C-symmetric complete impulse approximation for the pion electromagnetic form factor in the Covariant Spectator Theory

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(Dated: August 27, 2015)

The pion form factor is calculated in the framework of the charge-conjugation invariant Covariant Spectator Theory. This formalism is established in Minkowski space and the calculation is set up in momentum space. In a previous calculation we included only the leading pole coming from the spectator quark (referred to as the relativistic impulse approximation). In this paper we also include the contributions from the poles of the quark which interacts with the photon and average over all poles in both the upper and lower half planes in order to preserve charge conjugation invariance (referred to as the C-symmetric complete impulse approximation). We find that these contributions are significant at all values of the four-momentum transfer, \( Q^2 \), but do not alter the shape obtained from the spectator poles alone.

PACS numbers: 11.30.Rd, 12.38.Lg, 12.39.Pn, 14.40.Be

I. INTRODUCTION

With the 12 GeV accelerator upgrade at Jefferson Lab, the charged pion form factor \( F_\pi \) will be known with high precision up to momentum transfer \( Q^2 \approx 6 \text{ GeV}^2 \) [1]. This measurement will cover the interesting region where the product \( Q^2 F_\pi \), as a function of \( Q^2 \), reaches a maximum and afterwards flattens down, and will help resolve the current discrepancy between the results for the \( \pi^+\gamma\gamma \) transition form factor obtained by the Babar and Belle Collaborations.

Together with theoretical calculations, the new measurements will narrow the uncertainty for the minimum momentum transfer down to which the description based on asymptotic parton distribution functions is valid. One may also certainly expect that the forthcoming data will clarify the mismatch between physical reality and perturbative QCD predictions in the region around \( Q^2 \approx 6 \text{ GeV}^2 \) [2]. This paper focuses on the pion form factor in the small \( Q^2 \) region that is planned to be covered by the new experiment. Part of the interest on this region lies also in its vicinity to the timelike sector and on the extrapolation of the derivative of the form factor to \( Q^2 < 0 \). The knowledge of the pion form factor enters into the evaluation of baryon form factors in the region near \( Q^2 \approx 0 \), and its behavior in the timelike region helps the interpretation of dilepton production data from heavy ion collisions.

Theoretically, the pion is the consequence both of the non-perturbative character of a quark-antiquark bound state, and of Spontaneous Chiral-Symmetry Breaking (S\( \chi \)SB). Its role in nuclear structure and nuclear dynamics is crucial. The pion cloud is seen to contribute to the structure of the nucleon and its excitations, through the coupling to external photons. Also, the exchange of pions between nucleons dominates their interaction at large distances and is the primary origin of the tensor force that has a decisive influence on the structure of nuclei.

The non-perturbative dynamics of the pion and other hadronic systems has been addressed by constituent quark models [3–6] and QCD sum rules. These approaches do not provide a comprehensive and global view for both light and heavy mesons, and baryons, and they cannot avoid a delicate fine-tuning between many parameters. More recently, QCD simulations on the lattice [7, 8], light-front formulations of quantum field theory [9–11], as well as models based on the Dyson-Schwinger approach and mass gap equation [12–19], provide an integrated account of mesons and baryons.

In this paper we use the Covariant Spectator Theory (CST). In common with the Dyson-Schwinger framework, it generates a dynamical quark mass, which is a function of the momentum, and this dressed mass is consistent with the two-body quark-antiquark dynamics. Although the CST equations share with lattice QCD and Dyson-Schwinger equations this important dynamical consistency, in contrast to those approaches CST equations are solved in Euclidean space. Therefore in this formalism the extension of results from the spacelike to the timelike region does not imply further work or the use of a different representation.

In Ref. [20] we presented the first calculation of the pion form factor based on the CST-Bethe Salpeter equation (CST-BSE) and the CST-Dyson equation (CST-DE), using a dressed quark mass function calibrated to fit existing lattice QCD data [21]. In the present paper, as well as in Ref. [20], the CST interaction kernel in momentum space has the form of a \( \delta \)-function plus a covariant generalization of the linear confining interaction. The fact that this model satisfies chiral symmetry is worth...
emphasizing; this was ensured by choosing a relativistic generalization of the confining interaction $\mathcal{V}_L$ which decouples, in the chiral limit, from both the one body CST-DE equation and from the two-body CST-BSE. In the calculation of the pion form factor in Ref. [20] we used the relativistic impulse approximation (RIA). However, we know this approximation will break down for small pion masses, since under these conditions the two pole contributions of the struck quark, which are omitted in RIA, are not negligible. The contribution of these additional poles to the charged pion form factor are calculated here, completing the results obtained from the RIA. In addition, we average over all of the propagator poles in both the upper and lower half planes, making our calculation consistent with the charge-conjugation symmetric equations from which the pion vertex must be calculated. This improvement is referred to as the C-symmetric competitive impulse approximation (C-CIA). A still more exact result can be obtained by adding interaction currents, a dynamically calculated pion vertex function, and the full dressing of the quark current to the C-CIA. These are planned for future work (for more discussion, see the final section.

This paper is organized as follows: In Section II a brief review of the one- and two-body CST equations is given. In Section III we present the ingredients for the calculation of the triangle diagram, and in Section IV the formulas for the contributions of all the poles to the pion form factor. In Section V we present the results and compare to the ones obtained with RIA. Finally, Section VI presents a short summary and conclusions.

II. BRIEF REVIEW OF THE ONE- AND TWO-BODY CST EQUATIONS

In this section we briefly describe how the CST is applied to the description of quark-antiquark mesons. For technical details, and references to earlier work, see [20–25].

In the four-dimensional BSE [26] for heavy-light mesons, it is known [27] that cancellations occur between iterations of ladder diagrams and higher-order crossed-ladder diagrams in the complete kernel. Due to this, the omission of crossed-ladder diagrams and part of the pole contributions of the ladder diagrams from the kernel can give a better approximation to the exact BSE than the ladder approximation does. Also, efficiency in the summation of the series can be gained by recognizing these partial cancellations at the beginning. The fundamental idea of CST is then to reorganize the Bethe-Salpeter series into an equivalent form—the CST equation. This leads to a redefinition of both the complete kernel and the (off-mass-shell) two-particle propagators in the intermediate states. The resulting three-dimensional equation, the one-channel CST (or Gross) Bethe-Salpeter equation [28], CST-BSE for short, is manifestly covariant. An important feature is that unlike the BSE in ladder approximation, the CST-BSE also has a smooth nonrelativistic limit, defining a natural covariant extension of the quantum mechanical Dirac and Schrödinger equations to quantum field theory.

In the heavy-light mass case, the redefined quark propagators emerge from keeping only the positive-energy pole contribution from the heavy quark propagator in the energy loop integration. The heavy quark is then on its positive-energy mass shell. In the case of light quarks to obtain the CST-BSE, an explicit charge-conjugation-symmetrization is made. The vertex functions of $\pi^+$ and $\pi^-$ are connected by charge conjugation and, therefore, both positive- and negative-energy quark poles must be included.

The idea of symmetrizing over all quark poles generates the charge-conjugation-symmetric CST-BSE [21]:

\[
\Gamma(p_1, p_2) = -\frac{1}{2} Z_0 \int_k \left[ \mathcal{V}(\rho, \tilde{k}, \rho - P/2) \Lambda(\tilde{k}) \Gamma(\tilde{k}, \tilde{k} - P) S(\tilde{k} - P) + \mathcal{V}(\rho, \tilde{k}, P/2) S(\tilde{k} + P) \Gamma(\tilde{k} + P, \tilde{k}) \Lambda(\tilde{k}) \right. \\
\left. + \mathcal{V}(\rho, -\tilde{k}, -\tilde{k} + P) \Lambda(-\tilde{k}) \Gamma(-\tilde{k}, -\tilde{k} + P) S(-\tilde{k} + P) + \mathcal{V}(\rho, -\tilde{k}, P/2) S(-\tilde{k} + P) \Gamma(-\tilde{k} + P, -\tilde{k}) \Lambda(-\tilde{k}) \right] \\
\equiv i \int_{k_0} \mathcal{V}(\rho, \kappa) S(\kappa + P/2) \Gamma(\kappa + P/2, \kappa - P/2) S(\kappa - P/2),
\]

where the three-dimensional covariant integration volume element is

\[
\int_k = \int \frac{d^3k}{(2\pi)^3} \frac{m}{E_k},
\]\n
and

\[
i \int_{k_0} \equiv i \int \frac{d^4k}{(2\pi)^4} \left| k_0 \text{ propagator poles only} \right. = -\frac{1}{2} \sum \left. \int_k \text{ propagator pole terms} \right.
\]

