Dynamic Tsallis entropy for simple model systems

Nail R. Khusnutdinov, Renat M. Yulmetyev, and Natalya A. Emelyanova

Department of Theoretical Physics, Kazan State Pedagogical University, Mezhlauk 1, 420021, Kazan, Russia

In this paper we consider the dynamic Tsallis entropy and employ it for four model systems: (i) the motion of Brownian oscillator, (ii) the motion of Brownian oscillator with noise, (iii) the fluctuation of particle density in hydrodynamics limit as well as in (iv) ideal gas. We show that the small value of parameter nonextensivity $0 < q < 1$ works as non-linear magnifier for small values of the entropy. The frequency spectra become more sharp and it is possible to extract useful information in the case of noise. We show that the ideal gas remains non-Markovian for arbitrary values of $q$.

I. INTRODUCTION

There is a great interest towards to studying complex systems of physical, chemical, biological, physiological and financial origin by using different approaches and concepts [3, 6]. Significant role in these investigations plays concept of information entropy such as Shannon and Renyi entropy, Kolmogorov-Sinai entropy rate [4, 8, 9, 13, 17]. In the Refs. [20, 21, 22] the notion of dynamical information Shannon entropy [26] has been defined for complex systems. It was suggested to generalize the Shannon entropy by considering square of time correlation function (TCF) as a probability of state. Then the entropy becomes a function of time. The dynamic Shannon entropy has successfully used to obtain new information in dynamics of RR-intervals from human ECG, physiological activity of the individuals and short-time human memory [20, 21, 22]. The investigations reveal also the great role of frequency spectrum of dynamic Shannon entropy.

A new approach for entropy was suggested by Tsallis in Ref. [16] (the earlier works on this subject see in Ref. [2]). This entropy is characterized by one parameter $q$ which was called as nonextensive parameter. The natural boundary of this parameter is unit. It means that for $q \to 1$ the Tsallis entropy transforms to Shannon entropy. Nevertheless, it was often considered the cases with $q > 0$ and even the limit $q \to \infty$. There is/was huge activity for application of this new notion to various complex systems (see, for examples, Ref. [17] and reference therein). It was shown that this entropy gives a new kind of distribution. Many complex systems in nature are described by this non-Gaussian distribution with different values of parameter $q$.

The goal of this paper is generalization of dynamic Shannon entropy in the manner as Tsallis entropy generalizes the Shannon entropy. In other words we consider the Tsallis entropy with square of TCF as probability of state. We refer this entropy as dynamic Tsallis entropy (DTE). The real systems in nature have very complex behavior – the useful signal is lost in “see” of noises and random unexpected influences. For this reason we would like to consider the dynamic Tsallis entropy for model systems, which nevertheless have deep physical contents. Application of the present theory to the alive systems was considered in Ref. [23]. We employ the information Tsallis entropy and its frequency spectrum with different values of parameter $q$ for model systems. We exploit four well-known models. First one is the TCF of position of oscillator which performs the Brownian motion. In order to consider more real situation we use the motion of Brownian oscillator with noise in second model. The noise is modelled by generator of random numbers. Third model is the TCF of relaxation of particle density fluctuations in hydrodynamics limit (the Landau-Placzek formula). We consider a specific medium – Helium. The fourth model describes the relaxation of particle density fluctuations in ideal gas. In the later case we additionally calculate the spectrum of non-Markovity parameter for different $q$. The non-Markovity parameter and its spectrum firstly was suggested in Refs. [14, 15] and it was calculated for ideal gas in Ref. [1]. This parameter characterizes the statistical memory effects in systems. In present paper we use slightly different approach and define the non-Markovity parameter in terms of the entropy.

In all considered models the behavior of the dynamic Tsallis entropies and their frequency spectrum crucially depend on the parameter $q$. We observe that decreasing of this parameter $q \to 0$ allows us to enlarge the fine structure of entropies and to make their frequency spectrum more sharp.

The organization of this paper is as follows. In section II we discuss the well known hierarchy of Zwanzig-Mori’s kinetic equations and general properties of the DTE and define different kind of relaxation times. The
TCF of Brownian motion of oscillator is considered in section III and with noise in section IV. The formula of Landau and Placzek for relaxation of density fluctuations are exploited in section V. The TCF of density fluctuations in ideal gas is used in section VI. We finish our paper by concluding remarks in last section VII.

II. THE ZWANZIG-MORI HIERARCHY AND DYNAMIC TSALLIS ENTROPIES

At the beginning we shortly discuss the well known hierarchy of Zwanzig-Mori equations \[10, 24\]. Let us consider the dynamic variable \( \delta A(t) \). It may be, for example, the Fourier component of density fluctuation. This variable obeys to the Liouville equation:

\[
\frac{d\delta A(t)}{dt} = i\hat{L}\delta A(t). 
\]

Applying \( n \) times the Liouville operator \( \hat{L} \) to the initial variable \( \delta A(0) \) we obtain the infinite set of variables \( B_n(0) \)

\[
B_n(0) = \hat{L}^n\delta A(0),
\]

by using which and the Liouville equation we may obtain the initial dynamical variable in arbitrary moment of time

\[
\delta A(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} B_n.
\]

Applying the Gram-Schmidt orthogonalization procedure [11] to this set of functions we obtain the complete set of dynamic variables \( W_n \) which are orthogonal at the initial time

\[
\langle W^*_n W_l \rangle = \langle |W_n|^2 \rangle \delta_{n,l},
\]

where \( \langle \ldots \rangle \) denotes the statistical average over Gibbs ensemble. If the dynamic variable is evaluated by the Liouville’s operator, then the orthogonality is preserved at any moment of time due to the self-adjointness of the Liouville’s operator.

