Vanishing of Beta Function of Non Commutative $\phi^4$ Theory to all orders

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Abstract

The simplest non commutative renormalizable field theory, the $\phi^4$ model on four dimensional Moyal space with harmonic potential is asymptotically safe up to three loops, as shown by H. Grosse and R. Wulkenhaar, M. Disertori and V. Rivasseau. We extend this result to all orders.

I Introduction

Non commutative (NC) quantum field theory (QFT) may be important for physics beyond the standard model and for understanding the quantum Hall effect [1]. It also occurs naturally as an effective regime of string theory [2] [3].

The simplest NC field theory is the $\phi^4$ model on the Moyal space. Its perturbative renormalizability at all orders has been proved by Grosse, Wulkenhaar and followers [4][5][6][7]. Grosse and Wulkenhaar solved the difficult problem of ultraviolet/infrared mixing by introducing a new harmonic potential term inspired by the Langmann-Szabo (LS) duality [8] between positions and momenta.

Other renormalizable models of the same kind, including the orientable Fermionic Gross-Neveu model [9], have been recently also shown renormalizable at all orders

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and techniques such as the parametric representation have been extended to NCQFT \cite{10}. It is now tempting to conjecture that commutative renormalizable theories in general have NC renormalizable extensions to Moyal spaces which imply new parameters. However the most interesting case, namely the one of gauge theories, still remains elusive.

Once perturbative renormalization is understood, the next problem is to compute the renormalization group (RG) flow. It is well known that the ordinary commutative $\phi^4_4$ model is not asymptotically free in the ultraviolet regime. This problem, called the Landau ghost or triviality problem affects also quantum electrodynamics. It almost killed quantum field theory, which was resurrected by the discovery of ultraviolet asymptotic freedom in non-Abelian gauge theory \cite{11}.

An amazing discovery was made in \cite{12}: the non commutative $\phi^4_4$ model does not exhibit any Landau ghost at one loop. It is not asymptotically free either. For any renormalized Grosse-Wulkenhaar harmonic potential parameter $\Omega_{\text{ren}} > 0$, the running $\Omega$ tends to the special LS dual point $\Omega_{\text{bare}} = 1$ in the ultraviolet. As a result the RG flow of the coupling constant is simply bounded\footnote{The Landau ghost can be recovered in the limit $\Omega_{\text{ren}} \rightarrow 0$.}. This result was extended up to three loops in \cite{13}.

In this paper we compute the flow at the special LS dual point $\Omega = 1$, and check that the beta function vanishes at all orders using a kind of Ward identity inspired by those of the Thirring or Luttinger models \cite{14,15,16}. Note however that in contrast with these models, the model we treat has quadratic (mass) divergences.

The non perturbative construction of the model should combine this result and a non-perturbative multiscale analysis \cite{17,18}. Also we think the Ward identities discovered here might be important for the future study of more singular models such as Chern-Simons or Yang Mills theories, and in particular for those which have been advocated in connection with the Quantum Hall effect \cite{19,20,21}.

In this letter we give the complete argument of the vanishing of the beta function at all orders in the renormalized coupling, but we assume knowledge of renormalization and effective expansions as described e.g. in \cite{18}, and of the basic papers for renormalization of NC $\phi^4_4$ in the matrix base \cite{4,5,6}.

\section{Notations and Main Result}

We adopt simpler notations than those of \cite{12,13}, and normalize so that $\theta = 1$, hence have no factor of $\pi$ or $\theta$.

The propagator in the matrix base at $\Omega = 1$ is

$$C_{mn;kl} = G_{mn}\delta_{ml}\delta_{nk} ; \quad G_{mn} = \frac{1}{A + m + n} ,$$

(II.1)
where \( A = 2 + \mu^2 / 4, m, n \in \mathbb{N}^2 \) (\( \mu \) being the mass) and we used the notations
\[
\delta_{ml} = \delta_{m_1 l_1} \delta_{m_2 l_2}, \quad m + m = m_1 + m_2 + n_1 + n_2.
\]

There are two version of this theory, the real and complex one. We focus on the complex case, the result for the real case follows easily [13].

