FINITE TWO-DISTANCE TIGHT FRAMES

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ABSTRACT. A finite collection of unit vectors $S \subset \mathbb{R}^n$ is called a spherical two-distance set if there are two numbers $a$ and $b$ such that the inner products of distinct vectors from $S$ are either $a$ or $b$. When $a + b \neq 0$, we derive new structural properties of the Gram matrix of a two-distance set that also forms a tight frame for $\mathbb{R}^n$. One of the main results of this paper is a new correspondence between two-distance tight frames and certain strongly regular graphs. This allows us to use spectral properties of strongly regular graphs to construct two-distance tight frames. Several new examples are obtained using this characterization.

1. INTRODUCTION

This paper is devoted to new ideas of constructing spherical two-distance sets that at the same time form tight frames for $\mathbb{R}^n$.

1.1. Two-distance sets. A finite collection of unit vectors $S \subset \mathbb{R}^n$ is called a spherical two-distance set if there are two numbers $a$ and $b$ such that the inner products of distinct vectors from $S$ are either $a$ or $b$. If in addition $a = -b$, then $S$ defines a set of equiangular lines through the origin in $\mathbb{R}^n$. Equiangular lines form a classical subject in discrete geometry following foundational papers of Van Lint, Seidel, and Lemmens [18, 17]. The main results in this area are concerned with bounding the maximum size $g(n)$ of the spherical two-distance set in $n$ dimensions. A well-known general upper bound was obtained in the work of Delsarte et al. [10] who also constructed some examples of two-distance sets. Recently Musin [20] found the exact values of $g(n)$ for $7 \leq n \leq 39$ except the case $n = 23$. Barg and Yu [2] used the semidefinite programming method to resolve the case for dimension 23 as well as to obtain exact answers for $n \leq 93$ except the dimensions $n = 46, 78$. As far as constructions are concerned, the only known general method is rather trivial. Namely, let $e_1, \ldots, e_{n+1}$ be the standard basis in $\mathbb{R}^{n+1}$. The set

$$S = \{e_i + e_j, 1 \leq i < j \leq n + 1\}$$

forms a spherical two-distance set in the plane $x_1 + \cdots + x_{n+1} = 2$ (after scaling), and therefore $g(n) \geq n(n+1)/2, n \geq 2$. Isolated examples of two-distance sets were constructed in [10, 19]. The following theorem summarizes the state of the art for $g(n)$ including the results of all the papers cited above.

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Theorem 1.1 ([10] [19] [20] [2]). We have \( g(2) = 5, g(3) = 6, g(4) = 10, g(5) = 16, g(6) = 27, g(22) = 275 \),

\[
\frac{n(n+1)}{2} \leq g(n) \leq \frac{n(n+3)}{2} - 1, \quad n = 46, 78
\]

and 4465 \( \leq g(94) \leq 4492 \). If \( n \geq 95 \), then \( n(n+1)/2 \leq g(n) \leq n(n+3)/2 \) (The upper bound here can sometimes be improved to \( n(n+3)/2 - 1 \); see [2] for details).

This theorem shows that for most small values of the dimension \( n \), construction [10] gives an optimally sized two-distance set.

An important result about the cardinality of spherical two-distance sets is given by the following theorem.

Theorem 1.2 (Larman, Rogers, and Seidel, [16]). Let \( S \) be a spherical two-distance set in \( \mathbb{R}^n \).
If \( |S| > 2n+1 \) then the inner products \( a, b \) are related by \( b_k(a) = (ka - 1)/(k-1) \) where \( k \in \{2, \ldots, \lfloor (1+\sqrt{2n})/2 \rfloor \} \) is an integer.

The original result of [16] had \( 2n+3 \) in place of \( 2n+1 \), while the above improvement is due to Neumayer [22].

1.2. Finite unit-norm tight frames (FUNTFs). A finite collection of vectors \( S = \{x_i, i \in I\} \subset \mathbb{R}^n \) is called a finite frame for the Euclidean space \( \mathbb{R}^n \) if there are constants \( 0 < A \leq B < \infty \) such that for all \( x \in \mathbb{R}^n \)

\[
A ||x||^2 \leq \sum_{i \in I} |\langle x, x_i \rangle|^2 \leq B ||x||^2.
\]

If \( A = B \), then \( S \) is called an \( A \)-tight frame. If in addition \( ||x_i|| = 1 \) for all \( i \in I \), then \( S \) is a unit-norm tight frame or FUNTF. If at the same time \( S \) is a spherical two-distance set, we call it a two-distance tight frame. In particular, if the two inner products in \( S \) satisfy the condition \( a = -b \), then it is an equiangular tight frame or ETF.

The Gram matrix \( G \) of \( S \) is defined by \( G_{ij} = \langle x_i, x_j \rangle, 1 \leq i, j \leq N \), where \( N = |S| \). If \( S \) is a FUNTF for \( \mathbb{R}^n \), then it is straightforward to show that \( G \) has one nonzero eigenvalue \( \lambda = N/n \) of multiplicity \( n \) and eigenvalue 0 of multiplicity \( N - n \), [12].

