Superradiant instability of dyonic Reissner–Nordström black holes

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Abstract We investigate the superradiant instability of dyonic Reissner–Nordström (dRN) black holes with two charges under an electrically charged massive scalar perturbation. Two conditions for possessing a trapping well are obtained from analyzing asymptotic scalar potential and far-region wave functions. It is clear that the superradiant instability is not allowed in the dRN black holes because the conditions for a trapping well are not compatible with the superradiance condition.

1 Introduction

It was first mentioned that superradiant instability of rotating black hole with mass $M$ and angular momentum $J$ could be a continuous source of gravitational waves [1]. Ultralight bosons (axions or string axiverse) with mass $\mu \ll 1/M$ could trigger superradiant instability of the rotating black holes and form bosonic clouds which emit gravitational waves. These waves might be detected by gravitational wave observations [2,3]. Superradiant instability of astrophysical black holes has been used to constrain physical parameters because ultralight bosons might be a promising candidate for the dark matter and it is not easy to detect its evidence by collision experiment in accelerator [4,5].

On the other hand, superradiance may represent an amplification of a charged massive scalar impinging on a static Reissner–Nordström (RN) black hole with mass $M$ and charge $Q$ when frequency $\omega$ and charge $q$ of the scalar obey the superradiance condition $\omega < \omega_c = q\Phi_H (\Phi_H = qQ/r_+, \text{electric potential at horizon})$ [6]. We point out that the literatures [7–9] have used a shortened potential to show that the gravitational attraction between RN black hole and a charged massive scalar cannot provide a confinement mechanism, triggering the superradiant instability. Especially, it was shown in [7] that a lower bound of $\mu > qQ$ has been proposed for a necessary condition to get a trapping well, but this condition is not satisfied simultaneously by superradiant states. Furthermore, one would say that this inequality implies the Newton–Coulomb requirement for the gravitational force to exceed the electrostatic force. Recently, it was reported that one condition for no trapping well is given by an upper bound of $\mu M < qQ$ in the Kerr–Newman black hole whose non-rotating limit is the RN black hole [10]. We would like to mention that a correct potential was not used to derive this bound [11].

Some aspects of the superradiance has been investigated in the RN black hole [12] and the absorption cross section of a charged massive scalar [13] was obtained in the RN black hole. It is important to note that the superradiant instability is hard to arise naturally from a charged massive scalar propagation around the RN black holes. However, the superradiant instability of a charged massive scalar could be obtained in certain parameter space if a cavity is placed around the RN black hole [14–17]. This is understood as the charged black hole-mirror bomb. Here, numerical techniques have used to show the lower bound on $q (q > \mu)$, which may be considered as a necessary condition for the superradiant instability when a cavity is placed around the RN black hole [14].

In addition, a magnetic field surrounding a black hole could provide a confinement mechanism because it induces an effective mass $\mu_{\text{eff}} \propto B$ [18,19]. This implies that a magnetic field has an effect on the superradiance of a black hole, which may have astrophysical applications. Investigating a massless scalar perturbation in a magnetized Schwarzschild black hole with magnetic field $B$ leads to triggering the superradiant instability in a confining box of size $1/B$ [20]. Here, $Bm$ plays a role of mass $\mu_{\text{eff}}$ as $(Bm)^2$-term in the potential $V_{\text{eff}}(r, \theta)$.

In this connection, the superradiant stable regime for dyonic Reissner–Nordström (dRN) black holes with electric $Q_e$ and magnetic charge $Q_m$ was introduced to obtain an
upper bound of \((Q_e^2 + Q_m^2)/M^2 < 24/25\) by analyzing an effective potential \(V(r)\) in \([21]\). We note that this black hole becomes an asymptotically flat RN black hole when replacing \(Q_e^2 + Q_m^2\) by \(Q^2\), but it was suggested that the dRN black hole spacetime could provide a magnetic field. However, we wish to point out that a correct potential \(V_{\text{dRN}}(r)\) which differs by \(q^2Q_m^2/r^2\) from the scalar potential \(V_{\text{RN}}(r)\) around the RN black hole. It is important to clarify that an inclusion of a magnetic charge \(Q_m\) in \(q^2Q_m^2/r^2\) could not induce a magnetic field like \(\mu_{\text{eff}} \propto B\) in the scalar potential \(V_{\text{dRN}}(r)\). This is likely to upset the original motivation of introducing dRN black holes to get a magnetic field. However, we will obtain two conditions for getting a trapping well by analyzing all asymptotic potentials and far-region scalar wave functions. These conditions are given by \(M\mu^2 > qQ_{e\omega} (M\mu/qQ_e > \omega/\mu)\) and \(a < 0 [B/(2A) > k + 1/2]\). The former condition indicates the Newton–Coulomb requirement for the gravitational force to exceed the electrostatic force, while the latter represents the condition to get a quasibound state.

