Damping identification in multiple degree-of-freedom systems using an energy balance approach

Marco Prandina, John E Mottershead and Elvio Bonisoli

1 Department of Engineering, University of Liverpool, Brownlow Hill, Liverpool L69 3GH, United Kingdom
2 Department of Production Systems, Politecnico di Torino, Corso Duca degli Abruzzi, 24 – 10129, Torino, Italy
j.e.mottershead@liv.ac.uk

Abstract. The dissipation of energy in a vibrating structure is often neglected or oversimplified in engineering design. However, there are many cases where an accurate model of damping is necessary and an effective identification is required in order to predict instabilities or unwanted vibrations of the system. The paper presents a method which balances the energy input by a sinusoidal force with the energy dissipated by damping mechanisms in order to locate and identify damping. If a complete set of measurements is available, there is no need to know or to evaluate the mass and stiffness matrices and the only data needed are the time histories of the response of the structure excited with a set of single-frequency harmonic excitations. The method has been validated by numerical simulations and tested on a real structure, a clamped aluminium beam with a magnetic eddy current dashpot.

1. Introduction
Damping in multiple degree-of-freedom systems is often modeled as linear viscous damping, i.e. forces dependent on instantaneous velocities only. This is an approximation that may not be very representative of reality, since there are many mechanisms that remove energy from a mechanical system (material damping, friction, gas pumping at interfaces, energy radiation, etc.) and often they are nonlinear. However, if the damping is light the dynamical behavior is principally determined by the relatively large elastic or inertial forces so this approach is usually chosen for its mathematical convenience and can represent a valid approximation in many cases.

The literature on the subject includes several different methods to identify the linear viscous damping matrix in multiple degree-of-freedom systems. Some of these methods have been numerically compared by Srikantha Phani and Woodhouse [1] to bring out the merits and drawbacks of each method when dealing with some typical problems in damping identification such as modal incompleteness, noise, spatial truncation, modal overlap and computing cost. The results show that one of the drawbacks of many methods is the poor performance when spatial or modal incompleteness of the input occurs in the identification process. The authors’ recent review paper [2] compares the philosophy and performances of the main strategies in the identification of linear viscous damping, namely the perturbation method [3, 4], Lancaster’s formula [5, 6], the imaginary part of the inverse receptance matrix [7, 8] and the energy method [9, 10]. In [2] is shown that the last two strategies are in principle identical when dealing with linear viscous damping only and they perform better than the others when modal incompleteness is present in the measurements.
For these reasons, the method described in this paper is based on the energy strategy proposed by Liang [10] with some variations that allow the identification of damping without the need to estimate the mass and the stiffness matrices. However, the two matrices could be necessary in the case of spatial incompleteness in order to expand the measurements to all degrees of freedom.

2. Theory
The equations of motion of a \( n \) degree-of-freedom system can be written in the matrix form

\[
M\ddot{x} + Kx + D \cdot f(x, \dot{x}, \ddot{x}) = g(t)
\] (1)

Where \( M \in \mathbb{R}^{n \times n} \) is the mass matrix, \( K \in \mathbb{R}^{n \times n} \) is the stiffness matrix, \( M = M^T, K = K^T, D \in \mathbb{R}^{n \times n} \) represents one of the possible damping matrices multiplied by \( f(x, \dot{x}, \ddot{x}) \in \mathbb{R}^{n \times 1} \), a function of displacements, velocities or accelerations. \( g(t) \in \mathbb{R}^{n \times 1} \) is the excitation input vector. Premultiplying eq.(1) by \( \dot{x}^T \) and then integrating over time, an energy equation is obtained:

\[
\int_t^{t+T} \dot{x}^T M \ddot{x} dt + \int_t^{t+T} \dot{x}^T K \dot{x} dt + \int_t^{t+T} \dot{x}^T D \cdot f(x, \dot{x}, \ddot{x}) dt = \int_t^{t+T} \dot{x}^T g(t) dt.
\] (2)

In the case where the excitation force \( g(t) \) and the response \( x \) are periodic, the integration of all conservative components of eq.(2) is zero over a full cycle of periodic motion. If \( T \) is the period of the response, the sum of kinetic and potential energies over \( T \) is zero

\[
\int_t^{t+T} \dot{x}^T M \ddot{x} dt + \int_t^{t+T} \dot{x}^T K \dot{x} dt = 0
\] (3)

so eq.(2) becomes

\[
\int_t^{t+T} \dot{x}^T D \cdot f(x, \dot{x}, \ddot{x}) dt = \int_t^{t+T} \dot{x}^T g(t) dt.
\] (4)

