Infinite Families of Gauge-Equivalent $R$-Matrices and Gradations of Quantized Affine Algebras

Anthony J. Bracken $^a$, Gustav W. Delius $^b$, Mark D. Gould $^a$

and

Yao-Zhong Zhang $^a$

$^a$. Department of Mathematics, University of Queensland, Brisbane Qld 4072, Australia

$b$. Fakultät für Physik, Universität Bielefeld, 33501 Bielefeld, Germany

Abstract:

Associated with the fundamental representation of a quantum algebra such as $U_q(A_1)$ or $U_q(A_2)$, there exist infinitely many gauge-equivalent $R$-matrices with different spectral-parameter dependences. It is shown how these can be obtained by examining the infinitely many possible gradations of the corresponding quantum affine algebras, such as $U_q(A_1^{(1)})$ and $U_q(A_2^{(1)})$, and explicit formulae are obtained for those two cases. Spectral-dependent similarity (gauge) transformations relate the $R$-matrices in different gradations. Nevertheless, the choice of gradation can be physically significant, as is illustrated in the case of quantum affine Toda field theories.
1 Introduction

Quantized universal enveloping algebras (quantum algebras) provide a powerful tool for solving the spectral-dependent quantum Yang-Baxter equation (QYBE), which plays a central role in the study of integrable systems in many areas of physics. In particular, each solution is a spectral-dependent $R$-matrix which may define the Boltzmann weights of a solvable vertex model in statistical mechanics, or a scattering matrix in a quantum affine Toda theory.

In this paper, we present a method for constructing different spectral-dependent and gauge equivalent $R$-matrices associated with one and the same representation of a quantum algebra. The idea is to examine various gradations of a corresponding quantum affine algebra, including in particular the important homogeneous and principal gradations. We exploit techniques previously developed, which relate spectral-dependent $R$-matrices associated with a quantum algebra (such as $U_q(A_n)$) to the universal $R$-matrix of a corresponding quantum affine algebra (such as $U_q(A^{(1)}_n)$). Here we consider $n = 1, 2$, and construct infinitely many gauge equivalent $R$-matrices with different spectral-dependences, corresponding to the infinitely many different gradations of each of the quantum affine algebras $U_q(A^{(1)}_1)$ and $U_q(A^{(1)}_2)$.

Gauge equivalent $R$-matrices are known to lead to solvable statistical models which are essentially equivalent. In the case of quantum affine Toda theories, the choice of gradation is more significant, as we shall show.

2 Universal $R$-Matrix for $U_q(A^{(1)}_1)$ and $U_q(A^{(1)}_2)$

This section is devoted to a brief review of the construction of the universal $R$-matrix for $U_q(A^{(1)}_1)$ and for $U_q(A^{(1)}_2)$. Throughout the paper, we use the notations:

$$(ad_qx_\alpha)x_\beta = [x_\alpha, x_\beta] = x_\alpha x_\beta - q^{(\alpha,\beta)} x_\beta x_\alpha$$

$$\theta(q^h) = q^{-h}, \quad \theta(E_i) = F_i, \quad \theta(F_i) = E_i, \quad \theta(q) = q^{-1}$$

$$(n)_q = \frac{1 - q^{-n}}{1 - q^{-1}}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad q_\alpha = q^{(\alpha,\alpha)}$$

$$\exp_q(x) = \sum_{n \geq 0} \frac{x^n}{(n)_q!}, \quad (n)_q! = (n)_q(n - 1)_q \ldots (1)_q.$$

(2.1)

We start with the rank 2 case, and fix the normal ordering in the positive root system $\Delta_+$. 

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1. Introduction
2. Universal $R$-Matrix for $U_q(A^{(1)}_1)$ and $U_q(A^{(1)}_2)$
of $A_1^{(1)}$ as

$$\alpha, \alpha + \delta, \ldots, \alpha + n\delta, \ldots, \delta, 2\delta, \ldots, m\delta, \ldots, \ldots, (\delta - \alpha) + l\delta, \ldots, \delta - \alpha, \quad (2.2)$$

where $\alpha$, $\delta - \alpha$ are simple roots and $\delta$ is the minimal positive imaginary root. Then one finds \[9\] the universal $R$-matrix for $U_q(A_1^{(1)})$,

$$R = \prod_{n \geq 0} \exp_{q_n}((q - q^{-1})(E_{\alpha+n\delta} \otimes F_{\alpha+n\delta})) \cdot \exp \left( \sum_{n>0} n[n]_{q_n}^{-1}(q_\alpha - q_{-\alpha}^{-1})(E_{n\delta} \otimes F_{n\delta}) \right) \cdot \prod_{n \geq 0} \exp_{q_n}((q - q^{-1})(E_{(\delta-\alpha)+n\delta} \otimes F_{(\delta-\alpha)+n\delta})) \cdot q^{\frac{1}{2}h_0 \otimes h_0 + c \otimes d + d \otimes c}. \quad (2.3)$$

