The polynomial Hermite-Padé \( m \)-system for meromorphic functions on a compact Riemann surface

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Abstract. Given a tuple of \( m + 1 \) germs of arbitrary analytic functions at a fixed point, we introduce the polynomial Hermite-Padé \( m \)-system, which includes the Hermite-Padé polynomials of types I and II. In the generic case we find the weak asymptotics of the polynomials of the Hermite-Padé \( m \)-system constructed from the tuple of germs of functions \( f_1, f_2, \ldots, f_m \) that are meromorphic on an \( (m + 1) \)-sheeted compact Riemann surface \( \mathcal{R} \). We show that if \( f_j = f^l \) for some meromorphic function \( f \) on \( \mathcal{R} \), then with the help of the ratios of polynomials of the Hermite-Padé \( m \)-system we recover the values of \( f \) on all sheets of the Nuttall partition of \( \mathcal{R} \), apart from the last sheet.

Bibliography: 18 titles.

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§ 1. Introduction

First let \( f_{0,\infty}(z) \equiv 1, f_{1,\infty}(z), \ldots, f_{m,\infty}(z) \) be \( m + 1 \) arbitrary analytic germs at infinity (on the Riemann sphere \( \hat{\mathbb{C}} \)). Fix a natural number \( k \in \{1, \ldots, m\} \) and, for each \( n \in \mathbb{N} \), consider the tuple of \( \binom{m+1}{k} \) \( k \)th polynomials of the Hermite-Padé \( m \)-system of order \( n \), which are constructed from the tuple of germs \([1, f_{1,\infty}, \ldots, f_{m,\infty}]\) at \( \infty \) as follows. These are the polynomials \( P_{n;i_1,\ldots,i_k} \), \( 0 \leq i_1 < i_2 < \cdots < i_k \leq m \), such that \( \deg P_{n;i_1,\ldots,i_k} \leq (m+1-k)n \), at least one polynomial \( P_{n;i_1,\ldots,i_k} \equiv \neq 0 \) and, for each set of indices \( 0 < j_1 < \cdots < j_k \leq m \),

\[
P_{n;j_1,\ldots,j_k}(z) + \sum_{s=1}^{k} (-1)^s P_{n;0,j_1,\ldots,j_s-1,j_{s+1},\ldots,j_k}(z) f_{j_s,\infty}(z) = O\left(\frac{1}{z^{kn+1}}\right) \tag{1}
\]

as \( z \to \infty \). It is easy to see that condition (1) is a system of \( n(m+1)+1\binom{m}{k} \) homogeneous linear equations for the \( n(m+1-k)+1\binom{m+1}{k} = n(m+1)\binom{m}{k} + \binom{m+1}{k} \) unknown coefficients of the polynomials \( P_{n;i_1,\ldots,i_k} \). The coefficients of this system are linear expressions of the first \( (m+1)n \) Taylor coefficients of the germs \( f_{s,\infty} \) (with respect to the variable \( 1/z \)). Hence the polynomials \( P_{n;i_1,\ldots,i_k} \) always exist.

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but are not unique in general. (Note that an Hermite-Padé \( m \)-system can also be constructed from germs \( f_{j,\infty} \) which are meromorphic at \( \infty \). In this case we should increase the highest possible degree of the polynomials \( P_{n;i_1,\ldots,i_k} \) allowed. For example, it suffices to assume that \( \deg P_{n;i_1,\ldots,i_k} \leq (m + 1 - k)n + M \), where \( M \) is the highest order of the poles of the \( f_{j,\infty} \) at \( \infty \).)

It is clear that conditions (1) are linearly independent; however, the index 0 'plays a special role'. We can also give another, so-called homogeneous definition of the \( k \)th polynomials of the Hermite-Padé \( m \)-system, in which the conditions are no longer linearly independent: \( \deg P_{n;i_1,\ldots,i_k} \leq (m + 1 - k)n \), at least one polynomial \( P_{n;i_1,\ldots,i_k} \not\equiv 0 \), and for each set of indices \( 0 \leq i_0 < i_1 < \cdots < i_k \leq m \) we have

\[
\sum_{s=0}^{k} (-1)^s P_{n;i_0,\ldots,i_s-1,i_{s+1},\ldots,i_k}(z)f_{i_s,\infty}(z) = O\left(\frac{1}{z^{kn+1}}\right) \quad \text{as } z \to \infty. \tag{2}
\]

It is clear that all conditions (1) are contained in (2) (for \([i_0, i_1, \ldots, i_k] := [0, j_1, \ldots, j_k]\)). To see that conditions (1) are sufficient for (2) to hold also for \( i_0 \not\equiv 0 \), it suffices to substitute the expressions for all \( P_{n;i_0,\ldots,i_{s-1},i_{s+1},\ldots,i_k} \) obtained from (1) for \([j_1, \ldots, j_k] := [i_0, \ldots, i_{s-1}, i_{s+1}, \ldots, i_k]\) into (2) and then check that the right-hand side of (2) is 0 up to \( O(z^{-(kn+1)}) \). So definitions (1) and (2) are equivalent. In what follows we mostly use the first.

Now recall the definitions of the classical Hermite-Padé polynomials of types I and II. The Hermite-Padé polynomials of type I of order \( n \) constructed from a tuple of germs \([1, f_1,\infty, \ldots, f_m,\infty] \) at \( \infty \in \mathbb{C} \) are polynomials \( Q_{n,i}, 0 \leq i \leq m \), such that \( \deg Q_{n,i} \leq n \), at least one polynomial \( Q_{n,i} \not\equiv 0 \), and

\[
\sum_{j=0}^{m} Q_{n,j}(z)f_{j,\infty}(z) = O\left(\frac{1}{z^{m(n+1)}}\right) \quad \text{as } z \to \infty \tag{3}
\]

in a neighbourhood of infinity. The Hermite-Padé polynomials of type II of order \( n \) constructed for a tuple of germs \([1, f_1,\infty, \ldots, f_m,\infty] \) at \( \infty \in \mathbb{C} \) are polynomials \( q_{n,i}, 0 \leq i \leq m \), such that \( \deg q_{n,i} \leq mn \), at least one polynomial \( q_{n,i} \not\equiv 0 \), and

\[
q_{n,0}(z)f_{j,\infty}(z) - q_{n,j}(z) = O\left(\frac{1}{z^{n+1}}\right) \quad \text{as } z \to \infty \tag{4}
\]

in a neighbourhood of infinity for all \( j, 1 \leq j \leq m \).