Frames have been used in signal processing and have a large number of applications in sampling theory, wavelet theory, data transmission, and filter banks [6] [14] [15]. The study of ETFs was initiated by Strohmer and Heath [25] and Holmes and Paulsen [13]. In particular, [12] shows that equiangular tight frames give error correcting codes that are robust against two erasures. Bodmann et al. [5] show that ETFs are useful for signal reconstruction when all the phase information is lost. Sustik et al. [24] derived necessary conditions on the existence of ETFs as well as bounds on their maximum cardinality.

Benedetto and Fickus [4] introduced a useful parameter of the frame, called the frame potential. For our purposes it suffices to define it as \( FP(S) = \sum_{i,j=1}^N |\langle x_i, x_j \rangle|^2 \). For a two-distance frame we obtain

\[
\sum_{i,j=1}^N |\langle x_i, x_j \rangle|^2 = N + 2Na^2 + (N(N-1) - 2Na)b^2,
\]

where \( N_a = |\{ (i,j), i < j : \langle x_i, x_j \rangle = a \}| \). Moreover, if \( N > 2n+1 \), Theorem 1.2 implies that \( b = (ka - 1)/(k-1) \), where \( k \) is an integer between 2 and \( (1/2)(1 + \sqrt{2n}) \). This gives some information for a lower bound on \( FP(S) \), but fortunately, a more general and concrete result is known from [4].
Theorem 1.3. [4, Theorem 6.2] If \( N > n \) then
\[
FP(S) \geq \frac{N^2}{n}
\]  
with equality if and only if \( S \) is a tight frame.

1.3. Previous research on frames and strongly regular graphs. The classic connection between equiangular line sets, 2-graphs, and strongly regular graphs (Seidel et al. [23, 10]; see also [11]) has been recently addressed in the context of frame theory, particularly in the study of ETFs [25, 13, 26]. The starting point of these studies can be summarized as follows. Let \( L \) be an equiangular line set with angles \( a, -a \). Let us choose one vector on each of the lines of \( L \) (there are \( 2^N \) possible choices) and denote this set of vectors by \( X = \{x_1, \ldots, x_N\} \). Let \( G = X^T X \) be the Gram matrix of \( X \). Writing \( G = I + aS \), we define the Seidel matrix \( S \) of the set \( X \) as a symmetric matrix with off-diagonal entries equal to \( \pm 1 \). The Seidel matrix can be also thought of as an adjacency matrix of a graph on \( |X| \) vertices, where \( -1 \) denotes adjacency and \( 1 \) denotes non-adjacency.

Next we note that it is possible to choose the vectors in \( L \) so that some fixed vector, say \( x_1 \), has the same angle \( a \) to all the other vectors in the set \( X \). Indeed, if \( \langle x_1, x_i \rangle = a \), then we include \( x_i \) in \( X \), and otherwise if \( \langle x_1, x_i \rangle = -a \), we include \( -x_i \). This amounts to multiplying the matrix \( X \) by a diagonal matrix \( D \) with \( \pm 1 \) on the diagonal so that the new matrix \( S' \) takes the form
\[
S' = DSD = \begin{bmatrix} 0 & 1^T \\ 1 & S' \end{bmatrix}.
\]  
In the language of graphs this operation is called Seidel switching, and the result of this switching is a graph in which vertex \( v_1 \) is isolated from the rest of the vertices. Generally, there are \( 2^N \) graphs that are switching equivalent, and the collection of these graphs is called a switching class. According to (1.6), the spectrum of \( S' \) is the same as the spectrum of \( S \), so all the graphs in the switching class of \( X \) are co-spectral. The switching class of a graph is also known as a two-graph. By the above arguments, the spectrum of the two-graph is well-defined.

Now suppose that \( X \) is an ETF, then \( G \) has exactly two eigenvalues, namely \( N/n \) and 0, so the Seidel spectrum of the corresponding two-graph is \( \{\frac{1}{a}(-1 + N/n), -1/a\} \). Two-graphs with two eigenvalues are called regular, and one of the basic results about them is that each of the matrices \( S' \) defined in (1.6) is the Seidel adjacency matrix of a strongly regular graph [11, Thm. 11.6.1]. This enables one to use the known results about the existence of strongly regular graphs to construct new examples of ETFs. This line of thought was pursued in [13] and in particular in the recent work by Waldron [26], resulting in new examples of ETFs in \( \mathbb{R}^n \), \( n \leq 50 \). We note that some of the examples in [26] are two-distance frames, even though this paper did not emphasize the two-distance condition.