Eq.(4) is the basis of the identification method proposed. It represents the balance between the energy dissipated by the damping mechanisms on the left hand side of the equation and the energy input to the system on the right hand side. The unknowns are represented by the coefficient matrix \( D \) which can be parameterized in several different ways, depending on how the damping is modeled and the number of unknowns can vary substantially depending on the parameterization chosen. Each test provides one equation and by exciting the structure with a certain number of different forces and measuring the responses, it is possible to obtain multiple versions of eq.(4) so that the number of equations is equal or bigger than the number of unknown parameters in \( D \). The identification may then be carried out by solving an overdetermined inverse problem (using, for example, least squares techniques) and calculating the coefficients in \( D \).

An important aspect of this method is that many of the most common damping models can be expressed in the form \( D \cdot f(x, \dot{x}, \ddot{x}) \), such as the viscous damping or the Coulomb friction.

3. Diagonal viscous damping matrix
For a better understanding of how the method works, the simplest case of a system with diagonal viscous damping matrix is considered. In this case, the matrix \( D \) and the function \( f(x, \dot{x}, \ddot{x}) \) in eq.(4) can be written in the form:

\[
D = \text{diag}(c_i) \in \mathbb{R}^{n \times n}, f(x, \dot{x}, \ddot{x}) = \dot{x} \in \mathbb{R}^{n \times 1}
\] (5)
where $D$ is a constant $n \times n$ matrix, with positive damping coefficients $c_{i,i}$ in each $i^{th}$ degree of freedom of the diagonal and zero elsewhere, and $f(x, \dot{x}, \ddot{x})$ is the vector of instantaneous velocities $\dot{x}$ only. Eq.(4) becomes

$$\int_{t}^{t+T} \dot{x}^T \text{diag}(c_{i,i}) \dot{x} \, dt = \int_{t}^{t+T} \dot{x}^T g(t) \, dt$$  \hspace{1cm} (6)

or

$$c_{1,1} \int_{t}^{t+T} \dot{x}_1^2 \, dt + \cdots + c_{i,i} \int_{t}^{t+T} \dot{x}_i^2 \, dt + \cdots + c_{n,n} \int_{t}^{t+T} \dot{x}_n^2 \, dt = E_a$$  \hspace{1cm} (7)

By exciting the structure with $m$ different excitations, multiple versions of eq.(7) are obtained and then grouped into a single matrix equation in the form:

$$Ac = e$$  \hspace{1cm} (8)

where

$$A = \begin{bmatrix} \int_{t}^{t+T} \dot{x}_1^2 \, dt & & & \int_{t}^{t+T} \dot{x}_n^2 \, dt \\ \int_{t}^{t+T} \dot{x}_2^2 \, dt & \cdots & \int_{t}^{t+T} \dot{x}_n^2 \, dt \\ \vdots & \cdots & \cdots & \vdots \\ \int_{t}^{t+T} \dot{x}_1^2 \, dt & \cdots & \int_{t}^{t+T} \dot{x}_n^2 \, dt \end{bmatrix} \in \mathbb{R}^{m \times n}, \quad c = \begin{bmatrix} c_{1,1} \\ \vdots \\ c_{n,n} \end{bmatrix} \in \mathbb{R}^{n \times 1}, \quad e = \begin{bmatrix} E_{a(1)} \\ \vdots \\ E_{a(m)} \end{bmatrix} \in \mathbb{R}^{m \times 1}. \hspace{1cm} (9)$$

If $m \geq n$, eq.(8) can be solved to obtain the individual entries of the $D$ matrix by

$$c = (A^T A)^{-1} A^T e$$  \hspace{1cm} (10)

Using the pseudo-inverse of $A$ in an overdetermined system of equations, the solution $c$ is the best solution in a least squares sense. Eq.(8) of the diagonal viscous damping matrix example can also be solved forcing the non-negative definiteness of the $D$ matrix by using Non-Negative Least-Squares (NNLS) algorithms, giving more physical meaning to the results. If the measurement of velocities $\dot{x}_1, \ldots, \dot{x}_n$ is available and if it is possible to excite the structure with a sufficient number of different excitations $m$, the damping matrix $D$ can be easily estimated.

