On Membrane Interaction in Matrix Theory

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Abstract

We compute the interaction potential between two parallel transversely boosted wrapped membranes (with fixed momentum $p_-$) in $D = 11$ supergravity with compact light-like direction. We show that the supergravity result is in exact agreement with the potential following from the all-order Born-Infeld-type action conjectured to be the leading planar infra-red part of the quantum super Yang-Mills effective action. This provides a non-trivial test of consistency of the arguments relating Matrix theory to a special limit of type II string theory. We also find the potential between two (2+0) D-brane bound states in $D = 10$ supergravity (corresponding to the case of boosted membrane configuration in 11-dimensional theory compactified on a space-like direction). We demonstrate that the result reduces to the SYM expression for the potential in the special low-energy ($\alpha' \to 0$) limit, in agreement with previous suggestions. In appendix we derive the action obtained from the $D = 11$ membrane action by the world-volume duality transformation of the light-like coordinate $x^-$ into a 3-vector.

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1 Introduction

The remarkable correspondence between the super Yang-Mills (and thus matrix theory) and supergravity descriptions of interactions between branes was originally tested in the leading (one-loop) approximation (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8]). It was observed [4, 6] that to have precise agreement between the interaction potentials derived from SYM and supergravity one needs to take a large $N$ (large 0-brane charge) limit in the supergravity expression. In the simplest cases this can be effectively accomplished by subtracting the asymptotic value 1 from the harmonic function $H$ in the action of a D-brane probe moving in the supergravity background produced by a source brane. As was observed in [9] on the example of subleading term in the interaction potential between two D0-branes, the required supergravity expression can be obtained automatically by keeping $N$ finite but considering the $D = 10$ configuration of branes resulting from an M-theory configuration compactified on a light-cone direction $x^- = x_{11} - t$ as in the finite $N$ proposal of [10]. It was checked in [11], that a similar large $N$ or $H \rightarrow H - 1$ recipe is important also for the SYM-supergravity correspondence at the level of subleading (two-loop) term in the interaction potential between a D-brane and a BPS bound state of D-branes.

A suggestion about precisely which (‘low-energy’) limit of $D = 10$ supergravity should have a SYM description was made in [12, 13]. A related argument providing a kinematical explanation for the correspondence between $D = 11$ M-theory compactified on a light-like direction (with $p_- = \frac{N}{R}$) and a transverse p-torus ($p \leq 3$) and a low energy, short distance, weak coupling limit of a system of $N$ Dp branes in type II string theory on the dual p-torus described by super Yang-Mills theory was put forward in [14, 15].

According to [12, 13], the tree-level supergravity describing configurations with RR charges admits a low-energy limit in which it may have a SYM description. That limit does not formally require taking $N$ to be large, but to justify the possibility to ignore closed string loop and $\alpha'$ corrections one needs also to assume that $g_s$ is small and $g_s N$ is large. As was observed in [3, 15], the result of taking this low-energy limit in $D = 10$ supergravity expressions should be achieved automatically by starting with $D = 11$ expressions and taking the light-like direction to be compact (and fixing the value of $p_-$).

The observations made in [13, 14] as such do not imply the agreement between SYM (Matrix theory) and supergravity [16]. That agreement depends on certain special ‘non-renormalisation’ properties of a class of planar diagrams in string theory and thus in maximally supersymmetric large $N$ SYM theory [1, 8, 11].

One aim of the present paper to perform a test of the formal arguments in [13] and [15] on the example of all-order interaction between extended BPS branes. Our results provide also another test of the Born-Infeld ansatz [11] for the leading planar part of the

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1 The same supergravity potential is found either by plugging the $x^-$-reduced background into the D0-brane probe action in $D = 10$ or considering the graviton probe action in $D = 11$ and fixing the light-cone component of momentum $p_-$ [4].
perturbative SYM effective action which is central to this SYM-supergravity correspondence: that single universal expression happens to describe interaction potentials between various types of branes once one plugs in appropriate gauge field backgrounds.

We shall consider a potential between two parallel transverse $D = 11$ membranes having fixed values of the light-cone momentum. The corresponding configuration in $D = 10$ type II string theory is that of two parallel $(2+0)$-branes (bound states of D2-branes and D0-branes). We shall find that the $D = 11$ supergravity expression for the interaction potential between two membranes computed using the procedure similar to that in [3] (i.e. by smearing the background produced by the source membrane in the compact $x^-$ direction and fixing the $p_-$ component of the momentum of the probe membrane) is in exact agreement with the all-order potential following from the conjectured Born-Infeld (BI) type expression [11] for the leading large $N$, finite IR part of the quantum super Yang-Mills effective action.

At the same time, the $D = 10$ supergravity expression for the $(2 + 0) - (2 + 0)$ interaction potential is in correspondence with the SYM expression (and thus with the light-like compactified $D = 11$ expression) only in a special limit, which turns out to be precisely the low-energy limit of [13] ($\alpha' \to 0$, with ‘Yang-Mills’ parameters fixed). The result of taking this limit in the present example is no longer equivalent simply to the substitution $H \to H - 1$ of the harmonic function in the supergravity background as was the case in the previously discussed ‘brane – (bound state of branes)’ interactions [1, 11].

It should be stressed that this test of supergravity-SYM correspondence is non-trivial: though it may seem that the D2-probe action has already a BI form, part of the gauge field dependence is encoded in the curved space geometry produced by the source D2-brane, and it is only after taking the limit that the resulting action becomes the BI-type expression expected on the SYM side.

As we shall be using both the weak-coupling low-energy IIA string theory ($D = 10$ supergravity) picture and the light-like compactified M-theory ($D = 11$ supergravity) picture, let us first review the relation between the corresponding parameters [10, 9, 14, 15].

\begin{equation}
R_{11} = g_s (2\pi T)^{-1/2}, \quad M_{11} = (2\pi g_s)^{-1/3} (2\pi T)^{1/2}, \quad R_{11} M_{11}^3 = T \equiv \frac{1}{2\pi \alpha'}, \quad \text{(1.1)}
\end{equation}

of IIA theory compactified on a circle ($x_{11} = x_{11} + 2\pi R_{11}$) and a p-torus of volume $V_p$ and the parameters $R$, $M_P$ of M-theory with compact light-like direction ($x^- \equiv x^- + 2\pi R$) compactified on a p-torus of volume $V_p$ are related as follows [14, 13]

\begin{equation}
R_{11} M_{11}^2 = R M_P^2, \quad V_p M_{11}^p = V_p M_P^p, \quad x M_{11} = x M_P, \quad \text{(1.2)}
\end{equation}

