The inner and outer bounds on the capacity of the 3-user GBC-SKT with unequal state variances

Leila Ghabeli

Abstract
The “Gaussian BroadCast channel (GBC) with States non-causally Known at the Transmitter” (GBC-SKT) introduces Gaussian broadcast channel with additive i.i.d. Gaussian states non-causally known at the transmitter. In this model additive interferences on links are considered as states which are only non-causally known to the transmitter. Such cases are encountered in, e.g., wireless multiple-antenna multicasting scenario. Previously the approximate capacity of the K-user GBC-SKT has been studied where all states but one are assumed to have the same variance. In this work, the capacity of the 3-user GBC-SKT channel communicating only common messages is studied. All three states are assumed to have different variances. The key point is how to simultaneously pre-code the common message against three states. Two transmission strategies are proposed by combination of power-sharing and time-sharing methods and two inner bounds are derived. From analytical and simulation results it is seen that one inner bound outperforms the other inner bound. Moreover, the outer bound on the capacity of this channel is driven and the conditions on the state variances for which, the capacity of the 3-user GBC-SKT channel is derived to within 1 bpcu, are extracted.

1 INTRODUCTION

In the ‘Writing on Dirty Paper’ (WDP) channel [1], a channel with output \( Y = X + S + Z \) was examined, the state \( S \) and the noise \( Z \) are Gaussian random variables. If \( S \) is known to the encoder, it does not reduce the capacity of the channel and the capacity is shown to be the capacity of a standard Gaussian channel, even though \( S \) is unknown to the receiver.

The same coding technique as the one used for the WDP channel has also been used to attain the capacity of the Gaussian BroadCast (GBC) channel [2, 3] which is a degraded BroadCast (BC) channel. The ‘GBC with States non-causally Known at the Transmitter’ (GBC-SKT) is the combination of the GBC and Costa’s WDP channels. The 2-user GBC-SKT is studied in [4] where inner bounds combining Marton coding [5] and Gel’fand–Pinsker binning [6] are derived. The authors of [4] also derived the capacity for the model with statistically equivalent channel outputs in which only private messages are sent over the channel.

In [7] ‘Carbon Copy on Dirty Paper’ (CCDP) channel is considered. In this channel the BC receivers must decode the same common message by observing a channel output also containing a state sequence non-causally known at the transmitter. The approximate capacity for a specific choice of correlation among Gaussian distributed states has been derived in [8] for the 2-user GBC-SKT channel, and in [9] for the K-user GBC-SKT channel. In both problems they assume statistically equivalent outputs and the states with equal variances. In these works a tighter outer bound than [7] has been derived based on the fact that the capacity should be decreasing in state variance.

In [10], the general two-user compound BC is studied in which an encoder wants to transmit two private messages to two receivers while being oblivious to the actual channel realizations. An achievable rate region is derived based on the interference decoding, in which each receiver decodes its intended message and chooses to decode, or not, the interfering non-intended message. This decoding scheme is shown to be capacity achieving for a class of non-trivial compound BEC/BSC broadcast channels while the worst case of Marton’s inner bound based
on no interference decoding fails to achieve the capacity region.
The achievable rate region is derived based on multiple description
coding wherein the encoder transmits a common description as well as multiple dedicated private descriptions to the many possible channel realizations of the users. It is shown that MD coding yields larger inner bounds than the single description scheme for a class of compound MISO BC.

In [11] closed-form capacity bounds for an exponential version of the dirty paper channel is presented. This channel can be used to model the non-coherent Gaussian dirty paper channel. First, a superposition modulation-like technique is used to obtain a capacity upper bound, then a closed-form capacity lower bound is obtained by using a Costa-like strategy.

In [12] we derived the approximate capacity for the 2-user GBC-SKT channel considering the states with unequal variances and unequal channel gains, communicating both common and private messages. For the inner bound we divided the user message into two parts. For sending one part, interference is considered as noise while for sending the other part, the codewords are pre-coded against the states and they will be sent by the power-sharing strategy. For the outer bound we used the same idea as [9], the capacity should be decreasing in the state variance, but for the assumption of independent states with unequal variances. We obtained the gap between inner and outer bounds at most 1.25 bpcu for all channel parameters and independent Gaussian distributed channel states.

In [13] we extended the result of [12] to obtain the approximate capacity for the $K$-user GBC-SKT channel in which $(K - 1)$ of $K$ channel outputs are statistically equivalent, that is, they have the same channel gain and the same variance for the channel states. For this model we showed the approximate capacity is achieved to within 1.5 bpcu for all channel parameters and independent Gaussian distributed channel state information.

Here, we derive inner and outer capacity bounds on the 3-user GBC-SKT channel communicating common messages to three users. We assume the channel states to have different variances. We propose an outer bound and two inner bounds. For the inner bounds we propose two new approaches which use combinations of time- and power-sharing methods. The difference is about how and where interference can be considered as noise while for sending the other part, the code-words are pre-coded against the states and they will be sent by the power-sharing strategy. For the outer bound we used the same idea as [9], the capacity should be decreasing in the state variance, but for the assumption of independent states with unequal variances. We obtained the gap between inner and outer bounds at most 1.25 bpcu for all channel parameters and independent Gaussian distributed channel state information.

Throughout we use the same notations and the channel model as those defined in [13]. For completeness we rewrite them in this section.

$[1; a]$ for $a \in \mathbb{Z}$ stands for the set of integers $\{1, \ldots, a\}$. $[a]$ stands for the largest integer number which is not greater than $a$. $[a]^b$ stands for $\max(a, 0)$. Random variables are denoted by capital letters, for example, $X$. Vectors with length $N$ are shown with capital letters and a superscript which denotes the vector length, for example, $X^N$ stands for the sequence of letters $[X_1, \ldots, X_N]$.

