Tight Bounds for Approximate Carathéodory and Beyond

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Abstract

We give a deterministic nearly-linear time algorithm for approximating any point inside a convex polytope with a sparse convex combination of the polytope’s vertices. Our result provides a constructive proof for the Approximate Carathéodory Problem \cite{2}, which states that any point inside a polytope contained in the $\ell_p$ ball of radius $D$ can be approximated to within $\epsilon$ in $\ell_p$ norm by a convex combination of only $O\left(D^2 p/\epsilon^2\right)$ vertices of the polytope for $p \geq 2$. We also show that this bound is tight, using an argument based on anti-concentration for the binomial distribution.

Along the way of establishing the upper bound, we develop a technique for minimizing norms over convex sets with complicated geometry; this is achieved by running Mirror Descent on a dual convex function obtained via Sion’s Theorem.

As simple extensions of our method, we then provide new algorithms for submodular function minimization and SVM training. For submodular function minimization we obtain a simplification and (provable) speed-up over Wolfe’s algorithm, the method commonly found to be the fastest in practice. For SVM training, we obtain $O(1/\epsilon^2)$ convergence for arbitrary kernels; each iteration only requires matrix-vector operations involving the kernel matrix, so we overcome the obstacle of having to explicitly store the kernel or compute its Cholesky factorization.
1 Introduction

The (exact) Carathéodory Theorem is a fundamental result in convex geometry which states that any point \( u \) in a polytope \( P \subseteq \mathbb{R}^n \) can be expressed as a convex combination of \( n + 1 \) vertices of \( P \). Recently, Barman [2] proposed an approximate version and showed that it can be used to improve algorithms for computing Nash equilibria in game theory and algorithms for the \( k \)-densest subgraph in combinatorial optimization. Versions of the Approximate Carathéodory Theorem have been proposed and applied in different settings. Perhaps its most famous incarnation is as Maurey’s Lemma [23] in functional analysis. It states that if one is willing to tolerate an error of \( \epsilon \) in \( \ell_p \) norm, \( O(D^2p/\epsilon^2) \) vertices suffice to approximate \( u \), where \( D \) is the radius of the smallest \( \ell_p \) ball enclosing \( P \). The key significance of the approximate Carathéodory Theorem is that the bound it provides is dimension-free, and consequently allows us to approximate any point inside the polytope with a sparse convex combination of vertices.

The Approximate Carathéodory Problem

Given a polytope \( P \) contained inside the \( \ell_p \) ball of radius \( D \), and \( u \in P \), find a convex combination \( \sum_{i=1}^k x_i v_i \) of vertices \( v_i \) of \( P \) such that \( k = O(D^2p/\epsilon^2) \) and \( \left\| \sum_{i=1}^k x_i v_i - u \right\|_p \leq \epsilon \).

Both Barman’s proof and Maurey’s original proof start from a solution \( u = \sum_{i=1}^{n+1} \lambda_i v_i \) of the exact Carathéodory problem, interpret the coefficients \( \lambda_i \) of the convex combination as a probability distribution and generate a sparse solution by sampling from the distribution induced by \( \lambda \). Concentration inequalities are then used to argue that the average sampled solution is close to \( u \) in the \( \ell_p \)-norm. The proof is clean and elegant, but it leaves two questions: Is randomization really necessary for this proof? And, can we bypass a solution to the exact approximate Carathéodory problem and directly compute a solution to the approximate version?

The second question is motivated by the fact that computing the solution to the exact Carathéodory problem can be costly. In fact, this takes \( O(n^\omega) \) time even if the points \( v_i \) are known in advance. The situation becomes even worse for polytopes for which it is not desirable to maintain an explicit representation of all its vertices (e.g. the matching polytope or the matroid base polytope) since there may be exponentially many of them. In this case, even finding the \( n + 1 \) vertices whose convex hull contains \( u \) becomes significantly more difficult.

Our first contribution addresses those two questions by giving a constructive proof of the approximate Carathéodory Theorem. As a corollary, this gives the first nearly linear time deterministic algorithm for the approximate Carathéodory problem that does not require knowing \( u = \sum_{i=1}^{n+1} \lambda_i v_i \) in advance. Our algorithm runs in \( O(D^2p/\epsilon^2) \) iterations, each of which takes linear time.

Our second contribution is to provide a lower bound showing that the \( O(D^2p/\epsilon^2) \) factor is tight. This improves upon a lower bound of \( \Omega((D/\epsilon)^{p/(p-1)}) \) proved by Barman. Barman’s lower bound is tight up to constant factors for \( \ell_2 \) but leaves a significant gap for any \( p > 2 \). We prove our lower bound by exhibiting a polytope \( P \) in the radius-\( D \) \( \ell_p \) ball and a point \( u \) inside for which all convex combinations of \( O(D^2p/\epsilon^2) \) vertices are \( \epsilon \)-far from the \( u \) in the \( \ell_p \)-norm.

These are in principle the best results one can hope for. We also show that even though the dependence on \( \epsilon \) can’t be improved in general, it can be greatly improved in a special case. If \( u \) is far away from the boundary of \( P \), i.e., if the ball of radius \( r \) around \( u \) is contained in \( P \), then there exist a solution to the approximate Carathéodory problem with \( k = O(D^2p/\epsilon^2 \log(\epsilon)) \).

In order to achieve the positive results for approximate Carathéodory, we develop a technique for minimizing norms over convex sets with complicated geometry; this is achieved by running Mirror Descent on a dual convex function obtained via Sion’s Theorem. This technique may
be of independent interest. To show its potential, we note that simple extensions of our method result in new algorithms for submodular function minimization and SVM training. For submodular function minimization, we obtain a simplification and (provable) speed-up over Wolfe’s algorithm, the method commonly found to be the fastest in practice. For SVM training, we obtain \( O(1/\epsilon^2) \) convergence for arbitrary kernels; each iteration only requires matrix-vector operations involving the kernel matrix, so we overcome the obstacle of having to explicitly store the kernel or compute its Cholesky factorization. Next, we elaborate on our technique and then discuss these applications in more details.

1.1 Our techniques: mirror descent and lower bounds

Our new constructive proof of the approximate Carathéodory Theorem employs a technique from Convex Optimization called Mirror Descent, which is a generalization of Subgradient Descent. Both subgradient and mirror descent are first order methods that minimize arbitrary convex functions to an additive precision of \( \epsilon \) using only information about the subgradient of the function.

In particular, we formulate the approximate Carathéodory problem as:

\[
\min_{x \in \Delta} \|Vx - u\|_p
\]

where the columns of \( V \) are the vertices of \( P \) and \( \Delta \) is the unit simplex.

Our first instinct is to apply gradient or mirror descent to \( f(x) = \|Vx - u\|_p \). This fails to achieve any sort of sparseness guarantee since its gradient is generally not sparse and the new iterate \( x_{t+1} = x_t - \eta \nabla f(x_t) \) would not be either.

Inspired by algorithms for solving positive linear programs such as \([24, 34]\), we reformulate our problem as a saddle point problem \( \max_{y \in B_q(1)} \min_{x \in \Delta} y^\top(Vx - u) \), where \( B_q(1) \) is the \( \ell_q \)-ball. This can be viewed as a zero-sum game. Applying a generalization of the minimax theorem we can obtain a dual convex function.

We apply the mirror descent framework to this dual function. Mirror descent is a framework that needs to be instantiated by the choice of a mirror map, which plays a similar role as linking functions in Online Learning. We will provide an overview of Mirror Descent in the next section so that the paper is self contained. But for the reader familiar with Mirror Descent terminology, our mirror map is a truncated version of the square \( \ell_q \)-norm, where \( q \) is chosen such that \( \frac{1}{p} + \frac{1}{q} = 1 \).

The analysis is enabled by choosing the right function to optimize and the appropriate mirror map. At the very high level, our algorithm incrementally improves our current choice of \( x \) by expanding its support by 1 in each iteration. The desired sparsity of \( x \) then follows as we can show that the number of iterations is \( O(D^2p/\epsilon^2) \).

Our lower bound is inspired by a method proposed by Klein and Young \([17]\) for proving conditional lower bounds on the running time for solving positive linear programs. It again follows from interpreting the Carathéodory problem as a zero-sum game between a maximization and a minimization player. We construct a random instance such that with high probability the minimization player has a dense strategy with value close to zero, but for every sparse support, the maximization player can force the strategy to be \( \epsilon \)-far from zero with high probability. The lower bound follows from taking the union bound over the probabilities and applying the probabilistic method.

1.2 Applications

In the following, we discuss a number of applications of our results and techniques. While the first result is a straightforward use of our improved approximate Carathéodory theorem, the second result is a simple application of the mirror-descent technique, and the third one is a simple application of an extension of the technique to SVMs.

Warm-up: fast rounding in polytopes with linear optimization oracles. The most direct application of our approach is to efficiently round a point in a polytope whenever it admits
a good linear optimization oracle. An obvious such instance is given by the matroid polytope. Given an $n$-element matroid by $\mathcal{M}$ of rank $r$ and a fractional point $x^*$ inside its base polytope, our algorithm produces a sparse distribution $\mathcal{D}$ over matroid bases such that marginals are approximately preserve in expectation. More specifically, for any $p \geq 2$, $\mathcal{D}$ has a support of size \( \frac{\|x^*\|^p}{p^r} \), and $\|E_{x \sim \mathcal{D}}[x] - x^*\|_p \leq \epsilon$; furthermore, computing $\mathcal{D}$ requires only $O(nr^2/p^2\epsilon^2)$ calls to $\mathcal{M}$’s independence oracle.

**Submodular function minimization.** Fujishige’s minimum-norm point algorithm is the method typically employed by practitioners, to minimize submodular functions $[11, 1]$. Traditionally this has been implemented using variants of Wolfe’s algorithm $[32]$, which lacked a rigorous convergence analysis (it was only known to converge in exponential number of steps). Only recently Chakrabarty, Jain, and Kothari $[7]$ proved the first polynomial time bound for this method, obtaining an algorithm that runs in time $O\left((n^5 \cdot \mathcal{T} + n^7) F^2\right)$, where $\mathcal{T}$ is the time required to answer a single query to $f$, and $F$ is the maximum marginal difference in absolute value.

As our second application, we show that our technique can replace Wolfe’s algorithm in the analysis of $[7]$ obtaining an an $O\left(n^6 F^2 \mathcal{T}\right)$ time algorithm for exact submodular function minimization, and a $O(n^6 F^2 \mathcal{T}/k^4)$ for a $k$-additive approximation. We emphasize that those are not the best theoretical algorithm, but a simplification and a speed-up of the algorithm that is commonly found to be the fastest in practice.