The non-perturbative SYM–supergravity agreement in the case of infinite membrane scattering with $\Delta p_{11} \neq 0$ was demonstrated (for the leading $O(v^4)$ term) in [17, 18]. Related discussion of $v^4$ terms in the graviton scattering in Matrix theory compactified on 2-torus appeared in [19]. Here we will ignore non-perturbative instanton contributions in $d = 3$ SYM theory which are crucial for consistency of type IIB string interpretation in the limit of vanishing volume of 2-torus [19] but are not important in the present case.
where $x$ and $x$ are any transverse scales of the two theories. The sector with $N$ D0-branes in string theory or with momentum $p_- = \frac{N}{R}$ in M-theory is described at low energies by the $U(N)$ SYM theory on the dual p-torus with volume $\tilde{V}_p$ (dots stand for the standard adjoint scalar and fermionic terms)

$$S = -\frac{1}{4g_{YM}^2} \int d^{p+1}\tilde{x} \; \text{tr} \left( F_{ab} F_{ab} \right) + \ldots,$$

(1.3)

where $d^{p+1}\tilde{x} \equiv dt d^{p}\tilde{x}$ and

$$g_{YM}^2 = (2\pi)^{(p-1)/2} T^{(3-p)/2} \tilde{g}_s = (2\pi)^{-1/2} T^{3/2} \tilde{V}_p g_s = (RM_p^2)^3 \tilde{V}_p,$$

(1.4)

$$\tilde{V}_p = \frac{(2\pi)^p}{T^p V_p} = \frac{(2\pi)^p}{(RM_p^2)^p V_p}, \quad \tilde{V}_p = \frac{V_p}{g_s^2}.$$  

(1.5)

The M-theory parameters $R, M_p, V_p$ and thus $g_{YM}$ and $\tilde{V}_p$ remain finite in the limit $R_{11} \to 0$, $M_{11} \to \infty$ or $\alpha' \to 0$, $g_s \to 0$, $V_p \to 0$ \[14, 13\].

Consider two parallel sets of D-branes (‘probe’ and ‘source’ with charge $N$) separated by a distance $r$ and imagine computing perturbative string theory diagrams with one boundary on the probe brane and $L$ boundaries on the source brane. The large distance limit of such diagrams is expected to be described by the massless closed string (supergravity) modes, while the short distance limit – by the massless open string (SYM) modes. The contribution of the $L = 1$ (annulus) diagram (in a $F=$-const background representing, e.g., a velocity or a flux \[20\]) expanded in powers of $F$ has, symbolically, the structure $(n = 7 - p, \; r = \frac{x}{\alpha'}) \; Z_1 \sim \frac{f_1^{(1)}}{r^7} F^4 + \frac{\alpha'(r^2)}{r^{p+1}} F^6 + \ldots$. As was noted in \[4\], $f_1^{(1)}$ has trivial (factor) dependence on $\alpha'$ (i.e. is independent of $r$) and thus the leading $F^4/r^n$ term is the same for large and small $r$, so that the $F^4$ terms in the supergravity and SYM expressions should agree. The result of \[9\] implies that the 2-loop string diagram should give $Z_2 \sim \frac{f_2^{(2)}}{r^{p+6}} F^6 + \frac{\alpha'(r^2)}{r^{p+1}} F^8 + \ldots$, where $f_2^{(2)} = 0$ and $f_2^{(2)}$ has trivial dependence on $\alpha'$.

One may further conjecture \[11\] that, in general, $Z_L \sim \frac{f_L^{(L)}}{r^{p+2L+2}} F^{2L+2} + \frac{\alpha'(r^2)}{r^{p+1}} F^{2L+4} + \ldots$, where $f_L^{(L)}$ does not depend on $r$. The $\alpha' \to 0$ limit of the open string theory (with $F, r, g_{YM}^2 \sim \alpha'/(4-n)^2 \tilde{g}_s$ and modular UV cutoff being fixed) leads to SYM theory, so that the related conjecture about SYM theory is that the $F^{2L+2}$ terms appear in the large $N$, IR part of the SYM effective action only at $L$-th loop order, i.e. that all lower order $F^m, m < 2L + 2$ terms at $L$-th loop have vanishing coefficients. The assumption that non-planar string diagrams (i.e. diagrams with closed string loops) are not included is justified provided the string coupling $\tilde{g}_s$ is small; the assumption that higher-order supergravity $\alpha'$ corrections are not included is justified provided $N\tilde{g}_s \sim N\tilde{g}_{YM}^2$ is large, i.e. in the large $N$ limit. To ignore subleading terms at each loop order one needs to assume the low-energy limit, i.e. that $F^2/r^4 \ll 1$. To ensure that leading $F^{2L+2}/r^{L+4}$ terms are dominant, i.e. to be able to ignore, say, 1-loop SYM $F^4/r^{n+4}$ correction as compared to

\[3\]This is due to the fact that, as explained in \[21\], only short open string BPS multiplets contribute to the coefficient of $F^4$ term at one loop.
the 2-loop correction $N g_{YM}^2 F^6/r^{2n}$ one is to assume that $N g_{YM}^2/r^{n-4} \gg 1$ (which is the case if $N$ is large for fixed $r$).

Under the above assumptions, the sum of the leading large $N$ (planar) IR contributions to the quantum SYM effective action can be written as ($F$ is a gauge field background, $r$ is a scale of a scalar field background) \cite{11, 9, 22}

$$
\Gamma = \sum_{L=0}^{\infty} \Gamma^{(L)} = \frac{1}{2} \sum_{L=0}^{\infty} \int d^{p+1} \vec{x} \left( \frac{a_p}{r^{7-p}} \right)^L \left( g_{YM}^2 N \right)^{L-1} \hat{C}_{2L+2} (F) + \ldots ,
$$

where we included also the tree-level $L = 0$ term, $a_p = 2^{-p} \pi^{(p+1)/2} (7 - p)$, $\hat{C}_{2L+2} (F) \sim F^{2L+2}$ and dots stand for terms depending on covariant derivatives and commutators of the gauge field $F$ and scalars. It was conjectured in \cite{11} that $\hat{C}_{2L+2} (F) = \hat{STr} C_{2L+2} (F)$, where $C_{2k}$ have the same Lorentz index structure as the polynomials appearing in the expansion of the abelian BI action and $\hat{STr}$ is a modified symmetrized trace that reduces to the adjoint representation trace for some simple (abelian) backgrounds $F$. For $L = 0, 1$ the trace $\hat{STr}$ is equal to the standard symmetrized trace in the adjoint representation; for $L = 2$ its structure was determined using indirect considerations in \cite{11}.

