A RIEMANN-HILBERT PROBLEM FOR SKEW-ORTHOGONAL POLYNOMIALS

VIRGIL U. PIERCE

Abstract. We find a local \((d+1) \times (d+1)\) Riemann-Hilbert problem characterizing the skew-orthogonal polynomials associated to the partition function of the Gaussian Orthogonal Ensemble of random matrices with a potential function of degree \(d\). Our Riemann-Hilbert problem is similar to a local \(d \times d\) Riemann-Hilbert problem found by Kuijlaars and McLaughlin characterizing the bi-orthogonal polynomials. This gives more motivation for finding methods to compute asymptotics of high order Riemann-Hilbert problems, and brings us closer to finding asymptotics of the skew-orthogonal polynomials.

Keywords: Skew-orthogonal polynomials; Riemann-Hilbert problems

1. Introduction

The partition function of random matrices for the Gaussian Unitary Ensemble (GUE) is associated with a family of orthogonal polynomials \(P_j(x)\) given by the conditions

\[ \int_{\mathbb{R}} P_j(x)P_k(x)e^{-NV(x)}dx = h_j \delta_{jk}, \]

where \(V(x)\) is an even degree polynomial with positive leading coefficient, and \(P_j(x) = x^j + \mathcal{O}(x^{j-1})\) for \(x \to \infty\) \([7, 15, 25]\). Define the orthonormal polynomials \(\hat{P}_j(x) = P_j(x)/h_j^{1/2}\). The asymptotic expansion of the classical families of orthogonal polynomials were computed by \([30]\). The orthonormal polynomials satisfy a three term recursion relation that, under changes in \(V\), satisfies the Toda lattice hierarchy. They may be represented as Hankel determinants \([27]\).

To compute the orthogonal polynomials numerically it is helpful to use the following result: Define the moments

\[ m_i = \int_{\mathbb{R}} x^i e^{-NV(x)}dx. \]

Let \(M\) be the symmetric matrix \(M_{ij} = m_{i+j}\). Then factor \(M = QDR\) using the QR-algorithm. The columns of \(R^{-1}\) give the coefficients of the orthogonal polynomials.

One may show that the zeros of the orthogonal polynomials are real, interlace, and their distribution converges to a measure \(\mu(V)\) which is given as the solution to a minimization problem \([9]\).

The asymptotic expansions of \(P_N\) and \(P_{N-1}\) are computed by realizing them as the unique solutions of a \(2 \times 2\) Riemann-Hilbert problem (see \([9, 12, 13, 22]\)). These expansions were assembled and used to prove the existence of an asymptotic expansion of \(\frac{1}{N} \log (Z_N(V))\) for a cone of \(V\), in \([15]\).

Similar studies have been started regarding multi-matrix models and multi-orthogonal polynomials \([8]\). The simplest case is that of bi-orthogonal polynomials given by two families of polynomials \(\{P_k(x)\}\) and \(\{Q_k(x)\}\) satisfying

\[ \int_{\mathbb{R}^2} P_j(x)Q_k(y) \exp \left[ -N(V(x) + W(y)) - 2N\tau xy \right] dx dy = h_j \delta_{jk}, \]

where \(\tau\) is a non-zero coupling constant, \(V(x)\) and \(W(y)\) are polynomials of even degree with positive leading coefficient, \(P_j(x) = x^j + \mathcal{O}(x^{j-1})\) for \(x \to \infty\), and \(Q_k(y) = y^k + \mathcal{O}(y^{k-1})\) for \(y \to \infty\) \([14]\). The bi-orthogonal polynomials satisfy a recursion relation that, under changes in \(V\), satisfies the Full-Kostant Toda lattice hierarchy. One may compute them as determinants \([14]\). They may be characterized in terms of Riemann-Hilbert problems (see \([0, 14, 20, 24, 33]\)).

