On a class of invariant coframe operators with
application to gravity

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Abstract

Let a differential 4D-manifold with a smooth coframe field be given. Consider the operators
on it that are linear in the second order derivatives or quadratic in the first order derivatives of
the coframe, both with coefficients that depend on the coframe variables. The paper exhibits the
class of operators that are invariant under a general change of coordinates, and, also, invariant
under the global SO(1,3)-transformation of the coframe. A general class of field equations is
constructed. We display two subclasses in it. The subclass of field equations that are derivable
from action principles by free variations and the subclass of field equations for which spherical-
symmetric solutions, Minkowskian at infinity exist. Then, for the spherical-symmetric solutions,
the resulting metric is computed. Invoking the Geodesic Postulate, we find all the equations
that are experimentally (by the 3 classical tests) indistinguishable from Einstein field equations.
This family includes, of course, also Einstein equations. Moreover, it is shown, explicitly, how
to exhibit it. The basic tool employed in the paper is an invariant formulation reminiscent of
Cartan’s structural equations. The article sheds light on the possibilities and limitations of the
coframe gravity. It may also serve as a general procedure to derive covariant field equations.

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1 Introduction

In the framework of the Einstein theory the gravity field is described as geometrical property of a four-dimensional pseudo-Riemannian manifold (differential manifold endowed with a pseudo-Euclidean metric). In this manifold the evolution of the metric is described by a field equation which is covariant i.e. invariant under the group of diffeomorphic transformations of the manifold. The metric is used as the basic building block. Thus, component-wise, the field equation is a system of 10 differential equations for 10 components of the metric tensor. Furthermore, Einstein looked for the second order PDE’s, linear in the principle part. It is interesting to note that Einstein in General Relativity as well as his later writings on unified field theories did not rely the Action Principle.

After the establishment of General Relativity E. Cartan [1] introduced the notion of a frame to differential geometry. He, then, showed that the basic constructs of classical differential geometry can be obtained via his Repere Mobile. Einstein [2], [3] applied Cartan’s ideas for the definition of a teleparallel space. He attempted, by that, to construct a unified theory of gravity and electromagnetism. Weitzenböck [4] investigated of the geometric structure of teleparallel spaces. Theories based on this geometrical structure are used for alternative models of gravity and also to describe the spin properties of matter. For the recent investigations in this area see Refs. [5] to [9], and [10] to [17].

For an account of teleparallel spaces in the metric-affine framework see [11], [12].

In [19] an alternative gravity model based on teleparallel spaces was suggested. Let us exhibit here a short account of it. There, the field equation was taken to be

\[ \Box \vartheta^a = \lambda \vartheta^a, \]  

(1.1)

where \( \Box = d \ast d \ast \ast d \ast d \) is the Hodge - de Rham Laplacian. It turns out that there exists a unique (up to a constant, which is identified with the mass) spherical-symmetric, asymptotically flat and static solution:

\[ \vartheta^0 = e^{-\frac{m}{r}}dt \quad \vartheta^i = e^{\frac{m}{r}}dt \quad i = 1, 2, 3, \]  

(1.2)

The resulting line element is

\[ ds^2 = e^{-2\frac{m}{r}}dt^2 - e^{2\frac{m}{r}}(dx^2 + dy^2 + dz^2). \]  

(1.3)
The metric defined by (1.3) is the celebrated Yilmaz-Rosen metric \cite{21,22,23,24}. This metric is experimentally (by three classical tests) indistinguishable from the Schwarzschild metric. In \cite{19} it was shown that (1.1) is derivable from a constrained variational principle. U. Muench, F. Gronwald and F.W. Hehl \cite{20} placed the model in the area of the various teleparallel theories. They also demonstrated that the equation (1.1) can not be derived from a variational principle, by unconstrained variations.

In the present paper we study the structure of the invariant differential coframe operators on a teleparallel manifold. We construct a general field equation on the coframe variable. That is covariant and SO(1, 3) invariant. It is a system of 16 PDE's for 16 coframe variables. The equations are linear in the second order derivatives and quadratic in the first derivatives. It turns out that in contrast to the metric gravity there exists on a teleparallel space a wide class of invariant field equation.

We also study the structure of field equations that can be derived from a quadratic Lagrangian by free variations. We show that these equations form a subclass of the class above. We also show that the Einstein field equation is a unique symmetric equation in this subclass.

Another subclass of equations is defined by the conformity to observations i.e. confirmation with the three classical tests of gravity.

The approach can also be useful for the establishment of alternative field equations for other fields. All the computations are carried out, explicitly, in a covariant manner. This introduces a great computational simplicity.

2 Invariant objects on the teleparallel space

Consider a 4D-differential manifold \( M \) endowed with a smooth coframe tetrad \( \vartheta^a \), \( a = 0, 1, 2, 3 \). This is a basis of the cotangent space \( \Lambda^1 := T^*_x M \) at an arbitrary point \( x \in M \). Let the vector space \( \Lambda^1 \) be endowed with the Lorentzian metric \( \eta_{ab} = \eta^{ab} = \text{diag}(1, -1, -1, -1) \). We will refer to the triad \( \{ M, \vartheta^a, \eta_{ab} \} \) as a teleparallel space. A hyperbolic metric on the manifold \( M \) is defined by the coframe \( \{ \vartheta^a(x) \} \) as

\[
g = \eta_{ab} \vartheta^a \otimes \vartheta^b. \quad (2.1)
\]
The coframe \( \{ \vartheta^a(x) \} \) is pseudo-orthonormal with respect to the metric \( g \)

\[
g^{\mu\nu} \vartheta^a_\mu \vartheta^b_\nu = \eta^{ab}.
\]

Let us exhibit our basic construction.

Consider the exterior differential of the basis 1-forms

\[
d\vartheta^a = \vartheta^a_\beta,_{\alpha} dx^\alpha \wedge dx^\beta = \frac{1}{2} C^a_{bc} \vartheta^{bc}.
\]  

Where, for uniqueness, the coefficients \( C^a_{bc} \) are antisymmetric:

\[
C^a_{bc} = -C^a_{cb}.
\]

We will refer to the coefficients \( C^a_{bc} \) as to 3-indexed \( C \)-objects. These coefficients can be written explicitly as

\[
C^a_{bc} = e_c^{(e_b^d \vartheta^a_d)}.
\]

Their contraction i.e.

\[
C_a = C^m_{bm} = -C^m_{mb}
\]

will be referred to as the 1-indexed \( C \)-object.

Their explicit expression is

\[
C_a = e_m^{(e_a^d \vartheta^m_d)}.
\]

These \( C \)-objects represent diffeomorphic covariant and \( SO(1,3) \) invariant generalized derivatives of the coframe field \( \vartheta^a \). In [28] it is shown that the coderivative of the coframe can also be represented by the \( C \)-objects.

In order to construct the invariant generalized second order derivative we consider the exterior differential of the \( C \)-objects.

\[
dC^a_{mn} = B^a_{mnp} \vartheta^p.
\]

The components of this 1-form \( B^a_{mnp} \) will be referred to as 4-indexed \( B \)-objects. Again, the \( B^a_{mnp} \) are scalars. Observe that they are antisymmetric in the middle indices

\[
B^a_{mnp} = -B^a_{npm}.
\]  

\(^1\)We use here and later abbreviation \( \vartheta^{ab\ldots} = \vartheta^a \wedge \vartheta^b \wedge \ldots \).
The explicit expression of 4-indexed $B$-object is

$$B^a_{bcd} = e_d e^a \left( e_c \epsilon_b d \theta^a \right). \quad (2.6)$$

As we will see later the field equation should include only the 2-indexed values. These can be obtained by a contraction of 4-indexed $B$-objects with $\eta_{ab}$. Because of the antisymmetry the only possible contractions are

$$^1B_{ab} = B_{abm}^m, \quad (2.7)$$
$$^2B_{ab} = B^m_{mab}, \quad (2.8)$$
$$^3B_{ab} = B^m_{mab}, \quad (2.9)$$

where the indices are raised and lowered via the $\eta_{ab}$, for example, $B_{abm}^m = \eta^{mk} \eta_{ap} B_{pbn}^k$.

This contracted objects will be referred to as 2-indexed $B$-objects. Observe that one of them, namely $^3B_{ab}$, is antisymmetric and two others have generally non-zero symmetric and antisymmetric parts.

Note, also, that the exterior differential of the 1-indexed $C$-object can be expressed by the 2-indexed $B$-object

$$dC_a = dC^m_{am} = B^m_{amp} \eta^p = -^2B_{ap} \eta^p. \quad (2.10)$$

We will also include in the general equation the scalar (non-indexed) objects constructed from the $B^a_{nmp}$. By the antisymmetry of the middle indices only one non trivial full contraction $B$ of the quantities $B^a_{nmp}$ is possible (up to a sign): 

$$B = B^a_{ab} \eta^b = B^a_{abc} \eta^{bc} = ^1B^a_a = ^2B^a_a. \quad (2.11)$$

The fully contracted $B$-object with no index attached will be referred to as the scalar $B$-object.

The coframe $\vartheta^a$ is usually used as a device to express physical variables like metric and connections. We take the view that the $\vartheta^a$ may serve as physical variables, as well. As such we construct field equations for it. Let us list the conditions that we are impose on these equations.

- The field equation should be global $SO(1, 3)$-invariant and tensorial diffeomorphic covariant.

For that the $B$ and $C$-objects may served as building blocks. Thus we construct it from the $B$ and $C$-objects.
• The coframe field $\vartheta^a$ has 16 independent components, these components are the independent dynamical variables. Thus, if we write the equation in a scalar form, it should be a 2-indexed equation.

• We are interested only in partial differential equations of the second order. Moreover, we are looking for equations that are linear in the $B$-objects (so that an approximation by the wave operators is possible).

