Fock representations of non-centrally extended super-diffeomorphism algebras

T. A. Larsson

Dannemoragatan 10
S-113 44 Stockholm, Sweden
email: tal@hdd.se

Abstract

A class of Fock representations of non-central extensions of the super-diffeomorphism algebra in \((N+1|M)\) dimensions is constructed, by superization of the paper [physics/9705040]. The representations act on trajectories in \((N|M)\)-dimensional superspace, the extra dimension being the parameter along the trajectory. The restrictions to various subalgebras are considered. In particular, the centrally extended superconformal algebra is obtained by restriction to the contact superalgebra \(K(1|1)\). This shows that one of the basic assumptions in superstring theory (the distinguished nature of the superconformal algebra) is incorrect.

1 Introduction

The diffeomorphism group (and its algebra \(diff(N+1)\)) in \(N+1\)-dimensional space-time plays a crucial role in classical physics; suffice it to say, that local differential geometry and general relativity may be phrased in the language of \(diff(N+1)\) modules and intertwiners. However, quantum field theory
is only invariant under the Poincaré group, because until recently it was not known how to build projective $\text{diff}(N+1)$ Fock modules. The problem is that normal ordering of tensor densities gives rise to infinities; also, no central extension exists when $N > 0$. Recall that a module is projective if it admits a group action up to a phase; on the Lie algebra level, this corresponds to a representation of an abelian extension of $\text{diff}(N+1)$. If the phase is local, the resulting extension must transform non-trivially under the diffeomorphism algebra, i.e. it is non-central. Only if the phase is globally constant, the extension is equivalent to a central one.

The first example of a projective Fock representation of $\text{diff}(N+1)$ was found by Eswara-Rao and Moody \cite{6}. Analogous representations of current algebras were previously discovered by the same group \cite{5} \cite{7} \cite{14}. Subsequently the present author uncovered the geometrical meaning of their construction, and greatly generalized it \cite{11}. It turns out that the algebra acts on trajectories in space, the extra time dimension being the parameter along the trajectory. The word is trajectory, not string, because we deal with one-dimensional objects in space-time. It is remarkable that although space and time are treated in a completely different fashion, a proper $\text{diff}(N+1)$ realization on trajectories is obtained. Because the realization is non-linear, normal ordering gives rise to a non-central extension of $\text{diff}(N+1)$; the extension does distinguish between space and time. Dzhumadildaev \cite{4} has classified extensions of $\text{diff}(N+1)$ by irreducible modules (i.e. tensor densities), but only some of the extensions that I found are covered by his theorem. The point is that it is not sufficient to consider irreducible modules; interesting extensions also arise when we consider reducible but indecomposable modules. One example is provided by the modules defined by the relations \eqref{3.36} and \eqref{3.37} below.

The super-diffeomorphism algebra $\text{diff}(N+1|M)$ is the algebra of first-order differential operators in $(N+1|M)$-dimensional super space-time (alternative names: algebra of super vector fields $\text{vect}(N+1|M)$, generalized Witt algebra $W(N+1|M)$). The classical representations of this algebra and its various subalgebras have been worked out in several papers \cite{3} \cite{9} \cite{10} \cite{12} \cite{13}. For information about its bosonic counterpart, see e.g. \cite{8} \cite{17}. Some superalgebras possess a central extension \cite{1} \cite{2} \cite{13} \cite{14}; for a classification see \cite{9}.

The purpose of the present paper is to superize the construction in \cite{11}. It turns out that $\text{diff}(N+1|M)$ has the same type of projective Fock represen-
tations as \( \text{diff}(N+1) \); superization simply amounts to a judicious insertion of minus signs. After some preliminaries in section 2, the main theorem is given in section 3. We prove that there is a classical realization on trajectories in theorem 3.1, and then normal order to describe the extension in theorem 3.4. Section 4 is devoted to subalgebras of \( \text{diff}(N+1|M) \). Clearly, for every subalgebra \( \mathfrak{h} \in \text{diff}(N+1|M) \), we obtain by restriction a Fock representation of an extension of \( \mathfrak{h} \). In general this extension is non-central, but under special circumstances it may reduce to a central extension (or even no extension at all). In this way the centrally extended super-conformal algebra is obtained. This means that the common belief that there is an exceptional algebraic structure underlying string theory is simply wrong. Section 5 contains a brief statement of the corresponding representations for gauge superalgebras, i.e. the higher-dimensional analog of the Kac-Moody superalgebra.

2 Preliminaries

Consider a superalgebra \( \mathfrak{g} \) with basis \( J^a \), where \( \deg J^a \equiv \deg a = 0 \) if \( a \) is even (bosonic) and \( \deg a = 1 \) if \( a \) is odd (fermionic). Let the symbol \((-)^a = (-1)^{\deg a} \). These symbols satisfy an algebra modulo 2: \( a^2 = a \), \( 2a = 0 \). A superalgebra satisfies the condition of graded skewness,

\[
[J^a, J^b] = (-)^{ab}[J^b, J^a],
\]

and the super-Jacobi identity,

\[
(-)^{ac}[J^a, [J^b, J^c]] + (-)^{ba}[J^b, [J^c, J^a]] + (-)^{bc}[J^c, [J^a, J^b]] = 0.
\]

In terms of structure constants, the brackets are

\[
[J^a, J^b] = i f^{ab}_c J^c,
\]

where \( f^{ab}_c = 0 \) unless \( a + b + c = 0 \) mod 2, and

\[
f^{ba}_c = (-)^{ab} f^{ab}_c,
\]

\[
(-)^{ac} f^{bc}_d f^{ad}_e + (-)^{ab} f^{ca}_d f^{bd}_e + (-)^{bc} f^{ac}_d f^{cd}_e = 0.
\]
The supertrace of a matrix $A = (A^{a}_b)$ is $\text{str}(A) = (-)^{Aa + aA^a_b} \text{str}(BA) = (-)^{AB} \text{str}(AB)$. Henceforth, let $\mathfrak{g}$ be a finite-dimensional superalgebra with a graded symmetric Killing metric $\delta^{ab} \propto \text{str}(J^a J^b)$, satisfying $[J^a, \delta^{bc}] = 0$, i.e.

$$
\delta^{ba} = (-)^{ab} \delta^{ab},
(-)^{ac} \delta^{ad} f^{bc}_d = (-)^{ab} \delta^{bd} f^{ca}_d = (-)^{bc} \delta^{ad} f^{ab}_d.
$$

The associated Kac-Moody superalgebra $\hat{\mathfrak{g}}$ reads

$$
[J^a(s), J^b(t)] = i f^{ab} c f^c(s) \delta(s - t) + \frac{k}{2\pi i} \tilde{\delta}^{ab} \delta(s - t),
$$

where $s, t \in S^1$. It is the unique central extension of $\text{map}(1, \mathfrak{g})$, the superalgebra of maps from $S^1$ to $\mathfrak{g}$.

The Virasoro algebra $Vir$ with central charge $c$ has the three equivalent forms

$$
[L_m, L_n] = (n - m) L_{m+n} - \frac{c}{12}(m^3 - m) \delta_{m+n},
$$

$$
[L(s), L(t)] = (L(s) + L(t)) \delta(s - t) + \frac{c}{24\pi i} (\tilde{\delta}(s - t) + \delta(s - t)).
$$

$$
[L_\xi, L_n] = L_{[\xi, n]} + \frac{c}{24\pi i} \int dt \dot{\delta}(\xi(t) \eta(t) - \dot{\xi}(t) \eta(t)).
$$

where $m, n \in \mathbb{Z}$, $s, t \in S^1$, $L_\xi = \int dt \xi(t) L(t)$ and $[\xi, \eta] = \xi \dot{\eta} - (-)^{\xi \eta} \dot{\xi} \eta$. Note our sign convention, which is appropriate for lowest (as opposed to highest) weight representations. The Virasoro algebra is bosonic; time possesses no useful graded generalization. It is compatible with $\hat{\mathfrak{g}}$ in the sense that

$$
[L(s), J^b(t)] = J^b(s) \delta(s - t).
$$

Consider $(N+1|M)$-dimensional super space-time with coordinates $x^\mu$ and partial derivatives $\partial_\mu = \partial / \partial x^\mu$, where greek indices $\mu, \nu = 0, 1,.., N, N + 1,..N + M$, deg $x^\mu = \deg \partial_\mu = \deg \mu$. The coordinate $t = x^0$ is called time. Further, we use latin indices $i, j = 1,.., N, N + 1,..N + M$ for $(N|M)$-dimensional superspace, excluding the time label 0. Let the first $N+1$ coordinates be bosonic (including time), and let the remaining $M$ coordinates be fermionic. Thus, deg $\mu = 0, \mu = 0, 1,.., N$ and deg $\mu = 1, \mu = N+1,.., N+M$. Let $(-)^\mu = (-1)^{\deg \mu}$. The notation is consistent because time is bosonic:
To this super space-time we associate a super-Heisenberg algebra with generators $q^\mu$ and $p_\nu$, satisfying

$$[p_\nu, q^\mu] = \delta^\mu_\nu, \quad [q^\mu, q^\nu] = [p_\mu, p_\nu] = 0.$$  \hfill (2.9)

Note that the brackets are graded: $[q^\mu, p_\nu] = (-)^{\mu\nu}[p_\nu, q^\mu]$.

The brackets of $gl(N+1|M)$ are

$$[T^\mu_\nu, T^\sigma_\tau] = \delta^\sigma_\nu T^\mu_\tau - (-)^{(\mu+\nu)(\sigma+\tau)}\delta^\mu_\tau T^\sigma_\nu.$$ \hfill (2.10)

The fundamental $gl(N+1|M)$ representations are contravariant vectors, covariant vectors, and densities of weight $\kappa$.

$$[T^\mu_\nu, u^\sigma] = \delta^\sigma_\nu u^\mu, \quad [T^\mu_\nu, v_\tau] = -(-)^{(\mu+\nu)\tau}\delta^\mu_\tau v_\nu, \quad [T^\mu_\nu, w] = -\kappa(-)^\mu\delta_\nu^\mu w,$$ \hfill (2.11)

respectively. The action on a general tensor density is given by tensoring.

The associated Kac-Moody algebra $gl(N+1|M)$ reads

$$[T^\mu_\nu(s), T^\sigma_\tau(t)] = (\delta^\sigma_\nu T^\mu_\tau(s) - (-)^{(\mu+\nu)(\sigma+\tau)}\delta^\mu_\tau T^\sigma_\nu(s))\delta(s - t) - \frac{1}{2\pi i}(k_1(-)^\mu\delta^\mu_\tau \delta^\sigma_\nu + k_2(-)^{\mu+\sigma}\delta^\mu_\nu \delta^\sigma_\tau)\delta(s - t)$$

$$[L(s), T^\mu_\nu(t)] = T^\mu_\nu(t)\delta(s - t)$$ \hfill (2.12)

There are two independent central charges $k_1$ and $k_2$, because $gl(N+1|M) \cong sl(N+1|M) \oplus gl(1)$.

The special subalgebra $sl(N+1|M)$ consists of operators satisfying $T^\mu_\mu \equiv 0$. The fundamental representations are as in (2.11), except that $\kappa = 0$. The Kac-Moody algebra $sl(N+1|M)$ has only one independent central extension, and the brackets are as in (2.12) with $k_1 = -(N + 1 - M)k_2$.

