Existence of solutions for a class of IBVP for nonlinear hyperbolic equations

Svetlin Georgiev Georgiev1,2 · Mohamed Majdoub1,2

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Abstract
We study a class of initial boundary value problems of hyperbolic type. A new topological approach is applied to prove the existence of non-negative classical solutions. The arguments are based upon a recent theoretical result.

Keywords Hyperbolic equations · Positive solution · Fixed point · Cone · Sum of operators

Mathematics Subject Classification 47H10 · 58J20 · 35L15

1 Introduction
This paper concerns global existence of classical solutions of one-dimensional nonlinear wave equations with initial and mixed boundary conditions. More precisely, we investigate the following IBVP

\[
\begin{aligned}
&u_{tt} - u_{xx} = f(t, x, u), \quad t \geq 0, \quad x \in [0, L], \\
&u(0, x) = u_0(x), \quad x \in [0, L], \\
&u_t(0, x) = u_1(x), \quad x \in [0, L], \\
&u(t, 0) = u_x(t, L) = 0, \quad t \geq 0,
\end{aligned}
\]

(1.1)

where \( L > 0, f : [0, \infty) \times [0, L] \times \mathbb{R} \to \mathbb{R} \) is continuous and \( u_0 \in C^2([0, L]), u_1 \in C^1([0, L]) \) are the initial data.

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Svetlin Georgiev Georgiev
svetlingeorgiev1@gmail.com

Mohamed Majdoub
mmajdoub@iau.edu.sa

1 Department of Mathematics, College of Science, Imam Abdulrahman Bin Faisal University, P. O. Box 1982, Dammam, Saudi Arabia

2 Basic and Applied Scientific Research Center, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, 31441 Dammam, Saudi Arabia
Mixed boundary value problems arise in several areas of applied mathematics and physics, such as gas dynamics, nuclear physics, chemical reaction, studies of atomic structures, and atomic calculation. Therefore, mixed problems have attracted much interest and have been studied by many authors. See [1, 5, 6] and references cited therein. In our case, the Eq. (1.1) describes the interaction of solitary waves in elastic rods, the dynamics of one-dimensional internal gravity waves in an incompressible stratified fluid.

In [13] the author investigate the following mixed problem

\[
\begin{align*}
\begin{cases}
u_t - \nu_{xx} &= f(u), & x > 0, \quad t > 0, \\
u_x + \nu_{tt} &= 0, & x = 0, \quad t > 0, \\
u &= \psi_0, \quad u_t = \psi_1, & x > 0, \quad t = 0,
\end{cases}
\end{align*}
\]

where \(|\gamma| \geq 1\), under the compatibility conditions

\[
\begin{align*}
\psi'_0(0) + \gamma \psi_1(0) &= 0, \\
\psi'_1(0) + \gamma(\psi'_0(0) + F(\psi_0(0))) &= 0, \\
\psi''_0(0) + \gamma \psi''_1(0) &= 0.
\end{align*}
\]

If \(f \in C^1(\mathbb{R})\), \(\gamma \neq 1\), \(\psi_j \in C^{2,j}([0, \infty))\), \(j = 0, 1\), it is proved that there exists an open neighborhood \(U\) of \(\{0\} \times [0, \infty)\) such that (1.2) has exactly one solution \(u \in C^3(U)\). Moreover, if \(f \in C^2(\mathbb{R})\), \(\psi_j \in C^{3,j}([0, \infty))\) for \(j = 0, 1\), then there exists an open neighborhood \(U\) of \(\{0\} \times [0, \infty)\) such that (1.2) has exactly one solution \(u \in C^3(U)\). The method used in [13] is mainly based on conservation laws.

The following mixed problem is investigated in [15]

\[
\begin{align*}
\begin{cases}
u_t - \nu_{xx} + g(u) &= f(x, t), & (x, t) \in (0, L) \times (0, T), \\
u(x, 0) &= \phi(x), \quad u_t(x, 0) = \psi(x), & x \in [0, L], \\
u_x(0, t) &= F(u(0, t)) + \alpha(t), \quad u_x(L, t) = \beta(t)u(L, t) + \gamma(t), & t \in [0, T],
\end{cases}
\end{align*}
\]

where \(g, f, \phi, \psi, \alpha, \beta, \gamma\) and \(F\) are given functions. For \(f \in C([0, L] \times [0, T])\), \(g \in C(\mathbb{R})\), \(F \in C^1(\mathbb{R})\), \(\phi \in C^2([0, L])\), \(\psi \in C^1([0, L])\), \(\alpha, \beta, \gamma \in C^1([0, T])\), necessary conditions for solvability of the problem (1.3) in the class \(C^2([0, L] \times [0, T])\) are given in [15]. To prove the main results in [15], the authors reduce (1.3) to an equivalent system of Volterra type in the class of continuous functions.

Note also that (1.3) was studied in [4, 14, 16] in the energy space using Fourier method.

In the present paper we propose a new approach based on the fixed point theory on cones for the sum of two operators. Before stating our main result we precise the assumptions made on the nonlinearity and the initial data. We suppose that \(f\) is continuous and satisfies

\[
0 \leq f(t, x, u) \leq \sum_{j=1}^{l} c_j(t, x) |u|^{p_j}, \quad (t, x, u) \in [0, \infty) \times [0, L] \times \mathbb{R},
\]

where \(p_j > 0\), \(c_j \in C([0, \infty) \times [0, L])\), \(j \in \{1, \ldots, l\}\), \(l \in \mathbb{N}\). For the initial data \(u_0, u_1\) we make the following assumption.
\[
\begin{align*}
& u_0 \in C^2([0,L]), \; u_1 \in C^1([0,L]), \quad u_0(0) = u_0(0) = 0, \; u_0(0) = u_1(0) = 0, \\
& 0 \leq u_0, u_1 < r \quad \text{on} \quad [0,L], \quad u_0 > 0 \quad \text{on} \quad \left[\frac{L}{3}, \frac{2}{3}\right],
\end{align*}
\]  

(1.5)

where \( r \in (0,1) \).

Our main result reads as follows.

**Theorem 1.1** Suppose that assumptions (1.4)–(1.5) are fulfilled. Then the IBVP (1.1) has at least one non-negative solution \( u \in C^2([0, \infty) \times [0,L]) \).

The set up of the paper is as follows. In the next Section we give some useful tools and preliminary results. In Sect. 3, we prove our main result. In the last section, Sect. 4, we give an example.

## 2 Background and preliminary results

Let \( X \) be a real Banach space.

