Characterizations of some classes of finite $\sigma$-soluble $P\sigma T$-groups

-Dedicated to Professor J.C. Beidleman on the occasion of his 80-th birthday

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Abstract

Let $\sigma = \{\sigma_i|i \in I\}$ be some partition of the set of all primes $\mathbb{P}$ and $G$ a finite group. $G$ is said to be $\sigma$-soluble if every chief factor $H/K$ of $G$ is a $\sigma_i$-group for some $i = i(H/K)$.

A set $\mathcal{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ if every member $H \neq 1$ of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $i \in I$ such that $\sigma_i \cap \pi(G) \neq \emptyset$. A subgroup $A$ of $G$ is said to be $\sigma$-permutable in $G$ if $G$ has a complete Hall $\sigma$-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$.

We obtain characterizations of finite $\sigma$-soluble groups $G$ in which $\sigma$-permutability is a transitive relation in $G$.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi = \{p_1, \ldots, p_n\} \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If $n$ is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing $n$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$. $G$ is said to be a $D_\pi$-group if $G$ possesses a Hall $\pi$-subgroup $E$ and every $\pi$-subgroup of $G$ is contained in some conjugate of $E$.

In what follows, $\sigma$ is some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i|i \in I\}$, where $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; $\Pi$ is always supposed to be a subset of the set $\sigma$ and $\Pi' = \sigma \setminus \Pi$.

By the analogy with the notation $\pi(n)$, we write $\sigma(n)$ to denote the set $\{\sigma_i|\sigma_i \cap \pi(n) \neq \emptyset\}$; $\sigma(G) = \sigma(|G|)$. $G$ is said to be: $\sigma$-primary [1] if $G$ is a $\sigma_i$-group for some $i$; $\sigma$-decomposable [2] or $\sigma$-nilpotent [3] if $G = G_1 \times \cdots \times G_n$ for some $\sigma$-primary groups $G_1, \ldots, G_n$; $\sigma$-soluble [1] if every chief factor of $G$ is $\sigma$-primary; a $\sigma$-full group of Sylow type [1] if every subgroup $E$ of $G$ is a $D_{\sigma_i}$-group for every $\sigma_i \in \sigma(E)$. Note in passing, that every $\sigma$-soluble group is a $\sigma$-full group of Sylow type [4].

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A set \( \mathcal{H} \) of subgroups of \( G \) is a **complete Hall \( \sigma \)-set** of \( G \) \([\mathbb{I}, \mathbb{J}]\) if every member \( \neq 1 \) of \( \mathcal{H} \) is a Hall \( \sigma_i \)-subgroup of \( G \) for some \( \sigma_i \in \sigma \) and \( \mathcal{H} \) contains exactly one Hall \( \sigma_i \)-subgroup of \( G \) for every \( \sigma_i \in \sigma(G) \).

Recall also that a subgroup \( A \) of \( G \) is said to be **\( \sigma \)-subnormal** in \( G \) \([\mathbb{I}]\) if there is a subgroup chain

\[
A = A_0 \leq A_1 \leq \cdots \leq A_n = G
\]
such that either \( A_{i-1} \trianglelefteq A_i \) or \( A_i/(A_{i-1})_{A_i} \) is \( \sigma \)-primary for all \( i = 1, \ldots, n \).

**Definition 1.1.** We say that a subgroup \( A \) of \( G \) is said to be **\( \sigma \)-quasinormal** or **\( \sigma \)-permutable** in \( G \) if \( G \) possesses a complete Hall \( \sigma \)-set and \( A \) permutes with each Hall \( \sigma_i \)-subgroup \( H \) of \( G \), that is, \( AH = HA \) for all \( i \in I \).

**Remark 1.2.** Using Theorem B in \([\mathbb{I}]\), it is not difficult to show that if \( G \) possesses a complete Hall \( \sigma \)-set \( \mathcal{H} \) such that \( AH^x = H^xA \) for all \( H \in \mathcal{H} \) and all \( x \in G \), then \( A \) is \( \sigma \)-permutable in \( G \).

**Remark 1.3.** (i) In the classical case when \( \sigma = \sigma^0 = \{2, 3, \ldots\} \): \( G \) is \( \sigma^0 \)-soluble (respectively \( \sigma^0 \)-nilpotent) if and only if \( G \) possesses a complete Hall \( \sigma \)-set and \( A \) permutes with each Hall \( \sigma_i \)-subgroup \( H \) of \( G \), that is, \( AH = HA \) for all \( i \in I \).

(ii) In the other classical case when \( \sigma = \sigma^\pi = \{\pi, \pi'\} \): \( G \) is \( \sigma^\pi \)-soluble (respectively \( \sigma^\pi \)-nilpotent) if and only if \( G \) possesses a complete Hall \( \pi \)-set and \( A \) permutes with each Hall \( \pi_i \)-subgroup \( H \) of \( G \), that is, \( AH = HA \) for all \( i \in I \).

(iii) In fact, in the theory of \( \pi \)-soluble groups (\( \pi = \{p_1, \ldots, p_n\} \)) we deal with the partition \( \sigma = \sigma^{0\pi} = \{\pi_1, \ldots, \pi_n, \pi'\} \) of \( \mathbb{P} \). Note that \( G \) is \( \sigma^{0\pi} \)-soluble (respectively \( \sigma^{0\pi} \)-nilpotent) if and only if \( G \) is \( \pi \)-soluble (respectively \( G = O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G) \)). A subgroup \( A \) of a \( \pi \)-soluble group \( G \) is \( \sigma^{0\pi} \)-permutable in \( G \) if and only if \( A \) permutes with all Hall \( \pi' \)-subgroups and with all Sylow \( p \)-subgroups of \( G \) for all \( p \in \pi \). Note also that a subgroup \( A \) of \( G \) is \( \sigma^{0\pi} \)-subnormal in \( G \) if and only if it is **\( \mathfrak{S} \)-subnormal** in \( G \) in the sense of Kegel \([\mathbb{J}]\), where \( \mathfrak{S} \) is the class of all \( \pi' \)-groups.

