KARHUNEN-LOÉVE EXPANSION OF BROWNIAN MOTION FOR APPROXIMATE SOLUTIONS OF LINEAR STOCHASTIC DIFFERENTIAL MODELS USING PICARD ITERATION
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Abstract: This work presents an application of the Picard Iterative Method (PIM) to a class of Stochastic Differential Equations where the randomness in the equation is considered in terms of the Karhunen-Loéve Expansion finite series. Two applicable numerical examples are considered to illustrate the convergence of the approximate solutions to the exact solutions and also to check the efficiency of the method. The results obtained show clearly that accuracy will be more visible by increasing the number of terms in the iteration. Thus, it is recommended for nonlinear financial models of different classes of Stochastic Differential Equations.

Keywords: linear Stratonovich SDEs; Karhunen-Loéve expansion; approximate solution; PIM; Brownian motion.

2010 AMS Subject Classification: 60H10, 37L55, 41A58

1. INTRODUCTION

Stochastic Differential Equations (SDEs) are used to model random phenomena. Recently, SDEs have become a typical model for quantifying financial products like options, stocks, interest rates,
and some other underlying assets [1]. Merton’s work brought phenomena advancement in the application of SDEs to financial problems [2]. Several other areas, such as population dynamic, option pricing, and so on have been discussed in [3]. The methods for solving different classes of stochastic differential equations are presented in [4, 5], but the explicit solutions are limited for such equations [6]. Picard Iteration Method (PIM) is one of the effective numerical methods that can be used for solving differential equations. It transforms a differential equation into an integral equation. This method can be used to get the approximate analytical solutions of linear and nonlinear class of differential equations. Youssef, in [7] used Picard’s Iterative Method with Gauss-Seidel Method to solve some IVP. Xiaohui et al., in [8] applied PIM to singular fractional differential equation. In [9], Edeki et al., used PIM and Differential Transforms Method to obtain the solutions of linear and nonlinear differential equations. Rach, in his research work, used Adomian Decomposition Method (ADM) and PIM [10]. ADM and PIM were compared by Bellomo and Sarafyan [11], Antonis et al., used Picard’s for the approximation of stochastic differential equations by applying it to Libor models [12]. Little research has been done on the application of PIM to stochastic differential models. Therefore, in this paper, our concern is to modify this numerical method by considering the Stratonovich linear stochastic differential models where the random path of the SDEs are being taken care of by the Karhunen-Loève expansion of Brownian Motion.

The rest of the work is organized as follows. In section 2, linear SDEs are discussed, the method of solution is discussed in section 3, two numerical examples are given in section 4, and lastly, a concluding remark is presented in section 5.

2. LINEAR STOCHASTIC DIFFERENTIAL MODEL

Considering the Stratonovich SDE of the form:

\[
\begin{align*}
\frac{dX}{dt} &= f(X, t) dt + \sum_{j=1}^{d} \sigma_j(X, t) \circ dW_j(t) \\
X(0) &= X_0,
\end{align*}
\]  

(2.1)
where, $f$ and $\sigma$ are the drift and volatility coefficients respectively, $X(t) = X \in \mathbb{R}^n$ represents the stochastic process and $W(t)$ is the standard Brownian motion (Wiener process), also the dimension of the Brownian Motion is denoted with $d$. We remark as follows that $W = \{W(t), 0 \leq t \leq T\}$ is a 1-dimensional Brownian motion [13].

Basically, the SDE in (2.1) can be written in Itô form as:

$$dX = f(t, X)dt + \sum_{j=1}^{d} \sigma_j(t, X)dW^j(t)$$

$$X(0) = X_0.$$ (2.2)

In integral form, the SDE in (2.1) is written as:

$$X(t) = X_0 + \int_0^t f(s, X(s))ds + \int_0^t \sigma(s, Y(s))dW(s).$$ (2.3)

The first integral and second integral in (2.3) are known as the Riemann-Stiltjes and stochastic integral, respectively.

3. METHODS OF SOLUTION

This section presents the concepts of the Picard Iterative Method (PIM) and Karhunen-Loeve Expansion (K-LE).

