Matrix Theory Black Holes and the Gross Witten Transition

L. Susskind\textsuperscript{1}

\textsuperscript{1}Department of Physics, Stanford University
Stanford, CA 94305-4060

Large N gauge theories have so called Gross-Witten phase transitions which typically can occur in finite volume systems. In this paper we relate these transitions in supersymmetric gauge theories to transitions that take place between black hole solutions in general relativity. The correspondence between gauge theory and gravitation is through matrix theory which represents the gravitational system in terms of super Yang Mills theory on finite tori. We also discuss a related transition that was found by Banks, Fischler, Klebanov and Susskind.
1. The Gross Witten Transition

According to matrix theory [2], there is a duality between Super Yang Mills theory on a spatial d-torus and 11 dimensional supergravity compactified on a d+1 torus*. Our knowledge of the two theories sometimes overlaps and this allows us to test the duality. More often, we know something about one theory which leads to predictions about the other. In this paper we will use knowledge about black holes to gain information about Gross Witten type transitions [3] in 3+1 dimensional super Yang Mills theory compactified on a 3-torus of size $\Sigma$. In particular we will see that such a transition exists and that the transition temperature is a function of the ‘t Hooft coupling $g^2_{ym}N$ which scales like

$$T_3 \sim \frac{1}{\Sigma \sqrt{g^2_{ym}N}}$$

for large values of $g^2_{ym}N$. Detailed features of the transition could in principle be predicted from classical solutions of supergravity. A similar argument has been given by Witten for the case of compactification on a sphere. In this case the duality is between super Yang Mills theory and supergravity in $AdS_5 \times S_5$ [4].

We begin with a brief review of Gross Witten transitions. Normally systems of small numbers of degrees of freedom or systems in finite volume do not exhibit sharp phase transitions or singularities in the partition function. However in the large $N$ limit such transitions become possible. In fact the Gross Witten transition was first found in the theory of a single plaquette in lattice gauge theory [3].

The plaquette is described by a unitary $N \times N$ matrix $U$. The energy is

$$E = -\frac{1}{g^2} \text{tr} U$$

and the partition function is

$$Z = \int dU \exp \left[ \frac{\beta}{g^2} \text{tr} U \right]$$

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* We include the longitudinal 11th direction among the compact directions. Going to large $N$ effectively decompactifies this direction but this is not the main focus of this paper
The coupling $g$ is assumed to satisfy ’t Hooft scaling

$$g^2N = \lambda$$

(1.4)

with $\lambda$ fixed. The integral over unitary matrices can be replaced by an integral over the $N$ eigenvalues $\alpha_i = e^{i\theta_i}$ of $U$. The measure for the integration is a certain determinant $\Delta$ whose essential properties we will describe. The partition function is

$$Z = \int d\alpha \exp \left[ \frac{N\beta}{\lambda} \sum_i \alpha_i + \log \Delta \right]$$

(1.5)

The log of $\Delta$ is a sum over pairs of $\alpha'$s of a two body repulsive potential which diverges logarithmically when the two eigenvalues approach one another.

$$\log \Delta \sim \sum \log |\theta_i - \theta_j|$$

(1.6)

The first term in the brackets in eq (1.4) may be thought of as an attraction of the eigenvalues to the point $\alpha = 1$. The equilibrium behavior can be estimated by looking for a stationary point of the action which occurs when the forces on an eigenvalue balance. The result at is a “droplet” of eigenvalues with a density of order $N$. The size of the droplet depends only on $\frac{\beta}{\lambda}$ and it has sharp edges as $N \to \infty$. For small $\lambda$ the droplet is small and as $\lambda$ increases its size increases until at some critical value of $\frac{\beta}{\lambda}$ the edges of the distribution meet at $\theta = \pi$. At this point there is a singularity. This is the Gross Witten transition. For this very simple case the temperature of the transition scales like

$$T_c \sim \frac{1}{\lambda} = \frac{1}{g^2N}$$

(1.7)

In eq (1.7) we see the characteristic dependence of the transition temperature on the ’t Hooft coupling $g^2N$ which serves as the expansion parameter for perturbation theory in the large $N$ limit.