1.4. Contributions of this paper. Motivated by the research on ETFs, in this paper we study frames that are at the same time two-distance sets and FUNTFs. Assume that the values of the inner product between distinct vectors in \( S \) are either \( a \) or \( b \). We prove that the distance distribution of the frame with respect to any vector is the same (i.e., the Gram matrix \( G \) contains the same number of \( a's \) in every row). Using this fact, we establish a new relation between two-distance FUNTFs and strongly regular graphs, different from the connection discussed above, and find several examples of two-distance FUNTFs using this correspondence. In the particular case of ETFs our connection enables us to recover the earlier examples in [26] as well as obtain some new examples of ETFs. We also make a few remarks on the parameters of ETFs and strongly regular graphs.

\footnote{In the language of frame theory \( S (S') \) is called the (reduced) signature matrix of the frame.}
2. Characterization of Two-Distance FUNTFs

First we show that the set of points \([1,1]\) forms a FUNTF.

**Proposition 2.1.** The set of all midpoints of the edges of a regular simplex in \(\mathbb{R}^{n+1}\) \([1,1]\) forms a two-distance FUNTF for \(\mathbb{R}^n\).

**Proof.** Suppose that \(S\) is given by \([1,1]\), then the inner products of distinct vectors in \(S\) are either 1 or 0. Let

\[ N_{11} = |\{(i, j) : i < j, \langle e_i + e_j, e_i + e_j \rangle = 1\}| \]

Observe that \((i, j)\) is contained in this set if and only if \(i = 1\) or \(i = 2\), and we obtain \(N_{11} = 2(n-1)\).

By symmetry, the value \(N_{11}\) does not depend on the choice of the fixed vector \(e_1 + e_2\), so the total number of (unordered) pairs of vectors in \(S\) with inner product 1 equals

\[ N_1 = \frac{1}{2} \binom{n+1}{2} \quad N_{11} = \frac{1}{2} (n-1)n(n+1). \]

The pairs of distinct vectors not counted in \(N_1\) are orthogonal, and their count is

\[ N_0 = \left(\frac{n(n+1)/2}{2}\right) - N_1 = \frac{1}{8}(n-2)(n-1)n(n+1). \]

Now let us project the vectors of \(S\) on the plane \(x_1 + \cdots + x_{n+1} = 2\) and scale the result to place them on the unit sphere around the point \(\frac{2}{\sqrt{n}}(1, 1, \ldots, 1)\). By Theorem 1.2 the obtained vectors have pairwise inner products that are either \(a = (n-3)/(2(n-1))\) or \(b = -2/(n-1)\). This information suffices to compute the frame potential, and we obtain

\[ FP(S) = N + 2N_1a^2 + 2N_0b^2 = \frac{N^2}{n}. \]

The frame potential meets the lower bound \((1,5)\) with equality, which implies that \(S\) forms a FUNTF for \(\mathbb{R}^n\). \(\square\)

In the remainder of this section we prove several characterization results for two-distance FUNTFs. Let \(S \subset \mathbb{R}^n, |S| = N\) be a two-distance set with inner products \(a\) and \(b\), \(b < a\), and let \(N_a = |\{(i, j) : i < j, x_i, x_j \in S, \langle x_i, x_j \rangle = a\}|\). We note that Theorems 1.2 and 1.3 give some necessary conditions for the existence of a two-distance FUNTF with the parameters \(n, N, a, N_a\). However, we did not find them to be particularly useful, so we do not list them here.

The following theorem gives the value of \(N_a\) for a two-distance non-equiaangular FUNTF.

**Theorem 2.2.** Let \(G\) be the Gram matrix of a two-distance FUNTF \(S \subset \mathbb{R}^n\) with inner products \(a\) and \(b\) such that \(a + b \neq 0\). Then every column of \(G\) contains the same number of entries \(a\) and \(b\), and the count of \(a\)'s is given by

\[ N_a = \frac{N + 1 - (N - 1)b^2}{a^2 - b^2}. \]

**Proof.** \(G\) is similar to a diagonal matrix of order \(N\) with \(n\) nonzero entries \(\lambda = N/n\) on the diagonal. Therefore, \(G^2 - \lambda G = 0\), so \(G^2 = \lambda G\) and the \((G^2)_{ii} = \lambda\) since \(G_{ii} = 1\). We also have \((G^2)_{ij} = \sum_{j=1}^{N} G_{ij}^2\), so the norm of every row and of every column is the same and equals \(\sqrt{\lambda}\).

Now let \(N_a\) be the number of entries \(a\) in any fixed column. Then

\[ 1 + a^2 N_a + b^2 (N - 1 - N_a) = \frac{N}{n}. \]

This implies our claim. \(\square\)
If \( a = -b \), then the statement of the theorem does not hold. Indeed, consider the set \( S = \{x_1, \ldots, x_{28}\} \) of 28 vectors in \( \mathbb{R}^7 \) constructed according to (1.1). By Theorem 1.2 the inner products between distinct vectors in \( S \) are \( \pm 1/3 \), so they form a set of equiangular lines. For any given vector \( x \in S \) we have \(|\{y \in S : \langle x, y \rangle = 1/3\}| = 12\) and \(|\{y \in S : \langle x, y \rangle = -1/3\}| = 15\). Now consider the set \( S' = \{-x_1, x_2, \ldots, x_{28}\} \) which is also a FUNTF with inner products \( \pm 1/3 \), but the first column of \( G \) contains 12 entries equal to \(-1/3\), which is different from all the other columns.

