A Proof of the Gutzwiller Semiclassical Trace Formula
Using Coherent States Decomposition

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Abstract

The Gutzwiller trace formula links the eigenvalues of the Schrödinger operator $\hat{H}$ as Planck’s constant goes to zero (the semiclassical régime) with the closed orbits of the corresponding classical mechanical system. Gutzwiller gave a heuristic proof of this trace formula, using the Feynman integral representation for the propagator of $\hat{H}$. Later, using the theory of Fourier integral operators, mathematicians gave rigorous proofs of the formula in various settings. Here we show how the use of coherent states allows us to give a simple and direct proof.

1 Introduction

Our goal in this paper is to give a simple proof of the “semiclassical Gutzwiller trace formula”. The pioneering works in quantum physics of Gutzwiller [17] (1971) and Balian-Bloch [4] [5] (1972-74) showed that the trace of a quantum observable $\hat{A}$, localized in a spectral neighborhood of size $O(\hbar)$ of an energy $E$ for the quantum Hamiltonian $\hat{H}$, can be expressed in terms of averages of the classical observable $A$ associated with $\hat{A}$ over invariant sets for the flow of the classical Hamiltonian $H$ associated with $\hat{H}$. This is related to the spectral asymptotics for $\hat{H}$ in the semi-classical limit, and it can be understood as a “correspondence principle” between classical and quantum mechanics as Planck’s constant $\hbar$ goes to zero.

Between 1973 and 1975 several authors gave rigorous derivations of related results, generalizing the classical Poisson summation formula from $d^2/d\theta^2$ on the circle to elliptic operators on compact manifolds: Colin de Verdière [8], Chazarain [7], Duistermaat-Guillemin [14]. The first article is based on a parametrix construction for the associated heat equation, while the second two replace this with a parametrix, constructed as a Fourier integral operator, for the associated wave equation. More recently, papers by Guillemin-Uribe (1989), Paul-Uribe (1991, 1995), Meinrenken (1992) and Dozias (1994) have developed the necessary tools from microlocal analysis in a nonhomogeneous...
(semiclassical) setting to deal with Schrödinger-type Hamiltonians. Extensions and simplifications of these methods have been given by Petkov-Popov [31], and Charbonnel-Popov [6].

The coherent states approach presented here seems particularly suitable when one wishes to compare the phase space quantum picture with the phase space classical flow. Furthermore, it avoids problems with caustics, and the Maslov indices appear naturally. In short, it implies the Gutzwiller trace formula in a very simple and transparent way, without any use of the global theory of Fourier integral operators. In their place we use the coherent states approximation (gaussian beams) and the stationary phase theorem.

The use of gaussian wave packets is such a useful idea that one can trace it back to the very beginning of quantum mechanics, for instance, Schrödinger [35] (1926). However, the realization that these approximations are universally applicable, and that they are valid for arbitrarily long times, has developed gradually. In the mathematical literature these approximations have never become textbook material, and this has lead to their repeated rediscovery with a variety of different names, e.g. coherent states and gaussian beams. The first place that we have found where they are used in some generality is Babich [4] (1968) (see also [2]). Since then they have appeared, often as independent discoveries, in the work of Arnaud [1] (1973), Keller [24] (1974), Heller [20] (1975, 1987), Ralston [33],[34] (1976,1982), Hagedorn [18] (1980-85), and Littlejohn [25] (1986) – and probably many more that we have not found. Their use in trace formulas was proposed by Wilkinson [36] (1987). The propagation formulas of [18] were extended in Combescure-Robert [10], with a detailed estimate on the error both in time and in Planck’s constant. The early application of these methods in [2] was for the construction of quasi-modes, and this has been pursued further in [33] and Paul-Uribe [27]. There have also been recent applications to the pointwise behaviour of semiclassical measures [28].

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2 The Semiclassical Gutzwiller Trace Formula

We consider a quantum system in $L^2(\mathbb{R}^n)$ with Hamiltonian

\[ \hat{H} = -\hbar^2 \Delta + V(x), \]

where $\Delta$ is the Laplacian in $L^2(\mathbb{R}^n)$ and $V(x)$ a real, $C^\infty(\mathbb{R}^n)$ potential. The corresponding Hamiltonian for the classical motion is

\[ H(q,p) = p^2 + V(q), \]

and for a given energy $E (\in \mathbb{R})$ we denote by $\Sigma_E$ the “energy shell”

\[ \Sigma_E := \{(q,p) \in \mathbb{R}^{2n} : H(q,p) = E\}. \]
More generally we shall consider Hamiltonians \( \hat{H} \) obtained by the \( \hbar \)-Weyl quantization of the classical Hamiltonian \( H \), so that

\[
\hat{H} = \text{Op}_w(\hbar)(H),
\]

where

\[
\text{Op}_w(\hbar)(H)\psi(x) = (2\pi \hbar)^{-n} \int_{\mathbb{R}^{2n}} H \left( \frac{x+y}{2}, \xi \right) \psi(y) e^{i\frac{(x-y)\cdot \xi}{\hbar}} dyd\xi
\]

The Hamiltonian \( H \) is assumed to be a smooth, real valued function of \( z = (x, \xi) \in \mathbb{R}^{2n} \), and to satisfy the following global estimates

- (H.0) there exist non-negative constants \( C, m, C_\gamma \) such that

\[
|\partial_\gamma^z H(z)| \leq C_\gamma <H(z)>, \quad \forall \ z \in \mathbb{R}^{2n}, \forall \gamma \in \mathbb{N}^n
\]

- (H.1) \( <H(z)> \leq C <H(z')> \cdot <z-z'>^m, \quad \forall \ z, z' \in \mathbb{R}^{2n} \)

where we have used the notation \( <u> = (1 + |u|^2)^{1/2} \) for \( u \in \mathbb{R}^m \).

