Reconstruction of Slow-roll $F(R)$ Gravity Inflation from the Observational Indices

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In this work, we introduce a bottom-up $F(R)$ gravity reconstruction technique, in which we fix the observational indices and we seek for the $F(R)$ gravity which may realize them. Particularly, as an exemplification of our method, we shall assume that the scalar-to-tensor ratio has a specific form, and from it we shall reconstruct the $F(R)$ gravity that may realize it, focusing on special values of the parameters in order to obtain analytical results. The observational indices we study are compatible with the latest observational data, and we discuss how the functional form of the observational indices may affect the viability of the model.

PACS numbers: 04.50.Kd, 95.36.+x, 98.80.-k, 98.80.Cq, 11.25.-w

I. INTRODUCTION

Describing the inflationary era \cite{1-4} in a consistent way is unarguably one of the streamline tasks of modern cosmology. Lately, considerable effort is given in describing the early-time acceleration era by using modified gravity in its various forms \cite{5-8}, with $F(R)$ gravity having a prominent role among all modified gravities. The $F(R)$ gravity framework is a concise and appealing theoretical framework, which in conjunction with the simplicity, renders $F(R)$ gravity one of the most important theories of modified gravity. In addition, the latest Planck data \cite{9} and also the BICEP2/Keck-Array data \cite{10}, indicate that the $R^2$ inflationary model is compatible with the observational data, so many researchers turn their focus on $F(R)$ gravity descriptions of inflation.

Due to the importance of the inflationary era in our primordial Universe, finding a consistent and observationally acceptable description is a compelling task. In this line of research, with this paper we shall approach the $F(R)$ gravity inflationary era by using a bottom-up approach, in which we shall fix the observational indices and we shall seek for the vacuum $F(R)$ gravity that can realize such an evolution, in the slow-roll approximation, with the observational indices being compatible with the latest observational data. Particularly, we shall assume that the scalar-to-tensor ratio $r$ has a specific form, and by using well-known reconstruction techniques \cite{11}, we shall investigate which $F(R)$ gravity can realize such an evolution, always in the slow-roll approximation. The slow-roll approximation actually simplifies the expressions that yield the observational indices in the pure $F(R)$ gravity case. The examples we shall present in order to exemplify our technique, lead to analytic results, but this is not the general case. Also our bottom-up approach will enable us to find the exact form of the Hubble rate as a function of the cosmic time, which corresponds to the set of the observational indices which we fixed to have a specific form. Moreover, as we will show, the spectral index of the primordial curvature perturbations and the scalar-to-tensor ratio are compatible with the current observational data. Finally, we briefly discuss various functional forms of the observational indices and we examine the viability of each scenario, in the context of slow-roll vacuum $F(R)$ gravity. A different approach to ours which uses an inverse reconstruction scheme was presented by A. Starobinsky in \cite{12}.

This letter is organized as follows: In section II, we present in brief all the essential information for the vacuum $F(R)$ gravity theory which are necessary for the calculations that follow. In section III, we employ well known reconstruction techniques and also we develop our own technique, which will enable us to find which $F(R)$ gravity may realize a given set of observational indices compatible with the observational data. We focus eventually on the cases that analytical results can be obtained. In section IV we discuss the viability of various cosmological scenarios for various functional forms of the scalar-to-tensor ratio. Finally the conclusions follow in the end of the paper.

In this paper, the cosmological geometric background will be assumed a flat Friedmann-Robertson-Walker (FRW) geometric background, in which case the line element is,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2 ,$$

\hspace{1cm} (1)
where \( a(t) \) is the scale factor of the Universe. With regard to the metric connection, we shall assume that it is a metric compatible, symmetric and torsion-less connection, the Levi-Civita connection.