The generating functional is:
\[
Z(\eta, \bar{\eta}) = \int d\phi d\bar{\phi} e^{-S(\bar{\phi}, \phi) + F(\bar{\eta}, \eta; \phi, \bar{\phi})}
\]
\[
F(\bar{\eta}, \eta; \bar{\phi}, \phi) = \bar{\phi} \eta + \bar{\eta} \phi
\]
\[
S(\bar{\phi}, \phi) = \bar{\phi} X \phi + \phi X \bar{\phi} + A \bar{\phi} \phi + \frac{\lambda}{2} \bar{\phi} \phi \bar{\phi}
\]
where traces are implicit and the matrix \( X_{mn} \) stands for \( m \delta_{mn} \).

\( S \) is the action and \( F \) the external sources.

We denote \( \Gamma^4(0, 0, 0, 0) \) the amputated one particle irreducible four point function and \( \Sigma(0, 0) \) the amputated one particle irreducible two point function with external indices set to zero. The wave function renormalization is \( \partial_L \Sigma = \partial_R \Sigma = \Sigma(1, 0) - \Sigma(0, 0) \) [13]. Our main result is:

**Theorem** The equation:
\[
\Gamma^4(0, 0, 0, 0) = \lambda (1 - \partial_L \Sigma(0, 0))^2
\]
holds up to irrelevant terms to all orders of perturbation, either as a bare equation with fixed ultraviolet cutoff, or as an equation for the renormalized theory. In the latter case \( \lambda \) should still be understood as the bare constant, but reexpressed as a series in powers of \( \lambda_{ren} \).

**III Ward Identities**

Let \( U = e^{iA} \) with \( A \) small. We consider the “right” change of variables:
\[
\phi^U = \phi U; \bar{\phi}^U = U^\dagger \bar{\phi}.
\]

There is a similar “left” change of variables. The variation of the action is, at first order:
\[
\delta S = \phi U X U^\dagger \bar{\phi} - \phi X \bar{\phi} \approx \lambda (\phi X \bar{\phi} - \phi X A \bar{\phi})
\]
\[
= \lambda A (X \bar{\phi} \phi - \phi \phi X)
\]

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and the variation of the external sources is:

\[ \delta F = U^\dagger \tilde{\phi} \eta - \tilde{\eta} \phi + \bar{\eta} \phi U - \bar{\eta} \phi \approx -iA \tilde{\phi} \eta + i \bar{\eta} \phi A \]

(III.7)

We obviously have:

\[ \frac{\delta \ln Z}{\delta A_{ba}} = 0 = \frac{1}{Z(\bar{\eta}, \eta)} \int d\tilde{\phi} d\phi (-\frac{\delta S}{\delta A_{ba}} + \frac{\delta F}{\delta A_{ba}}) e^{-S+F} \]

\[ = \frac{1}{Z(\bar{\eta}, \eta)} \int d\tilde{\phi} d\phi e^{-S+F} (-[X \tilde{\phi} \phi - \tilde{\phi} \phi X]_{ab} + [-\tilde{\phi} \eta + \bar{\eta} \phi]_{ab}) . \]  

(III.8)

We now take \( \partial_{\eta} \partial_{\bar{\eta}}|_{\eta=\bar{\eta}=0} \) on the above expression. As we have at most two insertions we get only the connected components of the correlation functions.

\[ 0 = \langle \partial_{\eta} \partial_{\bar{\eta}} (-[X \tilde{\phi} \phi - \tilde{\phi} \phi X]_{ab} + [-\tilde{\phi} \eta + \bar{\eta} \phi]_{ab}) e^{F(\bar{\eta}, \eta)}|_{0} \rangle > c , \]