**Remark 2.1**

i) \( H(q,p) = p^2 + V(q) \) satisfies (H.0), if \( V(q) \) is bounded below by some \( a > 0 \) and satisfies the property (H.0) in the variable \( q \).

ii) The technical condition (H.0) implies in particular that \( \hat{H} \) is essentially self-adjoint on \( L^2(\mathbb{R}^m) \) for \( \hbar \) small enough and that \( \chi(\hat{H}) \) is a \( \hbar \)-pseudodifferential operator if \( \chi \in C_0^\infty(\mathbb{R}) \) (see [21]).

Let us denote by \( \phi_t \) the classical flow induced by Hamilton’s equations with Hamiltonian \( H \), and by \( S(q,p; t) \) the classical action along the trajectory starting at \( (q, p) \) at time \( t = 0 \), and evolving during time \( t \):

\[
S(q,p; t) = \int_0^t (p_s \cdot \dot{q}_s - H(q_s, p_s)) \, ds
\]

where \( (q_t, p_t) = \phi_t(q, p) \), and dot denotes the derivative with respect to time. We shall also use the notation: \( \alpha_t = \phi_t(\alpha) \) where \( \alpha = (q, p) \in \mathbb{R}^{2n} \), is a phase space point.

An important role in what follows is played by the “linearized flow” around the classical trajectory, which is defined as follows. Let

\[
H''(\alpha_t) = \left. \frac{\partial^2 H}{\partial \alpha^2} \right|_{\alpha = \alpha_t}
\]

be the Hessian of \( H \) at point \( \alpha_t = \phi_t(\alpha) \) of the classical trajectory. Let \( J \) be the symplectic matrix

\[
J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
\]

where 0 and \( I \) are respectively the null and identity \( n \times n \) matrices. Let \( F(t) \) be the \( 2n \times 2n \) real symplectic matrix solution of the linear differential equation

\[
\begin{cases}
\dot{F}(t) = J H''(\alpha_t) F(t) \\
F(0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = I
\end{cases}
\]
$F(t)$ depends on $\alpha = (q,p)$, the initial point for the classical trajectory, $\alpha_t$.

Let $\gamma$ be a closed orbit on $\sum_E$ with period $T_\gamma$, and let us denote simply by $F_\gamma$ the matrix $F_\gamma = F(T_\gamma)$. $F_\gamma$ is usually called the “monodromy matrix” of the closed orbit $\gamma$. Of course, $F_\gamma$ does depend on $\alpha$, but its eigenvalues do not, since the monodromy matrix with a different initial point on $\gamma$ is conjugate to $F_\gamma$. $F_\gamma$ has 1 as eigenvalue of algebraic multiplicity at least equal to 2. In all that follows, we shall use the following definition

**Definition 2.2** We say that $\gamma$ is a nondegenerate orbit if the eigenvalue 1 of $F_\gamma$ has algebraic multiplicity 2.

Let $\sigma$ denote the usual symplectic form on $\mathbb{R}^{2n}$

$$\sigma(\alpha,\alpha') = p \cdot q' - p' \cdot q \quad \alpha = (q,p); \quad \alpha' = (q',p')$$

($\cdot$ is usual scalar product in $\mathbb{R}^n$). We denote by $\{\alpha_1, \alpha_1'\}$ the eigenspace of $F_\gamma$ belonging to the eigenvalue 1, and by $V$ its orthogonal complement in the sense of the symplectic form $\sigma$

$$V = \left\{ \alpha \in \mathbb{R}^{2n} : \sigma(\alpha, \alpha_1) = \sigma(\alpha, \alpha_1') = 0 \right\}.$$  \hspace{1cm} (11)

Then, the restriction $P_\gamma$ of $F_\gamma$ to $V$ is called the (linearized) “Poincaré map” for $\gamma$.

In some cases the Hamiltonian flow will contain manifolds of periodic orbits with the same energy. When this happens, the periodic orbits will necessarily be degenerate, but the techniques we use here can still apply. The precise hypothesis for this (“Hypothesis C”) will be given in Section 4. Following Duistermaat and Guillemin we call this a “clean intersection hypothesis”, but it is more explicit than other versions of this assumption. Since the statement of the trace formula is simpler and more informative when one does assume that the periodic orbits are nondegenerate, we will give that formula here.

We shall now assume the following. Let $(\Gamma_E)_T$ be the set of all periodic orbits on $\sum_E$ with periods $T_\gamma$, $0 < |T_\gamma| \leq T$ (including repetitions of primitive orbits and assigning negative periods to primitive orbits traced in the opposite sense). Then we require:

- (H.1) There exists $\delta E > 0$ such that $H^{-1}(\{E - \delta E, E + \delta E\})$ is a compact set of $\mathbb{R}^{2n}$ and $E$ is a noncritical value of $H$ (i.e. $H(z) = E \Rightarrow \nabla H(z) \neq 0$).

- (H.2) For any $T > 0$, $(\Gamma_E)_T$ is a discrete set, with periods $-T \leq T_{\gamma_1} < \cdots < T_{\gamma_N} \leq T$.

- (H.3) All $\gamma$ in $(\Gamma_E)_T$ are nondegenerate, i.e. 1 is not an eigenvalue for the corresponding “Poincaré map”, $P_\gamma$.

We can now state the Gutzwiller trace formula. Let $\tilde{A} = OP_h^w(A)$ be a quantum observable, such that $A$ satisfies the following

- (H.4) there exists $\delta \in \mathbb{R}$, $C_\gamma > 0$ ($\gamma \in \mathbb{N}^{2n}$), such that

$$|\partial_\gamma A(z)| \leq C_\gamma < H(z) >^\delta \quad \forall z \in \mathbb{R}^{2n},$$

- (H.5) $g$ a $C^\infty$ function whose Fourier transform $\tilde{g}$ is of compact support with $\text{Supp} \tilde{g} \subset [-T, T]$
and let $\chi$ be a smooth function with a compact support contained in $|E - \delta E, E + \delta E|$, equal to 1 in a neighborhood of $E$. Then the following “regularized density of states” $\rho_A(E)$ is well defined

$$\rho_A(E) = \operatorname{Tr} \left( \chi(\hat{H}) \hat{A} \chi(\hat{H}) g \left( \frac{E - \hat{H}}{|\hat{h}|} \right) \right)$$

