Computational Higher Type Theory IV: Inductive Types

Evan Cavallo∗ Carnegie Mellon University
Robert Harper† Carnegie Mellon University

January, 2018

Abstract

This is the fourth in a series of papers extending Martin-Löf’s meaning explanation of dependent type theory to higher-dimensional types. In this installment, we show how to define cubical type systems supporting a general schema of cubical inductive types, inductive types whose constructors may take dimension parameters and may have specified boundaries. Using this schema, we are able to specify and implement many of the higher inductive types which have been postulated in homotopy type theory, including homotopy pushouts, the torus, W-quotients, truncations, and arbitrary localizations. We also construct one indexed inductive type, the fiber family of a term. Using the fiber family, it is possible to define an identity type whose eliminator satisfies an exact computation rule on the reflexivity constructor. We believe that the techniques used to construct the fiber family could be straightforwardly combined with our schema for inductive types in order to give a schema for indexed cubical inductive types.

The addition of higher inductive types and identity types makes computational higher type theory a model of homotopy type theory, capable of interpreting almost all of the constructions in the HoTT Book [41] (with the exception of general indexed inductive types and inductive-inductive types). This is the first such model with an explicit canonicity theorem stating that all closed terms of boolean type evaluate either to true or to false.

1 Introduction

Parts I and II of this series [3, 2] introduced dependent cubical realizability, a mathematical meaning explanation for higher-dimensional type theory based on the Cartesian cube category (see e.g., [18, 14, 7, 15]). Part III [6] extended this meaning explanation to include an infinite hierarchy of univalent universes and non-Kan pretypes such as an exact equality type. A paper of Angiuli et al. [5], published between Parts II and III, provides a more accessible introduction to dependent cubical realizability, and the paper Angiuli et al. [4] develops a formal Cartesian cubical type theory incorporating ideas from this series and the parallel work of Bezem et al. [11], Cohen et al. [16]. The interactive proof assistant RedPRL [40] implements a proof refinement logic for computational higher type theory based on dependent refinement rules [37, 38]. RedPRL currently includes reasoning principles for all of the types of Parts I-III as well as certain instances of the schema we develop here.

This paper adds cubical inductive types (CITs) to computational higher type theory, and is divided into two sub-parts. In Part IV.A, we show how to construct cubical type systems supporting a general class of CITs, which is sufficiently expressive to implement many of the higher inductive types previously postulated in the HoTT Book [41] and elsewhere. In Part IV.B, we step outside of this class and construct an indexed inductive type, the homotopy fiber family.

∗ecavallo@cs.cmu.edu
†rwh@cs.cmu.edu
A cubical inductive type is a type inductively generated by constructors which may take dimension parameters and which may have specified boundaries. We think of a constructor taking an $n$ dimension parameters as an $n$-cube, with its boundary describing how it is attached to the type under construction. The classic example, constructed already in Part I, is the following presentation of a circle [41, §6.4]:

\[
\begin{array}{c}
\text{base} \\
\downarrow \\
\text{loop}^x
\end{array}
\]

The circle is generated by a 0-dimensional “point” constructor `base` and a 1-dimensional “path” constructor `loop`. The constructor `loop` depends on a dimension parameter $r$, which we think of as ranging from 0 to 1. It has also specified boundaries `loop^0 \mapsto base` and `loop^1 \mapsto base`. Being inductively generated, the circle supports an eliminator which, given a point in a type $A$ and a loop at that point, constructs a map from the circle into $A$.

Going beyond base types like the circle or torus, we can use cubical inductive types to express homotopy-theoretic constructions on existing types. For example, we can define the homotopy pushout [41, §6.8] of a span of types $A \xleftarrow{e} C \xrightarrow{f} B$ as a CIT II generated by point constructors $a : A \gg \text{left}(a) \in \Pi$ and $b : B \gg \text{right}(b) \in \Pi$ and a path constructor $c : \Pi \gg \text{glue}^c(e) \in \Pi$ with boundaries $\text{glue}^0(P) \mapsto \text{left}(F(P))$ and $\text{glue}^1(P) \mapsto \text{right}(G(P))$.

Using Part III’s universes $U^\text{Kan}_j$ of Kan types, we will be able to define a pushout type constructor as a parameterized CIT $A : U^\text{Kan}_j, B : U^\text{Kan}_j, C : U^\text{Kan}_j, F : C \to A, G : C \to B \gg \Pi(A; B; C; F; G) \in U^\text{Kan}_j$.

The most interesting CTIs are those with recursive constructors. Here, the traditional example is the $(−1)$-truncation of a type [41, §6.9]. Given a type $A$, its $(−1)$-truncation $\|A\|$ has all of the elements of $A$ and an additional path constructor connecting each pair of elements of $\|A\|$. Thus, $\|A\|$ collapses the homotopical structure of $A$. We can define the $(−1)$-truncation $\|−\|$ as a parameterized CIT $A : U^\text{Kan}_j \gg \|A\| \in U^\text{Kan}_j$ with a point constructor $a : A \gg \text{pt}(a) \in \|A\|$ and a recursive path constructor $t_0 : \|A\|, t_0 : \|A\| \gg \text{path}^x(t_0; t_1) \in \|A\|$ with boundaries $\text{path}^0(N_0; N_1) \mapsto N_0$ and $\text{path}^1(N_0; N_1) \mapsto N_1$. Here, the boundary of $\text{path}$ is given not by previously defined constructors, but by terms which are recursive arguments to $\text{path}$. The $(−1)$-truncation and its higher analogues the $n$-truncations are all examples of localizations [33].

Our goal in Part IV.A is to define a schema for cubical inductive types which suffices to define all of these examples but which is relatively simple; we do not aim for utmost generality. As in the non-higher case, our language of argument types consists of strictly positive type operators. We do not allow constructors to take arguments in a path type of the inductive being defined, e.g., one cannot directly define a CIT $A$ with constructors $c_0 \in A$ and $p : \text{Path}_A(c_0, c_0) \gg c_1(p) \in \text{Path}_A(c_0, c_0)$. While there are interesting CTIs that could be defined in such a way, such as the higher $n$-truncations, there are typically alternative routes, so we choose to dodge the complexity of dependent argument types. Likewise, we choose the simplest sensible language of boundary terms, consisting of constructor forms for the inductive type, $\lambda$-abstraction, and function application. The limitations of this approach are discussed with our examples in Section 6.

In Part IV.B, we define the family of homotopy fibers as an example of an indexed inductive type. Given types $A, B$ and a map $F \in A \to B$, their homotopy fiber family is a family $b : B \gg \text{fib}(A; B; F; b) \in U^\text{Kan}_j$ which is generated by a single constructor $a : A \gg \text{refl}(a) \in \text{fib}(A; B; F; F(a))$. While the family has no explicit higher-dimensional constructors, it nonetheless has higher-dimensional structure, which must be present in order for the family to support transport between indices. In the special case that $F$ is the diagonal map.

---

1. When we define the pushout type using our schema, the constructor `glue` will be annotated with additional information; we are suppressing this for the moment.
2. In a parameterized inductive type, all constructors are indexed by the parameters and introduce elements uniformly in all indices. We distinguish parameterized inductive types from the more general indexed inductive types, where a constructor may introduce elements at a particular index.
\( \lambda a. \langle a, a \rangle \in A \to A \times A \), the homotopy fiber family is the Martin-Löf identity type family \( \text{Id}_A \) \cite{29}. Unlike the equivalent type \( \text{Path}_A \) defined in Parts I-III, this type supports an exact computation rule for the eliminator applied to a reflexivity term. We believe that our technique for implementing this indexed inductive type is orthogonal to the rest of the paper, and that the schema could be straightforwardly adapted to handle general indexed cubical inductive types.

**Terminology** We speak of cubical inductive types as a specific realization of the general and vague concept of higher inductive type. In the same way that our cubical type theory is a higher type theory, i.e., a type theory for reasoning explicitly about higher-dimensional objects, cubical inductive types are higher inductive types: types generated by explicit higher-dimensional constructors.

Our cubical inductive types occupy a specific niche on two axes. First, as the name suggests, their constructors are \( n \)-cubes; this is not the only choice. For example, the higher inductive types of the HoTT Book could be called globular inductive types (GITs). A globular inductive type is a type generated by constructors which map into either the type itself or into its iterated path/identity types. Although there is an obvious correspondence between 1-dimensional CITs and 1-dimensional GITs, the situation is murkier at higher dimensions. As an example, we can define the torus as a CIT generated by a point constructor \( \text{base} \), two path constructors \( \text{loop}^a \) and \( \text{loop}^b \), and a 2-dimensional “square” constructor \( \text{surf}^{x,y} \), with boundaries specified like so:

\[
\begin{array}{c}
\text{base} \\
\text{loop}^y \\
\text{loop}^x \\
\text{base}
\end{array} \quad \begin{array}{c}
\text{loop}^x \\
\text{surf}^{x,y} \\
\text{loop}^y \\
\text{base}
\end{array} \quad \begin{array}{c}
\text{loop}^y \\
\text{loop}^x \\
\text{base} \\
\text{base}
\end{array}
\]

With GITs, we need a different way of specifying the square constructor \( \text{surf} \). We might define the torus \( T \) as generated by a constructor \( \text{base} \) in \( T \), constructors \( \text{loop}^a, \text{loop}^b \) in \( \text{Path}_T(\text{base, base}) \), and a constructor \( \text{surf} \) in \( \text{Path}_{\text{Path}_T(\text{base, base})}(\text{loop}^a \cdot \text{loop}^b, \text{loop}^b \cdot \text{loop}^a) \), but it is not immediately clear that this is equivalent to our CIT definition. Converting between GITs and CITs becomes increasingly difficult with increasing constructor dimensionality. For GITs in particular, stating an eliminator becomes more difficult with higher-dimensional constructors.

The second distinction we want to make is between what we will call Path-HITs and Id-HITs. In computational higher type theory, the path type \( \text{Path}_A(M, N) \), whose values are terms with a bound dimension variable, is meaningfully distinct from the identity type \( \text{Id}_A(M, N) \), which we define in Part IV.B as the indexed inductive type generated by reflexivity. Although the two types are homotopy equivalent, they are not the same. In particular, there is no known constructive univalent type theory where \( \text{Path} \) supports the eliminator for \( \text{Id} \) with its exact computation rule (see online discussion \cite{19}). The same \( \text{Path}/\text{Id} \) distinction exists in Cohen et al. \cite{16}, but with a different definition of the identity type due to Swan \cite{39}. Path-HITs and Id-HITs, then, are higher inductive types specified in terms of \( \text{Path} \) and \( \text{Id} \) respectively. Our cubical inductive are Path-HITs, as their higher constructors produce elements of path types. In contrast, the higher inductive types of the HoTT Book are Id-HITs, as their constructors produce elements of identity types. This distinction seems to be orthogonal to the shape distinction; for example, Licata and Brunerie \cite{26} define an indexed inductive type of squares and specify the torus as a cubical inductive type in terms of that type. Awodey \cite{7} has given a cubical model in the Cartesian cube category in which \( \text{Id} \) and \( \text{Path} \) types coincide, but which is not known to support univalent universes.

Of course, any Path-HIT is homotopy equivalent to the corresponding Id-HIT, but again there may be differences at the level of exact equality. The eliminator for an Id-HIT as specified in the HoTT Book satisfies an exact computation rule on point constructors, but the corresponding “computation” rules for higher constructors hold only up to identification \cite[§6.2]{41}. For Id-HITs, it is not clear whether it is sensible
to ask for exact computation rules for higher constructors, for reasons described in the HoTT Book. In contrast, our Path-HITs satisfy exact computation rules for constructors of any dimensionality. By way of the Path $\simeq \text{Id}$ equivalence, a 1-dimensional Path-HIT is an Id-HIT in the sense of the HoTT Book, but the converse implication fails because we cannot obtain these exact equations.

**Related Work** The concept of inductive types with higher-dimensional constructors originated at the 2011 Oberwolfach meeting, in discussions between Andrej Bauer, Peter Lumsdaine, Mike Shulman, and Michael Warren (see [41, §6 Notes]). Since then, there has been an abundance of work seeking to make the concept precise, none of which has been the final word. (This paper makes no pretensions to that throne.) The HoTT Book presents many examples of higher inductive types, including types with recursive constructors, indexed inductive types, and inductive-inductive types, but only sketches a general schema [41, §6.13].

In the non-higher setting, schemata for inductive types in dependent type theory [30, 17, 20, 21] typically provide inductive types which are fixed-points of strictly positive operators, that is, the syntactic class of type operators in which the type variable never occurs in the domain of a function type. As we move to the higher setting, we can ask whether this class of argument types remains sufficient for general use. Moreover, we have the additional dimension of boundary terms: when we give a path constructor, what can its endpoints be?

Beginning in 2014, Sojakova [34, 35, 36] introduced the class of W-quotients (also called W-suspensions) and showed that they could be characterized as homotopy-initial algebras, building on work on ordinary inductive types in HoTT [8, 9]. A W-quotient is generated by a recursive 0-constructor (à la W-types [30]) and a recursive 1-constructor which connects instances of the point constructor. W-quotients suffice to define types such as pushouts and modular arithmetic types, but cannot directly be used to define types like truncations with constructors whose boundaries are not point constructors. (However, since pushouts can be written as W-quotients, some of these can be encoded indirectly; see below.)

In 2016, Altenkirch et al. [1] gave a universal property internal to a type theory satisfying UIP (uniqueness of identity proofs) which characterizes a very broad class of higher inductive types with 0- and 1-constructors, including indexed inductive and inductive-inductive types. They do not restrict to a syntactic grammar of argument types and boundary terms, instead allowing arbitrary functors and natural transformations. This definition far outstrips ours in generality, but in relying on UIP is incompatible with univalence.

Also in 2016, van der Weide et al. [42, 10] gave a schema for HITs with 0- and 1-constructors in formal Martin-Löf type theory, using a syntactic grammar of argument types and boundary terms. This schema allows for recursive constructors, so can be used to define HITs like the $(\neg 1)$-truncation directly. Moeneclaey et al. [31, 22] give a similar schema which also includes HITs with (globular) 2-constructors, and give an interpretation in the groupoid model. These schemata are the closest in spirit to our own, which is based on a similar syntactic grammar. Our grammar builds on theirs with support for (cubical) $n$-constructors for all $n$ and recursive arguments indexed by non-recursive types, the latter of which makes it possible to define a general localization type directly.

Another line of work seeks to reduce more complex HITs to simpler ones. Van Doorn [43] and Kraus [25] gave two different constructions of the $(\neg 1)$-truncation from non-recursive HITs, each obtaining the truncation as the sequential homotopy colimit of an $\omega$-indexed sequence of types. (Homotopy colimits indexed by $\omega$ can be defined using pushouts and a natural numbers type.) Rijke [32] later gave a construction of $n$-truncations in general, using a definition of the image of a term which is again constructed via pushouts and a sequential colimit. We expect that many HITs can be defined in this way. On the other hand, the complexity of these definitions makes them unwieldy for computational purposes, and they generally support “computation” rules for path constructors only up to a path. Moreover, Lumsdaine and Shulman [28, §9] give an example of a HIT which cannot be constructed from pushouts and the natural numbers (and indeed is not constructible in ZF).

On the semantic side, Lumsdaine and Shulman [28] developed the notion of cell monad with parameters, a sort of semantic specification for a higher inductive type, and gave a class of model categories for which all such higher inductive types exist. (This work was prefigured in 2012 in an unpublished note [27].) This class does not obviously correspond to a particular syntactic schema, but includes all of the examples we
present in Section 6. However, their work does not allow boundary terms to use the fibrant structure of the type being defined; we allow homogeneous composition in boundary terms, though not coercion. Also, for reasons related to fibrant replacement, their approach suffers from size issues. For example, a pushout type may not lie in the same universe as its constituent parts. In our setting, we can be more careful about the free fibrant structure we add; it is not clear to us whether this is possible at their level of generality.

One of the central motivations for investigating cubical type theory, particularly in the work of Brunerie and Licata [14, 26], was as a convenient language for specifying and proving theorems about higher inductive types. Even in traditional homotopy type theory, cubes proved to be a useful organizing principle. It also seemed that a primitive cubical type theory would allow for eliminators with exact computation rules on path constructors, which was believed to be problematic in standard homotopy type theory [41, §6.2]. Bezem et al. [11] gave the first constructive model of type theory in cubical sets, but this model is believed to be incompatible with HITs due to a lack of diagonals in the cube category. Cohen et al. [16] dodged this issue by adding diagonals (and reversals and connections) to the cube category, defining a univalent formal type theory with a circle and (−1)-truncation type. Huber [23] then proved a canonicity result for this type theory. The previous parts of this series [3, 2, 6] include a circle type, which satisfies a canonicity theorem by definition. Angiuli et al. [4] defines a formal Cartesian cubical type theory with a suspension type, and their formalized model generalizes this to a pushout type. Iasev [24] has proposed a type theory with an interval type supporting a general class of data types with conditions (and dual records with conditions), which is quite similar in spirit to our schema. For a broader overview of the development of cubical type theory, we refer the reader to Angiuli et al. [4].

We now turn to Part IV.B, where we define the homotopy fiber family and gesture towards indexed inductive types more generally. As previously mentioned, Altenkirch et al. [1] have given a schema including indexed inductive types in a type theory with UIP. However, the more closely related work in this area concerns the definition of an identity type in univalent cubical type theory with an exact computation rule for the eliminator. (As mentioned above, it is not known how to give a cubical type theory with univalence where Path has this property.) In 2014, Swan [39, §9] gave an algebraic weak factorization system for (a category equivalent to) the cubical sets of Bezem et al. [11], and showed how to use this to define an identity type with an exact computation rule for the eliminator applied to reflexivity. The idea is to define the identity type as a subset of the factorization of the diagonal, the restriction to a subset being necessary to ensure stability under substitution. This is quite conceptually similar to our definition, but the specifics of Swan’s construction less obviously generalize beyond identity types. Following Swan’s ideas, Cohen et al. [16, §9.1] defined an identity type for cubical sets with diagonals and connections, with elements of the identity type being elements of the path type paired with an element of the face lattice on which they are degenerate. This definition is farther in spirit from ours, and again seems to be limited to identity types.

**Future Work** We plan to implement the types presented in this paper in the proof assistant RedPRL, giving greater attention to issues of efficiency. Once this is complete, we hope to use the system to prove computational results from homotopy type theory, the standard challenge problem being the execution of a program computing the cardinal of $\pi_4(S^1)$ formalized by Brunerie [13, Appendix B]. We are also interested in attempting to internalize our schema and prove generic results on the homotopy of higher inductive types, exploiting the analogy between HITs and cell complexes.

**Acknowledgments** The authors would like to thank Carlo Angiuli, Daniel Gratzer, Kuen-Bang Hou (Favonia), Dan Licata, Ed Morehouse, Anders Mörtberg, and Jonathan Sterling for their comments and insights, and Steve Awodey for advice and discussions which were essential to the genesis of Part IV.B. Of course, this paper would not exist without the previous installments Angiuli et al. [3], Angiuli and Harper [2], and Angiuli et al. [6], and we are indebted to the other lines of work on cubical type theories, particularly Brunerie and Licata [14, 26] and Coquand et al. [11, 16].

The authors gratefully acknowledge the support of the Air Force Office of Scientific Research through MURI grant FA9550-15-1-0053. Any opinions, findings and conclusions or recommendations expressed in

---

3See notes at [http://www.cs.cmu.edu/~ecavallo/notes/muri17.pdf](http://www.cs.cmu.edu/~ecavallo/notes/muri17.pdf) for a more detailed comparison of the two constructions.


this material are those of the authors and do not necessarily reflect the views of the AFOSR.

2 Computational Higher Type Theory

This series studies cubical type systems, which are systems for establishing properties of a cubical operational semantics. A cubical operational semantics consists of a grammar of programs, cubical in the sense that they may contain dimension terms, and a deterministic set of rules describing how to execute such programs. These rules are specified by way of two judgments: $\mathit{M \text{ val}}$ (“$M$ is a value”) and $M \rightarrow M’$ (“$M$ steps to $M’$”).

A cubical type system, roughly speaking, is a collection of types, where a type is a named (higher-dimensional) partial equivalence relation on values satisfying certain conditions. This paper uses the definitions of type and cubical type system given in Part III; in this section, we recapitulate the definitions necessary for our purposes.

Dimensions A dimension term is either 0, 1, or one of a fixed set of dimension variables. We use $r,s,t$ for dimension terms, $x,y,z$ for dimension variables, and $\varepsilon$ for 0 or 1. Dimension variables are structural: we have weakening, contraction, and exchange. A dimension context $\Psi = (x_1, \ldots, x_n)$ is a list of dimension variables. We say that $r \dim \Psi$ holds when $r \in \{0,1\} \cup \Psi$. We write $\mathsf{FD}(M)$ for the set of dimension variables occurring in a term $M$, and say that $M \tm \Psi$ when $\mathsf{FD}(M) \subseteq \Psi$. A dimension substitution $\psi : \Psi’ \rightarrow \Psi$ assigns some $\psi(x) \dim \Psi’$ to every $x \in \Psi$. Given $r \dim \Psi$, we write $(r/x) : \Psi \rightarrow (\Psi, x)$ for the substitution which replaces $x$ with $r$. For $\psi : \Psi’ \rightarrow \Psi$ and $\psi’ : \Psi’’ \rightarrow \Psi’$, we write $\psi \psi’ : \Psi’’ \rightarrow \Psi$ for their composition. Given $\psi : \Psi’ \rightarrow \Psi$ and $M \tm \Psi$, the substituted term $\psi M \tm \Psi’$ is obtained by replacing each $x$ occurring in $M$ with $\psi(x)$. We refer to the terms $\psi M$ for various $\psi$ as the aspects of $M$. When $M \psi \rightarrow M’ \psi$ for all $\psi$, we write $M \rightarrow_{\text{co}} M’$ and say that $M$ stably steps to $N$.

$\Psi$-Relations To capture the denotations of every aspect of a type in context $\Psi$, we introduce the notion of $\Psi$-relation. A $\Psi$-relation $\alpha = (\alpha_\psi)_\psi : \Psi’ \rightarrow \Psi$ is a collection of relations indexed by substitutions into $\Psi$, where for each $\psi : \Psi’ \rightarrow \Psi$, the relation $\alpha_\psi$ relates terms in context $\Psi’$. We abbreviate $\alpha_{\text{id}}(M, M’)$ as $\alpha(M, M’)$. For $\psi : \Psi’ \rightarrow \Psi$, the $\Psi’$-relation $\alpha_\psi$ is defined by $(\alpha_\psi)_\psi = \alpha_\psi$. A $\Psi$-relation is stable if $\alpha_\psi(M, M’)’$ implies $\alpha_{\psi \psi’}(M \psi’, M’ \psi’’)$ for all $\psi’ : \Psi’’ \rightarrow \Psi’$.

We will write $\alpha_{\psi}(M_1, \ldots, M_n)$ to mean that $\alpha_\psi(M_i, M_j)$ holds for all $i,j \leq n$; for $\Psi$-PERs, where there is no possibility for confusion, we will simply write $\alpha_\psi(M_1, \ldots, M_n)$. A value $\Psi$-relation is one which relates only values; these will serve as the denotations of types. The lattices of $\Psi$-relations and value $\Psi$-relations are complete. Given a monotone operator $\mathcal{F}$ on these lattices, we write $\mu \mathcal{F}$ and $\nu \mathcal{F}$ for its least and greatest fixed-points respectively.

Given a value $\Psi$-relation $\alpha$, we define the $\Psi$-relation $\mathsf{TM}(\alpha)$, its coherent extension to terms, by

\[
\mathsf{TM}(\alpha)_\psi(M, M’) \iff \begin{cases} 
\forall \psi_1 : \Psi_1 \rightarrow \Psi, \psi_2 : \Psi_2 \rightarrow \Psi_1, \exists M_1, M_1’, M_2, M_2’, M_{12}, \mathsf{MV}_1; M_1 \wedge M_1’ \wedge M_2 \wedge M_2’ \wedge M\psi_1 \psi_2 \wedge M_{12} \wedge M’\psi_1 ; M_1’ \wedge M_2’ \wedge M’\psi_1 \psi_2 \wedge M_{12}’ \wedge 
\alpha_{\psi_1 \psi_2}(M_2, M_2’) \land \alpha_{\psi_1 \psi_2}(M_1, M_1’) \land \alpha_{\psi_1 \psi_2}(M_{12}, M_{12}’)
\end{cases}
\]

The reader can safely ignore this unpleasant definition, because we will only interact with $\mathsf{TM}$ through an interface validated in Appendix A. The basic intuition is that $\mathsf{TM}(\alpha)_\psi(M, M’)’$ holds when, for any pair of dimension substitutions $\psi_1, \psi_2$, $M$ and $M’$ compute to values related by $\alpha$ no matter how these substitutions are interleaved with evaluation. If $\alpha$ is a $\Psi$-PER, then so is $\mathsf{TM}(\alpha)$, and $\mathsf{TM}(\alpha)$ is always stable. A $\Psi$-relation is value-coherent if $\alpha_\psi(V, V’)’$ implies $\mathsf{TM}(\alpha)_\psi(V, V’)’$ for all $\psi, V, V’$.

The following lemma, the first in our interface to $\mathsf{TM}$, is used to prove introduction rules.

Lemma A.2 (Introduction). Let $\alpha$ be a value $\Psi$-PER. If for all $\psi : \Psi’ \rightarrow \Psi$, either $\alpha_\psi(M \psi, M’ \psi)$ or $\mathsf{TM}(\alpha)_\psi(M \psi, M’ \psi)$, then $\mathsf{TM}(\alpha)(M, M’)’$. 

6
The next, a head expansion lemma, is used to prove computation rules, both for eliminators and for the boundaries of introduction forms. Roughly, if a term $M'$ is in $\alpha$, and a term $M$ steps to $M'$ at all aspects modulo equality in $\alpha$, then $M$ and $M'$ are equal in $\alpha$.

**Lemma A.3** (Coherent expansion). Let $\alpha$ be a value $\Psi$-PER and let $M, M' \triangleright tm \{\Psi\}$. If for all $\psi : \Psi' \to \Psi$, there exists $M''$ such that $M \psi \to^* M''$ and $\text{TM}(\alpha)_{\psi}(M'', M' \psi)$, then $\text{TM}(\alpha)(M, M')$.

A constraint $\xi = \{r, r'\}$ specifies an (un-ordered) equation on dimension terms; we say that $\models \xi$ holds for $\xi = \{r, r'\}$ if $r = r'$. We will write constraints as $\{r, r'\}$ rather than $\{r, r'\}$. A constraint context $\Xi = (\xi_1, \ldots, \xi_n)$ is an ordered list of constraints; we say that $\models \Xi$ holds if $\models \xi_i$ holds for all $\xi_i \in \Xi$. We say $\Xi$ is valid if either $\models \xi$ for some $\xi \in \Xi$ or there exists some $r$ such that both $(r = 0) \in \Xi$ and $(r = 1) \in \Xi$. This technical condition was introduced in Part III in order to ensure certain canonicity properties of zero-dimensional terms. Validity of a constraint context $\Xi$ is a conservative approximation of the property that for all closing substitutions $\psi : \to \Psi$ we have $\models \xi'$ for some $\xi \in \Xi$. For a $\Psi$-relation $\alpha$, $\psi : \Psi' \to \Psi$, and a constraint context $\Xi$ with $\text{FD}(\Xi) \subseteq \Psi'$, we define $\alpha_{\psi | \Xi}(M, M') :\models \forall \psi' : \Psi'' \to \Psi'.(\models \Xi \psi' (M, M'))$. For a $\Psi$-relation $\alpha$ and $\Xi$ with $\text{FD}(\Xi) \subseteq \Psi$, we define a $\Psi$-relation $(\alpha | \Xi)$ by $(\alpha | \Xi)_{\psi}(M, M') :\models \alpha_{\psi | \Xi}(M, M')$. It is convenient to have a variant of the head expansion lemma for restricted relations.

**Corollary A.4** (Restricted expansion). Let $\alpha$ be a value $\Psi$-PER and $\Xi$ be a constraint context. Let $\alpha$ be a value $\Psi$-PER and let $M, M' \triangleright tm \{\Psi\}$. If for all $\psi : \Psi' \to \Psi$ with $\models \Xi \psi$, there exists $M''$ such that $M \psi \to^* M''$ and $\text{TM}(\alpha)_{\psi}(M'', M' \psi)$, then $\text{TM}(\alpha)\{\Xi\}(M, M')$.

For elimination rules, we need a notion of a dependent $\Psi$-relation. We say that $\beta = \beta[-\gamma]$ is a $\Psi$-relation over a $\Psi$-relation $\alpha$ when $\beta_{\psi}[M]$ is a $\Psi$-relation for all $\psi : \Psi' \to \Psi$ and $\alpha_{\psi}[\{\}]$, such that

1. $\beta_{\psi}[M] \psi' = \beta_{\psi' \psi}[M']$ for all $\psi : \Psi' \to \Psi$, $\psi' : \Psi'' \to \Psi'$, and $\alpha_{\psi}[\{\}]$,

2. $\beta_{\psi}[M] = \beta_{\psi}[M']$ for all $\psi : \Psi' \to \Psi$ and $\alpha_{\psi}[\{\}, M, M']$.

Again, we will abbreviate $\beta_{\psi}[M]$ as $\beta(M)$. If $\alpha$ is a $\Psi$-relation, $\beta$ is a $\Psi$-relation over $\alpha$, and $\vdash N, N' \triangleright tm \{\Psi\}$, we write $\alpha : \beta : N, N' \triangleright tm \{\Psi\}$ to mean that $\beta_{\psi}[M](N \psi[M/a], N' \psi[M'/a])$ holds for all $\psi : \Psi' \to \Psi$ and $\alpha_{\psi}(M, M')$.

When we prove elimination rules, we can reduce the problem of proving the eliminator is well-typed on terms to proving it is well-typed on values if the eliminator is eager.

**Definition A.5.** We say that $\vdash N \triangleright tm \{\Psi\}$ is eager if for all $\psi : \Psi' \to \Psi$ and $M \triangleright tm \{\Psi\}$, we have $N \psi[M/a] \downarrow W$ iff there exists $V \triangleright tm \{\Psi\}$ such that $M \Downarrow V$ and $N \psi[V/a] \Downarrow W$.

