Characterizations of Connections for Positive Operators

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Abstract

An axiomatic theory of operator connections and operator means was investigated by Kubo and Ando in 1980. A connection is a binary operation for positive operators satisfying the monotonicity, the transformer inequality and the joint-continuity from above. In this paper, we show that the joint-continuity assumption can be relaxed to some conditions which are weaker than the separate-continuity. This provides an easier way for checking whether a given binary operation is a connection. Various axiomatic characterizations of connections are obtained. We show that the concavity is an important property of a connection by showing that the monotonicity can be replaced by the concavity or the midpoint concavity. Each operator connection induces a unique scalar connection. Moreover, there is an affine order isomorphism between connections and induced connections. This gives a natural viewpoint to define any named means.

Keywords: operator connection, operator mean, operator monotone function

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1. Introduction

Throughout, let $\mathcal{H}$ denote an infinite-dimensional Hilbert space. Let $B(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$ and $B(\mathcal{H})^+$ its positive cone. Equip $B(\mathcal{H})$ with the usual positive semidefinite ordering. Unless otherwise stated, any limit in $B(\mathcal{H})$ is taken in the strong-operator topology.

The concept of means is one of the most familiar concepts in mathematics. It is proved to be a powerful tool from theoretical as well as practical points of view. The theory of scalar means was developed since the ancient Greek by the Pythagoreans (via the method of proportions, see [17]) until the last century by many famous mathematicians. The theory of connections and means for matrices and operators started when the concept of parallel sum was introduced.

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in [1] for analyzing electrical networks. The parallel sum of two positive definite matrices (or invertible positive operators) $A$ and $B$ is defined by

$$(A, B) \mapsto (A^{-1} + B^{-1})^{-1}.$$ 

For general positive operators, use continuity:

$$(A, B) \mapsto \lim_{\epsilon \downarrow 0} (A^{-1} + \epsilon B)^{-1}, \quad A_{\epsilon} \equiv A + \epsilon I, B_{\epsilon} \equiv B + \epsilon I.$$ 

The harmonic mean, denoted by $!$, for positive operators is the twice parallel sum. The geometric mean of two positive semidefinite matrices (or positive operators) is defined and studied in [3]:

$$A \# B = \lim_{\epsilon \downarrow 0} A_{\epsilon}^{1/2} (A_{\epsilon}^{-1/2} B_{\epsilon} A_{\epsilon}^{-1/2})^{1/2} A_{\epsilon}^{1/2}$$

where $A_{\epsilon} \equiv A + \epsilon I, B_{\epsilon} \equiv B + \epsilon I$. In [3], geometric means and harmonic means played crucial roles in the study of concavity and monotonicity of many interesting maps between matrix spaces. Another important mean in mathematics, namely the power mean, was considered in [4].

A study of operator means in an abstract way was given by Kubo and Ando [13]. Let $\mathcal{K}$ be a Hilbert space, here we do not assume that $\dim \mathcal{K} = \infty$. A connection is a binary operation $\sigma$ on $B(\mathcal{K})^+$ such that for all positive operators $A, B, C, D$:

(M1) **monotonicity:** $A \leq C, B \leq D \implies A \sigma B \leq C \sigma D$

(M2) **transformer inequality:** $C (A \sigma B) C \leq (CAC) \sigma (CBC)$

(M3) **joint-continuity from above:** for $A_n, B_n \in B(\mathcal{K})^+$, if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \sigma B_n \downarrow A \sigma B$.

Typical examples of connection are the sum $(A, B) \mapsto A + B$ and the parallel sum. A mean is a connection $\sigma$ such that $I \sigma I = I$. The followings are examples of means in practical usage:

- **$t$-weighted arithmetic means:** $A \wedge_t B = (1 - t)A + tB$
- **$t$-weighted geometric means:** $A \#_t B = A^{1/2}(A^{-1/2}B A^{-1/2})^t A^{1/2}$
- **$t$-weighted harmonic means:** $A \dagger_t B = [(1 - t)A^{-1} + tB^{-1}]^{-1}$
- **logarithmic mean:** $(A, B) \mapsto A^{1/2} f(A^{-1/2}B A^{-1/2})A^{1/2}$ where $f(x) = (x - 1)/\log x$.