The time correlation function \( M_0 \) of variable \( W_0 = \delta A \) is defined as follows:

\[
M_0(t) = \frac{\langle W_0(0)^*W_0(t) \rangle}{\langle |W_0|^2 \rangle} = \frac{\langle W_0^* \exp(i\hat{L}t)W_0 \rangle}{\langle |W_0|^2 \rangle}.
\]

It is well known that this TCF obeys to the infinite hierarchy of the Zwanzig-Mori’s kinetic equations

\[
\frac{dM_n(t)}{dt} = i\omega_0^{(n)} M_n(t) - \Omega_{n+1}^2 \int_0^t dt' M_{n+1}(t') M_n(t - t'),
\]

(1)

where

\[
\omega_0^{(n)} = \frac{\langle W_n^* \hat{L}W_n \rangle}{\langle |W_n|^2 \rangle}, \quad \Omega_n^2 = \frac{\langle |W_n|^2 \rangle}{\langle |W_{n-1}|^2 \rangle}
\]

and

\[
M_n(t) = \frac{\langle W_n^* \exp(i\hat{L}_2^{(n)}t)W_n \rangle}{\langle |W_n|^2 \rangle}.
\]

(2)

The operator \( \hat{L}_2^{(n)} \) is defined in following manner:

\[
\hat{L}_2^{(n)} = P_{n-1}P_{n-2}\ldots P_0\hat{L}P_1\ldots P_{n-2}P_{n-1}
\]

in terms of projectors \( P_n = 1 - \Pi_n \), where the \( \Pi_n \) is projector for state \( W_n \):

\[
\Pi_n = \frac{W_n\langle W_n^* \rangle}{\langle |W_n|^2 \rangle}.
\]
The functions $M_n$ for $n \geq 1$ are not, in fact, usual TCFs because the operator $\exp(i\sum_{22}^{(n)}t)$ is not operator of evolution. Nevertheless, we will refer for these functions as TCF’s for the next dynamic variables.

The functions $M_n$ are considered as functions characterizing the statistical memory of the system. In order to describe quantitatively the non-Markovity of hierarchy, the parameter of non-Markovity and its spectrum were introduced in Refs. [14, 15]. The spectrum [24] of this parameter is defined as ratio of two neighboring relaxation times

$$\epsilon_n = \frac{\tau_n}{\tau_{n+1}}.$$  

The relaxation time $\tau_n$ of function $M_n$ was defined as real part of the Laplace image (see Eq.(4)) of this function at zero point:

$$\tau_n = \Re\bar{M}_n(0) = \Re \int_0^\infty M_n(t) dt,$$

where $\Re$ means a real part. Because of the function $M_{n+1}$ is an integral core of integro-differential equation (4) for $M_n$, then this parameter [22] compares the integral core with function itself. More precisely we compare squares under TCFs. If parameter $\epsilon_n$ is around unit, this level (level means $n$) is non-Markovian: we can not transform integro-differential equation for $M_n$ to differential one, and vice versa, if this parameter tends to infinity the core has sharp peak and we may transform the integro-differential equation for $M_n$ to differential one. In this case there is no memory (integral) in this level. In this paper we use slightly different definition for relaxation time (see Sec.VI).

Applying the Laplace transformation

$$\tilde{M}_n(s) = \int_0^\infty dt e^{-st} M_n(t)$$

we transform this hierarchy to the infinite system of algebraic equations

$$\tilde{M}_n(s) = [s - i\omega_0^{(n)} + \Omega_n^{2}\bar{M}_{n+1}(s)]^{-1},$$

by using which we may express $M_n$ in term of zero TCF $M_0$.

In statistical physics of non-equilibrium systems the time correlation function (TCF) acts as the function of distribution and pair correlation and can be used to calculate different thermodynamical parameters and the spacial structure of the system [27]. For many physical discrete systems it is impossible to find a distribution function. For this reason it seems optimal to obtain the TCF for investigations of complex systems with the help of integro-differential equations which are based on small increments of time and independent variables.

The set of the measured parameters of the complex system may be represented as a set of fluctuations [23]

$$Z = \{\zeta(T), \zeta(T + \tau), \zeta(T + 2\tau), \cdots, \zeta(T + k\tau), \cdots, \zeta(T + \tau N - \tau)\},$$

where $\zeta$ represents the fluctuation $\delta x$ of some quantity $x$.

Let us define the sampling by length $k$, which starts at the moment $T + m\tau$ by the relation

$$\zeta_{m+k} = \{\zeta(T + m\tau), \cdots, \zeta(T + m\tau + (k-1)\tau)\}.$$

The operator which projects the sampling by length $k$ on the sampling at the initial moment of time has the form

$$\Pi = \frac{\langle \zeta_k^0 | \zeta_k^0 \rangle}{\langle \zeta_k^0 | \zeta_k^0 \rangle},$$

where angle brackets mean the scalar product (time average). The utilization of this projection operator allows us to represent the sampling as a sum of two independent parts

$$\zeta_{m+k} = \zeta_{m+k}^' + \zeta_{m+k}^'' ,$$

where

$$\zeta_{m+k}^' = \Pi \zeta_{m+k}^0 = \zeta_k^0 M_0(t),$$

$$\zeta_{m+k}^'' = (1 - \Pi) \zeta_{m+k}^0 = \zeta_{m+k}^0 - \zeta_k^0 M_0(t).$$

(7a)  

(7b)
and TCF \( M_0(t) \) is defined by \[2\].

It is easy to see that

\[
\langle (\zeta_{m+k}^n)^2 \rangle = \langle (\zeta_{m+k}^m)^2 \rangle + \langle (\zeta_{m+k}^n)^2 \rangle,
\]

which is the consequence of orthogonality of \[16\] and \[17\].