The $K$-user (GBC-SKT) is depicted in Figure 1. The channel outputs are obtained as

$$Y_k^N \leftarrow a_k X^N + S_k^N + Z_k^N, \quad k \in [1; K],$$

for

$$1 = a_1 \leq a_2 \leq \cdots \leq a_K,$$

and $Z_k^N$, an iid Gaussian random variable with zero mean and variance $N$, at all receivers, that is, for all $k$, and $S_1^N, \ldots, S_K^N$ are iid jointly Gaussian vectors with zero mean and variance $Q_k$ and $S_i^N \perp S_j^N, i \neq j$. Note that $a_k$’s are taken positive without loss of generality: since both the state and the noise distributions are symmetric, the sign of the channel gain can be chosen freely. Additionally, the channel input sequence $X^N$ is subject to the
power constraint:

\[ \frac{1}{N} \mathbb{E}[|X|^2] = P. \] (3)

Having non-causal knowledge of the channel states, the transmitter wishes to reliably communicate the common message \( W^r \in [1; 2^{NR_0}] \) to all receivers and the private message \( W^k \in [1; 2^{NR_k}] \) to receiver \( k \in [1; K] \).

## 3 | PREVIOUS RESULTS

In this section we only review previous results of the author in [13]. We have given complete review in that reference.

### 3.1 | 2-user GBC-SKT channel

The 2-user GBC-SKT with \( R_0 \neq 0, R_1 \neq 0, R_2 \neq 0 \), and \( a_1 \geq a_2 \geq 1 \) has been considered. The approximate capacity of this channel has been determined. The results is stated in the following theorem:

**Theorem 3.1.** [13, Thm. 3] The approximate capacity for the general 2-user GBC-SKT: Consider the 2-user GBC-SKT with unequal channel gains and independent states with unequal variances. An outer bound to the capacity of this channel is obtained as

\[
R_k = \frac{1}{2} \log \left( 1 + \frac{a_k^2 \beta_k P}{1 + \beta_k P} \right), k = [2; K],
\] (6a)

\[
R_0 + R_1 \leq \frac{1}{2} \log \left( 1 + \frac{\beta P}{1 + \beta P} \right),
\] (4b)

\[
2R_0 + R_1 + R_2 \leq \log \left( 1 + \frac{(P - Q)^+}{1 + Q} \right)
\] + \( \frac{1}{2} \log (1 + \min \{P, Q\}) + \frac{1}{2} \log a_2^2 + 2, \) (4c)

\[\text{for } \beta \in [0, 1], \beta = 1 - \beta, \text{ and } Q = \max \{Q_1, Q_2\}. \] The variance of noise is assumed to be equal to 1. The exact capacity is to within a gap of 1.25 bpcu from the outer bound shown in (4).

### 3.2 | K-user GBC-SKT channel

The approximate capacity of the general \( K \)-user GBC-SKT with two different values for channel gains and two different values for the variances of independent states, that is, we consider the following assumptions,

\[ 1 = a_1 = a_2 = \cdots = a_{K-1} < a_K, \] (5a)

\[ Q_1 \leq Q_2 = \cdots = Q_K = Q. \] (5b)

**Theorem 3.2.** [13, Thm. 4] The approximate capacity for a special class of the \( K \)-user GBC-SKT with independent states satisfying (5), then an outer bound to the capacity of this channel is obtained as

\[
R_k = \frac{1}{2} \log \left( 1 + \frac{a_k^2 \beta_k P}{1 + a_k^2 \sum_{\ell=k+1}^{K} \beta_{\ell} P} \right), k = [2; K],
\] (6a)

\[
R_0 + R_1 \leq \frac{1}{2} \log \left( 1 + \frac{\beta_1 P}{1 + \sum_{\ell=2}^{K} \beta_{\ell} P} \right),
\] (6b)

\[
(K-1)R_0 + \sum_{k=1}^{K-1} R_k \leq \frac{K-1}{2} \log \left( 1 + \frac{(P - Q)^+}{1 + Q} \right)
\] + \( \frac{1}{2} \log (1 + \min \{P, Q\}) + K - 1, \) (6c)

\[
\frac{KR_0 + \sum_{k=1}^{K} R_k}{2} \leq \frac{K}{2} \log \left( 1 + \frac{(P - Q)^+}{1 + Q} \right)
\] + \( \frac{1}{2} \log (1 + \min \{P, Q\}) + \frac{1}{2} \log a_K^2 + K, \) (6d)

\[\text{for } \beta_{\ell} \in [0, 1], \sum_{\ell=1}^{K} \beta_{\ell} = 1. \] The variance of noise is assumed to be equal to 1. The exact capacity is to within a gap of 1.5 bpcu from the outer bound in (6).

For the outer bound we use the fact that the capacity should be decreasing in states as suggested in [14] to obtain a tighter outer bound than that proposed in [7] and [15].

For the inner bound, we proved the achievability of curves \( A \delta A^\prime, B B', \) and point \( C \) as shown in Figure 2 and then considered the total rate region as time sharing between these regions.
We allowed the “pre-coded against state” codewords to have different power depending on state powers and channel gains, which attains higher rates than simple time-sharing strategy.

The gap between curve $BB'$ and the outer bound is 1.25 bpcu, and the gap between point $C$ and the outer bound (4c) is 2 bpcu.

For the curve $AA'$, which is shown with the green color in Figure 2, we obtain the exact capacity. This result includes the result of [4, Theorem 2] which only considered communicating private messages.

As is seen, the difficult point to not achieve the capacity is in communicating the common message to the users, so, in this paper, to more clarify the problem, we only focus on the transmission of common messages.

In [13] we considered two different variances for channel states and we divided the power $P$ between two codewords as $[P - Q_1^+]$ and $\min[P, Q_1]$ to send the common message. In this paper we consider three different variances for channel states and we need to do more complex power dividing between codewords as explained in the next section.

