Stochastic state-transition-change process and particle physics

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Abstract The description of particle phenomena is important for basically all fields of physics. The contemporary theoretical statistical description of a particle phenomenon typically requires knowledge of the Hamiltonian of the system, cross sections or other information concerning various microscopic processes. However, this knowledge may not be available and need to be determined first on the basis of experimental data. Moreover, the diverse hitherto theoretical approaches are either restricted to relatively very small class of particle phenomena or are not easily usable for analysis of experimental data. Therefore, new state-transition-change (STC) stochastic process is proposed. STC stochastic process satisfies Markov property and opens up new possibilities for obtaining in unified way deeper insight to experiments having independent outcomes. With its help it is possible to determine probability density functions characterizing states of a system and probabilities of transitions of the states on the basis of experimental data. For example, in the context of particle physics it allows to describe in unified way particle decays, random motion of particles, particle–matter interactions (such as transmission of light, laser beam, through various optical elements), as well as particle–particle interactions (collisions). Not all particle experiments satisfy the assumptions of STC stochastic process, but broad class of particle experiments can be analyzed with its help.

1 Introduction

Understanding of particle phenomena is essential as they concern, in one way or another, basically every field of physics. Several theoretical approaches have been developed to describe particle transport phenomena. Often an equation describing a transport phenomena originally introduced in one field of physics has been later used in another area of physics due to similarities of many transport phenomena [1, 2].

Theoretical descriptions of transport phenomena may be divided into two groups: macroscopic and microscopic, each has its own advantages and disadvantages. The first group tries to describe observed transport phenomena on macroscopic level. For example, continuum mechanics is based on the concept of continuum rather than on discrete particles (corpuscles) to model certain type of transport phenomena. The concept of continuum is used in fluid mechanics to derive, e.g., the Navier–Stokes equations to characterize motion of various viscous fluid substances. Fluid mechanics is based on the concept of momentum transfer [3]. Several transport equations used to describe momentum, heat and mass transfer are summarized in [3]. The equation of radiative transfer deals with transfer of energy in a medium which absorbs, emits or scatters radiation [4] (for applications in astrophysics see also [5]). The radiative transfer and heat transfer are specific cases of energy transfer. There are other transport equations modeling transport phenomena as an overall macroscopic effect using the concept of, e.g., momentum, energy or mass flow taking into account transport of individual particles only partially or not at all.

The second group of transport theories\(^1\) is based on the concept of particles. Time dependent position and momentum of one particle can be described with the help of Newton’s second law of motion (law of force) or Hamilton’s equations. These equations describe motion of particles deterministically. Let us focus on statistical descriptions of particle transport phenomena.

Consider a group of \(N\) particles. Hamiltonian formulation of mechanics allows to derive the Liouville equation. The equation merges classical mechanics with probability theory, and it represents one of the basic equations of statistical mechanics. The Liouville equation is the evolution equation of the probability density function of finding particle at given position, momentum and time. It is necessary to know both the Hamiltonian of the system and initial position and momentum of each of the particles to determine the

\(^1\) The term kinetic theory [6] is sometimes [7] regarded as the aspect of statistical mechanics that is concerned with the derivation and study of equations for the particle phase space density or distribution function (called kinetic equations or transport equations). The term transport theory is in [7] restricted to mathematical discipline concerned with the solutions of kinetic equations and applications of the solutions to the study of particle transport processes. We will not use this distinction. The term kinetic theory will not be used at all and the term transport theory will be used in very general sense denoting any theory which describes transport phenomena, not necessary related directly to particles, and try to provide some insight into transport processes.

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probability density function. Determining the probability density function may be, therefore, very difficult or not possible at all in many cases (especially if \( N \) is high).

The Liouville equation can be reduced, under certain conditions, to the Boltzmann equation also called Boltzmann transport equation [8]. It has been introduced in thermodynamics by L. Boltzmann in 1872; it has, therefore, long history. Using the Boltzmann equation one may obtain less detailed statistical description of the physical system than using the Liouville equation, but one does not need to know the initial positions and momenta of all the particles (the number of degrees of freedoms is significantly reduced). The Boltzmann equation requires to know the external force acting on the particles. The interactions between particles, if not neglected, are described by a collision term which is in general not easy to determine. Analytical solution of the Boltzmann equation is known only in simpler cases; various numerical methods have been developed in the past to find (at least approximate) solution of the Boltzmann equation in more complicated cases [8, 9].

The Boltzmann equation or similar equations (including relativistic and quantum mechanical analogues) are widely used for statistical description of many transport phenomena. Theoretical description of charged particle transport in gases and condensed matter is summarized in detail in [10]. The equation forms a basis of molecular thermodynamics [11]. Electron transport in semiconductors is modeled with the help of the Boltzmann equation, too [12]. The Vlasov–Fokker–Planck equation, special form of the Boltzmann equation, found its application in plasma modeling [13–15].

Another widely known equation which is closely connected to statistics and probabilities is the Schrödinger equation. It requires knowledge of the Hamiltonian of the system (or the Hamiltonian needs to be determined on the basis of experimental data). Its application to particle transport phenomena may be, however, very delicate already in the case of simple physical phenomena such as (elastic) proton-proton scattering at high energies (more comments to description of this collision process will be given later in Sect. 5.4).

Monte Carlo (MC) methods represent another broad group of theoretical statistical methods used for description of particle transport phenomena. To simulate a random process, i.e., sample from a probability density function for an event, it uses random numbers or pseudorandom numbers. MC methods are summarized in [16]. To make a MC simulation one first needs to identify basic physical processes. Determination of the probability density functions of the processes typically requires detailed knowledge of cross sections of various interaction types, mean free paths, spatial distribution and types of scattering centers, etc.

It is worth to mention statistical optics [17]. It deals with statistical properties of light waves. Its application to data concerning propagation of light phenomena is based on several strong assumptions (such as introduction of complex amplitudes, various approximations concerning the amplitudes etc.). It does not fully explore the concept of quantum of light-photon (discovered by Planck [18, 19] and Einstein [20] at the beginning of twentieth century), and it can be hardly used for other particle types and in different fields than optics.

The contemporary theoretical statistical methods used for description of particle transport phenomena have something in common. They all require knowledge of the Hamiltonian of the studied system, various cross sections or other information which, however, in many cases is not a priory known and need to be determined first, using a suitable method, on the basis of experimental data. For example, a particle from a beam passes through a medium with certain probability and, if it passes through, its properties may or may not change according to the corresponding conditional probability density function. The contemporary widely used theoretical statistical methods (see also the statistical mechanics [21]) do not allow to easily determine these conditional probabilities. Moreover, e.g., the probability density function determined on the basis of the Boltzmann equation is function of position, momentum and time (i.e., 7 variables in total); however, different variables may be essential for explanation of measured characteristics of given particle transport phenomenon.

Particle transport may be regarded as stochastic (random) process, a system is transition through some states described by random variables. Brownian motion (random walk) is a well-known example of stochastic process. Another well-known stochastic process used in context of particle physics (decay of particles) is Poisson process. In this paper the theory of stochastic processes will be used to extend and unify description of many particle transport phenomena.

This paper is structured as follows. The statement of the problem is further discussed in Sect. 2. New state-transition-change (STC) stochastic process is introduced in Sect. 3 where several formulae describing the process are derived, too. General guidelines for analysis of experimental data using the probabilistic model are in Sect. 4. Several examples of applications of the stochastic process to experimental data of particle phenomena are summarized in Sect. 5. Sections 5.1 to 5.4 show that four very different types of particle phenomena (particle decay, particle motion, particle–matter interaction and particle–particle interaction) can be all described with the help of the STC process. Concluding remarks can be found in Sect. 6.

2 Problem statement

Definition 2.1 (particles) By particles we mean any countable set of object such as photons, neutrons, electrons, ions, atoms, molecules, etc.
Sequential arrangement of many particle transport experiments. A beam of particles passing through control surfaces $\Sigma_i$, $i \in (0, 1, \ldots, M)$, which define $M$ transport segments.

**Definition 2.2 (Control surface)** Two dimensional surface (flat or curved) which one can imagine anywhere in space is called control surface. It may or may not correspond to physical surface of some matter objects or media (such as prisms, foils, glass, water, mirror, ...).

Let us consider a particle beam passing through a sequence of control surfaces $\Sigma_i$, $i \in (0, 1, \ldots, M)$, as schematically shown in Fig. 1. Particles can have very different properties, see Definition 2.1. The control surfaces (see Definition 2.2) may or may not be orthogonal to the beam. The control surfaces define $M$ transport segments as shown in Fig. 1. $i$-th transport segment, $i \in (0, \ldots, M−1)$, may contain any medium, field or free space (i.e., anything) between two control surfaces $\Sigma_i$ and $\Sigma_{i+1}$.

There are many particle transport phenomena which have this sequential arrangement or more general arrangement containing splitting and merging of beams as it is very common in many laser (optical) systems. In the case of light beams there are many optical components widely used to change properties of the beam (prisms, polarizers, Faraday rotators, lenses, beam splitters, mirrors, various electro-optic and magneto-optic devices, etc.). In the case of charged particle beams various magnets and other devices used for beam steering are commonly denoted as optical components, too.

Interaction of a particle with $i$-th transport segment may depend on various variables (parameters) characterizing the transport segment, e.g., rotation of an optical component (its spatial orientation with respect to the beam), an electric or magnetic field applied to it, temperature of a medium, etc. These variables may depend, e.g., on time or other variables.

Various detector technologies and experimental methods are used to measure quantities characterizing both beam particles passing through the control surfaces $\Sigma_i$ and properties of a medium placed between $\Sigma_i$ and $\Sigma_{i+1}$. These quantities are typically measured as functions of only some of the variables characterizing state of a system (consisting of input or output particles and the transport segment), in limited intervals of their values, and often without the normalization etc. i.e., it is often difficult or even impossible to determine all characteristics of the system on experimental ground.

More experimental information about the system (particle beam and transport segments) may be obtained by changing the order of the transport segments in the sequence and measuring the quantities for each permutation of the transport segments. A lot of useful experiments data about given transport segment (“sample”) may be obtained if it is irradiated by beams of different properties and the beam properties of the output (reflected or transmitted) beams are measured. The transport segments which precede the given sample modify the properties of the input source beam. The transport segments following the sample in the sequence allow better analysis of the properties of the output beam which interacted with the sample.

