FOLIATIONS OF ASYMPTOTICALLY FLAT 3-MANIFOLDS BY STABLE CONSTANT MEAN CURVATURE SPHERES

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Abstract. Let $(M, g)$ be an asymptotically flat Riemannian 3-manifold. We provide a short new proof based on Lyapunov-Schmidt reduction of the existence of an asymptotic foliation of $(M, g)$ by constant mean curvature spheres. In the case where the scalar curvature of $(M, g)$ is non-negative, we prove that the leaves of this foliation are the only large stable constant mean curvature spheres that enclose the center of $(M, g)$. This had been shown previously under more restrictive assumptions and using a different method by S. Ma. We also include a new proof of the fact that the geometric center of mass of the foliation agrees with the Hamiltonian center of mass of $(M, g)$.

1. Introduction

Let $(M, g)$ be a complete asymptotically flat Riemannian 3-manifold. We refer to Appendix A for definitions and conventions related to such manifolds that we use in this paper.

Asymptotically flat manifolds arise as initial data for the Einstein field equations modeling isolated gravitational systems. They have been studied extensively in the context of mathematical relativity. It is known that the asymptotic geometry of such initial data is intricately tied to the geometry of large stable constant mean curvature spheres in $(M, g)$.

The goal of this paper is to provide short, conceptually simple new proofs of some of these results that work for general, asymptotically flat Riemannian 3-manifolds with non-negative scalar curvature. To describe these results and related work, recall from [1] that the mass of $(M, g)$ is given by

$$m = \lim_{\lambda \to \infty} \frac{1}{16 \pi} \sum_{i,j=1}^{3} \frac{1}{\lambda} \int_{S_{\lambda}(0)} x^i [(\partial_j g)(e_i, e_j) - (\partial_i g)(e_j, e_j)] d\bar{\mu}$$

where the computation is carried out in an asymptotically flat chart (57) of $(M, g)$. The bar indicates that a geometric quantity is computed with respect to the Euclidean background metric $\bar{g}$. Similarly, the Hamiltonian center of mass of $(M, g)$ is defined by $C = (C^1, C^2, C^3)$ where

$$C^\ell = \lim_{\lambda \to \infty} \frac{1}{16 \pi m} \frac{1}{\lambda} \int_{S_{\lambda}(0)} \left[ \sum_{i,j=1}^{3} x^i x^j [(\partial_i g)(e_i, e_j) - (\partial_j g)(e_j, e_j)] - \sum_{i=1}^{3} x^i g(e_i, e_\ell) - x^\ell g(e_i, e_i) \right] d\bar{\mu}$$

provided the limits on the right-hand side exist for $\ell = 1, 2, 3$; see [25].

A two-sided surface $\Sigma \subset M$ is said to have constant mean curvature if its mean curvature $H(\Sigma)$
equals a scalar. We survey general properties of such surfaces in Appendix B. D. Christodoulou and S.-T. Yau have shown in [11] that the quasi-local Hawking mass of a stable constant mean curvature sphere is non-negative if \((M, g)\) has non-negative scalar curvature. This observation suggests that the geometry of stable constant mean curvature spheres encodes information on the strength of the gravitational field in the domain they enclose. G. Huisken and S.-T. Yau have shown that the asymptotic region of an asymptotically flat manifold that is asymptotic to Schwarzschild (58) is foliated by stable constant mean curvature spheres; see also the work [28] of R. Ye in this context. They have also established a characterization result for the leaves of this foliation, which has been sharpened by J. Qing and G. Tian in [24]. We provide further details on these and related contributions in Appendix G.

Outline of related results. Remarkably, it turns out that many of these results also hold when \((M, g)\) is merely asymptotically flat (57). The following result in this direction has been proved by C. Nerz; see also the discussion on [21, p. 947]. For the statement, we define the area radius \(\lambda(\Sigma) > 0\) of a closed surface \(\Sigma \subset M\) by
\[
4 \pi \lambda(\Sigma)^2 = |\Sigma|
\]
and the inner radius \(\rho(\Sigma)\) of such a surface by
\[
\rho(\Sigma) = \sup \{ r > 0 : B_r \cap \Sigma = \emptyset \}.
\]
We refer to Appendix A for the definition of the sets \(B_r\).

**Theorem 1** ([23, Theorems 5.1, 5.2, and 5.3]). Let \((M, g)\) be \(C^2\)-asymptotically flat (57) with \(m \neq 0\). There exists \(H_0 > 0\) and a family
\[
\{ \Sigma(H) : H \in (0, H_0) \},
\]
where \(\Sigma(H) \subset M\) is a sphere with constant mean curvature \(H\), that forms a foliation of the complement of a compact subset of \(M\). The spheres \(\Sigma(H)\) are stable if and only if \(m > 0\).

Moreover, given \(\delta > 0\), there exists a compact set \(K \subset M\) such that every stable constant mean curvature sphere \(\Sigma \subset M\) enclosing \(K\) with
\[
\delta \lambda(\Sigma) < \rho(\Sigma)
\]
satisfies \(\Sigma = \Sigma(H)\) for some \(H \in (0, H_0)\).

For some settings, a stronger characterization of the leaves of the foliation (3) than that stated in Theorem 1 has been obtained. The following global uniqueness result has been established by S. Ma in [21].

**Theorem 2** ([21, Theorem 1.1]). Suppose that \((M, g)\) is \(C^4\)-asymptotically flat of rate \(\tau = 1\) with mass \(m \neq 0\). There exists a compact set \(K \subset M\) such that every stable constant mean curvature sphere \(\Sigma \subset M\) that encloses \(K\) belongs to the foliation (3).

**Remark 3.** A. Carlotto and R. Schoen [6] have constructed asymptotically flat manifolds with positive mass and non-negative scalar curvature that contain a Euclidean half-space. Given any number
V > 0, such manifolds contain infinitely many stable constant mean curvature spheres whose enclosed volume is equal to V. Note that these spheres are neither isoperimetric nor do they enclose the center of (M, g); see [10].

Remark 4. The assumption τ = 1 in Theorem 2 is needed for arguments in [21, §4 and §5].

In the case where (M, g) is asymptotically flat of rate τ ∈ (1/2, 1] and g satisfies additional asymptotic symmetries, L.-H. Huang has proven the following semi-global uniqueness result in [17]. We review the so-called Regge-Teitelboim conditions in Appendix A.

Theorem 5 ([17, Theorem 2]). Let τ ∈ (1/2, 1]. Suppose that (M, g) is C^5-asymptotically flat of rate τ with m > 0 and that (M, g) satisfies the C^5-Regge-Teitelboim conditions (59) of rate τ. Given s > 1 with

\[ s < \frac{4 + 2 \tau}{5 - \tau}, \]

there exists a compact set K ⊂ M such that every stable constant mean curvature sphere Σ ⊂ M that encloses K with

\[ \lambda(\Sigma) < \rho(\Sigma)^s \]

belongs to the family (3).

Remark 6. The pinching condition (6) prevents a sequence \{Σ_i\}_{i=1}^\infty of large stable constant mean curvature spheres Σ_i ⊂ M from drifting too quickly with respect to the center of (M, g); see Figure 1.
O. Chodosh, Y. Shi, H. Yu, and the first-named author have shown in [10] that the leaves of the foliation (3) are globally unique as isoperimetric surfaces provided \((M,g)\) has non-negative scalar curvature; see [29] for an alternative proof by H. Yu. More precisely, they are the unique surfaces of least area for the volume they enclose.

The foliation (3) leads to a notion of a geometric center of mass \(C\) of \((M,g)\) with components given by

\[
C_{\ell}^{CMC} = \lim_{H \to 0} |\Sigma(H)|^{-1} \int_{\Sigma(H)} x^\ell \, d\mu
\]

provided the limits on the right-hand side of (7) exist for \(\ell = 1, 2, 3\). C. Nerz has shown that this geometric center of mass agrees with the Hamiltonian center of mass of \((M,g)\) provided \(g\) satisfies certain asymptotic symmetries.

**Theorem 7** ([23, Theorem 6.3]). Suppose that \((M,g)\) is \(C^3\)-asymptotically flat with \(m \neq 0\) and that \((M,g)\) satisfies the weak \(C^2\)-Regge-Teitelboim conditions (60). The limit in (2) exists if and only if the limit in (7) exists, in which case, \(C = C_{CMC}\).

**Remark 8.** If \((M,g)\) is \(C^3\)-asymptotically flat and satisfies the \(C^2\)-Regge-Teitelboim conditions, then the Hamiltonian center of mass (2) of \((M,g)\) is well-defined; see [16, Theorem 2.2].

We also mention the important previous results of L.-H. Huang [17, 16] and of J. Corvino and H. Wu [12] on this problem.

We survey the methods used by L.-H. Huang, S. Ma, and C. Nerz in Appendix H.

**Outline of the results.** Our contributions in this paper are threefold.

First, we use the method of Lyapunov-Schmidt reduction to give a conceptually simple and relatively short proof of Theorem 1.

Second, we expand on the work in the asymptotically Schwarzschild setting of S. Brendle and the first-named author [4] and of O. Chodosh and the first-named author [8] to investigate the global uniqueness of large stable constant mean curvature spheres in \((M,g)\). In the case where the scalar curvature is non-negative, this approach enables us to extend Theorem 2 of S. Ma to all decay rates \(\tau \in (1/2, 1]\) in the following way.

**Theorem 9.** Let \((M,g)\) be \(C^2\)-asymptotically flat of rate \(\tau > 1/2\) with \(R \geq 0\) and \(m > 0\). There exists a compact set \(K \subset M\) such that every stable constant mean curvature sphere \(\Sigma \subset M\) that encloses \(K\) satisfies \(\Sigma = \Sigma(H)\) for some \(H \in (0, H_0)\).

**Remark 10.**

(i) Unlike in Theorems 1 and 5, no centering assumption on the surface \(\Sigma\) and no asymptotic symmetries on the metric \(g\) are required in Theorem 9.

(ii) If \((M,g)\) is not flat \(\mathbb{R}^3\), the assumption \(m > 0\) in Theorem 9 follows from the positive mass theorem; see [26] and [2, Theorem 6.3].

(iii) If \((M,g)\) contains no properly embedded totally geodesic flat planes along which the ambient scalar curvature vanishes, then every stable constant mean curvature sphere \(\Sigma \subset M\) with sufficiently large enclosed volume is disjoint from \(K\); see [5, Theorem 1.10].
(iv) Note that the short proof of Theorem 9 given in this paper is essentially self-contained except for the use of an estimate on the Hawking mass due to G. Huisken and T. Ilmanen [19] in the proof of Lemma 28.

In view of Remark 3, Theorem 9 completes the characterization of large stable constant mean curvature spheres in asymptotically flat Riemannian 3-manifolds with non-negative scalar curvature under general decay assumptions on the metric. In the case where the scalar curvature is allowed to change sign, we obtain the following improvement of the uniqueness results stated in Theorem 1 and Theorem 5.

**Theorem 11.** Let \((M, g)\) be \(C^2\)-asymptotically flat of rate \(\tau \in (1/2, 1)\) with \(m \neq 0\) and suppose that \(R = O(|x|^{-5/2-\tau})\) as \(x \to \infty\). Let \(s > 1\) with

\[
s < 1 + \frac{3}{4} \frac{2\tau - 1}{1 - \tau}.
\]

There exists a compact set \(K \subset M\) such that every stable constant mean curvature sphere \(\Sigma \subset M\) that encloses \(K\) and such that

\[
\lambda(\Sigma) < \rho(\Sigma)^s
\]

satisfies \(\Sigma = \Sigma(H)\) for some \(H \in (0, H_0)\).

**Remark 12.**

(i) The condition (9) is weaker than (4).

(ii) Unlike in Theorem 5, we impose no further assumptions on the asymptotic symmetries of \(g\) in Theorem 11.

(iii) The centering assumption (9) is less restrictive than (6) if and only if

\[
\tau > \frac{\sqrt{553} - 17}{12} \approx 0.55.
\]

(iv) The bound (8) seems to be the best possible for the method developed in this paper; see (48).

(v) If \(m < 0\), Theorem 11 and Theorem 1 imply that there are no large stable constant mean curvature spheres in \((M, g)\) that enclose \(K\) and satisfy (9).

Third, we also obtain a new proof of Theorem 7 in the case where \(g\) satisfies slightly stronger asymptotic symmetries. Under these assumptions, Theorem 7 was first proven by L.-H. Huang in [17].

