VERSALITY IN TORIC GEOMETRY

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ABSTRACT. We study deformations of affine toric varieties. The entire deformation theory of these singularities is encoded by the so-called versal deformation. The main goal of our paper is to construct the homogeneous part of some degree $-R$ of this, i.e. a maximal deformation with prescribed tangent space $T^1(-R)$ for a given character $R$. To this aim we use the polyhedron obtained by cutting the rational cone defining the affine singularity with the hyperplane defined by $[R = 1]$. Under some length assumptions on the edges of this polyhedron, we provide the versal deformation for primitive degrees $R$.

1. INTRODUCTION

Understanding the deformation theory of a toric variety $X$ and its boundary $\partial X$ is useful for several classification projects. For example, smoothings of such singularities are used to compactify moduli spaces of surfaces of general type. In line with the recent interest in classifying Fano manifolds using Mirror Symmetry [10, 18, 12], it is conjectured that all low dimensional smooth Fano varieties can be degenerated to a singular Fano toric variety [11]. By the comparison theorem of Kleppe from [15], see also [9, Section 2.1] for an overview, understanding deformations of affine toric varieties implies understanding deformations of projective toric varieties as well.

The versal base space of an affine toric singularity inherits a torus action, and thus a lattice grading. Our aim is to construct a maximal deformation in a given primitive degree $-R$. The rational cone defining the toric singularity, together with the degree $-R$, can be entirely reconstructed from a rational polyhedron. For isolated Gorenstein toric singularities the whole versal deformation is concentrated in a single degree (the “Gorenstein degree”). Assuming also smoothness in codimension two, the corresponding polyhedron is a lattice polytope with primitive edges. The versal deformation for such toric singularities was obtained in [1]. In this paper we drop both the Gorenstein and the smoothness in codimension two assumptions, so the versal base space may have several non-trivial graded components. As a special case, we obtain yet another point of view for the deformations of 2-dimensional cyclic quotient singularities ([16, 8, 19]).

We work over an algebraically closed field $k$ of characteristic 0. Let $N$ and $M$ be dual lattices and let $\sigma \subseteq (N \oplus \mathbb{Z}) \otimes \mathbb{R}$ be a polyhedral cone. The associated affine toric variety is

$$X = TV(\sigma) = \text{Spec } k[S],$$

where $S = \sigma^{\vee} \cap (M \oplus \mathbb{Z})$. The tangent space of the deformation functor of $X$ is $T^1_X = \text{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$. It is a $k$-vector space with an $M$-grading induced by the torus action. For every $R \in M \oplus \mathbb{Z}$ we denote

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by \( T^1_X(-R) \) the graded component of \( T^1_X \) of degree \(-R\). Taking a cross-cut of \( \sigma \) with the affine hyperplane \([R = 1] = \{ a \in \sigma : \langle a, R \rangle = 1 \}\) we obtain a rational polyhedron \( P \). Our goal is to start from a polyhedron \( P \) as above and construct a maximal deformation for \( X \) in degree \(-R\).

Our approach is to use the polyhedron \( P \) to construct a pair of monoids \( \tilde{T} \subseteq \tilde{S} \) which fit into the following Cartesian diagram

\[
\begin{array}{ccc}
X = \text{Spec} \ k[S] & \longrightarrow & \text{Spec} \ k[\tilde{S}] \\
\downarrow R & & \downarrow \tilde{R} \\
\mathbb{A}_k^g = \text{Spec} \ k[\mathbb{N}] & \longrightarrow & \text{Spec} \ k[\tilde{T}] \\
\end{array}
\]

with \( \dim \text{Spec} \ k[\tilde{T}] = \dim_k T^1_X(-R) + 1 \) and the property that \( k[\tilde{S}] \) is a flat \( k[\tilde{T}] \)-module. The monoid \( \tilde{T} \) is a generalization of the Minkowski scheme of a lattice polytope and the monoid \( \tilde{S} \) is a generalization of the monoid corresponding to a tautological cone, see [1]. The main result of [7] is that the pair \((\tilde{T}, \tilde{S})\) is a universal extension of the pair \((\mathbb{N}, R, S)\). Our hope is that this universal extension allows one to construct the versal deformation of \( X \).

Here is the idea how to produce a deformation diagram for \( X \) from the diagram (1). Assume that \( \tilde{T} \) is generated in degree 1, i.e. by elements \( t_0, \ldots, t_g \in \tilde{T} \) mapping to 1 via the map \( \tilde{T} \rightarrow \mathbb{N} \) from (1). From the above assumption we obtain an embedding \( \text{Spec} \ k[\tilde{T}] \hookrightarrow \text{Spec} \ k[u_0, \ldots, u_g] = \mathbb{A}^g_k \) such that the composition \( \Delta : \mathbb{A}^g_k = \text{Spec} \ k[t] \hookrightarrow \text{Spec} \ k[\tilde{T}] \hookrightarrow \mathbb{A}^g_{k+1} \) equals the diagonal morphism \( 1 \mapsto \frac{1}{1} \). This means that the corresponding map of \( k \)-algebras \( k[u_1, \ldots, u_g] \rightarrow k[t] \) is given by \( u_i \mapsto t \).

In particular, this embedding is linear, and we may consider the quotient

\[
\mathbb{A}^g_k \longrightarrow \mathbb{A}^{g+1}_k / \Delta := \mathbb{A}^{g+1}_k / (k \cdot 1) = \text{Spec} \ k[u_i - u_j : 0 \leq i, j \leq g].
\]

For any given closed subscheme \( \overline{\mathcal{M}} \subseteq \mathbb{A}^{g+1}_k / \Delta \) with \( \mathcal{M} := \ell^{-1}(\overline{\mathcal{M}}) \subseteq \text{Spec} \ k[\tilde{T}] \subseteq \mathbb{A}^{g+1}_k \) we obtain the following commutative diagram:

\[
\begin{array}{ccc}
X & \longrightarrow & \tilde{X} & \longrightarrow & \text{Spec} \ k[\tilde{S}] \\
\downarrow R & & \downarrow \tilde{R} & & \downarrow \tilde{R} \\
\mathbb{A}^g_k & \longrightarrow & \mathcal{M} & \longrightarrow & \text{Spec} \ k[\tilde{T}] \\
\downarrow & & \downarrow & & \downarrow \ell \\
0 & \longrightarrow & \overline{\mathcal{M}} & \longrightarrow & \mathbb{A}^{g+1}_k / \Delta.
\end{array}
\]

The double arrow indicates that there is a maximal closed subscheme \( \overline{\mathcal{M}} \subseteq \mathbb{A}^g_k \) meeting the requirement \( \ell^{-1}(\overline{\mathcal{M}}) \subseteq \text{Spec} \ k[\tilde{T}] \). The point of the whole construction is the direct interplay between the schemes \( \text{Spec} \ k[\tilde{T}] \) and \( \overline{\mathcal{M}} \) – both refer to different moduli problems. While we will see in Subsection 6.1 that \( \text{Spec} \ k[\tilde{T}] \) is a base space for a deformation of \( R^{-1}(0) \), we obtain with \( \overline{\mathcal{M}} \) a base space for a deformation of \( X \). Indeed, this is a consequence of the following two facts: first, the map \( \tilde{R} : \tilde{X} \rightarrow \mathcal{M} \) inherits flatness from \( R : \text{Spec} \ k[\tilde{S}] \rightarrow k[\tilde{T}] \). Second, since \( \mathcal{M} = \ell^{-1}(\overline{\mathcal{M}}) \) is a full preimage, the lower left square in diagram (2) is Cartesian with a flat projection \( \ell : \mathcal{M} \rightarrow \overline{\mathcal{M}} \). We define \( \tilde{X} := \tilde{R}^{-1}(\mathcal{M}) \).

The full details for this are given in Section 6. The main result of this paper is the following theorem.
Theorem 1.1. For all compact edges \(d\) of \(P\) assume that the sub-monoid \(\widetilde{T}_d \subset \widetilde{T}\) is generated by degree 1 elements. Then the maximal \(\mathcal{M} \subseteq \mathbb{A}^2_k\) with \(\ell^{-1}(\mathcal{M}) \subseteq \text{Spec } k[\widetilde{T}]\) yields the deformation diagram, which is maximal with prescribed tangent space \(T^1(-R)\). That is, the family \(\widetilde{X} \to \mathcal{M}\) cannot be extended to a larger deformation of \(X\) without enlarging the ambient linear space of the base.

Note that Theorem 1.1 has been shown in [1] and [4] for the special case of \(X\) lacking singularities in codimension two, which is a very special case of \(\widetilde{T}_d\) being generated by degree 1 elements for all compact edges \(d\) of \(P\), see Section 6. Our main result holds more general and is obtained with different techniques than in [1] and [4].

The first part of the paper (Sections 2 to 5) focuses on the monoids \(\widetilde{T}\) and \(\widetilde{S}\). The main results there are the explicit descriptions of the generators of \(\widetilde{T}\) and \(\widetilde{S}\) (Proposition 3.13 and Corollary 4.6, respectively), and of the relations among them (Section 5.4 and Prop 5.10). An important feature is that the generators of \(\widetilde{T}\) can be computed knowing only the compact edges of \(P\). In Section 6 we return to algebraic geometry, and introduce our main result: Theorem 6.2. Sections 7 to 9 are dedicated to the proof of the main result, which is obtained by proving that the obstruction map is injective.

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2. Preliminaries

In this section we recall the construction of \(\widetilde{T} \subseteq \widetilde{S}\) from [7]. For our main result we will then exploit new properties of the monoids \(\widetilde{T}\) and \(\widetilde{S}\): their generators (cf. Section 3), their relation to flatness (cf. Section 4) and the syzygies of the corresponding semigroup rings (cf. Section 5).

2.1. The setup. Throughout the paper \(N\) is a lattice of finite rank, that is \(N \cong \mathbb{Z}^n\) for some \(n \in \mathbb{N}\), and \(M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z})\) the dual lattice of \(N\). Let \(P \subseteq N_\mathbb{R} := N \otimes \mathbb{R}\) be a rational, convex polyhedron. We embed \(P\) in an affine hyperplane of height one of \(N_\mathbb{R} \oplus \mathbb{R}\) and take the cone over it:

\[
\sigma := \text{cone}(P, 1) \subseteq N_\mathbb{R} \oplus \mathbb{R}.
\]

We define the monoid \(S := \sigma^\vee \cap (M \oplus \mathbb{Z})\). This monoid contains the lattice-primitive element \(R = (0, 1)\) spanning the discrete ray \(T := \mathbb{N} \cdot R\). Thus, starting from \(P\) we construct the pair of monoids \(T \subseteq S\). Our objective is to study the deformations of the affine toric variety

\[
X := \mathbb{T}V(\sigma) := \text{Spec } k[S].
\]

To this aim, we will use co-Cartesian extensions, cf. [7, Definition 3.1]. The main result of [7] was the construction of a universal co-Cartesian extension

\[
\begin{array}{ccc}
\widetilde{T} & \longrightarrow & \widetilde{S} \\
\pi_T & & \downarrow \pi_S \\
T & \longrightarrow & S.
\end{array}
\]

The polyhedron \(P\) may not be bounded, meaning that its tail cone

\[
\text{tail}(P) := \{a \in P - P : a + P \subseteq P\}
\]

may be not trivial. Every element \(c \in M_\mathbb{R}\) is a linear form on \(N_\mathbb{R}\), and \(c\) is bounded below on \(P\) if \(c \in \text{tail}(P)^\vee \subseteq M_\mathbb{R}\). It is easy to see that the minimum is achieved at some vertex. For every
$c \in \text{tail}(P)^\vee$ we choose and fix one such vertex $v(c)$. While this choice is not unique, the value of the upcoming numbers $\eta(c) \in \mathbb{Q}$ and $\eta_{\mathbb{Z}}(c) \in \mathbb{Z}$ will not depend on it.

**Definition 2.1.** For every linear form $c \in \text{tail}(P)^\vee$ define

$$\eta(c) := -\min_{v \in P} \langle v, c \rangle = -\langle v(c), c \rangle \in \mathbb{R}.$$  

This is not always an integer, and we denote the round up to the next integer by $\eta_{\mathbb{Z}}(c) := \lceil \eta(c) \rceil \geq \eta(c)$.

Note that $\sigma^\vee = \{ [c, \eta(c)] : c \in \text{tail}(P)^\vee \} + \mathbb{R}_{\geq 0} \cdot [0, 1]$. The Hilbert basis of $S$ has the form

$$(3) \quad \{ s_1 = [c_1, \eta_{\mathbb{Z}}(c_1)], \ldots, s_r = [c_r, \eta_{\mathbb{Z}}(c_r)], R := [0, 1] \},$$

with uniquely determined elements $c_i \in \text{tail}(P)^\vee \cap M$.

### 2.2. Short edges.

We denote the set of vertices and the set of compact edges of the polyhedron $P$ by

$$\text{Vert}(P) = \{ v^1, \ldots, v^n \} \quad \text{and} \quad \text{edge}(P) = \{ d^1, \ldots, d^p \},$$

respectively. If an edge $d^\nu$ connects the vertices $v^i, v^j$, then we will also denote it by $d^\nu = d^ij = [v^i, v^j]$. Alternatively, we might equip it with an orientation by either understanding it as a vector $d^ij = v^j - v^i \in \mathbb{N}_\mathbb{R}$ or as a half open segment $d^\nu = [v^i, v^j]$ which is, of course, no longer compact.

**Definition 2.2.** To each bounded half open edge $d = [v, w]$ of $P$ we associate the positive integer

$$g_d := \min \{ g \in \mathbb{Z}_{\geq 1} : \text{the affine line through } gv \text{ and } gw \text{ contains lattice points} \} \geq 1.$$  

We call $d = [v, w]$ a short half open edge if

$$\# \{ g_d \cdot [v, w] \cap N \} \leq g_d - 1.$$  

Moreover, we call $d = [v, w]$ a short edge if both $[v, w]$ and $[w, v]$ are short half open edges.

In particular, the vertex $v$ of a short half open edge $[v, w]$ never belongs to the lattice $N$. Moreover, if at least one of the half open edges $[v, w]$ or $[w, v]$ is short, then $\ell(w - v) < 1$ where $\ell$ denotes the lattice length – this is defined as the homogeneous function on $N_\mathbb{R}$ such that any primitive element of $N$ has lattice length one.

**Example 2.3.** 1) In the one-dimensional case, that is when $P = d = [v, w] \subset \mathbb{R}$ we always have $g_d = 1$. The edge $d$ is short if and only if $[v, w] \cap \mathbb{Z} = \emptyset$. In particular, the edge $d = [-\frac{1}{m}, \frac{1}{n}] \subset \mathbb{R}$ with $n, m \in \mathbb{N}$ is never short.

2) Take $d = \left[ (-\frac{1}{6}, \frac{1}{2}), (\frac{2}{5}, \frac{1}{3}) \right] \subset \mathbb{R}^2$. We need to multiply $d$ with 2 to produce lattice points on the affine line and thus $g_d = 2$. Since $\# \{ g_d P \cap N \} = 2$ we see that both half open edges are not short.

3) Take $d = \left[ (\frac{1}{2}, 1), (\frac{3}{4}, \frac{5}{2}) \right] \subset \mathbb{R}^2$. Also in this case we have $g_d = 2$, so $g_d \cdot d = \left[ (1, 2), (\frac{3}{4}, \frac{5}{2}) \right]$, which contains exactly one lattice point. So both half-open edges are short.

It is well-known that the set of Minkowski summands of scalar multiples of $P$ carries the structure of a convex, polyhedral cone $C(P)$, i.e. each $\xi \in C(P)$ represents a Minkowski summand $P_\xi$, see [1, Section 2.2]. Note that $C_{\text{lin}}(P) := C(P) - C(P) \subset \mathbb{R}^P$ is a linear subspace with coordinates $t_{\nu}$ encoding the dilation of the compact edges. It is defined by the equations

$$(4) \quad \sum_{d^\nu \in \epsilon} \delta_{\epsilon}(d^\nu) \cdot t_{\nu} \cdot d^\nu = 0$$

where $\epsilon \leq P$ runs through all compact 2-dimensional faces and $\delta_{\epsilon}(d^\nu) \in \{ 0, 1, -1 \}$ is chosen such that the edges $\delta_{\epsilon}(d^\nu) \cdot d^\nu \in N_\mathbb{R}$ form a cycle along the boundary of $\epsilon$. 

Definition 2.4. If \( t_{ij} \) denotes the dilation factor for the compact edge \([v^i, v^j] \leq P\) and \( s_i \) is the coordinate on \( \mathbb{R}^m \) referring to the vertex \( v^i \), then we define

\[
T(P) := \{(t, s) \in C^\text{lim}(P) \oplus \mathbb{R}^m : \begin{array}{ll}
s_i &= 0 \quad \text{if } v^i \in N, \\
s_i &= s_j \quad \text{if } [v^i, v^j] \leq P \text{ with } [v^i, v^j] \cap N = \emptyset, \text{ and} \\
s_i &= t_{ij} \quad \text{if } [v^i, v^j] \text{ is a half open short edge}\end{array}\}.
\]

Note that the vector space \( T(P) \) contains a distinguished element \((1, 1, 0) = [P]\) which is defined by \( s_i := 0 \) for \( v^i \in N \) and \( s_j := 1 \) and \( t_{ij} := 1 \) for all remaining coordinates. In the upcoming sections we will often deal with the dual vector space \( T^*(P) \), where elements \( s_i, t_{ij} \in T^*(P) \) form a generating set. We could easily omit the elements \( s_i = 0 \) for \( v^i \in N \). However, while they are just zero, there existence will simplify some formulae. Let

\[
\pi : T^*(P) \to \mathbb{R}
\]

be the map that sends the generators \( t_{ij} \in T^*(P) \) to 1 and \( s_i \in T^*(P) \) to 1 or 0 depending on \( v^i \notin N \) or \( v^j \in N \), respectively. Note that this map is well-defined.

