Emergent universe and the phantom tachyon model

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Abstract
In this work, I have considered that the universe is filled with normal matter and a phantom field (or tachyonic field). If the universe is filled with a scalar field, Ellis et al have shown that an emergent scenario is possible only for $k = +1$, i.e. for a closed universe. Here I have shown that the emergent scenario is possible for a closed universe if the universe contains the normal tachyonic field. But for a phantom field (or tachyonic field), the negative kinetic term can generate the emergent scenario for all values of $k (=0, \pm 1)$. From recently developed statefinder parameters, the behaviour of different stages of the evolution of the emergent universe has been studied. The static Einstein universe and the stability analysis have been briefly discussed for both phantom and tachyon models.

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Recently, Ellis and Maartens [1] have considered a cosmological model where inflationary cosmologies exist in which the horizon problem is solved before inflation begins, there is no big-bang singularity, no exotic physics is involved and the quantum gravity regime can even be avoided. The inflationary universe emerges from a small static state that has within it the seeds for the development of the microscopic universe and it is called the emergent universe scenario (i.e., modern version and extension of the Lemaître–Eddington universe). They have shown that the emergent scenario is possible only for $k = +1$, i.e., for the closed model. The universe has a finite initial size with a finite amount of inflation occurring over an infinite time in the past and with inflation then coming to an end via reheating in the standard way. There are several features for the emergent universe: (i) the universe is almost static at the finite past ($t \to -\infty$) and isotropic, homogeneous at large scales, (ii) it is ever existing and there is no timelike singularity, (iii) the universe is always large enough so that the classical description of spacetime is adequate, (iv) the universe may contain exotic matter so that
the energy conditions may be violated, (v) the universe is accelerating, etc. An interesting example of this scenario has been given by Ellis et al [2] for a closed universe model with a minimally coupled scalar field $\phi$ and a special form of potential $V(\phi)$ with energy density $\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi)$ and pressure $p = \frac{1}{2} \dot{\phi}^2 - V(\phi)$ and possibly some ordinary matter with equation of state $p = w\rho$ where $-\frac{1}{3} \leq w \leq 1$. There are several works on an emergent universe scenarios [3–5]. Mukherjee et al [6] have considered a general framework for emergent universe model contains a fluid which has an EOS $p = A\rho - B\sqrt{\rho}$ where $A$ and $B$ are constants. Also Campo et al [7] have studied an emergent universe model in the context of self-interacting Brans–Dicke theory. Very recently, Banerjee et al [8] have obtained the emergent universe in the brane-world scenario.

In this work, I have considered that the universe is filled with normal matter and a phantom field [9] (or tachyonic field [10]) instead of the normal scalar field. The Lagrangian of the tachyonic field $\phi$ with potential $V(\phi)$ can be written as $L = -V(\phi)\sqrt{1 - \epsilon \dot{\phi}^2}$ [10]. The phantom field [9] has the property that it has a negative kinetic term so that the ratio between pressure and energy density is always less than $-1$. Here $\epsilon = +1$ represents the normal tachyon and $\epsilon = -1$ represents the phantom tachyon [10]. So my main motivation is that if the universe is filled with a phantom field (or tachyonic field) instead of the normal scalar field, then the emergent scenario is possible for flat, open and closed models.

For a FRW spacetime, the line element is
\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right), \] (1)
where $a(t)$ is the scale factor and $k(=0, \pm 1)$ is the curvature scalar. Now consider the Hubble parameter ($H$) and the deceleration parameter ($q$) in terms of the scale factor as
\[ H = \frac{\dot{a}}{a}, \quad q = -\frac{a\ddot{a}}{a^2} = -1 - \frac{H}{H^2}. \] (2)

We consider that the universe contains normal matter and a phantom field (or tachyonic field). The Einstein equations for the spacetime given by equation (1) are
\[ 3H^2 + \frac{3k}{a^2} = \rho_m + \rho_\phi \] (3)
and
\[ 2\dot{H} + 3H^2 + \frac{k}{a^2} = -(p_m + p_\phi), \] (4)
where $\rho_m$ and $p_m$ are the energy density and pressure of the normal matter connected by the equation of state
\[ p_m = w\rho_m, \quad -1 \leq w \leq 1 \] (5)
and $\rho_\phi$ and $p_\phi$ are the energy density and pressure due to the phantom field (or tachyonic field).