In this work, wish to investigate the superradiant (un)stable regime for dRN black holes by employing the correct scalar potential \(V_{\text{dRN}}(r) = U_1(r)\) which differs by \(q^2Q_m^2/r^2\) from the scalar potential \(V_{\text{RN}}(r)\) around the RN black hole. We will obtain two conditions for getting a magnetic field by analyzing all asymptotic potentials and far-region scalar wave functions. These conditions are given by \(M\mu^2 > qQ_{e\omega} (M\mu/qQ_e > \omega/\mu)\) and \(a < 0 [B/(2A) > k + 1/2]\). The former condition indicates the Newton–Coulomb requirement for the gravitational force to exceed the electrostatic force, while the latter represents the condition to get a quasibound state.

On the other hand, two conditions for no trapping well are \(M\mu^2 < qQ_{e\omega} (M\mu/qQ_e < \omega/\mu < 1)\) and \(a > 0 [B/(2A) < k + 1/2]\). The former represents the Coulomb–Newton requirement for the electrostatic force to exceed the gravitational force, while the latter represents the condition to get a bound state. Furthermore, we explore how the known bounds of \(M\mu > qQ (M\mu < qQ)\) arise from the pseudo-Newtonian potential for a charged particle with mass \(\mu\) and charge \(q\) moving around the RN black holes with mass \(M\) and charge \(Q\) \([22]\).

\[d^2s_{\text{dRN}}^2 = g_{\mu\nu}dx^\mu dx^\nu = -f_{\text{dRN}}(r)dt^2 + \frac{dr^2}{f_{\text{dRN}}(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\]

with the dRN metric function and its electromagnetic potential

\[f_{\text{dRN}}(r) = 1 - \frac{2M}{r} + \frac{Q_e^2 + Q_m^2}{r^2},\]

\[\vec{A} = \left[ -\frac{Q_e}{r}, 0, 0, Q_m (\cos \theta \mp 1) \right].\]

where the upper minus (lower plus) denotes the north-half sphere (south-half sphere) of the dRN black hole. Here, the line element Eq. (1) is stationary and spherically symmetric. In this case, the outer and inner horizons are given by

\[r_\pm = M \pm \sqrt{M^2 - Q_e^2 - Q_m^2},\]

where the condition \((Q_e^2 + Q_m^2)/M^2 \leq 1\) is necessary for the existence of a horizon including an extremal black hole. For \(Q_m = 0\), one finds the RN black hole with mass \(M\) and charge \(Q_e\). Also, if one replaces \(Q_e^2 + Q_m^2\) by \(Q^2\), one finds the RN black hole with \(f_{\text{RN}}(r) = 1 - 2M/r + Q^2/r^2\).

An electrically charged massive scalar perturbation \(\Phi\) on the background of dRN black holes is described by

\[(\tilde{\nabla}^\mu - iq\vec{A}^\mu)(\tilde{\nabla}_\mu - iq\vec{A}_\mu)^i \Phi - \mu^2 \Phi = 0,\]

where \(q\) denotes an electric charge of \(\Phi\) and \(\mu\) represents a mass of \(\Phi\).

Taking into account the background (1), it is useful to separate the scalar perturbation into modes

\[\frac{1}{\sin \theta} \partial_\theta \left( \sin \theta \partial_\theta Y_1(\theta) \right) + \left[ \lambda_1 - \frac{m^2 + 2(mqQ_m + q^2Q_m^2)(1 - \cos \theta)}{\sin^2 \theta} \right] Y_1(\theta) = 0,\]

\[\frac{1}{\sin \theta} \partial_\theta \left( \sin \theta \partial_\theta Y_2(\theta) \right) + \left[ \lambda_2 - \frac{m^2 - 2(mqQ_m + q^2Q_m^2)(1 + \cos \theta)}{\sin^2 \theta} \right] Y_2(\theta) = 0.\]

Requiring that \(\{Y_1(\chi), Y_2(\eta)\}\) with \(\chi, \eta = \cos \theta\) be finite at north and south poles, the solutions to Eqs. (6) and (7) are given by

\[Y_1(\chi) = c_1(1 - \chi)^{\frac{m}{2}}(1 + \chi)^{-\frac{m + 2qQ_m}{2}} \times \left[ a_1, \beta_1, 1 + m; \frac{1 - \chi}{2} \right] F_1,\]

\[Y_2(\eta) = c_2(1 - \eta)^{-\frac{m - 2qQ_m}{2}}(1 + \eta)^{\frac{m}{2}} \times \left[ a_2, \beta_2, 1 - m + 2qQ_m; \frac{1 - \eta}{2} \right] F_1,\]

with the hypergeometric functions \(F_1[\cdots]\) and their coefficients

\[a_1/\beta_1 = \frac{1 - 2qQ_m \pm \sqrt{1 + 4\Lambda_1}}{2}\].
\[ \alpha_2/\beta_2 = \frac{1 + 2q Q_m + \sqrt{1 + 4\lambda_2}}{2}. \]  

