Solutions for the MaxEnt problem with symmetry constraints

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Received: 29 July 2018 / Accepted: 29 July 2019 / Published online: 13 August 2019 © Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
In this paper, we deal with the situation in which the unknown state of a quantum system has to be estimated under the assumption that it is prepared obeying a known set of symmetries. We present a system of equations and an explicit solution for the problem of determining the MaxEnt state satisfying these constraints. Our approach can be applied to very general situations, including symmetries of the source represented by Lie and finite groups.

Keywords Maximum entropy principle · Symmetries in quantum mechanics · Quantum state estimation

1 Introduction

The principle of maximum entropy (MaxEnt principle) is a powerful technique for estimating states of probabilistic models [1–4]. This principle states that the most suitable probability distribution compatible with the known data is the one with largest entropy [1,2]. It finds applications in many fields of research [4–20]. In particular, it has become very useful in quantum information theory for estimating quantum
states [17,21,22]. It can also be extended to a very general family of probabilistic models [23].

In this paper, we want to apply the MaxEnt principle to physical systems with symmetry constraints. Symmetries play a key role in the study of physical systems. They are mathematically described by group transformations that leave some physical properties unchanged. The problem of determining the MaxEnt state for generalized probabilistic models under quite general symmetry constraints was studied in [24]. In that paper, the MaxEnt approach was reformulated allowing the inclusion of group transformations representing physical symmetries, and the conditions for the existence of the solution were determined.

In this work, we take a step further by presenting an analytical expression for the MaxEnt state with symmetry constraints, suitable for a vast family of quantum models. This can be useful for the problem of quantum state estimation in quantum information processing problems. As is well known, the number of measurements needed to perform a complete quantum tomography on an unknown state, increases with the dimension of the system [25,26]. Thus, this procedure becomes inviable in practice, specially for multiqubit states, for which the number of measurements needed scales exponentially with the number of particles. This situation gives a major relevance to the quantum state estimation techniques that use fewer measurements. In this paper, we deal with the situation in which the state estimation using MaxEnt is performed under the assumption that the state is prepared obeying a known set of symmetries. This assumption reduces the complexity of the problem in a substantial way [24]. Our approach can be applied to very general situations, including symmetries of the source represented by both, Lie and finite groups.

The paper is organized as follows. In Sect. 2, we revise the traditional version of the MaxEnt estimation problem, and we reformulate it by including additional symmetry constraints. In Sect. 3, we discuss a classical example that shows the need of a systematic treatment of the problem. Moreover, we consider the general case of the classical MaxEnt estimation problem for a finite system with symmetry constraints. In Sect. 4, we present the solution to the quantum MaxEnt problem with symmetry constraints. Finally, in Sect. 5, we draw some conclusions.

2 Principle of maximum entropy

The MaxEnt principle states that the probability distribution which best fits the available information is the one that maximizes the entropy (also called missing information). This principle was first introduced by Jaynes in two seminal papers [1,2], where he emphasized a natural correspondence between statistical mechanics and information theory. He argued that classical and quantum statistical mechanics can be formulated on the basis of information theory if the probability distribution or the density operator is obtained from the MaxEnt principle [1–3].

In classical mechanics, the probability distribution is defined over the phase space of the system and its entropy is given by Shannon’s entropy. In quantum mechanics, states are described by density operators acting on a Hilbert space and the entropy of the system is given by von Neumann’s entropy, which can be considered as a
natural noncommutative generalization of Shannon’s entropy \[27,28\]. While in this work we use the Shannon and von Neumann entropies, it is important to remark that there exist other entropic quantities that find many applications in diverse fields of research \[27,29–37\]. In the case that the available information is given by the mean value of some set of observables, the probability distribution can be obtained using Lagrange multipliers. In the classical case, these observables are functions over phase space, and in the quantum case they are self-adjoint operators acting on a Hilbert space.

The classical and quantum versions of the MaxEnt problem can be stated as follows\(^1\):

- **Classical MaxEnt**
  Given \(n\) observables \(A_i\) (\(1 \leq i \leq n\)), with \(m\) outcomes \(A_i(j)\) (\(1 \leq j \leq m\)), determine the probability distribution \(p_j\) which maximizes the Shannon entropy

\[
H_S = - \sum_{j=1}^{n} p_j \ln p_j,
\]

and satisfies the constraints

\[
\langle A_i \rangle = \sum_{j=1}^{m} p_j A_i(j) = a_i, \quad \forall \ i = 1, \ldots, n.
\]

The solution is given by (see, for example, [1])

\[
p_j = \frac{e^{\sum_{i=1}^{n} \lambda_i A_i(j)}}{Z}, \quad 1 \leq j \leq m.
\]

where \(Z = \sum_{j=1}^{m} e^{\sum_{i=1}^{n} \lambda_i A_i(j)}\) is the partition function, and the Lagrange multipliers \(\lambda_i\) are given by the relation

\[
a_i = \frac{\partial}{\partial \lambda_i} \ln Z, \quad 1 \leq i \leq n.
\]

- **Quantum MaxEnt**
  Given \(n\) observables \(\hat{A}_i\) (\(1 \leq i \leq n\)), determine the density matrix \(\hat{\rho}\) which maximizes the von Neumann entropy

\[
H_{VN} = -\text{Tr}(\hat{\rho} \ln \hat{\rho}),
\]

and satisfies the constraints

\[
\langle \hat{A}_i \rangle = \text{Tr}(\hat{\rho} \hat{A}_i) = a_i, \quad \forall \ i = 1, \ldots, n.
\]

\(^1\) We restrict ourselves to finite cases in order to simplify the exposition. For a more general treatment, see [3].
The solution is given by (see, for example, [2])

\[ \hat{\rho} = \frac{e^{\sum_{i=1}^{n} \lambda_i \hat{A}_i}}{Z}, \]  

(7)

with \( Z = \text{Tr}(e^{\sum_{i=1}^{n} \lambda_i \hat{A}_i}) \), and the Lagrange multipliers are given by the relations

\[ a_i = \frac{\partial}{\partial \lambda_i} \ln Z, \quad 1 \leq i \leq n. \]  

