PROBABILISTIC GALOIS THEORY OVER \( p \)-ADIC FIELDS

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Abstract. We estimate several probability distributions arising from
the study of random, monic polynomials of degree \( n \) with coefficients in
the integers of a general \( p \)-adic field \( K_p \) having residue field with \( q = p^f \)
elements. We estimate the distribution of the degrees of irreducible
factors of the polynomials, with tight error bounds valid when \( q > n^2 + n \).
We also estimate the distribution of Galois groups of such polynomials,
showing that for fixed \( n \), almost all Galois groups are cyclic in the limit
\( q \to \infty \). In particular, we show that the Galois groups are cyclic with
probability at least \( 1 - \frac{1}{q} \). We obtain exact formulas in the case of \( K_p \)
for all \( p > n \) when \( n = 2 \) and \( n = 3 \).

1. Introduction

The study of the Galois groups of polynomials has a long history. In 1892
Hilbert [18] showed that for an irreducible polynomial \( f(X,t) \in (\mathbb{Q}[X])[t] \)
there are infinitely many rational numbers \( x \) for which \( f(x,t) \) is irreducible
in \( \mathbb{Q}[t] \). In 1936 van der Waerden [29] gave a quantitative form of this
assertion. Consider the set of degree \( n \) monic polynomials with integer
coefficients restricted to a box \( |a_i| \leq B \). Van der Waerden showed that
a polynomial drawn at random from this set has Galois group \( S_n \) with
probability going to 1 as \( B \to \infty \). (See Section 1.3 for more details). The
study of the distribution of Galois groups for polynomials with coefficients
drawn from some given probability distribution is now termed probabilistic
Galois theory.

In this paper we address problems analogous to that of van der Waerden
for \( p \)-adic fields. We investigate the distribution of Galois groups of monic
polynomials of fixed degree \( n \) with coefficients drawn from the integers \( \mathcal{O}_p \)
of a general \( p \)-adic field \( K_p \), i.e. a finite extension of \( \mathbb{Q}_p \). Quantitative
questions parallel to van der Waerden’s study concern, firstly, the probability
a randomly drawn \( p \)-adic polynomial is irreducible, secondly, the probability
such a polynomial has irreducible factors of given degrees, and thirdly, the
distribution of Galois groups of such polynomials. One can ask about the

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probability of occurrence of each kind of Galois group for such polynomials. In this paper we will find bounds concerning such probabilities, and exact formulas for degrees $n = 2$ and $n = 3$.

Our main results are:

1. Estimates for the probability distribution of splitting types of monic degree $n$ polynomials with coefficients drawn from the integers $\mathfrak{O}_p$ in a $p$-adic field $K_p$. These include exact formulas if one conditions the polynomials to have discriminant relatively prime to $p$. (Theorems 1.1 and 1.2);

2. Estimates for the probability distribution of Galois groups of monic degree $n$ polynomials with coefficients drawn from the integers $\mathfrak{O}_p$ in a $p$-adic field $K_p$ (Theorem 1.4);

3. The existence of a limiting distribution for the distributions of the splitting types, resp. the Galois groups, as $q \to \infty$, where $q = p^f_p$ is the order of the residue class field of $K_p$ (Theorems 1.3 and 1.5);

4. Exact probability calculations for (1), (2) above for the case of any $p$-adic field $K_p$ for degrees $n = 2, 3$ and all primes $p > n$ (see Section 4).

Below we state the results in more detail. These results reveal both some interesting parallels and some sharp differences with the distributions for polynomials with integer coefficients; we give a comparison in Section 1.3.

1.1. Factorization of $p$-adic Polynomials. We let $K_p$ denote a finite extension of the $p$-adic field $\mathbb{Q}_p$, and we let $\mathfrak{O}_p$ denote the integers in the field $K_p$. Two invariants of $K_p$ are its ramification index $e = e_p$ and its residue class field degree $f = f_p$. These are given by $(p)\mathfrak{O}_p = (\pi^e)\mathfrak{O}_p$, where $\pi = \pi_p$ is a uniformizing parameter for the maximal ideal $\mathfrak{p} := \pi\mathfrak{O}_p$, and the residue class field given by $\mathbb{F}_q = \mathfrak{O}_p/\pi\mathfrak{O}_p$ with $q = p^f_p$. We normalize the additive $p$-adic Haar measure $h_p$ on $K_p$ to assign mass 1 to the $p$-adic integers $\mathfrak{O}_p$.

For fixed positive integer $n$, any unique factorization domain $R$, and any monic degree $n$ polynomial $f(x) \in R[x]$, we can factor $f(x)$ uniquely as

$$f(x) = \prod_{i=1}^k g_i(x)^{e_i},$$

where the $e_i$ are positive integers and the $g_i(x)$ are distinct, monic, irreducible, and non-constant. Following the framework of Bhargava [2], we define the splitting type of such a polynomial to be the formal symbol

$$\mu(f) := (\deg(g_1)^{e_1}, \deg(g_2)^{e_2}, \ldots, \deg(g_k)^{e_k})$$

where $k$ is the number of distinct irreducible factors of $f(x)$. Here we order the degrees in decreasing order. We then define $T_n$ to be the set of all
possible splitting types above. Thus \( T_3 = \{(111), (21), (3), (1^21), (1^3)\} \). Next we define \( T_n^* \) to be the set of splitting types such that all the exponents are equal to 1. Thus \( T_3^* = \{(111), (21), (3)\} \). The latter sets are the ones relevant to this paper.

The class \( T_n^* \) labels the possible splitting types of square-free polynomials. Each element \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \in T_n^* \) with \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_k \) describes a partition of \( n \), has associated a unique conjugacy class \( C_\mu \subset S_n \), the symmetric group on \( n \) elements. The conjugacy class is the set of all elements of \( S_n \) whose cycle lengths are equal to the numbers \( \mu_1, \ldots, \mu_k \) ordered in nonincreasing order.

For a splitting type \( \mu \), let \( c_i(\mu) \) count the number of terms \( \mu_j \) which are equal to \( i \).

We let \( P_{n,p}(\mu) = P_{n,p}(\mu; K_p) \) denote the set of tuples \( (a_0, a_1, \ldots, a_{n-1}) \in (\mathbb{O}_p)^n \) for which the polynomial \( f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \) has splitting type \( \mu \in T_n^* \), so in particular \( f(x) \) is square-free. This set is a full measure subset of \( (\mathbb{O}_p)^n \). We will also write \( f(x) \in P_{n,p}(\mu) \) if the tuple of coefficients of \( f(x) \) are in \( P_{n,p}(\mu) \).

We next let \( P_{n,p}^*(\mu) \) be the subset of \( P_{n,p}(\mu) \) whose associated discriminants are not divisible by \( p \), i.e. their associated polynomials have modulo \( p \) reductions which are square-free. Theorem \ref{thm:main} below determines the measure of \( P_{n,p}^*(\mu) \). To state this result, we also use the number \( M(q; m) \) with \( q = p^f \) which counts the number of monic, irreducible polynomials of degree \( m \) in \( \mathbb{F}_q[x] \). By a formula of Gauss this number is

\[
M(q; m) := \frac{1}{m} \sum_{d|m} \mu(d)q^{m/d} = \frac{1}{m} \sum_{d|m} \mu(d)q^{m/d}.
\]

Thus \( M(q; 1) = q, M(q; 2) = \frac{1}{2}(q^2 - q) \), etc. For any positive integer \( n \) and \( p \)-adic extension \( K_p \) we will denote

\[
P_{n,p}^* := \bigcup_{\mu \in T_n} P_{n,p}^*(\mu),
\]

which corresponds to the set of monic degree \( n \) polynomials \( f(x) \in \mathbb{O}_p[x] \) whose discriminant is not divisible by \( p \). In the following result we adopt the convention that \( \binom{A}{0} = 1 \) for any non-negative integer \( A \), and \( \binom{A}{B} = 0 \) whenever \( B > A \).

**Theorem 1.1.** Fix any integer \( n \geq 2 \) and prime \( p \), and let \( K_p \) be a finite extension of \( \mathbb{Q}_p \), with residue class degree \( q = p^f \). The probability that a random monic, degree \( n \) polynomial \( f(x) \in \mathbb{O}_p[x] \) has coefficients belonging to \( P_{n,p}^* \) is

\[
\nu_{n,p}(P_{n,p}^*) = 1 - \frac{1}{q}.
\]
For any splitting type $\mu \in T_n^*$, we have

$$\nu_{n,p}(P_{n,p}^*(\mu)) = \frac{1}{q^n} \prod_{i=1}^{n} \left( M(q; i) \right).$$

This result is based on the fact that if the modulo $p$ reduction of a monic polynomial in $\mathcal{O}_p[x]$ is square-free, then the splitting type of the polynomial is the same as the splitting type of its modulo $p$ reduction. The probability $1 - \frac{1}{q}$ of a monic polynomial over $\mathbb{F}_q[X]$ having square-free factorization was noted by Zieve (see Lemma 3.1). The formula for the number of square-free polynomials over $\mathbb{F}_q$ having a given $\mu$ was noted by Cohen [4, p.256]. The formula is valid for all $n$ and $q$, including those $q \leq n$.

For any element $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in T_n^*$, let $\nu_n(\mu)$ be the proportion of elements in $S_n$ which are in the conjugacy class $C_\mu$. This is the probability distribution on $T_n^*$ given by

$$\nu_n(\mu) := \frac{|C_\mu|}{|S_n|} = \frac{|C_\mu|}{n!}. \quad (1)$$

A well known explicit formula for $\nu_n(\mu)$ in terms of $\mu_1, \ldots, \mu_k$ is given in Lemma 3.2. The distribution $\nu_n(\mu)$ is exactly the distribution associated with the Chebotarev density theorem for Artin symbols in $S_n$-extensions of $\mathbb{Q}$, compare [19, pp. 410-411].

Next, we obtain bounds on these probabilities when $q$ is sufficiently large compared to $n$, as follows.

**Theorem 1.2.** Fix any positive integer $n \geq 2$ and any splitting type $\mu \in T_n^*$. Consider monic polynomials with coefficients in the ring of integers $\mathcal{O}_p$ of a finite extension field $K_p$ over $\mathbb{Q}_p$, with residue class field $\mathbb{F}_q$ with $q = p^f$.

1. If $n \geq 3$ and $q \geq n^2$ then we have

$$|\nu_{n,p}(P_{n,p}^*(\mu)) - \nu_n(\mu)| < \frac{n^2}{q - n^2} \nu_n(\mu).$$

2. If $n = 2$ and $q \geq n^2 + n$ we have

$$|\nu_{n,p}(P_{n,p}^*(\mu)) - \nu_n(\mu)| < \frac{n^2 + n}{q - (n^2 + n)} \nu_n(\mu).$$

In Theorem 1.1 the probabilities $\nu_{n,p}(P_{n,p}^*(\mu))$ depend only of the parameter $q$ and the splitting type $\mu$, as do the bounds obtained in Theorem 1.2 on these distributions.