(10)

Furthermore, imposing the finiteness of \((1 - \chi)^{m/2}\) and \((1 + \eta)^{m/2}\) leads to \(m \geq 0\). Also, considering the convergence of two \(F_1[\cdots]\) under \(|(1 - \chi)/2| \leq 1\) and \(|(1 - \eta)/2| \leq 1\), one finds the quantization condition for magnetic charge, the constraint on \(\lambda_1\) and \(\lambda_2\), and the upper bound on the magnetic charge, respectively,

\[ q Q_m = \text{integer}, \quad \lambda_1 = \lambda_2 = I(l + 1), \quad q^2 Q_m^2 < I(l + 1). \]  

(11)

On the other hand, the radial equation for \(R_I(r)\) with \(\Delta = r^2 f_{dRN}(r)\) is given by

\[ \Delta \hat{\psi}_r (\Delta \hat{\psi}_r (r)) + \hat{U}(r) R_I(r) = 0 \]  

(12)

with

\[ \hat{U}(r) = (\omega r^2 - q Q_e r)^2 - \Delta \left[ \mu^2 r^2 + I(l + 1) - q^2 Q_m^2 \right]. \]  

(13)

Here, the last term of \(q^2 Q_m^2\) is an additional contribution from the angular part. It is worth noting that Eq. (12) is usually used to obtain exact solutions. In this direction, the radial solution to the Klein–Gordon equation for a massive charged scalar propagating around the Kerr–Newman black hole is given by the confluent Heun function \(\text{HeunC}(\alpha, \pm \beta, \gamma, \delta, \eta; z)\) [24, 25]. Two near-horizon and asymptotic solutions derived from the confluent Heun function were compared with the Whittaker’s form which is obtained asymptotically from the confluent Kummer and Tricomi functions as well as in the near-horizon.

Now, introducing a tortoise coordinate \(r_*\) defined by

\[ r_* = \int \frac{dr}{f_{dRN}(r)} = r + \frac{r^2}{r_+ - r_o} \ln (r - r_+) \]  

\[ - \frac{r^2}{r_+ - r_o} \ln (r - r_+), \]  

(14)

one may derive the Schrödinger-type equation when setting \(\Psi_I(r) = r R_I(r)\) from Eq. (12)

\[ \frac{d^2 \Psi_I(r_*)}{dr_*^2} + \left[ \omega^2 - V_{dRN}(r) \right] \Psi_I(r_*) = 0, \]  

(15)

where the potential \(V_{dRN}(r)\) is found to be

\[ V_{dRN}(r) = \omega^2 - \left( \omega - \frac{Q_e}{r} \right)^2 + f_{dRN}(r) \]  

\[ \times \left[ \mu^2 + \frac{l(l + 1)}{r^2} - \frac{q^2 Q_m^2}{r^2} + \frac{2(Mr - Q_e^2)}{r^2} \right]. \]  

(16)

Here, the last term in \([\cdots]\) comes from the introduction of the tortoise coordinate \(r_*\). We wish to point out asymmetric roles of two charges \(Q_e\) and \(Q_m\). We observe that an electric charge \(Q_e\) appears in the second term because an electrically charged massive scalar perturbation \(\Phi\) is introduced as a probe field. On the other hand, a magnetic charge \(Q_m\) appears as the third term in \([\cdots]\). Comparing it with the mass term \(\mu^2\) leads to the fact that an inclusion of \(Q_m\) does not induce a mass term. It may play the role of a centrifugal term when comparing with a centrifugal \(l(l + 1)/r^2\)-term. Thus, it is interesting to mention that \(V_{dRN}(r)\) differs by the third term from the potential for an electrically charged massive scalar in the RN black hole background with mass \(M\) and charge \(Q\) as [12–17]

\[ V_{RN}(r) = \omega^2 - \left( \omega - \frac{Q}{r} \right)^2 + f_{RN}(r) \]  

\[ \times \left[ \mu^2 + \frac{l(l + 1)}{r^2} + \frac{2(Mr - Q^2)}{r^2} \right]. \]  

(17)

Taking the asymptotic form of Eq. (15) and its near-horizon form, one finds plane-wave solutions

\[ \Psi^{\infty}(r_*) \sim e^{-i\sqrt{\omega^2 - \mu^2} r_+} (\cdots) + R e^{i\sqrt{\omega^2 - \mu^2} r_+} (\cdots), \]  

(18)

\[ r_+ \to +\infty (r \to \infty), \]  

\[ \Psi^{-\infty}(r_*) \sim T e^{-i(\omega - \omega_c) r_+} (\cdots), \]  

(19)

\[ r_+ \to -\infty (r \to r_+), \]  

where \(T (R)\) are the transmission (reflection) amplitudes. Here, we may have \(V(r) \to r_+ = \omega^2 - (\omega - \omega_c)^2\) with \(\omega_c = q Q_e/r_+\) in the near-horizon limit. Imposing the flux conservation, we obtain a relation between reflection and transmission coefficients as

\[ |R|^2 = 1 - \frac{\omega - \omega_c}{\sqrt{\omega^2 - \mu^2}} |T|^2, \]  

(20)

which means that the scalar waves with \(\omega > \mu\) propagate to infinity and the superradiant scattering may occur \((\to, |R| > |T|)\) when \(\omega < \omega_c\) (superradiance condition) is satisfied.