**Definition 2.1** A mapping \( K : X \to X \) is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

The concept for \( k \)-set contraction is related to that of the Kuratowski measure of non-compactness which we recall for completeness.

**Definition 2.2** Let \( \Omega_X \) be the class of all bounded sets of \( X \). The Kuratowski measure of noncompactness \( \alpha : \Omega_X \to [0, \infty) \) is defined by

\[
\alpha(Y) = \inf \left\{ \delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad \text{and} \quad \text{diam}(Y_j) \leq \delta, \; j \in \{1,\ldots,m\} \right\},
\]

where \( \text{diam}(Y_j) = \sup \{ \|x - y\|_X : x, y \in Y_j \} \) is the diameter of \( Y_j, j \in \{1,\ldots,m\} \).

For the main properties of measure of noncompactness we refer the reader to [7].

**Definition 2.3** A mapping \( K : X \to X \) is said to be \( k \)-set contraction if there exists a constant \( k \geq 0 \) such that

\[
\alpha(K(Y)) \leq k \alpha(Y)
\]

for any bounded set \( Y \subset X \).

Obviously, if \( K : X \to X \) is a completely continuous mapping, then \( K \) is 0-set contraction (see [10]).

**Definition 2.4** Let \( X \) and \( Y \) be real Banach spaces. A mapping \( K : X \to Y \) is said to be expansive if there exists a constant \( h > 1 \) such that

\[
\|Kx - Ky\|_Y \geq h\|x - y\|_X
\]
for any \( x, y \in X \).

**Definition 2.5** A closed, convex set \( P \) in \( X \) is said to be cone if

1. \( ax \in P \) for any \( a \geq 0 \) and for any \( x \in P \),
2. \( x, -x \in P \) implies \( x = 0 \).

The following Proposition will be used to prove our main result. We refer the reader to [9] for more detail.

**Proposition 2.1** Suppose that \( P \) is a cone in \( X \). Let \( \Omega \) be a subset of \( P \) and \( U \) be a bounded open subset of \( P \) with \( 0 \in U \). Assume that the mapping \( T : \Omega \subset P \to X \) be such that \((I - T)\) is Lipschitz invertible with constant \( \gamma > 0 \), \( F : U \to X \) is a \( k \)-set contraction with \( 0 \leq k < \gamma^{-1} \), and \( F(U) \subset (I - T)(\Omega) \). If

\[
Fx \neq (I - T)(\lambda x) \quad \text{for all} \quad x \in \partial U \cap \Omega, \quad \lambda \geq 1 \quad \text{and} \quad \lambda x \in \Omega,
\]

then the mapping \( T + F \) has at least one fixed point in \( U \cap \Omega \).

In order to apply Proposition 2.1 and prove our main result, we will make the following assumptions.

There exist positive constants \( \epsilon, A, R \) and \( b_1 \) such that

\[
\begin{aligned}
&\epsilon, A \in (0, 1), \quad 4A < \epsilon, \quad R \geq r, \quad b_1 > 1, \\
&\epsilon r + 4\left(r + \sum_{j=1}^{l} r^{(j)}\right)A \leq (\epsilon - 4A)R, \\
&4\left(r + 2R + \sum_{j=1}^{l} r^{(j)}\right)A < \frac{1}{b_1}.
\end{aligned}
\tag{2.1}
\]

Denote

\[ B_1 = \max\{1, 2L, 2L^2, 2L^3, 2L^4\}. \]

There exist a non-negative function \( g \in C([0, \infty) \times [0, L]) \) and \( m \in (0, 1) \) such that

\[
\begin{aligned}
&B_1(1 + t + t^2 + t^3 + t^4) \int_0^1 \int_0^L g(t_1, x_1)dx_1dt_1 \leq A, \\
&B_1(1 + t + t^2 + t^3 + t^4) \int_0^1 \int_0^L g(t_1, x_1) \int_0^t \int_0^L c_j(t_2, x_2)dt_2dx_2dx_1dt_1 \leq A, \quad j \in \{1, \ldots, l\}, \\
&\frac{1 - m}{4} \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (2 - t_1)^2(L - x_1)^2g(t_1, x_1) \int_{\frac{1}{2}}^1 x_2u_0(x_2)dx_2dx_1dt_1 \geq \frac{A}{b_1}.
\end{aligned}
\tag{2.2}
\]

In the last section we will give an example for constants \( \epsilon, A, r, R, m, b_1 \) and for a function \( g \) that satisfy (2.1) and (2.2).

Let \( E = C^2([0, \infty) \times [0, L]) \) be endowed with the norm

\[
\|u\| = \|u\|_{\infty} + \left\| \frac{\partial u}{\partial t} \right\|_{\infty} + \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{\infty} + \left\| \frac{\partial u}{\partial x} \right\|_{\infty} + \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{\infty}.
\]
provided it exists, where \( \|v\|_{\infty} = \sup_{(t,x) \in [0,\infty) \times [0,L]} |v(t,x)|. \)

**Lemma 2.1**  Let \( u \in E \) be a solution to the integral equation

\[
0 = -\frac{1}{4} \int_0^t (t-t_1)^2 \int_0^t (x-x_1)^2 g(t_1, x_1) \int_0^{t_1} (t_1 - t_2) u(t_2, x_1) dt_2 dx_1 dt_1 \\
+ \frac{1}{4} \int_0^t (t-t_1)^2 \int_0^t (x-x_1)^2 g(t_1, x_1) \int_0^{t_1} x_2 \left( -u(t_1, x_2) + u_0(x_2) + tu_1(x_2) \right) dx_2 dx_1 dt_1 \\
+ \int_0^t (t-t_1) f(t_1, x_2, u(t_2, x_2)) dt_2 \right) dx_2 dx_1 dt_1 \\
+ \frac{1}{4} \int_0^t (t-t_1) \int_0^t x_1 (x-x_1)^2 g(t_1, x_1) \int_0^{t_1} x_1 \left( -u(t_1, x_2) + u_0(x_2) + tu_1(x_2) \right) dx_2 dx_1 dt_1 \\
+ \int_0^t (t-t_1) f(t_1, x_2, u(t_2, x_1)) dt_2 \right) dx_2 dx_1 dt_1, \quad (t, x) \in [0, \infty) \times [0, L].
\]

Then \( u \) solves the IBVP (1.1).

