1$, by Lemma 4.1, Corollary 4.5, and Lemma 4.6, we have

$$
||L_1||_{E_i^k} \leq ||Q_{s+1}^{h} U_s Q_s^k R_{i,s+1}^k r_{i,s}^k||_a + ||Q_{s+1}^{h} U_s Q_s^k R_{i,s+1}^k r_{i,s}^k||_a = ||Q_{s+1}^{h} P_U^{k} U_s Q_s^k R_{i,s+1}^k r_{i,s}^k||_a + ||Q_{s+1}^{h} P_U^{k} U_s Q_s^k R_{i,s+1}^k r_{i,s}^k||_a \leq CH ||R_{i,s+1}^k r_{i,s}^k||_a + CH ||Q_s^k R_{i,s+1}^k r_{i,s}^k||_a.
$$

Note that

$$
||Q_s^k R_{i,s+1}^k r_{i,s}^k||_a \leq \frac{\lambda^k}{\lambda^k - A^h} ||Q_s^k (B_{i,s})^{-1} Q^H r_{i,s}^k||_b \leq CH^2 ||(B_{i,s})^{-1} Q^H r_{i,s}^k||_b \leq \frac{CH^2}{\lambda^k - A^h} ||Q^H r_{i,s}^k||_b \leq CH^2 ||(\lambda^k - A^k) Q_s^k u_i^k||_b \leq CH^2 ||g_i^k||_a.
$$

By (5.29), (5.33) and (5.34), we obtain

$$
||L_1||_{E_i^k} \leq CH^3 ||g_i^k||_a + CH^5 ||g_i^k||_a \leq CH^3 ||g_i^k||_a.
$$

For $L_2$ in (5.32), we deduce

$$
||L_2||_{E_i^k} \leq ||Q_{s+1}^{h} U_s Q_s^k R_{i,s+1}^k r_{i,s}^k||_a + ||Q_{s+1}^{h} U_s Q_s^k \tilde{S}_{i,s+1}^k r_{i,s}^k||_a = ||Q_{s+1}^{h} P_U^{k} U_s Q_s^k R_{i,s+1}^k r_{i,s}^k||_a + ||Q_{s+1}^{h} P_U^{k} U_s Q_s^k S_{i,s+1}^k r_{i,s}^k||_a \leq CH ||S_{i,s+1}^k r_{i,s}^k||_a + CH ||Q_s^k S_{i,s+1}^k r_{i,s}^k||_a.
$$

In addition, by Lemma 2.3, Lemma 4.7, and the Poincaré inequality in $V^{(l)}$, we get

$$
||Q_s^k \tilde{S}_{i,s+1}^k r_{i,s}^k||_a \leq \lambda^k ||\tilde{S}_{i,s+1}^k r_{i,s}^k||_a \leq C\lambda^k H^6 \sum_{l=1}^N ||(B_{i,s})^{-1} Q^{(l)} r_{i,s}^k||_b \leq CH^6 \sum_{l=1}^N ||Q^{(l)} r_{i,s}^k||_b \leq CH^6 ||g_i^k||_a.
$$

By (5.29), (5.33) and (5.34), we obtain

$$
||L_1||_{E_i^k} \leq CH^3 ||g_i^k||_a + CH^5 ||g_i^k||_a \leq CH^3 ||g_i^k||_a.
$$

For $L_2$ in (5.32), we deduce

$$
||L_2||_{E_i^k} \leq ||Q_{s+1}^{h} U_s Q_s^k R_{i,s+1}^k r_{i,s}^k||_a + ||Q_{s+1}^{h} U_s Q_s^k \tilde{S}_{i,s+1}^k r_{i,s}^k||_a = ||Q_{s+1}^{h} P_U^{k} U_s Q_s^k R_{i,s+1}^k r_{i,s}^k||_a + ||Q_{s+1}^{h} P_U^{k} U_s Q_s^k S_{i,s+1}^k r_{i,s}^k||_a \leq CH ||S_{i,s+1}^k r_{i,s}^k||_a + CH ||Q_s^k S_{i,s+1}^k r_{i,s}^k||_a.
$$

In addition, by Lemma 2.3, Lemma 4.7, and the Poincaré inequality in $V^{(l)}$, we get

$$
||Q_s^k \tilde{S}_{i,s+1}^k r_{i,s}^k||_a \leq \lambda^k ||\tilde{S}_{i,s+1}^k r_{i,s}^k||_a \leq C\lambda^k H^6 \sum_{l=1}^N ||(B_{i,s})^{-1} Q^{(l)} r_{i,s}^k||_b \leq CH^6 \sum_{l=1}^N ||Q^{(l)} r_{i,s}^k||_b \leq CH^6 ||g_i^k||_a.
$$

By (5.29), (5.33) and (5.34), we obtain

$$
||L_1||_{E_i^k} \leq CH^3 ||g_i^k||_a + CH^5 ||g_i^k||_a \leq CH^3 ||g_i^k||_a.
$$

For $L_2$ in (5.32), we deduce

$$
||L_2||_{E_i^k} \leq ||Q_{s+1}^{h} U_s Q_s^k R_{i,s+1}^k r_{i,s}^k||_a + ||Q_{s+1}^{h} U_s Q_s^k \tilde{S}_{i,s+1}^k r_{i,s}^k||_a = ||Q_{s+1}^{h} P_U^{k} U_s Q_s^k R_{i,s+1}^k r_{i,s}^k||_a + ||Q_{s+1}^{h} P_U^{k} U_s Q_s^k S_{i,s+1}^k r_{i,s}^k||_a \leq CH ||S_{i,s+1}^k r_{i,s}^k||_a + CH ||Q_s^k S_{i,s+1}^k r_{i,s}^k||_a.
$$

In addition, by Lemma 2.3, Lemma 4.7, and the Poincaré inequality in $V^{(l)}$, we get

$$
||Q_s^k \tilde{S}_{i,s+1}^k r_{i,s}^k||_a \leq \lambda^k ||\tilde{S}_{i,s+1}^k r_{i,s}^k||_a \leq C\lambda^k H^6 \sum_{l=1}^N ||(B_{i,s})^{-1} Q^{(l)} r_{i,s}^k||_b \leq CH^6 \sum_{l=1}^N ||Q^{(l)} r_{i,s}^k||_b \leq CH^6 ||g_i^k||_a.
$$
Combining (5.30), (5.36) and (5.37), we obtain
\[ \|L_2\|_{E^k_i} \leq CH^3\|g^k_i\|_a + CH^4\|g^k_i\|_a \leq CH^3\|g^k_i\|_a. \] (5.38)

