Lipschitz equivalence of self-similar sets with touching structures

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Abstract
Lipschitz equivalence of self-similar sets is an important area in the study of fractal geometry. It is known that two dust-like self-similar sets with the same contraction ratios are always Lipschitz equivalent. However, when self-similar sets have touching structures the problem of Lipschitz equivalence becomes much more challenging and intriguing at the same time. So far, all the known results only cover self-similar sets in \( \mathbb{R} \) with no more than three branches. In this study we establish results for the Lipschitz equivalence of self-similar sets with touching structures in \( \mathbb{R} \) with arbitrarily many branches. Key to our study is the introduction of a geometric condition for self-similar sets called substitutable.

Keywords: Lipschitz equivalence, self-similar sets, touching structure, martingale convergence theorem, graph-directed sets, substitutable

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1. Introduction

1.1. Motivation
A fundamental concept in fractal geometry is dimension. It is often used to differentiate fractal sets, and when two sets have different dimensions (Hausdorff dimensions, box dimensions or other dimensions) we often consider them to be unalike. However two compact sets, even with the same dimension, may in fact be quite different in many ways. Thus it is natural to seek a
suitable quality that would allow us to tell whether two fractal sets are ‘similar’. Generally, Lipschitz equivalence is thought to be such a quality.

It has been pointed out in [5] that, while topology may be regarded as the study of equivalence classes of sets under homeomorphism, fractal geometry is sometimes thought of as the study of equivalence classes under bi-Lipschitz mappings. More restrictive maps such as isometry tend to lead to poor and rather uninteresting equivalent classes, while far less restrictive maps such as general continuous maps take us completely out of geometry into the realm of pure topology (see [7]). Bi-Lipschitz maps offer a good balance, which leads to categories that are interesting and intriguing both geometrically and algebraically.

There have been many works done in the field of Lipschitz equivalence of two fractal sets. Some earlier fundamental results were obtained by Cooper and Pignataro [1], David and Semmes [2], and Falconer and Marsh [4,5]. Recently, based on these works and motivated by problem 11.16 in [2], Rao, Ruan, Wang, Xi, Xiong and their collaborators obtained a series of results; see, e.g., [13–15, 20–25]. One main tool in these papers is the graph-directed method introduced in [14]. More recently, Lau and Luo [9] introduced the hyperbolic boundary method to investigate the Lipschitz equivalence of the totally disconnected self-similar set. There are also some other related works. Xi [19] discussed the nearly Lipschitz equivalence of self-conformal sets. Mattila and Saaranen [11] studied the Lipschitz equivalence of Ahlfors–David regular sets. Deng et al [3] and Llorente and Mattila [10] discussed the bi-Lipschitz embedding of fractal sets. Feng and Wang [6] studied the structure of generating iterated function systems (IFSs) of Cantor sets.

Let $E$ and $F$ be two compact subsets of $\mathbb{R}^d$. A bijection $f : E \to F$ is said to be bi-Lipschitz if there exist two positive constants $c$ and $c'$ such that

$$c|x - y| \leq |f(x) - f(y)| \leq c'|x - y|,$$

$\forall x, y \in E$. (1)

$E$ and $F$ are said to be Lipschitz equivalent, denoted by $E \sim F$, if there exists a bi-Lipschitz map $f$ from $E$ to $F$.

We recall some basic notations in fractal geometry. Given a family of similitude $\Phi_i(x)$, $i = 1, \ldots, n$, on $\mathbb{R}^d$, where each $\Phi_i$ has contraction ratio $\rho_i$ with $\rho_i < 1$, there exists a unique nonempty compact subset $E$ of $\mathbb{R}^d$ such that $\bigcup_{i=1}^n \Phi_i(E) = E$, see [8]. The set of maps $\{\Phi_i(x), i = 1, \ldots, n\}$ is called an IFS and $E$ is called the attractor, or the invariant set, of the IFS. We also call $E$ a self-similar set, since every $\Phi_i$ is a similitude. If $\Phi_i(E) \cap \Phi_j(E) = \emptyset$ for any distinct $i$ and $j$, the IFS $\{\Phi_i\}$ is then said to satisfy the strong separation condition (SSC), and $E$ is said to be dust-like.

Given $\rho_1, \ldots, \rho_n \in (0, 1)$ with $\sum_{i=1}^n \rho_i^d < 1$, we call $\rho = (\rho_1, \ldots, \rho_n)$ a contraction vector (in $\mathbb{R}^d$). We denote by $D(\rho)$ the family of all dust-like self-similar sets with contraction vector $\rho$ (here the ambient dimension $d$ is implicitly fixed). The following property is well known; see, e.g., [14].

**Proposition 1.1.** Any two sets in $D(\rho)$ are Lipschitz equivalent.

There are examples where two different contraction vectors $\rho_1$ and $\rho_2$ lead to Lipschitz equivalent families $D(\rho_1)$ and $D(\rho_2)$. For example, Rao et al [13] have completely classified the Lipschitz equivalence of dust-like families $D(\rho)$ where $\rho = (\rho_1, \rho_2)$, and one of the results is that $\rho_1 = (\lambda, \lambda^5)$ and $\rho_2 = (\lambda^2, \lambda^3)$ lead to Lipschitz equivalence families whenever the resulting self-similar sets are dust-like. The paper [13] and some earlier studies such as [1, 2, 5] have explored the impact of algebraic properties of the contraction vectors on Lipschitz equivalence, yielding a number of intriguing results showing the links.

Nevertheless one should not overlook the importance of geometric properties of the underlying IFSs on Lipschitz equivalence of self-similar sets. Relating to this point is an interesting problem proposed by David and Semmes [2, problem 11.16].
Problem 1.1. Let $S_i(x) := x/5 + (i - 1)/5$ be a contractive map from $[0, 1]$ to $[0, 1]$ where $i \in \{1, \ldots, 5\}$. Let $M$ and $M'$ be the attractor of the IFS $\{S_1, S_3, S_5\}$ and the IFS $\{S_1, S_3, S_3\}$, respectively. Are $M$ and $M'$ Lipschitz equivalent?

We call $M$ the $\{1, 3, 5\}$-set and $M'$ the $\{1, 4, 5\}$-set. Clearly, $M$ is dust-like and $M'$ has a certain touching structure; see figure 1. In this problem, the contraction ratios are all identical, so the difference lies entirely in the geometry of the two IFSs. David and Semmes conjectured that $M \not\sim M'$. However, by examining graph-directed structures of the attractors and introducing techniques to study Lipschitz equivalence on these structures, Rao et al [14] proved that in fact $M \sim M'$.

Notice that all contractive maps in the above problem have the same contraction ratio, $1/5$. Some similar works has been done in higher dimensional cases; see, e.g., [9, 17, 18, 23].

A follow-up study by Xi and Ruan [21] exploits the interplay of algebraic properties of contraction vectors and geometric properties of IFSs. It considers the following generalization of the $\{1, 3, 5\}$--$\{1, 4, 5\}$ problem.

Problem 1.2. Let $\rho = (\rho_1, \rho_2, \rho_3)$ be a contraction vector (in $\mathbb{R}$). Let $\Phi_i(x) = \rho_i x + d_i$, $i \in \{1, 2, 3\}$, where $d_1 = 0$, $d_3 = 1 - \rho_3$ and $\rho_1 < d_2 < 1 - \rho_2 - \rho_3$ (e.g. $d_2 = \rho_1 + (1 - \rho_1 - \rho_2 - \rho_3)/2$). Let $\Psi_1 = \Phi_1$, $\Psi_3 = \Phi_3$ and $\Psi_2(x) = \rho_2 x + t_2$ with $t_2 = 1 - \rho_2 - \rho_3$. Let $M_\rho$ and $M'_\rho$ be the attractor of $\{\Phi_1, \Phi_2, \Phi_3\}$ and $\{\Psi_1, \Psi_2, \Psi_3\}$, respectively. See figure 2 for their initial configuration. Are $M_\rho$ and $M'_\rho$ Lipschitz equivalent?

Somewhat surprisingly, the Lipschitz equivalence of the two sets is completely determined by the algebraic property of $\rho_1$ and $\rho_3$ and independent of $\rho_2$. It is shown in [21] that $M_\rho \sim M'_\rho$ if and only if $\log \rho_1 / \log \rho_3 \in \mathbb{Q}$.

The above example is nevertheless a very special case. It is natural to exploit such algebraic and geometric connections further in more general settings, which is the aim of this paper. Given the complexity of even establishing the result for problem 1.2, this may appear to be a very daunting task. Fortunately, by defining a suitable convergent partition sequence of $T$ (see corollary 2.6) and introducing a new geometric notion called substitutable, we are able to prove a number of results in this direction.

Throughout this paper we assume that $\rho = (\rho_1, \ldots, \rho_n)$ is a contraction vector (in $\mathbb{R}$) with $n \geq 3$. Let $D \in \mathcal{D}(\rho)$. By proposition 1.1, we may assume without loss of generality that $D$ is the attractor of the IFS $\{\Phi_i(x) = \rho_i x + d_i\}_{i=1}^n$, where $\Phi_1([0, 1])$, $\ldots$, $\Phi_n([0, 1])$ are equally spaced closed subintervals of $[0, 1]$ arranged from left to right, normalized so that the left endpoint of $\Phi_1([0, 1])$ is 0 and the right end of $\Phi_n([0, 1])$ is 1.

We are interested in the Lipschitz equivalence of $D$ with the attractor $T$ of another IFS $\{\Psi_i(x) = \rho_i x + t_i\}_{i=1}^n$ having the same contraction vector $\rho$ but with translations $\{t_i\}$ that...
may result in some of the subintervals $\Psi_1([0, 1]), \ldots, \Psi_n([0, 1])$ touching one another (but not overlapping). More precisely, the IFS $\{\Psi_i(x) = \rho_i x + t_i\}_{i=1}^n$ satisfies the following three properties.

1. The subintervals $\Psi_1([0, 1]), \ldots, \Psi_n([0, 1])$ are spaced from left to right without overlapping, i.e. their interiors do not intersect.
2. The left endpoint of $\Psi_1([0, 1])$ is 0 and the right endpoint of $\Psi_n([0, 1])$ is 1.
3. There exists at least one $i \in \{1, 2, \ldots, n-1\}$, such that the intervals $\Psi_i([0, 1])$ and $\Psi_{i+1}([0, 1])$ are touching, i.e., $\Psi_i(1) = \Psi_{i+1}(0)$.

Denote by $T$ the attractor of the IFS $\{\Psi_i\}_{i=1}^n$. Figure 3 gives an example of $[\Phi_i]$ and $[\Psi_i]$, respectively. In this paper we present necessary conditions and sufficient conditions for $D \sim T$.

1.2. Notations

First some commonly used basic notations. Denote $\Sigma_n := \{1, 2, \ldots, n\}$ and $\Sigma_n^\ast := \bigcup_{m \geq 1} \Sigma_n^m = \bigcup_{m \geq 1} \{1, 2, \ldots, n\}^m$. We shall call any $i \in \Sigma_n$ a letter and $i = i_1 \cdots i_m \in \Sigma_n^m$ a word of length $|i| := m$. $i_1$ and $i_m$ are called the first letter and last letter of $i$, respectively. We define $\rho_i = \rho_{i_1} \cdots \rho_{i_m}$, $\Psi_i = \Psi_{i_1} \circ \cdots \circ \Psi_{i_m}$ and $T_i = \Psi_i(T)$. $T_i$ is called a cylinder of the IFS $\{\Psi_i\}$ for $i$. Similarly, we define $\Phi_i = \Phi_{i_1} \circ \cdots \circ \Phi_{i_m}$ and the cylinder $D_i = \Phi_i(D)$.

Specific to this study we introduce also other notations. A letter $i \in \Sigma_n$ is a (left) touching letter if $\Psi_i([0, 1])$ and $\Psi_{i+1}([0, 1])$ are touching, i.e., $\Psi_i(1) = \Psi_{i+1}(0)$. We use $\Sigma_T \subset \Sigma_n$ to denote the set of all (left) touching letters. Note that one may view $\Sigma_T + 1$ to be the set of all right touching letters. For simplicity we shall drop the word ‘left’ for $\Sigma_T$. Let $\alpha$ and $\beta$ be the maximal integer such that $\bigcup_{i=1}^{\alpha} \Psi_i([0, 1])$ and $\bigcup_{i=\beta}^{n} \Psi_i([0, 1])$ is an interval, respectively.

