The group of quasi-isometries of the real line cannot act effectively on the line

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We prove that the group $\text{QI}^+(\mathbb{R})$ of orientation-preserving quasi-isometries of the real line is a left-orderable, nonsimple group, which cannot act effectively on the real line $\mathbb{R}$.

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1 Introduction

A function $f : X \to Y$ between metric spaces $X$ and $Y$ is a quasi-isometry if there exist real numbers $K \geq 1$ and $C \geq 0$ such that

$$\frac{1}{K}d(x_1, x_2) - C \leq d(f(x_1), f(x_2)) \leq Kd(x_1, x_2) + C$$

for any $x_1, x_2 \in X$, and $d(\text{Im } f, y) \leq C$ for any $y \in Y$. Two quasi-isometries $f$ and $g$ are called equivalent if they are of bounded distance; ie $\sup_{x \in X} d(f(x), g(x)) < \infty$. The quasi-isometry group $\text{QI}(X)$ is the group of all equivalence classes $[f]$ of quasi-isometries $f : X \to X$ under composition. The notion of quasi-isometries is one of the fundamental concepts in geometric group theory. In this note, we consider the quasi-isometry group $\text{QI}(\mathbb{R})$ of the real line. Gromov and Pansu [3, Section 3.3B] noted that the group of bi-Lipschitz homeomorphisms has a full image in $\text{QI}(\mathbb{R})$. Sankaran [9] proved that the orientation-preserving subgroup $\text{QI}^+(\mathbb{R})$ is torsion-free and many large groups, like Thompson groups and free groups of infinite rank, can be embedded into $\text{QI}^+(\mathbb{R})$.

Recall that a group $G$ is left-orderable if there is a total order $\leq$ on $G$ such that $g \leq h$ implies $fg \leq fh$ for any $f \in G$. We will prove the following.

Theorem 1.1 The quasi-isometry group $\text{QI}^+(\mathbb{R})$ — or $\text{QI}([0, +\infty))$ — is not simple.
Theorem 1.2  The quasi-isometry group $\text{QI}^+(\mathbb{R})$ — or $\text{QI}([0, +\infty))$ — is left-orderable.

Theorem 1.3  The quasi-isometry group $\text{QI}^+(\mathbb{R})$ cannot act effectively on the real line $\mathbb{R}$.

Other (uncountable) left-orderable groups that cannot act on the line are been known. For example, the germ group $G_\infty(\mathbb{R})$, due to Mann [4] and Rivas; and the compact supported diffeomorphism group $\text{Diff}_c(\mathbb{R}^n)$ for $n > 1$, due to Chen and Mann [1].

2  The group structure of $\text{QI}(\mathbb{R})$

Let $\text{QI}(\mathbb{R}_+)$ (resp. $\text{QI}(\mathbb{R}_-)$) be the quasi-isometry group of the ray $[0, +\infty)$ (resp. $(-\infty, 0]$), viewed as subgroup of $\text{QI}(\mathbb{R})$ fixing the negative (resp. positive) part.

Lemma 2.1  $\text{QI}(\mathbb{R}) = (\text{QI}(\mathbb{R}_+) \times \text{QI}(\mathbb{R}_-)) \rtimes \langle t \rangle$, where $t \in \text{QI}(\mathbb{R})$ is the reflection $t(x) = -x$ for any $x \in \mathbb{R}$.

Proof  Sankaran [9] proves that the group $\text{PL}_\delta(\mathbb{R})$ consisting of piecewise linear homeomorphisms with bounded slopes has a full image in $\text{QI}(\mathbb{R})$. Since every homeomorphism $f \in \text{PL}_\delta(\mathbb{R})$ is of bounded distance to the map $f - f(0) \in \text{PL}_\delta(\mathbb{R})$, we see that the subgroup

$$\text{PL}_{\delta,0}(\mathbb{R}) = \{ f \in \text{PL}_\delta(\mathbb{R}) \mid f(0) = 0 \}$$

also has full image in $\text{QI}(\mathbb{R})$. Let

$$\text{PL}_{\delta,+}(\mathbb{R}) = \{ f \in \text{PL}_\delta(\mathbb{R}) \mid f(x) = x, x \leq 0 \},$$

$$\text{PL}_{\delta,-}(\mathbb{R}) = \{ f \in \text{PL}_\delta(\mathbb{R}) \mid f(x) = x, x \geq 0 \}.$$

Since $\text{PL}_{\delta,+}(\mathbb{R}) \cap \text{PL}_{\delta,-}(\mathbb{R}) = \{ \text{id}_\mathbb{R} \}$, we see that $\text{PL}_{\delta,+}(\mathbb{R}) \times \text{PL}_{\delta,-}(\mathbb{R})$ has a full image in $\text{QI}^+(\mathbb{R})$, the orientation-preserving subgroup of $\text{QI}(\mathbb{R})$. It’s obvious that $\text{PL}_{\delta,+}(\mathbb{R})$ (resp. $\text{PL}_{\delta,-}(\mathbb{R})$) has a full image in $\text{QI}(\mathbb{R}_+)$ (resp. $\text{QI}(\mathbb{R}_-)$). Therefore, $\text{QI}(\mathbb{R}) = (\text{QI}(\mathbb{R}_+) \times \text{QI}(\mathbb{R}_-)) \rtimes \langle t \rangle$.  