Proposition 2.5. For any rational polyhedron \( P \) and for \( R = [0, 1] \in M \) we have

\[
T_{X,\sigma}^1(-R) = (T(P) \otimes \mathbb{R})/k \cdot (1, 1, 0).
\]

Proof. Essentially, this corresponds to [3, Theorem 2.5]. One has just to check that the equations called \( G_{jk} \) in [3, (2.6)] coincide with those in the definition of the \( \mathbb{R} \)-vector space \( T(P) \).

2.3. The lattice structure in \( T(P) \).

Definition 2.6. We define the subgroup \( T_{\mathbb{Z}}(P) \subset T(P) \) by

\[
(t, s) \in T_{\mathbb{Z}}(P) : \iff \begin{cases} s_i \in \mathbb{Z} & \forall v^i \in \text{Vert}(P), \\
(t_{ij} - s_i)v^i - (t_{ij} - s_j)v^j \in N & \forall [v^i, v^j] \in \text{edge}(P) \end{cases}
\]

Clearly \( T_{\mathbb{Z}}(P) \) is a subgroup of \( T(P) \), thus it is torsion-free and Abelian. Moreover, it is easy to see that it is a free Abelian group satisfying \( T_{\mathbb{Z}}(P) \otimes \mathbb{R} = T(P) \), see also [7, Lemma 5.14]. Using the dual lattice \( T_{\mathbb{Z}}^*(P) \), the two conditions of Definition 2.6 can be rephrased as:

\[
s_i \in T_{\mathbb{Z}}^*(P), \quad L_{ij} := (t_{ij} - s_i) \otimes v^i - (t_{ij} - s_j) \otimes v^j \in T_{\mathbb{Z}}^*(P) \otimes \mathbb{Z} N.
\]

2.4. The main monoids. For the upcoming constructions we need to choose and fix a reference vertex \( v_* \in P \). We establish the following convention, which may require shifting \( P \) by a lattice vector.

Convention 2.7. Whenever \( v_* \in P \) belongs to the lattice \( N \), we assume that \( v_* = 0 \).

For every \( c \in \text{tail}(P)^V \) we choose a path \( v_* = v_0, v_1, \ldots, v_k = v(c) \) along the compact edges of \( P \), and split \( -\eta(c) \) from Definition 2.1 as a sum in the following way:

\[
-\eta(c) = \langle v(c), c \rangle = \langle v_*, c \rangle + \sum_{j=1}^{k} \langle (v_j - v_{j-1}), c \rangle.
\]

This leads us to the next definition.
Definition 2.8. For every $c \in \text{tail}(P)^\vee \cap M$ we define $\bar{\eta}(c) \in T^*_Z(P)$ as
\[
\bar{\eta}(c) := - \langle v_*, c \rangle \cdot s_{v_*} - \sum_{j=1}^k \langle (v_j - v_{j-1}), c \rangle \cdot t_{j-1,j}.
\]
Note that the first summand, i.e. the $s_{v_*}$ part, vanishes if $v_* \in N$.

It is easy to see that the definition of $\bar{\eta}(c) \in T^*_Z(P)$ does neither depend on the choice of the vertices $v_*$ and $v(c)$, nor on the choice of the path connecting $v_*$ and $v(c)$. Note also that, due to Convention 2.7, $\bar{\eta}(c)$ is always a lifting of $\eta(c)$ via the map $\pi$ from the equation (5).

Definition 2.9. For every $c \in \text{tail}(P)^\vee \cap M$ we define
\[
\bar{\eta}_Z(c) := \bar{\eta}(c) + (\eta_Z(c) - \eta(c)) \cdot s_{v(c)} + \sum_{j=1}^k L_{j-1,j}(c) \in T^*_Z(P),
\]
where $L_{j-1,j}(c) := \langle L_{j-1,j}, c \rangle$. Moreover, for $c_1, c_2 \in \text{tail}(P)^\vee \cap M$ we measure convexity via
\[
\bar{\eta}_Z(c_1, c_2) := \bar{\eta}_Z(c_1) + \bar{\eta}_Z(c_2) - \bar{\eta}_Z(c_1 + c_2) \in T^*_Z(P).
\]
The main monoids which provide a universal extension in [7, Theorem 8.2], and which we will analyse in order to produce a maximal deformation, are the following.

Definition 2.10. For every rational polyhedron $P$, in the above notation, define:
\[
\bar{T} := \text{Span}_N\{[0, \bar{\eta}_Z(c_1, c_2)] : c_1, c_2 \in \text{tail}(P)^\vee \cap M\}
\]
\[
\bar{S} := \bar{T} + \text{Span}_N\{[c, \bar{\eta}_Z(c)] : c \in \text{tail}(P)^\vee \cap M\},
\]
with $\bar{T} \hookrightarrow \bar{S} \subset M \oplus T^*_Z(P)$. These two objects fit in the following diagram
\[
\begin{array}{ccc}
\bar{T} & \xrightarrow{\pi_T} & \bar{S} \\
\downarrow{\pi_T} & & \downarrow{\pi_S} \\
N & \xrightarrow{t_*} & S
\end{array}
\]
with vertical maps induced by $t_*, s_v \mapsto 1$ for $v \not\in N$ and $s_v \mapsto 0$ for $v \in N$. We call $\pi_T : \bar{T} \to T = N$ the degree map.

In particular, for $c_1, c_2, c \in \text{tail}(P)^\vee \cap M$ and $\bar{t} \in \bar{T}$, we have
\[
\bar{\eta}_Z(c_1, c_2) \xrightarrow{\pi_T} \eta_Z(c_1) + \eta_Z(c_2) - \eta_Z(c_1 + c_2) \in N, \quad \text{and}
\]
\[
\bar{t} + [c, \eta_Z(c)] \xrightarrow{\pi_S} [0, \pi_T(\bar{t})] + [c, \eta_Z(c)] \in S.
\]

3. **Explicit generators of $\bar{T}$**

We start this section by analysing the monoid $\bar{T}$. We denote the set of non-lattice vertices of $P$, that is the set of vertices that are not contained in $N$, with $\text{Vert}_{\notin N}(P)$. We will write $\text{Vert}_{\notin N}(P)$ for the set of lattice vertices, so $\text{Vert}_{\notin N}(P) = \text{Vert}(P) \setminus \text{Vert}_{\in N}(P)$. Moreover, for real numbers $z \in R$ we will quite often use the following notation:
\[
\{z\} := \lfloor z \rfloor - z.
\]
In particular, the following lemma trivially holds.

**Lemma 3.1.** For each $z_1, z_2 \in R$ there is either $\{z_1 + z_2\} = \{z_1\} + \{z_2\}$ or $\{z_1 + z_2\} + 1 = \{z_1\} + \{z_2\}$.

Let $c \in M$. For a compact edge with vertices $v^i$ and $v^j$ let $d^{i,j} := v^j - v^i$ be the oriented edge. From the elements appearing in the following definition we will get later the explicit generators of $\bar{T}$.
Definition 3.2. Let $c \in M$. Assume that $\langle c, d^{ij} \rangle \geq 0$. Then we define
\[
\tilde{t}(c, d^{ij}) := \langle c, d^{ij} \rangle t_{ij} + \{\langle c, v^i \rangle, \langle c, v^j \rangle \} s_{v^i} - \{\langle c, v^i \rangle, \langle c, v^j \rangle \} s_{v^j}.
\]
Moreover, we set $\tilde{t}(c, -d^{ij}) := \tilde{t}(c, d^{ij})$. In particular, the $t_{ij}$-coefficient is always non-negative.

Note that in the previous definition we do not restrict only to $c \in M \cap \text{tail}(P)^{\vee}$, but allow any $c \in M$.

Remark 3.3. For $\langle c, d^{ij} \rangle = \langle c, v^i - v^j \rangle \geq 0$ the following holds:
\[
\tilde{t}(-c, d^{ij}) = \tilde{t}(-c, -d^{ij}) = \tilde{t}(-c, v^i - v^j) = \langle c, d^{ij} \rangle t_{ij} + \{\langle c, v^i \rangle, \langle c, v^j \rangle \} s_{v^i} - \{\langle c, v^i \rangle, \langle c, v^j \rangle \} s_{v^j}.
\]
In particular, $\tilde{t}(-c, d^{ij}) = \tilde{t}(c, d^{ij}) - s_{v^i} + s_{v^j}$ unless $\langle c, v^i \rangle, \langle c, v^j \rangle \in \mathbb{Z}$. If one of these is integral, then the corresponding $s_{v^i}$ or $s_{v^j}$ has to be omitted in the previous relation.

Remark 3.4. One should compare the previous definition with that of the elements $L_{ij} \in T^*_\mathbb{Z}(P) \otimes \mathbb{Z} N$ of Subsection 2.3. Indeed, for a given $c \in \text{tail}(P)^{\vee} \cap M$ being non-negative on $d^{ij}$, we have
\[
-L_{ij}(c) = \langle c, d^{ij} \rangle t_{ij} - \langle c, v^i \rangle s_{v^i} + \langle c, v^j \rangle s_{v^j},
\]
i.e. this differs from $\tilde{t}(c, d^{ij})$ by the integral $\lceil \{\langle c, v^i \rangle, \langle c, v^j \rangle \} s_{v^i} - \{\langle c, v^i \rangle, \langle c, v^j \rangle \} s_{v^j}$.

Lemma 3.5. If $d = v^i - v^j$ is an oriented edge of $P$, and if $c \in M$ such that $\langle c, d \rangle = 0$, then $\tilde{t}(c, d) = 0$.

Proof. If $g_d = 1$, then $\langle c, v^i \rangle = \langle c, v^j \rangle = \langle c, w \rangle \in \mathbb{Z}$, where $w$ is a lattice point lying on the line passing through $v^i$ and $v^j$ ($w$ exists, since $g_d = 1$). If $g_d \geq 2$, then $s_i = s_j$ by definition. Together with $\langle c, v^i \rangle = \langle c, v^j \rangle$ this shows the claim. \hfill $\Box$

Each path along compact edges determines an element of $\mathbb{Z}^r$, whose entries count how often and from which direction we passed through an edge ($r$ is the total number of compact edges). While this element does not suffice to recover the original path completely, we will, nevertheless, call it a path, too.

Definition 3.6. For $a, c \in \text{tail}(P)^{\vee}$ we define (as in [1]) the following paths on the 1-skeleton of $P$:
\[
\begin{align*}
\Lambda(a) &:= [\text{some path } v_\ast \leadsto v(a) = [\lambda_1(a), \ldots, \lambda_r(a)] \in \mathbb{Z}^r, \text{ and} \\
\mu^a(c) &:= [\text{some path } v(a) \leadsto v(c) \text{ such that } \mu_i^a(c) \langle c, d^i \rangle \leq 0 \forall d^i = [\mu_1^a(c), \ldots, \mu_r^a(c)] \in \mathbb{Z}^r].
\end{align*}
\]
Moreover, we define the path $\Lambda^{\vee}(c) := \Lambda(a) + \mu^c(a)$, which is a special path from $v_\ast$ to $v(c)$ that depends on $a$.

Remark 3.7. Note that $\lambda_i(a)$ or $\mu_i^a(c)$ are not uniquely defined. If $d_1, \ldots, d_k$ are oriented edges going from $v_\ast$ to $v(a)$, i.e. it holds that $v(a) = v_\ast + \sum_{i=1}^k d_i$, then we can choose $\lambda_i(a) = 1$ for $i = 1, \ldots, k$ and $\lambda_i(a) = 0$ for other $i$.

The following lemma is crucial in connecting the generic generators $\eta(c, c_2)$ of $\tilde{T}$ from Definition 2.10, with the specific elements $\tilde{t}(c, d)$, which will provide an explicit finite set of generators of $\tilde{T}$ (cf. Proposition 3.14).

Lemma 3.8. Let $c_1, c_2 \in \text{tail}(P)^{\vee} \cap M$ and let $c = c_1 + c_2$. For $j = 1, 2$ we write $\mu^j(c) := \mu^{c_j}(c)$ and $\Lambda^{\vee}(c) := \Lambda^{\vee}(c)$. It holds that
\[
\eta\tilde{E}(c_1, c_2) = \sum_{j=1}^2 \{\eta(c_j)\} \cdot s_{v(c_j)} - \{\eta(c)\} \cdot s_{v(c)} - \sum_{j, \nu} \mu^j(c) \langle c_j, d^\nu \rangle t_{\nu}.
\]
Proof. We pick the path $\lambda^j(c)$ from $v_*$ to $v(c_j)$ and compute
\[
\tilde{\eta}_\mathbb{Z}(c_1, c_2) = \sum_{j=1}^{2} (\eta_{\mathbb{Z}}(c_j) - \eta(c_j)) \cdot s_{v(c_j)} - (\eta_{\mathbb{Z}}(c) - \eta(c)) \cdot s_{v(c)} - \\
- \sum_{\nu} \left( \sum_{j} \lambda_{\nu}^j(c_j, d^\nu) - \lambda_{\nu}(c, d^\nu) \right) t_{\nu} = \sum_{j=1}^{2} \left( (\eta_{\mathbb{Z}}(c_j) - \eta(c_j)) \cdot s_{v(c_j)} - \sum_{\nu} \mu_{\nu}^j(c_j, d^\nu) t_{\nu} \right) - (\eta_{\mathbb{Z}}(c) - \eta(c)) \cdot s_{v(c)}.\]
\[\Box\]

**Lemma 3.9.** For each vertex $v^j$ and $m \in M$ there exist $m_j \in \text{tail}(P)^\vee \cap M$ such that $v(m + m_j) = v^j$ and $\langle v, m_j \rangle \in \mathbb{Z}$ for all vertices $v \in P$, even for those not in $N$.

**Proof.** Let us take $\tilde{m}_j \in \text{tail}(P)^\vee \cap M$ such that $\langle v_j, \tilde{m}_j \rangle < \langle v, \tilde{m}_j \rangle$ for all other vertices $v \neq v_j$. Then we take $m_j := k\tilde{m}_j$ such that $k$ is big enough that $v(m + m_j) = v^j$ and that additionally $\langle v, m_j \rangle \in \mathbb{Z}$ for all vertices $v \in P$. Since the vertices of $P$ have rational coordinates, such an $m_j$ exists.

**Lemma 3.10.** It holds that $\tilde{t}(c, d^j) \in \tilde{T}$.

**Proof.** Without loss of generality we assume that $\langle c, d^j \rangle \geq 0$. With the same argument as in Lemma 3.9 there exists an element $\tilde{c}_j \in \text{tail}(P)^\vee \cap M$ which is perpendicular to the edge $d^j$, and such that $\tilde{c}_j - c \in \text{tail}(P)^\vee \cap M$ with $v(\tilde{c}_j) = v(c_j - c) = v^j$ and $\langle \tilde{c}_j, v^j \rangle = \langle c_j, v^j \rangle \in \mathbb{Z}$. Note that, by assumption we have $\langle -c, v^j \rangle \leq \langle -c, v^i \rangle$. We fix such a $\tilde{c}_j$. Again by Lemma 3.9 there exists also $\tilde{c}_i \in \text{tail}(P)^\vee \cap M$ such that $v(\tilde{c}_i) = v(c_i + \tilde{c}_j - c) = v^j$ and $\langle \tilde{c}_i, v^j \rangle = \langle c_i, v^j \rangle \in \mathbb{Z}$. Thus
\[
\eta(-c + \tilde{c}_j) = \langle c, v^j \rangle, \quad \eta(c_i) = 0, \quad \text{and} \quad \eta(-c + \tilde{c}_i + \tilde{c}_j) = \langle c, v^i \rangle.
\]

Lemma 3.8 gives us
\[
\tilde{\eta}_\mathbb{Z}(-c + \tilde{c}_j, c_i) = \{ \eta(-c + \tilde{c}_j) \} \cdot s_{v^j} + \{ \eta(c_i) \} \cdot s_{v^i} - \{ \eta(-c + \tilde{c}_i + \tilde{c}_j) \} \cdot s_{v^i} - \{ -c + \tilde{c}_i, d^j \} t_{ij}.
\]
Since $\langle d^j, \tilde{c}_j \rangle = 0$, it follows by (7) that $\tilde{t}(c, d^j) = \tilde{\eta}_\mathbb{Z}(-c + \tilde{c}_j, c_i) \in \tilde{T}$. \[\Box\]

**Example 3.11.** Let $P = [v^1, v^2] = [-\frac{a_1}{b_1}, \frac{a_2}{b_2}] \subset \mathbb{R}$ with $a_i, b_i > 0$ and such that $\frac{a_1}{b_1}, \frac{a_2}{b_2} \in \mathbb{Q} \setminus \mathbb{Z}$. We denote by $m = \text{lcm}(b_1, b_2)$ and set $v_* := v^1$. So we have $M = \mathbb{Z}$. For $k \in \mathbb{Z}_{\geq 0}$ we have by definition
\[
\tilde{\eta}_\mathbb{Z}(k) = \left[ \frac{a_1 k}{b_1} \right] s_1 \quad \text{and} \quad \tilde{\eta}_\mathbb{Z}(-k) = -\frac{a_1 k}{b_1} s_1 + k \left( \frac{a_1}{b_1} + \frac{a_2}{b_2} \right) t + \left[ \frac{a_2 k}{b_2} \right] s_2.
\]
Thus, denoting by $\ell(P) = v^2 - v^1$ the length of $P$, we compute for $k \in \{1, \ldots, m\}$ that
\[
\tilde{\eta}_\mathbb{Z}(-k, m) = k \cdot \ell(P) t_{12} + \left[ \frac{a_2 k}{b_2} \right] s_2 - \left[ \frac{a_1 k}{b_1} \right] s_1,
\]
\[
\tilde{\eta}_\mathbb{Z}(-m, k) = k \cdot \ell(P) t_{12} + \left[ \frac{a_1 k}{b_1} \right] s_1 - \left[ \frac{a_2 k}{b_2} \right] s_2,
\]
and thus we see that
\[
\tilde{t}(k, v^2 - v^1) = \begin{cases} 
\tilde{\eta}_\mathbb{Z}(-k, m) & \text{if } k \in \{1, 2, \ldots, m\} \\
\tilde{\eta}_\mathbb{Z}(-m, k) & \text{if } k \in \{-1, -2, \ldots, -m\}.
\end{cases}
\]

The next lemma follows immediately from the definitions.
Lemma 3.12. Let \( v^0, v^1, \ldots, v^k \) be a path along the compact edges of \( P \) from vertex \( v^0 \) to vertex \( v^k \).