II. ESSENTIAL FEATURES OF \( F(R) \) GRAVITY

Let us briefly recall some basic features of \( F(R) \) gravity, which are necessary for our presentation, for reviews on this topic see [3] [8]. The gravitational action of \( F(R) \) gravity in vacuum is equal to,

\[
S = \frac{1}{2\kappa^2} \int \! d^4x \sqrt{-g} F(R),
\]

where \( \kappa^2 \) stands for \( \kappa^2 = 8\pi G = \frac{1}{M_p^2} \) and also \( M_p \) is the Planck mass. By using the metric formalism, we vary the action with respect to the metric tensor \( g_{\mu\nu} \), and the gravitational equations read,

\[
F'(R) R_{\mu\nu}(g) - \frac{1}{2} F(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F'(R) + g_{\mu\nu} \Box F'(R) = 0,
\]

which can be cast as follows,

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{\kappa^2}{F'(R)} \left( T_{\mu\nu} + \frac{1}{\kappa^2} \left( \frac{F(R) - RF'(R)}{2} g_{\mu\nu} + \nabla_\mu \nabla_\nu F'(R) - g_{\mu\nu} \Box F'(R) \right) \right).
\]

Note that the prime in Eq. (4) denotes differentiation with respect to the Ricci scalar \( R \). For the FRW metric of Eq. (1), the cosmological equations read,

\[
0 = - \frac{F(R)}{2} + 3 \left( H^2 + \dot{H} \right) F'(R) - 18 \left( 4H^2 \dot{H} + H \ddot{H} \right) F''(R),
\]

\[
0 = \frac{F(R)}{2} - \left( \dot{H} + 3H^2 \right) F'(R) + 6 \left( 8H^2 \ddot{H} + 4H^2 + 6H \dot{H} + H \dddot{H} \right) F''(R) + 36 \left( 4H \dot{H} + H \dddot{H} \right)^2 F'''(R),
\]

where \( H \) denotes the Hubble rate \( H = \dot{a}/a \) and the Ricci scalar for the FRW metric is equal to \( R = 12H^2 + 6\dot{H} \).

III. VACUUM \( F(R) \) GRAVITY FROM THE OBSERVATIONAL INDICES

In this section, by using a bottom-up approach, we shall investigate how a viable set of the observational indices \( n_s \) and \( r \) can be realized by an \( F(R) \) gravity in the context of the slow-roll approximation, where \( n_s \) is the power spectrum of the primordial curvature perturbations and \( r \) is the scalar-to-tensor ratio. By using a well-known reconstruction technique, we shall investigate which \( F(R) \) gravity can realize a given set of observational indices, starting from the scalar-to-tensor ratio. In order to illustrate how the method works, we shall use a specific quantitative example for the scalar-to-tensor ratio. It is important to note that the slow-roll approximation shall be considered to hold true during our calculations. In this case, the dynamics of inflation is quantified perfectly by the generalized slow-roll indices \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \) [3] [13] [16]. The first slow-roll parameter \( \epsilon_1 \) controls the duration of the inflationary era and more importantly if it occurs in the first place, and it is equal to \( \epsilon_1 = -\frac{\dot{H}}{H^2} \). In the case of vacuum \( F(R) \) gravity in the context of the slow-roll approximation, the slow-roll parameters can be approximated as follows [3] [13] [16],

\[
\epsilon_2 = 0, \quad \epsilon_1 \simeq -\epsilon_3, \quad \epsilon_4 \simeq \frac{F_{RRR}}{F_R} \left( 24\dot{H} + 6\frac{\dot{H}}{H} \right) - 3\epsilon_1 + \frac{\dot{\epsilon}_1}{H\epsilon_1},
\]

where \( F_R = \frac{dF}{dR} \), and \( F_{RRR} = \frac{d^3F}{dR^3} \). In addition, the spectral index of the primordial curvature perturbations of the vacuum \( F(R) \) gravity, and the corresponding scalar-to-tensor ratio, are equal to [3] [13] [16],

\[
n_s \simeq 1 - 6\epsilon_1 - 2\epsilon_4, \quad r = 48\epsilon_1^2.
\]