In Eq. (1) $\Gamma(p_1, p_2)$ is the $(4 \times 4)$ bound-state vertex function with $p_1 = \rho + \frac{P}{2}$ and $p_2 = -\rho + \frac{P}{2}$ the four-momenta of the outgoing quark and antiquark (respec-
tively); \(k_1 = \kappa + \frac{p}{2}\) and \(-k_2 = -\kappa + \frac{p}{2}\) are the intermediate four-momenta for the quark and antiquark (respectively); \(P\) is the total bound-state momentum, \(\rho\) and \(\kappa\) are the relative momenta; \(\hat{k} = (E_k, k)\) is the on-shell four-momentum with \(E_k = \sqrt{m^2 + \mathbf{k}^2}\); \(\mathcal{V}(\rho, \kappa) \equiv \mathcal{V}(\rho, \kappa; P)\) is the interaction kernel with the general structure

\[
\mathcal{V}(\rho, \kappa) \mathcal{X} \equiv \sum_i \mathcal{V}_i(\rho, \kappa) \mathcal{O}_i \mathcal{X} \mathcal{O}_i, \tag{4}
\]

where the sum \(i = \{S, P, V, A, T\}\) is over the five possible invariant structures that could contribute: scalar, pseudoscalar, vector, axial-vector, and tensor. The dressed quark propagator, \(S(k)\), projection operator, \(\Lambda(k)\), and the numerator function, \(N(k)\), are given by

\[
S(k) = \frac{1}{m_0 - \hat{\mathbf{k}} + \Sigma(k) - i\epsilon}, \tag{5}
\]

\[
\Lambda(k) = \frac{M(k^2) + \hat{\mathbf{k}}}{2M(k^2)} = \frac{N(k)}{2M(k^2)}, \tag{6}
\]

where \(M(k^2)\) the dressed quark mass function, \(m_0\) the bare quark mass, and \(\Sigma(k)\) is the quark self-energy. For consistency, \(\Sigma(k)\) is the solution of the one-body CST-Dyson equation (CST-DE) using the same interaction kernel \(\mathcal{V}\) that dresses the quark-antiquark vertex. One obtains [21]

\[
\Sigma(p) = \frac{1}{2} \int_0^\infty \left\{ \mathcal{V}(p, \hat{\mathbf{k}}) \Lambda(\hat{\mathbf{k}}) + \mathcal{V}(p, -\hat{\mathbf{k}}) \Lambda(-\hat{\mathbf{k}}) \right\}
\]

\[
\equiv -i \int_0^\infty \mathcal{V}(p, \mathbf{k}) S(k) . \tag{7}
\]

The self-energy is written

\[
\Sigma(p) = A(p^2) + \rho B(p^2) \tag{8}
\]

and then

\[
S(p) = Z(p^2) \frac{M(p^2) + \rho}{M^2(p^2) - p^2 - i\epsilon} \equiv Z(p^2) \frac{N(k)}{D(k^2)}, \tag{9}
\]

where \(D(k^2) = M^2(p^2) - p^2 - i\epsilon\) is the denominator of the propagator, and the mass function \(M(p^2)\) and the wave function normalization \(Z(p^2)\) are

\[
M(p^2) = \frac{A(p^2) + m_0}{1 - B(p^2)},
\]

\[
Z(p^2) = \frac{1}{1 - B(p^2)}, \tag{10}
\]

and \(Z_0 \equiv Z(m^2)\). For \(\Sigma(p) = 0\), \(S(p)\) becomes the bare propagator denoted as \(S_0(p)\). For the models we are using, \(B(p^2) = 0\) and \(Z(p^2) = 1\).

III. THE TRIANGLE DIAGRAM AND INGREDIENTS

A. The triangle diagram in charge-conjugation-invariant CST

The elastic electromagnetic form factor for a positively charged \(\pi^+\) consisting of a \(u\) and a \(\bar{d}\) quark is obtained from the pion current, which—\(\text{in impulse approximation}—\)is the sum of two triangle diagrams, in which the photon couples either to the \(u\) or the \(\bar{d}\) quark. The two diagrams are depicted in Fig. 1.

![Fig. 1](image)

In the charge-conjugation invariant formulation of CST [21, 25] the \(\pi^+\) current is given by

\[
J_{\pi^+}^\mu(P_+, P_-) = e F_\pi(Q^2) (P_+ + P_-)^\mu = i \frac{2 e}{3} \int_{k_0}^\infty \text{tr} \left[ \tilde{\Gamma}(k, p_+) S(p_+) J^\mu(p_+, p_-) S(p_-) \Gamma(p_-, k) S(k) \right] - e \frac{1}{3} \int_{k_0}^\infty \text{tr} \left[ \tilde{\Gamma}(k, p_-) S(p_-) J^\mu(p_-, p'_+) S(p'_+) \Gamma(p'_+, k) S(k) \right], \tag{11}
\]
where \( p_\pm = k + P_\pm, \ p'_\pm = k - P_\pm \), and \( j^\mu(p_+, p_-) \) is the dressed current for off-shell quarks (defined below). We assume equal masses for the \( u \) and \( d \)-quarks, so the \( u \) and \( d \) propagators are identical. Using this it has been shown in our earlier paper [20] that the second contribution to the form factor can be transformed into the first one, and the two can be added together. This gives

\[
J^\mu_{x+}(P_+, P_-) = i\epsilon \int_{k_0} \text{tr} \left[ \tilde{F}(k, p_+) S(p_+) j^\mu(p_+, p_-) S(p_-) \Gamma(p_-, k) S(k) \right].
\] (12)

As in Eq. (1), the “\( k_0 \)” in Eq. (11) stands for the charge-conjugation invariant CST prescription of how to perform the \( k_0 \)-contour integration. It requires taking all quark propagator-pole contributions of the \( u \) and the \( \bar{d} \) quark into account. The triangle diagram has six propagator poles in the complex \( k_0 \) plane, three positive-energy poles in the lower- and three negative-energy poles in the upper-half plane. The “\( k_0 \)” prescription require averaging over these poles. In the Breit frame, where

\[
P_\pm = (P_0, 0, \pm \frac{1}{2} Q)
\]
\[
q = (0, 0, Q)
\] (13)

with \( P_0 = \sqrt{\mu^2 + \frac{1}{4} Q^2} \), \( \mu \) the pion mass and \( Q \) the photon momentum transfer, the poles of the spectator \( d \)-quark are located at \( k_0 = \eta E_k - \eta i \kappa \), with \( \eta = \pm 1 \), \( E_k = \sqrt{m^2 + k^2} \), and the poles of the active \( u \)-quark with momenta \( p_- \) and \( p_+ \) are at

\[
k_0 = \pm E_\eta - P_0 \mp i \kappa ,
\]
\[
k_0 = \pm E_\eta + P_0 \mp i \kappa ,
\] (14)

respectively, where

\[
E_\pm = \sqrt{m^2 + k^2_{\perp} + \left( k_2 \pm \frac{Q}{2} \right)^2}.
\] (15)

The active poles can be collectively written as \( k_0 = \eta E_k - P_0 - \eta i \kappa \), with \( \eta = \pm 1 \) and \( \eta' = \mp 1 \). Since the square roots in the last two expressions are positive, recalling that \( p_\pm = k + P_\pm \) means that \( \eta = 1 \) denotes the positive-energy poles and \( \eta = -1 \) denotes the negative-energy poles of the struck \( u \)-quark.