(III.9)

which gives:

\[ < \frac{\partial(\tilde{\phi} \phi)_{ab}}{\partial \eta} \frac{\partial(\tilde{\phi} \phi)_{ab}}{\partial \bar{\eta}} > c = < \frac{\partial(\tilde{\phi} \phi)_{ab}}{\partial \eta} \frac{\partial(\tilde{\phi} \phi)_{ab}}{\partial \bar{\eta}} > c - < \frac{\partial(\tilde{\phi} \phi)_{ab}}{\partial \eta} \frac{\partial(\tilde{\phi} \phi)_{ab}}{\partial \bar{\eta}} > c > (III.10) \]

Using the explicit form of \( X \) we get:

\[ (a - b) < [\tilde{\phi} \phi]_{ab} \frac{\partial(\tilde{\phi} \phi)_{ab}}{\partial \eta} \frac{\partial(\tilde{\phi} \phi)_{ab}}{\partial \bar{\eta}} > c = < \delta_{a \beta} \phi_{ab} \phi_{\mu \nu} > c - < \delta_{b \mu} \phi_{ab} \phi_{\alpha \beta} > c \]

(III.11)

and for \( \bar{\eta}_{\beta \alpha} \eta_{\mu \nu} \) we get:

\[ (a - b) < [\tilde{\phi} \phi]_{ab} \phi_{\alpha \beta} \phi_{\mu \nu} > c = < \delta_{a \beta} \phi_{ab} \phi_{\mu \nu} > c - < \delta_{b \mu} \phi_{ab} \phi_{\alpha \beta} > c \]

(III.12)

We now restrict to terms in the above expressions which are planar with a single external face, as all others are irrelevant. Such terms have \( \alpha = \nu, a = \beta \) and \( b = \mu \). The Ward identity reads:

\[ (a - b) < [\tilde{\phi} \phi]_{ab} \phi_{\nu \alpha} \phi_{\bar{\nu} \bar{\alpha}} > c = < \phi_{ab} \phi_{\nu \alpha} > c - < \phi_{ab} \phi_{\nu \alpha} > c \]

(III.13)

(repeated indices are not summed). There is a similar Ward identity obtained with the left transformation and a \( \phi \tilde{\phi} \) insertion.

Deriving once more we get:

\[ (a - b) < [\tilde{\phi} \phi]_{ab} \partial_{\bar{\eta}_{1}} (\tilde{\phi} \phi) \partial_{\eta_{2}} (\tilde{\phi} \phi) > c = \]

\[ < \partial_{\bar{\eta}_{1}} (\tilde{\phi} \phi) \partial_{\eta_{2}} (\tilde{\phi} \phi) > c > +1 \leftrightarrow 2 . \]
Take $\bar{\eta}_1 \beta_\alpha$, $\eta_1 \nu\mu$, $\bar{\eta}_2 \delta_\gamma$ and $\eta_2 \sigma_\rho$. We get:

$$\begin{align*}
(a - b) &< [\bar{\phi}\phi]_{ab}\phi\beta_{\mu\nu}\phi_\gamma\delta_{\rho\sigma} >_c \\
&= <\phi_{a\beta}\bar{\phi}_\mu\nu\delta_{a\beta}\phi_\gamma\delta_{b\sigma} >_c - <\phi_{a\beta}\bar{\phi}_\mu\nu\phi_\gamma\delta_{a\sigma}\delta_{b\rho} >_c \\
&+ <\phi_{a\beta}\phi_\mu\nu\delta_{a\beta}\phi_\gamma\delta_{b\mu} >_c - <\phi_{a\beta}\phi_\mu\nu\phi_\gamma\delta_{a\rho}\delta_{b\sigma} >_c .
\end{align*}$$

Again neglecting all terms which are not planar with a single external face leads to

$$(a - b) < \phi_{aa}[\bar{\phi}\phi]_{ab}\bar{\phi}_b\phi_\delta\delta_{6a} >_c = <\phi_{ab}\bar{\phi}_b\phi_\delta\delta_{6a} >_c - <\phi_{aa}\phi_{b\delta}\delta_{b\delta}\delta_{6a} >_c .$$

Clearly there are similar identities for $2p$ point functions for any $p$.

