The spectral radius of graphs without long cycles

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Abstract

Nikiforov conjectured that for a given integer \( k \geq 2 \), any graph \( G \) of sufficiently large order \( n \) with spectral radius \( \mu(G) \geq \mu(S_{n,k}) \) (or \( \mu(G) \geq \mu(S_{n,k}^+) \)) contains \( C_{2k+1} \) or \( C_{2k+2} \) (or \( C_{2k+2} \)), unless \( G = S_{n,k} \) (or \( G = S_{n,k}^+ \)), where \( C_\ell \) is a cycle of length \( \ell \) and \( S_{n,k} = K_k \vee K_{n-k} \), the join graph of a complete graph of order \( k \) and an empty graph on \( n-k \) vertices, and \( S_{n,k}^+ \) is the graph obtained from \( S_{n,k} \) by adding an edge in the independent set of \( S_{n,k} \). In this paper, a weaker version of Nikiforov’s conjecture is considered, we prove that for a given integer \( k \geq 2 \), any graph \( G \) of sufficiently large order \( n \) with spectral radius \( \mu(G) \geq \mu(S_{n,k}) \) (or \( \mu(G) \geq \mu(S_{n,k}^+) \)) contains a cycle \( C_\ell \) with \( \ell \geq 2k+1 \) (or \( C_\ell \) with \( \ell \geq 2k+2 \)), unless \( G = S_{n,k} \) (or \( G = S_{n,k}^+ \)). These results also imply a result of Nikiforov given in [Theorem 2, The spectral radius of graphs without paths and cycles of specified length, LAA, 2010].

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1 Introduction

In this paper, all graphs considered are simple and finite. Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). For \( x \in V(G) \), let \( N_G(x) \) be the set of neighbors of \( x \) in \( G \) and let \( N_G[x] = N_G(x) \cup \{ x \} \). In particular, \( d_G(x) = |N_G(x)| \) is the degree of \( x \) in \( G \). For non-empty subset \( S \subseteq V \), let \( G[S] \) be the subgraph of \( G \) induced by \( S \) and write \( e_G(S) \) for \( e(G[S]) \). For disjoint subsets \( X \) and \( Y \) of \( V(G) \), let \( E_G(X,Y) \) be
the set of edges with one end in \( X \) and the other in \( Y \) and let \( e_G(X, Y) = |E_G(X, Y)| \). We write \( E_G(x, Y) \) instead of \( E_G(\{x\}, Y) \) for convenience. All the subscripts defined here will be omitted if \( G \) is clear from the context.

Given a graph \( G \), let \( A(G) \) and \( \mu(G) \) denote the adjacency matrix and the largest eigenvalue of \( A(G) \), called the spectral radius of \( G \). For a pair of adjacent vertices \( u, v \) of \( G \), define
\[
P_v(u) = \{ x \mid x \in N(u) \setminus \{v\}, \text{ but } x \not\in N(v) \}
\]
to be the set of private neighbors of \( u \) with respect to \( v \). Define \( G_{u \to v} \) to be the graph obtained from \( G \) by deleting the edges joining \( u \) to vertices in \( P_v(u) \) and adding new edges connecting \( v \) to vertices in \( P_v(u) \), that is \( V(G_{u \to v}) = V(G) \) and \( E(G_{u \to v}) = (E(G) \setminus E_G(u, P_v(u))) \cup E_G(v, P_v(u)) \).

As Turán type problems ask for maximum number of edges in graphs of given order not containing a specified family of subgraphs, Brualdi-Solheid-Turán type problems ask for maximum spectral radius of graphs of given order not containing a specified family of subgraphs. A survey about the subject can be found in [4]. In this paper, we mainly concern a Brualdi-Solheid-Turán type conjecture proposed by Nikiforov [3]. Let \( S_{n,k} \) be the graph obtained by joining every vertex of a complete graph of order \( k \) to every vertex of an independent set of order \( n - k \), that is \( S_{n,k} = K_k \lor K_{n-k} \), the join graph of \( K_k \) and \( K_{n-k} \), and let \( S_{n,k}^+ \) be the graph obtained from \( S_{n,k} \) by adding a single edge to the independent set of \( S_{n,k} \). For \( n = 2 \) and \( k = 1 \), we set \( S_{2,1} = S_{2,1}^+ = K_2 \). Write \( P_\ell \) and \( C_\ell \) for a path and cycle of order \( \ell \) and \( P_{\geq \ell} \) and \( C_{\geq \ell} \) for a path and cycle of order at least \( \ell \). In [3], Nikiforov proved the following theorem and proposed a related conjecture as follows.

**Theorem 1** (Nikiforov, 2010). Let \( k \geq 2 \), \( n > 2^4k \) and \( G \) be a graph of order \( n \).

(a) If \( \mu(G) \geq \mu(S_{n,k}) \), then \( G \) contains a \( P_{2k+2} \) unless \( G = S_{n,k} \).

(b) If \( \mu(G) \geq \mu(S_{n,k}^+) \), then \( G \) contains a \( P_{2k+3} \) unless \( G = S_{n,k}^+ \).

**Conjecture 2** (Nikiforov, 2010). Let \( k \geq 2 \) and \( G \) be a graph of sufficiently large order \( n \).

(a) If \( \mu(G) \geq \mu(S_{n,k}) \), then \( G \) contains a \( C_{2k+1} \) or \( C_{2k+2} \) unless \( G = S_{n,k} \).

(b) If \( \mu(G) \geq \mu(S_{n,k}^+) \), then \( G \) contains a \( C_{2k+2} \) unless \( G = S_{n,k}^+ \).

A first step to attack Conjecture 2 was given by Yuan, Wang and Zhai in [5], they proved that Conjecture 2 (a) holds when \( k = 2 \). It seems that it is harder to attack Conjecture 2 (b) than to (a), and there is few result known as we have checked.
In this paper, we prove the following theorem, which is weaker than Conjecture 2, but is stronger than Theorem 1 (Clearly, Theorem 1 is a corollary of Theorem 3 and we also give a better lower bound for $n$ in Theorem 3).

**Theorem 3.** Let $k \geq 2$ and $n \geq 13k^2$ and let $G$ be a graph of order $n$.

(a) If $\mu(G) \geq \mu(S_{n,k})$, then $G$ contains a $C_{2k+1}$ unless $G = S_{n,k}$.

(b) If $\mu(G) \geq \mu(S_{n,k}^+)$, then $G$ contains a $C_{2k+2}$ unless $G = S_{n,k}^+$.