The theory of operator monotone functions plays a crucial role in Kubo-Ando theory of connections and means. A continuous real-valued function $f$ on an interval $I$ is called an operator monotone function if one of the following equivalent conditions holds:

(i) $A \leq B \implies f(A) \leq f(B)$ for all Hermitian matrices $A, B$ of all orders whose spectrums are contained in $I$;
(ii) $A \leq B \implies f(A) \leq f(B)$ for all Hermitian operators $A, B \in B(\mathcal{H})$ whose spectra are contained in $I$ and for some infinite-dimensional Hilbert space $\mathcal{H}$;

(iii) $A \leq B \implies f(A) \leq f(B)$ for all Hermitian operators $A, B \in B(\mathcal{K})$ whose spectra are contained in $I$ and for all Hilbert spaces $\mathcal{K}$.

This concept was introduced in [15]; see also [5, 11, 12]. Denote by $OM(\mathbb{R}^+)$ the set of operator monotone functions from $\mathbb{R}^+ = [0, \infty)$ to itself.

In [13], a connection $\sigma$ on $B(\mathcal{H})^+$ can be characterized as follows:

- There is an $f \in OM(\mathbb{R}^+)$ satisfying
  $$f(x)I = I \sigma(xI), \quad x \in \mathbb{R}^+. \quad (1)$$

- There is an $f \in OM(\mathbb{R}^+)$ such that
  $$A \sigma B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}, \quad A, B > 0. \quad (2)$$

- There is a finite Borel measure $\mu$ on $[0, \infty]$ such that
  $$A \sigma B = \alpha A + \beta B + \int_{(0, \infty)} \frac{\lambda + 1}{2\lambda} \{ (\lambda A) \backslash B \} d\mu(\lambda) \quad (3)$$

  where the integral is taken in the sense of Bochner, $\alpha = \mu(\{0\})$ and $\beta = \mu(\{\infty\})$.

In fact, the functions $f$ in (1) and (2) are unique and the same. We call $f$ the representing function of $\sigma$. From the integral representation (3), every connection $\sigma$ is concave in the sense that

$$\sigma(tA + (1-t)B) \geq t(\sigma(A) + (1-t)\sigma(B)) \quad (4)$$

for all $A, B \geq 0$ and $t \in (0, 1)$. Moreover, the map $\sigma \mapsto f$ is an affine order-isomorphism between the set of connections and $OM(\mathbb{R}^+)$. Here, the order-isomorphism means that when $\sigma_1 \mapsto f_i$ for $i = 1, 2$, $A \sigma_1 B \leq A \sigma_2 B$ for all $A, B \in B(\mathcal{H})^+$ if and only if $f_1(x) \leq f_2(x)$ for all $x \in \mathbb{R}^+$.

The mean theoretic approach has various applications. The concept of means can be used to obtain the monotonicity, concavity and convexity of interesting maps between matrix algebras or operator algebras (see the original idea in [3]). The fact that the map $f \mapsto \sigma$ is an order isomorphism can be used to obtain simple proofs of operator inequalities concerning means. For example, the arithmetic–geometric–logarithmic–harmonic means inequalities are obtained from applying this order isomorphism to the scalar inequalities

$$\frac{2x}{1+x} \leq x^{1/2} \leq \frac{x-1}{\log x} \leq \frac{1+x}{2}, \quad x > 0, x \neq 1.$$

The concavity of general connections serves simple proofs of operator versions of Hölder inequality, Cauchy-Schwarz inequality, Minkowski’s inequality, Aczel’s
inequality, Popoviciu’s inequality and Bellman’s inequality (e.g. [16]). The fa-
mous Furuta’s inequality and its generalizations are obtained from axiomatic
properties of connections (e.g. [8, 9, 10]). Kubo-Ando theory can be applied
to matrix and operator equations since harmonic and geometric means can be
viewed as solutions of certain operator equations. See some examples of applica-
tions in [2, 14]. It also plays an important role in noncommutative information
theory. A relative operator entropy was defined in [6] to be the connection cor-
responding to the operator monotone function \( x \mapsto \log x \). See more information
in [5, Chapter IV] and therein references.

