Entropy of monomial algebras and derived categories

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Let $A$ be a finitely presented associative monomial algebra. We study the category $qgr(A)$ which is a quotient of the category of graded finitely presented $A$-modules by the finite-dimensional ones. As this category plays a role of the category of coherent sheaves on the corresponding noncommutative variety, we consider its bounded derived category $D^b(qgr(A))$. We calculate the categorical entropy of the Serre twist functor on $D^b(qgr(A))$ and show that it is equal to the (natural) logarithm of the entropy of the algebra $A$ itself. Moreover, we relate these two kinds of entropy with the topological entropy of the Ufnarovski graph of $A$ and the entropy of the path algebra of the graph. If $A$ is a path algebra of some quiver, the categorical entropy is equal to the logarithm of the spectral radius of the quiver’s adjacency matrix.

1 Introduction

Let $A$ be a finitely presented monomial algebra

$$A = \frac{k\Gamma}{(F)},$$

that is, a quotient of a path algebra $k\Gamma$ for a finite quiver $\Gamma = (\Gamma_0, \Gamma_1)$ by an ideal generated by a fixed finite set $F$ of words (paths).

The (algebraic) entropy of a graded algebra $A$ is the exponential measure of its growth [28, 31],

$$h_{alg}(A) = \lim_{n \to \infty} \sqrt[n]{\dim A_n},$$

where the graded component $A_n$ is the span of all paths of length $n$ in $A$. The logarithm $\log h_{alg}$ of this entropy in the case of monomial algebra equals to the entropy $h(L)$ of the (regular) language [22] consisting of the nonzero paths in $A$, or to the topological entropy of the corresponding subshift [1]. Moreover, $h(L)$ is equal to the topological entropy $h(Q_A)$ of the Ufnarovski graph $Q_A$ of $A$. In this connections, the entropy of the algebra measures also the complexity both of the language $L$ and of the graph $Q_A$.

On the other hand, one can consider $A$ as a coordinate ring of a noncommutative variety. We denote by $Gr(A)$ and $gr(A)$ the category of $\mathbb{Z}$-graded right modules and its subcategory of the finitely presented right modules, respectively. Denote by $Tors(A)$ and $tors(A)$ their full subcategories of all torsion modules (which are sums of finite-dimensional ones). Then the quotient category $Qgr(A) := Gr(A)/Tors(A)$ plays the role of the of the category of quasicoherent sheaves on the noncommutative variety defined by $A$ [2]. Moreover, as $A$ is coherent (see [31] and Section 3 below), $gr(A)$ is an abelian Serre subcategory, so that one can define the category of coherent sheaves as $qgr(A) := gr(A)/tors(A)$ [31]. Consider the bounded derived category $D^b(qgr(A))$ and the Serre twist functor on it.

The notion of the entropy for (exact) endofunctors of triangulated categories (having a split generators) has been introduced by Dimitrov, Haiden, Katzarkov, and Kontsevich [3]. For some endomorphisms of projective varieties, the categorical entropy of the induced endofunctors of the derived category of coherent sheaves is connected with topological entropy [18, 24, 39]. The entropy and its value at zero is calculated for functors on the derived categories of algebraic varieties in [3, 7, 8, 23, 25, 26, 32].

Received 1 Month 20XX; Revised 11 Month 20XX; Accepted 21 Month 20XX

Communicated by A. Editor

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Here we provide a calculation of the categorical entropy for the category $D^b(qgr(A))$ associated to the noncommutative variety defined by the coordinate ring $A$. In particular, we show that the entropy of the Serre twist functor is a constant and relate it to other type of entropy of topological and algebraic origin associated to the monomial algebra $A$.

For definitions and details related to the above versions of the entropy, see Section 2. We will prove the following connection between them:

**Theorem 1.1** (Theorems 5.5 and 5.3). The categorical entropy $h_t(D^b(qgr(A)), S)$ of the Serre twist functor $S$ is a constant which is equal to

$$h_t(D^b(qgr(A)), S) = \log h_{alg}(A).$$

In particular, we see that the algebraic entropy $h_{alg}(A)$ is a derived invariant of the noncommutative variety associated to $A$. It would be interesting to describe other noncommutative coherent algebras with this property. Note that the above entropy is related with the other kinds of entropy associated to $A$ as follows.

**Corollary 1.2** (Propositions 2.4 and 5.5). For a monomial algebra $A$, we have the equalities

$$h_t(D^b(qgr(A)), S) = h_{top}(X_F) = h(L) = \log h_{alg}(A)$$

$$= h(Q_A) = \log h_{alg}(kQ_A) = h_t(D^b(qgr(kQ_A)), S) = \log \rho(Q_A),$$

where the entropies $h, h_{top}, h_{alg}, h_t$ relate the language of nonzero monomials of positive length in $A$, the subshift $X_F$ associated to $A$, the Ufnarovski graph $Q_A$ of $A$, the path algebra $kQ_A$, and the spectral radius $\rho(Q_A)$ of the adjacency matrix of the graph $Q_A$.

If $A = k\Gamma$ is the path algebra, then the equalities are degenerated to

$$h_t(D^b(qgr(k\Gamma)), S) = \log h_{alg}(k\Gamma) = \log \rho(\Gamma),$$

since in this case $Q_A = \Gamma$.

Our approach is the following. Holdaway and Smith [15] have established an equivalence of the category $Qgr(A)$ with the analogous category $Qgr(kQ_A)$ for the quiver algebra of the Ufnarovski graph $Q_A$ (see also the paper of Holdaway and Sisoda [14] for the same result in a more general setting of non-connected algebras). This equivalence is induced by a map $f : A \rightarrow Q_A$ referred here as Holdaway–Smith map. We show that this map induces also an equivalence of categories $qgr A \simeq qgr(kQ_A)$. Smith have described the category $qgr$ for quiver algebras [31]. In particular, such category is semisimple. We use this fact to calculate the entropy $h_t(D^b(qgr(A)), S) = h_t(D^b(qgr(kQ_A)), S)$.

In Section 2 we recall various definitions of entropy used in this paper. In particular, we give the definitions of the entropy of a graded algebra, the topological entropy of subshifts, formal languages, and directed graphs.

To describe the relations of this last entropy with the previous ones, we should define and describe the category $qgr A$ for a finitely presented monomial algebra $A$. First, we recall the fact the $A$ is coherent in Section 5. A proof [30] of this fact is given in a less general situation than the one considered here (that is, for the connected monomial algebras in place of the quotients of quiver algebras). Here we give another self-contained proof. We call an algebra with a fixed multiplicative ordering on the monomials a right Groebner finite if it has finite Groebner basis of relations and each right-sided ideal in it has finite Groebner basis as well (Definition 3.3). Then we follow [3] to define the categorical entropy for the triangulated categories.

**Proposition 1.3** (Corollary 3.4). Each right Groebner finite algebra is right coherent.

As each finitely presented monomial algebra is right Groebner finite (Lemma 3.1), we deduce that such an algebra is coherent (Corollary 3.5). Note that $A$ is coherent in the general non-graded sense.

Since $A$ is coherent, the category $qgr A$ exists. Moreover, we have

**Proposition 1.4** (Proposition 4.6). Let $R$ be a finitely generated right graded coherent algebra. Then there is a natural equivalence of categories $qgr R \equiv fp(Qgr R)$, where $fp$ denotes the subcategory of finitely presented objects.
Let  \( m \) denote the minimal number where \( N \) is a subword of \( u \). Remark 2.7 below. For \( n \), the minimal number where \( h \) is a subword of \( u \), the minimal number \( s \) such that \( X \) is a direct summand of \( O^{*} \) in \( D^{B}(C) \).

In the next Section 4 we use some properties of the category \( qgr(kQ_{A}) \) established by Smith \[35\] (mainly, the semi-simplicity) to establish Theorem 5.4. The key lemma is the following. For the definition of complexity used in it, we refer the reader to \[3\] or to Subsection 2.4.