**Support vector machines.** Training support vector machines (SVMs) can also be formulated as minimizing a convex function. We show that our technique of converting a problem to a saddle point formulation and solving the dual via Mirror Descent can be applied to the problem of training $\nu$-SVMs. This is based on a formulation introduced by Schölkopf, Smola, Williamson, and Bartlett $[27]$. Kitamura, Takeda and Iwata $[16]$ show how SVMs can be trained using Wolfe’s algorithm. Replacing Wolfe’s algorithm by Mirror Descent we obtain an $\epsilon$-approximate solution in time $O\left(\max(\frac{2}{\epsilon}, \|K\|)/\left(\nu n \epsilon^2\right)\right)$, where $K$ is the kernel matrix. Whenever the empirical data belongs to the unit $\ell_2$ ball, this yields a constant number of iterations for polynomial and RBF kernels. Our method does not need to explicitly store the kernel matrix, since every iteration only requires a matrix-vector multiplication, and the entries of the matrix can be computed on-the-fly as they are needed. In the special case of a linear kernel, each iteration can be implemented in time linear in input size, yielding a nearly-linear time algorithm for linear SVM training.

### 1.3 Related work

As previously mentioned, the Approximate Carathéodory Theorem was been independently discovered many times in the past. The earliest record is perhaps due to Novikoff $[21]$ in 1962 who showed that the $\ell_2$ version of Approximate Carathéodory can be obtained as a byproduct of the analysis of the Perceptron Algorithm (as pointed out by $[4]$). Maurey $[23]$ proves it in the context of functional analysis. We refer to the appendix of $[5]$ for the precise statement of Maurey’s lemma as well as a self-contained proof. Farias et al $[9]$ studies it for the special case of the the bipartite matching polytope. Barman $[2]$ study the $\ell_p$ case and provides several applications to game theory and combinatorial optimization.

Related to the Approximate Carathéodory problem is the question studied by Shalev-Shwartz, Srebro and Zhang of minimizing the loss of a linear predictor while bounding the number of features used by the predictor. Their main result implies a gradient-descent based algorithm for the $\ell_2$-version of the Approximate Carathéodory Theorem but is only able to produce $O(1/\epsilon^p)$ for $p > 2$. A different optimization approach to Approximate Carathéodory is done by Garber and Hazan $[12]$ who solve the optimization problem $\min_{x \in P} \|x - u\|_2^2$ using Frank-Wolfe methods, also obtaining the $\ell_2$ version of the result.

Finally, the literature on Mirror Descent is too large to survey, but we refer to the book by
Ben-Tal and Nemirovski [3] for a comprehensive overview, including a discussion of the \( \ell_q \) square mirror map. In Online Learning a variant of this mirror map has been used in Gentile’s \( p \)-norm algorithms [13].

2 Preliminaries

2.1 Notation
Given a point \( x \in \mathbb{R}^d \), we define its \( \ell_p \)-norm as \( \|x\|_p = \left( \sum_{i=1}^{d} |x_i|^p \right)^{1/p} \) for \( 1 \leq p < \infty \) and its \( \ell_\infty \) norm by \( \|x\|_\infty = \max_i |x_i| \). Given a norm \( \|\cdot\| \), we denote by \( B_{\|\cdot\|}(r) = \{x \in \mathbb{R}^d; \|x\| \leq r \} \). For \( \ell_p \) and \( \ell_\infty \) norms, we denote the balls simply by \( B_p(r) \) and \( B_\infty(r) \).

Given a norm \( \|\cdot\| \), we define its dual norm \( \|\cdot\|_* \) as \( \|y\|_* = \max_{x: \|x\| = 1} y^\top x \) in such a way that Hölder’s inequality holds with equality: \( y^\top x \leq \|y\|_* \cdot \|x\| \). The dual norm of the \( \ell_p \) norm is the \( \ell_q \) norm for \( \frac{1}{p} + \frac{1}{q} = 1 \).

Given a vector \( x \), let its support \( \text{supp}(x) = \{i|x_i \neq 0\} \) represent the number of nonzero coordinates of \( x \).

2.2 Approximate Carathéodory problem
The (exact) Carathéodory Theorem is a fundamental result in linear algebra which bounds the maximum number of points necessary to describe a point in the convex hull of a set. More precisely, given a finite set of points \( X \subseteq \mathbb{R}^d \) and a point \( u \in \text{conv}(X) \) := \( \{x \in \mathbb{R}^d; \sum_{x \in X} \lambda_x \cdot x; \sum_{x \in X} \lambda_x = 1, \lambda_x \geq 0\} \), there exist \( d+1 \) points in \( x_1, \ldots, x_{d+1} \in X \) such that \( u \in \text{conv}\{x_1, \ldots, x_{d+1}\} \). On the plane, in particular, every point in the interior of a convex polygon can be written as a convex combination of three of its vertices.

The approximate version of the Carathéodory theorem bounds the number of points necessary to describe a point \( u \in \text{conv}(X) \) approximately. Formally, given a norm \( \|\cdot\| \), an additive error parameter \( \epsilon \) and a set of points \( X \subseteq B_{\|\cdot\|}(1) \subseteq \mathbb{R}^d \), for every \( u \in \text{conv}(X) \) we want \( k \) points \( x_1, \ldots, x_k \in X \) such that there exists \( u' \in \text{conv}\{x_1, \ldots, x_k\} \) and \( \|u - u'\| \leq \epsilon \).

A general result of this type is given by Maurey’s Lemma [23]. For the case of \( \ell_p \) norms, \( p \geq 2 \), Barman [2] showed that \( k \leq 4p/\epsilon^2 \) points suffice. A notable aspect of this theorem is that the bound is independent of the dimension of the ambient space.

2.3 Convex functions
We give a brief overview on the theory of convex functions. For a detailed exposition we refer readers to [26].

Subgradients. A function \( f : Q \subseteq \mathbb{R}^d \to \mathbb{R} \) defined on a convex domain \( Q \) is said to be convex if every point \( x \in Q \) has a non-empty subgradient \( \partial f(x) = \{g \in \mathbb{R}^d; f(y) \geq f(x) + g^\top(y-x), \forall y \in Q\} \).

Geometrically, this means that a function is convex iff it is the maximum of all its supporting hyperplanes, i.e. \( f(x) = \max_{x_0, g \in \partial f(x_0)} f(x_0) + g^\top(x-x_0) \). When there is a unique element in \( \partial f(x) \) we call it the gradient and denote it by \( \nabla f(x) \). We will sometimes abuse notation and refer to \( \nabla f(x) \) as an arbitrary element of \( \partial f(x) \) even when it is not unique.

Strong convexity and smoothness. We say that a function \( f : Q \subseteq \mathbb{R}^d \to \mathbb{R} \) is \( \mu \)-strongly convex with respect to norm \( \|\cdot\| \) if for all \( x, y \in Q \) and all subgradients \( g \in \partial f(x) \):

\[
    f(y) - f(x) - g^\top(y-x) \geq \frac{1}{2} \mu \|y-x\|^2
\]

A function is said to be \( \sigma \)-smooth with respect to the \( \|\cdot\| \) if for all \( x, y \in Q \) and \( g \in \partial f(x) \):

\[
    f(y) - f(x) - g^\top(y-x) \leq \frac{1}{2} \sigma \|y-x\|^2
\]
Bregman divergence and the Hessian. Every continuously differentiable \( f \) induces a concept of ‘distance’ known as the Bregman divergence: given \( x, y \in Q \), we define \( D_f(y \| x) := f(y) - f(x) - \nabla f(x) ^ \top (y - x) \) as the second order error when computing \( f(y) \) using the linear approximation of \( f \) around \( x \). The fact that \( f \) is convex guarantees \( D_f(y \| x) \geq 0 \).

If the subgradient of \( f \) is unique everywhere, we can define \( \mu \)-strong convexity and \( \sigma \)-smoothness with respect to the Bregman divergence, as \( D_f(y \| x) \geq \frac{1}{2} \sigma \| x - y \|^2 \) and \( D_f(y \| x) \leq \frac{1}{2} \mu \| x - y \|^2 \).

If \( f \) is also twice-differentiable, a simple way to compute its strong convexity and smoothness parameters is by bounding the \( \| \cdot \| \)-eigenvalues of the Hessian. If \( \mu \cdot \| w \|^2 \leq w ^ \top \nabla ^2 f(x) w \leq \sigma \cdot \| w \|^2 \) for all \( x \in Q \) and \( w \neq 0 \), then \( f \) is \( \mu \)-strongly convex and \( \sigma \)-smooth. This is because:

\[
D_f(y \| x) = \int _0 ^1 \| \nabla f(x + (y - x) t) - \nabla f(x) \|^2 (y - x) dt = \int _0 ^1 \int _0 ^s (y - x) ^ \top \nabla ^2 f(x + (y - x) s) (y - x) ds dt
\]

Lipschitz constant. We say that a convex function is \( \rho \)-Lipschitz with respect to norm \( \| \cdot \| \) if \( \| \nabla f(x) \|_* \leq \rho \). Note that \( \rho \)-Lipschitz continuity requires a bound on the dual norm, since:

\[
| f(y) - f(x) | = \left| \int _0 ^1 \nabla f(x + t(y - x)) ^ \top (y - x) dt \right| \leq \int _0 ^1 \| \nabla f(x + t(y - x)) \|_* \cdot \| y - x \| dt \leq \rho \cdot \| y - x \|
\]

Fenchel duality. It is useful to write a convex function as the maximum of its supporting hyperplanes. One way to do that is using the Fenchel transform. When defining Fenchel transforms, it is convenient to identify a function \( f : Q \to \mathbb{R} \) to its extension \( \hat{f} : \mathbb{R} ^ d \to \mathbb{R} \cup \{ \infty \} \) such that \( \hat{f}(x) = f(x) \) for \( x \in Q \) and \( \hat{f}(x) = \infty \) otherwise. Given that identification, we can define the Fenchel transform of a function \( f \) as the function \( f^* : \mathbb{R} ^ d \to \mathbb{R} \cup \{ \infty \} \) given by \( f^*(z) = \sup _{x \in \mathbb{R} ^ d} z ^ \top x - f(x) \). If \( f \) is convex, the Fenchel transformation is self-invertible, i.e., \((f^*)^* = f \) or equivalently: \( f(x) = \max _z z ^ \top x - f^*(z) \). Notice that the previous expression is a way to write any convex function as a maximum over linear functions in \( x \) parametrized by \( z \). The Fenchel inequality \( f(x) + f^*(z) \geq z ^ \top x \) follows directly from the definition of the Fenchel transform.

Envelope Theorem. When writing a convex function \( f = \max _i f_i \) as a maximum of other convex function (typically linear functions), the Envelope Theorem gives a way to compute derivatives. Its statement is quite intuitive: since gradients are local objects, the gradient of \( f \) at a certain point is the gradient of the function \( f_i \) being maximized at that point. Formally, if \( f(x) = \max _z g(x, z) \) where \( g(x, z) \) is convex in \( x \) for every fixed \( z \), then if \( f(x_0) = g(x_0, z_0) \), then \( \partial _x g(x_0, z_0) \subseteq \partial f(x_0) \).