(8)

In this work, we are going to consider the MaxEnt problem with additional constraints given by symmetries represented by the action of a group \( G \). More specifically, we aim to solve the following problems:

- **Classical MaxEnt with symmetries**
  Given a group \( G \) and \( n \) observables \( A_i \) \((1 \leq i \leq n)\), with \( m \) outcomes \( A_i(j) \) \((1 \leq j \leq m)\), determine the probability distribution \( p_j \) (i.e., \( \sum_{j=1}^{m} p_j = 1 \) and \( p_j \geq 0 \) for \( 1 \leq j \leq m \)) which maximizes the Shannon entropy and satisfies the constraints

\[ \langle A_i \rangle = \sum_{j=1}^{m} p_j A_i(j) = a_i, \quad \forall i = 1, \ldots, n. \]  

(9)

\[ p_{g(j)} = p_j, \quad \forall g \in G, \quad \forall j = 1, \ldots, m. \]  

(10)

- **Quantum MaxEnt with symmetries**
  Given a group \( G \) representing a physical symmetry and \( n \) observables \( \hat{A}_i \) \((1 \leq i \leq n)\), determine the density matrix \( \hat{\rho} \) which maximizes the von Neumann entropy and satisfies the constraints

\[ \langle \hat{A}_i \rangle = \text{Tr}(\hat{\rho} \hat{A}_i) = a_i, \quad \forall i = 1, \ldots, n. \]  

(11)

\[ \hat{U}_g \hat{\rho} \hat{U}_g^\dagger = \hat{\rho}, \quad \forall g \in G, \]  

(12)

where \( \hat{U}_g \) is the unitary representation of the group element \( g \).

The conditions for the existence of the solutions of these two problems (under more general constraints) were discussed in [24]. In the following sections, we look for solutions to the classical and quantum MaxEnt problems with symmetry constraints given by a group \( G \).

### 3 Motivation: classical MaxEnt

In this section, we consider a classical and discrete example of the MaxEnt problem with and without symmetry constraints, in order to illustrate the differences between both cases. This simple example serves as a motivation for the systematic treatment
of the problem that we present for the quantum case in the next section. Moreover, we consider the general case of the classical MaxEnt problem for a finite system with symmetry constraints.

A fair dice must be manufactured in such a way that all faces are equivalent—from a physical point of view and for all practical purposes. In that case, there is a symmetry group that leaves invariant the probabilities of all outcomes, equal to 1/6. The action of an element of this group is represented by a permutation of the outcomes.

Now suppose that the dice is fabricated in such a way that the sixth face is heavier than the others, and the remaining faces are designed in an equivalent way. It is not hard to imagine an scenario in which such a breaking of the symmetry makes the sixth face more likely than the others, while faces 2, 3, 4 and 5 are equally likely. Then, the symmetry group of the loaded dice is reduced to a subgroup of the full symmetry group. This subgroup, which we call \( G \), leaves invariant the probabilities of faces 2, 3, 4 and 5, and thus, it is formed by all possible permutations of these outcomes. In what follows, we are going to show that, without taking into account the constraints of the dice given by the symmetry group \( G \), the MaxEnt solution for the probabilities is not adequate.

The set of outcomes of the die is given by \( \Lambda = \{1, 2, 3, 4, 5, 6\} \), and \( p_1, \ldots, p_6 \) are the corresponding probabilities. We consider the observable \( A \) taking the values \( A(j) = j \) for all \( j \in \Lambda \). The constraint given by the mean value \( \langle A \rangle \) of observable \( A \) is \( \sum_{j=1}^{6} j p_j = a \), where \( a \) is a real number between 1 and 6. A fair dice satisfies \( \langle A \rangle = 3.5 \).

On the one hand, we consider the situation in which we only know that the loaded dice has mean value \( \langle A \rangle = a \). If we consider the solution of the MaxEnt problem without taking into account the symmetry constraints (i.e., without considering that the faces 2, 3, 4 and 5 are all equally likely), according to Eq. (3), we obtain the following probabilities:

\[
p_j = \frac{e^{j\lambda}}{Z}, \quad 1 \leq j \leq m,
\]

with \( Z = \sum_{j=1}^{6} e^{j\lambda} \), and the Lagrange multiplier \( \lambda \) is given by Eq. (4),

\[
a = \frac{\sum_{j=1}^{6} j e^{j\lambda}}{Z}.
\]

It should be noted that for \( a \neq 3.5 \), we obtain \( \lambda \neq 0 \), and therefore, all the \( p_j \)'s are different from each other. This solution does not satisfy the symmetry constraint.

On the other hand, we consider that we know that there is a symmetry in the fabrication process of the dice implying that \( p_2 = p_3 = p_4 = p_5 \). This symmetry is described by the group \( G \) generated by all possible transformations which permute outcomes 2, 3, 4, 5. Under these conditions, the MaxEnt problem consists of obtaining the probability distribution \( p_1, p_2 = p_3 = p_4 = p_5, p_6 \) which maximizes the Shannon entropy

\[
H_S = -p_1 \ln p_1 - 4p_2 \ln p_2 - p_6 \ln p_6,
\]

and satisfies the constraints
\[ \sum_{j=1}^{6} p_j = p_1 + 4p_2 + p_6 = 1, \quad \langle A \rangle = a = p_1 + 14p_2 + 6p_6. \]  

(16)

To solve this problem, we can use the method of Lagrange multipliers. We define the Lagrangian function

\[ \mathcal{L}(p_1, p_2, p_6) = -p_1 \ln p_1 - 4p_2 \ln p_2 - p_6 \ln p_6 + \lambda_0 (p_1 + 4p_2 + p_6 - 1) \]

\[ + \lambda (p_1 + 14p_2 + 6p_6 - a), \]

and we find the stationary points, i.e., we solve the following equations:

\[ \frac{\partial \mathcal{L}}{\partial p_1} = -\ln p_1 - 1 + \lambda_0 + \lambda = 0, \]  