Direct calculations yield

\[
\langle (\zeta_{m+k}^n)^2 \rangle = \langle (\zeta_k^0)^2 \rangle M_0(t)^2.
\]

For stationarity processes, when dispersion does not depend on time, we obtain

\[
\langle (\zeta_{m+k}^n)^2 \rangle = \langle (\zeta_k^0)^2 \rangle M_0(t)^2 = \langle (\zeta_{m+k}^n)^2 \rangle(1 - M_0(t)^2).
\]

Therefore the mean-square value of the fluctuations is presented as a sum of two parts

\[
\langle (\zeta_{m+k}^n)^2 \rangle = \langle (\zeta_k^0)^2 \rangle M_0(t)^2 + \langle (\zeta_{m+k}^n)^2 \rangle(1 - M_0(t)^2),
\]

or in a generalized form

\[
\langle (\zeta_{m+k}^n)^2 \rangle = \langle (\zeta_k^0)^2 \rangle M_0(t)^2 + \langle (\zeta_{m+k}^n)^2 \rangle(1 - M_0(t)^2).
\]

The above expression \[9\] has a standard form of a mean value

\[
\langle (\zeta_{m+k}^n)^2 \rangle = \langle (\zeta_k^0)^2 \rangle P(t) + \langle (\zeta_{m+k}^n)^2 \rangle Q(t),
\]

where \( P(t) \) is the probability of the state and \( Q(t) = 1 - P(t) \). By analogy with this formula we define

\[
P_n(t) = |M_n(t)|^2 \quad (Q_n(t) = 1 - |M_n(t)|^2), \quad n \geq 0
\]

as the probability of the creation (annihilation) of correlation of fluctuations (or memory) for the \( n \)th level of relaxation (see, for details Refs. \[10\]).

In the recent work of authors \[22\] the dynamical informational Shannon entropy for the study of complex systems was suggested:

\[
S_n(t) = -\sum_{i=c,a} P_i(t) \ln P_i(t) = -|M_n(t)|^2 \ln |M_n(t)|^2 - \{1 - |M_n(t)|^2\} \ln \{1 - |M_n(t)|^2\}, \quad (12a)
\]

where

\[

\begin{align}
S_n^c(t) &= -|M_n(t)|^2 \ln |M_n(t)|^2, \quad (12b) \\
S_n^a(t) &= -\{1 - |M_n(t)|^2\} \ln \{1 - |M_n(t)|^2\}. \quad (12c)
\end{align}
\]

Here the \( S_n^c(t) \) is the entropy for the stochastic channels of memory creation, and \( S_n^a(t) \) is entropy for the stochastic channels of memory annihilation.

The infinite set of TCFs \( M_n(t) \) produces the infinite set of entropies which are defined by relations

\[
S_n^a[t] = S[M_n(t)], \quad S_n^{nc}[t] = S^c[M_n(t)], \quad S_n^{na}[t] = S^a[M_n(t)].
\]

Because of the TCFs are always smaller then unit then all of these entropies are positive.

We also generalize our definitions of entropy due to Tsallis by introducing parameter of nonextensivity \( q \):

\[

\begin{align}
S_q(t) &= -\frac{(1 - |M_0(t)|^2)^q - 1 + |M_0(t)|^{2q}}{q - 1}, \quad (13a) \\
S_q^c(t) &= -\frac{|M_0(t)|^{2q} - |M_0(t)|^2}{q - 1}, \quad (13b) \\
S_q^a(t) &= -\frac{(1 - |M_0(t)|^2)^q - 1 + |M_0(t)|^2}{q - 1}. \quad (13c)
\end{align}
\]

The infinite set entropies are defined by relations

\[
S_q^a[t] = S_q[M_n(t)], \quad S_q^{nc}[t] = S_q^c[M_n(t)], \quad S_q^{na}[t] = S_q^a[M_n(t)].
\]
We also calculate the frequency spectra \( \tilde{S}_n[\nu], \tilde{S}_{nc}[\nu], \tilde{S}_\alpha[\nu], \tilde{S}_{q}[\nu], \tilde{S}_{q^{nc}}[\nu], \tilde{S}_q^{na}[\nu] \) of these functions and define the spectra of the relaxation times by relations

\[
\tau_n = \tilde{S}_n[\nu]_{\nu=0}, \quad \tau_{nc} = \tilde{S}_{nc}[\nu]_{\nu=0}, \quad \tau_{na} = \tilde{S}_q^{na}[\nu]_{\nu=0},
\]

We define the frequency spectrum as Fourier transformation of these quantities by relation

\[
\tilde{M}_n(\nu) = \int_{-\infty}^{+\infty} M_n(\tau) e^{-2\pi i\nu \tau} d\tau.
\]

To calculate the Fourier transformation of all quantities we use package for numerical Fourier transformation from package "Mathematica". We tabulate functions \( M_n(\tau) \) with step \( \Delta = 1/40 \) in range \( \tau = (0, 60) \). The amount of points \( N = 2401 \). Then the Fourier transform

\[
\tilde{M}_n(\nu) = \int_{-\infty}^{+\infty} M_n(\tau) e^{-2\pi i\nu \tau} d\tau = 2 \int_{0}^{+\infty} M_n(\tau) \cos(2\pi \nu \tau) d\tau
\]

is represented as a function of discrete frequencies \( \nu = \frac{k}{N\Delta} \) \( (k = 0, \ldots, N - 1) \)

\[
\tilde{M}_n(\nu) = 2\Delta \Re \sum_{k=0}^{N-1} M_n(k\Delta) e^{-2\pi i k \nu / N} - M_n(0).
\]  