As an example, take $K_p = \mathbb{Q}_p$, and $\mu = (n)$, which is the splitting type of a polynomial $f(x)$ which is irreducible over $\mathbb{Q}_p$. Theorem 1.1 asserts in this case that $\nu_{n,p}(P_{n,p}^*(\mu)) = \frac{M_{p^n}}{n!}$. In this case $\nu_n(\mu) = \frac{1}{n}$ and Theorem 1.2 says that on the range $n^2 + n < p < \infty$ one has $|\nu_{n,p}(P_{n,p}^*(\mu)) - \frac{1}{n}| = O(\frac{1}{p^2})$, with the implied constant in the $O$-symbol depending on $n$. In fact, the formula for $M(p; n)$ yields the stronger convergence rate $|\nu_{n,p}(P_{n,p}^*(\mu)) - \frac{1}{n}| = O(p^{-\frac{n}{2}})$. 
Another example with $K_p = \mathbb{Q}_p$ is $\mu = (1, 1, \ldots, 1)$ ($n$ times), which is the event that the polynomial splits completely into distinct linear factors over $\mathbb{Q}_p$. In this case $\nu_n(\mu) = \frac{1}{n}$. Now $\nu_n(\mu(p, \mu)) = \frac{1}{p^k}$ and one can see (by noting that $\frac{1}{n!} \cdot (p-1) \cdots (p-n+1) = 1 + O\left(\frac{1}{p}\right)$) that $\frac{a}{p} > \nu_n((1, 1, \ldots, 1)) - \frac{1}{n!} > \frac{b}{p}$ for some constants $a$ and $b$. Thus in this case the $O\left(\frac{1}{p}\right)$ estimate of Theorem 1.2 is the correct order of magnitude of the error as $p \to \infty$.

Theorem 1.3 immediately gives a convergence result for the distribution as $q = p^f \to \infty$.

**Theorem 1.3.** Fix any positive integer $n \geq 2$ and any $\mu \in T_n$. Consider monic polynomials with coefficients in the ring of integers $\mathcal{O}_p$ of a finite extension field $K_p$ over $\mathbb{Q}_p$, with residue class field $\mathbb{F}_q$ with $q = p^f$. Then letting the $p$-adic field $K_p$ vary, with either or both of $p$ and $f$ changing in any way such that $q \to \infty$, one has

$$\nu_n(\mu(p, \mu)) = \nu_n(\mu) \text{ as } q \to \infty.$$

In consequence

$$\nu_n(\mu(p, \mu)) = \nu_n(\mu) \text{ as } q \to \infty.$$

Two extreme cases of this result are the limit $q = p^f$ with $p$ fixed and $f \to \infty$, and the limit $q = p$ with $p$ varying and $p \to \infty$.

1.2. *Galois groups of $p$-adic Polynomials.* We next consider the distribution of Galois groups of random $p$-adic polynomials. Every polynomial $f(x)$ in $\mathcal{O}_p[x]$ has an associated splitting field $K_f$ over $K_p$ and Galois group $G_f = \text{Gal}(K_f/K_p)$. It is well known that the group $G_f$ is solvable (see Frey and Manin [16, p. 59]). Additionally, we can realize $G_f$ as a subgroup of the symmetric group $S_n$ by choosing an ordering of the roots of $f$, and identifying an element of $G_f$ with its permutation action on the roots. The embedding of $G_f$ into $S_n$ is only unique up to conjugation, because of the explicit choice of order of the roots of $f$. For any subgroup $G \subset S_n$, let $C_G$ denote the collection of subgroups of $S_n$ which are conjugate to $G$, and we denote $|C_G|$ the number of subgroups of $S_n$ conjugate to $G$. Let $P_{n,p}(G)$ be the set of tuples $(a_0, \ldots, a_{n-1}) \in (\mathcal{O}_p)^n$ for which the polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ has splitting field with Galois group in $C_G$; we again will write $f(x) \in P_{n,p}(G)$ in this case. Likewise, let $P_{n,p}^*(G)$ be the subset of $P_{n,p}(G)$ where the discriminants of the associated polynomials are not divisible by $p$.

We now define a probability distribution $\tilde{\nu}_n(G)$ on conjugacy classes of subgroups of $S_n$ as follows. For any cyclic subgroup $G \subset S_n$, there is a splitting type $\mu_G \in T^*_n$ such that for any $\sigma \in C_G \subset S_n$ the cyclic group $\langle \sigma \rangle$ is conjugate to $G$. We therefore define

$$\tilde{\nu}_n(G) := \nu_n(\mu_G).$$

For any non-cyclic group $G \subset S_n$ we set $\tilde{\nu}_n(G) = 0$. We term this distribution the *Erdős-Turan distribution* associated to $S_n$, because the distributions
\(\nu_n\) were studied for their own sake in a series of papers by Erdős and Turán \([9, 10, 11, 12, 13, 14, 15]\) over the period 1965-1972. Erdős and Turan studied properties of the order of elements of \(S_n\). In particular they studied the distributions of the size of the order; the number of prime divisors of the order; the size of the maximal prime divisor of the order, and related quantities.

**Theorem 1.4.** Fix a positive integer \(n \geq 2\) and let \(G\) be a subgroup of \(S_n\), defined up to conjugacy in \(S_n\). Consider monic polynomials with coefficients in the ring of integers \(O_p\) of a finite extension field \(K_p\) over \(\mathbb{Q}_p\), with residue class field \(\mathbb{F}_q\) with \(q = p^f\).

1. If \(G\) is cyclic, \(n \geq 3\), and \(q \geq n^2\) then we have
   \[
   \left| \nu_{n,p}(P_n^*(G)) - \tilde{\nu}_n(G) \right| < \frac{n^2}{q - n^2} \tilde{\nu}_n(G).
   \]
   In consequence
   \[
   \left| \nu_{n,p}(P_n(G)) - \tilde{\nu}_n(G) \right| < \frac{n^2}{q - n^2} \tilde{\nu}_n(G) + \frac{1}{q}.
   \]

   If \(G\) is cyclic, \(n = 2\), and \(q \geq n^2 + n\) then we have
   \[
   \left| \nu_{n,p}(P_n^*(G)) - \tilde{\nu}_n(G) \right| < \frac{n^2 + n}{q - (n^2 + n)} \tilde{\nu}_n(G).
   \]
   In consequence
   \[
   \left| \nu_{n,p}(P_n(G)) - \tilde{\nu}_n(G) \right| < \frac{n^2 + n}{q - (n^2 + n)} \tilde{\nu}_n(G) + \frac{1}{q}.
   \]

2. For non-cyclic \(G\) we have:
   \[
   \sum_{G \subseteq S_n \atop G \text{ non-cyclic}} \frac{\nu_{n,p}(P_n(G))}{|C_G|} \leq \frac{1}{q}.
   \]

In Theorem 1.4 the values \(\nu_{n,p}(P_n(G))\) might a priori depend on the particular field \(K_p\) and on the splitting type \(\mu\), but the bounds given above depend only on \(q\) and \(\mu\). We also note that if \(G\) is a non-cyclic group then \(\nu_{n,p}(P_n^*(G)) = 0\) because a monic, \(p\)-adic polynomial whose modulo \(p\) reduction is square-free necessarily has cyclic Galois group.

As an example of these distributions, consider for \(K_p = \mathbb{Q}_p\), the subgroups of \(S_4\) given by \(G_i = \langle \sigma_i \rangle\) for \(i = 1, 2\) with \(\sigma_1 = (12)\) and \(\sigma_2 = (12)(34)\). Here \(G_1\) and \(G_2\) are both cyclic of order 2, but are not conjugate subgroups inside \(S_4\). We have \(\tilde{\nu}_4(G_1) = \frac{6}{24}\) while \(\tilde{\nu}_4(G_2) = \frac{3}{24}\). If one instead considers the abstract group \(\mathbb{Z}/2\mathbb{Z}\) up to isomorphism type, then we may set \(\tilde{\nu}_4(\mathbb{Z}/2\mathbb{Z}) = \frac{2}{24}\), defining it to be the proportion of elements having order 2 in \(S_4\).

Theorem 1.4 has an immediate consequence, concerning the approach to a limiting distribution as the field \(K_p\) varies in any way such that \(q = p^f \to \infty\).
Theorem 1.5. Fix a positive integer \( n \geq 2 \) and let \( G \) be a subgroup of \( S_n \), defined up to conjugacy in \( S_n \). Consider monic polynomials with coefficients in the ring of integers \( \mathcal{O}_p \) of a finite extension field \( K_p \) over \( \mathbb{Q}_p \), with residue class field \( \mathbb{F}_q \) with \( q = p^f \). Then letting the \( p \)-adic field \( K_p \) vary, with either or both of \( p \) and \( f \) changing in any way such that \( q \to \infty \), the distributions converge to the Erdős-Turan distribution, i.e.

\[ \nu_{n,p}(P_{n,p}^*(G)) \to \tilde{\nu}_n(G) \] as \( q \to \infty \).

In particular the probability of a non-cyclic Galois group occurring becomes 0 in this limit. For some \( p \)-adic fields, non-cyclic Galois groups occur with positive probability as soon as \( n \geq 3 \), but they do not contribute to the limiting probabilities as \( q \to \infty \).

To shed some information on these probabilities, in Section 4 we explicitly calculate for the probabilities \( \nu_{n,p}(P_{n,p}^*(G)) \) for \( n \in \{2, 3\} \), any \( p \)-adic field \( K_p \) with prime \( p > n \), and \( G \) any subgroup of \( S_n \). Note that for \( n = 2, 3 \) the isomorphism type and conjugacy type of all subgroups of \( S_n \) coincide. The calculations in Section 4 reveal that for \( n = 2, 3 \) the following hold:

1. The splitting types of polynomials of degree \( n = 2, 3 \) are rational functions of \( q = p^f \) for \( p > n \);
2. The probabilities of Galois groups are not always rational functions of \( q \). For example, \( S_3 \)-extensions of \( \mathbb{Q}_p \) never occur for \( p \equiv 1 \pmod{3} \), but occur with positive probability for each \( p \equiv 2 \pmod{3} \), with \( p > 3 \) (\( S_3 \)-extensions occur for \( p = 2 \) as well, by an argument not given here).

1.3. Comparison with polynomials over \( \mathbb{Z} \): Galois groups. We compare and contrast the results above concerning Galois groups with known results on the distribution of Galois groups of random polynomials with integer coefficients.

In 1936, van der Waerden [29] showed that, for a suitable notion of “random,” a random monic degree \( n \) polynomial in \( \mathbb{Z}[x] \) has Galois group \( S_n \) (and hence is irreducible) with 100% probability.

