Estimation the Vasicek interest rate model driven by fractional Lévy processes with application

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Abstract:-

In this article, we present that fractional Lévy processes which is very an important field in both probability theory and its application in recent years. The fractional Brownian motion is suggested as the fractional Lévy processes in this article. We will make parameters estimate of the Vasicek process driven by fractional Brownian motion, that represented the short memory parameter \( 0 < H < \frac{1}{2} \) and the long memory parameter \( \frac{1}{2} < H < 1 \). So, our aim is to study the behavior of stochastic Vasicek Interest driven by fractional Brownian motion. We use maximum likelihood to estimate the drift, diffusion and Hurst parameters and generally the fractional Lévy processes. We illustrate our methods, and show the behavior of stochastic parameters using simulation and real data (ISX60).

Keywords:- Fractional Lévy Processes, Long memory, Short memory, Maximum likelihood Function, Hurst parameter.

1. Introduction:-

Fractional Brownian motion has become very important tool in modern probability and statistic modeling [12]. Fractional Lévy processes (FLp) by Mandelbort-Van Ness representation were first defined by, and the theory was developed by [9]. Hence it is the popular model with many application, there is one parameters namely Hurst parameter \( H \in (0,1) \). For Hurst parameter \( H < 1/2 \) the increments are negatively correlated and \( H > 1/2 \) a process has long memory dependence property, for and \( H = 1/2 \) the increments are independent i.e. it is called standard Brownian motion. Also fractional Lévy processes always show (long memory process), but the class of generalized fractional Lévy processes (GFLP) can be modeled both of them (a short and a long memory) [5].
Our problem is the possibility to make estimate the parameters of Vasicek Interest Rate driven by type of Brownian motion such as fractional Lévy processes (fractional Brownian motion) and estimate the parameters of the stochastic differential equation (SDE).

In literature review, Marquardt. T. (2006) [9], shows that the moving average (MA) integral representation of fractional Brownian motion (FBm), a class of (FLP) fractional Lévy processes is introduced by replacing the Brownian motion by the general Lévy processes, and offer different methods of construction (FLP), FLP has a like (second order) structure, such as (FBm) and depend on Lévy processes. Tikkanmäki, H. (2010) [12], in this study prove a fractional Lévy processes presented as an integral via different integral transformations have a same finite dimensional, also he present relations between different fractional Lévy processes and analyze the properties. Wang and Chen. (2017) [14] show (FBm), with Hurst index H ∈ (0,1) and let S= [S_t, t ≥ 0] be an inverse a-stable subordinator independent of W^H, where W^H = [W^H(t), t ∈ R] is a real value fractional Brownian motion. In this paper note a inverse stationary subordintor fractional Brownian motion (SFBm) Z^H = { Z^H(t),t ≥ 0} is definite by Z^H(t) = W^H(S_t). The aim of this paper is to choose large deviation results for the process Z^H. Al-Saadony M., et al. (2017) [2], show in this study the formula of stochastic differential equation (SDE) is driven by fractional Lévy process (FLp), when H ∈ (0,1) and estimate the parameters of the drift and volatility. Tanaka., K. (2019) [11], presented the estimation of the drift parameters in the fractional Vasicek model (FVm) from a continuous record of observations. The aim of this paper is to estimate the maximum likehood of k and α, and develop the asymptotic distributions for a maximum likelihood (Mle) under three cases, k > 0 and H ∈ (0,1/2), k = 0 and H ∈ (0,1), and k < 0 and H ∈ (0,1).

The paper is organized as follows: Section 2, we present fractional Lévy process and some important definition about it, added to definition of long memory and short memory. Section 3, we present Vasicek process driven by fractional Lévy process and using the maximum likelihood method. Section 4, we present Hurst parameter and using the maximum likelihood method. Section 5, we present results from simulation and real data. We conclude our paper in section 6.