where $c = h_\alpha + h_{\delta-\alpha}$ and the Cartan-Weyl generators, $E_\gamma$, $F_\gamma = \theta(E_\gamma)$, $\gamma \in \Delta_+$, are given by

$$E_\delta = [(\alpha, \alpha)]_{q^{-1}}^{-1}[E_\alpha, E_{\delta-\alpha}]_{q}, \quad E_{\alpha+n\delta} = (-1)^n \left( \text{ad}_q \tilde{E}_\delta \right)^n E_\alpha$$

$$E_{(\delta-\alpha)+n\delta} = \left( \text{ad}_q \tilde{E}_\delta \right)^n E_{\delta-\alpha}, \quad \ldots, \quad \tilde{E}_{n\delta} = [(\alpha, \alpha)]_{q^{-1}}^{-1}[E_{\alpha+(n-1)\delta}, E_{\delta-\alpha}]_q$$

$$\tilde{E}_{n\delta} = \sum_{p_1+2p_2+\ldots+np_n=n} \frac{(q^{(\alpha,\alpha)} - q^{-(\alpha,\alpha)})^{\sum p_i-1}}{p_1! \ldots p_n!}(E_\delta)^{p_1}(E_{2\delta})^{p_2} \ldots (E_{n\delta})^{p_n}. \quad (2.4)$$

The order in the product (2.3) coincides with the chosen normal order.

Turning to the rank 3 case, we fix a normal order in the positive root system $\Delta_+$ of $A_2^{(1)}$ as

$$\alpha, \alpha + \beta, \alpha + \delta, \alpha + \beta + \delta, \ldots, \alpha + m_1\delta, \alpha + \beta + m_2\delta, \ldots, \ldots,$$

$$\beta, \beta + \delta, \ldots, \beta + m_3\delta, \ldots, \delta, 2\delta, \ldots, k\delta, \ldots, \ldots, (\delta - \beta) + l_1\delta, \ldots, \delta - \beta, \ldots,$$

$$\ldots, (\delta - \alpha) + l_2\delta, (\delta - \alpha - \beta) + l_3\delta, \ldots, \ldots, \delta - \alpha, \delta - \alpha - \beta, \quad (2.5)$$

where $m_i, k, l_i \geq 0, \ i = 1, 2, 3$. Then one can show \[10\] (see also \[1\]) that the universal $R$-matrix for $U_q(A_2^{(1)})$ is given by

$$R = \prod_{\gamma < \delta} \exp_{q_\gamma}((q - q^{-1})(E_\gamma \otimes F_\gamma)) \cdot \exp \left( \sum_{n>0} \sum_{i,j=1}^2 C_{ij}^n(q)(q - q^{-1})(E_{n\delta}^{(i)} \otimes F_{n\delta}^{(j)}) \right) \cdot \prod_{\gamma > \delta} \exp_{q_\gamma}((q - q^{-1})(E_\gamma \otimes F_\gamma)) \cdot q^{\sum_{i,j=1}^2 (a_{\text{sym}}^{-1})^{ij}h_{ij} \otimes h_{ij} + c \otimes d + d \otimes c}, \quad (2.6)$$
where \( c = h_0 + h_\psi \) with \( \psi = \alpha + \beta \) being the highest root of \( A_2^{(1)} \) and \( (a_{ij}^{(\text{sym}}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \);

\[
(C_{ij}^n(q)) = \left( C_{ji}^n(q) \right) = \frac{n}{[n]_q} \frac{[2]_q^2}{q^{2n} + 1 + q^{-2n}} \begin{pmatrix} q^n + q^{-n} & (-1)^n \\ (-1)^n & q^n + q^{-n} \end{pmatrix}; \tag{2.7}
\]

and the Cartan-Weyl generators, \( E_\gamma, \ F_\gamma = \theta(E_\gamma), \ \gamma \in \Delta_+, \) are given by \((\alpha_i = \alpha, \beta, \alpha + \beta \) below\)

\[
E_{\alpha + \beta} = [E_\alpha, E_\beta]_q, \quad E_{\delta - \alpha} = [E_\beta, E_{\delta - \alpha - \beta}]_q, \quad E_{\delta - \beta} = [E_\alpha, E_{\delta - \alpha - \beta}]_q
\]

\[
\tilde{E}^{(i)}_\alpha = \left[ (\alpha_i, \alpha_i) \right]^{-1} [E_{\alpha_i}, E_{\delta - \alpha}]_q, \quad E_{\alpha_i + \alpha} = (-1)^n \left( \text{ad}_q \tilde{E}^{(i)}_\alpha \right)^n E_{\alpha_i}
\]

\[
E_{\delta - \alpha_i + \alpha} = \left( \text{ad}_q \tilde{E}^{(i)}_\alpha \right)^n E_{\delta - \alpha_i}, \quad \ldots, \quad \tilde{E}^{(i)}_{\alpha_i} = \left[ (\alpha_i, \alpha_i) \right]^{-1} [E_{\alpha_i + (n - 1)\delta, E_{\delta - \alpha_i}]_q
\]

