I. INTRODUCTION

Quantum many-body systems give rise to a number of intriguing phenomena such as quantum phase transitions, topological insulators, or high-temperature superconductivity. Yet, their description is a formidable challenge as the dimension of the Hilbert space grows exponentially with number of constituents. The huge number of degrees of freedom thus renders an exact description in general infeasible, even if one resorts to numerical approaches. Exceptions to this intractability are quantum systems that do not explore their entire Hilbert space, where numerical optimization approaches such as the Density Matrix Renormalization Group become efficient descriptions. Alternatively one may aim for only obtaining the information of interest about the quantum state of the entire system and try to find accurate and efficient approximations for the sought quantities. Mean-field approaches can be understood as representatives of this strategy as they only predict properties of a single constituent of the many-body system.

Equations of motion for the part of the quantum state that is of interest to the researcher have been derived in the context of open quantum systems where the density matrix of the entire system is split into a ‘relevant’ part describing the system and a complementary ‘irrelevant’ part with the help of the Mori projector.

Here, we introduce an approach to the calculation of local properties of a quantum many-body system by defining a time dependent projection operator that may be viewed as a generalization of the Mori projector. Based on this projector, we are able to derive an integro-differential equation that shares similarities with a Nakajima-Zwanzig equation and exactly describes the dynamics of the reduced density matrix of one subsystem (or a cluster of subsystems) of a quantum many-body system.

Our theory is thus capable of describing stationary states and dynamical evolutions for any situation in which one is only interested in the physics of a part of the system under study. We thus expect applications of our theory to be very useful for quantum few- and many-body systems. Most notably, it efficiently predicts non-equilibrium dynamics for very long times, applies to two- and higher-dimensional systems in the same way as to one-dimensional ones, and can directly and efficiently calculate stationary states of many-body systems with dissipation. Besides these applications our technique generalizes the theory of open quantum systems to scenarios beyond the Markov or thermal equilibrium regimes.

In many experimental situations, the samples will however suffer from decoherence and dissipation. Hence dissipative and driven-dissipative quantum many-body systems are currently receiving enormous interest in the search for strongly correlated steady states and non-equilibrium analogs of quantum phase transitions.

As a second test we thus apply our method to driven and dissipative quantum many-body systems and find that it predicts the values of local quantities with very good accuracy.

The remainder of the paper is organized as follows. In section we introduce the self-consistent Mori projector and derive an exact equation of motion for the
to the celebrated Mori projector. As we aim at deriving an equation of motion for the reduced density matrices $\rho_n(t) = \text{Tr}_R R(t)$ only, we take $\rho_{\delta_0}(t)$ to read,

$$\rho_{\delta_0}(t) = \bigotimes_{n \neq n_0} \rho_n(t) \quad \text{with} \quad \rho_n(t) = \text{Tr}_R R(t).$$

In contrast to standard approaches, the reference density matrix $\rho_{\delta_0}(t)$, is in our approach determined consistently from the state $R(t)$ of the entire setup. This general approach requires to allow for a time dependence of the projector $P_{t_0}^{\delta_0}$. A more general choice for $\rho_{\delta_0}$ could be $\rho_{\delta_0} = \text{Tr}_{n_0} R(t)$, where $\text{Tr}_{n_0}$ denotes the trace over the degrees of freedom of constituent $n_0$. This would however lead to equations of motion where local quantities depend on non-local ones and thus not lead to the same reduction of the complexity of the description as $P_{t_0}^{\delta_0}$.

Using the projector $P_{t_0}^{\delta_0}$, see Eq. (2), we derive an exact equation of motion for the part of the density matrix $R(t)$ that is relevant for our interests. The only assumption made in the derivation is that the initial state factorizes with respect to the considered subsystems, $R(t_0) = \bigotimes_n \rho_n(t_0)$. Taking the trace $\text{Tr}_{\delta_0}$ we find,

$$\dot{\rho}_{\delta_0}(t) = \mathcal{L}_{\delta_0} \rho_{\delta_0}(t) + \text{Tr}_{\delta_0} \mathcal{L}_I P_{t_0}^{\delta_0} R(t)$$

$$+ \text{Tr}_{\delta_0} \mathcal{L}_I \int_{t_0}^{t} dt' \mathcal{D}(t,t') C_{t'} \mathcal{L}_I P_{t'}^{\delta_0} R(t'),$$

where the action of $C_t = 1 - \sum_{n=1}^{N} P_n$ is to extract correlations and $\mathcal{D}(t,t') = \tilde{T} \exp\{\int_{t}^{t'} dt'' (\mathcal{L}_0 + C_{t''} \mathcal{L}_I)\}$. $\tilde{T}$ orders any product of operators such that the time arguments increase from right to left. The derivation of Eq. (4) is presented in the Appendix, see $\textbf{A}$ and $\textbf{B}$.

Equation (4) exactly describes the dynamics of reduced density matrices for individual subsystems of a quantum many-body system of arbitrary size and geometry. This many-body system may even be open so that its dynamics is not necessarily unitary. Eq. (4) may be viewed as a generalization of the celebrated Nakajima-Zwanzig equation for open systems. It takes the correlations between subsystems explicitly into account via the action of the projection operator $C_t$. Moreover, whenever the total state $R$ is pure, the growth of the von Neumann entropy of the states $\rho_n(t)$, as described by Eq. (4), accounts for the entanglement that is built up between each individual subsystem and its surrounding. Since the first line of Eq. (4) for this case describes a unitary evolution, entanglement between subsystems is only taken into account via the second line of Eq. (4).

Yet, despite only describing the local quantities of the constituent of interest, Eq. (4) is still very demanding to solve in full generality. A viable way for finding its solution is thus to expand it as a Dyson series in powers of the interaction $\mathcal{L}_I$, see Appendix $\textbf{C}$. Here we keep terms up to second order in $\mathcal{L}_I$, which reduces the complexity of the dynamical map $\mathcal{D}(t,t')$ considerably, and end up

FIG. 1. Illustration of our approach for a one-dimensional lattice. We consider a quantum many-body system where each subsystem (QS) has some unitary and potentially some non-unitary dynamics (indicated by the rate $\gamma$). The subsystems are coupled via the interaction $\mathcal{L}_I$. Within our theory we pick one QS of interest and trace out the remaining constituents.

reduced density matrices of individual subsystems of a quantum many-body system. In section $\textbf{III}$ we show the applicability and accuracy of our method for both, the unitary dynamics in closed quantum systems and non-unitary dynamics in one-dimensional and two-dimensional open quantum systems. Finally we give our conclusions and an outlook in section $\textbf{IV}$.

II. SELF-CONSISTENT PROJECTION OPERATOR THEORY

We consider a quantum many-body system of $N$-partite structure for which the density matrix $R(t)$ of the entire system obeys the Liouville equation of motion,

$$\dot{R}(t) = \mathcal{L} R(t),$$

where the dot denotes a time derivative and $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ with $\mathcal{L}_0 = \sum_{n=1}^{N} \mathcal{L}_n$. Here, the superoperators $\mathcal{L}_n$ describe the free dynamics of the $n$-th constituent and $\mathcal{L}_I$ accounts for the interaction between any of the $N$ parts. Importantly, $\mathcal{L}$ may feature non-unitary terms as we are only interested in the properties of one subsystem, say subsystem $n_0$, and discard information about the remaining subsystems, see Fig. 1 for an illustration. For these aims it suffices to determine the reduced density matrix $\rho_{n_0}(t) = \text{Tr}_{\delta_0} R(t)$, where $\text{Tr}_{\delta_0}$ denotes the trace over all $N$ constituents but the $n$-th. To derive an equation of motion for $\rho_{n_0}(t)$ we define the time dependent projection operator

$$P_{t_0}^{\delta_0}(\cdot) = \rho_{\delta_0}(t_0) \otimes \text{Tr}_{\delta_0}(\cdot),$$

where $\rho_{\delta_0}(t)$ is a reference density matrix for the degrees of freedom that have been traced out. $P_{t_0}^{\delta_0} R(t) = \rho_{\delta_0}(t) \otimes \rho_{n_0}(t)$ yields our quantity of interest $\rho_{n_0}(t)$. In the language of open quantum systems, one would call the subsystem $n_0$ the “system” and the remaining subsystems, $n \neq n_0$ the “environment” and for cases where $\rho_{\delta_0}$ is constant, the projector $P_{t_0}^{\delta_0}$ would reduce...
with,
\[
\rho_{n_0}(t) = \mathcal{L}_{n_0} \rho_{n_0}(t) + \sum_{n=1} Z \Tr_n \mathcal{L}_{<n_0,n>\rho_n(t) \otimes \rho_{n_0}(t)} \]
\[
+ \sum_{n=1} Z \Tr_n \mathcal{L}_{<n_0,n>} \int_0^t dt' K_{<n_0,n>}(t, t') \rho_n(t') \otimes \rho_{n_0}(t').
\]

