Static Solutions for $4^{th}$ Order Gravity

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The Lichnerowicz and Israel theorems are extended to higher order theories of gravity. In particular it is shown that Schwarzschild is the unique spherically symmetric, static, asymptotically flat, black-hole solution, provided the spatial curvature is less than the quantum gravity scale outside the horizon. It is then shown that in the presence of matter (satisfying certain positivity requirements), the only static and asymptotically flat solutions of General Relativity that are also solutions of higher order gravity are the vacuum solutions.

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I. INTRODUCTION

Whilst General Relativity (GR) remains the most successful classical theory we have, one may question whether it is possible for there to be corrections to it. Indeed we expect GR to break down as we approach the Planck scale where the quantum theory of gravity (whatever it is) will become dominant. In this sense GR is not a full physical theory, only an excellent approximation to some (presumably) complicated underlying theory, with the approximation getting better and better as the scales we consider are further and further from the Planck scale$^1$.

Over the past century there have been many important theorems proved about the solutions and structure of GR and an important question is whether these theorems are also valid in theories that deviate from GR at some scale. In this paper we will look in particular at two theorems, firstly the Lichnerowicz theorem$^2$, which tells us that the only static, asymptotically flat, geodesically complete, vacuum, solution to Einstein’s equations is flat space-time. We then consider the Israel theorem$^3$ which demonstrates that the only static, asymptotically flat, vacuum space-time, which contains past and future event horizons (that intersect on a surface that is topologically $S^2$) is given by the Schwarzschild metric$^2$. This ‘no-hair’ theorem is a striking result in classical GR and extensions of it to more general gravity theories, in particular as we approach the scales on which GR is expected to be violated, would provide us with insight into the transition from GR to full quantum gravity.

The action for GR is the well-known Einstein-Hilbert action which is linear in the Ricci scalar $R$ (here for simplicity we neglect the cosmological constant). A natural extension of

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$^1$ It is important to note that here the Planck scale can include both very small (UV) scales and very large (IR) scales (see for example$^4$).

$^2$ The Israel theorem also shows that the Reissner-Nordström metric must be the solution if the vacuum is replaced by an electromagnetic field.
this is to include terms of higher powers of curvature invariants to find,

\[ S_{\text{Grav}} = \int d^4x \sqrt{-g} \left( \frac{\gamma}{\kappa^2} R - \alpha_0 \hbar R^{abcd} R_{abcd} - \alpha \hbar R^{ab} R_{ab} + \beta \hbar R^2 + \mathcal{O}(R^3) \right), \quad (1.1) \]

where we have written only terms that are only 2nd order in the curvature. Throughout we will use Latin indices to label space-time components \( a, b, \ldots = 0, \ldots, 3 \), our signature is \((-\hbar, +, +, +)\) and our conventions for curvature are \( R_{abc}^d = \partial_b \Gamma_{d ac}^\dagger + \ldots \) and \( R_{ab} = R_{c a b} \).

In Eq. (1.1) the coefficients \( \alpha_0, \alpha, \beta \) and \( \gamma \) are dimensionless number and \( \kappa^2 = 32\pi G \), so GR is recovered simply by setting the coefficients of the higher order terms to zero \( (\alpha_0 = \alpha = \beta = 0) \) and taking \( \gamma = 2 \). Also note that the presence of the \( \hbar \) in Eq. (1.1) is only due to dimensions i.e. this remains a classical theory. If these correction terms are motivated from some quantum gravity theory, then we may expect the coefficients \( \alpha_0, \alpha \) and \( \beta \) to be order unity, however in general they can take any value.

In fact, in 4-dimensions, the Gauss-Bonnet combination of second order curvature invariants integrates to a topological invariant, i.e.

\[ R_{abcd} R_{abcd} - 4 R_{ab} R_{ab} + R^2 = \text{total divergence}, \quad (1.2) \]

which we can use to eliminate the \( R_{abcd} R_{abcd} \) term from Eq. (1.1). Thus, up to second order in the curvature, the most general correction to GR comes from the action,

\[ S_{\text{Grav}} = \int d^4x \sqrt{-g} \left( \frac{\gamma}{\kappa^2} R - \alpha \hbar R^{ab} R_{ab} + \beta \hbar R^2 \right). \quad (1.3) \]

This is the theory that we will work with and in the following we refer to it as 4th order gravity, since the equations of motion involve, at most, 4th derivatives of the metric. Historically, theories of this type, with \( \gamma = 0 \) were considered for many reasons, not least because of their improved behaviour under renormalisation, however it was shown that without a term linear in \( R \), the theory (for \( \alpha = 0 \)) does not couple to matter correctly \[4\]. The theory, with general \( \gamma \) was first considered in \[3\] and later in \[6\], where a perturbative analysis around the Schwarzschild solution was performed, whilst the initial value formulation of the theory was given in \[7\]. The consequence for black holes in higher dimensions has been consider in \[8\] and the cosmological implications of 4th order gravity were investigated in \[9\] \[12\], with the extension to include (tensor) perturbations appearing only recently \[13\]. The special case in which the correction terms are conformal (when \( \alpha = 3\beta \), which gives the Bach-Einstein equations) has received considerable attention (see for example \[14\]) and has recently appeared in the context of non-commutative geometry \[15\], with much work going into restricting the parameters of this form of the theory via cosmological and astrophysical observations \[16\] \[20\]. More generally, corrections to GR of this type are expected from String Theory (see for example \[21\]), renormalization group techniques \[22\] and Loop Quantum Gravity \[23\].

In addition there has been a great deal of phenomenological work on the cosmological consequences of higher order gravity theories, usually in the form of \( f(R) \) theories (see for example \[24\] \[26\]). Despite this it is important to note that theories defined by Eq. (1.3) cannot represent true physical theories for arbitrary coefficients due to the presence of ghosts \[4\] \[27\] \[28\]. Not only are ghost a problem for the quantisation of the theory, they also lead to a break-down of causality at the classical level. Note however that with
certain restrictions on the coefficients (e.g. $\alpha = 0$ and $\beta < 0$), all the difficulties associated with these ghosts can be eliminated. Here we will take the (conservative) view that, just as for GR, 4th order gravity should be considered only as an approximation to the underlying, full theory (which is presumable free of ghosts etc.).

In this paper we shall demonstrate three key results, the first two: an extension of the Lichnerowicz (Section III) and Israel (Section IV) theorems to 4th order gravity (for space-times satisfying some restriction on the magnitude of the spatial curvature) suffice to demonstrate that Schwarzschild is the unique, spherically symmetric, static, asymptotically flat, vacuum solution to the theory. The final result (Section V) considers non-vacuum solutions to the theory and demonstrates that for static, asymptotically flat space-times with matter satisfying certain positivity requirements, it is only the vacuum solutions of GR and 4th order gravity that agree; all other solutions of GR fail to be solutions of 4th order gravity.

II. STATIC SOLUTIONS OF THE EQUATIONS OF MOTION

Varying the action given in Eq. (1.3) with respect to the metric, one finds,

\[
H_{ab} = (\alpha - 2\beta) R_{;a;b} - \alpha \Box R_{ab} - \frac{1}{2} (\alpha - 4\beta) g_{ab} \Box R + 2\alpha R^{cd} R_{acbd}
\]

\[
-2\beta RR_{ab} - \frac{1}{2} g_{ab} (\alpha R^{cd} R_{cd} - \beta R^2) - \frac{\gamma}{\hbar \kappa^2} \left( R_{ab} - \frac{1}{2} g_{ab} R \right) = -\frac{1}{2\hbar} T_{ab} .
\]  

(2.1)

Considering the vacuum, the trace part gives the Klein-Gordon equation for the Ricci scalar,

\[
(\alpha - 3\beta) \left( \Box - \frac{\gamma \kappa^{-2}}{2 (\alpha - 3\beta)} \right) R = 0 .
\]  

(2.2)

It is well known that there are no static, asymptotically constant solutions of the Klein-Gordon equation for a scalar field (see for example [29]), however since the method used to prove this result will be the one that is later extended to the full equation of motion, Eq. (2.1), we give it explicitly here.