Let $\text{Homeo}_+(\mathbb{R})$ be the group of orientation-preserving homeomorphisms of the real line. Two functions $f, g \in \text{Homeo}_+(\mathbb{R})$ are of bounded distance if

$$\sup_{|x| \geq M} |f(x) - g(x)| < \infty$$

for a sufficiently large real number $M$. This means when we study elements $[f]$ in $\text{QI}(\mathbb{R})$, we don’t need to care too much about the function values $f(x)$ for $x$ with small
The group of quasi-isometries of the real line cannot act effectively on the line absolute values. We will implicitly use this fact in the following context. As PL_{d}(\mathbb{R}) has a full image in QI(\mathbb{R}) (by Sankaran [9]), we take representatives of quasi-isometries which are homeomorphisms in the rest of the article.

2.1 QI(\mathbb{R}+) is not simple

Let QI(\mathbb{R}+) be the quasi-isometry group of the half-line [0, +∞). Note that the quasi-isometry group QI^{+}(\mathbb{R}) = QI(\mathbb{R}+) × QI(\mathbb{R}) and QI(\mathbb{R}+) \cong QI(\mathbb{R}), by Lemma 2.1. Let \(H = \{[f] \in QI(\mathbb{R}+) \mid \lim_{x \to \infty}(f(x) - x)/x = 0\}\). Theorem 1.1 follows from the following theorem.

**Theorem 2.2** H is a proper normal subgroup of QI(\mathbb{R}+). In particular, QI(\mathbb{R}+) is not simple.

**Proof** For any \([f], [g] \in H\),
\[
\frac{f(g(x)) - x}{x} = \frac{f(g(x)) - g(x) g(x)}{x} + \frac{g(x) - x}{x}.
\]
Since \(g\) is a quasi-isometry, we know that \((1/K)x - C \leq g(x) - g(0) \leq Kx + C\). Therefore, \(1/K - 1 \leq g(x)/x \leq K + 1\) for sufficiently large \(x\). When \(x \to \infty\), we have \(g(x) \to \infty\). This means \((f(g(x)) - g(x))/g(x) \to 0\). Therefore, \((f(g(x)) - x)/x \to 0\) as \(x \to \infty\). This proves that \([fg] \in H\).

Note that
\[
\frac{|f^{-1}(x) - x|}{x} = \frac{|f^{-1}(x) - f^{-1}(f(x))|}{x} \leq \frac{K|x - f(x)| + C}{x}.
\]
Therefore,
\[
\lim_{x \to \infty} \frac{|f^{-1}(x) - x|}{x} = 0.
\]
This means \([f^{-1}] \in H\) and that \(H\) is a subgroup.

For any quasi-isometric homeomorphism \(g \in Homeo(\mathbb{R}+)\) and any \([f] \in H\),
\[
\frac{g^{-1}(f(g(x))) - x}{x} = \frac{g^{-1}(f(g(x))) - g^{-1}(g(x))}{x} = \frac{g^{-1}(f(g(x))) - g^{-1}(g(x)) g(x)}{g(x)}.
\]
Note that when \(x \to \infty\), the function \(g(x)/x\) is bounded. Let \(y = g(x)\). We have
\[
\frac{|g^{-1}(f(y)) - g^{-1}(y)|}{y} \leq \frac{K|f(y) - y| + C}{y} \to 0, \quad x \to \infty.
\]
Therefore, \([g^{-1}fg] \in H\).
It’s obvious that the function $f$ defined by $f(x) = 2x$ is not an element in $H$. The function defined by $g(x) = x + \ln(x + 1)$ gives a nontrivial element in $H$. Thus $H$ is a proper normal subgroup of $\text{QI}(\mathbb{R}_+)$. 

**Lemma 2.3** Let 

$$W(\mathbb{R}) = \{ f \in \text{Diff}(\mathbb{R}) \mid \sup_{x \in \mathbb{R}} |f(x) - x| < \infty, \sup_{x \in \mathbb{R}} |f'(x)| < \infty \}$$

be the group consisting of diffeomorphisms with bounded derivatives and of bounded distance from the identity. Define a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = e^{x}$ when $x \geq 1$, $h(x) = -h(-x)$ when $x \leq -1$, and $h(x) = e^{x}$ when $-1 \leq x \leq 1$. Then $hf h^{-1}$ is a quasi-isometry for any $f \in W(\mathbb{R})$.

**Proof** For any $f \in W(\mathbb{R})$ and sufficiently large $x > 0$, its derivative satisfies that 

$$|hf h^{-1}(x)'| = |(e^{f(ln x)})'|$$

$$= |(x e^{f(ln x) - ln x})'|$$

$$= |e^{f(ln x) - ln x}(1 + f'(ln x) - 1)|$$

$$= |e^{f(ln x) - ln x} f'(ln x)|$$

$$\leq e^{\sup_{x \in \mathbb{R}} |f(x) - x|} \sup_{x \in \mathbb{R}} |f'(x)|.$$ 

The case for negative $x < 0$ can be calculated similarly. This proves that $hf h^{-1}$ is a quasi-isometry. 

The following result was proved by Sankaran [9].

**Corollary 2.4** The quasi-isometry group $\text{QI}(\mathbb{R})$ contains $\text{Diff}_Z(\mathbb{R})$ (the lift of $\text{Diff}(S^1)$ to $\text{Homeo}(\mathbb{R})$).

