ON THE MODIFIED AFFINE HECKE ALGEBRAS AND QUIVER HECKE ALGEBRAS OF TYPE A

JUN HU AND FANG LI

Abstract. We introduce some modified forms for the degenerate and non-degenerate affine Hecke algebras of type $A$ such that their finite dimensional module categories are equivalent to the finite dimensional modules categories over the original affine Hecke algebras. These are certain subalgebras living inside the inverse limit of cyclotomic Hecke algebras. Many classical results (including faithful polynomial representations, standard bases and description of the centers) for the original affine Hecke algebras are generalized to these modified affine Hecke algebras. We construct some explicit algebra isomorphisms between some generalized Ore localizations of the modified affine Hecke algebras and of the quiver Hecke algebras of type $A$, which generalize Brundan-Kleshchev’s isomorphisms between the cyclotomic Hecke algebras and the cyclotomic quiver Hecke algebras of type $A$. As applications, we give a categorical equivalence for quiver Hecke algebras, a simplicity result for the convolution products of simple modules and prove a conjecture for the center of cyclotomic quiver Hecke algebras in the cases of linear quivers and of some special cyclic quivers.

1. Introduction

Let $\mathbb{Z}$ be the set of integers and $\mathbb{N}$ the set of non-negative integers. Let $e \in \{0, 2, 3, 4, \cdots\}$ be a fixed integer and $I := \mathbb{Z}/e\mathbb{Z}$. Let $\Gamma_e$ be the quiver with vertex set $I = \mathbb{Z}/e\mathbb{Z}$ and edges $i \rightarrow i + 1$, for all $i \in I$. Following [10, Chapter 1], attach to $\Gamma_e$ the standard Lie theoretic data of a Cartan matrix $(a_{ij})_{i,j \in I}$, simple roots $\{\alpha_i \mid i \in I\}$, fundamental weights $\{\Lambda_i \mid i \in I\}$ and positive root lattice $Q^+ = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$. For each $\alpha = \sum_{i \in I} k_i \alpha_i \in Q^+$ we define $\text{ht}(\alpha) := \sum_{i \in I} k_i$. We set $Q^+_n := \{\beta \in Q^+ \mid \text{ht}(\beta) = n\}$ for each $n \in \mathbb{N}$.

Let $P^+ := \bigoplus_{i \in I} \mathbb{N} \Lambda_i$ be the dominant weight lattice, $\ell \in \mathbb{N}$ and $k_1, \cdots, k_\ell \in \mathbb{Z}/e\mathbb{Z}$. We define

\begin{equation}
\Lambda := \Lambda_{k_1} + \cdots + \Lambda_{k_\ell} \in P^+
\end{equation}

and call $\ell$ the level of $\Lambda$. In this paper we shall consider both the non-degenerate and the degenerate settings as follows.

In the non-degenerate setting, we assume that $K$ is a field, $1 \neq q \in K^\times$ such that either $e$ is the minimal positive integer $k$ which satisfies that $1 + q + q^2 + \cdots + q^{k-1} = 0$; or $e = 0$ and there is no such positive integer $k$. In this case, let $H_n^A(q)$ be the non-degenerate cyclotomic Hecke algebra of type $A$ over $K$ with Hecke parameter $q$ and cyclotomic parameters $q^{a_1}, \cdots, q^{a_\ell}$ (cf. [1], [5]). By definition, $H_n^A(q)$ is generated by $T_0, T_1, \ldots, T_{n-1}$ which satisfy the following relations:

\begin{align*}
(T_0 - q^{a_1}) \cdots (T_0 - q^{a_\ell}) &= 0, \\
T_0^q T_k T_0 &= T_k T_0 T_k, \\
(T_i - q)(T_i + 1) &= 0, \quad T_i T_k = T_k T_i, \quad 1 \leq i < n, 0 \leq k < n, |i - k| > 1, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad 1 \leq i \leq n - 2.
\end{align*}

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Let $L_1 := T_0$ and $L_{i+1} = q^{-1} T_i L_i T_i$ for $1 \leq i < n$. The elements $L_1, L_2, \ldots, L_n$ are called the **Jucys–Murphy operators** of $\mathcal{H}_n^\Lambda(q)$.

In the degenerate setting, we assume that $e = 0$ or $e$ is a prime number and $K$ is field with char $K = e$. In this case, let $H_n^\Lambda$ be the **degenerate cyclotomic Hecke algebra** of type $A$ over $K$ with cyclotomic parameters $\kappa_1 \cdot 1_K, \ldots, \kappa_\ell \cdot 1_K$ (cf. [7], [13]). By definition, $H_n^\Lambda$ is generated by $s_1, \ldots, s_{n-1}, L_1, \ldots, L_n$ which satisfy the following relations:

\[
(L_1 - \kappa_1 \cdot 1_K) \cdots (L_1 - \kappa_\ell \cdot 1_K) = 0,
\]

\[
s_i^2 = 1, s_is_k = s_ks_i, \quad \text{for } 1 \leq i, k < n, |i - k| > 1,
\]

\[
s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \quad \text{for } 1 \leq i \leq n - 2,
\]

\[
L_iL_k = L_kL_i, \quad s_iL_i = L_is_i, \quad \text{for } 1 \leq i < n, 1 \leq k, l \leq n, l \neq i, i + 1,
\]

\[
L_{i+1} = s_iL_is_{i+1} + s_i, \quad \text{for } 1 \leq i < n.
\]

The elements $L_1, L_2, \ldots, L_n$ are called the **Jucys–Murphy operators** of $H_n^\Lambda$. Note that in general $L_1s_1L_1s_1 \neq s_1L_1s_1L_1$ in $H_n^\Lambda$.

For any $\beta \in Q_n^+$, we define

\[
I^\beta := \{i = (i_1, \ldots, i_n) \in I^n | \alpha_{i_1} + \cdots + \alpha_{i_n} = \beta \}.
\]

For each $i \in I^n$, Brundan and Kleshchev have introduced in [4, §3.1, §4.1] an idempotent in $\mathcal{H}_n^\Lambda(q)$ and an idempotent in $H_n^\Lambda$ which (by abuse of notations) are both denoted by $e(i)$. Let $\beta \in Q_n^+$. We set $e(\beta) := \sum_{i \in I^n} e(i)$. Then $e(\beta)$ is either equal to 0 or a block idempotent of $\mathcal{H}_n^\Lambda(q)$ (resp., of $H_n^\Lambda$). We set

\[
J_\beta := \{\beta \in Q_n^+ | e(\beta) \neq 0\}.
\]

By [20] and [3], both the blocks of $\mathcal{H}_n^\Lambda(q)$ and of $H_n^\Lambda$ are parameterized by $\beta \in J_\beta$. For any block idempotents $e(\beta)$ of $\mathcal{H}_n^\Lambda(q)$ and of $H_n^\Lambda$, we define

\[
\mathcal{H}_n^\Lambda(q) := e(\beta) \mathcal{H}_n^\Lambda(q), \quad H_n^\Lambda := e(\beta) H_n^\Lambda,
\]

which are the block subalgebras corresponding to $\beta$ of $\mathcal{H}_n^\Lambda(q)$ and of $H_n^\Lambda$ respectively.

Let $\mathcal{A}_\beta := \mathcal{A}_\beta(\Gamma_e)$ be the quiver Hecke algebra associated to $\Gamma_e$ and $\beta \in Q_n^+$ introduced by Khovanov and Lauda [12], and by Rouquier [26]. This algebra has been a hot topic in recent years and plays an important role in the theory of categorification of quantum groups. By definition, $\mathcal{A}_\beta$ is generated by the elements $\{\psi_1, \ldots, \psi_{n-1}\} \cup \{y_1, \ldots, y_n\} \cup \{e(l) \mid i \in I^\beta\}$ which satisfy certain relations. We refer the readers to Section 2 for the list of all the relations. Let $\mathcal{B}_\beta := \mathcal{B}_\beta(\Gamma_e)$ be the quotient of $\mathcal{A}_\beta$ by the two-sided ideal generated by

\[
y_1^{(\Lambda, \alpha_i^e)} e(1), \quad i \in I^\beta.
\]

We call the algebra $\mathcal{B}_\beta$ the **cyclo tonic quiver Hecke algebra** of type $A$ associated to $\beta$ and $\Lambda$. When the context is clear, we use the same letters to denote both the KLR generators of $\mathcal{B}_\beta$ and the KLR generators of $\mathcal{A}_\beta$, and use the same letter $e(1)$ to denote both the idempotents of $\mathcal{H}_n^\Lambda(q)$ and of $H_n^\Lambda$ and the KLR idempotent of $\mathcal{A}_\beta$.

1.5. **Theorem** (Brundan-Kleshchev [4, Theorem 1.1]). Let $\beta \in Q_n^+$ and $\mathcal{A}_\beta^\Lambda \in \{\mathcal{H}_n^\Lambda(q), H_n^\Lambda\}$. Then there is an isomorphism of $K$-algebras $\theta_\Lambda : \mathcal{B}_\beta^\Lambda \cong \mathcal{A}_\beta^\Lambda$ that...
sends $e(i) \mapsto e(i)$, for all $i \in I^\beta$ and
\[
y_r e(i) \mapsto \begin{cases} (1 - q^{-1} L_r)e(i), & \text{if } H^\Lambda_\beta = H^\Lambda_\beta(q), \\ (L_r - i_r)e(i), & \text{if } H^\Lambda_\beta = H^\Lambda_\beta, \end{cases}
\]
\[
\psi_k e(i) \mapsto \begin{cases} (T_k + P_k(i))Q_k(i)^{-1}e(i), & \text{if } H^\Lambda_\beta = H^\Lambda_\beta(q), \\ (s_k + P_k(i))Q_k(i)^{-1}e(i), & \text{if } H^\Lambda_\beta = H^\Lambda_\beta, \end{cases}
\]
where $1 \leq r \leq n$, $1 \leq k < n$, $P_k(i), Q_k(i) \in K[y_k, y_{k+1}]$ are certain polynomials introduced in [4, (3.22), (3.27–3.29), (4.27), (4.33–4.35)]. In particular, $H^\Lambda_\beta \neq 0$ if and only if $\sum_{i \in I^\beta} e(i) \neq 0$ in $R^\Lambda_\beta$, and $H^\Lambda_\beta \neq 0$ if and only if $e(\beta) \neq 0$ in $R^\Lambda_\beta$.

Let $H_\alpha(q)$ and $H_n$ be the non-degenerate and the degenerate type $A$ affine Hecke algebra respectively. Then $H^\Lambda_n(q)$ and $H^\Lambda_n$ are isomorphic to the quotients of $H_\alpha(q)$ and $H_n$ by the two-sided ideals generated by $(X_1 - q^{\kappa_1}) \cdots (X_1 - q^{\kappa_\ell})$ and $(x_1 - \kappa_1 \cdot 1_K) \cdots (x_1 - \kappa_\ell \cdot 1_K)$ respectively. We refer the readers to Section 2 for more details and unexplained notations. There are many similarities on the structure and representation theory between the algebras $H^\Lambda_n(q) \in \{H^\Lambda_n(q), H^\Lambda_n\}$ and $H^\Lambda_\beta$, and between the algebras $H_\alpha \in \{H_\alpha(q), H_n\}$ and $R^\Lambda_n := \oplus_{\beta \in Q_n^+} R^\Lambda_\beta$. As both of the algebras $H_n$ and $R_\beta$ are $\mathbb{Z}$-graded, almost all the results on the representations of $H_n$ and $H^\Lambda_n$ have $\mathbb{Z}$-graded versions in the representation theory of $H_n$ and $H^\Lambda_n$, see [7, [13, [14, [17, [23] and the references therein. In view of Brundan–Kleshchev’s isomorphism Theorem 1.5, one can think of $R^\Lambda_\beta$ as a $\mathbb{Z}$-graded lift $H^\Lambda_\beta$.

It is natural to ask if there is a similar isomorphism directly on the level of the affine Hecke algebra $H_n$ and the quiver Hecke algebra $R_n$, which is the starting point of this work. In this paper, for each $\beta \in Q_n^+$, we introduce some modified forms $\tilde{H}_\beta \in \{\tilde{H}_\beta(q), \tilde{H}_\beta\}$ for both the non-degenerate and the degenerate type $A$ affine Hecke algebras. We construct some explicit $\mathbb{K}$-algebra isomorphisms $\theta : R^\Lambda_\beta \cong \tilde{H}_\beta(q), \theta' : \tilde{H}_\beta \cong \tilde{H}_\beta$, between the generalized Ore localizations $\tilde{H}_\beta(q), \tilde{H}_\beta, \tilde{H}_\beta$ (or $\tilde{H}_\beta$) of $\tilde{H}_\beta(q)$, $\tilde{H}_\beta$ and $\tilde{H}_\beta$ respectively. These modified affine Hecke algebras $\tilde{H}_\beta$ and their generalized Ore localizations are certain subalgebras of the inverse limit of the cyclotomic Hecke algebras $H^\Lambda_n$ which contains the images of the “blocks” of the type $A$ affine Hecke algebra $H_n$, and they are closely related to the original type $A$ affine Hecke algebra $H_n$ in that every finite dimensional module over $H_n$ which belongs to the block labelled by $\beta$ naturally becomes a module over $\tilde{H}_\beta$ and this correspondence gives rise to a category equivalence. Many classical results (including faithful polynomial representations, standard bases and description of the centers) for the original affine Hecke algebras are generalized to these modified affine Hecke algebras, see Proposition 3.39, Lemma 3.45 and Lemma 3.59. Moreover, our isomorphism between the generalized Ore localized forms of $\tilde{H}_\beta$ and of $\tilde{H}_\beta$ descend to Brundan–Kleshchev’s isomorphism between $H^\Lambda_n$ and $R^\Lambda_n$ after taking finite dimensional quotients.

The isomorphisms $\theta, \theta'$ give a conceptual and direct way to connect the structure and representation theory of the type $A$ affine Hecke algebras with that of the quiver Hecke algebras. In particular, one can identify the convolution products in the category of finite dimensional modules over affine Hecke algebras with the convolution product in the category of finite dimensional modules over quiver Hecke algebras. We give three applications of our main results in this paper. The first one is an equivalence of categories for quiver Hecke algebras and tensor products of its non-unital quiver Hecke subalgebras; the second one is a simplicity result for the convolution products of simple modules over quiver Hecke algebras; and
the third one is the proof of a conjecture on the center of cyclotomic quiver Hecke algebras in the linear quiver cases and some special cyclic quiver cases. The explicit isomorphisms $\theta, \theta'$ also open the possibility to find out the (affine) Hecke algebras forms for the (cyclotomic) quiver Hecke algebras associated to any simply-laced Dynkin quiver other than type $A$.

We note that Rouquier has presented a similar isomorphism between different localized from of $H_n$ and $R_n$ in the preprint [27, 3.15, 3.18]. Our isomorphism $\theta, \theta'$ are different with Rouquier’s isomorphisms and the algebra $\hat{H}_\beta$ we introduced in this paper does not appear in [27]. In another preprint [31], Webster have proved that certain completion of $H_n$ and of $R_n$ are isomorphic to each other.

The content of this paper is organised as follows. In Section 2, we recall some preliminary knowledge about the non-degenerate and the degenerate type $A$ affine Hecke algebras, the (cyclotomic) quiver Hecke algebras, as well as their cyclotomic quotients. In Section 3, we introduce the modified forms of the affine Hecke algebras of type $A$. We construct faithful representations, standard bases and describe the centers for these modified affine Hecke algebras. We also introduce some generalized Ore localization for these modified affine Hecke algebras and quiver Hecke algebras. The general definition of the so-called generalized Ore localization, which is a generalization of the classical construction of the Ore localization with respect to a right denominator set, is given in the appendix of this paper. The main results (Theorem 4.1 and 4.2) are given in Section 4, where we set up isomorphisms between these generalized Ore localization for modified affine Hecke algebras and the generalized Ore localization for quiver Hecke algebras. The main idea in the proof of is to embed the generalized Ore localizations for these modified affine Hecke algebras (resp., for the quiver Hecke algebras) into the inverse limits of cyclotomic Hecke algebras (resp., of the cyclotomic quiver Hecke algebras) and lift Brundan–Kleshchev’s isomorphisms. In Section 5 we give three main applications of Theorem 4.1 and 4.2. Firstly, we give a categories equivalence result (Corollary 5.6) for quiver Hecke algebras and the tensor products of its non-unital quiver Hecke subalgebras. Secondly, we obtain a simplicity result (Corollary 5.8) for the convolution products of simple modules over quiver Hecke algebras. Thirdly, we prove a conjecture (Proposition 5.10) for the center of the cyclotomic quiver Hecke algebra associated to a linear quiver when $\text{char } K = 0$, which claims that the center of the corresponding quiver Hecke algebras maps surjectively onto the center of the cyclotomic quiver Hecke algebra. For the cyclic quiver we prove the same statement under the assumption that the length of the cyclic quiver is a prime number $p > 0$ and $\text{char } K = p$.

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2. Preliminary

In this section, we shall recall some preliminary knowledge about the non-degenerate and the degenerate affine Hecke algebras of type $A$, their cyclotomic quotients and the (cyclotomic) quiver Hecke algebras associated to $\Gamma_e$ and $\beta \in \mathbb{Q}^+$. In particular, we shall fix some choices of the polynomials $P_k(i), Q_k(i) \in K[y_k, y_{k+1}]$ in the construction of Brundan–Kleshchev’s isomorphism Theorem 1.5.
Recall that \( \ell, n \in \mathbb{N} \) and \( e \in \{0, 2, 3, \ldots\} \) and \( \kappa_1, \ldots, \kappa_\ell \in I := \mathbb{Z}/e\mathbb{Z} \). Let \( K \) be a field. In the non-degenerate setting, we assume that \( 1 \neq q \in K^\times \) such that \( e \) is the minimal positive integer \( k \) satisfying \( 1 + q + q^2 + \cdots + q^{k-1} = 0 \); or \( e = 0 \) and there is no such positive integer \( k \). Let \( \mathcal{H}_n(q) \) be the non-degenerate type \( A \) affine Hecke algebra over \( K \). By definition, \( \mathcal{H}_n(q) \) is the unital associative \( K \)-algebra with generators \( T_1, \ldots, T_{n-1}, X_1^{\pm 1}, \ldots, X_{n}^{\pm 1} \) and relations:

\[
(T_i - q)(T_i + 1) = 0, \quad 1 \leq i < n, \tag{2.1}
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 1 \leq i \leq n - 2, \tag{2.2}
\]

\[
T_i T_k = T_k T_i, \quad |i - k| > 1, \tag{2.3}
\]

\[
X_i^{\pm 1} X_k^{\pm 1} = X_k^{\pm 1} X_i^{\pm 1}, \quad 1 \leq i, k \leq n, \tag{2.4}
\]

\[
X_k X_k^{-1} = 1 = X_k^{-1} X_k, \quad 1 \leq k \leq n, \tag{2.5}
\]

\[
T_i X_k = X_k T_i, \quad k \neq i, i + 1, \tag{2.6}
\]

\[
X_{i+1} = q^{-1} T_i X_i T_i, \quad 1 \leq i < n. \tag{2.7}
\]

Note that one can also replace the last relation above with the following:

\[
X_{i+1} T_i = T_i X_i + (q - 1) X_i, \quad 1 \leq i < n. \tag{2.8}
\]

The non-degenerate type \( A \) cyclotomic Hecke algebra \( \mathcal{H}_n^\Lambda(q) \) introduced in Section 1 is isomorphic to the quotient of \( \mathcal{H}_n(q) \) by the two-sided ideal generated by

\[
(X_1 - q^{\kappa_1})(X_1 - q^{\kappa_2}) \cdots (X_1 - q^{\kappa_\ell}). \tag{2.9}
\]

Under this isomorphism, \( T_0 \) is identified with the image of \( X_1 \) in \( \mathcal{H}_n^\Lambda(q) \), and each \( L_i \) is identified with the image of \( X_i \) in \( \mathcal{H}_n^\Lambda(q) \) for \( 1 \leq i \leq n \). For each \( 1 \leq j < n \), we still use \( T_j \) to denote the image of \( T_j \) in \( \mathcal{H}_n^\Lambda(q) \).