(17)

\[ \frac{\partial \mathcal{L}}{\partial p_2} = -4 \ln p_2 - 4 + 4\lambda_0 + 14\lambda = 0, \]  

(18)

\[ \frac{\partial \mathcal{L}}{\partial p_6} = -\ln p_6 - 1 + \lambda_0 + 6\lambda = 0. \]  

(19)

It is easy to see that the probability distribution is given by

\[ p_1 = \frac{e^{\lambda}}{Z}, \quad p_j = \frac{e^{3.5\lambda}}{Z} \quad j = 2, \ldots, 5, \quad p_6 = \frac{e^{6\lambda}}{Z}, \]  

(20)

with

\[ Z = e^{1-\lambda_0} = e^{\lambda} + 4e^{3.5\lambda} + e^{6\lambda}, \]  

and \( \lambda \) is obtained from the relation

\[ a = \frac{\partial}{\partial \lambda} \ln Z \]

\[ a = \frac{e^{\lambda} + 14e^{3.5\lambda} + 6e^{6\lambda}}{Z}. \]  

(21)

In this way, we arrive at a situation in which the solution of the MaxEnt problem can be different if we introduce information concerning its symmetries.

If we want to deal with more complex problems, a systematic treatment is needed. In what follows, we discuss the general case of the classical MaxEnt problem with symmetry constraints.

We consider a classical physical system with a finite set of states given by \( \Lambda = \{1, \ldots, m\} \), with \( p_1, \ldots, p_m \) being the corresponding probabilities. Moreover, we consider \( n \) observables \( A_i \), with \( m \) outcomes \( A_i(j) \) \((1 \leq j \leq m)\), and mean values

\[ \langle A_i \rangle = \sum_{j=1}^{m} p_j A_i(j) = a_i. \]

The system also has a symmetry constraint given by a group \( G \), which implies

\[ p_j = p_{g(j)}, \quad \forall \ g \in G, \quad \forall \ j = 1, \ldots, m. \]  

(22)

The MaxEnt principle implies maximizing Shannon entropy

\[ H_S = -\sum_{j=1}^{m} p_j \ln p_j, \]  

(23)

under the following constraints:
\[ p_j = p_{g(j)}, \quad \forall \ g \in G, \quad \forall \ j = 1, \ldots, m, \quad (24) \]

\[ 1 = \sum_{j=1}^{m} p_j, \quad (25) \]

\[ \langle A_i \rangle = \sum_{j=1}^{m} p_j A_i(j) = a_i, \quad \forall \ i = 1, \ldots, n. \quad (26) \]

The symmetry constraints (24) allow to regroup the probabilities \( p_j \) in sets of equal probabilities. That means that there are \( r \) sets of indexes \( J_l \subseteq \{1, \ldots, m\} \), with \( 1 \leq l \leq r \), such that

\[ \bigcup_{l=1}^{r} J_l = \{1, \ldots, m\}, \quad J_l \cap J_{l'} = \emptyset, \quad l \neq l' \quad (27) \]

and \( p_j = p_{j'} \), if and only if, there is a set \( J_l \) such that \( j, j' \in J_l \).

Let \( d_l \) be the cardinality of \( J_l \), and for each \( l = 1, \ldots, r \), we define \( q_l = p_j \), with \( j \) an arbitrary index in \( J_l \). Now, we can express the Shannon entropy and the constraint equations (24)–(26) in terms of \( q_l \),

\[ H_S = - \sum_{j=1}^{m} p_j \ln p_j = - \sum_{l=1}^{r} d_l q_l \ln q_l \]

\[ 1 = \sum_{j=1}^{m} p_j = \sum_{l=1}^{r} d_l q_l \]

\[ \langle A_i \rangle = a_i = \sum_{j=1}^{m} p_j A_i(j) = \sum_{l=1}^{r} q_l \sum_{j \in J_l} A_i(j) = \sum_{l=1}^{r} d_l q_l \tilde{A}_i(l), \quad \forall \ i = 1, \ldots, n, \]

with \( \tilde{A}_i(l) = \frac{1}{d_l} \sum_{j \in J_l} A_i(j) \).

To maximize the Shannon entropy, we use the method of Lagrange multipliers. The Lagrangian function is given by

\[ L(q_1, \ldots, q_r) = - \sum_{l=1}^{r} d_l q_l \ln q_l + \lambda_0 \left( \sum_{l=1}^{r} d_l q_l - 1 \right) + \sum_{i=1}^{n} \lambda_i \left( \sum_{l=1}^{r} d_l q_l \tilde{A}_i(l) - a_i \right). \quad (28) \]

The equations for the stationary points are the following ones:

\[ \frac{\partial L}{\partial q_l} = -d_l \ln q_l - d_l + d_l \lambda_0 + d_l \sum_{i=1}^{n} \lambda_i \tilde{A}_i(l) = 0. \quad (29) \]
It is easy to see that the probability distribution is given by

\[ q_l = \frac{e^{\sum_{i=1}^{n} \lambda_i \hat{A}_i(l)}}{Z}, \quad \text{with} \quad Z = e^{1-\lambda_0} = \sum_{l=1}^{r} d_l e^{\sum_{i=1}^{n} \lambda_i \hat{A}_i(l)}, \tag{30} \]

and the Lagrange multipliers \( \lambda_i \) are obtained from the equations

\[ a_i = \frac{\sum_{l=1}^{r} d_l \hat{A}_i(l) e^{\sum_{i=1}^{n} \lambda_i \hat{A}_i(l)}}{Z}. \tag{31} \]

The classical MaxEnt problem illustrates the relevance of taking into account the symmetry constraints for estimating probabilities. In the next section, we discuss the quantum case.

4 Quantum MaxEnt with symmetries

In this section, we study the quantum MaxEnt problem with symmetry constraints. We consider a quantum physical system with Hilbert space \( \mathcal{H} \) and \( n \) observables \( \hat{A}_i \). Their mean values are given by \( \text{Tr}(\hat{\rho} \hat{A}_i) = a_i \), with \( \hat{\rho} \) an unknown quantum state.