We need to stress that the expressions in Eq. (8) hold true only in the context of the slow-roll limit, where \( \epsilon_1, \epsilon_4 \ll 1 \) and this is an important assumption, which as we demonstrate it is satisfied.
At this point, let us exemplify our bottom-up reconstruction method by using a characteristic example, and to this end, let us assume that the scalar-to-tensor ratio $r$ is equal to,

$$ r = \frac{c^2}{(q + N)^2}, $$

(9)

where $N$ is the e-foldings number and $c, q$ are arbitrary parameters for the moment. As we now demonstrate, the choice (9) can lead to a viable inflationary cosmology. By using the expression in Eq. (8) for the scalar-to-tensor ratio $r$, we obtain that,

$$ r = \frac{48 \dot{H}(t)^2}{H(t)^4}, $$

(10)

and by expressing the above expression in terms of the e-foldings number $N$, by using the following,

$$ \frac{d}{dt} = H \frac{d}{dN}, $$

(11)

the scalar-to-tensor ratio in terms of $H(N)$ is,

$$ r = \frac{48H'(N)^2}{H(N)^2}, $$

(12)

where the prime now indicates differentiation with respect to $N$. By combining Eqs. (9) and (12), we obtain the differential equation,

$$ \sqrt{48} \frac{H'(N)}{H(N)} = \frac{c}{(q + N)}, $$

(13)

which can be solved and the solution is,

$$ H(N) = \gamma (N + q)^{\frac{\gamma}{\sqrt{3}}}. $$

(14)

The spectral index $n_s$ can be calculated in terms of $N$, however it is worth providing the expression in terms of the cosmic time, which is,

$$ n_s \simeq 1 + \frac{4\dot{H}(t)}{H(t)^2} - \frac{2\ddot{H}(t)}{H(t)H(t)} + \frac{F_{RRR}}{F_R} \left( 24\dot{H} + \frac{\ddot{H}}{H} \right), $$

(15)

so by using (14) and also the following expression,

$$ \frac{d^2}{dt^2} = H^2 \frac{d^2}{dN^2} + H \frac{dH}{dN} \frac{d}{dN}, $$

(16)

the spectral index in terms of the e-foldings number is equal to,

$$ n_s \simeq 1 + \frac{4H'(N)}{H(N)} - \frac{2(H(N)H''(N) + H'(N)^2)}{H(N)H'(N)} + \frac{F_{RRR}}{F_R} \left( 24H(N)H'(N) + 6H(N)H''(N) + 6H'(N)^2 \right), $$

(17)

where the prime indicates differentiation with respect to the e-foldings number. Finally, by substituting Eq. (14), the spectral index becomes equal to,

$$ n_s = 1 + \frac{c}{\sqrt{3(N + q)}} - \frac{cN}{\sqrt{3(N + q)^2}} - \frac{cq}{\sqrt{3(N + q)^2}} + \frac{2N}{(N + q)^2} + \frac{2q}{(N + q)^2} + $$

$$ \frac{c^2\gamma^2F_{RRR}(N + q)^{\frac{\gamma}{\sqrt{3}} - 2}}{8F_R} + \frac{5\sqrt{3}c\gamma^2F_{RRR}(N + q)^{\frac{\gamma}{\sqrt{3}} - 1}}{2F_R}. $$

(18)

As we shall demonstrate later on in this section, the observational indices (14) and (18) can become compatible with the Planck data [2] and even with the BICEP2/Keck-Array data [10], but we need first to investigate which $F(R)$ gravity can produce the inflationary era quantified by Eqs. (14) and (18), in order to find the analytic form of the last
two terms in Eq. (18). As we shall see, if the parameter \( c \) is appropriately chosen, an analytic expression for \( F(R) \) can be obtained. In order to find the \( F(R) \) gravity which realizes the observational indices (14) and (18), we shall employ the reconstruction technique of Ref. [11], so the cosmological equation appearing in Eq. (15), can be rewritten in the form,

\[
-18 \left( 4H(t)^2 \dot{H}(t) + H(t) \ddot{H}(t) \right) F_{RR}(R) + 3 \left( H^2(t) + \dot{H}(t) \right) F_R(R) - \frac{F(R)}{2} = 0 ,
\]

where \( F'(R) = \frac{dF(R)}{dR} \). The reconstruction method introduced in Ref. [11] uses the e-folding number \( N \), which in terms of the scale factor \( a \) is,

\[
e^{-N} = \frac{a_0}{a} ,
\]

and in the following we set \( a_0 = 1 \). By writing the FRW equation of Eq. (19) in terms of the e-foldings number \( N \), we obtain,

\[
-18 \left( 4H^3(N)H'(N) + H^2(N)(H')^2 + H^3(N)H''(N) \right) F_{RR}(R)
+ 3 \left( H^2(N) + H(N)H'(N) \right) F_R(R) - \frac{F(R)}{2} = 0 ,
\]

