Existence of least energy nodal solution for a Schrödinger-Poisson system in bounded domains

Claudianor O. Alves† and Marco A.S. Souto‡
Universidade Federal de Campina Grande
Unidade Acadêmica de Matemática e Estatística
CEP:58429-900, Campina Grande - PB, Brazil.

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Abstract
We prove the existence of least energy nodal solution for a class of Schrödinger-Poisson system in a bounded domain \( \Omega \subset \mathbb{R}^3 \) with nonlinearity having a subcritical growth.

1 Introduction

This paper was motivated by some works that have appeared in recent years concerning with the nonlinear Schrödinger-Poisson system

\[
\begin{cases}
-i\frac{\partial \psi}{\partial t} = -\Delta \psi + \phi(x) \psi - |\psi|^{p-2}\psi \text{ in } \Omega, \\
-\Delta \phi = |\psi|^2 \text{ in } \Omega, \\
\phi = \psi = 0 \text{ on } \partial \Omega,
\end{cases}
\]

\( (NSP) \)

where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with smooth boundary, \( 2 < p < 2^* = 6 \) and \( \psi : \overline{\Omega} \rightarrow \mathbb{C} \) and \( \phi : \overline{\Omega} \rightarrow \mathbb{R} \) are unknown functions.

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†C.O. Alves was partially supported by CNPq/Brazil 303080/2009-4, coalves@dme.ufcg.edu.br
‡M.A.S. Souto was supported by CNPq/Brazil 304652/2011-3, marco@dme.ufcg.edu.br
The first equation in (NSP), called Schrödinger equation, describes quantum (non-relativistic) particles interacting with the electromagnetic field generated by the motion. An interesting Schrödinger equation class is when the potential \( \phi(x) \) is determined by the charge of wave function itself, that is, when the second equation in (NSP) (Poisson equation) holds.

Knowledge of the solutions for the elliptic equation

\[
\begin{aligned}
-\Delta u + \phi u &= f(u) \text{ in } \Omega, \\
-\Delta \phi &= u^2 \text{ in } \Omega, \\
u, \phi &= 0 \text{ on } \partial\Omega
\end{aligned}
\]  

(SP)

has a great importance in the study of stationary solutions \( \psi(x,t) = e^{-it}u(x) \) of (NSP) and it contains two kinds of nonlinearities: the first one is \( \phi(x)u \) and concerns the interaction with the electric field. This term is nonlocal, since the electrostatic potential \( \phi(x) \) depends also on the wave function. The second nonlinearity is \( f(u) \). For more information involving physical situations where (SP) appears, we cite the papers of Benci-Fortunato [9], Bokanowski & Mauser [11], Mauser [24], Ruiz [26], Ambrosetti-Ruiz [4] and S’anchez & Soler [28].

An important fact involving system (SP) is that this class of system can be transformed into a Schrödinger equation with a nonlocal term (see, for instance, [5, 18, 26, 29]), which allows to use variational methods. Effectively, by the Lax-Milgram Theorem, given \( u \in H^1_0(\Omega) \), there exists a unique \( \phi = \phi_u \in H^1_0(\Omega) \) such that

\[-\Delta \phi = u^2.\]

By using standard arguments, we have that \( \phi_u \) verifies the following properties (for a proof see [15, 26, 29]):

**Lemma 1.1** For any \( u \in H^1_0(\Omega) \), we have

i) there exists \( C > 0 \) such that \( ||\phi_u|| \leq C||u||^2 \) and

\[
\int_\Omega |\nabla \phi_u|^2 \, dx = \int_\Omega \phi_u u^2 \, dx \leq C||u||^4 \quad \forall u \in H^1_0(\Omega);
\]

where \( ||u|| = \int_\Omega |\nabla u|^2 \, dx \).

ii) \( \phi_u \geq 0 \ \forall u \in H^1_0(\Omega) \);

iii) \( \phi_u = t^2 \phi_u, \ \forall t > 0 \text{ and } u \in H^1_0(\Omega) \);
iv) if \( u_n \to u \) in \( H^1_0(\Omega) \), then \( \phi_{u_n} \to \phi_u \) in \( H^1_0(\Omega) \) and
\[
\lim_{n \to +\infty} \int_\Omega \phi_{u_n} u_n^2 dx = \int_\Omega \phi_u u^2 dx.
\]

Therefore, \((u, \phi) \in H^1_0(\Omega) \times H^1_0(\Omega)\) is a solution of \((SP)\) if, and only if, \( \phi = \phi_u \) and \( u \in H^1_0(\Omega) \) is a weak solution of the nonlocal problem
\[
\begin{align*}
-\Delta u + \phi_u u &= f(u) \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]

\((P)\)