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Gaussian Symplectic Ensemble (GSE) (1.3). There is an ambiguity in conditions (1.4)-(1.7) in the sense that \( \tilde{W} \) the bi-orthogonal polynomials has been found by [5]. Another high order Riemann-Hilbert problem characterizing the case when \( \deg(P_j) \) is given by the conditions (1.6) and (1.7) where \( \delta_{kj} \) and \( \beta \) are real [14]. One finds numerically that the zeros of the bi-orthogonal polynomials are real [14].

One may show that the zeros of the bi-orthogonal polynomials are real [14]. One finds numerically that they interlace and the suspicion is that their distributions converge to a pair of measures.

The bi-orthogonal polynomials are solutions of a non-local \( 2 \times 2 \) Riemann-Hilbert problem (see [6, 14]). Recently a local \( d \times d \) Riemann-Hilbert problem that determines the bi-orthogonal polynomials was found in the case when \( \deg(W) = d \) by [21]. The advantage of this local problem is that there is reason to believe it is more amenable to asymptotic analysis, see [23]. Another high order Riemann-Hilbert problem characterizing the bi-orthogonal polynomials has been found by [3].

Define the skew-inner products

\[
\langle f, g \rangle_1 = \int_{\mathbb{R}^2} f(x)g(y)e^{\beta N(x-y)} dx dy,
\]

where

\[
\epsilon(x) = \begin{cases} 
-1 & : x < 0 \\
1 & : x > 0
\end{cases}
\]

(1.3)

\[
\langle f, g \rangle_4 = \int_{\mathbb{R}} [f(x)g'(x) - f'(x)g(x)] e^{\beta N(x)} dx.
\]

The partition functions of random matrices for the Gaussian Orthogonal Ensemble (GOE) (\( \beta = 1 \)) or Gaussian Symplectic Ensemble (GSE) (\( \beta = 4 \)) are associated with skew-orthonormal polynomials \( \{p_j(x)\} \) given by the conditions

\[
\langle p_{2k}(x), p_{2j}(y) \rangle_\beta = 0
\]

(1.4)

\[
\langle p_{2k}(x), p_{2j+1}(y) \rangle_\beta = h_j \delta_{kj}
\]

(1.5)

\[
\langle p_{2k+1}(x), p_{2j}(y) \rangle_\beta = -h_j \delta_{kj}
\]

(1.6)

\[
\langle p_{2k+1}(x), p_{2j+1}(y) \rangle_\beta = 0,
\]

(1.7)

where \( p_{2j}(x) = x^{2j} + O(x^{2j-1}) \), and \( p_{2j+1}(x) = x^{2j+1} + O(x^{2j-1}) \) for \( x \to \infty \) (see [2, 10, 11, 17, 19, 24, 26, 28, 31, 32, 34]). There is an ambiguity in conditions (1.4)-(1.7) in the sense that \( \hat{p}_{2j+1} = p_{2j+1} + q \beta \).

The skew-orthonormal polynomials to be \( \hat{p}_{2j}(x) = p_{2j}(x)/\langle p_{2j+1} \rangle_\beta^{1/2} \), and \( \hat{p}_{2j+1}(x) = p_{2j+1}(x)/\langle p_{2j+1} \rangle_\beta^{1/2} \).

The skew-orthonormal polynomials satisfy a recursion relation that, under changes in \( V \), satisfies the Pfaff lattice hierarchy. For example if \( V = V_0(x) + t_j x^j \) then it can be shown that the recursion relation is of the form

\[
x \tilde{p}(x) = L \tilde{p}(x),
\]

where \( \tilde{p}(x) = (\tilde{p}_0(x), \tilde{p}_1(x), \tilde{p}_2(x), \ldots)^T \) and

\[
L = \begin{pmatrix}
0 & 1 & 0 & 0 \\
b_1 & d_1 & a_1 & 0 \\
c_1 & c_1 & -d_1 & 1 \\
f_1 & c_2 & b_2 & d_2 \\
& & & & \\
& & & & \\
& & & & \\
& & & & 
\end{pmatrix}.
\]
There is a generalized Gaussian elimination algorithm which factors the skew-symmetric matrix $V$.

