An expedition to the islands of stability in the first-order causal hydrodynamics

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Abstract

The recently proposed connection between the Lorentz invariance of stability and the speed of signal propagation has been tested for a first-order relativistic dissipative hydrodynamic theory. The fact that the stability situation in different reference frames agrees with each other only as long as the signal propagation respects causality, has been explicitly established for the theory, which is microscopically derived from the covariant kinetic equation in general hydrodynamic frames with arbitrary momentum-dependent interactions.

1. Introduction

Since the inception of the special theory of relativity (STR), there has been a consensus that faster-than-light communication is not possible. To our best knowledge, it also has not been achieved in any experiment. However, several theories and phenomena related to superluminal communication have been proposed or studied, including tachyons [1], neutrinos, quantum nonlocality, etc. The superluminal signal propagations are not permitted according to STR because, in a Lorentz-invariant theory, it could be used to transmit information into the past. This complicates causality leading to logical paradoxes of the "kill your own grandfather" type. On the other hand, examples of matter of very high density allowing wave modes with superluminal group velocities have been known for a long time [2]. The early formulations of the relativistic dissipative hydrodynamics (a.k.a the first-order hydrodynamic theories because the dissipative corrections to the energy-momentum tensor were constructed from the first-order gradient of fluid variables) by Eckart and Landau are also known to be inherently acausal due to the spacelike nature of the characteristics [3, 4]. The acausality problem was later cured by Mueller-Israel-Stewarts (MIS), where the entropy flux includes the higher order dissipative terms [5]. At the same time, people also realized that the causality of relativistic hydrodynamics is closely associated with the stability of the fluid under small perturbations. Usually, any small perturbation in fluid propagates with a characteristic speed of sound and transports energy from the source of the perturbation to distant points; the amplitude of the perturbation usually diminishes due to the dissipation of kinetic energy into heat which increases the total entropy, and the system reaches a new thermodynamic stable state. In some cases, the small perturbation may grow exponentially, eventually leading to instability in the fluid [6]. Energy conservation, however, enforces that the instability cannot grow indefinitely unless the temperature of the system tends to infinity at the same time [7]. The situation for relativistic fluids became further complicated since it was found that a state of fluid can be stable to perturbations in one Lorentz frame and unstable in another, which clearly violates the principle of relativity. Hence one must make sure that any consistent relativistic hydrodynamic formulation must testify to the two benchmark criteria to be physically acceptable: (i) signal propagation must be subluminal to respect the causality, and (ii) the system must be stable in all Lorentz frames against any perturbations from the equilibrium state. The close correlation between them has been studied for MIS theory [8–10], where the theory is shown to be unstable because of superluminal modes. In the context of the recently proposed first-order stable-causal (BDNK) theory, [11] gives a theorem that asserts that stability in the fluid’s local rest frame implies stability in any Lorentz boosted frame, provided that the system is causal and strong hyperbolic. Finally, in [12], the authors provide an intuitive understanding of this puzzle. They show that in presence of dissipation two observers can disagree on whether a state is stable or unstable iff the perturbations can exit the light cone. In conclusion, the stability of a state is a Lorentz-invariant property of dissipative theories iff the principle of causality holds. Motivated by these studies, we explicitly demonstrate here the said connection for a stable-causal first-order theory, derived from a first principle kinetic theory and valid in general hydrodynamic frames and with arbitrary momentum-dependent interactions. In [13], a first-order stable-causal relativistic theory (henceforth called BMR approach for brevity) has been derived, motivated from the works of [14] (BDNK theory) that presents a pathology-free first-order relativistic theory via hydrodynamic field redefinition. The BMR approach showed that along with the general hydrodynamic frame choice the momentum dependence of microscopic interaction rate are also imperative for producing a causal and stable first-order relativistic theory. In the current work, we have checked that for BMR approach, all microscopic parameters considered to define the hydrodynamic frame and momentum dependence of the interaction rate, the linear stability analysis
can differ from one Lorentz frame to another, if and only if the signal propagation exits the light cone. Stability is Lorentz invariant for all time-like perturbations; hence, testing it in one reference frame suffices for causal wave propagation.