Kubo-Ando definition of a connection is a binary operation satisfying axioms
(M1), (M2) and (M3). In this work, we show that some of the axioms can be
weakened. Moreover, we provide alternative sets of axioms involving concavity
property. This gives a direct tool for studying operator inequalities.

Consider the following axioms:

(M3') for each \( A, X \in B(\mathcal{H})^+ \), if \( A_n \downarrow A \), then \( A_n \sigma X \downarrow A \sigma X \) and \( I \sigma A_n \downarrow I \sigma A \);

(M3'') for each \( A, X \in B(\mathcal{H})^+ \), if \( A_n \downarrow A \), then \( X \sigma A_n \downarrow X \sigma A \) and \( A_n \sigma I \downarrow A \sigma I \);

(M4) concavity: \( (tA + (1-t)B) \sigma (tA' + (1-t)B') \geq t(A \sigma A') + (1-t)(B \sigma B') \)
for \( t \in (0,1) \);

(M4') midpoint concavity: \( (A + B)/2 \sigma (A' + B')/2 \geq [(A \sigma A') + (B \sigma B')]/2 \).

Note that condition (M3') is one of the axiomatic properties of abstract
solidarity introduced in [7]. The definition of a connection can also be relaxed
as follows:

Theorem 1.1. Let \( \sigma \) be a binary operation on \( B(\mathcal{H})^+ \). Then the following
statements are equivalent:

(i) \( \sigma \) is a connection;
(ii) \( \sigma \) satisfies (M1), (M2) and (M3');
(iii) \( \sigma \) satisfies (M1), (M2) and (M3'').

Condition (M3') or (M3'') is clearly weaker, and easier to verify, than the
joint-continuity assumption (M3) in Kubo-Ando definition.

A connection can be axiomatically defined as follows. Fix the transformer
inequality (M2). We can freely replace the monotonicity (M1) by the concavity
(M4) or the mid-point concavity (M4'). At the same time, we can use (M3') or
(M3'') instead of the joint-continuity (M3).

Theorem 1.2. Let \( \sigma \) be a binary operation on \( B(\mathcal{H})^+ \) satisfying (M2). Then
the following statements are equivalent:

(1) \( \sigma \) is a connection;
(2) \( \sigma \) satisfies (M4) and (M3);
(3) \( \sigma \) satisfies (M4) and (M3');
(4) \( \sigma \) satisfies (M4) and (M3'');
(5) \( \sigma \) satisfies (M4') and (M3);
(6) \( \sigma \) satisfies (M4') and (M3'');
(7) \( \sigma \) satisfies (M4') and (M3'').

This theorem gives different viewpoints of Kubo-Ando connections. It shows the importance of the concavity property of a connection. Moreover, it asserts that the concepts of monotonicity and concavity are equivalent under suitable conditions.

Theorem 1.1 and Theorem 1.2 are established in Sections 2 and 3, respectively. Each operator connection induces a unique scalar connection on \( \mathbb{R}^+ \). Furthermore, there is an affine order isomorphism between connections and induced connections. This gives a natural way to define any named mean. For example, a geometric mean on \( B(\mathcal{H})^+ \) is the mean on \( B(\mathcal{H})^+ \) that corresponds to the usual geometric mean on \( \mathbb{R}^+ \). The correspondence between connections and induced connections will be discussed in details in Section 4.

2. Relaxing the definition of connection

In this section, we show that the joint-continuity assumption in the definition of connection can be relaxedly defined by (M3') or (M3''), which are weaker than (M3). Let \( \sigma : B(\mathcal{H})^+ \times B(\mathcal{H})^+ \to B(\mathcal{H})^+ \) be a binary operation.