**Lemma A.6** (Elimination). Let $\alpha$ be a value-coherent $\Psi$-PER and $\beta$ be a value $\Psi$-PER over $\text{TM}(\alpha)$. Suppose $\alpha : \beta : N, N' \triangleright tm \{\Psi\}$ for some $\gamma \subseteq \alpha$. If $\vdash N, N' \triangleright tm \{\Psi\}$ are eager, then $\alpha : \text{TM}(\gamma) \Rightarrow \text{TM}(\beta)[\alpha](N, N')$.

**Cubical type systems**  Per Part III, a candidate cubical type system $\tau$ is a relation $\tau(\Psi, A_0, A'_0, \varphi)$ ranging over dimension contexts $\Psi$, term values $A_0, A'_0 \triangleright tm \{\Psi\}$, and relations $\varphi$ on values in context $\Psi$. We think of $\tau(\Psi, A_0, A'_0, \varphi)$ as saying that the $\Psi$-terms $A_0$ and $A'_0$ are (equal) names for the value relation $\varphi$. In the same way that $\text{TM}$ extends value $\Psi$-relations to terms, we have an operator $\text{PTy}$ extending a candidate cubical type system from values and relations to terms and $\Psi$-relations. If $\tau$ is a cubical type system, then $\text{PTy}(\tau)(\Psi, A, A', \alpha)$ is a relation ranging over contexts $\Psi$, terms $A, A' \triangleright tm \{\Psi\}$, and value $\Psi$-relations $\alpha$, defined by

$$\text{PTy}(\tau)(\Psi, A, A', \alpha) :\models \left\{ \forall \psi_1 : \Psi_1 \to \Psi, \psi_2 : \Psi_2 \to \Psi_1, \exists A_1, A'_1, A_2, A'_2, A_{12}, A'_{12}. \right.$$ 

$$A_{12} \downarrow A_1 \land A_2 \downarrow A'_1 \land \psi_1 \downarrow A_1 \land \psi_2 \downarrow A'_1 \land A_{12} \land A'_{12} \land A_{12} \land A'_{12} \land \tau(\Psi_2, A_2, A'_2, \alpha_{\psi_1 \psi_2}) \land \tau(\Psi_2, A_2, A'_2, \alpha_{\psi_1 \psi_2}) \land \tau(\Psi_2, A_2, A'_2, \alpha_{\psi_1 \psi_2}) \right\}$$

where $\tau(\Psi, \{A_1, \ldots, A_n\}, \varphi)$ is defined to hold when $\tau(\Psi, A_i, A_j, \varphi)$ holds for all $i, j \leq n$. The intent is for $\text{PTy}(\tau)(\Psi, A, A', \alpha)$ to hold when the aspects of $A$ and $A'$ coherently name the $\Psi$-relation $\alpha$. For our purposes, we will only need to know the following.

7
Proposition 2.1. If \( \tau(\Psi', A\psi, A'\psi, \alpha_\psi) \) holds for all \( \psi : \Psi' \to \Psi \), then \( \mathbf{PTy}(\tau)(\Psi, A, A', \alpha) \) holds.

As with \( \mathbf{Tm} \), \( \mathbf{PTy}(\tau) \) is stable: \( \mathbf{PTy}(\tau)(\Psi, A, A', \alpha) \) implies \( \mathbf{PTy}(\tau)(\Psi', A\psi, A'\psi, \alpha_\psi) \). A cubical type system is a candidate cubical type system such that

1. if \( \tau(\Psi, A, A', \varphi) \) and \( \tau(\Psi, A, A', \varphi') \), then \( \varphi = \varphi' \),
2. if \( \tau(\Psi, A, A', \varphi) \), then \( \varphi \) is a partial equivalence relation,
3. \( \tau(\Psi, -, -, \varphi) \) is a partial equivalence relation for all \( \Psi \) and \( \varphi \),
4. if \( \tau(\Psi, A_0, A'_0, \varphi) \), then \( \mathbf{PTy}(\tau)(\Psi, A_0, A'_0, \alpha) \) for some \( \alpha \).

Now, fixing a cubical type system \( \tau \), we can define the central judgments of computational higher type theory. The judgment \( A \simeq A' \) \( \text{type}_{\Psi} \) is defined to hold when \( \mathbf{PTy}(\tau)(\Psi, A, A', \alpha) \) for a value-coherent \( \Psi \)-relation \( \alpha \). We abbreviate \( A \simeq A' \) \( \text{type}_{\Psi} \) as \( A \equiv \Psi \). We write \( \llbracket A \rrbracket \) for the \( \Psi \)-relation \( \alpha \), which is unique when it exists. Presupposing \( A \equiv \Psi \), the judgment \( M \simeq M' \) \( A \equiv \Psi \) is defined to hold when \( \mathbf{Tm}(\llbracket A \rrbracket)(M, M') \) holds. We abbreviate \( M \simeq M \) \( A \equiv \Psi \) as \( M \in A \equiv \Psi \). As with relations, we define restricted versions of these judgments:

1. \( A \simeq A' \) \( \text{type}_{\Psi} \) \( M \equiv \Xi \) holds when \( A\psi \simeq A'\psi \) \( \text{type}_{\Psi} \) \( \llbracket A \rrbracket \) for all \( \psi : \Psi' \to \Psi \) with \( \llbracket = \Xi \rrbracket \),
2. Presupposing \( A \equiv \Psi \) \( \text{type}_{\Psi} \) \( M \equiv M' \equiv M \) \( A \equiv \Psi \) \( \Xi \) holds when \( M\psi \simeq M'\psi \) \( A\psi \equiv \Psi \) \( \llbracket = \Xi \rrbracket \) for all \( \psi : \Psi' \to \Psi \) with \( \llbracket = \Xi \rrbracket \).

We refer to these \( \simeq \) judgments as exact equality judgments so as to distinguish them from homotopy, which is sometimes called equality in the HoTT literature.

A pretype \( A \) is a (Kan) type when it satisfies the five Kan conditions, which require that \( A \) supports well-defined composition and coercion operators. The first three Kan conditions concern the homogeneous composition operator \( \mathbf{hcom} \), and the last two concern the coercion operator \( \mathbf{coe} \). Intuitively, a pretype \( A \) is \( \mathbf{hcom} \)-Kan when the operator \( \mathbf{hcom}_A \) implements a homogeneous composition operation for \( A \). Given endpoints \( r, r' \) and collection of tube faces \( \xi_i \to y.N_i \) in \( A \), homogeneous composition takes a cap \( M \) in \( A \) which lines up with each term \( N_i(r/y) \) under the corresponding constraint \( \xi_i \), and constructs a composite \( \mathbf{hcom}_A^{r \to r'}(M; \xi_i \to y.N_i) \) which lines up with each term \( N_i(r'/y) \) under \( \xi_i \). We imagine \( \mathbf{hcom} \) as sliding \( M \) from \( y = r \) to \( y = r' \) within the “tube” created by the terms \( \xi_i \to y.N_i \). As such, we require that \( \mathbf{hcom}_A^{r \to r'}(M; \xi_i \to y.N_i) \) be equal to \( M \) when \( r = r' \). As an example, considering the term \( \mathbf{hcom}_A^{0 \to y}(M; x = 0 \to y.N_0, x = 1 \to y.N_1) \), we have a diagram

\[
\begin{array}{c}
\xymatrix{
\mathbf{hcom}_A^{0 \to y}(M; \xi_1) \ar[d]_{\mathbf{hcom}_A^{-1}} \ar[r] & N_1 \\
\mathbf{hcom}_A^{0 \to y}(M; \xi_0) \ar[u]_{\mathbf{hcom}_A} & N_0 \\
}\\
x \\
y
\end{array}
\]

in \( A \).

Definition 2.2. Given \( A, A' \) \( \mathbf{tm \Psi} \) and a value \( \Psi \)-PER \( \alpha \), we say that \( (A, A', \alpha) \) are equally \( \mathbf{hcom} \)-Kan if for all \( \psi : \Psi' \to \Psi \), \( r, r' \) \( \text{dim} \Psi' \), valid constraint contexts \( \Xi = \xi \), and

(a) \( \mathbf{Tm}(\alpha)_{\psi}(M, M') \),
(b) \( \mathbf{Tm}(\alpha)_{\psi,i}(\xi_i, N_i) \) for all \( i, j \),
(c) \( \mathbf{Tm}(\alpha)_{\psi,i}(N_i(r/y), M) \) for all \( i \),
we have

K1. \( Tm(\alpha)_{\Psi}(hcom_{A_{\Psi}}(M; \xi_{i} \leftarrow y.N_{i}), hcom_{A'_{\Psi}}(M'; \xi'_{i} \leftarrow y.N'_{i})) \),

K2. \( Tm(\alpha)_{\Psi}(hcom_{A_{\Psi}}(M; \xi_{i} \leftarrow y.N_{i}), M) \),

K3. \( Tm(\alpha)_{\Psi|\xi_{i}}(hcom_{A_{\Psi}}(M; \xi_{i} \leftarrow y.N_{i}), N_{i}(r'/y)) \) for all i.

A pretype A is coe-Kan when the operator \( coe_{A_{\Psi}} \) implements coercion \( \psi: (\Psi, y) \rightarrow \Psi \). Coercion transports elements from one aspect of A to another: if M is an element of \( A_{\Psi}(r/y) \), then \( coe_{A_{\Psi}}(M) \) is an element of \( A_{\Psi}(r'/y) \). Naturally, we require that \( coe_{A_{\Psi}}(M) \) is equal to M itself.

**Definition 2.3.** We say that \((A, A', \alpha)\) are equally coe-Kan if for all \( \psi: (\Psi, y) \rightarrow \Psi, r, r' \) dim \([\Psi']\), and \( Tm(\alpha)_{\psi(r/y)}(M, M') \), we have

K4. \( Tm(\alpha)_{\psi(r'/y)}(coe_{y,A_{\Psi}}(M), coe_{y,A'_{\Psi}}(M')) \),

K5. \( Tm(\alpha)_{\psi(r/y)}(coe_{y,A_{\Psi}}(M), M) \).

We say that \((A, A', \alpha)\) are equally Kan when they are equally hcom and equally coe-Kan. Presupposing \( A \equiv A'_{\text{type}_{\Psi}} \), the judgment \( A \equiv A'_{\text{type}_{\Psi}} \) is defined to hold when \((A, A', [A])\) are equally Kan. Using the operators hcom and coe, we can define a heterogeneous composition operator com, which composes along a type line:

\[
\text{com}_{y,A}(M; \xi_{i} \leftarrow y.N_{i}) \rightarrow hcom_{y,A'(r'/y)}(coe_{y,A}(M); \xi_{i} \leftarrow y.coe_{y,A'}(N_{i})).
\]

**Proposition 2.4.** If \((A, A', \alpha)\) are equally Kan, then for all \( \psi: (\Psi, y) \rightarrow \Psi, r, r' \) dim \([\Psi']\), valid constraint contexts \( \Xi = \xi_{i} \), and

(a) \( Tm(\alpha)_{\psi(r/y)}(M, M') \),
(b) \( Tm(\alpha)_{\psi|\xi_{i}}(N_{i}, N'_{i}) \) for all i, j,
(c) \( Tm(\alpha)_{\psi(r'/y)|\xi_{i}}(N_{i}(r/y), M) \) for all i,

we have

1. \( Tm(\alpha)_{\psi(r'/y)}(\text{com}_{y,A}(M; \xi_{i} \leftarrow y.N_{i}), \text{com}_{y,A'}(M'; \xi_{i} \leftarrow y.N'_{i})) \),
2. \( Tm(\alpha)_{\psi(r/y)}(\text{com}_{y,A'}(M; r \leftarrow y.N_{i}), M) \),
3. \( Tm(\alpha)_{\psi(r'/y)|\xi_{i}}(\text{com}_{y,A'}(M; \xi_{i} \leftarrow y.N_{i}), N_{i}(r'/y)) \) for all i.

Finally, the judgments on closed terms are extended to open term judgments defined by simultaneous induction on context length:

1. \( \Gamma \equiv \Gamma' \text{ctx}_{\kappa} \) \([\Psi] \) holds for \( \Gamma = a_{i} : A_{i} \) and \( \Gamma' = a_{i} : A'_{i} \) when \( a_{i} : A_{1}, \ldots, a_{i-1} : A_{i-1} \gg A_{i} \equiv A'_{i} \) type_{\kappa} \([\Psi] \) holds for all i,
2. Presupposing \( \Gamma \text{ctx}_{\preff} \) \([\Psi] \), \( \overrightarrow{M}_{i} \equiv \overrightarrow{M'_{i}} \in \Gamma \) \([\Psi] \) holds when \( M_{i} \equiv M'_{i} \in A_{i}[M_{1}, \ldots, M_{i-1}/a_{1}, \ldots, a_{i-1}] \) \([\Psi] \) holds for all i, where \( \Gamma = a_{i} : A_{i} \),
3. Presupposing \( \Gamma \text{ctx}_{\preff} \) \([\Psi] \), \( \gg A \equiv A' \) type_{\kappa} \([\Psi] \) holds when \( A_{\psi}[\overrightarrow{M_{i}/a_{i}}] \equiv A'_{\psi}[\overrightarrow{M'_{i}/a_{i}}] \) type_{\kappa} \([\Psi'] \) holds for all \( \psi: \Psi \rightarrow \Psi \) and \( M_{i} = M'_{i} \in \Gamma_{\psi} \) \([\Psi'] \).
4. Presupposing \( \Gamma \text{ctx} \Psi \), \( \Gamma \gg N \equiv N' \in A [\Psi] \) holds when \( N \psi [\overrightarrow{M_i/\overrightarrow{a_i}}] = N' \psi [\overrightarrow{M_i'/\overrightarrow{a_i'}}] \in A \psi [\overrightarrow{M_i/\overrightarrow{a_i}}] [\Psi'] \) for all \( \psi : \Psi' \rightarrow \Psi \) and \( \overrightarrow{M_i} = \overrightarrow{M_i'} \in \Gamma \psi [\Psi'] \).

In this paper, we will say very little about the details of constructing a cubical type system. Rather, we will define the \( \Psi \)-PERs we want to use and assume we are working in a cubical type system which names them. Performing the actual construction requires no more machinery than is needed to construct a type system with a circle and dependent sums, so we refer the interested reader to Part III, which covers the construction of cubical type systems in great detail.

**Part IV.A**

**Cubical Inductive Types**

In this part, we define a schema for cubical inductive types, construct \( \Psi \)-PERs \( \iota(\mathcal{K}) \) realizing each instance \( \mathcal{K} \) of the schema, and show that these \( \Psi \)-PERs satisfy expected introduction, composition, coercion, elimination, and computation rules. The schema is defined in Section 3; argument types and boundary terms are specified by a small formal type theory. In Section 4, we describe what it means for a \( \Psi \)-relation \( \alpha \) to support an instance \( \mathcal{K} \) and show that the introduction rules hold for any such \( \alpha \). Section 5 defines the cubical inductive type \( \iota(\mathcal{K}) \) generated by a schema \( \mathcal{K} \) as the least \( \Psi \)-relation supporting \( \mathcal{K} \), then proves composition, coercion, and elimination rules for \( \iota(\mathcal{K}) \). In Section 6, we show how to encode several previously postulated higher inductive types in our schema and discuss potential improvements.

### 3 A schema for cubical constructors

We begin by defining our schema, an instance of which specifies the constructors of a cubical inductive type. The space of well-typed codes is carved out by means of five mutually inductively defined judgments.

\[
\begin{align*}
\mathcal{K} &\equiv \mathcal{K}' \ \text{constrs} [\Psi] & &\text{list of constructors} \\
\mathcal{K} \vdash C \equiv C' \ \text{constr} [\Psi] & &\text{constructor} \\
\Lambda \equiv \Lambda' \ \text{atype} [\Psi] & &\text{argument type} \\
\Theta \equiv \Theta' \ \text{actx} [\Psi] & &\text{argument context} \\
\mathcal{K}; \Theta \vdash M \equiv M' : A [\Psi] & &\text{boundary term}
\end{align*}
\]

These judgments are extended to the open forms \( \Gamma \gg \mathcal{K} \equiv \mathcal{K}' \ \text{constrs} [\Psi], \Gamma \gg \mathcal{K} \vdash C \equiv C' \ \text{constr} [\Psi], \Gamma \gg \Lambda \equiv \Lambda' \ \text{atype} [\Psi], \Gamma \gg \Theta \equiv \Theta' \ \text{actx} [\Psi], \) and \( \Gamma \gg \mathcal{K}; \Theta \vdash M \equiv M' : A [\Psi] \) in the usual functional fashion. For example, \( \Gamma \gg \Theta \equiv \Theta' \ \text{actx} [\Psi] \) is defined to hold for \( \Gamma = a_i : A_i \) when \( \Theta \psi [\overrightarrow{M_i/\overrightarrow{a_i}}] = \Theta' \psi [\overrightarrow{M_i'/\overrightarrow{a_i'}}] \) acts \( \Psi' \) holds for all \( \psi : \Psi' \rightarrow \Psi \) and \( \overrightarrow{M_i} = \overrightarrow{M_i'} \in \Gamma \psi [\Psi'] \). Because the open judgments are defined in terms of the closed judgments by semantic quantification, we will get parameterized inductive types “for free.” Note, however, that quantification over argument contexts \( \Theta \) will be defined formally.

We use \( \Gamma, \Delta \) and \( \gamma, \delta, \rho \) for ordinary term contexts and context variables (lists of term variables), and \( \Theta, \varphi \) and \( \theta, \varphi \) for argument contexts and argument context variables. We reserve \( p, q \) for boundary term variables; other letters denote ordinary term variables.

**Definition 3.1.** The grammars of constructor lists, constructors, argument types, argument contexts, and boundary terms are given by

\[
\begin{align*}
\mathcal{K} &::= \bullet \mid [\mathcal{K}, \ell : C] \\
C &::= (\Gamma; \gamma; \Theta; \overrightarrow{x_i}, \xi_k \mapsto \gamma, \theta, M_k) \text{ where } |\Gamma| = |\gamma| \text{ and } |\Theta| = |\theta| \\
A &::= X \mid (a : A) \rightarrow A \\
\Theta &::= \cdot \mid \Theta, p : A \rightarrow \cdot \\
M &::= \text{intro}^\mathcal{K}(\overrightarrow{M_n : M_j}) \mid \text{fcom}^{r \rightarrow r'}(\overrightarrow{M_i}, \overrightarrow{x_i} \mapsto y.M_k) \mid \lambda a.M \mid \text{app}(M, M).
\end{align*}
\]
A constructor list consists of entries of the form \( \ell : (\Gamma; \gamma, \Theta; \xi_k \mapsto \gamma, \theta, M_k) \), where \( \ell \) is some label drawn from a fixed set \( L \), \( \Gamma \) is a term context specifying the nonrecursive arguments of the constructor, \( \Theta \) is an argument term context specifying the recursive arguments of the constructor (which may depend on \( \gamma : \Gamma \)), \( |\xi_k| \) is the dimensionality of the constructor, each \( \xi_k \) is a constraint under which the constructor should reduce, and the corresponding \( \gamma, \theta, M_k \) is the boundary term to which it should reduce (which may depend on \( \gamma : \Gamma \) and \( \theta : \Theta \)). We will assume that each label \( \ell \) has an associated arity specifying the cardinalities \( |\ell|, |\xi_k|, |\xi_k| \), but will leave this implicit. We write \( \text{dom} K \) for the list of labels occurring in \( K \), and abbreviate \( \ell \in \text{dom} K \) as \( \ell \in K \). Given \( \ell \in K \), we write \( K[\ell] \) for the constructor bearing the label \( \ell \) in \( K \) and \( K_{<\ell} \) for the prefix of \( K \) preceding the entry labelled \( \ell \). We write \( K \sqsubseteq K' \) to mean that \( K \) is a prefix of \( K' \).

The argument types are the types permitted for recursive arguments. An argument type is either the indeterminant \( X \), which stands for the inductive type itself, or a dependent function \((a:A)\to A\) from an ordinary (non-recursive) type into an argument type. Thus, every argument type has the form \((a_1:A_1)\to \cdots \to (a_n:A_n)\to X\) for some context \( \Gamma = a_i : A_i \). The arguments to a constructor are therefore given by a strictly positive type operator, as is standard for non-higher inductive types \([20, 21]\). Note that argument types can depend on ordinary term variables but not argument term variables.

The boundary terms are the terms which inhabit argument types and which can be used to specify the behavior of a constructor on its boundary. In the indeterminant type \( X \), we have

\[
\text{fc} \quad \text{intro} \quad \text{intro}^\gamma(M_i : N_i)
\]

for the constructor terms \( \text{intro}^\gamma \). These boundary terms correspond to the \( \text{fcom} \) and \( \text{intro} \) terms which will make up an inductive type (we will say more about \( \text{fcom} \) later on). In the function argument type, we have \( \lambda \)-abstraction \( \lambda a.M \), and we can eliminate \( X \) from this type with the corresponding application \( \text{app}(M, N) \). We write \( M \text{ bnd } \Psi \) to mean that \( M \) is a free boundary term with \( \text{FD}(M) \subseteq \Psi, L(M) \) for the set of labels occurring on constructors in the boundary term \( M \), and \( \text{FB}(M) \) for the set of boundary term variables occurring in \( M \) (not including labels or boundary variables which occur inside of ordinary terms within \( M \)).

Although we could reuse the ordinary term language for argument types and boundary terms, we keep the two separate in order to stress the conceptual distinction. There is no operational semantics of argument types or boundary terms, and their formal type theory is not required to be extensional; they serve merely as a language for describing real terms parametric in an indeterminant type.

We now make all of this precise by defining, mutually inductively, the typing judgments for constructor lists, constructors, argument types, argument contexts, and boundary terms.

**Definition 3.2.A.** The judgment \( K \equiv K' \) \( \text{consts } \Psi \) is defined to hold when either

1. \( K = K' = \cdot \) (the empty list), or
2. \( K = [K_1, \ell : C] \) and \( K' = [K'_1, \ell : C'] \) where \( K_1 \equiv K'_1 \) \( \text{consts } \Psi \), \( \ell \not\in K_1 \), and \( K_1 \vdash C \equiv C' \) \( \text{consts } \Psi \).

The prefix judgment \( K \sqsubseteq K' \) \( \text{consts } \Psi \) is defined to hold when \( K \equiv K'' \) \( \text{consts } \Psi \) for some \( K' \subseteq K'' \).

**Definition 3.2.B.** Presupposing \( K \) \( \text{consts } \Psi \), the judgment \( K \vdash C \equiv C' \) \( \text{consts } \Psi \) is defined to hold when \( C = (\Gamma; \gamma, \Theta; \xi_k \mapsto \gamma, \theta, M_k) \) and \( C' = (\Gamma'; \gamma, \Theta'; \xi_k \mapsto \gamma, \theta, M'_k) \) where

1. \( \Gamma = \Gamma' \) \( \text{ctx}_\text{can } \Psi \),
2. \( \gamma : \Gamma \Rightarrow \Theta \equiv \Theta' \) \( \text{actx } \Psi \),
3. \( \text{FD}(\xi_k) \subseteq \{\xi_i\} \) and if \( |\xi_i| > 0 \) then \( \xi_k \) is valid,
4. \( \gamma : \Gamma \Rightarrow K'_1 ; \Theta ; M_k \equiv M'_1 ; X ; [\Psi, \xi_i, \xi_i] \) for all \( k, l \).

**Definition 3.2.C.** The judgment \( A \equiv A' \) \( \text{atype } \Psi \) is inductively defined by the following rules.

\[
\frac{X \equiv X \text{ atype } \Psi}{A \equiv A' \text{ atype } \Psi} \quad \frac{A \equiv A' \text{ type}_\text{can } \Psi \quad a : A \Rightarrow B \equiv B' \text{ atype } \Psi}{(a:A) \to B \equiv (a:A') \to B' \text{ atype } \Psi}
\]

As usual, the restricted judgment \( A \equiv A' \psi \) \( \text{atype } \Psi' \) is defined to hold when \( A \psi \equiv A' \psi \) \( \text{atype } \Psi' \) holds for all \( \psi : \Psi' \to \Psi \) such that \( \models \Xi \psi \).
Definition 3.2.D. For $\Theta = \overrightarrow{p_i : A_i}$ and $\Theta' = \overrightarrow{p_i : A'_i}$, the judgment $\Theta \equiv \Theta'$ actx $[\Psi]$ is defined to hold when $A_i \equiv A'_i$ atype $[\Psi]$ holds for all $i$.

Definition 3.2.E. Presupposing $K$ constrs $[\Psi]$, $\Theta$ actx $[\Psi]$, and $A$ atype $[\Psi]$, the boundary term equality judgment $K; \Theta \vdash m \equiv m' : A [\Psi]$ is inductively defined by the rules shown in Figure 1.

Proposition 3.3. If $K \leq K'$ constrs $[\Psi]$ and $K; \Theta \vdash m \equiv m' : A [\Psi]$, then $K'; \Theta \vdash m \equiv m' : A [\Psi]$.

Note that here we are inductively defining an open judgment form $K; \Theta \vdash m \equiv m' : A [\Psi]$; this judgment form is not defined from the judgment form $K; m \equiv m' : A [\Psi]$ by functionality. On the other hand, the judgment form $\Gamma \triangleright \triangleright K; \Theta \vdash m \equiv m' : A [\Psi]$ is defined in terms of the form $K; \Theta \vdash m \equiv m' : A [\Psi]$ by functionality; where the “$\triangleright\triangleright$-open” form occurs in Figure 1, one should imagine it replaced by its definition.

We will use the pushout and $(-1)$-truncation types as running examples. Let $A, B, C$ type$_{\text{Kan}} [\Psi]$ and $G \in C \rightarrow A [\Psi]$, $F \in C \rightarrow B [\Psi]$ be given. Using $\emptyset$ to denote empty lists, the pushout type for this span can be encoded in our schema as

$$K_{H(A; B; C; a.F ; a.G)} : \begin{cases} \text{left} : (A; a.\emptyset; \emptyset.\emptyset) \\ \text{right} : (B; b.\emptyset; \emptyset.\emptyset) \\ \text{glue} : (C; c.\emptyset; x. x) & x = 0 \mapsto c.\emptyset.\text{intro}_\emptyset^{\emptyset} (\text{app}(F, c); \emptyset), \\ & x = 1 \mapsto c.\emptyset.\text{intro}_{\emptyset}^{\emptyset} (\text{app}(G, c); \emptyset) \end{cases}$$

For $A$ type$_{\text{Kan}} [\Psi]$, we encode its $(-1)$-truncation as

$$K_{||A||} : \begin{cases} \text{pt} : (A; a.\emptyset; \emptyset.\emptyset) \\ \text{path} : (\emptyset; \emptyset.(X, X); x) & x = 0 \mapsto \emptyset.(t_0, t_1).t_0, \\ & x = 1 \mapsto \emptyset.(t_0, t_1).t_1 \end{cases}$$

While these examples hardly exercise our schema to its limits, most of the issues we will encounter in the course of defining inducitives will be visible in one of the two. We will run through a number of more complex examples in Section 6.

We now proceed to define the interpretation of argument types and boundary terms at a given instantiation of the indeterminant $X$. We first define these as untyped operations, then establish a typing rule.

Definition 3.4. For a value $\Psi$-PER $\alpha$ and a value $\Psi$-relation $\beta$ over $\text{Tm}(\alpha)$, define a value $\Psi$-relation $\text{Pi}(\alpha, \beta)$ by

$$\text{Pi}(\alpha, \beta) := \left\{ (\lambda b.N, \lambda b.N') \mid \forall \psi', M, M'. \right.$$ $\left. \text{Tm}(\alpha\psi\psi')(M, M') \implies \text{Tm}(\beta_{\psi\psi}[M]) (N\psi'[M/b], N'\psi'[M'/b]) \right\}.$

Definition 3.5 (Argument type interpretation). Given $K$, an argument type $B$, and a value $\Psi$-relation $\alpha$, we define a value $\Psi$-relation $\llbracket B \rrbracket(\alpha)$, the semantic interpretation of $B$ at $\alpha$, by recursion on the structure of $B$:

$$\llbracket X \rrbracket(\alpha) := \alpha$$

$$\llbracket (b : B \rightarrow C) \rrbracket(\alpha) := \text{Pi}[\llbracket B \rrbracket, (\psi, N) \mapsto \llbracket C \rrbracket[\psi][N/b]](\alpha\psi).$$

For any open term $A$ $\text{tm} [\Psi]$, we define an open term $\llbracket B \rrbracket(A)$ $\text{tm} [\Psi]$, the syntactic interpretation of $B$ at $A$, by

$$\llbracket X \rrbracket(A) := A$$

$$\llbracket (b : B \rightarrow C) \rrbracket(A) := (b : B \rightarrow \llbracket C \rrbracket(A)).$$

Proposition 3.6. In a cubical type system with all dependent function types, if

1. $A \equiv A'$ type$_{\text{Kan}} [\Psi]$,
2. $B \equiv B'$ atype $[\Psi]$,

then $\llbracket B \rrbracket(A) \equiv \llbracket B' \rrbracket(A')$ type$_{\text{Kan}} [\Psi]$ and $\llbracket B \rrbracket(A) = \llbracket B' \rrbracket(\llbracket A \rrbracket)$. 

