QUANTUM EINSTEIN GRAVITY AS A TOPOLOGICAL FIELD THEORY

Andrew Toon\textsuperscript{1}
University of Oxford
Department of Theoretical Physics
1 Keble Road
Oxford, OX1 3NP
England.

Abstract

General covariance in quantum gravity is seen once one integrates over all possible metrics. In recent years topological field theories have given us a different route to general covariance without integrating over all possible metrics. Here we argue that Einstein quantum gravity may be viewed of as a topological field theory as long as a certain constraint from the path integral measure is satisfied.

\textsuperscript{1}Supported by the Royal Society England.
1 Introduction

Topological quantum field theories first made their impact by the way they reproduce topological invariants of certain manifolds [1,2]. From a physical point of view it is hoped that these theories may have something to do with quantum gravity. That is, one imagines that the initial phase of the Universe is in an unbroken phase where there is no space-time metric. The Universe we see today would then correspond to the so-called broken phase where, via some mechanism, the initial unbroken phase develops some metric dependence there by generating a space-time structure where physics can operate.

Something along these lines is witnessed in 2+1 dimensions where 2+1 dimensional quantum gravity can be given a gauge theoretic interpretation in terms of a Chern-Simons theory with gauge group $ISO(2,1)$, $SO(3,1)$ or $SO(2,2)$, depending on whether the cosmological constant $\lambda$ is zero, positive or negative [3,4]. This theory has two natural phases depending on whether a certain operator has zero modes or not and it is argued that each of these phases corresponds to the unbroken and broken phases of gravity [4,5].

Unfortunately, if one tries to give a similar analysis to 3+1 dimensional quantum gravity, one soon fails for the simple reason that quantum gravity in 3+1 dimensions has no natural gauge theoretic interpretation like its 2+1 dimensional counterpart. There do exist various 3+1 dimensional topological field theories but at present they seem to have nothing to do with quantum gravity [6].

Rather then to try and make contact with gravity starting from a topological field theory we try here to formally start with quantum Einstein gravity and attempt to make contact with topological field theory. That is, we argue that Einstein quantum gravity can formally be given an interpretation as a topological field theory.

This should come as no great surprise since in the path integral approach to quantum gravity one integrates over all possible metrics, in a given topology, and so one expects to extract only topological information. But exactly what topology does one integrate all possible metrics over and how does one formally do this in a quantum theory of gravity?

To answer this question we turn first of all to topological field theories. Here we learn what needs to be done for the case of quantum gravity in order that the theory as an interpretation as a topological field theory. The crucial observation being the role played by the so-called gauge fixing metric. We then turn to the gauge fixing procedure of Einstein quantum gravity and argue that this can be given a formal structure that mimics the proof of topological invariance in topological quantum

---

2 Strictly speaking a topological field theory is only an approximation to a quantum theory of gravity since we expect to sum over all possible topologies.
field theories based on BRST invariance. We then worry about the path integral measure which must be independent of the gauge fixing metric for the proof of topological invariance to be legal. From this we derive a constraint on the particle content of a theory for topological invariance to be preserved in the quantum theory. We finally finish with a conclusion.

2 Topological Field Theories

In this section we will review the relevant facts of topological field theories in order to discover what is needed in the case of quantum gravity for it to have an interpretation as a topological field theory. As we will see, the crucial point is in the gauge fixing. It is this procedure that really tells us what we mean by a topological field theory.

Consider then an $n$-dimensional smooth manifold $M_n$. We assume that we can construct some classical action $S$ on $M_n$ which is general coordinate invariant. That is, $S$ does not contain any space-time metric of $M_n$. To evaluate, for example, the partition function of $S$ we must gauge fix. The important point here is in order to fully gauge fix the theory we must choose some metric $g_{ij}$ on $M_n$. The fully gauged fixed quantum action $S_q$ then takes the form