Here, \( K_{<n_0,n>}(t, t') = e^{(t-t')/(\mathcal{L}_{n_0} + \mathcal{L}_n)_{\sigma^<n_0,n> \mathcal{L}_{<n_0,n>}} \}
with \( C^<_{n_0,n,n'} = 1 - \rho_n(t') \otimes \Tr_n - \rho_n(t) \otimes \Tr_n\). \( \mathcal{L}_{<n_0,n>}\) denotes the interaction between subsystems \( n_0 \) and \( n \), and \( Z \) the coordination number of the lattice.

Equation (5) is a nonlinear integro-differential equation for the reduced density matrices \( \rho_n(t) \) of individual subsystems and can be integrated numerically using standard techniques.\(^{20}\) In particular for large systems where one can assume translation invariance, \( \rho_n(t) = \rho_{n_0}(t) \) for all \( n \), Eq. (5) reduces to an equation for \( \rho_{n_0}(t) \) only. For these cases our approach thus achieves a similar reduction of computational complexity as Gutzwiller type mean-field calculations. Yet, despite this efficiency it is remarkably more accurate than mean-field as we show for a series of examples below. Before discussing applications of Eq. (5) we comment on some of its properties.

We start by noting that whereas one could derive an exact equation for \( \rho_{n_0} \) for any choice of the reference state \( \rho_{n_0} \), the quality of the approximation in Eq. (5) depends on the ansatz for \( \rho_{n_0} \). Here, our time dependent and self-consistent projector, defined in Eq. (2), appears to be the best ansatz to account for a highly dynamical quantum many-body environment. Moreover, a Markov approximation as applied in open system theory is here not applicable since reduced density matrices of the ‘system’ and its surrounding are treated on an equal footing and correlation functions of the surrounding can thus not be expected to decay faster than the dynamics generated by the coupling between subsystems. Consequently, our approach does not require any separation of the time scales. In fact, for models with time scale separations, which are the basis of approaches using adiabatic elimination or approximate Schrieffer-Wolff transformations,\(^{21}\) Eq. (5) reduces to a standard master equation.\(^{10}\)

Finally, the first two terms on the right hand side of Eq. (4) are equivalent to the mean-field or Gutzwiller approach which has been exploited with remarkable success in equilibrium physics\(^{11,17,23}\) and was the starting point for recent investigations of non-equilibrium systems.\(^{20,22,29}\) Mean-field can thus be understood as an approximation to linear order in \( \mathcal{L}_I \) for the dynamics of single site reduced density matrices. Our theory therefore forms a systematic generalization of mean-field approaches. As the non-Markovian properties, the explicit consideration of correlations via the projector \( \mathcal{L}_I \), and entanglement between subsystems are only present in terms of higher than linear order in \( \mathcal{L}_I \), Eq. (5) yields a different quality of approximation than mean-field. We will show in the sequel that this is indeed the case.

### III. APPLICATIONS AND ACCURACY TESTS

We now turn to test the accuracy of Eq. (5) in applications to one- or two-dimensional systems where either exact solutions or very accurate t-DMRG simulations are available for comparison. In doing so we focus on a lattice of two-level systems or spins and extensions thereof which include coherent drives and relaxation of individual spins. This model is described by Eq. (1) with \( \hbar = 1 \),
\[
\mathcal{L}_I(\cdot) = -i[H_n, \cdot] + D_n(\cdot) \quad \text{and} \quad L_I(\cdot) = -i[H_L, \cdot],
\]
where \( H_n = \Delta \sigma_n^I \sigma_n + (\Omega/2)(\sigma_n^I \sigma_n + \sigma_n^I \sigma_n^\dagger) \) and \( D_n(\cdot) = (\gamma/2)\langle \sigma_n^I(\cdot)\sigma_n^I - \sigma_n^I(\cdot)\sigma_n^I + \sigma_n^I(\cdot)\sigma_n^I \rangle \). Here, \( \sigma_n = \langle 0_n | 1_n \rangle \) is the de-excitation operator on site \( n \). We have written the Hamiltonian in a rotating frame such that, \( \Delta = \omega - \omega_L \) is the detuning between spin transition frequency \( \omega \) and drive frequency \( \omega_L \). \( J \) is the tunneling rate between nearest-neighbor sites indicated by the notation \( < n, m > \) and \( \Omega \) the drive amplitude. This model allows us to study both, the unitary dynamics of a closed system as well as stationary states of driven-dissipative systems.

#### A. Unitary dynamics of closed systems

In a first example we consider a one-dimensional closed system version of Eq. (6) with \( \omega_L = \Omega = \gamma = 0 \) and periodic boundary conditions that is initially prepared in a pure state with one excitation in every second site and none otherwise, \( |\psi_0\rangle = |1_2, 0_1, 0_3, 0_4, \ldots\rangle \). As this model has an exact solution\(^{20}\) we use it to test the accuracy of Eq. (5), see Appendix B for its explicit form for the model of Eq. (6).

For this homogeneous model with staggered initial conditions all initially occupied sites (denoted A-sites) and all initial empty sites (B-sites) have the same reduced density matrices \( \rho_A(t) \) respectively \( \rho_B(t) \). We thus denote operators acting on A-sites (B-sites) \( \sigma_A \) \( \sigma_B \). The exact result for \( \langle \sigma_A^I(\cdot)\sigma_A(\cdot) \rangle \) is shown in dashed black in Fig. 2(a). For the present setup, Eq. (5) leads to two coupled equations for \( \rho_A(t) \) and \( \rho_B(t) \). \( \text{Tr}\{\sigma_A^I(\cdot)\rho_A(t)\} \), as resulting from this c-MoP calculation is shown in dash-dotted brown in Fig. 2(a). One can also group two adjacent spins together and consider the resulting spin dimer as one subsystem described by a reduced density matrix \( \rho_{AB} \). This procedure leads to a cluster version of our approach where all clusters are here initially in the same state \| 1_A, 0_B \rangle. \( \text{Tr}\{\sigma_A^I(\cdot)\rho_{AB}(t)\} \), as resulting from this cluster c-MoP calculation is shown in dotted purple in Fig. 2(a). One can also consider larger clusters, e.g. \( \text{Tr}\{\sigma_A^I(\cdot)\rho_{ABCD}(t)\} \) for 4-site clusters described by \( \rho_{ABCD} \) and initially prepared in \| 1_A, 0_B, 1_C, 0_D \rangle. Results from 4-site and 8-site cluster c-MoP calculations are shown in dashed-dotted blue and solid red in Fig. 2(a).
We observe that the accuracy of our approach is excellent for short times, but as expected eventually deteriorates for longer times. The same properties can be seen for correlations $\langle \sigma_A^+ \sigma_B(t) \rangle$ in Fig. 2(b). Remarkably, the time range in which the approximation is highly accurate grows significantly as one applies the c-MoP approach to increasingly larger clusters. This tendency suggests that even substantially larger time ranges should become accessible as one increases the cluster size further. To appreciate this perspective one should compare the time range in which the approximation is highly accurate with $\tau = 40$, which is the largest time range that we are able to accurately describe here (although only for local quantities) to those reached with t-DMRG approaches ($Jt = 6$) on high performance computing clusters.\textsuperscript{20} We note that mean-field terms vanish in our example, $\text{Tr} \{\sigma_i \rho_n(t)\} = 0$ for all $t$, due to excitation number conservation. A mean-field calculation for clusters of $m$ lattice sites will thus be identical to the result for an $m$-site open boundary lattice, and therefore be inaccurate.

