GUAGE THEORY FORMULATION OF THE $C = 1$ MATRIX MODEL: SYMMETRIES AND DISCRETE STATES

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ABSTRACT

We present a non-relativistic fermionic field theory in 2-dimensions coupled to external gauge fields. The singlet sector of the $c = 1$ matrix model corresponds to a specific external gauge field. The gauge theory is one-dimensional (time) and the space coordinate is treated as a group index. The generators of the gauge algebra are polynomials in the single particle momentum and position operators and they form the group $W_{1+\infty}^\text{(+)}$. There are corresponding Ward identities and residual gauge transformations that leave the external gauge fields invariant. We discuss the realization of these residual symmetries in the Minkowski time theory and conclude that the symmetries generated by the polynomial basis are not realized. We motivate and present an analytic continuation of the model which realises the group of residual symmetries. We consider the classical limit of this theory and make the correspondence with the discrete states of the $c = 1$ (Euclidean time) Liouville theory. We explain the appearance of the $SL(2)$ structure in $W_{1+\infty}^\text{(+)}$. We also present all the Euclidean classical solutions and the classical action in the classical phase space. A possible relation of this theory to the $N = 2$ string theory and also self-dual Einstein gravity in 4-dimensions is pointed out.
1. Introduction

In this paper we explore the $c = 1$ matrix model with the optimism that such a study may hint at some general symmetry principles of string theory.

Let us begin by recalling some of the known results of the $c = 1$ matrix model and the $d = 2$ critical string theory relevant to this paper. In the double scaling limit the matrix model in the $U(N)$-invariant sector is equivalent to the theory of free fermions in one space and one time dimensions interacting with a background potential $V(x) = -\frac{1}{2}x^2$ [1-5]. This system has a natural and exact description in terms of a two-dimensional non-relativistic field theory of fermions [6,7,8,9]. The elementary low-energy excitation of this model is a particle-hole pair which has a massless dispersion relation [10,11] and has been identified with the tachyon of the continuum theory [12,6]. In the continuum theory, the physical spectrum corresponds to excitations around a specific tachyon background [13]. Scattering amplitudes of these tachyons have been calculated by several authors [14,15,8,16,17]. These results are in agreement with the calculation of the scattering amplitude in two dimensional critical string theory [18] (where the liouville mode acts as a spatial dimension) with a flat metric and a linear dilaton.

Besides the massless tachyon the spectrum of the two dimensional critical string theory (in the above mentioned background) has an infinite tower of discrete states labelled by two non-negative integers $(r, s)$ [19-21] These states are the gauge invariant content of the higher spin states. In higher dimensions one is left with fields after all the gauge degrees of freedom are removed from the higher spin gauge fields. However, in two dimensions one is left with a discrete set of states. The existence of such states in the matrix model has been most clearly indicated by a study of the two point function of time dependent operators [22].

In what follows we study the one dimensional matrix model in an enlarged framework which enables us to organize the above facts. Hopefully this frame-
work will throw light on the understanding of the two dimensional black hole [23,24,25,26] in the matrix model and other important issues in string theory such as symmetry principles and a background independent formulation.

Let us summarise our results.

In section 2, we review the infinite number of conservation laws [6, 7] in the fermion field theory and write down the conserved currents. We show that the conservation laws are a reflection of global symmetries in which the fermion field not only gets multiplied by some function of $x$ but at the same time gets acted on by the translation operators involving powers of $-i\partial/\partial x$. This observation motivates us to consider (section 3) a fermion field theory coupled to external gauge fields such that the theory has a symmetry group which include arbitrary combinations of translation and phase multiplication on the fermi field (to be precise, $\delta \psi(x,t) \rightarrow \sum_{n=0}^{\infty} \epsilon_n(x,t)p^n\psi = \sum_{m,n=0}^{\infty} \epsilon_{mn}(t)x^m p^n \psi(x,t)$). Interestingly the above set of gauge transformations are precisely the set of all unitary transformations in the quantum mechanical Hilbert space of a single particle! Clearly such transformations form a group (in section 5 we identify the Lie algebra to be $W^{(+)}_{1+\infty}$). We write down the gauge-invariant lagrangian in the quantum mechanical notation. We show that the $c = 1$ double scaled matrix model corresponds to a particular choice of the background gauge field. In section 4 we derive the Ward identities of gauge invariance. In section 5 we consider a fixed arbitrary background and find the special gauge transformations that leave the background gauge field invariant. These do not lead to Ward identities but rather act as symmetry transformations. Their generators are constants of motion for an arbitrary time dependent background and as expected satisfy the $W^{(+)}_{1+\infty}$ algebra. We specifically study the background gauge field $\bar{A} = -1/2(p^2 - x^2)$, which corresponds to the matrix model and present the specific time-dependence of the special gauge transformations.

In section 6, we discuss the algebra of the above symmetry transformations in the fermion Fock space. We find that in the Minkowski theory where the single
particle wave functions are parabolic cylinder functions, the unitary group as generated by the differential operators \( x^m p^n \) does not have a well defined action on the Hilbert space, meaning that each element of this basis transforms the single particle wavefunctions out of the Hilbert space. In sections 7 and 8, we motivate and present a specific analytic continuation of the theory, viz. \( t \to it \) and \( p \to -ip \) (and \( A(t) \to -iA(t) \)) which enables the construction of a representation of the symmetry algebra generated by the basis \( x^m p^n \). The background gauge field in this case is \( \bar{A} = 1/2(p^2 + x^2) \), the ordinary harmonic oscillator hamiltonian. We show that all the states of the fermion field theory can be generated by acting with the symmetry generators on the Fermi sea. We emphasize that the analytically continued fermion theory does not contain any state with continuous energy: even the tachyon spectrum is discrete.

In section 9 we discuss the classical limit as \( g_{st} \to 0 \) using the fermi fluid picture recently emphasized by Polchinski[1, 17]. We show that the classical limit of the expectation value of the generator \( W^{(r,s)} \) in a given state \( |f> \), described by a profile \( f(x,p) = 0 \) of the fermi fluid, can be expressed as the phase space average of a function \( V_{rs}(\tau, t - \theta) \):

\[
V_{rs}(\tau, \theta) = \exp(-2\tau) \exp(-\tau(r + s) + i\theta(r - s))
\]

where \( x = \exp(-\tau) \cos \theta, \ p = \exp(-\tau) \sin \theta \) parametrise the single-fermion phase space. We also discuss how \( SL(2) \) naturally arises in the discussion of the single-particle phase space, and present a simple understanding of the representation theory in classical terms. Finally, we present a complete list of the Euclidean classical solutions of the theory and propose this space of solutions as the classical phase space. We identify the classical limit of the field theory operators \( W_{rs} \) as functions \( w_{rs} \) on this space. We calculate the Poisson brackets of these functions; we show that all other functions on the phase space are functions of these basic observables \( w_{rs} \) and hence the knowledge of \( \{w_{rs}, w_{mn}\}_{\text{PB}} \) constitutes a complete specification of the symplectic form on this phase space. Since the hamiltonian in
this phase space is also known (it is simply $w_{11}$) we have a complete specification of the classical physics. We present the classical action in this phase space.

In section 10, we consider perturbing the model by the operators $\sum_{rs} g_{rs} W^{(r,s)}$ which is a gauge-inequivalent deformation of the background gauge field. We briefly discuss a scenario for a background independent formulation of the theory.

In section 11, we point out the possible connection of this work with that of Vafa and Ooguri [32] on the $N = 2$ string field theory and also self-dual Einstein gravity in 4-dimensions.

While this work was in progress we received several papers which partly overlap with this work [33,34, 35]. The gauge group discussed in this paper has also been independently discussed by S. Rajeev [36].

2. Fermion field theory

The mapping of the one dimensional matrix model onto a theory of non-relativistic fermions in a potential is a most fortunate yet mysterious circumstance, especially when viewed from the viewpoint of the two dimensional string theory. The double scaled fermion field theory is described by the action

\[ S = \int dt \ dx \psi^+(x,t) \left[ i\partial_t + \frac{1}{2} \partial_x^2 + \frac{1}{2} x^2 \right] \psi(x,t) \]  

(2.1)

Since the fermions are non-interacting the energy of each fermion is separately conserved. This implies that the sum over any power of the individual energies is also conserved. These conserved charges are given by $Q_n = \int dx \psi^+(x,t) h^n \psi(x,t)$, $n = 0, 1, 2 \cdots$ where $h = \frac{1}{2}(p^2 - x^2)$ and $p = -i \partial_x$ [6, 7]. It is easy to find local conservation laws with charge and current densities given by [27, 28,29]

\[ J_n^0 = \psi^+(x,t) h^n \psi(x,t), \quad J_n^1 = \frac{i}{2} (\partial_x \psi^+(x,t) h^n \psi(x,t) - \psi^+(x,t) \partial_x h^n \psi(x,t)) \]  

(2.2)

which satisfy $\partial_t J_n^0 - \partial_x J_n^1 = 0$ by the equations of motion. These conservation laws
are statements of the global symmetries of the fermion action (2.1)

\[ \delta \psi(x, t) = i \sum_{n=0}^{\infty} \alpha_n h^n \psi(x, t) \quad \delta \psi^+(x, t) = -i \sum_{n=0}^{\infty} \alpha_n h^n \psi^+(x, t) \] (2.3)

For reasons which will be clear later, we will now enlarge the framework of our discussion by considering a model in which these symmetries are *gauged*. This means we introduce gauge fields which couple to these currents and assign transformation rules to them so that the transformations in (2.3) are symmetries with parameters \( \alpha_n(x, t) \) which are arbitrary functions of \( x \) and \( t \). The transformations in (2.3), when \( \alpha_n \)'s are constants, involve specific linear combinations of operators of the form \( x^m \partial_x^n \) acting on the fermion fields. We shall consider instead a model in which the gauge transformations involve arbitrary combinations of these operators.