Let us summarize here some analytical results. In the limit \( q \to 1 \) the formulas \( \tilde{13} \) transform to the equations \( \tilde{12} \). For this reason we may refer for entropies \( \tilde{12} \) as entropies \( \tilde{13} \) at the point \( q = 1 \). The entropy \( S_q^{nc} \) \( \tilde{12} \) amounts to maximum value, \( (2^{1-q} - 1)/(1 - q) \), at the point \( |M_q(t)|^2 = \frac{1}{2} \). For small values of \( |M_q(t)|^2 \) the dependence of entropies \( S_q^{nc} \), \( S_q^{nc} \) changes drastically on value of \( q \). For \( q \ll 1 \) we have

\[
S_q \approx |M_q(t)|^{2q}, \quad S_q^{nc} \approx |M_q(t)|^{2q}, \quad S_q^{na} \approx |M_q(t)|^{2q}.
\]  

These expression are valid for all cases: \( |M_q(t)|^2 \ll q, |M_q(t)|^2 \gg q \) and \( |M_q(t)|^2 \sim q \). For comparison we have for small values of \( |M_q(t)|^2 \): \( S \approx -|M_q(t)|^2 \ln |M_q(t)|^2 \). Therefore \( S_q^{nc} < S_q, S_q, S \) and we observe that decreasing of \( q \) does as magnifier for small values of \( |M_q(t)|^2 \).

For \( q \gg 1 \) and still \( |M_q(t)|^2 \ll 1 \) we have to consider three cases \( q|M_q(t)|^2 \ll 1, \ q|M_q(t)|^2 \sim 1 \) and \( q|M_q(t)|^2 \gg 1 \). We have correspondingly

\[
S_q \approx |M_q(t)|^2, \quad S_q^{nc} \approx \frac{1}{q} |M_q(t)|^2, \quad S_q^{na} \approx |M_q(t)|^2, \quad [q|M_q(t)|^2 \ll 1]
\]

\[
S_q \approx -\frac{1}{q} [(1 - |M_q(t)|^2)^q - 1], \quad S_q^{nc} \approx \frac{1}{q} |M_q(t)|^2, \quad S_q^{na} \approx -\frac{1}{q} [(1 - |M_q(t)|^2)^q - 1], \quad [q|M_q(t)|^2 \sim 1]
\]

\[
S_q \approx \frac{1}{q}, \quad S_q^{nc} \approx \frac{1}{q} |M_q(t)|^2, \quad S_q^{na} \approx \frac{1}{q}, \quad [q|M_q(t)|^2 \gg 1] .
\]

We observe that for sufficiently large \( q \) and small \( |M_q(t)|^2 \ll 1 \) we have: \( S_q, S_q^{nc}, S_q^{na} \ll S \). Therefore we observe that increasing of \( q \) do as demagnification lens for small values of \( |M_q(t)|^2 \). We note that the entropies equal to zero for arbitrary \( q \) for zero value of \( M_q(t) \) as well as for \( M_q(t) = 1 \). It is easy to see, if the TCF \( M_q(t) \) possesses an extremum at some time \( t_0 \) the entropies have extremum, too. The structure of extremums of TCF generates the same structure of extremums of entropies. After this general frameworks let us consider specific models.

III. THE MOTION OF BROWNIAN OSCILLATOR

Let us consider the Brownian oscillator which models the particle with internal oscillatory degree of freedom with frequency \( \omega_0 \). The friction coefficient \( \beta \) describes the movement of the particle in a medium. We may use the general theory of Zwanzig and Mori \( \tilde{10}, \tilde{24} \) for this particle because its position and velocity are random quantities.
The position and velocity of the Brownian oscillator is subject for equations

\[
\frac{dx}{dt} = v, \quad (16)
\]

\[
\frac{dv}{dt} = -2cv - \omega_0^2x + \frac{F(t) + f(t)}{m}, \quad (17)
\]

where \( c = \beta/2m \), \( F(t) \) – external force, \( f(t) \) – the Langevin random force. The random quantities \( x, v, f \) are described by the following average values:

\[
\langle x \rangle = 0, \quad \langle v \rangle = 0, \quad \langle f \rangle = 0, \quad (18)
\]

\[
\langle x^2 \rangle = \frac{T}{m\omega_0^2}, \quad \langle v^2 \rangle = \frac{T}{m}, \quad \langle xv \rangle = 0,
\]

\[
\langle xf \rangle = 0, \quad \langle vf \rangle = 0, \quad \langle f(t)f(t') \rangle = 2T \beta \delta(t - t'),
\]

where \( T \) – temperature in the units of the energy, \( m \) – the mass of Brownian particle.

We use the fluctuation of particle’s position \( x_0 = x(0) \) as the initial dynamic variable. The next orthogonal dynamic variables are calculated with help of the following recurrent relations:

\[
W_1 = \{i\hat{L} - \omega_0^{(0)}\} W_0,
\]

\[
W_n = \{i\hat{L} - \omega_0^{(n-1)}\} W_{n-1} + \Omega_n^2 W_{n-2}, \quad n > 1, \quad (19)
\]

where the frequencies \( \omega_0^{(n)} \) and \( \Omega_n \) are defined by equations

\[
\omega_0^{(n)} = \frac{\langle W_n^* i\hat{L} W_n \rangle}{\langle |W_n|^2 \rangle}, \quad \Omega_n^2 = \frac{\langle |W_n|^2 \rangle}{\langle |W_{n-1}|^2 \rangle}.
\]