where the primes stand for \( H' = dH/dN \) and \( H'' = d^2H/dN^2 \). By using the function \( G(N) = H^2(N) \), the differential equation (21) can be cast as follows,

\[
-9G(N(R)) \left( 4G'(N(R)) + G''(N(R)) \right) F_{RR}(R) + \left( 3G(N) + \frac{3}{2} G'(N(R)) \right) F_R(R) - \frac{F(R)}{2} = 0 ,
\]

where \( G'(N) = dG(N)/dN \) and \( G''(N) = d^2G(N)/dN^2 \). Also the Ricci scalar can be expressed in terms of the function \( G(N) \) as follows,

\[
R = 3G'(N) + 12G(N) .
\]

Thus, by solving the differential equation (22), we can find the \( F(R) \) gravity which may realize a cosmological evolution. Now we shall make use of the reconstruction technique we just presented in order to find the \( F(R) \) gravity which realizes the observational indices (14) and (18). In our case, the function \( G(N) \) is,

\[
G(N) = \gamma^2 (N + q) \frac{\sqrt{\gamma}}{\sqrt{R}} ,
\]

and consequently, the algebraic equation (23) takes the following form,

\[
12\gamma^2 (N + q) \frac{\sqrt{\gamma}}{\sqrt{R}} + \frac{1}{2} \sqrt{3} \gamma^2 (N + q) \frac{1}{\sqrt{R}} - 1 = R .
\]

In general it is quite difficult to obtain a general solution to this equation, however is \( c \) is chosen appropriately, it is possible to obtain even full analytic results. For example if \( c = \sqrt{12} \), the results have a fully analytic form. In the following we shall investigate only the case with \( c = \sqrt{12} \), in which case the algebraic equation (23) becomes,

\[
3\gamma^2 + 12\gamma^2 N + 12\gamma^2 q = R ,
\]

so the function \( N(R) \) is equal to,

\[
N(R) = \frac{-3\gamma^2 - 12\gamma^2 q + R}{12\gamma^2} .
\]

By combining Eqs. (21) and (27) the differential equation (22) in this case becomes,

\[
-36\gamma^4 \left( \frac{-3\gamma^2 - 12\gamma^2 q + R}{12\gamma^2} + q \right) F''(R) + \frac{1}{4} \left( 3\gamma^2 + R \right) F'(R) - \frac{F(R)}{2} = 0 ,
\]

which can be solved analytically, and the solution is,

\[
F(R) = \frac{3}{2} \sqrt{3} \gamma^3 \delta + \frac{\delta R^2}{2\sqrt{3} \gamma} - 3\sqrt{3}\gamma \delta R + \mu \left( R - 3\gamma^2 \right)^{3/2} L^{3/4} \left( \frac{1}{12} \left( \frac{R}{\gamma^2} - 3 \right) \right) ,
\]
where the function $L_\alpha^n(x)$ is the generalized Laguerre Polynomial and also $\delta$ and $\mu$ are arbitrary integration constants. The existence of the Laguerre polynomial term, imposes the constraint $R < 3\gamma^2$, however in this case the term containing the root becomes complex. Hence in order to avoid inconsistencies, we set $\mu = 0$, and hence the resulting $F(R)$ gravity is,

$$\frac{3}{2}\sqrt{3\gamma^3} \delta + \frac{\delta R^2}{2\sqrt{3\gamma}} - 3\sqrt{3\gamma} \delta R,$$

(30)

which is a variant form of the Starobinsky model [17]. By requiring the coefficient of $R$ to be equal to one, $\delta$ must be equal to $\delta = -\frac{1}{3\sqrt{3\gamma}}$, hence the resulting $F(R)$ gravity during the slow-roll era is,

$$F(R) = R - \frac{\gamma^2}{2} - \frac{R^2}{18\gamma^2}.$$

(31)

We can find the Hubble rate as a function of the cosmic time, by solving the differential equation,

$$\dot{N} = H(N(t)),$$

(32)

where $H(N)$ is given in Eq. (14), and the resulting evolution is,

$$H(t) = \frac{1}{4} \left( \Lambda^2 - 4q + \gamma^2 t^2 - 2\gamma \Lambda t \right),$$