**Proof**  We differentiate trice in \( t \) and then trice in \( x \) the Eq. (2.3) and we get

\[
0 = -g(t, x) \int_0^t (t-t_1) u(t_1, x) dt_1 \\
+ g(t, x) \int_0^t x_1 \left( -u(t, x_1) + u_0(x_1) + tu_1(x_1) \right) dx_1 \\
+ \int_0^t (t-t_1) f(t_1, x_1, u(t_1, x_1)) dt_1 \right) dx_1 \\
+ xg(t, x) \int_0^L \left( -u(t, x_1) + u_0(x_1) + tu_1(x_1) \right) dx_1 \\
+ \int_0^t (t-t_1) f(t_1, x_1, u(t_1, x_1)) dt_1 \right) dx_1,
\]

\((t, x) \in [0, \infty) \times [0, L], \) whereupon

\[
0 = -\int_0^t (t-t_1) u(t_1, x) dt_1 \\
+ \int_0^t x_1 \left( -u(t, x_1) + u_0(x_1) + tu_1(x_1) \right) dx_1 \\
+ \int_0^t (t-t_1) f(t_1, x_1, u(t_1, x_1)) dt_1 \right) dx_1 \\
+ x \int_0^L \left( -u(t, x_1) + u_0(x_1) + tu_1(x_1) \right) dx_1 \\
+ \int_0^t (t-t_1) f(t_1, x_1, u(t_1, x_1)) dt_1 \right) dx_1,
\]

\((t, x) \in [0, \infty) \times [0, L]. \) Now we differentiate the last equation with respect to \( t \) and we find
\[ 0 = -\int_0^t u(t_1, x) dt_1 + \int_0^x x_1 \left( -u_t(t, x_1) + u_1(x_1) + \int_0^t f(t_1, x_1, u(t_1, x_1)) dt_1 \right) dx_1 \tag{2.5} \]

\[ + x \int_x^L \left( -u_t(t, x_1) + u_1(x_1) + \int_0^t f(t_1, x_1, u(t_1, x_1)) dt_1 \right) dx_1, \]

\((t, x) \in [0, \infty) \times [0, L],\) which we differentiate in \(t\) and we arrive at

\[ 0 = -u(t, x) + \int_0^x x_1 \left( -u_t(t, x_1) + f(t, x_1, u(t, x_1)) \right) dx_1 + x \int_x^L \left( -u_t(t, x_1) + f(t, x_1, u(t, x_1)) \right) dx_1, \tag{2.6} \]

\((t, x) \in [0, \infty) \times [0, L].\) Now we differentiate with respect to \(x\) the last equation and we find

\[ 0 = -u_x(t, x) + \int_0^x \left( -u_{tx}(t, x) + f(t, x, u(t, x)) \right) dx_1 + x \int_x^L \left( -u_{tx}(t, x) + f(t, x, u(t, x)) \right) dx_1 \tag{2.7} \]

\[(t, x) \in [0, \infty) \times [0, L].\) Now we differentiate the last equation with respect to \(x\) and we find

\[ 0 = -u_{xx}(t, x) + u_{tx}(t, x) - f(t, x, u(t, x)), \quad (t, x) \in [0, \infty) \times [0, L].\]

We put \(t = 0\) in (2.4) and we find

\[ 0 = \int_0^x x_1 (-u(0, x_1) + u_0(x_1)) dx_1 + x \int_x^L (-u(0, x_1) + u_0(x_1)) dx_1, \quad x \in [0, L], \]

which we differentiate in \(x\) and we get

\[ 0 \equiv x(-u(0, x) + u_0(x)) + \int_x^L (-u(0, x_1) + u_0(x_1)) dx_1 - x(-u(0, x) + u_0(x)) \]

\[ = \int_x^L (-u(0, x_1) + u_0(x_1)) dx_1, \quad x \in [0, L], \]

again we differentiate in \(x\) and we find

\[ u(0, x) = u_0(x), \quad x \in [0, L]. \]
Now we put $t = 0$ in (2.5) and we get

$$0 = \int_0^x x_1(-u_t(0, x_1) + u_1(x_1))dx_1 + x \int_x^L (-u_t(0, x_1) + u_1(x_1))dx_1, \quad x \in [0, L],$$

which we differentiate twice in $x$ and we find

$$u_t(0, x) = u_1(x), \quad x \in [0, L].$$

Now we put $x = 0$ in (2.6) and we get

$$u(t, 0) = 0, \quad t \in [0, \infty).$$

We put $x = L$ in (2.7) and we find

$$u_x(t, L) = 0, \quad t \in [0, \infty).$$

This completes the proof.$\Box$

For $u \in E$ and $(t, x) \in [0, \infty) \times [0, L]$, define

$$Gu(t, x) = -\frac{1}{4} \int_0^t (t-t_1)^2 \int_0^x (x-x_1)^2 g(t_1, x_1) \int_0^{t_1} (t_2-t_1)u(t_2, x_1)dt_2dx_1dt_1,$$

$$F_1u(t, x) = \int_0^x x_1(-u(t, x_1) + u_0(x_1) + tu_1(x_1))$$
$$+ \int_0^t (t-t_1)f(t_1, x_1, u(t_1, x_1))dt_1 dx_1,$$

$$F_2u(t, x) = \int_x^L \left(-u(t, x_1) + u_0(x_1) + tu_1(x_1)ight)$$
$$+ \int_0^t (t-t_1)f(t_1, x_1, u(t_1, x_1))dt_1 dx_1,$$

$$F_3u(t, x) = F_1u(t, x) + xF_2u(t, x),$$

$$Fu(t, x) = \frac{1}{4} \int_0^t \int_0^x (t-t_1)^2(x-x_1)^2g(t_1, x_1)F_3u(t_1, x_1)dx_1dt_1.$$
Lemma 2.2 Suppose (1.4) and (1.5) are fulfilled. Then, for \( u \in C([0, \infty) \times [0, L]) \), \( |u| \leq r \) on \([0, \infty) \times [0, L]\), we have

\[
|F_1 u(t, x)| \leq 2L^2 r(1 + t) + L t \sum_{j=1}^l r_j^p \int_0^L \int_0^t c_j(t_1, x_1) dt_1 dx_1,
\]

\[
|F_2 u(t, x)| \leq 2rL(1 + t) + t \sum_{j=1}^l r_j^p \int_0^L \int_0^t c_j(t_1, x_1) dt_1 dx_1,
\]

and

\[
|F_3 u(t, x)| \leq 4L^2 r(1 + t) + 2L t \sum_{j=1}^l r_j^p \int_0^L \int_0^t c_j(t_1, x_1) dt_1 dx_1.
\]