We say that \( G \) is a **\( P\sigma T \)-group** \([\mathbb{I}]\) if \( \sigma \)-permutability is a transitive relation in \( G \), that is, if \( K \) is a \( \sigma \)-permutable subgroup of \( H \) and \( H \) is a \( \sigma \)-permutable subgroup of \( G \), then \( K \) is a \( \sigma \)-permutable subgroup of \( G \). In the case when \( \sigma = \sigma^0 \), a \( P\sigma T \)-group is also called a **\( PST \)-group** \([\mathbb{J}]\). Note that if \( G = (Q_8 \times C_3) \wr (C_7 \times C_3) \) (see \([\mathbb{J}] \text{ p. 50}\)), where \( Q_8 \times C_3 = SL(2, 3) \) and \( C_7 \times C_3 \) is a non-abelian group of order 21, then \( G \) is not \( PST \)-group but \( G \) is a \( P\sigma T \)-group, where \( \sigma = \{\{2, 3\}, \{2, 3\}'\} \).

The description of \( PST \)-groups was first obtained by Agrawal \([\mathbb{I}0]\), for the soluble case, and by Robinson in \([\mathbb{I}I]\), for the general case. In the further publications, authors (see, for example, the
recent papers [12]–[22] and Chapter 2 in [6]) have found out and described many other interesting characterizations of soluble $\mathcal{PST}$-groups.

The purpose of this paper is to study $\sigma$-soluble $\mathcal{PST}$-groups in the most general case (i.e., without any restrictions on $\sigma$). In view of Theorem B in [1], $G$ is a $\mathcal{PST}$-group if and only if every $\sigma$-subnormal subgroup of $G$ is $\sigma$-permutable. Being based on this result, here we prove the following revised version of Theorem A in [1].

**Theorem A.** Let $D = G^{\mathcal{N}_\sigma}$. If $G$ is a $\sigma$-soluble $\mathcal{PST}$-group, then the following conditions hold:

(i) $G = D \rtimes M$, where $D$ is an abelian Hall subgroup of $G$ of odd order, $M$ is $\sigma$-nilpotent and every element of $G$ induces a power automorphism in $D$;

(ii) $O_{\sigma_i}(D)$ has a normal complement in a Hall $\sigma_i$-subgroup of $G$ for all $i$.

Conversely, if Conditions (i) and (ii) hold for some subgroups $D$ and $M$ of $G$, then $G$ is a $\mathcal{PST}$-group.

In this theorem, $G^{\mathcal{N}_\sigma}$ denotes the $\sigma$-nilpotent residual of $G$, that is, the intersection of all normal subgroups $N$ of $G$ with $\sigma$-nilpotent quotient $G/N$.

**Corollary 1.4.** If $G$ is a $\sigma$-soluble $\mathcal{PST}$-group, then every quotient and every subgroup of $G$ are $\mathcal{PST}$-groups.

In the case when $\sigma = \sigma^0$, we get from Theorem A the following

**Corollary 1.5.** Let $D = G^{\mathcal{N}_\sigma}$ be the nilpotent residual of $G$. If $G$ is a soluble $\mathcal{PST}$-group, then $D$ is an abelian Hall subgroup of $G$ of odd order and every element of $G$ induces a power automorphism in $D$.

In the case when $\sigma = \sigma^\pi$, we get from Theorem A the following

**Corollary 1.6.** $G$ is a $\pi$-separable $\mathcal{PST}$-group if and only if the following conditions hold:

(i) $G = D \rtimes M$, where $D$ is an abelian Hall subgroup of $G$ of odd order, $M$ is $\pi$-decomposable and every element of $G$ induces a power automorphism in $D$;

(ii) $O_\pi(D)$ has a normal complement in a Hall $\pi$-subgroup of $G$;

(iii) $O_{\pi'}(D)$ has a normal complement in a Hall $\pi'$-subgroup of $G$.

In the case when $\sigma = \sigma^{0\pi}$, we get from Theorem A the following

**Corollary 1.7.** $G$ is a $\pi$-soluble $\mathcal{PST}$-group if and only if the following conditions hold:

(i) $G = D \rtimes M$, where $D$ is an abelian Hall subgroup of $G$ of odd order, $M = O_{p_1}(M) \times \cdots \times O_{p_n}(M) \times O_{\pi'}(M)$ and every element of $G$ induces a power automorphism in $D$;

(ii) $O_{\pi'}(D)$ has a normal complement in a Hall $\pi'$-subgroup of $G$.

A natural number $n$ is said to be a $\Pi$-number if $\sigma(n) \subseteq \Pi$. A subgroup $A$ of $G$ is said to be: a Hall $\Pi$-subgroup of $G$ [1] if $|A|$ is a $\Pi$-number and $|G : A|$ is a $\Pi'$-number; a $\sigma$-Hall subgroup of $G$ if $A$ is a Hall $\Pi$-subgroup of $G$ for some $\Pi \subseteq \sigma$. 

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The proof of Theorem A is based on many results and observations of the paper [1]. We use also the remarkable result of the paper by Alejandre, Ballester-Bolinches and Pedraza-Aguilera [14] (see also Theorem 2.1.8 in [6]) that in a soluble PST-group $G$ any two isomorphic chief factors are $G$-isomorphic. Finally, in the proof of Theorem A, the following fact is useful, which is possibly independently interesting.

**Theorem B.** Let $G$ have a normal $\sigma$-Hall subgroup $D$ such that: (i) $G/D$ is a $P\sigma T$-group, and (ii) every $\sigma$-subnormal subgroup of $D$ is normal in $G$. If $G$ is a $\sigma$-full group of Sylow type, then $G$ is a $P\sigma T$-group.

**Corollary 1.8** (See Theorem A in [1]). Let $G$ have a normal $\sigma$-Hall subgroup $D$ such that: (i) $G/D$ is $\sigma$-nilpotent, and (ii) every subgroup of $D$ is normal in $G$. Then $G$ is a $P\sigma T$-group.

In the case when $\sigma = \sigma^0$, we get from Theorem B the following

**Corollary 1.9** (Agrawal [10, Theorem 2.4]). Let $G$ have a normal Hall subgroup $D$ such that: (i) $G/D$ is a PST-group, and (ii) every subnormal subgroup of $D$ is normal in $G$. Then $G$ is a PST-group.

Some other applications of Theorems A and B and some other characterizations of $\sigma$-soluble $P\sigma T$-groups we discuss in Section 4.

## 2 Some preliminary results

In view of Theorem B in [4], the following fact is true.