In the degenerate setting, we assume that either \( e = 0 \) or \( e \) is a prime number and \( \text{char } K = e \). Let \( H_n \) be the degenerate type \( A \) affine Hecke algebra over \( K \). By definition, \( H_n \) is the unital associative \( K \)-algebra with generators \( s_1, \ldots, s_{n-1}, x_1, \ldots, x_n \) and relations:

\[
s_i^2 = 1, \quad 1 \leq i < n, \tag{2.10}
\]

\[
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad 1 \leq i \leq n - 2, \tag{2.11}
\]

\[
s_i s_k = s_k s_i, \quad |i - k| > 1, \tag{2.12}
\]

\[
x_i x_k = x_k x_i, \quad 1 \leq i, k \leq n, \tag{2.13}
\]

\[
x_i x_k = x_k x_i, \quad k \neq i, i + 1, \tag{2.14}
\]

\[
x_{i+1} = s_i x_i s_i + s_i, \quad 1 \leq i < n. \tag{2.15}
\]

Note that one can also replace the last relation above with the following:

\[
x_{i+1} s_i = s_i x_i + 1, \quad 1 \leq i < n. \tag{2.16}
\]

Then the degenerate type \( A \) cyclotomic Hecke algebra \( H_n^\Lambda \) introduced in Section 1 is isomorphic to the quotient of \( H_n \) by the two-sided ideal generated by

\[
(x_1 - \kappa_1 \cdot 1_K)(x_1 - \kappa_2 \cdot 1_K) \cdots (x_1 - \kappa_\ell \cdot 1_K). \tag{2.17}
\]

Under this isomorphism, each \( L_i \) is identified with the image of \( x_i \) in \( H_n^\Lambda \) for \( 1 \leq i \leq n \). For each \( 1 \leq j < n \), we still use \( s_j \) to denote the image of \( s_j \) in \( H_n^\Lambda \). Inside both \( H_n \) and \( H_n^\Lambda \), the subalgebra generated by \( s_1, \ldots, s_{n-1} \) is isomorphic to the symmetric group algebra associated to the symmetric group \( S_n \) on \( n \) letters (with \( s_r \) being identified with the permutation \((r, r+1)\) for each \( r \)).
Let \( \{ t_k | 1 \leq k \leq n \} \) be a set of \( n \) algebraically independent indeterminates over \( K \). Let \( \mathcal{P}_n := K[t_1^{\pm 1}, \cdots, t_n^{\pm 1}] \) and \( P_n := K[t_1, \cdots, t_n] \). Clearly there is a natural left action of \( \mathfrak{S}_n \) on \( \mathcal{P}_n \), \( T_n \) and \( P_n \) respectively.

For any \( f \in \mathcal{P}_n \), \( g \in P_n \), \( 1 \leq r < n \) and \( 1 \leq k \leq n \), we define

\[
\begin{align*}
X_k^{\pm 1} * f &= t_k^{\pm 1} f, \\
T_r * f &= (t_{r+1} - q t_r) s_r(f) - f + q f,
\end{align*}
\]

and

\[
\begin{align*}
x_k * g &= t_k g, \\
s_r * g &= -s_r(g) - g + s_r(g),
\end{align*}
\]

The following results are well known, see [22].

2.20. **Lemma.** Let \( n \in \mathbb{S}_n \) and \( 1 \leq r \leq n \). Then

\[
T_w X_r = X_{w(r)} T_w + \sum_{u > w} g_u(X_1, \cdots, X_n) T_u,
\]

where \( g_u(X_1, \cdots, X_n) \in K[X_1, \cdots, X_n] \) for each \( u \). A similar statement also holds if we replace \( T_w, X_r \) by \( w, x_r \) respectively.

**Proof.** This follows from an induction on \( \ell(w) \). \( \square \)

2.21. **Lemma.** The above rules extend uniquely to a faithful representation \( \rho_q \) of \( \mathcal{H}_n(q) \) on \( \mathcal{P}_n \) as well as a faithful representation \( \rho_1 \) of \( H_n \) on \( P_n \).

2.22. **Lemma.** The elements in the following set

\[
\{ X_1^{a_1} \cdots X_n^{a_n} T_w \mid w \in \mathbb{S}_n, a_1, \cdots, a_n \in \mathbb{Z} \}
\]

are \( K \)-linearly independent and form a basis of \( \mathcal{H}_n(q) \). Similarly, the elements in the following set

\[
\{ x_1^{a_1} \cdots x_n^{a_n} w \mid w \in \mathbb{S}_n, a_1, \cdots, a_n \in \mathbb{N} \}
\]

are \( K \)-linearly independent and form a basis of \( H_n \).

Let \( * \) be the \( K \)-algebra anti-isomorphism of \( \mathcal{H}_n(q) \) which is defined on generators by \( T_i^* := T_i, \ X_k^* := X_k \) for \( 1 \leq i < n, 1 \leq k \leq n \). By abuse of notations, we also use \( * \) to denote the \( K \)-algebra anti-isomorphism of \( H_n \) which is defined on generators by \( s_i^* := s_i, \ x_k^* := x_k \) for \( 1 \leq i < n, 1 \leq k \leq n \). Applying the anti-isomorphism \( * \), we see that the elements in the following set \( \{ T_w X_1^{a_1} \cdots X_n^{a_n} \mid w \in \mathbb{S}_n, a_1, \cdots, a_n \in \mathbb{Z} \} \) are \( K \)-linearly independent and form another basis of \( \mathcal{H}_n(q) \); and the elements in the following set \( \{ w x_1^{a_1} \cdots x_n^{a_n} \mid w \in \mathbb{S}_n, a_1, \cdots, a_n \in \mathbb{N} \} \) are \( K \)-linearly independent and form another basis of \( H_n \).

Note that the subalgebra of \( \mathcal{H}_n(q) \) generated by \( X_1^{\pm 1}, \cdots, X_n^{\pm 1} \) is canonically isomorphic to the Laurent polynomial \( K \)-algebra \( \mathcal{P}_n \), while the subalgebra of \( H_n \) generated by \( x_1, \cdots, x_n \) is canonically isomorphic to the polynomial \( K \)-algebra \( P_n \).

2.23. **Lemma** (Bernstein). The center of \( \mathcal{H}_n(q) \) is equal to the set of symmetric Laurent polynomials in \( X_1^{\pm 1}, \cdots, X_n^{\pm 1} \), while the center of \( H_n \) is equal to the set of symmetric polynomials in \( x_1, \cdots, x_n \).

Let \( \mathcal{H}_n^+(q) \) be the \( K \)-subalgebra of \( \mathcal{H}_n(q) \) generated by \( T_1, \cdots, T_n-1, X_1, \cdots, X_n \). Then the elements in the following set

\[
\{ X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n} T_w \mid w \in \mathbb{S}_n, a_1, \cdots, a_n \in \mathbb{N} \}
\]

form a \( K \)-basis of \( \mathcal{H}_n^+(q) \).
2.25. **Lemma.** The \( K \)-algebra \( \mathcal{H}^+_n(q) \) is isomorphic to the abstract \( K \)-algebra defined by generators \( T_1, \ldots, T_{n-1}, X_1, \ldots, X_n \) and relations (2.1), (2.2), (2.3), (2.6), (2.7) together with the relations \( X_i X_k = X_k X_i \), \( 1 \leq i, k \leq n \).

**Proof.** Let \( \mathcal{H}^+ \) be the abstract \( K \)-algebra which is defined by generators \( T_1, \ldots, T_{n-1}, X_1, \ldots, X_n \) and relations (2.1), (2.2), (2.3), (2.6), (2.7) together with the relations \( X_i X_k = X_k X_i \), \( 1 \leq i, k \leq n \). To prove that \( \mathcal{H}^+ \cong \mathcal{H}^+_n(q) \), it suffices to show that the elements in \( \mathcal{H}^+ \) which are of the form (2.24) form a \( K \)-basis of \( \mathcal{H}^+ \).

It is easy to see that the elements in \( \mathcal{H}^+ \) which are of the form (2.24) generates \( \mathcal{H}^+ \) as a \( K \)-linear space. Moreover, the following formulae

\[
X_k \cdot f := t_k f, \quad T_r \cdot f := (t_{r+1} - q t_r) \frac{s_r(f) - f}{t_{r+1} - t_r} + q f,
\]

also defines a representation \( \rho^+_q \) of \( \mathcal{H}^+ \) on \( P_n \). Using this representation \( \rho^+_q \) it is easy to check that the elements in \( \mathcal{H}^+ \) which are of the form (2.24) are \( K \)-linearly independent and hence form a \( K \)-basis of \( \mathcal{H}^+ \), as required. It turns out that \( \rho^+_q \) is a faithful polynomial representation of \( \mathcal{H}^+ \cong \mathcal{H}^+_n(q) \). \( \square \)

2.26. **Definition.** Suppose that \( \beta \in \mathbb{Q}_n^+ \). Define the quiver Hecke algebra \( \mathcal{R}_\beta \) to be the unital associative \( K \)-algebra with generators

\[
\{ \psi_1, \ldots, \psi_{n-1} \} \cup \{ y_1, \ldots, y_n \} \cup \{ e(i) \mid i \in I^\beta \}
\]

and relations

\[
e(i) e(j) = \delta_{ij} e(i), \quad \sum_{i \in I^\beta} e(i) = 1,
\]

\[
y_r e(i) = e(i) y_r, \quad \psi_r e(i) = e(s_r i) \psi_r, \quad y_r y_s = y_s y_r,
\]

\[
\psi_r y_{r+1} e(i) = (y_r \psi_r + \delta_{r, r+1}) e(i), \quad y_{r+1} \psi_r e(i) = (\psi_r y_r + \delta_{r, r+1}) e(i), \quad \text{if } s \neq r, r + 1,
\]

\[
\psi_r y_s = y_s \psi_r, \quad \text{if } |r - s| > 1,
\]

\[
\psi_r^2 e(i) = \begin{cases} 0, & \text{if } i_r = i_{r+1}, \\ (y_{r+1} - y_r) e(i), & \text{if } i_r \rightarrow i_{r+1}, \\ (y_r - y_{r+1}) e(i), & \text{if } i_r \leftarrow i_{r+1}, \\ (y_{r+1} - y_r)(y_r - y_{r+1}) e(i), & \text{if } i_r \Rightarrow i_{r+1}, \\ e(i), & \text{otherwise}, \end{cases}
\]

\[
\psi_r \psi_{r+1} e(i) = \begin{cases} (\psi_r \psi_r \psi_{r+1} + 1) e(i), & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \\ (\psi_r \psi_{r+1} \psi_{r+1} - 1) e(i), & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\ (\psi_r \psi_{r+1} \psi_{r+1} + y_r - 2 y_{r+1} + y_{r+2}) e(i), & \text{if } i_r = i_{r+2} \Rightarrow i_{r+1}, \\ \psi_{r+1} \psi_r \psi_{r+1} e(i), & \text{otherwise}. \end{cases}
\]

for \( i, j \in I^\beta \) and all admissible \( r \) and \( s \).

Let \( \mathcal{R}^\beta_n \) be the quotient of \( \mathcal{R}_\beta \) by the two-sided ideal generated by

\[
y_1^{(\Lambda, \alpha_{i_r}^\nu)} e(i), \quad i \in I^\beta.
\]

The algebra \( \mathcal{R}^\beta_n \) is called the type \( A \) **cyclotomic quiver Hecke algebra** associated to \( \beta \) and \( \Lambda \).

Let \( i \in I^n \) and \( r \) be an integer with \( 1 \leq r < n \). Recall the definition of \( P_r(i) \) given in [4, (3.22), (4.27)]: if \( i_r = i_{r+1} \) then \( P_r(i) = 1 \); if \( i_r \neq i_{r+1} \) and in the
non-degenerate setting, then
\begin{equation}
(2.28) \quad P_r(i) := \frac{1 - q}{1 - q^{r+1}} \left\{ 1 + \frac{y_r - y_{r+1}}{1 - q^{r+1-i_r}} + \sum_{k \geq 1} \frac{y_r - y_{r+1}}{1 - q^{r+1-i_r}} \left( q^{r+1-i_r} - q^{r-i_r} \right)^k \right\};
\end{equation}
while if \( i_r \neq i_{r+1} \) and in the degenerate setting, then
\begin{equation}
(2.29) \quad P_r(i) := \frac{1}{i_r - i_{r+1}} \left\{ 1 + \sum_{k \geq 1} \left( \frac{y_r - y_{r+1}}{i_{r+1} - i_r} \right)^k \right\}.
\end{equation}
Since each \( y_k \) is nilpotent (cf. [4, Lemma 2.1]), it follows that the above sums are actually both finite sums. The Brundan–Kleshchev’s isomorphism in Theorem 1.5 between \( \mathcal{H}_\beta^A \) and \( \mathcal{S}_\beta^A \) depends on the choice of certain polynomials \( Q_r(i) \) for \( 1 \leq r < n \), see [4, (3.27–3.29),(4.33–4.35)]. Instead of following Brundan–Kleshchev’s choice given in [4, (3.30), (4.36)], we make a different choice for our purpose. In the degenerate setting, we set
\begin{equation}
(2.30) \quad Q_r(i) := \begin{cases} 1 + y_{r+1} - y_r, & \text{if } i_{r+1} = i_r; \\ 1 + \sum_{k \geq 1}(y_{r+1} - y_r)^k, & \text{if } i_r = i_{r+1} + 1; \\ P_r(i) - 1, & \text{if } i_r \neq i_{r+1}, i_{r+1} + 1. 
\end{cases}
\end{equation}
In the non-degenerate setting, following Stroppel–Webster [29, (27)], we set
\begin{equation}
(2.31) \quad Q_r(i) := \frac{1 - q + q y_{r+1} - y_r}{1 - q} \left( 1 + \sum_{k \geq 1} \left( \frac{y_{r+1} - 2 y_r}{1 - q} \right)^k \right), \quad \text{if } i_{r+1} = i_r; \\
\frac{1 - q}{1 - q} \left( 1 + \sum_{k \geq 1} \left( \frac{y_{r+1} - 2 y_r}{1 - q} \right)^k \right), \quad \text{if } i_r = i_{r+1} + 1; \\
\frac{P_r(i) - 1}{1}, \quad \text{if } i_r \neq i_{r+1}, i_{r+1} + 1.
\end{equation}
Note that in both (2.30) and (2.31),
\begin{equation}
(2.32) \quad Q_r(i) = \frac{P_r(i) - 1}{y_r - y_{r+1}} \text{ whenever } i_r = i_{r+1} + 1.
\end{equation}
Since \( y_1, \ldots, y_n \) are nilpotent elements in \( \mathcal{H}_\beta^A \), the sums in (2.28), (2.29), (2.30) and (2.31) are always finite sums. One can verify that the definitions in both (2.30) and (2.31) satisfy the requirement in [4, (3.27–3.29),(4.33–4.35)]. Thus they can be used to define Brundan–Kleshchev’s isomorphism in Theorem 1.5.

Henceforth, we shall use these particular choices of Brundan–Kleshchev’s isomorphisms to identify \( \mathcal{H}_\beta^A(q) \) and \( \mathcal{S}_\beta^A \) in the non-degenerate setting and to identify \( H_\beta^A \) and \( \mathcal{S}_\beta^A \) in the degenerate setting.

3. The modified forms of affine Hecke algebras and their generalized Ore localization

The purpose of this section is to introduce the modified forms for both the non-degenerate and the degenerate affine Hecke algebras of type A. Many classical results (including faithful polynomial representations, standard bases and description of the centers) for the original affine Hecke algebras are generalized to these modified affine Hecke algebras. The definitions of these modified forms make use of inverse limit of cyclotomic Hecke algebras and certain generalized Ore localizations with respect to some multiplicative closed subsets (cf. Appendix A).

A **partition** of an integer \( m \) is a weakly decreasing sequence \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \) of non-negative integers such that \( |\lambda| = \sum \lambda_i = m \). A **multipartition**, or \( \ell \)-partition, of \( n \) is an ordered \( \ell \)-tuple \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \) of partitions such that
\[ |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| = n. \] In this case, we write \( \lambda \vdash n \). The **diagram** of \( \lambda \) is the set
\[ [\lambda] = \{ (k, r, c) \mid r \geq 1, 1 \leq c \leq \lambda_c^{(k)} \text{ and } 1 \leq k \leq \ell \}. \]

A **standard \( \lambda \)-tableau** is a map \( t : [\lambda] \to \{1, 2, \ldots, \ell\} \) such that for \( s = 1, \ldots, \ell \) the entries in each row of \( t^{(s)} \) increase from left to right and the entries in each column of \( t^{(s)} \) increase from top to bottom. Let \( \text{Std}(\lambda) \) be the set of standard \( \lambda \)-tableaux.

Let \( \lambda \) be a multipartition of \( n \). For any \( 1 \leq k \leq n \) and \( t \in \text{Std}(\lambda) \), we define
\[ \text{res}_t(k) := \kappa_t + c - r \in \mathbb{Z}/e\mathbb{Z}, \]
whenever the node occupied by the integer \( k \) in \( t \) is \((l, r, c) \in [\lambda] \). For any \( t \in \text{Std}(\lambda) \), we define the \( n \)-tuple \( (\text{res}_t(1), \ldots, \text{res}_t(n)) \in \mathbb{Z}^n \) to be the residue sequence of \( t \).

Let \( \mathcal{H}_n \in \{ \mathcal{H}_n(q), \mathcal{H}_n^\Lambda \} \) and \( \mathcal{H}_n^\Lambda \in \{ \mathcal{H}_n^\Lambda(q), \mathcal{H}_n^\Lambda \} \), where \( \Lambda = \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_\ell} \in P^+ \). For each \( i \in I \), we define
\[ q_i := \begin{cases} q_i^+, & \text{if } q \neq 1; \\ q_i, & \text{if } q = 1. \end{cases} \]

Let \( i = (i_1, \ldots, i_n) \in \mathbb{Z}^n \) be an arbitrary residue sequence. Following [18, §3.1], for any \( 1 \leq r \leq n \), \( j \in I \) with \( j \neq i_r \), we choose \( N > \dim \mathcal{H}_n^\Lambda = \ell^n n! \) (where \( \ell \) is the level of \( \Lambda \)), and define
\[ L_{i_r, j} := \begin{cases} 1 - \left( \frac{q_i^+ - L_{i_r, j}^+}{q_i^+ - q_j^+} \right)^N, & \text{if } q \neq 1; \\ 1 - \left( \frac{L_{i_r, j}^-}{q_j - q_j^+} \right)^N, & \text{if } q = 1. \end{cases} \]

We define
\[ I_0(\Lambda, n) := \left\{ i \in I \mid i = \text{res}_t(k) \text{ for some } 1 \leq k \leq n, t \in \text{Std}(\lambda) \right\} \]
and \( \Lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \vdash n \).

For any finite subset \( J \subseteq I \) such that \( J \supseteq I_0(\Lambda, n) \), we define
\[ L_\Lambda(i) := \prod_{i_r, j \in J} L_{i_r, j} \in \mathcal{H}_n^\Lambda, \quad X_{\Lambda, i} := \prod_{i_r, j \notin J} X_{i_r, j} \in \mathcal{H}_n. \]

The point of the above definition lies in that \( J \) is a finite set so that the definition of \( L_\Lambda(i) \) makes sense.

**3.1. Lemma.** With the notations as above, we have that \( e(i) = \prod_{r=1}^n L_r(i)^N \). In particular, it does not depend on the choice of \( J \) and \( N \). Moreover, if we set
\[ E_{\Lambda, i} := \prod_{r=1}^n X_{\Lambda, i} \in \mathcal{H}_n, \]
then \( \pi_\Lambda(E_{\Lambda, i}) = e(i) \).

**Proof.** The proof is essentially the same as the proof of [18, Corollary 3.9]. Note that Ge Li [18, Corollary 3.9] use the set \( I \) instead of the finite subset \( J \) in the definition of \( L_\Lambda(i) \) and he consider only the case when \( e > 0 \). However, by [8, Lemma 4.1, Theorem 5.8], we know that \( e(i) \neq 0 \) in \( \mathcal{H}_n^\Lambda \) if and only if \( i = 1^1 \) for some \( t \in \text{Std}(\lambda) \) and \( \Lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \vdash n \). Therefore, the same argument used in the proof of [18, Corollary 3.9] applies equally well to the proof of the current lemma. \( \Box \)

For any \( \Lambda, \Lambda' \in P^+ \), we define \( \Lambda > \Lambda' \) if \( \Lambda - \Lambda' \in P^+ \). Then \( (P^+, >) \) becomes a directed poset. If \( \Lambda > \Lambda' \) in \( P^+ \), then there is a canonical surjective homomorphism
\[ \pi_{\Lambda, \Lambda'} : \mathcal{H}_n^\Lambda \twoheadrightarrow \mathcal{H}_n^{\Lambda'}, \]
such that \( \pi_{\Lambda'} = \pi_{\Lambda, \Lambda'} \circ \pi_{\Lambda} \), where
\[ \pi_{\Lambda} : \mathcal{H}_n \to \mathcal{H}_n^\Lambda \]
To avoid confusion, we put a tilde on the notation of every generator of the algebra $\sum$ where $b$.  

Lemma. Without loss of generality, we can assume that $z$ is similar. Without loss of generality, we can assume that $\tilde{z} > \ell$, where $\tilde{z}$ is the level of $\tilde{\Lambda}$. To avoid confusion, we put a tilde on the notation of every generator of the algebra $\mathcal{H}_{\tilde{\Lambda}}$. We take an integer $\tilde{N}$ such that $\tilde{N} > \ell n!$. Then $\tilde{N} > \ell^* n!$. Since $\pi_{\tilde{\Lambda},\tilde{\Lambda}}(L_\ell(I)\tilde{N}) = L_\ell(I)\tilde{N}$, it follows that $\pi_{\tilde{\Lambda},\tilde{\Lambda}}$ sends the idempotent $\tilde{e}(I)$ of $\mathcal{H}_{\tilde{\Lambda}}^+$ to the idempotent $e(I)$ of $\mathcal{H}_{\Lambda}^+$. As a result, we can deduce that for each $i \in I^n$, there exists an idempotent $\tilde{e}(I) \in \mathcal{H}_{\tilde{\Lambda}}^+$ such that the canonical image of $\tilde{e}(I)$ in $\mathcal{H}_{\Lambda}^+$ is equal to $e(I)$ for each $\Lambda \in P^+$.  