For \( L_3 \) in (5.32), by Lemma 4.2, Remark 5.2 and (5.15), we have
\[ \|L_3\|_{E^k_i} \leq \|Q^k_{s+1}P_{U^k_i}Q^k_{s+1}(B^k_i)^{-1}r^k_{i,s+1}\|_a + \|Q^k_{s+1}P_{U^k_i}Q^k_{s+1}(B^k_i)^{-1}r^k_{i,s+1}\| \]
\[ \leq CH\|(B^k_i)^{-1}r^k_{i,s+1}\|_a + CH\|Q^k_{s}(B^k_i)^{-1}r^k_{i,s+1}\| \]
\[ \leq CH\|T_{0,i,s+1}r^k_{i,s+1}\|_a + CH\|Q^k_{s}\sum_{l=1}^{N} T_{i,s+1}r^k_{i,s+1}\|_a + CH\|Q^k_{s}\sum_{l=1}^{N} T_{i,s+1}r^k_{i,s+1}\|_a \]
\[ + CH\|Q^k_{s}\sum_{l=1}^{N} T_{i,s+1}r^k_{i,s+1}\|_a \leq CH\|e^k_{i,s+1}\|_{E^k_i}, \] (5.39)

which, together with (5.32), (5.35), (5.38), yields
\[ \|\alpha^k_{i,s+1}Q^k_{s+1}(B^k_i)^{-1}r^k_{i,s+1}\|_{E^k_i} \leq CH\|e^k_{i,s+1}\|_{E^k_i} + CH^3\|g^k_i\|_a. \] (5.40)

Finally, combining (5.28), (5.31) and (5.40), we may complete the proof of this theorem. \( \square \)

**Theorem 5.6** It holds that
\[ \|e^k_{i,s+1}\|_{E^k_i} \leq c_0(H)(1 - C\frac{\delta}{H})\|e^k_{i,s+1}\|_{E^k_i} + CH^2\|g^k_i\|_a, \quad i = 1, 2, ..., s, \]
where \( H \)-dependent constant \( c_0(H) \) \( (\equiv 1 + \frac{CH}{1-C\frac{\delta}{H}}) \) decreases monotonically to 1, as \( H \to 0 \).

**Proof.** By (5.8), Theorem 5.2 and Theorem 5.5, we have
\[ \|e^k_{i,s+1}\|_{E^k_i} \leq \|I_{1,i}e^k_{i,s+1}\|_{E^k_i} + \|I_{2,i}e^k_{i,s+1}\|_{E^k_i} = \|I_{1,i}e^k_{i,s+1}\|_{E^k_i} + \|I_{2,i}e^k_{i,s+1}\|_{E^k_i} \]
\[ \leq (1 - C\frac{\delta}{H})\|e^k_{i,s+1}\|_{E^k_i} + CH\|e^k_{i,s+1}\|_{E^k_i} + CH^2\|g^k_i\|_a \]
\[ \leq \{(1 - C\frac{\delta}{H}) + CH\}\|e^k_{i,s+1}\|_{E^k_i} + CH^2\|g^k_i\|_a \]
\[ = c_0(H)(1 - C\frac{\delta}{H})\|e^k_{i,s+1}\|_{E^k_i} + CH^2\|g^k_i\|_a, \]
where \( c_0(H) \) \( (\equiv 1 + \frac{CH}{1-C\frac{\delta}{H}}) \) decreases monotonically to 1, as \( H \to 0 \). \( \square \)

**Remark 5.4** By Remark 5.2, (5.34), (5.37), (5.39) and the facts that \( (B^k_i)^{-1} = \tilde{R}_{i,s+1} + \tilde{s}_{i,s+1} \) and \( r^k_i = r^k_i + r^k_{i,s+1} + r^k_{i,s+1} \), we have
\[ \|Q^k_{s+1}(B^k_i)^{-1}r^k_i\|_a \leq C\|Q^k_{s+1}(B^k_i)^{-1}r^k_i\|_a \leq CH\|e^k_{i,s+1}\|_{E^k_i} + CH^2\|g^k_i\|_a. \]

Similarly, by Remark 5.2 (5.15) and (5.31), we get
\[ \|Q^k_{s+1}(B^k_i)^{-1}r^k_i\|_a \leq C\|Q^k_{s+1}(B^k_i)^{-1}r^k_i\|_a \leq CH\|e^k_{i,s+1}\|_{E^k_i} + CH^2\|g^k_i\|_a. \]
5.2 The proof of the main result

In this subsection, based on Theorem 5.6 in previous subsection, we first give an estimate for

\[ \sum_{i=1}^{s} (Rq(\tilde{u}_i^{k+1}) - Rq(Q_s^h u_i^k)), \]

and then present a rigorous proof of Theorem 5.1.

**Lemma 5.7** It holds that

\[ \sum_{i=1}^{s} (Rq(\tilde{u}_i^{k+1}) - Rq(Q_s^h u_i^k)) \leq \gamma_0 \sum_{i=1}^{s} (\lambda_i^k - Rq(Q_s^h u_i^k)) + CH \sum_{i=1}^{s} (\lambda_i^k - \lambda_i^k), \]  

where \( \gamma_0 = (1 - C^2 \frac{\delta}{H})^2 + CH. \)

**Proof.** Firstly, by the fact that \( \|\tilde{u}_i^{k+1}\|_{E_i^k}^2 = \|\tilde{c}_i^{k+1}\|_{E_i^k}^2 + \|Q_s^h \tilde{u}_i^{k+1}\|_{E_i^k}^2 \), we deduce

\[
\|\tilde{c}_i^{k+1}\|_{E_i^k}^2 = b((A^h - \lambda_i^k)\tilde{u}_i^{k+1}, \tilde{u}_i^{k+1}) - b((A^h - \lambda_i^k)Q_s^h \tilde{u}_i^{k+1}, Q_s^h \tilde{u}_i^{k+1}) \\
= (Rq(\tilde{u}_i^{k+1}) - Rq(Q_s^h u_i^k))\|\tilde{c}_i^{k+1}\|_{E_i^k}^2 + \|Rq(Q_s^h u_i^k) - \lambda_i^k\|\tilde{u}_i^{k+1}\|_{E_i^k}^2 \\
+ (\lambda_i^k - Rq(Q_s^h \tilde{u}_i^{k+1}))\|Q_s^h \tilde{u}_i^{k+1}\|_{E_i^k}^2 \\
= (Rq(\tilde{u}_i^{k+1}) - Rq(Q_s^h u_i^k))\|\tilde{c}_i^{k+1}\|_{E_i^k}^2 + \|Rq(Q_s^h u_i^k) - \lambda_i^k\|\tilde{u}_i^{k+1}\|_{E_i^k}^2 \\
+ (Rq(Q_s^h \tilde{u}_i^{k+1}) - Rq(Q_s^h u_i^k))\|Q_s^h \tilde{u}_i^{k+1}\|_{E_i^k}^2,
\]

