A Background Independent Formulation of Noncritical String Theory

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ABSTRACT

Using the string field theory recently proposed by the authors and collaborators, we give a background independent formulation of rational noncritical string theories with $c \leq 1$. With a little modification of the string field Hamiltonians previously constructed, we obtain string field theories which include various rational noncritical string theories as classical backgrounds.
1 Introduction

Constructing a field theory of string is one of the most important problems in string theory\textsuperscript{[1]}. One of the advantages of string field theories is that they give a background independent formulation of string theory. In the first quantized string theory, only perturbative analyses around a solution of the classical equations of motion are possible. The theory heavily depends on which classical background is chosen. The problem is that innumerable classical solutions are known now. In order to answer questions such as which of the classical solutions (or which superposition of them) describes the real world, we should have a background independent formulation of string theory.

Recently a new class of string field theories of noncritical strings are formulated by the authors and collaborators \textsuperscript{[2][3][4][5][6][7]}. Other recent developments are in \textsuperscript{[8]}. It is natural to ask if such a formulation provides a background independent one. The purpose of the present work is to answer this question.

Let us briefly explain the formalism of our string field theory. We introduce the creation (annihilation) operator $\Psi_{i}^{\dagger}(l)$ of a string. The argument $l$ here denotes the length of the string and the subscript $i$ does the other degrees of freedom. They satisfy the following commutation relations:

$$[\Psi_{i}(l), \Psi_{j}^{\dagger}(l')] = \delta_{ij}\delta(l - l').$$  \hspace{1cm} (1)

With the vacuum state $|0\rangle$, satisfying $\Psi_{i}(l)|0\rangle = 0$, one can express the $N$-string amplitude as

$$\lim_{D \to \infty} <0|e^{-DH}\Psi_{i_{1}}^{\dagger}(l_{1}) \cdots \Psi_{i_{N}}^{\dagger}(l_{N})|0>.$$  \hspace{1cm} (2)

Here $H$ is the Hamiltonian of the string field theory. The Schwinger-Dyson (S-D) equations for the string field are written as

$$\lim_{D \to \infty} \partial_{D} <0|e^{-DH}\Psi_{i_{1}}^{\dagger}(l_{1}) \cdots \Psi_{i_{N}}^{\dagger}(l_{N})|0> = 0.$$  \hspace{1cm} (3)

The classical equations of motion of a field theory is nothing but the classical limit of the S-D equations. Therefore, we expect that eq.(3) with $N = 1$ corresponds to the classical equation of motion of our string field theory, when only the string tree level contributions are included. The classical backgrounds are the solutions to this tree level S-D equation. We would like to construct a Hamiltonian with which we can realize various first quantized string theories as the solutions to the tree level S-D equation.

In section 2, we will construct the string field Hamiltonian which includes the multicritical points of the one matrix model as the classical backgrounds. In section 3, we will give a formulation in which $c \leq 1$ rational string theories are realized as the classical backgrounds. Section 4 is devoted to the discussions.
2 A Background Independent Formulation of the Multicritical One Matrix Models

The one matrix model has multicritical points [9], each of which corresponds to a \((2, 2k-1)\) conformal field theory coupled to quantum gravity. These multicritical points can be considered as the classical backgrounds of one string theory. In this section, we would like to construct a string field Hamiltonian which has these multicritical theories as classical backgrounds.

Let us recall the Hamiltonian for the second critical point, i.e. \(c = 0\) case [2]. In this case, the only degrees of freedom of a string is its length and we need no subscript \(i\) for the string creation and annihilation operators in eq.(1). The string field Hamiltonian becomes

\[
\mathcal{H} = \int_0^\infty dl_1 \int_0^\infty dl_2 \Psi^\dagger(l_1)\Psi^\dagger(l_2)\Psi(l_1 + l_2)(l_1 + l_2) + g \int_0^\infty dl_1 \int_0^\infty dl_2 \Psi^\dagger(l_1 + l_2)\Psi(l_1)\Psi(l_2)l_1l_2 + \int_0^\infty dl \rho(l)\Psi(l). \tag{4}
\]

