Synthetic Approach to the Singularity Problem

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Abstract

We try to convince the reader that the categorical version of differential geometry, called Synthetic Differential Geometry (SDG), offers valuable tools which can be applied to work with some unsolved problems of general relativity. We do this with respect to the space-time singularity problem. The essential difference between the usual differential geometry and SDG is that the latter enriches the real line by introducing infinitesimal of various kinds. Owing to this geometry acquires a tool to penetrate “infinitesimally small” parts of a given manifold. However, to make use of this tool we must switch from the category of sets to some other suitable category. We try two topoi: the topos $G$ of germ determined ideals and the so-called Basel topos $B$. The category of manifolds is a subcategory of both of them. In $G$, we construct a simple model of a contracting sphere. As the sphere shrinks, its curvature increases, but when the radius of the sphere reaches infinitesimal values, the curvature becomes infinitesimal and the singularity is avoided. The topos $B$, unlike the topos $G$, has invertible infinitesimal and infinitely large nonstandard natural numbers. This allows us to see what happens when a function “goes through a singularity”. When changing from the category of sets to another topos, one must be ready to switch from classical logic to in-
tuitionistic logic. This is a radical step, but the logic of the universe
is not obliged to conform to the logic of our brains.

1 Introduction

One of still pending problems of general relativity is the problem of space-time singularities. Although there is a common belief that this problem will be resolved by the quantum theory of gravity, when finally discovered, so far it remains only a belief. And it cannot be excluded that, vice versa, digging into the singularity problem could help finding the correct quantum gravity theory. After all, both these problems are concerned with space-time on the smallest possible scale. One should also not forget that the singularity problem is highly interesting from the mathematical point of view, and it constitutes a good testing field for new mathematical tools.

One of such tools, only recently applied to general relativity (see, for instance [4, 8, 9, 10, 14, 17]), is category theory. It is obvious that general relativity in its standard formulation is done in the category of sets and functions between sets (SET category), just as all macroscopic physical theories (usually without explicitly specifying this assumption), and most often in its subcategory of smooth manifolds and smooth maps between them (\(\mathcal{M}\) category). However, when we approach a level of “very small” quantities – as, for example, in our search for quantum gravity and in the singularity problem – the situation can drastically change. Topos formulation of quantum mechanics (see, [1, 6]) can serve as a motivation for applying similar approach to some unsolved problems of general relativity. Category theorists have elaborated a categorical version of differential geometry, the so-called Synthetic Differential Geometry (SDG) (see, [16]) which almost exactly parallels the usual differential geometry employed in relativistic calculations. The essential difference consists in the fact that in SDG infinitesimals appear which substantially enrich the usual real line. Owing to this fact geometry acquires a tool to penetrate infinitesimally small portions of a given manifold (a manifold’s “germs”, to be defined below) which in the usual approach are invisible (in SET they simply do not exist). This creates an invaluable opportunity for physical applications. However, to make use of this opportunity we must change from SET to a suitable category. And our working hypothesis is that the fundamental level (below Planck’s threshold) is structured by some other (than SET) category. Of course, some correspondence between SET and this
category should exist. The minimum requirement is that the manifold category \( \mathcal{M} \) must be a subcategory of this new category. More than one choices are possible, and in the following we shall experiment with at least two topoi.

In the present paper, we apply the above strategy to the singularity problem in general relativity. We aim at blazing a trail rather than directly attack the final solution. The paper is addressed to relativity professionals who might not be in deep acquainted with category theory. This is why the categorical material is presented in as soft manner as possible (only section 6 is more technical than descriptive). In section 2, for the sake of completeness and eventual category theorist readers, we briefly sketch the singularity problem in general relativity. In section 3, we introduce infinitesimals and show, on the one hand, how they change the perspective of doing differential geometry and, on the other, how they enforce a modification of logic. To make this new perspective to work, we must put the entire problem into the structural environment of a concrete category. We do this in section 4. A few possibilities are open. Our choice, mainly for simplicity reasons, is the topos \( \mathcal{G} \) of germ determined ideals. It is generated by a subcategory \( \mathcal{G} \) of the category \( \mathbb{L} \) of loci, which we also define and briefly discuss. The category \( \mathcal{M} \) of manifolds sits fully and faithfully in \( \mathcal{G} \). Section 4 is based on our earlier result [12] that on any infinitesimal neighbourhood the components of the Riemann curvature tensor are infinitesimal (if not zero). We illustrate this with the help of a simple model of a contracting sphere. If we look at the contraction process “from outside” (in topos SET), as the sphere shrinks, its curvature increases and diverges at zero. However, from the “inside perspective” (in topos \( \mathcal{G} \)), when the radius reaches infinitesimal dimensions, the components of the curvature become infinitesimal, and the “singularity” can be avoided. The model is not quite satisfactory since it does not show how the components of curvature from extremely large (almost “infinite”) suddenly become infinitesimal. To gain an insight into this process, we change, in section 6, to another topos that is subtle enough to enable us to see what happens when a function diverges. This is the so-called Basel topos \( \mathcal{B} \), generated by a subtler Grothendieck topology on \( \mathbb{L} \) than that in the case of \( \mathcal{G} \). Unlike the topos \( \mathcal{G} \), it has invertible infinitesimals and infinitely large nonstandard natural numbers. They form a tool allowing us to manage divergences and, in some cases, ”go through a singularity”. Finally, in section 7, we make some general comments and a few remarks concerning future perspectives.
2 Space-Time Singularities