The rest of the paper is arranged as follows. In Section 2, we give some preliminary facts and lemmas which will be used in the proof of Theorem 3. In Section 3, the proof of Theorem 3 is given and in the last section, we discuss some open problems.

## 2 Preliminaries

First we recall some notation not defined in the above section, if $G$ and $H$ are two graphs, we write:

- $e(G)$ for $|E(G)|$;
- $c(G)$ for the circumference (the length of a longest cycle in a graph) of $G$;
- $x(G)$ for the eigenvector of $A(G)$ corresponding to the largest eigenvalue $\mu(G)$;
- $f(G) = \prod_{u \in V(G)} d(u)$;
- $H \subseteq G$ for $H$ being a subgraph of $G$;
- $G + H$ for the graph on vertex set $V(G) \cup V(H)$ and $E(G) \cup E(H)$;
- $G - X$ for the subgraph of $G$ induced by $V(G) \setminus X$, where $X \subseteq V(G)$.

The following two facts were taken from [3] and proved in [1].

**Fact 1.** Let $k \geq 1$, $n > 3k$ and $G$ be a connected graph of order $n$. If

$$e(G) \geq e(S_{n,k}) = kn - (k^2 + k)/2$$

then $G$ contains a $P_{2k+2}$, unless there is equality in (1) and $G = S_{n,k}$.

**Fact 2.** Let $k \geq 1$, $n > 3k$ and $G$ be a connected graph of order $n$. If

$$e(G) \geq e(S_{n,k}^+) = kn - (k^2 + k)/2 + 1$$

then $G$ contains a $P_{2k+3}$, unless there is equality in (2) and $G = S_{n,k}^+$.

The following classical result was given in [2].
**Fact 3.** Let $\ell \geq 1$ and $G$ be a graph of order $n$. If

$$e(G) > (\ell - 2)n/2,$$

then $G$ contains a $P_\ell$.

The following fact can be checked directly from the structure of $S_n$ and $S_n^+$.

**Fact 4.** Let $n \geq 2k$. If a graph $G$ contains a subgraph $S_n^+$ (or $S_n^+$), then $G$ has a path $P_{\geq 2k-1}$ (or $P_{\geq 2k}$) with two ends in the class $V(K_k)$.

In the following we give four technical lemmas which will be used in the proof of Theorem 3.

**Lemma 4.** Let $k \geq 1$, $t \geq 2$. If $G$ is a graph with a partition $A \cup B$ of $V(G)$ such that $G[A] \cong K_t$, $e(A, B) \geq k|B|$ and $B$ has at least one vertex $b$ with $|N(b) \cap A| > k$, and $|B| > kt$, then $G$ has a path $P_{\geq 2k+1}$ with both ends in $A$ and $|V(P_{\geq 2k+1}) \cap B| \geq k$.

**Proof.**

**Claim 1.** There exist $k$ vertices $b_1, b_2, \ldots, b_k$ in $B$ such that $|N(b_i) \cap A| > k$ and $|N(b_i) \cap A| \geq k$ for $i = 2, \ldots, k$.

Let $S = \{x \mid x \in B$ and $|N(x) \cap A| \geq k\}$. Clearly, $b \in S$ and $|N(b) \cap A| > k$. So we can choose $b_1 = b$ and it is sufficient to show that $|S| \geq k$. If not then we have

$$k|B| \leq e(A, B) \leq |S||A| + (|B| - |S|)(k - 1) < k|B|,$$

the last inequality holds because $|B| > tk$, the contradiction implies the claim.

**Claim 2.** $G$ contains a path $P_{\geq 2k+1}$ with both ends in $A$ and $\{b_1, \ldots, b_k\} \subset V(P_\ell)$.

By induction on $k$. For $k = 1$, the claim is clearly true. Now suppose $k > 1$ and the statement holds for $k - 1$. Choose $a_k$ from $A$ such that $a_kb_k \in E(G)$ and let $A' = A \setminus \{a_k\}$ and $B' = B \setminus \{b_k\}$. Let $G'$ be the subgraph of $G$ induced by $A' \cup B'$. Thus $G'[A'] \cong K_{t-1}$, $|N_{G'}(b_i) \cap A'| \geq |N_G(b_i) \cap A| - 1 \geq k - 1$ for $i = 2, \ldots, k - 1$, and $|N_{G'}(b_1) \cap A'| \geq |N_G(b_1) \cap A| - 1 > k - 1$. By induction hypothesis, $G'$ contains a path $P_{\ell'} \geq 2k - 1$, with two ends, say $a_0$ and $a_{k-1}$, in $A'$ and $\{b_1, \ldots, b_{k-1}\} \subset B'$. If $\ell' \geq 2k$ then $P_{\ell'} + a_{k-1}a_k$ is the desired path. Hence assume $\ell' = 2k - 1$ and so $|V(P_{\ell'}) \cap A'| \leq k$. If one of $a_0, a_{k-1}$ belongs to $N_G(b_k)$, say $a_{k-1}b_k \in E(G)$, then $P_{\ell'} + a_{k-1}b_ka_k$ is a path with length $\ell' + 2 \geq 2k + 1$, as claimed. So, assume none of $a_0, a_{k-1}$ belongs to $N_G(b_k)$. Since $|N_G(b_k) \cap A| \geq k$, there is at least one vertex $a'_k \in N_G(b_k) \cap A$ such that $a'_k \notin V(P_{\ell'}) \cap A$. Therefore, $P_{\ell'} + a_{k-1}a'_kb_ka_k$ is a path with length $\ell' + 3 > 2k + 1$, as claimed. This completes the proof of the lemma.
**Remark.** In particular, the result is still true if we replace the condition “\( e(A, B) \geq k|B| \) and \( B \) has at least one vertex \( b \) with \(|N(b) \cap A| > k \)” by “\( e(A, B) > k|B| \)” in Lemma 4.

**Lemma 5.** Let \( G \) be a graph of order \( n \) and \( uv \in E(G) \). If \( P_v(u) \neq \emptyset \) and \( P_u(v) \neq \emptyset \) then either \( G' = G_{u \to v} \) or \( G' = G_{v \to u} \) has the property that \( c(G') \leq c(G) \), \( \mu(G') \geq \mu(G) \), and \( f(G') < f(G) \).

**Proof.** Let \( x = x(G) \). Without loss of generality, assume \( x_u \leq x_v \). We show that \( G' = G_{u \to v} \) is the desired graph.