On the other hand, one may choose the bound state condition \((\omega < \mu)\) to have an exponentially decaying scalar as it tends to zero at infinity

\[ \Psi^{b, \infty}(r_*) \sim e^{-\sqrt{\mu^2 - \omega^2} r} \to 0. \]  

(21)

Solving Eq. (12) directly, the frequency \(\omega\) is permitted to be complex (small complex modification) as

\[ \omega = \omega_R + i\omega_I. \]  

(22)

Here the sign of \(\omega_I\) determines the solution which is decaying \((\omega_I < 0)\) or growing \((\omega_I > 0)\) in time.

Finally, we mention two cases for a electrically charged massive scalar propagating around the dRN black holes according to the potential analysis.

Case (i) superradiant stability: \(\omega < \omega_c\) and \(\omega < \mu\) without a positive trapping well.
Case (ii) superradiant instability: $\omega < \omega_c$ and $\omega < \mu$ with a positive trapping well. Solving the radial equation (12) directly leads to the real frequency ($\omega_R$) and the imaginary one ($\omega_I$). In this case, one could describe the above cases again: Case (i): $\omega_I < 0$ and $\omega_R < \omega_c$. The solution is stable (decaying in time).
Case (ii): $\omega_I > 0$ and $\omega_R < \omega_c$. The solution is unstable (growing in time).

3 Potential analysis

First of all, we start with an explicit form of the scalar potential (16) as

$$
\tilde{V}_{dRN}(r) = \mu^2 - \frac{2(M\mu^2 - q Q_e \omega)}{r} + \frac{l(l+1) + (Q_e^2 + Q_m^2)(\mu^2 - q^2)}{r^2} - \frac{2M[l(l+1) - q^2 Q_m^2 - 1]}{r^3} + \frac{(Q_e^2 + Q_m^2)[l(l+1) - q^2 Q_m^2 - 2] - 4M^2}{r^4} + \frac{6M(Q_e^2 + Q_m^2)}{r^5} - \frac{2(Q_e^2 + Q_m^2)^2}{r^6},
$$

(23)

where the first line is generated from $\tilde{U}(r)$ in the radial equation (13) solely, the second one comes from both $\tilde{U}(r)$ and the introduction of tortoise coordinate $r_s$, and the third line is generated from introducing tortoise coordinate $r_s$. It is important to mention that the potential in the first line plays an essential role in analyzing the superradiant (in)stability. An asymptotic form of the potential is given by

$$
V_{aadR}(r) = \mu^2 - \frac{2(M\mu^2 - q Q_e \omega)}{r} \equiv 1 - \frac{B}{r},
$$

(24)

which is obtained in the asymptotic region of $r \rightarrow \infty$ from (23). Here, we note the absence of magnetic charge $Q_m$. This Newton–Coulomb potential could be used to find the condition to get a trapping well as

$$
V'_{aadR}(r) > 0 \rightarrow M \mu^2 > q Q_e \omega (B > 0).
$$

(25)

This indicates the Newton–Coulomb requirement for the gravitational (attractive) force to exceed the electrostatic (repulsive) force. On the other hand, the condition for no trapping well is realized as

$$
V'_{aadR}(r) < 0 \rightarrow M \mu^2 < q Q_e \omega (B < 0),
$$

(26)

which describes the Coulomb-Newton requirement for the electrostatic force to exceed the gravitational force. The case of $V'_{aadR}(r) = 0$ corresponds to the absence of the Newton–Coulomb-term ($0/r$).

However, Eq. (25) [Eq. (26)] is not a sufficient condition for determining a trapping well [no trapping well]. This is so because a case of $V'_{aadR}(r) > 0$ includes three potential types: potential with a trapping well (local maximum), potential with a tiny well, and an increasing potential without any extrema. Also, the case of $V'_{aadR}(r) < 0$ implies either a potential without trapping well or a potential with a local maximum. Therefore, we have to find the other conditions to specify a trapping well. For this purpose, we are necessary to introduce the far-region potential appeared in the large $r$ region from (23) as

$$
V_{aadR}(r) = \mu^2 - \frac{2(M\mu^2 - q Q_e \omega)}{r} + \frac{l(l+1) + (Q_e^2 + Q_m^2)(\mu^2 - q^2)}{r^2}
$$

$$
\equiv \frac{\mu^2 - B}{r} + \frac{C}{r^2},
$$

(27)