2. Fractional Lévy processes:-

A fractional Lévy process is decomposition of Lévy process and (FBp). In this thesis we suggest that the fractional Lévy process considered as with fractional Brownian motion (FBm) [9]. The fractional Brownian motion (FBm), B^H is generalized of standard Brownian motion (SBm), when H= 0.5, a Hurst parameter (H) is between (0,1), i.e., H ∈ (0,1). We can write the period of (H) in details as follows:

1. If H < 0.5, then it is called” short memory process”.


2. If $H = 0.5$, then it is called ”standard Brownian motion”.

3. If $H > 0.5$, then it is called ”long memory process”.

2.1 Definition// Fractional Lévy Process:

Let $L = (L(t), t \in \mathbb{R})$ be a zero mean two-sided Lévy process with $E[L(1)^2] < \infty$ and without the Brownian component as shown in Marquardt [9], so $d \in (0,0.5)$. A stochastic process is:

$$L^d_t = \frac{1}{\Gamma(1 + d)} \int_{-\infty}^{\infty} [(t - s)^d - (-s)^d] L(ds), \quad t \in \mathbb{R}$$

is called ”fractional Lévy process” (FLp), where

$L(t) = L_1(t), t \geq 0.$

$L(t) = -L_2(-t), t < 0.$

$L_1(t), \text{ and } [L_2(-t)]$ are two-independent copies of the one-sided Lévy process.

Now, we will give a short definition for long and short memory as follows:

2.2 Definition // Short memory:-

Let $X(t), t \in \mathbb{Z}$ be a real value time series. The autocovariance function $\delta(h,t)$ is defined by [13],

$$\delta(h,t) = \text{Cov} (X(t), X(t + h)), \quad t, h \in \mathbb{Z}.$$ 

If $\delta(h,t)$ is independent of $\delta(t)$, and $\text{Var} (X(t)) < \infty$, we can say $X(t)$ is second order stationary.

We will define $\delta(h,t) = \delta(h)$. For second-order stationary processes, we also write a autocorrelation function (ACF) $\rho(h)$ of $X(t)$, as follows:

$$\rho(h) = \text{Corr}(X(t), X(t + h)) = \frac{\delta(h)}{\delta(0)}, \quad t, h \in \mathbb{Z}.$$ 

An autocorrelation function will decay to zero, when the lag is increasing. Although, for the large family of processes including (ARMA) process, the auto-correlation decay exponentially

$$|\rho(h)| \leq Cr^h, \quad C > 0, \quad 0 < r < 1$$

(2)

These processes are said to have short memory.
Figure 1: Represents an illustration of the path of fractional Brownian motions when $H = 0.25$ and this process called short memory.

2.3 Definition // Long memory:

The key defining characteristic of a long memory is that, its autocorrelation slower than that in equation (2). Beran (1994) [13] presented the definition of long memory process with autocorrelation function, as follows:

$$
\rho(h) \sim C_p h^{2d-1}, \quad C_p \neq 0, \quad 0 < d < 0.5, \quad \text{as } h \to \infty
$$

(3)

$d$ is the so-called (long memory) parameter which controls the speed of decay of the autocorrelation. The definition (2.3) in equation (3) of long memory are duplicate. There also exists alternative definitions of long memory process. Definitions are also used in the literature (see, e.g., Palma (2007)):

1. $\sum_{h=-\infty}^{\infty} \delta(h)$
2. $\sum_{h=-n}^{n} \delta(h) \sim n^{2d} L_1(n)$, as $n \to \infty, 0 < d < 0.5.$
3. $\delta(h) \sim h^{2d-1} L_2(h)$, as $h \to \infty, 0 < d < 0.5.$

There is another definition of long memory process [12].

2.4 Definition:

$X = \{X_t\}_{t \in \mathbb{R}}$, represent long memory process that is the stationary stochastic process and

$$
\delta_x(h) = \text{cov}(X_{t+h}, X_t), \quad h \in \mathbb{R}.
$$

It is autocovariance function. If there exist $0 < d < 0.5$, and the constant $c_\delta > 0$. Then:

$$
\lim_{h \to 0} \frac{\delta_x(h)}{h^{2d-1}} = c_\delta.
$$

(4)

Thus, $x$ is the stationary process with a long memory [9], [13].
Figure 2: Represents an illustration of the path of fractional Brownian motions when $H = 0.75$ and this process called long memory.