**Proof** For any $f \in \text{Diff}_Z(\mathbb{R})$, we have $f(x + 1) = f(x) + 1$ for any $x \in \mathbb{R}$. This means $\sup_{x \in \mathbb{R}} |f(x) - x| < +\infty$. Since $f(x) - x$ is periodic, we know that $f'(x)$ is bounded. Suppose that $f(x) > x$ for some $x \in [0, 1]$. Take $y_n = e^{x+n}$ for $n > 0$. Let $h$ be the function defined in Lemma 2.3. We have 

$$|hf h^{-1}(y_n) - y_n| = |e^{f(x+n)} - e^{x+n}| = |e^{f(x)} - e^{x}|e^n \to \infty,$$

which means $[hf h^{-1}] \neq [\text{id}] \in \text{QI}(\mathbb{R})$. 

**Lemma 2.5** $\text{QI}(\mathbb{R})$ contains the semidirect product $\text{Diff}_Z(\mathbb{R}) \ltimes H$. 

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**Proof** Since $H$ is normal, it’s enough to prove that $\text{Diff}_Z(\mathbb{R}) \cap H = \{e\}$, the trivial subgroup. Actually, for any $f \in \text{Diff}_Z(\mathbb{R})$, the conjugate $hfh^{-1}$ is a quasi-isometry as in the proof of Corollary 2.4. If $hfh^{-1} \in H$, then

$$
\lim_{x \to \infty} \frac{hfh^{-1}(x)}{x} = \lim_{x \to \infty} \frac{x e^{f(\ln x) - \ln x}}{x} = \lim_{x \to \infty} e^{f(\ln x) - \ln x} = 1.
$$

Since $f(x) - x$ is periodic, we know that $f(\ln x) = \ln x$ for any sufficiently large $x$. But this means that $f(y) = y$ for any $y$, so $f$ is the identity.

**2.2 Affine subgroups of QI(\mathbb{R})**

**Lemma 2.6** The quasi-isometry group $\text{QI}(\mathbb{R}^+)$ (actually, the semidirect product $\text{Diff}_Z(\mathbb{R}) \ltimes H$) contains the semidirect product $\mathbb{R}_{>0} \ltimes \bigoplus_{i \in \mathbb{R}_{\geq 1}} \mathbb{R}$, generated by $A_t$ and $B_{i,s}$ for $t \in \mathbb{R}_{>0}, i \in \mathbb{R}_{\geq 1} = [1, \infty)$ and $s \in \mathbb{R}$ satisfying

$$
A_t B_{i,s} A^{-1}_t = B_{i,ts^{-1}}, \quad B_{i,s1} B_{i,s2} = B_{i,s1+s2},
$$

$$
A_{t1} A_{t2} = A_{t1t2}, \quad B_{i,s1} B_{j,s2} = B_{j,s2} B_{i,s1},
$$

for any $t_1, t_2 \in \mathbb{R}_{>0}, i, j \in \mathbb{R}_{\geq 1}$ and $s_1, s_2 \in \mathbb{R}$.

**Proof** Let

$$
A_t(x) = tx, \quad t \in \mathbb{R}_{>0},
$$

$$
B_{i,s}(x) = x + sx^{\frac{1}{i+1}}, \quad s \in \mathbb{R},
$$

for $x \geq 0$. We define $A_t(x) = B_{i,s}(x) = x$ for $x \leq 0$. Since the derivatives

$$
A'_t(x) = t, \quad B'_{i,s}(x) = 1 + \frac{s}{i+1} x^{\frac{i}{i+1}}
$$

are bounded for sufficiently large $x$, we know that $A_t$ and $B_{i,s}$ are quasi-isometries. For any $x \geq 1,$

$$
A_t B_{i,s} A_t^{-1}(x) = A_t B_{i,s} \left( \frac{x}{t} \right) = A_t \left( \frac{x}{t} + s \left( \frac{x}{t} \right)^{\frac{1}{i+1}} \right) = x + st^{\frac{1}{i+1}} x^{\frac{1}{i+1}} = B_{i,ts^{\frac{1}{i+1}}}(x).
$$

For any $x \geq 1,$

$$
B_{i,s1} B_{i,s2}(x) = B_{i,s1} (x + s_2 x^{\frac{1}{i+1}}) = x + s_2 x^{\frac{1}{i+1}} + s_1 (x + s_2 x^{\frac{1}{i+1}})^{\frac{1}{i+1}}
$$

and

$$
|B_{i,s1} B_{i,s2}(x) - B_{i,s1+s2}(x)| = |s_1((x + s_2 x^{\frac{1}{i+1}})^{\frac{1}{i+1}} - x^{\frac{1}{i+1}})| \leq \left| s_1 \frac{s_2 x^{\frac{1}{i+1}}}{x^{\frac{1}{i+1}}} \right| \leq |s_1 s_2|
$$

by Newton’s binomial theorem. This means that $B_{i,s1} B_{i,s2}$ and $B_{i,s1+s2}$ are of bounded distance. It is obvious that $A_{t1} A_{t2} = A_{t1t2}$.
When \( i < j \) are distinct natural numbers,
\[
|B_{i,s_1} B_{j,s_2}(x) - B_{j,s_2} B_{i,s_1}(x)| \\
= |x + s_2 x^{1 \over i + 1} + s_1 (x + s_2 x^{1 \over j + 1})^{1 \over j + 1} - (x + s_1 x^{1 \over i + 1} + s_2 (x + s_1 x^{1 \over i + 1})^{1 \over i + 1})| \\
= |s_1 ((x + s_2 x^{1 \over j + 1})^{1 \over j + 1} - x^{1 \over i + 1}) + s_2 (x^{1 \over j + 1} - (x + s_1 x^{1 \over i + 1})^{1 \over i + 1})| \\
\leq \left| \frac{s_2 x^{1 \over j + 1}}{x^{1 \over i + 1}} \right| + \left| \frac{s_1 x^{1 \over i + 1}}{x^{1 \over j + 1}} \right| \\
\leq 2|s_1 s_2|
\]
for any \( x \geq 1 \). This proves that images \([A_1], [B_{i,s}] \in \text{QI}\(\mathbb{R}_{\geq 0}\)\) satisfy the relations. By abuse of notation, we still denote the classes by the same letters.

