Extracting $\pi\pi$ $S$-wave scattering lengths from cusp effect in heavy quarkonium dipion transitions

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Abstract

Charge-exchange rescattering $\pi^+\pi^- \rightarrow \pi^0\pi^0$ leads to a cusp effect in the $\pi^0\pi^0$ invariant mass spectrum of processes with $\pi^0\pi^0$ in the final state which can be used to measure $\pi\pi$ $S$-wave scattering lengths. Employing a non-relativistic effective field theory, we discuss the possibility of extracting the scattering lengths in heavy quarkonium $\pi^0\pi^0$ transitions. The transition $\Upsilon(3S) \rightarrow \Upsilon(2S)\pi^0\pi^0$ is studied in details. We discuss the precision that can be reached in such an extraction for a certain number of events.

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1 Introduction

Being much lighter than all the other hadrons, the pions play a unique role in the strong interactions. They are the pseudo-Goldstone bosons of the spontaneous chiral symmetry breaking in quantum chromodynamics (QCD). Thus, to a large extent, the interaction between pions are governed by spontaneous and explicit chiral symmetry breaking. The $\pi\pi$ scattering problem already has a long history, which began about half a century ago [11]. At low energies, the strength of the $\pi\pi$ $S$-wave interaction is described by the scattering lengths, which can shed light on the fundamental properties of QCD. The scattering lengths can be calculated in chiral perturbation theory (ChPT) [2,3], the low-energy effective theory of QCD, to a given order in the chiral expansion. Combining two-loop ChPT with Roy equations, the $\pi\pi$ scattering lengths were predicted with a high precision [4,5]. For instance, the difference between the isospin $I = 0$ and $I = 2$ $S$-wave scattering lengths was predicted to be $(a_0 - a_2)M_{\pi^+} = 0.265 \pm 0.004$. A similar result of $0.262 \pm 0.006$ was obtained in Ref. [6] using dispersion relations without input from ChPT.

Experimentally, the $\pi\pi$ scattering lengths can be measured in several ways. The angular distributions of $K_{L\Sigma}$ decay is sensitive to the $\pi\pi$ phase shifts which are related with the scattering lengths. The first experiment along these lines was carried out by the Geneva-Saclay Collaboration in the seventies of the last century [7]. A similar method was recently employed by the E865 and NA48/2 Collaborations [8,9,10,11].

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In this paper, we investigate the possibility of extracting the $\pi\pi$ scattering lengths using the cusp effect in heavy quarkonium dipion transitions. These transitions are among the most important decay modes of the heavy quarkonium states below open heavy-flavor thresholds. Taking the process $\psi' \to J/\psi\pi^0\pi^0$ as an example, the branching fraction is $17.75 \pm 0.34\%$ [24], and the BESIII and CLEO-c Collaborations have already accumulated huge data samples in this channel. In particular, the BESIII Collaboration has acquired a sample of 106 million $\psi'$ events, and this number is still increasing [26]. Because of the Watson final-state theorem [27], it is possible to learn about the $\pi\pi$ interaction from the dipion transitions. There were suggestions of studying the $\pi\pi$ scattering phase shifts in the $\psi' \to J/\psi\pi^+\pi^-$ transitions [28,29,30]. There are also huge data samples for the bottomonium states which were collected in the $B$-factories, and more are expected to come from the next-generation high-luminosity $B$-factories [31,32]. In view of this situation, it is interesting to explore the cusp effect in heavy quarkonium transitions with two neutral pions in the final state. Because the isospin symmetry is well conserved here, one has $B(\psi' \to J/\psi\pi^+\pi^-)/B(\psi' \to J/\psi\pi^0\pi^0) \approx 2$ [24], and similarly for the other heavy quarkonium dipion transitions. This is similar to the $\eta' \to \eta\pi\pi$, which will make the charge-exchange rescattering effect more important than those in the processes $K_L \to 3\pi$ and $\eta \to 3\pi$. In addition, it was found in Ref. [23] that two-loop rescattering is highly suppressed in the $\eta' \to \eta\pi\pi$ process due to the approximate isospin symmetry, so that the cusp in the $\pi^0\pi^0$ distribution...
is completely dominated by one-loop contributions. The same conclusion should hold in our case. Therefore, it is safe to work up to only one-loop order. From the theoretical point of view, since the interaction between a heavy quarkonium and pion is highly Okubo-Zweig-Iizuka (OZI) suppressed, we may further simplify the problem by neglecting this type of contributions.

