Finite size analysis of the pseudo specific heat in SU(2) gauge theory

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We investigate the pseudo specific heat of SU(2) gauge theory near the crossover point on $4^4$ to $16^4$ lattices. Several different methods are used to determine the specific heat. The curious finite size dependence of the peak maximum is explained from the interplay of the crossover phenomenon with the deconfinement transition occurring due to the finite extension of the lattice. In this context we calculate the modulus of the lattice average of the Polyakov loop on symmetric lattices and compare it to the prediction from a random walk model.

1. INTRODUCTION

The pseudo specific heat $C_V$ of SU(2) gauge theory was already investigated in the beginning of Monte Carlo lattice studies. It is known to have a peak near $\beta = 4/g^2 \approx 2.2$, in the crossover region between strong and weak coupling behaviour. A first finite size analysis by Brower et al. [1] on $4^4$ to $10^4$ lattices revealed a strange dependence of the peak on the volume $V = (N_\sigma \alpha)^4$ of the lattice. Here, $\alpha$ is the lattice spacing and $N_\sigma$ the number of points in each direction. The location of the peak shifts with increasing volume from smaller $\beta$-values to larger and then to smaller ones again; the peak maximum decreases with increasing volume. Such a behaviour is unknown for any ordinary phase transition. The nature and origin of the peak remained therefore unclear, though a connection to the nearby endpoint of the first order critical line in the $(\beta, \beta_{-\text{adjoint}})$-plane was proposed by Bhanot and Creutz [2]. Recently, new calculations for this extended SU(2) model were performed [3]. Likewise, a new study of $C_V$ with higher statistics and also on larger lattices, utilizing the analysis techniques now available, seems appropriate. In addition, since symmetric lattices are used to simulate zero temperature physics, it is important to estimate remaining finite temperature effects which may be seen in the pseudo specific heat.

2. METHODS TO CALCULATE $C_V$

We use the standard Wilson action for SU(2)

$$S = \beta \cdot \sum_{x,\mu\nu} P_{\mu\nu}(x) ,$$

where

$$P_{\mu\nu}(x) = 1 - \frac{1}{2} \text{Tr} U_{\mu\nu}(x) ,$$

is the plaquette or energy and $U_{\mu\nu}(x)$ is the plaquette link operator. The sum extends over all independent forward plaquettes. There are $N_P = 6N_\sigma^4$ such plaquettes. We denote the lattice average of the plaquettes by $P$

$$P = \frac{1}{N_P} \sum_{x,\mu\nu} P_{\mu\nu}(x) .$$

The pseudo specific heat is then defined by

$$C_V = \frac{d\langle P \rangle}{d(1/\beta)} = -\beta^2 \frac{d\langle P \rangle}{d\beta} .$$

There are three methods to determine $C_V$:

i) one measures the plaquette expectation values $\langle P \rangle$ as a function of $\beta$ and calculates the numerical derivative at $\beta_M = \beta + \Delta\beta/2$ from

$$C_V(\beta_M) = -\frac{\beta_M^2}{\Delta\beta} \langle P \rangle(\beta + \Delta\beta) - \langle P \rangle(\beta) ;$$

ii) one measures the variance of the plaquettes, which is proportional to $C_V$

$$C_V = \beta^2 N_P(\langle P^2 \rangle - \langle P \rangle^2) ;$$

or,
iii) one calculates the sum of plaquette-plaquette correlations

\[ C_V = \beta^2 \sum_{x', \mu', \nu'} \left( \langle P_{\mu\nu}(x) P_{\mu'\nu'}(x') \rangle - \langle P \rangle^2 \right) . \]  

(7)

As it should be, all three methods are in complete consistence with each other. The most straightforward way is, of course, to calculate the variance of \( \langle P \rangle \). The density of states method (DSM) may then be used to interpolate between the points. The variance becomes, however, definitely too small, if the plaquettes are measured during and not after each update, because that leads to local correlations among the plaquettes.

Figure 1. The pseudo specific heat \( C_V \) calculated from the variance ( squares), the numerical derivative (circles) and the DSM interpolation, calculated on a \( 6^4 \) lattice.

In Fig. 1 we show for comparison the results from methods i) and ii) for a \( 6^4 \) lattice. Here, on the average 90-120 thousand measurements were made every fifth update at each \( \beta \)–value. An update consisted of one heatbath and two overrelaxation steps.