3.1 Picard Iterative Method (PIM)

Let us consider the differential equation of this form:

$$\begin{cases}
x' = f(t, x), \\
x(0) = x_0.
\end{cases}$$ (3.1)

First order differential equations fall under this type of equation, and PIM is one of the suitable methods for handling this type of differential equation. Now, by integrating both sides of (3.1), we get:

$$\int_0^t x'(s)ds = \int_0^t f(s, x(s))ds,$$ (3.2)

therefore, following the basic concept of calculus (3.2) becomes:
\[ x(t) - x(0) = \int_0^t f(s, x(s)) ds, \]
\[ \Rightarrow x(t) = x(0) + \int_0^t f(s, x(s)) ds. \quad (3.3) \]

Since \( x(t) \) is appearing on both sides of equation (3.3) for arbitrary \( t \), we therefore adopt this iterative process by choosing an initial condition:

\[ x(0) = x_0, \text{ for } n \geq 1, n \in \mathbb{Z}^+: \]
\[ \therefore x_{n+1} = x_0 + \int_0^t f(s, x_n(s)) ds. \quad (3.4) \]

In what follows, the approximation of (3.1) is:

\[ x(t) = \lim_{n \to \infty} x_{n+1}(t), \text{ as } n \to \infty. \]

The existence and uniqueness of this method have been studied by [9, 14].

### 3.2 Karhunen-Loève Expansion (K-LE) finite series of Brownian Motion

The K-L expansion is an extension of the Fourier Transform (FT). Its analysis is from deterministic functions to probabilistic form. It is also referred to as a bi-orthogonal stochastic process expansion [15]. Stefano [16], in his work, used the K-L expansion to characterize the Brownian Motion (Wiener process). In this expansion [16], \( \{X(t, w), t \in T\} \) represents a stochastic process in terms of sequence of identically and independent sample variables \( \{z_i, i \in N\} \).

The Wiener process, \( W(t) \) has a trajectory belonging to \( L^2([0,T]) \) for almost all \( W's \).

Therefore, The K-L expansion for Brownian Motion can be represented as:

\[
\begin{cases}
W(t) = W(t, \omega) = \sum_{i=0}^{\infty} z_i(w) \Psi_j(t), & 0 \leq t \leq T, \\
\Psi_j(t) = \frac{2\sqrt{2T}}{(2j+1)} \sin \left( \frac{(2j+1)\pi t}{2T} \right),
\end{cases}
\]
\[ (3.5) \]

where \( \Psi_j(t) \) form a basis of orthogonal function [6, 16].
So, simplifying (3.5) with $T = 1$ gives:

$$W(t) = \sqrt{2} \sum_{j=0}^{\infty} \left[ \sin \left( (j + 0.5) \pi t \right) \right] z_j.$$ \hspace{1cm} (3.6)

Thus,

$$dW(t) = \sqrt{2} \sum_{j=0}^{\infty} \left( \cos \left( (j + 0.5) \pi t \right) \right) z_j.$$ \hspace{1cm} (3.7)

Replacing the random path of SDE (2.1) with (3.7), we obtain:

$$\begin{cases} 
 dX(t) = f(X,t)dt + \sigma(X,t) \left( \sum_{j=0}^{5} z_j \Psi_j(t) \right), \\
 X(0) = X_0.
\end{cases} \hspace{1cm} (3.8)$$

For the purpose of this research work, $z_j$ is generated using a mathematical computer software.

### 4. Numerical Examples

**Example 4a:** Consider the Linear Stratonovich Stochastic Differential Equation [12]:

$$\begin{cases} 
 dX(t) = \beta X(t) \circ dW(t), \\
 X(0) = 1.
\end{cases} \hspace{1cm} (4.1)$$

The exact solution of (4.1) is

$$X_{\text{exact}} = X(0) \exp(\beta W(t)), \quad t \in [0, T].$$

Next, we re-express (4.1) in an integral form of (3.4):

$$X_{n+1} = X_0 + \int_0^t f\left(s, X_n(s)\right)dt$$

$$X_0 = 1$$

$$X_1 = 1 + \int_0^t \left( \beta X_0 dW_s \right) ds,$$

$$= 1 + \int_0^t dW_s ds, \quad \beta = 1,$$

$$= 1 + \int_0^t \sqrt{2} \sum_{j=0}^{\infty} \left( \cos \left( (j + 0.5) \pi t \right) \right) z_j ds.$$
\[ X_2 = 1 + \int_0^t \left( (X_1) dW_s \right) ds \]
\[ = 1 + \beta \int_0^t (X_1) dW_s ds, \]
\[ = 1 + \int_0^t (X_1) \sqrt{2} \sum_{j=0}^{\infty} \left( \cos \left( (j + 0.5) \pi t \right) \right) z_j ds \]