$$S_q[\Phi_r, g_{ij}] = S[\Phi_r] + \delta_Q V[\Phi_r, g_{ij}]. \tag{1}$$

Here, $\Phi_r$ ($r = 1, 2, ...$) are the fields of the theory including matter, gauge, ghost and auxiliary fields etc. The second term on the right hand side of equation (1) is the ghost plus gauge fixing term associated with all the gauge invariance of $S[\Phi_r]$. $\delta_Q$ stands for the related nilpotent BRS-transformation and thus the entire gauge fixing plus ghost term is written as a BRST variation of some functional $V[\Phi_r, g_{ij}]$. The partition function for the theory is thus given by:

$$Z(M_n, g_{ij}) = \int D[\Phi] \exp iS_q[\Phi_r, g_{ij}]. \tag{2}$$

What we mean by a topological field theory is that, for example, the partition function only depends on the gauge fixing metric $g_{ij}$ topologically. That is:

$$\frac{\delta}{\delta g_{ij}} Z(M_n, g_{ij}) = \int D[\Phi] \exp(iS_q) \frac{\delta}{\delta g_{ij}} (\delta_Q V)$$

$$= \int D[\Phi] \exp(iS_q) \delta_Q (\frac{\delta}{\delta g_{ij}} V) = 0, \tag{3}$$

We will ignore such issues as Gribov ambiguities in this paper.
by BRST invariance. What is also very important is the assumption that the path integral measure $D[\Phi]$ is metric independent. We will return to this point shortly.

Our main point in this section has been the following. By a topological field theory we mean that the partition function, for example, of the theory does not depend explicitly on the gauge fixing metric $g_{ij}$ that appears only in the gauge fixing and ghost terms of the quantum action. It depends on it only topologically in that a small variation of the metric $g_{ij}$ will not change $Z(M_n, g_{ij})$. It is this aspect of topological field theories that we claim to observe in Einstein quantum gravity.

3 Gauge Fixing in Quantum Gravity

Einstein quantum gravity is a non-renormalisable theory at least pertubatively. It may, however, be a sensible theory in the non-perturbative regime. With this view in mind, we wish to argue that the full non-perturbative partition function of Einstein quantum gravity has a structure similar to that of the previous section. That is, Einstein quantum gravity has an interpretation as a topological field theory with respect to some gauge fixing metric $g_{ij}$.

Our starting point is the usual classical action for Einstein gravity in $n$ space-time dimensions:

$$S = \int \sqrt{G}(R + \lambda),$$

where $R$ is the scalar curvature, $\lambda$ is the cosmological constant and $G = \det G_{ij}$ with $G_{ij}$ being the usual Einstein metric of general relativity. Due to the invariance of $S$ under diffeomorphims given by:

$$\delta f^i G_{ij} = G_{ik} \partial_j f^k + G_{kj} \partial_i f^k + (\partial_k G_{ij}) f^k,$$

$f^i$ being an arbitrary infinitesimal contravariant vector, we must gauge fix equation (4) in order to quantise the theory. In order to mimic the corresponding case in topological field theories we first make the following observation.

We regard the metric $G_{ij}$ of equation (4) to be a rank two symmetric tensor field on some manifold with metric $g_{ij}$ which we regard as being a gauge fixing metric. Since the classical action (4) is independent of $g_{ij}$, which will only appear in the gauge fixing terms, we can trivially regard the classical action (4) to be independent of $g_{ij}$.

For the gauge fixing condition $F_i$, we will choose the familiar one given by $F_i = D^j G_{ij} + \frac{1}{2} \alpha b_i$ where $b_i$ is a Lagrange multiplier field, $D^j$ is the gravitational covariant derivative with respect to some reference (gauge fixing) metric $g_{ij}$ and $\alpha$ is a gauge fixing parameter. It is now clear that the full action of the gauge fixing plus FP
ghosts is given by [7,8]:

\[ S_{g_{f+FP}} = \delta Q \int \sqrt{g} \left( c^i (D^i G_{ij} + \frac{1}{2} \alpha b_i) \right), \]  

where \( \delta Q \) is the associated BRS-transformation given by:

\[ \delta Q G_{ij} = c^k \partial_k G_{ij} + G_{ij} \partial_k c^k + G_{kj} \partial_i c^k, \]
\[ \delta Q c^i = c^k \partial_k c^i, \]
\[ \delta Q \bar{c}^i = b^i, \]
\[ \delta Q b^i = 0. \]  