Now, we would like to mention that the existence of solutions for problem \((P)\) can be made via variational methods, because if the nonlinearity \( f \) belongs to \( C^1(\mathbb{R}, \mathbb{R}) \) and satisfies
\[(f_1) \quad \lim_{s \to 0} \frac{f(s)}{s} = 0;\]
\[(f_2) \quad \lim_{|s| \to +\infty} \frac{f(s)}{s^p} = 0,\]
the Lemma [1.1] gives that the functional \( J : H^1_0(\Omega) \to \mathbb{R} \) given by
\[
J(u) = \frac{1}{2} ||u||^2 + \frac{1}{4} \int_\Omega \phi_u u^2 dx - \int_\Omega F(u) dx,
\]
where
\[
F(s) = \int_0^s f(t) dt,
\]
belongs to \( C^1(H^1_0(\Omega), \mathbb{R}) \) and
\[
J'(u)v = \int_\Omega \nabla u \nabla v dx + \int_\Omega \phi_u uv dx - \int_\Omega f(u) v dx \quad \forall u, v \in H^1_0(\Omega).
\]

Hence, critical points of \( J \) are the weak solutions for nonlocal problem \((P)\).

From the above commentaries, we have that system \((SP)\) has a nontrivial solution if, and only if, \((P)\) has a nontrivial solution. This way, in the last years, many authors that studied the system \((SP)\) have focused their attention on problem \((P)\) aiming to establish existence and nonexistence of solutions, multiplicity of solutions, ground state solutions, radial and nonradial solutions, semiclassical limit and concentrations of solution for the
case where $\Omega = \mathbb{R}^N$, see the papers of Azzollini & Pomponio [5], Cerami & Vaira [13], Coclite [14], D’Aprile & Mugnai [15, 16], d’Avenia [17], Ianni [20], Kikuchi [19], and Zhao & Zhao [29]. For the case where $\Omega$ is a bounded domain, we would like to cite the papers of Siciliano [18], Ruiz & Siciliano [27] and Pisani & Siciliano [25]. In all those papers, the solutions found are nonnegative. However, related to nodal (or sign-changing) solution, we found few papers, see for example, Ianni [21] and Kim & Seok [22]. In [21] and [22] the existence of nodal solutions have been established at balls centered origin or in whole $\mathbb{R}^3$.

Motivated by papers above, we are interested in finding nodal solution for system $(SP)$, by assuming only that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary. Once that we will apply variational methods and term $\int_\Omega \phi u u^2 dx$ is homogeneous of degree 4, the corresponding Ambrosetti-Rabinowitz condition on $f$ is the following:

(AR) There exists $\theta > 4$ such that

$$0 < \theta F(s) \leq sf(s) \quad \forall s \in \mathbb{R} \setminus \{0\}.$$  

This condition is important not only to ensure that the functional $J$ has the mountain pass geometry, but also to guarantee that the Palais-Smale, or Cerami, sequences associated with $J$ are bounded. We recall that (AR) implies a weaker condition: there exist $\theta > 4$ and $C_1, C_2 > 0$ such that

$$F(s) \geq C_1|s|^{\theta} - C_2, \quad \forall s \in \mathbb{R}.$$  

(1.1)

However, we consider here another much weaker one, namely,

($f_3$) $\lim_{s \to \pm\infty} \frac{F(s)}{s^4} = +\infty$.

Moreover, we also assume that the nonlinearity $f$ satisfies

($f_4$) $\frac{f(s)}{s^3}$ is increasing in $|s| > 0$.

**Remark 1.2** The condition ($f_4$) implies that $H(s) = sf(s) - 4F(s)$ is a non-negative function, increasing in $|s|$ with

$$sH'(s) = s^2 f'(s) - 3f(s)s > 0$$

for any $|s| > 0$. 

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Our main result is the following

**Theorem 1.3** Suppose that \( f \) satisfies \((f_1)-(f_4)\). Then problem \((P)\) possesses at least energy nodal solution, which has precisely two nodal domains.

In the proof of Theorem 1.3 we prove that functional \( J \) assumes a minimum value on the nodal set

\[ \mathcal{M} = \{ u \in \mathcal{N} : J'(u)u^+ = J'(u)u^- = 0 \text{ and } u^\pm \neq 0 \} \]

where \( u^+ = \max\{u(x),0\}, u^-(x) = \min\{u(x),0\} \) and

\[ \mathcal{N} = \{ u \in H^1_0(\Omega) \setminus \{0\} : J'(u)u = 0 \} . \]

More precisely, we prove that there is \( w \in \mathcal{M} \) such that

\[ J(w) = \inf_{u \in \mathcal{M}} J(u). \]

After, motivated by argument used in Bartsch, Weth & Willem [6], we use a deformation lemma to prove that \( w \) is a critical point of \( J \), and so, \( w \) is a least energy nodal solution for \((SP)\) with exactly two nodal domains.

