On the Generalization Properties of Differential Privacy

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Abstract

A new line of work, started with Dwork et al. [2], studies the task of answering statistical queries using a sample and relates the problem to the concept of differential privacy [4]. By the Hoeffding bound, a sample of size $O(\log k/\alpha^2)$ suffices to answer $k$ non-adaptive queries within error $\alpha$, where the answers are computed by evaluating the statistical queries on the sample. This argument fails when the queries are chosen adaptively (and can hence depend on the sample). Dwork et al. showed that if the answers are computed with $(\epsilon, \delta)$-differential privacy then $O(\epsilon)$ accuracy is guaranteed with probability $1 - O(\delta^2)$. Using the Private Multiplicative Weights mechanism [5], they concluded that the sample size can still grow polylogarithmically with the $k$.

Very recently, Bassily et al. [1] presented an improved bound and showed that (a variant of) the private multiplicative weights algorithm can answer $k$ adaptively chosen statistical queries using sample complexity that grows logarithmically in $k$. However, their results no longer hold for every differentially private algorithm, and require modifying the private multiplicative weights algorithm in order to obtain their high probability bounds.

We greatly simplify the results of Dwork et al. and improve on the bound by showing that differential privacy guarantees $O(\epsilon)$ accuracy with probability $1 - O(\delta \log(1/\epsilon)/\epsilon$). It would be tempting to guess that an $(\epsilon, \delta)$-differentially private computation should guarantee $O(\epsilon)$ accuracy with probability $1 - O(\delta)$. However, we show that this is not the case, and that our bound is tight (up to logarithmic factors). Our bound also improves the guarantees of the private multiplicative weights algorithm such that the sample complexity grows logarithmically in $k$.

While the bounds of Bassily et al. have a slightly better dependence on the accuracy parameter, our bounds are proven as a property of differential privacy and hold for every differentially private computation of a predicate.
1 Introduction

A new line of work, started with Dwork et al. [2], studies the task of answering $k$ adaptively chosen statistical queries w.r.t. an unknown distribution using i.i.d. sample from it. Consider a data analyst interested in learning properties of an unknown distribution $D$ over a domain $X$. The analyst interacts with $D$ through a data curator $A$ holding a database $S = (s_i)$ containing i.i.d. samples from $D$. The interaction is adaptive, where at every round the analyst specifies a query $q : X \rightarrow \{0, 1\}$ and receives an answer $a_q(S)$ that (hopefully) approximates $q(D) = \mathbb{E}_{x \sim D}[q(x)]$. Dwork et al. focused on the question of how many samples should $S$ contain to enable $A$ to answer accurately.

Let $k$ be the number of queries the analyst makes. Using the Hoeffding bound, it is easy to see that if all $k$ queries are fixed before interacting with $A$, then $A$ could simply answer every query with its empirical average on $S$, i.e., letting $a_q(S) = \frac{|\{i : q(s_i) = 1\}|}{|S|}$. Having $|S| = O\left(\frac{\log(k)}{\alpha^2}\right)$ would then suffice for making all $k$ answers $\alpha$-approximate. This approach would however fail for the case where queries are chosen adaptively, and, surprisingly, Dwork et al. [2] showed that it is possible to answer $k$ adaptively chosen queries if $|S|$ grows polylogarithmically with $k$, i.e., $|S| = O\left(\frac{\log(k)^3}{\alpha^3} \sqrt{\log |X|}\right)$. Very recently, Bassily et al. [1] presented an improved bound showing that $\tilde{O}\left(\frac{\log(k)}{\alpha^3} \sqrt{\log |X|}\right)$ samples suffice. Both results are obtained via a connection to differential privacy [4], a mathematical definition for privacy providing strong guarantees. Loosely speaking, differential privacy bounds the effect of every single entry in the database on the outcome distribution of the algorithm.

Definition 1.1 ([4]). A randomized algorithm $M : X^n \rightarrow Y$ is $(\epsilon, \delta)$ differentially private if for every two datasets $S, S' \in X^n$ that differ on one row, and every set $T \subseteq Y$, we have

$$\Pr[M(S) \in T] \leq e^\epsilon \cdot \Pr[M(S') \in T] + \delta.$$ 

By now a rich literature has shown that many data analysis tasks are compatible with differential privacy. One task of special interest is to design differentially private algorithms capable of accurately estimating the empirical average of a given predicate on the (private) input database. This task has proven to be very challenging (although it is trivial without the privacy requirement), and has been the subject of a long line of works. The state of the art mechanism for this task is the Private Multiplicative Weights mechanism (PMW) by Hardt and Rothblum [5]. The PMW accurately estimates the empirical average of $k$ adaptively chosen predicates using a database of size $O\left(\frac{1}{\alpha^2} \log(k) \sqrt{\log |X|}\right)$.

