Planscherel Measure on $E_q(2)$

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Abstract

Following the construction of the invariant integral and the scalar product for the quantum Euclidean group $E_q(2)$, we obtained the full matrix elements of its unitary irreducible representations from $SU_q(2)$ by contraction and then derived the Planscherel measure.

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1 Introduction

If the space–time is not a differentiable manifold but a non–commutative geometry there may be interesting new physical effects. Because of the nonexistence of sufficiently developed differential calculus on the non–commutative geometry to study physical effects one needs new mathematical tools. Since on the other hand the quantum group spaces are natural examples for the non–commutative geometries it is of interest to develop an algorithm that can be employed in physical applications. In that direction we have recently investigated the Green function on $SU_q(2)$ [1]. Purpose of the present work is to present some tools to be used in the spectral analysis on group $E_q(2)$ or corresponding non–commutative plane $E_q(2)/U(1)$.

In the following section we first review the known invariant integral on $SU_q(2)$ and then present the similar construction for $E_q(2)$.

In Section III the scalar product on $E_q(2)$ is introduced.

Section IV is devoted to the derivation of the matrix elements of $E_q(2)$ irreducible representations from $SU_q(2)$ by contraction.

In Section V we obtain the Planscherel measure on $E_q(2)$.

Most of the known formulae as well as methods which we employ are presented in the Appendix.
2 Invariant Integral on $E_q(2)$

Let us start by reviewing the construction of the invariant integral for $SU_q(2)$ which will guide in $E_q(2)$ case.

**Review of the invariant integral on $SU_q(2)$**

The coordinate functions $x, u, x^*, u^*$ generating the $*$–Hopf algebra $A(SU_q(2))$ satisfy the following relations [2]

$$
\begin{align*}
  u(x) &= qxu, \quad x^* u = q u^* x, \quad uu^* = u^* u, \\
  xx^* + uu^* &= 1, \quad x^* x + q^2 uu^* = 1.
\end{align*}
$$

In $q \to 1$ limit the above relations define the three dimensional sphere $S^3$ which is the topological manifold of $SU(2)$. In $q \neq 1$ case we consider irreducible $*$–representation $\pi$ of the associative algebra $A(SU_q(2))$ in the Hilbert space $L^2(Z)$ with the orthonormal basis $\{|n\rangle\}_{n \in Z}$ [3]

$$
\begin{align*}
  \pi(x) |n\rangle &= (1 - q^{2n})^{1/2} |n - 1\rangle, \\
  \pi(u) |n\rangle &= q^n |n\rangle, \\
  \pi(u^*) |n\rangle &= q^n |n\rangle, \\
  \pi(x^*) |n\rangle &= (1 - q^{2n+2})^{1/2} |n + 1\rangle
\end{align*}
$$

(II.2)

for $0 < q < 1$. We associate a vector $v \in L^2(Z)$ to each point of the “topological space” $SU_q(2)$. Then

$$
 f_v = \langle v | \pi(f) | v \rangle
$$

(II.3)

gives the value of any function $f \in A(SU_q(2))$ at this point. Thus by topological space of the quantum group $SU_q(2)$ we mean the carrier space $L^2(Z)$ of the $*$–representation of $A(SU_q(2))$. By the subspace $X_q$ of $SU_q(2)$ we mean the subspace $H'$ of the Hilbert space $L^2(Z)$. The empty set in $SU_q(2)$ is the empty set in $L^2(Z)$. Two subspaces $X_q$ and $X'_q$ of $SU_q(2)$ are said to have zero intersection ($X_q \cap X'_q = \emptyset$) if the corresponding subspaces $H$ and $H'$ of the Hilbert space $L^2(Z)$ has zero intersection. The union and intersection of subspaces $X_q, X'_q \subset SU_q(2)$ are understood as the union and intersection of corresponding subspaces $H, H' \subset L^2(Z)$.
Consider the linear map \( \mu : SU_q(2) \to [0, \infty) \) defined by
\[
\mu(X_q) = (1 - q^2) \sum_{n \in J} \langle n \mid \pi(\mu_S) \mid n \rangle,
\] (II.4)
where \( J \subset \mathbb{Z} \), such that the vectors \( \mid n \rangle \), \( n \in J \) span the subspace \( H \) of \( \mathcal{L}^2(Z) \) corresponding to the subspace \( X_q \) of \( SU_q(2) \), \( \mu_S \in A(SU_q(2)) \). Since the left hand side of the above expression is positive definite the operator \( \pi(\mu_S) \) must be self–adjoint and positive definite in \( \mathcal{L}^2(Z_0) \). To make the linear map (II.4) a measure on \( SU_q(2) \) we have to impose the additivity condition
\[
\mu(\bigcup_j X_{qj}) = \sum_j \mu(X_{qj}) \quad (II.5)
\]
for the disjoint subspaces \( X_{qj} \) of \( SU_q(2) \). Inspecting (II.2) and (II.4) we conclude that \( \mu_S \) is a polynomial of \( \xi = uu^* \). The measure on \( SU_q(2) \) is then given by
\[
\mu(X_q) = (1 - q^2) \sum_{n \in J} \langle n \mid \pi(\mu_S(\xi)) \mid n \rangle = (1 - q^2) \sum_{n \in J} \mu_S(q^{2n}).
\] (II.6)