For \( c \in M \) let

\[
\delta_i(c) := \begin{cases} 
-1, & \text{if } \langle c, v^i - v^{i-1} \rangle \leq 0, \\
1, & \text{if } \langle c, v^i - v^{i-1} \rangle > 0.
\end{cases}
\]

We then have

\[
\sum_{i=1}^k \delta_i(c) \bar{t}(c, v^i - v^{i-1}) = -\{(c, v^0)\} s_{v^0} + \sum_{i=1}^k \{\langle c, v^i - v^{i-1} \rangle t_{i-1,i} + \{(c, v^k)\} s_{v^k}.
\]

Proposition 3.13. The following set generates \( \tilde{T} \):

\[
\{\bar{t}(c, d^{ij}) : d^{ij} \in \text{edge}(P), \; c \in \text{tail}(P)^\vee \cap M \} \cup \{s_v : v \in \text{Vert}(P)\}.
\]

Proof. Let \( c_1, c_2 \in \text{tail}(P)^\vee \cap M \) and let \( c = c_1 + c_2 \). As in Lemma 3.12 we compute that

\[
\sum_{\nu=1}^r \bar{t}(-c_j, \mu^j(c) d^{\nu}) = -\{(\{c_j, v(c)\}) s_{v(c)} + \{\{c_j, v(c)\} \} s_{v(c)} - \sum_{\nu=1}^r \mu^j(c) \{d^{\nu}, c_j\} t_{\nu},
\]

where \( \nu \) runs through all the edge indices \( 1, \ldots, r \). Thus using Lemma 3.8 we see that

\[
\bar{\eta} c_1, c_2) = \sum_{j=1}^r \sum_{\nu=1}^r \bar{t}(-c_j, \mu^j(c) d^{\nu}) = -\{\eta(c)\} s_{v(c)} + \sum_{j=1}^r \{\{c_j, v(c)\} \} s_{v(c)}.
\]

Since \( \{\eta(c)\} = -\{c_1 + c_2, v(c)\} \) by definition, we see by Lemma 3.1 that

\[
\bar{\eta}(c_1, c_2) = \sum_{j=1}^r \sum_{\nu=1}^r \bar{t}(c_j, \mu^j(c) d^{\nu}) + n s_{v(c)},
\]

where \( n \) is either 0 or 1, and both actually do appear. From this description we can easily see that \( s_v \in \bar{T} \) for \( v \in \text{Vert}(P) \): take \( c_1 = c_2 \) with \( v(c_1) = v \) and such that we get \( n = 1 \) above. Since \( \bar{t}(c, d^{ij}) \in \bar{T} \) by Lemma 3.10, the equation (11) concludes the proof.

The proof of the next proposition gives an explicit finite set of generators of \( \bar{T} \). We will use this for the proof of versality. Finite generation was also proven in [7, Proposition 7.7] with different methods.

Let \( d = w - v \) be an oriented edge and let

\[
k := \min\{|\langle c, d \rangle| : c \in M, \langle c, d \rangle \neq 0\}.
\]

Let \( c \in M \) be such that \( \langle c_1, d \rangle = k \). We define \( m_1 \) to be the minimal natural number such that \( m_1 \langle c_1, v \rangle, m_1 \langle c_1, w \rangle \in \mathbb{Z} \).

Proposition 3.14. The set

\[
\{s_v, s_w\} \cup \{\bar{t}(k_1 \cdot c_1, d) : k_1 = \pm 1, \ldots, \pm m_1\}
\]

generates \( \text{Span}_\mathbb{Z}\{\bar{t}(c, d) : c \in M\} \subset \bar{T} \).

Proof. We choose an arbitrary element \( c \in \text{tail}(P)^\vee \cap M \) and write \( c = r_1 c_1 + c_2 \) for some \( r_1 \in \mathbb{Z} \) and \( c_2 \in M \) such that \( \langle c_2, d \rangle = 0 \). Let \( r_i \in \{1, \ldots, m_1\} \) be such that \( r_1 + n m_1 = r_1 \) for some \( n_1 \in \mathbb{N} \). Without loss of generality we assume that \( \langle c, d \rangle, \langle c_1, d \rangle \geq 0 \). We obtain that

\[
(13) \quad \bar{t}(c, d) - n_1 \bar{t}(m_1 c_1, d) - \bar{t}(r_1 c_1, d) = \left(\{\langle c, w \rangle\} - \{\langle r_1 c_1, w \rangle\}\right) s_w - \left(\{\langle c, v \rangle\} - \{\langle r_1 c_1, v \rangle\}\right) s_v.
\]

If \( g_d = 1 \), then as in Lemma 3.5 we have

\[
\langle c_2, v \rangle = \langle c_2, w \rangle = \langle c_2, n \rangle \in \mathbb{Z},
\]

where \( n \in \mathbb{N} \). Thus \( \{\langle c, w \rangle\} = \{\langle r_1 c_1, w \rangle\} \) and the same for \( v \), which proves the claim.

If \( g_d \geq 2 \), then \( s_v = s_w \) and thus the equation (13) is equal to \( n s_v = n s_w \) for some \( n \in \mathbb{N} \), from which the claim follows. \( \square \)
Corollary 3.15. The monoid $\overline{T}$ is finitely generated.

Example 3.16. Let us consider the one-dimensional polyhedron $P = [v, w] \subset \mathbb{R}$, with $v = -\frac{1}{2}$ and $w = \frac{1}{2}$. Embedding $P$ in height one in $\mathbb{R}^2$ and dualizing produces the cone

$$\sigma^\vee = \text{Span}_{\mathbb{R}_{\geq 0}}\{(-1, 2), (1, 2)\} \subseteq \mathbb{R}^2.$$ 

So the semigroup is $S = \sigma^\vee \cap \mathbb{Z}^2$. The Hilbert basis, i.e. the set of minimal generators of $S$, equals

$$(14) \quad \{(-2, 1), (-1, 1), (0, 1), (1, 1), (2, 1)\}.$$ 

Since $P$ is free from short half open edges, we obtain $T(P) = \mathbb{R}^3$ with coordinates $(t, s_1, s_2)$. The oriented edge is $d = w - v = 1$ and we claim that

$$(15) \quad s_1, s_2, \overline{t}(1, d) = t + \frac{1}{2}s_2 - \frac{1}{2}s_1, \quad \overline{t}(-1, d) = t - \frac{1}{2}s_2 + \frac{1}{2}s_1$$

is the minimal generating system of $\overline{T}$. Besides the elements given in (15), according to (12) from the proof of Proposition 3.14, we should also take $\overline{t}(2, d)$ and $\overline{t}(-2, d)$ as generators. However, we have that

$$\overline{t}(2, d) = \overline{t}(-2, d) = 2t = \overline{t}(1, d) + \overline{t}(-1, d),$$

which concludes our claim. So the generating set presented in (12) is finite, but not necessarily minimal.

Example 3.17. Let us revisit the polyhedron $P = \text{conv} \{(-\frac{1}{3}, \frac{1}{3}), (\frac{2}{5}, \frac{1}{3})\} \subset \mathbb{R}^2$ from Example 2.3.2. In this case $s := s_1 = s_2$ and thus we get the finitely generated semigroup

$$\overline{T} = \text{Span}_\mathbb{N} \{ s, \frac{5}{6}t + \frac{1}{6}s, \frac{10}{6}t - \frac{4}{6}s \}.$$ 

So far we have two generating sets for the semigroup $\overline{T}$: the original one $\{\overline{t}_\mathbb{Z}(c_1, c_2)\}$ from Subsection 2.4, and the more recent one $\{\overline{t}(c, d) \mid s_v\}$ from Proposition 3.13. The latter are rather local gadgets; they just deal with one compact edge $d = w - v$. The following sub-monoid reflects this.

Definition 3.18. For a compact edge $d = [v, w]$ of $P$ we define the sub-monoid $\overline{T}_d \subset \overline{T}$ as

$$\overline{T}_d := \text{Span}_\mathbb{N}\{s_v, s_w, \overline{t}(c, d) : c \in M\}.$$ 

We are going to discuss the degree of $\overline{t}(c, d) \in \overline{T}$ now. Assume, for the following that $\langle c, d \rangle \geq 0$, i.e. that $\langle c, w \rangle \geq \langle c, v \rangle$, or even, because of Lemma 3.5, $\langle c, w \rangle > \langle c, v \rangle$. While it is clear that the degree of $\overline{t}(c, d)$ equals

$$\lceil \langle c, w \rangle \rceil - \lceil \langle c, v \rangle \rceil \geq 0,$$

we will provide a different characterization. For this, we will generalize the notion of short edges from Definition 2.2 in Subsection 2.2.

Definition 3.19. We call $d = [v, w]$ a $k$-short (half open) edge if

$$\# \{g_d \cdot [v, w] \cap N\} < (k + 1) \cdot g_d.$$ 

We call $d = [v, w]$ a $k$-short edge if both half open edges $[v, w]$ and $[w, v]$ are $k$-short. In particular, 0-shortness means the old plain shortness.
Remark 3.20. There is a quite subtle relationship between the notion of \( k \)-shortness and the true lattice length \( \ell := \ell(d) \in \mathbb{Q}_{\geq 0} \) of an edge \( d \). We have the following implications:

\[
\ell \leq (k + 1) - \frac{1}{g_d} \Rightarrow d \text{ is } k \text{-short} \Rightarrow \ell < k + 1.
\]

These two implications are not inverse to each other; the worst case appears for \( g = 1 \). There, the first expressions just means \([\leq k]\).

Recall the degree map \( \pi \) from Definition 2.10.

Proposition 3.21. Let \( d \) be a \( k \)-short compact edge of \( P \) which is not \((k - 1)\)-short. Then,

\[
\min \left\{ \pi(\bar{t}(c,d)) : c \in M \text{ with } \langle c,w \rangle \neq \langle c,v \rangle \right\} = k.
\]

Proof. Assume first that \( g_d = 1 \) and denote by \( v^1, v^2, \ldots, v^\ell \), the sequence of lattice points in the half open interval \([v,w)\) with increasing \( c \)-value. Then, the assumption means \( \ell = k \). Moreover, denote by \( v^0 \) and \( v^{\ell+1} \) the adjacent lattice points, hence located outside \([v,w)\). Then, we have \( \bar{t}(c,v^1) \) and \( \bar{t}(c,v^\ell) + 1 \leq \bar{t}(c,w) \leq \bar{t}(c,v^{\ell+1}) \). This implies

\[
\bar{t}(c,v) \geq \bar{t}(c,v^\ell) + 1 - \bar{t}(c,v^1) \geq \ell = k.
\]

On the other hand, let \( c \) be a special element of \( M \) such that \( \bar{t}(c,v^{i+1} - v^i) = 1 \). Then all the inequalities in the previous three lines turn into equalities.

Let us turn to the case of \( g := g_d \geq 2 \). Again, we name the lattice points \( v^1, v^2, \ldots, v^\ell \), but now inside the half open interval \([gv,gw)\); the assumption of the proposition means \( k \cdot g \leq \ell < (k + 1) \cdot g \). We denote by \( g^* \) the first index \( i \) such that \( g^* = g \). This relation remains valid for all \( i \in (g^* + g\mathbb{Z}) \) among \( \{1, \ldots, \ell\} \), i.e. for \( i = g^* + \nu g \) with \( \nu = 0, \ldots, \nu^* := \lfloor g^*/g \rfloor \). Now, similarly to the \( g = 1 \) case, we obtain

\[
\frac{1}{g} \bar{t}(c, v^*) \leq \frac{1}{g} \bar{t}(c, gw) \leq \frac{1}{g} \bar{t}(c, v^{\ell+1}).
\]

This implies

\[
\bar{t}(c,v) - \bar{t}(c,v^1) \geq \bar{t}(c, v^*+\nu g) + 1 - \bar{t}(c,v^*) \geq \nu^* + 1 = \lfloor \frac{\ell + g - g^*}{g} \rfloor \geq \ell = k.
\]

To show that this minimal value can be achieved, we choose again \( c \) in such a way that \( \bar{t}(c,v^{i+1} - v^i) = 1 \). Similarly to the first case, this yields always equality signs until \( \bar{t}(c,w) - \bar{t}(c,v) = \lfloor \frac{\ell + g - g^*}{g} \rfloor \).

However, since we may adjust our \( c \) such that it leads to \( g^* = g \), the claim is proven.

Corollary 3.22. If \( \bar{t}(c,d^{ij}) \neq 0 \) then the degree of \( \bar{t}(c,d^{ij}) \) is strictly bigger than 0. In particular the kernel of the map \( \pi_T = \pi : \hat{T} \rightarrow T = \mathbb{N} \) is 0.

Proof. We already know that the degree \( \pi(\bar{t}(c,d^{ij})) \) is non-negative. Moreover, by Definition 3.19, there is a unique \( k \in \mathbb{N} \) such that the open half edge \( d^{ij} \) is precisely \( k \)-short, i.e. not \((k - 1)\)-short. Then, Proposition 3.21 implies that the degree is at least \( k \), and it remains to treat the case \( k = 0 \).

However, if \( d^{ij} \) is 0-short, i.e. short, then we know that \( t_{ij} = s_i \) which already solves the case \( g_d \geq 2 \), since we have the equation \( s_i = s_j \) anyway. Indeed, having the equations \( s_i = t_{ij} = s_j \), then the elements \( \bar{t}(c,d^{ij}) \) and \( \pi(\bar{t}(c,d^{ij})) \) are essentially equal, i.e. the vanishing of the latter implies that of
the former.
Finally, if \( q_d = 1 \), then the shortness of \([v^i, v^j]\) immediately implies the shortness of \((v^i, v^j)\), unless \( v^j \in N \). However, the latter means \( \{\langle c, v^j \rangle \} = 0 \) and \( s_j = 0 \), and we are done again.

\[ \square \]

4. Free pairs

In this section we introduce the notion of free pair. In Subsection 4.1 we connect it with free and flat modules. The results of Subsection 4.2 appear in [7] as well; here we provide a slightly different perspective based on the results from Section 3.

**Definition 4.1.** Let \( T \subset S \) be two sharp monoids, i.e. commutative semigroups with identity satisfying \( S \cap (-S) = \{0\} \). The boundary of \( S \) relative to \( T \) is defined as

\[ \partial_T S = \{ s \in S \mid (s - T) \cap S = \{s\} \}. \]

We say that \( T \subset S \) form a free pair \((T, S)\) if the addition map \( a : (\partial_T S) \times T \to S \) is bijective.

For any free pair, we write the unique decomposition of every element \( s \in S \) as

\[ s = \partial(s) + \lambda(s) \quad \text{with} \quad \partial(s) \in \partial_T S \quad \text{and} \quad \lambda(s) \in S. \]

**Example 4.2.** When \( T \subset S \) is the pair of semigroups associated to a rational polyhedron introduced in Section 2.1, we have by [7, Proposition 2.10 and Remark 5.3] that the pair \((T, S)\) is a free pair with

\[ \partial_T S = \{ \{c, \eta_Z(c)\} \mid c \in \text{tail}(P)^\partial \cap M \}. \]

4.1. The relation to free modules. Let \( k \) be any field. Then, the inclusion \( \iota : T \hookrightarrow S \) gives rise to an embedding of semigroup algebras \( k[T] \subseteq k[S] \).

**Proposition 4.3.** Assume that the addition map \( a : \partial_T S \times T \to S \) is surjective. Then the pair \((T, S)\) is free if and only if \( k[S] \) is a free \( k[T] \)-algebra, and this holds if and only if \( k[S] \) is flat over \( k[T] \).

**Proof.** If \((T, S)\) is a free pair, then the bijection \( a : \partial_T S \times T \to S \) provides an isomorphism of \( k[T] \)-modules \( \bigoplus_{s \in \partial_T S} k[T] \cdot \chi^s \cong k[S] \), i.e. \( k[S] \) is a free \( k[T] \)-module.