Theorem 1.6 (van der Waerden). For integer \( B \geq 1 \) let \( \mathcal{F}_{n,B} \) be the set of monic, degree \( n \) polynomials in \( \mathbb{Z}[x] \) with all coefficients in the box \([-B + 1, B] \); there are \((2B)^n\) such polynomials. Let \( P_n(B) \) denote the proportion of polynomials in \( \mathcal{F}_{n,B} \) which are irreducible and have Galois group \( S_n \). Then with \( c = \frac{1}{6(n-2)} \), we have \( 1 - P_n(B) \ll B^{-c/\log \log B} \) as \( B \to \infty \).

In 1973 Gallagher [17] applied the large sieve to obtain a better bound, showing that \( P_n(B) = 1 - O\left(\frac{\log B}{\sqrt{B}}\right) \) as \( B \to \infty \). Recently Dietmann [8] obtained a further improvement on the remainder term, to \( P_n(B) = 1 - O_\epsilon\left(B^{-(2-\sqrt{2})+\epsilon}\right) \). Zywina [30, Theorem 1.6] obtained an improvement of
the error term for Galois groups not equal to $S_n$ or $A_n$. In the direction of number fields, in 1979 Cohen [5] extended this result to number fields in several directions, the simplest of which considers polynomials with coefficients in the ring of integers of a number field $k$, and considered $S_n$-extensions of $k$ that are obtained by adjoining the roots of such polynomials. See Cohen [6] for further generalizations of this result.

We make the following comparison between random polynomials in $\mathbb{Q}[x]$ and random polynomials in $\mathbb{Z}[x]$, summarized in the following Table 1. Its entries on the $p$-adic side are based on Theorems 1.1 - 1.5.

| Integer Polynomials                          | $p$-Adic Polynomials |
|----------------------------------------------|----------------------|
| Degree $n$, monic polynomials with integer coefficients bounded by $B$ | Degree $n$, monic polynomials with $p$-adic integer coefficients |
| Probability the polynomial is irreducible approaches 1 | Probability the polynomial is irreducible approaches $\frac{1}{n}$ |
| Limiting distribution of Galois groups is probability 1 of being $S_n$ | Limiting distribution of Galois groups is probability 1 of being a cyclic subgroup of $S_n$ |
| The limit of distributions is taken as $B \to \infty$ | The limit of distributions is taken as $q \to \infty$ |

Table 1. Summary of random integer polynomial results and random $p$-adic polynomial results.

In a different quantitative direction, considering number fields instead of polynomials, Bhargava [2] recently studied the question of counting the number of degree $n$ number fields having field discriminant at most $X$ and having Galois closure with Galois group $S_n$. He advanced conjectures for their number being asymptotic to $c_n X$ as $X \to \infty$, for specific constants $c_n$, and for the fraction of these fields having a given splitting behavior above a fixed prime $p$. He showed these conjectures hold for $n \leq 5$, using results of Davenport and Heilbronn [7] for the case $n = 3$, and using his own work ([1], [3]) on discriminants of quartic and quintic rings for the cases $n = 4, 5$. Bhargava’s work is partially based on $p$-adic mass formulas for étale extensions of local fields. His work can also be viewed as investigating splitting properties of primes in extensions of local fields.

In a separate paper with J. C. Lagarias we consider the splitting behavior above a fixed prime ($p$) of monic $S_n$-polynomials over $\mathbb{Z}[X]$, randomly drawn as in the van der Waerden model above. We determine the limiting probability distributions as $B \to \infty$ for all $n$ and $p$. Rather interestingly the resulting distributions do not match those predicted in the conjectures of Bhargava [2] above for the fraction of $S_n$-extensions of $\mathbb{Q}$ of discriminant below a bound $X$ having prescribed splitting behavior over a fixed prime.
p. In particular, the probabilities do not agree in some cases where Bhargava’s conjectures are proved; this shows the random number field model and random polynomial models are truly different.

1.4. Contents of paper. In section 2 we prove preliminary lemmas. Then we prove Theorems 1.1, 1.2, 1.4 and 1.5 in section 3. Finally in section 4 we calculate the probabilities $\nu_{n,p}(P_{n,p}(G))$ for $n \in \{2, 3\}$.

2. Preliminary Results

In this section we prove some preliminary results which will be used in the main proofs in sections 3 and 4.

We first remark on the measurability of all sets for which we are computing probabilities. These sets are subsets of coefficients $(a_0, a_1, ..., a_{n-1}) \in (\mathbb{O}_p)^n$. The condition for a monic polynomial $f(x)$ to have a repeated root is that the coefficients belong to the discriminant locus $\mathcal{V}_{n,p} := \{(a_0, a_1, ..., a_{n-1}) \in (\mathbb{O}_p)^n : \text{Disc}(f) = 0\}$, where $\text{Disc}(f)$ is a certain multivariate polynomial in the $a_i$’s which has integer coefficients. The set $\mathcal{V}_{n,p}$ is a closed subset of $(\mathbb{O}_p)^n$ having measure zero in the product measure; thus its complement $(\mathbb{O}_p)^n \setminus \mathcal{V}_{n,p}$ is a full measure open subset of $(\mathbb{O}_p)^n$. We now observe that all sets that we consider have the property that their intersection with $(\mathbb{O}_p)^n \setminus \mathcal{V}_{n,p}$ is an open set in $(\mathbb{O}_p)^n$. This is a consequence of Krasner’s lemma ([23, Prop. 5.4]), which states that if $a, b \in \mathbb{K}_p$ are such that $|a - b| < |a - \sigma(a)|$ for any element $\sigma \in \text{Gal}(\mathbb{K}_p/\mathbb{K}_p)$ with $\sigma(a) \neq a$, then $a \in \mathbb{K}_p(b)$. Now, if the roots of our monic polynomial $f(x) \in \mathbb{K}_p(x)$ are all distinct, then since the roots of a $p$-adic polynomial are continuous functions of the coefficients $a_i$ (viewed in an appropriate extension field of $\mathbb{K}_p$), a sufficiently small perturbation of the coefficients will induce a small enough perturbation of the roots for Krasner’s lemma to guarantee no change in the splitting type of the polynomial and of the splitting field generated by the roots. This verifies the open set property, which in turn implies that all the sets we consider are measurable for the product measure on $(\mathbb{O}_p)^n$.

The following lemmas present some properties of $p$-adic random variables which are distributed according to the Haar measure. We show that the modulo $p$ reduction of a random polynomial in $\mathbb{O}_p[x]$ is a random polynomial in $\mathbb{F}_q[x]$.

Lemma 2.1. A random variable $X$ on $\mathbb{O}_p$ is distributed according to the normalized Haar measure if and only if its modulo $p^n$ reduction is uniform for all integers $n > 0$.

Proof. The Haar measure $h_p$, normalized so that $h_p(\mathbb{O}_p) = 1$, is the unique probability measure on $\mathbb{O}_p$ which is additively invariant [25, Theorem 1-8].
Thus for any \( n > 0 \) and any \( a, b \in \mathcal{O}_p \) it holds that
\[
h_p(a + p^n\mathcal{O}_p) = h_p(b + p^n\mathcal{O}_p).
\]
This means that for all \( n > 0 \), a \( p \)-adic random variable \( X \) which is distributed according to the Haar measure is uniform modulo \( p^n \).

Conversely, if \( X \) is a \( p \)-adic random variable which is uniform modulo \( p^n \) for any \( n > 0 \), then the probability \( X \) is in \( a + p^n\mathcal{O}_p \) is equal to the probability it lies in \( b + p^n\mathcal{O}_p \) for any \( a, b \in \mathcal{O}_p \). A basis of the topology of \( \mathcal{O}_p \) are the open sets \( a + p^n\mathcal{O}_p \) for any \( a \in \mathcal{O}_p \) and any \( n > 0 \). Since \( X \) is additively invariant with respect to these, it is additively invariant. Thus by uniqueness, it must be distributed with respect to the Haar measure. \( \square \)

**Lemma 2.2.** If \( X \) and \( Y \) are independent random variables on \( \mathcal{O}_p \), and \( X \) is distributed according to the normalized Haar measure of \( \mathcal{O}_p \), then \( X + Y \) is as well.

**Proof.** By the previous lemma, we need only show that \( X + Y \) is uniformly distributed modulo \( p^n \) for all \( n \geq 1 \). Since we have shown in the previous lemma that \( X \) is uniformly distributed modulo \( p^n \), it must be that \( X + Y \) is as well. \( \square \)

We will also need the following lemma to facilitate the calculation of \( \nu_{2,p}(P_{2,p}(G)) \) and \( \nu_{3,p}(P_{3,p}(G)) \) done in section 4.

**Lemma 2.3.** For a fixed integer \( n > 1 \) any prime \( p \nmid n \) and any \( p \)-adic field \( K_p \), let \( f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \) with the \( a_i \) independent \( p \)-adic random variables distributed according to the Haar measure. Then the polynomial
\[
f \left( x - \frac{a_{n-1}}{n} \right) = x^n + A_{n-2}x^{n-2} + A_{n-3}x^{n-3} + \cdots + A_0
\]
is such that \( (A_0, \ldots, A_{n-2}) \in (\mathcal{O}_p)^{n-1} \) is distributed with respect to the Haar measure on \( (\mathcal{O}_p)^{n-1} \).

Additionally for any group \( G \), the probability that \( G \) is isomorphic to the Galois group of \( x^n + A_{n-2}x^{n-2} + A_{n-3}x^{n-3} + \cdots + A_0 \) is equal to \( \nu_{n,p}(P_{n,p}(G)) \).
Likewise, the probability that \( \mu \in T_n \) is the splitting type of \( x^n + A_{n-2}x^{n-2} + A_{n-3}x^{n-3} + \cdots + A_0 \) is equal to \( \nu_{n,p}(P_{n,p}(\mu)) \).