which yields

\[
(Rq(\tilde{u}_i^{k+1}) - Rq(Q_s^h u_i^k))\|\tilde{c}_i^{k+1}\|_{E_i^k}^2 = \{(\|\tilde{c}_i^{k+1}\|_{E_i^k}^2 + \|\tilde{c}_i^{k+1}\|_{E_i^k}^2) + (\lambda_i^k - Rq(Q_s^h u_i^k))\|\tilde{u}_i^{k+1}\|_{E_i^k}^2 \\
+ (Rq(Q_s^h \tilde{u}_i^{k+1}) - Rq(Q_s^h u_i^k))\|Q_s^h \tilde{u}_i^{k+1}\|_{E_i^k}^2 =: J_1 + J_2.
\]

Secondly, we estimate \( J_1 \) and \( J_2 \) in (5.43) one by one. For \( J_1 \), by Lemma 4.7 and Theorem 5.6, we get

\[
J_1 \leq \|\tilde{c}_i^{k+1}\|_{E_i^k}^2 + (\lambda_i^k - Rq(Q_s^h u_i^k))\|\tilde{c}_i^{k+1}\|_{E_i^k}^2 \\
\leq \{(1 - C^2 \frac{\delta}{H}) + CH\|\tilde{c}_i^{k+1}\|_{E_i^k}^2 + CH\|g_i^{k}\|_{a}^2 \\
+ CH^2 [C\|\tilde{c}_i^{k+1}\|_{E_i^k}^2 + CH \|g_i^{k}\|_{a}^2] \\
\leq \gamma_0 \|\tilde{c}_i^{k+1}\|_{E_i^k}^2 + CH^2 \|g_i^{k}\|_{a}^2 \leq \gamma_0 (\lambda_i^k - Rq(Q_s^h u_i^k)) + CH^2 \|g_i^{k}\|_{a}^2.
\]

For convenience, denote by \( u_i^k := Q_s^h Q_{s - 1}^h (B_{s - 1}^h)^{-1} T_i^k \). For \( J_2 \) in (5.43), by (5.4), we deduce

\[
b(Q_s^h \tilde{u}_i^{k+1}, Q_s^h \tilde{u}_i^{k+1}) = b(Q_s^h u_i^k, Q_s^h u_i^k) + 2\alpha_i^k b(Q_s^h w_i^k, w_i^k) + (\alpha_i^k)^2 b(w_i^k, w_i^k),
\]

and

\[
a(Q_s^h \tilde{u}_i^{k+1}, Q_s^h \tilde{u}_i^{k+1}) = a(Q_s^h u_i^k, Q_s^h u_i^k) + 2\alpha_i^k a(Q_s^h w_i^k, w_i^k) + (\alpha_i^k)^2 a(w_i^k, w_i^k),
\]

which yields

\[
J_2 = a(Q_s^h \tilde{u}_i^{k+1}, Q_s^h \tilde{u}_i^{k+1}) - Rq(Q_s^h u_i^k)\|Q_s^h \tilde{u}_i^{k+1}\|_{E_i^k}^2 \\
= 2\alpha_i^k b((A^h - \lambda_i^k)Q_s^h u_i^k, w_i^k) + 2\alpha_i^k b((A^h - \lambda_i^k)Q_s^h u_i^k)Q_s^h u_i^k, w_i^k) \\
+ (\alpha_i^k)^2 b((A^h - Rq(Q_s^h u_i^k))u_i^k, w_i^k) =: J_{2,1} + J_{2,2} + J_{2,3}.
\]
For $J_{2,1}$ in (5.45), by the Cauchy-Schwarz inequality, Lemma 4.4, Lemma 4.6, Lemma 4.7 and Remark 5.4 we get

$$J_{2,1} \leq C|\lambda_k^h - \lambda_k^t\hat{Q}_h^k u_t^i||\lambda_i||w_t^k||_b \leq C\theta_h^k||g_t^k||_a||(B_t^k)^{-1}r_t^k||_b$$
$$\leq CH||w_t^k||_a(\lambda_i + CH^2||g_t^k||_a) \leq CH(\lambda_i - Rq(Q_h^k u_t^i)) + CH||g_t^k||_a^2. \tag{5.46}$$

Similarly, for $J_{2,2}$ in (5.45), we have

$$J_{2,2} \leq C(\lambda_k^h - Rq(Q_h^k u_t^i)||w_t^k||_b \leq C\theta_h^k(\lambda_k^h - Rq(Q_h^k u_t^i)||(B_h^k)^{-1}r_h^k||_b$$
$$\leq CH^2(\lambda_k^h - Rq(Q_h^k u_t^i)) + CH||g_t^k||_a^2. \tag{5.47}$$

For $J_{2,3}$ in (5.45), we deduce

$$J_{2,3} = (\alpha_k^h)^2(||w_t^k||_a^2 + (\lambda_k^h - Rq(Q_h^k u_t^i)||w_t^k||_a^2 - \lambda_k^t||w_t^k||_a^2)$$
$$\leq C(\lambda_k^h + \lambda_k^t - Rq(Q_h^k u_t^i)||w_t^k||_a^2 \leq C(\theta_h^k)^2||(B_h^k)^{-1}r_h^k||_b^2$$
$$\leq CH^2(\lambda_k^h - Rq(Q_h^k u_t^i)) + CH||g_t^k||_a^2,$$

which, together with (5.45), (5.46), (5.47), yields

$$J_2 = (Rq(Q_h^h u_t^k) - Rq(Q_h^k u_t^i)||w_t^k||_a \leq CH(\lambda_k^h - Rq(Q_h^k u_t^i)) + CH||g_t^k||_a^2. \tag{5.48}$$

Finally, combining (5.43), (5.44) and (5.48), we have

$$(Rq(\tilde{u}_t^k) - Rq(Q_h^k u_t^i)||\tilde{u}_t^k||_a \leq \{\gamma_0(\lambda_k^h - Rq(Q_h^k u_t^i)) + CH^2||g_t^k||_a^2$$
$$\quad + \{CH(\lambda_k^h - Rq(Q_h^k u_t^i)) + CH||g_t^k||_a^2\}$$
$$\leq \gamma_0(\lambda_k^h - Rq(Q_h^k u_t^i)) + CH||g_t^k||_a^2. \tag{5.49}$$

Since \(\frac{1}{||u_t^k||_b} = 1\), we obtain

$$Rq(\tilde{u}_t^k) - Rq(Q_h^k u_t^i) \leq \gamma_0(\lambda_k^h - Rq(Q_h^k u_t^i)) + CH||g_t^k||_a^2. \tag{5.49}$$

Taking summation over $i$ in (5.49) and using Lemma 4.3, we complete the proof of this lemma.