Define the skew-symmetric matrix of moments $J$,

$$
J = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}.
$$

(1.8)

Then $L$ satisfies the differential equation

$$
\frac{dL}{dt_j} = -\pi_k(L^j), L,
$$

where

$$
\pi_k(M) = M_+ - JM_+^T J + \frac{1}{2}(M_0 - JM_0^T J),
$$

and $M_\pm$ is projection onto the upper (resp. lower) $2 \times 2$ block triangular parts of $M$ and $M_0$ is projection onto the diagonal $2 \times 2$ blocks of $M$ \([2 \ 3 \ 4]\).

Define the Pfaffian of a skew-symmetric matrix $M$ by

$$
\det(M) = [\text{pf}(M)]^2.
$$

Define the skew-symmetric matrix of moments

$$
M_{ij} = \langle x^i, y^j \rangle_b.
$$

Then one finds Pfaffian formulas for the skew-orthogonal polynomials (see \([2 \ 3 \ 4]\)):

$$
p_{2j}(x) = \text{pf} \begin{pmatrix}
0 & M_{01} & M_{02} & \ldots & M_{02j} & 1 \\
-M_{01} & 0 & M_{12} & \ldots & M_{12j} & x \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-M_{02j} & -M_{12j} & -M_{22j} & \ldots & 0 & x^{2j} \\
-1 & -x & -x^2 & \ldots & -x^{2j} & 0
\end{pmatrix}
$$

and

$$
p_{2j+1}(x) = \text{pf} \begin{pmatrix}
0 & M_{01} & M_{02} & \ldots & 1 & M_{02j+1} \\
-M_{01} & 0 & M_{12} & \ldots & x & M_{12j+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -x & -x^2 & \ldots & 0 & x^{2j+1} \\
-M_{02j+1} & -M_{12j+1} & -M_{22j+1} & \ldots & -x^{2j+1} & 0
\end{pmatrix}.
$$

To compute the skew-orthogonal polynomials numerically it is helpful to use the following representation. There is a generalized Gaussian elimination algorithm which factors the skew-symmetric matrix $M$ as

$$
M = LD^{1/2}JD^{1/2}L^T,
$$

where $L$ is lower $2 \times 2$ block triangular matrix with the $2 \times 2$ block identity on the diagonal, $J$ is given in (1.8), and $D$ is a diagonal matrix with multiples of the $2 \times 2$ block identity on the diagonal. The rows of $L^{-1}$ give the coefficients of the skew-orthogonal polynomials. Numerical computation of skew-orthogonal polynomials with $V(x) = \frac{1}{2}x^2 + tx^4$ lead to the conjecture that the zeros of the even polynomials are real, interlace, and their distribution converges to some measure; and that the zeros of the odd polynomials are real, interlace, and their distribution converges to some measure.

The leading order of the asymptotics of the skew-orthogonal polynomials have been computed by \([16]\). In section \([2\) we will compute a local $(d + 1) \times (d + 1)$ Riemann-Hilbert problem, where $\deg(V) = d$ (similar to that of \([21\)) for the skew-orthogonal polynomials with respect to the skew-inner product \([1.2\). This Riemann-Hilbert problem may be tractable, at least in the sense that the bi-orthogonal one is, and allow one to rigorously compute the asymptotic expansion of the skew-orthogonal polynomials. The problem we find determines $p_{2k}$ uniquely but only finds $p_{2k+1}$ up to a multiple of $p_{2k}$. An equivalent Riemann-Hilbert problem for the skew-orthogonal polynomials with respect to the skew-inner product \([1.3\ has not been found.
2. **RIEMANN-HILBERT PROBLEM**

We will now work with the inner product (2.1) however outside of the random matrix context we may use the simpler expression

\[
\langle f, g \rangle_1 = \int f(x) g(y) \epsilon(x-y)e^{-V(x)-V(y)} dx dy,
\]

with no loss of generality. Assume that \( V \) is a polynomial of degree \( d \).

