Compact-like abelian groups without non-trivial quasi-convex null sequences*

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Abstract

In this paper, we study precompact abelian groups \( G \) that contain no sequence \( \{x_n\}_{n=0}^{\infty} \) such that \( \{0\} \cup \{\pm x_n \mid n \in \mathbb{N}\} \) is infinite and quasi-convex in \( G \), and \( x_n \rightarrow 0 \). We characterize groups with this property in the following classes of groups:
(a) bounded precompact abelian groups;
(b) minimal abelian groups;
(c) totally minimal abelian groups;
(d) \( \omega \)-bounded abelian groups.

We also provide examples of minimal abelian groups with this property, and show that there exists a minimal pseudocompact abelian group with the same property; furthermore, under Martin’s Axiom, the group may be chosen to be countably compact minimal abelian.

1. Introduction

This note is a sequel to [7], and its aim is to provide an answer to Problem I posed in that paper.

One of the main sources of inspiration for the theory of topological groups is the theory of topological vector spaces, where the notion of convexity plays a prominent role. In this context, the reals \( \mathbb{R} \) are replaced with the circle group \( \mathbb{T} := \mathbb{R}/\mathbb{Z} \), and linear functionals are replaced by characters, that is, continuous homomorphisms to \( \mathbb{T} \). By making substantial use of characters, Vilenkin introduced the notion of quasi-convexity for abelian topological groups as a counterpart of convexity in topological vector spaces (cf. [30]).

Let \( \pi : \mathbb{R} \rightarrow \mathbb{T} \) denote the canonical projection. Since the restriction \( \pi_{|(-\frac{1}{2}, \frac{1}{2})} : (-\frac{1}{2}, \frac{1}{2}) \rightarrow \mathbb{T} \) is a bijection, we often identify in what follows, par abus de language, a number \( a \in (-\frac{1}{2}, \frac{1}{2}) \) with its image (coset) \( \pi(a) = a + \mathbb{Z} \in \mathbb{T} \). We put \( \mathbb{T}_m := \pi([-\frac{1}{4m}, \frac{1}{4m}]) \) for all \( m \in \mathbb{N}\setminus\{0\} \). According to standard notation in this area, we use \( \mathbb{T}_+ \) to denote \( \mathbb{T}_1 \). For an abelian topological group \( G \), we denote by \( \hat{G} \) the Pontryagin dual of \( G \), that is, the group of all characters of \( G \) endowed with the compact-open topology.
Definition 1.1. For \( E \subseteq G \) and \( A \subseteq \hat{G} \), the polar of \( E \) and the prepolar \( A \) are defined as
\[
E^\circ = \{ \chi \in \hat{G} \mid \chi(E) \subseteq T_+ \} \quad \text{and} \quad A^\circ = \{ x \in G \mid \forall \chi \in A, \chi(x) \in T_+ \}.
\]
The set \( E \) is said to be quasi-convex if \( E = E^\circ \).

Obviously, \( E \subseteq E^\circ \) holds for every \( E \subseteq G \). Thus, \( E \) is quasi-convex if and only if for every \( x \in G \setminus E \) there exists \( \chi \in E^\circ \) such that \( \chi(x) \notin T_+ \). The set \( Q_G(E) := E^{\circ\circ} \) is the smallest quasi-convex set of \( G \) that contains \( E \), and it is called the quasi-convex hull of \( E \).

Definition 1.2. A sequence \( \{ x_n \}_{n=0}^\infty \subseteq G \) is said to be quasi-convex if \( S = \{ 0 \} \cup \{ \pm x_n \mid n \in \mathbb{N} \} \) is quasi-convex in \( G \). We say that \( \{ x_n \}_{n=0}^\infty \) is non-trivial if the set \( S \) is infinite, and it is a null sequence if \( x_n \longrightarrow 0 \).

Examples 1.3. It turns out that most “common” groups contain a non-trivial quasi-convex null sequence:
(a) For every prime \( p \), the group \( \mathbb{J}_p \) of \( p \)-adic integers admits a non-trivial quasi-convex null sequence contained in the subgroup \( \mathbb{Z} \) (cf. \([6, 1.4]\), \([8, \text{Theorem D}]\), and \([7, \text{Theorem B}]\)).
(b) For every prime \( p \), \( \mathbb{T} \) admits a non-trivial quasi-convex null sequence contained in the \( p \)-component of \( \mathbb{T} \) (cf. \([6, 1.1]\), \([8, \text{Theorem B}]\), and \([7, \text{Theorem C}]\)).
(c) If \( \{ m_k \}_{k=1}^\infty \) is a sequence of integers such that \( m_k \geq 4 \) for every \( k \in \mathbb{N} \), then \( \bigoplus_{k=0}^\infty \mathbb{Z}_{m_k} \) admits a non-trivial quasi-convex null sequence contained in the subgroup \( \bigoplus_{k=0}^\infty \mathbb{Z}_{m_k} \) (cf. \([7, 5.5]\)).

In our paper \([7]\), we characterized the locally compact abelian groups that admit no non-trivial quasi-convex null sequences as follows.

Theorem 1.4. ([7, Theorem A]) For every locally compact abelian group \( G \), the following statements are equivalent:
(i) \( G \) admits no non-trivial quasi-convex null sequences;
(ii) one of the subgroups \( G[2] = \{ g \in G \mid 2g = 0 \} \) or \( G[3] = \{ g \in G \mid 3g = 0 \} \) is open in \( G \);
(iii) \( G \) contains an open compact subgroup of the form \( \mathbb{Z}_\kappa \) or \( \mathbb{Z}_\kappa \) for some cardinal \( \kappa \).
Furthermore, if \( G \) is compact, then these conditions are also equivalent to:
(iv) \( G \cong \mathbb{Z}_\kappa \times F \) or \( G \cong \mathbb{Z}_\kappa \times F \), where \( \kappa \) is some cardinal and \( F \) is a finite abelian group;
(v) one of the subgroups \( 2G \) and \( 3G \) is finite.

We also asked whether it was possible to replace the class of locally compact abelian groups with a different class that contains all compact abelian groups (cf. \([7, \text{Theorem A}]\)). In this note, we present several classes, and characterizations of groups in these classes that admit no non-trivial quasi-convex null sequences. To that end, we recall some compactness-like properties.

Definition 1.5. Let \( G \) be a (Hausdorff) topological group.
(a) \( G \) is precompact if it can be covered by finitely many translates of any neighborhood of the identity, or equivalently, if it is a dense subgroup of a compact group \( \hat{G} \), its completion;
(b) \( G \) is minimal if there is no coarser (Hausdorff) group topology (cf. \([29]\) and \([12]\));
(c) \( G \) is totally minimal if every (Hausdorff) quotient group \( G \) is minimal (cf. \([10]\));
(d) \( G \) is pseudocompact if every real-valued continuous function on \( G \) is bounded (cf. [15, 3.10]);
(e) \( G \) is countably compact if every countable open cover of \( G \) admits a finite subcover (cf. [15, 3.10]);
(f) \( G \) is \( \omega \)-bounded if every countable subset of \( G \) is contained in a compact subgroup of \( G \).

By the celebrated Prodanov-Stoyanov Theorem, every minimal abelian group is precompact (cf. [26] and [27]), and thus the relationships among the aforementioned properties can be described as follows.

\[ \omega \text{-bounded} \implies \text{countably compact} \implies \text{pseudocompact} \implies \text{precompact} \]

\[ \text{totally minimal} \implies \text{minimal} \implies \text{minimal abelian} \implies \text{precompact} \]

For greater clarity, all of these implications, except for the very last one, hold without the assumption that the group in question is abelian.

The classes of groups studied in this paper overlap with those investigated in [7] only to the smallest possible extent, because every group that is both precompact and locally compact is actually compact. Thus, the present note is complementary to our work in [7]. Our first two results demonstrate the level of complexity of the problem of finding non-trivial quasi-convex null sequences when one leaves the class of locally compact abelian groups. Indeed, in the absence of local compactness, Theorem 1.4 may fail even in the presence of strong compactness-like properties.

**Theorem A.** Let \( p = 2 \) or \( p = 3 \), and let \( \kappa \) be an infinite cardinal.
(a) There exists a minimal abelian group \( G \) of exponent \( p^2 \) such that \( |pG| = \kappa \) and \( G \) admits no non-trivial quasi-convex null sequences.
(b) If \( \kappa^\omega = \kappa \), then there exists a minimal pseudocompact abelian group of exponent \( p^2 \) such that \( |pG| = \kappa \) and \( G \) admits no non-trivial quasi-convex null sequences.

In part (b) of Theorem A, \( pG \) is a pseudocompact group (being a continuous image of \( G \)), and thus \( |pG| = \kappa \) must satisfy certain constraints that the size of every infinite pseudocompact homogeneous space does: \( \kappa \geq \aleph_0 \), and \( \kappa \) cannot be a strong limit cardinal of countable cofinality (cf. [14, 1.2, 1.3(a)]). We note that both of these conditions follow from the hypothesis \( \kappa^\omega = \kappa \). Under the Generalized Continuum Hypothesis (GCH), \( \kappa^\omega > \kappa \) for a cardinal \( \kappa \geq \aleph_0 \) if and only if \( \kappa \) is a strong limit cardinal of countable cofinality. Therefore, under GCH, the hypothesis \( \kappa^\omega = \kappa \) is not only sufficient but also necessary for the existence of a group \( G \) as in Theorem A(b).

**Theorem B.** Under Martin’s Axiom (or the Continuum Hypothesis), there exists a countably compact minimal abelian group of exponent 4 such that \( 2G \) is infinite and \( G \) admits no non-trivial quasi-convex null sequences.