This assumption is equivalent to the following conjecture for the derivative and commutator term independent part of the large $N$ effective action of maximally supersymmetric SYM theory \cite{11} (see also \cite{13, 23})

$$
\Gamma = -\frac{1}{2N g_{YM}^2} \int d^{p+1} \vec{x} \hat{STr} \left( H_p^{-1} \left[ \sqrt{-\det(\eta_{ab} I + H_p^{1/2} F_{ab})} - I \right] \right) ,
$$

where

$$
H_p = \frac{a_p N g_{YM}^2}{r^{7-p}} .
$$

This ansatz is consistent with general one-loop \cite{7, 8} and some special two-loop \cite{26, 9} perturbative calculations in SYM theory. Its correctness is supported by the fact that this single expression provides a universal description of interaction potentials between various (bound states of) branes computed using supergravity methods: (i) the $F^4$ term in (1.7) gives the leading order ($1/r^7$) potentials for BPS branes with different amounts of supersymmetry (see, e.g., \cite{3, 4, 5, 6, 7}) as well as for near-extremal branes \cite{27} and non-supersymmetric configurations of branes \cite{23, 29}; (ii) the $F^6$ term in (1.7) gives subleading ($1/r^2 (7-p)$) terms in the interaction potentials between brane configurations with 1/2, 1/4 and 1/8 fraction of supersymmetry \cite{3, 11}.

Moreover, eq. (1.7) reproduces the exact (all-order) supergravity interaction potential between two 0-branes \cite{11} (or two Dp-branes) and a 0-brane and a non-marginal bound state ($p + ... + 0$) of D-branes \cite{11, 11}, in particular, the potential between a 0-brane and a (2+0)-brane or between a graviton and a transversely boosted membrane in $D = 11$ \cite{11}.

Other related arguments supporting the correctness of the BI ansatz (1.7) were given in $$
\text{Explicitly, } C_0 = 1, \quad C_2 = -\frac{1}{4} F^2, \quad C_4 = -\frac{1}{8} [F^4 - \frac{1}{6} (F^2)^2], \quad C_6 = -\frac{1}{8} [F^6 - \frac{3}{8} F^4 F^2 + \frac{1}{32} (F^2)^3], \ldots
$$

where $F^k = F_{a_1 a_2} F_{a_2 a_3} \ldots F_{a_k a_1}$. 


In particular, in the case of \( p = 3 \), the quantum \( N = 4, D = 4 \) SYM effective action is expected to have special symmetry, reflecting the fact that the exact conformal invariance of this theory is spontaneously broken only by the adjoint scalar scale \( r \). Indeed, the abelian version of (1.7) (its \( \partial X_m \) dependent part) was shown to possess a kind of special conformal symmetry \[30,31\].

Below we shall subject (1.7) to a further non-trivial test: we shall show that it reproduces the all-order supergravity interaction potential between two parallel \((2+0)\) branes in type IIA theory or two transversely boosted membranes in M-theory (with fixed values of \( p_- \)). This agreement is much less obvious than in the \( 0-(2+0) \) or graviton - membrane interaction case \[11\] and depends on details of the light-like compactification prescription in \( D = 11 \) or details of the low energy limit in \( D = 10 \).

In section 2 we shall find the explicit form of the interaction potential between two wrapped membranes in Matrix theory as implied by the BI ansatz (1.7) for the SYM effective action. In section 3 we shall compare the SYM result with the all-order expression for the interaction potential in supergravity found using probe-source method. We shall first consider the \((2+0)-(2+0)\) interaction potential in \( D = 10 \) supergravity and show its agreement with the SYM expression in the special low energy limit of \[13\]. We shall then compute the interaction potential between two transversely boosted wrapped membranes in \( D = 11 \) supergravity with compact light-like direction and demonstrate that performing the Legendre transformation \( \dot{x}^- \rightarrow p_- = \text{fixed} \) (which, in the present membrane context, is a special case of the \( d = 3 \) world-volume scalar-vector duality) leads to the expression for the interaction potential coinciding with the SYM (BI) expression. The \( D = 11 \) supergravity derivation is more straightforward than the \( D = 10 \) supergravity one, as it does not involve any special limit. This illustrates the advantage of the light-like compactification procedure of \[10,9,15\].

As is well known, the scalar-vector duality \( x_{11} \rightarrow A_m \) transforms the standard \( D = 11 \) membrane action \[32\] (in a \( x_{11}\)-independent \( D = 11 \) supergravity background) into the D2-brane \( d = 3 \) BI action (in a generic \( D = 10 \) supergravity background) \[33,17\]. In Appendix we discuss the duality transformation of the membrane action in the case when it is the light-like coordinate \( x^- \) that is rotated into a vector.

## 2 Membrane–membrane potential from super Yang-Mills theory

In the Matrix theory context, one is supposed to start with a system of 0-branes in type IIA string theory on a torus \( V_2 \) and view 2-branes as their ‘collective excitations’. Making \( T \)-duality which interchanges the numbers of D2-branes and D0-branes, let us consider the interaction of the two \((2+0)\) bound states wrapped over the dual torus of volume \( \tilde{V}_2 \): one – ‘probe’– with the 2-brane number \( \tilde{n}_2 = n_0 \), the 0-brane number
\( \tilde{n}_0 = n_2 \) and the velocity in the direction 9, and another – ‘source’ – with the 2-brane number \( \tilde{N}_2 = N_0 \) and the 0-brane number \( \tilde{N}_0 = N_2 \). The corresponding pure gauge field background can be represented by the following gauge field strength matrices in the fundamental representation of \( u(N) \) (\( N = n_0 + N_0 \))

\[
\tilde{F}_{09} = \begin{pmatrix} \mathbf{v}_1 I_{n_0 \times n_0} & 0 \\ 0 & 0 I_{N_0 \times N_0} \end{pmatrix}, \quad \tilde{F}_{12} = \begin{pmatrix} f_1 I_{n_0 \times n_0} & 0 \\ 0 & f_2 I_{N_0 \times N_0} \end{pmatrix},
\]

where the charges and the fluxes are related by

\[
2\pi n_2 = n_0 \tilde{V}_2 f_1, \quad 2\pi N_2 = N_0 \tilde{V}_2 f_2.
\]