**B. Stationary states of driven-dissipative systems**

Instead of numerically integrating Eq. (5) we now focus on the physically very interesting scenario of steady states in driven-dissipative quantum many-body systems.\textsuperscript{14,15,21} We thus consider the model (6) with $\Omega \neq 0$ and $\gamma \neq 0$, assume translation invariance and periodic boundary conditions such that $\rho_n(t) = \rho_{n0}(t)$ for all $n$, and drop site indices in the following. Equation (5) can be simplified significantly if one is only interested in the steady state solution $\rho_{ss} = \lim_{t \to \infty} \rho_{n0}(t)$ since the action of the integral kernel $\kappa_{\langle n_0,n \rangle}(t,t')$ on $\rho_n(t') \otimes \rho_{n0}(t')$ vanishes for $|t-t'|$ sufficiently large. For $t \to \infty$ one can thus approximate $\rho_n(t') \approx \rho_{ss}$ in the right

hand side of Eq. (5) and extract an algebraic equation for $\rho_{ss}$ (see Appendix E),

$$0 = (\mathcal{L}_s^{\text{LT}} + \mathcal{L}_s^{\text{MF}} + \mathcal{L}_s^{\text{PP}}) \rho_{ss},$$

where $\mathcal{L}_s^{\text{LT}} \propto J_0$, $\mathcal{L}_s^{\text{MF}} \propto Z J_1$ and $\mathcal{L}_s^{\text{PP}} \propto Z J_2$. The explicit expressions for these time independent superoperators are given in the Eq. (E4,E5,E6,E7). Since $\mathcal{L}_s^{\text{MF}}$ and $\mathcal{L}_s^{\text{PP}}$ depend on $\rho_{ss}$, Eq. (7) is a nonlinear algebraic equation for $\rho_{ss}$, the roots of which can be found numerically with standard routines. One can follow the same procedure for subsystems formed by clusters of two, three or more adjacent lattice sites, which results in a cluster c-MoP calculation.

We test Eq. (7) for subsystems consisting of single sites and two-site clusters by comparing its solution, for which we denote single-site reduced density matrices by $\rho_{ss}^{\text{MF}}$ and $\rho_{ss}^{\text{MF-cl}}$ respectively, to t-DMRG integrations of Eq. (1) for the one-dimensional model in Eq. (6) with $N = 21$ lattice sites and open boundary conditions.\textsuperscript{14,31} From this t-DMRG numerics, which integrated Eq. (1) for a time range $T = 20/\gamma$ using a second order Trotter expansion with steps $\delta t = 10^{-3}/\gamma$, we extract the reduced density matrix for the central site $n_0 = 11$, denoted by $\rho_{ss}^{\text{DMRG}}$. For comparison we also consider mean-field results for single sites and two-site clusters, denoted by $\rho_{ss}^{\text{MF}}$ and $\rho_{ss}^{\text{MF-cl}}$ respectively, and results of standard perturbation theory to second order in the interaction, denoted by $\rho_{ss}^{\text{PT}}$, to show the dramatic quantitative and qualitative improvement of c-MoP over these approaches. For this purpose we compare expectation values of on-site observables and calculate the trace distance $D(\rho_1,\rho_2) = \frac{1}{2} \left| \rho_1 - \rho_2 \right|$ with $|A| = \sqrt{AA'^*}$ between the t-DMRG result, $\rho_1 = \rho_{ss}^{\text{DMRG}}$, and the approximations, $\rho_2 = \rho_{ss}^{\text{MF}}, \rho_{ss}^{\text{MF-cl}}, \rho_{ss}^{\text{PT}}, \rho_{ss}^{\text{C-MoP}}$ or $\rho_{ss}^{\text{C-MoP-cl}}$.

Figure 3(a) shows the occupation number $\text{Tr} \{ \sigma_A^+ \sigma_{ss} \}$, whereas Figs. 3(b) and (c) show the real and imaginary

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{(Color online) Application to unitary dynamics of closed quantum many-body systems. (a) $\langle \sigma_A^+ \sigma_A \rangle(t)$: exact value (dashed black), single-site c-MoP (dash-dotted brown), 2-site cluster c-MoP (dotted purple), 4-site cluster c-MoP (dash-dotted blue), and 8-site cluster c-MoP (solid red). (b) $\langle \sigma_A^+ \sigma_B \rangle(t)$: exact value (dashed black), 2-site cluster c-MoP (dotted purple), 4-site cluster c-MoP (dash-dotted blue), and 8-site cluster c-MoP (solid red). The real part of $\langle \sigma_A^+ \sigma_B \rangle(t)$ vanishes for all $t$.}
\end{figure}
parts of $\text{Tr}\{\sigma \rho_{ss}\}$ for t-DMRG, c-MoP and mean-field calculations. We find a very good agreement between the c-MoP results (red) and t-DMRG results (green), which again improves significantly for two-sites clusters (solid red lines) compared to individual lattice sites (dashed red lines). The mean-field results however deviate from the t-DMRG results to an extend which makes them unreliable over a large parameter range for both, single-site (black dotted) as well as two-site cluster (black dashed-dotted) versions. These findings are further illustrated by Fig. 3(d) which shows the distances $D(\rho_{ss}^{\text{DMRG}}, \rho_{ss}^{\text{MF}})$ (dotted black), $D(\rho_{ss}^{\text{DMRG}}, \rho_{ss}^{\text{c-MoP}})$ (dash-dotted black), $D(\rho_{ss}^{\text{DMRG}}, \rho_{ss}^{\text{c-MoP}})$ (dashed red), $D(\rho_{ss}^{\text{DMRG}}, \rho_{ss}^{\text{c-MoP}})$ (solid red), and $D(\rho_{ss}^{\text{DMRG}}, \rho_{ss}^{\text{PT}})$ (blue) for the same parameters as (a).

### C. Comparison of accuracy for stationary states of one- and two-dimensional lattices

To further elucidate the versatility of our approach, we here examine its accuracy for stationary states of two-dimensional lattices in comparison to one-dimensional chains. Fig. 4 compares the solutions of single-site c-MoP approach, a single-site mean-field approach and a numerically exact approach for small lattices of one ($Z = 2$) and two ($Z = 4$) dimensions. Trace distances between mean-field approximations and exact solutions are plotted in the upper row whereas the lower row shows trace distances between c-MoP approximations and exact solutions. Figs. 4(a) and (c) show $D(\rho_{ss}^{\text{c-MoP}}, \rho_{ss}^{\text{PT}})$ and $D(\rho_{ss}^{\text{c-MoP}}, \rho_{ss}^{\text{c-MoP}})$ respectively for $Z = 2$, $\Delta = 0.5\gamma$, and $N = 3$ with periodic boundary conditions as functions of $ZJ/\gamma$ and $\Omega/\gamma$. Figs. 4(b) and (d) in turn show $D(\rho_{ss}^{\text{c-MoP}}, \rho_{ss}^{\text{PT}})$ and $D(\rho_{ss}^{\text{c-MoP}}, \rho_{ss}^{\text{c-MoP}})$ respectively for $Z = 4$, $\Delta = 0.5\gamma$, and $N = 5$ with periodic boundary conditions as functions of $ZJ/\gamma$ and $\Omega/\gamma$. The lattice sizes $N = 3$ in one dimension and $N = 5$ in two dimensions are chosen because these are the minimal lattice sizes where each lattice site has distinct left and right neighbors which close the lattice in periodic boundary conditions in each dimension. Nonetheless both lattices are small enough to allow for full numerical solutions for their stationary states.