Our next result shows that the values of \( a \) and \( b \) for a two-distance FUNTF can be found directly, without recourse to Theorem 1.2.

**Proposition 2.3.** Let \( S \) be a non-equiaangular two-distance tight frame in \( \mathbb{R}^n \) of cardinality \( N \) with inner product values \( a \) or \( b \). Then

\[
\frac{N}{n(a-aN-1)} \quad \text{or} \quad \frac{(N-n)(1-a)}{N-n(a(N-1)+1)}.
\]

**Proof.** Theorem 2.2 implies that \( 1 \) is \((1, \ldots, 1)\) is an eigenvector of the Gram matrix \( G \) with eigenvalue 0 or \( N/n \). Suppose it is the former, then \( G \cdot 1 = 0 \), so the sum of entries in every row is 0. This implies that \( 1 + aN_a + (N - 1 - N_a)b = 0 \), so from (2.1) we obtain the first of the two options for \( b \) in the statement.

Now suppose that \( G \cdot 1 = \frac{N}{n} \cdot 1 \), so the sum of entries of \( G \) in any given row equals \( N/n \). Repeating the calculation performed for the first case, we obtain the second of the two possibilities for \( b \). □

Finally, we note one more necessary condition for the existence of a two-distance tight frame implied by Theorem 1.2.

**Proposition 2.4.** Let \( S \) be a two-distance non-equiaangular FUNTF in \( \mathbb{R}^n \) with \( N \) vectors, inner products \( a \) and \( b \) and \( N_a \) entries \( a \) in each row of the Gram matrix. Suppose that \( N > 2n + 1 \), then

\[
\frac{N(N-1)(ka-1)^2}{2(k-1)^2} - \frac{N_a}{(k-1)^2}((2k-1)a^2-2ka+1) = \frac{N(N-n)}{2n}, \tag{2.2}
\]

where \( k \in \{2, \ldots, [(1+\sqrt{2n})/2]\} \).

**Proof.** Indeed, let \( S \) be such a frame. Using the value of the frame potential found in (1.4) together with \( b = (ka-1)/(k-1) \), we obtain

\[
F_{N_a}(a) := \sum_{i<j}^N \langle x_i, x_j \rangle^2 = N_a a^2 + \left(\frac{N(N-1)}{2} - N_a\right)\left(\frac{ka-1}{k-1}\right)^2, \tag{2.3}
\]

which is the same as the left-hand side of (2.2). At the same time, since \( FP(S) = N^2/n = 2F_{N_a}(a) + N \). Consequently, \( F_{N_a}(a) = \frac{N(N-n)}{2n} \) which conclude the proof. □

These necessary conditions on the parameters are really useful for small values of \( n \). Indeed, if \( n \leq 12 \), then \( k \) can take only the value 2, which enables us to rule out many sets of parameters.

3. Two-distance FUNTFs and strongly regular graphs

Connections between equiangular line sets and ETFs on the one side and strongly regular graphs on the other are well known and have been used in the literature to characterize the sets of parameters of ETFs [11, Ch. 11], [26]. In this section we extend this connection by relating two-distance (non equiangular) FUNTFs and strongly regular graphs in a new way.

We begin with a sufficient condition for the existence of two-distance FUNTFs. Let \( S \) be such a frame. The Gram matrix of any two-distance set with inner products \( a, b \) can be written as \( G = \)
\( I + a\Phi_1 + b\Phi_2 \), where \( \Phi_1 \) and \( \Phi_2 \) are the corresponding indicator matrices. Letting \( J \) to be the matrix of all ones, we can write
\[
G = (1 - b)I + (a - b)\Phi_1 + bJ. \quad (3.1)
\]
Note that \( J^2 = NJ \) and \( \Phi_1 J = N_aJ \), where \( N_a \) is given by (2.1), and
\[
G^2 = \frac{N}{n}G = \frac{N}{n}(1 - b)I + (a - b)\Phi_1 + bJ.
\]
Therefore, squaring (3.1), we obtain a quadratic equation for \( \Phi_1 \):
\[
(a - b)^2\Phi_1^2 + (a - b)\left(2 - 2b - \frac{N}{n}\right)\Phi_1 + \left(2b(1 - b) + b^2N + 2(a - b)b/a - \frac{N}{n}b\right)J
+ (1 - b)\left(1 - b - \frac{N}{n}\right)I = 0 \quad (3.2)
\]
Note moreover that \( b \) is a function of \( a \) as described in Theorem 1.2. Therefore, we obtain the following claim.