• The nonlinear part of field equation is taken to be quadratic in the $C$-objects. Thus, if the coframe field $\vartheta^a$ is dimensionless, all the parts of the equation have the same dimension and all the free parameters are dimensionless.

In the sequel we will develop a general procedure for getting all the field equations that are constructed by the $C$ and $B$ objects. It is a wide class. In particular we get all the equations derived by the variation of a general quadratic Lagrangian. The Einstein equation is a particular case of our general field equation. The procedure above allows us to treat also the situation where the metric tensor $g$, as defined by (2.1), is the primary physical variable. In this case there are 10 independent variables which are the combinations of the components of the coframe $\vartheta^a$. These variables satisfy 10 field equations that turn out to be the Einstein equations. This will be worked out explicitly in the sequel.

Let us write the leading (second order) part of the equation as a linear combination of the two-indexed $B$-objects:

\[ L_{ab} = \beta_1 B_{(ab)} + \beta_2 B_{(ab)} + \beta_3 B_{ab} + \beta_4 \eta_{ab} B + \beta_5 B_{[ab]} + \beta_6 B_{[ab]}, \]

(2.12)

where the symbols $(ab)$ and $[ab]$ mean, consequently, symmetrization and antisymmetrization of the indices. The coefficients $\beta_i$ are free numerical constants.

(2.12) is the general 2-indexed tensorial expression constructed by the 4-indexed object $B^a_{bcd}$ by combination of contraction and transpose operators. The scalar $B$ is transformed into a two-indexed object by multiplying it with $\eta_{ab}$. Note that the terms in (2.12) are not independent. Their number will be reduced.

The general quadratic part of the equation can be constructed as a linear combination of 2-indexed terms of type $C \times C$ contracted by the Minkowskian metric $\eta_{ab}$. Consider all the possible combi-
nations of the indices and take into account the antisymmetry of the $C$-objects to get the following list of independent two-indexed terms:

\[ (1) A_{ab} := C_{abm}C^m, \quad (2.13) \]
\[ (2) A_{ab} := C_{mab}C^m \quad \text{antisymmetric object}, \quad (2.14) \]
\[ (3) A_{ab} := C_{amn}C^m C^b_{mn} \quad \text{symmetric object}, \quad (2.15) \]
\[ (4) A_{ab} := C_{amn}C^m C^b_n, \quad (2.16) \]
\[ (5) A_{ab} := C_{man}C^m C^b_n \quad \text{symmetric object}, \quad (2.17) \]
\[ (6) A_{ab} := C_{man}C^m C^b_n \quad \text{symmetric object}, \quad (2.18) \]
\[ (7) A_{ab} := C_a C_b \quad \text{symmetric object}. \quad (2.19) \]

In addition to the 2-indexed $A$-objects the general field equation may also include their traces multiplied by $\eta_{ab}$. These traces of 2-indexed objects are scalar $SO(1,3)$ invariants:

\[ (1) A := (1) A_a^a = -(7) A_a^a, \quad (2.20) \]
\[ (2) A := (3) A_a^a = (6) A_a^a, \quad (2.21) \]
\[ (3) A := (4) A_a^a = (5) A_a^a. \quad (2.22) \]

The trace of the antisymmetric object $^2A_{ab}$ is zero.

It is easy to see that not all of the objects $^iA_{ab}, ^iB_{ab}$ are independent. Starting from the relation

\[ d\delta^a = 0 \]

we obtain:

\[ d(C^a_{mn}\delta^{mn}) = B^a_{mnp}\delta^{mnp} + C^a_{mn}\delta^{mn} \delta = (B^a_{knp} + C^a_{mn}C^m_{pk})\delta^{pkn} = 0. \]

This equation results in a 4-indexed Bianchi identity of the first-order

\[ [B^a_{knp} + C^a_{m[n}C^m_{pk]} = 0], \quad (2.23) \]

where [knp] and [npk] are the antisymmetrization of the respective indices. Taking the unique non-vanishing contraction of the relation (2.23) we obtain

\[ B^a_{kna} + C^a_{m[n}C^m_{ak]} = 0. \]
Using the antisymmetry of the $B$-objects in the middle indices the first term in the l.h.s. of this relation results in

$$B^a_{[kna]} = 2\left(B^a_{kna} + B^a_{nak} + B^a_{akn}\right) = 2\left(2^{(2)}B_{[kn]} + (3)B_{kn}\right).$$

As for the second part

$$C^a_{m[n}C^m_{ak]} = 2\left(C^a_{mn}C^m_{ak} + C^a_{ma}C^m_{kn} + C^a_{mk}C^m_{na}\right) = 2^{(2)}A_{kn}.$$

Thus (2.23) reduces to a 2-indexed Bianchi identity of the first-order

$$2 \cdot (2^{(2)}B_{[kn]} + (3)B_{kn} + (2)A_{kn} = 0. \tag{2.24}$$

$(2)B_{[kn]}$ is a linear combination of $(3)B_{kn}$ and $(2)A_{kn}$. Consequently, $\beta_6$ in (2.12) can be taken to be zero.

The general quadratic part of an equation can be written as

$$Q_{ab} = \alpha_1^{(1)}A_{(ab)} + \alpha_2^{(2)}A_{ab} + \alpha_3^{(3)}A_{ab} + \alpha_4^{(4)}A_{(ab)} + \alpha_5^{(5)}A_{ab} + \alpha_6^{(6)}A_{ab} + \alpha_7^{(7)}A_{ab} + \alpha_8^{(1)}A_{[ab]} + \alpha_9^{(4)}A_{[ab]} + \eta_{ab}\left(\alpha_{10}^{(1)}A + \alpha_{11}^{(2)}A + \alpha_{12}^{(3)}A\right), \tag{2.25}$$

where $\alpha_i$, $i = 1, \ldots, 12$ are free dimensionless parameters.

Let us summarize the construction by

**Theorem 2.1**: The most general 2-indexed system of equations satisfying the following conditions

1. diffeomorphic covariant and $SO(1,3)$-invariant,

2. linear in the second order derivatives and quadratic in the first order derivatives with coefficients depending on the coframe variables,

3. obtained from the quantities $C^a_{bc}$ and $B^a_{bcd}$ by contractions and transpose

is

$$L_{ab} + Q_{ab} = 0, \tag{2.26}$$

where the linear leading part $L_{ab}$ is defined by (2.12) while the quadratic part $Q_{ab}$ is defined by (2.25).
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The field equation (2.26) is a system of 16 equations and it can be \(SO(1,3)\) invariantly decomposed into three independent equations:

The trace equation

\[ L^a_a + Q^a_a = 0, \]  
(2.27)

The traceless symmetric equation

\[ \left( L_{(ab)} - \frac{1}{4} L^m_m \eta_{ab} \right) + \left( Q_{(ab)} - \frac{1}{4} Q^m_m \eta_{ab} \right) = 0, \]  
(2.28)

And the antisymmetric equation

\[ L_{[ab]} + Q_{[ab]} = 0. \]  
(2.29)

Again, the brackets \((ab), [ab]\) mean, respectively, symmetrization and antisymmetrization.

The trace equation (2.27) can be explicitly written as

\[(\beta_1 + \beta_2 + 4\beta_4)B + (\alpha_1 - \alpha_7 + 4\alpha_{10}) (1)A + (\alpha_3 + \alpha_6 + 4\alpha_{11}) (2)A + \]
\[(\alpha_4 + \alpha_5 + 4\alpha_{12}) (3)A = 0. \]  
(2.30)

As for the traceless symmetric equation

\[ \beta_1 \overline{(1)B}_{(ab)} + \beta_2 \overline{(2)B}_{(ab)} + \alpha_1 \overline{(1)A}_{(ab)} + \alpha_3 \overline{(3)A}_{ab} + \alpha_4 \overline{(4)A}_{(ab)} + \]
\[ + \alpha_5 \overline{(5)A}_{ab} + \alpha_6 \overline{(6)A}_{ab} + \alpha_7 \overline{(7)A}_{ab} = 0, \]  
(2.31)

where the bar means

\[ \overline{M}_{ab} = M_{ab} - \frac{1}{4} \eta_{ab} M^m_m. \]  
(2.32)

The antisymmetric equation is

\[ \beta_3 (3)B_{ab} + \beta_5 (1)B_{[ab]} + \beta_6 (2)B_{(ab)} + \alpha_2 (2)A_{ab} + \alpha_8 (1)A_{[ab]} + \alpha_9 (4)A_{[ab]} = 0. \]  
(2.33)

This way a general family of field equations for the coframe field \(\vartheta^a\) was constructed. Every equation in the family is invariant under diffeomorphic transformations of the coordinate system. It is invariant under global \(SO(1,3)\)-transformations of the coframe as well. In the following sections we impose additional two conditions:
• An action condition. That means that the equation is derivable from a suitable action.

• Long-distance approximation conditions. That means that the equation has a solution confirming with the observed data.

3 Quadratic Lagrangians

One of the basic tools to derive field equations is the variational principle. By that a suitable Lagrangian is chosen and its variation is equated to zero. In the teleparallel approach the coframe field $\vartheta^a$ is the basic field variable, while the Lagrangian $\mathcal{L}$ is a differential 4-form. A general Lagrangian density for the coframe field $\vartheta^a$ (quadratic in the first order derivatives and linear in the second order derivatives) can be expressed as a linear combination of scalar $A$- and $B$-objects.

$$\mathcal{L} = \frac{1}{\ell^2} \left( \mu_0 B + \mu_1 (1)A + \mu_2 (2)A + \mu_3 (3)A \right) \ast 1, \quad (3.1)$$

where $\ell$ is a length-dimensional constant, $B$ is the second order scalar object, defined by (2.11) and $(i)A$ with $i = 1, 2, 3$ are quadratic scalar objects defined by (2.21-2.22).