\[
\tilde{E}^{(i)}_{\alpha_i} = \sum_{p_1 + 2p_2 + \ldots + np_n = n} \frac{(q^{(\alpha_i, \alpha_i)} - q^{-2(\alpha_i, \alpha_i)}) \sum_{p_{i_1}} p_{i_1} \ldots \sum_{p_{i_n}} p_{i_n}}{p_1! \ldots p_n!} (E_{\alpha_i}^{(i)} p_{i_1} (E_{\alpha_i}^{(i)} p_{i_2} (E_{\alpha_i}^{(i)} p_{i_3} (E_{\alpha_i}^{(i)} p_{i_4} (E_{\alpha_i}^{(i)} p_{i_5} (E_{\alpha_i}^{(i)} p_{i_6} (E_{\alpha_i}^{(i)} p_{i_7} (E_{\alpha_i}^{(i)} p_{i_8} (E_{\alpha_i}^{(i)} p_{i_9} (E_{\alpha_i}^{(i)}) p_{i_{10}} \right) \tag{2.8}
\]

Once again, the order in the product \( \prod \) is defined by that in \( \prod \).

### 3 Infinitely Many Gauge-Equivalent R-Matrices for \( U_q(A_1) \)

It can be shown that, for any \( z \in \mathbb{C}^\times \), there exist algebra homomorphisms \( \text{ev}_z : U_q(A_1^{(1)}) \to U_q(A_1) \) given by

\[
\text{ev}_z(E_\alpha) = z^{s_1} E_\alpha, \quad \text{ev}_z(F_\alpha) = z^{-s_1} F_\alpha, \quad \text{ev}_z(h_\alpha) = h_\alpha, \quad \text{ev}_z(c) = 0
\]

\[
\text{ev}_z(E_{\delta - \alpha}) = z^{s_0} F_\alpha, \quad \text{ev}_z(F_{\delta - \alpha}) = z^{-s_0} E_\alpha, \quad \text{ev}_z(h_{\delta - \alpha}) = -h_\alpha, \tag{3.1}
\]

Each homomorphism \( \text{ev}_z \) defines a corresponding gradation \((s_0, s_1)\) of \( U_q(A_1^{(1)}) \). \( s_0 \) and \( s_1 \) are arbitrary real numbers.

Following similar lines to those developed earlier \[\text{[7]}\], we derive from \( \prod \) the spectral-dependent universal \( R \)-matrix for \( U_q(A_1) \), corresponding to the gradation \((s_0, s_1)\):

\[
R^{(s_0, s_1)}(u) = \prod_{n \geq 0} \exp_{q_n} \left( (q - q^{-1}) u^{(s_1 + s_0)n + s_1} \left( q^{-n h_\alpha} E_\alpha \otimes F_\alpha q^{n h_\alpha} \right) \right)
\]

\[
\cdot \exp \left( \sum_{n > 0} n \frac{[n]_q^{-1} (q_\alpha - q^{-1} \alpha_\alpha) u^{(s_1 + s_0)n} (E_{n \delta}^{(1)} \otimes F_{n \delta}^{(1)})} \right)
\]

\[
\cdot \prod_{n \geq 0} \exp_{q_n} \left( (q - q^{-1}) u^{(s_1 + s_0)n + s_0} \left( F_\alpha q^{-n h_\alpha} \otimes q^{n h_\alpha} E_\alpha \right) \right) \cdot q^{\frac{1}{2} h_\alpha \otimes h_\alpha}, \tag{3.2}
\]
where \( E'_{n\delta} \) and \( F'_{n\delta} \) are determined by the following equalities of formal power series:

\[
\begin{align*}
1 + (q_\alpha - q_\alpha^{-1}) \sum_{k=1}^{\infty} \tilde{E}_{k\delta} u^k &= \exp \left( (q_\alpha - q_\alpha^{-1}) \sum_{l=1}^{\infty} \tilde{E}'_{l\delta} u^l \right), \\
1 - (q_\alpha - q_\alpha^{-1}) \sum_{k=1}^{\infty} \tilde{F}_{k\delta} u^{-k} &= \exp \left( -(q_\alpha - q_\alpha^{-1}) \sum_{l=1}^{\infty} \tilde{F}'_{l\delta} u^{-l} \right), \\
\tilde{E}_{n\delta} &= [2] q^{-1} (-1)^{n-1} q^{-(n-1)h_\alpha} (E_\alpha F_\alpha - q^{-2} F_\alpha E_\alpha), \\
\tilde{F}_{n\delta} &= [2] q^{-1} (-1)^{n-1} q^{-(n-1)h_\alpha} (F_\alpha E_\alpha - q^{2n} E_\alpha F_\alpha).
\end{align*}
\]

(3.3)

