Genuine \((k, m)\)-threshold controlled teleportation and its security

Xin-Wen Wang\(^1\), Da-Chuang Li\(^2\), and Guo-Jian Yang\(^1\)\(^*\)

\(^1\)Department of Physics, and Applied Optics Beijing Area Major Laboratory, Beijing Normal University, Beijing 100875, China
\(^2\)School of Physics and Material Science, Anhui University, Hefei 230039, China

We propose genuine \((k, m)\)-threshold controlling schemes for controlled teleportation via multi-particle entangled states, where the teleportation of a quantum state from a sender (Alice) to a receiver (Bob) is under the control of \(m\) supervisors such that \(k\) \((k \leq m)\) or more of these supervisors can help Bob recover the transferred state. By construction, anyone of our quantum channels is a genuine multipartite entangled state of which any two parts are inseparable. Their properties are compared and contrasted with those of the well-known Greenberger-Horne-Zeilinger, W, and linear cluster states, and also several other genuine multipartite entangled states recently introduced in literature. We show that our schemes are secure against both Bob’s dishonesty and supervisors’ treacheries. For the latter case, the game theory is utilized to prove that supervisors’ cheats can be well prevented. In addition to their practical importance, our schemes are also useful in seeking and exploring genuine multipartite entangled states and opening another perspective for the applications of the game theory in quantum information science.

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I. INTRODUCTION

In quantum information science, information is encoded in quantum states. Quantum information processing is in fact the manipulation or (and) transfer of quantum states. Quantum teleportation\(^1\) is a typical quantum information processing task, which functions as transferring a quantum state from one site to another one via previously shared entanglement assisted by classical communications and local operations. Quantum teleportation can not only be directly used to realize quantum communication but also construct a primitive of a quantum computer\(^2\). Quantum teleportation has been realized in many experiments (see e.g., \(^3\)). Since the end of last century, a new quantum teleportation idea, i.e., controlled teleportation (CT), has been attracting much interest\(^4, 5, 6, 7, 8, 9, 10, 11\). CT functions as teleporting a quantum state from a sender’s (Alice) site to a receiver’s (Bob) site under the control of multiple supervisors (Charlie 1, Charlie 2, \(\cdots\)). In other words, Alice and Bob need the cooperation of Charlies in order to realize the teleportation (communication) successfully. A CT scheme has already been demonstrated in an optical experiment\(^12\).

CT is useful in the context of networked quantum communication, quantum computation, and cryptographic conference\(^6, 13, 14, 15, 16, 17, 18\). For instance, CT can be used as a secret sharing to hide a quantum state as a secret\(^6, 15\). In addition, CT has many similarities with the secure multi-party quantum computation (MPQC) protocol\(^16\) which allows multiple players to compute an agreed quantum circuit where each player has access only to its own quantum input. A MPQC protocol has two phases, sharing phase and reconstruction phase. In the sharing phase, dealers provide many agents with their initial state; in the reconstruction phase, one agent is designated to reconstruct the final state of the protocol with the help of the other ones. CT may have other interesting applications, such as in opening a credit account on the agreement of multiple managers in a quantum network.

The previous CT schemes\(^4, 5, 6, 7, 8, 9, 10, 11\) are focused on the \((m, m)\)-threshold controlling schemes where the achievement of teleportation is conditioned on the collaboration of all the supervisors. In other words, it is impossible to realize teleportation between Alice and Bob if anyone of Charlies does not cooperate for subjective or objective reasons. However, a more general CT scheme should consider the \((k, m)\)-threshold

* E-mail: yanggj@bnu.edu.cn
case \((k \leq m)\) where \(k\) or more of the supervisors can help Bob successfully recover the transferred state, but less than \(k\) of them cannot. Recently, different \((k, m)\)-threshold controlling schemes were discussed in Refs. [15, 19]. The scheme in Ref. [19] needs lowering the fidelity of teleportation and its successful probability for enduring the uncooperation of part of supervisors. In Ref. [13], authors pointed out that a \((k, m)\)-threshold controlling scheme can be constructed by using secret sharing. That is, the teleportation is controlled by a classical key which is shared by the supervisors such that \(k\) or more of them can recover the key. However, as mentioned in Ref. [19], a classical key can be easily copied, and Charlies cannot stop Bob from recovering Alice’s original state if Bob manages to obtain at least \(k\) shares of the key without consent of Charlies. More importantly, the classical \((k, m)\)-threshold controlling scheme can not prevent Charlies’ cheats as will be shown. They also proposed another \("(k, m)\)-threshold"\) CT scheme which is a combination of a \((m, m)\)-threshold CT scheme and the \((k, m)\)-threshold secret sharing scheme mentioned above. Evidently, it is not a genuine \((k, m)\)-threshold controlling scheme, because Bob still needs the assistance of all the supervisors for recovering Alice’s original state. In principle, a \((k, m)\)-threshold controlling scheme can be constructed by using the quantum polynomial codes [20] as mentioned in Ref. [15]. However, it needs the supervisors and Bob to come together and perform nonlocal operations (multi-particle joint operations).

In this article, we propose genuine \((k, m)\)-threshold controlling schemes for CT. In these schemes, the supervisors (Charlies) only need to perform single-particle measurements and announce their outcomes. If the recipient receives \(k\) correct outcomes, he or she can reconstruct the original state that the sender wants to transfer by appropriate local operations. We first consider the CT of a single-particle state via a multipartite entangled sate. Then the CT of an \(n\)-particle state can be directly realized by using \(n\) such multipartite entangled states. However, the directly generalized method requires considerably large auxiliary particle resources and local operations, as well as classical communications, especially when \(n\) is very large. We propose a much more economical scheme for CT of an arbitrary \(n\)-particle state with a single multipartite entangled state. By construction, our quantum channels are genuine multipartite entangled states in which any two parts are inseparable. Their properties are compared and contrasted with those of the well-known Greenberger-Horne-Zeilinger, W, and linear cluster states, and also several other genuine multipartite entangled states recently introduced in literature. We show that our schemes are secure against both Bob’s dishonesty and supervisors’ treacheries. For the latter case, the game theory is utilized to prove that supervisors’ cheats can be well prevented. In addition to the potential applications in networked quantum communication and quantum computation, our schemes are also useful in seeking and exploring genuine multipartite entangled states and opening another perspective for the applications of the game theory in quantum information science.

The paper is organized as follows. In Sec. II, we describe the \((k, m)\)-threshold CT protocols, and briefly analyze the features of the entanglement channels. In Sec. III, we discuss the security of our schemes against Bob’s dishonesty and supervisors’ treacheries. Concluding remarks appear in Sec. IV.

