The merger history of primordial-black-hole binaries

You Wu$^{1,*}$

$^1$School of Mathematics and Physical sciences, Hunan University of Arts and Science, Changde, 415000, China

As a candidate of dark matter, primordial black holes (PBHs) have attracted more and more attentions as they could be possible progenitors of the heavy binary black holes (BBHs) observed by LIGO/Virgo. Accurately estimating the merger rate of PBH binaries will be crucial to reconstruct the mass distribution of PBHs. It was pointed out the merger history of PBHs may shift the merger rate distribution depending on the mass function of PBHs. In this paper, we use 10 BBH events from LIGO/Virgo O1 and O2 observing runs to constrain the merger rate distribution of PBHs by accounting the effect of merger history. It is found that the second merger process makes subdominant contribution to the total merger rate, and hence the merger history effect can be safely neglected.

Keywords: primordial black holes, merger rate, merger history

I. INTRODUCTION

The direct detection of gravitational wave (GW) from a binary black hole (BBH) coalescence [1] has opened a new window of astronomy. Over the past few years, ten BBH mergers have been reported by LIGO/Virgo during the O1 and O2 observing runs [1–7]. The progenitors of these BBHs are still unknown and under intensively investigation (see e.g. [8–27]). These LIGO/Virgo BBHs present a much heavier mass distribution (in particular the source-frame primary mass of GW170729 event can be as heavy as 50.2$^{+16.2}_{-10.2}$Msun [7]) than that inferred from X-ray observations [28–31], which would challenge the formation and evolution mechanisms of astrophysical black holes. One possible explanation for LIGO/Virgo BBHs is the primordial black holes (PBHs) [8–10] formed through the gravitational collapse of the primordial density fluctuations [32, 33], which may accompany the induced GWs [34–37]. On the other hand, PBHs can also be a candidate of cold dark matter (CDM), and the abundance of PBHs in CDM has been constrained by various experiments [34, 36, 38–56].

In order to account for the LIGO/Virgo BBHs, the merger rate of PBH binaries has been estimated to be 17 $\sim$ 288 Gpc$^{-3}$ yr$^{-1}$ [55]. One should notice that theoretically there exist some uncertainties in estimating the merger rate distribution of PBH binaries, and the estimation has been continuing improved. The merger rate of PBH binaries with monochromatic mass function has already been given in [9, 57] for the case where two neighboring PBHs having sufficiently small separation can form a binary in the early Universe due to the torque from the third nearest PBH. These binaries would then evolve and coalesce within the age of the Universe and finally explain the merger events observed by LIGO/Virgo [9]. Later, the merger rate estimation is improved in [15] by taking into account the torques exerted by all CDM (including all the PBHs and linear density perturbations), but it is also assumed that all PBHs have the same mass. It is also pointed out in [15] that the effects such as encountering with other PBHs, tidal field from the smooth halo, the the baryon accretion are subdominant and can be neglected when estimating the merger rate.

Various attempts have been made to estimate the merger rate distribution of PBH binaries when PBHs have an extended mass function [10, 55, 58, 59]. In particular, a formalism to estimate the effect of merger history of PBHs on merger rate distribution has been developed in [60], and it is argued that the multiple-merger effect may not be ignored if PBHs have a power-law or a log-normal mass function by choosing some specific parameter values of the mass function. An accurate estimation of the merger rate will be crucial to infer the event rate of LIGO/Virgo BBHs and constrain the abundance of PBHs in CDM either through null detection of sub-solar mass PBH binaries or the null detection of stochastic GW background (SGWB) from PBH binaries.

In this paper, we will use the public available GW data from LIGO/Virgo O1 and O2 observations to estimate the merger rate distribution of PBH binaries with a general mass function assuming all of LIGO/Virgo BBHs are or primordial-origin. We find that the merger history effect makes no significant contribution to the merger rate of PBHs and can be safely ignored. The rest of this paper is organized as follows. In Sec. II, we review the calculation of merger rate distribution accounting for the merger history effect. In Sec. III, we elaborate the data analysis method used to infer the PBH populations from LIGO/Virgo data. In Sec. IV, we present the results for PBHs with a power-law and a log-normal mass function respectively. Finally, we summarize and discuss our results in Sec. V.