In the present relativistic paradigm space-time is modeled by a pair \((M, g)\) where \(M\) is a (four-dimensional) manifold and \(g\) a Lorentz metric on \(M\), with suitable differentiability conditions guaranteeing mathematical consistency and physically realistic interpretation. One should additionally assure that no regular points are removed from \(M\), i.e. that space-time \((M, g)\) could not be extended with all differentiability conditions suitably preserved. If all these conditions are satisfied, and space-time is in some sense incomplete, this means that “somewhere in it” there is a singularity or, in other words, that this space-time is singular.

All these conditions should be made precise, and this is far from being straightforward. In the “canonical” singularity theorems \([11]\) space-time incompleteness is usually understood as (timelike or null) geodesic incompleteness. However, from the physical point of view, it would be desirable to take into account not only geodesics but also all curves (of bounded acceleration\(^1\)). Such a construction was proposed by Schmidt \([20]\), but it soon turned out that in physically realistic cases (such as closed Friedman and Schwarzschild solutions) it leads to a pathological behaviour. Several remedies were proposed to cure the situation \([5, 13]\). However, in the present work there is no need to go deeper into this side of the problem.

A space-time \((M', g')\) is said to be an extension of \((M, g)\) if there is an isometric embedding \(\theta : M \rightarrow M'\) such that \(\theta\) is onto and \(\theta(M)\) is a proper subset of \(M'\). A space-time is said to be extendible if it has an extension. Any space-time can be extended until it cannot be further extended, and if this is the case, such a space-time is called maximal. When we are hunting for singularities, we are looking for incomplete maximal space-times. The point is, however, that an extension is not, in general, unique. The most often obstacles are failures of various degrees of differentiability required to assure the existence of unique extensions (for other obstacles see \([3, p. 9]\)).

Various degrees of differentiability are used in this context. Clarke \([3, pp. 63-64]\) argues that in order to extend a space-time through a suspected singularity one must go beyond the differentiability class assumed for this space-time. For instance, if one works, within a given space-time, with \(C^k\)-functions, for any \(k\), then when attempting to make an extension, one must

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\(^1\)A curve of an unbounded acceleration could hardly be imagined to represent the history of a physically realistic object.
assume at least $C^{0,\alpha}$ differentiability$^2$.

When working with singular space-times, we want not only to know whether a singularity exists or not, but also how physics behaves on approaching the singularity. In this respect Riemann curvature tensor is more important than Lorentz metric. Unbounded metric could be a purely coordinate dependent effect whereas unbounded curvature tensor signifies unbounded tidal forces. This raises the problem of various degrees of differentiability of curvature tensor components. However, one should proceed here carefully since an unbounded curvature alone, does not indicate the existence of a singularity.

As we can see, in the singularity problem many things depend on the differentiability class of mathematical objects, that is to say on how these objects behave in a “very small neighbourhood”. If we want to radically push forward the singularity problem, we should look for a method of dealing with space-time on the smallest possible scale. This is exactly what the so-called Synthetic Differential Geometry (SDG) is about.

### 3 The SDG Strategy

We begin with enriching the usual real line $\mathbb{R}$ by assuming the existence of infinitesimals, such as

$$D := \{ x \in \mathbb{R} | x^2 = 0 \},$$

$x \in D$ is so small (but not necessarily equal to zero) that $x^2 = 0$. The real line, enriched in this way, will be denoted by $R$. Let us notice the following consequence of the existence of $D$. Suppose we want to compute the derivative of the function $f(x) = x^2$ at $x = c$. For $d \in D$ we have

$$f(c + d) = (c + d)^2 = c^2 + 2cd + d^2 = c^2 + (2c)d.$$

The linear part of the latter expression can be safely identified with the derivative of $f(x)$ at $c$, $f'(c) = 2c$. We change this example into the rule by assuming

$^2C^{k,\alpha}$ differentiability means that $k$’th derivatives of a function must be Hölder continuous with exponent $\alpha$. A function $f$ is said to be Hölder continuous with exponent $\alpha$ if, for each compact domain $U$ of $f$, there is a constant $K$ such that $|f(x) - f(y)| < K|x - y|^\alpha$, for all $x, y \in U$.  

