

Research Article

On a Diophantine Inequality with $s$ Primes

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Let $2 < c < \delta$. In this study, for prime numbers $p_1, \ldots, p_s$ and a sufficiently large real number $N$, we prove the Diophantine inequality:

$$|p_1^s + \cdots + p_s^s - N| < N^{-\frac{294}{123}s/\left((\delta/5)^{s/2}\right)}$$

(1)

for any $s \geq 5$. When $s = 5$, this result improves a previous result.

1. Introduction

Suppose that $k \geq 1$ is an integer and $c > 1$ is not an integer. Let $\epsilon$ be a small positive number. The Waring–Goldbach problem is to study the solvability of the Diophantine equality:

$$N = p_1^k + \cdots + p_s^k,$$

(1)

in prime numbers $p_1, \ldots, p_s$. In [1], the author studied the analog of the Waring–Goldbach problem. For any sufficiently large real number $N$, let $H(c)$ denote the smallest natural number $s$ such that the Diophantine inequality,

$$|p_1^s + \cdots + p_s^s - N| < \epsilon,$$

(2)

is solvable in prime numbers $p_1, \ldots, p_s$. In [1], the author proved that

$$\limsup_{c \to \infty} \frac{H(c)}{c \log c} \leq 4.$$  

(3)

In [1], the author also obtained that $H(c) \leq 5$, for $1 < c < (3/2)$. Later, the result was improved in [2–6]. Now, the best result for $H(c) \leq 5$ is $2 < c < (52/25)$ by Li and Cai [3].

In [4], the authors first proved that $H(c) \leq 4$, for $1 < c < (81/68)$. Later, the result was improved in [7]. Now, the best result for $H(c) \leq 4$ is $1 < c < (1193/889)$ by Zhang and Li [8]. When $s = 3$ in inequality (2), Tolev [9] obtained the result $1 < c < (27/26)$. Afterwards, the range of $c$ was enlarged by several authors in [10–16]. Now, the best result for $s = 3$ is $1 < c < (43/36)$ by Cai [12]. When $s = 2$, in inequality (2), Laporta obtained $1 < c < (15/14)$ in [17]. Laporta’s result was improved by some authors in [4, 18, 19]. Now, the best result for $r = 2$ is $1 < c < (59/44)$ by Li and Cai [19].

In this paper, we focus on the Diophantine inequality (2) and prove the following result.

Theorem 1. Let $2 < c < \delta$. For any sufficiently large real number $N$, let $\epsilon = N^{-\frac{294}{123}s/\left((\delta/5)^{s/2}\right)}$ and let $B_0(N, s)$ denote the number of solutions of the Diophantine inequality:

$$|p_1^s + \cdots + p_s^s - N| < \epsilon,$$

(4)

in prime numbers $p_1, p_2, \ldots, p_s$, where $\delta = (294 - 210s)/(123 - 97s)$ and $s \geq 5$. We can obtain

$$B_0(N, s) \gg \frac{cN^{(s/\log c)^{1/2}}}{\log N},$$

(5)

For $s = 5$ in Theorem 1, we can get better result than Li and Cai [3].

Corollary 2. Under the notations of Theorem 1, for $2 < c < (378/181)$, we have
Lemma 3. We have
\[
\int_{-\infty}^{\infty} f^\dagger(x)e(-xN)\Phi(x)dx \gg \varepsilon x^{1-c}. \tag{12}
\]
Proof. Similar to Lemma 6 of [16], we have

Let \( \Lambda(n) \) denote von Mangoldt’s function. We write \( \omega \sim \Omega \) if the range of \( \omega \) is \( \Omega < \omega \leq 2\Omega \). \( a(\omega) \ll b(\omega) \) means that \( a(\omega) = O(b(\omega)) \).

2. Some Lemmas

In order to prove our theorem, we need the following lemmas.

Lemma 1. Let \( r \) be a positive integer. There exists a function \( \phi(y) \) which is \( r = [\log X] \) times continuously differentiable and satisfies

\[
\phi(y) = 1, \quad \text{for } |y| \leq a - b,
\]

\[
0 < \phi(y) < 1, \quad \text{for } a - b < |y| < a + b,
\]

and its Fourier transformation,

\[
\Phi(x) = \int_{-\infty}^{\infty} e(-xy)\phi(y)dy,
\]

satisfies

\[
|\Phi(x)| \leq \min\left(2a, \frac{1}{\pi|y|}, \frac{1}{\pi|y|} \left(\frac{r}{2\pi|y|b}\right)^r\right). \tag{10}
\]

Proof. This can be found in Piatetski-Shapiro [1]. \( \square \)

Lemma 2. Let \( a(l) \) be a sequence of complex numbers; then, for \( L, Q \geq 1 \), we have

\[
\left| \sum_{L \leq l \leq 2L} a(l) \right|^2 \leq \left(2 + \frac{L}{Q}\right) \sum_{|l| < Q} \left(1 - \frac{|l|}{Q}\right) \sum_{L \leq \mu l + \nu - 2L} a(l + q)a(l - q). \tag{11}
\]

where \( \overline{z} \) denotes the conjugate of the complex number \( z \).