By means of the measure \( \mu \) on the quantum group \( SU_q(2) \) we introduce the linear functional \( \psi : A(SU_q(2)) \to \mathbb{C} \) as
\[
\psi(f) = (1 - q^2) \sum_{n=0}^{\infty} \langle n \mid \pi(f \mu_S) \mid n \rangle.
\] (II.7)
If \( f \) is the function of \( \xi \) only we can rewrite (II.7) as
\[
\psi(f) = \int_0^1 f(\xi) d_q^2 \mu_S(\xi).
\] (II.8)
On the other hand the invariant integral on \( SU_q(2) \) is
\[
\psi(f) = \int_0^1 f(\xi) d_q^2 \xi.
\] (II.9)
Comparing to (II.8) we see that if
\[
\mu_S(\xi) = \xi,
\] (II.10)
the linear functional (II.7) defines the invariant integral on $SU_q(2)$. Hence, the invariant measure and invariant integral on the quantum group $SU_q(2)$ are given by
\[
\mu(X_q) = (1 - q^2) \sum_{n \in J} q^{2n} \tag{II.11}
\]
and
\[
\psi(f) = (1 - q^2) \sum_{n=0}^{\infty} \langle n \mid \pi(f \xi) \mid n \rangle \tag{II.12}
\]
respectively. Here $J \subset \mathbb{Z}$, such that the vectors $|n\rangle$, $n \in J$ span the subspace $H$ of $L^2(Z)$ corresponding to the subspace $X_q$ of $SU_q(2)$.

**Invariant integral on $E_q(2)$**

In fashion parallel to $SU_q(2)$ we can define the invariant measure on $E_q(2)$. The irreducible $*$-representation $\pi$ of the algebra of polynomials on $E_q(2)$ is constructed in the Hilbert space $L^2(S)$ of square integrable functions on the circle $S$. In the orthonormal basis $|j\rangle = \frac{1}{\sqrt{2\pi}} e^{ij\psi}$, $-\infty < j < \infty$ we have
\[
\pi(z) |j\rangle = q^{-j} |j-1\rangle, \quad \pi(z^*) |j\rangle = q^{-j-1} |j+1\rangle, \quad \pi(\delta) |j\rangle = |j-2\rangle. \tag{II.13}
\]
Thus as topological space the quantum Euclidean group $E_q(2)$ is equivalent to the Hilbert space $L^2(S)$. The measure on $E_q(2)$ is given by
\[
\mu(X_q) = (1 - q^2) \sum_{j \in J} \langle j \mid \pi(\mu_E) \mid j \rangle, \tag{II.14}
\]
where $\mu_E$ is polynomial of coordinate functions $z$, $z^*$, $\delta^{\pm 1}$; and $J \subset \mathbb{Z}$ such that the vectors $|j\rangle$, $j \in J$, span a basis in the subspace $H \subset L^2(S)$ corresponding to the subspace $X_q \subset E_q(2)$. Since $E_q(2)$ can be obtained from $SU_q(2)$ by contraction we define $\mu_E$ from $\mu_S$ of (II.11) as
\[
\mu_E = \lim_{r \to \infty} (r^2 \mu_S(u_0 u_0^* \frac{z z^*}{r^2})) = z z^* = \rho^2. \tag{II.15}
\]
Thus the invariant measure on $E_q(2)$ is given by
\[
\mu(X_q) = (1 - q^2) \sum_{j \in J} \langle j \mid \pi(\rho^2) \mid j \rangle = N \sum_{j \in J} q^{-2j}. \tag{II.16}
\]
By the virtue of the invariant measure we define the invariant integral on \(E_q(2)\) as

