Traveling Wave for Reaction Diffusion Equation with Spatio-Temporal Delay*

By

Zhihong ZHAO and Erhua RONG

(University of Science & Technology Beijing and Shanxi agricultural University, P.R. China)

Abstract. In this paper, we study the traveling wave fronts of reaction diffusion system with spatio-temporal delay. With some particular delay kernel, the nonlocal equation is reduced to a system of singularly perturbed ODEs. It is proved, by use of geometric singular perturbation analysis and Fredholm theory, that the traveling wave front persists when the delay is suitably small and it is qualitatively similar to those of the undelayed reaction diffusion system.

Key Words and Phrases. Reaction diffusion system, Spatio-temporal delay, Traveling wave front, Singular perturbation analysis, Fredholm theory.

2010 Mathematics Subject Classification Numbers. 35K10, 35K57.

1. Introduction

The theory of traveling wave solutions of delayed reaction diffusion equations is one of the fast developing areas of modern mathematics, e.g. [1, 2, 3, 4, 5, 6, 7, 8].

In recent years, reaction diffusion equations with spatio-temporal nonlocal terms in the form of convolution of a kernel have been proposed in many mathematical models. This type of equations are considered to be more realistic than the usual kind of reaction diffusion models. For the detailed derivation of convolution kernel and related references, we refer to [9, 10, 11]. Subsequently, the study of traveling wave front for different models becomes a very active research subject. See [12, 13, 14, 15, 16] and the references therein.

Owing to this motivation, in this paper, we are concerned with the existence of traveling wave front for the following reaction diffusion system with spatio-temporal delay:

* Supported by the NNSF of PR China (11071014, 11071205, 11001032), the Fundamental Research Funds for the Central Universities (06108114).
(1.1) \[
\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + F(u(x, t), \int_{-\infty}^{+\infty} G(x - y, t - s)k(t - s)g(u(y, s))dsdy),
\]
where \( t \in \mathbb{R}, x \in \mathbb{R}, \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) is a continuously differentiable function. The function \( k(t) \) satisfying \( \int_0^{+\infty} k(s)ds = 1 \) is used to weight the distributed delay, \( G \) is a weighting function describing the distribution at position \( x \) and time \( t \). The diffusion coefficient of \( u \) is 1. Then \( G \) must satisfy
\[
\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial x^2}, \quad G(x, 0) = \delta(x),
\]
where \( \delta \) is the Dirac function. So \( G \) is the fundamental solution of heat equation, that is,
\[
G(x, t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right).
\]

Two specific cases of delay kernel function \( k(t) \), which have been widely used, are weak generic kernel:
\[
k(t) = \frac{1}{\tau} e^{-t/\tau}, \quad \tau > 0,
\]
and strong generic kernel:
\[
k(t) = \frac{t}{\tau^2} e^{-t/\tau}, \quad \tau > 0.
\]

In the literatures, the existence of traveling wave of (1.1) with some particular \( F \) and \( g \) was studied. Here are some examples: \( F(u, w) = u - uw, \) \( g(u) = u \) in [12], \( F(u, w) = u[1 + au - (1 + a)w], \) \( g(u) = u \) in [7], \( F(u, w) = -au + b[1 - u]w, \) \( g(u) = u \) in [16], \( F(u, w) = ru[1 - a_1w - a_2w^2], \) \( g(u) = u \) in [14].

The purpose of this paper is to tackle the existence of traveling wave solutions of reaction diffusion equation with spatio-temporal delay (1.1), where delay is suitably small. In particular, we obtain the existence of traveling wave front of delayed reaction diffusion equation. Furthermore, the qualitative behavior of traveling wave fronts is described.