Throughout the manuscript, we have used natural unit \( (\hbar = c = k_B = 1) \) and flat space-time with mostly negative metric signature \( g^{\mu\nu} = \text{diag} (1, -1, -1, -1) \).

2. BMR theory

In [13], a first-order, relativistic, stable and causal hydrodynamic theory has been derived in general frames from the Boltzmann transport equation, where the system interactions are introduced via the microscopic particle momenta captured through momentum-dependent relaxation time approximation (MDRTA) [15]. The basic idea is to estimate the out-of-equilibrium one-particle distribution function \( f(x, p) \) for general hydrodynamic frame and arbitrary momentum dependent interactions. \( f(x, p) \) can be decomposed in two components: one is the homogeneous solution which sets the hydrodynamic frame choice and the second is the inhomogeneous solution that is controlled by the system interaction. To extract the homogeneous part of the solution, we use the matching conditions (constraints that set the thermodynamic fields to their equilibrium values even in the presence of dissipation). For estimating the inhomogeneous part, the relativistic transport equation, \( p^\mu \partial_\mu f(x, p) = -\mathcal{L}[f] \), is solved where \( \mathcal{L}[f] \) is the linearized collision operator with \( f \) as the deviation in particle distribution due to dissipation. \( p^\mu \) and \( x \) denote the particle four-momenta and space-time variable respectively. We employ here the MDRTA method to explicitly linearize the collision term, with the relaxation time of single particle distribution function \( \tau_R \) expressed as a power law of scaled particle energy \( (E_p = p^\mu u_\mu / T) \) in comoving frame, \( \tau_R = \tau^0_R (x)(E_p)^{\lambda} \). We propose an appropriate collision operator (that satisfies all the conservation properties, adjoint properties and summation invariant properties) for a semi-orthogonal, monic polynomial basis of the distribution function, instead of the Anderson-Witting type relaxation kernel,

\[
\mathcal{L}_{\text{MDRTA}}[\phi] = \left( \frac{(p \cdot u)}{\tau_R} \right) f^{(0)} (1 \pm f^{(0)}) \left( \phi - \bar{\phi} \right) \left( \frac{E_p}{\tau_R} \phi \bar{E}_p \right) \left( \frac{E_p}{\tau_R} \bar{E}_p \phi \right) - \frac{E_p}{\tau_R} \frac{\partial}{\partial E_p} \left\{ \frac{E_p}{\tau_R} \phi \bar{E}_p \left( \frac{E_p}{\tau_R} \bar{E}_p \phi \right) \right\} - \frac{E_p}{\tau_R} \frac{\partial}{\partial E_p} \left\{ \frac{E_p}{\tau_R} \phi \bar{E}_p \left( \frac{E_p}{\tau_R} \bar{E}_p \phi \right) \right\} + \frac{E_p}{\tau_R} \frac{\partial}{\partial E_p} \left\{ \frac{E_p}{\tau_R} \phi \bar{E}_p \left( \frac{E_p}{\tau_R} \bar{E}_p \phi \right) \right\}.
\]

(1)

This collision operator preserves the conservation laws microscopically. By virtue of that, the right hand side of the transport equation singularly excludes the zero modes of the linearized collision operator. Because of the fact, the left hand side of the transport equation is not necessarily needed to be orthogonal to zero modes and hence the covariant time derivatives appearing on the left hand side of the transport equation are not necessarily required to be exchanged by the spatial gradients. As a result, to solve the equation, we can safely apply a perturbative method keeping the time derivatives over the fundamental thermodynamic quantities alive (necessarily for a stable-causal theory) in the first-order thermodynamic field corrections as the following,

\[
\begin{align*}
\begin{pmatrix} \delta N(x, t) \\ \delta E^\mu(x, t) \\ \delta P^{\mu}(x, t) \end{pmatrix} &= \begin{pmatrix} v_1 & DT & v_2 \\ \epsilon_1 & \tau_1 & \epsilon_2 \\ \pi_1 & v_3 & \pi_1 \end{pmatrix} \begin{pmatrix} \phi \cdot u \end{pmatrix} \begin{pmatrix} v_1 \\ \epsilon_2 \\ \pi_2 \end{pmatrix} \left( \mu \right), \\
\begin{pmatrix} \phi^{(1)} \mu \\ \phi^{(1)} \nu \end{pmatrix} &= \begin{pmatrix} \theta_1 & \pi_1 & \theta_3 \end{pmatrix} \begin{pmatrix} \nabla^\mu \phi \end{pmatrix} + \frac{\tau_R}{\gamma} \nabla^\mu \phi.
\end{align*}
\]