**IV Proof of the Theorem**

We will denote $G^4(m, n, k, l)$ the connected four point function restricted to the planar single-border case, where $m, n, ...$ are the indices of the external borders in the correct cyclic order). $G^2(m, n)$ is the corresponding connected planar single-border two point function and $G_{ins}(a, b; ...)$ the planar single-border connected functions with one insertion on the left border where the matrix index jumps from $a$ to $b$. 
All the identities we use, either Ward identities or the Dyson equation of motion can be written either for the bare theory or for the theory with complete mass renormalization, which is the one considered in [13]. In the first case the parameter $A$ in (II.1) is the bare one, $A_{\text{bare}}$, and there is no mass subtraction. In the second case the parameter $A$ in (II.1) is $A_{\text{ren}} = A_{\text{bare}} - \Sigma(0, 0)$, and every two point 1PI subgraph is subtracted at 0 external indices.

Let us prove first the Theorem in the mass-renormalized case, then in the next subsection in the bare case. Indeed the mass renormalized theory is the one used in [13]: it is free from any quadratic divergences, and remaining logarithmic sub-divergences in the ultra violet cutoff can then be removed easily by passing to the “useful” renormalized effective series, as explained in [13].

We analyze a four point connected function $G^4(0, m, 0, m)$ with index $m \neq 0$ on the right borders. This explicit break of left-right symmetry is adapted to our problem.

Consider a $\bar{\phi}$ external line and the first vertex hooked to it. Turning right on the $m$ border at this vertex we meet a new line. If we cut it the graph may fall into two disconnected components having either 2 and 4 or 4 and 2 external lines ($G^4_{(1)}$ and $G^4_{(2)}$ in Fig. 2) or it may remain connected, in which case the new line was part of a loop ($G^4_{(3)}$ in Fig. 2). Accordingly

$$G^4(0, m, 0, m) = G^4_{(1)}(0, m, 0, m) + G^4_{(2)}(0, m, 0, m) + G^4_{(3)}(0, m, 0, m). \quad (IV.16)$$

The second term $G^4_{(2)}$ is zero after mass renormalization of the two point insertion since it has a two point subgraph with zero external border. We will prove that $G^4_{(1)} + G^4_{(3)}$ yields $\Gamma^4 = \lambda(1 - \partial \Sigma)^2$ after amputation of the four external propagators.

Start with $G^4_{(1)}$. It is of the form:

$$G^4_{(1)}(0, m, 0, m) = \lambda C_{0m} G^2(0, m) G^2_{\text{ins}}(0, 0; m). \quad (IV.17)$$

By the Ward identity we have:

$$G^2_{\text{ins}}(0, 0; m) = \lim_{\zeta \to 0} G^2_{\text{ins}}(\zeta, 0; m) = \lim_{\zeta \to 0} \frac{G^2(0, m) - G^2(\zeta, m)}{\zeta} = -\partial_L G^2(0, m), \quad (IV.18)$$

and as $\partial_L C^{-1} = \partial_R C^{-1} = 1$ and $G^2(0, m) = [C^{-1}_{0m} - \Sigma(0, m)]^{-1}$ one has:

$$G^4_{(1)}(0, m, 0, m) = \lambda C_{0m} C^2_{0m} [1 - \partial_L \Sigma(0, m)] \frac{C_{0m} [1 - \partial_L \Sigma(0, m)]}{1 - C_{0m} \Sigma(0, m) [1 - \partial_L \Sigma(0, m)]^2} = \lambda (C^D_{0m})^4 C_{0m}^2 \left[1 - \partial_L \Sigma(0, m)\right]. \quad (IV.19)$$

\[These mass subtractions need not be rearranged into forests since 1PI 2point subgraphs never overlap non trivially.\
The self energy is (again up to irrelevant terms (5)):

$$\Sigma(m, n) = \Sigma(0, 0) + (m + n)\partial_L \Sigma(0, 0)$$  \hspace{1cm} (IV.20)