II. \((k, m)\)-THRESHOLD CONTROLLING SCHEME FOR CONTROLLED TELEPORTATION

A. A brief review of the teleportation scheme with a Bell state

Quantum teleportation was first proposed by Bennett et al. [1]. In their original scheme, the state to be teleported is an arbitrary single-particle state given by

\[
|\psi\rangle_T = \alpha|0\rangle_T + \beta|1\rangle_T
\]

with \(|\alpha|^2 + |\beta|^2 = 1\), and the quantum channel shared by the sender Alice and the receiver Bob is an EPR singlet state. In fact, the quantum channel can be anyone of the four Bell basis states

\[
|\mathcal{B}^1\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB},
\]

\[
|\mathcal{B}^2\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)_{AB},
\]
applies the required Pauli rotation to transform the state of his particle instance, \( |B \rangle \).

Note that the four Bell states can be transformed into each other by local operations on one particle. For instance, \( |B^1\rangle_{AB} = \sigma_x^B |B^2\rangle_{AB} = \sigma_y^B |B^3\rangle_{AB} = \sigma_z^B |B^4\rangle_{AB} \), where \( \sigma^j (j = x, y, z) \) are the conventional Pauli matrices given by

\[
\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

As an example, we assume that the quantum channel is \( |B^1\rangle_{AB} \). Then the state of the total system is

\[
|\Psi\rangle_{total} = |\psi\rangle_T \otimes |B^1\rangle_{AB}
\]

\[
= \frac{1}{2} \left[ |B^1\rangle_{TA} (|0\rangle_B + |1\rangle_B) + |B^2\rangle_{TA} (|0\rangle_B - |1\rangle_B) + |B^3\rangle_{TA} (|1\rangle_B + |0\rangle_B) + |B^4\rangle_{TA} (|1\rangle_B - |0\rangle_B) \right]
\]

\[
= \frac{1}{2} \left[ |B^1\rangle_{TA}|\psi\rangle_B + \sigma_x^B |B^1\rangle_{TA}\sigma_x^B |\psi\rangle_B + \sigma_y^A |B^1\rangle_{TA}\sigma_y^B |\psi\rangle_B + \sigma_z^A |B^1\rangle_{TA}\sigma_z^B |\psi\rangle_B \right].
\]

Alice performs a Bell-basis measurement on particles \( T \) and \( A \) and broadcasts the outcomes, after which Bob applies the required Pauli rotation to transform the state of his particle \( B \) into an accurate replica of the original state of Alice’s particle \( T \). The one-to-one correspondence between Alice’s possible measurement outcomes and the required Pauli rotations can be easily obtained from Eq. (4). It can be easily proved that if the quantum channel is another Bell state \( |B^j\rangle_{AB} (j = 2, 3, 4) \), the state of the total system can also be expanded as

\[
|\Psi\rangle_{total} = |\psi\rangle_T \otimes |B^j\rangle_{AB}
\]

\[
= \frac{1}{2} \left[ |B^j\rangle_{TA}|\psi\rangle_B + \sigma_x^B |B^j\rangle_{TA}\sigma_x^B |\psi\rangle_B + \sigma_y^A |B^j\rangle_{TA}\sigma_y^B |\psi\rangle_B + \sigma_z^A |B^j\rangle_{TA}\sigma_z^B |\psi\rangle_B \right].
\]

Thus the one-to-one correspondence between Alice’s possible measurement outcomes and the required Pauli rotations can always be easily obtained.

B. \((k, m)\)-threshold controlled teleportation for an arbitrary single-particle state

Before discussing the \((k, m)\)-threshold schemes, we first give a general description on the basic idea of CT. Assume that there is a community which is composed of \( m + 2 \) members, Alice, Bob, Charlie 1, Charlie 2, \( \ldots \), and Charlie \( m \). The members are distributed in a network and connected by a quantum channel, i.e., a multipartite entangled state, and one or more classical channels (can be considered as the conventional classical communication facilities). One of Alice and Bob is the sender of a quantum state (the carrier of quantum information), and the other one is the receiver. Charlies act as the supervisors who can decide whether or not to allow Alice and Bob to carry out the teleportation. In a word, the teleportation of a quantum state between Alice and Bob is supervised by Charlies and needs their approval. Without loss of generality, we assume Alice is the sender and Bob is the receiver. In order to realize the CT of the single-particle state \( |\psi\rangle_T \), the quantum channel shared by them can be in the form of

\[
|\Phi\rangle_{2+m} = x_1 |B^1\rangle_{AB} |\phi^1\rangle_{C_1C_2\ldots C_m} + x_2 |B^2\rangle_{AB} |\phi^2\rangle_{C_1C_2\ldots C_m}
+ x_3 |B^3\rangle_{AB} |\phi^3\rangle_{C_1C_2\ldots C_m} + x_4 |B^4\rangle_{AB} |\phi^4\rangle_{C_1C_2\ldots C_m},
\]
where $\sum_{i=1}^{4} |x_i|^2 = 1$, $|\phi_i\rangle_{C_1C_2\cdots C_m}$ are normalized and their forms depend on the concrete schemes but should satisfy $\langle \phi^i|\phi^i\rangle = \delta_{ii'}$ and can be distinguished by local measurements and classical communications. Here, particle $A$ belongs to Alice, particle $B$ to Bob, and particle $C_j$ to Charlie $j$ ($j = 1, 2, \cdots, m$). It has been shown in the above subsection that anyone of the four Bell states can be competent for realizing the teleportation of the state $|\psi\rangle_T$. However, Alice and Bob can carry out the teleportation only if they can ascertain which Bell state their subsystem is in. With the quantum channel $|\Phi\rangle_{2+m}$, the identification of the Bell states can be achieved by the following method: Charlies make measurements with appropriate bases on their own particles and inform Bob the outcomes; then Bob can distinguish the states $\{|\phi^i\rangle_{C_1C_2\cdots C_m}, i = 1, 2, 3, 4\}$ and thus can identify the Bell states. The one-to-one correspondence between their subsystem is in. With the quantum channel $|\Phi\rangle_{2+m}$, the identification of the Bell states can be achieved by the following method: Charlies make measurements with appropriate bases on their own particles and inform Bob the outcomes; then Bob can distinguish the states $\{|\phi^i\rangle_{C_1C_2\cdots C_m}, i = 1, 2, 3, 4\}$ and thus can identify the Bell states. The one-to-one correspondence between $\{|\Phi\rangle\}$ and $\{|\phi^i\rangle_{C_1C_2\cdots C_m}\}$ is clearly shown in Eq. (9). Without the cooperation of Charlies, the subsystem of Alice and Bob will be in the mixed state 

$$
\rho_{AB} = \text{tr}_{C_2\cdots C_m}(\langle\Phi\rangle_{2+m}|\Phi\rangle) = |x_1|^2|B^1\rangle_{AB}|B^1\rangle + |x_2|^2|B^2\rangle_{AB}|B^2\rangle + |x_3|^2|B^3\rangle_{AB}|B^3\rangle + |x_4|^2|B^4\rangle_{AB}|B^4\rangle.
$$

The mixed state cannot be used to implement perfect teleportation [21].