1.1 Differential Privacy and Generalization

The results in [2] and [1] (for accurately answering $k$ adaptively chosen statistical queries) are obtained by showing that when the PMW is applied to a database containing i.i.d. samples from a distribution $D$, its answers are also accurate w.r.t. $D$ (and not just with respect to the database). In fact, Dwork et al. [2] showed that a predicate computed with differential privacy automatically provide generalization. Bassily et al. [1] improved the generalization parameters, but their results no longer hold for every differentially private computation. We improve on Dwork et al. [2] and get parameters that are (almost) as good as [1], but what we get works for every differentially private
computation of predicates. Specifically, we show that the following high probability bound holds for any differentially private computation:

**Theorem 1.2.** Let \( A : X^n \rightarrow 2^X \) be an \((\epsilon, \delta)\)-differentially private algorithm that operates on a database of size \( n \geq 2^{\frac{2}{\epsilon}} \ln(\frac{8}{\delta}) \) and outputs a predicate \( h : X \rightarrow \{0, 1\} \). Let \( D \) be a distribution over \( X \), let \( S \) be a database containing \( n \) i.i.d. elements from \( D \), and let \( h \leftarrow A(S) \). Then,

\[
\Pr[|h(S) - h(D)| > 13\epsilon] \leq \frac{2\delta}{\epsilon} \ln \left( \frac{2}{\epsilon} \right),
\]

where \( h(S) \) is the empirical average of \( h \) on \( S \), and \( h(D) \) is the expectation of \( h \) over \( D \).

In words, if \( A \) is a differentially private algorithm operating on a database containing \( n \) i.i.d. samples from \( D \), then \( A \) cannot (with significant probability) identify a predicate that behaves differently on the sample \( S \) and on \( D \).

We begin with the following expectation bound. Consider an algorithm \( A \) operating on \( \ell \) sub-databases \( \vec{S} = (S_1, S_2, \ldots, S_\ell) \), where every \( S_i \) contains i.i.d. samples from \( D \). Algorithm \( A \) outputs a predicate \( h \) and an index \( 1 \leq i \leq \ell \), and succeeds if \( h(D) \) is far from \( h(S_i) \). That is, algorithm \( A \) is aimed at identifying a sub-database \( S_i \) and a predicate \( h \) that behaves differently on \( S_i \) and on \( D \). We first show that no differentially private algorithm can succeed in this task in expectation. That is, if \( A \) is differentially private, then the expectations \( \mathbb{E}_{\vec{S}, A}[h(D)] \) and \( \mathbb{E}_{\vec{S}, A}[h(S_i)] \) are close.

This expectation bound for algorithms that operate on \( \ell \) sub-databases is then transformed into a high probability bound on private algorithms that operate on a single database: Assume the existence of a differentially private algorithm \( B \) that operates on a database \( S \) containing i.i.d. samples from \( D \) and, with probability \( \beta \), outputs a hypothesis \( h \) s.t. \( h(S) \) is far from \( h(D) \). We can use \( B \) to construct an algorithm \( A \) operating on \( \ell \approx \frac{1}{\beta} \log(\frac{1}{\beta}) \) sub-databases that contradicts our expectation bound. Specifically, algorithm \( A \) applies \( B \) on every sub-database, and obtains \( \ell \) predicates \( H = \{h_1, \ldots, h_\ell\} \). Since \( \ell \approx \frac{1}{\beta} \log(\frac{1}{\beta}) \), w.h.p. at least one of the \( h_i \)'s behaves differently on \( S_i \) and on \( D \), and \( B \) can identify such an \( h_i \in H \) using standard differentially private tools. This will contradict our expectation bound.

### 1.2 Application to Private Learning

Another application of Theorem 1.2 is in the context of private learning [6]. A private learning algorithm for a concept class \( C \) is a PAC learner for \( C \) that is also differentially private. A typical proof that an algorithm is a private learner consists of proving three properties: (1) The algorithm is differentially private, (2) The algorithm identifies a hypothesis with small empirical error, and (3) The output hypothesis has good generalization. Theorem 1.2 can be used to prove that the 3rd property follows from the previous two.