\[
\psi(f) = (1 - q^2) \sum_{j=-\infty}^{\infty} \langle j | \pi(f \rho^2) | j \rangle,
\]  

(II.17)

provided that the left–hand side is finite. If \(f(g) = f(\rho)\) the above expression can be rewritten by means of \(q\)–integral as

\[
\psi(f) = \int_{-\infty}^{\infty} f(\rho) d_q(\rho^2).
\]  

(II.18)

The invariant measure and invariant integral on the quantum group \(E_q(2)\) are then given by

\[
\mu(X_q) = (1 - q^2) \sum_{j \in J} q^{-2j}
\]  

(II.19)

and

\[
\psi(f) = (1 - q^2) \sum_{j=-\infty}^{\infty} \langle j | \pi(f \rho^2) | j \rangle
\]  

(II.20)

respectively. The subset \(J \subset Z\) is defined such that the vectors \(|j\rangle, j \in J\) span the subspace \(H \subset \mathcal{L}^2(S)\) corresponding to the subspace \(X_q\) of \(E_q(2)\).

3 Scalar Product on \(E_q(2)\)

Let \(\Phi(E_q(2))\) be the set of analytic functions on \(E_q(2)\) such that

\[
\psi(ff^*) < \infty,
\]  

(III.1)

where \(\psi\) is the invariant integral (II.20) on \(E_q(2)\).

Recall that the homomorphism

\[
\phi(z) = 0, \quad \phi(z^*) = 0, \quad \phi(\delta) = t
\]  

(III.2)

defines the quantum subgroup \(U(1) \subset E_q(2)\). We have then the decomposition

\[
\Phi(E_q(2)) = \bigoplus_{ij \in Z} \Phi[i, j]
\]  

(III.3)

6
where
\[
\Phi[i, j] = \{ f \in \Phi(E_q(2)) : L(f) = t^i \otimes f; \quad R(f) = f \otimes t^j \} \tag{III.4}
\]
and
\[
L = (\phi \otimes id) \circ \Delta, \quad R = (id \otimes \phi) \circ \Delta. \tag{III.5}
\]
The subspace \( \Phi[i, j] \subset \Phi(E_q(2)) \) consists of the elements of the following form
\[
f_{ij}(g) = \delta^j z^i j f_{ij}(\rho^2), \quad \text{for } i \geq j \tag{III.6}
\]
and
\[
f_{ij}(g) = \delta^j (z^*)^i j f_{ij}(\rho^2), \quad \text{for } i \leq j, \tag{III.7}
\]
where \( \rho^2 = zz^* \).

By means of the invariant integral we introduce in \( \Phi(E_q(2)) \) the bilinear forms
\[
(f, f')_L = \psi(f^* f') \tag{III.8}
\]
and
\[
(f', f)_R = \psi(f' f^*) \tag{III.9}
\]
related to each other as
\[
(f, f')_L = (\tau(f'), f)_R, \tag{III.10}
\]
where \( \tau \) is the automorphism in \( E_q(2) \) defined as
\[
\tau(z) = qz, \quad \tau(z^*) = q^{-2}z^*, \quad \tau(\delta) = q^{-4}\delta \tag{III.11}
\]
By the virtue of (III.6) and (III.7) we get
\[
\tau(f) = q^{-2(i+j)} f \tag{III.12}
\]
for \( f \in \Phi[i, j] \). To prove the identity (III.10) we use the following representation for the invariant integral
\[
\psi(f) = (1 - q^2) Tr(f \rho^2), \tag{III.13}
\]
which is the result of (II.20). In this representation the equality (III.10) reads
\[
Tr(f^* f' \rho^2) = Tr(\tau(f') f^* \rho^2). \tag{III.14}
\]
Due to the decomposition (III.3) it is enough to verify that (III.13) is valid for \( f \in \Phi[i, j] \) and \( f' \in \Phi[i', j'] \). The latter can easily be verified by using (III.6), (III.7), (III.12) and (B.7).