Define

\[
w_j(x) = \int y^j \epsilon(x-y)e^{-V(x)-V(y)} dy
\]

and

\[
\mathcal{W}(x) = e^{-2V(x)}.
\]

Let

\[
\langle f, g \rangle_2 = \int f(x) g(x) \mathcal{W}(x) dx.
\]

The skew inner product (2.1) is non-degenerate in the sense that the matrix \( M_{ij} = \langle x^i, y^j \rangle_1 \) is skew diagonalizable; that is it can be written as

\[
M = L J L^T
\]

where \( L \) is lower \( 2 \times 2 \) block triangular with non-zero multiples of the \( 2 \times 2 \) identity matrix on the diagonal, and \( J \) is given by (1.8). An equivalent condition is that the principle \( 2n \times 2n \) minors of \( M \) are non-singular.

The Pfaffian of these principle minors is proportional to a multi-integral of a positive function, hence is non-zero. This non-degeneracy gives the existence and uniqueness of the skew-orthogonal polynomials.

The family of skew-orthogonal polynomials with respect to (2.1) is characterized by the conditions: \( \langle p_{2k}(x), y^j \rangle_1 = 0 \) and \( \langle p_{2k+1}(x), y^j \rangle_1 = 0 \) for \( 0 \leq j \leq 2k-1 \), \( p_{2k}(x) = x^{2k} + O(x^{2k-1}) \), and \( p_{2k+1}(x) = x^{2k+1} + O(x^{2k-1}) \).

Define

\[
\pi_{j+d-1}(y) = \frac{d}{dy} \left( y^j e^{-V(y)} \right) e^{V(y)}.
\]

This function is a polynomial of degree \( j + d - 1 \) in \( y \). The fundamental theorem of calculus and the definition of \( \epsilon(x-y) \) implies that

\[
\langle f(x), \pi_{j+d-1}(y) \rangle_1 = 2 \langle f(x), x^j \rangle_2.
\]

Therefore we find that the \( 2k \) orthogonality conditions on \( p_{2k}(x) \) become: the \( d - 1 \) conditions

\[
\langle p_{2k}(x), y^j \rangle_1 = 0, \quad 0 \leq j \leq d - 2
\]

together with the \( 2k - d + 1 \) conditions

\[
\langle p_{2k}(x), x^j \rangle_2 = 0, \quad 0 \leq j \leq 2k - d.
\]

For \( p_{2k+1}(x) \) we find the same conditions (with the same ranges on \( j \)).

Consider a pair of Riemann-Hilbert problems: Find a \( (d + 1) \times (d + 1) \) matrix valued function \( Y(z) \) satisfying:

1. \( Y \) is analytic on \( \mathbb{C}/\mathbb{R} \).
2. \( Y \) has boundary values

\[
Y_\pm(x) = \lim_{\epsilon \to 0^\pm} Y(x + i\epsilon)
\]

for \( x \in \mathbb{R} \)
3. The boundary values of \( Y \) satisfy the matrix equation

\[
Y_+ = Y_- M
\]

where

\[
M = \begin{pmatrix}
1 & \mathcal{W}(x) & w_0(x) & \ldots & w_{d-2}(x) \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1
\end{pmatrix}.
\]
4. (a) In the Even problem $Y$ satisfies the asymptotic boundary value

$$\begin{pmatrix}
2k & z^{-2k+d-1} \\
\vdots & \ddots \\
2k & z^{-1}
\end{pmatrix}$$

as $|z| \to \infty$.

(b) In the Odd problem $Y$ satisfies the asymptotic boundary value

$$\begin{pmatrix}
2k+1 & z^{-2k+d-1} \\
\vdots & \ddots \\
2k+1 & z^{-1}
\end{pmatrix}$$

as $|z| \to \infty$.

Let

$$C(f)(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(x)}{x - z} \, dx$$

be the Cauchy Transform of $f(x)$.