The description of the low temperature behavior as a droplet is a bit misleading. In fact the combination of repulsive short range forces and attraction to the
origin creates a lattice with regularly spaced eigenvalues that could be described as a crystal. In fact even when the distribution spreads over the circle at high temperature, a strong degree of local crystalline order is preserved due to the repulsive forces.

In this paper we will present evidence that Gross Witten transitions occur in 3+1 dimensional toroidally compactified super Yang Mills theory. In this case the matrix valued variables that replace the plaquette variable $U$ are the Wilson loops around the cycles of the torus.

2. Matrix Black Holes

In [1] matrix theory [2] was applied to the behavior of black holes in M Theory compactified on a 3-torus of size $L$. Strictly speaking the theory is compactified on a 4 torus but by passing to the large $N$ limit while keeping $L$ fixed, the 11th direction is effectively decompactified. For the problem of black holes the decompactification can be quantified as follows. Begin with a configuration with entropy $S$. Here we assume that $S \gg 1$. For $N < S$ the configuration described in [1] behaves like a 10 dimensional near extremal black hole with D0-brane charge. Alternatively it can be thought of as an 11 dimensional black string wound around the 11th direction. It is homogeneous in the 11th direction. As $N$ increases, an instability is encountered. The black string breaks and forms a localized blob in the 11th direction [6]. The blob becomes an 11 dimensional Schwarzschild black hole. The transition from homogeneous black string to localized black hole occurs at $N \sim S$. In fact one finds that at this point, the free energy of the black string and Schwarzschild black hole are equal. We will return to this transition in sect 3 but for now we are interested in the ”black string” region $N << S$ where the black hole is uniform in $x^{11}$.

Let us consider the behavior of the black hole as we vary the size of the 3-torus $L$, keeping $N$ large but fixed. It is convenient to think of compactification as periodic identification and to imagine a transverse lattice of black holes with spacing $L$. It is obvious that for very large $L$ the compactification can be ignored and the black hole is localized somewhere in the torus with a radius much smaller
than $L$. In this region the black hole behaves as a 10 dimensional near extremal D0-brane black hole in uncompactified transverse space. It is so far from its periodic images that they can be ignored. On the other hand for small enough $L$ the black hole will merge with its images and a uniform homogeneous configuration on the 3 torus will result. In fact a sharp transition must take place [7]. To see this consider the horizon. For very large $L$ the horizon will form a small sphere localized in the torus. As $L$ is decreased the horizons of the images will eventually touch and the black holes will merge. The change in topology of the horizon is discrete and signals a singularity in its area and thus in the thermodynamic quantities.

The approximate location of the phase transition can be found either by thermodynamic or by solving the gravitational equations and determining the point at which the horizon changes topology. We will use thermodynamics. The thermodynamic relation between free energy and temperature for a near extremal D0-brane black hole is

$$ F = \frac{T^4 N^2 l_{st}^6}{L^3} \quad (2.1) $$

where $l_{st}$ is the string length scale.

For a D0-brane black hole in uncompactified ($L \to \infty$) space one finds [5]

$$ F = T^{14/5} N^{7/5} l_{st}^{9/5} g_{st}^{-3/5} \quad (2.2) $$

where $g_{st}$ is the string coupling constant.

To find the transition we equate (2.1) and (2.2) to give

$$ T_c = \sqrt{\frac{L^3}{l_{st}^3 g_{st} N}} \quad (2.3) $$

* Throughout this paper all irrelevant numerical constants will be set to 1
3. Gauge Theory Interpretation of the Transition

According to Matrix theory the system we are describing is dual to super Yang Mills theory compactified on a 3-torus. The parameters of the super Yang Mills theory are a coupling constant \( g_{\text{ym}} \) and a compactification radius \( \Sigma \). We also introduce two M-Theory quantities; \( R \) and \( l_{11} \). \( R \) is the compactification radius of the 11 direction and \( l_{11} \) is the 11 dimensional planck length. The various quantities are related by

\[
R = l_{st} g_{st} \tag{3.1}
\]

\[
l_{11} = l_{st} g_{st}^{1/3} \tag{3.1}
\]

and

\[
g_{\text{ym}}^2 = \frac{l_{11}^3}{L^3} \tag{3.2}
\]

\[
\Sigma = \frac{l_{11}^3}{LR} \tag{3.2}
\]