Given a cylinder $T_i$ and a non-negative integer $k$, we can define respectively the level $(k+1)$ left touching patch and the level $(k+1)$ right touching patch of $T_i$ to be

$$L_k(T_i) = \bigcup_{j=1}^{\alpha} T_{[i]j}, \quad R_k(T_i) = \bigcup_{j=n-\beta+1}^{n} T_{[i]j},$$

where $[i]^{\ell}$ is defined to be the word $\underbrace{\ell \cdots \ell}_{k}$ for any $\ell \in \{1, \ldots, n\}$, with $i[1]^k j$ the concatenation of $i$, $[1]^k$ and the letter $j$ (similarly for $i[n]^k j$). We remark that $L_0(T_i) = \bigcup_{j=1}^{\alpha} T_{ij}$ and $R_0(T_i) = \bigcup_{j=n-\beta+1}^{n} T_{ij}$.

Now comes the main notation we introduce for this paper. A letter $i \in \Sigma_T$ is called left substitutable if there exist $j \in \Sigma_n^\ast$ and $k, k' \in \mathbb{N}$ such that diam $L_k(T_{i+1}) = \text{diam} L_k(T_{ij})$ and the last letter of $j$ does not belong to $\{1\} \cup (\Sigma_T + 1)$. Geometrically, this simply means that a certain left touching patch of the cylinder $T_{i+1}$ has the same diameter as that of some left touching patch of a cylinder $T_{ij}$, and as a result we can substitute one of the left touching patches by the other without disturbing the other neighbouring structures in $T$ because they have the same diameter. The actual substitution is performed in the proof of our main theorem. Similarly, $i \in \Sigma_T$ is called right substitutable if there exist $j \in \Sigma_n^\ast$ and $k, k' \in \mathbb{N}$ such that...
diam $R_k(T_i) = \text{diam } R_k'(T(i+1)_j)$ and the last letter of $j$ does not belong to $\{n\} \cup \Sigma_T$. We say that $i \in \Sigma_T$ is substitutable if it is left substitutable or right substitutable.

**Remark 1.1.** Both left and right substitutable properties can be characterized algebraically as well. By definition, it is easy to check that

$$\rho_{i+1} \rho_{i}^k = \rho_{i} \rho_{i}^k \rho_j, \quad \text{(3)}$$

while

$$\rho_n \rho_{i}^k = \rho_{i+1} \rho_{i}^k \rho_j. \quad \text{(4)}$$

**Example 1.1.** Let $\Psi_1, \Psi_2, \Psi_3$ be defined as in problem 1.2 and let $T$ be its attractor. Clearly $\Sigma_T = \{2\}$, $\alpha = 1$ and $\beta = 2$. Assume that $\log \rho_1 / \log \rho_3 \in \mathbb{Q}$, i.e., there exist $u, v \in \mathbb{Z}^+$ such that $\rho_1^u = \rho_3^v$. Pick $k = v + 1$, $k' = 0$ and $j = 2[1]^u$. It is easy to check that (4) holds for $i = 2$ and the last letter of $j$ is $1 \not\in \{3\} \cup \Sigma_T$. Thus the touching letter 2 is right substitutable. See figure 4 for a graphical illustration.

### 1.3. Statement of results

We establish several results in this paper. First we prove the following necessary condition for $D \sim T$, regardless of the geometric configuration of the IFS $\{\Psi_i\}$.

**Theorem 1.2.** Assume that $D \sim T$. Then $\log \rho_1 / \log \rho_n \in \mathbb{Q}$.

Later in the paper we give an example showing that the condition $\log \rho_1 / \log \rho_n \in \mathbb{Q}$ is not sufficient for $D \sim T$. We shall also see later that if $\log \rho_i / \log \rho_j \in \mathbb{Q}$ for all $i, j \in \{1, \ldots, n\}$ then $D \sim T$. To go deeper, we must take into account the geometric information of the IFS $\{\Psi_i\}$. The main theorem of the paper is the following.

**Theorem 1.3.** Assume that $\log \rho_1 / \log \rho_n \in \mathbb{Q}$. Then, $D \sim T$ if every touching letter for $T$ is substitutable.

As indicated in example 1.1, the IFS $\{\Psi_i\}$ in problems 1.2 has a single touching letter $\Sigma_T = \{2\}$ and the letter is substitutable. Thus the Lipschitz equivalences in problems 1 and 2 follow directly from theorem 1.3.

**Corollary 1.4 ([14, 21]).** Let $\rho = (\rho_1, \rho_2, \rho_3)$ and $M_\rho$ and $M'_\rho$ be sets defined in problem 1.2. Then $M_\rho \sim M'_\rho$ if and only if $\log \rho_1 / \log \rho_3 \in \mathbb{Q}$. In particular, let $M$ and $M'$ be sets defined in problem 1.1. Then $M \sim M'$.

Theorem 1.3 allows us to establish a more general corollary. The argument used to show the substitutability in example 1.1 is easily extended to prove the following corollary.

**Corollary 1.5.** $D \sim T$ if one of the following conditions holds:
Lemma 2.1. For any $i, j \in \{1, n, \alpha\} \cup (\Sigma_T + 1)$.

The following result, which we wish to state as a theorem because of the simplicity of its statement, is a direct corollary of corollary 1.5.

Theorem 1.6. Assume that $\log \rho_i / \log \rho_j \in \mathbb{Q}$ for all $i, j \in \{1, \ldots, n\}$. Then $D \sim T$.

The rest of the paper will be devoted to proving the stated results. In section 2 we prove theorems 1.2, and in section 3 we prove theorem 1.3. The example showing that $\log \rho_i / \log \rho_j \in \mathbb{Q}$ is not sufficient for $D \sim T$ will be given in the appendix.

2. Necessary condition for $D \sim T$

2.1. Bi-Lipschitz map related to a dust-like self-similar set

In this subsection, we shall discuss the property of bi-Lipschitz map $f : E \rightarrow F$, where $E$ is a nonempty compact subset of $\mathbb{R}^d$ and $F$ is a dust-like self-similar subset of $\mathbb{R}^d$ with contraction vector $(\rho_1, \ldots, \rho_n)$. We assume that $f$ satisfies (1), where $c$ and $c'$ are positive constants.

For any nonempty subsets $A, B \subset \mathbb{R}^d$, we define $d(A, B) = \inf \{|x - y| : x \in A, y \in B\}$. The diameter of $A$ is defined to be $\text{diam} A := \sup \{|x - y| : x, y \in A\}$. If $d(A, B \setminus A) > 0$, we say that $A$ is a $B$-separate set. If $d(A, B \setminus A) \geq \lambda \cdot \text{diam} A$ for some $\lambda > 0$, we say that $A$ is a $(B, \lambda)$-separate set.

We now present a lemma which is similar to [5, lemma 3.2] and [21, lemma 10].

Lemma 2.1. For any $\lambda > 0$, there exists a positive integer $n_0$ such that for any $(E, \lambda)$-separate set $A \subset E$ there exist $k, j_1, \ldots, j_p \in \Sigma_n^*$ such that $F_{k, j_1}, \ldots, F_{k, j_p}$ are disjoint and

$$f(A) = \bigcup_{r=1}^{p} F_{k, j_r} \subset F_k,$$

where each $|j_r| = n_0$.

Proof. Given an $(E, \lambda)$-separate set $A \subset E$, let $F_k$ be the smallest cylinder containing $f(A)$. Then, it is clear that there exists a positive constant $\delta$ dependent only on $F$ such that $\text{diam} F_k \leq \delta \text{diam} f(A)$. For a detailed proof, please see, e.g., [5, lemma 3.1]. Thus, by (1), we have

$$\text{diam} F_k \leq \delta \text{diam} f(A) \leq \delta c \text{diam} A.$$ 

Let $n_0$ be the smallest positive integer satisfying $\bar{\rho}^n \delta c' \leq \frac{\delta}{2}$, where $\bar{\rho} = \max\{\rho_1, \ldots, \rho_n\}$. Then,

$$\text{diam} F_{k, j} \leq \bar{\rho}^n \delta \text{diam} F_k \leq \bar{\rho}^n \delta c' \text{diam} A \leq \frac{\delta}{2} \text{diam} A.$$ 

Assume that $F_{k, j} \cap f(A) \neq \emptyset$; we shall prove $F_{k, j} \subset f(A)$ by showing $d(z, f(E \setminus A)) > 0$ for any $z \in F_{k, j}$.

Pick $x \in f^{-1}(F_{k, j} \cap f(A))$; we have

$$d(f(x), f(E \setminus A)) \geq d(f(A), f(E \setminus A)) \geq c \cdot d(A, E \setminus A) \geq c\lambda \text{diam} A.$$ 

Thus, for any $z \in F_{k, j}$, using (6) and (7), we have

$$d(z, f(E \setminus A)) \geq d(f(x), f(E \setminus A)) - \text{diam} F_{k, j} \geq c\lambda \text{diam} A - \frac{c\lambda}{2} \text{diam} A > 0.$$ 

This completes the proof of the lemma. □
Remark 2.1. By the proof, we can require that $F_k$ is the smallest cylinder containing $f(A)$. Under this restriction, $k$ is uniquely determined by $A$. Consequently, the set $\{j_1, \ldots, j_p\}$ is also uniquely determined by $A$ and $n_0$.

2.2. Construction of $(T, \lambda)$-separate sets

Let $\emptyset$ be the empty word. We say that the length of $\emptyset$ is 0. Define $\Psi_\emptyset = \Phi_\emptyset = \text{id}$ and $\rho_0 = 1$.

By the definition of $T$, we know that there exists $i$ such that $\Psi_i(1) = \Psi_{i+1}(0)$. We pick one such $i$ and denote it by $i_0$. Without loss of generality, we assume that $\rho_1 > \rho_0$. For positive integer $k$, we define $\tau(k)$ to be the unique positive integer satisfying

$$\rho_k \rho_1 < \tau(k) \leq \rho_k^2.$$  

It is clear that $\tau(k) \geq k$ and $\tau(k)$ is increasing with respect to $k$. We define

$$C^k = R_k(T_{i_0}) \cup L_{\tau(k)}(T_{i_0+1}).$$

We remark that $C^1 \supset C^2 \supset \cdots$.

We shall adopt the notation $\preceq$ throughout this paper. Let $A$ be a given index set. Given two sequences of positive real numbers $(a_i)_{i \in A}$ and $(b_i)_{i \in A}$ indexed by $A$, we denote $(a_i) \preceq (b_i)$ if there exist positive constants $c_1, c_2$ independent of $i$ such that $c_1 a_i \leq b_i \leq c_2 a_i$ for all $i \in A$. For convenience of statement in the proofs, we shall often write $a_i \asymp b_i$ for all $i \in A$, or simply $a_i \asymp b_i$ if there is no confusion about the index set.

Lemma 2.2. There exists $\lambda > 0$, such that $C^k$ is $(T, \lambda)$-separate for all $k$.

Proof. By (8), for all $k$ we have

$$\text{diam } C^k = \text{diam } R_k(T_{i_0}) + \text{diam } L_{\tau(k)}(T_{i_0+1})$$

$$= \rho_k \rho_1 \cdot \text{diam } \left( \bigcup_{j=n-\beta+1}^n T_j \right) + \rho_{i_0+1} \rho_1^{\tau(k)} \cdot \text{diam } \left( \bigcup_{j=1}^a T_j \right) \asymp \rho_k^2. \quad (9)$$

On the other hand, it is clear that the distance of $C^k$ and $T \setminus C^k$ is equal to the minimum of the following two distances: $d(R_k(T_{i_0}), T_{\{n|n{(n-\beta)}\}})$ and $d(L_{\tau(k)}(T_{i_0+1}), T_{(i_0+1)|1{(\alpha+3)}})$. Since

$$d(R_k(T_{i_0}), T_{\{n|n{(n-\beta)}\}}) = d(T_{\{n|n{(n-\beta)}\}}, T_{\{n|n{(n-\beta)}\}}) = \rho_k \rho_1 \cdot d(T_{n-\beta+1}, T_{n-\beta}),$$

and

$$d(L_{\tau(k)}(T_{i_0+1}), T_{(i_0+1)|1{(\alpha+3)}}) = \rho_{i_0+1} \rho_1^{\tau(k)} \cdot d(T_{\alpha}, T_{\alpha+1}),$$

we know from (8) that $d(C^k, T \setminus C^k) \asymp \rho_k^2$ for all $k$. Combining this with (9), we have $d(C^k, T \setminus C^k) \asymp \text{diam } C^k$ for all $k$. Hence, there exists $\lambda > 0$ such that $d(C^k, T \setminus C^k) \geq \lambda \cdot \text{diam } C^k$ for all $k$. \hfill $\square$

For all $i \in \Sigma^*_n \cup \{\emptyset\}$ and $k \in Z^+$, we define

$$C^k_i = \Psi_i(C^k).$$

It is clear that $\text{diam } C^k_i = \rho_i \cdot \text{diam } C^k$ and $d(C^k_i, T \setminus C^k_i) = \rho_i \cdot d(C^k, T \setminus C^k)$. Thus, $C^k_i$ is $(T, \lambda)$-separated, where $\lambda$ is defined as in lemma 2.2. For any $k \in Z^+$, we define

$$C_k = \{C^k_i : |i| + j = k \text{ where } i \in \Sigma^*_n \cup \{\emptyset\} \text{ and } j \in Z^+\}.$$
Suppose that $\text{Nonlinearity} \geq 2$. If $j_1 = j_2$, we have $|i_1| = |i_2|$ and $i_1 \neq i_2$, so $A \cap B = \emptyset$. Thus, without loss of generality, we assume that $j_1 < j_2$.