In the conventional CT schemes which use the Greenberger-Horne-Zeilinger (GHZ)-type entangled states as the quantum channel, two terms of $\{x_i, i = 1, 2, 3, 4\}$ are set to zero, and the other two are not and their corresponding $|\phi^i\rangle_{C_1C_2\cdots C_m}$ states are different Dicke states. For example, the quantum channel $|\Phi\rangle_{2+m}$ is a GHZ state $|\text{GHZ}\rangle_{2+m} = (1/\sqrt{2})\left(|000\cdots 0\rangle + |1111\cdots 1\rangle\right)_{ABC_1C_2\cdots C_m}$, then $x_3 = x_4 = 0$, $x_1 = x_2 = 1/\sqrt{2}$. $|\phi^1\rangle_{C_1C_2\cdots C_m} = (1/\sqrt{2m})\left[\sum_{l=0}^{m-1} S_{2m}^{2l+1}(-1)^l + 1\right] + (m-2l-1)$, and $|\phi^2\rangle_{C_1C_2\cdots C_m} = (1/\sqrt{2m})\left[\sum_{l=0}^{m-1} S_{2m}^{2l+1}(-1)^l + 1\right] + (m-2l-1)$, where $\{\pm\} = \{(0,0\pm 1)/\sqrt{2}\}$. $S_i^j = m!/|\tilde{l}!(m-\tilde{l})!| (\tilde{l} = 2l, 2l+1)$ is the combinational coefficient, $\{-\}^i_0^+ + \{\}^0_0^\odot(m-\tilde{l})$ denotes that $\tilde{l}$ particles are in the state $\{\}^0_0$ and $\tilde{l}$ particles are in the state $\{-\}$. When $m$ is odd $m^2 = (m-1)/2$, otherwise, $m^2 = m/2 - 1$ and $m^2 = m/2$. That is, $|\phi^1\rangle_{C_1C_2\cdots C_m}$ and $|\phi^2\rangle_{C_1C_2\cdots C_m}$ are the Dicke states with even $\{-\}$ and odd $\{-\}$, respectively. Thus Charlies can perform single-particle measurements on their own particles with the basis $\{\pm\}$ and inform Bob the outcomes, and Bob can identify the Bell states with the outcomes, even or odd Charlies can perform single-particle measurements on their own particles with the basis $\{\pm\}$ and inform Bob the outcomes, and Bob can identify the Bell states with the outcomes, even or odd.

Evidently, such a CT scheme is a $(m, m)$-threshold controlling scheme, i.e., Alice and Bob can implement the teleportation if and only if all Charlies agree and cooperate.

Now, let us move on to the $(k, m)$-threshold controlling scheme. For simplicity, we first consider the case $k = 1$. That is, Alice and Bob can realize successfully teleportation if anyone of Charlies cooperate with them. We can set $x_3 = x_4 = 0$, $x_1 = x_2 = 1/\sqrt{2}$. $|\phi^1\rangle_{C_1C_2\cdots C_m} = |00\cdots 0\rangle_{C_1C_2\cdots C_m}$, and $|\phi^2\rangle_{C_1C_2\cdots C_m} = |11\cdots 1\rangle_{C_1C_2\cdots C_m}$ in Eq. (9). Then the quantum channel is 

$$
|\Phi\rangle_{2+m} = \frac{1}{\sqrt{2}}\left(|B^1\rangle_{AB}|00\cdots 0\rangle_{C_1C_2\cdots C_m} + |B^2\rangle_{AB}|11\cdots 1\rangle_{C_1C_2\cdots C_m}\right)
$$

$$
= \frac{1}{2}\left(|000\cdots 0\rangle + |001\cdots 1\rangle + |110\cdots 0\rangle - |111\cdots 1\rangle\right)_{ABC_1C_2\cdots C_m}.
$$

(7)

It can be seen that if anyone of Charlies performs a measurement on his particle with the basis $\{0\}, \{1\}$ (i.e., in the z direction) and informs Bob the outcome, Bob can know particles $A$ and $B$ are in the Bell state $|B^1\rangle_{AB}$ for the outcome $|0\rangle$ or $|B^2\rangle_{AB}$ for $|1\rangle$. In other words, anyone of Charlies suffices to help Alice and Bob achieve the teleportation of the state $|\psi\rangle_T$. However, if all of Charlies do not collaborate with them, they cannot achieve the teleportation. Note that any combination of $\{00\cdots 0\}, \{11\cdots 1\}$ with two of the four Bell states can construct a quantum channel which can realize the $(1, m)$-threshold CT mentioned above. For instance, we can also construct a suitable quantum channel by setting $x_1 = x_2 = 0$, $|\phi^3\rangle_{C_1C_2\cdots C_m} = |00\cdots 0\rangle_{C_1C_2\cdots C_m}$, and $|\phi^4\rangle_{C_1C_2\cdots C_m} = |11\cdots 1\rangle_{C_1C_2\cdots C_m}$ in Eq. (9).

For the case $k > 1$, the quantum channel can be constructed as 

$$
|\Phi^k\rangle_{2+m} = \frac{1}{\sqrt{2}}\left(|B^1\rangle_{AB}|\phi^1\rangle_{C_1C_2\cdots C_m} + |B^2\rangle_{AB}|\phi^2\rangle_{C_1C_2\cdots C_m}\right)
$$

$$
|\phi^1\rangle_{C_1C_2\cdots C_m} = |00\cdots 0\rangle_{C_1C_2\cdots C_m}
$$

$$
|\phi^2\rangle_{C_1C_2\cdots C_m} = \frac{1}{\sqrt{k-1+m-k+1}}|k-1, m-k+1\rangle_{C_1C_2\cdots C_m},
$$

(8)

where $S_m^{k-1} = m!/((m-k+1)!(k-1)!)$. $\{k-1, m-k+1\}$ is the combinational coefficient, $|k-1, m-k+1\rangle_{C_1C_2\cdots C_m}$ denotes
all the totally symmetric states including \( k - 1 \) zeros and \( m - k + 1 \) ones. For example, \( m = 3 \) and \( k = 2 \), then \( |\psi|^2_{C_1C_2C_3} = \frac{1}{\sqrt{3}}(|011⟩ + |101⟩ + |110⟩)_{C_1C_2C_3} \). As a matter of fact, \( |\psi|^2_{C_1C_2⋯C_m} \) is then a symmetric Dicke state with \( m - k + 1 \) excitations. By the way, the symmetric six-qubit Dicke state with three excitations has recently been realized in experiment \[24\]. Note that when \( k = 1 \), the state of Eq. (8) reduces to that of Eq. (7). We consider that \( l \leq m \) of Charlies perform single-particle measurements on their own particles with the basis \( \{|0⟩, |1⟩\} \). There are two cases. (a) \( l \geq k \), if all of them get the outcome \( |0⟩ \), the subsystem of particles \( A \) and \( B \) collapses into \( |B^1⟩_{AB} \); otherwise, it collapses into \( |B^2⟩_{AB} \). (b) \( l < k \), if all of them get the outcome \( |0⟩ \), the subsystem of particles \( A \) and \( B \) collapses into a mixed state of \( |B^1⟩_{AB} \) and \( |B^2⟩_{AB} \). Thus we can conclude that \( k \) or more of Charlies can help Alice and Bob deterministically distinguish between the two Bell states \( |B^1⟩_{AB} \) and \( |B^2⟩_{AB} \), while less than \( k \) of them cannot. In other words, Alice can deterministically teleport the state \( ψ⟩_T \) to Bob if and only if \( k \) or more of Charlies collaborate with them. The procedure of such a CT protocol is as follows.