On the other hand, if \( s, s' \in \partial_T S \) and \( t, t' \in T \) with \( s + t = s' + t' \) and \( s \neq s' \), then we consider the exact sequence of \( k[T] \)-modules

\[ \bigoplus_{i \in I} k[T] \cdot e_i \xrightarrow{\sum_i (t_i, t'_i)} k[T] \oplus k[T] \xrightarrow{(t, t')} k[T] \]

where \( I \) parametrizes a generating set \( \{ (\chi^t, \chi^{t'}) \} \) of \( \ker \left( \begin{pmatrix} \chi^t & 0 \\ 0 & \chi^{t'} \end{pmatrix} \right) \), i.e. it exhibits the minimal pairs \((t_i, t'_i) \in T^2 \) satisfying \( t_i + t' = t'_i + t \). Tensorizing with \( \otimes_{k[T]} k[S] \) replaces \( k[T] \) with \( k[S] \) in the above sequence, and we obtain the new element

\[ (\chi^s, \chi^{s'}) \in \ker \left( \begin{pmatrix} \chi^{t'} \\ -\chi^t \end{pmatrix} \right) \otimes \text{id}_{k[S]} \].

However, this element cannot be in the image of the first map \( \sum_{i \in I} (t_i, t'_i) \otimes \text{id}_{k[S]} \). Otherwise, there is an element \( s'' \in S \) such that \((t_i + s'', t'_i + s'') = (s, s') \) for some \( i \). But then, the defining property of \( \partial_T S \) would imply that \( t_i = t'_i = 0 \) and \( s = s'' = s' \). Hence, \( k[S] \) is not flat over \( k[T] \). \( \square \)
Remark 4.4. In [17, Section 11] it was shown by cohomological methods that for so-called affine semigroups \( T \), i.e. for those being subsemigroups of some \( \mathbb{Z}^n \), the \( \mathbb{Z}^n \)-graded flat \( k[T] \)-modules are direct sums of degree shifts of localizations of \( k[T] \). This fits well to the consequence of Proposition 4.3 stating that \( k[S] \) is flat over \( k[T] \) if and only if it is free (with basis \( \partial T S \)).

4.2. The monoid \( \tilde{S} \). Now we will also start analysing the monoid \( \tilde{S} \), from Definition 2.10. We will show that \((\tilde{T}, \tilde{S})\) is a free pair (see Corollary 4.7) from which it follows that \( k[\tilde{S}] \) is a free \( k[\tilde{T}] \)-module by Proposition 4.3. Recall the notation \( T = \mathbb{N}, S = \text{cone}(P)^\vee \cap M \) and recall the two maps \( \pi T \colon \tilde{T} \to T \) and \( \pi S \colon \tilde{S} \to S \) from Definition 2.10.

Lemma 4.5. The monoid \( \tilde{S} \) decomposes as \( \tilde{S} = \tilde{T} + \text{Span}_\mathbb{N} \{ [c_1, \eta_\mathbb{Z}(c_1)], \ldots, [c_r, \eta_\mathbb{Z}(c_r)] \} \), where \( \{ [c_1, \eta_\mathbb{Z}(c_1)], \ldots, [c_r, \eta_\mathbb{Z}(c_r)] \} \) is the Hilbert basis of \( S \).

Proof. Let \( c \in \text{tail}(P)^\vee \cap M \). Then \( [c, \eta_\mathbb{Z}(c)] \in \tilde{S} \), i.e. we know that
\[
[c, \eta_\mathbb{Z}(c)] = \sum_{i=1}^r \lambda_i [c_i, \eta_\mathbb{Z}(c_i)]
\]
for certain \( \lambda_i \in \mathbb{N} \). Every element of \( \tilde{S} \) can be written as \( \tilde{s} = [c, \eta_\mathbb{Z}(c)] + \tilde{t} \) for some \( \tilde{t} \in \tilde{T} \). We will prove that
\[
[c, \eta_\mathbb{Z}(c)] = \sum_{i=1}^r \lambda_i [c_i, \eta_\mathbb{Z}(c_i)].
\]
Indeed, from (17) we have \( c = \sum_{i=1}^k \lambda_i c_i \). So it is enough to prove that \( \tilde{t} := \eta_\mathbb{Z}(c) - \sum_{i=1}^k \eta_\mathbb{Z}(c_i) = 0 \). This follows since \( \tilde{t} \in \tilde{T} \) and \( \pi T (\tilde{t}) = \eta_\mathbb{Z}(c) - \sum_{i=1}^k \eta_\mathbb{Z}(c_i) \), which is zero by (17). Since \( \ker(\pi T) = 0 \), by Corollary 3.22, we indeed have \( \tilde{t} = 0 \). Thus equation (18) holds.

Corollary 4.6. The monoid \( \tilde{S} \) is finitely generated. Its generators are the generators of \( \tilde{T} \) and
\[
\{ \tilde{s}_1 := [c_1, \eta_\mathbb{Z}(c_1)], \ldots, \tilde{s}_r := [c_r, \eta_\mathbb{Z}(c_r)] \} \subset \partial \tilde{T} \tilde{S}.
\]

Corollary 4.7. The pair \((\tilde{T}, \tilde{S})\) is free and we have an isomorphism \( \pi S : \partial \tilde{T} \tilde{S} \iso \partial T S \).

Proof. Using Corollary 3.22 we can easily check that
\[
\partial \tilde{T} \tilde{S} = \{ [c, \eta_\mathbb{Z}(c)] : c \in \text{tail}(P)^\vee \cap M \}.
\]
We get then the isomorphism \( \pi S : \partial \tilde{T} \tilde{S} \iso \partial T S \) using the description of \( \partial T S \) in the equation (16).

To prove that \((\tilde{T}, \tilde{S})\) is free let us assume that \( \tilde{b}_1 + \tilde{t}_1 = \tilde{b}_2 + \tilde{t}_2 \), with \( \tilde{b}_i \in \partial \tilde{T} \tilde{S} \) and \( \tilde{t}_i \in \tilde{T} \). Applying the map \( \pi S : \tilde{S} \to S \) we obtain
\[
\pi S(\tilde{b}_1) + \pi S(\tilde{t}_1) = \pi S(\tilde{b}_2) + \pi S(\tilde{t}_2).
\]
We have \( \pi(\tilde{t}_1), \pi(\tilde{t}_2) \in T \) and using the isomorphism on the boundaries we get \( \pi S(\tilde{b}_1), \pi S(\tilde{b}_2) \in \partial T S \). Since \((T, S)\) is a free pair, we have that \( \pi S(\tilde{b}_1) = \pi S(\tilde{b}_2) \). Again by the isomorphism on the boundaries we obtain \( \tilde{b}_1 = \tilde{b}_2 \), and thus \( \tilde{t}_1 = \tilde{t}_2 \), so the decomposition is unique.

5. Syzygies of the free pair \((\tilde{T}, \tilde{S})\)

5.1. Binomial equations. Recall from (3) the Hilbert basis \( \{ s_1, \ldots, s_r \} \) of \( S \), and the liftings \( \tilde{s}_i \in \tilde{S} \) of the \( s_i \in S \) from (19). Let \( \tilde{t}_0, \ldots, \tilde{t}_g \) be a set of generators of \( \tilde{T} \), and thus from Corollary 4.6 it follows that \( \tilde{t}_0, \ldots, \tilde{t}_g, \tilde{s}_1, \ldots, \tilde{s}_r \) generate \( \tilde{S} \). Let us introduce also the following notation
For each element \( 1 \), \( K. \) Altmann, A. Constantinescu, and M. Filip

\[ k[S] = k[t, x_1, \ldots, x_r]/\mathcal{I}_S, \]
\[ k[T] = k[u_0, \ldots, u_g]/\mathcal{I}_{\tilde{T}}, \]
\[ k[\tilde{S}] = k[u_0, \ldots, u_g, x_1, \ldots, x_r]/\mathcal{I}_{\tilde{S}}. \]

**Definition 5.1.** For \( k = (k_1, \ldots, k_r) \in \mathbb{N}^r \) let \( x^k := \prod_{i=1}^r x_i^{k_i} \), and let

\[ \partial(k) := \partial(\sum_{j=1}^r k_j c_j, \eta(z(c_j))) \in \partial_T S, \quad \lambda(k) := \lambda(\sum_{j=1}^r k_j c_j, \eta(z(c_j))) \in T, \]
\[ \tilde{\partial}(k) := \tilde{\partial}(\sum_{j=1}^r k_j c_j, \eta(z(c_j))) \in \partial_{\tilde{T}} \tilde{S}, \quad \tilde{\lambda}(k) := \tilde{\lambda}(\sum_{j=1}^r k_j c_j, \eta(z(c_j))) \in \tilde{T}. \]

Note that the isomorphism \( \partial_{\tilde{T}} \tilde{S} \cong \partial_T S \) from Corollary 4.7 sends \( \tilde{\partial}(k) \) to \( \partial(k) \). We will identify the two and write \( \partial(k) = \tilde{\partial}(k) \). Note also that the map \( \pi_T \) from Definition 2.10 maps \( \tilde{\lambda}(k) \) to \( \lambda(k) \).

**Definition 5.2.** For each element \( s \in S \) (resp. \( \tilde{s} \in \tilde{S} \)) we fix a representation \( s = a_0 R + \sum_{i=1}^r a_i s_i \) (resp. \( \tilde{s} = \sum_{j=0}^q n_j \tilde{t}_j + \sum_{i=1}^r a_i \tilde{s}_i \)) with \( \sum_{i=1}^r a_i \tilde{s}_i = \sum_{i=1}^r a_i s_i \) inside \( \partial_T S \rightarrow \partial_T S \). Define

\[ x^s := t^{a_0} \prod x_i^{a_i}, \quad x^{\tilde{\partial}(s)} := \prod x_i^{a_i}, \quad u^{\tilde{\lambda}(s)} := \prod x_i^{n_j}. \]

In particular we can present \( \partial(k) = \tilde{\partial}(k) \) as an element of \( \mathbb{N}^r \). We define the binomials

\[ f_k := x^k - x^{\partial(k)} k^{\lambda(k)}, \quad F_k := x^k - x^{\partial(k)} u^{\tilde{\lambda}(k)}. \]

**Lemma 5.3.** The binomials \( f_k \) generate the ideal \( I_S = \ker(\varphi : k[t, x_1, \ldots, x_r] \rightarrow k[S]) \) and the binomials \( F_k \) generate the ideal \( \ker(\tilde{\varphi} : k[\tilde{T}][x_1, \ldots, x_r] \rightarrow k[\tilde{S}]) \).

**Proof.** Let us only prove the second statement, the first one follows analogously. By construction we have \( F_k \in \ker(\tilde{\varphi}) \). Since \( \ker(\tilde{\varphi}) \) is \( S \)-homogeneous, the kernel is spanned by binomials of the form

\[ u^n x^k - x^{\partial(a+k)} u^{\tilde{\lambda}(a+k)} = u^n F_k, \]

where \( a \in \mathbb{N}^r \), which concludes the proof. \( \square \)

5.2. Lifting syzygies. We start with a general lemma which will turn out useful.

**Lemma 5.4.** For any free pair \( (T, S) \) and for any \( w_1, w_2 \in S \) we have

\[ \partial(w_1 + w_2) = \partial(\partial(w_1) + \partial(w_2)), \]
\[ \lambda(w_1 + w_2) - \lambda(w_1) - \lambda(w_2) = \lambda(\partial(w_1) + \partial(w_2)). \]

**Proof.** To conclude it is enough to apply the unique decomposition of \( w_1 + w_2 \) in the following:

\[ \partial(w_1 + w_2) + \lambda(w_1 + w_2) = w_1 + w_2 \]
\[ = \partial(w_1) + \partial(w_2) + \lambda(w_1) + \lambda(w_2) \]
\[ = \partial(\partial(w_1) + \partial(w_2)) + \lambda(\partial(w_1) + \partial(w_2)) + \lambda(w_1) + \lambda(w_2). \]

\( \square \)

Let \( \mathcal{R} \) denote the kernel of the map

\[ \psi : \bigoplus_{k \in \mathbb{N}^r} k[t, x_1, \ldots, x_r] e_k \xrightarrow{\psi} \mathcal{I}_S \subset k[t, x_1, \ldots, x_r]. \]

Thus \( \mathcal{R} \) is the module of linear relations between the \( f_k \).
**Definition 5.5.** For every $a, k \in \mathbb{N}^r$ we define

$$R_{a,k} := e_{a+k} - x^a e_k - \lambda(k) e_{\partial(k)+a}.$$ 

To check that $R_{a,k} \in \mathcal{R}$ we compute:

$$\psi(R_{a,k}) = x^{a+k} - x^{\partial(a+k)} \lambda(a+k) - x^a \left( x^k - x^{\partial(k)} \lambda(k) \right) - \lambda(k) \left( x^{\partial(k)+a} - x^{\partial(a+k)} \right) \lambda(k) = 0,$$

where the last equality we obtain by Lemma 5.4.

**Lemma 5.6.** The module $\mathcal{R}$ is spanned by $R_{a,k}$ for $a, k \in \mathbb{N}^r$.

**Proof.** Let $R = \sum g_i e_{k_i} \in \mathcal{R}$ be a homogeneous relation in $S$-degree $w$. Computing modulo $R_{a,k}$, we can always replace $x^a e_k$ by $e_{a+k} - \lambda(k) e_{\partial(k)+a}$. So we may assume that each $g_i = \alpha_i t^{a_i}$. Moreover, we can assume that there exists an index such that $a_i = 0$ (otherwise, divide $R$ by the minimal power of $t$). Let $R_0 = \sum a_i = 0 \alpha_i t^{a_i} e_{k_i}$. We have $\psi(R_0) = \sum a_i = 0 \alpha_i \left( x^k - x^{\partial(k)} \lambda(k) \right) = 0$, and furthermore, each $a_i x^k$ must cancel with $\lambda(k)$ such that $\lambda(k_j) = 0$ for each $j$ with $a_j = 0$. This, together with $S$-homogeneity, implies that $x^{\partial(k_j)} = x^{\partial(w)}$ for all $j$ with $a_j = 0$. Thus actually $R_0$ is the empty sum, contradicting the existence of an $a_i = 0$ in $R$. \[\square\]

Let $\widetilde{\mathcal{R}}$ denote the kernel of the map

$$\tilde{\psi} : \bigoplus_{k \in \mathbb{N}^r} k[u_0, \ldots, u_g, x_1, \ldots, x_r] E_k \xrightarrow{E_k \mapsto F_k} IS \subset k[u_0, \ldots, u_g, x_1, \ldots, x_r].$$

Thus $\widetilde{\mathcal{R}}$ is the module of linear relations between $F_k$.

**Definition 5.7.** For each $a, k \in \mathbb{N}^r$ we define the relation among the generators of $\widetilde{S}$ given in Lemma 5.3:

$$\widetilde{R}_{a,k} = E_{a+k} - x^a E_k - u^{\lambda(k)} E_{\partial(k)+a}.$$ 

As we did for $R_{a,k}$ we also compute in this case that

$$(20) \quad \tilde{\psi}(\widetilde{R}_{a,k}) = x^{\partial(a+\partial(k))} u^{\lambda(a+\partial(k)) + \lambda(k)} - x^{\partial(a+k)} u^{\lambda(a+k)},$$

which is equal to 0 in $k[\widetilde{T}][x_1, \ldots, x_r]$ by Lemma 5.4. In particular, $R_{a,k}$ lifts to $\widetilde{R}_{a,k}$.

### 5.3. Explicit description of $\widetilde{\lambda}(k)$. We write $k = (k_1, \ldots, k_r) \in \mathbb{N}^r$ and $c = \sum_{i=1}^r k_i c_i$, where the $c_i \in M$ are the elements appearing in the Hilbert basis of $S$, see (3). Recall the elements $\tilde{\lambda}(k)$ and $\partial(k)$ from Definition 5.1.

**Lemma 5.8.** For all $k \in \mathbb{N}^r$ we have $\partial(k) = [c, \eta(c)]$, $\tilde{\lambda}(k) = [c, \bar{\eta}(c)]$ and

$$\tilde{\lambda}(k) = \bar{\eta}(c) := [0, \sum_{i=1}^r k_i \eta(c_i) - \eta(c)].$$

**Proof.** We have $\sum_{i=1}^r k_i [c_i, \bar{\eta}(c)] = [0, \sum_{i=1}^r k_i \eta(c_i) - \eta(c)] + [c, \bar{\eta}(c)]$ with $[c, \bar{\eta}(c)] \in \widetilde{S}$ and $[0, \sum_{i=1}^r k_i \eta(c_i) - \eta(c)] \in \widetilde{T}$, which concludes the proof. \[\square\]
By Definition 5.2 we treat \( \partial(k) \) as an element of \( \mathbb{N}^r \), say \( \partial(k) = (b_1, \ldots, b_r) \in \mathbb{N}^r \). This means that
\[
\partial(k) = [c, \eta_{\mathbb{Z}}(c)] = \sum_j b_j [c_j, \eta_{\mathbb{Z}}(c_j)].
\]
(21)

Recall the definition of \( v(c) \) from Section 2.1 and the paths \( \lambda(a), \mu^j(a), \lambda^j(a) \) from Definition 3.6.

**Lemma 5.9.** It holds that \( \sum_{j=1}^r \lambda^j(v)(c) b_j(c_j, d') = \lambda(c, d') \) for each compact edge \( d' \).

**Proof.** Let \( F \) be the face of \( P \) where \( c \) attains its minimum and let \( F_j \) be the face of \( P \) where \( c_j \) attains its minimum. Then \( b_j \neq 0 \) only for those \( j \) such that \( F_j \subset F \), from which the proof easily follows. \( \square \)

The following description of \( \widetilde{\lambda}(k) \) will be important in Section 7.