On applying (3.2) to the concrete representation on the tensor product space \( V_{1/2} \otimes V_{1/2} \), where \( V_{1/2} \) carries the fundamental representation of \( U_q(A_1) \), we get an infinite family of \( R \)-matrices with different spectral-dependences, corresponding to the infinite number of gradations \((s_0, s_1)\),

\[
R_{1/2,1/2}^{(s_0,s_1)}(u) = f_q(u) \cdot \begin{pmatrix}
1 & q^{-1} (1-u^{s_1+s_0}) & u^{s_1}(1-q^{-2}) \\
q^{-1} (1-u^{s_1+s_0}) & 1-q^{-2} u^{s_1+s_0} & u^{s_1}(1-q^{-2}) \\
1-q^{-2} u^{s_1+s_0} & u^{s_1}(1-q^{-2}) & 1-q^{-2} u^{s_1+s_0}
\end{pmatrix},
\]

(3.4)

where \( f_q(u) \) is an irrelevant overall scalar factor,

\[
f_q(u) = q^{1/2} \cdot \exp \left( \sum_{n=0}^{\infty} \frac{q^n - q^{-n} u^{(s_1+s_0)n}}{n} \right),
\]

(3.5)

which will be ignored in what follows. It is readily checked directly that each of the \( R \)-matrices \((12)\) satisfies the parameter-dependent QYBE.

Some remarks are in order:

(i) Semi-classical limit:

\[
R_{1/2,1/2}^{(s_0,s_1)}(u) = I + \left( \frac{1}{2} \log q \right) P_{1/2,1/2}^{(s_0,s_1)}(u) + \mathcal{O} \left( \left( \frac{1}{2} \log q \right)^2 \right),
\]

\[
P_{1/2,1/2}^{(s_0,s_1)}(u) = \begin{pmatrix}
\frac{1+u^{s_1+s_0}}{1-u^{s_1+s_0}} & 4u^{s_1} & \frac{1+u^{s_1+s_0}}{1-u^{s_1+s_0}} \\
-\frac{1+u^{s_1+s_0}}{1-u^{s_1+s_0}} & \frac{4u^{s_1}}{1-u^{s_1+s_0}} & -\frac{1+u^{s_1+s_0}}{1-u^{s_1+s_0}} \\
\frac{4u^{s_0}}{1-u^{s_1+s_0}} & -\frac{1+u^{s_1+s_0}}{1-u^{s_1+s_0}} & \frac{1+u^{s_1+s_0}}{1-u^{s_1+s_0}}
\end{pmatrix}.
\]

(3.6)

Eq.(14) defines the corresponding rational solutions of the classical Yang-Baxter equation (CYBE).

(ii) Homogeneous gradation \( s_1 = 0, s_0 = 1 \): This reproduces a result well known in the literature
we denote it by $R_{1/2,1/2}(u)$:

$$R_{1/2,1/2}(u) = \begin{pmatrix}
1 & \frac{q^{-1}(1-u)}{1-q^{-2}u} & \frac{1-q^{-2}}{1-q^{-2}u} \\
\frac{u(1-q^{-2})}{1-q^{-2}u} & 1 & \frac{1-q^{-2}}{1-q^{-2}u} \\
\frac{1-q^{-2}}{1-q^{-2}u} & \frac{q^{-1}(1-u)}{1-q^{-2}u} & 1
\end{pmatrix}. \quad (3.7)
$$

(iii) Principal gradation $s_1 = s_0 = 1$: This produces the symmetric form of $R$-matrix (see, cf. [11]); we denote it by $R_{1/2,1/2}(u)$:

$$R_{1/2,1/2}(u) = \begin{pmatrix}
1 & \frac{q^{-1}(1-u^2)}{1-q^{-2}u^2} & \frac{u(1-q^{-2})}{1-q^{-2}u^2} \\
\frac{u(1-q^{-2})}{1-q^{-2}u^2} & 1 & \frac{1-q^{-2}}{1-q^{-2}u^2} \\
\frac{1-q^{-2}}{1-q^{-2}u^2} & \frac{q^{-1}(1-u^2)}{1-q^{-2}u^2} & 1
\end{pmatrix}. \quad (3.8)
$$

(iv) It is important to note that, with the exception of $R_{1/2,1/2}(u)$, none of these $R$-matrices can be directly obtained by solving the Jimbo equations [3] or by using Yang-Baxterization procedures developed previously [4][5][6]. In particular, it is only for the homogeneous gradation that $PR_{1/2,1/2}(s_0, s_1)$ commutes with all generators of $U_q(A_1)$, where $P$ is the operator that permutes the two spaces in the tensor product $V_{1/2} \otimes V_{1/2}$.