II. MERGER RATE DISTRIBUTION OF PBHS

In this section, we will briefly review the calculation of merger rate density by closely following [60]. See also

* youwuphy@gmail.com. Project 11847107 supported by National Natural Science Foundation of China.
After some cumbersome derivations as presented in [60], the density of PBHs, \( n_{\text{PBH}} \), at cosmic time \( t \) is given by [60]

\[
0.85 f_{\text{PBH}} P(m) \, dm,
\]

where \( f_{\text{PBH}} \) is the fraction of PBHs in CDM, and the coefficient 0.85 accounts for the fraction of CDM in non-relativistic matter. Similar to [60], one may define a quantity \( m_{\text{PBH}} \) as

\[
\frac{1}{m_{\text{PBH}}} = \int \frac{P(m)}{m} \, dm.
\]

Furthermore, the present average number density of PBHs with mass \( m \) in the present total average number density of PBHs, \( F(m) \), can be obtained by [60]

\[
F(m) = P(m) \frac{m_{\text{PBH}}}{m}.
\]

After some cumbersome derivations as presented in [60], one can get the total merger rate density, \( R(t, m_i, m_j) \), of PBHs at cosmic time \( t \) with masses \( m_i M_\odot \) and \( m_j M_\odot \) to be

\[
R(t, t, m_i) = \sum_{n=1}^{m_i} R_n(t, m_i, m_j),
\]

where \( R_n(t, m_i, m_j) \) is the merger rate density in the \( n \)-th merger process. Then the total merger rate can be obtained by

\[
R(t) = \int R(t, m_i, m_j) \, dm_i \, dm_j = \sum_{n=1}^{m_i} R_n(t),
\]

where

\[
R_n(t) = \int R_n(t, m_i, m_j) \, dm_i \, dm_j.
\]

As demonstrated in [60], \( R_{n+1}(t, m_i, m_j) \) is not necessarily smaller than \( R_n(t, m_i, m_j) \) (see Fig. 7 and Fig. 8 in [60]). However, \( R_{n+1}(t) \) should be smaller than \( R_n(t) \) as expected [60]. Here, we only consider the merger history up to second-merger process. The merger rate density of first-merger process, \( R_1(t, m_i, m_j) \), in Eq. (5) is given by [60]

\[
R_1(t, m_i, m_j) = \int \hat{R}_1(t, m_i, m_j, m_l) \, dm_l,
\]

where

\[
\hat{R}_1(t, m_i, m_j, m_l) = 1.32 \times 10^6 \times \left( \frac{t}{t_0} \right)^{-\frac{34}{37}} \left( \frac{f_{\text{PBH}}}{m_{\text{PBH}}} \right)^{\frac{35}{37}} \times m_l^{\frac{-34}{37}} (m_i m_j)^{\frac{34}{37}} (m_i + m_j)^{\frac{35}{37}} F(m_i) F(m_j) F(m_l).
\]

The merger rate density of second-merger process, \( \mathcal{R}_2(t, m_i, m_j) \), in Eq. (5) is given by [60]

\[
\mathcal{R}_2(t, m_i, m_j) = \frac{1}{2} \int \mathcal{R}_2(t, m_i - m_e, m_e, m_j, m_l) \, dm_l \, dm_e
\]

\[
+ \frac{1}{2} \int \mathcal{R}_2(t, m_j - m_e, m_e, m_i, m_l) \, dm_l \, dm_e,
\]

where

\[
\mathcal{R}_2(t, m_i, m_j, m_k, m_l) = 1.59 \times 10^4 \times \left( \frac{t}{t_0} \right)^{-\frac{31}{37}} \left( \frac{f_{\text{PBH}}}{m_{\text{PBH}}} \right)^{\frac{69}{37}} \times m_i^{\frac{84}{37}} (m_i + m_j)^{\frac{34}{37}} (m_i + m_j + m_k)^{\frac{35}{37}} \times F(m_i) F(m_j) F(m_k) F(m_l).
\]