Case 1. Assume that $i_2 = \emptyset$. Then $|i_1| + j_1 = j_2 = k$. Let $m = |i_1|$. Then $m \geq 1$. Notice that

$$B = C^k = R_1(T_n) \cup L_{\tau(k)}(T_{i+1}) = \left( \bigcup_{\ell = m-\beta+1}^{\infty} T_{i_0[n]^\ell} \right) \cup \left( \bigcup_{\ell = 1}^{\infty} T_{(i_0+1)[n]^\ell} \right).$$

From $A = C_{i_1}^1 \subset T_{i_1}$ and $\tau(k) \geq k > m$, we know that $A \cap B = \emptyset$ if $i_1 \notin [i_0][n]^{m-1}$, $(i_0 + 1)[n]^{m-1}$.

If $i_1 = i_0[n]^{m-1}$, we have $A \subset \Psi_{[n]^{m-1}}(C^1)$. Thus

$$\max A \leq \Psi_{[n]^{m-1}(i_0+1)x_1}(1) \leq \Psi_{[n]^{m-1}(i_0+1)x_1}(1) \leq \Psi_{[n]^{m-1}(i_0+1)x_1}(1) = \max A,$$

so that $A \cap B = \emptyset$. Similarly, if $i_1 = (i_0 + 1)[n]^{m-1}$, using $A \subset \Psi_{[n]^{m-1}}(C^1)$ and $\tau(k) \geq k > m$, we can easily see that $\max B < \Psi_{[n]^{m-1}(i_0+1)x_1}(1) < \min A$, so $A \cap B = \emptyset$.

Assume that $i_2 \neq \emptyset$. Let $u \in \Sigma^*_n \cup \{\emptyset\}$ be the word with the maximal length which satisfies $i_1 = u_1$ and $i_2 = u_2$ for some $i_1', i_2' \in \Sigma^*_n \cup \{\emptyset\}$. If $u = i_2$, by the result of case 1, we have $A \cap B = \Psi_u(C_{i_1'}^1 \cap C_{i_2'}^1) = \emptyset$. If $u \neq i_2$, both $i_1'$ and $i_2'$ belong to $\Sigma^*_n$ with $i_1'(1) \neq i_2'(1)$, where $i_1'(1)$ and $i_2'(1)$ are the first letters of $i_1'$ and $i_2'$, respectively. Notice that $C_{i_1'}^1 \subset T_{i_1'}(1)$ and $C_{i_2'}^1 \subset T_{i_2'}(1)$. Thus $A \cap B = \Psi_u(C_{i_1'}^1 \cap C_{i_2'}^1) = \emptyset$.

**Lemma 2.4.** For any $A \subset C_u$ and $B \subset C_v$ with $u > v$, we have either $A \cap B = \emptyset$ or $A \subset B$.

**Proof.** Suppose that $A = C_{i_1}^1$ and $B = C_{i_2}^1$. If $i_1 = i_2 = \emptyset$, it is clear that $A \subset B$. If $i_1 = \emptyset$ and $i_2 \in \Sigma^*_n$, from lemma 2.3, we have $A \cap B = C^u \cap C^{i_1} \subset C^u \cap C_{i_2}^1 = \emptyset$. Thus, we can assume that $i_1 \in \Sigma^*_n$ in the following.

Given $i \in \Sigma^*_n$ and $\ell \in \mathbb{Z}^+$, it is easy to check that we have either $C_i^1 \cap R_\ell(T_{i_0}) = \emptyset$ or $C_i^1 \subset R_\ell(T_{i_0})$, while $C_0^1 \subset R_\ell(T_{i_0})$ if only if one of the following holds: (1) $i = i_0[n]^\ell$ and $i_0 \geq n - \beta + 1$, or (2) $i = i_0[n]^\ell u_\ell$ for some $\ell \in [n - \beta + 1, \ldots, n]$ and $u \in \Sigma^*_n \cup \{\emptyset\}$. Similarly, we have either $C_0^1 \subset L_{\tau(1)}(T_{i_0}) = \emptyset$ or $C_0^1 \subset L_{\tau(1)}(T_{i_0})$. It follows that we have either $C_i^1 \cap C^\ell = \emptyset$ or $C_i^1 \subset C^\ell$.

If $i_2 = \emptyset$, since $A \subset C_{i_1}^1$, we know from the above discussion that the lemma holds in this case.

If $i_2 \neq \emptyset$, let $u \in \Sigma^*_n \cup \{\emptyset\}$ be the word with the maximal length which satisfies $i_1 = u_1$ and $i_2 = u_2$ for some $i_1', i_2' \in \Sigma^*_n \cup \{\emptyset\}$. If one of $i_1'$ and $i_2'$ is equal to $\emptyset$, then using the above discussions we have either $A \cap B = \emptyset$ or $A \subset B$. If both $i_1'$ and $i_2'$ belong to $\Sigma^*_n$, then using the same method as in case 2, we can conclude that $A \cap B = \emptyset$. □

Let $P$ be a family of finitely many compact subsets of $T$. If $\bigcup_{A \in P} A = T$ and the union is disjoint, we call $P$ a partition of $T$ and define $\|P\| = \max_{A \in P} \text{diam} A$. Let $P$ and $P'$ be two partitions of $T$. If for any $A \in P$ there exist $j \in \mathbb{Z}^+$ and $A_1', \ldots, A_{\ell'} \in P'$ such that $A = \bigcup_{j=1}^{\ell} A_j'$, then $P'$ is called a refinement of $P$.

Let $\{P_k\}$ be a sequence of partitions of $T$. $\{P_k\}$ is called hierarchical if $P_{k+1}$ is a refinement of $P_k$ for any $k$. $\{P_k\}$ is called convergent if it is hierarchical and $\lim_{k \to \infty} \|P_k\| = 0$.

Denote by $A$ the cardinality of a set $A$. Given a bounded subset $B$ of $\mathbb{R}$, we define $\text{Conv}(B)$ to be the minimal closed interval containing $B$. 

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Given $k \in \mathbb{Z}^+$ and a compact subset $F$ of $T$, we define $C_k(F) = \{A : A \in C_k \text{ and } A \subset F\}$. Furthermore, we define $S$ to be the family of disjoint compact subsets of $F$ with the minimal cardinality such that $C_k(F) \subset S, \bigcup_{B \in S} B = F$ and $\text{Conv}(B_1 \cup B_2) \cap \bigcup_{A \in C_k(F)} A \neq \emptyset$ for any two distinct sets $B_1, B_2$ in the family $S$. We denote this $S$ by $S_k(F)$.

Now we inductively construct $\{S_k\}$ as follows. Define $S_1 = S_1(T)$ and $S_{k+1} = \bigcup_{F \in S_k} S_{k+1}(F)$ for $k \geq 1$. Clearly, $S_1 = \{[0, a] \cap T, C^1, [b, 1] \cap T\}$, where $a = \Psi_{\iota_\alpha}(\alpha - \beta)(1)$ and $b = \Psi_{\iota_{\alpha+1}\iota_{\alpha+1}}(0)$.

**Lemma 2.5.** $\{S_k\}$ is a hierarchical partition sequence of $T$ such that $C_k \subset S_k$ for all positive integers $k$.

**Proof.** It suffices to show that $C_k \subset S_k$ for all $k$. We shall prove this by induction.

Clearly, $C_1 \subset S_1$. Assume that $C_k \subset S_k$ for all $k \leq m$ for some given $m \in \mathbb{Z}^+$. Now we shall show that $C_{m+1} \subset S_{m+1}$. Clearly, we only need to prove that the following property holds.

For any $A \in C_{m+1}$, there exists $B \in S_m$ such that $A \subset B$. (10)

Given $A \in C_{m+1}$, since $A \subset T$ and $S_m$ is a partition of $T$, there exists $B \in S_m$ such that $A \cap B \neq \emptyset$. Denote this $B$ by $B_m$. Notice that, by definition, $S_{m+1}$ is a refinement of $S_m$ for all $k$. Hence there exists $\{B_k\}_{k=1}^{m+1}$ such that $B_{k+1} \subset B_k$ and $B_k \in S_k$ for all $k = 1, 2, \ldots, m$. It follows from $A \cap B_m \neq \emptyset$ that $A \cap B_k \neq \emptyset$ for all $k = 1, 2, \ldots, m$.

If $A \cap C^1 \neq \emptyset$, using lemma 2.4, we have $A \subset C^1$. Since all sets in the family $S_1$ are disjoint, we have $B_1 = C^1$, so $A \subset B_1$. If $A \cap C^1 = \emptyset$, using $\text{Conv}(A) \cap C^1 = \emptyset$ and $S_1 = S_1(T)$, there exists $E \in S_1 \setminus \{C^1\}$ such that $A \subset E$. Since all sets in $S_1$ are disjoint, we have $B_1 = E$, so $A \subset B_1$. Hence, we always have $A \subset B_1$. Thus (10) holds if $m = 1$.

Now we consider the case where $m \geq 2$. Suppose that there exists $B' \in C_2$ such that $A \cap B' \neq \emptyset$. From lemma 2.4, we have $A \subset B'$. By the inductive assumption, we have $C_2 \subset S_2$. Thus, noticing that all sets in $S_2$ are disjoint, we have $B_2 = B'$, so $A \subset B_2$. Suppose that $A \cap \bigcup_{F \in C_2} F = \emptyset$. Then we have $\text{Conv}(A) \cap \bigcup_{F \in C_2} F = \emptyset$. Thus, from $A \subset B_1$, we can easily see that there exists $B' \in S_2(B_1) \setminus C_2(B_1) \subset S_2$ such that $A \subset B'$. Since all sets in $S_2$ are disjoint, we have $B_2 = B'$, so $A \subset B_2$. Repeating this procedure, we can see that $A \subset B_k$ for $k = 1, 2, \ldots, m$.

**Corollary 2.6.** There exists a convergent partition sequence $\{T_k\}$ of $T$ such that $C_k \subset T_k$ for all positive integers $k$.

**Proof.** Let $E$ be a compact subset of $T$. An open interval $(a, b)$ is said to be a gap of $E$ if $a, b \in E$ and $(a, b) \cap E = \emptyset$. We call $b - a$ the length of the gap $(a, b)$. Let $\delta$ be a positive real number. We define $\mathcal{G}(E, \delta) = \{(a, b) : (a, b) \text{ is a gap of } E \text{ such that } b - a \geq \delta\}$. Define $\mathcal{J}(E, \delta)$ to be the family of all connected components of $\text{Conv}(E) \setminus \bigcup_{F \in \mathcal{G}(E, \delta)} F$. Define $\mathcal{P}(E, \delta) = \{A \cap E : A \in \mathcal{J}(E, \delta)\}$. Then $\mathcal{P}(E, \delta)$ is a partition of $E$.