As to the theoretical description of particle transport phenomena. As mentioned in the introduction there are several theoretical approaches used for description of particle transport phenomena but they have limited region of applicability. They are used typically only for a specific class of particle transport phenomena and provide only partial insight. Unified description is missing. They are many particle transport phenomena which have never been satisfactorily explained at all. Transmission of particles through a medium is often considered as probabilistic process but the probability (density) functions characterizing the transitions have never been determined (except very special cases). Transmission of light trough 3 linear polarizers may serve as an example which has small number of optical elements, the transmission of light can be measured as function of rotation angles of the polarizers, but its contemporary theoretical description is not satisfactory.

### 3 Probability model

Section 3.1 summarizes for convenience basic relations and definitions concerning probability and probability density functions. It also introduces basic notation which will be used later. Control event defined in Sect. 3.1.6 and density of states of an event defined in Sect. 3.1.7 are the only terms which are not widely used, or may have different meaning.

Section 3.2 starts with standard definitions concerning stochastic processes [22–24]. The general concept of density of states of an event is then used to derive several relations describing newly introduced stochastic process.
3.1 Probability

3.1.1 Probability, basic relations

**Definition 3.1 (Probability)** The probability \( P(A) \) of an event \( A \) is the proportion (relative frequency) of times (denoted as \( N(A) \)) that the event is expected to occur when an experiment is repeated a large number of times \( N \) under identical conditions

\[
P(A) = \frac{N(A)}{N}.
\]  

(1)

**Definition 3.2 (Conditional probability)** Let \( A \) and \( B \) be two events. If \( P(A) \neq 0 \) then the conditional probability of \( A \) given \( B \) is defined as the ratio

\[
P(B \mid A) = \frac{P(A \cap B)}{P(A)}
\]  

(2)

where \( P(A) \) is the unconditional probability of \( A \), and \( P(A \cap B) \) is the probability that both events \( A \) and \( B \) occur (\( \cap \) denotes the intersection). It holds (for the reason of symmetry)

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)}.
\]  

(3)

(assuming \( P(B) \neq 0 \)).

**Proposition 3.1** Let \( A_0, A_1, \ldots, A_{n-1} \) where \( n \geq 2 \) be a sequence of events. If \( P(A_0) \neq 0 \) then

\[
P(A_1 \cap \ldots \cap A_{n-1} \mid A_0) = P(A_{n-1} \mid A_0 \cap \ldots \cap A_{n-2})P(A_{n-2} \mid A_0 \cap \ldots \cap A_{n-3})\ldots P(A_2 \mid A_0 \cap A_1)P(A_1 \mid A_0).
\]  

(4)

**Proof** Equation (2) implies the chain rule (\( n \geq 2 \))

\[
P(A_0 \cap \ldots \cap A_{n-1}) = P(A_{n-1} \mid A_0 \cap \ldots \cap A_{n-2})P(A_{n-2} \cap \ldots \cap A_{n-3})\ldots P(A_2 \mid A_0 \cap A_1)P(A_1 \mid A_0).
\]  

(5)

Using Eq. (5) recursively one obtains

\[
P(A_0 \cap \ldots \cap A_{n-1}) = P(A_{n-1} \mid A_0 \cap \ldots \cap A_{n-2})P(A_{n-2} \mid A_0 \cap \ldots \cap A_{n-3})\ldots P(A_1 \mid A_0)P(A_0).
\]  

(6)

This allows to factorize the probability \( P(A_0 \cap \ldots \cap A_{n-1}) \) with the help of conditional probabilities. By changing the order of the events one obtains different expressions how the probability \( P(A_0 \cap \ldots \cap A_{n-1}) \) can be factorized. The factorization of the probability can be chosen according to given use case.

It holds (see Eq. (2))

\[
P(A_0 \cap (A_1 \cap \ldots \cap A_{n-1})) = P(A_1 \cap \ldots \cap A_{n-1} \mid A_0)P(A_0).
\]  

(7)

Using Eqs. (7) and (5) one obtains

\[
P(A_1 \cap \ldots \cap A_{n-1} \mid A_0)P(A_0) = P(A_{n-1} \mid A_0 \cap \ldots \cap A_{n-2})P(A_0 \cap \ldots \cap A_{n-2}).
\]  

(8)

Taking into account the assumption that \( P(A_0) \neq 0 \) and dividing Eqs. (6) and (7) by \( P(A_0) \) one obtains Eq. (4). □

**Remark 3.1** Let \( A \) and \( B \) be two events. If \( B \) can be expressed as union of \( n \geq 2 \) events, then Proposition 3.1 allows factorization of conditional probability \( P(B \mid A) \) into a product of \( n \) probabilities.

**Definition 3.3** Probability \( P(A) \) of an event \( A \) can depend on one or more variables. In this case one can write \( P_A(\ldots) \) where “...” denotes the variables (arguments of the functions). Similar notation can be introduced for conditional probability and other functions of an event (given possibly another event).

3.1.2 Integration over multiple variables

Let \( x \) represents several variables, i.e., let \( x = (x_0, x_1, \ldots, x_{n-1}) \) be finite ordered set of \( n \) variables. Multiple integral of a function \( f(x) \) over its domain may be then written as

\[
\int x f(x)dx = \int_{x_{n-1}} \ldots \int_{x_1} \int_{x_0} f(x_0, x_1, \ldots, x_{n-1})dx_0dx_1\ldots dx_{n-1}
\]  

(9)

where

\[
dx = dx_0dx_1\ldots dx_{n-1}.
\]  

(10)
If \( x_1 = ((x_0)_1, (x_1)_1, \ldots, (x_{n-1})_1) \) and \( x_2 = ((x_0)_2, (x_1)_2, \ldots, (x_{n-1})_2) \) then one may define

\[
\int_{x_1}^{x_2} f(x) \, dx = \int_{(x_{n-1})_1}^{(x_{n-1})_2} \cdots \int_{(x_1)_1}^{(x_1)_2} \int_{(x_0)_1}^{(x_0)_2} f(x_0, x_1, \ldots, x_{n-1}) \, dx_0 \, dx_1 \cdots dx_{n-1}.
\] (11)

If a variable has discrete values then the corresponding integration needs to be replaced by sum.

We may further define an abbreviation for \( n \)-dimensional interval

\[(x_1, x_2) = ((x_0)_1, (x_0)_2) \times ((x_1)_1, (x_1)_2) \times \cdots \times ((x_{n-1})_1, (x_{n-1})_2)\]

and infinitesimal \( n \)-dimensional interval

\[(x, x + dx) = ((x_0) + dx_0, (x_1) + dx_1) \times \cdots \times ((x_{n-1}) + dx_{n-1})\]

with a convention that for a discrete variable the corresponding interval contains only one value.

**Remark 3.2** A simple rule for understanding an expression (operation) containing variables represented by \( x \) is the following. One may first imagine that \( x \) represents only one continuous variable. If the variable is discrete then the expression needs to be replaced by analog expression for discrete variable (replacing an integral by sum, infinitesimally small interval by a point, ...). If \( x \) represents more than one variable then the expression needs to be generalized for multiple variables. If \( n = 0 \) then the integration of a function over \( x \) may be removed from the expression. Integration theory provides deeper reasoning of these rules, see chapter 2 in [24].

**Remark 3.3** In the context of probability theory functions which are typically non-negative (being product of probability functions and probability density functions having finite normalization) are often integrated over many variables. The Fubini-Tonelli theorem can be used to change order of the integrations and to prove the existence of the multidimensional integrals.

### 3.1.3 Probability density function

**Definition 3.4** Let us denote the non-random variables specifying an event \( A \) as \( X^{NR}(A) \) and the random variables of \( A \) as \( X(A) \).

**Definition 3.5** Infinitesimal interval of continuous random variables of an event \( A \) is denoted as \( dX(A) \) or, equivalently, as \( d(A) \), i.e.,

\[
d(A) = dX(A).
\] (14)

If the event \( A \) has no continuous random variable then

\[
d(A) = 1.
\] (15)

**Definition 3.6** If \( X(A) \) contains at least one continuous random variable then the probability \( P(A) \) can be expressed with the help of probability density function \( \rho \)

\[
P(A) = \rho(A) \, d(A).
\] (16)

\( P(A) \) is infinitesimally small probability (it will not be denoted as \( dP(A) \)).

**Proposition 3.2** If

\[
\int_{X(A)} P(A) = 1
\] (17)

then the probability density function is normalized to unity

\[
\int_{X(A)} \rho(A) \, d(A) = 1.
\] (18)

**Proof** By inserting Eqs. (16) to (17) one obtains Eq. (18). \( \square \)

**Definition 3.7** In the case of conditional probability of event \( A \) given event \( B \) one can introduce conditional probability density function \( \rho \)

\[
P(A \mid B) = \rho(A \mid B) \, d(A \mid B)
\] (19)

and the random variables of event \( A \) given event \( B \) are those which are random variables of event \( A \) and not variables of event \( B \). The infinitesimal interval \( d(A \mid B) \) corresponds to the continuous random variables of event \( A \) given event \( B \), i.e., it does not contain the infinitesimal intervals corresponding to the random variables of event \( B \). If the event \( A \) given event \( B \) does not depend on any continuous random variable then

\[
d(A \mid B) = 1.
\] (20)
Proposition 3.3 If $B \neq \emptyset$ and
\[ \int_{\mathcal{X}(A)} P(A \mid B) = 1 \] (21)
then the probability density function is normalized to unity
\[ \int_{\mathcal{X}(A|B)} \rho(A \mid B) d(A \mid B) = 1. \] (22)

Proof By inserting Eqs. (19) to (21) one obtains Eq. (22). □

Remark 3.4 In the case of probabilities depending on continuous random variables one can replace symbol $P$ in an equation like Eq. (2), containing multiplication of probabilities) by $\rho$ (resp. $d$) to obtain analogical expressions for probability density functions (resp. infinitesimal intervals). Probability density functions may be factorized similarly as one can factorize probability functions depending on discrete random variables, see Sect. 3.1.1. For example, it holds (see Eq. (8))
\[ d(A_1 \cap \ldots \cap A_{n-1} \mid A_0) d(A_0) = d(A_{n-1} \mid A_0 \cap \ldots \cap A_{n-2}) d(A_0 \cap \ldots \cap A_{n-2}). \] (23)
and (if $\rho(A_0) \neq 0$, see Proposition 3.1)
\[ \rho(A_1 \cap \ldots \cap A_{n-1} \mid A_0) = \rho(A_1 \mid A_0) \rho(A_2 \mid A_0 \cap A_1) \ldots \rho(A_{n-2} \mid A_0 \cap \ldots \cap A_{n-2}) \rho(A_{n-1} \mid A_0 \cap \ldots \cap A_{n-2}). \] (24)