Lemma 1.5 (Lemma 6.4). Let \( O \) be a generator and \( X \) be an object of a semi-simple abelian category \( C \). Then the complexity \( \delta_{i}(O, X) \) of \( X \) relative to \( O \) in the category \( D^{B}(C) \) is a constant which is equal to \( r_{kO}(X) \), that is, the minimal number \( s \) such that \( X \) is \( r_{kO}(X) \) in \( D^{B}(C) \).

Since the categorical entropy is defined in terms of complexity, we use the lemma to deduce in Theorem 6.5 that \( h_{t}(D^{B}(\text{qgr}(kQ_{A}))), S) = \log h_{\text{alg}}(kQ_{A}) \). In the view of the above equivalence \( qgr(A) \equiv qgr(kQ_{A}) \), Theorem 1.4 now follows. In Section 7 we consider a particular example.

2 Algebraic entropy, topological entropy and category-theoretical entropy

2.1 Algebraic entropy of a graded algebra

Let \( A = \bigoplus_{m \geq 0} A_{m} \) be a graded algebra over a field \( k \). Assume that each graded component \( A_{m} \) is finite-dimensional as a vector space over \( k \), so that \( a_{m} = \dim_{k} A_{m} < +\infty \). Newman, Schneider and Shalev \[28\] defined the algebraic entropy of \( A \) by

\[
\text{h}_{\text{alg}}(A) := \lim_{m \to +\infty} \sqrt[m]{a_{m}}.
\]

In other words, it is the exponent of growth of the algebra \( A \) \[31\].

Note that if the algebra \( A \) is generated in degrees 0 and 1 (like the algebras considered in this paper), then the sequence \( \{a_{m}\} \) is sub-multiplicative: \( a_{m} a_{p} \geq a_{m+p} \). By Fekete’s Lemma, in this case the above limit exists:

\[
\text{h}_{\text{alg}}(A) = \lim_{m \to +\infty} \sqrt[m]{a_{m}} = \inf_{m \geq 0} \sqrt[m]{a_{m}}.
\]

Note that the radius of convergence of the Hilbert series \( H_{A}(z) = \sum_{m} a_{m} z^{m} \) is \( r = 1/\text{h}_{\text{alg}}(A) \).

2.2 Topological entropy of formal languages and subshifts

Topological entropy was first introduced in 1965 by Adler, Konheim and McAndrew in [1]. For a compact space \( X \), let \( U \) be an open cover of \( X \). The entropy of \( U \) is

\[
H(U) = \log N(U),
\]

where \( N(U) = \min \{ |\mathcal{V}| : \mathcal{V} \text{ is a finite subcover of } U \} \). Here \( \log \) denotes the logarithm to some fixed base, see Remark 2.7 below. For \( m \in \mathbb{Z}_{> 0} \) and open covers \( U_{1}, \cdots, U_{m} \) let

\[
U_{1} \vee \cdots \vee U_{m} = \{ \bigcap_{i=1}^{m} U_{i} : U_{i} \in U_{i} \}.
\]

Let \( f : X \to X \) be a continuous self-map. The topological entropy of \( f \) with respect to \( U \) is

\[
H_{\text{top}}(f, U) = \lim_{n \to +\infty} \frac{H(U \vee f^{-1}(U) \vee \cdots \vee f^{-n+1}(U))}{n}.
\]

The topological entropy of \( f \) is

\[
h_{\text{top}}(f) := \sup \{ H_{\text{top}}(f, U) : U \text{ is an open cover of } X \}.
\]

We will consider the entropy of a subshift, which is a special case of the topological entropy.

Let \( G = \{ x_{1}, x_{2}, \cdots, x_{n} \} \) be an alphabet, \( G^{+} \) the set of words in \( G \) with finite length and \( G^{*} := G^{+} \cup \{ \epsilon \} \), where \( \epsilon \) denotes the empty word. We write the usual concatenation of words as \( uv \) for \( u, v \in G^{+} \), and \( v \prec u \) if \( v \) is a subword of \( u \). Under word concatenation, \( G^{*} \) forms a free monoid generated by \( G \) with identity given by the
empty word \(e\). All subsets of \(G^\infty\) for a finite alphabet \(G\) will automatically adopt the subspace topology, where \(G\) is a discrete space and \(G^\infty\) is endowed with the product topology.

Let \(F\) be a finite subset of \(G^+\) which we call a set of forbidden words. The shift of finite type \(X_F\) associated to \(F\) is the space

\[
X_F = \{x \in G^n | u < x \Rightarrow u \notin F\}.
\]

Words in \(X_F\) are called legal. We write \(L_n\) for the set of legal words of length \(n\). The topological entropy of \(X_F\) is

\[
h_{\text{top}}(X_F) := \lim_{n \to +\infty} \frac{1}{n} \log |L_n|.
\]

Note that \(X_F\) is a closed subspace of \(G^\infty\) and the the self-map \(\sigma : X_F \to X_F\) defined as

\[
\sigma : (a_0, a_1, \ldots, a_k, \ldots) \mapsto (a_1, a_2, \ldots, a_k, \ldots)
\]

is continuous. Then \(h_{\text{top}}(X_F) = h_{\text{top}}(\sigma)\). We are grateful to one of the anonymous reviewers who put our attention to this remark.

The formal language \(L = \bigcup_n L_n\) is the set of all nonzero paths of positive length in the monomial algebra \(A\). Its entropy is defined to be

\[
h(L) = \log \lim_{n \to +\infty} \sqrt[n]{|L_n|},
\]

see [22].

**Proposition 2.1.** Let \(A\) be a graded monomial algebra of the form

\[
A = k\Gamma/(F),
\]

where \(\Gamma = (\Gamma_0, \Gamma_1)\) is a finite quiver with \(\Gamma_1 = \{x_1, x_2, \ldots, x_n\}\), and \((F)\) denotes the ideal generated by a finite set \(F = \{w_1, \ldots, w_k\}\) of words in the alphabet \(\Gamma_1\). We denote by \(X_F\) the shift of finite type associated to \(F\), and by \(L\) the formal language as above. Then

\[
h_{\text{top}}(X_F) = h(L) = \log h_{\text{alg}}(A).
\]

**Proof.** We have

\[
h_{\text{alg}}(A) := \lim_{n \to +\infty} \sqrt[n]{\dim_k A_n} = \lim_{n \to +\infty} \sqrt[n]{\dim_k |L_n|}.
\]

Hence

\[
h_{\text{top}}(X_F) = \lim_{n \to +\infty} \frac{1}{n} \log |L_n| = \lim_{n \to +\infty} \log \sqrt[n]{\dim_k L_n} = h(L) = \log h_{\text{alg}}(A). \quad \Box
\]

### 2.3 Graph entropy

Let \(G = (G_0, G_1)\) be a directed graph. Suppose that \(G\) is finite, and let \(A_G\) denotes the adjacency matrix of \(G\). Then the topological entropy of the graph \(h(G)\) is defined to be the topological entropy \(h_{\text{top}}(X_F)\) of the subshift \(X_{G_1}\) of the paths in \(G\), that is,

\[
h(G) := \lim_{n \to +\infty} \frac{\log a_n}{n},
\]

where \(a_n\) the number of paths of length \(n\). It is well-known that

\[
h(G) = \log r(A_G),
\]

where \(r(A_G)\) is the spectral radius of \(A_G\). Since the entropy of the subshift \(X_{G_1}\) is equal to the algebraic entropy of the path algebra \(kG\), we have the equality

\[
h(G) = \log h_{\text{alg}}(kG).
\]

Note that there are various definitions of entropy for infinite graphs (such as the topological entropy, loop entropy, block entropy, and other). There is also another version of the entropy defined in terms of the associated C*-algebra. Still, in the case of finite graphs, all these entropies give the same value \(h(G) = \log r(A_G)\), see [16] and references therein.
2.4 Category-theoretical entropy

For triangulated categories and thick subcategories we refer to [38] and [27].