The proofs of Theorems A and B are presented in §3 and they are based on “lifting” known examples of precompact abelian groups of exponent \( p \) that admit no non-trivial convergent sequences at all (cf. [16], [5, Theorem 3], [13, 8.1 & 9], and [18]). Although Theorem B may appear to suggest a negative answer to [7, Problem II], it is possible to establish a criterion in the spirit of Theorem 1.4 for precompact groups of a finite exponent. To that end, recall that a subset \( A \) of a topological space \( X \) is **sequentially open** if for every convergent sequence \( \{x_n\} \subseteq X \) such that \( \lim x_n \in A \), one has \( x_n \in A \) for all but finitely many \( n \).
**Theorem C.** For every bounded precompact abelian group $G$, the following statements are equivalent:

(i) $G$ admits no non-trivial quasi-convex null sequences;
(ii) one of the subgroups $G[2] = \{g \in G \mid 2g = 0\}$ or $G[3] = \{g \in G \mid 3g = 0\}$ is sequentially open in $G$.

The proof of Theorem C is presented in §4.

Certainly, the entire problem of searching for non-trivial quasi-convex null sequences is meaningless in a group that admits no non-trivial convergent sequences at all. While $\omega$-bounded groups are known to have non-trivial convergent sequences, our interest in minimal abelian groups is also motivated by a recent result of Shakhmatov, which guarantees the existence of non-trivial convergent sequences in minimal abelian groups (cf. [28, 1.3]).

**Theorem D.** For every minimal abelian group $G$, the following statements are equivalent:

(i) $G$ admits no non-trivial quasi-convex null sequences;
(ii) $G \cong P \times F$, where $P$ is a minimal bounded abelian $p$-group ($p \leq 3$) admitting no non-trivial quasi-convex null sequences, and $F$ is a finite abelian group;
(iii) one of the subgroups $G[2] = \{g \in G \mid 2g = 0\}$ or $G[3] = \{g \in G \mid 3g = 0\}$ is sequentially open in $G$;
(iv) $G$ contains a sequentially open compact subgroup of the form $\mathbb{Z}_2^\kappa$ or $\mathbb{Z}_3^\kappa$ for some cardinal $\kappa$.

The following result shows that much of Theorem 1.4 can be salvaged by imposing stronger compactness-like properties.

**Theorem E.** The following statements are equivalent for every abelian group $G$ that is $\omega$-bounded or totally minimal:

(i) $G$ admits no non-trivial quasi-convex null sequences;
(ii) one of the subgroups $G[2] = \{g \in G \mid 2g = 0\}$ or $G[3] = \{g \in G \mid 3g = 0\}$ is open in $G$;
(iii) one of the subgroups $2G$ and $3G$ is finite.

Furthermore, if $G$ is totally minimal, then these conditions are also equivalent to:
(iv) $G \cong \mathbb{Z}_2^\kappa \times F$ or $G \cong \mathbb{Z}_3^\kappa \times F$, where $\kappa$ is some cardinal and $F$ is a finite abelian group.

Theorems A and B show that in Theorem E one cannot weaken “$\omega$-bounded” to “countably compact and minimal” (or to “pseudocompact and minimal”), and “totally minimal” to “minimal” in Theorem E. Therefore, Theorem D is the best one can achieve in the class of minimal abelian groups. The proofs of Theorems D and E are presented in §5. The totally minimal case of Theorem E relies on intermediate steps in the proof of Theorem D. One of the main ingredients of Theorem D is the following result.

**Theorem F.** Let $\{q_n\}_{n=0}^{\infty}$ be a sequence of positive integers, and put $b_n = q_0 \cdots q_n$ for every $n \in \mathbb{N}$. If $q_n \geq 8$ for every $n \in \mathbb{N}$, then $\left\{\frac{1}{b_n}\right\}_{n=0}^{\infty}$ is a quasi-convex sequence in $\mathbb{T}$.

Theorem F implies that the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by elements of prime order admits a non-trivial quasi-convex null sequence. We note that the condition $q_n \geq 8$ in Theorem F is unnecessarily restrictive, and can be replaced with $q_n \geq 5$; however, the proof of the latter is more complicated and longer, and Theorem F is sufficient for establishing Theorem D. The proof of Theorem F is presented in §6.
Finally, we turn to posing some open problems. As we mentioned earlier, under GCH, the hypothesis $\kappa^\omega = \kappa$ is not only sufficient but also necessary for the existence of a group $G$ as in Theorem A(b). Thus, in ZFC, it is not possible to prove Theorem A(b) without the assumption $\kappa^\omega = \kappa$.

Problem I. Let $p = 2$ or $p = 3$, and let $\kappa$ be a cardinal such that $\kappa \geq \mathfrak{c}$ and $\kappa^\omega > \kappa$. Is it consistent that there exists a minimal pseudocompact abelian group of exponent $p^2$ such that $|pG| = \kappa$ and $G$ admits no non-trivial quasi-convex null sequences?

We show in [3] that if one of the subgroups $G[2]$ or $G[3]$ is sequentially open in (locally) precompact group $G$, then it contains no non-trivial quasi-convex null sequences (Proposition 3.1). Theorems C, D, and E state that the converse of this implication is also true in each of the classes of bounded precompact, minimal, totally minimal, and $\omega$-bounded abelian groups. Thus, it is natural to ask whether the implication remains reversible in general, without assuming some compact-like properties.

Problem II. Let $G$ be a (locally) precompact group, and suppose that $G$ admits no non-trivial quasi-convex null sequences. Is one of the subgroups $G[2]$ or $G[3]$ sequentially open in $G$?

2. Preliminaries

In this section, we provide a few well-known definitions and results that we rely on in the paper. We start off by recalling some terminology from duality theory. Let $H$ be a subgroup of an abelian topological group $G$. The annihilator of $H$ in $\hat{G}$ is the subgroup $H^\perp := \{\chi \in \hat{G} | \chi(H) = \{0\}\}$. The subgroup $H$ is dually closed in $G$ if $H = \bigcap \{\ker \chi | \chi \in H^\perp\}$. Since $H^\perp = H^\circ$ for every subgroup, $H$ is dually closed in $G$ if and only if it is quasi-convex in $G$. The subgroup $H$ is dually embedded in $G$ if every continuous character of $H$ has an extension to a continuous character of $G$, that is, the restriction homomorphism $\hat{G} \to \hat{H}$ is surjective.

Examples 2.1.
(a) If $H$ is an open subgroup of the abelian topological group $G$, then $H$ is dually closed and dually embedded in $G$ (cf. [23, 3.3]).
(b) If $H$ is a closed subgroup of a locally compact abelian group $L$, then $H$ is dually closed and dually embedded in $L$ (cf. [24, Theorems 37 and 54]).
(c) If $H$ is a dense subgroup of an abelian topological group $G$, then $H$ is dually embedded in $G$.
(d) Every subgroup of a locally compact group is dually embedded in it.

Lemma 2.2. Let $G$ be an abelian topological group.
(a) If $H$ is a dually embedded subgroup of $G$, then $Q_H(S) = Q_G(S) \cap H$ for every subset $S$ of $H$.
(b) If $H$ is a dually closed and dually embedded subgroup of $G$, then $Q_H(S) = Q_G(S)$ for every subset $S$ of $H$.

Lemma 2.2 is similar to [7, 5.1], and its proof, which relies on the following general property of the quasi-convex hull, is provided here only for the sake of completeness.

Lemma 2.3. ([21, 1.3(e)], [6, 2.7]) If $f : G \to H$ is a continuous homomorphism of abelian topological groups, and $E \subseteq G$, then $f(Q_G(E)) \subseteq Q_H(f(E))$. 
Proof of Lemma 2.2. Let \( S \subseteq H \) be a subset.

(a) Let \( \iota : H \to G \) denote the inclusion map. By Lemma 2.3,
\[
Q_H(S) \subseteq \iota^{-1}(Q_G(S)) = Q_G(S) \cap H.
\]
To show the reverse inclusion, let \( h \in Q_G(S) \cap H \). If \( \chi \in \hat{H} \) is such that \( \chi(S) \subseteq \mathbb{T}_+ \), then \( \chi = \psi|_H \) for some \( \psi \in \hat{G} \) (as \( H \) is dually embedded in \( G \)), and one has \( \psi(S) = \chi(S) \subseteq \mathbb{T}_+ \). Consequently, \( \chi(h) = \psi(h) \in \mathbb{T}_+ \), because \( h \in Q_G(S) \). This shows that \( Q_G(S) \cap H \subseteq Q_H(S) \), as required.

(b) Since \( H \) is dually closed, \( Q_G(S) \subseteq Q_G(H) = H \). Consequently, the statement follows from part (a).

Examples 2.1 combined with Lemma 2.2 yields the following consequences.

Corollary 2.4. Let \( G \) be a (locally) precompact abelian group, and \( H \) a closed subgroup. Then \( H \) is dually closed and dually embedded in \( G \), and if \( \{x_n\} \subseteq H \) is a quasi-convex sequence, then \( \{x_n\} \) is quasi-convex in \( G \).

Proof. Both \( G \) and \( H \) are subgroups of the completion \( \hat{G} \), which is locally compact. Thus, by Example 2.1(d), \( G \) and \( H \) are dually embedded in \( \hat{G} \). In particular, \( H \) is dually embedded in \( G \). By Example 2.1(b), \( \text{cl}_G H \) is dually closed in \( \hat{G} \). Therefore, by Lemma 2.2(a),
\[
Q_G(H) = Q_G(H) \cap G \subseteq Q_G(\text{cl}_G H) \cap G \subseteq (\text{cl}_G H) \cap G = H,
\]
because \( H \) is closed in \( G \). This shows that \( H \) is dually closed in \( G \). The second statement follows now by Lemma 2.2(b).

Corollary 2.5. Let \( G \) be a (locally) precompact abelian group, \( H \) a (not necessarily closed) subgroup, \( \{x_n\} \subseteq G \) a quasi-convex sequence such that \( x_n \in H \) for infinitely many \( n \in \mathbb{N} \), and \( \{x_{n_k}\} \) the subsequence of \( \{x_n\} \) consisting of all members that belong to \( H \). Then \( \{x_{n_k}\} \) is quasi-convex in \( H \).