If we concentrate on the region near the $\pi\pi$ threshold, the three-momenta of all the final particles are small in comparison with their masses. Thus, a nonrelativistic effective field theory (NREFT) can be employed. The processes to be considered are similar to the $\eta' \rightarrow \eta\pi\pi$. We will follow here the NREFT framework developed and applied in Refs. [20, 23] and refer to these papers and references therein for more details. This method was firstly used in the study of cusp effect in $K \rightarrow 3\pi$, and then extended to other reactions such as e.g. $\eta' \rightarrow \eta\pi\pi$ and $\eta \rightarrow 3\pi$.

Our paper is organized as follows. The framework of NREFT will be briefly introduced in Section 2, where the necessary terms in the effective Lagrangians are listed. The low-energy constants entering the tree-level production amplitude are determined in Section 3 by matching to a relativistic description of the decay for the transition $\Upsilon(3S) \rightarrow \Upsilon(2S)\pi^0\pi^0$ in the framework of unitarized chiral perturbation theory. In the same section, we generate several sets of synthetic data using the Monte Carlo method, and investigate the accuracy of the extraction of the $\pi\pi$ scattering lengths from these synthetic data. A short summary is presented in Section 4, and an estimate of the $J/\psi\pi$ scattering length is attempted in Appendix A.

2 Nonrelativistic effective field theory

Here and in what follows, we consider $\psi'$ dipion transitions as an example. The method can be easily extended to bottomonium case. Let us describe the power counting scheme in the NREFT, which is essentially a nonrelativistic velocity counting. The masses of all involved particles are counted as $O(1)$. Since we focus on the region close to the thresholds of two pions, even pions can be dealt with nonrelativistically. The heavy quarkonium in the final state is also nonrelativistic because its three-momentum in the rest frame of the decaying particle does not exceed 500 MeV. We, therefore, count all these three-momenta as quantities of order $O(\epsilon)$. The kinetic energy $T_i = p_i^0 - M_i$ is then counted as $O(\epsilon^2)$. Another expansion parameter used in this scheme is the $\pi\pi$ scattering length, denoted by $a_{\pi\pi}$. Here one relies on the fact that low-energy interactions between two pions are weak due to their Goldstone boson nature. In principle, $J/\psi\pi$ scattering should also be taken into account. However, it should be suppressed according to the OZI rule because the $J/\psi$ and pion do not have any common valence quark. A rough estimation of the $J/\psi\pi$ scattering length carried out in the appendix yields $|a_{J/\psi\pi}| \leq 0.02$ fm which is consistent with the preliminary lattice result $a_{J/\psi\pi} = (-0.01 \pm 0.01)$ fm [33]. Thus, the $J/\psi\pi$ scattering length is at least one order of magnitude smaller than $a_0$ and $a_2$. The bottomonium-pion scattering length would be even smaller. The situation here is, therefore, similar to the one in the process $\eta' \rightarrow \eta\pi\pi$, where $\eta\pi$ interaction was found to play a minor role in the $\pi\pi$ cusp structure, and its effect can be largely absorbed into the polynomial production amplitude [23]. Thus, in the following, we will not take into account the heavy quarkonium-pion scattering.

The relevant effective Lagrangians contain two parts

$$L_{eff} = L_\psi + L_{\pi\pi}. \quad (1)$$
Here, the first term describes the production mechanism and reads up to $O(\epsilon^2)$

$$
\mathcal{L}_\psi = \frac{1}{2} \sum_{n=0}^{1} G_n \left( \psi_i^\dagger (W_{J/\psi} - M_{J/\psi})^n J_i \Phi_0 \Phi_0 + h.c. \right) 
+ \sum_{n=0}^{1} H_n \left( \psi_i^\dagger (W_{J/\psi} - M_{J/\psi})^n J_i \Phi_+ \Phi_- + h.c. \right) + \cdots ,
$$

where $W_{J/\psi} = \sqrt{M^2_{J/\psi} - \triangle}$, with $\triangle$ being the Laplacian. At this order, the production is purely $S$-wave, while the $D$-wave contribution starts from $O(\epsilon^4)$. $\pi\pi$ interaction is described by [21]:

$$
\mathcal{L}_{\pi\pi} = 2 \sum_{k=0,\pm} \Phi_k^\dagger W_k (i\partial_t - W_k) \Phi_k 
+ C_x (\Phi_0^\dagger \Phi_0^\prime \Phi_+ \Phi_- + h.c.) + \frac{1}{4} C_{00} (\Phi_0^\dagger \Phi_0^\prime \Phi_0 \Phi_0 + h.c.) 
+ D_x \left( (\Phi_0^\dagger)_{\mu} (\Phi_0^\prime)_{\mu} \Phi_+ \Phi_- + (\Phi_0^\dagger)_{\mu} (\Phi_+^\dagger)_{\mu} (\Phi_-^\dagger)_{\mu} + h.c. \right) 
+ \frac{1}{4} D_{00} \left( (\Phi_0^\dagger)_{\mu} (\Phi_0^\prime)_{\mu} \Phi_0 \Phi_0 + (\Phi_0^\dagger)_{\mu} (\Phi_0^\prime)_{\mu} (\Phi_0^\prime)_{\mu} \Phi_0 + h.c. \right) + \cdots ,
$$