**Remark 2.1.** The transformer inequality (M2) implies

- **congruence invariance**: \( C(A\sigma B)C = (CAC)\sigma(CBC) \) for \( A, B \geq 0 \) and \( C > 0 \);
- **positive homogeneity**: \( \alpha(A\sigma B) = (\alpha A)\sigma(\alpha B) \) for \( A, B \geq 0 \) and \( \alpha \in (0, \infty) \).

We say that \( \sigma \) satisfies property (P) if

\[
P(A\sigma B) = (PA)\sigma(PB) = (A\sigma B)P
\]

for any projection \( P \in B(\mathcal{H})^+ \) commuting with \( A, B \in B(\mathcal{H})^+ \). A function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) and \( \sigma \) are said to satisfy property (F) if for any \( x \in \mathbb{R}^+ \),

\[
f(x)I = I\sigma(xI).
\]

**Lemma 2.2.** Let \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) be an increasing function. If \( \sigma \) satisfies the positive homogeneity, (M3') and (F), then \( f \) is continuous.

**Proof.** To show that \( f \) is right continuous at each \( x \in \mathbb{R}^+ \), consider a sequence \( x_n \in \mathbb{R}^+ \) such that \( x_n \downarrow x \). Then by (M3')

\[
f(x_n)I = I\sigma(x_nI) \downarrow I\sigma(xI) = f(x)I,
\]
i.e. $f(x_n) \downarrow f(x)$. To show that $f$ is left continuous at each $x > 0$, consider a sequence $x_n > 0$ such that $x_n$ is increasing and $x_n \to x$. Then $x_n^{-1} \downarrow x^{-1}$ and

$$\lim x_n^{-1}f(x_n) = \lim x_n^{-1}(I \sigma x_n I) = \lim (x_n^{-1}I) \sigma I = (x^{-1}I) \sigma I = x^{-1}(I \sigma x)f(x)$$

That is $x \mapsto x^{-1}f(x)$ is left continuous and so is $f$. □

**Lemma 2.3.** Let $\sigma$ be a binary operation on $B(\mathcal{H})^+$ satisfying (M3') and (P). If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function where $\sigma$ and $f$ satisfy (F), then $f(A) = I \sigma A$ for any $A \in B(\mathcal{H})^+$.

**Proof.** First consider $A \in B(\mathcal{H})^+$ in the form $\sum_{i=1}^m \lambda_i P_i$ where $\{P_i\}_{i=1}^m$ is an orthogonal family of projections with sum $I$ and $\lambda_i > 0$ for all $i = 1, \ldots, m$. Since each $P_i$ commutes with $A$, we have by the property (P) that

$$I \sigma A = \sum P_i (I \sigma A) = \sum P_i \sigma P_i A = \sum P_i \sigma \lambda_i P_i = \sum f(\lambda_i) P_i = f(A).$$

Now, consider $A \in B(\mathcal{H})^+$. Then there is a sequence $A_n$ of strictly positive operators in the above form such that $A_n \downarrow A$. Then $I \sigma A_n \downarrow I \sigma A$ and $f(A_n)$ converges strongly to $f(A)$. Hence, $I \sigma A = \lim I \sigma A_n = \lim f(A_n) = f(A)$. □

Denote by $BO(M1, M2, M3')$ the set of binary operations satisfying axioms (M1), (M2) and (M3'). Similar notations are applied for other axioms.

**Proof of Theorem 1.1:** We have known that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (i). Let $\sigma \in BO(M1, M2, M3')$. As in [13], the conditions (M1) and (M2) imply that $\sigma$ satisfies (P) and there is a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ subject to (F). If $0 \leq x_1 \leq x_2$, then by (M1)

$$f(x_1) I = I \sigma (x_1 I) \leq I \sigma (x_2 I) = f(x_2) I,$$

i.e. $f(x_1) \leq f(x_2)$. Then the assumption (M3') is sufficient to guarantee that $f$ is continuous by Lemma 2.2. Lemma 2.3 results in $f(A) = I\sigma A$ for all $A \geq 0$. Now, (M1) and the fact that $\dim \mathcal{H} = \infty$ yield that $f$ is operator monotone. The uniqueness of $f$ is obvious. Thus, we establish a well-defined map $\sigma \in BO(M1, M2, M3') \mapsto f \in OM(\mathbb{R}^+)$ such that $\sigma$ and $f$ satisfy (F).