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**Axiom 1.** For any \( g : D \to R \), there exists a unique \( b \in R \) such that
\[
\forall d \in D, g(d) = g(0) + d \cdot b.
\]
Let us notice that Axiom 1 states that the graph of \( g \) coincides with a fragment of the straight line through \( (0, g(0)) \) and the slope \( b \). And we can define the derivative of any function \( f : R \to R \) at \( c \) to be \( f'(c) = b \). The important consequence of Axiom 1 is that every function has a derivative. In fact, in SDG all functions are differentiable (our presentation here follows [21]).

If this is so, all our problems with differentiability and extendibility in dealing with space-time singularities are eliminated with one stroke. But a high price is to be paid for such a step.

Let us consider the function
\[
g(d) = \begin{cases} 
1 & \text{if } d \neq 0 \\
0 & \text{if } d = 0
\end{cases}
\]
From Axiom 1 we have: \( D \neq \{0\} \). Therefore, by the law of excluded middle, there exists \( d_0 \neq 0 \) in \( D \) and, on the strength of Axiom 1,
\[
g(d_0) = g(0) + d_0 \cdot b
\]
which lead to \( 1 = g(d_0) = d_0 \cdot b \), and after squaring we get \( 1 = 0 \).

There is only one strategy to save our “radical way” of dealing with space-time singularities – to block the law of excluded middle. And indeed the SDG is founded on this strategy. It works on the basis of weakening classical logic to the intuitionistic logic (in which the law of excluded middle is not valid). Of course, one cannot change logic at will. This would lead to the complete mental anarchy. A modified logic requires a correct structural environment that would not only justify but also enforce the correct modification of logic. In the case of SDG this environment is provided by suitable topoi.

There are two approaches to constructing SDG: one can first formulate it in the form of an axiomatic system and then look for suitable topoi as its models [16], or one first studies special cases of some interesting topoi and then generalises obtained results to the form of suitable axioms [19]. Since we are only exploring possibilities of this method as far as the singularity problem is concerned, we work with a topos that seems best suited to this end, the one that generalises the category \( \mathbb{M} \) of manifolds (and smooth maps) in such a way that it includes manifolds with (at least some of) singularities. Only in section 6 we shall explore possibilities of another topos.
4 Categorical Environment of Singularities

Let $A$ be an $R$-algebra, i.e. a commutative unitary algebra equipped with a homomorphism $R \rightarrow A$. A $C^\infty$-ring is an $R$-ring such that, for each smooth map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ there is a smooth map $A(f) : A^n \rightarrow A^m$ which preserves compositions, identities and projections. A homomorphism of $C^\infty$-rings ($C^\infty$-homomorphism) is a ring homomorphism $\phi : A \rightarrow B$ such that, for each smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, one has

$$\phi^m \circ A(f) = B(f) \circ \phi^n$$

where $\phi^n : A^n \rightarrow B^n$, $\phi^m : A^m \rightarrow B^m$.

The crucial example of a $C^\infty$-ring is $C^\infty(M)$ where $M$ is a smooth manifold. Moreover, it can be shown [19, p. 24] that if $M$ is a manifold, $C^\infty(M)$ is a finitely presented $C^\infty$-ring. Let us remind that a $C^\infty$-ring $A$ is finitely presented if it is isomorphic to a $C^\infty$-ring of the form $C^\infty(\mathbb{R}^m)/I$ where $I$ is a finitely generated ideal.

We are now ready to define the category of loci, our “first approximation” to determine the correct structural environment for the proposed categorical approach to the singularity problem. The category of loci, denoted by $\mathbb{L}$, is the opposite (dual) category of the category of finitely presented $C^\infty$-rings as objects and $C^\infty$-homomorphisms as morphisms. Thus the objects are the same as in the category of finitely presented $C^\infty$-rings (for formal clarity we will distinguish an object $A$ and its formal dual $lA$) with arrows just reversed.

The category of loci contains the category of manifolds. More precisely, there is a functor $s : \mathbb{M} \rightarrow \mathbb{L}$ given by $s(M) = lC^\infty(M)$, where $M \in \mathbb{M}$, which is full and faithful [19, p. 60]. It turns out that manifolds, put in the new environment, reveal new properties which were switched off in the old environment (in the category $\mathbb{M}$).