Note that conditions (1), which define the first polynomials of the Hermite-Padé \( m \)-system (that is, for \( k = 1 \)) coincide (up to a sign) with conditions (4), which define the Hermite-Padé polynomials of type II. So the first polynomials of the Hermite-Padé \( m \)-system are precisely the Hermite-Padé polynomials of type II, that is, \( P_{n;i} \equiv q_{n,i} \). At the same time condition (1), which defines the \( m \)th polynomials of the Hermite-Padé \( m \)-system (that is, for \( k = m \)), is actually the same as condition (3), which defines the Hermite-Padé polynomials of type I. More precisely, if we set \( P_{n;0,1,\ldots,j-1,j+1,\ldots,m} := (-1)^j Q_{n,j} \), then the left-hand sides of (1) for \( k = m \) and (3) will be equal and the orders of contact with 0 of their right-hand sides will be \( m(n+1) \) and \( mn + 1 \), respectively. Thus the polynomials \((-1)^j Q_{n;j}\) automatically satisfy (1), that is, the Hermite-Padé polynomials of type I (with odd polynomials taken with opposite sign) are a particular case of the \( m \)th polynomials of the
Hermite-Padé \(m\)-system. Moreover, in what follows we will mostly be interested in the so-called weak asymptotics of polynomials of the Hermite-Padé \(m\)-system (in the spirit of the classical Stahl theorem for ordinary Padé polynomials; see [17] and [1]), hence the above discrepancy by the fixed quantity \(m - 1\) in the orders of contact is immaterial for us. Only a few constructions generalizing Hermite-Padé polynomials of types I and II are presently known. One example is polynomials of mixed type (see [9] and [10]), but these do not include the Hermite-Padé polynomials themselves.

Remark 1. Like the Hermite-Padé polynomials of types I and II, a polynomial Hermite-Padé \(m\)-system can be constructed from any tuple of \(m + 1\) germs at an arbitrary point \(z_0\) on the Riemann sphere \(\hat{C}\), not only at \(z_0 = \infty\). Namely, the \(k\)th polynomials of the Hermite-Padé \(m\)-system at a point \(z_0 \in \mathbb{C}\) are defined by a relation similar to (1) (or (2)), where the left-hand side is the same, and the right-hand side is

\[
\frac{z^{kn+1}}{\left(z - z_0\right)^{m(m+1)+1}} \quad \text{as} \quad z \to z_0.
\]

However, it will be more convenient for us to consider an Hermite-Padé \(m\)-system constructed from analytic germs at infinity. Nevertheless, all the results discussed below are also true in the general case, with the appropriate changes in wording.

We study the weak asymptotics of the \(k\)th polynomials of the Hermite-Padé \(m\)-system (1) introduced above in the case when the \(f_{j,\infty}\) are the germs of meromorphic functions \(f_j\) on a compact \((m+1)\)-sheeted Riemann surface \(\mathcal{R}\), and \(\mathcal{R}\) satisfies a certain additional condition. More precisely, let \(\mathcal{R}\) be a compact Riemann surface, let \(\pi: \mathcal{R} \to \hat{C}\) be an \((m+1)\)-sheeted holomorphic branched covering of the Riemann sphere \(\hat{C}\), \(m \geq 1\), and let \(\Sigma\) be the set of critical values of the projection \(\pi\). Points on \(\mathcal{R}\) are denoted by symbols in boldface and their projections by symbols in lightface (for example, \(z \in \mathcal{R}\), and \(\pi(z) = z\)). We denote the space of meromorphic functions on \(\mathcal{R}\) by \(\mathcal{M}(\mathcal{R})\). Let \(f_1, f_2, \ldots, f_m \in \mathcal{M}(\mathcal{R})\) be functions such that \(1, f_1, f_2, \ldots, f_m\) are independent over the field of rational functions \(\mathbb{C}(z)\). Let \(\circ\) be an arbitrary point on \(\mathcal{R}\) that is not a critical point of the projection \(\pi\). We assume without loss of generality that \(\circ \in \pi^{-1}(\infty)\) and write \(\infty^{(0)} := \circ\). It should be noted that the case when \(\infty \in \Sigma\) is not excluded, that is, \(\infty\) can be a critical value of \(\pi\), but \(\infty^{(0)} \notin \pi^{-1}(\infty)\) must be a noncritical point of the mapping \(\pi\). Let \(f_{1,\infty}(z), \ldots, f_{m,\infty}(z)\) be the meromorphic germs of \(f_1(z), \ldots, f_m(z)\) at \(\infty^{(0)}\), respectively. More precisely, \(f_{j,\infty}(z) := f_j(\pi_0^{-1}(z))\), where \(\pi_0^{-1}\) is the inverse mapping of \(\pi\) in a neighbourhood of \(\infty^{(0)}\). For the simplicity of presentation we assume that the \(f_{j,\infty}(z)\) are holomorphic at \(\infty\), that is, they have no poles at this point. In what follows we consider only polynomials of the Hermite-Padé \(m\)-system (1) constructed for such a tuple of germs \([1, f_1, \ldots, f_m, \infty]\) at \(\infty\). Hence, unless otherwise stated, \(P_{n_1, j_1, \ldots, j_k}\) denotes the corresponding \(k\)th polynomial of the Hermite-Padé \(m\)-system constructed from this tuple of germs. It should be emphasized that in the definition of an Hermite-Padé \(m\)-system and in the definition of the covering \(\pi\), \(m\) is the same.

In this paper we find the limit distribution of the zeros and the asymptotics of the ratios of \(k\)th polynomials of the Hermite-Padé \(m\)-system constructed from the above tuple of germs of functions (which are meromorphic on the Riemann surface \(\mathcal{R}\)), under the following additional condition on \(\mathcal{R}\). We can assume that the
Riemann surface $\mathcal{R}$ is the standard compactification of the Riemann surface of an $(m+1)$-valued global analytic function (GAF) $w(\cdot)$ defined in the domain $\hat{\mathbb{C}} \setminus \Sigma$, that is, $z = (z, w(z))$. We define the surface $\tilde{\mathcal{R}}_{[k]}$ to be the standard compactification of the Riemann surface of all possible unordered tuples of $k$ distinct germs of the function $w(\cdot)$ considered at the same point $z \in \hat{\mathbb{C}} \setminus \Sigma$ (for more details, see §4). We assume that $\tilde{\mathcal{R}}_{[k]}$ is connected (this condition is discussed in §6).

Note that the distribution of the zeros and the asymptotic behaviour of the ratios of Hermite-Padé polynomials of type I constructed from the tuple of germs under consideration (that is, in fact, for the $m$th polynomials of the Hermite-Padé $m$-system) were rigourously justified in [8], and for Hermite-Padé polynomials of type II (that is, for the first polynomials in the system) this was done in [12] (under a certain condition of ‘general position’). The surfaces $\tilde{\mathcal{R}}_{[1]}$ and $\tilde{\mathcal{R}}_{[m]}$ are isomorphic to $\mathcal{R}$ (see §6), hence they are always connected. Thus, in our paper we, in particular, reprove the result in [8] and establish the result in [12] in the most general case. In our investigations, as in [8], we use the basic ideas of Nuttall’s approach (see [11] and [12]). Note that our proofs are close in spirit to the proofs of the corresponding results in [8], and the main tools in [8] were the methods of potential theory on compact Riemann surfaces.

We also apply our results here to the problem of the recovery of the values of an algebraic function from a prescribed germ of it. In particular, we show that if $f_j = f_j^1$ for some $f \in \mathcal{M}(\mathcal{R})$ and the surface $\tilde{\mathcal{R}}_{[k]}$ is connected, then the ratio $P_{n;0,\ldots,k-2,k}/P_{n;0,\ldots,k-1}$ of two $k$th polynomials of the Hermite-Padé $m$-system restores asymptotically (as $n \to \infty$) the sum of the values of $f$ on the first $k$ sheets of the Nuttall partition of the Riemann surface $\mathcal{R}$.