### 3. Gauge theory of the group of unitary transformations in a Hilbert space

The framework hinted at in the previous section may be best described in the following way. The fermion field \( \psi(x, t) \) is viewed as a vector in a Hilbert space \( \mathcal{H} \) such that

\[ \psi(x, t) \equiv \psi_x(t) = <x|\psi(t)> \]

The index \( x \) labels the component of the vector. In the following we shall sometimes denote \( |\psi(t)> \) by \( \psi(t) \). Now consider the action of unitary operators \( \mathcal{U} \) on \( \psi \), \( \psi \rightarrow \mathcal{U}\psi \). This is clearly a symmetry of the free fermion theory

\[ S_0 = \int dt <\psi(t)|i\partial_t|\psi(t)> = \int dt \int dx \psi^+_x(t)i\partial_t\psi_x(t) \] (3.1)

The symmetry may be gauged by introducing a self-adjoint gauge field \( A(t) \) (\( A(t) = \)
The action

\[ S = \int dt \langle \psi(t) | i \partial_t + A(t) | \psi(t) \rangle \]  

is then gauge invariant under the transformations

\[ |\psi(t)\rangle \rightarrow U(t)|\psi(t)\rangle \]

\[ A(t) \rightarrow U(t) A(t) U^+(t) + iU(t) \partial_t U^+(t) \]  

Clearly the set of unitary transformations form a group. \( U \) may be parameterized as \( U = \exp(i\epsilon) \) where \( \epsilon = \epsilon^+ \).

We can realize the hilbert space \( \mathcal{H} \) in terms of the space of functions on the real line \( \mathbb{R}^1 \). Then \( A(t) \) can be considered as a function of the basic operators \( x \) and \( p \) which satisfy the commutation relations \([x, p] = i\hbar\). The components of \( A(t) \) are obtained by expanding it in terms of the set of self adjoint operators \((x^m p^n + p^n x^m)\); \( m, n \geq 0 \).

\[ A(t; x, p) = \sum_{m,n=0}^{\infty} A_{mn}(t)(x^m p^n + p^n x^m) \]  

The operators \( \hat{l}_{mn} = x^m p^n \) form a closed algebra:

\[ [\hat{l}_{mn}, \hat{l}_{rs}] = \sum_p (-i)^p (C(n, p)r_p - C(s, p)m_p) \hat{l}_{m+r-p, n+s-p} \]  

Here \( n_p \equiv n(n-1) \cdots (n-p+1), C(n, p) = n_p/p! \) when \( n \geq p \), and 0 otherwise. The sum over \( p \) is through non-negative integers, there are only finite number of terms because the coefficients vanish for large enough \( p \) by the above definitions. This algebra is actually isomorphic to the algebra called \( W_{1+\infty}^+ \) (see section 5).
we choose the representation $p = -i\partial_x$ then

$$A_{xy}(t) = <x|A(t; x, p)|y> = \sum_{m,n=0}^{\infty} A_{mn}(t)[(x^m + y^m)(i\partial_x)^n]\delta(x - y) \quad (3.6)$$

A similar expansion holds for the parameter $\epsilon$ of gauge transformations, where $U \sim 1 + i\epsilon$ and $\epsilon$ is hermitian. One has

$$\epsilon_{xy} = <x|\epsilon(t)|y> = \sum_{n=0}^{\infty} [(\epsilon_n(x) + \epsilon_n(y))(i\partial_x)^n]\delta(x - y) \quad (3.7)$$

We also define $\epsilon_{mn}$ by

$$\epsilon_n(x) = \sum_{m=0}^{\infty} \epsilon_{mn}x^m$$

The infinitesimal gauge transformation (3.3) may be written in component form as

$$\delta\psi(x, t) = i \int dy \epsilon_{xy}(t)\psi(y, t) \quad \delta\psi^+(x, t) = -i \int dy \psi^+(y, t)\epsilon_{yx}(t) \quad (3.8)$$

and

$$\delta A_{xy}(t) = i\mathcal{D}\epsilon = i(i\partial_t\epsilon(t) + [A(t), \epsilon(t)])_{xy} \quad (3.9)$$

In terms of these components the action becomes

$$S = \int dt \ dx \ dy \ \psi_x^+(t)[i\partial_t\delta(x - y) + A_{xy}(t)]\psi_y(t) \quad (3.10)$$

The fermionic field theory description of the singlet sector of the $d = 1$ matrix model is related to this gauge theory in the following way. The fermion field theory (2.1) may be easily seen to be the gauge theory (3.10) in a specific and fixed
background gauge field given by \( \bar{A} \)

\[
\bar{A}(t; x, p) = -\hbar = -\frac{1}{2}(p^2 - x^2) \tag{3.11}
\]

or, in terms of components

\[
\bar{A}_{xy}(t) = \langle x| A(t; x, p)|y\rangle = \frac{1}{2}(\partial_x^2 + x^2)\delta(x - y) \tag{3.12}
\]

Thus we shall be concerned with the above formulated gauge theory in \textit{fixed} backgrounds.

In the classical limit as \( \hbar \to 0 \) (\( \hbar \) plays the role of string coupling) we have the usual correspondence. The commutator \([x, p] = i\hbar\) is replaced by the Poisson bracket and \( x \) and \( p \) are coordinates on a phase space. Operators like the gauge fields then become functions on the classical phase space and the algebra of unitary transformations goes over to the algebra of area preserving diffeomorphisms of the phase space. This algebra is generated via Poisson brackets, the generators \( l_{mn} = x^n p^m, \; n, m \geq 0 \) satisfy

\[
\{l_{mn}, l_{rs}\}_{PB} = (ns - rm)l_{n+r-1, m+s-1} \tag{3.13}
\]

4. Ward Identities

We now turn to some consequences of the gauge symmetries, viz. Ward identities. Consider the functional integral:

\[
\mathcal{Z}[\bar{A}] = \int \mathcal{D}\bar{\psi}_x(t)\mathcal{D}\psi_x(t)e^{\frac{i}{\hbar}\int dt<\bar{\psi}(t)|i\partial_t + \bar{A}|\psi(t)>} \tag{4.1}
\]

where \( \bar{A} \) is an arbitrary background gauge field. Ward identities are a consequence of the invariance of the fermion measure under gauge transformations. We regard
the fermion measure to be invariant because we do not expect any anomalies for non-relativistic fermions. Since the action is gauge invariant, we have the identity

\[ Z[\bar{A}] = Z[\bar{A} + i\bar{D}\epsilon] \]  \hspace{1cm} (4.2)

where the covariant derivative \(\bar{D}\) has been defined in (3.9) and the bar means that the gauge field in \(\bar{D}\) is the background gauge field \(\bar{A}\).

Introducing the notation

\[ R_{yx}(t) = \frac{\delta}{\delta A_{xy}(t)}Z \]  \hspace{1cm} (4.3)

(4.2) can be written as a differential equation

\[ \int dt\, dx\, dy\, \bar{D}\epsilon_{xy}(t) R_{yx}(t) = 0 \]  \hspace{1cm} (4.4)

For gauge transformations which do not keep the background gauge field invariant, i.e. \(\bar{D}\epsilon \neq 0\), we can integrate by parts in (4.4) and arrive at the Ward identities or "transversality conditions"

\[ i\partial_t \bar{R}_{yx} + [\bar{A}(t), \bar{R}(t)]_{yx} = 0 \]  \hspace{1cm} (4.5)

In the special case when \(\bar{A} = -(p^2 - x^2)/2\) the fermion theory corresponds to the standard double scaled matrix model. These identities then imply an infinite set of relations between correlation functions of that model. In this paper we shall not pursue further equation (4.5) especially with regard to important questions of boundary conditions and the ability to explicitly evaluate the correlation functions.
5. Residual Gauge Symmetry: $W_{1+\infty}^{(+)}$

Given a particular background gauge field $\bar{A}(t)$, a generic gauge transformation $\epsilon(t)$ would not leave the gauge field invariant. It is an interesting question to ask what is the set of gauge transformations $\epsilon(t)$ that do leave the background invariant. Clearly such gauge transformations would form a subgroup of the original gauge transformations, for instance in case of the 'tHooft-Polyakov monopole the residual subgroup was a $U(1)$ subgroup of the original gauge group $SO(3)$.