Firstly, we notice that there is no bistability for $\rho_{ss}^{\text{PT}}$ in the whole parameter range. In the bistable regions of the mean-field approximation we have here chosen the branch which is closer to the exact solution. Secondly, we find a remarkable quantitative improvement of c-MoP over mean-field especially for regions where the on-site parameter $\Omega$ is comparable to the tunneling, i.e. $\Omega \approx ZJ$. For $\Omega \gg ZJ$ both approximations become very good as the dynamics is dominated by the on-site Liouvillian
FIG. 4. (Color online) Performance of the method in terms of trace distances from exact solutions for small systems. (a) and (c): $D(\rho_{ss}^{1D}, \rho_{ss}^{MF})$ and $D(\rho_{ss}^{1D}, \rho_{ss}^{cM})$ respectively for $Z = 2, \Delta = 0.5\gamma$, and $N = 3$ with periodic boundary conditions as functions of $ZJ/\gamma$ and $\Omega/\gamma$. (b) and (d): $D(\rho_{ss}^{2D}, \rho_{ss}^{MF})$ and $D(\rho_{ss}^{2D}, \rho_{ss}^{cM})$ respectively for $Z = 4, \Delta = 0.5\gamma$, and $N = 5$ with periodic boundary conditions as functions of $ZJ/\gamma$ and $\Omega/\gamma$. In the bistable regions of the mean-field approximation we have chosen the branch which is closer to the exact solution.

In the opposite case of $ZJ \gg \Omega$ the steady state of Eq. (1) of the main text is close to the vacuum state which is here a product state leading to high accuracy for both approximations. Finally, we find that both approaches become more accurate for a two dimensional lattice, see Fig. 4(b) and (d), where mean-field however still remains unsatisfactory.

IV. CONCLUSIONS AND OUTLOOK

In conclusion we have derived an exact equation of motion for the reduced density matrices of subsystems of quantum many-body systems. When expanded in powers of the interaction between subsystems or clusters of subsystems, our equation leads to a highly efficient and very accurate approximation of the dynamics of local quantities. Although only of the same computational complexity, our equation is a significant qualitative and quantitative improvement of mean-field approaches and can also be straightforwardly extended to describe clusters of subsystems. The accuracy of the approach improves very fast as the size of the clusters is increased and the convergence of the results with increasing cluster size is a control handle for verifying their reliability. The method gives access to correlations of lengths below or equal to the cluster size but an extension to the calculation of longer range correlations is straightforward since the considered clusters can also be composed of non-adjacent subsystems.

For unitary dynamics it is capable of covering time ranges comparable to those accessible with cutting edge t-DMRG calculations for small cluster sizes already. Moreover it takes into account additional terms that appear to be highly relevant in two-dimensional lattices, which makes it a highly promising candidate for the description of these systems and their still illusive physics. For steady states, it reduces to a simple algebraic equation that provides a promising technique for exploring phase diagrams of driven-dissipative systems. When combined with techniques developed it will allow to efficiently compute output spectra of photonic quantum many-body systems. Some intriguing tasks for future research would be to extend the c-MoP approach to larger size clusters and to investigate higher order terms in the expansion of Eq. (4).
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Appendix A: Time dependent Mori projector and its relation to the theory of open quantum systems

We consider a physical setup with an N-partite structure, where we are only interested in the degrees of freedom of one subsystem of the full setup. Our approach is inspired by the theory of open quantum systems. We thus refer to our subsystem of interest as the “system” and to the remaining N−1 subsystems as the “environment”.

The latter will be traced out in our considerations. The dynamics of the entire setup is described by Eq. (1) of the main text, which we restate here for completeness,

\[ \dot{R}(t) = LR(t) = \left( \sum_{n=1}^{N} L_n + L_I \right) R(t) = \left( L_{n_0} + L_{\theta_0} + \sum_{n=1}^{Z} L_{\subset n_0, n>} + L_{Iy_0} \right) R(t). \] (A1)

To make the relation to open quantum system approaches more transparent, we have here grouped the superoperators into four parts. One part given by \( L_{n_0} \) only acts on the “system”, i.e. the subsystem of interest. Another part \( L_{\theta_0} \equiv \sum_{n=1, n \neq n_0} L_n \), which describes the dynamics of each of the remaining N−1 subsystems in the environment of the subsystem of interest. And finally, two parts contained in the superoperator \( L_I \), where \( \sum_{n=1}^{Z} L_{\subset n_0, n>} \) denotes the pairwise interaction of the subsystem without \( n_0 \) with \( Z \leq N-1 \) different subsystems \( n \), and where \( L_{Iy_0} \) accounts for any interaction between subsystems excluding the subsystem of interest. By assigning \( L_{Iy_0} \) to the interaction part \( L_I \), we treat all \( N \) subsystems on an equal footing. Moreover, we do not only consider unitary dynamics, where \( \mathcal{L}(\cdot) = -i[H_{\cdot}, \cdot] \). Nevertheless, all superoperators shall be the generators of completely positive, trace preserving maps of Lindblad type.

As we are only interested in “system” observables, it is sufficient to know the reduced density matrix of subsystem \( n_0 \) given by \( \rho_{n_0}(t) = \text{Tr}_{\bar{n}_0} \{ R(t) \} \). The guiding idea of our approach is thus to introduce a projector \( P \), similar to the Mori projector, which projects the full density matrix onto a relevant fraction \( R_{\text{rel}}(t) = PR(t) \) with \( P(\cdot) = \rho_{\theta_0} \otimes \text{Tr}_{\theta_0}(\cdot) \). In strong contrast to open system theory, we however introduce a time dependent projection operator defined in Eq. (2) of the main text, that is,

\[ P^{\text{no}}(\cdot) = \rho_{\theta_0}(t) \otimes \text{Tr}_{\theta_0}(\cdot), \] (A2)

where the density matrix \( \rho_{\theta_0}(t) \) is given by a factorized state of the reduced density matrices of all \( N-1 \) environmental constituents, i.e.

\[ \rho_{\theta_0}(t) = \bigotimes_{n \neq n_0} \rho_n(t) \text{ with } \rho_n(t) = \text{Tr}_{\bar{n}} R(t). \] (A3)

The term ‘relevant’ indicates that \( PR(t) \) contains all information needed to determine the exact expectation value of any system operator \( A_{n_0} \), i.e. \( \langle A_{n_0}(t) \rangle = \text{Tr}\{ A_{n_0} PR(t) \} \).

We emphasize that the environmental density matrix \( \rho_{\theta_0}(t) \) or rather each reduced matrix \( \rho_n(t) \) is determined consistently from the evolution \( R(t) \) of the full physical setup. Thus, we speak of a self-consistent Mori projector approach as both the reduced density matrix of the system \( \rho_n(t) \) and the environmental state \( \rho_{\theta_0}(t) \) are determined consistently with the full dynamics given by \( R(t) \). Similar to standard open system theory our derivation also employs the complement of \( P^{\text{no}}_t \) which projects out the irrelevant part of the density matrix \( R_{\text{irr}}(t) = Q^{\text{no}}_t R(t) \) and is given by

\[ Q^{\text{no}}_t(\cdot) = 1 - P^{\text{no}}_t(\cdot), \] (A4)

where \( \mathbb{1} \) is the identity mapping. As usual, we find the complementarity of the two subspaces \( R_{\text{rel}}(t) \) and \( R_{\text{irr}}(t) \), i.e. \( P^{\text{no}}_t Q^{\text{no}}_t = Q^{\text{no}}_t P^{\text{no}}_t = 0 \) and \( \mathbb{1} = P^{\text{no}}_t + Q^{\text{no}}_t \), and both projectors share the characteristic features \( (P^{\text{no}}_t)^2 = P^{\text{no}}_t \) and \( (Q^{\text{no}}_t)^2 = Q^{\text{no}}_t \).