Now, we define $\delta_k = \max\{\text{diam} A : A \in C_k\}$ for $k \in \mathbb{Z}^+$. Then the sequence $\{\delta_k\}_{k=1}^{\infty}$ is decreasing and $\lim_{k \to \infty} \delta_k = 0$. Define $T_k = \bigcup_{F \in \mathcal{P}(E, \delta_k)} F$ for all $k$. Clearly, $T_k$ is also a hierarchical partition sequence of $T$ with $C_k \subset T_k$ for all $k$. Also, for all $F \in T_k, \text{Conv}(F)$ does not contain any gap of $T$ whose length is greater than $\delta_k$. Thus, from $C^1_{k+1} \subset T_k, C^1_k \subset T_{k+1}$, we can easily see that

$$\|T_k\| < 2 \cdot \max\{\text{diam} T_i : |i| = k - 1\} + \delta_k$$

for all $k \geq 2$ so that $\lim_{k \to \infty} \|T_k\| = 0$. It follows that the corollary holds.\qed
2.3. Martingales and the proof of theorem 1.2

From corollary 2.6, we can prove theorem 1.2 by using a similar method to that in [21]. Assume that $f : T \to D$ is bi-Lipschitz, which satisfies inequalities in (1) for all $x, y \in T$. Let $s$ be the common Hausdorff dimension of $T$ and $D$.

By lemmas 2.1 and 2.2, there exists an integer $n_0$ such that for any $i \in \Sigma_n^* \cup \{\emptyset\}$ and $k \in \mathbb{Z}^+$ there exist $j, j_1, \ldots, j_p \in \Sigma_n^*$ such that $D_{j_{j_1}}, D_{j_{j_2}}, \ldots, D_{j_{j_p}}$ are disjoint and

$$f(C_i^k) = \bigcup_{r=1}^p D_{j_{j_r}} \subset D_j,$$

where each $|j_r| = n_0$. Furthermore, by remark 2.1, we can require $D_k$ to be the smallest cylinder containing $f(C_i^k)$. We denote this $j$ by $j(i, k)$ and define $\gamma_{i,k} = \sum_{r=1}^p \rho_{j_r}^i$. Then

$$\mathcal{H}^s(f(C_i^k)) = \mathcal{H}^s(D_{j(i,k)}) \cdot \gamma_{i,k}, \quad (12)$$

and

$$D_{j(i,k)} \subset D_{j(i',k)} \quad \text{if} \quad C_i^k \subset C_{i'}^k. \quad (13)$$

Define

$$\mathcal{M} = \left\{ \sum_{j \in A} \rho_j^i \mid A \subset \{1, \ldots, n\}^{n_0} \right\}. \quad (14)$$

Then $\gamma_{i,k} \in \mathcal{M}$ for all $i$ and $k$.

Let $\{T_k\}$ be a convergent partition sequence of $T$ as defined in corollary 2.6. We define

$$g_k(A) = \frac{\mathcal{H}^s(f(A))}{\mathcal{H}^s(A)}, \quad A \in T_k.$$  

Notice that for any $A \in T_k$ we can decompose $A$ by $A = \bigcup_{i=1}^j A_i$, where $A_1, \ldots, A_j \in T_{k+1}$. Then,

$$\sum_{i=1}^j \frac{\mathcal{H}^s(A_i)}{\mathcal{H}^s(A)} g_{k+1}(A_i) = \sum_{i=1}^j \frac{\mathcal{H}^s(A_i)}{\mathcal{H}^s(A)} \frac{\mathcal{H}^s(f(A_i))}{\mathcal{H}^s(A_i)} = \frac{\mathcal{H}^s(f(A))}{\mathcal{H}^s(A)} = g_k(A). \quad (15)$$

Let $\mathcal{F}_k$ be the sigma-field generated by $T_k$, and define

$$g_k(x) = g_k(A)$$

for $x \in A$ where $A \in T_k$. Then $g_k$ is an $\mathcal{F}_k$-measurable function. By (15), we know that $(g_k, \mathcal{F}_k)$ is a martingale. Furthermore, by (1), we have $c^t \leq g_k(x) \leq (c')^t$ for any $k$ and any $x \in T$. Therefore, the martingale convergence theorem implies that

$$g_k(x) \to g(x) \quad \text{as} \quad k \to \infty, \quad \text{for} \quad \mathcal{H}^s\text{-almost all} \quad x \quad \text{in} \quad T, \quad (16)$$

where $g$ is $\mathcal{F}$-measurable, with $\mathcal{F}$ the sigma-field generated by $\bigcup_{k=1}^\infty \mathcal{F}_k$.

Define $\mu_i = \rho_i^i$ for all $i = 1, \ldots, n$. For any $i = i_1 \cdots i_j \in \Sigma_n^*$, we define $\mu_i = \prod_{k=i_1}^{i_j} \mu_{i_k}$. Denote $\mu_L = \sum_{j=1}^q \mu_j$ and $\mu_R = \sum_{j=q+1}^n \mu_j$. We have the following lemma.

**Lemma 2.7.** Given $\ell \in \mathbb{Z}^+$, the set

$$\left\{ \frac{\mathcal{H}^s(D_{j(i',t)})}{\mathcal{H}^s(D_{j(i,t)})} \right\} : 1 \leq t < t' < \ell, \quad i \in \Sigma_n^*$$

and the set

$$\left\{ \frac{\mathcal{H}^s(C_i)}{\mathcal{H}^s(C_i')}, 1 \leq t' < t < \ell, \quad i \in \Sigma_n^* \right\}$$

are finite.

**Proof.** For any $i \in \Sigma_n^*$ and $1 \leq t' < t < \ell$, we have

$$\frac{g_{i+t'}(C_i)}{g_{i+t'}(C_i')} = \frac{\mathcal{H}^s(D_{j(i,t)})}{\mathcal{H}^s(D_{j(i,t')})} \cdot \frac{\gamma_{s,t}}{\gamma_{s,t'}} \cdot \frac{\mathcal{H}^s(C_i')}{\mathcal{H}^s(C_i)}. \quad (17)$$
By (13), $D_{ji(t)} \subset D_{ji(t)}'$, so there exist non-negative integers $k_1, k_2, \ldots, k_n$ such that
\[ \frac{\mathcal{H}^i(D_{ji(t)})}{\mathcal{H}^i(D_{ji(t)'})} = \mu_1^{k_1} \mu_2^{k_2} \cdots \mu_n^{k_n}. \]
(18)

Notice that $f$ is bi-Lipschitz and $\text{card } M < +\infty$. By (12), for all $i \in \Sigma_n^*$ and $1 \leq t' < t \leq \ell$, we have
\[ \mathcal{H}^i(D_{ji(t)}) \times \mathcal{H}^i(f(C'_i)) \times \mathcal{H}^i(C'_i) \times \mathcal{H}^i(f(C'_i)) \times \mathcal{H}^i(D_{ji(t)'}) \times \mathcal{H}^i(D_{ji(t)'}). \]

It follows that the set \( \frac{\mathcal{H}^i(C'_i)}{\mathcal{H}^i(C'_i')} : 1 \leq t' < t \leq \ell, \ i \in \Sigma_n^* \) is finite. By definition, it is easy to see that $\frac{\mathcal{H}^i(C'_i)}{\mathcal{H}^i(C'_i')}$ is independent of $i$, so the set \( \frac{\mathcal{H}^i(C'_i)}{\mathcal{H}^i(C'_i') : 1 \leq t' < t \leq \ell, i \in \Sigma_n^*} \) is also finite. Thus the lemma holds.

Lemma 2.8. There exist a constant $M > 0$ and infinitely many $(p_1, p_2, q_1, q_2, k_1, \ldots, k_n) \in (\mathbb{Z}^+)^4 \times \mathbb{N}$ with $q_1 \neq q_2$ and \( \max \{|p_2 - p_1|, |q_2 - q_1|, k_1, \ldots, k_n\} \leq M \) such that
\[ \mu_{i_0}^q \mu_R + \mu_{i_0+1}^p \mu_L \mu_{i_0}^q \mu_R + \mu_{i_0+1}^p \mu_L = \mu_1^{k_1} \mu_2^{k_2} \cdots \mu_n^{k_n}. \]
(19)

Proof. Let $\Sigma_n^\infty = \{i_1 i_2 \cdots i_m | i_m \in \{1, \ldots, n\} \text{ for all } m \}$. For each $x \in T$, there exists $i_1 \cdots i_m \in \Sigma_n^\infty$ such that $\{x\} = \bigcap_{m \geq 1} \psi_{i_m \cdots i_1}(T)$. We call $i_1 \cdots i_m$ the address of $x$. We remark that the address of $x$ may be not unique. However, if we define $\tilde{T}$ to be the set of all points in $T$ with unique address, then $\mathcal{H}^i(\tilde{T}) = \mathcal{H}^i(T)$ by the definition of $T$. For each $x \in \tilde{T}$ with address $i_1 \cdots i_m$, we define $\sigma(x)$ to be the point with address $i_2 \cdots i_m$. It is easy to check that $\sigma(x) \in \tilde{T}$ for all $x \in \tilde{T}$.

Let $\tilde{F}$ be the sigma-field generated by $\{T_i \cap \tilde{T} : i \in \Sigma_n^\infty\}$. Define $\nu(A) = \mathcal{H}^i(A \cap \tilde{T})/\mathcal{H}^i(\tilde{T})$, $\forall A \in \tilde{F}$. Then $\sigma : (\tilde{T}, \tilde{F}, \nu) \rightarrow (T, F, \nu)$ is measure preserving. Fix $p \geq \text{card } M + 1$ in the proof of the lemma, where $M$ is defined by (14). Given $q \in \mathbb{Z}^+$, by the Poincaré recurrence theorem, $\nu$-almost all $x \in C^{pq} \cap \tilde{T}$, i.e. for $\mathcal{H}^i$-almost all $x \in C^{pq} \cap \tilde{T}$, there is an integer sequence $0 < n_1(x, q) < n_2(x, q) < \cdots$ such that $\sigma^{n_i(x,q)}(x) \in C^{pq} \cap \tilde{T}$ for all $i$. Thus, from (16), we can pick a point $x_q \in C^{pq} \cap \tilde{T}$ with $\sigma^{n_i(x,q)}(x_q) \in C^{pq} \cap \tilde{T}$ for each $i$ and $g_i(x_q) \rightarrow g(x_q)$ as $k \rightarrow \infty$.

Let $i_1 \cdots i_m$ be the address of $x_q$. Define $i_k = i_1 i_2 \cdots i_k$. Then
\[ x_q = \psi_{i_k}(\sigma^{n_i(x,q)}(x_q)) \in \psi_{i_k}(C^{pq}) \subset C_{i_k}^{pq}. \]
For any $t, t' \in \mathbb{Z}^+ \cap [1, p]$ with $t' < t$, from $x_q \in C_{i_k}^{pq} \subset C_{i_k}^{pq-1} \subset C_{i_k}^{pq-1} + r$, we have
\[ \frac{g_{i_k}(p-1+r, x_q)}{g_{i_k}(p-1-r, x_q)} = \frac{g_{i_k}(p-1+r, C_{i_k}^{pq-1})}{g_{i_k}(p-1-r, C_{i_k}^{pq-1})}. \]

By lemma 2.7, for fixed $q \geq 1$, the set \( \{g_{i_k}(p-1+r, x_q) : k \geq 1, 1 \leq t' < t \leq p \} \) is finite. Combining this with \( \lim_{k \rightarrow \infty} g_j(x_q) = g(x_q) \), we can take $k$ large enough that $g_{i_k}(p-1+r, x_q) = 1$. Since $p \geq \text{card } M + 1$, we can take $i_k > i_j$ such that $Y_{i_k, p(q-1)+r_q} = Y_{i_j, p(q-1)+r_q}$. Thus, by (17) and (18), there exist non-negative integers $(k_i(q))_{1 \leq i \leq n}$ such that
\[ \frac{\mu_{i_0}^{p(q-1)+r_q} \mu_R + \mu_{i_0+1}^p \mu_L}{\mu_{i_0}^{p(q-1)+r_q} \mu_R + \mu_{i_0+1}^p \mu_L} = \mu_1^{k_1(q)} \mu_2^{k_2(q)} \cdots \mu_n^{k_n(q)}. \]
From lemma 2.7 and (18), we know that \( \{k_t(q)\}_1 \) are bounded. Also, we have 
\[
| (p(q - 1) + t_q) - (p(q - 1) + t_q') | \leq p - 1.
\]
From
\[
\rho_1 \tau(p(q - 1) + t_q) \leq \rho_n p(q - 1) + t_q \leq \rho_n \tau(p(q - 1) + t_q') \leq \rho_1
\]
for all \( q \), we know that \( \{\tau(p(q - 1) + t_q) - \tau(p(q - 1) + t_q')\}_1 \) are bounded.