Remarks:

1. We can also use Theorem 3.2 for the case $Q_i \neq Q_j$, $i, j \in K$, that is, $Q_1 < Q_2 < \ldots < Q_K$ by replacing $Q_K$ instead of $Q$ in (6), it means that we consider the worst case and thus the rate that is achievable for the worst case is also achievable for the other cases. For example, for 3-user GBC-SKT with $Q_1 < Q_2 < Q_3$ communicating only common messages, by substituting $K = 3$, $R_1 = R_2 = R_3 = 0$, $Q = Q_3$, and also considering noise variance $N \neq 1$ in (6), the following rate is achievable:

$$R_0 = \frac{1}{2} \log \left(1 + \frac{[P - Q_1^+]}{N + Q_3}\right) + \frac{1}{6} \log \left(1 + \min \{P, Q_1\}\right).$$

We use (7) for comparison in simulation section in this paper. We also use the rate obtained by the time-sharing method for comparison which is always an inner bound for such problems.

$$R_0 = \frac{1}{6} \log (1 + P).$$

4 | THE CAPACITY RESULT FOR THE 3-USER GBC-SKT

In this section we derive two inner bounds and one outer bound on the capacity of the 3-user GBC-SKT communicating common messages to three users, that is, $R_1 = R_2 = R_3 = 0$, and we assume $R_0 = R$ in the rest of document. The channel gains are assumed to be one, that is, $\alpha_1 = \alpha_2 = \alpha_3 = 1$. We assume Gaussian distributed independent states to have different variances. Without loss of generality, we consider $Q_1 < Q_2 < Q_3$.

**Theorem 4.1.** The first inner bound: For the 3-user GBC-SKT with independent states with unequal variances, the following rate is achieved:

$$R = \frac{1}{2} \log \left(1 + \frac{[P - Q_1^+]}{N + Q_3}\right) + \frac{\alpha}{2} \log \left(1 + \frac{\min \{P, Q_3\}}{N}\right).$$

where

$$\alpha = \begin{cases} \log \left(1 + \frac{\min \{P, Q_3\}}{N}\right) - \frac{1}{2} \log \left(1 + \frac{Q_3}{N}\right), & P \geq Q_2, \\ \frac{1}{3}, & P < Q_2. \end{cases}$$

**Proof.** The transmission strategy consists of both one time-sharing and two power-sharing phases. We divide the transmission time unit into three parts with duration $\alpha$, $\alpha/2$, and $\bar{A}/2$.

At duration $\alpha$ we divide power $P$ into two parts $[P - Q_1^+]$ and $\min[P, Q_1]$. Note that

$$[P - Q_1^+] + \min[P, Q_1] = P.$$ 

At duration $\bar{A}$ we divide power $P$ into three parts $[P - Q_3^+]$, $\min[P, Q_3] - Q_3^+]$, and $\min[P, Q_3]$. Note that

$$[P - Q_3^+] + \min[P, Q_3] - Q_3^+] + \min[P, Q_3] = P.$$ 

The strategy is graphically represented in Figure 3.

1. At duration $\alpha$, we transmit power $P$ with two codewords $X_{SAN3}^N$ with power $[P - Q_1^+]$, $X_{SAN1}^N$ with power $\min[P, Q_1]$. (SAN stands for ‘state as noise’ and PAN stands for ‘pre-coded against the state sequences’).
2. At duration $\alpha/2$, we transmit power $P$ with three codewords $X_{SAN3}^N$, $X_{SAN2}^N$, and $X_{SAN1}^N$ with power $\min[P, Q_3] - Q_3^+]$, and $\min[P, Q_3]$. Note that here we again transmit $X_{SAN1}^N$ with power $\min[P, Q_3]$, no need to use $Q_1$. The reason was illustrated in [12] for the 2-user GBC-SKT channel.

The codewords are defined as follows:

1) The power-sharing codeword for three users, $X_{SAN3}^N$, with power $[P - Q_1^+]$, carries the message $W_{SAN3}$ with rate $R_{SAN3}$ which treats all state sequences as additional noise. Since the state variance at the third receiver is the largest, the codeword $X_{SAN3}^N$ can be decoded at all users if its rate is
The coding scheme I for sending common message for \( K = 3 \) is limited by

\[
R_{\text{SAN3}} = \frac{1}{2} \log \left( 1 + \frac{[P - Q_3]^+}{N + Q_3} \right). \quad (11)
\]

2) The power-sharing and time-sharing codewords for users 1 and 2, \( X_{\text{SAN2}}^N \) with power \([P, Q_2] - Q_2^+\), carries the message \( W_{\text{SAN2}} \) with rate \( R_{\text{SAN2}} \), transmitted in \( \bar{\alpha} \) of time and treats state sequences \( S_1, S_2 \) as additional noise. Since the state variance at the second user is larger than the state variance at the first user, the codeword \( X_{\text{SAN2}}^N \) can be decoded at users 1, 2 if its rate is limited by,

\[
R_{\text{SAN2}} = \frac{\bar{\alpha}}{2} \log \left( 1 + \frac{[P, Q_2] - Q_2^+}{N + Q_2} \right). \quad (12)
\]

3) The power-sharing and time-sharing codewords, \( X_{\text{PAS3}}^N \), \( X_{\text{PAS2}}^N \), and \( X_{\text{PAS1}}^N \), with powers \([P, Q_3], [P, Q_2], \) and \([P, Q_2] \), which are pre-coded against the state sequences \( S_1^N, S_2^N \), and \( S_1^N \), and are transmitted on \( \bar{\alpha}, \bar{\alpha}/2, \) and \( 3/2 \) portions of the three channel uses, respectively, where \( \bar{\alpha} = 1 - \alpha \).