**B. Pion vertex function**

One input for the pion form factor calculation is the pion vertex function \( \Gamma \). Instead of solving the full CST-BSE, which will be the subject of another paper, we use the approximated pion vertex function:

\[
\Gamma(p_1, p_2) = G_0 h(p_1^2) h(p_2^2) \gamma^5 L(\rho^2)
\] (16)

where \( p_1 = \rho + P/2, \ p_2 = \rho - P/2 \) [where \( P = (P_0, P') \)] with \( P'_0 = \sqrt{\mu^2 + P^2} \) is the pion four-momentum, \( h(p^2) \) a strong quark form factor normalized to \( h(m_\pi^2) = 1 \), (where \( m_\pi \) the dressed quark mass in the chiral limit), and \( G_0 \) is the inverse norm of the wave function. The additional structure function \( L(\rho^2) \) is a placeholder for the dynamically calculated pion vertex function, to be eventually obtained from the solution of the CST wave equation for the pion bound state. All we currently know is that \( L(\rho^2) = 1 \) in the chiral limit, and with this choice the form (16) coincides with the one introduced in Ref. [20] and can be understood as a finite-pion-momentum extension of the chiral-limit pion vertex function in the pion rest frame

\[
\Gamma(\rho, \rho) = G_0 h(\rho^2) \gamma^5
\] (17)

which is obtained from solving the pion CST-BSE in the chiral-limit with a delta-function kernel of the form

\[
\mathcal{V}(\rho, \kappa) = \frac{C}{2m} H(p_1, p_2; \hat{k}, k_2)(2\pi)^3 E_k \delta^3(\rho - \kappa) (\gamma^\mu \otimes \gamma_\mu)
\]
\[
\to \frac{1}{2} h^2(\rho^2)(2\pi)^3 E_k \delta^3(\rho - \kappa) (\gamma^\mu \otimes \gamma_\mu),
\] (18)

with \( C \) the strength (with \( C \to m_\chi \) in the chiral limit, see Ref. [21]), and with \( H \) a shorthand notation for a product of four form factors

\[
H(p_1, p_2; \hat{k}, k_2) = h(p_1^2) h(p_2^2) h(m_\pi^2) h(k_2^2)
\to h^2(\rho^2).
\] (19)

The first line of Eq. (19) gives the result for \( H \) in the case when particle 1 on-shell in the initial state [so that \( \hat{k}_1 = (E_k, k), k_2^2 = m^2, \hat{k}_2 = (E_k - P_{\rho 0}, k - P') \)] and for both particles off-shell in the final state. The second lines of the two Eqs. (18) and (19) give the results in the chiral limit in the rest frame, when \( m_\eta \to 0, m \to m_\chi, \mu \to 0 \). Note that a linear confining part of the interaction has been omitted from (18); it was shown in Ref. [21] that this part does not contribute to the pseudoscalar bound-state equation in the chiral limit.

From here on, all quantities are assumed to be given in the chiral limit, except for the pion mass which is kept finite.

**C. Quark current**

In a consistent pion form factor calculation, the dressed quark current should also be calculated from solving a (inhomogeneous) four-channel CST-BSE. This will, however, be the subject of a future paper and we use, for simplicity, the Ansätze proposed in Ref. [20], which uses the framework introduced by Riska and Gross [29] where strong quark form factors are attached to the interaction vertices, as described in Eqs. (18) and (19). These form
The functions can be equivalently moved to the quark propagators, leading to so-called damped propagators

$$\tilde{S}(p) = h^2(p^2)S(p),$$

reduced vertex functions

$$\Gamma_R(p_1, p_2) = h^{-1}(p_1^2)\Gamma(p_1, p_2)h^{-1}(p_2^2) = G_0\gamma_5,$$

and reduced currents

$$j_R^\mu(p', p) = h^{-1}(p'^2)j^\mu(p', p)h^{-1}(p^2).$$

In order to ensure current conservation in this framework, the reduced current must satisfy the Ward-Takahashi identity involving the damped propagators,

$$q^\mu j_R(p', p) = \tilde{S}^{-1}(p) - \tilde{S}^{-1}(p').$$

The simplest possible solution to this equation for point-like quarks (i.e. quark from factors equal to 1 and no magnetic moment) is given by [20, 30]

$$j_R^\mu(p', p) = f(p', p)\gamma^\mu + \delta(p', p)\Lambda(-p')\gamma^\mu + \delta(p, p')\gamma^\mu\Lambda(-p) + g(p', p)\Lambda(-p')\gamma^\mu\Lambda(-p).$$

The functions $f, \delta, \text{and} g$, fixed through the Ward-Takahashi identity, are

$$g(p', p) = 4MM'\frac{(h^2 - h'^2)}{h^2h'^2(p'^2 - p^2)},$$

$$\delta(p', p) = 2M'(M' - M)\frac{1}{h'^2(p'^2 - p^2)},$$

$$f(p', p) = \frac{M^2 - p^2}{h^2(p'^2 - p^2)} - \frac{M'^2 - p'^2}{h'^2(p'^2 - p^2)},$$

where we have introduced the short-hand notation $h = h(p^2), h' = h(p'^2), M = M(p^2)$, and $M' = M(p'^2)$. The mass function $M(p^2)$, calculated from the chiral limit CST-DE using the kernel in Eq. (18), is equal to $m_\chi, h^2(p^2)$, which reduces the off-shell form factors $f, \delta, \text{and} g$ to

$$g(p', p) \rightarrow -2\delta(p', p) \rightarrow 4m_\chi^2\frac{(h^2 - h'^2)}{p'^2 - p^2},$$

$$f(p', p) \rightarrow \frac{1}{4}g(p', p) + \frac{p'^2h^2 - p^2h'^2}{h^2h'^2(p'^2 - p^2)}.$$

In order to study the influence of the running dressed quark mass $M(p^2)$ on the pion form factor, we will also consider in this paper the case of fixed dressed quark masses by setting $\tilde{M}(p^2) = m_\chi$. In this case the off-shell form factors reduce to

$$g(p', p) \rightarrow \frac{4m_\chi^2}{h^2h'^2}\frac{(h^2 - h'^2)}{p'^2 - p^2},$$

$$\delta(p', p) \rightarrow 0,$$

$$f(p', p) \rightarrow \frac{m_\chi^2 - p^2}{h^2(p'^2 - p^2)} - \frac{m_\chi^2 - p'^2}{h'^2(p'^2 - p^2)}.$$

IV. CONTRIBUTIONS FROM THE QUARK CURRENT

Substituting (20), (21), and (22) into the pion current (12) gives

$$J_{\pi+}^{\mu}(P_+, P_-) = i\epsilon\int_{k_0} G_0^2 \frac{h^2 h_+ L_+ L_-}{D D_+ D_-} \times \text{tr} \left[ \gamma^5 N(p_+)j_R^\mu(p_+, p_-)N(p_-)\gamma^5\Lambda(k) \right] = J_\pi^\mu,$$

where used the short-hand notation $D = M^2 - k^2$ and $D_\pm = M_\pm^2 - p_\pm^2$ for the denominators of the propagators [with $M = M(k^2)$ and $M_\pm = M(p_\pm^2)$], and $h = h(k^2)$ and $h_\pm = h(p_\pm^2)$ for the strong form factors of spectator and active quarks (respectively), and $L_\pm = L(\rho_\pm^2)$ (with $\rho_\pm = k \pm \frac{1}{2} P_\pm$), where these additional structure functions are presumed to be functions of the relative momentum, but they might also depend on $p_\pm$ individually.

Next we insert (24) for the the quark current, which gives three contributions to the pion current associated with the off-shell form factors $f, \delta, \text{and} g$:

$$J_\pi^\mu = J_{\pi+}^\mu + J_{\pi-}^\mu + J_\pi^\mu_0.$$

In this section we analyze these contributions separately and calculate for each of them the contributions from the spectator and the active poles.

