Finite time blow-up results for the damped wave equations with arbitrary initial energy in an inhomogeneous medium

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Abstract
In this paper we consider the long time behavior of solutions of the initial value problem for the damped wave equation of the form

$$u_{tt} - \rho(x)^{-1} \Delta u + u_t + m^2 u = f(u)$$

with some $\rho(x)$ and $f(u)$ on the whole space $\mathbb{R}^n$ ($n \geq 3$).

For the low initial energy case, which is the non-positive initial energy, based on concavity argument we prove the blow up result. As for the high initial energy case, we give out sufficient conditions of the initial datum such that the corresponding solution blows up in finite time.

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Keywords: Wave equations; Blowing up; High initial energy; Damping term; Inhomogeneous medium

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1 Introduction

In this paper our aim is to study a class of wave equations in the following form

\[
\begin{align*}
\begin{cases}
    u_{tt} - \rho(x)^{-1} \Delta u + u_t + m^2 u &= f(u), \quad (t, x) \in [0, T) \times \mathbb{R}^n \\
    u(0, x) &= u_0(x), \quad x \in \mathbb{R}^n \\
    u_t(0, x) &= u_1(x), \quad x \in \mathbb{R}^n
\end{cases}
\end{align*}
\]

(1.1)

where \(\Delta\) is Laplacian operator on \(\mathbb{R}^n\) \((n \geq 3)\), \(u_0(x)\) and \(u_1(x)\) are real valued functions, \(m\) is a real constant (the case, \(m = 0\), is called as the mass free case; \(m \neq 0\) as the mass case), \(\rho(x)\) satisfies the following condition

\((H)\) \(\rho(x) > 0\) for every \(x \in \mathbb{R}^n\), \(\rho \in C^{0, \gamma}(\mathbb{R}^n)\) with \(\gamma \in (0, 1)\), and \(\rho \in L^{n/2}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)\).

The wave equations (1.1) appear in applications in various areas of mathematical physics \([1][21][27]\), as well as in geophysics and ocean acoustics, where, for example, the coefficient \(\rho(x)\) represents the speed of sound at the point \(x \in \mathbb{R}^N\) \([12]\), in other words, \(\rho(x) \neq \text{constant}\) implies that the medium is inhomogeneous, where the sound travels.

For the nonlinear power, throughout the paper we make the following assumption: the nonlinear power \(f(s)\) satisfies that there exists some constant \(\epsilon > 0\) such that

\[f(s)s \geq (2 + \epsilon) F(s)\]

for any \(s \in \mathbb{R}\), where

\[F(\zeta) = \int_0^\zeta f(\kappa) d\kappa.\]

(1.3)

For such nonlinear power, it was first stated for abstract wave equations with \(\rho(x) = 1\) by Levine \([14]\). And then Cazenave \([4]\) also considered it for Klein-Gordon equations.

Before going any further, we first briefly introduce some research works for the wave equation (1.1) with \(\rho(x) = \text{constant} \neq 0\) (without loss of generality let \(\rho(x) = 1\)), obviously it does not satisfy the assumption \((H)\). For the general nonlinear power \(f(u)\) with \((1.2)\) it was firstly considered for some abstract wave equations in \([11]\), where Levine proved the blow up result when the initial energy was negative. But mostly the results about Cauchy problem for the wave equation were investigated the typical form of nonlinear power as

\[f(u) = |u|^{p-1}u\]

(1.4)

where \(1 < p < \frac{n+2}{n-2}\). Here we note that the above power satisfies the condition \((1.2)\). For the power \((1.4)\), the wave equations with damping term were studied by many authors. It is well known that the local solution blows up in finite time when the initial energy is negative. For global existence and nonexistence of solutions for Cauchy problem of the equation (1.1) with \(\rho(x) = 1\), \((1.4)\) and (possibly nonlinear) damping term, we here refer to \([6][7][8][15][16][18][20][25]\).

In special, recently the wave equation with damping term was considered in \([17]\), where Levine and Todorova showed that for arbitrarily positive initial energy there are choices of initial datum such that the local solution blows up in finite time. Subsequent Todorova and Vitillaro \([24]\) established more precise result..
about the existence of initial values such that the corresponding solution blows up in finite time for arbitrarily high initial energy. More recently, Gazzola and Squassina [5] established sufficient conditions of initial datum with arbitrarily positive initial energy such that the corresponding solution blows up in finite time for the wave equation with linear damping and (1.4) in the mass free case on a bounded Lipschitz subset of \( \mathbb{R}^n \). And the author [26] establish the blow up result with arbitrarily positive initial energy for some Klein-Gordon equations on the whole space \( \mathbb{R}^n \).

For the case \( \rho(x) = 1 \) and \( m = 0 \), we also note that it was considered that \( f(u) = |u|^p \) by some authors. Here we just refer to the papers [23][9] and some papers cited therein.