These special gauge transformations satisfy

$$\mathcal{D}\epsilon = i\partial_t \epsilon + [\bar{A}(t), \epsilon] = 0 \quad (5.1)$$

The general solution of (5.1) is given by

$$\epsilon(t) = U(t)\epsilon(0)U(t)^{-1} \quad (5.2)$$

where $U(t) = T \exp i \int^t A(t') dt'$ is a unitary operator and $\epsilon(0) \equiv \epsilon(x, p)$ is as yet an arbitrary operator on the single particle Hilbert space.

The generators of the special gauge transformations in the field theory are given by

$$W[\epsilon, t] = \int dx \psi^+ (x, t) \epsilon_{xy}(t) \psi (y, t) = \langle \psi(t)|\epsilon(t)|\psi(t) \rangle \quad (5.3)$$

Using (5.1) and the equation of motion of $\psi(x, t)$, it is easy to check the expected result $dW[\epsilon, t]/dt = 0$.

We now make the specific choice of expanding it in terms of the basis of generators $\hat{l}_{mn} = x^m p^n$. This basis is well defined and general enough. In particular it includes the single particle Hamiltonian and the generators of the Lie algebra...
$SL(2)$. In terms of this basis we have $W(\epsilon(t)) = \sum_{mn} \tilde{\epsilon}_{mn} W_{mn}(t)$ where

$$W_{mn}(t) = \int dx \psi^+(x,0) \hat{\imath}_{mn}(t) \psi(x,0) \quad (5.4)$$

and we have defined $\hat{\imath}_{mn}(t) = U_t \hat{i}_{mn} U_t^{-1}$. The constants of motion satisfy the same algebra as that of the single-particle generators $\hat{i}_{mn}$. Therefore the residual symmetry algebra around any background is the same.

We now discuss the case of the background (3.11) which is explicitly time-independent. In that case $U_t = \exp(i\bar{A}t)$ and we can look for solutions of the form $\epsilon(t) = \exp(i\bar{A}t) \epsilon(0)$.

Equation (5.1) then becomes the eigenvalue problem

$$\text{ad} \bar{A} \cdot \epsilon = E \epsilon \quad (5.5)$$

where $\text{ad} \bar{A}$ denotes the adjoint action of $\bar{A}$. It is convenient to introduce the eigenoperators $d_\pm \equiv \frac{1}{\sqrt{2}}(x \pm p)$ such that

$$\text{ad} \bar{A} \cdot d_\pm = \pm id_\pm \quad (5.6)$$

Now since the action of $\text{ad} \bar{A}$ is associative, we have

$$\text{ad} \bar{A} \cdot (d_\pm)^m = \pm i m \ (d_\pm)^m \quad (5.7)$$

where $m$ is a non-negative integer. A solution of (5.1) is, therefore, labelled by two positive integers and can be written as

$$\epsilon_{xy}^r(t) \equiv \epsilon^r(t;x,y) = \frac{1}{2\dagger}\{E^r_+, E^s_-\}_x \delta(x-y) \quad (5.8)$$

where we have defined

$$E_\pm \equiv e^{\mp t} d_\pm \quad (5.9)$$

The hermitian charges that generate the special gauge transformations are
given by (5.3) as
\[
W^{(r,s)} = \int dx \psi_+^+(t) \epsilon_{rs}(t; x, y) \psi_+(t)
\] (5.10)

They can be expressed as a linear combination of the $W_{mn}$ of equation (5.4).

Let us now make correspondence with similar algebras that exist in the literature. The operators $W^{(r,s)}$ form a closed algebra under commutation, which is identical to the algebra of the single particle operators $\epsilon_{rs}(t)$. This is the algebra $W_{1+\infty}^{(+)}$. The generators of the standard $W_{1+\infty}^{(+)}$ algebra are linear combinations of the $W^{(r,s)}$. These can be constructed following the construction of $W_{1+\infty}$ algebra as an enveloping algebra of a $U(1)$ Kac-Moody algebra with a derivation given in [30]. In the present case we take the modes of the $U(1)$ current to be $E_m^r$, $m \geq 0$ and for the derivation we take $i$ times the single particle hamiltonian,
\[
h = \frac{i}{2} (p^2 - x^2) = -\frac{i}{2} \{ E_+, E_- \}.
\]
These satisfy the requirements of the construction given in [30]. Then using the notation $V^j_m$ for the $W_{1+\infty}$ generators where the spin is $(j+2)$ and the mode is $m$ we have the recursion relation which determines them
\[
V^j_m = (ih + \frac{m}{2})V^{j-1}_m + \frac{j^2(j^2 - m^2)}{4(4j^2 - 1)} V^{j-2}_m
\] (5.11)
where
\[
V^{-1}_m = E_m^-
\] (5.12)
and
\[
V^0_m = (ih + \frac{m}{2})E_m^- = -\frac{i}{2} \{ E_+, E_-^{m+1} \}
\] (5.13)
For example
\[
V^1_m = -\frac{1}{2} \{ E_+^2, E_-^{m+2} \} - \frac{1}{3} (m + 1)(m + 2)E_m^-
\] (5.14)
It is clear that in this way we can express any $V^j_m$ as a linear combination of the $\epsilon_{rs}$. Since $r$ and $s$ are restricted to be non-negative we see that the $V^j_m$ thus obtained are restricted to $j \geq -1$ and $m \geq -j + 1$. In this way we can construct the standard $W_{1+\infty}^{(+)}$ algebra from linear combinations of $W^{(r,s)}$. 

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We mention some other important properties of the operators $W^{(r,s)}$.

(i) $W^{(r,s)}$ are time independent. It turns out these conservation laws are implied by the local conservation of the following currents

\[ J^0_{r,s}(x,t) = \psi^+ (x,t) \epsilon_{rs} \psi(x,t) \]
\[ J^1_{r,s}(x,t) = \frac{i}{2} [\partial_x \psi^+ (x,t) \epsilon_{rs} \psi(x,t) - \psi^+ (x,t) \partial_x \epsilon_{rs} \psi(x,t)] \]
\[ \partial_t J^0_{rs} - \partial_x J^1_{r,s} = 0 \] (5.15)

(ii) For $r = s$, $J^\mu_{r,r}$ can be expressed as linear combinations of the currents given in (2.2). The corresponding charges are the set $Q_r = W^{(r,r)}$. Clearly these charges commute among themselves

\[ [W^{(r,r)}, W^{(r',r')}] = 0 \] (5.16)

and $-W^{(1,1)}$ is the hamiltonian.

(iii) The $SL(2)$ in $W^{(+)}_{1+\infty}$:

The operators $W^{(r,s)}$ have an interesting $SL(2)$ structure. The $SL(2)$ structure has appeared in the discussions of the continuum theory in [22]. In fact, it can be shown that linear combinations of the $W^{(r,s)}$ fall into $SL(2)$ multiplets. The $SL(2)$ is generated by

\[ J_+ = \frac{i}{2} W^{(2,0)}, \quad J_- = \frac{i}{2} W^{(0,2)}, \quad J_3 = -\frac{i}{2} W^{(1,1)} \] (5.17)

They satisfy the standard algebra

\[ [J_+, J_-] = -2J_3, \quad [J_3, J_\pm] = \pm J_\pm \] (5.18)

and the quadratic casimir is

\[ C_2 = J_3^2 - \frac{1}{2} (J_+ J_- + J_- J_+) \] (5.19)

The set of operators that form an $(n+1)$ dimensional representation of the $SL(2)$ may be constructed as follows. One starts with the operator $W^{(n,0)}$. The next
member of the set is given by the commutator \([J_-, W^{(n,0)}]}\). Taking a further commutator gives the next member \([J_-, [J_-, W^{(n,0)}]}]\) and so on. One can easily show that the last operator in the chain is \(W^{(0,n)}\) and so the chain stops after \(n\) steps. The resulting set of \((n+1)\) operators forms the \((n+1)\) dimensional representation of \(SL(2)\) by construction. (One could have equivalently started with \(W^{(0,n)}\) and obtained the above multiplet by the repeated action of \(J_+\)).

Let us denote a member of a given multiplet by \(W^{(r,s)}\). This is obviously a linear combination of the \(W^{(r,s)}\). For example,

\[
W^{(n,0)} = W^{(n,0)}, \quad W^{(n,1)} = W^{(n,1)}, \\
W^{(n,2)} = W^{(n,2)} - 2n(n - 1)W^{(n-2,0)} \ldots
\]

(5.20)

It may be easily checked that the \(J_3\) eigenvalue of \(W^{(r,s)}\) is \((\frac{r-s}{2})\) while the quadratic casimir on it gives the eigenvalue \((\frac{r+s}{2})(\frac{r+s}{2} + 1)\). The operators \(J_3\) and the quadratic Casimir act by adjoint action on \(W^{(r,s)}\). Thus the two numbers that specify any member \(W^{(r,s)}\) of a given \(SL(2)\) multiplet are \((r-s)\) and \((r+s)\) respectively.

6. On representation of \(W^{(+)}_{1+\infty}\) in the fermion field theory in an inverted harmonic oscillator potential

In the previous section we considered the algebraic structure of the group of residual gauge transformations. Now we consider the important question of its representation in the fermionic field theory.