where

$$
(\Phi_k)_{\mu} = (P_k)_{\mu} \Phi_k, \quad (P_k)_{\mu} = (W_k, -i\nabla)
$$

$$
(\Phi_k^\dagger)_{\mu} = (P_k^\dagger)_{\mu} \Phi_k^\dagger, \quad (P_k^\dagger)_{\mu} = (W_k, i\nabla)
$$

and $W_k = \sqrt{M^2_k - \triangle}$. In Eq. (3), the couplings $C_x, C_{00}, D_x$ and $D_{00}$ can be obtained by matching the NREFT amplitude to the effective range expansion of $\pi\pi$ scattering amplitudes [10].

$$
T^I(s, t) = 32\pi \sum_{l=0}^{\infty} (2l + 1) t^l_I (s) P_l(z),
$$

$$
\text{Re } t^l_I (s) = q_{ab}^2 \left[ a^2_l + b^2_l q^2_{ab} + O(q_{ab}^4) \right],
$$

where $t^l_I$ is the partial wave amplitude with angular momentum $l$ and isospin $I$, $P_l(z)$ are the Legendre polynomials with $z = \cos \theta$, where $\theta$ is the scattering angle in the center-of-mass system, and $q_{ab} = [\lambda(\lambda, s^2, M^2_{\pi^0})/s]^{1/2}/2$ is the center-of-mass momentum with $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ac)$ being the Källén function. We thus have the following relations,

$$
C_x = \frac{16\pi}{3} M_{\pi^+} (a_0 - a_2) \left( 1 + \frac{\xi}{3} \right), \quad C_{00} = \frac{16\pi}{3} M_{\pi^+} (a_0 + 2a_2) (1 - \xi),
$$

$$
D_x = \frac{4\pi}{3} M_{\pi^+} (b_2 - b_0), \quad D_{00} = \frac{4\pi}{3} M_{\pi^+} (b_0 + 2b_2),
$$

where $\xi = (M^2_{\pi^+} - M^2_{\pi^0}) / M^2_{\pi^+}$, and the isospin breaking in the $S$-wave scattering lengths has been considered at leading order in ChPT [34]. We have used the phase convention such that $|\pi^+\rangle = -|1, +1\rangle$. Because we only consider $S$-wave scattering here, we have denoted the $I = 0$ and $I = 2$ scattering lengths by $a_0$ and $a_2$ for brevity in the above equations.
In this section we explore the possibility to extract the scattering lengths from the heavy quarkonium dipion transitions. A known feature of the reaction $\psi' \to J/\psi \pi^0 \pi^0$ via tree diagram and $\pi \pi$ rescattering diagrams.

Figure 1: $\psi' \to J/\psi \pi^0 \pi^0$ via tree diagram and $\pi \pi$ rescattering diagrams.

In this paper, we work only up to one-loop order. Neglecting the $J/\psi \pi$ interaction as explained above, the diagrams need to be considered are shown in Fig. 1. With the momenta defined as

$$\psi'(P_{\psi'}) \to \pi^0(p_1) \pi^0(p_2) J/\psi(p_3),$$

and $s_i = (P_{\psi'_i} - p_i)^2$ for $i = 1, 2, 3$, the transition amplitudes at the tree and one-loop level are

$$T^{\text{tree}} = \left[G_0 + G_1(p_3^0 - M_J)\right] \vec{e}_{\psi'} \cdot \vec{e}_J,$$

$$T^{1\text{-loop}} = 2 \left[C_x + D_x(s_3 - 4M_{\pi^0}^2)\right] \left[H_0 + H_1(p_3^0 - M_J)\right] J_{+-}(s_3) \vec{e}_{\psi'} \cdot \vec{e}_J$$

$$+ \left[C_{00} + D_{00}(s_3 - 4M_{\pi^0}^2)\right] \left[G_0 + G_1(p_3^0 - M_J)\right] J_{00}(s_3) \vec{e}_{\psi'} \cdot \vec{e}_J,$$

respectively, where $J_{ab}$ is a nonrelativistic loop integral defined as

$$J_{ab}(D^2) = \int \frac{d^Dl}{i(2\pi)^D} \frac{1}{2w_a(l)(w_a(l) - l_0)} \frac{1}{2w_b(P - l)(w_b(P - l) - P_0 + l_0)},$$

with $w(l) = \sqrt{M^2 + l^2}$. Within the nonrelativistic power counting scheme, the loop integral measure is counted as $O(e^5)$, and each of the two propagators is of order $O(e^2)$. Thus, the loop integrals $J_{+-}$ and $J_{00}$ are of order $O(e)$. Using dimensional regularization and taking $D = 4$, we obtain