A direct application of this theorem is in computing the gradients of the Fenchel dual: \( \nabla f^*(z) = \arg \min _{x} \{ z ^ \top x - f(x) \} \) and \( f^*(z) = z ^ \top \nabla f^*(z) - f(\nabla f^*(z)) \).

Smoothness and strong convexity duality. Finally, we will use the following duality theorem:

**Theorem 2.1.** The function \( f : Q \to \mathbb{R} \) is a \((1/\sigma)\)-strongly convex function with respect to \( \| \cdot \| \) if and only if its Fenchel dual \( f^* : \mathbb{R} ^ d \to \mathbb{R} \) is a \( \sigma \)-smooth with respect to \( \| \cdot \|_* \).

Proof. Here we prove that \( \sigma \)-strong convexity of a function implies \((1/\sigma)\)-smoothness of its dual, since this is the direction we will use. We refer to [14, 28] for a proof of the converse.

Fix \( z_1, z_2 \in \mathbb{R} ^ d \) and let \( y_i \in \partial f^*(z_i) = \arg \max _{y \in Q} z_i ^ \top y - f(y) \). Since \( f \) is strongly convex, there in an unique maximum, so we can write \( y_i = \nabla f^*(z_i) \). Also, \( f^*(z_i) = z_i ^ \top y_i - f(y_i) \). Since the Fenchel transform is self-dual, \( f(y_i) = \max _z y_i ^ \top z - f^*(z) = z_i ^ \top y_i - f^*(z_i) \). In particular, this means that \( z_i \in \partial f(y_i) \).

Using the strong-convexity of \( f \), we can write:

\[
\begin{align*}
f(y_2) - f(y_1) - z_1 ^ \top (y_2 - y_1) &\geq \frac{1}{2\sigma} \| y_1 - y_2 \|^2 \\
f(y_1) - f(y_2) - z_2 ^ \top (y_1 - y_2) &\geq \frac{1}{2\sigma} \| y_1 - y_2 \|^2
\end{align*}
\]
Summing the expressions above and applying Holder’s inequality, we get:
\[
\frac{1}{\sigma} \|y_1 - y_2\|^2 \leq (z_2 - z_1)^\top(y_2 - y_1) \leq \|z_1 - z_2\|_* \cdot \|y_1 - y_2\|
\]
Therefore:

\[
\sigma \cdot \|z_1 - z_2\|_* \geq \|y_1 - y_2\| = \|\nabla f^*(z_1) - \nabla f^*(z_2)\|
\]
which implies the smoothness bound:
\[
D_{f^*}(z_2\|z_1) = \int_0^1 [\nabla f^*(z_1 + t(z_2 - z_1)) - \nabla f^*(z_1)]^\top(z_2 - z_1)dt \leq \frac{1}{2}\sigma \|z_1 - z_2\|_*^2.
\]

\[\square\]

### 2.4 A primer on Mirror Descent

For the sake of completeness, we will present here an elementary exposition of the Mirror Descent Framework, which is used in our proof. For a complete exposition we refer to Nemirovskii [3] or Bubeck [6].

The goal of Mirror Descent is to minimize a convex function \( f : Q \subseteq \mathbb{R}^d \to \mathbb{R} \) with Lipschitz constant \( \rho \) with respect to norm \( \|\cdot\| \). To motivate Mirror Descent, it is useful to think of dot products \( y^\top x \) as a product of vectors in two different vector spaces, which can be thought as vectors vs linear forms or column vectors vs row vectors. In the spirit of Hölder’s inequality, we can think of \( x \) as living in the \( \mathbb{R}^d \) space equipped with \( \|\cdot\| \) norm while \( y \) lives in \( \mathbb{R}^d \) equipped with the dual norm \( \|\cdot\|_* \). When we approximate \( f(y) - f(x) \approx \nabla f(x)^\top(y - x) \), the second term is a dot-product of a vector in the domain \( y - x \), which we call the primal space and measure using \( \|\cdot\| \) norm and a gradient vector, which we call the dual space and measure with dual norm \( \|\cdot\|_* \).

Keeping the discussion in the previous paragraph in mind, we can revisit the most intuitive method to minimize convex functions: gradient descent. The gradient descent method consists in following the directions of steepest descent, which is the direction opposite to the gradient. This leads to an iteration of the type: \( y_{t+1} = y_t - \eta \cdot \nabla f(y_t) \). In the view of primal space and dual space, this iteration suddenly looks strange, because one is summing a primal vector \( y_t \) with a dual vector \( \nabla f(y_t) \) which live in different spaces. In some sense, the gradient descent for Lipschitz convex functions only makes sense in the \( \ell_2 \) norm, in which \( \|\cdot\| = \|\cdot\|_* \) (see the subgradient descent method in [20]).

This motivated the idea of a map \( M : \mathbb{R}^d \to Q \) connecting the primal and the dual space. The idea in the mirror descent algorithm is to keep two vectors \((y_t, z_t)\) one in the primal space and one in the dual space. In each iteration we compute \( \nabla f(y_t) \), obtaining a dual vector and update:

\[
z_{t+1} = z_t - \eta \nabla f(y_t) \quad \quad y_{t+1} = M(z_{t+1})
\]

It is convenient in the analysis to think of this map as the gradient of a convex function \( M = \nabla \omega^* \). In the usual setup, we define the mirror map, which is a convex function \( \omega : Q \to \mathbb{R} \), \( \sigma^{-1} \)-strongly convex with respect to \( \|\cdot\| \). Let \( \omega^* : \mathbb{R}^d \to \mathbb{R} \) be the Fenchel-dual \( \omega^*(z) = \sup_{y \in Q} z^\top(y - \omega(y)) \) which is a \( \sigma \)-smooth convex function with respect to \( \|\cdot\|_* \) by Theorem 2.1.

Notice that \( \omega^* \) is defined as a maximum over linear functions of \( z \) indexed by \( y \). The result known as the envelope theorem states that \( \nabla \omega^*(z_0) \) is the gradient of the linear function maximized at \( z_0 \). Therefore: \( \nabla \omega^*(z_0) = y \in \arg \max_y \{ z_0^\top(y - \omega(y)) \} \). This in particular implies that \( \nabla \omega^*(z) \in Q \) since \( \omega(y) = \infty \) for \( y \notin Q \).

Using the definition of \( \omega \) and \( \omega^* \) we can define the Mirror Descent iteration as:

\[
z_{t+1} = z_t - \eta \nabla f(y_t) \quad \quad y_{t+1} = \nabla \omega^*(z_{t+1})
\]

**Theorem 2.2.** In the setup described above with \( D = \max_{z \in Q} D_\omega(z\|z_0) \), \( \eta = \epsilon / \sigma \rho^2 \) then in \( T \geq 2D\sigma \rho^2 / \epsilon^2 \) iterations, it holds that \( \frac{1}{T} \sum_t \nabla f(y_t)^\top(y_t - y) \leq \epsilon, \forall y \in Q \).
Proof. The idea of the proof is to bound the growth of \( \omega^*(z_t) \) using smoothness property of \( \omega^* \):
\[
\omega^*(z_t) \leq \omega^*(z_0) + \sum_{t=0}^{T-1} \nabla \omega^*(z_t)^\top (z_{t+1} - z_t) + \frac{\sigma}{2} \|z_{t+1} - z_t\|_\sigma^2
\]
\[
= \omega^*(0) - \sum_{t=0}^{T-1} \eta \nabla y_t \nabla f(y_t) + \frac{\sigma}{2} \eta^2 \|\nabla f(y_t)\|_\sigma^2
\]
By the Fenchel inequality \( \omega^*(z_t) \geq z_t^\top y - \omega(y) = (z_0 - \sum_t \eta \nabla f(y_t))^\top y - \omega(y^*) \) for all \( y \in Q \).
Combining with the previous inequality and re-arranging the terms, we get:
\[
\eta \sum_t \nabla f(y_t)^\top (y_t - y) \leq \omega(y) + \omega^*(z_0) - \nabla \omega(y_0)^\top y + \frac{\sigma}{2} \eta^2 \rho^2 T
\]
The gradient of \( \omega^*(z_0) = \sup_y z_0^\top y - \omega(y) \) corresponds by the envelope theorem to \( y \) maximizing
\( z_0^\top y - \omega(y) \). Therefore, since \( y_0 = \nabla \omega^*(z_0), \omega^*(z_0) = z_0^\top y_0 - \omega(y_0) \).
Substituting \( \omega^*(z_0) \) in the above expression and using the definition of Bregman divergence, we get:
\[
\eta \sum_t \nabla f(y_t)^\top (y_t - y) \leq D_\omega(y\|y_0) + \frac{\sigma}{2} \eta^2 \rho^2 T
\]
Rearranging the terms and using that \( D_\omega(y\|y_0) \leq D \), we obtain:
\[
\frac{1}{T} \sum_t \nabla f(y_t)^\top (y_t - y) \leq \frac{D}{\eta T} + \frac{\sigma \eta \rho^2}{2} = \sqrt{\frac{2D\sigma \rho^2}{T}} \text{ for } \eta = \sqrt{\frac{2D}{T\sigma \rho^2}}
\]
So for \( T \geq \frac{2\sigma D \rho^2}{\eta^2} \), \( \frac{1}{T} \sum_t \nabla f(y_t)^\top (y_t - y) \leq \epsilon \). \( \square \)

**Corollary 2.3.** In the conditions of the previous theorem, for \( \bar{y}_t = \frac{1}{T} \sum_{t=1}^T y_t \), \( f(\bar{y}_t) - f^* \leq \epsilon \),
where \( f^* = \min_{y \in Q} f(y) \)

**Proof.** Let \( y^* = \arg \min_{y \in Q} f(y) \). Applying the previous theorem with \( y = y^* \) we get:
\[
f(\bar{y}_t) - f(y^*) \leq \frac{1}{T} \sum_t f(y_t) - f(y^*) \leq \frac{1}{T} \sum_t \nabla f(y_t)^\top (y_t - y^*) \leq \epsilon
\]
where both inequalities follow from convexity of \( f \). \( \square \)

## 3 Nearly linear time deterministic algorithm

In this section, we present a nearly linear time deterministic algorithm for the approximate Carathéodory Problem. Barman’s original proof [2] involves solving the exact Carathéodory problem, i.e. writing \( u = \sum x \cdot \lambda_x \), interpreting \( \lambda \) as a probability distribution over \( X \), sampling \( k \) points from \( X \) according to \( \lambda \) and arguing using concentration bounds (Khintchine inequality to be precise) that the expectation \( \mathbb{E} \|u - \frac{1}{k} \sum_{i=1}^k x_i\|_p \leq \epsilon \). From an algorithmic point of view, this requires: (i) solving a linear program to compute \( \lambda \); (ii) using randomization to sample \( x_i \). Our main theorem shows that neither is necessary. There is a linear time deterministic algorithm that doesn’t require a solution \( \lambda \) to the exact Carathéodory problem.