Now we return to the equation (1.1) with some general \( \rho(x) \). For the linear case, \( f(u) = 0 \), Eidus [3] first studied the existence of solutions for linear wave equation (1.1). Then Karachalios and Stavrakakis [10] studied the existence of the solution of the damped wave equation (1.1) with some nonlinear power. And they [11] also established the results about the global existence and blow up of solutions for the equation (1.1) with (H) and (1.4) in the free mass case by potential wall method, which was firstly developed by Sattinger [22]. Their blow up result was under the condition that the initial energy was negative. Recently, for the equation (1.1) with (H) and (1.4) Zhou [28] investigated the global existence and blow up result including the mass free case and mass case. Zhou established the blow up result when the initial energy was less than a positive constant, which dependent on the function \( \rho(x) \) (H). But in all the above works, the high initial energy case was not considered for the equation (1.1) under the assumption (H). Moreover, there is no one who consider the equation (1.1) with some general nonlinear power, for example, like (1.2).

The main purpose of this paper is to establish the blow up result for the equation (1.1) with (H) and (1.2) when the initial energy is high. Based on a concavity argument, which was originally introduced by Levine [14][15], we first establish the blow up result when the initial energy is non-positive. Thus we extend the blow up results [11] and [28]. Note that, when the initial energy vanishes, the blow up result was also established in [28], where it needed the assumption \( \rho(x) \in L^1(\mathbb{R}^n) \) with (H) and \( \int \rho(x) u_0(x) u_1(x) dx \geq 0 \). Here in some sense (See Remark 2.4) we improve the blow up result in [28]. As for the arbitrarily positive initial energy case, we establish the blow up result under some conditions of \( (u_0, u_1) \) on the whole space \( \mathbb{R}^n \). To the best of our knowledge this is the first blow up result with high initial energy for the equation (1.1) with (H). The work is motivated by [5]. But Our proof is different with [5], and very simple. Additionally, our proof is also valid for the case \( \rho(x) = \text{constant} \neq 0 \). So at last we also make some remarks on the case \( \rho(x) = \text{constant} \neq 0 \). We find that, if \( \rho(x) \) satisfies (H) then the mass \( m \) does not affect the blow up result, but if \( \rho = \text{constant} \neq 0 \) it will affect the blow up result, that is, if \( m = 0 \) and \( \rho(x) = 1 \) then the blow up result is obtained only on a bounded subset of \( \mathbb{R}^n \). Indeed, by (H) we see that \( \rho(x) \) will rapidly enough decreasing at infinity, thus it make us possibly consider the equation (1.1) on the whole space \( \mathbb{R}^n \) in the mass free case.

The paper is composed of three sections. In the next section we will denote some notations, and state our main results. The last section is the main part, we prove the blow up results there. In addition, some remakes are made on the case of \( \rho(x) = 1 \).
2 Principal Result

In order to state our main results, we briefly mention here some facts, notations and known results. We denote by \( \| \cdot \|_q \) the \( L^q(\mathbb{R}^n) \) norm for \( 1 \leq q \leq \infty \), and we define the spaces: \( H^1(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n); \| u \|_{H^1(\mathbb{R}^n)} = \|(1-\Delta)^{1/2}u\| < \infty \} \), and \( H^1_0(\mathbb{R}^n) = \{ u \in H^1(\mathbb{R}^n); \text{supp}(u) \text{ is compact in } \mathbb{R}^n \} \). For simplicity we will denote \( f_{\mathbb{R}^n} \) by \( f \). The notation \( t \to T^- \) means \( t < T \) and \( t \to T \).

As [13], we introduce the function space \( X^{1,2}(\mathbb{R}^n) \), which is defined as the closure of \( C^\infty_0 \) functions with respect to the energy norm \( \| u \|_{X^{1,2}} := \int |\nabla u(x)|^2 dx \), that is,

\[
X^{1,2}(\mathbb{R}^n) = \{ u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) : \nabla u \in (L^2(\mathbb{R}^n))^n \}. \tag{2.1}
\]

And it is known that \( X^{1,2} \) is embedding continuously in \( L^{\frac{2n}{n-2}} \), which means that there exists a constant \( k > 0 \) such that

\[
\| u \|_{X^{1,2}} \leq k\| u \|_{X^{1,2}}. \tag{2.2}
\]

Following (2.2), we have the following inequality [2].

**Lemma 2.1** Suppose \( \rho \in L^p(\mathbb{R}^n) \) and \( n \geq 3 \). Then there exists a constant \( \alpha > 0 \) such that

\[
\int |\nabla u(x)|^2 dx \geq \alpha \int \rho(x)|u(x)|^2 dx \tag{2.3}
\]

for every \( u \in C^\infty_0(\mathbb{R}^n) \). Moreover, \( \alpha = k^{-2}\| \rho \|^{-1}_{\frac{n}{2}} \) where \( k \) is defined in (2.2).