Proof. Both \( G \) and \( H \) are subgroups of the completion \( \hat{G} \), which is locally compact. Thus, by Example 2.1(d), \( G \) and \( H \) are dually embedded in \( \hat{G} \). In particular, \( H \) is dually embedded in \( G \). Put \( S := \{\pm x_n \mid n \in \mathbb{N}\} \cup \{0\} \) and \( S' := \{\pm x_{n_k} \mid k \in \mathbb{N}\} \cup \{0\} \). By Lemma 2.2(a),
\[
Q_H(S') = Q_G(S') \cap H \subseteq Q_G(S) \cap H = S \cap H.
\]
It follows from the definition of the subsequence \( \{x_{n_k}\} \) that \( S \cap H = S' \), as desired.

We turn now to minimality and total minimality.

Theorem 2.6. Let \( G \) be an abelian group with completion \( \hat{G} \). Then:

(a) ([29, Theorem 2], [25, 3.31], [2, Propositions 1 and 2], [22, 3.31]) \( G \) is minimal if and only if \( G \) is precompact and \( G \cap H \neq \{0\} \) for every non-trivial closed subgroup \( H \) of \( \hat{G} \);

(b) ([10, 3.31]) \( G \) is totally minimal if and only if \( G \) is precompact and \( G \cap H \) is dense in \( H \) for every closed subgroup of \( \hat{G} \).

Recall that the socle of \( \text{soc}(A) \) of an abelian group \( A \) is the subgroup of torsion elements whose order is square-free, or equivalently, the direct sum of the subgroups \( A[p] := \{x \in A \mid px = 0\} \), where \( p \) ranges over all primes. We put \( \text{tor}(A) \) for the torsion subgroup of \( A \).
Corollary 2.7. Let $G$ be an abelian group with completion $\tilde{G}$.

(a) If $G$ is minimal, then $\text{soc}(\tilde{G}) \subseteq G$.

(b) $([11, 4.3.4])$ If $G$ is totally minimal, then $\text{tor}(\tilde{G}) \subseteq G$.

Furthermore, if $\tilde{G}$ is a bounded compact abelian group, then the converse of (a) is also true.

Definition 2.8. ([9], [11, p. 141]) A compact abelian group is an exotic torus if it contains no subgroup that is topologically isomorphic to the $\mathbb{J}_p$ ($p$-adics) for some prime $p$.

The notion of exotic torus was introduced by Dikranjan and Prodanov in [9], who also provided, among other things, the following characterization for such groups.

Theorem 2.9. ([9], [7, 2.6]) A compact abelian group $K$ is an exotic torus if and only if it contains a closed subgroup $B$ such that

(i) $K/B \cong \mathbb{T}^n$ for some $n \in \mathbb{N}$, and

(ii) $B = \prod_p B_p$, where each $B_p$ is a compact bounded abelian $p$-group.

Furthermore, if $K$ is connected, then each $B_p$ is finite.

Recall that a topological group is pro-finite if it is the (projective) limit of finite groups, or equivalently, if it is compact and zero-dimensional. For a prime $p$, a topological group $G$ is called a pro-$p$-group if it is the (projective) limit of finite $p$-groups, or equivalently, if it is pro-finite and $x^{p^n} \to e$ for every $x \in G$ (or, in the abelian case, $p^n x \to 0$).

Theorem 2.10. ([1], [20, Corollary 8.8(ii)], [11, 4.1.3]) Let $G$ be an abelian pro-finite group. Then $G = \prod_p G_p$, where each $G_p$ is a pro-$p$-group.

Finally, we note for the sake of clarity that in our notation, $\mathbb{N} = \{0, 1, 2, \ldots\}$, that is, $0 \in \mathbb{N}$.

3. Counterexamples

Theorem A. Let $p = 2$ or $p = 3$, and let $\kappa$ be an infinite cardinal.

(a) There exists a minimal abelian group $G$ of exponent $p^2$ such that $|pG| = \kappa$ and $G$ admits no non-trivial quasi-convex null sequences.

(b) If $\kappa^\omega = \kappa$, then there exists a minimal pseudocompact abelian group of exponent $p^2$ such that $|pG| = \kappa$ and $G$ admits no non-trivial quasi-convex null sequences.

Theorem B. Under Martin’s Axiom (or the Continuum Hypothesis), there exists a countably compact minimal abelian group of exponent 4 such that $2G$ is infinite and $G$ admits no non-trivial quasi-convex null sequences.

In this section, we prove Theorems A and B.

Proposition 3.1. Let $G$ be a (locally) precompact abelian group such that $G[2]$ or $G[3]$ is sequentially open in $G$. Then $G$ admits no non-trivial quasi-convex null sequences. In particular, if $2G$ or $3G$ admits no non-trivial null sequences, then $G$ admits no non-trivial quasi-convex null sequences.
Lemma 3.2. ([7] 5.3) If $G$ is an abelian topological group of exponent 2 or 3, then $G$ admits no non-trivial quasi-convex null sequences.

**Proof of Proposition 3.1.** Suppose that $G[p]$ is sequentially open in $G$, where $p = 2$ or $p = 3$, and let $\{x_n\} \subseteq G$ be a quasi-convex null sequence. Then $x_n \in G[p]$ for all but finitely many $n \in \mathbb{N}$. Let $\{x_{n_k}\}$ denote the subsequence of $\{x_n\}$ consisting of all members that belong to $G[p]$. By Corollary 2.5, $\{x_{n_k}\}$ is a quasi-convex null sequence in $G[p]$. Therefore, by Lemma 3.3, $\{x_{n_k}\}$ is trivial, and hence $\{x_n\}$ is trivial.

In order to show the second statement, we observe that $pG$ is a continuous homomorphic image of $G/G[p]$. Thus, if $pG$ has no non-trivial null sequences, then $G/G[p]$ has no non-trivial null sequences either, and therefore $G[p]$ is sequentially open in $G$. □

Lemma 3.3. Let $\mathcal{P}$ be a topological property that is an inverse invariant of open perfect maps, $p$ a prime number, and $D$ a non-trivial precompact abelian group of exponent $p$ with property $\mathcal{P}$. Then there exists a minimal abelian group of exponent $p^2$ with property $\mathcal{P}$ such that $pG \cong D$.

**Proof.** Since $D$ has exponent $p$, so does its completion $\tilde{D}$, and thus $\tilde{D} \cong \mathbb{Z}_p^\lambda$ for some cardinal $\lambda$ (cf. [11] 4.2.2). Put $K := \mathbb{Z}_p^\lambda$, and let $f : K \to pK \cong \tilde{D}$ denote the continuous homomorphism defined by $f(x) = px$. Since $K$ is compact, the map $f$ is open and perfect. Consequently, $G := f^{-1}(D)$ has property $\mathcal{P}$, the exponent of $G$ is $p^2$ (because $pG = f(G) = D$ is non-trivial), and $G$ is dense in $K$. In particular, $K = \tilde{G}$, and thus $\text{soc}(G) = K[p] = \text{ker} f \subseteq G$. Therefore, by Corollary 2.7(a), $G$ is minimal. □

**Proof of Theorem A.** (a) Let $\kappa$ be an infinite cardinal, and let $D$ denote the direct sum $\mathbb{Z}_p^{(\kappa)}$ equipped with the Bohr-topology. By Lemma 3.3 (with $\mathcal{P}$ the trivial property), there exists a minimal abelian group $G$ of exponent $p^2$ such that $pG \cong D$. By a well-known theorem of Flor, the Bohr-topology of a discrete abelian group admits no non-trivial convergent sequences (cf. [16]). Hence, by Proposition 3.1, $G$ admits no non-trivial quasi-convex null sequences. This completes the proof, because $|pG| = |\mathbb{Z}_p^{(\kappa)}| = \kappa$.

(b) Let $\kappa$ be an infinite cardinal such that $\kappa^{\omega} = \kappa$. By a theorem of Dijkstra and van Mill (cf. [5] Theorem 3); see also [17] 5.8), the compact group $\mathbb{Z}_p^\kappa$ admits a subgroup $D$ such that:

1. $D$ is dense in the $G_\delta$-topology of $\mathbb{Z}_p^\kappa$;
2. $D$ contains no non-trivial convergent sequences; and
3. $|D| = \kappa$.

It follows from property (1) that $D$ is pseudocompact (cf. [4] 1.2). Since pseudocompactness is an inverse invariant of open perfect maps (cf. [15] 3.10.H), by Lemma 3.3, there exists a pseudocompact minimal group $G$ of exponent $p^2$ such that $pG \cong D$. Hence, by Proposition 3.1, $G$ admits no non-trivial quasi-convex null sequences. This completes the proof, because $|pG| = |D| = \kappa$ by property (3). □

**Proof of Theorem B.** Van Douwen showed that under MA, there exists an infinite countably compact abelian group $D$ of exponent 2 that admits no non-trivial convergent sequences (cf. [13] 8.1), and he also observed that under CH, a construction of Hajnal and Juhász yields a group $D$ with the same properties (cf. [13] and [13] 9]). Since countable compactness is an inverse invariant of perfect maps (cf. [15] 3.10.10), by Lemma 3.3, there exists a countably compact minimal group $G$ of exponent 4 such that $2G \cong D$. Hence, by Proposition 3.1, $G$ admits no non-trivial quasi-convex null sequences. □
4. Bounded precompact groups without non-trivial quasi-convex null sequences

**Theorem C.** For every bounded precompact abelian group $G$, the following statements are equivalent:

(i) $G$ admits no non-trivial quasi-convex null sequences;

(ii) one of the subgroups $G[2] = \{g \in G \mid 2g = 0\}$ or $G[3] = \{g \in G \mid 3g = 0\}$ is sequentially open in $G$.