(2.8)

Proof Let \( u \in C([0, \infty) \times [0, L]) \) and \( |u| \leq r \) on \([0, \infty) \times [0, L]\). Then, using (1.4) and (1.5), we get

\[
|F_1 u(t, x)| = \left| \int_0^x x_1 \left( -u(t, x_1) + u_0(x_1) + tu_1(x_1) \right) \right. \\
\left. + \int_0^t (t - t_1) f(t_1, x_1, u(t_1, x_1)) dt_1 \right| dx_1
\]

\[
\leq \int_0^x x_1 \left( |u(t, x_1)| + |u_0(x_1)| + t |u_1(x_1)| \right) dx_1 \\
\left. + \int_0^t (t - t_1) |f(t_1, x_1, u(t_1, x_1))| dt_1 \right| dx_1
\]

\[
\leq L \int_0^L \left( (2 + t)r + t \sum_{j=1}^l \int_0^t c_j(t_1, x_1) |u(t_1, x_1)|^p dt_1 \right) dx_1
\]

\[
\leq L^2 (2 + t)r + L t \sum_{j=1}^l r_j^p \int_0^L \int_0^t c_j(t_1, x_1) dt_1 dx_1
\]

\[
\leq 2L^2 r(1 + t) + L t \sum_{j=1}^l r_j^p \int_0^L \int_0^t c_j(t_1, x_1) dt_1 dx_1
\]

and
Proof Using (2.1) and the first inequality of (2.2), we have the following estimates

\[
|F_2u(t, x)| \leq \int_x^L \left( u(t, x_1) + u_0(x_1) + tu_1(x_1) \right) dx_1 \\
+ \int_0^t (t - t_1) f(t_1, x_1, u(t_1, x_1)) dt_1 \left|\int_0^L (u(t_1, x_1)|t_1|) dt_1 dx_1 \right| \\
\leq \int_x^L \left( |u(t, x_1)| + |u_0(x_1)| + t|u_1(x_1)| \right) dx_1 \\
+ \int_0^t (t - t_1) |f(t_1, x_1, u(t_1, x_1))| dt_1 \left|\int_0^L (u_1(x_1)) dt_1 dx_1 \right| \\
\leq \int_0^L (2 + t)r + t \sum_{j=1}^i \int_0^L c_j(t_1, x_1) |u(t_1, x_1)|^{\rho_j} dt_1 dx_1 \\
\leq \int_0^L 2L(2 + t) + t \sum_{j=1}^i \int_0^L c_j(t_1, x_1) dt_1 dx_1 dx_1,
\]

\((t, x) \in [0, \infty) \times [0, L]\). Hence, we get (2.8). This completes the proof.

Lemma 2.3 Suppose (1.4), (1.5), (2.1) and (2.2) are fulfilled. Then, for \(u \in E\) and \(\|u\| \leq r\), we have

\[
\|Gu\| \leq 4rA, \\
\|Fu\| \leq 4 \left( r + \sum_{j=1}^i \rho_j \right) A.
\]

Proof Using (2.1) and the first inequality of (2.2), we have the following estimates

\[
|Gu(t, x)| = \left| -\frac{1}{4} \int_0^t \int_0^x (t - t_1)^2 (x - x_1)^2 g(t_1, x_1) \int_0^{t_1} (t_1 - t_2) u(t_2, x_1) dt_2 dx_1 dt_1 \right| \\
\leq \frac{1}{4} \int_0^t \int_0^x (t - t_1)^2 (x - x_1)^2 g(t_1, x_1) \int_0^{t_1} (t_1 - t_2) u(t_2, x_1) dt_2 dx_1 dt_1 \\
\leq \frac{r}{4} \int_0^t \int_0^x (t - t_1)^2 (x - x_1)^2 g(t_1, x_1) dr_1 dx_1 dt_1 \\
\leq \frac{r}{4} t^2 L^2 \int_0^t \int_0^L g(t_1, x_1) dx_1 dt_1 \\
\leq \frac{rB_1 t^4}{4} \int_0^t \int_0^L g(t_1, x_1) dx_1 dt_1 \\
\leq rA,
\]

and
\[
\left| \frac{\partial}{\partial t} Gu(t, x) \right| = -\frac{1}{2} \int_0^t \int_0^x (t - t_1)(x - x_1)^2 g(t_1, x_1) \int_0^{t_1} (t_1 - t_2)u(t_2, x_1) \, dt_2 \, dx_1 \, dt_1 \\
\leq \frac{1}{2} \int_0^t \int_0^x (t - t_1)(x - x_1)^2 g(t_1, x_1) \int_0^{t_1} (t_1 - t_2)u(t_2, x_1) \, dt_2 \, dx_1 \, dt_1 \\
\leq \frac{r}{2} \int_0^t \int_0^x r_1^2(t - t_1)(x - x_1)^2 g(t_1, x_1) \, dx_1 \, dt_1 \\
\leq \frac{r}{2} L^2 \int_0^t \int_0^L g(t_1, x_1) \, dx_1 \, dt_1 \\
\leq r B_1 t^2 \int_0^t \int_0^L g(t_1, x_1) \, dx_1 \, dt_1 \\
\leq r A,
\]

and

\[
\left| \frac{\partial^2}{\partial t^2} Gu(t, x) \right| = -\frac{1}{2} \int_0^t \int_0^x (x - x_1)^2 g(t_1, x_1) \int_0^{t_1} (t_1 - t_2)u(t_2, x_1) \, dt_2 \, dx_1 \, dt_1 \\
\leq \frac{1}{2} \int_0^t \int_0^x (x - x_1)^2 g(t_1, x_1) \int_0^{t_1} (t_1 - t_2)u(t_2, x_1) \, dt_2 \, dx_1 \, dt_1 \\
\leq \frac{r}{2} \int_0^t \int_0^x r_1^2(x - x_1)^2 g(t_1, x_1) \, dx_1 \, dt_1 \\
\leq \frac{r}{2} L^2 \int_0^t \int_0^L g(t_1, x_1) \, dx_1 \, dt_1 \\
\leq r B_1 t^2 \int_0^t \int_0^L g(t_1, x_1) \, dx_1 \, dt_1 \\
\leq r A,
\]

and

\[
\left| \frac{\partial}{\partial x} Gu(t, x) \right| = -\frac{1}{2} \int_0^t \int_0^x (t - t_1)^2(x - x_1) g(t_1, x_1) \int_0^{t_1} (t_1 - t_2)u(t_2, x_1) \, dt_2 \, dx_1 \, dt_1 \\
\leq \frac{1}{2} \int_0^t \int_0^x (t - t_1)^2(x - x_1) g(t_1, x_1) \int_0^{t_1} (t_1 - t_2)u(t_2, x_1) \, dt_2 \, dx_1 \, dt_1 \\
\leq \frac{r}{2} \int_0^t \int_0^x r_1^2(t - t_1)^2(x - x_1) g(t_1, x_1) \, dx_1 \, dt_1 \\
\leq \frac{r}{2} L^2 \int_0^t \int_0^L g(t_1, x_1) \, dx_1 \, dt_1 \\
\leq r B_1 t^2 \int_0^t \int_0^L g(t_1, x_1) \, dx_1 \, dt_1 \\
\leq r A,
\]

and
Thus,

\[ \|Gu\| \leq 4rA. \]