The terms in the expression (3.1) are completely independent. They are diffeomorphic covariant and rigid $SO(1, 3)$ invariant. Let us show that the linear combination (3.1) is equivalent to the gauge invariant translation Lagrangian of Rumpf [26] (up to his $\Lambda$-term).

$$V = \frac{1}{2\ell^2} \sum_{I=1}^{3} \rho_I^{(I)} V, \quad (3.2)$$

where

$$^{(1)}V = \ast d\vartheta^a \wedge \ast d\vartheta_a, \quad (3.3)$$
$$^{(2)}V = \left( d\vartheta_a \wedge \vartheta^a \right) \wedge \ast (d\vartheta_b \wedge \vartheta^b), \quad (3.4)$$
$$^{(3)}V = \left( d\vartheta_a \wedge \vartheta^b \right) \wedge \ast\left( d\vartheta_b \wedge \vartheta^a \right). \quad (3.5)$$

Indeed, the first term (3.3) can be rewritten as

$$^{(1)}V = C^a_{\ mn} \vartheta^m \wedge \ast C^p_{\ qn} \vartheta^p = 2C_{amn}C^{amn} \ast 1 = 2 \ (2)A \ast 1. \quad (3.6)$$
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As for the second term (3.4)

\[(2) V = C_{amn} \varphi^{amn} \wedge C_{bpq} * \varphi^{bpq} = -2C_{amn}(C^{amn} + C^{mna} + C^{nam}) * 1 \]
\[= 2(2(3)A - (2)A) * 1. \quad (3.7)\]

The third term (3.5) takes the form

\[(3) V = C_{amn} \varphi^{mnb} \wedge C_{bpq} * \varphi^{pqa} = -4C_{mC}C^{m} * 1 = 2(2(1)A - A(2)A) * 1. \quad (3.8)\]

Thus the coefficients of (3.1) are the linear combinations of the coefficients of (3.2).

As for the second derivative term \(B\),

\[B * 1 = B^a_{ab} * 1 = \eta^{bc}B^a_{abc} * 1 = \eta^{bc}(e_c \Delta dC^a_{ab}) * 1 \]
\[= -d^b \wedge *d(C^a_{ab}) = -d(C^a_{ab}) \wedge *d^b \]
\[= -d(C^a_{ab} * d^b) + C^a_{ab}d * d^b = d(C_b * d^b) - C_b d * d^b. \]

Using the relation

\[d * d^b = -C^b * 1 \quad (3.9)\]

we obtain

\[B * 1 = d(C_a * d^a) + C_a C^a * 1 = d(C_a * d^a) - (1)A * 1. \quad (3.10)\]

It is well known that total derivatives do not contribute to the field equation. Thus we can neglect the \(B\)-term in the Lagrangian (3.1).

The comparison of (3.1) and (3.2) yields

\[
\begin{aligned}
\mu_1 &= 2\rho_3, \\
\mu_2 &= \rho_1 - \rho_2 - \rho_3, \\
\mu_3 &= 2\rho_2.
\end{aligned}
\quad (3.11)
\]

Therefore, the Rumpf Lagrangian (3.2) is the most general quadratic Lagrangian.

In [20], the Einsteinian theory of gravity is reinstated from the Lagrangian (3.2) by letting

\[
\rho_1 = 0, \quad \rho_2 = -\frac{1}{2}, \quad \rho_3 = 1.
\quad (3.12)
\]

Thus, correspondingly

\[
\mu_1 = 2, \quad \mu_2 = -\frac{1}{2}, \quad \mu_3 = -1.
\quad (3.13)
\]

This way we have shown that the general translation invariant Lagrangian can be expressed by the scalar \(A\)-objects.
4 The action generated field equation

It is natural to expect that the field equation, derived from the Lagrangian above can be expressed in terms of the $A$ and $B$-objects. The free variations of the Rumpf Lagrangian (3.2) yield the field equation due to Kopczyński [8]. Let us express it in the following form (Cf. [20]):

$$- 2\ell^2 \Sigma_a = 2\rho_1 d \ast d\vartheta_a - 2\rho_2 \vartheta_a \wedge d \ast (d\vartheta^b \wedge \vartheta_b) - 2\rho_3 \vartheta_b \wedge d \ast (\vartheta_a \wedge d\vartheta^b)$$

$$+ \rho_1 \left[ e_a \mathbf{J}(d\vartheta^b \wedge \ast d\vartheta_b) - 2(e_a \mathbf{J}d\vartheta^b) \wedge \ast d\vartheta_b \right]$$

$$+ \rho_2 \left[ 2d\vartheta_a \wedge \ast (d\vartheta^b \wedge \vartheta_b) + e_a \mathbf{J}(d\vartheta^c \wedge \vartheta_c \wedge \ast (d\vartheta^b \wedge \vartheta_b)) - 2(e_a \mathbf{J}d\vartheta^b) \wedge \ast (d\vartheta^c \wedge \vartheta_c) \right]$$

$$+ \rho_3 \left[ 2d\vartheta_b \wedge \ast (\vartheta_a \wedge d\vartheta^b) + e_a \mathbf{J}(\vartheta_c \wedge d\vartheta^b \wedge \ast (d\vartheta^c \wedge \vartheta_b)) - 2(e_a \mathbf{J}d\vartheta^b) \wedge \ast (d\vartheta^c \wedge \vartheta_b) \right],$$

(4.1)

where $\Sigma_a$ depends on matter fields.

By Appendix A this equation can be rewritten as

$$- 2\ell^2 \Sigma_a = \rho_1 \left( - 2 (1)B_{ab} - 2 (1)A_{ab} - (3)A_{ab} - \frac{1}{2} (2)A\eta_{ab} + 4 (6)A_{ab} \right) \ast \vartheta^b$$

$$\rho_2 \left( 4 (1)B_{[ab]} + 2 (3)B_{ab} + 4 (1)A_{[ab]} + 2 (2)A_{[ab]} + 3 (3)A_{ab} \right. + \frac{1}{2} (2)A\eta_{[ab]} - (3)A\eta_{ab} + 2 (5)A_{ab} - 2 (6)A_{ab} \right) \ast \vartheta^b +$$

$$\rho_3 \left( 2 (1)B_{ab} + 2 (2)B_{ab} - 2B\eta_{ab} + 2 (1)A_{ab} - 3 (1)A\eta_{ab} + \frac{1}{2} (2)A\eta_{ab} + (3)A_{ab} - 2 (6)A_{ab} \right) \ast \vartheta^b.$$

(4.2)

Observe, that, of all the objects defined in (2.26) only the objects $(2)A_{(ab)}$ and $(4)A_{ab}$ are missing in (4.2).

The calculations above can be summarized by

**Theorem 4.1:** The field equation generated by the variation of Rumpf Lagrangian are expressed by combination of the structural $A$ and $B$-objects as in (4.2).
Consider the special case, when the antisymmetric part of equation (4.2) is identically zero. Extracting the antisymmetric part of equation (4.2) we obtain

\[
\ell^2 \star (\vartheta_a \wedge \Sigma_b - \vartheta_b \wedge \Sigma_a) = \rho_1 \left( -2 (1)B_{[ab]} - 2 (1)A_{[ab]} \right) + \\
\rho_2 \left( 4 (1)B_{[ab]} + 2 (3)B_{[ab]} + 4 (1)A_{[ab]} + 2 (2)A_{[ab]} \right) + \\
\rho_3 \left( 2 (1)B_{[ab]} + 2 (2)B_{[ab]} + 2 (1)A_{[ab]} \right). \tag{4.3}
\]

Impose the symmetry condition on the matter current \(\Sigma_a\)

\[
\vartheta_a \wedge \Sigma_b = \vartheta_b \wedge \Sigma_a. \tag{4.4}
\]

Substitute the Bianchi identity (2.24), to get

\[
(-2\rho_1 + 4\rho_2 + 2\rho_3) (1)B_{[ab]} + (2\rho_2 + \rho_3) (3)B_{[ab]} + \\
(-2\rho_1 + 4\rho_2 + 2\rho_3) (1)A_{[ab]} + (2\rho_2 + \rho_3) (2)A_{[ab]} = 0. \tag{4.5}
\]

The l.h.s. of this equation is identically zero if and only if

\[
\rho_1 = 0, \quad 2\rho_2 + \rho_3 = 0. \tag{4.6}
\]

By homogeneity we obtain that the system (4.6) is equivalent to the system (3.12). This is the case for the teleparallel equivalent of the Einsteinian gravity. In this case the metric is an independent field variable and the field equation is restricted to a system of 10 independent equations. Thus we have shown that Einstein equation is the unique symmetric field equation that can be derived from a quadratic Lagrangian.

### 5 Diagonal static ansatz

Another subclass of the general field equation (2.26) can be constructed by with the requirement to have a solution which is confirmed by the observational data. We restrict ourselves to the three classical gravity tests, namely the Mercury perihelion shift, the light ray shift and the red shift. The experimental results can be described by a metric element

\[
ds^2 = -F dt^2 + G(dx^2 + dy^2 + dz^2),
\]
with

\[
F = 1 - \frac{2m}{r} + \frac{4m^2}{r^2} + \cdots,
\]

\[
G = 1 + \frac{2m}{r} + \cdots.
\]

The contribution of the third order term in the temporal component and the second order term in the spatial component can not be experimentally detected.