12
Structural

$\text{K; } \Theta; p : B \vdash p \equiv p : B$ \hspace{1cm} \text{HYP}

$\text{K; } \Theta; \vdash M \equiv M' : B$ \hspace{1cm} \text{SYM}

$\text{K; } \Theta; \vdash M \equiv M' : B$ \hspace{1cm} \text{TRANS}

Constructors

$\text{K[\ell]} = (\Gamma; \gamma, \Phi; x_i, \xi_k \mapsto \gamma \cdot \varphi \cdot M_k)$

$\text{intro}_i \text{ dim } [\Psi]$

$\text{P}_n \mapsto \text{P}_n' \in \Gamma [\Psi]$

$\text{K; } \Theta; \vdash \text{intro}_i (\text{P}_n; N_j) \equiv \text{intro}_i (\text{P}_n'; N_j') : X [\Psi]$ \hspace{1cm} \text{intro}_I

$\text{K[\ell]} = (\Gamma; \gamma, \Phi; x_i, \xi_k \mapsto \gamma \cdot \varphi \cdot M_k)$

$\text{intro}_i \text{ dim } [\Psi]$

$\text{P}_n \mapsto \text{P}_n' \in \Gamma [\Psi]$

$\text{K; } \Theta; \vdash \text{intro}_i (\text{P}_n; N_j) \equiv M_k (\text{P}_n'/\gamma)[N_j'/\varphi] : X [\Psi]$ \hspace{1cm} \text{intro}_B

Composition

$\forall i, j \quad \text{K; } \Theta; \vdash N_i \equiv N_j' \equiv N_i : X [\Psi] [\xi_i]$

$\text{K; } \Theta; \vdash \text{fcom} \equiv \text{fcom}': \forall i \quad \text{K; } \Theta; \vdash N_i(r/y) \equiv M : X [\Psi] [\xi_i]$ \hspace{1cm} \text{fcom-I}

$\forall i, j \quad \text{K; } \Theta; \vdash N_i \equiv N_j : X [\Psi] [\xi_i, \xi_j]$

$\text{K; } \Theta; \vdash N_i(r/y) \equiv M : X [\Psi] [\xi_i]$ \hspace{1cm} \text{fcom-C}

$\forall i, j \quad \text{K; } \Theta; \vdash N_i \equiv N_j : X [\Psi] [\xi_i, \xi_j]$

$\text{K; } \Theta; \vdash N_i(r/y) \equiv M : X [\Psi] [\xi_i]$ \hspace{1cm} \text{fcom-T}

Functions

$A \text{ type}_{\text{Kan}} [\Psi]$ \hspace{1cm} \text{a : A } \Rightarrow \text{K; } \Theta; \vdash N \equiv N' : B [\Psi]$ \hspace{1cm} \text{I}

$\text{K; } \Theta; \vdash \text{app}(N, M) \equiv \text{app}(N', M') : B[M/a] [\Psi]$ \hspace{1cm} \text{E}

$\text{K; } \Theta; \vdash \text{app}(\lambda a. N, M) \equiv N[M/a] : B [\Psi]$ \hspace{1cm} \text{E}

$\text{K; } \Theta; \vdash \text{app}(\lambda a. (\text{app}(M, a)), (a : A) \Rightarrow B [\Psi]$ \hspace{1cm} \text{E}

Figure 1: Boundary term typing rules
The interpretation function for boundary terms acts on open terms; given an open boundary term \( \theta \vdash M \text{ bnd } [\Psi] \) with \( \text{FB}(M) \subseteq \theta \) (but which may contain any ordinary term variables), we get an ordinary open term \( \delta \vdash \{\theta,M\}^K(\delta) \text{ tm } [\Psi] \) where \( |\delta| = |\theta| \). (When we give a typing rule in Lemma 4.8, the types of \( \delta \) will be the interpretations of the types of \( \theta \).) The indeterminant \( X \) can never occur in a boundary term, so there is no need to supply the instantiating type explicitly to \( \succeq (-,-) \).

**Definition 3.7** (Boundary term interpretation). Given \( K \), an open boundary term \( \theta \vdash M \text{ bnd } [\Psi] \) with \( L(M) \subseteq \text{dom } K \) and \( \text{FB}(M) \subseteq \theta \), and \( \bar{N}_j \) \text{ tm } [\Psi] \) with \( |\bar{N}_j| = |\theta| \), we define a term \( \{\theta,M\}^K(\bar{N}_j) \) \text{ tm } [\Psi] \), the interpretation of \( \theta.M \) at constructor list \( K \) and term variables \( \bar{N}_j \), by

\[
\begin{align*}
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\end{align*}
\]

\[
\begin{align*}
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\{\theta,\ell \}\text{tm } [\Psi] & \vdash K \to \text{tm } [\Psi] \\
\end{align*}
\]

Note that real intro terms carry additional annotations which are not tracked in formal intro terms, namely the constructor data for the label and the prefix preceding it in \( K \). This information is necessary for the operational semantics, as an intro term must be able to compute its boundary.

**Proposition 3.8** (Basic facts on boundary interpretation).

(a) \( \{\theta,M\}^K(\bar{N}_j)_a = \{\theta,M[a]\}^K(\bar{N}_j)_a \).

(b) If \( \theta, p \vdash M \text{ bnd } [\Psi] \) and \( \theta \vdash N \text{ bnd } [\Psi] \), then \( \{\theta,M[N/p]\}^K(\bar{N}_j) = \{\theta,p,M\}^K(\bar{N}_j,\{\theta,N\}^K(\bar{N}_j)) \).

(c) If \( K \subseteq K' \) then \( \{\theta,M\}^K(\bar{N}_j) = \{\theta,M\}^{K'}(\bar{N}_j) \).

We will establish a typing rule for this operation in Lemma 4.8 after having defined what it means for a \( \Psi \)-relation to support a constructor list \( K \). (The reader may wish to refer forward to this lemma for intuition.)

4 Algebras

Now that we have laid out the language of constructor lists, we can say what it means for a \( \Psi \)-relation \( \alpha \) to support a particular constructor list \( K \). This will be the case when \( \alpha \) is closed under operators \( \text{FCOM}(-) \) and \( \text{INTRO}_{K,l}(-) \) for all \( l \in K \). When applied to a \( \Psi \)-relation, these operators respectively collect the \( \text{FCOM} \) and intro terms that can be built from elements of that relation.

An \( \text{FCOM} \) term is a *formal (or free) composite*, which we add to an inductive type in order to satisfy the \( \text{hcom} \)-Kan conditions. In a circle, for example, it is not enough to include only base and loop\(^*\); we also need values representing arbitrary composites of loops. Formal compositions provide a universal solution to this problem: \( \text{FCOM} \) implements homogeneous composition in a way which satisfies the equations required by the \( \text{hcom} \)-Kan conditions and nothing more.

**Definition 4.1.** For a value \( \Psi \)-relation \( \alpha \), define a value \( \Psi \)-relation \( \text{FCOM}(\alpha) \) as generated by

1. \( \text{FCOM}(\alpha)_\psi(\text{FCOM}^{\rightarrow\rightarrow r'}(M;\xi_i \mapsto y.N_i),\text{FCOM}^{\rightarrow\rightarrow r'(M';\xi_i \mapsto y.N_i'))) \) whenever

(a) \( \xi_i \) is valid,

(b) \( \xi_i \) for all \( i \),

(c) \( r \neq r' \),

14
\(\forall i \neq r \neq r'\) for all \(i, j\) whenever \(i < j\), \(\forall\).

\[
\begin{array}{c|c}
\text{(∀i) } \not\models r \neq r' & \text{(∀i) } r = r' \\
\text{fcom}^{r ightarrow r'}(M; \xi_i \rightarrow y. N_i) \text{ val} & \text{fcom}^{r ightarrow r'}(M; \xi_i \rightarrow y. N_i) \rightarrow M \\
\models \xi_i & \forall j < i \ not \ xi_j \\
\text{fcom}^{r ightarrow r'}(M; \xi_i \rightarrow y. N_i) \rightarrow N_i(r'/y)
\end{array}
\]

Figure 2: Operational semantics of fcom

\[
C = (\Gamma; \gamma. \Theta; \overline{x}_i. \xi_k \rightarrow \gamma. \theta. M_{\theta}) \quad (\forall k) \not\models \xi_k(\overline{r}_i/\overline{x}_i)
\]
\[
\text{intro}\_{\mathcal{K}, \ell, \mathcal{C}}^P(P_n; N_j) \text{ val}
\]
\[
C = (\Gamma; \gamma. \Theta; \overline{x}_i. \xi_k \rightarrow \gamma. \theta. M_{\theta}) \quad \models \xi_k(\overline{r}_i/\overline{x}_i) \quad (\forall l < k) \not\models \xi_l(\overline{r}_i/\overline{x}_i)
\]
\[
\text{intro}\_{\mathcal{K}, \ell, \mathcal{C}}^P(P_n; N_j) \rightarrow (\theta. M_{\theta}(\overline{r}_i/\overline{x}_i)[P_n/\gamma])^\mathcal{C}(N_j)
\]

Figure 3: Operational semantics of intro

Constructor terms for a label \(\ell \in \mathcal{K}\) take the form \(\text{intro}\_{\mathcal{K}, \ell, \mathcal{C}}^P(P_n; N_j)\), where \([\mathcal{K'}, \ell : \mathcal{C}]\) constrs \([\Psi]\) is an annotation exactly equal to the prefix of \(\mathcal{K}\) up to and including \(\ell\), \(\overline{r}_i\) are the dimension parameters to the constructor, \(\overrightarrow{P}_n\) are the non-recursive arguments, and \(\overrightarrow{N}_j\) are the recursive arguments. When one of the boundary constraints in \(C\) is satisfied, the intro term steps to the corresponding boundary term, which is computed from the information in \(\mathcal{K}'\) and \(C\).

**Definition 4.2.** For a value \(\Psi\)-relation \(\alpha\), \(\mathcal{K}\) constrs \([\Psi]\) and \(\ell \in \mathcal{K}\), define a value \(\Psi\)-relation INTRO\(_{\mathcal{K}, \ell}(\alpha)\) as generated by

1. **INTRO\(_{\mathcal{K}, \ell}\)\((\alpha)\)\((\text{intro}\_{\mathcal{K}, \ell, \mathcal{C}}^P(P_n; N_j), \text{intro}\_{\mathcal{K}, \ell, \mathcal{C}'}^P(P'_n; N'_j))\) whenever**
   
   \(\begin{align*}
   &\text{a) } K\psi[\ell] = (\Gamma; \gamma. \Theta; \overline{x}_i. \xi_k \rightarrow \gamma. \theta. M_{\theta}) \text{ where } \Theta = \rho_j : B_j, \\
   &\text{b) } (K\psi)_{<\ell} \equiv K' \equiv K'' \text{ constr } [\Psi'], \\
   &\text{c) } (K\psi)_{<\ell} \vdash K\psi[\ell] \equiv C' \equiv C'' \text{ constr } [\Psi'], \\
   &\text{d) } \not\models \xi_k(\overline{r}_i/\overline{x}_i) \text{ for all } k, \\
   &\text{e) } \overrightarrow{P}_n \equiv \overrightarrow{P}'_n \in \Gamma [\Psi], \\
   &\text{f) } \{B_j[P_n/\gamma]\}(\alpha\psi)(N_j, N'_j) \text{ for all } j.
   \end{align*}\n\)

**Definition 4.3.** For a constructor list \(\mathcal{K}\), define a monotone operator \(F_{\mathcal{K}}\) on value \(\Psi\)-relations:

\[
F_{\mathcal{K}}(\alpha) := \text{FCOM}(\alpha) \cup \bigcup_{\ell \in \mathcal{K}} \text{INTRO}_{\mathcal{K}, \ell}(\alpha)
\]
We say that $\alpha$ supports $\mathcal{K}$ if $\mathcal{F}_\mathcal{K}(\alpha) \subseteq \alpha$. In other words, $\alpha$ supports $\mathcal{K}$ if $\alpha$ is an algebra for the functor $\mathcal{F}_\mathcal{K}$ in the category of value $\Psi$-relations and inclusions.

**Definition 4.4.** For any operator $\mathcal{F}$ on value $\Psi$-relations, define a monotone operator $\mathcal{F}^\ast$ by $\mathcal{F}^\ast(\alpha) := \alpha \cup \mathcal{F}(\alpha)$ and an operator $\mathcal{F}^\ast$ by $\mathcal{F}^\ast(\alpha) := \mu(\beta \mapsto \alpha \cup \mathcal{F}(\beta))$. Note that $\mathcal{F}(\alpha) \subseteq \mathcal{F}(\alpha) \subseteq \mathcal{F}^\ast(\alpha)$.

We now prove introduction rules for the term relations $\text{Term}(\text{FCom}?(\alpha))$ and $\text{Term}(\text{Intro}_{\mathcal{K},\ell}?(\alpha))$. Here, we are checking that the evaluation of $\text{FCom}$ and $\text{Intro}$ terms is coherent with respect to the equality prescribed by $\text{FCom}?(\alpha)$ and $\text{Intro}_{\mathcal{K},\ell}?(\alpha)$, respectively. To see why we use $\text{FCom}?(\alpha)$ rather than $\text{FCom}(\alpha)$, consider that a term in $\text{Term}(\text{FCom}(\alpha))$ must evaluate to an $\text{FCom}$ value at all dimensions, whereas the term $\text{FCom}^\rightarrow_{\ell r'}(M; \xi_i \mapsto y.N_i)$ will sometimes evaluate to $M$ or some $N_i(r'/y)$.

**Lemma 4.5 (FCom-I).** For any $\Psi$-PER $\alpha$, $\psi : \Psi' \rightarrow \Psi$, $\xi$ valid, and

1. $\text{Term}(\alpha)(\psi)(M, M')$,
2. $\text{Term}(\alpha)(\psi|_{\xi_i})(N_i, N'_i)$ for all $i, j$,
3. $\text{Term}(\alpha)(\psi|_{\xi_i})(N_i(r'/y), M)$ for all $i$,

we have

(a) $\text{Term}(\alpha)(\psi|_{\xi_i})(\text{FCom}^\rightarrow_{\ell r'}(M; \xi_i \mapsto y.N_i), N_i(r'/y))$ for all $i$,

(b) $\text{Term}(\alpha)(\psi|_{r=r'})(\text{FCom}^\rightarrow_{\ell r'}(M; \xi_i \mapsto y.N_i), M)$,

(c) $\text{Term}(\text{FCom}?(\alpha))(\psi)(\text{FCom}^\rightarrow_{\ell r'}(M; \xi_i \mapsto y.N_i), \text{FCom}^\rightarrow_{\ell r'}(M'; \xi_i \mapsto y.N'_i))$.

**Proof.** We prove the three statements in order.

(a) By Corollary A.4. Let $\psi' : \Psi'' \rightarrow \Psi'$ be given with $\models \xi_i \psi'$. Take $j$ to be least such that $\models \xi_j \psi'$. We have $\text{FCom}^\rightarrow_{\ell r'}(M; \xi_i \mapsto y.N_i).\psi' \rightarrow N_j(r'/y)\psi'$ and $\text{Term}(\alpha)(\psi|_{r'=\psi'})(N_j(r'/y)\psi', N_i(r'/y)\psi')$ by assumption.

(b) By Corollary A.4. Let $\psi' : \Psi'' \rightarrow \Psi'$ be given with $r\psi' = r'\psi'$. Either $\not\models \xi_i \psi'$ for all $i$, or there exists $i$ least such that $\models \xi_i \psi'$. In the former case, we have $\text{FCom}^\rightarrow_{\ell r'}(M; \xi_i \mapsto y.N_i).\psi' \rightarrow M\psi'$ and $\text{Term}(\alpha)(\psi|_{r'=\psi'})(M\psi')$ by assumption. In the latter case, we have $\text{FCom}^\rightarrow_{\ell r'}(M; \xi_i \mapsto y.N_i).\psi' \rightarrow N_j(r'/y)\psi'$ and $\text{Term}(\alpha)(\psi|_{r'=\psi'})(N_j(r'/y)\psi', M\psi')$ by assumption.

(c) We go by Lemma A.2. Let $\psi' : \Psi'' \rightarrow \Psi'$ be given. We divide into three cases.

- There exists $i$ such that $\models \xi_i \psi'$.
  Then $\text{Term}(\text{FCom}?(\alpha))(\psi|_{\psi'})(\text{FCom}^\rightarrow_{\ell r'}(M; \xi_i \mapsto y.N_i).\psi', \text{FCom}^\rightarrow_{\ell r'}(M'; \xi_i \mapsto y.N'_i).\psi')$ by reducing with (a) on each side and applying assumption 2.

- $r\psi' = r'\psi'$ and $\not\models \xi_i \psi'$ for all $i$.
  Then $\text{Term}(\text{FCom}?(\alpha))(\psi|_{\psi'})(\text{FCom}^\rightarrow_{\ell r'}(M; \xi_i \mapsto y.N_i).\psi', \text{FCom}^\rightarrow_{\ell r'}(M'; \xi_i \mapsto y.N'_i).\psi')$ by reducing with (b) on each side and applying assumption 1.

- $\not\models \xi_i \psi'$ for all $i$ and $r\psi' \neq r'\psi'$.
  Then $\text{FCom}?(\alpha)(\psi|_{\psi'})(\text{FCom}^\rightarrow_{\ell r'}(M; \xi_i \mapsto y.N_i).\psi', \text{FCom}^\rightarrow_{\ell r'}(M'; \xi_i \mapsto y.N'_i).\psi')$ holds by definition of $\text{FCom}(\alpha)$.

For intro terms, the proof of the introduction rule is a bit more complicated. As the boundary of an intro term can step to the interpretation of a boundary term $\text{M}_k$, the proof of its coherence relies on the type-correctness of boundary term interpretation. On the other hand, the correctness of boundary term interpretation relies on the introduction rule for intro terms. We will therefore prove these two by a sort of mutual induction.
Definition 4.6. We say that the property Interp$(n)$ holds for some $n \in \mathbb{N}$ if for every
1. $\mathcal{K} \equiv \mathcal{K}'$ constrs $[\Psi]$ with $|\text{dom } \mathcal{K}| = n$,
2. $\mathcal{K}; \theta : \Theta \vdash m \equiv m' : b$ $[\Psi]$ where $\Theta = p_j : b_j$,
3. $\Psi$-relation $\alpha$ supporting $\mathcal{K}$,
4. $\text{TM}(\langle \mathcal{B}_j \mathcal{B}_j \rangle (\alpha \psi))$ $(N_j, N_j')$ for all $j$,
we have $\text{TM}(\langle \mathcal{B}_j \mathcal{B}_j \rangle (\alpha \psi)) (\langle \theta, \mathcal{M} \rangle \mathcal{K} (N_j), (\theta, \mathcal{M}' \mathcal{K}' (N_j'))$.

Lemma 4.7 (intro-I). Let $\mathcal{K}$ constrs $[\Psi]$, $\ell \in \mathcal{K}$, and a $\Psi$-PER $\alpha$ supporting $\mathcal{K}_{<\ell}$ be given, and suppose that Interp$(\mathcal{K}_{<\ell})$ holds. For all $\psi : \Psi' \rightarrow \Psi$, if
1. $\mathcal{K}\psi[\ell] = (\Gamma; \gamma, \Theta; \tilde{x}_k, \xi_k \rightarrow \gamma, \theta, \mathcal{M}_k)$ where $\Theta = p_j : B_j$,
2. $(\mathcal{K}\psi)_{<\ell} \equiv \mathcal{K}' \equiv \mathcal{K}''$ constrs $[\Psi']$,
3. $\mathcal{K}\psi \vdash \mathcal{K}\psi[\ell] \equiv C' \equiv C''$ constr $[\Psi']$,
4. $\tilde{P}_n \vdash \tilde{P}_n \in \Gamma [\Psi']$,
5. $\text{TM}(\langle \mathcal{B}_j \mathcal{B}_j \rangle (\alpha \psi)) (N_j, N_j')$ for all $j$,
then
\[ (a) \quad \text{TM}(\alpha \psi)_{\xi_k \langle \tilde{r}_i / \tilde{x}_j \rangle} ((\theta, \mathcal{M}_k \langle \tilde{r}_i / \tilde{x}_j \rangle) (\mathcal{P}_n / \gamma) [\mathcal{K}' (\tilde{N}_j)]) \text{ for all } k, \]
\[ (b) \quad \text{TM}(\text{INTRO}_{\mathcal{K}, \ell} (\alpha \psi))_{\tilde{x}_m} ((\theta, \mathcal{M}_k \langle \tilde{r}_i / \tilde{x}_j \rangle) (\mathcal{P}_n / \gamma) [\mathcal{K}' (\tilde{N}_j)]) \text{.} \]

Proof. We prove the two statements in sequence.

(a) By Corollary A.4. Let $\psi' : \Psi' \rightarrow \Psi'$ be given with $\models \xi_k \langle \tilde{r}_i / \tilde{x}_j \rangle \psi'$. Take $l$ to be least such that $\models \xi_l \langle \tilde{r}_i / \tilde{x}_j \rangle \psi'$. Then $\text{INTRO}_{\mathcal{K}, \ell} (\alpha \psi)_{\tilde{x}_m} (\mathcal{P}_n / \gamma) [\mathcal{K}' (\tilde{N}_j)] (\tilde{N}_j) \psi'$. By Interp$(\mathcal{K}_{<\ell})$, the assumption that $\alpha$ supports $\mathcal{K}_{<\ell}$, and $\mathcal{K}'; \Theta \vdash M_k \equiv M_l : X [\Psi' / \xi_k, \xi_l]$ we have
\[ \text{TM}(\alpha \psi)_{\psi'} ((\theta, \mathcal{M}_k \langle \tilde{r}_i / \tilde{x}_j \rangle) (\mathcal{P}_n / \gamma) [\mathcal{K}' (\tilde{N}_j)] (\tilde{N}_j) \psi'). \]

(b) By Lemma A.2. Let $\psi' : \Psi' \rightarrow \Psi'$ be given; we have two cases.
- There exists $k$ such that $\models \xi_k \langle \tilde{r}_i / \tilde{x}_j \rangle \psi'$. Then
\[ \text{TM}(\text{INTRO}_{\mathcal{K}, \ell} (\alpha \psi)_{\psi'})_{\tilde{x}_m} ((\theta, \mathcal{M}_k \langle \tilde{r}_i / \tilde{x}_j \rangle) (\mathcal{P}_n / \gamma) [\mathcal{K}' (\tilde{N}_j)] \psi', \theta, \mathcal{M}_k \langle \tilde{r}_i / \tilde{x}_j \rangle (\mathcal{P}_n / \gamma) [\mathcal{K}' (\tilde{N}_j)] \psi') \]
by first reducing with (a) on either side, then applying Interp$(\mathcal{K}_{<\ell})$ with $\gamma : \Gamma \Rightarrow \mathcal{K}'; \Theta \vdash M_k \equiv M_k : X [\Psi' / \xi_k]$.
- $\not\models \xi_k \langle \tilde{r}_i / \tilde{x}_j \rangle \psi'$ for all $k$.

Then we have $\text{INTRO}_{\mathcal{K}, \ell} (\alpha \psi)_{\psi'} ((\theta, \mathcal{M}_k \langle \tilde{r}_i / \tilde{x}_j \rangle) (\mathcal{P}_n / \gamma) [\mathcal{K}' (\tilde{N}_j)] \psi', \theta, \mathcal{M}_k \langle \tilde{r}_i / \tilde{x}_j \rangle (\mathcal{P}_n / \gamma) [\mathcal{K}' (\tilde{N}_j)] \psi')$ by the definition of $\text{INTRO}_{\mathcal{K}, \ell} (\alpha \psi)$. 

Lemma 4.8 (Boundary interpretation typing). Interp$(n)$ holds for all $n \in \mathbb{N}$. That is, for all
1. $\mathcal{K} \equiv \mathcal{K}'$ constrs $[\Psi]$,
2. $\mathcal{K}; \theta : \Theta \vdash m \equiv m' : b$ $[\Psi]$ where $\Theta = p_j : b_j$. 

3. \(\Psi\)-\(\alpha\) supporting \(K\),

4. \(\text{Tm}((\llparenthesis B_j \rrparenthesis^0(\alpha))(N_j,N'_j))\) for all \(j\),

we have \(\text{Tm}((\llparenthesis B_j \rrparenthesis^0(\alpha))((\theta \cdot M)^K(N_j), (\theta \cdot M')^{K'}(N'_j)))\).

Proof. By strong induction on \(n\). Suppose that \(\text{Interp}(m)\) holds for all \(m < n\). We then go by an inner induction on the derivation of \(K ; \theta : \Theta \vdash M \equiv M' : B \Psi\). The proof is entirely routine, so we will omit it, mentioning only that the \(\text{fcom-}\) and \(\text{intro}_{\ell}\)-related cases use Lemmas 4.5 and 4.7 respectively. In the \(\text{intro}_{\ell}\) case, the use of Lemma 4.7 is justified because \(|K_{<\ell}| < |K| = n\).

\[\square\]

5 Inductive types

5.1 Formation

Definition 5.1. Given \(K\), define the inductive \(\Psi\)-relation generated by \(K\) by \(i(K) := \mu \Psi \mu_{\Psi F\alpha}\). By definition, \(i(K)\) is the least \(\Psi\)-relation which supports \(K\). It is easy to check that \(i(K)\) is a \(\Psi\)-\(\alpha\).

Definition 5.2. If \(K \equiv K'\), we say that the cubical type system \(\tau\) has their inductive type if \(\tau(\Psi', \text{ind}(K)\psi, \text{ind}(K')\psi, i(K)\psi)\) holds for all \(\psi : \Psi' \rightarrow \Psi\).

With the definition of \(i(K)\) in hand, it is no more difficult to insert into a cubical type system than any other type. For details, we refer the reader to Part III.

Proposition 5.3. There exists a cubical type system which has the inductive type for every \(K \equiv K'\).

As quantification over \(\Gamma\) in \(\Gamma \gg K \equiv K'\) is defined semantically, the type system described in the previous proposition also has all parameterized inductive types expressible in our schema. It is equally simple to define universes which are closed under all parameterized inductive types expressible in our schema, as universes are defined in the same manner as cubical type systems.

For the remainder of this section, we fix \(K \equiv K'\) and assume that \(\tau\) has their inductive type as well as all dependent function types.

Lemma 5.4. \(\text{PTy}(\tau)(\Psi, \text{ind}(K), \text{ind}(K'), i(K))\).

Proof. By Proposition 2.1. \[\square\]

5.2 Introduction

Theorem 5.5 (I). \(i(K)\) is value-coherent.

Proof. By Lemmas 4.5 and 4.7, as \(\text{FCOM}(i(K)) \subseteq i(K)\) and \(\text{INTRO}_{\ell}\) holds for every \(\ell \in K\).

Corollary 5.6. \(\text{ind}(K) \equiv \text{ind}(K')\).

By definition of \(\text{Tm}\), \(\text{ind}(K)\) has the following canonicity property: for closed terms \(M, M'\), \(M = M' \in \text{ind}(K)\) implies that \(M \downarrow V\) and \(M' \downarrow V'\) for some \(V\) and \(V'\) with \(i(K)(V, V')\). In other words, every pair of equal terms in \(\text{ind}(K)\) evaluates to either a pair of equal \(\text{fcom}\) values or a pair of equal intro values. Because every valid constraint context \(\Xi\) with \(\text{FD}(\Xi) = \emptyset\) contains some \(\xi \in \Xi\) with \(\models \xi\), there are no \(\text{fcom}\) or higher-dimensional intro values in an empty dimension context. Thus, we can also make the stronger guarantee that \(M = M' \in \text{ind}(K)\) implies that \(M\) and \(M'\) evaluate to a pair of equal zero-dimensional intro values.
Returning to our examples, let us now introduce a few abbreviations. For pushouts, we define
\[
\Pi(A; B; C; F; G) := \text{ind}(K_{\Pi(A; B; C; F; G)})
\]
\[
\text{left}(M) := \text{intro}^o_{\text{left}}(M; \emptyset)
\]
\[
\text{right}(N) := \text{intro}^o_{\text{right}}(N; \emptyset)
\]
\[
\text{glue}^\tau(P) := \text{intro}^o_{\text{glue}}(P; \emptyset),
\]
where we have omitted annotations on both the intro terms and their abbreviations for the sake of concision. For truncations, we define
\[
\|A\| := \text{ind}(K_{\|A\|})
\]
\[
\text{pt}(M) := \text{intro}^o_{\text{pt}}(M; \emptyset)
\]
\[
\text{path}^\tau(N_0; N_1) := \text{intro}^o_{\text{path}}(\emptyset; (N_0, N_1)).
\]

5.3 Composition

\[
\text{hcom}^\tau_{\text{ind}(K)}(M; \xi_i \mapsto y.N_i) \rightarrow \text{fcom}^\tau_{\text{ind}(K')}^\tau(M; \xi_i \mapsto y.N_i)
\]

Figure 4: Operational semantics of hcom_{\text{ind}(K)}

Lemma 5.7 (H). $\text{ind}(K) \cong \text{ind}(K')$ type_pre $[\Psi]$ are equally hcom-Kan.

Proof. We have $\text{hcom}^\tau_{\text{ind}(K)}(M; \xi_i \mapsto y.N_i) \rightarrow_{\Delta} \text{fcom}^\tau_{\text{ind}(K')}^\tau(M; \xi_i \mapsto y.N_i)$. By Lemma A.3, it therefore suffices to show that K1-3 hold when hcom_{\text{ind}(K)} and hcom_{\text{ind}(K')} are replaced with fcom. This is true by Lemma 4.5 and the fact that FCOM?($\iota(K)$) $\subseteq \iota(K)$.

5.4 Coercion

As we turn to coercion, we transition from introduction-like operators (fcom and intro) to elimination-like operators (coe and elim). These are operators which evaluate an argument of type $\text{ind}(K)$ and analyze its form. For each, we will apply the same methodology in order to prove a typing rule. First, we define a $\Psi$-PER $\sigma \subseteq \iota(K)$ which by definition is the largest sub-relation of values on which the operator is well-behaved, where the meaning of “well-behaved” is operator-dependent. Using Lemma A.6, we extend this well-behavedness property to terms in $\text{TM}(\sigma)$. Finally, we show that $\sigma$ supports $K$, which implies that $\sigma = \iota(K)$ and therefore that terms in $\iota(K)$ are well-behaved. This last step generally consists in first proving $\beta$-reduction rules for fcom and intro terms, then using these to prove the main theorem. It is in the $\beta$-reduction proofs that we actually encounter the operational semantics.