(i) Alice performs a Bell-basis measurement on particles \( T \) and \( A \), and informs Bob the outcome, one of \( \{|B^1⟩_{TA}, |B^2⟩_{TA}, |B^3⟩_{TA}, |B^4⟩_{TA} \} \).

(ii) Bob sends his petition to Charlies.

(iii) Charlies talk over whether or not to allow Bob to recover the original state of Alice’s particle \( T \). If more than a certain number of Charlies (e.g., \( 2/3 \) of them) vote for allowing, a collective decision should be made that permitting Bob to recover Alice’s original state. Then all Charlies should perform single-particle measurements on their own particles with the basis \( \{|0⟩, |1⟩\} \) and broadcast their outcomes.

(iv) According to Alice’s and Charlies’ measurement outcomes, Bob performs a corresponding Pauli rotation on particle \( B \) and recovers Alice’s original state on it.

Note that we need all of Charlies instead of \( k \) of them to broadcast their outcomes in step (iii) is based on the consideration that there may exist treacherous Charlies who will cheat Bob and send him the false measurement outcomes. The detailed proof for the security of our scheme against Charlies’ cheats will be given in Sec. III.

C. \((k, m)\)-threshold controlled teleportation for an arbitrary multi-particle state

As a direct generalization of the teleportation of a single-particle state, teleportation of an arbitrary \( n \)-particle state

\[
|ψ⟩_{T_1T_2⋯T_n} = \sum_{j_1, j_2⋯j_n = 0}^1 y_{j_1j_2⋯j_n} |j_1j_2⋯j_n⟩_{T_1T_2⋯T_n}
\]  

(9)
can be achieved with \( n \) Bell states. In fact, the teleportation of a two-particle state with two Bell states has already been demonstrated in an optical experiment \[24\].

Thus, one can use \( n \) copies of the state of Eq. (9) to realize the CT of an arbitrary \( n \)-particle state. Also, we can directly use \( n \) copies of the state \( |Φ^1⟩_{2+m} \) or \( |Φ^2⟩_{2+m} \) [see Eqs. (7) and (8)] to accomplish the \((k, m)\)-threshold CT of \( |ψ⟩_{T_1T_2⋯T_n} \). However, this method requires considerably large auxiliary particle resources and local operations, as well as classical communications, especially when the number of “teleported” qubits is very large. Particularly, each Charlie needs to hold \( n \) controlling particles, perform \( n \) single-particle measurements, and send Bob \( n \) bits of classical information about the measurement outcomes.

We now propose a much more economical way to implement the \((k, m)\)-threshold CT of an arbitrary \( n \)-particle state. The quantum channel is the multipartite entangled state

\[
|Φ^k⟩_{2n+m} = \frac{1}{\sqrt{2}} \left( \prod_{i=1}^{n} |B^1⟩_{A_iB_i} ⊗ |00⋯0⟩_{C_1C_2⋯C_m} + \prod_{i=1}^{n} |B^2⟩_{A_iB_i} ⊗ \frac{1}{\sqrt{S_m}} |k - 1, m - k + 1⟩_{C_1C_2⋯C_m} \right),
\]  

(10)
where particles $A_i$ are held by Alice, $B_i$ held by Bob. In order to successfully implement the teleportation, Alice and Bob need Charlies to help them identify the two sequences of Bell states. Particularly, the procedure is as follows.

(i) Alice performs a sequence of Bell-basis measurements on the pairs of particles $\{(T_i, A_i), i = 1, 2, \cdots, n\}$, and informs Bob the outcomes.

(ii) and (iii) are the same as that of the CT protocol for a single-particle state.

(iii) According to Alice’s and Charlies’ measurement outcomes, Bob applies the corresponding Pauli rotations on particles $\{B_i, i = 1, 2, \cdots, n\}$ and reconstructs the state of Eq. (9).

As shown above, regardless of the number of qubits to be teleported, the proposed approach only requires that each supervisor holds one particle, performs one single-particle measurement on his or her particle, and send one bit of classical message to the receiver Bob. Therefore, compared with the directly generalized method mentioned above, this method is much more economical, because the required auxiliary particle resources, the number of measurements, and the quantity of classical communications are greatly reduced.

We notice that any two of the four Bell states can be distinguished by local (single-particle) measurements with appropriate measurement bases and classical communications. For instance, we can distinguish between the two sets $\{|B^1\rangle, |B^2\rangle\}$ and $\{|B^3\rangle, |B^4\rangle\}$ by using the measurement basis $\{|0\rangle, |1\rangle\}$, which can be evidently seen from Eq. (2). In order to show how to distinguish between the two sets $\{|B^1\rangle, |B^3\rangle\}$ and $\{|B^2\rangle, |B^4\rangle\}$ by local measurements and classical communication, we rewrite them as

\[
\begin{align*}
|B^1\rangle &= \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle), \\
|B^2\rangle &= \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle), \\
|B^3\rangle &= \frac{1}{\sqrt{2}}(|++\rangle - |--\rangle), \\
|B^4\rangle &= \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle).
\end{align*}
\]

Obviously, if two participants perform, respectively, a single-particle measurement on different particles with the basis $\{|\pm\rangle\}$, they can discriminate between the two sets $\{|B^1\rangle, |B^3\rangle\}$ and $\{|B^2\rangle, |B^4\rangle\}$ by exchanging the outcomes. That is, if their outcomes are anticorrelated, the state of the whole system is initially in the set $\{|B^1\rangle, |B^3\rangle\}$, otherwise, it is in the set $\{|B^1\rangle, |B^3\rangle\}$. With this method, Alice and Bob can measure anyone of $n$ pairs of particles $\{(A_i, B_i), i = 1, 2, \cdots, n\}$ and identify the states of the other $n - 1$ pairs of particles in the quantum channel of Eq. (10). Then Alice and Bob can realize the teleportation of an $n$-particle state with a high fidelity when $n$ is large, out of the control of Charlies. Especially, when the $n$-particle state $|\psi\rangle_{T_1T_2\cdots T_n}$ [see Eq. (9)] is separable, such as $y_{j_1j_2\cdots j_n} = y_{j_1}y_{j_2}\cdots y_{j_n}$, Alice and Bob can realize perfect teleportation of $n - 1$ qubits information escaping from the control of Charlies.