**Theorem 13** ([17, Theorem 1]). Suppose that \((M, g)\) is \(C^3\)-asymptotically flat with \(m \neq 0\), that \((M, g)\) satisfies the \(C^2\)-Regge-Teitelboim conditions, and that \(R = o(|x|^{-3})\) as \(x \to \infty\). Then the limits in (2) and (7) exist and \(C = C_{CMC}\).

**Remark 14.** Note that the condition (60) assumed in Theorem 7 is weaker than the condition (59) assumed in Theorem 13. Analyzing the center of mass (7) under the weaker assumptions of Theorem 7 appears to be beyond our method’s reach. In particular, our stronger assumptions are required to control the error terms in the estimate (55).
Outline of our arguments. To prove Theorems 1 and 13, we expand upon the method of Lyapunov-Schmidt reduction used in [4, 8, 13].

Let $\delta \in (0, 1/2)$. We use the implicit function theorem to construct surfaces $\Sigma_{\xi, \lambda}$ as perturbations of the Euclidean coordinate spheres

$$S_\lambda(\lambda \xi) = \{ x \in \mathbb{R}^3 : |x - \lambda \xi| = \lambda \}$$

where $\xi \in \mathbb{R}^3$ with $|\xi| < 1 - \delta$ and $\lambda > 1$ is large such that $\text{vol}(\Sigma_{\xi, \lambda})$ does not depend on $\xi$ and such that $\Sigma_{\xi, \lambda}$ is a constant mean curvature sphere if and only if $\xi$ is a critical point of the function $G_\lambda$ defined by

$$G_\lambda(\xi) = \lambda^{-1}|\Sigma_{\xi, \lambda}|.$$

Using an integration by parts observed in [8], we show that

$$G_\lambda(\xi) = G_\lambda(0) + 4\pi m |\xi|^2 + G_{2, \lambda}(\xi)$$

where $G_{2, \lambda}(\xi) = o(1)$ as $\lambda \to \infty$. If $m \neq 0$, it follows that $G_\lambda$ has a unique critical point $\xi(\lambda)$ with $|\xi(\lambda)| < 1/2$ as $\lambda \to \infty$. This proves Theorem 1. To prove Theorem 13, we observe that

$$\lambda \xi(\lambda) = |\Sigma(\lambda)|^{-1} \int_{\Sigma(\lambda)} x^f \, d\mu + o(1)$$

if $(M, g)$ satisfies the $C^2$-Regge-Teitelboim conditions where $\Sigma(\lambda) = \Sigma_{\xi(\lambda), \lambda}$. Moreover, we show that $\lambda (\hat{D}G_{2, \lambda})|_{\xi(\lambda)}$ is essentially proportional to the Hamiltonian center of mass $C$ provided that $\lambda > 1$ is sufficiently large and $(M, g)$ satisfies the $C^2$-Regge-Teitelboim conditions.

The proofs of Theorem 9 and Theorem 11 are based on curvature estimates and an integration by parts that have been observed and used in a related context by O. Chodosh and the first-named author in [8].

Using an estimate proven by D. Christodoulou and S.-T. Yau in [11] together with global arguments developed by G. Huisken and T. Ilmanen in [19], we obtain an improved global $L^2$-estimate for the traceless second fundamental $\hat{h}(\Sigma)$ of a stable constant mean curvature sphere $\Sigma \subset M$. Such an estimate has also been used by O. Chodosh and the first named-author in [9]. This allows us to prove curvature estimates for $\Sigma$ that are slightly stronger than those available in the literature. In particular, these estimates improve when the scalar curvature of $(M, g)$ is non-negative. We then suppose, for a contradiction, that there exists a sequence $\{\Sigma_i\}_{i=1}^{\infty}$ of large stable constant mean curvature spheres $\Sigma_i \subset M$ that enclose the center of $(M, g)$ and do not belong to the foliation (3). In light of (10), we may assume that

$$\lim_{i \to \infty} \rho(\Sigma_i) = \infty \quad \text{and} \quad \lim_{i \to \infty} \lambda(\Sigma_i)^{-1} \rho(\Sigma_i) = 0.$$  

Since $H = H(\Sigma_i)$ is constant, we have

$$0 = \int_{\Sigma_i} H \, g(a, \nu) \, d\mu - H(\Sigma_i) \int_{\Sigma_i} g(a, \nu) \, d\mu$$

for every $a \in \mathbb{R}^3$ with $|a| = 1$. Using integration by parts, that $\Sigma_i$ encloses the center of $(M, g)$, and (11), we show that the right-hand side of (12) equals $4\pi m + E_i$ for some error term $E_i$ provided $a \in \mathbb{R}^3$ is chosen appropriately. If $R \geq 0$, or, alternatively, if $R = O(|x|^{-5/2-\tau})$ as $x \to \infty$ and
\( \lambda(\Sigma_i) = o(\rho(\Sigma_i)^s) \) as \( i \to \infty \) where \( s > 1 \) satisfies (8), the curvature estimates imply that \( E_i = o(1) \). This is incompatible with (12) so Theorem 9 and Theorem 11 follow.

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2. Proof of Theorem 1

The method of Lyapunov-Schmidt reduction has been used by S. Brendle and the first-named author in [4] and by O. Chodosh and the first-named author in [8] to study large constant mean curvature spheres that do not enclose the center of a Riemannian 3-manifold that is asymptotic to Schwarzschild (58). In [13], the authors have used the method of Lyapunov-Schmidt reduction to study so-called large area-constrained Willmore spheres in asymptotically Schwarzschild manifolds. Here, we adapt this approach to study constant mean curvature spheres that enclose the center of a general asymptotically flat 3-manifold.

In this section, we assume that \( g \) is a complete Riemannian metric on \( \mathbb{R}^3 \) with integrable scalar curvature and that there is \( \tau \in (1/2, 1] \) with

\[
(13) \quad g = \bar{g} + \sigma \quad \text{where} \quad \partial_J \sigma = O \left( |x|^{-\tau - |J|} \right)
\]

as \( x \to \infty \) for every multi-index \( J \) with \( |J| \leq 2 \).

Given \( \xi \in \mathbb{R}^3 \) and \( \lambda > 1 \), we abbreviate

\[
S_{\xi, \lambda} = S_\lambda(\lambda \xi) = \{ x \in \mathbb{R}^3 : |x - \lambda \xi| = \lambda \}.
\]

Given \( u \in C^\infty(S_{\xi, \lambda}) \), we define the map

\[
\Phi^u_{\xi, \lambda} : S_{\xi, \lambda} \to \mathbb{R}^3 \quad \text{given by} \quad \Phi^u_{\xi, \lambda}(x) = x + u(x) (\lambda^{-1} x - \xi).
\]

We denote by

\[
\Sigma_{\xi, \lambda}(u) = \Phi^u_{\xi, \lambda}(S_{\xi, \lambda})
\]

the Euclidean graph of \( u \) over \( S_{\xi, \lambda} \). We tacitly identify functions defined on \( \Sigma_{\xi, \lambda}(u) \) with functions defined on \( S_{\xi, \lambda} \) by precomposition with \( \Phi^u_{\xi, \lambda} \); see e.g. Proposition 15.

Let \( \delta \in (0, 1/2) \). In the following proposition, we use \( \Lambda_0(S_{\xi, \lambda}) \) and \( \Lambda_1(S_{\xi, \lambda}) \) to denote the constant functions and first spherical harmonics viewed as subspaces of \( C^\infty(S_{\xi, \lambda}) \), respectively. We use \( \perp \) to denote the orthogonal complements of these spaces in \( C^\infty(S_{\xi, \lambda}) \) with respect to the Euclidean \( L^2 \)-inner product. In the estimate (14), \( \bar{D} \), the dash, and \( \bar{\nabla} \) denote differentiation with respect to \( \xi \in \mathbb{R}^3 \), \( \lambda \in \mathbb{R} \), and \( x \in S_{\xi, \lambda} \), respectively.

**Proposition 15.** There are constants \( \lambda_0 > 1 \) and \( \epsilon > 0 \) depending on \((M, g)\) and \( \delta \in (0, 1/2) \) such that for every \( \xi \in \mathbb{R}^3 \) with \( |\xi| < 1 - \delta \) and \( \lambda > \lambda_0 \) there exists a function \( u_{\xi, \lambda} \in C^\infty(S_{\xi, \lambda}) \) such that the following holds. There holds \( u_{\xi, \lambda} \perp \Lambda_1(S_{\xi, \lambda}) \) and, as \( \lambda \to \infty \),

\[
|u_{\xi, \lambda}| + \lambda |\bar{\nabla} u_{\xi, \lambda}| + \lambda^2 |\bar{\nabla}^2 u_{\xi, \lambda}| = o(\lambda^{1/2}),
\]

\[
(\bar{D} u)_{(\xi, \lambda)} = o(\lambda^{1/2}),
\]

\[
\lambda \ u'_{(\xi, \lambda)} = o(\lambda^{1/2})
\]

(14)
uniformly for all $\xi \in \mathbb{R}^3$ with $|\xi| < 1 - \delta$. The surface 
\[ \Sigma_{\xi, \lambda} = \Sigma_{\xi, \lambda}(u_{\xi, \lambda}) \]
has the properties
- $H \in \Lambda_0(S_{\xi, \lambda}) \oplus \Lambda_1(S_{\xi, \lambda})$,
- \( \text{vol}(S_{2\lambda}(0)) - \text{vol}(\Sigma_{\xi, \lambda}) = \frac{28\pi}{3} \lambda^3 \).
Moreover, if $\Sigma_{\xi, \lambda}(u)$ with $u \perp \Lambda_1(S_{\xi, \lambda})$ is such that
- $H \in \Lambda_0(S_{\xi, \lambda}) \oplus \Lambda_1(S_{\xi, \lambda})$,
- \( \text{vol}(S_{2\lambda}(0)) - \text{vol}(\Sigma_{\xi, \lambda}(u)) = \frac{28\pi}{3} \lambda^3 \),
- $|u| + \lambda |\nabla u| + \lambda^2 |\nabla^2 u| < \epsilon \lambda$,
then $u = u_{\xi, \lambda}$.

**Proof.** This follows from the implicit function theorem and scaling; see e.g. [13, Proposition 17].

To capture the variational nature of the constant mean curvature equation on the families of surfaces $\{\Sigma_{\xi, \lambda} : |\xi| < 1 - \delta\}$ from Proposition 15, we consider the reduced area function 
\[ G_\lambda : \{\xi \in \mathbb{R}^3 : |\xi| < 1 - \delta\} \to \mathbb{R} \]
given by 
\[ G_\lambda(\xi) = \lambda^{-1} |\Sigma_{\xi, \lambda}|. \]

**Lemma 16.** Given $\delta \in (0, 1/2)$, there is $\lambda_0 > 1$ such that for every $\lambda > \lambda_0$ and $\xi \in \mathbb{R}^3$ with $|\xi| < 1 - \delta$ the following holds. The sphere $\Sigma_{\xi, \lambda}$ has constant mean curvature if and only if $\xi$ is a critical point of $G_\lambda$.

**Proof.** This follows as in e.g. [13, Lemma 21].

In the following two lemmas, we compute the asymptotic expansion of $G_\lambda$ as $\lambda \to \infty$.

**Lemma 17.** Let $a \in \mathbb{R}^3$ with $|a| = 1$. There holds, as $\lambda \to \infty$,
\[ (\bar{D}_a G_\lambda)|_\xi = \frac{1}{2} \int_{\Sigma_{\xi, \lambda}} \left[ \bar{D}_a \text{tr} \sigma - (\bar{D}_a \sigma)(\bar{\nu}, \bar{\nu}) - 2 \lambda^{-1} \text{tr} \bar{g}(a, \bar{\nu}) \right] d\bar{\mu} + o(1) \]
uniformly for all $\xi \in \mathbb{R}^3$ with $|\xi| < 1 - \delta$.