The last term in the above is the tadpole term which describes the process where a string with zero length disappears. \(\rho(l)\) is taken to be \(3\delta''(l) - \frac{3}{4}t\delta(l)\) in which \(t\) is the cosmological constant. The form of \(\rho(l)\) is fixed to reproduce the disk amplitude for \(c = 0\) string theory. The S-D equation eq.(3) for the disk amplitude \(f(l)\) is [2]

\[
l \int_0^l dl' \left( \int_0^l dl'' f(l')f(l-l') + \rho(l) \right) = 0, \tag{5}
\]

or

\[-\partial_\zeta (\tilde{f}(\zeta))^2 + \tilde{\rho}(\zeta) = 0, \tag{6}\]

in the Laplace transformed form. Here

\[
\tilde{f}(\zeta) = \int_0^\infty d\zeta e^{-\zeta l} f(l), \quad \tilde{\rho}(\zeta) = \int_0^\infty d\zeta e^{-\zeta l} \rho(l). \tag{7}
\]

In order for the \(c = 0\) disk amplitude \(\tilde{f}(\zeta) = (\zeta - \frac{1}{2}\sqrt{7})\sqrt{\zeta + \sqrt{t}}\) to be the solution for eq.(6), we should choose

\[
\tilde{\rho}(\zeta) = 3\zeta^2 - \frac{3}{4}t, \tag{8}
\]

or

\[
\rho(l) = \delta''(l) - \frac{3}{4}t\delta(l). \tag{9}
\]

Now let us first consider how one can construct the string field Hamiltonian corresponding to a multicritical point of the one matrix model. In such a case, the disk amplitude \(\tilde{f}(\zeta)\) is of the form

\[
\sqrt{Q(\zeta)}, \tag{10}\]

\(Q(\zeta) = \zeta^2 - \frac{3}{4}t\).
where $Q(\zeta)$ is a polynomial of $\zeta$. $Q(\zeta)$ is of degree $2k - 1$ for the $k$-th critical point. Therefore, it is conceivable that one can obtain the Hamiltonian by just changing the $\rho$ in eq.(4) as

$$\tilde{\rho}(\zeta) = Q'(\zeta). \quad (11)$$

However this does not work. One can see that most easily in the following manner. Let us decompose the Hamiltonian in eq.(4) as

$$H = \int_0^\infty dl[T(l) + \rho(l)]\Psi(l), \quad (12)$$

where

$$T(l) = \int_0^l dl', \Psi^\dagger(l')\Psi^\dagger(l - l') + g \int_0^\infty dl', \Psi^\dagger(l + l')\Psi(l')l'. \quad (13)$$

Then the S-D equation for the string field is equivalent to

$$<v|[lT(l) + \rho(l)] = 0, \quad (14)$$

with

$$<v| = \lim_{D \to \infty} <0|e^{-DH}. \quad (15)$$

In order for eq.(14) to be integrable, the operator $lT(l) + \rho(l)$ should form a closed Lie algebra. In [3] it was shown that this condition is related to a sort of the residual general coordinate invariance on the worldsheet. The commutator of $lT(l) + \rho(l)$ was calculated to be

$$[l_1T(l_1) + \rho(l_1), l_2T(l_2) + \rho(l_2)] = gl_1l_2[l_1 - l_2][T(l_1 + l_2) + \rho(l_1 + l_2)] - gl_1l_2[l_1 - l_2]\rho(l_1 + l_2). \quad (16)$$

The integrability requires that the last term on the right hand side should vanish. In the Laplace transformed form, this means

$$\frac{\partial_{\zeta_1}\partial_{\zeta_2}(\partial_{\zeta_1} - \partial_{\zeta_2})}{\zeta_1 - \zeta_2} \int_{\zeta_2}^{\zeta_1} d\zeta' \tilde{\rho}(\zeta') = 0. \quad (17)$$

This equation is satisfied only when $\tilde{\rho}(\zeta)$ is a polynomial of $\zeta$ with degree less than 4. Then the algebra in eq.(16) is equivalent to the Virasoro algebra. Therefore the S-D equation eq.(14) is consistent only for the first and the second critical points.