The representation (III.13) of the invariant integral allows us to prove that the bilinear forms (III.8) and (III.9) are scalar products in \( \Phi(E_q(2)) \). For that purpose we put \( F = f \rho \) and \( F' = f' \rho \) in (III.8) and using (III.13) we get

\[
(f, f')_L = (1 - q^2) Tr(F^*F').
\] (III.15)

The left hand side of the above equality defines the scalar product in the space of Hilbert–Schmidt type operators. Thus the bilinear form (III.8) defines the scalar product in \( \Phi(E_q(2)) \). In a similar fashion one can show that bilinear form (III.9) is also scalar product.

From (III.6), (III.7), (III.13) and (II.20) we have

\[
(f_1, f_2)_{L,R} = 0,
\] (III.16)

for \( f_1 \in \Phi[i, j], f_1 \in \Phi[i', j'] \) such that \( (i, j) \neq (i', j') \). Thus the decomposition (III.3) is orthogonal with respect to the scalar products (III.8) and (III.9).

4 Matrix Elements of the Irreducible Representations of \( E_q(2) \) from \( SU_q(2) \) by Contraction

Unitary representations of \( E_q(2) \) are previously studied [4]. We now show that the matrix elements of the unitary irreducible representations of \( E_q(2) \) can be obtained from the ones of \( SU_q(2) \) by contraction. This example shows that there exists \( q \)-analog of contraction procedure of classical groups representations which was investigated in [5].

In Appendix C we review the derivation of the matrix elements of the unitary irreducible representations of \( E(2) \) from \( SU(2) \) by the following contraction procedure

\[
t^p_{ij}(\phi, \rho, \zeta) = \lim_{l \to \infty} t^l_{ij}(\phi, \rho/l, \zeta - \phi),
\] (IV.1)
where one puts $\theta = \rho \rho / l$ in the decomposition (C.5) of $SU(2)$. The quantum analog of the above procedure is

$$t_i^p = \lim_{l \to \infty} t_i^l (x_0, y_0, \frac{pv_0}{l}, \frac{pu_0}{l}),$$  \hspace{1cm} (IV.2)$$

where $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ and $t_i^j (x, y, v, u)$ are the matrix elements of the unitary irreducible representation of $SU_q(2)$ (see [3] and references therein):

$$t_i^l = \lambda^{l+i-j} v_i^j \phi_1 (q^{-2l+i-j}; q^2 l+1; q^2 l-1, q^2 uv),$$  \hspace{1cm} (IV.3)$$

with $i + j \leq 0$ and $j \leq i$ and

$$\lambda^{l+j} = q^{l+i-j} \frac{[l+j]! [l-j]!}{[l-i]! [l+i]!}.$$

(IV.4)

Here $\phi_1$ and $[,]_q$ are the $q$–hypergeometric function and the $q$–binomial coefficients respectively. We first calculate

$$\lim_{l \to \infty} \lambda^{l+i-j} x_0^i v_0^j = \frac{q^{m^2+2m+i+j}}{[i-j]} p^{i-j} x_0^i v_0^j$$  \hspace{1cm} (IV.5)$$

and

$$\lim_{l \to \infty} \phi_2 (q^{-2l+i-j}; q^2 l+1; q^2 l-1, q^2 ; p^{2 v_0 v_0^*}[l]) = [i-j] \sum_{k=0}^{\infty} (-1)^k \frac{[k]![k+i-j]}{[k]!(k+i-j)!} (q^{i-j} p^{2 z z^*})^k.$$  \hspace{1cm} (IV.6)$$

We then combine them to arrive at

$$t_i^p (g) = (iq^{-1/2} i^{-j} \delta^{-j/2} (p z^*)^{i-j} \mathcal{J}_{i-j} (p^2 zz^*) \delta^{-j/2}, \hspace{1cm} i \geq j.$$  \hspace{1cm} (IV.7)$$

