Relative error bound using Gershgorin circles for non diagonal perturbations of well seperated matrices

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Abstract

We call a matrix as a well seperated if all its Gershgorin circles are away from the unit circle and they are seperated from each other. This article is a study on the region of relative errors in approximating eigenvalues between two well seperated matrices having same main diagonal entries.

Keywords: Eigenvalues, Relative error bounds, Matrix Perturbation,Circle inversion

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1 Introduction

The relative error in approximating the eigenvalue $\lambda$ of $A$ by $\tilde{\lambda}$ of $B$ is $\frac{|\lambda - \tilde{\lambda}|}{|\lambda|}$. There are several relative error bounds \[2\] for the matrix perturbation. Like the well known Bauer-Fike \[1\] type of bound

$$\frac{|\lambda - \tilde{\lambda}|}{|\lambda|} \leq \kappa(X)\|A^{-1}(A - B)\|.$$ 

Here $\kappa(X) = \|X\|\|X^{-1}\|$, is the condition number of eigenvector matrix $X$ of $A$ and $B$ is the perturbed matrix. $\lambda$ and $\tilde{\lambda}$ are the eigenvalues of the matrix $A$ and $B$. However these bounds are difficult to compute, since the eigenvector matrix is not known a priori. In this article, we use the knowledge of Gershgorin circles \[6\] to obtain the bounds.

For an $n \times n$ square matrix $A$, the circle in the complex plane with center at $A(i, i)$ and radius $\sum_{j=1}^{n} |A(i, j)|$ is called the $i$ th Gershgorin circle. The radius of the circle can be chosen as the minimum of $\sum_{j=1}^{n} |A(i, j)|$ or $\sum_{i=1}^{n} |A(i, j)|$.

Gershgorin circle theorem : When all the Gershgorin circles of a matrix are disjoint, then each circle contains exactly one eigenvalue. When two or more circles overlap, then the union of the region contain that many eigenvalues.

Thus Gershgorin circle theorem gives an easily computable bounding region for the eigenvalue in the complex plane. Here analogously using the theorem we try to obtain the region of relative error in perturbing the matrix, but retaining
Consider the matrices $A$ and $B$ which have the same diagonal elements and the Gershgorin circles formed by these diagonal entries are well separated and they lie away from the unit circle in the complex plane.

The expression for relative error involves $\frac{1}{\lambda}$, and it can be related to inversion mapping in geometry (Inversive geometry [5]).

Inversion with respect to a circle of radius $R$ having center at origin: The point $p$ in the Euclidean plane is mapped to the point $q$ on the ray joining origin to $p$ such that: the distance between origin to $p$ ($r_p$) and distance between the origin to $q$ ($r_q$) satisfy the relation,

$$R^2 = r_q r_p.$$

In this article the inversion mapping is always with respect to the unit circle ($R = 1$).

A circle is mapped into a circle under the inversion (Chapter 8 on transformations in [4]). The center of the inverted circle, original circle and the origin lie on a straight line.

The complex map $z \to \frac{1}{z}$ is called reciprocation in geometry and it is the reflection with respect to real axis after doing the circle inversion.

## 2 On the relative errors:

Let $ae^{i\alpha} + r_1 e^{i\theta}$ be a Gershgorin circle for matrix $A$, where $ae^{i\alpha}$ correspond to the diagonal entry of the matrix and $r_1$ is the absolute sum of non diagonal (row or column) entries. Let this circle contain the eigenvalue $\lambda$. Then the circle corresponding to $\frac{1}{\lambda}$ is inside the unit circle and has the center

$$\frac{1}{2} \left( \frac{1}{a + r_1} + \frac{1}{a - r_1} \right) e^{-i\alpha} = \frac{a}{(a + r_1)(a - r_1)} e^{-i\alpha}. \quad (1)$$

The radius of the circle is given as

$$\frac{1}{2} \left( \frac{1}{a - r_1} - \frac{1}{a + r_1} \right) = \frac{r_1}{(a + r_1)(a - r_1)}. \quad (2)$$

This can be also seen from the inverse circle mapping. However the inverted circle in this case is also reflected about the real axis due to the complex conjugation.

Suppose the matrix $A$ is approximated by the matrix $B$ by truncating the diagonals of $A$, the $\tilde{\lambda}$ corresponding to the eigenvalue $\lambda$ lies inside the circle $ae^{i\alpha} + r_2 e^{i\eta}$. The relative error in this approximation is given by

$$\frac{|\lambda - \tilde{\lambda}|}{|\lambda|} = |1 - z|. \quad (3)$$
Here the ratio is inside the region given by product of two circles. Which is,
\[
(a e^{i\alpha} + r_2 e^{i\eta}) \left( \frac{a}{(a + r_1)(a - r_1)} e^{-i\alpha} + \frac{r_1}{(a + r_1)(a - r_1)} e^{i\theta} \right).
\] (4)

The product of two circles is in general known as cartesian oval \[3\], which is a curve of genus 1. On further simplifications (note that \(\theta\) and \(\eta\) are arbitrary and independent),
\[
(a + r_2 e^{i\eta}) \left( \frac{a + r_1 e^{i\theta}}{(a + r_1)(a - r_1)} \right).
\] (5)

This gives the region of \(z\), thus we have,
\[
z - 1 = \frac{r_1^2}{(a + r_1)(a - r_1)} + \frac{a(r_1 e^{i\theta} + r_2 e^{i\eta}) + r_1 r_2 e^{i(\theta + \eta)}}{(a + r_1)(a - r_1)}.
\] (6)

Since \(a, r_1, r_2 > 0\), we have
\[
|z - 1| \leq \frac{r_1^2 + a(r_1 + r_2) + r_1 r_2}{a - r_1} = \frac{r_1 + r_2}{a - r_1}.
\] (7)

This serves as the upper bound for relative error. Approximating the region by circle: By looking at the maximum and minimum values of the product of two circles, we can approximate the quadratic oval by the circle with center
\[
\frac{1}{2} \left( \frac{a + r_2}{a - r_1} + \frac{a - r_2}{a + r_1} \right) = \frac{a^2 + r_1 r_2}{(a + r_1)(a - r_1)},
\] (8)

and radius,
\[
\frac{1}{2} \left( \frac{a + r_2}{a - r_1} - \frac{a - r_2}{a + r_1} \right) = \frac{a(r_1 + r_2)}{(a + r_1)(a - r_1)}.
\] (9)

So \(z - 1\) is inside the circle of same radius, but center is shifted to,
\[
\frac{a^2 + r_1 r_2}{(a + r_1)(a - r_1)} - 1 = \frac{r_1^2 + r_1 r_2}{(a + r_1)(a - r_1)}.
\] (10)

3 Numerical examples

The example shown in Figure 1 is a matrix with strictly positive entries on the diagonal, random Gaussian symmetric entries in non diagonal. It is taken care that the circles are disjoint and away from the origin. It can be noted that, in Figure 1 the center of the region overlaps with the relative error.

In Figure 2 a upper Hessenberg matrix with positive entries is taken with disjoint Gershgorin circles. It can be noted that the center of the regions themselves form an upperbound to the relative error in this case.
Figure 1: Logarithmic relative error, bound and the center of the region for symmetric matrix of dimension 100 with half the Gerschgorian radius

Figure 2: Logarithmic relative error, bound and the center of the region for upper Hessenberg matrix of dimension 100 with half the Gerschgorian radius
4 Conclusion

Upper bound for the relative error for non diagonal perturbation of a well separated diagonally dominant matrix is given in terms of Gershgorin circle parameters. Numerical experiments show the goodness in the approximation of the region of relative error (quadratic oval) by a circle.

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