**Theorem II.1.** If the space-time is static ($\mathcal{L}_t g_{ab} = 0$) and $D_a R \to 0$ sufficiently fast at infinity, then $(\Box - m^2) R = 0$, for $m \neq 0 \in \mathbb{R}$ implies $R = 0$. If the space-time is static and $R \to 0$ sufficiently fast at infinity the equation $\Box R = 0$ implies $R = 0$.

**Proof.** We decompose the metric into components parallel and perpendicular to the constant time space-like hyper-surfaces $S$ via $g_{ab} = h_{ab} - \frac{1}{\lambda} t_a t_b$, where $t_a$ is the time-like Killing field, which has norm $t^a t_a = -\lambda$ and $h_{ab}$ is spatial metric on $S$. Note the following useful identities

\[
\mathcal{L}_t R = t^b \nabla_b R = 0 ,
\]  

(2.3)

\[
\nabla_a t_b = \frac{1}{2\lambda} \left( t_b \nabla_a \lambda - t_a \nabla_b \lambda \right) ,
\]  

(2.4)

3 This result holds for black-holes in which the spatial curvature outside the horizon is small relative to the scale set between $\gamma \kappa^{-2}$ and $\hbar \alpha$. If these corrections come from some quantum gravity theory, this restriction is essentially that the spatial curvature be small compared to the quantum gravity scale.
and
\[ t^a \nabla_a \lambda = 0 . \] (2.5)

Using these one can readily show that,
\[ g^{ab} \nabla_a \nabla_b R - m^2 R = D^a D_a R + \frac{1}{2\lambda} (D^a \lambda) (D_a R) - m^2 R = 0 , \] (2.6)
where \( D_a \) is the spatial covariant derivative compatible with \( h_{ab} \). Multiplying this equation by \( \lambda^{1/2} R \) and integrating it over the entire spatial slice, one finds
\[ \int_S \sqrt{h} d^3x \left[ \lambda^{1/2} R D^a D_a R + \frac{1}{2} \lambda^{-1/2} R (D^a \lambda) (D_a R) - m^2 \lambda^{1/2} R^2 \right] = 0 . \] (2.7)

Integrating the first term by parts this becomes,
\[ \int_S \sqrt{h} d^3x \left[ D^a \left( \lambda^{1/2} R D_a R \right) - \lambda^{1/2} (D^a R) (D_a R) - m^2 \lambda^{1/2} R^2 \right] = 0 . \] (2.8)

If the spatial slice is asymptotically constant i.e. \( D_a R \to 0 \) sufficiently fast at infinity, then the boundary term vanishes. The second and third terms in the integrand of Eq. (2.8) are manifestly negative definite and hence each term must vanish at every spatial point. If \( m \neq 0 \) this implies \( R = 0 \), for the case of \( m = 0 \) this gives \( D_a R = 0 \) and hence we additionally require that \( R \to 0 \) at infinity in order to find \( R = 0 \).

Note that here we require only that \( D_a R \to 0 \) (or \( R \to 0 \)) sufficiently fast at infinity, which is satisfied by asymptotically constant (or flat) space-times, but is in general a weaker condition. In the following we will consider asymptotically constant (or flat) space-times since they are of more interest, however all the results hold also for the weaker condition given above.

The basic method of this proof is the following. Decompose the metric onto (and perpendicular to) constant time space-like hyper-surfaces and use this to write the 4-dimensional covariant derivatives as spatial, 3-dimensional covariant derivatives plus correction terms. Multiply the resulting equation by a suitable factor and integrate over the space-like hyper-surface. Prove that, up to boundary terms, the integrand has a definite sign and hence vanishes at each spatial point. This is precisely the same method that we will use to prove that, with some restrictions on the magnitude of the spatial curvature on \( S \), there are no static, asymptotically constant solutions to the trace-free part of Eq. (2.1), which, using theorem (II.1) can be written as,
\[ \hbar \alpha \left( \Box R_{ab} - \frac{\gamma}{\hbar \alpha \kappa^2} R_{ab} + \frac{1}{2} g_{ab} R^{cd} R_{cd} - 2 R^{cd} R_{acbd} \right) = 0 , \] (2.9)
for \( \alpha - 3\beta \geq 0 \). Since Eq. (2.9) is rather more complicated than Eq. (2.2), we proceed by first proving a series of lemmas 4.

Lemma II.2. If the space-time is static and asymptotically constant, then \( (\Box - m^2) R_{ab} = 0 \), for \( m \neq 0 \in \mathbb{R} \) implies \( R_{ab} = 0 \). If the space-time is static and asymptotically flat, then

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4 A similar line of argument has used in \[29\].
\[ \Box R_{ab} = 0, \text{ implies } R_{ab} = 0. \]

**Proof.** We begin by noting several identities, which are proved in Appendix VI.

\[
\begin{align*}
t^a \nabla_a t^b \nabla_b R_{cd} &= 2 \left[ (R_{ac} \nabla_d t^b) (\nabla_b t^a) + R_{ab} (\nabla_c t^b) (\nabla_d t^a) \right] \tag{2.10} \\
(\nabla_d t^c) (\nabla_a t^d) &= -\frac{1}{4\lambda^2} t^c t_a (\nabla_d \lambda) (\nabla^d \lambda) + \frac{1}{4\lambda} (\nabla^c \lambda) (\nabla_a \lambda), \tag{2.11} \\
\frac{1}{2\lambda^2} h^{ab} (\nabla_a \lambda) t^c t^d \nabla_b R_{cd} &= \frac{1}{2\lambda} (\nabla^a \lambda) \left( \nabla_a^{(3)} R \right), \quad (2.12)
\end{align*}
\]

and

\[
\frac{1}{\lambda} t^c t^d h^{ab} \nabla_a h^c_b \nabla_e R_{cd} = D^a D^{(3)}_a R, \tag{2.13}
\]

where \((3) R\) is the 3-dimensional Ricci scalar formed from the spatial metric \(h_{ab}\). Now, decomposing \(g_{ab} = h_{ab} - \frac{1}{\lambda} t_a t_b\) one finds,

\[
(\Box - m^2) R_{ab} = h^{ab} \nabla_a h^c_b \nabla_c R_{cd} - \frac{1}{\lambda} t^a t^b \nabla_a \nabla_b R_{cd} - m^2 R_{cd} = 0. \quad (2.14)
\]

Noting the staticity of \(R_{ab}\),

\[
\mathcal{L}_t R_{ab} \equiv t^a \nabla_a R_{cd} + R_{ad} \nabla_c t^a + R_{ca} \nabla_d t^a = 0, \tag{2.15}
\]

and using Eq. (2.10) and Eq. (2.11), one can show (see Appendix VI) that Eq. (2.14) becomes,

\[
\begin{align*}
&h^{ab} \nabla_a h^c_b \nabla_e R_{cd} + \frac{1}{2\lambda} h^{ab} (\nabla_a \lambda) (\nabla_b R_{cd}) + \frac{1}{4\lambda^3} \left[ t_c t^a (\nabla_b \lambda) (\nabla^b \lambda) - \lambda (\nabla_c \lambda) (\nabla^a \lambda) \right] R_{cd} \\
&- \frac{1}{2\lambda^2} \left[ t^a \nabla_d \lambda - t_d \nabla^a \lambda \right] \left[ t^b \nabla_c \lambda - t_c \nabla^b \lambda \right] R_{ba} + \frac{1}{4\lambda^3} \left[ t_d t^a (\nabla_b \lambda) (\nabla^b \lambda) - \lambda (\nabla_d \lambda) (\nabla^a \lambda) \right] R_{ac} \\
&- m^2 R_{cd} = 0. \quad (2.16)
\end{align*}
\]