Now consider that there is no interaction between normal matter and the phantom field (or tachyonic field), so the normal matter and the phantom field (or tachyonic field) are separately conserved. The energy conservation equations for normal matter and the phantom field (or tachyonic field) are
\[ \dot{\rho}_m + 3H(p_m + \rho_m) = 0 \] (6)
and
\[ \dot{\rho}_\phi + 3H(p_\phi + \rho_\phi) = 0. \] (7)
From equation (6) we have the expression for energy density of matter as
\[ \rho_m = \rho_0 a^{-3(1+w)}, \]  
(8)
where \( \rho_0 \) is the integration constant. For the emergent universe, the scale factor can be chosen as [6]
\[ a = a_0(\beta + e^{\alpha t})^n, \]  
(9)
where \( a_0, \alpha, \beta \) and \( n \) are positive constants. So the Hubble parameter and its derivatives are given by
\[
H = \frac{n\alpha e^{\alpha t}}{\beta + e^{\alpha t}}, \quad \dot{H} = \frac{n\beta \alpha^2 e^{\alpha t}}{\beta + e^{\alpha t}}^2, \quad \ddot{H} = \frac{n\beta \alpha^3 e^{\alpha t}(\beta - e^{\alpha t})}{(\beta + e^{\alpha t})^3}.
\]
(10)

Here \( H \) and \( \dot{H} \) are both positive, but \( \ddot{H} \) changes sign at \( t = \frac{1}{\alpha} \log \beta \). Thus \( H, \dot{H} \) and \( \ddot{H} \) all tend to zero as \( t \to -\infty \). On the other hand as \( t \to \infty \) the solution gives asymptotically a de Sitter universe.

For the above choice of the scale factor, the deceleration parameter \( q \) (see figure 1) can be simplified to the form
\[ q = -1 - \frac{\beta}{n e^{\alpha t}}. \]  
(11)

- **Phantom field.** The energy density \( \rho_\phi \) and pressure \( p_\phi \) due to the phantom field \( \phi \) are given by
\[ \rho_\phi = -\frac{1}{2} \dot{\phi}^2 + V(\phi), \]  
(12)
and
\[ p_\phi = -\frac{1}{2} \dot{\phi}^2 - V(\phi), \]  
(13)
where \( V(\phi) \) is the relevant potential for the phantom field \( \phi \).

From (3), (4), (12) and (13), we get
\[ \dot{\phi}^2 = 2\ddot{H} + (w + 1)\rho_m - \frac{2k}{a^2} \]  
(14)
and
\[ V(\phi) = \ddot{H} + 3H^2 + \frac{1}{2}(w - 1)\rho_m + \frac{2k}{a^2}. \]  
(15)
From equations (5) and (10), it can be seen that $\dot{H}$ and $(1 + w)$ are always positive. Now the first and second terms of equation (14) are always positive. So from equation (14), it is seen that $\dot{\phi}^2 > 0$ for $k = 0, -1$. But for $k = +1$, $\dot{\phi}^2$ may or may not be positive. In this case, $\dot{\phi}^2$ will be positive if $2\dot{H} + (w + 1)\rho_m > \frac{2a_0^2}{\alpha}$ holds. So for the phantom model, the emergent scenario is possible for flat, open and closed types of universes while for the normal scalar field model the emergent scenario is possible only for the closed universe [2].

From equations (3)–(13), one gets the expressions for $\phi$ and $V$ as

$$
\phi = \int \sqrt{(1 + w)\rho_0 a_0^{-3(1+w)}(\beta + e^{\alpha t})^{-3n(1+w)} + 2n\beta a^2 e^{\alpha t}(\beta + e^{\alpha t})^{-2} - 2k a_0^{-2}(\beta + e^{\alpha t})^{-2n}} dt
$$

(16)

and

$$
V = \frac{1}{2} (w - 1)\rho_0 a_0^{-3(1+w)}(\beta + e^{\alpha t})^{-3n(1+w)} + na^2 e^{\alpha t}(\beta + 3n e^{\alpha t})(\beta + e^{\alpha t})^{-2}
$$

$$
+ 2k a_0^{-2}(\beta + e^{\alpha t})^{-2n}.
$$

(17)

Now it is very difficult to express the phantom field $\phi$ in a closed form, so the potential function $V$ cannot be expressed in terms of $\phi$ explicitly. Now from the numerical investigations, I have plotted $V$ against $\phi$ for some particular values of arbitrary constants ($\alpha = 2$, $\beta = 4$, $w = 1/3$, $n = 4$, $\rho_0 = 3$ and $a_0 = 1$) in figure 2. From the figure, it can be seen that $V$ always increases as $\phi$ increases from negative (at early universe) to positive value (at late universe).