**Lemma 2.1.** If $G$ is $\sigma$-soluble, then $G$ is a $\sigma$-full group of Sylow type.

**Lemma 2.2** (See Corollary 2.4 and Lemma 2.5 in [1]). The class of all $\sigma$-nilpotent groups $\mathfrak{N}_\sigma$ is closed under taking products of normal subgroups, homomorphic images and subgroups. Moreover, if $E$ is a normal subgroup of $G$ and $E/E \cap \Phi(G)$ is $\sigma$-nilpotent, then $E$ is $\sigma$-nilpotent.

**Lemma 2.3** (See Proposition 2.2.8 in [23]). If $N$ is a normal subgroup of $G$, then $(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma} N/N$.

**Lemma 2.4** (See Knyagina and Monakhov [24]). Let $H$, $K$ and $N$ be pairwise permutable subgroups of $G$ and $H$ be a Hall subgroup of $G$. Then $N \cap HK = (N \cap H)(N \cap K)$.

**Lemma 2.5** (See Lemma 2.8 in [1]). Let $A$, $K$ and $N$ be subgroups of $G$. Suppose that $A$ is $\sigma$-subnormal in $G$ and $N$ is normal in $G$.

1. If $N \leq K$ and $K/N$ is $\sigma$-subnormal in $G/N$, then $K$ is $\sigma$-subnormal in $G$.
2. $A \cap K$ is $\sigma$-subnormal in $K$.
3. If $A$ is a $\sigma$-Hall subgroup of $G$, then $A$ is normal in $G$.
4. If $H \neq 1$ is a Hall $\Pi$-subgroup of $G$ and $A$ is not a $\Pi'$-group, then $A \cap H \neq 1$ is a Hall $\Pi$-subgroup of $A$. 

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(5) \( AN/N \) is \( \sigma \)-subnormal in \( G/N \).

(6) If \( K \) is a \( \sigma \)-subnormal subgroup of \( A \), then \( K \) is \( \sigma \)-subnormal in \( G \).

Lemma 2.6 (See Lemmas 2.8, 3.1 and Theorem B in [1]). Let \( H, K \) and \( R \) be subgroups of \( G \). Suppose that \( H \) is \( \sigma \)-permutable in \( G \) and \( R \) is normal in \( G \). Then:

(1) \( H \) is \( \sigma \)-subnormal in \( G \).

(2) The subgroup \( HR/R \) is \( \sigma \)-permutable in \( G/R \).

(3) If \( K \) is a \( \sigma_i \)-group, then \( K \) is \( \sigma \)-permutable in \( G \) if and only if \( O^{\sigma_i}(G) \leq N_G(K) \).

(4) If \( G \) is a \( \sigma \)-full group of Sylow type and \( H \leq K \), then \( H \) is \( \sigma \)-permutable in \( K \).

(5) If \( G \) is a \( \sigma \)-full group of Sylow type, \( R \leq K \) and \( K/R \) is \( \sigma \)-permutable in \( G/R \), then \( K \) is \( \sigma \)-permutable in \( G \).

(6) \( H/H_G \) is \( \sigma \)-nilpotent.

Lemma 2.7. The following statements hold:

(i) \( G \) is a \( P\sigma T \)-group if and only if every \( \sigma \)-subnormal subgroup of \( G \) is \( \sigma \)-permutable in \( G \).

(ii) If \( G \) is a \( P\sigma T \)-group, then every quotient \( G/N \) of \( G \) is also a \( P\sigma T \)-group.

Proof. (i) This follows from Lemmas 2.5(6) and 2.6(1).

(ii) Let \( H/N \) be a \( \sigma \)-subnormal subgroup of \( G/N \). Then \( H \) is a \( \sigma \)-subnormal subgroup of \( G \) by Lemma 2.5(1), so \( H \) is \( \sigma \)-permutable in \( G \) by hypothesis and Part (i). Hence \( H/N \) is \( \sigma \)-permutable in \( G/N \) by Lemma 2.6(2). Hence \( G/N \) is a \( P\sigma T \)-group by Part (i).

The lemma is proved.

3 Proofs of Theorems A and B

Proof of Theorem B. Since \( G \) is a \( \sigma \)-full group of Sylow type by hypothesis, it possesses a complete Hall \( \sigma \)-set \( \mathcal{H} = \{H_1, \ldots, H_t\} \) and a subgroup \( H \) of \( G \) is \( \sigma \)-permutable in \( G \) if and only if \( HH_i^x = H_i^x H \) for all \( H_i \in \mathcal{H} \) and \( x \in G \). We can assume without loss of generality that \( H_i \) is a \( \sigma_i \)-group for all \( i = 1, \ldots, t \).

Assume that this theorem is false and let \( G \) be a counterexample of minimal order. Then \( D \neq 1 \) and for some \( \sigma \)-subnormal subgroup \( H \) of \( G \) and for some \( x \in G \) and \( k \in I \) we have \( HH_k^x \neq H_k^x H \) by Lemma 2.7(i). Let \( E = H_k^x \).

(1) The hypothesis holds for every quotient \( G/N \) of \( G \).

It is clear that \( G/N \) is a \( \sigma \)-full group of Sylow type and \( DN/N \) is a normal \( \sigma \)-Hall subgroup of \( G/N \). On the other hand,

\[
(G/N)/(DN/N) \cong G/DN \cong (G/D)/(DN/D),
\]
so \((G/N)/(DN/N)\) is a \(P\sigma T\)-group by Lemma 2.7(ii). Finally, let \(H/N\) be a \(\sigma\)-subnormal subgroup of \(DN/N\). Then \(H = N(H \cap D)\) and, by Lemma 2.5(1), \(H\) is \(\sigma\)-subnormal in \(G\). Hence \(H \cap D\) is \(\sigma\)-subnormal in \(D\) by Lemma 2.5(2), so \(H \cap D\) is normal in \(G\) by hypothesis. Thus \(H/N = N(H \cap D)/N\) is normal in \(G/N\). Therefore the hypothesis holds for \(G/N\).

(2) \(H_G = 1\).

Assume that \(H_G \neq 1\). Clearly, \(H/H_G\) is \(\sigma\)-subnormal in \(G/H_G\). Claim (1) implies that the hypothesis holds for \(G/H_G\), so the choice of \(G\) implies that \(G/H_G\) is a \(P\sigma T\)-group. Hence

\[(H/H_G)(EH_G/H_G) = (EH_G/H_G)(H/H_G)\]

by Lemma 2.7(i). Therefore \(HE = EH\), a contradiction. Hence \(H_G = 1\).

(3) \(DH = D \times H\).