We find two theorems:

**Theorem 2.1 (Even).** The Riemann-Hilbert Problem above with asymptotic condition (2.6) has a unique solution given by the $(d+1) \times (d+1)$ matrix

$$Y_{2k}(z) = \begin{pmatrix}
p_{2k}^{(0)} & C(p_{2k} W) & C(p_{2k} w_0) & \ldots & C(p_{2k} w_{d-2}) \\
p_{2k-1}^{(0)} & \ldots & \ldots \\
p_{2k-1}^{(1)} & \ldots & \ldots \\
p_{2k-1}^{(d-1)} & \ldots & \ldots \\
p_{2k-1} & \ldots & \ldots \\
\end{pmatrix}$$

where $p_{2k-1}^{(0)} = \alpha_k p_{2k-2}$ and $p_{2k-1}^{(m)}$, $m > 0$ are polynomials of degree at most $2k-1$ which will be specified in the proof. The other entries of $Y_{2k}(z)$ are Cauchy transforms of $p_{2k-1}^{(m)} W$, and $p_{2k-1}^{(m)} w_n$, $0 \leq n < d-1$.

**Theorem 2.2 (Odd).** The Riemann-Hilbert Problem above with asymptotic condition (2.7) has general solution given by the $(d+1) \times (d+1)$ matrix

$$Y_{2k+1}(z) = \begin{pmatrix}
p_{2k+1}^{(0)} & C(p_{2k+1} W) & C(p_{2k+1} w_0) & \ldots & C(p_{2k+1} w_{d-2}) \\
p_{2k} & \ldots & \ldots \\
p_{2k}^{(1)} & \ldots & \ldots \\
p_{2k}^{(d-1)} & \ldots & \ldots \\
p_{2k} & \ldots & \ldots \\
\end{pmatrix}$$

where $p_{2k+1} = p_{2k+1} + \alpha_k p_{2k}$, $p_{2k}^{(0)} = b_{k,0} p_{2k} + c_{k,0} p_{2k-2}$, and $p_{2k}^{(m)} = b_{k,m} p_{2k} + c_{k,m} p_{2k-1}^{(m)}$. The other entries of $Y_{2k+1}(z)$ are Cauchy transforms of $p_{2k}^{(m)} W$, and $p_{2k}^{(m)} w_n$, $0 \leq n < d-1$. In particular the general solution is determined up to a multiple of $p_{2k}$ in each row.

3. Proof of Theorems 2.1 and 2.2

Uniqueness of the solution in theorem 2.1 follows in the standard way, by checking that both the jump condition (2.3) and the asymptotic condition (2.6) have determinant one.

To prove that $Y_{2k}$ is a solution of the Riemann-Hilbert problem with asymptotic condition (2.6), one checks that:
• $Y_1$ must be an analytic function of degree $2k$, hence a polynomial of degree $2k$.
• $Y_2$ satisfies the jump condition
$$Y_{2,+} = Y_{11} W(x) + Y_{2,-}$$
across $\mathbb{R}$. The Plemelj formula implies that
$$Y_{2}(z) = C(Y_{11} W)(z).$$
For this function to have the asymptotics
$$Y_{2}(z) = O(z^{-2k+d-2})$$
the $2k - d + 1$ conditions in (2.4) must be satisfied.
• $Y_{1(j+3)}$ (for $0 < j < d - 2$) satisfies the jump condition
$$Y_{1(j+3),+} = Y_{11} w_j(x) + Y_{1(j+3),-}$$
across $\mathbb{R}$. The Plemelj formula implies that
$$Y_{1(j+3)}(z) = C(Y_{11} w_j)(z).$$
For this function to have the asymptotics
$$Y_{1(j+3)}(z) = O(z^{-2})$$
the condition (2.3) must be satisfied.
Hence $Y_{11}(z)$ is the polynomial $p_{2k}(z)$ satisfying conditions (2.3) and (2.4).