The straightforward calculations give

\[
\omega_0^{(0)} = \frac{\langle x(0)v(0) \rangle}{\langle |x(0)|^2 \rangle} = 0,
\]

\[
W_1 = (i\hat{L} - \omega_0^{(0)})W_0 = v(0),
\]

\[
\omega_0^{(1)} = \frac{\langle v(0)dv(0)/dt \rangle}{\langle |v(0)|^2 \rangle} = -2c,
\]

\[
\Omega_1^2 = \frac{\langle |v(0)|^2 \rangle}{\langle |x(0)|^2 \rangle} = \omega_0^2,
\]

\[
W_2 = \frac{f(0)}{m}.
\]

Taking into account the equations of motion (15), one obtains the time correlation function (TCF)

\[
M_0(t) = \frac{\langle x(t)x(0) \rangle}{\langle |x(0)|^2 \rangle} = \frac{r_- e^{r_+ t} - r_+ e^{r_- t}}{r_- - r_+}, \quad (20)
\]

where \( r_{\pm} = -\left[c \pm \sqrt{c^2 - \omega_0^2}\right] \).

To find the TCF for next levels we exploit the Zwanzig-Mori infinite chain of equations given by Eq. (1). The Laplace-image of initial TCF has the following form:

\[
\tilde{M}_0(s) = \frac{s - (r_- + r_+)}{(s - r_-)(s - r_+)}.
\]

With \( n = 0 \) we obtain from chain (15) the Laplace-image of first memory function

\[
\tilde{M}_1(s) = \frac{1}{s - (r_- + r_+)}.
\]
Taking the inverse Laplace transform we obtain the first memory function

\[ M_1(t) = e^{(r_+ + r_-)t} = e^{-2ct}. \]

The calculation in closed form of the memory functions on the next relaxation levels requires the following detailed elaboration of the structure of Langevin force \( f \). For example, in order to calculate the memory function \( M_2(t) \) we need for frequency \( \Omega_2 \) which may be found by setting the mean value of square of Langevin force. For this reason we restrict our consideration by first two levels of relaxation.

We consider the TCF of Brownian oscillator in the case of small damping \( p = c/\omega_0 \ll 1 \). In this case the TCF \( M_0(t) \) has the following form:

\[ M_0(t) = \left\langle x(t)x(0)\right\rangle = \cos(2\pi \nu t)e^{-c|t|} \quad (21) \]

and describes the motion of Brownian oscillator with frequency \( \omega = 2\pi \nu \gg c = \beta/2m \), where \( m \) is the mass of oscillator, and \( \beta \) is the friction coefficient of Brownian particle \( \text{[12]} \).

There are two parameters \( \nu \) and \( c \) which characterize the frequency oscillation and relaxation damping of TCF, respectively. For simplicity we consider dimensionless time \( t \), frequency of oscillation \( \nu \) and damping parameter \( c \). The Fourier transform of square of this TCF has the following form

\[ \widetilde{M}_0^2(\nu) = \frac{c}{2(c^2 + \pi^2 \nu^2)} + \frac{c}{4(c^2 + \pi^2(\nu - 2\nu')^2)} + \frac{c}{4(c^2 + \pi^2(\nu + 2\nu')^2)}. \]

We consider frequency spectrum of square of TCF because the entropies are expressed in terms of square of TCF. Therefore there are three maximums at \( \nu = 0, \nu = \pm 2\nu' \). In the limit of zero damping \( c \to 0 \) we obtain

\[ \widetilde{M}_0^2(\nu) = \frac{1}{2}\delta(\nu) + \frac{1}{4}\delta(\nu - 2\nu') + \frac{1}{4}\delta(\nu + 2\nu') \]

by using well-known formula:

\[ \pi\delta(x) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2 + \varepsilon^2}. \]

In this case the spectrum consists of three lines at \( \nu = 0, \nu = \pm 2\nu' \).

In Fig.1 we reproduce the entropies \( \text{[12]} \) and in Fig.2 we show the spectra of them for \( \nu' = 0.1 \) and for \( c = 0, 0.01, 0.1, 1 \). There is a peak in spectra at double frequency \( \nu = 2\nu' = 0.2 \) for arbitrary small but non-zero damping \( c \). For zero damping \( c = 0 \) this peak disappears in total entropy. There are peaks in \( \hat{S}^c \) and \( \hat{S}^a \) at \( \nu = 0.2 \) but with opposite sign. We note that the increasing of the damping leads to smearing the fine structure of entropies.

Calculation the entropies \( \text{[13]} \) shows that the small values of \( q \) "works" as non-linear magnifier for small value of \( M_1^0 \). To illustrate this fact we reproduce in Fig.3 the entropies \( \text{[13]} \) and \( \text{[12]} \) for \( q = 1, 0.1, \nu' = 0.1, c = 0.1 \). First of all, as expected from Eq. \( \text{[15]} \) the smaller entropy the greater magnification. Second, the entropy \( \hat{S}_q^a \) does not change sufficiently. The great variation prove \( \hat{S}_q^a \) and \( \hat{S}_q^c \). The decreasing of \( q \) leads to increasing the value of peaks for great frequencies and makes peaks more sharp. The small \( q \) makes better sharpness of frequency spectra. This is not the case for \( \hat{S}_q^a \). It does not change great.