3.1.4 Implying events

Proposition 3.4 Let event $A$ implies event $B$, i.e.,
\[ A = A \cap B \] (25)
which is equivalent to
\[ A \subseteq B. \] (26)

It holds
\[ P(A) = P(A \cap B), \] (27)
\[ P(A) = P(B)P(A \mid B), \] (28)
\[ N(A) = N(B)P(A \mid B). \] (29)

Proof Equation (27) follows from Eq. (25). Equations (27) and (3) imply Eq. (28). The number of occurrences of event $A$ in an experiment given by Eq. (29) follows from Eqs. (3) and (1). □

3.1.5 Mutually independent events

Definition 3.8 (Mutually independent events) Two events $A$ and $B$ are mutually independent if and only if it holds
\[ P(A \cap B) = P(A)P(B). \] (30)

If two events are not mutually independent then they are mutually dependent. More generally, a family $\{A_n\}_{n \in \mathbb{N}}$ of events is called mutually independent if it holds for any finite set of indices $i_1 < \ldots < i_r$ where $i_j \in \mathbb{N}$ (1 $\leq j \leq r$)
\[ P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}) = P(A_{i_1})P(A_{i_2})\ldots P(A_{i_r}). \] (31)

Remark 3.5 The word mutually is often omitted for the sake of simplicity.
3.1.6 Control event

**Assumption 3.1** Given set (event) is not specified by any random continuous variable.

**Proposition 3.5** If an event B satisfies Assumption 3.1 then it holds

\[ d(B) = 1. \]  (32)

**Proof** It follows from Definition 3.5.

**Definition 3.9** (Control set (event)) A control set (event) B is an event satisfying Assumption 3.1.

**Proposition 3.6** Given event A implying control event B then

\[ d(A) = d(A | B). \]  (33)

**Proof** In Eq. (28) (see Proposition 3.4) one can replace symbol \( P \) by \( d \), see Remark 3.4, it yields

\[ d(A) = d(B)d(A | B). \]  (34)

Control event B according to Definition 3.9 satisfies Eq. (32) (see Proposition 3.5). Eqs. (32) and (34) imply Eq. (33).

**Proposition 3.7** Let B be a control set and A be a subset of B. It holds

\[ N(A) = N(B)\rho(A | B)d(A). \]  (35)

**Proof** Equation (29) (see Proposition 3.4) and Eq. (19) imply

\[ N(A) = N(B)P(A | B) \]

\[ = N(B)\rho(A | B)d(A | B). \]  (36) (37)

Equations (37) and (33) (see Proposition 3.6) imply Eq. (35).

3.1.7 Density of states

**Definition 3.10** (Density of states) Density of states represented by event A is the ratio

\[ \text{dos} (A) = \frac{N(A)}{d(A)}. \]  (38)

**Proposition 3.8** Let B be a control set and A be a subset of B. It holds

\[ \text{dos} (A) = N(B)\rho(A | B). \]  (39)

**Proof** It follows from Eqs. (38) and (35) (see Definition 3.9 and Proposition 3.7).

**Remark 3.6** The Eq. (39) can be used as alternative definition of density of states corresponding to event A if another event B satisfying Eq. (25) is given.

**Proposition 3.9** Integration of \( \text{dos} (A) \) over all its random variables given a control event B (which may reduce the limits of integration) yields number of occurrences of the control event B

\[ N(B) = \int_{X(A|B)} \text{dos} (A)d(A). \]  (40)

**Proof** Integration of Eq. (39) (see Proposition 3.8) over \( X(A | B) \) and using Eq. (33) (see Proposition 3.6) imply Eq. (40).

**Remark 3.7** \( N(B) \) represents normalization of the density \( \text{dos} (A) \) according to Proposition 3.9. Integration of \( \text{dos} (A) \) over just some random variables of A (given control event B) one obtains density of event corresponding to the random variables over which it is not integrated.

**Remark 3.8** Many quantities like particle number flux, number density (number of specified objects per unit volume), density of energy states of a system, etc. are commonly used in physics to describe density of states of various physical systems. Definition 3.10 represents unification and generalization of these quantities used in physics in special cases.
Proposition 3.10 Let $B$ be a control set. If $N(B) \neq 0$ then

$$\rho(A \mid B) = \frac{\text{dos}(A)}{\int_{X(A|B)} \text{dos}(A) \text{d}A(A)} \quad (41)$$

Proof If $N(B) \neq 0$ then one can divide Eq. (39) by $N(B)$ being equal to the right-hand side of Eq. (40). □

Proposition 3.11 If $A$ is an event satisfying Assumption 3.1, then

$$\text{dos}(A) = N(A) \quad (42)$$

Proof Definition 3.10 and Eq. (32) (see Proposition 3.5) imply Eq. (42). □

3.2 Stochastic STC processes

3.2.1 Generalities

Definition 3.11 (Index set) Let $I$ be an index set with the Borel $\sigma$-algebra $I$ (the smallest $\sigma$-algebra on $I$ containing all open sets or, equivalently, closed sets).

Definition 3.12 (Probability space) A probability space is triple $(\Omega, \mathcal{F}, P)$, where

1. $\Omega$ is a sample space, i.e., the set of all possible outcomes (also called realizations).
2. $\mathcal{F}$ is an event space, i.e., $\sigma$-algebra on $\Omega$ (admissible subsets of $\Omega$, a subset called event being a set of outcomes in the sample space).
3. $P$ is a probability function which assigns each event in the event space a probability, a number between 0 and 1.

Definition 3.13 (Stochastic process) A stochastic process is a family $X = \{X_i : i \in I\}$ of random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$ and, for fixed index $i$, taking their values $X_i$ in given space $S_i$ which must be measurable with respect to some $\sigma$-algebra $S_i$ of admissible subsets. i.e., Assumption 3.3 holds, see below.

Definition 3.14 (States and state space) The values of random variables $X_i$ for fixed $i \in I$ represent “states” and the set of all states corresponding to all the possible values of the random variables $X_i$ is called “state space” $S_i$ of the stochastic process for fixed $i$.

Assumption 3.2 (One common state space) There is one common state space $S$ for all $i \in I$.

Assumption 3.3 (Different state spaces) State spaces $S_i$ may or may not be the same for all $i \in I$.

Remark 3.9 (Different state spaces) Definition of a stochastic process is sometimes a priori restricted by Assumption 3.2. Definition 3.13 of stochastic process based on Assumption 3.3 is more general. Moreover, in the case of Assumption 3.3 a one-to-one correspondence of the states from one state space to another state space may or may not exist. If a state $s_i \in S_i$ transitioned to a state $s_j \in S_j$ ($i \in I$, $j \in I$, $i \neq j$), then the values of the random variables $X_i$ may or may not differ from the values of random variables $X_j$. Moreover, there may or may not be different numbers of variables in $X_i$ and $X_j$, or the numbers may be the same but their meaning may be completely different, or some of the random variables may or may not have the same domains.

Definition 3.15 (Non-random variables) Let $X_i^{NR}$ be non-random variables (parameters) influencing whether a system will be in a state $s_i \in S_i$ for given index $i \in I$.

Remark 3.10 (Dependence of functions on non-random variables) Values of random variables $X_i$ and non-random variables $X_i^{NR}$ (parameters) specify a state $s_i$ in state space $S_i$ (and vice versa). A system can transition from one state to another state. Therefore, a function of a subset of states in state space $S_i$ may in general depend also on all the non-random variables further specifying states of the system before they transitioned to a state in the subset. The dependence on the non-random parameters may or may not be written explicitly in the following.

Definition 3.16 (Sample function) For each $\omega \in \Omega$, the function $i \in I \rightarrow X_i(\omega_i) \in S_i$ is called the sample function, a realization or outcome (sometimes called also $\omega$-trajectory, or $\omega$-sample path), of the stochastic process $\{X_i : i \in I\}$. $\omega_i$ is one of the states $s_i \in S_i$.

Definition 3.17 (Branching process) Stochastic process $\{X_i : i \in I\}$ is called branching process if it holds for all indexes $i \in I$: if a state $s_i$ in state space $S_i$ transition to another state space $S_j$ ($j \in I$, $i \neq j$) then there is no other state space from which a state could transition to the state space $S_j$.

Remark 3.11 State space $S_i$ is a control set (event), see Definition 3.9. Density of states introduced in Sect. 3.1.7 is, therefore, suitable for quantifying densities of states $s_i \in S_i$. 
**Definition 3.18** Let us define (the newly defined functions being on the left-hand sides)

\[ \rho_{S_{i}}(X_{i}) = \rho_{s_{i}|S_{i}}(X_{i}) \]  
\[ N_{i} = N_{S_{i}} \]  
\[ \text{dos}_{i}(X_{i}) = \text{dos}_{s_{i}}(X_{i}) \]

for all \( i \in I \).

**Proposition 3.12** Let \( \{X_{i} : i \in I\} \) be stochastic process given by Definition 3.13. If \( S_{i} \neq \emptyset \) then

\[ \int_{X_{i}} \rho_{S_{i}}(X_{i})dX_{i} = 1 \]  
\[ \text{dos}_{i}(X_{i}) = N_{i}\rho_{S_{i}}(X_{i}) \]  
\[ N_{i} = \int_{X_{i}} \text{dos}_{i}(X_{i})dX_{i} \]

for all \( i \in I \).

**Proof** 1. Proposition 3.3 and Eq. (43), and definitions of \( s_{i} \) and \( S_{i} \) (see Definition 3.14) imply Eq. (46).
2. Using Eq. (39) for \( s_{i} \) which is a subset of \( S_{i} \), and Eq. (43), one obtains Eq. (47).
3. By integrating Eq. (47) over \( X_{i} \) and using the normalization condition given by Eq. (46) one obtains Eq. (48) (see also Eq. (40)).

### 3.2.2 STC process

**Assumption 3.4** *(No “lost” transitions)* If \( S_{j} \) is a set of all possible states \( s_{j} \) into which a system in any of the states \( s_{i} \in S_{i} \) can possibly transition then

\[ P(S_{j} | s_{i}) = 1 \]

for all states \( s_{i} \) and for all \( i \in I \) and \( j \in I \). I.e., the system in given state \( s_{i} \) always transition to a state in \( S_{j} \).