We also establish an estimate for $\sum_{i=1}^{s}(\lambda_k^h - Rq(\tilde{u}_t^k))$, where $\tilde{\lambda}_i^{k+1}$ is defined in (5.6). In order to make the proof of our main result neat, we put the proof of following lemma (Lemma 5.8) in Appendix.

**Lemma 5.8** *It holds that*

$$\sum_{i=1}^{s}(\lambda_k^h - Rq(\tilde{u}_t^k)) \leq CH \sum_{i=1}^{s}(\lambda_k^h - Rq(Q_h^k u_t^i)) + CH^5 \sum_{i=1}^{s}(\lambda_k^h - \lambda_k^t).$$

Now we are in a position to prove the main result of this paper.

**Proof of Theorem 5.1** By Lemma 5.7 and Lemma 5.8, we get

$$\sum_{i=1}^{s}(\lambda_k^h + \lambda_k^t) = \sum_{i=1}^{s}(\lambda_k^h - Rq(\tilde{u}_t^k)) + \sum_{i=1}^{s}(Rq(\tilde{u}_t^k) - Rq(Q_h^k u_t^i)) + \sum_{i=1}^{s}(Rq(Q_h^k u_t^i) - \lambda_k^t)$$
$$\leq \{CH \sum_{i=1}^{s}(\lambda_k^h - Rq(Q_h^k u_t^i)) + CH^5 \sum_{i=1}^{s}(\lambda_k^h - \lambda_k^t)\} + \gamma_0 \sum_{i=1}^{s}(\lambda_k^h - Rq(Q_h^k u_t^i)) +$$
$$+ CH \sum_{i=1}^{s}(\lambda_k^h - \lambda_k^t) + \sum_{i=1}^{s}(Rq(Q_h^k u_t^i) - \lambda_k^t)$$
$$\leq \gamma_0 \sum_{i=1}^{s}(\lambda_k^h - Rq(Q_h^k u_t^i)) + \sum_{i=1}^{s}(Rq(Q_h^k u_t^i) - \lambda_k^t) + CH \sum_{i=1}^{s}(\lambda_k^h - \lambda_k^t).$$
Considering $Rq(Q^h s u^h_k) \leq \lambda^k_i$ and $\lambda^k_{i+1} \leq \hat{\lambda}^k_{i+1}$, we deduce
\[
\sum_{i=1}^{s} (\hat{\lambda}^{k+1}_{i} - \lambda^k_i) \leq \gamma_0 \sum_{i=1}^{s} (\lambda^k_i - \lambda^h_i) + (1 - \gamma_0) \sum_{i=1}^{s} (Rq(Q^h s u^h_k) - \lambda^h_i) +
\]
\[
+ CH \sum_{i=1}^{s} (\lambda^h_i - \lambda^h_i) \leq \gamma \sum_{i=1}^{s} (\lambda^h_i - \lambda^h_i),
\]
where $\gamma = \max\{\gamma_0, 1 - \gamma_0\} = \gamma_0 = (1 - C^4 \frac{a}{\pi^2})^2 + CH = c(H)(1 - C^4 \frac{a}{\pi^2})^2$. Here, without loss of generality, let $\gamma_0 \geq \frac{1}{2}$. The $H$-dependent constant $c(H) := 1 + \frac{CH}{(1 - C^4 \frac{a}{\pi^2})}$ decreases monotonically to 1, as $H \to 0$.

Combining Lemma 4.4, Corollary 4.5 and (5.1), we may prove (5.2) and (5.3), which completes the proof of this theorem.

6 Numerical experiments

In this section, we present several numerical experiments in two and three dimensional eigenvalue problems to support our theoretical findings. For the stopping criterion of the proposed method, we choose the accuracy of $\sum_{i=1}^{s} |\lambda^{k+1}_i - \lambda^k_i| < tol = 1e^{-10}$.

6.1 2D Laplacian eigenvalue problem

In this subsection, we shall present some numerical results of 2D Laplacian eigenvalue problems in convex and L-shaped domains.

Example 6.1 We consider the Laplacian eigenvalue problem in $(0, \pi)^2$ and use the triangle $P_1$-conforming finite element to compute the first $s$ eigenpairs. First, we choose an initial uniform partition $\mathcal{J}_H$ in $\Omega$ with the number of subdomains $N = 512$, and coarse grid size $H = \frac{\sqrt{\pi}}{4}$. We refine uniformly the grid layer by layer and fix the ratio $\frac{\alpha}{H} = \frac{1}{4}$. Next, we test the optimality and scalability of our algorithm.
Before analyzing numerical results, we first introduce some notations used in Table 1 which have the same meanings as following tables. We denote by \( d.o.f. \) degrees of freedom, by \( it. \) the number of iterations and by \( \text{stop.} \) the total error between two adjacent iterative eigenvalues \( \sum_{i=1}^{s} |\lambda_{k+1}^{i} - \lambda_{k}^{i}| \) when exiting the outer loop in our two-level BPJD algorithm. It is shown in Table 1 that the number of iterations of the proposed method keeps stable when \( d.o.f. \to +\infty \), which illustrates that our method is optimal. It is seen from Figure 1 that all of the curves of the total error \( \sum_{i=1}^{s} |\lambda_{k+1}^{i} - \lambda_{k}^{i}| \) with different degrees of freedom coincide, which verifies that the convergence rate of the proposed method is independent of \( h \). In order to test the scalability of our algorithm, we set \( d.o.f. = 16769025 \) and the ratio \( \delta = \frac{1}{4} \) to observe the relationship between the number of iterations and the number of subdomains.

It is obvious to see in Table 2 that the number of iterations decreases, as the number of subdomains increases, which shows that our algorithm is scalable. More intuitively, it is observed in Figure 2 that curves of \( \sum_{i=1}^{s} |\lambda_{k+1}^{i} - \lambda_{k}^{i}| \) with different subdomains are almost parallel, which illustrates that our algorithm has a good scalability. Although our theoretical analysis only holds for convex cases, our algorithm still works very well for nonconvex cases. We present some numerical results for the 2D Laplacian eigenvalue problem in L-shape domain.