A. $f$-term contribution

The $f$ contribution to the pion current is

$$J_\pi^{f,\mu} = i\epsilon\int_{k_0} G_0^2 \frac{h^2 h_+^2 h^2 L_+ L_- f(p_+, p_-)}{D D_+ D_-} \times \text{tr} \left[ N(p_+)\gamma^\mu N(p_-)\gamma^5\Lambda(-k) \right] = i\epsilon\int_{k_0} G_0^2 \frac{h^2 L_+ L_- \Delta(k_+, k_-)N_0}{(p_+^2 - p_-^2)D},$$

where

$$\Delta(p_+, p_-) = \frac{h_+^2}{M_+^2 - p_+^2} - \frac{h^2}{M_-^2 - p_-^2},$$

and the trace is
\[ N^f = 4k^\mu(p_- \cdot p_+ - M_- M_+) + 4p_+^\mu(MM_+ - k \cdot p_+) + 4p_-^\mu(MM_- - k \cdot p_-) + 4k_0^\mu(MM_+ + MM_- - k \cdot p_- - k \cdot p_+) + 4P_0^\mu(MM_+ + MM_- - k \cdot p_- - k \cdot p_+) \]
\[ + 2q^\mu(MM_- - M_{-M} + k \cdot p_+ - k \cdot p_-). \]  

(37)

Note that \( p_{z_{\pm}}^2 = (k_0 + P_0)^2 - (k \pm \frac{1}{2}q)^2 \), so that \( p_+^2 - p_-^2 = -2k_z Q \), and excluding the \( N^f \) factor the integrand is even in \( k \). The \( q^\mu \) term in \( N^f \) is odd in \( k_0 \) and hence this term integrates to zero – a consequence of current conservation. For the \( k^\mu \) term in \( N^f \) we introduce \( 2P_0^\mu \equiv (P_+ + P_-)^\mu \) and use the relation \( k^\mu \rightarrow P_0^\mu k_0/P_0 \), which holds since the rest of the integrand is even in \( k \). Then, the remaining terms reduce to

\[ N^f = 4P_0^\mu \frac{k_0}{P_0} (MM_+ - M_- M_+ + MM_- - k^2 + P_+ \cdot P_-) + 4P_0^\mu (MM_+ + MM_- - 2k^2 - 2k \cdot P_0) \]
\[ = 4P_0^\mu \left[ (MM_+ + MM_-) \left( 1 + \frac{k_0}{P_0} \right) - 2k^2 \left( 1 + \frac{k_0}{2P_0} \right) \right] + \frac{k_0}{P_0} - 2k_0 P_0 \]
\[ = 8P_0^\mu \left\{ M \left[ \frac{M_+ + M_-}{2} \right] - k^2 + \left[ M(M_+ + M_-) + M_+ M_- - k^2 \right] \right\} \frac{k_0}{2P_0} - \frac{\mu^2 k_0}{2P_0} . \]  

(38)

The \( f \) contribution to the form factor then becomes

\[ F_{\pi}^f(Q^2) = -i G_0^2 \int_{k_0} h_0^2L_\Delta \Delta(p_+ + p_-) \chi(Q) \frac{P_0}{k_z Q P_0} , \]  

(39)

where

\[ \chi(k_0, Q) = \frac{1}{2} \left\{ P_0 \left[ M(M_+ + M_-) - 2k^2 \right] + k_0 \left[ M(M_+ + M_-) - M_+ M_- - k^2 \right] \right\} . \]  

(40)

For fixed dressed quark masses, \( M = M_+ = M_- \rightarrow m_\chi \), and \( \chi(Q) \) reduces to

\[ \chi(k_0, Q) \rightarrow 2P_0 + k_0 - \frac{\mu^2 k_0}{m_\chi^2 - k^2} . \]  

(41)

Note that the spectator pole vanishes if \( \mu = 0 \), so that, in this limit, the entire result comes from the active quark poles in \( \Delta \). Next we perform the \( k_0 \) integration.

1. Spectator pole contribution

First we calculate the spectator pole contribution to the \( f \) contribution (39). Averaging over the poles at \( k_0 = \eta E_k \) gives

\[ F_{\pi}^f(Q^2) = C_0^2 \int \frac{d^3k}{(2\pi)^3} \sum_{\eta = -1}^{1} \sum_{\eta' = -1}^{1} \frac{\eta' \chi_\eta(Q) h_{\eta' \eta}^2 h_{\eta' \eta}^2 L_\Delta \Delta}{2P_0 k_z Q (M_{\eta' \eta}^2 - p_{\eta' \eta}^2)} . \]  

(42)

where we use the notation \( p_{\eta' \eta}^2 \) for the active quark momenta squared, \( p_{\eta' \eta}^2 \), evaluated at the spectator poles at \( k_0 = \eta E_k \), such that

\[ p_{\eta' \eta}^2 = (\eta E_k + P_0)^2 - \left( k + \eta' \frac{1}{2}q \right)^2 = m_\chi^2 + \mu^2 + 2\eta E_k P_0 - \eta' k_z Q . \]  

(43)

The relative momenta that appear as arguments of the \( L \) functions becomes

\[ \rho_{\eta' \eta}^2 = (k + \frac{1}{2} P_0)^2 = k^2 + k \cdot P_0 + \frac{\mu^2}{4} \]
\[ = m_\chi^2 + \frac{\mu^2}{4} + \eta E_k P_0 - \eta' k_z Q . \]  

(44)

With this notation we introduce the abbreviations

\[ h_{\eta' \eta} = h(p_{\eta' \eta}^2) \]
\[ M_{\eta' \eta} = M(p_{\eta' \eta}^2) \]
\[ L_{\eta' \eta} = L(p_{\eta' \eta}^2) \]  

(45)

and

\[ \chi(\eta E_k, Q) \equiv \chi_\eta(Q) \]
\[ = \frac{P_0}{2E_k} \left[ m_\chi(M_{\eta' \eta} + M_{-\eta}) - 2m_\chi^2 \right] \]
\[ + \frac{1}{2} \eta \left[ m_\chi(M_{\eta' \eta} + M_{-\eta}) - M_{+\eta} M_{-\eta} - m_\chi^2 - \mu^2 \right] . \]  

(46)

For fixed quark masses, \( \chi_\eta(Q) \rightarrow -\eta \mu^2/2 \), and the contribution to the form factor simplifies to

\[ F_{\pi}^{f,s}(Q^2) \rightarrow -C_0^2 \int \frac{d^3k}{(2\pi)^3} \frac{\mu^2}{4P_0 k_z Q} \sum_{\eta = -1}^{1} \sum_{\eta' = -1}^{1} \frac{\eta' \eta \eta' \eta \eta \eta \eta \eta \eta \eta \eta \eta \eta}{m_\chi^2 - p_{\eta' \eta}^2} \]  

(47)

In Appendices A 2c and A 2d we analyze, for fixed quark masses, the small and large \( Q^2 \) behavior of the spectator contribution to the \( f \) term.

2. Active pole contributions

Next we calculate the active pole contributions, averaging over poles in the upper half plane at \( k_0 = -E_k - P_0 \) and in the lower half plane at \( k_0 = E_k - P_0 \), collectively
at \( k_0 = \eta E_{q'} - P_0 \), and identified by the subscripts \( \{ \eta' \eta \} \). In these cases we need the spectator momentum \( k \), the active particle with momentum \( p_q \), or the relative momentum \( p_{q'} \), all at the active particle pole \( \{ \eta' \eta \} \). At these poles we use the notation

\[
\begin{align*}
\bar{k}_\eta^2 &= (\eta E_\eta - P_0)^2 - k^2 \\
&= m_\eta^2 + \mu^2 - 2\eta E_\eta P_0 + \eta' k_z Q + \frac{1}{2} Q^2 \\
\bar{m}_\eta^2 &= m_\eta^2 \\
(p_\eta^2)^{\eta' \eta} &= (\eta E_\eta - \frac{1}{2} P_0)^2 - k_\eta^2 - \left(k_z - \frac{1}{4} \eta' Q\right)^2 \\
&= m_\eta^2 + 2\eta k_z Q - \bar{m}_\eta^2 \\
(p_{\eta'}^2)^{\eta' \eta} &= \left(\eta E_{\eta'} - \frac{1}{2} P_0\right)^2 - k_{\eta'}^2 - \left(k_z + \frac{1}{4} \eta' Q\right)^2 \\
&= \frac{1}{2} k_{\eta'}^2 + m_{\eta'}^2 - \frac{1}{4} \mu^2 \\
(p_{\eta - \eta'}^2)^{\eta' \eta} &= \left(\eta E_{\eta - \eta'} - \frac{1}{2} P_0\right)^2 - k_{\eta - \eta'}^2 - \left(k_z - \frac{1}{4} \eta' Q\right)^2 \\
&= (\rho_{\eta'}^2)^{\eta' \eta} + \eta' k_z Q. \tag{48}
\end{align*}
\]

Since \((\rho_{\eta'}^2)^{\eta' \eta} = m_{\eta'}^2\), we need only have a simplified notation for \((p_{\eta - \eta'}^2)^{\eta' \eta}\), leading to the following abbreviations

\[
\begin{align*}
\bar{M}_{\eta' \eta} &= M(\bar{k}_\eta^2) \\
\bar{M}_{\eta'} &= M(\bar{m}_{\eta'}^2) \tag{49}
\end{align*}
\]