This strategy attains the following achievable rate,

\[
R_{\text{PAS1}} = R_{\text{PAS2}} = \frac{\bar{\alpha}}{4} \log \left( 1 + \frac{[P, Q_2]}{N} \right), \quad (13a)
\]

\[
R_{\text{PAS3}} = \frac{\alpha}{2} \log \left( 1 + \frac{[P, Q_3]}{N} \right), \quad (13b)
\]

According to Figure 3, it is seen that at duration \( \alpha \), the following rate is achievable:

\[
R_1 = R_{\text{SAN3}} + R_{\text{SAN2}} + R_{\text{PAS3}} = \frac{1}{2} \log \left( 1 + \frac{[P - Q_3]^+}{N + Q_3} \right) + \bar{\alpha} \frac{1}{2} \log \left( 1 + \frac{[P, Q_2]}{N} \right) \quad + \frac{1}{4} \log \left( 1 + \frac{[P, Q_2]}{N} \right), \quad (14)
\]

at duration \( \bar{\alpha} \), the following rate is achievable:

\[
R_2 = R_{\text{SAN3}} + R_{\text{SAN2}} + R_{\text{PAS1}} = \frac{1}{2} \log \left( 1 + \frac{[P - Q_3]^+}{N + Q_3} \right) + \bar{\alpha} \frac{1}{2} \log \left( 1 + \frac{[P, Q_3] - Q_3^+}{N + Q_2^+} \right) \quad + \frac{1}{4} \log \left( 1 + \frac{[P, Q_2]}{N} \right), \quad (15)
\]

By increasing \( \alpha \), (14) increases and (15) decreases, so maximum of the achievable rate at any time occurs when \( R_1 = R_2 \) which results

\[
\bar{\alpha} \frac{1}{2} \log \left( 1 + \frac{[P, Q_3]}{N} \right) = \bar{\alpha} \left( \frac{1}{2} \log \left( 1 + \frac{[P, Q_3] - Q_3^+}{N + Q_2^+} \right) \right) + \frac{1}{4} \log \left( 1 + \frac{[P, Q_2]}{N} \right), \quad (16)
\]

and by solving (16), \( \alpha \) is obtained as shown in (17).

\[
\alpha = \frac{\log \left( 1 + \frac{[P, Q_2]}{N} \right) + \frac{1}{2} \log \left( 1 + \frac{[P, Q_3]}{N} \right)}{\log \left( 1 + \frac{[P, Q_3] - Q_3^+}{N + Q_2^+} \right) + \log \left( 1 + \frac{[P, Q_3] - Q_3^+}{N + Q_2^+} \right) + \frac{1}{2} \log \left( 1 + \frac{[P, Q_2]}{N} \right)} + \frac{1}{2} \log \left( 1 + \frac{[P, Q_2]}{N} \right). \quad (17)
\]

1. For \( P < Q_2 < Q_3 \) we have

\[
\frac{1}{2} \log \left( 1 + \frac{[P, Q_3] - Q_3^+}{N + Q_2^+} \right) = 0, \quad (18a)
\]

\[
\frac{1}{2} \log \left( 1 + \frac{[P, Q_3]}{N} \right) = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right), \quad (18b)
\]

\[
\frac{1}{2} \log \left( 1 + \frac{[P, Q_2]}{N} \right) = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right), \quad (18c)
\]

substituting (18) in (17) results \( \alpha = \frac{1}{3} \).
For $P \geq Q_2$ we have

$$\log \left( 1 + \frac{[\min(P, Q_3) - Q_2]^+}{N + Q_2} \right) + \frac{1}{2} \log \left( 1 + \frac{\min(P, Q_3)}{N} \right)$$

$$= \log \left( 1 + \frac{\min(P, Q_3) - Q_2}{N} \right) + \frac{1}{2} \log \left( 1 + \frac{Q_2}{N} \right)$$

$$= \log (N + \min(P, Q_3)) - \frac{1}{2} \log (N + Q_2) - \frac{1}{2} \log (N)$$

$$- \frac{1}{2} \log (N + Q_2) + \frac{1}{2} \log (N)$$

$$= \log \left( 1 + \frac{\min(P, Q_3)}{N} \right) - \frac{1}{2} \log \left( 1 + \frac{Q_2}{N} \right), \quad (19)$$

substituting (19) in (17) results (10).

**Theorem 4.2.** The second inner bound: The following rate is achievable for the 3-user GBC-SKT with independent states with unequal variances:

$$R = \frac{1}{2} \log \left( 1 + \frac{[P - Q_3]^+}{N + Q_3} \right) + \frac{\beta}{2} \log \left( 1 + \frac{\min(P, Q_3) - Q_2}{N} \right) + \frac{1}{6} \log \left( 1 + \frac{\min(P, Q_3)}{N} \right), \quad (20)$$

where $\beta$ is as shown in (21).

$$\beta = \begin{cases} \frac{\log \left( 1 + \frac{\min(P, Q_3)}{N} \right) - \log \left( 1 + \frac{Q_2}{N} \right)}{ \log \left( 1 + \frac{[\min(P, Q_3) - Q_2]^+}{N} \right) + \log \left( 1 + \frac{\min(P, Q_3)}{N} \right) - \log \left( 1 + \frac{Q_2}{N} \right) }, & P \geq Q_2, \\ 0, & P < Q_2. \end{cases} \quad (21)$$

$$\beta = \begin{cases} \log \left( 1 + \frac{[\min(P, Q_3) - Q_2]^+}{N + Q_2} \right) - \frac{1}{6} \log \left( 1 + \frac{[\min(P, Q_3) - Q_2]^+}{N + Q_2} \right) + \frac{1}{2} \log \left( 1 + \frac{\min(P, Q_3) - Q_2}{N + Q_2} \right), & P \geq Q_2, \\ 0, & P < Q_2. \end{cases} \quad (22)$$

**Proof.** The transmission strategy consists of one power-sharing and two time-sharing phases. We divide power $P$ into three parts $[P - Q_3]^+$, $[\min(P, Q_3) - Q_2]^+$, and $\min(P, Q_3)$. We consider two divisions for the transmission time unit: $\{\beta, 3\beta\}$ and $(1/3, 1/3, 1/3)$.