In order to obtain a metric of such type we begin with a diagonal static ansatz

\[
\vartheta^0 = e^f \, dx^0, \quad \vartheta^m = e^g \, dx^m, \quad m, n = 1, 2, 3, \quad (5.1)
\]

where \(f\) and \(g\) are two arbitrary functions of the spatial coordinates \(x, y, z\). Substitute of (5.1) into equation (2.26) to get

**Theorem 5.1:** All the possible solutions of the equation

\[
L_{ab} + Q_{ab} = 0
\]

of the form (5.1) are determined by the solutions of

\[
\mu_1 \Delta f + \mu_2 \Delta g = \mu_3 (\nabla f \cdot \nabla f) + \mu_4 (\nabla g \cdot \nabla g) + \mu_5 (\nabla f \cdot \nabla g), \quad (5.2)
\]

\[
\eta_{mn} (\nu_1 \Delta f + \nu_2 \Delta g) + \nu_3 f_{mn} + \nu_4 g_{mn} =
\]

\[
\eta_{mn} (\nu_5 (\nabla f \cdot \nabla f) + \nu_6 (\nabla g \cdot \nabla g) + \nu_7 (\nabla f \cdot \nabla g)) +
\]

\[
\nu_8 f_m f_n + \nu_9 g_m g_n + \nu_10 f_m g_n + \nu_1f_n g_m, \quad (5.3)
\]
where \( \mu_1, \ldots, \mu_5 \) and \( \nu_1, \ldots, \nu_{11} \) determined by

\[
\begin{align*}
\mu_1 &= \beta_1 + \beta_4, \\
\mu_2 &= 2\beta_4, \\
\mu_3 &= \alpha_1 - 2\alpha_3 - \alpha_4 - \alpha_6 + \alpha_{10} - 2\alpha_{11} - \alpha_{12}, \\
\mu_4 &= 2\beta_4 + 4\alpha_{10} - 4\alpha_{11} - 2\alpha_{12}, \\
\mu_5 &= \beta_1 + \beta_4 + 2\alpha_1 + 4\alpha_{10}, \\
\nu_1 &= \beta_4, \\
\nu_2 &= \beta_1 + 2\beta_4, \\
\nu_3 &= 0, \\
\nu_4 &= \beta_1 - 2\beta_2, \\
\nu_5 &= \alpha_{10} - 2\alpha_{11} - \alpha_{12}, \\
\nu_6 &= \beta_1 + 2\beta_4 + 2\alpha_1 - 2\alpha_3 - \alpha_4 - \alpha_6 + 4\alpha_{10} - 4\alpha_{11} - 2\alpha_{12}, \\
\nu_7 &= \beta_4 + \alpha_1 + 4\alpha_{10}, \\
\nu_8 &= \alpha_5 + \alpha_6 + \alpha_7, \\
\nu_9 &= -\beta_1 + \beta_2 + 2\alpha_1 - 2\alpha_3 - \alpha_4 + 2\alpha_5 + \alpha_6 + 4\alpha_7, \\
\nu_{10} &= -\beta_2 - \alpha_2 + 2\alpha_7, \\
\nu_{11} &= \alpha_2 + \alpha_7.
\end{align*}
\] (5.4)

The detailed computations are carried in Appendix B. Only 12 coefficients in (5.4) are independent.

Indeed, in addition to the relations \( \nu_3 = 0 \) and \( 2\nu_1 = \mu_2 \) the \( \mu_i \) and \( \nu_i \) have to satisfy

\[
-2\mu_1 + 4\mu_3 + 2\mu_5 + 4\nu_2 + \nu_5 = 0
\] (5.5)

and

\[
3\mu_1 - \mu_2 - 2\mu_4 - \mu_5 + 2\nu_2 + 4\nu_5 = 0.
\] (5.6)

Observe that (5.3) is a system of 4 equations. Thus, the system (5.2) and (5.3) is over-determined.

This means that the coefficients \( \mu_k \) and \( \nu_k \) have to be chosen so that two independent equation for \( f \) and \( g \) are left. This can be done in a variety of ways.

Let us turn to a special case of the spherical symmetry.
Theorem 5.2: If \( f \) and \( g \) are functions of the radial coordinate \( r = (x^2 + y^2 + z^2)^{1/2} \) then (5.2) and (5.3) read

\[
\begin{align*}
\mu_1 f'' + \mu_2 g'' + 2\frac{1}{r}(\mu_1 f' + \mu_2 g') &= \mu_3 f'^2 + \mu_4 g'^2 + \mu_5 f'g' \\
\nu_1 f'' + \nu_2 g'' + \frac{1}{r}[2\nu_1 f' + (2\nu_2 + \nu_4)g'] &= \nu_5 f'^2 + \nu_6 g'^2 + \nu_7 f'g' \\
\nu_4 g'' + \frac{1}{r}\nu_4 g' &= \nu_8 f'^2 + \nu_9 g'^2 + (\nu_{10} + \nu_{11}) f'g'
\end{align*}
\] (5.7)

This is obtained by a direct substitution of spherical-symmetric ansatz in (5.2) and (5.3). Observe that in this case the terms \( f_{mn}, g_{mn}, f_m f_n, g_m g_n \) and \( f_m g_n \) all contain \( x_m x_n \) as a factor. Thus the equation (5.3) is decomposed into two distinct equations. Note that the system (5.7) is still over-determined - three ODE for two independent variables \( f \) and \( g \). An obvious way to reduce (5.7) to two equations is to take \( \nu_4 = \nu_8 = \nu_9 = \nu_{10} + \nu_{11} = 0 \).

6 Approximate solutions

In order to confirm the field equation (2.26) with the observed data we construct an approximate solution of the restricted system (5.7). Correspondingly we consider the long-distance approximation of the functions \( f \) and \( g \). This means that the weak field on a distance far greater than the mass of the body (in the natural system of units) is studied. Take the Taylor expansion of the functions \( f \) and \( g \):

\[
\begin{align*}
f &= 1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots, \quad (6.1) \\
g &= 1 + \frac{b_1}{r} + \frac{b_2}{r^2} + \cdots \quad (6.2)
\end{align*}
\]

So, the first equation of the system (5.7) takes the form

\[
\begin{align*}
\mu_1(6\frac{a_2}{r^4} + \cdots) + \mu_2(6\frac{b_2}{r^4} + \cdots) + 2\frac{1}{r}\left(\mu_1(-2\frac{a_2}{r^3} + \cdots) + \mu_2(-2\frac{b_2}{r^3} + \cdots)\right) &= \\
\mu_3\left(\frac{a_1}{r^2} + 2\frac{a_2}{r^3} + \cdots\right) + \mu_4\left(\frac{b_1}{r^2} + 2\frac{b_2}{r^3} + \cdots\right)^2 + \mu_5\left(\frac{a_1}{r^2} + 2\frac{a_2}{r^3} + \cdots\right)\left(\frac{b_1}{r^2} + 2\frac{b_2}{r^3} + \cdots\right).
\end{align*}
\]

Thus, up to \( O\left(\frac{1}{r^6}\right) \)

\[
2\mu_1 a_2 + 2\mu_2 b_2 = \mu_3 a_1^2 + \mu_4 b_1^2 + \mu_5 a_1 b_1.
\] (6.3)
As for the second equation of the system (5.7)

\[ \nu_1 \left( \frac{a_2}{r^4} + \cdots \right) + \nu_2 \left( \frac{b_2}{r^4} + \cdots \right) + \frac{1}{r} \left[ 2\nu_1 \left( -\frac{a_2}{r^3} + \cdots \right) + (2\nu_2 + \nu_4) \left( -\frac{b_2}{r^3} + \cdots \right) \right] \]

\[ = \nu_5 \left( \frac{a_1}{r^2} + \frac{a_2}{r^3} + \cdots \right)^2 + \nu_6 \left( \frac{b_1}{r^2} + \frac{b_2}{r^3} + \cdots \right)^2 + \nu_7 \left( \frac{a_1}{r^2} + \frac{a_2}{r^3} + \cdots \right) \left( \frac{b_1}{r^2} + \frac{b_2}{r^3} + \cdots \right). \]

Thus

\[ \nu_4 b_1 = 0, \quad (6.4) \]

\[ 2\nu_1 a_2 + 2\nu_2 b_2 - 2\nu_4 b_2 = \nu_5 a_1^2 + \nu_6 b_1^2 + \nu_7 a_1 b_1, \quad (6.5) \]

The third equation of the system (5.7) is

\[ \nu_4 \left( \frac{b_1}{r^3} + \frac{b_2}{r^4} + \cdots \right) + \frac{1}{r} \nu_4 \left( -\frac{b_1}{r^2} - \frac{b_2}{r^3} + \cdots \right) = \nu_8 \left( -\frac{a_1}{r^2} - \frac{a_2}{r^3} + \cdots \right)^2 + \nu_9 \left( -\frac{b_1}{r^2} - \frac{b_2}{r^3} + \cdots \right)^2 + (\nu_{10} + \nu_{11}) \left( -\frac{a_1}{r^2} - \frac{a_2}{r^3} + \cdots \right) \left( -\frac{b_1}{r^2} - \frac{b_2}{r^3} + \cdots \right). \]

Consequently

\[ \nu_4 b_1 = 0 \quad (6.6) \]

\[ 4\nu_4 b_2 = \nu_8 a_1^2 + \nu_9 b_1^2 + (\nu_{10} + \nu_{11}) a_1 b_1. \quad (6.7) \]

The classical gravity tests (Mercury perihelion shift, light ray shift and the red shift) can be described by the following choice of the coefficients

\[ a_1 = -m, \quad b_1 = m, \quad a_2 = m^2, \quad b_2 = km^2, \quad (6.8) \]

where \( m \) is the mass of the Sun in dimensionless units. As for \( k \), it is an arbitrary dimensionless constant. This really means that \( \beta_2 \) is free. For such choice of it follows that

\[
\begin{cases}
\nu_4 = 0, \\
2\mu_1 + 2k\mu_2 = \mu_3 + \mu_4 - \mu_5, \\
2\nu_1 + 2\nu_2 k = \nu_5 + \nu_6 - \nu_7, \\
\nu_8 + \nu_9 = \nu_{10} + \nu_{11}
\end{cases}
\quad (6.9)
\]
By (5.4) rewrite these relations by the coefficients $\alpha_i, \beta_k$. 

