Mathematical Simulation of the Problem of the Pre-Critical Sandwich Plate Bending in Geometrically Nonlinear One Dimensional Formulation

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Abstract. In this paper we consider the geometrically nonlinear problem of determining the equilibrium position of a sandwich plate consisting of two external carrier layers and located between transversely soft core, connected with carrier layer by means of adhesive joint. We investigate the generalized statement of the problem. For its numerical implementation we offer a two-layer iterative process and investigate the convergence of the method. Numerical experiments are carried out for the model problem.

1. Introduction

Multilayer structures, in particular plates, are widely used in various fields of modern technology, aerospace, aviation, shipbuilding; industrial, civil and transport construction, chemical and power engineering [1–6]. The interest in layered plates associated primarily with the fact that they have a set of properties and features that are qualitatively different from conventional constructions. In this paper the construction and study of a generalized statement of geometrically nonlinear problem of the bending of sandwich plate with transversal-soft core are carried out. To solving the problem an iterative method is proposed and its convergence is investigated. In the study correctness of a generalized statement and the convergence of method we use the pseudomonotone operators apparatus. Note that the generalized statement of physically non-linear and geometrically linear problem in the form of saddle problems, as well as a method for its solution were considered in [7–10]. The study of nonlinear problems of the shells theory, including the approximate methods for their solution in [11–18] is carried out.

2. Problem statement

In this paper we consider the problem of determining the stress-strain state of an infinitely long sandwich plate with transversal-soft core. The width of the plate is \(a\), the core thickness is \(2h\), the thickness of the carrier layers are equal \(2h_k\), where \(k\) is the number of the layer. To describe the stress-strain state in the carrier layers are used the equations of Kirchhoff-Love model, in the core the equations of elasticity theory, simplified by the accepted model of transversely soft layer and integrated over its thickness with the satisfaction of the of conjugation conditions of the layers on the
displacements [19–21]. According to [19–21], we introduce the following notations (here and in what follows we assume that \( k = 1, 2 \)), \( X^1_{(k)}, X^3_{(k)} \) are surface load components reduced to the middle surface of the \( k \)-th layer, \( u^{(k)} \) and \( w^{(k)} \) are bending and axial displacements of the middle surface points of \( k \)-th layer, respectively, \( T_{11}^{(k)}, M_{11}^{(k)} \) are the inner membrane forces and bending moments in the \( k \)-th layer respectively. The plate edges are assumed fixed, so that the conditions \( u^{(k)}(x) = 0, \ d w^{(k)}/dx = 0 \) are satisfied at \( x = 0, \ x = a \). Let us consider the problem in geometrically nonlinear statement, i.e.,

\[
T_{11}^{(k)} = B_{(k)} \left( d u^{(k)}/dx + \frac{1}{2} (d w^{(k)}/dx)^2 \right),
\]

\[
M_{11}^{(k)} = D_{(k)} q^2 w^{(k)} / dx^2,
\]

where \( B_{(k)} = 2 h_{(k)} E^{(k)} / (1 - \nu_{12}^{(k)} \nu_{21}^{(k)}) \) is the stiffness of \( k \)-th layer on the tension-compression, \( E^{(k)} \) and \( \nu_{12}^{(k)}, \nu_{21}^{(k)} \) are the first-order elastic modulus and Poisson's coefficients of the material of the carrier \( k \)-th layer, \( D_{(k)} = B_{(k)} h_{(k)}^2 / 3 \) is the bending stiffness of \( k \)-th layer. Let \( U = (w^{(1)}, w^{(2)}, u^{(1)}, u^{(2)}) \) be a displacements vector of points of the middle surface of the \( k \)-th layer, \( q^1 \) be a tangential stresses in the core. For \( q^1 \) we assume the boundary conditions

\[
q^1(0) = q^1(a) = 0
\]

are satisfied. In [19–21] to describe the stress-strain state of sandwich plate following strain energy functional was constructed

\[
L(U, q^1) = P(U, q^1) - A_s(U, q^1) - A_q(U, q^1),
\]

where

\[
P(U, q^1) = \frac{1}{2} \int_0^a \left\{ \sum_{k=1}^2 \left[ B_{(k)} \left( d u^{(k)}/dx + \frac{1}{2} (d w^{(k)}/dx)^2 \right)^2 + D_{(k)} (d^2 w^{(k)}/dx^2)^2 \right] + c_1 (q^1)^2 + c_2 (d q^1/dx)^2 + c_3 (w^{(2)} - w^{(1)})^2 \right\} dx
\]

is the strain potential, \( c_1 = 2h / G_{13}, \ c_2 = 2h^3 / 3 E_3, \ c_3 = E_3 / (2h), \ G_{13}, E_3 \) are the transverse shear and compression modules of a core,

