Topological term in the non-linear $\sigma$ model of the SO(5) spin chains

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Abstract

We show that there is a topological (Berry phase) term in the non-linear $\sigma$ model description of the SO(5) spin chain. It distinguishes the linear and projective representations of the SO(5) symmetry group, in exact analogy to the well-known $\theta$-term of the SO(3) spin chain. The presence of the topological term is due to the fact that $\pi_2(SO(5)) = Z$. We discuss the implication of our results on the spectra of the SO(5) spin chain, and connect it with a recent solvable SO(5) spin model which exhibits valence bond solid ground state and edge degeneracy.

**Key words:** Non-linear sigma model, topological terms, spin chains
1 Introduction

The effects of topological terms on the dynamics of Goldstone modes and the quantum number of solitons and instantons in non-linear $\sigma$ (NL$\sigma$) models have a long history, and continue to attract strong interests from the physics community\cite{1}. For example, in one spatial dimension (1D), quantum SO(3) spin chains have fundamentally different low energy properties, depending on whether the site representation is linear (spin integer) or projective (spin half-odd integer). For nearest neighbor isotropic exchange interaction, the former cases always have excitation gaps while the latter are gapless - the well known Haldane conjecture\cite{2}. Aside from the usual stiffness terms, the (1+1)D NL$\sigma$ models for these spin chains contain a topological (Berry phase) term (also known as the $\theta$ term)\cite{2}. When the space-time configuration of the Neel order parameter wraps the target space\cite{3} ($S^2$) $n$ times, the Berry phase factor is $+1$ for integer spin chains, while it is $(-1)^n$ for half-odd integer spin chains\cite{2}.

Using an algebraic approach, Chen, Gu and Wen\cite{4} recently generalized an idea of Ref.\cite{5} and argued that, in one spatial dimension, a gapped ground state which is invariant under translation and the global symmetry operation (we refer to this type of state as “totally symmetric” in the following) is obtainable when the site representation of the global symmetry group (which can be discrete) is linear. When the site representation is projective, a totally symmetric ground state must be gapless. In a projective representation $D$, the matrix product $D(g_1)D(g_2)$, where $g_{1,2}$ are group elements, can differ from $D(g_1g_2)$ by a phase factor $e^{i\theta(g_1,g_2)}$. For the SO(3) group, an integer spin forms the linear representation while a half-odd integer spin forms the projective representation. Hence the spectral difference between the translational invariant integer and half-odd integer SO(3) spin chains constitutes a special example of the results of Ref.\cite{4}. Thus there are two ways to view the difference between integer and half-odd integer SO(3) spin chain: one is geometrical (the Berry’s phase)\cite{2} and the other is algebraic\cite{4}.

SO(5) is a rank-2 classical Lie group. It also has linear and projective representations. For example, in the vector representation, the generators of the Lie algebra are given by\cite{6}

$$
(L_{ab})_{jk} = -i\delta_{a,j}\delta_{b,k} + i\delta_{a,k}\delta_{b,j}, \quad (1)
$$

where $a, b, = 1,..,5$, and $i, j = 1,..,5$. Two consecutive $\pi$ rotations generated by, e.g., $L_{12}$ give

$$
U_{12}(\pi)U_{12}(\pi) = I_{5 \times 5}. \quad (2)
$$
The spinor representation, on the other hand, is given by

\[ L_{ab} = i[\Gamma_a, \Gamma_b]/4, \quad (3) \]

where \( \Gamma_{a,b} \) are the \( 4 \times 4 \) gamma matrices (e.g., \( \Gamma_1 = -\sigma_y \otimes \sigma_x \), \( \Gamma_2 = -\sigma_y \otimes \sigma_y \), \( \Gamma_3 = -\sigma_y \otimes \sigma_z \), \( \Gamma_4 = \sigma_x \otimes I_{2 \times 2} \), \( \Gamma_5 = \sigma_z \otimes I_{2 \times 2} \)). In this case it is simple to check that two consecutive \( \pi \) rotations generated by \( L_{12} \) yield

\[ U_{12}(\pi)U_{12}(\pi) = -I_{4 \times 4}. \quad (4) \]

Thus the vector representation is linear, while the spinor representation is projective. According to Ref.\[3\], a spin chain of the former type can have a totally symmetric ground state with a gapped spectrum, while a spin chain of the latter type has to be gapless if it is totally symmetric. In the following, we will seek for the geometric (Berry’ phase) difference between the two cases.