\[
\begin{align*}
\beta_1 & = 2\beta_2, \\
6\beta_2 + (1 + 4k)\beta_4 & = -\alpha_1 - 2\alpha_3 - \alpha_4 - \alpha_6 + \alpha_{10} - 6\alpha_{11} - 3\alpha_{12}, \\
2\beta_2(2k - 1) + (4k + 1)\beta_4 & = \alpha_1 - 2\alpha_3 - \alpha_4 - \alpha_6 + \alpha_{10} - 6\alpha_{11} - 3\alpha_{12}, \\
2\alpha_1 - 2\alpha_3 - \alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 & = 0.
\end{align*}
\]

(6.10)

Or

\[
\begin{align*}
\beta_1 & = 2\beta_2, \\
2(k - 2)\beta_2 & = \alpha_1, \\
6\beta_2 + (1 + 4k)\beta_4 & = -\alpha_1 - 2\alpha_3 - \alpha_4 - \alpha_6 + \alpha_{10} - 6\alpha_{11} - 3\alpha_{12}, \\
2\alpha_1 - 2\alpha_3 - \alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 & = 0.
\end{align*}
\]

(6.11)

In order to eliminate the arbitrary constant $k$ from the system (6.11) we first consider the special case

\[
\beta_2 = 0.
\]

(6.12)

The system (6.11) takes now the form

\[
\begin{align*}
\beta_1 & = \beta_2 = \alpha_1 = 0, \\
(1 + 4k)\beta_4 & = -2\alpha_3 - \alpha_4 - \alpha_6 + \alpha_{10} - 6\alpha_{11} - 3\alpha_{12}, \\
-2\alpha_3 - \alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 & = 0.
\end{align*}
\]

(6.13)

(6.14)

(6.15)

The equation (6.15) gives no information, since it contains the arbitrary constant $k$. The remaining equations (6.13) and (6.15) do not constitute a viable physical system. Indeed, in this case, the traceless symmetric equation (2.31) in this case has no a leading (second derivatives) part. Thus one can not get an approximation by a wave equation for small fluctuations of the coframe and consequently of the metric tensor.

In the case $\beta_2 \neq 0$ we use the homogeneity of the system (6.11) to take $\beta_2 = 1$. Thus the system (6.11) is rewritten

\[
\begin{align*}
\beta_1 & = 2, \\
\beta_2 & = 1, \\
(9 + 2\alpha_1)\beta_4 & = -6 - \alpha_1 - 2\alpha_3 - \alpha_4 - \alpha_6 + \alpha_{10} - 6\alpha_{11} - 3\alpha_{12}, \\
2\alpha_1 - 2\alpha_3 - \alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 & = 0.
\end{align*}
\]

(6.16)
Theorem 6.1: Any operator that satisfies (6.16) have solutions that can not be experimentally distinguished, by the three classical tests, from Schwarzschild solution.

This way we obtain a wide class of field equations which are confirmed by three classical tests.

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Appendices

A Field equation from action

In order to rewrite the field equation (4.1) in terms of the $A, B, C$-variables we use the following formulas

\begin{align*}
  d^+(\vartheta^a) &= C^a, \\
  d^+(\vartheta^{ab}) &= C^a \vartheta^b - C^b \vartheta^a - C^m_{ab} \vartheta^m, \\
  d^+(\vartheta^{abc}) &= C^a \vartheta^{bc} - C^b \vartheta^{ac} + C^c \vartheta^{ab} - C^m_{bc} \vartheta^{am} + C^m_{ac} \vartheta^{bm} + C^m_{ab} \vartheta^{cm}, \\
  d^+(\vartheta^{abcd}) &= 0.
\end{align*}

For the first second derivative term in equation (4.1) we obtain

\begin{align*}
  2\rho_1 d + d\vartheta_a &= \rho_1 d\left(C_{abc} * \vartheta^{bc}\right) = \rho_1 B_{abc} \vartheta^m \wedge * \vartheta^{bc} + \rho_1 C_{abc} d * \vartheta^{bc} \\
  &= \rho_1 B_{abc} (\delta^m_b * \vartheta^c - \delta^m_c * \vartheta^b) + \rho_1 C_{abc} d * \vartheta^{bc} \\
  &= 2\rho_1 B_{amc} \vartheta^m + \rho_1 C_{abc}(C^b * \vartheta^c - C^c * \vartheta^b - C^{bc}_m * \vartheta^m) \\
  &= \rho_1(2B_{amc} + C_{abc}C^b - C_{acb}C^b - C_{amb}C^{bm}) * \vartheta^c \\
  &= -\rho_1 \left(2(1)B_{ab} + 2(1)A_{ab} + (3)A_{ab}\right) * \vartheta^b
\end{align*}

The second term in (4.1) takes the form
\[-2\rho_2 \vartheta_a \wedge d \ast (d \vartheta_b \wedge \vartheta^b) = -\rho_2 \vartheta_a \wedge d(C_{bmn} \ast \vartheta^{mn}) \]
\[-\rho_2 \left(B_{bmn}^k \vartheta_{ak} \wedge \ast \vartheta^{mbn} + C_{bmn} \vartheta_a \wedge d \ast \vartheta^{mbn} \right) = \rho_2 B_{bmn}^k \vartheta_a \wedge \ast (\delta^n_k \vartheta^{nb} - \delta^n_a \vartheta^{nb} + \delta_k^b \vartheta^{mn}) + \rho_2 C_{bmn} \vartheta_a \wedge \ast \left(C^m \vartheta^n - C^m \vartheta^{nb} + C^b \vartheta^{mn} + \left(C_k^{nb} \vartheta^mk - C_k^{mb} \vartheta^{nk} + C_k^{mn} \vartheta^{bk} \right) \right) \]
\[= \rho_2 B_{bmn}^k \vartheta_a \wedge \ast \left(\delta^n_k \vartheta^{nb} - \delta^n_a \vartheta^{nb} + \delta_k^b \vartheta^{mn} \right) + \rho_2 C_{bmn} \vartheta_a \wedge \ast \left[C^m \vartheta^n - C^m \vartheta^{nb} + C^b \vartheta^{mn} \right] \]
\[\left(\delta_k^a \vartheta^{b} - \delta_k^b \vartheta^{a} + \delta^a_k \vartheta^{b} \right) + \rho_2 C_{bmn} \vartheta_a \wedge \ast \left(-\vartheta^p \right) + \rho_2 \left(C_{bmn}^k \vartheta_a \wedge \ast \left(C_k^{nb} \vartheta^mk - C_k^{mb} \vartheta^{nk} + C_k^{mn} \vartheta^{bk} \right) \right) \]
\[= 2\rho_2 (B_{nka}^k + B_{an}^k + B_{kan}^k) \ast \vartheta^n + 2\rho_2 (C_{bna} + C_{ban} + C_{nab}) \ast \vartheta^n + 2\rho_2 \left(C_{mbn} C_{k}^{mn} + C_{amn} C_{k}^{mn} \right) \ast \vartheta^k \]
\[= \rho_2 \left(2^{(1)} B_{[ab]} + 2^{(3)} B_{ab} + 4^{(1)} A_{[ab]} + 2^{(2)} A_{[ab]} + 2^{(3)} A_{ab} + 4^{(4)} A_{[ab]} \right) \ast \vartheta^b \]

(A.6)

Note, that the first five terms in the brackets are antisymmetric matrices while the last one, namely \(^{(3)}A_{ab}\), is symmetric.

The third term in equation (4.1) is

\[-2\rho_3 \vartheta_b \wedge d \ast (\vartheta_a \wedge d \vartheta^b) = -\rho_3 \vartheta_b \wedge d\left(C_{bmn} \ast \vartheta_{amn} \right) = \]
\[-\rho_3 B_{bmnk} \vartheta_{bk} \wedge \ast \vartheta_{amn} - \rho_3 C_{bmn} \vartheta_b \wedge d \ast \vartheta_{amn} \]
\[= \rho_3 B_{bmn}^k \ast \left[e^b_j (e^k_j \vartheta_{amn}) \right] + \rho_3 C_{bmn} \ast \left(e_b^j d^k \vartheta_{amn} \right) \]
\[= \rho_3 B_{bmn}^k \ast \left(e^b_j \left(\delta^n_k \vartheta_{mn} - \delta^n_m \vartheta_{an} + \delta^n_a \vartheta_{mn} \right) \right) \]
\[+ \rho_3 C_{bmn}^k \ast \left(e^b_j \left(C_a \vartheta_{mn} - C_m \vartheta_{an} + C_n \vartheta_{am} + C_{k \mn} \vartheta_{ak} - C_{k \an} \vartheta_{mk} + C_{k \amn} \vartheta_{nk} \right) \right) \]
\[= \rho_3 B_{bmn}^k \ast \left(\delta^n_a \vartheta_{mn} - \delta^n_m \vartheta_{an} + \delta^n_m \vartheta_{an} \right) \]
\[+ \rho_3 C_{bmn}^k \ast \left(C_a \delta^n_a \vartheta_{mn} - C_m \delta^n_m \vartheta_{an} + C_n \delta^n_n \vartheta_{am} + C_{k \mn} \vartheta_{ak} - C_{k \an} \vartheta_{mk} + C_{k \amn} \vartheta_{nk} \right) \]
\[= 2\rho_3 \left(B_{bmn}^k \vartheta_{bn} + B_{a}^{mn} \vartheta_{am} + B_{b}^{mb} \vartheta_{m} \right) \vartheta^b + \rho_3 \left(C_{bmn}^k \vartheta_{bn} + C_{c}^{mb} \vartheta_{cm} + C_{a}^{mn} \vartheta_{cm} \right) \vartheta^b \]
\[= \rho_3 \left(2^{(1)} B_{ab} + 2^{(2)} B_{ba} - 2 B_{[ab]} + 4^{(1)} A_{(ab)} - 2^{(1)} A_{(ab)} \right) \ast \vartheta^b \]