By Lemma 2.5(2), \(H \cap D\) is \(\sigma\)-subnormal in \(D\). Hence \(H \cap D\) is normal in \(G\) by hypothesis, which implies that \(H \cap D = 1\) by Claim (2). Lemma 2.5(2) implies also that \(H\) is \(\sigma\)-subnormal in \(DH\). But \(H\) is a \(\sigma\)-Hall subgroup of \(DH\) since \(D\) is a \(\sigma\)-Hall subgroup of \(G\) and \(H \cap D = 1\). Therefore \(H\) is normal in \(DH\) by Lemma 2.5(3), so \(DH = D \times H\).

**Final contradiction.** Since \(D\) is a \(\sigma\)-Hall subgroup of \(G\), then either \(E \leq D\) or \(E \cap D = 1\). But the former case is impossible by Claim (3) since \(HE \neq EH\), so \(E \cap D = 1\). Therefore \(E\) is a \(\Pi^i\)-subgroup of \(G\), where \(\Pi = \sigma(D)\). By the Schur-Zassenhaus theorem, \(D\) has a complement \(M\) in \(G\). Then \(M\) is a Hall \(\Pi^i\)-subgroup of \(G\) and so for some \(x \in G\) we have \(E \leq M^x\) since \(G\) is a \(\sigma\)-full group of Sylow type. On the other hand, \(H \cap M^x\) is a Hall \(\Pi^i\)-subgroup of \(H\) by Lemma 2.5(4) and hence \(H \cap M^x = H \leq M^x\). Lemma 2.5(2) implies that \(H\) is \(\sigma\)-subnormal in \(M^x\). But \(M^x \simeq G/D\) is a \(P\sigma T\)-group by hypothesis, so \(HE = EH\) by Lemma 2.7(i). This contradiction completes the proof of the theorem.

**Sketch of the proof of Theorem A.** Since \(G\) is \(\sigma\)-soluble by hypothesis, \(G\) is a \(\sigma\)-full group of Sylow type by Lemma 2.1. Let \(\mathcal{H} = \{H_1, \ldots, H_n\}\) be a complete Hall \(\sigma\)-set of \(G\). We can assume without loss of generality that \(H_i\) is a \(\sigma_i\)-group for all \(i = 1, \ldots, n\).

First suppose that \(G\) is a \(P\sigma T\)-group. We show that Conditions (i) and (ii) hold for \(G\) in this case. Assume that this is false and let \(G\) be a counterexample of minimal order. Then \(D \neq 1\).

(1) If \(R\) is a non-identity normal subgroup of \(G\), then Conditions (i) and (ii) hold for \(G/R\) (Since the hypothesis holds for \(G/R\) by Lemma 2.7(ii), this follows from the choice of \(G\)).

(2) If \(E\) is a proper \(\sigma\)-subnormal subgroup of \(G\), then \(E^{\mathfrak{H}_\sigma} \leq D\) and Conditions (i) and (ii) hold for \(E\).

Every \(\sigma\)-subnormal subgroup \(H\) of \(E\) is \(\sigma\)-subnormal in \(G\) by Lemma 2.5(6), so \(H\) is \(\sigma\)-permutable in \(G\) by Lemma 2.7(i). Thus \(H\) is \(\sigma\)-permutable in \(E\) by Lemma 2.6(4). Therefore \(E\) is a \(\sigma\)-soluble \(P\sigma T\)-group by Lemma 2.7(i), so Conditions (i) and (ii) hold for \(E\) by the choice of \(G\). Moreover, since \(G/D \in \mathfrak{H}_\sigma\) and \(\mathfrak{H}_\sigma\) is a hereditary class by Lemma 2.2, \(E/E \cap D \simeq ED/D \in \mathfrak{H}_\sigma\) and so
$E^{\Omega_p} \leq E \cap D \leq D$.

(3) $D$ is nilpotent.

(4) $D$ is a Hall subgroup of $G$. Hence $D$ has a $\sigma$-nilpotent complement $M$ in $G$.

Suppose that this is false and let $P$ be a Sylow $p$-subgroup of $D$ such that $1 < P < G_p$, where $G_p \in \text{Syl}_p(G)$. We can assume without loss of generality that $G_p \leq H_1$.

$(a^0)$ $D = P$ is a minimal normal subgroup of $G$.

Let $R$ be a minimal normal subgroup of $G$ contained in $D$. Since $D$ is nilpotent by Claim (3), $R$ is a $q$-group for some prime $q$. Moreover, $D/R = (G/R)^{\Omega_q}$ is a Hall subgroup of $G/R$ by Claim (1) and Lemma 2.3. Suppose that $PR/R \neq 1$. Then $PR/R \in \text{Syl}_p(G/R)$. If $q \neq p$, then $P \in \text{Syl}_p(G)$. This contradicts the fact that $P < G_p$. Hence $q = p$, so $R \leq P$ and therefore $P/R \in \text{Syl}_p(G/R)$ and we again get that $P \in \text{Syl}_p(G)$. This contradiction shows that $PR/R = 1$, which implies that $R = P$ is the unique minimal normal subgroup of $G$ contained in $D$. Since $D$ is nilpotent, a $p$-complement $E$ of $D$ is characteristic in $D$ and so it is normal in $G$. Hence $E = 1$, which implies that $R = D = P$.

$(b^0)$ $D \notin \Phi(G)$. Hence for some maximal subgroup $M$ of $G$ we have $G = D \times M$ (This follows from $(a^0)$ and Lemma 2.2 since $G$ is not $\sigma$-nilpotent).

$(c^0)$ If $G$ has a minimal normal subgroup $L \neq D$, then $G_p = D \times (L \cap G_p)$. Hence $O_{pq}(G) = 1$.

Indeed, $DL/L \simeq D$ is a Hall subgroup of $G/L$ by Claim (1) and lemma 2.3. Hence $G_pL/L = DL/L$, so $G_p = D \times (L \cap G_p)$. Thus $O_{pq}(G) = 1$ since $D < G_p$ by Claim $(a^0)$.

$(d^0)$ $V = C_G(D) \cap M$ is a normal subgroup of $G$ and $C_G(D) = D \times V \leq H_1$.

In view of Claims $(a^0)$ and $(b^0)$, $C_G(D) = D \times V$, where $V = C_G(D) \cap M$ is a normal subgroup of $G$. Moreover, $V \simeq DV/D$ is $\sigma$-nilpotent by Lemma 2.2. Let $W$ be a $\sigma_1$-complement of $V$. Then $W$ is characteristic in $V$ and so it is normal in $G$. Therefore we have $(d^0)$ by Claim $(c^0)$.