3. Vasicek Process driven by Fractional Lévy Process:-

We will replace Vasicek process driven by Brownian motion, by a fractional Lévy process [2]. as follows:

$$dX_t = K(b - x_t) \, dt + \sigma \, dL^H_t$$ \hspace{1cm} K and $\sigma > 0$ \hspace{1cm} (5)

$L^H_t$ when $t \in \mathbb{R}$, is a fractional Lévy process (FLp) with the parameter $H$, which is also called a self similarity or Hurst parameter $(0 < H < 1)$ [11]. An fractional Lévy Process $L^H_t$ is a zero mean Gaussian process. When $H= 0.5$, then $L^0.5_t = B_t$ which a standard Brownian motion and a (FLp) becomes a standard Lévy model. In this case, it can be shown that fundamental martingale $M^H_t$ becomes the standard Brownian motion and the maximum likelihood estimation.

3.1 Maximum likelihood estimation for Fractional Lévy Processes:-

We construct the maximum likelihood estimation (Mle) for unknown two parameters $K$ and $b$ when $\sigma > 0$. Our objects is to appreciate the parameter $\sigma$, which is difficult to estimate. See for example, Kubilius et al. (2018) . For $K > 0$ and $H= 0.5$ in model (9), as $T \to \infty$, we have

$$\sqrt{T}(R_T - K) \xrightarrow{b} N(0,2K),$$

$$\sqrt{T}(b_T - b) \xrightarrow{b} N(0, \frac{\sigma^2}{K^2})$$

Where,
\begin{equation}
R_T = \frac{(X_T - X_0) \int_0^T X_t \, dt - T \int_0^T X_t \, dL_t^H}{T \int_0^T X_t^2 \, dt - (\int_0^T X_t \, dt)^2},
\end{equation}

\begin{equation}
\hat{b}_T = \frac{(X_T - X_0) \int_0^T X_t^2 \, dL_t^H - \int_0^T X_t \, dX_t \int_0^T X_t \, dL_t^H}{(X_T - X_0) \int_0^T X_t \, dL_t^H - T \int_0^T X_t \, dL_t^H},
\end{equation}

Where, \( \int_0^T X_t \, dL_t \) is interpreted as an Itô integral [11].

4. Hurst Parameter:

A fractional Brownian motion has become very important tool in modern probability and statistical model [12]. It fractional processes has been starting from a fractional Brownian motion which is presented by "Kolmogorov" in (1940) [9], and his popularity by "Mandelbort" and " Van Ness" in (1968). We Remember that a fractional Brownian motion (FBm) with Hurst parameter index \( H \in (0,1) \), if it is a mean zero continuous Gaussian process with \( B_0^H = 0 \) normal distribution increments, [11],[6] and [8]:

\begin{equation}
E[B^H, B^H_s] = R_H(t,s) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right), \quad \forall \ t,s \geq 0
\end{equation}

and

\begin{equation}
B^H(t) = \int_0^t R_H(t,s) \, dB(s),
\end{equation}

where, B is Brownian motion when \( H = 0.5 \), \((t,s) \in \mathbb{R} \). We discussed in this section, estimation of self-similarity index or Hurst index. Our aim here is to discuss some ways of estimation of Hurst index (H). The parameter K in equation (5) is often referred to as a persistence parameter, when \( K > 0 \), \( x_t \) is stationary and ergodic. In this case, \( b \) is an unconditional mean of \( x_t \) and \( K \) is a mean-reverting parameter. When \( K < 0 \), \( x_t \) is explosive and hence non-ergodic. When \( K = 0 \), \( x_t \) is null-recurrent and a drift term \( K(b - x_t) \, dt \) disappears, so \( b \) is superfluous in this case.