The background (2.1) can be interpreted as the finite-\( N \) Matrix theory configuration describing the interaction of two \( D = 11 \) membranes wrapped over the torus of volume \( V_2 \), with the light-cone momenta of the probe and the source membranes given by

\[
p_− = \frac{n_0}{R} = \frac{T_2^{(1)} V_2}{f_1} = \frac{m_1}{f_1}, \quad P_− = \frac{N_0}{R} = \frac{T_2^{(2)} V_2}{f_2} = \frac{m_2}{f_2}.
\]

Here the tensions are

\[
T_2^{(1)} = \frac{n_2 M_P^3}{2\pi}, \quad T_2^{(2)} = \frac{N_2 M_P^3}{2\pi},
\]

and the dimensionless fluxes \( f_n \) and the velocity of the probe membrane are (see (1.2))

\[
f_n = \frac{f_n}{R M_P^3}, \quad v = \frac{\mathbf{v}}{R M_P^3}.
\]

The relations of the type (2.3), i.e.

\[
p_− = \frac{T_2 V_2}{f} = \frac{m}{f}, \quad m \equiv T_2 V_2,
\]

follow from the interpretation of the SYM (Matrix theory) Hamiltonian as the light-cone energy \([2, 10]\) (see (1.3), (1.4))

\[
E_τ = \frac{N \tilde{V}_2 f^2}{2 (g_{YM}^2)_{2+1}} = \frac{m^2}{2p_-}.
\]

In the context of comparison with the \( D = 10 \) string theory (supergravity) we shall have instead of (2.3) the following fluxes and velocity

\[
f_n = \frac{f_n}{T}, \quad v = \frac{\mathbf{v}}{T},
\]

so that the \( D = 10 \) and \( D = 11 \) parameters are related as in \([1, 2]\)

\[
f_n = \frac{M_P}{M_{11}} f_n, \quad v = \frac{M_P}{M_{11}} v.
\]

\(^5\)Note that in view of (1.1), (1.2) the rescaling factor in (2.5) is equal to \( R M_P^3 = T \frac{M_P}{M_{11}} = T (\frac{R}{M_P})^{1/2} \).
It is useful to subtract the traces and describe the background \((2.1)\) by the \(su(N)\) matrices \(F_{ab}\) which are proportional to the same matrix \(J_0\) as in \([1]\)

\[
F_{09} = f_{09}J_0 = v J_0 , \quad F_{12} = f_{12}J_0 = (f_1 - f_2)J_0 , \quad J_0 \equiv \frac{1}{n_0 + N_0} \begin{pmatrix} N_0 I_{n_0 \times n_0} & 0 \\ 0 & -n_0 I_{N_0 \times N_0} \end{pmatrix} , \quad \text{tr } J_0 = 0 .
\]

(2.10)

Since all of the components of \(F_{ab}\) are proportional to the same matrix, the trace \(\hat{\text{STr}}\) in \((1.7)\) reduces simply to the trace in the adjoint representation. Using that \(\hat{\text{Tr}} J_0^{2k} = 2n_0 N_0\), one finds that polynomials constructed out of powers of \(F_{ab}\) have the structure

\[
\text{STr}\left[C_{2k}(F_{ab})\right] = \text{Tr}\left[C_{2k}(F_{ab})\right] = 2n_0 N_0 C_{2k}(f_{ab}) .
\]

(2.12)

Substituting the background \((2.10)\) into \(\Gamma (1.7)\) (and replacing \(N\) in \((1.7)\), \((1.8)\) by \(N_0\) to facilitate comparison with the supergravity probe-method expression for the potential which is linear in the probe’s charge) we get

\[
\Gamma = -\frac{n_0}{g_{\text{YM}}^2} \int d^3\tilde{x} H_2^{-1} \left[ \sqrt{1 - H_2 v^2} \left[ 1 + H_2 (f_1 - f_2)^2 \right] - 1 \right] , \quad H_2 \equiv \frac{3N_0 g_{\text{YM}}^2}{4\pi r^5} .
\]

(2.13)

The SYM scalar scale \(r\) will be related to the scales \(r\) and \(r\) in the \(D = 10\) and \(D = 11\) supergravity expressions according to (cf. \((1.2)\), \((2.9)\))

\[
r = \frac{r}{T} , \quad r = \frac{r}{RM_P} , \quad r = \frac{M_{P}}{M_{11}} r .
\]

(2.15)

Below we shall reproduce the expression \((2.13)\) as the action for a probe membrane moving in a supergravity background of a source membrane. The dependence of the interaction potential on the difference of fluxes or on the difference of the values of \(p_-\) component of the momentum (cf. \((2.6)\)) which is expected on the \(D = 11\) kinematics grounds, will not be obvious a priori in the exact supergravity expression derived using the probe-source method. It will appear only after taking appropriate limit in the \(D = 10\) supergravity expression for the \((2 + 0)\) – \((2 + 0)\) interaction potential, or after a duality (Legendre) transformation fixing \(p_-\) in the \(D = 11\) supergravity result for the membrane action.

### 3 Membrane interaction from supergravity

Our starting point will be the \(D = 11\) supergravity background produced by a BPS membrane source \([34]\). Applying a boost along \(x_{11}^\prime = x_{11} \cosh \beta - t \sinh \beta , \quad t' = t \cosh \beta - x_{11} \sinh \beta\), we get

\[
\text{ds}_{11}^2 = K^{1/3} \left[ K^{-1} \left( - dt'^2 + dy_1^2 + dy_2^2 + dx_{11}^2 + dx_i dx_i \right) \right] ,
\]

(3.1)

\(^6\text{Given a diagonal matrix in the fundamental representation of }u(N)\text{ with entries }a_i\text{ the corresponding matrix in the adjoint representation has entries }a_i - a_j.\text{ This implies that }J_0\text{ has }2n_0 N_0\text{ non-vanishing diagonal elements equal to }\pm 1.\)
\[ \mathcal{K} = 1 + \mathcal{W} , \quad \mathcal{W} = \frac{Q}{r^6} , \quad Q = \frac{8N_2}{M_5^6} , \quad (3.2) \]

\[ C_{r1y2} = \mathcal{K}^{-1} - 1 , \quad C_{r1y2} = (\mathcal{K}^{-1} - 1) \cosh \beta , \quad C_{11y1y2} = -(\mathcal{K}^{-1} - 1) \sinh \beta . \quad (3.3) \]