The results in this paper were partially announced in [7]. The paper is organized as follows. The main results are formulated in §2. In §3, under the assumption that all surfaces $\tilde{\mathcal{R}}_{[k]}$, $k = 1, \ldots, m$, are connected, we show how we can use the polynomial Hermite-Padé $m$-system to recover asymptotically the values of an arbitrary function $f \in \mathcal{M}(\mathcal{R})$ on all sheets of the Nuttall partition of $\mathcal{R}$, apart from the last sheet, from the germ of $f$. In §4 we give a rigorous definition of the surface $\tilde{\mathcal{R}}_{[k]}$. Next we give an equivalent definition of the $k$th polynomials of an Hermite-Padé $m$-system, in terms of the surface $\tilde{\mathcal{R}}_{[k]}$ and certain special meromorphic functions on it, which are constructed from the original functions $f_1, \ldots, f_m$. We prove of our main results, Theorems 1 and 2, in §5. In §5.1 we introduce the necessary definitions, prove auxiliary results and fix normalizations. Theorem 1 is proved in §5.2 and Theorem 2, in §5.3. In §6 we discuss the condition in Theorems 1 and 2 that the surface $\tilde{\mathcal{R}}_{[k]}$ be connected. In particular, in §6 we give a sufficient condition for the connectedness of all surfaces $\tilde{\mathcal{R}}_{[k]}$, $k = 1, \ldots, m$.

§2. The statements of the main results

Following Nuttall [11], [12], we introduce a partition of $\mathcal{R}$ into sheets (for more details, see [8]). Let $u(z)$ be a harmonic function on $\mathcal{R} \setminus \pi^{-1}(\infty)$ with the following logarithmic singularities at the points in the set $\pi^{-1}(\infty)$:

\begin{align}
  u(z) &= -m \log |z| + O(1), \quad z \to \infty^{(0)}, \\
  u(z) &= \log |z| + O(1), \quad z \to \pi^{-1}(\infty) \setminus \infty^{(0)}. 
\end{align}  \tag{5
The function \( u \) always exists and is defined up to an additive constant (it can be constructed explicitly using standard bipolar Green’s functions; for more details, see formula (23) in [8]). Note that some authors (for instance, see [14] and [2]) denote \( u \) by \( g \) and call it the \( g \)-function of \( \mathcal{R} \). However, we use the notation \( g \) for bipolar Green’s functions; see (44).

Remark 2. We emphasize that the definition of the function \( u \) is also consistent for \( \infty \in \Sigma \). (If \( \infty \in \pi^{-1}(\infty) \) is a critical point of \( \pi \) of order \( N - 1 \), then using a local coordinate \( \zeta : \Omega \to \{ \eta : |\eta| < \delta \} \), \( \zeta(\infty) = 0 \), in a neighbourhood \( \Omega \) of it, the second condition in (5) can be written in the form \( u(\zeta^{-1}(\eta)) = -N \log |\eta| + O(1) \) as \( \eta \to 0 \).

Let \( z \in \mathbb{C} \), and let \( u_0(z), \ldots, u_m(z) \) be the values of \( u \) at the points in \( \pi^{-1}(z) \), written in nondecreasing order (for \( z \in \Sigma \) the value is repeated \( s + 1 \) times, where \( s \) is the order of the corresponding point in \( \pi^{-1}(z) \) as a critical point of \( \pi \)):

\[
\begin{align*}
    u_0(z) \leq u_1(z) \leq \cdots \leq u_m(z) \leq u_m(z).
\end{align*}
\] (6)

If \( u_{j-1}(z) < u_j(z) < u_{j+1}(z) \) (for \( j = 0 \) we consider only the inequality \( u_0(z) < u_1(z) \), and for \( j = m \), only the inequality \( u_{m-1}(z) < u_m(z) \)), then we include the point \( z^{(j)} \in \pi^{-1}(z) \) such that \( u(z^{(j)}) = u_j(z) \) in the set \( \mathcal{R}^{(j)} \) (the \( j \)th sheet of the surface \( \mathcal{R} \), \( j = 0, \ldots, m \)). Otherwise, points in \( \pi^{-1}(z) \) are not included in \( \mathcal{R}^{(j)} \). For \( z = \infty \) we replace \( u(z) \) by \( u(z) - \log |z| \) in (6). Thus the sheets \( \mathcal{R}^{(j)} \) are formally defined by

\[
\begin{align*}
    \mathcal{R}^{(0)} &:= \{ z \in \mathcal{R} : 0 < u_1(z) - u(z) \}, \\
    \mathcal{R}^{(j)} &:= \{ z \in \mathcal{R} : u_{j-1}(z) - u(z) < 0 < u_{j+1}(z) - u(z) \}, \quad j = 1, \ldots, m - 1, \\
    \mathcal{R}^{(m)} &:= \{ z \in \mathcal{R} : u_{m-1}(z) - u(z) < 0 \}.
\end{align*}
\] (7)

It follows from the definition that the \( \mathcal{R}^{(j)} \) are (generally speaking, disconnected) pairwise disjoint open subsets of \( \mathcal{R} \) and the projection \( \pi : \mathcal{R}^{(j)} \to \pi(\mathcal{R}^{(j)}) \) is biholomorphic. In what follows, the point in \( \mathcal{R}^{(j)} \) lying over \( z \in \hat{\mathbb{C}} \) is denoted by \( z^{(j)} \). The boundary of the sheet \( \mathcal{R}^{(j)} \) is denoted by \( \partial(\mathcal{R}^{(j)}) \). Since \( u_1(z) - u_0(z) \to +\infty \) as \( z \to \infty \), the point \( \infty^{(0)} \) selected originally (at which we consider the germs of the functions \( f_j \)) always lies on the sheet \( \mathcal{R}^{(0)} \); this is consistent with our notation for points on sheets. It is clear that no critical point of the projection \( \pi \) lies in any \( \mathcal{R}^{(j)} \).