In order to obtain the sufficient number $m$, different force configurations are necessary to obtain the full range of responses at a chosen frequency. Different frequencies are also necessary because the various damping terms become dominant when different modes are excited whereas varying the amplitude of the force is particularly important when trying to identify nonlinearities.

Unfortunately in real structures it is practically impossible to obtain the full vector of velocities since it is not feasible to measure the infinitely many degrees of freedom of a large structure. For this reason, in the case of spatial incompleteness, i.e. a limited number of degrees of freedom measured, a method to expand the measurements of few degrees of freedom to the whole structure is necessary. In the numerical simulations, the method described in [11] is used, based on the mode shapes of the undamped finite element model of the system. This method needs the knowledge of the stiffness and mass matrix so one of the main advantages of the identification method proposed is nullified. However, using engineering knowledge, the degrees of freedom where damping is most likely to be located can be chosen in advance and then they can be only used to identify damping.

Even if in simple cases it would be possible to measure all the degrees of freedom, the number of tests to have $m > n$ could be probably too big and expensive. In the case where $m < n$, solving eq.(8) becomes an underdetermined problem and a minimum-angle criterion is used to select the most representative degrees of freedom to locate and identify damping.
4. Location of damping using a minimum angle criterion

Once the matrix $\mathbf{A}$ of eq.(8) is obtained, either from an expansion of the available measurements or populated with selected degrees of freedom only, and $m < n$ there is still the problem of solving an underdetermined problem. There are many ways of making the problem overdetermined: reducing the number of unknowns by changing the parameterization of the damping matrix, producing extra measurements using different force configurations or reducing the size of $\mathbf{A}$ by selecting the most representative columns using a criterion that allows the location of main damping sources. The minimum-angle criterion gave good results in many simulations. This angle-based method was used in a slightly different way in a model updating example in [12]. It is known from the theory of this identification method that the vector $\mathbf{e}$ is a linear combination of some columns of the matrix $\mathbf{A}$. These columns $\mathbf{a}_i$ correspond to the degrees of freedom where the damping matrix has non-zero elements whereas the columns corresponding to the degrees of freedom where zeros are present does not give any contribution to the total energy dissipated. An angle $\theta$ between two vectors $\mathbf{a}_i$ and $\mathbf{e}$ can be calculated using the relationship

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}_i^T \mathbf{e} \sqrt{\mathbf{a}_i^T \mathbf{a}_i \cdot \mathbf{e}^T \mathbf{e}}}{\sqrt{\mathbf{a}_i^T \mathbf{a}_i} \cdot \sqrt{\mathbf{e}^T \mathbf{e}}}ight)$$

(11)

Among all the columns of matrix $\mathbf{A}$, a certain number $s$ of columns which have the smallest angle with the energy vector $\mathbf{e}$ is selected. These columns are the most parallel to the vector $\mathbf{e}$ so they are likely to be the best representative of the vector in a one-dimensional approximation. After selecting these $s$ columns, the concept of angle between subspaces is introduced [13]. The angle between two matrices (in this case a matrix containing some columns of $\mathbf{A}$ and a matrix containing the $\mathbf{e}$ vector only) can be calculated with the method now described: consider the two matrices $\mathbf{A}' \in \mathbb{R}^{n \times s}$ and $\mathbf{e} \in \mathbb{R}^{n \times 1}$, an orthogonal basis for these subspaces can be obtained by the QR algorithm

$$\mathbf{A}' = \mathbf{Q}_A \mathbf{R}_A, \mathbf{e} = \mathbf{Q}_e \mathbf{R}_e$$