**Definition 2.2.** Let \( C \) be a triangulated category. We say a full triangulated subcategory \( L \) is thick if it is closed under taking direct summands.

**Definition 2.3.** An object \( O \) of \( C \) is called a generator if the smallest thick subcategory of \( C \) containing \( O \) is equal to \( C \) itself.

**Definition 2.4.** ([3], Definition 2.1) Let \( C \) be a triangulated category with a generator \( O \). Let \( E \) be an object of a triangulated category \( T \). There is an object \( E' \) and a tower of distinguished triangles

\[
\begin{array}{ccccccc}
E_0 & \to & E_1 & \to & E_2 & \cdots & E_{p-1} & \to & E_p \cong E \oplus E' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
O[n_1] & \to & O[n_2] & \to & \cdots & \to & O[n_p]
\end{array}
\]

(2)

with \( E_0 = 0, p \geq 0 \), and \( n_i \in \mathbb{Z} \). Let \( t \) be a real number. To each tower of distinguished triangles of the form (2) we associate the exponential sum \( \sum_{i=1}^{p} e^{e_{i,t}} \). Let \( S_t \subset \mathbb{R} \) be the set of all such sums for a given \( t \). The complexity of \( E \) with respect to \( O \) is the function \( \delta_t(O,E) : \mathbb{R} \to [0,+\infty] \) of \( t \), given by \( \delta_t(O,E) = \inf S_t \).

**Lemma 2.5.** (Subadditivity) \( \delta_t(O,E_1 \oplus E_2) \leq \delta_t(O,E_1) + \delta_t(O,E_2) \).

**Definition 2.6.** Let \( F : C \to C \) be an exact endofunctor of a triangulated category \( C \) with generator \( O \). The entropy of \( F \) is the function \( h_t(F) : R \to [-\infty, +\infty] \) of \( t \) given by

\[
h_t(C,F) = \lim_{n \to \infty} \frac{1}{n} \log \delta_t(O,F^n)O).
\]

It is shown in [3, Lemma 2.5] that \( h_t(C,F) \) is well-defined, i.e., the limit defining \( h_t(C,F) \) exists and is independent of the choice of generator \( O \).

**Remark 2.7.** In the above definitions of topological entropy of formal languages and subshift in Subsection 2.2 and of graph entropy in Subsection 2.3 the log symbol denotes the logarithm to an arbitrary fixed base \( a > 1 \). In contrast, in Definition 2.6 due to [3] the symbol \( \log \) denotes the natural logarithm. We are grateful to an anonymous referee who pointed out this fact. If one needs to use \( \log_a \) in place of the natural logarithm here, one should replace the exponents \( e^{e_{i,t}} \) by \( a^{e_{i,t}} \) in Definition 2.4 of complexity. In the view of this remark, below we assume that all our logarithms are to the same base.

3 Finitely presented monomial algebras are coherent

It is proved in [3] that each connected finitely presented monomial algebra is coherent. Here we give a self-contained proof of this fact in the more general setting of monomial quotients of path algebras. In particular, we give a new proof of key Lemma 3.1 which establishes that each finitely presented monomial algebra is right Groebner finite, that is, each finitely generated right ideal in it admits finite Groebner basis.

Let us firstly recall standard facts from noncommutative Groebner bases theory. Note that the noncommutative Groebner bases are also known as Groebner–Shirshov bases. We use a version of the theory suitable for quiver algebras, see [7, 11]. Our terminology is closed to the one from [37].

In this section, let \( A = k\Gamma/I \) be a quotient of a path algebra \( k\Gamma \) for a finite quiver \( \Gamma = (\Gamma_0, \Gamma_1) \). Initially, we do not assume that \( A \) is monomial. The paths of \( \Gamma \) and the unit are called monomials; the monomials form a multiplicative submonoid of \( k\Gamma \). Let \( \leq \) denote an arbitrary multiplicative well-order on the monomials such that \( 1 \) is the minimal element. An example of such order is the degree-lexicographical one (then \( m_1 < m_2 \) iff either \( \deg m_1 < \deg m_2 \) or \( \deg m_1 = \deg m_2 \) and \( m_1 \) is less then \( m_2 \) lexicographically). Then a monomial \( m \in k\Gamma \) is called normal if its image \( m + I \) in \( A \) cannot be presented as a linear combination of strictly less monomials. Then each element of \( f \in A \) can uniquely be presented as \( f = N(f) + I \), where the normal form \( N(f) \in k\Gamma \) is a linear combination of normal monomials \( \sum \alpha_i m_i \), where for nonzero \( f \in A \) one can assume that \( \alpha_i \in k^\times \) and \( m_i \) are normal monomial. Here the largest monomial \( m_i \) is called the leading monomial of \( f \), and the corresponding
summand \( \alpha_1 m_1 \) is called the leading term of \( f \). They are denoted by \( \text{LM}(f) \) and \( \text{LT}(f) \), respectively. The element \( \alpha_t \in k \) here is called the leading coefficient of \( f \).

We identify each element \( f \in A \) with its normal form \( N(f) \in k \Gamma \). So, we extend the map \( N : f \mapsto N(f) \) to a \( k \)-linear projector \( N : k \Gamma \rightarrow k \Gamma \). The multiplication of (the normal forms of) elements of the quotient algebra \( A \) is denoted by \( * \), so that we put
\[
a * b = N(ab)
\]
for all \( a, b \in k \Gamma \).

A subset \( G \subset I \) of the two-sided ideal \( I \) is called a Groebner basis of \( I \), if for each \( f \in I \) there exist two monomials \( p, q \in k \Gamma \) such that \( \text{LM}(f) = p \text{LM}(q) \) for some \( g \in G \). A Groebner basis is always a generating set of the ideal \( I \). In particular, if \( I \) is generated by a set \( M = \{m_1, m_2, \ldots \} \) of monomials, then \( M \) is a Groebner basis of \( I \).

Similarly, a subset \( H \) of a two-sided (respectively, right-sided) ideal \( J \) of the quotient algebra \( A = k \Gamma/\Gamma \) is called a Groebner basis of \( J \) if for each \( g \in J \) there are normal monomials \( p, q \) (resp., a single normal monomial \( q \)) such that \( \text{LM}(g) = p \text{LM}(q) \) (resp., \( \text{LM}(f) = \text{LM}(q) \)). In the case of right ideal, a Groebner basis of the ideal \( J \) is also a Groebner basis of it as a right submodule of \( A \) in the sense of [11, Prop. 4.2].

A version of the next Lemma for connected monomial algebras is given in [21, Th. 1].

**Lemma 3.1.** Suppose that the ideal \( I \) of relations of the algebra \( A = k \Gamma/\Gamma \) is generated by a finite set of monomials. Suppose that \( J \) is a finitely generated right-sided ideal in \( A \). Then \( J \) admits a finite Groebner basis (with respect to some monomial ordering).

More precisely, if the defining monomial relations of \( A \) have degrees at most \( l + 1 \) and the generators of \( J \) have degrees at most \( d \), then for each monomial ordering which refines the partial order by the degrees of monomials (for example, the degree-lexicographical order) there exists a Groebner basis of \( J \) such that all its elements have degrees at most \( d + l \). The elements of this basis have the form \( hm \), where \( \text{deg} h \leq d \) and \( m \) is a normal monomial.

**Proof.** Let \( M \) denotes the minimal set of the monomial generators of \( I \), and let \( F = \{f_1, \ldots, f_s\} \) be the minimal generating set of \( J \) such that \( \text{deg} f_i \leq d \) for all \( i \). Consider the set \( G \) consisting of all elements of \( J \) having the form \( g = hm \), where \( \text{deg} h \leq d \) and \( m \) is a normal monomial (maybe, empty) such that \( \text{deg} h + \text{deg} m \leq d + l \).