**Proposition 3.1.** Suppose that the values of \( N, n, \) and \( a \) are fixed. Let \( \Phi_1 \) be a symmetric 0-1 matrix with the same number of 1s in every row that satisfies equation (3.2). Then there exists a two-distance FUNTF for \( \mathbb{R}^n \) with \( N \) vectors whose Gram matrix is given by (3.1).

Conversely, to each two-distance FUNTF for \( \mathbb{R}^n \) with \( N \) vectors and inner products \( a, b \) is associated such a symmetric matrix \( \Phi_1 \).

**Proof.** The first part follows from the fact that given a matrix \( \Phi_1 \) that satisfies these conditions, we can find a valid Gram matrix \( G \) and therefore, construct the configuration \( S \).

The converse is straightforward. \(\square\)

This approach can be sometimes used to construct a 2-distance FUNTF. Consider the following example.

**Example 3.2.** Let \( n = 4, N = 10, a = 1/6 \) and \( b = -2/3 \), then (3.2) takes the form \( \Phi_1^2 + \Phi_1 - 2I - 4J = 0 \). This gives the following relation for the entry \( (i, j) (i \neq j) \) of \( \Phi_1 \):
\[
(\Phi_1^2)_{ij} + (\Phi_1)_{ij} - 4 = 0. \quad (3.3)
\]
From (2.1) we find that \( N_a = 6 \). Without loss of generality assume that the first row of \( \Phi_1 \) is 0111111000. Eq. (3.3) yields constraints on the rows 2 to 10 of \( \Phi_1 \) : for instance, \( (\Phi_1^2)_{12} = \sum_{i=1}^{10} (\Phi_1)_{1,i}(\Phi_1)_{2,i} = 3 \), so we can assume that the second row of \( \Phi_1 \) has the form 1011100110. Proceeding in this way, we can construct the rows of \( \Phi_1 \) by trial and error. In this example, this approach succeeds, yielding the matrix
\[
\Phi_1 = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}
\]

Now the Gram matrix \( G \) of a two-distance FUNTF \( S \) is found from (3.1), and the vectors of \( S \) can be found by constructing a 10 \( \times \) 4 matrix \( F \) such that \( FFT = G \).
This approach works well in small examples, but becomes computationally difficult as \( n \) and \( N \) increase because of the exponentially increasing search complexity. This motivates us to seek other methods of constructing the matrix \( \Phi_1 \) that satisfies (3.2). Taking inspiration from the connection between ETFs and strongly regular graphs, e.g., [26], we use properties of the adjacency matrix of the graph as a tool for finding \( \Phi_1 \).

A regular graph of degree \( k \) on \( v \) vertices is called strongly regular if every two adjacent vertices have \( \lambda \) common neighbors and every two non-adjacent vertices have \( \mu \) common neighbors. Below we use the notation \( \text{srg}(v, k, \lambda, \mu) \) to denote such strongly regular graph.

**Theorem 3.3** ([11], p.219), [7], p.117]. Let \( G \) be a graph on \( v \) vertices that is neither complete nor edgeless. Then \( G \) is strongly regular with the parameters \( (v, k, \lambda, \mu) \) if and only if its adjacency matrix \( A \) satisfies the equation

\[
A^2 + (\mu - \lambda)A - \mu J + (\mu - k)I = 0. \tag{3.4}
\]

For instance, the construction in Example 3.2 can be obtained from the adjacency matrix of strongly regular graph \( \text{srg}(10, 6, 3, 4) \) because in this case Equation (3.4) coincides with (3.2).

**Example 3.4.** Consider another example for \( N = 25 \). There exists a strongly regular graph \( \text{srg}(25, 8, 3, 2) \) whose adjacency matrix therefore satisfies the equation

\[
A^2 - A - 2J - 6I = 0. \tag{3.5}
\]

Aiming at constructing a two-distance FUNTF in \( \mathbb{R}^8 \) with \( N = 25 \) vectors and inner products \( a = 3/8 \) and \( b = -1/4 \), we note from (3.2) that its matrix \( \Phi_1 \) should satisfy Eq. (3.5). Using the adjacency matrix \( A \) it is easy to construct the vectors of the frame \( S \).

Concluding, if a two-distance FUNTF and a strongly regular graph give rise to the same matrix equation, then one of these objects exists if and only if so does the other. We obtain the following result whose proof is immediate by comparing Equations (3.2) and (3.4).