Another motivation for us to study the Berry’s phase of the SO(5) spin chain is a recent exactly solvable 1D SO(5) spin model (in the vector representation) proposed by Tu et. al.\[7\]. The ground state is a translational invariant matrix product state, i.e., a valence bond solid state\[8\], and the excitation spectrum has a gap. Moreover, when the chain is subjected to open boundary condition, there are edge states. These properties are reminiscent of those of integer SO(3) spin chains\[9\].

With the advance of cold atom physics, the SO(5) spin chain might not be a purely academic model anymore. An SO(5) symmetric spin chain can in principle be realized experimentally when the hyperfine spin-3/2 cold fermions on an 1D optical lattice form the Mott-insulating state\[10\]. At quarter filling (one fermion per site), the effective spin chain is in the spinor representation, while for half-filling (two fermions per site) it is in the SO(5) vector representation. Therefore the idea presented here might one day be tested experimentally.

2 Model formulation

Let us start by considering the following SO(5) invariant Hamiltonian

\[ H = \sum_i \left[ J_1 \left( \sum_{a<b} L^i_{ab} L^{i+1}_{ab} \right) + J_2 \left( \sum_{a<b} L^i_{ab} L^{i+1}_{ab} \right)^2 \right] \quad (5) \]
where $L_{ab}$'s are the SO(5) generators and $J_{1,2} > 0$. When $L_{ab}$ are given by Eq.(1), the ground state is translational invariant and the spectrum has a gap in the parameter range $1/9 < J_2/J_1 < 1/3$. Naively, one would not expect the NL$\sigma$ model action of this model to contain a topological term. This is because in contrast to SO(3) spin chain where the space-time dimension $(1+1)$ matches the dimension of the target space of the order parameter ($S^2$), for the SO(5) spin chain the target space dimension is much larger than the space-time dimension.

To understand the structure of the target space for the SO(5) spin chain, we need to know how the presence of SO(5) “magnetic” moment breaks the global SO(5) symmetry. For that purpose, it is sufficient to consider the following mean-field theory where non-linear terms in $L_{ab}$ are decoupled into linear ones with order parameter given by $\langle L_{ab}^i \rangle = (-1)^i m_{ab}$

$$H_{MF} = \left( -2J_1 + 2J_2 \Delta^2 \right) \sum_{i,a<b} (-1)^i m_{ab} L_{ab}^i + \sum_i \left( J_1 \Delta^2 - 3J_2 \Delta^4 \right),$$

where $\Delta^2 = \sum_{a<b} m_{ab}^2$. The question at hand is, for a fixed total magnitude of $m_{ab}$ (i.e. fixed $\sum_{a<b} m_{ab}^2$), what is the most energetically favorable ratio between different components of $m_{ab}$. This can be answered by diagonalizing a single-site Hamiltonian

$$H_1 = -\sum_{a<b} m_{ab} L_{ab}.$$  

and see what ratio gives the lowest ground state energy. (Of course we need to remember the sign of $m_{ab}$ change from site to site.)

In the following, we study the two irreducible representations given by Eqs.(1) and (3). First we consider the vector representation, Eq.(1). It is straightforward to show the energy spectrum of $H_1$ is $E = (-\Delta_1, -\Delta_2, 0, \Delta_2, \Delta_1)$ where $\Delta_{1,2} = \sqrt{A \pm \sqrt{A^2 + B - C}}$ and

$$A = \sum_{a<b} m_{ab}^2 / 2 = \Delta^2 / 2,$$

$$B = 2 \sum_{a<b<c<d} (m_{ac} m_{ad} m_{bc} m_{bd} - m_{ab} m_{ad} m_{bc} m_{cd} + m_{ab} m_{ac} m_{bd} m_{cd}),$$

$$C = \sum_{a<b} \sum_{a<c<d} m_{ab}^2 m_{cd}^2 (1 - \delta_{bc})(1 - \delta_{bd}).$$

The single site ground state energy reaches the minimum when $B = C$, where the energy spectrum of $H_1$ is $E = \{-\Delta, 0, 0, 0, \Delta\}$. Now we ask how is the SO(5) symmetry broken under such a condition. Let us use $H_1$ as one of the two Cartan
Fig. 1. (color on-line) The root (upper row) and weight (lower row) patterns of the vector (a) and spinor (b) representation of SO(5). The x and y coordinates of each dot correspond to the eigenvalue of $H_1$ and $H_2$. The red arrows indicate how the raising/lowering operators in the SO(5) Lie algebra change the eigenvalues of $H_1$ and $H_2$. The central dots in the root patterns are doubly degenerate.