The super-diffeomorphism algebra $diff(N+1|M)$ is the algebra of first-order differential operators (i.e. vector fields) in $(N+1|M)$-dimensional super space-time. Locally, such a vector field takes the form $\xi = \xi^\mu(x)\partial_\mu$. The brackets read

$$[\xi, \eta] = \xi^\mu \partial_\mu \eta^\nu \partial_\nu - (-)^{\xi^\mu \eta^\nu} \partial_\nu \xi^\mu \partial_\mu = -(-)^{\xi^\mu}[\eta, \xi]$$ \hfill (2.13)
The divergence of a vector field is
\[
\text{div} \xi = (-)^{\xi_{\mu+1}} \partial_{\mu} \xi^\mu. \tag{2.14}
\]

\(\text{diff}(N+1|M)\) has generators \(\mathcal{L}_\xi\) and brackets
\[
[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi, \eta]}. \tag{2.15}
\]

The classical representations are tensor densities, corresponding to the following expression for \(\mathcal{L}_\xi\).
\[
\mathcal{L}_\xi = \xi^\mu(q) p_\mu + (-)^{(\xi+\mu)^{\nu+1}} \partial_\nu \xi^\mu(q) T^\nu_\mu, \tag{2.16}
\]

where the \(T^\nu_\mu\) satisfy \(gl(N+1|M)\) \(\text{(2.11)}\). One shows by direct calculation that \(\text{(2.12)}\) is satisfied. The representations inherited from \(\text{(2.11)}\) are
\[
[\mathcal{L}_\xi, \Phi^\mu(x)] = -\xi^\mu(x) \partial_\mu \Phi^\nu(x) + (-)^{(\xi+\nu+1)} \partial_\nu \xi^\mu(x) \Phi^\mu(x),
\]
\[
[\mathcal{L}_\xi, \Phi_\nu(x)] = -\xi^\mu(x) \partial_\mu \Phi_\nu(x) - (-)^{(\xi+\nu+1)} \partial_\nu \xi^\mu(x) \Phi_\mu(x),
\]
\[
[\mathcal{L}_\xi, \Phi(x)] = -\xi^\mu \partial_\mu \Phi(x) - \kappa \text{div} \xi(x) \Phi(x), \tag{2.17}
\]

respectively. We sometimes need a alternative version of \(\text{(2.17)}\), acting on \(\Psi^\nu = (-)^{\nu} \Phi^\nu\) and \(\Phi_\nu = (-)^{\nu} \Phi_\nu\).
\[
[\mathcal{L}_\xi, \Psi^\mu(x)] = -\xi^\mu(x) \partial_\mu \Psi^\nu(x) + (-)^{(\xi+\nu+1)} \partial_\nu \xi^\mu(x) \Psi^\mu(x),
\]
\[
[\mathcal{L}_\xi, \Psi_\nu(x)] = -\xi^\mu(x) \partial_\mu \Psi_\nu(x) - (-)^{(\xi+\nu+1)} \partial_\nu \xi^\mu(x) \Psi_\mu(x). \tag{2.18}
\]

The actions on a general tensor densities can be obtained by tensoring, if we keep the extra signs in mind. Thus, if \(\Phi(x)\) and \(\Psi(x)\) are two fields (with indices suppressed),
\[
[\mathcal{L}_\xi, \Phi(x) \otimes \Psi(x)] = [\mathcal{L}_\xi, \Phi(x)] \otimes \Psi(x) + (-)^{\xi \Phi} \Phi(x) \otimes [\mathcal{L}_\xi, \Psi(x)] \tag{2.19}
\]

Explicitly, the transformation law for a tensor field, with all upper indices placed in front of the lower ones, reads
\[
[\mathcal{L}_\xi, \Phi^{\sigma_1 \cdots \sigma_p \tau_1 \cdots \tau_q}(x)] = -\xi^\mu(x) \partial_\mu \Phi^{\sigma_1 \cdots \sigma_p \tau_1 \cdots \tau_q}(x) - \kappa \text{div} \xi(x) \Phi^{\sigma_1 \cdots \sigma_p \tau_1 \cdots \tau_q}(x)
\]
\[
+ \sum_{i=1}^p (-)^{(\sigma_1 + \cdots + \sigma_i)(\mu + \sigma_i)} \xi^\mu \partial_{\mu+\sigma_i} \Phi^{\sigma_1 \cdots \sigma_p \tau_1 \cdots \tau_q}(x)
\]
\[
- \sum_{j=1}^q (-)^{(\sigma_1 + \cdots + \sigma_p + \tau_1 + \cdots + \tau_j + \mu)(\mu + \tau_j)} \xi^\mu \partial_{\tau_j} \Phi^{\sigma_1 \cdots \sigma_p \tau_1 \cdots \tau_q}(x). \tag{2.20}
\]
\[ \text{deg } \Phi^{\sigma_1 \ldots \sigma_p \tau_1 \ldots \tau_q}(x) = \sigma_1 + \ldots + \sigma_p + \tau_1 + \ldots + \tau_q. \]

If \( \Phi^\mu \) and \( \Psi_\nu \) are vector fields, their contraction \( \Phi^\nu(x)\Psi_\nu(x) = (-)^{\nu\nu}\Psi_\nu(x)\Phi^\nu(x) \) is a scalar field. The contraction of higher tensors is defined analogously, but care has to be taken with signs for non-adjacent indices. A tensor is graded symmetric if \( \Phi^{\mu} \) and \( \Psi_\nu \) are vector fields, their contraction \( \Phi^\nu(x)\Psi_\nu(x) = (-)^{\nu\nu}\Psi_\nu(x)\Phi^\nu(x) \) is a scalar field. The contraction of higher tensors is defined analogously, but care has to be taken with signs for non-adjacent indices. A tensor is graded symmetric if \( \Phi^{\nu\mu}(x) = (-)^{\mu\nu}\Phi^{\nu\mu}(x) \) and graded skewsymmetric if \( \Phi^{\nu\mu}(x) = (-)^{\mu\nu}\Phi^{\nu\mu}(x) \). For any set of smearing functions \( f^{\tau_1 \ldots \tau_q} \), define

\[ \Phi(f) = \int d^{N+M+1}x \ f^{\tau_1 \ldots \tau_q} \sigma_1 \ldots \sigma_p(x) \Phi^{\mu_1 \ldots \mu_p \tau_1 \ldots \tau_q}(x). \]

Clearly, \( \text{deg } \Phi(f) = \text{deg } f \). It transforms as \( [\mathcal{L}_\xi, \Phi(f)] = \Phi(\ell_\xi f) \), where

\[ \ell_\xi f^{\tau_1 \ldots \tau_q} \sigma_1 \ldots \sigma_p(x) = \xi^\mu (x) \partial_\mu f^{\tau_1 \ldots \tau_q} \sigma_1 \ldots \sigma_p(x) + (1 - \kappa) \text{div } \xi(x) f^{\tau_1 \ldots \tau_q} \sigma_1 \ldots \sigma_p(x) \]

\[ + \sum_{i=1}^{p} (-)^{\mu + \tau_i} (n + \sigma_i) \partial_\sigma_i \xi^\mu(x) f^{\tau_1 \ldots \tau_q} \sigma_1 \ldots \sigma_p(x) \]

\[ - \sum_{j=1}^{q} (-)^{\mu + \tau_j} (n + \mu + \tau_j) \partial_\mu \xi^{\tau_j}(x) f^{\tau_1 \ldots \tau_q} \sigma_1 \ldots \sigma_p(x). \]

### 3 Realization on trajectories

A trajectory in \((N|M)\)-dimensional superspace is simply a vector-valued function of time, \( q^i(t) \), which satisfies the graded Heisenberg algebra together with its canonically conjugate momentum \( p_j(t) \).

\[ [p_j(s), q^i(t)] = \delta_j^i \delta(s - t), \quad [q^i(s), q^j(t)] = [p_i(s), p_j(t)] = 0. \]

By differentiating with respect to \( t \), we obtain the useful relation

\[ [p_j(s), \dot{q}^i(t)] = -\delta_j^i \delta(s - t). \]

The trajectories may formally be extended to super space-time, by defining time components \( q^0(t) \) and \( p_0(t) \) as

\[ q^0(t) = t, \quad p_0(t) = -\dot{q}^0(t)p_0(t). \]

Clearly, \( \text{deg } q^0 = \text{deg } p_0 = 0 \). The oscillators \( q^\mu(t) = (q^0(t), q^i(t)) \) and \( p_\nu(t) = (p_0(t), p_j(t)) \) satisfy the following algebra.

\[ [p_\nu(s), q^\mu(t)] = (\delta_\nu^\mu - \delta_\nu^0 \dot{q}^\mu(s)) \delta(s - t), \]
\[ [p_\nu(s), \dot{q}^\mu(t)] = -(\delta_\nu^\mu - \delta_\nu^0 \dot{q}^\mu(s))\delta(s-t), \]
\[ [q^\mu(s), q^\nu(t)] = [q^\mu(s), \dot{q}^\nu(t)] = 0 \]
\[ [p_\mu(s), p_\nu(t)] = \left(\delta_\mu^0 p_\nu(s) + \delta_\mu^0 p_\nu(t)\right)\delta(s-t). \]

(3.4)

Note that \( p_0(t) \) satisfy \( \text{diff}(1) \) (2.7), and that \( \dot{q}^0(t) = 1, \ddot{q}^0(t) = \dot{q}^\mu p_\mu(t) = 0 \). Eq. (3.4) is formally the same as in the bosonic case. The only place where the super nature of these relations enter is that the brackets are graded, e.g.
\[ [q^\mu(t), p_\nu(s)] = (-)^{\mu\nu}[p_\nu(s), q^\mu(t)]. \]

(3.5)

It follows that
\[ [p_\mu(s), f(q(t))] = (\partial_\mu f(q(t)) - \delta_\mu^0 \dot{f}(q(t)))\delta(s-t). \]

(3.6)

We assume that super space-time is periodic in the temporal direction. This means that \( \int dt \dot{f}(t) = 0 \) for every (operator-valued) function; in particular, if \( \theta \) is a fermionic coordinate, \( \int dt \dot{\theta}(t) = \int d\theta 1 = 0 \) can be interpreted as a property of the Berezin integral. Moreover, every function can be expanded in a Fourier series. The algebra (3.4) has a natural Fock module, which is obtained from the universal enveloping algebra by introducing a vacuum which is annihilated by all negative frequency modes.

**Theorem 3.1** Let \( L(t) \) satisfy \( \text{diff}(1) \) and let \( T^\nu_\mu(t) \) satisfy \( \text{map}(1, \text{gl}(N+1|M)) \).

Then the following expression provides a realization of \( \text{diff}(N+1|M) \).

\[
\mathcal{L}_\xi = \int dt \xi^\mu(q(t))p_\mu(t) + \xi^0(q(t))L(t) + (-)^{(\xi+\mu)\nu\nu+\mu} \partial_\nu \xi^\mu(q(t))T^\nu_\mu(t)
\]
\[ = \int dt \xi^\mu(q(t))p_\mu(t) - \xi^0(q(t))\dot{q}^\mu(t)p_\mu(t)
\]
\[ + \xi^0(q(t))L(t) + (-)^{(\xi+\mu)\nu\nu+\mu} \partial_\nu \xi^\mu(q(t))T^\nu_\mu(t), \]

(3.7)

where \( \xi^\mu(q(t)) = \xi^\mu(t, q^1(t), ..., q^{N+M}(t)) \).