The components of the metric in terms of the light-cone coordinates \( x^\pm = x_{11} \pm t \) or \( \tau = \frac{1}{2} x^+ \) and \( x^- \) are

\[ g_{\tau \tau} = e^{-2\beta} \mathcal{K}^{-2/3} (\mathcal{K} - 1) , \quad g_{--} = \frac{1}{4} e^{2\beta} \mathcal{K}^{-2/3} (\mathcal{K} - 1) , \]
\[ g_{r-} = \frac{1}{2} (1 + \mathcal{K}) \mathcal{K}^{-2/3} , \quad g_{y_1y_2} = \mathcal{K}^{-2/3} , \quad g_{ij} = \delta_{ij} \mathcal{K}^{1/3} . \quad (3.4) \]

We shall consider either space-like \((x_{11} \equiv x_{11} + 2\pi R_{11})\) or light-like \((x^- \equiv x^- + 2\pi R)\) compactification and smear the membrane background in the compact ‘transverse’ direction (this corresponds to fixing the component of the 11-dimensional momentum of the source membrane). Since the above membrane solution is a BPS one, this is equivalent to ‘smearing’ the harmonic function \( \mathcal{K} \) in the compact direction. In the case of the space-like compactification we get

\[ \mathcal{W} \rightarrow \mathcal{W}_{11} = \sum_{n=-\infty}^{\infty} (x_{11}^2 + r^2)^{-3} = \mathcal{W}_{11} = \sum_{n=-\infty}^{\infty} \left( (x_{11} + 2\pi n R_{11}) \cosh \beta - t \sinh \beta r^2 + r^2 \right)^{-3} \]
\[ = \mathcal{W}_{11} = \frac{Q_{11}}{2\pi R_{11} \cosh \beta} \int_{-\infty}^{\infty} \frac{dx_{11}}{(r^2 + x_{11}^2)^{3/2}} = \frac{3Q_{11}}{16R_{11} \cosh \beta r^5} , \quad (3.5) \]
\[ \mathcal{K}_{11} = 1 + \mathcal{W}_{11} = 1 + \frac{3Q_{11}}{16R_{11} \cosh \beta r^5} , \quad Q_{11} = \frac{8N_2}{M_5^6} . \quad (3.6) \]

In the case of compactification on the light-like direction \( x^- \)

\[ \mathcal{W} \rightarrow \mathcal{W}_- = \frac{Q}{\pi R e^\beta} \int_{-\infty}^{\infty} \frac{dx^-}{r^2 + (x^-)^2} = \frac{3Q}{8Re^\beta r^5} , \quad (3.7) \]
\[ \mathcal{K}_- = 1 + \mathcal{W}_- = 1 + \frac{3Q e^{-\beta}}{8R r^5} . \quad (3.8) \]

### 3.1 (2 + 0)-(2 + 0) interaction in \( D = 10 \) supergravity

The \( D = 10 \) type IIA supergravity background representing the bound state \((2 + 0)\) of \( N_0 \) D0-branes and \( N_2 \) D2-branes wrapped over a torus of volume \( V_2 \) can be obtained, e.g., by compactifying the boosted M2-brane background along the spatial direction \( x_{11} \)

\[ ds_{10}^2 = K^{1/2}[-K^{-1}dt^2 + K^{-1}d(y_1^2 + dy_2^2) + dx_idx_i] , \]
\[ e^{2\phi} = K^{3/2}K^{-1} , \quad C_{\tau y_1y_2} = -\sin \theta \ W K^{-1} , \]
\[ C_t = -\cos \theta \ W K^{-1} , \quad B_{y_1y_2} = \sin \theta \ \cos \theta \ W K^{-1} . \]
\[ K = 1 + W, \quad K' = 1 + W \sin^2 \theta = K_{11}, \quad (3.9) \]

where
\[
W = \frac{Q_0^{(2)}}{r^5} \sqrt{1 + f_2^2}, \quad \cos \theta = \frac{1}{\sqrt{1 + f_2^2}}, \quad \sin \theta = \frac{f_2}{\sqrt{1 + f_2^2}}, \quad (3.10) \]

\[
f_2 = \frac{Q_2}{Q_0^{(2)}} = \frac{V_2 T N_2}{2 \pi N_0} = \frac{2 \pi N_2}{V_2 T N_0}, \quad Q_2 = \frac{3 N_2 g_s}{2 (2 \pi)^{1/2} T^{5/2}}, \quad Q_0^{(2)} = \frac{3 (2 \pi)^{1/2} N_0 g_s}{2 T^{7/2} V_2}. \]

The angle \( \theta \) is related to the 11-dimensional boost parameter \( \beta \) by \( \sin \theta = (\cosh \beta)^{-1} \). The limit of \( \sin \theta = 0 \) \((K' = 1)\) or \( f_2 \to 0 \) \((N_0 \gg N_2)\) corresponds to the 0-brane background smeared over the volume \( V_2 \) \((Q_0^{(2)}\) is the effective charge parameter of 0-brane background). The limit of \( \sin \theta = 1 \) \((K = K')\) or \( f_2 \to \infty \) \((N_2 \gg N_0)\) corresponds to the pure 2-brane background.

Having in mind comparison with Matrix theory, we have presented the \((2 + 0)\) background from the ‘0-brane point of view’, i.e. as a modification (due to the presence of a D2-brane charge) of the D0-background smeared over the torus \( V_2 \). To establish the correspondence with the SYM theory one is then to consider the T-dual theory. T-duality along the two directions of the torus transforms the original theory with coupling \( g_s \) and \((0 + 2) \((N_0, N_2)\) bound state wrapped over \( V_2 \) into the dual theory with coupling \( \tilde{g}_s \) \((V_2/g_s^2 = \tilde{V}_2/\tilde{g}_s^2)\) and the \((2 + 0) \((\tilde{N}_2 = N_0, \tilde{N}_0 = N_2)\) bound state wrapped over the dual torus with volume \( \tilde{V}_2 = (2 \pi/T)^2 V_2^{-1} \).

Applying T-duality along \((y_1, y_2)\) one finds that the T-dual background has (apart from the change of sign of \( B_{mn} \)) exactly the same form as \((3.9)\) but with \( \sin \theta \leftrightarrow \cos \theta \), i.e., in particular, with \[ K' \to \tilde{K}' = 1 + W \cos^2 \theta. \quad (3.11) \]

As one might expect, this transformation is equivalent to replacing \( f_2 \) by \( \tilde{f}_2 = 1/f_2 \) or \( N_0 \leftrightarrow N_2, V_2 \to \tilde{V}_2 \) (as well as changing \( Q_0^{(2)} \to Q_2 \) in \( W \) \((3.10)\) as \( W \) is to remain invariant).