(12)

where $\mathbf{Q}_A$ and $\mathbf{Q}_e$ are orthogonal matrices of dimension $n \times s$ and $n \times 1$ respectively and $\mathbf{R}_A$ and $\mathbf{R}_e$ are upper triangular. If $q < m$ there will be $q$ principal angles between the subspaces, $\theta_i$, which are computed from the singular value decomposition of $\mathbf{Q}_A^T \mathbf{Q}_e$. Thus

$$\cos(\theta_i) = \sigma_i(\mathbf{Q}_A^T \mathbf{Q}_e)$$

(13)

where $\sigma_i(\cdot)$ indicates the $i^{th}$ singular value. Again, it is now possible to select a number $s$ of subsets of two columns of $\mathbf{A}$ that have the smallest angle with $\mathbf{e}$. These subsets will be created using combinations of one of the previously selected vectors with the other columns of $\mathbf{A}$. This method can then be applied for bigger subsets of three columns (keeping the two selected before) and so on. When the angle between these subsets and $\mathbf{e}$ is sufficiently small, the selection can be stopped and only the selected columns will be used to solve eq.(8). In this way, matrix $\mathbf{A}$ becomes of size $n \times s$ and it can be solved with least-square techniques provided $s < m$. Other constraints can be considered in the method to obtain results with more physical meaning like, for example, the non-negative definiteness of the viscous damping matrix. The minimum-angle selection corresponds to the spatial location of damping in a real structure. By choosing the columns of $\mathbf{A}$ the method allows the location of the main damping sources in the system. However, the angle-selection itself does not guarantee the selection of the correct degrees of freedom but just one of the possible locations from an energy point of view.

5. Numerical simulation

The results obtained from a simple example on a clamped ten-element cantilever beam with four viscous dashpots (Figure 1) are shown.
5.1. Case 1
The simulated structure dimensions are \(4 \times 40 \times 560\) mm, the material is aluminium and the four dashpots are located at degrees of freedom number 3, 5, 13 and 17 with values respectively equal to 0.01, 0.5, 0.1 and 1 Ns/m. The cantilever beam is excited with sinusoidal forces at frequencies close to those of the first eight natural modes. The signals from three accelerometers, contaminated with random noise and applied on degrees of freedom 7, 11 and 19 are measured. The full vector of velocities is then derived using the expansion described in [11] and the location and identification of damping is obtained by solving the energy equation using Non-Negative Least-Squares (NNLS) algorithm after selecting the columns of matrix \(A\) using the minimum-angle criterion.

![Figure 1 – Dashpots configuration and accelerometers location](image)

In this example the viscous damping matrix is a \(20 \times 20\) matrix with four positive values on the diagonal corresponding to the dashpots. The actual number of unknowns of the identification problem is 400, but since the dashpots are connected to the ground and there are no sources of damping between two elements, the number of unknowns can be reduced to 20 (the diagonal). The matrix \(A\) described in the previous section will be size \(8 \times 20\), where each row of the matrix corresponds to one of the eight different excitations. This example was chosen in order to simulate a case where the number of excitations is smaller than the number of degrees of freedom of the structure and the measurements were taken in the wrong locations. The identified damping coefficients are summarized in Table 1. \(N\) represents the number of columns of matrix \(A\) selected to identify the damping, \(\theta\) is the minimum angle between the selected columns and the vector \(e\) of input energies.

| \(N\) | DOF of identified dashpots | Identified damping coefficients (Ns/m) | \(\theta\) |
|------|---------------------------|-------------------------------------|------|
| 1    | - - - 17 - - - - -        | 1.084 12.557                        |      |
| 2    | - - 5 - 17 - - -          | 0.581 - 1.042 1.029                 |      |
| 3    | - - 5 13 17 - - -         | 0.506 0.124 0.989 0.263             |      |
| 4    | 3 5 13 17 - - -           | 0.01 0.501 0.099 1.002 0.001        |      |
| Exact| 3 5 13 17 - - -           | 0.01 0.5 0.1 1 0                    |      |

These results, considering the presence of noise and the small amount of data measured, are reasonably good. Especially the location of the main sources of damping is always correct and it can be noticed that the sources of damping are selected from the biggest to the smallest when increasing the number of dashpots to identify. This is an important aspect since even if the number of columns selected is smaller than the real number of dashpots of the structure the method can still identify the most representative dashpots and can be considered the most reasonable approximation at that level.
5.2. Case 2
In the second case the only difference is the value of the four damping coefficients, all set to 0.1 Ns/m. The results are shown in Table 2.