One may also check that the BRST operator \( \delta Q \) is nilpotent. That is:

\[ \delta^2 Q = 0, \]  

for all fields. Following the arguments of section (2), it is now clear that the formal partition function of Einstein quantum gravity is topological with respect to the gauge fixing metric \( g_{ij} \). That is, given that:

\[ Z(g_{ij}) = \int D[G_{ij}] D[b^i] D[c^i] D[\bar{c}^i] \exp i(S + S_{g_{f+FP}}), \]  

and assuming that the path integral measure is independent of \( g_{ij} \), then:

\[ \frac{\delta}{\delta g_{ij}} Z(g_{ij}) = 0. \]  

Just like the case of section (2) regarding topological field theories, we see that the formal partition function of pure Einstein quantum gravity has exactly the same structure with respect to some gauge fixing metric \( g_{ij} \).

### 4 The path integral measure

We showed in the previous section that the formal partition function of Einstein quantum gravity does not depend on a choice of gauge fixing metric \( g_{ij} \). In other words, Einstein quantum gravity is topological with respect to \( g_{ij} \). What was crucial in the above observation is the assumption that the path integral measure of the theory be independent of \( g_{ij} \). This, as we shall now discuss, is in general not true.
Consider any general coordinate invariant field theory in \( n \)-dimensions with quantum action given by equation (1). The partition function is then not the naive partition function of equation (2) but is given by:

\[
\tilde{Z}(M_n, g_{ij}) = \int \tilde{D}[\Phi] \exp iS_q[\Phi_r, g_{ij}],
\]

(11)

where \( \tilde{D}[\Phi] \) stands for an appropriate general coordinate invariant measure over the full set of fields \( \Phi_r \). Fujikawa [9,10] as proposed the following choice of the measure for being general coordinate invariant:

\[
\tilde{D}[\Phi] = \prod_x \prod_r d\tilde{\Phi}_r(x),
\]

(12)

where for any given field component \( \Phi_r \):

\[
\tilde{\Phi}_r(x) = g^{\alpha_r}(x)\Phi_r(x),
\]

(13)

\( g(x) = \det g_{ij} \).

The \( \alpha_r \) are constants which depend upon the tensor nature of the field as well as the number of space-time dimensions. Some examples being:

\[
\alpha_r = \begin{cases} 
\frac{1}{4} & \text{for each scalar field} \\
\frac{n-2}{4n} & \text{for each component of a covariant vector field} \\
\frac{n+2}{4n} & \text{for each component of a contravariant vector} \\
\frac{n-4}{4n} & \text{for each component of a covariant tensor field of rank two.}
\end{cases}
\]

(14)

It is now clear that:

\[
\prod_x d\tilde{\Phi}_r(x) = \prod_x d[g^{\alpha_r}(x)\Phi_r(x)] = \prod_x [g^{\alpha_r\sigma_r}(x)d\Phi_r(x)],
\]

(15)

where the signature \( \sigma_r \) is +1 (-1) for commuting (anti-commuting) fields. Thus, the general covariant Fujikawa measure becomes:

\[
\tilde{D}[\Phi] = \prod_x [g(x)]^K(\prod_{r,y} d\Phi_r(y)),
\]

(16)

where:

\[
K = \sum_r \sigma_r \alpha_r,
\]

(17)

is an index which measures the \( g \)-metric dependence of the path integral measure. The partition function (11) thus becomes:

\[
\tilde{Z}(M_n) = (\prod_x [g(x)]^K)Z(M_n),
\]

(18)

where \( Z(M_n) \) is the partition function using the naive measure \( \prod_{x,r} d\Phi_r(x) \). Thus for the topological invariance to be preserved at the quantum level with respect to the metric \( g \), we require:

\[
K = \sum_r \sigma_r \alpha_r = 0.
\]

(19)
5 Path integral measure and quantum gravity

In section (3) we argued that Einstein quantum gravity has an interpretation as a topological field theory with respect to some gauge fixing metric $g_{ij}$. What was crucial in this observation was the assumption that the path integral measure be independent of the gauge fixing metric $g_{ij}$. The last section, however, showed that for this to be true, we require the index $K$ to vanish. For pure Einstein quantum gravity in $n$-dimensions this is not the case.