To see this, let us first denote $lC^\infty(\mathbb{R})$ by $R$, and analogously $lC^\infty(\mathbb{R}^n)$ by $R^n$. Let us also define the germ of $R$ at a point $p$ as

$$\Delta_p = \bigcap \{s(U) | p \in U \text{ open in } \mathbb{R}\} \cong lC^\infty_p(\mathbb{R}),$$

and analogously for $p \in \mathbb{R}^n$,

$$\Delta_p = \bigcap_{p \in U} s(U) \cong lC^\infty_p(\mathbb{R}^n).$$

\[3\text{In the following, we consider manifolds with a countable basis; in particular they are paracompact and can be embedded in a closed subspace of } \mathbb{R}^n \text{ for some } n.\]
Let $lA$ be any locus with $A = C^\infty(\mathbb{R}^n)/I$, and let $p$ be a point of $lA$, i.e. a map $1 \to lA$ in $\mathbb{L}$, where $1 = R^0 = lC^\infty(\mathbb{R}^0)$ is the one point locus. Then $\Delta_p \cap lA$, where $\Delta_p \subset R^n$, is, by definition the germ of $lA$ at $p$. By applying this to a manifold $M \in \mathbb{M}$, we obtain

$$\Delta_p \cap s(M) \cong lC^\infty_p(M) = \bigcap \{s(U)|p \in U \text{ open in } M\},$$

the germ of $M$ (strictly speaking of $s(M)$) at $p$. And the germ of a function $M \to \mathbb{R}$ at $p$ is now simply the restriction of $f$ to the germ of $M$ at $p$.

Here we have a comment on this important result: "Thus, whereas the usual category of manifolds $\mathbb{M}$ is too small to contain spaces which are germs of manifolds at some point, these ‘very small submanifolds’ do exists in $\mathbb{L}$" [19, p. 64]. It is rather obvious that taking into account the existence of these “very small submanifolds” should change the strategy of dealing with singularities.

And what about higher classes of differentiability? Let us define

$$D_k(n) = l(C^\infty_0(\mathbb{R}^n)/(x^{k+1})) = \{x \in R^n|x^\alpha = 0, \forall \alpha \text{ such that } |\alpha| = k + 1\}.$$

We immediately have

$$D(n) := D_1(n) \subset D_2(n) \subset D_3(n) \subset \ldots;$$

All these infinitesimals are contained in $\Delta_0^n \subset R^n$ where $\Delta_0 = \bigcap_{n \in \mathbb{N}} s(-\frac{1}{n}, \frac{1}{n}) \subset R$.

Let $f : R^n \to R$ be a map in $\mathbb{L}$. It corresponds to a smooth function $F : \mathbb{R}^n \to \mathbb{R}$, and the $k$-jet $j^k_n(F)$ of $F$, i.e. the equivalence class of $F|_0$ modulo $m^{k+1}$, is just the restriction $f|_{D_k(n)}$.

Of course, all these infinitesimals can be defined not only at 0 of $R^n$ but also at any point $p$ of $s(M)$.

Smoothness of manifolds in $\mathbb{L}$ is “exact” to such an extent that any manifold in $\mathbb{L}$ has the tangent space when restricted to an infinitesimal neighbourhood. To be more precise, let $M \in \mathbb{M}$ and let $TM$ be the total space of the tangent bundle. Then we have

$$s(M)^D \cong s(TM)$$

with the projection

$$ev_0 : s(M)^D \to s(M).$$
It is not a surprise that the tangent vector at a point \( p \in M \) is just a point in \( L \), namely \( v : 1 \to s(M)^D \), such that \( ev_0 \circ v = p \) [19, pp.67-68].

In spite of all its advantages, the category \( L \) of loci is not a good environment for doing differential geometry. In general, it has no exponentials of the form \( l(A)^{l(B)} \), it is not Cartesian closed\(^4\), and consequently not even a topos. This makes it difficult to adapt techniques known from the category \( \text{SET} \) of sets and maps between sets to use them in \( L \). We improve the situation in the following way.

Let us form the category \( \text{SET}^{L^{op}} \) the objects of which are functors from the category \( L^{op} \), opposite with respect to \( L \), to \( \text{SET} \), and morphisms are natural transformations between these functors. The category \( \text{SET}^{L^{op}} \) has the presheaf structure. It contains the category \( L \), the inclusion being given by the Yoneda embedding

\[ Y : L \hookrightarrow \text{SET}^{L^{op}}, \]

\[ Y(lA) = \text{Hom}_L(-, lA). \]

It preserves all exponentiations \( L \) possesses and is Cartesian closed. The category \( M \) of manifolds sits (fully and faithfully) in \( \text{SET}^{L^{op}} \). To see this it is enough to make the composition \( Y \circ s \). However, we are still not quite happy with this categorical environment for manifolds. The category \( \text{SET}^{L^{op}} \) has some “unwanted” properties (for instance the enriched real line \( R \) is in it non-Archimedean\(^5\)). To straighten them out we must impose the correct topology. This is done by converting presheaf into a sheaf. There are a few ways of doing that. The following way seems best suited for our purposes. Roughly speaking, we restrict the category \( \text{SET}^{L^{op}} \) to those functors “which believe that open covers of \( M \) are covers in \( \text{SET}^{L^{op}} \)”.\(^{[19, p. 97]} \)

More precisely, we consider a subcategory, let us call it \( G \), of \( L \), the objects of which are duals \( l(C^\infty(\mathbb{R}^n)/I) \) of finitely generated \( C^\infty \)-rings, \( I \) being a germ determined ideal. We define a suitable Grothendieck topology on \( G \).\(^6\) This topology (being subcanonical) gives rise to the category of sheaves on \( G \) and we obtain a (Grothendieck) topos, denoted by \( G \) and called the topos

\(^4\)Only duals of Weil algebras have exponentials in \( L \); loosely speaking, only if \( l(B) \) in \( l(A)^{l(B)} \) is “sufficiently small”.