(12)

Note that (H.1) implies that the spectrum of $\hat{H}$ is purely discrete in a neighborhood of $E$ so that $\rho_A(E)$ is well defined. Then we have the following,

**Theorem 2.3**: Assume (H.0)-(H.3) are satisfied for $H$, (H.4) for $A$ and (H.5) for $g$. Then the following asymptotic expansion holds true, modulo $O(\bar{\hbar}^\infty)$,

$$\rho_A(E) \equiv (\pi)^{-n/2} \bar{g}(0) \bar{h}^{-(n-1)} \int_{\Sigma_E} A(\alpha) d\sigma_E(\alpha) + \sum_{k \geq -n+2} c_k(\bar{g}) \bar{h}^k$$

(13)

$$+ \sum_{\gamma \in (\Gamma E)_T} (2\pi)^{n/2-1} \left\{ \bar{g}(T_\gamma) \frac{e^{i(S_\gamma/\bar{h} + \sigma_\gamma \pi/2)}}{|\det(I - P_\gamma)|^{1/2}} \int_0^{T^*_\gamma} A(\alpha_s) ds + \sum_{j \geq 1} d_j^T(\bar{g}) \bar{h}^j \right\}$$

where $A(\alpha)$ is the classical Weyl symbol of $\hat{A}$,

$T^*_\gamma$ is the primitive period of $\gamma$,

$\sigma_\gamma$ is the Maslov index of $\gamma$ ( $\sigma_\gamma \in \mathbb{Z}$ ),

$S_\gamma = \oint pdq$ is the classical action along $\gamma$,

$c_k(\bar{g})$ are distributions in $\hat{g}$ with support in $\{0\}$,

$d_j^T(\bar{g})$ are distributions in $\hat{g}$ with support $\{T_\gamma\}$ and $d\sigma_E$ is the Liouville measure on $\Sigma_E$:

$$d\sigma_E = \frac{d\Sigma_E}{|\nabla H|} \quad (d\Sigma_E \text{ is the Euclidean measure on } \Sigma_E)$$

**Remark 2.4** We can include more general Hamiltonians depending explicitly in $\bar{h}$,

$$H = \sum_{j=1}^K \bar{h}^j H^{(j)} \quad \text{such that } H^{(0)} \text{ satisfies (H.0) and for } j \geq 1,$$

$$|\partial^\gamma H^{(j)}(z)| \leq C_{\gamma,j} < H^{(0)}(z) >$$

(14)

It is useful for applications to consider Hamiltonians like $H^{(0)} + \bar{h}H^{(1)}$ where $H^{(1)}$ may be, for example, a spin term. In that case the formula (13) is true with different coefficients. In particular the first term in the contribution of $T_\gamma$ is multiplied by $\exp \left( -i \int_0^{T^*_\gamma} H^{(1)}(\alpha_s) ds \right)$.

**Remark 2.5** For Schrödinger operators we only need smoothness of the potential $V$. In this case the trace formula (13) is still valid without any assumptions at infinity for $V$ when we restrict ourselves to a compact energy surface, assuming $E < \liminf_{|x| \to \infty} V(x)$. Using exponential decrease of the eigenfunctions we can prove that, modulo an error term of order $O(\bar{h}^{+\infty})$, the potential $V$ can be replaced by a potential $\tilde{V}$ satisfying the assumptions of the Remark (2.1).
3 Preparations for the Proof

We shall make use of “coherent states” which can be defined as follows. Let
\[ \psi_0(x) = (\hbar \pi)^{-n/4} \exp \left( -\frac{|x|^2}{2\hbar} \right), \]  
be the ground state of the \( n \)-dimensional harmonic oscillator, and for \( \alpha = (q,p) \in \mathbb{R}^{2n} \),
\[ \mathcal{T}(\alpha) = \exp \left\{ \frac{i}{\hbar} (p \cdot x - q \cdot \hbar D_x) \right\} \]  
is the Weyl-Heisenberg operator of translation by \( \alpha \) in phase space where \( D_x = \frac{\partial}{\partial x} \). We also denote by
\[ \phi_\alpha = \mathcal{T}(\alpha)\psi_0 \]  
the usual coherent states centered at the point \( \alpha \). Then it is known that any operator \( B \) with a symbol decreasing sufficiently rapidly is in trace class (see [15]), and its trace can be computed by
\[ \operatorname{Tr} B = (2\pi \hbar)^{-n} \int <\phi_\alpha, B\phi_\alpha> \, d\alpha. \]  
The regularized density of states \( \rho_A(E) \) can now be rewritten as
\[ \rho_A(E) = (2\pi)^{-n-1} \hbar^{-n} \int \tilde{g}(t) \, e^{iEt/\hbar} <\phi_\alpha, \hat{A}_\chi U(t) \phi_\alpha> \, dt \, d\alpha \]  
where \( U(t) \) is the quantum unitary group :
\[ U(t) = e^{-it\hat{H}/\hbar} \]  
and \( \hat{A}_\chi = \chi(\hat{H}) \hat{A}_\chi(\hat{H}) \).

Our strategy for computing the behavior of \( \rho_A(E) \) as \( \hbar \) goes to zero is first to compute the bracket
\[ m(\alpha,t) = <\hat{A}_\chi \phi_\alpha, U(t) \phi_\alpha>, \]  
where we drop the subscript \( \chi \) in \( A_\chi \) for simplicity. It is useful to rewrite (15) as
\[ \psi_0 = \Lambda_\hbar \bar{\psi}_0, \]  
where \( \Lambda_\hbar \) is the following scaling operator:
\[ (\Lambda_\hbar \psi)(x) = \hbar^{-n/4} \psi(x \hbar^{-1/2}) \quad \text{and} \quad \bar{\psi}_0(x) = \pi^{-n/4} \exp \left(-\frac{|x|^2}{2}\right). \]  

First of all we shall use the following lemma, giving the action of an \( \hbar \)-pseudodifferential operator on a Gaussian.