Now, given $f \in OM(\mathbb{R}^+)$, we construct $\sigma$ as in [13]:

$$A \sigma B = \alpha A + \beta B + \int_{(0, \infty)} \frac{\lambda + 1}{2\lambda} \{(\lambda A)! B\} d\mu(\lambda) \quad (5)$$

where $\mu$ is the corresponding measure of $f$, $\alpha = \mu(\{0\})$ and $\beta = \mu(\{\infty\})$. Then $\sigma$ satisfies (M1), (M2), (M3') and (F). This shows that the map $\sigma \mapsto f$ is surjective.
To show the injectivity of this map, let $\sigma_1, \sigma_2 \in BO(M_1, M_2, M_3')$ be such that $\sigma_i \mapsto f$ where, for each $t \geq 0$,

$$I \sigma_i(xI) = f(x)I, \quad i = 1, 2.$$  

Since $\sigma_i$ satisfies the property (P), we have $I \sigma_i A = f(A)$ for $A \geq 0$ by Lemma 2.3. Since $\sigma_i$ satisfies the congruence invariance, we have that for $A > 0$ and $B \geq 0$,

$$A \sigma_i B = A^{1/2}(I \sigma_i A^{-1/2} B A^{-1/2}) A^{1/2} = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}.$$  

By the limiting argument, we see that (M3') implies $\sigma_1 = \sigma_2$.

Thus there is a bijection between $OM(\mathbb{R}^+)$ and $BO(M_1, M_2, M_3')$. Every element in $BO(M_1, M_2, M_3')$ has an integral representation (5). Since the harmonic mean possesses (M3), so is any element in $BO(M_1, M_2, M_3')$.

(iii) $\Rightarrow$ (i). We can develop the analogous results when (M3') is replaced by (M3'') by swapping “left” and “right.” Indeed, we establish a one-to-one correspondence between $\sigma \in BO(M_1, M_2, M_3'')$ and $g \in OM(\mathbb{R}^+)$, where

$$g(x)I = (xI) \sigma I, \quad x \in \mathbb{R}^+.$$  

Here, (6) plays the same role as property (F) in the proof of (ii) $\Rightarrow$ (i).

**Remark 2.4.** The representing function of a connection $\sigma$ in Kubo-Ando theory can be shown to be the function $f \in OM(\mathbb{R}^+)$ satisfying one of the following equivalent conditions for each $x \in \mathbb{R}^+$:

(i) $f(x)I = I \sigma (xI)$;
(ii) $f(x)P = P \sigma (xP)$ for all projections $P$;
(iii) $f(x)A = A \sigma (xA)$ for all $A > 0$;
(iv) $f(x)A = A \sigma (xA)$ for all $A \geq 0$.

There is also a one-to-one correspondence between connections $\sigma$ and operator monotone functions $g$ on $\mathbb{R}^+$ satisfying (6). Note that $g$ is the representing function of the transpose of $\sigma$. Indeed, the relationship between the representing function $f$ and the function $g$ in (6) is given by

$$g(x) = xf(1/x).$$

3. Characterizations of connections  

In this section, we give various characterizations of connections. In order to prove Theorem 1.2, we need the following lemmas.

**Lemma 3.1.** If $\sigma \in BO(M_2, M_4')$, then for each $A, B, C, D \geq 0$,

(i) $(A \sigma B) + (C \sigma D) \leq (A + C) \sigma (B + D)$;
(ii) $A \leq B$ implies $A \sigma I \leq B \sigma I$ and $I \sigma A \leq I \sigma B$. 

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Proof. As in Remark 2.1, (M2) implies the positive homogeneity. The fact (i) follows from the midpoint concavity (M4′) and positive homogeneity. If \( A \preceq B \), then by (i),
\[
I \sigma B = (I + 0) \sigma (A + B - A) \geq (I \sigma A) + (0 \sigma (B - A)) \geq I \sigma A.
\]
Similarly, \( B \sigma I \geq A \sigma I \).