4.1 Lie groups

We start with a system that has a symmetry given by a continuous group \( G \). We will work with the unitary representation of \( G \) in the Hilbert space, and we assume that it is a connected Lie group. Hence, any \( g \in G \) will be represented by a unitary operator \( \hat{U}_g \), and—as it is well known—we can write \( \hat{U}_g = e^{i \hat{Q}} \), where \( \hat{Q} \) is a self-adjoint operator [38, 12.37].

We are looking for states \( \hat{\rho} \) which are invariant under the action of the symmetry group. Therefore, they have to satisfy the condition

\[ \hat{U}_g \hat{\rho} \hat{U}_g^\dagger = \hat{\rho}, \quad \forall \ g \in G. \tag{32} \]

Since the Lie group is connected, the above condition is also valid for elements of the form \( U_g = e^{i \hat{Q} t} \), with \( t \) a real parameter belonging to some interval. By considering elements of this form and using the Taylor expansion, we obtain:

\[ \hat{U}_g \hat{\rho} \hat{U}_g^\dagger - \hat{\rho} = e^{it \hat{Q}} \hat{\rho} e^{-it \hat{Q}} - \hat{\rho} = it [\hat{Q}; \hat{\rho}] + o(t^2) = 0, \tag{33} \]

where \([\hat{Q}; \hat{\rho}]\) is the commutator between \( \hat{Q} \) and \( \hat{\rho} \). Condition (33) is valid for all values of the parameter \( t \) and all elements \( \hat{Q} \) of the Lie algebra \( \mathfrak{g} \) associated with \( G \). As the polynomial functions in the expansion are linearly independent, this condition can only be satisfied if
\[
[\hat{\rho}; \hat{Q}] = 0, \quad \forall \hat{Q} \in \mathfrak{g}.
\] (34)

If we consider a set of generators \(\{\hat{Q}_k\}_{k \in I}\) (\(I\) a set of indexes) of the Lie algebra \(\mathfrak{g}\), the condition (34) can be expressed as follows:

\[
[\hat{\rho}; \hat{Q}_k] = 0, \quad \forall \hat{Q}_k \in \{\hat{Q}_k\}_{k \in I}.
\] (35)

Therefore, the quantum MaxEnt problem with symmetry constraints can be reformulated as follows:

- **Quantum MaxEnt with symmetries** Given a connected unitary Lie group \(G\), and \(n\) observables \(\hat{A}_i\) \((1 \leq i \leq n)\), determine the density matrix \(\hat{\rho}\) which maximizes the von Neumann entropy

\[
H_{\text{VN}} = -\text{Tr}(\hat{\rho} \ln \hat{\rho}),
\] (36)

and satisfies the constraints

\[
\langle \hat{A}_i \rangle = \text{Tr}(\hat{\rho} \hat{A}_i) = a_i, \quad \forall i = 1, \ldots, n,
\] (37)

\[
[\hat{\rho}; \hat{Q}_k] = 0, \quad \forall \hat{Q}_k \in \{\hat{Q}_k\}_{k \in I},
\] (38)

where \(\{\hat{Q}_k\}_{k \in I}\) is a set of generators of the Lie Algebra associated with \(G\).

In what follows, we give an explicit solution of this problem for quantum systems represented by finite-dimensional Hilbert spaces. The infinite-dimensional case is more complicated and will be treated elsewhere.

Let \(m\) be the dimension of the Hilbert space \(\mathcal{H}\) and \(\{\hat{O}_j\}_{1 \leq j \leq m^2}\) a basis for the space of linear operators \(\mathcal{L}(\mathcal{H})\). First, we note that

\[
[\hat{\rho}; \hat{Q}_k] = 0 \iff \text{Tr}([\hat{\rho}; \hat{Q}_k] \hat{O}_j) = 0, \quad \forall j = 1, \ldots, m^2.
\] (39)

Then, using the cyclic property of the trace, we obtain \(\text{Tr}([\hat{\rho}; \hat{Q}_k] \hat{O}_j) = \text{Tr}(\hat{\rho}[\hat{Q}_k; \hat{O}_j])\).

Therefore, we conclude

\[
[\hat{\rho}; \hat{Q}_k] = 0 \iff \text{Tr}(\hat{\rho}[\hat{Q}_k; \hat{O}_j]) = 0, \quad \forall j = 1, \ldots, m^2.
\] (40)

In particular, it is possible to choose the basis \(\{\hat{O}_j\}_{1 \leq j \leq m^2}\) in such a way that all the operators \(\hat{O}_i\) are Hermitian. Moreover, since the generators of the Lie Algebra \(\hat{Q}_k\) are Hermitian, the commutators \([i \hat{Q}_k; \hat{O}_j]\) are also Hermitian. Then,

\[
[\hat{\rho}; \hat{Q}_k] = 0 \iff \text{Tr}(\hat{\rho}[i \hat{Q}_k; \hat{O}_j]) = 0, \quad \forall j = 1, \ldots, m^2,
\] (41)

with \(\{\hat{O}_j\}_{1 \leq j \leq m^2}\) an Hermitian basis of \(\mathcal{L}(\mathcal{H})\). Since \([i \hat{Q}_k; \hat{O}_j]\) are Hermitian operators, the conditions \(\text{Tr}(\hat{\rho}[i \hat{Q}_k; \hat{O}_j]) = 0\) can be considered as constraints for the mean values of a family of auxiliary observables \(\{[i \hat{Q}_k; \hat{O}_j]\}_{k \in I; j = 1, \ldots, m^2}\). It should
be noted that we have proved that the symmetries constraints can be rewritten as linear constraints.