In order to express the higher critical points, we should change the form of the Hamiltonian eq.(4) without spoiling the Virasoro algebra. The structures of the processes corresponding to the splitting and the joining of the strings are very firm and it is difficult to change them. The only modification one can think of is to increase the power of $l$ in front of $T(l)$ in eq.(12):

$$H = \int_0^\infty dl[l^nT(l) + \rho(l)]\Psi(l). \quad (18)$$

\footnote{Notice that a kind of coherent state representation of $T(l)$ was used in [3].}
This modification means that when we consider the propagation of a string with length $l$, we take the time variable $D$ to be $ln^{-1} \times$ (geodesic distance) instead of the geodesic distance itself. Then the S-D equation becomes

$$<v| [lnT(l) + \rho(l)] = 0. \quad (19)$$

The disk amplitude of the $k$-th multicritical point is reproduced if $\tilde{\rho}(\zeta)$ is a polynomial of degree $2k-1-n$. The integrability condition is satisfied if

$$\partial^n_\zeta \partial^n_\zeta (\partial_\zeta - \partial_\zeta) \int_\zeta d\zeta_1 \int_\zeta d\zeta_2 \cdots \int_\zeta d\zeta_n \tilde{\rho}(\zeta_n) = 0, \quad (20)$$

namely if $\tilde{\rho}(\zeta)$ is a polynomial of degree less than $n+3$. Therefore if one chooses $n \geq k-1$, it seems that the Hamiltonian in eq.(18) describes the $k$-critical points of the one matrix model.

Indeed one can show that one can derive the Virasoro constraints for the $k$-th critical point from the S-D equation eq.(19) with $\tilde{\rho}(\zeta)$ of degree $2k-1-n$ and $n \geq k-1$. Let us define the generating functional of the loop amplitudes as

$$Z(J) = <v| e^{-H} e^{\int dl J(l) \Psi^\dagger(l)} |0> . \quad (21)$$

Eq.(19) is rewritten in terms of $Z(J)$ as

$$[ln \{ \int_0^l dl' \frac{\delta^2}{\delta J(l') \delta J(l - l')} + g \int_0^\infty dl' d\gamma J(l') \frac{\delta}{\delta J(l + l')} \} + \rho(l)] Z(J) = 0. \quad (22)$$

This equation should be equivalent to the Virasoro constraints which are relations between the correlation functions of the local operators $O_r$, $r = n + \frac{1}{2}, \ n = 0, 1, 2, \cdots$, with which the string creation operator is expanded as

$$\Psi^\dagger(l) = g \sum_{r>0} \frac{l^r}{\Gamma(r+1)} O_r. \quad (23)$$

In order to extract such informations, we should choose $J(l)$ so that

$$\int_0^\infty dl J(l) l^r = g^{-1} \Gamma(r+1) x_r, \quad (24)$$

where $x_r$ is the source for $O_r$. Moreover, we should divide $Z(J)$ into the universal and the nonuniversal parts as $Z_{\text{non}} Z_{\text{univ}}$, where

$$Z_{\text{non}} = \exp \{ \int dl P(l) J(l) + \frac{1}{2} \int dldl' J(l) J(l') C_{\text{non}}(l, l') \}. \quad (25)$$

\footnote{Here we take a convention different from \cite{2} but rather follow \cite{3}. In the following, the indices $r, s$ should be understood as denoting half odd integers.}
\( P(l) \) is the nonuniversal contribution from the disk amplitude, which is of the form

\[
\sum_{r>0} c_r \frac{l^{-r}}{\Gamma(-r+1)},
\]

(26)

for the higher critical points. \( C_{\text{non}}(l, l') \) takes the same form for all the critical points:

\[
C_{\text{non}}(l, l') = \frac{g}{2\pi} \frac{\sqrt{l'}}{l + l'}.
\]

(27)

The universal part \( Z_{\text{univ.}} \) should be considered as the generating function of correlators:

\[
Z_{\text{univ.}}(x) = \langle e^{\sum_r x_r O_r} \rangle.
\]