$(e^0)$ $G_p \neq H_1$.

Assume that $G_p = H_1$. Let $Z$ be a subgroup of order $p$ in $Z(G_p) \cap D$. Then, since $O^{q_1}(G) = O^q(G)$, $Z$ is normal in $G$ by Lemmas 2.6(3) and 2.7(i). Hence $D = Z < G_p$ by Claim $(a^0)$ and so $D < C_G(D)$. Then $V = C_G(D) \cap M \neq 1$ is a normal subgroup of $G$ and $V \leq H_1 = G_p$ by Claim $(d^0)$. Let $L$ be a minimal normal subgroup of $G$ contained in $V$. Then $G_p = D \times L$ is a normal elementary abelian subgroup of $G$ by Claim $(e^0)$. Therefore every subgroup of $G_p$ is normal in $G$ by Lemma 2.6(3). Hence $|D| = |L| = p$. Let $D = \langle a \rangle$, $L = \langle b \rangle$ and $N = \langle ab \rangle$. Then $N \not\leq D$, so in view of the $G$-isomorphisms

$$DN/D \simeq N \simeq NL/L = G_p/L = DL/L \simeq D$$

we get that $G/C_G(D) = G/C_G(N)$ is a $p$-group since $G/D$ is $\sigma$-nilpotent by Lemma 2.2. But then Claim $(e^0)$ implies that $G$ is a $p$-group. This contradiction shows that we have $(e^0)$.

**Final contradiction for (4).** In view of Theorem A in [3], $G$ has a $\sigma_1$-complement $E$ such that $EG_p = G_pE$. Let $V = (EG_p)^{\Omega_p}$. By Claim $(e^0)$, $EG_p \neq G$. On the other hand, since $D \leq EG_p$ by
Claim $(a^0)$, $EG_p$ is $\sigma$-subnormal in $G$ by Lemma 2.5(1). Therefore Claim (2) implies that $V$ is a Hall subgroup of $EG_p$ and $V \leq D$, so for a Sylow $p$-subgroup $V_p$ of $V$ we have $|V_p| \leq |P| < |G_p|$. Hence $V$ is a $p'$-group and so $V \leq C_G(D) \leq H_1 = G_p$ by Claim $(a^0)$. Thus $V = 1$. Therefore $EG_p = E \times G_p$ is $\sigma$-nilpotent and so $E \leq C_G(D) \leq H_1$. Hence $E = 1$ and so $D = 1$, a contradiction. Thus, $D$ is a Hall subgroup of $G$. Hence $D$ has a complement $M$ in $G$ by the Schur-Zassenhaus theorem and $M \cong G/D$ is $\sigma$-nilpotent by Lemma 2.2.

(5) $H_i = O_{\sigma_i}(D) \times S$ for each $\sigma_i \in \sigma(D)$.

First assume that $N = O^{\pi_i}(D) \neq 1$. Since $D$ is nilpotent by Claim (3), $N$ is a $\sigma'_i$-group. Moreover, $G/N$ is a $P\sigma T$-group by Lemma 2.7(ii) and so the choice of $G$ implies that

$$H_i \cong H_iN/N = (O_{\sigma_i}(D/N)) \times (V/N) = (O_{\sigma_i}(D)N/N) \times (V/N).$$

Since $D$ is a Hall subgroup of $H_i$ by Claim (4), $DN/N$ is a Hall subgroup of $H_iN/N$ and so $V/N$ is a Hall subgroup of $H_iN/N$. Hence $V/N$ is characteristic in $H_iN/N$. On the other hand, since $D/N = (G/N)^{\pi_i}$ is $\sigma$-nilpotent by Lemma 2.2, $H_iN/N$ is normal in $G/N$ and so $V/N$ is normal in $G/N$. The subgroup $N$ has a complement $S$ in $V$ by the Schur-Zassenhaus theorem. Thus $H_i \cap V = H_i \cap NS = S(H_i \cap N) = S$ is normal in $H_i$.

Now assume that $O^{\pi_i}(D) = 1$, that is, $D$ is a $\sigma_i$-group. Then $H_i$ is normal in $G$, so all subgroups of $H_i$ are $\sigma$-permutably in $G$ by Lemmas 2.5(6), 2.7(i) and hypothesis. Since $D$ is a normal Hall subgroup of $H_i$, it has a complement $S$ in $H_i$. Lemma 2.6(3) implies that $D \leq O^{\pi_i}(G) \leq N_G(S)$. Hence $H_i = D \times S$.

(6) Every subgroup $H$ of $D$ is normal in $G$. Hence every element of $G$ induces a power automorphism in $D$.

Since $D$ is nilpotent by Claim (3), it is enough to consider the case when $H \leq O_{\sigma_i}(D) = H_i \cap D$ for some $\sigma_i \in \sigma(D)$. Claim (5) implies that $H_i = O_{\sigma_i}(D) \times S$. It is clear that $H$ is subnormal in $G$, so $H$ is $\sigma$-permutably in $G$. Therefore

$$G = H_iO^{\pi_i}(G) = (O_{\sigma_i}(D) \times S)O^{\pi_i}(G) = SO^{\pi_i}(G) \leq N_G(H)$$

by Lemma 2.6(3).

(7) If $p$ is a prime such that $(p - 1, |G|) = 1$, then $p$ does not divide $|D|$. Hence the smallest prime in $\pi(G)$ belongs to $\pi(|G : D|)$. In particular, $|D|$ is odd.

Assume that this is false. Then, by Claim (6), $D$ has a maximal subgroup $E$ such that $|D : E| = p$ and $E$ is normal in $G$. It follows that $C_G(D/E) = G$ since $(p - 1, |G|) = 1$. Hence $G/E = (D/E) \times (ME/E)$, where $ME/E \cong M \cong G/D$ is $\sigma$-nilpotent. Therefore $G/E$ is $\sigma$-nilpotent. But then $D \leq E$, a contradiction. Hence we have (7).

(8) $D$ is abelian.

In view of Claim (6), $D$ is a Dedekind group. Hence $D$ is abelian since $|D|$ is odd by Claim (7).
From Claims (4)–(8) we get that Conditions (i) and (ii) hold for G.