$$J_{+-}(s_3) = \frac{-1}{16\pi} \sqrt{\frac{4M_{\pi^0}^2 - s_3}{s_3}}, \text{ when } s_3 \leq 4M_{\pi^0}^2,$$

$$J_{+-}(s_3) = \frac{i}{16\pi} \sqrt{\frac{s_3 - 4M_{\pi^0}^2}{s_3}}, \text{ when } s_3 > 4M_{\pi^0}^2,$$

$$J_{00}(s_3) = \frac{i}{16\pi} \sqrt{\frac{s_3 - 4M_{\pi^0}^2}{s_3}}.$$

One observes that $J_{+-}$ has a nonanalyticity at the $\pi^+\pi^-$ threshold which gives rise to a cusp effect in the $\pi^0\pi^0$ invariant mass distribution. The expression for the decay amplitude up to $O(a_{\pi\pi}e^2)$ reads:

$$T = \left[G_0 + G_1(p_3^0 - M_J) + 2C_x H_0 J_{+-}(s_3) + C_{00} G_0 J_{00}(s_3)\right] \vec{e}_{\psi'} \cdot \vec{e}_J.$$

In the isospin limit, we have $H_0 = G_0$ in our phase convention, and it will be used in the following.

### 3 Extraction of the scattering lengths

In this section we explore the possibility to extract the $\pi\pi$ scattering lengths from the heavy quarkonium dipion transitions. A known feature of the reaction $\psi' \to J/\psi\pi\pi$ is that the kinematical region
around the the ππ threshold we are interested in is strongly suppressed so that it only corresponds to a tiny fraction of the total events, see e.g. the BES and CLEO data for the \( \psi' \rightarrow J/\psi \pi^+ \pi^- \) [35, 36]. A more promising reaction is the \( \Upsilon(3S) \rightarrow \Upsilon(2S)\pi^0\pi^0 \), which we will concentrate on. The updated data came from the CLEO Collaboration [37], and their analysis is based on a \( \Upsilon(3S) \) yield about \( 5 \times 10^6 \).

### 3.1 Chiral unitary approach

In order to show the cusp effect in the \( \Upsilon(3S) \rightarrow \Upsilon(2S)\pi^0\pi^0 \), it is necessary to determine the values of \( G_0 \) and \( G_1 \). This will be achieved via matching to parameters entering the relativistic decay amplitude which can be fixed from fitting to the experimental data of the \( \pi^0\pi^0 \) invariant mass spectrum. A precise determination is presently not possible due to the bad data quality. However, a rough estimate of the ratio \( G_0/G_1 \) can be obtained. We employ a simple parametrization of the tree-level relativistic decay amplitude [38]

\[
N \epsilon \cdot \epsilon' (s_3 - C),
\]

where \( N \) is an overall normalization constant, and \( \epsilon \) (\( \epsilon' \)) is the polarization vector of \( \Upsilon(2S) \) (\( \Upsilon(3S) \)). The cusp effect shows up only when the \( \pi\pi \) FSI is considered. This may be taken into account using the chiral unitary approach (CHUA) [39, 40, 41, 42, 43], which has been used in studying the dipion transitions among heavy quarkonium states in Refs. [44, 45, 46]. In the CHUA, the \( \pi\pi S \)-wave scattering amplitude after taking into account isospin symmetry is given by

\[
T^0_0 (s_3) = V^0_0 (s_3) \left[ 1 - G(s_3)V^0_0 (s_3) \right]^{-1},
\]

where the \( 2 \times 2 \) matrix \( V^0_0 (s_3) \) contains the \( S \)-wave projected \( \pi^0\pi^0 \rightarrow \pi^0\pi^0, \pi^+\pi^- \rightarrow \pi^+\pi^- \) and \( \pi^0\pi^0 \rightarrow \pi^+\pi^- \) amplitudes derived from the lowest order chiral perturbation theory with virtual photons [47].

\[
V^0_0 (s_3) = \frac{1}{F_\pi^2} \left( \frac{M_{\pi^0}^2}{s_3 - M_{\pi^0}^2} \right) \left( \frac{(s_3 - M_{\pi^0}^2)/\sqrt{2}}{s_3 + 4(M_{\pi^+}^2 - M_{\pi^0}^2)/2} \right),
\]

Figure 2: Fit to the parametrization of \( \pi\pi \) phase shifts introduced in Ref. [6] (solid line), where the dashed line represents the CHUA results.
Figure 3: (a) Comparison of the best fit (histogram) to the \( \pi^0\pi^0 \) invariant mass spectrum of the \( \Upsilon(3S) \to \Upsilon(2S)\pi^0\pi^0 \) data (points with error bars) measured in Ref. [37]. The fit is done by integrating the distribution bin-by-bin. The solid smooth curve is the invariant mass spectrum calculated using the best fit parameters and multiplied by an arbitrary normalization constant. (b) The phase space subtracted spectrum around the \( \pi^+\pi^- \) threshold.