\(^5\)\( R \) is Archimedean if for every \( x \in R \) there exists \( n \in \mathbb{N} \) such that \( x < n \).

\(^6\)Going into details would blow up the limits of the present paper and, anyway, they are not essential to understand our main argument. The interested reader could consult [19, pp. 98-101].
of germ determined ideals. In what follows, we regard it as providing the correct categorical environment for our analysis.

The standard theory of general relativity is formulated within the SET category. It is clear that we cannot just jump to another category. We should rather carefully establish the relationship between categories SET and $G$, and see how in this relationship the category $M$ is situated. The general setting is given by the following diagram

$$M \hookrightarrow G \overset{\Gamma}{\rightarrow} \text{SET}$$

As we remember, the category $M$ sits in $\text{SET}^{\text{op}}$ fully and faithfully, and $G$ is just a suitable restriction of the latter\(^7\). The functor $\Gamma$, called the global sections functor, is defined by $\Gamma(F) = F(1)$ where $1$ is the terminal object of the category. This functor has the left adjoint functor $\Delta : \text{SET} \rightarrow G$, called the “constant set” functor, and $\Delta(S), S \in \text{SET}$, is just an ordinary set\(^8\).

5 Singularities and Curvature

In this section, we present a simplified (toy) model illustrating the interaction between singularities and curvature. Strict results would require more detailed analysis which is under way. The usual $C^\infty$-manifold concept has its SDG generalisation as the formal manifold concept. It is just a smooth manifold equipped with an infinitesimal extension\(^9\). In the following, we consider formal manifolds in the category $G$. In [12] we have constructed an infinitesimal version of $n$-dimensional formal manifold, i.e. a formal manifold $M$ with local maps of the form $\{(D_\infty)^n_i \rightarrow M | i \in I\}$ where $D_\infty^n = \bigcup_{k=1}^{\infty} D_k(n)$, and we have shown that the curvature tensor $\widehat{\mathcal{R}}$ of any locally $D_\infty^n$-formal manifold, assumes only infinitesimal values in the object $D_k(m)$ for some $k \in \mathbb{N}$ and $m > n, m, n \in \mathbb{N}$. This result has important consequences for the singularity problem. We illustrate them with the help of the following simplified model.

Let us consider a model for an evolving universe given by

$$S^3 \times \mathbb{R} \subset \mathbb{R}^4$$

\(^7\)The composition $Y \circ s$ is customarily also denoted by $s$.

\(^8\)Formally, $\Delta(S)(lA) = S$ for all $lA \in \text{L}^{\text{op}}$.

\(^9\)A smooth $n$-dimensional manifold $M$, as an object in a suitable category, is a formal manifold if it is equipped with an open cover $\phi_i : \{U_i \rightarrow M\}$ by formally etalé monomorphisms, where $U_i$ are model objects, i.e. formal etalé subobjects in $R^n$ [15].
where, in analogy with the closed Friedman-Lemaître world model, \( \mathbb{R} \) can be interpreted as a cosmic time, and \( S^3 \) as a 3-dimensional instantaneous time section. Let us suppose that the smooth evolution \( S^3 \times \mathbb{R} \), described below, respects the standard smoothness of \( \mathbb{R}^4 \), i.e. the unique smooth structure on \( \mathbb{R}^4 \) in which the product \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) is a smooth product.

Let us farther assume that the diameter \( \rho_{S^3} \) of \( S^3 \) shrinks to zero size (i.e., to a point in \( \mathbb{R}^4 \)) which we call “singularity” and situate it at, say, \( x_0 = 0, x_0 \in \mathbb{R} \). Topologically, we have a cone over \( S^3 \) with the vertex at the singularity. We should remember that it is a simple cone singularity rather than a curvature singularity met in standard cosmological models. If we delete the vertex together with its open neighbourhood, the cone is a standard smooth open submanifold of \( \mathbb{R}^4 \). If we leave the vertex in the picture, we obtain a toy model of a contracting universe ending its evolution in the singularity.

This is how the evolution looks like in the category SET (which the standard relativistic cosmology tacitly assumes). How this evolution can be described when regarded as happening inside the topos \( \mathcal{G} \)? We have

\[
(S^3_{\mathcal{G}} \times_{\mathcal{G}} \mathbb{R}) \hookrightarrow \mathbb{R}^4.
\]

Let us notice that \( S^3_{\mathcal{G}} \neq S^3 \) because now \( S^3_{\mathcal{G}} \) is enriched by infinitesimals. Since in \( \mathcal{G} \) infinitesimals appear, we can call it a fundamental or micro level, whereas SET will represent the macro level.