For $i \leq j$ on the other hand one obtains

$$t_i^p (g) = (-iq^{1/2} i^{-j} \delta^{-j/2} \mathcal{J}_{j-i} (p^2 zz^*) (p z)^{2-i-j} \delta^{-j/2}.$$  \hspace{1cm} (IV.8)$$

In the above formulae $\mathcal{J}_j$ are the $q$–Bessel functions given by

$$\mathcal{J}_j (x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]!(k+j)!} (q^{-j} x)^k.$$  \hspace{1cm} (IV.9)$$
Before closing this section for the sake of completeness we like to present the already known formulae for the right and left representations of the quantum algebra $U_q(e(2))$ obtained from the representation of $U_q(su(2))$ by contraction [8]

\[ \mathcal{R}(E_{\pm})t_{ij}^p = pt_{i\pm 1,j}^p, \quad \mathcal{R}(k)t_{ij}^p = q^{-1}t_{ij}^p \quad (IV.10) \]

and

\[ \mathcal{L}(E_{\pm})t_{ij}^p = pt_{i,j+1}^p, \quad \mathcal{L}(k)t_{ij}^p = q^{-j}t_{ij}^p. \quad (IV.11) \]

5 Planscherel Measure on $E_q(2)$

The comultiplication

\[ \Delta : \Phi(E_q(2)) \rightarrow \Phi(E_q(2)) \otimes \Phi(E_q(2)) \quad (V.1) \]

defines the regular representations of the quantum group $E_q(2)$ in $\Phi(E_q(2))$. Since the scalar product in $\Phi(E_q(2))$ is defined by means of the invariant integral this representation is unitary. The linear space $\Phi(E_q(2))$ is common invariant dense domain for the set of linear operators $\mathcal{R}(\phi)$, $\mathcal{L}(\phi)$, $\phi \in U_q(e(2))$. Since the representation $\Phi(E_q(2))$ is unitary the representatives of $\mathcal{R}(E_+E_-)$, $\mathcal{R}(H)$, $\mathcal{L}(H)$ will be at least symmetric operators in $\Phi(E_q(2))$. From (IV.10), (IV.11) as well as from

\[ \mathcal{R}(E_+E_-)t_{ij}^p = p^2t_{ij}^p \quad (V.2) \]

we see that the matrix elements $t_{ij}^p$ are eigenfunctions of these operators. Thus, the eigenfunctions $t_{ij}^p$ and $t_{ij'}^{p'}$ corresponding to different eigenvalues are orthogonal

\[ (t_{ij}^p, t_{ij'}^{p'})_L = c_{ij}^p(p)\delta_{ii'}\delta_{jj'}\delta(p - p') \quad (V.3) \]

\[ (t_{ij}^p, t_{ij'}^{p'})_R = c_{ij}^p(p)\delta_{ii'}\delta_{jj'}\delta(p - p') \quad (V.4) \]

where $c_{ij}^p(p)$ and $c_{ij}^p(p)$ are the normalization constants. Due to the existence of the delta function the matrix elements of the unitary irreducible representations of $E_q(2)$ in $\Phi(E_q(2))$ do not belong to $\Phi(E_q(2))$. This is natural feature of the non–compact groups.
Since \( \tau(t_{ij}^p) = q^{2(i+j)}t_{ij}^p \) from (III.10) and the above orthogonality conditions we obtain

\[
c_{ij}^r(p) = q^{2(i+j)}c_{ij}^r(p). \tag{V.5}
\]

From the explicit expressions (IV.7) and (IV.8) for the matrix elements we have

\[
t_{ij}^{p_0} = \beta(t_{ij}^p) \tag{V.6}
\]

with \( \beta \) being the automorphism in the quantum group \( E_q(2) \) defined as

\[
\beta(z) = p_0 z, \quad \beta(\delta) = \delta, \quad \beta(z^*) = p_0 z^*; \quad p_0 \in (0, \infty). \tag{V.7}
\]

To preserve the unitarity of \( E_q(2) \) representations we choose \( p_0 \) to be real and positive. Using the explicit expressions for the Hermitian forms (III.8) and (III.9) we arrive at

\[
(t_{ij}^p, t_{i'j'}^{p'})_{R,L} = p_0^2 (t_{ij}^{p_0}, t_{i'j'}^{p_0})_{R,L} \tag{V.8}
\]

which implies

\[
c_{ij}^{r,l}(p) = p_0 c_{ij}^{r,l}(p_0) \tag{V.9}
\]

for any \( p_0 \in (0, \infty) \) and \( i, j \in \mathbb{Z} \). Thus the normalization coefficients can be represented as

\[
c_{ij}^{r,l}(p) = \frac{1}{p} a_{ij}^{r,l}, \tag{V.10}
\]

where the coefficients \( a_{ij}^{r,l} \) do not depend on \( p \).