In this section, we prove Theorem C. Recall that a set $\{f_1, \ldots, f_n\}$ of non-zero elements in an abelian group $G$ is independent if whenever $\sum_{i=1}^n l_i f_i = 0$ for some $l_i \in \mathbb{Z}$, then $l_i f_i = 0$ for every $i$, or equivalently, if $\langle f_1, \ldots, f_n \rangle = \langle f_1 \rangle \oplus \cdots \oplus \langle f_n \rangle$.

**Lemma 4.1.** Let $\{f_1, \ldots, f_n\}$ be an independent subset of a (locally) precompact abelian group $G$. If $4 \leq o(f_i) < \infty$ for every $1 \leq i \leq n$, then $X := \{0\} \cup \{\pm f_1, \ldots, \pm f_n\}$ is quasi-convex in $G$.

**Proof.** Set $m_k := o(f_k)$ for $1 \leq k \leq n$, $m_k = 4$ for $k > n$, and $P := \prod_{k=1}^\infty \mathbb{Z}_{m_k}$. For every $k \geq 1$, put $e_k := (0, \ldots, 0, 1, 0, \ldots)$, with 1 at the $k$-th coordinate and zero elsewhere. The authors showed in an earlier work that $S' := \{0\} \cup \{ \pm e_k \mid k \geq 1\}$ is quasi-convex in $P$ (cf. [7], 5.5). Thus, by Corollary 2.4 and Lemma 2.2(a), $S := \{0\} \cup \{ \pm e_k \mid 1 \leq k \leq n\}$ is quasi-convex in $H := \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ (because $S = S' \cap H$).

Put $F := (f_1, \ldots, f_n) = \langle f_1 \rangle \oplus \cdots \oplus \langle f_n \rangle$. Clearly, $F$ is finite. Consequently, the homomorphism $\varphi : F \to H$ defined by $\varphi(f_i) = e_i$ for $1 \leq i \leq n$ is a topological isomorphism. Therefore, $X = \varphi^{-1}(S)$ is quasi-convex in $F$. Hence, by Corollary 2.4 and Lemma 2.2(b), $X$ is quasi-convex in $G$. \hfill $\square$

**Proposition 4.2.** Let $E$ and $F$ be finite abelian groups, and suppose that $\exp E \geq 4$. Then every dense subgroup $A \leq F \times E^\omega$ contains a non-trivial null sequence that is quasi-convex both in $A$ and in $F \times E^\omega$.

**Proof.** For every positive integer $n$, let $\pi_n : F \times E^\omega \to F \times E^n$ denote the canonical projection of the first $n+1$ coordinates. Pick $y \in E$ such that $o(y) = \exp E$. Since $A$ is dense in $F \times E^\omega$, one has $\pi_n(A) = F \times E^n$. Thus, for every $n$, we may pick $x_n \in A$ such that $\pi_n(x_n) = (0, \ldots, 0, y)$. We claim that $\{x_n\}$ is a quasi-convex null sequence in $F \times E^n$.

**Step 1.** We show by induction on $n$ that the set $\{\pi_n(x_1), \ldots, \pi_n(x_n)\}$ is independent in $F \times E^n$. For $n = 1$, the statement is trivial, because $\pi_1(x_1)$ is non-zero. Assume now that the statement holds for $n$, and suppose that $\sum_{i=1}^{n+1} l_i \pi_{n+1}(x_i) = 0$ for $l_i \in \mathbb{Z}$. Then $\sum_{i=1}^{n+1} l_i \pi_n(x_i) = 0$, and thus $\sum_{i=1}^n l_i \pi_n(x_i) = 0$, because $\pi_n(x_{n+1}) = 0$. By the inductive hypothesis, it follows that $l_i \pi_n(x_i) = 0$ for $1 \leq i \leq n$. The $(i+1)$-th coordinate of $\pi_n(x_i)$ is $y$, and so $o(y) | l_i$ for $1 \leq i \leq n$. Therefore, $l_i x_i = 0$ for $1 \leq i \leq n$, because $o(y) = \exp E$. Hence, $l_{n+1} \pi_{n+1}(x_{n+1}) = - \sum_{i=1}^n l_i \pi_{n+1}(x_i) = 0$, as required.

**Step 2.** Put $S := \{0\} \cup \{\pm x_n \mid n \in \mathbb{N}\}$. By Lemma 4.1, $\pi_n(S) = \{0\} \cup \{\pm \pi_n(x_1), \ldots, \pm \pi_n(x_n)\}$ is quasi-convex in $F \times E^n$, because $o(\pi_n(x_i)) = o(y) = \exp E \geq 4$. Thus, by Lemma 2.3, $Q_{F \times E^\omega}(S) \subseteq \pi_n^{-1}(\pi_n(S)) = S + \ker \pi_n$.
for every $n$. Therefore,

$$Q_{F \times E^\omega}(S) \subseteq \bigcap_{n=1}^{\infty} (S + \ker \pi_n) = \operatorname{cl}_{F \times E^\omega} S,$$

because $\{\ker \pi_n\}_{n=1}^{\infty}$ is a base for the topology of $F \times E^\omega$ at zero. Since $x_k \in \ker \pi_n$ for every $k > n$, it follows that $\{x_n\}$ is a null sequence, and $S$ is closed in $F \times E^\omega$. Hence, by (1), $S$ is quasi-convex in $F \times E^\omega$, as desired. By Corollary 4.3, this implies that $\{x_n\}$ is quasi-convex in $A$ as well. □

Corollary 4.3. Let $K$ be an infinite bounded compact metrizable abelian group that contains no open compact subgroup of the form $\mathbb{Z}_2^\omega$ and $\mathbb{Z}_3^\omega$. Then every dense subgroup $A$ of $K$ contains a non-trivial null sequence that is quasi-convex in both $A$ and $K$.

**Proof.** Since $K$ is bounded compact abelian, it is topologically isomorphic to a product of finite cyclic groups (cf. [11, 4.2.2]). The number of the factors is countably infinite, because $K$ is metrizable and infinite, and the number of non-isomorphic factors is finite, as $K$ is bounded. Thus, $K \cong F \times \mathbb{Z}_{m_1}^\omega \times \cdots \times \mathbb{Z}_{m_l}^\omega$, where $F$ is a finite abelian group and $m_1, \ldots, m_l$ are distinct integers. Consequently, for $E := \mathbb{Z}_{m_1}^\omega \times \cdots \times \mathbb{Z}_{m_l}^\omega$, one has $K \cong F \times E^\omega$. Since $K$ contains no open compact subgroup of the form $\mathbb{Z}_2^\omega$ and $\mathbb{Z}_3^\omega$, clearly $E \neq \mathbb{Z}_2$ and $E \neq \mathbb{Z}_3$. Therefore, $\exp(E) \geq 4$, and hence the statement follows by Proposition 4.2. □

Corollary 4.4. Let $A$ be a bounded precompact metrizable abelian group. If the subgroups $A[2]$ and $A[3]$ are not open in $A$, then $A$ contains a non-trivial null sequence that is quasi-convex in both $A$ and the completion $\tilde{A}$ of $A$.

**Proof.** Put $K := \tilde{A}$. Since $A$ is metrizable and bounded, so is $K$. One has $A[p] = K[p] \cap A$, and thus $K[2]$ and $K[3]$ are not open in $K$. In particular, $K$ is infinite, and it contains no open compact subgroup of the form $\mathbb{Z}_2^\omega$ and $\mathbb{Z}_3^\omega$. Hence, the statement follows from Corollary 4.3. □

Lemma 4.5. If $A$ is a bounded precompact abelian group generated by a null sequence $\{w_n\}$, then $A$ is metrizable.

**Proof.** Let $m$ denote the exponent of $A$. Clearly, every character $\chi \in \hat{A}$ is completely determined by the values $\chi(w_n)$ taken at the generators of $A$. Since $\{w_n\}$ is a null sequence and $\chi$ is continuous, $\chi(w_n) \to 0$ in $\mathbb{T}$, and $\chi(x_n)$ belongs to the cyclic subgroup of order $m \mathbb{T}$, because $mA = 0$. Consequently, $\chi(x_n) = 0$ for all but finitely many indices $n$. Therefore, $\hat{G}$ is countable. Hence, $G$ is metrizable (cf. [19, 2.12]). □

**Proof of Theorem.** (ii) ⇒ (i): This implication holds even without the assumption that $G$ is bounded, and has already been shown in Proposition 3.1.

(i) ⇒ (ii): Suppose that neither $G[2]$ nor $G[3]$ is sequentially open in $G$. Then there are sequences $\{y_n\}$ and $\{z_n\}$ in $G$ that witness that $G[2]$ and $G[3]$ are not sequentially open. In other words, $y_n \to y_0 \in G[2]$, but $y_n \notin G[2]$ for infinitely many indices $n$, and $z_n \to z_0 \in G[3]$, but $z_n \notin G[3]$ for infinitely many indices $n$. By replacing $\{y_n\}$ with $\{y_n - y_0\}$ and $\{z_n\}$ with $\{z_n - z_0\}$, we may assume that $y_n \to 0$ and $z_n \to 0$.

Let $\{w_n\}$ denote the alternating sequence $y_1, z_1, y_2, z_2, \ldots$. Clearly, $\{w_n\}$ is a null sequence, and $w_n \notin G[2]$ for infinitely many indices $n$ and $w_n \notin G[3]$ for infinitely many indices $n$. Let $A$
that contrary to our assumption.

Closed in indices by Lemma 4.5, denote the subgroup of \( G \) generated by \( \{ w_n \} \). Then \( A \) is a bounded precompact abelian group, and by Lemma 4.5, \( A \) is metrizable. Although \( w_n \to 0 \), one has \( w_n \notin A[2] = G[2] \cap A \) for infinitely many indices \( n \) and \( w_n \notin A[3] = G[3] \cap A \) for infinitely many indices \( n \). Thus, the subgroups \( A[2] \) and \( A[3] \) are not open in \( A \). Therefore, by Corollary 4.4, there is a non-trivial null sequence \( \{ x_n \} \subseteq A \) such that \( \{ x_n \} \) is quasi-convex both in \( A \) and the completion \( \tilde{A} \) of \( A \).