The operator $\text{coe}_{\text{ind}(K)}$ is eager, so its behavior is determined by its behavior on values. On fcom terms, this behavior is fairly straightforward: a $\text{coe}_{\text{ind}(K)}$ on an fcom value pushes inside the fcom, becoming an fcom of $\text{coe}_{\text{ind}(K)}$ terms. One might expect the same to be the case for intro terms. Consider, however, our pushout example. In the glue case, this naive operational semantics would have
\[
\text{coe}^\tau_{\Pi(A; B; C; F; G)}(\text{glue}^\tau(P)) \rightarrow \text{glue}^\tau(\text{coe}^\tau_{\Pi(A; B; C; F; G)}(P)) \text{ val}
\]
If we substituted 0 for $x$ before evaluating, we would get
\[
\text{coe}^\tau_{\Pi(A; B; C; F; G)}(\text{glue}^0(P)) \rightarrow \text{coe}^\tau_{\Pi(A; B; C; F; G)}(\text{left(app}(F, P))) \rightarrow \text{left}(\text{coe}^\tau_{\Pi(A; B; C; F; G)}(\text{app}(F(r/z), P))) \text{ val}
\]
If, on the other hand, we substituted 0 for $x$ after evaluating, we would instead get
\[
\text{glue}^0(\text{coe}^{r'}_{z.C}(P)) \mapsto \text{left}(\text{app}(F(r'/z), \text{coe}^{r''}_{z.C}(P))) \text{ val.}
\]

For $\text{coe}^{r''}_{z.C}(\text{II}(A;B;C;F;G))(\text{glue}^x(P))$ to be well-typed, it must give equal results no matter how we interleave dimension substitution and evaluation. Thus, we would need $\text{coe}^{r''}_{z.C}(\text{app}(F(r'/z), P))$ and $\text{app}(F(r'/z), \text{coe}^{r''}_{z.C}(P))$ to be equal in $A$. Unfortunately, this is not generally the case.\(^4\)

Luckily, all maps do commute with $\text{coe}$ up to a path: the $y$-line $\text{coe}^{y, r'}\psi (\text{app}(F(y/z), \text{coe}^{y, r'}\psi (P)))$ connects $\text{app}(F(r'/z), \text{coe}^{y, r'}\psi (P))$ (at $y = r'$) to $\text{coe}^{y, r'}\psi (\text{app}(F(r'/z), P))$ (at $y = r$), shifting the $\text{coe}$ out from inside of $F$. We can therefore fix the behavior of $\text{coe}$ on $\text{glue}$ by wrapping the output glue term in an $\text{fcom}$, whose tube faces slide the $\text{coe}$ outside of the boundary terms at $x = 0$ and $x = 1$:

\[
\text{coe}^{r''}_{z.C}(\text{fcom}^{r'' \rightarrow r}(\text{glue}^x(\text{coe}^{r''}_{z,C}(P))) \mapsto \text{fcom}^{r'' \rightarrow r}(\text{glue}^x(\text{coe}^{r''}_{z,C}(P))); \\
\text{fcom}^{r'' \rightarrow r}(\text{glue}^x(\text{coe}^{r''}_{z,C}(P)));
\]

This is the approach we take for higher constructors in the general case.

In order to coerce the non-recursive arguments of an $\text{intro}$ term, we define a meta-operation $\text{mcoe}$ in Figure 5, which implements coercion for lists of terms inhabiting a dependent context. It is straightforward to derive typing rules for $\text{mcoe}$ from the $\text{coe}$-Kan conditions for $\Gamma$, following the proofs of the $\text{coe}$-Kan conditions for dependent product types in Part III:

**Proposition 5.8.** Let $\Gamma = \Gamma' \text{ctx}_{\text{Kan}} [\Psi]$. For any $\psi : (\Psi', z) \rightarrow \Psi, r, r' \text{ dim } [\Psi']$, and $\overline{M_n} = \overline{M_n} \in \Gamma \psi(r/z) [\Psi]$, we have

\[
\text{MK}4. \quad \text{mcoe}^{r''}_{z,\Gamma}(\overline{M_n}) = \text{mcoe}^{r''}_{z,\Gamma'}(\overline{M_n}) \in \Gamma \psi(r'/z) [\Psi],
\]

\[
\text{MK}5. \quad \text{mcoe}^{r''}_{z,\Gamma}(\overline{M_n}) = \overline{M_n} \in \Gamma \psi(r/z) [\Psi].
\]

For recursive arguments, we need the following proposition, which we can prove following the $\text{coe}$-Kan condition proofs for dependent function types in Part III.

**Proposition 5.9.** Let $B \equiv B' \text{ atype } [\Psi]$. If for terms $A, A'$ and a $\Psi$-PER $\alpha$ we know that $(A, A', \alpha)$ are equally $\text{coe}$-Kan, then $(\{B\}(A), \{B'\}(A'), \{B\}(\alpha))$ are equally $\text{coe}$-Kan.

We will now define the relation $\sigma \subseteq i(\mathcal{K})$ consisting of values on which $\text{coe}_{z,\text{ind}(\mathcal{K})}$ and $\text{coe}_{z,\text{ind}(\mathcal{K}')}^\alpha$ are well-behaved and proceed to show that it contains all of $i(\mathcal{K})$.

**Definition 5.10.** Given a value $\Psi$-relation $\alpha$, define a value $\Psi$-relation $\text{COE}^{-1}(\alpha)$ by saying that, for any $\psi : \Psi' \rightarrow \Psi$, $\text{COE}^{-1}(\alpha)_\psi(V, V')$ holds when $i(\mathcal{K})_\psi(V, V')$ and, for all $\psi' : (\Psi', z) \rightarrow \Psi$ and $r, r' \text{ dim } [\Psi']$ with $\psi'(r/z) = \psi$, we have

4. $\text{TM}(\alpha)(\psi'(r/z), \text{coe}^{r''}_{z,\text{ind}(\mathcal{K})}(W), \text{coe}^{r''}_{z,\text{ind}(\mathcal{K})}(W'))$ for all $W, W' \in \{V, V'\}$, and

5. $\text{TM}(\alpha)(\psi'(r/z), \text{coe}^{r''}_{z,\text{ind}(\mathcal{K})}(W), W)$ for all $W \in \{V, V'\}$.

Define the value $\Psi$-PER $\sigma := \nu(\text{COE}^{-1})$ to be the greatest fixed-point of $\text{COE}^{-1}$.

\(^4\)As an example, take $A := D \rightarrow D$ for some $D$, and $F := \lambda c. \lambda d. d \in C \rightarrow D \rightarrow D$. The term $\text{coe}^{r''}_{z,D}(\text{app}(F(r'/z), P))$ evaluates to $\lambda d. \text{coe}^{r''}_{z,D}(\text{coe}^{r''}_{z,D}(d))$, while $\text{app}(F(r'/z), \text{coe}^{r''}_{z,C}(P))$ evaluates to $\lambda d. d$. To see why these will not be equal in general, take $D$ to be the path type of an inductive type. There, coercions introduce value $\text{fcom}$ terms, so there is no chance that $\text{coe}^{r''}_{z,D}(\text{coe}^{r''}_{z,D}(N))$ will be equal to $N$ in general. One could also take $D$ to be the homotopy fiber type defined in Part IV.B, in which coercions introduce value $\text{fco}$ terms.
To see that \( \sigma \) is \( \text{coe-\text{-}Kan} \), we need to extend the properties that hold of values in \( \text{COE}^{-1}(\alpha) \) by definition to terms in \( \text{TM}(\text{COE}^{-1}(\alpha)) \).

**Lemma 5.11** (Extension to terms). Let \( \alpha \) be a \( \Psi \)-PER. For any \( \psi : (\Psi', z) \to \Psi \) and \( r, r' \) \( \text{dim } [\Psi'] \), if \( \text{TM}(\text{COE}^{-1}(\alpha))_{\psi}(M, M') \), then

4. \( \text{TM}(\alpha)(\psi^{r/z})(\text{coe}_{z,\text{ind}(\Psi')}(M), \text{coe}_{z,\text{ind}(\Psi')}(M')) \) and

5. \( \text{TM}(\alpha)(\psi^{r/z})(\text{coe}_{z,\text{ind}(\Psi')}(M), M) \).

**Proof.** To show that this is true, it suffices to show that for every \( \psi : (\Psi', z) \to \Psi \) and \( r, r' \) \( \text{dim } [\Psi'] \), we have

4. \( \text{a} : \text{TM}(\text{COE}^{-1}(\alpha)(\psi^{r/z})) \supseteq \text{TM}(\alpha)(\psi^{r/z})(\text{coe}_{z,\text{ind}(\Psi')}(\alpha), \text{coe}_{z,\text{ind}(\Psi')}(\alpha)) \) and

5. \( \text{a} : \text{TM}(\text{COE}^{-1}(\alpha)(\psi^{r/z})) \supseteq \text{TM}(\alpha)(\psi^{r/z})(\text{coe}_{z,\text{ind}(\Psi')}(\alpha), \alpha) \).

Since \( \text{coe}_{z,\text{ind}(\Psi)}(-), \text{coe}_{z,\text{ind}(\Psi')}(-) \), and the identity are eager, we can apply Lemma A.6 to reduce these to showing

4. \( \text{a} : \text{COE}^{-1}(\alpha)(\psi^{r/z}) \supseteq \text{TM}(\alpha)(\psi^{r/z})(\text{coe}_{z,\text{ind}(\Psi')}(\alpha), \text{coe}_{z,\text{ind}(\Psi')}(\alpha)) \) and

5. \( \text{a} : \text{COE}^{-1}(\alpha)(\psi^{r/z}) \supseteq \text{TM}(\alpha)(\psi^{r/z})(\text{coe}_{z,\text{ind}(\Psi')}(\alpha), \alpha) \).

These are true by definition of \( \text{COE}^{-1} \).
Corollary 5.12. \( (\text{ind}(K), \text{ind}(K'), \sigma) \) are equally \( \text{coe-Kan} \).

We now prove two \( \beta \)-rules for \( \text{coe}_{z, \text{ind}(K)} \). Here, we are essentially showing that the operational semantics rules for \( \text{coe} \) are coherent with respect to the equality on \( \text{ind}(K) \). This is therefore the meat of the proof. We prove the \( \text{coe-fcom-} \beta \) rule in sufficient generality to later establish Lemma 5.16, which we will use in both the fcom and intro cases of the final theorem.

Lemma 5.13 (coe-fcom-\( \beta \)). Let \( \alpha \) be a \( \Psi \)-PER. For any \( (\Psi', z) : \psi \rightarrow \Psi, r, r' \dim [\Psi'] \), and \( \xi_i \) valid, if

1. \( \text{Tm}(\text{COE}^{-1}(\alpha))_{\psi(r/z)}(M) \),
2. \( \text{Tm}(\text{COE}^{-1}(\alpha))_{\psi(r/z)}(N_i, N'_i) \) for all \( i, j \),
3. \( \text{Tm}(\text{COE}^{-1}(\alpha))_{\psi(r/z)}(N_i(s/y), M) \) for all \( i \),

then, abbreviating \( \text{fcom} := \text{fcom}^{\rightarrow s}((M; \xi_i \mapsto y.N_i)) \), we have

\[
\text{Tm}(\text{Fcom?}((\alpha))_{\psi(r/z)}(\text{coe}_{z, \text{ind}(K')}(\text{fcom}), \text{fcom}^{\rightarrow s}((\text{coe}_{z, \text{ind}(K')}(\text{fcom}), (N_i) : (\xi_i \mapsto y.\text{coe}_{z, \text{ind}(K')}(N_i))))).
\]

Proof. By Lemma A.3. Let \( \psi' : \Psi'' \rightarrow \Psi' \) be given. We have three cases.

1. There exists a least \( i \) such that \( \models \xi_i\psi' \).

Then \( \text{coe}_{z, \text{ind}(K')}(\text{fcom})\psi' \longrightarrow \text{coe}_{z, \text{ind}(K')}(\text{fcom})\psi' \). By Lemma 5.11 and Lemma 4.5(a), the right-hand side is equal to \( \text{fcom}^{\rightarrow s}((\text{coe}_{z, \text{ind}(K')}(\text{fcom}))(\xi_i \mapsto y.\text{coe}_{z, \text{ind}(K')}(N_i))\psi' \). Then \( \text{Tm}(\text{Fcom?}((\alpha))_{\psi(r/z)}(\xi_i \mapsto y.\text{coe}_{z, \text{ind}(K')}(N_i))\psi' \).

2. \( s\psi' = s'\psi' \) and \( \not\models \xi_i\psi' : \Psi'' \rightarrow \Psi' \) for all \( i \).

Then \( \text{coe}_{z, \text{ind}(K')}(\text{fcom})\psi' \rightarrow \text{coe}_{z, \text{ind}(K')}(\text{fcom})\psi' \). By Lemma 5.11 and Lemma 4.5(b), the right-hand side is equal to \( \text{fcom}^{\rightarrow s}((\text{coe}_{z, \text{ind}(K')}(\text{fcom}))(\xi_i \mapsto y.\text{coe}_{z, \text{ind}(K')}(N_i))\psi' \). Then \( \text{Tm}(\text{Fcom?}((\alpha))_{\psi(r/z)}(\xi_i \mapsto y.\text{coe}_{z, \text{ind}(K')}(N_i))\psi' \).

3. \( s\psi' \neq s'\psi' \) and \( \not\models \xi_i\psi' : \Psi'' \rightarrow \Psi' \) for all \( i \).

Then \( \text{coe}_{z, \text{ind}(K')}(\text{fcom})\psi' \rightarrow \text{fcom}^{\rightarrow s}((\text{coe}_{z, \text{ind}(K')}(\text{fcom}))(\xi_i \mapsto y.\text{coe}_{z, \text{ind}(K')}(N_i))\psi' \), and the right-hand side is in \( \text{Tm}(\text{Fcom?}((\alpha))_{\psi(r/z)}(\xi_i \mapsto y.\text{coe}_{z, \text{ind}(K')}(N_i))\psi' \). \( \square \)

For intro terms, \( \text{coe} \) has two separate reduction rules, one for 0-constructors and one for higher constructors.

Lemma 5.14 (coe-intro-\( \beta_0 \)). Let \( \alpha \) be a \( \Psi \)-PER such that \( (\text{ind}(K), \text{ind}(K'), \alpha) \) are \( \text{coe-Kan} \). For all \( \psi : (\Psi', z) \rightarrow \Psi \) and \( r, r' \dim [\Psi'] \), if

1. \( [K_1, \ell : C] \leq K\psi(r/z) \) constrs \( [\Psi'] \),
2. \( \alpha \) supports \( K_1 \),
3. \( K\psi[\ell] = (\Gamma; \gamma, \Theta; \emptyset, \emptyset) \) where \( \Theta = p_j : B_j \),
4. \( \bar{P}_n \in \Gamma(r/z) [\Psi] \),
5. \( \text{Tm}(\text{B}_j(r/z)[\bar{P}_n/\gamma])((\alpha\psi(r/z))(N_j)) \) for all \( j \),
then, abbreviating

\[ K_1^s := (\mathcal{K}_\psi(s/z))_{< \ell} \]

\[ C^s := (\mathcal{K}_\psi(s/z))[\ell] \]

\[ \overrightarrow{P_n} := \text{mcoe}_{\gamma, \Gamma}^r(\overrightarrow{P_n}) \]

\[(\forall j) N_j^s := \text{coe}_{z, [n_j]/\Gamma}^r(\mathcal{N}_j)\]

and

\[ \text{intro} := \text{intro}_{K_1^s, \ell, C^s}(\overrightarrow{P_n}; N_j^s) \]

\[ O := \text{intro}_{K_1^s, \ell, C^s}(\overrightarrow{P_n}; N_j^s) \]

we have \( Tm(\text{INTRO}_{\mathcal{K}_\psi}(\alpha))_{\psi(r/z)}(\text{coe}_{z, \text{ind}(\mathcal{K}_\psi)}^r(\text{intro}), O) \).

**Proof.** First, observe the following:

1. By Proposition 5.8, we have (a) \( \overrightarrow{P_n} \in \Gamma(y/z) \) \([\Psi', y]\) and (b) \( \overrightarrow{P_n} = \overrightarrow{P_n} \in \Gamma(r/z) \) \([\Psi']\).

2. By the assumption that \( (\text{ind}(\mathcal{K}), \text{ind}(\mathcal{K}'), \alpha) \) are equally \( \text{coe}-\text{Kan} \) and Proposition 5.9, we know that

\[ \langle \{ \text{coe}_{\gamma, \mathcal{K}'}^r(\text{intro}) \rangle(\alpha(\psi(y/z)))(N_j^s) \text{ for all } j, \]

\[ \langle \text{coe}_{\gamma, \mathcal{K}}^r(\text{intro})(\alpha(\psi(r/z)))(N_j^s) \text{ for all } j. \]

Supplying 1(a) and 2(a) to Lemma 4.7, we get \( Tm(\text{INTRO}_{\mathcal{K}_\psi}(\alpha))_{\psi(r/z)}(\text{intro}_{K_1^s, \ell, C^s}(\overrightarrow{P_n}; N_j^s)) \). The equality then follows from Lemma A.3. \( \Box \)

**Lemma 5.15 (coe-intro-\( \beta \geq 0 \)).** Let \( \alpha \) be a \( \Psi \)-PER such that \((\iota(\mathcal{K}), \iota(\mathcal{K}'), \alpha) \) are \( \text{coe}-\text{Kan} \). For all \( \psi : (\Psi', z) \rightarrow \Psi \) and \( r, r' \) \( \text{dim} \) \([\Psi']\), if

1. \([\mathcal{K}_1, \ell : \mathcal{C}] \leq \mathcal{K}_\psi(r/z) \) constrs \([\Psi']\),

2. \( \alpha \) supports \( \mathcal{K}_1 \),

3. \( \mathcal{K}_\psi[\ell] = (\Gamma; \gamma, \Theta; x, \xi_k \mapsto \gamma, \theta, \mathcal{M}_k) \) where \( \Theta = \overrightarrow{y}, B_j \),

4. \( \overrightarrow{P_n} \in \Gamma(r/z) \) \([\Psi]\),

5. \( Tm(\{ B_j(r/z)|\overrightarrow{P_n}|(\alpha(\psi(r/z)))(N_j) \text{ for all } j, \]

then, abbreviating

\[ K_1^s := (\mathcal{K}_\psi(s/z))_{< \ell} \]

\[ C^s := (\mathcal{K}_\psi(s/z))[\ell] \]

\[ \overrightarrow{P_n} := \text{mcoe}_{\gamma, \Gamma}^r(\overrightarrow{P_n}) \]

\[(\forall j) N_j^s := \text{coe}_{z, [n_j]/\Gamma}^r(\mathcal{N}_j)\]

\[(\forall k) M_k^y := \text{coe}_{z, \text{ind}(\mathcal{K}_\psi)}^r(\Theta, \mathcal{M}_k, \overrightarrow{r}/\overrightarrow{s})(\overrightarrow{P_n}/\gamma)^{\mathcal{K}(s/z)}(\overrightarrow{N_j^s})\]

and

\[ \text{intro} := \text{intro}_{K_1^s, \ell, C^s}(\overrightarrow{P_n}; N_j^s) \]

\[ O := \text{fcom}^{r'}(\text{intro}_{K_1^s, \ell, C^s}(\overrightarrow{P_n}; N_j^s); \xi_k(\overrightarrow{r}/\overrightarrow{s}) \rightarrow y.M_k^y), \]

we have \( Tm(\text{FCOM}?(\text{INTRO}_{\mathcal{K}_\psi}(\alpha))_{\psi(r/z)}(\text{coe}_{z, \text{ind}(\mathcal{K}_\psi)}^{r'}(\text{intro}), O) \).
Proof. In addition to the equalities from the previous proof, we also know the following.

3. By the above, Lemma 4.8 (using the assumption that \( \alpha \) supports \( K_{<\ell} \)), and the assumption that \((\text{ind}(K), \text{ind}(K'))\) are equally coe-Kan, we have

(a) \( \text{TM}(\alpha\psi(y/z)) \mid \xi, \xi_1(\overline{\psi/(\overline{z})}) \) for all \( k, l \),
(b) \( \text{TM}(\alpha\psi(r'/z)) \mid \xi_1(\overline{\psi/(\overline{z})}) \) for all \( k, l \),
(c) \( \text{TM}(\alpha\psi(r'/z)) \mid \xi_1(\overline{\psi/(\overline{z})}) \) for all \( k, l \),

4. By Lemmas 4.5 and 4.7 and the above, we have

(a) \( \text{TM}(\text{FM}?(\text{INTRO}_{K,K'?}(\alpha\psi(r'/z)))) \mid \xi, \xi_1(\overline{\psi/(\overline{z})}) \) for all \( k, l \),
(b) \( \text{TM}(\text{FM}?(\text{INTRO}_{K,K'?}(\alpha\psi(r'/z)))) \mid \xi, \xi_1(\overline{\psi/(\overline{z})}) \) for all \( k, l \).

Note that we use here the assumption that \( \text{FD}(\overline{\psi/(\overline{z})}) \subseteq \{\overline{\psi/(\overline{z})}\} \). We now proceed by Lemma A.3. Let \( \psi' : \Psi' \rightarrow \Psi' \) be given; we have two cases.

- There exists a least \( k \) such that \( \models \xi_1(\overline{\psi/(\overline{z})}) \psi' \).
  Then \( \text{coe}^{-r'_{\xi_1}}(\text{introduction}) \psi' \mapsto \text{coe}^{-r'_{\xi_1}}(\text{introduction})(\overline{\psi/(\overline{z})}) \psi' \). We apply 4(a).
- There is no such \( k \).
  Then \( \text{coe}^{-r'_{\xi_1}}(\text{introduction}) \psi' \mapsto \text{O} \psi' \). We apply 4(b). \( \square \)

Lemma 5.16. For any \( \Psi \)-PER \( \alpha \), we have \( \text{FM}?(\text{COE}^{-1}(\alpha)) \subseteq \text{COE}^{-1}(\text{FM}?(\alpha)) \).

Proof. Let \( \alpha \) be given, and suppose that \( \text{FM}?(\text{COE}^{-1}(\alpha)) \) for some \( \psi : \Psi' \rightarrow \Psi \). Then \( V = \text{fm}^{-s''} \psi(M; \xi_i \mapsto y.N_i) \) and \( V' = \text{fm}^{-s''} \psi(M'; \xi_i \mapsto y.N_i) \) where \( \models \xi_i \) for all \( i, r \neq r' \), and

1. \( \text{TM}(\text{COE}^{-1}(\alpha)) \psi(M, M') \),
2. \( \text{TM}(\text{COE}^{-1}(\alpha)) \psi ; y; \xi, \xi_1(N_i, N_i') \) for all \( i, j \),
3. \( \text{TM}(\text{COE}^{-1}(\alpha)) \psi ; x; \xi, (N_i(s/y), M) \) for all \( i \).

To show \( \text{COE}^{-1}(\text{FM}?(\alpha)) \subseteq \{V, V'\} \), we need to show that \( \psi''(V, V') \) holds and that for every \( r, r' \) \text{dim} \(|\Psi'|, \psi' : (\Psi', z) \rightarrow \Psi \) with \( \psi'(r/z) = \psi \), and \( W, W' \subseteq \{V, V'\} \), we have

4. \( \text{TM}(\text{FM}?(\alpha)) \psi'(r/z) \text{coe}^{-r'_{\xi_1}}(W) \), \text{coe}^{-r'_{\xi_1}}(W') \) and
5. \( \text{TM}(\text{FM}?(\alpha)) \psi'(r/z) \text{coe}^{-r'_{\xi_1}}(W) \), \text{coe}^{-r'_{\xi_1}}(W') \).

We know that \( \psi''(V, V') \) holds because \( \psi''(V, V') \) supports \( K \). We prove the other two statements as follows.

4. By Lemma 5.13, we have

- \( \text{TM}(\text{FM}?(\alpha)) \psi'(r/z) \text{coe}^{-r'_{\xi_1}}(W) \), \text{coe}^{-r'_{\xi_1}}(M; \xi_i \mapsto y.\text{coe}^{-r'_{\xi_1}}(N_i)) \) and
- \( \text{TM}(\text{FM}?(\alpha)) \psi'(r/z) \text{coe}^{-r'_{\xi_1}}(W') \), \text{coe}^{-r'_{\xi_1}}(M; \xi_i \mapsto y.\text{coe}^{-r'_{\xi_1}}(N_i')) \).

The two right-hand side terms are equal in \( \text{TM}(\text{FM}?(\alpha)) \) by 1-3, Lemma 5.11, and Lemma 4.5.

24
5. Again, we have
\[
\mathrm{TM}(F\mathrm{COM}(\alpha))\psi'(r/z)\left(\text{coe}_{z,\text{ind}(K,\psi')}^{\rightarrow}(W), \text{fcom}\left(\text{coe}_{z,\text{ind}(K,\psi')}^{\rightarrow}(M); \xi \mapsto y.\text{coe}_{z,\text{ind}(K,\psi')}^{\rightarrow}(N))\right)
\]
by Lemma 5.13, and the right-hand side is equal to \(W\) in \(\mathrm{TM}(F\mathrm{COM}(\alpha))\) by 1-3, Lemma 5.11, and Lemma 4.5.

**Theorem 5.17.** \(\sigma\) supports \(K\).

**Proof.** We prove that \(\sigma\) supports every prefix \(K_1 \subseteq K\) by induction on the form of \(K_1\).

1. \(K_1 = \bullet\).

Then we have to show \(F\mathrm{COM}(\sigma) \subseteq \sigma\). We prove the equivalent statement that \(F\mathrm{COM}(\sigma) \subseteq \sigma\). By definition of \(\sigma\), it suffices to show that \(F\mathrm{COM}(\sigma)\) is a post-fixed-point of \(\mathrm{COE}^{-1}\), i.e., that \(F\mathrm{COM}(\sigma) \subseteq \mathrm{COE}^{-1}(F\mathrm{COM}(\sigma))\). As \(\sigma = \mathrm{COE}^{-1}(\sigma)\), this follows from Lemma 5.16.

2. \(K_1 = [K_2, \ell : \ldots]\).

Then we have to show that \(F_{K_2}(\sigma) \cup \mathrm{INTRO}_{K,\ell}(\sigma) \subseteq \sigma\). By induction hypothesis we know \(F_{K_2}(\sigma) \subseteq \sigma\), so it remains to show \(\mathrm{INTRO}_{K,\ell}(\sigma) \subseteq \sigma\). We prove the stronger statement that \(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)) \subseteq \sigma\).

By the universal property of \(\sigma\), it suffices to show \(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)) \subseteq \mathrm{COE}^{-1}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)))\).

(Here, the addition of \(\mathrm{Fcom}^*\) has given us leeway to fit a new \(\mathrm{fcom}\) in the output.) By the universal property of \(\mathrm{Fcom}^*\) and definition of \(\mathrm{INTRO}_{K,\ell}\), it is then enough to show that

\[(a) \ \sigma \subseteq \mathrm{COE}^{-1}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma))),\]
\[(b) \ \mathrm{INTRO}_{K,\ell}(\sigma) \subseteq \mathrm{COE}^{-1}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma))),\]
\[(c) \ F\mathrm{COM}(\mathrm{COE}^{-1}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)))) \subseteq \mathrm{COE}^{-1}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma))).\]

We prove these in turn.

(a) This holds because \(\sigma = \mathrm{COE}^{-1}(\sigma) \subseteq \mathrm{COE}^{-1}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)))\).

(b) Suppose \(\mathrm{INTRO}_{K,\ell}(\sigma)\psi(V, V')\) holds for some \(\psi\). To show \(\mathrm{COE}^{-1}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)))\psi(V, V')\), we need to show that \(\imath(K)\psi(V, V')\) holds and that for every \(r, r'\) \(\dim[\Psi], \psi' : (\Psi', z) \to \Psi\) with \(\psi'(r/z) = \psi\), and \(W, W' \in \{V, V'\}\), we have

\[4. \ \mathrm{TM}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)))\psi'(r/z)\left(\text{coe}_{z,\text{ind}(K,\psi')}^{\rightarrow}(W), \text{fcom}\left(\text{coe}_{z,\text{ind}(K,\psi')}^{\rightarrow}(W'); \xi \mapsto y.\text{coe}_{z,\text{ind}(K,\psi')}^{\rightarrow}(W')\right)\right)\]
\[5. \ \mathrm{TM}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)))\psi'(r/z)\left(\text{coe}_{z,\text{ind}(K,\psi')}^{\rightarrow}(W), W'\right)\]

We know that \(\imath(K)\psi(V, V')\) holds because \(\sigma \subseteq \imath(K)\) and \(\imath(K)\) supports \(K\). We prove the other two statements as follows.

4. By either Lemma 5.14 or Lemma 5.15, we have that

- \(\mathrm{TM}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)))\psi'(r/z)\left(\text{coe}_{z,\text{ind}(K,\psi')}^{\rightarrow}(W), O\right)\),

- \(\mathrm{TM}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)))\psi'(r'/z)\left(\text{coe}_{z,\text{ind}(K,\psi')}^{\rightarrow}(W'), O'\right)\)

hold, where \(O\) and \(O'\) are as defined in the appropriate lemma. The right-hand sides of these equations are themselves equal in \(\mathrm{TM}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)))\) by Corollary 5.12 and Lemmas 4.5 and 4.7, using the induction hypothesis that \(\sigma\) supports \(K_2\) for Lemma 4.7.

5. Again, we have \(\mathrm{TM}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)))\psi'(r/z)\left(\text{coe}_{z,\text{ind}(K,\psi')}^{\rightarrow}(W), O\right)\) (this time with \(r'\) replaced with \(r\) in \(O\)), and the right-hand side is equal to \(W\) in \(\mathrm{TM}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)))\) by Corollary 5.12 and Lemmas 4.5 and 4.7.

(c) We have \(F\mathrm{COM}(\mathrm{COE}^{-1}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma)))) \subseteq \mathrm{COE}^{-1}(F\mathrm{COM}(F\mathrm{COM}^*(\mathrm{INTRO}_{K,\ell}(\sigma))))\) by Lemma 5.16, and the result follows because \(F\mathrm{COM} \circ F\mathrm{COM}^* = F\mathrm{COM}^*\).

**Corollary 5.18 (C).** \(\text{ind}(K) \cong \text{ind}(K')\) type_{\text{pre}} \(\Psi\) are equally coe-Kan.

**Proof.** By Theorem 5.17, we have \(\sigma = \imath(K)\). The result follows from Corollary 5.12.
5.5 Elimination

For elimination, we separate our presentation into two parts. First, we specify the data which is provided to the eliminator for \( \text{ind}(K) \). Second, we prove the typing rules for said eliminator.

5.5.1 Elimination data

**Definition 5.19.** The grammar of elimination lists is given by

\[
\mathcal{E} ::= \cdot | [\mathcal{E}, \ell : \mathcal{E}_1, \gamma, \delta, \rho, R] \quad (\text{where } |\delta| = |\rho|)
\]

We say that an elimination list \( \mathcal{E} \) matches a constructor list \( K \) if either \( K = \cdot \) and \( \mathcal{E} = \cdot \), or \( K = [K_1, \ell : (\Gamma; \gamma, \Theta; \xi \mapsto \gamma, \delta, \rho, R)] \) and \( \mathcal{E} = [\mathcal{E}_1, \ell : \mathcal{E}_2, \gamma, \delta, \rho, R] \) where \( K_1 \) matches \( \mathcal{E}_1 \) and \( |\theta| = |\delta| = |\rho| \). As with constructor lists, we write \( \mathcal{E}[\ell] \) for the entry at label \( \ell \), \( \mathcal{E}_{< \ell} \) for the prefix preceding \( \ell \), and \( \mathcal{E} \subseteq \mathcal{E}' \) to mean that \( \mathcal{E} \) is a prefix of \( \mathcal{E}' \).