Recall that the elements of \( A \) (in particular, the element \( g \)) are identified with their normal forms, so that the equality \( g = hm \) for normal forms implies that the elements \( h \) and \( m \) of \( k \Gamma \) are normal and \( hm = h \ast m \).

Then the set \( G \) has the following properties.

(a) \( G \) is a generating set of \( J \), since \( F \subset G \).

(b) If \( g = hm \) as above such that \( \text{deg} m \geq l \), then for each monomial \( q \) we have either \( g \ast q = 0 \) or \( g \ast q = gq \).

(c) Moreover, for each \( g = hm \) as above with no additional restrictions on \( \text{deg} m \) and each normal monomial \( q \) we have either \( g \ast q = 0 \) or \( g \ast q = g'q' \), where \( g' \in G \) and \( g' \) is some right divisor of \( q \).

Indeed, assume that \( g \ast q \neq 0 \). If \( q = bc \) and \( \text{deg} m + \text{deg} b \geq l \), then we have \( g \ast q = (g \ast b)c = g'c \), where \( g' = g \ast b \). Otherwise, \( g'q' \in G \).

(d) If \( g_1, g_2 \in G \) and \( \text{LM}(g_1) = \text{LM}(g_2)m \) for some monomial \( m \), then the element \( g = g_1 + \alpha g_2 \ast m \) belongs to \( G \) for each \( \alpha \in k \).

(e) \( G \) is a Groebner basis of \( J \).

Indeed, let \( f \) be an arbitrary element of \( J \). We should prove that \( \text{LM}(f) \) is left divisible by some \( \text{LM}(g) \) with \( g \in G \). By (a) and (c), \( f \) can be presented as a sum
\[
f = \sum_{i=1}^{N} g_i p_i,
\]
where \( g_i \in G \), the elements \( p_i \) are normal, and \( g_i \ast p_i = g_ip_i \). Let the monomial \( \text{max}_k \text{LM}(g_i p_i) \) is called the leading monomial of the presentation. Among all such representations for \( f \), let us choose the one with the least possible leading monomial.

Let \( \text{LM}(g_1 p_1) \geq \text{LM}(g_2 p_2) \geq \cdots \geq \text{LM}(g_N p_N) \) are listed in decreasing order. If \( \text{LM}(f) = \text{LM}(g_1 p_1) = \text{LM}(g_1) \text{LM}(p_1) \), then \( \text{LM}(f) \) is left divisible by \( \text{LM}(g_1) \). Otherwise, there is \( n \) such that \( \text{LM}(g_1 p_1) = \cdots = \text{LM}(g_n p_n) > \text{LM}(g_{n+1} p_{n+1}) \geq \cdots \) and \( \text{LT}(g_1 p_1) + \cdots + \text{LT}(g_n p_n) = 0 \). We can assume also that \( \text{LM}(g_1) \geq \cdots \geq \text{LM}(g_{n+1}) \) for \( i \in [2, n] \), we have in addition \( \text{LM}(g_1) \text{LM}(p_1) = \text{LM}(g_1) \text{LM}(p_1) \), so that there is a normal monomial \( m_1 \) such that \( \text{LM}(g_1) = \text{LM}(g_1)m_1 \), and \( \text{LM}(p_1) = m_1 \text{LM}(p_1) \). For completeness, we put also \( m_1 = 1 \).

Let \( \alpha_i \) denotes the leading coefficients of \( p_i \).

Consider the element
\[
s = \sum_{i=1}^{n} g_i \text{LT}(p_i) = \sum_{i=1}^{n} \alpha_i g_i m_1 \text{LM}(p_1) = g \text{LM}(p_1),
\]
where the element $g = \sum_{i=1}^{n} \alpha_i g_i m_i$ belongs to $G$ by (d). Since the sum of the leading terms in the above presentation of $s$ is zero, the leading monomial $\text{LM}(s) = \text{LM}(g) \text{LM}(p_1)$ is less then the leading monomial $\text{LM}(g_1) \text{LM}(p_1)$ of the presentation (3) for $f$.

Then we get another presentation for $f$,

$$f = \sum_{i=1}^{n} g_i (p_i - \text{LT}(p_i)) + g \text{LM}(p_1) + \sum_{i=n+1}^{N} g_ip_i$$

such that its leading monomial is less than the one of the presentation (3). A contradiction.

Now, note that the set of all possible leading monomials of the elements of $G$ is finite. Let $G'$ be any finite subset of $G$ such that for each $g \in G$, there is an element $g' \in G'$ such that $\text{LM} g$ is left divisible by $\text{LM} g'$. Then the set $G' \cup F$ is a finite Groebner basis of $J$ which satisfies the conditions of Lemma.

**Proposition 3.2.** Let $A = k\Gamma / I$ a quotient algebra of a path algebra $k\Gamma$, where the ideal $I$ in $k\Gamma$ has finite Groebner basis. Assume that a right-sided ideal $J$ in the algebra $A$ also has finite Groebner basis. Then $J$ is finitely presented as a right $A$-module. □

**Proof.** Let $S = \{s_1, \ldots, s_m\}$ and $G = \{g_1, \ldots, g_s\}$ be minimal Groebner bases for $I$ and $J$, respectively. Let $D$ be the set of all proper right divisors of the leading terms of the elements of $S$. Let $A$ be the associated monomial algebra $A = k\Gamma / (\text{LM}(S))$. Then the right annihilator $\text{Ann}_A w$ in $A$ of a normal monomial $w$ is generated by some elements of $D$.

Suppose that $m \in \text{Ann}_A \text{LM}(g_i)$ for some $g_i \in G$. Then the leading monomial of the element $g_i \ast m$ of $J$ is less than $g_i m$. This leading monomial is left divisible by some $\text{LM}(g_i) _1$; say, this leading monomial is $\text{LM}(g_i) _1 m_1$. Next, the leading monomial of the element

$$g_i \ast m - g_i \ast m_1$$

is left divisible by some $\text{LM}(g_i _2 )$, etc. After a finite number $N$ of steps, we get zero. So, we obtain a presentation

$$g_i \ast m = \sum_{j=1}^{N} g_i \ast m_j,$$

(4)

where each for each $j$ we have $\text{LM}(g_i _j ) < g_i m$.

Since $J$ is generated by $s$ elements, we have a short exact sequence

$$0 \to \Omega \to A^s \xrightarrow{\pi} J \to 0$$

for some submodule $\Omega \subset A^s$. We should prove that $\Omega$ is finitely generated. Let $\tilde{g}_1, \ldots, \tilde{g}_s$ be the free generators of the above free module $P = A^s$. We claim that $\Omega$ is generated by the elements

$$S(i, m) = \tilde{g}_i m - \sum_{j=1}^{N} \tilde{g}_i \ast m_j,$$

where $g_i$ runs over $G$, $m$ runs over the monomial generators of $\text{Ann}_A \text{LM}(g_i)$, and the sum arises from (4). Note that the above map $\pi : A^s \to J$ sends $\tilde{g}_i$ to $g_i$, so that (3) ensures that each $S(i, m)$ belongs to $\Omega$.

To prove this claim, let us extend the monomial order to the free module $P$ via the $k$-linear injection $P \to k\Gamma$ which sends $\tilde{g}_i$ to $\text{LM}(g_i)$. It is indeed an injection since $G$ is a minimal Groebner basis, so that the monomials $\text{LM}(g_i)$ are not left divisible by each other. Each element of $\Omega$ has the form

$$\omega = \sum_{i} \tilde{g}_i b_i,$$

where $\sum_i g_i \ast b_i = 0$. Assume that $\omega$ does not belong to the submodule $\Omega_S$ generated by the elements $S(i, m)$ and that its leading monomial is the least one among all elements having this property.