Since \( J \) has the nonlocal term \( \int_\Omega \phi u^2 dx \), if \( u \) is a nodal solution for \( J \), we have that

\[ J'(u^+)^u = -\int_\Omega \phi u^-(u^+)^2 < 0 \text{ and } J'(u^-)u^- = -\int_\Omega \phi u^+(u^-)^2 < 0 . \]

From this, some arguments used to prove the existence of nodal solutions for problem like

\[ \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0, & \text{on } \partial \Omega \end{cases} \quad (P_1) \]

can not be used, and so, a careful analysis is necessary in a lot of estimates, see Section 2 for details.

Before to conclude this introduction, we would like to cite the papers of Alves [1], Alves & Soares [2, 3], Bartsch, Weth and Willem [6], Bartsch & Weth [7], Bartsch, Liu & Weth [8], Castro, Cossio & Neuberger [12], Zou [30] and their references, where existence of nodal solution has been studied for problem related to \((P_1)\).

The paper is organized as follows. In Section 2, we show some estimates involving functions that change sign, with the most of them being new for problem \((P)\). The Section 3 is devoted to prove the main result Theorem 1.3.
2 Important estimates

In what follows, we denote by \( \mathcal{N} \) the Nehari manifold associated with \( J \), that is,

\[
\mathcal{N} = \{ u \in H^1_0(\Omega) \setminus \{0\} : J'(u)u = 0 \}.
\]

A critical point \( u_0 \neq 0 \) of \( J \) is a ground state of (\( P \)) if

\[
J(u_0) = \inf_{\mathcal{N}} J(u).
\]

Since we are looking for least energy nodal solutions (or sign-changing solutions), our goal is to prove the existence of a critical point for \( J \) in the set

\[
\mathcal{M} = \{ u \in \mathcal{N} : J'(u)u^+ = J'(u)u^- = 0 \text{ and } u^\pm \neq 0 \}.
\]

Let us start with some technical lemmas.

**Lemma 2.1** There exists \( \rho > 0 \) such that

(i) \( J(u) \geq ||u||^2/4 \) and ||\( u || \geq \rho, \forall u \in \mathcal{N} \);

(ii) \( ||w^\pm|| \geq \rho, \forall w \in \mathcal{M} \).

**Proof:** From (\( f_4 \)) and Remark 1.2, for any \( u \in \mathcal{N} \)

\[
4J(u) = 4J(u) - J'(u)u = ||u||^2 + \int_{\Omega} [uf(u) - 4F(u)]dx \geq ||u||^2
\]

and so,

\[
J(u) \geq ||u||^2/4 \quad \forall u \in \mathcal{N}.
\]

From (\( f_1 \)) and (\( f_2 \)), there is \( C > 0 \) such that

\[
f(s)s \leq \frac{\lambda_1}{2}s^2 + Cs^6, \text{ for all } s \in \mathbb{R}.
\]

where \( \lambda_1 \) is the first eigenvalue of \((-\Delta, H^1_0(\Omega)). Since J'(u)u = 0, \)

\[
||u||^2 < ||u||^2 + \int_{\Omega} \phi_a |u|^2 dx = \int_{\Omega} uf(u)dx \leq \frac{\lambda_1}{2} \int_{\Omega} u^2 dx + C \int_{\Omega} u^6 dx.
\]

Then by Sobolev embeddings,

\[
||u||^2 < \frac{1}{2}||u||^2 + \hat{C}||u||^6,
\]
from where it follows that

\[ ||u|| \geq \rho \quad \forall u \in \mathcal{N}, \]

where \( \rho = \left( \frac{1}{2C} \right)^{\frac{1}{2}} \), finishing the proof of (i).

If \( w \in \mathcal{M} \), we have that \( J'(w)w^\pm = 0 \). Then, a simple computation gives \( J'(w^\pm)w^\pm < 0 \), which implies

\[ ||w^\pm||^2 < ||w^\pm||^2 + \int_\Omega \phi_{w^\pm}(w^\pm)^2 dx < \int_\Omega f(w^\pm)w^\pm dx. \]

As in the item (i), we can deduce that \( ||w^\pm|| \geq \rho. \)

**Lemma 2.2** If \((w_n)\) is a bounded sequence in \( \mathcal{M} \) and \( p \in (2, 6) \), we have

\[ \lim \inf_n \int_\Omega |w_n^\pm|^p dx > 0. \]

**Proof:** From \((f_1)\) and \((f_2)\), given \( \varepsilon > 0 \) there exists \( C > 0 \) such that

\[ f(s)s \leq \varepsilon \lambda_1 s^2 + C|s|^p + \varepsilon s^6, \quad \text{for all } s \in \mathbb{R}. \]