III. INFERENCE ON PBH MASS DISTRIBUTION FROM GW DATA

Given a general mass function of PBHs \( P(m|\theta) \) which satisfy the normalization condition of Eq. (1), the time (or redshift) dependent merger rate can be obtained by Eq. (6), namely

\[
R(t|\theta) = \int R(t, \lambda|\theta) \, d\lambda,
\]

where \( \lambda = \{m_1, m_2\} \), and \( \theta \) are the parameters that characterize the mass function and will be inferred from GW data. For instance, \( \theta = \{\alpha, M\} \) for the power-law PDF (see Eq. (20)) and \( \theta = \{m_\epsilon, \sigma\} \) for the log-normal PDF (see Eq. (23)). The local merger rate density then reads [55]

\[
R(t_0, \lambda|\theta) = R_0 p(\lambda|\theta),
\]

where \( R_0 = R(t_0|\theta) \) is the local merger rate, and \( p(m_1, m_2|\theta) \) is the population distribution of BBH mergers. Note that Eq. (13) guarantees \( p(m_1, m_2|\theta) \) is normalized, namely

\[
\int p(\lambda|\theta) \, d\lambda = 1.
\]

Given the GW data, \( d = (d_1, \ldots, d_N) \), which consist of \( N \) BBH merger events, we aim to extract the population parameters \( \{\theta, R_0\} \) from \( d \). In order to do that, it is necessary to perform the hierarchical Bayesian inference on the BBHs’ mass distribution [3, 61–66]. In this work, we will use the data of ten BBHs [3, 7] reported by LIGO/Virgo O1 and O2 observations, and hence \( N = 10 \). The posterior samples of these BBHs are public available from [67]. Because the standard priors on masses for each event in LIGO/Virgo analysis are taken to be uniform [3, 7], the likelihood of an individual event \( p(d_i|\lambda) \) is proportional to the posterior of that event \( p(\lambda|d_i) \). The total
likelihood for an inhomogeneous Poisson process can be evaluated as [63–66]

\[ p(d|\theta, R_0) \propto R_0^N e^{-R_0} \beta(\theta) \prod_i \int d\lambda p(d_i|\lambda) p(\lambda|\theta) , \tag{15} \]

where \( \beta(\theta) \) is defined as

\[ \beta(\theta) \equiv \int d\lambda \; VT(\lambda) \; p(\lambda|\theta) , \tag{16} \]

in which \( VT(\lambda) \) is the sensitive spacetime volume [61, 62] of LIGO. We adopt the semi-analytical approximation from [61, 62] to estimate \( VT \), where we use the “IMRPhenomPv2” waveform to simulate the BBH templates and neglect the effect of spins for BHs. Furthermore, the threshold signal-to-noise ratio (SNR) of detection for a single-detector is set to 8, which corresponds to a network SNR threshold of around 12.

Assuming the prior distributions \( p(\theta, R_0) \) are uniform for \( \theta \) parameters and log-uniform for local merger rate \( R_0 \) [4, 61], namely

\[ p(\theta, R_0) \propto \frac{1}{R_0} , \tag{17} \]

the posterior probability distribution \( p(\theta, R_0|d) \) can be directly calculated by

\[ p(\theta, R_0|d) \propto p(d|\theta, R_0) \; p(\theta, R_0) . \tag{18} \]

The marginalized posterior \( p(\theta|d) \) can then be readily obtained by integrating over \( R_0 \) in Eq. (18), namely

\[ p(\theta|d) \propto \left[ \beta(\theta) \right]^{-N} \prod_i \int d\lambda \; p(d_i|\lambda) \; p(\lambda|\theta) . \tag{19} \]

This marginalized posterior has been widely used in previous population inferences [3, 4, 55, 61, 62, 68]. In the following section, we will utilize the posterior (18) to infer the population parameters \( \{ \theta, R_0 \} \) by considering two concrete mass distributions, a power-law PDF and a log-normal PDF, respectively.

IV. RESULTS

A. Power-law mass function

We now consider a power-law mass function of PBHs as [69]

\[ P(m) = \frac{\alpha - 1}{M} \left( \frac{m}{M} \right)^{-\alpha} , \tag{20} \]

where \( m > M \), and \( \alpha > 1 \) is the power-law index. Note that \( \theta = \{ \alpha, M \} \) and the free parameters are \( \{ \theta, R_0 \} = \{ \alpha, M, R_0 \} \) in this case. Using Eq. (3) and Eq. (4), it is easily to get

\[ m_{\text{pbh}} = M \frac{\alpha}{\alpha - 1} . \tag{21} \]

\[ F(m) = \frac{\alpha}{m} \left( \frac{m}{M} \right)^{-\alpha} . \tag{22} \]