A clause \( \ell : (x : X, \gamma, \delta, \rho, R) \) in an elimination list specifies the behavior of the eliminator on intro terms labelled \( \ell \). The inputs \( x, \gamma, \) and \( \delta \) supply the dimension parameters, non-recursive arguments, and recursive arguments respectively. The input \( \rho \) supplies the results of the recursive calls, which are executed on the terms \( \delta \). Recall that each recursive argument is of type \( \{B\}(\text{ind}(K)) \) for some \( B \). We will therefore apply the dependent functorial action of \( B \) on the eliminator to each recursive argument. To state the output type of this functional action, which is implemented in the operational semantics by the operator \( \text{func} \), we will need a dependent analogue of \( \{B\}(\cdot) \).

**Definition 5.20** (Dependent argument type interpretation). Let an argument type \( B \), a value \( \Psi \)-relation \( \alpha \), and a value \( \Psi \)-relation \( \beta \) over \( \text{Tm}(\alpha) \) be given. We define a value \( \Psi \)-relation \( \{B\}_a(\beta) \) over \( \{B\}(\alpha) \) by recursion on the structure of \( B \):

\[
\begin{align*}
\{X\}_d(\beta)_\Psi[N] &:= \beta_N[N] \\
\{(b : B) \to c\}_d(\beta)_\Psi[N] &:= \Pi(\{B\}_\Psi, (\psi', M) \mapsto \{c\}_d(\psi'(M/b))_\Psi \cdot \text{app}(N, M))
\end{align*}
\]

For any terms \( A \text{ tm } [\Psi], a \vdash D \text{ tm } [\Psi], \) and \( N \text{ tm } [\Psi], \) we define a term \( \{B\}_a(a.B; N) \) by

\[
\begin{align*}
\{X\}_d(a.A; N) &:= D[N/a] \\
\{(b : B) \to c\}_d(a.A; N) &:= (b : B) \to \{c\}_d(a.A; \text{app}(N, b))
\end{align*}
\]

**Proposition 5.21.** If

1. \( B \equiv B' \) atype \( [\Psi] \),
2. \( A \text{ type}_\text{Kan } [\Psi], \) and
3. \( h : A \to \to D \equiv D' \text{ type}_\text{Kan } [\Psi], \)

then

- 1. \( b : \{B\}(A) \Rightarrow \{B\}_a(h.D; b) \equiv \{B'\}_a(h.D'; b) \) type\( \text{Kan } [\Psi], \) and
- 2. \( [b, \{B\}_a(h.D; b)] \equiv \{B\}_a([h.D]) \) as \( \Psi \)-relations over \( \text{Tm}(\{B\}(\Psi)) \), where we extend \( [\cdot] \) to dependent types in the natural way.

In order for an elimination list to be well-typed, we will want its cases to satisfy coherence conditions: the \( \xi_k \) boundary of the case for a term \( \text{intro}^{\mathcal{E}_1, \ell \mathcal{C}}_{K_1, \xi_k}([P_n; N_f]) \) should line up with the case for the term’s boundary. For example, we expect the data required to eliminate from the pushout \( \Pi(A; B; C; F; G) \) into a motive \( h.D \) to consist of

1. \( a : A \to R_{\text{left}} \in D[\text{left}(a)/h] [\Psi], \)
|   |   |
|---|---|
| $M \mapsto M'$ | $\text{elim}_{h,D}(M; \mathcal{E}) \mapsto \text{elim}_{h,D}(M'; \mathcal{E})$ |
| $(\forall i) \not\ni r_i \quad r \neq r'$ | $\text{com}^{\to \gamma', \gamma} \left( (\theta; M; \xi_1 \mapsto y.N_i); \mathcal{E} \right)$ |

$$C = (\Gamma; \gamma.\Theta; \xi_k \mapsto \gamma.\theta.M) \quad \mathcal{E}[\ell] = \bar{x_i} : \gamma.\delta.\rho.R \quad \Theta = p_j : B_j \quad (\forall k) \not\ni \xi_k(\bar{r}_i / \bar{x_i})$$

$$\text{elim}_{h,D}(\text{intro}_{K,\mathcal{E}}^\gamma M; \mathcal{E}) \mapsto R(\bar{r}_i / \bar{x_i})[P_n / \gamma][N_j / \delta][\text{func}_{h,D}(\bar{r}_i / \bar{x_i})]^\gamma [h.\text{elim}_{h,D}(h; \mathcal{E}); N_j] / \rho]$$

Figure 6: Operational semantics of \text{elim}

2. $b : B \Rightarrow R_{\text{right}} \in D[\text{right}(b)/h] [\Psi]$,  
   3. $c : C \Rightarrow R_{\text{glue}} \in D[\text{glue}^c(c)/h] [\Psi, x]$, such that  
      (a) $c : C \Rightarrow R_{\text{glue}}(0/x) \equiv R_{\text{left}}[\text{app}(F, c)/a] \in D[\text{glue}^0(c)/h] [\Psi]$,  
      (b) $c : C \Rightarrow R_{\text{glue}}(1/x) \equiv R_{\text{right}}[\text{app}(G, c)/a] \in D[\text{glue}^1(c)/h] [\Psi]$,  

   The boundary conditions on $R_{\text{glue}}$ ensure that the eliminator evaluates coherently with respect to the equality on $h.D$. Likewise, the eliminator for $\| A \|$ should require  

1. $a : A \Rightarrow R_{\text{pt}} \in D[\text{pt}(a)/h] [\Psi]$,  
   2. $t_0 : \| A \|, t_1 : \| A \|, r_0 : D[t_0/h], r_1 : D[t_1/h] \Rightarrow R_{\text{path}} \in D[\text{path}^r(t_0, t_1)/h] [\Psi, x]$ such that  
      (a) $t_0 : \| A \|, t_1 : \| A \|, r_0 : D[t_0/h], r_1 : D[t_1/h] \Rightarrow R_{\text{path}}(0/x) \equiv r_0 \in D[\text{path}^0(t_0, t_1)/h] [\Psi]$,  
      (b) $t_0 : \| A \|, t_1 : \| A \|, r_0 : D[t_0/h], r_1 : D[t_1/h] \Rightarrow R_{\text{path}}(1/x) \equiv r_1 \in D[\text{path}^1(t_0, t_1)/h] [\Psi]$,  

   To formulate these conditions in general, we will also need a dependent analogue of the interpretation of boundary terms.

**Definition 5.22** (Dependent boundary interpretation). Let a constructor list $K$, matching elimination list $\mathcal{E}$, term $h \vdash D \text{ tm } [\Psi]$, and terms $N_j$ and $S_j$ be given. For any $\theta \vdash M \text{ bnd } [\Psi]$ with $\text{FB}(M) \subseteq \theta, L(M) \subseteq \text{dom} \mathcal{E}$ and $|\theta| = |N_j| = |S_j|$, we define a term $(\theta; M)^{K,\mathcal{E}}_{h,D}(N_j; S_j)$ by

$$(\theta; \text{intro}^\gamma_{K,\mathcal{E}}(P_n; \bar{N}_j))^{K,\mathcal{E}}_{h,D}(N_j; S_j) := S_j$$

$$(\theta; \text{app}(N, M))^{K,\mathcal{E}}_{h,D}(N_j; S_j) := \text{app}(\theta; N)^{K,\mathcal{E}}_{h,D}(N_j; S_j), M)$$

$$(\theta; \text{com}^{\to \gamma', \gamma} \left( (\theta; M; \xi_1 \mapsto y.N_i); \mathcal{E} \right)^{K,\mathcal{E}}_{h,D}(N_j; S_j) := \text{com}^{\to \gamma', \gamma} \left( (\theta; M; \xi_1 \mapsto y.N_i); \mathcal{E} \right)^{K,\mathcal{E}}_{h,D}(N_j; S_j)$$

We will establish a typing rule for dependent boundary interpretation in Lemma 5.25 after defining a typing judgment for elimination lists.

**Proposition 5.23** (Basic facts on boundary interpretation).
1. \( \Theta.\mathsf{M} \mid^{K,E} \mathcal{N}_j \mid \mathcal{S}_j \mid \mathcal{P}\mathcal{A} = \Theta.\mathsf{M} \mid^{K[E/P[A]]} \mathcal{N}_j \mid \mathcal{S}_j \mid \mathcal{P}\mathcal{A} \).

2. If \( \Theta, p \vdash \mathcal{M} \mathcal{A} \mid \mathcal{P} \mathcal{A} \mathcal{B} \) and \( \Theta \vdash \mathcal{N} \mathcal{A} \mathcal{B} \), then
   \( \Theta.\mathsf{M} \mid^{K,E} \mathcal{N}_j \mid \mathcal{S}_j \mid \mathcal{P}\mathcal{A} = \Theta.\mathsf{M} \mid^{K[E/P[A]]} \mathcal{N}_j \mid \mathcal{S}_j \mid \mathcal{P}\mathcal{A} \).

3. If \( \mathcal{K} \subseteq \mathcal{K}' \) and \( \mathcal{E} \subseteq \mathcal{E}' \), then
   \( \Theta.\mathsf{M} \mid^{K,E} \mathcal{N}_j \mid \mathcal{S}_j \mid \mathcal{P}\mathcal{A} = \Theta.\mathsf{M} \mid^{K[E/E'] \mathcal{N}_j \mid \mathcal{S}_j \mid \mathcal{P}\mathcal{A} \).
where $\mathcal{K}[\ell] = (\Gamma; \gamma. \Phi; x_i: \xi_k \mapsto \gamma. \varphi. M_k)$, $\Phi = q_i : C_i$, $E[\ell] = x_i. \gamma. \delta. p. R$, and $E'[\ell] = x_i. \gamma. \delta. p. R'$. We know that $\gamma : \Gamma, \delta : \{C_i\}(\text{ind}(\mathcal{K})), \rho : \{C_i\}_d(h, D; \delta_i) \Rightarrow R \Rightarrow R' \in D[\mathcal{R}^\mathcal{E}_{K < \ell, E}[\mathcal{K}][\ell](\gamma; \delta)/h] [\Psi, x_i]$ by $E \equiv E' : \mathcal{K} \to h, D [\Psi]$. We also have:

- $P_n = P'_n \in \Gamma [\Psi]$ from the premises of ($\text{intro}_E$-I),
- $\{\theta. N_l\}_h[K]^{\ell} (\overrightarrow{N}_j) \equiv \{\theta. N_l'\}_h[K]^{\ell} (\overrightarrow{N}_j) \in \{C_i\}[P_n/\gamma] [\text{ind}(\mathcal{K})]$ for each $l$ by the premises of ($\text{intro}_E$-I) and Lemma 4.8,
- $\{\theta. N_l\}_h[D, \gamma, \delta, S_j] \equiv \{\theta. N_l'\}_h[D, \gamma, \delta, S_j] \in \{C_i\}[P_n/\gamma] [d(h, D; \delta_i)] [\Psi, \overrightarrow{N}_j]$ for each $l$ by induction hypothesis.

Plugging $\overrightarrow{r}_i$ and these equations into $R, R'$, we get the equality of $\{\theta. M\}_h[D, \gamma, \delta, S_j]$ and $\{\theta. M'\}_h[D, \gamma, \delta, S_j]$ in $D[\mathcal{R}^\mathcal{E}_{K < \ell, E}[\mathcal{K}][\ell](P_n; \theta. N_l)[\mathcal{K}](\overrightarrow{N}_j)/h]$.

($\text{intro}_E$-B) Then $M = \mathcal{R}^\mathcal{E}_{K < \ell, E}[\mathcal{K}][\ell](\overrightarrow{r}_i; \overrightarrow{N}_j)$, $M' = M_k(\overrightarrow{r}_i/\overrightarrow{x}_i)[P_n/\gamma] [\overrightarrow{N}_j/\varphi]$, and $B = X$, so

- $\{\theta. M\}_h[D, \gamma, \delta, \overrightarrow{S}_j] = R(\overrightarrow{r}_i/\overrightarrow{x}_i)[P_n/\gamma] [\theta. N_l][\mathcal{K}](\overrightarrow{N}_j) / \delta][\theta. N_l'][\mathcal{K}](\overrightarrow{N}_j) / \rho],$
- $\{\theta. M'\}_h[D, \gamma, \delta, \overrightarrow{S}_j] = \{\theta. M_k(\overrightarrow{r}_i/\overrightarrow{x}_i)[P_n/\gamma] [\overrightarrow{N}_j/\varphi][\theta. N_l][\mathcal{K}](\overrightarrow{N}_j) / \rho],$
- $\{\theta. N_l\}_h[D, \gamma, \delta, \overrightarrow{S}_j] = \mathcal{R}^\mathcal{E}_{K < \ell, E}[\mathcal{K}][\ell](\overrightarrow{r}_i; \overrightarrow{N}_j)$, which by iterated application of Proposition 5.23(b) is equal to $\{\overrightarrow{\varphi. M_k(\overrightarrow{r}_i/\overrightarrow{x}_i)[P_n/\gamma] [\overrightarrow{N}_j/\varphi][\theta. N_l][\mathcal{K}](\overrightarrow{N}_j) / \rho],$
- $\{\theta. N_l\}_h[D, \gamma, \delta, \overrightarrow{S}_j] = \mathcal{R}^\mathcal{E}_{K < \ell, E}[\mathcal{K}][\ell](\overrightarrow{N}_j) / h][\Psi, \overrightarrow{r}_i]$, and $\xi_k$.

By induction hypothesis, we know that $\{\overrightarrow{\varphi. M_k(\overrightarrow{r}_i/\overrightarrow{x}_i)[P_n/\gamma] [\overrightarrow{N}_j/\varphi][\theta. N_l][\mathcal{K}](\overrightarrow{N}_j) / \rho],$ is equal at this type to $\{\overrightarrow{\varphi. M_k}_h[D, \gamma, \delta, \overrightarrow{S}_j]$, which is the same as $\{\overrightarrow{\varphi. M_k}_h[D, \gamma, \delta, \overrightarrow{S}_j]$ by Proposition 5.23. We also have:

- $P_n = P'_n \in \Gamma [\Psi]$ from the premises of ($\text{intro}_E$-B),
- $\{\theta. N_l\}_h[K]^{\ell} (\overrightarrow{N}_j) \equiv \{\theta. N_l'\}_h[K]^{\ell} (\overrightarrow{N}_j) \in \{C_i\}[P_n/\gamma] [\text{ind}(\mathcal{K})]$ for each $l$ by the premises of ($\text{intro}_E$-B) and Lemma 4.8,
- $\{\theta. N_l\}_h[D, \gamma, \delta, S_j] \equiv \{\theta. N_l'\}_h[D, \gamma, \delta, S_j] \in \{C_i\}[P_n/\gamma] [d(h, D; \delta_i)] [\Psi, \overrightarrow{N}_j]$ for each $l$ by induction hypothesis.

Plugging these into the above equation, we get the equality of $\{\theta. M\}_h[D, \gamma, \delta, S_j]$ and $\{\theta. M'\}_h[D, \gamma, \delta, S_j]$ in $D[\mathcal{R}^\mathcal{E}_{K < \ell, E}[\mathcal{K}][\ell](P_n; \theta. N_l)[\mathcal{K}](\overrightarrow{N}_j)/h]$.

5.5.2 Elimination rules

We will now embark on the proofs of the elimination rules. For the remainder of this section, we assume a fixed $h : \text{ind}(\mathcal{K}) \Rightarrow D \Rightarrow \text{D}$ type $\text{Functorial Action} \text{func}$, which is used to call elin on the recursive arguments of an intro term.

**Lemma 5.26 (Dependent functorial action).** If

1. $B \equiv B'$ atype $[\Psi]$,
2. $\alpha$ is a $\Psi$-$\text{PER}$ and $\beta$ is a $\Psi$-$\text{PER}$ over $\text{Tm}(\alpha)$,
3. \( a : \text{Tm}(\alpha) \supseteq \text{Tm}(\beta)[a](R,R') \),

4. \( \text{Tm}(\text{[B]}(\alpha))(N,N') \),

then \( \text{Tm}(\text{[B]}(\beta))(N,N') \),

**Proof.** By induction on the derivation of \( B \equiv B' \) atype \( \Psi \). The operational semantics rules for \( \text{func} \) are all stable, so the proof is completely routine. \[ \square \]

As with \( \text{coe} \), we will prove the elimination typing rule by first defining a subrelation \( \sigma \subseteq i(K) \) on which the eliminator is well-behaved, then showing that the eliminator satisfies \( \text{fcom}-\beta \) and \( \text{intro}-\beta \) rules, then using these \( \beta \)-rules to show that \( \sigma \) is closed under \( \text{FCOM} \) and \( \text{INTRO}_{\ell} \). The \( \text{elim} \) proof is in some ways conceptually simpler than the \( \text{coe} \) proof, because we are no longer mapping back into the inductive type.

**Definition 5.27.** We define a value \( \Psi-\text{PER} \sigma \subseteq i(K) \) by

\[
\sigma_{\psi}(V,V') := i(K)_{\psi}(V,V') \land \forall W,W' \in \{V,V'\}. (\text{elim}_{h.D\psi}(W;\psi) \equiv \text{elim}_{h.D'\psi}(W';\psi') \in D\psi[V/h] [\Psi']).
\]

**Lemma 5.28** (Extension to terms). \( h : \text{Tm}(\sigma) \supseteq \text{Tm}([D])(\text{elim}_{h.D}(h;\psi),\text{elim}_{h.D'}(h;\psi')). \)

**Proof.** By Lemma A.6, as the eliminator is eager and \( \text{ind}(K) \) is value-coherent. \[ \square \]

**Lemma 5.29** (\( \text{fcom}-\beta \)). For any \( \psi : \Psi' \rightarrow \Psi \), if

1. \( \text{Tm}(\sigma)_{\psi}(M) \),

2. \( \text{Tm}(\sigma)_{\psi|x_i,x_j}(N_i,N_j) \) for all \( i,j \),

3. \( \text{Tm}(\sigma)_{\psi|x_i}(M,N_i) \) for all \( i \),

then, abbreviating \( \text{fcom}^* := \text{fcom}^{\rightarrow-s}(M;\xi_i \leftarrow y.N_i) \), we have

\[
\text{elim}_{h.D\psi} (\text{fcom}^{\rightarrow-s};\psi) \equiv \text{com}^{\rightarrow-s'h} (\text{elim}_{h.D}(M;\psi);\xi_i \leftarrow y.\text{elim}_{h.D}(N_i;\psi')) \in D\psi[\text{fcom}^{\rightarrow-s}/h] [\Psi'].
\]

**Proof.** This proof is identical in structure to that of Lemma 5.13 except for the replacement of \( \text{fcom} \) with \( \text{com} \) in the output, so we leave it to the reader. \[ \square \]

As with the interleaved proofs of Lemmas 4.7 and 4.8, we will extract a sub-lemma of the \( \text{intro}-\beta \) rule establishing a property of the term interpretation functions. In this case, the property is a sort of \( \beta \)-rule for the eliminator applied to boundary terms.

**Definition 5.30.** We say that the property \( \text{ElimBnd}(n) \) holds for some \( n \in \mathbb{N} \) if for every \( \psi : \Psi' \rightarrow \Psi \), if

1. \( K_1 \subseteq K \psi \) constrs \( [\Psi] \) with \( |K_1| = n \),

2. \( K_1;\theta : \Theta \vdash M : B \ [\Psi] \) where \( \Theta = \overline{p_j : B_j} \),

3. \( \text{Tm}(\text{[B]}_j)(\sigma)_{\psi}(N_j) \) for each \( j \),

then

\[
\text{func}_\theta (h.\text{elim}_{h.D\psi}(h;\psi);(\theta.M)|_{K \psi}(N_j)) = (\theta.M)|_{K \psi,h.D\psi} (N_j;\text{func}_\theta (h.\text{elim}_{h.D\psi}(h;\psi);N_j))
\]

in \( \text{[B]}_d(h.D\psi;\theta.M)|_{K \psi}(N_j) \) at \( \Psi' \).

**Lemma 5.31** (\( \text{intro}-\beta \)). Let \( \ell \in K \) and suppose that \( \text{ElimBnd}([K_{<\ell}]) \) holds. For any \( \psi : \Psi' \rightarrow \Psi \), if

1. \( K \psi[\ell] = (\Gamma;\gamma:\Theta;x_k;\xi_k \leftarrow \gamma.\theta.M_k) \) where \( \Theta = \overline{p_j : B_j} \),

then

\[
\text{func}_\theta (h.\text{elim}_{h.D\psi}(h;\psi);(\theta.M)|_{K \psi}(N_j)) = (\theta.M)|_{K \psi,h.D\psi} (N_j;\text{func}_\theta (h.\text{elim}_{h.D\psi}(h;\psi);N_j))
\]

in \( \text{[B]}_d(h.D\psi;\theta.M)|_{K \psi}(N_j) \) at \( \Psi' \).
2. \((K\psi)_{<\ell} \equiv K_1\) constrs \([\Psi']\),

3. \((K\psi)_{<\ell} \vdash K_1[\ell] \equiv C\) constrs \([\Psi']\),

4. \(P_n = \overline{P_n} \in \Gamma\) \([\Psi']\),

5. \([B_j]|P_n/\gamma\rangle (\alpha\psi) (N_j, N_j')\) for all \(j\).

then, abbreviating

\[
\text{intro} := \text{intro}_{K_1,E;C}(P_n; N_j)
\]

\[
O := R(\overline{r_i}/\overline{x_i})[P_n/\gamma][N_j/\delta][\overline{\text{func}}_{B_j}[P_n/\gamma]_{E,D}(h,\text{elim}_{h,D}(h;\psi); N_j)/\rho],
\]

we have \(\text{elim}_{h,D}(\text{intro}; \varepsilon\psi) \equiv O \in D\psi[\text{intro}/h] \,[\Psi']\).

Proof. By Lemma A.3. Let \(\psi' : \Psi'' \rightarrow \Psi'\) be given. We have two cases.

- There exists a least \(k\) such that \(\vdash \xi_k(\overline{r_i}/\overline{x_i})\psi'\).
  
  Then \(\text{elim}_{h,D}(\text{intro}; \varepsilon\psi) \psi' \rightarrow \text{elim}_{h,D}(\{\theta,M_k(\overline{r_i}/\overline{x_i})[P_n/\gamma]\}^{K_1}(N_j); \varepsilon\psi) \psi'\).

  By ElimBnd \((K_{<\ell})\), we have

  \[
  \text{elim}_{h,D}(\{\theta,M_k(\overline{r_i}/\overline{x_i})[P_n/\gamma]\}^{K_1}(N_j); \varepsilon\psi) \psi' \\
  \equiv \{\theta,M_k(\overline{r_i}/\overline{x_i})[P_n/\gamma]\} \text{func}_{B_j}[P_n/\gamma]_{E,D}(h,\text{elim}_{h,D}(h;\psi); N_j)\psi'
  \]

  in a type which is equal to \(D\psi[\text{intro}/h] \psi'\) by Lemma 4.7(a). The right-hand side of this equation is equal to \(O\psi' \in D\psi[\text{intro}/h] \psi'\) by the assumptions on \(R\) in \(E : K \rightarrow h,D \, [\Psi']\) and Lemmas 5.26 and 5.28.

- \(\nabla \xi_k(\overline{r_i}/\overline{x_i})\psi'\) for all \(k\).

  Then \(\text{elim}_{h,D}(\text{intro}; \varepsilon\psi) \psi' \rightarrow O\psi'\), and the reduct is in \(D\psi[\text{intro}/h] \psi'\) by the assumptions on \(R\) in \(E : K \rightarrow h,D \, [\Psi']\) and Lemmas 5.26 and 5.28.

\[\square\]

Lemma 5.32. ElimBnd \((n)\) holds for all \(n \in \mathbb{N}\).

Proof. By induction on \(n\) and the derivation of \(K_1\psi; \theta : \Theta \vdash M : B \, [\Psi']\). Assume ElimBnd \((m)\) holds for all \(m < n\). We will prove a few representative cases. We abbreviate \(S_j := \text{func}_{B_j}(h,\text{elim}_{h,D}(h;\psi); N_j)\) to save space.

(Hyp) Then \(M = \theta_j\) and \(B = B_j\) for some \(j\), so

1. \(\theta,M\) \(\psi_1(N_j) = N_j\),

2. \(\theta,M\) \(\psi_1,\psi_2(N_j; S_j) = \text{func}_{B_j}(h,\text{elim}_{h,D}(h;\psi); N_j)\),

and we want to show \(\text{func}_{B_j}(h,\text{elim}_{h,D}(h;\psi); N_j) = \text{func}_{B_j}(h,\text{elim}_{h,D}(h;\psi); N_j) \in \{B_j\}_d(h,D; N_j) \,[\Psi']\).

This follows from Lemmas 5.26 and 5.28 and the assumption \(\text{TM}(\{B_j\}((\psi_1))(N_j)\).

(intro-1) Then \(M = \text{intro}_{\ell}(P_n; N_1)\) and \(B = X\), so

- \(\theta,M\) \(\psi_1(N_j) = \text{intro}_{(K_{<\ell}; \psi_1)}(P_n; \theta_1; N_1)\)

- \(\theta,M\) \(\psi_1,\psi_2(N_j; S_j) = R(\overline{r_i}/\overline{x_i})[P_n/\gamma][\theta_1; N_1][\psi_1(N_j)/\delta][\theta_1; N_1][\psi_2(N_j)/\rho][\psi_2(N_j)/\rho] \}

31
where \( \ell \in K_1 \), \( K_\psi[\ell] = (\Gamma; \gamma; \Phi; \overline{x_i}; \xi_k \mapsto \gamma; \varphi; M_k) \) with \( \Phi = q_l : C_l \), and \( \xi_\psi[\ell] = \overline{x_i} \cdot \gamma; \delta; \rho; R \). In this case, we want to show that

\[
\text{func}_\chi(h, \text{elim}_{h, D_\psi}(h; E_\psi); \text{intro}^{\overline{n}}_{(K_\psi) \langle \ell, \ell \rangle, K, \xi_\psi[\ell]}(\overline{P_\ell}; \langle \theta, N_j \rangle^{K_\psi} (N_j))) = \quad
\]

\[
R(\overline{\tau_i} / \overline{x_i})[\overline{P_\ell} / \gamma] [[\theta, N_j]^{K_\psi} (N_j)] / \delta [[\theta, N_j]^{K_\psi, E_\psi} (N_j); S_j] / \rho]
\]

in \( D_\psi[\text{intro}^{\overline{n}}_{(K_\psi) \langle \ell, \ell \rangle, K, \xi_\psi[\ell]}(\overline{P_\ell}; \langle \theta, N_j \rangle^{K_\psi} (N_j))/h] \) at \( \Psi' \). Since the operational semantics of \( \text{func}_\chi \) is stable, we can reduce the term on the left-hand side to \( \text{elim}_{h, D_\psi}(\text{intro}^{\overline{n}}_{(K_\psi) \langle \ell, \ell \rangle, K, \xi_\psi[\ell]}(\overline{P_\ell}; \langle \theta, N_j \rangle^{K_\psi} (N_j)); E_\psi) \) by Lemma A.3. By Lemma 5.31 (as \( |K_\ell| < n \)), this is in turn equal to

\[
R(\overline{\tau_i} / \overline{x_i})[\overline{P_\ell} / \gamma] [[\theta, N_j]^{K_\psi} (N_j)] / \delta [\text{func}_{c_l}(\overline{P_\ell} / \gamma)(h, \text{elim}_{h, D_\psi}(h; E_\psi); \langle \theta, N_j \rangle^{K_\psi} (N_j)) / \rho] \quad (*)
\]

in \( D_\psi[\text{intro}^{\overline{n}}_{(K_\psi) \langle \ell, \ell \rangle, K, \xi_\psi[\ell]}(\overline{P_\ell}; \langle \theta, N_j \rangle^{K_\psi} (N_j))/h] \) at \( \Psi' \). Finally, we can apply the inner induction hypothesis to each \( K_\psi; \varnothing \vdash N_l : C_l[\overline{P_\ell} / \gamma] [\Psi'] \) to get

\[
\text{func}_{c_l}(\overline{P_\ell} / \gamma)(h, \text{elim}_{h, D_\psi}(h; E_\psi); \langle \theta, N_j \rangle^{K_\psi} (N_j)) \cong \langle \theta, N_j \rangle^{K_\psi, E_\psi} (N_j; S_j)
\]

in \( \langle [\overline{c_l}(\overline{P_\ell} / \gamma)]d(h, D_\psi; \langle \theta, N_j \rangle^{K_\psi} (N_j)) \rangle \) for each \( l \). Replacing the left-hand sides of these equations by the right in the \( \rho \) position in (\(*\)), which gives an equation in \( D_\psi[\text{intro}^{\overline{n}}_{(K_\psi) \langle \ell, \ell \rangle, K, \xi_\psi[\ell]}(\overline{P_\ell}; \langle \theta, N_j \rangle^{K_\psi} (N_j))/h] \) by the typing of \( R \) in \( E : K \rightarrow h, D [\Psi] \), brings us to our destination.

The proof of the final theorem is essentially mechanical: we prove the eliminator is well-behaved on each possible input by referring to the appropriate \( \beta \) rule.

**Theorem 5.33.** \( \sigma \) supports \( K \).

**Proof.** We need to show that \( \text{Fcom}(\sigma) \subseteq \sigma \) and that \( \text{Intro}_{K, \ell}(\sigma) \subseteq \sigma \) for every \( \ell \in K \).

1. \( \text{Fcom}(\sigma) \subseteq \sigma \).

Suppose we have \( \text{Fcom}(\sigma) \varphi(V, V') \). For any \( W, W' \in \{V, V'\} \), each of the terms \( \text{elim}_{h, D_\psi}(W; E_\psi) \) and \( \text{elim}_{h, D_\psi}(W'; E_\psi) \) is equal to a reduct per Lemma 5.29, and these reducts are equal by Lemma 5.28 and the typing rule for \( \text{com} \).

2. \( \text{Intro}_{K, \ell}(\sigma) \subseteq \sigma \).

Suppose we have \( \text{Intro}_{K, \ell}(\sigma) \varphi(V, V') \). For any \( W, W' \in \{V, V'\} \), each of the terms \( \text{elim}_{h, D_\psi}(W; E_\psi) \) and \( \text{elim}_{h, D_\psi}(W'; E_\psi) \) is equal to a reduct per Lemma 5.31, and these reducts are equal by Lemma 5.28, Lemma 5.26, and \( E \equiv E' : K \rightarrow h, D [\Psi] \).

**Corollary 5.34** (E). \( h : \text{ind}(K) \Rightarrow \text{elim}_{h, D}(h; E) \cong \text{elim}_{h, D'}(h; E') \in D [\Psi] \).