**Proof.** Using that $\bar{D} \text{vol}(\Sigma_{\xi, \lambda}) = 0$, we obtain
\[ (\bar{D}_a G_\lambda)|_\xi = \int_{\Sigma_{\xi, \lambda}} [H - 2 \lambda^{-1}] g(a + \lambda^{-1} (\bar{D}_a u)|_{(\xi, \lambda)} \bar{\nu}(S_{\xi, \lambda}), \nu) d\mu. \]

Using (62), (14), and Lemma 38, we obtain
\[ H(\Sigma_{\xi, \lambda}) = H(S_{\xi, \lambda}) + o(\lambda^{-3/2}) = 2 \lambda^{-1} + o(\lambda^{-3/2}). \]
In conjunction with (14) and (15), we find
\[ (\bar{D}_a G_\lambda)|_\xi = \int_{\Sigma_{\xi, \lambda}} [H - 2 \lambda^{-1}] g(a, \nu) d\mu + o(1). \]
The first variation formula implies that
\[ \int_{\Sigma_{\xi, \lambda}} [H - 2 \lambda^{-1}] g(a, \nu) d\mu = \int_{\Sigma_{\xi, \lambda}} [\text{div} a - g(D_v a, \nu) - 2 \lambda^{-1} g(a, \nu)] d\mu. \]
Using (14) and (13), we obtain that
\[ \int_{\Sigma_{\xi,\lambda}} [\text{div} a - g(D_{\nu} a, \nu) - 2 \lambda^{-1} g(a, \nu)] d\mu = \int_{S_{\xi,\lambda}} [\text{div} a - g(D_{\nu} a, \nu) - 2 \lambda^{-1} g(a, \nu)] d\mu + o(1). \]
In conjunction with (17) and (18), we conclude that
\[ (\bar{D}_a G_\lambda)|_\xi = \int_{S_{\xi,\lambda}} [\text{div} a - g(D_{\nu} a, \nu) - 2 \lambda^{-1} g(a, \nu)] d\mu + o(1). \]
The claim now follows from Lemma 38.

**Lemma 18.** Let \( \delta \in (0, 1/2) \). There holds, as \( \lambda \to \infty \),
\[ G_\lambda(\xi) = G_\lambda(0) + 4 \pi m |\xi|^2 + o(1), \]
\[ (\bar{D}G_\lambda)|_\xi = 8 \pi m \xi + o(1) \]
uniformly for all \( \xi \in \mathbb{R}^3 \) with \( |\xi| < 1 - \delta \).

**Proof.** Let \( a \in \mathbb{R}^3 \) with \( |a| = 1 \). Using Lemma 17 and Lemma 36, we have
\[ (\bar{D}_a G_\lambda)|_\xi = \frac{1}{2} \lambda^{-1} \int_{S_{\xi,\lambda}} \left[ \bar{g}(a, x - \lambda \xi) \left[ \bar{D}_\nu \text{tr} \sigma - (\text{div} \sigma)(\bar{\nu}) + \sigma(\bar{\nu}, a) - \bar{g}(a, \bar{\nu}) \text{tr} \sigma \right] \right] d\bar{\mu} + o(1). \]
Note that, by (77),
\[ \text{div} \left( \sum_{j=1}^3 \left[ \bar{D}_{e_j} \text{tr} \sigma - (\text{div} \sigma)(e_j) \right] \bar{g}(a, \lambda^{-1} x - \xi) + \lambda^{-1} [\sigma(a, e_j) - \bar{g}(a, e_j) \text{tr} \sigma] e_j \right) \]
\[ = -R \bar{g}(a, \lambda^{-1} x - \xi) + O(|x|^{-2-2\tau}). \]
Using the divergence theorem and that the scalar curvature is integrable, we find that
\[ (\bar{D}_a G_\lambda)|_\xi = \frac{1}{2} \bar{g}(a, \xi) \int_{S_{\lambda}(0)} [(\text{div} \sigma)(\bar{\nu}) - \bar{D}_\nu \text{tr} \sigma] d\bar{\mu} + (2 \lambda^{-1}) \int_{S_{\lambda}(0)} \left[ \bar{g}(a, x) \left[ \bar{D}_\nu \text{tr} \sigma - (\text{div} \sigma)(\bar{\nu}) \right] + \sigma(\bar{\nu}, a) - \bar{g}(a, \bar{\nu}) \text{tr} \sigma \right] d\bar{\mu} + o(1). \]
The assertion of the lemma follows from Lemma 35 and integration.

**Proof of Theorem 1.** Let \( \delta = 1/2 \). Depending as to whether \( m > 0 \) or \( m < 0 \), Lemma 18 implies that \( G_\lambda \) is strictly radially increasing respectively decreasing on \( \{ \xi \in \mathbb{R}^3 : |\xi| = 1/2 \} \) for every \( \lambda > 1 \) sufficiently large. In particular, \( G_\lambda \) has a critical point \( \xi(\lambda) \in \mathbb{R}^3 \) with \( |\xi(\lambda)| < 1/2 \). According to Lemma 16, \( \Sigma(\lambda) = \Sigma_{\xi(\lambda),\lambda} \) is a constant mean curvature sphere.

Using Lemma 18, we find that
\[ \xi(\lambda) = o(1) \]
as \( \lambda \to \infty \) and that
\[ |\bar{D}^2 G_\lambda| \geq 4 \pi |m| \]
for every sufficiently large $\lambda > 1$. Arguing as in the proof of Lemma 18, we find that

\begin{equation}
\tilde{D}G'_\lambda = o(\lambda^{-1})
\end{equation}

as $\lambda \to \infty$ uniformly for all $\xi \in \mathbb{R}^3$ with $|\xi| < 1/2$. Differentiating the equation $(\tilde{D}G_\lambda)_{\xi(\lambda)} = 0$ and using (19), (20), and (21), we find that

\[ \xi'(\lambda) = \left[ (\tilde{D}^2G_\lambda)_{\xi(\lambda)} \right]^{-1} (\tilde{D}G'_\lambda)_{\xi(\lambda)} = o(\lambda^{-1}). \]

Arguing as in [13, Proposition 48], we conclude that the spheres \( \{ \Sigma(\lambda) : \lambda > \lambda_0 \} \) form a foliation of the complement of a compact set provided that $\lambda_0 > 1$ is sufficiently large.

Next, recall from (16) that

\[ H(\Sigma(\lambda)) = \bar{H}(\Sigma(\lambda)) + o(\lambda^{-3/2}) = 2\lambda^{-1} + o(\lambda^{-3/2}). \]

Moreover, there holds

\[ H(\Sigma(\lambda))' = -2\lambda^{-2} + o(\lambda^{-5/2}). \]

It follows that $\lambda \mapsto H(\Sigma(\lambda))$ is strictly decreasing on $(\lambda_0, \infty)$ provided that $\lambda_0 > 1$ is sufficiently large.

Recall from (63) that $\Sigma(\lambda)$ is stable if and only if

\[ Lf \geq 0 \]

for every $f \in C^\infty(\Sigma(\lambda))$ with

\[ \int_{\Sigma(\lambda)} f \, d\mu = 0. \]

Lemma 38 and (14) imply that

\begin{equation}
\text{proj}_{\Lambda_0(S_{\xi(\lambda)}, \lambda)} f = o(\lambda^{-3/2} \| f \|_{L^2(\Sigma(\lambda))})
\end{equation}

for every such $f$. Using (61), (13), and Corollary 34, we find that

\begin{equation}
\lambda^2 Lf \geq 2f
\end{equation}

for every $f \in \Lambda_0(S_{\xi(\lambda)}, \lambda) \oplus \Lambda_1(S_{\xi(\lambda)}, \lambda)^\perp$ provided that $\lambda > 1$ is sufficiently large. Recall that $\xi(\lambda)$ is a critical point of $G_\lambda$ and that

\[ \tilde{D}^2G_\lambda \begin{cases}
\geq & 8\pi m \text{ Id} + o(1) \quad \text{if } m > 0, \\
\leq & 8\pi m \text{ Id} + o(1) \quad \text{if } m < 0;
\end{cases} \]

see Lemma 18. In conjunction with Lemma 25 and (14), it follows that

\begin{equation}
\lambda^3 \int_{\Sigma(\lambda)} f \, Lf \, d\mu \begin{cases}
\geq & [8\pi m + o(1)] \int_{\Sigma(\lambda)} f^2 \, d\mu \quad \text{if } m > 0, \\
\leq & [8\pi m + o(1)] \int_{\Sigma(\lambda)} f^2 \, d\mu \quad \text{if } m < 0
\end{cases}
\end{equation}

for every $f \in \Lambda_1(S_{\xi(\lambda)}, \lambda)$. Moreover, using (61), we find that

\begin{equation}
\text{proj}_{\Lambda_0(S_{\xi(\lambda)}, \lambda)^\perp} Lf = o(\lambda^{-5/2} f)
\end{equation}
for every $f \in \Lambda_0(S_{\xi(\lambda)}, \lambda)$. Assembling (22-25), we see that $\Sigma(\lambda)$ is stable for every $\lambda > 1$ sufficiently large if and only if $m > 0$.

Finally, for the uniqueness statement, we apply Lemma 18 to find that $G_\lambda$ is convex on $\{ \xi \in \mathbb{R}^3 : |\xi| < 1 - \delta \}$ provided that $\lambda > 1$ sufficiently large. In particular, $G_\lambda$ has at most one critical point. Using Remark 32 and (74), we may now argue as in the proof of [4, Theorem 2] that every stable constant mean curvature sphere $\Sigma \subset M$ with $\delta \lambda(\Sigma) < \rho(\Sigma)$ and $\rho(\Sigma) > 1$ sufficiently large satisfies $\Sigma = \Sigma(\lambda)$ for some $\lambda > \lambda_0$. $\square$

3. PROOF OF THEOREM 9

In this section, we assume that $g$ is a complete Riemannian metric on $\mathbb{R}^3$ with non-negative and integrable scalar curvature and that there is $\tau \in (1/2, 1]$ with $g = \bar{g} + \sigma$ where $\partial J \sigma = O(|x|^{-\tau-|J|})$ as $x \to \infty$ for every multi-index $J$ with $|J| \leq 2$.

Let $\{\Sigma_i\}_{i=1}^\infty$ be a sequence of stable constant mean curvature spheres $\Sigma_i \subset \mathbb{R}^3$ that enclose $B_1(0)$ with

\[ \lim_{i \to \infty} \rho(\Sigma_i) = \infty \]

and

\[ \rho(\Sigma_i) = o(\lambda(\Sigma_i)) \]

as $i \to \infty$. By [27, Lemma 1.1], Lemma 27, and (27),

\[ \sup_{x \in \Sigma_i} |x| = O(\lambda(\Sigma_i)). \]

Let $x_i \in \Sigma_i \cap S_{\rho(\Sigma_i)}(0)$. Passing to a subsequence, we may assume that there is $\xi \in \mathbb{R}^3$ with $|\xi| = 1$ and

\[ \lim_{i \to \infty} |x_i|^{-1} x_i = -\xi. \]

Lemma 19. The surfaces $\frac{1}{2} H(\Sigma_i) \Sigma_i$ converge to $S_1(\xi)$ in $C^1$ in $\mathbb{R}^3$.

Proof. We may assume that $\xi = e_3$. Let $a_i \in \mathbb{R}^3$ with $|a_i| = 1$ and $a_i \perp x_i, e_3$. Let $R_i \in SO(3)$ be the unique rotation with $R(a_i) = a_i$ and $R(x_i) = |x_i| e_3$. By (29), $\lim_{i \to \infty} R_i = \text{Id}$.

Let $\gamma_i > 0$ be largest such that there is a smooth function $u_i : \{y \in \mathbb{R}^2 : |y| \leq \gamma_i\} \to \mathbb{R}$ with

\[ ||(\nabla u_i)|y| \leq 1, \]

\[ (y, \rho(\Sigma_i) + u_i(y)) \in R_i(\Sigma_i) \]

for all $y \in \mathbb{R}^2$ with $|y| \leq \gamma_i$. Clearly, $\gamma_i > 0$ and $(\nabla u_i)|_0 = 0$. It follows that

\[ 4 |(y, \rho(\Sigma_i) + u_i(y))| \geq |y| + \rho(\Sigma_i) \]

and

\[ |(\nabla^2 u_i)|y| \leq 8 |\bar{h}(R_i(\Sigma_i))(y, \rho(\Sigma_i) + u_i(y))| \]
for every $y \in \mathbb{R}^2$ with $|y| \leq \gamma_i$. Moreover, Lemma 29, (74), the improved curvature estimates in Remark 32, and (28) imply that

$$
\bar{h}(R_i(\Sigma_i)) = \frac{1}{2} H(\Sigma_i) \bar{g}|_{R_i(\Sigma_i)} + O(|x|^{-1-\tau}) + O(|x|^{-1} H(\Sigma_i)^{1/2})
$$

(33)

$$
= \frac{1}{2} H(\Sigma_i) \bar{g}|_{R_i(\Sigma_i)} + O(|x|^{-3/2}).
$$

Combining (32), (33), (31), we have

$$
|\langle \nabla^2 u_i \rangle_y| \leq 4 H(\Sigma_i) + O((|y| + \rho(\Sigma_i))^{-3/2}).
$$

Integrating,

$$
|\langle \nabla u_i \rangle_y| \leq 4 |y| H(\Sigma_i) + O(\rho(\Sigma_i)^{-1/2}).
$$

(34)

It follows that $\frac{1}{2} H_i \gamma_i \geq \frac{1}{16}$ for all $i$ sufficiently large. (34) also shows that, given $\epsilon > 0$, there is $\delta > 0$ such that

$$
|\nu(R_i(\Sigma_i)) - c_3| \leq \epsilon \quad \text{on} \quad \{(y, \rho(\Sigma_i) + u_i(y)) : y \in \mathbb{R}^2 \text{ with } \frac{1}{2} H_i |y| \leq \delta\}.
$$

Finally, Lemma 27, [27, Theorem 3.1], and (33) imply that $\frac{1}{2} H(\Sigma_i) R_i(\Sigma_i)$ converges to $S_1(\hat{\xi})$ in $C^2$ locally in $\mathbb{R}^3 \setminus \{0\}$ where $\hat{\xi} \in \mathbb{R}^3$; see also [24, Lemma 3.1] and [9, Proposition 2.2]. The preceding argument shows that $\hat{\xi} = \xi$ and that the convergence is in $C^1$ in $\mathbb{R}^3$. \hfill \Box

**Proof of Theorem 9.** Suppose, for a contradiction, that the conclusion of Theorem 9 fails. Using Theorem 1, it follows that there is a sequence $\{\Sigma_i\}_{i=1}^{\infty}$ of stable constant mean curvature spheres $\Sigma_i \subset \mathbb{R}^3$ enclosing $B_1(0)$ that satisfies (26) and (27).