When the Hurst parameter \( H > 1/2 \) and, if \( K > 0 \) non-ergodic [1] borrowing a idea of Hu and Nualart (2010) and Hu et al. (2017), Xiao and Yu (2019a) considered two methods, least square estimation (LSE) and a ergodic-type estimation of \( K \) and \( b \). When \( H \geq 1/2 \) and \( K = 0 \) or, if \( K < 0 \) ergodic [1], Xiao and Yu (2019a) consider the (LSE) method. Xiao and Yu (2019b) extends a results of Xiao and Yu (2019a) [11] from the case, where \( H \in (\frac{1}{2},1) \) to where \( H \in (\frac{1}{2},0) \). Lohvinenko and Ralchenko (2017) [7] considered a maximum likelihood (ML) estimation of \( K \) and \( b \), when \( K > 0 \) and \( H \in (\frac{1}{2},1) \). fractional Vasicek interest rate process, [15] and [8] can be written:

\begin{equation}
dX_t = K(b - x_t) \, dt + \sigma \, dB_t^H \quad K, b and \sigma > 0
\end{equation}
\( B^H = [B^H_t]_{t \geq 0} \) is a fractional Brownian motion with Hurst parameter, \( H \in (0, 1/2) \). The solution of the fractional stochastic differential equation (SDE) can be found by applying the fractional Itô lemma to \( e^{Kt} x_t \),

\[
d(e^{Kt} x_t) = (K e^{Kt} x_t + e^{Kt} K(b - x_t)) \, dt + \sigma \, dB^H_t
\]

Integrating both sides,

\[
e^{Kt} x_t - x_0 = K b \int_0^t e^{Ks} \, ds + \sigma \int_0^t e^{Ks} dB^H_s
\]

\[
x_t = e^{-Kt} x_0 + K (1 - e^{-Kt}) + \sigma \int_0^t e^{K(s-t)} dB^H_s
\]

Where, a stochastic integral is understood as the positive integral. To solve the \( \sigma \int_0^t e^{-K(t-s)} dB^H_s \) term, we will use ergodic theorem [15], and elementary Calculus, when \( H \in (0, 1/2) \) i.e.,

\[
\sigma \int_0^t e^{-K(t-s)} dB^H_s = \sigma^2 H (2H - 1) \int_0^t \int_0^\infty e^{-K(t+s)} |t-s|^{2H-2} \, ds \, ds
\]

\[
= \sigma^2 K^{-2H} H \Gamma (2H) \Delta^{2H-2}
\]

\[
= \sigma^2 K^{-2H} H (2H - 1) \Delta^{2H-2}
\]

If \( 0 < H < 1/2 \), using ergodic theorem.

\[
x_t = x_0 e^{-Kt} + (1 - e^{-Kt}) b + \sigma K e^{-Kt} + \int_{-\infty}^0 e^{Ks} dB^H_s \, ds.
\]

Then,

\[
\sigma \int_{-\infty}^t e^{-K(t-s)} dB^H_s = \sigma^2 K^{1-2H} (2H - 1) \Delta^{2H-2}
\]

4.1 Maximum Likelihood Estimation for Hurst parameter:-

Now, assume that we have a sequence of \( B^H_1, B^H_2, \ldots, B^H_N \), which are presented of a fractional Brownian motion \( B^H_t \) at time instants \( 1, 2, \ldots, N \). Then, there are many important estimation methods for estimating the Hurst parameter as seen [3]. In this work, our main is to provide estimation of the Hurst parameter (\( 0 < H < 1 \)) so, we put

\[
X_1 = B^H_1, X_2 = B^H_2, \ldots, X_N = B^H_N.
\]

Because fractional Brownian motion (FBm) has a stationary increments,

\[
Y_1 = X_1, \quad Y_2 = X_2 - X_1, \quad Y_3 = X_3 - (X_1 + X_2), \ldots, Y_N = X_N - X_{N-1}
\]