Appendix B: Exact Nakajima-Zwanzig type equation for one subsystem

To derive an exact equation of motion for one subsystem, we first derive a set of equations for the two complements \( P^{\text{no}}_t R(t) \) and \( Q^{\text{no}}_t R(t) \) of the full density matrix \( R(t) \), then state a formal solution for the irrelevant part and finally deduce a closed equation for the relevant part. For time independent projectors, the analog of this equation is known as the Nakajima-Zwanzig equation. On our way we will exploit the identity \( \mathbb{1} = P^{\text{no}}_t + Q^{\text{no}}_t \), the full dynamics given by Eq. (A1) and the relation \( P^{\text{no}}_t Q^{\text{no}}_t = 0 \). We start with the equation of motion for the relevant fraction of the full density matrix

\[ \dot{R}_{\text{rel}} = \dot{P}^{\text{no}}_t R(t) + P^{\text{no}}_t \dot{R}(t) = \dot{P}^{\text{no}}_t \mathbb{1} R(t) + P^{\text{no}}_t \mathcal{L} R(t) = P^{\text{no}}_t (\mathbb{1} + Q^{\text{no}}_t \mathcal{L}) R(t) + P^{\text{no}}_t \mathcal{L} (P^{\text{no}}_t + Q^{\text{no}}_t) R(t) = (P^{\text{no}}_t + Q^{\text{no}}_t \mathcal{L}) P^{\text{no}}_t R(t) + P^{\text{no}}_t \mathcal{L} Q^{\text{no}}_t R(t). \] (B1)

In contrast to standard open system theory we do not only have a formal time dependence in the projectors but also a new term \( P^{\text{no}}_t P^{\text{no}}_t R(t) \) arising from the explicit time dependence of the environmental state \( \rho_{\theta_0}(t) \). Next, we use the operator equality \( Q^{\text{no}}_t = -\dot{P}^{\text{no}}_t \) to obtain an equation of motion for the irrelevant part of the full density matrix. In analogy to Eq. (B1) we find

\[ \dot{R}_{\text{irr}} = (-\dot{P}^{\text{no}}_t + Q^{\text{no}}_t \mathcal{L}) P^{\text{no}}_t R(t) + Q^{\text{no}}_t \mathcal{L} Q^{\text{no}}_t R(t). \] (B2)
We proceed with the treatment of the environmental density matrix and its time derivative. Therefore, we employ the properties of a trace preserving generator given for all superoperators from Eq. (A1). In particular, we will use that \(\text{Tr}_n \mathcal{L}_n(\cdot) = 0\), \(\forall n\) and find
\[
\dot{P}_t^{n_0} \rho_t^{n_0}(t) = \rho_{t_0}(t) \otimes \rho_{t_0}(t) = \sum_{m \neq n_0} \rho_{m}(t) \otimes \bigotimes_{n \neq m} \rho_{n}(t)
\]
\[
= \sum_{m \neq n_0} \text{Tr}_m \left( \hat{R}(t) \right) \otimes \bigotimes_{n \neq m} \rho_{n}(t)
\]
\[
= \sum_{m \neq n_0} \text{Tr}_m \left\{ \left( \sum_{j=1}^{N} \mathcal{L}_j + \mathcal{L}_t \right) R(t) \right\} \otimes \bigotimes_{n \neq m} \rho_{n}(t)
\]
\[
= \sum_{m \neq n_0} \left[ \text{Tr}_m \{ \mathcal{L}_t R(t) \} + \mathcal{L}_m \rho_m(t) \right] \otimes \bigotimes_{n \neq m} \rho_{n}(t)
\]
\[
= \mathcal{L}_{t_0} P_t^{n_0} R(t) + \mathcal{P}^{n_0} \mathcal{L}_t (P_t^{n_0} + Q_t^{n_0}) R(t),
\]
(B3)

where, in analogy to the projector defined in Eq. (A2), we have introduced the projector
\[
P_t^{n_0}(\cdot) \equiv \sum_{m \neq n_0} P_t^m(\cdot) = \sum_{m \neq n_0} \bigotimes_{n \neq m} \rho_{n}(t) \otimes \text{Tr}_m \{\cdot\}. \quad \text{(B4)}
\]

We observe that for an \(N\)-partite physical setup we have an ensemble of \(N\) projectors \(P_t^{n_0}\). By picking one part, \(n_0\), of the full setup as the "system" of interest, we have picked \(P_t^{n_0 = n_0} = P_t^{n_0}\) as our projector of interest. Now, we find a connection between all \(N\) projectors due to the explicit time dependence of the environmental reference state \(\rho_{t_0}(t)\). Note, that the projector \(P_t^{n_0}\), see Eq. (B4), depends on \(P_t^{n_0} R(t)\) via \(\rho_{t_0}(t) = \text{Tr}_t \{ P_t^{n_0} R(t) \}\). However, there is no dependence on \(Q_t^{n_0} R(t)\). This feature allows us to find a closed equation for the relevant part of the density matrix.

To this end we employ \(\mathcal{L}_{n_0} P_t^{n_0}(\cdot) = \mathcal{L}_{t_0} P_t^{n_0}(\cdot)\) and restate the equation of motion for the irrelevant part of the full density matrix. Starting from Eq. (B2) we end up with
\[
\frac{d}{dt} (Q_t^{n_0} R(t)) = \mathcal{C}_t \mathcal{L}_t P_t^{n_0} R(t)
\]
\[
+ (\mathcal{C}_t \mathcal{L}_t + \mathcal{L}_0) Q_t^{n_0} R(t), \quad \text{(B5)}
\]

where we utilize the abbreviations \(\mathcal{L}_0 = \sum_{n=1}^{N} \mathcal{L}_n\) and \(\mathcal{C}_t = -\mathcal{P}_t^{n_0} + Q_t^{n_0} = I - \sum_{n=1}^{N} P_t^{n}\). Here the projector \(\mathcal{C}_t\) projects onto the correlations contained in the object it is applied on only. Before we state the formal solution of Eq. (B5), it is convenient to introduce a shorthand notation for the time-propagator \(\hat{D}(t, t') = \hat{T} \exp \{ i \int_{t'}^{t} dt'' \left[ \mathcal{C}_t'' \mathcal{L}_t + \mathcal{L}_0'' \right] \}\) including the time-ordering operator \(\hat{T}\) which orders any product of superoperators such that the time arguments increase from right to left(10). By iteratively integrating Eq. (B5) we are able to cast the formal solution of \(Q_t^{n_0} R(t)\) for a given state \(R(t_0)\) at an initial time \(t_0\), into the form
\[
Q_t^{n_0} R(t) = \int_{t_0}^{t} dt' \hat{D}(t, t') \mathcal{C}_t' \mathcal{L}_t P_t^{n_0} R(t')
\]
\[
+ \hat{D}(t, t_0) Q_{t_0}^{n_0} R(t_0). \quad \text{(B6)}
\]

In the context of this work, we focus on a physical situation with a factorized initial state, i.e. \(Q_{t_0}^{n_0} R(t_0) = 0\). This assumption is natural for the driven-dissipative scenarios we here consider as one of the applications of our approach. Whenever the driving is switched on at time \(t_0\), the initial state of a dissipative quantum many body system is mostly given by the vacuum which is typically a factorized many body state. We insert the formal solution of Eq. (B5) into Eq. (B1) and trace over the environmental degrees of freedom to obtain the exact equation for the reduced density matrix of the system,
\[
\dot{\rho}_{n_0}(t) = \mathcal{L}_{t_0} \rho_{t_0}(t) + \sum_{n=1}^{Z} \text{Tr}_{t_0} \{ \mathcal{L}_{<n_0,n>} P_t^{n_0} R(t) \}
\]
\[
+ \sum_{n=1}^{Z} \text{Tr}_{t_0} \{ \mathcal{L}_{<n_0,n>} \int_{t_0}^{t} dt' \hat{D}(t, t') \mathcal{C}_t' \mathcal{L}_t P_t^{n_0} R(t') \}, \quad \text{(B7)}
\]

which is a closed equation for the relevant part of the full density matrix and identical to Eq. 4 in the main text. By tracing over the environment the term \(P_t^{n_0} R(t)\) in Eq. (B1) drops out as \(\text{Tr}_{t_0} \{ \hat{P}_t^{n_0} \} = 0\). Note, that we have also replaced \(\text{Tr}_{t_0} \mathcal{L}_t(\cdot)\) by \(\sum_{n=1}^{Z} \text{Tr}_{t_0} \mathcal{L}_{<n_0,n>} \). This can be easily understood as \(\mathcal{L}_{t_0}\), see Eq. (A1), is a generator of a trace preserving map acting solely on the environmental Hilbert space, and hence \(\text{Tr}_{t_0} \mathcal{L}_{t_0} = 0\). In the case of vanishing system-environment interaction, i.e. \(\mathcal{L}_{<n_0,n>} = 0\), we recover the free dynamics of the system being completely independent of the environment. Eq. (B7) may be viewed as a generalization of the prominent Nakajima-Zwanzig equation for open systems. In contrast to the standard Nakajima-Zwanzig equation there is however a dependence on \(P_t^{n_0} R(t)\) contained in the integral kernel and the dynamical map \(\hat{D}(t, t')\). We emphasize that this is a direct consequence of a time dependent or rather a self-consistent Mori projector ansatz.