\( S^3_{\mathcal{G}} \times_{\mathcal{G}} \mathbb{R} \) becomes a locally \( D^4_{\infty} \)-infinitesimal formal manifold. If \( S^3_{\mathcal{G}} \) contracts, its 3-curvature grows, but when its radius reaches infinitesimal size, the components of the curvature become infinitesimal (if not zero). In this way, the conic singularity is avoided (and the evolution can be prolonged beyond 0 of \( R \)).

We have an interesting result: from the macro point of view (SET perspective), the world has the initial singularity; from the micro point of view (\( \mathcal{G} \) perspective) the evolution is smooth (no singularity).

We have also the functor \( \Gamma : \mathcal{G} \to \text{SET} \). It tells us how the macro-image of the world emerges out of the micro-level. \( \Gamma \) acts in this way that it evaluates the functor \( F : \mathbb{L}^{\text{op}} \to \text{SET} \) (in \( \mathcal{G} \)) at one point locus \( 1 = l(C^\infty(\mathbb{R})/x) \), \( \Gamma(F) = F(1) \). Owing to the fact that \( \Gamma \) has a left adjoint \( \Delta \), it enjoys nice properties from the point of view of category theory (it preserves inverse limits [19, pp. 104-105]).
6 Passing through some Divergences of Curvature in a Smooth Topos - An Example

In spite of its attractiveness, the simplified model presented in the preceding section has one serious disadvantage. It does not tell us how the curvature tending to infinity suddenly becomes infinitesimal (or whether there is no obstacle preventing such an outcome). In the present section, we address this problem, but to do so we must choose another topos whose inner environment is sensitive enough to see what happens on approaching the singularity. The topos we will work with is called the Basel topos, denoted by \( \mathcal{B} \). It is constructed in the following way. On the category \( \mathcal{L} \) of loci we define a suitable Grothendieck topology. \( \mathcal{L} \) with this topology is called the site \( \mathcal{B} \), and the Basel topos \( \mathcal{B} \) is the topos of sheaves on \( \mathcal{B} \). The Basel topos has much more complex structure than the topos \( \mathcal{G} \), but many properties of \( \mathcal{G} \) are incorporated in \( \mathcal{B} \). In particular, the category \( \mathcal{M} \) of manifolds also sits fully and faithfully in \( \mathcal{B} \). However, this richer structure of \( \mathcal{B} \) is indispensable to deal with such notions as convergence or “going to a limit”.

The space of infinitesimals in \( \mathcal{B} \) is given by

\[
\Delta = \{ x \in \mathbb{R} | \forall n \in \mathbb{N} \ - \frac{1}{n+1} < x < \frac{1}{n+1} \}. 
\]

It has two subspaces

\[
\mathbb{I} = \{ x \in \Delta | x \text{ is invertible} \},
\]

and

\[
\Delta = \{ x \in \mathbb{R} | x \text{ is not invertible} \}. 
\]

In \( \mathcal{B} \) we have \( D \subset \Delta \subset \Delta \subset \mathbb{R} \) and \( \mathbb{I} \neq \emptyset \). This is the consequence of the existence in \( \mathcal{B} \) the natural numbers \( \mathbb{N} \), called also the smooth natural numbers, which are end-extension of the standard \( \mathbb{N} \), i.e. \( \mathbb{N} \preceq \mathbb{N} \). Then \( \mathbb{I} = \{ x \in \mathbb{R} | \exists n \in \mathbb{N} \setminus \mathbb{N} (x < \frac{1}{n+1} \vee x > \frac{-1}{n+1}) \} \) and \( \Delta = \{ x \in \mathbb{R} | \forall n \in \mathbb{N}(-\frac{1}{n+1} < x < \frac{1}{n+1}) \}. \) Since \( \mathbb{N} \preceq \mathbb{N} \) and \( \Delta_0 = \bigcap_{n \in \mathbb{N}} s(-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R} \), it is now clear that also \( \Delta \preceq \Delta_0 \).

Let us now consider a real function \( f : \mathbb{R} \to \mathbb{R} \) in \( \mathcal{B} \) which has divergent behaviour near \( 0 \in \mathbb{R} \), i.e. \( \lim_{x \to 0^+} f(x) = +\infty \). In our example of a 3-sphere

\[\text{Consisting of finite open covers and projections [19, pp. 285-286].}\]
with its radius shrinking to zero at \(0 \in \mathbb{R}\), the scalar curvature of the sphere is simply given by \(f(x) = \frac{1}{x^2}\). In general, we can have a function of arbitrary fast divergence. We will show that the smooth topos \(\mathcal{B}\) gives us a tool allowing for smoothly overpassing such divergences.