Our algorithm is based on Mirror Descent. The idea is to formulate the Carathéodory problem as an optimization problem. Inspired by early positive Linear Programming solvers such as the one of Plotkin, Shmoys and Tardos [24], we convert this problem to a saddle point problem and then solve the dual using Mirror Descent. Using Mirror Descent to solve the dual guarantees a sparse primal certificate that would act as the desired convex combination.

Recall that we are given a finite set of points \( X = \{v_1, v_2, \ldots, v_m\} \subseteq B_p(1) \) and \( u \in \text{conv}(X) \). Our goal is to produce a sparse convex combination of the points in \( X \) that is \( \epsilon \)-close to \( u \) in the \( \ell_p \)-norm. Dropping the sparsity constraint for now, we can formulate this problem as:
\[
\min_{x \in \Delta} \|Vx - u\|_p \quad \text{(P-CARA)}
\]
where $V$ is a $d \times m$ matrix where the columns are the vectors $v_1, \ldots, v_m$ and $\Delta = \{ x \in \mathbb{R}^d \mid \sum_i x_i = 1, x \geq 0 \}$ is the unit simplex in $d$-dimensions. We refer to $\text{P-CARA}$ as the primal Carathéodory problem. This problem can be converted to a saddle point problem by noting that we can write the $\ell_p$ norm as $\|x\|_p = \max_y \|y\|_q = 1 y^\top x$ for $\frac{1}{p} + \frac{1}{q} = 1$. So we can reformulate the problem as:

$$\min_{x \in \Delta} \max_{y \in B_q(1)} y^\top (V x - u) \quad \text{(S-CARA)}$$

Sion’s Theorem [30] is a generalization of Von Neumann’s minimax theorem that allows us to swap the order of minimization and maximization for any pair of compact convex sets. This leads to dual version of the Carathéodory problem:

$$\max_{y \in B_q(1)} \left( \tilde{f}(y) := \min_{x \in \Delta} y^\top (V x - u) \right) \quad \text{(D-CARA)}$$

The function $\tilde{f}$ is concave, since it is expressed as a minimum over linear functions in $y$ parametrized by $x$. Maximizing a concave function is equivalent to minimizing a convex function. To keep the minimization terminology, which is more standard in optimization, we write:

$$- \min_{y \in B_q(1)} \left( f(y) := \max_{x \in \Delta} y^\top (u - V x) \right) \quad \text{(D-CARA’)}$$

**Sparse solution by solving the dual.** Since $u \in \text{conv}(X)$, there is a vector $x \in \Delta$ such that $u = V x$. Hence, the optimal solution for $\text{P-CARA}$ is zero and therefore are the solution of all equivalent formulations. Even though we know the optimal solution, it makes sense to optimize $f(y)$ since in the process we can obtain an $\epsilon$-approximation in a few number of iterations. If each iteration updates only one coordinate, then we are guaranteed to obtain an approximation with sparsity equal to the number of iterations. As it will become clear in a second, while the updates of variable $y$ are not sparse, the dual certificate produced by Mirror Descent will be sparse.

To make this statement precise, consider the gradient of $f$, which can be obtained by an application of the envelope theorem: $\nabla f(y) = u - V x$ for $x = \arg \max_{x \in \Delta} y^\top (u - V x)$. This problem corresponds to maximizing a linear function over the simplex, so the optimal solution is a corner of the simplex. In other words, $\nabla f(y) = u - v_i$ where $i = \arg \max_y [- (V^\top y)]$. Finally, we can use the Mirror Descent guarantee in Theorem 2.2 to bound the norm of the average gradient. We make this precise in the proof of the following theorem.

**Remark 3.1.** In fact $V$ does not even have to be explicitly given. All we need is to solve $i = \arg \max_y [- (V^\top y)]$. When $V$ is explicitly given, this can be done in $dn$ time by picking the best vertex. Sometimes, especially in combinatorial optimization, we have a polytope (whose vertices are $V$) represented by its constraints. Our result states that for these alternate formulations, we can still obtain a sparse representation efficiently if we can solve linear optimization problems over it fast. This observation will be important for our application to submodular minimization.

**Theorem 3.2.** Consider a $(1/\sigma)$-strongly convex function $\omega : B_q(1) \rightarrow \mathbb{R}$ with respect to the $\ell_q$-norm, $D = \max_{y \in B_q(1)} D_\omega(y\|0)$ and $T \geq 8D\sigma^2/\epsilon^2$. Let $y_1, \ldots, y_T$ be the $T$ first iterates of the Mirror Descent algorithm (Theorem 2.2) with mirror map $\nabla \omega^*$ minimizing function $f$ in $\text{D-CARA’}$. If $\nabla f(y_t) = u - v_i(t)$, then

$$\left\| u - \frac{1}{T} \sum_{t=1}^T v_i(t) \right\|_p \leq \epsilon.$$

**Proof.** We consider the space $y \in B_q(1)$ equipped with the $\ell_q$ norm. To apply the Mirror Descent framework, we need first to show that the dual norm (the $\ell_p$-norm, in this case) of the gradient is
bounded. This is easy, since in the approximate Carathéodory problem, \( v_i \in B_p(1) \), so \( \| \nabla f(y) \|_p = \| u - v_i \|_p \leq \| u \|_p + \| v_i \|_p \leq 2 \). So we can take \( p = 2 \) in Theorem 2.2.

Since \( f(y) = \max_{x \in \Delta} y^\top(u - Vx) \) and \( \nabla f(y) = (u - Vx) \) for \( x \in \arg \max_{x \in \Delta} y^\top(u - Vx) \), then \( f(y) = \nabla f(y)^\top y \). Also, since \( f(y) \) can be written as \( \| u - Vx \|_\infty \), clearly \( f(y) \geq 0 \) for all \( y \). Plugging those two facts in the guarantee of Theorem 2.2, we get:

\[
\epsilon \geq \frac{1}{T} \sum_{t=1}^{T} \nabla f(y)(t^\top)(y_t - y) = \frac{1}{T} \sum_{t=1}^{T} [f(y(t)) - \nabla f(y)(t^\top)y] \geq \left[ -\frac{1}{T} \sum_{t=1}^{T} \nabla f(y(t)) \right]^\top y, \forall y \in B_q(1)
\]

Taking the maximum over all \( y \in B_q(1) \) we get:

\[
\left\| u - \frac{1}{T} \sum_{t=1}^{T} v_i(t) \right\|_p = \left\| \frac{1}{T} \sum_{t=1}^{T} \nabla f(y(t)) \right\|_p = \max_{y \in B_q(1)} \left[ -\frac{1}{T} \sum_{t=1}^{T} \nabla f(y(t)) \right]^\top y \leq \epsilon
\]

\[\square\]

To complete the picture, we need to provide a function a \((1/\sigma)\)-strongly convex function \( \omega : B_q(1) \to \mathbb{R} \) with a small value of \( \sigma \cdot \max_{y \in B_q(1)} D_\omega(y(0)). \)

**Proposition 3.3.** For \( 1 < q \leq 2 \), the function \( \omega : B_q(1) \to \mathbb{R} \), \( \omega(y) = \frac{1}{2} \| y \|_q^2 \) is \((q - 1)\)-strongly convex with respect to the \( \ell_q \) norm and \( \max_{y \in B_q(1)} D_\omega(y(0)) = \frac{1}{2} \).

**Proof.** We want to bound \( \omega(y) - \omega(x) - g^\top(y - x) \) for all \( g \in \partial \omega(x) \). For all \( x \) in the interior of the ball \( B_q(1) \) there is a unique subgradient which we represent by \( \nabla \omega(x) \). In the border of \( B_q(1) \), however, there are multiple subgradients. First we claim that we need only to bound \( \omega(y) - \omega(x) - \nabla \omega(x)^\top(y - x) \) where \( \nabla \omega(x) \) denotes the gradient of the function \( \frac{1}{2} \| y \|_q^2 \). In order to see that, notice that if \( g \) is a subgradient in a point \( x \) and \( y \in B_q(1) \) then:

\[
\omega(x + t(y - x)) - \omega(x) - g^\top(y - x) \geq 0
\]

by the definition of subgradient. Dividing the expression by \( t \) and taking the limit when \( t \to 0 \), we get:

\[
\nabla \omega(x)^\top(y - x) \geq g^\top(y - x), \text{ so in particular: } \omega(y) - \omega(x) - g^\top(y - x) \geq \omega(y) - \omega(x) - \nabla \omega(x)^\top(y - x).
\]

This observation allows us to bound the strong convexity parameter of \( \omega \) by looking at the \( \| \cdot \|_q \) eigenvalues of the Hessian of \( \omega \). In particular, we will show that for all \( w \in \mathbb{R}^d \), \( w^\top \nabla^2 \omega(y)w \geq (q - 1) \| w \|_q^2 \).

To make the notation simpler, we define \( \text{POW} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \) as \( \text{POW}(y, p) = ([|y_i|^p \cdot \text{sgn}(y_i)]_i) \).

This allows us to represent \( \nabla \| y \|_q \) in a succinct form: since

\[
\partial_i \| y \|_q = \frac{1}{q} (\| y \|_q^q)^{\frac{1}{q} - 1} x_i q \cdot \text{sgn}(x_i) = \| y \|_q^{1-q} x_i \| y \|_q^{q-1} \text{sgn}(y_i)
\]

so we can write \( \nabla \| y \|_q = \| y \|_q^{1-q} \cdot \text{POW}(y, q - 1) \). Therefore:

\[
\nabla \omega(y) = \nabla \left[ \frac{1}{2} \| y \|_q^2 \right] = \| y \|_q \cdot \nabla \| y \|_q = \| y \|_q^{2-q} \cdot \text{POW}(y, q - 1)
\]

Now, to compute the Hessian, we have:

\[
\nabla^2 \omega(y) = (2 - q) \| y \|_q^{2 - 2q} \cdot \text{POW}(y, q - 1) \text{POW}(y, q - 1)^\top + (q - 1) \| y \|_q^{2-q} \text{DIAG}([|y_i|^{q-2}])
\]

where \( \text{DIAG}([|y_i|^{q-2}]) \) is the diagonal matrix with \( x_i^{q-2} \) in the diagonal. Using the fact that \( 1 < q \leq 2 \),
we can write:
\[ w^T \nabla^2 \omega(y) w = (2 - q) \cdot \| y \|_q^{2-q} \cdot [\text{Pow}(y, q) - 1]^T w]^2 + (q - 1) \cdot \| y \|_q^{2-q} \sum_i |y_i|^{q-2} w_i^2 \]
\[ \geq (q - 1) \left( \sum_i |y_i|^{q-2} w_i^2 \right) \cdot \left( \sum_i |y_i|^{q-2} w_i^2 \right) \]
\[ = (q - 1) \left[ \left( \sum_i |y_i|^{q(q-2)/2} \right)^{\frac{2-q}{q}} \cdot \left( \sum_i |y_i|^{q(q-2)/2} w_i^2 \right)^{\frac{2-q}{q}} \right] \]
\[ = (q - 1) \cdot \left( \sum_i w_i^q \right)^{\frac{2-q}{q}} = (q - 1) \cdot \| w \|_q^2 \]

Finally, we need to show how to compute the Fenchel dual \( \omega^* \) and the mirror map \( \nabla \omega^* \) efficiently:

**Proposition 3.4.** The Fenchel dual of the function \( \omega \) defined in Proposition 3.3 can be computed explicitly:

\[ \omega^*(z) = \begin{cases} \frac{1}{2} \| z \|_p^2 & \text{if } \| z \|_p \leq 1 \\ \| z \|_p - \frac{1}{2} & \text{if } \| z \|_p > 1 \end{cases} \]

Also, \( \nabla \omega^*(z) = \phi(z) \cdot \min(1, \| z \|_p) \) where \( \phi(z) \) is a vector with \( \ell_q \)-norm 1 such that \( z^\top \phi(z) = \| z \|_p \).