**Proof.** For any \( 0 \leq i \leq n - 2 \)
\[
A_i = a_i + \rho_i(a_{i+1}, \ldots, a_{n-1}) = a_i + \psi_i(A_{i+1}, \ldots, A_{n-2}, a_{n-1})
\]
where \( \rho_i \) and \( \psi_i \) are polynomials with \( \mathcal{O}_p \)-coefficients depending only on \( n \) and \( i \). Note that if \( p \nmid n \), then there may be an \( i \) with \( A_i \) not in \( \mathcal{O}_p \). Since \( a_i \) is independent of \( \rho_i(a_{i+1}, \ldots, a_{n-1}) \) for every \( 0 \leq i \leq n - 2 \) and \( a_i \) is distributed according to the Haar measure, then by Lemma 2.2 the \( A_i \) are identically distributed \( p \)-adic random variables with respect to the Haar measure.
To see that $A_i$ is independent of $A_{i+1}, \ldots, A_{n-2}$ for every $0 \leq i \leq n - 3$ notice that given any $b_1, \ldots, b_{n-2} \in \mathcal{O}_p$ and any $n \geq 2$

\[
\text{Prob} \left[ A_i \equiv b_i \pmod{p^n} \mid A_{i+j} \equiv b_{i+j} \pmod{p^n} \right. \left. \text{for all } 1 \leq j \leq n + 2 - i \right] = \text{Prob} \left[ a_i \equiv b_i - \psi_i(b_{i+1}, \ldots, b_{n-2}, a_{n-1}) \pmod{p^n} \right] = \frac{1}{q^n}.
\]

Finally, note that the Galois groups and splitting types of $f(x)$ and $f(x - \frac{a_{n-1}}{n})$ are the same as the two polynomials are linear transformations of each other. Thus the induced distributions on Galois groups and splitting types are the same. □

The following lemma will be used to bound products which appear in several of the later proofs.

**Lemma 2.4.** Fix $n$ a positive integer. Given a real number $r > n$ and real numbers $x_1, \ldots, x_n$ with $\max_{1 \leq i \leq n} |x_i| \leq \frac{1}{r}$ then

\[
\left| \prod_{i=1}^{n} (1 + x_i) - 1 \right| < \frac{n}{r - n}.
\]

**Proof.** We have that

\[
\left( 1 - \frac{1}{r} \right)^n \leq \prod_{i=1}^{n} (1 + x_i) \leq \left( 1 + \frac{1}{r} \right)^n.
\]

Note that since $r \geq 1$ none of the terms are negative. So it suffices to prove that

\[
\left( 1 + \frac{1}{r} \right)^n - 1 \leq \frac{n}{r - n}
\]

and

\[
1 - \left( 1 - \frac{1}{r} \right)^n \leq \frac{n}{r - n}.
\]

Since $r > n$, the we can express $\frac{n}{r - n}$ as the sum of a geometric series:

\[
\frac{n}{r - n} = \frac{\frac{n}{r}}{1 - \frac{r}{n}} = \frac{n}{r} + \frac{n^2}{r^2} + \frac{n^3}{r^3} + \cdots
\]

The left side of equation (2) can be expanded as

\[
\left( 1 + \frac{1}{r} \right)^n - 1 = \left( \frac{n}{1} \right) \frac{1}{r} + \left( \frac{n}{2} \right) \frac{1}{r^2} + \cdots + \left( \frac{n}{n} \right) \frac{1}{r^n}.
\]

Since each $\binom{n}{i} < n^i$ we have that

\[
\left( \frac{n}{1} \right) \frac{1}{r} + \left( \frac{n}{2} \right) \frac{1}{r^2} + \cdots + \left( \frac{n}{n} \right) \frac{1}{r^n} < \frac{n}{r} + \frac{n^2}{r^2} + \cdots + \frac{n^n}{r^n} < \frac{n}{r - n}.
\]
Likewise for equation (3) we can expand the left side as
\[ 1 - \left( 1 - \frac{1}{r} \right)^n = \left( \frac{n}{1} \right) \frac{1}{r} - \left( \frac{n}{2} \right) \frac{1}{r^2} + \cdots - (-1)^n \left( \frac{n}{n} \right) \frac{1}{r^n} \]
\[ \leq \left( \frac{n}{1} \right) \frac{1}{r} + \left( \frac{n}{2} \right) \frac{1}{r^2} + \cdots + \left( \frac{n}{n} \right) \frac{1}{r^n} \]
\[ < \frac{n}{r - n}. \]

\[ \square \]

3. Distribution of Galois Groups

In this section we analyze the distribution of splitting types and of Galois groups of random, monic, degree \( n \) polynomials over a general finite field \( \mathbb{F}_q[x] \), with \( q = p^f \). We then use the results to prove Theorems 1.1 to 1.5 concerning splitting types and Galois groups for \( p \)-adic polynomials.

**Lemma 3.1.** Fix an integer \( n > 1 \). Let \( q = p^f \) where \( p \) is any prime and \( f \geq 1 \). Then the probability that a random, monic, degree \( n \) polynomial \( f(x) \in \mathbb{F}_q[x] \) is square free equals \( 1 - \frac{1}{q} \).

**Proof.** (We are indebted to Mike Zieve for the following proof.) We will define a surjective \( q \)-to-1 map \( \psi \) from the set of monic, degree \( n \) polynomials in \( \mathbb{F}_q[x] \) which are not square-free to the set of monic, degree \( n - 2 \) polynomials in \( \mathbb{F}_q[x] \). For any \( f(x) \in \mathbb{F}_q[x] \) which is monic, degree \( n \), and has a repeated root, we can factor it uniquely as \( f(x) = g(x)(h(x))^2 \) where \( g(x), h(x) \in \mathbb{F}_q[x] \) are monic, \( g(x) \) is square-free, and \( \text{deg}(h) > 0 \). Define
\[ \psi(f)(x) = g(x) \left( \frac{h(x) - h(0)}{x} \right)^2. \]
This is a polynomial, because \( h(x) - h(0) \) is a non-zero polynomial with constant term 0, so it is divisible by \( x \). Additionally, \( \psi(f)(x) \) has degree \( n - 2 \) because it is the quotient of a degree \( n \) polynomial and \( x^2 \).

Given an \( r(x) \in \mathbb{F}_q[x] \) which is monic and degree \( n - 2 \), there is a unique factorization of it as \( r(x) = g(x)(h(x))^2 \) where \( g(x), h(x) \in \mathbb{F}_q[x] \) are monic (and if \( h(x) \) is constant then \( h(x) = 1 \)) and \( g(x) \) is square-free. Then \( \psi(f)(x) = r(x) \) if and only if \( f(x) = g(x)(xh(x) + c)^2 \) for some \( c \in \mathbb{F}_q \).

Since we have that \( \psi \) is \( q \)-to-1, and there are \( q^{n-2} \) monic degree \( n - 2 \) polynomials in \( \mathbb{F}_q[x] \), there must be \( q.q^{n-2} = q^{n-1} \) monic degree \( n \) polynomials with repeated roots in \( \mathbb{F}_q[x] \). Thus the proportion of these to all monic degree \( n \) polynomials in \( \mathbb{F}_q[x] \) is
\[ \frac{q^{n-1}}{q^n} = \frac{1}{q}. \]
So the probability a random, monic, degree \( n \) polynomial in \( \mathbb{F}_q[x] \) does not have repeated roots (and hence is square-free) is

\[
1 - \frac{1}{q}.\]

\[\square\]

The following result is well known [28, p. 28]:

**Lemma 3.2.** Given a fixed positive integer \( n \), and splitting type \( \mu \in T_n^* \), the probability of a uniformly drawn element of \( S_n \) falling in conjugacy class \( C_\mu \) is

\[
\nu_n(\mu) := \frac{C_\mu}{n!} = \prod_{i=1}^{n} \frac{i^{-c_i(\mu)}}{c_i(\mu)!}.
\]

We now compute the probability that a random, monic, degree \( n \) polynomial in \( \mathbb{F}_q[x] \) is square-free and has prescribed splitting type. Recall that for any prime power \( q \) there are \( M(q; i) = \sum_{d|i} \mu(i/d)q^d \) monic, irreducible polynomials in \( \mathbb{F}_q[x] \) of any degree \( i \geq 1 \) [26, p. 13]. In what follows we adopt the convention that \( \binom{A}{0} = 1 \) for any non-negative integer \( A \), and \( \binom{A}{B} = 0 \) whenever \( B > A \).

**Lemma 3.3.** Fix a positive integer \( n \geq 2 \), a prime power \( q = p^f \), and a splitting type \( \mu \in T_n^* \). Let \( Q_{n,q}(\mu) \) be the number of monic, square-free, degree \( n \) polynomials in \( \mathbb{F}_q[x] \) which have \( \mu \) as their splitting type. Then

\[
Q_{n,q}(\mu) = \prod_{i=1}^{n} \left( \frac{M(q; i)}{c_i(\mu)} \right).
\]

Additionally, if \( n \geq 3 \) and \( q > n^2 \), then

\[
\left| \frac{1}{q^n} Q_{n,q}(\mu) - \nu_n(\mu) \right| < \frac{n^2}{q - n^2} \nu_n(\mu).
\]

If \( n = 2 \) and \( q > n^2 + n \), then

\[
\left| \frac{1}{q^2} Q_{n,q}(\mu) - \nu_n(\mu) \right| < \frac{n^2 + n}{q - (n^2 + n)} \nu_n(\mu).
\]

**Proof.** Since \( Q_{n,q}(\mu) \) counts the number of ways of choosing combinations of distinct irreducible polynomials of prescribed degrees, we have that

\[
Q_{n,q}(\mu) = \prod_{i=1}^{n} \left( \frac{M(q; i)}{c_i(\mu)} \right). \tag{4}
\]

Note that

\[
\frac{M(q; i)}{q^i} = \frac{1}{i} (1 + \varepsilon(i, q))
\]
with

\[ |\varepsilon(i, q)| \leq \begin{cases} 0 & \text{if } i = 1 \\ \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{i-1}} & \text{if } i \geq 2 \end{cases} \]

This is because for \( i \geq 2 \)

\[ |\varepsilon(i, q)| = \left| \frac{1}{q^i} \sum_{d<i \atop d|i} \mu(m/d)q^{d'} \right| \leq \frac{1}{q^i} \sum_{d=1}^{i-1} q^{d'} = \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^i}. \]

With this, we can rewrite equation (4) as

\[
\frac{Q_{n,q}(\mu)}{q^n} = \prod_{i=1}^{n} \frac{1}{c_i(\mu)!} \prod_{\ell=0}^{c_i(\mu)-1} \frac{(M(q; i) - \ell)}{q^{\ell}} = \prod_{i=1}^{n} \frac{i - c_i(\mu)}{c_i(\mu)!} \prod_{\ell=0}^{c_i(\mu)-1} \left( 1 + \varepsilon(i, q) - i\ell \right). \]

Note that since the sum \( \sum_{i=1}^{n} ic_i(\mu) = n \) we have for all \( \ell \) with \( 0 \leq \ell \leq c_i(\mu) - 1 \) that \( i\ell < n \). Also note that for \( 0 \leq \ell \leq c_i(\mu) - 1 \) we can write

\[ M(q; i) = \frac{1}{i} (1 + \bar{\varepsilon}_\ell(i, q)) \]

with, for \( i \geq 2 \) and \( q \geq n \geq 2 \)

\[ |\bar{\varepsilon}_\ell(i, q)| \leq |\varepsilon(i, q)| + \frac{i\ell}{q^i} < \frac{1}{q-1} + \frac{n}{q^2} \leq \frac{1}{q-1} + \frac{1}{q} \leq \frac{2}{q} + \frac{1}{q(q-1)}, \]

and if \( i = 1 \) then

\[ |\bar{\varepsilon}_\ell(i, q)| = \frac{\ell}{q} \leq \frac{n}{q}. \]

Thus by Lemma 3.2

\[ \left| \frac{1}{q^n} Q_{n,q}(\mu) - \nu_n(\mu) \right| = \nu_n(\mu) \left| \prod_{i=1}^{n} \prod_{\ell=0}^{c_i(\mu)-1} (1 + \bar{\varepsilon}_\ell(i, q)) - 1 \right|. \]  

When \( n \geq 3 \) all the terms \( \bar{\varepsilon}_\ell(i, q) \) have absolute value less than or equal to \( \frac{2}{q} \). When \( n = 2 \) all the \( \bar{\varepsilon}_\ell(i, q) \) have absolute value less than or equal to \( \frac{n}{q} \).