Looking back at equation (9), we see that the full set of dynamical fields that appear in the formal expression for the partition function of Einstein quantum gravity is:

$$\{G_{ij}, b^i, c^i, \bar{c}^i\}. \quad (20)$$

Here $G_{ij}$ is the metric that appears in the Einstein action which we are regarding as a rank two symmetric tensor field on a manifold with metric $g_{ij}$, $b^i$ is a Lagrange multiplier field which enforces the gauge fixing constraint and $c^i, \bar{c}^i$ are the corresponding ghost, anti-ghost fields. For this set of fields we find that $K$ is given by:

$$K = \frac{1}{8} (n^2 - 5n - 8), \quad (21)$$

and we remind the reader that $G_{ij}$ has $n(n+1)/2$ components and a vector field has $n$ components in $n$ space-time dimensions. It is easy to check that $K$ does not equal zero for any integer value of $n$. Thus, it appears that Einstein quantum gravity in $n$ dimensions does not have an interpretation as a topological field theory in any space-time dimension.\footnote{We stress here that we are talking about Einstein quantum gravity in the second order formalism with metric $G_{ij}$ and not the first order formalism in terms of the vierbein and spin connection fields.}

The above observation may be a blessing in disguise since many feel that a true quantum theory of gravity should be a theory of everything in that it should also tell us about the matter and field content of our Universe. Our idea then is to add some other fields to pure Einstein gravity, without spoiling the quantum topological invariance arguments of section (3), such that the index $K$ is forced to be zero [11].

Lets us specialise to the physically interesting case of $n = 4$. If we couple to Einstein gravity a pure gauge theory with gauge group $G$ our classical action becomes:

$$S = \int \sqrt{G} ((R + \lambda) + F^{ij} F_{ij}), \quad (22)$$

where $F^{ij}$ is the associated field strength of the gauge field $A_i$. After introducing the appropriate Lagrange multiplier and ghost fields for the gauge fixing of $A_i$ with
respect to the gauge group \( G \), the formal partition function for this theory is now:

\[
Z = \int D[G_{ij}]...D[A_i]... \exp iS_q,
\]

(23)

where the dots after the \( G_{ij} \) measure represent the fields needed in the gauge fixing of diffeomorphism invariance and the dots after the \( A_i \) measure represent the fields needed in the gauge fixing of the gauge field \( A_i \) with respect to the gauge group \( G \). Recall also that the gauge fixing of the gauge field \( A_i \), as well as the inclusion of its kinetic term, is performed with respect to the Einstein metric \( G_{ij} \) and thus the proof of topological invariance with respect to the gauge fixing metric \( g_{ij} \) of section (3) is maintained.

We see that \( K \) now takes the value:

\[
K = -\frac{6}{4} + \frac{\dim(G)}{4},
\]

(24)

where \( \dim(G) \) is the dimension of the gauge group \( G \), in this case, the dimension of the adjoint representation of \( G \). We clearly see that for \( K \) to be zero \( \dim(G) = 6 \). That is, if we couple a pure gauge theory with \( \dim(G) = 6 \) to Einstein gravity, the full quantum theory has an interpretation as a topological field theory with respect to the gauge fixing metric \( g_{ij} \).

There are a number of choices of gauge group \( G \) such that \( \dim(G) = 6 \). An obvious choice is the product of six \( U(1)'s \) but a more interesting one is \( G = SO(4) \sim SU(2) \times SU(2) \). We thus see, in particular, that if we couple a pure \( SO(4) \) gauge theory to Einstein gravity in \( n = 4 \) space-time dimensions then \( K \) is forced to be zero and thus the full quantum theory has an interpretation as a topological field theory with respect to the gauge fixing metric \( g_{ij} \).