**Proof.** By Theorem 5.33, we have \( \text{i}(K) = \sigma \). Apply Lemma 5.28.

## 6 Examples

In this section, we show how to encode various inductive types in our schema. We will also discuss opportunities for optimizations and alternative constructions in special cases.

For the sake of readability, we use \( \emptyset \) rather than \( \cdot \) to denote empty lists and omit these where unambiguous. We will write constructor operators simply as \( \ell \) and \( \ell' \) rather than \( \text{intro} \) and \( \text{Intro}_{K, \ell, E} \) and leave the reader to infer the annotations.

32
6.1 Weak W-types

Let \( A \) type\( \text{Kan} \)[\( \Psi \)], \( a : A \gg B \) type\( \text{Kan} \)[\( \Psi \)] be given. We can define their W-type [30] as \( W(A; a.B) := \text{ind}(\mathcal{K}_{W(A; a.B)}) \) where

\[
\mathcal{K}_{W(A; a.B)} := \left[ \text{sup} : (A; a.B \to X; \emptyset; \emptyset) \right]
\]

and derive a typing rule for the eliminator:

\[
W\text{-elim}_{h, D}(M; a.g.r.R) := \text{elim}_{h, D}(M; [\text{sup} : \emptyset.a.g.r.R])
\]

\[
h : W(A; a.B) \gg D \text{ type}\( \text{Kan} \)[\( \Psi \)] \quad M \in W(A; a.B)[\Psi]
\]

\[
a : A, g : B \to W(A; a.B), r : (b.B) \to D[\text{app}(g, b)/h] \gg R \in D[\text{sup}(a; g)/h][\Psi]
\]

We call these weak W-types because they have free hcom-Kan structure, so contain fcom values at higher dimensions. For constructor lists with only zero-dimensional constructors, we can define a strict inductive type by instead reducing hcoms, as the sides of an open box will always be of the same constructor form. In the case of W-types, we would first define projection functions

\[
\begin{align*}
\text{proja}(M) & \quad \text{proja}(\text{sup}(M; G)) \\
\text{projb}(M; N) & \quad \text{projb}(M'; N)
\end{align*}
\]

which satisfy typing rules

\[
\begin{align*}
M & \in W(A; a.B)[\Psi] \\
\text{proja}(M) & \in A[\Psi] \\
M & \in W(A; a.B)[\Psi] \\
N & \in B[\text{proja}(M)/a][\Psi] \\
M & \in W(A; a.B)[\Psi] \\
M & \equiv \text{sup}(\text{proja}(M); \lambda b.\text{projb}(M; b)) \in W(A; a.B)[\Psi]
\end{align*}
\]

We would then define homogeneous composition in \( W(A; a.B) \) by

\[
H = \text{hcom}_{\text{ind}}(\text{proja}(M); \xi_i \mapsto y.\text{proja}(N_i))
\]

\[
\text{hcom}_{W(A; a.B)}(M; \xi_i \mapsto y.N_i) \mapsto\sup(H[r'/y]; \text{com}_{\text{ind}}(H/a) \mapsto W(A; a.B)(\lambda b.\text{projb}(M; b); \xi_i \mapsto \lambda b.\text{projb}(N_i; b)))
\]

For strict W-types, we can leave the motive annotation \( h.D \) off the eliminator, since this is only used in the reduction rule for value fcoms. It is not so hard to see how this definition would generalize to arbitrary zero-dimensional instances of our schema.\(^5\)

\(^{5}\)Handling multiple constructors does require some wrangling. In general, one reduces the cap argument \( M \) of the hcom to a value, then decides which projection operator to apply to the tube based on its value. Note that we cannot also evaluate the tube arguments, as we only know they will terminate when their respective constraints hold.

33
6.2 Torus

The most natural way to define the torus in the cubical setting is à la Licata and Brunerie [26, §IV.E]. We set \( T := \text{ind}(\mathcal{K}_T) \) where

\[
\mathcal{K}_T := \begin{pmatrix}
\text{base} & : & (\emptyset; \emptyset; \emptyset; \emptyset; \emptyset) \\
\text{loopa} & : & (\emptyset; \emptyset; \emptyset; x. \ x = 0 \mapsto \text{base}, \ x = 1 \mapsto \text{base}) \\
\text{loopb} & : & (\emptyset; \emptyset; \emptyset; y. \ y = 0 \mapsto \text{base}, \ y = 1 \mapsto \text{base}) \\
\text{surf} & : & (\emptyset; \emptyset; \emptyset; x, y. \ x = 0 \mapsto \text{loopb}^y, \ y = 0 \mapsto \text{loopa}^x, \ x = 1 \mapsto \text{loopb}^y, \ y = 1 \mapsto \text{loopa}^x)
\end{pmatrix}
\]

The eliminator is then given by

\[
\text{T-elim}_{h.D}(M; R_{\text{base}}, x.R_{\text{loopa}}, y.R_{\text{loopb}}, x.y.R_{\text{surf}}) := \text{elim}_{h.D}
\left(
\begin{pmatrix}
\text{base} & : & \emptyset; \emptyset; \emptyset; R_{\text{base}} \\
\text{loopa} & : & x. \emptyset; \emptyset; R_{\text{loopa}} \\
\text{loopb} & : & y. \emptyset; \emptyset; R_{\text{loopb}} \\
\text{surf} & : & (x, y). \emptyset; \emptyset; R_{\text{surf}}
\end{pmatrix}
\right)
\]

and satisfies the typing rule

\[
\begin{align*}
R_{\text{base}} & \in D[\text{base}/h] [\Psi] & R_{\text{loopa}} & \in D[\text{loopa}^x/h] [\Psi, x] & (\forall \varepsilon) \ R_{\text{loopa}}(\varepsilon/x) & \doteq R_{\text{base}} & \in D[\text{loopa}^\varepsilon/h] [\Psi] \\
R_{\text{loopb}} & \in D[\text{loopb}^y/h] [\Psi, y] & (\forall \varepsilon) \ R_{\text{loopb}}(\varepsilon/y) & \doteq R_{\text{base}} & \in D[\text{loopb}^\varepsilon/h] [\Psi] & R_{\text{surf}} & \in D[\text{surf}^{x,y}/h] [\Psi, x, y] \\
(\forall \varepsilon) \ R_{\text{surf}}(\varepsilon/x) & \doteq R_{\text{loopb}} & \in D[\text{surf}^{x,y}/h] [\Psi, y] & (\forall \varepsilon) \ R_{\text{surf}}(\varepsilon/y) & \doteq R_{\text{surf}} & \in D[\text{surf}^{x,y}/h] [\Psi, x]
\end{align*}
\]

We can also define the torus in a “globular” style more reminiscent of the HoTT Book’s definition [41, §6.6]. We take the same specifications for base, loopa, loopb, but change surf to

\[
\text{surf} : \begin{pmatrix}
x = 0 & \mapsto & \text{base}, \\
x = 1 & \mapsto & \text{base}, \\
y = 0 & \mapsto & \text{fcom}^{0\rightarrow 1}(\text{loopa}^x; \begin{cases} x = 0 & \mapsto z. \text{base}, \\
x = 1 & \mapsto z. \text{loopb}^x \end{cases}), \\
y = 1 & \mapsto & \text{fcom}^{0\rightarrow 1}(\text{loopb}^x; \begin{cases} x = 0 & \mapsto z. \text{base}, \\
x = 1 & \mapsto z. \text{loopb}^x \end{cases})
\end{pmatrix}
\]

With this definition, the coherence conditions on surf in the typing rule for the eliminator become

- \( R_{\text{surf}}(\varepsilon/x) \doteq R_{\text{base}} \in D[\text{surf}^{x,y}/h] [\Psi, y] \) for \( \varepsilon = 0, 1 \),

- \( R_{\text{surf}}(0/y) \doteq \text{com}_{D[F]/h}^{0\rightarrow 1}(R_{\text{loopa}}; \begin{cases} x = 0 & \mapsto R_{\text{base}}, \\
x = 1 & \mapsto R_{\text{loopb}} \end{cases}) \in D[\text{surf}^{x,0}/h] [\Psi, x] \)
  where \( F = \text{fcom}_{D/F}^{0\rightarrow z}(\text{loopa}^x; x = 0 \mapsto z. \text{base}, \ x = 1 \mapsto z. \text{loopb}^x) \),

- \( R_{\text{surf}}(1/y) \doteq \text{com}_{D[F]/h}^{0\rightarrow 1}(R_{\text{loopb}}(y/x); \begin{cases} x = 0 & \mapsto R_{\text{base}}, \\
x = 1 & \mapsto R_{\text{loopb}}(y/x) \end{cases}) \in D[\text{surf}^{x,1}/h] [\Psi, x] \)
  where \( F = \text{fcom}_{D[F]/h}^{0\rightarrow z}(\text{loopb}^x; x = 0 \mapsto z. \text{base}, \ x = 1 \mapsto z. \text{loopa}^x) \).

The torus is a closed inductive type: it has no parameters and no free dimension variables. For such types, we can optimize by making coercion trivial:

\[
\text{coeq}_{\varepsilon, \varepsilon'}^M(M) \rightarrow M.
\]

For zero-dimensional closed inductive types, such as \textsf{bool} or \textsf{nat}, we can go even further and make composition trivial as well:

\[
\text{hcom}_{\textsf{nat}}^{\varepsilon, \varepsilon'}(M; z. \xi_i \mapsto y. N_i) \rightarrow M.
\]
6.3 W-Quotients

W-quotients [36, §3.2] extend W-types by adding a recursive path constructor. Path constructor elements connect point constructor elements as specified by two provided functions. Let $A$ type$_{Kan}$ [$\Psi$], $a : A \gg B$ type$_{Kan}$ [$\Psi$], $C$ type$_{Kan}$ [$\Psi$], and $F_0, F_1 \in C \to A$ [$\Psi$] be given. We define $WQ(A; a.B; C; F_0; F_1) : = \text{ind}(\mathcal{K}_{WQ})$ where

$$\mathcal{K}_{WQ} = \left[ \begin{array}{c} \text{sup} : (A; a.B \to X; \emptyset; \emptyset), \\ \text{cell} : (C; c.(B[F_0(c)/a] \to X, B[F_1(c)/a] \to X); x \mapsto c.(g_0, g_1).sup(app(F_0, c); g_0), \\ \quad \quad x = 1 \mapsto c.(g_0, g_1).sup(app(F_1, c); g_1) \end{array} \right]$$

The eliminator is given by

$$WQ\text{-elim}_{h.D}(M; a.g.r.R_{sup}, x.c.g_0.g_1.r_0.r_1.R_{cell}) : = \text{elim}_{h.D}(M; \begin{array}{c} \text{sup} : \emptyset; a.g.r.R_{sup}, \\ \text{cell} : x.c.(g_0, g_1).(r_0, r_1).R_{cell} \end{array})$$

Abbreviating $WQ(A; a.B; C; F_0; F_1)$ as $WQ$, the eliminator satisfies the typing rule

$$h : WQ \gg D \text{ type}_Kan \Psi \quad M \in WQ \Psi, \quad a : A, g : B \to WQ, r : (b.B) \to D[app(g, b)/h] \gg R_{sup} \in D[sup(a; g)/h]\Psi, \quad \Gamma_{cell} = (c : C, g_\varepsilon : B[app(F_\varepsilon, c)/a] \to WQ, r_\varepsilon : (b.B[app(F_\varepsilon, c)/a] \to D[app(g_\varepsilon, b)/h])$$

$$\Gamma_{cell} \gg R_{cell}(0/x) \equiv R_{sup}[app(F_0, c); g_0, r_0/a, g, r] \in D[cell^0(c; g_0; g_1)/h]\Psi, \quad \Gamma_{cell} \gg R_{cell}(1/x) \equiv R_{sup}[app(F_1, c); g_1, r_1/a, g, r] \in D[cell^1(c; g_0; g_1)/h]\Psi$$

$WQ\text{-elim}_{h.D}(M; a.h.r.R_{sup}, x.c.h_0.h_1.r_0.r_1.R_{cell}) \in D[M/h]\Psi$.

W-quotients carve out a space of higher inductive types which are in a certain sense recursive only at the level of points. Although the cell constructor does take recursive arguments, the recursive arguments of a cell term are fully determined by the 0-dimensional sup elements at its boundary. The form of the 1-dimensional constructor can therefore vary only in the type $C$ of its non-recursive parameter and the functions $F_0, F_1$ which form the non-recursive part of the boundary term.

6.4 Higher truncations

Encoding the higher truncations [41, §7.3] in our schema requires some indirection. One option is to use a hub-and-spokes construction as in the HoTT Book. Assuming we have already defined the n-spheres, we would then define the $n$-truncation as $\|A\|_n : = \text{ind}(\mathcal{K}_{\|A\|_n})$ where

$$\mathcal{K}_{\|A\|_n} : = \left[ \begin{array}{c} \text{pt} : (A; a.\emptyset; \emptyset.\emptyset), \\ \text{hub} : (\emptyset; \emptyset; S^{n+1} \to X; \emptyset.\emptyset), \\ \text{spoke} : (S^{n+1}; s.S^{n+1} \to X; x \mapsto 0 \mapsto s.f.hub(f), x = 1 \mapsto s.f.app(f, s) \end{array} \right]$$

The idea of this definition is to construct $\|A\|_n$ by recursively contracting every $(n+1)$-sphere to a hub point. We can define the eliminator as

$$\text{trunc-elim}_{h.D}(M; a.R_{pt}, f.r.f.R_{hub}, s.f.r.f.R_{spoke}) : = \text{elim}_{h.D}(M; \begin{array}{c} \text{pt} : \emptyset; a.\emptyset; \emptyset.R_{pt}, \\ \text{hub} : \emptyset; \emptyset; f.r.f.R_{hub}, \\ \text{spoke} : x.s.f.r.f.R_{spoke} \end{array})$$

which satisfies the typing rule

$$h : \|A\|_n \gg D \text{ type}_Kan \Psi \quad M \in \|A\|_n \Psi, \quad a : A \gg R_{pt} \in D[pt(a)/h]\Psi, \quad f : S^{n+1} \to \|A\|_n, r_f : (s.S^{n+1}) \to D[app(f, s)/h] \gg R_{hub} \in D[hub(f)/h]\Psi, \quad \Gamma_{spoke} = (s : S^{n+1}, f : S^{n+1} \to \|A\|_n, r_f : (s.S^{n+1}) \to D[app(f, s)/h])$$

$$\Gamma_{spoke} \gg R_{spoke} \in D[spoke^0(s; f)/h]\Psi, \quad \Gamma_{spoke} \gg R_{spoke}(0/x) \equiv R_{hub} \in D[spoke^0(s; f)/h]\Psi, \quad \Gamma_{spoke} \gg R_{spoke}(1/x) \equiv app(f, s) \in D[spoke^1(s; f)/h]\Psi$$

$$\text{trunc-elim}_{h.D}(M; a.R_{pt}, f.r.R_{hub}, s.f.r.f.R_{spoke}) \in D[M/h]\Psi$$

35
\section{Localization}

Given a family of maps \( i : I \to F_i \in S_i \to T_i \) \([\Psi]\), a type \( A \) \( \text{type}_{\text{Kan}}[\Psi] \) is \( F \)-\textit{local} if precomposition by \( F_i \) gives an equivalence between \( T_i \to A \) and \( S_i \to A \) for all \( i \). The \textit{localization} \( \mathcal{L}_F(A) \) of an type \( A \) \( \text{type}_{\text{Kan}}[\Psi] \) at \( F \) is the universal \( F \)-local type with a map \( A \to \mathcal{L}_F(A) \). Shulman [33] constructs localization as a higher inductive type which we can encode in our schema.

\[
\mathcal{K}_L :=
\begin{cases}
\text{loc} : (A; \_; \_; \_), \\
\text{ext} : ((i : I; t : T_i); (i; _); S_i \to X; \_; \_), \\
\text{ext}' : ((i : I; t : T_i); (i; _); S_i \to X; \_; \_), \\
\text{rtr} : (i : I; s : S_i); (i; _); S_i \to X; x; \\
\begin{array}{l}
x = 0 \iff (i; s).g.\text{app}(g, s), \\
x = 1 \iff (i; s).g.\text{ext}(i; \text{app}(F_i, s); g).
\end{array}
\end{cases}
\]

The constructor \text{loc} includes \( A \) in \( \mathcal{L}_F(A) \). The constructors \text{ext} and \text{rtr} give a right inverse to precomposition by \( F_i \) for each \( i \), while the constructors \text{ext}' and \text{rtr}' give a left inverse. Per [41, §4.3], this data makes \( \_ \circ F_i \) an equivalence for each \( i \). We will not write out the eliminator for this inductive type, but it is not hard to see that any function from \( A \) into an \( F \)-local type factors through \( \mathcal{L}_F(A) \).

\section{Making an example of a non-example}

Lumsdaine and Shulman [28, §9] give an example of a higher inductive type which is not modeled in ZF and therefore cannot be encoded using only pushouts and natural numbers, adapting a result of Blass [12, §9]. This type also cannot be encoded in our schema, as it requires the definition of a boundary term by natural number recursion. However, we can encode it if we extend the specification language with a natural number recursor. Assume we are defining a cubical type system which contains the strict natural numbers type \( \text{nat} \) defined in Part III (or a weak natural numbers type defined using our schema) and \((-1)\)-truncations. We extend the boundary term language with

\[
\left( M : \text{nat} \right) \quad \left( K; \Theta \vdash N : A \right) \quad \left( a : \text{nat} \right) \\
\quad \vdash K; \Theta \vdash \text{natrec}(M; M, a.p.M)
\]

and add the following rules to the formal type system.

\[
\begin{align*}
M \equiv M' & \in \text{nat} \quad K; \Theta \vdash N \equiv N' : A \quad a \vdash \text{nat} \quad K; \Theta, p : A \vdash Q \equiv Q' : A \\
K; \Theta & \vdash \text{natrec}(M; N, a.p.Q) \equiv \text{natrec}(M'; N', a.p.Q') : A \\
K; \Theta & \vdash \text{natrec}(z; N, a.p.Q) \equiv N : A \\
M \in \text{nat} & \quad K; \Theta \vdash N : A \\
& \quad a \vdash \text{nat} \quad K; \Theta, p : A \vdash Q : A \\
& \quad \vdash K; \Theta \vdash \text{natrec}(s(M); N, a.p.Q) \equiv Q[M/a] \text{natrec}(M; N, a.p.Q)/p : A \\
K; \Theta & \vdash \text{natrec}(s(M); N, a.p.Q) \equiv Q[M/a] \text{natrec}(M; N, a.p.Q)/p : A
\end{align*}
\]

(We could instead add a full-fledged formal natural numbers type \( \text{nat} \), but this is unnecessary). We extend the interpretation functions by

\[
\left( \theta.\text{natrec}(M; N, a.p.Q) \right)^K[N_j] := \text{natrec}(M; \theta.N)^K[N_j], a.r.(\theta.p.Q)^K[N_j, r])
\]

and

\[
\left( \theta.\text{natrec}(M; N, a.p.Q) \right)^{K,E}_h[D](N_j; S_j) := \\
\text{natrec}(M; \theta.N)^{K,E}_h[D](N_j; S_j), a.r.(\theta.p.Q)^{K,E}_h[D](N_j, \theta.\text{natrec}(a; N, a.p.Q)^{K,E}_h[D](N_j; S_j, r)).
\]

\[\]
We will not list the many constructors of the inductive type $F$ here. Suffice to say that the addition of \texttt{natrec} is needed to encode the constructor (4) in [28, §9], specifically in the definition of the functions $h_k : \texttt{nat} \rightarrow X$ by recursion on $k$.

This blind spot in our schema is of course not limited to natural numbers, but arises whenever one wishes to define a boundary term by recursion on some element of a positive type. In this paper, we have seen that our language nonetheless suffices to define the majority of constructs with established uses in homotopy type theory; it remains to be seen whether this will change as new applications come to light.
Part IV.B
One Indexed Inductive Type

In this section, we extend our investigation to indexed inductive types. An \textit{indexed inductive type} is a family $a : A \ni B$ \texttt{type}_\texttt{kan} [Ψ] which is generated by constructors which have the form $c : C \ni N \ni B[M/a] [Ψ]$ for some $c : C \ni M \ni A [Ψ]$. In other words, constructors introduce elements in a specified index/fiber of the family. This is a broader class than the parameterized inductive types, whose constructors always have the form $a : A, c : C \ni N \ni B [Ψ]$, i.e., introduce elements uniformly in all fibers of the family.

Rather than present another schema and go through the work of Part IV.A all over again, we will present a single example: the family of homotopy fibers of a map.\footnote{The concept of and proof theory for the homotopy fiber family were introduced to the first author by Steve Awodey via notes from a 2012 seminar talk at Carnegie Mellon University.}

In this part, we define the type \texttt{fib} in this section, we extend our investigation to indexed inductive types. An \textit{indexed inductive type} is a family $a : A \ni B$ \texttt{type}_\texttt{kan} [Ψ] which is generated by constructors which have the form $c : C \ni N \ni B[M/a] [Ψ]$ for some $c : C \ni M \ni A [Ψ]$. In other words, constructors introduce elements in a specified index/fiber of the family. This is a broader class than the parameterized inductive types, whose constructors always have the form $a : A, c : C \ni N \ni B [Ψ]$, i.e., introduce elements uniformly in all fibers of the family.

Rather than present another schema and go through the work of Part IV.A all over again, we will present a single example: the family of homotopy fibers of a map.\footnote{The concept of and proof theory for the homotopy fiber family were introduced to the first author by Steve Awodey via notes from a 2012 seminar talk at Carnegie Mellon University.}

In this part, we define the type \texttt{fib} in this section, we extend our investigation to indexed inductive types. An \textit{indexed inductive type} is a family $a : A \ni B$ \texttt{type}_\texttt{kan} [Ψ] which is generated by constructors which have the form $c : C \ni N \ni B[M/a] [Ψ]$ for some $c : C \ni M \ni A [Ψ]$. In other words, constructors introduce elements in a specified index/fiber of the family. This is a broader class than the parameterized inductive types, whose constructors always have the form $a : A, c : C \ni N \ni B [Ψ]$, i.e., introduce elements uniformly in all fibers of the family.

Rather than present another schema and go through the work of Part IV.A all over again, we will present a single example: the family of homotopy fibers of a map.\footnote{The concept of and proof theory for the homotopy fiber family were introduced to the first author by Steve Awodey via notes from a 2012 seminar talk at Carnegie Mellon University.}

In this part, we define the type \texttt{fib} in this section, we extend our investigation to indexed inductive types. An \textit{indexed inductive type} is a family $a : A \ni B$ \texttt{type}_\texttt{kan} [Ψ] which is generated by constructors which have the form $c : C \ni N \ni B[M/a] [Ψ]$ for some $c : C \ni M \ni A [Ψ]$. In other words, constructors introduce elements in a specified index/fiber of the family. This is a broader class than the parameterized inductive types, whose constructors always have the form $a : A, c : C \ni N \ni B [Ψ]$, i.e., introduce elements uniformly in all fibers of the family.

Rather than present another schema and go through the work of Part IV.A all over again, we will present a single example: the family of homotopy fibers of a map.\footnote{The concept of and proof theory for the homotopy fiber family were introduced to the first author by Steve Awodey via notes from a 2012 seminar talk at Carnegie Mellon University.}
• $V = \text{refl}(M)$ and $V' = \text{refl}(M')$ where
  1. $\alpha_\psi'(M, M')$,
  2. $\beta_\psi'(F \psi'[M/a], N \psi')$,
• $\text{FCOM}(\gamma_\psi[N]) \psi'(V, V')$ (see Section 4),
• $V = \text{fcoe}_{x,p'}(O)$ and $V' = \text{fcoe}_{x,p'}(O')$ where
  1. $r \neq r'$,
  2. $\beta_\psi \cdot x(P, P')$,
  3. $\beta_\psi(P(r/x), N \psi')$,
  4. $\gamma_\psi(P(r/x))(O, O')$.

We define a $\Psi$-relation $f(\alpha, \beta, a.F)$ over $\text{TM}(\beta)$, the homotopy fiber family of $(\alpha, \beta, a.F)$, as the least fixed-point of $\mathcal{F}$. (As value $\Psi$-relations over $\text{TM}(\beta)$ ordered by fiber-wise inclusion form a complete lattice, this is well-defined.)

**Proposition 7.2.** If $a : \text{TM}(\alpha) \gg \beta(F, F')$, then $f(\alpha, \beta, a.F) = f(\alpha, \beta, a.F')$.

**Definition 7.3.** Given $A \doteq A'$ type$_{\text{Can}}[\Psi]$, $B \doteq B'$ type$_{\text{Can}}[\Psi]$, and $a : A \gg F \doteq F' \in B[\Psi]$, we say that the cubical type system $\tau$ has their homotopy fiber family if
\[
\tau(\Psi', \text{fib}(A\psi; B\psi; a.F\psi; N), \text{fib}(A'\psi; B'\psi; a.F'\psi; N'), f([A], [B], a.F)\psi[N])
\]
holds for every $\psi : \Psi' \rightarrow \Psi$ and $N \doteq N' \in B\psi[\Psi]$.

For the remainder of this section, we fix $A \doteq A'$ type$_{\text{Can}}[\Psi]$, $B \doteq B'$ type$_{\text{Can}}[\Psi]$, and $a : A \gg F \doteq F' \in B[\Psi]$, and assume that $\tau$ has their homotopy fiber family. In order to improve readability, we will group the arguments of $\text{fib}(A; B; a.F; N)$ as $\text{fib}((A; B; a.F); N)$, so that we can abbreviate $\text{fib}(A\psi; B\psi; a.F\psi; N) \equiv \text{fib}((A; B; a.F)\psi; N)$.

**Proposition 7.4.** $\text{PTy}(\tau)(\Psi', \text{fib}((A; B; a.F)\psi; N), \text{fib}((A'\psi; B'\psi; a.F')\psi; N'), f([A], [B], a.F)\psi[N])$ holds for every $\psi : \Psi' \rightarrow \Psi$ and $N \doteq N' \in B\psi[\Psi]$.

*Proof.* By Proposition 2.1.

### 7.2 Introduction

The homotopy fiber family is made up of refl, fcom, and fcoe terms. The first two should be self-explanatory by now, so let us focus on fcoe. An fcoe is a fiber (or free or formal) coercion, which implements transport between indices of the homotopy fiber family. If $O$ is in some $\text{fib}(A; B; a.F; N(r/x))$, then $\text{fcoe}_{x,N}^{r\mapsto r'}(O)$ will be in $\text{fib}(A; B; a.F; N(r'/x))$. An fcoe is a value unless $r = r'$, in which case it reduces away. Note that fcoe only transports between fibers of the family: we cannot directly use it to get from $\text{fib}(A; B; a.F; N)(r/x)$ to $\text{fib}(A; B; a.F; N)(r'/x)$ when $A, B$, or $F$ mentions $x$. The key insight is that we can derive a full-fledged coe operation using only fcoe and the Kan conditions for $A$ and $B$.

| $\text{refl}(M)$ val | $r \neq r'$ | $\text{fcoe}_{x,p'}^{r\mapsto r'}(O)$ val | $\text{fcoe}_{x,p'}^{r\mapsto r'}(O) \mapsto O$ |
|----------------------|-------------|--------------------------------|---------------------------------|

**Figure 7:** Operational semantics of refl and fcoe

39
Lemma 7.5 (refl-I). For \( \psi : \Psi' \rightarrow \Psi \) and \( N \in B\psi [\Psi'] \), if

1. \( M \equiv M' \in A\psi [\Psi'] \),
2. \( F\psi[M/a] \equiv N \in B\psi [\Psi'] \),

then \( \text{TM}(\mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N])(\text{refl}(M), \text{refl}(M')) \).

Proof. By Lemma A.2.

Lemma 7.6 (fcom-I). For \( \psi : \Psi' \rightarrow \Psi \), \( N \in B\psi [\Psi'] \), and \( \xi \), valid, if

1. \( \text{TM}(\mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N])(O, O') \),
2. \( \text{TM}(\mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N])\mid \text{id}_{\Psi' \rightarrow \Psi}(\xi, \xi) (Q_i, Q_j) \) for all \( i, j \),
3. \( \text{TM}(\mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N])\mid \text{id}_{\Psi' \rightarrow \Psi}(Q_i \rightarrow y, O) \) for all \( i \),

then

(a) \( \text{TM}(\mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N]) \mid \text{id}_{\Psi' \rightarrow \Psi}(\xi \rightarrow y.Q_i) \) for all \( i \),
(b) \( \text{TM}(\mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N]) \mid \text{id}_{\Psi' \rightarrow \Psi}(O; \xi \rightarrow y.Q_i), O) \),
(c) \( \text{TM}(\mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N]) \mid \text{id}_{\Psi' \rightarrow \Psi}(O; \xi \rightarrow y.Q_i), \text{fcom}_{\Psi' \rightarrow \Psi}(O'; \xi \rightarrow y.Q_i)) \).

Proof. By Lemma 4.5, since \( f(A, B, a.F)\psi[N] \) supports \( \ast \) by definition.

Lemma 7.7 (fcoe-I). For \( \psi : \Psi' \rightarrow \Psi \) and \( N \in B\psi [\Psi] \), if

1. \( P \equiv P' \in B [\Psi, x] \),
2. \( P(r'/x) \equiv N \in B [\Psi'] \),
3. \( \text{TM}(\mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N])(O, O') \),

then

(a) \( \text{TM}(\mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N]) \mid r = r') (\text{fcoe}_{\Psi' \rightarrow \Psi}(O), O) \),
(b) \( \text{TM}(\mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N]) (\text{fcoe}_{\Psi' \rightarrow \Psi}(O), \text{fcoe}_{\Psi' \rightarrow \Psi}(O')) \).

Proof.

(a) By Corollary A.4. Let \( \psi' : \Psi' \rightarrow \Psi' \) with \( r') = r' \) be given. Then \( \text{fcoe}_{\Psi' \rightarrow \Psi}(O) \psi' \rightarrow \psi' \), and we know that \( \text{TM}(\mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi'[N \psi']) \psi' \psi' \) holds by assumptions 2 and 3.

(b) By Lemma A.2. Let \( \psi' : \Psi' \rightarrow \Psi' \) be given. We have two cases.