Is a stationary Gaussian time sequence. A covariance of this sequence is,
\[ \rho_H(k) = \varepsilon(Y_k Y_{k+1}) = \varepsilon(Y_m Y_{m+k}) \]
\[ = \frac{1}{2} (k + 1)^{2H} + (k - 1)^{2H} - 2k^{2H} \]

(9)

We can find an estimator of \( H \) by (Mle). A maximum likelihood estimator is the commonly using estimator in the theory of statistic. Denote \( \delta_N = (Y_1, Y_2, ..., Y_N)^T \), and the covariance matrix of \( \delta_N \),

\[ \sum(\gamma) = (\sigma_{ij})_{1 \leq i \leq n}, \quad \text{where} \quad \sigma_{ij} = \rho_\gamma(|j - i + 1|). \]

A likelihood function \( L(\gamma) \) [3],

\[ L(\gamma) = \frac{1}{(2\pi)^{n/2}} \det \left( \sum(\gamma) \right)^{-1} \exp\left( -\frac{1}{2} \sum(\gamma)^{-1} \delta_N \right) \]

(10)

Thus a maximum likelihood estimator \( \hat{\gamma}_n \), is a maximum of the following function:

\[ \hat{\gamma}_n = \arg \max \left( -\frac{1}{2} \left[ \log \det \left( \sum(\gamma) \right) + \delta_N^T \sum(\gamma)^{-1} \delta_N \right] \right), \]

or, \( \hat{\gamma}_n \) is given by a following equation:

\[ \frac{d}{d\gamma} \log \det \left( \sum(\gamma) \right) + \delta_N^T \frac{d}{d\gamma} \sum(\gamma)^{-1} \delta_N = 0. \]

(11)

1. If \( H > 1/2 \), the covariance of the fractional Brownian motion [8] can be written:

\[ R_H(t, s) = m_H \frac{1}{2} t^{2H - 1} \int_s^t |u - s|^{H - 3/2} u^{H - 1/2} \, du, \]

(12)

where [4],

\[ m_H = \left( \frac{H(2H - 1)}{\beta(2 - 2H, H - \frac{1}{2})} \right)^{1/2}, t > s. \]

2. If \( H < 1/2 \),

\[ R_H(t, s) = c_H \left[ \frac{t}{s} \right]^{H - \frac{1}{2}} \left( t - s \right)^{H - \frac{1}{2}} - \left( H - \frac{1}{2} \right) \frac{s^{2H} - 1}{2} \int_s^t (u - s)^{H - \frac{1}{2}} u^{H - \frac{3}{2}} \, du \]

(13)

with \( c_H = \left[ \frac{2H}{(1 - 2H)\beta(1 - 2H, H + \frac{1}{2})} \right]^{1/2} \) and \( t > s \).

we obtain [3] and [4]:

1. If \( H \in \left( \frac{1}{2}, 1 \right) \sum_{n=1}^\infty \text{Cov} (B_n^H - B_{n+1}^H) = \infty. \)
2. If \( H \in \left( 0, \frac{1}{2} \right) \sum_{n=1}^\infty \text{Cov} B_n^H - B_{n+1}^H < \infty. \)
Maximum likelihood estimation for Hurst parameter and expectation, covariance in the three cases is:

When \( \frac{1}{2} < H < 1 \), and two real valued measureable function

\[
E\left[ \int_{-\infty}^{\infty} f(s) dB^H_s \right] \int_{-\infty}^{\infty} g(s) dB^H_s = H(2H - 1) \int_{-\infty}^{\infty} f(s) g(t)|s - t|^{2H-2} \, ds \, dt
\]

Where \( f(s) = g(s) = e^{Ks} \), \( \int_{-\infty}^{\infty} f(s) dB^H_s = \int_{-\infty}^{\infty} g(s) dB^H_s \) is two integrals [7].

\[
R_H(s,t) = \text{Cov}(B^H_s B^H_t) = H(2H - 1)|s - t|^{2H-2} \quad s, \, t \in \mathbb{R}
\]