Since \( x_v \geq x_u > 0 \), we have
\[
\mu(G') \geq \frac{x^T A(G') x}{x^T x} \geq \frac{x^T A(G) x}{x^T x} = \mu(G).
\]
Since \( d_{G'}(u) = d_{G'}(u) + |P_v(u)| \), \( d_{G}(v) = d_{G'}(u) + |P_u(v)| \), and \( d_{G'}(v) = d_{G'}(u) + |P_v(u)| + |P_u(v)| \), we have
\[
\frac{f(G')}{f(G)} = \frac{d_{G'}(u)d_{G'}(v)}{d_{G}(u)d_{G}(v)} < 1,
\]
that is \( f(G') < f(G) \).

Let \( C' = c_1c_2 \ldots c_kc_1 \) be a cycle of length \( \ell \) in \( G' \). We show that \( G \) has a cycle \( C \) of length at least \( \ell \). If \( v \notin V(C') \) then \( C = C' \) also is a cycle in \( G \) and we are done. Now suppose that \( v \in V(C') \) and let \( a \) and \( b \) be the two neighbors of \( v \) on \( C' \). If \( a \notin P_v(u) \) and \( b \notin P_v(u) \) then \( C = C' \) also is a cycle in \( G \) and we are done. Hence, without loss of generality, assume \( a \in P_v(u) \).

**Case 1.** \( b \notin P_u(v) \).

Hence \( b \in N_G(u) \). If \( u \notin V(C') \), set \( C = (C' - \{va, vb\}) \cup \{ua, ub\} \), then \( C \) is a cycle of length \( \ell \) in \( G \), as claimed. Hence assume \( u \in V(C') \). Let \( c \) and \( d \) be the two neighbors of \( u \) on \( C' \). Note that, for all \( x \in N_G'(u) \), \( x \in N_G(v) \). Set \( C = (C' - \{va, vb, uc, ud\}) \cup \{ua, ub, vc, vd\} \). Thus \( C \) is a cycle of length \( \ell \) in \( G \), as claimed.

**Case 2.** \( b \in P_u(v) \).

If \( u \notin V(C') \), set \( C = (C - \{av\}) \cup \{au, uv\} \), then \( C \) is a cycle of length \( \ell + 1 \) in \( G \), as claimed. Hence assume \( u \in V(C') \) and let \( c \) be the neighbor of \( u \) such that \( a, v, u, c \) are arranged in clockwise on \( C' \). Set \( C = (C' - \{av, uc\}) \cup \{au, vc\} \). Then \( C \) is a cycle of length \( \ell \) in \( G \), as claimed.

The proof is completed. \( \square \)
Remark: Since \( f(G) \) is limited, after finite steps, the moving neighbor operations in Lemma 5 will stop at a graph \( H \) with the property that \( \mu(H) \geq \mu(G) \), \( c(H) \leq c(G) \) and, for any edge \( uv \in E(H) \), \( P_v(u) = \emptyset \) or \( P_u(v) = \emptyset \).

**Lemma 6.** Given two positive integers \( a, b \) and a nonnegative symmetric irreducible matrix \( A \) of order \( n \), let \( \mu \) be the largest eigenvalue of \( A \) and let \( \mu' \) be the largest root of the polynomial \( f(x) = x^2 - ax - b \). Define \( B = f(A) \) and let \( B_j = \sum_{i=1}^{n} B_{ij} \) for \( j = 1, 2, \ldots, n \). If \( B_j \leq 0 \) for all \( j = 1, 2, \ldots, n \), then \( \mu \leq \mu' \) with equality holds if and only if \( B_j = 0 \) for all \( j = 1, 2, \ldots, n \).

**Proof.** Let \( x = (x_1, x_2, \ldots, x_n) \) be a positive eigenvector of \( A \) corresponding to \( \mu \) with \( \sum_{i=1}^{n} x_i = 1 \), and let \( 1 \) be the vector of dimension \( n \) with all entries 1. On one hand,

\[
1(Bx^T) = 1(\mu^2 - \alpha \mu - b)x^T = \mu^2 - \alpha \mu - b.
\]

On the other hand,

\[
(1B)x^T = (B_1, B_2, \ldots, B_n)x^T = \sum_{i=1}^{n} B_i x_i \leq 0.
\]

Hence, we have \( \mu^2 - \alpha \mu - b \leq 0 \), which implies that \( \mu \leq \mu' \) and the equality holds if and only if \( B_j = 0 \) for all \( j = 1, 2, \ldots, n \).

**Lemma 7.** Given integer \( m(\geq 1) \), let \( H \) be a graph with components \( A_1, \cdots, A_{m-1} \) and let \( G = (H + S_{tm,1}^+) \land K_1 \) and \( G' = (H + S_{t_m^+,1}^+ + S_{t_{m+1}^+,1}^+) \land K_1 \). If \( t_m' + t_{m+1}' = tm \) then \( \mu(G) > \mu(G') \).

**Proof.** The follow claim can be checked directly from the definitions of eigenvalue and eigenvector.

**Claim 3.** Let \( G \) is a connected graph and \( x \) be a positive eigenvector of \( A(G) \) corresponding to \( \mu(G) \). For any \( uv \in E(G) \), if \( P_v(u) = \emptyset \) and \( P_u(v) \neq \emptyset \) then \( x_v > x_u \), and if \( P_v(u) = P_u(v) = \emptyset \) then \( x_v = x_u \).

Let \( x = x(G') \) be the eigenvector of \( A(G') \) corresponding to \( \mu(G') \). Let \( u_1 \) and \( u_2 \) be centers of \( S_{tm,1}^+ \) and \( S_{tm+1,1}^+ \), respectively. By Claim 3 we have \( x_{u_1} \geq x_v \) for any \( v \in V(S_{tm,1}^+) \) and \( x_{u_2} \geq x_v \) for any \( v \in V(S_{tm+1,1}^+) \). Without loss of generality, assume \( x_{u_1} \geq x_{u_2} \). Note that \( G \) can be seen as the graph obtained from \( G' \) by deleting all the edges of \( E(S_{tm+1,1}^+) \) and adding new edges connecting \( u_1 \) to all vertices of \( V(S_{tm+1,1}^+) \). Clearly,

\[
\mu(G') = xA(G')x^T \leq xA(G)x^T \leq \mu(G),
\]
the equality holds only if \( x_{u_2} = x_{u_1} \). This implies that \( x \) is not an eigenvector of \( A(G) \) corresponding to \( \mu(G) \), otherwise, we have \( x_{u_1} > x_{u_2} \) by Claim 3. Therefore, \( \mu(G') < \mu(G) \). \( \square \)

3 Proof of Theorem 3

Clearly, \( \mu = \mu(S_{n,k}) \) is the largest root of the polynomial \( f(x) = x^2 - (k-1)x - k(n-k) \). Let \( G \) be a \( C_{2k+2} \)-free graph of order \( n \) with maximum spectral radius. By the remark of Lemma 5, we may assume \( P_u(v) = \emptyset \) or \( P_v(u) = \emptyset \) for any edge \( uv \in E(G) \). For \( u \in V(G) \), let \( Y_u = V(G) \setminus N(u) \), \( S_u = N(u) \cap N(Y_u) \), and \( T_u = N(u) \setminus S_u \). Let \( s_u = |S_u|, t_u = |T_u| \). The following claim holds.