Thus when \( n \geq 3 \) and \( q \geq n^2 \) we can apply Lemma 2.4 with \( r = \frac{n}{q} \geq n \) and conclude that

\[ \left| \frac{1}{q^n} Q_{n,q}(\mu) - \nu_n(\mu) \right| < \nu_n(\mu) \frac{n^2}{q - q^2}. \]

When \( n = 2 \) and \( q \geq n^2 + n \) we can apply Lemma 2.4 with \( r = \frac{q}{n+1} \geq n \) and conclude that

\[ \left| \frac{1}{q^n} Q_{n,q}(\mu) - \nu_n(\mu) \right| < \nu_n(\mu) \frac{n^2 + n}{q - (n^2 + n)}. \]

\[ \square \]
We now have the necessary tools to prove Theorem 1.1 through Theorem 1.5.

of Theorem 1.4. By Lemma 2.1 we have over $K_p$ that the modulo $p$ reduction of a degree $n$, monic, $p$-adic, random polynomial is equidistributed among the monic, degree $n$ polynomials in $\mathbb{F}_q[x]$, with $q = p^f$. Thus $\nu_{n,p}(P_{n,p}^*)$ is equal to the probability a random monic, degree $n$ polynomial in $\mathbb{F}_q[x]$ is square-free. By Lemma 3.1 this probability is $1 - \frac{1}{q}$, thus

$$\nu_{n,p}(P_{n,p}^*) = 1 - \frac{1}{q}. $$

Hensel’s lemma [24, p. 129] implies that if $f(x) \in \mathbb{F}_p[x]$ is monic and has modulo $p$ reduction $\overline{f}(x)$ which is square-free, then the splitting types $\mu(\overline{f})$ and $\mu(f)$ are equal. Given any splitting type $\mu \in T_n^*$, Lemma 3.3 shows that for any prime power $q = p^f$, the probability that $\overline{f}$ is square-free and has splitting type $\mu$ equal to $\frac{1}{q} \prod_{i=1}^n \left( \frac{M(q;i)}{c_i(\mu)} \right)$. This is $\nu_{n,p}(P_{n,p}^*(\mu))$. □

of Theorem 1.2. Lemma 3.3 shows that

$$\left| \frac{1}{q^n} \prod_{i=1}^n \left( \frac{M(q;i)}{c_i(\mu)} \right) - \nu_n(\mu) \right| < \begin{cases} \frac{n^2 + n}{q - (n^2 + n)} \nu_n(\mu) & \text{if } n = 2 \text{ and } pq \geq n^2 + n, \\ \frac{n^2}{q - n^2} \nu_n(\mu) & \text{if } n \geq 3 \text{ and } q \geq n^2. \end{cases} \quad (7)$$

Substituting in $\nu_{n,p}(P_{n,p}^*(\mu)) = \frac{1}{q^n} \prod_{i=1}^n \left( \frac{M(q;i)}{c_i(\mu)} \right)$ from Theorem 1.1 completes the proof. □

of Theorem 1.3. As $q = p^f$ grows without bound both terms on the right side of equation (7) goes to zero on the order of $\frac{1}{q}$ and so $\nu_{n,p}(P_{n,p}^*(\mu)) \to \nu_n(\mu)$ as $q \to \infty$. □

of Theorem 1.4. Let $f(x)$ be a monic, degree $n$ polynomial in $\mathbb{F}_p[x]$ with square-free reduction modulo $p$. Then the Galois group of the splitting field of $f(x)$ over $K_p$ is cyclic. Furthermore, if $\mu$ is the splitting type of $f(x)$, its Galois group is generated by an element of $C_\mu \subset S_n$. Therefore, the Galois group of $f(x)$ is conjugate to a given cyclic subgroup $G \subset S_n$ if and only if the splitting type of $f(x)$ is equal to $\mu_G$.

So for a cyclic group $G \subset S_n$,

$$\nu_{n,p}(P_{n,p}^*(G)) = \nu_{n,p}(P_{n,p}^*(\mu_G)).$$

If $n \geq 3$, and $q > n^2$, we have by an application of Theorem 1.1 and the definition of $\tilde{\nu}_n(G)$ that

$$\left| \nu_{n,p}(P_{n,p}^*(G)) - \tilde{\nu}_n(G) \right| = \left| \nu_{n,p}(P_{n,p}^*(\mu_G)) - \nu_n(\mu_G) \right| < \frac{n^2}{q - n^2} \nu_n(\mu_G) = \frac{n^2}{q - n^2} \tilde{\nu}_n(G). \quad (8)$$

(9)
If \( n = 2 \) and \( q > n^2 + n \) we instead have that
\[
|\nu_n,p(P_{n,p}(G)) - \tilde{\nu}_n(G)| < \frac{n^2 + n}{q - (n^2 + n)} \tilde{\nu}_n(G).
\] (10)

We also showed in Theorem 1.1 that the measure of all degree \( n \), monic polynomials in \( \mathcal{O}_p[x] \) with discriminant divisible by \( p \) is equal to \( \frac{1}{q} \). Thus the probability that a polynomial has cyclic Galois group and has discriminant divisible by \( p \) is at most \( \frac{1}{q} \). Combining this with equations (8), and (10) by the triangle inequality we have that
\[
|\nu_n,p(P_{n,p}(G)) - \tilde{\nu}_n(G)| < \begin{cases} 
\frac{n^2 + n}{q - (n^2 + n)} \tilde{\nu}_n(G) + \frac{1}{q} & \text{if } n = 2 \text{ and } q > n^2 + n, \\
\frac{n^2 + n}{q - n^2} \tilde{\nu}_n(G) + \frac{1}{q} & \text{if } n \geq 3 \text{ and } q > n^2.
\end{cases}
\]

Finally, the only way a monic polynomial \( f(x) \in \mathcal{O}_p[x] \) can have non-cyclic Galois group is if its modulo \( p \) reduction is not square-free. By Lemma 3.1 the probability of this occurring is at most \( \frac{1}{q} \). Thus \( \sum_G \nu_{n,p}(P_{n,p}(G)) \leq \frac{1}{q} \).

of Theorem 1.5. This is immediate from the bounds obtained in Theorem 1.1.

4. LOW DEGREE CASES

We obtain exact formulas for degrees \( n = 2 \) and \( n = 3 \) for general \( p \)-adic fields \( K_p \), when \( p > n \). We compute \( \nu_{n,p}(\mathcal{P}_{n,p}(\cdot)) \) for all splitting types and all Galois groups, noting that exact formulas for \( \nu_{n,p}(P_{n,p}^*(\cdot)) \) were given in Theorem 1.1. The restriction that \( p > n \) guarantees that no splitting field of a degree \( n \) polynomial can be wildly ramified. The case \( n = 2 \) was previously analyzed for \( \mathbb{Q}_p \) in the 1999 Ph.D. thesis of Limmer [20, Corollary, p. 27]. The proof given here is based on similar ideas but differs in some details.

4.1. Degree \( n = 2 \).

Theorem 4.1. Let \( p > 2 \) be any prime, and let \( K_p \) be any finite extension of \( \mathbb{Q}_p \) with residue field \( \mathbb{F}_q \). Then for monic quadratic polynomials with \( \mathcal{O}_p \) coefficients we have
\[
\nu_{2,p}(P_{2,p}(id)) = \nu_{2,p}(P_{2,p}(1, 1)) = \frac{1}{2} - \frac{1}{2q + 2},
\]
\[
\nu_{2,p}(P_{2,p}(\mathbb{Z}/2\mathbb{Z})) = \nu_{2,p}(P_{2,p}(2)) = \frac{1}{2} + \frac{1}{2q + 2}.
\]

Proof. By Lemma 2.3 we can reduce to studying random polynomials of the form \( f(x) = x^2 + A \in \mathcal{O}_p[x] \). If \( 2r \) is the largest even power of \( \pi_p \) which divides \( A \), which is a \( q^{-2r} - q^{-2r-2} \) event, then \( g(x) := \pi_p^{-2r} f (\pi_p^r x) \) is a polynomial in \( \mathcal{O}_p[x] \). Furthermore, \( g(x) \) and \( f(x) \) have the same splitting type and Galois group. The coefficients of \( g(x) = x^2 + B \) are distributed...
with respect to the Haar measure conditional on the fact that the constant term is not divisible by $\pi_p^2$.

If we let $Y$ denote the conditional probability that $x^2 + B \in \mathcal{O}_p[x]$ factors into two linear polynomials given that $B \notin \pi_p^2 \mathcal{O}_p$, then we have the following equality:

$$\nu_2(p_2(p(1, 1))) = \nu_2(p_2(p(id))) = Y \sum_{i=0}^{\infty} \left( \frac{1}{q^{2i}} - \frac{1}{q^{2i+2}} \right) = Y.$$

We then have to study the distribution on Galois groups induced by the random polynomial $x^2 + B \in \mathcal{O}_p[x]$, when $B \not\in \pi_p \mathcal{O}_p$. Since we are studying the conditional probability, we will only be calculating measure of subsets of $\mathcal{O}_p \setminus \pi_p^2 \mathcal{O}_p$, and so will need to divide our measures by $1 - q^{-2}$ to obtain probabilities. If $B \not\in \pi_p \mathcal{O}_p$, which is a probability $1 - \frac{1}{q}$ event, then the Galois group is trivial (and has splitting type $(1, 1)$) if $B$ is a quadratic residue modulo $(p)$, and it is $\mathbb{Z}/2\mathbb{Z}$ (with splitting type $(2)$) otherwise; both of these are equally likely events. If $B \in \pi_p \mathcal{O}_p \setminus \pi_p^2 \mathcal{O}_p$, which is a probability $\frac{1}{q} - \frac{1}{q^2}$ event, then the extension $K_f$ must be ramified hence the Galois group is $\mathbb{Z}/2\mathbb{Z}$. Thus we have that

$$Y = \frac{1}{2} \left( \frac{1}{1 - \frac{1}{q}} + 0 \left( \frac{1}{q} - \frac{1}{q^2} \right) \right) = \frac{1}{2} \left( \frac{1}{1 + \frac{1}{q}} \right) = \frac{1}{2} - \frac{1}{2(q + 1)}.$$

We also wish to analyze the case where the splitting field of a polynomial is unramified over $K_p$. The splitting field can sometimes be unramified, even when $\pi_p$ divides the discriminant of the polynomial. Let $R_{2,p}$ be the set of coefficients $(a_2, a_1, a_0)$ of monic quadratic polynomials in $\mathcal{O}_p[x]$ whose splitting fields are unramified over $K_p$. For any $\mu \in T_n$, let $R_{2,p}(\mu)$ be the subset of $R_{2,p}$ associated to polynomials with splitting type $\mu$.