We prove that the subgroup generated by \([B_{i,s} \mid i \in \mathbb{R}_{\geq 1}, s \in \mathbb{R}\) is the infinite direct sum \( \bigoplus_{i \in \mathbb{R}_{\geq 1}} \mathbb{R} \). It’s enough to prove that \( B_{i_1,s_1}, B_{i_2,s_2}, \ldots, B_{i_k,s_k} \) are \( \mathbb{Z} \)-linearly independent for distinct \( i_1, i_2, \ldots, i_k \) and nonzero \( s_1, s_2, \ldots, s_k \in \mathbb{R} \). This can directly checked. For integers \( n_1, n_2, \ldots, n_k \), suppose that \( B^{n_1}_{i_1,s_1} \circ B^{n_2}_{i_2,s_2} \circ \cdots \circ B^{n_k}_{i_k,s_k} = \text{id} \in \text{QI}(\mathbb{R}_{\geq 0}) \). We have
\[
\sup_{x \in \mathbb{R}_{\geq 0}} \left| B^{n_1}_{i_1,s_1} \circ B^{n_2}_{i_2,s_2} \circ \cdots \circ B^{n_k}_{i_k,s_k}(x) - x \right| \\
= \sup_{x \in \mathbb{R}_{\geq 0}} \left| n_k s_k x^{1 \over i_k + 1} + n_{k-1} s_{k-1} (x + n_k s_k x^{1 \over i_k + 1})^{1 \over i_{k-1} + 1} + \cdots + n_1 s_1 (x + \cdots)^{1 \over i_1 + 1} \right| \\
< +\infty,
\]
which implies \( n_1 = n_2 = \cdots = n_k = 0 \), considering the exponents.

The subgroup \( \mathbb{R}_{>0} \rtimes (\bigoplus_{i \in \mathbb{R}_{\geq 1}} \mathbb{R}) \) lies in \( \text{Diff}_\mathbb{Z}(\mathbb{R}) \rtimes H \) by the following construction. Let \( a_t, b_{i,s} : \mathbb{R} \to \mathbb{R} \) be defined by \( a_t(x) = x + \ln t \) and \( b_{i,s}(x) = \ln (e^x + se^{1 \over i + 1}) \) for \( t \in \mathbb{R}_{>0}, i \in \mathbb{R}_{\geq 1} \) and \( s \in \mathbb{R} \). It can be directly checked that \( a_t \in \text{Diff}_\mathbb{Z}(\mathbb{R}) \) and \( b_{i,s} \in W(\mathbb{R}) \) (defined in Lemma 2.3). Let \( h(x) = e^x \). A direct calculation shows that \( ha_t h^{-1} = a_t \) and \( hb_{i,s} h^{-1} = B_{i,s} \), as elements in \( \text{QI}(\mathbb{R}_+) \).

\section{Left-orderability}

The following is well known; for a proof, see [7, Proposition 1.4]:

\textbf{Lemma 3.1} \ A group \( G \) is left-orderable if and only if, for every finite collection of nontrivial elements \( g_1, \ldots, g_k \), there exist choices \( \varepsilon_i \in \{1, -1\} \) such that the identity is not an element of the semigroup generated by \( \{g_i^{\varepsilon_i} \mid i = 1, 2, \ldots, k\} \).
The proof of Theorem 1.2 follows a similar strategy used by Navas to prove the left-orderability of the group $G_\infty$ of germs at $\infty$ of homeomorphisms of $\mathbb{R}$; cf [2, Remark 1.1.13] or [4, Proposition 2.2].

**Proof of Theorem 1.2** It’s enough to prove that $\text{QI}(\mathbb{R}_+)$ is left-orderable. Let $f_1, f_2, \ldots, f_n \in \text{QI}(\mathbb{R}_+)$ be any finitely many nontrivial elements. Note that any $1 \neq [f] \in \text{QI}(\mathbb{R}_+)$ has $\sup_{x>0} |f(x) - x| = \infty$. This property doesn’t depend on the choice of $f \in [f]$. Without confusion, we still denote $[f]$ by $f$. Choose a sequence $\{x_{1,k}\} \subset \mathbb{R}_+$ such that $\sup_{k \in \mathbb{N}} |f_1(x_{1,k}) - x_{1,k}| = \infty$. For each $i > 1$, we have either $\sup_{k \in \mathbb{N}} |f_i(x_{1,k}) - x_{1,k}| = \infty$ or $\sup_{k \in \mathbb{N}} |f_i(x_{1,k}) - x_{1,k}| \leq M$ for a real number $M$. After passing to subsequences, we assume for each $i = 1, 2, \ldots, n$ that either $f_i(x_{1,k}) - x_{1,k} \to +\infty$, $f_i(x_{1,k}) - x_{1,k} \to -\infty$ or $\sup_{k \in \mathbb{N}} |f_i(x_{1,k}) - x_{1,k}| \leq M$. We assign $\varepsilon_i = 1$ for the first case and $\varepsilon_i = -1$ for the second case. For the third case, let