(v) The various $R_{1/2,1/2}(s_0, s_1)$ are related to $R_{1/2,1/2}(u)$ by spectral-dependent similarity (gauge) transformations,

$$S(u)R_{1/2,1/2}(s_0, s_1)S^{-1}(u) = R_{1/2,1/2}(u^{s_1+s_0})$$

$$S(u) = \text{diag}(1, u^{-s_1/2}, u^{s_1/2}, 1), \quad (3.9)$$

Note the change in the spectral parameter on the right hand side. This defines a gauge symmetry of the spectral-dependent QYBE. Thus the differences between the $R$-matrices may be regarded as having their origins in different gradations of the same algebra $U_q(A_{1}^{(1)})$. The $R$-matrices (3.7) and (3.8) have different limits when $u \to 0$, which can no longer be transformed into each other by a similarity transformation, and the associated braid group generators are inequivalent [11][12].
4 Infinitely Many Gauge-Equivalent $R$-Matrices for $U_q(A_2)$

We now turn to $U_q(A_2^{(1)})$. In this case, for any $z \in \mathbb{C}^\times$, there exist algebra homomorphisms $\text{ev}_z: U_q(A_2^{(1)}) \to U_q(A_2)$ given by

\[
\begin{align*}
\text{ev}_z(E_\alpha) &= z^{s_1} E_\alpha, \quad \text{ev}_z(F_\alpha) = z^{-s_1} F_\alpha, \quad \text{ev}_z(h_\alpha) = h_\alpha \\
\text{ev}_z(E_\beta) &= z^{s_2} E_\beta, \quad \text{ev}_z(F_\beta) = z^{-s_2} F_\beta, \quad \text{ev}_z(h_\beta) = h_\beta \\
\text{ev}_z(E_{\delta-\alpha-\beta}) &= z^{s_0} F_{\alpha+\beta} q^{(h_\beta-h_\alpha)/3}, \quad \text{ev}_z(F_{\delta-\alpha-\beta}) = z^{-s_0} q^{(h_\alpha-h_\beta)/3} F_{\alpha+\beta} \\
\text{ev}_z(h_{\delta-\alpha-\beta}) &= -h_{\alpha+\beta}, \quad \text{ev}_z(c) = 0,
\end{align*}
\]

where $s_0$, $s_1$ and $s_2$ are arbitrary real numbers and define the gradation of $U_q(A_2^{(1)})$.

Carrying out long but similar calculations to those given previously [1], we derive from (2.6) an infinite family of $R$-matrices associated with the fundamental representation of $U_q(A_2)$, corresponding to the infinitely many different gradations $(s_0, s_1, s_2)$:

\[
R_{(3),(3)}^{(s_0,s_1,s_2)}(u) = g_q(u) \cdot \left\{ e_{11} + e_{55} + e_{99} + \frac{q^{-1}(1 - u^{s_1+s_2+s_0})}{1 - q^{-2} u^{s_1+s_2+s_0}} (e_{22} + e_{33} + e_{44} + e_{66} + e_{77} + e_{88}) + \frac{1 - q^{-2}}{1 - q^{-2} u^{s_1+s_2+s_0}} (u^{s_1} e_{24} + u^{s_1+s_2} e_{37} + u^{s_2} e_{68} + u^{s_2+s_0} e_{42} + u^{s_0} e_{73} + u^{s_1+s_0} e_{86}) \right\},
\]

$e_{ij}$ is the matrix satifying $(e_{ij})_{kl} = \delta_{ik} \delta_{jl}$ and

\[
g_q(u) = q^{2/3} \cdot \exp \left( \sum_{n>0} \frac{q^{2n} - q^{-2n}}{q^{2n} + 1 + q^{-2n}} \frac{u^{(s_1+s_2+s_0)n}}{n} \right)
\]

is an irrelevant scalar factor which, like the other scalar factors, will be ignored in what follows.

These various $R$-matrices all satisfy the parameter-dependent QYBE, and are apparently all associated with 15-vertex models, solvable in principle. The following remarks are in order:

(i) Semi-classical limit:

\[
R_{(3),(3)}^{(s_0,s_1,s_2)}(u) = I + \left( \frac{1}{2} \log q \right) r_{(3),(3)}^{(s_0,s_1,s_2)}(u) + O \left( \left( \frac{1}{2} \log q \right)^2 \right),
\]

\[
r_{(3),(3)}^{(s_0,s_1,s_2)}(u) = \frac{1 + u^{s_1+s_2+s_0}}{1 - u^{s_1+s_2+s_0}} \left( e_{11} + e_{55} + e_{99} - e_{22} - e_{33} - e_{44} - e_{66} - e_{77} - e_{88} \right) + \frac{4}{1 - u^{s_1+s_2+s_0}} (u^{s_1} e_{24} + u^{s_1+s_2} e_{37} + u^{s_2} e_{68} + u^{s_2+s_0} e_{42} + u^{s_0} e_{73} + u^{s_1+s_0} e_{86}).
\]
This defines the corresponding rational solutions of the CYBE.

(ii) Homogeneous gradation \( s_1 = s_2 = 0, \ s_0 = 1 \): This produces the \( R \)-matrix

\[
R^{h}_{(3),(3)}(u) = e_{11} + e_{55} + e_{99} + \frac{q^{-1}(1-u)}{1-q^{-2}u} (e_{22} + e_{33} + e_{44} + e_{66} + e_{77} + e_{88}) + \\
+ \frac{1-q^{-2}}{1-q^{-2}u} (e_{24} + e_{37} + e_{68} + u(e_{42} + e_{73} + e_{86})) , \tag{4.5}
\]

which is exactly the well-known solution due to Jimbo \[2\].