Claim 4. For any \( u \in V(G) \), \( G[S_u] \cong K_{s_u}, K_{s_u} \cup \overline{K_{t_u}} \subseteq G[N(u)] \), and \( e_G(T_u, Y_u) = 0 \).

By definition of \( T_u \), \( e_G(T_u, Y_u) = 0 \). If there are two vertices \( v, w \) in \( S_u \) (or a vertex \( v \in S_u \) and a vertex \( w \in T_u \)) such that \( vw \notin E(G) \), then there is at least one vertex \( x \in Y_u \) with \( vx \in E(G) \), so \( P_u(v) \neq \emptyset \) (clearly, \( x \in P_u(v) \)) and \( P_v(u) \neq \emptyset \) (as \( w \in P_v(u) \)), a contradiction. Hence \( G[S_u] \cong K_{s_u} \) and \( K_{s_u} \cup \overline{K_{t_u}} \subseteq G[N(u)] \).

By the above claim and \( G \) contains no \( C_{2k+2} \), it is an easy task to check that the following claim holds.

Claim 5. For any \( u \in V(G) \), we have

1. \( s_u \leq 2k \);
2. \( \min\{s_u, t_u\} \leq k \); in particular, if \( s_u \geq t_u \) then \( d(u) \leq 2k + 1 \).

Proof of Theorem 3 (a)

Suppose \( G \) is a \( C_{2k+1} \)-free graph with \( \mu(G) \geq \mu = \mu(S_{n,k}) \) and \( G \neq S_{n,k} \). Let \( A = A(G), B = f(A) = A^2 - (k - 1)A - k(n - k)I, \) and \( B_u = \sum_{1 \leq i \leq n} B_{iu} \) for every \( u \in V(G) \). By definition, for every \( u \in V(G) \),

\[
B_u = e(N(u), Y_u) + 2e(N(u)) - (k - 2)d(u) - k(n - k).
\]  \( (3) \)

By Lemma 6 there must be a vertex \( u \in V(G) \) such that \( B_u \geq 0 \). To get a contradiction, we will show that either \( B_u \leq 0 \) and the equality holds if and only if \( G \cong S_{n,k} \) or \( B_u > 0 \) but \( \mu(G) < \mu \).

Case 1. \( G[N(u)] \) is connected.
Subcase 1.1. \( d(u) > 3k \).

By Claim 5, \( s_u < t_u \) and hence \( \min\{s_u, t_u\} = s_u \leq k \). This implies that

\[
e(N(u), Y_u) = e(S_u, Y_u) \leq k|Y_u|.
\] (4)

Since \( G \) contains no \( C_{2k+1} \), \( G[N(u)] \) contains no \( P_{2k} \). By Fact 1, we have

\[
e(N(u)) \leq (k - 1)d(u) - (k^2 - k)/2,
\] (5)

with equality if and only if \( G[N(u)] \cong S_{d(u), k-1} \). By (3), we have

\[
B_u \leq k|Y_u| + 2(k - 1)d(u) - (k^2 - k) - (k - 2)d(u) - k(n - k) = 0,
\]

and the equality holds if and only if the equalities hold in (4) and (5) if and only if \( s_u = k - 1 \) and \( Y_u = \emptyset \) if and only if \( G \cong S_{n,k} \), as desired.

Subcase 1.2. \( d(u) \leq 3k \).

First we claim that

\[
e(S_u, Y_u) > (k - 1)|Y_u|.
\] (6)

If not, then

\[
B_u \leq (k - 1)|Y_u| + d(u)(d(u) - 1) - (k - 2)d(u) - k(n - k)
\]

\[
= (d(u) - k + 1)^2 + k - n
\]

\[
\leq (2k + 1)^2 + k - n
\]

\[
< 0,
\]

the last inequality holds because \( n \geq 13k^2 \).

Inequality (3) implies that \( s_u \geq k \). Note that \( s_u \leq 2k \). Hence \( |Y_u| = n - d(u) - 1 \geq n - 3k - 1 > 9k^2 > s_u(k - 1) \). By Lemma 4, there is a path \( P \) of length at least \( 2k - 1 \) with two ends in \( S_u \) and \( |V(P) \cap Y_u| \geq k - 1 \). If \( |S_u \cup T_u| > k \), then it is an easy task to find a cycle of length at least \( 2k + 1 \) containing \( u \) and \( V(P) \), a contradiction. Therefore, \( s_u = k \) and \( t_u = 0 \). Note that \( d(u) = s_u = k \), \( |Y_u| = n - k - 1 \), and \( e(N(u), Y_u) \leq s_u|Y_u| = k|Y_u| \), we have

\[
B_u \leq k(n - k - 1) + k(k - 1) - (k - 2)k - k(n - k) = 0,
\]

and the equality holds if and only if \( G \cong S_{n,k} \) (otherwise, \( E(Y_u) \neq \emptyset \), hence \( S_{n-1,k}^{+} \subseteq G[S_u \cup Y_u] \) and so we can find a path \( P_{2k} \) with two ends in \( S_u \) by Fact 4, thus we have a cycle of length \( 2k + 1 \) containing \( V(P) \cup \{u\} \), a contradiction).
**Case 2.** $G[N(u)]$ is disconnected.

This implies that $S_u = Y_u = \emptyset$ and hence $d(u) = n - 1$ and $u$ is the only vertex of $G$ with $G[N(u)]$ being disconnected. Let $A_1, \ldots, A_t$ be all the components of $G - \{u\}$ and $n_i = |V(A_i)|$ for $i = 1, \ldots, t$.