### IV. THE MOTION OF BROWNIAN OSCILLATOR WITH NOISE

The real signal from alive systems often contains noise (see Ref. \( \text{[22]} \)). For this reason we suggest the model of Brownian oscillator with noise. We consider the following model of TCF:

\[ M_0(t) = R(t) \cos(2\pi \nu t)e^{-c|t|}, \quad (22) \]

where \( R(t) \) denotes the random numbers in interval \((-1, 1)\) and \( R(0) = 1 \). Therefore the function \( R(t) \) makes random the amplitude of oscillation but this is not the case for frequency and damping parameters. At the beginning we know the frequency of oscillations. The TCF has more complicate form which is more close to real dates. The random numbers describe a noise which usually appears in an experiment. The time dependence of entropy is much more complicate but nevertheless in this case the DTE works better. We observe the appearing peaks in places, which we know the peaks must be, but they disappeared in noise. There is another observation: the noise is better for frequency spectrum. The peaks for small frequencies looks better (see Fig. 4).
FIG. 1: The plots of entropies $S$ for frequency $\nu' = 0.1$ and $c = 0(a), 0.01(b), 0.1(c), 1(d)$ ($\nu'$ and $c$ are oscillation and damping parameters of TCF respectively). It is seen that the time behavior of total $S$ and single channels $S^c$ and $S^a$ of DTEs for various damping regimes reveal a stochastic ordering of time correlation.

V. DENSITY FLUCTUATIONS IN HYDRODYNAMICAL LIMIT

The TCF of density fluctuations in hydrodynamical limit was calculated by Landau and Placzek \cite{12}. It describes scattering of light in liquid in hydrodynamics limit when $k \to 0$. The TCF has the following form

$$M_0(t) = \alpha e^{-\gamma k^2 t} \cos \theta_s k t + (1 - \alpha) e^{-\sigma k^2 t},$$

where $\gamma = \frac{1}{2\rho} \left( \frac{4}{3} \eta + \zeta + \kappa \left[ \frac{1}{c_v} - \frac{1}{c_p} \right] \right)$, $\sigma = \frac{\kappa}{\rho c_p}$, $\alpha = c_v/c_p$. Here the $c_v, c_p, \kappa, \eta, \zeta, \rho, \theta_s$ are specific heat capacities in units of mass at constant volume and constant pressure, the coefficient of thermal conductivity, the coefficient of shear viscosity and volume viscosity, the mass density and sound velocity, correspondingly.

The spectrum of this TCF contains three peaks. The central Rayleigh peak at zero frequency describes isothermal propagation of sound. Two symmetric peaks at frequencies $\omega = \pm \theta_s k$ describe adiabatic propagation of sound with damping (Brillouin doublet).

It is more suitable to define dimensionless time $\tau$ by relation $\tau = \theta_s k t / 2\pi$. Then the position of Brillouin doublet will be at the dimensionless frequency $\nu = \omega / 2\pi = 1$, and TCF will take the following form

$$M_0(\tau) = \alpha e^{-\gamma k^2 \tau} \cos 2\pi \tau + (1 - \alpha) e^{-\sigma k^2 \tau},$$

where $\gamma = \frac{1}{2\rho \sigma_s} \left( \frac{4}{3} \eta + \zeta + \kappa \left[ \frac{1}{c_v} - \frac{1}{c_p} \right] \right)$, $\sigma = \frac{\kappa}{\rho c_p \sigma_s}$, $\alpha = c_v/c_p$. We consider a specific medium – Helium at temperature $T = 20^\circ C$ and pressure $p = 1 b$. In this case we have $\alpha \approx 0.56, \theta_s \approx 272 \text{ m/c, } \gamma \approx 6 \cdot 10^{-9} \text{ m}^{-1}, \sigma \approx 7 \cdot 10^{-9} \text{ m}^{-1}$. We make calculations for $k = 2 \cdot 10^7 \text{ m}^{-1}$. 
FIG. 2: The plots of spectra of entropies (12) for $\nu' = 0.1$ and $c = 0(a), 0.01(b), 0.1(c), 1(d)$. One can see that a stochastic ordering of time correlation in previous Fig.1 reduces to an appearance of specific peculiarities in low frequency region. Weak damping ($c = 0.1$) results in amplification of characteristic frequency peaks at $\nu = 0.2$ and $\nu = 0.4$ whereas zero damping ($c = 0$) leads to disappearance of specific frequency peaks.

In the Fig.5 we reproduce time and frequency dependencies entropies defined before for two values of parameter $q = 1, 0.1$. There is periodicity over $\tau$ with unit period which gives the appearance specific peaks in the frequency spectrum (Brillouin doublet). We observe the same picture as in previous section. Decreasing the parameter of nonextensivity $q$ leads to increasing small peaks in entropies $S_{qn}$ and $S_{qcm_n}$, whereas quantity $S_{qam_n}$ is changed insufficiently. The frequency spectrum entropies $S_{qn}$ and $S_{qcm_n}$ becomes more sharp. The shape of frequency spectrum $S_{qam_n}$, in fact, does not change.

The non-Markovity parameter of this system has been calculated earlier in Refs. [7, 19]. It was shown that in hydrodynamical limit $k \rightarrow 0$ the spectrum of non-Markovity parameter has a form of alternating Markovian and non-Markovian levels.

VI. IDEAL GAS

Let us consider the Fourier transformation of the fluctuation of the particle number density of a system

$$\delta\rho_k(t) = \frac{1}{V} \sum_{l=1}^N \exp(ikr_l) - \frac{N}{V} \delta_{k,0}.$$
FIG. 3: The plots of entropies for $q = 0.1$ and $q = 1$ and their spectra for $\nu' = 0.1, c = 0.1$. Comparison of dynamic Shannon ($q = 1$) and Tsallis ($q = 0.1$) entropies shows the amplification of DTE for small values of TCF. Therefore one can conclude that DTE acts as a magnifier for small values of TCF.