(28)

where $C$-term plays a crucial role in making a trapping well. We observe from $V_{aadR}(r)$ that an inclusion of magnetic charge $Q_m$ plays an equal role of the electric charge $Q_e$, but it does not play the role of a magnetic field like $\mu \propto B$. It is worth noting that two charges of the dRN black holes induce repulsive (attractive) effects to the scalar potential for $\mu > q$ ($\mu < q$). However, such effects are limited because the condition for the existence of a horizon is given by $(Q_e^2 + Q_m^2)/M^2 \leq 1$. This can be seen easily from a centrifugal potential ($l(l+1)/r^2$-term) which may have a greatly repulsive effect on making a trapping well for large $l$. Actually, there is no way to make a trapping well (local minimum) if one keeps the Newton–Coulomb potential $V_{aadR}(r)$ only. Here, we have an extremal point ($r_e$)

$$
V_{aadR}(r_e) = 0 \rightarrow r_e = \frac{l(l+1) + (Q_e^2 + Q_m^2)(\mu^2 - q^2)}{M \mu^2 - q Q_e \omega}
$$

$$
= \frac{2C}{B} > r_+, \quad (29)
$$

which becomes either a local minimum or a local maximum, located far from the outer horizon.

Hereafter, we choose $M = 1$ such that $M\mu$ becomes $\mu$ for simplicity. We wish to introduce five distinct potentials to test their presence of a trapping well. We expect that these potentials offer a complete classification of all potentials. First of all, we consider a potential appeared in (left) Fig. 1. It includes no trapping well, being consistent with $V_{aadR}(r) < 0$. It is curious to note that (left) Fig. 2 corresponds to a potential with a trapping well (●), implying $V'_{aadR}(r) > 0$. On the other hand, it is interesting to explore a secret nature of a potential of (left) Fig. 3. In this case, we observe apparently that there is no trapping wells. But, the condition of $V'_{aadR}(r) > 0$ may imply a trapping well. So, it seems that $V_{aadR}(r) > 0$ is not compatible with our expectation of no trapping well. To resolve it, we note that
Fig. 1 (Left) Potential $V_{dRN}(r)$ as function of $r \in [r_+ = 1.77, 100]$ with $M = 1, Q_e = 0.4, Q_m = 0.5, \omega = 0.02, q = 2, l = 2, \mu = 0.05$. We have no trapping well here. We note $\omega < \mu$ and $\omega < \omega_c (= 0.45)$ to meet a superradiant stability with $V_{adr}''(r) < 0$. (Right) Far-region scalar function $\Psi_2(r)$ and its asymptotic form $\Psi_2^A(r)$ with confluent hypergeometric $U[2.94, 5.3, 0.1r]$ represent bound states.

Fig. 2 (Left) Potential $V_{dRN}(r)$ as function of $r \in [r_+ = 1.99, 200]$ with $M = 1, Q_e = 0.01, Q_m = 0.05, \omega = 2.95, q = 20, l = 13, \mu = 3$. A blue dot (●) denotes a local minimum at $r = 17.9$, implying a trapping well. Here, one has $V_{adr}''(r) > 0$. We note that $\omega_c (= 0.1) < \omega$ does not satisfy the superradiance condition. (Right) Far-region scalar function $\Psi_{13}(r)$ and its asymptotic form $\Psi_{13}^A(r)$ as functions of $r \in [21.5, 200]$ represent quasibound states.

Fig. 3 (Left) Potential $V_{dRN}(r)$ as function of $r \in [r_+ = 1.62, 100]$ with $M = 1, Q_e = 0.6, Q_m = 0.5, \omega = 0.2, q = 2, l = 3, \mu = 0.5$. We note $\omega < \mu$ and $\omega < \omega_c (= 0.74)$ to meet a superradiant stability with $V_{adr}''(r) > 0$. Here, no trapping well is found apparently. (Right) $V_{dRN}(r) \simeq V_{adr}(r)$ have a tiny well (●) located at $r = 971$, $V_{adr}(r)$ approaches them for $r > 971$.

Interestingly, (left) Fig. 5 shows a different potential with $V_{adr}'(r) < 0$, but it has a local maximum (●) located at $r = 142$ in $V_{dRN}(r) \simeq V_{adr}(r)$ [see (right) Fig. 5]. It looks like an upside-down figure of (right) Fig. 3. A final potential with $V_{adr}'(r) > 0$ appeared in (Left) Fig. 7 represents an increasing function approaching $\mu^2 = 32.5$, implying no trapping well. Here, $V_{adr}(r)$ includes a well inside the outer...
suggested that one finds a decreasing function $U[a, b; cr]$ for $a > 0$, whereas one has an increasing function $U[a, b; cr]$ for $a < 0$. Plugging Eq. (37) into Eq. (34) leads to an asymptotic wave function as

$$
\Psi_1^A(r) \simeq e^{-Ar} \left(2Ar \right)^{n/2} \times \left[ 1 - \frac{(k + 1/2 - B/2A)(-k + 1/2 - B/2A)}{(2Ar)} \right]. \quad (38)
$$

Here, we observe that taking into account $e^{-Ar}$, a quasibound state may appear for $\frac{B}{2A} > 0$, whereas a bound state may appear for $\frac{B}{2A} < 0$. In addition, considering the first derivative $U'[a, b; cr] = -acU[1 + a, 1 + b; cr]$ with respect to $r(c > 0)$, it implies that the condition for a trapping well is given by $U'[a, b; cr] > 0$, indicating $a < 0$. On the other hand, the condition for no trapping well takes the upper bound $U'[a, b; cr] < 0$, showing $a > 0$.