**Proposition 5.10.** For \( k = (k_1, \ldots, k_r) \), \( \partial(k) = (b_1, \ldots, b_r) \) and \( c = \sum_{i=1}^r k_i c_i \), it holds that
\[
\widetilde{\lambda}(k) = \sum_j (k_j - b_j) \left( \sum_{\nu} \delta_{j,\nu}(c) \bar{\lambda}^j(c) c_j + \{ c_j, v* \} s_{v*} \right) \in \bar{T},
\]
where
\[
\delta_{j,\nu}(c) := \begin{cases} 1 & \text{if } \langle c_j, \lambda^j(c) d' \rangle > 0 \\ 0 & \text{if } \langle c_j, \lambda^j(c) d' \rangle \leq 0. \end{cases}
\]

**Proof.** In the definition of \( \widetilde{\lambda}(k) \) let us pick the path \( \lambda^j(c) \) from \( v_+ \) to \( v(c) \). We compute
\[
\widetilde{\lambda}(k) = \sum_j k_j \left( \eta_{\mathbb{Z}}(c_j) - \eta(c_j) \right) \cdot s_{v(c_j)} + \sum_{\nu} \left( \sum_j k_j \lambda^j(c) c_j - \lambda_{\nu}(c) \right) t_{\nu} + \sum_{\nu} \lambda^j(c) c_j t_{\nu},
\]
where in the last equality we used the equation (21) and Lemma 5.9. By Lemma 3.12 we see that
\[
\sum_{\nu} \delta_{j,\nu}(c) \bar{\lambda}^j(c) c_j = \left( \eta_{\mathbb{Z}}(c_j) - \eta(c_j) \right) \cdot s_{v(c_j)} - \{ c_j, v* \} s_{v*} + \sum_{\nu} \lambda^j(c) d'(c_j) t_{\nu},
\]
from which we conclude the proof. \( \square \)

6. The deformation diagram

6.1. The free pair \((\bar{T}, \bar{S})\) yields a deformation of a hyperplane section. The injection \( T \hookrightarrow S \) yields a morphism \( R : X = \text{Spec } k[S] \to A_1^L \). Its zero-fiber \( Z := R^{-1}(0) \subseteq X \) equals \( \text{Spec } k[\partial_T S] \) where the definition of the \( k \)-vector space \( k[\partial_T S] \) is straightforward, and it becomes a \( k \)-algebra via the multiplication law saying that for \( s, s' \in \partial_T S \) we set
\[
\chi^s \cdot \chi^{s'} := \begin{cases} \chi^{s+s'} & \text{if } s + s' \in \partial_T S, \text{ i.e. if } \lambda(s + s') = 0 \\ 0 & \text{if } \lambda(s + s') > 0. \end{cases}
\]

**Example 6.1.** Let us consider Example 3.16. Here, the equations for \( Z \subseteq A_2^L \) are 
\[
z_i z_{-j} = 0 \quad (i, j = 1, 2) \quad \text{and} \quad z_1^2 = z_2^2 = 0,
\]
where \( z_i \) is the coordinate corresponding to the Hilbert basis element \((i, 1)\) in (14). Hence, \( Z \) is the union of two orthogonal double lines.
We have the commutative diagram
\[
\begin{array}{c}
Z = R^{-1}(0) \quad X \quad \text{Spec} \, k[\mathcal{S}] \\
\downarrow \quad \downarrow R \quad \downarrow \bar{R} \\
0 \quad \mathbb{A}^1_k \quad \text{Spec} \, k[\bar{T}].
\end{array}
\]

By Proposition 4.3, all vertical maps are flat, and both squares are Cartesian diagrams. That is, both \( R : X \to \mathbb{A}^1_k \) and \( \bar{R} : \text{Spec} \, k[\mathcal{S}] \to \text{Spec} \, k[\bar{T}] \) are deformations of \( Z = R^{-1}(0) \).

6.2. Deformations of \( X \) instead of \( Z \). From now on we assume that \( \bar{T} \) is generated by degree 1 elements. Lemma 6.7 below offers a geometric interpretation of this condition.

There is an alternative possibility to produce a deformation diagram out of the right hand square of the diagram in Subsection 6.1:

\[
\begin{array}{c}
Z \quad X \quad \tilde{X} \quad \text{Spec} \, k[\mathcal{S}] \\
\downarrow \quad \downarrow R \quad \downarrow \bar{R} \quad \downarrow \bar{R} \\
0 \quad \mathbb{A}^1_k \quad \mathcal{M} \quad \text{Spec} \, k[\bar{T}] \quad \mathbb{A}^{g+1}_k \\
\downarrow \quad \downarrow \ell \quad \downarrow \ell \quad \downarrow \ell \\
0 \quad \mathcal{M} \quad \tilde{\mathcal{M}} \quad \text{Spec} \, k[\bar{T}] \quad \mathbb{A}^{g+1}_k / \Delta.
\end{array}
\]

We described the most important part of the diagram (22) already in Introduction, see diagram (2). The double arrow between \( \text{Spec} \, k[\bar{T}] \) and \( \tilde{\mathcal{M}} \) is supposed to indicate that there is a maximal closed subscheme \( \tilde{\mathcal{M}} \subseteq \mathbb{A}^g_k \) meeting the requirement \( \ell^{-1}(\mathcal{M}) \subseteq \text{Spec} \, k[\bar{T}] \). One obtains the ideal providing this distinguished maximal \( \mathcal{M} \) as follows: write all (binomial) equations \( f(u_0, \ldots, u_g) \) from the ideal of \( \text{Spec} \, k[\bar{T}] \subseteq \mathbb{A}^{g+1}_k \) in coordinates \( u_0, T_1, \ldots, T_g \) with \( T_i := u_0 - u_i (i = 1, \ldots, g) \), such as
\[
f(u_0, \ldots, u_g) = \sum_{l \geq 0} f_l(T_1, \ldots, T_g) \cdot u_0^l.
\]

Then by definition, the ideal of \( \tilde{\mathcal{M}} \) is generated by the coefficients \( f_l(T_1, \ldots, T_g) \in k[T_1, \ldots, T_g] \).

Recall the sub-monoids \( \bar{T}_d \subseteq \bar{T} \) from Definition 3.18. The main result of this paper is the following.

**Theorem 6.2.** Let \( X \) be a toric variety from our setup in Section 2.1. Assume that \( \bar{T}_d \) is generated by degree 1 elements for all compact edges \( d \) of \( P \). Then the maximal \( \bar{\mathcal{M}} \subseteq \mathbb{A}^g_k \) with \( \ell^{-1}(\mathcal{M}) \subseteq \text{Spec} \, k[\bar{T}] \) yields a maximal deformation with prescribed tangent space \( T^1_X(-R) \subseteq T^1_X \).

**Remark 6.3.** Theorem 6.2 has been shown in [1] and [4] for the special case of \( X \) lacking singularities in codimension two. In the combinatorial language of polytopes this means that all two faces \( \langle a^i, a^j \rangle \) are smooth, i.e. \( a^i \) and \( a^j \) are the base of \( \langle a^i, a^j \rangle \cap N \). Lemma 6.7 will show to what extent Theorem 6.2 is a generalization of this case.

Clearly the assumption that \( \bar{T}_d \) is generated by degree 1 elements for all \( d \) implies that \( \bar{T} \) is generated by degree 1 elements by the description of the generators of \( \bar{T} \) in Proposition 3.13.

**Remark 6.4.** Note that from Proposition 2.5 we see that \( \mathcal{M} \) has indeed \( T^1_X(-R) \) as its tangent space. Moreover, the assumption \( \bar{T}_d \) is generated by degree 1 elements for all \( d \) implies that all edges are 1-short by Proposition 3.21. This implies that \( T^1_X(-kR) = 0 \) for \( k \geq 2 \) by Proposition 2.5.
Example 6.5. Let \( P = \text{conv}\{ (0,0), (2,0), (2,1), (1,2), (0,1) \} \subset \mathbb{R}^2 \) be the lattice polygon:

\[
\begin{array}{c}
\text{d}_4 \\
\text{d}_5 \\
\text{d}_1 \\
\text{d}_2 \\
\text{d}_3
\end{array}
\]

The generators of \( \overline{T} \) are \( 2t_1, t_2, t_3, t_4, t_5 \) (they correspond to the five edges). The closing condition (4) on 2-faces gives us that \( 2t_1 = t_3 + t_4 \), which implies that \( \overline{T} \) is generated by degree 1 elements but on the other hand we see that \( T_{d_1} \subset \overline{T} \) is not generated by degree 1 elements (it is generated by \( 2t_1 \)).

Remark 6.6. Example 6.5 shows that the condition that \( \overline{T}_d \) are generated by degree 1 elements is slightly stronger than the condition that \( T \) is generated by degree 1 elements. See also Remark 7.8 why we impose this stronger condition.

Lemma 6.7. For a compact edge \( d = [v,w] \) of \( P \) assume that one of the following holds:

a) \( d \) is a short edge;

b) one vertex of \( d \) lies in \( N \) and the lattice length of \( d \) is strictly smaller than 2;

c) \( g_d \geq 2 \) and \( d \) is 1-short;

d) \( g_d = 1 \) and \( d \) is 1-short and the lattice length of \( d \) is bigger than 1;

e) there is an isomorphism of the lattice \( N \) that maps the edge \( d \) to the edge with vertices \( (-\frac{1}{n},0,\ldots,0) \) and \( (\frac{1}{m},0,\ldots,0) \), \( n,m \in \mathbb{N} \) and \( n,m \neq 1 \).

Then \( \overline{T}_d \) is generated by degree 1 elements.

Proof. Let \( c' \in M \) be such that

\[
\langle c',d \rangle = \min\{ | \langle c,d \rangle | : c \in M, \langle c,d \rangle \neq 0 \}.
\]

Note that this value appeared in the proof of Proposition 3.14, where we described the generators of \( \overline{T} \).

If a) holds the claim trivially follows.

If b) holds we may assume that \( w \in N \). Then we can easily see that the semigroup \( \overline{T}_d \) is generated by two elements, namely \( s_v \) and \( \overline{t}(c',d) \). Since the lattice length of \( d \) is strictly smaller than 2 we see that \( \overline{t}(c',d) \) has degree 1 by Proposition 3.21, from which the claim follows.

If c) holds, then \( s_v = s_w \) and as in b) we can easily verify that \( \overline{T}_d \) is generated by two elements, namely \( s_v = s_w \) and \( \overline{t}(c',d) \). By Proposition 3.21 we see that the degree of \( \overline{t}(c',d) \) is 1.

If d) holds, then we can easily check that \( \overline{T}_d \) is generated by four elements: \( s_v, s_w, \overline{t}(c',d), \overline{t}(-c',d) \), which have degree 1 by Proposition 3.21.

Let us now assume that only e) holds. It is enough to show that for \( d = [v,w] = [-\frac{1}{n},\frac{1}{m}] \subset \mathbb{R} \) the semigroup \( S' = \text{Span}_{\mathbb{N}}\{ s_v, s_w, \overline{t}(c,d) : c \in M \} \) is generated by degree 1 elements. This is clear by explicit description of the generators described in Proposition 3.14: let us first consider the case when \( m,n \neq 1 \). In this case \( S' \) is isomorphic to the following monoid (see also Example 6.12 for a geometric picture): let the polytope \( Q \) be the convex hull of the vertices \( (0,0), (0,1), (n-1,0), (m-1,1) \). Let \( C(Q) \) be the cone over this polytope, i.e. generated by \( (0,0,1), (0,1,1), (n-1,0,1), (m-1,1,1) \). We will show that the monoid \( \mathbb{Z}^3 \cap C(Q) \) is isomorphic to \( S' \). Indeed, the isomorphism is given by

\[
s_v \mapsto (0,0,1), \quad s_w \mapsto (0,1,1), \quad \overline{t}(a,d) \mapsto (a,1,1), \quad \overline{t}(-b,d) \mapsto (b,0,1),
\]

for \( a = 1,\ldots,n-1 \) and \( b = 1,\ldots,m-1 \). From this we conclude the proof. \( \square \)
Remark 6.8. The compact edges of $P$ correspond to the two dimensional cyclic quotient singularities. Thus Lemma 6.7 can be phrased using this language as well, see [5, Section 2].

Example 6.9. Let $d = [-\frac{2}{3}, \frac{1}{4}] \subset \mathbb{R}$. Then $\tilde{T}_d$ is generated by degree 1 elements, namely by
\[
s_1, s_2, \quad \tilde{t}(1, d) = \frac{11}{12}t + \frac{3}{4}s_2 - \frac{2}{3}s_1, \quad \tilde{t}(-1, d) = \frac{11}{12}t + \frac{1}{3}s_1 - \frac{1}{4}s_2.
\]

Thus we see that the list in Lemma 6.7 is not exhaustive, i.e. $\tilde{T}_d$ is generated by degree 1 elements but it does not appear on the list.

Example 6.10. Let us consider $P = [v_1, v_2] = [-\frac{3}{5}, \frac{1}{9}] \subset \mathbb{R}$. Here we have only one edge, which is 1-short and we will show that $\tilde{T}$ is not generated by degree 1 elements. The degree 1 elements are
\[
s_1, s_2, \quad \tilde{t}(1, d) = \frac{4}{5}t + \frac{4}{5}s_2 - \frac{3}{5}s_1, \quad \tilde{t}(-1, d) = \frac{4}{5}t + \frac{2}{5}s_1 - \frac{1}{5}s_2.
\]

We see that we can not write the element
\[
\tilde{t}(3, d) = \frac{12}{5}t + \frac{2}{5}s_2 - \frac{4}{5}s_1
\]
as a sum of degree 1 elements in (23), thus $\tilde{T}$ is not generated by degree 1 elements. We have $g_d = 1$ and the lattice length of $d$ is smaller than 1 thus none of the conditions in Lemma 6.7 is satisfied for this example.

Example 6.11. Let us continue with Example (3.16). Let us denote $k[\tilde{T}] = k[u_1, u_2, u_A, u_B]$, where the variables correspond to the minimal generating set of $\tilde{T}$, written in (15) (here $A = \tilde{t}(1, d)$ and $B = \tilde{t}(-1, d)$). We only have the following binomial equation
\[
u_A u_1 - u_B u_2 = 0.
\]

Writing $u_0 = u_1$ and $T_i = u_0 - u_i$ for $i = 2, A, B$, turn this equation into
\[
(u_0 - T_A)u_0 - (u_0 - T_B)(u_0 - T_2) = 0
\]
from which we get
\[
u_0(-T_A + T_B + T_2) - T_2T_B = 0.
\]
The equations of our versal base space are
\[
T_A = T_2 + T_B, \quad T_2T_B = 0
\]
and thus
\[
\overline{M} = \text{Spec } k[T_2, T_B]/(T_2T_B),
\]
i.e. $\overline{M}$ equals the union of two lines. These two lines correspond to the two Minkowski decompositions
\[
P = P_1 + P_2 = Q_1 + Q_2,
\]
where $P_1$ is the vertex $-\frac{1}{2}$, $P_2 = [0, 1]$, $Q_1 = [-\frac{1}{2}, 0]$ and $Q_2 = [0, \frac{1}{2}]$, see also [7, Section 9], where those Minkowski decompositions were called lattice friendly Minkowski decompositions.

Example 6.12. Let $P = [v, w] = [-\frac{1}{2}, \frac{1}{3}] \subset \mathbb{R}$ and thus $\sigma = \text{Span}_{\mathbb{R}}\{(-1, 2), (1, 3)\} \subseteq \mathbb{R}^2$. Dualizing we obtain free embedding of monoids
\[
T = \mathbb{N} \longrightarrow \text{Span}_{\mathbb{R}}\{(-3, 1), (2, 1)\} \cap \mathbb{Z}^2 = S.
\]
The Hilbert basis of $S$, i.e. the set of minimal generators, equals
\[
\{(−3, 1), (−2, 1), (−1, 1), (0, 1), (1, 1), (2, 1)\}.
\]
Since $P$ is free from short half open edges, we obtain $\mathcal{T}(P) = \mathbb{R}^3$ with coordinates $(t, s_v, s_w)$. We see that the elements
\[
A = \overline{t}(1, d) = \frac{5}{6} t + \frac{2}{3} s_w - \frac{1}{2} s_v, \quad B_1 = \overline{t}(-1, d) = \frac{5}{6} t + \frac{1}{2} s_v - \frac{1}{3} s_w, \quad B_2 = \overline{t}(2, d) = \frac{5}{3} t - \frac{2}{3} s_w,
\]
together with $s_v$ and $s_w$ generate $\overline{T}$. Thus $\overline{T}$ is the set of lattice points of the cone over the quadrangle $A \to B_2 \to B_1 \to B_2$.

The generators obey the affine relations
\[
A + s_v = B_1 + s_w \quad \text{and} \quad 2B_1 = B_2 + s_v,
\]
which induce the following binomial equations:
\[
\begin{align*}
(u_A - u_{B_1} - u_w) & \quad \text{and} \quad u_{B_1}^2 - u_{B_2} - u_v.
\end{align*}
\]
After writing $u_0 = u_{B_1}$ and $T_i = u_0 - u_i$ for $i = v, w, A, B$, these equations turn into
\[
(u_0 - T_A)(u_0 - T_v) - u_0(u_0 - T_w) = 0, \quad u_0^2 - (u_0 - T_{B_2})(u_0 - T_v) = 0.
\]
and thus after some computation we obtain
\[
u_0(T_w - T_v - T_A) + T_A T_v = 0, \quad u_0(T_v + T_{B_2}) - T_{B_2} T_v = 0.
\]
We can write
\[
T_w = T_v + T_A, \quad T_{B_2} = -T_v
\]
and we end up with
\[
\overline{M} = \text{Spec } \mathbb{k}[T_v, T_A]/(T_v^2, T_A T_v),
\]
i.e. $\overline{M}$ equals the line with an embedded point. This line is corresponding to the Minkowski decomposition $P = [-\frac{1}{2}, 0] + [0, \frac{1}{2}]$.