We set

\[
\begin{align*}
    F_j &:= \{ z \in \hat{\mathbb{C}} : u_{j-1}(z) = u_j(z) \}, \quad j = 1, \ldots, m, \\
    F &:= \bigcup_{j=1}^{m} F_j.
\end{align*}
\] (8)

It was shown in Appendix 1 to [8] that the sets \( F_j \) and \( \partial(\mathcal{R}^{(j)}) \) are (real) one-dimensional piecewise analytic sets without isolated points. The precise definition of a piecewise analytic set was also given in [8]. Informally speaking, this means that such a set is the closure of the union of a finite number of analytic arcs featuring a certain regularity at the endpoints. In particular, this implies that the sets \( F_j \) have empty interior, which immediately yields the equalities \( \pi(\partial\mathcal{R}^{(j)}) = F_j \cup F_{j+1} \) for \( j = 1, \ldots, m - 1 \), and also \( \pi(\partial\mathcal{R}^{(0)}) = F_1 \) and \( \pi(\partial\mathcal{R}^{(m)}) = F_m \).
In the same way as in [8], on the set \( \widehat{\mathbb{C}} \setminus F \) we define a matrix \( A \) by

\[
A(z) := \begin{pmatrix}
1 & f_1(z^{(0)}) & \cdots & f_m(z^{(0)}) \\
1 & f_1(z^{(1)}) & \cdots & f_m(z^{(1)}) \\
& \cdots & \cdots & \cdots \\
1 & f_1(z^{(m)}) & \cdots & f_m(z^{(m)})
\end{pmatrix}
\]  \tag{9}

(we number the rows and columns of \( A \) by the integers 0 to \( m \)). Clearly, \( \det A \in \mathcal{M}(\widehat{\mathbb{C}} \setminus F) \) (is a meromorphic function on \( \widehat{\mathbb{C}} \setminus F \)). It is easy to see that \( (\det A)^2 \) extends to a meromorphic function on the whole of \( \widehat{\mathbb{C}} \) (crossing an arc in \( F \) results only in interchanging some rows of \( A \), so that \( \det A \) can only change sign). In addition, \( \det A \neq 0 \) since the functions \( f_1, f_2, \ldots, f_m \) are independent over \( \mathbb{C}(z) \).

For arbitrary \( 0 \leq j_1 < \cdots < j_k \leq m \) we let \( M_{j_1 \cdots j_k}(z) \) denote the minor of \( A \) corresponding to the columns with numbers \( j_1, \ldots, j_k \) and rows with numbers \( 0, 1, \ldots, k-1 \). It follows from the definition that \( M_{j_1 \cdots j_k} \in \mathcal{M}(\widehat{\mathbb{C}} \setminus F) \). Moreover, if \( \mathcal{K}_k \) is connected, then \( M_{j_1 \cdots j_k} \) does not vanish identically in any domain in \( \widehat{\mathbb{C}} \setminus F \) (see Proposition 2). Then for any \( 0 \leq j_1 < \cdots < j_k \leq m \) and \( 0 \leq i_1 < \cdots < i_k \leq m \) the ratio \( M_{j_1 \cdots j_k}(z)/M_{i_1 \cdots i_k}(z) \) is a meromorphic function on \( \widehat{\mathbb{C}} \setminus F_k \). In fact, crossing an arc in \( F \) results in interchanging certain rows of the matrix \( A \) (whose numbers correspond to the sheets with common boundary projecting onto this arc). Moreover, the rows with numbers 0 to \( k-1 \) cannot interchange with rows with numbers \( k \) to \( m \) when we cross \( F \setminus F_k \) (since \( u_k \neq u_{k-1} \) on \( F \setminus F_k \), and so the sheets with numbers 0 to \( k-1 \) have no common boundary points with the sheets with numbers \( k \) to \( m \)). So when we cross \( F \setminus F_k \), either all minors \( M_{j_1 \cdots j_k}(z) \) remain unchanged or all of them change sign simultaneously. Hence the functions \( M_{j_1 \cdots j_k}(z)/M_{i_1 \cdots i_k}(z) \) glue together into meromorphic functions on \( F \setminus F_k \).

We introduce some notation we adhere to in what follows. We denote weak* convergence by \( \rightharpoonup^* \) (indicating the space in which it is considered if necessary). We use \( \overset{\text{cap}}{\rightharpoonup} \) to denote convergence in (logarithmic) capacity (indicating the set on which it is considered if necessary). Let \( d\sigma := (i/2\pi)(dz \wedge d\overline{z})/(1 + |z|^2)^2 \) be the normalized area form of the spherical metric on \( \widehat{\mathbb{C}} \). In order to speak about the asymptotic behaviour of the \( k \)th polynomials of the Hermite-Padé \( m \)-system, we need to fix their normalization. Therefore, along with the polynomials \( P_{n; i_1 \cdots i_k} \) (see (1)) we will also use the polynomials \( P^*_{n; i_1 \cdots i_k} := c_{n; i_1 \cdots i_k} P_{n; i_1 \cdots i_k} \) (the \( c_{n; i_1 \cdots i_k} \) are positive constants) for which the functions \( \log |P^*_{n; i_1 \cdots i_k}| \) are spherically normalized:

\[
\int_{\widehat{\mathbb{C}}} \log |P^*_{n; i_1 \cdots i_k}| \, d\sigma = 0. \tag{10}
\]

Note that relation (1) does not hold for the \( P^*_{n; i_1 \cdots i_k} \) in general. We also denote the standard analogue of the Laplace operator on Riemann surfaces by \( dd^c \); in the general case this operator transforms currents of degree 0 into currents of degree 2 and, in a local coordinate \( \zeta = x + iy \), it acts on smooth functions \( \varphi \) by \( dd^c \varphi = (\varphi_{xx} + \varphi_{yy}) \, dx \, dy = \Delta \varphi \, dx \, dy \). (For the properties of the \( dd^c \) and results in potential theory on compact Riemann surfaces that we require, see [8], Appendix 2, and [5]. We refer to these facts retaining the corresponding notation where possible.)
So recall that the $P_{n;j_1,\ldots,j_k}$ are the $k$th polynomials of the Hermite-Padé $m$-system constructed from the tuple of germs $[f, f_1, \ldots, f_m]$ at $\infty$ of the functions $f_j$, $j = 1, \ldots, m$, which are meromorphic on the Riemann surface $\mathcal{R}$. Then the following results hold for these polynomials.

**Theorem 1.** Assume that the surface $\tilde{\mathcal{R}}_{[k]}$ constructed in terms of $\pi$ is connected. Then the following hold.

1) There exists $L \in \mathbb{N}$ such that, for any neighbourhood $V$ of the compact set $F_k$ and all sufficiently large $n$: $n > N = N(V)$, of the polynomials $P_{n;i_1,\ldots,i_k}$ have at most $L$ zeros outside $V$.

2) For any $p \in [1, \infty)$, as $n \to \infty$,

$$\frac{1}{n} \log |P_{n;i_1,\ldots,i_k}(z)| \to -\sum_{s=0}^{k-1} u_s(z) \text{ in } L^p(\hat{\mathbb{C}}, d\sigma) \quad (11)$$

where the function $\sum_{s=0}^{k-1} u_s(z)$ is spherically normalized: $\int_{\hat{\mathbb{C}}} \sum_{s=0}^{k-1} u_s(z) d\sigma = 0$.