### 1.3 Comparison with [2]

Our result is similar in spirit to that of Dwork et al. [2], who also gave a high probability bound that holds for any differentially private algorithm. Such bounds involve two parameters – the accuracy parameter \( \alpha \) (bounding the difference between \( h(D) \) and \( h(S) \)), and the confidence parameter \( \beta \) (guaranteeing that the bound on \( |h(D) - h(S)| \) holds with all but probability \( \beta \)). The confidence parameter in our bound is significantly improved over [2]: For an \((\epsilon, \delta)\)-differentially private computation, our bound guarantees \( 13\epsilon \)-accuracy with all but probability \( \frac{2\delta}{\epsilon} \ln(\frac{2}{\epsilon}) \), whereas
their bound guarantees 4ε-accuracy with all but probability 8(δ)ε. A typical setting of parameters for differentially private algorithms would be to let ε be a small constant, and let δ be a negligible function of the database size. For example, if ε = 0.01 and δ = 10−10 then the confidence parameter in our bound would be ≈ 10−7, whereas the confidence parameter of [2] would be bigger than 1. Specifically, in order to get a confidence parameter β, the results of [2] require that δ ≤ (β/8)1/ε whereas our results only require that δ ≤ 2ln(2/ε).

It would be tempting to guess that an (ε, δ)-differentially private computation should guarantee O(ε) accuracy with probability 1 − O(δ). However, we show that this is not the case, and that our bound is tight (up to logarithmic factors).

2 Preliminaries

Consider a distribution D over a domain X, a database S ∈ X*, and a predicate h : X → {0, 1}. Throughout this work, we use the convention that h(S) is the average of h on S, and that h(D) = Prx∼D[h(x) = 1].

2.1 Differential Privacy

Our results rely on a number of basic facts about differential privacy. We will present algorithms that access their input database using several differentially private mechanisms and use the following composition theorem to prove their overall privacy guarantee.

Lemma 2.1 (Composition, e.g. [3]). Let A1 : Xn → R1 be (ε1, δ1)-differentially private. Let A2 : Xn × R1 → R2 be (ε2, δ2)-differentially private for any fixed value of its second argument. Then the composition A(S) = A2(S, A1(S)) is (ε1 + ε2, δ1 + δ2)-differentially private.

One primitive we will use is the exponential mechanism of McSherry and Talwar [7]. A quality function q : X* × F → R defines an optimization problem over the domain X and a finite solution set F: Given a database S ∈ X*, choose f ∈ F that (approximately) maximizes q(S, f). The exponential mechanism solves such an optimization problem by choosing a random solution where the probability of outputting any solution f increases exponentially with its quality q(S, f). Specifically, we say that q has sensitivity Δ if for every f ∈ F and for every two neighboring databases S, S′ ∈ Xn, it holds that |q(S, f) − q(S′, f)| ≤ Δ. The exponential mechanism outputs each f ∈ F with probability ∝ exp(ε · q(S, f)/2Δ).

Proposition 2.2 (Properties of the Exponential Mechanism).

1. The exponential mechanism is ε-differentially private.

2. Let q be a quality function with sensitivity at most 1. Fix a database S ∈ Xn and let OPT = maxf∈F{q(S, f)}. Let t > 0. Then exponential mechanism outputs a solution f with q(S, f) ≤ OPT − tn with probability at most |F| · exp(−tn/2).

3 From Expectation to High Probability Bounds

Let A be an (ε, δ)-differentially private algorithm that outputs a predicate, and consider a database S sampled i.i.d. from some distribution D. Our goal is to show that if h ← A(S), then with high
probability $h(D)$ is close to the empirical average $h(S)$, where the probability is over the choice of $S$ and the random coins of $A$. This is in spite of $h$ being chosen based on $S$. Our main theorem is the following.