**Lemma 3.1** Assume that \( A \) satisfies (H.0). Then we have
\[ \hat{A}_\chi \phi_\alpha = \sum_{\gamma} \hbar^{\frac{|\gamma|}{2}} \frac{\partial^{|\gamma|} A(\alpha)}{\gamma!} \Psi_{\gamma,\alpha} + O(\hbar^{\infty}) \]  

in $L^2(\mathbb{R}^n)$, where $\gamma \in \mathbb{N}^{2n}$, $|\gamma| = \sum_1 2n \gamma_i! = \prod_1 2n \gamma_i!$ and
\[ \Psi_{\gamma,\alpha} = T(\alpha) \Lambda h Op^w_1(z^\gamma) \tilde{\psi}_0. \] (25)

where $Op^w_1(z^\gamma)$ is the 1-Weyl quantization of the monomial:
\[ (x,\xi)^\gamma = x^{\gamma'} \xi^{\gamma''}, \quad \gamma = (\gamma',\gamma'') \in \mathbb{N}^{2n}. \]

This lemma is easily proved using a scaling argument and Taylor expansion for the symbol $A$ around the point $\alpha$. Thus $m(t,\alpha)$ is a linear combination of terms like
\[ m_{\gamma}(\alpha,t) = <\Psi_{\gamma,\alpha}, U(t) \varphi_\alpha>. \] (26)

Now we compute $U(t) \varphi_\alpha$, using the semiclassical propagation of coherent states result as it was formulated in Combesure-Robert [10]. We recall that $F(t)$ is a time dependent symplectic matrix (Jacobi matrix) defined by the linear equation (3). $Met$ denotes the metaplectic representation of the linearized flow $F$ (see for example Folland [15]), and the $\hbar$-dependent metaplectic representation is defined by
\[ Met_h(F) = \Lambda h^{-1} Met(F) \Lambda_h \] (27)

We will also use the notation
\[ \delta(\alpha,t) = \int_0^t p_s \cdot q_s ds - tH(\alpha) - \frac{p_t \cdot q_t - p \cdot q}{2} \] (28)

From Theorem (3.5) of [10] (and its proof) we have the following propagation estimates in the $L^2$-norm: for every $N \in \mathbb{N}$ and every $T > 0$ there exists $C_{N,T}$ such that
\[ ||U(t) \varphi_\alpha - \exp \left( \frac{i\delta(\alpha,t)}{h} \right) T(\alpha_t) Met_h(F(t)) \Lambda h P_N(x, D_x, t, h) \tilde{\psi}_0|| \leq C_{N,T} h^N \] (29)

where $P_N(t,h)$ is the $(h,t)$-dependent differential operator defined by
\[ P_N(x, D_x, t, h) = I + \sum_{(k,j) \in I_N} h^{k/2-j} p_{kj}^w(x, D_t) \]

with $I_N = \{(k,j) \in \mathbb{N} \times \mathbb{N}, 1 \leq j \leq 2N-1, k \geq 3j, 1 \leq k - 2j < 2N\}$ (30)

where the differential operators $p_{kj}^w(x, D_t)$ are products of $j$ Weyl quantization of homogenous polynomials of degree $k_s$ with $\sum_{1 \leq s \leq j} k_s = k$ (see [10] Theorem (3.5) and its proof). So that we get
\[ p_{kj}^w(x, D_t) \tilde{\psi}_0 = Q_{kj}(x) \tilde{\psi}_0(x) \] (31)

where $Q_{kj}(x)$ is a polynomial (with coefficients depending on $(\alpha,t)$) of degree $k$ having the same parity as $k$. This is clear from the following facts: homogeneous polynomials have a definite parity, and Weyl quantization behaves well with respect to symmetries: $Op^w(A)$ commutes to the parity operator $\Sigma f(x) = f(-x)$ if and only if $A$ is an even
symbol and anticommutes with $\Sigma$ if and only if $A$ is an odd symbol) and $\tilde{\psi}_0(x)$ is an even function. So we get

$$m(\alpha, t) = \sum_{(j,k) \in IN, |\gamma| \leq 2N} c_{k,j,\gamma} \hbar^{k+|\gamma|/2} \cdot j \exp \left( \frac{i \delta(\alpha, t)}{\hbar} \right) \cdot \left\langle T(\alpha) \Lambda_\hbar Q\gamma \tilde{\psi}_0, T(\alpha_t) \Lambda_\hbar Q_{k,j} \text{Met}(F(t))\tilde{\psi}_0 \right\rangle + O(\hbar^N)$$

(32)

where $Q_{k,j}$ respectively $Q_\gamma$ are polynomials in the $x$ variable with the same parity as $k$ respectively $|\gamma|$. This remark will be useful in proving that we have only entire powers in $\hbar$ in (13), even though half integer powers appear naturally in the asymptotic propagation of coherent states. By an easy computation we have

$$\left\langle T(\alpha) \Lambda_\hbar Q\gamma \tilde{\psi}_0, T(\alpha_t) \Lambda_\hbar Q_{k,j} \text{Met}(F(t))\tilde{\psi}_0 \right\rangle = \exp \left( -i \frac{1}{2\hbar} \sigma(\alpha, \alpha_t) \right) \left\langle T_1 \left( \frac{\alpha - \alpha_t}{\sqrt{\hbar}} \right) Q\gamma \tilde{\psi}_0, Q_{k,j} \text{Met}(F(t))\tilde{\psi}_0 \right\rangle$$

(33)

where $T_1(\cdot)$ is the Weyl translation operator with $\hbar = 1$.