Therefore, for finite-dimensional models, the quantum MaxEnt problem with symmetry constraints is that of determining a density matrix $\hat{\rho}$ which maximizes the von Neumann entropy and satisfies the constraints

$$\langle \hat{A}_i \rangle = \text{Tr}(\hat{\rho} \hat{A}_i) = a_i, \quad \forall \, i = 1, \ldots, n,$$

$$\langle [i \hat{Q}_k; \hat{O}_j] \rangle = \text{Tr}(i \hat{\rho} [\hat{Q}_k; \hat{O}_j]) = 0, \quad \forall \, k \in I, \, \forall \, j = 1, \ldots, m^2. \quad (42)$$

The advantage of this formulation of the problem is that the solution is straightforward. Since the extra conditions are also mean values constraints, the solution has the same form as the standard MaxEnt problem. The explicit solution is given by

$$\hat{\rho} = \frac{e^{\sum_{i=1}^{n} \lambda_i \hat{A}_i + \sum_{k \in I} \sum_{j=1}^{m^2} \gamma_{k,j} [i \hat{Q}_k; \hat{O}_j]}}{Z} \quad \text{Z}, \quad (44)$$

where \{ $\hat{Q}_k$ $\}_{k \in I}$ is a set of generators of the Lie algebra, \{ $\hat{O}_j$ $\}_{1 \leq j \leq m^2}$ is an Hermitian basis of $\mathcal{L}(\mathcal{H})$ and $Z = \text{Tr}(e^{\sum_{i=1}^{n} \lambda_i \hat{A}_i + \sum_{k \in I} \sum_{j=1}^{m^2} \gamma_{k,j} [i \hat{Q}_k; \hat{O}_j]})$. The Lagrange multipliers satisfy the relations

$$a_i = \frac{\partial}{\partial \lambda_i} \ln Z, \quad 1 \leq i \leq n, \quad (45)$$

$$0 = \frac{\partial}{\partial \gamma_{k,j}} \ln Z, \quad 1 \leq j \leq m^2, \, k \in I. \quad (46)$$

In Sect. 4.3., we illustrate with some examples how this method works.

### 4.2 Finite groups

In this section, we focus on a system whose symmetries are represented by a finite group $G$. Again, we will work with the unitary representation of $G$ in the Hilbert space $\mathcal{H}$. Since $G$ is a finite group, each element $\hat{U} \in G$ can be written as

$$\hat{U} = \hat{U}_k \ldots \hat{U}_1 \quad (47)$$

where the $\hat{U}_k$ are the generators of the group, with $k \in I$ and $I$ a set of indexes. The condition $\hat{U} \hat{\rho} \hat{U}^\dagger = \hat{\rho}$ for all $\hat{U} \in G$ implies that $\hat{U}_k \hat{\rho} \hat{U}_k^\dagger = \hat{\rho}$ for all generators of the group. The converse is also true: If $\hat{U}_k \hat{\rho} \hat{U}_k^\dagger = \hat{\rho}$ for all generator $\hat{U}_k$, then, for an arbitrary element $\hat{U} \in G$, we have

$$\hat{U} \hat{\rho} \hat{U}^\dagger = \hat{U}_k \ldots \hat{U}_k \hat{\rho} \hat{U}_k \ldots U_k^\dagger = \hat{U}_k \ldots \hat{U}_{k-1} \hat{\rho} U_{k-1}^\dagger \ldots \hat{U}_{k-1} \hat{\rho} U_{k-1} \ldots \hat{U}_k \hat{\rho} \hat{U}_k \ldots \hat{U}_1 = \hat{\rho} \quad (48)$$

Therefore, the state is invariant under all elements of the group if, and only if, it is invariant under the set of generators.
Then, proceeding similarly as in Sect. 4.1, we choose an hermitian basis \( \{ \hat{O}_j \}_{1 \leq j \leq m^2} \) for the space of linear operators \( \mathcal{L}(\mathcal{H}) \) (with \( m \) the dimension of \( \mathcal{H} \)), and we note that

\[
[\hat{\rho}; \hat{U}_k] = 0 \iff \text{Tr}(\hat{\rho} [i \hat{U}_k; \hat{O}_j]) = 0, \quad \forall j = 1, \ldots, m^2, \tag{49}
\]

Therefore, the quantum MaxEnt problem with a symmetry given by a finite group is equivalent to determine a density matrix \( \hat{\rho} \) which maximizes the von Neumann entropy and satisfies the following constraints:

\[
\langle [i \hat{U}_k; \hat{O}_j] \rangle = \text{Tr}(i \hat{\rho} [\hat{U}_k; \hat{O}_j]) = 0, \quad \forall k \in I, \quad \forall j = 1, \ldots, m^2. \tag{50}
\]

These equations define the conditions that come from group invariance. Extra conditions must be added if we impose additional average value constraints.

### 4.3 Examples

In this section, we discuss some applications to illustrate our method.

- **Qubit**
  
  We considered the simplest quantum system: one qubit. Since the state space of a qubit is homotopic to a sphere, this example gives a graphical representation of how this method works.

  Suppose that we want to determine an unknown state \( \hat{\rho} \) of a qubit system, knowing that it is invariant under rotations along the \( \hat{z} \)-axis. The generator of the group of rotations along the \( \hat{z} \)-axis is \( \hat{J}_z = \hbar \hat{\sigma}_z \), with \( \hat{\sigma}_z \) the Pauli matrix given by

\[
\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{51}
\]

  Therefore, we have to find a state satisfying the condition

\[
e^{-\frac{i}{\hbar} \hat{J}_z \theta} \hat{\rho} e^{\frac{i}{\hbar} \hat{J}_z \theta} = \hat{\rho}, \quad \forall \theta \in [0, 2\pi], \tag{52}\]

  or equivalently, using Eq. (41),

\[
\text{Tr}(\rho [i \hat{\sigma}_z; \hat{O}_j]) = 0, \quad \forall \hat{O}_j \in \{ \hat{O}_j \}_{1 \leq j \leq 4}, \tag{53}
\]

  with \( \{ \hat{O}_j \}_{1 \leq j \leq 4} \) a basis of the space complex matrices \( \mathbb{C}^{2 \times 2} \). If we choose \( \hat{O}_1 = \hat{I} \), \( \hat{O}_2 = \hat{\sigma}_x \), \( \hat{O}_3 = \hat{\sigma}_y \) and \( \hat{O}_4 = \hat{\sigma}_z \), the only nontrivial commutators are the following:

\[
[i \hat{\sigma}_z, \hat{\sigma}_x] = -2\hat{\sigma}_y, \quad [i \hat{\sigma}_z, \hat{\sigma}_y] = 2\hat{\sigma}_x. \tag{54}\]