Since \(L_+ L_-\) always occur as a symmetric product, they will always have the form

\[
L_{\eta' \eta} = L \left[(\rho_{\eta'}^2)^{\eta' \eta}\right] L \left[(\rho_{\eta - \eta'}^2)^{\eta' \eta}\right]. \tag{50}
\]

At each of these poles \( \chi(k_0, Q) \) becomes

\[
\begin{align*}
\tilde{\chi}_{\eta' \eta}(Q) &= \chi(\eta E_{q'} - P_0, Q) \\
&= \frac{1}{M_{\eta' \eta}^2 - \bar{k}_\eta^2} \left\{ P_0 \left[ M_{\eta' \eta}(\bar{M}_{\eta'} + m_\chi) - 2\bar{k}_{\eta'}^2 \right] \\
&+ (\eta E_{q'} - P_0) \left[ \bar{M}_{\eta' \eta}(\bar{M}_{\eta'} + m_\chi) - \bar{M}_{\eta'} m_\chi - \bar{k}_{\eta'}^2 - \mu^2 \right] \right\}. \tag{51}
\end{align*}
\]

The \( f \) contribution of the active poles to the form factor then becomes

\[
F_{\pi f}(Q^2) = G_\pi^2 \int \frac{d^3 k}{(2\pi)^3} \sum_{\eta' = -1}^{1} \sum_{\eta = -1}^{1} \frac{\eta' \tilde{\chi}_{\eta' \eta}(Q) \bar{k}_{\eta' \eta}^2 L_{\eta' \eta}^2}{4k_z E_{q'} P_0} \tag{52}
\]

where \( \tilde{h}_{\eta' \eta} = h(\bar{k}_{\eta' \eta}^2) \). In Appendices A 2a and A 2b we analyze the small and large \( Q^2 \) behavior of this contribution for fixed dressed quark masses.

**B. \( \delta \)-term contribution**

Next we look at the contributions from the \( \delta \) terms in the quark current. Because the \( \delta \) form factors are proportional to the difference of the mass functions, there are no contributions to this term in the case of fixed quark masses. For running quark masses the \( \delta \)-term contribution is given by

\[
J_{\pi}^{\delta, \mu} = i e \int_{k_0} \frac{G_\pi^2 h_{\pi}^2 h_{\pi}^2 L_+ L_-}{D_+ D_-} \text{tr} \left\{ N(p_+) \right. \\
&\times \left[ \left( \delta(p_+, p_-) \Lambda(-p_+ \gamma^\mu + \delta(p_-, p_+) \gamma^\mu \Lambda(-p_-) \right) \\
&\left. \times N(p_-) N(-k) \right] \right\} = -i e \int_{k_0} \frac{G_\pi^2 h_{\pi}^2 L_+ L_-}{2k_z Q D_-} \delta M \left\{ \frac{h_{\pi}^2 N_{\pi}^\delta}{D_-} + \frac{h_{\pi}^2 N_{\pi 0}^\delta}{D_+} \right\} \tag{53}
\]

where \( \delta M = M_+ - M_- \) and the traces are

\[
\begin{align*}
N_{\pi}^\delta &= \text{tr} \left\{ \gamma^\mu (M_+ - \bar{p}_+) (M - \bar{k}) \right\} \\
&= 4k^\mu (M_+ - M_-) + 4P_0^\mu M + 2q^\mu M, \\
N_{\pi 0}^\delta &= \text{tr} \left\{ (M_+ + \bar{p}_+) \gamma^\mu (M - \bar{k}) \right\} \\
&= 4k^\mu (M_+ - M_-) + 4P_0^\mu M - 2q^\mu M. \tag{54}
\end{align*}
\]

As before, the coefficient of the \( q^\mu \) term is odd in \( k_z \), and thus integrates to zero, a consequence of current conservation. Since the factor multiplying \( k^\mu \) is even in \( k_z \), we can substitute \( k^\mu \to k_0 P_0^\mu / P_0 \) in the trace terms, which gives

\[
\begin{align*}
N_{\pi}^\delta &\to 4P_0^\mu \left[ (M - M_+) \frac{k_0}{P_0} + M \right], \\
N_{\pi 0}^\delta &\to 4P_0^\mu \left[ (M - M_-) \frac{k_0}{P_0} + M \right]. \tag{55}
\end{align*}
\]

Then the contribution to the form factor becomes

\[
F_{\pi}^\delta(Q^2) = -i \int_{k_0} \frac{G_\pi^2 h_{\pi}^2 L_+ L_-}{k_z Q P_0} \frac{\delta M}{M^2 - k^2} \times \{ \Sigma_1(p)(P_0 + k_0) - \Sigma_2(p) k_0 \}, \tag{56}
\]

where

\[
\Sigma_1(p) = M \left[ \frac{h_{\pi}^2}{M_+^2 - p_-^2} + \frac{h_{\pi}^2}{M_-^2 - p_+^2} \right], \\
\Sigma_2(p) = M \left[ \frac{M_+ h_{\pi}^2}{M_+^2 - p_+^2} + \frac{M_- h_{\pi}^2}{M_-^2 - p_-^2} \right]. \tag{57}
\]

Next we perform the \( k_0 \) integration. Averaging over the spectator poles at \( k_0 = \eta E_k \) gives the spectator contribution
and averaging over active quark poles at $k_0 = \eta E_k - P_0$ gives

$$F_\pi^\delta,\alpha(Q^2) = - G_0^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{4k_z Q P_0} \sum_{\eta=1}^1 \sum_{\eta'=-1}^1 \eta' \eta h_{\eta'\eta}^2 L_{+,\eta} L_{-,\eta} \left( \frac{\bar{M}_{\eta'} - M_{\eta'}}{M_{\eta'\eta}^2 - \bar{M}_{\eta'\eta}^2} \right) \left[ m_\chi P_0 + \eta E_k (m_\chi - M_{\eta'}) \right].$$

(59)

C. $g$-term contribution

The remaining contribution comes from the fully off-shell $g$ term in the quark current, which reads

$$J^{g,\mu}_\pi(x) = i e \int_{k_0} \frac{G_0^2 p_0}{4k_z Q P_0} \frac{h_0^2}{D_+ D_-} g(p_+, p_-) \times \text{tr} \left\{ N(p_+) \Lambda(-p_+) \gamma^\mu \Lambda(-p_-) N(p_-) \right\}$$

$$= -i e \int_{k_0} \frac{2 G_0^2 h_0^2 (h_0^2 - h_0^2) L_+ L_-} {k_z Q D} k^\mu. \quad (60)$$

There is no contribution from the active quark poles to this term, because the denominators $D_+$ and $D_-$ cancel with $N(p_+) \Lambda(-p_+)$ and $\Lambda(-p_-) N(p_-)$, respectively. Hence, the only contribution comes from the spectator poles. Excluding the factor of $k^\mu$, the integrand is even in $k$. Hence, we can replace $k^\mu \rightarrow k_0 P_0^{\mu}/P_0$ and the $g$-term contribution to the pion form factor becomes

$$F_\pi^g(Q^2) = - i G_0^2 \int_{k_0} h_0^2 (h_0^2 - h_0^2) L_+ L_- k_0 \frac{1}{k_z Q D P_0}. \quad (61)$$

Then, averaging over the poles at $k_0 = \eta E_k$ gives

$$F_\pi^g(Q^2) = G_0^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{4k_z Q P_0} \times \sum_{\eta=1}^1 \sum_{\eta'=-1}^1 \eta' \eta h_{\eta'\eta}^2 L_{+,\eta} L_{-,\eta}. \quad (62)$$

It is interesting to observe that the $g$-term contribution is identical for fixed and running quark masses, because the dependence on the mass function cancels out. In Appendices A1a and A1b we analyze the small and large $Q^2$ behavior of this contribution.