**Theorem 4.2.** Fix a prime $p > 2$ and let $K_p$ be any finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Then

$$\nu_2(p_2(R_{2,p})) = 1 - \frac{1}{q + 1}.$$  

Here

$$\nu_2(p_2(R_{2,p}(1, 1))) = \frac{1}{2} - \frac{1}{2q + 2}$$
$$\nu_2(p_2(R_{2,p}(2))) = \frac{1}{2} - \frac{1}{2q + 2}.$$

**Proof.** As in the previous theorem, we reduce to analyzing polynomials of the form $x^2 + B \in \mathcal{O}_p[x]$ with $B \not\in \pi_p^2 \mathcal{O}_p$. Whenever $B \not\in \pi_p \mathcal{O}_p$, the modulo $p$ reduction of $x^2 + B$ is a square-free polynomial and the splitting field of
The polynomial. Let $R$ in $O$, unramified over $K$. The roots is trivial, conditioned on requiring that the field extension be unramified, is, for $p > 3$ and any $p$-adic field $K_p$, $\nu(R_{2,p}) = 1 - \frac{q^2 - 1}{q^2 - 1} = 1 - \frac{1}{q + 1}$.

This result shows that the probability $P_{2,p}^{unr}$ that the field generated by the roots is trivial, conditioned on requiring that the field extension be unramified, is, for $p > 3$ and any $p$-adic field $K_p$, $P_{2,p}^{unr} = \frac{\nu(R_{2,p}(1, 1))}{\nu(R_{2,p})} = \frac{1}{2}$.

This conditional probability is independent of $p$ and $K_p$.

4.2. Degree $n = 3$. We use the notation $\left(\frac{a}{b}\right)$ to denote the Legendre symbol.

**Theorem 4.3.** Fix a prime $p > 3$ and let $K_p$ be any finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Then for monic cubic polynomials with $\Omega_p$ coefficients we have

\[ \nu_3(P_{3,p}(id)) = \nu_3(P_{3,p}(1, 1)) = \nu_3(P_{3,p}(2, 1)) = \nu_3(P_{3,p}(3)) = \frac{1}{3} - \frac{q^2 + q + 1}{3(q^4 + q^3 + q^2 + q + 1)} + \frac{1}{2} \left(1 + \left(\frac{q}{3}\right)\right) \frac{q^2 + 1}{q^4 + q^3 + q^2 + q + 1}, \]

and

\[ \nu_3(P_{3,p}(\mathbb{Z}/3\mathbb{Z})) = \nu_3(P_{3,p}(S_3)) = \frac{1}{3} - \frac{q^2 + q + 1}{3(q^4 + q^3 + q^2 + q + 1)} + \frac{1}{2} \left(1 \left(\frac{q}{3}\right)\right) \frac{q^2 + 1}{q^4 + q^3 + q^2 + q + 1}. \]

We also wish to analyze the case where the splitting field of a polynomial is unramified over $K_p$. This can occur even when $\pi_p$ divides the discriminant of the polynomial. Let $R_{3,p}$ be the set of coefficients of monic cubic polynomials in $\Omega_p[x]$ whose splitting fields are unramified over $K_p$. For any $\mu \in T_n$, let $R_{3,p}(\mu)$ be the subset of $R_{3,p}$ associated to polynomials with splitting type $\mu$. 
Theorem 4.4. Fix a prime $p > 3$ and let $K_p$ be any finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Then

$$\nu_{3,p}(R_{3,p}) = 1 - \frac{q^2 + q + 1}{q^4 + q^3 + q^2 + q + 1}$$

and

$$\nu_{3,p}(R_{3,p}(1, 1, 1)) = \frac{1}{6} - \frac{3q^4 + 3q^3 + 2q^2 + 2q + 1}{6(q + 1)(q^4 + q^3 + q^2 + q + 1)},$$

$$\nu_{3,p}(R_{3,p}(2, 1)) = \frac{1}{2} + \frac{q^4 - 3q^3 - 2q^2 - q - 1}{2(q + 1)(q^4 + q^3 + q^2 + q + 1)},$$

$$\nu_{3,p}(R_{3,p}(3)) = \frac{1}{3} - \frac{q^2 + q + 1}{3(q^4 + q^3 + q^2 + q + 1)}.$$

Note as well that by definition

$$\nu_{3,p}(R_{3,p}) = \nu_{3,p}(R_{3,p}(1, 1, 1)) + \nu_{3,p}(R_{3,p}(2, 1)) + \nu_{3,p}(R_{3,p}(3)).$$

This result shows that the probability $P_{3,p}^{unr}$ that the field generated by the roots is trivial, conditioned on requiring that the field extension be unramified, is, for $p > 3$ and any $p$-adic field $K_p$,

$$P_{3,p}^{unr} = \frac{\nu_{3,p}(R_{3,p}(1, 1, 1))}{\nu_{3,p}(R_{3,p})} = \frac{q^2 - q + 1}{6(q^2 + 2q + 1)} = 1 - \frac{q}{2(q^2 + 2q + 1)}.$$

In order to prove these results, we will need the following lemma.

Lemma 4.5. Fix a prime $p > 3$, let $K_p$ be any finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, and fix $a \in (\mathcal{O}_p/(p)\mathcal{O}_p)^*$. Then for any $n \geq 1$ there are $q^{2n-2}$ polynomials $g(x) = x^3 + Ax + B \in \mathcal{O}_p/(p)^n\mathcal{O}_p[x]$ which satisfy $g(x) \equiv (x + a)^2(x - 2a) \mod (p)$. If $n$ is even then

$$\frac{q^{n-1}}{2} \left( \frac{q^{n-1} - q}{q + 1} \right) + q^{n-1}$$

of these polynomials $g(x)$ are expressible as the product of three linear factors, and if $n$ is odd then

$$\frac{q^{n-1}}{2} \left( \frac{q^{n-1} - 1}{q + 1} \right) + q^{n-1}$$

of these polynomials are expressible as the product of three linear factors.

Let $A, B \in \mathcal{O}_p$ such that $x^3 + Ax + B \equiv (x + a)^2(x - 2a) \mod (p)$. The ratio of the measure in $(\mathcal{O}_p)^2$ of the subset of such polynomials that factor into three linear factors in $\mathcal{O}_p[x]$ to the measure of all such $(A, B)$ is exactly

$$\frac{1}{2q+2}.$$

Proof. There are $q^{2n-2}$ factorizations $(x - \alpha)(x - \beta)(x - \gamma) \in \mathcal{O}_p/(p)^n\mathcal{O}_p[x]$ with $\alpha + \beta = -\gamma$ and $\alpha \equiv \beta \equiv a \mod (p)$. However $\mathcal{O}_p/(p)^n\mathcal{O}_p[x]$ is not a unique factorization domain. We call two different triples $(\alpha, \beta, \gamma), (\alpha', \beta', \gamma') \in \mathcal{O}_p/(p)^n\mathcal{O}_p$ equivalent if $(x - \alpha)(x - \beta)(x - \gamma) = (x - \alpha')(x - \beta')(x - \gamma')$. 


\( \gamma' \). To count the total number of distinct polynomials \( x^3 + ax + b \in \mathcal{O}_p/(p)^n \mathcal{O}_p[x] \) which are expressible as a product of three linear factors, we will count equivalence classes of triples \((\alpha, \beta, \gamma) \in \mathcal{O}_p/(p)^n \mathcal{O}_p\) with \((\alpha, \beta, \gamma) \equiv (a, a, -2a) \mod\ (p)\), and \(\alpha + \beta + \gamma = 0\).

If two such triples \((\alpha, \beta, \gamma)\) and \((\alpha', \beta', \gamma')\) are equivalent, then \(\gamma\) must equal \(\gamma'\). This is because Hensel’s lemma guarantees a unique lift of \((x - 2a)\) to \(\mathcal{O}_p/(p)^n \mathcal{O}_p[x]\). We are allowed to apply Hensel’s lemma because the derivative of \(x^3 + ax + b\) is \((x + a)(3x - 3a)\). Plugging in \(x = 2a\) we get \(9a^2\) which is not zero because \(p > 3\).

Additionally, for the triples to be equivalent, the coefficients of the two polynomials must be equal. In particular

\[
(x - \alpha)(x + \alpha + \gamma)(x - \gamma) = (x - \alpha')(x + \alpha' + \gamma)(x - \gamma).
\]

So the constant terms must agree, which is equivalent to:

\[
\alpha \gamma (\alpha + \gamma) = \alpha' \gamma (\alpha' + \gamma)
\]
\[
\iff \alpha (\alpha + \gamma) = \alpha' (\alpha' + \gamma)
\]
\[
\iff (\alpha - \alpha') (\alpha' - \beta) = 0.
\]

Also the linear terms must be equal:

\[
\alpha^2 + \alpha \gamma + \gamma^2 = (\alpha')^2 + \alpha' \gamma + \gamma^2,
\]

which is clearly equivalent to the same condition.

For any integer \(r \in [1,n - 1]\) and any of the \(q^n - 1\) choices of \(\alpha\), there are \(q^{n-r} - q^{n-r-1}\) different elements \(\beta\) so that \(\pi_p^r\) is the largest power of \(\pi_p\) dividing \(\alpha - \beta\). If \(r < \frac{n}{2}\) then \(\alpha - \alpha' (\beta - \alpha') \equiv 0 \mod\ (p)^n\) if and only if \(\alpha' \equiv \alpha \mod\ (p)^{n-r}\) or \(\alpha' \equiv \beta \mod\ (p)^{n-r}\); there are \(2q^r\) such \(\alpha'\). If \(r \geq \frac{n}{2}\), then \((\alpha - \alpha')(\beta - \alpha') \equiv 0 \mod\ (p)^n\) if and only if \(\alpha' \equiv \alpha \equiv \beta \mod\ (p)\left[\frac{n}{2}\right]\), and there are \(q\left[\frac{n}{2}\right]\) such \(\alpha' \in \mathcal{O}_p/(p)^n \mathcal{O}_p\).