Appendix C: Expansion and Born Approximation

Regardless of the chosen projector the resulting Nakajima-Zwanzig type equation is often exceedingly difficult to solve in full generality and approximations are usually needed. Obviously, equation (B7) can be expanded as a Dyson series in powers of the system-environment interaction \(\mathcal{L}_t\).
\[
\dot{\rho}_{n_0}(t) = \mathcal{L}_{n_0} \rho_{n_0}(t) + \sum_{n=1}^{Z} \text{Tr}_{d_0} \mathcal{L}_t P_{t}^{n_0} R(t) + \text{Tr}_{d_0} \mathcal{L}_t \int_{t_0}^{t} dt' e^{\mathcal{L}_0(t-t')} \mathcal{C}_t \mathcal{L}_t P_{t'}^{n_0} R(t') + \sum_{m=3}^{\infty} \mathcal{Y}_m, \tag{C1}
\]

where the \( m \)-th order term reads,

\[
\mathcal{Y}_m = \text{Tr}_{d_0} \mathcal{L}_t \int_{t_0}^{t} dt' \int_{t'}^{t} dt_{m-1} e^{\mathcal{L}_0(t-t_{m-1})} \mathcal{C}_{t_{m-1}} \mathcal{L}_t \int_{t'}^{t_{m-1}} dt_{m-2} e^{\mathcal{L}_0(t_{m-1}-t_{m-2})} \mathcal{C}_{t_{m-2}} \mathcal{L}_t \times \cdots \times \int_{t_{m-2}}^{t_{m-1}} dt_2 e^{\mathcal{L}_0(t_3-t_2)} \mathcal{C}_{t_2} \mathcal{L}_t e^{\mathcal{L}_0(t_2-t')} \mathcal{C}_t \mathcal{L}_t P_{t'}^{n_0} R(t'), \tag{C2}
\]

with a time ordering as \( t_0 \leq t' \leq t_2 \leq \ldots \leq t_{m-1} \leq t \). In order to understand the physical processes described by the \( m \)-th order correction, for \( m \geq 2 \), it is convenient to read Eq. (C2) from right to left. There are always \( m \) different chronologically ordered points in time \( \{t_0, t_1, \ldots, t_m\} \), with \( t_1 = t' \) and \( t_m = t \). At each point in time \( t_j \), for \( 1 \leq j \leq m-1 \), we find an interaction vertex described by the superoperator \( \mathcal{C}_t \mathcal{L}_t \). The interaction vertices are linked via interaction-free time evolution \( e^{\mathcal{L}_0(t_j+1-t_j)} \). At the very right end of Eq. (C2) we find the factorized state \( P_{t'}^{n_0} R(t') \) which, by maintaining the order in \( \mathcal{L}_t \), can be rewritten into \( P_{t'}^{n_0} R(t_1) = e^{\mathcal{L}_0(t_1-t_0)} P_{t'}^{n_0} R(t_0) = e^{\mathcal{L}_0(t_1-t_0)} R(t_0) \). Therefore, the term \( \mathcal{Y}_m \) describes a process where the initially factorized state \( R(t_0) \) evolves to the first vertex point at \( t_1 = t' \) where the action of the superoperator \( \mathcal{L}_t \) builds up correlations which are then projected out by the action of \( \mathcal{C}_t \). The resulting operator given by \( A \equiv \mathcal{C}_t \mathcal{L}_t e^{\mathcal{L}_0(t_1-t_0)} R(t_0) \) solely contains the correlated part between any constituents of the entire \( N \)-partite setup which have been built up by the action of \( \mathcal{L}_t \). For instance, if \( \mathcal{L}_t \) describes nearest-neighbor interactions on a lattice, then the resulting operator \( A \) would contain all correlations between any nearest-neighbor pair on the entire lattice.

Similarly, this process continues from one point in time \( t_j \) to the following one \( t_{j+1} \) successively building up correlations until the last point in time \( t = t_m \) is reached. The interaction vertex at the present time \( t = t_m \), however, is not described by the superoperator \( \mathcal{C}_t \mathcal{L}_t \) but rather by \( \text{Tr}_{d_0} \mathcal{L}_t \equiv \text{Tr}_{d_0} P_{t}^{n_0} \mathcal{L}_t \). Clearly, the superoperator \( P_{t}^{n_0} \mathcal{L}_t \) projects out the relevant part of the dynamics of the reduced density matrix \( \rho_{n_0}(t) \). In summary, we conclude that the \( m \)-th order corrections for \( m \geq 2 \) contain the influence of correlations which arise due to non-Markovian memory effects in the system-environment interaction, where the role of the system is taken by the subsystem with constituent number \( n_0 \). In turn, the terms up to first order in \( \mathcal{L}_t \) do neither contain any correlations nor non-Markovian memory effects. Hence, we expect a large improvement in the quality of the approximations by going from first order in \( \mathcal{L}_t \) to second order in \( \mathcal{L}_t \).

Motivated by these insights we apply the so-called Born approximation\(^{10}\) which takes all terms up to second order into account. We are able to directly formulate Eq. (B7) in Born approximation by dropping all terms proportional to \( \mathcal{L}_t \) from the exponent of the dynamical map \( \mathcal{D}(t, t') \) and finally arrive at an equation to which we will refer as the c-MoP equation,

\[
\dot{\rho}_{n_0}(t) = \mathcal{L}_{n_0} \rho_{n_0}(t) + \sum_{n=1}^{Z} \text{Tr}_n \{ \mathcal{L}_{<n_0,n>} \rho_n(t) \otimes \rho_{n_0}(t) \} + \sum_{n=1}^{Z} \text{Tr}_n \mathcal{L}_{<n_0,n>} \int_{t_0}^{t} dt' \mathcal{D}_{<n_0,n>}(t, t') \mathcal{C}_{t'}^{<n_0,n>} \mathcal{L}_{<n_0,n>} \rho_n(t') \otimes \rho_{n_0}(t'), \tag{C3}
\]

with \( \mathcal{D}_{<n_0,n>}(t, t') \equiv e^{(t-t') (\mathcal{L}_{n}+\mathcal{L}_{n_0})} \) for the dynamical map describing the free evolution of the “system” and the \( n \)-th constituent, and \( \mathcal{C}_{t'}^{<n_0,n>} \equiv 1 - \rho_n(t') \otimes \text{Tr}_n - \rho_{n_0}(t') \otimes \text{Tr}_{n_0} \). This equation is identical to equation (5) in the main text. Interestingly, in the Born approximation we only find “non-mixing” terms proportional to \( \mathcal{L}_{<n_0,n>} \mathcal{L}_{<n_0,m>} \delta_{n,m} \) and all possible terms containing \( \mathcal{L}_{t_{j_0}} \) vanish as well. In fact, this is the consequence of a factorized environmental reference state, see Eq. (A3).