Since \( w_n \in \mathcal{M} \), by Lemma 2.1

\[ \rho^2 \leq ||w_n^\pm||^2 < \int_\Omega w_n^+ f(w_n^+) dx \leq \varepsilon \lambda_1 \int_\Omega (w_n^+)^2 dx + C \int_\Omega |w_n^+|^p dx + \varepsilon \int_\Omega (w_n^+)^6 dx, \]

that is,

\[ \rho^2 \leq \varepsilon \left( \lambda_1 \int_\Omega (w_n^+)^2 dx + \int_\Omega (w_n^+)^6 dx \right) + C \int_\Omega |w_n^+|^p dx. \]

Using the boundedness of \((w_n)\), there is \( C_1 \) such that

\[ \rho^2 \leq \varepsilon C_1 + C \int_\Omega |w_n^+|^p dx. \]

Fixing \( \varepsilon = \frac{\rho^2}{2C_1} \), we get

\[ \int_\Omega |w_n^+|^p dx \geq \frac{\rho^2}{2C}, \]

showing that

\[ \lim \inf_n \int_\Omega |w_n^+|^p dx \geq \frac{\rho^2}{2C} > 0. \]
Lemma 2.3 Let $v \in H_0^1(\Omega)$ with $v^\pm \neq 0$. Then, there are $t, s > 0$ such that $J'(tv^+ + sv^-)v^+ = 0$ and $J'(tv^+ + sv^-)v^- = 0$.

Proof: It what follows, we consider the vector field

$$V(s, t) = (J'(tv^+ + sv^-)(tv^+), J'(tv^+ + sv^-)(sv^-)).$$

from $(f_1) - (f_3)$, a straightforward computation yields that there are $0 < r < R$ such that

$$J'(rv^+ + sv^-)(rv^+), J'(tv^+ + rv^-)(rv^-) > 0, \forall s, t \in [r, R]$$

and

$$J'(Rv^+ + sv^-)(Rv^+), J'(tv^+ + Rv^-)(Rv^-) < 0, \forall s, t \in [r, R].$$

Now, the lemma follows applying Miranda theorem [23].

Hereafter, for $v \in H_0^1(\Omega)$ with $v^\pm \neq 0$, we consider the functions $h^v : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ given by

$$h^v(t, s) = J(tv^+ + sv^-)$$

and $\Phi^v : [0, +\infty) \times [0, +\infty) \to \mathbb{R}^2$ defined as

$$\Phi^v(t, s) = \left(\frac{\partial h^v}{\partial t}(t, s), \frac{\partial h^v}{\partial s}(t, s)\right) = \left(J'(tv^+ + sv^-)v^+, J'(tv^+ + sv^-)v^-\right).$$

Since $f$ is a $C^1$ function, it follows that $\Phi^v$ is also a $C^1$ map. Moreover, it is easy to check that if $(t, s)$ is a critical point of $h^v$, then

$$h^v(t, s) = h^v(t, s) - \frac{1}{4} \langle \nabla h^v(t, s), (t, s) \rangle$$

$$= \frac{1}{4} t^2 ||v^+||^2 + \frac{1}{4} \int_{\Omega} [f(tv^+)tv^+ - 4F(tv^+)] dx +$$

$$\frac{1}{4} s^2 ||v^-||^2 + \frac{1}{4} \int_{\Omega} [f(sv^-)tv^- - 4F(sv^-)] dx.$$
Lemma 2.4 If \( w \in \mathcal{M} \),

(a) \( h^w(t, s) < h^w(1, 1) = J(w) \), for all \( s, t \geq 0 \) such that \( (s, t) \neq (1, 1) \);

(b) \( \det(\Phi^w)'(1, 1) > 0 \).

Proof: Once that \( w \in \mathcal{M} \), we have

\[
|w| + \int_\Omega \phi_w(w^+)^2 dx + \int_\Omega \phi_w(w^-)^2 dx = \int_\Omega f(w^+)w^+ dx
\]

and

\[
|w| - \int_\Omega \phi_w(w^-)^2 dx + \int_\Omega \phi_w(w^+)^2 dx = \int_\Omega f(w^-)w^- dx.
\]

These equalities imply that \((1, 1)\) is a critical point of \( h^w \). On the other hand, condition \((f_3)\) leads to the limit

\[
\lim_{{|\langle t, s \rangle | \to \infty}} h^w(t, s) = -\infty,
\]

which implies \( h^w \) assumes a global maximum in some \((a, b)\).