Since the completion \( \tilde{A} \) is a closed subgroup of the completion \( \tilde{G} \) of \( G \), by Corollary 2.4, \( \{ x_n \} \) is also quasi-convex in \( \tilde{G} \). Hence, by Corollary 2.5, \( \{ x_n \} \) is quasi-convex in \( G \), because \( \{ x_n \} \subseteq A \subseteq G \).

5. Compact-like abelian groups that admit no non-trivial quasi-convex null sequences

**Theorem D.** For every minimal abelian group \( G \), the following statements are equivalent:

(i) \( G \) admits no non-trivial quasi-convex null sequences;
(ii) \( G \cong P \times F \), where \( P \) is a minimal bounded abelian \( p \)-group \((p \leq 3)\) admitting no non-trivial quasi-convex null sequences, and \( F \) is a finite abelian group;
(iii) one of the subgroups \( G[2] = \{ g \in G \mid 2g = 0 \} \) or \( G[3] = \{ g \in G \mid 3g = 0 \} \) is sequentially open in \( G \);
(iv) \( G \) contains a sequentially open compact subgroup of the form \( \mathbb{Z}_2^\kappa \) or \( \mathbb{Z}_3^\kappa \) for some cardinal \( \kappa \).

**Theorem E.** The following statements are equivalent for every abelian group \( G \) that is \( \omega \)-bounded or totally minimal:

(i) \( G \) admits no non-trivial quasi-convex null sequences;
(ii) one of the subgroups \( G[2] = \{ g \in G \mid 2g = 0 \} \) or \( G[3] = \{ g \in G \mid 3g = 0 \} \) is open in \( G \);
(iii) one of the subgroups \( 2G \) and \( 3G \) is finite.

Furthermore, if \( G \) is totally minimal, then these conditions are also equivalent to:

(iv) \( G \cong \mathbb{Z}_2^\kappa \times F \) or \( G \cong \mathbb{Z}_3^\kappa \times F \), where \( \kappa \) is some cardinal and \( F \) is a finite abelian group.

In this section, we present the proofs of Theorems D and E. Since the latter relies on the former one, we prove Theorem D first.

**Proposition 5.1.** Let \( G \) be a minimal abelian group that admits no non-trivial quasi-convex null sequences. Then the completion \( \tilde{G} \) of \( G \) contains no closed subgroup \( H \) such that \( \text{soc}(H) \) contains a non-trivial null sequence that is quasi-convex in \( H \).

**Proof.** Let \( \{ x_n \} \subseteq \text{soc}(H) \) be a non-trivial null sequence that is quasi-convex in \( H \). Since \( H \) is closed in \( \tilde{G} \), by Corollary 2.4, \( \{ x_n \} \) is quasi-convex in \( \tilde{G} \). By Corollary 2.7(a), \( \text{soc}(\tilde{G}) \subseteq G \), and thus \( \{ x_n \} \subseteq G \). Therefore, by Corollary 2.5, \( \{ x_n \} \) is a non-trivial quasi-convex null sequence in \( G \), contrary to our assumption.

**Proposition 5.2.** Let \( G \) be a minimal abelian group that admits no non-trivial quasi-convex null sequences. Then the completion \( \tilde{G} \) of \( G \) contains no subgroups that are topologically isomorphic to:

(a) \( \mathbb{J}_p \) for some prime \( p \),
(b) \( \mathbb{T} \), or
(c) \( \prod_{k=1}^{\infty} \mathbb{Z}_{r_k} \) for square-free numbers \( r_k > 3 \).
PROOF. (a) Assume that $H$ is a subgroup of $\tilde{G}$ that is topologically isomorphic to $\mathbb{J}_p$ for some prime $p$. By Theorem 2.6(a), there is $y \in G \cap H$ such that $y \neq 0$. Since $\text{cl}_G(y) \cong \mathbb{J}_p$, by replacing $H$ with $\text{cl}_G(y)$ if necessary, we may identify $\langle y \rangle$ with $\mathbb{Z}$ in $\mathbb{J}_p$. Thus, by Example 1.3(a), $H$ admits a non-trivial quasi-convex sequence $\{x_n\}$ such that $\{x_n\} \subseteq G \cap H$. Since $H$ is closed in $\tilde{G}$, by Corollary 2.4, $\{x_n\}$ is quasi-convex in $\tilde{G}$. Therefore, by Corollary 2.5, $\{x_n\}$ is a non-trivial quasi-convex null sequence in $\tilde{G}$, contrary to our assumption.

(b) Assume that $H$ is a subgroup of $\tilde{G}$ that is topologically isomorphic to $\mathbb{T}$. Then $H$ is closed in $\tilde{G}$, and by Example 6.7, $H$ admits a non-trivial quasi-convex null sequence $\{x_n\}$ such that $\{x_n\} \subseteq \text{soc}(H) \cong \text{soc}(\mathbb{T})$. By Proposition 5.1, the statement follows.

(c) Assume that $H$ is a subgroup of $\tilde{G}$ that is topologically isomorphic to $\prod_{k=1}^\infty \mathbb{Z}_{r_k}$, where $r_k > 3$ and $r_k$ is square-free for every $k$. Then $H$ is closed in $\tilde{G}$, and by Example 1.3(c), $H$ admits a non-trivial quasi-convex null sequence $\{x_n\}$ such that $\{x_n\} \subseteq \text{soc}(H) \cong \bigoplus_{k=1}^\infty \mathbb{Z}_{r_k}$. The statement follows now by Proposition 5.1.

Lemma 5.3. Let $K$ be a compact abelian group that contains no subgroups that are topologically isomorphic to $\mathbb{T}$, $\mathbb{J}_p$ for some $p$, or $\prod_{k=1}^\infty \mathbb{Z}_{r_k}$ for square-free numbers $r_k > 3$. Then $K = K_p \times F$, where $K_p$ is a compact bounded abelian $p$-group, $p \leq 3$, and $F$ is a finite abelian group.

PROOF. Step 1. Suppose that $K$ is a pro-finite group. Since $K$ contains no subgroups that are topologically isomorphic to $\mathbb{J}_p$ for some prime $p$, it is an exotic torus. The group $K$ has no connected quotients, because it is pro-finite, and thus, by Theorem 2.9, $K = \prod_{p} K_p$, where each $K_p$ is a compact bounded abelian $p$-group. Consequently, each $K_p$ is topologically isomorphic to a product of finite cyclic $p$-groups (cf. [11, 4.2.2]), and $K_p$ is infinite if and only if it contains a subgroup that is topologically isomorphic to $\mathbb{Z}_p^\omega$. By our assumption, $K$ contains no such subgroups for $p > 3$. Hence, $K_p$ is finite for $p > 3$.

Put $K' := \prod_{p > 3} K_p$. If $K'$ is infinite, then there are infinitely many primes $p_k > 3$ such that $K_{p_k} \neq 0$. Consequently, $K'$ (and thus $K$) contains a subgroup that is topologically isomorphic to the product $\prod_{k=1}^\infty \mathbb{Z}_{p_k}$, contrary to our assumption. This shows that $K'$ is finite.

Finally, if both $K_2$ and $K_3$ are infinite, then $K$ contains a subgroup that is topologically isomorphic to $\mathbb{Z}_2^\omega \times \mathbb{Z}_3^\omega \cong \mathbb{Z}_6^\omega$, contrary to our assumption. Hence, one of $K_2$ and $K_3$ is finite, and either $K = K_2 \times F$, where $F := K_3 \times K'$ is finite, or $K = K_3 \times F$, where $F := K_2 \times K'$ is finite.

Step 2. In the general case, we show that $K$ is pro-finite. To that end, let $C$ be the connected component of $K$. Since $K$ is an exotic torus, so is $C$, and by Theorem 2.9, $C$ contains a closed subgroup $B$ such that $B = \prod_p B_p$, where each $B_p$ is a finite $p$-group, and $C/B \cong \mathbb{T}^n$ for some $n \in \mathbb{N}$.

In particular, $B$ is a compact pro-finite group that satisfies the conditions of this lemma. Thus, by what we have shown so far, $B' := \prod_{p > 3} B_p$ is finite, and therefore $B = B_2 \times B_3 \times B'$ is finite. Consequently, by Pontryagin duality, $\hat{B} \cong \hat{C}/B$ is finite (cf. [24, Theorem 54]), and $B_2 \cong C/B = \mathbb{Z}^n$ (cf. [24, Theorem 37]). This implies that $\hat{C}$ is finitely generated. On the other hand, $\hat{C}$ is torsion free, because $C$ is connected (cf. [24, Example 73]), which means that $\hat{C} = \mathbb{Z}^n$ and $C \cong \mathbb{T}^n$. By our
assumption, however, $K$ contains no subgroup that is topologically isomorphic to $T$. Hence, $n = 0$ and $C = 0$. This shows that $K = B$ is pro-finite, and the statement follows from Step 1.

**Corollary 5.4.** Let $G$ be a minimal abelian group that admits no non-trivial quasi-convex null sequences. Then the completion $\tilde{G}$ of $G$ is a bounded compact abelian group, and $\tilde{G} = K_p \times F$, where $K_p$ is a compact bounded abelian $p$-group, $p \leq 3$, and $F$ is a finite abelian group.