This function can be explicitly computed as: \( \phi(z)_i = \text{sgn}(z_i) \cdot |z_i|^{p-1} / \| z \|_p^{p-1} \).

**Proof.** By the definition of Fenchel duality:

\[ \omega^*(z) = \max_{y \in B_q(1)} z^T y - \frac{1}{2} \| y \|_q^2 = \max_{0 \leq \lambda \leq 1} \left[ \max_{\| \hat{y} \|_q = 1} \lambda z^T \hat{y} - \frac{1}{2} \lambda^2 \right] = \max_{0 \leq \lambda \leq 1} \lambda \| z \|_p - \frac{1}{2} \lambda^2 \]

where the second equality follows from writing \( y = \lambda \hat{y} \) for \( 0 \leq \lambda \leq 1 \) and \( \| \hat{y} \|_p = 1 \). The optimal value of \( \hat{y} \) is \( \phi(z) \). The expression \( \lambda \| z \|_p - \frac{1}{2} \lambda^2 \) is maximized at \( \lambda = \| z \|_p \). Since \( \lambda \) is restricted to lie between 0 and 1, the optimal \( \lambda \) must be \( \min(1, \| z \|_p) \).

If \( \| z \|_p \leq 1 \), \( \lambda = \| z \|_p \) and \( \omega^*(z) = \frac{1}{2} \| z \|_p^2 \). If \( \| z \|_p > 1 \), then \( \lambda = 1 \) and \( \omega^*(z) = \| z \|_p - \frac{1}{2} \).

By the envelope theorem, \( \nabla \omega^*(z) = \hat{y} \cdot \lambda = \phi(z) \cdot \min(1, \| z \|_p) \). Now, it simple to check that \( \phi \) has the desired properties:

\[ \| \phi(z) \|_q^2 = \sum_i |z_i|^{q(p-1)} / \| z \|_p^{q(p-1)} = \sum_i |z_i|^{p} / \| z \|_p^{p-1} = 1 \]

\[ z^T \phi(z) = \sum_i z_i^p / \| z \|_p^{p-1} = \| z \|_p \]

Combining the previous results, we obtain:

**Theorem 3.5.** Given \( n \) points \( v_1, \ldots, v_n \in B_p(1) \subseteq \mathbb{R}^d \) with \( p \geq 2 \) and \( u \in \text{conv}\{v_1, \ldots, v_n\} \), there is a deterministic algorithm of running time \( O(nd \cdot p/\epsilon^2) \) that outputs a multiset \( v_i(1), \ldots, v_i(k) \) for \( k = 4(p-1)/\epsilon^2 \) such that \( u' = \frac{1}{k} \sum_{t=1}^k v_i(t) \) and \( \| u' - u \|_p \leq \epsilon \).
Proof. The number of iterations of Mirror Descent (and consequently the sparsity bound) $T = 4p/\epsilon^2$ can be obtained by substituting $D = 1/2$ and $\sigma^{-1} = (q - 1)^{-1} = p - 1$ from Proposition 3.3 in Theorem 3.2.

For the running time, notice that the time per iteration is dominated by the computation of the subgradient of $f$. The most expensive step is the computation of $V^Ty$ which takes $dn$ operations, which is the size of matrix $V$.

\[ \square \]

3.1 Improved bound when $u$ is far from the boundary

Theorem 3.6. Let $P$ be a polytope contained inside the unit $\ell_p$ ball, and a point $u \in P$. If $B_p(r) \subseteq P$, then there exists $x \in 2(1 - \epsilon/r) \cdot \Delta$ supported at $k = O\left(\frac{p}{\epsilon^2} \cdot \log \frac{r}{\epsilon} \right)$ coordinates such that $\left\| \sum_{i \in \text{supp}(x)} x_i v_i - u \right\|_p \leq \epsilon$.

Proof. Let $\text{ApproxCara}(u)$ represent the convex combination $x$ of vertices of $P$ returned by the algorithm from Theorem 3.5, when receiving as input the vertices of $P$ and the point $u$, and solving for precision $r/2$. Then let $e_0 = u$, $x_i = \text{ApproxCara}(e_{i-1})$, $e_i = 2(e_{i-1} - V x_i)$, for $i \in \{1, \ldots, \beta\}$ where $\beta = \log(r/\epsilon)$. Note that $\|e_i\|_p = 2\|e_{i-1} - V x_i\|_p \leq 2 \cdot \frac{r}{2} \leq r$, hence $e_i \in P$, so the input to $\text{ApproxCara}$ is always well defined. Let $\overline{x} = \sum_{i=1}^{\beta} \frac{1}{2^{i-1}} \cdot x_i \in \left(\sum_{i=1}^{\beta} 2^{-(i-1)}\right) \Delta = \frac{2^\beta - 1}{2^{\beta} - 1} \cdot \Delta = 2(1 - \epsilon/r) \Delta$.

Let us bound the error when approximating $u$ with $V \overline{x}$:

\[ \|V \overline{x} - u\|_p = \left\| V \left( \sum_{i=1}^{\beta} \frac{1}{2^{i-1}} \cdot x_i \right) - u \right\|_p = \left\| \sum_{i=1}^{\beta} \frac{1}{2^{i-1}} \cdot V x_i - e_0 \right\|_p = \left\| \sum_{i=2}^{\beta} \frac{1}{2^{i-1}} V x_i + V x_1 - e_0 \right\|_p \]

\[ = \frac{1}{2^{\beta-1}} \left\| \sum_{i=2}^{\beta} \frac{1}{2^{i-1}} V x_i - e_{\beta-1} \right\|_p = \frac{1}{2^{\beta-1}} \|V x_\beta - e_{\beta-1}\|_p \leq \frac{1}{2^{\beta-1}} \cdot \frac{r}{2} \]

Each of the $\beta = \log(r/\epsilon)$ iterations requires a call to $\text{ApproxCara}$ for precision $r/2$, which produces a solution with sparsity $O(p/r^2)$. Hence $\overline{x}$ will have $O\left(\frac{p}{r^2} \cdot \log \frac{r}{\epsilon} \right)$ nonzero coordinates. \[ \square \]

Corollary 3.7. If $u \in P$ satisfies $B_p(u, r) \subseteq P \subseteq B_p(u, 1)$, $r \geq 2\epsilon$, then there exists $x \in \Delta$ supported at $k = O\left(\frac{p}{\epsilon^2} \cdot \log \frac{r}{\epsilon} \right)$ coordinates such that $\|\sum_{i \in \text{supp}(x)} x_i v_i - u\|_p \leq \epsilon$.

Proof. Let $v_i' = v_i - u$ for all $i$. This corresponds to translating $P$ such that $u$ is placed at the origin. By the triangle inequality, this at most doubles the radius of the origin-centered $\ell_p$ ball circumscribing the polytope. Applying Theorem 3.6 we obtain a vector $x \in 2(1 - \epsilon/r) \Delta$ such that $\left\| \sum_{i \in \text{supp}(x)} x_i v_i' \right\|_p \leq \epsilon$. Let $x_i' = x_i/\|x\|_1 \in \Delta$. This satisfies $\left\| \sum_{i \in \text{supp}(x)} x_i' v_i' \right\|_p \leq \epsilon/\|x\|_1$. Hence $\left\| \sum_{i \in \text{supp}(x)} x_i' v_i' \right\|_p = \left\| \sum_{i \in \text{supp}(x)} x_i' (v_i - u) \right\|_p = \left\| \sum_{i \in \text{supp}(x)} x_i' v_i - u \right\|_p \leq \frac{\epsilon}{2(1 - \epsilon/r)} \leq \epsilon$. \[ \square \]

4 Lower bound

We showed that if $V$ is a $d \times n$ matrix whose columns are contained in the unit $\ell_p$ ball, then for any $x \in \Delta_n$ there is $\tilde{x} \in \Delta_n$ with $|\text{supp}(\tilde{x})| \leq O(p/\epsilon^2)$ such that $\|Ax - Ax'\|_p \leq \epsilon$, where $\text{supp}(x) = \{i | x_i \neq 0\}$. 

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In this section we argue that no dimension independent bound better then $O(p/e^2)$ is possible. This shows that the sparsity bound in Theorem 3.5 is tight and improves Barman’s $\Omega(1/e^{p(p-1)})$ lower bound [2]. Formally, we show that:

**Theorem 4.1.** There exists a constant $K$ such that for every $p \geq 2$ and $n \geq n_0(p)$, there exists $n \times n$ matrix $V$ with columns of unit $\ell_p$ norm, and a point $u = Vx$, $x \in \Delta_n$, such that for all $\bar{x} \in \Delta$ with sparsity $|\text{supp}(\bar{x})| \leq Kp/e^2$, one has that $\|V \bar{x} - u\|_p \geq 2\epsilon > \epsilon$.

In other words, even though $u$ is a convex combination of columns of $V$, every $(Kp/e^2)$-sparse convex combination of columns of $V$ has distance at least $\epsilon$ from $u$ in the $\ell_p$-norm. We first present a simple and constructive $\Omega(1/e^2)$ lower bound and later a tight $\Omega(p/e^2)$ lower bound based on the probabilistic method.

### 4.1 A simple lower bound $\Omega(1/e^2)$

The lower bound relies on Sylvester’s construction of Hadamard matrices, which are defined for all values of $n$ that are powers of 2.

Sylvester’s recursive construction follows by defining $H_1 = [1]$, and for every $n$ that is a power of 2:

$$H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$$

**Proposition 4.2.** The Sylvester matrix $H_n$ defined as above is Hadamard. In other words, $H_{ij} = \pm 1$ for all $i, j$ and $H^TH = nI$ (its columns are mutually orthogonal).