We are in a position to describe far-region and asymptotic wave function according to the potential classification. Considering the potential without trapping well whose asymptotic derivative is negative ($V''_\text{اذكر}(r) < 0$) [(left) Fig. 1], (right) Fig. 1 shows that all $\Psi_2(r)$, $\Psi_3^A(r)$, and $U[2.94, 5.3; 0.1r]$ are decreasing functions with a negative $U''[2.94, 5.3; 0.1r]$. They describe bound state. This case represents no trapping well clearly.

Let us observe a radial mode $\Psi_{13}(r)$ and its asymptotic mode $\Psi_{13}^A(r)$ for the potential with a trapping well ($V''_\text{اذكر}(r) > 0$) [see (left) Fig. 2]. As is shown in (right) Fig. 2, they represent quasi-bound states. In this case, one has an increasing function $U[-2.35, 27.9; 1.1r]$ and its increasing derivative $U'[-2.35, 27.9; 1.1r]$ appeared in (left) Fig. 4. This case shows a trapping well clearly.

Choosing a potential [(left) Fig. 3] without trapping well, all $\Psi_1(r)$, $\Psi_3^A(r)$, and $U[3.6, 7.3; 0.92r]$ are decreasing functions with a negative $U'[3.6, 7.3; 0.92r]$. They describe bound state. We note again that this potential with $V''_\text{اذكر}(r) > 0$ includes a tiny well located at $r = 971$ [see (right) Fig. 3]. After analyzing far-region and asymptotic wave functions, it turns out that the tiny well could be neglected effectively.

We observe the far-region and asymptotic wave functions corresponding to a local maximum potential [(left) Fig. 5] with $V''_\text{اذكر}(r) < 0$. As is shown in (left) Fig. 6, all functions are complex. Their real parts represent oscillating functions, while $\text{Re} [\Psi_1^0(r)]$ represents a negative bound state. This is so because of $C(-39) < -0.25$. This belongs to an unwanted case.

For an increasing potential [(left) Fig. 7] with $V''_\text{اذكر}(r) > 0$, one has a half of a peak (quasibound state, $\Psi_4(r) \simeq \Psi_4^A(r)$) [see (right) Fig. 7]. As is shown in Fig. 8, the confluent hypergeometric function $U[-0.0008, 10.6; 10.5r]$ is constant nearly and thus, its first derivative $U'[−0.0008, 10.6;
Fig. 4 (Left) Hypergeometric function \( U[-1.45, 27.9, 1.1r] \) and its first derivative \( U'[-1.45, 27.9, 1.1r] \) represent (left) Fig. 2. (Right) Far-region scalar function \( \Psi_3(r) \) and its asymptotic form \( \Psi_3^A(r) \) with \( U[3.6, 7.3, 0.92r] \) represent bound states appeared in (left) Fig. 3.

Fig. 5 Local maximum potential \( V_{\text{dR}}(r) \) as function of \( r \in [r_+ = 1.77, 100] \) with \( M = 1, Q_e = 0.4, Q_m = 0.5, \omega = 0.02, q = 10, l = 5, \mu = 0.05 \). We note \( \omega < \mu \) and \( \omega < \omega_c (= 2.26) \) with \( V_{\text{sadR}}(r) < 0 \). (Right) \( V_{\text{dR}}(r) \) have a local maximum (●) located at \( r = 142 \). \( V_{\text{sadR}}(r) \) approaches them for \( r > 142 \).

Fig. 6 (Left) Oscillating function \( \text{Re}[\Psi_{10}(r)] \) with \( \text{Re}[U[2.19 + 3.78i, 1 + 6.56i; 0.92r]] \) and \( \text{Re}[U'[2.19 + 3.78i, 1 + 6.56i; 0.92r]] \) for (left) Fig. 5. (Right) Its asymptotic wave function \( \text{Re}[\Psi_0^A(r)] \) represents a negative bound state.

10.5\( r \) is zero nearly \( (a \approx 0) \). This case represents a boundary between trapping well \( (a < 0) \) and no trapping well \( (a > 0) \).