The proof is as in [11], theorem 3.1, except for the extra signs.  \( \square \)
Hence the following transformation law defines a \( \text{diff}(N+1|M) \) representation.

\[
[\mathcal{L}_\xi, \phi^{\sigma_1 \ldots \sigma_p \tau_1 \ldots \tau_q}(t)] = -\xi^0(q(t))\dot{\phi}^{\sigma_1 \ldots \sigma_p \tau_1 \ldots \tau_q}(t) - \lambda \dot{\xi}^0(q(t))\phi^{\sigma_1 \ldots \sigma_p \tau_1 \ldots \tau_q}(t) + i\omega \xi^0(q(t))\phi^{\sigma_1 \ldots \sigma_p \tau_1 \ldots \tau_q}(t) - \kappa \text{div}(\xi(q(t)))\phi^{\sigma_1 \ldots \sigma_p \tau_1 \ldots \tau_q}(t)
+ \sum_{i=1}^{p} (-)^{(\sigma_1 + \ldots + \sigma_i)(\mu + \sigma_i)}\partial_\mu \xi^{\sigma_i}(q(t))\phi^{\sigma_1 \ldots \mu \ldots \sigma_p \tau_1 \ldots \tau_q}(t)
- \sum_{j=1}^{q} (-)^{(\sigma_1 + \ldots + \sigma_p + \tau_1 + \ldots + \tau_j + \mu)(\mu + \tau_j)}\partial_{\tau_j} \xi^{\mu}(q(t))\phi^{\sigma_1 \ldots \sigma_p \tau_1 \ldots \mu \ldots \tau_q}(t).
\]  \hfill (3.8)

We call \( \phi^{\sigma_1 \ldots \sigma_p \tau_1 \ldots \tau_q}(t) \) a primary trajectory field of type \( \mathcal{P}(\lambda, \omega; \kappa, p, q) \). The trajectory itself transforms as

\[
[\mathcal{L}_\xi, q^\nu(t)] = \xi^\nu(q(t)) - \xi^0(q(t))\dot{q}^\nu(t).
\]  \hfill (3.9)

Its time derivative is a primary trajectory field of type \( \mathcal{P}(1, 0; 0, 1, 0) \). From

\[
[\mathcal{L}_\xi, \dot{q}^\nu(t)] = (-)^{\mu(\mu + \xi + \nu)}\partial_\mu \xi^{\nu}(q(t))\dot{q}^\mu(t) - \dot{\xi}^0(q(t))\dot{q}^\nu(t) - \xi^0(q(t))\dot{q}^\nu(t),
\]  \hfill (3.10)

it follows that \((-)^\nu\dot{q}^\nu(t)\) transforms as in (3.8).

To understand the meaning of theorem 3.1, consider its restriction to the spatial subalgebra generated by time-independent vector fields.

\[
\mathcal{L}_\xi = \int dt \; \xi^i(q(t))p_i(t) + (-)^{(\xi + i)j + i}\partial_j \xi^i(q(t))T^j_i(t)
\]  \hfill (3.11)

This is recognized as the action of infinitesimal diffeomorphisms on extended objects in superspace. There is no need here to limit ourselves to one-dimensional objects; \( t \) could very well have several components. From the algebraic point of view, this action is highly reducible; in fact, for every value of \( t \) we have an independent tensor density (2.16), and thus (3.11) describes a continuous direct sum of tensor densities.

However, for one-dimensional extended objects two miracles occur. First, we can extend the action to \( (N+1|M) \)-dimensional super space-time by means of (3.7). Now \( t \) must be one-dimensional because the time derivative appears, both in \( p_0(t) = -\dot{q}^i(t)p_i(t) \) and in the right-hand side of (2.7).

This realization is no longer obviously reducible, although it still is reducible.
in a more subtle manner. For a trajectory field of type $\mathcal{P}(1, \omega; \kappa, p, q)$ and smearing functions $f^{\tau_1...\tau_q, \sigma_1...\sigma_p}(x)$, define

$$
\phi(f) = \int dt \ f^{\tau_1...\tau_q, \sigma_1...\sigma_p}(q(t)) \phi^{\sigma_1...\sigma_p, \tau_1...\tau_q}(t).
$$

(3.12)

It can be shown that $\phi(f)$ transforms in the same fashion as $\Phi(f)$ (2.21), provided that $\Phi(x)$ has the weight $\kappa - 1$. This phenomenon, which was called correspondance in [11], implies that this type of trajectory field contains a tensor field subrealization.

The second miracle is that one-dimensional objects admit normal ordering, which gives rise to a superalgebra extension. We now proceed to calculate it. Split the delta function into positive and negative energy parts.

$$
\delta^>(t) = \frac{1}{2\pi} \sum_{m>0} e^{-imt}, \quad \delta^<(t) = \frac{1}{2\pi} \sum_{m<0} e^{-imt}.
$$

(3.13)

Lemma 3.2 ([11], Lemma 5.1)

i. $\delta^>(t)\delta^<(-t) - \delta^>(-t)\delta^<(t) = \frac{1}{2\pi i} \delta(t)$

ii. $\delta^>(t)\dot{\delta}^<(-t) - \delta^>(-t)\dot{\delta}^<(t) = \frac{1}{4\pi i} (\ddot{\delta}(t) + i\dot{\delta}(t))$

iii. $\dot{\delta}^>(t)\dot{\delta}^<(-t) - \dot{\delta}^>(-t)\dot{\delta}^<(t) = \frac{1}{12\pi i} (\dddot{\delta}(t) + \ddot{\delta}(t))$

Introduce

$$
\tilde{\xi}^i(t) \equiv \xi^i(q(t), \dot{q}(t)) = \xi^i(q(t)) - \xi^0(q(t))\dot{q}^i(t),
$$

(3.14)

and

$$
\chi^{\tilde{\xi}^i}_{\xi^j}(t, s) \equiv [p^\tilde{\xi}_j, \tilde{\xi}^i(s)] = \partial_j\tilde{\xi}^i(s)\delta^{\tilde{\xi}}(t - s) + (-)^{\tilde{\xi}}\delta^j\xi^0(s)\dot{\delta}^{\tilde{\xi}}(t - s).
$$

(3.15)

Moreover, set $\chi^{\tilde{\xi}^i}_{\xi^j}(t, s) = \chi^{\tilde{\xi}^i}_{\xi^j}(t, s) + \chi^{\tilde{\xi}^i}_{\xi^j}(t, s)$. 

10
Lemma 3.3 The expressions defined in (3.14) satisfy the following relations.

\[ (-\xi^{i+\nu}_\mu) = \text{div } \xi - \dot{\xi}^0 \]  (3.16)

\[ (-\xi^{(i+\nu)}_\mu) = (-\xi^{(i+\nu)}_\mu) \partial_\mu \xi^0_0 \cdot \partial_\mu \eta^\nu \]

\[ (-\xi^{(i+\nu)}_\mu) = (-\xi^{(i+\nu)}_\mu) \partial_\mu \xi^0_0 \cdot \partial_\mu \eta^\nu \]

\[ + \frac{d}{dt}(\xi^0_0 \cdot \eta^\nu) \]  (3.17)

Proof: We use that \( \xi^0_0 \equiv 0 \). Eq. (3.16) thus equals

\[ (-\xi^{(i+\nu)}_\mu) = (-\xi^{(i+\nu)}_\mu) \partial_\mu \xi^0_0 \cdot \partial_\mu \eta^\nu \]

whereas (3.17) becomes

\[ (-\xi^{(i+\nu)}_\mu) = (-\xi^{(i+\nu)}_\mu) \partial_\mu \xi^0_0 \cdot \partial_\mu \eta^\nu \]

\[ (-\xi^{(i+\nu)}_\mu) = (-\xi^{(i+\nu)}_\mu) \partial_\mu \xi^0_0 \cdot \partial_\mu \eta^\nu \]

\[ + \frac{d}{dt}(\xi^0_0 \cdot \eta^\nu) \]  (3.19)

Normal ordering amounts to the replacement

\[ \tilde{\xi}^i(t) p_i(t) \rightarrow : \tilde{\xi}^i(t) p_i^\nu(t) := \tilde{\xi}^i(t) p_i^\nu(t) + (-\xi^{i+\nu}_\mu) p_i^\nu(t) \tilde{\xi}^i(t) \]  (3.20)

Moreover, it also affects the generators of \( dif f(1) \) and \( map(1, gl(N+1|M)) \), replacing these algebras by their central extensions.

Theorem 3.4 Let \( L(t) \) satisfy the Virasoro algebra Vir (2.7) with central charge \( c \) and let \( T_\mu^\nu(t) \) satisfy the Kac-Moody super-algebra \( gl(N+1|M) \) (2.12) with central charges \( k_1 \) and \( k_2 \). The generators

\[ \mathcal{L}_\xi = \int dt :\xi^\mu(q(t)) p_\mu(t) : + \xi^0(q(t)) L(t) + (-\xi^{(i+\nu)}_\mu) \partial_\mu \xi^0(q(t)) T_\mu^\nu(t) \]

\[ = \int dt \tilde{\xi}^i(q(t), \dot{q}(t)) p_i^\nu(t) + (-\xi^{i+\nu}_\mu) \tilde{\xi}^i(q(t), \dot{q}(t)) \]

\[ + \xi^0(q(t)) L(t) + (-\xi^{(i+\nu)}_\mu) \partial_\mu \xi^0(q(t)) T_\mu^\nu(t) \]  (3.21)
satisfy the superalgebra

$$\left[ \mathcal{L}_\xi, \mathcal{L}_\eta \right] = \mathcal{L}_{[\xi,\eta]} + \text{ext}(\xi,\eta).$$  \tag{3.22}$$

The extension is

$$\text{ext}(\xi,\eta) = \frac{1}{2\pi i} \int dt \left\{ (1 + k_1)(-)^{(\xi+\eta)\nu} \partial_\nu \dot{\xi}^\mu(q(t)) \partial_\mu \dot{\eta}^\nu(q(t)) 
+ k_2 \text{div} \dot{\xi}(q(t)) \text{div} \eta(q(t)) 
+ (-)^{(\xi+\eta)\nu} \partial_\nu \xi^0(q(t)) \dot{q}^\rho(t) \partial_\rho \dot{\eta}^\nu(q(t)) - \dot{q}^\rho(t) \partial_\rho \dot{\xi}^\nu(q(t)) \partial_\nu \eta^0(q(t)) 
- \dot{\xi}^0(q(t)) \dot{q}^\rho(t) \partial_\rho \eta^0(q(t)) + \dot{q}^\rho(t) \partial_\rho \dot{\xi}^0(q(t)) \eta^0(q(t)) 
+ \frac{1}{2} \text{div} \dot{\xi}(q(t)) \eta^0(q(t)) - \frac{1}{2} \dot{\xi}^0(q(t)) \text{div} \eta(q(t)) 
- (2 - \frac{1}{12} \xi^0(q(t)) \eta^0(q(t)) 
- \frac{1}{12} \xi^0(q(t)) \eta^0(q(t)) 
+ \frac{i}{2} (\text{div} \dot{\xi}(q(t)) \eta^0(q(t)) - \dot{\xi}^0(q(t)) \text{div} \eta(q(t)))) \right\},$$  \tag{3.23}$$

where \( \dot{f}(q(t)) = \dot{q}^\rho(t) \partial_\rho f(q(t)) \), \( \text{div} \xi \) was defined in \( \text{(2.14)} \) and \( q^\mu(t) \) transforms as in \( \text{(3.9)} \).