In addition, the weighted space \( L^2_{\rho}(\mathbb{R}^n) \) is defined to be the closure of \( C^\infty_0(\mathbb{R}^n) \) functions with respect to the inner product

\[
(u, v)_{L^2_{\rho}} := \int \rho(x)u(x)v(x)dx, \tag{2.4}
\]

and norm

\[
\| u \|_{L^2_{\rho}} = (u, u)_{L^2_{\rho}}. \tag{2.5}
\]

For local existence of solutions of the equation (1.1), we state

**Theorem 2.2** Under the assumption (H). Let the initial datum \( (u_0, u_1) \in X^{1,2}(\mathbb{R}^n) \times L^2_{\rho}(\mathbb{R}^n) \), and \( f \) satisfying the following conditions: \( f(0) = 0 \) and

\[
|f(\lambda_1) - f(\lambda_2)| \leq c(|\lambda_1|^{p-1} + |\lambda_2|^{p-1})|\lambda_1 - \lambda_2| \tag{2.6}
\]

for all \( \lambda_1, \lambda_2 \in \mathbb{R} \), some constant \( c > 0 \), and

\[
1 < p < \frac{n}{n-2} \text{ when } n \geq 3. \tag{2.7}
\]

Then there exists a unique local solution \( u(t, x) \) of the equation (1.1) on a maximal time interval \([0, T_{\max})\) satisfying

\[
u \in C([0, T_{\max}); X^{1,2}(\mathbb{R}^n)) \text{ and } u_t \in C([0, T_{\max}); L^2_{\rho}), \tag{2.8}\]

\[
u(0, x) = u_0(x) \text{ and } u_t(0, x) = u_1(x). \tag{2.9}\]
In addition, \( u(t, x) \) satisfies

\[
E(0) - E(t) = \int_0^t \|u_\tau(x, \cdot)\|^2_{L^2} d\tau \geq 0 \tag{2.10}
\]

every \( t \in [0, T_{\text{max}}) \), where

\[
E(t) = \frac{1}{2} \int (\rho(x)|u(t, x)|^2 + |\nabla u(t, x)|^2 + m^2\rho(x)|u(t, x)|^2 - 2\rho(x)F(u(t, x))) dx,
\]

where \( F(u) \) is defined in (1.3).

This result can be proved by Banach fixed point theorem. The proof follows from the weighted-norm Lebesgue space of the corresponding theorem for the wave equations of Kirchhoff type [19].

If \( T_{\text{max}} < \infty \), then the local solution is said to blow up in finite time \( T_{\text{max}} \). Otherwise, \( T_{\text{max}} = \infty \), the corresponding local solution is global.

Next we state our first blow up result for the equation (1.1) with (H) and (1.2) in the non-positive initial energy case.

**Theorem 2.3** Under the assumptions (H) and (1.2). If the nonzero initial datum \((u_0, u_1) \in X_{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) satisfies

\[
E(0) < 0, \tag{2.12}
\]

or

\[
\int \rho(x)u_0(x)u_1(x) dx \geq 0 \text{ if } E(0) = 0, \tag{2.13}
\]

then the corresponding local solution of the equation (1.1) blows up in finite time \( T_{\text{max}} < \infty \), that is,

\[
\lim_{t \to T_{\text{max}}} \int \rho(x)|u(t, x)|^2 dx = \infty. \tag{2.14}
\]

**Remark 2.4** In the case \( E(0) = 0 \), Zhou [28] also established the blow up result for the equation (1.1) with (H) and (1.4) under some another assumptions as \( \int \rho(x)u_0(x)u_1(x) dx \geq 0 \) and \( \rho \in L^1(\mathbb{R}^n) \). But in the above theorem, we remove \( \rho \in L^1(\mathbb{R}^n) \). Thus, in this sense we improve the result [28].

To state our main blow up result for the arbitrarily positive initial energy case, we introduce a function as follows

\[
I(u) = \int (|\nabla u(x)|^2 + m^2\rho(x)|u(x)| - \rho(x)f(u(x))u(x)) dx. \tag{2.15}
\]