### Table 1: \( N = 512, \delta = \frac{1}{4}, s = 19 \)

| \( d.o.f. \) | 16129 | 65025 | 261121 | 1046529 | 4190209 | 16769025 |
|---|---|---|---|---|---|---|
| \( \lambda_{i} \) | 21(it.) | 22(it.) | 22(it.) | 22(it.) | 22(it.) | 22(it.) |
| \( \lambda_{1} = 2 \) | 2.00030120 | 2.00007300 | 2.00001882 | 2.00000471 | 2.00000118 | 2.00000029 |
| \( \lambda_{2} = 5 \) | 5.00129490 | 5.00032372 | 5.00008093 | 5.00002023 | 5.00000506 | 5.00000126 |
| \( \lambda_{3} = 5 \) | 5.00201852 | 5.00050458 | 5.00012614 | 5.00003154 | 5.00000788 | 5.00000197 |
| \( \lambda_{4} = 8 \) | 8.00481845 | 8.00120474 | 8.00030119 | 8.00007530 | 8.00001882 | 8.00000471 |
| \( \lambda_{5} = 10 \) | 10.00592410 | 10.00148092 | 10.00037022 | 10.00009256 | 10.00002314 | 10.00000578 |
| \( \lambda_{6} = 10 \) | 10.00592615 | 10.00148105 | 10.00037023 | 10.00009256 | 10.00002314 | 10.00000578 |
| \( \lambda_{7} = 13 \) | 13.00904908 | 13.00226266 | 13.00056569 | 13.00014142 | 13.00003536 | 13.00000884 |
| \( \lambda_{8} = 13 \) | 13.01514849 | 13.00378646 | 13.00056567 | 13.00014142 | 13.00003536 | 13.00000884 |
| \( \lambda_{9} = 17 \) | 17.01592318 | 17.00397968 | 17.00099485 | 17.00014142 | 17.00003536 | 17.00000884 |
| \( \lambda_{10} = 17 \) | 17.01592615 | 17.00397968 | 17.00099485 | 17.00014142 | 17.00003536 | 17.00000884 |
| \( \lambda_{11} = 18 \) | 18.02436417 | 18.00609718 | 18.00152468 | 18.00038119 | 18.00009530 | 18.00002383 |
| \( \lambda_{12} = 20 \) | 20.02650464 | 20.00662628 | 20.00165658 | 20.00041414 | 20.00010354 | 20.00002588 |
| \( \lambda_{13} = 20 \) | 20.02655291 | 20.00666292 | 20.00165677 | 20.00041414 | 20.00010354 | 20.00002588 |
| \( \lambda_{14} = 25 \) | 25.03383780 | 25.00846626 | 25.00211699 | 25.00052927 | 25.00013232 | 25.00003308 |
| \( \lambda_{15} = 25 \) | 25.03797117 | 25.01447495 | 25.00361190 | 25.00090297 | 25.00022574 | 25.00005644 |
| \( \lambda_{16} = 26 \) | 26.03646327 | 26.00911235 | 26.00227787 | 26.00056945 | 26.00014236 | 26.00003559 |
| \( \lambda_{17} = 26 \) | 26.03646513 | 26.00911246 | 26.00227788 | 26.00056945 | 26.00014236 | 26.00003559 |
| \( \lambda_{18} = 29 \) | 29.05779711 | 29.01279949 | 29.00319937 | 29.00079981 | 29.00019995 | 29.00004999 |
| \( \lambda_{19} = 29 \) | 29.05793749 | 29.01333488 | 29.00333317 | 29.00083326 | 29.00028031 | 29.00005298 |
| \( \text{stop.} \) | 8.7853e-11 | 3.1262e-11 | 3.5083e-11 | 3.5370e-11 | 3.8950e-11 | 4.2572e-11 |

### Table 2: \( N = 512, 2048, 8192, \delta = \frac{1}{4}, s = 19 \)

| \( N \) | \( d.o.f. \) | \( it. \) |
|---|---|---|
| 512 | 16769025 | 22 |
| 2048 | 16769025 | 18 |
| 8192 | 16769025 | 16 |
Example 6.2 We consider the Laplacian eigenvalue problem in L-shape domain \((-\pi, \pi)^2 \setminus [0, \pi) \times (-\pi, 0]\) and use the triangle \(P_1\)-conforming finite element to compute the first \(s\) eigenpairs. First, we choose an initial uniform partition \(\mathcal{J}_H\) in \(\Omega\) with \(N = 384\), \(H = \frac{\sqrt{2\pi}}{2}\). We refine uniformly the grid layer by layer and fix the ratio \(\frac{\delta}{H} = \frac{1}{4}\). Next, we also test the optimality and scalability of our algorithm.
It is known that some eigenfunctions of the Laplacian eigenvalue problem have singularities at the re-entrant corner but our algorithm still works very well. The number of iterations of our algorithm keeps stable in Table 3 as \( \text{d.o.f.} \to +\infty \), i.e., the convergence rate of our algorithm is independent of \( h \). The number of iterations decreases, as the number of subdomains increases in Table 4, which verifies that our algorithm is scalable.

### 6.2 3D Laplacian eigenvalue problem

In order to illustrate that our theoretical analysis still holds for 3D cases, we design two experiments to verify it.

**Example 6.3** We consider the Laplacian eigenvalue problem in \((0, \pi)^3\) and use the trilinear conforming finite element to compute the first \( s \) eigenpairs. First, we choose an initial uniform partition \( J_H \) in \( \Omega \) with \( N = 512, H = \frac{\pi}{32} \). We refine uniformly the grid layer by layer and fix the ratio \( \frac{\delta}{H} = \frac{1}{4} \). Next, we test the optimality and scalability of our algorithm.
Table 5: \( N = 512, \frac{\delta}{\pi} = \frac{1}{2}, s = 20 \)