Proof. This is Lemma 2 of Fouvry and Iwaniec [20]. \( \square \)

Proof. Similar to Lemma 6 of [16], we have

\[
H = \int_{-\infty}^{\infty} f^\dagger(x)e(-xN)\Phi(x)dx
\]

\[
= \sum_{X/2}^{X} \prod_{X/2}^{X} e(x(t_1^* + \cdots + t_s^* - N))\Phi(x)dxdt_1\cdots dt_s
\]

\[
\geq \sum_{X/2}^{X} \prod_{X/2}^{X} e(x(t_1^* + \cdots + t_s^* - N))\Phi(x)dxdt_1\cdots dt_s
\]

\[
\geq \sum_{X/2}^{X} \prod_{X/2}^{X} e(x(t_1^* + \cdots + t_s^* - N))\Phi(x)dxdt_1\cdots dt_s
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\]

\[
\geq \sum_{X/2}^{X} \prod_{X/2}^{X} e(x(t_1^* + \cdots + t_s^* - N))\Phi(x)dxdt_1\cdots dt_s
\]

where * means that \( |t_1^* + \cdots + t_s^* - N| < (4\varepsilon/5) \), and \( \lambda \) and \( \mu \) satisfy

For comparison, \( 2 < c < (378/181) = 2.0837977, \ldots, (52/25) = 2.08 \).

Notation 1. In this paper, let \( \delta = (294 - 210s)/(123 - 97s) \) and \( \delta_1 = (131 - 97s)/(123 - 97s), s \geq 5 \). We also assume

\[
2 < c < \delta,
\]

\[
X = \left( \frac{3N}{10} \right)^{1/c},
\]

\[
\eta = \frac{1}{1000}(\delta - c),
\]

\[
\kappa = X^{1-c-\eta},
\]

\[
\epsilon = N^{-(9/10c)(\delta - c)},
\]

\[
\chi = X^{\delta - c},
\]

\[
a = \frac{9\epsilon}{10},
\]

\[
b = \frac{\epsilon}{10},
\]

\[
S(x) = \sum_{(X/2) < p \leq X} e(xp^c)\log p,
\]

\[
I(x) = \int_{X/2}^{X} e(xt^c)dt.
\]
\[
\left(\frac{1}{2}\right)^{1/c} < \mu < \lambda < \left(\frac{1}{2} \left(2 - \frac{1}{2c}\right)\right)^{1/c} < 1,
\]

\[
\mathcal{M} = \left[\frac{X}{2}, X\right] \cap \left[\left(N - \frac{4e}{5} - t_1 - \cdots - t_{s-1}\right)^{1/c}, \left(N + \frac{4e}{5} - t_1 - \cdots - t_{s-1}\right)^{1/c}\right],
\]

where \( t_i \) are defined in Lemma 7 of [16].

Lemma 7.\(\) For \( \kappa < \lambda \), we can obtain by the mean-value theorem, where \( \mathcal{M} \) is Lemma 3 of Li and Cai [3].

Proof.\(\) This is Theorem 7 of Tolev [16].

Lemma 5.\(\) Let \( 0 \leq k \leq (1/2) \leq l \leq 1 \) and let \( |B| > 0 \). For any exponent pair \((k, l)\) and \( \Omega \leq \Omega' \leq 10\Omega \), we have

\[
\sum_{\Omega \leq \omega \leq \Omega'} e(B\omega^k) \ll (|B|\Omega')^{-k} \Omega^{l-k} + \frac{\Omega}{|B|\Omega^{c}}.
\]

Proof.\(\) This is Lemma 3 of Li and Cai [3].

Lemma 6.\(\) For \( |x| < \kappa \), we have

\[
S(x) = I(x) + O\left(\exp\left(-\log^{(1/5)} X\right)\right).
\]

Proof.\(\) This is Lemma 14 of Tolev [16].