Now we consider the polytope $P$ formed by the convex hull of the normalized columns of $H$. One can easily check that for the construction above the uniform combination of columns is $H \cdot \frac{1}{n} = e_1$ where $1$ is the vector of all 1’s and $e_1$ is the unit vector in the direction of the 1-th component. We show that $e_1$ is at distance greater than $\epsilon$ from the convex hull of any $o(1/e^2)$ columns of $H$.

**Theorem 4.3.** Let $H_n$ be a $n \times n$ Sylvester matrix, and let $P$ be the convex hull of the columns of $\tilde{H} := H/n^{1/p}$. Let $u = \tilde{H} \cdot \frac{1}{n} = e_1/n^{1/p} \in P$. Then any $x \in \Delta_n$ satisfying

$$\|\tilde{H}x - u\|_p \leq \epsilon$$

has sparsity $|\text{supp}(x)| \geq \min(1/e^2, n)$.

**Proof.** Let $x \in \Delta_n$ be $k$-sparse, i.e. $|\text{supp}(x)| = k$, such that $\|\tilde{H}x - u\|_p \leq \epsilon$. We would like to lower bound the sparsity $k$ in terms of $\epsilon$ and $p$.

We will use two main ingredients in the proof: the first is the power mean inequality which states that for any vector $x \in \mathbb{R}^n$, $(\frac{1}{n} \sum_i x_i^t)^{1/t}$ is non-decreasing in $t$. In particular, this implies that $\|x\|_t \cdot n^{-1/t}$ is non-decreasing\(^1\). The second fact we will use is that for every vector $\|x\|_2^2 \leq \|x\|_2^2 \cdot |\text{supp}(x)|$. This follows from the Cauchy-Schwarz inequality: $\|x\|_1^2 = \left(\sum_{i \in \text{supp}(x)} x_i \cdot 1\right) \leq \left(\sum_{i \in \text{supp}(x)} x_i^2\right) \cdot \left(\sum_{i \in \text{supp}(x)} 1\right) = \|x\|_2^2 \cdot |\text{supp}(x)|$. Combining both results give us a bound involving the 2-norm of the error:

$$\epsilon \geq \|\tilde{H}x - u\|_p = \frac{1}{n^{1/p}} \cdot \|Hx - e_1\|_p \geq \frac{1}{n^{1/2}} \cdot \|Hx - e_1\|_2$$

where the last step follows from the power-mean inequality. Squaring both sides, we get:

$$\epsilon^2 \geq \frac{1}{n} (Hx - e_1)^\top (Hx - e_1) = \frac{1}{n} \left[ x^\top H^\top Hx - 2e_1^\top Hx + 1 \right] = \|x\|_2^2 - \frac{1}{n} \left( \frac{\|x\|_2^2}{|\text{supp}(x)|} - \frac{1}{n} \right) = \frac{1}{k} - \frac{1}{n}$$

\(^1\)This can be seen by computing the derivative of $M_t(x) = (\frac{1}{n} \sum_i x_i^t)$ with respect to $t$ and showing it is non-positive.
We used the fact that $e_1^\top H x = \|x\|_1 = 1$ since the top row of $H$ consists of only 1’s. Hence $k \geq (e^2 + 1/n)^{-1} \geq 1/\max (e^2, 1/n) = \min (1/e^2, n)$.

### 4.2 Tight lower bound $\Omega(p/e^2)$

We now establish a tight lower bound via a probabilistic existence argument inspired by the construction of Klein and Young [17]. The example used to exhibit the lower bound is very simple. The proof of its validity, however, is quite involved and requires a careful probability analysis. We first give an overview before delving into the details.

**Overview.** Recall the formulation S-CARA of the Carathéodory problem as a saddle point problem described in Section 3. If we translate all points such that $u = 0$, then we can write the problem as:

$$\min_{x \in \Delta} \max_{y \in B_q(1)} y^\top V x$$

which can be seen as a game between a player controlling $x$ and $y$. The approximate Carathéodory theorem states that (under the conditions from Theorem 3.5), if the value of the game is zero, then the $x$-player has a $k$-sparse strategy that guarantees that the value of the game is at most $\epsilon$ for $k = O(p/e^2)$.

For the lower bound, our goal is to construct an instance of this game with value $v$ such that for all $k$-sparse strategies of the $x$-player with $k < Cp/e^2$, the $y$-player can force the game to have a value strictly larger than $v + \epsilon$.

**Probabilistic Construction:** We define the matrix $V$ such that $V = n^{-1/p} \cdot A$ where $A$ is an $n \times n$ matrix with random ±1 entries, i.e., each entry of $A$ is chosen at random from $\{-1, +1\}$ independently with probability $1/2$. Note the the $\ell_p$ norm of the columns of $A$ is equal to 1, so we are in the conditions of Theorem 3.5. Then we will show that the following events happen with high probability:

1. The center of the polytope defined by the columns of $V$ is $\epsilon$-close to zero, i.e., $\|V \cdot 1/n\|_p \leq \epsilon$.

2. For each set $S$ of $k$-coordinate, if $x$ is restricted to only use those coordinates, the $y$-player can force the value of the game to be at least $2\epsilon$. We prove so by exhibiting a strategy for the $y$-player such that $y^\top V$ is at least $2\epsilon$ for all coordinates in $S$.

After we bound the probabilities of the previous events, the result follows by taking the union bound over all the $\binom{n}{k}$ possible subsets $S$ of cardinality $k$. This will show that with nonzero probability a matrix will be constructed, for which the $y$ player will always be able force $y^\top V x \geq 2\epsilon$, no matter what $o(p/e^2)$-sparse strategy player $x$ chooses.

**Bounding probabilities.** All the lemmas and theorems from this of this section are in the conditions of the previous paragraphs: $A$ and $V$ are the random matrices previously defined, and $S$ is a fixed subset of $x$-coordinates of size $k$.

**Lemma 4.4.** If the $x$-player plays the uniform strategy, then $E \left[ \|V \cdot 1/n\|_p \right] \leq \sqrt{p/n}$, and

$$\mathbb{P} \left[ \|V \cdot 1/n\|_p \geq \epsilon \right] \leq \sqrt{\frac{p}{n \epsilon^2}} \text{ for } n \geq p/e^2.$$ 

**Proof.** The bound on the expectation follows from Khintchine’s inequality, which states that for any given vectors $u_1, \ldots, u_m \in \mathbb{R}^n$ and iid uniform $\{-1, +1\}$-variables $r_i$

$$E \left[ \left\| \sum_i r_i u_i \right\|_p \right] \leq \sqrt{p} \cdot \left( \sum_i \|u_i\|^2_p \right)^{1/2}.$$
Proof. Applying Lemma 4.7. There are constants $C, c > 0$ such that for all $\lambda \leq \sigma$,
\[
\mathbb{P} \left[ \left| \sum_{i=1}^{k} X_i - \rho \cdot k \right| \leq \lambda \sigma \right] \geq c \exp \left( -C\lambda^2 \right)
\]

Lemma 4.6 (Anti-concentration for the binomial distribution). In the same conditions as in the previous lemma, there exist constants $\tilde{C}, \tilde{c} > 0$ such that for all $\lambda \leq \sigma$,
\[
\mathbb{P} \left[ \left| \sum_{i=1}^{k} X_i - \rho \cdot k \right| \geq \lambda \sigma \right] \geq \tilde{c} \exp \left( -\tilde{C}\lambda^2 \right)
\]

Lemma 4.7. There are $r = \Omega(n \exp(-O(ke^2)))$ good rows with probability at least $1 - \exp(-\Omega(n \exp(-O(ke^2))))$.

Proof. Applying Lemma 4.6 with $\rho = 1/2$ and $\lambda = 4\epsilon \sigma$ we obtain that the probability that a row is at least $k \left( \frac{1}{2} + \epsilon \right) k$ $+1$’s (or $-1$’s, due to symmetry) is at least $\exp(-O(ke^2))$. So in expectation, there are $n \exp(-O(ke^2))$ good rows, so the result follows by applying the Chernoff bounds with $\rho = \exp(-O(ke^2))$ and $\lambda = c\sigma$.

With high probability there will be at least $r = \Omega(n \exp(-O(ke^2)))$ good rows for the $y$-player to play. We want to argue that if the $y$-player can find $r$ good rows, then he can play $y_i = r^{-1/q}$ for each good row $i$, and 0 otherwise, and he will leave an $x$-player restricted to choosing only columns from a subset $S$ without any good option to play.

Lemma 4.8. Let $S$ be a fixed subset of columns of $A$, and let $A_S$ be the matrix whose columns are the columns of $A$ that belong to $S$. Conditioning on $A_S$ having $r$ good rows, with probability at least $1 - k \exp \left( -\Omega(r\epsilon^2) \right)$, every column in $S$ contains at least $r(1/2 + \epsilon/2)$ $+1$’s in the $r$ good rows.

Proof. Sample matrix $A$ according to the following procedure: in the first phase, sample each entry of $A$ uniformly and independently from $\{-1, +1\}$. In the second phase, for each row, shuffle the entries in $S$ (i.e. for each row, sample a random permutation of $S$ and apply to the entries corresponding to those columns). In the first phase we can decide which rows are good, call those $R$. Conditioning on the first phase, and fixing a column $j \in S$, the entries $A_{ij}$ for $i \in R$ are
independent and uniform from \{-1, +1\}; the probability of \(A_{ij}\) being 1 is at least \(\frac{1}{2} + \epsilon\), since this entry is a random entry from a good row sampled in the first phase. The result follows by applying the Chernoff bound with \(\rho = \frac{1}{2} + \epsilon\) and \(\lambda = (\sigma \epsilon) / (\rho \cdot (1 - \rho))\).

Now, we combine all the events discussed so far using the union bound:

**Lemma 4.9.** Fix \(\epsilon\) and \(k\). For sufficiently large \(n\), there is a matrix \(A\) such that \(V = n^{-1/p} A\) satisfies \(\|V \cdot \vec{1}/n\|_p \leq \epsilon\), and for every subset \(S\) of \(k\) rows, there is a subset \(R\) of \(r = \Omega(n \exp(-O(ke^2)))\) rows such that for all \(i \in S\), \(\sum_{j \in R} A_{ij} \geq \epsilon r\).

**Proof.** The proof follows from the probabilistic method. For each subset \(S\), with probability at least \(1 - \exp(-\Omega(n \exp(-O(ke^2))))\) there are \(r\) good rows (Lemma 4.7) and with probability \(1 - k \exp(-\Omega(re^2)) = 1 - k \exp(-\Omega(n^2 \exp(-O(ke^2))))\) there are at least \((\frac{1}{2} + \epsilon) r\) many +1s in each column corresponding to the \(r\) rows (Lemma 4.8), causing \(\sum_{j \in R} A_{ij} \geq \epsilon r\). The probability that both events occur can be bounded by \(1 - O(k \exp(-\Omega(n^2 \exp(-O(ke^2))))))\). Applying the union bound over all \(\binom{n}{k}\) subsets \(S\), we get:

\[
1 - \binom{n}{k} O(k \exp(-\Omega(n^2 \exp(-O(ke^2))))) \geq 1 - \exp(k \log n - O(\epsilon^2 n \exp(-O(ke^2))))
\]

goese to one as \(n \to \infty\) for any fixed \(k\) and \(\epsilon\). Also, as \(n \to \infty\) the probability that \(\|V(\frac{1}{n})\vec{1}\|_p \leq \epsilon\) also goes to 1.