**Theorem 3.5.** A non-equiangular two-distance FUNTF \( (N, n, \lambda, \mu) \) of cardinality \( N \) in \( \mathbb{R}^n \) exists if and only if there exists a strongly regular graph with the parameters \( (v, k, \lambda, \mu) \) where

\[
\begin{align*}
    v &= N, \quad k = c_2 + c_3, \quad \lambda = c_1 + c_2, \quad \mu = c_2, \\
    c_1 &= -\frac{2 - 2b - \frac{N}{a-b}}{a-b}, \quad c_2 = \frac{2b(1-b) + b^2 N + 2(a-b)bNa - \frac{N}{a-b}}{(a-b)^2}, \\
    c_3 &= -\frac{(1-b)(1-b - \frac{N}{a})}{(a-b)^2}
\end{align*}
\]

The four parameters of an \( \text{srg}(v, k, \lambda, \mu) \) are not independent and must obey the following relation [11], p.119]:

\[
k(k - 1 - \lambda) = \mu(v - k - 1).
\]

Together with the values of \( k, \lambda, \mu \) in (3.6) this implies the following result.

**Corollary 3.6.** A two-distance FUNTF with the parameters \( (N, n, \lambda, \mu) \) exists only if

\[
(c_2 + c_3)(c_3 - c_1 + 1) = c_2(N - 1 - c_2 - c_3).
\]

**Constructing two-distance FUNTFs for small dimensions.** The approach outlined above suggests a way of constructing two-distance FUNTFs using the tables of known strongly regular graphs (see, e.g., [7], pp.143ff.] and the online tables [8]). Below in Table 1 we list examples...
obtained in this way for dimensions $4 \leq n \leq 10$, cardinality $N \leq 50$, and inner products satisfying $b = 2a - 1$.

Rows of the table labelled by * indicate that the FUNTFs in these rows have the largest possible cardinality as two-distance sets (cf. [13]). In the last two rows we list putative parameters of two-distance FUNTFs that would give rise to strongly regular graphs with the parameters $(50, 28, 18, 12)$. To the best of our knowledge, the existence of such graphs constitutes an open question. At the same time, (1.3) implies that spherical two-distance sets in dimensions $n = 7$ and 8 have cardinality at most $N = 36$. This implies that graphs srg(50, 28, 18, 12) do not exist, which apparently was not known until this paper [8].

Table 1 includes several new examples of two-distance FUNTFs. For instance, the frame with the parameters $(N, n, a, b) = (25, 8, \frac{3}{8}, -\frac{1}{2})$ can be constructed from the srg(25, 8, 3, 2), which is a product of two copies of $K_5$ (a complete graph on 5 vertices), etc.

### Table 1. Two-distance FUNTFs from graphs. The rows marked ‘new’ provide new examples of two-distance FUNTFs.

| $N$ | $n$ | $N_a$ | $a$ | Quadratic equation | srg | comments |
|-----|-----|-------|-----|-------------------|-----|----------|
| 9   | 4   | 1/4   |     | $A^2 + A - 2J - 2I = 0$ | srg(9,4,1,2) | new      |
| 9   | 5   | 2/5   |     | $A^2 + A - 2J - 2I = 0$ | srg(9,4,1,2) | new      |
| 10  | 4   | 1/6   |     | $A^2 + A - 4J - 2I = 0$ | srg(10,6,3,4) | Lisoněk [19] |
| 10  | 3   | 1/3   |     | $A^2 + A - J - 2I = 0$  | srg(10,3,0,1) | ETF      |
| 10  | 6   | 1/3   |     | $A^2 + A - 4J - 2I = 0$ | srg(10,6,3,4) | ETF      |
| 10  | 3   | 4/9   |     | $A^2 + A - J - 2I = 0$  | srg(10,3,0,1) | new      |
| 15  | 5   | 1/4   |     | $A^2 - 4J - 4I = 0$      | srg(15,8,4,4) | Construction [11] |
| 15  | 6   | 3/8   |     | $A^2 - 4J - 4I = 0$      | srg(15,8,4,4) | new      |
| 16  | 5   | 1/5   |     | $A^2 - 6J - 4I = 0$      | srg(16,10,6,6) | Lisoněk |
| 16  | 6   | 1/3   |     | $A^2 - 2J - 4I = 0$      | srg(16,6,2,2) | ETF      |
| 16  | 10  | 1/3   |     | $A^2 - 6J - 4I = 0$      | srg(16,10,6,6) | ETF      |
| 16  | 7   | 3/7   |     | $A^2 - 2J - 4I = 0$      | srg(16,6,2,2) | new      |
| 21  | 6   | 3/10  |     | $A^2 - 4J - 6I = 0$      | srg(21,10,5,4) | Construction [11] |
| 21  | 7   | 2/5   |     | $A^2 - A - 4J - 6I = 0$  | srg(21,10,5,4) | new      |
| 25  | 8   | 3/8   |     | $A^2 - A - 2J - 6I = 0$  | srg(25,8,3,2) | new      |
| 25  | 9   | 4/9   |     | $A^2 - A - 2J - 6I = 0$  | srg(25,8,3,2) | new      |
| *27 | 6   | 1/4   |     | $A^2 - 2A - 8J - 8I = 0$ | srg(27,16,10,8) | Lisoněk [19] |
| *27 | 7   | 5/14  |     | $A^2 - 2A - 8J - 8I = 0$ | srg(27,16,10,8) | new      |
| *28 | 7   | 12/13 |     | $A^2 - 2A - 4J - 8I = 0$ | srg(28,12,6,4) | ETF      |
| 28  | 7   | 18/13 |     | $A^2 - 2A - 10J - 8I = 0$| srg(28,18,12,10)| ETF |
| 28  | 8   | 5/12  |     | $A^2 - 2A - 4J - 8I = 0$ | srg(28,12,6,4) | new      |
| 28  | 9   | 13/27 |     | $A^2 - 2A - J - 8I = 0$  | srg(28,9,3,1) | new      |
| *36 | 8   | 14/14 |     | $A^2 - 3A - 4J - 10I = 0$| srg(36,14,7,4) | Construction [11] |
| 36  | 9   | 3/7   |     | $A^2 - 3A - 4J - 10I = 0$| srg(36,14,7,4) | new      |
| 36  | 10  | 2/5   |     | $A^2 - 2A - 2J - 8I = 0$ | srg(36,10,4,2) | new      |
| *45 | 9   | 16/16 |     | $A^2 - 4A - 4J - 12I = 0$| srg(45,16,8,4) | Construction [11] |
| 45  | 10  | 7/16  |     | $A^2 - 4A - 4J - 12I = 0$| srg(45,16,8,4) | new      |
| 50  | 7   | 28/17 |     | $A^2 - 6A - 12J - 16I = 0$| srg(50,28,18,12) | does not exist |
| 50  | 8   | 3/8   |     | $A^2 - 6A - 12J - 16I = 0$| srg(50,28,18,12) | does not exist |