Let $a \in \mathbb{R}^3$ with $|a| = 1$. Clearly,

$$
\int_{\Sigma_i} H \, g(a, \nu) \, d\mu = H(\Sigma_i) \int_{\Sigma_i} g(a, \nu) \, d\mu.
$$

On the one hand, Lemma 39 implies that

$$
g(a, \nu) \, d\mu = [\bar{g}(a, \bar{\nu}) + \tilde{g}(a, \bar{\nu}) \, \text{tr} \sigma + O(|x|^{-2})] \, d\bar{\mu}
$$

uniformly on $\Sigma_i$ as $i \to \infty$. Moreover, by the divergence theorem,

$$
\int_{\Sigma_i} \tilde{g}(a, \bar{\nu}) \, d\bar{\mu} = 0.
$$

In conjunction with Lemma 29, (74), and Lemma 26, we obtain

$$
H(\Sigma_i) \int_{\Sigma_i} g(a, \nu) \, d\mu = \frac{1}{2} \int_{\Sigma_i} \bar{H} \tilde{g}(a, \bar{\nu}) \, \text{tr} \sigma \, d\bar{\mu} + o(1).
$$

(36)

On the other hand, by the first variation formula, we have

$$
\int_{\Sigma_i} H \, g(a, \nu) \, d\mu = \int_{\Sigma_i} [\text{div} a - g(D_a a, \nu)] \, d\mu.
$$

(37)

By Lemma 39,

$$
[\text{div} a - g(D_a a, \nu)] \, d\mu = \frac{1}{2} [\bar{D}_a \text{tr} \sigma - (\bar{D} a \sigma)(\bar{\nu}, \bar{\nu}) + O(|x|^{-1-2\tau})] \, d\bar{\mu}
$$
uniformly on $\Sigma_i$ as $i \to \infty$. In conjunction with Lemma 26, we find

$$\int_{\Sigma_i} [\text{div} \, a - g(Dv_a, \nu)] \, d\mu = \frac{1}{2} \int_{\Sigma_i} [\bar{D}_a \text{tr} \sigma - (\bar{D}_a) \bar{v}, \bar{v})] \, d\bar{\mu} + o(1).$$

Assembling (35–38) and using Lemma 37 as well as (74), we conclude that

$$0 = \int_{\Sigma_i} [\bar{D}_a \text{tr} \sigma - (\bar{d} \text{div} \, \sigma)(\bar{v})] \, \bar{g}(a, \nu) \, d\mu + \frac{1}{2} H(\Sigma_i) \int_{\Sigma_i} [\sigma(a, \nu) - \text{tr} \sigma \bar{g}(a, \nu)] \, d\bar{\mu}$$

$$+ O \left( \int_{\Sigma_i} |\bar{h}||\sigma| \, d\bar{\mu} \right) + o(1).$$

Note that

$$\int_{\Sigma_i} |\bar{h}||\sigma| \, d\bar{\mu} = O \left( \int_{\Sigma_i} |\bar{h}| |x|^{-\tau} \, d\bar{\mu} \right) = o(1)$$

by Remark 32 and Lemma 26. Let $z_i \in \Sigma_i$ with $\bar{v}(z_i) = -|x_i|^{-1} x_i$ and

$$\xi_i = \frac{1}{2} H(\Sigma_i) z_i - \bar{v}(z_i).$$

It follows from Lemma 19 that $4|z_i| \geq H_i$ for all $i$ sufficiently large and

$$|\xi_i| = 1 + o(1).$$

We define the map $E_i : \Sigma_i \to \mathbb{R}^3$ by

$$E_i = \bar{v}(\Sigma_i) - \frac{1}{2} H(\Sigma_i) x + \xi_i.$$

Using Remark 32 and (74), we have

$$\nabla E_i = O(|x|^{-3/2}).$$

Integrating and using Lemma 19, this gives $E_i = O(|x|^{-1/2})$. In conjunction with (39), (40), and Lemma 26, we obtain

$$0 = \int_{\Sigma_i} [\bar{D}_a \text{tr} \sigma - (\bar{d} \text{div} \, \sigma)(\bar{v})] \, \bar{g}(a, H(\Sigma_i) x - \xi_i) \, d\mu + \frac{1}{2} H(\Sigma_i) \int_{\Sigma_i} [\sigma(a, \nu) - \bar{g}(a, \nu) \text{tr} \sigma] \, d\bar{\mu}$$

$$+ o(1).$$

As in the proof of Lemma 18, (77) gives

$$\text{div} \left( \sum_{j=1}^{3} \left[ [D_{e_j} \text{tr} \sigma - (\text{div} \, \sigma)(e_j)] \, \bar{g}(a, H(\Sigma_i) x - \xi_i) + \frac{1}{2} H(\Sigma_i) [\sigma(a, e_j) - \bar{g}(a, e_j) \text{tr} \sigma] \, e_j \right] 
= -R \bar{g}(a, H(\Sigma_i) x - \xi_i) + O(|x|^{-2+2\tau}).$$

Using the divergence theorem and that $R$ is integrable, we find

$$0 = \bar{g}(a, \xi_i) \int_{S_H(\Sigma_i) - 1(0)} \left[ (\text{div} \, \sigma)(\bar{v}) - \bar{D}_a \text{tr} \sigma \right] \, d\bar{\mu}$$

$$+ \frac{1}{2} H(\Sigma_i) \int_{S_H(\Sigma_i) - 1(0)} \left[ \bar{g}(a, x) \, [\bar{D}_a \text{tr} \sigma - (\text{div} \, \sigma)(\bar{v})] + \sigma(\bar{v}, a) - \bar{g}(a, \bar{v}) \text{tr} \sigma \right] \, d\bar{\mu}$$

$$+ o(1);$$
Figure 2. An illustration of the proof of Theorem 9. Using the divergence theorem, the flux integrals in (35) on $\Sigma_i$ can be computed on the sphere $S_{H(\Sigma_i)}^{-1}(0)$. The cross marks the origin in the asymptotically flat chart.

see Figure 2. In conjunction with Lemma 35, we conclude that
\[ 0 = 16\pi m \tilde{g}(a, \xi_i) + o(1) \]
for every $a \in \mathbb{R}^3$ with $|a| = 1$. This is incompatible with (41). \hfill \Box

4. PROOF OF THEOREM 11

In this section, we assume that $g$ is a complete Riemannian metric on $\mathbb{R}^3$ and that there is $\tau \in (1/2, 1]$ with
\[ g = \tilde{g} + \sigma \quad \text{where} \quad \partial_J \sigma = O \left( |x|^{-\tau - |J|} \right) \quad \text{and} \quad R = O(|x|^{-5/2-\tau}) \]
as $x \to \infty$ for every multi-index $J$ with $|J| \leq 2$.

Let $\{\Sigma_i\}_{i=1}^{\infty}$ be a sequence of stable constant mean curvature spheres $\Sigma_i \subset \mathbb{R}^3$ that enclose $B_1(0)$ with
\[ \lim_{i \to \infty} \rho(\Sigma_i) = \infty \quad \text{and} \quad \rho(\Sigma_i) = o(\lambda(\Sigma_i)) \]
as $i \to \infty$.

Let $x_i \in \Sigma_i \cap S_{\rho(\Sigma_i)}(0)$. Passing to a subsequence, we may assume that there is $\xi \in \mathbb{R}^3$ with $|\xi| = 1$ and $\lim_{i \to \infty} |x_i|^{-1} x_i = -\xi$.

Lemma 20. Suppose that, for some $s > 1$,
\[ \lambda(\Sigma_i) = O(\rho(\Sigma_i)^s). \]
The surfaces $\frac{1}{2} H(\Sigma_i)$ $\Sigma_i$ converge to $S_1(\xi)$ in $C^1$ in $\mathbb{R}^3$.

Proof. Using (28) and (44), we have
\[ \sup_{x \in \Sigma_i} |x| = O(\rho(\Sigma_i)^s). \]
In conjunction with Lemma 29, (74), and Remark 32, we obtain
\begin{equation}
\bar{h}(\Sigma_i) = \frac{1}{2} H_i \bar{g}|_{\Sigma_i} + O(|x|^{-1-(1/4+\tau'/2)\lambda}) + O(|x|^{-1} H_i^{1/2}).
\end{equation}

We may now argue as in the proof of Lemma 19 using (45) instead of (33).

\[\square\]

\textbf{Proof of Theorem 11.} The proof is similar to that of Theorem 9. We only point out the necessary modifications.

Let \( s > 1 \) be as in (8). If the conclusion of Theorem 11 fails, there is a sequence \( \{\Sigma_i\}_{i=1}^{\infty} \) of stable constant mean curvature spheres \( \Sigma_i \subset \mathbb{R}^3 \), each enclosing \( B_1(0) \), that satisfies (43) and \( \lambda(\Sigma_i) = O(\rho(\Sigma_i)^{s}) \). In particular,
\begin{equation}
\lambda(\Sigma_i) = o(\rho(\Sigma_i)^{s'})
\end{equation}
for every \( s' > s \). Let \( \tau' \in (1/2, \tau) \) be such that
\begin{equation}
s < 1 + \frac{3}{4} \frac{2 \tau - 1}{1 - \tau'} + \frac{1}{2} \frac{\tau' - \tau}{1 - \tau'}.
\end{equation}

It follows from (46) and (47) that
\begin{equation}
O(\lambda(\Sigma_i)^{1-\tau'} \rho(\Sigma_i)^{-1/4-\tau/2-(\tau-\tau')}) = o(1),
\end{equation}
\begin{equation}
O(\lambda(\Sigma_i)^{1-\tau'} \rho(\Sigma_i)^{-1/4-\tau/2-(\tau-\tau')/2}) = o(1).
\end{equation}

Compared to the proof of Theorem 9, to obtain (40), we now use Remark 32 to estimate
\begin{align*}
|\hat{h}| |\sigma| &= O(|x|^{-1-\tau} \rho(\Sigma_i)^{-1/4-\tau/2}) + O(|x|^{-1-\tau} \lambda(\Sigma_i)^{-1/2}) \\
&= O(\lambda(\Sigma_i)^{1-\tau'} |x|^{-2-(\tau-\tau')} \rho(\Sigma_i)^{-1/4-\tau/2}) + O(|x|^{-3/2-\tau}).
\end{align*}

Using Lemma 26 and (48), we obtain
\[\int_{\Sigma_i} |\hat{h}| |\sigma| \, d\bar{\mu} = O(\lambda(\Sigma_i)^{1-\tau'} \rho(\Sigma_i)^{-1/4-\tau/2-(\tau-\tau')}) + O(\rho(\Sigma_i)^{1/2-\tau}) = o(1).\]

Instead of (42), we now apply Remark 32 and (74) to estimate
\[\nabla E_i = O(|x|^{-1} \rho(\Sigma_i)^{-1/4-\tau/2}) + O(|x|^{-1} \lambda(\Sigma_i)^{-1/2}).\]

Lemma 20 and integration give
\begin{align*}
E_i &= O(|x|^{(\tau-\tau')/2} \rho(\Sigma_i)^{-1/4-\tau/2}) + O(|x|^{(\tau-\tau')/2} \lambda(\Sigma_i)^{-1/2}) \\
&= O(\lambda(\Sigma_i)^{1-\tau'} |x|^{-1+\tau-(\tau-\tau')/2} \rho(\Sigma_i)^{-1/4-\tau/2}) + O(|x|^{-1/2+(\tau'-\tau)/2}).
\end{align*}