### Appendix D: Application to a spin lattice

#### 1. Mean-field terms

The right hand side of Eq. (C3) shows three terms, which are zeroth, first and second order in \( \mathcal{L}_{<n_0,n>} \), respectively. The zeroth order term \( \mathcal{L}_{n_0} \rho_{n_0}(t) \) denotes the free evolution of the system which becomes exact for cases without interactions between the constituents,
\( \mathcal{L}_I = 0 \). The first and zeroth order terms taken together are equivalent to the well-known mean-field approximation. For the model given in Eq. (9) of the main text with \( \mathcal{L}_{<n_0>n}() = -i[H_I,()] \) and \( H_I = -J[\sigma_n \sigma_0^\dagger + \text{H.c.}] \), the equation of motion up to first order in \( J \) reads

\[
\dot{\rho}_{n_0}(t) = \mathcal{L}_{n_0}\rho_{n_0}(t)
\]

\[ + iJ \sum_{n=1}^{Z} [\sigma_{n_0} \text{Tr}_n\{\sigma_n^\dagger \rho_n(t)\} + \text{H.c.}, \rho_{n_0}(t)] \]

\[ = \mathcal{L}_{n_0}\rho_{n_0}(t) + iJ \sum_{n=1}^{Z} [\sigma_{n_0}(\sigma_n^\dagger)(t) + \text{H.c.}, \rho_{n_0}(t)]. \] (D1)

The same equation can be obtained by a formal replacement \( \sigma_n \sigma_0^\dagger \rightarrow \sigma_n(\sigma_0^\dagger) \) in the full dynamics given by Eq. (A1), where the operator-valued interaction \( \sigma_n \sigma_0^\dagger \) is replaced by a coupling of the system to a classical field \( \Phi \) with \( \Phi = \langle \sigma_0^\dagger \rangle \). We thus find that for a general setup described by Eq. (A1), the mean-field approximation can be understood as the first two leading terms of an exact Nakajima-Zwanzig type equation with a time dependent and self consistent Mori projector and that the second order term goes beyond mean-field implying that the choice of the time dependent Mori projector is well motivated.

2. Born Terms

Next, we want to determine the structure of the terms in Eq. (C3) which are of second order in \( \mathcal{L}_{<n_0>n} \). In the following, we will refer to these terms as the Born terms. We start with the two terms originating from the integrand \( \mathcal{D}(t,t')Q_{t'}^{n_0} \mathcal{L}_I P_{t'}^{n_0} R(t') \rightarrow (1 - \rho_n(t') \otimes \text{Tr}_n) \mathcal{L}_{<n_0>n} \rho_n(t') \otimes \rho_n(t') \), see Eq. (B7) and Eq. (C3). By choosing \( t_0 = 0 \) and substituting \( \tau = t - t' \), we find for the model given in Eq. (6) of the main text,

\[
\sum_{n=1}^{Z} \text{Tr}_{y_n} \mathcal{L}_{<n_0>n} \int_{t_0}^{t} dt' \mathcal{D}(t,t') Q_{t'}^{n_0} \mathcal{L}_I P_{t'}^{n_0} R(t') = -J^2 \sum_{n=1}^{Z} \sum_{j \in \{+,-\}} \int_{0}^{t} d\tau d^j_n(\tau,t) [\sigma_n^j, e^{\tau \mathcal{L}_n} [\sigma_n, \rho_n(t - \tau)]] + \text{H.c.}
\]

\[
- J^2 \sum_{n=1}^{Z} \sum_{j \in \{+,-\}} \int_{0}^{t} d\tau s^j_n(\tau,t) [\sigma_n^j, e^{\tau \mathcal{L}_n} \rho_n(t - \tau) \sigma_n] + \text{H.c.},
\] (D2)

with the correlation functions of the environment

\[
d^j_n(\tau,t) = \text{Tr}_n \{(\sigma_n^j)^{\dagger} e^{\tau \mathcal{L}_n} \sigma_n \rho_n(t - \tau)\} - \text{Tr}_n \{(\sigma_n^j)^{\dagger} e^{\tau \mathcal{L}_n} \sigma_n \rho_n(t - \tau)\} \text{Tr}_n \{\sigma_n^\dagger \rho_n(t - \tau)\},
\]

\[
s^j_n(\tau,t) = \text{Tr}_n \{(\sigma_n^j)^{\dagger} e^{\tau \mathcal{L}_n} [\sigma_n^\dagger, \rho_n(t - \tau)]\}
\] (D3)

and a sum \( \sum_{j \in \{+,-\}} \) which runs over all possible combinations of operators \( \sigma^j \equiv \sigma \) and \( \sigma^j \equiv \sigma^\dagger \). Note that we have here restricted \( \mathcal{L}_{<n_0,n>} \) to a unitary case which is in general not necessary.

Apart from the time dependence in the state of the environment \( \rho_n(t - \tau) \), these terms appear in the standard open system approach as well. Within the Born-Markov approximation they generate the standard Lindblad type terms which describe effects like spontaneous emission or incoherent thermal pumping. We emphasize that these terms are only effected by correlation functions of purely environmental type. In contrast, the Born term generated by the actual time dependence of the environmental state, hence by the integrand \( P_{t'}^{n_0} \mathcal{L}_I P_{t'}^{n_0} R(t) \) in Eq. (B7), displays a dependence on correlation functions of system type

\[
- J^2 \sum_{n=1}^{Z} \sum_{j \in \{+,-\}} \int_{0}^{t} d\tau d^j_n(\tau,t)[\sigma_n^j, e^{\tau \mathcal{L}_n} \rho_n(t - \tau)] + \text{H.c.},
\]

\[
- J^2 \sum_{n=1}^{Z} \sum_{j \in \{+,-\}} \int_{0}^{t} d\tau s^j_n(\tau,t)[\sigma_n^j, e^{\tau \mathcal{L}_n} \rho_n(t - \tau)] + \text{H.c.},
\] (D4)

where we find a correlation function which includes both system and environmental state dependence given by

\[
h^j_n(\tau,t) = i \text{Tr}_{n_0} \{(\sigma_{n_0}^j)^{\dagger} \rho_{n_0}(t - \tau)\} \times \text{Tr}_n \{\sigma_n^\dagger e^{\tau \mathcal{L}_n} [\sigma_n^\dagger, \rho_n(t - \tau)]\}. \] (D5)
indicating that the physical interaction process has to take a route starting from the system leading over to the environment and back to the system again.

**Appendix E: Steady State equations**

The Nakajima-Zwanzig equation has the structure of an integro-differential equation. The change of the density matrix \( \rho_{n_0}(t) \) depends not only on the current state \( \rho_{n_0}(t) \) but also on its past history \( \rho_{n_0}(t - \tau) \). Within the Markov approximation the integral kernel or rather the environmental correlation function can be very well approximated by a temporal delta function \( \delta(t - \tau) \) resolving the integro-differential structure. In our approach however, the reduced density matrices of the system and the environment are treated on an equal footing and correlation functions of the environment can not be expected to decay faster than the dynamics generated by the system-environment coupling which renders a Markov approximation inappropriate.

Under certain conditions, however, the equation for the steady state of the density matrix, i.e. \( \rho_{n_0}^{ss} \equiv \lim_{t \to \infty} \rho_{n_0}(t) \), does not exhibit the integro-differential structure. We emphasize that, so far, these are the first restrictions that we apply in the time-dependent and self-consistent Mori projector ansatz. We assume that the superoperator \( \mathcal{L}_{n_0} \) or rather \( \mathcal{L}_n \) shall describe a master equation in Lindblad form instead of a unitary evolution. Furthermore, it shall have a unique steady state to which all states relax in the limit \( t \to \infty \). For the model given in Eq. (6) of the main text, the existence of such a unique steady state for a truncated Hilbert space follows directly from Spohn’s theorem. Then, if such a unique steady state exists, the dynamical map \( e^{\tau \mathcal{L}_n} \) asymptotically maps all operators onto the same operator for each trace class, i.e. \( \lim_{\tau \to \infty} e^{\tau \mathcal{L}_n} A = \lim_{\tau \to \infty} e^{\tau \mathcal{L}_n} B \) for all operators \( A \) and \( B \) with \( \text{Tr} \{ A \} = \text{Tr} \{ B \} \). This implies that all commutators \( [A, B] \) vanish under the action of the dynamical map \( e^{\tau \mathcal{L}_n} \) in the long time limit since \( \lim_{\tau \to \infty} e^{\tau \mathcal{L}_n} AB = \lim_{\tau \to \infty} e^{\tau \mathcal{L}_n} BA \). Therefore, we can conclude that the correlation functions \( s(\tau, t - \tau) \) and \( h(\tau, t - \tau) \) given in Eq. (D3) and Eq. (D5), respectively, vanish to zero in the limit \( \tau \to \infty \) for all \( t \geq 0 \). The correlation function \( d(\tau, t - \tau) \) in Eq. (D2) vanishes as well in this limit, as it can be understood as the correlated part of a two-time correlation function. That is, it is of the form \( \langle A_{\eta_0}(t - \tau)A_{\eta_0}(t) \rangle - \langle A_{\eta_0}(t) \rangle \langle A_{\eta_0}(t - \tau) \rangle \), which vanishes for \( \tau \to \infty \) for relaxing systems that have a unique steady state.