We now project this equation onto the spatial slice, \(S\) with \(h^{c}_c h^{d}_d\) and define the projection of \(R_{ab}\) onto \(S\) as \(\bar{R}_{ab} \equiv h^{c}_a h^{d}_b R_{cd}\) to find,

\[
D^a D_a \bar{R}_{ef} + \frac{1}{2\lambda} (D_a \lambda) (D^a \bar{R}_{ef}) - \frac{1}{4\lambda^2} (D_c \lambda) (D^a \lambda) \bar{R}_{af} - \frac{1}{2\lambda^3} (D_f \lambda) (D_e \lambda) t^a t^b R_{ab}
- \frac{1}{4\lambda^2} (D^a \lambda) (D_f \lambda) \bar{R}_{ea} - m^2 \bar{R}_{ef} = 0. \quad (2.17)
\]

Noting that the extrinsic curvature of the spatial slice vanishes (as can be shown by direct calculation) the Codacci and Gauss equations reduce to,

\[
\begin{align*}
h^{a}_a \nabla_c \bar{R}_{bc} &= 0, \tag{2.18} \\
\frac{1}{\lambda} t^a t^b R_{ab} &= (3) R, \quad (2.19)
\end{align*}
\]
where we have use Theorem (II.1) to set \( R = 0 \). Using these relations in Eq. (2.17) we find,

\[
D^a D_a \tilde{R}_{cd} + \frac{1}{a \lambda} (D_a \lambda) (D^a \tilde{R}_{cd}) - \frac{1}{4 \lambda^2} (D_c \lambda) (D^a \lambda) \tilde{R}_{ad} - \frac{1}{2 \lambda^2} (D_d \lambda) (D_c \lambda) (3) R \\
- \frac{1}{4 \lambda^2} (D^a \lambda) (D_d \lambda) \tilde{R}_{ca} - m^2 \tilde{R}_{cd} = 0 .
\]  

(2.20)

Multiplying this by \( \lambda^{1/2} \tilde{R}^{cd} \) and integrating over the 3-dimensional spatial slice gives,

\[
\int_S \sqrt{h} d^3 x \left[ D^a \left( \lambda^{1/2} \tilde{R}^{cd} D_a \tilde{R}_{cd} \right) - \lambda^{1/2} (D^a \tilde{R}^{cd}) (D_a \tilde{R}_{cd}) - \frac{1}{2} \lambda^{1/2} \tilde{R}^{cd} \tilde{R}_{ad} (D_c \lambda) (D^a \lambda) \\
- \frac{1}{2} \lambda^{-3/2} (3) R \tilde{R}^{cd} (D_d \lambda) (D_c \lambda) - \lambda^{1/2} m^2 \tilde{R}_{cd} \tilde{R}^{cd} \right] = 0 .
\]  

(2.21)

We now project Eq. (2.16) perpendicular to the space-like hyper-surface, with \( t^b t^c \) and again use the Codacci and Gauss equations (Eq. (2.18) and Eq. (2.19) respectively) to find,

\[
t^{c d} h^{a b} \nabla_a h_b \nabla_e R_{cd} + \frac{1}{2 \lambda} t^{c d} h^{a b} (\nabla_a \lambda) \nabla_b R_{cd} - \frac{1}{2 \lambda} (\nabla_b \lambda) (\nabla^b \lambda)^{(3)} R \\
- \frac{1}{2 \lambda} (\nabla^a \lambda) (\nabla^b \lambda) R_{a b} - \lambda m^2 (3) R = 0 .
\]  

(2.22)

Multiplying this by \( \lambda^{-1/2} (3) R \), integrating over the 3-dimensional hyper-surface and using the identities given in Eq. (2.12) and Eq. (2.13), gives,

\[
\int_S \sqrt{h} d^3 x \left[ D^a \left( \lambda^{1/2} (3) RD_a (3) \tilde{R} \right) - \lambda^{1/2} (D^a (3) \tilde{R}) (D_a (3) \tilde{R}) \\
- \frac{1}{2} \lambda^{-3/2} (D_a \lambda) (D^a \lambda) (3) R \right] \cdot \left[ - \frac{1}{2} \lambda^{-3/2} (D^a \lambda) (D^b \lambda) \tilde{R}_{ab} (3) R - \lambda^{1/2} m^2 (3) R \right] = 0 .
\]  

(2.23)

Finally, adding Eq. (2.21) and Eq. (2.23) one finds,

\[
\int_S \sqrt{h} d^3 x \left[ D^a \left( \lambda^{1/2} \tilde{R}^{cd} D_a \tilde{R}_{cd} + \lambda^{1/2} (3) RD_a (3) \tilde{R} \right) - \lambda^{1/2} (D^a \tilde{R}^{cd}) (D_a \tilde{R}_{cd}) \\
- \lambda^{1/2} (D^a (3) \tilde{R}) (D_a (3) \tilde{R}) - \frac{1}{2} \lambda^{-3/2} (3) R \tilde{R}_{cd} D_c \lambda + (3) RD_d \lambda \right] \cdot \left[ \tilde{R}_{ad} D^a \lambda + (3) RD_d \lambda \right] \\
- \lambda^{1/2} m^2 \left( \tilde{R}^{cd} \tilde{R}_{cd} + (3) R \right)^2 \right] = 0 .
\]  

(2.24)

This is sufficient to prove the desired result, however there is a simplification that can be made by noting that the contracted Bianchi identity \( \nabla^a (\tilde{R}_{ab} - \frac{1}{2} g_{ab} \tilde{R}) = 0 \) implies that

\[
\tilde{R}_{dc} D^d \lambda + (3) RD_c \lambda = -2 \lambda D^a \tilde{R}_{ac} .
\]  

(2.25)
Substituting this into Eq. (2.24) explicitly removes the derivatives of \( \lambda \) from the integrand and ensures that \( \lambda \) only appears with positive powers. This will be crucial when we consider space-times with (null) horizons.

In any case, asymptotic constancy i.e. \( D_a R_{cd} \to 0 \) at infinity, ensures that the boundary terms in Eq. (2.24) vanish. All the remaining terms are negative definite and hence each term must independently vanish at every spatial point. For \( m \neq 0 \) this implies \((3) R = 0\) and \( \bar{R}_{cd} = 0 \), which together with theorem (II.1) and Eq. (2.18) implies \( R_{ab} = 0 \). For the \( m = 0 \) case we find that \( D_a^{(3)} \neq 0 \) and \( \bar{R}_{cd} = 0 \), which together with theorem (II.1) and Eq. (2.18) implies \( R_{ab} = 0 \).

This result is easily extended to include the third term in Eq. (2.9) by using the fact that, by Theorem (II.1) we can set \( R = 0 \).

**Lemma II.3.** If the space-time static and asymptotically constant, then the pair of equations \( R = 0 \) and \((\square - m_1^2) R_{ab} + m_2 g_{ab} R^{cd} R_{cd} = 0\), for \( m_1 \neq 0 \) and \( m_2 \in \mathbb{R} \) imply \( R_{ab} = 0 \), whilst for \( m_1 = 0 \), additionally requiring the space-time to be asymptotically flat implies \( R_{ab} = 0 \).

*Proof.* From the equation \((\square - m_1^2) R_{ab} + m_2 g_{ab} R^{cd} R_{cd} = 0\) it follows that,

\[
\left[ \lambda^{1/2} \bar{R}^{cd} h^a_c h^b_d + \lambda^{-1/2} \ (3) R t^a t^b \right] \left[ (\square - m_1^2) R_{ab} + m_2^2 g_{ab} R^{cd} R_{cd} \right] = 0 \, .
\] (2.26)

Recalling that \( t_a t^a = -\lambda \) and noting that \( h^{ab} \bar{R}_{ab} = (3) R \) we see that Eq. (2.26) reduces to,

\[
\left[ \lambda^{1/2} \bar{R}^{cd} h^a_c h^b_d + \lambda^{-1/2} \ (3) R t^a t^b \right] (\square - m_1^2) R_{ab} = 0 \, ,
\] (2.27)

which is independent of the value of \( m_2 \). By Lemma (II.2), for \( m_1 \neq 0 \), this implies \( R_{ab} = 0 \), provided the space-time is static and asymptotically constant, whilst for \( m_1 = 0 \) and the space-time being static and asymptotically flat implies \( R_{ab} = 0 \). Indeed integrating Eq. (2.27) over the spatial slice exactly gives Eq. (2.24).