First of all, we claim that \( a, b > 0 \). If \( b = 0 \), we have

\[
J(aw^+) \geq J(tw^+), \quad \forall t > 0
\]

and then \( J'(aw^+)(aw^+) = 0 \), or equivalently,

\[
a^2|w^+|^2 + a^4 \int_\Omega \phi_w(w^+)^2 dx = \int_\Omega f(aw^+)aw^+ dx.
\]

Since \( J'(w^+)w^+ = J'(w)w^+- \int_\Omega \phi_w(w^+)^2 dx < 0 \), we derive

\[
|w^+|^2 + \int_\Omega \phi_w(w^+)^2 dx < \int_\Omega f(w^+)w^+ dx
\]

and so,

\[
\left(1 - \frac{1}{a^2}\right) |w^+|^2 < \int_\Omega \left( \frac{f(w^+)w^+}{(w^+)^4} - \frac{f(aw^+)aw^+}{(aw^+)^4} \right) (w^+)^4 dx.
\]
If \( a > 1 \) the left side in this inequality is positive while, from \((f_4)\), the right side is negative. This information gives that \( a \leq 1 \). Now, combining the Remark 1.2 with the fact that \( a \leq 1 \), we get

\[
h^w(a,0) = J(aw^+) = J(aw^+) - \frac{1}{4} J'(aw^+)(aw^+) = \]
\[
= \frac{1}{4} a^2||w^+||^2 + \frac{1}{4} \int_\Omega [f(aw^+)aw^+ - 4F(aw^+)]dx
\]
\[
\leq \frac{1}{4} ||w^+||^2 + \frac{1}{4} \int_\Omega [f(w^+)w^+ - 4F(w^+)]dx\]
\[
< \frac{1}{4} ||w^+||^2 + \frac{1}{4} \int_\Omega [f(w^+)w^+ - 4F(w^+)]dx + \]
\[
+ \frac{1}{4} ||w^-||^2 + \frac{1}{4} \int_\Omega [f(w^-)w^- - 4F(w^-)]dx =
\]
\[
= J(w) - \frac{1}{4} J'(w)w = J(w) = h(1,1)
\]
that is,

\[
h^w(a,0) < h^w(1,1)
\]

which is absurd, because \((a,0)\) is a global maximum point for \( h^w \). The same type of argument works to show that \( a \neq 0 \), and the proof of claim is done.

The second claim is \( 0 < a, b \leq 1 \). In fact, since \((a,b)\) is another critical point of \( h^w \),

\[
a^2||w^+||^2 + a^4 \int_\Omega \phi_{w^+}(w^+)^2dx + a^2b^2 \int_\Omega \phi_{w^-}(w^+)^2dx = \int_\Omega f(aw^+)aw^+dx
\]

and

\[
b^2||w^-||^2 + b^4 \int_\Omega \phi_{w^-}(w^-)^2dx + a^2b^2 \int_\Omega \phi_{w^+}(w^-)^2dx = \int_\Omega f(bw^-)bw^-dx.
\]

Without loss of generality, we will suppose that \( a \geq b \). From this,

\[
a^2||w^+||^2 + a^4 \int_\Omega \phi_{w^+}(w^+)^2dx + a^4 \int_\Omega \phi_{w^-}(w^+)^2dx \geq \int_\Omega f(aw^+)aw^+dx
\]

leading to

\[
\left( \frac{1}{a^2} - 1 \right) ||w^+||^2 \geq \int_\Omega \left( \frac{f(aw^+)aw^+}{(aw^+)^4} - \frac{f(w^+)w^+}{(w^+)^4} \right) (w^+)^4 dx.
\]
If \( a > 1 \) the left side in this inequality is negative, but from \((f_4)\), the right side is positive, thus we can deduce that \( a \leq 1 \).

To conclude the proof of item \((a)\), we will show that \( h^w \) does not have global maximum in \([0,1] \times [0,1] \setminus \{(1,1)\}\). From definition of \( h^w \),

\[
h^w(a, b) = \frac{1}{4} a^2 ||w^+||^2 + \frac{1}{4} \int_{\Omega} [f(aw^+) aw^+ - 4F(aw^+)] dx + \frac{1}{4} b^2 ||w^-||^2 + \frac{1}{4} \int_{\Omega} [f(bw^-) bw^- - 4F(bw^-)] dx.
\]

Then, if \( 0 < a, b \leq 1 \) and \((a, b) \neq (1,1)\),

\[
h^w(a, b) < \frac{1}{4} ||w^+||^2 + \frac{1}{4} \int_{\Omega} [f(w^+) w^+ - 4F(w^+)] dx + \frac{1}{4} ||w^-||^2 + \frac{1}{4} \int_{\Omega} [f(w^-) w^- - 4F(w^-)] dx = h^w(1,1)
\]

showing that, \( h^w(a, b) < h^w(1,1) \)

and thereby, the proof of item \((a)\) is complete.