For any $i \in I^n$, it is clear that if $\Lambda \in P^+$ such that $\Lambda \geq \Lambda_i + \cdots + \Lambda_k$, then $i = i^t$ for some $t \in \text{Std}(\Lambda)$ and $\Lambda = (\lambda^{(1)}, \cdots, \lambda^{(t)}) \vdash n$, where $\ell$ is the level of $\Lambda$. In particular, $e(I) \neq 0$ in $\mathcal{H}_{\Lambda}^+$.  

3.4. Lemma. Let $i \in I^n$ and $z \in \mathcal{H}_n$. Then $e(I) \neq 0$. Furthermore, if $e(I)z = 0$ or $z\tilde{e}(I) = 0$ in $\mathcal{H}_{\Lambda}^+$, then $z = 0$ in $\mathcal{H}_n$. 

Proof. The first part of the lemma follows from the discussion in the paragraph above this lemma. It remains to prove the second part of this lemma. Without loss of generality we assume that $\tilde{e}(I)z = 0$.  

We only prove the statement for the non-degenerate case as the degenerate case is similar. Without loss of generality, we can assume that $z \in \pi(\mathcal{H}_{\Lambda}^+(q))$. We fix an $Z \in \mathcal{H}_{\Lambda}^+(q)$ such that $\pi(Z) = z$. Suppose that $z \neq 0$. Then $Z \neq 0$. We choose $\ell > n$ and $\Lambda := m_1 \Lambda_{a_1} + \cdots + m_k \Lambda_{a_k} \in P^+$, where $m_1, \cdots, m_k$ is $Z^{n!}$ such that $\sum_{i=1}^k m_i = \ell$, $a_1, \cdots, a_k$ are distinct elements in $I$ such that $a_1 = i_1$, and $\Lambda > \Lambda_{i_1} + \cdots + \Lambda_{a_k}$. Furthermore, since $Z$ is prefixed, we can assume that each $m_i$ is sufficiently large such that 

$$Z \in K\text{-Span}\left\{f(X_1-q^{a_1}, \cdots, X_n-q^{a_k})T_w \mid w \in S_n, f(t_1, \cdots, t_n) \in K[t_1, \cdots, t_n] \right\} \quad \text{with } \deg f < m_i - n! \forall 1 \leq i \leq k. \right\}$$

By assumption, we have that $e(I)z = 0$. As a result, 

$$E_{n,\Lambda}(I)Z \in (X_1 - q^{a_1})^{m_1} \cdots (X_1 - q^{a_k})^{m_k}.$$ 

By construction, we have that 

$$E_{n,\Lambda}(I) = \pm \prod_{r=1}^n \prod_{i_r \neq j \in I_0(\Lambda,n)} (X_r - q^{b_{r,j}})$$

where $b_{r,j} \in \mathbb{N}$ for each pair of $r, j$. 

Using Lemma 2.20 we see that for any \( y \in \langle (X_1 - q^{a_1})^{m_1} \cdots (X_1 - q^{a_k})^{m_k} \rangle \), \( y \) must live inside the \( K \)-linear subspace spanned by some elements of the form
\[
(\chi_1, \ldots, \chi_n, g_w (X_1, \ldots, X_n))_i T_w,
\]
where \( w \in S_n, g_w (X_1, \ldots, X_n) \in K[X_1, \ldots, X_n] \) such that \( b_{s,j} \geq m_j - n \) for any \( 1 \leq i \leq k \) and \( 1 \leq j \leq k \). However, this is impossible by (3.5), (3.7), Lemma 2.20 and Lemma 2.22. This completes the proof of the lemma when \( \epsilon (i)z = 0 \).

Finally, if \( z \epsilon (i) = 0 \), the lemma follows from a similar argument. \( \square \)

3.9. **Definition.** Let \( \beta \in Q^+_n \). In the non-degenerate setting, we define \( \mathcal{H}_\beta (q) \) to be the \( K \)-subalgebra of \( \lim_{\lambda \to \lambda} \mathcal{H}_\lambda(q) \) generated by the following elements:
\[
\mathcal{T}_k \epsilon (i), \quad x^{\pm 1} \epsilon (i), \quad \epsilon (i), \quad i \in I^\beta, \quad 1 \leq k < n, \quad 1 \leq r \leq n.
\]

In the degenerate setting, we define \( \mathcal{H}_\beta \) to be the \( K \)-subalgebra of \( \lim_{\lambda \to \lambda} \mathcal{H}_\lambda^\Lambda \) generated by the following elements:
\[
\mathcal{T}_k \epsilon (i), \quad x^{\pm 1} \epsilon (i), \quad \epsilon (i), \quad i \in I^\beta, \quad 1 \leq k < n, \quad 1 \leq r \leq n.
\]

Let \( \mathcal{H}_\beta \in \{ \mathcal{H}_\beta (q), \mathcal{H}_\beta \} \). We set \( \tilde{\epsilon} (\beta) := \sum_{i \in I^\beta} \tilde{\epsilon} (i) \in \mathcal{H}_\beta \). Then \( \tilde{\epsilon} (\beta) \) is the identity element of \( \mathcal{H}_\beta \).

3.10. **Lemma** (Non-degenerate cases). In the non-degenerate case, the following relations hold:
\[
(3.11) \quad \hat{X}_k^{\pm 1} \epsilon (i) = \epsilon (i) \hat{X}_k^{\pm 1}, \quad \epsilon (i) \epsilon (j) = \delta_{ij} \epsilon (i),
\]
\[
\epsilon (i) \mathcal{T}_r (\hat{X}_{r+1} - \hat{X}_r) \epsilon (i) = (q - 1) \epsilon (i) \hat{X}_{r+1} \epsilon (i), \quad \epsilon (i) \mathcal{T}_r \hat{X}_r \epsilon (i) = \epsilon (i) \hat{X}_r \epsilon (i), \quad \epsilon (i) \mathcal{T}_r \hat{X}_{r-1} \epsilon (i) = \epsilon (i) \hat{X}_{r-1} \epsilon (i), \quad \text{if } i \in I^\beta, \text{ } i_r \neq i_{r+1}, \quad \text{if } i \in I^\beta, \text{ } i \neq j, \text{ } f \in K[\hat{X}_1^{\pm 1}, \ldots, \hat{X}_n^{\pm 1}],
\]
\[
\epsilon (i) f \epsilon (j) = 0, \quad \text{if } i, j \in I^\beta, \text{ } i \neq j, \text{ } f \in K[\hat{X}_1^{\pm 1}, \ldots, \hat{X}_n^{\pm 1}],
\]
\[
\epsilon (i) \mathcal{L}_r \epsilon (j) = 0, \quad \text{if } i, j \in I^\beta, \text{ } i \notin \{j, s, j\},
\]
\[
\epsilon (i) (\mathcal{T}_r - q) (\mathcal{T}_r + 1) \epsilon (j) = 0, \quad \epsilon (i) \mathcal{T}_r \mathcal{T}_r + 1 \mathcal{T}_r \epsilon (j) = \epsilon (i) \mathcal{T}_r \mathcal{T}_r + 1 \mathcal{T}_r \epsilon (j), \quad \epsilon (i) X_k^{\pm 1} X_k^{\pm 1} \epsilon (j) = \epsilon (i) X_k^{\pm 1} X_k^{\pm 1} \epsilon (j), \quad \epsilon (i) \mathcal{T}_r \hat{X}_r \epsilon (j) = \epsilon (i) \mathcal{T}_r \hat{X}_r \epsilon (j), \quad \epsilon (i) \mathcal{T}_r \hat{X}_{r+1} \epsilon (j) = \epsilon (i) \mathcal{T}_r \hat{X}_{r+1} \epsilon (j), \quad \text{if } i, j \in I^\beta, \text{ } |a - k| > 1 \text{ and } i, j \in I^\beta, \text{ } a, b < n, 1 \leq i < n - 1.
\]
\[
(3.17) \quad \epsilon (i) \mathcal{T}_r \hat{X}_k \epsilon (j) = \epsilon (i) \hat{X}_k \mathcal{T}_r \epsilon (j), \quad \text{if } i \neq b, b + 1 \text{ and } i, j \in I^\beta,
\]
where \( 1 \leq k \leq n, \quad 1 \leq r, a, b < n, 1 \leq i < n - 1. \)

**Proof.** The relations (3.15), (3.16) and (3.17) follow directly from (2.1)–(2.7), (2.8). The remaining relations follows from (2.28), (2.31), (2.32) and Theorem 1.5. For example, assume that \( i_r \neq i_{r+1} \) and in the non-degenerate setting, in order to show that \( \epsilon (i) \mathcal{T}_r (\hat{X}_{r+1} - \hat{X}_r) \epsilon (i) = (q - 1) \epsilon (i) \hat{X}_{r+1} \epsilon (i) \), it suffices (by (3.3)) to show that for any \( \lambda \in P^+, \epsilon (i) \mathcal{T}_r (L_{r+1} - L_r) \epsilon (i) = (q - 1) \epsilon (i) L_{r+1} \epsilon (i) \).
In fact, we have that
\[
e(i)T_r(L_{r+1} - L_r) e(i) = e(i)\left(\psi_r Q_r(i) e(i) - P_r(i) e(i)\right)(L_{r+1} - L_r) = -e(i)P_r(i) e(i)(L_{r+1} - L_r)
\]
\[
= \frac{1 - q}{1 - q^{r-r+1}} \left\{ 1 + \frac{y_r - y_{r+1}}{1 - q^{r-r+1}} + \sum_{k \geq 1} \frac{y_r - y_{r+1}}{1 - q^{r-r+1}} \left( \frac{q^{r+1} y_{r+1} - q^{r} y_r}{q^{r+1} - q^{r}} \right)^k \right\}
\]
\[
\times (L_{r+1} - L_r) e(i)
\]
\[
= \frac{(q - 1)q^{r+1}(1 - y_{r+1})}{q^{r+1}(1 - y_r) - q^{r}(1 - y_r)} (L_{r+1} - L_r) e(i) = (q - 1)L_{r+1} e(i),
\]
as required. The other equalities can be verified in a similar way. \hfill \square

3.18. Lemma (Degenerate cases). In the degenerate case, the following relations hold:

\[
\hat{x}_k e(i) = \hat{e}(i) \hat{x}_k, \quad \hat{e}(i) \hat{e}(j) = \delta_k \hat{e}(i),
\]

\[
\begin{cases}
\hat{e}(i) \hat{s}_r(\hat{x}_{r+1} - \hat{x}_r) e(i) = \hat{e}(i), \\
\hat{e}(i) \hat{s}_r e(i) = \hat{e}(i) \hat{s}_r \hat{e}(i), \\
\hat{e}(i) \hat{s}_r \hat{x}_{r+1} e(i) = \hat{e}(i) \hat{s}_r x_{r+1} \hat{e}(i),
\end{cases}
\]

if \( i \in I^\beta, r \neq r+1, \)

\[
\hat{e}(i) f \hat{e}(j) = 0, \quad \text{if } i, j \in I^\beta, i \neq j, f \in K[\hat{x}_1, \ldots, \hat{x}_n],
\]

\[
\hat{e}(i) \hat{s}_r \hat{e}(j) = 0, \quad \text{if } i, j \in I^\beta, i \notin \{j, \beta, j\},
\]

\[
\begin{cases}
\hat{e}(i) \hat{s}_{i+1} \hat{e}(j) = 0, \\
\hat{e}(i) \hat{s}_r \hat{s}_{i+1} \hat{s}_r \hat{e}(j) = \hat{e}(i) \hat{s}_r \hat{s}_{i+1} \hat{s}_r \hat{e}(j), \\
\hat{e}(i) \hat{x}_r \hat{x}_{r+1} \hat{e}(j) = \hat{e}(i) \hat{x}_r \hat{x}_{r+1} \hat{e}(j),
\end{cases}
\]

if \( i, j \in I^\beta, \)

\[
\hat{e}(i) \hat{s}_a \hat{s}_b \hat{e}(j) = \hat{e}(i) \hat{s}_a \hat{s}_b \hat{e}(j), \quad \text{if } |a - k| > 1 \text{ and } i, j \in I^\beta,
\]

\[
\hat{e}(i) \hat{s}_b \hat{x}_k \hat{e}(j) = \hat{e}(i) \hat{x}_k \hat{s}_b \hat{e}(j), \quad \text{if } k \neq b, b+1 \text{ and } i, j \in I^\beta,
\]

where \( 1 \leq k \leq n, 1 \leq r, a, b < n, 1 \leq i < n - 1. \)

Proof. The relations (3.23), (3.24) and (3.25) follow directly from (2.10)–(2.15), (2.16). The remaining relations follows from (2.29), (3.30), (3.32) and Theorem 1.5. \hfill \square

For each \( \Lambda \in P^+, \) if \( r \neq r+1, \) then in the non-degenerate setting,
\[
(L_r - L_s) e(i) = q^{r} (1 - y_r) - q^{s} (1 - y_s) = \left( (q^{r} y_r - q^{s} y_s) + (q^{r} y_r - q^{s} y_s) \right) e(i),
\]
which is invertible in \( e(i) \mathcal{H}_n^\Lambda(q) e(i) \) because \( (q^{r} y_r - q^{s} y_s) e(i) \) is nilpotent. Moreover, its inverse can be expressed as a power series on \( (q^{r} y_r - q^{s} y_s) e(i). \) It follows that \( (\hat{X}_r - \hat{X}_s) e(i) \) is actually an invertible element in \( \hat{e}(i) \left( \lim_{\Lambda} \mathcal{H}_n^\Lambda(q) \right) \hat{e}(i) \) (regarded as an algebra with the identity element \( \hat{e}(i) \)). We denote its inverse by \( (\hat{X}_r - \hat{X}_s)^{-1} \hat{e}(i). \) Similarly, the element \( (\hat{x}_r - \hat{x}_s) \hat{e}(i) \) is an invertible element in \( \lim_{\Lambda} \hat{H}_n^\Lambda. \) We denote its inverse by \( (\hat{x}_r - \hat{x}_s)^{-1} \hat{e}(i). \)
3.26. **Definition.** In the non-degenerate setting, we define the **modified non-degenerate affine Hecke algebra** \( \tilde{H}_\beta(q) \) of type \( A \) to be the subalgebra of \( \left( \bigcup_{\lambda} \mathcal{H}^{\lambda}_\beta(q) \right) \) generated by \( \mathcal{H}_\beta(q) \) and all the elements of the form

\[
(\tilde{X}_r - \tilde{X}_s)^{-1}\tilde{e}(i), \quad \text{where} \quad 1 \leq r < s, i \in I^\beta \text{ and } i_r \neq i_s.
\]

In the degenerate setting, we define the **modified degenerate affine Hecke algebra** \( \hat{H}_\beta(q) \) of type \( A \) to be the subalgebra of \( \left( \bigcup_{\lambda} H^{\lambda}_\beta \right) \) generated by \( H_\beta \) and all the elements of the form

\[
(\hat{x}_r - \hat{x}_s)^{-1}\hat{e}(i), \quad \text{where} \quad 1 \leq r < s, i \in I^\beta \text{ and } i_r \neq i_s.
\]

3.27. **Remark.**

1) Replacing \( \mathcal{H}_n(q) \) by \( \mathcal{H}_n^+(q) \) in the definition of \( \tilde{H}_\beta(q) \) and \( X_1^{\pm 1}, \ldots, X_n^{\pm 1} \) by \( X_1, \ldots, X_n \), we can get an algebra which will be denoted by \( \tilde{H}_\beta(q) \). It is clear that \( \tilde{H}_\beta(q) \) is a \( K \)-subalgebra of \( \mathcal{H}_\beta(q) \).

2) Inside the algebra \( \mathcal{H}_\beta(q) \), we can rewrite the first relation in (3.12) as

\[
\hat{e}(i)\hat{T}_r\hat{e}(i) = (q - 1)\hat{e}(i)\hat{X}_{r+1}(\hat{X}_{r+1} - \hat{X}_r)^{-1}\hat{e}(i), \quad \text{if} \quad i_r \neq i_{r+1}.
\]

Similarly, inside the algebra \( \hat{H}_\beta \), we can rewrite the first relation in (3.20) as

\[
\hat{e}(i)\hat{s}_r\hat{e}(i) = \hat{e}(i)(\hat{x}_{r+1} - \hat{x}_r)^{-1}\hat{e}(i), \quad \text{if} \quad i_r \neq i_{r+1}.
\]

Let \( \tilde{\mathcal{H}}_\beta \) be the \( K \)-algebra anti-isomorphism of \( \mathcal{H}_\beta \) which is uniquely determined by

\[
\hat{e}(i)^* := \hat{e}(i), \quad (\hat{e}(i)\hat{f}(j))^* := \hat{e}(i)\hat{f}^*\hat{e}(j), \quad \forall i, j \in I^\beta, f \in \mathcal{H}_n.
\]

\[
(\hat{X}_r - \hat{X}_s)^{-1}\hat{e}(i)^* = (\hat{X}_r - \hat{X}_s)^{-1}\hat{e}(i), \quad (\hat{x}_r - \hat{x}_s)^{-1}\hat{e}(i)^* = (\hat{x}_r - \hat{x}_s)^{-1}\hat{e}(i).
\]

3.30. **Definition.** Let \( i \in I^\beta \). If \( \tilde{\mathcal{H}}_\beta = \mathcal{H}_\beta(q) \), then we define

\[
\tilde{M}(i) := \left\{ \hat{e}(i) \right\} \bigcup \left\{ (\hat{X}_r - \hat{X}_s)\hat{e}(i) \left| 1 \leq r < s \leq n, i_r \neq i_s \right. \right\}.
\]

If \( \tilde{\mathcal{H}}_\beta = \tilde{H}_\beta \), then we define

\[
\tilde{M}(i) := \left\{ \hat{e}(i) \right\} \bigcup \left\{ (\hat{x}_r - \hat{x}_s)\hat{e}(i) \left| 1 \leq r < s \leq n, i_r \neq i_s \right. \right\}.
\]

Let \( \hat{\Sigma}(i) \) be the multiplicative closed subset generated by the elements in \( \tilde{M}(i) \).

3.31. **Lemma.** All the assumptions and conditions in Lemma A1 are satisfied if we take

\[
A = \mathcal{H}_\beta(q), \quad A_0 := K[\hat{X}_1^{\pm 1}, \ldots, \hat{X}_n^{\pm 1}], \quad \{e_i\}_{i=1}^m := \{\hat{e}(i)\} i \in I^\beta, \quad S_j = \{\hat{\Sigma}(i)\} i \in I^\beta.
\]

In particular, we can construct the generalized Ore localization of \( \mathcal{H}_\beta(q) \) with respect to \( (A_0, \{e_i\}_{i=1}^m, \{S_j\}_{j=1}^m) \). Moreover the resulting generalized Ore localization is canonically isomorphic to \( \mathcal{H}_\beta(q) \). A similar statement holds if we replace \( \mathcal{H}_\beta(q), \hat{X}_k^{\pm 1} \) and \( \tilde{H}_\beta(q) \) by \( \hat{H}_\beta, \hat{x}_k \) and \( \tilde{H}_\beta \) respectively.
Proof. We only consider the non-degenerate case as the degenerate case is similar. In view of Lemma A1, it suffices to verify the assumptions (O1) and (O2). Let \(1 \leq r < s \leq n\). Since \(\mathcal{H}_q^\Lambda (q) \subseteq \lim_{\Lambda} \mathcal{H}_n^\Lambda (q)\) and if \(i_r \neq i_s\) then \((\hat{X}_r - \hat{X}_s)\hat{e}(i)\) is invertible in \(\hat{e}(i) \left( \lim_{\Lambda} \mathcal{H}_n^\Lambda (q) \right) \hat{e}(i)\) (by the discussion in the paragraph above Definition 3.26), it follows that the assumption (O1) is satisfied.

It remains to verify the assumption (O2). Let \(1 \leq r < k \leq n\) and \(i \in I^\beta\) with \(i_r \neq i_k\). First, it is clear that \((\hat{X}_r - \hat{X}_k)\hat{e}(i)\) commutes with any \(\hat{X}_s\hat{e}(j)\). It remains to consider the following seven cases:

**Case 1.** \(i_r \neq i_{s+1}\). In this case, then by (3.12) and (3.17),
\[
(\hat{X}_r - \hat{X}_k)\hat{e}(i)\hat{e}(i)\hat{T}_s\hat{e}(i) = \hat{e}(i)\hat{T}_s\hat{e}(i)(\hat{X}_r - \hat{X}_k)\hat{e}(i).
\]

**Case 2.** \(i_r \neq i_{s+1}\) and \(k = r + 1\). In this case, by (3.12),
\[
(\hat{X}_r - \hat{X}_{r+1})\hat{e}(i)\hat{e}(i)\hat{T}_r\hat{e}(s,i) = \hat{e}(i)(\hat{X}_r - \hat{X}_{r+1})\hat{T}_r\hat{e}(s,i) = \hat{e}(i)\hat{T}_r(\hat{X}_r - \hat{X}_{r+1})\hat{e}(s,i)
\]
where \((\hat{X}_r - \hat{X}_{r+1})\hat{e}(s,i) \in \hat{M}(s,i)\) because \(i_r \neq i_k\).