The final step that is required in order to allow us to consider Eq. (2.9) is to extend the above Lemmas to include a term of the form \( R_{ab} R_{acbd} \). To do this note that by using the definition of the Riemann tensor,

\[
(\nabla_a \nabla_b - \nabla_b \nabla_a) R_{cd} = R_{abc}^\ e R_{ed} + R_{abd}^\ e R_{ce} \, ,
\] (2.28)

and the fact that the contracted Bianchi identity, for \( R = 0 \), gives,

\[
\nabla_c R_b^\ e = \frac{1}{2} \nabla_b R = 0 \, ,
\] (2.29)

we can write (for \( R = 0 \)),

\[
R^{cd} R_{acbd} = -\nabla_c \nabla_a R_b^\ e + R_{ac} R_b^\ e \, .
\] (2.30)

Thus we see that this term is in fact closely related to \( \square R_{ab} \) and can be dealt with in a similar manner.
Lemma II.4. For a static space-time the following relation holds:

\[
\int \sqrt{\text{h}^3} \, x \left[ (\lambda^{1/2} \bar{R}^{ef} h^a_e h^b_f + \lambda^{-1/2} \, (3) \, R \right] h^{5b} R_{acbd} R^{cd] =
\int \sqrt{\text{h}^3} \, x \left[ D^a \left( -\lambda^{1/2} \bar{R}^{ef} D_e \bar{R}_{af} + \lambda^{1/2} \, (3) \, R D_d \bar{R}_{ab} \right) + \lambda^{1/2} \left( D_a \bar{R}_{ef} \right) \left( D^e \bar{R}^{af} \right) \\
+ \lambda^{1/2} \left( D^a \bar{R}_{ac} \right) \left( D_d \bar{R}^{ec} \right) - 2 \lambda^{1/2} \left( D^c \bar{R}_{ca} \right) \left( D^a \, (3) \, R \right) + \lambda^{1/2} \left[ \bar{R}^a_b \bar{R}^b_c \bar{R}^{ec} - \left( (3) \, R \right) \right] \right] .
\]  

(2.31)

Proof. To show this we project Eq. (2.30) onto the spatial surface, \( S \), with \( h^a_e h^b_f \) and use

the identity

\[
h^a_e h^b_f \nabla_e \nabla_a R^c_{bd} = h^a_e h^b_f \left( \bar{f}^{cd} - \frac{1}{\lambda} \, t^c t^d \right) \nabla_e \nabla_a R_{bd} \\
= D_a D_e \bar{R}^d - \frac{1}{4 \lambda^2} \left( D_e \lambda \right) \left( D_f \lambda \right) \left( 3 \right) \, R - \frac{1}{4 \lambda^2} \left( D_e \lambda \right) \left( D^d \lambda \right) \bar{R}_{fd} \\
+ \frac{1}{2 \lambda} \left( D^f \lambda \right) \left( 3 \right) \, R + \frac{1}{2 \lambda} \left( D^a \lambda \right) \left( D_e \bar{R}_{fa} \right)
\]

(2.32)

where the second equality comes from the tedious but straightforward algebra coming from bringing the \( h^b f t^d \) inside both covariant derivatives and repeatedly using Eq. (2.4), Eq. (2.5), Eq. (2.18) and Eq. (2.19).

Similarly we project Eq. (2.30) along \( t^a t^b \) to find,

\[
t^a t^b R^{cd} R_{acbd} = - t^a t^b \nabla_e \nabla_a R^c_{bd} + R_{ae} R^c_{bd} t^a t^b \\
= \frac{1}{2} \left[ \left( D_a \bar{R}^a_b \right) \left( D_b \lambda \right) + \bar{R}^{ab} D_a D_b \lambda + \left( (3) \, R D_a D^a \lambda \right) - \lambda \left[ (3) \, R \right]^2 \right] ,
\]

(2.34)

where the second equality follows from the staticity of the space-time and again, repeated use of Eq. (2.4), Eq. (2.5), Eq. (2.18) and Eq. (2.19).
Multiplying Eq. (2.34) by $\lambda^{-1/2}$ and integrating over $S$, one finds,

$$
\int_S \sqrt{h} d^3 x \left[ \lambda^{-1/2} (3) R^a t^b R^c dR_{abcd} \right] = 
\int_S \sqrt{h} d^3 x \left[ D^a \left( \frac{1}{2} \lambda^{-1/2} (3) R R_{ab} D^b \lambda + \frac{1}{2} \lambda^{-1/2} (3) \bar{R} D^a \lambda \right) \right.
+ \frac{1}{4} \lambda^{-3/2} (3) \bar{R}^a (D_a \lambda) (D_b \lambda) + \frac{1}{4} \lambda^{-3/2} (3) R (D^a \lambda) (D_a \lambda)
- \frac{1}{2} \lambda^{-1/2} R^{ab} (D^a (3) R) (D_b \lambda) - \lambda^{1/2} (3) R (D^a (3) R) (D_a \lambda) - \lambda^{1/2} \left[ (3) R \right]^3 \right].
\tag{2.35}
$$

Finally, adding Eq. (2.33) and Eq. (2.35) and using the contracted Bianchi identity given in Eq. (2.25) demonstrates the desired result, Eq. (2.31).

III. GENERALISED LICHNEROWICZ THEOREM FOR 4th ORDER GRAVITY

Using the lemmas proved in the previous section, we will now extend the Lichnerowicz theorem to 4th order gravity (for $\alpha \neq 0$ and $\alpha \geq 3\beta$), provided the spatial scalar curvature satisfies certain bounds.

**Theorem III.1.** Consider a static space-time, with a spatial slice that is topologically $\mathbb{R}^3$, in which the spatial curvature everywhere satisfies the following two conditions:

$$
m^2 - (3) R \geq 0, \\
\bar{R}^a \bar{R}^b (m^2 + \mathcal{R}) \geq 0, 
\tag{3.1}
$$

where $\mathcal{R}$ is defined as,

$$
\mathcal{R} \equiv \frac{\bar{R}^a \bar{R}^b \bar{R}^c \bar{R}^a}{\bar{R}^a \bar{R}^b \bar{R}^b \bar{R}^b}. 
\tag{3.2}
$$

Let the space-time be asymptotically constant, unless both of the inequalities in Eq. (3.1) are saturated, in which case let the space-time be asymptotically flat. Then for such a space-time the equations $R = 0$ and

$$
\Box R_{ab} - m^2 R_{ab} + \frac{1}{2} g_{ab} R^{cd} R_{cd} - 2 R^{cd} R_{acbd} = 0, 
\tag{3.3}
$$

imply $R_{ab} = 0$.

**Proof.** The equation,

$$
\Box R_{ab} - m^2 R_{ab} + \frac{1}{2} g_{ab} R^{cd} R_{cd} - 2 R^{cd} R_{acbd} = 0, 
\tag{3.4}
$$

implies,

$$
I \equiv \left( \lambda^{1/2} \bar{R}^e D^f h_e t^f + \lambda^{-1/2} (3) R t^a t^b \right) \left( \Box R_{ab} - m^2 R_{ab} + \frac{1}{2} g_{ab} R^{cd} R_{cd} - 2 R^{cd} R_{acbd} \right) = 0. 
\tag{3.5}
$$
By Lemmas (II.3) and (II.4) and Eq. (2.24) and Eq. (2.25) of Lemma (II.2) the integral of Eq. (3.5) over the constant time slice, $S$ is,

$$
\int_S \sqrt{h} d^3x = \int_S \sqrt{h} d^3x \left\{ D^a \left[ \lambda^{1/2} (^{(3)} R) D_{cd}^{(3)} R + \tilde{R}^{cd} D_a \tilde{R}_{cd} - \tilde{R}^{ef} D_e \tilde{R}_{af} - (^{(3)} R) D^b \tilde{R}_{ba} \right] \right.
- \lambda^{1/2} \left( D^a \tilde{R}^{cd} \right) \left( D_a \tilde{R}_{cd} \right) - 2 \lambda^{1/2} \left( D_a \tilde{R}_{bc} \right) \left( D^b \tilde{R}^{ac} \right) - \lambda^{1/2} \left[ D^a \left( ^{3} R \right) - 2 D^e \tilde{R}^{a}_e \right] \left[ D^a_{(3)} - 2 D^b \tilde{R}_{ab} \right] 
- \lambda^{1/2} \left[ \tilde{R}_{cd} \tilde{R}^{cd} \left( m^2 + \mathcal{R} \right) + \left( ^{3} R \right)^2 \left( m^2 - ^{(3)} R \right) \right] \left\} = 0. \right.
$$