Therefore up to irrelevant terms:

$$C_{0m}D = \frac{1}{m[1 - \partial_L \Sigma(0, 0)] + A_{ren}}$$  \hspace{1cm} (IV.21)

and

$$\frac{C_{0m}}{C_{0m}D} = 1 - \partial_L \Sigma(0, 0) + \frac{A_{ren}}{m + A_{ren}} \partial_L \Sigma(0, 0).$$  \hspace{1cm} (IV.22)

Figure 3: Two point insertion and opening of the loop with index $p$

For the $G_{(3)}^4(0, m, 0, m)$ one starts by “opening” the face which is “first on the right” with the $p$ index. For bare Green functions this reads:

$$G_{(3)}^4,\text{bare}(0, m, 0, m) = C_{0m} \sum_p G_{\text{ins}}^{4,\text{bare}}(p, 0; m, 0, m).$$  \hspace{1cm} (IV.23)

Passing to mass renormalized Green functions one sees that if the face $p$ belonged to a 1PI two point insertion in $G_{(3)}^4$ this 2 point insertion disappears on the right hand side of eq. (IV.23) (see fig. 3)! In the equation for $G_{(3)}^4(0, m, 0, m)$ one must therefore add its missing counterterm, so that:

$$G_{(3)}^4(0, m, 0, m) = C_{0m} \sum_p G_{\text{ins}}^{4}(0, p; m, 0, m) - CT_{\text{lost}}.$$

The part of the self energy with non trivial right border is called $\Sigma^R$, so that the difference $\Sigma - \Sigma^R$ is the generalized left tadpole $\Sigma^L_{\text{tadpole}}$. The missing mass counterterm must have a right $p$ face, so it is restricted to $\Sigma^R$ and is:

$$CT_{\text{lost}} = C_{0m} \Sigma^R(0, 0)G^4(0, m, 0, m).$$  \hspace{1cm} (IV.25)
We compute the value of this counterterm again by opening its right face $p$ and using the Ward identity (III.13). We get:

$$\Sigma^R(0, 0) = \frac{1}{C_{00}^D} \sum_p G_{ins}^2(0, p; 0) = \frac{1}{C_{00}^D} \sum_p \frac{1}{p} [G^2(0, 0) - G^2(p, 0)] = \sum_p \frac{1}{p} \left(1 - \frac{C_{0p}^D}{C_{00}^D}\right).$$  \hspace{1cm} \text{(IV.26)}$$

A similar equation can be written also for $\Sigma^R(0, 1)$:

$$\Sigma^R(0, 1) = \sum_p \frac{1}{p} \left(1 - \frac{C_{1p}^D}{C_{01}^D}\right).$$ \hspace{1cm} \text{(IV.27)}$$

We then conclude that:

$$CT_{lost} = C_{0m} G^4(0, m, 0, m) \sum_p \frac{1}{p} \left(1 - \frac{C_{0p}^D}{C_{00}^D}\right).$$ \hspace{1cm} \text{(IV.28)}$$

But by the Ward identity (III.16):

$$C_{0m} \sum_p G^4_{ins}(0, p; m, 0, m) = C_{0m} \sum_p \frac{1}{p} \left(G^4(0, m, 0, m) - G^4(p, m, 0, m)\right) \hspace{1cm} \text{(IV.29)}$$

so that subtracting (IV.26) from (IV.29) computes:

$$G^4_{(03)}(0, m, 0, m) = -C_{0m} \sum_p \frac{1}{p} (G^4(p, m, 0, m) + G^4(0, m, 0, m) \frac{C_{0p}^D}{C_{00}^D}).$$  \hspace{1cm} \text{(IV.30)}$$

The first term in eq (IV.30) is irrelevant, having at least three denominators linear in $p$. We rewrite the last term, using (IV.26), (IV.27) starting with:

$$\partial_R \Sigma(0, 0) = \partial_R \Sigma^R(0, 0) = \Sigma^R(0, 1) - \Sigma^R(0, 0) = \sum_p \frac{1}{p} \left(\frac{C_{p0}^D}{C_{00}^D} - \frac{C_{p1}^D}{C_{01}^D}\right).$$ \hspace{1cm} \text{(IV.31)}$$