**Proof of Theorem D.** (i) $\Rightarrow$ (ii): Let $G$ be a minimal abelian group that admits no non-trivial quasi-convex null sequences. By Corollary 5.4, $\tilde{G} = K_p \times F$, where $K_p$ is a compact bounded abelian $p$-group ($p \leq 3$), and $F$ is a finite abelian group. Without loss of generality, we may assume that $F$ contains no $p$-elements. Let $e = p^m m$ be the exponent of $\tilde{G}$, where $m$ and $p$ are coprime. Then $p^\alpha G$ is dense in $p^\alpha \tilde{G} = F$, and thus $F = p^\alpha G \subseteq G$. Therefore, for $P := G \cap K_p$, one has $G = P \times F$, and $P$ is a bounded abelian $p$-group. Since $P$ is a closed subgroup of $G$, by Corollary 2.4, every quasi-convex sequence in $P$ is also quasi-convex in $G$, and so $P$ admits no non-trivial quasi-convex null sequences. In order to show that $P$ is minimal, let $H$ be a closed subgroup of $P \subseteq K_p$. Then, in particular, $H$ is a closed subgroup of $\tilde{G}$. Consequently, by Theorem 2.6(a),

$$P \cap H = (G \cap K_p) \cap H = G \cap H \neq \{0\},$$

because $G$ is minimal. Hence, by Theorem 2.6(a), $P$ is minimal. (It is well known that closed central subgroups of minimal groups are minimal, but in this case, a direct proof was also available.)

(ii) $\Rightarrow$ (iii): By Theorem 2.6(a), $P$ is precompact. Thus, by Theorem 2.6, $P[2]$ or $P[3]$ is sequentially open in $P$. Since $F$ is finite, $P$ is open in $G$. Therefore, $P[2]$ or $P[3]$ is sequentially open in $G$. Hence, $G[2]$ or $G[3]$ is sequentially open in $G$.

(iii) $\Rightarrow$ (iv): Since $G$ is minimal, by Corollary 2.7(a), $G[p] = \tilde{G}[p]$ for every prime $p$, and in particular $G[p]$ is compact. Thus, $G[p] \cong \mathbb{Z}_p^{\kappa_p}$ for some cardinal $\kappa_p$ for every prime $p$.

(iv) $\Rightarrow$ (i): If $G$ contains a sequentially open compact subgroup of the form $\mathbb{Z}_2^{\kappa_2}$ or $\mathbb{Z}_3^{\kappa_3}$, then $G[2]$ or $G[3]$ is sequentially open in $G$, and therefore the statement follows by Proposition 5.1.

We turn now to the case where $G$ is $\omega$-bounded.

**Proposition 5.5.** Let $G$ be an $\omega$-bounded abelian group that admits no non-trivial quasi-convex null sequences. Then:

(a) $G$ is bounded;

(b) the subgroup $G_p$ of $p$-elements is finite for $p > 3$;

(c) at least one of $G_2$ and $G_3$ is finite;

(d) $2G_2$ and $3G_3$ are finite.

The proof of Proposition 5.5 is based on a well-known result of Comfort and Robertson.

**Theorem 5.6.** (3.7.4) Every pseudocompact abelian torsion group is bounded.

**Proof of Proposition 5.5.** (a) Let $x \in G$. Since $G$ is $\omega$-bounded, $\langle x \rangle$ is contained in a compact subgroup of $G$, and thus $K := \text{cl}_G \langle x \rangle$ is compact. If $x$ has an infinite order, then $2K$ and $3K$ are infinite, and so by Theorem 1.4, $K$ admits a non-trivial quasi-convex null sequence $\{x_n\}$. Since $K$ is a closed subgroup of $G$, by Corollary 2.4, $\{x_n\}$ is a non-trivial quasi-convex null sequence in $G$, contrary to our assumption. This shows that every element in $G$ has a finite order; in other
words, $G$ is a torsion group. Since $G$ is $\omega$-bounded, in particular, it is pseudocompact. Hence, by Theorem 5.6, $G$ is bounded.

(b) Let $p$ be a prime, and suppose that $G_p$ is infinite. Then $G[p]$ contains a countably infinite subset $S$, which in turn is contained in a compact subgroup $K$, because $G$ is $\omega$-bounded. Since $K[p]$ is an infinite compact group of exponent $p$, it is topologically isomorphic to $\mathbb{Z}_p^\lambda$ for some infinite cardinal $\lambda$ (cf. [11, 4.2.2]). In particular, $G$ contains a closed subgroup $H$ that is topologically isomorphic to $\mathbb{Z}_p^\omega$.

If $p > 3$, then by Example 1.3(c), $H$ admits a non-trivial quasi-convex null sequence, and since $H$ is closed in $G$, this sequence will also be quasi-convex in $G$ according to Corollary 2.4 contrary to our assumption. This shows that $G_p$ is finite for $p > 3$.

(c) Assume that both $G_2$ and $G_3$ are infinite. Then, by what we have seen so far, $G$ contains a subgroup $H_2$ that is topologically isomorphic to $\mathbb{Z}_2^\omega$, and a subgroup $H_3$ that is topologically isomorphic to $\mathbb{Z}_3^\omega$. Thus, $H := H_2 + H_3$ is topologically isomorphic to $\mathbb{Z}_6^\omega$. By Example 1.3(c), $H$ admits a non-trivial quasi-convex null sequence, and since $H$ is closed in $G$, this sequence will also be quasi-convex in $G$ according to Corollary 2.4 contrary to our assumption. This shows that at least one of $G_2$ and $G_3$ is finite.

(d) Let $p = 2$ or $p = 3$, and assume that $pG_p$ is infinite. Then, in particular, $(pG_p)[p]$ is infinite, and so there is a countably infinite subset $S$ of $G$ such that $pS$ is infinite and $p^2S = 0$. Since $G$ is $\omega$-bounded, $S$ is contained in a compact subgroup $K$ of $G$. By replacing $K$ with $K[p^2]$ if necessary, we may assume that $K$ has exponent $p^2$, and so $K$ is topologically isomorphic to $\mathbb{Z}_p^{\lambda_1} \times \mathbb{Z}_p^{\lambda_2}$ for some cardinals $\lambda_1$ and $\lambda_2$. As $pS \subseteq pK \cong \mathbb{Z}_p^{\lambda_2}$ is infinite, $\lambda_2$ is infinite, and thus $G$ contains a subgroup $H$ that is topologically isomorphic to $\mathbb{Z}_p^{\omega}$. By Example 1.3(c), $H$ admits a non-trivial quasi-convex null sequence, and since $H$ is closed in $G$, this sequence will also be quasi-convex in $G$ according to Corollary 2.4 contrary to our assumption. This shows that $2G_2$ and $3G_3$ are finite.

We are now ready to prove Theorem 6

**Proof of Theorem 6.** By Proposition 3.1, the implication (ii) $\Rightarrow$ (i) holds for every precompact group, and obviously so does the equivalence (ii) $\Leftrightarrow$ (iii). The implication (iv) $\Rightarrow$ (iii) is also clear. Thus, it suffices to prove that (i) $\Rightarrow$ (iii), and if $G$ is totally minimal, (i) $\Rightarrow$ (iv).

Suppose that $G$ is $\omega$-bounded and admits no non-trivial quasi-convex null sequences. Then, by Proposition 5.5(a), $G$ is bounded, and so $\tilde{G}$ is bounded. Thus, $\tilde{G}$ is a product of its subgroups of $p$-elements, and therefore $G \cong G_2 \times G_3 \times \cdots \times G_p$, where $p_1, \ldots, p_l > 3$ are prime factors of the exponent of $G$. By Proposition 5.5(b), each $G_{p_i}$ is finite. By Proposition 5.5(c), one of $G_2$ and $G_3$ is finite, and so $G \cong G_F \times F$, where $p = 2$ or $p = 3$, and $F$ is a finite abelian group of order coprime to $p$. By Proposition 5.5(d), $pG_p$ is finite, and hence $pG \cong pG \times F$ is finite, as desired.

Suppose that $G$ is totally minimal and admits no non-trivial quasi-convex null sequences. Then, by Corollary 5.4, the completion $\tilde{G}$ of $G$ is bounded. Thus, by Corollary 2.7(b), $\tilde{G} = G$, and so $G$ is compact. Hence, both (iii) and (iv) follow from Theorem 1.4

6. Sequences of the form $\left\{ \frac{1}{b_n} \right\}_{n=0}^\infty$ in $\mathbb{T}$

**Theorem 6.** Let $\left\{ q_n \right\}_{n=0}^\infty$ be a sequence of positive integers, and put $b_n = q_0 \cdots q_n$ for every $n \in \mathbb{N}$. If $q_n \geq 8$ for every $n \in \mathbb{N}$, then $\left\{ \frac{1}{b_n} \right\}_{n=0}^\infty$ is a quasi-convex sequence in $\mathbb{T}$. 

In this section, we prove Theorem\textsuperscript{F}. Although in Theorem\textsuperscript{F} itself we require \( q_n \geq 8 \), a number of intermediate statements remain true under no conditions at all or weaker conditions imposed upon the sequence \( \{q_n\} \). Consequently, we consider \( \{q_n\}_{n=0}^{\infty} \) (and thus \( \{b_n\}_{n=0}^{\infty} \)) a fixed sequence of positive integers throughout this section, but make no further assumptions about their properties; instead, we impose conditions on the \( \{q_n\} \) in each statement as needed. Finally, we set \( X := \{0\} \cup \{ \pm \frac{1}{b_n} \mid n \in \mathbb{N} \} \).

The next theorem is in the vein of [8, 4.3] and [7, 3.2], and plays an important role in the proof of Theorem\textsuperscript{F}.

**Theorem 6.1.** If \( q_{k+1} \geq 4 \) whenever \( q_k = 7 \), then \( Q_{\mathbb{T}}(X) \subseteq \left\{ \sum_{i=0}^{\infty} \varepsilon_i \mid \varepsilon_i \in \{-1, 0, 1\} \right\} \).

The proof of Theorem\textsuperscript{6.1} requires two preparatory steps. First, as we have seen in [8] and [7], finding a convenient way to represent elements of \( \mathbb{T} \) is a useful tool in calculating quasi-convex hulls of sequences. Let \( \{d_i\}_{i=0}^{\infty} \) be an increasing sequence of positive integers such that \( |d_i|/d_{i+1} \) for every \( i \in \mathbb{N} \). Then \( z \in \mathbb{T} \) (which, as we stated in the Introduction, is identified with \( (-\frac{1}{2}, \frac{1}{2}) \)) can be expressed in the form \( z = \sum_{i=0}^{\infty} \frac{c_i}{d_i} \), with \( c_i \) integers such that \( |c_i| \leq \frac{d_i}{2d_{i+1}} \). (We consider \( d_{-1} = 1 \).)