**Assumption 3.5** *(Possibility of no transition)* If \( S_{j} \) is a set of all possible states \( s_{j} \) into which a system in any of the states \( s_{i} \in S_{i} \) can possibly transition then the probability \( P(S_{j} | s_{i}) \), that the system in a state \( s_{i} \in S_{i} \) transitions to a state in \( S_{j} \), is in the interval from 0 to 1 (not necessarily in the whole interval)

\[ 0 \leq P(S_{j} | s_{i}) \leq 1 \]

for all states \( s_{i} \) and for all \( i \in I \) and \( j \in I \). I.e., the system in given state \( s_{i} \) may or may not transition to a state in \( S_{j} \) with the probability \( P(S_{j} | s_{i}) \).

**Remark 3.12** Stochastic processes discussed in literature are often based on restrictive Assumption 3.4. Assumption 3.5 is generalization of Assumption 3.4 which does not introduce any restriction on a system which can be modeled using a stochastic process based on Assumption 3.5.

**Assumption 3.6** *(Independence of realizations)* Realizations (outcomes) \( \omega \in \Omega \) are mutually independent.

**Remark 3.13** Assumption 3.6 is a strong requirement. There is, of course, significant loss of generality when description of stochastic processes is limited by this assumption. On the other side, the assumed independence strongly simplifies description of stochastic processes. Many processes (not only) in physics satisfy this assumption.

In the example of transport of particles discussed in Sect. 2, the Assumption 3.6 excludes phenomena when, e.g., a bunch of particles is absorbed by a medium such that transmission of any other particle through the medium is influenced by it. For example, the absorbed particles may change the properties of the medium. This is the case of, e.g., stimulated emission of photons where probabilities of transmission of two photons through a medium at different times may differ in dependence on the number of already absorbed photons in the medium (state of the medium at given time).

**Remark 3.14** Assumption 3.6 does not imply that if a state \( s_{i_1} \) transitions to a state \( s_{i_2} \) and the state \( s_{i_2} \) transitions to a state \( s_{i_3} \) (\( i_1 \neq i_2 \neq i_3, i_1 \in I, i_2 \in I, i_3 \in I \)) then the conditional probabilities of the two transitions are independent (they may or may not be independent).
Assumption 3.7 (Markov property) Let \( \{X_i : i \in I\} \) be stochastic process given by Definition 3.13. Let the process have Markov property. For example, if index set \( I \) is totally ordered \( \leq \) and \( (\mathcal{F}_i, i \in I) \) is filtration then
\[
P(s_{i_2} \in S_{i_2} | \mathcal{F}_{i_1}) = P(s_{i_2} \in S_{i_2} | s_{i_1})
\] (51)
for each \( i_2, i_1 \in I \) with \( i_1 < i_2 \).

Remark 3.15 Roughly speaking, Assumption 3.7 means that probability of future state of system depends on its present state, but not on the past states in which the system was. This property is also known as memoryless. Some processes are Markovian and some are not. The Markov property can appear to be violated when, e.g., not all important variables are not taken into account in description of a system (i.e., if states of the system are not sufficiently specified). The Markov property can appear to be violated also due to insufficient amount of data (information about the system). Assumption 3.7 cannot be made without a loss of generality. Markovian processes are subject of research with long history, they have found applications in many fields.

Definition 3.19 (STC process) Let us define stochastic process (see Definition 3.13) which satisfies Assumptions 3.3 and 3.5 to 3.7. This process is called state-transition-change (STC) process.

3.2.3 STC sequence

Definition 3.20 (Number of transitions) An evolving system can transition through sequence of \( M + 1 \) states where \( M \) is the number of the transitions, i.e., positive integer.

Definition 3.21 (STC sequence) Let \( \{X_i : i \in I\} \) be stochastic process given by Definition 3.19. Let the index set \( I \) be totally ordered sequence \((0, \ldots, M)\) and \( N_0 \neq 0 \). The process satisfies Assumptions 3.3 and 3.5 to 3.7.

Remark 3.16 If \( N_0 = 0 \) then there is no initial state in \( S_0 \) which could transition to \( S_1 \), etc. Therefore, one can assume \( N_0 \neq 0 \) without lost of generality. If \( N_i = 0 \) for a fixed \( i \) which satisfies \( 0 < i \leq M \) then \( N_j = 0 \) for all \( j \in I \) for which \( i \leq j \leq M \) (i.e., the system never transition to a state \( s_j \)).

Proposition 3.13 Let \( \{X_i : i \in I\} \) be stochastic process given by Definition 3.21. If \( N_0 \neq 0 \) then
\[
\frac{N_i}{N_0} = \int_{X_i} \frac{\text{dos}(X_i)}{N_0} \, dX_i
\] (52)
for all \( i \in I \).

Proof Dividing Eq. (48) by \( N_0 \) \((N_0 \neq 0)\) one obtains Eq. (52).

Definition 3.22 Let \( \{X_i : i \in I\} \) be stochastic process given by Definition 3.21. One can introduce the following functions (being on the left-hand sides of the equations)
\[
P_{T,i}(X_i) = P_{S_{i+1}|S_i}(X_i) \quad i = 0, \ldots, M - 1
\] (53)
\[
\rho_{C,i}(X_i, X_{i+1}) = \rho_{S_{i+1}|S_i \cap S_{i+1}}(X_i, X_{i+1}) \quad i = 0, \ldots, M - 1.
\] (54)

Proposition 3.14 Let \( \{X_i : i \in I\} \) be stochastic process given by Definition 3.21. It holds
\[
\rho_{S_{i+1}|S_i}(X_{i+1}) = \int_{X_i} \rho_{S_i \cap S_{i+1}|S_i}(X_i, X_{i+1}) \, dX_i.
\] (55)

Proof It holds
\[
P_{S_{i+1}|S_i}(X_{i+1}) = \rho_{S_{i+1}|S_i}(X_{i+1}) \, dX_{i+1}
\] (56)
\[
= \int_{X_i} P_{S_i \cap S_{i+1}|S_i}(X_i, X_{i+1}) \, dX_{i+1}
\] (57)
\[
= \int_{X_i} \rho_{S_i \cap S_{i+1}|S_i}(X_i, X_{i+1}) \, dX_{i+1} \, dX_i.
\] (58)
Equations (56) and (58) imply Eq. (55).

Proposition 3.15 Let \( \{X_i : i \in I\} \) be stochastic process given by Definition 3.21. It holds
\[
N_{S_{i+1}} = N_{S_i} P_{S_{i+1}|S_i}
\] (59)
and
\[
N_{S_{i+1}} \leq N_{S_i}.
\] (60)
Proof Equations (59) and (60) follow from Assumption 3.5.

**Theorem 3.1** (Transformation of density of states) Let \( \{X_i : i \in I\} \) be stochastic process given by Definition 3.21. It holds

\[
\text{dos}_{s_{i+1}}(X_{i+1}) = \int_{X_i} \text{dos}_s(X_i) P_{s_{i+1}|s_i}(X_i) \rho_{s_{i+1}|s_i \cap S_{i+1}}(X_i, X_{i+1}) dX_i, 
\]

or, equivalently, (using different notation, see Eqs. (45), (53) and (54))

\[
\text{dos}_{i+1}(X_{i+1}) = \int_{X_i} \text{dos}_i(X_i) P_{T, i}(X_i) \rho_{C, i}(X_i, X_{i+1}) dX_i.
\]

**Proof** If \( S_i = \emptyset \) for given \( i \in I \) then \( S_j = \emptyset \) and \( \text{dos}_s(X_j) = 0 \) for all \( j \geq i \) \((j \in I)\). I.e., Eqs. (61) and (62) hold for all \( j \geq i \). Let us, therefore, assume without the lost of generality that \( S_j \neq \emptyset \) for all \( i \in I \).

The probability \( P(s_{i} \cap s_{i+1} \mid S_j) \) can be factorized into 3 conditional probabilities (see Assumptions 3.3 and 3.5 to 3.7 and the chain rule given by Eq. (4))

\[
P(s_{i} \cap s_{i+1} \mid S_j) = P(s_{i} \mid S_j) P(S_{i+1} \mid s_{i}) P(s_{i+1} \mid s_{i} \cap S_{i+1}).
\]

The factorization allows to derive several important formulae. Two out of the three conditional probabilities on the right-hand side of Eq. (63) can be infinitesimal probabilities (if they are specified by at least one random continuous variable); they may be expressed with the help of the corresponding probability density functions \( \rho \):

\[
\rho(s_{i} \cap s_{i+1} \mid S_j) = \rho(s_{i} \mid S_j) \rho(S_{i+1} \mid s_{i}) \rho(s_{i+1} \mid s_{i} \cap S_{i+1}).
\]

Equation (64) can be rewritten in different notation (writing explicitly the arguments of the functions)

\[
\rho_{s_{i+1}|S_i}(X_i, X_{i+1}) = \rho_{s_{i+1}|S_i}(X_i) P_{S_{i+1}|s_i}(X_i) \rho_{s_{i+1}|s_i \cap S_{i+1}}(X_i, X_{i+1}).
\]

Integrating Eq. (65) over \( X_i \), multiplying it by \( N_{S_j} \) and using Eq. (55) (see Proposition 3.14) yields

\[
N_{S_j} \rho_{s_{i+1}|S_i}(X_{i+1}) = N_{S_j} \int_{X_i} \rho_{s_{i+1}|S_i}(X_i) P_{S_{i+1}|s_i}(X_i) \rho_{s_{i+1}|s_i \cap S_{i+1}}(X_i, X_{i+1}) dX_i.
\]

For the purpose of using Proposition 3.1 one can introduce events \( A_0 = S_i, A_1 = S_{i+1} \) and \( A_2 = s_{i+1} \). \( S_i \neq \emptyset \) implies \( P(A_0) \neq 0 \) (see Definition 3.21). It holds \( A_1 = A_1 \cap A_0 \). The chain rule given by Proposition 3.1 for probability \( P(A_1 \cap A_2 \mid A_0) \) implies

\[
\rho_{s_{i+1}|S_i}(X_{i+1}) = P_{S_{i+1}|s_i}(X_{i+1}) \rho_{s_{i+1}|s_i \cap S_{i+1}}(X_{i+1}).
\]

Multiplying Eq. (67) by \( N_{S_j} \) and using Eq. (59) gives

\[
N_{S_j} \rho_{s_{i+1}|S_i}(X_{i+1}) = N_{S_j} \rho_{s_{i+1}|s_i \cap S_{i+1}}(X_{i+1}).
\]

Equations (66) and (68) imply

\[
N_{S_j} \rho_{s_{i+1}|S_i}(X_{i+1}) = \int_{X_i} \left( N_{S_j} \rho_{s_{i+1}|S_i}(X_i) P_{S_{i+1}|s_i}(X_i) \rho_{s_{i+1}|s_i \cap S_{i+1}}(X_i, X_{i+1}) dX_i.
\]

With the help of the definition of density of states, see Eq. (39), one can write

\[
\text{dos}_{s_{i}}(X_i) = N_{S_j} \rho_{s_{i+1}|S_i}(X_i)
\]

\[
\text{dos}_{s_{i+1}}(X_{i+1}) = N_{S_j} \rho_{s_{i+1}|s_i \cap S_{i+1}}(X_{i+1}).
\]

Equations (69) and (70) to (71) imply Eq. (61).