The rest part of this paper is organized as follows. Section 2 is devoted to some preliminary discussions. Then in section 3, the existence of a traveling wave front for delayed reaction diffusion system (1.1) is justified by employing the geometrical singular perturbation theory and Fredholm theory. And finally, in Section 4, our main result is applied to the Nagumo equation with delay.
2. Preliminary

In this section, we introduce some known results about system (1.1) without delay. The corresponding undelayed equation reads

\begin{equation}
\label{eq:2.1}
t_i = u_{xx} + F(u, g(u)).
\end{equation}

A traveling wave front is a solution \( u(x, t) = \tilde{u}(z) \), \( z = x + ct \), velocity \( c > 0 \). Substituting \( u(x, t) = \tilde{u}(x + ct) \) into (2.1), we obtain the corresponding wave equation

\[ \tilde{u}'' - c\tilde{u}' + F(\tilde{u}, g(\tilde{u})) = 0, \]

which can be transformed into a first-order system

\begin{equation}
\begin{cases}
    x_1' = x_2, \\
    x_2' = cx_2 - F(x_1, g(x_1)).
\end{cases}
\end{equation}

Without loss of generality, we assume real number \( K > 0 \). Furthermore, we assume, throughout the remainder of this paper, the following hypotheses hold.

\begin{enumerate}
\item[(H1)] \( F \) and \( g \) are continuously differentiable and satisfy the conditions

\[ F(0, g(0)) = F(K, g(K)) = 0, \quad F(u, g(u)) \neq 0 \quad \text{in} \quad (0, K). \]

\item[(H2)] Let \( f(u) = F(u, g(u)) \), then

\[ f'(0)f'(K) \neq 0. \]

\end{enumerate}

Summarizing the relevant conclusions of system (2.1) \([17, 18, 19, 20]\), we have

**Proposition 2.1.** Assume that (H1) and (H2) hold. We suppose further one of the following conditions is satisfied:

\begin{enumerate}
\item[(H3)] If \( f'(0) > 0 \), then \( f(u) > 0 \), speed \( c \) satisfies \( c \geq 2\sqrt{k_1} \) where

\[ k_1 = \sup_{0 < u < K} \frac{f(u)}{u}. \]

\item[(H4)] If \( f'(0) < 0 \), then \( f(u) < 0 \), speed \( c \) satisfies \( c \geq 2\sqrt{k_2} \) where

\[ k_2 = \sup_{0 < u < K} \frac{f(u)}{u - K}. \]
\end{enumerate}

Then system (2.2) has a unique heteroclinic orbit \( \gamma_c^0 \) connecting the singular points \((0, 0)\) and \((K, 0)\) and the corresponding traveling wave of system (2.1) is strictly monotonical, thus, this traveling wave is traveling wave front. Namely, if
(H3) hold,  
\[
\lim_{z \to -\infty} \tilde{u}(z) = 0, \quad \lim_{z \to +\infty} \tilde{u}(z) = K,
\]
if (H4) hold,  
\[
\lim_{z \to -\infty} \tilde{u}(z) = K, \quad \lim_{z \to +\infty} \tilde{u}(z) = 0.
\]

Next, our main result is the following

**Theorem 2.2.** Suppose (H1), (H2) hold. If (H3) or (H4) holds, then for sufficiently small \( \tau > 0 \), system (1.1) has a traveling wave solution \( u(z, c, \tau) \) connecting the equilibria 0 and K with wave speed c. \( u(z, c, \tau) \) lies in some small neighborhood of traveling wave front \( \tilde{u}(z, c) \) of system (2.1).

3. Existence of traveling wave front for delay equation

In this section, we focus our study on the case of weak generic delay kernel and prove Theorem 2.2. Let \( w(x, t) \) denote the spatio-temporal delay term, i.e.,

\[
w(x, t) = \frac{1}{\tau} \int_{-\infty}^{+\infty} \int_{-\infty}^{y} G(x - y, t - s)e^{-s/\tau}g(u(y, s))dyds.
\]

Then by straightforward computation, we have

\[
\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{1}{\tau}(g(u) - w).
\]

Thus, the original system (1.1) can be reformulated as

\[
\begin{aligned}
&u_t = u_{xx} + F(u, w), \\
&w_t = w_{xx} + \frac{1}{\tau}(g(u) - w).
\end{aligned}
\tag{3.1}
\]

Obviously, system (3.1) is not a delay differential system. The delay in system (1.1) now plays its role through the parameter \( \tau \).