V. RESULTS

In our previous paper, Ref. [20], on the pion form factor in the relativistic impulse approximation, we have used strong quark form factors of the simple form

$$h(p^2) = \left( \frac{\Lambda^2 - m_\chi^2}{\Lambda^2 - p^2} \right)^2. \quad (63)$$

Here $\Lambda$ is a mass parameter. Both, $m_\chi$ and $\Lambda$ are determined by a fit of the quark mass function $M(p^2) = m_\chi h^2(p^2)$ at negative $p^2$ to the lattice QCD data [31] extrapolated to the chiral limit [21]. The fit gives $\Lambda = 2.04$ GeV and $m_\chi = 0.308$ GeV. Note that $h(p^2)$ of (63) has a pole at $\Lambda^2 = p^2$. Our previous calculation only explored the values of $p_0^2$ at the RIA spectator particle pole (at $k_0 = -E_k$), which, from Eq. (43), is bounded by

$$p_0^2 \leq \frac{\Lambda^2}{\alpha^2} \leq m_\chi^2 + m_\chi^2. \quad (64)$$

The contributions from the active particle poles and the spectator pole at $k_0 = E_k$ included in the present C-CIA calculation will probe the structure of the form factor $h$ at large positive $p^2$ and will depend on the definition of $h$ in this region. To study this sensitivity, we adopt a generalization of the piecewise form proposed in [21]:

$$h(p^2) = \begin{cases} \left( \frac{\Lambda^2 - m_\chi^2}{\Lambda^2 - p^2} \right)^2 & \text{if } p^2 < s_+ \\ N(\alpha) \left( \frac{\alpha^2 \Lambda^2 - m_\chi^2}{\alpha^2 \Lambda^2 + p^2 - 2s_+} \right)^2 & \text{if } p^2 > s_+ , \end{cases} \quad (65)$$

where $s_+ < \Lambda^2$ is some fixed value (given below), and the normalization factor

$$N(\alpha) = \left( \frac{(\alpha^2 \Lambda^2 - m_\chi^2)(\alpha^2 \Lambda^2 - s_+)}{(\alpha^2 \Lambda^2 - m_\chi^2)(\Lambda^2 - s_+)} \right)^2, \quad (66)$$

ensures that $h(p^2)$ is continuous at $p^2 = s_+$. Note that this definition of $h(p^2)$ in the region $p^2 > s_+$ ensures that it has no pole at $p^2 = \Lambda^2$, or anywhere in the region if $\alpha^2 \Lambda^2 > s_+$. Varying the value of the parameter $\alpha$ allows us to study the sensitivity of the form factor to the definition of $h(p^2)$ in this region (where it is not constrained by lattice data). If $\alpha = 1$, the new definition (65) is identical to the one proposed in [21], and we emphasize that when $p^2 < s_+$, the form (65) is identical to Eq. (63) used in our previous calculations of the pion form factor in the RIA [21], and therefore the results of these calculations remain unchanged by using (65). We choose $s_+ = (\Lambda - m_\chi)^2/4 = 0.752$ GeV. For $\alpha = 1$, $h(p^2)$ is symmetric about $p^2 = s_+$. Fig. 2 shows the mass function $M(p^2) = m_\chi h^2(p^2)$ for $\alpha = 0.5, 1$, and $3$ together with the lattice data extrapolated to the chiral limit.

We have calculated the form factor for both, fixed dressed quark masses by setting $M(p^2) \rightarrow m_\chi$ and running dressed quark masses, with $M(p^2) = m_\chi h^2(p^2)$. In both cases we used the same strong quark form factors (which appear also in the pion vertex functions and in
the quark current) as defined by Eq. (65). Our pion form factor depends on the pion mass $\mu$. All results presented here are obtained with $L(\rho^2) = 1$, but we have also used other choices for $L(\rho^2)$ in order to study the sensitivity of our pion form factor on the vertex function. In Fig. 3 we compare the spectator and active quark pole contributions, calculated with running quark masses, and different values of $\mu = 0.14$, 0.42, and 0.6 GeV. Also shown in this figure is the asymptotic function $\frac{1}{3} Q^{-2}$ which gives a good fit to the pion form factor at large $Q^2$, as shown in Fig. 4. Comparing the slopes of this asymptotic form to the form factor curves shows that all of the contributions have the correct $1/Q^2$ fall-off at large $Q^2$. Note that both the spectator and active pole contribution are normalized by the same factor to ensure that $F_{\pi}^s(0) + F_{\pi}^a(0) = 1$.

Figure 5 shows the ratio $F_{\pi}^s(Q^2)/F_{\pi}^a(Q^2)$ for fixed and running quark masses, and different values of $\mu$. Note the surprising result that these fixed and running mass calculations are very close to each other over the entire $Q^2$ range, with the possible exception for large $\mu$ and large $Q^2$, where the running mass results are larger by about 10%.

We have also looked at the sensitivity of the pion form factor to the shape of the $h$ form factor in the timelike $p^2$ region, where it is not constrained by lattice data. In particular, we have varied the parameter $\alpha$ between 0.5 and 3. We find that the pion form factor is quite insensitive to these variations; only the active quark pole contributions at large $Q^2$ display any sensitivity at all. Fig. 6 shows the results for $\mu = 0.42$ GeV.

\section{VI. SUMMARY AND CONCLUSIONS}

The present study of the pion form factor using the Covariant Spectator Theory (CST) is the continuation of our first work \cite{20}. This previous calculation was done in the relativistic impulse approximation (RIA), which included only one quark pole contribution to the triangle diagram for the pion current, the negative-energy spectator pole. In the present work we take all six quark pole contributions into account. These are the positive and negative-energy spectator poles and the positive and negative-energy poles of the quark that interacts with the photon. The latter are referred to as the active poles. The inclusion of both, positive and negative-energy poles is necessary in order to preserve charge-conjugation invariance in CST \cite{21,25}, hence we refer to the present calculation as the $C$-symmetric complete impulse approx-
FIG. 4. The data for the pion form factor shown against the simple model $F_π(Q^2) = \frac{1}{3}Q^{-2}$ (solid line).

FIG. 5. The ratio $F_π^s/F_π^a$ for fixed (dashed lines) and running (solid lines) quark masses, and different values of $\mu$. The pairs of curves, from top to bottom are the results obtained with $\mu = 0.60$ (brown), 0.42 (orange) and 0.14 GeV (purple).

For the calculation of the pion form factor in C-CIA we use the same off-shell quark current as introduced in [20]. It satisfies the Ward-Takahashi identity and therefore our pion current is conserved. Our result for the pion form factor in C-CIA depends, beside the momentum-transfer squared, $Q^2$, also on the pion mass, $\mu$, similar to our previous RIA result of Ref. [20]. As expected from the pole analysis therein, we find for small $\mu$, that the active quark contributions are as important as the spectator contributions, over the whole range of $Q^2$. For large $\mu$ and large $Q^2$, the active pole contributions are smaller than the spectator contributions by about 30%. However—and somewhat in contrast to our first expectation—we find for small $\mu$, that the shapes of our C-CIA form factor and our previous RIA result are the same. In fact, the spectator and active pole contributions are nearly identical, not only in shape but also in magnitude, and differ only when both $\mu$ and $Q^2$ are large. This is definitely a surprising result. Nevertheless, it confirms the corrected large-$Q^2$ behavior obtained in [20], which remains unchanged by the inclusion of the active poles. We emphasize that the pion vertex function, as introduced in Ref. [20], is a very simple Ansatz, that can be understood as a finite-pion-momentum extension of the pion vertex function in the chiral limit. It turns out that the equivalence between spectator and active pole contributions is actually the result of using $L(\rho^2) = 1$. Indeed, by using a different Ansatz for $L(\rho^2)$ we were able to obtain also different shapes for active and spectator contributions.

We have also studied the effect of the running dressed quark masses on the pion form factor. We find that for small $\mu$ the results for the pion form factors calculated with running and with fixed dressed quark masses are nearly identical, and only for larger $\mu$ and $Q^2$ we observe some differences, which are, however, quite small. This is in contrast to Dyson-Schwinger approaches. The insensitivity remains by using a pion vertex function different from the choice $L(\rho^2) = 1$.

Related to this we have also investigated the sensitivity of the pion form factor on the strong quark form factors. In the C-CIA, the quark form factors, and accordingly the quark mass functions, are tested over the whole $p^2$ range. In particular, at positive $p^2$ where their shapes are
not constrained by lattice QCD data, this offers the possibility to obtain important information about the mass function in this region. However, we find that the shape of the pion form factor is quite insensitive to the shape of the strong quark form factors, and only the active poles for large $\mu$ and $Q^2$ show a small dependence. Moreover, this insensitivity persists through using a different pion vertex function.