### Table 2 – Located and identified damping coefficients (case 2)

| N | DOF of identified dashpots | Identified damping coefficients (Ns/m) | θ  |
|---|-----------------|---------------------------------|---|
| 1 | - - 19 | - - - | 0.107 6.505 |
| 2 | - - 13 | - - | 0.151 0.059 0.404 |
| 3 | - 5 15 | 17 - | 0.212 0.127 0.055 0.124 |
| 4 | 3 5 13 | 17 - | 0.101 0.098 0.099 0.1 0.001 |
| Exact | 3 5 13 | 17 | 0.1 0.1 0.1 0.1 0 |

In this case, the location of the dashpots is not always correct. However, considering for example the case with N = 3, the degree of freedom 15 is close to the correct one 13 and the value of damping coefficients are similar considering the different locations. In particular, the damping coefficient of the dashpot in 5 is almost the sum of the dashpots in 3 and 5 of the original system. Considering the fact that we are using three dashpots to represent a four-dashpot system and it was assumed absolutely no knowledge about the location and values of the dashpots, these are still reasonably good results. Even the wrong solutions still hold some useful information. Moreover the identified system can be considered as an energy equivalent approximation of the original system. This can be noticed by comparing the damping ratio $\zeta$ of the first 10 modes of the identified system with the original system (Table 3).

### Table 3 – Damping ratios of the first 10 modes (case 2)

| Mode | Correct $\zeta$ | $\zeta (N = 1)$ | $\zeta (N = 2)$ | $\zeta (N = 3)$ |
|------|----------------|----------------|----------------|----------------|
| 1    | 0.014092 | 0.013534 | 0.014096 | 0.014092 |
| 2    | 0.001496 | 0.002160 | 0.001495 | 0.001496 |
| 3    | 0.001024 | 0.000772 | 0.000894 | 0.001035 |
| 4    | 0.000338 | 0.000395 | 0.000305 | 0.000341 |
| 5    | 0.000138 | 0.000240 | 0.000149 | 0.000134 |
| 6    | 0.000190 | 0.000162 | 0.000193 | 0.000181 |
| 7    | 0.000114 | 0.000117 | 0.000118 | 0.000110 |
| 8    | 0.000057 | 0.000089 | 0.000049 | 0.000048 |
| 9    | 0.000106 | 0.000068 | 0.000073 | 0.000105 |
| 10   | 0.000085 | 0.000044 | 0.000061 | 0.000087 |

6. Experimental results
The experimental setup consists of a clamped aluminium beam (Figure 2a) with the same dimensions of the numerical simulation. A small wing is attached to the side of the beam and an eddy current dashpot (Figure 2b) consisting of two rare-earths NdFeB magnets placed above and below the wing which generates the magnetic field that creates damping. This kind of damping is a close approximation to classical viscous damping [14, 15]. There is no contact between surfaces so that friction is avoided. The damping can be varied by varying the air gap between the magnets and the wing; a gap of 2 mm is chosen and a preliminary experiment on a trolley coming down from a slope identified a viscous damping coefficient equal to 1.515 Ns/m. In the first part of the experiment, the structure without magnetic dashpot is excited with a set of 16 different excitations with frequencies close to those of the first 8 modes. The complete set of accelerations in 10 equally spaced degrees of freedom through the length of the beam is measured and an energy-equivalent viscous damping matrix is identified as the offset structural damping. Then, the experiment is repeated with the damping
device attached on degree of freedom 9. The purpose of the exercise is to show that the method described above is capable of correctly locating and identifying this damping value.

Since there is no spatial incompleteness, no expansion is needed so the procedure consists in these five steps: measuring the accelerations and the input force, deriving the velocities from the accelerations, calculating the matrix $A$ of integrals and the vector $e$ of energies, subtracting from $e$ the energies dissipated from the offset structural damping to find the effective energies dissipated by the damping device only and then solving eq.(8) for $c$ using the Non-Negative Least-Squares (NNLS) solver.