Converting to traveling wave form, by writing

\[
u(x, t) = U(z), \quad w(x, t) = W(z), \quad z = x + ct, \quad c > 0,
\]

we get

\[
\begin{aligned}
cU' &= U'' + F(U, W), \\
cW' &= W'' + \frac{1}{\tau}(g(U) - W),
\end{aligned}
\]
where \( \dot{r} = \frac{d}{dz} \). Let
\[
V_1 = U', \quad V_2 = W',
\]
the system becomes
\[
\begin{aligned}
U' &= V_1, \\
V_1' &= cV_1 - F(U, W), \\
W' &= V_2, \\
V_2' &= cV_2 - \frac{1}{\tau} (g(U) - W).
\end{aligned}
\tag{3.2}
\]

This system has the equilibria of form
\[
(U, V_1, W, V_2) = (0, 0, g(0), 0), \quad (U, V_1, W, V_2) = (K, 0, g(K), 0).
\]

Since the delay is small, we introduce the small parameter
\[
\varepsilon = \sqrt{\tau},
\]
and define new variables
\[
u_1 = U, \quad u_2 = V_1, \quad w_1 = W, \quad w_2 = \varepsilon V_2,
\]
then system (3.2) becomes further
\[
\begin{aligned}
u_1' &= u_2, \\
u_2' &= cu_2 - F(u_1, w_1), \\
\varepsilon w_1' &= w_2, \\
\varepsilon w_2' &= \varepsilon cw_2 - (g(u_1) - w_1),
\end{aligned}
\tag{3.3}
\]
which will subsequently be referred as the slow system.

When \( \varepsilon = 0 \), system (3.3) is system (2.2). From Proposition 2.1, we know that system (3.3) with \( \varepsilon = 0 \) have a traveling wave front. Note when \( \varepsilon = 0 \), system (3.3) was not a dynamic system in \( \mathbb{R}^4 \). This problem can be overcome by the transformation \( z = \eta \), under which system (3.3) is transformed into
\[
\begin{aligned}
\dot{u}_1 &= \varepsilon u_2, \\
\dot{u}_2 &= \varepsilon (cu_2 - F(u_1, w_1)), \\
\dot{w}_1 &= w_2, \\
\dot{w}_2 &= \varepsilon cw_2 - (g(u_1) - w_1),
\end{aligned}
\tag{3.4}
\]
where the dots denote differentiation with respect to \( \eta \), and (3.4) be referred as the fast systems.

The slow system and fast system are equivalent when \( \varepsilon > 0 \). When \( \varepsilon = 0 \), the slow system (3.3) has its dynamical behaviour only on
\[
\mathscr{M}_0 = \{(u_1, u_2, w_1, w_2) \in \mathbb{R}^4 : w_1 = g(u_1), w_2 = 0\},
\]
which is a two-dimensional submanifold of $\mathbb{R}^4$. If $\mathcal{M}_0$ is normally hyperbolic, then for sufficiently small $\varepsilon > 0$, we can apply the geometric singular perturbation theory of Fenichel [21] to obtain a two-dimensional invariant manifold $\mathcal{M}_\varepsilon$ for system (3.3). Therefore, it suffices to study the flow of slow system restricted to $\mathcal{M}_\varepsilon$.

To verify normal hyperbolicity, we need to verify that the linearization of the fast system (3.4), restricted to $\mathcal{M}_0$, has exactly $2(\dim \mathcal{M}_0)$ eigenvalues with zero real part. The linearization of (3.4) restricted to $\mathcal{M}_0$ is given by

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
g'(u_1) & 0 & 1 & 0
\end{pmatrix},
$$

which has four eigenvalues 0, 0, 1 and $-1$. Thus, $\mathcal{M}_0$ is normally hyperbolic. By geometric singular perturbation theory, we know there exists an invariant manifold $\mathcal{M}_\varepsilon$, close to $\mathcal{M}_0$, for system (3.3) for $\varepsilon > 0$ sufficiently small. In fact, $\mathcal{M}_\varepsilon$ can be expressed in the form

$$
\mathcal{M}_\varepsilon = \{(u_1, u_2, w_1, w_2) \in \mathbb{R}^4 : w_1 = g(u_1) + h_1(u_1, u_2, \varepsilon), w_2 = h_2(u_1, u_2, \varepsilon)\},
$$