However, this drawback can be avoided by the following methods. We can establish two sequences of states chosen from the four Bell states for the $n$ pairs of particles $\{(A_i, B_i)\}$, and make one-to-one correspondence between them and the two states $|00\cdots 0\rangle_{C_1C_2\cdots C_m}$ and $\{(1/\sqrt{S^k_m})|k-1, m-k+1\rangle\}_{C_1C_2\cdots C_m}$. That is, we can use the following entangled state, instead of $\Phi^k_{2n+m}$, to act as the quantum channel:

\[
|\Phi^k\rangle_{2n+m} = \frac{1}{\sqrt{2}} \left( \prod_{i=1}^{n} |B^{s_i}\rangle_{A_iB_i} \otimes |00\cdots 0\rangle_{C_1C_2\cdots C_m} \\
+ \prod_{i=1}^{n} |B^{r_i}\rangle_{A_iB_i} \otimes \frac{1}{\sqrt{S^k_m}}|k-1, m-k+1\rangle_{C_1C_2\cdots C_m} \right),
\]

where $r_i$ ($s_i$) = 1, 2, 3, or 4. Note that Alice and Bob can know the $n$ pairs of particles $\{(A_i, B_i)\}$ are in the sequence of states $\prod_{i=1}^{n} |B^{s_i}\rangle_{A_iB_i}$ or $\prod_{i=1}^{n} |B^{r_i}\rangle_{A_iB_i}$, if and only if they know the particles $\{C_j, j = 1, 2, \cdots, m\}$ are in the state $|00\cdots 0\rangle_{C_1C_2\cdots C_m}$ or $\{(1/\sqrt{S^k_m})|k-1, m-k+1\rangle\}_{C_1C_2\cdots C_m}$ by Charlies’ help. In other words, they
cannot ascertain which sequence of states the subsystem of their \( n \) pairs of particles is in without the cooperation of Charlies. The teleportation of an arbitrary \( n \)-particle state can also be implemented by using a genuine \( 2n \)-particle entangled state as shown in Refs. \[25, 26, 27\]. Thus the quantum channel of the \((k, m)\)-threshold CT of an \( n \)-particle state can also be constructed as the following form for avoiding the aforementioned drawback:

\[
|\Phi^{m k}_{2n + m} = \frac{1}{\sqrt{2}} \left( |MES\rangle_{A_1 \cdots A_n B_1 \cdots B_n} \otimes \frac{1}{\sqrt{S_{m-1}}} |k-1, m-k+1\rangle_{C_1 C_2 \cdots C_m} + \sigma_{A_1}^{j_1} \cdots \sigma_{A_n}^{j_n} |MES\rangle_{A_1 \cdots A_n B_1 \cdots B_n} \otimes |00 \cdots 0\rangle_{C_1 C_2 \cdots C_m} \right),
\]

where \(|MES\rangle_{A_1 \cdots A_n B_1 \cdots B_n}\) is a genuine \(2n\)-particle entangled state showed in Eq. (18) of Ref. \[25\] (for \( n = 2 \)) or Eq. (10) of Ref. \[26\], \( j_i = 0, x, y, \) or \( z \) \((i = 1, 2, \cdots, n)\) with \( \sigma^0 \) being the two-dimensional identity operator. However, when \( n \geq 4 \), \(|MES\rangle_{A_1 \cdots A_n B_1 \cdots B_n}\) was not explicitly constructed in Ref. \[26\]. As shown in Ref. \[25\], \(|MES\rangle_{A_1 \cdots A_n B_1 \cdots B_n}\) can be replaced by a \(2n\)-qubit cluster state \(|\text{Cluster}\rangle_{A_1 B_1 \cdots A_n B_n} = (|0\rangle_{A_1} + |1\rangle_{A_1} \sigma_{B_1}^z)(|0\rangle_{B_1} + |1\rangle_{B_1} \sigma_{A_2}^z) \cdots (|0\rangle_{A_n} + |1\rangle_{A_n} \sigma_{B_n}^z)(|0\rangle_{B_n} + |1\rangle_{B_n})\) \[28\]. Then the quantum channel reads

\[
|\Phi^{m k}_{2n + m} = \frac{1}{\sqrt{2}} \left( |\text{Cluster}\rangle_{A_1 B_1 \cdots A_n B_n} \otimes \frac{1}{\sqrt{S_{m-1}}} |k-1, m-k+1\rangle_{C_1 C_2 \cdots C_m} + \sigma_{A_1}^{j_1} \cdots \sigma_{A_n}^{j_n} |\text{Cluster}\rangle_{A_1 B_1 \cdots A_n B_n} \otimes |00 \cdots 0\rangle_{C_1 C_2 \cdots C_m} \right).
\]

Note that \( \sigma_{A_1}^{j_1} \cdots \sigma_{A_n}^{j_n} \) in Eqs. (13) and (14) cannot be set to \( \sigma_{A_1}^{0} \cdots \sigma_{A_n}^{0} \), i.e., \( j_i \) cannot be simultaneously equal to zero.

## D. The features of the entanglement channels

It is known that the complexity of multipartite entanglement increases greatly with the increase of the number of parties involved. So far, the properties of multipartite entanglement are not very clear. The classification and quantification of genuine three-qubit \[29\] and four-qubit \[30\] entangled states were intensively studied. The classification and quantification of genuine entangled states involving more than four qubits were also discussed \[31, 32\]. Although several typical multipartite entangled states, such as GHZ states \[22\], W states \[29\], and cluster states \[28\], were presented, the inequivalent types of genuine multipartite entangled states for more than four particles are still very vague. It will need a long-term effort to well understand the entanglement involving many parties. To seek for genuine multipartite entangled states we can resort to particular quantum schemes since sharing a unique entanglement may allow ones to do some things that ones cannot otherwise do. Teleportation is a well example, with which some genuine multipartite entangled states were found \[8, 25, 26\]. Obviously, all the states \(|\Phi^{k}_{2n + m}\) [see Eq. (8)], \(|\Phi^{k}_{2n + m}\) [see Eq. (10)], \(|\Phi^{k}_{2n + m}\) [see Eq. (12)], \(|\Phi^{m k}_{2n + m}\) [see Eq. (13)], and \(|\Phi^{m k}_{2n + m}\) [see Eq. (14)], which act as the quantum channels in our \((k, m)\)-threshold CT schemes, are genuine multipartite entangled states, because any bipartite cut in them is inseparable \[32\]. Here, we roughly show the relationships or differences between them and other genuine multipartite entangled states presented in literature.