(iii) Principal gradation \( s_1 = s_2 = s_0 = 1 \): This defines the \( R \)-matrix

\[
R^{p}_{(3),(3)}(u) = e_{11} + e_{55} + e_{99} + \frac{q^{-1}(1-u^3)}{1-q^{-2}u^3} (e_{22} + e_{33} + e_{44} + e_{66} + e_{77} + e_{88}) + \\
+ \frac{u(1-q^{-2})}{1-q^{-2}u^3} (e_{24} + e_{68} + e_{73} + u(e_{37} + e_{42} + e_{86})) . \tag{4.6}
\]

(iv) Our results suggest that, associated with the fundamental representation of \( U_q(A_2) \), there is no fully symmetric \( R \)-matrix. However, there is an “almost” symmetric \( R \)-matrix corresponding to the gradation \( g3 = (0, 1, 1) \),

\[
R^{g3}_{(3),(3)}(u) = e_{11} + e_{55} + e_{99} + \frac{q^{-1}(1-u^2)}{1-q^{-2}u^2} (e_{22} + e_{33} + e_{44} + e_{66} + e_{77} + e_{88}) + \\
+ \frac{u(1-q^{-2})}{1-q^{-2}u^2} (e_{24} + e_{37} + e_{42} + e_{73}) + \frac{1-q^{-2}}{1-q^{-2}u^2}(e_{68} + u^2 e_{86}) . \tag{4.7}
\]

(v) Of all the \( R \)-matrices (20), only \( R^{h}_{(3),(3)}(u) \) can be directly derived by solving the Jimbo equations \[2\] or by using the Yang-Baxterization method \[5\] \[6\] since others do not have the intertwining property for the usual two coproducts of \( U_q(A_2) \).

(vi) The matrices \( R^{(s_0,s_1,s_2)}_{(3),(3)}(u) \) and \( R^{h}_{(3),(3)}(u) \) can be transformed into each other by similarity (gauge) transformations,

\[
S(u)R^{(s_0,s_1,s_2)}_{(3),(3)}(u)S^{-1}(u) = R^{h}_{(3),(3)}(u^{s_1+s_2+s_0}) ,
\]

\[
S(u) = e_{11} + e_{55} + e_{99} + u^{-s_1/2} e_{22} + u^{s_1/2} e_{44} + \\
+ u^{-(s_1+s_2)/2} e_{33} + u^{(s_1+s_2)/2} e_{77} + u^{-s_2/2} e_{66} + u^{s_2/2} e_{88} . \tag{4.8}
\]

Eq.(4.8) defines a gauge symmetry of the QYBE. This implies the gauge transformations have their origins in different gradations of the same algebra.

(vii) When the spectral parameter goes to zero, we obtain

\[
R^{h}_{(3),(3)}(u = 0) = e_{11} + e_{55} + e_{99} + q^{-1}(e_{22} + e_{33} + e_{44} + e_{66} + e_{77} + e_{88})
\]
\[(1 - q^{-2})(e_{24} + e_{37} + e_{68})\]

\[R^p_{(3),(3)}(u = 0) = e_{11} + e_{55} + e_{99} + q^{-1}(e_{22} + e_{33} + e_{44} + e_{66} + e_{77} + e_{88})\]

\[R^g_{(3),(3)}(u = 0) = e_{11} + e_{55} + e_{99} + q^{-1}(e_{22} + e_{33} + e_{44} + e_{66} + e_{77} + e_{88}) + (1 - q^{-2})e_{68}\]

These limiting forms are not related by similarity transformations and define three inequivalent braid group generators [12]. (In particular, they have differing numbers of linearly independent eigenvectors corresponding to the eigenvalue \(q^{-1}\).) While the first of these is the universal \(R\)-matrix for \(U_q(A_2)\) in the fundamental representation, the relationship of the second and third ones to this quantum algebra is unclear.

5 Changing Gradations: General Case

We have seen that quantum \(R\)-matrices with different spectral parameter dependence can be obtained from the universal \(R\)-matrix of the associated quantum affine algebra by choosing different gradations. The change from one gradation to another can be achieved through spectral parameter dependent similarity transformations, examples of which we have seen in (3.9) and (4.8).

One strong point of our method is that we obtain the spectral dependent \(R\)-matrices in a universal form (i.e., as an element of \(U_q(\hat{G}) \otimes U_q(\hat{G})\) and thus representation independent, see eq. (3.2)). We can also write the transformation between different gradations in a universal form as follows.