Now we introduce our main blow up result for the equation (1.1) in the arbitrarily positive initial energy case, as far as we know, which is the first blow up result for the equation (1.1) with (H) on the whole space \( \mathbb{R}^n \).
Theorem 2.5 Under the assumptions (H) and (1.2). If the initial datum 
\((u_0, u_1) \in X^{1,2}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)\) satisfies

\[ E(0) > 0, \]
\[ I(u_0) < 0, \]  
\[ \int \rho(x) u_0(x) u_1(x) dx \geq 0, \]  
\[ \|u_0\|_{L^2_\rho}^2 > \frac{2(2+\epsilon)}{m^2 \epsilon} E(0) \text{ when } m \neq 0, \]  
\[ \|u_0\|_{L^2_\rho}^2 > \frac{2(2+\epsilon)}{\min\{1, \alpha\} \epsilon} E(0) \text{ when } m = 0, \]

where \(\epsilon\) and \(\alpha\) are stated in (1.2) and Lemma 2.1, respectively. Then the corresponding local solution of the equation (1.1) blows up in finite time \(T_{\max} < \infty\), that is,

\[ \lim_{t \to T_{\max}} \int \rho(x) |u(t, x)|^2 dx = \infty. \]

Remark 2.6 We note that, for the case \(E(0) < 0\), by (2.11) and (2.10) it is valid that \(I(u(t, \cdot)) < 0\) for every \(t \in [0, T_{\max})\).

Reading Theorem 2.3, 2.5 and Remark 2.6, naturally one considers how about the local solution when the initial data satisfies \(E(0) > 0\) and \(I(u_0) > 0\). Indeed, for this case, being similar as the argument with \(m = 0\) and \(f(u) = |u|^{p-1}u\) [11], by a potential wall method we can also obtain the global existence of solutions of the equation (1.1) with (H) and (1.2) when the positive initial energy is small enough. Here we omit it. Furthermore, it is still open that whether there exists a global solution for wave equations when the initial energy is arbitrarily high.

3 Proof of the main theorems

In this section, we prove Theorem 2.3 and 2.5 based on concavity argument. Next we first claim two lemmas. The following lemma is basic.

**Lemma 3.1** Let \(T > 0\) and \(H(t)\) be a Lipschitzian function over \([0, T)\). Assume that \(H(0) \geq 0\) and

\[ \frac{d}{dt} H(t) + H(t) > 0 \]  
for every \(t \in [0, T)\). Then \(H(t) > 0\) for every \(t \in (0, T)\).

Following the way [5], by Lemma 3.1 we can obtain next lemma. For the convenient of readers and completeness of the paper we here still give out a proof.

**Lemma 3.2** Assume that \((u_0, u_1) \in X^{1,2}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)\) satisfies

\[ \int \rho(x) u_0(x) u_1(x) dx \geq 0. \]  

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If the corresponding local solution \((u(t, x), u_t(t, x)) \in C([0, T_{\text{max}}], X^{1, 2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n))\) is such that
\[
I(u(t, \cdot)) < 0
\] (3.3)
for every \(t \in [0, T_{\text{max}}]\), then \(\|u(t, \cdot)\|_{L^2}^2\) is strictly increasing on \([0, T_{\text{max}}]\).

**Proof.** Since \(u(t, x)\) is the local solution of the equation (1.1), then we easily have
\[
\frac{1}{2} \frac{d^2}{dt^2} \int \rho(x)|u(t, x)|^2 dx = \int \rho(x)|u_t(t, x)|^2 dx + \int \rho(x)u_t(t, x)u_t(t, x) - \int (m^2 \rho(x)|u(x)|^2 + |\nabla u(t, x)|^2 - \rho(x)u(t, x)f(u(t, x))) d^2
\] (3.4)
for every \(t \in [0, T_{\text{max}}]\).

As a result, it follows
\[
\frac{d^2}{dt^2} \int \rho(x)|u(t, x)|^2 dx + \frac{d}{dt} \int \rho(x)|u(t, x)|^2 dx = 2(\|u(t, x)\|_{L^2}^2 - I(u(t, \cdot)))
\] (3.5)
By (3.3) and the above equation we have
\[
\frac{d^2}{dt^2} \int \rho(x)|u(t, x)|^2 dx + \frac{d}{dt} \int \rho(x)|u(t, x)|^2 dx > 0
\] (3.5)
for every \(t \in [0, T_{\text{max}}]\).

Here we let \(H(t) = \frac{d}{dt} \int \rho(x)|u(t, x)|^2 dx\), then as [11] we see that the function \(H(t)\) is Lipschitzian function over \([0, T_{\text{max}}]\). Thus, from Lemma 3.1 and (3.5) it follows that \(\|u(t, x)\|_{L^2}^2\) is strictly increasing on \([0, T_{\text{max}}]\).

**Proof of Theorem 2.3** We first define the following auxiliary function
\[
G(t) = \int \rho(x)|u(t, x)|^2 dx + \int_0^t \|u(\tau, \cdot)\|_{L^2}^2 d\tau + (T_0 - t)\|u_0\|_{L^2}^2 + \zeta(T_1 + t)^2
\] (3.6)
where the constants, \(T_0 > 0, T_1 > 0, \zeta > 0\), will be determined later.