  Therefore, we have two symmetry constraint equations,

\[
\langle \hat{\sigma}_x \rangle = \text{Tr}(\hat{\rho} \hat{\sigma}_x) = 0, \quad \langle \hat{\sigma}_y \rangle = \text{Tr}(\hat{\rho} \hat{\sigma}_y) = 0. \tag{55}\]
From the general solution given in Eq. (44), we obtain the density operator which maximizes the entropy and satisfies the symmetry constraints,

\[ \hat{\rho} = \frac{e^{\gamma_x \hat{\sigma}_x + \gamma_y \hat{\sigma}_y}}{Z}, \quad Z = \text{Tr}(e^{\gamma_x \hat{\sigma}_x + \gamma_y \hat{\sigma}_y}), \quad (56) \]

and, according with Eq. (46), the Lagrange multipliers are given by the relations

\[ 0 = \frac{\partial}{\partial \gamma_x} \ln Z, \quad 0 = \frac{\partial}{\partial \gamma_y} \ln Z. \quad (57) \]

or equivalently,

\[ 0 = \frac{\partial Z}{\partial \gamma_x}, \quad 0 = \frac{\partial Z}{\partial \gamma_y}. \quad (58) \]

In order to calculate the explicit expression of the quantum MaxEnt state, we use the following relation (see equation (2.231) of [39]):

\[ e^{\gamma_x \hat{\sigma}_x + \gamma_y \hat{\sigma}_y + \lambda \hat{\sigma}_z} = \frac{e^\gamma + e^{-\gamma}}{2} \hat{I} + \frac{e^\gamma - e^{-\gamma}}{2\gamma} (\gamma_x \hat{\sigma}_x + \gamma_y \hat{\sigma}_y), \quad \gamma = \sqrt{\gamma_x^2 + \gamma_y^2}. \quad (59) \]

The partition function is given by \( Z = e^\gamma + e^{-\gamma} \), and Eq. (58) takes the form

\[ \frac{\gamma_x}{\gamma} (e^\gamma - e^{-\gamma}) = 0, \quad \frac{\gamma_y}{\gamma} (e^\gamma - e^{-\gamma}) = 0. \quad (60) \]

Finally, replacing expressions (59), (60) and the partition function expression into the quantum MaxEnt (56), we obtain \( \hat{\rho} = \frac{\hat{I}}{2} \).

This solution is in agreement with what it was expected: The states in the Bloch sphere which are invariant under our symmetry are situated along the zeta axis, and the MaxEnt state is situated in the center of the sphere.

Now, we can see how this result is modified if we add an extra condition for the mean value of some observable, for example \( \langle \hat{\sigma}_z \rangle = \text{Tr}(\hat{\rho} \hat{\sigma}_z) = a \), with \(-1 < a < 1\).

From the general solution given in Eqs. (44)–(46), we obtain

\[ \hat{\rho} = \frac{e^{\lambda \hat{\sigma}_z + \gamma_x \hat{\sigma}_x + \gamma_y \hat{\sigma}_y}}{Z}, \quad Z = \text{Tr}(e^{\lambda \hat{\sigma}_z + \gamma_x \hat{\sigma}_x + \gamma_y \hat{\sigma}_y}), \quad (61) \]

\[ a = \frac{\partial}{\partial \lambda} \ln Z, \quad 0 = \frac{\partial}{\partial \gamma_x} \ln Z, \quad 0 = \frac{\partial}{\partial \gamma_y} \ln Z. \quad (62) \]

Again, we use the relation

\[ e^{\gamma_x \hat{\sigma}_x + \gamma_y \hat{\sigma}_y + \lambda \hat{\sigma}_z} = \frac{e^\gamma + e^{-\gamma}}{2} \hat{I} + \frac{e^\gamma - e^{-\gamma}}{2\gamma} (\gamma_x \hat{\sigma}_x + \gamma_y \hat{\sigma}_y + \lambda \hat{\sigma}_z), \quad (63) \]
with $\gamma = \sqrt{\gamma_x^2 + \gamma_y^2 + \lambda^2}$.

The partition function is given by $Z = e^\gamma + e^{-\gamma}$, and Eq. (62) takes the form

$$\frac{\lambda}{\gamma} (e^\gamma - e^{-\gamma}) = a, \quad \frac{\gamma_x}{\gamma} (e^\gamma - e^{-\gamma}) = 0, \quad \frac{\gamma_y}{\gamma} (e^\gamma - e^{-\gamma}) = 0. \quad (64)$$

Finally, replacing expressions (63), (64) and the partition function expression into the quantum MaxEnt state (61), we obtain

$$\hat{\rho} = \frac{1}{2} \left( \hat{I} + a\hat{\sigma}_z \right).$$

This result is in agreement with what one would expect by appealing to a geometrical argument. It should be stressed that in the last case, the maximization of entropy is unnecessary, because there is only one possible state compatible with the constraints.

- **Qutrit**

The second example is a three-dimensional quantum system. We want to estimate the state of the system, knowing that it is invariant under rotations along the $\hat{z}$-axis. The generator of the rotations group along the $\hat{z}$ axis is $\hat{J}_z$,

$$\hat{J}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (65)$$

Therefore, we have to find a state satisfying the condition

$$e^{-\frac{i}{\hbar} \hat{J}_z \theta} \hat{\rho} e^{\frac{i}{\hbar} \hat{J}_z \theta} = \hat{\rho}, \quad \forall \theta \in [0, 2\pi], \quad (66)$$

or equivalently, using our method

$$\text{Tr}(\hat{\rho}[i\hat{J}_z, \hat{O}_j]) = 0, \quad \forall \hat{O}_j \in \{\hat{O}_j\}_{1 \leq j \leq 9}, \quad (67)$$