**Remark 3.17** The key step to proof Theorem 3.1 is to factorize the probability \( P(s_{i} \cap s_{i+1} \mid S_j) \) into 3 terms, see Eq. (63). This equation is then basically only integrated over all the random variables characterizing the input states, and several basic relations concerning probabilities are used together with the definition of density of states.

**Remark 3.18** The factorization of the probability \( P(s_{i} \cap s_{i+1} \mid S_j) \) into 3 terms (see Eq. (63)) is one of the key ideas of simplifying and unifying descriptions of various systems corresponding to Assumptions 3.3 and 3.5 to 3.7. The first term in Equation (63) corresponds to conditional probability of system being in a fixed input state \( s_i \) given that it is in \( i \)-th state space \( S_i \) (here the concept of density of states is essential). The second term corresponds to the conditional probability of transition of the state \( s_i \) to \((i+1)\)-th state space \( S_{i+1} \) given that the system was in the state \( S_i \). The third term corresponds to the conditional probability of change of the input state \( s_i \) to given output state \( s_{i+1} \) given that the input state transitioned to the state space \( S_{i+1} \). Experimental data may constrain differently the individual probabilities, i.e., functions \( \text{dos}_s(X_i), P_{T, i}(X_i) \) and \( \rho_{C, i}(X_i, X_{i+1}) \). The factorization simplifies taking into account additional knowledge (assumptions) about the evolving system. It The factorization into the 3 terms is also the motivation for calling the stochastic process state-transition-change (STC) process.
**Remark 3.19** Theorem 3.1 is of major importance as it shows how to transform density of states $\text{dos}_{S_{i}}(X_{i})$ (characterizing the input states) to density of states $\text{dos}_{S_{i+1}}(X_{i+1})$ (characterizing the output states).

**Proposition 3.16** Let $\{X_{i} : i \in I\}$ be stochastic process given by Definition 3.21. It holds

$$\frac{\text{dos}_{i+1}(X_{i+1})}{N_{0}} = \int_{X_{i}} \frac{\text{dos}_{i}(X_{i})}{N_{0}} P_{T,i}(X_{i}) \rho_{C,i}(X_{i}, X_{i+1}) dX_{i}$$

(72)

where the initial density of states corresponding to $i = 0$ and divided by $N_{0}$ is

$$\frac{\text{dos}_{0}(X_{0})}{N_{0}} = \rho_{S,0}(X_{0})$$

(73)

**Proof** Equation (62) can be divided by $N_{0}$ ($N_{0} \neq 0$) which yields Eq. (72). Equation (47) implies Eq. (73). \qed

**Remark 3.20** The right-hand side of Eq. (73) does not depend on the value of $N_{0}$. Equation (72), therefore, implies that the relative density of states $\text{dos}_{i}(X_{i})/N_{0}$ does not depend on the value of $N_{0}$ for any $i \geq 0$ ($i \in I$). When the input probability density function $\rho_{S,0}(X_{0})$ in dependence on values of $X_{0}$ corresponding to the state space $S_{0}$ is given (see Eq. (73)) then Eq. (72) represents recursive formula for obtaining the relative densities of events $\text{dos}_{i}(X_{i})/N_{0}$ ($i = 1, \ldots, M$).

**Corollary 3.1** Let $\{X_{i} : i \in I\}$ be stochastic process given by Definition 3.21. It holds

$$\frac{N_{i+1}}{N_{0}} = \int_{X_{i}} \frac{\text{dos}_{i}(X_{i})}{N_{0}} P_{T,i}(X_{i}) dX_{i}$$

(74)

where $\text{dos}_{0}(X_{0})/N_{0}$ is given by Eq. (73).

**Proof** By integrating Eq. (72) over $X_{i+1}$ (under the assumptions of Proposition 3.16), using Eq. (76) (see Proposition 3.18), and Eq. (48) for $i + 1$, one obtains Eq. (74). \qed

**Remark 3.21** It means that the relative number of events $N_{i+1}/N_{0}$ given by Corollary 3.1 does not depend on function $\rho_{C,i}(X_{i}, X_{i+1})$ characterizing the $i$-th transition; it, however, depends on functions $\rho_{C,j}(X_{j}, X_{j+1})$ for $j = 0, \ldots, i - 1$ as they are needed to calculate $\text{dos}_{j}(X_{j})/N_{0}$ for any $j = 0, \ldots, i - 1$ with the help of Proposition 3.16. It also means that it is not possible to determine $\rho_{C,i}(X_{i}, X_{i+1})|_{i=M-1}$ corresponding to the last $(M - 1)$-th transition on the basis of measured ratio $N_{i}/N_{0}$, even if the ratios are known for all $i = 0, \ldots, M$ (i.e., including the last state space $S_{M}$).

**Proposition 3.17** Let $\{X_{i} : i \in I\}$ be stochastic process given by Definition 3.21. If $N_{0} \neq 0$ then

$$1 \geq \frac{N_{i}}{N_{0}} \geq \cdots \geq \frac{N_{M-1}}{N_{0}} \geq \frac{N_{M}}{N_{0}} \geq 0$$

(75)

**Proof** According to Assumption 3.5 a state $s_{i}$ may or may not transition to state space $S_{i+1}$. It implies Eq. (75). \qed

**Proposition 3.18** Let $\{X_{i} : i \in I\}$ be stochastic process given by Definition 3.21. It holds

$$\int_{X_{i+1}} \rho_{S_{i+1}|\cap S_{i+1}}(X_{i}, X_{i+1}) dX_{i+1} = 1$$

(76)

**Proof** No variable represented by $X_{i}$ is random variable of the conditional probability (density) functions $\rho_{S_{i+1}|\cap S_{i+1}}(X_{i}, X_{i+1})$ and $P_{S_{i+1}|S_{i+1}}(X_{i}, X_{i+1})$ given event $s_{i} \cap S_{i+1}$. Only random variables $X_{i+1}$ are random variables of these functions. If $N_{0} \neq 0$ for event $s_{i} \cap S_{i+1}$ then

$$\int_{X_{i+1}} P_{S_{i+1}|s_{i} \cap S_{i+1}}(X_{i}, X_{i+1}) = 1.$$  

(77)

Equations (22) and (77) (see Proposition 3.3) imply Eq. (76). \qed

**Corollary 3.2** Let $\{X_{i} : i \in I\}$ be stochastic process given by Definition 3.21. It holds

$$N(S_{i+1}) = \int_{X_{i}} N(s_{i}) P_{S_{i+1}|s_{i}}(X_{i}) dX_{i}.$$  

(78)

**Proof** Integration of Eq. (61) over $X_{i+1}$ (see Theorem 3.1), Propositions 3.9 and 3.18 imply Eq. (78). \qed

**Proposition 3.19** Let $\{X_{i} : i \in I\}$ be stochastic process given by Definition 3.21. The probability that system transitioned to any of the states $s_{i+1}$ in $S_{i+1}$ given it was in any of the states $s_{i}$ in $S_{i}$ is

$$P_{S_{i+1}|S_{i}} = \int_{X_{i}} \rho_{s_{i}|S_{i}}(X_{i}) P_{S_{i+1}|s_{i}}(X_{i}) dX_{i}.$$  

(79)
Proof It holds
\[ P_{S_{i+1}|S_i} = \int_{X_i} \int_{X_{i+1}} \rho_{S_{i+1}|S_i}(X_i, X_{i+1}) dX_{i+1} dX_i. \] (80)
Integrating Eq. (65) over \( X_{i+1} \) and \( X_i \) (under Assumptions 3.3 and 3.5 to 3.7) and using Eqs. (80) and (46) imply Eq. (79). □

**Proposition 3.20** Let \( \{X_i : i \in I\} \) be stochastic process given by Definition 3.21. It holds
\[ P_{S_{i+1}|S_i} = 1 \] (81)
if and only if
\[ P_{S_{i+1}|S_i}(X_i) = 1. \] (82)
It holds
\[ 0 \leq P_{S_{i+1}|S_i} < 1 \] (83)
if and only if
\[ P_{S_{i+1}|S_i}(X_i) < 1. \] (84)

**Proof 1** Integrating Eq. (65) over \( X_{i+1} \) (under Assumptions 3.3 and 3.5 to 3.7) and using Proposition 3.18 one obtains
\[ \int_{X_i} \rho_{S_{i+1}|S_i}(X_i, X_{i+1}) dX_{i+1} = \int_{X_i} \rho_{S_{i+1}|S_i}(X_i) P_{S_{i+1}|S_i}(X_i) \rho_{S_{i+1}|S_i}(X_i, X_{i+1}) dX_{i+1} \]
\[ = \rho_{S_{i+1}|S_i}(X_i) P_{S_{i+1}|S_i}(X_i) \int_{X_{i+1}} \rho_{S_{i+1}|S_i}(X_i, X_{i+1}) dX_{i+1} \]
\[ = \rho_{S_{i+1}|S_i}(X_i) P_{S_{i+1}|S_i}(X_i). \] (87)
If the probability \( P_{S_{i+1}|S_i}(X_i) < 1 \) then
\[ 0 \leq \rho_{S_{i|S_i}}(X_i) P_{S_{i+1}|S_i}(X_i) < \rho_{S_{i|S_i}}(X_i). \] (88)
Integration of this inequality over \( X_i \) and Eq. (46) imply
\[ 0 \leq \int_{X_i} \rho_{S_{i|S_i}}(X_i) P_{S_{i+1}|S_i}(X_i) dX_i < 1. \] (89)
This inequality can hold if and only if \( P_{S_{i+1}|S_i}(X_i) < 1. \)
The equality
\[ \int_{X_i} \rho_{S_{i|S_i}}(X_i) P_{S_{i+1}|S_i}(X_i) dX_i = 1 \] (90)
holds if and only if \( P_{S_{i+1}|S_i}(X_i) = 1. \) □

**Proof 2** If Eq. (82) holds then Eqs. (79) and (46) imply Eq. (81). Equation (46) is normalized to unity if integrated over \( X_i \) (see Eq. (46)). If Eq. (81) holds then Eq. (79) can hold if and only if Eq. (82) holds.