(A.7)
On a class of invariant coframe operators with application to gravity

The first quadratic term in equation (4.1) is

\[
\rho_1 e_a \mathcal{J}(d\vartheta^b \wedge *d\vartheta_b) = \frac{1}{4} \rho_1 C^b_{mn} C^p_{bq} e_a \mathcal{J}(\vartheta^m \wedge *\vartheta_q) = \frac{1}{4} \rho_1 C^b_{mn} C^p_{bq} (\delta^m_\rho \delta^q_\sigma - \delta^m_\rho \delta^q_\sigma) * \vartheta_a
\]

\[= -\frac{1}{2} \rho_1 C^b_{bm} C^b_{mn} * \vartheta_a = -\frac{1}{2} \rho_1 (2) A_{ab} * \vartheta^b \tag{A.8}\]

The second quadratic term in equation (4.1) is

\[-2 \rho_1(e_a \mathcal{J}d\vartheta^b) \wedge *d\vartheta = -\frac{1}{2} \rho_1 C^b_{mn} C^p_{bq} (e_a \mathcal{J}\vartheta^m) \wedge *\vartheta_q \tag{A.9}\]

\[= -\frac{1}{2} \rho_1 C^b_{mn} C^p_{bq} (\delta^m_\rho \vartheta_q - \delta^m_\rho \vartheta_q - \delta^m_\rho \vartheta_q - \delta^m_\rho \vartheta_q) \]

\[= -\frac{1}{2} \rho_1 C^b_{mn} C^p_{bq} (\delta^m_\rho \vartheta_q - \delta^m_\rho \vartheta_q - \delta^m_\rho \vartheta_q - \delta^m_\rho \vartheta_q) \]

\[= 4 \rho_1 C_{bap} C^{bap} * \vartheta_q = 4 \rho_1 (6) A_{ab} * \vartheta^b \tag{A.10}\]

The third quadratic term in equation (4.1) is

\[2 \rho_2 d\vartheta_a \wedge *(d\vartheta^b \wedge \vartheta_b) = \frac{1}{2} \rho_2 C^b_{mn} C^{apq} \vartheta^p \wedge *\vartheta_{mn} = \frac{1}{2} \rho_2 C^b_{mn} C^{apq} \vartheta^p \wedge *\vartheta_{mn} \]

\[= -\frac{1}{2} \rho_2 C^b_{mn} C^{apq} \vartheta^p \wedge (\delta^m_\rho \vartheta_{mn} - \delta^m_\rho \vartheta_{mn} + \delta^m_\rho \vartheta_{mn}) \]

\[= -\frac{1}{2} \rho_2 C^b_{mn} C^{apq} \vartheta^p \wedge (\delta^m_\rho \vartheta_{mn} - \delta^m_\rho \vartheta_{mn} + \delta^m_\rho \vartheta_{mn}) \]

\[= -\frac{1}{2} \rho_2 C^b_{mn} C^{apq} \vartheta^p \wedge (\delta^m_\rho \vartheta_{mn} - \delta^m_\rho \vartheta_{mn} + \delta^m_\rho \vartheta_{mn}) \]

\[= -\frac{1}{2} \rho_2 C^{apq} (C^{apq} - C^{apq} - C^{apq} - C^{apq}) \]

\[= -\rho_2 C^{apq} (C^{apq} - C^{apq} - C^{apq} - C^{apq}) \]

\[= \rho_2 (3) A_{ab} - 2 (4) A_{ab} * \vartheta^b \tag{A.11}\]

The fourth quadratic term in equation (4.1) using the previous one takes the form

\[\rho_2 e_a \mathcal{J}(d\vartheta^c \wedge \vartheta_c \wedge *(d\vartheta^b \wedge \vartheta_b)) = \frac{1}{2} \rho_2 (3) A_{mn} - 2 (4) A_{mn} e_a \mathcal{J}(\vartheta^m \wedge *\vartheta^n) \]

\[= \frac{1}{2} \rho_2 (3) A_{mn} - 2 (4) A_{mn} \eta^{mn} \wedge \vartheta_a = \frac{1}{2} \rho_2 (2) A - 2 (3) A \eta_{ab} * \vartheta^b \tag{A.12}\]

The fifth quadratic term in equation (4.1) is

\[-2 \rho_2(e_a \mathcal{J}d\vartheta^c) \wedge \vartheta_b \wedge *(d\vartheta^c \wedge \vartheta_c) = -\frac{1}{2} \rho_2 C^b_{mn} C^{apq} (e_a \mathcal{J}\vartheta^m) \wedge \vartheta_b \wedge *\vartheta^{apq} \]

\[= \frac{1}{2} \rho_2 C^b_{mn} C^{apq} \vartheta^p \wedge *(e_b \mathcal{J}\vartheta^{apq}) \]

\[= \frac{1}{2} \rho_2 C^b_{mn} C^{apq} \vartheta^p \wedge *(e_b \mathcal{J}\vartheta^{apq}) \]

\[= \frac{1}{2} \rho_2 C^b_{mn} C^{apq} \vartheta^p \wedge *(e_b \mathcal{J}\vartheta^{apq}) \]

\[= \frac{1}{2} \rho_2 C^b_{mn} C^{apq} \vartheta^p \wedge *(e_b \mathcal{J}\vartheta^{apq}) \]

\[= \rho_2 C^b a \wedge (C_{bmn} \vartheta^c - C_{bnq} \vartheta^q - C_{mnb} \vartheta^c + C_{nbq} \vartheta^p + C_{bmq} \vartheta^q - C_{bmn} \vartheta^p) \]

\[= 2 \rho_2 C^b a \wedge (C_{bmn} + C_{nbm} + C_{mnb}) \]

\[= 2 \rho_2 (4) A_{ab} + (5) A_{ab} - (6) A_{ab} * \vartheta^b \tag{A.13}\]
The sixth quadratic term in equation (4.1) is

\[ 2\rho_3 d\vartheta_b \wedge *(\vartheta_d \wedge d\vartheta^b) = \frac{1}{2} \rho_3 C_{bmn} C^{bpq} \vartheta^{mn} \wedge * \vartheta_{apq} = -2C_{bmn} C^{bpq} \vartheta^m \wedge *(e^n J \vartheta_{apq}) \]

\[ = -\frac{1}{2} \rho_3 C_{bmn} C^{bpq} \vartheta^m \wedge *(\delta^a_p \vartheta_{pq} - \delta^m_p \vartheta_{aq} + \delta^m_q \vartheta_{ap}) \]

\[ = -\frac{1}{2} \rho_3 C_{bmn} C^{bpq} \left( \delta^m_a (\delta^m_p \vartheta_q - \delta^m_q \vartheta_p) - \delta^m_p (\delta^m_q \vartheta_a - \delta^m_q \vartheta_a) + \delta^m_q (\delta^m_a \vartheta_p - \delta^m_a \vartheta_a) \right) \]

\[ = -\rho_3 C^{bpq} \left( C_{bpa} \vartheta_q + C_{bqp} \vartheta_a + C_{baq} \vartheta_p \right) = \rho_3 \left( (2) A_{aba} - (6) A_{ab} \right) \eta^b \]

(A.14)

The seventh quadratic term in equation (4.1) can be calculated using the relation (3.8)

\[ \rho_3 e_a (\vartheta_c \wedge d\vartheta^b \wedge *(d\vartheta^c \wedge \vartheta_b)) = \frac{1}{2} \rho_3 \left( -2 (1) A_{ab} + (2) A_{ab} \right) \eta^b \]

(A.15)

The eighth quadratic term in equation (4.1) is

\[ -2\rho_3 (e_a \partial_d \vartheta^b) \wedge \vartheta_c \wedge *(d\vartheta^c \wedge \vartheta_b) = \frac{1}{2} \rho_3 C^{bpq} C^c_{mn} \left( (e_c J \vartheta^{mn}) \wedge *(e_b J \vartheta_{pq}) \right) \]

\[ = \rho_3 C^{bpq} C^c_{mn} \left( e^n J (\delta^c_p \vartheta_{qb} - \delta^c_q \vartheta_{pb} + \delta^c_b \vartheta_{pq}) \right) = 2\rho_3 \left( C_{ban} C_{mnb} + C_a C_b + C_{man} C^{mn} \right) \]

\[ = 2\rho_3 \left( (1) A_{ba} - (6) A_{ab} + (7) A_{ab} \right) \eta^b \]

(A.16)

The substitution of (2.13-2.22) in (3.1) results in (3.2).

**B A Diagonal Static Ansatz**

Consider a diagonal static coframe

\[ \vartheta^0 = e^f dx^0, \quad \vartheta^m = e^g dx^m, \]

(B.1)

where \( f \) and \( g \) are two arbitrary functions of the spatial coordinates \( x, y, z \). Compute the exterior derivative of the coframe

\[ d\vartheta^0 = e^f f_m dx^m \wedge dt = e^{-g} f_m \vartheta^{m0}, \]

\[ d\vartheta^k = e^g g_m dx^m \wedge dx^k = e^{-g} g_m \vartheta^{mk}. \]

2Greek indices run from 0 to 3 while Roman indices run from 1 to 3.
Thus the non-vanishing $C$-objects take the form:

\[ C^0_{m0} = -C^0_{0m} = e^{-g} f_m, \]  
\[ C^k_{mn} = e^{-g} (g_m \delta^k_n - g_n \delta^k_m) \]  

or, lowering the indices \[ C^0_{m0} = -C^0_{0m} = e^{-g} f_m \] \hspace{1cm} (B.4) \]
\[ C^k_{mn} = -C^k_{nm} = e^{-g} (g_m \eta_{kn} - g_n \eta_{km}) \] \hspace{1cm} (B.5) \]

The one-indexed $C$-objects $C_\alpha = C^{\beta}_{\alpha \beta}$ are

\[ C_0 = C^k_{0k} = 0 \] \hspace{1cm} (B.6) \]

and

\[ C_m = C^0_{m0} + C^n_{mn} = e^{-g} f_m + e^{-g} (g_m \delta^n_n - g_n \delta^n_m) \]
\[ = e^{-g} (f_m + 2g_m) \] \hspace{1cm} (B.7) \]