According to the above prescription, the quasibound state could be achieved when the first argument of \( U[a, b; cr] \) is negative as

\[
a < 0 \quad \Rightarrow \quad \frac{B}{2A} > k + \frac{1}{2}
\]

which corresponds to the other condition to possess a trapping well. On the other hand, the bound state could be found when the first argument of \( U[a, b; cr] \) is positive as

\[
a > 0 \quad \Rightarrow \quad \frac{B}{2A} < k + \frac{1}{2},
\]

which is regarded as the other condition for no trapping well. The boundary case is obtained when \( a \) is zero nearly,

\[
a \approx 0 \quad \Rightarrow \quad \frac{B}{2A} = k + \frac{1}{2}.
\]
The former condition shows
\[
M \mu^2 > q Q_e \omega [V'_{\text{adR}}(r) > 0, \ B > 0] \quad \text{and} \quad \frac{B}{2A} > k + \frac{1}{2}.
\]
(42)

which differs slightly from the necessary condition for a trapping well \( \frac{M \mu}{q Q_e} > 1 \) in [7,15]. This is so because of the bound state condition \( \frac{\omega}{\mu} < 1 \). On the other hand, two conditions for no trapping well are given by

\[
M \mu^2 < q Q_e \omega [V'_{\text{adR}}(r) < 0, \ B < 0] \quad \text{and} \quad \frac{B}{2A} < k + \frac{1}{2},
\]
(44)

where the former condition could be written as [27]

\[
\frac{M \mu}{q Q_e} < \frac{\omega}{\mu} < 1,
\]
(45)

which is different slightly from the condition for no trapping well as \( \frac{M \mu}{q Q_e} < 1 \) appeared in Ref. [10].

5 Geodesic analysis in the RN geometry

In this section, we wish to explain how the known bounds of \( \frac{M \mu}{q Q_e} > 1 (\frac{M \mu}{q Q_e} < 1) \) arise from the pseudo-Newtonian potential for a charged particle with mass \( \mu \) and charge \( q \) in the RN black holes with mass \( M \) and charge \( Q \) [22]. The RN black hole (\( g_{\mu \nu} \) and \( \Lambda_{\mu} \)) is easily recovered from the dRN black hole (1) by resetting \( Q_e = Q \) and \( Q_m = 0 \). We start with the Lagrangian for a charged particle moving in the RN geometry

\[
L = \frac{1}{\hat{g}_{\mu \nu}} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - \frac{q}{\mu} \Lambda_{\mu} \frac{dx^\mu}{d\lambda}
\]
(46)

Finally, we obtain two conditions for getting a trapping well as

\[
M \mu^2 > q Q_e \omega [V'_{\text{adR}}(r) > 0, \ B > 0] \quad \text{and} \quad \frac{B}{2A} > k + \frac{1}{2}.
\]

The former condition shows

\[
M \mu > \frac{\omega}{\mu},
\]
(43)

which is different slightly from the condition for no trapping well as \( \frac{M \mu}{q Q_e} < 1 \) appeared in Ref. [10].

with \( \lambda = \tau / \mu \) the proper time per unit mass. The radial equatorial (\( \theta = \pi/2 \)) motion for a charged particle takes the form

\[
\left( \frac{dr}{d\lambda} \right)^2 = E^2 - V_g(r),
\]
(47)

where \( E = \partial L / \partial \dot{\lambda} \) is the conserved energy of the particle and \( V_g(r) \) is the geodesic potential defined as

\[
V_g(r) = \mu^2 - \frac{2(M \mu^2 - q Q E)}{r} + \frac{L^2 + Q^2(\mu^2 - q^2)}{r^2} - \frac{2ML^2}{r^3} + \frac{Q^2 L^2}{r^4},
\]
(48)

where \( L = \partial L / \partial \dot{\phi} \) is the conserved projection of the particle’s angular momentum on the axis of the black hole. In association with \( E \leftrightarrow \omega \), \( Q \leftrightarrow Q_e \) with \( Q_m = 0 \), and \( L^2 \leftrightarrow l(l + 1) \), the geodesic potential \( V_g(r) \) matches the scalar potential \( V_{\text{RN}}(r) \) exactly when neglecting the terms from introducing the tortoise coordinate \( r_* \) in \( V_{\text{RN}}(r) \).

Fig. 7 (Left) Increasing potential \( V_{\text{RN}}(r) \simeq V_{\text{adR}}(r) \simeq V_{\text{adR}}(r) \) as function of \( r \in [r^+ = 1.81, 50] \) with \( M = 1, Q_e = 0.43, Q_m = 0.4, \omega = 2.2, q = 5, l = 4, \mu = 5.7 \). A local minimum of \( V_{\text{adR}}(r) \) is found inside the outer horizon at \( r = r^+ (\bullet) \). We note \( \omega < \omega_e (\approx 4.75) \) and \( \omega < \mu \) with \( V'_{\text{adR}}(r) > 0 \). (Right) Far-region \( \Psi_4(r) \) and its asymptotic form \( \Psi_4^+(r) \simeq \Psi_4(r) \) represent a half of a peak.