$$S_1 = \{f_i \mid \sup_{k \in \mathbb{N}} |f_i(x_{1,k}) - x_{1,k}| \leq M\}.$$ 

Note that $f_1 \notin S_1$. Choose $f_{i_0} \in S_1$ if $S_1$ is not empty. We choose another sequence $\{x_{2,k}\}$ such that $\sup_{k \in \mathbb{N}} |f_{i_0}(x_{2,k}) - x_{2,k}| = \infty$. Similarly, after passing to a subsequence, we have for each $f \in S_1$ that either $f(x_{2,k}) - x_{2,k} \to +\infty$, $f(x_{2,k}) - x_{2,k} \to -\infty$ or $\sup_{k \in \mathbb{N}} |f(x_{2,k}) - x_{2,k}| \leq M’$ for another real number $M’$. Assign $\varepsilon_i = 1$ for the first case and $\varepsilon_i = -1$ for the second case. Continue this process to define $S_2, S_3, \ldots$ and choose sequences $\{x_{i,k}\}, i = 3, 4, \ldots$ to assign $\varepsilon_i$ for each $f_i$. Note that the process will stop at $n$ times, as the number of elements without assignment is strictly decreasing.

For an element $f \in \text{QI}(\mathbb{R}_+)$ satisfying $f(x_i) - x_i \to \infty$ as $i \to \infty$ for some sequence $\{x_i\}$, we assume that $f(x_i) - x_i > 0$ for each $i$. Since $f$ and $f^{-1}$ are orientation-preserving,

$$f^{-1}(x_i) - x_i = -(x_i - f^{-1}(x_i))$$

$$= -(f^{-1}(f(x_i)) - f^{-1}(x_i)) \leq -\left(\frac{1}{K}(f(x_i) - x_i) - C\right) \to -\infty.$$

Let $w = f_{i_1}^{\varepsilon_{i_1}} \cdots f_{i_m}^{\varepsilon_{i_m}} \in \{f_1, f_2, \ldots, f_n\}$ be a nontrivial word. If $\{i_1, \ldots, i_m\} \not\subseteq S_1$, we have $w(x_{1,k}) - x_{1,k} \to \infty$. Otherwise, $\sup_{k \in \mathbb{N}} |w(x_{1,k}) - x_{1,k}| < \infty$. Suppose that $\{i_1, \ldots, i_m\} \subset S_t$, but $\{i_1, \ldots, i_m\} \not\subseteq S_{t+1}$ with the assumption that $S_0 = \{f_1, f_2, \ldots, f_n\}$. We have $w(x_{t+1,k}) - x_{t+1,k} \to \infty$ as $k \to \infty$. This proves that $w \neq 1 \in \text{QI}(\mathbb{R}_+)$. Therefore, $\text{QI}(\mathbb{R}_+)$ is left-orderable by Lemma 3.1. □

**Lemma 3.2** The group $\text{QI}(\mathbb{R}_+)$ is not locally indicable.
Proof Note that $\text{QI}(\mathbb{R}_+)$ contains the lift $\overline{\Gamma}$ of $\text{PSL}(2, \mathbb{R}) < \text{Diff}(S^1)$ to $\text{Homeo}(\mathbb{R})$ (Corollary 2.4). But this lift $\overline{\Gamma}$ contains a subgroup $\Gamma = \langle f, g, h : f^2 = g^3 = h^7 = fgfh \rangle$, the lift of the $(2, 3, 7)$–triangle group. There are no nontrivial maps from $\Gamma$ to $(\mathbb{R}, +)$; for more details see [2, page 94].

4 The quasi-isometric group cannot act effectively on the line

The following was proved by Mann [4, Proposition 6].

Lemma 4.1 Consider the affine group $\mathbb{R}_{>0} \ltimes \mathbb{R}$, generated by $A_t$ and $B_s$ for $t \in \mathbb{R}_{>0}$ and $s \in \mathbb{R}$ satisfying

$$A_t B_s A_t^{-1} = B_{ts}, \quad B_{s_1} B_{s_2} = B_{s_1 + s_2}, \quad A_t A_{t_2} = A_{t_1 t_2}.$$ 

The affine group $\mathbb{R}_{>0} \ltimes \mathbb{R}$ cannot act effectively on the real line $\mathbb{R}$ by homeomorphisms with $A_t$ a translation for each $t$.

Proof Suppose that $\mathbb{R}_{>0} \ltimes \mathbb{R}$ acts effectively on the real line $\mathbb{R}$ with each $A_t$ a translation. After passing to an index-2 subgroup, we assume that the group is orientation-preserving. If $B_1$ acts freely on $\mathbb{R}$, then it is conjugate to the translation $T : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x + 1$. In such a case, we have $A_2 T A_2^{-1} = T^2$. Therefore, $A_2^{-1}(x + 2) = A_2^{-1}(x) + 1$ for any $x$. Since $A_2^{-1}$ maps intervals of length 2 to an interval of length 1, it is a contracting map, and thus has a fixed point.