**Tachyonic field.** The energy density $\rho_\phi$ and pressure $p_\phi$ due to the tachyonic field $\phi$ have the expressions

$$
\rho_\phi = \frac{V(\phi)}{\sqrt{1 - \epsilon \phi^2}}
$$

(18)

and

$$
p_\phi = -V(\phi)\sqrt{1 - \epsilon \phi^2},
$$

(19)
where $V(\phi)$ is the relevant potential for the tachyonic field $\phi$. It can be seen that $\frac{\partial V}{\partial \phi} = -1 + \epsilon \phi^2 > -1$ or $< -1$ according to normal tachyon ($\epsilon = +1$) or phantom tachyon ($\epsilon = -1$). From the tachyonic field, I have the expression for $\phi^2$ as

$$\phi^2 = \frac{2k}{\epsilon} - 2H - (1 + w)\rho_m. \quad (20)$$

From equations (5) and (10), it is seen that $\dot{\phi}^2$ is always positive. From equation (20), it is seen that when $\epsilon = -1$, $\dot{\phi}^2$ is always positive for $k = 0, -1$. But for $k = +1$, $\dot{\phi}^2$ may or may not be positive when $\epsilon = -1$. In this case, $\dot{\phi}^2$ will be positive if $2H + (w + 1)\rho_m > \frac{2k}{\epsilon}$ holds. Now when $\epsilon = +1$, $\dot{\phi}^2$ will be positive if $2H + (w + 1)\rho_m < \frac{2k}{\epsilon}$ holds. So from the above discussion, it may be concluded that (i) if the universe contains the normal tachyonic field ($\epsilon = +1$), the emergent scenario is possible only for the closed universe and (ii) if the universe contains the phantom tachyonic field ($\epsilon = -1$), the emergent scenario is possible for flat, open and closed universes.

From equations (18)–(20), one gets the expressions for $\phi$ and $V$ as

$$\phi = \int \left[ \frac{k\alpha^2(\beta + e^{\omega t})^{-2n} - 2n\beta\alpha^2 e^{\omega t}(\beta + e^{\omega t})^{-2} - (1 + w)\rho_0 a_0^{-\frac{3(1+w)}{\epsilon}} (\beta + e^{\omega t})^{-3n(1+w)}}{3k\alpha^2(\beta + e^{\omega t})^{-2n} + 3n^2\alpha^2 e^{\omega t}(\beta + e^{\omega t})^{-2} - \rho_0 a_0^{-\frac{3(1+w)}{\epsilon}} (\beta + e^{\omega t})^{-3n(1+w)}} \right] dt \quad (21)$$

and

$$V = \left[ n\alpha^2 e^{\omega t}(2\beta + 3n e^{\omega t})(\beta + e^{\omega t})^{-2} + k\alpha^2(\beta + e^{\omega t})^{-2n} - w\rho_0 a_0^{-\frac{3(1+w)}{\epsilon}} (\beta + e^{\omega t})^{-3n(1+w)} \right] \times \left[ 3k\alpha^2(\beta + e^{\omega t})^{-2n} + 3n^2\alpha^2 e^{\omega t}(\beta + e^{\omega t})^{-2} - \rho_0 a_0^{-\frac{3(1+w)}{\epsilon}} (\beta + e^{\omega t})^{-3n(1+w)} \right] \quad (22)$$

Now it is very difficult to express the phantom field $\phi$ in a closed form, so the potential function $V$ cannot be expressed in terms of $\phi$ explicitly. Now from the numerical investigations, I have plotted $V$ against $\phi$ for some particular values of arbitrary constants (i) ($\alpha = 1, \beta = 0.5, w = 1/3, n = 1, \rho_0 = 3, a_0 = 1$ and $k = 1$) in figure 3 for the normal

![Figure 3](image-url)
tachyon model ($\epsilon = -1$) and $\alpha = 2$, $\beta = 4$, $w = 1/3$, $n = 4$, $\rho_0 = 3$ and $a_0 = 1$ in figure 4
for the phantom tachyon model ($\epsilon = 1$). From the figures, it can be seen that $V$ always
increases as $\phi$ increases from negative to positive value.