**Case 3.** \(i_a \neq i_{a+1}\), either \(a = k\) or \(a = k - 1\) and \(k > r + 1\). In this case, by (3.12) and (3.17),
\[(\hat{X}_r - \hat{X}_k)\hat{e}(i)\hat{T}_a\hat{e}(s,i) = \hat{e}(i)(\hat{X}_r - \hat{X}_k)\hat{T}_a\hat{e}(s,i) = \hat{e}(i)\hat{T}_a(\hat{X}_r - \hat{X}_k)\hat{e}(s,i),
\]
where \((\hat{X}_r - \hat{X}_k)\hat{e}(s,i) \in \hat{M}(s,i)\) because \(i_r \neq i_k\).

**Case 4.** \(i_a \neq i_{a+1}\), either \(a = r - 1\) or \(a = r\) and \(r < k - 1\). In this case, by (3.12) and (3.17),
\[(\hat{X}_r - \hat{X}_k)\hat{e}(i)\hat{T}_a\hat{e}(s,i) = \hat{e}(i)(\hat{X}_r - \hat{X}_k)\hat{T}_a\hat{e}(s,i) = \hat{e}(i)\hat{T}_a(\hat{X}_r - \hat{X}_k)\hat{e}(s,i),
\]
where \((\hat{X}_r - \hat{X}_k)\hat{e}(s,i) \in \hat{M}(s,i)\) because \(i_r \neq i_k\).

**Case 5.** \(i_a = i_{a+1}\), either \(a = r - 1\) or \(a = r\) and \(r < k - 1\). In this case, by (3.12),
\[
(\hat{X}_r - \hat{X}_k)(\hat{X}_{a(r)} - \hat{X}_k)\hat{e}(i)\hat{T}_a\hat{e}(i) = \hat{e}(i)(\hat{X}_r - \hat{X}_k)(\hat{X}_{a(r)} - \hat{X}_k)\hat{T}_a\hat{e}(i)
\]
\[
= \hat{e}(i)\hat{T}_a(\hat{X}_r - \hat{X}_k)(\hat{X}_{a(r)} - \hat{X}_k)\hat{e}(i),
\]
where \((\hat{X}_{a(r)} - \hat{X}_k)(\hat{X}_r - \hat{X}_k)\hat{e}(i) \in \hat{S}(i)\) because \(i_{a+1} = i_a \neq i_k\).

**Case 6.** \(i_a = i_{a+1}\), either \(a = k\) or \(a = k - 1\) and \(k > r + 1\). In this case, by (3.12),
\[
(\hat{X}_r - \hat{X}_k)(\hat{X}_r - \hat{X}_{a(k)})\hat{e}(i)\hat{T}_a\hat{e}(i) = \hat{e}(i)(\hat{X}_r - \hat{X}_k)(\hat{X}_r - \hat{X}_{a(k)})\hat{T}_a\hat{e}(i)
\]
\[
= \hat{e}(i)\hat{T}_a(\hat{X}_r - \hat{X}_{a(k)})(\hat{X}_r - \hat{X}_k)\hat{e}(i),
\]
where \((\hat{X}_r - \hat{X}_{a(k)})(\hat{X}_r - \hat{X}_k)\hat{e}(i) \in \hat{S}(i)\) because \(i_{a+1} = i_a \neq i_r\).

**Case 7.** \(a \notin \{r, r - 1, k, k - 1\}\). In this case, then by (3.17),
\[
(\hat{X}_r - \hat{X}_k)\hat{e}(i)\hat{T}_a\hat{e}(s,i) = \hat{e}(i)\hat{T}_a\hat{e}(i)(\hat{X}_r - \hat{X}_k)\hat{e}(s,i).
\]
where \((\hat{X}_r - \hat{X}_k)\hat{e}(s,i) \in \hat{M}(i)\) because \(i_r \neq i_k\).

Finally, applying the anti-involution * of \(\mathcal{H}_q^\beta (q)\) to the above seven cases, we see that the assumption (O2) is satisfied. Therefore, this lemma is a direct consequence of Lemma A1. \qed
Let $\beta \in Q^+_n$. For each $i \in I^\beta$, let $\{ t_k(i) | 1 \leq k \leq n \}$ be a set of $n$ algebraically independent indeterminates over $K$. We define

$$\text{Pol}_{\beta} = \bigoplus_{i \in I^\beta} \text{Pol}_n(i),$$

where

$$\text{Pol}_n(i) := \begin{cases} K[t_1(i)^{\pm 1}, \ldots, t_n(i)^{\pm 1}], & \text{if } \mathcal{H}_\beta = \mathcal{H}_\beta(q); \\ K[t_1(i), \ldots, t_n(i)], & \text{if } \mathcal{H}_\beta = \hat{H}_\beta. \end{cases}$$

Let $\mathcal{H}_\beta(i)$ be the localisation of $\text{Pol}_n(i)$ with respect to the following multiplicative closed subset

$$\{(t_r(i) - t_s(i))^k | 1 \leq r \neq s \leq n, k \in \mathbb{Z}_{\geq 0}\}.$$  

We set

$$\mathcal{H}_\beta := \bigoplus_{i \in I^\beta} \mathcal{H}_\beta(i).$$

The symmetric group $S_n$ acts on $\mathcal{H}_\beta$ by taking $t_k(i)$ to $t_{w(k)}(w^{-1})$, and $(t_r(i) - t_s(i))^k$ to $(t_{w(r)}(w))^{-2} - (t_{w(s)}(w))^k$, where $w \in S_n$, $k \in \mathbb{Z}$. In particular, the translation $s_k$ maps $t_a(i)$ to $t_a(s_k(i))$, if $a \neq k, k+1$; $t_k(i)$ to $t_{k+1}(s_k(i))$, and $t_{k+1}(i)$ to $t_k(s_k(i))$.

Recall that $\{ t_k | 1 \leq k \leq n \}$ is a set of $n$ algebraically independent indeterminates over $K$. Let $\mathcal{P}_n$ be the localisation of $K[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ if $\mathcal{H}_\beta = \mathcal{H}_\beta(q)$, or the localisation of $K[t_1, \ldots, t_n]$ if $\mathcal{H}_\beta = \hat{H}_\beta$, with respect to the following multiplicative closed subset

$$\{(t_r - t_s)^k | 1 \leq r \neq s \leq n, k \in \mathbb{Z}_{\geq 0}\}.$$  

Let

$$\theta_i : \mathcal{P}_n \cong \text{Pol}_n(i)$$

be the canonical isomorphism induced by the map $t_k^{\pm 1} \mapsto t_k(i)^{\pm 1}$ for each $1 \leq k \leq n$. For each $f \in \mathcal{P}_n$, we set

$$f_i := \theta_i(f) \in \text{Pol}_n(i).$$

The symmetric group $S_n$ acts on $\mathcal{P}_n$ by taking $t_k$ to $t_{w(k)}$, and $(t_r - t_s)^k$ to $(t_{w(r)} - t_{w(s)})^k$, where $w \in S_n$, $k \in \mathbb{Z}$. For any $f \in \mathcal{P}_n$, we have that $w(f_1) = (w(f))_{w_1}$ for any $w \in S_n, 1 \in I^\beta$.

For any $i, j \in I^\beta, f \in \mathcal{P}_n, 1 \leq r < n$ and $1 \leq k \leq n$, we define

$$X_k^{\pm 1} \hat{e}(i) \cdot f_i := t_k(i)^{\pm 1} f_i,$$

$$\hat{e}(j) \cdot f_i := \delta_{ij} f_i,$$  

$$\hat{T}_r \hat{e}(i) \cdot f_i := \left( \frac{t_{r+1} - q \tau_r}{t_{r+1} - t_r} s_r(f) \right)_{s_i, i} + (q - 1) \frac{t_{r+1} f_{t_r} - t_r f_{t_{r+1}}}{t_{r+1} - t_r} f_i,$$

and

$$\hat{e}_k \hat{e}(i) \cdot f_i := t_k(i) f_i,$$

$$\hat{e}(j) \cdot f_i := \delta_{ij} f_i,$$  

$$\hat{s}_r \hat{e}(i) \cdot f_i := \left( \frac{t_{r+1} - t_r - 1}{t_{r+1} - t_r} s_r(f) \right)_{s_i, i} + \frac{1}{t_{r+1} - t_r} f_i,$$

(3.37)  

(3.38)

3.39. Proposition. Let $\mathcal{H}_\beta \in \{ \mathcal{H}_\beta(q), \hat{H}_\beta \}$. The above rules extend uniquely to a faithful representation $\hat{H}_\beta$ of $\mathcal{H}_\beta$ on $\text{Pol}_\beta$. 

Proof. We only consider the non-degenerate case as the degenerate case is similar. We divide the proof into four steps:

Step 1. For any \( i \in F^3, f \in \mathcal{P}_n \), we define

\[
\begin{aligned}
X^{\pm_k}_i \cdot f_i &= \delta_0 t_k(i)^{\pm_k} f_i, \\
T_r \cdot f_i &= \left( \frac{t_{r+1} - qt_r s_r(f)}{t_{r+1} - t_r} \right)_{s,i} + (q-1) \frac{t_{r+1} - t_r}{t_{r+1} - t_r} f_i,
\end{aligned}
\tag{3.40}
\]

We claim that the formulae (3.40) extends to a well-defined representation \( \tilde{\rho}_n \) of \( \mathcal{H}_n(q) \) on \( \text{Pol}_\beta \).

We need to verify the defining relations for \( \mathcal{H}_n(q) \). The only non-trivial relation that need to be checked is the braid relations and the quadratic relations. In other words, we need to prove \((T_r - q)(T_r + 1)f_i = 0\) and

\[
T_{r+1}T_r T_{r+1}f_i = T_r T_{r+1}T_r f_i, \quad 1 \leq r < n - 1.
\tag{3.41}
\]

The first equality follows from a direct and easy verification. For the second one, it can be proved by a brutal force calculation via comparing the coefficients of

\[
\begin{aligned}
f_i, (s_r(f))_{s,i}, (s_{r+1}(f))_{s_{r+1},i}, (s_r s_{r+1}(f))_{s,s_{r+1}i}, \\
(s_{r+1} s_r(f))_{s_{r+1}s_r,i}, (s_r s_{r+1} s_r(f))_{s_{r+1}s_r s_r,i}.
\end{aligned}
\]

on both sides of (3.41). Most of the check is an easy job except for the coefficient of \( f_i \). In fact, we can get the following coefficient \( C_1 \) of \( f_i \) appearing in the LHS of (3.41):

\[
C_1 = (q - 1)^3 \frac{t_{r+2}(i)^2 t_{r+1}(i)}{(t_{r+2}(i) - t_{r+1}(i))^2(t_{r+1}(i) - t_r(i))} \times t_{r+1}(i) - t_r(i) \times (q-1)t_{r+2}(i)
\]

while the coefficient \( C_2 \) of \( f_i \) appearing in the RHS of (3.41) is as follows:

\[
C_2 = (q - 1)^3 \frac{t_{r+1}(i)^2 t_{r+2}(i)}{(t_{r+1}(i) - t_r(i))^2(t_{r+2}(i) - t_{r+1}(i))} \times t_{r+1}(i) - t_r(i) \times (q-1)t_{r+2}(i)
\]

We want to prove that \( C_1 = C_2 \). It suffices to show that

\[
(q - 1)^2 t_{r+1}(i) t_{r+2}(i)(t_{r+1}(i) - t_r(i)) (t_{r+1}(i) - t_r(i)) -
\]

\[
(t_{r+1}(i) - q t_{r+1}(i))(t_{r+2}(i) - t_r(i)(t_{r+1}(i) - t_r(i))^2
\]

\[
= (q - 1)^2 (t_{r+1}(i))^2 (t_{r+2}(i) - t_{r+1}(i))(t_{r+2}(i) - t_r(i)) -
\]

\[
(t_{r+1}(i) - q t_{r+1}(i))(t_{r+2}(i) - t_r(i))(t_{r+2}(i) - t_{r+1}(i))^2.
\]

We regard the above equality as an equation on the indeterminate \( t_{r+1}(i) \) with degree \( \leq 2 \). Set \( t_{r+1}(i) = t_r(i), t_{r+1}(i), 0 \), we always get an identity. This implies that it must be an identity forever. This proves that \( C_1 = C_2 \) as required. This completes the proof of our claim.

Step 2. We claim that representation \( \tilde{\rho}_n \) constructed in Step 1 is faithful.

In fact, comparing the formulae (3.40) and (2.18), we see that for any \( i \in F^3 \) and \( f \in \mathcal{P}_n \),

\[
\sum_{j \in F^3} \theta_j^{-1}(z \cdot f_i) = z \ast f,
\]

where by convention we understand that \( \theta_j^{-1}(x) := 0 \) whenever \( x \in \text{Pol}_n(j) \) with \( j \neq i \). As a result, the faithfulness of \( \tilde{\rho}_n \) follows from the faithfulness of \( \rho_q \).
Step 3. We claim that representation $\tilde{\rho}_n$ can factor through the surjection $\mathcal{H}_n(q) \twoheadrightarrow \mathcal{H}_n^\rho(q)$ so that it induces a faithful representation $\tilde{\rho}_n$ of $\mathcal{H}_n(q)$ on $\text{Pol}_\beta$. To prove this, it suffices to show that $\ker \hat{\pi}$ acts as 0 on $\text{Pol}_\beta$.

We fix $a \in I$. In the rest of the proof of this lemma, we shall regard any polynomial in $K[t_1(j), \cdots, t_n(j)]$ as a polynomial in $K[t_1(j) - q^a, t_2(j) - q^a, \cdots, t_n(j) - q^a]$ and only consider its degree with respect to the indeterminates $t_1(j) - q^a$, $t_2(j) - q^a, \cdots, t_n(j) - q^a$. Suppose that $z \in \ker \hat{\pi}$ and $\tilde{\rho}_\beta(z) \neq 0$. Then there exists some $f \in \mathcal{T}_n$, $k \in I^3$ such that $zf_k \neq 0$. We can write
\begin{equation}
zf_k = \sum_{j \in I^3} g_{2,j} f_j,
\end{equation}
where for each $j \in I^3$, $g_{1,j}, g_{2,j} \in K[t_1(j), \cdots, t_n(j)] = K[t_1(j) - q^a, t_2(j) - q^a, \cdots, t_n(j) - q^a]$ and (by the formula (3.40))
\begin{equation}
g_{1,j} \neq 0 \text{ only if } \deg g_{2,j} \leq n!.
\end{equation}
We set
\begin{equation}
d_0 := \max\{\deg g_{1,j} j \in I^3, g_{1,j} \neq 0\}.
\end{equation}
Now $\pi(z) = 0$ implies that $\pi_\Lambda(z) = 0$ for any $\Lambda \in P^+$. We define $\tilde{\Lambda} := (d_0 + 2n! + 1)\Lambda$. Since $\pi_\Lambda(z) = 0$, we can deduce that $z$ lives inside the two-sided ideal of $\mathcal{H}_n(q)$ generated by $(X_1 - q^a)^{d_0 + 2n! + 1}$. However, this is impossible by the formula (3.40), (3.42), (3.43) and (3.44).

Step 4. Now every elements of $\mathcal{H}_\beta^\rho(q)$ can be written uniquely as $\sum_{i \in I^3} z_i \hat{e}(i)$. We define
\begin{equation}
\rho_\beta \left( \sum_{i \in I^3} z_i \hat{e}(i) \right)(f_j) := \sum_{i \in I^3} \rho_n(z_i) \delta_{ij}(f_j), \quad j \in I^3, f \in K[t_1^{n+1}, \cdots, t_n^{n+1}].
\end{equation}
Applying Lemma 3.4, we see that $\rho_\beta$ is a well-defined representation of $\mathcal{H}_\beta^\rho(q)$ on $\text{Pol}_\beta$. The faithfulness of $\rho_\beta$ follows from the faithfulness of $\tilde{\rho}_n$.

Finally, for any $1 \leq r < n$, $t_r(j) - t_r + 1(j)$ is invertible in $\text{Pol}_n(j)$. It follows from Lemma 3.31 and Lemma A1 that the representation $\tilde{\rho}_\beta$ of $\mathcal{H}_\beta^\rho(q)$ on $\text{Pol}_\beta$ can be extended uniquely to the representation $\tilde{\rho}_\beta$ of $\mathcal{H}_\beta^\rho(q)$ on $\text{Pol}_\beta$ which is given exactly by the formula (3.37). The faithfulness of $\tilde{\rho}_\beta$ follows from the faithfulness of $\tilde{\rho}_n$. \hfill \Box

Let $i \in I^3$. For each $w \in S_n$, we fix a reduced expression $s_{j_1}, s_{j_2}, \cdots, s_{j_k}$ of $w$, and we define
\begin{align*}
\hat{w}_1 := & (\hat{e}(wi)\hat{s}_{j_1}\hat{e}(s_{j_1}wi))(\hat{e}(s_{j_1}wi)\hat{s}_{j_2}\hat{e}(s_{j_2}s_{j_1}wi)) \cdots (\hat{e}(s_{j_1}i)\hat{s}_{j_k}\hat{e}(1)), \\
\hat{T}_{w,1} := & (\hat{e}(wi)\hat{T}_1\hat{e}(s_{j_1}wi))(\hat{e}(s_{j_1}wi)\hat{T}_2\hat{e}(s_{j_2}s_{j_1}wi)) \cdots (\hat{e}(s_{j_1}i)\hat{T}_k\hat{e}(1)).
\end{align*}

3.45. **Lemma.** The elements in the following set
\begin{equation}
\left\{ \hat{T}_{w,1}x_{a_1} \cdots \hat{x}_{a_n} \prod_{1 \leq r < s \leq n, 1 \leq r \neq s} (\hat{x}_r - \hat{x}_s)^{-br,s} \hat{e}(i) \middle| w \in S_n, i \in I^3, b_{r,s} \in \mathbb{N}, a_1, \cdots, a_n \in \mathbb{Z}, b_{r,s} > 0 \text{ only if either} \right. \\
\left. a_r = 0 \geq a_s \text{ or } a_r > 0 = a_s \right\}
\end{equation}
form a $K$-basis of $\mathcal{H}_\beta^\rho(q)$, and the elements in the following set
\begin{equation}
\left\{ \hat{w}_1\hat{x}_{a_1} \cdots \hat{x}_{a_n} \prod_{1 \leq r < s \leq n} (\hat{x}_r - \hat{x}_s)^{-br,s} \hat{e}(i) \middle| w \in S_n, i \in I^3, b_{r,s} \in \mathbb{N}, a_1, \cdots, a_n \in \mathbb{N}, b_{r,s} > 0 \text{ only if } a_s = 0 \right\}
\end{equation}
form a $K$-basis of $\hat{H}_\beta$. 

\textbf{PROOF.}
Proof. We only prove (3.46) as (3.47) can be proved in a similar way. Using Lemma 2.22 and the relation (3.28) it is easy to see that the elements in the set (3.46) is a K-linear generators of $\mathcal{H}_\beta(q)$. It remains to prove that they are $K$-linearly independent. To this end, by Proposition 3.39, it suffices to show that their images under $\rho_\beta$ are $K$-linearly independent.

By Lemma 3.4 and Lemma 3.10, \{\hat{e}(i)|i \in I^\beta\} is a set of pairwise nonzero orthogonal idempotents. Suppose that the elements in (3.46) are $K$-linearly dependent. Then we can find $i, j \in I^\beta$ and a subset $J \subseteq \mathfrak{S}_n$, such that $wi = j$ for any $w \in J$, and

$$
\sum_{w \in J} c_{w, (a, b)} \hat{e}(j) \hat{T}_{w, i} \hat{X}^{a_1}_1 \cdots \hat{X}^{a_n}_n \prod_{1 \leq r < s \leq n} (\hat{X}_r - \hat{X}_s)^{-b_{r, s}} \hat{e}(i) = 0,
$$

where $a = (a_1, \cdots, a_n) \in \mathbb{Z}_n$, $b = (b_{r, s})$, and $b_{r, s} > 0$ only if $a_r = 0 \geq a_{r+1}$ or $a_r > 0 = a_{r+1}$ for any $1 \leq r < n$, and $0 \neq c_{w, (a, b)} \in K$ for any 3-tuple $(w, a, b)$. Furthermore, we assume that (3.48) is chosen such that

$$
\# \{(w, a, b)|w \in J, c_{w, (a, b)} \neq 0\} \text{ is minimal.}
$$

For each $w \in J$, we set

$$
J(w) := \{(a, b)|c_{w, (a, b)} \neq 0\}.
$$

By assumption $J \neq \emptyset$. Then

$$
\sum_{w \in J} \hat{e}(j) \hat{T}_{w, i} \hat{e}(i) \left( \sum_{(a, b) \in J(w)} c_{w, (a, b)} \hat{X}^{a_1}_1 \cdots \hat{X}^{a_n}_n \prod_{1 \leq r < s \leq n} (\hat{X}_r - \hat{X}_s)^{-b_{r, s}} \hat{e}(i) \right) \hat{e}(i) = 0.
$$

We divide the remaining proof into two steps:

Step 1. We claim that if

$$
0 \neq f \in K[\hat{X}_1^{\pm 1}, \hat{X}_r - \hat{X}_s)^{-1}|1 \leq i \leq n, 1 \leq r < s \leq n, i_r \neq i_s],
$$

then $\sum_{w \in J} \hat{e}(j) \hat{T}_{w, i} \hat{e}(i) f \neq 0$.