Thus, for any of the \(q^{n-1} q\left[\frac{n}{2}\right]\) triples \((\alpha, \beta, \gamma)\) with \(\pi_p\left[\frac{n}{2}\right] | \alpha - \beta\), there are \(q\left[\frac{n}{2}\right]\) equivalent triples \((\alpha', \beta', \gamma')\). So there are only \(q^{n-1}\) distinct polynomials with \(\pi_p\left[\frac{n}{2}\right] | \alpha - \beta\).

For any \(1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor\) there are \(q^{n-1}(q^{n-r} - q^{n-r-1})\) triples \((\alpha, \beta, \gamma)\) with \(\pi_p^r | \alpha - \beta\) and \(\pi_p^{r+1} \nmid \alpha - \beta\). Every such triple is equivalent to \(q^r\) other triples \((\alpha', \beta', \gamma')\). So there are \(q^{n-r-1}(q^{n-r} - q^{n-r-1})\) distinct polynomials with \(\pi_p^r | \alpha - \beta\) and \(\pi_p^{r+1} \nmid \alpha - \beta\).
So the total number of equivalence classes of triples \((\alpha, \beta, \gamma)\) is

\[
q^{n-1} + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} q^{n-r-1} (q^{n-r} - q^{n-r-1}) = q^{n-1} + q^n (q^{n-1} + q^{n-2}) \frac{1 - q^2 \lfloor \frac{n}{2} \rfloor}{1 - q^{-2}}
\]

\[
= \begin{cases} 
\frac{q^{n-1} - 1}{2} \frac{q^{n-1} - 1}{q + 1} + q^{n-1} & \text{if } n \text{ is odd} \\
\frac{q^{n-1} - 1}{2} \frac{q^{n-1} - 1}{q + 1} + q^{n-1} & \text{if } n \text{ is even} 
\end{cases}
\]

The limit of the probabilities a polynomial \(x^3 + Ax + B \in \mathcal{O}_p/(p)^n \mathcal{O}_p[x]\) factors into three linear polynomials, given that \(x^3 + Ax + B \equiv x^3 + ax + b \pmod{(p)}\) is equal to the desired ratio in \((\mathcal{O}_p)^2\). From the previous part of the lemma, when \(n\) is odd this limit is

\[
\lim_{n \to \infty} q^{-2n+2} \left[ \left( \frac{q^{n-1}}{2} \right) \left( \frac{q^{n-1} - 1}{q + 1} \right) + q^{n-1} \right] = \frac{1}{2q + 2}
\]

When \(n\) is even the limit is

\[
\lim_{n \to \infty} q^{-2n+2} \left[ \frac{q^{n-1}}{2} \left( \frac{q^{n-1} - 1}{q + 1} \right) + q^{n-1} \right] = \frac{1}{2q + 2}
\]

\(\square\)

of Theorem 4.3. The proof of this theorem proceeds much like Theorem 4.1. We again use Lemma 2.3 to reduce to the case of \(f(x) = x^3 + Ax + B \in \mathcal{O}_p[x]\). If \(A \in \pi_p^2 \mathcal{O}_p\) and \(B \in \pi_p^3 \mathcal{O}_p\) then the polynomial \(\pi_p^{-3} f(\pi_p x)\) is in \(\mathcal{O}_p[x]\) and has the same splitting type and Galois group as \(f(x)\). Thus if \(m\) is the largest integer such that \(\pi_p^{3m} B\) and \(\pi_p^{2m} A\), which is a \(q^{-5m} - q^{-5m - 5}\) event, then we can instead study \(\pi_p^{-3m} f(\pi_p^m x)\). Thus, like in Theorem 4.1 if \(Y_G\) is the conditional probability that \(f(x)\) has Galois group \(G\) given that \(\pi_p^2 \nmid A\) or \(\pi_p^3 \nmid B\), then

\[
\nu_{3,p}(P_{3,p}(G)) = Y_G \sum_{i=0}^{\infty} \left( \frac{1}{q^{5i}} - \frac{1}{q^{5i+5}} \right) = Y_G.
\]

The same equality holds if we instead studied \(Y_\mu\), the conditional probability that \(f(x)\) had splitting type \(\mu\). Note that the measure of the conditional probability space is \(1 - q^{-5}\).

We use the Newton polygon of the polynomial [21, p. 144, Prop. 6.3] with uniformizing parameter \(\pi_p\) to determine the ramification of the extension generated by \(x^3 + Ax + B\). The polygon is the upper convex hull of the points \((0, \text{ord}_p(B))\), \((1, \text{ord}_p(A))\), and \((3, 0)\). The denominators of the slopes of this polygon correspond to the degree of ramification of the roots of the polynomial, and therefore to the degree of ramification of the splitting field over \(K_p\). We will divide the set of possible pairs \((A, B)\) into cases, and compute the unconditional probability of each case occurring. At the end
we will divide these probabilities by $1 - q^{-5}$ to arrive at the conditional probabilities, $Y_G$ for each subgroup $G \subset S_3$ and $Y_\mu$ for each $\mu \in T_3^*$. 

**Case 1:** $B \in \pi_p \mathcal{O}_p$ and $A \notin \pi_p \mathcal{O}_p$: The probability of choosing a polynomial in this case is $\frac{1}{q} \left(1 - \frac{1}{q}\right)$.

In this case the reduction modulo $(p)$ is $x(x^2 + a)$ where $a \equiv A \mod (p)$. Then this factors into three distinct linear polynomials with probability $\frac{1}{2}$ and $x^2 + a$ is irreducible with probability $\frac{1}{2}$ depending on whether $a$ is a quadratic residue or not. Since the reduction is always square-free, the splitting field of the polynomial is unramified and the splitting type of the quadratic residue or not. Since the reduction is always square-free, the splitting field of the polynomial is unramified and the splitting type of the polynomial must factor as an irreducible quadratic and a linear, and so the splitting type must be $(2, 1)$ and the Galois group must be $\mathbb{Z}/2\mathbb{Z}$. Therefore, this case contributes the following distribution:

$$
\mu(f) = (1, 1, 1) \text{ with probability } \frac{1}{2} \left(\frac{1}{q} \left(1 - \frac{1}{q}\right)\right)
$$

$$
\mu(f) = (2, 1) \text{ with probability } \frac{1}{2} \left(\frac{1}{q} \left(1 - \frac{1}{q}\right)\right)
$$

$$
Gal(K_f/K_p) = Id \text{ with probability } \frac{1}{2} \left(\frac{1}{q} \left(1 - \frac{1}{q}\right)\right)
$$

$$
Gal(K_f/K_p) = \mathbb{Z}/2\mathbb{Z} \text{ with probability } \frac{1}{2} \left(\frac{1}{q} \left(1 - \frac{1}{q}\right)\right).
$$

**Case 2:** $B \in \pi_p^2 \mathcal{O}_p$ and $A \in \pi_p \mathcal{O}_p - \pi_p^2 \mathcal{O}_p$: The probability of choosing a polynomial in this case is $q^{-2} (q^{-1} - q^{-2})$.

If $B \in \pi_p^2 \mathcal{O}_p$ and $A \in \pi_p \mathcal{O}_p - \pi_p^2 \mathcal{O}_p$ then there would be ramification of degree two, as the slopes of the Newton polygon would be $-\text{ord}_{\pi_p}(B) + 1$ and $-\frac{1}{2}$, where $\text{ord}_{\pi_p}(B)$ is the number of powers of $\pi_p$ which divide $B$. Thus the extension would always be a degree 2 extension with Galois group $\mathbb{Z}/2\mathbb{Z}$. This is because two of the roots are in a quadratic ramified extension, and the third is in an unramified extension. Thus the polynomial must factor as an irreducible quadratic and a linear, and so the splitting type must be $(2, 1)$ and the Galois group must be $\mathbb{Z}/2\mathbb{Z}$. Therefore, this case contributes the following distribution:

$$
\mu(f) = (2, 1) \text{ with probability } \frac{1}{2} q^{-2} \left(q^{-1} - q^{-2}\right)
$$

$$
Gal(K_f/K_p) = \mathbb{Z}/2\mathbb{Z} \text{ with probability } \frac{1}{2} q^{-2} \left(q^{-1} - q^{-2}\right).
$$

**Case 3:** $B \in \pi_p^2 \mathcal{O}_p - \pi_p^3 \mathcal{O}_p$ and $A \in \pi_p^3 \mathcal{O}_p$. The probability of choosing a polynomial in this case is $(q^{-2} - q^{-3}) q^{-2}$.

In this case, the slope of the Newton polygon is $-\frac{2}{3}$. Thus there are three degrees of ramification, and $f(x)$ is has splitting type $(3)$. If the discriminant of $f(x)$ is a square then, then the Galois group is $\mathbb{Z}/3\mathbb{Z}$, otherwise it is $S_3$. The discriminant is $-4A^3 - 27B^2$, thus $\pi_p^4$ divides the discriminant. After
factoring that out, we are left with \( \frac{1}{\pi_p^3}(-4A^3 - 27B^2) \) which is not in \( \pi_p \mathcal{O}_p \).

To determine if it is a square, we look at its modulo \((p)\) reduction which is congruent to \(-27B^2/\pi_p^3\). Thus it is a square if and only if \(-27\) is a square modulo \((p)\). If \(f_p\) is even, then every integer is a square modulo \((p)\). Otherwise, \(-27\) is a square if and only if \(-27\) is a square modulo \(p\), which is if and only if \(p \equiv 1 \pmod{3}\). Combining these two facts, \(-27\) is a square modulo \((p)\) if and only if \(q \equiv 1 \pmod{3}\). Therefore, this case contributes the following distribution:

\[
\mu(f) = (3) \text{ with probability } (q^{-2} - q^{-3}) q^{-2},
\]

\[
\text{Gal}(K_f/K_p) = \mathbb{Z}/3\mathbb{Z} \text{ with probability } (q^{-2} - q^{-3}) q^{-2}, \quad \text{if } q \equiv 1 \pmod{3},
\]

\[
\text{Gal}(K_f/K_p) = S_3 \text{ with probability } (q^{-2} - q^{-3}) q^{-2}, \quad \text{if } q \equiv 2 \pmod{3}.
\]

**Case 4:** \(B \in \pi_p \mathcal{O}_p - \pi_p^2 \mathcal{O}_p\) and \(A \in \pi_p \mathcal{O}_p\). The probability of choosing a polynomial in this case is \((q^{-1} - q^{-2}) q^{-1}\).