(28)

In the Laplace transformed form, eq.(22) becomes the following equation for \( Z_{\text{univ.}} \):

\[
[(\frac{\partial}{\partial \zeta})^n_{\sum_{r>0}} \zeta^{-m-2} L_m + \sum_{r+s \geq 2} c_r c_s \zeta^{r+s-2} + 2g \sum_{r-s \geq 2} \zeta^{-s-2} c_r \frac{\partial}{\partial x_s} - g \sum_{r \geq 2} c_r \sum_{j=1}^{r-\frac{3}{2}} \zeta^{r-j-\frac{3}{2}} \int_0^\infty dr' J(r') \frac{r'-j+\frac{3}{2}}{\Gamma(-j+\frac{3}{2})} + \tilde{\rho}(\zeta)]Z_{\text{univ.}} = 0.
\]

(29)

Here \( L_m \)'s denote the Virasoro operators in [10] and

\[
L_m = g \sum_{r+s=m} \frac{\partial^2}{\partial x'_r \partial x'_s} + \sum_{r-s=m} r x'_r \frac{\partial}{\partial x'_s} + \frac{1}{4g} \sum_{r-s=m} r s x'_r x'_s + \frac{1}{16} \delta_{m,0},
\]

(30)

in our notation. In order for eq.(29) to be equivalent to the Virasoro constraints, the terms except for the first term in eq.(29) should cancel with each other. This happens only when \( c_r = 0 \) for \( r \geq n + \frac{5}{2} \) and

\[
(-\partial_\zeta)^n_{\sum_{r+s \geq 2}} c_r c_s \zeta^{r+s-2} + \tilde{\rho}(\zeta) = 0.
\]

(31)

Then the Virasoro constraints are derived from eq.(19).

For the \( k \)-th critical points with the disk amplitude eq.(10) with \( \tilde{Q}(\zeta) \) a polynomial of degree \( 2k - 1 \), \( \tilde{P}(\zeta) \) is of the form \( c_{k+\frac{1}{2}} \zeta^{k-\frac{1}{2}} + \cdots \), with \( c_{k+\frac{1}{2}} \neq 0 \). We should choose \( n \geq k - 1 \) and \( \tilde{\rho}(\zeta) \) becomes a polynomial of degree \( 2k - 1 - n \).

An interesting consequence of these considerations is the following. If one takes \( n > 2k - 1 \), \( \rho = 0 \). In this case, the disk amplitude \( f(l) \) satisfies the S-D equation

\[
\partial_\zeta^n \left( f(\zeta)^2 \right) = 0.
\]

(32)
Although there are no tadpole terms, the disk amplitude can be nonzero because of the integration constant of this equation. However, the disk amplitude for the $k$-th critical point is not the only solution of this equation. An amplitude of the form

$$\tilde{f}(\zeta) = \sqrt{Q(\zeta)},$$

(33)

with $Q(\zeta)$ a polynomial of degree less than $n/2$ is also a solution. Therefore we can choose $f(l)$ to be the disk amplitude for the $j$-th critical point with $j < n/2$. Then the above arguments imply that all the string amplitudes coincide with the amplitudes of the $j$-th critical point. Hence the string field theory with the Hamiltonian in eq.(18) with $\rho = 0$ has the $j$-th critical point as a classical background. Therefore this Hamiltonian gives a background independent formulation of $(2, 2j - 1)$ string theories with $j < n/2$.

We would like to conclude this section with the following remark about the nature of the “background” we treated in this section. Although we have a background independent formulation of $(2, 2j - 1)$ string theories, the partition function depends on which background is chosen[11], even if it is calculated nonperturbatively. As was discussed in [12], these backgrounds are rather superselection sectors. In our formalism, this fact manifests itself as follows. For the Hamiltonian eq.(18) with $\rho = 0$, expression of the amplitudes in eq.(2) does not have a definite meaning if one does not choose the background. Indeed, with only three string interaction terms, we can not expand it perturbatively. However once one chooses a background with which one shifts $\Psi^\dagger$, one obtains a Hamiltonian with a kinetic term and one can expand the amplitude perturbatively. Therefore the expression eq.(2) is well-defined only after a background is chosen.