Proof: We begin by considering \( \mathcal{L}_\xi^0 = \int dt :\dot{\xi}(t)p_i(t) : \).

$$\left[ \mathcal{L}_\xi, \mathcal{L}_\eta^0 \right] = \iint dsdt \left[ \dot{\xi}(s)p_i^< (s) + (-)^{\xi+i}p_i^> (s) \dot{\xi}(s), \right.
\left. \dot{\eta}^j(t)p_j^> (t) + (-)^{\eta+j}p_j^> (t) \dot{\eta}^j(t) \right]$$

$$= \iint dsdt \left[ \dot{\xi}(s) \chi^{<j}_{\eta i} (s,t)p_j^> (t) 
+ (-)^{\xi(\eta+j)+ij} \eta^j(t)(-)^{(\xi+i)j} \chi^{<j}_{\xi i} (t,s)p_i^< (s) 
+ (-)^{\eta+j} \left\{ (-)^{ij} \dot{\xi}(s)p_j^> (t) \chi^{<j}_{\eta i} (s, t) 
+ (-)^{in} (-)^{j(\xi+i)} \chi^{<i}_{\xi j} (t, s) \eta^j (t)p_i^< (s) \right\} \right].$$
\[ +(-)^{\xi+i+1}\{(\xi^{i})^{\eta} \chi_{\xi}^{+j}(s, t)p_{j}^{\leq}(t)\tilde{\xi}^{i}(s) \]
\[ +(-)^{(\xi+i)(\eta+j)}p_{i}(s)\tilde{\eta}^{j}(t)(-(-)^{(\xi+j)}\chi_{\xi}^{+i}(t, s)) \]
\[ +(-)^{\xi+i+i+j}p_{j}(s)(-(-)^{j(\xi+i)}\chi_{\xi}^{+j}(t, s))\tilde{\eta}^{j}(t) \]
\[ +(-)^{\xi+j(\eta+i)}p_{j}(t)\chi_{\eta}^{+j}(s, t)\tilde{\xi}^{i}(s) \}. \tag{3.24} \]

Of these eight terms, the third can be rewritten as
\[ (-)^{(n+j+i)}\{(\xi^{i})^{\eta} \chi_{\eta}^{+j}(s, t)\tilde{\xi}^{i}(s) - (-)^{(\xi+j)}\chi_{\xi}^{+j}(t, s)\chi_{\eta}^{+i}(s, t) \} \tag{3.25} \]
and the fifth as
\[ (-)^{(\xi+i)(\eta+i)}\{(\xi^{i})^{\eta} \chi_{\eta}^{+j}(s, t)\tilde{\xi}^{i}(s)p_{j}^{\leq}(t) + \chi_{\eta}^{+j}(s, t)\chi_{\eta}^{+i}(t, s) \} \tag{3.26} \]

Hence
\[ [\mathcal{L}_{\xi}^{0}, \mathcal{L}_{\eta}^{0}] = \int ds dt \tilde{\xi}^{i}(s)\chi_{\eta}^{+j}(s, t)p_{j}^{\leq}(t) - (-)^{\xi^{i} \eta^{j}}(t)\chi_{\xi}^{+j}(t, s)p_{i}^{\leq}(s) \]
\[ +(-)^{(\xi^{i} \eta^{j})}p_{j}^{\leq}(t)\tilde{\xi}^{i}(s)\chi_{\eta}^{+j}(s, t) - (-)^{\eta^{i} \eta^{j}(t)}p_{i}^{\leq}(s)\tilde{\xi}^{i}(s)(s, t) \]
\[ +\tilde{\xi}^{i}(s)\chi_{\eta}^{+j}(s, t)p_{j}^{\leq}(t) - (-)^{(\xi+i)(\eta+i)}p_{i}^{\leq}(s)\tilde{\eta}^{j}(t)\chi_{\xi}^{+i}(t, s) \]
\[ -(-)^{(\xi+i)(\eta+i)}p_{j}^{\leq}(s)\tilde{\eta}^{j}(t)\chi_{\xi}^{+j}(t, s) + (-)^{(\xi^{i} \eta^{j})}p_{j}^{\leq}(t)\tilde{\xi}^{i}(s)\chi_{\eta}^{+j}(s, t) \]
\[ -(-)^{(\xi^{i} \eta^{j})}\chi_{\eta}^{+j}(t, s)\tilde{\xi}^{i}(s)(s, t) - (-)^{\xi^{i} \eta^{j}}(t)\chi_{\xi}^{+j}(t, s)\tilde{\xi}^{i}(s) \}. \tag{3.27} \]

The regular piece is
\[ \int ds dt \tilde{\xi}^{i}(s)\chi_{\eta}^{+j}(s, t)p_{j}^{\leq}(t) + (-)^{(\xi^{i} \eta^{j})}p_{j}^{\leq}(t)\tilde{\xi}^{i}(s)\chi_{\eta}^{+j}(s, t) - (-)^{\eta^{i} \xi^{j}}(t)\eta \tag{3.28} \]

We focus on the first term.
\[ \int ds dt \tilde{\xi}^{i}(s)\chi_{\eta}^{+j}(s, t)p_{j}^{\leq}(t) - (-)^{\xi^{i} \eta^{j}} \\eta \]
\[ = \int ds dt \tilde{\xi}^{i}(s)(\partial_{\eta}^{\eta^{j}}(t)\delta(s - t) + (-)^{\mu^{i} \eta^{j}}(t)\partial_{\mu}^{\eta^{j}}(s - t)p_{j}^{\leq}(t) - (-)^{\xi^{i} \eta^{j}} \\eta \]
\[ = \int \{(-)^{\xi^{i} \eta^{j}}(\delta^{j} - \eta^{j} \eta^{i}) - (-)^{\eta^{i} \xi^{j} \eta^{j}}\}p_{j}^{\leq}(t) - (-)^{\xi^{i} \eta^{j}} \\eta. \tag{3.29} \]
which equals $\mathcal{L}_0^{\xi, \eta}$. This could have been anticipated from theorem 3.1. We here suppressed the integration variable in the single integral, because no confusion is possible. The extension $\text{ext}_0(\xi, \eta)$ becomes

$$
\iint dsdt \; - (\xi^{\eta+j}) \chi_{\xi_j}(t, s) \chi_{\eta_j}(s, t) + (-)\xi^{\eta+(\xi^{\eta+j})i} \chi_{\eta_j}(s, t) \chi_{\xi_j}(t, s)
$$

$$
= - \iint dsdt \; (-)\xi^{\eta+j} \partial \xi_i(s) \delta^x(t-s) + (-)\xi^{\eta+j} \partial \eta_i(s) \delta^y(t-s) \times
\times \cot_j(t) \delta^x(s-t) + (-)\cot_j(t) \delta^y(s-t) \delta^x(s-t) - (-)\xi^{\xi} \eta \leftrightarrow \eta
$$

$$
= - \iint dsdt \; (-)\xi^{\eta+j} \partial \xi_i(s) \partial \eta_i(t) \delta^x(t-s) \delta^x(s-t) + (-)\xi^{\eta+j} \partial \xi_i(s) \eta_i(t) \delta^x(t-s) \delta^x(s-t)
$$

$$
+ (-)\xi^{\eta+j} \partial \xi_i(s) \eta_i(t) \delta^x(t-s) \delta^y(s-t) - (-)\xi^{\xi} \eta \leftrightarrow \eta
$$

$$
= \frac{1}{2\pi i} \iint dsdt \; (-)\xi^{\eta+j} \partial \xi_i(s) \partial \eta_i(t) \delta^x(t-s)
$$

$$
+ \frac{1}{2} (-)\xi^{\eta+j} \partial \xi_i(s) \partial \eta_i(t) \delta^x(t-s) - i\delta^x(t-s)
$$

$$
- \frac{1}{2} (-)\xi^{\eta+j} \partial \xi_i(s) \eta_i(t) \delta^x(t-s) + i\delta^x(t-s)
$$

$$
- \frac{N-M}{6} (\xi^{\eta+j} \partial \xi_i(s) \eta_i(t) \delta^x(t-s) + \delta^x(t-s))
$$

$$
= \frac{1}{2\pi i} \iint dsdt \; (-)\xi^{\eta+j} \partial \xi_i(s) \partial \eta_i(t) \delta^x(t-s)
$$

$$
- \frac{1}{2} (-)\xi^{\eta+j} \partial \xi_i(s) \partial \eta_i(t) \delta^x(t-s) - \frac{1}{2} (-)\xi^{\eta+j} \partial \xi_i(s) \eta_i(t) \delta^x(t-s)
$$

$$
+ \frac{i}{2} (-)\xi^{\eta+j} \partial \xi_i(s) \partial \eta_i(t) \delta^x(t-s) + (-)\xi^{\eta+j} \partial \xi_i(s) \eta_i(t) \delta^x(t-s)
$$

$$
\equiv \mathcal{L}_0^{\xi, \eta} + \text{ext}_0(\xi, \eta)
$$

where we used Lemma 3.2 and the fact that $(-)\xi^i = N-M$. Now consider the full algebra.

$$
[\mathcal{L}_\xi, \mathcal{L}_\eta] \equiv [\mathcal{L}_\xi, \mathcal{L}_\eta] + \text{ext}(\xi, \eta)
$$

$$
= \mathcal{L}_{[\xi, \eta]} + \text{ext}_0(\xi, \eta) + \frac{c}{24\pi i} \iint dsdt \; \xi^0(s) \eta^0(t) (\delta(t-s) + \delta(s-t))
$$

$$
- \frac{1}{2\pi i} \iint dsdt \; \partial \xi^\mu(s) \partial_r \eta^\nu(t) (-)\xi^{\eta+j} \partial \xi_i(s) \eta_i(t) \delta^x(t-s) \delta^y(s-t) \times
$$

$$
\frac{N-M}{6} (\xi^{\eta+j} \partial \xi_i(s) \eta_i(t) \delta^x(t-s) + \delta^x(s-t))
$$

$$
\equiv \mathcal{L}_\xi \mathcal{L}_\eta + \text{ext}_0(\xi, \eta)
$$
Thus,
\begin{equation}
\text{ext}(\xi, \eta) = \text{ext}_0(\xi, \eta) + \frac{1}{2\pi i} \int \frac{c}{12} (\dot{\xi}^0 \eta^0 - \dot{\xi}^0 \eta^0) + k_1(-)(\xi + \eta + \nu \mu) \partial_\nu \partial_\mu \hat{\delta}^\nu - k_2(-)(\xi + \mu + \nu \eta) \partial_\mu \partial_\nu \hat{\delta}^\nu. \tag{3.32}
\end{equation}

The result now follows by means of lemma 3.3. As a consistency check we note that the extension satisfies \(\text{ext}(\eta, \xi) = -(-)(\xi) \text{ext}(\eta, \xi)\). \(\square\)

The superalgebra described in this theorem is not a Lie superalgebra, because the right-hand side is not linear in \(q^\mu(t)\). Rather, it is a graded associative algebra, and the bracket must be interpreted as the graded commutator. However, it is easy to rewrite the extension in linear form, by introducing a sufficient number of new generators. We then obtain an abstract Lie superalgebra extension of \(\text{diff}(N+1|M)\).