Plugging these equations into (2.3) we find the transition temperature satisfies

\[
\Sigma T_c = (g_{\text{ym}}^2 N)^{-1/2} \tag{3.3}
\]

Note that the right side of eq (3.3) depends only on the 't Hooft coupling constant \( g_{\text{ym}}^2 N \) as we would expect for a Gross Witten type transition. On the left side we the scale invariant function of the temperature that we can construct from the Yang Mills parameters. This reflects the exact conformal invariance of 3+1 dimensional super Yang Mills theory with 16 real supersymmetries.

In order to see that this transition is analogous to the Gross Witten transition on a single plaquette, let us recall the description of the black hole (for \( N << S \)) given in [1]. It was found that the D0-branes form a regular lattice on the 3-torus. Indeed the 't Hooft parameter \( g_{\text{ym}}^2 N = \frac{N l_{11}^3}{L^4} \) is the density of D0-branes. The origin of the lattice like structure in both the 1-plaquette example and the much more sophisticated super Yang Mills theory is the same; eigenvalue repulsion [8].

In interpreting the transition as a Gross Witten transition the most important point is that the location of the D0-branes on the torus of size \( L \) is given in Matrix
theory by the eigenvalues of the Wilson loops around the 3 cycles. Thus, just as for the single plaquette, the transition is one in which the distribution of Wilson loop eigenvalues begins at high temperatures by filling the torus with a crystal lattice and passes to a localized blob at low temperatures.

However the picture described above only holds over part of the $N, L$ plane. As the temperature is lowered it is possible that another transition intervenes before the Gross Witten transition. This is the transition reported in [1] in which the 10 dimensional black hole (11 D black string) condenses into a blob in the 11 direction and becomes an 11 dimensional schwarzschild black hole. For want of a better name the transition will be referred to as the BFKS transition. The BFKS transition occurs at the point where the entropy is equal to $N$. The critical temperature for this transition satisfies

$$\Sigma T_{bfks} = N^{-\frac{1}{3}}$$

(3.4)

The gross Witten transition will occur only if $T_c > T_{bfks}$ or

$$\frac{L^9}{l_{11}^9} > N$$

(3.5)

Like the GW transition, the BFKS transition also separates a homogeneous phase from a blob-like phase. At high temperature the black hole system will not only be homogeneous in the 3-torus of size $L$ but also the longitudinal compact direction of size $R$. If the inequality (3.5) is violated then as the temperature is lowered the black object will become localized in the longitudinal direction. This transition is not of the Gross Witten type in the 3+1 gauge theory but as shown in [1] its existence is easily understood in the D3-brane description of the super Yang Mills theory as a transition which occurs when the thermal wavelength exceeds the size of the effective quantization volume. For a more complete explanation the reader is referred to [1]. However in the SYM description it is not clear that this transition is a sharp one. The gravitational description however, makes it clear that a singularity occurs. The point is once again that when the system goes from black string to black hole, the horizon topology suddenly changes. This sudden change signals a singularity in the area and therefore the entropy.
REFERENCES

1. T. Banks, W. Fischler, I. Klebanov and L. Susskind, hep-th/9709091.
2. T. Banks, W. Fischler, S. Shenker and L. Susskind, hep-th/9610043.
3. D. J. Gross, E. Witten, Phys. Rev. D21:446-453, 1980.
4. Edward Witten, hep-th/9802150.
5. Igor R. Klebanov, Leonard Susskind, Schwarzschild Black Holes in Various Dimensions from Matrix Theory, hep-th/9709108.
6. Gary T. Horowitz, Emil J. Martinec, Comments on Black Holes in Matrix Theory, hep-th/9710217.
7. Gary T. Horowitz, Joseph Polchinski, A Correspondence Principle for Black Holes and Strings, hep-th/9612146.
8. G. Polhemus, Statistical Mechanics of Multiply Wound D-Branes, hep-th/9612130.