Define \( p_1 = \tau(p(q - 1) + t_q'), q_1 = p(q - 1) + t_q', p_2 = \tau(p(q - 1) + t_q), q_2 = p(q - 1) + t_q \).
From \( 1 \leq t_q' < t_q \leq p \), we have \( q_1, q_2 \in \mathbb{Z}^+ \cap [p(q - 1) + 1, pq] \) with \( q_1 \neq q_2 \). Since \( q \) can be arbitrary chosen in \( \mathbb{Z}^+ \), we finally obtain infinitely many solutions of (19). The lemma is proved. \( \square \)

Now, we can prove theorem 1.2.

**Proof of theorem 1.2.** By lemma 2.8, there exist infinitely many solutions \((p_1, p_2, q_1, q_2, k_1, \ldots, k_n)\) of (19) with \( q_1 \neq q_2 \) such that \((p_2 - p_1), (q_2 - q_1), \{k_t(q)\}_1\) are constants. It follows that there are infinitely many \((p_1, q_1) \in (\mathbb{Z}^+)^2\) and constants \(\mu_0, \mu_0^2, \mu_0^3, \ldots, \mu_0^n \) such that the following equation holds:
\[
\frac{(\mu_0^0, \mu_0^1)^2 \mu_0^2 \mu_0^3 \ldots \mu_0^n}{(\mu_0^0, \mu_0^1)^2 \mu_0^2 \mu_0^3 \ldots \mu_0^n} = \mu_0^1 \mu_0^2 \ldots \mu_0^n.
\]
Assume that \(\frac{\mu_0^0, \mu_0^1}{\mu_0^2 \mu_0^3} \neq \frac{\mu_0^0, \mu_0^1}{\mu_0^2 \mu_0^3} \). Then there is a constant \( \delta \) such that \( \frac{\mu_0^0, \mu_0^1}{\mu_0^2 \mu_0^3} = \delta \) for infinitely many \((p_1, q_1) \in (\mathbb{Z}^+)^2\). Take \((p_1, q_1), (p_1', q_1') \in (\mathbb{Z}^+)^2\) such that \((p_1, q_1) \neq (p_1', q_1')\) and \(\frac{\mu_0^0, \mu_0^1}{\mu_0^2 \mu_0^3} = \delta = \frac{\mu_0^0, \mu_0^1}{\mu_0^2 \mu_0^3}\). It follows that \(\mu_0^0, \mu_0^1, \mu_0^2, \mu_0^3, \ldots, \mu_0^n\) and \(\mu_0^0, \mu_0^1, \mu_0^2, \mu_0^3, \ldots, \mu_0^n\), which implies that \(\log \rho_1 / \log \rho_n \in \mathbb{Q}\).

Assume that \(\frac{\mu_0^0, \mu_0^1, \mu_0^2, \mu_0^3}{\mu_0^4, \mu_0^5, \mu_0^6} = \frac{\mu_0^0, \mu_0^1, \mu_0^2, \mu_0^3}{\mu_0^4, \mu_0^5, \mu_0^6}\). Then \(\mu_0^0, \mu_0^1, \mu_0^2, \mu_0^3, \ldots, \mu_0^n\), so \(\log \rho_1 / \log \rho_n \in \mathbb{Q}\). \( \square \)

3. **Sufficient condition for \( D \sim T \)**

3.1. **Graph-directed sets corresponding to \( D \) and \( T \)**

Now we recall the notion of **graph-directed set**. Let \( G = (V, \Gamma) \) be a directed graph and \( d \) a positive integer. Suppose that for each edge \( e \in \Gamma \) there is a corresponding similarity \( S_e : \mathbb{R}^d \to \mathbb{R}^d \) with ratio \( r_e \). Assume that for each vertex \( i \in V \) there exists an edge starting from \( i \), and assume that \( r_{e_1} \cdots r_{e_k} < 1 \) for any cycle \( e_1 \cdots e_k \). Then there exists a unique family \( \{K_i\}_{i \in V} \) of compact subsets of \( \mathbb{R}^d \) such that for any \( i \in V \)
\[
K_i = \bigcup_{j \in V} \bigcup_{e \in \mathcal{E}_{ij}} S_e(K_j),
\]
where \( \mathcal{E}_{ij} \) is the set of edges starting from \( i \) and ending at \( j \). In particular, if the union in (20) is disjoint for any \( i \), we call \( \{K_i\}_{i \in V} \) dust-like graph-directed sets on \( (V, \Gamma) \). For details on graph-directed sets, please see [12, 16].

Similarly to theorem 2.1 in [14], we have the following lemma.

**Lemma 3.1.** Suppose that \( \{K_i\}_{i \in V} \) and \( \{K'_i\}_{i \in V} \) are dust-like graph-directed sets on \( (V, \Gamma) \) satisfying (20) and \( K'_i = \bigcup_{j \in V} \bigcup_{e \in \mathcal{E}_{ij}} S'_e(K'_j) \). If similarities \( S_e \) and \( S'_e \) have the same ratio for each \( e \in \Gamma \), then \( K_i \sim K'_i \) for each \( i \in V \).
Definition 3.1. Assume that \( K = \{K_i\}_{i=1}^m \) and \( K' = \{K'_i\}_{i=1}^m \) are two families of compact subsets of \( \mathbb{R}^d \), where \( m \geq 2 \) is a given positive integer. We say that two compact subsets \( A \) and \( B \) of \( \mathbb{R}^d \) have the same dust-like decomposition w.r.t. \( K \) and \( K' \), denoted by dec \( K(A) = \text{dec } K'(B) \), if there exist a positive integer \( t \geq 2 \) and positive integers \( j_1, j_2, \ldots, j_t \in \{1, \ldots, m\} \) such that

\[
A = \bigcup_{i=1}^{t} S_i(K_j) \quad \text{and} \quad B = \bigcup_{i=1}^{t} S'_i(K'_j),
\]

where the above two unions are disjoint, while for each \( i \), \( S_i \) and \( S'_i \) are similarites from \( \mathbb{R}^d \) to \( \mathbb{R}^d \) with the same ratio.

Clearly, the following lemma is a weak version of lemma 3.1.

Lemma 3.2. Suppose that \( K = \{K_i\}_{i=1}^m \) and \( K' = \{K'_i\}_{i=1}^m \) are two families of compact subsets of \( \mathbb{R}^d \). If for each \( 1 \leq i \leq m \) \( \text{dec } K(K_i) = \text{dec } K'(K'_i) \), then \( K_i \sim K'_i \) for each \( i = 1, \ldots, m \).

Given \( i \in \Sigma^*_n \) and a non-negative integer \( k \), we recall that

\[
L_k(T_i) = \bigcup_{j=1}^{\alpha} T_{[i][1^\alpha]j}, \quad R_k(T_i) = \bigcup_{j=-\beta+1}^{n} T_{[i][\alpha]j}
\]
as defined in (2). Now we define

\[
L_k(D_i) = \bigcup_{j=1}^{\alpha} D_{[i][1^\alpha]j}, \quad R_k(D_i) = \bigcup_{j=-\beta+1}^{n} D_{[i][\alpha]j}.
\]

In the rest of this section, we shall always assume that \( \log \rho_1/\log \rho_n \in \mathbb{Q} \) and every touching letter is substitutable. Thus, for any \( i \in \Sigma_T \), there exist \( j_i \in \Sigma^*_n \) and \( k_i, k'_i \in \mathbb{N} \), such that one of the following holds:

1. \( \text{diam } L_k(T_j) = \text{diam } L_{k'}(T_{j'}) \) and the last letter of \( j_i \) does not belong to \( \{1\} \cup (\Sigma_T + 1) \),
2. \( \text{diam } R_k(T_j) = \text{diam } R_{k'}(T_{j'}) \) and the last letter of \( j_i \) does not belong to \( \{n\} \cup \Sigma_T \).

From \( \log \rho_1/\log \rho_n \in \mathbb{Q} \), there exist \( p, q \in \mathbb{Z}^* \) such that \( \rho_1^p = \rho_n^q \). Notice that we can choose \( p, q \) large enough so that \( p, q > \max(k_i' + |j_i| : i \in \Sigma_T) \).

If \( i \) is a left substitutable letter, from

\[
\max(L_k(D_{j_i})) \leq \max(L_0(D_{[i][1^\alpha]})) < \min(R_0(D_{[i][\alpha]})) < \min(L_0(D_{[i][\alpha]})) = \min(R_q(D_i))
\]
we have \( L_k(D_{j_i}) \cap R_q(D_i) = \emptyset \). Using the same method, we can obtain

\[
L_k'(T_{[i][1^\alpha]j}) \cap R_{q'}(T_j) = L_k'(D_{[i][\alpha]j}) \cap R_{q'}(D_i) = \emptyset.
\]

Similarly, for any right substitutable touching letter \( i \), we have \( L_p(D_{j_{i+1}}) \cap R_{q'}(D_{(i+1)[1^\alpha]j_{i+1}}) = \emptyset \) and

\[
L_{q'}(T_{(i+1)[1^\alpha]j_{i+1}}) \cap R_{q'}(T_{[i+1][1^\alpha]j_{i+1}}) = L_{q'}(D_{(i+1)[1^\alpha]j_{i+1}}) \cap R_{q'}(D_{j_{i+1}}) = \emptyset.
\]

We shall fix \( p, q \) in this section. We also remark that the full restriction on \( p, q \) will be used later in order to apply corollary 3.9.

Define \( I_i = \Psi_i([0, 1]) \) for any \( i \in \Sigma^*_n \). Given a positive integer \( m \), let \( J_m = \bigcup_{i=0}^{m} I_i \) and \( J_m, \ldots, J_{m,c_1} \) be the connected components of \( J_m \), spaced from left to right. We define \( \Lambda_i = \{j | I_j \subset J_i\} \) for \( i = 1, \ldots, c_1 \). Define

\[
T_i^{(1)} = \bigcup_{j \in \Lambda_i} T_j, \quad D_i^{(1)} = \bigcup_{j \in \Lambda_i} D_j, \quad \forall i = 1, \ldots, c_1.
\]
We remark that \( L_\ell(T_i) = \Psi_i(T^{(1)}_i) \) and \( R_\ell(T_i) = \Psi_i(T^{(0)}_i) \) for any \( i \in \Sigma_n^* \) and \( k \in \mathbb{N} \).

For any touching letter \( i \), we define
\[
T_i^{(2)} = R_0(T_i) \cup L_0(T_{i+1}), \quad D_i^{(2)} = R_0(D_i) \cup L_0(D_{i+1}),
\]
\[
T_i^{(3)} = R_\ell(T_i) \cup L_\ell(T_{i+1}), \quad D_i^{(3)} = R_\ell(D_i) \cup L_\ell(D_{i+1}).
\]

Furthermore, for any touching letter \( i \), if \( i \) is left substitutable, we define
\[
T_i^{(4)} = R_\ell(T_i) \cup L_\ell(T_{i+1}), \quad D_i^{(4)} = L_\ell(D_i) \cup R_\ell(D_{i+1}).
\]

Otherwise, we define
\[
T_i^{(4)} = R_\ell(T_i) \cup L_\ell(T_{i+1}), \quad D_i^{(4)} = L_\ell(D_i) \cup R_\ell(D_{i+1}).
\]

Define \( T = \{T\} \cup \{T^{(1)}_i : i = 1, \cdots, c_1\} \cup \{T^{(3)}_i : i \in \Sigma_T, j = 2, 3, 4\} \) and \( D = \{D\} \cup \{D^{(1)}_i : i = 1, \cdots, c_1\} \cup \{D^{(3)}_i : i \in \Sigma_T, j = 2, 3, 4\} \). We shall show that each corresponding pair in \( T \) and \( D \) has the same dust-like decomposition w.r.t. \( T \) and \( D \).

### 3.2. The family \( T^* \)

The following lemma is easy to check.

**Lemma 3.3.** Given \( i \in \Sigma_n^* \) and \( j \in \{1, \ldots, c_1\} \), \( \Psi_i(T^{(1)}_j) \) is not \( T \)-separate if and only if one of the following conditions holds:

1. \( j = 1 \) and there exist \( i' \in \Sigma_n^* \cup \{\emptyset\} \), \( k \in \Sigma_T + 1 \), \( \ell \in \mathbb{N} \) such that \( i = i'k[1]^\ell \);
2. \( j = c_1 \) and there exist \( i' \in \Sigma_n^* \cup \{\emptyset\} \), \( k \in \Sigma_T \) and \( \ell \in \mathbb{N} \) such that \( i = i'k[\ell]^\ell \).