(2)

Here, \( \delta N, \delta E, \delta P, \phi^{(1)} \) respectively denote dissipative corrections over particle number density, energy density, pressure, energy flow and particle current. The field corrections include first order derivatives over all the fundamental thermodynamic quantities such as temperature \( T \), chemical potential \( \mu \) and hydrodynamic four-velocity \( u^\mu \). The field correction coefficients turn out to be elaborate functions of the frame indices \( i, j, k \) and the parameter \( \Lambda \) of MDRTA (for details see [13]). These field corrections along with \( \pi^{(1)} = 2n \pi^{(2)} \) (\( \eta \) is shear viscosity), constitute the first order out of equilibrium particle four-flow, \( N^{(1)} = (n + \delta n^{(1)}) \phi^{(1)} + \phi^{(2)} \) and energy-momentum tensor, \( T^{\mu\nu} = (\epsilon + \delta \epsilon^{(1)}) \phi^{(1)} + \phi^{(2)} \) and \( \pi^{(1)} \), respectively. Here, \( n, \epsilon \) and \( P \) indicate particle number density, energy density and pressure at their local equilibrium values. We observed that both the conventional frame choices (Landau (\( i, j, k = 1, 2, 3 \)) and Eckart (\( i, j, k = 1, 2, 0 \))) and the momentum independent \( \tau_R (\Lambda = 0) \) lead to an acausal and unstable theory.

3. Stability and causality analysis with boosted background

In [13], the stability and causality of BMR theory have been investigated (with an affirmation that the theory is indeed stable and causal with a critical dependence on the system interaction) in the Lorentz rest frame of the fluid. But it has been observed a number of times [16, 17] that such an analysis in local rest frame is often inadequate (even sometimes misleading) because with a boosted background the stability situation can alter drastically, even completely new modes can appear due to the degree of the dispersion polynomial changes. We linearize the conservation equations for small perturbations of fluid variables around the hydrostatic equilibrium, \( \psi(t, x) = \psi_0 + \delta \psi(t, x) \), with the fluctuations expressed in their plane wave solutions via a Fourier transformation \( \delta \psi(t, x) \rightarrow e^{i(\omega \tau - k \cdot x)} \delta \phi(\omega, k) \), (subscript 0 indicates global equilibrium). The background fluid is now considered to be boosted along x-axis with a constant velocity \( v, u^0 = \gamma (1, v, 0, 0) \) with \( \gamma = 1/\sqrt{1 - v^2} \). The corresponding velocity fluctuation is \( \delta u^\mu = (\gamma v \delta u^0, \gamma v \delta u^0, \delta u^0, \delta u^0) \) which again gives \( u^\mu \delta u_\mu = 0 \) to maintain velocity normalization. The dispersion relation for linear perturbations can be obtained in the boosted frame by giving the transformations, \( \omega \rightarrow \gamma (\omega - kv) \) and \( k^\mu \rightarrow \gamma (\omega - kv)^2 - \omega^2 + k^2 \) to the local rest frame. The resulting dispersion relation polynomial turns out to be too mathematically cumbersome (as expected) and it is only possible to
extract the modes in limiting situations. The transverse or shear channel at low wave number limit \((k \to 0)\) gives only one non-hydrodynamic (gapped) frequency mode,

\[
\omega^\perp = \frac{\omega_0 + P_0}{\gamma - \eta v^2} + O(k),
\]