In fact, multiplying some monomial of the form $\hat{X}_1^{c_1} \hat{X}_2^{c_2} \cdots \hat{X}_n^{c_n} \prod_{1 \leq r < s \leq n} (\hat{X}_r - \hat{X}_s)^{c_{r, s}}$ on the RHS of (3.48) if necessary, we can assume without loss of generality that $0 \neq f \in K[\hat{X}_1, \cdots, \hat{X}_n]$ and $f$ is a $K$-linear combination of some monomials of the form $\hat{X}_1^{c_1} \cdots \hat{X}_n^{c_n}$ with $0 < c_1 < c_2 < \cdots < c_n$. We fix one such $n$-tuple $(c_1, \cdots, c_n)$.

let "<" be the Bruhat partial order on $\mathfrak{S}_n$. Let $w_{0, J}$ be a maximal element under "<" in the set $J$. We now using the representation $\hat{\rho}_\beta$, it follows from (3.40) that $t_{w_{0, J}(1)} \hat{e}(j) t_{w_{0, J}(n)} \hat{e}(i) f$ must appear with non-zero coefficient in $\left( \sum_{w \in J} \hat{e}(j) \hat{T}_{w, e}(i) f \right) (1)$. This implies that $\sum_{w \in J} \hat{e}(j) \hat{T}_{w, e}(i) f \neq 0$ as required.

Step 2. In view of the result we proved in Step 1, we can get that for each $w \in J$,

$$
\sum_{(a, b) \in J(w)} c_{w, (a, b)} \hat{X}^{a_1}_1 \cdots \hat{X}^{a_n}_n \prod_{1 \leq r < s \leq n} (\hat{X}_r - \hat{X}_s)^{-b_{r, s}} \hat{e}(i) = 0.
$$

Equivalently, by Lemma 3.4, it suffices to show that

$$
\sum_{(a, b) \in J(w)} c_{w, (a, b)} t_1^{a_1} \cdots t_n^{a_n} \prod_{1 \leq r < s \leq n} (t_r - t_s)^{-b_{r, s}} = 0.
$$

If $b_{r, s} = 0$ for any $1 \leq r < s \leq n$, then it is clear that we will get a contradiction. Otherwise, we set

$$
s_0 := \max \{s|b_{r, s} > 0, 1 \leq r < s \leq n, i_r \neq i_s, (a, b) \in J(w)\},
$$

$$
N_0 := \max \{b_{r, s}|b_{r, s} > 0, 1 \leq r < s \leq n, (a, b) \in J(w)\}.
$$
We fix an \(1 \leq r_0 < s_0 \leq n\) such that \(b_{r_0, s_0} = N_0\). We multiply \((t_{r_0} - t_{s_0})^{N_0}\) on both sides of (3.50) and then specialize \(t_{s_0} := t_{r_0}\). Then by our construction, the fact that \(a_{r_0} = 0 \geq a_{s_0}\) or \(a_{r_0} > 0 = a_{s_0}\) and (3.49), we can deduce that
\[
(3.51) \quad b_{r_0, s_0} = N_0 \text{ whenever } c_{w, b} \neq 0.
\]

Therefore, multiplying \((t_{r_0} - t_{s_0})^{N_0}\) on both sides of (3.50) we can reduce the proof of (3.50) to the following inequality:
\[
(3.52) \quad \sum_{(a, b) \in J(w)} c_{w, a} t_1^{a_1} \cdots t_n^{a_n} \prod_{1 \leq r < s < s_0, i_r \neq i_s} (t_r - t_s)^{-b_{r,s}} = 0.
\]

Next we define
\[
s_1 := \max\{s | b_{r,s} > 0, 1 \leq r < s \leq s_0, (r, s) \neq (r_0, s_0), i_r \neq i_s, (a, b) \in J(w)\},
\]
\[N_1 := \max\{b_{r,s} | b_{r,s} > 0, 1 \leq r < s_1, (a, b) \in J(w)\}.
\]

We fix an \(1 \leq r_1 < s_1 \leq n\) such that \(b_{r_1, s_1} = N_1\). Then we repeat the above argument. After a finite step, we shall find a unique \(b\) such that the following inequality:
\[
(3.53) \quad \sum_{(a, b) \in J(w)} c_{w, a} t_1^{a_1} \cdots t_n^{a_n} = 0.
\]

which is again a contradiction. This completes the proof of the lemma. \(\square\)

Applying the anti-isomorphism *, we see that the elements in the following set
\[
\{ \hat{e}(w) \hat{X}_1^{a_1} \cdots \hat{X}_n^{a_n} \prod_{1 \leq r < s \leq n, i_r \neq i_s} (\hat{X}_r - \hat{X}_s)^{-b_{r,s}} \mid w \in \mathfrak{S}_n, i \in I^\beta, b_{r,s} \in \mathbb{N}, \ \begin{array}{l} a_1, \ldots, a_n \in \mathbb{Z}, b_{r,s} > 0 \text{ only if either} \\
 a_r = 0 \geq a_s \text{ or } a_r > 0 = a_s 
\end{array} \}
\]
form a \(K\)-basis of \(\hat{\mathcal{H}}_{\beta}(q)\), and the elements in the following set
\[
\{ \hat{e}(w) \hat{x}_1^{a_1} \cdots \hat{x}_n^{a_n} \prod_{1 \leq r < s \leq n, i_r \neq i_s} (\hat{x}_r - \hat{x}_s)^{-b_{r,s}} \mid w \in \mathfrak{S}_n, i \in I^\beta, b_{r,s} \in \mathbb{N}, a_1, \ldots, a_n \in \mathbb{N}, \\
 b_{r,s} > 0 \text{ only if } a_s = 0 \}
\]
form a \(K\)-basis of \(\hat{H}_\beta\).

3.54. Corollary. For any \(i \in I^\beta, i \neq f \in \hat{\mathcal{H}}_{\beta}(q)\hat{e}(i), 0 \neq g \in \hat{e}(i)\hat{\mathcal{H}}_{\beta}(q), \ 0 \neq h \in K[\hat{X}_1^\pm, \ldots, X_n^\pm], \text{ we have that } f \hat{e}(i)h \neq 0 \neq h \hat{e}(i)g. \text{ The same is true if we replace } \hat{\mathcal{H}}_{\beta}(q) \text{ and } K[\hat{X}_1^\pm, \ldots, X_n^\pm] \text{ by } \hat{H}_\beta \text{ and } K[x_1, \ldots, x_n] \text{ respectively.}\)

Proof. This follows directly from Lemma 3.45. \(\square\)

Recall the definition of the subalgebra \(\hat{\mathcal{H}}_{\beta}(q)\) of \(\hat{\mathcal{H}}_{\beta}(q)\) in Remark 3.27. By a natural restriction of (3.37) to the subalgebra \(\hat{\mathcal{H}}_{\beta}(q)\), we can also get a faithful representation \(\rho_\beta^+\) of \(\hat{\mathcal{H}}_{\beta}(q)\) on \(\oplus_{i \in I^\beta} \hat{P}_n(i)\), where \(\hat{P}_n(i)\) is the localization of \(K[t_1(i), \ldots, t_n(i)]\) with respect to (3.34) for each \(i\). In a similar way one can prove that the elements in the following set
\[
\{ \hat{e}(w(i)) \hat{T}_{w,i} \hat{X}_1^{a_1} \cdots \hat{X}_n^{a_n} \prod_{1 \leq r < s \leq n, i_r \neq i_s} (\hat{X}_r - \hat{X}_s)^{-b_{r,s}} \hat{e}(i) \mid w \in \mathfrak{S}_n, i \in I^\beta, b_{r,s}, a_1, \ldots, a_n \in \mathbb{N}, \\
 b_{r,s} > 0 \text{ only if } a_s = 0 \}
\]
form a \(K\)-basis of \(\hat{\mathcal{H}}_{\beta}(q)\).
3.56. **Corollary.** Suppose that \( i \in I^\beta, i_r = i_{r+1} \). Then in the non-degenerate case, inside \( \mathcal{H}_\beta(q) \), we have that
\[
(3.57) \quad \hat{e}(i) \hat{T}_r \hat{e}(i) \hat{T}_r^{-1} \hat{e}(i) = \hat{e}(i) = \hat{e}(i) \hat{T}_r^{-1} \hat{e}(i) \hat{T}_r \hat{e}(i),
\]
while in the degenerate case, inside \( \hat{H}_\beta \), we have that
\[
(3.58) \quad \hat{e}(i) \hat{s}_r \hat{e}(i) \hat{s}_r \hat{e}(i) = \hat{e}(i).
\]
\[\]
**Proof.** This follows from Proposition 3.39, the formulae given in (3.37) and (3.38) and some direct verifications. Equivalently, one can also deduce the corollary from the first relation in (3.15) and the first relation in (3.23). □

For any \( 1 \leq k \leq n, i \in I^\beta \) and \( w \in S_n \), we define \( w(\hat{X}_k \hat{e}(i)) := \hat{X}_{w(k)} \hat{e}(w(i)) \). By Lemma 3.4, this is well-defined and extends uniquely to an action of \( S_n \) on the set of polynomials in \( \{ \hat{X}_k \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta \} \) and on the set of polynomials in \( \{ \hat{x}_k \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta \} \) respectively. For any \( i, j \in I^n \), we write \( i \sim j \) whenever \( j = \sigma i \) for some \( \sigma \in S_n \). Let \( I^\beta / \sim \) be a set of representatives with respect to the equivalence relation \( \sim \). Let \( i \in I^\beta / \sim, 1 \leq a < b \leq n \) with \( i_a \neq i_b \). We define
\[
S_n(i, a, b) := \{ \sigma \in S_n | \sigma a = i, \sigma(a) = a, \sigma(b) = b \}.
\]
Let \( D_n(i, a, b) \) be a set of left coset representatives of \( S_n(i, a, b) \) in \( S_n \). Note that for any \( d \in D_n(i, a, b) \), if \( d \hat{i} = i \) then \( (d(a), d(b)) \notin \{(a, b), (b, a)\} \) because \( i_a \neq i_b \). The next result describes the center for the modified affine Hecke algebras \( \mathcal{H}_\beta(q) \) and \( \hat{H}_\beta \).

3.59. **Lemma.** Let \( \beta \in Q_n^+ \). The center \( Z(\mathcal{H}_\beta(q)) \) of \( \mathcal{H}_\beta(q) \) is equal to
\[
\left\{ \left( \sum_{i \in I^\beta / \sim} \sum_{1 \leq a < b \leq n} \prod_{d \in \mathcal{D}(i, a, b)} (\hat{X}_d \hat{e}(a) - \hat{x}_d \hat{e}(b))^{-a_i} \hat{e}(k) \right) f \left| \begin{array}{c}
a_i \in \mathbb{N}, \forall i \in I^\beta / \sim, f \text{ is a symmetric polynomial in } \{ \hat{X}_k \hat{e}(i), \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta \} \end{array} \right. \right\},
\]
while the center \( Z(\hat{H}_\beta) \) of \( \hat{H}_\beta \) is equal to
\[
\left\{ \left( \sum_{i \in I^\beta / \sim} \sum_{1 \leq a < b \leq n} \prod_{d \in \mathcal{D}(i, a, b)} (\hat{x}_d \hat{e}(a) - \hat{x}_d \hat{e}(b))^{-a_i} \hat{e}(k) \right) f \left| \begin{array}{c}
a_i \in \mathbb{N}, \forall i \in I^\beta / \sim, f \text{ is a symmetric polynomial in } \{ \hat{x}_k \hat{e}(i), \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta \} \end{array} \right. \right\}.
\]
\[\]
**Proof.** We only prove the lemma in the non-degenerate case, while the degenerate case is similar.

Suppose that \( z = \sum_{j \in I^\beta} f(j) \hat{e}(j) \) is a symmetric polynomial in \( \{ \hat{X}_k \hat{e}(i), \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta \} \), where \( f(j) \in K[\hat{X}_k, \ldots, \hat{X}_n] \) for each \( j \). Then \( f(s_r(j)) = s_r(f(j)) \) for any \( j \in I^\beta \) and any \( 1 \leq r < n \).

By the relations (2.6), (2.7), it is easy to see that for any \( f \in K[\hat{X}_k, \ldots, \hat{X}_n] \),
\[
(3.60) \quad \hat{e}(i) \hat{T}_r \hat{e}(s_r(i)) f - s_r(f) \hat{e}(i) \hat{T}_r \hat{e}(s_r(i)) = (q - 1) \hat{e}(i) \hat{X}_{r+1} - f - s_r(f) \hat{X}_r \hat{e}(s_r(i)), \quad 1 \leq r < n.
\]
To show that \( z \in Z(\mathcal{H}_\beta(q)) \), the only nontrivial relations that needed to be checked are the following two relations:
\[
(3.61) \quad z \hat{e}(j) \hat{T}_r \hat{e}(s_r(j)) = \hat{e}(j) \hat{T}_r \hat{e}(s_r(j)) z, \quad z \hat{e}(j) \hat{T}_r \hat{e}(j) = \hat{e}(j) \hat{T}_r \hat{e}(j) z.
\]
where \( j \in I^\beta, 1 \leq r < n \). Equivalently, we need to check that
\[
(3.62) \quad f(j) \hat{e}(j) \hat{T}_r \hat{e}(s_r(j)) = \hat{e}(j) \hat{T}_r \hat{e}(s_r(j)) f(j), \quad f(j) \hat{e}(j) \hat{T}_r \hat{e}(j) = \hat{e}(j) \hat{T}_r \hat{e}(j) f(j).
\]
If \( j_r \neq j_{r+1} \), then the second relation follows from (3.12), while the first relation follows from (3.60).

If \( j_r = j_{r+1} \), then it suffices to check the second relation. In this case it again follows from (3.60) and the condition that \( f(j) = s_r f(j) \) whenever \( j = s_r j \).

Therefore, (3.62) always holds. So \( z \in Z(\mathcal{H}_\beta(q)) \) as required. By the result we obtained above and the definition of \( D_n(i,a,b) \), we can deduce that

\[
\sum_{i \in I^\beta} \sum_{1 \leq a < b \leq n \atop i_a \neq i_b} \prod_{d \neq k \in I^\beta} (\hat{X}_{d(a)} - \hat{X}_{d(b)})^{-a_i} \hat{e}(k) \in Z(\mathcal{H}_\beta(q)).
\]

It follows that

\[
\left( \sum_{i \in I^\beta} \sum_{1 \leq a < b \leq n \atop i_a \neq i_b} \prod_{d \neq k \in I^\beta} (\hat{X}_{d(a)} - \hat{X}_{d(b)})^{-a_i} \hat{e}(k) \right) z \in Z(\mathcal{H}_\beta(q)),
\]

as required.

Conversely, suppose that \( z = \sum_{w} \hat{e}(w) \hat{T}_{a,w} \hat{f}_w \in Z(\mathcal{H}_\beta(q)) \), where

\[
f_w \in K[X_1^{\pm 1} e(i), \ldots, X_n^{\pm 1} e(i), (X_r - X_s)^{-1} \hat{e}(i)] \text{ for } 1 \leq r < s \leq n, i_r \neq i_s.
\]

Since \( \hat{e}(i) z = z \hat{e}(i) \), we can rewrite \( z \) as

\[
z = \sum_{i \in I^\beta, w \in \Theta_n \atop w(i) = 1} \hat{e}(i) \hat{T}_{a,w} \hat{f}_w \hat{e}(i) \in Z(\mathcal{H}_\beta(q)).
\]

Suppose that \( z \notin K[X_1^{\pm 1} e(i), \ldots, X_n^{\pm 1} e(i), (X_r - X_s)^{-1} \hat{e}(i)] \text{ for } 1 \leq r < s \leq n, i_r \neq i_s \). Let \( u \) be maximal with respect to the Bruhat partial order “\( < \)” such that \( f_u \neq 0, u(i) = 1 \) and \( u \neq 1 \). Then \( u(r) \neq r \) for some \( 1 \leq r \leq n \). By definition of center, we have that \( \hat{X}_r z = z \hat{X}_r \). However, by an easy induction based on Lemma 2.20, we can get that

\[
\hat{X}_r \hat{e}(i) \hat{T}_{a,w} \hat{f}_w \hat{e}(i) = \hat{e}(i) \hat{T}_{a,w} \hat{f}_w \hat{e}(i) = \hat{e}(i) \left( \hat{T}_{a,w} \hat{X}_{u-r} + \sum_{w' < w} \hat{T}_{a,w'} \hat{g}_{w'} \right) \hat{e}(i),
\]

where \( g_{w'} \in K[X_1^{\pm 1} e(i), \ldots, X_n^{\pm 1} e(i)] \). It follows that the coefficient of \( \hat{e}(i) \hat{T}_{a,w} \hat{f}_w \hat{e}(i) \) is different in \( \hat{X}_r z \) and \( z \hat{X}_r \), a contradiction. Therefore,

\[
z \in K[X_1^{\pm 1} e(i), \ldots, X_n^{\pm 1} e(i), (X_r - X_s)^{-1} \hat{e}(i)] \text{ for } 1 \leq r < s \leq n, i_r \neq i_s.
\]

We divide the remaining proof into two cases:

**Case 1.** Suppose that \( z = \sum_{i \in I^\beta} f(i) \hat{e}(i) \), where \( f(i) \in K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \) for each \( i \in I^\beta \).

In this case, for any \( j \in I^\beta \) and \( 1 \leq r < n \), we have that \( z \hat{e}(j) \hat{T}_r \hat{e}(s_r j) = \hat{e}(j) \hat{T}_r \hat{e}(s_r j) z \). It follows that \( \hat{e}(j) \hat{T}_r f(s_r j) \hat{e}(s_r j) = \hat{e}(j) \hat{T}_r f(j) \hat{e}(s_r j) \hat{e}(s_r j) \) for each \( j \in I^\beta \).

If \( s_r j \neq j \), then the relations (2.6), (2.7) imply that

\[
f(j) \hat{e}(j) \hat{T}_r f(s_r j) \hat{e}(s_r j) = \hat{e}(j) f(j) \hat{T}_r \hat{e}(s_r j) = \hat{e}(j) \hat{T}_r f(j) \hat{e}(s_r j),
\]

Applying Lemma 3.45, we can deduce that \( f(s_r j) = s_r(f(j)) \).

If \( s_r j = j \), then as \( \hat{e}(j) \hat{T}_r \hat{e}(j) = \hat{e}(j) \hat{T}_r \hat{e}(j) z \) and hence

\[
\hat{e}(j) f(j) \hat{T}_r \hat{e}(j) = \hat{e}(j) \hat{T}_r f(j) \hat{e}(j)
\]

it follows from (3.60) that

\[
\hat{e}(j) \left( f(j) - s_r(f(j)) \right) \hat{T}_r \hat{e}(j) = \hat{e}(j) \hat{T}_r f(j) \hat{e}(j) - \hat{e}(j) s_r(f(j)) \hat{T}_r \hat{e}(j)
\]

\[
= (q - 1) \hat{X}_{r+1} \frac{f(j) - s_r(f(j))}{\hat{X}_{r+1} - \hat{X}_r} \hat{e}(j).
\]
Now applying Lemma 3.45 and noting that \( q - 1 \neq 0 \), we can deduce that \( f(j) - s_r(f(j)) = 0 \).

Therefore, we conclude that \( f(s_rj) = s_r(f(j)) \) for any \( j \in I^\beta \) and any \( 1 \leq r < n \). This implies that \( z \) is a symmetric polynomial in \( \{ \hat{X}_k^{\pm 1} \hat{e}(i), \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta \} \) as required.

Case 2. Suppose that \( z \notin K[\hat{X}_1^{\pm 1}\hat{e}(i), \cdots, \hat{X}_n^{\pm 1}\hat{e}(i) | i \in I^\beta] \).

Then \( n > 1 \) and

\[
z = \sum_{I^\beta} (\hat{X}_r - \hat{X}_s)^{-r(b_r,s)g_{r,s}(i)}\hat{e}(i),
\]

where \( g_{r,s}(i) \in K[\hat{X}_1^{\pm 1}, \cdots, \hat{X}_n^{\pm 1}] \) for each \( i \in I^\beta \), \( b_r,i \in \mathbb{N} \) for each \( r \), and at least one of these \( b_r,i \) is positive, and \( g_{r,s}(i) \) is coprime to \( \hat{X}_r - \hat{X}_s \) whenever \( b_r,i > 0 \).