4. Equiangular Tight Frames

In this section we examine the approach of this paper for the case of ETFs, i.e., the case when $b = -a$. In this case Theorem 1.2 implies that $a = 1/(2k - 1)$ as long as the cardinality of the
ETF satisfies $N > 2n + 1$. At the same time, Theorem 3.3 does not apply in this case, so it may be possible to obtain ETFs from strongly regular graphs, but the existence of graphs does not form a necessary condition.

We say that $S$ is an $(N, n, a)$ ETF in $\mathbb{R}^n$ if it has cardinality $N$ and inner products $a$ and $-a$. We begin with a necessary condition for the existence of ETFs.

**Proposition 4.1.** An $(N, n, a = \frac{1}{2k-1})$ ETF with $N > 2n + 1$ vectors exists only if

$$(N - n)(2k - 1)^2 = (N - 1)n, k = 2, 3, \ldots, \lfloor (1 + \sqrt{2n})/2 \rfloor.$$  

**Proof.** The quantity $F_N(a)$ in (2.3) (essentially, the frame potential) in this case equals $N(N - n)/2n$. At the same time, since $S$ forms an equiangular line set, we have $F_N(a) = N(N - 1)/2(2k - 1)^2$.

Thus if an ETF in $n$ dimensions exists, its cardinality can be found from this proposition. We list all the possible parameters of ETFs in $\mathbb{R}^n$, $n \leq 60$ in Table 2. Two instances in the table, for $n = 17$ and $n = 54$, lead to matrix equations for $\Phi_1$ with no solutions, so our approach is invalid.

In the remainder of this section we discuss one possible approach to the construction of ETFs. Let $S$ be an ETF of cardinality $N$ with inner products $a$ and $-a$. Assume that the distance distribution of $S$ with respect to any vector in it is the same, i.e., the number $\{|y \in S : \langle x, y \rangle = a \}$ does not depend on $x \in S$. Then the Gram matrix $G(S)$ has the same number of entries equal to $a$ in every row. Since the eigenvalues of $G$ are $N/n$ and 0, we have

$$G \cdot \mathbf{1} = \frac{N}{n} \quad \text{or} \quad G \cdot \mathbf{1} = 0,$$

or in other words

$$aN_n - a(N - 1 - N_n) + 1 = \frac{N}{n} \quad \text{or} \quad 0.$$  

This gives just two possibilities for the value of $N_n$.

With this, it becomes possible to link ETFs and strongly regular graphs. For instance, taking $(N, n, a) = (36, 15, \frac{1}{5})$, we find that $N_n$ is either 15 or 21. Now (3.2) implies that the matrix $\Phi_1$ is a root of the quadratic equation

$$A^2 - 6J - 9I = 0 \quad \text{or} \quad A^2 - 12J - 9I = 0.$$  

If it is the former, then recalling (3.4), we conclude that $\Phi_1$ is the adjacency matrix of the graph srg(36,15,6,6). In the second case the parameters of the graph are (36, 21, 12, 12). If either of these graphs exists, it gives rise to an ETF (36, 15, $\frac{1}{5}$).

In Table 2 we list the parameters of strongly regular graphs that are found using the above approach for all the possible parameters of ETFs with $n \leq 60$. The parameters of ETFs for $n \leq 47$ are cited from [9], which did not pursue the connection with strongly regular graphs.