\[ X_3 = 1 + \int_0^t \left( (X_2) dW_s \right) ds \]
\[ = 1 + \beta \int_0^t (X_2) dW_s ds, \]
\[ = 1 + \int_0^t (X_2) \sqrt{2} \sum_{j=0}^{\infty} \left( \cos \left( (j + 0.5) \pi t \right) \right) z_j ds \]

\[ X_4 = 1 + \int_0^t \left( (X_3) dW_s \right) ds \]
\[ = 1 + \beta \int_0^t (X_3) dW_s ds, \]
\[ = 1 + \int_0^t (X_3) \sqrt{2} \sum_{j=0}^{\infty} \left( \cos \left( (j + 0.5) \pi t \right) \right) z_j ds \]

\[ X_5 = 1 + \int_0^t \left( (X_4) dW_s \right) ds \]
\[ = 1 + \beta \int_0^t (X_4) dW_s ds, \]
\[ = 1 + \int_0^t (X_4) \sqrt{2} \sum_{j=0}^{\infty} \left( \cos \left( (j + 0.5) \pi t \right) \right) z_j ds \]

\[ \vdots \]

\[ X_{n+1} = X(0) + \int_0^t \left( (X_n) dW_s \right) ds. \]

**Example 4b:** Consider the Linear Stochastic Differential Equation [18]:

\[
\begin{cases}
    dX(t) = \frac{1}{2} X(t) dt + X(t) \circ dW(t), \\
    X(0) = 1.
\end{cases}
\]

(4.3)

The Stratonovich exact solution of (4.3) is given as

\[ X_{\text{exact}} = \exp \left( \frac{t}{2} + B(t) \right). \]

(4.4)
Next, we re-express (4.3) in an integral form as in (3.4):

\[ X_{n+1} = X_0 + \int_0^t f(s, X_n(s)) \, dt. \]  

(4.5)

Thus, the following are obtained:

\[ X_0 = 1, \]

\[ X_1 = 1 + \frac{1}{2} \int_0^t X_0 + \int_0^t (X_0 \, dW_s) \, ds, \]

\[ = 1 + \frac{1}{2} \int_0^t X_0 + \int_0^t (X_0) \left( \sqrt{2} \sum_{j=0}^{\infty} \cos \left( (j + 0.5) \pi t \right) z_j \right) \, ds. \]

\[ X_2 = 1 + \frac{1}{2} \int_0^t X_1 + \int_0^t (X_1 \, dW_s) \, ds, \]

\[ = 1 + \frac{1}{2} \int_0^t X_1 + \int_0^t (X_1) \left( \sqrt{2} \sum_{j=0}^{\infty} \cos \left( (j + 0.5) \pi t \right) z_j \right) \, ds. \]

\[ X_3 = 1 + \frac{1}{2} \int_0^t X_2 + \int_0^t (X_2 \, dW_s) \, ds, \]

\[ = 1 + \frac{1}{2} \int_0^t X_2 + \int_0^t (X_2) \left( \sqrt{2} \sum_{j=0}^{\infty} \cos \left( (j + 0.5) \pi t \right) z_j \right) \, ds. \]

\[ X_4 = 1 + \frac{1}{2} \int_0^t X_3 + \int_0^t (X_3 \, dW_s) \, ds, \]

\[ = 1 + \frac{1}{2} \int_0^t X_3 + \int_0^t (X_3) \left( \sqrt{2} \sum_{j=0}^{\infty} \cos \left( (j + 0.5) \pi t \right) z_j \right) \, ds. \]

\[ X_5 = 1 + \frac{1}{2} \int_0^t X_4 + \int_0^t (X_4 \, dW_s) \, ds, \]

\[ = 1 + \frac{1}{2} \int_0^t X_4 + \int_0^t (X_4) \left( \sqrt{2} \sum_{j=0}^{\infty} \cos \left( (j + 0.5) \pi t \right) z_j \right) \, ds. \]

\[ X_6 = 1 + \frac{1}{2} \int_0^t X_5 + \int_0^t (X_5 \, dW_s) \, ds, \]

\[ = 1 + \frac{1}{2} \int_0^t X_5 + \int_0^t (X_5) \left( \sqrt{2} \sum_{j=0}^{\infty} \cos \left( (j + 0.5) \pi t \right) z_j \right) \, ds, \]

\[ \vdots \]

\[ X_{n+1} = X(0) + \int_0^t (X_n \, dW_s) \, ds. \]