Compute the exterior derivative of the $C$-objects:

\[ dC^0_{m0} = e^{-2g} (f_{mk} - f_m g_k) \vartheta^k \]
\[ dC^k_{mn} = e^{-2g} \left( (g_{mp} - g_m g_p) \delta^k_n - (g_{np} - g_n g_p) \delta^k_m \right) \vartheta^p. \]

Thus the non-vanishing four-indexed $B$-objects are

\[ B^0_{m0k} = e^{-2g} (f_{mk} - f_m g_k), \] \hspace{1cm} (B.8) \]
\[ B^k_{mnp} = e^{-2g} \left( (g_{mp} - g_m g_p) \delta^k_n - (g_{np} - g_n g_p) \delta^k_m \right), \] \hspace{1cm} (B.9) \]

or, lowering the first index

\[ B^0_{m0k} = -B^0_{0mk} = e^{-2g} (f_{mk} - f_m g_k) \] \hspace{1cm} (B.10) \]
\[ B^k_{knp} = e^{-2g} \left( (g_{mp} - g_m g_p) \eta_{kn} - (g_{np} - g_n g_p) \eta_{km} \right) \] \hspace{1cm} (B.11) \]

\(^3\text{Recall that we use the signature } (+, -, -, -).\)
The first two-indexed B-object is \((1)B_{\alpha\beta} = B_{\alpha\beta\gamma\delta}\). Thus
\[
(1)B_{00} = B_{00mk}\eta^{mk} = -e^{-2g}(f_{mk} - f_{m}g_{k})\eta^{mk} = e^{-2g}(\triangle f - \nabla f \nabla g). \tag{B.12}
\]
\[
(1)B_{0m} = (1)B_{k0} = 0 \tag{B.13}
\]
\[
(1)B_{km} = B_{kmnp}\eta^{np} = e^{-2g}\left((g_{mp} - g_{m}g_{p})\eta_{kn} - (g_{np} - g_{n}g_{p})\eta_{km}\right)\eta^{np}
= e^{-2g}\left[\eta_{km}(\triangle g - \nabla g \nabla g) + (g_{mk} - g_{m}g_{k})\right]. \tag{B.14}
\]

We use here and later the following notations
\[
\triangle f = f_{11} + f_{22} + f_{33} = -\eta^{mn}f_{mn}, \tag{B.15}
\]
\[
\nabla f \nabla g = f_{1}g_{1} + f_{2}g_{2} + f_{3}g_{3} = -\eta^{mn}f_{m}g_{n}. \tag{B.16}
\]

The second two-indexed B-objects \((2)B_{\alpha\beta} = B_{\gamma\delta\alpha\beta}\eta^{\gamma\delta}\) are calculated to be
\[
(2)B_{00} = (2)B_{0n} = (2)B_{n0} = 0, \tag{B.17}
\]
\[
(2)B_{np} = B_{0np} + B_{kmnp}\eta^{km} = -e^{-2g}(f_{np} - f_{n}g_{p}) + e^{-2g}\left((g_{mp} - g_{m}g_{p})\eta_{kn} - (g_{np} - g_{n}g_{p})\eta_{km}\right)\eta^{km}
= e^{-2g}\left((g_{np} - g_{n}g_{p}) - 3(g_{np} - g_{n}g_{p}) - (f_{np} - f_{n}g_{p})\right)
= -e^{-2g}\left(2(g_{np} - g_{n}g_{p}) + (f_{np} - f_{n}g_{p})\right) \tag{B.18}
\]

The antisymmetric B-object \((3)B_{\alpha\beta} = B_{\gamma\delta\alpha\beta}\eta^{\gamma\delta}\) has the components
\[
(3)B_{0m} = (3)B_{m0} = 0, \tag{B.19}
\]
\[
(3)B_{mn} = B_{kmnp}\eta^{kp} = e^{-2g}\left((g_{mp} - g_{m}g_{p})\eta_{kn} - (g_{np} - g_{n}g_{p})\eta_{km}\right)\eta^{kp}
= e^{-2g}\left((g_{mn} - g_{m}g_{n}) - (g_{nm} - g_{n}g_{m})\right) = 0. \tag{B.20}
\]

The full contraction of the quantities \(B_{\alpha\gamma\alpha\beta\eta^{\gamma\delta}}\) gives
\[
B = (1)B_{00} + (1)B_{km}\eta^{km} = e^{-2g}(\triangle f - \nabla f \nabla g) + e^{-2g}\left[\eta_{km}(\triangle g - \nabla g \nabla g) + (g_{mk} - g_{m}g_{k})\right]\eta^{km}
= e^{-2g}\left[2(\triangle g - \nabla g \nabla g) + (\triangle f - \nabla f \nabla g)\right]. \tag{B.21}
\]
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Calculate the first two-indexed $A$-object $(1)A_{\alpha\beta} = C_{\alpha\beta\mu}C^\mu$

$$(1)A_{00} = C_{00m}C^m = e^{-2g}\eta^{mn}f_m(f_n + 2g_n)$$

$$= e^{-2g}(\nabla^2 f + 2\nabla f \nabla g). \quad \text{(B.22)}$$

$$(1)A_{0n} = (1)A_{m0} = 0. \quad \text{(B.23)}$$

$$(1)A_{mn} = C_{mn0}C^0 + C_{mnk}C^k = e^{-2g}(g_n\eta_{mk} - g_k\eta_{mn})(f_p + 2g_p)\eta^{kp}$$

$$= e^{-2g}\left(g_n f_m + 2g_n g_m + \eta_{mn}(2\nabla^2 g + \nabla f \nabla g)\right). \quad \text{(B.24)}$$

For the second antisymmetric two-indexed $A$-object $(2)A_{\alpha\beta} = C_{\mu\alpha\beta}C^\mu$:

$$(2)A_{00} = C_{\mu00}C^\mu = 0, \quad \text{(B.25)}$$

$$(2)A_{0m} = (2)A_{m0} = 0, \quad \text{(B.26)}$$

$$(2)A_{mn} = C_{0mn}C^0 + C_{kmn}C^k = e^{-2g}(g_m\eta_{kn} - g_k\eta_{mn})(f_p + 2g_p)\eta^{kp}$$

$$= e^{-2g}\left(g_m(f_n + 2g_n) - g_n(f_m + 2g_m)\right)$$

$$= e^{-2g}(g_m f_n - g_n f_m). \quad \text{(B.27)}$$

For the third symmetric two-indexed $A$-object $(3)A_{\alpha\beta} = C_{\alpha\mu\nu}C^\mu C^\nu$:

$$(3)A_{00} = C_{00m}C^0_0 + C_{000}C^0 + C_{0mn}C^m_0 + C_{0mn}C^m_0$$

$$= 2e^{-2g}f_m f_n \eta^{mn} = -2e^{-2g}\nabla^2 f, \quad \text{(B.28)}$$

$$(3)A_{0m} = 0, \quad \text{(B.29)}$$

$$(3)A_{mn} = C_{m00}C^n_0 + 2C_{m0k}C^n_k + C_{mpq}C^p_0 C^q_0$$

$$= e^{-2g}(g_p\eta_{mn} - g_q\eta_{mp})(g_r\eta_{ns} - g_s\eta_{nr})\eta^{pr}\eta^{qs}$$

$$= -2e^{-2g}(\eta_{mn}\nabla^2 g + g_m g_n). \quad \text{(B.30)}$$

For a fourth two-indexed $A$-object $(4)A_{\alpha\beta} = C_{\alpha\mu\nu}C^\mu C^\nu$:

$$(4)A_{00} = C_{00m}C^0_0 + C_{000}C^0 + C_{0mn}C^m_0 + C_{0mn}C^m_0$$

$$= C_{00m}C^{00m} + C_{0m0}C^{m00} + C_{0mn}C_{m0n} = e^{-2g}f_m f_n \eta^{mn}$$

$$= -e^{-2g}\nabla^2 f, \quad \text{(B.31)}$$
For the symmetric object

\[(4)A_{0\alpha} = (4)A_{\alpha 0} = 0,\]  

\[(4)A_{ab} = C_{a0n}C^{0\ n}_b + C_{am0}C^{m\ 0}_b + C_{amn}C^{m\ n}_b\]  
\[= C_{a0n}C_{0bm}\eta^{mn} + C_{am0}C_{b0n}\eta^{mn} + C_{amn}C_{pbq}\eta^{pm}\eta^{qn}\]  
\[= e^{-2g}(g_{m\eta_{bn}} - g_{n\eta_{bm}})(g_{bn}\eta_{pq} - g_{q\eta_{bp}})\eta^{pm}\eta^{qn}\]  
\[= -e^{-2g}(\eta_{ab}\nabla^2 g + g_{a}g_{b}),\]  

(B.33)

The fifth symmetric two-indexed A-object \((5)A_{\alpha\beta} = C_{\mu\nu\alpha}C_{\beta\mu\nu}\) is

\[(5)A_{00} := C_{00n}C^{0\ n}_0 + C_{m0n}C^{m\ 0}_0 = 0,\]  
\[\begin{equation} (5)A_{0\alpha} = 0, \end{equation}\]  

(B.35)

\[(5)A_{ab} = C_{a00}C^{0\ 0}_b + C_{0an}C^{n\ 0}_b + C_{ma0}C^{0\ a}_b + C_{man}C^{m\ a}_b\]  
\[= C_{0a0}C_{b00} + C_{0an}C_{nb0}\eta^{mn} + C_{ma0}C_{0bn}\eta^{mn} + C_{man}C_{pbn}\eta^{pm}\eta^{qn}\]  
\[= e^{-g}f_{a}e^{-g}f_{b} + e^{-g}(g_{a}\eta_{mn} - g_{n}\eta_{ma})e^{-g}(g_{b}\eta_{pq} - g_{q}\eta_{pb})\eta^{pm}\eta^{qn}\]  
\[= e^{-2g}(f_{a}f_{b} + 2g_{a}g_{b}).\]  