**Theorem 3.1.** Let $A : X^n \rightarrow 2^X$ be an $(\epsilon, \delta)$-differentially private algorithm that operates on a database of size $n \geq \frac{26}{\delta} \ln \left( \frac{74}{\alpha \beta} \right)$ and outputs a predicate $h : X \rightarrow \{0, 1\}$. Let $D$ be a distribution over $X$, let $S$ be a database containing $n$ i.i.d. elements from $D$, and let $h \leftarrow A(S)$. Then,

$$\Pr \left[ |h(S) - h(D)| > 13\epsilon \right] \leq \frac{2\delta}{\epsilon} \ln \left( \frac{2}{\epsilon} \right).$$

**Remark 3.2.** In Theorem 3.1 we derived the accuracy and confidence parameters from the privacy parameters $\epsilon, \delta$. Alternatively, for every $\alpha, \beta > 0$, if $A$ is an $(\epsilon, \delta) \leq \frac{\alpha \delta}{26 \ln(12/\alpha)}$-differentially private algorithm that operates on a database $S$ containing $n \geq \frac{676}{\alpha \beta} \ln(\frac{74}{\alpha \beta})$ i.i.d. samples from $D$ and outputs a predicate $h$, then $\Pr \left[ |h(S) - h(D)| > \alpha \right] \leq \beta$.

As an example for using Theorem 3.1, let $C$ be a class of predicates, let $\alpha, \beta > 0$ and let $n \geq \frac{676}{\alpha \beta} \ln(\frac{74}{\alpha \beta})$. Assume that $A$ is an algorithm that operates on a labeled database $S$ of size $n$ and outputs a predicate $h$ (not necessarily in $C$). Furthermore, assume that with probability at least $(1 - \beta)$ the output $h$ is s.t. error$_S(h) \leq \alpha + \min_{f \in C} \{\text{error}_S(f)\}$, where error$_S(f)$ is the empirical error of $f$ on $S$. Theorem 3.1 now states that if $A$ is $(\epsilon, \delta)$-differentially private for $\epsilon = \frac{\alpha}{15}$ and $\delta = \frac{\alpha \beta}{26 \ln(12/\alpha)}$ then $A$ is a $(2\alpha, 2\beta)$-PAC learner for $C$.

Similarly to [2] and [1], we start the proof of Theorem 3.1 by showing that $h(S)$ and $h(D)$ are close in expectation.

**Lemma 3.3.** Let $A : (X^*)^\ell \rightarrow 2^X \times \{1, 2, \ldots, \ell\}$ be an $(\epsilon, \delta)$-differentially private algorithm that operates on $\ell$ sub-databases and outputs a predicate $h : X \rightarrow \{0, 1\}$ and an index $i \in \{1, 2, \ldots, \ell\}$. Let $D$ be a distribution over $X$, let $\vec{S} = (S_1, \ldots, S_\ell)$ where every $S_i$ is a database containing i.i.d. elements from $D$, and let $(h, i) \leftarrow A(\vec{S})$. Then

$$\mathbb{E}_{\vec{S} \sim D} \left[ \mathbb{E}_{(h, i) \sim A(\vec{S})} \left[ h(S_i) \right] \right] \leq \ell \delta + 2\epsilon + \mathbb{E}_{\vec{S} \sim D} \left[ \mathbb{E}_{(h, i) \sim A(\vec{S})} \left[ h(D) \right] \right].$$

In Claim 3.4 below, we will show that this limits the ability of $A$ to identify a sub-database $S_i$ and a predicate $h$ s.t. $h(S_i)$ is far from $h(D)$. Note that $\ell = 1$ gives the standard case where $A$ operates on a single database.

**Proof.** We use $\vec{x} \sim \vec{S}$ to denote $\vec{x} = (x_1, \ldots, x_\ell)$ where each $x_i \sim S_i$. Hence,

$$\mathbb{E}_{\vec{S} \sim D} \left[ \mathbb{E}_{(h, i) \sim A(\vec{S})} \left[ h(S_i) \right] \right] = \mathbb{E}_{\vec{S} \sim D} \left[ \mathbb{E}_{(h, i) \sim A(\vec{S})} \left[ \mathbb{E}_{\vec{x} \sim \vec{S}} \left[ h(x_i) \right] \right] \right] = \mathbb{E}_{\vec{S} \sim D} \left[ \mathbb{E}_{\vec{x} \sim \vec{S}} \left[ \Pr_{A} \left[ \exists i \text{ s.t. } A(\vec{S}) = (h, i) \text{ and } h(x_i) = 1 \right] \right] \right]$$

$$= \mathbb{E}_{\vec{S} \sim D} \left[ \mathbb{E}_{\vec{x} \sim \vec{S}} \left[ \sum_{i=1}^\ell \Pr_{A} \left[ A(\vec{S}) = (h, i) \text{ and } h(x_i) = 1 \right] \right] \right]$$

$$= \mathbb{E}_{\vec{S} \sim D} \left[ \sum_{i=1}^\ell \Pr_{A} \left[ A(\vec{S}) = (h, i) \text{ and } h(x_i) = 1 \right] \right].$$