This representation, however, need not be unique: For example, if \( d_0 = 3 \) and \( d_1 = 6 \), then \( \frac{1}{6} \) can be expressed with \( c_0 = 0 \) and \( c_1 = 1 \), but also with \( c_0 = 1 \) and \( c_1 = -1 \). In order to eliminate this anomaly, we say that the representation of \( z \) is \textit{standard} if the following conditions are satisfied:

(i) \( c_i \in \mathbb{Z} \) and \( |c_i| \leq \frac{d_i}{2d_{i+1}} \) for all \( i \in \mathbb{N} \);

(ii) \( \left| z - \sum_{i=0}^{k} \frac{c_i}{d_i} \right| \leq \frac{1}{2d_{k+1}} \) for every \( k \in \mathbb{N} \);

(iii) if \( \left| z - \sum_{i=0}^{k} \frac{c_i}{d_i} \right| = \frac{1}{2d_k} \) for some \( k \), then \( \left| \frac{c_k}{d_k} \right| < \left| z - \sum_{i=0}^{k-1} \frac{c_i}{d_i} \right| \).

(In the aforementioned example, \( c_0 = 0 \) and \( c_1 = 1 \) is a standard representation of \( \frac{1}{6} \), but \( c_0 = 1 \) and \( c_1 = -1 \) is not a standard one.) In order to formulate an analogue of [7, 2.1, 2.2, 2.4], we introduce our own rounding functions: For \( x \in \mathbb{R} \), we put

\[
[x] := \min\{n \in \mathbb{Z} \mid x < n\}, \quad \lceil x \rceil := \max\{n \in \mathbb{Z} \mid n \leq x\}, \quad \lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n < x\}.
\]

We note that these are not the usual definitions of the floor and ceiling functions (as we use strict inequality in both).

**Lemma 6.2.** Let \( z = \sum_{i=0}^{\infty} \frac{c_i}{d_i} \in \mathbb{T} \) be a standard representation.

(a) If \( mz \in \mathbb{T}_+ \) for all \( m = 1, \ldots, \lceil \frac{d_0}{6} \rceil \), then \( c_0 \in \{-1, 0, 1\} \).

(b) If \( mz \in \mathbb{T}_+ \) for all \( m = 1, \ldots, \lceil \frac{d_0}{6} \rceil \) and \( d_0 \neq 7 \), then \( c_0 \in \{-1, 0, 1\} \).

(c) If \( mz \in \mathbb{T}_+ \) for all \( m = 1, \ldots, \lceil \frac{d_0}{4} \rceil \) and for \( m = d_0 - 1 \), then \( c_0 \in \{-1, 0, 1\} \).

**Proof.** (a) Put \( l = \lceil \frac{d_0}{6} \rceil \). Since \( mz \in \mathbb{T}_+ \) for all \( m = 1, \ldots, l \), one has \( z \in \{1, \ldots, l\}^{\omega} = \mathbb{T}_l \), and thus \( |z| \leq \frac{1}{4l} < \frac{3}{2d_0} \). Therefore,

\[
\left| \frac{c_0}{d_0} \right| \leq |z| + \left| z - \frac{c_0}{d_0} \right| < \frac{3}{2d_0} + \frac{1}{2d_0} = \frac{2}{d_0}.
\]

Hence, \( |c_0| < 2 \), as desired.
(b) For \( d_0 = 2 \) and \( d_0 = 3 \), the conclusion is trivial. If \( d_0 \neq 6, 7 \), then one has \( [\frac{d_0}{7}] \leq [\frac{d_0}{4}] \), and the statement follows from part (a). Suppose that \( d_0 = 6 \). Then it is given that \( z \in \mathbb{T}_+ \), and thus \( |z| \leq \frac{1}{2} \). If \( |z - \frac{c_0}{d_0}| < \frac{1}{2d_0} \), then

\[
\left| \frac{c_0}{d_0} \right| = |z| + \left| z - \frac{c_0}{d_0} \right| < \frac{1}{4} + \frac{1}{2d_0} = \frac{2}{d_0},
\]

and thus \( |c_0| < 2 \). If \( |z - \frac{c_0}{d_0}| = \frac{1}{2d_0} \), then \( |\frac{c_0}{d_0}| \leq |z| \leq \frac{1}{4} \), because the representation of \( z \) is standard. Hence, \( |c_0| < 2 \), as desired.

(c) If \( d_0 \neq 7 \), then the statement follows from (b), and so we may suppose that \( d_0 = 7 \). Then it is given that \( z, 6z \in \mathbb{T}_+ \), which means that

\[
z \in \{1, 6\}^6 = \mathbb{T}_6 \cup (-\frac{1}{6} + \mathbb{T}_6) \cup (\frac{1}{6} + \mathbb{T}_6).
\]

Thus, \( |z| \leq \frac{5}{24} \). Therefore,

\[
\left| \frac{c_0}{d_0} \right| = |z| + \left| z - \frac{c_0}{d_0} \right| \leq \frac{5}{24} + \frac{1}{14} = \frac{47}{168} < \frac{48}{168} = \frac{2}{d_0}.
\]

Hence, \( |c_0| < 2 \). \qed

The second preparatory step to precede the proof of Theorem 6.1 involves finding characters in \( X^\circ \). Let \( \eta_0 : \mathbb{T} \to \mathbb{T} \) denote the identity homomorphism, and for \( k \geq 1 \), set \( \eta_k := b_{k-1} \eta_0 \).

**Lemma 6.3.** If \( q_k \geq 4 \), then \( m \eta_k \in X^\circ \) for \( m = 1, \ldots, \lfloor \frac{q_k}{4} \rfloor \). If in addition \( q_{k+1} \geq 4 \), then \( m \eta_k \in X^\circ \) also for \( m = q_k - 1 \).

**Proof.** Fix \( n \in \mathbb{N} \). If \( n < k \), then \( \eta_k(\frac{1}{b_n}) = \frac{b_{k-1}}{b_k} \equiv 0 \) and thus \( m \eta_k(\frac{1}{b_n}) \in \mathbb{T}_+ \).

If \( n = k \), then \( \eta_k(\frac{1}{b_n}) = \frac{b_{k-1}}{b_k} = \frac{1}{q_k} \), and so \( m \eta_k(\frac{1}{b_n}) = \frac{m}{q_k} \in \mathbb{T}_+ \) for \( m = 1, \ldots, \lfloor \frac{q_k}{4} \rfloor \) and \( m = q_k - 1 \).

If \( n > k \), then \( \eta_k(\frac{1}{b_n}) = \frac{b_{k-1}}{b_k} = \frac{1}{q_k q_k \ldots q_k} \) and \( m \eta_k(\frac{1}{b_n}) = \frac{m}{q_k q_k \ldots q_k} \). Consequently, \( m \eta_k(\frac{1}{b_n}) \in \mathbb{T}_+ \) for \( m = 1, \ldots, \lfloor \frac{q_k}{4} \rfloor \). If \( q_k + 1 \geq 4 \), then \( \frac{q_{k+1}-1}{q_k q_k \ldots q_k} \leq \frac{1}{q_{k+1}} \leq \frac{1}{4} \), and hence \( (q_k - 1) \eta_k(\frac{1}{b_n}) \in \mathbb{T}_+ \). \qed

We are now ready to prove Theorem 6.1.

**Proof of Theorem 6.1.** Let \( x \in Q_T(X) \), and let \( x = \sum_{i=0}^{\infty} c_i b_i \) be a standard representation of \( x \). Fix \( k \in \mathbb{N} \), and put \( d_i := \frac{b_{k-1}}{b_{k-1}} = q_k \ldots q_{k+i} \) for every \( i \in \mathbb{N} \). (As usual, we consider \( b_{-1} = 1 \).) Then \( z := \eta_k(x) = b_{k-1} x \equiv 1 \sum_{i=0}^{\infty} c_{i+k} d_i \) is a standard representation of \( z \). Furthermore, if \( m \eta_k \in X^\circ \), then \( m z = m \eta_k(x) \in \mathbb{T}_+ \).

If \( q_k < 4 \), then \( |c_k| \leq \frac{q_k}{2} < 2 \), and there is nothing to prove. So, we may assume that \( q_k \geq 4 \). Thus, by Lemma 6.3, \( m \eta_k \in X^\circ \) for \( m = 1, \ldots, \lfloor \frac{q_k}{4} \rfloor \). Therefore, \( m z \in \mathbb{T}_+ \) for \( m = 1, \ldots, \lfloor \frac{q_k}{4} \rfloor \). Therefore, if \( q_k \neq 7 \), then by Lemma 6.2(b), the first coefficient of \( z \), that is \( c_k \), satisfies \( c_k \in \{-1, 0, 1\} \). If \( q_k = 7 \), then by our assumption, \( q_{k+1} \geq 4 \), and by Lemma 6.3, \( (q_k - 1) \eta_k \in X^\circ \). Consequently, by Lemma 6.2(c), the first coefficient of \( z \), that is \( c_k \), satisfies \( c_k \in \{-1, 0, 1\} \). \qed

The next lemma is somewhat similar to [7] 3.3, both in its content and its role in the proof of Theorem 6.1.
Lemma 6.4. Let \( k_1, k_2 \in \mathbb{N} \) be such that \( k_1 < k_2 \). Then \( \left[ \frac{q_{k_1}}{4} \right] \eta_{k_1} + \left[ \frac{q_{k_2}}{4} \right] \eta_{k_2} \in X^\circ \).