In the second term of the above equation one can change the $C_{p1}^D$ in $C_{p0}^D$ at the price of an irrelevant term. Using (IV.22) we have:

$$\partial_R \Sigma(0, 0) = -[1 - \partial_{L} \Sigma(0, 0)] \sum_p \frac{1}{p} C_{p0}^D$$  \hspace{1cm} \text{(IV.32)}$$
hence

\[ G^4_{(3)}(0, m, 0, m; p) = -G^4(0, m, 0, m) \frac{A_{\text{ren}} \partial_R \Sigma(0, 0)}{(m + A_{\text{ren}})(1 - \partial_L \Sigma(0, 0))} . \]  (IV.33)

Using (IV.19), (IV.22) and (IV.33), equation (IV.16) rewrites as:

\[ G^4(0, m, 0, m) \left( 1 + \frac{A_{\text{ren}} \partial_R \Sigma(0, 0)}{(m + A_{\text{ren}})(1 - \partial_L \Sigma(0, 0))} \right) = \lambda_{\text{bare}} \left( C_{0,m}^D \right)^4 \left( 1 - \partial_L \Sigma(0, 0) + \frac{A_{\text{ren}}}{m + A_{\text{ren}}} \partial_L \Sigma(0, 0) \right) \left[ 1 - \partial_L \Sigma(0, m) \right] . \]  (IV.34)

Multiplying (IV.34) by \[ [1 - \partial_L \Sigma(0, 0)] \] and amputating four times proves (II.4), hence the theorem.

\[ \square \]

IV.1 Bare identity

Let us explain now why the main theorem is also true as an identity between bare functions, without any renormalization, but with ultraviolet cutoff.

Using the same Ward identities, all the equations go through with only few differences:

- we should no longer add the lost mass counterterm in (IV.25)
- the term \( G^4_{(2)} \) is no longer zero.
- equation (IV.22) and all propagators now involve the bare \( A \) parameter.

But these effects compensate. Indeed the bare \( G^4_{(2)} \) term is the left generalized tadpole \( \Sigma - \Sigma^R \), hence

\[ G^4_{(2)}(0, m, 0, m) = C_{0,m} \left( \Sigma(0, m) - \Sigma^R(0, m) \right) G^4(0, m, 0, m) . \]  (IV.35)

Equation (IV.22) becomes up to irrelevant terms

\[ \frac{C_{0,m}^{\text{bare}}}{C_{0,m}^{D,\text{bare}}} = 1 - \partial_L \Sigma(0, 0) + \frac{A_{\text{bare}}}{m + A_{\text{bare}}} \partial_L \Sigma(0, 0) - \frac{1}{m + A_{\text{bare}}} \Sigma(0, 0) \]  (IV.36)

The first term proportional to \( \Sigma(0, m) \) in (IV.35) combines with the new term in (IV.36), and the second term proportional to \( \Sigma^R(0, m) \) in (IV.35) is exactly the former “lost counterterm” (IV.25). This proves (II.4) in the bare case.

V Conclusion

Since the main result of this paper is proved up to irrelevant terms which converge at least like a power of the infrared cutoff, as this infrared cutoff is lifted towards
infinity, we not only get that the beta function vanishes in the ultraviolet regime, but that it vanishes fast enough so that the total flow of the coupling constant is bounded. The reader might worry whether this conclusion is still true for the full model which has $\Omega_{\text{ren}} \neq 1$, hence no exact conservation of matrix indices along faces. The answer is yes, because the flow of $\Omega$ towards its ultra-violet limit $\Omega_{\text{bare}} = 1$ is very fast (see e.g. [13], Sect II.2).

The vanishing of the beta function is a step towards a full non-perturbative construction of this model without any cutoff, just like e.g. the one of the Luttinger model [23, 15]. But NC $\phi^4$ would be the first such four dimensional model, and the only one with non-logarithmic divergences. Tantalizingly, quantum field theory might actually behave better and more interestingly on non-commutative than on commutative spaces.

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