3) As $n \to \infty$,

$$\frac{1}{n} \frac{dd^e \log |P_{n;i_1,\ldots,i_k}(z)|}{dz^c} \to -dd^c \left( \sum_{s=0}^{k-1} u_s(z) \right) \text{ in } C(\hat{\mathbb{C}})^* \quad (12)$$

**Theorem 2.** Assume that the surface $\tilde{\mathcal{R}}_{[k]}$ constructed in terms of $\pi$ is connected. Then for each compact set $K \subset \mathbb{C} \setminus F_k$, as $n \to \infty$,

$$\frac{P_{n;j_1,\ldots,j_k}(z)}{P_{n;i_1,\ldots,i_k}(z)} \stackrel{\text{cap}}{\to} \frac{M_{j_1,\ldots,j_k}(z)}{M_{i_1,\ldots,i_k}(z)}, \quad z \in K. \quad (13)$$

Moreover, for an arbitrary $\varepsilon > 0$

$$\text{cap}\left\{ z \in K : \left| \frac{P_{n;j_1,\ldots,j_k}(z)}{P_{n;i_1,\ldots,i_k}(z)} - \frac{M_{j_1,\ldots,j_k}(z)}{M_{i_1,\ldots,i_k}(z)} \right|^{1/n} e^{u_k(z) - u_{k-1}(z)} \geq 1 + \varepsilon \right\} \to 0. \quad (14)$$

**§ 3. Recovering the values of a meromorphic function on $\mathcal{R}$ from a germ of it via the Hermite-Padé $m$-system**

In this section we consider the following problem. Let $f \in \mathcal{M}(\mathcal{R})$. Assume that the germ of a fixed multivalued analytic function $f(\pi^{-1}(z))$ at a point $z_0 \in \hat{\mathbb{C}}$ is defined in terms of its Taylor series. (Without loss of generality we assume that $z_0 = \infty$.) The question is how to recover the values of $f$ constructively in ‘as large a domain as possible’ on $\mathcal{R}$. The most obvious way is to use Weierstrass continuation by re-expanding the Taylor series at points ‘close’ to the boundary of the disc of convergence. However, this method is not constructive (see [6]). Another way is to use Padé approximation. By Stahl’s theorem (see [17]) Padé approximants recover the values of $f$ in a domain $D$ on $\mathcal{R}$ which projects one-to-one onto $\pi(D) = \hat{\mathbb{C}} \setminus S$, where $S$ is the Stahl compact set, consisting of a finite number of analytic arcs (featuring a certain regularity at the endpoints). So we can say that the Padé...
Thus, $M_{0,1,\ldots,k-1}(z) := \det \begin{pmatrix} 1 & f(z^{(0)}) & \cdots & f^{k-1}(z^{(0)}) \\ 1 & f(z^{(1)}) & \cdots & f^{k-1}(z^{(1)}) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & f(z^{(k-1)}) & \cdots & f^{k-1}(z^{(k-1)}) \end{pmatrix}$

and

$M_{0,1,\ldots,k-2,k}(z) := \det \begin{pmatrix} 1 & f(z^{(0)}) & \cdots & f^{k-2}(z^{(0)}) & f^k(z^{(0)}) \\ 1 & f(z^{(1)}) & \cdots & f^{k-2}(z^{(1)}) & f^k(z^{(1)}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & f(z^{(k-1)}) & \cdots & f^{k-2}(z^{(k-1)}) & f^k(z^{(k-1)}) \end{pmatrix}$.

Thus, $M_{0,1,\ldots,k-1}(z)$ is a Vandermonde determinant:

$M_{0,1,\ldots,k-1}(z) = \prod_{0 \leq i < j < k} (f(z^{(i)}) - f(z^{(j)})),$

and $M_{0,1,\ldots,k-2,k}(z)$ is the determinant of the matrix obtained from the same Vandermonde matrix by increasing the powers of all elements in the last column by 1, that is,

$M_{0,1,\ldots,k-2,k}(z) = \sum_{s=0}^{k-1} f(z^{(s)}) \prod_{0 \leq i < j < k} (f(z^{(i)}) - f(z^{(j)}))$.

Therefore, $M_{0,1,\ldots,k-2,k}(z)/M_{0,1,\ldots,k-1}(z) = \sum_{s=0}^{k-1} f(z^{(s)})^k$. The next result now follows from Theorem 2.

**Corollary 1.** Assume that the surface $\tilde{R}^{(k)}$ constructed in terms of $\pi$ is connected. Let $f \in \mathcal{M}(\mathcal{R})$ and let the functions $1, f, f^2, \ldots, f^m$ be independent over $\mathbb{C}(z)$. Set $f_j := f^j$. Then for any compact set $K \subset \mathbb{C} \setminus F_k$, as $n \to \infty$,

$\frac{P_{n;0,1,\ldots,k-2,k}(z)}{P_{n;0,1,\ldots,k-1}(z)} \underset{\text{cap}}{\to} \sum_{s=0}^{k-1} f(z^{(s)}), \quad z \in K. \quad (15)$

Moreover, for an arbitrary $\varepsilon > 0$,

$\text{cap} \left\{ z \in K \mid \frac{P_{n;0,1,\ldots,k-2,k}(z)}{P_{n;0,1,\ldots,k-1}(z)} - \sum_{s=0}^{k-1} f(z^{(s)})^n e^{u_k(z) - u_{k-1}(z)} \geq 1 + \varepsilon \right\} \to 0. \quad (16)$
Assume that the projection $\pi$ is such that the surfaces $\tilde{R}_k[R]$ are connected for $k = 1, \ldots, m$. (We show in Statement 4 that this property holds for all $\pi$ satisfying the condition that all critical points of $\pi$ are of the first order and there is at most one critical point of $\pi$ over each $z \in \hat{C}$, so that the class of such $\pi$ is quite wide.) Now, evaluating the $k$th polynomials of the Hermite-Padé $m$-system for all $k = 1, \ldots, m$ and considering the ratios $P_{n:0,1,\ldots,k-2,k}/P_{n:0,1,\ldots,k-1}(z)$ we can asymptotically recover the sums $\sum_{s=0}^{k-1} f(z^{(s)})$ outside $\pi^{-1}(F_k)$ in succession. Hence (since $F := \bigcup_{j=1}^m F_j$) we also recover the values of $f$ on all Nuttall sheets of $R$, apart from $R^{(m)}$, outside the set $\pi^{-1}(F)$. Since to evaluate the $k$th polynomials of an Hermite-Padé $m$-system of order $n$ it suffices to know the first $(m+1)n$ Taylor coefficients of the germs defining the polynomials (see definition (1)), our approximants $P_{n:0,1,\ldots,k-2,k}/P_{n:0,1,\ldots,k-1}(z)$ are found constructively. Note that the idea of using suitable polynomials to recover the sum of the values of the function $f$ on the first $k$ Nuttall sheets, rather than the values on sheets themselves, was expressed for the first time in [18].