Here \( F_\pi \) is the pion decay constant and the difference between the charged and neutral pion masses is taken into account. \( G(s_3) = \text{diag}\{G_{00}(s_3), G_{+-}(s_3)\} \) is a diagonal matrix with

\[
G_{00(+-)}(s_3) = -\frac{1}{16\pi^2} \left[ \tilde{a}(\mu) + \log \frac{M_{\pi^0(\pi^0)}^2}{\mu^2} + \sigma_{0(\pi)} \log \left( \frac{\sigma_{0(\pi)} + 1}{\sigma_{0(\pi)} - 1} \right) \right]
\]

denoting the usual scalar loop function. Here, \( \sigma_{0(\pi)} = \sqrt{1 - 4M_{\pi^0(\pi^0)}^2/s_3} \), and \( \tilde{a}(\mu) \) is a subtraction constant introduced to regularize the loop \([42, 43]\). In order for the \( \pi\pi \) FSI to be consistent with \( \pi\pi \) scattering, the value of \( \tilde{a}(\mu) \) may be fixed by reproducing the S-wave \( \pi\pi \) phase shifts in the isoscalar channel, \( \delta^0_1(s_3) \). We fit to the parametrization of \( \delta^0_1(s_3) \) introduced in Ref. [6], which is given by Eq. (6) in that paper, and the central values of the parameters \( B_i \) in the parametrization are used. Isospin breaking effects are neglected in the fit. The fit range is chosen to be from the \( \pi\pi \) threshold up to 340 MeV, which contains all the available phase space of the \( \Upsilon(3S) \to \Upsilon(2S)\pi^0\pi^0 \). Our best fit is shown in Fig. 2 and yields \( \tilde{a}(1 \text{ GeV}) = -0.930 \). With this value of \( \tilde{a}(1 \text{ GeV}) \), we can fix the value of \( C \) in Eq. (15) by fitting to the \( \pi^0\pi^0 \) invariant mass spectrum of the \( \Upsilon(3S) \to \Upsilon(2S)\pi^0\pi^0 \) data measured in Ref. [37]. The decay amplitude in the CHUA is given by

\[
N\epsilon \cdot e'(s_3 - C) \left[ 1 + G_{00}(s_3)T^0_0(s_3)_{11} + \sqrt{2}G_{+-}(s_3)T^0_0(s_3)_{21} \right],
\]

where \( T^0_0(s_3)_{11(21)} \) refer to the unitarized amplitudes for the \( \pi^0\pi^0 \to \pi^0\pi^0 \) \( (\pi^+\pi^- \to \pi^0\pi^0) \) defined in Eq. (16). The best fit with \( \chi^2/\text{dof} = 1.44 \) is shown in Fig. 3(a). A small cusp at the \( \pi^+\pi^- \) threshold shows up, which is more apparent in the phase-space-subtracted invariant mass spectrum in Fig. 3(b). From the fit, we obtain

\[
C = -0.0197^{+0.0167}_{-0.0116} \text{ GeV}^2 = -1.01^{+0.86}_{-0.50} M_{\pi^+}^2.
\]

Matching \( G_{0,1} \) to \( N \) and \( C \) in Eq. (15) leads to the relations
\[ G_0 = N \left( (M - m_3)^2 - C \right), \quad G_1 = -2NM. \] (20)

Thus, the ratio is determined to be
\[ \frac{G_0}{G_1} = -4.37^{+0.81}_{-0.56} \text{ MeV}. \] (21)

It is small because \( G_0 \) contains the mass difference of the two heavy quarkonia while \( G_1 \) is proportional to the mass of the initial state. Using the central value and adopting the central values of the scattering lengths summarized in Ref. [25], \( a_0 = 0.2196 \) and \( a_2 = -0.0444 \) in units of \( M_{\pi^+}^{-1} \), the cusp effect in the NREFT is plotted in Fig. 4. Certainly, if charge-exchange rescattering is switched off, the cusp would disappear as shown by the dashed line. When integrating the spectrum below the threshold of \( \pi^+\pi^- \), the cusp effect resulted from charge-exchange rescattering will reduce the number of events in this region by about 9% with respect to the tree-level contribution. The values of this quantity in the processes \( K^+ \rightarrow \pi^0\pi^0\pi^+ \), \( \eta' \rightarrow \eta\pi\pi \) and \( \eta \rightarrow 3\pi \) are about 13%, 8% and less than 2%, respectively [23, 48].