Define
\[
T^{(1)} = \left\{ \Psi_i(T^{(1)}_j) : i \in \Sigma_n^* \cup \{\emptyset\} \right\} \quad \text{and} \quad j \in \{1, \ldots, c_1\} \text{ such that } \Psi_i(T^{(1)}_j) \text{ is } T \text{-separate},
\]
\[
T^{(k)} = \left\{ \Psi_i(T^{(k)}_j) : i \in \Sigma_n^* \cup \{\emptyset\} \right\} \quad \text{and} \quad j \in \Sigma_T, \quad k = 2, 3.
\]

It is clear that all sets in \( T^{(2)} \) and \( T^{(3)} \) are \( T \)-separate. Define
\[
T^* = \{ A \} \mid A \text{ is a disjoint union of finitely many (} \geq 2 \text{) sets in the class } T^{(1)} \cup T^{(2)} \cup T^{(3)} \}
\]

**Remark 3.1.** Assume that \( A \in T^* \) with \( A \subset (0, 1) \). Then \( \Psi_i(A) \in T^* \) for all \( i \in \Sigma_n^* \).

Let \( \Sigma_n^\infty \) be the set defined in the proof of lemma 2.8. Given \( i = i_1 \cdots i_m \cdots \in \Sigma_n^\infty \), there exists a unique point \( x \in T \) such that
\[
\{ x \} = \bigcap_{m=1}^\infty \Phi_{i_1 \cdots i_m}(0, 1).
\]

We denote this unique \( x \) by \( \pi_T(i) \). Then \( \pi_T : \Sigma_n^\infty \rightarrow T \) is a surjection. Similarly, we can define \( \pi_D : \Sigma_n^\infty \rightarrow D \) by
\[
\{ \pi_D(i) \} = \bigcap_{m=1}^\infty \Phi_{i_1 \cdots i_m}(0, 1), \quad \forall i = i_1 \cdots i_m \cdots \in \Sigma_n^\infty.
\]

Since \( D \) is dust-like, \( \pi_D \) is a bijection. By the definition of \( \pi_T \) and \( \pi_D \), it is easy to check that
\[
\pi_D \circ \pi_T^{-1}(\Psi_i(T^{(k)})) = \Phi_{i}(D^{(k)}), \quad \forall \Psi_i(T^{(k)}), k = 1, 2, 3. \quad (22)
\]

Using this fact, we have the following lemma.

**Lemma 3.4.** Let \( A \in T^* \). Then \( \text{dec } T(A) = \text{dec } \pi_D(\pi_T^{-1}(A)) \).
Proof. Assume that \( A = \bigcup_{i=1}^{m} A_i \), where \( m \geqslant 2 \), \( A_j \in T^{(1)} \cup T^{(2)} \cup T^{(3)} \) for all \( i \) and the union is disjoint. Then
\[
\pi_D \circ \pi_T^{-1}(A) = \bigcup_{i=1}^{m} \pi_D \circ \pi_T^{-1}(A_i),
\]
and the union is disjoint. Thus the lemma follows from (22).

3.3. Graph-directed decomposition and the proof of theorem 1.3

Lemma 3.5. \( \text{dec}_{\gamma}(T) = \text{dec}_{\gamma}(D) \) and \( \text{dec}_{\gamma}(T^{(i)}_1) = \text{dec}_{\gamma}(D^{(i)}_1) \) for \( i = 1, \ldots, c_1 \).

Proof. From \( T = \bigcup_{i=1}^{c_1} T^{(i)}_1 \) and \( D = \bigcup_{i=1}^{c_1} D^{(i)}_1 \), we have \( \text{dec}_{\gamma}(T) = \text{dec}_{\gamma}(D) \). Given \( i = 1, \ldots, c_1 \), Notice that
\[
T^{(i)}_1 = \bigcup_{j \in \Lambda_i} T_j = \bigcup_{j \in \Lambda_i} \Psi_j(T) = \bigcup_{j \in \Lambda_i} \bigcup_{k=1}^{c_1} \Psi_j(T^{(i)}_k). \tag{23}
\]
Let \( b(i) \) and \( e(i) \) be the minimal and maximal elements in \( \Lambda_i \), respectively. If \( b(i) = e(i) \), then
\[
T^{(i)}_1 = \bigcup_{k=1}^{c_1} \Psi_{b(i)}(T^{(i)}_k) \in T^*.
\]
since each \( \Psi_{b(i)}(T^{(i)}_k) \) is \( T \)-separate in this case. If \( b(i) < e(i) \), then for any \( b(i) \leqslant j < e(i) \),
\[
\Psi_j(T^{(i)}_k) \cup \Psi_{j+1}(T^{(i)}_k) = T^{(2)}_j,
\]
and the other \( \Psi_{j}(T^{(i)}_k) \) in (23) are \( T \)-separate, so they belong to \( T^{(1)} \). Thus \( T^{(i)}_1 \) also belongs to \( T^* \) in this case. By lemma 3.4, \( T^{(i)}_1 \) and \( D^{(i)}_1 = \pi_D \circ \pi_T^{-1}(T^{(i)}_1) \) have the same dust-like decomposition w.r.t. \( T \) and \( D \).

Remark 3.2. It follows from the proof of the above lemma that \( T^{(i)}_1 \subset T^* \).

Lemma 3.6. Given \( i \in \Sigma_n^* \) and two non-negative integers \( u, \nu \) with \( u < \nu \), we have
\[
\text{dec}_{\gamma}(L_u(T_i) \setminus L_{\nu}(T_i)) = \text{dec}_{\gamma}(L_u(D_i) \setminus L_{\nu}(D_i)), \tag{24}
\]
and
\[
\text{dec}_{\gamma}(R_u(T_i) \setminus R_{\nu}(T_i)) = \text{dec}_{\gamma}(R_u(D_i) \setminus R_{\nu}(D_i)). \tag{25}
\]
Proof. Without loss of generality, we only show that (24) holds. It is clear that
\[
\pi_D \circ \pi_T^{-1}(L_u(T_i) \setminus L_{\nu}(T_i)) = L_u(D_i) \setminus L_{\nu}(D_i).
\]
Thus, from lemma 3.4 and noticing that
\[
L_u(T_i) \setminus L_{\nu}(T_i) = \bigcup_{i=1}^{n-1} \left(L_k(T_i) \setminus L_{k+1}(T_i) \right),
\]
it suffices to show that \( L_k(T_i) \setminus L_{k+1}(T_i) \in T^* \) for all \( k \in \mathbb{N} \).

Given \( k \in \mathbb{N} \), assume that \( 1 \notin \Sigma_T \), i.e. \( \alpha = 1 \). Then
\[
L_k(T_i) \setminus L_{k+1}(T_i) = \bigcup_{j=2}^{n} L^{(i)}_{j-1} \setminus L^{(i)}_{j} = \bigcup_{j=2}^{c_1} \Psi_{j}^{(i)}(T^{(i)}_k).
\]
Notice that \( \Psi_j^{(i)}(T^{(i)}_k) \) is \( T \)-separate in this case. By remark 3.2, \( L_k(T_i) \setminus L_{k+1}(T_i) \in T^* \).
Lemma 3.7. Assume that $1 \in \Sigma_T$, i.e. $\alpha \geq 2$. Then $L_k(T_i) \setminus L_{k+1}(T_i) = \Psi_{ij}(A)$, where

$$A = \left( \bigcup_{j=\alpha+1}^{n} T_{ij} \right) \cup \left( \bigcup_{t=2}^{\alpha} T_{i} \right) = \left( \bigcup_{j=2}^{\alpha} \Psi_1(T_{ij}^{(1)}) \right) \cup \left( \bigcup_{t=2}^{\alpha} \bigcup_{j=1}^{\alpha} \Psi_{t}(T_{ij}^{(1)}) \right).$$

(26)

For each $1 \leq \ell \leq \alpha - 1$, $\Psi_{\ell}(T_{i}^{(1)}) \cup \Psi_{\ell+1}(T_{i}^{(1)}) = T_{i}^{(2)}$. Furthermore, the other $\Psi_1(T_{ij}^{(1)})$ and $\Psi_{t}(T_{ij}^{(1)})$ in the right-hand side of (26) are $T$-separate. Thus $A \in T^*$. By remark 3.1, $L_k(T_i) \setminus L_{k+1}(T_i) \in T^*$.

\[\square\]

From this fact, we have the following lemma.

**Lemma 3.7.** $\text{dec}_T(T_i^{(2)}) = \text{dec}_D(D_i^{(2)})$ for any $i \in \Sigma_T$.

**Proof.** For each touching letter $i$, we have

$$T_i^{(2)} = (R_0(T_i) \setminus R_q(T_i)) \cup (L_0(T_{i+1}) \setminus L_p(T_{i+1})) \cup T_i^{(3)},$$

and

$$D_i^{(2)} = (R_0(D_i) \setminus R_q(D_i)) \cup (L_0(D_{i+1}) \setminus L_p(D_{i+1})) \cup D_i^{(3)},$$

where the unions are disjoint. Thus the lemma follows from lemma 3.6. \[\square\]

Given $i = i_1 i_2 \cdots i_m, j = j_1 j_2 \cdots j_m \in \Sigma_n^*$ with the same length, we denote $i < j$ if there exists $1 \leq k \leq m$ such that $i_k < j_k$ and $i_t = j_t$ for $1 \leq t < k$. We denote $i \leq j$ if $i < j$ or $i = j$.

Given $i, j \in \Sigma_n^*$ with $i < j$, we say that $(i, j)$ is a joint pair if $k \leq i$ for every $k \in \Sigma_n^*$ with $k < j$.

**Lemma 3.8.** Let $i, j \in \Sigma_n^*$ with $\Psi_i(0) < \Psi_j(1)$. Suppose that $[\Psi_i(0), \Psi_j(1)] \cap T$ is $T$-separate and $\max|i|, |j| \leq \min|p, q|$. Then $[\Psi_i(0), \Psi_j(1)] \cap T \in T^*$.

**Proof.** Given $k \in \Sigma_n^*$, we define the middle part of $T_k$ to be

$$M(T_k) = \bigcup_{j=\alpha+1}^{n-\beta} T_{kj}.$$

It is clear that $M(T_k) = T_k \setminus \{L_0(T_k) \cup R_0(T_k)\}$. We remark that $M(T_k) = \emptyset$ for all $k$ if $\alpha = n - \beta$. If $\alpha < n - \beta$, it is clear that $M(T_k) \in T^*$ for all $k$.

Without loss of generality, we may assume that $|i| \leq |j|$. Define $k = |j| - |i|$. Then $[\Psi_{|i|}(0), \Psi_{|j|}(1)] \cap T$ is $T$-separate since $\Psi_{|i|}(0) = \Psi_i(0)$. Thus, noticing that the lemma holds if $i = j$, we assume that $i < j$ in the remainder of the proof.

Let $m$ be the common length of $i$ and $j$. Define $\Sigma(i, j) = \{k \in \Sigma_n^* : i \leq k \leq j\}$. Then

$$[\Psi_i(0), \Psi_j(1)] \cap T = \bigcup_{k \in \Sigma(i, j)} T_k.$$  

(27)

Now we arbitrarily pick a joint pair $(u, v)$ with $u, v \in \Sigma(i, j)$. Notice that

$$T_u = L_0(T_u) \cup R_0(T_u) \cup M(T_u), \quad T_v = L_0(T_v) \cup R_0(T_v) \cup M(T_v).$$

If $R_0(T_u)$ is $T$-separate, we have $R_0(T_u) \in T^{(1)}$. Also, in this case, we must have that $L_0(T_u)$ is $T$-separate so that $L_0(T_u) \in T^{(1)}$. It follows that there exists $A(u, v) \in T^*$ such that

$$T_u \cup T_v = L_0(T_u) \cup R_0(T_v) \cup A(u, v),$$

(28)

where the union is disjoint.

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If \( R_0(T_u) \) is not \( T \)-separate, we define \( s \) to be the maximal non-negative integer which satisfies \( u = u'[n]^s \) for some \( u' \in \Sigma^*_n \). Let \( u' = u_1u_2 \cdots u_{m_1} \). Then \( u_{m_1} \in \Sigma_T \) and \( v = v'[1]^r \), where \( v' = u_1 \cdots u_{m_1-s-1}(u_{m_1-s} + 1) \). It is clear that \( s < \min(p, q) \) since \( |u| = |u'| \leq \min(p, q) \). Notice that
\[
R_0(T_u) = R_{q - s}(T_u) \cup \left( R_0(T_u) \backslash R_{q - s}(T_u) \right),
\]
where the unions are disjoint and \( R_0(T_u) \backslash R_{q - s}(T_u), L_0(T_u) \backslash L_{p - s}(T_u) \in \mathcal{T}^* \) by lemma 3.6.