As is well-known, the fermionic field theory (2.1) can be built entirely from the knowledge of the single-particle states. These are given by parabolic cylinder functions [2,8]. The ground state \(|\Omega\rangle\) of the field theory is obtained by filling all the single-particle levels upto the fermi level \(\mu\). Let us call the distribution of single-particle energy levels \(\rho(E)\). Using this we can easily calculate the energy of the ground state; in fact we can compute the eigenvalues of all the commuting generators \(W^{(r,r)}\) which are linear combinations of the form \(\sum_{\ell=0}^r c^\ell \int \psi^+(x,t)\hbar^l \psi(x,t),\)
where \( c_r^l \) are constants, in the ground state by computing moments of \( \rho(E) \). The results are, after subtracting an infinite constant:

\[
e_r = \sum_{l=0}^{r} c_r^l \int_\mu^\infty dE \rho(E) E^l
\]

Now consider the action of the generator \( W^{(r,s)}, r \neq s \) on the ground state:

\[
|r, s> = W^{(r,s)}|\Omega>
\]  

(6.1)

Using the commutation relation \([W^{(1,1)}, W^{(r,s)}] = i(r - s)W^{(r,s)}\), which is basically a reflection of the single-particle relation (5.6), we see that

\[
W^{(1,1)}|r, s> = (e_1 + i(r - s))|r, s>
\]  

(6.2)

Hence the Hamiltonian of the theory, which is just \(-W^{(1,1)}\), has a complex eigenvalue in the state \(|r, s>\). Since on the other hand we can explicitly construct all the states of the field theory in terms of multiple electron-hole excitations and they all have real energies, (remember parabolic cylinder functions have real energy eigenvalues), this means that the state \(|r, s>\) cannot be expressed as a linear combination of the complete set of states in the field theory; in other words the \( W^{(+)}_{1+\infty} \) transformation of the ground state takes it out of the Hilbert space.

Let us explain the last comment in a little more detail. Let us expand the second quantised fermi field in terms of parabolic cylinder wave-functions and the corresponding creation-annihilation operators of the single-particle states. Now the action of \( W^{(r,s)} \) on the ground state would generically involve applying the operators \( d_\pm = \frac{1}{\sqrt{2}} \left( x \mp i \frac{\partial}{\partial x} \right) \) on parabolic cylinder functions. If one looks up the asymptotic behaviour of parabolic cylinder functions with energy eigenvalue \( \lambda \in \mathbb{R} \), one finds that they behave like \( \psi \sim 1/\sqrt{x} \exp if(x) \) where \( f(x) = \text{constant} \ x^2 + \text{constant} \ \lambda \log(x) + \cdots \). The oscillatory behaviour is characteristic of the fact that
these represent scattering states. It is easy to see that the action of the operators $d_{\pm}$ amounts to changing the eigenvalue $\lambda$ by $\pm i$ which implies that even though $\psi$ vanishes at $\pm \infty$ the boundary conditions of the new eigenfunctions obtained this way are different. For example the action of $d_+$ results in a new wavefunction which blows up at $\infty$. Clearly such a wave-function cannot be expressed as a linear combination of $\psi(\lambda, x)$’s for real $\lambda$’s. We say that $d_+$ does not have a well defined action on the Hilbert space of the parabolic cylinder functions. The same is true for a generic differential operator formed as a finite linear combination of $d'_+, d''_-$ or equivalently $x^m p^n$.

At this point let us recall the discussion after equation (5.2). There the operator $\epsilon(x, p)$ was unspecified and we made a choice of expanding it in terms of $\hat{l}_{mn}$. Notwithstanding the virtues of this basis one can certainly imagine a class of operators $\epsilon(x, p)$, closed under commutation, which have well defined action in the Hilbert space of parabolic cylinder functions. An example are the operators $f_{\alpha\beta}(x, p) = \exp(i\alpha x + i\beta p)$ for $\alpha$ and $\beta$ arbitrary real numbers. From the viewpoint of the classical limit (which will be subsequently discussed in section 9) one is distinguishing between generators of canonical transformations which are expressed as linear combinations of the sets $F = \{f_{\alpha\beta}(x, p) = \exp(i\alpha x + i\beta p)\}$ and $L = \{\hat{l}_{mn}(x, p) = x^m p^n\}$.

In the next section we shall see that the basis $L$ indeed has a well defined action in the Hilbert space of Hermite functions, basically because these functions represent bound states and have an exponential decay at infinity.
7. Analytic continuation of the fermion field theory

In this section we motivate and discuss the analytic continuation of the fermion field theory in which the algebra of residual gauge transformations can be realised in the $L$ basis.

Let us recall that the fermion field theory (2.1) can be expressed as a Feynman path integral over the classical trajectories of the fermions, which are governed by the action

$$iS_M = i \int dt(\frac{\dot{x}^2}{2} + \frac{x^2}{2}) \quad (7.1)$$

The classical equation of motion $d^2x/dt^2 - x = 0$ is solved by the hyperbolic functions $x(t) = A \cosh(t + \theta)$. The canonical momentum is given by $p(t) = \dot{x}(t) = A \sinh(t + \theta)$. The hamiltonian is then defined by $h(x, p) = p\dot{x} - (\dot{x}^2 + x^2)/2$, and evaluates to $h(x, p) = (p^2 - x^2)/2 = -A^2/2$. Hence constant energy trajectories in phase space are given by hyperbolas.

Now consider the analytic continuation of time $t \rightarrow it$ to a Euclidean picture. The Euclidean action corresponding to the Minkowski action (7.1) is

$$S_E = \int dt(-\frac{\dot{x}^2}{2} + \frac{x^2}{2}) \quad (7.2)$$

The classical equation of motion $\partial_t^2 x + x = 0$ (simple harmonic oscillator) are solved by periodic functions $x(t) = A \cos(t + \theta)$. The canonical momentum is given by $p = \partial h/\partial \dot{x} = -\dot{x} = A \sin(t + \theta)$ and the hamiltonian is $h(x, p) = p\dot{x} + \frac{1}{2}(\dot{x}^2 - x^2) = -1/2(p^2 + x^2) = -1/2A^2$. The constant energy trajectories are given by circles of radius $A$.

From the above discussion we deduce that the standard analytic continuation of time $t \rightarrow it$ in the coordinate space formulation corresponds to the analytic continuation $t \rightarrow it$ and $p \rightarrow -ip$ in the phase space formulation. The analytic continuation is illustrated in Figs 1a and 1b.
In the quantum theory the orbits are appropriately quantised and the ground state is obtained by filling the fermi sea. In figures 1a and 1b we have indicated this by the shaded regions $x^2 - p^2 \leq \mu$ and $x^2 + p^2 \geq \mu$ respectively. $\mu$ is the fermi energy and defines the string coupling $g_{st} = 1/\mu$. In the classical limit $g_{st} \to 0$ the states of the field theory can be described in terms of a fermi fluid and we shall investigate in detail the classical solutions (instantons) of the Euclidean field theory in terms of motion of this fluid in section 9.

We wish to emphasize that in quantising the classical phase space (Fig 1b) obtained by the analytic continuation $t \to it$ and $p \to -ip$, we are going beyond the standard Euclidean continuation of quantum mechanics. The reason for this is that in the standard Euclidean continuation, the quantum hamiltonian, obtained by the transfer matrix method, does not change. The analytic continuation we have performed changes the quantum hamiltonian from that of the inverted harmonic oscillator $h = 1/2(p^2 - x^2)$ to that of the ordinary harmonic oscillator $h = -1/2(p^2 + x^2)$. The final result of our analytic continuation is identical ( upto the overall sign of the hamiltonian) to the result obtained by Gross and Milkovic [3] who regarded the inverted harmonic oscillator as the usual harmonic oscillator with an imaginary frequency. The verity of their procedure and also ours is well supported by the fact the correlation function calculation in Danielssen-Gross [22] agrees completely with that of Moore [8].

We end this section by discussing the single-particle levels and the fermi sea. The single-particle levels are the hermite functions $H_n(x) = \langle x|n \rangle$ and the energy levels are $E_n = -(n+1/2), n = 0, 1, 2, \cdots$. The fermi sea is filled up from $n = \infty (E_n = -\infty)$ to some level $n = n_F (E_n \equiv E_F = -\mu)$. The semiclassical limit as $\mu \to \infty$ was described before (fig 1b). It is worth commenting that in Fig 1b the number of unoccupied levels is also of order $\mu$. This should be contrasted with the fact that the number of unoccupied levels in Fig 1a (Minkowski picture) is actually infinite. Hence the analytic continuation seemingly reduces the number of degrees of freedom. This seems to be the analogue for functional integrals of a similar phenomenon that occurs in the evaluation of real integrals as a sum over
poles.