From the unitarity of the left regular representation we have

\[
(\mathcal{R}(E_n^\pm t_{ij}^p, t_{i'j'}^{p'})_R = (t_{ij}^p, \mathcal{R}(E_n^\mp t_{i'j'}^{p'})_R. \tag{V.11}
\]

The above relation implies

\[
a_{i,j}^{r,l} = a_{i \pm n,j}^{r,l} \tag{V.12}
\]

for any \( n \in \mathbb{Z}_0 \). Thus the coefficients \( a_{i,j}^{r,l} \) do not depend on the first subindex \( i \).

\[
a_{i,j}^{r,l} = c_{j}^{r} \tag{V.13}
\]

In a similar fashion using the unitarity of the right regular representation we get

\[
a_{i,j}^{r,l} = c_{i}^{l} \tag{V.14}
\]
which is independent of its second subindex. By the virtue of (V.3) we have
\[ c^l_i = q^{2(i+j)}c^r_j, \]  
which is solved by \( c^l_i = cq^{2i} \) and \( c^r_j = cq^{-2j} \) with \( c = constant \). Thus the matrix elements of the unitary irreducible representations satisfy the following orthogonality conditions
\[ (t^p_{ij}, t^{p'}_{i'j'})_R = \frac{cq^{-2j}}{p} \delta_{ii'} \delta_{jj'} \delta(p - p'), \]  
and
\[ (t^p_{ij} | t^{p'}_{i'j'})_L = \frac{cq^{2i}}{p} \delta_{ii'} \delta_{jj'} \delta(p - p'), \]  
which implies that the Planscherel measure on \( E_q(2) \) is \( p \).

For any function \( f \in \Phi(E_q(2)) \) which can be expressible as the linear combination of the matrix elements \( t^p_{ij} \) we have
\[ f = \frac{1}{c} \int_0^\infty dp \sum_{i,j=-\infty}^\infty q^{2j} \hat{f}^p_{ij} t^p_{ij}, \]  
where
\[ \hat{f}^p_{ij} = (f, t^p_{ij})_R. \]
Appendix

A. Quantum Group $SU_q(2)$ and Algebra $U_q(su(2))$

The quantum group $SU_q(2)$ or the $*$–Hopf algebra $A(SU_q(2))$ is the algebra of polynomials of the coordinate functions $x, x^*, u$ and $u^*$ satisfying the relations (II.1) and the coalgebra operations

$$
\Delta(x) = x \otimes x - qu \otimes u^*, \quad \Delta(u) = x \otimes u + u \otimes x^*,
$$

(A.1)

$$
\varepsilon(x) = 1, \quad \varepsilon(u) = 0
$$

(A.2)

and the antipode

$$
S(x) = x^*, \quad S(x^*) = x, \quad S(u) = -qu, \quad S(v) = -q^{-1}v.
$$

(A.3)

The quantum algebra $U(su_q(2))$ is generated by the elements

$$
\mathcal{E}_\pm, \quad k_\pm = q^{\pm H/4}
$$

(A.4)

satisfying the relations

$$
[\mathcal{E}_+, \mathcal{E}_-] = \frac{k^2 - k^{-2}}{q - q^{-1}}, \quad k\mathcal{E}_\pm = q^{\pm 1}\mathcal{E}_\pm k
$$

(A.5)

involutions

$$
(\mathcal{E}_\pm)^* = \mathcal{E}_{\mp}, \quad k^* = k
$$

(A.6)

and co–algebra operations

$$
\Delta_U(\mathcal{E}_\pm) = \mathcal{E}_\pm \otimes k + k^{-1} \otimes \mathcal{E}_\pm, \quad \Delta_U(k) = k \otimes k,
$$