Associated with a given gradation of \(U_q(\hat{G})\) there is an algebra homomorphism \(D(s)_z: U_q(\hat{G}) \to U_q(\hat{G}) \otimes \mathbb{C}(z, z^{-1})\) defined by

\[D(s)_z(E_i) = z^{s_i} E_i, \quad D(s)_z(F_i) = z^{-s_i} F_i, \quad D(s)_z(h_i) = h_i, \quad i = 0, \cdots, r\] (5.1)

We call \(z\) the spectral parameter. The \(s_i\) are arbitrary real numbers. The homogeneous gradation is given by \(s_i = \delta_{i0}\) and below the homomorphism corresponding to this gradation will be denoted as \(D^h_z\).

We define the operator

\[T(s)(z) = z^{\chi(s)}, \quad \chi(s) \in \mathcal{H}_0\] (5.2)

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where $\mathcal{H}_0$ is the subspace of the Cartan subalgebra generated by the $h_i$, $i = 1, 2, \cdots, r$, i.e. without $h_0$. This operator transforms the homogeneous gradation as follows

$$
T^{(s)}(z) D^h_2(E_i) (T^{(s)}(z))^{-1} = z^{(\chi^{(s)}, \alpha_i) + \delta_0} E_i,
$$

$$
T^{(s)}(z) D^h_2(F_i) (T^{(s)}(z))^{-1} = z^{-(\chi^{(s)}, \alpha_i) - \delta_0} F_i,
$$

$$
T^{(s)}(z) D^h_2(h_i) (T^{(s)}(z))^{-1} = h_i, \quad i = 0, 1, \cdots, r \tag{5.3}
$$

We observe that this can be rewritten as

$$
T^{(s)}(z) D^h_2(a) (T^{(s)}(z))^{-1} = D_z^{(s)}(a), \quad \forall a \in U_q(\hat{G}) \tag{5.4}
$$

with $z' = z^{1/\mu}$, provided that the $s_i$ are related to $\mu$ and $\chi^{(s)}$ as follows:

$$
s_i = \mu(\chi^{(s)}, \alpha_i), \quad i = 1, 2, \cdots, r
$$

$$
s_0 = \mu(1 + (\chi^{(s)}, \alpha_0)) \tag{5.5}
$$

Thus, by solving these equations for $\mu$ and $\chi^{(s)}$ we find from (5.2) the operator $T^{(s)}(z)$ which relates the homogeneous gradation to an arbitrary gradation according to (5.4). Note that such a change of gradation is accompanied by a change of spectral parameter from $z$ to $z^{1/\mu}$. To determine $\mu$, we take a linear combination of equations (5.3) and the equation $\sum_{i=0}^{r} n_i \alpha_i = 0$, where $n_i$ are the Kac-labels ([13], p.54), and find $\mu = \sum_{i=0}^{r} n_i s_i$.

The spectral parameter dependent universal R-matrices $R^{(s)}(u)$ can be obtained from the universal R-matrix $R$ of $U_q(\hat{G})$ as

$$
R^{(s)}(u) = (D_u^{(s)} \otimes I)R. \tag{5.6}
$$

They are related to the spectral parameter dependent R-matrix in the homogeneous gradation by

$$
R^h(u^\mu) = \left(T^{(s)}(u^\mu) \otimes 1\right)^{-1} R^{(s)}(u) \left(T^{(s)}(u^\mu) \otimes 1\right). \tag{5.7}
$$

Note that the universal spectral parameter dependent R-matrices $R^{(s)}(u)$ are elements of $U_q(\hat{G}) \otimes U_q(\hat{G})$. It is only after the specialization to some finite dimensional representations $\pi^\lambda, \pi^{\lambda'}$ of $U_q(\hat{G})$

$$
R^{(s)}_{\lambda\lambda'}(u) = (\pi^\lambda \otimes \pi^{\lambda'}) R^{(s)}(u) \tag{5.8}
$$
that $R_{\lambda\lambda'}^{(s)}(u)$ can be viewed as a spectral dependent R-matrix of the quantum algebra $U_q(G)$. This is so because the $U_q(G)$ modules $V(\lambda)$ and $V(\lambda')$ are automatically also (possibly reducible) modules of $U_q(G)$. However only some $U_q(G)$-modules are also $U_q(\hat{G})$-modules. We call these modules "affinizable". Spectral dependent $U_q(G)$ R-matrices exist only for affinizable modules. For an investigation of affinizable modules see [14].

It is easily checked that eq. (3.9) and eq. (4.8) can be obtained from this by specializing to the particular representations.

6 Gradations in Quantum Affine Toda Theories

Even though the spectral parameter dependent $R$-matrices for different gradations are related by similarity transformations, they are not necessarily equivalent physically. A nice example of this is furnished by the quantum affine Toda field theories.

It is well-known that associated to every $\hat{G}$ there is a 1+1 dimensional affine Toda field theory [5], denoted $T(\hat{G})$. It is described by the field equations

$$\Box \phi = \frac{\sqrt{-1}}{\beta} \sum_{i=0}^{r} n_i \alpha_i e^{\sqrt{-1/\beta} \alpha_i \cdot \phi}$$

(6.1)

The field $\phi(x, t)$ takes values in $H_0$, the subspace of the Cartan subalgebra generated by the $h_i$, $i = 1, \ldots, r$, i.e. without $h_0$. $\beta$ is the coupling constant and the $\alpha_i$ the simple roots. For $\hat{G} = A_1^{(1)}$ eq.(6.1) specializes to the sine-Gordon (or affine Liouville) equation. The field equations (6.1) have soliton solutions.