(33)

where $\Lambda > 0$ is an integration constant. Hence, the resulting evolution is a quasi-de Sitter evolution, if $\Lambda$ is chosen to be quite large so that it dominates the evolution at the early-time era, in which case $H(t) \simeq \frac{\Lambda}{4}$. Also it is trivial to see that $\dot{a} > 0$, so the solution (33) describes an inflationary era. Finally, let us now demonstrate if the resulting cosmology is compatible with the Planck data. Firstly, let us see how the spectral index becomes in view of Eq. (31) and due to the fact that $F_{RRR} = 0$, the spectral index becomes,

$$n_s = 1 + \frac{c}{\sqrt{3(N + q)}} - \frac{cN}{\sqrt{3(N + q)^2}} - \frac{c q}{\sqrt{3(N + q)^2}} + \frac{2N}{(N + q)^2} + \frac{2q}{(N + q)^2}.$$

(34)

By using the value of $c$, namely $c = \sqrt{12}$, and also for $N = 60$ and $q = -118$, the observational indices become,

$$n_s \simeq 0.9658, \quad r \simeq 0.00346842.$$

(35)

Recall that the 2015 Planck data constrain the observational indices as follows,

$$n_s = 0.9644 \pm 0.0049, \quad r < 0.10,$$

(36)

and also, the latest BICEP2/Keck-Array data [10] constrain the scalar-to-tensor ratio as follows,

$$r < 0.07,$$

(37)

at 95% confidence level. Hence, the observational indices (35) are compatible to both the Planck and the BICEP2/Keck-Array data.

Hence, by using a bottom-up approach, we found in an analytic way the $F(R)$ gravity which may realize a viable set of observational indices ($n_s, r$). In principle, more choices for the observational indices are possible, although in most of the cases, semi-analytic results will be obtained, due to the complexity of the differential equation (22).

Another simple case that can be analyzed analytically is the case for which,

$$r = \frac{1}{\beta^2},$$

(38)

in which case, by solving the differential equation [10], we obtain,

$$H(N) = \gamma e^{\frac{N}{\beta^2}}.$$

(39)

Then, we can easily find that, the function $G(N)$ is,

$$G(N) = \gamma^2 e^{\frac{N}{2\beta^2}},$$

(40)
and in effect, the algebraic equation (23) becomes,

\[ 12 \gamma^2 e^{\frac{N}{2 \beta}} + \frac{\sqrt{3} \gamma^2 e^{\frac{N}{2 \beta}}}{2 \beta} = R, \]

so the function \( N(R) \) is equal to,

\[ N(R) = 2 \sqrt{3} \beta \ln \left( \frac{2 \beta R}{(24 \beta + \sqrt{3}) \gamma^2} \right). \]

By combining Eqs. (40) and (42) the differential equation (22) in the case at hand becomes,

\[ -36 \gamma^4 \left( -3 \left( \frac{8 \sqrt{3} \beta + 1}{24 \beta + \sqrt{3}} \right)^2 \right) F''(R) + \left( \frac{12 \beta + \sqrt{3}}{2 (24 \beta + \sqrt{3})} \right) F'(R) - \frac{F(R)}{2} = 0, \]

which can also be solved analytically, and the resulting \( F(R) \) gravity is,

\[ F(R) = C_1 R^\mu + C_2 R^\nu, \]

where \( C_1 \) and \( C_2 \) are integration constants and also \( \mu \) and \( \nu \) stand for,

\[ \mu = \frac{96 \beta^2 + \sqrt{24 \beta + \sqrt{3}} \sqrt{384 \sqrt{3} \beta^2 - 36 \beta + 912 \beta^2 - 32 \sqrt{3} \beta + 1}}{32 \sqrt{3} \beta + 4}, \]

\[ \nu = \frac{96 \beta^2 - \sqrt{24 \beta + \sqrt{3}} \sqrt{384 \sqrt{3} \beta^2 - 36 \beta + 912 \beta^2 - 32 \sqrt{3} \beta + 1}}{32 \sqrt{3} \beta + 4}. \]