(A.7)

$$
\varepsilon_U(\mathcal{E}_\pm) = 0, \quad \varepsilon_U(k) = 1
$$

(A.8)

and antipode

$$
S_U(\mathcal{E}_\pm) = -q^{\pm 1}\mathcal{E}_\pm, \quad S_U(k) = k^{-1}.
$$

(A.9)

The quantum algebra $U_q(su(2))$ is in the non–degenerate duality with the quantum group $SU_q(2)$

B. $E_q(2)$ from $SU_q(2)$ by Contraction
Substituting
\[ x \rightarrow x_0, \quad u \rightarrow \frac{1}{r}u_0, \] (B.1)
in the formulas (II.1) and (A.1)→(A.3) we get the relations
\[ u_0u_0^* = u_0^*u_0, \quad u_0x_0 = qx_0u_0, \quad qu_0x_0^* = x_0^*u_0, \]
\[ x_0x_0^* + \frac{1}{r^2}u_0u_0^* = 1, \quad x_0^*x_0 + \frac{q^2}{r^2}u_0u_0^* = 1, \] (B.2)
the coalgebra operations
\[ \Delta x_0 = x_0 \otimes x_0 - \frac{q}{r^2} u_0 \otimes u_0^*, \quad \Delta u_0 = x_0 \otimes u_0 + u_0 \otimes x_0^*, \] (B.3)
\[ \varepsilon(x_0) = 1, \quad \varepsilon(u_0) = 0 \] (B.4)
and the antipode
\[ S(x_0) = x_0^*, \quad S(u_0) = -qu_0. \] (B.5)
Taking \( r \rightarrow \infty \) limit in the above formulae and putting
\[ z = iqx_0u_0, \quad \delta = x_0^2 \] (B.6)
we arrive at
\[ zz^* = q^{-2}z^*z, \quad \delta = \frac{q^2}{r^2} \delta z, \quad z^* \delta = q^2 \delta z^*, \] (B.7)
\[ \delta^* = \delta^{-1}, \] (B.8)
\[ \Delta(z^*) = z^* \otimes 1 + \delta^{-1} \otimes z^*, \quad \Delta(z) = z \otimes 1 + \delta \otimes z, \quad \Delta(\delta) = \delta \otimes \delta, \] (B.9)
\[ \varepsilon(\delta) = 1, \quad \varepsilon(z) = 0, \] (B.10)
and
\[ S(\delta) = \delta^{-1}, \quad S(z) = -\delta^{-1}z, \quad S(z^*) = -\delta z^*. \] (B.11)
The above relations define the quantum Euclidean group \( E_q(2) \) [7].
Due to the duality one obtains \( U_q(e(2)) \) from \( U_q(su(2)) \) also by contraction. Substituting
\[ E_{\pm} \rightarrow rE_{\pm}, \quad k \rightarrow k \] (B.12)
in (A.5)→(A.9) and taking \( r \rightarrow \infty \) limit one gets the relations
\[ [E_+, E_-] = 0, \quad kE_{\pm} = q^{\pm 1}E_{\pm}k, \] (B.13)
the involution
\[ E^*_+ = E_+, \quad k^* = k, \]  
(B.14)

the coalgebra operations
\[ \Delta_U(E_+) = E_+ \otimes k + k^{-1} \otimes E_+, \quad \Delta_U(k) = k \otimes k, \]  
(B.15)

\[ \varepsilon_U(E_+) = 0, \quad \varepsilon_U(k) = 1 \]  
(B.16)

and the antipode
\[ S_U(E_+) = -q^{\pm 1} E_+, \quad S_U(k) = k^{-1}, \]  
(B.17)

C. Unitary Representations of E(2) from SU(2) by Contraction

Matrix elements \( t_{kj}^p \), \( p \in (0, \infty), \ -\infty < k, \ j < \infty \), of the unitary irreducible representation of \( E(2) \) has the following integral representation
\[ t_{kj}^p(g_E) = \frac{e^{-i(k\phi + j(\zeta - \phi))}}{2\pi} \int_0^{2\pi} e^{ip\rho \cos \psi} e^{i(k-j)\psi} d\psi, \]  
(C.1)

where
\[ g_E = \begin{pmatrix} \cos \zeta & -\sin \zeta & \rho \cos \phi \\ \sin \zeta & \cos \zeta & \rho \sin \phi \\ 0 & 0 & 1 \end{pmatrix}. \]  
(C.2)