The proof of item \((b)\) is the following. By a simple calculation

\[
\det(\Phi^w)'(1,1) = G(w^+)G(w^-) - 4 \left[ \int_{\Omega} \phi_{w^-}(w^+)^2 dx \right]^2
\]

where

\[
G(v) = \int_{\Omega} [f'(v)v^2 - f(v)v] dx - 2 \int_{\Omega} \phi_v v^2 dx.
\]

From Remark 1.2

\[
G(v) \geq 2 \left[ \int_{\Omega} f(v)v dx - \int_{\Omega} \phi_v v dx \right].
\]

Once that

\[
\int_{\Omega} f(w^+)w^+ dx - \int_{\Omega} \phi_{w^+}(w^+)^2 dx = ||w^+||^2 + \int_{\Omega} \phi_{w^-}(w^+)^2 dx
\]

and

\[
\int_{\Omega} \phi_{w^-}(w^+)^2 dx = \int_{\Omega} \phi_{w^+}(w^-)^2 dx,
\]

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we have that
\[ G(w^+) > 2 \int_{\Omega} \phi_{w^+} (w^+) dx \]
and
\[ G(w^-) > 2 \int_{\Omega} \phi_{w^-} (w^+) dx. \]
Combining the above informations, it follows that \( \det(\Phi_w)'(1, 1) > 0. \)

**Corollary 2.5** Let \( v \in H^1_0(\Omega) \) be a function verifying
\[ v^+ \neq 0 \text{ and } J'(v)v^\pm \leq 0. \]
Then, there are \( t, s \in [0, 1] \) such that
\[ tv^+ + sv^- \in \mathcal{M}. \]

**Proof.** An immediate consequence of the arguments used in the proof of Lemma 2.4.

3 Existence of least energy nodal solution.

In this section, our main goal is to prove the Theorem 1.3. In what follows, we denote by \( c_0 \) the infimum of \( J \) on \( \mathcal{M} \), that is,
\[ c_0 = \inf_{v \in \mathcal{M}} J(v). \]
From Lemma 2.1(i), we deduce that \( c_0 > 0. \)

Let \( (w_n) \) be a sequence in \( \mathcal{M} \) such that
\[ \lim_{n} J(w_n) = c_0. \]
Still from Lemma 2.1(i), \( (w_n) \) is a bounded sequence. Hence, without loss of generality, we can suppose that there is \( w \in H^1_0(\Omega) \) verifying
\[ w_n \rightharpoonup w \text{ in } H^1_0(\Omega), \]
\[ w_n \to w \text{ in } L^p(\Omega) \quad \forall \, p \in [1, 2^*). \]
and 

\[ w_n(x) \to w(x) \text{ a.e. in } \Omega. \]

The condition \((f_2)\) combined with the compactness lemma of Strauss [10, Theorem A.I, p.338] gives

\[ \lim_{n} \int_{\Omega} |w_n^\pm|^p dx = \int_{\Omega} |w^\pm|^p dx, \]

\[ \lim_{n} \int_{\Omega} w_n^\pm f(w_n^\pm) dx = \int_{\Omega} w^\pm f(w^\pm) dx \]

and

\[ \lim_{n} \int_{\Omega} F(w_n^\pm) dx = \int_{\Omega} F(w^\pm) dx, \]

from where it follows together with Lemma 2.2 that \(w^\pm \neq 0\). Then, by Lemma 2.3 there are \(t, s > 0\) verifying

\[ J'(tw^+ + sw^-)w^+ = 0 \text{ and } J'(tw^+ + sw^-)w^- = 0. \]

Next, we will show that \(t, s \leq 1\). Since \(J'(w_n)w_n^\pm = 0\),

\[ ||w_n^+||^2 + \int_{\Omega} \phi_{w_n^+}(w_n^+)^2 dx + \int_{\Omega} \phi_{w_n^-}(w_n^+)^2 dx = \int_{\Omega} f(w_n^+)w_n^+ dx \]

and

\[ ||w_n^-||^2 + \int_{\Omega} \phi_{w_n^-}(w_n^-)^2 dx + \int_{\Omega} \phi_{w_n^+}(w_n^-)^2 dx = \int_{\Omega} f(w_n^-)w_n^- dx. \]

Taking the limit in the above equalities, we obtain

\[ ||w^+||^2 + \int_{\Omega} \phi_{w^+}(w^+)^2 dx + \int_{\Omega} \phi_{w^-}(w^+)^2 dx \leq \int_{\Omega} f(w^+)w^+ dx \]

and

\[ ||w^-||^2 + \int_{\Omega} \phi_{w^-}(w^-)^2 dx + \int_{\Omega} \phi_{w^-}(w^-)^2 dx \leq \int_{\Omega} f(w^-)w^- dx. \]

Once that

\[ J'(tw^+ + sw^-)(tw^+) = J'(tw^+ + sw^-)(sw^-) = 0, \]