8. Representation of $W_{1+\infty}^{(+)}$ in the analytically continued fermion field theory

Most of the algebraic steps of section 5 in the single-particle quantum mechanics can be repeated with the substitution $p \to -ip$. In particular $d_\pm = \frac{1}{\sqrt{2}}(x \pm p) \to -\frac{i}{\sqrt{2}}(ix \pm p)$. Let us denote $p - ix = a$ and $p + ix = a^\dagger$ in the standard way. Since $t \to it$, the gauge field $A$ is also analytically continued $A \to -iA$ so that $dt$ $A$ is left unchanged. Hence the residual gauge transformation (similarly to (5.1)) satisfies

$$i\partial_t \epsilon + [\bar{A}, \epsilon] = 0 \quad (8.1)$$

Substituting $\bar{A} = (p^2 + x^2)/2 = -h$ in (8.1) we get

$$i\partial_t \epsilon - [h, \epsilon] = 0 \quad (8.2)$$

Once again, we write $\epsilon(t) = \exp(iEt)\epsilon(0)$ and get the eigenvalue problem

$$-adh.\epsilon = E\epsilon \quad (8.3)$$

Now since $(p^2 + x^2)/2 = (a^\dagger a + aa^\dagger)/4$ therefore $-adh.a = a$ and $-adh.a^\dagger = -a^\dagger$. Hence

$$-adh.a^r a^s = (r - s)a^r a^s \quad (8.4)$$

Denote $a^r a^s = \epsilon_{rs}$ and note that $\epsilon_{rs} = \epsilon_{sr}^\dagger$. The general solution of (8.2) is now given by

$$\epsilon(t) = \sum_{r,s} e^{i(r-s)t} \epsilon_{rs} \alpha_{r,s} \quad (8.5)$$

where $\alpha_{r,s}^* = \alpha_{s,r}$ ensuring that $\epsilon^\dagger = \epsilon$. 

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The generators of the residual gauge transformation in the fermi field theory are given by

\[ W^{(r,s)} = \int dx \psi^+(x,t) \epsilon_{rs} \psi(x,t), \quad W^{(r,s)\dagger} = W^{(s,r)} \] (8.6)

They are constants of motion. The parameters \( \epsilon_{rs} = a^r a^{\dagger s} \) again satisfy the \( W_{1+\infty}^+ \) algebra and imply the same for \( W^{(r,s)} \).

We can realise \( W^{(r,s)} \) in terms of the constituent modes of the fermion field which can now be expanded in terms of hermite polynomials

\[ \psi(x,t) = \sum_{n=0}^{\infty} c_n e^{-\frac{1}{2}x^2} H_n(x) \]

so that

\[ W^{(r,s)} = \sum_{m,n=0}^{\infty} <n|a^r a^{\dagger s}|m> c_m^\dagger c_n \]

\[ = \sum_{n=0}^{\infty} <n+s-r|a^r a^{\dagger s}|n> c_{n+s-r}^\dagger c_n \] (8.7)

Note that \( \epsilon_{rs} = a^r a^{\dagger s} \) have well defined action on the Hilbert space of Hermite polynomials.

Now consider the action of \( W^{(r,s)} \) on the ground state \( |\Omega> \), which we constructed in section 7. Identical to our discussion in section 5, we see that the vacuum is an eigenstate of all \( W^{(r,r)} \), with eigenvalues which are moments of the level density \( \rho(E) \), which were originally calculated within this formulation in Gross-Milkovic [3]. For \( r \neq s \), we see that

\[ W^{(r,s)} |\Omega> = 0, \quad r < s \] (8.8)

and the state \( |r,s> = W^{(r,s)} |\Omega> \), \( r > s \) is an eigenstate of the hamiltonian \( W^{(1,1)} \) with eigenvalue \( e_1 + (r-s) \),

\[ W^{(1,1)} |r,s> = e_1 + (r-s) |r,s>, \quad r > s \geq 0 \] (8.9)

It is important to note that we can indeed generate all the states of the fermi field theory by acting with the \( W^{(r,s)} \)'s on the fermi sea \( |\Omega> \) sufficient number of.
times. The reason is that all the states of the field theory can be generated from the ground state by exciting fermions from below the fermi level to above it: if the initial state of the fermion was $n$ and the final state $m$ then the elementary excitation is given by the operator $E_{mn} \equiv c^\dagger_m c_n$. A generic state of the field theory can therefore be written as a certain number of these $E$-operators acting on the ground state $|\Omega \rangle$. Now equation (8.7) expresses the $W^{(r,s)}$ as a linear combination of the $E_{mn}$'s, it is easy to show that the $E_{mn}$'s can also be expressed as linear combinations of the $W^{(r,s)}$'s (i.e. the equation (8.7) is invertible). The easiest way to prove this is to consider the basic boson operator of the theory: the bilocal operator $\Phi(x, y) \equiv \Psi(x)\Psi(y)$. The $c^\dagger c$ can be clearly constructed out of the $\Phi(x, y)$ by taking appropriate moments, on the other hand $\Phi(x, y) = \sum_{r,s}(-1)^sC(r, s)y^r x^s(\frac{\partial}{\partial x})^r\Psi(x)$, $C(r, s) = r!/((r - s)!s!)$ which means that moments of $\Phi(x, y)$ can be constructed out of the generators $W^{(r,s)}$.

The upshot of the above remarks is that the Fock space of the (analytically continued) fermion field theory constitutes one irreducible representation of the algebra $W_{1+\infty}$. (Clearly any state can be reached from any other by the operators $E_{mn}$ hence by the operators $W^{(r,s)}$).

The other important remark is that since we have described a complete list of states in the theory, there are no tachyon states in the spectrum with continuously varying energy-momentum. This however is not in conflict with the fact that the analytically continued theory meaningfully defines the correlation functions of external tachyons with continuous momentum.
9. The Classical Limit

In this section we describe the semi-classical limit of the fermi field theory in terms of a fermi fluid in the single-particle phase space \([1,17]\). In this limit states of the single-particle theory are described by small cells of area \(2\pi\hbar\) in the 2dim phase space, and states of the field theory are described by specifying which of these single-particle states are occupied. Remembering that each single-particle state can accommodate only one fermion, a state in the field theory corresponds to a region of the 2dim phase space uniformly filled with a fermi fluid (for simplicity we shall consider a single connected region). If we describe the curve bounding this region by the equation \(f(x,p) = 0\) the function \(f\) would then specify the state of the field theory in the semi-classical limit.\(^\star\) Of course, in general the fluid profile \(f\) would evolve in time, because each fermion would evolve according to the single-particle hamiltonian, tracing out circles (in the case of the harmonic oscillator with angular frequency 1) and unless the profile is a circle to start with, it will change from \(f(p,q) = 0\) to \(f_t(p,q) = 0\). The dynamics of the classical field theory consists in solving the motion of the fluid profiles. In section (9.3) we will explicitly solve and classify all the classical solutions of the euclidean field theory.

We described how states are represented semiclassically. How about operators? Since the fermi field theory is quadratic, it is enough to consider a generic fermion bilinear \(G \equiv \int dx\psi^+(x,t)g(x, -i\partial/\partial x, t)\psi(x,t)\). In the semi-classical picture this simply measures the total amount of \(g(x,p,t)\) carried by all fermions in the fluid. In other words, if we consider a state of the field theory \(|f>\) described by a fluid profile \(f\) then we have the Thomas-Fermi correspondence

\[
< G(t) >_f = < f | \int dxg(x, -i\partial/\partial x, t)|f > \sim \int \int_{R_t} dxdpg(x, p, t)
\]  

(9.1)

\(^\star\) We should remark that the states which are appropriate for describing the semi-classical limit are coherent states; in fact in our problem they are coherent states of the \(W\)-algebra; the reason is, as we shall see later we can co-ordinatise the classical phase space in terms of the classical values of the \(W\)-generators - which in particular means that different non-commuting generators should have sufficiently well-defined values, which is typical of coherent states of a group.
where $R_f$ denotes the region bounded by the fluid profile at time $t \quad f_t(x,p) = 0$.

As we remarked before, the states $|f>$ are coherent states.

We can see how most quantities in the classical field theory are related to quantities in the single-particle phase space. It is appropriate therefore to understand the $W$-algebra and in particular the sub-algebra $SL(2)$ that we found in the quantum theory in this 2dim classical phase space.

9.1. **Representation of $SL(2)$ and the $W$-algebra in 2dim phase space**

The basic reason why the above algebras appeared in our quantum theory is that the single-particle classical phase space naturally carries a representation of $w_\infty$ and even more naturally of $SL(2)$; the former classically is the set of area-preserving diffeomorphisms in two dimensions and it is precisely in two dimensions that the set of area-preserving diffeomorphisms is also the one which preserves the poisson bracket. Therefore $w_\infty$ is simply the algebra of all canonical transformations. $SL(2)$ by definition is the set of real two-by-two matrices of determinant one, in other words they are linear transformations which preserve area. Thus $SL(2)$ transformations are canonical transformations, and therefore the $SL(2)$ subalgebra of $w_\infty$ must be simply those subset of canonical transformations which act linearly on the coordinates $(x,p)$! We shall see this now explicitly.