Let \(h = h_{\mu_1..\mu_n}(x) dx^{\mu_1} \circ .. \circ dx^{\mu_n}\) be a graded symmetric \(n\)-tensor and let \(g = g_\mu(x) dx^\mu\) be a one-tensor. Define the operators
\begin{align*}
S_n(h) &= -\frac{1}{2\pi i} \int dt \dot{q}^{\mu_1}(t) .. \dot{q}^{\mu_n}(t) h_{\mu_1..\mu_n}(q(t)) \\
R_n(g, h) &= -\frac{1}{2\pi i} \int dt \ddot{q}^{\mu_1}(t) .. \ddot{q}^{\mu_n}(t) g_\mu(q(t)) h_{\nu_1..\nu_n}(q(t)). \tag{3.33}
\end{align*}

We now proceed somewhat differently from [11], in order to keep better track of the minus signs. Define kernels \(S^{\nu_1..\nu_n}(x)\) and \(R^{\nu_1..\nu_n}(x)\) by
\begin{align*}
S_n(h) &= -\int d^{N+M+1}x S^{\nu_1..\nu_n}(x) h_{\nu_1..\nu_n}(x) \\
R_n(g, h) &= -\int d^{N+M+1}x R^{\mu_1..\nu_n}(x) g_\mu(x) h_{\nu_1..\nu_n}(x). \tag{3.34}
\end{align*}

Both fields are graded symmetric in \(\nu_1..\nu_n\). To show that these definitions are consistent, i.e. that (3.33) and (3.34) transform identically, we integrate by parts and throw away the boundary terms. Thus, we assume that the relations
\begin{equation}
\int dt \dot{f}(t) = \int d^{N+M+1}x \partial_\mu F(x) = 0 \tag{3.35}
\end{equation}
hold for all functions $f(t)$ and $F(x)$. The kernels must satisfy the following relations.

\[
[\mathcal{L}_\xi, S_n^{\nu_1...\nu_n}(x)] = -\xi^\mu(x)\partial_\mu S_n^{\nu_1...\nu_n}(x) - \text{div}\xi(x)S_n^{\nu_1...\nu_n}(x)
+ \sum_{j=1}^n (-)\mu^{(\nu_1+...+\nu_{j-1}+\nu_j)}\partial_\mu \xi^{\nu_j}(x)S_n^{\nu_1...\nu_n}(x)
- (n-1)(-)^0\xi^\mu\partial_\mu \xi^0(x)S_n^{\nu_1...\nu_n}(x),
\]

\[
(-)^0\partial_\nu S_n^\nu(x) = 0,
\]

\[
S_n^{\nu_1...\nu_n}(x) = S_n^{\nu_1...\nu_n}(x),
\]

(3.36)

and

\[
[\mathcal{L}_\xi, R_n^{\sigma[\nu_1...\nu_n]}(x)] = -\xi^\mu(x)\partial_\mu R_n^{[\nu_1...\nu_n]}(x) - \text{div}\xi(x)R_n^{[\nu_1...\nu_n]}(x)
+ (-)^n\mu^{(\xi+\sigma+\mu)}\partial_\mu \xi^{\nu_j}(x)R_n^{[\nu_1...\nu_n]}(x)
+ \sum_{j=1}^n (-)\mu^{(\nu_1+...+\nu_{j-1}+\nu_j)}\partial_\mu \xi^{\nu_j}(x)R_n^{[\nu_1...\nu_n]}(x)
- (n+1)(-)^0\xi^\mu\partial_\mu \xi^0(x)R_n^{[\nu_1...\nu_n]}(x)
- (-)^0\xi^\mu\partial_\mu \xi^0(x)R_n^{[\nu_1...\nu_n]}(x)
+ (-)\mu^{(\nu_1+...+\nu_{j-1}+\nu_j)}\partial_\mu \xi^{\nu_j}(x)R_n^{[\nu_1...\nu_n]}(x)
- (-)^0\mu^{(\nu_1+...+\nu_{j-1}+\nu_j)}\partial_\mu \xi^{\nu_j}(x)R_n^{[\nu_1...\nu_n]}(x),
\]

\[
(-)^\nu_j\partial_\mu S_n^{\nu_1...\nu_n}(x) = \sum_{j=1}^n (-)^\nu_j\partial_\nu S_n^{\nu_1...\nu_{j-1}+\nu_j+\nu_1}(x),
\]

\[
R_n^{[\nu_1...\nu_n]}(x) = R_n^{[\nu_1...\nu_n]}(x),
\]

\[
R_n^{[\nu_1...\nu_n]}(x) = 0,
\]

(3.37)

where the check mark $\check{\nu}_j$ denotes omission. The subsidiary conditions follow from $\int dt \dot{f}(q(t)) = 0$, $\dot{q}^0(t) = 1$, $\int dt \frac{d}{dt} \check{q}^{(\nu_1)}(t)\check{q}^{(\nu_2)}(t)h^{(\nu_3...\nu_n)}(q(t)) = 0$, $\check{q}^{0}(t) = 1$, and $\check{q}^{0}(t) = 0$, respectively. The extension can now be rewritten as

\[
\text{ext}(\xi, \eta) = \frac{1}{2\pi i} \int dt \left\{ (1 + k_1)(-)^{\xi^\eta+\nu}\dot{q}^\mu \partial_\mu \xi^\rho \partial_\rho \eta^\nu
+ k_2 q^\mu \partial_\mu \text{div}\xi \text{div}\eta
+ (-)^{\xi^\eta+\nu}(\dot{q}^{(\rho+\nu+\xi)} q^\nu \eta^\rho \partial_\nu \xi^\mu \partial_\mu \eta^\nu
\right\}
\]
graded Fock module, and pick arbitrary lowest-weight modules for provide a realization of it. Further, if we represent (3.4) on the natural define an abstract Lie superalgebra, and the expressions in (3.21) and (3.33) tended super-diffeomorphism algebra.

\[ (-\sigma^\rho q^\sigma \partial_\rho \partial_\sigma \xi^\mu \partial_\nu \eta^0 - (-\sigma^\rho q^\sigma \partial_\rho \partial_\sigma \xi^\mu \partial_\nu \eta^0 \right.
\]
\[ + (\omega^\rho q^\sigma \partial_\rho \partial_\sigma \xi^\mu \partial_\nu \eta^0) + \frac{1}{2}(\omega^\rho q^\sigma \partial_\rho \partial_\sigma \xi^\mu \partial_\nu \eta^0 - \frac{1}{2}(\omega^\rho q^\sigma \partial_\rho \partial_\sigma \xi^\mu \partial_\nu \eta^0)
\]
\[ - (2 - \frac{c + 2(N - M)}{12})(\omega^\rho q^\sigma \partial_\rho \partial_\sigma \xi^\mu \partial_\nu \eta^0)
\]
\[ + (\omega^\rho q^\sigma \partial_\rho \partial_\sigma \xi^\mu \partial_\nu \eta^0 - \frac{c + 2(N - M)}{12} q^\rho \partial_\rho \xi^\mu \eta^0)
\]
\[ + i(\omega^\rho q^\sigma \partial_\rho \eta^0 - \frac{c + 2(N - M)}{12} q^\rho \partial_\rho \xi^\mu \eta^0)
\]
\[ \int d^{N+M+1}x \left\{ (1 + k_1)(\xi + \eta^0) S_1^\rho(x) \partial_\rho \partial_\sigma \xi^\mu(x) \partial_\nu \eta^0(x)
\]
\[ + k_2 \xi S_1^\rho(x) \partial_\rho \partial_\sigma \xi^\mu(x) \partial_\nu \eta^0(x)
\]
\[ + (\xi + \eta^0) (\omega^\rho S_2^\rho(x) \partial_\rho \partial_\sigma \xi^\mu(x) \partial_\nu \eta^0(x)
\]
\[ - (\omega^\rho S_2^\rho(x) \partial_\rho \partial_\sigma \xi^\mu(x) \partial_\nu \eta^0(x)
\]
\[ - (\omega^\rho S_2^\rho(x) \partial_\rho \partial_\sigma \xi^\mu(x) \partial_\nu \eta^0(x)
\]
\[ + (\omega^\rho S_2^\rho(x) \partial_\rho \partial_\sigma \xi^\mu(x) \partial_\nu \eta^0(x)
\]
\[ + \frac{1}{2}(\omega^\rho q^\sigma \partial_\rho \partial_\sigma \xi^\mu(x) \partial_\nu \eta^0(x) - \partial_\rho \partial_\sigma \xi^\mu(x) \partial_\nu \eta^0(x)
\]
\[ - (2 - \frac{c + 2(N - M)}{12})(\omega^\rho q^\sigma \partial_\rho \partial_\sigma \xi^\mu(x) \partial_\nu \eta^0(x)
\]
\[ + (\omega^\rho q^\sigma \partial_\rho \partial_\sigma \xi^\mu(x) \partial_\nu \eta^0(x)
\]
\[ - (2 - \frac{c + 2(N - M)}{12})(\omega^\rho q^\sigma \partial_\rho \partial_\sigma \xi^\mu(x) \partial_\nu \eta^0(x)
\]
\[ + \int \right\} (3.38)
\]

Eqs. (3.22), (3.36), (3.37) and (3.38), together with the conditions
\[
[S_m^{\mu_1 \cdots \mu_m}(x), S_n^{\nu_1 \cdots \nu_n}(y)] = \left[ S_m^{\mu_1 \cdots \mu_m}(x), R_n^{\nu_1 \cdots \nu_n}(y) \right]
\]
\[
= [R_m^{\sigma_1 \cdots \sigma_m}(x), R_n^{\nu_1 \cdots \nu_n}(y)] = 0,
\]
provide a realization of it. Further, if we represent (3.4) on the natural graded Fock module, and pick arbitrary lowest-weight modules for Vir and \( gl(N+1|M) \), we obtain a lowest-energy module for the non-centrally extended super-diffeomorphism algebra.
4 Subalgebras

A projective representation of $\text{diff}(N+1|M)$ yields by restriction a projective representation of its subalgebras, i.e. superalgebras whose generators admit a realization as first-order differential operators on super space-time. Such algebras are described in [3][9]. The bosonic case is classical, and can be found e.g. in [8].

4.1 Temporal subalgebra $\text{diff}(1)$

The temporal subalgebra is generated by space-independent vector fields $\xi = \xi^0(t)\partial_0$. Eq. (3.21) becomes

$$\mathcal{L}_\xi = \int dt \, \xi^0(t)(-\dot{q}^i(t)p_i(t): + L(t)) + \xi^0(t)T^0_0(t)$$

$$= \int dt \, \xi^0(t)L'(t)$$

(4.1)

where

$$L'(t) \equiv -\dot{q}^i(t)p_i(t): + L(t) - \dot{T}^0_0(t)$$

(4.2)

generates a Virasoro algebra. The extension (3.23) is

$$\text{ext}(\xi, \eta) = \frac{1}{2\pi i} \int (k_1 + k_2 + \frac{c + 2(N-M)}{12})\dot{\xi}^0\dot{\eta}^0 - \frac{c + 2(N-M)}{12}\xi^0\eta^0.$$ (4.3)

Hence the temporal subalgebra is a Virasoro algebra with central charge $c_{\text{Temp}} = c + 2(N-M) + 12(k_1 + k_2)$.

