Abstract: We consider the discrete Hammersley-Aldous-Diaconis process (HAD) and the totally asymmetric simple exclusion process (TASEP) in \( \mathbb{Z} \). The basic coupling induces a multiclass process which is useful in discussing shock measures and other important properties of the processes. The invariant measures of the multiclass systems are the same for both processes, and can be constructed as the law of the output process of a system of multiclass queues in tandem; the arrival and service processes of the queueing system are a collection of independent Bernoulli product measures. The proof of invariance involves a new coupling between stationary versions of the processes called a multi-line process; this process has a collection of independent Bernoulli product measures as an invariant measure. Some of these results have appeared elsewhere and this paper is partly a review, with some proofs given only in outline. However we emphasize a new approach via dual points: when the graphical construction is used to construct a trajectory of the TASEP or HAD process as a function of a Poisson process in \( \mathbb{Z} \times \mathbb{R} \), the dual points are those which govern the time-reversal of the trajectory. Each line of the multi-line process is governed by the dual points of the line below. We also mention some other processes whose multiclass versions have the same invariant measures, and we note an extension of Burke’s theorem to multiclass queues which follows from the results.

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1. Introduction

The macroscopic evolution of the totally asymmetric simple exclusion process (TASEP) \cite{20, 21} and the Hammersley-Aldous-Diaconis process (HAD) \cite{17, 1} can be described by the Burgers equation \cite{27, 28}. This equation admits shock solutions which have a microscopic counterpart. A crucial tool for the study of these shocks is Liggett’s basic coupling \cite{19, 20} of two or more copies of the process; see for example \cite{4, 7, 10, 15, 31}. These coupled processes have a natural interpretation as a process with two or more classes of particle (which is also of interest from a combinatorial viewpoint \cite{2, 5}). Such coupled and multiclass processes, and their invariant distributions, are the subject of this paper.
Our basic objects are the TASEP and a discrete-space version of the HAD process\textsuperscript{[8][16]}. These are continuous-time Markov processes taking values in the state-space $\mathcal{X} = \{0,1\}^\mathbb{Z}$; for a configuration $\eta \in \mathcal{X}$, we say that $\eta$ has a particle at $x$ (or that $x$ is occupied) if $\eta(x) = 1$, and that $\eta$ has a hole at $x$ (or that $x$ is empty) if $\eta(x) = 0$. In the TASEP, each particle tries to jump one site to the left at rate 1, succeeding if the site to its left is empty. In the HAD process, each empty site summons at rate 1 the nearest particle to its left.

In the HAD process, each empty site summons at rate 1 the nearest particle to its left. However, in the TASEP, each particle tries to jump one site to the left at rate 1, succeeding if the site to its left is empty.

The graphical construction allows us to realize either the TASEP or the HAD process as a function of a rate 1 Poisson process on $\mathbb{R} \times \mathbb{Z}$ (the process of “marks” or “points” or “bells”) and the initial configuration. Under the basic coupling, processes started at different initial conditions $\eta$ are coupled by using the same realization of the Poisson points. Under the basic coupling, processes started at different initial conditions $\eta$ are coupled by using the same realization of the Poisson points. Suppose the initial configurations are ordered, in that $\eta^1(x) \leq \eta^2(x) \cdots \leq \eta^n(x)$ for all $x \in \mathbb{Z}$. Then this ordering is preserved by the dynamics of the coupling, and we obtain a coupled process taking values in the space $\mathcal{X}^{n\uparrow}$ of ordered configurations, defined by

$$\mathcal{X}^{n\uparrow} = \{(\eta^1, \ldots, \eta^n) \in \mathcal{X}^n : \eta^1(x) \leq \cdots \leq \eta^n(x) \text{ for all } x \in \mathbb{Z}\}. \quad (1)$$

We define a map $R: \mathcal{X}^{n\uparrow} \mapsto \mathcal{Y}_n = \{1, 2, \ldots, n+1\}^\mathbb{Z}$, taking an ordered configuration $\eta$ into a multiclass configuration $\xi = R\eta$, by

$$\xi(x) = n + 1 - \sum_{k=1}^n \eta^k(x). \quad (2)$$

The multiclass configuration $\xi$ labels each site $i$ with a class. The larger the number of $\eta$ particles at site $i$, the lower is its class. If a site is occupied in all $n$ marginals of $\eta$, then $\xi(x) = 1$ and we say that $x$ contains a first-class particle. If $\eta^1(x) = 0$ but $\eta^2(x) = 1$ then $x$ contains a second-class particle ($\xi(x) = 2$), and so on. When $x$ is empty in all the marginals of $\eta$, $\xi$ gives value $n + 1$ at $i$. Conventionally, we regard such sites with value $n + 1$ as holes (this is also how they are displayed in the figures in this paper), though they could equivalently be regarded as particles of class $n + 1$. A coupled process $(\eta_t)$ taking values in $\mathcal{X}^{n\uparrow}$ induces a multiclass process $(\xi_t) = (R\eta_t)$; since $R$ is a bijection, the two are essentially equivalent. In the case of the TASEP, the dynamics of the multiclass process have a natural interpretation as a TASEP in which lower-numbered particles have priority over higher-numbered particles, and can jump from the right to displace them; these dynamics are related to various sorting algorithms.

The Bernoulli product measures $\nu^\rho$ on $\mathcal{X}$ of density $\rho \in (0,1)$ are invariant for both the HAD process and the TASEP. We look for invariant measures for the corresponding coupled processes, whose $k$th marginal is $\nu^{\rho^k}$ for each $1, 2, \ldots, k$, for fixed densities $0 < \rho^1 < \cdots < \rho^n < 1$. The family of invariant measures $\pi = \pi^{(\rho^1), \ldots, \rho^n}$ obtained for the coupled process is the same for the TASEP and for the discrete HAD process, and also for various other particle systems with values in $\mathcal{X}$. To each such $\pi$ corresponds a distribution $\mu$ on $\mathcal{Y}_n$, which is invariant for the multiclass process. The form of $\mu$ can be obtained as the stationary output process of a system of multiclass $./M/1$ queues in tandem. A striking property of these invariant measures is that a sample $\eta$ from $\pi$ can be obtained as a
deterministic function $T$ of a configuration $\alpha \in \mathcal{X}^n$ with distribution $\nu = \nu^1 \times \cdots \times \nu^n$, a product of Bernoulli product measures.

To show that $\pi$ and $\mu$ are invariant for the coupled and multiclass process respectively, one can perform a generator-like computation as [2, 4, 9, 29] did for the 2-class TASEP. We proceed otherwise, introducing a new process $\alpha_t = (\alpha^1_t, \ldots, \alpha^n_t)$ on $\mathcal{X}^n$ called a multi-line process and showing that (a) the product of product measures $\nu$ is invariant for the multi-line process and (b) the process $T\alpha_t$ is the coupled process. See [12] for the TASEP and [11] for an analogous construction for the continuous-space HAD process. We review the method in this paper, and describe an approach to the multi-line process via dual points, rather different to the approach of [12].

Stationary versions of the TASEP and the HAD process can be constructed as deterministic functions of the Poisson process $\omega$. For each density $\rho$ and almost all Poisson point configurations $\omega$, there exists a unique trajectory of the process $(\eta_t, t \in \mathbb{R})$ with time-marginal distribution $\nu^\rho$ and governed by $\omega$. This was proved for the continuous-space HAD process in [11] and the proof extends to the cases discussed here. The $\omega$ points and the trajectory of the process induce new points $D^\rho(\omega)$, defined — roughly speaking — as the points governing the time-reversed trajectory. They are called dual points and, just as $\omega$ itself, form a Poisson process on $\mathbb{R} \times \mathbb{Z}$. In addition, the dual points with time coordinate less than $t$ are independent of the particle configuration at time $t$. These properties have been shown for the continuous-space version of the HAD by Cator and Groeneboom [3]; see also [11]. We show them here for both the TASEP and the discrete HAD process. The proof is a bit more complicated because a trajectory in the discrete-space case does not determine all the Poisson points governing it. To overcome this problem we augment the state-space and introduce spin-flip processes to mark the Poisson points missed by the trajectories. See Proposition 5 for the HAD and Proposition 10 for the TASEP. These are the main new results of this paper.