If $B_1$ has a nonempty fixed point set $\text{Fix}(B_1)$, choose $I$ to be a connected component of $\mathbb{R} \setminus \text{Fix}(B_1)$. Suppose that $A_2(x) = x + a$, a translation by some real number $a > 0$. Since $A_2 = A_{2/n}^{n}$, we have $A_{2/n}(x) = x + a/n$ for each positive integer $n$. For each $n$, let $F_n = A_{2/n} B_1 A_{2/n}^{-1}$. Since $A_{2/n} B_1 A_{2/n}^{-1}$ commutes with $B_1$, we see that $F_n \text{Fix}(B_1) = \text{Fix}(B_1)$. This means that either $F_n(I) = I$ or $F_n(I) \cap I = \emptyset$. Since $F_n(x) = B_1(x - a/n) + a/n$ for any $x \in \mathbb{R}$, we know that $F_n(I) = I$ for sufficiently large $n$. Without loss of generality, we assume that $I$ is of the form $(x, y)$ or $(-\infty, y)$. Choose a sufficiently large $n$ such that $y - a/n \in I$. We have

$$A_{2/n} B_1 A_{2/n}^{-1}(y) = B_1 \left(y - \frac{a}{n}\right) + \frac{a}{n} \neq y,$$

which is a contradiction to the fact that $F_n(I) = I$.

Definition 4.2 A topologically diagonal embedding of a group $G < \text{Homeo}(\mathbb{R})$ is a homomorphism $\phi : G \to \text{Homeo}_+(\mathbb{R})$ defined as follows. Choose a collection of disjoint open intervals $I_n \subset \mathbb{R}$ and homeomorphisms $f_n : \mathbb{R} \to I_n$. Define $\phi$ by $\phi(g)(x) = f_n g f_n^{-1}(x)$ when $x \in I_n$ and $\phi(g)(x) = x$ when $x \notin I_n$.
The following is similar to a result proved by Militon [6].

**Lemma 4.3** (Militon [6]) Let \( \Gamma = \text{PSL}_2(\mathbb{R}) \) and \( \widetilde{\Gamma} \subset \text{Homeo}^+(\mathbb{R}) \) be the lift of \( \Gamma \) to the real line. Any effective action \( \phi: \widetilde{\Gamma} \hookrightarrow \text{Homeo}^+(\mathbb{R}) \) of \( \widetilde{\Gamma} \) on the real line \( \mathbb{R} \) is a topological diagonal embedding.

**Proof** After passing to an index-2 subgroup, we assume the action is orientation-preserving. Let \( \tau: \mathbb{R} \to \mathbb{R} \) be the translation \( x \mapsto x + 1 \). Suppose that \( \text{Fix}(\phi(\tau)) \neq \emptyset \). Note that \( \tau \) lies in the center of \( \widetilde{\Gamma} \). The quotient group \( \Gamma = \widetilde{\Gamma}/\langle \tau \rangle \) acts on the fixed point set \( \text{Fix}(\phi(\tau)) \). For any \( f \in \Gamma \) and \( x \in \text{Fix}(\phi(\tau)) \), we denote the action by \( f(x) \) without confusion. Choose any torsion-element \( f \in \Gamma \) and any \( x \in \text{Fix}(\phi(\tau)) \). We must have \( x = f(x) \), for otherwise \( x < f(x) < f^2(x) < \cdots < f^k(x) \) for any \( k \). Since \( \Gamma \) is simple, we know that the action of \( \widetilde{\Gamma} \) on \( \text{Fix}(\tau) \) is trivial. For each connected component \( I_i \subset \mathbb{R} \setminus \text{Fix}(\phi(\tau)) \), we know that \( \tau|_{I_i} \) is conjugate to a translation. The group \( \Gamma = \widetilde{\Gamma}/\langle \tau \rangle \) acts on \( I_i/\langle \phi(\tau) \rangle = S^1 \). A result of Matsumoto [5, Theorem 5.2] says that the group \( \widetilde{\Gamma} \) is conjugate to the natural inclusion \( \text{PSL}_2(\mathbb{R}) \hookrightarrow \text{Homeo}^+(S^1) \) by a homeomorphism \( g \in \text{Homeo}^+(S^1) \). Therefore, the group \( \phi(\widetilde{\Gamma})|_{I_i} \) is conjugate to the image of the natural inclusion \( \widetilde{\Gamma} \hookrightarrow \text{Homeo}^+(\mathbb{R}) \).

For a real number \( a \in \mathbb{R} \), let

\[
t_a: \mathbb{R} \to \mathbb{R}, \quad x \mapsto x + a
\]

be the translation. Denote by \( A = \langle t_a : a \in \mathbb{R} \rangle \), the subgroup of translations in the lift \( \widetilde{\Gamma} \) of \( \text{PSL}_2(\mathbb{R}) \).

**Corollary 4.4** For any injective group homomorphism \( \phi: \widetilde{\Gamma} \to \text{Homeo}(\mathbb{R}) \), the image \( \phi(A) \) is a continuous one-parameter subgroup; i.e. \( \lim_{a \to a_0} \phi(t_a) = \phi(t_{a_0}) \) for any \( a_0 \in \mathbb{R} \).

**Proof** If \( \phi \) is injective, the previous lemma says that \( \phi \) is a topological diagonal embedding. Therefore, \( \phi(A) \) is continuous.

We will need the following elementary fact.

**Lemma 4.5** Let \( \phi: (\mathbb{R}, +) \to (\mathbb{R}, +) \) be a group homomorphism. If \( \phi \) is continuous at any \( x \neq 0 \), then \( \phi \) is \( \mathbb{R} \)-linear.

**Proof** For any nonzero integer \( n \), we have \( \phi(n) = n\phi(1) \) and \( \phi(1) = \phi(\frac{1}{n}n) = n\phi(\frac{1}{n}) \). Since \( \phi \) is additive, we have \( \phi(m/n) = m\phi(\frac{1}{n}) = \frac{m}{n}\phi(1) \) for any integers \( m, n \neq 0 \).
For any nonzero real number \( a \in \mathbb{R} \), choose a rational sequence \( r_i \to a \). When \( \phi \) is continuous, we have that \( \phi(r_i) \to \phi(a) \) and \( \phi(r_i) = r_i \phi(1) \to a \phi(1) = \phi(a) \). \( \Box \)

The following is the classical theorem of Hölder: a group acting freely on \( \mathbb{R} \) is semi-conjugate to a group of translations; see Navas [8, Section 2.2.4].