We have also investigated the plaquette correlations. We find in general a rapid fall with \( R = x' - x \), the correlation length is of order 1. The plaquettes \( P_{\mu\nu}(x) \) and \( P_{\mu'\nu'}(x') \) may be in parallel or orthogonal planes. At the peak \( (\beta \approx 2.23) \) we find that the total contribution of the orthogonal correlations is about 30% higher than that of the parallel correlations, whereas far away from the peak, at \( \beta = 2.70 \), the contributions are essentially equal.

3. FINITE SIZE DEPENDENCE OF \( C_V \)

In Fig. 2 we compare the results for \( C_V \) from lattices with \( N_\sigma = 4, 6, 8, 12 \) and 16. The general behaviour already found in [1] is fully confirmed. On the other hand, we see that there is no further finite size dependence in the peak region, if \( N_\sigma \geq 8 \). This suggests, that the finite size dependence of the smaller lattices is related to a

Figure 2. The pseudo specific heat \( C_V \) vs. \( \beta \) on \( N_\sigma^4 \) lattices. Part (a) shows the results for \( N_\sigma = 4 \) and 6 from the DSM interpolation, for \( N_\sigma = 8, 12 \) and 16 the measured points were connected by straight lines. Part (b) shows the peak region for \( N_\sigma = 8, 12 \) and 16 in detail.
different phenomenon. Indeed, the critical point for the $N_f = 4$ finite temperature deconfinement transition is at $\beta_c = 2.30$, very close to the crossover peak position. Since we are using periodic boundary conditions for all directions, the approach to the critical point corresponding to $N_\sigma$ will influence the plaquette expectation values. To check this, we have calculated the lattice average $L$ of the Polyakov loop $L(\vec{x})$

\[
L = \frac{1}{N_\sigma^3} \sum_{\vec{x}} L(\vec{x}) ; \quad L(\vec{x}) = \frac{1}{2} \text{Tr} \prod_{t=1}^{N_\sigma} U_{\vec{x},t}.
\]  (8)

As can be seen from Fig. 3, the expectation value of the modulus of $L$ is not zero on symmetric lattices, not even in the strong coupling limit, i.e. we have finite temperature effects also on symmetric lattices. Well below $\beta_c(N_\sigma)$ the quantity $\langle |L| \rangle$ is a constant. With increasing $\beta$ it starts to increase already before the transition point. It is obvious, that due to the nearby transition points the crossover peaks of the $4^4$ and $6^4$ lattices are stronger distorted than those of the larger lattices, where only the right shoulders of the peaks are slightly influenced. We may find the $N_\sigma$-dependence of $\langle |L| \rangle$ at $\beta = 0$ from a simple random walk model, where the modulus of a sum over $N_\sigma^3$ equal random variables is proportional to $N_\sigma^{-3/2}$. Indeed, we obtain from a simulation at $\beta = 0$ the following relation for $SU(2)$

\[
\langle |L| \rangle_{\beta=0} = 0.400 \cdot N_\sigma^{-3/2}.
\]  (9)

The $\langle |L| \rangle$-values, which we calculated at $\beta = 2.0$ are shown in Fig. 3 together with the results for $\beta = 0$. At $\beta = 2.0$ all lattices apart from the $4^4$ lattice have already reached the strong coupling value. Deviations from this value indicate then the onset of finite temperature effects.

Our final conclusion is, that the crossover peak is not the result of an ordinary phase transition. For large lattices the peak is at $\beta_{co} = 2.23(2)$, its height is $C_{V,co} = 1.685(10)$.

![Figure 3](image-url)

Figure 3. The expectation value of the modulus of $L$ for $N_\sigma = 4, 6, 8, 12$ and 16 vs. $\beta$. The broken vertical lines show the locations of the corresponding finite temperature phase transitions.

![Figure 4](image-url)

Figure 4. The logarithm of $\langle |L| \rangle$ vs. $\ln N_\sigma$ on $4^4 - 16^4$ lattices at $\beta = 0$ (solid line), at $\beta = 2.0$ (circles) and from an estimate from the $\langle |L| \rangle$ value at $N_\sigma = 1$ (dashed line).

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