§ 4. The Riemann surface $\tilde{R}_k[R]$ and the definition of the $k$th polynomials of the Hermite-Padé $m$-system in terms of this surface

First of all, fix $k \in \{1, \ldots, m\}$. We introduce a compact Riemann surface $\tilde{R}_k[R]$ associated with $R$, which is in general disconnected. Along with $\tilde{R}_k[R]$ we introduce a branched covering $\tilde{\pi}: \tilde{R}_k[R] \to \hat{C}$, which is constructed from the projection $\pi$. Since $\pi: R \to \hat{C} (z \mapsto z)$ is an $(m+1)$-sheeted holomorphic branched covering of $\hat{C}$, and since $\Sigma$ is the set of critical values of the projection $\pi$, we can regard $R$ as the standard compactification of the Riemann surface $R'$ of an $(m+1)$-valued global analytic function (GAF) $w(z)$ in the domain $\hat{C} \setminus \Sigma$, so that $z = (z, w(z))$. Informally speaking, $\tilde{R}_k[R]$ is the standard compactification of the Riemann surface $R'_k[R]$ of all unordered $k$-tuples of distinct germs of the function $w(\cdot)$ at the same point $z \in \hat{C} \setminus \Sigma$. More precisely, $\tilde{R}_k[R]$ consists of the pairs $(z, \{w_{\tilde{1}}^z, \ldots, w_{\tilde{k}}^z\})$, where $z \in \hat{C} \setminus \Sigma$ and $\{w_{\tilde{1}}^z, \ldots, w_{\tilde{k}}^z\}$ is an unordered $k$-tuple of distinct germs of $w(\cdot)$ at the point $z$. We introduce the structure of a Riemann surface on this set similarly to the procedure of constructing the Riemann surface of a GAF. Thus, by a neighbourhood of a point $(z_0, \{w_{\tilde{1}}^{z_0}, \ldots, w_{\tilde{k}}^{z_0}\})$ we mean the set of points $(z, \{w_{\tilde{1}}^z, \ldots, w_{\tilde{k}}^z\})$ such that: 1) $z \in B_{z_0}(\delta)$, where $B_{z_0}(\delta)$ is the disc with centre $z_0$ and radius $\delta$ such that the germs $w_{\tilde{1}}^{z_0}, \ldots, w_{\tilde{k}}^{z_0}$ are holomorphic in $B_{z_0}(\delta)$; 2) there exists a bijection between the elements of the tuples $\{w_{\tilde{1}}^{z_0}, \ldots, w_{\tilde{k}}^{z_0}\}$ and $\{w_{\tilde{1}}^z, \ldots, w_{\tilde{k}}^z\}$, such that the corresponding germs are direct analytic continuations of each other. (Since $w(\cdot)$ is an $(m+1)$-valued function, there exists at most one such bijection.) It is clear that the Riemann surface $R'_k[R]$ is a (disconnected, in general) $(m+1)k$-sheeted holomorphic covering space of $\hat{C} \setminus \Sigma$ (with natural projection $z: (z, \{w_{\tilde{1}}^z, \ldots, w_{\tilde{k}}^z\}) \mapsto z$). Next, using the standard procedure of compactifying a finite-sheeted covering of the Riemann sphere $\hat{C}$ with a finite number of punctures, from $\tilde{R}_k[R]$ we obtain a (disconnected, in general) compact Riemann surface $\tilde{R}_k[R]$ and a holomorphic covering $\tilde{\pi}: \tilde{R}_k[R] \to \hat{C}$ (branched at the points in $\Sigma$), which extend the original covering $\pi: R \to \hat{C}$.
z: \( \mathfrak{R}'_{[k]} \rightarrow \hat{\mathbb{C}} \setminus \Sigma \). We denote points on \( \mathfrak{R}'_{[k]} \) by symbols in boldface with ‘tildes’; their projections will, as before, be denoted by the corresponding symbols in lightface (for example, \( \tilde{z} \in \mathfrak{R}'_{[k]} \) and \( \tilde{\pi}(\tilde{z}) = z \)). We emphasize that, unlike Theorems 1 and 2, in this section the surface \( \mathfrak{R}'_{[k]} \) may be disconnected.

For \( z \in \hat{\mathbb{C}} \setminus F \) all inequalities in (6) are strict, hence \( \pi^{-1}(z) = \{z^{(0)}, z^{(1)}, \ldots, z^{(m)}\} \) for such \( z \), where \( z^{(j)} \in \mathfrak{R}^{(j)} \). So the Riemann surface \( \mathfrak{R}_{[k]} \) decomposes into \( (m+1 \choose k) \) disjoint sheets \( \mathfrak{R}_{[k]}^{(j_1 j_2 \ldots j_k)} \), \( 0 \leq j_1 < j_2 < \cdots < j_k \leq m \), over \( \hat{\mathbb{C}} \setminus F \). The sheet \( \mathfrak{R}_{[k]}^{(j_1 j_2 \ldots j_k)} \) consists of the unordered tuples \( \{w^{z^{(j_1)}}(\cdot), w^{z^{(j_2)}}(\cdot), \ldots, w^{z^{(j_k)}}(\cdot)\} \) considered for all \( z \in \hat{\mathbb{C}} \setminus F \), where \( w^{z^{(j)}}(\cdot) \) is the germ of \( w(\cdot) \) at \( z^{(j)} \). It is clear that on all sheets the mapping \( \tilde{\pi}: \mathfrak{R}_{[k]}^{(j_1 j_2 \ldots j_k)} \rightarrow \hat{\mathbb{C}} \setminus F \) is biholomorphic. The point on the sheet \( \mathfrak{R}_{[k]}^{(j_1 j_2 \ldots j_k)} \) lying over \( z \in \hat{\mathbb{C}} \setminus F \) is denoted by \( \tilde{z}^{(j_1 j_2 \ldots j_k)} \). Note that for \( z \in \hat{\mathbb{C}} \setminus F_k \) we have \( u_{k-1}(z) < u_k(z) \) in (6). Therefore, the unordered tuple \( \{w^{z^{(0)}}(\cdot), w^{z^{(1)}}(\cdot), \ldots, w^{z^{(k-1)}}(\cdot)\} \) is defined over \( z \in \hat{\mathbb{C}} \setminus F_k \). Moreover, extending it along all possible paths in \( \hat{\mathbb{C}} \setminus F_k \), we always obtain the same tuple. Hence the sheet \( \mathfrak{R}_{[k]}^{(01 \ldots k-1)} \) is defined over \( \hat{\mathbb{C}} \setminus F_k \) (and not only over \( \hat{\mathbb{C}} \setminus F \)) and the projection \( \tilde{\pi}: \mathfrak{R}_{[k]}^{(01 \ldots k-1)} \rightarrow \hat{\mathbb{C}} \setminus F_k \) is biholomorphic on it.