**Possible maximally sized ETFs:** We note that for several of the sets of parameters that correspond to open cases in Table 2 their cardinality matches the best known upper bound on the size of equiangular lines set in that dimension (the semidefinite programming, or SDP, bound of [3]). Specifically, this applies to $n = 19, 20, 42, 45, 46$. For instance, in the case of $n = 42$ the SDP bound gives $N = 288$ and $a = 1/7$ (it is not known whether a set of 288 equiangular lines in $\mathbb{R}^{42}$ exists). Using our approach, we observe that such a set could be constructed from srg(288,140,76,60) and srg(288,164,100,84). Unfortunately, neither of these two graphs is known to exist (or not). For two of the sets of graph parameters listed in the table, the graphs are known not to exist; however, this is not sufficient to claim the nonexistence of the corresponding ETFs.
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### Table 2. Parameter sets of ETFs for $n \leq 60$

| $k$ | $n$ | $N$ | $\alpha$ | comments |
|-----|-----|-----|----------|----------|
| 2   | 5   | 10  | 1/3      | srg(10,3,0,1) (Y) |  |
|     |     |     |          | srg(10,6,3,4) (Y) |  |
| 2   | 6   | 16  | 1/3      | srg(16,6,2,2) (Y) |  |
|     |     |     |          | srg(16,10,6,6) (Y) |  |
| 2   | 7   | 28  | 1/3      | srg(28,12,6,4) (Y) |  |
|     |     |     |          | srg(28,18,12,10) (Y) |  |
| 3   | 15  | 36  | 1/5      | srg(36,15,6,6) (Y) |  |
|     |     |     |          | srg(36,21,12,12) (Y) |  |
| 3   | 17  | 51  | 1/5      | does not exist [9] |  |
| 3   | 19  | 76  | 1/5      | srg(76,45,28,24)(o) |  |
|     |     |     |          | srg(76,35,18,14)(o) |  |
| 3   | 20  | 96  | 1/5      | srg(96,45,24,18) (o) |  |
|     |     |     |          | srg(96,57,36,30) (N) |  |
| 3   | 21  | 126 | 1/5      | srg(126,60,33,24) (Y) |  |
|     |     |     |          | srg(75,48,48,39) (Y) |  |
| 3   | 22  | 176 | 1/5      | srg(176,85,48,34) (Y) |  |
|     |     |     |          | srg(176,105,68,54) (Y) |  |
| 3   | 23  | 276 | 1/5      | srg(276,135,78,54) (Y) |  |
|     |     |     |          | srg(276,165,108,84) (N) |  |
| 4   | 28  | 64  | 1/7      | srg(64,28,12,12) (Y) |  |
|     |     |     |          | srg(64,36,20,20) (Y) |  |
| 4   | 33  | 99  | 1/7      | does not exist [9] |  |
| 4   | 35  | 120 | 1/7      | srg(120,56,28,24) (Y) |  |
|     |     |     |          | srg(120,68,40,36) (Y) |  |
| 4   | 37  | 148 | 1/7      | srg(148,70,36,30) (o) |  |
|     |     |     |          | srg(148,84,50,44) (o) |  |
| 4   | 41  | 246 | 1/7      | srg(246,140,85,72) (o) |  |
|     |     |     |          | srg(246,119,64,51) (o) |  |
| 4   | 42  | 288 | 1/7      | srg(288,140,76,60) (o) |  |
|     |     |     |          | srg(288,164,100,84) (o) |  |
| 4   | 43  | 344 | 1/7      | srg(344,168,92,72) (Y) |  |
|     |     |     |          | srg(344,196,120,100) (o) |  |
| 4   | 45  | 540 | 1/7      | srg(540,266,148,144) (o) |  |
|     |     |     |          | srg(540,308,190,156) (N) |  |
| 5   | 45  | 100 | 1/9      | srg(100,45,20,20) (Y) |  |
|     |     |     |          | srg(100,55,30,30) (Y) |  |
| 4   | 46  | 736 | 1/7      | srg(736,364,204,156) (o) |  |
|     |     |     |          | srg(736,420,260,212) (o) |  |
| 4   | 47  | 1128| 1/7      | does not exist [21] |  |
| 5   | 51  | 136 | 1/9      | srg(136,63,30,28) (Y) New |  |
|     |     |     |          | srg(136,75,42,40)(Y) |  |
| 5   | 54  | 160 | 1/9      | does not exist [9] |  |
| 5   | 57  | 190 | 1/9      | srg(190,90,45,40) (o) |  |
|     |     |     |          | srg(190,105,60,55) (o) |  |

The label ‘o’ means that the existence of an SRG with these parameters is an open problem. ‘Y’ means that the corresponding ETF or a graph is known to exist and ‘N’ means that the SRG does not exist. The cases of $n = 17, 54$ result in matrix equations (3.2) that have no solutions, so our method does not apply.