Lemma 3.2. If \( \sigma \in BO(M_2, M_4′) \), then \( \sigma \) satisfies (P).

Proof. Let \( P \) be a projection such that \( AP = PA \) and \( BP = PB \). We have
\[
A = PAP + (I - P)A(I - P) \quad \text{and} \quad B = PBP + (I - P)B(I - P).
\]
Then by Lemma 3.1 (i) and (M2)
\[
A \sigma B \geq (PAP \sigma PBP) + ((I - P)A(I - P) \sigma (I - P)B(I - P)) \geq P(A \sigma B)P + (I - P)(A \sigma B)(I - P).
\]
Consider \( C = A \sigma B - P(A \sigma B)P - (I - P)(A \sigma B)(I - P) \). Then \( C \) is positive and \( PCP = 0 = (I - P)C(I - P) \), which implies \( C^{1/2}P = 0 = C^{1/2}(I - P) \). Hence, \( CP = 0 = C(I - P) \) and \( C = 0 \), meaning that
\[
A \sigma B = P(A \sigma B)P + (I - P)(A \sigma B)(I - P).
\]
It follows that \( P(A \sigma B) = P(A \sigma B)P = (A \sigma B)P \). Furthermore, inequalities (7) and (8) become equalities, which implies \( P(A \sigma B)P = (PAP) \sigma (PBP) = (PA \sigma PB) \).

Lemma 3.3. If \( \sigma \in BO(M_2, M_4′) \), then there exists a unique binary operation \( \tilde{\sigma} \) on \( \mathbb{R}^+ \) subject to the same properties and
\[
(xI) \sigma (yI) = (x \tilde{\sigma} y)I, \quad x, y \in \mathbb{R}^+.
\]

Proof. Note that any projection on \( \mathcal{H} \) commutes with \( xI \) and \( yI \) for any \( x, y \in \mathbb{R}^+ \). By Lemma 3.2, \((xI) \sigma (yI)\) commutes with any projection in \( B(\mathcal{H}) \). The spectral theorem implies that \((xI) \sigma (yI)\) is a nonnegative multiple of identity, i.e. there exists a \( k \in \mathbb{R}^+ \) such that \((xI) \sigma (yI) = kI\). If there is a \( k' \in \mathbb{R}^+ \) such that \((xI) \sigma (yI) = k'I\), then \( k' = k \). Hence, each connection \( \sigma \) on \( B(\mathcal{H}^+) \) induces a unique binary operation \( \tilde{\sigma} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying (9). It is routine to check that \( \tilde{\sigma} \) satisfies (M2) and (M4′).

Proposition 3.4. If \( \sigma \in BO(M_2, M_3', M_4') \), then there exists a unique \( f \in OM(\mathbb{R}^+) \) satisfying (F). In fact, \( f(x) = 1 \tilde{\sigma} x \) for \( x \in \mathbb{R}^+ \).