For the other Cartan generators $H_2$, we choose it to be a different linear combination of $L_{ab}$ so that $\text{Tr}(H_1 H_2) = 0$. The root and weight diagrams are shown in Fig. (1a). Here the x and y coordinates of the dots are the eigenvalues of $H_1$ and $H_2$, respectively. The arrows in the root diagrams indicate how the raising/lowering operators in the SO(5) Lie algebra change the eigenvalues of $H_{1,2}$. $H_2$ and the raising/lowering operators generate an SO(3) subgroup which commutes with $H_1$. As the result, the SO(5) symmetry is broken down to SO(3)×SO(2), where the SO(2) is generated by $H_1$ itself.

Second we consider the spinor representation, Eq. (3). It is straightforward to show that the energy spectrum of $H_1$ is $E = (-\Delta_1, -\Delta_2, \Delta_2, \Delta_1)$ with $\Delta_{1,2} = \sqrt{A \pm \sqrt{C - B} / \sqrt{2}}$. In this case the single site ground state energy reaches the minimum when $A = \sqrt{C - B}$, where the energy spectrum of $H_1$ is $E = \{-\Delta, 0, 0, \Delta\}$. The root and weight patterns are shown in Fig. (1b). Again the SO(5) symmetry is broken down to SO(3)×SO(2).

After fixing the ratio of $m_{ab}$, we assume that the low energy fluctuations correspond to smooth space-time dependent SO(5) rotations of such an order parameter pattern. The NL$\sigma$ model precisely describes such smooth fluctuations. The order
parameter lives on the manifold \( \frac{SO(5)}{SO(3) \times SO(2)} \), which has \( \mathbb{Z} \) as its second homotopy group. Thus the corresponding NL\( \sigma \) model may contain a topological term, which can lead to a spectral difference between the vector and spinor representations.

3 The single-site Berry’s phase

To study the possible topological term, we begin by analyzing the Berry’s phase of a single SO(5) spin described by the following time-dependent Hamiltonian

\[
H_1(t) = -\sum_{a < b} m_{ab}(t)L_{ab},
\]

where \( m_{ab}(t) \) satisfy the constraints: (a) \( \sum_{a < b} m_{ab}^2 = 1 \), and (b) \( H_1(t) \) possesses \( SO(3) \times SO(2) \) symmetry. Both constraints can be satisfied by starting with a reference Hamiltonian \( H_{1,0} \) satisfying (a) and (b) and perform time-dependent SO(5) conjugation, i.e.,

\[
H_1(t) = U^\dagger(t) H_{1,0} U(t).
\]

As usual, the Berry’s phase is given by the loop integral of the Berry connection. We can use Stoke’s theorem to convert this loop integral to an areal integral over a disk with the loop as the boundary. The advantage of doing so is the Berry curvature rather than the Berry connection appears in the latter integral. This makes the integral involving only local quantities and gauge invariant:

\[
S_B = \frac{i}{2} \int_0^1 du \int dt \, \epsilon^{\mu\nu} \text{Tr} F_{\mu\nu}.
\]

In the above \( F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) \) and \( A_\mu = -i \langle \Omega | \partial_\mu | \Omega \rangle \). Here \( | \Omega(t, u) \rangle \) is the ground state of

\[
H_1(t, u) = U^\dagger(t, u) H_{1,0} U(t, u).
\]

In Eq. (12), \( U(t, u) \) is the extension of the \( U(t) \) in Eq. (10) to the disk. Because \( \pi_1(\text{SO(5)/SO(3) } \times \text{SO(2)})=0 \), we can always construct the extension so that \( U(u = 1, t) = U(t) \) and \( U(u = 0, t) = U_0 \), where \( U_0 \) is a certain reference \( \text{SO(5)} \) element.