In similar way it is possible to proof the equivalence of Eqs. (83) and (84). □

**Remark 3.22** Propositions 3.19 and 3.20 show that some “probability density functions” may not be normalized to unity (compare it to, e.g., Eq. (46) and Proposition 3.18 where the probability density functions are normalized to unity). This is consequence of Assumption 3.5 and that given process is a sequence, i.e., a branching process.

### 3.2.4 STC sequence—special case 1

**Assumption 3.8** The probability density functions \( \rho_{C,i}(X_i, X_{i+1}) \) (defined by Eq. (54)) do not depend on \( X_i \) for \( i = 0, \ldots, M - 1 \) (i.e., all transitions).

**Definition 3.23** *(STC sequence—case 1)* Let \( \{X_i : i \in I\} \) be stochastic process given by Definition 3.21. Let it satisfy Assumption 3.8. I.e., the process satisfies Assumptions 3.3 and 3.5 to 3.8.

**Definition 3.24** The probability density function \( \rho_{C,i}(X_i, X_{i+1}) \) satisfying the Assumption 3.8 and corresponding to the \( i \)-th transition may be denoted as \( \rho_{C,i}(X_{i+1}). \) It holds \( i = 0, \ldots, M - 1 \)
\[ \rho_{C,i}(X_{i+1}) = \rho_{C,i}(X_i, X_{i+1}). \] (91)
Corollary 3.3  Let \{X_i : i \in I\} be stochastic process given by Definition 3.23. It holds (for any \(i = 0, \ldots, M - 1\))

\[
\frac{N_{i+1}}{N_0} \rho_{S,i+1}(X_{i+1}) = \rho_{C,i}(X_{i+1}) \left[ \int_{X_0} \rho_{S,0}(X_0) P_{T,0}(X_0) dX_0 \right] \\
\prod_{j=1}^{i} \left[ \int_{X_j} \rho_{C,j-1}(X_j) P_{T,j}(X_j) dX_j \right]
\]

(92)

with a convention \(\prod_{j=1}^{i} [...] = 1\) if \(i = 0\).

Proof  Taking into account the Assumptions 3.3 and 3.5 to 3.7 and 3.8, using Eq. (74) (see Corollary 3.1 and Assumptions 3.3 and 3.5 to 3.7) together with Definitions 3.18, 3.22 and 3.24 one obtains

\[
\frac{N_1}{N_0} \rho_{S,1}(X_1) = \rho_{C,1}(X_2) \left[ \int_{X_0} \rho_{S,0}(X_0) P_{T,0}(X_0) dX_0 \right]
\]

(93)

\[
\frac{N_2}{N_0} \rho_{S,2}(X_2) = \rho_{C,2}(X_3) \left[ \int_{X_0} \rho_{S,0}(X_0) P_{T,0}(X_0) dX_0 \right] \\
\left[ \int_{X_1} \rho_{C,1}(X_2) P_{T,1}(X_1) dX_1 \right]
\]

(94)

\[
\frac{N_3}{N_0} \rho_{S,3}(X_3) = \rho_{C,3}(X_4) \left[ \int_{X_0} \rho_{S,0}(X_0) P_{T,0}(X_0) dX_0 \right] \\
\left[ \int_{X_1} \rho_{C,1}(X_2) P_{T,1}(X_1) dX_1 \right] \\
\left[ \int_{X_2} \rho_{C,2}(X_3) P_{T,2}(X_2) dX_2 \right]
\]

(95)

etc. It implies Eq. (92) with the convention \(\prod_{j=1}^{i} [...] = 1\) if \(i = 0\). Equation (92) can be proven also by mathematical induction. □

3.2.5 STC sequence—special case 2

Assumption 3.9  The probability density functions \(\rho_{C,i}(X_i, X_{i+1})\) are the same for \(i \in \{0, \ldots, M - 1\}\) (i.e., all transitions).

Definition 3.25  (STC sequence—case 2) Let us define stochastic process \{X_i : i \in I\} as in Definition 3.23 and let it satisfy also Assumption 3.9. I.e., the process satisfies Assumptions 3.3 and 3.5 to 3.9.

Definition 3.26  The probability density function \(\rho_{C,i}(X_i, X_{i+1})\) satisfying both the Assumptions 3.8 and 3.9 may be denoted as \(\rho_{\tilde{C}}\) and it holds in this case

\[
\rho_{\tilde{C}}(X_{i+1}) = \rho_{C,i}(X_i, X_{i+1}).
\]

(96)

Corollary 3.4  Let \{X_i : i \in I\} be stochastic process given by Definition 3.25. It holds (for any \(i = 0, \ldots, M - 1\))

\[
\frac{N_{i+1}}{N_0} \rho_{S,i+1}(X_{i+1}) = \rho_{\tilde{C}}(X_{i+1}) \left[ \int_{X_0} \rho_{S,0}(X_0) P_{T,0}(X_0) dX_0 \right] \\
\prod_{j=1}^{i} \left[ \int_{X_j} \rho_{\tilde{C}}(X_j) P_{T,j}(X_j) dX_j \right]
\]

(97)

where \(\prod_{j=1}^{i} [...] = 1\) if \(i = 0\).

Proof  Equation (92) (see Corollary 3.3 and Assumptions 3.3 and 3.5 to 3.8) can be simplified under Assumption 3.9 and using Eq. (96) (see Definition 3.26); it implies Eq. (97).

Remark 3.23  The quantities \(\frac{N_{i+1}}{N_0} \rho_{S,i+1}(X_{i+1})\) as functions of \(X_{i+1}\) and given by Eq. (97) have the same shapes (given by \(\rho_{\tilde{C}}\)) for all \(i = 0, \ldots, M - 1\); the functions differ only in their normalizations.

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3.2.6 STC sequence—special case 3, no random variables

Assumption 3.10 Both $X_i$ and $X_{i+1}$ do not contain any random variable for all $i = 0, \ldots, M - 1$.

Definition 3.27 (Single STC process without random variables) Let $\{X_i : i \in I\}$ be stochastic process given by Definition 3.21. Let it satisfy Assumption 3.10. The process satisfies Assumptions 3.3 and 3.5 to 3.7 and 3.10.

Proposition 3.21 Let $\{X_i : i \in I\}$ be stochastic process given by Definition 3.27. It holds

$$N(s_{i+1}) = N(s_i)P_{S_{i+1}|s_i}(X_i),$$

i.e., (see Eqs. (44) and (53))

$$N_{i+1} = N_iP_{T,i}(X_i).$$

Proof Definition 3.10 and Assumption 3.10 imply that the densities of states $s_i$ and $s_{i+1}$ are

$$\text{dos}(s_i) = N(s_i).$$

It holds

$$N(S_{i+1}) = N(s_{i+1}).$$

It holds according to Corollary 3.2 and Eq. (100) (under Assumptions 3.3 and 3.5 to 3.7)

$$N(S_{i+1}) = \int_{X_i} N(s_j)P_{S_{i+1}|s_i}(X_i)dX_j.$$

$X_i$ does not represent any random variable, therefore, there is effectively no integration in Eq. (102). Equations (101) and (102) imply Eq. (98).

Remark 3.24 The proof of Proposition 3.21 shows how the simple and expected result represented by Eq. (99) is derived under special conditions from much more general Corollary 3.2.

3.3 Place-transition net of (stochastic) processes

Remark 3.25 Dynamics of a system can be described with the help of a state-transition net, also called place-transition (PT) or Petri net. A PT net is directed bipartite graph that has two types of elements: places and transitions.

One can define control sets (see Definition 3.9) and some quantities characterizing states of the system in the control sets (e.g., density of states corresponding to each of the control sets, see Definition 3.10). The quantities can represent places and transformations of the quantities represent the transitions between different places. The transitions can be specified with the help of Theorem 3.1, Corollary 3.2 and Proposition 3.21; or some other transformations if needed.

Various types of PT nets are studied in mathematics and computer science; one can take advantage of the achieved results and developed tools. Description of dynamics of a system using a PT net substantially simplifies modeling (simulation) of the system if it consists of many places and transitions between them. Modeling of a system using a PT net provides higher level of abstraction and it helps to visualize complexity of the places and transitions.

Remark 3.26 As to the example in Sect. 2 it is possible to define transformations corresponding to splitting of a beam as well as merging of two or more beams into one. Description of the splitting (using a beam splitter) is relatively straightforward. It is necessary to define transformation of, e.g., density of states corresponding to a control set before the split and a control set corresponding to given split beam (it can be done for all the split beams individually, the control sets must be disjoint). If particle beam can only split and do not merge then the stochastic process is branching process defined by Definition 3.17. Description of merging of two or more beams is more delicate to describe, but it can be done, too. Laser systems (having typically many components and including concurrent processes) can be, therefore, represented by a PT net.

4 General guidelines for analysis of experimental data

Analysis of experimental data using a probabilistic model can be divided into several steps (being analogous to an analysis of data using any other model). We will focus mainly on using STC process.
4.1 Collection of data and model formulation

Gathering various measured data concerning a system and its initial inspection is the first step in formulation of a (probabilistic) model. Measured data often point to symmetries, even without analysing them in detail using sophisticated theoretical methods. The symmetries typically simplify formulation of the model. One may define a place-transition net as described in Sect. 3.3. In the case of a stochastic STC process it is necessary to define state spaces $S_i$ and both random $X_i$ and non-random $X^{NR}_i$ variables characterizing states $s_i \in S_i$ for all indexes $i$ in an index set $I$. On the basis of experimental data it must be possible to determine the number of states in state spaces $S_i$ or density of states in state spaces $S_i$ as a function of the variables (the more detailed experimental information is available the better).