We then have
\[
G'(t) = \frac{d}{dt} G(t)
\]
\[
= 2 \int \rho(x)u_t(t, x)u_t(t, \cdot) d\tau + \|u(t, \cdot)\|_{L^2}^2 - \|u_0\|_{L^2}^2 + 2\zeta(T_1 + t)
\]
\[
= 2 \int \rho(x)u_t(t, x)u_t(t, \cdot) d\tau + 2 \int_0^t (u(\tau, \cdot), u_t(\tau, \cdot))_{L^2} d\tau + 2\zeta(T_1 + t)
\] (3.7)
and
\[
\frac{1}{2} G''(t) = \int \rho(x)|u_t(t, x)|^2 dx + \int \rho(x)u(t, x)u_{tt}(t, x) d\tau + \int \rho(x)u(t, x)u_t(t, x) d\tau + \zeta
\]
\[
= \int \rho(x)|u_t(t, x)|^2 dx + \int \rho(x)u(t, x)f(u(t, x)) dx - \int (|\nabla u(t, x)|^2 + m^2 \rho(x)|u(t, x)|^2) dx + \zeta
\] (3.8)
**Case I:** $E(0) < 0$. By (1.2), (2.10) and (2.11), we see that

$$
\int \rho(x)F(u(t,x))dx = \frac{1}{2} \int \left( m^2 \rho(x)|u(t,x)|^2 + \rho(x)|u_t(t,x)|^2 + |\nabla u(t,x)|^2 \right) dx
-E(0) + \int_0^t \|u_\tau(\tau,x)\|_{L^2}^2 d\tau
\leq \frac{1}{2} \int \rho(x)u(t,x)f(u(t,x))dx
$$

for all $t \in [0,T_{\text{max}})$.

Thus, from (3.8) it follows that

$$
G''(t) \geq (4 + \epsilon) \int \rho(x)u_t(t,x)^2 dx + 2(2 + \epsilon) \int_0^t \|u_\tau(\tau,x)\|_{L^2}^2 d\tau
+ \epsilon \int (|\nabla u(t,x)|^2 + m^2 \rho(x)|u(t,x)|^2) dx - 2(2 + \epsilon)E(0) + 2\zeta.
$$

We now let the constant $\zeta$ satisfy

$$
0 < \zeta \leq -2E(0).
$$

Then it follows from (3.11)

$$
-2(2 + \epsilon)E(0) + 2\zeta \geq (4 + \epsilon)\zeta.
$$

Obviously, $G''(t) > 0$ on $[0,T_{\text{max}})$. Moreover, we can take $T_1 > 0$ large sufficiently such that

$$
G'(0) = 2 \int \rho(x)u_0(x)u_1(x) + 2BT_1 > 0,
$$

and

$$
\frac{\epsilon}{2} \left( \int \rho(x)u_0(x)u_1(x)dx + \zeta T_1 \right) > \int \rho(x)u_0(x)^2 dx.
$$

Thus, by (3.13) we see that $G(t) > 0$, $G'(t) > 0$ and $G''(t) > 0$ for every $t \in [0,T_{\text{max}})$. That is, $G(t)$ and $G'(t)$ is strictly increasing on $[0,T_{\text{max}})$. Then we let

$$
A = \int \rho(x)|u(t,x)|^2 dx + \int_0^t \|u(\tau,\cdot)\|_{L^2}^2 d\tau + \zeta(T_1 + t)^2
$$

$$
B = \frac{1}{2} G'(t)
$$

$$
C = \int \rho(x)|u_t(t,x)|^2 dx + \int_0^t \|u_\tau(\tau,\cdot)\|_{L^2}^2 d\tau + \zeta.
$$

For every $t \in [0,T_0]$ we obviously have

$$
G(t) \geq A,
$$

and by (3.10) and (3.12)

$$
G''(t) \geq (4 + \epsilon)C.
$$
We now let $T_0$ sufficiently large and satisfy
\[ T_0 \geq \frac{4G(0)}{\epsilon G'(0)}. \]  
(3.20)
Noting the inequalities \(3.6\), \(3.7\) and \(3.14\), we see that the definition of $T_0$ as \(3.20\) is reasonable.

And suppose the solution $u(t, x)$ exists on $[0, T_0]$. Then it follows that
\[ G''G(t) - \frac{4 + \epsilon}{4} (G'(t))^2 \geq (4 + \epsilon)(AC - B^2) \]  
(3.21)
for every $t \in [0, T_0]$. By a simple computation we see that
\[ A\delta^2 - 2B\delta + C = \int \rho(x)(su(t, x) + u_t(t, x))^2 \, dx \]
\[ + \int_0^t \|su(\tau, \cdot) + u_\tau(\tau, \cdot)\|_{L^2}^2 \, d\tau + \zeta(s(T_1 + t) + 1)^2 \]
\[ \geq 0 \]  
(3.22)
for every $s \in \mathbb{R}$ and $t \in [0, T_0]$, which means that $(2B)^2 - 4AC \leq 0$.