(1)
Given $\vec{S} = (S_1, \ldots, S_\ell)$ and an element $z \in X$, we define $\vec{S}^{(x_i:z)}$ to be the same as $\vec{S}$, except that $x_i \in S_i$ is replaced with $z$. Note that $\vec{S}$ and $\vec{S}^{(x_i:z)}$ are neighboring. Hence,

$$
(1) \leq \mathbb{E}_{\vec{S} \sim D} \left[ \mathbb{E}_{\vec{x} \sim \vec{S}} \left[ \sum_{i=1}^{\ell} e^\epsilon \cdot \Pr_{A} \left[ A(\vec{S}^{(x_i:z)}) = (h, i) \text{ and } h(x_i) = 1 \right] \right] + \delta \right]
$$

$$
= \ell \delta + e^\epsilon \cdot \sum_{i=1}^{\ell} \mathbb{E}_{\vec{S} \sim D} \left[ \Pr_{A} \left[ A(\vec{S}^{(x_i:z)}) = (h, i) \text{ and } h(x_i) = 1 \right] \right]
$$

Now note that every $\vec{S}^{(x_i:z)}$ above contains i.i.d. samples from $D$, and that $x_i$ is independent of $\vec{S}^{(x_i:z)}$. Hence,

$$
(2) = \ell \delta + e^\epsilon \cdot \sum_{i=1}^{\ell} \mathbb{E}_{\vec{S} \sim D} \left[ \Pr_{A} \left[ \exists i \text{ s.t. } A(\vec{S}) = (h, i) \text{ and } h(x_i) = 1 \right] \right]
$$

$$
= \ell \delta + e^\epsilon \cdot \mathbb{E}_{\vec{S} \sim D} \left[ \Pr_{A} \left[ \exists i \left( h(i) \leq h(D) \right) \right] \right]
$$

Our goal is to transform the expectation bound in Lemma 3.3 into a high probability bound. The first step is to transform it into the following “low probability bound”:

**Claim 3.4.** Let $A : (X^*)^{\ell} \rightarrow 2^X \times \{1, 2, \ldots, \ell\}$ be an $(\epsilon, \delta)$-differentially private algorithm that operates on $\ell$ sub-databases and outputs a predicate $h : X \rightarrow \{0, 1\}$ and an index $i \in \{1, 2, \ldots, \ell\}$. Let $D$ be a distribution over $X$, let $\vec{S} = (S_1, \ldots, S_\ell)$ where every $S_i$ is a database containing i.i.d. elements from $D$, and let $(h, i) \leftarrow A(\vec{S})$. Then,

$$
\Pr[h(S_i) \leq h(D) + \ell \delta + 4\epsilon] \geq \epsilon,
$$

where the probability is over the choice of $\vec{S}$ and random coins of $A$.

**Proof.** Assume towards contradiction that $A(\vec{S})$ outputs a pair $(h, i)$ s.t. $h(S_i) > \ell \delta + 4\epsilon + h(D)$ with probability greater than $(1 - \epsilon)$. Then,
\[ \mathbb{E}_{\vec{S},(h,i)}[h(S_i)] = \sum_{\vec{S},(h,i)} h(S_i) \cdot \Pr[\vec{S},(h,i)] \]

\[ = \sum_{\vec{S},(h,i): h(S_i) > \ell + 4\epsilon + h(\mathcal{D})} h(S_i) \cdot \Pr[\vec{S},(h,i)] + \sum_{\vec{S},(h,i): h(S_i) \leq \ell + 4\epsilon + h(\mathcal{D})} h(S_i) \cdot \Pr[\vec{S},(h,i)] \]

\[ > \sum_{\vec{S},(h,i): h(S_i) > \ell + 4\epsilon + h(\mathcal{D})} (\ell \delta + 4\epsilon + h(\mathcal{D})) \cdot \Pr[\vec{S},(h,i)] \]

\[ > (1 - \epsilon)(\ell \delta + 4\epsilon) + \sum_{\vec{S},(h,i): h(S_i) > \ell + 4\epsilon + h(\mathcal{D})} h(\mathcal{D}) \cdot \Pr[\vec{S},(h,i)] \]