By (48) and Lemma 26,
\[0 = \int_{\Sigma_i} [\hat{D}_\rho \text{tr} \sigma - (\text{d} \text{div} \sigma)(\bar{\nu})] \bar{g}(a, \frac{1}{2} H(\Sigma_i) x - \xi_i) \, d\bar{\mu} + \frac{1}{2} H(\Sigma_i) \int_{\Sigma_i} [\sigma(a, \bar{\nu}) - \bar{g}(a, \bar{\nu}) \text{tr} \sigma] \, d\bar{\mu} + o(1).
\]

The argument concludes exactly as in the proof of Theorem 9. \[\square\]
Lemma 21. There holds, as \( \lambda \to \infty \),

\[
|u^o| + \lambda |\nabla u^o| + \lambda^2 |\nabla^2 u^o| = o(\lambda^{1/2} |\xi(\lambda)|) + o(\lambda^{-1/2}).
\]

Proof. Using (14), (62), and (49), we obtain

\[
Lu = H(\Sigma(\lambda)) - H(S(\lambda)) + o(\lambda^{-5/2}).
\]

Note that \( H(\Sigma(\lambda))^o = 0 \). Using Lemma 38, the decay assumptions (49) and (51), as well as Taylor’s theorem, we find that

\[
H(S(\lambda))^o = o(\lambda^{-3/2} |\xi(\lambda)|) + o(\lambda^{-5/2}).
\]

Likewise, using (61), (49), (51), (14), and Taylor’s theorem, we find

\[
Lu = L^o u + o(\lambda^{-3/2} |\xi(\lambda)|) + o(\lambda^{-5/2})
\]
where $L^e f = (Lf^e)^o + (Lf^o)^o$. Note that $L^e : \Lambda_1(S(\lambda)) \rightarrow \Lambda_1(S(\lambda))$ maps odd functions to odd functions and even functions to even functions. In view of (61), we obtain, as $\lambda \rightarrow \infty$,

$$L^e f = -\Delta f - 2 \lambda^{-2} f + o(\lambda^{-5/2})$$

uniformly for all $f \in H^2(S(\lambda))$ with $||f||_{H^2(S(\lambda))} = 1$. We see from Lemma 33 that $L^e$ is invertible. The assertion now follows from the estimates above using that $\text{proj}_{\Lambda_1(S(\lambda)))} u = 0$.

**Lemma 22.** There holds, as $\lambda \rightarrow \infty$,

$$\lambda \int_{\Sigma(\lambda)} H g(a, \nu) \, d\mu = \lambda \int_{S(\lambda)} H g(a, \nu) \, d\mu + o(1) + o(\lambda |\xi(\lambda)|).$$

**Proof.** By the first variation formula,

$$\int_{\Sigma(\lambda)} H g(a, \nu) \, d\mu = \int_{\Sigma(\lambda)} [\text{div } a - g(D_\nu a, \nu)] \, d\mu.$$ 

Using (51), (14), and Taylor’s theorem, we obtain

$$\int_{\Sigma(\lambda)} [\text{div } a - g(D_\nu a, \nu)] \, d\mu - \int_{S(\lambda)} [\text{div } a - g(D_\nu a, \nu)] \, d\mu = \int_{\Sigma(\lambda)} [\tilde{\text{div }} a - \tilde{g}(\tilde{D}_\nu a, \tilde{\nu})] \, d\tilde{\mu} - \int_{S(\lambda)} [\tilde{\text{div }} a - \tilde{g}(\tilde{D}_\nu a, \tilde{\nu})] \, d\tilde{\mu} + o(\lambda^{-1}).$$

Here and below, the tilde indicates that a quantity is computed with respect to the metric (50). Using (52), (49), and Taylor’s theorem, we find

$$\int_{\Sigma(\lambda)} [\tilde{\text{div }} a - \tilde{g}(\tilde{D}_\nu a, \tilde{\nu})] \, d\tilde{\mu} = \int_{\Sigma(\lambda), \lambda(u^e)} [\tilde{\text{div }} a - \tilde{g}(\tilde{D}_\nu a, \tilde{\nu})] \, d\tilde{\mu} + o(\lambda^{-1}) + o(|\xi(\lambda)|).$$

Using (49), (16), and (14), we have

$$\int_{\Sigma(\lambda), \lambda(u^e)} [\tilde{\text{div }} a - \tilde{g}(\tilde{D}_\nu a, \tilde{\nu})] \, d\tilde{\mu} = \int_{\Sigma_0, \lambda(u^e)} [\tilde{\text{div }} a - \tilde{g}(\tilde{D}_\nu a, \tilde{\nu})] \, d\tilde{\mu} + \int_0^1 \left( \int_{\Sigma_0, \lambda(s u^e)} \left[ \lambda \tilde{D}_\nu \tilde{\text{div }} a - \lambda \tilde{g}(\tilde{D}_\nu a, \tilde{\nu}) + 2 [\tilde{\text{div }} a - \tilde{g}(\tilde{D}_\nu a, \tilde{\nu})] \tilde{g}(\xi(\lambda), \tilde{\nu}) + \tilde{g}(\tilde{D}_\xi a, \tilde{\nu}) + g(\tilde{D}_\nu a, \xi^T)] \, d\tilde{\mu} \right) \, ds$$

$$+ o(|\xi(\lambda)|).$$

In conjunction with (49), (14), and Taylor’s theorem, we obtain that

$$\int_{\Sigma(\lambda), \lambda(u^e)} [\tilde{\text{div }} a - \tilde{g}(\tilde{D}_\nu a, \tilde{\nu})] \, d\tilde{\mu} - \int_{S(\lambda)} [\tilde{\text{div }} a - \tilde{g}(\tilde{D}_\nu a, \tilde{\nu})] \, d\tilde{\mu}$$

$$= \int_{\Sigma_0, \lambda(u^e)} [\tilde{\text{div }} a - \tilde{g}(\tilde{D}_\nu a, \tilde{\nu})] \, d\tilde{\mu} - \int_{S(\lambda)} [\tilde{\text{div }} a - \tilde{g}(\tilde{D}_\nu a, \tilde{\nu})] \, d\tilde{\mu}$$

$$+ o(|\xi(\lambda)|).$$
By symmetry, we have
\[
\int_{\Sigma_{0,\lambda}(\nu^c)} [\tilde{\text{div}} a - \tilde{g}(\tilde{D}_a a, \tilde{\nu})] \, d\tilde{\mu} = 0 \quad \text{and} \quad \int_{\Sigma_{0}(\lambda)} [\tilde{\text{div}} a - \tilde{g}(\tilde{D}_a a, \tilde{\nu})] \, d\tilde{\mu} = 0.
\]
Assembling these estimates, the assertion follows. \(\square\)

**Lemma 23.** There holds, as \(\lambda \to \infty\),
\[
\lambda \int_{\Sigma(\lambda)} H g(a, \nu) \, d\mu = \frac{1}{2} \lambda \int_{\Sigma(\lambda)} [\tilde{D}_a \text{tr} \sigma - (\tilde{D}_a \sigma)(\tilde{\nu}, \tilde{\nu})] \, d\tilde{\mu} + o(1) + o(\lambda |\xi(\lambda)|).
\]

**Proof.** Let \(\mathcal{M}\) be the space of \(C^2\)-asymptotically flat metrics on \(\mathbb{R}^3\). Given \(\lambda > 1\), we define the functional \(F_\lambda : \{\xi \in \mathbb{R}^3 : |\xi| < 1/2\} \times \mathcal{M} \to \mathbb{R}\) by
\[
F_\lambda(\xi, g) = \lambda \int_{\Sigma_{\xi,\lambda}} [\text{div} a - g(D_o a, \nu)] \, d\mu.
\]
Note that \(F\) is differentiable with respect to the first variable and smooth with respect to the second variable. Moreover,
\[
(53) \quad F_\lambda(0, \tilde{g}) = 0
\]
for every \(g \in \mathcal{M}\). Using Taylor’s theorem, we have
\[
F_\lambda(\xi(\lambda), \tilde{g}) = F_\lambda(\xi(\lambda), \tilde{g}) + (D_\sigma F_\lambda)|_{(\xi(\lambda), \tilde{g})} + \frac{1}{2} (D^2_\sigma F_\lambda)|_{(\xi(\lambda), \tilde{g})} + \frac{1}{6} (D^3_\sigma F_\lambda)|_{(\xi(\lambda), \tilde{g})} + o(1)
\]
where \(D\) indicates differentiation with respect to the second variable. On the one hand, (49), (51), and Taylor’s theorem imply that
\[
(D^2_\sigma F_\lambda)|_{(\xi(\lambda), \tilde{g})} + (D^3_\sigma F_\lambda)|_{(\xi(\lambda), \tilde{g})} = (D^2_\sigma F_\lambda)|_{(\xi(\lambda), \tilde{g})} + (D^3_\sigma F_\lambda)|_{(\xi(\lambda), \tilde{g})} + o(1)
\]
\[
= (D^2_\sigma F_\lambda)|_{(0, \tilde{g})} + (D^3_\sigma F_\lambda)|_{(0, \tilde{g})} + o(1) + o(\lambda |\xi(\lambda)|).
\]
On the other hand, (53) implies that
\[
F_\lambda(\xi(\lambda), \tilde{g}) = (D^2_\sigma F_\lambda)|_{(0, \tilde{g})} = (D^3_\sigma F_\lambda)|_{(0, \tilde{g})} = 0.
\]
Finally, Lemma 38 implies that
\[
(D_\sigma F)|_{(\xi(\lambda), \tilde{g})} = \frac{1}{2} \lambda \int_{\Sigma(\lambda)} [\tilde{D}_a \text{tr} \sigma - (\tilde{D}_a \sigma)(\tilde{\nu}, \tilde{\nu})] \, d\tilde{\mu}.
\]
Assembling these estimates, the assertion follows. \(\square\)

**Lemma 24.** There holds, as \(\lambda \to \infty\),
\[
\lambda \int_{\Sigma(\lambda)} H g(a, \nu) \, d\mu = \int_{\Sigma(\lambda)} \tilde{g}(a, \tilde{\nu}) \, \text{tr} \sigma \, d\tilde{\mu} + o(1) + o(\lambda |\xi(\lambda)|).
\]

**Proof.** Since \(H(\Sigma(\lambda))\) is constant, we have
\[
\lambda \int_{\Sigma(\lambda)} H g(a, \nu) \, d\mu = \lambda H(\Sigma(\lambda)) \int_{\Sigma(\lambda)} g(a, \nu) \, d\mu.
\]
Arguing as in the proof of Lemma 22, we obtain
\[
\int_{\Sigma(\lambda)} g(a, \nu) \, d\mu = \int_{\Sigma(\lambda)} g(a, \nu) \, d\mu + o(1) + o(\lambda |\xi(\lambda)|).
\]
Arguing as in the proof of Lemma 23 and using Lemma 38, it follows that
\[
\int_{S(\lambda)} g(a, \nu) \, d\mu = \frac{1}{2} \int_{S(\lambda)} \bar{g}(a, \bar{\nu}) \, \tr \sigma \, d\bar{\mu} + o(1) + o(\lambda |\xi(\lambda)|).
\]
Recall from (16) that \(\lambda H(\Sigma(\lambda)) - 2 = o(\lambda^{-1/2})\). Using (51), we find
\[
\int_{S(\lambda)} \bar{g}(a, \bar{\nu}) \, \tr \sigma \, d\bar{\mu} = o(\lambda^{1/2}).
\]
Similarly, using Taylor’s theorem and symmetry, we obtain
\[
\int_{S(\lambda)} \bar{g}(a, \bar{\nu}) \, \tr \tilde{\sigma} \, d\bar{\mu} = \int_{S_{\lambda}(0)} \bar{g}(a, \bar{\nu}) \, \tr \tilde{\sigma} \, d\bar{\mu} + o(\lambda^{3/2} |\xi(\lambda)|) = o(\lambda^{3/2} |\xi(\lambda)|).
\]
Assembling these estimates, the assertion follows. □