**ACKNOWLEDGMENTS**

E.B. acknowledges the warm hospitality of the Jefferson-Lab Theory Center. This work received financial support from Fundação para a Ciência e a Tecnologia (FCT) under Grants No. PTDC/FIS/113940/2009, No. CFTP-FCT (PEst-OE/FIS/U/0777/2013), and No. SFRH/BPD/100578/2014. This work was also partially supported by the European Union under the HadronPhysics3 Grant No. 283286, and by Jefferson Science Associates, LLC, under U.S. DOE Contract No. DE-AC05-06OR23177. All diagrams have been drawn with JaxoDraw, Ref. [32].

Appendix A: Small and large $Q$ behavior of the pion form factor

1. $g$-term contribution

Here we investigate the small and large $Q$ behavior of the $g$-term contribution $F_\pi^g(Q^2)$, Eq. (61).

a. Small $Q$ behavior

It is easy to obtain the small $Q$ behavior if $\mu \neq 0$ (we have studied the $\mu = 0$ case, and the form factor diverges for all $Q$). Then, to order $Q$,

$$h_{\eta'' \eta} \simeq h_{\eta''} - \eta' k_z Q \frac{d h_{\eta''}}{d \eta''},$$

where

$$p_{\eta''} = \lim_{Q \to 0} p_{\eta'' \eta} = m_\chi + \mu^2 + 2 \eta \mu E_k,$$

and

$$\hat{h}_{\eta''} = h(p_{\eta''}^2).$$

(Here the “hat should not be confused with a on-shell four-vector.) Note that $p_{\eta''}^2 \to \eta\eta$ as $k \to \infty$, only if $\mu \neq 0$. Since $\hat{h}_{\eta''}$ does not depend on $\eta'$, the $\eta'$ sum contributes a factor of 2, and the small $Q$ limit becomes

$$\lim_{Q \to 0} F_\pi^g(Q^2) = -G_0^2 \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\mu} \sum_{\eta''} \eta \frac{d \hat{h}_{\eta''}^2}{d \eta''},$$

where we have used that $P_0 \to \mu$ as $Q \to 0$.

b. Large $Q$ behavior

At large $Q$ the arguments of $h$ can only be small (and hence the integrals large) if $k_z$ is very large (so that $k_z \sim E_k$). Expanding the arguments for large $k_z$ and $Q$ gives

$$p_{\eta'' \eta}' = m_\chi + \eta' + Q \left( \eta \frac{m_\chi^2 + k_z^2}{2 |k_z|} - \eta' k_z \right)$$

$$+ \frac{2 \eta' Q k_z}{Q} \equiv p_{\eta''}' \eta''.$$

For $p_{\eta''}^2$ to be small at large $k_z$ requires that the sign of $k_z$ be chosen so that $\eta k_z - \eta' k_z = 0$, which will be true only for positive $k_z$ if $\eta = \eta'$, and negative $k_z$ if $\eta = -\eta'$. Hence the integrals for $\eta = \eta'$ run from $0 \leq k_z < \infty$, and for $\eta = -\eta'$ from $-\infty < k_z \leq 0$, and with this understanding, the arguments go to

$$p_{\eta''}^2 \to \begin{cases} m_\chi + \mu^2 + \eta Q \left( \frac{m_\chi^2 + k_z^2}{2 |k_z|} + \frac{2 \eta' k_z}{Q^2} \right), & \eta = \eta' \\ m_\chi + \mu^2 - \eta Q \left( \frac{m_\chi^2 + k_z^2}{2 |k_z|} + \frac{2 \eta' k_z}{Q^2} \right), & \eta = -\eta'. \end{cases}$$

We obtain

$$\lim_{Q \to \infty} F_\pi^g(Q^2) = \frac{R_\pi^g}{Q^2}$$

where, transforming $k_z = Q k_z''$,

$$R_\pi^g = G_0^2 \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\mu} \sum_{\eta''} \int_0^\infty dk_z'' h^2 \left[ \left\{ \int_0^\infty \frac{dk_z''}{k_z''} h^2 \left[ p_{\eta'' \eta''}^2 \right] - \int_{-\infty}^0 \frac{dk_z''}{k_z''} h^2 \left[ p_{\eta'' \eta''}^2 \right] \right] \right]$$

$$\times \left\{ \int_0^\infty \frac{dk_z''}{k_z''} h^2 \left[ p_{\eta'' \eta''}^2 \right] \right\}$$

$$\times \left\{ \int_0^\infty \frac{dk_z''}{k_z''} h^2 \left[ p_{\eta'' \eta''}^2 \right] \right\}$$

where we introduced the convenient notation

$$p_{\eta'' \eta''}^2 = m_\chi^2 + 2 \eta' E_k \left( k_z'' + \frac{1}{2} k_z'' \right) + n \frac{m_\chi^2 + k_z''^2}{2 |k_z''^2|},$$

and, in the in the second equation, we changed the sign of $k_z''$, allowing the two terms to be combined into one (with a factor of 2). Note that the singularity that was at $k_z = 0$ is now suppressed by the property $h(\pm \infty) = 0$, and that, because $\mu \neq 0$ the integrals converge at both $k_z'' = 0$ and $k_z'' = \infty$. The integrals peak at

$$k_z''_{\text{peak}} = \sqrt{\frac{m_\chi^2 + k_{\perp}^2}{2 \mu}}.$$

For future reference below we note that if $\eta = -1$, and $k_z'' = 1/2$, the argument becomes

$$k_z''_{\text{peak}} = \sqrt{\frac{m_\chi^2 + k_{\perp}^2}{2 \mu}}.$$
2. f-term contribution

Here we investigate, for fixed quark masses, the small and large $Q$ behavior of the $f$-term contribution $F^f_\pi(Q^2)$, Eq. (39).

a. Small $Q$ behavior of active pole contributions

The small $Q$ behavior can be obtained by expanding $\hat{h}^2$ near $Q = 0$ to order $Q$

\[
\hat{h}_{\eta'\eta}^2 \simeq \hat{h}_{\eta}^2 - \frac{\eta'Qk_z}{E_k} \left( E_k - \eta \mu \right) \frac{d\hat{h}_{\eta}^2}{dp_{-\eta}^2}, \quad (A12)
\]

where $\hat{p}_{\eta}^2$ and $\hat{h}_{\eta}$ were defined in Eq. (A2). Using

\[
\tilde{\chi}_{\eta'\eta}(0) = \eta E_k + \mu - \frac{\mu(\eta E_k - \mu)}{2\eta E_k - \mu} = \frac{2E_k^2}{2\eta E_k - \mu}, \quad (A13)
\]

which is nonsingular as long as $2m_\chi > \mu$, and taking the $Q \to 0$ limit in the rest of (52) gives

\[
\lim_{Q \to 0} F^{f,\alpha}_{\pi}(Q^2) = C_6^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mu} \left( 1 - \frac{\eta(\eta E_k - \mu)}{2\eta E_k - \mu} \frac{d\hat{h}_{\eta}^2}{dp_{-\eta}^2} \right), \quad (A14)
\]

where the two identical $\eta'$ terms have been combined, giving a factor of 2. Changing $\eta \to -\eta$ in the sum gives a result which reduces to Eq. (A4) if $\mu = 0$

\[
\lim_{Q \to 0} F^f_{\pi}(Q^2) = -C_6^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mu} \left( 1 - \frac{\eta(\eta E_k + \mu)}{2\eta E_k + \mu} \frac{d\hat{h}_{\eta}^2}{dp_{\eta}^2} \right), \quad (A15)
\]

b. Large $Q$ behavior of the active pole contributions

The large $Q$ behavior of (52) follows from arguments similar to those used for the study of $F^g_{\pi}$. Anticipating that the dominant contributions come from large $k_z$, expand the arguments as follows

\[
\tilde{k}_{\eta'\eta}^2 \simeq \frac{1}{2} \left( m_\chi^2 + \mu^2 - \eta Q|k_z| + \frac{1}{2} \eta'Q|k_z| + \frac{1}{2} \eta'Q \right) - \eta Q \left( \frac{m_\chi^2 + k_z^2}{2|k_z + \frac{1}{2} \eta'Q|} + \frac{2\mu^2|k_z + \frac{1}{2} \eta'Q|}{Q^2} \right). \quad (A16)
\]