(3)

Noting that
\[ \epsilon > \sum_{\vec{S},(h,i): h(S_i) \leq \ell + 4\epsilon + h(\mathcal{D})} \Pr[\vec{S},(h,i)] \geq \sum_{\vec{S},(h,i): h(S_i) \leq \ell + 4\epsilon + h(\mathcal{D})} h(\mathcal{D}) \cdot \Pr[\vec{S},(h,i)], \]

we get

\[ > (1 - \epsilon)(\ell \delta + 4\epsilon) + \sum_{\vec{S},(h,i): h(S_i) > \ell + 4\epsilon + h(\mathcal{D})} h(\mathcal{D}) \cdot \Pr[\vec{S},(h,i)] \]

\[ = (1 - \epsilon)(\ell \delta + 4\epsilon) - \epsilon + \sum_{\vec{S},(h,i)} h(\mathcal{D}) \cdot \Pr[\vec{S},(h,i)] \]

\[ \geq \ell \delta + 2\epsilon + \mathbb{E}_{\vec{S},(h,i)}[h(\mathcal{D})]. \]

This contradicts Lemma 3.3.

We now transform the above “low probability bound” for private algorithms that operate on \( \ell \) databases into a high probability bound for private algorithms that operate on a single database.

**Lemma 3.5.** Let \( A : X^n \rightarrow 2^X \) be an \((\epsilon, \delta)\)-differentially private algorithm that operates on a database of size \( n \geq \frac{\delta}{\epsilon} \ln\left(\frac{2}{\delta}\right) \) and outputs a predicate \( h : X \rightarrow \{0, 1\} \). Let \( \mathcal{D} \) be a distribution over \( X \), let \( S \) be a database containing \( n \) i.i.d. elements from \( \mathcal{D} \), and let \( h \leftarrow A(S) \). Then,

\[ \Pr[h(S) \leq 13\epsilon + h(\mathcal{D})] \geq 1 - \frac{\delta}{\epsilon} \ln\left(\frac{2}{\epsilon}\right), \]

where the probability is over the choice of \( \vec{S} \) and random coins of \( A \).

**Proof.** Assume towards contradiction that with probability greater than \( \frac{\delta}{\epsilon} \ln\left(\frac{2}{\epsilon}\right) \) algorithm \( A \) outputs a predicate \( h \) s.t. \( h(S) > 13\epsilon + h(\mathcal{D}) \). We now use \( A \) to construct the following algorithm \( B \) that contradicts claim 3.4.
Let \( D \) be a distribution over \( X \) with \( \epsilon \), \( \delta \)-differential privacy guarantees \((\epsilon, \delta)\)-differential privacy. By composition, algorithm \( B \) is \((2\epsilon, \delta)\)-differentially private.

Now consider applying \( B \) on databases \( \bar{S} = (S_1, \ldots, S_\ell), T \) containing i.i.d. samples from some distribution \( D \). Observe that \( T \) is independent of \( h_1, \ldots, h_\ell \). Hence, for \(|T| \geq \frac{1}{2\epsilon} \ln(\frac{8}{\delta}) = \frac{1}{2\epsilon} \ln(\frac{8}{\delta})\), the Hoeffding bound ensures that except with probability \( \frac{\delta}{\epsilon} \), for all \( i \) it holds that \(|h_i(T) - h_i(D)| \leq \epsilon\).

By our assumption on \( A \) (and since \( \ell = \frac{\delta}{\epsilon} \)), with probability greater than \((1 - \frac{\delta}{\epsilon})\) there exists an \( i \) s.t. \( h_i(S_i) > 13\epsilon + h_i(D) \). Hence, with probability greater than \((1 - \frac{\delta}{\epsilon})\) there exists an \( i \) s.t. \( h_i(S_i) > 12\epsilon + h_i(T) \). Assuming that this is the case, for \( n \geq \frac{2}{\epsilon} \ln(\frac{4\ell}{\delta}) = \frac{2}{\epsilon} \ln(\frac{4\ell}{\delta})\), the exponential mechanism ensures that with probability at least \((1 - \frac{\delta}{\epsilon})\) the chosen \((h_i, i)\) is s.t. \( h_i(S_i) > 11\epsilon + h_i(T) \).