**Theorem 4.10** (Carathéodory lower bound). There is a matrix \(V\) whose columns have unit \(\ell_p\) norm such that \(\|V \cdot \vec{1}/n\|_p \leq \epsilon\), and for every \(x \in \Delta\), \(|\text{supp}(x)| \leq k = O(p/\epsilon^2),\|Vx\|_p \geq 2\epsilon\).

**Proof.** Let \(V\) be the matrix obtained in Lemma 4.9. From there, we have that \(\|V \cdot \vec{1}/n\|_p \leq \epsilon\).

Now, fix any \(x \in \Delta\) with \(|\text{supp}(x)| \leq k\), and let \(S\) be the set of columns corresponding to the support of \(x\). Let also \(R\) be the set of rows for which \(\sum_{j \in R} A_{ij} \geq \epsilon r\) for all \(i \in S\). Now, define \(y \in B_q(1)\) such that \(y_i = r^{1/q}\) if \(i \in R\), and \(y_i = 0\) otherwise:

\[
\|Vx\|_p \geq y^\top Vx = n^{-1/p} \cdot (y^\top A)x = \frac{r \epsilon}{r^{1/q} \cdot n^{1/p}} = \epsilon \left(\frac{r}{n}\right)^{1/p}
\]

We want to choose the parameters such that \(\left(\frac{r}{n}\right)^{1/p} \geq 2\). Substituting \(r = \Omega(n \exp(-O(ke^2)))\):

\[
\left(\frac{r}{n}\right)^{1/p} = \exp\left(-O\left(\frac{ke^2}{p}\right)\right)
\]

If \(k \leq C \cdot \frac{p}{\epsilon^2}\) for a suitable constant \(C\), we get \(\|Vx\|_p \geq 2\epsilon\).

## 5 Applications

The approach presented in the previous sections can be easily generalized or directly applied to a series of applications. Here we identify three representative applications to illustrate the usefulness of our approach. We note that there are many other possible applications in combinatorial optimization, game theory and machine learning, where a convex combination is often maintained as a subroutine of the algorithm.

### 5.1 Warm-up: fast rounding in polytopes with linear optimization oracles

The most direct application of our approach is to efficiently round a point in a polytope whenever it admits a fast linear optimization oracle. An natural such instance is given by the matroid polytope. We denote a \(n\)-element matroid by \(M\) and its rank by \(r\).
Proposition 5.1. There is an algorithm which, given a fractional point \( x^* \) contained inside the base polytope of a matroid \( M \), and a norm parameter \( p \geq 2 \), produces a distribution \( D \) over matroid bases supported on \( O \left( \frac{2^{n^2/p}}{\epsilon^2} \right) \) points, such that \( \| E_{x \sim D} [x] - x^* \|_p \leq \epsilon \). Furthermore the algorithm requires \( O \left( nr^{2/p}p^2/\epsilon^2 \right) \) calls to \( M \)'s independence oracle.

Proof. The result follows from applying Theorem 3.2 for \( x^* \) in the convex hull of the characteristic vectors for matroid bases. Note that each of these vectors has sparsity \( r \) so their \( p \) norm is precisely \( r^{1/p} \). Hence we have the desired sparsity for the support of \( D \). Each iteration requires maximizing a linear function over the bases of the polytope, which can be done using the standard greedy algorithm, and requires \( O(n) \) calls to the independence oracle. \qed

Of course, there are other nice polytopes where the existence of an efficient linear optimization oracle offers advantages. To this aspect, we mention the \( s-t \)-flow polytope (i.e. the convex hull of all \( s-t \) paths), whose oracle is implemented with a single shortest path computation. This enables us to speed up the path stripping subroutine in the Raghavan-Thompson randomized rounding algorithm for approximating minimum congestion integral multicommodity flows [25]. As described in [25] the algorithm takes \( O(m^2) \), which can be improved to near linear time by carefully using link-cut trees [15]. By contrast, approximate Carathéodory provides a lightweight algorithm for producing an approximate decomposition into integral paths, without the need of complicated data structures.

Proposition 5.2. There is an algorithm which, given a fractional \( s-t \)-flow \( f^* \) routing one unit of demand in \( G \), and a norm parameter \( p \geq 2 \), produces a distribution \( D \) over \( s-t \)-paths supported on \( O \left( \frac{2^{n^2/p}}{\epsilon^2} \right) \) points, such that \( \| E_{f \sim D} [f] - f^* \|_p \leq \epsilon \). Furthermore the algorithm requires \( O \left( \frac{2^{n^2/p}p^2}{\epsilon^2} \right) \) shortest path computations.

In the setting of Raghavan-Thompson, fixing \( p = \Theta(\log n) \) yields an approximate path stripping routine that runs in time \( \tilde{O}(m/\epsilon^2) \).

5.2 Submodular function minimization

Submodular function minimization is a primitive that has been studied in combinatorial optimization [19], and that has become increasingly more popular in machine learning literature [18]. In submodular minimization we are given a function \( f \) defined over the subsets of a finite ground set \( X \), satisfying \( f(S) + f(T) \geq f(S \cup T) + f(S \cap T) \) for all \( S, T \subseteq X \), and have to find a subset of \( X \) minimizing the value of \( f \). Examples of submodular functions include matroid rank functions, graph cut functions, and entropy functions.

The most theoretically efficient algorithms for the problem require the use of the ellipsoid algorithm or complex combinatorial procedures. For this reason, practitioners typically employ the method called minimum-norm point algorithm, devised by Fujishige [10, 11, 1]. Traditionally this has been implemented using variants of Wolfe’s algorithm [32], but no provable convergence analysis has been available until recently. In [7] Chakrabarty et al. provided a robust version of Fujishige’s theorem, showing that minimizing the 2-norm of a point in the base polyhedron of \( f \) to accuracy \( 1/n^2 \) yields exact submodular function minimization. Their approach requires \( O \left( n^{5 \cdot T + n^5} F^2 \right) \) time, where \( T \) is the time required to answer a single function value query to \( f \), and \( F = \max_{i \in S \subseteq X} |f(S) - f(S \setminus \{i\})| \) is the maximum marginal difference in absolute value.

Our contribution is to show that our algorithm for the approximate Carathéodory problem from Section 3 can replace Wolfe’s algorithm. This immediately yields an \( \tilde{O} \left( n^{6}F^2 \cdot T \right) \) time algorithm for exact submodular function minimization, and a \( \tilde{O} \left( n^{6}F^2/k^4 \cdot T \right) \) time algorithm for a \( k \)-additive approximation. The first algorithm enjoys a better running time than Wolfe’s algorithm but has a
worse oracle complexity. We leave it as an open problem if it is possible to combine ingredients of both algorithms to achieve a running time of \( \tilde{O}(n^5 F^2 \cdot T) \).

**Proposition 5.3.** Let \( f : 2^X \to \mathbb{Z} \) be a submodular function. There exists an algorithm returning an exact minimizer of \( f \) in time \( \tilde{O}(n^6 F^2 \cdot T) \), respectively a \( k \)-additive approximation to the minimizer in time \( \tilde{O}(n^6 F^2/k^3 \cdot T) \).

Before proving the statement, we define the base polyhedron \( B_f \) of a submodular function, and state the robust Fujishige’s theorem.

**Definition 5.4.** Given a submodular function \( f \) defined on subsets of a ground set \( X \), its base polyhedron is defined as \( B_f = \{ x \in \mathbb{R}^n : x(A) \leq f(A), \forall A \subset X, \text{ and } x(X) = f(X) \} \).

The following lemma, due to Edmonds, is required for an efficient implementation of the step problem in mirror descent. It turns out that in our setting, the step problem requires optimizing a linear function over the base polyhedron \( B_f \), which can be done efficiently using only \( O(n) \) queries to \( f \).

**Lemma 5.5** (Linear optimization over the submodular base polyhedron). Given \( c \in \mathbb{R}^n \), renumber the indices such that \( c_i \leq c_{i+1} \) for all \( i \). Setting \( q_i = f([i]) - f([i-1]) \) yields the solution to the optimization problem \( \min \{ c^\top x : x \in B_f \} \).

**Theorem 5.6** (robust version of Fujishige’s Theorem [7]). Let \( x \in B_f \) such that \( x^\top (x-z) \leq (k/n)^2 \) for all \( z \in B_f \). Then \( x \) can be used to recover a \( k \)-additive approximation to the minimizer of \( f \) in \( \tilde{O}(n) \) time.

**Proof of Proposition 5.3.** Consider the problem \( \min_{x \in \Delta} \| Vx \|_2 \) where the columns of \( V \) are vertices of \( B_f \). Note that the squared 2-norm of the columns of \( V \) is upper bounded by \( nF^2 \). Using the algorithm from Section 3 to solve the problem to accuracy \( (k/n)^2 \) yields \( O(n^5 F^2/k^3) \) iterations, each of them involving a linear optimization problem over the submodular base polyhedron. As seen in Lemma 5.5, this requires sorting the vector of weights, and querying the submodular function \( O(n) \) times. Hence the algorithm runs in \( \tilde{O}(n^6 F^2 T/k^4) \). Setting \( k \) to any constant smaller than 1 yields an exact solution, since by Theorem 5.6, we obtain a set \( S \) satisfying \( f(S) \leq f(S^*) + k < f(S^*) + 1 \) and \( f \) takes values over \( \mathbb{Z} \).

### 5.3 SVM training

Support vector machines (SVM) are an extremely popular classification method, and have found ample usage in machine learning, with applications ranging from finance to neuroscience. In the era of big data it is crucial for any such method to be able to train on huge datasets. While a number of implementations (Liblinear [8], P-packSVM [35], Pegasos [29]) achieve excellent convergence rates in the case of linear SVM’s, handling arbitrary kernels raises a significantly harder problem. Liblinear and Pegasos achieve \( O(\log(1/\epsilon)) \), respectively \( O(1/\epsilon) \) convergence rate, but cannot be extended beyond linear kernels. The \( \epsilon \) dependence for P-packSVM scales as \( O(1/\epsilon^2) \), but it requires knowing the Cholesky factorization of the kernel matrix in advance. In our case, a simple extension of the method described in Section 3 gives \( O(1/\epsilon^2) \) convergence, while only requiring matrix-vector multiplications involving the kernel matrix. So Cholesky factorization is no longer required, and the matrix does not need to be stored explicitly. In the case of linear SVM’s, our method runs in nearly linear time.