Proof. Define \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) by \( x \mapsto 1 \tilde{\sigma} x \). The function \( f \) is well-defined, unique and satisfying (F) by Lemma 3.3. If \( 0 \leq x_1 \leq x_2 \), then Lemma 3.1 (ii) implies
\[
f(x_1)I = I \sigma (x_1I) \leq I \sigma (x_2I) = f(x_2)I,
\]
i.e. \( f(x_1) \leq f(x_2) \). The continuity of \( f \) is assured by Lemma 2.2. Then Lemma 2.3 implies \( f(A) = I \sigma A \) for all \( A \succeq 0 \). If \( A, B \in B(\mathcal{H})^+ \) are such that \( A \preceq B \), then \( f(A) = I \sigma A \preceq I \sigma B = f(B) \), again by Lemma 3.1 (ii). Since \( \mathcal{H} \) is infinite dimensional, \( f \) is operator monotone.
Proof of Theorem 1.2: We have known that (1) implies (2)-(7). It suffices to show that (6) implies (1). Assume that \( \sigma \in BO(M_2, M_3', M_4') \). Our aim is to construct a bijection between \( BO(M_2, M_3', M_4') \) and \( OM(\mathbb{R}^+) \). Proposition 3.4 assures that the map \( \sigma \in BO(M_2, M_3', M_4') \mapsto f \in OM(\mathbb{R}^+) \) where \( \sigma \) and \( f \) satisfy the property (F) is well-defined. This map is surjective via the same method as the construction in Theorem 1.1. The injectivity of this map can be proved by using the same argument as in the proof of Theorem 1.1. Here, the property (P) of \( \sigma \in BO(M_2, M_3', M_4') \) is fulfilled by Lemma 3.2. Hence, we are allowed to consider only the binary operations constructed from operator monotone functions on \( \mathbb{R}^+ \). Thus, \( \sigma \) takes the form (5). By passing the properties (M1) and (M3) of the harmonic mean through the integral representation, \( \sigma \) also satisfies those properties. \( \square \)

4. Induced connections

In this section, we consider the relationship between connections and their induced connections.

Each connection \( \sigma \) on \( B(\mathcal{H})^+ \) induces a unique connection \( \bar{\sigma} \) on \( \mathbb{R}^+ = B(\mathbb{C})^+ \) satisfying

\[
(x \bar{\sigma} y)I = (xI) \sigma (yI), \quad x, y \in \mathbb{R}^+.
\]

We call \( \bar{\sigma} \) the induced connection of \( \sigma \). Using the positive homogeneity of \( \sigma \) and Lemma 2.3, we have

\[
x \bar{\sigma} y = xf(y/x) = xf(y/x), \quad x, y > 0.
\]  \( (10) \)

**Proposition 4.1.** Each connection \( \sigma \) on \( B(\mathcal{H})^+ \) gives rise to an operator monotone function \( x \mapsto 1 \bar{\sigma} x \) on \( \mathbb{R}^+ \). Moreover, any operator monotone function on \( \mathbb{R}^+ \) arises in this form.

**Proof.** The correspondence between connections on \( B(\mathcal{H})^+ \) and operator monotone functions on \( \mathbb{R}^+ \) allows us to consider only operator monotone functions on \( \mathbb{R}^+ \) constructing from connections on \( B(\mathcal{H})^+ \). Proposition 3.4 shows that these operator monotone functions take the form \( x \mapsto 1 \bar{\sigma} x \). \( \square \)

**Proposition 4.2.** Each binary operation \( \sigma \) on the center

\[
\mathbb{R}^+ I = \{ kI : k \in \mathbb{R}^+ \}
\]

of \( B(\mathcal{H})^+ \) can be uniquely extended to a connection on \( B(\mathcal{H})^+ \).

**Proof.** Let \( \tau, \eta \) be two connections on \( B(\mathcal{H})^+ \) which are extensions of \( \sigma \). Let \( f, g \) be representing functions of \( \tau, \eta \), respectively. Then for \( x \geq 0 \),

\[
f(x)I = I \tau (xI) = I \sigma (xI) = I \eta (xI) = g(x)I,
\]

i.e. \( f = g \). Hence, \( \tau = \eta \). \( \square \)
Theorem 4.3. The map \( \sigma \mapsto \tilde{\sigma} \) from the set of connections on \( B(\mathcal{H})^+ \) to the set of connections on \( \mathbb{R}^+ \) such that

\[
(x \tilde{\sigma} y)I = (xI) \sigma (yI), \quad x, y \in \mathbb{R}^+,
\]

is an affine order isomorphism.

Proof. To show that this map is surjective, let \( \eta \) be a connection on \( \mathbb{R}^+ \). Define a binary operation \( \tilde{\sigma} \) on the center \( \mathbb{R}^+I \) of \( B(\mathcal{H})^+ \) by

\[
(xI) \tilde{\sigma} (yI) = (x \eta y)I, \quad x, y \in \mathbb{R}^+.
\]

Extend it to a connection on \( B(\mathcal{H})^+ \) by Proposition 4.2.