Lemma 7.\(\) For \( \kappa < |x| < \chi \), we have

\[
\int_{|\omega| |x| < \chi} |S(x)| |\Phi(x)| \, dx \ll (X^{4c-\epsilon} + X^2)X^{3\eta}.
\]

Proof.\(\) This is (50) of Zhai and Cao [5].

Lemma 8.\(\) Let \( a \) and \( \beta \) be real numbers such that \( a \neq 0, 1, 2 \) and \( \beta \neq 0, 1, 2, 3 \). Set

\[
T(\Omega, \Gamma) = \sum_{\Omega \leq \omega \leq 2\Omega} \sum_{1 \leq l \leq 2l} a(\omega)b(\gamma)e\left(\frac{F\omega^\alpha\gamma^\beta}{\Omega^\alpha\Gamma^\beta}\right),
\]

where \( F \gg \Omega^2, \Gamma \geq \Omega, \) and \( |a(\omega)|, |b(\gamma)| \leq 1 \). Then, we have

\[
T(\Omega, \Gamma) \ll \left(F^{-1}1^{1/4} \Omega^{7/10}1^{13/16} + F^{1/2}\Omega^{93/104}1^{23/26}
\right) + F^{1/4}1^{1/128}\Omega^{57/60}1^{65/64}
\]

\[
\left.O^{65/62}\Gamma\right)(\Omega^\alpha\Gamma^\beta),
\]

\[
\text{Proof.}\quad \text{This is Theorem 1 of Baker and Weingartner [6].}\]

Lemma 9.\(\) Let \( 3 < Q < K < W < X \). Suppose that \( W - (1/2) \in \mathbb{N}, X \geq 64W^2UQ, W \geq 4Q^2, K^3 \geq 32X \). Assume further that \( F(n) \) is a complex-valued function such that \( |F(n)| \leq 1 \). Then, the sum

\[
\sum_{X/\log^{K/2}X} \Lambda(n)F(n),
\]

can be written in \( O(\log^{10} X) \) sums.

Proof.\(\) This is Lemma 3 of Heath-Brown [21].

3. The Estimate of \( S(x) \)

In this section, we draw our attention to the estimate of exponential sums, which also has lots of applications (e.g., see [22–35]). Suppose that \( |a(\omega)| \ll \omega^a \) and \( |b(\gamma)| \ll \gamma^a \); then, we estimate the exponential sums in the following two forms. Type I:

\[
\sum_{\omega \in \Omega, \gamma \in \Gamma} \sum_{x \in X} a(\omega)b(\gamma)F(x^\alpha\gamma^\beta),
\]

and Type II:

\[
\sum_{\omega \in \Omega, \gamma \in \Gamma} \sum_{x \in X} a(\omega)b(\gamma)F(x^\alpha\gamma^\beta).
\]

Lemma 10.\(\) For complex number sequences \( a(\omega) \) and \( b(\gamma) \), suppose that \( |a(\omega)| \ll 1, |b(\eta)| \ll 1, \kappa \leq |x| \leq \chi, X^{36} \ll T \ll X^{(438 \sim 1788)/369 \sim 291}, \Omega^\alpha=G \); then, we have

\[
S_{\Omega^\alpha} = \sum_{x=\Omega^\alpha \gamma \in \Gamma} a(\omega)b(\gamma)e\left(x^\alpha(\gamma\gamma)^\beta\right) \ll X^{\delta_\alpha, \gamma}.\]

Proof.\(\) By Cauchy’s inequality and Lemma 2 with \( M = \Omega^{(36) \sim (291 \sim 1788)/369} \), we obtain
Suppose that we choose the exponent pair $(\frac{5}{3}, \frac{4}{3})$. Let
\[
\lambda^\ast \ll \frac{\Omega}{M} \sum_{\gamma \sim \Gamma} \sum_{\gamma^m \leq m} \left( 1 - \frac{m}{M} \right)
\]
\[
\times \sum_{\omega \sim \Omega} a(\omega + m) a(\omega - m) e \left( \frac{(y + m)^c - (y - m)^c}{\gamma} \right)
\]
where
\[
S^\ast = \sum_{\gamma \sim \Gamma} \sum_{\gamma^m \leq m} \sum_{\omega \sim \Omega} e \left( f(\omega, \gamma, m) \right)
\]
with $f(\omega, \gamma, m) = x a^c (y + m)^c - (y - m)^c$. By Lemma 5, we choose the exponent pair $((1/4), (11/14))$. Then, we can obtain
\[
S^\ast \ll \sum_{1 \leq m \leq M} \sum_{\gamma \sim \Gamma} \left( \left( \frac{m}{\gamma} \right)^{1/4} \left( \frac{1}{\gamma} \right)^{11/14} \right) + \frac{1}{\gamma} \sum_{\gamma \sim \Gamma} \frac{1}{\gamma} \log M
\]
\[
\ll \Omega M^{15/14} \chi^{1/4} X^{(c-1)/4} X^{11/14} \Gamma^{-1/4} \Gamma + \Gamma \chi^{1-c} \log M
\]
\[
\ll \chi. \tag{28}
\]
Now, Lemma 10 follows from (26) and (28).