In all, with probability greater than \((1 - \epsilon)\), the returned \((h_i, i)\) is such that \( h_i(S_i) > 11\epsilon + h_i(T) > 2\ell\delta + 8\epsilon + h_i(D) \). This contradicts Claim 3.4.

In Lemma 3.5 we bounded the probability that the empirical average is significantly larger than the expectation. Similar arguments could be used to bound the reverse direction (i.e., that the empirical average is significantly smaller than the expectation), proving Theorem 3.1.

In Section 3 we showed that \((\epsilon, \delta)\)-differential privacy guarantees \( O(\epsilon) \) accuracy with probability \( 1 - O(\delta \log(1/\epsilon)/\epsilon) \). It would be tempting to guess that \((\epsilon, \delta)\)-differential privacy should guarantee \( O(\epsilon) \) accuracy with probability \( 1 - O(\delta) \). As we will now see, this is not the case, and our results are tight (up to logarithmic factors).

### 4 Negative Results

In Section 4 we showed that \((\epsilon, \delta)\)-differential privacy guarantees \( O(\epsilon) \) accuracy with probability \( 1 - O(\delta \log(1/\epsilon)/\epsilon) \). It would be tempting to guess that \((\epsilon, \delta)\)-differential privacy should guarantee \( O(\epsilon) \) accuracy with probability \( 1 - O(\delta) \). As we will now see, this is not the case, and our results are tight (up to logarithmic factors).
Theorem 4.1. Let $U$ be the uniform distribution over $[0, 1]$. For every $\alpha > \delta$ there exists a $(0, \delta)$-differentially private algorithm $A$ such that the following holds. If $S$ is a database containing $n \geq \frac{1}{\alpha}$ i.i.d. samples from $U$, and if $h \leftarrow A(S)$ then

$$\Pr[h(S) \geq h(U) + \alpha] \geq \frac{\delta}{2\alpha}.$$  

Proof. Consider the following simple algorithm, denoted as $B$. On input a database $S$, output $S$ with probability $\delta$, and otherwise output the empty database. Clearly, $B$ is $(0, \delta)$-differentially private. Now construct the following algorithm $A$.

Algorithm $A$

**Input:** $\frac{1}{\alpha}$ databases of size $\alpha n$ each: $\vec{S} = (S_1, \ldots, S_{1/\alpha})$.

1. For $1 \leq i \leq \frac{1}{\alpha}$ let $\hat{S}_i = B(S_i)$.
2. Return $h : [0, 1] \to \{0, 1\}$ where $h(x) = 1$ iff $\exists i$ s.t. $x \in \hat{S}_i$.

As $B$ is $(0, \delta)$-differentially private, and as $A$ only applies $B$ on disjoint databases, we get that $A$ is also $(0, \delta)$-differentially private.

Suppose $\vec{S} = (S_1, \ldots, S_{1/\alpha})$ contains i.i.d. samples from $U$, and let $h \leftarrow A(\vec{S})$. Observe that $h$ evaluates to 1 only on a finite number of points from $[0, 1]$, and hence, we have that $h(U) = 0$. Next note that $h(\vec{S}) = \alpha \cdot |\{i : \hat{S}_i = S_i\}|$. Therefore, if there exists an $i$ s.t. $\hat{S}_i = S_i$ then $h(\vec{S}) \geq h(U) + \alpha$.

The probability that this is not the case is at most

$$(1 - \delta)^{1/\alpha} \leq e^{-\delta/\alpha} \leq 1 - \frac{\delta}{2\alpha},$$

ans thus, with probability at least $\frac{\delta}{2\alpha}$, algorithm $A$ outputs a predicate $h$ s.t. $h(S) \geq h(U) + \alpha$. \qed

In particular, using Theorem 4.1 with $\alpha = \epsilon$ shows that the confidence parameter in Theorem 3.1 is tight (up to logarithmic factors).

5 Answering $k$ Adaptively Chosen Statistical Queries

Consider an $(\epsilon, \delta)$-differentially private mechanism $A$ that holds a database $S \sim \mathcal{D}$ and answers $k$-adaptively chosen statistical queries specified by an analyst: $h_1, h_2, \ldots, h_k$. Dwork et al. observed that since the analyst only interacts with $S$ through $A$, the queries themselves are the result of an $(\epsilon, \delta)$-differentially private computation on $S$. Hence, by the results of the previous section, for every single query $h_j$ with high probability it holds that $h_j(S) \approx h_j(\mathcal{D})$. We now show that this holds simultaneously for all of the queries.