For this case the initial TCF is calculated exactly and it has the following form:

$$M_0(t) = \frac{\langle \delta \rho_k(t) \delta \rho_k(0)^* \rangle}{\langle |\delta \rho_k(0)|^2 \rangle} = e^{-t^2/\tau_r^2}, \quad \tau_r^2 = 2m/k^2T.$$

For this variable the all frequencies $\omega_0^{(n)}$ are equal to zero. The main relaxation frequencies in Eq. have the simple form

$$\Omega_n^2 = n\Omega_1^2, \quad \Omega_1^2 = k^2T/m.$$
Let us rescale time $t \rightarrow \tau = t \Omega_1$, and the Laplace transformation parameter $s \rightarrow c = s / \Omega_1$, and the Laplace images $\tilde{M}_n(s) \rightarrow \tilde{m}_n(s) = \Omega_1 \tilde{M}_n(s)$. In this case the hierarchy (1) has the following form

$$\tilde{m}_{n+1}(c) = \frac{1}{n+1} \left\{ \frac{1}{\tilde{m}_n(c)} - c \right\}.$$  

The Laplace image $\tilde{m}_0(c)$ may be found in close form:

$$\tilde{m}_0(c) = e^{\frac{c^2}{2}} \sqrt{\frac{\pi}{2}} \text{Erfc}[\frac{c}{\sqrt{2}}],$$  \hspace{1cm} (23)

where Erfc$(x) = 1 - \text{Erf}(x)$ – additional probability integral.

The inverse transformation may be represented in the following form:

$$M_n(\tau) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{ic} \tilde{m}_n(c) dc,$$

where $\sigma$ is greater than real part of zeros of $\tilde{m}_n(c)$. By using the expression (23) we may set $\sigma = 0$ and by changing $c \rightarrow ix$ we obtain

$$M_n(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix} \tilde{m}_n(ix) dx,$$

or in manifest form

$$M_1(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix} \left[ \frac{1}{\tilde{m}_0(ix)} - ix \right] dx,$$

$$M_2(\tau) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} e^{ix} \left[ \frac{1}{\tilde{m}_1(ix)} - ix \right] dx.$$

These formulas we will calculate numerically. The first three memory functions and their spectrum are plotted in Fig.6 In Fig. 7 we reproduce the plot of the entropy $S^n$ and their frequency spectrum for $n = 0, 1, 2$.

It is not difficult to show that the entropy $S^n$ (12) amounts to maximum value, ln 2, at the point $|M_n|^2 = \frac{1}{2}$. For $n = 0$ the position of maximum is at the point $\tau = \sqrt{\ln 2} \approx 0.832$. The greater $n$, the smaller time of
FIG. 6: The first normalized memory functions $M_n(\tau)$ and their frequency spectrum $\hat{M}_n(\nu)$ for ideal gas.

FIG. 7: The plot the time dependence of entropy and its frequency spectrum for different relaxation levels $n = 0, 1, 2$ for ideal gas.

maximum. The relaxation times $\tau_0 \approx 1.6453$, $\tau_1 = 1.0688$, $\tau_2 = 0.8630$. The parameter of non-Markovity, $\epsilon_n = \frac{\tau_n}{\tau_{n+1}}$

has the following values $\epsilon_0 \approx 1.54$, $\epsilon_1 \approx 1.24$. Note that these values very close to that calculated in paper [19] directly for TCFs, $\epsilon_0 \approx 1.57$, $\epsilon_1 \approx 1.27$.

In the Fig. [8] we give the plots of entropy $S_n^q$ and their spectrums for $q = 1/2$ and $q = 3/2$. In order to show what happens if we will vary the value of $q$ we reproduce in Fig. [9] the entropy $S_n^q$ and its spectrum for $n = 2$ and for $q = 0.5, 1.1.5$. To show the dependence $S_n^q$ for more wide range of $q$ in Fig. [10] we give entropies for more wide range of $q$.

By using these three quantities we define seven different spectra of non-Markovity parameter:

$$\epsilon_{qn} = \frac{\tau_{qn}}{\tau_{qn+1}}, \epsilon_{qcn} = \frac{\tau_{qcn}}{\tau_{qcn+1}}, \epsilon_{qan} = \frac{\tau_{qan}}{\tau_{qan+1}}$$  \hspace{1cm} (24a)

$$\epsilon_{qan} = \frac{\tau_{qan}}{\tau_{qan+1}}, \epsilon_{qacn} = \frac{\tau_{qan}}{\tau_{qan+1}}, \epsilon_{qacn} = \frac{\tau_{qacn}}{\tau_{qacn+1}}, \epsilon_{qan} = \frac{\tau_{qan}}{\tau_{qan+1}}$$  \hspace{1cm} (24b)

The plots of all quantities are shown in Fig. [11]. As expected at the beginning more interesting situations are possible for small values of $q$. For great value of $q$ all lines tend to constants.

VII. CONCLUSION

Let us summarize our results. We considered the dynamic Tsallis entropy and applied it for model systems. In accordance of Refs. [20, 21, 22] the square of TCF is regarded as probability of dynamic state. We used the time dependent entropies as well as their frequency spectra. We considered four model TCFs which, nevertheless, have physical sense. The first and second models describe the motion of Brownian oscillator without and with noise. The noise is modelled by generator of random numbers. Third model is the Landau-Placzek model
for TCF of particle density fluctuations in hydrodynamic limit. The last model is ideal gas, the relaxation of particle density fluctuations.

In all model we considered different values of parameter of nonextensivity \( q \). All entropies have the same structure zeros and extremums as square of TCF. The magnitude of extremums sufficiently depends on value of \( q \). Small values of \( q \) work as non-linear magnifier: the smaller magnitude of entropy the greater magnification. Great values of \( q \) do in opposite way as demagnified lens. Concerning the frequency spectra of entropies we observe that for small values of \( q \) the peaks become more sharp and more large. For systems with noise this property works better. It is possible to reveal peaks even if they are lost in noise.