3 A Background Independent Formulation of $c \leq 1$ String Theories

We have shown that our string field theory provides a background independent formulation of $(2, 2k - 1)$ string theories. In this section, we would like to generalize this to the whole $(p, q)$ string theories. In the Liouville theory approach, these theories can be considered as 2D string theories in various dilaton and tachyon backgrounds. Therefore it is reasonable to look for a string field theory which realizes these as classical backgrounds.

In order to do so, we will use the string field theories constructed in [3]. In that paper, we considered a string theory in which the matter degrees of freedom are expressed by an RSOS-like lattice model. Now the string fields $\Psi_i(l)$ have the subscript $i$ which indicates a node of a Dynkin-like diagram. The Hamiltonian is

$$\mathcal{H} = \sum_i \int_0^\infty dl_1 \int_0^\infty dl_2 \Psi^\dagger_i(l_1)\Psi^\dagger_i(l_2)\Psi_i(l_1 + l_2)(l_1 + l_2)$$
Here, \( C_{ij} \) is the connectivity matrix, where \( C_{ij} = 1 \) when the heights \( i \) and \( j \) are linked on the Dynkin-like diagram and it vanishes otherwise. In [3] we have shown that the S-D equation derived from this Hamiltonian satisfies the integrability condition. The Hamiltonian can be decomposed as

\[
\mathcal{H} = \sum_i \int_0^\infty dl_i T_i(l) \Psi_i(l),
\]

and the operator \( T_i(l) \) satisfies the decoupled Virasoro algebra:

\[
[T_i(l_1), T_j(l_2)] = -g(l_1 - l_2)T_i(l_1 + l_2)\delta_{ij}.
\]

This is satisfied for any symmetric \( C_{ij} \). If one chooses \( C_{ij} \) to be the connectivity matrix of an \( ADE \) or \( \tilde{A}\tilde{D}\tilde{E} \)-type Dynkin diagram, one can show that the disk amplitude of a \( c \leq 1 \) string theory satisfies the tree level S-D equation [3][13].

The facts given above were the only justification in [3] for taking the Hamiltonian in eq.(34) as the one which describes the \( c \leq 1 \) string theories. Recently [14] it is shown that the \( W \) constraints can be derived from this Hamiltonian. Therefore now the Hamiltonian in eq.(34) is on firm ground.

We would like to obtain a background independent formulation of string theories using this Hamiltonian. Following the previous section, let us first modify the Hamiltonian as

\[
\mathcal{H} = \sum_i \int_0^\infty dl^n T_i(l) \Psi_i(l).
\]

The S-D equations for the disk amplitudes \( f_i(l) \) become

\[
\partial_\xi^n [(\tilde{f}_i(\zeta))^2 + \sum_j C_{ij} \int_C \frac{d\zeta'}{2\pi i} \frac{\tilde{f}_i(\zeta') \tilde{f}_j(-\zeta')}{\zeta - \zeta'}] = 0.
\]

Here \( C \) denotes the contour in the complex \( \zeta' \) plane described in Fig.1. In [3], we claimed that for \( n = 1 \) these equations have a solution of the form

\[
\tilde{f}_i(\zeta) = v_i[(\zeta + \sqrt{\zeta^2 - t})^\alpha + (\zeta - \sqrt{\zeta^2 - t})^\alpha],
\]

where \( \alpha \) and \( v_i \) satisfy

\[
\sum_j C_{ij} v_j = -2\cos(\pi\alpha) v_i.
\]

If this is a solution for the \( n = 1 \) case, it is of course a solution for the \( n > 1 \) case. In order to prove that eq.(34) gives a solution to eq.(38), we should perform
the deformation of contour depicted in Fig.1. Strictly speaking, this deformation is allowed if \( n > 2\alpha \). Therefore we will take \( n \) to be large enough here to make this deformation well-defined.