Next, using (2.1) and the first and second inequalities of (2.2), we get

\[
\left| \frac{\partial^2}{\partial x^2} Gu(t,x) \right| = \left| -\frac{1}{2} \int_0^t \int_0^x (t-t_1)^2 g(t_1, x_1) \int_0^{t_1} (t_1-t_2)u(t_2, x_1)dt_2 dx_1 dt_1 \right|
\leq \frac{1}{2} \int_0^t \int_0^x (t-t_1)^2 g(t_1, x_1) \int_0^{t_1} (t_1-t_2)u(t_2, x_1)dt_2 dx_1 dt_1
\leq \frac{r}{2} \int_0^t \int_0^x (t-t_1)^2 g(t_1, x_1)dx_1 dt_1
\leq \frac{r}{2} t^4 \int_0^t \int_0^L g(t_1, x_1)dx_1 dt_1
\leq rB_1 t^4 \int_0^t \int_0^L g(t_1, x_1)dx_1 dt_1
\leq rA, \quad (t,x) \in [0, \infty) \times [0,L].
\]
\[
\frac{\partial}{\partial t} Fu(t, x) = \frac{1}{2} \int_0^t \int_0^x (t - t_1)(x - x_1)^2 g(t_1, x_1) F_3 u(t_1, x_1) dx_1 dt_1 \\
\leq \frac{1}{2} \int_0^t \int_0^x (t - t_1)(x - x_1)^2 g(t_1, x_1) |F_3 u(t_1, x_1)| dx_1 dt_1 \\
\leq 2L^2 r \int_0^t \int_0^x (1 + t_1)(x - x_1)^2 g(t_1, x_1) dx_1 dt_1 \\
+ L \sum_{j=1}^l r^p_j \int_0^t \int_0^x t_1(t - t_1)(x - x_1)^2 g(t_1, x_1) \int_0^t \int_0^{t_1} c_j(t_2, x_2) dx_2 dx_1 dt_1 \\
\leq 2L^4 r(t + t^2) \int_0^t \int_0^L g(t_1, x_1) dx_1 dt_1 \\
+ \frac{L^3}{2} \sum_{j=1}^l r^p_j t \int_0^t \int_0^L g(t_1, x_1) \int_0^L \int_0^{t_1} c_j(t_2, x_2) dx_2 dx_1 dt_1 \\
\leq B_1 r(t + t^2) \int_0^t \int_0^L g(t_1, x_1) dx_1 dt_1 \\
+ B_1 \sum_{j=1}^l r^p_j t \int_0^t \int_0^L g(t_1, x_1) \int_0^L \int_0^{t_1} c_j(t_2, x_2) dx_2 dx_1 dt_1 \\
\leq \left( r + \sum_{j=1}^l r^p_j \right) A, \quad (t, x) \in [0, \infty) \times [0, L],
\]

and

\[
\frac{\partial^2}{\partial t^2} Fu(t, x) = \frac{1}{2} \int_0^t \int_0^x (x - x_1)^2 g(t_1, x_1) F_3 u(t_1, x_1) dx_1 dt_1 \\
\leq \frac{1}{2} \int_0^t \int_0^x (x - x_1)^2 g(t_1, x_1) |F_3 u(t_1, x_1)| dx_1 dt_1 \\
\leq 2L^2 r \int_0^t \int_0^x (1 + t_1)(x - x_1)^2 g(t_1, x_1) dx_1 dt_1 \\
+ L \sum_{j=1}^l r^p_j \int_0^t \int_0^x t_1(t - t_1)(x - x_1)^2 g(t_1, x_1) \int_0^t \int_0^{t_1} c_j(t_2, x_2) dx_2 dx_1 dt_1 \\
\leq 2L^4 r(1 + t) \int_0^t \int_0^L g(t_1, x_1) dx_1 dt_1 \\
+ \frac{L^3}{2} \sum_{j=1}^l r^p_j t \int_0^t \int_0^L g(t_1, x_1) \int_0^L \int_0^{t_1} c_j(t_2, x_2) dx_2 dx_1 dt_1 \\
\leq B_1 r(1 + t) \int_0^t \int_0^L g(t_1, x_1) dx_1 dt_1 \\
+ B_1 \sum_{j=1}^l r^p_j t \int_0^t \int_0^L g(t_1, x_1) \int_0^L \int_0^{t_1} c_j(t_2, x_2) dx_2 dx_1 dt_1 \\
\leq \left( r + \sum_{j=1}^l r^p_j \right) A, \quad (t, x) \in [0, \infty) \times [0, L],
\]
and

\[ \frac{\partial}{\partial x} Fu(t, x) = \frac{1}{2} \int_0^t \int_0^x (t - t_1)^2 (x - x_1) g(t_1, x_1) F_3 u(t_1, x_1) \, dx_1 \, dt_1 \]

\[ \leq \frac{1}{2} \int_0^t \int_0^x (t - t_1)^2 (x - x_1) |F_3 u(t_1, x_1)| \, dx_1 \, dt_1 \]

\[ \leq 2L^2 r \int_0^t \int_0^x (1 + t_1) (t - t_1)^2 (x - x_1) \, dx_1 \, dt_1 \]

\[ + L \sum_{j=1}^I r^p_j \int_0^t \int_0^x t_1 (t - t_1)^2 (x - x_1) \, dx_1 \, dt_1 \]

\[ \leq 2L^3 (r^2 + r^3) \int_0^t \int_0^x g(t_1, x_1) \, dx_1 \, dt_1 \]

\[ + L \sum_{j=1}^I r^p_j t^3 \int_0^t \int_0^x g(t_1, x_1) \int_0^L \int_0^{t_1} c_j (t_2, x_2) \, dt_2 \, dx_2 \, dx_1 \, dt_1 \]