- \( r') = r' \)
  Then \( \mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N \psi'] \psi' \psi' \) and \( \mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N \psi'] \psi' \psi' \) by (a), so that \( \mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N \psi'] \psi' \psi' \) holds by our typing assumptions and transitivity.
- \( r' \neq r' \)
  Then \( \mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F)\psi[N \psi'] \psi' \psi' \) by definition of \( \mathcal{f}(\llbracket A \rrbracket, \llbracket B \rrbracket, a.F) \).

Theorem 7.8 (I). \( \mathcal{f}(A, B, a.F)\psi[N] \) is value-coherent for all \( \psi : \Psi' \rightarrow \Psi \) and \( N \in B\psi [\Psi'] \).

Proof. By Lemmas 7.5 to 7.7.

Corollary 7.9. \( \text{fib}(A; B; a.F)\psi[N] \equiv \text{fib}(A'; B'; a.F\psi[N]) \) type \( \text{pre} [\Psi] \) for every \( \psi : \Psi' \rightarrow \Psi \) and \( N \equiv N' \in B\psi [\Psi'] \).
7.3 Composition

Lemma 7.10. If $\text{fib}((A; B; a.F); N) \Rightarrow \text{fib}((A'; B'; a.F'); N')$ type$_{\text{pre}} \ [\Psi']$ are equally hcom-Kan for every $\psi : \Psi' \to \Psi$ and $N = N' \in B \psi \ [\Psi']$.

Proof. By Lemmas 4.5 and A.3.

7.4 Coercion

We now want to implement $\text{fib}_x, \text{fib}_y, \text{fib}_z$ using the tools we have at hand: $\text{coe}_x, \text{coe}_y, \text{coe}_z, \text{fcoe}_x, \text{fcoe}_y, \text{fcoe}_z$. It will be useful to introduce an auxiliary operator $\text{tcoe}_{x,A}, \text{tcoe}_{y,B}, \text{tcoe}_{z,N}$, which will implement what we call total space coercion. This will satisfy the typing rule

$$\frac{A \text{ type}_{\text{Kan}} \ [\Psi, x] \quad B \text{ type}_{\text{Kan}} \ [\Psi, x] \quad a : A \to a.F \in B \ [\Psi, x]}{\text{tcoe}_{x,A}(A; B; a.F; N) \in \text{fib}((A; B; a.F)(r/x); \text{coe}_{x,B}^r(N)) \ [\Psi]}$$

To see why we call $\text{tcoe}$ total space coercion, consider the total space of the fiber family, the type $(b:B) \times \text{fib}(A; B; a.F; b)$. Suppose we know that this type is Kan, and say we have $(N, O) \in (b:B) \times \text{fib}(A; B; a.F; b)(r/x) \ [\Psi]$. By definition of $\text{coe}$ in a dependent sum type, we then have

$$\text{coe}_{x,(b:B) \times \text{fib}(A; B; a.F; b)}((N, O)) \Rightarrow (\text{coe}_{x,B}^r(N), \text{coe}_{x,\text{fib}(A; B; a.F; b)}^r(b))(O)$$

The second component of the reduct then must have type $\text{fib}((A; B; a.F)(r/x); \text{coe}_{x,B}^r(N))$, exactly the output type of $\text{tcoe}_{x,A}(A; B; a.F; N)$.

Once we have defined $\text{tcoe}$, it will be easy to define $\text{coe}$:

$$\frac{\text{fib}((A; B; a.F); N)(r/x)}{\text{tcoe}_{x,(A; B; a.F)}^r(-) \Rightarrow \text{fib}((A; B; a.F)(r/x); \text{coe}_{x,B}^r(N(r/x)))}$$

$$\text{coe}_{x,\text{fib}(A; B; a.F); N}(-) \Rightarrow \text{fib}((A; B; a.F); N)(r/x)$$

$$\frac{\text{fib}((A; B; a.F); N)(r/x)}{\text{fcoe}_{x,A}^r(-) \Rightarrow \text{fib}((A; B; a.F)(r/x); \text{coe}_{x,B}^r(N(r/x)))}$$

Intuitively, we can decompose a coercion in the fiber family by first coercing in the total space, then coercing the result into the proper fiber.

As shown in Figure 9, $\text{tcoe}$ is an eager operator. On $\text{fcom}$ and $\text{fcoe}$ terms, $\text{tcoe}$ simply pushes inside. The more interesting case is $\text{fcom}$. We cannot simply take $\text{tcoe}_{x,A}^r(M \in B) \Rightarrow \text{refl}(\text{fcom}_{x,A}^r(M))$, because this reduct lives in the fiber $F(r/x)[\text{coe}_{x,A}^r(M/a)]$ rather than the desired fiber $\text{fcom}_{x,A}^r(F(r/x)[M/a])$. However, we can define an $x$-line $\text{coe}_{x,A}^r(F[\text{coe}_{x,A}^r(M/a)])$ which is equal to the former at $x = r'$ and the latter at $x = r$, so we obtain a term of the correct type by applying an $\text{fcom}$ from $r'$ to $r$ along this line.

Much as with other eager operators in Part IV.A, we will prove typing rules for $\text{tcoe}$ by defining a $\Psi$-PER $\sigma$ over $\text{fib}(B)$ of values on which $\text{tcoe}$ is well-behaved and then showing that $\sigma = \text{fib}(\text{fib}(B), a.F)$. As with $\text{fib}$, we will write $\text{tcoe}_{x,A}(A; B; a.F; B; \psi)(A; B; a.F; \psi)$ as an abbreviation for $\text{tcoe}_{x,A}(A; B; a.F; B; \psi)(A; B; a.F; \psi)$.
Figure 9: Operational semantics of $\text{coe}_{\text{fib}}(A; B; a. F; N)$

Definition 7.11. We define a value $\Psi$-PER over $\text{TM}([B])$ by saying that, for any $\psi : \Psi' \rightarrow \Psi$, $N \in B \psi [\Psi]$, and $\psi' : \Psi'' \rightarrow \Psi'$, $\sigma_{\psi}[N] \psi'(V, V')$ holds iff $\mathcal{E}([A], [B], a. F) \psi[N] \psi'(V, V')$ and for all $r, r'$ dim $[\Psi''], \psi_x : (\Psi'', x) \rightarrow \Psi$ with $\psi_x(r/x) = \psi'_x$, and $W, W' \in \{V, V'\}$, we have

1. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(W) = \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

2. $\text{coev}^\rightarrow_{x. B \psi_x}(N \psi') \text{coe}_{\text{fib}}(\Psi', x) \psi_x(W) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

3. $\text{coev}^\rightarrow_{x. B \psi_x}(N \psi') \text{coe}_{\text{fib}}(\Psi', x) \psi_x(W) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

4. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(W) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

5. $\text{coev}^\rightarrow_{x. B \psi_x}(N \psi') \text{coe}_{\text{fib}}(\Psi', x) \psi_x(W) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$.

We now extend the properties of values in $\sigma$ to terms in $\text{TM}(\sigma)$.

Lemma 7.12. If $N \in B \psi [\Psi]$ and $\text{TM}(\sigma_{\psi}[N]) \psi'(O, O')$, then for all $r, r'$ dim $[\Psi'']$ and $\psi_x : (\Psi'', x) \rightarrow \Psi$ with $\psi_x(r/x) = \psi'_x$, we have

1. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) = \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

2. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

3. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

4. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

5. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$.

Proof. It suffices to show that for all $\psi : \Psi' \rightarrow \Psi$, $N \in B \psi [\Psi]$, $\psi' : \Psi'' \rightarrow \Psi'$, $r, r'$ dim $[\Psi'']$ and $\psi_x : (\Psi'', x) \rightarrow \Psi$ with $\psi_x(r/x) = \psi'_x$, we have

1. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) = \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

2. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

3. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

4. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

5. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$.

As $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(\_)$, and the identity map are all eager, Lemma A.6 reduces this to showing that

1. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) = \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

2. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

3. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

4. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$,

5. $\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \in \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']$.

which follows from the definition of $\sigma$.

Next, we prove $\beta$ rules.

Lemma 7.13 (tcoe-refl-β). For all $\psi : \Psi' \rightarrow \Psi$, $N \in B \psi [\Psi']$, $\psi' : \Psi'' \rightarrow \Psi'$, $r, r'$ dim $[\Psi'']$ and $\psi_x : (\Psi'', x) \rightarrow \Psi$ with $\psi_x(r/x) = \psi'_x$, if

$$
\text{coe}_{\text{fib}}(\Psi', x) \psi_x(O) \rightarrow \text{fib}((A; B; a. F) \psi_x(r/x); \text{coe}_{x. B \psi_x}(N \psi')) [\Psi'']
$$
1. $M \in A \psi' \ [\Psi']$,
2. $F \psi'[M/a] \doteq N \in B \psi' \ [\Psi']$, 
then
\[ \text{tcoe}_{x.(A;B:a.F)\psi_x}(\text{refl}(M)) \doteq \text{fcoe}_{x.(A;B:a.F)\psi_x}(\text{coe}_{x.A\psi_x}(M/a)) \text{ (refl(coe}_{x.A\psi_x}(M))) \]
in fib((A; B; a.F)\psi_x(r'/x); \text{coe}_{x.B\psi_x}(N\psi')) \ at \ \Psi'$.  

Proof. As \[ \text{tcoe}_{x.(A;B;a.F)\psi_x}(\text{refl}(M)) \doteq \text{fcoe}_{x.(A;B;a.F)\psi_x}(\text{coe}_{x.A\psi_x}(M/a)) \text{ (refl(coe}_{x.A\psi_x}(M))), \] it suffices by Lemma 7.5 to show that the right-hand side is in the desired type. We have
\[ \text{coe}_{x.A\psi_x}(M) \in A \psi_x(r'/x) \ [\Psi'] \] by K4 for A, so
\[ \text{refl(coe}_{x.A\psi_x}(M)) \in \text{fib((A; B; a.F)\psi_x(r'/x); F \psi_x(r'/x)[\text{coe}_{x.A\psi_x}(M)]\ [\Psi'])} \] by Lemma 7.5.

Now let us examine the motives of the outer \text{fcoe} terms. By K4 for A and B, we have
\[ \text{coe}_{x.B\psi_x}(F \psi_x[\text{coe}_{x.A\psi_x}(M)/a]) \in B \psi_x(r'/x) \ [\Psi', x]. \]
Moreover, K5 for A and B tells us that
\[ \text{coe}_{x.B\psi_x}(F \psi_x[\text{coe}_{x.A\psi_x}(M)/a])(r'/x) \doteq F \psi_x(r'/x)[\text{coe}_{x.A\psi_x}(M)/a] \in B \psi_x(r'/x) \ [\Psi'] \] and
\[ \text{coe}_{x.B\psi_x}(F \psi_x[\text{coe}_{x.A\psi_x}(M)/a])(r'/x) \doteq \text{coe}_{x.B\psi_x}(F \psi_x(r'/x)[M/a]) \in B \psi_x(r'/x) \ [\Psi']. \]
Applying Lemma 7.7, we can conclude that \[ \text{fcoe}_{x.(A;B:a.F)\psi_x}(\text{coe}_{x.A\psi_x}(\text{refl(coe}_{x.A\psi_x}(M)))) \] is in fib((A; B; a.F)\psi_x(r'/x); \text{coe}_{x.B\psi_x}(F \psi_x(r'/x)[M/a])), which is equal to the desired type by the hypothesis \[ F \psi_x(r'/x)[M/a] \doteq N \in B \psi' \ [\Psi']. \]

**Lemma 7.14 (tcoe-fcom-β).** For all $\psi : \Psi' \to \Psi$, $N \in B \psi' \ [\Psi']$, $\psi' : \Psi'' \to \Psi'$, $r, r' \dim [\Psi'']$ and $\psi_x : (\Psi'', x) \to \Psi$ with $\psi_x(r/x) = \psi'\psi$, if
1. $\text{TM}(\sigma_x[N])\psi'(O)$,
2. $\text{TM}(\sigma_x[N])\psi_x[Q_i, Q_j] \ (\forall i, j)$,
3. $\text{TM}(\sigma_x[N])\psi_x[Q_i(s/y), O] \ (\forall i)$,
then
\[ \text{tcoe}_{x.(A;B;a.F)\psi_x}(\text{fcom}^{x \to s}(\text{O}; \xi_i \leftrightarrow y.Q_i)) \doteq \text{fcom}^{x \to s}(\text{tcoe}_{x.(A;B;a.F)\psi_x}(\text{O}); \xi_i \leftrightarrow y.\text{tcoe}_{x.(A;B;a.F)\psi_x}(Q_i)) \]
in fib((A; B; a.F)\psi_x(r'/x); \text{coe}_{x.B\psi_x}(N\psi')) \ at \ \Psi'$.  

Proof. The proof is identical in structure to that of Lemma 5.13 and thus omitted. 

**Lemma 7.15 (tcoe-fcoe-β).** For all $\psi : \Psi' \to \Psi$, $N \in B \psi' \ [\Psi']$, $\psi' : \Psi'' \to \Psi'$, $r, r' \dim [\Psi'']$ and $\psi_x : (\Psi'', x) \to \Psi$ with $\psi_x(r/x) = \psi'\psi$, if
1. $P \in B \psi' \ [\Psi'']$,
2. $P(s/x) \doteq N\psi' \in B \psi' \ [\Psi']$,
3. $\text{TM}(\sigma_x[P(s/x)])(O)$,
then
\[\text{tcoe}_{x,(A;B;a.F)}^r(\text{fcoe}_{y,p}^{s,s'}(O)) = \text{fcoe}_{y,x.B\psi_x}^{s,s'}(P)(\text{tcoe}_{x,(A;B;a.F)}^r(\psi_x(O)))\]
in \(\text{fib}((A;B;a.F)\psi_x(r'/x); \text{coe}_{x,B\psi_x}^{r,-r'}(N\psi'))\) at \(\Psi''\).

**Proof.** By Lemma A.3. Let \(\Psi'' : \Psi \to \Psi''\) be given. We have two cases.

1. \(s\psi'' = s'\psi''\).

Then \(\text{tcoe}_{x,(A;B;a.F)}^r(\text{fcoe}_{y,p}^{s,s'}(O))\psi'' \mapsto \text{tcoe}_{x,(A;B;a.F)}^r(\psi_x(O))\psi''\), so it suffices to show
\[\text{tcoe}_{x,(A;B;a.F)}^r(\psi_x(O))\psi'' = \text{fcoe}_{y,x.B\psi_x}^{s,s'}(P)(\text{tcoe}_{x,(A;B;a.F)}^r(\psi_x(O)))\psi''\]
in \(\text{fib}((A;B;a.F)\psi_x(r'/x); \text{coe}_{x,B\psi_x}^{r,-r'}(N\psi'))\psi''\), which follows from Lemma 7.12 and Lemma 7.7(a).

2. \(s\psi'' = s''\psi''\).

Then \(\text{tcoe}_{x,(A;B;a.F)}^r(\text{fcoe}_{y,p}^{s,s'}(O))\psi'' \mapsto \text{fcoe}_{y,x.B\psi_x}^{s,s'}(P)(\text{tcoe}_{x,(A;B;a.F)}^r(\psi_x(O)))\psi''\), and the reduce is in \(\text{fib}((A;B;a.F)\psi_x(r'/x); \text{coe}_{x,B\psi_x}^{r,-r'}(N\psi'))\) by Lemma 7.12 and Lemma 7.7(b).

**Theorem 7.16 (Total space coercions).** Let \(\psi : (\Psi', x) \to \Psi, r, r' \text{ dim } [\Psi'], \text{ and } N \in B\psi(r/x) [\Psi']\) be given. If

1. \(O \Rightarrow O' \in \text{fib}((A;B;a.F)\psi(r/x); N) [\Psi'],\)

then

(a) \(\text{tcoe}_{x,(A;B;a.F)}^{r,r'}(O) \Rightarrow \text{tcoe}_{x,(A;B';a.F)}^{r',r'}(O') \in \text{fib}((A;B;a.F)\psi(r/x); \text{coe}_{x,B\psi_x}^{r,-r'}(N)) [\Psi'],\)

(b) \(\text{tcoe}_{x,(A;B;a.F)}^{r,r'}(O) \Rightarrow O \in \text{fib}((A;B;a.F)\psi(r/x); N) [\Psi'],\)

**Proof.** By Lemma 7.12, it suffices to show that \(\#([A], [B], a.F) \subseteq \sigma\). By definition of \(\#([A], [B], a.F)\), this means showing that \(\sigma\) is closed under \(F\). Suppose we are given \(\psi : \Psi' \to \Psi, N \in B\psi [\Psi'], \psi' : \Psi'' \to \Psi',\) and \(F(\sigma)\psi[N]_{\psi'}(V, V')\). We want to show that \(\sigma_{\psi}[N]_{\psi'}(V, V')\), so let \(\psi_x : (\Psi'', x) \to \Psi, r, r' \text{ dim } [\Psi'']\) with \(\psi_x(r/x) = \psi_x r\), and \(W, W' \in \{V, V'\}\) be given. We have three cases.

- \(W = \text{refl}(M)\) and \(W' = \text{refl}(M')\) where
  1. \(M = M' \in A\psi\psi' [\Psi'],\)
  2. \(F\psi\psi'[M/a] \Rightarrow N \in B\psi\psi' [\Psi'']\).

Then by Lemma 7.13, it suffices to show
\[\text{fcoe}_{x,\text{coe}_{x,B\psi_x}^{r,r'}(F\psi_x[\text{coe}_{x,A\psi_x}^{r,r'}(M)/a])}(\text{refl}(\text{coe}_{x,A\psi_x}^{r,r'}(M)))\]
in \(\text{fib}((A;B;a.F)\psi_x(r'/x); \text{coe}_{x,B\psi_x}^{r,-r'}(N\psi'))\) at \(\Psi''\) and
\[\text{fcoe}_{x,\text{coe}_{x,B\psi_x}^{r,r'}(F\psi_x[\text{coe}_{x,A\psi_x}^{r,r'}(M)/a])}(\text{refl}(\text{coe}_{x,A\psi_x}^{r,r'}(M))) \Rightarrow \text{refl}(M)\]
in \(\text{fib}((A;B;a.F)\psi; N)\psi'\) at \(\Psi''\). For the first equation, we refer to the proof of Lemma 7.13; the binary version of the argument that the reducts are well-typed gives their equality. For the second equation, observe that
\[ \text{Theorem 7.17} \quad \text{fib}((A; B; a.F)\psi; N) \cong \text{fib}((A'; B'; a.F')\psi; N') \text{ for every } \psi : \Psi' \to \Psi \text{ and } N \cong N' \in B\psi' [\Psi']. \]
Proof. Let $\psi : \Psi' \to \Psi$, $N = N' \in B\psi [\Psi']$, $\psi' : (\Psi'', x) \to \Psi'$, $r, r' \in \dim [\Psi'']$ be given, and suppose we have $O \doteq O' \in \text{fib}(\langle A; B; a.F \rangle \psi; N) \langle r/x \rangle [\Psi'']$. Since

$$\text{coe}_{x.B; \psi; \psi}[O] \xrightarrow{\text{K4 for } B, B',} \text{coe}_{x.B; \psi; \psi}[N] \langle r/x \rangle [\Psi'']$$

it suffices to show that the Kan conditions hold after these reductions. By Theorem 7.16, we have

$$\text{coe}_{x,B; \psi; \psi}[O] \doteq \text{coe}_{x,B; \psi; \psi}[O'] \in \text{fib}(\langle A; B; a.F \rangle \psi; \langle r'/x \rangle ; \text{coe}_{x,B; \psi; \psi}[N] \langle r'/x \rangle [\Psi''])$$

As for the motive of the outer $\text{coe}$, we have

\begin{itemize}
    \item $\text{coe}_{B; \psi; \psi}[N] \doteq \text{coe}_{B; \psi; \psi}[N'] \in B\psi [\Psi'; x] \doteq \text{K4 for } B, B'$,
    \item $\text{coe}_{B; \psi; \psi}[N] \langle r/x \rangle \doteq \text{coe}_{B; \psi; \psi}[N] \langle r'/x \rangle \in B\psi [\Psi'; r' \times x] \doteq \text{K4 for } B, and
    \item $\text{coe}_{B; \psi; \psi}[N] \langle r/x \rangle \doteq N' \langle r'/x \rangle \in B\psi [\Psi'; r'/x] \doteq \text{K5 for } B.
\end{itemize}

Hence, we have

$$\text{coe}_{x,B; \psi; \psi}[O] \doteq \text{coe}_{x,B; \psi; \psi}[O'] \in \text{fib}(\langle A; B; a.F \rangle \psi; \langle r'/x \rangle ; \text{coe}_{x,B; \psi; \psi}[N] \langle r'/x \rangle [\Psi''])$$

in $\text{fib}(\langle A; B; a.F \rangle \psi; \langle r'/x \rangle ; N\psi \langle r'/x \rangle [\Psi''])$ at $\Psi''$ by Lemma 7.7. Thus the fourth Kan condition holds. For the fifth, we have

$$\text{coe}_{x,B; \psi; \psi}[O] \doteq \text{coe}_{x,B; \psi; \psi}[O'] \doteq O$$

in $\text{fib}(\langle A; B; a.F \rangle \psi; \langle r'/x \rangle ; N\psi \langle r'/x \rangle [\Psi''])$ at $\Psi''$ by Theorem 7.16 and Lemma 7.7. $\square$

7.5 Elimination

The elimination rule for $b : B \gg \text{fib}(A; B; a.R; b)$ $\text{type}_{\text{Kan}} [\Psi]$ expresses that the free Kan type family generated by refl.

$$b : B, f : \text{fib}(A; B; a.F; b) \gg D \text{ type}_{\text{Kan}} [\Psi]$$

$$N \in B[\Psi] \quad O \in \text{fib}(A; B; a.F; N)[\Psi] \quad a : A \gg R \in D[F, \text{refl}(a)/b, f][\Psi]$$

$$\textstyle \frac{J_{b,f,D,N}(O; a.R) \doteq J_{b,f,D',N'}(O'; a.R') \in D[N, O/b, f][\Psi]}{J_{b,f,D,N}(O; a.R) \doteq J_{b,f,D',N'}(O'; a.R') \in D[N, O/b, f][\Psi]}$$

The name $J$ is of course a nod to the eliminator for the identity type, and $J$ indeed coincides with said eliminator in the case of $\text{fib}(A; A \times A; a.(a, a); -)$. The operational semantics of $J_{b,f,D,N}(-; a.R)$ is exceedingly simple. On a refl term, it steps to the provided $R$. On an fcom term, it steps to a com in the target type of $J$’s applied to each argument of the fcom, as with the eliminator in Part IV.A. Likewise, on a fcoe term it steps to a coe in the target type of a J applied to the argument of the fcoe.

For the remainder of this section, fix

1. $b : B, f : \text{fib}(A; B; a.F; b) \gg D \doteq D' \text{ type}_{\text{Kan}} [\Psi]$, $\quad a : A \gg R \in D[F, \text{refl}(a)/b, f][\Psi]$

2. $J_{b,f,D,N}(O; a.R) \doteq J_{b,f,D',N'}(O'; a.R') \in D[N, O/b, f][\Psi]$

**Definition 7.18.** We define a value $\Psi$-PER $\sigma$ over $\beta$ by saying that, for any $\psi : \Psi' \to \Psi$, $N \in B\psi [\Psi']$, and $\psi' : (\Psi'', x) \to \Psi'$, $\sigma_{\psi}[N] \psi(V, V')$ holds iff $\langle [A], [B], a.R \rangle [N] \psi(V, V')$ and

1. for all $N'$ with $N\psi' \doteq N' \in B\psi [\Psi'']$ and $W, W' \in \{V, V'\}$, we have

$$J_{b,f,D,\psi'; N}(W; a.S \psi'') \doteq J_{b,f,D',\psi'; N'}(W'; a.S') \in D[\Psi/N/b] \psi'[W/f][\Psi'']$$
Lemma 7.19. If \( \text{Ty}(\sigma[tN]) \models \phi'(O, O') \), then for all \( N' \) with \( N \psi' = N' \in B \psi \psi' [\Psi'] \), we have
\[
J_{b,f,D;N}(O \rightarrow a.R) \implies J_{b,f,D;N}(O' \rightarrow a.R)
\]
\[
J_{b,f,D;N}(\text{refl}(M) \rightarrow a.R) \implies J_{b,f,D;N}(\text{refl}(M/a) \rightarrow R[M/a])
\]
\[
\text{com}^{\rightarrow \leftarrow}(O; \xi_i \rightarrow y.Q_i; a.R) \implies \text{com}^{\rightarrow \leftarrow}(O; \xi_i \rightarrow y.J_{b,f,D;N}(Q_i; a.R))
\]
\[
J_{b,f,D;N}(\text{fco}(\rightarrow \leftarrow)(O); a.R) \implies \text{fco}(\rightarrow \leftarrow)(O); a.R)
\]
\[
J_{b,f,D;N}(\text{co}(\rightarrow \leftarrow)(O); a.R) \implies \text{co}(\rightarrow \leftarrow)(O); a.R)
\]

Lemma 7.20 (J-refl-\( \beta \)). For any \( \psi : \Psi' \rightarrow \Psi \) and \( N \in B \psi [\Psi'] \), if
1. \( M \in A \psi [\Psi] \)
then \( J_{b,f,D;N}(\text{refl}(M) \rightarrow a.S\psi)(O, a.S') \) is \( D\psi[N/b][\Psi'] \rightarrow \Psi' \).

Proof. By Lemma A.3.

Lemma 7.21 (J-fco-\( \beta \)). For any \( \psi : \Psi' \rightarrow \Psi \) and \( N \in B \psi [\Psi'] \), if
1. \( \text{Ty}(\sigma[tN])(O) \)
2. \( \text{Ty}(\sigma[tN])[id_\Psi; y][\xi_i, \xi_j](Q_i, Q_j) \) for all \( i, j \),
3. \( \text{Ty}(\sigma[tN])[id_\Psi[\xi_i](Q_i, y), O) \) for all \( i \),
then
\[
J_{b,f,D;N}(\text{fco}(\rightarrow \leftarrow)(O; \xi_i \rightarrow Q_i; a.S\psi)) \implies \text{fco}(\rightarrow \leftarrow)(O; \xi_i \rightarrow y.J_{b,f,D;N}(Q_i; a.S))
\]

in \( D\psi[fco(\rightarrow \leftarrow)(O; \xi_i \rightarrow Q_i)/a] \rightarrow \Psi' \).

Proof. By Lemma 7.19, \( b : B; f : \text{fib}(A; B; a.R; b) \rightarrow D \text{type}_\text{Kan} [\Psi] \), Lemma 7.6, and the typing rule for \( \text{com} \) (see [2, Theorem 35]), the right-hand side of this equation is in the desired type. The rest of the proof is identical in structure to that of Lemma 5.13 and thus omitted.

Lemma 7.22 (J-coe-\( \beta \)). For any \( \psi : \Psi' \rightarrow \Psi \) and \( N \in B \psi [\Psi'] \), if
1. \( P \in B \psi [\Psi', y] \),
2. \( P[\psi'/x] \) is \( N \in B \psi [\Psi'] \),
3. \( \text{Ty}(\sigma[tP[x]])(O) \),
then \( J_{b.f.D;N}(\text{fcoe}_{x,P}^{r,r'}(O);a.S\psi) \rightarrow \text{coe}_{x,P}^{r,r'}(O) \rightarrow (J_{b.f.D;P;r/x}(O;a.S\psi)) \in D[\text{fcoe}_{x,P}^{r,r'}(O)/h] \ [\Psi'] \).

**Proof.** By Lemma 7.19, \( b:B, f:fib(A;B;a.R; b) \gg D \text{ type}_{\text{Kan}} [\Psi], \) Lemma 7.7, and K4 for \( D, \) the right-hand side of this equation is in the desired type. We first prove the following claim.

(a) \( J_{b.f.D;N}(\text{fcoe}_{x,P}^{r,r'}(O);a.S\psi) \rightarrow J_{b.f.D;N}(O;a.S\psi) \in D[\text{fcoe}_{x,P}^{r,r'}(O)/h] \ [\Psi' \mid r = r'] \).

By Lemma A.3 and Lemma 7.19.

Now we prove the main result by Lemma A.3. Let \( \psi' : \Psi'' \rightarrow \Psi' \) be given. We have two cases.

1. \( r\psi' = r'\psi'. \)

Then by (a), it suffices to show that \( \text{coe}_{x,P}^{r,r'}(O) \rightarrow (J_{b.f.D;P;r/x}(O;a.S\psi))\psi' \) is equal to \( J_{b.f.D;N}(O;a.S\psi)\psi' \) in \( D[O/h] \). This holds by K5 for \( D. \)

2. \( r\psi' \neq r'\psi'. \)

Then \( J_{b.f.D;N}(\text{fcoe}_{x,P}^{r,r'}(O);a.S\psi) \rightarrow \text{coe}_{x,P}^{r,r'}(O) \rightarrow (J_{b.f.D;P;r/x}(O;a.S\psi)). \)

**Theorem 7.23.** \( f([A], [B], a.R) \subseteq \sigma. \)

**Proof.** It suffices to show that \( F(\sigma) \subseteq \sigma. \) The proof is similar in structure to those of Theorems 5.33 and 7.16: for each of the clauses defining \( F(\sigma), \) we reduce each side of the equation with the appropriate \( \beta\)-rule and prove equality of the reducts using our typing assumptions, induction hypotheses, and Kan conditions for \( D, D'. \) We therefore leave the details to the reader.

**Corollary 7.24** (E). \( b:B, f:fib(A;B;a.R; b) \gg J_{b.f.D;H}(f;a.S) \equiv J_{b.f.D';H}(f;a.S') \in D [\Psi]. \)

**Proof.** By Theorem 7.23 and Lemma 7.19.
Appendix

A Lemmas

Definition A.1. For a \( \Psi \)-relation \( \alpha \), define a \( \Psi \)-relation \( \alpha^g \) by

\[
\alpha^g(M,M') : \iff M \downarrow V \land M' \downarrow V' \land \alpha(V,V').
\]

Lemma A.2 (Introduction). Let \( \alpha \) be a value \( \Psi \)-PER. If for all \( \psi : \Psi' \rightarrow \Psi \), either \( \alpha^g(M,\psi) \) or \( \text{Tm}(\alpha)_{\psi}(M,\psi) \), then \( \text{Tm}(\alpha)(M,M') \).

Proof. Let \( \psi_1 : \Psi_1 \rightarrow \Psi \) and \( \psi_2 : \Psi_2 \rightarrow \Psi_1 \) be given. We divide into three cases.

(a) \( \alpha\psi_1(M_1\psi_1, M'_1\psi_1) \) and \( \alpha\psi_1\psi_2(M_1\psi_1\psi_2, M'_1\psi_1\psi_2) \).