where $h_1$, $h_2$ depend smoothly on $\varepsilon$ and

$$
h_1(u_1, u_2, 0) = h_2(u_1, u_2, 0) = 0.
$$

By substituting into slow system (3.3), we obtain the partial differential equations

$$
\varepsilon \left[ g'(u_1)u_2 + \frac{\partial h_1}{\partial u_1} u_2 + \frac{\partial h_1}{\partial u_2} (cu_2 - F(u_1, g(u_1) + h_1)) \right] = h_2,
$$

$$
\varepsilon \left[ \frac{\partial h_2}{\partial u_1} u_2 + \frac{\partial h_2}{\partial u_2} (cu_2 - F(u_1, g(u_1) + h_1)) \right] = \varepsilon h_2 + h_1.
$$

Since $g$ and $h$ are zero when $\varepsilon = 0$, we expand them into the form of Taylor series with respect to $\varepsilon$,

$$
h_1(u_1, u_2, \varepsilon) = \varepsilon h_1^1(u_1, u_2) + \varepsilon^2 h_1^2(u_1, u_2) + \cdots;
$$

$$
h_2(u_1, u_2, \varepsilon) = \varepsilon h_2^1(u_1, u_2) + \varepsilon^2 h_2^2(u_1, u_2) + \cdots.
$$

Substituting (3.7) into (3.5) (3.13), expanding $F(u_1, g(u_1) + h_1)$ at $(u_1, g(u_1))$ and comparing coefficients of $\varepsilon$ and $\varepsilon^2$, we obtain

$$
h_1^1 = 0, \quad h_1^2 = g''(u_1)u_2^2 + F(u_1, g(u_1) + h_1)
$$

$$
h_2^1 = g'(u_1)u_2, \quad h_2^2 = 0.$$
The slow system (3.3), restricted to \( \mathcal{M}_\varepsilon \), is therefore given by

\[
\begin{align*}
    u'_0 &= u_2, \\
    u'_2 &= cu_2 - F(u_1, g(u)) + h_1(u_1, u_2, \varepsilon),
\end{align*}
\] (3.9)

where \( h_1 \) is given by (3.7) and (3.8). When \( \varepsilon = 0 \), it is easy to verify that (3.9) is a nondelay system and for any \( \varepsilon > 0 \) system (3.9) has equilibrium points \( (u_1, u_2) = (0, 0), (K, 0) \). Since \( \mathcal{M}_\varepsilon \) is smooth, the vectorfield in (3.9) is smooth for sufficiently small \( \varepsilon \). We can use the theorems of continuous dependence and implicit function to get the existence of heteroclinic orbit for small \( \varepsilon \), see [22].

Next, we would get the existence of heteroclinic orbit from the viewpoint of functional analysis. System (3.9) can be rewritten into the following form

\[
\begin{align*}
    u'_0 &= u_2, \\
    u'_2 &= cu_2 - F(u_1, g(u)) + \varepsilon^2 G(u_1, u_2, \varepsilon),
\end{align*}
\] (3.10)

Let \( X^0 := (u^0_1, u^0_2)^T \) be a solution of (2.2). To solve (3.10) for \( \varepsilon > 0 \) sufficiently small, we set

\[
X = X^0 + \varepsilon^2 V,
\]
where \( V(z, \varepsilon) = (\phi(z, \varepsilon), \psi(z, \varepsilon))^T \). Substituting \( X \) into (3.10), we have

\[
V' = A(z)V + \tilde{G}(X^0, 0) + o(\varepsilon^2)
\] (3.11)

where

\[
A(z) = \begin{pmatrix}
0 & 1 \\
-f'(u^0_1) & c
\end{pmatrix}, \quad f(u) = F(u, g(u)), \quad \tilde{G}(X^0, 0) = \begin{pmatrix}
0 \\
G(X^0, 0)
\end{pmatrix}.
\]

We shall show that this system has a solution satisfying \( V(\pm \infty, \varepsilon) = 0 \). Let \( L = d/dz - A(z) \), then the persistence question becomes the solvability of the equation

\[
LV = \tilde{G}(X^0, 0) + o(\varepsilon^2).
\]