\[
A_s(U, q^1) = \int_0^a \sum_{k=1}^2 \left[ X^1_{(k)} u^{(k)} + M^1_{(k)} d w^{(k)}/dx + X^3_{(k)} w^{(k)} \right] dx
\]

is the work of given external forces and moments, \( M^1_{(k)} \) is the surface moment of external forces, reduced to the middle surface of the \( k \)-th layer, and

\[
A_q(U, q^1) = \int_0^a \left[ (u^{(1)} - u^{(2)}) - \sum_{k=1}^2 H_{(k)} d w^{(k)}/dx + c_1 q^1 - c_2 d q^1/ dx^2 \right] q^1 dx
\]

is the work of unknown tangential stresses on corresponding displacements. It has been established [22] that the solution of the problem of the sandwich plate equilibrium is a stationary point of the functional \( L \).
3. Generalized statement of the problem

Let $V_k = W^{(k)}_2 ((0, a))$ be the Sobolev spaces $[23, 24]$ with inner products $(u, \eta)_k = \int_0^a d^k u / dx^k \cdot d^k \eta / dx^k \, dx$, $V_q$ be the Sobolev space of functions with compact support on $(0, a)$ having the first generalized derivative, integrable with the square, with the inner product $(\gamma, \zeta)_\gamma = \int_0^a [c_2 y(x) \gamma(x) + c_3 d \gamma / dx \, d \zeta / dx ] \, dx$. We denote the inner product in $V$ through $(\cdot, \cdot)_V$.

Using the Sobolev embedding theorem and the Rellich-Kondrashov theorem $[23, 24]$, it is easy to verify that the functional $L$ is well defined on $q V V$ $\times$ $\times$. We obtain the equations for the stationary points of the functional $L$. For this purpose we find the Gateaux its derivatives $[24, 25]$. Let us rewrite the functional $L$ in the form $L(U, q^j) = \Phi_0(U) + \Phi_1(U, q^j) - \Phi_2(q^j)$, where

\[
\Phi_0(U) = \frac{1}{2} \int_0^a \left\{ \sum_{k=1}^2 B_{(k)} \left( \frac{d u^{(k)}}{dx} + \frac{1}{2} \left( \frac{d w^{(k)}}{dx} \right)^2 \right)^2 + D_{(k)} \left( \frac{d^2 w^{(k)}}{dx^2} \right)^2 \right\} \, dx + c_3 (w^{(2)} - w^{(1)})^2 \, dx - \int_0^a \sum_{k=1}^2 \left[ X_{(k)}^1 u^{(k)} + M_{(k)}^1 d w^{(k)} / dx + X_{(k)}^3 w^{(k)} \right] \, dx,
\]

\[
\Phi_1(U, q^j) = \int_0^a \left[ \sum_{k=1}^2 H_{(k)} \frac{d w^{(k)}}{dx} + (u^{(2)} - u^{(1)}) \right] q^j \, dx, \quad \Phi_2(q^j) = \frac{1}{2} \int_0^a \left\{ c_1 (q^j)^2 + c_2 \left( \frac{d q^j}{dx} \right)^2 \right\} \, dx
\]

Defining the Gateaux derivatives we have that the stationary points of the functional $L$ are the solutions of variational equations (integral identities)

\[
\frac{a}{2} \int_0^a \left[ \sum_{k=1}^2 B_{(k)} \left( \frac{d u^{(k)}}{dx} + \frac{1}{2} \left( \frac{d w^{(k)}}{dx} \right)^2 \right)^2 \right] \frac{d \eta^{(k)}}{dx} \, dx + \frac{a}{2} \int_0^a \sum_{k=1}^2 B_{(k)} \left( \frac{d u^{(k)}}{dx} + \frac{1}{2} \left( \frac{d w^{(k)}}{dx} \right)^2 \right) \frac{d w^{(k)}}{dx} \, dx + \frac{a}{2} \int_0^a d^2 w^{(k)} / dx^2 \, dx + \frac{a}{2} \int_0^a \sum_{k=1}^2 \left[ X_{(k)}^1 \eta^{(k)} + M_{(k)}^1 d \eta^{(k)} / dx + X_{(k)}^3 \eta^{(k)} \right] \, dx
\]

\[
+ c_3 \int_0^a (w^{(2)} - w^{(1)}) (z^{(2)} - z^{(1)}) \, dx + \int_0^a \left\{ \sum_{k=1}^2 H_{(k)} \frac{d z^{(k)}}{dx} + (\eta^{(2)} - \eta^{(1)}) \right\} q^j \, dx = 0 \quad \forall Z \in V,
\]

\[
\int_0^a \left\{ \sum_{k=1}^2 H_{(k)} \frac{d w^{(k)}}{dx} + (u^{(2)} - u^{(1)}) + c_2 q^j \right\} y + c_2 d q^j / dx \, dy / dx \, dx = 0 \quad \forall y \in V_i
\]