(3.6)

Asymptotic constancy ensures that the boundary terms vanish, whilst the final two terms are negative definite if the spatial curvature satisfies the properties,

$$
m^2 - ^{(3)} R \geq 0, \quad \tilde{R}^a_b \tilde{R}^b_a \left( m^2 + \mathcal{R} \right) \geq 0. \quad (3.7)
$$

Finally, as shown in Appendix (VI), the combination

$$
\left( D^a \tilde{R}^{cd} \right) \left( D_a \tilde{R}_{cd} \right) + 2 \left( D_a \tilde{R}_{bc} \right) \left( D^b \tilde{R}^{ac} \right) \geq 0 \quad (3.8)
$$

for all $\tilde{R}_{ab}$. Thus each term in the integrand of Eq. (3.6) is negative definite and hence required to vanish. Thus $D_e \tilde{R}_{ab} = 0$ and if either of the inequalities in the expressions given in Eq. (3.7) fail to be saturated in any open region, Eq. (3.6) implies $R_{ab} = 0$. If the expressions in Eq. (3.7) are saturated everywhere, then asymptotic flatness is required to imply $R_{ab} = 0$ everywhere.

The equations of motion of 4th order gravity (with $\alpha - 3 \beta \geq 0$), imply $R = 0$ (by Theorem (II.1)) and (from Eq. (2.9))

$$
h \alpha \left( \Box R_{ab} - m^2 R_{ab} + \frac{1}{2} g_{ab} R^{cd} R_{cd} - 2 R^{cd} R_{abcd} \right) = 0, \quad (3.9)
$$

with $m^2 = \gamma \kappa^{-2} h^{-1} \alpha^{-1}$. Thus, provided the spatial curvature everywhere obeys the bounds,

$$
^{(3)} R \leq \frac{\gamma}{h \alpha \kappa^2}, \quad \mathcal{R} \geq \frac{\gamma}{h \alpha \kappa^2}, \quad (3.10)
$$

there are no, non-trivial, static, asymptotically constant vacuum solutions to 4th order gravity (with $\alpha - 3 \beta \geq 0$ and $\alpha \neq 0$).

**IV. GENERALISED ISRAEL THEOREM FOR 4TH ORDER GRAVITY**

The result proved in Section (III) demonstrates that there are no static, vacuum solutions to 4th order gravity for space-times which have a spatial section that is topologically $\mathbb{R}^3$, but one may ask whether a similar result holds for space-times in which the spatial slice contains a boundary. In particular, in order to consider space-times with a (static) black-hole, we
need to allow for a null, interior boundary to our space-time i.e. the equal time hyper-surfaces become null. At first sight the presence of boundary terms in Eq. (3.6) would appear to be a significant difficulty for such a space-time, however on the horizon, the normal to the spatial hyper-surface, \( n_a \) becomes null i.e. \( n^a n_a = \lambda = 0 \) and hence the boundary terms vanish. Thus, if the space-time is asymptotically constant (which ensures the boundary terms at infinity also vanish) and all interior boundaries are null, the result of Section (III) will still apply.

**Theorem IV.1.** Consider a static space-time, in which the spatial curvature everywhere satisfies the following two conditions:

\[
(3)R \leq \frac{\gamma}{\hbar \alpha \kappa^2}, \quad \mathcal{R} \geq -\frac{\gamma}{\hbar \alpha \kappa^2},
\]

Let the space-time be asymptotically constant, unless both of the inequalities in Eq. (4.1) are saturated, in which case let the space-time be asymptotically flat. Let the spatial slice be bounded by a null surface that is topologically \( S^2 \). Then the only solution to 4th order gravity (with \( \alpha \neq 0 \) and \( \alpha - 3\beta \geq 0 \), in the region exterior to the null surface, is \( R_{ab} = 0 \).

**Proof.** As in Theorem (III.1) the equations of motion for 4th order gravity imply, (Eq. (3.6))

\[
\int \sqrt{\mathcal{h}} \, d^3x \ (I) = \int \sqrt{\mathcal{h}} \, d^3x \left\{ D^a \left[ \lambda^{1/2} \left( (3)R D_a^3 \right) R + \bar{R}^c d D_a \bar{R}_{cd} - \bar{R}^e f D_e \bar{R}_{af} - (3) R D^b \bar{R}_{ba} \right] \right. \\
- \lambda^{1/2} \left( D^a \bar{R}^c d \right) \left( D_a \bar{R}_{cd} \right) - 2 \lambda^{1/2} \left( D_a \bar{R}^c e \right) \left( D^b \bar{R}^{ac} \right) - \lambda^{1/2} \left[ D^a (3) R - 2 D^c \bar{R}_{ac} \right] \left[ D^c (3) R - 2 D^b \bar{R}_{ab} \right] \\
- \lambda^{1/2} \left( \bar{R}_{cd} \bar{R}^c d (\gamma \kappa^{-2} \alpha^{-1} + \mathcal{R}) + (3) R \right)^2 \left( \gamma \kappa^{-2} \alpha^{-1} - (3) R \right) \} = 0.
\]

At infinity the boundary terms vanish because the spatial section is asymptotically constant, whilst on the interior boundary they vanish because \( \lambda = 0 \). Thus, just as in Theorem (III.1), provided the 3-dimensional curvature satisfies the conditions

\[
\frac{\gamma}{h \alpha \kappa^2} - (3) R \geq 0, \\
\frac{\gamma}{\hbar \alpha \kappa^2} + \mathcal{R} \geq 0,
\]

each term in the integrand of Eq. (4.2) is negative definite and hence vanishes. If either of these inequalities is not saturated everywhere, asymptotic constancy is sufficient to imply \( R_{ab} = 0 \), otherwise asymptotic flatness is required.

Since 4th order gravity implies \( R_{ab} = 0 \) (for asymptotically constant and static space-times), provided

\[
(3)R \leq \frac{\gamma}{h \alpha \kappa^2}, \\
\mathcal{R} \geq -\frac{\gamma}{\hbar \alpha \kappa^2},
\]

(4.3)
even in the presence of null boundaries, it follows that the only spherically symmetric solution is the (exterior of the) Schwarzschild metric.

It is important to note here that this result relies on the existence of a null boundary to the equal time hyper-surfaces and the existence of a vacuum \((T_{ab} = 0)\) everywhere between this horizon and infinity. This is true for a (static, vacuum) space-time containing an event horizon, however the result cannot be applied to the exterior of a general spherical object that does not contain such a horizon. In particular this result does not imply that the metric outside a spherically symmetric distribution of matter will be Schwarzschild, unless that matter is entirely contained within the event horizon. This agrees with the (perturbative) results of (for example) \cite{6}.

V. NON-VACUUM SOLUTIONS

In Sections (III) and (IV) we showed that for \(T_{ab} = 0\), the static, asymptotically constant solutions of GR are also solutions to 4\(^{th}\) order gravity (with \(\alpha \neq 0\) and \(\alpha - 3\beta \geq 0\), at least in the case of space-times that have a spatial topology of \(\mathbb{R}^3\) (Theorem (III.1)) or space-times that have internal null boundaries (Theorem (IV.1)), and whose spatial curvature everywhere satisfies Eq. (4.3). A natural question that one may ask is whether this connection between the solutions of GR and 4\(^{th}\) order gravity extends to the non-vacuum case i.e. \(T_{ab} \neq 0\). We will now demonstrate that it does not. In fact below we show that for, static, asymptotically flat space-times containing a barotropic fluid with equation of state \(\omega \geq -3^{-1/3} \approx -0.6933\), only the vacuum solutions of the two theories coincide, with all other solutions being distinct.