**Subcase 2.1.** There is some $i$ with $n_i > 3k$.

Without loss of generality, assume $n_1 > 3k$. By Fact 1, $e(A_1) \leq (k - 1)n_1 - (k^2 - k)/2$, and by Fact 3, $\sum_{j=2}^{t} e(A_j) \leq (k - 1)(n - n_1 - 1)$. Therefore, we have

$$e(N(u)) = \sum_{i=1}^{t} e(A_i) \leq (k - 1)(n - 1) - (k^2 - k)/2.$$  

By (3), we have

$$B_u \leq 2(k - 1)(n - 1) - (k^2 - k) - (k - 2)(n - 1) - k(n - k) = 0.$$  

If $B_u < 0$ then we are done. Hence assume $B_u = 0$. If $\mu(G) = \mu = \mu(S_{n,k})$, then Lemma 6 implies that each $B_v = 0$ for all $v \in V(G)$. Note that $u$ is the only vertex with $G[N(u)]$ being disconnected. Case 1 implies that if there is a vertex $v$ with $G[N(v)]$ being connected and $B_v = 0$, then $G \cong S_{n,k}$, as desired.

**Subcase 2.2.** For all $i \in \{1, \ldots, t\}$, $n_i \leq 3k$.

Let $g(x) = x^2 - kx - (k - 1/2)(n - k)$ and reset $B = g(A) = A^2 - kA - (k - 1/2)(n - k)I$. Let $B_v = \sum_{1 \leq i \leq n} B_{iv}$ for every $v \in V(G)$. Hence

$$B_v = 2e(N(v)) + e(N(v), Y_v) - (k - 1)d(v) - (k - 1/2)(n - k). \quad (7)$$

Let $\mu'$ be the largest root of the polynomial $g(x)$. Note that $\mu = \mu(S_{n,k})$ is the largest root of $f(x) = x^2 - (k - 1)x - k(n - k)$. By simple computation, we have $\mu' < \mu$ when $n \geq 13k^2$. By Lemma 6 to prove $\mu(G) \leq \mu'(< \mu(S_{n,k}))$, it is sufficient to show $B_v \leq 0$ for any $v \in V(G)$.

By Fact 3, $e(N(u)) \leq (k - 1)(n - 1)$. By (7) and $n \geq 13k^2$, we have

$$B_u \leq 2(k - 1)(n - 1) - (k - 1)(n - 1) - (k - 1/2)(n - k) < 0.$$  

For every $i \in \{1, \ldots, t\}$ and $v \in V(A_i)$, since $n_i \leq 3k$, we have $d(v) \leq 3k$. By Fact 3, $e(N(v)) \leq (k - 1)d(v)$. Note that $u \in N(v)$ and $d(u) = n - 1$, we have
where the equality holds if and only if $Y$.

Subcase 1.2. $d(u) > 3k$.

By Claim 5, $\min\{s_u, t_u\} = s_u \leq k$. This implies that

$$e(N(u), Y_u) = e(S_u, Y_u) \leq k|Y_u|.$$  

Since $G$ is $C_{2k+2}$-free, $G[N(u)]$ does not contain a $P_{2k+1}$. By Fact 2

$$e(N(u)) \leq (k-1)d(u) - (k^2-k)/2 + 1,$$

with equality holds if and only if $G[N(u)] \cong S_{d(u),k-1}^+$. By (3) and note that $|Y_u| = n - d(u) - 1$, we have

$$B_u \leq k|Y_u| + 2(k-1)d(u) - (k^2-k) + 2 - (k-2)d(u) - k(n-k) = 2,$$

where the equality holds if and only if $Y_u = \emptyset$ (otherwise, $s_u = k$, contradicts to $G[N(u)] \cong S_{d(u),k-1}^+$ and hence $G \cong S_{d(u),k-1}^+$; and $B_u = 1$ if and only if $e(N(u), Y_u) = e(S_u, Y_u) = k|Y_u| - 1$ and $G[N(u)] \cong S_{d(u),k-1}^+$ if and only if $|S_u| = k-1$ and $|Y_u| = 1$, that is $G$ is a subgraph of $S_{n,k}^+$ with $e(G) < e(S_{n,k}^+)$ and so we have $\mu(G) < \mu(S_{n,k}^+)$.

Subcase 1.2. $d(u) \leq 3k$.  

This completes the proof of Theorem 3 (a).

Proof of Theorem 3 (b)

Now suppose that there is a connected $C_{2k+2}$-free graph $G$ of order $n$ such that $\mu(G) \geq \mu(S_{n,k}^+)$ and $G \neq S_{n,k}^+$. Without loss of generality, we assume that $G$ has maximum spectral radius among all of such graphs of order $n$. Let $A = A(G)$ be the adjacent matrix of $G$, and let $B = f(A) = A^2 - (k-1)A - k(n-k)I$ and $B_u = \sum_{1 \leq i \leq n} B_{iu}$ for $u \in V(G)$. By Lemma 6, there must exist some vertex $u \in V(G)$ such that $B_u > 0$. In the following, we will find a contradiction, that is we show that either $B_u \leq 0$ or $B_u > 0$ but $\mu(G) \leq \mu(S_{n,k}^+)$ with equality if and only if $G \cong S_{n,k}^+$.

Case 1. $G[N(u)]$ is connected.

Subcase 1.1. $d(u) > 3k$.

By Claim 5, $\min\{s_u, t_u\} = s_u \leq k$. This implies that

$$e(N(u), Y_u) = e(S_u, Y_u) \leq k|Y_u|.$$  

Since $G$ is $C_{2k+2}$-free, $G[N(u)]$ does not contain a $P_{2k+1}$. By Fact 2

$$e(N(u)) \leq (k-1)d(u) - (k^2-k)/2 + 1,$$

with equality holds if and only if $G[N(u)] \cong S_{d(u),k-1}^+$. By (3) and note that $|Y_u| = n - d(u) - 1$, we have

$$B_u \leq k|Y_u| + 2(k-1)d(u) - (k^2-k) + 2 - (k-2)d(u) - k(n-k) = 2,$$

where the equality holds if and only if $Y_u = \emptyset$ (otherwise, $s_u = k$, contradicts to $G[N(u)] \cong S_{d(u),k-1}^+$ and hence $G \cong S_{d(u),k-1}^+$; and $B_u = 1$ if and only if $e(N(u), Y_u) = e(S_u, Y_u) = k|Y_u| - 1$ and $G[N(u)] \cong S_{d(u),k-1}^+$ if and only if $|S_u| = k-1$ and $|Y_u| = 1$, that is $G$ is a subgraph of $S_{n,k}^+$ with $e(G) < e(S_{n,k}^+)$ and so we have $\mu(G) < \mu(S_{n,k}^+)$.