By an induction on the number of terms, we can find a set of positive integers \( \{a_i | i \in I^\beta / \sim \} \) and \( f \in K[\hat{X}_1^{\pm 1}\hat{e}(i) | 1 \leq k \leq n, i \in I^\beta] \) such that

\[
z = \left( \sum_{I^\beta} \sum_{1 \leq a < b \leq n} \prod_{d \in D_n(i,a,b)} (\hat{X}_{d(a)} - \hat{X}_{d(b)})^{-a_1}\hat{e}(k) \right) f
\]

Now \( z \in Z(\mathcal{H}_\beta(q)) \) and

\[
\sum_{I^\beta / \sim} \sum_{1 \leq a < b \leq n} \prod_{d \in D_n(i,a,b)} (\hat{X}_{d(a)} - \hat{X}_{d(b)})^{-a_1}\hat{e}(k) \in Z(\mathcal{H}_\beta(q)),
\]

we can deduce that \( f \in Z(\mathcal{H}_\beta(q)) \) because it is easy to see (by Lemma 3.45) that \( \sum_{I^\beta / \sim} \sum_{1 \leq a < b \leq n} \prod_{d \in D_n(i,a,b)} (\hat{X}_{d(a)} - \hat{X}_{d(b)})^{-a_1}\hat{e}(k) \) is not a zero divisor. Using the result which proved in Case 1, we see that \( f \) is symmetric in \( \{ \hat{X}_k^{\pm 1}\hat{e}(i), \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta \} \). Hence the lemma follows. \( \Box \)

### 3.63 Corollary.
Let \( \beta \in Q^+ \). Then \( K[\hat{X}_1\hat{e}(i), \cdots, \hat{X}_n\hat{e}(i), \hat{e}(i) | i \in I^\beta] \cap Z(\mathcal{H}_\beta(q)) \) is equal to the set of symmetric polynomials in \( \{ \hat{X}_k\hat{e}(i), \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta \} \), and \( K[\hat{x}_1\hat{e}(i), \cdots, \hat{x}_n\hat{e}(i), \hat{e}(i) | i \in I^\beta] \cap Z(\mathcal{H}_\beta) \) is equal to the set of symmetric polynomials in \( \{ \hat{x}_k\hat{e}(i), \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta \} \).

Proof. This follows from the proof of Case 1 in Lemma 3.59. \( \Box \)

### 3.64 Corollary.
Let \( \beta \in Q^+ \). The center \( Z(\mathcal{H}_\beta^+) \) of \( \mathcal{H}_\beta^+ \) is

\[
\left\{ \sum_{I^\beta / \sim} \sum_{1 \leq a < b \leq n} \prod_{d \in D_n(i,a,b)} (\hat{X}_{d(a)} - \hat{X}_{d(b)})^{-a_1}\hat{e}(k) f \mid a_1 \in \mathbb{N}, \forall i \in I^\beta / \sim, f \ is \ a \ symmetric \ polynomial \ in \ \{ \hat{X}_k\hat{e}(i), \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta \} \right\}.
\]

Proof. This follows from (3.55) and a similar argument used in the proof Lemma 3.59. \( \Box \)

In the rest of this section, we are going to enlarge the rings \( \mathcal{H}_\beta(q), \mathcal{H}_\beta \) and \( \mathcal{H}_\beta^+ \) so that certain elements become locally invertible in the bigger rings. To this end, we will use again the generalized Ore localization which is introduced in the appendix of this paper.
3.65. Definition. Let \( j \in \mathbb{I}^\beta \). If \( \bar{\mathcal{H}}_\beta = \mathcal{H}_\beta(q) \), then we define

\[
M_n(j) := \{ \hat{e}(j) \} \cup \left\{ (\hat{X}_r - q^b \hat{X}_s)\hat{e}(j) \mid 1 \leq r \neq s \leq n, b \in \mathbb{I}, j_r \neq b + j_s \right\}.
\]

If \( \mathcal{H}_\beta = \hat{H}_\beta \), then we define

\[
M'_n(j) := \{ \hat{e}(j) \} \cup \left\{ (\hat{x}_r - \hat{x}_s - b)\hat{e}(j) \mid 1 \leq r \neq s \leq n, b \in \mathbb{I}, j_r \neq b + j_s \right\}.
\]

Let \( \Sigma_n(j) \) and \( \Sigma'_n(j) \) be the multiplicative closed subset generated by the elements in \( M_n(j) \) and \( M'_n(j) \) respectively.

3.66. Definition and Theorem. All the assumptions and conditions in Lemma A1 are satisfied if we take

\[
A = \mathcal{H}_\beta(q), \quad A_0 := K[\hat{X}_k, (\hat{X}_r - \hat{X}_s)^{-1} \mid 1 \leq k \leq n, 1 \leq r < s \leq n],
\]

\[
\{e_i\}_{i=1}^n := \{ \hat{e}(i) \mid i \in \mathbb{I}^\beta \}, \quad \{S_j\}_{j=1}^m := \{ \Sigma_n(j) \mid j \in \mathbb{I}^\beta \}.
\]

Moreover, the resulting generalized Ore localization \( \mathcal{H}_\beta(q) \) is canonically isomorphic to the subalgebra of \( \varprojlim_{\mathcal{A}} H_n^\lambda \) generated by \( \mathcal{H}_\beta(q) \) and the elements in the following subset

\[
(\hat{X}_r - q^b \hat{X}_s)^{-1}\hat{e}(j) \quad 1 \leq r \neq s \leq n, b \in \mathbb{I}, j \in \mathbb{I}^\beta, j_r \neq b + j_s \}
\]

(3.67)

A similar statement holds if we replace \( \Sigma_n(j), \mathcal{H}_\beta(q) \) and \( \mathcal{H}_\beta(q) \) by \( \Sigma'_n(j), \hat{H}_\beta \) and \( \hat{H}_\beta \) respectively. In particular, the generalized Ore localization \( \hat{H}_\beta \) is canonically isomorphic to the subalgebra of \( \varprojlim_{\mathcal{A}} H_n^\lambda \) generated by \( \hat{H}_\beta \) and the elements in the following subset

\[
(\hat{x}_r - \hat{x}_s - b)^{-1}\hat{e}(j) \quad 1 \leq r \neq s \leq n, b \in \mathbb{I}, j \in \mathbb{I}^\beta, j_r \neq b + j_s \}
\]

(3.68)

Proof. This follows from a similar argument in the proof of Lemma 3.31. \( \square \)

3.69. Definition. Let \( j \in \mathbb{I}^\beta \). In the non-degenerate setting, we define

\[
\hat{M}_n(j) := \left\{ (1 - y_r - q^b(1 - y_s))e(j), (1 - y_s)e(j) \mid 1 \leq r \neq s \leq n, 0 \neq b \in \mathbb{I} \right\}.
\]

while in the degenerate setting, we define

\[
\hat{M}'_n(j) := \left\{ (b + y_r - y_s)e(j) \mid 1 \leq r \neq s \leq n, 0 \neq b \in \mathbb{I} \right\}.
\]

Let \( \hat{\Sigma}_n(j) \) and \( \hat{\Sigma}'_n(j) \) be the multiplicative closed subsets generated by the elements in \( \hat{M}_n(j) \) and in \( \hat{M}'_n(j) \) respectively.

In a similar way as Theorem 3.66, we are going to use Lemma A1 to construct, in the non-degenerate setting, a bigger ring which contains \( \mathcal{S}_\beta \) and the elements in following subset

\[
(1 - y_r - q^b(1 - y_s))^{-1}e(j), (1 - y_s)^{-1}e(j) \quad 1 \leq r \neq s \leq n, j \in \mathbb{I}^\beta, 0 \neq b \in \mathbb{I} \}
\]

(3.70)
and in the degenerate setting, a bigger ring which contains \( R_\beta \) and the elements in following subset

\[
(3.71) \quad \left\{ (b + y_r - y_s)^{-1} e(j) \mid 1 \leq r \neq s \leq n, j \in I^3, 0 \neq b \in I \right\}.
\]

3.72. Definition and Theorem. All the assumptions and conditions in Lemma A1 are satisfied if we take

\[
A = R_\beta, \quad A_0 := K[y_1, \cdots, y_n], \quad \{e_i\}_{i=1}^m := \{e(i) \mid i \in I^3\},
\]

\[
\{S_j\}_{j=1}^m := \{\hat{\Sigma}_n(j) \mid j \in I^3\}.
\]

In particular, we can embedded \( R_\beta \) into \( \hat{R}_\beta := A[S_1, \cdots, S_m] \) which is generated by elements in \( R_\beta \) together with the elements in the subset (3.70). A similar statement holds if we replace \( \hat{\Sigma}_n(j), R_\beta, R_\beta \) and (3.70) by \( \hat{\Sigma}_n(j), \bar{R}_\beta, \bar{R}_\beta \) and (3.71) respectively.

Proof. In view of the defining relations for the quiver Hecke algebra \( R_\beta \), it suffices to show that for any \( i \in I^3, 1 \leq k < n \) with \( i_k = i_{k+1} \), and 1 \( r \neq s \leq n \), if 0 \( \neq b \in I \) then

\[
(3.73) \quad e(i)\psi_k e(i)\hat{\Sigma}_n(i) \cap ((1 - y_{\sigma(r)}) - q^b(1 - y_{\sigma(s)}))e(i)R_\beta e(i) \neq \emptyset,
\]

and

\[
(3.74) \quad e(i)\psi_k e(i)\hat{\Sigma}_n(i) \cap (1 - y_r) e(i)R_\beta e(i) \neq \emptyset.
\]

We define

\[
G(Y) := \prod_{\sigma \in \{1, s_k\}} \left( (1 - y_{\sigma(r)}) - q^b(1 - y_{\sigma(s)}) \right).
\]

Our assumption ensures that \( G(Y) \) is symmetric on \( y_k \) and \( y_{k+1} \), it is clear that \( G(Y) \) commutes with \( \psi_k \) and \( G(Y) e(i) \in \hat{\Sigma}_n(i) \). Note that \( G(Y) e(i) \) has \((1 - y_r) - q^b(1 - y_s))e(i) \) as a left factor. It follows that

\[
e(i)\psi_k e(i)G(Y) e(i) = e(i)\psi_k G(X) e(i) = e(i)G(Y) e(i)\psi_k e(i) = G(Y) e(i)\psi_k e(i) 
\]

\[
\in e(i)\psi_k e(i)\hat{\Sigma}_n(i) \cap ((1 - y_r) - q^b(1 - y_s))e(i)R_\beta e(i).
\]

This proves (3.73), while (3.74) can be proved in a similar way. Hence we prove the first half of the theorem. The second half of the theorem can be proved in a similar way.

For each \( w \in \mathcal{S}_n \), we fix a reduced expression \( s_{j_1} \cdots s_{j_k} \) of \( w \) and define

\[
\psi_w := \psi_{j_1} \cdots \psi_{j_k}.
\]

3.75. Lemma. (cf. [12]) The elements in the following set

\[
\left\{ \psi_w y_1^{a_1} \cdots y_n^{a_n} e(i) \mid w \in \mathcal{S}_n, i \in I^3, a_1, \cdots, a_n \in \mathbb{N} \right\}
\]

form a \( K \)-basis of \( R_\beta \).
4. The main results

The purpose of this section is to give the main results (Theorem 4.1 and 4.2) of this paper and their proofs.

Let $\beta \in \mathbb{Q}^+_n$. By some abuse of notations, we define $e(\beta) := \sum_{i \in I^\beta} e(i) \in \mathcal{H}_\beta^\Lambda$. Then $e(\beta) \neq 0$ if and only if $\mathcal{H}_\beta^\Lambda \neq 0$ and if and only if $\mathcal{R}_\beta^\Lambda \neq 0$. Henceforth, we assume that $e(\beta) \neq 0$. Let $p(\Lambda) : \mathcal{R}_\beta \twoheadrightarrow \mathcal{R}_\beta^\Lambda$ be the naturally defined surjective algebra homomorphism.

Let $i \in I^\beta$. Let us consider the image in $\mathcal{R}_\beta^\Lambda$ of the elements in $\hat{M}_s(i)$ and $\hat{M}'_s(i)$. Let $0 \neq b \in I$, $1 \leq r \neq s \leq n$. In the non-degenerate case, since $y_r, y_s$ are nilpotent elements in $\mathcal{R}_\beta^\Lambda$ and commutes with each other, it follows that $(1 - y_r) - q^i(1 - y_s)$ is invertible in $\mathcal{R}_\beta^\Lambda$ as $q^b \neq 1$. Thus

$$(1 - y_r) - q^i(1 - y_s) c(i) \left( (1 - y_r) - q^i(1 - y_s) \right)^{-1} = e(i).$$

In the degenerate case, the discussion is similar. It follows that the map $p(\Lambda)$ naturally induces a surjective algebra homomorphism $p_1(\Lambda) : \mathcal{R}_\beta \twoheadrightarrow \mathcal{R}_\beta^H$ and a surjective algebra homomorphism $p_2(\Lambda) : \mathcal{R}_\beta' \twoheadrightarrow \mathcal{R}_\beta^H$.

Recall that there is a natural surjective algebra homomorphism $H_\beta(q) \twoheadrightarrow \mathcal{H}_\beta^\Lambda(q)$ in the non-degenerate case and a natural surjective algebra homomorphism $H_\beta \twoheadrightarrow H_\beta^\Lambda$ in the degenerate case, and we have denoted both surjective maps by the same symbol $p_\Lambda$.

The following two theorems are the main results of this paper.

4.1. Theorem. In the non-degenerate case, there is a $K$-algebra isomorphism $\theta : \mathcal{R}_\beta \cong \mathcal{H}_\beta(q)$, such that $e(i) \mapsto \hat{e}(i)$, $y_s e(i) \mapsto \hat{e}(i)(1 - q^{-i} X_s)\hat{e}(i)$ and

$$\psi_r e(i) \mapsto \begin{cases} q^{i_r} (T_r + 1)(\hat{X}_r - q\hat{X}_{r+1})^{-1}\hat{e}(i), & \text{if } i_r = i_{r+1}; \\ q^{-i_r} (T_r(\hat{X}_r - \hat{X}_{r+1}) + (q - 1)\hat{X}_{r+1})\hat{e}(i), & \text{if } i_r = i_{r+1} + 1; \\ (\hat{T}_r(\hat{X}_{r+1} - \hat{X}_r) + (1 - q)\hat{X}_{r+1}) \\ \times (\hat{X}_r - q\hat{X}_{r+1})^{-1}\hat{e}(i), & \text{otherwise.} \end{cases}$$

for any $i \in I^\beta$, $1 \leq s \leq n$ and $1 \leq r < n$.

The inverse map $\eta$ is given by:

$$\eta(\hat{e}(i)) = e(i), \quad \eta(\hat{X}_s \hat{e}(i)) = q^i(1 - y_s)\hat{e}(i), \quad \eta(\hat{X}_s^{-1}\hat{e}(i)) = q^{-i}(1 - y_{s+1})^{-1}\hat{e}(i),$$

and $\eta(\hat{T}_r \hat{e}(i))$ is equal to $\psi_r (1 - q + qy_{r+1} - y_r) e(i) - e(i)$ if $i_r = i_{r+1}$; or

$$\left(q\psi_r e(i) - (q - 1)(1 - y_{r+1}) e(i)\right) \left(q(1 - y_r) - (1 - y_{r+1})\right)^{-1} e(i),$$

if $i_r = i_{r+1} + 1$; or otherwise

$$\psi_r (q^{i_r} - q^{i_{r+1}+1} - q^{i_r} y_r + q^{i_{r+1}+1}y_{r+1})(q^{i_{r+1}} - q^{i_r} + q^{i_r} y_r - q^{i_{r+1}+1}y_{r+1})^{-1} e(i)$$

$$- (1 - q)q^{i_{r+1}}(1 - y_{r+1})(q^{i_{r+1}} - q^{i_r} + q^{i_r} y_r - q^{i_{r+1}+1}y_{r+1})^{-1} e(i).$$

4.2. Theorem. In the degenerate case, there is a $K$-algebra isomorphism $\theta' : \mathcal{R}_\beta' \cong \tilde{H}_\beta$, such that $e(i) \mapsto \tilde{e}(i)$, $y_s e(i) \mapsto \tilde{e}(i)(\hat{x}_s - i_s)\tilde{e}(i)$ and

$$\psi_r e(i) \mapsto \begin{cases} (s_r + 1)(1 + \hat{x}_{r+1} - \hat{x}_r)^{-1}\hat{e}(i), & \text{if } i_r = i_{r+1}; \\ \left(s_r(\hat{x}_r - \hat{x}_{r+1}) + 1\right)\hat{e}(i), & \text{if } i_r = i_{r+1} + 1; \\ s_r(\hat{x}_r - \hat{x}_{r+1} + 1) \\ \times (1 + \hat{x}_{r+1} - \hat{x}_r)^{-1}\hat{e}(i), & \text{otherwise.} \end{cases}$$

for any $i \in I^\beta$, $1 \leq s \leq n$ and $1 \leq r < n$. 
The inverse map $\eta$ is given by:

$$\eta(\hat{e}(i)) = e(i), \quad \eta(\check{x}_s\hat{e}(i)) = (y_s + i_s)e(i),$$

and $\eta(\check{s}_r\hat{e}(i))$ is equal to $\psi_r(1 + y_{r+1} - y_r)e(i) - e(i)$ if $i_r = i_{r+1}$; or

$$\left(\psi_r e(i) - e(i)\right)(1 - y_{r+1} + y_r)^{-1} e(i),$$

if $i_r = i_{r+1} + 1$; or otherwise

$$\psi_r \left(1 - i_r + i_{r+1} + y_{r+1} - y_r\right)\left(i_r - i_{r+1} - y_{r+1} + y_r\right)^{-1} e(i) \quad \text{or} \quad (i_r - i_{r+1} - y_{r+1} + y_r)^{-1} e(i).$$

By a natural restriction of the canonical map $\varprojlim_{\Lambda} \mathcal{H}_\beta^\Lambda(q) \to \mathcal{H}_\beta^\Lambda(q)$ to the subalgebra $\hat{\mathcal{H}}_\beta(q)$, we get a surjective algebra homomorphism $\pi_{1,\Lambda} : \hat{\mathcal{H}}_\beta(q) \to \mathcal{H}_\beta^\Lambda(q)$ such that for any $i \in I^\beta, 1 \leq r < n, 1 \leq k \leq n,$

$$\pi_{1,\Lambda}(\hat{e}(i)) = e(i), \quad \pi_{1,\Lambda}(\check{T}_r e(i)) = T_r e(i), \quad \pi_{1,\Lambda}(X_k \hat{e}(i)) = X_k e(i),$$

in the non-degenerate case. Similarly, we have a well-defined surjective algebra homomorphism $\pi_{2,\Lambda} : \hat{\mathcal{H}}_\beta \to \mathcal{H}_\beta^\Lambda(q)$ such that for any $i \in I^\beta, 1 \leq r < n, 1 \leq k \leq n,$

$$\pi_{2,\Lambda}(\hat{e}(i)) = e(i), \quad \pi_{2,\Lambda}(\check{s}_r \hat{e}(i)) = s_r e(i), \quad \pi_{2,\Lambda}(\check{x}_r \hat{e}(i)) = x_k e(i),$$

in the degenerate case.

By a natural restriction of the canonical map $\varprojlim_{\Lambda} \mathcal{H}_\beta^\Lambda(q) \to \mathcal{H}_\beta^\Lambda(q)$ to the subalgebra $\hat{\mathcal{H}}_\beta(q)$, we get a surjective algebra homomorphism $\pi_{1}(\Lambda) : \hat{\mathcal{H}}_\beta(q) \to \mathcal{H}_\beta^\Lambda(q)$ in the non-degenerate case. Similarly, we have a well-defined surjective algebra homomorphism $\pi_{2}(\Lambda) : \hat{\mathcal{H}}_\beta \to \mathcal{H}_\beta^\Lambda$ in the degenerate case.

Recall the definition of $\mathcal{H}_\beta(q)$ in Remark 3.27. By a natural restriction of the canonical map $\varprojlim_{\Lambda} \mathcal{H}_\beta^\Lambda(q) \to \mathcal{H}_\beta^\Lambda(q)$ to the subalgebra $\hat{\mathcal{H}}_\beta(q)$, we get a surjective homomorphism $\pi_{+}(\Lambda) : \hat{\mathcal{H}}_\beta^+(q) \to \mathcal{H}_\beta^\Lambda(q)$ from $\hat{\mathcal{H}}_\beta^+(q)$ onto $\mathcal{H}_\beta^\Lambda(q)$. This surjection coincides with the composition of the natural surjective homomorphism $\sigma_{1,\Lambda}$ from $\hat{\mathcal{H}}_\beta(q)$ onto $\mathcal{H}_\beta^\Lambda(q)$ with the natural injection $\iota$ from $\hat{\mathcal{H}}_\beta^+(q)$ into $\mathcal{H}_\beta(q)$. As a result, we get the following corollary.