### 3.2 Monte Carlo simulations

An important question is to what precision the scattering lengths can be extracted from the considered process. To explore this issue, we will first generate artificial data using the Monte Carlo (MC) method. The von Neumann rejection method is employed to select the random data points that follow the normalized distribution of the \( \pi^0\pi^0 \) in range between 270 MeV and 290 MeV predicted in the NREFT. The MC data are generated using the central value of \( G_0/G_1 \) given in Eq. (21), and \( a_0 - a_2 = 0.2640 \) and \( a_2 = -0.0444 \) in units of \( M_{\pi^+}^{-1} \) as input. These data can then be divided into a number of bins with the statistical errors given by the square root of the number of events in each bin. Varying the MC event numbers and the bin widths, one may investigate the impact of the event numbers as well as the experimental energy resolution on the precision of the extraction of the scattering lengths.
We tried a number of different combinations of the event numbers and bin widths. Figure 5 shows the ones with about $6 \times 10^4$, $6 \times 10^5$ and $6 \times 10^6$ events in the range of [270, 290] MeV. We then fit the $\pi^0\pi^0$ invariant mass distribution calculated using Eq. (14) to the MC data. $G_0/G_1$ is fixed to the same value used in the data generation. The free parameters are an overall normalization constant, $a_0 - a_2$ and $a_2$. The results of the fits are collected in Table I where the uncertainties only reflect the statistical errors in the fit. Because of the random fluctuation in the data generating process, the best fit values are not guaranteed to be the same as the input. An interesting observation is that the precision of the extraction seems to be quite insensitive to the bin widths, at least up to 2 MeV. Comparing the extracted values with the input $M_{\pi^+}(a_0 - a_2) = 0.2640$ and $M_{\pi^+}a_2 = -0.0444$, one sees that the precision of the extracted value of $a_0 - a_2$ can reach 10 - 20% for $6 \times 10^4$ events in the range of [270, 290] MeV. For more events, the precision is better by a factor of around $\sqrt{N'/N}$, with $N'$ and $N$ the new and old event numbers, as it should be. From Table I one sees that the statistical precision of Ref. [19], $M_{\pi^+}(a_0 - a_2) = 0.2571 \pm 0.0048$ (stat.), may be reached with $3 \times 10^6$ events. The spectrum is rather insensitive to $a_2$ such that the uncertainty is about 50% for $6 \times 10^6$ events. In fact, because $a_2$, independent of $a_0 - a_2$, only enters through the $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ rescattering, its effect can be largely absorbed into the polynomial production amplitudes. We have checked that if $G_0/G_1$ was
released as an additional free parameter, one would not get any useful information on \(a_2\) any more. Furthermore, the uncertainty of \(a_0 - a_2\) would also increase by a factor of about 2 to 2.5. In fact, \(G_0/G_1\) also contributes to the \(\Upsilon(3S) \rightarrow \Upsilon(2S)\pi^+\pi^-\), and its value can be extracted by measuring the \(\pi^+\pi^-\) spectrum in parallel, and thus not completely free. Notice that if we use \(a_0\) instead of \(a_0 - a_2\) as a free parameter, the errors would be much larger. We should stress that to the precision at per cent level, radiative corrections, which is most important in the \(\pi^+\pi^-\) threshold region, should be taken into account \[49, 23\]. Nevertheless, we expect that the precision that can be achieved would not get worsened as the photon-exchange can be taken into account by a simple replacement of the \(\pi^+\pi^-\) loop function \[23\]. Since our aim is to explore the possibility of extracting \(a_0 - a_2\), they will not be considered here.

From the CLEO data of the \(\Upsilon(3S) \rightarrow \Upsilon(2S)\pi^0\pi^0\), the events in the range of \([270, 290]\) MeV correspond to around 15% of the total yield of the process. Thus, \(6 \times 10^4\), \(6 \times 10^5\), \(3 \times 10^6\) and \(6 \times 10^6\) events in \([270, 290]\) MeV require the yields of the \(\Upsilon(3S) \rightarrow \Upsilon(2S)\pi^0\pi^0\) of about \(4 \times 10^5\), \(4 \times 10^6\), \(2 \times 10^7\) and \(4 \times 10^7\), respectively. Given that the branching fraction of \(\Upsilon(3S) \rightarrow \Upsilon(2S)\pi^0\pi^0\) is \((1.85 \pm 0.14)\%\) \[24\], at least 2 billion \(\Upsilon(3S)\) events have to be accumulated in order to obtain \(6 \times 10^6\) events in the range of \([270, 290]\) MeV. Other factors like the detecting efficiency will increase the number even further.