where we define \(\theta = -\theta_i\). The other one is a hydrodynamic (gapless) mode that vanishes at spatially homogeneous limit \((k = 0)\). Hence, from the criteria that the imaginary part of frequency must be positive definite to give rise to exponentially decaying perturbations, the stability condition for the shear channel becomes \(\theta > \eta v^2\). Here, we recall the asymptotic causality condition for the shear channel in Lorentz rest frame (LRF) was \(\theta > \eta\) [13] which readily reproduces the stability condition for all \(0 < v < 1\). With boosted background, the group velocity of the propagating shear mode becomes

\[
v^\perp = \lim_{k \to \infty} \frac{dv}{dk} = \left(\frac{\sqrt{\left(\theta c^2/v^2\right)}}{1 + \frac{\sqrt{\left(\theta c^2/v^2\right)}}{v}}\right),
\]

which is subluminal as long as the asymptotic causality condition in LRF is satisfied.

In longitudinal or sound channel the situation becomes quite involved since the number of modes significantly increases. At the asymptotic limit \(k \to \infty\), the dispersion relation becomes

\[
H_{00}(\omega^2)^2 + H_{01}(\omega^2)^3k + H_{02}(\omega^2)^4k^2 + H_{03}(\omega^2)^5k^3 + H_{04}(\omega^2)^6k^4 + H_{05}k^6 = 0.
\]

The coefficients \(H_{00...5}\) are elaborate functions of frame indices \(i, j, k\), the MDRTA parameter \(\Lambda\) and the boost velocity \(v\). We will give here the explicit expressions of only those coefficients which are exclusively required for establishing causality and stability. In the limit of large \(k\), an expansion of the form, \(\omega^2 = \frac{\Lambda}{k} + \sum_{n>0} a_n k^{-n}\) can be used as a solution [18].

The values of the asymptotic group velocity \(v^\parallel\) are then given by the sixth order polynomial, \(H_{06}(v^\parallel)^6 + H_{05}(v^\parallel)^5k + H_{04}(v^\parallel)^4k^2 + H_{03}(v^\parallel)^3k^3 + H_{02}(v^\parallel)^2k^4 + H_{01}v^\parallel k^5 + H_{00}k^6 = 0\). Here, for simplicity, in Fig. (1), we extract the value of \(v^\parallel\) in LRF for two different general frames, \((i, j, k) = (4, 3, 2)\) and \((i, j, k) = (6, 4, 4)\), for which the sixth order polynomial reduces to a cubic equation, \((\Lambda^2)^3 + X_2(\Lambda^2)^2 + X_1(\Lambda^2) + X_0 = 0\). So we conclude, that more general the frame situation becomes (farther away from Landau or Eckart frame where we can not find any subluminal \(v^\parallel\) for any LRF value), the system reaches causality with smaller values of \(\Lambda\). But that never reaches \(\Lambda = 0\) (momentum independent RTA) whatever large frame indices are. It can be shown, that for the causal parameter sets shown in Fig.(1), the theory is asymptotically causal for any boosted frame as well, i.e. the original sixth order polynomial always has a subluminal root, as suggested by the principle of relativity.

Next we turn to the stability analysis of the longitudinal modes. The dispersion relation at \(k \to 0\) limit turns out to be,

\[
G_{06}(\omega^2)^6 + G_{05}(\omega^2)^5 + G_{04}(\omega^2)^4 + G_{03}(\omega^2)^3 + G_{02}(\omega^2)^2(ik) + G_{01}(\omega^2)k^2 + G_{00}k^3 = 0.
\]

Eq.(6) gives rise to three hydrodynamic and three nonhydrodynamic modes. The hydro modes are given by, \(\omega^\parallel = (a_{1,2,3})k\), where the group velocities can be estimated from the equation, \(G_{03}a^3 + G_{02}a^2 - G_{01}a - G_{00} = 0\). The non-hydro modes (growing ones give rise to instability) are given by the following polynomial equation,

\[
G_{06}(\omega^2)^6 - iG_{05}(\omega^2)^5 - G_{04}(\omega^2)^4 + iG_{03} = 0.
\]

Using Routh-Hurwitz stability criteria, we find the following
conditions for stability of the non-hydro modes,

\[ G_6, G_5, G_3 > 0, G_{B2} = (G_1 G_3 - G_2 G_6)/G_3 > 0. \]  (8)