**Lemma 11.** Let $a(\omega)$ be a sequence of complex numbers. Suppose that $|a(\omega)| \ll 1$, $\kappa \leq |x| \leq \chi$, and $\Omega \ll X^{(6029-22313)/3075-24250}$, then, we have
\[
S_I = \sum_{\omega \sim \Omega} \sum_{\gamma \sim \Gamma} a(\omega) e(\omega x)^c \ll X^{\delta_1 + \gamma}. \tag{29}
\]

**Proof.** If $\Omega \ll X^{(41-273)/123-973}$, by Lemma 5, we choose the exponent pair $((1/6), (2/3))$. Then, we obtain
\[
|S_{II}|^2 \ll \sum_{\gamma \sim \Gamma} \Omega \sum_{\gamma^m \leq m} \left( 1 - \frac{m}{M} \right)
\]
\[
\times \sum_{\omega \sim \Omega} a(\omega + m) a(\omega - m) e \left( \frac{(y + m)^c - (y - m)^c}{\gamma} \right)
\]
\[
\ll \Omega \sum_{\gamma \sim \Gamma} \left( \Omega^{c} \gamma + \sum_{\gamma^m \leq m} \left( 1 - \frac{m}{M} \right) \right)
\]
\[
\times \sum_{\omega \sim \Omega} a(\omega + m) a(\omega - m) e \left( \frac{(y + m)^c - (y - m)^c}{\gamma} \right)
\]
\[
\ll \frac{X^2}{M} + \frac{X}{M} |S^\ast|, \tag{26}
\]
where
\[
S^\ast = \sum_{1 \leq m \leq M} \sum_{\gamma \sim \Gamma} \sum_{\omega \sim \Omega} e \left( f(\omega, \gamma, m) \right), \tag{27}
\]
with $f(\omega, \gamma, m) = x a^c (y + m)^c - (y - m)^c$. By Lemma 9, we reduce the estimation of $S^\ast$ to the estimations of type I sums:
\[
\sum_{\omega \sim \Omega} \sum_{\gamma \sim \Gamma} a(\omega) F(\omega \gamma), \quad \Gamma > W, \tag{36}
\]
and type II sums:
\[
\sum_{\omega \sim \Omega} \sum_{\gamma \sim \Gamma} a(\omega) F(\omega \gamma), \quad Q < \Gamma < W, \tag{37}
\]
and estimate (34) follows from Lemmas 10 and 11.

**4. Proof of the theorem**

Let
\[
B(N) = \sum_{\gamma \sim \Gamma} \sum_{\gamma^m \leq m} \left( \log p_\gamma \right)^c \phi \left( p_\gamma + \ldots + p_\gamma^c - N \right), \tag{38}
\]
and
\[
B_\gamma(N) = \sum_{\gamma \sim \Gamma} \sum_{\gamma^m \leq m} \left( \log p_\gamma \right)^c \phi \left( p_\gamma^c - N \right).
\]
By the definition of \( \phi(y) \) in Lemma 1, we have
\[
B(N, s) \geq B_1(N, s). \tag{39}
\]

By the inverse Fourier transformation formula, we obtain
\[
B_1(N, s) = \int_{-\infty}^{\infty} S'(x) e(-xN)\Phi(x)dx = D_1(N) + D_2(N) + D_3(N), \tag{40}
\]
where
\[
D_1(N) = \int_{-\infty}^{\infty} S'(x) e(-xN)\Phi(x)dx,
D_2(N) = \int_{|x| \leq X} S'(x) e(-xN)\Phi(x)dx, \tag{41}
D_3(N) = \int_{|x| > X} S'(x) e(-xN)\Phi(x)dx.
\]

Let
\[
H_1(N) = \int_{-\infty}^{\infty} I'(x) e(-xN)\Phi(x)dx, \tag{42}
H(N) = \int_{-\infty}^{\infty} I'(x) e(-xN)\Phi(x)dx.
\]

Then, we have
\[
D_1(N) = H(N) + (H_1(N) - H(N)) + (D_1(N) - H_1(N)). \tag{43}
\]

From Lemma 10 in [36], we have \( I(x) \ll X^{-\varepsilon} |x|^{-1} \).