Matrix elements \( t_{ij}^l \), \( l \in \frac{1}{2} Z_0, \ -l \leq i, j \leq l \), of the unitary irreducible representation of \( SU(2) \) are given by
\[ t_{kj}^l(g_S) = e^{-i(k\phi + j\phi')} P_{kj}^l(\cos \theta), \]  
(C.3)

where \( P_{kj}^l \) is the Jacobi polynomial which has the following integral representation
\[ P_{kj}^l(\cos \theta) = \frac{1}{2\pi} \sqrt{\frac{(l - j)!(l + j)!}{(l - k)!(l + k)!}} \int_0^{2\pi} d\psi e^{ij\psi}(i \sin(\theta/2)e^{i\psi/2} + \cos(\theta/2)e^{-i\psi/2})^{l-k} \\
(i \sin(\theta/2)e^{-i\psi/2} + \cos(\theta/2)e^{i\psi/2})^{l+k} \]  
(C.4)
and
\[
\begin{pmatrix}
\cos \theta/2 & i e^{i(\phi-\phi')/2} \sin \theta/2 \\
i e^{i(\phi'+\phi)/2} \sin \theta/2 & e^{-i(\phi+\phi')/2} \cos \theta/2
\end{pmatrix}
\]
(C.5)

Putting \( \theta = p \rho/l \) in (C.4) for \( l >> 1 \) we have
\[
P_{kj}^l(\cos(p \rho/l)) = \frac{1}{2\pi} \int_0^{2\pi} d\psi e^{i(j-k)\psi} d\psi \left( 1 + \frac{ip\rho}{2l} e^{-i\psi/2} \right)^{l-k} \left( 1 + \frac{ip\rho}{2l} e^{i\phi/2} \right)^{l+k}.
\]
(C.6)

By the virtue of \( \lim_{l \to \infty} (1 + x/l)^l = e^x \) we obtain
\[
\lim_{l \to \infty} P_{kj}^l(\cos(p \rho/l)) = \frac{1}{2\pi} \int_0^{2\pi} e^{ip\rho \cos \psi} e^{i(k-j)\psi} d\psi.
\]
(C.7)

Hence the matrix elements of the unitary irreducible representation of the Euclidean group \( E(2) \) with weight \( p \) can be obtained from those of \( SU(2) \) by the following contraction formula
\[
t_{kj}^p(\phi, \rho, \zeta) = \lim_{l \to \infty} t_{kj}^l(\phi, p \rho/l, \zeta - \phi).
\]
(C.8)
References

[1] Ahmedov, H and Duru, I. H., J. Phys. A: Math. Gen., 31, 5742 (1998).

[2] Faddev, L. D. and Takhtajan, L. A., Lect. Notes Phys., 246, 166 (1986).

[3] Vaksman, L. L. and Soibelman, Y. S., Funkt. Anal. i Prilozhen., 22, 1 (1988).

[4] Vaksman, L. L., Korogodski, L. I., Dokl. Akad. Nauk SSSR, 304, 1036 (1989); Vaksman, L. L., Dokl. Akad. Nauk SSSR, 306, 269 (1989); Bonechi, F., Ciccoli, N., Giachetti, R., Sorace, E., Tarlini, M., Commun. Math. Phys., 175, 161, (1996).

[5] Inönü, E. and Wigner, E. P., Proc. Nat. Acad. Sci., 39, 510 (1956).

[6] Vilenkin, N. Ja. and Klimyk, A. O., Representation of Lie Groups and Special Functions, 3, Dordrecht: Kluwer Akad. Publ., 1992.

[7] Celeghini, E., Giachetti, R., Sorace, E., Tarlini, M., J. Phys. A : Math. Gen., 21, 2548 (1990).

[8] Dobrowski, L. and Sobczyk, J, Lett. Math. Phys., 32, 249 (1994).

[9] Vilenkin, N. Ja., Special Functions and Theory of Group Representations, Translation of Math. Monogr. 22, Amer. Math. Soc. Providence, Rhode Island.