We begin with the state \(|\Phi^{1}_{2m}\). When \( m = 1 \), \(|\Phi^{1}_{2m+1} = (1/2)(|00\rangle_{AB} + |11\rangle_{AB}) = |C\rangle_{C_2 C_3} \) is a three-qubit GHZ state; when \( m = 2 \), \(|\Phi^{1}_{2m+2} = (1/2)(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)_{ABC,C_2} \) is just a four-qubit linear cluster state \[28\]. As to \( m > 2 \), \(|\Phi^{1}_{2m+2} = \text{l.u.}(G)_{2m+2}\), where “l.u.” indicates that the equality holds up to local unitary transformations on one or more of the qubits and

\[
|G\rangle_{2m+2} = (|0\rangle_A + |1\rangle_A \sigma_B^z)(|0\rangle_B + |1\rangle_B \sigma_{C_1}^z) \otimes (|0\rangle_{C_1} + |1\rangle_{C_1} \sigma_{C_2}^z \cdots \sigma_{C_m}^z) \prod_{i=2}^{m} (|0\rangle_{C_i} + |1\rangle_{C_i})
\]

is a graph state \[32\] shown in Fig. 1. Obviously, when \( m > 2 \), \(|\Phi^{1}_{2m+2}\) is inequivalent to the well-known GHZ, W, and linear cluster states, in terms of stochastic local operations and classical communications (SLOCC). By
the way, many schemes for generating multi-qubit graph states were presented (see e.g., \[34\]), and the six-qubit graph states are already achievable in the optical experiment \[35\].

\[\Phi^{k}_{2+m}\] As shown in Ref. \[28\], two states with different \(\Phi^{k}_{2+m}\) disentangled state means a product state of all particles is the minimum number of local measurements such that, for all measurement outcomes, the state is completely disentangled. For pure states, a completely disentangled state means a product state of all \(N\) particles \[28\]. Evidently, for all \(N\)-qubit states \(0 \leq P_{e} \leq N-1\). As shown in Ref. \[28\], two states with different \(P_{e}\) are SLOCC inequivalent, but the inverse case needs further investigation. We now discuss the three cases as follows. (a) \(k < m\) and \(k \neq m/2\). We can prove that \(P_{e}(\Phi^{k}_{2+m}) = k + 1\) is different from \(P_{e}(|GHZ\rangle_{2+m}) = 1, P_{e}(|W\rangle_{2+m}) = m + 1, and P_{e}(|Cluster\rangle_{2+m}) = [(m + 2)/2] \[28\]. Thus the state \(\Phi^{k}_{2+m}\) is SLOCC inequivalent to the corresponding GHZ, W, and linear cluster states. (b) \(k = m/2\). \(P_{e}(\Phi^{k}_{2+m}) = m/2 + 1 = P_{e}(|Cluster\rangle_{2+m})\). The relation of \(\Phi^{k}_{2+m}\) and \(|Cluster\rangle_{2+m}\) needs further investigation. (c) \(k = m\). \(P_{e}(\Phi^{m}_{2+m}) = m + 1 = P_{e}(|W\rangle_{2+m})\). Then we cannot distinguish between \(\Phi^{m}_{2+m}\) and \(|W\rangle_{2+m}\) by this method. However, we notice that \(\Phi^{m}_{2+m}\) belongs to the GHZ-W-type entangled states recently proposed by Chen et al. \[36\], and thus does not belong to the W-type states. On the other hand, \(\text{tr}_{C_{1}...C_{m}}(\Phi^{m}_{2+m}(\Phi^{m}_{2})) = \frac{1}{2}|00\rangle_{AB}|00\rangle + \frac{1}{2}|11\rangle_{AB}|11\rangle\) is a separable state and \(\text{tr}_{C_{1}...C_{m}}(|W\rangle_{2+m}(W)) = \frac{m}{m+2}|00\rangle_{AB}|00\rangle + \frac{2}{m+2}|B^{3}\rangle_{AB}|B^{3}\rangle\) is a partially mixed entangled state, which also justifies the conclusion that \(\Phi^{m}_{2+m}\) and \(|W\rangle_{2+m}\) are SLOCC inequivalent. By the way, a scheme for generating a GHZ-W-type state has been proposed lately \[14\]. Similarly, we can prove that all the states \(|\Phi^{k}_{2+ m}\rangle\) \[see Eq. \[3\]\], \(|\Phi^{k}_{2+n}\rangle\) \((or \ |\Phi^{k}_{2+n}\rangle), |GHZ\rangle_{2+n+m}, |W\rangle_{2+n+m}, |Cluster\rangle_{2+n+m}\) are generally SLOCC inequivalent to each other.

Now, let us pay attention to the states \(|\Phi^{m}_{2+n}\rangle\) and \(|\Phi^{m}_{2+n}\rangle\). In the state \(|\Phi^{m}_{2+n}\rangle\), \(|MES\rangle_{A_{1}...A_{n-1}B_{1}B_{2}}\) is explicitly constructed for \(n = 2\) \[25\] and \(n = 3\) \[26\], respectively. That is,

\[
|M ES\rangle_{A_{1}A_{2}B_{1}B_{2}} = \frac{1}{2\sqrt{2}}(|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle)_{A_{1}A_{2}B_{1}B_{2}},
\]

\[
|M ES\rangle_{A_{1}A_{2}A_{3}B_{1}B_{2}B_{3}} = \frac{1}{2\sqrt{2}}(|000000\rangle + |010110\rangle + |111010\rangle + |100010\rangle + |011011\rangle + |001101\rangle + |101111\rangle + |111101\rangle)_{A_{1}A_{2}A_{3}B_{1}B_{2}B_{3}},
\]

Both the states were proved to be SLOCC inequivalent to the corresponding GHZ and W states \[25\] \[26\]. By the way, a scheme for generating \(|MES\rangle_{A_{1}A_{2}B_{1}B_{2}}\) has been proposed recently \[37\]. We notice that \(|\Phi^{m}_{1}\rangle_{4+1}\) is SLOCC equivalent to the state of Eq. (17) of Ref. \[3\] which was constructed also for implementing (1,1)-threshold CT of a two-particle state. In addition, \(|\Phi^{m}_{1}\rangle_{4+2}\) can be transformed into \(|MES\rangle_{A_{1}A_{2}B_{1}B_{2}}\) by local operations with \(C_{1}\) and \(C_{2}\) replaced by \(A_{3}\) and \(B_{3}\), respectively; \(|\Phi^{m}_{1}\rangle_{2n+1}\) is a \((2n + 1)\)-qubit linear cluster state. It can be proved that \(P_{e}(|\Phi^{m}_{2n+m}\rangle) = P_{e}(|\Phi^{m}_{2n+m}\rangle) = n + k\). Thus when \(k \neq m/2\) \((m > 1)\), \(|\Phi^{m}_{2n+m}\rangle\) and \(|\Phi^{m}_{2n+m}\rangle\) are SLOCC inequivalent to the corresponding GHZ, W, and linear cluster states. As to the case \(k = m/2\), \(|\Phi^{m}_{2n+m}\rangle\) and \(|\Phi^{m}_{2n+m}\rangle\) are also SLOCC inequivalent to the corresponding GHZ and W states, but the relation of them and linear cluster states needs further investigation.