| \( \lambda_i \) | 14(it.) | 20(it.) | 25(it.) | 30(it.) | 35(it.) |
|---------------|---------|---------|---------|---------|---------|
| \( \lambda_1 \) = 3 | 3.00965062 | 3.00060244 | 3.00015060 | 3.00003765 | 3.00000000 |
| \( \lambda_2 \) = 6 | 6.05809793 | 6.00361542 | 6.00090366 | 6.00022590 | 6.00000000 |
| \( \lambda_3 \) = 6 | 6.05809793 | 6.00361542 | 6.00090366 | 6.00022590 | 6.00000000 |
| \( \lambda_4 \) = 6 | 6.05809793 | 6.00361542 | 6.00090366 | 6.00022590 | 6.00000000 |
| \( \lambda_5 \) = 9 | 9.10654523 | 9.00662840 | 9.00165671 | 9.00041415 | 9.00000000 |
| \( \lambda_6 \) = 9 | 9.10654523 | 9.00662840 | 9.00165671 | 9.00041415 | 9.00000000 |
| \( \lambda_7 \) = 9 | 9.10654523 | 9.00662840 | 9.00165671 | 9.00041415 | 9.00000000 |
| \( \lambda_8 \) = 9 | 9.10654523 | 9.00662840 | 9.00165671 | 9.00041415 | 9.00000000 |
| \( \lambda_9 \) = 11 | 11.26956430 | 11.00662840 | 11.00165671 | 11.00041415 | 11.00000000 |
| \( \lambda_{10} \) = 11 | 11.26956430 | 11.00662840 | 11.00165671 | 11.00041415 | 11.00000000 |
| \( \lambda_{11} \) = 12 | 12.15499254 | 11.00662840 | 11.00165671 | 11.00041415 | 11.00000000 |
| \( \lambda_{12} \) = 14 | 14.31801161 | 12.00662840 | 11.00165671 | 11.00041415 | 11.00000000 |
| \( \lambda_{13} \) = 14 | 14.31801161 | 12.00662840 | 11.00165671 | 11.00041415 | 11.00000000 |
| \( \lambda_{14} \) = 14 | 14.31801161 | 12.00662840 | 11.00165671 | 11.00041415 | 11.00000000 |
| \( \lambda_{15} \) = 14 | 14.31801161 | 12.00662840 | 11.00165671 | 11.00041415 | 11.00000000 |
| \( \lambda_{16} \) = 14 | 14.31801161 | 12.00662840 | 11.00165671 | 11.00041415 | 11.00000000 |
| \( \lambda_{17} \) = 14 | 14.31801161 | 12.00662840 | 11.00165671 | 11.00041415 | 11.00000000 |
| \( \lambda_{18} \) = 17 | 17.36645892 | 17.00662840 | 11.00165671 | 11.00041415 | 11.00000000 |
| \( \lambda_{19} \) = 17 | 17.36645892 | 17.00662840 | 11.00165671 | 11.00041415 | 11.00000000 |
| \( \lambda_{20} \) = 17 | 17.36645892 | 17.00662840 | 11.00165671 | 11.00041415 | 11.00000000 |
| stop. | 8.9527e-11 | 3.6276e-11 | 5.3570e-11 | 2.8785e-11 | 4.5785e-11 |

Table 6: \( N = 512, 4096, \frac{\delta}{\pi} = \frac{1}{2}, s = 20 \)

| \( N \) | d.o.f. | it. |
|-------|-------|-----|
| 512   | 16581375 | 17  |
| 4096  | 16581375 | 15  |

It is seen from Table 5 that the number of iterations of our algorithm keeps stable nearly, as \( d.o.f. \rightarrow +\infty \), which shows that our algorithm is optimal. To verify the scalability of the method, we set \( d.o.f. = 16581375 \) and observe the number of iterations for \( N = 512, 4096 \). Numerical results in Table 6 show the number of iterations decreases as \( N \) increases, which means that the proposed method has a good scalability. Next, we also present some numerical results for three dimensional L-shape domain.

Example 6.4 We consider the Laplacian eigenvalue problem in \((0, 2\pi) \times (0, 2\pi) \times (0, \pi) \setminus [\pi, 2\pi) \times [\pi, 2\pi) \times (0, \pi)\) and use the trilinear conforming finite element to compute the first \( s \) eigenpairs. First, we choose an initial uniform partition \( J_H \) in \( \Omega \) with \( N = 1536, H = \frac{\pi}{3} \). We refine uniformly the grid layer by layer and fix the ratio \( \frac{\delta}{\pi} = \frac{1}{2} \). Next, we also test the optimality and scalability of our algorithm.

It is observed from Table 7 that the number of iterations keeps stable nearly when \( d.o.f. \rightarrow +\infty \), which verifies that the method is optimal for nonconvex domain. In addition, if we observe Table 7 carefully, we may find that some of eigenvalues are close to each other \((\lambda_{17} \approx \lambda_{18} \approx \lambda_{19})\) and our algorithm works still very well, which illustrates that the convergence rate in our two-level BPJD method is not adversely affected by the gap among the clustered eigenvalues. It is obvious to see that the number of iterations decreases as the number of subdomains increases in Table 8 which shows that our algorithm is scalable.
Table 7: \( N = 1536, \frac{\delta}{\pi} = \frac{1}{2}, s = 20 \)

| d.o.f. | 10575 | 91295 | 758079 | 6177407 |
|--------|-------|-------|--------|----------|
| \( \lambda_1 \) | 1.98468171 | 1.97908729 | 1.97745736 | 1.97695688 |
| \( \lambda_2 \) | 2.54796654 | 2.54184555 | 2.54031441 | 2.53993134 |
| \( \lambda_3 \) | 3.00965062 | 3.00241034 | 3.00060244 | 3.00015060 |
| \( \lambda_4 \) | 4.01258997 | 3.99650322 | 3.99248898 | 3.99148580 |
| \( \lambda_5 \) | 4.26520929 | 4.24231072 | 4.23602425 | 4.23422553 |
| \( \lambda_6 \) | 5.03132902 | 4.99115134 | 4.98047034 | 4.97770994 |
| \( \lambda_7 \) | 5.25609401 | 5.21633525 | 5.20604572 | 5.20330844 |
| \( \lambda_8 \) | 5.59641384 | 5.55390960 | 5.54323739 | 5.54068439 |
| \( \lambda_9 \) | 6.05809793 | 6.01447439 | 6.00361542 | 6.00090366 |
| \( \lambda_{10} \) | 6.05809793 | 6.01447439 | 6.00361542 | 6.00090366 |
| \( \lambda_{11} \) | 6.05809793 | 6.01447439 | 6.00361542 | 6.00090366 |
| \( \lambda_{12} \) | 6.05809793 | 6.01447439 | 6.00361542 | 6.00090366 |
| \( \lambda_{13} \) | 6.81375190 | 6.76357620 | 6.75061942 | 6.74719395 |
| \( \lambda_{14} \) | 7.06103728 | 7.00866727 | 6.99550196 | 6.99223885 |
| \( \lambda_{15} \) | 7.31636600 | 7.25437477 | 7.23903723 | 7.23497859 |
| \( \lambda_{16} \) | 7.71618710 | 7.64799074 | 7.62979692 | 7.62547071 |
| \( \lambda_{17} \) | 8.30454131 | 8.22839930 | 8.20905870 | 8.20215738 |
| \( \lambda_{18} \) | 8.34143977 | 8.32586694 | 8.30907121 | 8.30461491 |
| \( \lambda_{19} \) | 8.37606474 | 8.28281881 | 8.25955996 | 8.25360450 |
| \( \lambda_{20} \) | 8.66084944 | 8.58088775 | 8.56083735 | 8.55893933 |
| \( stop. \) | 7.5267e-11 | 5.4056e-11 | 1.8214e-11 | 3.6381e-11 |

Table 8: \( N = 1536, 12288, \frac{\delta}{\pi} = \frac{1}{2}, s = 20 \)

| N     | d.o.f. | it. |
|-------|-------|-----|
| 1536  | 6177407 | 18  |
| 12288 | 6177407 | 15  |

7 Conclusions

In this paper, based on a domain decomposition method, we propose a parallel two-level BPJD method for computing multiple and clustered eigenvalues. The method is proved to be optimal, scalable and cluster robust. Numerical results verify our theoretical findings.