6.3. The Obstruction map. From [13, Section 4] and [1, Section 7] (see also [14, Section 10]) we recall the definition of the obstruction map, which is the main tool for proving Theorem 6.2. As in Subsection 5.4, let $\mathcal{R}$ be the module of linear relations between $f_k$, which are the generators of $I_S$. The module $\mathcal{R}$ contains the submodule $\mathcal{R}_0$ of the so-called Koszul relations.

**Definition 6.13.** Let $S$ be the monoid defined by $P$ and $X = \text{Spec } \mathbb{k}[S]$. We define
\[
T_X^2 := \frac{\text{Hom}(\mathcal{R}/\mathcal{R}_0, k[S])}{\text{Hom}(\bigoplus_{k \in \mathbb{N}^r} k[x, t] f_k, k[S])}.
\]
Let \( k[\overline{T}] = k[u_0, \ldots, u_g]/\mathcal{I}_{\overline{T}} \), where \( \mathcal{I}_{\overline{T}} = (p_1, \ldots, p_k) \), for some homogenous polynomials \( p_i \). We will write for simplicity \( T \) for the list of variables \( T_1, T_2, \ldots, T_g \). Every degree \( d \) homogenous polynomial \( p \in k[u_0, \ldots, u_g] = k[u_0, T] \), with \( T_i = u_0 - u_i \), can be uniquely written as

\[
p = \sum_{n=1}^{d} p^{(n)}(T) u_0^{d-n},
\]

where \( p^{(n)}(T) \) is homogenous of degree \( n \). In Subsection 6.2 we saw that the equations of \( \mathcal{M} \) are given by the ideal

\[
\mathcal{J} := (p^{(n)}_1(T), \ldots, p^{(n)}_k(T) : n \in \mathbb{N}).
\]

**Definition 6.14.** We call \( p^{(n)}(T) \) the degree \( n \) part of \( p \). Let us consider the ideal

\[
\mathcal{J} := \mathcal{J} \cdot (T_1, T_2, \ldots, T_g) + \mathcal{J}_1 k[T] \subset k[T],
\]

where \( \mathcal{J}_1 := (p^{(1)}_1(T), \ldots, p^{(1)}_k(T)) \) denotes the ideal generated by the degree one elements. Let \( W := \mathcal{J} / \mathcal{J} \) be a \( \mathbb{Z} \)-graded vector space \( W = \bigoplus_{n \geq 2} W_n \), where \( W_n \) contains the degree \( n \) parts of the polynomials \( p \in \mathcal{I}_{\overline{T}} \).

We have the exact sequence

\[
0 \longrightarrow W \longrightarrow k[T]/\overline{\mathcal{J}} \longrightarrow k[T]/\mathcal{J} \longrightarrow 0. \tag{24}
\]

Identifying \( t \) with \( u_0 \), the tensor product of (24) with \( k[x, u_0] \) yields

\[
0 \longrightarrow W \otimes_k k[t, x] \longrightarrow k[t, T, x]/\overline{\mathcal{J}} \cdot k[t, T, x] \longrightarrow k[t, T, x]/\mathcal{J} \cdot k[t, T, x] \longrightarrow 0. \tag{25}
\]

Using the notation from Subsection 5.4, let \( s = \sum_k s_k e_k \in \mathcal{R} \), which means \( s_k \in k[x, t] \) as well as \( \psi(s) = 0 \in k[x, t] \). In Section 5 we showed that we can lift \( s_k \) to \( k[u_0, \ldots, u_g, x] \), from which we obtain \( \tilde{s} \in \mathcal{R} \) such that

\[
o(s) := \sum \tilde{s}_k E_k \mapsto 0 \quad \text{in} \quad k[t, T, x]/\mathcal{J} \cdot k[t, T, x].
\]

In particular, each relation \( s \in \mathcal{R} \) induces some element \( o(s) \in W \otimes_k k[x, t] \), which is well defined after the additional projection to \( W \otimes_k k[S] \). This procedure describes a certain element

\[
o \in T_X^2 \otimes_k W \cong \text{Hom}(W^*, T_X^2)
\]

called the obstruction map (note that this notation was used in [13] and [1] while in [14] the obstruction map was defined to be the dual of this). From Section 5 (see equation (20)) we obtain that

\[
o(R_{a, k}) = \sum_{n \geq 1} x^{a(k + a)} t^n \otimes h_{n, a, k}(T), \tag{26}
\]

where \( h_{n, a, k}(T) \) is the degree \( n \) part of the polynomial

\[
u \overline{\lambda(a + \partial(k) + \overline{\lambda(k)}} - u \overline{\lambda(a + k)}.
\]

Note that in (26) we identified \( u_0 \) with \( t \) and we also use that \( x^{a(k + a)} = x^{a(k) + a} \) in \( k[\overline{S}] \) by Lemma 5.4. For our \( R = [0, 1] \in S \) and \( n \in \mathbb{N} \) let us denote by \( T^2(-nR) \) the degree \(-nR\) part of \( T_X^2 \).

To prove Theorem 6.2 it is enough to prove that the dual of the obstruction map, denoted by \( o^* \), is surjective (see e.g. [14, Section 10] for the proof of this statement and note that \( o^* \in \text{Hom}(T_X^2, W) \)). To do that we need to understand the equations of \( k[\overline{T}] \). There are two (obvious) types of equations
of \( k[T] \): the first one we call the loop equations and they are introduced in Section 7; the second type are the so called local equations introduced in Section 8. In Section 9 we prove that the loop and local equations are in fact all the equations of \( k[T] \) by introducing new generators of \( T \). In order to prove that the dual of the obstruction map is surjective we thus need to prove that the loop and local equations (as elements in \( W \)) are obtained in the image of \( o^* \). For the loop equations this is done in Section 7 (see Corollary 7.7) and for the local equations this is done in Section 8 (see Proposition 8.2).

7. The Loop Equations

In this section we generalize the results from [1, Section 7] to our setting. Since this section is long and more technical we give some guidance and motivation at the beginning. In [1] and [4] the proof of the versality relies on knowing the explicit equations of the versal base space. We do not have explicit equations but in fact we do not need them, we only need the bi-linearity property, cf. Lemma 7.2.

In Subsection 7.2 we analyze the \( T_X^\mu \)-module in more details and we introduce the submodule of \( (T_X^\mu)^\ast \), cf. (34), which is mapped to our loop equations by \( o^* \). We first describe the restriction of the map \( o^* \) to this submodule in Proposition 7.5 and prove that all the loop equations are in the image of \( o^* \) in Proposition 7.6 and Corollary 7.7.

7.1. Bi-linearity of the equations. For each \( c \in M \) and a closed path \( \mu \) we define \( S_{\mu,c}^+ \) (resp. \( S_{\mu,c}^- \)) to be the set of edges \( d^j \) of \( P \), such that \( \langle \mu d^j, c \rangle > 0 \) (resp. \( \langle \mu d^j, c \rangle \leq 0 \)). We see by (4) and Remark 3.3 that

\[
\sum_{d^j \in S_{\mu,c}^+} \bar{t}(\mu d^j, c) - \sum_{d^j \in S_{\mu,c}^-} \bar{t}(\mu d^j, c) = 0.
\]

We call the equations corresponding to (28) the loop equations and denote them by

\[
p(\mu, c) := \prod_{d^j \in S_{\mu,c}^+} u(\mu d^j, c^j) - \prod_{d^j \in S_{\mu,c}^-} u(\mu d^j, c^j).
\]

Remark 7.1. We view \( p(\mu, c) \) as a polynomial in the variables \( u_0, \ldots, u_g \), which correspond to the generators of \( T \). There are many different ways to present \( p(\mu, c) \) as a polynomial in \( u_0, \ldots, u_g \). Whenever we say that a property holds for \( p \in \mathcal{J} \) we mean that it holds for all possible presentations. In particular, the bi-linearity of \( p \) shown in Lemma 7.2 holds for all presentations of \( p \) as a polynomial in \( \mathcal{J} \).

Let \( p^{(n)}(\mu, c) \) denote the degree \( n \) part of the polynomial \( p(\mu, c) \).

Lemma 7.2. For \( m_1, m_2 \in \text{tail}(P)^\vee \cap M \) and a closed path \( \mu \) we have

\[
p^{(n)}(\mu, m_1 + m_2) = p^{(n)}(\mu, m_1) + p^{(n)}(\mu, m_2) \in W_n.
\]

For two closed paths \( \mu^1, \mu^2 \) and \( m \in \text{tail}(P)^\vee \cap M \) it holds that

\[
p^{(n)}(\mu^1 + \mu^2, m) = p^{(n)}(\mu^1, m) + p^{(n)}(\mu^2, m) \in W_n.
\]
Let us consider the first homology group of the complex $L$.

After defining $E_0$, we define $E_1 := \{ d^i \in \text{edge}(P) : \langle \mu_i, d^i \rangle < 0, \langle \mu_i d^i, m_1 \rangle > 0, \langle \mu_i d^i, m_2 \rangle < 0 \}$, $E_2 := \{ d^i \in \text{edge}(P) : \langle \mu_i d^i, m_1 \rangle > 0, \langle \mu_i d^i, m_2 \rangle < 0 \}$, $E_3 := \{ d^i \in \text{edge}(P) : \langle \mu_i d^i, m_1 \rangle < 0, \langle \mu_i d^i, m_2 \rangle > 0 \}$, and $E_4 := \{ d^i \in \text{edge}(P) : \langle \mu_i d^i, m_1 \rangle > 0, \langle \mu_i d^i, m_2 \rangle > 0, \langle \mu_i d^i, m_2 \rangle < 0 \}$.

Straightforward computation shows that for each non-lattice vertex $v \in P$ there exist $n_v, m_v \in \mathbb{N}$ such that in $\mathcal{I}$ the following holds:

\[
\begin{align*}
p(\mu, m_1 + m_2) & \prod_{d^i \in E_1 \cup E_2} u(\mu_i m_1, d^i) \prod_{d^i \in E_3 \cup E_4} u(\mu_i m_2, d^i) \prod_{v} u(s_v)^{n_v} = \\
& = \frac{1}{2} p(\mu, m_1) \left( \prod_{d^i \in S_{m_2} \mu} u(\mu_i m_2, d^i) + \prod_{d^i \in S_{m_2} \mu} u(\mu_i m_2, d^i) \right) \prod_{v} u(s_v)^{n_v} + \\
& = \frac{1}{2} p(\mu, m_2) \left( \prod_{d^i \in S_{m_1} \mu} u(\mu_i m_1, d^i) + \prod_{d^i \in S_{m_1} \mu} u(\mu_i m_1, d^i) \right) \prod_{v} u(s_v)^{n_v},
\end{align*}
\]

from which (30) follows after looking at the degree $n$ part of the above equation taken modulo $\mathcal{J}$. We can prove (31) in a similar way, so we omit the proof. \qed

7.2. The module $T^\mathbb{Z}_e$ revisited. We recall the following from [6, Section 5.5]. Let $\sigma = \text{cone}(P)$ be generated by $a^i \in N$ and let $E$ denote the Hilbert basis of $S = \sigma^\vee \cap (M \oplus \mathbb{Z})$. We consider the canonical surjection $p : \mathbb{Z}^E \to M \oplus \mathbb{Z}$. Its kernel is a $\mathbb{Z}$-module $L(E) := \ker p$ which encodes the relations among elements in $E$.

**Definition 7.3.** For $R \in M \oplus \mathbb{Z}$ consider

\[
E_{\sigma}^R := E_1^R := \{ e \in E : \langle a^i, e \rangle < \langle a^i, R \rangle \}.
\]

For a subface $\tau \leq \sigma$ we define $E_\tau^R := \bigcap_{a^i \in \tau} E_{\sigma}^R$ and $L(E_\tau^R) := L(E) \cap \mathbb{Z}E_\tau^R$. Moreover, for $p \in \mathbb{N}$ we define

\[
L(E_\tau^R)_p := \bigoplus_{\tau \leq \sigma, \dim \tau = p} L(E_\tau^R).
\]

After defining $L(E_0^R) := \bigcup_{\sigma} E_{\sigma}^R$ we get a complex $L(E_\tau^R)_\ast$ with the usual differentials. Let us define $L_k(E_\tau^R) := L(E_\tau^R)_\ast \otimes \mathbb{Z} k$.

We have an exact sequence

\[
0 \to L_k(E_\tau^R) \to k_{E_\tau}^R \to \text{Span}_k E_\tau^R \to 0.
\]

Let us consider the first homology group of the complex $L_k(E_\tau^R)_\ast$:

\[
H_1(L_k(E_\tau^R)_\ast) = \frac{\ker \left( \bigoplus_i L_k(E_i^R) \to L_k(E) \right)}{\text{Image} \left( \bigoplus_{(a^i, a^j) \leq \sigma} L_k(E_i^R \cap E_j^R) \to \bigoplus_i L_k(E_i^R) \right)},
\]
which is isomorphic to \( H_2(\text{Span}_k E^*_\bullet) \) since \( H_i(k^*E^*_\bullet) = 0 \) for \( i \geq 1 \). In [6, Section 5.5] was proven that

\[
H_1(L_k(E^*_\bullet)) \cong \mathbb{Z}^k \subset \left(T^2(\mathbb{R})\right)^*.
\]

Recall the elements \( c_1, \ldots, c_r \) appearing in the Hilbert basis of \( S \) and the paths \( \Delta(v), \Delta^c(v) \) and \( \Delta^c(a) \) for \( a, c \in \text{tail}(P)^\vee \) (cf. Definition 3.6). For each vertex \( a_v \) of \( P \) we get the corresponding generator \( a_i^\ast \) of \( \sigma \). For a vertex \( v \) of \( P \) and \( c \in \text{tail}(P)^\vee \) we define similar paths

\[
\Delta(v) := [\text{some path } v_0 \leadsto v] = [\lambda_1(v), \ldots, \lambda_r(v)] \in \mathbb{Z}^r,
\]

\[
\mu^c(v) := [\text{some path } v \leadsto v(c) \text{ such that } \mu_i^c(v) \langle d^i, c \rangle \leq 0 \forall d^i] = [\mu_1^c(v), \ldots, \mu_r^c(v)] \in \mathbb{Z}^r,
\]

\[
\lambda^c(v) := \Delta(v) + \mu^c(v).
\]

In particular, \( \Delta(a) = \Lambda(v(a)), \mu^c(a) = \mu^c(v(a)) \) and \( \lambda^c(a) = \lambda^c(v(a)) \). For \( n \in \mathbb{N}, n \geq 2 \), we define the map:

\[
\psi^{(n)}_{i} : L_k(E^*_\bullet) \longrightarrow W_n
\]

\[
q \longmapsto \sum_{j=1}^{r} q_j p^{(n)}(\Delta^c_j(v^i) - \Delta(v(c_j)), c_j).
\]

**Lemma 7.4.** The maps \( \psi^{(n)}_{i} \) induce the linear map \( \psi^{(n)} : H_1(L_k(E^*_\bullet)) \longrightarrow W_n. \)

**Proof.** We need to show that for every face \( \text{Span}_{E^\ast} \{ a_i, a_j \} \subset \sigma \) the maps \( \psi^{(n)}_{i} \) and \( \psi^{(n)}_{j} \) agree on \( L(E^*_{a_i} \cap E^*_{a_j}) \). Let us write \( \Delta^k(v) := \lambda^c(v) \) and compute using Lemma 7.2 that

\[
\psi^{(n)}_{i}(q) - \psi^{(n)}_{j}(q) = \sum_{k=1}^{r} q_k p^{(n)}(\Delta^k(v^i) - \Delta^k(v^j), c_k).
\]

Denoting by \( \rho^{ij} \) the path consisting of the single edge running from \( v^i \) to \( v^j \) we see by Lemma 7.2 that for \( q \in L(E^*_{a_i} \cap E^*_{a_j}) \) we have

\[
\psi^{(n)}_{i}(q) - \psi^{(n)}_{j}(q) = \sum_{k=1}^{r} q_k p^{(n)}(\Delta^k(v^i) - \Delta^k(v^j) + \rho^{ij}, c_k) + \sum_{k=1}^{r} q_k p^{(n)}(\mu^k(v^i) - \mu^k(v^j) - \rho^{ij}, c_k) = 0.
\]

Indeed, the first sum is zero since \( \sum_{k=1}^{r} q_k c_k = 0 \) and the second sum is zero since \( q \in L(E^*_{a_i} \cap E^*_{a_j}) \), from which we can easily compute that the degree of \( p(\mu^k(v^i) - \mu^k(v^j) - \rho^{ij}, c_k) \) is strictly smaller than \( n \), which concludes the proof.