Thus, by Lemma 1, we have
\[
|H_1(N) - H(N)| \ll \int_{|x| > X} |I(x)||\Phi(x)|dx
\ll X^{2-\varepsilon} \int_{|x| > X} |\Phi(x)||x|^{-1}dx
\ll \varepsilon X^{2-\varepsilon} \kappa^{-s+1}
\ll \varepsilon X^{2-\varepsilon} \log X.
\]

It follows from Lemmas 1, 4, and 6 that
\[
U(x) = \sum_{X/2 \leq n \leq X} e(xn^\varepsilon). \tag{47}
\]

We have
\[
D_1(N) \gg \varepsilon X^{2-\varepsilon}. \tag{46}
\]

Let
\[
|D_2(N)| = \left| \sum_{X/2 \leq p \leq X} (\log p) \int_{|x| \leq X} e(xp^\varepsilon)S^\varepsilon(x)\Phi(x)e(-xN)dx \right|
\leq \sum_{X/2 \leq p \leq X} (\log p) \int_{|x| \leq X} e(xp^\varepsilon)S^\varepsilon(x)\Phi(x)e(-xN)dx \tag{48}
\leq (\log X) \sum_{X/2 \leq n \leq X} \int_{|x| \leq X} e(xn^\varepsilon)S^\varepsilon(x)\Phi(x)e(-xN)dx.
\]

It follows from (48) and Cauchy’s inequality that
By Lemma 5 with the exponent pair \(((13/84), (55/84))\) (see in [37]), we obtain

\[
U(x) \ll \min \left( \left| x \right| X^{c/2} X^{1/2} + \frac{X}{|x|X^{c}X} \right). \tag{50}
\]

\[
\int_{|x| \geq \chi} |S(x)|^4 |\Phi(x)| |U(x - y)| \, dx \\
\ll \int_{|x| \leq \chi} |S(x)|^4 |\Phi(x)| |U(x - y)| \, dx \\
|X^c < |x - y| \leq 2\chi \\
+ \int_{|x| \leq \chi} |S(x)|^4 |\Phi(x)| \left( |x - y| X^{c/2} X^{1/2} + \frac{X}{|x - y| X} \right) \, dx \\
X^c < |x - y| \leq 2\chi \\
\ll \epsilon X^{((647 - 485s)/(123 - 97s)) + 16\eta} \int_{|x - y| \leq X^{c}} dx + (X^c)^{13/84} X^{1/2} \int_{|x| \leq \chi} |S(x)|^4 |\Phi(x)| \, dx \\
+ \epsilon X^{((647 - 485s)/(123 - 97s)) - c + 16\eta} \int_{X^c < |x - y| \leq 2\chi} \frac{1}{|x - y|} \, dx \\
\ll \epsilon X^{((647 - 485s)/(123 - 97s)) - c + 7\eta} + X^{(107 - 81\eta)/(123 - 97s)} \int_{|x| \leq \chi} |S(x)|^4 |\Phi(x)| \, dx,
\]

where Lemma 12 is used.

Now, by (49) and (51), we obtain
where Lemma 7 is used. From (52), we obtain

\[
D_2(N) \ll \varepsilon^{1/2} X (\frac{-972^2+228s-147}{2(123-97s)}+c/2)+10\eta
\]

\[
+ X(\frac{-972^3+333s-294}{2(123-97s)}+2\eta) \ll \frac{\varepsilon X^{s-c}}{\log X}
\]  

(53)

It follows from Lemma 1 that

\[
D_3(N) \ll \int_{|x|<X} |S'(x)| \Phi(x)|dx \ll X^s
\]

\[
\ll X^s \left( \frac{r}{2\pi|x|b} \right)^r \ll 1.
\]  

(54)

By (40), (46), (53), and (54),

\[
B_1(N,s) \gg \varepsilon X^{s-c}.
\]  

(55)

It follows from (39) and (55) that

\[
B_0(N,s) \geq \frac{B(N,s)}{\log X} \geq \frac{B_1(N,s)}{\log X} \gg \varepsilon N^{(s-c-1)/2}
\]

(56)

Now, by (56), the proof of the theorem is completed.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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