\[ \leq B_1 r (r^2 + r^3) \int_0^t \int_0^x g(t_1, x_1) \, dx_1 \, dt_1 \]

\[ + B_1 \sum_{j=1}^I r^p_j t^3 \int_0^t \int_0^x g(t_1, x_1) \int_0^L \int_0^{t_1} c_j (t_2, x_2) \, dt_2 \, dx_2 \, dx_1 \, dt_1 \]

\[ \leq \left( r + \sum_{j=1}^I r^p_j \right) A, \quad (t, x) \in [0, \infty) \times [0, L], \]

and

\[ \frac{\partial^2}{\partial x^2} Fu(t, x) = \frac{1}{2} \int_0^t \int_0^x (t - t_1)^2 g(t_1, x_1) F_3 u(t_1, x_1) \, dx_1 \, dt_1 \]

\[ \leq \frac{1}{2} \int_0^t \int_0^x (t - t_1)^2 g(t_1, x_1) |F_3 u(t_1, x_1)| \, dx_1 \, dt_1 \]

\[ \leq 2L^2 r \int_0^t \int_0^x (1 + t_1) (t - t_1)^2 g(t_1, x_1) \, dx_1 \, dt_1 \]

\[ + L \sum_{j=1}^I r^p_j \int_0^t \int_0^x t_1 (t - t_1)^2 g(t_1, x_1) \, dx_1 \, dt_1 \]

\[ \leq 2L^2 (r^2 + r^3) \int_0^t \int_0^x g(t_1, x_1) \, dx_1 \, dt_1 \]

\[ + L \sum_{j=1}^I r^p_j t^3 \int_0^t \int_0^x g(t_1, x_1) \int_0^L \int_0^{t_1} c_j (t_2, x_2) \, dt_2 \, dx_2 \, dx_1 \, dt_1 \]

\[ \leq B_1 r (r^2 + r^3) \int_0^t \int_0^x g(t_1, x_1) \, dx_1 \, dt_1 \]

\[ + B_1 \sum_{j=1}^I r^p_j t^3 \int_0^t \int_0^x g(t_1, x_1) \int_0^L \int_0^{t_1} c_j (t_2, x_2) \, dt_2 \, dx_2 \, dx_1 \, dt_1 \]

\[ \leq \left( r + \sum_{j=1}^I r^p_j \right) A, \quad (t, x) \in [0, \infty) \times [0, L]. \]
Therefore
\[ \|Fu\| \leq 4 \left( r + \sum_{j=1}^{l} r^j \right) A. \]

This completes the proof.

\[ \square \]

3 Proof of the main result

For \( u \in E \), define the mappings
\[
Tu(t, x) = (1 - \epsilon)u(t, x) + Gu(t, x),
\]
\[
Su(t, x) = \epsilon u(t, x) + Fu(t, x), \quad (t, x) \in [0, \infty) \times [0, L].
\]

Note that if \( u \in E \) is a fixed point of the mapping \( T + S \), then
\[
u(t, x) = Tu(t, x) + Su(t, x)
= (1 - \epsilon)u(t, x) + Gu(t, x) + \epsilon u(t, x) + Fu(t, x)
= u(t, x) + Fu(t, x) + Gu(t, x), \quad (t, x) \in [0, \infty) \times [0, L],
\]
or
\[
0 = Gu(t, x) + Fu(t, x), \quad (t, x) \in [0, \infty) \times [0, L].
\]

Therefore any fixed point \( u \in E \) of the mapping \( T + S \) is a solution of the IBVP (1.1). Define
\[
\tilde{P} = \{ u \in E : u(t, x) \geq 0, \quad (t, x) \in [0, \infty) \times [0, L] \},
\]

Let \( P \) be the set of all equi-continuous families in \( \mathcal{P} \) (an example for an equi-continuous family in \( \tilde{P} \) is the family \( \{(3 + \sin(t + n))(3 + \cos(x + n)), \quad t \in [0, \infty), \quad x \in [0, L] \}_{n \in \mathbb{N}} \)). Let also,
\[
\Omega = \{ u \in \mathcal{P} : \|u\| \leq R \},
\]

\[
U = \left\{ u \in \mathcal{P} : u(t, x) < u_0(x), \quad (t, x) \in (0, \infty) \times [0, L],
\]
\[
\quad u(t, x) < mu_0(x), \quad (t, x) \in [1, 2] \times [0, L], \quad \|u\| < r \right\}.
\]

Note that for \( u \in U \), we have \( F_1u \geq 0, \quad F_2u \geq 0, \quad Fu \geq 0 \). Hence, for \( u \in U \), we have
\[
F_1u(t, x) \geq \int_0^x x_1(-u(t, x_1) + u_0(x_1))dx_1, \quad (t, x) \in [0, \infty) \times [0, L],
\]

and
\[
Fu(t, x) \geq \frac{1}{4} \int_0^t \int_0^x (t - t_1)^2(x - x_1)^2 g(t_1, x_1) \int_0^{x_1} x_2(-u(t_1, x_2) + u_0(x_2))dx_2dx_1dt_1,
\]

(3.1)
\((t, x) \in [0, \infty) \times [0, L]\).

1. For \(u \in \Omega\), we have

\[
(I - T)u(t, x) = \epsilon u(t, x) - Gu(t, x), \quad (t, x) \in [0, \infty) \times [0, L].
\]

Then, for \(u \in \Omega\), using Lemma 2.3, we find

\[
\| (I - T)u \| \leq \epsilon \| u \| + \| Gu \|
\]

\[
\| (I - T)u \| \leq (\epsilon + 4A) \| u \|,
\]

\[
\| (I - T)u \| \geq \| u \| - \| Gu \|
\]

\[
\| (I - T)u \| \geq (\epsilon - 4A) \| u \|.
\]

Thus, \(I - T : \Omega \rightarrow E\) is Lipschitz invertible with a constant \(\gamma \in \left[ \frac{1}{\epsilon + 4A}, \frac{1}{\epsilon - 4A} \right]\).

2. Let \(u \in \overline{U}\). By Lemma 2.3, we have

\[
\| Su \| \leq \epsilon \| u \| + \| Fu \|
\]

\[
\| Su \| \leq \epsilon \| u \| + \sum_{j=1}^{l} r^{j} A.
\]

Therefore \(S : \overline{U} \rightarrow E\) is uniformly bounded. Since \(S : \overline{U} \rightarrow \mathbb{R}\) is continuous, we have that \(S(\overline{U})\) is equi-continuous and \(S : \overline{U} \rightarrow E\) is relatively compact. Therefore \(S : \overline{U} \rightarrow E\) is a 0-set contraction.