**7.3. The restriction of the obstruction map.**

**Proposition 7.5.** The map \( \sum_{n \geq 1} \psi^{(n)} \) is equal to the dual of the obstruction map \( o^* \) restricted to

\[
\bigoplus_{n \geq 1} H_1(L_k(E^*_\bullet)) \subset \left(T^2(\mathbb{R})\right)^*.
\]

**Proof.** Let \( a = (k_1^a, \ldots, k_n^a) \in \mathbb{N}^r \) and \( k = (k_1^k, \ldots, k_r^k) \in \mathbb{N}^r \). Let

\[
c_a := \sum_{j=1}^{r} k_i^a c_j,
\]

\[
c_k := \sum_{j=1}^{r} k_i^k c_j,
\]

\[
\sum_{n \geq 1} \psi^{(n)}(q) \in \left(T^2(\mathbb{R})\right)^*.
\]
where \( c_j \) appear in the Hilbert basis of \( S \), see (3). We denote \( \partial(\mathbf{a} + k) = (k_1^{\partial(\mathbf{a} + k)}, \ldots, k_r^{\partial(\mathbf{a} + k)}) \) and \( \partial(k) = (k_1^{\partial(k)}, \ldots, k_r^{\partial(k)}) \). Recall the linear relation \( R_{a,k} \) which can also be rewritten as

\[
R_{a,k} = x^{a+k} - x^{a+\partial(k)}t^{\lambda(k)} - x^{a}(x^k - x^{\partial(k)}t^{\lambda(c)}) + t^{\lambda(k)}x^{\partial(a+k)} - x^{\partial(k)+a}t^{\lambda(k)}(x^a - a + x^{\partial(k)+a}t^{\lambda(k)}+a).
\]

Let us denote \( s_a := \sum_{j=1}^r k_{j}^a s_j \) and \( s_k := \sum_{j=1}^r k_{j}^k s_j \), where \( s_j \) are the Hilbert basis elements, see (3).

Using [2, Theorem 3.5] we can find an element of

\[
\text{Hom}(R/R_0, W_0 \otimes O(X))
\]

representing \( \psi^{(n)} \). Using our notation we can easily verify that it sends relation \( R_{a,k} \) to

\[
\begin{cases}
\left( \psi^{(n)}_{v,\epsilon(c_k)}(k - \partial(k)) - \psi^{(n)}_{v,\epsilon(a+c_k)}(k - \partial(k)) \right)x^{a+k} & \text{if } \lambda(s_a + s_k) \geq n, \\
0 & \text{otherwise}.
\end{cases}
\]

Moreover, we define

\[
t'_{a+c_k,j} := \langle v_*, c_j \rangle s_{v_*} + \sum_{\nu} \delta_{\nu,\epsilon(c_k)} \tilde{t}(\lambda_{\nu}(c_k)c_j, d^\nu) \in T^*_z(P),
\]

\[
t'_{a+c_k,j} := \langle v_*, c_j \rangle s_{v_*} + \sum_{\nu} \delta_{\nu,\epsilon(c_a + c_k)} \tilde{t}(\lambda_{\nu}(c_a + c_k)c_j, d^\nu) \in T^*_z(P),
\]

where

\[
\delta_{\nu,\epsilon(c_k)} := \begin{cases} 1 & \text{if } \langle c_j, \lambda_{\nu}(c_k)d^\nu \rangle > 0, \\ -1 & \text{if } \langle c_j, \lambda_{\nu}(c_k)d^\nu \rangle \leq 0.
\end{cases}
\]

\[
\delta_{\nu,\epsilon(c_a + c_k)} := \begin{cases} 1 & \text{if } \langle c_j, \lambda_{\nu}(c_a + c_k)d^\nu \rangle > 0, \\ -1 & \text{if } \langle c_j, \lambda_{\nu}(c_a + c_k)d^\nu \rangle \leq 0.
\end{cases}
\]

Using Proposition 5.10 we see that the polynomial

\[
u \tilde{\lambda}(a+\partial(k)) + \tilde{\lambda}(k) - \nu \tilde{\lambda}(a+k),
\]

appearing in (27), equals \( u^{e_1} - u^{e_2} \), where

\[
e_1 = \sum_{j=1}^r ((k_j^a + k_j^{\partial(k)}) - k_j^{\partial(a+\partial(k))})t'_{a+c_k,j} + (k_j^a + k_j^{\partial(a+k)})t'_{c_k,j},
\]

\[
e_2 = \sum_{j=1}^r (k_j^a + k_j^{\partial(a+k)})t'_{a+c_k,j}.
\]

By the equation (21) we see that \( c_k = \sum_{j=1}^r k_j^k = \sum_{j=1}^r k_j^{\partial(k)} \) and by Lemma 5.4 it holds that

\[
\partial(a + \partial(k)) = \partial(a + k).
\]

Using also Lemma 7.2 we see that in \( W_n \) it holds that

\[
\psi^{(n)}_{v,\epsilon}(k - \partial(k)) - \psi^{(n)}_{v,\epsilon(a+c)}(k - \partial(k)) = \sum_{j=1}^r (k_j^k - k_j^{\partial(k)})p^{(n)}_{\epsilon}(\lambda_{\epsilon}(c_k) - \lambda_{\epsilon}(c_a + c_k), c_j).
\]

Since \( e_1 - e_2 = (k_j^c - k_j^{\partial(c)})t'_{a+c_k,j} - t'_{c_k,j} \), the proof now follows.

**Proposition 7.6.** The image of the map \( \psi^{(n)} \) contains the equations \( p^{(n)}(\epsilon, c) \) for all bounded 2-faces \( \epsilon \) and \( c \in \text{tail}(P)^\vee \cap M \).
Proof. Let us recall the isomorphism of homology groups $H_1(L_k(E^n_R)_\bullet) \cong H_2(\text{Span}_k E^n_R)$ explained after the exact sequence (33) ($n \in \mathbb{N}$). Let the rank of the lattice $M$ be $d - 1$ and thus $\text{Span}_k(M \oplus \mathbb{Z}) \cong k^d$. For $n \geq 2$ we have

$$H_2(\text{Span}_k E^n_R) = \frac{\ker\left[\bigoplus_{\langle a^i, a^j \rangle < P} k^d \to \bigoplus_{a^i < P} k^d\right]}{\text{Image}\left[\bigoplus_{\epsilon < P \text{dim} \epsilon = 2} \text{Span}_k(\cap_{a^i \in \epsilon} E^n_i) \to \bigoplus_{\langle a^i, a^j \rangle < P} k^d\right]}.$$ 

Indeed, $\text{Span}_k E^n_R \cong k^d$ clearly holds for all rays $a^i \in \sigma$ and

$$(37) \quad \text{Span}_k(E^n_{a^i} \cap E^n_{a^j}) \cong k^d$$

holds for all 2-faces $\text{Span}_{\mathbb{R} \geq a}\{a^i, a^j\}$ of $\sigma$ since the lattice length of all edges is smaller than 2 because the semigroups $\mathbb{T}_d$ are generated by degree 1 elements, see Proposition 3.21 and Remark 3.20.

Clearly it holds that

$$\ker\left[\bigoplus_{\langle a^i, a^j \rangle < P} k^d \to \bigoplus_{a^i < P} k^d\right] \cong \text{Image}\left[\bigoplus_{\epsilon < P \text{dim} \epsilon = 2} k^d \to \bigoplus_{\langle a^i, a^j \rangle < P} k^d\right]$$

since the complex $\bigoplus_{\tau < P, \text{dim} \tau = 1} k^d$ is acyclic in degrees $\geq 1$. Thus we have a surjection

$$g : \bigoplus_{\epsilon < P \text{dim} \epsilon = 2} k^d \longrightarrow H_2(\text{Span}_k E^n_R) \cong H_1(L_k(E^n_R)_\bullet).$$

In the following we will explicitly describe the map $g$. After choosing a 2-face with oriented edges $d^1, \ldots, d^m$ we represent $[c, \eta_Z(c)]$ as a linear combination of elements of $E^n_{a^i} \cap E^n_{a^i+1}$:

$$[c, \eta_Z(c)] = \sum j q_{i,j} [c^j, \eta_Z(c^j)] + q_i (\emptyset, 1),$$

and $q_{i,j} \neq 0$ implies $[c^j, \eta_Z(c^j)] \in E^n_{a^i} \cap E^n_{a^i+1}$. This corresponds to the lifting of an element from $\text{Span}_k E^n_R$ to $k^d E^n_R$. From this we get an element in

$$\ker\left(\bigoplus_i L(E^n_{a^i}) \longrightarrow L(E)\right)$$

whose $i$-th summand is the linear relation

$$\sum_j (q_{i,j} - q_{i-1,j}) [c^j, \eta_Z(c^j)] + (q_i - q_{i-1}) (\emptyset, 1) = 0.$$ 

Thus we explicitly describe the map $g$.

Now we will check that $(\psi^{(n)} \circ g)[c, \eta_Z(c)] = p^{(n)}(c, \epsilon)$ holds:

$$(\psi^{(n)} \circ g)[c, \eta_Z(c)] = \sum_{i=1}^m \sum_{j=1}^r (q_{i,j} - q_{i-1,j}) p^{(n)}(\Delta^{c^j}(v^i) - \Delta(v(c^j)), c^j)$$

$$= \sum_{i=1}^m \sum_{j=1}^r p^{(n)}(\Delta^{c^j}(v^i) - \Delta^{c^j}(v^{i+1}), q_{i,j} c^j),$$
where in the last equality we used bi-linearity of $p^{(n)}$ proven in Lemma 7.2. Now as in Lemma 7.4 we introduce the path $\rho^i$ consisting of the single edge running from $a^i$ to $a^{i+1}$ and compute that

$$
(\psi^{(n)} \circ g)[c, \eta_Z(c)] = \sum_{i=1}^m \sum_{j=1}^r p^{(n)}(\Delta(v^i) + \mu^j(v^i) - \Delta(v^{i+1}) - \mu^j(v^{i+1}), q_i, j c_j)
$$

$$
= \sum_{i=1}^m \sum_{j=1}^r p^{(n)}(\Delta(v^i) - \Delta(v^{i+1}) + \rho^i, \sum_{j=1}^r q_i, j c_j) + \sum_{i=1}^m \sum_{j=1}^r p^{(n)}(\mu^j(v^i) - \mu^j(v^{i+1}) - \rho^i, q_i, j c_j)
$$

$$
= \sum_{i=1}^m p^{(n)}(\Delta(v^i) - \Delta(v^{i+1}) + \rho^i, \sum_{j=1}^r q_i, j c_j).
$$

We used in the computation above that $p^{(n)}(\mu^j(v^i) - \Delta(v^{i+1}) - \rho^i, q_i, j c_j) = 0$, which can be verified the same way as in the proof of Lemma 7.2. From this it follows that

$$
(\psi^{(n)} \circ g)[c, \eta_Z(c)] = \sum_{i=1}^m p^{(n)}(\Delta(v^i) - \Delta(v^{i+1}) + \rho^i, c) = \sum_{i=1}^m p^{(n)}(\rho^i, c) = p^{(n)}(c, c).
$$

\[\blacksquare\]

**Corollary 7.7.** The image of $\sigma^*$ contains all the degree $n$ parts of all loop equations (29).

**Proof.** Let $p(\mu, c)$ be a loop equation. By Lemma 7.2 we can write its degree $n$ part as $\sum_{j=1}^k p^{(n)}(\epsilon_j, c)$, where $\epsilon_j$ are bounded 2-faces. We conclude by Propositions 7.5 and 7.6. \[\blacksquare\]

**Remark 7.8.** Note that in the proof of Proposition 7.6 we really need the assumption that the semigroups $T_d$ are generated by degree 1 elements. If we would only assume that $\bar{T}$ is generated by degree 1 elements, then the crucial equation (37) in the proof above might not be satisfied. See Example 6.5 and consider the edge $d_1$ that defines the 2-face $\text{Span}_{\mathbb{R}}\{a^1, a^2\}$ of $\sigma$. Here we have that $(a^1, R) = 1$ and thus

$$\text{Span}_{\mathbb{R}}(E_2^2 \cap E_2^2) \cong \text{Span}_{\mathbb{R}}((a^1) \perp \cap (a^2) \perp, R) \cong k^2.$$

The rank of $M \oplus \mathbb{Z}$ is 3 here so we see that the equation (37) is not satisfied in this example.

8. The local equations

Let us now explicitly write the set of generators from Proposition 3.14. For each compact edge $d_{ij} = v^j - v^i$ let

$$k_{ij} = \min\{|\langle c, d_{ij} \rangle| : c \in M, \langle c, d_{ij} \rangle \neq 0\}.$$

We choose $c_{ij} \in M$ such that $\langle c_{ij}, d_{ij} \rangle = k_{ij}$. We define $m_{ij}$ to be the minimal natural number such that $m_{ij}(c_{ij}, v^i), m_{ij}(c_{ij}, v^j) \in \mathbb{Z}$.

From Proposition 3.14 we see that the following set is a generating set for $\bar{T}$:

$$\{\bar{t}(kc_{ij}, d_{ij}), s_v : d_{ij} \in \text{edge}(P), k \in \{\pm 1, \ldots, \pm m_{ij}\}, v \in \text{Vert}(P)\}.$$

We fix an edge $d_{ij}$ and let $n$ be the maximal natural number, such that the degree of $\bar{t}(nc_{ij}, d_{ij})$ equals 1. Similarly, let $m$ be the maximal natural number, such that the degree of $\bar{t}(-mc_{ij}, d_{ij})$ is 1. Recall the definition of the sub-monoid $T_{ij} := \bar{T}_{ij}$ from Definition 3.18. Since $\bar{T}$ is generated by degree 1 elements, then the following elements are the minimal generators of $T_{ij}$:

$$s_i, s_j, \bar{t}(c_{ij}, d_{ij}), \ldots, \bar{t}(mc_{ij}, d_{ij}), \bar{t}(-c_{ij}, d_{ij}), \ldots, \bar{t}(-mc_{ij}, d_{ij}).$$

Let the polytope $Q_{ij}$ be the convex hull of the vertices $(0, 0), (0, 1), (n, 0), (m, 1)$. Let $C(Q_{ij})$ be the cone over this polytope, i.e., generated by $(0, 0, 1), (0, 1, 1), (n, 0, 1), (m, 1, 1)$. Denote the following
monoid by $T_{ij}' := \mathbb{Z}^3 \cap C(Q_{ij})$. From the description of these elements in Section 3, one can see that the monoid $T_{ij}'$ is isomorphic to $T_{ij}''$; the isomorphism is given by

$$s_1 \mapsto (0, 0, 1), \quad s_j \mapsto (0, 1, 1), \quad \tilde{t}(ac_{ij}, d^{ij}) \mapsto (a, 0, 1), \quad \tilde{t}(-bc_{ij}, d^{ij}) \mapsto (b, 1, 1),$$

for $a = 1, \ldots, n$ and $b = 1, \ldots, m$.

**Corollary 8.1.** The affine variety $\text{Spec } k[T_{ij}]$ is given by the equations

$$x_{ij}x_{k2} - x_{i1}x_{l2} = 0, \quad 1 \leq k_1 + k_2 = l_1 + l_2 \leq n,$$

$$y_{ij}y_{k2} - y_{i1}y_{l2} = 0, \quad 1 \leq k_1 + k_2 = l_1 + l_2 \leq m,$$

$$x_{i0}y_0 - xy_{x0} = 0, \quad 1 \leq k \leq \min\{n, m\},$$

where, for $k \geq 0$, $x_k$ and $y_k$ correspond to the generators $\tilde{t}(kc_{ij}, d^{ij})$ and $\tilde{t}(-kc_{ij}, d^{ij})$, respectively. In particular, in this notation, $x_0$ corresponds to $s_1$ and $y_0$ corresponds to $s_j$.

**Proposition 8.2.** The image of $o^*$ contains all the degree $n$ parts of all the equations in Corollary 8.1.

**Proof.** Let us first consider the case when $P$ is a bounded 1-dim polytope, i.e. a line segment with $0 \in \text{int}(P)$. More precisely, we assume that $P = d = [v, w] = [\frac{a_1}{b_1}, \frac{a_2}{b_2}]$ with $a_1, b_1 > 0$ and such that $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are not integers (the same setting as in Example 3.11). The general case follows easily from this one using the proof of Lemma 3.10. We have $\sigma^\vee = \langle (-b_2, a_2), (b_1, a_1) \rangle$ with the Hilbert basis equal to

$$\{[-b_2, \eta_\mathbb{Z}(-b_2)], \ldots, [-1, \eta_\mathbb{Z}(-1)], [1, \eta_\mathbb{Z}(1)], \ldots, [b_1, \eta_\mathbb{Z}(b_1)]\}.$$

Thus $r$ is equal to $b_1 + b_2$ in this case and we write $e_{-b_2}, e_{-b_2+1}, \ldots, e_{-1}, e_1, e_2, \ldots, e_{b_1}$ for the components of $a, k \in \mathbb{N}^r$.