Thus we see that
\[ G''G(t) - \frac{4 + \epsilon}{4} (G'(t))^2 \geq 0 \]  
(3.23)
for every $t \in [0, T_0]$. Since $\frac{4 + \epsilon}{4} > 1$, we put $\alpha = \frac{\epsilon}{4}$. Then we have
\[ \frac{d}{dt} G^{-\alpha}(t) = -\alpha G^{-\alpha-1}G'(t) < 0 \]  
(3.24)
\[ \frac{d^2}{dt^2} G^{-\alpha}(t) = -\alpha G^{-\alpha-2} \left[ G''G(t) - \frac{4 + \epsilon}{4} (G'(t))^2 \right] \leq 0 \]  
(3.25)
for every $t \in [0, T_0]$, which means that the function $G^{-\alpha}$ is concave. Obviously $G(0) > 0$, then from \(3.25\) it follows that the function $G^{-\alpha} \to 0$ when $t < T_{\text{max}}$ and $t \to T_{\text{max}}$. Noting the assumption that the solution exists on $[0, T_0]$, where $T_0$ is defined as \(3.20\), thus we see that there exists a finite time $T_{\text{max}} > 0$ such that
\[ \lim_{t \to T_{\text{max}}} \|u(t, \cdot)\|_{L^2}^2 = \infty, \]  
(3.26)
which implies that the corresponding solution $u(t, x)$ of the equation \(1.1\) blows up in finite time $T_{\text{max}} < \infty$.

**Case II:** $E(0) = 0$ and $\int \rho(x)u_0(x)u_1(x) \, dx \geq 0$. By \(2.10\) and \(2.11\) we have
\[ \int (|\nabla u(t, x)|^2 + m^2 \rho(x)|u(t, x)|^2) \, dx - 2 \int \rho(x)F(u(t, x)) \, dx \leq 0 \]  
(3.27)
for every $t \in [0, T_{\text{max}})$.

And noting the fact that $\int \rho(x)F(u(t,x))dx \neq 0$ we obtain by (1.2)

$$2 \int \rho(x)F(u(t,x))dx < \int \rho(x)f(u(t,x))u(t,x)dx.$$  \hspace{1cm} (3.28)

We then get

$$I(u(t,x)) < 0$$  \hspace{1cm} (3.29)

for every $t \in [0, T_{\text{max}})$.

Thus by (2.13) and Lemma 3.2 we see that $\|u(t,\cdot)\|_{L^2}^2$ is strictly increasing on $[0, T_{\text{max}})$.

In this case we still use the auxiliary function $G(t)$ as (3.6).

Thus, according to the proof of Case I, by (2.13) we see that $G'(t) > 0$, $G''(t) > 0$ on $(0, T_{\text{max}})$, that is to say, $G(t)$ and $G'(t)$ is strictly increasing over $[0, T_{\text{max}})$.

And as (3.10) we also have

$$G''(t) \geq (4 + \epsilon) \int \rho(x)|u_t(t,x)|^2dx + 2(2 + \epsilon) \int_0^t \|u(\tau,\cdot)\|_{L^2}^2 d\tau$$

$$+ \epsilon \left( \int |\nabla u(t,x)|^2 + m^2 \rho(x)|u(t,x)|^2 \right) dx + 2\zeta$$

$$\geq (4 + \epsilon) \int \rho(x)|u_t(t,x)|^2dx + 2(2 + \epsilon) \int_0^t \|u(\tau,\cdot)\|_{L^2}^2 d\tau$$

$$+ \epsilon(m^2 + \alpha) \int \rho(x)|u_0(x)|^2dx + 2\zeta,$$ \hspace{1cm} (3.30)

where the last inequality comes from Lemma 2.1 and Lemma 3.2.

Now we let the constant $\zeta$ satisfy

$$0 < \zeta \leq \frac{\epsilon}{2 + \epsilon} \left( \alpha + m^2 \right) \|u_0\|_{L^2}^2$$ \hspace{1cm} (3.31)

for the mass free case or the mass case, and the other positive constants, $T_0$ and $T_1$, be large such that

$$T_0 \geq \frac{4G(0)}{\epsilon G'(0)}$$ \hspace{1cm} (3.32)

$$\frac{\epsilon}{2} \left( \int \rho(x)u_0(x)u_1(x)dx + \zeta T_1 \right) > \int \rho(x)|u_0(x)|^2dx.$$ \hspace{1cm} (3.33)

Then by the same argument as Case I, we can claim that the corresponding local solution of the equation (3.1) blows up in finite time.