One way to construct a stationary trajectory $(\alpha_t, t \in \mathbb{R})$ of the multi-line process referred to above, whose marginal distribution at any fixed time $t$ is $\nu = \nu^1 \times \cdots \times \nu^n$, is as follows. The bottom line $(\alpha^1_t, t \in \mathbb{R})$ of the process is constructed as a the unique trajectory with marginal $\nu^{\rho^1}$ governed by the Poisson points $\omega$. The dual points $\omega^{n-1}$ of the bottom trajectory and the density $\rho^{n-1}$ are then used to construct the $(n-1)st$ line, and so on. Since $\alpha^k_t$, the $k$th marginal at time $t$, depends only on the Poisson points $\omega^k$ with time coordinate less than $t$, which are independent of $\alpha^{k-1}_t, \ldots, \alpha^1_t$, the resulting distribution of $\alpha_t \in \mathcal{X}^n$ is indeed the product distribution $\nu$. The process is time-invariant by construction.

In the case of the continuous-space HAD process, the multi-line process is closely related to the polynuclear growth model studied by Prähöfer and Spohn [24, 25]; see also [13, 14]. In those papers the points in $\omega$ and the dual points are called nucleation events and annihilating events respectively; see in particular Figure 2 in [13].

In Section 2 we construct the coupled invariant measure $\pi$ and the multiclass invariant measure $\mu$ as functions of a product of Bernoulli product measures. In Section 3 we review
the proof of the invariance of $\pi$ (or $\mu$) for the coupled (respectively, multiclass) HAD process; at the end of this section we mention a case of the long range exclusion process which is equivalent to the HAD process. In Section 4 we review the proof for the case of the TASEP. In Section 5 we give some examples of other dynamics for which the associated multiclass processes have the family of measures $\mu$ as invariant distribution, including certain "sequential" TASEPs defined in discrete time. We also mention various related cases for which $\mu$ is not invariant. Finally, in Section 6 we note a multiclass generalization of Burke’s theorem. The measures $\mu$ constitute fixed point arrival processes for a multiclass priority $/M/1$ queue, in the sense that the law of the output process of the queue is the same as the law of the input. This property can be deduced from the invariance of $\mu$ and the tandem queue construction used in the proof (although more direct proofs are also available; see [22]).

In summary, our main aims in this paper are as follows: (i) to review the results of [12] and [11] describing invariant measures of multiclass processes; (ii) to illustrate the application of the results and methods to various different particle systems; (iii) to describe an approach via dual points which adapts well to various different processes and which differs, for example, from the more specific arguments used in [12] for the TASEP case, and (iv) to emphasize the correspondence between the multiclass process with values in $Y_n$ and the coupled process with values in $X_{n\uparrow}$.

2. Multiclass invariant measures

In this section we will construct a family of measures on $X_{n\uparrow}$, which we will later show to be invariant for the coupled HAD and TASEP processes. For given particle densities $0 < \rho^1 < \cdots < \rho^n < 1$, we will construct a measure $\pi = \pi^{(n)}_{\rho^1, \ldots, \rho^n}$ on $X_{n\uparrow}$ whose $k$th marginal will have distribution $\nu^{\rho_k}$, Bernoulli product measure of density $\rho_k$ (that is, the measure under which each site is occupied independently with probability $\rho_k$).

Let $\eta$ have distribution $\pi$, and write $\xi = R\eta \in Y_n$ for the configuration obtained from the map defined at (2). Since $\pi$ is invariant for the coupled processes, the distribution $\mu = R\pi$ of $\xi$ will be invariant for the corresponding multiclass processes.

The invariant measure $\pi$ on $X_{n\uparrow}$ is constructed starting from product measure $\nu = \nu^{\rho^1} \times \cdots \times \nu^{\rho^n}$ on $X^n$. Let $\alpha = (\alpha^1, \ldots, \alpha^n) \in X^n$ be distributed according to this product measure $\nu$. We will interpret the particles of $\alpha$ as events in a system of queues in tandem in discrete time; the sites of $\mathbb{Z}$ now correspond to times in the queueing system.

2.1. Construction of the coupled invariant measure $\pi$. Consider a queueing server in discrete time, governed by $\alpha^1$ and $\alpha^2$ in the following way. The particles of $\alpha^1$ represent times at which a customer arrives at the queue, and the particles of $\alpha^2$ represent potential service times (that is, times at which a customer can depart from the system, if any are present). At time $i$, the queue length increases by 1 if $\alpha^1(i) = 1$ and $\alpha^2(i) = 0$; it stays the same if $\alpha^1(i) = \alpha^2(i)$; and it decreases by 1 if $\alpha^1(i) = 0$ and $\alpha^2(i) = 1$, unless it was
already 0 in which case it remains 0. Then if \( Z(j) \) is the queue length just after time \( j \), one has

\[
Z(j) = (Z(j - 1) + \alpha^1(j) - \alpha^2(j))^+, \quad j \in \mathbb{Z}.
\]  

(3)

Since \( \alpha^1 \) and \( \alpha^2 \) are distributed as independent Bernoulli product measures, with densities \( \rho^1 \) and \( \rho^2 \) respectively, and \( \rho^1 < \rho^2 \), the queue-length process \( Z \) is a positive recurrent Markov chain (in fact, its stationary distribution is geometric with parameter \( \rho^1 / \rho^2 \)), and there is an essentially unique way to construct a stationary process \( Z \) as a function of the input \( \alpha \), namely by

\[
Z(j) = \sup_{r \leq j} \left( \sum_{i=r}^{j} [\alpha^1(i) - \alpha^2(i)] \right)^+.
\]  

(4)

The evolution of \( Z(j) \) is illustrated in Figure 1. It can be seen that if an arrival and a potential service occur at the same time, a customer may spend no time in the system; this is the case of the last customer to arrive. coupled configuration is

The configuration \( D = D(\alpha^1, \alpha^2) \), representing the departure times from the queue, is then defined by

\[
D(j) = \begin{cases} 
1 & \text{if } \alpha^2(j) = 1 \text{ and } Z(j - 1) + \alpha^1(j) > 0 \\
0 & \text{otherwise}.
\end{cases}
\]  

(5)

The interpretation is that \( D(j) = 1 \) if a customer departs from the queue at time \( j \). Allowing the value \( \infty \) in \( D \), this definition of the operator \( D : \mathcal{X}^2 \mapsto \mathcal{X} \) makes sense for any \( \alpha^1 \) and \( \alpha^2 \), but in fact we will only use it in cases where \( \alpha^1 \) and \( \alpha^2 \) have independent Bernoulli product measures of densities \( \rho^1 \) and \( \rho^2 \) with \( \rho^1 < \rho^2 \). Then the queueing process is stable (since the rate of arrivals is lower than the rate of services). In queueing theory terminology the queue is called a discrete-time \( M/M/1 \) queue (where “1” indicates a single-server queue and “M” stands for memoryless, indicating that the arrival and service processes each have product measure).
We note the following useful properties of the operator $D$. The first two follow immediately from (4) and (5), while the third is Burke’s Theorem for a discrete-time $M/M/1$ queue [18]:

**Proposition 1.**

(i) $D(\alpha^1, \alpha^2) \leq \alpha^2$.