3. Let \(u \in \overline{U}\). For \(z \in \Omega\), define the mapping

\[
Lz(t, x) = Tz(t, s) + Su(t, s), \quad (t, x) \in [0, \infty) \times [0, L].
\]

For \(z \in \Omega\), we get

\[
\| Lz \| = \| Tz + Su \|
\]

\[
\leq \| Tz \| + \| Su \|
\]

\[
\leq (1 - \epsilon + 4A) R + \epsilon r + 4 \left( r + \sum_{j=1}^{l} r^{j} \right) A
\]

\[
\leq (1 - \epsilon + 4A) R + (\epsilon - 4A) R
\]

\[
= R,
\]

i.e., \(L : \Omega \rightarrow \Omega\). Next, for \(z_1, z_2 \in \Omega\), we have

\[
\| Lz_1 - Lz_2 \| = \| Tz_1 - Tz_2 \|
\]

\[
= \| T(z_1 - z_2) \|
\]

\[
\leq (1 - \epsilon + 4A) \| z_1 - z_2 \|.
\]

Therefore \(L : \Omega \rightarrow \Omega\) is a contraction mapping. Hence, there exists a unique \(z \in \Omega\) such that

\[
z = Lz
\]

or

\[
(I - T)z = Su.
\]
Consequently \( S(\overline{U}) \subset (I - T)(\Omega) \).

(4) Assume that there are \( u \in \partial U \) and \( \lambda \geq 1 \) such that

\[
Su = (I - T)(\lambda u), \quad \lambda u \in \Omega.
\]

We have

\[
eu + Fu = \epsilon \lambda u - G(\lambda u)
\]

or

\[
e(\lambda - 1)u = Fu + G(\lambda u).
\]

Since \( \lambda u \in \Omega \), we have that \( \|\lambda u\| \leq R \). Hence and Lemma 2.3, we obtain \( \|G(\lambda u)\| \leq 4RA \). Then

\[
e(\lambda - 1)r = e(\lambda - 1)\|u\|
\]

\[
= \|Fu - G(\lambda u)\|
\]

\[
\leq \|Fu\| + \|G(\lambda u)\|
\]

\[
\leq 4 \left( r + R + \sum_{j=1}^{l} r^{p_j} \right) A.
\]

Hence, for \( \lambda u \in \Omega \) and \( u \in \partial U \), using (3.1), we get

\[
4 \left( r + R + \sum_{j=1}^{l} r^{p_j} \right) A \geq e(\lambda - 1)\|u\|
\]

\[
= \|Fu + G(\lambda u)\|
\]

\[
\geq \|Fu\| - \|G(\lambda u)\|
\]

\[
\geq \sup_{(t,x) \in [0,\infty) \times [0,L]} Fu(t,x) - \|G(\lambda u)\|
\]

\[
\geq Fu(2,L) - \|G(\lambda u)\|
\]

\[
\geq \frac{1}{4} \int_{0}^{2} \int_{0}^{L} (2 - t)^{2}(L - x)^{2} g(t, x, t) dx dt
\]

\[
\times \int_{0}^{u} x_{2}(-u(t, x, x) + u_{0}(x)) dx_{3} x_{1} dt_{1} - 4AR
\]

\[
\geq \frac{1}{4} \int_{1}^{2} \int_{1}^{L} (2 - t)^{2}(L - x)^{2} g(t, x, t) dx_{2} x_{1} dt_{1} - 4AR
\]

\[
\geq \frac{1}{4} \int_{1}^{2} \int_{1}^{L} (2 - t)^{2}(L - x)^{2} g(t, x, t) dx_{2} x_{1} dt_{1} - 4AR
\]

\[
\geq \frac{1}{4} \int_{1}^{2} \int_{1}^{L} (2 - t)^{2}(L - x)^{2} g(t, x, t) dx_{2} x_{1} dt_{1} - 4AR
\]

\[
\geq \frac{1}{4} \int_{1}^{2} \int_{1}^{L} (2 - t)^{2}(L - x)^{2} g(t, x, t) dx_{2} x_{1} dt_{1} - 4AR
\]

\[
\geq \frac{1}{4} \int_{1}^{2} \int_{1}^{L} (2 - t)^{2}(L - x)^{2} g(t, x, t) dx_{2} x_{1} dt_{1} - 4AR
\]

\[
\geq \frac{A}{b_{1}} - 4AR.
\]
whereupon
\[ 4 \left( r + 2R + \sum_{j=1}^{l} r_j \right) A \geq \frac{A}{b_1} \]

or
\[ 4 \left( r + 2R + \sum_{j=1}^{l} r_j \right) \geq \frac{1}{b_1}. \]

This is a contradiction.

By (1)–(4) and Proposition 2.1, we conclude that the mapping \( T + S \) has a fixed point in \( U \). This completes the proof.

### 4 Example

Consider the following IBVP

\[
\begin{align*}
  u_{tt} - u_{xx} &= |u|^p, & t \geq 0, & x \in [0, 1], \\
  u(0, x) &= \frac{1}{10} x(1 - x)^2, & x \in [0, 1], \\
  u_t(0, x) &= \frac{1}{50} x(1 - x)^2, & x \in [0, 1], \\
  u(t, 0) = u_t(t, 1) &= 0, & t \geq 0,
\end{align*}
\]

where \( p > 1 \). Here

\[ f(t, x, u) = |u|^p, \quad L = l = 1, \quad B_1 = 2, \quad c_1(t, x) = 1, \quad (t, x) \in [0, \infty) \times [0, 1], \]

and

\[ u_0(x) = \frac{1}{10} x(1 - x)^2, \quad u_1(x) = \frac{1}{50} x(1 - x)^2, \quad x \in [0, 1]. \]

Now we will construct a function \( g \) so that (2.2) holds. Let

\[ h(t) = \log \frac{1 + r^t \sqrt{2} + t^8}{1 - r^t \sqrt{2} + t^8}, \quad l(t) = \arctan \frac{r^t \sqrt{2}}{1 - r^t \sqrt{2}}, \quad t \geq 0. \]