4.2 Spatial subalgebra $\text{diff}(N|M)$

The spatial subalgebra is generated by time-independent vector fields $\xi = \xi^i(\vec{x})\partial_i$. The arrow denotes a vector with spatial components only; $\vec{x}^0 = 0$. Using that $\xi^0 = \partial_0\xi^i = 0$, we find

$$\mathcal{L}_\xi = \int dt \, :\xi^i(\vec{q}(t))p_i(t): + (-)^{(\xi^0)^2+i}\partial_j\xi^i(\vec{q}(t))T^j_i(t).$$ (4.4)
The extension becomes
\[
\text{ext}(\xi, \eta) = \frac{1}{2\pi i} \int dt (1 + k_1)(-)^{(\xi + \eta + j)j} \partial_j \dot{\xi}^i(q(t)) \partial_i \dot{\eta}^j(q(t)) + k_2(-)^{\xi i + \eta j + j} \partial_j \dot{\xi}^i(q(t)) \partial_i \dot{\eta}^j(q(t)).
\] (4.5)

4.3 Special superdiffeomorphism algebra \(sdiff(N+1|M)\)

It can be shown that the divergence (2.14) of a vector field satisfies
\[
\text{div}([\xi, \eta]) = \xi^\mu \partial_\mu \text{div} \eta - (-)^{\xi \eta} \partial_\mu \text{div} \xi.
\] (4.6)

The special (or divergence-free) algebra \(sdiff(N+1|M)\) is generated by vector fields with vanishing divergence. In theorems 3.1 and 3.4, the matrices \(T^\mu_i(t) \in sl(N+1|M)\), and hence \(k_1 = (N - M + 1)k_2\) in (2.12). The extension is
\[
\text{ext}(\xi, \eta) = \frac{1}{2\pi i} \int dt \left\{ (1 + k_1)(-)^{(\xi + \eta + \nu)\nu} \partial_\nu \dot{\xi}^\mu(q(t)) \partial_\mu \dot{\eta}^\nu(q(t)) + (-)^{\xi \xi + \eta \eta} \dot{\xi}^\nu(q(t)) \partial_\nu \dot{\eta}^\mu(q(t)) \right.
\]
\[
- \xi^0(q(t)) \dot{q}^\nu(t) \partial_\nu \dot{\xi}^0(q(t)) - \dot{q}^\nu(t) \partial_\nu \dot{\xi}^0(q(t))
\]
\[
- \xi^0(q(t)) \dot{q}^\nu(q(t)) \partial_\nu \dot{\xi}^0(q(t)) + \dot{q}^\nu(t) \partial_\nu \dot{\xi}^0(q(t)) - \left(2 - \frac{c + 2(N - M)}{12}\right) \xi^0(q(t)) \dot{\eta}^0(q(t))
\]
\[
- \left \{ (c + 2(N - M)) \xi^0(q(t)) \dot{\eta}^0(q(t)) \right\}.
\] (4.7)

4.4 Hamiltonian algebras \(H(N+1|M)\) and \(H(N|M)\)

The Hamiltonian algebra \(H(N+1|M)\) preserves the constant graded skew-symmetric matrix \(\omega_{\sigma \tau}\), satisfying
\[
\omega_{\tau \sigma} = -(-)^{\sigma \tau} \omega_{\sigma \tau} (4.8)
\]

Actually, it is sufficient if \(\omega_{\sigma \tau}(x)\) is a closed two-form, but we do not need this generalization here, because it is always possible to choose Darboux coordinates locally. However, it would be necessary for global considerations.

Define the inverse matrix \(\omega^{\mu \nu}\) by
\[
\omega^{\mu \rho} \omega_{\rho \nu} = (-)^{\mu} \delta^\mu_\nu, \quad (-)^{\rho} \omega_{\nu \rho} \omega^{\rho \mu} = \delta^\mu_\nu \] (4.9)
In general, a two-form transforms as
\[ [\mathcal{L}_\xi, \omega_{\sigma\tau}(x)] = -\xi^\mu \partial_\mu \omega_{\sigma\tau}(x) - (-)^{\xi_\sigma+\sigma+\mu} \partial_\sigma \xi^\mu \omega_{\mu\tau}(x) \]
\[ - (-)^{\xi_\tau+\tau+\mu+\sigma(\mu+\tau)} \partial_\tau \xi^\mu \omega_{\sigma\mu}(x). \] (4.10)

The matrix \( \omega_{\sigma\tau} \) can be regarded as a constant two-form, provided \( \xi \) is a Hamiltonian vector field of the form
\[ \xi = H_f \equiv (-)^{f_{\mu+\nu}} \partial_\mu f(x) \omega^{\mu\nu} \partial_\nu, \] (4.11)
for \( f \) an arbitrary function. Such vector fields generate the Hamiltonian algebra \( H(N+1|M) \subset \text{diff}(N+1|M) \). It is easy to verify that (4.11) and (4.10) imply that \( [\mathcal{L}_\xi, \omega_{\sigma\tau}(x)] = 0 \), and that
\[ \{ H_f, H_g \} = H\{ f, g \}, \]
\[ \{ f, g \} = (-)^{f_{\mu+\nu}} \partial_\mu f \omega^{\mu\nu} \partial_\nu g. \] (4.12)

\( \{\cdot, \cdot\} \) is called the Poisson bracket. It satisfies the axioms of a Lie superalgebra, acting as a derivation of the associative product.
\[ \{ g, f \} = (-)^{f_{\mu+\nu}} \partial_\mu f \omega^{\mu\nu} \partial_\nu g. \] (4.13)

Conventionally, one sets \( \omega^{\mu\nu} = 0 \) if \( \text{deg } \mu + \text{deg } \nu = 1 \). However, it is only necessary to demand that \( \omega^{\mu\nu} \) be a Grassmann (anti-commuting) number in this case. It appears that by choosing \( \omega^{\mu\nu} \) purely Grassmann, the Leitesian algebra \( [12], [9] \), i.e. the odd analogue of the Hamiltonian algebra, is obtained.

Inserting (4.11) into (3.21) yields the following realization
\[ \mathcal{L}(H_f) = \int dt \ (-)^{f_{\mu+\nu}} \partial_\mu f(q(t)) \omega^{\mu\nu} p_\nu(t): \]
\[ + (-)^{f_{\mu}} \partial_\mu f(q(t)) \omega^{\mu\rho} L(t) + (-)^{f(\mu+\rho)+\nu} \partial_\rho \partial_\mu f(q(t)) \omega^{\nu \rho} T^\mu_\rho(t). \] (4.14)

We have \( \text{deg } H_f = \text{deg } f \) and \( H_f^\mu = (-)^{f_{\sigma+\mu}} \partial_\sigma f \omega^{\sigma\mu} \). Moreover,
\[ \text{div}(H_f) = (-)^{f_{\nu+\mu}+f_{\mu+\nu}} \partial_\nu \partial_\mu f \omega^{\mu\nu} = 0. \] (4.15)
The generators in (4.14) satisfy an extended Hamiltonian algebra,
\[ [\mathcal{L}(H_f), \mathcal{L}(H_g)] = \mathcal{L}(H_{(f,g)_{P.b.}}) + \text{ext}(f, g), \]  
(4.16)
where the extension is obtained by specialization of (3.23).

\[ \text{ext}(f, g) \equiv \text{ext}(H_f, H_g) \]
\[ = \frac{(-)^{f^\sigma + g^\tau}}{2\pi i} \int dt \, (1 + k_1)(-)^{(f+g)^\nu + \mu} \partial_\nu \partial_\sigma \hat{f}_\omega \sigma^\mu \partial_\mu \partial_\tau \hat{g}_\omega \tau^\nu \]
\[ + (-)^{(f+g)^\nu} \partial_\nu \partial_\sigma \hat{f}_\omega \sigma^\mu \partial_\mu \partial_\tau \hat{g}_\omega \tau^\nu - (-)^{\mu \nu} \hat{q}_\nu \partial_\sigma \hat{f}_\omega \sigma^\mu \partial_\mu \partial_\tau \hat{g}_\omega \tau^0 \]
\[ - \partial_\sigma \hat{f}_\omega \sigma^\mu \partial_\mu \partial_\tau \hat{g}_\omega \tau^0 + \hat{q}_\sigma \partial_\sigma \hat{f}_\omega \sigma^\mu \partial_\mu \partial_\tau \hat{g}_\omega \tau^0 \]
\[ - (2 - \frac{c + 2(N - M)}{12}) \partial_\sigma \hat{f}_\omega \sigma^\mu \partial_\mu \partial_\tau \hat{g}_\omega \tau^0 \]
\[ - \frac{c + 2(N - M)}{12} \partial_\sigma \hat{f}_\omega \sigma^\mu \partial_\mu \partial_\tau \hat{g}_\omega \tau^0. \]  
(4.17)

where \( f = f(q(t)) \) and \( g = g(q(t)) \).

The dimensions in the Hamiltonian algebra appear in pairs: a coordinate and its conjugate momentum. Since time is a distinguished dimension, it is natural to consider matrices satisfying \( \omega^{0\nu} = 0 \), and time-independent functions. This leads to considerable simplifications.

\[ \{f, g\}_{P.b.} = (-)^{f_i + j_i} \partial_i f(\vec{x}) \omega^{ij} \partial_j g(\vec{x}), \]
(4.18)
\[ H_f = (-)^{f_i + j_i} \partial_i f(\vec{x}) \omega^{ij} \partial_j, \]  
(4.19)
\[ \mathcal{L}(H_f) = \int dt \, (-)^{f_i + j_i} \partial_i f(\vec{q}(t)) \omega^{ij} \partial_j p_j(t): \]
\[ + (-)^{f(i+k)+j} \partial_k \partial_i f(\vec{q}(t)) \omega^{ij} T^k_j(t), \]  
(4.20)
\[ \text{ext}(f, g) = (1 + k_1) \frac{(-)^{f+g+j+i}}{2\pi i} \times \]
\[ \times \int dt \, \partial_j \partial_k \hat{f}(\vec{q}(t)) \omega^{ki} \partial_l g(\vec{q}(t)) \omega^{lj}. \]  
(4.21)

### 4.5 Contact algebra \( K(N+1|M) \)

Denote the Euler operator \( E = x^i \partial_i \) and \( \Delta = 2 - E \). Clearly, \( \deg \Delta = \deg E = \deg \partial_0 = 0 \). The contact algebra \( K(N+1|M) \) is

\[ [K_f, K_g] = K_{(f, g)_{K.b.}}, \]  
(4.22)
where

\[ \{ f, g \}_{K.b.} = \Delta (f(x)) \partial_0 g(x) - \partial_0 f(x) \Delta (g(x)) - \{ f, g \}_{P.b.}, \quad (4.23) \]

and the Poisson bracket is given by (4.18). \( \{ \cdot, \cdot \}_{K.b.} \) is called the contact bracket. It satisfies the axioms of a Lie superalgebra, but it is not a derivation of the associative product, due to an extra term.

\[ \{ g, f \}_{K.b.} = (-)^{fg} \{ f, g \}_{K.b.} \]
\[ \{ f, gh \}_{K.b.} = \{ f, g \}_{K.b.} h + (-)^{fg} \{ g, f \}_{K.b.} h + 2 \partial_0 f \ g h \]
\[ (-)^{fh} \{ f, \{ g, h \}_{K.b.} \}_{K.b.} + (-)^{gh} \{ h, \{ f, g \}_{K.b.} \}_{K.b.} + (-)^{fg} \{ g, \{ f, h \}_{K.b.} \}_{K.b.} = 0. \quad (4.24) \]

To verify that (4.23) defines a Lie algebra, the following formulas are useful.

\[ \partial_0 \Delta = \Delta \partial_0 \]
\[ \Delta \{ f, g \}_{P.b.} = \{ \Delta f, g \}_{P.b.} + \{ f, \Delta g \}_{P.b.} \]
\[ \Delta (fg) = \Delta (f) g + f \Delta (g) - 2 fg \quad (4.25) \]

\( K(N+1|M) \) is realized by contact vector fields

\[ K_f = \Delta (f(x)) \partial_0 - H_f + \partial_0 f(x) E \quad (4.26) \]

which is verified by direct computation. The components are

\[ K_f^0(x) = \Delta (f(x)) \]
\[ K_f^i(x) = -H_f^i(x) + \partial_0 f(x)x^i, \quad (4.27) \]

where

\[ H_f^i(x) = (-)^{f^{k+i}} \partial_{k} f(x) \omega^{ki}. \quad (4.28) \]

The realization on trajectories is now obtained by substituting (4.27) into (3.21), and making the replacements \( \partial_0 \mapsto p_0(t) = -\dot{q}^i(t) p_i(t), \ E \mapsto q^i(t)p_i(t) \).