(ii) If $\tilde{\alpha}^1 \leq \alpha^1$ then $D(\tilde{\alpha}^1, \alpha^2) \leq D(\alpha^1, \alpha^2)$.

(iii) If $\alpha^1$ and $\alpha^2$ have independent Bernoulli product measures of densities $\rho^1$ and $\rho^2$ with $\rho^1 < \rho^2$, then $D(\alpha^1, \alpha^2)$ also has Bernoulli product measure with density $\rho^1$.

We now define a sequence of operators $D^{(n)} : \mathcal{X}^n \mapsto \mathcal{X}$ as follows. Let $D^{(1)}(\alpha^1) = \alpha^1$, and then recursively for $n \geq 2$, let

$$D^{(n)}(\alpha^1, \alpha^2, \ldots, \alpha^n) = D \left( D^{(n-1)}(\alpha^2, \ldots, \alpha^n), \alpha^1 \right).$$ (6)

The configuration $D^{(n)}(\alpha^1, \alpha^2, \ldots, \alpha^n)$ represents the departure process from a system of $(n-1)$ queues in tandem. The arrival process to the first queue is $\alpha^1$. The service process of the $k$th queue is $\alpha^{k+1}$, for $k = 1, \ldots, n-1$. Finally, for $k = 2, \ldots, n-1$, the arrival process to the $k$th queue is given by the departure process of the $(k-1)$st queue. This is known as a system of $./M/1$ queues in tandem.

Note $D^{(2)}(\alpha^1, \alpha^2) = D(\alpha^1, \alpha^2)$. By applying Proposition 1 repeatedly, we obtain the following properties of $D^{(n)}$:

**Proposition 2.**

(i) $D^{(n)}(\alpha^1, \ldots, \alpha^n) \leq D^{(n-1)}(\alpha^2, \ldots, \alpha^n) \leq \cdots \leq \alpha^n$.

(ii) If $\alpha^1, \ldots, \alpha^n$ have independent Bernoulli product measures of densities $\rho^1 < \cdots < \rho^n$, then $D^{(n)}(\alpha^1, \ldots, \alpha^n)$ also has Bernoulli product measure with density $\rho^1$.

Now define the configuration $\eta = (\eta^1, \ldots, \eta^n)$ by

$$\eta^k = D^{(n-k+1)}(\alpha^k, \alpha^{k+1}, \ldots, \alpha^n).$$ (7)

From Proposition 2 we have that

(i) $\eta \in \mathcal{X}^{n\uparrow}$ (that is, $\eta^k \leq \eta^{k+1}$ for all $k = 1, \ldots, n-1$);

(ii) for each $k$, $\eta^k$ has marginal distribution $\nu^{\rho^k}$.

We then define the map $T : \mathcal{X}^n \mapsto \mathcal{X}^{n\uparrow}$ by $T\alpha = \eta$. The desired distribution $\pi$ on $\mathcal{X}^{n\uparrow}$ is the induced distribution of $\eta$ (that is, $\pi = T\nu$).

2.2. Construction of the multiclass measure $\mu$. Let $\eta$ have the distribution $\pi$ constructed above, which we will show to be invariant for the coupled HAD and TASEP processes. Let $\xi = R\eta$, where $R$ is the map defined at (2). Then $\mu = R\pi$, the distribution of $\xi$, will be invariant for the multiclass HAD and TASEP processes.
This multiclass invariant measure can also be described directly via a tandem queueing system with multiclass queues. (This direct construction described below is not necessary to understand the proofs of invariance in later sections, which are written in terms of the construction of \( \pi \) given in the previous section).

As before, the system will now contain \( n-1 \) queues. The arrivals to the first queue are again the particles of \( \alpha^1 \). For \( k = 2, \ldots, n-1 \), the arrivals to the \( k \)th queue correspond to the services of the \((k-1)\)st queue, and are given by the particles of \( \alpha^k \). These are partitioned into \( k-1 \) different classes; these classes will be served by the \( k \)th queue according to a priority policy, under which lower-numbered classes are served ahead of higher-numbered classes. A class \( r \) customer departing from queue \( k-1 \) becomes a class \( r \) customer arriving at queue \( k \); an unused service at queue \( k-1 \) becomes a class \( k \) customer arriving at queue \( k \). Finally, the services of the \((n-1)\)st queue are given by the particles of \( \alpha^n \), and will be assigned \( n \) different classes; this will yield the \( n \)-type multiclass configuration desired, distributed according to \( \mu \).

The partition of the particles of \( \alpha^k \) into \( k \) classes is written using configurations \( \beta^k_1, \ldots, \beta^k_k \) such that \( \beta^k_1 + \cdots + \beta^k_k = \alpha^k \). The configuration \( \beta^k_r \) represents the \( \alpha^k \)-particles of class \( r \), for \( r = 1, 2, \ldots, k \).

Of course \( \beta^1_1 = \alpha^1 \). Then for \( 2 \leq k \leq n \), we set
\[
\beta^k_1 = D(\beta^{k-1}_1, \alpha^k);
\beta^k_r = D(\beta^{k-1}_r, \alpha^k - \beta^k_1 - \cdots - \beta^k_{r-1}) \quad \text{for} \ 1 < r < k;
\beta^k_k = \alpha^k - D(\alpha^{k-1}, \alpha^k).
\]

When a customer departs from queue \( k \) (that is, when an \( \alpha^k \)-service occurs and there is at least one customer present), the customer which departs is the lowest-numbered one present (including one which may just have arrived). From the equations above for the \( \beta^k_r \), this may be understood as follows: the \( r \)th-class customers arriving at queue \( k \) experience a service process which corresponds to \( \alpha^k \) but with the service times used by customers of classes \( 1, \ldots, r-1 \) removed.

Now the multiclass configuration \( \xi \) is derived from the \( n \)th line in the natural way; for \( k = 1, 2, \ldots, n \), let \( \xi(j) = k \) if \( \beta^n_k(j) = 1 \); otherwise (i.e. if \( \alpha^n(j) = 0 \) let \( \xi(j) = n+1 \). Call
\[
\xi = M\alpha \in \mathcal{Y}_n = \{1, \ldots, n+1\}^Z
\]
the resulting multiclass configuration.

The construction is most easily understood from a picture; see Figure 2 for an example with \( n = 3 \).

The relations between the multiclass configuration \( \xi \), the multi-line configuration \( \alpha \) and the ordered (or coupled) configuration \( \eta = T\alpha \) is given by
\[
\xi = R\eta = R(T\alpha) = M\alpha.
\]
Figure 2. Construction of $\xi = M\alpha$. Here $n = 3$. Each particle on line $k$ represents a site $i$ such that $\alpha^k(i) = 1$; the particle carries the label $r$ such that $\beta^k_r(i) = 1$ (representing a particle of $r$th class). In this diagram and later, sites in the multiclass configuration $\xi$ with value $n + 1$ are treated as holes and left empty.

Figure 3. From independent to ordered configurations

This correspondence is illustrated in Figure 3. To establish formally this equivalence between the construction of the function $M$ here and the construction of the function $T$ in the previous section, one can check by induction that for any $1 \leq r \leq k \leq n$,

$$\beta^k_1 + \beta^k_2 + \cdots + \beta^k_r = D^{(k-r+1)}(\alpha^r, \ldots, \alpha^k).$$

Define $\mu = M\nu = RT\nu$. This is the distribution of $\xi = M\alpha$ when $\alpha$ consists of $n$ independent configurations with Bernoulli product distributions of parameters $0 < \rho^1 < \cdots < \rho^n < 1$.