The leading term of $\omega$ is equal to $\tilde{g}_i \text{LT}(b_i)$ for some $t = i$. Let $l + 1$ be the maximal length the monomials in $\text{LM}(S)$. Since $\pi(\omega) = 0$, we have $\text{LT}(b_i) \in \text{Ann}_A \text{LM}(g_i)$, so that $\text{LT} b_i = m_q$ for some normal monomial $m$ of length at most $l$ and some normal $q$. Using (4), we obtain another element

$$\omega' = \omega - S(i, m) q = \sum_{t \neq i} \tilde{g}_t b_t + \tilde{g}_i (b_i - \text{LT}(b_i)) + \sum_{j=1}^{N} \tilde{g}_i \ast (m_j \ast q),$$

which belongs to $\Omega \setminus \Omega_S$ and has less leading monomial than $\omega$. This contradiction shows that $\Omega = \Omega_S$. □
Definition 3.3. Fix an admissible monomial ordering on the paths of a quiver $\Gamma$. We call an algebra $A = k\Gamma/I$ right Groebner finite if the two-sided ideal $I$ of relations has finite Groebner basis, and each right-sided ideal in $A$ has finite Groebner basis as well.

Lemma 3.1 shows that each finitely presented monomial algebra is right Groebner finite. A more general class of Groebner finite algebras has been considered in [32].

Similar notions of Groebner finite algebras (for two-sided ideals) and Groebner coherent rings (in the commutative settings) have been under consideration in [23] and [26].

For a more restrictive property of universal coherence for a class of Groebner finite algebras, see [33, Prop. 4.10].

Proposition 3.2 implies Corollary 3.4. Each right Groebner finite algebra is right coherent.

In the view of Lemma 3.1, we get Corollary 3.5. Suppose that the ideal $I$ of the algebra $k\Gamma$ is generated by a finite set of monomials. Then the quotient algebra $A = k\Gamma/I$ is right and left coherent.

4 Locally coherent Grothendieck categories of graded modules

In this section, we collect some known facts about locally coherent Grothendieck categories. We use these facts to connect the properties of categories $\text{qgr}$ and $\text{Qgr}$ in Proposition 4.5.

Let $A$ be an abelian category, and let $S$ be a nonempty full subcategory of $A$. $S$ is a Serre subcategory provided that it is closed under extensions, subobjects, and quotient objects. For Serre subcategories and quotient categories we refer to [9], [12], [24], and [8].

If $S$ is a Serre subcategory of $A$ with the inclusion functor $i_* : S \to A$, then there exists an abelian category $A/S$ and an exact functor $j^* : A \to A/S$, which is dense and whose kernel is $S$. The category $A/S$ and the functor $j^*$ are characterized by the following universal property: For any exact functor $G : A \to B$ such that $S \subset \text{Ker}(G)$ there exists a unique functor $H : A/S \to B$ such that

$$\begin{align*}
A & \xrightarrow{G} B \\
A/S & \xrightarrow{H}
\end{align*}$$

commutes, that is, such that there exists a natural equivalence of functors $H \circ j^* \approx G$. If $H'$ is another such functor, then $H \approx H'$ ([3], Corollary 15.9).

A Serre subcategory $S$ of $A$ is called localizing provided that the functor $j^* : A \to A/S$ admits a right adjoint $j_* : A/S \to A$ which is called section functor. In this case, $j_*$ is fully faithful ([10], Prop. 2.2). If $A$ is a Grothendieck category, then a Serre subcategory $S$ is localizing if and only if $S$ is closed under arbitrary coproducts ([8], Theorem 15.11). If $A$ is a Grothendieck category and $S$ a localizing subcategory, then the quotient category $A/S$ is again a Grothendieck category.

Definition 4.1. Let $A$ be a Grothendieck category.

- (1) An object $X$ is of finite type if whenever there are subobjects $X_i \subset X$ for $i \in I$ satisfying

$$\sum_{i \in I} X_i = X,$$

then there is already a finite subset $J \subset I$ such that

$$\sum_{i \in J} X_i = X.$$

The subcategory of objects of finite type is denoted by $\text{fg}(A)$.

- (2) An object $X$ is finitely presented if it is of finite type and if for any epimorphism $p : Y \to X$ where $Y$ is of finite type, it follows that $\text{ker}(p)$ is also of finite type. The subcategory of objects of finite type is denoted by $\text{fp}(A)$.

- (3) An object $X$ is coherent if it is of finite type and for any morphism $f : Y \to X$ where $Y$ is of finite type $\text{ker}(f)$ is of finite type. The subcategory of coherent objects is denoted by $\text{coh}(A)$.
Definition 4.2. Let $\mathcal{A}$ be a Grothendieck category.

- (1) $\mathcal{A}$ is of finite type provided that $\mathcal{A}$ has a generating set of objects of finite type.
- (2) $\mathcal{A}$ is locally finitely presented provided that $\mathcal{A}$ has a generating set of finitely presented objects.
- (3) $\mathcal{A}$ is locally coherent provided that every object of $\mathcal{A}$ is a direct limit of coherent objects.

Theorem 4.3. ([13], Theorem 1.6) Let $\mathcal{A}$ be a locally finitely presented Grothendieck category, then the following conditions are equivalent:

- (1) $\mathcal{A}$ is locally coherent;
- (2) fp($\mathcal{A}$) is abelian;
- (3) fp($\mathcal{A}$) = coh($\mathcal{A}$).

Theorem 4.4. ([21], Theorem 2.6) Let $\mathcal{A}$ be a locally coherent category and suppose that $\mathcal{C}$ is a Serre subcategory of $\mathcal{A}$. Then the following are equivalent:

- (1) The inclusion functor $\mathcal{C} \to \mathcal{A}$ admits a right adjoint $t: \mathcal{A} \to \mathcal{C}$.
- (2) The quotient functor $q: \mathcal{A} \to \mathcal{A}/\mathcal{C}$ admits a right adjoint $s: \mathcal{A}/\mathcal{C} \to \mathcal{A}$.

If the above conditions (1)–(2) are satisfied, then the following are equivalent:

- (3) $t$ commutes with direct limits;
- (4) $s$ commutes with direct limits.

If the above conditions (1)–(4) are satisfied, then $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are locally coherent and the following diagram of functors commutes where the unlabeled functors are inclusions.

\[
\begin{array}{ccc}
\text{fp}(\mathcal{C}) & \xrightarrow{\text{fp}(\mathcal{A})} & \text{fp}(\mathcal{A})/	ext{fp}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{q} & \mathcal{A}/\mathcal{C}
\end{array}
\]

Moreover, there is a unique functor $f: \text{fp}(\mathcal{A})/\text{fp}(\mathcal{C}) \to \text{fp}(\mathcal{A}/\mathcal{C})$ such that $\text{fp}(q) = f \circ p$, where $p: \text{fp}(\mathcal{A}) \to \text{fp}(\mathcal{A})/\text{fp}(\mathcal{C})$ denotes the quotient functor. The functor $f$ is an equivalence.

Proposition 4.5. Let $\mathcal{A}$ be a finitely generated right graded coherent algebra. Then the inclusion functor $\text{gr}(\mathcal{A}) \to \text{Gr}(\mathcal{A})$ induces an equivalence of categories $\text{qgr} \mathcal{A} \equiv \text{fp}(\text{Qgr} \mathcal{A})$.