The strategy is graphically represented in Figure 4.

If $\beta \leq 1/3$

1. At duration $(0, \beta)$, we transmit power $P$ with three codewords $X_{SAN3}^N$ with power $[P - Q_3]^+$, $X_{SAN3}^N$ with power $[\min(P, Q_3) - Q_2]^+$, and $X_{SAN3}^N$ with power $\min(P, Q_3)$.

2. At duration $(\beta, 1/3)$, we transmit power $P$ with three codewords $X_{SAN3}^N$ with power $[P - Q_3]^+$, $X_{SAN2}^N$ with power $[\min(P, Q_3) - Q_2]^+$, and $X_{SAN3}^N$ with power $\min(P, Q_3)$.

3. At duration $(1/3, 2/3)$, we transmit power $P$ with three codewords $X_{SAN3}^N$ with power $[P - Q_3]^+$, $X_{SAN2}^N$ with power $[\min(P, Q_3) - Q_2]^+$, and $X_{SAN3}^N$ with power $\min(P, Q_3)$.

4. At duration $(2/3, 1)$, we transmit power $P$ with three codewords $X_{SAN3}^N$ with power $[P - Q_3]^+$, $X_{SAN2}^N$ with power $[\min(P, Q_3) - Q_2]^+$, and $X_{SAN3}^N$ with power $\min(P, Q_3)$.

If $1/3 < \beta \leq 2/3$

1. At duration $(0, 1/3)$, we transmit power $P$ with three codewords $X_{SAN3}^N$ with power $[P - Q_3]^+$, $X_{SAN3}^N$ with power $[\min(P, Q_3) - Q_2]^+$, and $X_{SAN3}^N$ with power $\min(P, Q_3)$.

2. At duration $(1/3, 1)$, we transmit power $P$ with three codewords $X_{SAN3}^N$ with power $[P - Q_3]^+$, $X_{SAN3}^N$ with power $[\min(P, Q_3) - Q_2]^+$, and $X_{SAN3}^N$ with power $\min(P, Q_3)$.

3. At duration $(2/3, 1)$, we transmit power $P$ with three codewords $X_{SAN3}^N$ with power $[P - Q_3]^+$, $X_{SAN2}^N$ with power $[\min(P, Q_3) - Q_2]^+$, and $X_{SAN3}^N$ with power $\min(P, Q_3)$.

If $2/3 < \beta \leq 1$
1. At duration $0, 1/3$, we transmit power $P$ with three codewords $X_{SAN3}^N$ with power $[P - Q_3]^+$, $X_{PAS30}^N$ with power $\min\{P, Q_3\} - Q_3^+$, and $X_{PAS3}^N$ with power $\min\{P, Q_3\}$.

2. At duration $(1/3, 2/3)$, we transmit power $P$ with three codewords $X_{SAN3}^N$ with power $[P - Q_3]^+$, $X_{PAS3}^N$ with power $\min\{P, Q_3\} - Q_3^+$, and $X_{PAS2}^N$ with power $\min\{P, Q_3\}$.

3. At duration $(2/3, \beta)$, we transmit power $P$ with three codewords $X_{SAN3}^N$ with power $[P - Q_3]^+$, $X_{PAS3}^N$ with power $\min\{P, Q_3\} - Q_3^+$, and $X_{PAS1}^N$ with power $\min\{P, Q_2\}$.

4. At duration $(\beta, 1)$, we transmit power $P$ with three codewords $X_{SAN3}^N$ with power $[P - Q_3]^+$, $X_{SAN2}^N$ with power $\min\{P, Q_3\} - Q_3^+$, and $X_{PAS1}^N$ with power $\min\{P, Q_2\}$.

The codewords are defined as follows:

1) The power-sharing codeword, $X_{SAN3}^N$ with power $[P - Q_3]^+$, carries the message $W_{SAN3}$ with the following rate

$$R_{SAN3} = \frac{1}{2} \log \left( 1 + \frac{[P - Q_3]^+}{N + Q_3} \right), \quad (23)$$

that should be decoded at three users.

2) The power-sharing and time-sharing codeword $X_{PAS30}^N$ for user 3 with power $\min\{P, Q_3\} - Q_3^+$ which is pre-coded against the state sequence $S_{3}^N$, and is transmitted in $\beta$ portion of time.

$$R_{PAS3} = \beta \log \left( 1 + \frac{\min\{P, Q_3\} - Q_3^+}{N + Q_2} \right), \quad (24)$$

that should be decoded at user 3.

3) The power-sharing and time-sharing codeword for users 1 and 2, $X_{SAN2}^N$ with power $\min\{P, Q_3\} - Q_2^+$, is transmitted in $\beta$ portion of time with following rate

$$R_{SAN2} = \beta \log \left( 1 + \frac{\min\{P, Q_3\} - Q_2^+}{N + Q_2} \right), \quad (25)$$

that should be decoded at users 1 and 2.