2.3. The case $n = 2$. We give a few words about the constructions above in the case $n = 2$. We now have a single queue with arrival process $\alpha^1$ and service process $\alpha^2$ (that is, a single $M/M/1$ queue). The particles of $\eta^2$ are simply the particles of $\alpha^2$, that is, the potential service times. The particles of $\eta^1$ are the subset of those times where a departure actually occurs, namely the particles of $D(\alpha^1, \alpha^2)$. The discrepancies between the two configurations $\eta^1$ and $\eta^2$ correspond to second-class particles in the two-class interpretation; that is, to sites $j$ where $\xi(j) = 2$. These correspond to unused services in the queue.
This construction was described by Angel [2] for the invariant measure of the two-class TASEP. The queueing interpretation can be found in Ferrari and Martin [12]. The measure so obtained has been first computed by Derrida, Janowsky, Lebowitz and Speer [4] and then described in other ways by [9, 20, 5] for the two-class TASEP.

3. Discrete HAD

The discrete-space Hammersley-Aldous-Diaconis process is a continuous-time Markov process taking values in a subset of \( \mathcal{X} \). At rate one each site \( j \) calls the closest particle to the left of \( j \) (including \( j \)) making it jump to \( j \). The generator of the process is

\[
L_H f(\eta) = \sum_{j \in \mathbb{Z}} [f(A_j \eta) - f(\eta)]
\]

where, writing \( i = i(\eta, j) = \max\{k \leq j : \eta(k) = 1\} \) for the closest occupied site of \( \eta \) to the left of \( j \),

\[
A_j \eta(k) = \begin{cases} 
\eta(k) & \text{if } k \neq i, j \\
1 & \text{if } k = j \\
0 & \text{if } k = i \text{ and } i < j
\end{cases}
\]

\[\eta\]

before jump

\[i(\eta, j)\]

after jump

\[A_j \eta\]

Figure 4. Discrete Hammersley process. Jumps occur at rate 1

**Harris graphical construction.** To construct the process we attach to each site an independent Poisson process of rate 1; these processes form a rate-1 Poisson process on \( \mathbb{R} \times \mathbb{Z} \). Bells ring at the points (or space-time events) of this process, which are represented by \* in Figure 5. The space of point configurations is called \( \Omega \) and single point configurations are called \( \omega \). When a bell rings at \( j \) (that is, at a time \( s \) such that \((j, s) \in \omega\)), the closest particle to the left jumps to \( j \). A possible point configuration and the resulting trajectory are illustrated in Figure 5. Such a construction is easily seen to be well-defined for the corresponding process in a finite region (since there are finitely many points of \( \omega \) in any finite time-interval). To define the process on all of \( \mathbb{Z} \), we need to exclude the possibility of particles escaping immediately to \(+\infty\), and should restrict to the state space

\[
\hat{\mathcal{X}} = \left\{ \eta \in \mathcal{X} : \lim_{r \to \infty} r^{-1/2} \sum_{j=1}^{r} \eta(j) = \infty \right\}.
\]

For details, see Seppäläinen [28], where the analogous construction for the continuous-space HAD is carried out.
Figure 5. Harris construction

If one calls \( \omega \) a realization of the points, then the configuration at time \( t \) of the process is a function called \( \phi \) of \( \omega \) and the initial configuration \( \eta_0 \):

\[
(t, \omega, \eta_0) \mapsto \phi(t, \omega, \eta_0) \tag{11}
\]

For given \( \eta_0 \) and \( \omega \), the process \((\eta_t, t \geq 0)\) defined by \( \eta_t = \phi(t, \omega, \eta_0) \) is called the HAD process governed by \( \omega \) with initial condition \( \eta_0 \). In fact one has

\[
\eta_t = \phi(t - s, \omega, \eta_s) \tag{12}
\]

for all \( 0 \leq s < t < \infty \).

The invariant measures of the HAD process are the Bernoulli product measures \( \nu^\rho \) with density \( \rho \in (0, 1] \) (and mixtures of them).

Using the Kolmogorov extension theorem, we can dispense with the initial condition and construct jointly the Poisson points \( \omega \) and an evolution \((\eta_t, t \in \mathbb{R})\) such that, for all \( t \), the marginal distribution of \( \eta_t \) is \( \nu^\rho \), and which satisfies (12) for all \( -\infty < s < t < \infty \). We again say that \((\eta_t, t \in \mathbb{R})\) is governed by \( \omega \). In fact, it turns out that the construction of such a bi-infinite trajectory is essentially unique, as soon as the particle density \( \rho \) is fixed:

**Proposition 3.** Let \( \rho \in (0, 1) \). Then there exists an essentially unique function \( H_\rho \) mapping elements \( \omega \) of \( \Omega \) to trajectories \((\eta_t, t \in \mathbb{R})\) such that:

1. The induced law of \((\eta_t, t \in \mathbb{R}) = H_\rho(\omega) \) is stationary in time.
2. The marginal law of \( \eta_t \) for each \( t \) is space-ergodic with particle density \( \rho \).
3. With probability 1, \((\eta_t, t \in \mathbb{R}) \) is a HAD evolution governed by \( \omega \).

(Here “essentially unique” means that if \( H_\rho' \) is another function satisfying the three conditions, then \( H_\rho(\omega) = H_\rho'(\omega) \) with probability 1). Then in fact the marginal law of \( \eta_t \) for each \( t \) is \( \nu^\rho \).
Proposition 3 can be proved following the approach of Ekhaus and Gray [6]; see Mountford and Prabhakar [23] and our proof for the continuous-space HAD in [11]. One might conjecture that a stronger statement holds: for almost all $\omega$, there exists a unique HAD trajectory $(\eta_t, t \in \mathbb{R})$ governed by $\omega$ such that, for all $t$, the configuration $\eta_t$ has particle density $\rho$.

**Coupling.** Different initial configurations $\eta_0^1, \ldots, \eta_0^n$ with the same Poisson bells $\omega$ produce a joint process whose marginals are the HAD process with those initial configurations:

$$\eta_t^k = \phi(t, \omega, \eta_0^k)$$

Hence for an initial condition $\eta_0 = (\eta_0^1, \ldots, \eta_0^n) \in \tilde{\mathcal{X}}^n$, we can describe the coupled HAD process by

$$\eta_t = \phi^{(n)}(t, \omega, \eta_0),$$

where the function $\phi^{(n)} : \mathbb{R} \times \Omega \times \tilde{\mathcal{X}}^n \mapsto \tilde{\mathcal{X}}^n$ is defined by

$$(\phi^{(n)}(t, \omega, \eta))^k = \phi(t, \omega, \eta^k).$$

The generator of the coupled process is given by

$$L_C f(\eta) = \sum_{j \in \mathbb{Z}} [f(C_j \eta) - f(\eta)]$$

where

$$C_j \eta = (A_j \eta^1, \ldots, A_j \eta^n).$$

for $A_j$ defined in (10).

If the initial configurations are ordered, that is, $\eta_0^k(i) \leq \eta_0^{k+1}(i)$, for all $i$, then $\eta_t^k \leq \eta_t^{k+1}$ for all $t$. Put another way, we can regard the coupled process as a process taking values in the space $\mathcal{X}^{n\uparrow}$ of ordered configurations. In Figure 6 we illustrate the jumps produced by a bell at site $j$ in such a case. The closest particle to the left of $j$ in each marginal jumps to $j$; the jumps are simultaneous.