4.3. Corollary. With the notations as above, we have that

$$\ker \pi_+(\Lambda) = \ker \sigma_{1,\Lambda} \cap \hat{\mathcal{H}}_\beta^+(q).$$

Recall that the elements $y_1, \beta, \ldots, y_n \in \mathcal{H}_\beta$ generate a $K$-subalgebra which is isomorphic to the polynomial $K$-algebra $K[t_1, \ldots, t_n]$. Let

$$e_m(y_1, \ldots, y_n) := \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} y_{i_1} \cdots y_{i_m} \in K[y_1, \ldots, y_n]^{\otimes n},$$

be the $m$-th elementary symmetric polynomial. It is well-known that for each $1 \leq k \leq n$,

$$y_k^n = \sum_{i=0}^{n-1} (-1)^{n-i-1} y_k^{n-i} (y_1, \ldots, y_n). \quad (4.4)$$

Let $m_n$ be the maximal ideal of $K[y_1, \ldots, y_n]$ generated by $y_1, \ldots, y_n$. Let $n_n := (m_n)^{\otimes n}$. Applying (4.4), we get that

$$\text{for any } k \in \mathbb{N}, \text{ there exists some } N(k) \in \mathbb{N}, \text{ such that } y_k^{N(k)} \text{ lives inside the two-sided ideal of } K[y_1, \ldots, y_n] \text{ generated by } (n_n)^k. \quad (4.5)$$
4.6. **Lemma.** For each \( \Lambda \in P^+ \), let \( I(\Lambda) \) be the two-sided ideal of \( \mathbb{R}_\beta \) generated by \( \{ y^{(\Lambda, \alpha_i)}_1 e(i) | i \in I^\beta \} \). Then
\[
\bigcap_\Lambda I(\Lambda) = \{0\},
\]
where the subscript runs through all \( \Lambda \in P^+ \).

**Proof.** Suppose that \( \bigcap_\Lambda I(\Lambda) \neq 0 \). Let 0 \( \neq z \in \bigcap_\Lambda I(\Lambda) \). Then there exists an integer \( k \in \mathbb{Z}^>0 \), such that for any \( j \in I^\beta \), we can write
\[
ze(j) = \sum_{i=1}^s \psi_{w_i} f_i e(j),
\]
where \( w_1, \cdots, w_s \in \mathfrak{S}_n \) are pairwise distinct, and \( f_i \in K[y_1, \cdots, y_n] \) such that \( \deg(f_i) < k \) for any \( 1 \leq i \leq s \).

Now we pick an integer \( N := N(k) \) as in (4.5). We take a special \( \Lambda \in P^+ \) such that \( (\Lambda, \omega_\beta) = N \) for any \( j \in I^\beta \). By assumption, \( z \in I(\Lambda) \), which implies that \( ze(\beta) \) lives inside the two-sided ideal of \( \mathbb{R}_\beta \) generated by \( y^{(\beta)} e(\beta) \). Hence by (4.5) \( ze(\beta) \) lives inside the two-sided ideal of \( \mathbb{R}_\beta \) generated by \( (n_n)^k e(\beta) \). However, this is a contradiction to (4.7) by Lemma 3.75 and the fact that \( n_n e(\beta) \) is central in \( \mathbb{R}_\beta \). \( \square \)

4.8. **Corollary.** We have the following natural injections:
\[
\mathbb{R}_\beta \hookrightarrow \tilde{\mathbb{R}}_\beta \hookrightarrow \varprojlim_{\Lambda} \mathbb{R}_\beta^\Lambda, \quad \mathbb{H}_\beta(q) \hookrightarrow \tilde{\mathbb{H}}_\beta(q) \hookrightarrow \varprojlim_{\Lambda} \mathbb{H}_\beta^\Lambda(q), \quad \tilde{H}_\beta \hookrightarrow \tilde{H}_\beta \hookrightarrow \varprojlim_{\Lambda} \mathbb{H}_\beta^\Lambda.
\]

**Proof.** The first injection follows from Lemma 4.6, while the other two injections follows directly from their definitions. \( \square \)

**Proof of Theorem 4.1 and 4.2:** By our choices of \( Q_s(i) \) in (2.30) and (2.31), it is easy to see that Brundan–Kleshchev’s isomorphisms \( \theta_\Lambda \) induces the isomorphism
\[
\theta_1 : \varprojlim_{\Lambda} \mathbb{R}_\beta^\Lambda \cong \varprojlim_{\Lambda} \mathbb{H}_\beta^\Lambda(q),
\]
in the non-degenerate case. We have the following diagrams:
\[
\begin{array}{ccc}
\mathbb{R}_\beta & \xrightarrow{\theta} & \mathbb{H}_\beta(q) \\
\varprojlim_{\Lambda} \mathbb{R}_\beta^\Lambda & \xrightarrow{\sim} & \varprojlim_{\Lambda} \mathbb{H}_\beta^\Lambda(q) \\
\mathbb{H}_\beta(q) & \xrightarrow{\eta} & \mathbb{H}_\beta \\
\varprojlim_{\Lambda} \mathbb{H}_\beta^\Lambda(q) & \xrightarrow{\sim} & \varprojlim_{\Lambda} \mathbb{H}_\beta^\Lambda \\
\mathbb{H}_\beta & \xrightarrow{\theta_1^{-1}} & \varprojlim_{\Lambda} \mathbb{H}_\beta^\Lambda
\end{array}
\]
where the vertical maps are the injections given in Corollary 4.8, and for the moment both \( \theta \) and \( \eta \) are only defined on a set of \( K \)-algebra generators. Note that the bottom maps are both \( K \)-algebra isomorphisms. In order to show that \( \theta \) and \( \eta \) can be extended to a pair of well-defined \( K \)-algebra homomorphisms, it is enough to check that the above diagrams commutes on a set of \( K \)-algebra generators of \( \mathbb{R}_\beta \) and \( \mathbb{H}_\beta(q) \) respectively.

To show the first diagram commutes on a set of \( K \)-algebra generators of \( \mathbb{R}_\beta \), it is suffices to show that
\[
\begin{align*}
\pi_1(\Lambda) \left( \theta \left( \psi_r e(i) \right) \right) &= \theta_\Lambda \left( \pi_1(\Lambda) \left( \psi_r e(i) \right) \right), \\
\pi_1(\Lambda) \left( \theta \left( y_s e(i) \right) \right) &= \theta_\Lambda \left( \pi_1(\Lambda) \left( y_s e(i) \right) \right), \\
\pi_1(\Lambda) \left( \theta \left( e(i) \right) \right) &= \theta_\Lambda \left( \pi_1(\Lambda) \left( e(i) \right) \right),
\end{align*}
\]
where $1 \in I^0, 1 \leq r < n, 1 \leq s \leq n$. The last two equalities are obvious true. It remains to verify the first equality. There are three cases:

**Case 1.** $i_r = i_{r+1}$. In this case,

\[
\pi_1(\Lambda)\left(\theta(\psi_r e(i))\right) = q^{i_r} (T_r + 1)(L_r - qL_{r+1})^{-1} e(i)
\]

\[
= (T_r + 1)(q^{-i_r} L_r - q^{-i_r} L_{r+1})^{-1} e(i)
\]

\[
= (T_r + 1)(1 - y_r - q + qy_{r+1})^{-1} e(i)
\]

\[
= (T_r + P_r(i))Q_r(i)^{-1} e(i) = \theta_\Lambda \left( p_1(\Lambda)(\psi_r e(i)) \right),
\]

as required.

**Case 2.** $i_r = i_{r+1} + 1$. In this case,

\[
\pi_1(\Lambda)\left(\theta(\psi_r e(i))\right) = q^{-i_r} (T_r (L_r - L_{r+1}) + (q - 1)L_{r+1}) e(i)
\]

\[
= (T_r(1 - q^{-1} - y_r + q^{-1} y_{r+1}) + (1 - q^{-1})(1 - y_{r+1})) e(i).
\]

By definition, in the non-degenerate case,

\[
P_r(i) = 1 + \frac{y_r - y_{r+1}}{1 - q^{-1}} - \frac{1}{1 - \frac{y_{r+1} - qy_r}{1 - q}} = 1 + \frac{q(y_{r+1} - y_r)}{1 - q - y_{r+1} + qy_r},
\]

\[
Q_r(i) = \frac{1}{1 - q^{-1} - \frac{1}{1 - \frac{y_{r+1} - qy_r}{1 - q}} = \frac{-q}{1 - q + qy_r - y_{r+1}}}
\]

Therefore,

\[
\theta_\Lambda \left( p_1(\Lambda)(\psi_r e(i)) \right) = (T_r + P_r(i))Q_r(i)^{-1} e(i)
\]

\[
= \pi_1(\Lambda)\left(\theta(\psi_r e(i))\right).
\]

**Case 3.** $i_r \notin \{i_{r+1}, i_{r+1} + 1\}$. In the non-degenerate case, we have that

\[
P_r(i) = \frac{1 - q}{1 - q^{i-r_{r+1}}} \left\{ 1 + \frac{y_r - y_{r+1}}{1 - q^{i-r_{r+1}} - \frac{1}{1 - \frac{q^{i-r_{r+1} + 1} - q^{i-r_{r+1}}}{q^{i-r_{r+1}}}} \right\}
\]

\[
= \frac{1 - q}{1 - q^{i-r_{r+1}}} \left\{ 1 + \frac{q^{i_r}(y_{r+1} - y_r)}{1 - q^{i-r_{r+1}} - q^{i_r} - q^{i-r_{r+1} + 1} + q^{i_r} y_{r+1}} \right\}
\]

\[
= \frac{(1 - q)q^{i_r+1}(1 - y_{r+1})}{q^{i_r+1} - q^{i_r} - q^{i_r+1} y_{r+1} + q^{i_r} y_r}
\]

\[
Q_r(i)^{-1} = (P_r(i) - 1)^{-1}
\]

\[
= \left( \frac{(1 - q)q^{i_r+1}(1 - y_{r+1})}{q^{i_r+1} - q^{i_r} - q^{i_r+1} y_{r+1} + q^{i_r} y_r} - 1 \right)^{-1}
\]

\[
= \left( \frac{q^{i_r+1} - q^{i_r} - q^{i_r+1} y_{r+1} + q^{i_r} y_r}{q^{i_r} - q^{i_r} y_r - q^{i_r+1} + q^{i_r+1} y_{r+1}} \right)^{-1}
\]

By definition,

\[
\pi_1(\Lambda)\left(\theta(\psi_r e(i))\right) = \left( T_r (L_{r+1} - L_r) + (1 - q)L_{r+1} \right) (L_r - qL_{r+1})^{-1} e(i)
\]

\[
= (T_r + P_r(i))Q_r(i)^{-1} e(i)
\]

\[
= \theta_\Lambda \left( p_1(\Lambda)(\psi_r e(i)) \right).
\]
This proves the claim for the first diagram. In a similar way, we can prove that the second diagram commutes on a set of \( K \)-algebra generators of \( \mathcal{H}_\beta(q) \). Therefore, \( \theta \) and \( \eta \) can be extended to a pair of well-defined \( K \)-algebra homomorphisms.

Finally, they are mutually inverse maps because it is easy to check that \( \theta \eta \) and \( \eta \theta \) are both equal to the identity map on a set of generators. This completes the proof of Theorem 4.1, while Theorem 4.2 can be proved in a similar way.

5. Some applications

The purpose of this section is to give some applications of Theorem 4.1 and 4.2. Throughout this section, we assume that \( K \) is an algebraically closed field.

For any \( K \)-algebra \( A \), we use \( A \text{-mod} \) to denote the category of finite dimensional left \( A \)-modules. We have the following surjective algebra homomorphisms:

\[
\pi_1(\Lambda) : \mathcal{H}_\beta(q) \to \mathcal{H}_\beta^A(q), \quad \pi_2(\Lambda) : \tilde{\mathcal{H}}_\beta \to \mathcal{H}_\beta^A(q), \quad \pi_{1,\Lambda} : \mathcal{H}_\beta(q) \to \mathcal{H}_\beta^A(q), \quad \pi_{2,\Lambda} : \tilde{\mathcal{H}}_\beta \to \mathcal{H}_\beta^A(q),
\]

Recall that \( I = \mathbb{Z}/e\mathbb{Z} \). Let \( \mathcal{H}_n(q) \text{-mod}_I \) be the full subcategory of \( \mathcal{H}_n(q) \text{-mod} \) such that all the eigenvalues of \( X_1 \) are in \( q^I \), \( \mathcal{H}_\beta(q) \text{-mod}_I \) (resp., \( \mathcal{H}_\beta(q) \text{-mod}_I \)) be the full subcategory of \( \mathcal{H}_\beta(q) \text{-mod} \) (resp., \( \mathcal{H}_\beta(q) \text{-mod}_I \)) such that all the eigenvalues of \( X_1 \) are in \( q^I \). Let \( H_n \text{-mod}_I \) be the full subcategory of \( H_n \text{-mod} \) such that all the eigenvalues of \( x_1 \) are in \( I \), and we define \( \tilde{H}_\beta \text{-mod}_I \) and \( \tilde{H}_\beta \text{-mod}_I \) in a similar way. Then we have the following natural inclusions:

\[
\mathcal{H}_\beta^A(q) \text{-mod} \subseteq \mathcal{H}_\beta(q) \text{-mod}_I = \mathcal{H}_\beta(q) \text{-mod}_I, \quad H_\beta^A \text{-mod} \subseteq \tilde{H}_\beta \text{-mod}_I = \tilde{H}_\beta \text{-mod}_I, \quad \mathcal{H}_\beta^A \text{-mod} \subseteq \tilde{\mathcal{H}}_\beta \text{-mod}_I \subseteq \tilde{\mathcal{H}}_\beta \text{-mod}.
\]

Recall that for any \( i = (i_1, \ldots, i_n), j = (j_1, \ldots, j_n) \in I^n, i \sim j \) wherever there exists some \( w \in S_n \) such that \( wi = j \). Recall also that the central characters of \( \mathcal{H}_n(q) \) are in bijection with the elements in the set \( I^n/\sim \) of \( S_n \)-orbits, and hence are in bijection with the elements \( \beta \in Q_+^n \). For any \( \beta \in Q_+^n \), let \( (\mathcal{H}_n(q) \text{-mod})[\beta] \) be the subcategory of \( \mathcal{H}_n \text{-mod} \) which is determined by the central character \( \chi_\beta \) of \( \mathcal{H}_n(q) \) corresponding to \( \beta \) (cf. [14]). In a similar way, we have the subcategory \( (H_n \text{-mod})[\beta] \) of \( H_n \text{-mod} \).

5.1. Lemma. We have that

\[
\mathcal{H}_n(q) \text{-mod}_I = \bigoplus_{\beta \in Q_+^n} (\mathcal{H}_n(q) \text{-mod})[\beta], \quad H_n \text{-mod}_I = \bigoplus_{\beta \in Q_+^n} (H_n \text{-mod})[\beta],
\]

\[
\mathcal{H}_\beta(q) \text{-mod}_I = \mathcal{H}_\beta(q) \text{-mod}_I = (\mathcal{H}_\beta(q) \text{-mod})[\beta], \quad \mathcal{H}_\beta \text{-mod} = \mathcal{H}_\beta \text{-mod},
\]

\[
\tilde{H}_\beta \text{-mod}_I = \tilde{H}_\beta \text{-mod}_I = (H_n \text{-mod})[\beta], \quad \tilde{\mathcal{H}}_\beta \text{-mod} = \tilde{\mathcal{H}}_\beta \text{-mod}.
\]

5.2. Lemma. We have that

\[
\tilde{\mathcal{H}}_\beta(q) \text{-mod}_I = \lim_{\lambda} (\mathcal{H}_\beta^A(q) \text{-mod}), \quad \tilde{H}_\beta \text{-mod}_I = \lim_{\lambda} (H_\beta^A \text{-mod}),
\]

\[
\tilde{\mathcal{H}}_\beta \text{-mod} = \lim_{\lambda} (\mathcal{H}_\beta^A \text{-mod}).
\]

Proof. We only prove the first and the third equalities as the second one can be proved in a similar way. For any finite dimensional module \( V \in \mathcal{H}_\beta(q) \text{-mod}_I \), we can find \( \ell \in \mathbb{N}, \kappa_1, \ldots, \kappa_\ell \in \mathbb{Z}/e\mathbb{Z}, \) such that

\[
(\tilde{X}_1 \bar{e}(\beta) - q^{\kappa_1}) \cdots (\tilde{X}_1 \bar{e}(\beta) - q^{\kappa_\ell})(v) = 0, \quad \forall v \in V,
\]

because \( K \) is algebraically closed. Set \( \Lambda := \sum_{i=1}^{\ell} \kappa_i. \) Then \( V \in \mathcal{H}_\beta^A(q) \text{-mod} \) as required.
Then the commutative diagram in the previous paragraph implies that 
\[ L \]
we define
where vertical maps are isomorphisms induced from \( \theta \)
(5.3)
Similar statements apply to the categories \( \mathbb{H}_n \), \( \mathbb{R}_\beta \) module over \( H \)
and \( H \) module over \( H \)
Henceforth, we shall use the above equalities to identify these categories. Let \( m, n \in \mathbb{N} \). If we shift the subscripts of each generator of \( \mathcal{H}_n(q) \) upward by \( m \) position, then we get an algebra \( \mathcal{H}_n^{(m)}(q) \) which is isomorphic to \( \mathcal{H}_n(q) \) and with standard generators \( T_{m+1}, \ldots, T_{m+n-1}, X_{m+1}, \ldots, X_{m+n} \). For each \( g \in \mathcal{H}_n(q) \), let \( g^{(m)} \) be its canonical image in \( \mathcal{H}_n^{(m)}(q) \). For any \( \alpha \in Q^+_m, \beta \in Q^+_n \) and \( i = (i_1, \ldots, i_m) \in I^\alpha, j = (j_1, \ldots, j_n) \in I^\beta \), we define the concatenation \( i \vee j := (i_1, \ldots, i_m, j_1, \ldots, j_n) \in I^{\alpha+\beta} \). Then the map
\[
f e(i) \otimes g(j) \mapsto f g^{(m)} e(i \vee j), \quad \forall f \in \mathcal{H}_n(q), g \in \mathcal{H}_n(q)
\]
can be naturally extended to a well-defined injective non-unital algebra homomorphism \( \mathcal{H}_n(q) \otimes \mathcal{H}_\beta(q) \hookrightarrow \mathcal{H}_{\alpha+\beta}(q) \). By definition, this injection also induces a natural injection
\[
\iota_{\alpha, \beta} : \mathcal{H}_\alpha(q) \otimes \mathcal{H}_\beta(q) \hookrightarrow \mathcal{H}_{\alpha+\beta}(q).
\]
In a similar way, the well-known non-unital injection \( \mathbb{H}_\alpha \otimes \mathbb{H}_\beta \hookrightarrow \mathbb{H}_{\alpha+\beta} \) naturally induces an injection
\[
\tilde{\mathbb{H}}_{\alpha} \otimes \tilde{\mathbb{H}}_{\beta} \hookrightarrow \tilde{\mathbb{H}}_{\alpha+\beta},
\]
which will still be denoted by \( \iota_{\alpha, \beta} \). We have the following commutative diagram of morphisms:
\[
\begin{array}{ccc}
\mathcal{H}_\alpha(q) \otimes \mathcal{H}_\beta(q) & \xrightarrow{\iota_{\alpha, \beta}} & \mathcal{H}_{\alpha+\beta}(q) \\
\downarrow{\iota} & & \downarrow{\iota} \\
\tilde{\mathbb{H}}_{\alpha} \otimes \tilde{\mathbb{H}}_{\beta} & \xrightarrow{\iota_{\alpha, \beta}} & \tilde{\mathbb{H}}_{\alpha+\beta}
\end{array}
\]
where vertical maps are isomorphisms induced from \( \theta \).

For any \( V \in \mathcal{H}_\alpha(q)\)-mod, let \( V^\theta \in \mathcal{H}_\alpha\)-mod such that \( V^\theta = V \) as a \( K \)-linear space and \( \tilde{\mathbb{H}}_{\alpha} \) acts on \( V^\theta \) through the isomorphism \( \theta \). For any \( V \in \mathcal{H}_\alpha(q)\)-mod, \( W \in \mathcal{H}_\beta(q)\)-mod, we have the following convolution products:
\[
V \circ W := \text{Ind}_{\alpha, \beta}^{\alpha+\beta} V \otimes W = \mathcal{H}_{\alpha+\beta}(q) \otimes \mathcal{H}_{\alpha}(q) \otimes \mathcal{H}_{\beta}(q) (V \otimes W) \in \mathcal{H}_{\alpha+\beta}(q)\)-mod,
\[
V^\theta \circ W^\theta := \text{Ind}_{\alpha, \beta}^{\alpha+\beta} V^\theta \otimes W^\theta = \mathbb{H}_{\alpha+\beta} \otimes \mathbb{H}_{\alpha} \otimes \mathbb{H}_{\beta} (V^\theta \otimes W^\theta) \in \mathbb{H}_{\alpha+\beta}\)-mod,
\]
Then the commutative diagram in the previous paragraph implies that
\[
(V \circ W)^\theta \cong V^\theta \circ W^\theta.
\]
(5.3)

Similar statements apply to the categories \( \mathbb{H}_\alpha\)-mod, \( \tilde{\mathbb{H}}_{\alpha}\)-mod. With these results in mind, one can translate verbatim most of the results in the representation theory of \( \mathcal{H}_n \) (say, in [7], [30]) into the results in the representation theory of \( \mathbb{H}_n \) (say, in [17]) and vice versa.