Our next aim is to give another definition of the \( k \)th polynomials of the Hermite-Padé \( m \)-system, in terms of certain new meromorphic functions on \( \mathfrak{R}_{[k]} \), which are constructed from the original functions \( f_1, \ldots, f_m \). For any sets of indices \( 0 \leq j_1 < j_2 < \cdots < j_k \leq m \) and \( 0 \leq l_1 < l_2 < \cdots < l_k \leq m \) we let \( M_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z) \) denote the minor of the matrix \( A \) in (9) corresponding to the columns numbered \( j_1, \ldots, j_k \) and rows numbered \( l_1, l_2, \ldots, l_k \). (In particular, \( M_{j_1 \ldots j_k}^{0 \ldots k-1} = M_{j_1 \ldots j_k} \)) Let \( A_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z) \) denote the cofactor of \( M_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z) \) (that is, \( A_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z) \) is the complementary minor of \( M_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z) \) taken with sign \((-1)^{j_1 + \cdots + j_k + l_1 + \cdots + l_k}) \). Like the matrix \( A \), the minors \( M_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z) \) and their cofactors \( A_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z) \) are defined for \( z \in \hat{\mathbb{C}} \setminus F \). We denote the matrices corresponding to the minors \( M_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z) \) and cofactors \( A_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z) \) by \( \|M_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z)\| \) and \( \|A_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z)\| \), respectively. It is well known (see [13], Theorem 1.2.4.1, for example) that, since \( \det A \neq 0 \), we have

\[
\|M_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z)\| \left\| \begin{array}{c} A_{j_1 \ldots j_k}^{l_1 \ldots l_k}(z) \\ \det A(z) \end{array} \right\| = \text{Id}, \tag{17}
\]

where \( \text{Id} \) is the \( (m+1 \choose k) \times (m+1 \choose k) \) identity matrix. For \( z \in \hat{\mathbb{C}} \setminus F \) we consider the matrix

\[
M_w(z) := \|w^j(z^{(l)})\|_{j=0}^m = \begin{pmatrix} 1 & w(z^{(0)}) & \ldots & w^m(z^{(0)}) \\ 1 & w(z^{(1)}) & \ldots & w^m(z^{(1)}) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & w(z^{(m)}) & \ldots & w^m(z^{(m)}) \end{pmatrix}. \tag{18}
\]

For each set of indices \( 0 \leq l_1 < l_2 < \cdots < l_k \leq m \) we let \( M_{w}^{l_1 \ldots l_k}(z) \) denote the minor of \( M_w(z) \) corresponding to the columns numbered \( 0, 1, \ldots, k-1 \) and rows numbered \( l_1, l_2, \ldots, l_k \). Note that \( M_{w}^{l_1 \ldots l_k}(z) \neq 0 \): this is a Vandermonde
Proposition 1. For each set of indices $d_1, \ldots, d_k$ the determinant, hence

$$M^{l_1, \ldots, l_k}_w(z) = \prod_{1 \leq i < j \leq k} (w(z^{(j)}) - w(z^{(i)})),$$

and $w(z^{(s)}) \neq w(z^{(t)})$ for $s \neq t$ and $z \in \mathbb{C} \setminus F$, by the definition of $w(z)$ as the algebraic function defining the surface $\mathfrak{R}$. Therefore, for each set of indices $0 \leq j_1 < j_2 < \cdots < j_k \leq m$ the following functions are well defined for $z \in \mathfrak{R}[k] \setminus \bar{\pi}^{-1}(F)$:

$$M_{j_1, \ldots, j_k}(z) := \frac{M^{l_1, \ldots, l_k}_w(z)}{\det A},$$

$$A_{j_1, \ldots, j_k}(z) := \frac{A^{l_1, \ldots, l_k}_w(z)M^{l_1, \ldots, l_k}_w(z)}{\det A}. \tag{19}$$

Proposition 1. For each set of indices $0 \leq j_1 < \cdots < j_k \leq m$ the functions $M_{j_1, \ldots, j_k}(z)$ and $A_{j_1, \ldots, j_k}(z)$ extend to meromorphic functions on the whole of the Riemann surface $\mathfrak{R}[k]$.

Proof. First we prove the conclusion of the proposition for the $M_{j_1, \ldots, j_k}(z)$. From definition (19) it is clear that the function $M_{j_1, \ldots, j_k}(z)$ is meromorphic on $\mathfrak{R}[k] \setminus \bar{\pi}^{-1}(F)$. Since, as noted above, $F$ is a one-dimensional piecewise analytic set without isolated points (that is, $F$ is in fact the closure of a finite number of analytic arcs), the set $\bar{\pi}^{-1}(F)$ has the same property. Hence it suffices to check that when two arcs in $\bar{\pi}^{-1}(F)$, both separating a sheet $\mathfrak{R}[k]$ from $\mathfrak{R}[k]$, intersect, $M_{j_1, \ldots, j_k}(z)$ and $M_{j_1, \ldots, j_k}(\bar{z})$ glue together into a single function. In fact, when we go over from $\mathfrak{R}[k]$ to $\mathfrak{R}[k]$ across some arc, in crossing the projection of this arc the minor $M^{l_1, \ldots, l_k}_w(z)$ is transformed into $M^{l_1, \ldots, l_k}_w(z)$ up to a sign, and $M^{l_1, \ldots, l_k}_w(z)$ is transformed into $M^{l_1, \ldots, l_k}_w(z)$ up to a sign. In addition, $M^{l_1, \ldots, l_k}_w(z)$ and $M^{l_1, \ldots, l_k}_w(z)$ do or do not change sign simultaneously, and this depends only on the gluing pattern of the sheets of the original surface $\mathfrak{R}$.

Now let us prove the proposition for $A_{j_1, \ldots, j_k}(z)$. By (17) we have

$$\|M_{j_1, \ldots, j_k}(z)\| \|A_{j_1, \ldots, j_k}(z)\| T = \text{Id}. \tag{21}$$

Thus, for $z \in \mathbb{C} \setminus F$ the values of $A_{j_1, \ldots, j_k}(z)$ on sheets of the surface $\mathfrak{R}[k]$ are components of the row with subscripts $j_1 < \cdots < j_k$ of the inverse matrix of the matrix whose columns are formed by the values of the functions $M_{i_1, \ldots, i_k}(z)$ on sheets of this surface. The functions $A_{j_1, \ldots, j_k}(z)$ defined in this way in terms of the meromorphic functions $M_{i_1, \ldots, i_k}(z)$, can always be extended to meromorphic functions on $\mathfrak{R}[k]$. (It is worth pointing out that it is sufficient for this purpose that the original functions $M_{j_1, \ldots, j_k}(z)$, which are meromorphic on $\mathfrak{R}[k]$, are merely linearly independent over $\mathbb{C}(z)$; this is equivalent to the condition that $\det \|M_{j_1, \ldots, j_k}(z)\| \neq 0$, that is, they are not necessarily minors of the matrix $A$.) To check this we must use the well-known formula, which expresses elements of the inverse matrix as the cofactors of elements of the original matrix divided by its determinant. Using this formula it is easy to verify that when the boundaries of sheets are crossed,
the functions \( A_{j_1,\ldots,j_k}(\widehat{z}) \) are glued into a single function. Note that the analogous property of a system of meromorphic functions associated with a Nuttall partition was pointed out in \([11]\); also see \([8]\), Proposition 2. The proof is complete.

Now we can give a new definition of the \( k \)th polynomials of the Hermite-Padé \( m \)-system.

**Theorem 3.** There exists \( p \in \mathbb{N} \cup \{0\} \) (independent of \( n \)) such that the \( k \)th polynomials of the Hermite-Padé \( m \)-system \( P_{n;i_1,\ldots,i_k} \) satisfy the relations

\[
\sum_{0 \leq i_1 < \cdots < i_k \leq m} P_{n;i_1,\ldots,i_k}(z)A_{i_1,\ldots,i_k}(\widehat{z}^{(l_1\ldots l_k)}) = O\left(\frac{1}{z^{nk+1-p}}\right) \quad \text{as } z \to \infty \tag{22}
\]

for all \( 0 < l_1 < l_2 < \cdots < l_k \leq m \).