A canonical transformation is generated by taking Poisson bracket with some function $f(x,p)$:

$$x \rightarrow x + \{x, f\}, \quad p \rightarrow p + \{p, f\} \quad (9.2)$$

A generic $f$ can be expanded in the basis of the monomials $f^{(r,s)} = x^r p^s$ The $W^{(r,s)}$ acts on coordinates as

$$W^{(r,s)} : (x,p) \rightarrow (x + \{x, f^{(r,s)}\}_{PB}, y + \{y, f^{r,s}\}_{PB})$$

or

$$\delta_{rs}(x,p) = (sx^r p^{s-1}, -rx^{r-1} p^s) \quad (9.3)$$

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When the function $f$ is quadratic, the canonical transformation is linear. In particular, if one takes the basis $f = x^2, -p^2, xp$ of the quadratics, then we find from (9.2) that the corresponding canonical transformations have the effect of multiplying the column vector $(x, p)^T$ by the Pauli matrices $\sigma_+, \sigma_-$ and $\sigma_3$ respectively. This explains very naturally why $SL(2)$ appears in $w_{\infty}$, and specifically as $W^{(2,0)}, W^{(1,1)}$ and $W^{(0,2)}$.

Commuting two $W$-flows corresponds to taking Poisson brackets of the corresponding generating functions. The $f^{(r,s)}$'s form a closed Poisson bracket algebra:

$$\{f^{r,s}, f^{p,q}\}_{PB} = (rq - sp)f^{r+p-1,s+q-1} \quad (9.4)$$

It's interesting to see how the special $SL(2)$ sub-algebra acts on the rest of the generators. (9.4) gives

$$\{x^2, x^r p^s\}_{PB} = 2sx^{r+1}p^{s-1}$$
$$\{p^2, x^r p^s\}_{PB} = -2rx^{r-1}p^{s+1} \quad (9.5)$$
$$\{xp, x^r p^s\}_{PB} = -(r - s)x^{r} p^{s}$$

The first observation is that, the degree of a polynomial (assumed homogeneous in $x, p$) is preserved under Poisson bracket with any of the $SL(2)$-generators. Thus, for instance the monomials $\{p^n, xp^{n-1}, \cdots, x^n p, x^n\}$ are transformed into one another by the $SL(2)$ action, where all of these have degree $n$. Clearly there are $2n+1$ such monomials of degree $n$, thus the dimension of the $SL(2)$ representation is $2n+1$. This tells us that the $j$-value (Casimir= $j(j+1)$) for the representation is $j = n$, simply the degree of the Polynomial (it is easy to see the above representation is irreducible, for instance the action of $x^2$ takes one down from any basis element to the next). For later use, we remark that the degree of a polynomial in $x, p$ is represented by the operator

$$J = x\partial/\partial x + p\partial/\partial p \quad (9.6)$$

Thus $J$ measures the $j$-spin of a representation.
Thus we conclude that the entire $W$-algebra splits into irreps of the $SL(2)$ with $r + s$ being the spin $j$ and $r - s$ being the $J_3$-value (remember in (9.5) the operator $xp$ plays the role of $J_3$ and the other two are the raising and lowering operators $J_{\pm}$).

In the next section, we shall consider the $W^{(r,s)}$ generators in the semi-classical field theory. From the next section, in order to correspond exactly to the earlier discussion in the quantum theory operators we shall redefine our $W^{(r,s)}$ in terms of functions

$$g^{(r,s)} = (p - ix)^r (p + ix)^s$$  \hspace{1cm} (9.7)

9.2. **Correspondence with vertex operators of Liouville theory**

Let us examine the classical limit of the generator $W^{(r,s)}$. By the Thomas-Fermi correspondence (9.1) we have

$$< W^{(r,s)}(t) > f \equiv w^{(r,s)}(t, f) = \int \int_{R_f} (p - ix)^r (p + ix)^s$$  \hspace{1cm} (9.8)

where we have put $g = g^{(r,s)}$ (equation (9.7)) in (9.1).

Introducing the polar co-ordinates $p = R \cos \theta, x = R \sin \theta$ (9.8) becomes

$$w^{(r,s)}(t, f) = \int \int_{R_f} d\theta dRRR^{r+s} e^{i(r-s)(t-\theta)}$$  \hspace{1cm} (9.9)

Parametrising the radius $R = \exp(-\tau), -\infty < \tau < \infty$, (9.9) becomes

$$w^{(r,s)}(t, f) = \int \int_{R_f} d\theta d\tau V^{(r,s)}(\tau, t - \theta)$$  \hspace{1cm} (9.10)
where

$$V^{(r,s)}(\tau, t) = e^{-2\tau-(r+s)\tau+i(r-s)t}$$ \hspace{1cm} (9.11)

It is intriguing to note that this is identical to the exponential part of the $r, s$ vertex operators in the $c = 1$ Liouville theory for one of the liouville dressings, where the time $t$ is taken to be Euclidean. Using (9.7) we see that the operator $J$ which measures the total $j$-spin is $-\partial/\partial \tau$. Since the $J$-spin coincides with Liouville momentum the identification of $\tau$ as the Liouville direction is natural. (To get the more standard expression we need to put $\tau = \phi/\sqrt{2}$ and rescale $t \rightarrow t/\sqrt{2}$).

Importantly we are finding here vertex operators with only quantised energies, implying in particular the absence of the tachyon vertex operators with continuously varying energy and momentum. The operators $r, 0$ and $0, s$ corresponds to the tachyon (energy$^2 +$ momentum$^2 = 0$) but only at integral values of the energy.

9.3. The classical solutions and the classical phase space of the field theory

We have argued in the beginning of this section that the semiclassical dynamics of the fermi field theory is determined by the motion of a fluid of uniform density in the single-particle phase space parametrised, for instance, by the equation $f = 0$ for its boundary (which we shall call its “profile”). This motion in turn is determined by how each fermi particle moves in the single-particle phase space under hamiltonian evolution. Let us ask the question how to solve for the motion of all possible profiles $f$ as a function of time. In other words, given a fluid boundary $f(x, p) = 0$ at time $t = 0$, what is the function $f_t$ such that $f_t(x, p) = 0$ describes the fluid boundary at time $t$.

The answer is suprisingly simple:

$$f_t(x, p) = f(x \cos t + p \sin t, -x \sin t + p \cos t)$$ \hspace{1cm} (9.12)

To prove it, one just needs to remember that all the fermions move in circles in the phase space under the single-particle hamiltonian at an identical angular speed
(which we have taken to be 1). Hence if you look from a rotating frame of angular speed one, all the particles would look static and so would the fluid boundary. In other words, the fluid moves like a rigid body in a two-dimensional plane rotating round the origin.

Let us point out one immediate consequence of the formula (9.12).

All classical solutions of the theory are periodic.

Proof: $f_t(x, p) = f_{t+2\pi}(x, p)$ irrespective of the initial function $f$. Q.E.D

Of course, this is intimately connected with the fact that we are talking here about the Euclidean field theory where the single-particle orbits have become periodic in phase space (instead of parabolic).

The space of classical solutions as the classical phase space:

Equation (9.12) gives in principle all the classical solutions of the system. Since the space of classical solutions is the most natural definition of the classical phase space of our field theory (we shall call this phase space $M$) let us try to understand it better. We can parametrise the space of functions $f(x, p)$ as

$$f(x, p) = \sum_{mn} a_{mn} x^m p^n = \sum_{mn} \alpha_{mn}(p + ix)^m(p - ix)^n, \quad \alpha_{nm} = (\alpha_{mn})^* \quad (9.13)$$

We can think of $\alpha_{mn}$ (or equivalently $a_{mn}$) as parametrising $^* M$.

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* a few remarks are in order: in general the fermi fluid may have several disconnected regions; this would seem to require several functions $f_1(x, p), f_2(x, p), \cdots$ to specify the boundaries of the different disconnected regions; however interestingly the union of the contours of $f_i(x, p) = 0, i = 1, 2, \cdots$ is the same as the contour of $f(x, p) = 0$ where $f(x, p) = \prod_i f_i(x, p)$. In other words we are not losing on generality when we describe the space of fermi fluids by a single function $f(x, p)$. The second point is, the $\alpha_{mn}$’s are actually constrained, by the requirement that the fermi fluid should always occupy the same volume. For our purposes, this would not be too important, because we will consider flows in this space that automatically preserve this constraint. Lastly, the $\alpha_{mn}$’s contain some redundant information since the function $f$ can be deformed preserving its zero contour; so different $\alpha_{mn}$’s may actually refer to the same point. A better coordinatisation is provided by the functions $w^{rs}$ on $M$ which we will describe shortly.
We would like to make a comment here about bosonisation. As Polchinski has shown, in case the fluid profile is quadratic in \( p \), i.e. \( f(x, p) = (p - p_+(x))(p - p_-(x)) \) the upper and lower boundaries \( p_{\pm}(x) \) behave as canonically conjugate variables of a classical bosonic field theory. The quadratic restriction on the profile, however, is rather unnatural, as is clear from the fact that such a restriction is not even preserved in time. For instance if we start with a profile \( f = p^2 + x^4 - 1 = 0 \) at time \( t = 0 \) after a quarter period \( t = \pi/2 \) the profile looks like \( f(t = \pi/2) = x^2 + p^4 - 1 = 0 \) which is no longer quadratic in \( p \). We would therefore like to propose the entire space of fluid profiles (all the \( \alpha_{mn} \)'s) as the correct bosonic variables for our problem. In retrospect, this is a rather natural route to bosonisation for a fermi system: (a) find the space of classical solutions, (b) identify it as a classical phase space and (c) try to quantise the variables of the system by replacing Poisson brackets with commutators (the observables are naturally bosonic in this procedure).