(14)

When \( 0 < H < \frac{1}{2} \), [4].

\[
E\left[ \int_{-\infty}^{\infty} f(s) dB^H_s \right] \int_{-\infty}^{\infty} g(s) dB^H_s = \sigma^2 H(2H - 1) \int_{-\infty}^{\infty} f(s) g(t)|s - t|^{2H-2} \, ds \, dt
\]

\[
R(s,t) = \text{Cov}(B^H_s B^H_t) = \sigma^2 H(2H - 1)|s - t|^{2H-2} \quad s, \, t \in \mathbb{R}
\]

(15)

When \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \), \( N = 1, 2, ..., K > 0 \), \( s \to \infty \) and \( (-\infty \leq a < b \leq c < d < \infty) \) then [4].

\[
E\int_{a}^{b} e^{Ks} dB^H_s \int_{c}^{d} e^{Kt} dB^H_s = H(2H - 1) \int_{a}^{b} e^{Ks} ( \int_{c}^{d} e^{Kt} (s - t)^{2H-2} \, ds \, dt)
\]

and

\[
R_H(s,t) = \text{Cov}(B^H_t B^H_s) = \frac{1}{2} \sigma^2 \sum_{n=1}^{N} K^{-2n} \left( \prod_{k=0}^{2n-1} (2H - k) \right) s^{2H-2n} + (s^{2H-2N-2})
\]

(16)

5. Simulation Study:

In this section, we will present the simulation study to show our process is driven by fractional Lévy process. We use R language as shown in [10], which is free software dealing with our process. We set up some parameters for our process, a sample of 1000, \( K = 3, b = 1, \sigma = 0.19, H = 0.22 \), the parameters of fractional Lévy process, and we use analyze simulation results for the maximum likelihood estimation of fractional Lévy process, the results are as follows:
Figure 3: The path of Vasicek interest rate driven by fractional Lévy processes with different values of the Hurst parameter (H). In general, we can describe three cases for a dependence of (FLp) in terms of (H) in the figure as follows:

If \( H = 0.5 \) the process is standard Brownian motion \( B \). If \( H = 0.75 \) the process has positively correlated increments and is called a long memory. If \( H = 0.25 \) the process has negatively correlated increments and is called a short memory.

Table 1: Parameters estimation of fractional Lévy processes, when the initial value is the Hurst parameter \((0 < H < \frac{1}{2})\) for simulation data.

| Parameters | Parameters estimator | Standard. Error | 2.5% | 97.5% |
|------------|----------------------|-----------------|------|-------|
| K          | 2.9999               | 0.388194681     | 2.971535391 | 3.02835461 |
| b          | 1.4999               | 2.1470459       | 1.34251871 | 1.65727813 |
| \( \sigma \) | 0.0499417            | 0.00779275      | 0.049370493 | 0.0505129907 |
| H          | 0.1581265            | 0.038426110     | 0.155309873 | 0.160943127 |

Through the results of the table and parameters mentioned of fractional Lévy processes when the initial value is the Hurst parameter \((0 < H < \frac{1}{2})\). We note that confidence periods were low for the (H) parameter value it is the best then the (\( \sigma \)) and the (b) parameter. The confidence intervals were somewhat high for the (K) parameter estimate.

Table 2: Parameters estimation of fractional Lévy processes, when the initial value is the Hurst parameter \((1/2 < H < 1)\) for simulation data.
| Parameters | Parameters estimator | Standard.Error | 2.5%        | 97.5%        |
|------------|----------------------|----------------|-------------|-------------|
| K          | 2.9999               | 1.66942337     | 2.877531528 | 3.122268473 |
| b          | 1.500432             | 1.46371352     | 1.393142028 | 1.607721973 |
| σ          | 0.07589113           | 0.01713226     | 0.074635339 | 0.77146921  |
| H          | 0.95076698           | 0.11148908     | 0.942594848 | 0.958939112 |

Through the results of the table and parameters mentioned of fractional Lévy processes when, the initial value is the Hurst parameter \( \frac{1}{2} < H < 1 \), we note that confidence periods were low for the (H) parameter value it is the best then, the (σ) and the (b) parameter. The confidence intervals were somewhat high for the (K) parameter estimate.