Using data of 10 BBHs observed by LIGO/Virgo O1 and O2 observations and performing the hierarchical Bayesian inference, we obtain \( \alpha = 2.41^{+1.00}_{-0.87} \), \( M = 7.4^{+1.4}_{-3.3} M_\odot \), and \( R_0 = 48^{+37}_{-23} \text{Gpc}^{-3} \text{yr}^{-1} \). It is then easy to infer the abundance of PBHs in CDM to be \( F_{\text{pbh}} = 2.8^{+1.8}_{-1.2} \times 10^{-3} \) from the posterior distribution of local merger rate \( R_0 \). The results of local merger rate and abundance of PBHs are consistent with the previous estimations, confirming that the main components of CDM should not be made of stellar mass PBHs. The posteriors of parameters \( \{ \theta, R_0 \} = \{ \alpha, M, R_0 \} \) are shown in Fig. 1.

![Fig. 1](image1.png)

FIG. 1. The marginalized one- and two-dimensional posterior distributions for parameters \( \{ \theta, R_0 \} = \{ \alpha, M, R_0 \} \) in the power-law mass function of PBHs, by using 10 BBH events from LIGO/Virgo O1 and O2 observing runs. The contours are at the 68% and 95% credible levels, respectively.

![Fig. 2](image2.png)

FIG. 2. The ratio of merger rate density from second-merger history to that from first-merger history, \( \mathcal{R}_2(t_0, m_1, m_2)/\mathcal{R}_1(t_0, m_1, m_2) \).

Fig. 2 shows the ratio of merger rate density from second-merger history to the one from first-merger history, namely \( \mathcal{R}_2(t_0, m_1, m_2)/\mathcal{R}_1(t_0, m_1, m_2) \), by fixing
\( \{ \theta, R_0 \} \) to their best-fit values. It is clearly that the correction of total merger rate density from second-merger history is less than 10%. It is then readily to calculate the ratio of merger rate from second-merger history to the one from first-merger history, \( R_2(t_0)/R_1(t_0) = 0.5\% \). Therefore we conclude that the merger history effect can be safely ignored when estimating the merger rate (density) of PBHs.

B. Log-normal mass function

We now consider a log-normal mass function of PBHs as \cite{70}

\[
P(m) = \frac{1}{\sqrt{2\pi \sigma m}} \exp \left( -\frac{\ln^2 (m/m_c)}{2\sigma^2} \right),
\]

where \( m_c \) presents the peak mass of \( mP(m) \), and \( \sigma \) denotes the width of the mass spectrum. Note that \( \theta = \{ m_c, \sigma \} \) and the free parameters are \( \{ \theta, R_0 \} = \{ m_c, \sigma, R_0 \} \) in this case. Using Eq. (3) and Eq. (4), it is easily to get

\[
m_{pbh} = m_c \exp \left( -\frac{\sigma^2}{2} \right),
\]

\[
F(m) = \frac{m_c}{\sqrt{2\pi \sigma m^2}} \exp \left( -\frac{\sigma^2}{2} - \frac{\ln^2 (m/m_c)}{2\sigma^2} \right).
\]

Using data of 10 BBHs observed by LIGO/Virgo O1 and O2 observations and performing the hierarchical Bayesian inference, we obtain \( m_c = 8.9^{+7.8}_{-7.3}M_\odot \), \( \sigma = 0.91^{+0.50}_{-0.42} \), and \( R_0 = 55^{+42}_{-27} \text{Gpc}^{-3} \text{yr}^{-1} \). It is then easy to infer the abundance of PBHs in CDM to be \( f_{pbh} = 2.6^{+6.8}_{-1.4} \times 10^{-3} \) from the posterior distribution of local merger rate \( R_0 \). The results of local merger rate and abundance of PBHs are consistent with the previous estimations, confirming that the main components of CDM should not be made of stellar mass PBHs. The posteriors of parameters \( \{ \theta, R_0 \} = \{ m_c, \sigma, R_0 \} \) are shown in Fig. 3.