Lemma 5.1. Let $A$ be an $(\epsilon, \delta)$-differentially private algorithm that operates on a database of size $n \geq \frac{2}{\epsilon} \ln \left(\frac{2k}{\delta}\right)$ and outputs $k$ predicates. Let $\mathcal{D}$ be a distribution over $X$, let $S$ be a database containing $n$ i.i.d. elements from $\mathcal{D}$, and let $\{h_1, h_2, \ldots, h_k\}$ denote the output of $A(S)$. Then,

$$\Pr[\exists j \text{ s.t. } |h_j(S) - h_j(\mathcal{D})| > 55\epsilon] \leq \frac{4\delta}{\epsilon} \ln \left(\frac{2}{\epsilon}\right),$$

where the probability is over the choice of $\vec{S}$ and random coins of $A$. 

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Proof. Assume towards contradiction that with probability greater than $\frac{4\delta}{\epsilon} \ln(\frac{2}{\epsilon})$ algorithm $A$ outputs $k$ predicates $h_1, \ldots, h_k$ s.t. there is a $j$ for which $|h_j(S) - h_j(D)| > 55\epsilon$. We now use $A$ to construct the following algorithm $B$ that contradicts Lemma 3.6.

Algorithm $B$

Input: Parameters $\epsilon, \delta$ and two databases of size $n$ each: $S, T$.

1. Let $A(S) = H = \{h_1, h_2, \ldots, h_k\}$.
2. Use the exponential mechanism with privacy parameter $\epsilon$ to choose $h \in H$ with large
   
   $q(S, T, h) \triangleq n \cdot |h(S) - h(T)|$.
3. Output $(h, 1)$.

First observe that the function $q$ defined is Step 2 is of sensitivity 1, and hence, applying the exponential mechanism on Step 2 preserves $(\epsilon, 0)$-differential privacy. By composition, algorithm $B$ is $(2\epsilon, \delta)$-differentially private.

Now consider applying $B$ on databases $S, T$ containing i.i.d. samples from some distribution $D$. Observe that $T$ is independent of $h_1, \ldots, h_k$. Hence, for $|T| \geq \frac{\epsilon^2}{4} \ln(\frac{2}{\epsilon})$, the Hoeffding bound ensures that except with probability $(1 - \delta)$, for all $j$ it holds that $|h_j(T) - h_j(D)| \leq \epsilon$.

By our assumption on $A$, with probability greater than $\frac{4\delta}{\epsilon} \ln(\frac{2}{\epsilon})$ there exists a $j$ s.t. $|h_j(S) - h_j(D)| > 55\epsilon$. Hence, with probability greater than $\frac{3\delta}{\epsilon} \ln(\frac{2}{\epsilon})$ there exists a $j$ s.t. $|h_j(S) - h_j(T)| > 54\epsilon$. Assuming that this is the case, for $n \geq \frac{3}{\epsilon^2} \ln(\frac{2}{\epsilon})$, the exponential mechanism ensures that with probability at least $(1 - \delta)$ the chosen $h$ is s.t. $|h(S) - h(T)| > 53\epsilon$.

All in all, with probability greater than $\frac{3\delta}{\epsilon} \ln(\frac{2}{\epsilon})$, the returned $h$ is such that $|h(S) - h(T)| > 53\epsilon$, and hence, $|h(S) - h(D)| > 52\epsilon$. This contradicts Lemma 3.6.

Using Lemma 5.1 with the Private Multiplicative Weights (PMW) mechanism of [5], yields the following corollary.

**Corollary 5.2.** When given an input database $S \in X^n$ containing i.i.d. samples from $D$, the PMW is $(\alpha, \beta)$-accurate w.r.t. $D$, provided that

$$n = O\left(\frac{1}{\alpha^3} \log \left(\frac{k}{\beta}\right) \log \left(\frac{1}{\alpha\beta}\right) \sqrt{\log |X|}\right).$$

We remark that the bound on $n$ achieved by [1] is slightly better than ours (by a polylog$(\frac{1}{\alpha\beta})$ factor). On the other hand, our analysis holds directly for the PMW itself, where in the analysis of [1] the PMW is used to construct another algorithm achieving their bound.

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