The disk amplitude for \((p, q)\) string theory has the form in eq.(39) with \( \alpha = p/q \) or \( q/p \). For example, if one takes the \( A_{m-1} \) Dynkin diagram, the values of \( \alpha \) satisfying eq.(40) are

\[
\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m} \mod 1. \tag{41}
\]

Therefore, taking \( n = 3 \), the string field Hamiltonian eq.(37) for the \( A_{m-1} \) Dynkin diagram possesses the unitary \((m, m+1)\) string theory as the classical background. Besides that, the disk amplitude for the \((p, m)\), \((n = 1, \cdots, m-1, m+2, \cdots, [3m/2])\) string theories are also the solutions to eq.(38). Hence, with \( n = 3 \) and taking various \( A \)-type Dynkin diagrams, we can cover all the \((p, q)\) string theories. It is also possible to take \( D, E \) or even \( \hat{A} \hat{D} \hat{E} \) Dynkin diagrams, and cover such nondiagonal and \( c = 1 \) models.

Of course, what we would like to do is not to construct the Hamiltonian for various models, but to construct a Hamiltonian which includes these models as classical backgrounds. In order to do so, we should notice the following fact. The S-D equations eq.(38) for \( A_{m-1} \) Dynkin diagram have other special solutions. If the disk amplitudes \( f_i(l), i = 1, \cdots, m'-1 \) give a solution to eq.(38) for \( A_{m'-1}, m' < m \) diagram, then it is trivial to see that

\[
f_i(l) = f'_i(l), \quad i = 1, \cdots, m'-1, \\
f_i(l) = 0, \quad i = m', \cdots, m-1, \tag{42}
\]

give a solution to the equation for the \( A_{m-1} \) diagram. It is in general true that a solution to the equations corresponding to a subdiagram gives a special solution to those corresponding to the original diagram.

This fact makes it possible for us to give a background independent formulation of noncritical string theories. Indeed, if one takes a big diagram, string field Hamiltonian corresponding to it includes the string theories corresponding to the subdiagrams as classical backgrounds. For example, taking the diagram as in Fig. 2, i.e. \( A_\infty \), one can obtain a string field theory which has all the \( A_m \) type models as classical backgrounds. We can go even further and taking the diagram as in Fig. 3, we can obtain a string field theory which has all the rational \( c \leq 1 \) string theories corresponding to \( ADE \) and \( \hat{A}\hat{D}\hat{E} \) Dynkin diagrams as classical backgrounds.

Thus the Hamiltonian in eq.(34) corresponding to the diagram in Fig. 3 is what we want. We should remark that there can be many other classical backgrounds if one takes the diagram in Fig. 3. At present we are not sure about the nature of these backgrounds and especially their relation to the \( c = 1 \) barrier in two dimensional quantum gravity[10].

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\( ^3 \)Here we define the notion “subdiagram” as a diagram whose nodes form a subset of those of the original diagram and whose links are all the links connecting these nodes in the original diagram.
4 Discussions

In this work, we have presented string field Hamiltonians which include various rational string theories as classical backgrounds. It is shown that the Hamiltonian in eq.(18) with $\rho = 0$ has $(2, 2j - 1)$ ($j < n/2$) string theories as classical backgrounds. All the $ADE$ and $\dot{A}\dot{D}\dot{E}$ type string theories are classical backgrounds of one string field theory described by the Hamiltonian in eq.(34), where $C_{ij}$ is taken to be the connectivity matrix of the big Dynkin diagram in Fig. 3.

We have not tried to construct a field theory which includes irrational backgrounds. For example, $A_\infty$ diagram corresponds to a rational $c = 1$ string theory. It can be considered as a string theory on a discrete line. If one can take a “continuum limit” of this theory, one may be able to obtain an irrational $c = 1$ background. Such a problem is left as a future problem.

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Figure Captions

Fig. 1 The integration contour $C$ in eq. (38) and its deformation.

Fig. 2 The $A_{\infty}$ Dynkin diagram.

Fig. 3 The diagram which includes all the $ADE$ and $\hat{A}\hat{D}\hat{E}$ Dynkin diagrams as subdiagrams.
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