Let us first approximate the divergent behaviour of \(f(x)\) near 0 in \(\text{SET}\) by the sequence \((g_k)\) of smooth functions on the whole of \(\mathbb{R}\), which are convergent to the distribution \(\delta\). Of course, the convergence is understood in the distributional sense. Our minimal requirement is that each \(g_k, k = 0, 1, 2, \ldots\), continuously prolongs the preceding function \(g_{k-1}(x), k = 1, 2, \ldots\); and we assume that \(g_0\) prolongs \(f(x)\) starting from some \(\tilde{x} > 0\). Let us consider a \(\delta\)-like sequence of functions, \((\delta_\epsilon(x))\).\(^{11}\) We build the sequence \((g_k)\) with the help of \((\delta_\epsilon(x))\).

However, if we want to prolong the function \(f(x)\) by a member \(\delta_\epsilon(x)\), say, of the sequence, and then by another and so on, it can happen that \(f(x)\) and \(\delta_\epsilon(x)\) do not meet for any \(x > 0\) and \(\epsilon > 0\), so that the prolongation would not be continuous. In such a case we can always take \(A \cdot \delta_\epsilon(x)\) with \(A \in \mathbb{R}\) such that for some \(x, \epsilon \in \mathbb{R}_{>0}\), \(f(x) > \delta_\epsilon(x) > 0\) and \(A \geq \frac{f(x)}{\delta_\epsilon(x)}\). Now, \(A \cdot \delta_\epsilon(x)\) meets \(f(x)\), and the sequence \((A \cdot \delta_\epsilon(x))\) is still a \(\delta\)-like sequence converging distributionally to the distribution \(A \cdot \delta\).

Having this in mind, let us define the family of functions (subsequent prolongations of \(f(x)\)) with \(A\) depending on \(k\)

\[
g_k(x) = \begin{cases} 
  f(x) & x \in [\eta_0, +\infty) \\
  A(k) \cdot \delta_k(x) & 0 < x \leq \eta_k, A(0) \cdot \delta_0(\eta_0) = f(\eta_0) \\
  A(l+1) \cdot \delta_{l+1}(\eta_l) = \delta_l(\eta_l), l = 1, 2, \ldots, k-1. \\
  \eta_{l+1} \leq \eta_l, l = 0, 1, 2, \ldots, k-1.
\end{cases}
\] (1)

One has \(\lim_{k \to \infty} \eta_k = 0\), \(\lim_{k \to \infty} g_k(x) = +\infty\) and \(g_k(x) = f(x), x \in [\eta_0, +\infty)\). In fact, all \(A(k), k > 0\), can be chosen equal to 1, since \(\delta_k(x) = \delta_{k+1}(x)\) for some \(x \in (0, \tilde{\eta}_k), k = 1, 2, \ldots, \) and \(\frac{\delta_{k+1}(x)}{\delta_k(x)} = 1\) at this \(x\). This follows from the behaviour of continuous functions in the \(\delta\)-like sequence. By choosing this \(x\) as \(\eta_k\), we have the new interval \((0, \eta_k]\) where at \(x = \eta_k\) both functions, \(\delta_k(x)\) and \(\delta_{k+1}(x)\), assume the same value. In this way, \(g_k(x), x \in (x_0, +\infty)\), is a continuous function for any \(x_0 > 0\) (in fact it can be smoothed out).

\(^{11}\)\(\lim_{\epsilon \to 0} \delta_{\epsilon}(x) \xrightarrow{\text{distrib.}} \delta (\lim_{k \to \infty} \delta_k(x) \xrightarrow{\text{distrib.}} \delta) \iff\) for all test functions \(\phi(x) \in S\) it holds \(\lim_{\epsilon \to 0} \int_{\mathbb{R}} \phi(x) \delta_{\epsilon}(x) dx = \phi(0)\) (\(\lim_{k \to \infty} \int_{\mathbb{R}} \phi(x) \delta_k(x) dx = \phi(0)\)).

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Increasing $k$ means decreasing $x_0$. However, no continuous function, for all $x \in [0, \infty)$, can be a limit of $g_k(x)$ when $k \to \infty$. The correct limit has to be distributional. To describe it, let us define “the $\delta$-approximation of infinity” by the prolongations of $f(x)$, i.e. the sequence of functions

$$
\tilde{g}_k(x) = \begin{cases} 
g_k(x) & x \in (\eta_k, +\infty) 
A\delta_k(x) & x \in (-\infty, \eta_k]
\end{cases}
$$

(2)

The distributional limit $\lim_{k \to \infty} \tilde{g}_k(x)$ is the pair: the regular distribution represented by the function $g_k$, $k \to \infty$ (the subsequent prolongations) for $x \in (0, \infty)$, and the $A\delta$-distribution, i.e. the distributional limit of the $\delta$-like sequence $\delta_k(x)$, $x \in (-\infty, \eta_k], \eta_k \to 0$.