The initial inspection of data helps also in determination of properties of the system which can be added to the model as additional assumptions (see, e.g., STC sequence given by Definition 3.23 and Sect. 3.2.6). The additional assumptions allows to derive more specialized formulae and perform various calculations analytically. Formulation of the model depends also on its purpose (e.g., introducing only limited number of variables typically does not allow to fully describe the system, but it may allow to determine some partial characteristics of the system, which cannot be measured directly, very efficiently if only essential variables are taken into account).

4.2 Check of applicability of the model to measured data (consistency)

Before making a sophisticated analysis of data using the model it is useful to make basic consistency checks. For example, it may be possible to test experimentally the basic Assumptions 3.6 and 3.7 of a STC process (see Definition 3.19). In the case of a STC sequence (see Definition 3.21) one can test Eq. (75) (i.e., one can test Assumption 3.5).

If these conditions are not satisfied then at least one of the assumptions of the model must be changed to remove the contradiction. On the other hand, if these conditions are satisfied then one may continue to analyse experimental data with the help of the probabilistic model.

4.3 Parameterizations of unknown functions

All the unknown function in the model can be parameterized (in consistence with the assumptions of the model). For example, STC sequence of $M$ transitions (see Definition 3.21) requires in the most general case to parameterize density of states $\rho_s(X_i)$ for all $i \in (0, \ldots, M)$, and probability $P_{r,i}(X_i)$ and $\rho_{c,i}(X_i, X_{i+1})$ for all $i \in (0, \ldots, M - 1)$.

4.4 Fitting of the model to data

When one tries to find the values of all the free parameters of a model using an optimization procedure then basically three possibilities may occur:

1. No solution is found under given set of assumptions. This can indicate a contradiction in the model (e.g., the parameterizations may exclude the possibility of finding the solution from the very beginning independently on the values of the free parameters, or at least one of the assumptions of the model is not correct); or it may be just very difficult optimization problem.
2. More than one solution is found.
3. Only one solution is found.

This must be taken into account in making correct conclusions from an analysis of experimental data with the help of the model. In some cases it may be very useful, or even necessary, to modify some of the assumptions of the model and try to determine the parameterized functions on the basis of experimental data again. This can help to better understand the whole system and make correct interpretation of the results. It is also important for proposing additional experiments to test the results. Constrained optimization methods allows to take into account various experimental information into account.

4.5 Separation of determination of probabilities from other additional assumptions

An analysis of experimental data with the help of a probabilistic model can be divided into two stages:

1. **determination of probability (density) functions** One may first try to find the unknown probability (density) functions. Their parameterizations can contain many free parameters to be flexible enough. If a parameterization is a priori too restrictive it may not allow finding solution to the optimization problem (the solution may be excluded from the very beginning). While the free parameters may not have physical meaning, the probability (density) functions do have meaning and should be able to describe experimental data under given assumptions without any contradiction. The point of this analysis aim is to explore, test, and understand well the basic probabilistic character of an evolving system.

2. **determination of physical mechanisms (causes)** One may try to find physical processes (causes) which could lead to the probability (density) functions which can be (in one way or another) fitted to experimental data. The free parameters of these physical processes may be fitted to the determined probability (density) functions. For example, in the case of transmission of particles...
through matter (see Sect. 2) one may try to express the determined probability (density) functions with the help of mean free paths and cross sections of various interactions as the free parameters, and try to find their values by fitting them to the already determined probability functions. In some cases, depending on the studied physical system, it may be possible to similarly study, e.g., the Hamiltonian of the system, forces responsible for transition of one state of the system to another. This analysis aim tries to describe the system in greater detail, but it requires deeper insight to the studied system.

The possibility of the separation of the whole analysis of experimental data with the help of a probabilistic model into two stages is essential for understanding various phenomena (especially those which are more complex). Determining first the unknown overall probability (density) functions and then trying to explain the physical mechanisms may be in many cases much more effective than trying to guess all the key physical processes (some of them may not be even known), calculating the corresponding probability (density) functions, and then trying to fit the free parameters using experimental data. On the other hand in some (simpler) cases good guess of the physical mechanisms may help to find the probability (density) functions describing the measured data much faster. In one way or another it is advantageous to study and understand how much each of the probability (density) functions is constrained by data and how much by some additional assumptions. This allows to tests different assumptions and interpretation possibilities of experimental data using falsification approach (see Sect. VII.A in [25] for further details). Many theoretical statistical approaches, see Sect. 1, do not easily allow division of the whole analysis of data into the two stages (they are often based on some strong assumptions from the very beginning, see the introduction).

4.6 Computational complexity

As to the computational complexity. To determine the parameterized probability (density) functions on the basis of experimental data it may be necessary to calculate several multidimensional integrals, see, e.g., one of the main formulae characterizing STC sequence given by Eq. (62). The complexity is increasing with increasing number of variables and transitions. The integrals can be calculated analytically only in relatively simple cases, in the other cases they need to be calculated numerically which can be very time consuming. In some cases it may be possible to significantly reduce the computational complexity if the integrals are simplified under some additional assumptions, see Sect.3.2.5 as an example.

5 Applications

The STC stochastic process developed in Sect. 3.2 can help to describe various physical phenomena. In the following several examples from different fields of physics will be discussed. The examples concerns particles in the general sense, see Definition 2.1.

5.1 Decay of particles

Let us consider \( N(t_0) \) decaying particles at time \( t_0 \).

**Definition 5.1** One of possible definitions of the exponential function is

\[
e^x = \lim_{k \to \infty} \left(1 + \frac{x}{k}\right)^k.
\]  

**Assumption 5.1** Decays of unstable particles in a systems of \( N(t) \) particles are mutually independent at any time \( t \), see Definition 3.8. I.e., the probability of decay of a particle is not influenced by presence of the other particles.

**Assumption 5.2** Probability of decay of a particle in time interval \((t, t + dt)\) does not depend on time \( t \). The probability is equal to \( \lambda dt \) where \( \lambda > 0 \) is called the decay constant.

**Definition 5.2** Let \( \{X_i : i \in I\} \) be stochastic process given by Definition 3.27 and let it satisfy also Assumptions 5.1 and 5.2 (Assumption 5.1 being equivalent to Assumption 3.6). Let us define sequence of \( M + 1 \) times \( t_i \) for \( i = 0, \ldots, M \) \( (t_i < t_{i+1} \) for any \( i = 0, \ldots, M - 1) \). Let us define time \( t \) being equal to \( t_M \). Let \( \Delta t = t_{i+1} - t_i \) be the same for any \( i = 0, \ldots, M - 1 \), i.e.,

\[
\Delta t_i = \frac{\lambda(t - t_0)}{M} \quad (104)
\]

\[
\Delta t = t_{i+1} - t_i. \quad (105)
\]

Let

1. variables \( X_i \) and \( X_i^{NR} \) be

\[
X_i = () \quad (106)
\]

\[
X_i^{NR} = (t_0, \ldots, t_i, \lambda). \quad (107)
\]

I.e., there are no random variables, only non-random variables.
2. state space $S_i$ contain states of individual particles, represented by variables $X_i$, i.e., undecayed particles at time $t_i$ (all particles which have not decayed until time $t_i$). The state space $S_0$ corresponds to $t_0$ and the state space $S_M$ corresponds to $t_M = t$.

**Corollary 5.1** (Exponential decay) Suppose we have $N(t_0) > 0$ unstable particles at time $t_0$ and Assumptions 5.1 and 5.2 are satisfied, then it holds
\[
N(t) = N(t_0)P(t)
\]
where
\[
P(t) = e^{-\lambda(t-t_0)}
\]
is probability of one particle not decaying in the time interval from $t_0$ to $t$, and $\lambda$ is positive constant called the decay constant.

**Proof 1** It follows from Assumptions 5.1 and 5.2 that the number of decayed particles at time $t$ is
\[
dN = -\lambda N\,dt
\]
where the minus sign expresses the decrease of the undecayed particles ($\lambda$ being positive constant). Equation (110) can be rearranged
\[
\frac{dN}{N} = -\lambda\,dt.
\]
Integrating both sides (left side in the limits from $N(t_0)$ to $N(t)$, and right side in the corresponding limits $t_0, t$) implies
\[
\ln \frac{N(t)}{N(t_0)} = -\lambda(t-t_0).
\]
Exponentiating both sides gives Eq. (108).

**Proof 2** Let $(X_i : i \in I)$ be stochastic process given by Definition 5.2. It holds (constant $\lambda$ will not be written explicitly as an argument of a function of number of particles or a probability function in the following)
\[
N_i = N(t_i)
\]
for each index $i = 0, \ldots, M$. The assumptions of Corollary 5.1 imply assumptions of Theorem 3.1. Using iteratively Eq. (99) (see Proposition 3.21) for $M$ transitions one obtains Eq. (108) where
\[
P(t) = \prod_{i=0}^{M-1} P_{T_{i,i}}(t_i, t_{i+1}).
\]
The conditional probability $P_{T_{i,i}}(t_i, t_{i+1})$ does not depend on parameters $t_0, \ldots, t_{i-1}$ given that the particle have not decayed until time $t_i$. Assumption 5.2 implies that probability $P_\Delta$ that a particle does not decay in time interval $(i, i + \Delta t)$ for fixed $\Delta t > 0$ does not depend on time, and the probability $P_{T_{i,i}}(t_i, t_{i+1})$ is the same for all $i = 0, \ldots, M - 1$ (see Eq. (105)). It holds
\[
P_\Delta = P_{T_{i,i}}(t_i, t_{i+1}).
\]
The probability $P_\Delta$ can be approximated by
\[
P_\Delta \approx 1 - \lambda \Delta t.
\]
The smaller $\Delta t$ the better the Assumption 5.2 is satisfied. It holds in the limit $M \to \infty$ and using Eqs. (114) and (115) to (104) one obtains
\[
P(t) = \lim_{M \to \infty} \left(1 - \frac{\lambda(t-t_0)}{M}\right)^M.
\]
Equations (117) and (103) imply Eq. (109).

**Remark 5.1** Equation (108) can be used for analysis of experimental data to determine the free parameter of the decay process, i.e., the decay constant $\lambda$.

**Remark 5.2** The proof 1 is standard proof which can be found in textbooks. The proof 2 demonstrates that the exponential decay can be derived with the help of STC process introduced in Sect. 3.2. The proof 2 also shows one way how one can use STS process and deal with continuous transitions (see the limit $M \to \infty$). STC process can help to study decay processes in more complicated cases when, e.g., more variables may be needed to describe the decay products, i.e., the output states (input and output states may differ, see Remark 3.9). Or, experimental data do not correspond to the exponential formulae given by Eq. (109) (not even in more general case when more decay constants corresponding to different types of particles in the sample are considered).