Proof. By definition, $\text{Gr}(\mathcal{A})$ is a locally coherent category with fp($\text{Gr}(\mathcal{A})$) = gr($\mathcal{A}$). Moreover, $\text{Tors}(\mathcal{A})$ is its Serre subcategory and fp($\text{Tors}(\mathcal{A})$) = tors($\mathcal{A}$). Since $\text{Tors}(\mathcal{A})$ is a localizing subcategory, the condition (1) of Theorem 4.4 is satisfied. Note that the right adjoint to the inclusion functor $\text{Tors}(\mathcal{A}) \to \text{Gr} \mathcal{A}$ is the functor $t: \text{Gr}(\mathcal{A}) \to \text{Tors}(\mathcal{A})$ which sends a module $M$ to its maximal torsion submodule $\text{Tors} M$.

Let us show that this functor $t$ commutes with direct limits. Let $(\Lambda, \leq)$ be a directed partially ordered set, $((M_\lambda)_{\lambda \in \Lambda}, \{f_\lambda\}_{\lambda \leq \mu})$ be a direct system in $\text{Gr}(\mathcal{A})$ indexed by $\Lambda$, and $M = \varinjlim M_\lambda$. We identify $T = \varinjlim \text{Tors} M_\lambda$ with a submodule of $M$. We need to prove that $T = \text{Tors} M$.

Since $T$ consists of torsion elements, we conclude that $T \subset \text{Tors} M$. Reversely, for $x \in \text{Tors} M$ let $n \geq 0$ be such that $x A_{\geq n} = 0$. The module $A_{\geq n}$ is finitely generated; let $Y$ be some finite generating set. Then, for each $y \in Y$ there exists $\lambda(y) \in \Lambda$ and $x_y \in M_{\lambda(y)}$ such that $x_y$ represents $x$ and $x_y y = 0$. Since $Y$ is finite, there exists $\mu \geq \max \{\lambda(y) | y \in Y\}$ and $x_\mu \in M_\mu$ such that $x_\mu = f_{\lambda(y) \mu}(x_y)$ for each $y \in Y$. Hence, $x_\mu$ represents $x$ and $x_\mu A_{\geq n} = x_\mu F A = 0$, so that $x_\mu \in \text{Tors} M_\mu$. It follows that $x \in T$. Thus, $T = \text{Tors} M$.

Hence the functor $t$ commutes with direct limits, and the condition (3) of Theorem 4.4 is satisfied. Therefore

$qgr \mathcal{A} \equiv \text{fp}(\text{Gr}(\mathcal{A})/\text{Tors}(\mathcal{A}))$ ($= \text{fp}(\text{Qgr}(\mathcal{A}))$).
5 Ufnarovski graph and Holdaway-Smith map

5.1 The Ufnarovski graph

In [36], Ufnarovskii associates to any monomial algebra $A$ a directed graph $Q_A$ which has the same growth. The Ufnarovskii graph is defined as follows.

Let $A = k\Gamma/(F)$ be a graded monomial algebra of the form (1). Words not in $(F)$ are called legal. The set of legal words is denoted $L$. We write $L_n$ for the set of legal words of length $n$. Words in $F$ are said to be forbidden.

Let $l + 1$ be the maximum length of a forbidden word, i.e.,

$$l + 1 := \max \{ t | F \cap (k\Gamma)_t \neq \emptyset \}.$$  

The Ufnarovskii graph of $A$ is denoted $Q_A = (Q_0, Q_1, s, t)$, where $Q_0$ is the set of vertices, $Q_1$ the set of arrows, and $s, t : Q_1 \rightarrow Q_0$ are maps which assign to each arrow $w$ its starting vertex $s(w)$ and its terminating vertex $t(w)$. Ufnarovskii graph is defined as follows.

$$Q_0 := L_l,$$

$$Q_1 := L_{l+1},$$

$$s(w) := \text{the unique word in } L_l \text{ such that } w \in s(w)L_1,$$

$$t(w) := \text{the unique word in } L_l \text{ such that } w \in L_1t(w).$$

Example 5.1. Let $A$ be the monomial algebra

$$A = \frac{k(x, y, z)}{(x^2, yx, zy, xz, z^2, y^2)}.$$  

The sets of legal words of length 3 and 4 are

$$Q_0 = \{xy^2, xy, y^2z, yzx, zxy, y^3\};$$

$$Q_1 = \{xy^2z, xyzx, y^2zx, yzx, zxy^2, zzyz, y^3z, xy^3\}.$$  

Hence, the Ufnarovskii graph $Q_A$ is given by

We label arrows in $Q_A$ by elements in $L_1$. The label attached to an arrow $w$ is the first letter of $w$. For example, the label attached to $zxy$ is $z$.

Example 5.2. The labeling for the Ufnarovskii graph in example 5.1 is

We label arrows in $Q_A$ by elements in $L_1$. The label attached to an arrow $w$ is the first letter of $w$. For example, the label attached to $zxy$ is $z$.  

Example 5.2. The labeling for the Ufnarovskii graph in example 5.1 is
Proposition 5.3.  
\[ h_{alg}(A) = h_{alg}(kQ_A). \]

Proof. If \( m > l \), then each word \( x_1x_2 \cdots x_m \) not in \( F \) is uniquely associated with a path labeled \( x_1x_2 \cdots x_{m-1} \) of length \( m - l \). This correspondence is bijective \([15, \text{Lemma 3.1}]) \[36\]. Hence
\[ \dim_k A_m = \dim_k (kQ_A)_{m-l}, \]
and we have
\[ h_{alg}(A) = \lim_{m \to +\infty} \sqrt{\dim_k A_m} = \lim_{m \to +\infty} \sqrt{\dim_k (kQ_A)_m} = h_{alg}(kQ_A). \]

5.2 The Holdaway–Smith homomorphism

Holdaway and Smith \([15]\) have defined an algebra homomorphism \( f : A \to kQ_A \), this homomorphism were denoted by \( f \). It is defined on the generators of \( A \) as follows.

Let \( f : k\Gamma \to kQ_A \) be the unique algebra homomorphism such that for all \( x \in L_1 \),
\[ f(x) = \begin{cases} \text{the sum of all arrows labeled } x, \\ 0, \text{if there are no arrows labeled } x. \end{cases} \]
Then the homomorphism \( f : k\Gamma \to kQ_A \) induces a homomorphism of graded algebras \( \bar{f} : A \to kQ_A \) \([15, \text{Proposition 3.3}]) \[53\].

Theorem 5.4. (\([15]\), Theorem 1.1) Let \( A \) be a graded monomial algebra of the form \([11]\). Let \( Q_A \) be its Ufnarovskii graph and view \( k \) as a left \( A \)-module through the homomorphism \( \bar{f} : A \to kQ_A \). Then \( - \otimes_A kQ_A \) induces an equivalence of categories \( \text{Qgr}(A) \equiv \text{Qgr}(kQ_A) \).

Theorem 5.5. Let \( A \) be a graded monomial algebra of the form \([11]\). Let \( Q_A \) be its Ufnarovskii graph and view \( k \) as a left \( A \)-module through the homomorphism \( \bar{f} : A \to kQ_A \). Then the functor \( - \otimes_A kQ_A \) induces an equivalence of categories \( \text{agr}(A) \equiv \text{agr}(kQ_A) \).

Proof. Since \( A \) and \( kQ_A \) are right coherent rings by Corollary \([53]\) we can apply Proposition \([53]\) and conclude that
\[ \text{agr}(A) \equiv \text{fp}((\text{Gr}(A)/\text{Tors}(A)) (= \text{fp}(\text{Qgr}(A))) \]
and
\[ \text{agr}(kQ_A) \equiv \text{fp}((\text{Gr}(kQ_A)/\text{Tors}(kQ_A)) (= \text{fp}(\text{Qgr}(kQ_A))). \]

The functor \( - \otimes_A kQ_A \) induces an equivalence of categories \( \text{Qgr}(A) \equiv \text{Qgr}(kQ_A) \) by Theorem \([53]\) hence the functor \( - \otimes_A kQ_A \) induces an equivalence of categories \( \text{fp}(\text{Qgr}(A)) \equiv \text{fp}(\text{Qgr}(kQ_A)) \), i.e., the functor \( - \otimes_A kQ_A \) induces an equivalence of categories
\[ \text{agr}(kQ_A) \equiv \text{agr}(A). \]

6 Relationships between algebraic, topological and category-theoretical entropies

Let \( A \) be a graded monomial algebra of the form \([11]\) over a field \( k \), \( kQ_A \) be the Ufnarovskii graph of \( A \). \( \pi : \text{Gr}(kQ_A) \to \text{Qgr}(kQ_A) \) be the projection functor.