We have

\[
\begin{align*}
  h'(t) &= \frac{1}{(1 + r^t \sqrt{2} + t^8)(1 - r^t \sqrt{2} + t^8)} \left( (4\sqrt{2}r^3 + 8r^7)(1 - r^t \sqrt{2} + t^8) \\
  &\quad - (1 + r^t \sqrt{2} + t^8)(-4\sqrt{2}r^3 + 8r^7) \right) \\
  &= \frac{1}{(1 + r^t \sqrt{2} + t^8)(1 - r^t \sqrt{2} + t^8)} \left( 4\sqrt{2}r^3 - 8r^7 + 4\sqrt{2}r^{11} + 8r^7 \\
  &\quad - 8\sqrt{2}r^{11} + 8r^{15} + 4\sqrt{2}r^3 - 8r^7 - 8\sqrt{2}r^{11} + 4\sqrt{2}r^{11} - 8r^{15} \right) \\
  &= - \frac{8\sqrt{2}r^3(t^8 - 1)}{(1 + r^t \sqrt{2} + t^8)(1 - r^t \sqrt{2} + t^8)}, \quad t \geq 0.
\end{align*}
\]
Thus,

\[
\sup_{t \geq 0} h(t) = h(1) = \log \frac{2 + \sqrt{2}}{2 - \sqrt{2}},
\]

\(h\) is an increasing function on \([0, 1]\) and it is a decreasing function on \([1, \infty)\). Next,

\[
l'(t) = \frac{1}{1 + t^{16}} \frac{4\sqrt{2}r^3 (1 - t^8) + 8t^3r^4 \sqrt{2}}{(1 - t^8)^2} - 4\sqrt{2}r^3 - 4\sqrt{2}r^{11} + 8\sqrt{2}r^{11} = \frac{4\sqrt{2}r^3 (1 + t^8)}{1 + t^{16}}, \quad t \geq 0.
\]

Therefore \(l\) is an increasing function on \([0, \infty)\). Note that, by l’Hopital’s rule, we have

\[
\lim_{t \to \infty} th(t) = 0
\]

\[
\lim_{t \to \infty} t^2 h(t) = 0
\]

\[
\lim_{t \to \infty} t^3 h(t) = 0
\]

\[
\lim_{t \to \infty} t^4 h(t) = 2\sqrt{2},
\]

and

\[
\lim_{t \to \infty} tl(t) = 0
\]

\[
\lim_{t \to \infty} t^2 l(t) = 0
\]

\[
\lim_{t \to \infty} t^3 l(t) = 0
\]

\[
\lim_{t \to \infty} t^4 l(t) = - \sqrt{2}.
\]

Consequently, there exists a constant \(B > 1\) such that

\[
(1 + t + t^2 + t^3 + t^4) \left( \frac{1}{16\sqrt{2}} \log \frac{1 + t^4 \sqrt{2} + t^8}{1 - t^4 \sqrt{2} + t^8} + \frac{1}{8\sqrt{2}} \arctan \frac{t^4 \sqrt{2}}{1 - t^8} \right) \leq B.
\]

Note that, by [17] (pp. 707, Integral 79), we have

\[
\int \frac{dz}{1 + z^4} = \frac{1}{4\sqrt{2}} \log \frac{1 + z\sqrt{2} + z^2}{1 - z\sqrt{2} + z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1 - z^2}.
\]

Take
\begin{align*}
\epsilon &= \frac{1}{2}, \quad b_1 = \frac{B(15)^2(2^{16} + 3^{16})(2^2 + 3^2)3^4}{2^5}, \quad A = \frac{1}{20b_1}, \\
m &= \frac{1}{2}, \quad r = \frac{4}{27}, \\
R &= \frac{2}{e - 4A}(er + 4(r + r^2)A).
\end{align*}

Then

$$0 \leq u_0(x) < r, \quad 0 \leq u_1(x) < r, \quad x \in [0, 1],$$

and

$$u_0(0) = u_{0x}(1) = u_1(0) = u_{1x}(1) = 0,$$

i.e., (1.5) holds. Also, (2.1) holds. Let

$$g(t, x) = \frac{A}{200B(1 + t^{16})(1 + r^2)}, \quad (t, x) \in [0, \infty) \times [0, 1].$$

\begin{align*}
B_1(1 + t + t^2 + t^3 + t^4) &\int_0^t \int_0^1 g(t_1, x_1)dx_1dt_1 \\
&\leq \frac{A}{100B}(1 + t + t^2 + t^3 + t^4) \int_0^t \frac{t_1^3}{1 + t_1^{16}}dt_1 \\
&\leq \frac{A}{100} \\
&\leq A
\end{align*}

and

\begin{align*}
B_1(1 + t + t^2 + t^3 + t^4) &\int_0^t \int_0^1 \int_0^1 g(t_1, x_1) \int_0^1 c_j(t_1, x_1)dt_2dx_2dx_1dt_1 \\
&\leq 2(1 + t + t^2 + t^3 + t^4) \int_0^t \int_0^1 t_1g(t_1, x_1)dx_1dt_1 \\
&\leq \frac{A}{100B}(1 + t + t^2 + t^3 + t^4) \int_0^t \frac{t_1^3}{1 + t_1^{16}}dt_1 \\
&\leq \frac{A}{100} \\
&\leq A,
\end{align*}

and
\[
\frac{1-m}{4} \int_1^2 \int_1^2 (2-t_1)^2 (1-x_1)^2 g(t_1, x_1) \int_1^2 x_2 u_0(x_2) dx_2 dx_1 dt_1 \\
= \frac{A}{1600B} \int_1^2 \int_1^2 (2-t_1)^2 (1-x_1)^2 \frac{t_1^3}{(1+t_1^{16})(1+t_1^2)} \int_1^2 x_2^2 (1-x_2)^2 dx_2 dx_1 dt_1 \\
\geq \frac{A}{1600B} \left( \frac{1}{2} \right) \left( \frac{1}{3} \right)^2 \left( \frac{1}{1+(\frac{1}{2})^{16}} \right) \left( \frac{1}{1+(\frac{1}{2})^{2}} \right) \left( \frac{1}{2} \right) \left( \frac{2}{3} \right) \left( \frac{1}{6} \right)^2 \\
= \frac{2^{18}}{B \left( \frac{15}{2} \right)^2 (2^{16} + 3^{16})(2^2 + 3^2)} \\
\geq \frac{2^5 A}{B \left( \frac{15}{2} \right)^2 (2^{16} + 3^{16})(2^2 + 3^2) 3^4} = \frac{A}{b_1}.
\]

Consequently (2.2) holds and the IBVP (4.1) has at least one non-negative solution \( u \in C^2([0, \infty) \times [0, 1]) \).

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