**Lemma 4.6** Let \( \Gamma \) be a group acting freely on the real line \( \mathbb{R} \). There is an injective group homomorphism \( \phi : \Gamma \to (\mathbb{R}, +) \) and a continuous nondecreasing map \( \varphi : \mathbb{R} \to \mathbb{R} \) such that
\[
\varphi(h(x)) = \varphi(x) + \phi(h)
\]
for any \( x \in \mathbb{R} \) and \( h \in \Gamma \).

**Corollary 4.7** Suppose that the affine group \( \mathbb{R}_{>0} \times \mathbb{R} = \langle a_t : t \in \mathbb{R}_{>0} \rangle \times \langle b_s : s \in \mathbb{R} \rangle \) acts on the real line \( \mathbb{R} \) by homeomorphisms satisfying
\begin{enumerate}
\item the action of the subgroup \( \mathbb{R} = \langle b_s : s \in \mathbb{R} \rangle \) is free;
\item for any fixed \( x \in \mathbb{R} \), \( a_t(x) \) is continuous with respect to \( t \in \mathbb{R}_{>0} \).
\end{enumerate}
Then \( \phi : \langle b_s : s \in \mathbb{R} \rangle \to (\mathbb{R}, +) \) be the additive map in Lemma 4.6 for \( \Gamma = \langle b_s : s \in \mathbb{R} \rangle \). Then \( \phi \) is an \( \mathbb{R} \)-linear map.

**Proof** Note that \( a_t b_s a_t^{-1} = b_t s \). We have
\[
\varphi(b_t s(x)) = \varphi(x) + \phi(b_t s).
\]
Since \( b_t s(x) = a_t b_s a_t^{-1}(x) \to b_s(x) \) when \( t \to 1 \), we have that
\[
\varphi(x) + \phi(b_t s) \to \varphi(b_s(x)) = \varphi(x) + \phi(b_s).
\]
This implies that \( \phi(b_t s) \to \phi(b_s) \) as \( t \to 1 \). For any nonzero \( x \in \mathbb{R} \) and sequence \( x_n \to x \),
\[
\phi(b_{x_n}) = \phi(b_{\frac{x_n}{x} x}) \to \phi(b_x).
\]
The map \( \phi \) is \( \mathbb{R} \)-linear by Lemma 4.5. \( \Box \)

**Theorem 4.8** Consider \( G = \mathbb{R}_{>0} \times \bigoplus_{i \in \mathbb{R}_{\geq 1}} \mathbb{R} \), generated by \( A_t \) and \( B_{i,s} \) for \( t \in \mathbb{R}_{>0}, i \in \mathbb{R}_{\geq 1} = [1, \infty) \) and \( s \in \mathbb{R} \) satisfying
\[
A_t B_{i,s} A_t^{-1} = B_{i,s} A_t^{i, t}, \quad B_{i,s_1} B_{i,s_2} = B_{i,s_1 + s_2},
\]
\[
A_{t_1} A_{t_2} = A_{t_1 + t_2}, \quad B_{i,s_1} B_{j,s_2} = B_{i,s_1 + s_2} B_{j,s_2}
\]
for any \( t_1, t_2 \in \mathbb{R}_{>0}, i, j \in \mathbb{R}_{\geq 1} \) and \( s_1, s_2 \in \mathbb{R} \). Then \( G \) cannot act effectively on the real line \( \mathbb{R} \) by homeomorphisms when the induced action of \( \langle A_t : t \in \mathbb{R}_{>0} \rangle \) is a topologically diagonal embedding of the translation subgroup \( (\mathbb{R}, +) \hookrightarrow \text{Homeo}(\mathbb{R}) \).
**Proof** Suppose that $G$ acts effectively on $\mathbb{R}$ with the induced action of $\langle A_t : t \in \mathbb{R}_{>0} \rangle$, a topologically diagonal embedding of the translation subgroup $(\mathbb{R}, +) \hookrightarrow \text{Homeo}(\mathbb{R})$. Let $I$ be a connected component of $\mathbb{R} \setminus \text{Fix}(\langle A_t, B_{i,s} : t \in \mathbb{R}_{>0}, i = 1, s \in \mathbb{R} \rangle)$.