Notation 6.1.  
• (1) We write \( e_i \) for the trivial path at vertex \( i \). The indecomposable projective left \( kQ_A \)-modules are \( P_i = (kQ_A)_{e_i}, \cdots, P_n = (kQ_A)_{e_n}. \)

• (2) We define \( O := \pi(kQ_A). \)

• (3) We write \( P_i = \pi(P_i) \) for the images of the indecomposable projectives in \( \text{Qgr}(kQ_A). \)

• (4) Given a graded \( A \)-module \( M \), we denote by \( S(M) \) the graded \( A \)-module with \( S(M)_d = M_{d+1}. \) This is called the Serre twist of \( M. \)
The Serre twist $S$ induces an autoequivalence of $\text{qgr}(kQ_A)$, which will be denoted by the same letter $S$. Since the equivalence of the categories $\text{qgr}$ in Theorem 5.5 commutes with $S$, we get

**Proposition 6.2.** Let $\mathcal{O}$ be as above.

- (1) ([35], Lemma 3.3) $\mathcal{O}$ is a projective object in $\text{Qgr}(kQ_A)$.
- (2) ([35], Proposition 3.6) $\mathcal{O}$ is a generator in $\text{qgr}(kQ_A)$.
- (3) ([35], Proposition 3.2) $\text{qgr}(kQ_A)$ is a semi-simple category.

If $\mathcal{E}$ is an object of an abelian category $\mathcal{A}$, let $\mathcal{E}^\bullet$ denote the complex

$$\mathcal{E}^\bullet = \cdots \xrightarrow{0} \xrightarrow{0} \mathcal{E} \xrightarrow{0} \xrightarrow{0} \mathcal{E} \xrightarrow{0} \cdots$$

which is $\mathcal{E}$ in degree zero, and 0 in other degrees. Then $H^0(\mathcal{E}^\bullet) = \mathcal{E}$, so $\mathcal{E} \mapsto \mathcal{E}^\bullet$ is a fully faithful embedding of $\mathcal{A}$ into $\text{D}_b(\mathcal{A})$, with left inverse given by the functor $H^0$. Usually we just identify $\mathcal{E}$ with $\mathcal{E}^\bullet$ and regard $\mathcal{A}$ as a full subcategory of $\text{D}_b(\mathcal{A})$.

**Lemma 6.3.** Let $\mathcal{A}$ be a semi-simple abelian category with a generator $\mathcal{O}$ and $\text{D}_b(\mathcal{A})$ be its derived category, then

- (1) $\mathcal{O}$ is also a generator in $\text{D}_b(\mathcal{A})$.
- (2) If there is a distinguished triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

in $\text{D}_b(\mathcal{A})$, then $H^n(B^\bullet)$ is a direct summand of $H^n(A^\bullet) \oplus H^n(C^\bullet)$ for each $n \in \mathbb{Z}$.
- (3) If there is a tower of distinguished triangles

$$E_0^\bullet \xrightarrow{i_1} E_1^\bullet \xrightarrow{i_2} E_2^\bullet \cdots \xrightarrow{i_{p-1}} E_{p-1}^\bullet \xrightarrow{i_p} E_p^\bullet$$

in $\text{D}_b(\mathcal{A})$ with $E_0^\bullet = 0$, $p \geq 0$, and $n_i \in \mathbb{Z}$. Then $H^n(E_p^\bullet)$ is a direct summand of $\mathcal{O}^\oplus s$, where $s$ is the number of $\mathcal{O}[-n]$.

**Proof.** (1) Since $\mathcal{A}$ is a semi-simple category, every object

$$M^\bullet = \cdots \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} \cdots \xrightarrow{d^{j-1}} M^j \xrightarrow{d^j} \cdots$$

in $\text{D}_b(\mathcal{A})$ is isomorphic to the direct sum of its cohomologies

$$\cdots \xrightarrow{0} H^{-1}(M^\bullet) \xrightarrow{0} H^{0}(M^\bullet) \xrightarrow{0} H^{1}(M^\bullet) \xrightarrow{0} H^{2}(M^\bullet) \xrightarrow{0} \cdots \xrightarrow{0} H^{j}(M^\bullet) \xrightarrow{0} \cdots$$

(see, for example, [17, 2.5]). Hence $\mathcal{O}$ is a generator in $\text{D}_b(\mathcal{A})$.

(2) There is a long exact sequence

$$\cdots \to H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(C^\bullet) \xrightarrow{\delta} H^{n+1}(A^\bullet) \to H^{n+1}(B^\bullet) \to H^{n+1}(C^\bullet) \to \cdots$$

in $\mathcal{A}$, hence $H^n(B^\bullet)$ is a direct summand of $H^n(A^\bullet) \oplus H^n(C^\bullet)$ for each $n$.

(3) Since

$$H^j(\mathcal{O}[t]) = \begin{cases} 0 & \text{for } j \neq -t, \\ \mathcal{O} & \text{for } j = -t, \end{cases}$$

we have $H^n(E_p^\bullet)$ is a direct summand of $\mathcal{O}^\oplus s$, where $s$ is the number of $\mathcal{O}[-n]$ by (2). □
Lemma 6.4. Let $\mathcal{O}$ be a generator and $X$ be an object of a semi-simple abelian category $\mathcal{A}$. Then
\[
\delta_t(\mathcal{O}, X) = \text{rk}_\mathcal{O}(X),
\]
where we identify the object $X$ of $\mathcal{A}$ with the object $X^*$ of $\mathcal{D}^b(\mathcal{A}).$ □

Proof. Suppose that there is an object $X' \in \mathcal{D}^b(\mathcal{A})$ and a tower of distinguished triangles
\[
\xymatrix{ X_0 & X_1 & X_2 & \cdots & X_{p-1} & X_p \ar[r] & X \oplus X' \\
\mathcal{O}[n_1] & \mathcal{O}[n_2] & \mathcal{O}[n_3] & \cdots & \mathcal{O}[n_p] }
\]
with $X_0 = 0$, $p \geq 0$, and $n_i \in \mathbb{Z}$. Then by Lemma 6.3 (3), $X$ is a direct summand of $\mathcal{O}^\oplus s$, where $s$ is the number of $\mathcal{O}[0]$, hence the complexity $\delta_t(\mathcal{O}, \mathcal{E}) \geq s \geq \text{rk}_\mathcal{O}(X)$.