3.1. Existence of the $p_{2k-1}^{(m)}$. The same arguments as above show that the lower rows of $Y_{2k}(z)$ are of the form
\begin{equation}
\left( p_{2k-1}^{(m)}, C(p_{2k-1}^{(m)} W), C(p_{2k-1}^{(m)} w_0), \ldots, C(p_{2k-1}^{(m)} w_{d-2}) \right).
\end{equation}
Consider the second row: The asymptotic conditions on $Y_{2k}(z)$ imply that $p_{2k-1}^{(0)}$ satisfies the following conditions as $|z| \to \infty$:
\begin{equation}
p_{2k-1}^{(0)}(z) = O(z^{2k-1}),
\end{equation}
\begin{equation}
C(p_{2k-1}^{(0)} W) = -\frac{1}{2\pi i} \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \langle p_{2k-1}^{(0)} x^j, x^j \rangle = z^{-2k+d-1} + O(z^{-2k+d-2}),
\end{equation}
\begin{equation}
C(p_{2k-1}^{(0)} w_n) = -\frac{1}{2\pi i} \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \int_{\mathbb{R}} p_{2k-1}^{(0)} w_n(x) x^j dx = O(z^{-2}), \quad 0 \leq n \leq d - 2.
\end{equation}
Condition (3.2) forces $p_{2k-1}^{(0)}$ to be a polynomial with degree at most $2k - 1$. Condition (3.3) is equivalent to
\begin{equation}
\langle p_{2k-1}^{(0)} x^j, x^j \rangle = \frac{1}{2} \langle p_{2k-1}^{(0)}(x), \pi_{j,d-1}(y) \rangle_1 = 0, \quad 0 < j < 2k - d - 1.
\end{equation}
The first $2k - d - 1$ equations in (3.3) together with the $d - 1$ conditions (3.4) imply that
$$p_{2k-1}^{(0)} = a_{2,0}^{(2k-1)} p_{2k-1} + a_{1,0}^{(2k-1)} p_{2k-2}.$$ The case $j = 2k - d - 1$ in (3.5) implies that
\begin{equation*}
0 = \langle p_{2k-1}^{(0)}(x), \pi_{2k-2}(y) \rangle_1 = a_{2,0}^{(2k-1)} \langle p_{2k-1}(x), \pi_{2k-2}(y) \rangle_1 + a_{1,0}^{(2k-1)} \langle p_{2k-2}(x), \pi_{2k-2}(y) \rangle_1 = a_{2,0}^{(2k-1)} \langle p_{2k-1}(x), y^{2k-2} \rangle_1,
\end{equation*}
from which we conclude that $a_{2,0}^{(2k-1)} = 0$. Finally one uses the leading order of (3.3) to find that
$$a_{1,0}^{(2k-1)} = -4\pi i \langle p_{2k-2}, y^{2k-1} \rangle_1^{-1}.$$ The non-degeneracy of the skew-inner product (2.1) guarantees that $\langle p_{2k-2}, y^{2k-1} \rangle_1 \neq 0$. 


Consider the other rows: The asymptotic conditions on \( Y_{2k}(z) \) imply that \( p_{2k-1}^{(m)} \), \( m > 0 \) satisfies the following conditions as \( |z| \to \infty \):

\[
(3.6) \quad p_{2k-1}^{(m)}(z) = \mathcal{O}(z^{2k-1}),
\]

\[
(3.7) \quad C(p_{2k-1}^{(m)} W) = -\frac{1}{2\pi i} \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} (p_{2k-1}^{(m)}, x^j)_2 = \mathcal{O}(z^{-2k+d-2}),
\]

\[
(3.8) \quad C(p_{2k-1}^{(m)} w_n) = -\frac{1}{2\pi i} \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \int_{\mathbb{R}} p_{2k-1}^{(m)} w_n(x) x^j dx = \begin{cases} \mathcal{O}(z^{-2}) & m \neq n, \\ z^{-1} + \mathcal{O}(z^{-2}) & m = n, \end{cases} 0 \leq n \leq d - 2.
\]

Condition (3.6) forces \( p_{2k-1}^{(m)} \) to be a polynomial with degree at most \( 2k - 1 \). Condition (3.7) is equivalent to

\[
(3.9) \quad (p_{2k-1}^{(m)}, x^j)_2 = 0, \quad 0 \leq j \leq 2k - d.
\]