FIG. 1: The graph state of Eq. (15). (a) \(m = 3\). (b) \(m = 4\). (c) \(m = 6\).
III. SECURITY OF THE \((k, m)\)-THRESHOLD CONTROLLED TELEPORTATION

Our \((k, m)\)-threshold CT schemes are secure against both Bob’s dishonesty and Charlies’ treacheries.

A. Security against Bob’s dishonesty

Bob may manage to recover Alice’s original state out of the control of Charlies. Thus, during the distribution of the quantum channel, he intercepts \(k\) or more of the particles \(\{C_i, i = 1, 2, \ldots, m\}\) and performs them single-particle measurements with the basis \(\{|0\rangle, |1\rangle\}\), and resends them or sends other \(k\) or more auxiliary particles to corresponding Charlies, respectively. By this way, Bob can ascertain the state of the subsystem of pairs of particles \(\{(A_j, B_j), j = 1, 2, \ldots\}\) and thus successfully recovers Alice’s original state without the cooperation of Charlies. However, the correlation among particles \(A_j, B_j,\) and \(C_i\) is disturbed or destroyed. We take \(k = 1\) as an example. If Bob performs a measurement on one of the particles \(\{C_i\}\) and directly resends it to corresponding Charlie, the subsystem of Charlies will be in a product state \(|00 \cdots 0\rangle_{C_1C_2\cdots C_m}\) or \(|11 \cdots 1\rangle_{C_1C_2\cdots C_m}\). Then there is no any correlation among particles \(\{C_i\}\). This case can be easily found by Charlies. If Bob sends other \(m\) auxiliary particles in a GHZ state \((1/2)(|00 \cdots 0\rangle + |11 \cdots 1\rangle)_{C'_1C'_2\cdots C'_m}\) to Charlies, the correlation between the subsystem of Alice and Bob and that of Charlies is destroyed. Thus such an action of Bob can also be detected. In fact, the correlation of any genuine multipartite entangled state will be disturbed or destroyed by any measurement on a subspace of it, and cannot be perfectly simulated by another entangled state involving less parties. As a consequence, Bob’s dishonest action can always be detected in our schemes. The detailed proof is so complicated and prolix, and will be given elsewhere. Note that Charlies should randomly choose a sufficient subset of quantum channels to check whether particles are intercepted during the distribution before carrying out the task of CT. The security checking process is similar to that of quantum secret sharing schemes (see, e.g. [33]). As a matter of fact, most of quantum communication schemes need ones to use this method to check the security of quantum channels against eavesdropper’s interception. Also, all the previous CT schemes \([4, 5, 6, 8, 9, 10, 15]\) are secure against Bob’s dishonesty if checking the security of quantum channels before carrying out the corresponding tasks.

B. Security against Charlies’ treacheries

When some Charlies are not satisfied with a collective decision, they may betray the community by three possible ways as follows. (a) They privately help Bob to reconstruct Alice’s original state. (b) They reject cooperating with Bob and making measurements on their particles. (c) They cheat Bob and send him the false measurement outcomes. We assume that any classical communication is open and insecure, and treacherous Charlies will be punished if their treacherous actions are detected. Then cases (a) and (b) will not occur. In the following, we show how case (c) can be prevented.

We first consider that there is only one treacherous Charlie, e.g., Charlie \(j\), who cheats Bob and sends him the false measurement outcome. That is, when Charlie \(j\) gets the measurement outcome \(|0\rangle\) he broadcasts \(|1\rangle\), when getting \(|1\rangle\) he broadcasts \(|0\rangle\). There are two cases. Case one: \(k < m\). If the real measurement outcome on the subsystem of particles \(\{C_i, i = 1, 2, \ldots, m\}\) is \(|00 \cdots 0\rangle_{C_1C_2\cdots C_m}\), then the broadcasted outcome is \(|00 \cdots 010 \cdots 0\rangle_{C_1C_2\cdots C_j\cdots C_{j+1}\cdots C_m}\) because Charlie \(j\) announced the opposite outcome. However, such an outcome should not appear when there is no treacherous Charlie. Thus the cheat action of Charlie \(j\) is exposed. If the real measurement outcome is one term of \((1/\sqrt{S_m^{-1}})(|k - 1, m - k + 1\rangle_{C_1C_2\cdots C_m}\) involving \(k - 1\) zeros and \(m - k + 1\) ones, then the broadcasted outcome involves \(k - 2\) or \(k\) zeros. In this case, Bob can also find that there exists a betrayer, although he cannot directly know which Charlie cheated him. In a word, Bob can always detect whether or not there exist treacherous Charlies who cheat him. The probability of exactly finding the cheat action of Charlie \(j\) is \(1/2\). Case two: \(k = m\). If the real measurement outcome is \(|00 \cdots 0\rangle_{C_1C_2\cdots C_m}\) or \(|00 \cdots 010 \cdots 0\rangle_{C_1C_2\cdots C_j\cdots C_{j+1}\cdots C_m}\), the broadcasted outcome is \(|00 \cdots 010 \cdots 0\rangle_{C_1C_2\cdots C_j\cdots C_{j+1}\cdots C_m}\) or
Then the cheat action of Charlie \( j \) cannot be found and Bob will obtain a wrong state instead of Alice’s original state. If the real measurement outcome is \(|00\cdots0\rangle_{C_1C_2\cdots C_m}\), then the broadcasted outcome is \(|00\cdots010\cdots0\rangle_{C_1C_2\cdots C_{j-1}C_jC_{j+1}\cdots C_m}\). However, such an outcome should not appear when there is no treacherous Charlie. Thus Bob can find that there exists a betrayer. In a nutshell, the probability of finding the existence of treacherous Charlie is \((S_{m}^{k-1} - 1)/(2S_{m}^{k-1})\). Note that they may randomly broadcast an artificial outcome without measurement. This way has no essential differences with the one discussed above.

For the case where there are \( l \) \((l < m)\) treacherous Charlies who send the false outcomes to Bob, when \( l \neq m - k + 1 \), the probability of finding the treacherous Charlies is one \((l \text{ is odd})\) or \(1 - S_{l}^{k}/S_{m}^{k} \) \((l \text{ is even})\); when \( l = m - k + 1 \), the probability is \((S_{m}^{k} - 1)/(2S_{m}^{k}) \) \((l \text{ is odd})\) or \((S_{m}^{k} - S_{l}^{k}/2 - 1)/(2S_{m}^{k}) \) \((l \text{ is even})\).