### 5.1 Real data Study:

We apply our method to real data (Homepage [www.isx-iq.net](http://www.isx-iq.net)), we have used Iraq’s Stock Market index (ISX60) for the period 2017-2019. We use R software as shown in [10], which is the free software using to analyses the data statistical. We use analyze simulation results for the maximum likelihood estimation of fractional Lévy process, the results are as follows:

![Figure 4: Represented the path the Vasicek process for Iraq's Stock Market (ISX60) index for the period 2017 to 2019.](image)

Figure 4: Represented the path the Vasicek process for Iraq's Stock Market (ISX60) index for the period 2017 to 2019.
5.2 Normal distribution Test:

This test was used for the purpose of ensuring that the data is not normally distributed, it is a test of Kolmgorov-Smirnov. Assuming that the null hypothesis $H_0$ assume a normal distribution of data, as for the alternative hypothesis $H_1$ assume that the data is not normally distributed.

Table (3): Refers to a hypothesis test that the distribution.

| The value of the test statistic K.S | P-value | The decision |
|-----------------------------------|---------|--------------|
| 10.567                            | $2.2 \times 10^{-16}$ ≈ 0 | Rejected $H_0$ |

This table (3) shows the Kolmgorov-Smirnov test of normal distribution of data. We notice through P-value is less than 5% which means the data are not distributed naturally. Moreover, this we accepted the alternative hypothesis and rejected the null hypothesis.

Table 4: Parameters estimation of fractional Lévy processes, when the initial value is the Hurst parameter ($0 < H < 1/2$) for real data.

| Parameters | Parameters estimator | Standard. Error | 2.5%       | 97.5%       |
|------------|----------------------|-----------------|------------|------------|
| $K$        | 1.0001131            | 2.9970594       | 0.78039368 | 1.219606333|
| $b$        | 0.2269208            | 0.5096934       | 0.189560354| 0.264281246|
| $\sigma$   | 0.7756801            | 1.1594334       | 0.690693813| 0.860666387|
| $H$        | 0.3618806            | 0.1985341       | 0.34732082 | 0.376433118|

Figure 5: The histogram of the (ISX60) index and red line is represented the density of this index, shows the path of the Vasicek Interest Rate for the real data. We note through the results of the process data are not distributed normally.
Clearly, through the results of the table (4.5) of fractional Lévy processes when the initial value is the Hurst parameter \((0 < H < 1/2)\). We note that confidence periods were low for the \(b\) parameter value is the best, the \((\sigma)\) and the \((H)\) parameters the confidence intervals were low and then the estimation of \((K)\) parameter.

Table 5: Parameters estimation of fractional Lévy processes, when the initial value is the Hurst parameter \((1/2 < H < 1)\) for real data.

| Parameters | Parameters estimator | Standard. Error | 2.5%   | 97.5%   |
|------------|----------------------|-----------------|--------|---------|
| \(K\)     | 1.0044298            | 5.1741591       | 0.62516474 | 1.383694861 |
| \(b\)     | 0.2278860            | 0.9135418       | 0.160923529 | 0.294848471 |
| \(\sigma\)| 2.4052194            | 4.8239822       | 2.051622258 | 2.758816543 |
| \(H\)     | 0.6088248            | 0.5609853       | 0.567704666 | 0.649944935 |

Through the results of the above table of fractional Lévy processes when the initial value is the Hurst parameter \((1/2 < H < 1)\). We note that confidence periods were low for the \(b\) parameter value it is the best then, the \((H)\) and the \((K)\) parameters the confidence intervals were low, therefore that confidence periods were high of \((\sigma)\).