The numerical results and error analysis are given in Tables 1-2. Also, a graph is plotted for each
example, respectively. Other solution methods for linear and or nonlinear differential models (ordinary or partial), including SDEs are referred [19-31].

| $t$ | $X_{\text{exact}}$ | $X_5(t)$ | $|X_{\text{exact}} - X_5|$ |
|-----|--------------------|----------|-----------------|
| 0.0 | 1.0000             | 1.0000   | 0.0000          |
| 0.1 | 1.5981             | 1.5981   | 0.0000          |
| 0.2 | 2.2045             | 2.2041   | 0.0004          |
| 0.3 | 2.2390             | 2.2385   | 0.0005          |
| 0.4 | 2.1791             | 2.1787   | 0.0004          |
| 0.5 | 3.0838             | 3.0804   | 0.0034          |
| 0.6 | 4.9279             | 4.8986   | 0.0293          |
| 0.7 | 5.1487             | 5.1139   | 0.0348          |
| 0.8 | 4.1756             | 4.1608   | 0.0148          |
| 0.9 | 4.2626             | 4.2464   | 0.0162          |
| 1.0 | 4.7210             | 4.6963   | 0.0247          |

**Fig. 1:** Showing the graph of example 4a
Table 2: Error analysis of example 4b

| $t$ | $X_{exact}$ | $X_{\delta}(t)$ | $|X_{exact} - X_{\delta}|$ |
|-----|-------------|-----------------|--------------------------|
| 0.0 | 1.0000      | 1.0000          | 0.0000                   |
| 0.1 | 1.6801      | 1.6801          | 0.0000                   |
| 0.2 | 2.4363      | 2.4362          | 0.0001                   |
| 0.3 | 2.6013      | 2.6011          | 0.0002                   |
| 0.4 | 2.6615      | 2.6613          | 0.0002                   |
| 0.5 | 3.9596      | 3.9574          | 0.0022                   |
| 0.6 | 6.6520      | 6.6294          | 0.0226                   |
| 0.7 | 7.3064      | 7.2742          | 0.0322                   |
| 0.8 | 6.2292      | 6.2118          | 0.0174                   |
| 0.9 | 6.6850      | 6.6620          | 0.0230                   |
| 1.0 | 7.7837      | 7.7433          | 0.0404                   |

Fig. 2: Showing the graph of example 4b
4. CONCLUDING REMARKS
In this research paper, we have considered a procedure for obtaining an approximate analytical solution of a class of SDEs. The approach is by Picard Iterative Method (PIM), which transforms the differential equation into an equivalent integral equation, provided that the Lipchitz condition is satisfied and also, by converting the entire random sample into piecewise-continuous polynomials. In comparing the results obtained, it showed that more accuracy would be attained with an increasing number of terms. So based on these, further investigation can be carried out on the higher order of different classes of SDEs.

ACKNOWLEDGEMENT
The support of Covenant University Centre for Research Innovation and Development (CUCRID) is deeply appreciated.

CONFLICT OF INTERESTS
The authors declare no conflict of interest.

REFERENCES
[1] V.J. Francina, On the existence of solutions of stochastic differential equations in finance, Int. J. Educ. Appl. Sci. Res. 5 (2014), 24-31.
[2] R. C. Merton, Optimum consumption and portfolio rules in a continuous time model, J. Econ. Theory, 4 (1971), 373-413
[3] P.E. Kloeden, E. Platen, Numerical solution of stochastic differential equations, Springer-Verlag Berlin Heidelberg, New York, 1992.
[4] B. Oksendal, Stochastic differential equations: An introduction with applications, 6th ed. Springer, Berlin Heidelberg, New York, 2003.
[5] T.T. Soong, Random differential equations in science and engineering, Academic Press, New York, 1973.
[6] A. Fakharzadeh, E. Hesamaeddini and M. Soleimanivareki, Multi-step stochastic differential transformation
method for solving some class of random differential equations, Appl. Math. Engi. Manage. Technol. 3(2015), 115-123.

[7] I.K. Youssef, Picard iteration algorithm combined with Gauss-Seidel technique for initial value problem, Appl. Math. Comput. 190 (2007), 345-355.