We set

$$m_{k,j,\gamma}(\alpha, t) = \left\langle T_1 \left( \frac{\alpha - \alpha_t}{\sqrt{\hbar}} \right) Q\gamma \tilde{\psi}_0, Q_{k,j} \text{Met}(F(t))\tilde{\psi}_0 \right\rangle$$

(34)

$$m_0(\alpha, t) = \left\langle T_1 \left( \frac{\alpha - \alpha_t}{\sqrt{\hbar}} \right) \tilde{\psi}_0, \text{Met}(F(t))\tilde{\psi}_0 \right\rangle$$

(35)

We compute $m_0(\alpha, t)$ first. We shall use the fact that the metaplectic group transforms Gaussian wave packets to Gaussian wave packets in a very explicit way. If we denote by $A, B, C, D$ the four $n \times n$ matrices of the block form of $F(t)$,

$$F(t) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(36)

it is clear, since $F$ is symplectic, that $U = A + iB$ is invertible. So we can define

$$M = VU^{-1}, \quad \text{where } V = (C + iD).$$

(37)

We have $(13$, Ch.4)

$$m_0(\alpha, t) = (\det U)^{-1/2} \pi^{-n/2} \cdot \int_{\mathbb{R}^n} \exp \left\{ \frac{i}{2} (M + iI) x \cdot x - \frac{i}{\sqrt{\hbar}} (x - q - q_t) \cdot (p - p_t + i(q - q_t)) \right\} dx$$

(38)

**Remark 3.2** In (38), $(z(t))_{c}^{1/2}$ has the following meaning: if $t \mapsto z(t)$ is a continuous mapping from $\mathbb{R}$ into $\mathbb{C} \setminus \{0\}$ such that $z(0) > 0$ then $(z(t))_{c}^{1/2}$ denotes the square root defined by continuity in $t$ starting from $\sqrt{z(0)} > 0$. Thus factor $(\det U)^{-1/2}$ in (38) records the winding of $\det U(t)$ at $t$ varies. This takes the place of the “Maslov line bundle” in this construction.
If we make the change of variables $x \mapsto (y - q_t)/\sqrt{\hbar}$ in (32) and hence in (38), then the formula for the regularized density of states in (19) takes the form

$$\rho_A(E) = \int_R dt \int_{R^{2n}} d\alpha \int_{R^n} a(t, \alpha, y, \hbar) e^{i\Phi_E(y, \alpha, t)} dy. \quad (39)$$

The phase function $\Phi_E$ is given by

$$\Phi_E(t, y, \alpha) = S(\alpha, t) + q \cdot p + \frac{1}{2} (y - q_t) \cdot M(t)(y - q_t) + \frac{i}{2} |y - q|^2 - y \cdot p + Et, \quad (40)$$

where $\cdot$ denotes the usual bilinear product in $\mathbb{C}^n$, and $\alpha = (q, p)$, $\alpha_t = \phi_t(\alpha)$ as before.

Our plan is to prove Theorem 2.3 by expanding (39) by the method of stationary phase. The necessary stationary phase lemma for complex phase functions can easily be derived from Theorem 7.7.5 in [22, Vol. 1]. There is also an extended discussion of complex phase functions depending on parameters in [22] leading to Theorem 7.7.12, but the form of the stationary manifold here permits us to use the following

**Theorem 3.3 (stationary phase expansion)** Let $O \subset IR^d$ be an open set, and let $a, f \in C^\infty(O)$ with $\Im f \geq 0$ in $O$ and supp $a \subset O$. We define

$$M = \{x \in O, \Im f(x) = 0, f'(x) = 0\},$$

and assume that $M$ is a smooth, compact and connected submanifold of $IR^d$ of dimension $k$ such that for all $x \in M$ the Hessian, $f''(x)$, of $f$ is nondegenerate on the normal space $N_x$ to $M$ at $x$.

Under the conditions above, the integral $J(\omega) = \int_{R^d} e^{i\omega f(x)a(x)} dx$ has the following asymptotic expansion as $\omega \to +\infty$, modulo $O(\omega^{-\infty})$,

$$J(\omega) \equiv \left(\frac{2\pi}{\omega}\right)^{\frac{d+k}{2}} \sum_{j \geq 0} c_j \omega^{-j}.$$  

The coefficient $c_0$ is given by

$$c_0 = e^{i\omega f(m_0)} \int_M \left[\det \left(\frac{f''(m)||N_m|}{i}\right)\right]^{-1/2} a(m) dV_M(m),$$

where $dV_M(m)$ is the canonical Euclidean volume in $M$, $m_0 \in M$ is arbitrary, and $[\det P]_*^{-1/2}$ denotes the product of the reciprocals of square roots of the eigenvalues of $P$ chosen with positive real parts. Note that, since $\Im f \geq 0$, the eigenvalues of $\frac{f''(m)||N_m|}{i}$ lie in the closed right half plane.

**Sketch of proof**: Using a partition of unity, we can assume that $O$ is small enough that we have normal, geodesic coordinates in a neighborhood of $M$. So we have a diffeomorphism

$$\chi : U \to O,$$
where \( \mathcal{U} \) is an open neighborhood of \((0, 0)\) in \( IR^k \times IR^{d-k} \), such that
\[
\chi(x', x'') \in M \iff x'' = 0
\]
and if \( m = \chi(x', 0) \in M \) we have
\[
\chi'(x', 0)(R^k_{x''}) = T_m M
\]
\[
\chi'(x', 0)(R^{d-k}_{x''}) = N_m M, \text{ normal space at } m \in M.
\]
So the change of variables \( x = \chi(x', x'') \) gives the integral
\[
J(\omega) = \int_{IR^d} e^{i\omega f(\chi(x', x''))} a(x', x'') |\det \chi'(x', x'')| dx' dx''.
\] (43)
The phase
\[
\hat{f}(x', x'') := f(\chi(x', x''))
\]
clearly satisfies
\[
\{\hat{f}'_{x''}(x', x'') = 0, \Im \hat{f}(x', x'') = 0\} \iff x'' = 0. \tag{44}
\]
Hence, we can apply the stationary phase Theorem 7.7.5 of [22], (Vol. 1), in the variable \( x'' \), to the integral (43), where \( x' \) is a parameter (the assumptions of [22] are satisfied, uniformly for \( x' \) close to 0). We remark that all the coefficients \( c_j \) of the expansion can be computed using the above local coordinates and Theorem 7.7.5.