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it follows that
\[ t^2 ||w^+||^2 + t^4 \int_{\Omega} \phi_{w^+}(w^+)^2 dx + t^2 s^2 \int_{\Omega} \phi_{(w^-)^2} dx = \int_{\Omega} f(tw^+)tw^+ dx \]
and
\[ s^2 ||w^-||^2 + s^4 \int_{\Omega} \phi_{w^-}(w^-)^2 dx + t^2 s^2 \int_{\Omega} \phi_{(w^+)^2} dx = \int_{\Omega} f(sw^-)sw^- dx. \]

Now, without loss of generality, we will suppose that \( s \geq t \). Under this condition,
\[ s^2 ||w^-||^2 + s^4 \int_{\Omega} \phi_{w^-}(w^-)^2 dx + s^4 \int_{\Omega} \phi_{w^-}(w^+)^2 dx \geq \int_{\Omega} f(sw^-)sw^- dx \]
and then
\[ \left( \frac{1}{s^2} - 1 \right) ||w^-||^2 \geq \int_{\Omega} \left( \frac{f(sw^-)sw^-}{(sw^-)^4} - \frac{f(w^-)w^-}{(w^-)^4} \right) (w^-)^4 dx. \]

If \( s > 1 \), the left side in this inequality is negative, but from \((f_4)\), the right side is positive, thus we must have \( s \leq 1 \), which also implies that \( t \leq 1 \).

Our next step is show that \( J(tw^+ + sw^-) = c_0 \). Recalling that \( tw^+ + sw^- \in M \), we derive that
\[ c_0 \leq J(tw^+ + sw^-) = J(tw^+ + sw^-) - \frac{1}{4} J'(tw^+ + sw^-)(tw^+ + sw^-). \]

Thus,
\[ c_0 \leq \left( J(tw^+) - \frac{1}{4} J'(tw^+)(tw^+) \right) + \left( J(sw^-) - \frac{1}{4} J'(sw^-)(sw^-) \right). \]

From Remark \[1.2\]
\[ J(tw^+) - \frac{1}{4} J'(tw^+)(tw^+) \leq J(w^+) - \frac{1}{4} J'(w^+)(w^+) \]
and
\[ J(sw^-) - \frac{1}{4} J'(sw^-)(sw^-) \leq J(w^-) - \frac{1}{4} J'(w^-)(w^-). \]

Hence,
\[ c_0 \leq \left( J(w^+) - \frac{1}{4} J'(w^+)(w^+) \right) + \left( J(w^-) - \frac{1}{4} J'(w^-)(w^-) \right). \]
Using Fatous’ Lemma combined again with Remark 1.2,
\[ c_0 \leq J(tw^+ + sw^-) \leq \liminf_n \left( J(w_n) - \frac{1}{4}J'(w_n)w_n \right) = \lim_n J(w_n) = c_0 \]
from where it follows that
\[ c_0 = J(tw^+ + sw^-). \]

Until this moment, we have proved that there exists a \( w_o = tw^+ + sw^- \in M \), such that \( J(w_o) = c_0 \). In what follows, let us denote \( w_o \) by \( w \), consequently
\[ J(w) = c_0 \text{ and } w \in M. \]

To conclude the proof of Theorem 1.3, we claim that \( w \) is a critical point for functional \( J \). If it is not true, there exist \( \alpha > 0 \) and \( v_0 \in H^{1}_0(\Omega) \) with \( ||v_0|| = 1 \) satisfying
\[ J'(w)v_0 = 2\alpha > 0. \]
Since \( J' \) is continuous, we fix \( r > 0 \) such that
\[ J'(v)v_0 > \alpha, v^\pm \neq 0, \text{ for all } v \in B_r(w) \subset H^1_0(\Omega). \]
From now on, fix \( D = (\xi, \chi) \times (\xi, \chi) \subset \mathbb{R}^2 \) with \( 0 < \xi < 1 < \chi \) such that
(i) \( (1,1) \in D \) and \( \Phi^w(t,s) = 0 \) in \( \overline{D} \) if, and only if, \( t = s = 1; \)
(ii) \( c_0 \notin h^w(\partial D); \)
(iii) \( \{tw^+ + sw^- : (t,s) \in \overline{D}\} \subset B_r(w); \)
where \( h^w \) and \( \Phi^w \) were defined in Lemma 2.4. Since \( J \) is continuous, we can fix \( r' > 0 \) such that
\[ B = B_{r'}(w) \subset B_r(w) \]
and
\[ B \cap \{tw^+ + sw^- : (t,s) \in \partial D\} = \emptyset. \]
Consider the continuous mapping \( \rho : H^1_0(\Omega) \to [0, +\infty), \) defined by
\[ \rho(u) = \text{dist}(u, B^c). \]
Moreover, set the bounded Lipschitz vector field \( V : H^1_0(\Omega) \to H^1_0(\Omega) \) given by
\[ V(u) = -\rho(u)v_0. \]
For each \( u \in H_0^1(\Omega) \), we denote by \( \eta(\tau) = \eta(\tau, u) \) the unique solution of ODE

\[
\begin{cases}
\eta'(\tau) = V(\eta(\tau)), & t > 0 \\
\eta(0) = u.
\end{cases}
\]