The coefficients are given by, 
\[ G_6 = \gamma^6 (A_6 - A_2^2 v^2 + A_1^2 v^4 \pm A_1^2 v^4 \pm A_0^2 v^4), \]
\[ G_5 = \gamma^5 (A_5 - A_2^2 v^2 + A_1^2 v^4), \]
\[ G_4 = \gamma^4 (A_4^2 - A_2^2 v^2 + A_1^2 v^4) \]
and \[ G_3 = \gamma^3 (A_3^2 - A_2^2 v^2). \]

The explicit mathematical expressions of the \( A \) functions in terms of \( (i, j, k) \) and \( \Lambda \), through which the frame situation and the momentum dependence of the interaction cross section duly control the stability and causality of the theory, are given in the Appendix.

Among the coefficients of (8), \( G_3 = \gamma^3 n_0 (\epsilon_0 + P_0) (1 - v^2 c^2_i) \) is always positive where \( c_i \) is the velocity of sound. In the parameter space spanned by \( (i, j, k) \), \( \Lambda \), and \( v \), the positivity of \( G_6, G_5 \) and \( G_{B2} \) ensures the stability of the system. In Fig.(2), system stability has been displayed for two frame conditions, \( (i, j, k) = (4, 3, 2) \) and \( (i, j, k) = (6, 4, 4) \), as a function of two independent parameters \( \Lambda \) and \( v \). Here, the shaded areas indicate that the system stability holds for those parameters sets. Here, we notice something very interesting. The \( \Lambda \) values for each frame, beyond which the asymptotic causality condition holds, \( v_g < 1 \). \( \Lambda_c = 6.65 \) for \( (i, j, k) = (4, 3, 2) \) and \( \Lambda_c = 2.9 \) for \( (i, j, k) = (6, 4, 4) \), see Fig.(1)), the system is also stable irrespective of the values of boost velocity as long as \( 0 < v < 1 \). This is the main result of this work. In the region where \( v_g > 1 \), we observe that there is no \( \Lambda \) value for which the system is either stable or unstable for all values of \( v \), whereas beyond the \( v_g = 1 \) line, the stability situation agrees in all reference frames.

We now analyse the stability situation at large wave-number limit. With a general boost velocity \( v \), at large \( k \), the shear channel dispersion relation gives the following modes,

\[ \omega_{1,2}^\pm = \frac{k}{1 \pm \sqrt{v^2}} \pm i \frac{(\epsilon_0 + P_0) (1 \pm \sqrt{\alpha^2})}{2\gamma(\theta - \eta v^2)} + O(1/k^2). \]  (9)

Clearly, with \( 0 < v < 1 \) and asymptotic causality condition \( \theta > \eta \), the imaginary part is always positive which ensures stability of the shear modes. The situation at sound channel becomes somewhat non-trivial. With general \( k \) and \( v \), it is a sixth-order polynomial of \( \omega \), with each coefficient itself as a polynomial over \( k \), where the coefficient of each power of \( k \) is again a polynomial of \( v \). This is only possible to solve at limiting situations. Hence, we present here the solution upto the order of \( k \), i.e., \( \omega^0 = a_0 + a_1 k \). \( a_0 \) is purely imaginary with three roots whose positivity has been checked by Routh-Hurwitz stability criteria in Eq.(7). We have checked that \( a_1 \) is a real quantity which does not contribute to the stability of the theory. In view of the non-trivial structure of the sound channel polynomial, we believe that this result might suffice for a long-wavelength effective theory such as relativistic hydrodynamics. It is in principle important to analyse also the larger \( k \) limit, but in practice it is beyond the possibilities of the current work.

So, from this analysis we can safely conclude here that stability is a Lorentz invariant property if and only if the signal propagation within the medium is causal. Our results corroborates with the finding of [12], where the authors addressed the physical origin of the relation between “causality” and “instability” using space-like events connected via perturbation mediating through a dissipative medium. Fig.(2) gives a very clear pictorial representation of similar argument for the BMR theory, where we have established that for a first order relativistic dissipative system in general frame and with momentum dependent interaction cross section, the principle of causality is requisite for the stability to be a Lorentz invariant property.