**Theorem V.1.** Consider an asymptotically flat solution to GR, \(^{GR}g_{ab}\), with matter satisfying,

\[
\int \sqrt{\det g} \, d^3x \, T^a_b \, T^b_c \, T^c_a \geq 0.
\]

Then \(^{GR}g_{ab}\) is a solution to 4\(^{th}\) order gravity (with \(\alpha \neq 0\) and \(\alpha \neq 3\beta\)) iff \(R_{ab} = 0\).

**Proof.** Consider a solution to the Einstein’s equations:

\[
\gamma \kappa^{-2} \left( R_{ab} - \frac{1}{2} g_{ab} R \right) = \frac{1}{2} T_{ab}.
\]

that is static and asymptotically flat. Substituting this into the equations of motion for 4\(^{th}\) order gravity, Eq. (2.1) we find,

\[
(\alpha - 2\beta) \, R_{ab} - \alpha \Box R_{ab} - \frac{1}{2} (\alpha - 4\beta) \, g_{ab} \Box R + 2\alpha R^{cd} R_{acbd} - 2\beta R^{ab} - \frac{1}{2} g_{ab} \left( \alpha R^{cd} R_{cd} - \beta R^2 \right) = 0.
\]

The trace of Eq. (5.3) is,

\[
(\alpha - 3\beta) \, \Box R = 0.
\]

Then by Theorem (III.1), for \(\alpha - 3\beta \neq 0\), this implies \(R = 0\). Thus Eq. (5.3) becomes,

\[
- \alpha \left[ \Box R_{ab} + \frac{1}{2} g_{ab} R^{cd} R_{cd} - 2 R^{cd} R_{acbd} \right] = 0.
\]
which by Theorem (III.1) implies (see Eq. (3.6)),

\[
\int \sqrt{h} d^3 x (I) = \int \sqrt{h} d^3 x \left\{ D^a \left[ \lambda^{1/2} \left( (3) R D^b D^c D^d \bar{R}_{cd} - \bar{R}_c^c \bar{R}_d^d (3) R D_{ab} \right) \right] \\
- \lambda^{1/2} \left( D^a \bar{R}^c_{cd} \right) \left( D_a \bar{R}_{cd} \right) - 2 \lambda^{1/2} \left( D_a \bar{R}_{be} \right) \left( D^b \bar{R}^{ac} \right) - \lambda^{1/2} \left[ D^a (3) R - 2 D^a \bar{R}_c^c \right] \left[ D^3_a - 2 D^b \bar{R}_{ab} \right] \\
- \lambda^{1/2} \left[ \bar{R}_c^c \bar{R}_a^a - (3) R \right] \right\} = 0 .
\]

(5.6)

Thus, if

\[
\int \sqrt{h} d^3 x \left[ R^a_b R^b_c R^c_a - (3) R \right] \geq 0 ,
\]

(5.7)

then each term in the integrand is negative definite and hence vanishes.

However it is easy to show that,

\[
R^a_b R^b_c R^c_a = g^{ad} g^{be} g^{cf} R_{bd} R_{ce} R_{fa} ,
\]

\[
= \bar{R}^a_b \bar{R}^b_c \bar{R}^c_a - (3) R \right) \}
\]

(5.8)

where the second equality follows from decomposing \( g^{ab} = h^{ab} - \frac{1}{4} t^a t^b \), using the Codacci and Gauss equations (Eq. (2.19) and Eq. (2.18)) and noting that \( \bar{R} = 0 \). Thus each term in the integrand of Eq. (5.6) is negative definite, and hence vanishes, provided,

\[
\int \sqrt{h} d^3 x \left[ R^a_b R^b_c R^c_a - (3) R \right] \geq 0 .
\]

(5.9)

Using Eq. (5.2) and the fact that \( R = 0 \) for these solutions, this condition can be written as

\[
\int \sqrt{h} d^3 x T^a_b T^b_c T^c_a \geq 0 .
\]

(5.10)

Thus, for matter satisfying Eq. (5.10), a static, asymptotically flat solution to GR is also a solution to 4th order gravity iff \( R_{ab} = 0 \), which, by Theorems (III.1) and (IV.1) corresponds to the (static, asymptotically flat) vacuum solutions in both GR and 4th order gravity. Thus, other than the vacuum, there are no static, asymptotically flat solutions to GR, with matter satisfying Eq. (5.10), that are also solutions to 4th order gravity.

If we consider a perfect fluid the energy momentum tensor can be written as \( T^a_b = \rho u^a u_b + P (\delta^a_b + u^a u_b) \), with \( u_a \) a unit time-like vector tangent to the observer’s world-line, \( P \) the fluid’s pressure and \( \rho \) its energy density. Eq. (5.10) then becomes,

\[
\int \sqrt{h} d^3 x (\rho^3 + 3P^3) \geq 0 ,
\]

(5.11)

Assuming the strong energy condition holds, \( \rho \geq 0 \), then this condition is trivial for all matter with positive pressure. If we consider a barotropic fluid with equation of state
parameter $\omega$ i.e. $P = \omega \rho$ then Eq. (5.10) reduces to,
\[
\int \sqrt{h} d^3 x (1 + 3\omega^3) \geq 0 .
\] (5.12)

This is, in particular, satisfied for $\omega \geq -3^{-1/3} \approx -0.6933$ everywhere, and hence for dust ($\omega = 0$) and radiation ($\omega = 1/3$) fluids.

Thus for a static, asymptotically flat, space-time containing a single, barotropic fluid for which $\omega \geq -3^{-1/3}$ everywhere, no solutions to GR remain (static and asymptotically flat) solutions to 4th order gravity.

One can also consider, for example, the energy-momentum tensor of a Maxwell field,
\[
T^a_b = \frac{1}{4\pi} \left( F^a_c F^c_b - \frac{1}{4} g^a_b F^d_e F^e_d \right),
\] (5.13)
with $F_{ab}$ the electromagnetic field tensor. For such a field, one can directly check that
\[
T^a_b T^b_c T^c_a = \frac{1}{64\pi^3} \left[ \frac{1}{8} (\text{Tr} F^2)^3 - \frac{3}{4} (\text{Tr} F^4) (\text{Tr} F^2) - \text{Tr} F^6 \right] \geq 0 ,
\] (5.14)
where the matrix $F$ is given by $F^a_b$ and hence is anti-symmetric and the final inequality follows from direct computation, for a general anti-symmetric matrix.

VI. CONCLUSIONS

In the Sections (III) and (IV) we demonstrated that there are no, non-trivial, static, asymptotically constant solutions to 4th order gravity, provided the 3-dimensional curvature satisfies the following two conditions:
\[
(3) R \leq \frac{\gamma}{\hbar \alpha \kappa^2} \quad \text{and} \quad R \geq -\frac{\gamma}{\hbar \alpha \kappa^2} ,
\] (6.1)
everywhere outside a null horizon. The meaning of these conditions is the following. We require that the 3-dimensional scalar curvature, measured by both $(3) R$ and $R$, to be everywhere smaller in magnitude than the scale set by the ratio between $\gamma \kappa^{-2}$ and $\hbar \alpha$. This is exactly the scale at which corrections to General Relativity will become significant. If these corrections are motivated by quantum corrections to general relativity then this will be the quantum gravity scale. Thus our results can interpreted as follows: there are no, non-trivial, static, asymptotically constant, vacuum solutions to 4th order gravity, so long as the 3-dimensional scalar curvature is everywhere less than the quantum gravity scale. In particular for black holes, this implies that the Schwarzschild solution is the unique, spherically symmetric, static, asymptotically constant vacuum solution to 4th order gravity, unless the 3-dimensional scalar curvature exceeds the quantum gravity scale outside the horizon, a condition that is easily achieved for macroscopic black-holes.

This also demonstrates the fact that the ‘no-hair’ theorem applies to 4th order gravity provided that the spatial curvature (outside the horizon) is everywhere less than the quantum gravity scale. Note however that the converse does not follow i.e. if the spatial curvature outside the horizon is greater than the quantum gravity scale, Theorem (IV.1) does not
imply that there are additional (static, asymptotically flat) solutions.