Suppose that there is an element $B_{1,s}$ having a fixed point $x \in I$ for some $s > 0$. Since $A_4B_{1,s}A_4^{-1} = B_{1,s'}$, we know that $A_4x \in \text{Fix}(B_{1,s}) = \text{Fix}(B_{1,s}^2)$. Since there are no fixed points in $I$ for $\langle A_t, B_{1,s} : t \in \mathbb{R}_{>0}, s \in \mathbb{R} \rangle$, we know that $\lim_{n \to \infty} A_4^n x \notin I$. This implies that $A_4$ has no fixed point in $I$. Since the group homomorphism $\langle A_t : t \in \mathbb{R}_{>0} \rangle \to \text{Homeo}(\mathbb{R})$ is a diagonal embedding, we see that each $A_t$ has no fixed point in $I$ and the action of $\langle A_t : t \in \mathbb{R}_{>0} \rangle$ on $I$ is conjugate to a group of translations. By Lemma 4.1, the affine group $\langle A_t, B_{1,s} : t \in \mathbb{R}_{>0}, s \in \mathbb{R} \rangle$ cannot act effectively on $I$. Suppose that $A_t B_{1,s'}$ acts trivially on $I$ for some $t > 0$ and $s' > 0$. We have that $A_t B_{1,s} = A_2 A_{s'}^{-2}(A_t B_{1,s'}) A_{s'}^{-1}_{2s'}$ acts trivially on $I$. But $A_t B_{1,s}(x) = A_t(x) = x$ implies that $t = 1$. Therefore, the element $B_{1,s}$ (and any $B_{1,t} = A_t A_{s'}^{-2} B_{1,s} A_{s'}^{-1}_{2s'}$ for $t \in \mathbb{R}_{>0}$) acts trivially on $I$. This means that the action of $\langle B_{1,s} : s \in \mathbb{R} \rangle$ on the connected component $I$ is either trivial or free. Since the action of $G$ is effective, there is a connected component $I_1$ on which $B_{1,s}$ acts freely. A similar argument shows that $B_{i,s'}$ acts freely on a component $I_i$ for each $i \in \mathbb{R}_{\geq 1}$ and any $s' \in \mathbb{R} \setminus \{0\}$.

Since $B_{i,s'}$ commutes with $B_{j,s}$, we have $B_{i,s'}(I_1) \subset \mathbb{R} \setminus \text{Fix}(\langle B_{j,s} : s \in \mathbb{R} \rangle)$. Moreover, $B_{i,s'}(I_j) \cap I_j$ is either $I_j$ or the empty set. Suppose that $I_i \cap I_j \neq \emptyset$ and the right end $b_i$ of $I_i$ lies in $I_j$. Choose $x \in I_i \cap I_j$. Note that $B_{j,s}(x, b_i) \cap [x, b_i] = \emptyset$ for any $s > 0$. This is impossible as $B_{j,s/n}(x) \to x$ as $n \to \infty$. Therefore, $I_i \cap I_j = I_l$ or is empty for distinct $i, j \in \mathbb{R}_{\geq 1}$. Since we have uncountably many $i \in \mathbb{R}_{>0}$, there must be some distinct $i, j \in \mathbb{R}_{\geq 1}$ such that $I_i = I_j$. This means that the subgroup $\mathbb{R} \oplus \mathbb{R}$ spanned by the $i,j$-components acts freely on $I_l$. Hölder’s theorem (Lemma 4.6) gives an injective group homomorphism $\phi : \mathbb{R} \oplus \mathbb{R} \to (\mathbb{R}, +)$ and a continuous nondecreasing map $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\varphi(h(x)) = \varphi(x) + \phi(h)$$

for any $x \in \mathbb{R}$. Since $\langle A_t : t \in \mathbb{R}_{>0} \rangle \to \text{Homeo}(\mathbb{R})$ is a topological embedding, we have that for any fixed $x \in \mathbb{R}$, $A_t(x)$ is continuous with respect to $t \in \mathbb{R}_{>0}$. By Corollary 4.7,

\footnote{Otherwise, $\lim_{n \to \infty} A_4^n x \notin I$. But $A_4 \lim_{n \to \infty} A_4^n x = \lim_{n \to \infty} A_4^n x$ for any $t > 0$ by the topologically diagonal embedding. For any $s'$, we have $B_{1,s'} = A_2 A_{s'}^{-2} B_{1,s} A_{s'}^{-1}_{2s'}$ and $B_{1,s'} \lim_{n \to \infty} A_4^n x = \lim_{n \to \infty} A_4^n x$. This would imply that $\lim_{n \to \infty} A_4^n x$ is a global fixed point of $\langle A_t, B_{1,s} : t \in \mathbb{R}_{>0}, s \in \mathbb{R} \rangle$.}
the restriction map $\phi|_R$ is $R$–linear for each direct summand $R$. This is a contradiction to the fact that $\phi$ is injective. Therefore, the group $G$ cannot act effectively.

**Proof of Theorem 1.3** Suppose that $\text{QI}^+(\mathbb{R})$ acts on the real line by an injective group homomorphism $\phi : \text{QI}^+(\mathbb{R}) \to \text{Homeo}(\mathbb{R})$. The group $\text{QI}^+(\mathbb{R})$ contains the semidirect product $\mathbb{R}_{>0} \ltimes (\bigoplus_{i \in \mathbb{Z}_{\geq 1}} \mathbb{R})$, by Lemma 2.6. The subgroup $\mathbb{R}_{>0}$ (as the image of the exponential map) is a homomorphic image of the subgroup $\mathbb{R} < \tilde{\Gamma}$, which is the lift of $\text{SO}(2)/\{\pm I_2\} < \text{PSL}_2(\mathbb{R})$ to $\text{Homeo}(\mathbb{R})$. Note that $\tilde{\Gamma}$ is embedded into $\text{QI}^+(\mathbb{R})$ (see Corollary 2.4 and its proof). By Lemma 4.3, any effective action of $\tilde{\Gamma}$ on the real line $\mathbb{R}$ is a topological diagonal embedding. This means that the action of $\mathbb{R}_{>0}$ is a topological diagonal embedding (Corollary 4.4). Theorem 4.8 shows that the action of $\mathbb{R}_{>0} \ltimes (\bigoplus_{i \in \mathbb{Z}_{\geq 1}} \mathbb{R})$ is not effective.

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The group of quasi-isometries of the real line cannot act effectively on the line

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