Observe that

1. if \( u \not\in \mathcal{B} \), \( \eta(\tau, u) = u \), for all \( t \);
2. if \( u \in \mathcal{B} \), \( \tau \mapsto J(\eta(\tau, u)) \) is decreasing and \( \eta(\tau, u) \in \mathcal{B} \), for all \( \tau > 0 \);
3. there exists \( \tau_o > 0 \) such that \( J(\eta(\tau, w)) \leq J(w) - ((r'\alpha)/2)\tau \), for all \( 0 \leq \tau \leq \tau_o \).

The item (1) is an immediate consequence from the definition of \( \rho \). The item (2) follows from the inequality

\[
J'(\eta(\tau))\eta'(\tau) \leq -\rho(\eta(\tau))\alpha < 0, \ \forall \eta(\tau) \in \mathcal{B}.
\]

To verify (3), fix \( \tau_o > 0 \) such that

\[
||\eta(\tau, w) - w|| \leq \frac{r'}{2}, \ \text{for all} \ |\tau| \leq \tau_o.
\]

Thus,

\[
\frac{d}{dt} J(\eta(\tau, w)) \leq -\rho(\eta(\tau))\alpha \leq -\frac{r'\alpha}{2}.
\]

Integrating in \([0, \tau_0]\), we have

\[
J(\eta(\tau_0, w)) \leq J(w) - \frac{r'\alpha}{2}\tau_0.
\]

Now, consider \( \gamma : \overline{D} \to H_0^1(\Omega) \) given by.

\[
\gamma(t, s) = \eta(\tau_o, tw^+ + sw^-).
\]

It is easy to see that

\[
\max_{(t,s) \in \overline{D}} J(\gamma(t, s)) < c_0,
\]

because

\[
J(\gamma(t, s)) \leq h^w(t, s) < c_0 \ \forall (t, s) \in \overline{D} \ \backslash \ \{(1, 1)\}
\]
and
\[ J(\gamma(1, 1)) \leq J(w) - ((r'\alpha)/2))\tau_o < c_0. \]
Consequently \( \gamma(D) \cap M = \emptyset. \)

On the other hand, setting \( \Psi : D \rightarrow \mathbb{R}^2 \) by
\[ \Psi(t, s) = (t^{-1}J'(\gamma(t, s))(\gamma(t, s)^+), s^{-1}J'(\gamma(t, s))(\gamma(t, s)^-)), \]
we derive that
\[ \Psi(t, s) = (J'(tw^+ + sw^-)w^+, J'(tw^+ + sw^-)w^-) = \Phi^w(t, s) \quad \forall (t, s) \in \partial D. \]

Then, using the Brouwer’s topological degree
\[ d(\Psi, D, (0, 0)) = d(\Phi^w, D, (0, 0)) = \text{sgn}(\det(\Phi^w)'(1, 1)) = 1 \]
which yields \( \Psi \) has a zero \((a, b)\) in \( D \). Thereby, there is \((a, b) \in D\) verifying
\[ J'(\gamma(a, b))(\gamma(a, b)^\pm) = 0, \]
that is, \( \gamma(a, b) \in M \) which is a contradiction. From this, \( w \) is a critical point of \( J \), and so, a nodal solution for problem \((P)\). Now, we will show that \( w \) has exactly two nodal domains, to this end, we assume by contradiction that
\[ w = u_1 + u_2 + u_3 \]
with
\[ u_i \neq 0, u_1 \geq 0, u_2 \leq 0 \quad \text{and} \quad \text{suppt}(u_i) \cap \text{suppt}(u_j) = \emptyset \quad i \neq j \quad (i, j = 1, 2, 3). \]

Setting \( v = u_1 + u_2 \), we see that \( v^\pm \neq 0 \). Moreover, using the fact that \( J'(w) = 0 \), it follows that
\[ J'(v)(v^\pm) \leq 0. \]

By Corollary 2.5 there are \( t, s \in (0, 1] \) such that
\[ tv^+ + sv^- \in M \]
or equivalently,
\[ tu_1 + su_2 \in M, \]
and so,
\[ J(tu_1 + su_2) \geq c_0. \]
On the other hand, repeating the same type of argument explored in the proof of Lemma 2.4 combined with the fact that $u_3 \neq 0$, we find

$$J(tu_1 + su_2) < J(w) = c_0,$$

obtaining a contradiction. This way, $u_3 = 0$, and $w$ has exactly two nodal domains.

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