Finally, in Section (V), we demonstrated that for matter satisfying the rather mild condition,

$$\int \sqrt{h} d^3x T_a^b T_c^d T_e^f \geq 0,$$

(6.2)

except for the vacuum, none of the static, asymptotically flat solutions to GR are solutions to 4th order gravity. In particular this includes space-times containing single, barotropic fluids with equations of state everywhere satisfying, \( \omega \geq -3^{-1/3} \), such as relativistic and non-relativistic Baryons and Fermions. A specific application of this is that the GR solutions for (static, asymptotically flat) stars will fail to be solutions in 4th order gravity (for the same matter distribution). This agrees with the results of [6], which gives the solutions to 4th order gravity for a particular, extended, spherically symmetric, system (calculated perturbatively and close to the origin) which is shown not to be Schwarzschild.

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Appendix

In this appendix we prove various useful (if rather laborious) identities. First consider \( t^a \nabla_a t^b \nabla_b R_{cd} \), for a static space-time. Using the staticity condition,

$$\mathcal{L}_t R_{ab} = 0,$$

$$\Rightarrow t^a \nabla_a R_{cd} = -2 R_{a(d} \nabla_c) t^a,$$

(6.3)

we can write,

$$t^a \nabla_a t^b \nabla_b R_{cd} = 2 \left[ (\nabla_b t^a) R_{a(d} (\nabla_c) t^b) + R_{ab} (\nabla_{(d} t^a) (\nabla_c) t^b) \right] - R_{bd} t^a \nabla_a \nabla_c t^b - R_{cb} t^a \nabla_a \nabla_d t^b.$$

(6.4)

Using \( t^c \nabla_c \lambda = 0 \) and hence that \( \nabla_a (t^c \nabla_c \lambda) = 0 \), we find

$$t^c \nabla_c \nabla_a \lambda = - (\nabla_a t^c) (\nabla_c \lambda).$$

(6.5)

Thus,

$$t^c \nabla_c \nabla_a t^d = \frac{1}{2\lambda} \left( t^c t^d \nabla_a \nabla_c \lambda - t_a t^c \nabla_c \nabla^d \lambda \right)$$

$$= \frac{1}{2\lambda} \left( -t^d (\nabla_a t^c) (\nabla_c \lambda) + t_a (\nabla^d t^c) (\nabla_c \lambda) \right) = 0,$$

(6.6)
where the final equality follows from expanding $\nabla_{a}t_{c} = \frac{1}{\lambda} (t_{[c} \nabla_{a]} \lambda)$. Using this in Eq. (6.4) gives,
\[
t^{a} \nabla_{a} t^{b} \nabla_{b} R_{cd} = 2 \left[ (\nabla_{b} t^{a}) \left( R_{a[d} \nabla_{c]} t^{b} \right) + R_{ab} \left( \nabla_{c} t^{b} \right) \left( \nabla_{d} t^{a} \right) \right],
\]
which is Eq. (2.10).

By using $\nabla_{a} t_{c} = \frac{1}{\lambda} (t_{[c} \nabla_{a]} \lambda)$ and recalling that $t^{a} \nabla_{a} \lambda = 0$, we find,
\[
(\nabla_{b} t^{c}) (\nabla_{a} t^{d}) = \frac{1}{4 \lambda^{2}} (t^{c} \nabla_{a} \lambda - t_{d} \nabla^{c} \lambda) \left( t^{d} \nabla_{a} \lambda - t_{a} \nabla^{d} \lambda \right),
\]
\[
= -\frac{1}{4 \lambda^{2}} t^{c} t^{d} (\nabla_{a} \lambda) (\nabla_{d} \lambda) + \frac{1}{4 \lambda} (\nabla^{c} \lambda) (\nabla_{a} \lambda),
\]
which is just Eq. (2.11).

In order to show Eq. (2.14) note that,
\[
\frac{1}{\lambda} t^{c} t^{d} \nabla_{b} R_{cd} = \nabla_{b} \left( \frac{1}{\lambda} t^{c} t^{d} R_{cd} \right) - R_{cd} \nabla \left( \frac{1}{\lambda} t^{c} t^{d} \right),
\]
Now we use the fact that for a static space time, the extrinsic curvature of the constant time hyper-surfaces vanishes (this can be checked by direct computation). Thus the Gauss equation (Eq. (2.19)) gives,
\[
\frac{1}{\lambda} t^{a} t^{b} R_{ab} = (3) R - R,
\]
whilst the Codacci equation (Eq. (2.18)) is simply $t^{a} h^{b}_{c} R_{ab} = 0$. A consequence of the Codacci equation and $t^{c} \nabla_{c} \lambda = 0$ is,
\[
t^{a} (\nabla^{b} \lambda) R_{ab} = 0,
\]
Using this and the Gauss equation in Eq. (6.9) gives,
\[
\frac{1}{\lambda} t^{c} t^{d} \nabla_{b} R_{cd} = \nabla_{b} (3) R - R.
\]
With this we directly find Eq. (2.14):
\[
\frac{1}{2 \lambda^{2}} h^{ab} (\nabla_{a} \lambda) t^{c} t^{d} \nabla_{b} R_{cd} = \frac{1}{2 \lambda} (\nabla^{a} \lambda) (\nabla_{a} (3) R - R).
\]
Similarly, by noting,
\[
\frac{1}{\lambda} t^{c} t^{d} h^{ab} \nabla_{a} h^{c}_{b} \nabla_{e} R_{cd} = h^{a} b \nabla_{a} \left( \frac{1}{\lambda} t^{c} t^{d} h^{e}_{b} \nabla_{e} R_{cd} \right) - h^{ab} \left[ \nabla_{a} \left( \frac{1}{\lambda} t^{c} t^{d} \right) \right] \nabla_{b} R_{cd},
\]
and using Eq. (6.12) we find,
\[
\frac{1}{\lambda} t^{c} t^{d} h^{ab} \nabla_{a} h^{c}_{b} \nabla_{e} R_{cd} = D_{a} D^{a} (3) R - R,
\]
which is Eq. (2.17).

A key equation of Lemma (II.2) is Eq. (2.16), which we derive here. By decomposing
\[ g_{ab} = h_{ab} - \frac{1}{\lambda} t_a t_b \] we have,

\[ \Box R_{ab} - m^2 R_{ab} = h^{ab} \nabla_a \nabla_b R_{cd} - \frac{1}{\lambda} t^a t^b \nabla_a \nabla_b R_{cd} - m^2 R_{cd} = 0 \tag{6.16} \]

Splitting the projection \( h^{ab} = h^{ac} h_c^b \) and bringing one of the 3-metrics, inside the covariant derivative the first term becomes

\[ h^{ab} \nabla_a \nabla_b R_{cd} = h^{ab} \nabla_a h^e_b \nabla_e R_{cd}, \tag{6.17} \]

where we used the fact that because \( \nabla_a g_{bc} = 0 \), we have

\[ \nabla_a h_{bc} = \nabla_a \left( \frac{1}{\lambda} t_b t_c \right), \tag{6.18} \]

and \( h^{ab} \nabla_a t_b = 0 \) by symmetry. For the second term in Eq. (6.16), we again use the fact that \( \nabla_c g_{ab} = 0 \) to find,

\[ \nabla_c h_{ab} = \nabla_c \left( \frac{1}{\lambda} t_a t_b \right), \quad \Rightarrow t^a \nabla_c t_a = -t^a \nabla_a t_c = -\frac{1}{2} \nabla_c \lambda, \tag{6.19} \]

where the second line follows by expanding the right hand side, taking the trace over \((a, b)\) and noting that \( \nabla (a t_b) = 0 \). Using Eqs. (6.7) and (6.17) in Eq. (6.16) gives,

\[ h^{ab} \nabla_a h^e_b \nabla_e R_{cd} + \frac{1}{2\lambda} h^{ab} (\nabla_a \lambda) (\nabla_b R_{cd}) \]

\[ + \frac{1}{4\lambda^3} \left[ t_c t^a (\nabla_b \lambda) (\nabla^b \lambda) - \lambda (\nabla_c \lambda) (\nabla^a \lambda) \right] R_{ad} \]

\[ - \frac{1}{2\lambda^2} \left[ t^a (\nabla_d \lambda) - t_d (\nabla^a \lambda) \right] \left[ t^b (\nabla_c \lambda) - t_c (\nabla^b \lambda) \right] R_{ba} \]

\[ + \frac{1}{4\lambda^3} \left[ t^a t_d (\nabla_b \lambda) (\nabla^b \lambda) - \lambda (\nabla^a \lambda) (\nabla_d \lambda) \right] R_{ca} - m^2 R_{cd} = 0, \tag{6.20} \]

which is Eq. (2.16).