The T-duality transforms also the \((0 + 2) \((n_0, n_2)\) probe wrapped over \( V_2 \) in the original theory into the \((2 + 0) \((\tilde{n}_2 = n_0, \tilde{n}_0 = n_2)\) probe wrapped over \( \tilde{V}_2 \) in the dual theory. The action of a D2-brane probe propagating in the dual type IIA supergravity background is (we use the static gauge; \( m, n = 0, 1, 2; i, j = 3, \ldots, 9 \))

\[
\tilde{I}_2 = -\tilde{T}_2 \int d^3 \tilde{x} \left[ e^{-\tilde{\phi}} \sqrt{-\det(\tilde{G}_{mn} + \tilde{G}_{ij} \partial_m x^i \partial_n x^j + \tilde{F}_{mn})} \right. \\
\left. \quad - \frac{1}{6} e^{mnk} \tilde{C}_{mnk} - \frac{1}{2} e^{mnk} 	ilde{C}_{m} \tilde{F}_{nk} \right], \quad (3.12) \]

where in the present context of T-dual theory

\[
\tilde{F}_{mn} \equiv T^{-1} \tilde{F}_{mn} + \tilde{B}_{mn}, \quad \tilde{T}_2 = \tilde{n}_2 \tilde{g}_s^{-1} (2 \pi)^{-1/2} T^{3/2} = \frac{n_0 (2 \pi T)^{1/2}}{g_s \tilde{V}_2}. \]

\[ \text{The transformation rules in [36] imply that } \tilde{C}_t = C_{ty_1 y_2} - C_t B_{y_1 y_2}, \text{ etc.} \]
To find the \((2 + 0)\) probe action one should introduce a constant magnetic field \(\tilde{F}_{12}\) proportional to \(\tilde{n}_0 = n_2\) (for a similar discussion of \(0 - (2 + 0)\) interaction see [3, 11]).

Substituting the \((2 + 0)\) source background \((T\)-dual of (3.3)\) into the action of the \((2 + 0)\) probe we get (we ignore the dependence on the spatial coordinates parallel to the brane)

\[
\tilde{I}_2 = -\tilde{T}_2 \int d^3\tilde{x} \left[ (K\tilde{K})^{-1/2} \sqrt{(1 - K\tilde{v}^2)(1 + K^{-1}\tilde{K}'^2\tilde{F}^2)} \right.
\]

\[
+ W\tilde{K}^{-1}\cos \theta + W\tilde{F}K^{-1}\sin \theta \right],
\]

where

\[
\tilde{F} = \tilde{F}_{12} = f_1 - W\tilde{K}^{-1}\sin \theta \cos \theta , \quad f_1 = T^{-1}\tilde{F}_{12} = \frac{2\pi \tilde{n}_0}{V_2 T \tilde{n}_2}, \quad v = \partial_x x_9. \tag{3.14}
\]

The system of parallel D2-branes wrapped over \(\tilde{V}_2\) at low energies is expected to be described by the SYM theory on \(\tilde{V}_2\) [3]. Let us show that in the low-energy or ‘Yang-Mills’ limit of \([13]\) this complicated-looking supergravity action indeed reduces to effective action (2.13) of SYM theory on \(\tilde{V}_2\) with the SYM coupling given in (1.4). Expressing the parameters in the action (3.13) in terms of the SYM parameters \(f_n, v, r\) and \(g_{YM}\) (see (2.8), (2.15))

\[
f_n = T^{-1}f_n, \quad v = T^{-1}v, \quad r = T^{-1}r, \quad g_{YM}^2 = (2\pi)^{1/2}T^{1/2}\tilde{g}_s = \frac{\tilde{n}_2 T^2}{T_2}, \tag{3.15}
\]

and taking the low-energy (or short-distance) limit \([13]\) \(T \to \infty\) (or \(\alpha' \to 0\)) with \(f_n, v, r, g_{YM}\) being fixed, we find that \(\sin \theta = 1 + O(T^{-2}), \cos \theta = T^{-1}f_2 + O(T^{-3})\) and

\[
K = 1 + H_2 T^2 \to H_2 T^2, \quad \tilde{K}' = 1 + H_2 T^2(1 + O(T^{-2})) \to H_2 T^2, \tag{3.16}
\]

\[
\tilde{F} = T^{-1}(f_1 - f_2) + O(T^{-3}), \quad H_2 \equiv \frac{3N_0 g_{YM}^2}{4\pi r_5}. \tag{3.17}
\]

Expanding (3.13) in powers of inverse string tension \(T^{-1}\), we finish with

\[
\tilde{I}_2 = -\frac{\tilde{T}_2}{T^2} \int d^3\tilde{x} \left[ H_2^{-1} \sqrt{(1 - H_2v^2)[1 + H_2(f_1 - f_2)^2]} \right.
\]

\[
+ T^2 + f_1 f_2 - \frac{1}{2}f_2^2 - H_2^{-1} + O(T^{-2}) \right] \tag{3.17}
\]

\[
= -\frac{\tilde{T}_2}{T^2} \int d^3\tilde{x} + \frac{\tilde{T}_2}{T^2} \int d^3\tilde{x} \left[ \frac{1}{2}v^2 - \frac{1}{2}f_1^2 - V + O(T^{-2}) \right], \quad V = O\left(\frac{1}{r^5}\right).
\]

Since \(\tilde{F}_{12} = \frac{\tilde{n}_0}{\tilde{V}_2} (\text{see } (3.13))\), the finite part of this action is equivalent to the SYM BI-type expression (2.13) (up to the constant ‘self-energy’ \(O(f^2)\) term).

We would like to stress that this result is non-trivial:

(i) though it may seem that the probe action (3.12) has already a BI-type form, it also contains a complicated dependence on the source flux parameter in the background
supergravity fields: the gauge field enters not only through $\tilde{F}$ but also through, e.g., the antisymmetric tensor field in $\tilde{F}$, i.e. the gauge field background is partially encoded in the geometry;

(ii) it is only in the special low energy limit ($T \to \infty$) that the two fluxes combine to form the difference appearing in the SYM-BI action (2.13).