**Remark 5.3** Corollary 5.1 can be, and it is standadly done, proven in yet another way, using Poisson process defined in chapter 5 in [22]. Different approaches in derivation of one formula help to better study the involved assumptions which can or cannot lead to the same results.
5.2 Motion of particles

Motion of particles (bodies) in presence of an external force field can be, in some cases, described with the help of Newton’s second law of motion. I.e., as deterministic process, if the initial conditions of the particles are precisely known. If the initial conditions are characterized by a probability density function then the process can be considered as random (stochastic).

As an example of nontrivial particle motion one may consider a source emitting particles of different speeds and at different times in presence of no external force. It may not be possible to detect the particles at the place where they are emitted. Their characteristics can be determined on the basis of time dependent particle fluxes which can be measured at different distances from the source. This is a subject of well-known time resolved velocity spectrometry (a type of time-of-flight method). It is shown in [26] under which assumptions STC process is reduced to the formulae of time resolved velocity spectrometry.

The results from [26] can be generalized. It is possible to introduce non-zero external force and other properties of particles (such as electric charge, mass, ...) which are characterized by corresponding probability density functions (not known precisely). Transport of particles may depend on these particle characteristics. This allows to study forces acting on particles having some properties specified by random variables. Inertial mass increase in dependence on velocity may be also studied on the basis of experimental data under these conditions [27].

STC stochastic process, therefore, opens up new possibilities of determination of particle characteristics (and forces) in situations where randomness of the particle motion must be taken into account.

5.3 Particle–matter interaction

5.3.1 Transmission of light through polarizers

The probabilistic model developed in Sect. 3 has been used in [28] for description of transmission of photon beam through 3 linear polarizers ($M = 3$). In this example of particle–matter interaction each photon from the beam has been characterized by one random variable, one degree of freedom characterizing direction of polarization and having meaning of an angle. Each polarizer in the sequence has been characterized by its rotation angle (non-random variable). One example of input data of ratio of number of transmitted photons to initial number of photons in front of and behind each of the 3 polarizers in dependence on their rotation angles has been analysed in [28]. The probability (density) functions $\pi_{STC}(X_1, X_2, X_3)$ (being function of both random and non-random variables) characterizing interaction of the photon beam with the polarizers have been determined with the help of the formulae concerning STC sequence discussed in Sects. 3.2.3 to 3.2.5.

This example of description of 3 polarizer experiment shows how the probabilistic character of transmission of light through other optical elements may be studied with the help of a (STC) stochastic process. This may be useful, e.g., in further development of various components used in laser systems (see Sect. 2). The probabilistic model, therefore, opens up new possibilities how to study polarization of light and particle–matter interactions in general.

5.3.2 Particle beam as diagnostic tool

Various medical as well as industrial radiographic methods use ionizing and non-ionizing radiation for diagnostic purposes of various samples. Sample of different states of matter (gaseous, solid, liquid or in even more exotic) are widely studied.

E.g., laser is essential tool in chemistry, see [29] where one can find many widely used experimental methods for measuring properties of a particle beam (laser) before and after an interaction with a sample (which can be basically arbitrary). STC stochastic process can help to extend possibilities of analyses of experimental data.

Let us mention also some regions where particle beams are used as diagnostic tools but where description of the system is delicate, with or without (STC) stochastic process. They concern mainly systems which are highly dynamical (involve many time dependent processes), non-linear (having complicated properties depending on time and other variables), not exactly reproducible (initial conditions may vary), non-homogeneous and exist only for short amount of time (this makes high demands on their measurement). Limited experimental statistical information is often available for these systems.

A short-lived localized plasma sample (quasi-neutral system of electrons and ions) may be produced in laboratory when high-power laser beam is focused onto a solid target. The created plasma sample may be irradiated by another (diagnostic) laser beam and the beam intensity of the transmitted diagnostic beam may be measured (e.g., by a CCD camera), as it was demonstrated at Prague Asterix Laser System (PALS) in [30]. The measurements performed in [30] shows how polarimetry and interferometry can be combined to obtain important experimental information about a plasma sample. The generated plasma is not homogeneous and, therefore, the intensity of diagnostic beam (particle flux) is measured in dependence on transversal coordinates.

Another plasma diagnostic method using a particle beam is ion deflectometry, see [31] (and, e.g., [32–34]). Ions carrying electric charge and traversing plasma sample interact very differently than neutral photons. It means that ion deflectometry may provide information about a plasma sample which cannot be determined with the help of the photon diagnostic beam (and vice versa).

Better understanding of, e.g., laser plasma interaction and various other related particle transport phenomena is important for many areas of physics including the context of inertial confinement fusion [35]. Sample of warm dense matter (WDM) can be
prepared and diagnosed in laboratory similarly as plasma sample, see [36] (a WDM sample is similarly dynamical physical system as the plasma sample).

STC stochastic process can be useful in studies of these “exotic” samples mainly by helping to determine properties of the diagnostic beams before and after their interactions with the beam, see for example Sect. 5.3.1; and also in determination of properties of particles emitted from the samples, see for example Sect. 5.2.

Description of the processes inside a sample caused by particles from a beam will often require dedicated models which do not correspond to the assumptions of STC process. This applies not only to processes inside the samples mentioned above, but also to modeling of, e.g., radiobiological mechanisms standardly divided into two consecutive stages: physical-chemical and biological [37]. In [37] Petri nets were used to describe chemical stage (combined effect of diffusion of cluster of radicals and chemical reactions). Probabilistic two-stage model (see Sect. 6.3. in [37]) allows to describe response of irradiated cell to damage caused by radiation. This example demonstrates that place-transition (Petri) nets and theory of probability make possible to study and model efficiently complex system divided into very diverse stages. Theory of stochastic processes can further extend possibilities of this approach in various areas of scientific research, see also Sect. 3.3.

5.4 Particle–particle interaction

The probabilistic description of particle transport discussed in Sect. 3 is useful also for description of particle–particle collisions. Number of particle–particle collisions corresponding to an event is equal to cross section of the event times luminosity. Concept of luminosity (for both fixed target and colliding beams experiments) is closely related to density of states of the beam(s) (i.e., particle densities), see [38]. Probabilistic theory is also used, e.g., for calculation of pileup probabilities for specific events per bunch-crossing as discussed in [39]. If the luminosity and the number of collisions corresponding to an event is measured then the cross section of the event can be determined experimentally.

E.g., in [40] one may find example of one of many measurements of elastic proton-proton differential cross section performed by TOTEM\(^2\) experiment at the Large Hadron Collider (LHC) at CERN at the center-of-mass energy of 8 TeV. The measurement of scattered protons moving close to the beam requires dedicated detector technology and experimental techniques, see [41] for details and future plans extending TOTEM physics program.

However, theoretical understanding of the measured cross sections is in many cases very delicate. Possibilities, limitations and open questions in contemporary models of elastic collisions of particles interacting strongly at short distances (hadrons) have been summarized in [42]. Theoretical description of elastic hadron collisions is more delicate than it may seem at first glance, even though it is kinematically the simplest collision process and a lot of elastic scattering data are available at various collision energies, see [43–45] for further details.

To overcome the difficulties summarized in [42] a probabilistic particle collision model has been developed and used for description of measured elastic proton-proton differential cross section at collision energy of 52.8 GeV and some preliminary results have been obtained, see sect. 6 in [46]. With the help of the model it has been possible to take into account, at given collision energy, density of input two particle states distinguished by impact parameter. It allowed to determine probabilities of total, elastic and inelastic strong (short-ranged) interaction in dependence on the values of impact parameter and sizes of the colliding particles. A final state of two particles scattered elastically has been characterized by (mean) elastic scattering angle as a function of impact parameter; the model has allowed to determine this dependence, too.

STC stochastic process introduced in Sect. 3.2 can provide deeper insight into the probabilistic collision model introduced in sect. 6 in [46] and also to the concept of luminosity. STC stochastic process is also essential for developing further the probabilistic collision model. For example, measured cross sections of various types of interactions (not only elastic differential cross sections) can be analyzed consistently with its help, too.

6 Conclusion

Applicability of contemporary statistical descriptions of particle phenomena, see Sect. 1, for analysis of experimental data is in many cases restricted to relatively small class of phenomena, based on several strong assumptions, requires Hamiltonian of the system or other information which may not be known and may be difficult to determine with the help of the hitherto theoretical methods.

To overcome these difficulties state-transition-change stochastic process has been introduced in Sect. 3.2. It describe various systems transitioning through sequence of states. Main assumptions restricting applicability of the process to data are Assumptions 3.6 and 3.7, i.e., Markov property and independence of outcomes. Theorem 3.1 represents key theorem simplifying description of these systems. With the help of STC process it is possible to formulate a probability model of various phenomena taking into account both available experimental data and addition assumptions (constrain). The approach allows separation of analysis of experimental data into two stages. In the first stage, one may try to determine the probability (density) functions characterizing evolution and states of the system. In the second stage, one may try to explain the established probability (density) functions on the basis of

\(^2\) The TOTEM acronym stands for TOTal, Elastic and diffractive cross section Measurement, see totem.web.cern.ch.
microscopic processes, as it has been discussed in Sect. 4. Moreover, the model may depend only on variables which are essential for description of given phenomenon. The possibility of separation of the whole analysis of data into two stages and using only essential variables for given purpose is essential for deeper understanding of many particle phenomena as it allows to test various assumptions (interpretation possibilities) effectively.

Analysis of experimental data using STC process combines probability theory, theory of stochastic processes, infinitesimal calculus and optimization theory. Place-transition (Petri) nets can be used to simplify description of more complex systems (possibly with the help of other theoretical models, based on different assumptions than the assumptions of STC process). STC process, therefore, extends possibilities of contemporary descriptions of many systems. Stochastic STC process is distinct from other stochastic processes already used in physics [47, 48]; it, therefore, broaden applicability of stochastic theory in physics.

The various examples discussed in Sect. 5 show that STC process can unify description of many phenomena, even if they may look very distinct at first glance. For example, with its help it is possible to describe both motion of particles having initial values described with the help of probability density function, see Sect. 5.2 and [26], and transmission of light through various optical elements, see Sect. 5.3.1 and [28]. The unified description of many particle phenomena is not only possible but also highly desirable in terms of developing an intrinsic understanding of these phenomena which are important, in one way or another, for all areas of physics.

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