Using the first order perturbation theory for wave functions, it is simple to show that

\[
\text{Tr} F_{\mu\nu} = -i \sum_k \frac{\langle \Omega | \partial_\mu H | k \rangle \langle k | \partial_\nu H | \Omega \rangle - (\mu \leftrightarrow \nu)}{(E_0 - E_k)^2}.
\]
Here $k$ labels the excited states. Since all Hamiltonians described by Eq. (12) are unitary conjugate of one another, they have the same eigenspectrum. Under that condition we have

$$\partial_\mu H = \sum_k (E_k - E_0)(|\partial_\mu k\rangle \langle k| + |k\rangle \langle \partial_\mu k|). \quad (14)$$

In writing down the above equation we have made a shift of the zero of energy so that $E_0 \to 0$. Substitute Eq. (14) into Eq. (13) and use the fact that $\langle \Omega|k\rangle = \langle k|\Omega\rangle = 0$ we find

$$\text{Tr} F_{\mu\nu} = -i\text{Tr}(Q[\partial_\mu Q, \partial_\nu Q]), \quad (15)$$

where $Q$ is the ground state projection operator

$$Q(t, u) = |\Omega\rangle \langle \Omega| = U(t, u)PU^\dagger(t, u), \quad (16)$$

where $P = |0\rangle \langle 0|$ is the ground state projector operator of $H_{1,0}$. Substituting Eq. (15) into Eq. (11) we obtain

$$S_B = \int_0^1 du \int dt \text{Tr}(Q[\partial_\mu Q, \partial_\nu Q]), \quad (17)$$

Eq. (17) actually applies for any target space. For example, in the case of SO(3)/SO(2), we have

$$U(t, u) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $z_{1,2}(t, u)$ satisfy

$$(\bar{z}_1, \bar{z}_2) \cdot \vec{\sigma} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \hat{n}(t, u).$$

Substitute the above two expressions into Eq. (16) and Eq. (17) we obtain

$$S_B = \frac{i}{2} \int_0^1 du \int dt (\hat{n} \cdot \partial_\mu \hat{n} \times \partial_\nu \hat{n}), \quad (18)$$

which is the well known expression for the Berry’s phase of a spin-1/2[2].

Because the dimension of our target space is eight, there are more than one disks having the loop in question as the boundary. Therefore, we need to ask whether Eq. (17) yields the same answer for the Berry phase when different extensions of $U(t) \to U(t, u)$ are used. The difference in the Berry phase using two different disks as extension can be calculated by integrating the Berry curvature over a
closed two dimensional surface formed by joining the two disks at their common boundary. The resulting closed surface has the topology of a 2-sphere. Because the second homotopy group of our target space is \( \mathbb{Z} \), all 2-spheres in the target space are topologically the multiple of a basic sphere. Hence all we need to check is whether the Berry curvature integral is integer multiple of \( 2\pi \) when \( t \) and \( u \) in Eq. (17) parameterize the basic 2-sphere\(^{[12]} \). In the following, we perform such a calculation.

For the vector representation, we choose \( H_{1,0} = L_{12} \), and pick \( U(t, u) \) so that

\[
U^\dagger(t, u)H_{1,0}U(t, u) = \hat{w}(t, u) \cdot \vec{L},
\]

where \( \vec{L} = (L_{13}, L_{23}, L_{12}) \),

\[
\hat{w}(t, u) = (\sin(\pi u) \cos \frac{2\pi t}{\beta}, \sin(\pi u) \sin \frac{2\pi t}{\beta}, \cos(\pi u)),
\]

and \( \beta \) is the period of the imaginary time. For the spinor representation, we take \( H_{1,0} = (L_{12} - L_{34})/\sqrt{2} \), and \( \vec{L} = (L_{13} + L_{24}, L_{14} - L_{23}, L_{12} - L_{34})/\sqrt{2} \).

Using Eq. (17), or equivalently Eqs. (11) and (13), we find that

\[
S_{\text{vector}}^{B, \text{basic sphere}} = 4\pi, \quad S_{\text{spinor}}^{B, \text{basic sphere}} = 2\pi.
\]

Thus the condition for the uniqueness of the Berry phase is satisfied. As we shall see, the difference between the vector and spinor Berry phase in Eq. (20) serves to distinguish the vector and spinor SO(5) chains.