Some aspects of the \( F(R) \) gravity (41) were investigated in Ref. [18]. By combining Eqs. (42) and (44), the term \( \frac{F_{pair}}{F_R} \) can easily be calculated, so the resulting expression for the spectral index \( n_s \) is,

\[ n_s = 1 + \frac{4 \beta^2 C_1 2^\nu e^{\frac{N}{2 \beta}} \left( \frac{\beta^2 e^{\frac{N}{2 \beta}}}{8 \beta^2} + \frac{5 \sqrt{3} \gamma^2 e^{\frac{N}{2 \beta}}}{2 \beta} \right) \left( (\mu - 2)(\mu - 1)\mu \left( (24 \beta - \sqrt{3})^\mu \gamma^2 e^{-\frac{N}{2 \beta}} \right) \right)}{\beta^\mu \left( \sqrt{3} - 24 \beta \right)^2 \gamma^4 \left( C_1 \mu 2^\nu \left( (24 \beta - \sqrt{3})^\mu \gamma^2 e^{-\frac{N}{2 \beta}} \right) + C_2 2^\nu \nu \right)} \]

\[ + \frac{4 \beta^2 C_2 2^\nu e^{\frac{N}{2 \beta}} \left( \frac{\beta^2 e^{\frac{N}{2 \beta}}}{8 \beta^2} + \frac{5 \sqrt{3} \gamma^2 e^{\frac{N}{2 \beta}}}{2 \beta} \right) \left( (\nu - 2)(\nu - 1)\nu \left( (24 \beta - \sqrt{3})^\nu \gamma^2 e^{-\frac{N}{2 \beta}} \right) \right)}{\beta^\nu \left( \sqrt{3} - 24 \beta \right)^2 \gamma^4 \left( C_1 \nu 2^\nu \left( (24 \beta - \sqrt{3})^\nu \gamma^2 e^{-\frac{N}{2 \beta}} \right) + C_2 2^\nu \nu \right)} \].

The parameter space contains a lot of free parameters, and specifically \( \beta, \gamma \) and the integration constants \( C_1 \) and \( C_2 \), hence the Planck and BICEP2/Keck-array data of Eqs. (30) and (37) can easily be achieved. Indeed, if for example we choose \( \beta = 5.6201 \), for \( N = 60 \) e-foldings and with the rest of the parameters being equal to one, the spectral index becomes \( n_s \approx 0.966983 \) and the scalar-to-tensor ratio becomes equal to \( r = 0.0316601 \), so compatibility with both the Planck (30) and the BICEP2/Keck-Array data (37) is achieved.

**IV. GENERAL FORMS OF THE OBSERVATIONAL INDICES: A CRITICAL DISCUSSION**

An interesting outcome may be obtained if we study the various forms that the observational indices may take. In principle, any combination of functions is allowed, however in this section we shall consider exponentials and logarithmic functions that may appear in the scalar-to-tensor ratio, and we shall study the implications of these functional forms of the scalar-to-tensor ratio. In principle, the resulting cosmology should be critically checked in order to validate that it describes an inflationary cosmology, exactly as we did in the previous section. However,
we shall attempt a superficial approach focusing only on the observational indices, to see their behavior when the aforementioned functions are used. Also we shall take into account the simplest functional forms of the observational indices, and we shall assume that the theory is a slow-roll $F(R)$ gravity. Let us start with the exponential case, and we assume that the scalar-to-tensor ratio has the following form,

$$r = \frac{\gamma}{e^{\beta N}},$$

(47)

and hence, since the scalar-to-tensor ratio in terms of $H(N)$ is given in Eq. (12), by combining Eqs. (12) and (47), we obtain the following differential equation,

$$\sqrt{48} H'(N) = \frac{\sqrt{\gamma}}{\exp \left( \frac{\beta N}{2} \right)},$$

(48)

which can be solved and the resulting Hubble rate is,

$$H(N) = ce^{-\frac{\frac{\sqrt{\gamma}}{\exp \left( \frac{\beta N}{2} \right)}}{cN}},$$

(49)

where $c$ is an integration constant. By substituting Eq. (49) in Eq. (17), the spectral index $n_s$ reads,

$$n_s \simeq 1 + \beta + \frac{\gamma c^2 F_{RRR} \sqrt{\gamma e^{\beta N}}}{8F_R} + \frac{5\sqrt{3} \sqrt{\gamma} c^2 F_{RRR} e^{\sqrt{\gamma e^{\beta N}}}}{2F_R}.$$