Since
\[
R_{q - s}(T_u) \cup L_{p - s}(T_u) = \Psi_{u_1 \cdots u_{m_1-s-1}}(R_q(T_{u_{m_1-s-1}})) \cup L_p(T_{u_{m_1-s-1}}) = \Psi_{u_1 \cdots u_{m_1-s-1}}(T_{u_{m_1-s-1}}^{(3)}) \in \mathcal{T}^*,
\]
we know that, in this case, there also exists \( A(u, v) \in \mathcal{T}^* \) such that (28) holds while the union is disjoint.

Notice that \( L_0(T_u) \) and \( R_0(T_j) \) are \( T \)-separate, so they are all in \( \mathcal{T}^* \). Using (27) and (28), we can see that the lemma holds.

**Corollary 3.9.** Given \( i \in [1, 2, \ldots, n] \), \( k \in \mathbb{N} \) and \( j \in \Sigma^*_n \) with \( k + |j| < \min(p, q) \). Assume that the last letter of \( j \) does not belong to \([1] \cup (\Sigma_T + 1) \). Then \( R_q(T_i) \backslash \left( R_q(T_i) \cup L_k(T_{[n]^p}) \right) \in \mathcal{T}^* \).

**Proof.** Let \( j = j_1j_2 \cdots j_m \). Then \( j_m > 1 \). Define \( u = j_1 \cdots j_{m-1}(j_m - 1) \). Let \( s \) be the maximal non-negative integer such that \( j = [n]^s j' \) for some \( j' \in \Sigma^*_n \). We remark that \( j' = n \) if \( j = [n]^1 \). Notice that \( s \leq |j| - 1 < q - 1 \). It is easy to check that
\[
R_q(T_i) \backslash \left( R_q(T_i) \cup L_k(T_{[n]^p}) \right)
\]
is equal to
\[
 \left( R_q(T_i) \backslash R_{2q-1}(T_i) \right) \cup \left( [a_1, b_1] \cap T \right) \cup \left( [a_2, b_2] \cap T \right) \cup \left( R_{2q+1}(T_i) \backslash R_{3q}(T_i) \right),
\]
where \( a_1 = \Psi_{[n]^p(a_1)}(0), b_1 = \Psi_{[n]^p(a_1)}(1), a_2 = \Psi_{[n]^p(a_2)}(0), b_2 = \Psi_{[n]^p(a_2)}(1) \) and the union is disjoint. Thus it suffices to show that \([a_1, b_1] \cap T \in \mathcal{T}^* \) and \([a_2, b_2] \cap T \in \mathcal{T}^* \). Using lemma 3.8, we have
\[
[a_1, b_1] \cap T = \Psi_{[n]^p(a_1)} \left( [\Psi_{n(a_1)}(0), \Psi_{n(a_1)}(1)] \cap T \right) \in \mathcal{T}^*.
\]
From \( |j' + k + 1 | \leq \min(p, q) \) and lemma 3.8, we have
\[
[a_2, b_2] \cap T = \Psi_{[n]^p(a_2)} \left( [\Psi_{n(a_2)}(0), \Psi_{n(a_2)}(1)] \cap T \right) \in \mathcal{T}^*.
\]

The following lemma is useful in the proof of lemma 3.12.

**Lemma 3.10.** For any left substitutable touching letter \( i \), we have
\[
\Psi_{[n]^p} \circ \Psi_{i}^{-1}(T_i^{(4)}(T_{[n]^p})) = R_{3q}(T_T) \cup L_{2p+k}(T_{[n]^p}),
\]
and
\[
\Phi_{[n]^p} \circ \Phi_{i}^{-1}(D_i^{(4)}) = L_{3q}(D_{[n]^p}) \cup R_{3q}(D_i).
\]

**Proof.** By definition of \( T_i^{(4)}(T_{[n]^p}) \), in order to prove (29), it suffices to show that
\[
\Psi_{[n]^p} \circ \Psi_{i}^{-1}(R_q(T_i)) = R_{3q}(T_T) \quad \text{and} \quad \Psi_{[n]^p} \circ \Psi_{i}^{-1}(L_k(T_{[n]^p})) = L_{2p+k}(T_{[n]^p}) + 1.
\]
It is clear that \( \text{diam} \Psi_{[n]^p} \circ \Psi_{i}^{-1}(R_q(T_i)) = \text{diam} R_{3q}(T_T) \). Notice that
\[
\max \Psi_{[n]^p} \circ \Psi_{i}^{-1}(R_q(T_T)) = \Psi_{[n]^p} \circ \Psi_{i}^{-1}(\Psi_{i}^{(1)}) = \Psi_{[n]^p} \circ \Psi_{i}^{-1}(1) = \Psi_{i}^{(1)} = 1.
\]

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which is equal to the maximum value of \( R_{3p}(T_i) \). It follows that \( \Psi_{[n]} \circ \Psi_i^{-1}(R_q(T_i)) = R_{3p}(T_i) \).

Notice that \( \Psi_{[n]} \circ \Psi_i^{-1}(L_k(T_{i+1})) = \operatorname{diam} L_{2p+k}(T_{i+1}) \). Since \( i \) is a touching letter, the minimum value of \( T_{i+1} \) is equal to \( \Psi_i(1) \). Thus

\[
\min \Psi_{[n]} \circ \Psi_i^{-1}(L_k(T_{i+1})) = \Psi_{[n]} \circ \Psi_i^{-1}(\Psi_i(1)) = \Psi_{[n]}(1) = \Psi_i(1),
\]

which is equal to the minimum value of \( L_{2p+k}(T_{i+1}) \). It follows that \( \Psi_{[n]} \circ \Psi_i^{-1}(L_k(T_{i+1})) = L_{2p+k}(T_{i+1}) \).

Similarly to the above, we can show that (30) holds from

\[
\Phi_{[n]} \circ \Phi_i^{-1}(L_k(D_{j,k})) = L_k(D_{[n]}) \quad \text{and} \quad \Phi_{[n]} \circ \Phi_i^{-1}(R_q(D_i)) = R_q(D_i),
\]

where the common minimum value of \( \Phi_{[n]} \circ \Phi_i^{-1}(L_k(D_{j,k})) \) and \( \Phi_{[n]} \circ \Phi_i^{-1}(L_k(D_{[n]})) \) is \( \Phi_{[n]}(0) \).

The following lemma is natural.

**Lemma 3.11.** Given \( i, j \in \Sigma^*_x \), if there exist \( u, v \in \mathbb{Z}^* \) such that \( L_u(T_i) \) is \( T \)-separate and \( \operatorname{diam} L_v(T_i) = \operatorname{diam} L_v(T_j) \), then \( \operatorname{diam} L_u(T_i) = \operatorname{diam} L_v(T_j) \).

**Proof.** Note that \( L_u(T_i) = \Psi_{[1]}(T_i^{(1)}) \) and \( L_v(D_j) = \Phi_{[1]}(D_j^{(1)}) \). By \( \operatorname{diam} L_u(T_i) = \operatorname{diam} L_v(T_j) \), we have \( \rho_u \rho_j = \rho_u \rho_j \), so the contraction ratios of \( \Psi_{[1]} \) and \( \Phi_{[1]} \) are the same. Thus the lemma follows from \( \operatorname{diam} L_u(T_i) = \operatorname{diam} L_v(T_j) \).

Based on the above lemmas, we can prove the following crucial lemma.

**Lemma 3.12.** \( \operatorname{diam} L_u(T_i) = \operatorname{diam} L_v(T_j) \) for any \( i \in \Sigma^*_T \) and \( j \in \Sigma^*_T \).

**Proof.** We only show that the lemma holds for every left substitutable touching letter. The proof for right substitutable touching letters is similar so we omit it. By lemma 3.10, we have

\[
T_i^{(3)} = \left( \Psi_{[n]} \circ \Psi_i^{-1}(T_i^{(4)}) \right) \cup \left( L_p(T_{i+1}) \setminus L_{2p+k}(T_{i+1}) \right) \cup L_k(T_{i+1}) \cup A_1,
\]

\[
D_i^{(3)} = \left( \Phi_{[n]} \circ \Phi_i^{-1}(D_i^{(4)}) \right) \cup \left( L_p(D_{i+1}) \setminus L_{2p+k}(D_{i+1}) \right) \cup L_k(D_{i+1}) \cup B_1,
\]

where

\[
A_1 = R_q(T_i) \setminus \left( L_k(T_{i+1}) \cup R_{3p}(T_i) \right), \quad B_1 = R_q(D_i) \setminus \left( L_k(D_{i+1}) \cup R_{3p}(D_i) \right).
\]

Notice that \( L_k(T_{i+1}) \cap R_{3p}(D_i) = \emptyset \) by (21). Since \( D \) is dust-like, the union in (32) is disjoint. By definition, the last letter of \( j_i \) does not belong to \([1] \cup (\Sigma^*_T + 1) \). Thus, by lemma 3.3, we know that \( L_k(T_{i+1}) = \Psi_{[n]} \circ \Psi_{[1]}(T_i^{(1)}) \) is \( T \)-separate. Hence, from \( L_k(T_{i+1}) \cap R_{3p}(T_i) = \emptyset \), we know that the union in (31) is disjoint.

From the definitions of \( k_i \) and \( j_i \), we know that \( \operatorname{diam} L_{2p+k}(T_{i+1}) = \operatorname{diam} L_k(T_{i+1}) \). It follows that \( \operatorname{diam} L_k(T_{i+1}) = \operatorname{diam} L_{2p+k}(T_{i+1}) \). By lemma 3.11 and noticing that \( L_k(T_{i+1}) = \operatorname{diam} L_{2p+k}(T_{i+1}) \), we have \( \operatorname{diam} L_k(T_{i+1}) = \operatorname{diam} L_{2p+k}(T_{i+1}) \).

Notice that \( j_i + k_i < \min[p, q] \) and \( L_k(T_{i+1}) \) is \( T \)-separate. By corollary 3.9, we have \( A_1 \in T^* \). It is clear that \( B_1 = \pi_p \circ \pi^{-1}_q(A_1) \). By lemma 3.4, \( \operatorname{diam} A_1 = \operatorname{diam} B_1 \).

Hence, from lemma 3.6, we have \( \operatorname{diam} L_k(T_{i+1}) = \operatorname{diam} L_{2p+k}(T_{i+1}) \).

Using lemma 3.10 again, we can decompose \( T_i^{(4)} \) and \( D_i^{(4)} \) into the following disjoint unions.

\[
T_i^{(4)} = \left( \Psi_{[n]} \circ \Psi_i^{-1}(T_i^{(4)}) \right) \cup \left( L_k(T_{i+1}) \setminus L_{2p+k}(T_{i+1}) \right) \cup L_k(T_{i+1}) \cup A_1,
\]

\[
D_i^{(4)} = \left( \Phi_{[n]} \circ \Phi_i^{-1}(D_i^{(4)}) \right) \cup \left( L_k(D_{i+1}) \setminus L_{2p+k}(D_{i+1}) \right) \cup L_k(D_{i+1}) \cup B_1,
\]
Proof of theorem 1.3. The theorem follows from lemmas 3.2, 3.5, 3.7 and 3.12. \qed

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Appendix. The necessary condition for the case that $\rho_1 = \rho_4$ and $\Sigma_T = \{2\}$

In this appendix, we assume that $n = 4$, $\rho_1 = \rho_4$, $\Sigma_T = \{2\}$ and $f : T \to D$ is bi-Lipschitz. Denote $s := \dim_H T = \dim_H D$. Denote $\mu_i := \rho_i^s$ for each $i$. We shall prove the following result.

Theorem A.1. Assume that $D \sim T$. Then $\mu_2$ and $\mu_3$ must be algebraically dependent, namely there exists a nonzero rational polynomial $P(x, y)$ such that $P(\mu_2, \mu_3) = 0$.