This all-order agreement between the supergravity and SYM results for the $(2 + 0) - (2 + 0)$ interaction potential provides a check of the consistency of the low energy SYM limit of [13] and of the BI ansatz (1.7).

As expected, a similar agreement if found also for the $(2 + 0) - (2 + 0)$ system in the ‘pre- T-duality’ picture, i.e. when one considers the two D2-branes and describes their D0-brane content by the magnetic fluxes directly related to the SYM fluxes. In this case the $2 + 0$ background is described by (3.9) with $f_2 \to 1/f_2$ (i.e. with $W = \frac{\mathcal{Q}_2}{f_2} \sqrt{1 + f_2^2}$, $\sin \theta = \frac{1}{\sqrt{1 + f_2^2}}$). Then $f_2$ and $f_1$ in the corresponding probe action ((3.13) with $\sin \theta \leftrightarrow \cos \theta$, $T_2 = n_2 g_s^{-1}(2\pi)^{-1/2} T^{3/2}$) are proportional to the SYM field strength, so that the above limit gives again (3.17).

3.2 Membrane – membrane interaction in $D = 11$ supergravity with compact light-like direction

We shall now demonstrate that the equivalence between the SYM and supergravity results for the membrane–membrane potential can be established directly (with no need to take the special limit of the supergravity expression) in the framework of the $D = 11$ supergravity with compact light-like direction, in agreement with previous suggestions [10, 9, 15].

The action of a M2-brane probe with tension $T_2^{(1)}$ (2.3) propagating in curved $D = 11$ background is (we use the static gauge with $\tau = \frac{1}{2} x^+$)

$$S_2 = - T_2^{(1)} \int d^3 x \left[ - \det(g_{mn} + \partial_m x^i \partial_n x^j g_{ij}) - C_{\tau x_1 x_2} \right].$$

In the case of the background produced by a boosted membrane source averaged in $x^-$ direction we get (see (3.4),(3.3),(3.8))

$$C_{\tau x_1 x_2} = C_{tx_1 x_2} \frac{dt}{d\tau} + C_{11 x_1 x_2} \frac{dx_{11}}{d\tau} = (e^{-\beta} - \frac{1}{2} \dot{x}^- e^\beta)(\mathcal{K}_-^{-1} - 1), \quad \dot{x}^- = \partial_\tau x^-, \quad \mathcal{K}_- = 1 + \mathcal{W}_-$$

and thus ($v_1 = \partial_\tau x_1, \quad \mathcal{K}_- = 1 + \mathcal{W}_-$)

$$S_2 = - T_2^{(1)} \int d^3 x \left[ \mathcal{K}_-^{-1} \sqrt{- \left[ e^{-2\beta} \mathcal{W}_- + (2 + \mathcal{W}_-) \dot{x}^- + \frac{1}{4} e^{2\beta} \mathcal{W}_- (\dot{x}^-)^2 + \mathcal{K}_- v_1^2 \right]} - (e^{-\beta} - \frac{1}{2} \dot{x}^- e^\beta)(\mathcal{K}_-^{-1} - 1) \right].$$

Since we are going to consider the probe with fixed value of the light-like momentum $p_-$, we are to perform, as in [3], the Legendre transformation $\dot{x}^- \to p_-$ and set $p_-$ to be
constant. From a more general point of view, this transformation is a special case of a
\( d = 3 \) world-volume duality transformation that rotates a scalar into a vector (and, in the
case of space-like \( x_{11} \)-compactification, relates the M2-brane action to the D2-brane action
(3.3)). The transformation \( x^{-} \rightarrow A_{m} \) is discussed in Appendix, where it is demonstrated
that \( p_{-} \) has the interpretation of the (inverse of) magnetic field strength.

Namely, let us assume that the membrane coordinates depend only on \( \tau \) and compute
\[
S'_{2} = \int d\tau L', \quad L' = L(x^{-}(p_{-})) - \dot{x}^{-} p_{-}.
\]
As a result,
\[
L' = 4e^{-2\beta}p_{-}\left(1 - \sqrt{\frac{1 - v^{2}W_{-}}{4e^{-2\beta}}}[1 + \frac{(2e^{-\beta}p_{-} - m_{1})^{2}W_{-}}{4e^{-2\beta}p_{-}^{2}}]\right)
+ 2e^{-\beta}(e^{-\beta}p_{-} - m_{1}), \quad m_{1} = T_{2}^{(1)}V_{2}.
\] (3.21)
Let us introduce (see (3.8),(3.2))
\[
H_{2} = \frac{W_{-}}{4e^{-2\beta}} = \frac{W_{-}}{f_{2}^{2}} = \frac{3QN_{0}}{16R^{2}T_{2}^{(2)}V_{2}r^{5}} = \frac{3\pi N_{0}}{R^{2}M_{5}^{3}V_{2}r^{5}},
\] (3.22)
and
\[
f_{2} = 2e^{-\beta} = \frac{m_{2}}{P_{-}}.
\] (3.23)

\( N_{0} \) is related to the boost (with parameter \( \beta \)) applied to the source membrane and \( P_{-} \) is
the momentum of the source. Both membranes are assumed to be wrapped over the torus
with volume \( V_{2} \). The relation (3.23) between the dimensionless flux \( f_{2} \) corresponding to
the source membrane and the boost \( \beta \) can be understood as follows: with the choice of
the time variable \( \tau = \frac{1}{2}x^{+} \), the light-cone Hamiltonian and momentum are
\[
E_{\tau} = E - P_{11} = me^{-\beta}, \quad P_{-} = \frac{1}{2}(E + P_{11}) = \frac{1}{2}me^{\beta},
\] (3.24)
so that (cf. (2.6)) \( f = \frac{m_{1}}{P_{-}} = 2e^{-\beta} \).

Using (3.22) and (3.23) and introducing the flux corresponding to the probe membrane
\[
f_{1} = \frac{m_{1}}{p_{-}},
\] (3.25)
we can rewrite the probe Lagrangian (3.21) in terms of \( f_{1} \) and \( f_{2} \)
\[
L' = \frac{m_{1}}{f_{1}} \left[ H_{2}^{-1} \left(1 - \sqrt{(1 - H_{2}v^{2})(1 + H_{2}(f_{1} - f_{2})^{2})}\right) + \frac{1}{2}r_{2}^{2} - f_{1}f_{2}\right]
= p_{-}(\frac{1}{2}v^{2} - \frac{1}{2}f_{1}^{2} - \mathcal{V}) = -\frac{m_{1}^{2}}{2p_{-}} + \frac{p_{-}v^{2}}{2} - p_{-} \mathcal{V}, \quad \mathcal{V} = O(\frac{1}{r^{5}}).
\] (3.26)
The constant terms here are in agreement with the general dual \( (x^{-} \rightarrow A_{m}) \) form of the
membrane action found in Appendix.