Let us work in the space \( L^2 \) of square integral continuous functions with inner production

\[
\int_{-\infty}^{+\infty} (x(z), y(z))dz,
\]

with \( (\cdot, \cdot) \) being the Euclidean inner product on \( \mathbb{R}^2 \). From the Fredholm theory, we know that (3.11) will have a solution if and only if

\[
\int_{-\infty}^{+\infty} (x(z), \tilde{G}(X^0, 0) + o(\varepsilon^2))dz = 0,
\]
for all function \( x(z) \in \mathbb{R}^2 \) in the kernel of the adjoint of operator \( L \). It is easy to verify that the adjoint operator \( L^* \) is given by

\[
L^* = -\frac{d}{dz} - A(z)^T,
\]

In order to compute \( \text{Ker} \, L^* \) we have to find all \( x(z) \) satisfying

\[
\frac{dx}{dz} = -A(z)^T x(z).
\]

We are interested in the solutions satisfying \( x(\pm \infty) = 0 \). Note that \( u_0^0(z) \) is a the solution of the unperturbed problem and it tends to 0 (or \( K \)) as \( z \to -\infty \), when \( f'(0) > 0 \) (or \( f'(K) > 0 \)). Let \( z \to -\infty \) in (3.12), we get the asymptotic system of (3.12):

\[
\frac{dx}{dz} = \begin{pmatrix} 0 & f'(0) \\ -1 & -c \end{pmatrix} x(z) \quad \text{or} \quad \frac{dx}{dz} = \begin{pmatrix} 0 & f'(K) \\ -1 & -c \end{pmatrix} x(z).
\]

The eigenvalues \( \lambda \) of system (3.13) satisfy

\[
\lambda^2 + c\lambda + f'(0) = 0 \quad \text{(or) } \lambda^2 + c\lambda + f'(K) = 0.
\]

By Proposition 2.1, we have

\[
c \geq 2 \sqrt{\sup_{0 < u < K} \frac{f(u)}{u}} \geq 2 \sqrt{f'(0)} \quad \text{or} \quad \lambda^2 + c\lambda + f'(K) = 0.
\]

The eigenvalues are both real and negative.

Now we show that only \( x(z) \equiv 0 \) is the solution of (3.12) satisfying \( x(\pm \infty) = 0 \). Clearly \( x(z) \equiv 0 \) is a solution to (3.12). Suppose \( w(z) \neq 0 \) is another solution of (3.12), then by the Picard theorem, \( w(z) \neq 0 \) for each \( z \in \mathbb{R} \). Therefore, \( \lim_{z \to -\infty} w(z) = 0 \) appears only when the asymptotic system (3.13) has a solution \( \tilde{w}(z) \neq 0 \) with \( \lim_{z \to -\infty} \tilde{w}(z) = 0 \). However, each solution \( \tilde{w}(z) \neq 0 \) of system (3.13) doesn’t satisfy \( \tilde{w}(-\infty) = 0 \). Therefore, \( x(z) \equiv 0 \) is the only solution to (3.12) satisfying \( x(\pm \infty) = 0 \). This means, of course, that the Fredholm orthogonality condition trivially holds and so solutions of (3.11) exist, which satisfy \( V(\pm \infty, \varepsilon) = 0 \). Therefore, a heteroclinic connection exists between (0,0) and (K,0) of (3.9) for sufficiently small \( \varepsilon > 0 \).

**Remark** 3.1. The Picard Theorem excludes the case that a solution \( x(z) \neq 0 \) outside a compact interval. At the same time, we notice that although the argument is \( \mathcal{L}^2 \) space, the solution \( x \) to (3.12) is in \( \mathcal{H}^1 \).
4. Applications

In this section, we shall give some applications of our main result obtained in the previous section. Theorem 2.2 reduces the problem of establishing the existence of traveling wave front to the verification of \((H1)\)–\((H3)\) or \((H1)\), \((H2)\) and \((H4)\).