Define $E_k = T_{[2^k]} \cup T_{[1^k]}$ for all $k \geq 0$. Define $E_1 = [T_1, T_4, E_0]$ and

$$E_{k+1} = F_1(E_k) \cup F_2(E_k) \cup (F_3(E_k) \setminus T_{[2^k]}) \cup (F_4(E_k) \setminus T_{[1^k]}) \cup E_k$$

for all $k \geq 1$. For example, the sets in the class $E_2$ are

$$T_{11}, T_{14}, F_1(E_0), T_{41}, T_{44}, F_4(E_0), T_{21}, F_2(E_0), T_{34}, F_3(E_0), E_1.$$

Remark A.1. Let $i_0 = 2$ and $\{C_k\}_{k=0}^\infty$ be defined as in section 2.2. Then $C_k = T_{[2^k]} \cup T_{[1^k]}$, and $\{E_k\}_{k=0}^\infty$ is a convergent partition of $T$. However, in this appendix, we shall not use these facts or the martingale convergent theorem.

It is clear that the following lemma holds.
Lemma A.2. Let \( \lambda_0 = \min \{ \frac{\text{diam}(A)}{A} : A \in E_1 \} \). Then each set in the family \( \bigcup_{k=1}^{\infty} E_k \) is \((T, \lambda_0)\)-separate.

For any \( A \in \bigcup_{k=1}^{\infty} E_k \), we define
\[
\tilde{g}(A) = \frac{\mathcal{H}^1(f(A))}{\mathcal{H}^1(A)}.
\]

Lemma A.3. The set \( \{ \tilde{g}(A) : A \in E_{k+1}, B \in E_k \text{ with } A \subseteq B, k \geq 1 \} \) is finite.

Proof. Notice that \( \frac{\mathcal{H}^1(E_{k+1})}{\mathcal{H}^1(E_k)} = \mu_1 \) for all \( k \). By induction, we can easily see that
\[
\left\{ \frac{\mathcal{H}^1(f(A))}{\mathcal{H}^1(B)} : A \in E_{k+1}, B \in E_k \text{ with } A \subseteq B \right\} = \left\{ \mu_1, \mu_2, \mu_3, \mu_2 + \mu_3, \frac{\mu_1 \mu_2 + \mu_3}{\mu_2 + \mu_3}, \frac{\mu_1 \mu_3}{\mu_2 + \mu_3} \right\}
\]
for all \( k \geq 1 \). On the other hand, using lemma 2.1, remark 2.1 and lemma A.2, and using the bi-Lipschitz property of \( f \), we can see that the set
\[
\left\{ \frac{\mathcal{H}^1(f(A))}{\mathcal{H}^1(f(B))} : A \in E_{k+1}, B \in E_k \text{ with } A \subseteq B, k \geq 1 \right\}
\]
is finite. Thus the lemma holds. \( \square \)

Now we have the following property by using lemma A.3. We remark that the proof of this property is same as the proof of lemma 4 in [22], which was restated in [13] for completeness (see the proof of lemma 2.4 therein). Thus we omit the proof.

Lemma A.4. There is a set \( A_0 \) in the family \( \bigcup_{k=1}^{\infty} E_k \) and a constant \( \delta > 0 \) such that \( \tilde{g}(A) = \delta \) for all \( A \in \bigcup_{k=1}^{\infty} E_k \) with \( A \subseteq A_0 \).

By the lemma, the restriction of \( f \) on \( A_0 \) is measure preserving up to a constant. Thus, if we choose \( i_0 \in \Sigma^+ \) such that \( \Psi_{i_0}(T) \subset A_0 \), then the restriction of \( f \) on \( \Psi_{i_0}(T) \) is also measure preserving up to a constant. Hence, without loss of generality, we assume that \( A_0 = \Psi_{i_0}(T) \) in the following.

By lemmas 2.1 and A.2, there exists an integer \( n_1 \) such that for any \( k \in \mathbb{Z}^+ \) there exist \( j, j_1, \ldots, j_p \in \Sigma^+ \) such that \( D_{j_1}, D_{j_2}, \ldots, D_{j_p} \) are disjoint and
\[
f(\Psi_{i_0}(E_k)) = \bigcup_{r=1}^{p} D_{j_{r}},
\]
where each \( |j_r| = n_1 \). Furthermore, by remark 2.1, we can require \( D_j \) to be the smallest cylinder containing \( f(\Psi_{i_0}(E_k)) \). We denote this \( j \) by \( j'(k) \) and define \( \gamma'_k = \sum_{r=1}^{p} \mu_{j_r} \). Define \( \mathcal{M}' = \left\{ \sum_{j \in A} \mu_j : A \subset \{1, \ldots, 4^n\} \right\} \).

Proof of theorem A.1. Notice that \( 2\mu_1 + \mu_2 + \mu_3 = 1 \). In order to prove the lemma, it suffices to show that there exists a polynomial \( P(x_1, x_2, x_3) \) with rational coefficients such that \( P(\mu_1, \mu_2, \mu_3) = 0 \) and \( P(1 - x_2 - x_3)/2, x_2, x_3) \) is not identically equal to 0.

Let \( x^* \) be the unique point in the set \( T_{(1)} \cap T_3 \). Assume that
\[
\{ f(\Psi_{i_0}(x^*)) \} = D_{i_1 i_2 \cdots i_k}.
\]

Given \( i = i_1 i_2 \cdots i_k \in \Sigma^+ \) and \( \ell \in \{1, 2, 3, 4\} \), we define
\[
N_{i}(i) = \text{card } \{ j : i_j = \ell, 1 \leq j \leq |i| \}.\]
Case 1. Assume that there are infinitely many \( k \) such that \( tk = 2 \). Then \( \lim_{q \to \infty} N_2(j'(q)) = \infty \). Notice that \( \text{card} \mathcal{M}_k < +\infty \). Thus, we can choose \( q_1, q_2 \) with \( 1 \leq q_1 < q_2 \) such that \( \gamma_q = \gamma_q' \) and

\[ N_2(j'(q_2)) > N_2(j'(q_1)). \]  

(33)

From \( \tilde{g} = \tilde{g}(\Psi_\mu(E_{q_1})) \) and \( \gamma_q = \gamma_q' \), we have

\[ \frac{\mathcal{H}^\ast(\Psi_\mu(E_{q_1}))}{\mathcal{H}^\ast(\Psi_\mu(E_{q_2}))} = \frac{\mathcal{H}^\ast(D_{j'(q_2)})}{\mathcal{H}^\ast(D_{j'(q_1)})}. \]

Thus, by (33),

\[ \mu_{q-q_1} = \mu_1 \cdot \mu_2 \cdot \mu_3, \]

with \( k_2 \in \mathbb{Z}^+ \) and \( k_1, k_3 \in \mathbb{N} \). Since the above equality does not hold if we plug in \( \mu_1 = 1/2, \mu_2 = \mu_3 = 0 \), we know that \( \mu_2 \) and \( \mu_3 \) are algebra dependent.

Case 2. Assume that there are infinitely many \( k \) such that \( tk = 3 \). Then using the same method as in case 1, we can obtain that \( \mu_2 \) and \( \mu_3 \) are algebra dependent.

Case 3. Assume that there are only finitely many \( k \) such that \( tk \in \{2, 3\} \). Then there exists \( q_0 \in \mathbb{Z}^+ \) such that \( N_2(j'(q)) = N_2(j'(q_0)) \) and \( N_2(j'(q)) = N_2(j'(q_0)) \) for all \( q \geq q_0 \).

By definition, \( \gamma_q = \gamma_q' \). Substituting \( \mu_3 \) by \( 1 - 2\mu_1 - \mu_2 \), we know that \( \gamma_q' = \gamma_q' \) is a polynomial of \( \mu_1 \) and \( \mu_2 \) with integral coefficients. By using the Euclidean algorithm, it is easy to see that there exist polynomials \( Q(\mu_1, \mu_2) \) and \( R(\mu_2) \) with rational coefficients such that \( \gamma'_q = (1 - 2\mu_1)Q(\mu_1, \mu_2) + R(\mu_2) \). Thus \( \gamma'_q = (\mu_2 + \mu_3)Q(\mu_1, \mu_2) + R(\mu_2) \), so

\[ \tilde{g}(\Psi_\mu(E_{q_0})) = \frac{\mathcal{H}^\ast(D_{j'(q_0)})(\mu_2 + \mu_3)Q(\mu_1, \mu_2) + R(\mu_2)}{\mathcal{H}^\ast(T)}. \]  

(34)

From \( E_{q_0} = T_{2[4^{q_0}]} \cup T_{3[1^{q_0}]} \), we know that \( \Psi_\mu(T_{2[4^{q_0}]} \cup T_{3[1^{q_0}]} \) \). Thus the smallest cylinder of \( D \) containing \( f(\Psi_\mu(T_{2[4^{q_0}]})) \) is a subset of the smallest cylinder of \( D \) containing \( f(\Psi_\mu(E_{q_0})) \). It follows that there exists a polynomial \( P \) of \( \mu_1, \mu_2, \mu_3 \) with integral coefficients, such that

\[ \tilde{g}(\Psi_\mu(T_{2[4^{q_0}]})) = \frac{\mathcal{H}^\ast(D_{j'(q_0)}) \cdot P(\mu_1, \mu_2, \mu_3)}{\mathcal{H}^\ast(T)}. \]  

(35)

By lemma A.4, the right hand sides of (34) and (35) are equal, so

\[ ((\mu_2 + \mu_3)Q(\mu_1, \mu_2) + R(\mu_2))\mu_1\mu_2 = (\mu_2 + \mu_3)P(\mu_1, \mu_2, \mu_3). \]  

(36)

Case 3.1. Assume that the polynomial \( R(x) \) is not identically equal to 0. Then there exists \( a \neq 0 \) such that \( R(a) \neq 0 \). Notice that (36) does not hold if we plug in \( \mu_1 = 1/2, \mu_2 = a, \mu_3 = -a \). Thus \( \mu_2 \) and \( \mu_3 \) are algebra dependent.

Case 3.2. Assume that the polynomial \( R(x) \) is identically equal to 0. Then from (34) we have

\[ \tilde{g}(\Psi_\mu(E_{q_0})) = \frac{\mathcal{H}^\ast(D_{j'(q_0)}) \cdot Q(\mu_1, \mu_2)}{\mathcal{H}^\ast(T)}. \]  

(37)

Since there are only finitely many \( k \) such that \( tk \in \{2, 3\} \), we can choose \( q_3 > q_0 \) such that in the right hand side of

\[ f(\Psi_\mu(E_{q_0})) = \bigcup_{r=1}^p D_{j_{r+}}, \]
where \( j, j_1, \ldots, j_p \) have the same meaning as above, the cylinder \( D_{j_{j_1}} \) containing \( \Psi_{ij}(x^*) \) satisfies \( N_2(j_{j_1}) = N_3(j_{j_1}) = 0 \). It follows that \( y_{q_3}' \neq 0 \) if we plug in \( \mu_1 = 1/2, \mu_2 = \mu_3 = 0 \).

Notice that
\[
\tilde{g}(\Psi_{ij}(E_{q_3})) = \frac{\mathcal{H}^s(D_{j_{j_1}}(q_3)) \cdot y_{q_3}'}{\mu_{k_i}(\mu_2 + \mu_3)\mu_1^{q_3} \cdot \mathcal{H}^s(T)}.
\]

By (37) and using lemma A.4, we have
\[
(\mu_2 + \mu_3)\mu_1^{q_3} \cdot \mathcal{Q}(\mu_1, \mu_2) = \frac{\mathcal{H}^s(D_{j_{j_1}}(q_3))}{\mathcal{H}^s(D_{j_{j_1}}(q_0))} \cdot y_{q_3}'.
\]

It follows from \( N_2(j_{j_1}(q_3)) = N_2(j_{j_1}(q_0)) \) and \( N_3(j_{j_1}(q_3)) = N_3(j_{j_1}(q_0)) \) that \( \frac{\mathcal{H}^s(D_{j_{j_1}}(q_3))}{\mathcal{H}^s(D_{j_{j_1}}(q_0))} = \mu_1^k \) for some \( k \in \mathbb{N} \). Thus the above equality does not hold if we plug in \( \mu_1 = 1/2, \mu_2 = \mu_3 = 0 \) so that \( \mu_2 \) and \( \mu_3 \) are algebraic dependent.

**Example A.1.** Let \( n = 4 \) and \( \Sigma_T = \{2\} \). Let \( \mu_2 = e/4, \mu_3 = 1/4 \) and \( \mu_1 = \mu_4 = (1 - \mu_2 - \mu_3)/2 \), where \( e \) is the Euler constant. Let \( \rho_i = \mu_i^2 \) for all \( i \). Then \( \dim_H D = \dim_H T = 1/2 \). By theorem A.1, \( D \not\sim T \).

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