Now we show that if Conditions (i) and (ii) hold for G, then G is a $P\sigma T$-group. Assume that this is false and let G be a counterexample of minimal order. Then $D \neq 1$ and, by Lemma 2.7(i), for some $\sigma$-subnormal subgroup $H$ of $G$ and for some $x \in G$ and $k \in I$ we have $HH^x_k \neq H^x_k H$. Let $E = H^x_k$.

(1$^0$) If $N$ is a minimal normal subgroup of $G$, then $G/N$ is a $P\sigma T$-group (Since the hypothesis holds for $G/N$, this follows from the choice of $G$).

(2$^0$) If $N$ is a minimal normal subgroup of $G$, then $EHN$ is a subgroup of $G$. Hence $E \cap N = 1$.

Claim (1$^0$) implies that $G/N$ is a $P\sigma T$-group. On the other hand, $EN/N$ is a Hall $\sigma_k$-subgroup of $G/N$ and, by Lemma 2.5(5), $HN/N$ is a $\sigma$-subnormal subgroup of $G/N$. Note also that $G/N$ is $\sigma$-soluble, so every two Hall $\sigma_k$-subgroups of $G/N$ are conjugate by Lemma 2.1. Thus,

$$(HN/N)(EN/N) = (EN/N)(HN/N) = EHN/N$$

by Lemma 2.7(i). Hence $EHN$ is a subgroup of $G$. Since $G$ is $\sigma$-soluble, $N$ is a $\sigma_j$-group for some $j$. Hence in the case $E \cap N \neq 1$ we have $j = k$, so $N \leq E$. But then $EHN = EH = HE$, a contradiction. Thus $E \cap N = 1$.

(3$^0$) $|\sigma(D)| > 1$.

Indeed, suppose that $\sigma(D) = \{\sigma_1\}$. Then $H_i/D$ is normal in $G/D$ since $G/D \simeq M$ is $\sigma$-nilpotent by hypothesis, so $H_i = D \times S$ is normal in $G$. The subgroup $S$ is also normal in $G$ since it is characteristic in $H_i$. On the other hand, Theorem B and the choice of $G$ imply that $S \neq 1$.

Let $R$ and $N$ be minimal normal subgroups of $G$ such that $R \leq D$ and $N \leq S$. Then $R$ is a group of order $p$ for some prime $p$ and $N$ is a $p'$-group since $D$ is a Hall subgroup of $H_i$. Hence $R \cap HN \leq O_p(HN) \leq P$, where $P$ is a Sylow $p$-subgroup of $H$, so $R \cap HN = R \cap H$. Claim (2$^0$) implies that $EHR$ and $EHN$ are subgroups of $G$. Therefore from Lemma 2.4 and Claim (2$^0$) we get that $R \cap EHN = R \cap E(HN) = (R \cap E)(R \cap HN) = R \cap H$. Hence

$$EHR \cap EHN = E(HR \cap EHN) = EH(R \cap EHN) = EH(R \cap H) = EH$$

is a subgroup of $G$. Thus $HE = EH$, a contradiction. Hence we have (3$^0$).

**Final contradiction.** Since $|\sigma(D)| > 1$ by Claim (3$^0$) and $D$ is nilpotent, $G$ has at least two minimal normal subgroups $R$ and $N$ such that $R, N \leq D$ and $\sigma(R) \neq \sigma(N)$. Then at least one of the subgroups $R$ or $N$, $R$ say, is a $\sigma_i$-group for some $i \neq k$. Hence $R \cap HN \leq O_{\sigma_i}(HN) \leq V$, where $V$ is a Hall $\sigma_i$-subgroup of $H$, since $N$ is a $\sigma_i$-group and $G$ is a $\sigma$-full group of Sylow type. Hence $R \cap HN = R \cap H$. Claim (2$^0$) implies that $EHR$ and $EHN$ are subgroups of $G$. Now, arguing similarly as in the proof of (3$^0$), one can show that $EHR \cap EHN = EH = HE$. This contradiction completes the proof of the fact that $G$ is a $P\sigma T$-group.

The theorem is proved.
4 Some other characterizations of $\sigma$-soluble $P\sigma T$-groups

Theorem A and Theorem B in [1] are basic in the sense that many other characterizations of $\sigma$-soluble $P\sigma T$-groups can be obtained by using these two results. As a partial illustration to this, we give in this section our next three characterizations of $\sigma$-soluble $P\sigma T$-groups.

1. Recall that $Z_\sigma(G)$ denotes the $\sigma$-hypercentre of $G$ [20], that is, the largest normal subgroup of $G$ such that for every chief factor $H/K$ of $G$ below $Z_\sigma(G)$ the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is $\sigma$-primary.

We say, following [6, p. 20], that a subgroup $H$ of $G$ is $\sigma$-hypercentrally embedded in $G$, if $H/H_G \leq Z_\sigma(G/H_G)$.

**Theorem 4.1.** Let $G$ be $\sigma$-soluble. Then $G$ is a $P\sigma T$-group if and only if every $\sigma$-subnormal subgroup of $G$ is $\sigma$-hypercentrally embedded in $G$.

**Proof.** Let $D = G^{\sigma_\sigma}$. First we show that if $G$ is a $P\sigma T$-group, then every $\sigma$-subnormal subgroup $H$ of $G$ is $\sigma$-hypercentrally embedded in $G$. Assume that this is false and let $G$ be a counterexample with $|G| + |H|$ minimal. Then $G/H_G$ is a $\sigma$-soluble $P\sigma T$-group by Lemma 2.7(ii) and $H/H_G$ is $\sigma$-subnormal in $G/H_G$ by Lemma 2.5(5). Hence the choice of $G$ implies that $H_G = 1$, so $H$ is $\sigma$-nilpotent by Lemma 2.6(6). Therefore every subgroup of $H$ is $\sigma$-subnormal in $G$ by Proposition 2.3 in [1] and Lemma 2.5(6). Assume that $H$ possesses two distinct maximal subgroups $V$ and $W$. Then $V, W \leq Z_\sigma(G)$ by minimality of $|G| + |H|$ since $V_G = 1 = W_G$, which implies that $H \leq Z_\sigma(G)$. Hence $H$ is a cyclic $p$-group for some $p \in \sigma_i$.