**Multiclass process.** From now on we indeed regard the coupled process as a process taking values in the space $\mathcal{X}^{n\uparrow}$ of ordered configurations. Now we can regard as first-class particles the sites occupied in all marginals, second-class particles those occupied from the second marginal but not the first, and so on. The multiclass process is defined in terms of the coupled process by $\xi_t := R \eta_t$. Since $R$ is a bijection from $\mathcal{X}^{n\uparrow}$ to $\mathcal{Y}_n$, this is also a Markov process (whose behavior is not completely intuitive; compare for example with the more natural behavior of the multiclass TASEP process considered in Section 4). The generator of the multiclass process can be written in terms of the generator of the coupled process by $L_{MC} = RL_C R^{-1}$, and the operator $R$ commutes with the dynamics of the coupled and multiclass processes. Figure 6 illustrates the correspondence between the two processes.
3.1. Invariance of $\mu$.

**Theorem 4.** Let $\alpha = (\alpha^1, \ldots, \alpha^n)$ have law $\nu$, product of Bernoulli product measures with densities $\rho^1 < \cdots < \rho^n$. Then $\pi$, the law of $T\alpha$, is invariant for the coupled HAD process $(\eta_t)$ and $\mu$, the law of $M\alpha$, is invariant for the multiclass HAD process $(\xi_t)$.

**Sketch of proof.** From \[\text{3.2}\], the statements are equivalent. Hence it suffices to show that the law of $\eta = T\alpha$ is invariant for the coupled process. We do it in two steps. First introduce new dynamics $\alpha_t = (\alpha^1_t, \ldots, \alpha^n_t)$ called the multi-line process, and then show:

1) The product measure $\nu$ is invariant for the multi-line process $\alpha_t$.

2) $T\alpha_t$ is the coupled process $\eta_t$.

These statements are Propositions \[\text{3.3}\] and \[\text{7}\] below. \[\square\]

In fact one can go on to show that this family of measures $\mu$, indexed by the densities $\rho^1, \ldots, \rho^n$, are the only extremal invariant measures for the multiclass process. The proof of such a result follows a coupling argument of Ekhaus and Gray \[\text{3}\] as implemented by Mountford and Prabakhar \[\text{23}\]. See \[\text{11}\] for the argument for the continuous-space HAD process.

3.2. Dual points. Given a particle density $\rho \in (0,1)$ and a realisation of the Poisson marks $\omega$, Proposition \[\text{3}\] provides a stationary HAD trajectory $(\eta_t) = H_\rho(\omega)$ governed by $\omega$ with time-marginal $\nu^\rho$.

We define another set of marks $\Delta_\rho(\omega)$, called dual points. These are given by the positions of the particles just before jumps:

$$\Delta_\rho(\omega) := \{(i(\eta_{t-}, j), t) : (j, t) \in \omega\}$$

(15)

where $i(\eta, j)$ is the position of the closest $\eta$ particle to the left of $j$, as defined after \[\text{3}\]. This includes “jumps” of null size, when $i = j$. 

**Figure 6.** Coupled and multiclass processes. Effect of a bell at $j$
The points $\omega$ and the dual points $\Delta_\rho(\omega)$ are illustrated in Figure 7. Since the dual points are located in the space-time positions just vacated by particles, they govern the time-reversal of the trajectory. More precisely: the time-reversed and space-reversed trajectory is a HAD trajectory governed by (the time- and space-reversal of) $\Delta_\rho(\omega)$. In visual terms: turning Figure 7 upside-down exchanges the roles of the stars and the circles.

![Figure 7](image)

**Figure 7.** The dual points of the trajectory of Figure 5 are represented by circles.

The law of the dual points $\Delta_\rho(\omega)$ is then also Poisson, just as the law of $\omega$ itself. This is the first part of the following result.

**Proposition 5.** Let $\omega$ be a Poisson process in $\mathbb{R} \times \mathbb{Z}$, and let $\Delta_\rho(\omega)$ be the dual points for the HAD trajectory with particle density $\rho$ governed by $\omega$. Then $\Delta_\rho(\omega)$ is also a Poisson process in $\mathbb{R} \times \mathbb{Z}$. Furthermore $\{ (x, s) \in \Delta_\rho(\omega) : s < t \}$, the set of dual points earlier than $t$, is independent of the configuration $\eta_t$.

**Proof.** The proof is in the spirit of Reich’s proof of Burke’s theorem, used by Cator and Groeneboom for the continuous space HAD. The idea is to consider the time-reversal of the process, and goes as follows. As commented above, the dual points govern the reverse process. By doing a generator calculation (or verifying an equivalent detailed-balance property) one obtains that the time-reversal of the equilibrium HAD process with density $\rho$ is again an equilibrium HAD process with density $\rho$, but now with jumps to the left. (Put another way, the time-reversal of the process has the same law as the space-reversal). Now we would like to conclude that that the dual points therefore must also be Poisson. However, in the discrete-space case, the problem is that the trajectory of the HAD process does not identify all the points which govern it; it is also necessary to keep track of the points producing null jumps, which are not visible from the trajectory alone. To overcome this, we will add an auxiliary spin-flip process.

Let $\gamma_t \in \mathcal{X}$ be the process which behaves as follows: when a bell rings at $j$, if there is a $\eta$ particle at $j$, then $\gamma(j)$ flips to $1 - \gamma(j)$. The process $(\eta_t, \gamma_t)$ is Markovian and has $\nu_\rho \times \nu^{1/2}$.
as invariant measure. Again, the time-reversed process defined by \( (\eta^*_t, \gamma^*_t) = (\eta_{-t}, \gamma_{-t}) \) has the same law as the space-reflection of \( (\eta_t, \gamma_t) \): the jumps of \( \eta^*_t \) go to the left and the law of the spin-flip \( \gamma^*_t \) remains the same. On the other hand, given a trajectory of \( ((\eta_t, \gamma_t), t \geq 0) \) one can identify \( \omega \) as the space-time points \((j, t)\) such that either an \( \eta \) particle arrives at site \( j \) at time \( t \) or \( \gamma \) flips at \( j \) at time \( t \). The points governing the reverse process are the time reflection of \( \Delta_\rho(\omega) \). Since the reverse process has the same law as the space-reflected HAD+spin-flip process, the points governing it must be Poisson.

For any \( t \), the dual points \( \{(j, s) \in \Delta_\rho(\omega) : s < t\} \) are the points governing the evolution of the reverse process on the time interval \((-t, \infty)\) starting at the configuration \((\eta^*_{-t}, \gamma^*_{-t})\), and are independent of this configuration. But this is just the configuration \((\eta_t, \gamma_t)\) so the independence holds as desired.

3.3. Multi-line HAD process. We now define a multi-line process \( \alpha_t = (\alpha^1_t, \ldots, \alpha^n_t) \) taking values in \( X^n \). It is again governed by a Poisson process \( \omega \) on \( \mathbb{R} \times \mathbb{Z} \).

Let \( \rho^1, \ldots, \rho^n \in (0, 1) \). Let \( \omega^n = \omega \), and, recursively for \( k = n-1, \ldots, 1 \), let \( \omega^k = \Delta_{\rho^{k+1}}(\omega^{k+1}) \). From Proposition \( \Box \) each \( \omega^k \) is a Poisson process of rate \( 1 \) on \( \mathbb{R} \times \mathbb{Z} \).

Now let the “\( k \)th line” of the process, \( (\alpha^k_t, t \in \mathbb{R}) \), be \( H_{\rho^k}(\omega^k) \), the HAD trajectory with density \( \rho^k \) governed by the points \( \omega^k \), as provided by Proposition \( \Box \). Thus each line of the process is a HAD trajectory governed by the dual points produced from the line below.