Since $Q$ can be chosen to be positive, the argument is finite at large $Q$ only if

\[
\eta|k_z + \frac{1}{2} \eta'Q| - \eta' \left| k_z + \frac{1}{2} \eta'Q \right| = 0 \quad (A17)
\]

which leads to the following conditions

\[
k_z > \frac{1}{2} \eta Q \quad \eta = \eta' \quad (A18)
\]

and

\[
k_z < \frac{1}{2} \eta Q \quad \eta = -\eta'. \quad (A19)
\]

Because of these conditions, when $\eta = \eta'$ it is convenient to shift the $k_z$ integration to $k_z \to k_z' - \frac{1}{2} \eta Q$, so that $k_z + \frac{1}{2} \eta Q \to k_z'$ and the region of integration is $k_z' > 0$, while if $\eta = -\eta'$ the shift is $k_z \to k_z' + \frac{1}{2} \eta Q$, so that $k_z - \frac{1}{2} \eta Q \to k_z'$ and the region of integration is $k_z' < 0$. With these transformations, the arguments become

\[
\tilde{k}_{\eta'}^2 \to \left\{ \begin{array}{ll}
\frac{m_\chi^2 + \mu^2 - \eta Q(\frac{m_\chi^2 + k_z^2}{2k_z} + \frac{2\mu^2k_z'}{Q^2}), \quad \eta = \eta'}{\frac{m_\chi^2 + \mu^2 + \eta Q(\frac{m_\chi^2 + k_z^2}{2k_z} + \frac{2\mu^2k_z'}{Q^2})}, \quad \eta = -\eta'}. \quad (A20)
\end{array} \right.
\]

Note that

\[
\tilde{k}_{\eta'}^2 = p_{\eta'}^2(\eta \to -\eta, \eta' \to -\eta'). \quad (A21)
\]

This means that the results obtained from the study of the $g$-term can be used here: the high $Q^2$ form factor is dominated by $k_z' > 0$ when $\eta = \eta'$ and $k_z' < 0$ when $\eta = -\eta'$. In these regions, after shifting $k_z \to k_z'$ in the original definition (51) (as discussed above), the large $Q$ behavior of $\tilde{\chi}_{\eta'\eta}$ becomes

\[
\tilde{\chi}_{\eta'\eta}(Q) \simeq \eta|k_z'| + \frac{1}{2} Q - \frac{2\mu^2Q|k_z'|(|\eta|k_z'| - \frac{1}{2}Q)}{Q^2D(\eta)}
\]

\[
= \frac{1}{Q^2D(\eta)} \left[ Q^2(\frac{m_\chi^2 + k_z^2}{2k_z}(|k_z'| + \frac{1}{2}\eta Q)
\right.
+ \left. 4\mu^2k_z'^2(|k_z'|- \frac{1}{2}\eta Q)) \right]
\]

\[
= \frac{Q}{D(\eta)} \left[ \frac{m_\chi^2 + k_z^2}{2k_z}(|k_z'| + \frac{1}{2}\eta Q)
\right.
+ \left. 4\mu^2k_z'^2(|k_z'|- \frac{1}{2}\eta Q)) \right]
\]

\[
= QR(\eta, |k_z'|) \quad (A22)
\]

where, in the second from last line, we anticipate the substitution $k_z' = Qk_z''$ to be made below, and

\[
Q^2D(\eta) = \eta Q^2(\frac{m_\chi^2 + k_z^2}{2k_z} + 4\mu^2k_z'^2 - 2\mu^2Q|k_z'|)
\]

\[
= Q^2 \left[ \eta(m_\chi^2 + k_z^2) + 4\mu^2k_z'^2 - 2\mu^2Q|k_z'| \right]. \quad (A23)
\]

Note that the reduced $\tilde{\chi}_R(\eta, |k_z'|) = \tilde{\chi}_R(\eta, k_z')$ if $k_z' > 0$ and $\tilde{\chi}_R(\eta, |k_z'|) = \tilde{\chi}_R(\eta, -k_z')$ if $k_z' < 0$. Furthermore, it has the symmetry $\tilde{\chi}_R(\eta, k_z'') = \tilde{\chi}_R(-\eta, -k_z'')$

Finally, at large $Q$ the result (52) becomes

\[
\lim_{Q \to \infty} F^f_{\pi}(Q^2) = \frac{R_{\pi}^{f,\pi}}{Q^2}. \quad (A24)
\]
where, making the substitutions discussed above,

\[
R^{f,α} = G_0^2 \int \frac{d^2 k_⊥}{(2π)^3} \sum_{α=-1}^{1} \int_0^∞ \frac{dk''_z}{k''_z} \tilde{X}(η, k''_z) h^2[p'_{∞,-η}^2 - η^2]
- \int_{-∞}^0 \frac{dk''_z}{k''_z} \tilde{X}(η, -k''_z) h^2[p'_{∞,-η}^2 - η^2]
\]

\[
= 2G_0^2 \int \frac{d^2 k_⊥}{(2π)^3} \sum_{α=-1}^{1} \int_0^∞ \frac{dk''_z}{k''_z} \tilde{X}(η, k''_z) h^2[p'_{∞,-η}^2 - η^2],
\]

(A25)

where, in the second line, we have combined the two integrals over \(k''_z\) by changing \(k''_z \rightarrow -k''_z\) in the second integral. When \(η = 1\), the integral has a principal value singularity at \(k''_z = 1\), which can be easily integrated over.

c. Small \(Q\) behavior of the spectator pole contribution

Evaluation of this contribution follows the discussion of the \(g\)-term contribution. We obtain immediately

\[
\lim_{Q \to 0} F^{f,s}_π(Q^2) = G_0^2 \int \frac{d^3 k}{(2π)^3} \frac{1}{2} \sum_{α=-1}^{1} \frac{η}{2ηE_k + μ} \left[ \frac{d^2 h}{dp^2_η} \right] \left[ \frac{h^2}{m^2_χ - p^2_η} \right]
\]

\[
= G_0^2 \int \frac{d^3 k}{(2π)^3} \frac{1}{2μ} \sum_{η=-1}^{1} \frac{η}{2ηE_k + μ} \times \left\{ \frac{d^2 h}{dp^2_η} - \frac{d^2 h}{dp^2_η} \right\}.
\]

(A26)

The value of the total form factor at \(Q = 0\) is obtained by adding this to (A4) and (A15) giving

\[
\lim_{Q \to 0} F_π(Q^2) = G_0^2 \int \frac{d^3 k}{(2π)^3} \frac{1}{2μ} \sum_{η=-1}^{1} \frac{η}{2ηE_k + μ} \times \left\{ \frac{d^2 h}{dp^2_η} - \frac{d^2 h}{dp^2_η} \right\}.
\]

(A27)

d. Large \(Q\) behavior of the spectator pole contribution

The large \(Q\) behavior also follows from the discussion of the \(g\)-term contribution. Using (A6) and repeating the steps leading to (A8) gives

\[
\lim_{Q \to ∞} F^{f,s}_π(Q^2) = \frac{R^{f,s}}{Q^2},
\]

where, transforming \(k_z = Qk''_z\), and using the definition (A9),

\[
R^{f,s} = -G_0^2 \int \frac{d^2 k_⊥}{(2π)^3} \frac{1}{2μ} \sum_{η=-1}^{1} \left\{ \int_0^∞ \frac{dk''_z}{k''_z} \frac{h^2[p'_{∞,-η}^2]}{m^2_χ - p^2_η} \right\}
\]

\[
- \int_0^∞ \frac{dk''_z}{k''_z} \frac{h^2[p'_{∞,-η}^2]}{m^2_χ - p^2_η} \right\}
\]

\[
= 2G_0^2 \int \frac{d^2 k_⊥}{(2π)^3} \int_0^∞ \frac{dk''_z}{(2π)^3} \sum_{η=-1}^{1} \frac{ημ^2 h^2[p'_{∞,η}^2]}{m^2_χ + k''_z^2 + 4μk_z^2(2k''_z + η)}.
\]

(A29)

Combining the results for \(R^{f,s}, R^{f,α}\), and \(R^g\) gives the total coefficient for the \(Q^{-2}\) fall off.

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