with $\{\hat{O}_j\}_{1 \leq j \leq 9}$ a basis of the space of matrices $\mathbb{C}^{3 \times 3}$. We choose the basis given by the identity $\hat{I}$ and the Gell–Mann matrices $\hat{\lambda}_i$ ($i = 1 \ldots 8$),

$$\hat{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\hat{\lambda}_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\hat{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{\lambda}_8 = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}. $$
In this case, the only nontrivial commutators are the following:

\[ [i \hat{J}_z, \hat{\lambda}_1] = -\hbar \hat{\lambda}_2, \quad [i \hat{J}_z, \hat{\lambda}_2] = \hbar \hat{\lambda}_1, \quad [i \hat{J}_z, \hat{\lambda}_4] = -2 \hbar \hat{\lambda}_5, \]
\[ [i \hat{J}_z, \hat{\lambda}_5] = 2 \hbar \hat{\lambda}_4, \quad [i \hat{J}_z, \hat{\lambda}_6] = -\hbar \hat{\lambda}_7, \quad [i \hat{J}_z, \hat{\lambda}_7] = \hbar \hat{\lambda}_6. \]  

(68)

Therefore, we have six symmetry constraint equations,

\[ \text{Tr}(\hat{\rho} \hat{\lambda}_i) = 0, \quad \text{for } i = 1, 2, 4, 5, 6, 7. \]  

(69)

From the general solution given in Eq. (44), we obtain the density operator which maximizes the entropy and satisfies the symmetry constraints,

\[ \hat{\rho} = \frac{e^{\hat{M}}}{Z}, \quad Z = \text{Tr}(e^{\hat{M}}), \]  

(70)

with

\[ \hat{M} = \gamma_1 \hat{\lambda}_1 + \gamma_2 \hat{\lambda}_2 + \gamma_4 \hat{\lambda}_4 + \gamma_5 \hat{\lambda}_5 + \gamma_6 \hat{\lambda}_6 + \gamma_7 \hat{\lambda}_7 \]
\[ = \begin{pmatrix} 0 & \gamma_1 - i \gamma_2 & \gamma_4 - i \gamma_5 \\ \gamma_1 + i \gamma_2 & 0 & \gamma_6 - i \gamma_7 \\ \gamma_4 + i \gamma_5 & \gamma_6 + i \gamma_7 & 0 \end{pmatrix}, \]  

(71)

and the Lagrange multipliers are given by

\[ 0 = \frac{\partial}{\partial \gamma_i} \ln Z, \quad \text{for } i = 1, 2, 4, 5, 6, 7. \]  

(72)

Since \( \hat{M} \) is an Hermitian matrix, it is diagonalizable. Then, there exist a unitary matrix \( \hat{U} \in \mathbb{C}^{3 \times 3} \) and a real diagonal matrix \( \hat{D} \in \mathbb{C}^{3 \times 3} \),

\[ \hat{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \]  

(73)

where \( d_1, d_2, d_3 \) are the eigenvalues of \( \hat{M} \), such that \( \hat{M} = \hat{U} \hat{D} \hat{U}^{-1} \).

Therefore, \( Z = \text{Tr}(e^{\hat{M}}) = e^{d_1} + e^{d_2} + e^{d_3} \), and Eq. (72) takes the form

\[ 0 = e^{d_1} \frac{\partial d_1}{\partial \gamma_i} + e^{d_2} \frac{\partial d_2}{\partial \gamma_i} + e^{d_3} \frac{\partial d_3}{\partial \gamma_i}, \quad \text{for } i = 1, 2, 4, 5, 6, 7. \]  

(74)

The explicit expression of the eigenvalues in terms of the Lagrange multipliers and their derivatives is cumbersome. However, it can be shown that the solution for the Lagrange multipliers is \( \gamma_1 = \gamma_2 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = 0 \). Therefore, the quantum MaxEnt state is \( \hat{\rho} = \hat{I}/3 \).

Now, we can see how this result is modified if we add an extra condition for the mean value of some observable, for example \( \langle \hat{J}_z \rangle = \text{Tr}(\hat{\rho} \hat{J}_z) = \hbar a \), with \( a \in \mathbb{R} \).

In this case, we have seven constraint equations.
\[ \text{Tr}(\hat{\rho} \hat{\lambda}_i) = 0, \quad \text{for } i = 1, 2, 4, 5, 6, 7, \quad \text{Tr} \left( \frac{\hat{\rho} \hat{J}_z}{\hbar} \right) = a. \quad (75) \]

Again, the general solution is given by

\[ \hat{\rho} = \frac{e^{\hat{M}}}{Z}, \quad Z = \text{Tr}(e^{\hat{M}}), \quad (76) \]

with \( \hat{M} = \gamma_0 \hat{J}_z + \gamma_1 \hat{\lambda}_1 + \gamma_2 \hat{\lambda}_2 + \gamma_4 \hat{\lambda}_4 + \gamma_5 \hat{\lambda}_5 + \gamma_6 \hat{\lambda}_6 + \gamma_7 \hat{\lambda}_7 \), and the Lagrange multipliers are given by

\[ a = \frac{\partial}{\partial \gamma_0} \ln Z, \quad 0 = \frac{\partial}{\partial \gamma_i} \ln Z, \quad \text{for } i = 1, 2, 4, 5, 6, 7. \quad (77) \]

In order to solve this problem, we use a numerical algorithm. We consider two cases, \( a = -1/2, 1/2 \), and we minimize the following expression:

\[ \Delta = \left[ \text{Tr} \left( \frac{\hat{\rho} \hat{J}_z}{\hbar} \right) - a \right]^2 + \sum_{i \in \{1,2,4,5,6,7\}} \left[ \text{Tr}(\hat{\rho} \hat{\lambda}_i) \right]^2, \]

taking into account the general form of \( \hat{\rho} \) given in Eq. (76). The results obtained are the following:

For \( a = -1/2 \):

\[ \hat{\rho} = \begin{pmatrix} 0.116 & 0 & 0 \\ 0 & 0.268 & 0 \\ 0 & 0 & 0.616 \end{pmatrix} \quad (78) \]

For \( a = 1/2 \):

\[ \hat{\rho} = \begin{pmatrix} 0.616 & 0 & 0 \\ 0 & 0.268 & 0 \\ 0 & 0 & 0.116 \end{pmatrix} \quad (79) \]

The above examples show that the state of maximum entropy departs from the maximally mixed state, depending on the conditions imposed on the mean values of the observables.