Table 6: Represent the mean squares error (Mse) for fractional Lévy processes.

| Models       | (Mse)          |
|--------------|----------------|
| when \(H \in (1/2, 1)\) | 3.494488034   |
| when \(H \in (0, 1/2)\)  | 0.02745487    |

We note from the table (6), the maximum likelihood estimation have less mean square error (Mse) when, \(H \in (0, 1/2)\) best than, when \(H \in (1/2, 1)\).

6. Discuss the Results:-
We have applied the stochastic differential equation (SDE) to simulation data and real data i.e., using parametric estimation (maximum likelihood). We have performed inference for the parameters of the Vasicek Interest Rate driven by fractional Lévy processes. We got the best estimated of the parameters \((K, b, \sigma, H)\). We can conclude that the table (6), the maximum
likelihood estimation have less (Mse) compared to other estimators. In addition to fractional Lévy process when, \((0 < H < 1/2)\), have less (Mse) best than the fractional Lévy process when, \((1/2 < H < 1)\) in our article, this is evident through periods of confidence.

References:

[1] Alazemi., F., Alsenafi., A., & Es-Sebaiy., K. (2020), " Parameter estimation for Gaussian mean-reverting Ornstein-Uhlenbeck processes of the second kind:Non-ergodic case", *Stochastic and Dynamics*, **19**, 5, 2050011( 25 ).

[2] Al-Saadony, M., Flaih, A., & Elsalloukh, H. (2017), " Change of Measure in Stochastic Differential Equations", *Statistical Sciences Journal*, **2**, Part 2.

[3] Biagini, F., Hu, Y., Oksendal, B., & Zhang, T. (2008), " Stochastic calculus for fractional Brownian motion applications", Department of Mathematics, Springer-Verlag London.

[4] Cheridito, P., Kawaguchchi, H., & Maejima, M. (2003), " Fractional Ornstein-Uhlenbeck Processes", *Journal URL, Vol.8 No.3*, (1-14 ).

[5] Kluppelberg., C. & Matsui., M. (2015), "Generalized Fractional Lévy processes with Fractional Brownian Motion", *Limit. Technische Universitat Munchen*, (1108-1131).

[6] Li., S. & Dong., Yi. (2018), " Parametric Estimation in the Vasicek-Type Model driven by Sub-Fractional Brownian motion", *Applied Mathematic*, **11**, **197**, doi:10.3390/a11120197.

[7] Lohvinenko, S., & Ralchenko, K.(2017), " Maximum likelihood estimation in the fractional vasicek model", *Lithuanian Journal of Statistic*, **56(1)**, (77-87).

[8] Lohvinenko, S., & Ralchenko, K.(2019), " Maximum likelihood estimation in the non-ergodic fractional vasicek model, moder stochastic," : theory and application, **6(3)**, (377-395).

[9] Marquardt, T. (2006)," Fractional Lévy processes with an Application to Long Memory Moving Average Processes", *Bernoulli*, **12(6)**, (1099-1126 ).

[10] R., Core Team. (2016), " R: A language and Environment for Statistical Computing, Vienna, Austria: R Foundation for Statistical Computing", ISBN 3-900051-07-0.

[11] Tanaka., K., Xiao., W., & Yu., J. (2019), " Maximum likelihood estimation for the fractional vasicek model", *Singapre management university*, (1-31).

[12] Tikanimiäki, H. (2010), "Fractional Lévy process as A result of Compact interval integral transformation", *Institute of mathematics, P.O.Box, 11100, FI-00076, Finland.

[13] Tong, Zhigang (2012), " Option Pricing with Long Memory Stochastic Valatility Models", Ottawa, Canada University, (First ed. ), (1-146 ).
[14] Wang., W. & Chen., Z. (2017), "Large deviations for subordinated fractional Brownian motion and applications", *Journal of Mathematical Analysis and Application*, (1-14).

[15] Xiao, W., Zhang, W., & Xu, W. (2011), "Parameter estimation for fractional Ornstein-Uhlenbeck processes at discrete observation", *Applied Mathematical Modelling*, 35, (4196-4207).