According to the above analysis, when there is only one Charlie who cheats Bob and sends him the false measurement outcome, his cheat action can be directly detected with probability 1/2. Because when the cheat action of any one of Charlies is found, he will be chastised, the case where one or more Charlies cheat Bob will not occur in practice. We now prove it by the game theory. Assume that there are \( l \) potential treacherous Charlies who are not satisfied with a collective decision that permitting Bob to reconstruct Alice’s original state. They will play a multi-player *Prisoners-Dilemma*-like game. The so-called Prisoners’ Dilemma game is as follows. Two or more perpetrators are caught by the police and are interrogated in separate cells without communication among them. Unfortunately, the police lacks enough proof to implicate them. The chief policeman now makes the following offer to each prisoner: if one of them confesses to the crime, but the others do not, then he or she will be commuted by \( r \) years \((s < r)\); if all of them confess, then everyone will be commuted by \( t \) years \((t < s < r)\). The objective of each player (prisoner) is to maximize his or her individual payoff. The catch of the dilemma is that confessing (i.e., they defect from each other) is the dominant strategy, that is, rational reasoning forces each player to defect, and thereby doing substantially worse than if they would all decide to cooperate (deny). In terms of the game theory, such a mutual defection is a Nash equilibrium because each of the players comes to the conclusion that he or she could not have done better by unilaterally changing his or her own strategy. In our scheme, if one of the potential treacherous Charlies sends Bob the false outcome, and the others do not, he will be detected and chastised, and they will achieve their purpose of preventing Bob from recovering Alice’s original state; if two or more of them send false outcomes, they can accomplish their purpose escaping from penalty; if all of them do not send false outcome, each will not be punished but they cannot achieve their aim. Thus each of potential treacherous Charlies wish their partners but not himself to send the false outcomes, because then he can accomplish his purpose but not be chastised. The rational reasoning and selfish gene force each Charlie to send correct outcome. This decision is a Nash equilibrium because each of Charlies could not do better by unilaterally changing his action.

In a word, our schemes are secure against Charlies’ cheats. It is worth pointing out that all previous CT schemes \([4, 5, 6, 7, 8, 9]\), including the scheme of Ref. \([15]\), are insecure when there exist treacherous Charlies. That is, the cheat action of Charlies cannot be detected. Then Bob may obtain a wrong state with very low fidelity instead of Alice’s original state when one or more Charlies send him the false measurement outcomes. For instance, we consider the CT of a single-particle state \(|\psi\rangle_{T}\) [see Eq. (1)] with a standard GHZ state. When there are odd Charlies who send the false measurement outcomes to Bob, he will get a wrong state with only the fidelity \( F = (|\alpha|^2 - |\beta|^2)^2 \).

**IV. CONCLUDING REMARKS**

In summary, we have proposed several \((k, m)\)-threshold controlling schemes for CT, where the teleportation of a quantum state Alice to Bob is under the control of \(m\) Charlies such that \(k \leq m\) or more of them can help Bob successfully recover the transferred state. We have also shown that our schemes are secure against both Bob’s dishonesty and Charlies’ treacheries. However, previous \((m, m)\)-threshold schemes cannot prevent Charlies’ cheats. The presented schemes have potential applications in networked quantum information processing. For example, they can be used to implement the \((k, m)\)-threshold quantum-secret-sharing without
nonlocal operation among receivers and additional limitation for $k$, following the idea of Ref. [15]. Our schemes are also useful to seek and explore genuine multipartite entangled states. We utilized the game theory to prove the security of our schemes against Charlie’s cheats. This implies that our schemes may open another perspective for the applications of the game theory.

Although we only discussed the case where the quantum channels are pure entangled states, suitable mixed entangled states may also be competent for the $(k, m)$-threshold CT. In fact, the general form of the pure-entangled-state channel of Eq. (6) can be replaced by the mixed-state channel

$$
\rho_{2+m} = |x_1|^2|B^1\rangle_{AB}\langle B^1| \otimes |\phi^1\rangle_{C_1C_2\ldots C_m} \langle \phi^1|
+ |x_2|^2|B^2\rangle_{AB}\langle B^2| \otimes |\phi^2\rangle_{C_1C_2\ldots C_m} \langle \phi^2|
+ |x_3|^2|B^3\rangle_{AB}\langle B^3| \otimes |\phi^3\rangle_{C_1C_2\ldots C_m} \langle \phi^3|
+ |x_4|^2|B^4\rangle_{AB}\langle B^4| \otimes |\phi^4\rangle_{C_1C_2\ldots C_m} \langle \phi^4|.
$$

(17)

Then corresponding mixed-state channels of the $(k, m)$-threshold CT can be constructed by the same methods as in Sec. II B and Sec. II C. With the forms of the states of Eqs. (6) and (17), one can construct different quantum channels for implementing $(k, m)$-threshold CT. Note that all the quantum channels should at least satisfy the following conditions. (a) They are symmetric under permutation of qubits $\{C_1, C_2, \ldots, C_m\}$. (b) The four states $\{|\phi^1\rangle_{C_1C_2\ldots C_m}, |\phi^2\rangle_{C_1C_2\ldots C_m}, |\phi^3\rangle_{C_1C_2\ldots C_m}, |\phi^4\rangle_{C_1C_2\ldots C_m}\}$ can not be fully distinguished unless $k$ of supervisors perform single-particle measurements on their own particles with appropriate bases and combine the measurement outcomes. In addition, different methods may be needed to discuss the security of concrete schemes.

As mentioned above, SaiToh et al. [15] also proposed a “$(k, m)$-threshold” CT scheme which is a combination of a $(m, m)$-threshold CT scheme and a $(k, m)$-threshold secret sharing scheme. In their scheme, however, the receiver Bob still needs receiving all of the supervisors’ correct measurement outcomes, i.e., needs the cooperation of all Charlies, for recovering the teleported state. Thus their scheme is not a genuine $(k, m)$-threshold controlling scheme and can not prevent Charlies’ cheats. They also mentioned that a $(k, m)$-threshold controlling scheme can be constructed by sharing a classical key among Charlies such that $k$ or more of them can recover the key. The distribution of the key can be achieved by quantum cryptography. However, they did not construct a concrete scheme. In addition, as shown in Ref. [15], a classical key can be easily copied, and Charlies cannot stop Bob from recovering Alice’s original state if Bob manages to obtain as least $k$ shares of the key without consent of Charlies. More importantly, the classical $(k, m)$-threshold controlling scheme can not prevent Charlies’ cheats. In principle, a $(k, m)$-threshold controlling scheme can be constructed by using the quantum polynomial codes [20] as mentioned in Ref. [15]. However, it needs Charlies and Bob to come together and perform nonlocal operations (multi-particle operations). In contrast, our schemes do not need performing nonlocal operations and are secure against Charlies’ cheats of sending false measurement outcomes.

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