Our approach is inspired from a reformulation of the training problem of Kitamura, Takeda, and Iwata [16], who present a method for SVM training based on Wolfe’s algorithm. Their algorithm relies on a dual formulation introduced by Schölkopf et al. [27] which can be easily reformulated as a convex problem over a product of two convex sets. More specifically, we are given empirical
data \((x_i, y_i) \in \mathcal{X} \times \{\pm 1\}, 1 \leq i \leq n\), along with a function that maps features to a Hilbert space \(\Phi: \mathcal{X} \to \mathcal{H}\), which determines a kernel function \(k(x, y) = \langle \Phi(x), \Phi(y) \rangle\). Let \(K \in \mathbb{R}^{n \times n}\), where \(K_{ij} = k(x_i, x_j)\), \(E_+ = \{e_i : y_i = +1\}\), \(E_- = \{e_i : y_i = -1\}\).

In [16], the \(\nu\)-SVM problem is reformulated as:

\[
\min_{\lambda_+, \lambda_-} \left( \lambda_+ - \lambda_- \right) \top K \left( \lambda_+ - \lambda_- \right)
\]
subject to

\[
\lambda_+ \in \text{RCH}_\eta(E_+),
\]

\[
\lambda_- \in \text{RCH}_\eta(E_-)
\]

where \(\eta = \frac{2}{\nu n}\) and \(\text{RCH}_\eta(A) := \{\lambda \in \mathbb{R}^n : \sum a_i \lambda_a |0 \leq \lambda_a \leq \eta, \sum a_i \lambda_a = 1\}\) is the restricted convex hull of set \(A\).

Our approach to solve this problem will be to rephrase it as a saddle point problem (similar to what was done for the approximate Carathéodory problem) and apply Mirror Descent, with a suitable Mirror Map, to solve the dual. Before doing that, we introduce a few useful definitions and facts:

**Definition 5.7.** Let \(K\) be a symmetric positive definite matrix. Then \(\|x\|_K := \sqrt{x \top K x}\).

**Proposition 5.8.** The dual norm of \(\|x\|_K\) is \(\|x\|_{K^{-1}}\). In other words \(\|x\|_K = \max_{y : \|y\|_{K^{-1}} \leq 1} \langle y, x \rangle\).

**Proof.** This can be verified using Lagrange multipliers: over the unit \(\|\cdot\|_{K^{-1}}\)-ball the term \(y \top x\) attains its maximum at \(y = Kx / \|Kx\|_{K^{-1}} = Kx / \sqrt{x \top K x}\). We can verify that for this choice of \(y\), \(y \top x = \frac{1}{\sqrt{x \top K x}} Kx = \|x\|_K\).

**Definition 5.9.** Let \(S_\eta\) = \{\lambda_+ - \lambda_- | \lambda_+ \in \text{RCH}_\eta(E_+), \lambda_- \in \text{RCH}_\eta(E_-)\}.

**Proposition 5.10** (Linear optimization over \(S_\eta\)). Linear optimization over \(S_\eta\) can be implemented in \(O(n)\) time.

**Proof.** The implementation of the linear optimization routine is done in near-linear time via a simple greedy algorithm. The first thing to notice is that the objective is separable, so it is sufficient to optimize separately on \(E_+\) and \(E_-\). This can be done easily, since we need to distribute one unit of mass over the coordinates that span \(E_+\) and one unit of mass over the coordinates that span \(E_-\), such that no coordinate receives more than \(\eta\). Therefore adding mass to the coordinates spanning \(E_+\) in increasing order of the weights \(y\), and vice-versa to those spanning \(E_-\) yields the optimal solution.

With these facts on hand, we can now proceed to describing our equivalent formulation as a saddle-point problem, which will then be solved using a similar method to the one we employed for the previous applications.

Note that instead of directly using the kernel matrix \(K\) in the formulation, we replace it with \(\tilde{K} = K + \epsilon I\). This only changes the value of the objective by at most \(\epsilon / 2\) and it has the advantage of making \(\tilde{K}\) positive semidefinite, since it is now guaranteed to be non-degenerate. This allows us to write the objective function \((\lambda_+ - \lambda_-) \top \tilde{K} (\lambda_+ - \lambda_-)\) as \(\|\lambda_+ - \lambda_-\|_{\tilde{K}}^2\). This formulation can be easily converted to a saddle point problem:

\[
\min_{\lambda \in S} \|\lambda\|_{\tilde{K}} = \min_{\lambda \in S_\eta} \max_{y : \|y\|_{K^{-1}} \leq 1} y \top \lambda = - \min_{\lambda \in S_\eta} \left( - \min_{\lambda \in S_\eta} y \top \lambda \right) = - \min_{\lambda \in S_\eta} f(y)
\]

for \(f(y) := - \min_{\lambda \in S_\eta} y \top \lambda\) defined over the \(\|\cdot\|_{K^{-1}}\)-ball.
The subgradients of $f$ are easy to compute, since they require a simple linear optimization over $S$:

$$\partial f(y) = - \arg \min_{\lambda \in S} y^\top \lambda$$

which can be done in time $\tilde{O}(n)$ using the greedy algorithm described in Proposition 5.10. The mirror map of choice for the domain $\{y : \|y\|_{K^{-1}} \leq 1\}$ will be $\omega : \{y : \|y\|_{K^{-1}} \leq 1\} \to \mathbb{R}$, $\omega(y) = \frac{1}{2} \|y\|_{K^{-1}}^2$, with

$$\omega^*(z) = \begin{cases} \frac{1}{2} \|z\|_{K}^2 & \text{if } \|z\|_{K} \leq 1 \\ \|z\|_{K} - \frac{1}{2} & \text{if } \|z\|_{K} > 1 \end{cases}.$$  

Also, similarly to Proposition 3.4, we have $\nabla \omega^*(z) = \hat{K} z \cdot \min(1, 1/\|z\|_{\hat{K}})$, hence $\|\nabla \omega^*(Z)\|_{\hat{K}^{-1}} \leq 1$.

The only thing left to do is to analyze the algorithm’s iteration count by bounding the strong convexity of $\omega$ and the Lipschitz constant of $f$. We will do this with respect to $\| \cdot \|_2$.

**Proposition 5.11.** $\omega$ is $\min \left( \frac{2}{\epsilon}, \left( \|K\| + \frac{\epsilon}{2} \right)^{-1} \right)$-strongly convex with respect to $\| \cdot \|_2$, where $\|K\|$ is the spectral norm of $K$.

**Proof.** Writing down the Hessian of the mirror map, we obtain $\nabla^2 \omega(y) = \hat{K}^{-1} = (K + \frac{\epsilon}{2} I)^{-1} \succeq \min \left( \frac{2}{\epsilon}, \left( \|K\| + \frac{\epsilon}{2} \right)^{-1} \right) I$. The reason for using $\hat{K}$ instead of $K$ in the formulation is now evident: if $K$ is not full rank, then $\omega$ is not strongly convex. Adding a small multiple of the identity forces all the eigenvalues of $\hat{K}$ to be at least $\epsilon/2$, and avoids the degeneracy where some of them may be zero. \hfill \square

**Proposition 5.12.** $f$ is $2\sqrt{\eta}$-Lipschitz with respect to $\| \cdot \|_2$.

**Proof.** We simply need to bound the 2-norm of the subgradient. By the construction presented in Proposition 5.10 the subgradient contains $2 \cdot \lfloor 1/\eta \rfloor$ nonzero coordinates, $2 \cdot \lfloor 1/\eta \rfloor$ of which are precisely $\eta$. This enables us to obtain a better upper bound than one would usually expect on the 2-norm of the subgradient, namely $\sqrt{2 \cdot \left( \frac{1}{\eta^2} \cdot \frac{\eta}{\epsilon} + \left( 1 - \frac{1}{\eta} \right) \cdot \frac{\eta^2}{\epsilon^2} \right)} \leq \sqrt{2 \cdot \left( \frac{2}{\eta} \cdot \frac{\eta^2}{\epsilon^2} \right)} = 2\sqrt{\eta}$. \hfill \square

**Proposition 5.13.** $\max_{y : \|y\|_K \leq 1} \frac{1}{2} \|y\|_K^2 \leq \frac{1}{2}$

Finally we can put everything together:

**Theorem 5.14.** An $\epsilon$-approximate solution to $\nu$-SVM can be found in $O \left( \eta \cdot \max \left( \frac{2}{\epsilon} \cdot \frac{\|K\|}{\epsilon}, \frac{1}{\epsilon^2} \right) \right) = O \left( \max \left( \frac{1}{\epsilon^2} \frac{\|K\|}{\epsilon}, \frac{1}{\nu \epsilon^2} \right) \right)$ iterations.

**Proof.** Follows from plugging in the parameters $\sigma = \min \left( \frac{\epsilon}{2}, \left( \frac{\|K\|}{\epsilon} + \frac{\epsilon}{2} \right)^{-1} \right)$, $L = 2\sqrt{\eta}$, $R = O(1)$ into the mirror descent algorithm. \hfill \square

At this point, it makes sense to analyze the performance of our algorithm for the most common choices of SVM kernels, which only requires bounding the spectral norm of the kernel matrix; for this purpose we will simply use the trace bound. The results are summarized in the table below. The last column of the table contains the number of iterations required to find a solution down to a precision of $\epsilon$, given that all the vectors $x_i$ belong to the unit $\ell_2$ ball.
| Kernel type                  | Upper bound on $\|K\|$      | Iteration count |
|-----------------------------|-----------------------------|-----------------|
| Polynomial (homogeneous):   | $n \cdot \max_i \|x_i\|^{2d}$ | $O(\max(\frac{1}{n\nu\sigma}, \frac{1}{\nu\sigma}))$ |
| Polynomial (inhomogeneous): | $n \cdot (1 + \max_i \|x_i\|_2^2)^d$ | $O(\max(\frac{1}{n\nu\sigma}, \frac{2d}{\nu\sigma}))$ |
| RBF:                        | $n$                         | $O(\max(\frac{1}{n\nu\sigma}, \frac{1}{\nu\sigma}))$ |
| Sigmoid:                    | $n$                         | $O(\max(\frac{1}{n\nu\sigma}, \frac{1}{\nu\sigma}))$ |

It is worth mentioning that each iteration requires $\tilde{O}(n)$ time for computing the subgradient, and a multiplication of the kernel matrix with a vector; one advantage is that the kernel matrix does not need to be explicitly stored, as its entries can be computed on the fly, whenever needed. In the case of linear kernels, this computation is implemented in linear time since $\tilde{K}z = [x_1|\ldots|x_n]^T[x_1|\ldots|x_n]z + \frac{1}{2}z$, which requires computing a linear combination $h = \sum_i x_i \cdot z_i$ of the vectors $x$, and $n$ dot products between vectors from the training set and $h$.

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