![Figure 2](image)

**Figure 2** – a) experimental setup b) eddy current magnetic dashpot.

Since the number of equations is bigger than the number of unknowns there is no need to select the columns of the matrix $A$ with the minimum-angle criterion. The results of the identification are summarized in Table 4

| Damping coefficients | Expected (Ns/m) | Identified (Ns/m) |
|----------------------|----------------|------------------|
| $c_1$                | 0              | 0                |
| $c_2$                | 0              | 0                |
| $c_3$                | 0              | 0                |
| $c_4$                | 0              | 0                |
| $c_5$                | 0              | 0                |
| $c_6$                | 0              | 0                |
| $c_7$                | 0              | 0                |
| $c_8$                | 0              | 0                |
| $c_9$                | 1.515          | 1.320            |
| $c_{10}$             | 0              | 0.032            |

The main source of damping is located in the correct degree of freedom and the value is slightly smaller than the expected one with an error of about 12%. The method also identified a small source of damping at the free end of the beam that appears negligible compared to the one in degree of freedom 9. Other experiments are currently running in order to validate the method experimentally by putting more dashpots in different locations. Some Coulomb friction devices are ready to be attached to the structure to check the performances with nonlinear sources of damping too.
7. Conclusions
An energy-based method to identify damping in multi-degree-of-freedom structures is presented and a minimum-angle criterion is adopted to locate the main sources of damping. The method is intended to be used practically in real structures and to be able to identify both linear and nonlinear damping in the form $D \cdot f(x, \dot{x}, \ddot{x})$. There is no need to estimate mass and stiffness matrices a priori, unless a complete set of measurements is not available. The method is validated by a numerical simulation and gives good results in the first experimental tests. Further tests are currently running to confirm the good performance of the method for both linear and nonlinear damping identification.

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References
[1] Srikantha Phani, A. and Woodhouse, J., Viscous damping identification in linear vibration, Journal of Sound and Vibration, vol. 303, pp. 475-500, 2007.
[2] Prandina, M., Mottershead, J. E., and Bonisoli, E., An assessment of damping identification methods, Journal of Sound and Vibration (submitted), 2008.
[3] Lees, A. W., Use of perturbation analysis for complex modes, Proceedings of the 17th International Modal Analysis Conference, pp. 779-784, 1999.
[4] Adhikari, S. and Woodhouse, J., Identification of damping: Part 1, viscous damping, Journal of Sound and Vibration, vol. 243, pp. 43-61, 2001.
[5] Lancaster, P., Expressions for Damping Matrices in Linear Vibration Problems, Journal of the Aerospace Sciences, vol. 28, pp. 256-256, 1961.
[6] Pilkey, D. F., Park, G., and Inman, D. J., Damping matrix identification and experimental verification, Proceedings of the SPIE’s 6th Annual International Symposium on Smart Structures and Materials, vol. 3672, pp. 350–357, 1999.
[7] Chen, S. Y., Ju, M. S., and Tsuei, Y. G., Estimation of mass, stiffness and damping matrices from frequency response functions, Journal of Vibration and Acoustics-Transactions of the Asme, vol. 118, pp. 78-82, 1996.
[8] Lee, J. H. and Kim, J., Development and validation of a new experimental method to identify damping matrices of a dynamic system, Journal of Sound and Vibration, vol. 246, pp. 505-524, 2001.
[9] Liang, J. W. and Feeny, B. F., Balancing energy to estimate damping parameters in forced oscillators, Journal of Sound and Vibration, vol. 295, pp. 988-998, 2006.
[10] Liang, J. W., Damping estimation via energy-dissipation method, Journal of Sound and Vibration, vol. 307, pp. 349-364, 2007.
[11] Prandina, M., Mottershead, J. E., and Bonisoli, E., Location and identification of damping parameters, in IMAC XXVII Conference and Exposition on Structural Dynamics (accepted), Orlando, Florida, USA 2008.
[12] Friswell, M., Mottershead, J. E., and Ahmadian, H., Combining subset selection and parameter constraints in model updating, Transactions of the ASME, vol. 120, pp. 854-859, 1998.
[13] Bjorck, A. and Golub, G. H., Numerical Methods for Computing Angles between Linear Subspaces, Mathematics of Computation, vol. 27, pp. 579-594, 1973.
[14] Nagaya, K., Kojima, H., Karube, Y., and Kibayashi, H., Braking Forces and Damping Coefficients of Eddy-Current Brakes Consisting of Cylindrical Magnets and Plate Conductors of Arbitrary Shape, Ieee Transactions on Magnetics, vol. 20, pp. 2136-2145, 1984.
[15] Bonisoli, E. and Vigliani, A., Passive effects of rare-earth permanent magnets on flexible conductive structures, Mechanics Research Communications, vol. 33, pp. 302-319, 2006.