4. Conclusion

The idea is that, causality violation can chronologically reorder a perturbation in different reference frames by the relativity of simultaneity so that in a dissipative medium, two observers can disagree on whether the perturbation is growing or decaying. This gives rise to instability in the medium, so causality is the key signature in this paradox. This phenomenon has been explicitly demonstrated in this work, where all the stability controlling factors, such as frame indices and interaction parameters, give rise to a stable system at all inertial frames iff they respect the asymptotic causality for the system. This undeniably reduces the task of checking stability in all the reference frames, and the local rest frame suffices for the analysis. This is a much desirable situation since, with a boosted background, the dispersion relation becomes way more complicated than in the local rest frame. Most importantly, this work singularly establishes the physical argument and the associated theorems for a coarse-grained system with momentum-dependent microscopic interactions from the first principle calculation. The analysis of causality in this work is based on the asymptotic causality condition, which states that the asymptotic group velocity \( v_g = \lim_{k \to \infty} \frac{d \text{Re} \omega(k)}{dk} \) must be subluminal (i.e. \( v_g \leq 1 \)). However, there are examples of acausal theories with subluminal group velocity, such as the diffusion equation: \( \delta_x T = \partial^2 \partial T \), with dispersion relation \( \omega(k) = ik^2 \) and \( v_g = 0 \leq 1 \). And yet, we know that the diffusion equation is acausal. We note that the “asymptotic causality condition” is a necessary condition for causality, but not sufficient [19]. A rigorous study of causality requires the study of characteristics and ensuring the strong hyperbolicity condition, which is beyond the scope of the present work.
5. Appendix

The $A$ functions mentioned in the main text are given by:

$$A_6 = F_{13} F_F F_{6} - F_{1} F_F , \quad (10)$$

$$A_5 = F_{1} F_F F_{3} F_{6} - (\epsilon_0 + \rho_0) F_{13} F_{13} - n_0 F_F F_{13} , \quad (11)$$

$$A_0^0 = (\epsilon_0 + \rho_0) F_{1} F_F + n_0 F_F - n_0 (\epsilon_0 + \rho_0) \rho_0 F_{13} , \quad (12)$$

$$A_1^5 = F_{13} F_F F_{6} - F_{3} F_{6} + F_3 (F_{11} F_{1} - F_{1} F_{12}) - F_3 (F_9 F_{13} - F_6 F_{14} - F_{10} F_{12}) - F_3 (F_{11} F_{17} - F_6 F_{16} - F_{17} F_{13}) \quad (13)$$

$$A_5^0 = n_0 (\epsilon_0 + p_0) , \quad (14)$$

$$A_1^4 = n_0 (F_6 F_{14} + F_7 F_{12} - F_9 F_{13} + F_{11} F_{16} - F_{17} F_{12}) + n_0 (\rho_0) F_{1} F_F + F_{12} F_{13} - F_3 F_{11} + F_3 F_{11} \quad (15)$$

$$A_4^5 = n_0 (\epsilon_0 + \rho_0) F_{1} F_F + b F_{12} F_3 F_5 + F_{12} F_3 F_5 , \quad (16)$$

$$A_1^3 = (\epsilon_0 + \rho_0) F_{1} F_F (a + b) \quad (17)$$

$$A_1^4 = (F_2 F_9 - F_4 F_7) + n_0 (F_6 F_{14} + F_9 F_{11} - F_7 F_{12}) + (\epsilon_0 + \rho_0) F_{1} F_F (F_{12} - F_{14}) + b (F_2 F_9 - F_4 F_7) \quad (18)$$

$$A_1^0 = (\epsilon_0 + \rho_0) F_{1} F_F (a + b) \quad (19)$$

$$A_0^0 = F_6 F_{14} (F_F F_{6} - F_3 F_6) \quad (20)$$

$$A_0^0 = F_6 F_{14} (F_F F_{6} - F_3 F_6) \quad (21)$$

These $A$'s are the functions of the first order field correction coefficients (given in Eq. (2) and (3) of the main text) as the following:

$$F_1 = \nu_1 f + \nu_3 c , \quad F_2 = \gamma_1 f + \gamma_3 c , \quad (22)$$

$$F_3 = \nu_1 g + \nu_3 d , \quad F_4 = \gamma_1 g + \gamma_3 d , \quad (23)$$

$$F_5 = \nu_2 \gamma_1 + \nu_3 \gamma_2 \quad (24)$$

$$F_7 = \theta_1 \gamma_1 + \theta_2 c , \quad F_8 = \theta_1 g + \theta_3 d \quad (25)$$

$$F_9 = \theta_1 g + \theta_3 d \quad (26)$$

$$F_11 = f (\pi_1 - \pi_2 + \pi_0) \quad (27)$$

$$F_{12} = g (\pi_1 - \pi_2 + \pi_0) \quad (28)$$

$$F_{13} = i (\pi_1 - \pi_2 + \pi_0) \quad (29)$$

$$F_{14} = (4 n/3) (\epsilon_0 + p_0) - \pi_1 (\epsilon_0 + p_0) \quad (30)$$

Here, we have used, $a = \left( \frac{n_0 + p_0}{n_0 + p_0} \right) \frac{\partial \mu}{\partial n_0} \epsilon_0 \epsilon_0$, $b = \left( \frac{n_0 + p_0}{n_0 + p_0} \right) \frac{\partial \mu}{\partial n_0} \rho_0 \rho_0$, $c = J_0 J_3 / (I_2 - I_1 F_2)$, $d = J_0 J_3 / (I_2 - I_1 F_2)$, $f = -J_2 J_3 / (I_2 - I_1 F_2)$, and $g = J_1 J_3 / (I_2 - I_1 F_2)$. We readily identify that, $\nu_1 = a + b$. The moment integrals used here are defined as, $I_n = \int d F_p \tilde{p}^{\mu} \tilde{p}^{\nu} \tilde{p}^{\rho} \tilde{p}^{\theta} \tilde{E}_p$, and $\Lambda^{\mu \nu} k \rho = \int d F_p \tilde{p}^{\mu} \tilde{p}^{\nu} \tilde{p}^{\rho} \tilde{p}^{\theta} \tilde{E}_p$. $z = m/T$ is the scaled rest mass of the particles.

Next, we give the explicit expressions of the first order field correction coefficients (scaled by $\tau^0_k$) in terms of the frame indices ($i, j, k$) and MDRATA parameter $\Lambda$, using the notation $D_{ij}^{\alpha \beta} = L_{ij}^{\alpha \beta} - L_{ij}^{\alpha \beta}$:

$$v_1 = -T I_{A_1^2} + \frac{\partial n}{\partial \mu} D_{ij}^{A_1^1} + T \frac{\partial n}{\partial T} D_{ij}^{A_1^0} \quad (31)$$

$$v_2 = -T \left[ \frac{1}{3} I_{A_1^2} + \frac{2}{3} \Lambda \right] + \frac{\partial n}{\partial \mu} \left[ \frac{1}{3} D_{ij}^{A_1^1} + \frac{2}{3} \Lambda \right] \quad (32)$$

$$v_3 = -T I_{A_1^2} + \frac{\partial n}{\partial \mu} D_{ij}^{A_1^1} + T \frac{\partial n}{\partial T} D_{ij}^{A_1^0} \quad (33)$$

$$v_4 = -T \left[ \frac{1}{3} I_{A_1^2} + \frac{2}{3} \Lambda \right] + \frac{\partial n}{\partial \mu} \left[ \frac{1}{3} D_{ij}^{A_1^1} + \frac{2}{3} \Lambda \right] \quad (34)$$

$$\eta = \frac{1}{2} T^2 K_{A_1} \quad (35)$$

These 14 first order transport coefficients, through their de-
pendence on frame parameters and interaction parameter, critically control the stability and causality analysis presented in the current work. For the analysis we have used \( T = 300 \text{ MeV} \) and \( m = 300 \text{ MeV} \).

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