Note also that, directly from the definition, \((\alpha^1_t, \ldots, \alpha^{n-1}_t)\) is a multi-line process with densities \( \rho^1, \ldots, \rho^{n-1} \) and governed by \( \omega^{n-1} \).

Proposition 6. The multi-line HAD process \((\alpha_t, t \in \mathbb{R})\) is stationary, and the distribution of \( \alpha_t \) for each \( t \) is the product measure \( \nu = \nu^{\rho^1} \times \cdots \times \nu^{\rho^n} \).

Proof. By construction, the process is stationary and the marginal distribution of \( \alpha^k_t \) is \( \nu^{\rho^k} \) for any \( k \) and \( t \). So we need to show that, for any fixed \( t \), the configurations \( \alpha^1_t, \alpha^2_t, \ldots, \alpha^n_t \) are independent.

Let \( 2 \leq k \leq n \). By Proposition \( \Box \) the configuration \( \alpha^k_t \) is independent of the set of dual points \((x, s) \in \Delta_{\rho^k}(\omega^k)\) such that \( s < t \).

But the process \((\alpha^k_s, s \leq t)\) can be constructed as a function of precisely this set of dual points; and then, recursively, also the processes \((\alpha^j_s, s \leq t)\) for each \( 1 \leq j \leq k \).

In particular, we can construct \( \alpha^k_{t-1}, \alpha^{k-2}_t, \ldots, \alpha^1_t \) from the given set of dual points.

Thus for all \( i \), the configuration \( \alpha^k_t \) is independent of \( \alpha^k_{t-1}, \alpha^{k-2}_t, \ldots, \alpha^1_t \). Hence all the \( \alpha^k_t \) are independent as desired.

Remark: The dynamics of the multi-line process governed by \( \omega \) can be explained in a more constructive (or “local”) way. Each bell \((j, s) \in \omega = \omega^n \) causes an \( \alpha^n \) particle to jump to \( j \), from \( j^n \) say. This creates a bell \((j^n, s) \in \omega^{n-1} \), which summons an \( \alpha^{n-1} \) particle to \( j^n \) from \( j^{n-1} \), causing a bell \((j^{n-1}, s) \in \omega^{n-2} \) and so on.
The time-reversal of this process, with respect to the equilibrium measure $\nu$, can be described in the same way but with left and right exchanged and also top and bottom exchanged. When a Poisson bell rings at site $i$, the closest $\alpha^1$ particle to the right of $i$ (including $i$) located at a site called $i^1$ jumps to $i$. Then a bell rings at site $i^1$ for $\alpha^2$, and so on. See Figure 8. An alternative proof to Proposition 6 is to show directly that the process so defined is the reverse process with respect to the product measure. We followed such a strategy for the case of the TASEP in [12].

Note also that in the definition of the multi-line process, and in Proposition 6, we don’t require the densities $\rho^k$ to be increasing.

Now we wish to show that the image of the multi-line process under the map $T$ is the coupled process.

**Proposition 7.** Let $0 < \rho^1 < \cdots < \rho^n < 1$, and let $(\alpha_t, t \in \mathbb{R})$ be the multiline HAD trajectory governed by $\omega$ with densities $\rho^1, \ldots, \rho^n$. Let $\eta_t = T\alpha_t \in X^{n\uparrow}$. Then $(\eta^k_t, t \in \mathbb{R})$ is the HAD trajectory governed by $\omega$, with particle density $\rho^k$.

**Sketch of Proof of Proposition 7.** From the definition of $T$, we have

$$\eta^k_t = D^{(n-k+1)}(\alpha^k_t, \ldots, \alpha^n_t).$$

From Proposition 2 we know that $\eta^k_t$ has distribution $\nu^\rho^k$. So we simply need to show that the RHS of (16) is a HAD trajectory governed by $\omega$.

Since $(\alpha^k, \ldots, \alpha^n)$ is itself just a multi-line process (with $n-k+1$ lines) governed by $\omega$, it is enough to show that, for any $n$, $D^{(n)}(\alpha^1_t, \ldots, \alpha^n_t)$ is a HAD trajectory governed by $\omega$.

We argue by induction. From the definitions of $D^{(n)}$ and of the multi-line process, the induction step is simple, using

$$D^{(n)}(\alpha^1_t, \ldots, \alpha^n_t) = D^{(2)}(D^{(n-1)}(\alpha^1_t, \ldots, \alpha^{n-1}_t), \alpha^n_t)$$

and the fact that $(\alpha^1_t, \ldots, \alpha^{n-1}_t)$ is an $(n-1)$-line multiline process governed by $\omega^{n-1}$, as observed just before Proposition 6.

**Figure 8.** Local construction of the multi-line HAD process and its time-reversal.
The base case $n = 2$ remains. We use the local description of the multi-line (in fact, two-line) process. Let $(\alpha_1, \alpha_2)$ be the two-line process governed by $\omega$. Each mark $(x, s)$ in $\omega = \omega^2$ produces a jump in the process $\alpha^2$ at time $s$, and a corresponding dual point $(x', s)$ which becomes a mark in $\omega_1$. This mark in $\omega_1$ produces a jump in the process $\alpha_1$ at time $s$. One needs to verify that the combination of the two jumps, in $\alpha_1$ and $\alpha_2$, leads to a single HAD jump in the process $D(\alpha_1, \alpha_2)$, equivalent to a mark at $x$. In the language used at (10), we need to show $D(A_j'\alpha_1, A_j\alpha_2) = A_jD(\alpha_1, \alpha_2)$. This is not difficult to do by checking a small number of cases. Arguing jump by jump in this way, one obtains that $D(\alpha_1, \alpha_2)$ is a HAD process governed by $\omega$ as required. We give a proof along these lines for the continuous-space case in [11], and the same argument works here.

Figure 9. Jumps in coupled and multiclass LREP due to bell at $j$. LREP particles are represented by squares and empty sites by balls.

**The long range exclusion process.** The long range exclusion process (LREP) was introduced by Spitzer [30]. At rate one, a particle located at site $x$ jumps to the first empty site found by a Markov chain with transition jumps $p(\ldots)$ starting at $x$. Consider empty sites of the HAD process as particles and particles as empty sites. The resulting process $\tilde{\eta}_t$ given by $\tilde{\eta}_t(x) = 1 - \eta_t(x)$ is the LREP with transition matrix $p(x, x - 1) = 1$ (Guiol [16]). The effect of a bell at $j$ is represented by the map $\tilde{A}_j\tilde{\eta}(i) := 1 - A_j\eta(i)$. In this simple case, the site is just the first empty place to the left. The multiclass LREP $(\tilde{\xi}_t)$ has the same distribution as the multiclass HAD process with classes reversed. The coupled LREP is defined by $\tilde{\eta}_t^k(x) = 1 - \eta_t^{n+1-k}(x)$, where $\eta_t = (\eta_1^t, \ldots, \eta_n^t)$ is the coupled HAD. The effect of the bell at $j$ in the coupled LREP is given by the map $\tilde{C}_j\tilde{\eta} = (\tilde{A}_j\tilde{\eta}^1, \ldots, \tilde{A}_j\tilde{\eta}^n)$. The multiclass LREP is given by $R\tilde{\eta} = \tilde{\xi}$. In the multiclass LREP when a bell rings at $j$, the particle at $j$ jumps to the closest particle to the left of it with higher class or empty; simultaneously this particle jumps to the closest site to its left with higher class or empty, and so on. The jumps finish when a particle jumps to an empty site. See Figure 9.
4. TASEP