**Proof of Theorem 13.** Using Lemmas 22, 23, and 24, we have
\[
\frac{1}{2} \int_{S(\lambda)} \left[ \bar{D}_a \tr \sigma - (\bar{D}_a \sigma)(\bar{\nu}, \bar{\nu}) - 2 \lambda^{-1} \bar{g}(a, \bar{\nu}) \, \tr \sigma \right] \, d\bar{\mu} = o(1) + o(\lambda |\xi(\lambda)|).
\]
Arguing as in the proof of Theorem 1 and using that \(R = o(|x|^{-3})\) as \(x \to \infty\), we now obtain the improved estimate
\[
\bar{g}(a, \lambda \xi(\lambda)) \int_{S_{\lambda}(0)} [(\tr \sigma)(\bar{\nu}) - \bar{D}_\rho \tr \sigma] \, d\bar{\mu}
+ \int_{S_{\lambda}(0)} \left[ \bar{g}(a, x) \left[ \bar{D}_\rho \tr \sigma - (\tr \sigma)(\bar{\nu}) \right] + \sigma(\bar{\nu}, a) - \bar{g}(a, \bar{\nu}) \, \tr \sigma \right] \, d\bar{\mu}
= o(1) + o(\lambda |\xi(\lambda)|) + o\left( \int_{B_{\lambda}(\lambda \xi) \Delta B_{\lambda}(0)} |x|^{-2} \, d\bar{\mu} \right)
= o(1) + o(\lambda |\xi(\lambda)|).
\]
Since \((M, g)\) satisfies the \(C^2\)-Regge-Teitelboim conditions, the Hamiltonian center of mass \(C = (C^1, C^2, C^3)\) exists; see [17, Theorem 2.2]. Using Lemma 35 and (2), we find that
\[
\lambda \xi(\lambda) = C + o(1) + o(\lambda |\xi(\lambda)|). \tag{55}
\]
In particular,
\[
\lambda \xi(\lambda) = O(1) \tag{56}
\]
and in fact \(\lambda \xi(\lambda) = C + o(1)\). Using (52), (51), and (56), we find that
\[
|\Sigma(\lambda)|^{-1} \int_{\Sigma(\lambda)} x \, d\mu = \lambda \xi(\lambda) + o(1).
\]
The assertion follows from these estimates. □

### A. Asymptotically flat manifolds

Let \(k \geq 2\) be an integer.

A metric \(g\) on \(\{ x \in \mathbb{R}^3 : |x| > 1/2 \}\) is called \(C^k\)-asymptotically flat if its scalar curvature is
integrable and if there are \( \tau \in (1/2, 1] \) and a symmetric \((0,2)\)-tensor \( \sigma \) such that
\[
g = \hat{g} + \sigma \quad \text{where} \quad \partial_J \sigma = O \left( |x|^{1-\tau-J} \right) \quad \text{as} \ x \to \infty \quad \text{for every multi-index} \ J \ 	ext{with} \ |J| \leq k.
\]
g is called \( C^k \)-asymptotic to Schwarzschild with mass \( m \in \mathbb{R} \), if, more strongly,
\[
g = \left( 1 + \frac{m}{2|x|} \right)^4 \hat{g} + \sigma \quad \text{where} \quad \partial_J \sigma = O \left( |x|^{-2-J} \right) \quad \text{as} \ x \to \infty \quad \text{for every multi-index} \ J \ 	ext{with} \ |J| \leq k.
\]

A connected complete Riemannian 3-manifold \((M, g)\) is said to be \( C^k \)-asymptotically flat if there is a non-empty compact subset of \( M \) whose complement is diffeomorphic to \( \{ x \in \mathbb{R}^3 : |x| > 1/2 \} \) and if the pull-back of \( g \) by this diffeomorphism is \( C^k \)-asymptotically flat. Such a diffeomorphism is called an asymptotically flat chart. We sometimes say that \((M, g)\) is \( C^k \)-asymptotically flat of rate \( \tau > 1/2 \) to emphasize the decay rate. If there is an asymptotically flat chart such that the pull-back metric takes the form (58), \((M, g)\) is called \( C^k \)-asymptotic to Schwarzschild with mass \( m \).

R. Bartnik has shown that the mass (1) of a \( C^2 \)-asymptotically flat manifold converges and does not depend on the choice of asymptotically flat chart; see [2, Theorem 4.2].

We usually fix a specific asymptotically flat chart as reference and write \( B_r \), where \( r \geq 1 \), to denote the open, bounded domain in \((M, g)\) whose boundary corresponds to \( S_r(0) \) with respect to this chart.

Given an integer \( \ell \geq 2 \), we say that \((M, g)\) satisfies the \( C^\ell \)-Regge-Teitelboim conditions, see [25], if there is \( \hat{\tau} \in (1/2, 1] \) with
\[
\partial J \hat{g} = O \left( |x|^{-1-\hat{\tau}-J} \right) \quad \text{and} \quad \hat{R} = O(|x|^{-7/2-\hat{\tau}})
\]
as \( x \to \infty \) where
\[
\hat{g} = \frac{1}{2} [g(x) - g(-x)] \quad \text{and} \quad \hat{R} = \frac{1}{2} [R(x) - R(-x)].
\]
Moreover, if there is \( \hat{\tau} \in (1/2, 1] \) with
\[
\partial J \hat{g} = O \left( |x|^{-1-2-\hat{\tau}-J} \right) \quad \text{and} \quad \hat{R} = O(|x|^{-3-\hat{\tau}}),
\]
as \( x \to \infty \) for every multi-index \( J \) with \(|J| \leq 2\), we say that \((M, g)\) satisfies the weak \( C^\ell \)-Regge-Teitelboim conditions; see [23, Definition 6.2].

L.-H. Huang has verified that the Hamiltonian center of mass (2) converges if \((M, g)\) is \( C^3 \)-asymptotically flat and satisfies the \( C^2 \)-Regge-Teitelboim conditions; see [16, Theorem 2.2] and [25]. The center of mass depends on the choice of asymptotically flat chart. Recall from [2, Corollary 3.2] that any two asymptotically flat charts are asymptotically equal up to a Euclidean isometry. If different charts are used, then the respective centers of mass are related by the same isometry as the asymptotically flat charts; see [18, Theorem 4.3].

\[ \text{B. First and second variation of area and volume} \]

In this section, we collect some useful results on the first and second variation of area and volume from [5, Appendix II] that are used throughout this paper.

Let \((M, g)\) be a Riemannian 3-manifold and \( \Sigma \subset M \) be a closed, two-sided surface with unit
normal $\nu$. We also assume that $\Sigma \cap \partial M = \emptyset$. Let $\epsilon > 0$ and $U \in C^\infty(\Sigma \times (-\epsilon, \epsilon))$ with $U(\cdot, 0) = 0$. Decreasing $\epsilon > 0$ if necessary, we obtain a smooth variation $\{\Sigma_s : s \in (-\epsilon, \epsilon)\}$ of embedded surfaces $\Sigma_s = \Phi_s(\Sigma_s)$ where

$$\Phi_s : \Sigma \to M \quad \text{is given by} \quad \Phi_s(x) = \exp_x(U(x, s) \nu(x)).$$

We denote the initial velocity and initial acceleration of the variation by $u(x) = \dot{U}(x, 0)$ and $v(x) = \ddot{U}(x, 0)$.

Recall that

$$Lf = -\Delta f - (|h|^2 + \text{Ric}(\nu, \nu)) f$$

denotes the linearization of the mean curvature operator $H$, where $\Delta$ is the non-positive Laplace operator on $\Sigma$ with respect to the induced metric, $h$ the second fundamental form of $\Sigma$, and $\text{Ric}$ the Ricci curvature of $(M, g)$. In particular,

$$Lu = \frac{d}{ds} \bigg|_{s=0} H(\Sigma_s) \circ \Phi_s.$$  

In the following lemma,

$$\text{vol}(\Sigma_s) = \begin{cases} \int_{\Sigma \times [0, s]} \Psi^*(dv) & \text{if } s \geq 0 \\
\int_{\Sigma \times [s, 0]} \Psi^*(dv) & \text{if } s \leq 0 \end{cases}$$

denotes the relative volume of $\Sigma_s$. Here, $\Psi : \Sigma \times (-\epsilon, \epsilon) \to M$ is given by $\Psi(x, s) = \Phi_s(x)$.

**Lemma 25.** There holds

$$\left. \frac{d}{ds} \right|_{s=0} |\Sigma_s| = \int_{\Sigma} Hu \, d\mu \quad \text{and} \quad \left. \frac{d}{ds} \right|_{s=0} \text{vol}(\Sigma_s) = \int_{\Sigma} u \, d\mu.$$  

Moreover,

$$\left. \frac{d^2}{ds^2} \right|_{s=0} |\Sigma_s| = \int_{\Sigma} \left[ u Lu + H^2 u^2 + Hv \right] \, d\mu \quad \text{and} \quad \left. \frac{d^2}{ds^2} \right|_{s=0} \text{vol}(\Sigma_s) = \int_{\Sigma} [Hu^2 + v] \, d\mu.$$  

Surfaces that are critical for the area functional among so-called volume-preserving variations have constant mean curvature and are therefore called constant mean curvature surfaces. A variation $\{\Sigma_s : |s| < \epsilon\}$ is called volume-preserving if $\text{vol}(\Sigma_s) = \text{vol}(\Sigma)$ for all $s \in (-\epsilon, \epsilon)$.

A constant mean curvature surface $\Sigma$ is stable if it passes the second derivative test for area among all volume-preserving variations. It can be shown that $\Sigma$ is stable, if and only if,

$$\int_{\Sigma} u Lu \, d\mu \geq 0$$

for every $u \in C^\infty(\Sigma)$ with

$$\int_{\Sigma} u \, d\mu = 0.$$
C. Curvature estimates for stable CMC spheres

In this section, we discuss the curvature estimates for stable constant mean curvature spheres that are needed in this paper. As in [9] and differently from e.g. [21], we rely on refined $L^2$-estimates where the Christodoulou-Yau estimate (64) is combined with the global estimate (65) on the Hawking mass found by G. Huisken and T. Ilmanen [19].

Throughout, we assume that $(M, g)$ is $C^2$-asymptotically flat of rate $\tau \in (1/2, 1]$. Let $\Sigma \subset M$ be a stable constant mean curvature sphere. Recall from [11, p. 13] that

$$\frac{2}{3} \int_{\Sigma} (|\tilde{h}|^2 + R) \, d\mu \leq 16 \pi - \int_{\Sigma} H^2 \, d\mu. \tag{64}$$

We need the following decay estimate.

**Lemma 26** ([20, Lemma 5.2]). Let $\Sigma \subset \mathbb{R}^3 \setminus \{0\}$ be a closed surface and $q > 2$. There is a constant $c(q) > 0$ such that

$$\rho(\Sigma)^q \int_{\Sigma} |x|^{-q} \, d\bar{\mu} \leq c(q) \int_{\Sigma} \bar{H}^2 \, d\bar{\mu}. \tag{67}$$

Let $\{\Sigma_i\}_{i=1}^{\infty}$ be a sequence of stable constant mean curvature spheres $\Sigma_i \subset M$ with

$$\lim_{i \to \infty} \rho(\Sigma_i) = \infty \quad \text{and} \quad \rho(\Sigma_i) = O(\lambda(\Sigma_i)).$$

We recall the following two lemmas.

**Lemma 27** ([21, Lemma 2.3]). There holds, as $i \to \infty$,

$$\int_{\Sigma_i} \bar{H}^2 \, d\bar{\mu} = 16 \pi + o(1). \tag{68}$$

*Proof.* This follows from (64) and Lemma 26; see [21] for details. \hfill □

**Lemma 28** ([9, Proposition D.1]). If $R \geq 0$, we have

$$16 \pi - \int_{\Sigma_i} H^2 \, d\mu \leq O(\lambda(\Sigma_i)^{-1}) \tag{65}$$

as $i \to \infty$. If $R = O(|x|^{-5/2 - \tau})$ as $x \to \infty$, we have

$$16 \pi - \int_{\Sigma_i} H^2 \, d\mu \leq O(\rho(\Sigma_i)^{-1/2 - \tau}) + O(\lambda(\Sigma_i)^{-1}). \tag{66}$$

*Proof.* The estimate (65) is proven in [9, Proposition D.1].

To obtain (66), we adapt the argument in [9, Proposition D.1] as follows.