Then \( M_1 \downarrow M_1\psi_1 \) and \( M'_1 \downarrow M'_1\psi_1 \) with \( \alpha\psi_1\psi_2(M_1\psi_1\psi_2, M'_1\psi_1\psi_2) \), so \( \alpha^g(M_1\psi_1\psi_2, M'_1\psi_1\psi_2) \).

(b) \( \alpha\psi_1(M_1\psi_1, M'_1\psi_1) \) and \( \text{Tm}(\alpha)\psi_1\psi_2(M_1\psi_1\psi_2, M'_1\psi_1\psi_2) \).

Then \( M_1 \downarrow M_1\psi_1 \) and \( M'_1 \downarrow M'_1\psi_1 \) with \( \text{Tm}(\alpha)\psi_1\psi_2(M_1\psi_1\psi_2, M'_1\psi_1\psi_2) \), so \( \alpha^g(M_1\psi_1\psi_2, M'_1\psi_1\psi_2) \).

By \( \text{Tm}(\alpha)\psi_1(M_1\psi_1, M'_1\psi_1) \), we have \( M_1 \downarrow M_1 \) and \( M'_1 \downarrow M'_1 \) with \( \alpha^g(M_1\psi_1\psi_2, M'_1\psi_1\psi_2) \).

Lemma A.3 (Coherent expansion). Let \( \alpha \) be a value \( \Psi \)-PER and \( M, M' \) tm \( \Psi \). If for all \( \psi : \Psi' \rightarrow \Psi \), there exists \( M'' \) such that \( M\psi \rightarrow^* M'' \) and \( \text{Tm}(\alpha)\psi(M'', \psi) \), then \( \text{Tm}(\alpha)(M,M') \).

Proof. (Or see [6, Lemma 41].) Let \( \psi_1 : \Psi_1 \rightarrow \Psi \) and \( \psi_2 : \Psi_2 \rightarrow \Psi_1 \) be given. By assumption, there exists \( M_1 \) such that \( M_1 \psi_1 \rightarrow^* M_1 \) and \( \text{Tm}(\alpha)\psi_1(M_1, M_1\psi_1) \). By \( \text{Tm}(\alpha)\psi_1(M_1, M_1\psi_1) \), we know that \( M_1 \downarrow V_1, M'_1 \downarrow V'_1, V_1\psi_2 \downarrow V_2, V_2, M_1\psi_2 \downarrow V'_2, \) and \( M'_1\psi_2 \downarrow V'_2 \) with \( \alpha\psi_1\psi_2(V_1, V_2, V'_1, V'_2) \). We also have some \( M_1 \) such that \( M_1\psi_2 \rightarrow^* M_1 \) and \( \text{Tm}(\alpha)\psi_1\psi_2(M_1, M_1\psi_2) \). By \( \text{Tm}(\alpha)\psi_1\psi_2(M_1, M_1\psi_2) \), we have \( M_1 \downarrow W_1, M'_1 \downarrow W'_1, V_1\psi_2 \downarrow V_2, V_2, M_1\psi_2 \downarrow W_2, \) and \( M'_1\psi_2 \downarrow W'_2 \) with \( \alpha\psi_1\psi_2(W_1, W_2, W'_1, W'_2) \).

By this data, we have \( M_1 \downarrow V_1, M'_1 \downarrow V'_1, V_1\psi_2 \downarrow V_2, V_2, M_1\psi_2 \downarrow W_2, \) and \( M'_1\psi_2 \downarrow W'_2 \) with \( \alpha\psi_1\psi_2(W_1, W_2, W'_1, W'_2) \).

Corollary A.4 (Restricted expansion). Let \( \alpha \) be a value \( \Psi \)-PER and \( \Xi \) be a constraint context. Let \( \alpha \) be a value \( \Psi \)-PER and let \( M, M' \) tm \( \Psi \). If for all \( \psi : \Psi' \rightarrow \Psi \) with \( \Xi \psi \), there exists \( M'' \) such that \( M\psi \rightarrow^* M'' \) and \( \text{Tm}(\alpha)\psi(M'', \psi) \), then \( \text{Tm}(\alpha) | \Xi (M,M') \).

Definition A.5. We say that \( a \vdash N \) tm \( \Psi \) is eager if for all \( \psi : \Psi' \rightarrow \Psi \) and \( M \) tm \( \Psi' \), we have \( N[a] \downarrow V \) iff there exists \( V \) tm \( \Psi' \) such that \( M \downarrow V \land N[a] \downarrow W \).

Lemma A.6 (Elimination). Let \( \alpha \) be a value-coherent \( \Psi \)-PER and \( \beta \) be a value \( \Psi \)-PER over \( \text{Tm}(\alpha) \). Suppose \( a \vdash \beta \rightarrow^* \Gamma[a](N,N') \) for some \( \gamma \subseteq \alpha \). If \( a \vdash N, N' \) tm \( \Psi \) are eager, then \( a : \text{Tm}(\alpha) \rightarrow \text{Tm}(\beta)[a](N,N') \).

Proof. Let \( \psi : \Psi' \rightarrow \Psi \) and \( M, M' \) tm \( \Psi' \) be given with \( \text{Tm}(\gamma)_{\psi}(M,M') \). We want to show that \( \text{Tm}(\beta^g_{\psi}(M,M'))(N[\psi(a), \alpha')(\psi'/a)) \) holds, so let \( \psi_1 : \Psi_1 \rightarrow \Psi' \) and \( \psi_2 : \Psi_2 \rightarrow \Psi_1 \) be given.

By \( \text{Tm}(\alpha)_{\psi}(M,M') \), we know there exist \( M_1\psi_1 \downarrow M_1 \) and \( M'_1\psi_1 \downarrow M'_1 \) such that \( \gamma_{\psi_1} (M_1, M'_1) \) holds and \( \gamma_{\psi_1} (M_1\psi_2, M'_1\psi_2, M_1'\psi_2, M'_1'\psi_2) \) holds. By assumption, \( \text{Tm}(\beta_{\psi_1}(M_1)) (N[\psi_1(M_1/\alpha), \alpha'\psi'(\psi'/a)) \) holds. Because \( \gamma_{\psi_1} (M_1, M'_1) \) implies \( \gamma_{\psi_1} (M_1, M'_1) \) and \( \alpha \) is value-coherent, we have \( \text{Tm}(\alpha)_{\psi}(M_1, M'_1) \), so we can adjust the index for \( \text{Tm}(\beta^g_{\psi}(M_1))(N[\psi_1(M_1/\alpha), \alpha'\psi'(\psi'/a)) \). From this, we have \( N[\psi_1(M_1/\alpha)] \downarrow N_1 \) and \( N' \psi_1[M_1'/\alpha'] \downarrow N'_1 \) with \( \beta_{\psi_1}(M_1, M'_1) \).

Now, from \( \gamma_{\psi_1\psi_2} (M_1\psi_2, M'_1\psi_2, M_1'\psi_2, M'_1'\psi_2) \), we know that \( M_1\psi_2 \downarrow M_2 \) and \( M_1\psi_2 \downarrow M'_2 \). By assumption, this implies \( \text{Tm}(\beta_{\psi_1\psi_2}(M_2))(N[\psi_1\psi_2(M_2/a), \alpha'\psi'(\psi'/a)) \). Again
because $\alpha$ is value-coherent, we can obtain $\text{tm}(\beta_{\psi}[M])_{\psi_1\psi_2}(N\psi \psi_1\psi_2[M_2/a], N\psi \psi_1\psi_2[M_12/a])$ by adjusting the index. In particular, $\beta_{\psi}[M]^{\psi_1\psi_2}(N\psi \psi_1\psi_2[M_2/a], N\psi \psi_1\psi_2[M_12/a])$ holds. As $a \vdash \text{tm} \ [\Psi]$ is eager, we know that $N\psi \psi_1\psi_2[M_2/a]$ and $N\psi \psi_1\psi_2[M_12/a]$ converge to the same value, as do $N\psi \psi_1\psi_2[M_12/a]$ and $N\psi \psi_1\psi_2[M_1\psi_2/a]$. Thus $\beta_{\psi}[M]^{\psi_1\psi_2}(N\psi \psi_1\psi_2[M_1\psi_2/a], N\psi \psi_1\psi_2[M_1\psi_2/a])$ holds. Similarly, we can show that $\beta_{\psi}[M]^{\psi_1\psi_2}(N\psi \psi_1\psi_2[M_1\psi_2/a], N\psi \psi_1\psi_2[M_1\psi_2/a])$ holds. Finally, we use transitivity of $\beta_{\psi}[M]^{\psi_1\psi_2}$ to find that $\beta_{\psi}[M]^{\psi_1\psi_2}(N_1\psi_2, N\psi \psi_1\psi_2[M_1\psi_2/a], N_1\psi_2, N\psi \psi_1\psi_2[M_1\psi_2/a])$ holds. \hfill \qed

**B** Operational semantics for inductive types

**Formation**

\[ \text{ind}(K) \] val

**Introduction**

\[ \frac{\forall i \not\models \xi_i \quad r \neq r'}{\text{fcom}^{r \leadsto r'}(M; \xi_i \mapsto y.N_i) \text{ val}} \quad \frac{\forall i \not\models \xi_i \quad r = r'}{\text{fcom}^{r \leadsto r'}(M; \xi_i \mapsto y.N_i) \rightarrow M} \]

\[ \frac{\models \xi_i \quad (\forall j < i) \not\models \xi_j}{\text{fcom}^{r \leadsto r'}(M; \xi_i \mapsto y.N_i) \rightarrow N_i(r'/y)} \quad C = (\Gamma; \gamma.\Theta; \overline{x_i.\xi_k \mapsto \gamma.\theta.M_k}) \quad \frac{(\forall k) \not\models \xi_k(\overline{r_i/x_i})}{\text{intro}^{\gamma}_{K,t,c}(P_n; N_j) \text{ val}} \]

\[ \frac{C = (\Gamma; \gamma.\Theta; \overline{x_i.\xi_k \mapsto \gamma.\theta.M_k}) \quad \models \xi_k(\overline{r_i/x_i}) \quad (\forall l < k) \not\models \xi_l(\overline{r_i/x_i})}{\text{intro}^{\gamma}_{K,t,c}(P_n; N_j) \rightarrow \langle \theta.M_k(\overline{r_i/x_i})[P_n/\gamma]\rangle^K(N_j)} \]

**Composition**

\[ \text{hcom}_{\text{ind}(K)}^{r \leadsto r'}(M; \xi_i \mapsto y.N_i) \rightarrow \text{fcom}^{r \leadsto r'}(M; \xi_i \mapsto y.N_i) \]

**Coercion**

\[ \text{mcoe}_{\gamma}^{r \leadsto r'}(\cdot) := \cdot \]

\[ \text{mcoe}_{\gamma;\Theta;A}^{r \leadsto r'}((\overline{M_n, M})) := (\text{mcoe}_{\gamma;\Theta}^{r \leadsto r'}(\overline{M_n}), \text{coe}_{A}^{r \leadsto r'}(\text{mcoe}_{\gamma;\Theta}^{r \leadsto r'}(\overline{M_n})/\gamma)(M)) \]
\[
\begin{align*}
M & \mapsto M' \\
\text{cof}_{\text{ind}(K)}(M) & \mapsto \text{cof}_{\text{ind}(K)}(M') \\
(\forall i) & \not\equiv \xi_i \\
\text{cof}_{\text{ind}(K)}(\text{fcom}^{\rightarrow} \Gamma (M; \xi_i \mapsto y. N_i)) & \mapsto \text{fcom}^{\rightarrow} \Gamma (\text{cof}_{\text{ind}(K)}(M); \xi_i \mapsto y. \text{cof}_{\text{ind}(K)}(N_i))
\end{align*}
\]

\[\Theta = p_j : B_j \quad (\forall s) \quad \Gamma P_n = \text{cof}_{\text{ind}(K)}(P_n) \quad (\forall s, j) \quad N_j^s = \text{cof}_{\text{ind}(K)}(N_j) \]

\[\text{func}_\kappa(h.R; N) \mapsto R[N/h] \quad \text{func}_{\text{b}B}(h.R; N) \mapsto \lambda b. \text{func}_\kappa(h.R; \text{app}(N, b))\]

\[
\begin{align*}
M & \mapsto M' \\
\text{elim}_{\text{b}, D}(M; \xi) & \mapsto \text{elim}_{\text{b}, D}(M'; \xi) \\
(\forall i) & \not\equiv \xi_i \\
\text{cof}_{\text{ind}(K)}(\text{fcom}^{\rightarrow} \Gamma (M; \xi_i \mapsto y. N_i); \xi) & \mapsto \text{fcom}^{\rightarrow} \Gamma (\text{cof}_{\text{ind}(K)}(M); \xi_i \mapsto y. \text{elim}_{\text{b}, D}(N_i; \xi))
\end{align*}
\]

\[\mathcal{C} = (\Gamma; \gamma.M_k \mapsto \gamma. \theta.M_k) \quad \mathcal{E}[\ell] = \frac{\gamma_i.\gamma.\delta.\rho.R}{\Theta = p_j : B_j} \quad (\forall k) \quad \not\equiv \xi_k \langle r_i \mid x_i \rangle
\]

\[
\begin{align*}
\text{func}_{\text{b}B}(h.R; N) & \mapsto \lambda b. \text{func}_\kappa(h.R; \text{app}(N, b))\]
\end{align*}
\]

C Proof theory for inductive types

Formation

\[\mathcal{K} \equiv \mathcal{K'} \text{ constrs } \Psi \quad \text{ind}(\mathcal{K}) \equiv \text{ind}(\mathcal{K'}) \text{ type}_{\text{kan}}(\Psi)\]

Introduction

\[
\begin{align*}
\mathcal{C} & \equiv \mathcal{C'} \text{ constrs } \Psi \quad [\mathcal{C}', \ell : \mathcal{C}] \equiv [\mathcal{C'}, \ell : \mathcal{C'}] \text{ constrs } \Psi \\
\mathcal{C} & = (\Gamma; \gamma.\theta.M_k \mapsto \gamma.\theta.M_k) \quad P_n \equiv P_n' \in \Gamma \quad N_j^s \equiv N_j^{s'} \in \{\Theta P_n/\gamma\}(\text{ind}(\mathcal{K}))(\Psi) \\
\text{intro}_{\mathcal{K}', \ell \in \mathcal{C}}(P_n; N_j') & \equiv \text{intro}_{\mathcal{K'}, \ell \in \mathcal{C}'}(P_n'; N_j') \in \text{ind}(\mathcal{K}) \quad \Psi
\end{align*}
\]

\[
\begin{align*}
\mathcal{C} & \equiv \mathcal{C'} \text{ constrs } \Psi \quad [\mathcal{C}', \ell : \mathcal{C}] \equiv [\mathcal{C'}, \ell : \mathcal{C'}] \text{ constrs } \Psi \\
\mathcal{C} & = (\Gamma; \gamma.\theta.M_k \mapsto \gamma.\theta.M_k) \quad P_n \equiv P_n' \in \Gamma \quad N_j \equiv N_j' \in \{\Theta P_n/\gamma\}(\text{ind}(\mathcal{K}))(\Psi) \\
\text{intro}_{\mathcal{K}', \ell \in \mathcal{C}}(P_n; N_j') & \equiv \{\Theta P_n/\gamma\}(N_j) \in \text{ind}(\mathcal{K}) \quad \Psi
\end{align*}
\]

51
Dependent functorial action

\[ \begin{align*}
\text{A type}_n (\Psi) & \quad \text{B ≡ B'} \text{ atype } (\Psi) \\
\text{h : A } & \triangleright B \text{ type}_n (\Psi) \\
\text{h : A } & \triangleright R \overset{R'}{\equiv} B \text{ } (\Psi) \\
N & \overset{N'}{\equiv} \text{ in } [\Psi] (A) \text{ } (\Psi)
\end{align*} \]

\[ \begin{align*}
\text{func}_n (h.R; N) & \overset{\text{n}}{\equiv} \text{func}_{n'} (h.R'; N') \text{ in } [\Psi] (B) \text{ } (\Psi)
\end{align*} \]

\[ \begin{align*}
\text{A type}_n (\Psi) & \quad \text{B type}_n (\Psi) \\
\text{h : A } & \triangleright R \overset{R'}{\equiv} B \text{ } (\Psi) \\
N & \overset{N'}{\equiv} A \text{ } (\Psi)
\end{align*} \]

\[ \begin{align*}
\text{func}_X (h.R; N) & \overset{\text{n}}{\equiv} R[R/N/h] \text{ in } B \text{ } (\Psi)
\end{align*} \]

Elimination

\[ \begin{align*}
\text{K ≡ K' constrs } & \quad \text{h : ind(K) } \triangleright D \overset{D'}{\equiv} \text{type}_n (\Psi) \\
\text{E ≡ E'} & \quad \text{K : K } \rightarrow \text{h.D } (\Psi)
\end{align*} \]

\[ \begin{align*}
\text{elim}_{h.D} (h; E) & \overset{\text{n}}{\equiv} \text{elim}_{h.D'} (h; E') \text{ in } D \text{ } (\Psi)
\end{align*} \]

\[ \begin{align*}
\text{K constrs } & \quad \text{[K', \ell : C] } \overset{\text{n}}{\equiv} \text{K constrs } (\Psi) \\
\text{E : K } & \rightarrow \text{h.D } (\Psi)
\end{align*} \]

\[ \begin{align*}
\text{intro } & \overset{\text{n}}{=} \text{intro}_{h.D'} (\ell, C)(\text{P}_n; N_j)
\end{align*} \]

\[ \begin{align*}
\text{elim}_{h.D} (\text{intro}; E) & \overset{\text{n}}{=} R[\text{r}] (\text{P}_n)[\text{N}_j][\delta] \text{func}_{n} (\text{P}_n; N_j) (h.\text{elim}_{h.D} (h; E); N_j)[\rho] \text{ in } \text{D}[\text{intro}/h] \text{ } (\Psi)
\end{align*} \]

D Operational semantics for fiber families

Formation

\[ \text{fib}(A; B; a.R; N) \text{ val} \]

Introduction

\[ \begin{align*}
\text{refl}(M) & \text{ val} \\
\text{r } & \neq r' \\
\text{fco}_{x, B'} (O) & \text{ val} \\
\text{fco}_{x, r'} (O) & \rightarrow
\end{align*} \]

Composition

\[ \text{hcom}_{\text{fib}(A; B; a.R; N)} (O; \xi_i \mapsto Q_i) \rightarrow \text{fcom}_{x, r'} (O; \xi_i \mapsto Q_i) \]

Coercion

\[ \text{co}_{x, \text{fib}(A; B; a.R; N)} (O) \rightarrow \text{fco}_{x, \text{r'}} (\text{tco}_{x, (A; B; a.R)} (O)) \]

\[ \text{tco}_{x, (A; B; a.R)} (O) \rightarrow \text{fco}_{x, \text{r'}(\text{tco}_{x, (A; B; a.R)} (O))} \]

\[ \text{tco}_{x, (A; B; a.R)} (\text{refl}(M)) \rightarrow \text{fco}_{x, \text{r'}(\text{tco}_{x, (A; B; a.R)} (\text{refl}(M)))} \]

\[ \text{tco}_{x, (A; B; a.R)} (\text{fcom}_{x, \text{r'}} (O; \xi_i \mapsto y.Q_i)) \rightarrow \text{fcom}_{x, \text{r'}} (\text{tco}_{x, (A; B; a.R)} (O; \xi_i \mapsto y.\text{tco}_{x, (A; B; a.R)} (Q_i))) \]

\[ \text{tco}_{x, (A; B; a.R)} (\text{fco}_{x, \text{r'}} (O)) \rightarrow \text{fco}_{x, \text{r'}} (\text{tco}_{x, (A; B; a.R)} (O)) \]
Elimination

\[ O \rightsquigarrow O' \]
\[ J_{b.f.D;N}(O; a.R) \rightsquigarrow J_{b.f.D;N}(O'; a.R) \]
\[ J_{b.f.D;N}(\text{refl}(M); a.R) \rightsquigarrow R[M/a] \]

\[ \text{com}^r_{\rightarrow r'}(y.D[N/b][\text{com}^r_{\rightarrow r'}(M; tcoe_{x.F'}(O'; y.Q/a); a.R)]) \]
\[ J_{b.f.D;N}(\text{fcoe}^r_{x.F'}(O); a.R) \rightsquigarrow \text{coe}^r_{x.D}[p/f][\text{fcoe}^r_{x.F'}(O); f](J_{b.f.D;P[r/x]}(O; a.R)) \]

E Proof theory for fiber families

Formation

\[ A = A' \text{ type}_{\text{Kan}} [\Psi] \quad B = B' \text{ type}_{\text{Kan}} [\Psi] \quad a : A \gg F = F' \in B [\Psi] \quad N = N' \in B [\Psi] \]
\[ \text{fib}(A; B; a.F; N) = \text{fib}(A'; B'; a.F'; N') \text{ type}_{\text{Kan}} [\Psi] \]

Introduction

\[ Q = Q' \in A [\Psi] \quad F[Q/a] = N \in B [\Psi] \]
\[ \text{refl}(Q) = \text{refl}(Q') \in \text{fib}(A; B; a.F; N) [\Psi] \]

Total space coercion

\[ A = A' \text{ type}_{\text{Kan}} [\Psi, x] \quad B = B' \text{ type}_{\text{Kan}} [\Psi, x] \quad a : A \gg F = F' \in B [\Psi, x] \quad N = N' \in B(r/x) [\Psi] \quad O = O' \in \text{fib}((A; B; a.F)(r/x); N) [\Psi] \]
\[ \text{tcoe}^{r}_{r'}(A; B; a.F)(O) = \text{tcoe}^{r}_{r'}(A'; B'; F')(O') \in \text{fib}((A; B; a.F)(r'/x); \text{coe}^{r}_{r'}(N')) [\Psi] \]

Elimination

\[ b : B; f : \text{fib}(A; B; a.F; b) \gg D = D' \text{ type}_{\text{Kan}} [\Psi] \quad N = N' \in B [\Psi] \quad O = O' \in \text{fib}(A; B; a.F; N) [\Psi] \quad a : A \gg R = R' \in D[F, \text{refl}(a)/b, f] [\Psi] \]
\[ J_{b.f.D;N}(O; a.R) \rightsquigarrow J_{b.f.D';N'}(O'; a.R') \in D[N, O/b, f] [\Psi] \]
\[ b : B; f : \text{fib}(A; B; a.F; b) \gg D \text{ type}_{\text{Kan}} [\Psi] \quad Q \in A [\Psi] \quad F[Q/a] = N \in B [\Psi] \quad O = O' \in \text{fib}(A; B; a.F; N) [\Psi] \quad a : A \gg R = R' \in D[F, \text{refl}(a)/b, f] [\Psi] \]
\[ J_{b.f.D;N}(\text{refl}(Q); a.R) = R[Q/a] \in D[F[Q/a], \text{refl}(Q)/b, f] [\Psi] \]

References

[1] Thorsten Altenkirch, Paolo Capriotti, Gabe Dijkstra, and Fredrik Nordvall Forsberg. Quotient inductive-inductive types. arXiv:1612.02346, December 2016.

[2] Carlo Angiuli and Robert Harper. Computational higher type theory II: Dependent cubical realizability. arXiv:1606.09638, June 2016.
[3] Carlo Angiuli, Robert Harper, and Todd Wilson. Computational higher type theory I: Abstract cubical realizability. arXiv:1604.08873, April 2016.

[4] Carlo Angiuli, Guillaume Brunerie, Thierry Coquand, Kuen-Bang Hou (Favonia), Robert Harper, and Daniel R. Licata. Cartesian cubical type theory. https://github.com/dlicata335/cart-cube, December 2017.

[5] Carlo Angiuli, Robert Harper, and Todd Wilson. Computational higher-dimensional type theory. In Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages, POPL 2017, Paris, France, January 18-20, 2017, pages 680–693, 2017.

[6] Carlo Angiuli, Kuen-Bang Hou (Favonia), and Robert Harper. Computational higher type theory III: Univalent universes and exact equality. arXiv:1712.01800, December 2017.

[7] Steve Awodey. A cubical model of homotopy type theory. arXiv:1607.06413, June 2016.

[8] Steve Awodey, Nicola Gambino, and Kristina Sojakova. Inductive types in homotopy type theory. In Proceedings of the 27th Annual IEEE Symposium on Logic in Computer Science, LICS 2012, Dubrovnik, Croatia, June 25-28, 2012, pages 95–104, 2012.

[9] Steve Awodey, Nicola Gambino, and Kristina Sojakova. Homotopy-initial algebras in type theory. Journal of the ACM, 63(6):51:1–51:45, January 2017.

[10] Henning Basold, Herman Geuvers, and Niels van der Weide. Higher inductive types in programming. J. UCS, 23(1):63–88, 2017.

[11] Marc Bezem, Thierry Coquand, and Simon Huber. A model of type theory in cubical sets. In 19th International Conference on Types for Proofs and Programs, TYPES 2013, April 22-26, 2013, Toulouse, France, pages 107–128, 2013.

[12] Andreas Blass. Words, free algebras, and coequalizers. Fundamenta Mathematicae, 117(2):117–160, 1983. URL http://eudml.org/doc/211359.

[13] Guillaume Brunerie. On the homotopy groups of spheres in homotopy type theory. PhD thesis, Université de Nice Sophia Antipolis, 2015.

[14] Guillaume Brunerie and Daniel R. Licata. A cubical infinite-dimensional type theory. Talk at Oxford Workshop on Homotopy Type Theory, November 2014.

[15] Ulrik Buchholtz and Edward Morehouse. Varieties of cubical sets. In Relational and Algebraic Methods in Computer Science - 16th International Conference, RAMiCS 2017, Lyon, France, May 15-18, 2017, Proceedings, pages 77–92, 2017.

[16] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtenberg. Cubical type theory: a constructive interpretation of the univalence axiom. arXiv:1705.07088, 2016.

[17] Robert L. Constable and N. P. Mendler. Recursive definitions in type theory. In Logics of Programs, Conference, Brooklyn College, June 17-19, 1985, Proceedings, pages 61–78, 1985.

[18] Thierry Coquand. Variations on cubical sets (diagonals version). Available from http://www.cse.chalmers.se/coquand/diag.pdf, April 2014.

[19] Thierry Coquand. Re: [HoTT] a cubical type theory. Mailing list post. groups.google.com/d/msg/homotopytypetheory/oXQe5u_Mmtk/3HEDk5g5uq4J, May 2015.

[20] Thierry Coquand and Christine Paulin. Inductively defined types. In COLOG-88, International Conference on Computer Logic, Tallinn, USSR, December 1988, Proceedings, pages 50–66, 1988.
[21] Peter Dybjer. Inductive families. *Formal Aspects of Computing*, 6(4):440–465, 1994.

[22] Peter Dybjer and Hugo Moeneclaey. Finitary higher inductive types in the groupoid model. In *Mathematical Foundations of Programming Semantics, 33rd International Conference*, Ljubljana, Slovenia, 2017.

[23] Simon Huber. Canonicity for cubical type theory. arXiv:1607.04156, July 2016.

[24] Valery Isaev. Homotopy type theory with an interval type. Available from https://valis.github.io/doc.pdf, 2014.

[25] Nicolai Kraus. Constructions with non-recursive higher inductive types. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS ’16, New York, NY, USA, July 5-8, 2016, pages 595–604, 2016.

[26] Daniel R. Licata and Guillaume Brunerie. A cubical approach to synthetic homotopy theory. In *30th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS 2015, Kyoto, Japan, July 6-10, 2015, pages 92–103, 2015.

[27] Peter LeFanu Lumsdaine and Michael Shulman. Semantics of higher inductive types. Unpublished note. http://uf-ias-2012.wikispaces.com/file/view/semantics.pdf/410646692/semantics.pdf, 2012.

[28] Peter LeFanu Lumsdaine and Michael Shulman. Semantics of higher inductive types. arXiv:1705.07088, May 2017.

[29] Per Martin-Löf. An intuitionistic theory of types: predicative part. In H.E. Rose and J.C. Shepherdson, editors, *Logic Colloquium ’73*, volume 80 of *Studies in Logic and the Foundations of Mathematics*, pages 73–118. North-Holland, 1975.

[30] Per Martin-Löf. Constructive mathematics and computer programming. In L.J. Cohen, J. Loś, H. Pfeiffer, and K.-P. Podewski, editors, *Logic, Methodology and Philosophy of Science*, volume VI, pages 153–175. North-Holland, 1982.

[31] Hugo Moeneclaey. A schema for higher inductive types of level one and its interpretation. Internship report, supervised by Peter Dybjer, ENS Paris-Saclay, 2016.

[32] Egbert Rijke. The join construction. arXiv:1701.07538, January 2017.

[33] Michael Shulman. Localization as an inductive definition. Blog post. https://homotopytypetheory.org/2011/12/06/inductive-localization/, December 2011.

[34] Kristina Sojakova. Higher inductive types as homotopy-initial algebras. arXiv:1402.0761, February 2014.

[35] Kristina Sojakova. Higher inductive types as homotopy-initial algebras. In *Proceedings of the 42nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, POPL 2015, Mumbai, India, January 15-17, 2015, pages 31–42, 2015.

[36] Kristina Sojakova. *Higher Inductive Types as Homotopy-Initial Algebras*. PhD thesis, Carnegie Mellon University, 2016.

[37] Arnaud Spiwack. *Verified Computing in Homological Algebra, A Journey Exploring the Power and Limits of Dependent Type Theory*. PhD thesis, École Polytechnique, 2011.

[38] Jonathan Sterling and Robert Harper. Algebraic foundations of proof refinement. arXiv:1703.05215, March 2017.
[39] Andrew Swan. An algebraic weak factorisation system on 01-substitution sets: A constructive proof. arXiv:1409.1829, September 2014.

[40] The RedPRL Development Team. RedPRL – the People’s Refinement Logic, 2018. URL http://www.redprl.org/.

[41] The Univalent Foundations Program. Homotopy Type Theory: Univalent Foundations of Mathematics. https://homotopytypetheory.org/book, Institute for Advanced Study, 2013.

[42] Niels van der Weide. Higher inductive types. Master’s thesis, Radboud University Nijmegen, 2016.

[43] Floris van Doorn. Constructing the propositional truncation using non-recursive hits. In Proceedings of the 5th ACM SIGPLAN Conference on Certified Programs and Proofs, Saint Petersburg, FL, USA, January 20-22, 2016, pages 122–129, 2016.