Appendix A

Proof of Theorem 2.2: Let \( \tilde{\imath} \) be an imaginary unit and \( \Gamma \) be a circle, which includes \( \mu_1, \mu_2, \ldots, \mu_s \) and \( \mu_{h1}, \mu_{h2}, \ldots, \mu_{hs} \), with \( \left( \frac{\mu_1 + \mu_s}{2}, 0 \right) \) as a center and \( \frac{\mu_1 - \mu_s + 1}{2} \) as a radius in complex plane \( \mathbb{C} \). Define

\[
Z := \frac{1}{2\pi i} \int_{\Gamma} (z - T)^{-1} dz, \quad Z^h := \frac{1}{2\pi i} \int_{\Gamma} (z - T^h)^{-1} dz.
\]

Therefore, we know that \( Z \) and \( Z^h \) are orthogonal projectors onto \( U_s \) and \( U_s^h \), respectively, with respect to \( b(\cdot, \cdot) \) (also \( a(\cdot, \cdot) \)).

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Figure 3: $\Gamma$ is a circle which includes $\mu_1, \mu_2, \ldots, \mu_s$ and $\mu_{1h}, \mu_{2h}, \ldots, \mu_{sh}$.

Combining (2.4), (2.7) and standard finite element error estimate, we have $||T - T^h||_a \leq C_1h$. Since $\dim(U_s) = s = \dim(U_{sh})$, by Remark 2.2, we get

\[
\theta_a(U_s, U_{sh}) = \sup_{u \in U_s, ||u||_a = 1} \frac{||u - Z^h u||_a}{\inf_{v \in U_{sh}, ||v||_a = 1} ||u - v||_a} = \sup_{u \in U_s, ||u||_a = 1} ||u - Z^h u||_a \leq ||Z - Z^h||_a \leq \frac{1}{2\pi} \int_{\Gamma} (z - T^h)^{-1} (T - T^h) (z - T)^{-1} dz ||_a
\]

\[
= \frac{1}{2\pi} \times (2\pi \times \frac{\mu_1 - \mu_{s+1}}{2}) \times \frac{1}{\mu_s - \mu_{s+1}} \times \frac{1}{\mu_s - \mu_{s+1}} \times ||T - T^h||_a
\]

\[
\leq C_1 \frac{2(\mu_1 - \mu_{s+1})}{(\mu_s - \mu_{s+1})(2\mu_s - \mu_s - \mu_{s+1})} h \to 0, \ h \to 0, \quad (A.2)
\]

which means that (2.10) holds. Combining the Aubin-Nitsche argument, we have $||T - T^h||_b \leq C_0h^2$. By the same argument as in the proof of (2.10), we may also obtain (2.11). \hfill \square

To give a rigorous proof of Lemma 5.8 in this paper, we first introduce the following lemma (For the detailed proof, see Lemma 5 in [19] or Lemma 2 of Appendix A in [20]). For any matrix $X$, we denote by $\text{D}X$ the diagonal part of $X$, $\bar{D}X = D - X$. And we denote by $\text{Tr}(X)$ the trace of the matrix $X$.

**Lemma A.1** Let $A (= \Lambda - \tilde{A})$ be a symmetric and $B (= I - \tilde{B})$ a symmetric positive definite matrix, where $\Lambda$ is a diagonal matrix and $I$ is the identity matrix. Then

\[
\text{Tr}(B^{-1}A) = \text{Tr}(D_B^{-1}DA) - \text{Tr}(A_1) + \text{Tr}(A_2) + \text{Tr}(A_3),
\]

where

\[
A_1 = D_B^{-1}\tilde{B}D_B^{-1}(\tilde{\Lambda} + D), \quad A_2 = D_B^{-1}D_BD_B^{-1}(\tilde{\Lambda} + D), \quad A_3 = D_B^{-1}\tilde{B}B^{-1}\tilde{B}D_B^{-1}A,
\]

and $D$ is any diagonal matrix.

**Proof of Lemma 5.8** We consider the auxiliary eigenvalue problem (5.6) resulting in

\[
\lambda^{k+1}_i = \tilde{\lambda}^{k+1}_i B\xi^{k+1}_i, \quad (A.3)
\]
where $A = (a(\tilde{u}_j^{k+1}, \tilde{u}_i^{k+1}))_{1 \leq i,j \leq s}$, $B = (b(\tilde{u}_j^{k+1}, \tilde{u}_i^{k+1}))_{1 \leq i,j \leq s}$ and $\xi^{k+1}$ is the coordinate of $\tilde{u}_i^{k+1}$ in the basis $\{\tilde{u}_j^{k+1}\}_{j=1}^s$. Define $z_i^k := Q^k_i(B^k_i)^{-1}r_i^k$, $i = 1, 2, \ldots, s$. Substituting (5.5) into $b(\cdot, \cdot)$ and $a(\cdot, \cdot)$, we have

$$b(\tilde{u}_j^{k+1}, \tilde{u}_i^{k+1}) = \delta_{ij} + \alpha_j^k \alpha_i^kb(z_j^k, z_i^k) =: \delta_{ij} + (\tilde{B})_{ij}, \quad (A.4)$$

and

$$a(\tilde{u}_j^{k+1}, \tilde{u}_i^{k+1}) = \lambda_j^k \delta_{ij} + \alpha_j^k a(u_j^k, z_i^k) + \alpha_j^k a(z_j^k, u_i^k) + \alpha_j^k \alpha_i^k a(z_j^k, z_i^k) =: \lambda_j^k \delta_{ij} + (\tilde{A})_{ij}. \quad (A.5)$$