**Proof.** Let \( n \in \mathbb{N} \). If \( n < k_2 \), then \( \eta_{k_2} \left( \frac{1}{b_n} \right) = \frac{b_{k_2} - 1}{b_n} \equiv 0 \), and thus

\[
\left( \left[ \frac{q_{k_1}}{4} \right] \eta_{k_1} + \left[ \frac{q_{k_2}}{4} \right] \eta_{k_2} \right) \left( \frac{1}{b_n} \right) \equiv 1 \left[ \frac{q_{k_1}}{4} \right] \eta_{k_1} \left( \frac{1}{b_n} \right) \in \mathbb{T},
\]

because \( \left[ \frac{q_{k_1}}{4} \right] \eta_{k_1} \in X^\circ \) by Lemma 6.3. Suppose now that \( k_2 \leq n \). Then

\[
\left| \left[ \frac{q_{k_1}}{4} \right] \eta_{k_1} \left( \frac{1}{b_n} \right) \right| \leq \frac{1}{q_{k_1} q_{k_2}} \leq \frac{1}{4 q_{k_2}} \quad \text{and}
\]

\[
\left| \left[ \frac{q_{k_2}}{4} \right] \eta_{k_2} \left( \frac{1}{b_n} \right) \right| \leq \frac{1}{4 q_{k_2}}.
\]

Therefore,

\[
\left| \left( \left[ \frac{q_{k_1}}{4} \right] \eta_{k_1} \pm \left[ \frac{q_{k_2}}{4} \right] \eta_{k_2} \right) \left( \frac{1}{b_n} \right) \right| \leq \frac{1}{4 q_{k_1}} + \frac{1}{4 q_{k_2}} \leq \frac{1}{4},
\]

and hence \( \left[ \frac{q_{k_1}}{4} \right] \eta_{k_1} \pm \left[ \frac{q_{k_2}}{4} \right] \eta_{k_2} \left( \frac{1}{b_n} \right) \in \mathbb{T} \). This shows that \( \left[ \frac{q_{k_1}}{4} \right] \eta_{k_1} \pm \left[ \frac{q_{k_2}}{4} \right] \eta_{k_2} \in X^\circ \). \( \square \)

We introduce further notations to facilitate calculations in \( \mathbb{T} \). Let \( z = \sum_{i=0}^{\infty} \frac{d_i}{q_i} \) be a standard representation of \( z \in \mathbb{T} \) with respect to a sequence \( \{d_i\}_{i=0}^{\infty} \) such that \( d_i | d_{i+1} \) for every \( i \in \mathbb{N} \). We put \( \Lambda(z) := \{ i \in \mathbb{N} \mid c_i \neq 0 \} \), \( q(z) := \min \{ d_{i+1} | i \in \Lambda(z) \} \), and \( S(x) := \frac{1}{q(x)-1} \). (As usual, we consider \( d_{-1} = 1 \).)

Lemma 6.5. Let \( x = \sum_{i=0}^{\infty} \frac{d_i}{q_i} \) be a standard representation of \( x \in \mathbb{T} \), with \( \varepsilon_i \in \{-1, 0, 1\} \). Then, for every \( k \in \Lambda(x) \), one has \( \frac{1}{q_k} (1 - S(x)) \leq |\eta_k(x)| \leq \frac{1}{q_k} (1 + S(x)) \).

**Proof.** One has \( \eta_k(x) = b_{k-1} x \equiv \sum_{i=0}^{\infty} \frac{\varepsilon_{k+i}}{q_k \cdots q_{k+i}} \), and thus one obtains (modulo 1)

\[
\left| \eta_k(x) - \varepsilon_k \right| \leq \frac{1}{q_k} \sum_{i=1}^{\infty} \frac{1}{q_{k+1} \cdots q_{k+i}} \leq \frac{1}{q_k} \sum_{i=1}^{\infty} \frac{1}{(q(x))^2} = \frac{S(x)}{q_k}.
\]

Therefore, \( \frac{1}{q_k} (1 - S(x)) \leq |\eta_k(x)| \leq \frac{1}{q_k} (1 + S(x)) \), as required. \( \square \)

Lemma 6.6. Suppose that \( q_{k+1} \geq 4 \) whenever \( q_k = 7 \), and let \( x \in Q_T(X) \). If \( k_1, k_2 \in \Lambda(x) \) are such that \( k_1 < k_2 \), then

\[
\left( \frac{\left[ \frac{q_{k_1}}{4} \right]}{q_{k_1}} + \frac{\left[ \frac{q_{k_2}}{4} \right]}{q_{k_2}} \right) (1 - S(x)) \leq \frac{1}{4}.
\]
PROOF. Put \( \chi := \varepsilon_k \left( \frac{q_{k_1}}{4} \right) \eta_{k_1} + \varepsilon_k \left( \frac{q_{k_2}}{4} \right) \eta_{k_2} \). By Lemma 6.4, \( \chi \in X^p \), and so \( \chi(x) \in \mathbb{T}_+ \). The conditions imposed upon \( \{ q_n \}_{n=0}^{\infty} \) guarantee that Theorem 6.1 is applicable, and thus \( x \) can be expressed in standard form as \( x = \sum_{i=0}^{\infty} \frac{a_i}{b_i} \), with \( \varepsilon_i \in \{-1, 0, 1\} \). In what follows, we use Lemma 6.5 to estimate \( \chi(x) \) from below.

If \( q_1(x) = 2 \) or \( q_2(x) = 3 \), then the statement is trivial, and so we may assume that \( q_i(x) \geq 4 \). Then, by Lemma 6.5, \( |\eta_{k_j}(x)| \leq \frac{1}{3} q_{k_j} \) \( (j = 1, 2) \), and thus

\[
\left| \frac{q_{k_1}}{4} |\eta_{k_1}(x)| \right| + \left| \frac{q_{k_2}}{4} |\eta_{k_2}(x)| \right| \leq \frac{2}{3}.
\]

Therefore,

\[
|\chi(x)| \leq \left| \frac{q_{k_1}}{4} |\eta_{k_1}(x)| \right| + \left| \frac{q_{k_2}}{4} |\eta_{k_2}(x)| \right| \leq \frac{2}{3}.
\]

This implies that we can perform the remaining calculations in \( [\frac{-2}{3}, \frac{2}{3}] \subset \mathbb{R} \), and \( \chi(x) \in \mathbb{T}_+ \). If and only if \( \frac{1}{4} \leq \chi(x) \leq \frac{1}{4} \). Since the first term of \( \varepsilon_k \eta_{k_j}(x) \) is \( \frac{1}{q_{k_j}} \), one has \( 0 \leq \varepsilon_k \eta_{k_j}(x) \). Consequently, by Lemma 6.5, \( \frac{1}{q_{k_j}} (1 - S(x)) \leq \varepsilon_k \eta_{k_j}(x) \), and hence

\[
\left( \frac{q_{k_1}}{4} + \frac{q_{k_2}}{4} \right) (1 - S(x)) \leq \varepsilon_k \left( \frac{q_{k_1}}{4} \right) \eta_{k_1} + \varepsilon_k \left( \frac{q_{k_2}}{4} \right) \eta_{k_2} (x) = \chi(x) \leq \frac{1}{4},
\]

as desired. \( \square \)

We are now ready to prove Theorem 6.1.

PROOF OF THEOREM 6.1. Let \( x \in Q_\mathbb{T}(X) \), and assume that \( x \notin X \). Then \( |\Lambda(x)| > 1 \), and so we may pick \( k_1, k_2 \in \Lambda(x) \) such that \( k_1 < k_2 \). As \( q_n \geq 8 \) for every \( n \in \mathbb{N} \), one has \( 1 - S(x) \geq \frac{6}{7} \), and thus, by Lemma 6.6,

\[
\left| \frac{q_{k_1}}{4} \right| + \left| \frac{q_{k_2}}{4} \right| \leq \frac{7}{24}.
\]

This inequality, however, does not hold with any \( q_{k_j} \geq 8 \). Hence, \( x \in X \), as desired. \( \square \)

Example 6.7. Let \( \{ p_n \}_{n=0}^{\infty} \) be an enumeration of all primes greater than 8, and put \( b_n = p_0 \cdots p_n \) for every \( n \in \mathbb{N} \). By Theorem 6.1, \( \{ \frac{1}{b_n} \}_{n=0}^{\infty} \) is quasi-convex in \( \mathbb{T} \), and since each \( b_n \) is square-free, \( \{ \frac{1}{b_n} \}_{n=0}^{\infty} \subset \text{soc}(\mathbb{T}) \).

Using the next lemma, one can lift Theorem 6.1 into \( \mathbb{R} \). Recall that in this note, \( \pi : \mathbb{R} \to \mathbb{T} \) denotes the canonical projection.

Lemma 6.8. ([8, 2.4]) Let \( Y \subset \mathbb{R} \) be a compact subset. If there is \( \alpha \neq 0 \) such that \( \alpha Y \subset \left( -\frac{1}{2}, \frac{1}{2} \right) \) and \( \pi(\alpha Y) \) is quasi-convex in \( \mathbb{T} \), then \( Y \) is quasi-convex in \( \mathbb{R} \).

Corollary 6.9. Let \( \{ x_n \}_{n=0}^{\infty} \subset \mathbb{R} \) be a null sequence in \( \mathbb{R} \) such that \( q_n := \frac{x_{n-1}}{x_n} \) are integers and \( q_n \geq 8 \) for every \( n \in \mathbb{N} \setminus \{0\} \). Then \( \{ x_n \}_{n=0}^{\infty} \) is quasi-convex in \( \mathbb{R} \).

PROOF. Put \( \alpha = \frac{1}{8q_0} \), \( q_0 = 8 \), and \( b_n = q_0 \cdots q_n \). Then \( \alpha x_n = \frac{1}{b_n} \), and thus, by Theorem 6.1, the sequence \( \{ \pi(\alpha x_n) \}_{n=0}^{\infty} \) is quasi-convex in \( \mathbb{T} \). Since \( \{ \alpha x_n \}_{n=0}^{\infty} \subset [\frac{-1}{8}, \frac{1}{8}] \), by Lemma 6.8, the sequence \( \{ x_n \}_{n=0}^{\infty} \) is quasi-convex in \( \mathbb{R} \), as required. \( \square \)
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