(B.36)

The symmetric A-object \((6)A_{\alpha\beta} = C_{\mu\nu\alpha\beta}C_{\mu\nu\beta}\) has the components

\[(6)A_{00} = C_{00n}C^{0\ n}_0 + C_{m0n}C^{m\ 0}_0 = C_{00n}C_{00m}\eta^{mn} + C_{m0n}C_{0p\eta^{pm}\eta^{qn}}\]  
\[= e^{-2g}f_{m}f_{n}\eta^{mn} = -e^{-2g}\nabla^2 f\]  

(B.37)

\[(6)A_{0\alpha} = C_{00n}C^{0\ a}_n + C_{n0m}C^{m\ a}_0 = 0\]  

(B.38)

\[(6)A_{ab} = C_{0a0}C^{0\ b}_0 + C_{ma0}C^{m\ b}_0 + C_{0an}C^{a\ 0}_b + C_{man}C^{mn\ b}_n\]  
\[= C_{0a0}C_{b00} + C_{ma0}C_{nb0}\eta^{mn} + C_{0an}C_{bn0}\eta^{mn} + C_{man}C_{pbn}\eta^{pm}\eta^{qn}\]  
\[= e^{-2g}f_{a}f_{b} + e^{-2g}(g_{a}\eta_{mn} - g_{n}\eta_{ma})(g_{b}\eta_{pq} - g_{q}\eta_{pb})\eta^{pm}\eta^{qn}\]  
\[= e^{-2g}(f_{a}f_{b} + g_{a}g_{b} - \eta_{ab}\nabla^2 g)\]  

(B.39)

For the symmetric object \((7)A_{\alpha\beta} = C_{\alpha}C_{\beta}\)

\[(7)A_{00} = (7)A_{0\alpha} = 0\]  

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\[(7) A_{ab} = e^{-2g}(f_a + 2g_a)(f_b + 2g_b) = e^{-2g}\left[f_a f_b + 2(f_a g_b + f_b g_a) + 4g_a g_b\right] \tag{B.41}\]

The traces of the \(A\)-matrices are

\[(1) A = (1) A^\alpha = C_\alpha ^\mu C_\mu = -C_\alpha C_\alpha \]
\[= -e^{-2g}\eta^{mn}(f_m + 2g_m)(f_n + 2g_n) = e^{-2g}(\nabla^2 f + 4\nabla f \nabla g + 4\nabla^2 g) \tag{B.42}\]
\[(2) A = (3) A^\alpha = C_\alpha ^\mu C_\mu ^\nu = -2e^{-2g}(\nabla^2 f + 2\nabla^2 g) \tag{B.43}\]
\[(3) A = (4) A^\alpha = C_\alpha ^\mu C_\mu ^\nu ^\alpha = -e^{-2g}(\nabla^2 f + 2\nabla^2 g) \tag{B.44}\]

The leading (second order) part of the equation (2.26) is a linear combination of two-indexed \(B\)-objects:

\[L_{\alpha \beta} = \beta_1 (1) B_{(\alpha \beta)} + \beta_2 (2) B_{(\alpha \beta)} + \beta_3 (3) B_{\alpha \beta} + \beta_4 \eta_{\alpha \beta} B + \]
\[\beta_5 (1) B_{[\alpha \beta]} + \beta_6 (2) B_{[\alpha \beta]} \tag{B.45}\]

The general quadratic part of the equation (2.26) is

\[Q_{\alpha \beta} = \alpha_1 (1) A_{(\alpha \beta)} + \alpha_2 (2) A_{\alpha \beta} + \alpha_3 (3) A_{\alpha \beta} + \alpha_4 (4) A_{(\alpha \beta)} + \alpha_5 (5) A_{\alpha \beta} + \]
\[\alpha_6 (6) A_{\alpha \beta} + \alpha_7 (7) A_{\alpha \beta} + \alpha_8 (1) A_{[\alpha \beta]} + \alpha_9 (4) A_{[\alpha \beta]} + \]
\[\eta_{\alpha \beta} \left(\alpha_{10} (1) A + \alpha_{11} (2) A + \alpha_{12} (3) A\right) \tag{B.46}\]

Using the calculations above the leading part is

\[L_{00} = \beta_1 e^{-2g}(\triangle f - \nabla f \nabla g) + \beta_4 e^{-2g}\left[2(\triangle g - \nabla g \nabla g) + (\nabla f - \nabla f \nabla g)\right] \]
\[= e^{-2g}\left[(\beta_1 + \beta_4) \triangle f + 2\beta_4 \triangle g - (\beta_1 + \beta_4) \nabla f \nabla g - 2\beta_4 \nabla g \nabla g\right] \tag{B.47}\]

\[L_{0m} = L_{m0} = 0 \tag{B.48}\]
The quadratic part of the equation takes the form

\[
L_{mn} = \beta_1 e^{-2g} \left[ \eta_{mn} (\nabla g - \nabla g \nabla g) + (g_{mn} - g_m g_n) \right] - \beta_2 e^{-2g} \left( 2(g_{mn} - g_m g_n) + (f_{mn} - f_m g_n) \right) + \beta_4 \eta_{mn} e^{-2g} \left[ 2(\nabla g - \nabla g \nabla g) + (\nabla f - \nabla f \nabla g) \right]
\]

\[
e^{-2g} \left[ (\beta_1 - 2\beta_2) g_{mn} + (\beta_1 + 2\beta_4) \eta_{mn} \nabla g - \beta_2 f_{mn} + \beta_4 \eta_{mn} \nabla f \nabla g \right] - (\beta_1 - \beta_2) g_m g_n + \beta_2 f_m g_n - (\beta_1 + 2\beta_4) \eta_{mn} \nabla g \nabla g - \beta_4 \eta_{mn} \nabla f \nabla g \right]
\]

(B.49)

The quadratic part of the equation takes the form

\[
Q_{00} = e^{-2g} \left[ \alpha_1 (\nabla^2 f + 2 \nabla f \nabla g) - 2\alpha_3 \nabla^2 f - \alpha_4 \nabla^2 f - \alpha_6 \nabla^2 f \
+ \alpha_{10} (\nabla^2 f + 4 \nabla f \nabla g + 4 \nabla^2 g) - 2\alpha_{11} (\nabla^2 f + 2 \nabla^2 g) \
- \alpha_{12} (\nabla^2 f + 2 \nabla^2 g) \right]
\]

\[
e^{-2g} \left[ (\alpha_1 - 2\alpha_3 - \alpha_4 - \alpha_6 + \alpha_{10} - 2\alpha_{11} - \alpha_{12}) \nabla^2 f + (4\alpha_{10} - 4\alpha_{11} - 2\alpha_{12}) \nabla^2 g + (2\alpha_1 + 4\alpha_{10}) \nabla f \nabla g \right]
\]

(B.50)

\[
Q_{0m} = Q_{m0} = 0
\]

(B.51)

\[
Q_{mn} = e^{-2g} \left[ \alpha_1 \left( g_m f_n + 2 g_n g_m + \eta_{mn} (2 \nabla^2 g + \nabla f \nabla g) \right) \
+ \alpha_2 (g_m f_n - g_n f_m) - 2\alpha_3 (\eta_{mn} \nabla^2 g + g_m g_n) \
- \alpha_4 (\eta_{mn} \nabla^2 g + g_m g_n) + \alpha_5 (f_m f_n + 2 g_m g_n) \
+ \alpha_6 (f_m f_n + g_m g_n - \eta_{mn} \nabla^2 g) \
+ \alpha_7 \left( f_m f_n + 2 (f_m g_n + g_m f_n) + 4 g_m g_n \right) \
+ \eta_{mn} \left( \alpha_{10} (\nabla^2 f + 4 \nabla f \nabla g + 4 \nabla^2 g) \right) \
- 2\alpha_{11} (\nabla^2 f + 2 \nabla^2 g) - \alpha_{12} (\nabla^2 f + 2 \nabla^2 g) \right]
\]

\[
e^{-2g} \left[ (\alpha_5 + \alpha_6 + \alpha_7) f_m f_n + (\alpha_2 + \alpha_7) g_m f_n + (-\alpha_2 + 2\alpha_7) g_n f_m + \
(2\alpha_1 - 2\alpha_3 - \alpha_4 + 2\alpha_5 + \alpha_6 + 4\alpha_7) g_m g_n + \
\eta_{mn} \left( (\alpha_{10} - 2\alpha_{11} - \alpha_{12}) \nabla^2 f + (\alpha_1 + 4\alpha_{10}) \nabla f \nabla g + \
(2\alpha_1 - 2\alpha_3 - \alpha_4 - \alpha_6 + 4\alpha_{10} - 4\alpha_{11} - 2\alpha_{12}) \nabla^2 g \right) \right]
\]

(B.52)
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Thus the field equation \( \text{(2.26)} \) reduces to the form

\[
\mu_1 \triangle f + \mu_2 \triangle g = \mu_3 \nabla^2 f + \mu_4 \nabla^2 g + \mu_5 (\nabla f \nabla g) \tag{B.53}
\]

\[
\eta_{mn} (\nu_1 \triangle f + \nu_2 \triangle g) + \nu_3 f_{mn} + \nu_4 g_{mn} = \\
\eta_{mn} (\nu_5 \nabla^2 f + \nu_6 \nabla^2 g + \nu_7 (\nabla f \nabla g)) + \\
\nu_8 f_m f_n + \nu_9 g_m g_n + \nu_{10} f_m g_n + \nu_{11} f_n g_m,
\]

where the numerical coefficients are \((5.4)\).

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