- Two qubits

The third example is a composite system of two spin 1/2 systems. We want to estimate the state of the system, knowing that the first spin is invariant under rotations along the \( \hat{z} \)-axis. The generator of the symmetry is \( \hat{J}_z \otimes \hat{I} \). Moreover, we consider an extra condition for the mean value of the \( \hat{x} \) component of the total spin of the system, i.e., \( \hat{J}_{1x} + \hat{J}_{2x} \),

\[ \langle \hat{J}_{1x} + \hat{J}_{2x} \rangle = \text{Tr} \left[ \hat{\rho} \left( \hat{J}_{1x} + \hat{J}_{2x} \right) \right] = \hbar a / 2. \quad (80) \]

Therefore, we have to find a state satisfying the conditions
\[
\text{Tr}(\hat{\rho}[i \hat{J}_z \otimes \hat{I}, \hat{O}_j]) = 0, \quad \forall \hat{O}_j \in \{\hat{O}_j\}_{1 \leq j \leq 16}, \quad (81)
\]

\[
\text{Tr}\left[\hat{\rho} (\hat{\sigma}_{1x} + \hat{\sigma}_{2x})\right] = a. \quad (82)
\]

with \(\{\hat{O}_j\}_{1 \leq j \leq 16}\) a basis of the space of matrices \(\mathbb{C}^{4 \times 4}\). If we choose the following basis

\[
\hat{O}_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{O}_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\hat{O}_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{O}_4 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\hat{O}_5 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{O}_6 = \begin{pmatrix}
0 & i & 0 & 0 \\
-i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\hat{O}_7 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{O}_8 = \begin{pmatrix}
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\hat{O}_9 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{O}_{10} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
-i & 0 & 0 & 0
\end{pmatrix}
\]

\[
\hat{O}_{11} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{O}_{12} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\hat{O}_{13} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad \hat{O}_{14} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
0 & -i & 0 & 0
\end{pmatrix}
\]

\[
\hat{O}_{15} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \hat{O}_{16} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & -i & 0
\end{pmatrix}
\]

we obtain the following equations

\[
\text{Tr}(\hat{\rho} \hat{E}_i) = 0, \quad \text{for } i = 1, \ldots, 8, \quad \text{Tr}[\hat{\rho} (\hat{\sigma}_{1z} + \hat{\sigma}_{2z})] = a, \quad (83)
\]
with the matrixes $\hat{E}_i$ ($i = 1, \ldots, 8$) given by

$\hat{E}_1 = \begin{pmatrix} 0 & 0 & -i & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\hat{E}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\hat{E}_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\hat{E}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$\hat{E}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\hat{E}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\hat{E}_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$, $\hat{E}_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

From the general solution given in Eq. (44), we obtain the density operator which maximizes the entropy and satisfies the constraint equations (83),

$$\hat{\rho} = e^{\hat{M}} / Z, \quad Z = \text{Tr}(e^{\hat{M}}), \quad (84)$$

with $\hat{M} = \gamma_0 (\hat{\sigma}_{1x} + \hat{\sigma}_{2x}) + \sum_{i=1}^{8} \gamma_i \hat{E}_i$, and the Lagrange multipliers are given by

$$a = \frac{\partial}{\partial \gamma_0} \ln Z, \quad 0 = \frac{\partial}{\partial \gamma_i} \ln Z, \quad \text{for } i = 1, \ldots, 8. \quad (85)$$

In order to solve this problem, we use again a numerical algorithm. We consider three cases, $a = -1/2, 0, 1/2$, and we minimize the following expression:

$$\Delta = \left[ \text{Tr}(\hat{\rho} (\hat{\sigma}_{1x} + \hat{\sigma}_{2x})) - a \right]^2 + \sum_{i=1}^{8} \left[ \text{Tr}(\hat{\rho} \hat{E}_i) \right]^2$$

taking into account the general form of $\hat{\rho}$ given in Eq. (84).

The results obtained are the following:

For $a = -1$:

$$\hat{\rho} = \begin{pmatrix} 0.25 & -0.25 & 0 & 0 \\ -0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0.25 & -0.25 \\ 0 & 0 & -0.25 & 0.25 \end{pmatrix} \quad (86)$$
For $a = 0$:

$$
\hat{\rho} = \begin{pmatrix}
0.25 & 0 & 0 & 0 \\
0 & 0.25 & 0 & 0 \\
0 & 0 & 0.25 & 0 \\
0 & 0 & 0 & 0.25
\end{pmatrix}
$$

(87)

For $a = 1$:

$$
\hat{\rho} = \begin{pmatrix}
0.25 & 0.25 & 0 & 0 \\
0.25 & 0.25 & 0 & 0 \\
0 & 0 & 0.25 & 0.25 \\
0 & 0 & 0.25 & 0.25
\end{pmatrix}
$$

(88)

This example shows that the group structure is richer as the number of particles grows, as is expressed by the action of the local groups.

5 Conclusions

In this work, we continued studying the problem posed in [24], namely the state estimation of probabilistic models with symmetries represented by groups of transformations. A complete tomography of a multiparticle state requires a number of measurements that grows exponentially with the number of particles. This number can be significantly reduced whenever some a priori information is available. Our approach can be useful for quantum state estimation problems in which this information is given in terms of symmetries of the system.

First, we revised the traditional version of the classical and quantum MaxEnt estimation problem, and we reformulated them including additional symmetry constraints. We presented the classical MaxEnt estimation problem for finite systems with symmetry constraints. Then, we considered the quantum MaxEnt problem for systems with finite dimension and with symmetries represented by Lie and finite groups.

Finally, we proved that the symmetry constraints can be restated as a set of linear equations, and we presented a solution. We illustrated how our method works by showing some examples.

Acknowledgements This research was funded by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET).

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