4 The lattice Berry’s phase, gapful versus gapless, and the edge states

We now extend the above single-site Berry phase analysis to the one dimensional lattice. If the order parameter is perfectly “antiferromagnetic” at all time, the ground state of \( H_1(t) \) are \( |\Omega(t)\rangle \) on the even sublattice and its conjugate \( |\bar{\Omega}(t)\rangle = R|\Omega(t)\rangle^* \) on the odd sublattice, where \( R \) is the operator that satisfies \( RL_{ab}^* R^{-1} = -L_{ab} \). For the vector representation, \( R \) is given by \( R_{15} = -R_{24} = R_{33} = -R_{42} = R_{51} = -1 \) and \( R_{ij} = 0 \) otherwise. For the spinor representation, \( R = -iI_{2\times2} \otimes \sigma_y \).

Using the above result, it is straightforward to show that \( \langle \Omega | L_{ab} | \Omega \rangle = -\langle \bar{\Omega} | L_{ab} | \bar{\Omega} \rangle \), and \( RU^* R^{-1} = U \) for all \( U \in \text{SO}(5) \). This, plus the invariance of the trace under matrix transposition, allows one to show that \( e^{i\nu} \text{Tr} Q \partial_\mu \bar{Q} \partial_\nu \bar{Q} = -e^{i\nu} \text{Tr} Q \partial_\mu \bar{Q} \partial_\nu \bar{Q} \). As the result, the Berry’s phases associated with neighboring sites tend to cancel.
each other. Let $r$ label the center of mass position of site $i$ and $i + 1$ for $i =$ odd, the total lattice Berry’s phase is equal to

$$S_{tot}^B = \sum_r \sum_{\epsilon = \pm 1} (-1)^{\epsilon - 1} \int_0^1 du \int dt Tr(Q_{r+\epsilon/2}[\partial_u Q_{r+\epsilon/2}, \partial_t Q_{r+\epsilon/2}]),$$

(21)

where $Q$ is a smooth function of spacial coordinates. Under such a condition, Eq. (21) has a continuum limit

$$S_{tot}^B = \frac{1}{2} \int dx \int dt Tr(\partial_t Q, \partial_x Q),$$

(22)

the factor of $1/2$ arises from the density of odd lattice sites. Under open boundary condition, Eq. (22) becomes

$$S_{tot}^B = \frac{1}{2} \int dx dt \left\{ Tr[Q[\partial_u Q, \partial_t Q]]_R - Tr[Q[\partial_u Q, \partial_t Q]]_L \right\},$$

(23)

where the subscript “R” and “L” labels the right and the left ends. This topological term together with the stiffness term from the energetics, constitute the NL$\sigma$ model for the SO(5) spin chain.

Eqs. (20) and (22) have important implications. The fact that the mapping from the space-time to the target space is classified by integer homotopy classes implies that the space-time order parameter configurations can be grouped into different topological sectors labeled by an integer topological invariant. This is similar to the SO(3) NL$\sigma$ model where the topological invariant, the Pontryagin index[13], is the number of times which the order parameter configurations cover the target space $S^2$. In our case there is an analogous integer topological index, which we will refer to as the Pontryagin index as well. Eq. (20) implies that for the vector SO(5) spin chain the Berry phase associated with the order parameter configuration having different Pontryagin indices are all the same because $\exp(i4\pi/2 \times \text{integer}) = +1$. Given the facts that (i) the topological term has no effect (hence the NL$\sigma$ model has only the stiffness terms), and (ii) the target space dimension is high, it is easy to believe that the vector SO(5) spin chain should have a quantum disordered, i.e., translational invariant gapped, phase. For the spinor SO(5) chain, however, the order parameter configurations with even Pontryagin index have the Berry phase $\exp(i2\pi/2 \times \text{even integer}) = +1$, while those with odd Pontryagin index have Berry phase $\exp(i2\pi/2 \times \text{odd integer}) = -1$. This result is exactly analogous to the Berry’s phase in the spin-$1/2$ representation of the SO(3) antiferromagnetic
Heisenberg chain. In view of the result of Ref. [4], we conclude that the above non-trivial Berry’s phase also implies the lack of an energy gap as long as the translation symmetry is unbroken.