By Theorem A, $G = D \rtimes M$, where $D$ is a Hall subgroup of $G$, $M$ is $\sigma$-nilpotent and every subgroup of $D$ is normal in $G$. Then $H \cap D = 1$ and so, in view of Lemma 2.1, we can assume without loss of generality that $H \leq M$. Lemma 2.7(i) implies that $H$ is $\sigma$-permutable in $G$, so

$$H^G = H^{DM} = H^{G^{\sigma_\sigma}(G)M} = H^M \leq M$$

by Lemma 2.6(3). Hence $H^G \cap D = 1$ and then, from the $G$-isomorphism $H^G D/D \simeq H^G$, we deduce that $H \leq H^G \leq Z_\sigma(G)$. Therefore $H$ is $\sigma$-hypercentrally embedded in $G$. This contradiction completes the proof of the necessity of the condition of the theorem.

**Sufficiency.** It is enough to show that if a $\sigma$-subnormal subgroup $H$ of a $\sigma$-soluble group $G$ is $\sigma$-hypercentrally embedded in $G$, then $H$ is $\sigma$-permutable in $G$. Assume that this is false and let $G$ be a counterexample with $|G| + |H|$ minimal. Since $G$ is $\sigma$-soluble, it is a $\sigma$-full group of Sylow type by Lemma 2.1. Therefore, in view of Lemma 2.6(5), $H_G = 1$ and so $H \leq Z_\sigma(G)$. It is clear that $Z_\sigma(G)$ is $\sigma$-nilpotent, so $H = H_1 \times \cdots \times H_t$ for some $\sigma$-primary groups $H_1, \ldots, H_t$. Moreover, the minimality of $|G| + |H|$ implies that $H = H_1$ is a $\sigma_i$-group for some $i$. Hence $H \leq N$, where $N$ is a Hall $\sigma_i$-subgroup of $Z_\sigma(G)$. Since $Z_\sigma(G)$ is $\sigma$-nilpotent, $N$ is characteristic in $Z_\sigma(G)$ and so $N$ is normal in $G$.

Let $1 = Z_0 < Z_1 < \cdots < Z_t = N$ be a chief series of $G$ below $N$ and $C_i = C_G(Z_i/Z_{i-1})$. Let
\( C = C_1 \cap \cdots \cap C_t \). Then \( G/C \) is a \( \sigma_i \)-group. On the other hand, \( C/C_G(N) \cong A \leq \text{Aut}(N) \) stabilizes the series \( 1 = Z_0 < Z_1 < \cdots < Z_t = N \), so \( C/C_G(N) \) is a \( \pi(N) \)-group by [25, Ch. A, 12.4(a)]. Hence \( G/C_G(N) \) is a \( \sigma_i \)-group and so \( O^{\sigma_i}(G) \leq C_G(N) \). But then \( O^{\sigma_i}(G) \leq C_G(H) \), so \( H \) is \( \sigma \)-permutable in \( G \) by Lemma 2.6(3). This contradiction completes the proof of the sufficiency of the condition of the theorem.

The theorem is proved.

In the case when \( \sigma = \sigma^0 \), we have \( Z_\sigma(G) = Z_\infty(G) \). Hence from Theorem 4.1 we get

**Corollary 4.2** (See Theorem 2.4.4 in [6]). Let \( G \) be soluble. Then \( G \) is a PST-group if and only if every subnormal subgroup \( H \) of \( G \) is hypercentrally embedded in \( G \) (that is, \( H/H_G \leq Z_\infty(G/H_G) \)).

2. We say, following [6, p. 68], that \( G \) satisfies property \( \mathcal{Y}_{\sigma_i} \) if whenever \( H \leq K \) are two \( \sigma_i \)-subgroups of \( G \), \( H \) is \( \sigma \)-permutable in \( N_G(K) \).

The idea of the next theorem goes back to the paper [15].

**Theorem 4.3.** Let \( G \) be \( \sigma \)-soluble. Then \( G \) is a \( P\sigma T \)-group if and only if \( G \) satisfies \( \mathcal{Y}_{\sigma_i} \) for all primes \( i \).

**Lemma 4.4.** Let \( K \leq H \) and \( N \) be subgroups of \( G \). Suppose that \( K \) is \( \sigma \)-permutable in \( H \) and \( N \) is normal in \( G \). Then \( KN/N \) is \( \sigma \)-permutable in \( HN/N \).

**Proof.** Let \( f : H/H \cap N \rightarrow HN/N \) be the canonical isomorphism from \( H/H \cap N \) onto \( HN/N \). Then \( f(K(H \cap N)/(H \cap N)) = KN/N \), so \( KN/N \) is \( \sigma \)-permutable in \( HN/N \) by Lemma 2.6(2).

The lemma is proved.

**Sketch of the proof of Theorem 4.3.** Necessity. Let \( H \leq K \) be two \( \sigma_i \)-subgroups of \( G \) and \( N = N_G(K) \). Then \( H \) is \( \sigma \)-subnormal in \( N \) by Lemma 2.5(6). On the other hand, Corollary 1.2 implies that \( N \) is a \( \sigma \)-soluble \( P\sigma T \)-group. Therefore \( H \) is \( \sigma \)-permutable in \( N \) by Lemma 2.7(i).

Sufficiency. It is enough to show that Conditions (i) and (ii) of Theorem A hold for \( G \). Assume that this is false and let \( G \) be a counterexample of minimal order. Since \( G \) is \( \sigma \)-soluble, it is a \( \sigma \)-full group of Sylow type by Lemma 2.1. Let \( D = G^{\text{sol}} \).

(1) Every proper subgroup \( E \) of \( G \) is a \( \sigma \)-soluble \( P\sigma T \)-group and \( E^{\text{sol}} \leq D \) (This follows from Lemmas 2.2, 2.3, 2.6(4) and the choice of \( G \)).

(2) \( G/N \) is a \( \sigma \)-soluble \( P\sigma T \)-group for every minimal normal subgroup \( N \) of \( G \).

Let \( H/N \leq K/N \) be two \( \sigma_i \)-subgroups of \( G/N \). Since \( G \) is \( \sigma \)-soluble, \( N \) is a \( \sigma_j \)-subgroup for some \( j \). Assume that \( j \neq i \). Then there are a Hall \( \sigma_i \)-subgroup \( V \) of \( H \) and a Hall \( \sigma_j \)-subgroup \( W \) of \( K \) such that \( V \leq W \) since \( G \) is a \( \sigma \)-full group of Sylow type. Then \( V \) is \( \sigma \)-permutable in \( N_G(W) \) by hypothesis, so \( H/N = VN/N \) is \( \sigma \)-permutable in \( N_G(W)N/N = N_G(WN/N) = N_G(NK/N) \) by Lemma 4.4. Similarly we get that \( H/N \) is \( \sigma \)-permutable in \( N_{G/N}(K/N) \) in the case when \( j = i \).