4 The stationary Phase Computation

In this section we compute the stationary phase expansion of (39) with phase \( \Phi_E \) given by (40). Note that \( a(t, \alpha, y, \hbar) \) is actually, according to (32), a polynomial in \( \hbar^{1/2} \) and \( \hbar^{-1/2} \). Hence the stationary phase theorem (with \( \hbar \) independent symbol \( a \)) applies to each coefficient of this polynomial.

The first order derivatives of \( \Phi_E(t, y, \alpha) \) (up to \( O((y-q)^2, (\alpha-\alpha_t)^2) \) terms) are given by
\[
\begin{align*}
\frac{\partial_t \Phi_E}{} &= E - H(\alpha) + (y - q_t) \cdot \dot{p}_{t} - \dot{q}_t \cdot M(y - q_t) \\
\frac{\partial_y \Phi_E}{} &= p_t - p + i(y - q) + M(y - q_t) \\
\frac{\partial_q \Phi_E}{} &= i(q - q_t) - tA(p - p_t) + (t C - t AM - iI)(y - q_t) \\
\frac{\partial_p \Phi_E}{} &= q - q_t + (t D - t BM - I)(y - q_t).
\end{align*}
\]
Furthermore, since \( F \) is symplectic, one has
\[
2 \Im \Phi_E = |y - q|^2 + |(A + iB)^{-1}(y - q_t)|^2.
\]

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This implies that $\Phi_E(y, \alpha, t)$ is critical on the set:

$$C_E = \{(y, \alpha, t) \in \mathbb{R}^n_y \times \mathbb{R}^{2n}_\alpha \times \mathbb{R}_t : y = q_t, \alpha_t = \alpha, H(\alpha) = E\}.$$  

Thus each component $\mathcal{M}_\gamma$ of $C_E$ has the form

$$\mathcal{M}_\gamma = \{(y, \alpha, t) = (q, \alpha, T(\alpha)) : \alpha = (p, q) \in \gamma, \alpha_{T(\alpha)} = \alpha, H(\alpha) = E\}. \quad (45)$$

We will assume that each $\gamma$ is a smooth compact manifold. One sees immediately that the manifolds $\gamma$ are unions of periodic classical trajectories of energy $E$. We will also assume a “clean intersection” hypothesis which we will state shortly. Thus we have assumed that

$$C_E = \{0\} \times \Sigma \cup \{\mathcal{M}_{\gamma_1}, \ldots, \mathcal{M}_{\gamma_N}\}, \quad (46)$$

where each $\mathcal{M}_{\gamma_k}$ has the form (45) with $\gamma_k$ in the fixed point set of the mapping $\alpha \mapsto \alpha_{T_k}$.

The first thing to check, in order to apply the stationary phase theorem is that the support of $\alpha$ in (39) can be taken as compact, up to an error $O(\bar{h}^\infty)$. We do this in the following way: let us recall some properties of $\bar{h}$-pseudodifferential calculus proved in \cite{21, 12}. The function $m(z) = \langle \hat{H}(z) \rangle$ is a weight function. In \cite{12} it is proved that $\chi(\hat{H}) = \hat{H}\chi$ where $H(\chi) \in S(m^{-k})$, for every $k$. More precisely, we have in the $\bar{h}$ asymptotic sense in $S(m^{-k})$,

$$H(\chi) = \sum_{j \geq 0} H_{\chi, j} h^j$$

and support $[H_{\chi,j}]$ is in a fixed compact set for every $j$ (see (H.5) and \cite{21} for the computations of $H_{\chi,j}$). Let us recall that the symbol space $S(m)$ is equipped with the family of semi-norms

$$\sup_{z \in \mathbb{R}^2n} m^{-1}(z) |\partial_\gamma \frac{\partial^\gamma}{\partial z^\gamma} u(z)|$$

Now we can prove the following lemma

**Lemma 4.1** There is a compact set $K$ in $\mathbb{R}^{2n}$ such that for

$$m(\alpha, t) = \langle \hat{A}_\chi \varphi_\alpha, U(t) \varphi_\alpha \rangle$$

we have

$$\int_{\mathbb{R}^{2n}/K} |m(\alpha, t)| d\alpha = O(\bar{h}^{+\infty}),$$

uniformly in every bounded interval in $t$.

**Proof:** Let $\tilde{\chi} \in C_0^\infty([E - \delta E, E + \delta E])$ such that $\tilde{\chi}\chi = \chi$. Using (H.4) and the composition rule for $h$-pseudodifferential operators we can see that $\hat{A}_\chi(\hat{H})$ is bounded on $L^2(\mathbb{R}^n)$. So there exists a $C > 0$ such that

$$|m(\alpha, t)| \leq C \| \tilde{\chi}(\hat{H}) \varphi_\alpha \|^2.$$

But we can write
|| \tilde{\chi}(\hat{H})\varphi_{\alpha} ||^2 = < \tilde{\chi}(\hat{H})^2 \varphi_{\alpha}, \varphi_{\alpha} > .

Let us introduce the Wigner function, \( w_{\alpha} \), for \( \varphi_{\alpha} \) (i.e. the Weyl symbol of the orthogonal projection on \( \varphi_{\alpha} \)). We have

\[
< \tilde{\chi}(\hat{H})^2 \varphi_{\alpha}, \varphi_{\alpha} > = (\pi \hbar)^{-n} \int H_{\chi^2}(z) w_{\alpha}(z) dz
\]

where

\[
w_{\alpha}(z) = (\pi \hbar)^{-n} e^{-\frac{|z-\alpha|^2}{\hbar}}.
\]

Using remainder estimates from [21] we have, for every \( N \) large enough,

\[
\hat{H}_{\chi^2} = \sum_{0 \leq j \leq N} H_{\chi^2,j} \hbar^j + \hbar^{N+1} R_N(\hbar)
\]

where the following estimate in Hilbert-Schmidt norm holds

\[
\sup_{0 < \hbar \leq 1} || R_N(\hbar) ||_{HS} < +\infty.
\]