4) Three power-sharing and time-sharing codewords, $X_{PAS1}^N$, $X_{PAS2}^N$ and $X_{PAS3}^N$, each with power $\min\{P, Q_2\}$, which are pre-coded against the state sequences $S_{1}^N$, $S_{2}^N$, and $S_{3}^N$, respectively, and they are transmitted on $\frac{1}{3}$ portions of the transmission time unit with the following rate,

$$R_{PAS1} = R_{PAS2} = R_{PAS3} = \frac{1}{6} \log \left( 1 + \frac{\min\{P, Q_3\}}{N} \right). \quad (26)$$

According to Figure 4, It is seen that at duration $\beta$, the following rate is achievable:

$$R_1 = R_{SAN3} + R_{PAS3} + R_{PAS3}$$

$$= \frac{1}{2} \log \left( 1 + \frac{[P - Q_3]^+}{N + Q_3} \right)$$

$$+ \frac{\beta}{2} \log \left( 1 + \frac{\min\{P, Q_3\} - Q_3^+}{N} \right)$$

$$+ \frac{1}{6} \log \left( 1 + \frac{\min\{P, Q_3\}}{N} \right). \quad (27)$$

and at duration $\bar{\beta}$, the following rate is achievable:

$$R_2 = R_{SAN3} + R_{SAN2} + R_{PAS1}$$

$$= \frac{1}{2} \log \left( 1 + \frac{[P - Q_3]^+}{1 + Q_3} \right)$$

$$+ \frac{\beta}{2} \log \left( 1 + \frac{\min\{P, Q_3\} - Q_3^+}{N + Q_2} \right)$$

$$+ \frac{1}{6} \log \left( 1 + \frac{\min\{P, Q_3\}}{N} \right). \quad (28)$$

Increasing $\beta$ increases (27) but decreases (28), so to have the maximum total rate, $\beta$ should be chosen such that $R_1 = R_2$ which results

$$\frac{\beta}{2} \log \left( 1 + \frac{[\min\{P, Q_3\} - Q_3^+]}{N} \right)$$

$$= \frac{\bar{\beta}}{2} \log \left( 1 + \frac{[\min\{P, Q_3\} - Q_3^+]}{N + Q_2} \right) \quad (29)$$

which yields (21) for $\beta$.

1. For $P < Q_3 < Q_2$ we have

$$\frac{1}{2} \log \left( 1 + \frac{[\min\{P, Q_3\} - Q_3^+]}{N + Q_2} \right) = 0, \quad (30a)$$

substituting (30a) in (22) results $\beta = 0$.

2. For $P \geq Q_3$ we have

$$\log \left( 1 + \frac{[\min\{P, Q_3\} - Q_3^+]}{N + Q_2} \right)$$

$$= \log \left( 1 + \frac{\min\{P, Q_3\} - Q_3^+}{N + Q_2} \right)$$

$$= \log (N + \min\{P, Q_3\}) - \log (N + Q_2)$$
where

\[
= \log(N + \min\{P, Q_3\}) - \log(N)
\]

\[
- \log(N + Q_3) + \log(N)
\]

\[
= \log\left(1 + \frac{\min\{P, Q_3\}}{N}\right) - \log\left(1 + \frac{Q_3}{N}\right),
\]

substituting (31) in (22) results (21).

Remarks:

1) For \(P < Q_3\), we have

\[
[P - Q_3]^+ = [\min\{P, Q_3\} - Q_3]^+ = 0,
\]

Substituting them in (9) and (20) yields the time-sharing rate,

\[
\frac{1}{6} \log(1 + P).
\]

2) For \(Q_3 = Q_2 < P\), we have \([\min\{P, Q_3\} - Q_3]^+ = 0\). Substituting it in (10) and (21) results \(\alpha = \frac{1}{\beta} = 0\). In this case Figures 3 and 4 will be the same and two proposed inner bounds and (7) coincide.

3) Assuming \(\min\{P, Q_3\} > Q_3\) in (10) and (21) results \(\alpha = \beta \approx \frac{1}{2}\). Substituting them in (9) and (20) yields \(R_1\) and \(R_2\), respectively,

\[
R_1 = \frac{1}{2} \log\left(1 + \frac{[P - Q_3]^+}{N + Q_3}\right) + \frac{1}{4} \log\left(1 + \frac{\min\{P, Q_3\}}{N}\right),
\]

\[
R_2 = \frac{1}{2} \log\left(1 + \frac{[P - Q_3]^+}{N + Q_3}\right) + \frac{1}{4} \log\left(1 + \frac{\min\{P, Q_3\}}{N}\right) + \frac{1}{6} \log\left(1 + \frac{\min\{P, Q_3\}}{N}\right),
\]

(32a)

(32b)

As we see from (32), \(R_2 > R_1\), that is, the second inner bound is greater than the first one for this case.

4) It is not possible to compare directly the two proposed inner bounds because of their complexities which is mostly due to the fractional expressions for parameters \(\alpha\) and \(\beta\). In Section 5, through some simulation results, we see that the second inner bound outperform the first inner bound. We can justify this fact by saying that for the codewords with power \(\min\{P, Q_3\}\) we obtain the same rate for three users which is seemingly the better strategy. It is also concluded from (32).

**Theorem 4.3.** The outer bound: The capacity of the 3-user GBC-SKT with independent states with unequal variances is upper bounded by

\[
R \leq \frac{1}{2} \log(1 + P + Q_3) + \frac{1}{2} - \frac{1}{6} \log(Q_3Q_2 + Q_3Q_1 + Q_1Q_2).
\]

(33)

**Proof.** Applying Fano’s inequality yields,

\[
3NR \leq \sum_{k=1}^{3} (H(Y_k^N) - H(Y_k^N | W_0^N, W_k^N)).
\]

(34)

For the positive entropy term in (34) we have,

\[
\sum_{k=1}^{3} H(Y_k^N) \leq \frac{N}{2} \sum_{k=1}^{3} \log 2\pi e (P + Q_k + 2\sqrt{PQ_k}) + 1
\]

(35a)

\[
\leq \frac{N}{2} \sum_{k=1}^{3} \log 2\pi e (2P + 2Q_k + 2)
\]

(35b)

\[
\leq \frac{3N}{2} \log 2\pi e (2P + 2Q_3 + 2)
\]

(35c)

\[
= \frac{3N}{2} \log (2\pi e (P + Q_3 + 1) + 1).
\]

(35d)

Equation (35b) follows from the fact that \((P - Q_k)^2 \geq 0\). Equation (35c) follows from \(Q_3 \geq Q_2 \geq Q_1\).