Subcase 1.2. $d(u) \leq 3k$.
With a same argument as the proof of inequality (6), we have
\[ e(N(u), Y_u) = e(S_u, Y_u) > (k - 1)|Y_u|. \]
So \( s_u \geq k \). Since \( s_u \leq 2k \), \( |Y_u| = n - d(u) - 1 > 9k^2 > s_u k > s_u(k - 1) \). By Lemma 4, there is a path \( P_{\geq 2k-1} \) with two ends in \( S_u \) and \( |V(P_{\geq 2k-1}) \cap Y_u| \geq k - 1 \).
If \( |S_u \cup T_u| > k + 1 \), then it is an easy task to find a cycle of length at least \( 2k + 2 \) containing \( u \) and \( V(P_{\geq 2k-1}) \), a contradiction. Therefore, \( s_u + t_u \leq k + 1 \).
If \( s_u = k \) and \( t_u = 0 \), then \( e(N(u)) = k(k - 1)/2 \). By (3), we have
\[ B_u \leq k(n - k - 1) + k(k - 1) - (k - 2)k - k(n - k) = 0. \]
If \( s_u = k \) and \( t_u = 1 \), then \( e(N(u)) = k(k + 1)/2 \). By (3), we have
\[ B_u \leq k(n - k - 2) + k(k + 1) - (k - 2)(k + 1) - k(n - k) = 2, \]
and equality holds if and only if \( e(S_u, Y_u) = k|Y_u| \) if and only if \( G \cong S_{n,k}^+ \) (Otherwise, \( E(G[\{Y_u\}]) \neq \emptyset \) and so \( G[S_u \cup Y_u] \) contains a subgraph \( S_{n-2,k}^+ \). By Fact 4, \( G[S_u \cup Y_u] \) has a path \( P_{\geq 2k} \) with two ends in \( S_u \) and so it is an easy task to find a cycle \( C_{\geq 2k+2} \) containing \( V(P_{\geq 2k}) \cup \{u\} \cup T_u \), a contradiction), as desired; and \( B_u = 1 \) if and only if \( e(S_u, Y_u) = k|Y_u| - 1 \) if and only if \( G \subseteq S_{n,k}^+ \) with \( e(G) < e(S_{n,k}^+) \) (Otherwise, \( G[S_u \cup Y_u] \) contains a subgraph \( S_{n-2,k}^+ \) - \( e \), where \( e \) is some edge joining a vertex in \( S_u \) to a vertex in \( Y_u \), similar as in Fact 4 we can find a path \( P_{\geq 2k} \) in \( G[S_u \cup Y_u] \) with two ends in \( S_u \) and so again we get a contradiction), therefore \( \mu(G) < \mu(S_{n,k}^+) \), as claimed.
Now suppose \( s_u = k + 1 \) and \( t_u = 0 \). Then \( e(N(u)) = k(k + 1)/2 \). If \( e(S_u, Y_u) < k|Y_u| - 1 \), then (3) implies that
\[ B_u \leq k(n - k - 2) - 2 + k(k + 1) - (k - 2)(k + 1) - k(n - k) = 0. \]
Hence assume \( e(S_u, Y_u) \geq k|Y_u| - 1 \). We first claim that there is no vertex \( y \) in \( Y_u \) such that \( e(y, S_u) = k + 1 \). If not, choose a vertex \( y \in Y_u \) such that \( e(y, S_u) \) is minimal among all vertices in \( Y_u \), then \( e(y, S_u) \leq k - 1 \) because \( e(S_u, Y_u) \leq k|Y_u| \). Hence
\[ e(S_u, Y_u \setminus \{y\}) = e(S_u, Y_u) - e(y, S_u) \geq k|Y_u| - 1 - (k - 1) = k|Y_u \setminus \{y\}|. \]
Note that \(|Y_u \setminus \{y\}| = n - k - 3 > (k + 1)k \). By Lemma 4, \( G[S_u \cup (Y_u \setminus \{y\})] \) contains a path \( P_{\geq 2k+1} \) with two ends in \( S_u \) and hence, combining the vertex \( u \), we get a cycle of length at least \( 2k + 2 \) in \( G \), a contradiction. By the claim and
Now choose a vertex $v$ from $S_u$ such that $d(v)$ is minimal among all vertices in $S_u$. Then $N(v) \cap Y_u \subseteq N(w) \cap Y_u$ for any $w \in S_u$ by the minimality of $d(v)$ and the assumption that $P_v(w) = \emptyset$ or $P_w(v) = \emptyset$. Let $S'_u = S_u \setminus \{v\}$ and $Y'_u = Y_u \setminus (N(v) \cap Y_u)$. We claim that

$$|Y'_u| \geq k + 1.$$  

If not, we have

$$e(S_u, Y_u) \geq (k + 1)(|Y_u| - k) = k|Y_u| + |Y_u| - k(k + 1) > k|Y_u|,$$

a contradiction. Therefore, $G[S'_u \cup Y'_u]$ contains a subgraph isomorphic to $S_{2k,k}$. By Fact 4, $G[S'_u \cup Y'_u]$ contains a path $P_{\geq 2k-1}$ with two ends, say $\{a, b\}$, in $S'_u$. Choose a vertex $z$ from $N(v) \cap Y_u$ so that $zb \in E(G)$, this can be done since $N(v) \cap Y_u \subseteq N(b) \cap Y_u$. So $P_{\geq 2k-1} + bzuva$ is a cycle of length at least $2k + 2$ in $G$, a contradiction.

**Case 2.** $G[N(u)]$ is not connected.

This implies that $S_u = Y_u = \emptyset$ and hence $d(u) = n - 1$. Let $A_1, \ldots, A_t$ be the components of $G - \{u\}$ and let $n_i = |V(A_i)|$. Without loss of generality, assume $n_1 \leq \ldots \leq n_t$ and let $s$ be the largest integer such that $n_s \leq 3k$. Set $H = A_1 + \cdots + A_s$ and $H' = A_{s+1} + \cdots + A_t$. Let $h = |V(H)|$. So $|V(H')| = n - 1 - h$.

If $V(H') \neq \emptyset$, since $n_i > 3k$ for $i \in \{s + 1, t\}$ and $G$ is $C_{\geq 2k+2}$-free, Fact 2 implies that

$$e(H') = \sum_{i=s+1}^{t} e(A_i)$$

$$\leq \sum_{i=s+1}^{t} \left[ (k - 1)n_i - (k^2 - k)/2 + 1 \right]$$

$$\leq (k - 1)|V(H')| - (k^2 - k)/2 + 1,$$

with equality holds if and only if $k = 2$ and $A_i \cong S_{n_i,1}^+$ for each $i = \{s + 1, \ldots, t\}$ or $k \geq 3$ and $H'$ has only one component.