Let \( \mathcal{H}_n \in \{ \mathcal{H}_n(q), H_n \} \). For any \( (a_1, \ldots, a_n) \in I^n \), following [7], [13] and [17], we define \( L(a_1, \ldots, a_n) := f_{a_1} \cdots f_{a_n} 1 \), where 1 denotes the trivial irreducible module over \( \mathcal{H}_n \cong K \), and \( f_k \) is defined as in [7]. Then \( L(a_1, \ldots, a_n) \) is an irreducible module over \( \mathcal{H}_n \). Two irreducible \( \mathcal{H}_n \)-modules \( L(a_1, \ldots, a_n), L(b_1, \ldots, b_n) \) lie in
the same block if and only if \((a_1, \cdots, a_n) \sim (b_1, \cdots, b_n)\), i.e., they differ by a permutation. Note that in general a given irreducible module \(L\) may be parameterized by several different tuples \((a_1, \cdots, a_n)\). By a similar procedure \([17]\), one can define the irreducible module \(\tilde{L}(a_1, \cdots, a_n) := \tilde{f}_{a_1} \cdots \tilde{f}_{a_n}1\) for the quiver Hecke algebra \(R_\beta\) for each \(n\)-tuple \((a_1, \cdots, a_n) \in I^g\), where \(\beta \in Q_n^+\).

5.4. **Definition.** Let \(\alpha = \sum_{i \in I} l_i \alpha_i\), \(\beta = \sum_{i \in J} k_i \alpha_i \in Q_n^+\). We say that \(\alpha, \beta\) are **weakly separated** if for any \(1 \leq i, j \leq n\) with \(j - i \in \{1, -1\}\), either \(l_i = 0\) or \(k_i = 0\). We say that \(\alpha, \beta\) are **separated** if for any \(1 \leq i, j \leq n\) with \(j - i \in \{0, 1, -1\}\), either \(l_i = 0\) or \(k_i = 0\).

The following result was mentioned in \([13, 6.1.3]\) as a remark in the degenerate setting. The full details of the proof are included in \([9]\).

5.5. **Lemma.** (cf. \([13, 6.1.3]\), \([9]\)) Let \(k \in \mathbb{N}\) and \(n_1, \cdots, n_k \in \mathbb{N}\) such that \(\sum_{i=1}^k n_i = n\). Let \(\beta_i \in Q_{n_i}^+\) for each \(1 \leq i \leq k\). Set \(\beta := \sum_{i=1}^k \beta_i\). Suppose that \(\beta_1, \cdots, \beta_k\) are pairwise separated, then there is an equivalence of categories:

\[
(H_n-\text{mod})[\beta] \sim (\mathcal{H}_{n_1} \boxtimes \cdots \boxtimes \mathcal{H}_{n_k})-\text{mod}[\beta_1, \cdots, \beta_k].
\]

As a first application of Theorem 4.1, 4.2, we get that

5.6. **Corollary.** Let \(k \in \mathbb{N}\) and \(n_1, \cdots, n_k \in \mathbb{N}\) such that \(\sum_{i=1}^k n_i = n\). Let \(\beta_i \in Q_{n_i}^+\) for each \(1 \leq i \leq k\). Set \(\beta := \sum_{i=1}^k \beta_i\). Suppose that \(\beta_1, \cdots, \beta_k\) are pairwise separated, then there is an equivalence of categories:

\[
R_\beta-\text{mod} \sim (R_{\beta_1} \boxtimes \cdots \boxtimes R_{\beta_k})-\text{mod}.
\]

**Proof.** This follows from Lemma 5.5, Theorem 4.1, 4.2 and (5.3).

We remark the proof of Lemma 5.5 used certain intertwining elements of affine Hecke algebras introduced in \([25, \text{Sect. 2}]\) and \([21, \text{Sect. 5.1}]\). Note that Kang, Kashiwara and Kim have introduced in \([11, (1.3.1)]\) certain intertwiners inside the quiver Hecke algebras. However, it seems that one can not mimic the proof of Lemma 5.5 directly to get a proof of Corollary 5.6 inside the theory of quiver Hecke algebras because of the equality \([11, \text{Lemma 1.3.1(i)}]\) (which only make a difference for \(v_a := v_{a+1} + v_a \neq v_{a+1}\)).

The degenerate case of the following result follows from \([13, 6.1.4]\) and an inductive argument. The non-degenerate case is similar. In both cases the argument used the categorical equivalence in Lemma 5.5.

5.7. **Lemma.** Let \(k \in \mathbb{N}\) and \(n_1, \cdots, n_k \in \mathbb{N}\) such that \(\sum_{i=1}^k n_i = n\). For each \(1 \leq i \leq k\), let \(\beta_i \in Q_{n_i}^+\) and \(L(a^{(i)})\) be an irreducible module over \(H_{n_i}^{\text{aff}}\), where \(a^{(i)} = (a_1^{(i)}, \cdots, a_{n_i}^{(i)}) \in I_{n_i}^g\). If for any \(1 \leq i \neq j \leq k\), \(\sum_{i=1}^k \beta_i\) are weakly separated, then \(L(a^{(1)}) \circ \cdots \circ L(a^{(k)})\) is an irreducible module over \(H_{n}^{\text{aff}}\).

The following result is the second application of Theorem 4.1, 4.2.

5.8. **Corollary.** Let \(k \in \mathbb{N}\) and \(n_1, \cdots, n_k \in \mathbb{N}\) such that \(\sum_{i=1}^k n_i = n\). For each \(1 \leq i \leq k\), let \(\beta_i \in Q_{n_i}^+\) and \(L(a^{(i)})\) be an irreducible module over \(R_{\beta_i}\), where \(a^{(i)} = (a_1^{(i)}, \cdots, a_{n_i}^{(i)}) \in I_{n_i}^g\). Set \(\beta := \sum_{i=1}^k \beta_i\). If for any \(1 \leq i \neq j \leq k\), \(a^{(i)}, \sum_{i=1}^k \beta_i\) are weakly separated, then \(L(a^{(1)}) \circ \cdots \circ L(a^{(k)})\) is an irreducible module over \(R_\beta\).

**Proof.** This follows from Lemma 5.7 and (5.3).

In particular, the above corollary gives a partial answer in type A to the question raised in \([15, \text{Problem 7.6(ii)}]\). It would be interesting to know whether the sufficient condition given in the above corollary is also necessary or not.
Let $A$ be a symmetric generalized Cartan matrix and $\Gamma$ be the associated quiver with no loops as defined in [26, §3.2.4]. Let $g$ be the associated Kac-Moody Lie algebra over $\mathbb{C}$ with $P^+$ being the set of dominant integral weights and $Q^+_n$ being the set of positive root lattice. Given $\Lambda \in P^+$ and $\beta \in Q^+_n$, let $\mathcal{R}_\beta(\Gamma)$ and $\mathcal{R}_\beta^A(\Gamma)$ be the corresponding quiver Hecke algebra and cyclotomic quiver Hecke algebra associated with $\Gamma$.

The following conjecture has been a folklore for some years.

5.9. Conjecture. Let $A$ be a symmetric generalized Cartan matrix. Then the center of $\mathcal{R}_\beta(\Gamma)$ maps surjectively onto the center of $\mathcal{R}_\beta^A(\Gamma)$.

As the third application of Theorem 4.1 and 4.2, we shall prove the linear quiver cases and certain special cyclic quiver cases of the above conjecture. We keep the notation $\mathcal{R}_\beta^A := \bigoplus_{\beta \in \mathcal{Q}_n^+} \mathcal{R}_\beta$ to denote the type $A$ cyclotomic quiver Hecke algebra that we studied in previous sections, which is the cyclotomic quiver Hecke algebra associated to the quiver $i \to i + 1$ for any $i \in \mathbb{Z}/e\mathbb{Z}$.

Let $m \in \mathbb{N}$. If $e = 0$ then we denote by $\Gamma(m)$ the subquiver which are labelled by the vertices $1, 2, \cdots, m$. This is a finite type $A$ Dynkin quiver. Let $\mathcal{R}_\beta^A(\Gamma(m))$ be the similarly defined cyclotomic quiver Hecke algebra associated to the subquiver $\Gamma(m)$.

The following result is reminiscent of two similar results for the centers of cyclotomic Hecke algebras [3, Theorem 1] and [24, 3.4]. Let $t_1, \cdots, t_n$ be $n$ indeterminates over $K$. Recall from (1.2) and Theorem 1.5 that $J := \{\beta \in Q^+_n|H^A_\beta \neq 0\} = \{\beta \in Q^+_n|\mathcal{R}_\beta \neq 0\}$. It is well-known that $J$ is a finite set, i.e., the number of blocks of $H^A_n$ is finite.

5.10. Proposition. Assume that either $e = 0 = \text{char} K$ or $e = p = \text{char} K$, where $K$ is the ground field and $p > 0$ is a prime number. Let $\Lambda \in P^+$ and $n \in \mathbb{N}$. Then the center of $\mathcal{R}_\Lambda$ maps surjectively onto the center of $\mathcal{R}^A_\Lambda$. A similar result holds for the cyclotomic quiver Hecke algebra $\mathcal{R}^A_n(\Gamma(m))$ associated to the finite type $A$ subquiver $\Gamma(m)$ when $e = 0$.

Proof. We define

\[
\tilde{\mathcal{R}}_J := \bigoplus_{\beta \in J} \tilde{\mathcal{R}}_\beta, \quad \mathcal{R}_J := \bigoplus_{\beta \in J} \mathcal{R}_\beta, \quad \tilde{H}_J := \bigoplus_{\beta \in J} \tilde{H}_\beta.
\]

By construction, $\mathcal{R}^A_n := \bigoplus_{\beta \in J} \mathcal{R}^A_\beta$, $H^A_n := \bigoplus_{\beta \in J} H^A_\beta$. We have the following commutative diagram of morphisms:

\[
\begin{array}{ccc}
\mathcal{R}^A_n & \overset{\sim}{\longrightarrow} & H^A_n \\
\downarrow \scriptstyle{p_2(\Lambda)} & & \downarrow \scriptstyle{\pi_2(\Lambda)} \\
\tilde{\mathcal{R}}_J & \overset{\theta'}{\longrightarrow} & \tilde{H}_J
\end{array}
\]

where the two vertical maps are both surjective homomorphisms.

By definition, there is a surjective homomorphism from $H_n$ onto $H^A_n$ such that the image of each $X_s$ is equal to the image of $X_s \sum_{\beta \in J} e(\beta)$ in $H^A_n$ under $\pi_2(\Lambda)$ for $1 \leq s \leq n$. Brundan has proved in [3, Theorem 1] that the center $Z(H_n)$ of $H_n$ maps surjectively onto the center of $H^A_n$. It follows that the center $Z(H^A_n)$ of $H^A_n$ is the set of symmetric polynomials in $L_1e(\beta), \cdots, L_ne(\beta)$ for each $\beta \in J$. Therefore, the following subset

\[
\left\{ f(\hat{X}_1, \cdots, \hat{X}_n)e(\beta) \mid \beta \in J, f(t_1, \cdots, t_n) \text{ is a symmetric polynomial} \right\}
\]

in $K[t_1, \cdots, t_n]$
of $\tilde{H}_J$ maps surjectively onto the center of $H^n_n$. Using the isomorphism $\theta'$, we see that the above displayed subset (which is contained in the center of $H_J$) is mapped by $(\theta')^{-1}$ into the center $Z(\tilde{\mathbb{A}}_J)$ of $\tilde{\mathbb{A}}_J$. Since
\[
(\theta')^{-1}(\{\tilde{X}_1 e(\beta), \ldots, \tilde{X}_n e(\beta), e(\beta)\}) \subseteq \mathbb{A},
\]
it follows from the previous commutative diagram that $Z(\tilde{\mathbb{A}}_J) \supseteq Z(\tilde{\mathbb{A}}_J) \cap \mathbb{A}$ must map surjectively onto the center of $\mathbb{A}$. Since $Z(\tilde{\mathbb{A}}_J) \subseteq Z(\mathbb{A})$, it follows that the center of $\mathbb{A}$ maps surjectively onto the center of $\mathbb{A}$. As a consequence, we also see that for each $\beta \in Q^n_+$,
\[
(5.11) \quad \text{the center of } \mathbb{A} \text{ maps surjectively onto the center of } \mathbb{A}.
\]
This proves the first part of the proposition.

We now consider the second part of the proposition. Given the cyclotomic quiver Hecke algebra $\mathbb{R}_n^\Lambda(\Gamma(m))$ associated to the finite type $A$ subquiver $\Gamma(m)$, there is an idempotent $e(m) \in \mathbb{R}_n^\Lambda$ such that $\mathbb{R}_n^\Lambda(\Gamma(m)) = e(m)\mathbb{R}_n^\Lambda e(m)$ and $e(m)\mathbb{R}_n^\Lambda(1 - e(m)) = 0 = (1 - e(m))\mathbb{R}_n^\Lambda e(m)$. In fact,
\[
e(m) = \sum_{i_1, \ldots, i_n \in \{1, 2, \ldots, m\}} e(i_1, \ldots, i_n),
\]
and $e(m)$ is the identity element of $\mathbb{R}_n^\Lambda(\Gamma(m))$. It follows that
\[
Z(\mathbb{R}_n^\Lambda(\Gamma(m))) = Z(\mathbb{R}_n^\Lambda) \cap e(m)\mathbb{R}_n^\Lambda e(m) = e(m)Z(\mathbb{R}_n^\Lambda)e(m),
\]
from which the second part of the proposition also follows. 

As a consequence, we see that Conjecture 5.9 holds for linear quivers provided that char $K = 0$ and for cyclic quiver of length $p$ provided that char $K = p > 0$.

Finally, let $e \in \{0, 2, 3, \ldots\}$ and $K$ be an arbitrary field. McGerty proved in [24, Theorem 2.5] that the center of the non-degenerate cyclotomic Hecke algebra $\mathcal{H}_2^\Lambda(q)$ is the set of symmetric polynomials in $L_1, L_2$. Let $J := \{\beta \in Q^n_+ | \mathbb{R}_n^\Lambda \neq 0\}$. Setting $n = 2$ and replacing $\tilde{\mathbb{A}}_J$, $\tilde{H}_J$, $H_J$ in the proof of Proposition 5.10 by $\tilde{\mathbb{A}}_J$, $\tilde{H}_J$, $H_J$ respectively, the same argument as the proof of Proposition 5.10 will show the following corollary (which gives a further evidence of Conjecture 5.9).

5.12. Corollary. Let $e \in \{0, 2, 3, \ldots\}$ and $K$ be an arbitrary field. Then the center of $\mathbb{A}$ maps surjectively onto the center of $\mathbb{A}$.

**Appendix A. The Generalized Ore Localization**

In this appendix, we want to generalize the classical construction of Ore localization with respect to a right denominator set to a more general situation as follows. Let $A$ be a (non-commutative) ring with identity 1, $A_0$ be a commutative subring of $A$. Let $e_1, \ldots, e_m$ be a complete set of pairwise orthogonal idempotents of $A$. That says, $\sum_{i=1}^m e_i = 1$ and $e_i e_j = \delta_{ij} e_i$ for any $i, j$. We assume further that $f e_i = e_i f$ for any $f \in A_0$ and $1 \leq i \leq m$. For each $1 \leq i \leq m$, let $S_i$ be a multiplicative closed subset in $e_i A_0 e_i$ with $e_i \in S_i$. We want to investigate certain generalized Ore conditions on $S_i$ under which the ring $\tilde{A}$ can be embedded into a larger ring $\tilde{\tilde{A}}$ such that

- (G1) for any $1 \leq i \leq m$ and any $s \in S_i$, $s$ is an invertible element in the unital ring $e_i A e_i$ (with identity element $e_i$); and
- (G2) each element in $\tilde{A}$ has both the form
\[
\sum_{1 \leq i, j \leq m} e_i u_{i, j, k} f_{i, j, k}^{-1},
\]
and the form
\[ \sum_{1 \leq i,j \leq m} g_{i,j,k}^{-1} b_{i,j,k} e_j, \]
where \( a_{i,j,k}, b_{i,j,k} \in A, f_{i,j,k}, g_{i,j,k} \in S_j \).

**A1. Lemma.** With the notations and assumptions as above, and assume further that the subsets \( \{S_i\}_{i=1}^m \) satisfy the following two conditions:

1. For any \( g, h \in A, s \in S_i \),
   \[ se_i g = 0 \implies e_i g = 0, \quad he_i s = 0 \implies he_i = 0. \]
   In particular, \( 0 \notin S_i \); and
2. For any \( 1 \leq i, j \leq m \) and any \( a, b \in e_i A e_i, s, t \in S_i \), there exist some \( b, c \in e_j A e_j, u \in S_j \) and \( v \in S_i \), such that \( au = bs, av = tc \).

Then there exists a ring \( A[S_1, \ldots, S_m] \) together with an injective ring homomorphism \( \varphi : A \rightarrow \tilde{A} : A[S_1, \ldots, S_m] \) such that both (G1) and (G2) hold, and for any ring homomorphism \( \psi : A \rightarrow B \) such that \( \psi(s) \) is invertible in \( \psi(e_i)B\psi(e_i) \) for every \( s \in S_i \) and \( 1 \leq i \leq m \), then there is a unique ring homomorphism \( \sigma : A[S_1, \ldots, S_m] \rightarrow B \) such that \( \sigma \varphi = \psi \). Moreover, if \( \psi \) is injective then \( \sigma \) is injective too.

**Proof.** We define
\[ A[S_1, \ldots, S_m] := \bigoplus_{1 \leq i,j \leq m} \left( e_i A e_i \times S_i \right)/\sim_{ij}, \]
where \( \sim_{ij} \) is an equivalence relation in \( e_i A e_i \times S_i \) defined as \( (a, s) \sim_{ij} (b, t) \) if \( at = bs \), where \( a, b \in e_j A e_j, s, t \in S_i \). Denote by \([a,s]\) the equivalence class containing \((a,s)\).

We define the addition and multiplication in an obvious way: for any \( a \in e_j A e_i, b \in e_k A e_l, s \in S_i, t \in S_l \):
1. In the case \( i = k, j = l \), \([a,s] + [b,t] := [(at + bs, st)]; \)
2. in the case \( i \neq k \) or \( j \neq l \), \([a,s] + [b,t] \) means a formal sum;

\[ [a,s] + [b,t] := \begin{cases} \{(ac, tu)\}, & \text{if } i = k \quad \text{where } bu = sc, u \in S_i, c \in e_i A e_i; \\ 0, & \text{if } i \neq k. \end{cases} \]

It is routine to check that the above definition is independent of the choice of the representing couples and \( A[S_1, \ldots, S_m] \) is a well-defined ring. The universal property of \( A[S_1, \ldots, S_m] \) follows from a similar (and more easy) argument as in the classical Ore localization (cf. [16, Corollary 10.11], [28, Proposition 1.4]). Finally, assume that \( \psi \) is injective. Suppose that \( \sigma(z) = 0 \), where
\[ z = \sum_{1 \leq i,j \leq m} a_{i,j,k} \varphi(f_{i,j,k})^{-1} \in A[S_1, \ldots, S_m], \quad a_{i,j,k} \in e_i A e_j, f_{i,j,k} \in S_j, \forall i,j,k. \]

Then for any \( i, j \),
\[ \sigma \left( \sum_k a_{i,j,k} \varphi(f_{i,j,k})^{-1} \right) = \sigma(e_i z e_j) = \sigma(e_i) \sigma(z) \sigma(e_j) = 0. \]

It follows that
\[ \psi \left( \sum_k a_{i,j,k} \prod_{l \neq k} f_{i,j,l} \right) = \sigma \left( \sum_k a_{i,j,k} \prod_{l \neq k} \varphi(f_{i,j,l}) \right) = 0. \]
Since $\psi$ is injective, we can see that $\sum_k a_{i,j,k} \prod_{l \neq k} f_{i,j,l} = 0$ and hence
\[
\sum_k a_{i,j,k} \varphi(f_{i,j,k})^{-1} = 0,
\]
for each $i, j$, which implies that $z = 0$ and hence $\sigma$ is injective. This completes the proof of the lemma. □

A2. **Definition.** With the notations and assumptions as in Lemma A1, we shall call the ring $A[S_1, \cdots, S_m]$ the generalized Ore localization of $A$ with respect to the data $(A_0, \{e_i\}_{i=1}^m, \{S_j\}_{j=1}^m)$. 

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School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081, P.R. China

E-mail address: junhu404@bit.edu.cn

Department of Mathematics, Zhejiang University, Hangzhou, 310027, P.R. China

E-mail address: fangli@zju.edu.cn