Now there is an \( R > 0 \) such that for every \( j \), we have \( \text{Supp}[H_{\chi^2,j}] \subseteq \{ z, |z| < R \} \). So the proof of the lemma follows from

\[
\int_{|z| \leq R, |\alpha| \geq R+1} e^{-\frac{|z-\alpha|^2}{\hbar}} d\alpha \leq C e^{-\frac{R}{\hbar}}.
\]

The next step is the computation of the Hessian of \( \Phi_E \) on a \( \mathcal{M}_{\gamma_k} \). After an easy but tedious computation, with the variables ordered as \( (t, y, p, q) \), the Hessian \( \Phi''_E \) is the following \((1 + 3n) \times (1 + 3n)\) matrix

\[
\Phi''_E = \begin{pmatrix}
H_p \cdot (H_q + MH_p) & -H_q - H_p M & -H_p (D - MB) & -H_p (C - MA) \\
-H_q - MH_p & M + iI & D - MB - I & C - MA - iI \\
-(tD - tBM)H_p & tD - tBM - I & tBMB - tDB & tBMA - tBC \\
-(tC - tAM)H_p & tC - tAM - iI & tAMB - tCB & tAMA - tCA + iI
\end{pmatrix}
\]

(47)

where \( H_p \) (resp. \( H_q \)) denotes the vector \( \partial_p H|_{\alpha = \alpha_i} \) (resp. \( \partial_q H|_{\alpha = \alpha_i} \)), \( A, B, C, D \), are the \( n \times n \) matrices given by (36), \(^tA\) the transpose of \( A \), and \( M \) is defined by (37). (Recall \( I \) is the identity matrix).

We are going to perform elementary row and column operations on (47) to compute the nullspace of \( \Phi''_E \), and the determinant of \( \Phi''_E \) restricted to the normal space to the critical manifold. To begin with we have \( H_1 = ^tR_0 \Phi''_ER_0 \) where
we get hence, since $F$

Thus, subtracting the appropriate multiples of the third row in $H_1$ on the right by

changes it to

The key simplification comes from (37) which gives $M = (C + iD)(A + iB)^{-1}$, and hence, since $F$ is symplectic

Thus, subtracting the appropriate multiples of the third row in $H_2$ from the other rows we get

Finally using the fourth row to remove the three upper entries in the second column, multiplying the third row by $-1$, interchanging the second and fourth rows, and the third and fourth columns, we arrive at the simple form

\[ H_4 = \begin{pmatrix} 0 & 0 & -H_q & -H_p \\ 0 & -2i(A + iB)^{-1} & 0 & 0 \\ H_p & 0 & A - I & B \\ -H_q & 0 & C & D - I \end{pmatrix}. \]
and \( H_4 = R_1 \Phi''_E R_2 \) where \( R_1 \) and \( R_2 \) can be computed by repeating the elementary row and column operations that we have performed on the identity matrix, and in particular \( \det R_1 = 1 \) and \( \det R_2 = (-1)^n \).

In order to apply the stationary phase theorem the null space of \( \Phi''_E \) must be the tangent space to the critical set \( C_E \). However, one can read off the null space of \( H_4 \) from (48)

\[
\text{Null } H_4 = R_2^{-1} \text{Null } \Phi''_E = \left\{ (\tau, 0, v, w) : (F - I) \begin{pmatrix} v \\ w \end{pmatrix} + \tau \begin{pmatrix} H_p \\ -H_q \end{pmatrix} = 0 \text{ and } H_q \cdot v + H_p \cdot w = 0 \right\}.
\]

This leads us to impose the following “clean flow condition”

**Hypothesis C:** Assume that \( D_E := \{ (r, t) \in \Sigma_E \times IR / \phi_t(\alpha) = \alpha \} \) is a submanifold of \( IR^{1+2n} \). Then we say the \( D_E \) satisfies the clean flow condition, if for any \( (\alpha, t) \in D_E \), the tangent space to \( D_E \) is given by

\[
T_{\alpha,t} D_E = \left\{ (v, w, \tau) \in IR^{1+2n} : (F - I) \begin{pmatrix} v \\ w \end{pmatrix} + \tau \begin{pmatrix} H_p \\ -H_q \end{pmatrix} = 0 \text{ and } H_q \cdot v + H_p \cdot w = 0 \right\}.
\]

Since \( C_E = \{ (y, \alpha, t) : (\alpha, t) \in D_E \text{ and } y = q \} \), the tangent space \( T_{y,\alpha,t} C_E \) equals

\[
\{ (\tau, v, w, v) : (F - I) \begin{pmatrix} v \\ w \end{pmatrix} + \tau \begin{pmatrix} H_p \\ -H_q \end{pmatrix} = 0 \text{ and } H_q \cdot v + H_p \cdot w = 0 \},
\]

and, assuming Hypothesis C, this does equal the null space of \( \Phi''_E \), since

\[
R_2 = \begin{pmatrix} \tau \\ 0 \\ v \\ w \end{pmatrix} = \begin{pmatrix} \tau \\ Av + Bw + \tau H_p \\ w \\ v \end{pmatrix} = \begin{pmatrix} \tau \\ v \\ w \\ v \end{pmatrix}
\]

for \( (\tau, v, w) \) as in (49). Therefore, if \( P \) denotes the orthogonal projection on the null space of \( \Phi''_E \), then \( \det(\Phi''_E + P) \) will be the determinant of the Hessian of the phase restricted to the normal space, and setting

\[
\tilde{P} = R_1 P R_2
\]

we have \( \det(H_4 + \tilde{P}) = -(1)^n \det(\Phi''_E + P) \). Hence the computations of our paper provide a proof for the existence of a Gutzwiller trace formula under Hypothesis C. However, as stated earlier, we will only carry out the computations for the case that \( \gamma \) consists of a single trajectory here. In this case Hypothesis C reduces to the assumption (H.3) of isolated nondegenerate periodic orbits, and we may complete the computation in the following way.