![Fig. 3. The marginalized one- and two-dimensional posterior distributions for parameters \( \{ \theta, R_0 \} = \{ m_c, \sigma, R_0 \} \) in the power-law mass function of PBHs, by using 10 BBH events from LIGO/Virgo O1 and O2 observing runs. The contours are at the 68% and 95% credible levels, respectively.](image)

![Fig. 4. The ratio of merger rate density from second-merger history to that from first-merger history, \( R_2(t_0, m_1, m_2)/R_1(t_0, m_1, m_2) \), by fixing \( \{ \theta, R_0 \} \) to their best-fit values. The correction to total merger rate density from second-merger history is larger as component masses are heavier. However, the ratio of merger rate from second-merger history to the one from first-merger history is negligible, namely \( R_2(t_0)/R_1(t_0) = 3.0\% \). This is because the major contribution to the merger rate are from the binaries with masses less than 50\( M_\odot \). Therefore the merger history effect can be safely ignored when estimating the merger rate of PBHs.](image)

V. CONCLUSION

In this paper, we use the public available GW data of 10 BBH events from LIGO/Virgo O1 and O2 observing runs to constrain the merger rate distribution of PBHs by accounting the effect of merger history. Considering two concrete mass functions of PBHs, a power-law PDF and a log-normal one, we demonstrate that the contribution of merger rate (density) from second-merger history to total merger rate (density) is subdominant, and hence the second-merger history effect can be safely ignored. As third-merger (and later merger) history will make even less contribution to the total merger rate (density), we conclude that the effect of merger history is subdominant and can be neglected when evaluating the merger rate of PBH binaries.

Furthermore, the results of local merger rate and abun-
dance of PBHs inferred from the updated analysis are consistent with the previous estimations, confirming that the main components of CDM should not be made of stellar mass PBHs.

ACKNOWLEDGMENTS

We would like to thank Zu-Cheng Chen and Lang Liu for useful conversations. This research has made use of data, software and/or web tools obtained from the Gravitational Wave Open Science Center [71] ([https://www.gw-openscience.org]), a service of LIGO Laboratory, the LIGO Scientific Collaboration and the Virgo Collaboration. LIGO is funded by the U.S. National Science Foundation. Virgo is funded by the French Centre National de Recherche Scientifique (CNRS), the Italian Istituto Nazionale della Fisica Nucleare (INFN) and the Dutch Nikhef, with contributions by Polish and Hungarian institutes.