Thus the proof of Theorem 2.3 is completed.

\[\square\]

In the following part we will process Theorem 2.5. Next lemma is the crux to prove Theorem 2.5.
Lemma 3.3 Under the assumptions on $\rho(x)$, $f(u)$ and $(u_0, u_1)$ in Theorem 2.6, then the corresponding local solution $(u(t, x), u_t(t, x)) \in C([0, T_{max}), X^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$ satisfies

$$I(u(t, \cdot)) < 0, \quad \text{(3.34)}$$

$$\|u(t, \cdot)\|^2_{L^2/\rho} > \frac{2(2 + \epsilon)}{m^2 \epsilon} E(0) \text{ when } m \neq 0, \quad \text{(3.35)}$$

$$\|u(t, \cdot)\|^2_{L^2/\rho} > \frac{2(2 + \epsilon)}{\min\{1, \alpha\} \epsilon} E(0) \text{ when } m = 0, \quad \text{(3.36)}$$

for every $t \in [0, T_{max})$.

**Proof.** Here the proof is by a contradiction argument. We assume that (3.34) is not true over $[0, T_{max})$, that is, there exists a time $T > 0$ such that

$$T = \min\{t \in (0, T_{max}); I(u(t, \cdot)) = 0\}. \quad \text{(3.37)}$$

**Case I:** $m \neq 0$. Since $I(u(t, \cdot)) < 0$ on $[0, T)$, by Lemma 3.2 we see that $\|u(t, \cdot)\|^2_{L^2/\rho}$ is strictly increasing over $[0, T)$, which implies that

$$\|u(t, \cdot)\|^2_{L^2/\rho} > \|u_0\|^2_{L^2/\rho} > \frac{2(2 + \epsilon)}{m^2 \epsilon} E(0), \quad \text{(3.38)}$$

for $m \neq 0$ and every $t \in (0, T)$.

And by the continuity of $\|u(t, \cdot)\|^2_{L^2/\rho}$ at $t$, we see that

$$\|u(T, \cdot)\|^2_{L^2/\rho} > \frac{2(2 + \epsilon)}{m^2 \epsilon} E(0). \quad \text{(3.39)}$$

On the other hand, by (2.10) and (2.11) we see that

$$m^2 \|u(T, \cdot)\|^2_{L^2/\rho} + \|\nabla u(T, \cdot)\|^2 - 2 \int \rho(x) F(u(T, x)) dx \leq 2 E(T) \leq 2 E(0). \quad \text{(3.40)}$$

Moreover, noting the assumption $I(u(T, \cdot)) = 0$ and (1.2), we then have

$$\|u(T, \cdot)\|^2_{L^2/\rho} + \|\nabla u(T, \cdot)\|^2 \geq (2 + \epsilon) \int \rho(x) F(u(T, x)) dx. \quad \text{(3.41)}$$

Combining (3.40) and (3.41) we then obtain

$$m^2 \|u(T, \cdot)\|^2_{L^2/\rho} + \|\nabla u(T, \cdot)\|^2 \leq \frac{2(2 + \epsilon)}{\epsilon} E(0). \quad \text{(3.42)}$$

Obviously there is a contradiction between (3.39) and (3.42). Thus we have proved that

$$I(u(t, \cdot)) < 0. \quad \text{(3.43)}$$

for every $t \in [0, T_{max})$.

By Lemma 3.2 we see that $\|u(t, \cdot)\|^2_{L^2/\rho}$ is strictly increasing on $t$ if $I(u(t, \cdot)) < 0$ for every $t \in [0, T_{max})$. Thus, (3.43) implies that

$$\|u(t, \cdot)\|^2_{L^2/\rho} > \frac{2(2 + \epsilon)}{\epsilon} E(0) \quad \text{(3.44)}$$

for every $t \in [0, T_{max})$.\[11\]
for every $t \in [0, T_{\text{max}})$.

Hereunto the proof for Case I, $m \neq 0$, is accomplished.

**Case II: $m = 0$.** As the argument for (3.39), we can also obtain

$$\|u(T, \cdot)\|_{L^2}^2 > \frac{2(2 + \epsilon)}{\min\{1, \alpha\}} E(0).$$

(3.45)

Since $m = 0$, then the inequality (3.42) is rewritten as

$$\|\nabla u(T, \cdot)\|^2 \leq \frac{2(2 + \epsilon)}{\epsilon} E(0).$$

(3.46)

By Lemma 2.1, we see that

$$\alpha \|u(t, x)\|_{L^2}^2 \leq \frac{2(2 + \epsilon)}{\epsilon} E(0).$$

(3.47)

Thus by (3.45) and (3.47) we obtain that the assumption, $I(u(T, \cdot)) = 0$, is wrong. That is to say, it is valid that $I(u(t, \cdot)) < 0$ for every $t \in [0, T_{\text{max}})$.

Similarly, we also get

$$\|u(t, \cdot)\|_{L^2}^2 > \frac{2(2 + \epsilon)}{\min\{1, \alpha\}} E(0)$$

(3.48)

for every $t \in [0, T_{\text{max}})$.