Let $\Sigma'_i \subset M$ be the minimizing hull of $\Sigma_i$; see [19, p. 371]. Using [19, (1.15)], we have

$$16 \pi - \int_{\Sigma_i} H^2 \, d\mu \leq 16 \pi - \int_{\Sigma'_i} H^2 \, d\mu. \tag{67}$$

Moreover, there holds

$$\lambda(\Sigma_i) = (1 + o(1)) \lambda(\Sigma'_i), \tag{68}$$

see [9, (26)], while, clearly,

$$\rho(\Sigma'_i) \geq \rho(\Sigma_i). \tag{69}$$
Let \( u_i \in C^{1,1}(M) \) be the proper weak solution of inverse mean curvature flow with initial data \( \Sigma_i \) in the sense of [19, p. 365]. Using the growth formula [19, (5.22)] for the right-hand side of (67), the co-area formula, and arguing as in [10, Appendix H], we find that

\[
16 \pi - \int_{\Sigma_i} H^2 \, d\mu \leq O(\lambda(\Sigma_i)^{-1}) + O\left(\int_{M \setminus B_{\rho(\Sigma_i)}} |Du_i| \, R \, d\mu\right).
\]

By (64) and Lemma 26, we have \( H(\Sigma_i) = O(\lambda(\Sigma_i)^{-1}) \). In conjunction with [19, (3.1)], [19, (1.15)], and the fact that \((M, g)\) is \(C^2\)-asymptotically flat, we find that

\[
16 \pi - \int_{\Sigma_i} H^2 \, d\mu = O(\lambda(\Sigma_i)^{-1}) + O(|x|^{-1})
\]
on \( M \), uniformly as \( i \to \infty \). Assembling (67-71) and using that \( R = (|x|^{-5/2-\tau}) \) as \( x \to \infty \), the assertion follows. \[ \square \]

**Lemma 29.** If \( R \geq 0 \), we have

\[
H(\Sigma_i) = 2 \lambda(\Sigma_i)^{-1} + O(\lambda(\Sigma_i)^{-2})
\]
as \( i \to \infty \). If \( R = O(|x|^{-5/2-\tau}) \) as \( x \to \infty \), we have

\[
H(\Sigma_i) = 2 \lambda(\Sigma_i)^{-1} + O(\rho(\Sigma_i)^{-1/2-\tau} \lambda(\Sigma_i)^{-1}) + O(\lambda(\Sigma_i)^{-2}).
\]

**Proof.** If \( R \geq 0 \), we find, using (64) and Lemma 28, that

\[
16 \pi - \int_{\Sigma_i} H^2 \, d\mu = O(\lambda(\Sigma_i)^{-1}).
\]
Alternatively, if \( R = O(|x|^{-5/2-\tau}) \), we use (64), Lemma 26, and Lemma 28 to find that

\[
16 \pi - \int_{\Sigma_i} H^2 \, d\mu = O(\rho(\Sigma_i)^{-1/2-\tau}) + O(\lambda(\Sigma_i)^{-1}).
\]
Moreover, we have

\[
16 \pi - \int_{\Sigma_i} H^2 \, d\mu = 16 \pi - 4 \pi H(\Sigma_i)^2 \lambda(\Sigma_i)^2
\]
for all \( i \). Note that \( H(\Sigma_i) > 0 \) by the maximum principle. The assertion follows. \[ \square \]

**Lemma 30.** If \( R \geq 0 \), we have

\[
\int_{\Sigma_i} |\hat{h}|^2 \, d\mu = O(\lambda(\Sigma_i)^{-1})
\]
as \( i \to \infty \). If \( R = O(|x|^{-5/2-\tau}) \) as \( x \to \infty \), we have

\[
\int_{\Sigma_i} |\hat{h}|^2 \, d\mu = O(\rho(\Sigma_i)^{-1/2-\tau}) + O(\lambda(\Sigma_i)^{-1}).
\]

**Proof.** This follows from (64), Lemma 28, and Lemma 26. \[ \square \]

**Proposition 31.** If \( R \geq 0 \), we have

\[
\hat{h} = O(|x|^{-1-\tau}) + O(|x|^{-1} \lambda(\Sigma_i)^{-1/2})
\]
as \( i \to \infty \). If \( R = O(|x|^{-5/2-\tau}) \) as \( x \to \infty \), we have

\[
\hat{h} = O(|x|^{-\tau} \rho(\Sigma_i)^{-1/4-\tau/2}) + O(|x|^{-1} \lambda(\Sigma_i)^{-1/2}).
\]
In either case, 
(74) \[ \tilde{H} = H + O(|x|^{-1-\tau}). \]

Proof. As shown in [21, Theorem 2.7], the Simons’ identity and the Sobolev inequality imply that 
\[ |\hat{\psi}|^2 = O(|x|^{-2-2\tau}) + O \left( |x|^{-2} \int \hat{h}^2 \, d\mu \right) \]
as \( i \to \infty \). In conjunction with Lemma 30, we obtain (72) and (73). Now, (74) follows from the estimate 
\( \tilde{H} = H + O(|x|^{-\tau} |h|) + O(|x|^{-1-\tau}) \). \( \square \)

Remark 32. It follows from Proposition 31 that, if \( R \geq 0 \), 
\( \hat{\psi} = O(|x|^{-1-\tau}) + O(|x|^{-1} \lambda(\Sigma_i)^{-1/2}) \)
as \( i \to \infty \). If \( R = O(|x|^{-5/2-\tau}) \) as \( x \to \infty \), then 
\( \hat{\psi} = O(|x|^{-1} \rho(\Sigma_i)^{-1/4-\tau/2} + O(|x|^{-1} \lambda(\Sigma_i)^{-1/2}) \).

D. The Laplace operator on the unit sphere

In this section, we collect some standard facts about the Laplace operator on the unit sphere.

Lemma 33. The eigenvalues of the operator 
\[ -\bar{\Delta} : H^2(S_1(0)) \to L^2(S_1(0)) \]
are given by 
\[ \{ \ell (\ell + 1) : \ell = 0, 1, 2, \ldots \}. \]

We denote the eigenspace corresponding to the eigenvalue \( \ell (\ell + 1) \) by 
\[ \Lambda_\ell(S_1(0)) = \{ f \in C^\infty(S_1(0)) : -\bar{\Delta} f = \ell (\ell + 1) f \}. \]

Recall that these eigenspaces are finite-dimensional and that 
\[ L^2(S_1(0)) = \bigoplus_{\ell=0}^\infty \Lambda_\ell(S_1(0)). \]
Moreover, \( \Lambda_0(S_1(0)) = \text{span}\{1\} \) and \( \Lambda_1(S_1(0)) = \text{span}\{x^1, x^2, x^3\} \).

Corollary 34. Let \( f \perp [\Lambda_0(S_1(0)) \oplus \Lambda_1(S_1(0))] \). There holds 
\[ \int_{S_1(0)} f \bar{\Delta} f \, d\mu \geq 4 \int_{S_1(0)} f^2 \, d\mu. \]

E. Mass and center of mass

In this section, we collect some observations on the flux integrals that define the mass (1) and the center of mass (2).

We assume that \( g \) is a complete Riemannian metric on \( \mathbb{R}^3 \) with integrable scalar curvature and that there is \( \tau \in (1/2, 1] \) with 
\[ g = \bar{g} + \sigma \quad \text{where} \quad \sigma = O \left( |x|^{-\tau-|J|} \right) \]
as \( x \to \infty \) for every multi-index \( J \) with \(|J| \leq 2 \).

**Lemma 35.** There holds, as \( \lambda \to \infty \),

\[
\frac{1}{16 \pi} \int_{S(X(0))} \left[ (\text{div} \sigma)(\nu) - \hat{D}_\nu \sigma \right] d\bar{\mu} = m + o(1)
\]

and

\[
\int_{S(X(0))} \left[ \left[ \hat{D}_\nu \sigma - (\text{div} \sigma)(\nu) \right] + \lambda^{-1} \left( \sum_{i=1}^{3} \sigma(\nu, e_i) e_i - \lambda^{-1} \text{tr} \sigma \nu \right) \right] d\bar{\mu} = o(1).
\]

**Proof.** To prove (75), we use the divergence theorem, (77), \( R \in L^1(M, g) \), and (1); see e.g. [19, Lemma 7.3].

To verify (76), we define the map \( F : (1, \infty) \to \mathbb{R}^3 \) by

\[
F(\lambda) = \lambda^{-1} \int_{S(X(0))} \left[ \left[ \hat{D}_\nu \sigma - (\text{div} \sigma)(\nu) \right] x + \sum_{i=1}^{3} \sigma(\nu, e_i) e_i - \text{tr} \sigma \nu \right] d\bar{\mu}.
\]

Using (77) and that \((M, g)\) is \( C^2 \)-asymptotically flat, we have

\[
\text{div} \left[ \left( \hat{D}_\nu \sigma - \text{div} \sigma \right) x + \sigma - \text{tr} \sigma \hat{g} \right] = Rx + o(|x|^{-2})
\]

as \( x \to \infty \). Using the divergence theorem and that \( R \in L^1(M, g) \), we conclude that

\[
F' = -\lambda^{-1} F + o(\lambda^{-1}) + h
\]

as \( \lambda \to \infty \) where

\[
\int_1^\infty |h(\lambda)| d\lambda < \infty.
\]

The assertion follows from integration. \( \square \)

The following integration by parts formula has been proven in [8].

**Lemma 36 ([8, p. 168-169]).** Let \( \xi \in \mathbb{R}^3 \) and \( \lambda > 0 \). There holds

\[
\int_{S(t, \lambda)} \left[ \hat{D}_a \sigma - (\hat{D}_a \sigma)(\nu, \nu) - 2 \lambda^{-1} \text{tr} \sigma \hat{g}(a, \nu) \right] d\bar{\mu} = \int_{S(t, \lambda)} \left[ \lambda^{-1} \hat{g}(a, x - \lambda \xi) \left[ \hat{D}_\nu \sigma - (\text{div} \sigma)(\nu) \right] + \lambda^{-1} \sigma(\nu, a) - \lambda^{-1} \hat{g}(a, \nu) \text{tr} \sigma \right] d\bar{\mu}
\]

for every \( a \in \mathbb{R}^3 \).

In the following lemma, we adapt Lemma 36 to a general closed surface \( \Sigma \subset \mathbb{R}^3 \).

**Lemma 37.** Let \( \Sigma \subset \mathbb{R}^3 \) be a closed surface. There holds

\[
\int_{\Sigma} \left[ \hat{D}_a \sigma - (\hat{D}_a \sigma)(\nu, \nu) \right] d\bar{\mu} = \int_{\Sigma} \left[ \hat{g}(a, \nu) \left[ \hat{D}_\nu \sigma - (\text{div} \sigma)(\nu) \right] + \frac{1}{2} \hat{H} \left[ \sigma(\nu, a) + \hat{g}(a, \nu) \text{tr} \sigma \right] \right] d\bar{\mu} + \int_{\Sigma} \left[ \hat{g}(a^\top \hat{h}, (\nu, \sigma)|_{\Sigma}) - \hat{g}(a, \nu) \hat{g}(\hat{h}, \sigma)|_{\Sigma} \right] d\bar{\mu}
\]

for every \( a \in \mathbb{R}^3 \).
Proof. Let \( \{f_1, f_2\} \) be a local Euclidean orthonormal frame for \( T\Sigma \). We have

\[
\begin{align*}
\bar{D}_a \sigma - (\bar{D}_a)\sigma(\bar{\nu}, \nu) &= \bar{g}(a, \nu) \left[ \bar{D}_\nu \sigma - (\bar{\nabla} \sigma)(\bar{\nu}) \right] \\
+ \bar{D}_a^\top \sigma - (\bar{D}_a^\top)\sigma(\bar{\nu}, \nu) + \bar{g}(a, \nu) \sum_{\alpha = 1}^2 (\bar{D}_a)\sigma(f_\alpha, \nu).
\end{align*}
\]

Note that

\[
\begin{align*}
\bar{D}_a^\top \sigma - (\bar{D}_a^\top)\sigma(\bar{\nu}, \nu) + \bar{g}(a, \nu) \sum_{\alpha = 1}^2 (\bar{D}_a)\sigma(f_\alpha, \nu) &= \bar{\nabla}_\Sigma \left( (\sigma a - \sigma(\bar{\nu}, \nu) a + \bar{g}(a, \nu) \sum_{i = 1}^3 \sigma(e_i, \nu) e_i \right) \\
&+ \frac{1}{2} \bar{H} \left[ \sigma(a, \nu) - \bar{g}(a, \nu) \sigma \right] + \bar{g}(a^\top \hat{\lambda}, (\bar{\nu}, \sigma)|_\Sigma) - \bar{g}(a, \nu) \bar{g}(\hat{\lambda}, \sigma|_\Sigma).
\end{align*}
\]

The assertion follows from these identities and the first variation formula. \( \square \)

F. SOME GEOMETRIC EXPANSIONS

We collect several geometric expansions needed in this paper. We assume that \((M, g)\) is \(C^2\)-asymptotically flat of rate \(\tau \in (1/2, 1)\). Recall that \(S_{\xi, \lambda} = \{x \in \mathbb{R}^3 : |x - \lambda \xi| = \lambda\}\).