The totally asymmetric simple exclusion process, or TASEP, is a continuous-time Markov process in $\mathcal{X}$ with the following dynamics. At rate 1 if there is a particle at site $i$, it jumps one unit to the left (if the site to the left is empty). We use here the same notations as in the case of the HAD process to describe the analogous quantities for the TASEP. The generator of the process is given by

$$Lf(\eta) = \sum_j [f(A_j\eta) - f(\eta)]$$

(17)

where here (differently from the HAD definition (10))

$$(A_j\eta)(k) = \begin{cases} 
\eta(k) & \text{if } k \notin \{j - 1, j\} \\
\max\{\eta(j - 1), \eta(j)\} & \text{if } k = j - 1 \\
\min\{\eta(j - 1), \eta(j)\} & \text{if } k = j.
\end{cases}$$

(18)

For the graphical construction of the process we will use a system of Poisson points or marks, this time on $\omega$ on $\mathbb{R} \times (\mathbb{Z} + \frac{1}{2})$ (so that the bells now ring between sites). When a bell rings at site $x$, and there is a particle at $x + \frac{1}{2}$ and a hole at $x - \frac{1}{2}$, the contents at sites $x + \frac{1}{2}$ and $x - \frac{1}{2}$ are “interchanged”. The construction induces again a function $\phi$ of

$$\eta_t = \phi(t, \omega, \eta)$$

(19)

Figure 10. Jump in TASEP due to bell at $i$

Figure 11. Graphical construction of TASEP. The * represent the events of the Poisson process $\omega$. 
\( \omega \) and the initial configuration \( \eta_0 \):
\[
(t, \omega, \eta) \mapsto \phi(t, \omega, \eta_0) \tag{19}
\]
and \( \eta_t = \phi(t, \cdot, \eta_0) \) is the TASEP with initial configuration \( \eta_0 \). We say that the points \( \omega \) govern the process \( \eta_t \). As at (12) one has
\[
\eta_t = \phi(t - s, \omega, \eta_s) \tag{20}
\]
for all \( 0 \leq s < t < \infty \).

The Bernoulli product measures \( \nu^\rho \) (and mixtures of them) are again invariant for the TASEP. (In addition, certain blocking measures are also invariant; these measures are concentrated on a single configuration with only particles to the left of some site and only holes to its right).

If a trajectory \((\eta_t, t \in \mathbb{R})\) satisfies (12) for all \(-\infty < s < t < \infty\) then we again say that it is governed by \( \omega \). The following result is analogous to Proposition 8:

**Proposition 8.** Let \( \rho \in (0, 1) \). Then there exists an essentially unique function \( H_\rho \) mapping elements \( \omega \) of \( \Omega \) to trajectories \((\eta_t, t \in \mathbb{R})\) such that:

1. The induced law of \((\eta_t, t \in \mathbb{R}) = H_\rho(\omega) \) is stationary in time.
2. The marginal law of \( \eta_t \) for each \( t \) is space-ergodic with particle density \( \rho \).
3. With probability 1, \((\eta_t, t \in \mathbb{R}) \) is a TASEP evolution governed by \( \omega \).

Then in fact the marginal law of \( \eta_t \) for each \( t \) is \( \nu^\rho \).

**Figure 12. Coupling in TASEP**

4.1. **Coupled and multiclass TASEP.** The basic coupling between \( n \) TASEPs with initial configurations \( \eta = \eta_0^1, \ldots, \eta_0^n \) is given by \( \eta_t = (\eta_t^1, \ldots, \eta_t^n) = \phi^{(n)}(t, \cdot, \eta_0) \), where \( \phi^{(n)}(t, \omega, \eta_0)^k = \phi(t, \omega, \eta_0^k) \). See figure 12 where the effects of three possible Poisson bells are indicated. If \( \eta_0^1 \leq \cdots \leq \eta_0^n \), then this ordering is preserved by the coupling. Thus we obtain a coupled process \((\eta_t)\), governed by \( \omega \), taking values in \( X^n \).

The equivalent multiclass process \((\xi_t)\) with values in \( Y_n \) is then obtained by putting \( \xi_t = R\eta_t \). The evolution of this multiclass TASEP is more intuitive than in the case of the HAD process. At the ring of the bell at \( x \), particles of lower class at site \( x + \frac{1}{2} \) jump over particles of higher class at \( x - \frac{1}{2} \), exchanging places. See four examples of jumps in Figure 13.
Theorem 9. Under the conditions of Theorem 4, the distribution of $\eta = T\alpha$ is invariant for the coupled TASEP $(\eta_t)$, and $\mu$, the law of $M\alpha$, is invariant for the multiclass TASEP $(\xi_t)$.

The strategy to prove this theorem is the same as for Theorem 4. The differences come in the definitions of dual points and of the multi-line process. The key steps are given by Proposition 11 and Proposition 12 which play the roles played by Propositions 6 and 7 in the case of the HAD process.

4.2. Dual points in TASEP. We now define the dual points for the case of TASEP. Let the density $\rho$ be fixed and let $\omega$ be a Poisson process on $\mathbb{R} \times (\mathbb{Z} + \frac{1}{2})$. Let $(\eta_t, t \in \mathbb{R})$ be the TASEP trajectory $H_\rho(\omega)$ governed by $\omega$, as provided by Proposition 8. Now define the dual points $\Delta_\rho(\omega)$ by

$$\Delta_\rho(\omega) = \{(x, t) \in \omega : \eta_t(x + \frac{1}{2}) = 1\} \cup \{(x + 1, t) : (x, t) \in \omega \text{ and } \eta_t(x + \frac{1}{2}) = 0\}. \quad (21)$$

See Figure 14.

Proposition 10. Let $\omega$ be a Poisson process in $\mathbb{R} \times \mathbb{Z}$, and let $\Delta_\rho(\omega)$ be the dual points for the TASEP trajectory with particle density $\rho$ governed by $\omega$. Then $\Delta_\rho(\omega)$ is also a Poisson process in $\mathbb{R} \times \mathbb{Z}$. Furthermore $\{(x, s) \in \Delta_\rho(\omega) : s < t\}$, the set of dual points earlier than $t$, is independent of the configuration $\eta_t$.

Proof. This is a kind of Burke’s theorem like Proposition 5. Again the TASEP trajectory does not determine the points of $\omega$, so we introduce spin-flip processes to mark the missed points. There are two types of points missed by the trajectory: those space-time points $(x, t)$ such that $\eta_t(x + \frac{1}{2}) = 0$ and those space-time points $(x, t)$ such that $\eta_t(x + \frac{1}{2}) = 1$. We mark these two types with different spin-flip processes. Let $\gamma_t$ be a process taking values in $\mathcal{X} = \{0, 1\}^\mathbb{Z}$ which behaves as follows: when there is an $\omega$ point at $(x, t)$, if there is no $\eta_t$ particle at $x + \frac{1}{2}$, then the value of $\gamma$ at $(x + \frac{1}{2}, t)$ flips so that...
Figure 14. The circles represent the dual points of the TASEP trajectory of Figure 11. As in Figure 7, turning the picture upside-down exchanges the roles of circles and stars.

\[ \gamma_t(x + \frac{1}{2}) = 1 - \gamma_{t-}(x + \frac{1}{2}) \]

Let \( \zeta_t \) be a spin-flip process taking values in \( \{0, 1\}^{Z+\frac{1}{2}} \) with the following behavior: when there is an \( \omega \) point at \((x, t)\), if both \( x - \frac{1}{2} \) and \( x + \frac{1}{2} \) are occupied by \( \eta_{t-} \) particles, then the value of \( \zeta \) at \((x, t)\) flips so that \( \zeta_t(x) = 1 - \zeta_{t-}(x) \). Given the evolution \((\eta_t, \gamma_t, \zeta_t), t \in \mathbb{R}\), the points \( \omega \) can be recovered by

\[
\omega = \{(x, t) : (\eta_{t-}(x + \frac{1}{2}), \gamma_{t-}(x + \frac{1}{2}), \zeta_{t-}(x)) \neq (\eta_t(x + \frac{1}{2})\eta_{t-}(x + \frac{1}{2}), \gamma_t(x + \frac{1}{2}), \zeta_t(x))\}
\]