We find

\[ \mathcal{L}(K_f) = \int dt : (-\Delta (f(t)) \dot{q}^i(t) - H_f^i(t) + \partial_0 f(t) \dot{q}^i(t)) p_i(t) : + \Delta (f(t)) L(t) + \partial_0 \Delta (f(t)) T_0^i(t) \]
\[ + (-)^{i} (\partial_0 H_f^i(t) + \partial_0^2 f(t) \dot{q}^i(t)) T_0^i(t) + (-)^{fj} \partial_j \Delta (f(t)) T_0^j(t) \]
\[ + (-)^{f+i} (-\partial_j H_f^i(t) + \partial_j \partial_0 f(t) \dot{q}^i(t) + \delta_j^i \partial_0 f(t)) T_2^j(t), \quad (4.29) \]
where \( f(t) = f(q(t)) \), \( H_f(t) = H_f(q(t)) \). The extension can be computed from (4.27) and (3.23), but the calculations are tedious and the result is not very illuminating.

Actually, there are two different notions of time in our representation of the contact algebra: one is treated specially in (3.21) and the other in (4.23). Although it is natural to identify these two times, as we have done above, this is not necessary. A more general representation of extended \( K(N+1|M) \) is obtained as follows. Introduce fixed constant vectors \( z_\mu \) and \( z^\mu \) with components only in the bosonic directions, satisfying

\[
z_\mu z^\mu = 1, \quad z_\mu \omega^{\mu\nu} = 0. \tag{4.30}
\]

The case above is recovered when \( z_\mu = \delta^0_\mu, z^\mu = \delta^0_\mu \). Then

\[
E = x^\mu \partial_\mu - z_\mu x^\mu z^\nu \partial_\nu, \quad \Delta = 2 - E \\
\{f, g\}_{K.b.} = \Delta(f(x)) z^\mu \partial_\mu g(x) - z^\mu \partial_\mu f(x) \Delta(g(x)) - \{f, g\}_{P.b.} \\
K_f = \Delta(f(x)) z^\mu \partial_\mu - H_f + z^\mu \partial_\mu f(x) E, \tag{4.31}
\]

where the Poisson bracket is given by (4.12) and \( H_f \) by (4.11). The action in Fock space is given by

\[
\mathcal{L}(K_f) = \int dt : \{\Delta(f(t)) z^\mu - (-)^{f+\mu} \partial_\nu f(t) \omega^{\nu\mu} + z^\nu \partial_\nu f(t)(q^\mu(t) - z_\sigma q^\sigma(t) z^\mu)\} p_\mu(t) : \\
+\{\Delta(f(t)) z^0 - (-)^{f+\mu} \partial_\nu f(t) \omega^{\nu\nu} + z^\nu \partial_\nu f(t)(t - z_\sigma q^\sigma(t) z^0)\} L(t) \\
+(-)^{(f+\rho)\sigma \rho \partial_\rho \{\Delta(f(t)) z^\mu - (-)^{f+\mu} \partial_\nu f(t) \omega^{\nu\mu} + z^\nu \partial_\nu f(t)(q^\mu(t) - z_\sigma q^\sigma(t) z^\mu)\} \mathcal{T}_\mu^\rho(t), \tag{4.32}
\]

where \( f(t) \equiv f(q(t)) \).

### 4.6 Superconformal algebra

As is known [3], the superconformal algebra is a central extension of the contact algebra \( K(1|1) \). Denote \( t = x^0 \) and \( \theta = x^1 \), \( \text{deg} t = 0 \), \( \text{deg} \theta = 1 \). A Fourier basis for functions in \((1|1)\) dimensions is given by

\[
\ell_m = \frac{1}{2i} e^{int}, \quad g_m = \theta e^{int}. \tag{4.33}
\]
Let $\omega^{11} = i$ be the only non-zero component of the matrix $\omega^{\mu \nu}$. The Poisson brackets are
\[
\{\theta, \theta\}_{P.b.} = i, \quad \{t, t\}_{P.b.} = \{t, \theta\}_{P.b.} = 0.
\]
(4.34)

With $\Delta = 2 - \theta \partial_\theta$, we have $\Delta(\ell_m) = 2\ell_m$, $\Delta(g_m) = g_m$. The functions $\ell_m$ and $g_m$ generate a centerless superconformal algebra under the contact bracket \[1.23\].
\[
\{\ell_m, \ell_n\}_{K.b.} = (n - m)\ell_{m+n},
\{\ell_m, g_n\}_{K.b.} = (n - \frac{m}{2})g_{m+n},
\{g_m, g_n\}_{K.b.} = 2\ell_{m+n}
\]
(4.35)

The corresponding contact vector fields are
\[
K(\ell_m) = e^{int}(-i \partial_t + \frac{m}{2} \theta \partial_\theta)
\]
\[
K(g_m) = e^{int}(\theta \partial_t - i \partial_\theta)
\]
(4.36)

Substitution of these vector fields into \[3.21\] yields
\[
L_m \equiv \mathcal{L}(K(\ell_m)) = \int dt \ e^{int}\left\{i: \hat{\theta}(t) p_\theta(t): + \frac{m}{2}: \theta(t) p_\theta(t): - i L(t)
+ m T_0^0(t) - i \frac{m^2}{2} \theta(t) T_1^0(t) + \frac{m}{2} T_1^1(t) \right\}
\]
\[
G_m \equiv \mathcal{L}(K(g_m)) = \int dt \ e^{int}\left\{i: \hat{\theta}(t) \theta(t) p_\theta(t): - i p_\theta(t) + \theta(t) L(t)
+ im \theta(t) T_0^0(t) - m T_1^0(t) - T_1^1(t) \right\}.
\]
(4.37)

Introduce
\[
\theta_m = \frac{1}{2\pi i} \int dt \ \theta(t)e^{int}
\]
\[
U_m = \frac{1}{2\pi i} \int dt \ \hat{\theta}(t)\theta(t)e^{int}
\]
\[
V_m = \frac{1}{2\pi} \int dt \ \hat{\theta}(t)\hat{\theta}(t)e^{int}
\]
\[
W_m = \frac{1}{2\pi} \int dt \ \hat{\theta}(t)\theta(t)\hat{\theta}(t)e^{int}.
\]
These operators satisfy the following Lie superalgebra.

\[
[L_m, L_n]_- = (n - m) L_{m+n} + (-am^3 + a'm)\delta_{m+n}
\]
\[
[L_m, G_n]_- = (n - \frac{m}{2}) G_{m+n} + (\alpha m^3 + \beta m^2 n)\theta_{m+n}
\]
\[
-\gamma(mn + \frac{m^2}{2})\theta_{m+n} - 2\gamma'm\theta_{m+n}
\]
\[
[G_m, G_n]_+ = 2L_{m+n} + (bm^2 + 2\gamma' - a')\delta_{m+n}
\]
\[+((2\alpha - e)(m^2 + n^2) + (2\beta - e)mn)U_{m+n} + e V_{m+n}
\]
\[-\gamma(m + n)U_{m+n} - 2\gamma'U_{m+n}
\]
\[
[L_m, \theta_n]_- = (n + \frac{3m}{2})\theta_{m+n}
\]
\[
[L_m, U_n]_- = (n + m)U_{m+n}
\]
\[
[L_m, V_n]_- = (n - m)V_{m+n} + (m^3 + \frac{m^2 n}{2})U_{m+n}
\]
\[
[L_m, W_n]_- = (n - \frac{m}{2})W_{m+n}
\]
\[
[G_m, \theta_n]_+ = -\delta_{m+n} + U_{m+n}
\]
\[
[G_m, U_n]_- = (2m + n)\theta_{m+n}
\]
\[
[G_m, V_n]_- = (n - 2m)W_{m+n} + (2m^3 + 3m^2 n + mn^2)\theta_{m+n}
\]
\[
[G_m, W_n]_+ = V_{m+n} - (2m^2 + mn)U_{m+n}
\]
\[
[\theta_m, \theta_n]_+ = [\theta_m, U_n]_- = [\theta_m, V_n]_- = [\theta_m, W_n]_+ = [U_m, U_n]_- = [U_m, V_n]_-
\]
\[
= [U_m, W_n]_- = [V_m, V_n]_- = [V_m, W_n]_- = [W_m, W_n]_+ = 0,
\]

where \(2\alpha - \beta = 2a - b/2\). The symmetry of the bracket has here been made explicit; \([\cdot, \cdot]_-\) is the commutator and \([\cdot, \cdot]_+\) the anti-commutator. It is straightforward to verify all super-Jacobi identities and graded anti-symmetry. All parameters except \(a\) can be removed by the following redefinition.

\[
L_m \mapsto L_m - \frac{a'}{2}\delta_m + \frac{\beta}{2}m^2 U_m,
\]
\[
G_m \mapsto G_m - \frac{e}{2}W_m + ((\beta - 2\alpha)m^2 + \gamma m + \gamma')\theta_m.
\]

Eq. (4.39) then contains a superconformal subalgebra with central charge 12a.

\[
[L_m, L_n]_- = (n - m)L_{m+n} + (-am^3 + a'm)\delta_{m+n}
\]
\[ [L_m, G_n]^- = (n - \frac{m}{2}) G_{m+n} \]
\[ [G_m, G_n]^+ = 2L_{m+n} + (4am^2 - a')\delta_{m+n}. \] (4.41)

The generators in (4.37) and (4.38) satisfy the algebra (4.39) with parameters
\[
\begin{align*}
    a &= -3 + \frac{3}{4}k_1 + \frac{1}{4}k_2 + \frac{c - 2}{12}, \\
    a' &= \frac{c - 2}{12}, \\
    \alpha &= -\frac{5}{4} + \frac{c - 2}{12}, \\
    \beta &= -\frac{k_1}{2} - \frac{k_2}{2} - 1, \\
    \gamma &= -\frac{1}{2}, \\
    \gamma' &= \frac{c - 2}{24}, \\
    e &= -2 + \frac{c - 2}{12}, \\
    b &= 2k_1. 
\end{align*} \] (4.42)

The central charge of (4.41) is 12a = -11 + 9k_1 + 3k_2 + c. Hence we have constructed a representation of the superconformal algebra for each Vir × gl(1|1) module. It is possible to make the projection \( \theta_m = 0, U_m = \delta_m \) in (4.39), because this choice makes these operators central. A motivation comes from the Berezin integral: \( \int dt \dot{\theta}(t)\theta(t) \approx \int d\theta \theta = 1 \).