[1] B. P. Abbott et al. (Virgo, LIGO Scientific), “Observation of Gravitational Waves from a Binary Black Hole Merger,” Phys. Rev. Lett. 116, 061102 (2016), arXiv:1602.03837 [gr-qc].
[2] B. P. Abbott et al. (Virgo, LIGO Scientific), “GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence,” Phys. Rev. Lett. 116, 241103 (2016), arXiv:1606.04855 [gr-qc].
[3] B. P. Abbott et al. (Virgo, LIGO Scientific), “Binary Black Hole Mergers in the first Advanced LIGO Observing Run,” Phys. Rev. X 6, 041015 (2016), arXiv:1606.04856 [gr-qc].
[4] B. P. Abbott et al. (VIRGO, LIGO Scientific), “GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2,” Phys. Rev. Lett. 118, 221101 (2017), arXiv:1706.01812 [gr-qc].
[5] B. P. Abbott et al. (Virgo, LIGO Scientific), “GW170608: Observation of a 19-solar-mass Binary Black Hole Coalescence,” Astrophys. J. 851, L35 (2017), arXiv:1711.05578 [astro-ph.HE].
[6] B. P. Abbott et al. (Virgo, LIGO Scientific), “GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence,” Phys. Rev. Lett. 119, 141101 (2017), arXiv:1709.09660 [gr-qc].
[7] B. P. Abbott et al. (LIGO Scientific, Virgo), “GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs,” (2018), arXiv:1811.12907 [astro-ph.HE].
[8] Simeon Bird, Ilias Cholis, Julian B. Muoz, Yacine Ali-Hamoud, Marc Kamionkowski, Ely D. Kovetz, Alvise Raccanelli, and Adam G. Riess, “Did LIGO detect dark matter?” Phys. Rev. Lett. 116, 201301 (2016), arXiv:1603.00464 [astro-ph.CO].
[9] Misao Sasaki, Teruaki Suyama, Takahiro Tanaka, and Shuichiro Yokoyama, “Primordial Black Hole Scenario for the Gravitational-Wave Event GW150914,” Phys. Rev. Lett. 117, 061101 (2016), arXiv:1603.08338 [astro-ph.CO].
[10] Zu-Cheng Chen and Qing-Guo Huang, “Merger Rate Distribution of Primordial-Black-Hole Binaries,” Astrophys. J. 864, 61 (2018), arXiv:1801.10327 [astro-ph.CO].
[11] Maya Fishbach, Daniel E. Holz, and Ben Farr, “Are LIGO’s Black Holes Made From Smaller Black Holes?” Astrophys. J. 840, L24 (2017), arXiv:1703.06869 [astro-ph.HE].
[12] Sebastien Clesse and Juan Garca-Bellido, “The clustering of massive Primordial Black Holes as Dark Matter: measuring their mass distribution with Advanced LIGO,” Phys. Dark Univ. 15, 142–147 (2017), arXiv:1603.05234 [astro-ph.CO].
[13] Fabio Antonini and Frederic A. Rasio, “Merging black hole binaries in galactic nuclei: implications for advanced-LIGO detections,” Astrophys. J. 831, 187 (2016), arXiv:1606.04889 [astro-ph.HE].
[14] Kohei Inayoshi, Ryosuke Hirai, Tomoya Kinugawa, and Kenta Hotokezaka, “Formation pathway of Population III coalescing binary black holes through stable mass transfer,” Mon. Not. Roy. Astron. Soc. 468, 5020–5032 (2017), arXiv:1701.04823 [astro-ph.HE].
[15] Yacine Ali-Hamoud, Ely D. Kovetz, and Marc Kamionkowski, “Merger rate of primordial black-hole binaries,” Phys. Rev. D96, 123523 (2017), arXiv:1709.06756 [astro-ph.CO].
[16] Rosalba Perna, Yi-Han Wang, Nathan Leigh, and Matteo Caniello, “On the Apparent Dichotomy Between the Masses of Black Holes Inferred via X-rays and via Gravitational Waves,” (2019), arXiv:1901.03345 [astro-ph.HE].
[17] Bradley J. Kavanagh, Daniele Gaggero, and Gianfranco Bertone, “Merger rate of a subdominant population of primordial black holes,” Phys. Rev. D98, 023536 (2018), arXiv:1805.09034 [astro-ph.CO].
[18] Carl L. Rodriguez, Meagan Morscher, Bharath Pattabiraman, Sourav Chatterjee, Carl-Johan Haster, and Frederic A. Rasio, “Binary Black Hole Mergers from Globular Clusters: Implications for Advanced LIGO,” Phys. Rev. Lett. 115, 051101 (2015), [Erratum: Phys. Rev. Lett.116,no.2,029901(2016)], arXiv:1505.00792 [astro-ph.HE].
[19] Carl L. Rodriguez, Sourav Chatterjee, and Frederic A. Rasio, “Binary Black Hole Mergers from Globular Clusters: Masses, Merger Rates, and the Impact of Stellar Evolution,” Phys. Rev. D93, 023535 (2016), arXiv:1602.02444 [astro-ph.HE].
[20] Dawoo Park, Chunghye Kim, Hyung Mok Lee, Yeong-Bok Bae, and Krzysztof Belczynski, “Black Hole Binaries Dynamically Formed in Globular Clusters,” Mon. Not. Roy.
Hole Abundance from the First Advanced LIGO Observation Run Using the Stochastic Gravitational-Wave Background,” Phys. Rev. Lett. 120, 191102 (2018), arXiv:1610.08725 [astro-ph.CO].

[52] B. P. Abbott et al. (LIGO Scientific, Virgo), “Search for sub-solar mass ultracompact binaries in Advanced LIGO’s first observing run,” Phys. Rev. Lett. 121, 231103 (2018), arXiv:1808.04770 [astro-ph.CO].

[53] Ryan Magee, Anne-Sylvie Deutsch, Phoebe McClincy, Chad Hanna, Christian Horst, Duncan Meacher, Cody Messick, Sarah Shandera, and Madeline Wade, “Methods for the detection of gravitational waves from subsolar mass ultracompact binaries,” Phys. Rev. D98, 103024 (2018), arXiv:1808.04772 [astro-ph.IM].