For ideal gas we additionally calculated the spectrum of parameter non-Markovity by using different definition of relaxation time. By using three kind of information Tsallis entropy we defined seven kind of spectrum of parameter non-Markovity. We observe that all of these parameters (except \( \epsilon_{qca} \)) are close to unit. It means that the ideal gas remains non-Markovian system for arbitrary value of \( q \). It is in qualitative agreement with Ref. [1]. The variation \( q \) from unit does not mean the appearance of new interactions. The system is became

FIG. 8: The plots of time dependent entropy \( S_q^n \) and their frequency spectrums for \( q = 1/2 \) and \( q = 3/2 \).

FIG. 9: The time dependent entropy \( S_q^n \) and their frequency spectrum for \( n = 2 \) and for \( q = 0.5, 1, 1.5 \).
FIG. 10: The time dependence of the entropy $S_q^n$ and its frequency spectrum for six values $q = 0.001, 0.1, 0.5, 1, 1.5, 10$ and for $n = 0$. One observe that the increasing Tsallis’ parameter $q$ leads to trample a quantity of $S_q$ in domain of short time. Due to this fact one can make sufficient amplification in domain of short time owing to variation parameter $q$.

ideal and non-Markovian from this point of view.

Our analysis allows us to conclude, that use of dynamic Tsallis’s entropy extends essentially possibilities of the stochastic description of model physical systems. Advantage of application of DTE is that it allows to strengthen or suppress fluctuations either in low-frequency, or in high-frequency areas of a spectrum. Similar supervision opens appreciable prospects in the field of the study of real complex systems of wildlife where dynamic states of physiological and pathological systems are very important.

Acknowledgments

This work was supported in part by the Russian Foundation for Basic Research Grants N 02-02-16146, 03-02-96250 and by the Russian Humanitarian Scientific Foundation Grant N 03-06-00218a.

[1] Abe S 1999 Physica A 269 403
[2] Aczél A., Daróczy Z. 1975 On measures of information and their characterizations (Academic Press, New York)
[3] Beck C and Schiglof F 1993 Thermodynamics of Chaotic Systems (Cambridge University Press, Cambridge)
[4] Goychuk I and Hanggi P 2000 Phys. Rev. E61 4272
[5] Katok A., Hasselblatt B. 1995 Introduction to the Modern Theory of Dynamical Systems (Cambridge University Press, Cambridge)
[6] Kaufman S A 1993 The Origins of Order – Self-Organization and Selection in Evolution (Oxford University Press, Oxford)
[7] Khusnutdinov N R and Yulmetyev R M 1995 Teor. i Matem. Fiz. 105 292 (Engl. Transl. 1995 Theor. Math. Phys. 105 1426)
[8] Klimontovich Yu L 1998 Phys. Scr. 58 549
[9] Latora V, Baranger M, Rapisarda A and Tsallis C 2000 Phys. Lett. A273 97
[10] Mori H 1965 Prog. Theor. Phys. 33 423; 34 765
[11] Reed M and Simon B 1972 Methods of Modern Mathematical Physics (New York: Academic)
[12] Résibois P and De Leener M 1977 Classical kinetic theory of fluids (New York: Wiley, 1977)
[13] Scapino D J, Sears M and Ferel R A 1972 Phys. Rev. B6 3409
[14] Shurygin V Yu, Yulmetyev R M, and Vorobjev V V 1990 Phys. Lett. 148A 199
[15] Shurygin V Yu and Yulmetyev R M 1991 Zh. Eksp. Teor. Fiz. 99 144 (Engl. Transl. 1991 Sov. Phys.-JETP 72 80)
[16] Tsallis C 1988 J. Stat. Phys. 52 479
[17] Tsallis C 1999 Braz. J. Phys. 29 1
[18] Vargaftik N B Handbook of Physical Properties of Liquids and Gases: Pure Substances and Mixtures (Hemisphere Pub; 2nd edition 1983)
[19] Yulmetyev R M and Khusnutdinov N R 1994 J. Phys. A27 5363
[20] Yulmetyev R M and Kleiner M Ya 1998 Nonlinear Phenomena in Complex Systems 1 80
[21] Yulmetyev R M and Gafarov F M 1999 Physica A273 416
Yulmetyev R M and Gafarov F M 1999 Physica A274 381
[22] Yulmetyev R M, Gafarov F M, Yulmetyeva D G and Emeljanova N A 2002 Physica A303 427
FIG. 11: The spectrum of all parameters of non-Markovity as function of $q$. The influence of parameter $q$ is most effectively for small values of $q \ll 1$. In all cases we observe strong non-Markovity and clearly marked effect of statistical memory. The only case where we may sufficiently increase the non-Markovity properties by decreasing parameter $q \to 0$ is cross term $c\alpha$ (see Eq. (24b)). For small enough $q \ll 1$ the non-Markovity parameter $\epsilon_{qca_n}$ may reach great value $\epsilon_{qca_n} \gg 1$. 
In literature there is standard notion of dynamic entropy (see e.g. Ref. [5]). We suggest and use the functions $S$ and $S_q$ which we call the dynamic Shannon entropy and dynamic Tsallis entropy.

This is not frequency spectrum. Here the spectrum means set of parameters $\epsilon_n$. 

[23] Ylmetyev R M, Emeljanova N A and Gafarov F M 2004 *Physica* A**341** 649
[24] Zwanzig R 1961 *Phys. Rev.* 124 1338
[25] Zwanzig R 1965 *Annual review of physical chemistry* 16 67
[26] In literature there is standard notion of dynamic entropy (see e.g. Ref. [5]). We suggest and use the functions $S$ and $S_q$ which we call the dynamic Shannon entropy and dynamic Tsallis entropy.
[27] This is not frequency spectrum. Here the spectrum means set of parameters $\epsilon_n$. 