For the negative entropy term in (34) we have,

\[
- \sum_{k=1}^{3} H(Y_k^N | W_0^N, W_k^N) \leq -H(Y_3^N, Y_2^N, Y_1^N | W_0^N, W_1^N, W_2^N, W_3^N)
\]

(36a)

\[
= -H(Y_3^N - Y_2^N, Y_3^N - Y_1^N, Y_2^N | W_0^N, W_1^N, W_2^N, W_3^N)
\]

(36b)

\[
= -H(Y_3^N - Y_2^N, Y_3^N - Y_1^N, Y_2^N | W_0^N, W_1^N, W_2^N, W_3^N)
\]

(36c)

\[
= -H(Y_3^N - S_3^N, S_3^N - S_1^N, Y_2^N | W_0^N, W_1^N, W_2^N, W_3^N)
\]

\[
- H(Y_3^N - S_3^N, S_3^N - S_1^N, W_0^N, Y_2^N | W_0^N, W_1^N, W_2^N, W_3^N)
\]

\[
\leq -H(Y_3^N - S_3^N, S_3^N - S_1^N, W_0^N, W_1^N, W_2^N, W_3^N)
\]

\[
- H(Y_3^N - S_3^N, S_3^N - S_1^N, W_0^N, W_1^N, W_2^N, W_3^N)
\]

\[
= -N \log \left(\frac{[Q_3 + Q_2 - Q_3]}{Q_3 + Q_1 + Q_3 + Q_2}\right)
\]

(36d)

In the above, (36a) follows from the fact that conditioning reduces entropy. Equation (36b) follows from the fact that the determinant of the following transformation matrix is equal to one.

\[
\begin{bmatrix}
Y_3^N - Y_2^N & Y_3^N - Y_1^N & Y_2^N - Y_1^N
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 0 & 1 & 1 & 0 & -1 & Y_3^N & Y_2^N & Y_1^N
\end{bmatrix}.
\]

(37)
For convenience to obtain (36c), we consider all noises to be identical.

Substituting (35d) and (36d) in (34) yields (33).

Remarks: In the following we consider two special classes of the 3-user GBC-SKT channel for which the gap between inner and outer bounds is 1 bpcu.

1. For \( Q_1, Q_2 \geq 1 \) and \( (Q_1 + Q_2) \geq (Q_1 + 2) \), the negative entropy term, (36d), simplifies to,

\[
-\frac{1}{6} \log (Q_1 Q_2 + Q_1 Q_1 + Q_1 Q_2)
\leq -\frac{1}{6} \log (Q_1 (Q_2 + 2) + 1)
= -\frac{1}{3} \log (Q_1 + 1).
\] (38a)

Equations (38a) and (35d) yield the following outer bound,

\[
R \leq \frac{1}{2} \log (P + Q_1 + 1) + \frac{1}{2} - \frac{1}{3} \log (Q_1 + 1).
\] (39a)

For \( P \geq Q_3 \), (39a) reduces to,

\[
R \leq \frac{1}{2} \log (P + P + 1) + \frac{1}{2} - \frac{1}{3} \log (Q_1 + 1)
\leq \frac{1}{2} \log (2P + 2) + \frac{1}{2} - \frac{1}{3} \log (Q_1 + 1)
\leq \frac{1}{2} \log (P + 1) + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} \log (Q_1 + 1)
= \frac{1}{2} \log \left( 1 + \frac{P - Q_1}{1 + Q_1} \right) + \frac{1}{6} \log (Q_1 + 1) + 1.
\] (40d)

For \( P < Q_3 \), (39a) reduces to,

\[
R \leq \frac{1}{2} \log (Q_3 + 1) + 1 - \frac{1}{3} \log (Q_1 + 1)
= \frac{1}{6} \log (Q_3 + 1) + 1.
\] (41b)

Combining two cases together we have the following equivalent form for the outer bound (39a),

\[
R \leq \frac{1}{2} \log \left( 1 + \frac{[P - Q_1]^+}{1 + Q_1} \right) + \frac{1}{6} \log (1 + \min\{P, Q_1\}) + 1.
\] (42a)

This rate can be achieved by sending one codeword with power \([P - Q_1]^+\) which considers state as noise and the other codeword with power \(\min\{P, Q_1\}\) pre-coded against each state and transmitted in \(\frac{1}{2}\) part of time. Comparing with the second inner bound, (20), it is seen that for \( \beta = 0 \), we can achieve the capacity within 1 bpcu.

2. For \( Q_1, Q_2 \geq \frac{1}{\sqrt{2} - 1} \), we have \((Q_1 Q_1 + Q_1 Q_2) \geq (Q_2 + Q_1 + 1)\) which yields

\[
-\frac{1}{6} \log (Q_1 Q_2 + Q_1 Q_1 + Q_1 Q_2)
\leq -\frac{1}{6} \log ((Q_1 + 1) (Q_2 + 1)).
\] (43)

Equations (35a) and (43) yield the following outer bound,

\[
R \leq \frac{1}{2} \log (P + Q_1 + 1) + \frac{1}{2} - \frac{1}{6} \log (Q_3 + 1)
= \frac{1}{2} \log \left( 1 + \frac{[P - Q_1]^+}{1 + Q_2} \right)
+ \frac{1}{3} \log \left( 1 + \frac{\min\{P, Q_2\} - Q_2^+}{1 + Q_2} \right)
+ \frac{1}{6} \log (1 + \min\{P, Q_1\}) + 1.
\] (44a)

Comparing with the second inner bound, (20), it is seen that for \( \beta = \frac{2}{3} \), we can achieve the capacity within 1 bpcu. Note that subject to \( Q_1 \geq Q_2 \geq Q_3 \), the condition

\[
Q_1 + Q_2 \geq \frac{1}{Q_1 - 1}
\]

is equivalent to the conditions \( Q_1 < 1 \) or \( Q_1 > \frac{1 + \sqrt{3}}{2} \).