(50)

It is easy to understand how the spectral index above behaves, even without knowing the exact form of the $F(R)$ gravity, since the presence of the exponentials renders the last two terms negligible, and hence the spectral index reads $n_s \simeq 1 + \beta$. The last expression for the spectral index cannot be compatible with the current observational data for any value of $\beta > 0$, hence vacuum $F(R)$ gravity theories which yield a scalar-to-tensor ratio of the form (47) do not yield a viable inflationary era.

Another class of models which we consider is the ones for which the scalar-to-tensor ratio has the following form,

$$r = \frac{\gamma}{\ln N},$$

(51)

and hence by combining Eqs. (12) and (51), we obtain the following differential equation,

$$\sqrt{48} H'(N) = \frac{\sqrt{\gamma}}{\ln N},$$

(52)

which can be solved analytically and the resulting Hubble rate is,

$$H(N) = ce^{\frac{\sqrt{\gamma \sqrt{\gamma} \text{Erfi} \left( \sqrt{\ln \left( \frac{N}{2} \right)} \right)}}{2F_R \ln \left( \frac{N}{2} \right)}}.$$

(53)

where $c$ is again an integration constant and the function Erfi($x$) is the imaginary error function. By substituting Eq. (53) in Eq. (17), the spectral index $n_s$ reads in this case,

$$n_s \simeq 1 + \frac{5\sqrt{3} \sqrt{\gamma} c^2 F_{RRR} \sqrt{\gamma} \text{Erfi} \left( \sqrt{\ln \left( \frac{N}{2} \right)} \right)}{2F_R \ln \left( \frac{N}{2} \right)} + \frac{\gamma c^2 F_{RRR} \sqrt{\gamma} \text{Erfi} \left( \sqrt{\ln \left( \frac{N}{2} \right)} \right)}{8F_R \ln \left( \frac{N}{2} \right)} + \frac{1}{N \ln \left( \frac{N}{2} \right)}.$$

(54)

In this case, if the parameter $c$ is appropriately chosen, and if the resulting $F(R)$ gravity yields the term $F_{RRR} < 0$, then the spectral index may be compatible with the observational data. However, it is not easy to find the exact form of the $F(R)$ gravity, since by applying the reconstruction method of the previous section, one may see that the resulting differential equations cannot be solved analytically, unless an approximation is used.

Hence in this section we showed that apart from the power-law functions of the $e$-foldings number, other functions may successfully describe a viable $F(R)$ gravity inflation, but analyticity is hard to achieve. Also, combinations of the exponential and logarithmic functions, with power law functions may also yield quite interesting results. One such application occurs for the constant-roll inflationary scenario, but we defer this to a future work.
V. CONCLUSIONS

In this letter we studied an $F(R)$ reconstruction technique by using a bottom-up approach, in which the observational indices are fixed. Particularly, we assumed a specific form for the scalar-to-tensor ratio, in the context of slow-roll $F(R)$ gravity, and we investigated from which $F(R)$ gravity this scalar-to-tensor ratio may be realized. Also we calculated the spectral index of the primordial curvature perturbations, if the scalar-to-tensor ratio has the assumed specific form. By using well-known reconstruction techniques, we were able to find the analytic form of the $F(R)$ gravity and in addition, the observational indices can be compatible with the observational data. We also studied several functional forms of the scalar-to-tensor ratio, such as exponential and logarithmic functions, and we discussed the behavior of the spectral index and in effect the viability of the theory.

In principle, the bottom-up approach we propose with this letter, can be applied to more complex functional dependencies of the scalar-to-tensor ratio, but it is compelling in the end to find the Hubble rate as a function of the cosmic time, and check explicitly whether the resulting evolution is an inflationary evolution. Finally, apart from the slow-roll reconstruction method we studied in this paper, one can also study the constant-roll generalization of our bottom-up reconstruction technique. This issue will be addressed in future work.

Acknowledgments

This work is supported by MINECO (Spain), FIS2016-76363-P and by CSIC I-LINK1019 Project (S.D.O).

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