### 4.7 M = 2 superconformal algebra

The superconformal algebra with two supersymmetries is a central extension of the contact algebra \( K(1|2) \). The coordinates are \( t = x^0, \theta = x^1, \bar{\theta} = x^2, \) deg \( t = 0, \deg \theta = \deg \bar{\theta} = 1 \). A Fourier basis for functions in (1|2) dimensions is given by
\[
\begin{align*}
    \ell_m &= \frac{1}{2i} e^{imt}, \\
    t_m &= i\theta \bar{\theta} e^{int}, \\
    g_m &= \theta e^{imt}, \\
    \bar{g}_m &= \bar{\theta} e^{imt}. 
\end{align*} \] (4.43)

Let \( \omega^{12} = i \) be the only non-zero component of the matrix \( \omega^{\mu\nu} \). The Poisson brackets are
\[
\begin{align*}
    \{\theta, \bar{\theta}\}_{\text{P.b.}} &= i, \\
    \{\theta, \theta\}_{\text{P.b.}} &= \{\bar{\theta}, \bar{\theta}\}_{\text{P.b.}} = \{t, t\}_{\text{P.b.}} = \{t, \theta\}_{\text{P.b.}} = \{t, \bar{\theta}\}_{\text{P.b.}} = 0. \] (4.44)
\]
Moreover, \( \Delta = 2 - \theta \partial_{\theta} - \bar{\theta} \partial_{\bar{\theta}} \) and
\[
\begin{align*}
\Delta(\ell_m) &= 2\ell_m, & \Delta(t_m) &= 0 \\
\Delta(g_m) &= g_m, & \Delta(\bar{g}_m) &= \bar{g}_m.
\end{align*}
\] (4.45)

The functions \( \ell_m, g_m, \bar{g}_m \) and \( t_m \) generate a centerless \( M = 2 \) superconformal algebra under the contact bracket (4.23).

\[
\begin{align*}
\{\ell_m, \ell_n\}_{K.b.} &= (n - m) \ell_{m+n} \\
\{\ell_m, t_n\}_{K.b.} &= nt_{m+n} \\
\{\ell_m, g_n\}_{K.b.} &= (n - \frac{m}{2}) g_{m+n} \\
\{t_m, g_n\}_{K.b.} &= g_m, \\
\{g_m, g_n\}_{K.b.} &= 2\ell_{m+n} + (n - m)t_{m+n} \\
\{g_m, \bar{g}_n\}_{K.b.} &= \{t_m, \bar{g}_n\}_{K.b.} = 0.
\end{align*}
\] (4.46)

The corresponding contact vector fields are
\[
\begin{align*}
K(\ell_m) &= e^{int}(-i\partial_t + \frac{m}{2}(\theta \partial_{\theta} + \bar{\theta} \partial_{\bar{\theta}})) \\
K(g_m) &= e^{int}(\theta \partial_t - i\partial_{\bar{\theta}} + im\theta \partial_{\bar{\theta}}) \\
K(\bar{g}_m) &= e^{int}(\bar{\theta} \partial_t - i\partial_{\theta} - im\theta \partial_{\theta}) \\
K(t_m) &= e^{int}(\theta \partial_{\theta} - \bar{\theta} \partial_{\bar{\theta}}).
\end{align*}
\] (4.47)

Substitution of these vector fields into (3.21) yields a Fock representation of an extension of \( K(1|2) \). The extension is readily found from (3.23). We have not explicitly calculated it, since the calculation is quite tedious and the result is not very illuminating. However, by analogy with the \( M = 1 \) case, we expect that the resulting non-central extension has a subalgebra isomorphic to the standard centrally extended \( M = 2 \) superconformal algebra.

5 Gauge superalgebras

**Theorem 5.1** (cf. [11], theorem 7.1) Let \( J^a(t) \) obey the Kac-Moody superalgebra \( \widehat{\mathfrak{g}} \) and let \( q^a(t) \) be a trajectory (5.1). Then
\[
\mathcal{J}_X = \int dt \ X_a(q(t))J^a(t) 
\] (5.1)
satisfies a Lie superalgebra extension of \( \text{map}(N+1|M, g) \), the algebra of maps from \((N+1|M)\)-dimensional super space-time to \(g\). The brackets are

\[
[J_X, J_Y] = J_{[X,Y]} - k \int d^{N+M+1}x \ (-)^{a(Y+b)} S_1^\mu(x) \partial_\mu X_\alpha(x) Y_\alpha(x) \delta^{ab}
\]

\[
[J_X, S_1^\nu(x)] = 0,
\]

where

\[
[X, Y]_c = (-)^{a(Y+b)} f^{a b} c X_a Y_b = -(-)^{X Y} [Y, X]_c.
\]

The Killing super-metric \( \delta^{ab} \) was defined in (2.5), and

\[
\int d^{N+M+1}x \ S_1^\nu(x) h_\nu(x) = \frac{1}{2\pi i} \int dt \ \dot{q}_\nu(t) h_\nu(q(t)),
\]

as in (3.3) and (3.34). Moreover, there is an intertwining action of \( \text{diff}(N+1|M) \):

\[
[L_\xi, J_Y] = J_{\xi Y},
\]

\[
[L_\xi, S_1^\nu(x)] = -\xi^\mu(x) \partial_\mu S_1^\nu(x) - \text{div} \xi(x) S_1^\nu(x) + (-)^{\mu(\xi+\nu)} \partial_\mu \xi^\nu(x) S_1^\mu(x),
\]

where \( \xi Y = \xi^\mu(x) \partial_\mu Y(x) \).

## 6 Discussion

To conclude, I have discovered a class of non-central extensions of the super-diffeomorphism and super-gauge algebras in any number of bosonic and fermionic dimensions, and constructed Fock representations thereof. To my knowledge this is the first time non-central extensions of Lie superalgebras have been described.

The superconformal algebra was believed to be an exceptional algebraic structure, because it is one of the few superalgebras admitting a central extension (an exhaustive list is given in [3]). However, the interesting property, both mathematically and physically, is that an algebra has projective Fock
representations, not that the resulting extension necessarily be central. In subsection 4.6 I proved that the superconformal algebra is not exceptional at all, but rather a quite ordinary (and indeed one of the simplest) subalgebras of the non-centrally extended super-diffeomorphism algebra.

This result has some bearing on string theory. Since there is no compelling experimental evidence in favour of string theory, the only motivation has been that it represents an exceptional mathematical structure. However, the present paper shows that there is nothing special about the underlying algebraic structure, the superconformal algebra, and hence I find it difficult to believe that string theory should be relevant to physics. Stated differently: the superconformal algebra is merely a subalgebra of the non-centrally extended super-diffeomorphism algebra in the lowest possible dimension. It seems unlikely that Nature should prefer such a trivial symmetry at her most fundamental level.

Another motivation for string theory is that it may provide a consistent theory of quantum gravity. Fortunately, \textit{diff} \((N+1)\) can achieve the same goal, in the following sense. Its Fock modules are projectively generally covariant (the gravity property), they are Fock spaces with energy bounded from below (the quantum property), and all matrix elements of normal ordered operators are manifestly finite (the consistency property). Moreover, I showed in \cite{11} that trajectory analogs of the Einstein and geodesic equations are well defined, thus making the connection to Einsteinian gravity closer.

Finally, there is also a philosophical motivation for studying space-time diffeomorphisms. Namely, from a passive point of view, a diffeomorphism is simply a coordinate transformation. Because every physical object must be invariantly defined, rather than being an artifact of the choice of coordinate system, it must transform consistently under arbitrary space-time diffeomorphisms, i.e. as a representation. This also contains information about the dynamics, since the Hamiltonian is itself a space-time diffeomorphism; it is the generator of rigid time translations. Thus, the \textit{diff} \((N+1)\) representation theory amounts to a classification of all physical objects. A list of unitary irreps is urgently needed.

References
[1] Ademollo, M., L. Brink, A. D’Adda, R. D’Auria, E. Napolitano, S. Sciuto, E. Del Giudice, P. Di Vecchia, S. Ferrara, F. Gliozzi, R. Musto, R. Pettorino and J. Schwarz, *Dual string with U(1) color symmetry*, Nucl. Phys. **B111**, 77–110 (1976).

[2] Ademollo, M., L. Brink, A. D’Adda, R. D’Auria, E. Napolitano, S. Sciuto, E. Del Giudice, P. Di Vecchia, S. Ferrara, F. Gliozzi, R. Musto and R. Pettorino, *Supersymmetric strings and color confinement*, Phys. Lett. **62B**, 105–110 (1976).

[3] Bernstein, J.N. and D.A. Leites, *Invariant differential operators and irreducible representations of the Lie superalgebras of vector fields*, Sel. Math. Sov. **1**, 143–160 (1981).

[4] Dzhumadildaev A., *Virasoro type Lie algebras and deformations*, Z. Phys. C **72**, 509–517 (1996).

[5] Eswara Rao, S., R.V. Moody and T. Yokonuma, *Lie algebras and Weyl groups arising from vertex operator representations*, Nova J. of Algebra and Geometry **1**, 15–57 (1992).

[6] Eswara Rao, S. and R.V. Moody, *Vertex representations for N-toroidal Lie algebras and a generalization of the Virasoro algebra*, Commun. Math. Phys. **159**, 239–264 (1994).

[7] Fabbri, M. and R.V. Moody, *Irreducible representations of Virasoro-toroidal Lie algebras*, Commun. Math. Phys. **159**, 1–13 (1994).

[8] Fuks, D.B., *Cohomology of infinite-dimensional Lie algebras*, New York: Consultants bureau (1986).

[9] Grozman, P., D. Leites and I. Shchepochkina, *Lie superalgebras of string theories*, hep-th/9702120 (1997).

[10] Kac, V.G., *Lie superalgebras*, Adv. Math. **26**, 8–96 (1977).

[11] Larsson, T.A., *Lowest-energy representations of non-centrally extended diffeomorphism algebras*, physics/9705040 (1997).

[12] Leites, D., *New Lie superalgebras and mechanics*, Sov. Math. Doklady **18**, 1277–1280 (1977).
[13] Leites, D., *Introduction to the supermanifold theory*, Russ. Math. Surveys **35**:1, 1–64 (1980).

[14] Moody, R.V., S. Eswara Rao and T. Yokonoma, *Toroidal Lie algebras and vertex representations*, Geom. Ded. **35**, 283–307 (1990).

[15] Neveu A. and J. Schwartz, *Factorizable dual model of pions*, Nucl. Phys. **B31**, 86–112 (1971).

[16] Ramond, P., *Dual theory for free fermions*, Phys. Rev. **D3**, 2415–2418 (1971).

[17] Rudakov, A. N., *Irreducible representations of infinite-dimensional Lie algebras of Cartan type*, Math. USSR Izv. **8**, 836–866 (1974).