Now if $\text{rk}_\mathcal{O}(X) = e$, then there is an epimorphism $\mathcal{O}^\oplus e \rightarrow X \rightarrow 0$, i.e., $X$ is a direct summand of $\mathcal{O}^\oplus e$. Hence we have a tower of distinguished triangles
\[
\xymatrix{ 0 & \mathcal{O} & \mathcal{O}^\oplus 2 & \cdots & \mathcal{O}^\oplus e-1 & \mathcal{O}^\oplus e \ar[r] & X \oplus X' \\
\mathcal{O}[0] & \mathcal{O}[0] & \mathcal{O}[0] & \cdots & \mathcal{O}[0] }
\]
Hence
\[
\delta_t(\mathcal{O}, X) \leq e = \text{rk}_\mathcal{O}(X),
\]
and the lemma is proved. □

Theorem 6.5. In the notation above,
\[
h_t(\mathcal{D}^b(\text{qgr}(kQ_A)), S) = \log h_{\text{alg}}(kQ_A).
\]
□

Proof. Denote $B = kQ_A$. Let $U$ be the set of all vertices $u$ of $Q = Q_A$ that have no outgoing infinite paths, that is, $\dim u B < \infty$, and let $Q'$ be the full subgraph of $Q_A$ spanned by $Q_0 \setminus U$. It follows from [23 Prop. 4.2] that $\text{qgr}(kQ_A) \simeq \text{qgr}(kQ')$; the equivalence, being induced by a homogeneous algebra map, is compatible with the degree shift. Then $h_t(\mathcal{D}^b(\text{qgr}(kQ_A)), S) = h_t(\mathcal{D}^b(\text{qgr}(kQ'))), S)$. On the other hand, the characteristic polynomials $\chi(\lambda)$ and $\chi'(\lambda)$ of the adjacency matrices $A Q$ and $A Q'$ are connected by the equality $\chi(\lambda) = \chi'(\lambda) \cdot (-\lambda)^{|\mathcal{F}|}$, so that
\[
h_{\text{alg}}(kQ_A) = \rho(A) = \rho(A') = h_{\text{alg}}(kQ').
\]
Theorem 6.5. Thus, it is sufficient to prove the theorem in the case of quiver $Q = Q'$ without sinks. Then the module $B_B$ and all projective $B$-modules are torsion free.

Let $m \geq 0$ and let $s = \delta_t(\mathcal{O}, \mathcal{O}(m))$. By Lemma 6.3 we have $s = \text{rk}_\mathcal{O} \mathcal{O}(m)$. Then $\mathcal{O}(m)$ is a direct summand in $\mathcal{O}^\oplus s$ in $\text{qgr} \mathcal{B}$, that is, $\mathcal{O}^\oplus s \simeq \mathcal{O}(m) \oplus \pi(M)$ for some torsion free module $M$. It follows that for $n >> 0$, we get the isomorphism of truncated modules in $\text{Gr} \mathcal{B}$
\[
B_{\geq n}^\oplus s \simeq B_{\geq n+m}[m] \oplus M_{\geq n}.
\]
If $b_t$ denotes $\dim B_t$, we have $b_{m+n} \leq s b_t$, so that $s \geq b_{m+n}/b_t$. It follows that there exists $q > 0$ such that $s \geq b_{m+(s+1)+q}/b_{m+s+q}$ for all $s \geq 0$. Then for each $N > 0$ we have $s_{m} \geq b_{mN+q}/b_{q}$. Thus,
\[
s \geq \lim_{N \rightarrow \infty} \left( \frac{b_{mN+q}}{\delta_q} \right)^{1/N} = \lim_{N \rightarrow \infty} \left( b_{mN+q}^{1/(mN+q)} \right)^{(mN+q)/N} = \lim_{n \rightarrow \infty} \left( b_n \right)^{m/n} = h_{\text{alg}}(B)^m.
\]
Therefore,
\[
h_t(\mathcal{D}^b(\text{qgr}(kQ_A)), S) = \sup_{m \rightarrow \infty} \frac{1}{m} \log \delta_t(\mathcal{O}, \mathcal{O}(m)) \geq \lim_{m \rightarrow \infty} \frac{1}{m} \log (h_{\text{alg}}(B)^m) = \log h_{\text{alg}}(B).
\]
To prove the opposite inequality, note that the condition $s = \operatorname{rk}_\mathbb{O} \mathcal{O}(m)$ implies that there is no decomposition
$$\mathcal{O}(m) \simeq \bigoplus_j P_j^{\oplus k_j}$$
with $k_j \leq s - 1$ for all $j$. Hence in the decomposition
$$B_{\geq m}[m] \simeq \bigoplus_j P_j^{\oplus k_j}$$
of the projective $B$-module $B_{\geq m}[m]$ into the direct sum of the simple projectives we have $k_j \geq s$ for some $j$. It follows that $b_m \geq s$. Thus,
$$h_t(D^b(\operatorname{ag}(kQ_A)), S) = \limsup_{m \to +\infty} \frac{1}{m} \log \delta_t(\mathcal{O}, \mathcal{O}(m)) \leq \limsup_{m \to +\infty} \frac{1}{m} \log b_m = \log h_{alg}(B).$$

7 An example

Let $A$ be the monomial algebra
$$A = \frac{k(x, y, z)}{(F)},$$
where $F = \{xz, yz\}$. We have $\dim_k(A_s) = 2s + 1$, therefore
$$h_{alg}(A) = \lim_{s \to +\infty} \sqrt{\dim_k(A_s)} = 2,$$
and
$$h_{top}(X_F) = \lim_{s \to +\infty} \frac{1}{s} \log |\dim_k(A_s)| = 1.$$

The sets of legal words of length 1 and 2 are
$$Q_0 = \{x, y, z\}$$
and
$$Q_1 = \{x^2, y^2, z^2, xy, zx, zy, yx\}.$$

Hence, the Ufnarovski graph $Q_A$ is given by

Let $N_x^s, N_y^s, N_z^s$ be the numbers of paths of length $n$ that start with $x, y, z$, we have
$$N_x^s = 2^s, N_y^s = 2^s, N_z^s = 2s + 1 - 1,$$
and
$$\dim_k(kQ_A)_s = N_x^s + N_y^s + N_z^s = 2^{s+2} - 1.$$ 

Therefore
$$h_{alg}(kQ_A) = \lim_{s \to +\infty} \sqrt{\dim_k(kQ_A)_s} = 2.$$

Let $C = D^b(\operatorname{ag}(kQ_A))$, and let $S : C \to C$ be the Serre twist functor. We write $e_x, e_y, e_z$ for the trivial paths at vertex $x, y, z$. The indecomposable projective left $kQ_A$-modules are $P_x = (kQ_A)e_x, P_y = (kQ_A)e_y, P_z = (kQ_A)e_z$. We write $P_x = \pi(P_x), P_y = \pi(P_y), P_z = \pi(P_z)$ for the images of the indecomposable projectives in $QGr(kQ_A)$. Then
$$P_x(1) \cong P_x \oplus P_y \oplus P_z; \ P_y(1) \cong P_x \oplus P_y \oplus P_z; \ P_z(1) \cong P_z.$$
We can use induction and get
\[ \mathcal{P}_x(s) \simeq \mathcal{P}_x^{2s-1} \oplus \mathcal{P}_y^{2s-1} \oplus \mathcal{P}_z^{2s-1}. \]
Then
\[ \mathcal{O}(s) \simeq (\mathcal{P}_x \oplus \mathcal{P}_y \oplus \mathcal{P}_z)(s-1) \simeq \mathcal{P}_x(1)(s-1) = \mathcal{P}_x(s) \simeq \mathcal{P}_x^{2s-1} \oplus \mathcal{P}_y^{2s-1} \oplus \mathcal{P}_z^{2s-1}. \]
Then the object \( \mathcal{O}^{2s-1} \) is a direct summand of \( \mathcal{O}(s) = S^s \mathcal{O} \); in turn, the object \( \mathcal{O}(s) \) is a direct summand in \( \mathcal{O}^{2s-1} \). It follows that \( 2s-1 \leq \text{rk} \mathcal{O}^{S^s \mathcal{O}} \leq 2^s - 1 \). By Lemma 6.4, we have
\[ h_t(S) = \lim_{n \to \infty} \frac{1}{n} \log \delta_t(\mathcal{O}, S^n \mathcal{O}) = \lim_{n \to \infty} \frac{1}{n} \log \text{rk} \mathcal{O} S^n \mathcal{O} = \log 2. \]

**Acknowledgement**

The authors are grateful to anonymous reviewers for their comments and suggestions, which helped us to improve the text and avoid a number of inaccuracies. The work of D. Piontkovski has been supported by the grant of the Russian Science Foundation, RSF 22-21-00912.

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