We now proceed as in the proof of Proposition 5. The process \((\eta_t, \gamma_t, \zeta_t), t \in \mathbb{R}\) is stationary with time-marginal product measure \( \nu^\rho \times \nu^{1/2} \times \nu^{1/2} \). The reverse process with respect to this measure is defined by \((\eta^*_t, \gamma^*_t, \zeta^*_t) = (\eta_{t-}, \gamma_{t-}, \zeta_{t-})\). Then the law of this time-reversal is the same as the law of the space-reversal. (In particular, the first coordinate of the reverse process performs a TASEP with jumps to the right, while the other two coordinates have the same spin flip distribution as the forward process.) The points governing the reverse process are the time-reflection of \( \Delta^\rho(\omega) \), which is therefore also a Poisson process on \( \mathbb{R} \times (\mathbb{Z} + \frac{1}{2}) \). Independence is shown as in Proposition 5.

4.3. Multi-line TASEP. We now define a multi-line TASEP \( \alpha_t = (\alpha_1^t, \ldots, \alpha_n^t) \) taking values in \( \mathcal{X}^n \) and governed by Poisson points \( \omega \) on \( \mathbb{R} \times (\mathbb{Z} + \frac{1}{2}) \). The definition is analogous to that of the multi-line HAD in Section 3.3.

Let \( \rho^1, \ldots, \rho^n \in (0,1) \). Let \( \omega^n = \omega \), and, recursively for \( k = n - 1, \ldots, 1 \), let \( \omega^k = \Delta_{\rho^{k+1}}(\omega^{k+1}) \). Now let the \( k \)th line of the process, \((\alpha_k^t, t \in \mathbb{R})\), be \( H_{\rho^k}(\omega^k) \), the TASEP trajectory with density \( \rho^k \) governed by the points \( \omega^k \), as provided by Proposition 8.

Using Proposition 10 an argument analogous to the proof of Proposition 6 now gives the following result:

Proposition 11. The multi-line TASEP \((\alpha_t, t \in \mathbb{R})\) is stationary, and the distribution of \( \alpha_t \) for each \( t \) is the product measure \( \nu = \nu^\rho^1, \ldots, \nu^\rho^n \).
The proof of Theorem 9 is completed by the following result, analogous to Proposition 7.

**Proposition 12.** Let $0 < \rho^1 < \cdots < \rho^n < 1$, and let $(\alpha_t, t \in \mathbb{R})$ be the multiline TASEP trajectory governed by $\omega$ with densities $\rho^1, \ldots, \rho^n$. Let $\eta_t = T \alpha_t \in X^{n+}$. Then $(\eta^k_t, t \in \mathbb{R})$ is the TASEP trajectory governed by $\omega$, with particle density $\rho^k$.

**About the proof of Proposition 12.** The induction argument is the same as for Proposition 7. The case-by-case checking for $n = 2$ must now be done for the TASEP dynamics. We have done this in [12].

5. OTHER DYNAMICS

There are other examples of dynamics on $X$ for which the associated multiclass processes also have the family of measures $\mu$ as invariant distributions.

For example, consider the sequential TASEP. This is a discrete-time Markov chain with values in $\mathbb{Z}$. At each time step, each particle tries to jump left with probability $p$, succeeding if the site to its left is empty. Updates are carried out sequentially from left to right (so for example a particle may jump into a space which is only vacated at the same time-step). Now the governing points $\omega$ have Bernoulli product measure on $\mathbb{Z} \times \mathbb{Z}$ (and the same is true of the dual points, appropriately defined). An analogous method of proof via a multi-line process shows that $\mu$ is invariant for the multiclass process.

The same is true for a form of sequential TASEP with updates from right to left. Now a particle may not jump immediately into a vacated space, but the same particle may jump several times at the same time-step (because of several neighbouring points $(x, t)$, $(x-1, t), \ldots$ in the governing configuration $\omega$). In fact this process is dual to the one in the previous paragraph, under exchange of hole and particle and of left and right. The measure $\mu$ is invariant for the multiclass version, and similarly in the case of various discrete-time versions of the HAD process.

However, consider instead the parallel TASEP, again in discrete time. All sites are updated simultaneously; now jumps are only allowed at sites containing a particle with a hole to its left before any other update occurred. In particular, jumps at two neighbouring sites cannot occur at the same time-step. In this case, the basic coupling does not even preserve ordering of configurations, so that the multiclass process cannot be defined in the same way. Note also that product measure $\nu^\rho$ is no longer invariant for the parallel TASEP.

Consider also the asymmetric simple exclusion process (ASEP) in continuous time, in which each particle tries to jump left at rate $p$ and right at rate $1 - p$. Product measure $\nu^\rho$ is invariant for the process; however, unless $p = 1$, the measure $\mu$ is no longer invariant for the multiclass process. A very interesting question is whether the invariant multiclass measures of these more general ASEPs could be constructed using an approach related to the one described here for the TASEP.
6. Multiclass Burke’s Theorem

For each $n$ and densities $0 < \rho^1 < \cdots < \rho^n < 1$, we have constructed a measure $\mu = \mu_{\rho^1, \ldots, \rho^n}(n)$ on $\mathcal{Y}_n = \{1, 2, \ldots, n+1\}^2$ which is invariant for the multiclass HAD process and the multiclass TASEP. A configuration from $\mu_{\rho^1, \ldots, \rho^n}(n)$ has density $\rho^1$ of first-class particles and density $\rho^k - \rho^{k-1}$ of $k$th class particles, for $k = 2, \ldots, n$.

Fix $n$ and $\rho^1, \ldots, \rho^n$, and let $m < n$. Let $\xi^{(n)}$ be distributed according to $\mu_{\rho^1, \ldots, \rho^n}(n)$, and $\xi^{(m)}$ according to $\mu_{\rho^1, \ldots, \rho^m}(m)$. Then a nice property of this family of distributions is that $\xi^{(m)}$ has the same distribution as $[\xi^{(n)}]^m$,

$$\xi^{(m)} \text{ has the same distribution as } [\xi^{(n)}]^m,$$

where $[\xi^{(n)}]^m$ is the truncated configuration defined by

$$[\xi^{(n)}]^m(i) = \min \{\xi^{(n)}(i), m+1\}.$$

Putting $n = m+1$, we obtain the following statement in the context of the tandem queueing system described in Section 2: the $m$-class input process to queue $m$ has the same law as the $m$-class departure process from the same queue. Thus the measure $\mu_{\rho^1, \ldots, \rho^m}(m)$ is a fixed point for a discrete-time $./M/1$ priority queue with $m$ classes (here $./M/1$ denotes a queue whose sequence of potential service times is a Bernoulli process).

This may be called a multiclass Burke’s theorem. The original form of Burke’s theorem, in this discrete-time setting, states that a Bernoulli process is a fixed point for a (one-class) $./M/1$ queue; this is the statement (23) specialized to the case $n = 2$ and $m = 1$ (see e.g. [26], and [18] for the discrete-time case). Property (23) is easy to deduce from results above, using the uniqueness of the invariant measure for the coupled process with given particle densities (see [11] for the equivalent argument in continuous time). A more direct proof can be found in [22], using properties of invariance of the law of the departure process from a tandem queueing system under interchange of the order of the queues.

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