Thus all the proof of Lemma 2.3 has been completed.

$\square$

**Proof of Theorem 2.5.** Here we still use the auxiliary function $G$, defined as (3.6). We have

$$G''(t) = 2 \int \rho(x)|u_t(t, x)|dx + 2 \int \rho(x)f(u(t, x))u(t, x, dx)$$

$$- 2 \int (|\nabla u(t, x)|^2 + m^2 \rho(x)|u(t, x)|^2)dx + 2\zeta$$

$$= 2 \int \rho(x)|u_t(t, x)|dx - 2I(u(t, \cdot)) + 2\zeta$$

(3.49)

By Lemma 3.3, we see that

$$G''(t) > 0$$

(3.50)

for every $t \in [0, T_{\text{max}})$.

And from (2.13) we see that $G'(t) > 0$ for every $t \in (0, T_{\text{max}})$. Thus, it comes that $G(t)$ and $G'(t)$ is strictly increasing on $[0, T_{\text{max}})$.

Obviously the inequality (3.9) is also valid here for every $t \in [0, T_{\text{max}})$. Then by Lemma 2.1, Lemma 3.2, (3.35), (3.36) and (3.49) we have

$$G''(t) \geq (4 + \epsilon) \int \rho(x)|u_t(t, x)|^2dx + 2(2 + \epsilon) \int_0^t \|u_r(\tau, \cdot)|_{L^2}^2 d\tau$$

$$+ \epsilon \int (|\nabla u(t, x)|^2 + m^2 \rho(x)|u(t, x)|^2)dx - 2(2 + \epsilon)E(0) + 2\zeta$$

$$\geq (4 + \epsilon) \int \rho(x)|u_t(t, x)|^2dx + 2(2 + \epsilon) \int_0^t \|u_r(\tau, \cdot)|_{L^2}^2 d\tau$$

$$+ \epsilon \int (m^2 + \alpha) \rho(x)|u_0(x)|^2dx - 2(2 + \epsilon)E(0) + 2\zeta$$

(3.51)
for every $t \in [0, T_{\text{max}}]$.

By (2.19) and (2.20) we see that
\[
\frac{\epsilon(m^2 + \alpha)}{2 + \epsilon} \int \rho(x)|u_0(x)|^2 dx - 2E(0) > 0,
\]
for every $m \in \mathbb{R}$.

We now let $\zeta$ satisfy
\[
0 < \zeta \leq \frac{\epsilon(m^2 + \alpha)}{2 + \epsilon} \int \rho(x)|u_0(x)|^2 dx - 2E(0)
\]
for every $m \in \mathbb{R}$, the other positive constants, $T_0$, and $T_1$, be large such that
\[
T_0 \geq \frac{4G(0)}{\epsilon G'(0)}.
\]
\[
\frac{\epsilon}{2} \left( \int \rho(x)u_0(x)u_1(x)dx + \zeta T_1 \right) > \int \rho(x)|u_0(x)|^2 dx.
\]

We next let $A, B, C$ denote the same terms as (3.15), (3.16) and (3.17), respectively.

And assume that the solution $u(t, x)$ exists on $[0, T_0]$. Then we have
\[
G(t) \geq A
\]
and
\[
G''(t) \geq (4 + \epsilon)C
\]
for every $t \in [0, T_0]$.

Thus by the same way as Theorem 2.3 it comes that
\[
G(t)G''(t) - \frac{4 + \epsilon}{4} (G'(t))^2 \geq (4 + \epsilon)(AC - B^2) \geq 0
\]

As the proof of Theorem 2.3, by a concavity argument we can also obtain that, there exists finite time $T_{\text{max}} < \infty$ such that
\[
\lim_{t \to T_{\text{max}}} \left\| u(t, \cdot) \right\|_{L^2}^2 = \infty.
\]
which implies the corresponding solution $u(t, x)$ of the equation (1.1) blows up in finite time $T_{\text{max}} < \infty$.

\[\square\]

**Remark 3.4** Reading the proof of Theorem 2.5, we can easily use a similar way to the case $\rho(x) = 1$ and $m \neq 0$ on the function space $H_0^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

**Remark 3.5** In the mass free case, we see that in the proof of Theorem 2.5 it is necessary to use Lemma 2.1, which may be called as a general Poincaré inequality. But it is well-known that Poincaré inequality is valid on a bounded set. That is to say, In the mass free case with $\rho(x) = 1$ we cannot obtain a blow up result as Theorem 2.5 on the whole space $\mathbb{R}^n$ when the initial energy is high. For this case, Gazzola and Squassina [5] have established the blow up result with arbitrarily positive initial energy on a bounded Lipschitz subset of $\mathbb{R}^n$. 

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