Now, suppose \( \sigma_i \mapsto \sigma \) for \( i = 1, 2 \). Let \( f_i \) be the representing function of \( \sigma_i \) for \( i = 1, 2 \). Then for \( x \in \mathbb{R}^+ \)

\[
f_1(x)I = I \sigma_1 (xI) = (1 \eta x)I = I \sigma_2 (xI) = f_2(x)I,
\]

i.e. \( f_1 = f_2 \) and \( \sigma_1 = \sigma_2 \). It is straightforward to check that this map is affine (i.e. it preserves nonnegative linear combinations) and order-preserving.

Corollary 4.4. A connection on \( B(\mathcal{H})^+ \) and its induced connection have the same representing function and the same representing measure. More precisely, given an operator monotone function

\[
f(x) = \alpha + \beta x + \int_{(0, \infty)} \frac{\lambda + 1}{2\lambda} (\lambda^! x) d\mu(\lambda),
\]

one has, for each \( A, B \in B(\mathcal{H})^+ \) and \( x, y \in \mathbb{R}^+ \),

\[
A \tilde{\sigma} B = \alpha A + \beta B + \int_{(0, \infty)} \frac{\lambda + 1}{2\lambda} (\lambda A^! B) d\mu(\lambda),
\]

\[
x \tilde{\sigma} y = \alpha x + \beta y + \int_{(0, \infty)} \frac{\lambda + 1}{2\lambda} (\lambda x^! y) d\mu(\lambda).
\]

Proof. Let \( \sigma \) be a connection and \( \tilde{\sigma} \) its induced connection. Then the correspondences between connections, induced connections, finite Borel measures and operator monotone functions imply that \( \sigma \) and \( \tilde{\sigma} \) have the same representing function and the same representing measure. Hence, \( \sigma \) has the integral representation (13). The formula (14) of \( \tilde{\sigma} \) can be computed by using the formula (10). The direct computation shows that the induced connection of the harmonic mean on \( B(\mathcal{H})^+ \) is the scalar harmonic mean.

Corollary 4.5. A connection is a mean if and only if the induced connection is a mean on \( \mathbb{R}^+ \).

Proof. Use Corollary 4.4 and the fact that a connection is a mean if and only if its representing function is normalized.
It is easy to see that the class of means on $B(\mathcal{H})^+$ becomes a convex set.

**Corollary 4.6.** The map $\sigma \mapsto \tilde{\sigma}$ establishes an affine order isomorphism between operator means on $B(\mathcal{H})^+$ and scalar means on $\mathbb{R}^+$.

**Proof.** It is an immediate consequence of Theorem 4.3 and Corollary 4.5. □

**Remark 4.7.** According to Corollary 4.6, we can naturally define any named means on $B(\mathcal{H})^+$ to be the corresponding means on $\mathbb{R}^+$.

**Example 4.8.** For each $p \in [-1, 1]$ and $\alpha \in [0, 1]$, the map

$$x \mapsto [(1 - \alpha) + \alpha x^p]^{1/p}$$

is an operator monotone function on $\mathbb{R}^+$ (when $p = 0$, it is understood that we take limit as $p$ approaches 0). Hence, it produces a mean on $\mathbb{R}^+$, given by

$$x \#_{p,\alpha} y = [(1 - \alpha)x^p + \alpha y^p]^{1/p}.$$  

This is the formula of the *quasi-arithmetic power mean* with parameter $(p, \alpha)$. Now, we define the quasi-arithmetic power mean for positive operators $A, B$ on $\mathcal{H}$ to be the mean on $B(\mathcal{H})^+$ corresponds to this scalar mean. The class of quasi-arithmetic power means contains many kinds of means: The mean $\#_{1,\alpha}$ is the $\alpha$-weighed arithmetic mean. The case $\#_{0,\alpha}$ is the $\alpha$-weighed geometric mean. The case $\#_{-1,\alpha}$ is the $\alpha$-weighed harmonic mean. The mean $\#_{p,1/2}$ is the power mean of order $p$.

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