So we have the \( \alpha_{mn} \)'s as the phase space variables of a classical bosonic system. By the Thomas-Fermi correspondence (9.1), we know the classical Hamiltonian as a function of the \( \alpha_{mn} \):

\[
H(\{\alpha_{mn}\}) = \int \int_{R_f} dx dp \left( p^2 + x^2 \right)/2
\]  

(9.14)

The functional dependence on the \( \alpha_{mn} \) comes from the integration region \( R_f \) specified by the boundary \( f = \sum \alpha_{mn} z^m \bar{z}^n = 0 \), where \( z = p + ix \), \( \bar{z} = p - ix \). This is a rather implicit-looking function; but we’ll find that we don’t need to know it exactly.

If we tried to quantise the space \( M \) of the \( \alpha_{mn} \)'s we would like to know, for instance, the classical orbits in the space \( M \), and how to compute Possion brackets of two functions \( F(\{\alpha_{mn}\}) \) and \( G(\{\alpha_{mn}\}) \).

**Classical orbits:**
We simply transcribe the evolution $f \rightarrow f_t$, described in (9.12), as $\alpha_{mn} \rightarrow \alpha_{mn}(t)$ ($\alpha_{mn}(t)$ defined by $f_t(x, p) = \sum_{mn} \alpha_{mn}(t) z^m \bar{z}^n$, thus $\alpha_{mn} = \alpha_{mn}(0)$). The answer is:

$$\alpha_{mn}(t) = \alpha_{mn}(0) \exp(i(m - n)t)$$  \hspace{1cm} (9.15)

This again is the list of all classical orbits in the Euclidian phase space. Note that all classical solutions are periodic and have integer frequencies.

**Poisson brackets:**

Equation (9.15) tells us about the Poisson bracket of the $\alpha_{mn}$ and $H(\{\alpha_{pq}\})$:

$$\{\alpha_{mn}, H\}_{PB} = \frac{\partial}{\partial t} \Big|_{t=0} \alpha_{mn}(t) = i(m - n)\alpha_{mn}$$  \hspace{1cm} (9.16)

Note that we have computed this bracket without either knowing the explicit functional form of $H(\{\alpha_{mn}\})$ or $\{\alpha_{mn}, \alpha_{pq}\}_{PB}$. The fermi fluid picture told us how to find the flow of the point $\alpha_{mn}$ under the hamiltonian, and that gave us the Poisson bracket directly.

If we can calculate the Poisson brackets of two arbitrary functions on $M$ by this method, we wont need to know the Poisson bracket $\{\alpha_{mn}, \alpha_{pq}\}_{PB}$.

By the Thomas-Fermi correspondence (9.8), each of the $w^{rs}$’s correspond to a specific function of the $\alpha_{mn}$’s. In the quantum theory, we saw that all observables can be built from the $W^{(r,s)}$’s and their products (since the basic fermion bilinear $c_m^\dagger c_n$ could be written in terms of the $W^{(r,s)}$’s). In the classical theory, that would mean that all functions on $M$ can be expressed as functions of the $w^{rs}$. Thus it is enough to compute the Poisson bracket $\{w^{rs}, w^{pq}\}_{PB}$.

To do this, think of the Poisson bracket as the result of commuting two independent flows. Each of these flows is induced on $M$ from a corresponding flow in the single-particle phase space $Q$. To be precise the evolution in $M$ under $w^{rs}$ is computed by how the fluid profile evolves when the single fermions are evolved
by a “hamiltonian” \((p - ix)^r(p + ix)^s\). (One can show that this implies a Poisson bracket \(\{\alpha_{mn}, w^{rs}\}_{PB} = ((m + 1)s - (n + 1)r)\alpha_{m+1-r,n+1-s}\). From this map between flows, it is easy to see that the Poisson bracket algebras are isomorphic.

We therefore have the result:

\[
\{w^{rs}, w^{pq}\}_{PB} = (rq - sp)w^{r+p-1,s+q-1}
\]  

(9.17)

Remembering that \(H = w^{11}\) and that all other functions on \(M\) are functions of the basic observables \(w^{rs}\), (9.17) specifies the classical physics completely. In principle we now have all the ingredients to quantize the theory by doing functional integral over the classical phase space.

Let us explain the last comment in a little more detail. Consider the simpler example of a finite \((n)\) dimensional phase space \(M_n\), with generalised coordinates \(\xi^i\). Suppose one has \(n\) independent functions \(f^i\) on \(M\) whose Poisson brackets are known, namely

\[
\{f^i, f^k\}_{PB} = \Omega^{ik}(\{f^i\})
\]

Also suppose the hamiltonian \(h(\{\xi^i\})\) depends on the \(\xi^i\) through the functions \(f^i\): \(h = h(f^i(\xi^k))\). To write down a quantum theory first one needs to construct the symplectic form, which can be easily shown to be \(\Omega = \Omega_{ik}df^i \wedge df^k\) where \(\Omega_{ik}\) and \(\Omega^{ik}\) are inverse matrices. Since the symplectic form is closed (can be derived from the Jacobi identity for Poisson brackets) locally one can find a one-form \(\theta = \theta_idf^i\) such that \(d\theta = \Omega\). The phase space functional integral can be written by defining the lagrangian \(\theta_i\dot{f}^i - h(\{f^i\})\) and integrating with respect to a measure consistent with the volume form \(\Omega = \Omega^{n/2}\).

In our problem, the \(w^{rs}\) are not independent functions, because the phase space \(M\) is not the group \(G\) (of area-preserving diffeomorphisms) itself, but rather a coset \(G/H\). Thus, for instance if one starts with a fermi fluid which corresponds to the filled fermi sea with the circular symmetry, then the action of all the generators \(w^{rr}\) keep it invariant. Therefore there is a non-trivial isotropy subgroup \(H\) formed
by the diagonal generators. A particular choice of independent functions could be $w_{rs}, r \neq s$. (For example on $S^2 = SO(3)/SO(2)$ the functions $J^\pm, J^3$ are not all independent, as evident from the particular representation $J^+ = z^2, J^- = z^2, J^3 = z\bar{z}$ which says $J^3 = \sqrt{J^+ J^-}$). Thus (9.17) gives us the matrix $\Omega_{ik}$ mentioned in the last paragraph which gets written as $\Omega^{rs,pq}(\{w_{kl}, k \neq l\}) = (rq - sp)w^{r+p-1,s+q-1}$.

In the space of the non-diagonal $w_{rs}$'s this matrix is invertible. Let's call the inverse matrix (counterpart of $\Omega_{ik}$) $\Omega_{rs,pq}(\{w_{kl}, k \neq l\})$ and the one-form $\theta_i$ as $\theta_{rs}(\{w_{kl}, k \neq l\})$. The phase space action $(p\dot{q} - h(p,q))$ is therefore given by

$$L = \theta_{rs}(\{w_{kl}\}) \frac{dw_{rs}}{dt} - w^{11}(\{w_{kl}\})$$

(9.18)

where by the set $\{w_{kl}\}$ we mean again the off-diagonal $w$'s and the function $w^{11}(\{w_{kl}\})$ is determined by its poisson bracket with the $w_{kl}$.

It is significant to note that our problem with a symplectic manifold $W_{1+\infty}^{(+)}/H$ can be formulated as problem involving motion on $W_{1+\infty}^{(+)}$. This is along the same lines as the monopole problem, which has a symplectic manifold $SU(2)/U(1)$, and can be formulated as motion on the group $SU(2)$ using a construction that mimics the Wess-Zumino term in higher dimensions [31]. Finally we remark that it would be of interest to study the above action for an arbitrary single particle Hamiltonian because these correspond to different background gauge fields. An important question to answer is which single partical Hamiltonian corresponds to the black-hole background.

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* As an example of this procedure, we can think of $S^2$ as coordinatised by $J^+ , J^-$. The matrix $\Omega^{ik}$ has non-zero elements $\Omega^{12} = -\Omega^{21} = J^3 = \sqrt{J^+ J^-}$ whose inverse is given by $\Omega^{12} = -\Omega^{21} = 1/J^3$. This gives $\theta = \sqrt{J^+} d(\sqrt{J^-})$ implying $L = \sqrt{J^+ J^-} - \sqrt{J^+} J^-$ for the problem when the hamiltonian is $J^3$. 