If $|V(H)| < 2k - 2$ and $V(H) \neq \emptyset$, then

$$e(H) \leq |V(H)|(|V(H)| - 1)/2 < (k - 1)h - 1.$$
If \( V(H) = \emptyset \) and \( k \geq 3 \) or \( k = 2 \) and there exists one component \( A_i \neq S_{n,k-1}^+ \) for some \( i \in \{s+1, \ldots, t\} \), then \( e(H') \leq (k-1)|V(H')| - (k^2 - k)/2 \) because \( G[N(u)] \) is disconnected. Therefore, we always have

\[
2e(N(u)) = 2e(H) + 2e(H') \leq 2(k-1)(n-1) - (k^2 - k).
\]

Note that \( |V(H)| < 2k - 2 \) implies that \( V(H') \neq \emptyset \). By (3),

\[
B_u \leq 2(k-1)(n-1) - (k^2 - k) - (k-2)(n-1) - k(n-k) = 0.
\]

If \( V(H) = \emptyset, k = 2 \) and each component \( A_i \cong S_{n,1}^+ \), then \( G = H' \wedge K_1 \). By Lemma 7 we have \( \mu(G) < \mu(S_{n,k}^+) \).

Therefore, in the following we assume \( |V(H)| \geq 2k - 2 \). Recall that \( \mu = \mu(S_{n,k}) \).

We claim that \(-1/3 \leq \frac{x-\mu}{n-1-\mu} \leq 1 \) for \( x \in \{0, 1, \ldots, n-1\} \). When \( x = n-1 \), we have \( \frac{x-\mu}{n-1-\mu} = 1 \). So it is sufficient to show that \( \frac{\mu}{n-1-\mu} \leq 1/3 \). Since \( \mu = (k-1)/2 + \sqrt{k(n-k) + (k-1)^2}/4 \leq (k-1)/2 + \sqrt{kn} \) and \( n \geq 13k^2 \), we have \( n \geq \sqrt{n/13k^2} \geq 5\sqrt{n}/k > 4\sqrt{n}/k + 2k \geq 4\mu + 1 \). This implies that \( \frac{\mu}{n-1-\mu} \leq 1/3 \).

**Subcase 2.1.** \( |V(H')| = \emptyset \).

Let

\[
g(x) = x^2 - (k-1)x - k(n-k) - \frac{x-\mu}{n-1-\mu}(n+k^2-k-1)
\]

and reset

\[
B = g(A) = A^2 - (k-1)A - k(n-k)I - \frac{n+k^2-k-1}{n-1-\mu}(A - \mu I).
\]

Let \( B_v = \sum_{1 \leq i \leq n} B_{iv} \) for each \( v \in V(G) \). Hence

\[
B_v = 2e(N(v)) + e(N(v),Y_v) - (k-2)d(v) - k(n-k) - \frac{d(v)-\mu}{n-1-\mu}(n+k^2-k-1). \tag{9}
\]

Clearly, \( \mu = \mu(S_{n,k}) \) is still the largest root of \( g(x) \). By Lemma 8 to show \( \mu(G) < \mu(S_{n,k}^+) \), it is sufficient to show \( B_v \leq 0 \) for any \( v \in V(G) \) as \( \mu(S_{n,k}) < \mu(S_{n,k}^+) \). Since \( G \) is \( C_{2k+2} \)-free, \( H \) is \( P_{2k+1} \)-free. By Fact 8 \( e(H) \leq (2k-1)h/2 \). By (9), we have

\[
B_u = 2e(H) - (k-2)(n-1) - k(n-k) - n - k^2 + k + 1 \\
\leq (k+1)(n-1) - k(n-k) - n - k^2 + k + 1 \\
= 0.
\]
For any $v \in V(G)$ with $v \neq u$, note that $G[N(v)]$ is connected and $d(v) \leq 3k$ because each component of $H$ has order at most $3k$. By Fact 3, $e(N(v)) \leq (2k-1)d(v)/2$. Suppose $v \in V(A_i)$ then $Y_v = (\cup_{j \neq i} V(A_j)) \cup (Y_v \cap V(A_i))$. Thus
\[
e(N(v), Y_v) \leq n - d(v) + (d(v) - 1)(n_i - d(v)).
\]
By (9), we have
\[
B_v \leq (2k-1)d(v) + n - d(v) + (d(v) - 1)(n_i - d(v))
- (k - 2)d(v) - k(n - k) + (n + k^2 - k - 1)/3
\leq kd(v) + n + (n_i - 1)^2/4 - k(n - k) + (n + k^2 - k - 1)/3
\leq 3k^2 + (3k - 1)^2/4 + (k^2 - k - 1)/3 + (4/3 - k)n
\leq 0
\]
where the first inequality holds since $\frac{x - \mu}{n - 1 - \mu} \geq -1/3$ for any $x \in \{0, 1, \ldots, n - 1\}$, and the last inequality holds since $n \geq 13k^2$, $k \geq 2$ and $d(v) \leq n_i \leq 3k$.

**Subcase 2.2.** $|V(H^*)| \neq \emptyset$.

Let
\[
g(x) = x^2 - (k - 1)x - k(n - k) - \frac{x - \mu}{n - 1 - \mu}(h + 2)
\]
and reset
\[
B = g(A) = A^2 - (k - 1)A - k(n - k)I - \frac{h + 2}{n - 1 - \mu}(A - \mu I).
\]
Let $B_v = \sum_{1 \leq i \leq n} B_{iv}$ for each $v \in V(G)$. Hence
\[
B_v = 2e(N(v)) + e(N(v), Y_v) - (k - 2)d(v) - k(n - k) - \frac{d(v) - \mu}{n - 1 - \mu}(h + 2). \quad (10)
\]
Clearly, $\mu = \mu(S_{n,k})$ is still the largest root of $g(x)$. Thus, to show $\mu(G) < \mu(S_{n,k})$, it is sufficient to show $B_v \leq 0$ for any $v \in V(G)$ with a same reason as in Subcase 2.1.