The physical picture behind the assumption of a relaxing system is that the “environment” itself is an open system coupled to a Markovian bath. Intuitively, it can be understood that the memory of such an environment has a finite range into the past because all the information that reaches the Markovian bath is lost forever. Practically, it means that the integral kernels in all the Born terms vanish to zero for large values of \( \tau \). Hence, let \( t^* \) be the time after which the \( \rho_{n_0}(t) \) reach their steady state such that \( \rho_{n_0}(t - \tau) \approx \rho_{n_0}^{ss} \) for \( t - \tau > t^* \), and let \( \tau^* \) be the time for which \( s(\tau, t - \tau), h(\tau, t - \tau) \) and \( d(\tau, t - \tau) \) have decayed to zero. Provided we chose \( t^* > \tau^* > t^* \) and assume the existence of a steady state, which does not necessarily need to be unique, we can replace \( \rho_{n_0}(t - \tau) \to \rho_{n_0}^{ss} \) for all \( n \) in the limit \( t \to \infty \) for all \( \tau \) with \( 0 \leq \tau < t \). We obtain,

\[
0 = \mathcal{L}_{n_0}\rho_{n_0}^{ss} + \mathcal{L}_{n_0}^{MF}\rho_{n_0}^{ss} + \mathcal{L}_{n_0}^{BT}\rho_{n_0}^{ss},
\]

(E1)

with \( \mathcal{L}_{n_0}\rho_{n_0}^{ss} \) describing the free evolution of the system, the mean-field term \( \mathcal{L}_{n_0}^{MF}\rho_{n_0}^{ss} \) which is the first order correction in the system-environment coupling strength \( J \),

\[
\mathcal{L}_{n_0}^{MF}\rho_{n_0}^{ss} = iJ \sum_{n=1}^{Z} \left[ \sigma_{n_0} \text{Tr}_n \{ \sigma_{n_0}^\dagger \rho_{n_0}^{ss} \} + \text{H.c.} , \rho_{n_0}^{ss} \right],
\]

(E2)

and the Born terms beyond mean-field with steady-state dependent correlation functions,

\[
\mathcal{L}_{n_0}^{BT}\rho_{n_0}^{ss} = \sum_{n_1, n_2}^{Z} \sum_{j \in \{ - , + \}} \int_0^\infty \! d\tau \ a_j^\dagger (\tau, \rho_{n_0}^{ss}) \left[ \sigma_{n_0}^j, e^{\tau \mathcal{L}_{n_0}} [\sigma_{n_0}, \rho_{n_0}^{ss}] \right] + \text{H.c.}
\]

\[
- \sum_{n_1, n_2}^{Z} \sum_{j \in \{ - , + \}} \int_0^\infty \! d\tau \ s_j^\dagger (\tau, \rho_{n_0}^{ss}) \left[ \sigma_{n_0}, e^{\tau \mathcal{L}_{n_0}} \rho_{n_0}^{ss} \sigma_{n_0} \right] + \text{H.c.}
\]

\[
- i \sum_{n_1, n_2}^{Z} \sum_{j \in \{ - , + \}} \int_0^\infty \! d\tau \ h_j^\dagger (\tau, \rho_{n_0}^{ss}, \rho_{n_0}^{ss}) \left[ \sigma_{n_0}, e^{\tau \mathcal{L}_{n_0}} \rho_{n_0}^{ss} \right] + \text{H.c.}
\]

(E3)

Eq. (E1) is not closed, yet, as it still depends on the environmental steady state \( \rho_{n_0}^{ss} \), which is in strong contrast
to standard open system theory\textsuperscript{10} not an a priori given state. At this point, however, one could switch the part of the ”system” and the ”environment” and obtain two coupled but closed algebraic equations. Or alternatively, one could apply the self-consistency condition $\rho_n \cong \rho_{no}$ whenever it can be justified for the physical situation under study.

Setting $\rho^{ss}_n = \rho_{ss}$ and dropping site indices, $\sigma_n \equiv \sigma$ and $\mathcal{L}_n \equiv \mathcal{L}_{LT}$, for all $n$ we thus find the nonlinear algebraic equation (7) of the main text, where the individual terms read:

$$
\mathcal{L}_{LT}\rho_{ss} = -i \left[ \Delta \sigma^\dagger \sigma + \frac{\Omega}{2} (\sigma^\dagger + \sigma), \rho_{ss} \right] + \frac{\gamma}{2} (2 \sigma \rho_{ss} \sigma^\dagger - \sigma^\dagger \sigma \rho_{ss} - \rho_{ss} \sigma^\dagger \sigma)
$$

(E4)

for the local terms of order $J^0$,

$$
\mathcal{L}_{MF}^{ss}\rho_{ss} = i Z J [\sigma \text{Tr}\{\sigma^\dagger \rho_{ss}\} + \text{H.c.}, \rho_{ss}],
$$

(E5)

for the mean-field terms of order $J^1$, and,

$$
\mathcal{L}_{RT}^{ss}\rho_{ss} = \left\{ -i Z J^2 \sum_j \int_0^\infty d\tau \ h_j(\tau, \rho_{ss}) \ [\sigma, e^{\tau \mathcal{L}_{LT}} \rho_{ss}] ight. \\
- Z J^2 \sum_j \int_0^\infty d\tau \ s_j(\tau, \rho_{ss}) \ [\sigma^\dagger, e^{\tau \mathcal{L}_{LT}} \rho_{ss} \ \sigma] \\
- Z J^2 \sum_j \int_0^\infty d\tau \ d_j(\tau, \rho_{ss}) \ [\sigma^\dagger, e^{\tau \mathcal{L}_{LT}} [\sigma, \rho_{ss}]] \right\} + \text{H.c.}
$$

(E6)

for the Born terms of order $J^2$. The sum $\sum_j$ with $j \in \{-, +\}$ runs over all possible combinations of operators $\sigma^\dagger \equiv \sigma$ and $\sigma^\dagger \equiv \sigma^\dagger$. Moreover, the steady-state dependent correlation functions are given by

$$
\begin{align*}
{h}_j(\tau, \rho_{ss}) &= i \text{Tr}\{\langle \sigma^\dagger \rangle^j \rho_{ss}\} \text{Tr}\{\sigma^j e^{\tau \mathcal{L}_{LT}} [\sigma^j, \rho_{ss}]\} \\
{s}_j(\tau, \rho_{ss}) &= \text{Tr}\{\langle \sigma^\dagger \rangle^j e^{\tau \mathcal{L}_{LT}} [\sigma^\dagger, \rho_{ss}]\} \\
d_j(\tau, \rho_{ss}) &= \text{Tr}\{\langle \sigma^\dagger \rangle^j e^{\tau \mathcal{L}_{LT}} \sigma^\dagger \rho_{ss}\} \\
&\quad - \text{Tr}\{\langle \sigma^\dagger \rangle^j e^{\tau \mathcal{L}_{LT}} \rho_{ss}\} \text{Tr}\{\sigma^\dagger \rho_{ss}\}.
\end{align*}
$$

(E7)

Appendix G: Bistability within single-site mean-field solutions

For the stationary states we find a bistability in the single-site mean-field solution due to the non-linear character of the mean-field equation, which is no longer present in the two-site cluster version, see Fig. 3 of the main text. The c-MoP equation [Eq. (7) of the main text] which is a non-linear algebraic equation as well, does not exhibit bistable behavior for any of the considered variants in the whole parameter range of our study. This is in agreement with Spohn’s theorem\textsuperscript{13,14} which suggests that the dynamics of a Lindblad type equation of motion\textsuperscript{25} just like Eq. (1) of the main text with the $\mathcal{L}_n$ and $\mathcal{L}_I$ as specified in Eq. (6), relaxes to a unique steady-state. Hence bistabilities resulting from mean-field calculation\textsuperscript{25} can in general not be attributed a physical existence.

\footnotesize
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