5 | SIMULATION RESULTS

In this section we evaluate the proposed inner and outer bounds through simulations. We assume \( N = 1 \). At each experiment we change one of the parameters \( P, Q_1, Q_2, Q_3 \) and fix the others. We compare proposed inner bounds, (9) and (20) with proposed outer bound, (33), and also with previous inner bounds, (7) and (8).

We consider three following scenarios:

1. \( P = 100, Q_1 = 1, Q_2 = 10 \): By changing \( Q_3 \) from 1.01 \( Q_2 \) to 0.99 \( Q_2 \), we sketch the inner bounds in Figure 5a. As previously discussed when \( Q_3 \approx Q_2 \), two inner bounds and (7) will be nearly the same. From Figure 5a we see that as \( Q_3 \) grows the gap between these bounds increases. The time-sharing method and the second inner bound yield the lowest and highest achievable rates, respectively. In Figure 5b we depict two proposed inner bounds and the outer bound. In Figure 5c the gap between the proposed outer bound and each of the proposed inner bounds is shown. We see that the gap is obtained at most 0.5 bpcu for this experiment.
bounds in Figure 6b and the gap between the proposed outer bound and each of the proposed inner bounds in Figure 6c. As we see, the gap is obtained at most 0.3 bpcu for this experiment.

3) $Q_1 = 1, Q_2 = 10, Q_3 = 50$: By changing $Q_3$ from $Q_1$ to $Q_3$, we sketch the inner bounds in Figure 7a. As we see from this figure, when $P$ grows, inner bounds increase and again the second proposed inner bound outperforms the others. We also depict the proposed outer and inner bounds in Figure 7b and the gap between the proposed outer bound and each of the proposed inner bounds in Figure 7c. As we see, the gap obtained is at most 0.3 bpcu for this experiment.
6 CONCLUSION

It this work we investigated the capacity of the 3-user Gaussian broadcast channel with states known non-causally at the transmitter (GBC-SKT). The variances of the states are assumed to be different. We study the case where the transmitter communicates only common messages to the users. In fact the difficult point to not achieve the capacity of the GBC-SKT channel is in communicating common messages. The key point in deriving the inner bound is how to pre-code the common message against the states which do not have the same variance. Here we derived two inner bounds and one outer bound on the capacity of the 3-user GBC-SKT. The novel point was in defining power-sharing and time-sharing codewords and arranging them such that we get to the higher achievable rates. Simulation results show that one of the proposed inner bounds always outperforms the other inner bound. We also find the conditions for which the gap between inner and outer bounds are at most 1 bpcu.

REFERENCES

1. Costa, M.H.M.: Writing on dirty paper. IEEE Trans. Inf. Theory 29(3), 439–441 (1983)
2. Bergmans, P.: Random coding theorem for broadcast channels with degraded components. IEEE Trans. Inf. Theory 19(2), 197–207 (1973)
3. Gallager, R.: Capacity and coding for degraded broadcast channels. Probl. Transm. Inf. 10(3), 3–14 (1974)
4. Steinberg, Y., Shamai, S.: Achievable rates for the broadcast channel with states known at the transmitter. In: International Symposium on Information Theory (ISIT), IEEE (2005)
5. Marton, K.: A coding theorem for the discrete memoryless broadcast channel. IEEE Trans. Inf. Theory 25(3), 306–311 (1979)
6. Gel’Fand, S.: Coding for channels with random parameters. Probl. Contr. Inform. Theory 9(1), 19–31 (1980)
7. Khisti, A., et al.: Carbon copying onto dirty paper. IEEE Trans. Inf. Theory 53(5), 1814–1827 (2007)
8. Rini, S., Shamai, S.: The impact of phase fading on the dirty paper coding channel. In: International Symposium on Information Theory (ISIT, IEEE, pp. 2287–2291 (2014)
9. Rini, S., Shamai, S.: On capacity of the dirty paper channel with fading in the strong fading regime. In: Information Theory Workshop (ITW), IEEE, pp. 561–565 (2014)
10. Benammar, S.S.M., Piantanida, P.: On the compound broadcast channel: Multiple description coding and interference decoding. IEEE Trans. Inf. Theory 66(1), 38–64 (2020)
11. Monemizadeh, M., Fehri, H.: Closed-form capacity bounds for the exponential version of the dirty paper channel. IEEE Trans. Wireless Comm. 9(7), 1080–1083 (2020)
12. Ghabeli, L., Rini, S.: On the capacity of the Gaussian broadcast channel with states known at the transmitter. In: Proceedings of the International Conference on the Science of Electrical Engineering (ICSEE), Israel, Eilat (2016)
13. Ghabeli, L.: On the capacity of a class of K-user Gaussian broadcast channel with states known at the transmitter. IET Commun. 12(7), 787–795 (2018)
14. Rini, S., Goldsmith, A.: A general approach to random coding for multi-terminal networks. In: Information Theory and Applications (ITA) Workshop, IEEE, pp. 1–9 (2013)
15. Piantanida, P., Shitz, S.S.: On the capacity of compound state-dependent channels with states known at the transmitter. In: International Symposium on Information Theory (ISIT), pp. 624–628. IEEE (2010)

How to cite this article: Ghabeli, L.: The inner and outer bounds on the capacity of the 3-user GBC-SKT with unequal state variances. IET Commun. 15, 1756–1766 (2021).
https://doi.org/10.1049/cmu2.12187