Since $G$ is $C_{2k+2}$-free, $H$ is $P_{2k+1}$-free. By Fact 3 $e(H) \leq (2k-1)h/2$. Hence
\[
2e(H) + 2e(H^*) \leq (2k - 1)h + 2(k - 1)(n - h - 1) - (k^2 - k) + 2
= 2(k - 1)(n - 1) + h - (k^2 - k) + 2.
\]
By (10), we have
\[
B_v = 2e(H) + 2e(H^*) - (k - 2)(n - 1) - k(n - k) - h - 2
= k(n - 1) - (k^2 - k) - k(n - k)
= 0.
\]
For any \( v \in V(G) \) with \( v \neq u \), we have that \( G[N(v)] \) is connected. If \( v \in V(H) \) then \( d(v) \leq 3k \) because each component of \( H \) has order at most \( 3k \). By Fact 3, \( e(N(v)) \leq (2k - 1)d(v)/2 \). If \( v \in V(A_i) \) for some \( i \in \{1, \ldots, s\} \), note that \( Y_v = (\cup_{j \neq i} V(A_j)) \cup (Y_v \cap V(A_i)) \), then
\[
eq n - d(v) + (d(v) - 1)(n_i - d(v)).
\]

By (10), we have
\[
B_v \leq (2k - 1)d(v) + n - d(v) + (d(v) - 1)(n_i - d(v))
- (k - 2)d(v) - k(n - k) + (h + 2)/3
\leq kd(v) + n + (n_i - 1)^2/4 - k(n - k) + (h + 2)/3
\leq 3k^2 + (3k - 1)^2/4 + k^2 + 2/3 + (4/3 - k)n
\leq 0
\]
where the first inequality holds since \( \frac{x - \mu}{n - 1 - \mu} \geq -1/3 \) for any \( x \in \{0, 1, \ldots, n - 1\} \), and the last inequality holds since \( n \geq 13k^2 \), \( k \geq 2 \), \( d(v) \leq n_i \leq 3k \) and \( h \leq n - 1 \).

Now assume \( v \in V(H') \).

Claim 6. We have \( k \geq 3 \).

If not, then \( k = 2 \). We first claim that each component \( A_i \) of \( H' \) has maximum degree \( n_i - 1 \). Otherwise, choose a vertex \( x \) with maximum degree in \( A_i \) and a vertex \( y \in N_{A_i}(x) \) such that \( y \) has a neighbor \( z \notin N_{A_i}[x] \), such a vertex exists since \( d_{A_i}(x) < n_i - 1 \). By the maximality of the degree of \( x \), there is at least one vertex \( z' \in N_{A_i}(x) \) but \( z' \notin N_{A_i}[y] \). This implies that \( P_y(x) \neq \emptyset \) and \( P_x(y) \neq \emptyset \), a contradiction. Hence \( A_i \) is a subgraph of \( S_{n_i, 1}^+ \), otherwise, \( A_i \) has a path \( P_{G} \) and hence \( G \) has a cycle \( C_{G} \), a contradiction. By the maximality of the spectral radius of \( G \) and Lemma 7 we may assume \( H' = A_t \cong S_{n_i, 1}^+ \). If \( h \geq 4 \), then, for any \( v \in V(H') \), we have
\[
B_v = 2e(N(v)) + e(N(v), Y_v) - (2 - 2)d(v) - 2(n - 2) + (h + 2)/3
\leq 2(n - 1 - h) + h - 2(n - 2) + (h + 2)/3
\leq 0.
\]
If \( h \leq 3 \), again by the maximality of radius of \( G \), we may assume \( H \cong K_h \). Since \( K_h \cong S_{h, 1}^+ \), we have \( G \cong S_{h, 1}^+ + S_{n_i, 1}^+ \). By Lemma 7, \( \mu(G) < \mu(S_{n, k}^+) \), a contradiction. The claim holds.
In the following, we assume $k \geq 3$. If $d(v) \leq 3k$, we claim that $e(N(v), Y_v) > (k - 1)|Y_v|$. If not, then $e(N(v), Y_v) \leq (k - 1)|Y_v|$. By (10),

$$B_v \leq d(v)(d(v) - 1) + (k - 1)|Y_v| - (k - 2)d(v) - k(n - k) + (h + 2)/3$$

$$\leq (2k + 1)^2 + k - n + (n + 1)/3$$

$$\leq 0$$

the inequality holds because $n \geq 13k^2$, a contradiction. The claim is true. Therefore, $e(N(v), Y_v) > (k - 1)|Y_v|$. With a same argument as in Subcase 1.2, we have $d(v) = s_v + t_v \leq k + 1$. Since $G$ is $C_{\geq 2k+2}$-free, there are at most $k$ points $x$ in $Y_v \cap V(H')$ such that $e(x, N(v)) = k + 1$. Hence

$$e(S_v, Y_v) = e(S_v, Y_v \cap V(H')) + e(u, V(H)) \leq k(n - 1 - h - d(v)) + k + h.$$

By (10),

$$B_v \leq d(v)(d(v) - 1) + k(n - 1 - h - d(v)) + h + k - (k - 2)d(v) - k(n - k) + (h + 2)/3$$

$$= 8/3 + k + (4/3 - k)h$$

$$\leq 0,$$

the last inequality holds since $h \geq 2k - 2$ and $k \geq 3$.

If $d(v) > 3k$, then Fact 2 implies that $e(N(v)) \leq d(v)(k - 1) - (k^2 - k)/2 + 1$ because $G$ is $C_{\geq 2k+2}$-free. By Claim 4 we have $s_v \leq k$. Hence

$$e(S_v, Y_v) = e(S_v, Y_v \cap V(H')) + e(u, V(H)) \leq k(n - 1 - h - d(v)) + h.$$

By (10), we have

$$B_v \leq 2d(v)(k - 1) - k^2 + k + 2 + k(n - 1 - h - d(v)) + h - (k - 2)d(v) - k(n - k) + (h + 2)/3$$

$$= 8/3 + 4h/3 - kh$$

$$\leq 0,$$

the last inequality holds since $h \geq 2k - 2$ and $k \geq 3$.

This completes the proof of Theorem 3 (b).
4 Concluding remarks

In this paper, we prove a stronger version of Theorem 1 and a weak version of Conjecture 2. We believe that the following result also is true.

**Conjecture 8.** Let $k \geq 2$, $L \geq 0$ and let $G$ be a graph of sufficiently large order $n$.

(a) If $\mu(G) \geq \mu(S_{n,k})$, then $G$ contains a $C_\ell$ with $\ell \in \{2k+1, 2k+2, \ldots, 2k+2+L\}$ unless $G = S_{n,k}$.

(b) If $\mu(G) \geq \mu(S^+_{n,k})$, then $G$ contains a $C_\ell$ with $\ell \in \{2k+2, \ldots, 2k+2+L\}$ unless $G = S^+_{n,k}$.

Theorem 3 states that Conjecture 8 holds for $L = n - 2k - 2$ and Conjecture 2 hopes that Conjecture 8 holds for $L = 0$.

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