In this case, the slope of the Newton polygon is \(-\frac{1}{3}\). Like before the splitting type is \((3)\) and the extension has Galois group \(\mathbb{Z}/3\mathbb{Z}\) or \(S_3\) depending again on whether or not \(q\) is a square modulo \(3\). Therefore, this case contributes the following distribution:

\[
\mu(f) = (3) \text{ with probability } (q^{-1} - q^{-2}) q^{-1},
\]

\[
\text{Gal}(K_f/K_p) = \mathbb{Z}/3\mathbb{Z} \text{ with probability } (q^{-1} - q^{-2}) q^{-1}, \quad \text{if } q \equiv 1 \pmod{3},
\]

\[
\text{Gal}(K_f/K_p) = S_3 \text{ with probability } (q^{-1} - q^{-2}) q^{-1}, \quad \text{if } q \equiv 2 \pmod{3}.
\]

**Case 5:** \(B \notin \pi_p \mathcal{O}_p\). The probability of choosing a polynomial in this case is \((1 - q^{-1})\).

In this case there is no ramification, and so the splitting field is unramified over \(K_p\) and the Galois group is completely determined by the splitting type of the polynomial. There are \(q(q - 1)\) different possible reductions modulo \((p)\), and we consider the probability of each of the four possible splitting types of the modulo \((p)\) reduction: \((3), (2, 1), (1, 1, 1), (1^2, 1)\).

**Case 5A:** \(B \notin \pi_p \mathcal{O}_p\), and \(\overline{f}(x)\) has splitting type \((3)\).

We count the number of irreducible polynomials in \(\mathbb{F}_q[x]\) of the form \(x^3 + Ax + B\). There are \(\frac{1}{3}(q^3 - q)\) monic, irreducible cubic polynomials in \(\mathbb{F}_q[x]\). We call two monic polynomials \(f, g \in \mathbb{F}_q[x]\) equivalent if there is a monic linear polynomial \(h \in \mathbb{F}_q[x]\) with \(f = g \circ h\). These equivalence classes are all of size \(q\), since there are \(q\) choices of \(h\) which all give distinct polynomials. All polynomials in the same equivalence class have the same splitting type. There is also a unique polynomial of the form \(x^3 + Ax + B\) in each equivalence class of cubics. This means that there are \(\frac{1}{3}(q^3 - 1)\) irreducible cubic polynomials of the form \(x^3 + Ax + B\) in \(\mathbb{F}_q[x]\). So the probability of being in this subcase, given that the polynomial is in Case 5 is \(\frac{2q + 1}{3q} = \frac{1}{3} + \frac{1}{3q}\).
Therefore, this case contributes the following distribution:

\[ \mu(f) = (3) \text{ with probability } (1 - q^{-1}) \left( \frac{1}{3} + \frac{1}{3q} \right) \]

\[ \text{Gal}(K_f/K_p) = \mathbb{Z}/3\mathbb{Z} \text{ with probability } (1 - q^{-1}) \left( \frac{1}{3} + \frac{1}{3q} \right) \]

\[ \text{Gal}(K_f/K_p) = S_3 \text{ with probability } 0. \]

**Case 5B: \( B \notin \pi_p \mathcal{O}_p \), and \( f(x) \) has splitting type \((2, 1)\).**

We count the number of polynomials \( x^3 + Ax + B \in \mathbb{F}_q[x] \) which have an irreducible quadratic factor. There are \( \frac{1}{2} (q^2 - q) \) irreducible monic quadratic polynomials in \( \mathbb{F}_q[x] \). For each \( x^2 + ax + b \in \mathbb{F}_q[x] \) which is irreducible, the linear polynomial \((x - a)\) is unique such that their product is of the prescribed form. Thus, for each monic irreducible quadratic polynomial in \( \mathbb{F}_q[x] \) there is a unique cubic of the form \( x^3 + Ax + B \) which it divides. Thus there are \( \frac{1}{2} (q^2 - q) \) polynomials \( x^3 + Ax + B \in \mathbb{F}_q[x] \) which have an irreducible quadratic factor. However, whenever \( a = 0 \) this will give that \( B = 0 \), which is not allowed. There are \( \frac{1}{2} (q - 1)^2 \) irreducible quadratics of the form \( x^2 + b \), and so we subtract these from the total count to see that there are \( \frac{1}{2} (q - 1)^2 \) polynomials of the prescribed form which have an irreducible quadratic factor. So the probability of being in this subcase, given that the polynomial is in Case 5 is \( \frac{q - 1}{2q} = \frac{1}{2} - \frac{1}{2q} \).

Therefore, this case contributes the following distribution:

\[ \mu(f) = (2, 1) \text{ with probability } (1 - q^{-1}) \left( \frac{1}{2} - \frac{1}{2q} \right) \]

\[ \text{Gal}(K_f/K_p) = \mathbb{Z}/2\mathbb{Z} \text{ with probability } (1 - q^{-1}) \left( \frac{1}{2} - \frac{1}{2q} \right). \]

**Case 5C: \( B \notin \pi_p \mathcal{O}_p \), and \( f(x) \) has splitting type \((1, 1, 1)\).**

We wish to count the number of polynomials \( x^3 + Ax + B \in \mathbb{F}_q[x] \) which factor as three distinct linear polynomials. These must factor as \((x + a)(x + b)(x - a - b)\) with the following restrictions:

\[ a \neq b, \ a \neq 0, \ b \neq 0, \ b \neq -a, \ b \neq -2a, \ a \neq -2b. \]

There are \( q - 1 \) possible choices for \( a \). Given any \( a \), there are \( q - 5 \) choices for \( b \) which satisfy the restrictions. Additionally, all six possible re-orderings of the triple \((a, b, -a - b)\) produce the same polynomial. So the total number of such polynomials is \( \frac{(q - 1)(q - 5)}{6} \). So the probability of being in this subcase, given that the polynomial is in Case 5 is \( \frac{q - 5}{6q} = \frac{1}{6} - \frac{5}{6q} \).

Therefore, this case contributes the following distribution:

\[ \mu(f) = (1, 1, 1) \text{ with probability } (1 - q^{-1}) \left( \frac{1}{6} - \frac{5}{6q} \right) \]

\[ \text{Gal}(K_f/K_p) = \text{id} \text{ with probability } (1 - q^{-1}) \left( \frac{1}{6} - \frac{5}{6q} \right). \]
Case 5D: \( B \not\in \pi_p \mathcal{O}_p \), and \( f(x) \) has splitting type \((1^2, 1)\).

Any polynomial of the form \( x^3 + Ax + B \in \mathbb{F}_q[x] \) which has a square factor must be of the form \((x + a)^2(x - 2a)\), with \( a \neq 0 \). There are \( q - 1 \) possible choices for \( a \), and so the probability of being in this subcase, given that the polynomial is in Case 5 is \( \frac{1}{q} \).

By Lemma 4.3, a polynomial in \( \mathcal{O}_p[x] \) whose reduction is in Case 5D has splitting type \((1^2, 1, 1)\) with probability \( \frac{1}{2q^2} \), and has splitting type \((2, 1)\) with probability \( 1 - \frac{1}{2q^2} \).

Therefore, this case contributes the following distribution:

\[
\mu(f) = (1, 1, 1) \text{ with probability } \frac{1}{q} (1 - q^{-1}) \left( \frac{1}{2q + 2} \right)
\]

\[
\text{Gal}(K_f/K_p) = \text{id} \text{ with probability } \frac{1}{q} (1 - q^{-1}) \left( \frac{1}{2q + 2} \right)
\]

\[
\mu(f) = (2, 1) \text{ with probability } \frac{1}{q} (1 - q^{-1}) \left( 1 - \frac{1}{2q + 2} \right)
\]

\[
\text{Gal}(K_f/K_p) = \mathbb{Z}/2\mathbb{Z} \text{ with probability } \frac{1}{q} (1 - q^{-1}) \left( 1 - \frac{1}{2q + 2} \right).
\]

Summing over all cases and dividing by \( 1 - \frac{1}{q^5} \) we get the stated equalities.

For example

\[
\nu_{3,p}(P_3,p(1, 1, 1)) = \left[ \frac{1}{1 - q^{-5}} \right] \left[ \frac{1}{2} \left( \frac{1}{q} \left( 1 - \frac{1}{q} \right) \right) + (1 - q^{-1}) \left( \frac{1}{6} - \frac{5}{6q} \right) + \frac{1}{q} (1 - q^{-1}) \left( \frac{1}{2q + 2} \right) \right],
\]

which are the totals from cases 1, 5C, and 5D, and

\[
\nu_{3,p}(P_3,p(3)) = \left[ \frac{1}{1 - q^{-5}} \right] \left( (q^{-2} - q^{-3}) q^{-2} + (q^{-1} - q^{-2}) q^{-1} \right) + (1 - q^{-1}) \left( \frac{1}{3} + \frac{1}{3q} \right),
\]

which are the totals from cases 3, 4, and 5A. \( \square \)

of Theorem 4.4 The proof of this theorem is identical to that of Theorem 4.3, except we will only be summing over the cases where the Newton polygon of the polynomial has integral slopes, which is equivalent to the polygon having unramified splitting field. This excludes cases 2, 3, and 4 from the proof of Theorem 4.3. Polynomials in cases 1 and 5 generate unramified extensions. Therefore we have that

\[
\nu_{3,p}(R_3,p) = \left[ \frac{1}{1 - q^{-5}} \right] \left[ \frac{1}{q} \left( 1 - \frac{1}{q} \right) + 1 - q^{-1} \right],
\]

which comes from the probability of a polynomial being in cases 1 or 5.
Likewise, we calculate $\nu_{3,p}(R_{3,p}(1, 1, 1)), \nu_{3,p}(R_{3,p}(2, 1))$, and $\nu_{3,p}(R_{3,p}(3))$ by summing over cases 1 and 5 and dividing by $1 - q^{-5}$. This gives

\[
\nu_{3,p}(R_{3,p}(1, 1, 1)) = \left[ \frac{1}{1 - q^{-5}} \right] \left[ \frac{1}{2} \left( \frac{1}{q} \left( 1 - \frac{1}{q} \right) \right) \right] + (1 - q^{-1}) \left( \frac{1}{6} - \frac{5}{6q} + \frac{1}{2q^2 + 2q} \right)
\]

\[
\nu_{3,p}(R_{3,p}(2, 1)) = \left[ \frac{1}{1 - q^{-5}} \right] \left[ \frac{1}{2} \left( \frac{1}{q} \left( 1 - \frac{1}{q} \right) \right) \right] + (1 - q^{-1}) \left( \frac{1}{2} - \frac{1}{2q} + \frac{1}{q} - \frac{1}{2q^2 + 2q} \right)
\]

\[
\nu_{3,p}(R_{3,p}(3)) = \left[ \frac{1}{1 - q^{-5}} \right] \left( 1 - q^{-1} \right) \left( \frac{1}{3} + \frac{1}{3q} \right)
\]

\[\square\]

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