The \( 2k - d + 1 \) conditions from (3.9) and the \( d - 2 \) conditions from (3.8) when \( m \neq n \) determine the \( 2k - 1 \) free variables in \( p_{2k-1}^{(m)} \) by solving a homogeneous linear problem. One only need check that

\[
\int_{\mathbb{R}} p_{2k-1}^{(m)} w_n(x) dx \neq 0.
\]

Suppose that this integral is zero, then \( \hat{p}_{2k} = p_{2k} + p_{2k-1}^{(m)} \) satisfies the same orthogonality conditions as the skew-orthogonal polynomial \( p_{2k} \), which contradicts the uniqueness of \( p_{2k} \). This concludes the proof of Theorem 2.1.

In fact we can say more: using (3.7) we see that we could write the \( p_{2k-1}^{(m)} \) in terms of \( d - 1 \) of the orthogonal polynomials with respect to \( W(x) \) given by (1.1).

3.2. The proof of Theorem 2.2 For the odd Riemann-Hilbert problem we note that the asymptotic condition (2.4) has determinant \( z \). So solutions will only be unique up to possibly \( d + 1 \) many degrees of freedom.

The proof that the first row of \( Y_{2k+1} \) has the form

\[
(\tilde{p}_{2k+1}(x), C(\tilde{p}_{2k+1} W), C(\tilde{p}_{2k+1} w_0), \ldots, C(\tilde{p}_{2k+1} w_{d-2})),
\]

follows as above. The asymptotics of the first row of \( Y_{2k} \) fit within those of \( Y_{2k+1} \) so we only get the first row of \( Y_{2k+1} \) up to a multiple of the first row of \( Y_{2k} \). Therefore we may write \( \tilde{p}_{2k+1} = p_{2k+1} + a_k p_{2k} \).

The lower rows of \( Y_{2k+1} \) have the form

\[
(p_{2k}^{(m)}, C(p_{2k}^{(m)} W), C(p_{2k}^{(m)} w_0), \ldots, C(p_{2k}^{(m)} w_{d-2})).
\]

We will find that the first entry of each row is only determined up to a multiple of \( p_{2k} \), as the orthogonality conditions on \( p_{2k} \) fit within those of \( p_{2k}^{(m)} \). This gives \( d + 1 \) free parameters in the general solution to the Riemann-Hilbert problem.

Consider the second row: The asymptotic conditions on \( Y_{2k+1}(z) \) imply that \( p_{2k}^{(0)} \) satisfies the following conditions as \( |z| \to \infty \):

\[
(3.10) \quad p_{2k}^{(0)} = \mathcal{O}(z^{2k}),
\]

\[
(3.11) \quad C(p_{2k}^{(0)} W) = -\frac{1}{2\pi i} \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} (p_{2k}^{(0)}, x^j)_2 = z^{-2k+d-1} + \mathcal{O}(z^{-2k+d-2}),
\]

\[
(3.12) \quad C(p_{2k}^{(0)} w_n) = -\frac{1}{2\pi i} \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \int_{\mathbb{R}} p_{2k}^{(0)} w_n(x) x^j dx = \mathcal{O}(z^{-2}), \quad 0 \leq n \leq d - 2.
\]

Condition (3.10) forces \( p_{2k}^{(0)} \) to be a polynomial with degree at most \( 2k \). Condition (3.11) is equivalent to

\[
(p_{2k}^{(0)}, x^j)_2 = \frac{1}{2} (p_{2k}^{(0)}(x), \pi_{j+d-1}(y))_1 = 0, \quad 0 \leq j \leq 2k - d - 1.
\]
The first $2k - d - 1$ equations in (3.13) together with the $d - 1$ conditions (3.12) imply that
\[ p^{(0)}_{2k} = b_{k, 0} p_{2k} + d_{k, 0} p_{2k-1} + c_{k, 0} p_{2k-2}. \]
The case $j = 2k - d - 1$ in (3.13) implies that
\[
0 = \langle p^{(0)}_{2k}(x), \pi_{2k-2}(y) \rangle_1
= b_{k, 0} \langle p_{2k}, \pi_{2k-2}(y) \rangle_1 + d_{k, 0} \langle p_{2k-1}, \pi_{2k-2}(y) \rangle_1 + c_{k, 0} \langle p_{2k-2}, \pi_{2k-2}(y) \rangle_1
= d_{k, 0} \langle p_{2k-1}, y^{2k-2} \rangle_1,
\]
from which we conclude that $d_{k, 0} = 0$. Finally one uses the leading order of (3.11) to choose $c_{k, 0}$ so that
\[ c_{k, 0} \langle p_{2k-2}, y^{2k-1} \rangle_1 = -4\pi i. \]
Non-degeneracy of the inner product (2.1) guarantees that this can be done. The $b_{k, 0}$ remains as a free parameter.