(3) \( D \) is nilpotent.
(4) $D$ is a Hall subgroup of $G$ and $H_i = O_{\sigma_i}(D) \times S$ for each $\sigma_i \in \sigma(D)$ (See Claims (4) and (5) in the proof of Theorem A and use Claims (1), (2) and (3)).

(5) Every subgroup $H$ of $D$ is normal in $G$. Hence every element of $G$ induces a power automorphism in $D$.

Since $D$ is nilpotent by Claim (3), it is enough to consider the case when $H \leq O_{\sigma_i}(D) = H_i \cap D$ for some $\sigma_i \in \sigma(D)$. Hence $H$ is $\sigma$-permutable in $G$ by hypothesis. Claim (4) implies that $H_i = O_{\sigma_i}(D) \times S$. Therefore $G = H_i\sigma^\alpha(G) = SO^\alpha(G) \leq N_G(H)$ by Lemma 2.6(3).

(6) $D$ is abelian of odd order (See Claims (7) and (8) in the proof of Theorem A and use Claim (5)).

The theorem is proved.

**Corollary 4.5** (Ballester-Bolinches and Esteban-Romero [15], see also Theorem 2.2.9 in [6]). $G$ is a soluble PST-group if and only if $G$ satisfies $\not\approx_p$ for all primes $p$.

**Proof.** It is enough to note that, as it was remarked at the beginning of the proof of Theorem 2.2.9 in [6], every group which satisfies $\not\approx_p$ for all primes $p$ is soluble.

3. We say that a subgroup $A$ of $G$ is $\sigma$-modular ($S$-modular in the case $\sigma = \sigma^0$) provided $G$ possesses a complete Hall $\sigma$-set and $\langle A, H \cap C \rangle = \langle A, H \rangle \cap C$ for every Hall $\sigma_i$-subgroup $H$ of $G$ and all $i \in I$ and $A \leq C \leq G$.

**Theorem 4.6.** Let $G$ be $\sigma$-soluble. Then $G$ is a $P\sigma T$-group if and only if every $\sigma$-subnormal subgroup $A$ of $G$ is $\sigma$-modular in every subgroup of $G$ containing $A$.

**Proof.** Since $G$ is $\sigma$-soluble, $G$ is a $\sigma$-full group of Sylow type by Lemma 2.1. Hence $G$ possesses a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ and, for each $\sigma_i \in \sigma(G)$, a subgroup $H$ of $G$ is a Hall $\sigma_i$-subgroup of $G$ if and only if $H = H_k^x$ for some $x \in G$ and $H_k \in \mathcal{H}$.

**Sufficiency.** Assume that this is false and let $G$ be a counterexample of minimal order. Then, in view of Lemma 2.7(i), $G$ has a $\sigma$-subnormal subgroup $A$ which is not $\sigma$-permutable in $G$. Hence, for some $H_i \in \mathcal{H}$ and $x \in G$, we have $AH_i^x \neq H_i^x A$. Note also that every proper $\sigma$-subnormal subgroup $E$ of $G$ is a $P\sigma T$-group. Indeed, $E$ is clearly $\sigma$-soluble and if $H$ is a $\sigma$-subnormal subgroup of $E$, then $H$ is $\sigma$-subnormal in $G$ by Lemma 2.6(6). Hence $H$ is $\sigma$-modular in every subgroup of $E$ containing $H$ by hypothesis. Thus the hypothesis holds for $E$ and so $E$ is a $P\sigma T$-group by the choice of $G$.

By definition, there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \triangleright A_i$ or $A_i/(A_{i-1})A_i$ is $\sigma$-primary for all $i = 1, \ldots, n$. We can assume without loss of generality that $M = A_n < G$. Then $M$ is a $P\sigma T$-group since $M$ is clearly $\sigma$-subnormal in $G$, so $A$ is $\sigma$-permutable in $M$ by Lemma 2.7(i). Moreover, the $\sigma$-modularity of $A$ in $G$ implies that

$$M = M \cap \langle A, H_i^x \rangle = \langle A, (M \cap H_i^x) \rangle.$$ 

On the other hand, by Lemma 2.5(4), $M \cap H_i^x$ is a Hall $\sigma_i$-subgroup of $M$, where $\{\sigma_i\} = \sigma(H_i)$. Hence $M = A(M \cap H_i^x) = (M \cap H_i^x)A$. If $H_i^x \leq M_G$, then $A(M \cap H_i^x) = AH_i^x = H_i^x A$ and so
\[ H_i^x \notin M_G. \]

Now note that \( H_i^x M = MH_i^x \). Indeed, if \( M \) is normal in \( G \), it is clear. Otherwise, \( G/M_G \) is \( \sigma \)-primary and so \( G = MH_i^x = H_i^x M \) since \( H_i^x \notin M_G \) and \( H_i \in \mathcal{H} \). Therefore

\[ H_i^x A = H_i^x (M \cap H_i^x) A = H_i^x M = MH_i^x = H_i^x (M \cap H_i^x) A = H_i^x A. \]

This contradiction completes the proof of the sufficiency of the condition of the theorem.

**Necessity.** In view of Lemma 2.6(4), it is enough to show that if \( A \) is a \( \sigma \)-subnormal subgroup of \( G \), then \( A \) is \( \sigma \)-modular in \( G \). First note that \( A \) is \( \sigma \)-permutable in \( G \) by Lemma 2.7(i). Therefore for every subgroup \( C \) of \( G \) containing \( A \), for every \( i \in I \), and for all Hall \( \sigma_i \)-subgroup \( H \) of \( G \) we have

\[ \langle A, H \cap C \rangle = A(H \cap C) = AH \cap C = \langle A, H \rangle \cap C, \]

so \( A \) is \( \sigma \)-modular in \( G \).

The theorem is proved.

From Theorem 4.6 we get the following characterization of soluble PST-groups.

**Corollary 4.7.** Let \( G \) be soluble. Then \( G \) is a PST-group if and only if every subnormal subgroup \( A \) of \( G \) is \( S \)-modular in every subgroup of \( G \) containing \( A \).

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