The construction of the sequence $(g_k(x))$ of the subsequent prolongations of $f(x)$ gives the translation of the divergence of $f(x)$ into the $\delta$-like divergence of $g_k(x)$. We call it the $\delta$-like divergence approximating the divergence of a function $f$. In SET there is no way to pass smoothly or continuously over such a distributional divergence. However, in $\mathcal{B}$ the $\delta$-like divergence disappears and is replaced by the smooth internal evolution over $\mathbb{R}$. This is accomplished by the following theorems.

Let $F_n$ be the internal space in $\mathcal{B}$ of test functions in dimension $n$ [19, p. 322].

**Theorem 1** ([19] Theorem 3.6, p. 324). Every distribution $\mu$ on $\mathbb{R}^n$ in $\mathcal{B}$ can be represented by a function (predistribution) $\mu_0 : \mathbb{R}^n_{\text{acc}} \to \mathbb{R}$ such that for all $f \in F_n$, $\mu(f) = \int f(x)\mu_0(x)\,dx$. Here $\mathbb{R}^n_{\text{acc}} = \{x \in \mathbb{R} | \exists n \in \mathbb{N}(-n < x < n)\}$ (called accessible reals).

The following theorem is based on the $\delta$-like sequence $\delta_\epsilon(x) = \frac{1}{\pi} \frac{\sin x/\epsilon}{x/\epsilon}$ which, in terms of $\epsilon \approx 1/n$ reads $f_n(x) = \frac{1}{\pi} \frac{\sin nx}{x}$.

**Theorem 2** ([19] Theorem 3.7.1, p. 325). For every $n \in \mathbb{N} \setminus \mathbb{N}$ ($n > \infty$) and every $f \in F_1$

$$
\int \frac{\sin nx}{\pi x} f(x)\,dx \simeq f(0).
$$

Thus we can always represent a $\delta$-distribution by the function $\delta_0 = \mu_0$ making $\delta$ the regular distribution in $\mathcal{B}$, where $N$ is the object of smooth natural numbers. This has tremendous consequences. The following theorem
makes the direct use of a regular function, internal in $\mathcal{B}$, on the level $n > \infty$, replacing the $\delta$-distribution in $g(x)$ of (2) in SET.

**Theorem 3.** The $\delta$-like divergence in SET approximating the divergence of a function $f$, $f \in C^0(\mathbb{R}_{>0})$, $\lim_{x \to 0^+} f(x) = +\infty$, is realized in $\mathcal{B}$ by the internal $s$-evolution, $g_B : \mathbb{R} \to \mathbb{R}$ smooth at 0, in $\mathcal{B}$ ($s$ — after “smooth”, i.e. evolution in terms of smooth natural numbers). □

One easily extends this result, based on Theorem 1, to the divergences generated by $f : \mathbb{R}^n \to \mathbb{R}$, $n > 1$. We see that in order to make a geometrical use of noninvertible infinitesimals from $D^n_\infty$, we must first “reach them” with the help of non-zero invertible infinitesimals from $\mathbb{I}$. Happily enough they exists in $\mathcal{B}$, and their inversions, although infinite, help to define new $s$-finiteness in $\mathcal{B}$.

### 7 Final Remarks

As we have remarked in the Introduction, this is only an introductory work, the aim of which is to convince the reader that the topic is worthwhile to be pursuit, rather than to present concrete results. New tools provided by SDG, non available in the standard approach, give us an opportunity to investigate what happens on “infinitesimally small neighbourhoods” and when various processes “go to infinity”. The first candidate in this line of research, taken up in the present work, is the singularity problem. As the next step one could investigate various kinds of singularities (strong curvature singularities, quasiregular singularities, etc.) and singularities in various solutions to Einstein’s equations (see [8]). Other open fields are: geodesic incompleteness, various space-time extensions, and space-time boundaries of different kinds; everything as seen in the light of SDG possibilities.

It seems worthwhile to try also categorical and SDG methods in the search for quantum gravity theory (see [7]), especially that the very intensive search with the help of other methods has so far given rather modest results.

A remarkable circumstance is that the categorical approach not only can contribute at solving, or at least elucidating, some existing problems, but also is able to uncover new facts and regularities. For instance, interaction between SET and another topos (such as $\mathcal{B}$), via some functors, can produce in $\mathbb{R}^4$ an exotic smooth structure, and since no such structure can have
vanishing Riemann tensor, this effect can lead to interesting results (see [2, 12]).

However, there is a price one has to pay for all these advantages: when changing from SET to another topos, one must be ready to switch from classical to intuitionistic logic. This is a radical step. Our brain has evolved through a long interaction with its macroscopic environment, the logical structure of which is shaped by the internal logic of the topos SET, i.e. classical logic. However, the logic of the entire universe, on all its levels, is not obliged to conform to the logic of our brains.

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