Lemma 38. Let \(\delta \in (0, 1/2)\) and \(a \in \mathbb{R}^3\) with \(|a| = 1\). There holds, as \(\lambda \to \infty\),

\[
\begin{align*}
\text{div } a &= \frac{1}{2} \bar{D}_a \sigma + O(\lambda^{-1 - 2\tau}), \\
\bar{g}(D_\nu a, \nu) &= \frac{1}{2} (\bar{D}_a)\sigma(\bar{\nu}, \nu) + O(\lambda^{-1 - 2\tau})
\end{align*}
\]

on \(S_{\xi, \lambda}\) uniformly for all \(\xi \in \mathbb{R}^3\) with \(|\xi| < 1 - \delta\). Moreover,

\[
\begin{align*}
\nu - \bar{\nu} &= -\frac{1}{2} \sigma(\bar{\nu}, \nu) \nu - \sum_{\alpha = 1}^2 \sigma(\bar{\nu}, f_\alpha) f_\alpha + O(\lambda^{-2\tau}), \\
\text{d} \mu &= \left[ 1 + \frac{1}{2} \left[ \text{tr } \sigma - \sigma(\bar{\nu}, \nu) \right] + O(\lambda^{-2\tau}) \right] \text{d} \bar{\mu}, \\
H - \bar{H} &= \lambda^{-1} \left[ 2 \sigma(\bar{\nu}, \nu) - \text{tr } \sigma \right] + \frac{1}{2} \left[ \bar{D}_\nu \sigma + (\bar{D}_\nu)\sigma(\bar{\nu}, \nu) - 2 (\bar{\nabla} \sigma)(\bar{\nu}) \right] \\
&\quad + \lambda^{-1} \sigma \ast \sigma \ast \bar{\nu} \ast \bar{\nu} + \lambda^{-1} \sigma \ast \bar{D} \sigma \ast \bar{\nu} + O(\lambda^{-1 - 3\tau}).
\end{align*}
\]

Here, \(\{f_1, f_2\}\) is a local Euclidean orthonormal frame for \(TS_{\xi, \lambda}\).

Lemma 39. Let \(\{\Sigma_i\}_{i = 1}^\infty\) be a sequence of surfaces \(\Sigma_i \subset M\) with \(\lim_{i \to \infty} \rho(\Sigma_i) = \infty\). Let \(a \in \mathbb{R}^3\) with \(|a| = 1\). There holds

\[
\begin{align*}
\text{div } a &= \frac{1}{2} \bar{D}_a \sigma + O(|x|^{-1 - 2\tau}), \\
\bar{g}(D_\nu a, \nu) &= \frac{1}{2} (\bar{D}_a)\sigma(\bar{\nu}, \nu) + O(|x|^{-1 - 2\tau})
\end{align*}
\]
as \( i \to \infty \). Moreover,

\[
\nu - \bar{\nu} = -\frac{1}{2} \sigma(\bar{\nu}, \bar{\nu}) \bar{\nu} - \sum_{\alpha=1}^{2} \sigma(\bar{\nu}, f_{\alpha}) f_{\alpha} + O(|x|^{-2\tau}),
\]

\[
d\mu = \left[ 1 + \frac{1}{2} \text{tr} \sigma - \sigma(\bar{\nu}, \bar{\nu}) \right] + O(|x|^{-2}) \, d\bar{\mu}.
\]

Here, \( \{f_1, f_2\} \) is a local Euclidean orthonormal frame for \( T\Sigma_i \).

**Lemma 40.** There holds, as \( x \to \infty \),

\[
(77) \quad R = \text{div div} \sigma - \bar{\Delta} \text{tr} \sigma + O(|x|^{-2-2\tau}).
\]

### G. Stable CMC spheres in asymptotically Schwarzschild manifolds

Let \((M, g)\) be \(C^4\)-asymptotic to Schwarzschild (58). In their pioneering paper [20], G. Huisken and S.-T. Yau have shown that there is a family

\[(78) \quad \{\Sigma(H) : H \in (0, H_0)\}, \]

\(H_0 > 0\), of stable constant mean curvature spheres \( \Sigma(H) \subset M \) with mean curvature \( H(\Sigma(H)) = H \) that forms a foliation of the complement of a compact subset of \( M \). Moreover, they have established a powerful result characterizing the leaves of this foliation; see [20, Theorem 5.1]. As they explain in [20, Theorem 4.2 and Remark 4.3], the foliation (78) gives rise to both a canonical asymptotic coordinate system of \((M, g)\) and a notion of geometric center of mass defined by

\[
C_{CMC} = (C_{CMC}^1, C_{CMC}^2, C_{CMC}^3)
\]

where

\[
C_{CMC}^{\ell} = \lim_{H \to 0} |\Sigma(H)|^{-1} \int_{\Sigma(H)} x^{\ell} \, d\mu.
\]

provided the limits on the right-hand side exist.

The results in [20] have been improved in various directions. J. Metzger has shown in [22] that the foliation (78) exists under weaker regularity assumptions on \((M, g)\). The characterization of the leaves of the foliation has been strengthened by J. Qing and G. Tian [24]. They show that the leaves are the only large stable constant mean curvature spheres that enclose the center of \((M, g)\). J. Metzger and the first-named author have shown in [15, Theorem 1.1] that the leaves are the only isoperimetric surfaces of their respective volume in \((M, g)\). In the case where the scalar curvature of \((M, g)\) is non-negative, the characterization of the leaves has been refined further by J. Metzger and the first-named author [14], by S. Brendle and the first-named author [4], by A. Carlotto, O. Chodosh, and the first-named author [5], by O. Chodosh and the first named-author [9, 8], and by the authors [13]. In particular, if the scalar curvature of \((M, g)\) is non-negative and satisfies a mild growth condition, then the leaves of the foliation (78) are the only large closed stable constant mean curvature surfaces; see [13, Theorem 49] and the discussion there. An alternative proof of the characterization result by J. Qing and G. Tian [24] in the case where the scalar curvature of \((M, g)\) is non-negative has been given by O. Chodosh and the first-named author in [9, Appendix C]. L.-H. Huang has shown in [16, Theorem 2] that the geometric center of mass (79) agrees with the Hamiltonian center of mass (2) of \((M, g)\) provided either one of them exists. We also note the
work of C. Cederbaum and C. Nerz, who have shown that the existence of the limits in (79) needs additional assumptions; see [7, p. 1624].

H. Related results in asymptotically flat 3-manifolds

In this section, we give an overview of the strategies used in the works of L.-H. Huang [17], of S. Ma [21], and of C. Nerz [23]. We compare these strategies to the methods used and developed in this paper.

The work of L.-H. Huang [17]. L.-H. Huang has established Theorem 1 under the additional assumption that \((M, g)\) satisfies the Regge-Teitelboim conditions in [17]. To construct the asymptotic foliation by constant mean curvature spheres, L.-H. Huang argues in two steps: First, the sphere \(S_{\xi, \lambda}\) is perturbed to a surface \(\tilde{\Sigma}_{\xi, \lambda}\) with \(\text{proj}_{\Lambda} H(\tilde{\Sigma}_{\xi, \lambda}) = O(\lambda^{-1-2\tau})\); see [17, Lemma 3.2]. This improves the estimate (16). The Regge-Teitelboim conditions ensure that certain error terms arising in this perturbation are sufficiently small; see [17, (3.6)]. Second, the spheres \(\tilde{\Sigma}_{\xi, \lambda}\) are perturbed to obtain a constant mean curvature sphere \(\Sigma(\lambda)\); see [17, Theorem 3.1]. L.-H. Huang observes that this second perturbation is obstructed unless \(m \neq 0\) and \(\xi \approx \lambda^{-1} C\), where \(C\) is the Hamiltonian center of mass; see [17, (3.17)]. We remark that (54) provides a reason for this obstruction. Note that Theorem 13 follows as a corollary from the construction.

To establish uniqueness, L.-H. Huang first derives curvatures estimates for large stable constant mean curvature spheres \(\Sigma \subset M\) satisfying (6) and proves a result similar to Lemma 19; see [17, Corollary 4.11]. Using an argument analogous to but slightly coarser than that in the proof of Theorem 11, L.-H. Huang obtains a preliminary estimate on the barycenter of \(\Sigma\). This implies that \(\Sigma\) is within the range of the implicit function theorem. It then follows from local uniqueness that \(\Sigma\) belongs to the asymptotic foliation; see [17, §4.4].

The work of S. Ma [21]. The proof of Theorem 2 by S. Ma [21] expands upon the method of J. Qing and G. Tian in [24].

Let \((M, g)\) be asymptotically flat and suppose that \(\{\Sigma_i\}_{i=1}^{\infty}\) is a sequence of stable constant mean curvature spheres \(\Sigma_i \subset M\) with \(\rho(\Sigma_i) \to \infty\) and \(\rho(\Sigma_i) = o(\lambda(\Sigma_i))\) as \(i \to \infty\). To begin with, S. Ma notes that

\[
0 = \int_{\Sigma_i} (H - \bar{H}) \bar{g}(a, \bar{\nu}) \, d\bar{\mu}
\]

for all \(i\) and \(a \in \mathbb{R}^3\) with \(|a| = 1\); see [21, (34)]. To obtain a contradiction, S. Ma shows that the right-hand side of (80) is essentially proportional to the mass of \((M, g)\) as \(i \to \infty\). To this end, S. Ma studies the contributions to the integral in (80) in the three parts of the surface \(\Sigma_i\) where

- \(|x| \approx \rho(\Sigma_i)|
- \(|x| \approx \lambda(\Sigma_i)|
- \rho(\Sigma_i) \ll |x| \ll \lambda(\Sigma_i)|

see [21, §5]. To obtain sufficient analytic control to estimate these contributions, S. Ma uses the fact that \(\{\nu(\Sigma_i)\}_{i=1}^{\infty}\) forms a sequence of almost harmonic maps.

By comparison, in our proof of Theorem 9 in Section 3, we start with identity (35), which differs...
slightly from (80). Note that the right-hand side of (35) stands in for the derivative of the function $G_\lambda$. In view of Lemma 18, we expect this derivative to be proportional to the mass of $(M, g)$. As shown in Proposition 31, the assumption of non-negative scalar curvature leads to strong analytic control of the surfaces $\Sigma_i$. This control, in conjunction with the integration by parts formula from Lemma 37, is sufficient to conclude the argument. In particular, we do not require the delicate analysis of the unit normal as an almost harmonic map in the region where $\rho(\Sigma_i) \ll |x| \ll \lambda(\Sigma_i)$ developed by S. Ma [21, §4] building on the work of J. Qing and G. Tian [24, §4].

**The work of C. Nerz** [23]. C. Nerz’s proof of Theorem 1 in [23] expands upon the continuity argument developed by J. Metzger in [22]. Given an asymptotically flat metric $g$ on $\{x \in \mathbb{R}^3 : |x| > 1\}$ with mass $m \neq 0$, C. Nerz defines the family of metrics $\{g_t : t \in [0, 1]\}$ where

$$g_t = t g + (1 - t) \left(1 + \frac{m}{2|x|}\right)^4 \bar{g};$$

see [23, §5]. Note that $g_1 = g$ and that $g_0$ is equal to Schwarzschild with mass $m$. Let $I \subset [0, 1]$ be the set of all $t \in [0, 1]$ such that the conclusion of Theorem 1 holds for $g_t$. The difficult step when proving that $I = [0, 1]$ is to show that $I$ is open. To this end, C. Nerz notes that the stability operator (61) of large, centered constant mean curvature spheres with non-vanishing Hawking mass is invertible, even if $(M, g)$ is only asymptotically flat; see [23, Proposition 4.7]. In Lemma 18, we supply a geometric reason for this invertibility.

Moreover, using the same kind of continuity argument, C. Nerz observes that the leaves of the foliation from Theorem 1 are asymptotically symmetric provided $(M, g)$ satisfies the weak Regge-Teitelboim conditions (60). This step should be compared to Lemma 21. To establish Theorem 7, C. Nerz goes on to prove that

$$\frac{d}{dt} C_{CMC}(g_t) = C$$

where $C_{CMC}(g_t)$ is the geometric center of mass (7) of $(M, g_t)$; see [23, §6].

To conclude uniqueness in Theorem 1, C. Nerz follows an idea developed by J. Metzger. More precisely, one may reverse the continuity argument (81) and use the local uniqueness of the inverse function theorem and the global uniqueness of constant mean curvature spheres in Schwarzschild [3]; see [23, §5].

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