Consider the other rows of $Y_{2k+1}$: The asymptotic conditions on $Y_{2k+1}(z)$ imply that $p^{(m)}_{2k}$, $m > 0$ satisfy the following conditions as $|z| \to \infty$:
\[(3.14)\]
\[ p^{(m)}_{2k} = O(z^{2k}), \]
\[(3.15)\]
\[ C(p^{(m)}_{2k} W) = -\frac{1}{2\pi i} \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \langle p^{(m)}_{2k}, x^j \rangle_2 = O(z^{-2k+d-2}), \]
\[(3.16)\]
\[ C(p^{(m)}_{2k} w_n) = -\frac{1}{2\pi i} \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \int_{\mathbb{R}} p^{(m)}_{2k} w_n(x) x^j dx = \begin{cases} \frac{O(z^{-2})}{z^{-1}} & m \neq n, \quad 0 \leq n \leq d - 2 \\ \frac{O(z^{-2})}{m = n} & \end{cases} \]
Condition (3.14) forces $p^{(m)}_{2k}$ to be a polynomial with degree at most $2k$. The conditions (3.14)(3.16) are satisfied by $p^{(m)}_{2k-1}$, which fixes $2k - 1$ of the free variables in $p^{(m)}_{2k}$. The orthogonality conditions satisfied by $p_{2k}$ fit within those of $p^{(m)}_{2k}$, hence we find that $p^{(m)}_{2k} = b_{k, m} p_{2k} + c_{k, m} p^{(m)}_{2k-1}$ with $c_{k, m}$ determined by (3.16) with $n = m$, and $b_{k, m}$ a free parameter. This concludes the proof of Theorem 2.2.

4. Conclusions

Both the bi-orthogonal and skew-orthogonal polynomials naturally split into a pair of families: the bi-orthogonal polynomials from their definition, the skew-orthogonal polynomials into the families of even and odd degree polynomials. Numerical computations show that in both cases the distribution of zeros of the two families appear to converge to separate (yet mutually dependent) measures.

The bi-orthogonal and skew-orthogonal polynomials are interesting generalizations of the classical theory of orthogonal polynomials. They appear in the generalizations of random matrix theory from the well studied GUE case. So far little is understood about the necessary asymptotics for both types. We have formulated a Riemann-Hilbert problem for the skew-orthogonal polynomials in the GOE case. The structure of this Riemann-Hilbert problem is close to that of [21] for the bi-orthogonal polynomials. An obvious problem with our formulation is that $p_{2k+1}$ is not uniquely determined. Experience with skew-orthogonal polynomials and the Pfaff lattice hierarchy shows that this is a standard ambiguity complicating the problems.

The corresponding Riemann-Hilbert problem for the skew-orthogonal polynomials associated to the GSE model has not been found. The equivalent to formula (2.2) does not line up with the Cauchy transform as nicely as in the $\beta = 1$ case.

Our conclusions are: High order Riemann-Hilbert problems warrant further investigation with the goal of finding methods to compute their asymptotics. In all studies of bi-orthogonal polynomials one should consider simultaneously the equivalent result for skew-orthogonal polynomials. One suspects that there is a connection between these two types of polynomials.

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DEPT. OF MATH, THE OHIO STATE UNIVERSITY
E-mail address: vpierce@math.ohio-state.edu