We consider the Nagumo equation with delay

\[
\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) \left( 1 - \int_{-\infty}^{+\infty} G(x-y,t-s) \right.
\]

\[
\times \left. k(t-s)u(y,s)dyds \right) (u(x,t) - \alpha),
\]

which is an important reaction diffusion equation for modeling traveling fronts in population dynamics [23]. This equation can be regarded as a model for the transmission of electrical pulses in a nerve axon [24]. The reaction term of the corresponding undelayed equation of (4.1) is \(F(u,w) = u(1-w)(u-\alpha)\) and \(f(u) = u(1-u)(u-\alpha)\). If \(0 < \alpha \leq 1/2\), it is easy to check

\[(H1) \ f \text{ is continuously differentiable, } f(0) = f(\alpha) = 0 \text{ and } f(u) \neq 0 \text{ in } (0,\alpha); \]

\[(H2) \ f'(0) = -\alpha < 0 \text{ and } f'(\alpha) = \alpha(1-\alpha) > 0; \]

**Theorem 4.1.** If \(0 < \alpha \leq 1/2\), for any sufficient small \(\tau > 0\), there exist speeds \(c \geq 2\sqrt{\alpha(1-\alpha)}\) such that system (4.1) has a traveling wave front solution \(u(x,t) = \phi(x+ct)\) connecting \(u_{-\infty} = \alpha\) to \(u_{+\infty} = \alpha\).

If \(1/2 \leq \alpha < 1\), let \(\hat{u}(x,t) = u(x,t) - \alpha\), then the system (4.1) becomes

\[
\frac{\partial \hat{u}(x,t)}{\partial t} = \frac{\partial^2 \hat{u}(x,t)}{\partial x^2} + \hat{u}(x,t)(\hat{u}(x,t) + \alpha)
\]

\[
\times \left( 1 - \alpha - \int_{-\infty}^{+\infty} \int_{-\infty}^{t} G(x-y,t-s)k(t-s)\hat{u}(y,s)dyds \right).
\]

The reaction term of the corresponding undelayed equation of (4.2) is \(F(u,w) = u(u+\alpha)(1-\alpha-w)\) and \(f(u) = u(u+\alpha)(1-\alpha-u)\).

\[(H1) \ f \text{ is continuously differentiable, } f(0) = f(1-\alpha) = 0 \text{ and } f(u) \neq 0 \text{ in } (0,1-\alpha); \]

\[(H2) \ f'(0) = \alpha(1-\alpha) > 0 \text{ and } f'(1-\alpha) = \alpha - 1 < 0; \]

**Theorem 4.2.** If \(1/2 \leq \alpha < 1\), for any sufficient small \(\tau > 0\), there exist speeds \(c \geq 2\sqrt{\alpha(1-\alpha)}\) such that system (4.2) has a traveling wave front solution \(\hat{u}(x,t) = \phi(x+ct)\) connecting \(\hat{u}_{-\infty} = 0\) to \(\hat{u}_{+\infty} = 1 - \alpha\), that is, system (4.1) has a traveling wave front solution \(u(x,t) = \phi(x+ct)\) connecting \(u_{-\infty} = \alpha\) to \(u_{+\infty} = 1\).
By [24, 17], we have that system \((4.1)_{\tau=0}\) has a unique wave front solution from 0 to 1 with wave speed \(c = (1 - 2\varepsilon)/\sqrt{2}\). Using the same argument as Theorem 2.2, we get

**Theorem 4.3.** For any sufficient small \(\tau > 0\), there exists a speed \(c = (1 - 2\varepsilon)/\sqrt{2}\) such that system \((4.1)\) has a traveling wave front solution \(u(x, t) = \phi(x + ct)\) connecting \(u_{-\infty} = 0\) to \(u_{+\infty} = 1\).