Fig. 8 (Left) Constant hypergeometric function \( U[-0.0008, 10.6, 10.5 r] \) and its derivative \( U'[-0.0008, 10.6, 10.5 r] \) \( \simeq 0 \) for (left) Fig. 7.
tions from which one can determine $E$ and $L$ in terms of $M, Q, q$ and $\mu$. This is a process for eliminating $E$ and $L$. For $q/\mu \ll 1$, one expands $E$ and $L$ as functions of $q/\mu$ with $\mu$ fixed: $E \approx E_0(r) + q E_1(r) + O(q/\mu)^2$ and $L \approx L_0(r) + q L_1(r) + O(q/\mu)^2$ with $E_0(r)$ the conserved energy for a neutral particle and $L_0(r)$ the conserved projection of the angular momentum for a neutral particle. The centripetal force $F_0(r)$ acting on the neutral particle is given by $F_0(r) = (L_0^2(r)/E_0^2(r))/r^3$.

On the other hand, the centripetal force acting on the charged particle is given by

$$F(r) = \frac{L^2(r)}{E^2(r)} \frac{1}{r^3} = F_0(r) \left( 1 + \frac{q L_1(r)}{\mu E_0(r)} \right)^2.$$

This pseudo-Newtonian potential $V_M(r) = -\int F(r) dr$ gives rise to the Manev potential of $V_M(r) = G/r + H/r^2$ with $G$ and $H$ constants. A classical Manev potential to model a circular orbit of a lightly charged particle ($q/\mu \ll 1$) can be obtained by seeking expansion over the powers of $M/r$. The leading and next-leading terms of this potential are given by

$$V_M(r) \approx \left( 1 - \frac{q Q}{M \mu} \right) \frac{M}{r} + \left( 2 - \frac{Q^2}{2M^2} - \frac{9qQ}{4M} \right) \frac{M^2}{r^2}.$$

We summarize the results for all asymptotic potential $V_{\text{aadr}}(r)$ and far-region wave function analysis in Table 1. One condition to have a trapping well $[V'_{\text{aadr}}(r) > 0 (B > 0)]$ includes three potential types: potential with a trapping well [(left) Fig. 2], potential with a tiny well [(left) Fig. 3], and an increasing potential representing a boundary [(left) Fig. 7]. Among these, the potential with a trapping well [(left) Fig. 2] is satisfied with the other condition of $a < 0(a = k + \frac{1}{2} - \frac{q}{\mu})$. The potential with a tiny well [(left) Fig. 3] goes with $a > 0$ and an increasing potential [(left) Fig. 7] has $a \approx 0$. This implies that two conditions for getting a trapping well are given by $V_{\text{aadr}}'(r) > 0 (B > 0)$ and $a < 0$. The former condition describes the Newton–Coulomb requirement for the gravitational force exceed the electrostatic force, while the latter is the condition to obtain a quasibound state. Here, it is important to point out that two conditions for a trapping well [(left) Fig. 2] and the superradiance condition of $\omega < \omega_c$ could not be satisfied simultaneously. This implies that the superradiant instability is not found naturally from a charged massive scalar propagating around the dRN black hole. However, the superradiant instability of an electrically charged massive scalar could be achieved in certain parameter space if a cavity is placed around the dRN black hole [14–17].

On the other hand, one condition for no trapping well $[V'_{\text{aadr}}(r) < 0 (B < 0)]$ involves two cases: potential without trapping well [(left) Fig. 1] and potential with a local maximum [(right) Fig. 5]. The potential without trapping well [(left) Fig. 1] is satisfied with the other condition of $a > 0$. The local maximum potential [(right) Fig. 5] implies $a = 2.2 + 6.2i$ (complex) and $C(= -39) < -0.25$. This means that two conditions for no trapping well are given by $V_{\text{aadr}}'(r) < 0 (B < 0)$ and $a > 0$. The former condition describes the Coulomb–Newton requirement for the electrostatic force exceed the gravitational force, while the latter is the condition to find a bound state.

Now, it is worth mentioning that one condition to have a trapping well Eq. (43) differs slightly from the known bound of $M_{\text{aadr}}^* > 1$ [7, 15], while one condition for no trapping well Eq. (45) is different slightly from the known bound of $M_{\text{aadr}}^* < 1$ [10]. We have explained how these known bounds arise from the pseudo-Newtonian potential for a charged particle with mass $\mu$ and charge $q$ moving around the RN black hole with mass $M$ and charge $Q$ [22].

Finally, we would like mention that the dRN black hole reduces to the RN black hole when replacing $Q^2 + Q_m^2$ by $Q^2$. Considering an electrically charged massive scalar perturbation around the RN black hole, one finds the corresponding scalar potential $V_{\text{RN}}(r)$ in Eq. (17) [12–17] which differs by $q^2 Q_m^2/r^2$ from $V_{\text{aadr}}(r)$ in Eq. (16). Even though we expect to obtain similar results as in Table 1, we have not found
such a complete analysis for superradiant instability around the RN black holes in the literatures.

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