This is an interesting observation in that an \( SO(4) \) gauge theory is essentially a spin connection field in \( n = 4 \) dimensions with Euclidean metrics. Perhaps the theory is telling us that a proper approach to quantum gravity is via the vierbein-spin connection formalism which has been so successful in 2+1 dimensions.

6 conclusion

In this paper an attempt has been made to make a connection between topological field theory and Einstein quantum gravity. Our approach has been to start off with Einstein quantum gravity and argue that this could be given an interpretation as a topological field theory with respect to some gauge fixing metric \( g_{ij} \). This being exactly what is meant by topological invariance in topological field theories. After worrying about the path integral measure it was discovered that the path integral
measure of pure Einstein quantum gravity in \( n \)-dimensions develops a dependence on the gauge fixing metric \( g_{ij} \). This ruins the claim that pure Einstein quantum gravity has an interpretation as a topological field theory with respect to the gauge fixing metric \( g_{ij} \). However, it was shown that if we couple a \( SO(4) \) pure gauge theory, for example, to Einstein gravity in \( n = 4 \) space-time dimensions then the corresponding quantum theory has a path integral measure independent of the gauge fixing metric and thus has an interpretation as a topological field theory with respect to the gauge fixing metric \( g_{ij} \).

This is, we believe, a very interesting observation. The main physical motivation for studying topological field theories is the belief that they have something to do with quantum gravity. In this paper we have argued that Einstein gravity, in its second order formalism, indeed as this interesting interpretation when coupled to other fields. If we again return to a \( n \)-dimensional space time and couple Einstein gravity to some gauge theory with gauge group \( G \) together with \( m \) scalar fields transforming under some representation of \( G \), then the general constraint that the corresponding path integral measure be independent of the gauge fixing metric \( g_{ij} \) becomes:

\[
K = \frac{1}{8}(n^2 - 5n - 8) + \text{dim}_{\text{adj}}(G)(\frac{n - 3}{4}) + \sum_{i=1}^{m} \frac{\text{dim}G_i}{4} = 0,
\]

where each scalar field transforms in some representation \( G_i \) under \( G \). This equation is a remarkable equation in that in a single constraint we see the number of space-time dimensions \( n \), the dimension of the adjoint representation \( \text{dim}_{\text{adj}} G \) of the gauge group \( G \) together with the possible dimensions of the representations \( G_i \) the scalar fields can transform under. Since there are group theoretic constraints between \( G \), \( \text{dim}_{\text{adj}}(G) \) and \( \text{dim}(G_i) \), one can imagine some classification of possible theories based on equation (25) such that the corresponding theory has an interpretation as a topological field theory.

Finally we would like to make a comment on including fermionic matter in the theory. This of course requires a theory of gravity in its first order formalism. That is, in terms of the vierbein and spin connection fields. In the quantum region, this theory will be quite different from the corresponding pure metric theory but we believe that such a theory can also be given a topological interpretation as in 2+1 dimensions. Indeed, the above observations suggest, at least in \( n = 4 \) space-time dimensions, the inclusion of a spin connection field. This will be discussed elsewhere.
References

[1] E. Witten, Commun. Math. Phys. 117 (1988) 353.

[2] D. Birmingham, M. Blau, M. Rakowski, G. Thompson, Phys. Rep. 209 (1991) 129.

[3] E. Witten, Nucl. Phys. B311 (1988/89) 46.

[4] E. Witten, Nucl. Phys. B323 (1989) 113.

[5] A. Toon, Phys. Rev. D47 (1993) 2435. “Background fields in 2+1 topological gravity”, to appear in Phys. Rev. D.

[6] G. Horowitz, Commun. Math. Phys. 125 (1989) 417.

[7] S. Adler, Rev. Mod. Phys. Vol. 54, No. 3 (1982) 729.

[8] Z. Bern, S. Blau, E. Mottola, Phys. Rev. D43 (1991) 1212.

[9] K. Fujikawa, Phys. Rev. Lett. 42 (1979) 1195.

[10] K. Fujikawa, O. Yasuda, Nucl. Phys. B245 (1984) 436.

[11] A. Toon “Particle content in topological field theories”, University of Oxford preprint: OUTP-92-30P.