**References**

[1] Schaaf, K., Asymptotic behavior and traveling wave solutions for parabolic functional differential equations, Trans. Amer. Math. Soc., 302 (1987), 587–615.
[2] Wu, J. and Zou, X., Traveling wave fronts of reaction-diffusion systems with delay, J. Dynam. Differential Equations, 13 (2001), 651–687.
[3] Zou, X. and Wu, J., Existence of traveling wave fronts in delayed reaction-diffusion system via monotone iteration method, Proc. Amer. Math. Soc., 125 (1997), 2589–2598.
[4] Ma, S., Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem, J. Differential Equations, 171 (2001), 294–314.
[5] Huang, J., Lu, G. and Ruan, S., Existence of traveling wave solutions in a diffusive predator-prey model, J. Math. Biol., 46 (2003), 132–152.
[6] Huang, J. and Zou, X., Existence of traveling wavefronts of delayed reaction-diffusion systems without monotonicity, Discrete Contin. Dyn. Syst., 9 (2003), 925–936.
[7] Wang, Z.-C., Li, W.-T. and Ruan, S., Travelling wave fronts in reaction-diffusion systems with spatiotemporal delays, J. Differential Equations, 222 (2006), 185–232.
[8] Faria, T., Huang, W. and Wu, J., Travelling waves for delayed reaction-diffusion equations with global response, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 462 (2006), 229–261.
[9] Gourley, S. A., So, J. and Wu, J., Nonlocality of reaction-diffusion equations induced by delay: Biological modeling and nonlinear dynamics, J. Math. Sci. (N.Y.), 124 (2004), 5119–5153.
[10] Britton, N. F., Aggregation and the competitive exclusion principle, J. Theoret. Biol., 136 (1989), 57–66.
[11] Britton, N. F., Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model, SIAM J. Appl. Math., 50 (1990), 1663–1688.
[12] Ashwin, P., Bartuccelli, M. V., Bridges, T. J. and Gourley, S. A., Travelling fronts for the KPP equation with spatio-temporal delay, Z. Angew. Math. Phys., 53 (2002), 103–122.
[13] Gourley, S. A. and Ruan, S., Convergence and travelling fronts in functional differential equations with nonlocal terms: A competition model, SIAM J. Math. Anal., 35 (2003), 806–822.
[14] Song, Y., Peng, Y. and Han, M., Travelling wavefronts in the diffusive single species model with Allee effect and distributed delay, Appl. Math. Comput., 152 (2004), 483–497.
[15] Ai, S., Traveling wave fronts for generalized Fisher equations with spatio-temporal delays, J. Differential Equations, 232 (2007), 104–133.
[16] Peng, Y., Zhang, T. and Tadé, M. O., Existence of travelling fronts in a diffusive vector disease model with spatio-temporal delay, Nonlinear Anal. Real World Appl., 11 (2010), 2472–2478.
[17] Hadeler, K. P. and Rothe, F., Travelling fronts in nonlinear diffusion equations, J. Math. Biol., 2 (1975), 251–263.
Traveling Wave for Reaction Diffusion Equation

[18] Fife, P. C., *Mathematical Aspects of Reacting and Diffusing Systems*, Lectures Notes in Biomathematics, 28, Springer-Verlag, Berlin-New York, 1979.

[19] Britton, N. F., *Reaction-Diffusion Equations and Their Applications to Biology*, Academic Press, Inc., London, 1986.

[20] Zhao, Z. and Ge, W., Traveling wavefront solutions for reaction-diffusion equation with small delay, Funkcial. Ekvac., 54 (2011), 225–236.

[21] Fenichel, N., Geometric singular perturbation theory for ordinary differential equations, J. Differential Equations, 31 (1979), 53–98.

[22] Gourley, S. A., Travelling fronts in the diffusive Nicholson’s blowflies equation with distributed delays, Math. Comput. Modelling, 32 (2000), 843–853.

[23] Murray, J. D., *Mathematical Biology*, Biomathematics, Springer-Verlag, Berlin, 1993.

[24] Gilding, B. H. and Kersner, R., *Travelling waves in nonlinear diffusion-convection reaction*, Progress in Nonlinear Differential Equations and their Applications, 60, Birkhäuser Verlag, Basel, 2004.

nuna adreso:
Zhihong Zhao
School of Mathematics and Physics
University of Science & Technology Beijing
Beijing 100083
P. R. China
E-mail: zhaozhihong01@aliyun.com

Erhua Rong
Shanxi Agricultural University
Taigu, Shanxi 030801
P. R. China

(Ricevita la 28-an de julio, 2012)
(Reviiziita la 1-an de aprilo, 2013)