The affine Toda theory $T(\hat{G})$ possesses symmetry generators $E_i, F_i, h_i$, $i = 0, 1, \ldots, r$, which generate the quantum affine algebra $U_q(\hat{G})$ [6]. Here $\hat{G}$ is the dual Lie algebra to $\hat{G}$, i.e., it is obtained by interchanging the roles of the roots and the coroots. The deformation parameter $q$ is determined by the coupling constant as $q = e^{-\sqrt{-1/\beta^2}}$. The central charge is zero.

We will now explain how the physically relevant gradation is determined by Lorentz invariance. In quantum theory each soliton solution of (6.1) gives rise to a one-particle state $|a, \theta>$ in the Hilbert space, where $a$ labels the particle and $\theta$ is the rapidity [1]. The defining property

\footnote{The rapidity is related to the two-momentum by $p_0 = m ch(\theta)$ and $p_1 = m sh(\theta)$, $m$ being the mass of the particle.}
of a particle is its behaviour under Lorentz transformations, which in two dimensions takes the form,

\[ L(\lambda) |a, \theta \rangle = |a, \theta + \lambda \rangle \]  

(6.2)

where \( L(\lambda) \) is the Lorentz generator. Also the transformation property of the symmetry generators can be determined and one finds \[16\]

\[ L(\lambda) E_i = e^{\lambda s_i} E_i L(\lambda), \quad L(\lambda) F_i = e^{-\lambda s_i} F_i L(\lambda), \quad L(\lambda) h_i = h_i L(\lambda), \]  

(6.3)

with

\[ s_i = \frac{2(\alpha_i, \alpha_i)}{\beta^2} - 1 \]  

(6.4)

Comparing (6.3) and (6.2) fixes the \( \theta \)-dependence of the action of the \( U_q(\hat{G}) \) generators on the soliton states

\[ E_i|a, \theta \rangle = e^{s_i \theta} \Pi(E_i)_{a}^{b} |b, \theta \rangle \]

\[ F_i|a, \theta \rangle = e^{-s_i \theta} \Pi(F_i)_{a}^{b} |b, \theta \rangle \]

\[ h_i|a, \theta \rangle = \Pi(h_i)_{a}^{b} |b, \theta \rangle \]  

(6.5)

where \( \Pi \) is a \( \theta \)-independent finite dimensional representation of \( U_q(\hat{G}) \). We recognize the representation (6.5) as the loop representation of \( U_q(\hat{G}) \) with spectral parameter \( e^\theta \) and gradation \( s_i \) given by (6.4). When \( \hat{G} \) is simply laced, this is just the principal gradation (up to a rescaling of the spectral parameter), but for non-simply laced theories, this is an unusual gradation.

The physical quantity which is most immediately determined by the quantum affine algebra symmetry is the scattering matrix which describes the transition from an incoming 2-soliton state to an outgoing 2-soliton state. As is explained in \[16\], this is proportional to the \( U_q(\hat{G}) \) \( R \)-matrix in the representation (6.3). To predict the correct scattering behaviour of the solitons it is thus essential to work with the \( R \)-matrix in the gradation determined by (6.4). Different gradations are not physically equivalent because physics singles out a particular basis in Hilbert space, namely that given by particle states. The non-standard gradation (6.4), taken together with the axioms of \( S \)-matrix theory such as crossing symmetry, leads to interesting effects in the non-simply laced case, as we will describe in \[17\]. In particular it determines the quantum mass-ratios of the solitons.
7 Concluding Remarks

We have found infinitely many spectral-dependent $R$-matrices corresponding to different gradations – including the important homogeneous gradation and principal gradation – of $U_q(A_1^{(1)})$ and $U_q(A_2^{(1)})$. These $R$-matrices are related to each other by similarity (gauge) transformations but can have quite different limits as the spectral parameter $u \to 0$, and the choice of gradation can be dictated by the physics in particular applications.

Our results suggest that the “hierarchies” of solutions of the QYBE, which are gauge equivalent for non-zero values of the spectral parameter, but which may be inequivalent in the limit $u \to 0$, have their origin in the gradations of the quantum affine algebras.

Acknowledgements:

We are grateful to Uwe Grimm, Valeriy N. Tolstoy and S.O.Warnaar for reading the manuscript and pointing out some errors in the original version, and to Valeriy N. Tolstoy for clarifying discussions. G.W.D. wishes to thank the Department of Mathematics of the University of Queensland in Brisbane for its hospitality. Y.Z.Z. wishes to thank Loriano Bonora, Günter von Gehlen and Zhong-Qi Ma for discussions. Financial support from the Australian Research Council is gratefully acknowledged.

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