Finally, we need to show that the terms given in Eq. (3.8) are positive i.e. that

\[ (D_a \tilde{R}_{bc}) (D^a \tilde{R}^{bc}) + 2 (D_a \tilde{R}_{bc}) (D^b \tilde{R}^{ac}) \geq 0. \tag{6.21} \]

To do this, consider an arbitrary tensor \( T_{abc} \) that is symmetric in the final two indices i.e. \( T_{abc} = T_{acb} \). Now we use a tetrad basis, \( e^i_a \), that is othonormal i.e. \( e^i_a e^a_j = \delta^i_j \), to decompose the (positive definite) metric as,

\[ h_{ab} = e^i_a e^j_b \delta_{ij}. \tag{6.22} \]

With this one can write,

\[ T_{abc} T_{abc} = h^{ad} h^{ef} T_{abc} T_{def} = T_{ijk} T_{mnp} \delta^i_j \delta^m_p \delta^k_p, \]

\[ T_{bac} T_{bac} = h^{ae} h^{bf} T_{abc} T_{def} = T_{ijk} T_{mnp} \delta^i_j \delta^m_p \delta^k_p. \tag{6.23} \]
We now form the combination,

\[ T^{abc}T_{abc} + 2T^{abc}T_{bac}, \]  

(6.24)

and expand out the Einstein summations. Since \( T_{abc} \) is a three index tensor, the three possibilities we need to consider are: the indices are the same, one is distinct and all are distinct. The first possibility clearly involves only positive terms. In the orthonormal tetrad basis given by Eq. (6.22) these are

\[ T^{abc}T_{abc} + 2T^{abc}T_{bac} = 3 \left( T_{111}T_{111} + T_{222}T_{222} + T_{333}T_{333} \right) + \ldots, \]  

(6.25)

where the dots remind us that there are other terms we need to account for. The terms in Eq. (6.24) that arise from one index being distinct, with the other two being the same, can, by using Eq. (6.22), be grouped to give,

\[ T^{abc}T_{abc} + 2T^{abc}T_{bac} = \sum_{i \neq j} \left( 2T_{iij} + T_{ijj} \right) \left( 2T_{iij} + T_{ijj} \right) + \ldots, \]  

(6.26)

where there is no implicit summation over repeated indices and again the dots indicate that there are considering only certain terms. Finally we need to account for the terms in Eq. (6.24) for which all the indices are distinct, which are given by,

\[ T^{abc}T_{abc} + 2T^{abc}T_{bac} = \sum_{i \neq j \neq k} 2 \left( T_{ijk} + T_{jik} + T_{kij} \right) \left( T_{ijk} + T_{jik} + T_{kij} \right) + \ldots, \]  

(6.27)

where again there is no implicit summation over repeated indices and the dots indicate that there are additional terms. In Eq. (6.26) and Eq. (6.27), crucial use has been made of the fact that \( T_{ijk} = T_{ikj} \). Finally, putting these together we find that, in this orthonormal tetrad basis,

\[ T^{abc}T_{abc} + 2T^{abc}T_{bac} = 3 \sum_i T_{iii}T_{iii} + \sum_{i \neq j} \left( 2T_{iij} + T_{ijj} \right) \left( 2T_{iij} + T_{ijj} \right) \]

\[ + \sum_{i \neq j \neq k} 2 \left( T_{ijk} + T_{jik} + T_{kij} \right) \left( T_{ijk} + T_{jik} + T_{kij} \right), \]  

(6.28)

where once again there is no implicit summation over repeated indices. Clearly each term in Eq. (6.28) is positive. It is worth noting here that this result hold in any number of dimensions, provided the metric, \( h_{ab} \), is positive definite.

One may be concerned by the fact that if \( T_{abc} \) were also anti-symmetric in the first two indices, the combination given on the left-hand-side of Eq. (6.28), would appear to be negative. However there are no three index tensors that are anti-symmetric in the first pair of indices and symmetric in the second pair, as can easily be checked.

\[ T_{abc} = -T_{bac} = -T_{cba} = T_{cab} = -T_{acb} = -T_{abc}. \]  

(6.29)

Eq. (6.28) holds for any three index tensor that is symmetric in its final pair of indices and in particular it holds for \( T_{abc} = D_a \bar{R}_{bc} \), thus,

\[ \left( D_a \bar{R}_{bc} \right) \left( D^a \bar{R}^{bc} \right) + 2 \left( D_a \bar{R}_{bc} \right) \left( D^b \bar{R}^{ac} \right) \geq 0, \]  

(6.30)
as required.

[1] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi, JHEP 0610 (2006) 014
[2] A. Lichnerowicz, “Theories Relativistes de la Gravitation et de l’Electromagnétisme,” (1955) Masson, Paris.
[3] W. Israel, Phys. Rev. 164 (1967) 1776.
[4] E. Pechlaner and R. Sexl, Commun. Math. Phys. 2, 165 (1966).
[5] C. Gregory, Phys. Rev. 72 (1947) 72.
[6] K. S. Stelle, Gen. Rel. Grav. 9 (1978) 353.
[7] D. R. Noakes, J. Math. Phys. 24 (1983) 1846.
[8] V. P. Frolov and I. L. Shapiro, Phys. Rev. D 80 (2009) 044034.
[9] A. A. Starobinsky, Phys. Lett. B 91 (1980) 99.
[10] A. A. Starobinsky and H-J. Schmidt, Class. Quantum. Grav. 4 (1987) 695.
[11] H. J. Schmidt, eConf C0602061 (2006) 12 [Int. J. Geom. Meth. Mod. Phys. 4 (2007) 209].
[12] K. I. Macrae and R. J. Riegert, Phys. Rev. D 24, 2555 (1981).
[13] W. Nelson, in prep.
[14] H. J. Schmidt, Astron. Nachr. 306 (1985) 67.
[15] A. H. Chamseddine, A. Connes and M. Marcolli, Adv. Theor. Math. Phys. 11 (2007) 991.
[16] W. Nelson and M. Sakellariadou, Phys. Rev. D 81 (2010) 085038.
[17] W. Nelson and M. Sakellariadou, Phys. Lett. B 680 (2009) 263.
[18] W. Nelson, J. Ochoa and M. Sakellariadou, [arXiv:1005.4276] [hep-th].
[19] W. Nelson, J. Ochoa and M. Sakellariadou, Phys. Rev. Lett. 105 (2010) 101602.
[20] D. Kolodrubetz and M. Marcolli, Phys. Lett. B 693 (2010) 166.
[21] C. Johnson, “D-Branes,” Cambridge University Press (2003).
[22] M. Reuter and F. Saueressig, Phys. Rev. D 65 (2002) 065016.
[23] M. Bojowald and A. Skirzewski, eConf C0602061 (2006) 03 [Int. J. Geom. Meth. Mod. Phys. 4 (2007) 25].
[24] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82 (2010) 451.
[25] A. de la Cruz-Dombriz, A. Dobado and A. L. Maroto, Phys. Rev. D 80 (2009) 124011.
[26] S. Nojiri and S. D. Odintsov, eConf C0602061 (2006) 06 [Int. J. Geom. Meth. Mod. Phys. 4 (2007) 115].
[27] S. Kawai, M. a. Sakagami and J. Soda, Phys. Lett. B 437, 284 (1998).
[28] A. De Felice, M. Hindmarsh and M. Trodden, JCAP 0608 (2006) 005.
[29] H.-J. Schmidt, Ann. der Physik, 44 (1987) 361.