Now we comment on the edge state of the SO(5) “valence bond solid state” in Ref. [7]. By tuning the ratio of $J_1$ and $J_2$ in Eq. (5), Tu et al were able to show that a short-range entangled, translational invariant matrix product state is the exact ground state. In addition, under the open boundary condition the ground state wavefunction becomes $4 \times 4 = 16$ fold degeneracy. According to Eq. (23), the boundary of a vector spin chain should exhibit the following Berry phase

$$\frac{1}{2} \int_0^1 du \int dt \text{Tr} [\partial_u Q, Q \partial_t Q].$$ (24)

When $Q(t, u)$ is a unit Pontryagin index order parameter configuration, the value of Eq. (24) is $\frac{1}{2} \times 4\pi = 2\pi$. This is consistent with the spinor Berry phase in Eq. (20). Therefore, the edge state of the vector SO(5) spin chain carries the spinor representation. Because the latter is 4-dimensional, each end of the chain independently yields a 4-fold degeneracy of the ground state, resulting in a total $4 \times 4 = 16$ fold degenerate ground state for the open chain.

Before closing a technical remark is in order. The readers might wonder what if the ratio between different components of $m_{ab}$ fluctuates away from the optimal value. When that happens the single site spectrum will become \{-$\Delta_1$, -$\Delta_2$, 0, $\Delta_2$, $\Delta_1$\} and \{-$\Delta_1$, -$\Delta_2$, $\Delta_2$, $\Delta_1$\} for the vector and spinor representation, respectively. In this case, the SO(5) symmetry is broken down to SO(2)$\times$SO(2). The second homotopy group of $SO(5)/SO(2) \times SO(2)$ is $\mathbb{Z} \oplus \mathbb{Z}$ rather than $\mathbb{Z}$. In other words, the image of the space-time in the target space is topologically the multiple of two basic spheres. As $\Delta_2 \to 0$, one of these spheres shrinks to a point. We have checked that so long as $\Delta_2$ is small, i.e., when the ratio between $m_{ab}$ does not deviate from the optimal value too drastically, the Berry phase is only sensitive to the Pontryagin index of the dominant (large) sphere. Hence all the results discussed earlier remain unchanged.

5 Conclusion

We have studied the Berry’s phase of the antiferromagnetic SO(5) spin chain, and shown the existence of a topological term in the non-linear $\sigma$ model description of the system. The quantum phase factor associated with this topological term differentiates the vector (linear) and spinor (projective) representations. We argue
that this leads to the spectral difference as long as the translation and SO(5) symmetry is unbroken. More specifically, the vector spin chain can have a totally symmetric ground state while having an energy gap. The spinor chain, on the other hand, must be gapless if there is no symmetry breaking. Under the open boundary condition, we find the boundary Berry’s phase of the vector spin chain is consistent with the derived edge degeneracy of an exactly solvable model. The present result can be straightforwardly generalized to other irreducible representations, leading to two classes of SO(5) spin chain: in one class the site representation is linear, and in the other the site representation is projective. While the first class can have a totally symmetric ground state and maintain a gapped spectrum, the second class must have gapless spectrum if there is no symmetry breaking.

Finally, we would like to make contact with relevant works in the literature. Eq. (17) appeared in Ref. [14] in discussing a super-symmetric version of the Hubbard model. Of course, the target space is entirely different. In Ref. [15] Affleck discussed the topological term of an SU(N) spin chains. In that case the target space is \( U(N)/U(m) \times U(N-m) \), which also has \( \mathbb{Z} \) as its second homotopy group. In Ref. [16], Read and Sachdev have discussed the effects of Haldane’s (SU(2) or SO(3)) Berry phase term in two space dimensions. Subsequently, they generalized this to the SU(N) group in Ref. [17]. The essential difference between the SU(N) and the SO(5) spin chains is their target spaces. Different target spaces have distinct topology. As far as we know, this is the first time one points out that the target space of the SO(5) spin chain also has second homotopy group \( \mathbb{Z} \), and constructs the Berry’s phase term for the NL\(\sigma\) model description. Of course, analogous to Refs. [16,17] one can ask what is the effect of this topological term in two space dimensions. We leave this question for future studies.

6 Acknowledgement

We thank Wu-Yi Hsiang, Xiao-Gang Wen, Zheng-Cheng Gu, Ari Turner, Xie Chen, and Geoffrey Lee for helpful discussions. DHL is supported by DOE grant number DE-AC02-05CH11231 and thanks for the hospitality for the MIT condensed matter theory group in the last stage of this work. GMZ and TX acknowledge the support of NSF-China and the National Program for Basic Research of MOST, China.
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