### Table 1: Results of fitting to various sets of MC data. The extracted scattering lengths are given in units of \(M_{\pi^+}^{-1}\).

| Bin width (MeV) | Events   | \(6 \times 10^4\) | \(6 \times 10^5\) | \(3 \times 10^6\) | \(6 \times 10^6\) |
|-----------------|----------|--------------------|--------------------|--------------------|--------------------|
| \(\chi^2/\text{dof}\) | \(a_0 - a_2\) | \(0.293 \pm 0.036\) | \(0.260 \pm 0.012\) | \(0.2717 \pm 0.0048\) | \(0.2661 \pm 0.0036\) |
| \(\chi^2/\text{dof}\) | \(a_0 - a_2\) | \(0.286 \pm 0.035\) | \(0.251 \pm 0.014\) | \(0.2722 \pm 0.0048\) | \(0.2621 \pm 0.0038\) |
| \(\chi^2/\text{dof}\) | \(a_0 - a_2\) | \(0.262 \pm 0.026\) | \(0.256 \pm 0.012\) | \(0.2659 \pm 0.0051\) | \(0.2693 \pm 0.0035\) |
| \(\chi^2/\text{dof}\) | \(a_0 - a_2\) | \(0.221 \pm 0.054\) | \(0.291 \pm 0.010\) | \(0.2658 \pm 0.0054\) | \(0.2661 \pm 0.0037\) |
| \(\chi^2/\text{dof}\) | \(a_0 - a_2\) | \(0.260 \pm 0.040\) | \(0.262 \pm 0.012\) | \(0.2592 \pm 0.0055\) | \(0.2632 \pm 0.0037\) |

4 Summary

In this paper, we investigate the possibility of extracting the \(\pi\pi\) \(S\)-wave scattering lengths using the cusp effect in heavy quarkonium transitions emitting two neutral pions. These processes are different from all the others for the cusp effect because all involved particles apart from the pions are heavy. This has a theoretical advantage that the cross-channel rescattering of a pion off a heavy quarkonium is weak due to the OZI suppression. Due to the approximate isospin symmetry, \(B(V' \rightarrow V\pi^+\pi^-)/B(V' \rightarrow V\pi^0\pi^0) \sim 2\) will lead to an enhanced cusp effect in \(\pi^0\pi^0\) invariant mass spectrum. Since we are dealing with the process where the relevant particles have low momenta, the framework of NREFT is adopted to calculate the decay amplitude which can be directly parameterized in terms of
the $\pi\pi$ threshold parameters. In the present analysis, we worked out the amplitude to order $O(a_{\pi\pi}^2)$.

We then focus on the $\Upsilon(3S) \to \Upsilon(2S)\pi^0\pi^0$, for which the parameters in the production amplitude are determined by matching to a fit to the experimental data based on the chiral unitary approach. In order to have a feeling on the achievable accuracy of the extraction of the scattering length, we generated a number of sets of artificial data using the Monte Carlo method. We then fitted these synthetic data to using the values of $a_0 - a_2$ and $a_2$ as free parameters. It is comforting to see that the resulting accuracy is insensitive to the bin width and energy resolution. A statistical precision of about $2\%$ and $1.5\%$ of $a_0 - a_2$ can be reached with $2 \times 10^7$ and $4 \times 10^7$ events of the $\Upsilon(3S) \to \Upsilon(2S)\pi^0\pi^0$, which corresponds to at least $1 \times 10^9$ and $2 \times 10^9$ $\Upsilon(3S)$ events, respectively. The precision can be worsened by a factor of about $2$ in reality because $G_0/G_1$ in the production amplitude cannot be fixed completely. However, measuring the $\Upsilon(3S) \to \Upsilon(2S)\pi^+\pi^-$ in parallel is very helpful in constraining $G_0/G_1$, and hence increasing the precision of the $a_0 - a_2$ extraction. The CLEO detector already recorded a sample of $(5.93 \pm 0.10) \times 10^6$ $\Upsilon(3S)$ decays [50], while this number is $1.08 \times 10^8$ for the BaBar detector [51]. With future high-luminosity $B$-factories, the sample can be one or two order-of-magnitude larger.

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A An estimate of the $J/\psi\pi$ scattering length

Before presenting the formalism, let us first roughly estimate the $J/\psi\pi$ scattering length. Certainly there are no scattering data available, but one may use the amplitude of $\psi'\pi \to J/\psi\pi$ as a reference for the $J/\psi\pi$ elastic scattering amplitude. Considering a process of scattering a pion off a charmonium, two possible mechanisms are shown in Fig. 6 (a) corresponds to the situation in which the charmonium emits two soft gluons which hadronize into pions. This mechanism can be described by using the method of QCD multipole expansion. The charmonium-pion scattering can also occur through intermediate charmed mesons, as depicted in Fig. 6(b), which represents a kind of non-multipole effect [52]. Noticing that the analytic structures of the amplitudes for these two mechanisms are different, one concludes that there is no double counting.