By (A.4), it is easy to check that $\tilde{B} \geq 0_{s \times s}$ and $B = (I + \tilde{B})$ is symmetric and positive definite. Moreover, by Lemma 4.3, Lemma 4.7 and Remark 5.4, we obtain

$$||D_B||_F \leq ||\tilde{B}||_F \leq \sqrt{s}||\tilde{B}||_2 \leq \sqrt{s}Tr(\tilde{B}) = \sqrt{s} \sum_{i=1}^s (\lambda_i^k)^2 ||z_i^k||_2^2 \leq C \sum_{i=1}^s ||(B^k_i)^{-1}r_i^k||_b^2$$

$$\leq C \sum_{i=1}^s \{C||e_{i,s+a}||_2^2 + CH^2||g_i^k||_2^2\} \leq C \sum_{i=1}^s (\lambda_i^k - Rq(Q^k_s a_i^k)) + CH^4 \sum_{i=1}^s (\lambda_i^k - \lambda_i^k), \quad (A.6)$$

where $|| \cdot ||_F$ and $|| \cdot ||_2$ denote the Frobenius norm and 2-norm of matrix, respectively. Using the same argument as in (A.6), we deduce

$$||D_B||_F^2 = ||\tilde{D}||_F^2 \leq \sum_{i,j=1}^s (\alpha_j^k \alpha_i^k)^2 ||b(z_j^k, z_i^k)||_2^2 \leq C \sum_{i,j=1}^s ||z_i^k||_2^2 ||z_j^k||_2^2$$

$$= C \sum_{i=1}^s ||z_i^k||_2^2 \sum_{j=1}^s ||z_j^k||_2^2 \leq CH^2 \sum_{i=1}^s (\lambda_i^k - Rq(Q^k_s a_i^k)) + CH^2 \sum_{i=1}^s (\lambda_i^k - \lambda_i^k). \quad (A.7)$$

By (A.5), it is easy to check that $A = \Lambda + \tilde{A}$ is a symmetric matrix, where $\Lambda = Diag(\lambda_1^k, \lambda_2^k, \ldots, \lambda_s^k)$. Moreover, by Lemma 4.3, Lemma 4.7 and Remark 5.4, we get

$$||D_A||_F^2 \leq ||\tilde{A}||_F^2 = \sum_{i,j=1}^s \{\alpha_j^k a(u_j^k, z_i^k) + \alpha_j^k a(z_j^k, u_i^k) + \alpha_j^k \alpha_i^k a(z_j^k, z_i^k)\}^2$$

$$\leq C \sum_{i=1}^s ||z_i^k||_a^2 + \sum_{i,j=1}^s ||z_j^k||_a^2 ||z_i^k||_2^2 \leq C(1 + \sum_{j=1}^s ||z_j^k||_a^2) \sum_{i=1}^s ||z_i^k||_2^2 \leq CH^2,$$

and

$$||A||_2 \leq Tr(A) = \sum_{i=1}^s \{\lambda_i^k + 2\alpha_j^k a(u_j^k, z_i^k) + (\alpha_j^k)^2 a(z_j^k, z_i^k)\} \leq \sum_{i=1}^s \{\lambda_i^k + C||z_i^k||_a + C||z_i^k||_2^2\} \leq C \sum_{i=1}^s \lambda_i^k + CH + CH^2 \leq C. \quad (A.8)$$

By (5.6) and (A.3), we deduce

$$\sum_{i=1}^s (\lambda_i^{k+1} - Rq(\tilde{u}_i^{k+1})) = Tr(B^{-1}A) - Tr(D_B^{-1}D_A). \quad (A.9)$$

Using Lemma 4.1, we have

$$Tr(B^{-1}A) - Tr(D_B^{-1}D_A) = -Tr(A_1) + Tr(A_2) + Tr(A_3) \leq |Tr(A_1)| + |Tr(A_2)| + |Tr(A_3)|, \quad (A.10)$$

where

$$A_1 = D_B^{-1}(-\tilde{B})D_B^{-1}(-\tilde{A}) = D_B^{-1}BD_B^{-1}\tilde{A},$$

$$A_2 = D_B^{-1}D_{-B}D_B^{-1}D_{-A} = D_B^{-1}DB_B^{-1}D_A,$$

$$A_3 = D_B^{-1}DB_B^{-1}D_B^{-1}A.$$
We first estimate the term $|\text{Tr}(A_1)|$ in (A.10). Since $D_B (\geq I)$, $D_B^{-1}\hat{B}D_B^{-1}$, $\hat{A}$ and $\hat{B}$ are symmetric, we get

$$|\text{Tr}(A_1)| = |\text{Tr}(D_B^{-1}\hat{B}D_B^{-1}\hat{A})| \leq ||D_B^{-1}\hat{B}D_B^{-1}||_F ||\hat{A}||_F \leq ||\hat{B}||_F ||\hat{A}||_F$$

$$\leq CH \sum_{i=1}^s (\lambda_i^k - Rq(Q_i^h u_i^k)) + CH^5 \sum_{i=1}^s (\lambda_i^k - \lambda_i^h). \quad (A.11)$$

Next, we estimate the term $|\text{Tr}(A_2)|$ in (A.10). As $D_B (\geq I)$, $D_B^{-1}D_B^{-1}$, $D_B^{-1}D_B^{-1}$, $D_B^{-1}$ and $D_B^{-1}$ are symmetric, we obtain

$$|\text{Tr}(A_2)| = |\text{Tr}(D_B^{-1}D_B^{-1}D_B^{-1}D_B^{-1})| \leq ||D_B^{-1}D_B^{-1}D_B^{-1}D_B^{-1}||_F ||D_B^{-1}D_B^{-1}||_F \leq ||D_B^{-1}D_B^{-1}||_F ||D_B^{-1}D_B^{-1}||_F$$

$$\leq CH \sum_{i=1}^s (\lambda_i^k - Rq(Q_i^h u_i^k)) + CH^5 \sum_{i=1}^s (\lambda_i^k - \lambda_i^h). \quad (A.12)$$

Finally, we estimate the term $|\text{Tr}(A_3)|$ in (A.10). By the same argument as in (A.12), we have

$$|\text{Tr}(A_3)| = |\text{Tr}(D_B^{-1}D_B^{-1}D_B^{-1}D_B^{-1}A)| \leq ||D_B^{-1}D_B^{-1}D_B^{-1}D_B^{-1}A||_F \leq ||D_B^{-1}D_B^{-1}D_B^{-1}D_B^{-1}||_F ||A||_F$$

$$\leq \sqrt{s}||D_B^{-1}||_F ||A||_2 \leq CH^2 \sum_{i=1}^s (\lambda_i^k - Rq(Q_i^h u_i^k)) + CH^6 \sum_{i=1}^s (\lambda_i^k - \lambda_i^h),$$

which, together with (A.9), (A.10), (A.11) and (A.12), completes the proof of this lemma. \qed

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