[8] X. Yang, Y. Liu, Picard iterative processes for initial value problems of singular fractional differential equations, Adv. Difference Equ. 2014 (2014), 102

[9] S.O. Edeki, A.A Opanuga, H.I Okagbue, On the iterative techniques for numerical solutions of linear and nonlinear differential equations. J. Math. Comput. Sci. 4 (2014), 716-727.

[10] R. Rach, On the Adomian Decomposition method and comparisons with Picard’s method, J. Math. Anal. Appl. 128 (1987), 480-483.

[11] N. Bellomo, D. Sarafyan, On Adomian Decomposition method and some comparisons with Picard’s method. J. Math. Anal. Appl. 123 (1987), 389-400.

[12] A. Papapantoleon, D. Skovmand, Picard approximation of stochastic differential equations and application to LIBOR models, arXiv:1007.3362 [q-fin.CP], 2010.

[13] J. G. Gaines, T.J. Lyons, Variable step size control in the numerical solution of stochastic differential equations, SIAM J. Appl. Math., 57(1997), 1455–1484.

[14] D. Gutermuth, Picard’s Existence and Uniqueness Theorem, Lecture Note.
https://www.scribd.com/document/362663692/Picard-2-2

[15] S. Ghorai, Picard's existence and uniqueness theorem, Picard's iteration, Lecture V.
http://home.iitk.ac.in/~sghorai/TEACHING/MTH203/ode5.pdf

[16] L. Wang, Karhunen-Loeve expansions and their applications, PhD Thesis, London School of Economics and Political Science, 2008.

[17] M.I. Stefano, Simulation and Inference for Stochastic Differential Equations, Springer, New York, 2008.

[18] H.D. Azodi, Application of DJ method to Ito stochastic differential equations, J. Linear Topol. Algebra, 8(3)(2019), 183- 189.

[19] G.O. Akinlabi, R.B. Adeniyi, E.A. Owoloko, The solution of boundary value problems Bibi with mixed boundary conditions via boundary value methods, Int. J. Circ. Syst. Sign. Process. 12 (2018), 1-6.
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[20] A. Bibi, F. Merahi, Adomian decomposition method applied to linear stochastic differential equations, Int. J. Pure Appl. Math. 118(3)(2018), 501-510.

[21] G.O. Akinlabi, R.B. Adeniyi, Sixth-order and fourth-order hybrid boundary value methods for systems of boundary value problems, WSEAS Trans. Math. 17(2018), 258-264.

[22] O. Gonzalez-Gaxiola, J. Ruiz de Chavez, S. O. Edeki, Iterative method for constructing analytical solutions to the Harry-DYM initial Value Problem, Int. J. Appl. Math. 4(2018), 627-640.

[23] S. Bhalekar, V. Daftardar-Gejji, Convergence of the new iterative method, Int. J. Differ. Equ. 2011 (2011), 989065.

[24] S.O. Edeki, O.O. Ugbebor, E.A. Owoloko, He’s Polynomials for Analytical Solutions of the Black-Scholes Pricing Model for Stock Option Valuation, Proc. World Congr. Eng. 2(2016), 1-3.

[25] K. Abbaoui, Y. Cherruault, Convergence of Adomians method applied to nonlinear equations, Math. Comput. Model. 20(9)(1994), 60-73.

[26] G.O Akinlabi, S.O Edeki, The Solution of Initial-value Wave-like Models via Perturbation Iteration Transform Method, Proc. Int. Multi Conf. Eng. Comput. Sci. 2(2017), 1-4.

[27] S.O. Edeki, G.O. Akinlabi, Zhou Method for the Solutions of System of Proportional Delay Differential Equations, MATEC Web of Conferences, 125(2017), 02001.

[28] D. Lesnic, The decomposition method for initial value problems, Appl. Math Comput.181(2006), 206-213.

[29] S.O. Edeki, G.O. Akinlabi, Coupled Method for Solving Time-Fractional Navier-Stokes Equation, Int. J. Circ. Syst. Sign. Process. 12(2018), 27-34.

[30] J.H. He, Homotopy perturbation technique, Engineering, 178(1999), 257-262.

[31] W.A. Robin, Solving differential equations using modified Picard iteration, Int. J. Math. Educ. Sci. Technol. 41(5)(2010), 649-665.