In the first mechanism, the difference between the transition ($\psi'\pi \to J/\psi\pi$) and elastic ($J/\psi\pi \to J/\psi\pi$) amplitudes is due to the charmonia-two-gluon vertex, which is proportional to a quantity called

![Figure 6: Schematic diagrams of the charmonium-pion scattering. Here the doubly-solid, dashed, solid and wiggly lines represent charmonia, pions, charmed mesons and gluons, respectively.](image)
Figure 7: Phase space subtracted invariant mass spectrum of the $\pi\pi$ system for the decay $\psi' \rightarrow J/\psi\pi^+\pi^-$ (in arbitrary units). The original data are taken from Ref. [35].

charmonium chromo-polarizability $\alpha_{c\bar{c}}$, the definition of which can be found in Ref. [53]. Because $\alpha_{\psi'}\alpha_{J/\psi} \geq |\alpha_{\psi'J/\psi}|^2$ [53], one may expect that the elastic amplitude is somewhat larger than the transition one, i.e., $|A(J/\psi\pi \rightarrow J/\psi\pi)(o)| \gtrsim |A(\psi'\pi \rightarrow J/\psi\pi)(o)|$ at the $J/\psi\pi$ threshold. In the second mechanism, the elastic $J/\psi\pi$ scattering amplitude is proportional to $g_2^2$, and the transition amplitude is proportional to $g_2 g'_2$, where $g_2(g'_2)$ are the $J/\psi(\psi')DD$ coupling constants. Neither of these coupling constants can be measured directly. Based on a vector dominance model, it was estimated in Ref. [54] that $g_2 = \sqrt{M_{J/\psi}/(M_D f_{J/\psi})}$ with $f_{J/\psi}$ the $J/\psi$ decay constant. Thus, one may estimate

$$g'_2 \approx \frac{f_{J/\psi}}{f_{\psi'}} \approx \left( \frac{\Gamma(J/\psi \rightarrow e^+e^-)}{\Gamma(\psi' \rightarrow e^+e^-)} \right)^{1/2} \approx 1.5,$$

which means $|A(\psi'\pi \rightarrow J/\psi\pi)(b)| \gtrsim |A(J/\psi\pi \rightarrow J/\psi\pi)(o)|$. Combining with the estimate for the first mechanism, it is reasonable to assume

$$|A(J/\psi\pi \rightarrow J/\psi\pi)| \sim |A(\psi'\pi \rightarrow J/\psi\pi)|$$

at the $J/\psi\pi$ threshold.

Because of crossing symmetry, the $\psi'\pi \rightarrow J/\psi\pi$ scattering amplitude is related to the $\psi' \rightarrow J/\psi\pi\pi$. Assuming that the amplitudes are constant, denoted by $\tilde{C}$, they are the same for scattering and decay processes. This assumption is definitely not realistic, but it can be used to place an upper limit of the $J/\psi\pi$ scattering length. The $J/\psi\pi$ threshold occurs when the $\pi\pi$ invariant mass is $\sqrt{s_3} = M_{\pi\pi} = 415$ MeV. In Fig. 7 we show the phase-space-subtracted invariant mass spectrum of the $\pi\pi$ system for the decay $\psi' \rightarrow J/\psi\pi^+\pi^-$. That is, the experimental data [35] are divided by $|\vec{p}_1^*||\vec{p}_3|$, where

$$|\vec{p}_1^*| = \frac{1}{2\sqrt{s_3}} \sqrt{\lambda(s_3, M_{\phi}^2, M_{\pi}^2)}, \quad |\vec{p}_3| = \frac{1}{2M_{\psi'}} \sqrt{\lambda(M_{\psi'}^2, s_3, M_{\pi}^2)},$$

From Fig. 7 one can see that the physical decay amplitude at $\sqrt{s_3} = 415$ MeV should be smaller than the assumed (nonrealistic) constant amplitude $|\tilde{C}|$. From the decay width of $\psi' \rightarrow J/\psi\pi^+\pi^-$, one
can extract the constant $|\tilde{C}| \approx 9.6$. Using Eq. (22), we get an approximate upper limit for the $J/\psi\pi$ $S$-wave scattering length

$$|a_{J/\psi\pi}| \lesssim \frac{|\tilde{C}|}{8\pi (M_J + M_\pi)} \approx 0.02 \text{ fm.}$$  \hspace{1cm} (24)

Similarly, using the measured decay width of the $\Upsilon(3S) \to \Upsilon(2S)\pi^+\pi^-$, we get

$$|a_{\Upsilon(2S)\pi}| \lesssim 0.01 \text{ fm.}$$  \hspace{1cm} (25)

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