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10. General background for the $c = 1$ matrix model:

Let us begin the discussion for any arbitrary background gauge field $\bar{A}(t)$. As we have seen in section 5, the generators (5.4) of the special gauge transformations that leave the background invariant are constants of motion. Now let us consider deforming the (analytically continued) model by these constants of motion:

$$S_\epsilon = \int dt \, <\psi(t)|i\partial_t + \bar{A}(t)\psi(t)> + \int dt \, <\psi(t)|\epsilon(t)\psi(t)> \quad (10.1)$$

where $\epsilon(t)$ satisfies

$$i\partial_t \epsilon + \text{ad} \bar{A}(t) \epsilon = 0$$

Deforming the model at the background $\bar{A}(t)$ according to (10.1) corresponds to a shift of the background $\bar{A} \rightarrow \bar{A} + \epsilon$. It is easy to see that $\bar{A} + \epsilon$ is not a gauge transform of $\bar{A}(t)$. In the case of the specific background $\bar{A} = (p^2 + x^2)/2$, the time dependence of $\epsilon(t)$ is explicitly known and hence (10.1) becomes

$$S_\epsilon = \int dtdx \psi^+(x,t)(i\partial_t + \frac{p^2 + x^2}{2})\psi(x,t) + \sum_{r,s} g_{rs} W^{(r,s)} \quad (10.2)$$

where

$$W^{(r,s)} = \int dtdx \exp i(r - s)t \, \psi^+(x,t)a^r a^s\psi(x,t) \quad (10.3)$$

and we have defined the couplings $g_{rs}$ by

$$\epsilon(t) = \sum_{r,s} g_{rs} \exp i(r - s)t a^r a^s$$

Note that these couplings (which correspond to higher spin fields in space time) are conjugate to the generators $W^{(r,s)}$ of the infinite dimensional algebra $W_{1+\infty}^{(+)}$.

As yet our formulation of the problem has been such that a background gauge field $\bar{A}(t)$ is given to us. The main question is what principle determines these backgrounds.
One possibility is to note that our specific background $\tilde{A} = (p^2 + x^2)/2$ led to a specific representation of the algebra $W_{1+\infty}^{(+)}$ whose explicit construction was given in section 8, using the eigenfunctions and eigenvalues of the operator $\tilde{A} = (p^2 + x^2)/2$. This points to the fact that for each background $\tilde{A}(t)$ there will be an explicit representation of $W_{1+\infty}^{(+)}$. Hence if by some means we can classify and understand all representations of $W_{1+\infty}^{(+)}$, then we will have classified and understood all the background gauge fields $\tilde{A}(t)$. We believe that this is an attractive scenario. It would be most interesting to discover a background or representation that for instance describes the black hole solution of two-dimensional string theory.

One way of approaching the problem of representation is by appealing to the classical phase space discussed in section 9.

11. Connection with higher dimensional field theories

It is clear from the above discussions that our gauge-invariant lagrangian $L(\psi, \psi^\dagger, A)$ is symmetric under the group $F(R, G)$ where $G$ is the group of unitary (in the classical limit, canonical) transformations on the single-particle hilbert space and $F(R, G)$ denotes all maps from the real line (time) to $G$. The periodic maps are a particular subgroup of this, which is of course the loop group $LG$. The loop group $LG$ is particularly relevant for the background of $A$ which corresponds to the harmonic oscillator potential, because in that case the classical solutions are periodic in time, and one classical solution is transformed to another by the action of $LG$ rather than the full group $F(R, G)$.

Now interestingly, the loop group $LG$ is also the group of transformations from one classical solution to another for $N = 2$ strings living in a 4-dimensional space of (2,2) signature[32]. In other words, the classical phase space of $N = 2$ string field theory carries a natural representation of $LG$; indeed each classical solution can be mapped onto a specific element of $LG$ which can be denoted by a function $f(t, x, p)$ where $f$ is an arbitrary function of $t$ and the two-dimensional plane $x, p$ (at each $t$, $f$ can be used to generate a canonical transformation on the two-dimensional
plane). Each solution of the $N = 2$ SFT can also be shown to correspond to a solution of self-dual Einstein gravity (with (2,2) signature).

Thus, we see that in terms of classical phase spaces, our fermion-gauge field system is closely related to $N = 2$ string field theory and also self-dual Einstein gravity in four dimensions.

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Note Added: Although the operators $W^{(r,s)}$ are not defined in the Minkowski theory, one could introduce a generalization of the loop operator which provides a ”regularized” definition of the $W^{(r,s)}$ much in the same way in which the loop operator provides a ”regularized” definition for the moments of the density operator [8]. This generalization of the loop operator is

$$W(p, q, t) = \frac{1}{2} \int dx \ e^{ipx} \psi^+(x + \frac{1}{2}q, t)\psi(x - \frac{1}{2}q, t)$$

Formally the $W^{(r,s)}$ can be obtained from this by taking appropriate number of derivatives with respect to $p$ and $q$ at $p = q = 0$. These operators satisfy the algebra

$$[W(p, q, t), W(p', q', t)] = i \sin \frac{1}{2}(pq' - p'q) \ W(p + p', q + q', t)$$

One can derive a closed set of Ward identities for these operators which are first order partial differential equations in $p, q, t$ and relate the $n + 1$ point function
to the $n$ point function. Since the one point function can be evaluated exactly using the parabolic cylinder functions, in principle the higher point functions can be obtained by solving the differential equations mentioned above. We have solved for the two point function of the $W(p, q, t)$’s this way and obtained the two point functions of the $W^{(r,s)}$’s. These show poles at precisely the expected energies.

One could also introduce a generating functional for the $n$ point functions of the $W(p, q, t)$’s by introducing sources in the fermionic action. One can show that this generating functional satisfies a Ward identity coming from the algebra of the $W(p, q, t)$’s. This Ward identity can be solved for the generating functional perturbatively in the sources. The Legendre transform of the generating functional is presumably the quantum effective action for the discrete states. The details of this will appear elsewhere.

Figure captions:

Fig 1a. Fermi surface: $x^2 - p^2 = \mu$.

Fig 1b. Fermi surface: $x^2 + p^2 = \mu$.

REFERENCES

1. E.Brezin, C.Itzykson, G.Parisi and J.Zuber, Comm. Math. Phys. 59 (1978) 35.

2. E. Brezin, V. Kazakov and Al.B. Zamolodchikov, Nucl. Phys. B338 (1990) 673.

3. D. Gross and M. Miljkovic’, Phys. Lett. 238B (1990) 217.

4. P. Ginsparg, J. Zinn-Justin, Phys. Lett. 240B (1990) 333.

5. G. Parisi, Phys. Lett. 238B (1990)

6. A.M. Sengupta and S.R. Wadia, Int. J. Mod. Phys. A6 (1991) 1961.

7. D. Gross and I. Klebanov, Nucl. Phys. B352 (1990) 671.
8. G. Moore, Rutgers Preprint RU-91-12 (1991).
9. D. Karabali and B. Sakita, City College Preprint, CCNY-HEP-91/2.
10. I. Kostov, Phys. Lett. 215B (1988) 499.
11. J. Shapiro, Nucl. Phys. B184 (1981) 218; A. Jevicki and B. Sakita, Nucl. Phys. B165 (1980) 511.
12. S.R. Das and A. Jevicki, Mod. Phys. Lett. A5 (1990) 1639.
13. J. Polchinski, Nucl. Phys. B346 (1990) 253.
14. G. Mandal, A. Sengupta and S.R. Wadia, Mod. Phys. Lett. A6 (1991) 1465.
15. K. Demeterfi, A. Jevicki and J.P. Rodrigues, Brown Preprint BROWN-HET-795 (1991).
16. D. Gross and I. Klebanov, Princeton University Preprint PUPT-1242 (1991).
17. J. Polchinski, Texas Preprint, UTTG-06-91 (1991).
18. P. Di Francesco and D. Kutasov, Princeton Preprint PUPT-1237 (1991).
19. V. V. Kac and D. Kazhdan, Adv. in Math. 34 (1979) 97; G. Segal, Comm. Math. Phys. 80 (1981) 301; M. Wakimoto and H. Yamada, Hiroshima Math. J. 16 (1986) 427.
20. A.M. Polyakov, Mod. Phys. Lett. A6 (1991) 635.
21. D. Gross, I. Klebanov and M.J. Newmann, Nucl. Phys. B350 (1990) 621.
22. U. Danielsson and D. Gross, Princeton Preprint PUPT-1258 (1991).
23. G. Mandal, A. Sengupta and S.R. Wadia, Mod. Phys. Lett. A6 (1991) 1685.
24. E. Witten, Phys. Rev. D44 (1991) 314.
25. M. Rocek, K. Schoutens, and A. Sevrin, IAS Preprint, IASSNS-HEP-91/14.
26. S. Elitzur, A. Forge and E. Rabinovici, Hebrew Univ. Preprint RI-143/90.
27. S.R. Das, A. Dhar, G. Mandal and S.R. Wadia (unpublished).
28. J. Avan and A. Jevicki, Brown Preprint BROWN-HET-801 (1991).

29. J. Polchinski, Texas Preprint UTTG-16-91 (1991).

30. C.N. Pope, L.J. Romans and X. Shen, “A brief History of $W_\infty$,” in *Strings 90*, ed. R. Arnowitt et al. (World Scientific, 1991).

31. A.P. Balachandran, G. Marmo, B.S. Skagerstam and A. Stern, ”Gauge Symmetries and Fiber Bundles”, Springer Lecture notes in Physics vol. 188.

32. H. Ooguri and C. Vafa, Univ. of Chicago and Harvard Preprint EFI-91/05, HUTP-91/A003 (January, 1991).

33. G. Moore and N. Seiberg, Rutgers Preprint.

34. J. Avan and A. Jevicki, Brown Preprint, Brown-HET-824 (1991).

35. E. Witten, IAS Preprint, IASSNS-HEP-91/51.

36. S. Rajeev, Phys. Lett. B209 (1988) 53; Phys. Rev. D42 (1990) 2779.