In 2003, Sahni et al [11] have introduced a pair of parameters $\{r, s\}$, called statefinder
parameters. In fact, trajectories in the $\{r, s\}$ plane corresponding to different cosmological
models demonstrate qualitatively different behaviour. The statefinder parameters can
effectively differentiate between different forms of dark energy and provide simple diagnosis
regarding whether a particular model fits into the basic observational data. The above
diagnostic pair has the following form:

$$r = \frac{\ddot{a}}{aH^3} \quad \text{and} \quad s = \frac{r - 1}{3(q - \frac{1}{2})}.$$  \hspace{1cm} (23)

For our model, the parameters $\{r, s\}$ can be explicitly written in terms of $t$ as

$$r = 1 + \frac{\beta [\beta + (3n - 1) e^{\alpha t}]}{n^2 e^{2\alpha t}}, \quad s = -\frac{2\beta [\beta + (3n - 1) e^{\alpha t}]}{3n [2\beta + 3n e^{\alpha t}]} e^{\alpha t}$$ \hspace{1cm} (24)

or, the relation between $r$ and $s$ has the form

$$[2\beta (r - 1) - 9s] [3(3n - 2)s + 2n \beta (r - 1)] - 36ns^2 (r - 1) = 0.$$ \hspace{1cm} (25)

Figure 5 shows the variation of $s$ with the variation of $r$ for $\alpha = 2$, $\beta = 4$ and $n = 4$.
For the emergent universe, $s$ is always negative and $r \geq 1$. The curve shows that the universe
starts from an asymptotic Einstein static era ($r \to \infty$, $s \to -\infty$) and goes to the $\Lambda$CDM
model ($r = 1, s = 0$).

The Einstein static universe is characterized by $k = +1$, $a = \text{constant}$. An initial Einstein
static state of the universe arises if the field $\phi$ starts out in an equilibrium position $V'(\phi_0) = 0$.
Now consider that $V$ is trivial, i.e., a flat potential. Now the general Einstein static model has
$\dot{\rho}_m = 0$ and satisfies the following conditions (using equations (3) and (4)):

$$\frac{1}{2} (1 + 3w) \rho_m - \phi_0^2 = V_0$$

and

$$(1 + w) \rho_m - \phi_0^2 = \frac{2}{a_0^2}$$
for the phantom model and
\[ \frac{1}{2} (1 + 3w) \rho_{\text{m}0} + \frac{(3\epsilon \phi_0^2 - 2)V_0}{\sqrt{1 - \epsilon \phi_0^2}} = 0 \]
and
\[ (1 + w) \rho_{\text{m}0} + \frac{\epsilon \phi_0^2 V_0}{\sqrt{1 - \epsilon \phi_0^2}} = \frac{2}{a_0^2} \]
for the tachyon model.

Now it is easy to show that if the field has no kinetic energy (\(\dot{\phi}_0 = 0\)), the equation of state is \(w = -1\) and if there is no fluid (\(\rho_{\text{m}0} = 0\)), the equation of state is \(w = -\frac{1}{3}\) for both phantom and tachyon models. Also it is easy to show that the pure scalar field case is equivalent to the case of \(w = 1\).

Now consider the effect of inhomogeneous density perturbations on the simple one-component fluid model. Following Gibbons [12] and Barrow et al [13], it is easy to show that the speed of sound \(c_s\) must satisfy the inequality \(c_s^2 > \frac{1}{5}\) for the stable case. Harison [14] has discussed stability for the radiation filled model and instability for the dust filled model. Thus, the Einstein static universe with a fluid that satisfies the above inequality, is neutral stable against adiabatic density perturbations of the fluid for inhomogeneous models with no scalar fields. Now consider scalar field perturbation with non-flat potential \(V(\phi)\) with the initial Einstein static state at \(\phi = \phi_0\). Following Barrow et al [13], it is easy to show that the stability is not significantly changed for scalar field perturbation in both phantom and tachyon models.

In this work, I have considered that the universe is filled with normal matter and a phantom field (or tachyonic field). The phantom field has the property that it has a negative kinetic term so that the ratio between pressure and energy density is always less than \(-1\). It has been seen that the emergent scenario is possible for flat, open and closed universes if the universe contains the normal phantom field or phantom tachyonic field. But if the universe contains the normal tachyonic field, the emergent scenario is possible only for the closed universe. In these cases, the field \(\phi\) starts from a negative value at an early stage and ends in a positive value at the late stage. Figures 2 and 4 show that \(V\) increases as \(\phi\) increases. For this emergent
universe, s is always negative and decreases as r increases. The \( \{r, s\} \) diagram in figure 5 shows that the evolution of emergent universe starts from an asymptotic Einstein static era \( (r \rightarrow \infty, s \rightarrow -\infty) \) and goes to the \( \Lambda \)CDM model \( (r = 1, s = 0) \).

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