Using the notation from Corollary 8.1 we have that

$$\eta_\mathbb{Z}(1) = \cdots = \eta_\mathbb{Z}(n) = 1, \quad \eta_\mathbb{Z}(-1) = \cdots = \eta_\mathbb{Z}(-m) = 1.$$

Let $k_1, k_2, l_1, l_2 \in \mathbb{N}$ be such that $1 \leq k_1 + l_1 = k_2 + l_2 \leq n$ and $k_1 \leq l_1, k_2 \leq l_2$. We define

$$k_1 := e_\mathbb{N} + e_{-k_1}, \quad a_1 := e_{-l_1}, \quad k_2 := e_\mathbb{N} + e_{-k_2}, \quad a_2 := e_{-l_2}.$$

For $i = 1, 2$ we have

$$\tilde{\lambda}(\tilde{a}_i) = \tilde{\lambda}(\tilde{a}_i + k_i) = \tilde{t}(k_i, d),$$

where the last equality was proven in Example 3.11. Moreover, for $i = 1, 2$ we have

$$\tilde{\lambda}(\tilde{a}_i + k_i) = \tilde{\lambda}(\tilde{a}_i + k_i) = \tilde{t}(l_i, d).$$

Since $\tilde{\lambda}(a_1 + k_1) = \tilde{\lambda}(a_2 + k_2)$ we obtain

$$o^*(R_{a_1, k_1} - R_{a_2, k_2}) = \left( u^{\tilde{\lambda}(a_1 + k_1)}u^{\tilde{\lambda}(k_1)} - u^{\tilde{\lambda}(a_1 + k_1)} \right) - \left( u^{\tilde{\lambda}(a_2 + k_2)}u^{\tilde{\lambda}(k_2)} - u^{\tilde{\lambda}(a_2 + k_2)} \right)$$

$$= u^{\tilde{t}(k_1, d)}u^{\tilde{t}(l_1, d)} - u^{\tilde{t}(k_2, d)}u^{\tilde{t}(l_2, d)},$$

from which we see that the $n$-th parts of the equations (39) are in the image of $o^*$. Defining

$$k_1' := e_{-m} + e_{k_1}, \quad a_1' := e_{l_1}, \quad k_2' := e_{-m} + e_{k_2}, \quad a_2' := e_{l_2}$$

gives us as above that

$$o^*(R_{a_1', k_1'} - R_{a_2', k_2'}) = u^{\tilde{t}(k_1, d)}u^{\tilde{t}(l_1, d)} - u^{\tilde{t}(k_2, d)}u^{\tilde{t}(l_2, d)},$$

respectively.\]
from which we see that the $n$-th parts of the equations (40) are in the image of $o^*$. Finally, denoting $m := \min\{m, n\}$ and for $k = 1, \ldots, m$ defining

$$k' := e_m + e_{-k}, \quad a_1 := e_1, \quad k_2'' := e_{-m} + e_k, \quad a_2 := e_{-1}$$

give us as above that the $n$-th parts of the equations (41) are in the image of $o^*$.

So to prove our main Theorem 6.2 we only need to show that the loop and local equations are generating all the equations of $k[\mathcal{T}]$. This is done in the next section.

9. NEW GENERATORS OF $\mathcal{T}$

9.1. Decomposing the 1-skeleton of $P$. We start with some graph theoretic considerations. Denote by $P^{(1)}$ the compact part of the 1-skeleton of the polyhedron $P$. It splits into the vertices and the interior parts of the compact edges. We extend this to an abstract graph $P'$ by adding abstract edges $e(v, w)$ between vertices $v, w \in P$ such that $[v, w] \leq P$ is an ordinary edge with $[v, w] \cap N = \emptyset$. This graph contains the following subsets:

(V) consisting of the vertices $v \in P$ being not contained in the lattice $N$, the new abstract edges $e(v, w)$, and of the short half open edges $[v, w]$ and

(D) consisting of the remaining open edges $(v, w)$, i.e. those such that neither $[v, w]$ nor $(v, w)$ is short.

While $D$ consists of isolated (open) edges, the set $V$ contains connected clusters, made from vertices and half open edges. This leads to the next step. We denote by

(A) the set of those connected components of $V$ that do not contain short half open edges, i.e. of those consisting only of (non-lattice) vertices $v$ and new abstract edges $e(v, w)$ and

(B) the set of remaining connected components of $V$. In particular, every component from $B$ contains at least one short half open edge.

Altogether, our graph $P' \supset P^{(1)}$ splits into a disjoint union of elements of $A$, $B$, $D$, and of the lattice vertices of the original polyhedron $P$. The latter set might be denoted by $N$.

**Example 9.1.** Let $P := \text{conv}\{v_1, v_2, v_3, v_4\}$ with $v_1 = (-1, 0)$, $v_2 = (1, 0)$, $v_3 = (\frac{3}{4}, \frac{1}{2})$, $v_4 = (-\frac{1}{6}, \frac{1}{2})$ and let

$$d_1 := (v_1, v_2), \quad d_2 := (v_2, v_3), \quad d_3 := (v_3, v_4), \quad d_4 := (v_4, v_1)$$

be the open edges between our vertices. We see that $[v_3, v_2]$ and $[v_4, v_1]$ are short half open edges and that there are no other short (half open) edges (see Example (2.3)). Thus

$$A = \{e(v_3, v_4)\}, \quad B = \{v_3, v_4, d_2, d_4\}, \quad D = \{d_1, d_3\}.$$ 

9.2. New variables. We are going to replace the old variables $s_v$ (for vertices $v \in P$) and $t_d$ (for edges $d \leq P$) by new ones. They will be denoted by $\rho^A_a$ (for $a \in A$), $\rho^B_b$ (for $b \in B$), and $\rho^D_d$ (for $d \in D$), and they are defined as follows:

($\rho^A$) If $a \in A$, then this cluster is not incident with any of the ordinary edges, but with some of the vertices $v \in P$. We denote $\rho^A_a := s_v$ (for any such $v$).

($\rho^D$) If $d \in D$, then this points to a single edge $d \leq P$. We set $\rho^D_d := t_d$.

($\rho^B$) This is the only type of the three clusters containing both vertices $v \in P$ and ordinary edges $d \in P$. If $b \in B$, then we set $\rho^B_b := s_v = t_d$ (for any such $v, d$ being contained in $b$).
If \( \mu \in \mathbb{Z}_p \) is induced from a closed path along the compact edges, e.g. from the boundary of a compact 2-face \( F \leq P \), then, for any \( c \in M \), we used to have the equation
\[
\sum_{\nu=1}^P \mu_\nu \cdot \langle c, d^\nu \rangle \cdot t_{d^\nu} = 0.
\]
This turns into the relation
\[
\sum_{d \in D} \mu_d \cdot \langle c, d \rangle \cdot \rho_d^D + \sum_{b \in B} \langle c, \sum_{d \in b} \mu_d d \rangle \cdot \rho_b^B = 0,
\]
which we will call the loop relation \( R(\mu, c) \).

That is, compared with [1], we keep the equations for \( t_d = \rho_d \) (appearing as \( \rho_d^B \) or \( \rho_d^D \)) along 2-faces \( F \). However, as in [4], some of the variables are forced to become equal (the former \( t_d = \rho_d^B \) corresponding to those \( d \) being contained in some joint \( b \in B \) become \( \rho_b^B \)), and now, beyond [4], we also have additional free variables \( \rho_a^A \) (for \( a \in A \)) not appearing in the loop relations.

9.3. **New generators for \( \tilde{T} \).** By Proposition 3.13, the semigroup \( \tilde{T} \) is generated by the elements \( s_v \) and \( \bar{t}(c, d) \) where \( v \in P \) are vertices, \( d \leq P \) compact edges, and \( c \in M \). Nevertheless, e.g. for proving that every relation comes either from loops or from local relations (obtained after fixing an edge), it is much easier to replace the generators \( \bar{t}(c, d) \) (and the \( s_v \)) by new ones being associated to the new coordinates introduced in Subsection 9.2.

(A) For each \( a \in A \), we define \( \bar{t}(a) := \rho_d^A \). In particular, these elements equal certain \( s_v \), i.e. they are contained in \( \tilde{T} \).

(B) For each \( b \in B \), we define \( \bar{t}(b) := \rho_b^B \). As in Case A, the cluster contains certain vertices \( v \in P \), i.e. \( \bar{t}(b) = s_v \in \tilde{T} \).

At this point, to keep track of the converse, we have already ensured that all variables \( s_v \) are among the new generators \( \bar{t}(a) \) or \( \bar{t}(b) \) \((a \in A, b \in B)\). Moreover, if \( d = \langle v, w \rangle \) is a short half open edge, then there are two cases:

First, if \( w \notin N \) (this is equivalent to \( \langle v, w \rangle \) being a short half open edge, too), then for \( c \in M \) with \( \langle c, d \rangle \geq 0 \) we know by Definition 3.2 that
\[
\bar{t}(c, d) = \langle c, d \rangle t_d + \{\langle c, w \rangle\} s_w - \{\langle c, v \rangle\} s_v = \left(\langle c, w \rangle - \langle c, v \rangle\right) \cdot \rho_b^B \in \mathbb{N} \cdot \rho_b^B
\]
with \( d, v, w \in b \) and \( b \in B \), hence \( t_d = s_v = s_w = \rho_b^B \).

Second, if \( w \in N \), then \( s_w = 0 \), hence
\[
\bar{t}(c, d) = \langle c, d \rangle t_d - \{\langle c, v \rangle\} s_v = \left(\langle c, w \rangle - \langle c, v \rangle\right) \cdot \rho_b^B \in \mathbb{N} \cdot \rho_b^B
\]
with \( d, v \in b \) and \( b \in B \), hence \( t_d = s_v = \rho_b^B \).

Thus, in both cases, the old \( \bar{t}(c, d) \) together with the elements \( s_v \), on the one hand, and the new elements \( \bar{t}(a) \) and \( \bar{t}(b) \), on the other, can be mutually expressed using just semigroup operations. It remains to treat the non-short edges – however, here we do not change anything at all:

(D) For each \( d \in D \) and \( c \in M \) we stay with the usual \( \bar{t}(c, d) \in \tilde{T} \). It can be expressed as
\[
\bar{t}(c, d) = \langle c, d \rangle \rho_d^D + \{\langle c, w \rangle\} \rho_w - \{\langle c, v \rangle\} \rho_v
\]
where \( \rho_v, \rho_w \) are either 0 (if the corresponding vertex is contained in \( N \)), or they are coordinates of some components from the sets \( A \) or \( B \).
It is now clear that our definitions imply that
\[ T = \langle \tilde{t}(a), \tilde{t}(b), \tilde{t}(c, d) : a \in A, \ b \in B, \ d \in D, \text{ and } c \in M \rangle \]
as a semigroup. That is, the new \( T \) still form a generating system. However, we will see in Subsection 9.4 that their mutual relations are easier to understand.

9.4. **The relations among the new generators.** The relations among the generators of \( T \) defined in Subsection 9.3 split into two types.

9.4.1. **The local relations.** We call relations among the new \( T \) local if and only if they are relations with integer coefficients among the elements \( \tilde{t}(c, d) \in T \) for a single, fixed edge \( d \in D \) (and finitely many \( c \in M \)).

9.4.2. **The loop relations.** Among the non-local, i.e. the global relations, there is a special class of so-called loop relations for any given closed path \( \mu \in \mathbb{Z}^p \) along the compact edges, e.g. for the boundary \( \mu = \partial F \) of any compact 2-face \( F \leq P \). For any \( c \in M \), the loop relation \( R(\mu, c) \) from Subsection (9.2) among the coordinates
\[ \sum_{d \in D} \mu_d \cdot \langle c, d \rangle \cdot \rho^D_{t} + \sum_{b \in B} \langle c, \sum_{d \in b} \mu_d d \rangle \cdot \rho^B_{b} = 0 \]
induces
\[ \sum_{d \in D} \mu_d \cdot \left( \tilde{t}(c, d) - \{ \langle c, w \rangle \} \rho_w + \{ \langle c, v \rangle \} \rho_v \right) + \sum_{b \in B} \langle c, \sum_{d \in b} \mu_d d \rangle \cdot \rho^B_{b} = 0. \]
Recalling that \( \rho_w, \rho_v \) (and \( \rho^B_{b} \)) belong to the classes \( A \) or \( B \), i.e. not to class \( D \), we may replace them by the corresponding \( t(\ldots) \), yielding the loop \( T \)-relation
\[ \sum_{d \in D} \mu_d \cdot \left( \tilde{t}(c, d) - \{ \langle c, w \rangle \} \tilde{t}(w) + \{ \langle c, v \rangle \} \tilde{t}(v) \right) + \sum_{b \in B} \langle c, \sum_{d \in b} \mu_d d \rangle \cdot \tilde{t}(b) = 0 \]
which we will call \( \tilde{R}(\mu, c) \).

**Proposition 9.2.** Any integral relation among the elements \( \tilde{t}(a), \tilde{t}(b), \) and \( \tilde{t}(c, d) \) with \( a \in A, \ b \in B, \ d \in D, \) and \( c \in M \) is an integral linear combination of local relations (9.4.1) and loop relations \( \tilde{R}(\mu, c) \) from (9.4.2).

**Proof.** Denote by \( \tilde{R} \) an arbitrary integral relation like
\[ \sum_{a \in A} \lambda_a \tilde{t}(a) + \sum_{b \in B} \lambda_b \tilde{t}(b) + \sum_{d \in D, c \in M} \lambda_{d,c} \tilde{t}(c, d) = 0. \]
Then we use (A), (B), and (D) of Subsection 9.3, i.e.
\[ \tilde{t}(a) = \rho^A_a, \]
\[ \tilde{t}(b) = \rho^B_b, \]
\[ \tilde{t}(c, d) = \langle c, d \rangle \rho^D_{d} + \{ \langle c, w \rangle \} \rho_w - \{ \langle c, v \rangle \} \rho_v \]
to write this as
\[ \sum_{a \in A} \lambda_a \rho^A_a + \sum_{b \in B} \lambda_b \rho^B_b + \sum_{d \in D, c \in M} \lambda_{d,c} \cdot \left( \langle c, d \rangle \rho^D_{d} + \{ \langle c, w \rangle \} \rho_w - \{ \langle c, v \rangle \} \rho_v \right) = 0 \]
with \( v = v(d), w = w(d) \in A \cup B \). Now, we know that this relation among the \( \rho \)-coordinates is the sum of certain loop relations \( R(\mu, c) \) from Subsection (9.2) — note that it suffices to take only \( \mu := \partial F \) for some compact 2-faces \( F \leq P \). We denote by \( \tilde{R}' \) the corresponding sum of the associated loop \( \tilde{t} \)-relations \( \tilde{R}(\mu, c) \). By construction, we know that the original \( \tilde{R} \) and the sum of loop relations \( \tilde{R}' \) coincide after being transformed to relations among the \( \rho \)-variables. Modding out the \( A \)- and \( B \)-variables, we can then spot a linear combination of local relations in the sense of (9.4.1) generating the difference.

Note that the point for everything working as it has been said is the triangular structure, i.e. the fact that \( \tilde{t}(c, d) \mapsto \langle c, d \rangle \rho^D_d + \{ \langle c, w \rangle \} \rho_w - \{ \langle c, v \rangle \} \rho_v \) involves only a single \( d \) and elements of \( A \cup B \) where the map \( \tilde{t} \mapsto \rho \) is trivial. \( \square \)

Thus we conclude the proof of Theorem 6.2. Indeed, from Corollary 7.7, Proposition 8.2 and Proposition 9.2 it follows the surjectivity of map \( \rho^* \), which proves the theorem.

REFERENCES

[1] K. Altmann: *The versal deformation of an isolated, toric Gorenstein singularity*, Invent. Math. 128 (1997), 443–479.
[2] K. Altmann: *Infinitesimal deformations and obstructions for toric singularities*, J. Pure Appl. Alg. 119 (1997), 211–235.
[3] K. Altmann: *One parameter families containing three-dimensional toric Gorenstein singularities*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge (2000), 21–50.
[4] K. Altmann, L. Kastner: *Negative deformations of toric singularities that are smooth in codimension 2*, Deformations of surface singularities, Bolyai Mathematical Society (2013).
[5] K. Altmann, J. Kollár: *The dualizing sheaf on first order deformations of toric surface singularities*, J. reine angew. Math. 753 (2019), 137–158.
[6] K. Altmann, A. B. Sletsjøe: *André-Quillen cohomology of monoid algebras*, J. Alg. 210 (1998), 1899–1911.
[7] K. Altmann, A. Constantinescu, M. Filip: *Polyhedra, lattice structures, and extensions of semigroups*, arXiv:2004.07377.
[8] J.A. Christophersen: *On the components and discriminant of the versal base space of cyclic quotient singularities*, in Singularity theory and its applications. Springer, Berlin, Heidelberg, (1991), 81–92.
[9] J. Christophersen, N. O. Ilen: *Hilbert schemes and toric degenerations for low degree Fano threefolds*, J. reine angew. Math. 717 (2016), 77–100.
[10] T. Coates, A. M. Kasprzyk, T. Prince: *Laurent Inversion*, Pure and Applied Mathematics Quarterly 15 (2019).
[11] T. Coates, A. Corti, S. Galkin, V. Golyshev, A. Kasprzyk: *Mirror symmetry and Fano manifolds*, In European congress of mathematics, Eur. Math. Soc. (2013), 285–300.
[12] A. Corti, M. Filip, A. Petracci: *Smoothing toric Gorenstein affine 3-folds, 0-mutable polynomials and mirror symmetry*, to appear soon.
[13] T. de Jong, D. van Straten: *On the deformation theory of rational surface singularities with reduced fundamental cycle*, J. Alg. Geom. 3 (1994), 117–172.
[14] T. de Jong, G. Pfister: *Local analytic geometry. Basic theory and applications*, Advanced lectures in mathematics, Vieweg (2000).
[15] J.O. Kleppe: *Deformations of graded algebras*, Math. Scand. 45, no. 2, (1979), 205–231.
[16] J. Kollár, N.I. Shepherd-Barron: *Threefolds and deformations of surface singularities*. Invent. math. 91, (1988), 299–338.
[17] E. Miller, B. Sturmfels: *Combinatorial commutative algebra*, Graduate Texts in Mathematics, Springer Verlag, New York (2005).
[18] T. Prince: *Smoothing Calabi-Yau toric hypersurfaces using the Gross-Siebert algorithm*, arXiv:1909.02140.
[19] J. Stevens: *On the versal deformation of cyclic quotient singularities*, in Singularity theory and its applications. Springer, Berlin, Heidelberg, (1991), 302–319.
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