[54] Sai Wang, Takahiro Terada, and Kazunori Kohri, “Prospective constraints on the primordial black hole abundance from the stochastic gravitational-wave backgrounds produced by coalescing events and curvature perturbations,” (2019), arXiv:1903.05924 [astro-ph.CO].

[55] Zu-Cheng Chen, Fan Huang, and Qing-Guo Huang, “Stochastic Gravitational-Wave Background from Binary Black Holes and Baryon Neutron Stars,” Astrophys. J. 871, 97 (2019), arXiv:1809.10360 [gr-qc].

[56] Zu-Cheng Chen and Qing-Guo Huang, “Distinguishing Primordial Black Holes from Astrophysical Black Holes by Einstein Telescope and Cosmic Explorer,” (2019), arXiv:1904.02396 [astro-ph.CO].

[57] Takashi Nakamura, Misa Sasaki, Takahiro Tanaka, and Kip S. Thorne, “Gravitational waves from coalescing black hole MACBH binaries,” Astrophys. J. 487, L139–L142 (1997), arXiv:astro-ph/9708060.

[58] Martti Raidal, Christian Spethmann, Ville Vaskonen, and Hardi Veerme, “Formation and Evolution of Primordial Black Hole Binaries in the Early Universe,” (2018), arXiv:1812.01930 [astro-ph.CO].

[59] Martti Raidal, Ville Vaskonen, and Hardi Veerme, “Gravitational Waves from Primordial Black Hole Mergers,” JCAP 1709, 037 (2017), arXiv:1707.01480 [astro-ph.CO].

[60] Lang Liu, Zong-Kuan Guo, and Rong-Gen Cai, “Effects of the merger history on the merger rate of primordial black hole binaries,” (2019), arXiv:1901.07672 [astro-ph.CO].

[61] B. P. Abbott et al. (Virgo, LIGO Scientific), “The Rate of Binary Black Hole Mergers Inferred from Advanced LIGO Observations Surrounding GW150914,” Astrophys. J. 833, L1 (2016), arXiv:1602.03842 [astro-ph.HE].

[62] B. P. Abbott et al. (Virgo, LIGO Scientific), “Supplement: The Rate of Binary Black Hole Mergers Inferred from Advanced LIGO Observations Surrounding GW150914,” Astrophys. J. Suppl. 227, 14 (2016), arXiv:1606.03939 [astro-ph.HE].

[63] Daniel Wysocki, Jacob Lange, and Richard O. Schwab, “Reconstructing phenomenological distributions of compact binaries via gravitational wave observations,” (2018), arXiv:1805.06442 [gr-qc].

[64] Maya Fishbach, Daniel E. Holz, and Will M. Farr, “Does the Black Hole Merger Rate Evolve with Redshift?” Astrophys. J. 863, L41 (2018), [Astrophys. J. Lett. 863, L41 (2018)], arXiv:1805.10270 [astro-ph.HE].

[65] Ilya Mandel, Will M. Farr, and Jonathan R. Gair, “Extracting distribution parameters from multiple uncertain observations with selection biases,” (2018), arXiv:1809.02063 [physics.data-an].

[66] Eric Thrane and Colm Talbot, “An introduction to Bayesian inference in gravitational-wave astronomy: parameter estimation, model selection, and hierarchical models,” (2018), arXiv:1809.02293 [astro-ph.IM].

[67] https://www.gw-openscience.org/.

[68] Maya Fishbach and Daniel E. Holz, “Where Are LIGO’s Big Black Holes?” Astrophys. J. 851, L25 (2017), arXiv:1709.08584 [astro-ph.HE].

[69] Bernard J. Carr, “The Primordial black hole mass spectrum,” Astrophys. J. 201, 1–19 (1975).

[70] Alexandre Dolgov and Joseph Silk, “Baryon isocurvature fluctuations at small scales and baryonic dark matter,” Phys. Rev. D47, 4244–4255 (1993).

[71] Michele Vallisneri, Jonah Kanner, Roy Williams, Alan Weinstein, and Branson Stephens, “The LIGO Open Science Center,” Proceedings, 10th International LISA Symposium: Gainesville, Florida, USA, May 18-23, 2014. J. Phys. Conf. Ser. 610, 012021 (2015), arXiv:1410.4839 [gr-qc].