To compute \( \det(H_4 + \tilde{P}) \) we will use a special basis \( B \). We denote by \( E_\lambda \) the (algebraic) eigenspace of \( F \) belonging to the eigenvalue \( \lambda \). Then under assumption (H.3)
\[ \dim \bigoplus_{\lambda \neq 1} E_\lambda = 2n - 2 \]
\[ \dim E_1 = 2 \]
and \( \sigma(E_\lambda, E_1) = 0 \) for \( \lambda \neq 1 \) where \( \sigma \) is the symplectic form, as in (10). Let \((z_1, z_2)\) be a basis for \( E_1 \) with
\[ z_1 = (2H^2_p + H^2_q)^{-1/2}(H_p, -H_q), \]
and \((F - I)z_2 = \beta z_1\). Let \( m_1, \cdots m_{2n-2} \) be a (real) basis for the span of \( \bigoplus_{\lambda \neq 1} E_\lambda \), and let \( e_0, \cdots e_n \) be the Euclidean basis for \( \mathbb{R}^{n+1} \). Then we take \( B \) to be the basis
\[ \{(e_0, 0) \cdots (e_n, 0)\} \cup \{(0, m_1) \cdots (0, m_{2n-2})\} \cup \{(0, z_1), (0, z_2)\}. \]

Since the vector \( \tilde{P}(0, z_1) \) spans the range of \( \tilde{P} \) and \( H_4(0, z_1) = 0 \), we can use column operations to remove the contribution of \( \tilde{P} \) from all columns of the matrix \( H_4 + \tilde{P} \) with respect to \( B \), except the one corresponding to \( z_1 \). Then we can use column operations to remove all entries in the \( z_1 \)- and \( z_2 \)-columns corresponding to the basis vectors \((e_1, 0) \cdots (e_n, 0)\), and \((0, m_1) \cdots (0, m_{2n-2})\). Note that this does not change the entries in the first row of the matrix, since \( \sigma(z_1, m_j) = 0, j = 1, \ldots 2n - 2 \). After these simplifications which do not change the determinant, the matrix of \( H_4 + \tilde{P} \) with respect to \( B \) becomes:
\[
\begin{pmatrix}
  0 & 0 & 0 & b \\
  0 & -2i(A + iB)^{-1} & 0 & 0 \\
  0 & 0 & P_\gamma - I & 0 \\
  a & 0 & 0 & \Omega
\end{pmatrix}
\]

The vector \( a \) is just \( ((2H^2_p + H^2_q)^{1/2}, 0, \cdots 0) \) and
\[ b = \left(x, -(2H^2_p + H^2_q)^{1/2}\sigma(z_1, z_2)\right). \]

Therefore the determinant of \( H_4 + \tilde{P} \) equals
\[
(-i)^n \left[ \det \frac{A + iB}{2} \right]^{-1} \det(P_\gamma - I) \det \tilde{\Omega},
\]
where
\[
\tilde{\Omega} = \begin{pmatrix}
  0 & b \\
  a & \Omega
\end{pmatrix} = \begin{pmatrix}
  0 & x & -(2H^2_p + H^2_q)^{1/2}\sigma(z_1, z_2) \\
  (2H^2_p + H^2_q)^{1/2} & x & x \\
  0 & c & 0
\end{pmatrix}.
\]

Here \( x \) is used for entries that do not enter the calculation, and \( c \) is the component of \( \tilde{P}(0, z_1) \) along the basis vector \( z_2 \).
To compute $c$ and finish the computation of the determinant, we first compute $\tilde{P}(0, z_1)$. Writing $z_1 = (v, w)$, we have

$$\tilde{P} \begin{pmatrix} 0 \\ 0 \\ v \\ w \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x & x \\ x & -w + Bv - Aw \\ -w + Bv - Aw & 3v - Cw + Dv \end{pmatrix}. \tag{55}$$

We let $\tilde{P}_1 z_1$ denote the last $2n$ components of $\tilde{P}(0, z_1)$. Since $^tFJz_1 = Jz_1$, the normalization in the definition of $z_1$ gives, $\sigma(z_1, \tilde{P}_1 z_1) = 1$. Therefore, if $\tilde{P}_1 z_1 = cz_2 + dz_1$ we clearly have $c = \sigma(z_1, z_2)^{-1}$. Thus (54) yields

$$\det \tilde{\Omega} = - (2H_p^2 + H_q^2) \tag{56}$$

and, combining this with (53) and (56), we have

$$\det \Phi''_{E|N(M,\gamma)} = (-1)^{n-1}(-i)^n \det \left( \frac{U}{2} \right)^{-1} |(0, H_p, -H_q, H_p)|^2 \cdot \det(P_\gamma - I). \tag{57}$$

Using (42) and (57), we conclude

$$d_0 =$$

$$\tilde{g}(T_\gamma) e^{iS_n/\hbar} \int_0^{T_\gamma} \left[ \frac{(-1)^{1-n} |(0, \dot{q}_s, \dot{p}_s, \dot{q}_s)|^2 \det(P_\gamma - I)}{\det \left( \frac{U}{2} \right)} \right]^{-1/2} \left( \det \frac{U}{2} \right)^{-1/2} c A(\alpha_s) dV(s).$$

Using $|(0, \dot{q}_s, \dot{p}_s, \dot{q}_s)|^{-1} dV(s) = ds$ we get the result for $d_0$ in (13). Since $\det(P_\gamma - I) = (-1)^{\sigma'} |\det(P_\gamma - I)|$, where $\sigma'$ is the number of real eigenvalues of $P_\gamma$ which are greater than 1, we see that

$$\left[ \frac{(-1)^{1-n} \det(P_\gamma - I)}{\det \left( \frac{U}{2} \right)} \right]^{-1/2} \left( \det \frac{U}{2} \right)^{-1/2} c = \pm i^{n-1+\sigma'} |\det(P_\gamma - I)|^{-1/2}. \tag{58}$$

Note that the role of the Maslov index in (13) is to determine the sign in (58) and $\sigma_\gamma$ in (13) is either $n - 1 + \sigma'$ or $n + 1 + \sigma'$.

The other coefficients, $d_j$ are spectral invariants which have been studied by Guillemin and Zelditch. In principle we can compute them using this explicit approach. This completes the proof of Theorem 2.3.

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