Using, finally, the relations (2.5),(2.15),(2.14) we observe that
\[
H_{2}v^{2} = H_{2}v^{2}, \quad H_{2}(f_{1} - f_{2})^{2} = H_{2}(f_{1} - f_{2})^{2}, \quad \frac{m_{1}}{f_{1}H_{2}} = \frac{n_{0}V_{2}}{g_{\text{YM}}^{2}H_{2}},
\] (3.27)
and thus conclude that (3.26) is equivalent to the $D = 10$ action (3.17) as well as to the SYM expression for the potential in (2.13).

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Appendix A Light-like scalar – vector duality transformation of membrane action

The $d = 3$ duality transformation $x_{11} \rightarrow A_m$ is known to relate the flat-space Nambu-type membrane action $\int \sqrt{-\text{det} \, 3 \left( \partial_m x^M \partial_n x^M \right)}$ and the Born-Infeld D-membrane action $\int \sqrt{-\text{det} \, 3 \left( \partial_m x^\alpha \partial_n x^\alpha + F_{mn} \right)} [33]$. If one considers instead the duality transformation $x^- \rightarrow A_m$ the result is quite different as we find below. This new dual action can probably be viewed as a special singular limit of the curved-space D2-brane BI action. We suspect it may have some interesting applications, apart from being the free part of the action (3.26) derived in section 3.2.

Let us start with the membrane action in flat $D = 11$ background ($ds_{11}^2 = dx^+ dx^- + dx_i dx_i$), choosing the static gauge with $\tau = \frac{1}{2} x^+$ ($m, n = (\tau, a), \ a, b = 1, 2$)

$$S_2 = -\frac{1}{2} T_2 \int d^3 x \sqrt{-\text{det} \, 3 \, h_{mn}} \rightarrow \frac{1}{2} T_2 \int d^3 x \left[ U h_{\tau\tau} \text{det}_2 (h_{ab} - h_{\tau a} h_{\tau b}) - U^{-1} \right]. \quad (A.1)$$

Here $U$ is an auxiliary field introduced to ‘linearise’ the square root and

$$h_{\tau\tau} = 2 \partial_{\tau} x^- + \partial_{\tau} x^i \partial_{\tau} x^i, \quad h_{\tau a} = \partial_{\tau} x^- + \partial_d x^i \partial_{\tau} x^i, \quad h_{ab} = \delta_{ab} + \partial_a x^i \partial_b x^i.$$ To perform the duality transformation $x^- \rightarrow A_m$ we are to replace $\partial_m x^-$ by a vector $\Lambda_m$, add the Lagrange multiplier term

$$-\frac{1}{2} T_2 \int d^3 x \, \epsilon^{mnk} \Lambda_m F_{nk} = -T_2 \int d^3 x \, (\Lambda_\tau F + \Lambda_a F^a), \quad (A.2)$$

$$F_{nk} = \partial_n A_k - \partial_k A_n, \quad F = F_{12}, \quad F^a = \epsilon^{ab} F_{br},$$

and ‘intergate out’ $\Lambda_m$. Since the induced metric $h_{mn}$ depends on $\partial_m x^-$ only linearly, it is useful to redefine $\Lambda_m \rightarrow \lambda_m$ ($\Lambda_\tau = \frac{1}{2} \lambda_\tau - \frac{1}{2} \partial_{\tau} x^i \partial_{\tau} x^i, \ \Lambda_a = \lambda_a - \partial_{\tau} x^i \partial_a x^i$) so that $h_{\tau\tau} = \lambda_\tau, \ h_{\tau a} = \lambda_a$. Then

$$S'_2 = \frac{1}{2} T_2 \int d^3 x \left[ U \lambda_\tau \text{det}_2 (h_{ab} - \lambda_{\tau}^{-1} \lambda_a \lambda_b) - U^{-1} - (\lambda_\tau - \partial_{\tau} x^i \partial_{\tau} x^i) F - 2(\lambda_a - \partial_{\tau} x^i \partial_a x^i) F^a \right]. \quad (A.3)$$

Since

$$\text{det}_2 (h_{ab} - \lambda_{\tau}^{-1} \lambda_a \lambda_b) = h (1 - \lambda_{\tau}^{-1} h_{ab} \lambda_a \lambda_b), \quad h \equiv \text{det}_2 h_{ab},$$

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it is now easy to integrate out \( \lambda_a \),

\[
S'_2 = \frac{1}{2} T_2 \int d^3 x \left[ U h \lambda_r - F \lambda_r - U^{-1} + U^{-1} h^{-1} h_{ab} F^a F^b + \partial_x x^i \partial_{\tau} x^i F + 2 \partial_x x^i \partial_a x^i F^a \right]. \tag{A.4}
\]

Solving for \( \lambda_r \) gives \( U h = F \); eliminating \( U \), we finally obtain

\[
S'_2 = \frac{1}{2} T_2 \int d^3 x \left[ - F^{-1} \det_2 (\delta_{ab} + \partial_a x^i \partial_b x^i) + F \partial_x x^i \partial_{\tau} x^i + 2 F^a \partial_a x^i \partial_a x^i \right. \tag{A.5}
\]

\[
+ F^{-1} F^a F^b (\delta_{ab} + \partial_a x^i \partial_b x^i) \right],
\]

or, equivalently,

\[
S'_2 = \frac{1}{2} T_2 \int d^3 x \left[ - F^{-1} \det_2 (\delta_{ab} + \partial_a x^i \partial_b x^i) + F^{-1} F^a F_a + F^{-1} (F^a \partial_a x^i + F \partial_{\tau} x^i)^2 \right]. \tag{A.6}
\]

The last term can be rewritten also as \((\frac{1}{2} \epsilon^{m n k} F_{m n} \partial_k x^i)^2\). When \( \partial_a x^i = 0 \) and \( F_{m n} = \text{const} \) this action gives the first two terms in the action (3.26) \((S'_2 = f d\tau L'_2)\) with \( p_-= T_2 V_2 F \).

The magnetic field \( F \) is thus the inverse of the flux \( f \) (3.25), which is proportional to the SYM flux (2.5).

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