Summing the integral identities, we obtain the variational equation

\[
b((U, q^j), (Z, y)) = ((f_1, 0), (Z, y))_{y \times q} \quad \forall (Z, y) \in W = V \times V_q.
\]

Again using the Sobolev embedding theorem and the Rellich-Kondrashov theorem $[23, 24]$ it is easy to verify that the form $b(\cdot, \cdot)$ defined on $W \times W$ is linear and bounded to the second argument, and therefore, generates an operator $A : W \to W$, the right side (9) generates an element $F \in W$. Thus (3) can be written as operator equation

\[
A(U, q^j) = F.
\]
Recall that the operator \( A_0 : V \to V' \) is called pseudomonotone \([26, 27]\) if it is bounded and the weak convergence of the sequence \( \{ u_k \}_{k=1}^{\infty} \) in \( V \) to \( u^* \) and the inequality
\[
\limsup_{k \to +\infty} \langle A_0 u_k, u_k - u^* \rangle \leq 0
\]
imply that the following relationship holds
\[
\liminf_{k \to +\infty} \langle A_0 u_k, u_k - \eta \rangle \geq \langle A_0 u^*, u^* - \eta \rangle
\]
for all \( \eta \). The following result holds.

**Theorem 1.** Operator \( A \) is continuous, coercive and pseudomonotone.

From Theorem 1 and general results of the theory of monotone operators \([25, 26]\) it implies that the following theorem is true.

**Theorem 2.** The problem (10) has at least one solution.

### 4. Iterative method

To solve the problem (10) in analogy to \([28–32]\) we propose the following iteration process has been. Let \( (U_n, q_n^1) \) be any element from \( W \). For \( n = 0, 1, 2, \ldots \) we find \( (U_n, q_n^1) \) as a solution of the problem

\[
J((U_{n+1}, q_{n+1}^1) - (U_n, q_n^1)) = \tau(F - A(U_n, q_n^1)), \tag{11}
\]

where \( J : W \to W \) is a duality operator \([26]\), \( \tau > 0 \) is an iterative parameter.

Following \([33–36]\), we can prove the following theorem on the convergence of the iterative method.

**Theorem 3.** There is a constant \( \tau_0 \) such that for \( 0 < \tau < \tau_0 \) the sequence \( (U_n, q_n^1) \) weakly converges to some solution of the problem (10) as \( n \to +\infty \).

The proposed method (11) for solving the problem have been implemented numerically. Software package in Matlab environment was developed and carried out calculations for the model problem. For the model problem, numerical experiments were carried out. Iteration parameter was chosen empirically. The calculations were performed for the following characteristics: \( a = 1 \) cm, \( h_{(1)} = h_{(2)} = 0.005 \) cm, \( h = 0.05 \) cm, \( G_{13} = 15 \) MPa, \( E_3 = 25 \) MPa, \( X_{(1)}^3 = 0.0319 \) MPa, \( X_{(2)}^3 = 0 \), \( E^{(k)} = 7 \cdot 10^4 \) MPa, \( \nu_{(1)}^{(k)} = \nu_{(2)}^{(k)} = 0.3 \), \( X_{(k)}^1 = 0 \), \( M_{(k)}^1 = 0 \), \( k = 1, 2 \). The results of numerical experiments are shown in figures 1–3.

![Figure 1. Axial displacements in carrier layers.](image-url)

It should be noted that formulated for \( q^1 \) boundary conditions (1) correspond to the presence of, at the edges \( x = 0 \), \( x = a \) the diaphragms, this leads to the formation of the maximum transverse
tangential stresses in sections the filler a distance of about its thickness $2h$, as observed in figure 2. The limiting the free end sections $x = 0$, $x = a$ displacements in axial direction leads to the formation to the formation of a significant in magnitude membrane forces $T_{(1)}^{11}$, $T_{(2)}^{11}$ in the carrying layers $T_{(1)}^{11}$, $T_{(2)}^{11}$, the force $T_{(i)}^{11}$ of which is a contraction in cross-section $x = a/2$. For this reason, in the neighborhood of this section should be expected buckling carrier layers in a mixed form. It is easy to verify that the first two equations of (1) implies that $T_{(1)}^{11} + T_{(2)}^{11} = \text{const}$, in the implementation of which can be verified on the basis of the results shown in figure 3.

![Figure 2. Tangential stresses in core.](image1)

![Figure 3. Membrane forces in carrier layers.](image2)

5. Conclusion
In this paper, we construct a generalized statement for the geometrically nonlinear problem of the bending of a sandwich plate in a one-dimensional formulation in the form of an operator equation in the Sobolev space. The solvability of the equation is proved on the basis of general results of the monotone operators theory. To solve the problem, a two-layer iterative process is proposed and its convergence is investigated. The results of numerical experiments, their correspondence to the physical picture of the problem, showed both the adequacy of the mathematical model and the effectiveness of the proposed iterative method. The results of the work can be used in the design of the sandwich structures.
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