**Proof.** Let \( P_{n;i_1,\ldots,i_k}(z) \) be the \( k \)th polynomials of the Hermite-Padé \( m \)-system, that is, the solutions of (1). Set

\[
\widehat{R}_n(\widehat{z}) := \sum_{0 \leq i_1 < \cdots < i_k \leq m} P_{n;i_1,\ldots,i_k}(z)A_{i_1,\ldots,i_k}(\widehat{z}). \tag{23}
\]

Since \( A_{i_1,\ldots,i_k}(\widehat{z}) \in \mathcal{M}(\widehat{R}_k) \) (see Proposition 1), we also have \( \widehat{R}_n \in \mathcal{M}(\widehat{R}_k) \). We need to check that the right-hand side of (22) holds for \( \widehat{R}_n \). We look at the identities

\[
\widehat{R}_n(\widehat{z}^{(l_1\ldots l_k)}) = \sum_{0 \leq i_1 < \cdots < i_k \leq m} P_{n;i_1,\ldots,i_k}(z)A_{i_1,\ldots,i_k}(\widehat{z}^{(l_1\ldots l_k)}) \tag{24}
\]

for all sets of indices \( 0 \leq l_1 < l_2 < \cdots < l_k \leq m \) as a system of homogeneous linear equations with respect to the \( P_{n;i_1,\ldots,i_k}(z) \). The matrices \( \|M_{i_1,\ldots,i_k}(\widehat{z}^{(l_1\ldots l_k)})\| \) and \( \|A_{i_1,\ldots,i_k}(\widehat{z}^{(l_1\ldots l_k)})\|^T \) are mutually inverse (see (21)), hence solving this system we have

\[
P_{n;i_1,\ldots,i_k}(z) = \sum_{0 \leq l_1 < \cdots < l_k \leq m} M_{i_1,\ldots,i_k}(\widehat{z}^{(l_1\ldots l_k)})\widehat{R}_n(\widehat{z}^{(l_1\ldots l_k)}). \tag{25}
\]

Plugging these expressions for the \( P_{n;i_1,\ldots,i_k} \) into (1), we see that for any fixed set of indices \( 0 \leq j_1 < \cdots < j_k \leq m \),

\[
\sum_{0 \leq l_1 < \cdots < l_k \leq m} \left[ M_{j_1,\ldots,j_k}(\widehat{z}^{(l_1\ldots l_k)})
\right.
\left. + \sum_{s=1}^{k} (-1)^s M_{0,j_1,\ldots,j_{s-1},j_{s+1},\ldots,j_k}(\widehat{z}^{(l_1\ldots l_k)})f_{j_s,\infty}(z) \right] \widehat{R}_n(\widehat{z}^{(l_1\ldots l_k)}) = O\left(\frac{1}{z^{kn+1}}\right) \tag{26}
\]

as \( z \to \infty \). Substituting the explicit expression for \( M_{i_1,\ldots,i_k}(\widehat{z}^{(l_1\ldots l_k)}) \) from (19) into (26) and taking into account that \( f_{j_s,\infty}(z) = f_{j_s}(z^{(0)}) \) in a neighbourhood of
infinity, we obtain
\[
\sum_{0 \leq l_1 < \cdots < l_k \leq m} \left[ M_{j_1, \ldots, j_k}^{l_1, \ldots, l_k} (z) \right] \\
+ \sum_{s=1}^{k} (-1)^s M_{0,j_1,\ldots,j_{s-1},j_{s+1},\ldots,j_k}^{l_1,\ldots,l_k} (z) f_{j_s} (z^{(0)}) \right] \frac{\tilde{R}_n (z^{(l_1 \ldots l_k)})}{M_w^{l_1,\ldots,l_k} (z)} = O \left( \frac{1}{z^{kn+1}} \right) 
\tag{27}
\]
as \(z \to \infty\). Note that in square brackets in (27) we have the expansion of the minor \(M_{0,j_1,\ldots,j_k}^{l_1,\ldots,l_k} (z)\) of \(A\) with respect to the zeroth row. (In particular, for \(l_1 = 0\) the expression in square brackets is zero.) Hence (27) is equivalent to the following:
\[
\sum_{0 < l_1 < \cdots < l_k \leq m} \frac{M_{0,j_1,\ldots,j_k}^{0,l_1,\ldots,l_k} (z)}{M_w^{l_1,\ldots,l_k} (z)} \tilde{R}_n (z^{(l_1 \ldots l_k)}) = O \left( \frac{1}{z^{kn+1}} \right) \text{ as } z \to \infty. 
\tag{28}
\]

Let \(O_\infty = \{ z \in \mathbb{C} : |z| > \delta \}\) be a neighbourhood of infinity such that \(O_\infty \cap F_1 = \emptyset\). (Recall that \(F_1\) is the projection of the boundary of the sheet \(\mathfrak{R}^{(0)}\) of the original surface \(\mathfrak{R}\).) Then the sheets \(\mathfrak{R}_k^{(0s2\ldots sk)}\) and \(\mathfrak{R}_k^{(l_1\ldots l_k)}\) have no common boundary points over \(O_\infty\) for \(l_1 > 0\). Hence \(\mathfrak{R}_k^{(0s2\ldots sk)} \cap \pi^{-1}(O_\infty) = O_\infty^0 \cup O_\infty^1\), where \(O_\infty^0 := \pi^{-1}(O_\infty) \cap \bigcup_{0 \leq l_1 < \cdots < l_k \leq m} \mathfrak{R}_k^{(0l_2\ldots l_k)}\) and \(O_\infty^1 := \pi^{-1}(O_\infty) \cap \bigcup_{0 < l_1 < \cdots < l_k \leq m} \mathfrak{R}_k^{(l_1\ldots l_k)}\). For \(0 < j_1 < \cdots < j_k \leq m\) we define functions \(B_{j_1,\ldots,j_k}\) on \(O_\infty^1 \setminus \pi^{-1}(F)\) by
\[
B_{j_1,\ldots,j_k} (z^{(l_1 \ldots l_k)}) := \frac{M_{0,j_1,\ldots,j_k}^{0,l_1,\ldots,l_k} (z)}{M_w^{l_1,\ldots,l_k} (z)}. 
\tag{29}
\]
Since 1) for fixed \(0 < l_1 < \cdots < l_k \leq m\) the function \(B_{j_1,\ldots,j_k} (z^{(l_1 \ldots l_k)})\) is identically equal to the expression in square brackets in (26), 2) by Proposition 1 all functions \(M_{i_1,\ldots,i_k} (z)\) are meromorphic on \(\mathfrak{R}_k\) (and in particular, on \(O_\infty^1\), and 3) the germs \(f_{j,\infty}\) are holomorphic in \(O_\infty\), the functions \(B_{j_1,\ldots,j_k} (z)\) extend to meromorphic functions on \(O_\infty^1\).

Now we look at the expressions (28) for arbitrary \(0 < j_1 < \cdots < j_k \leq m\) as a system of homogeneous linear equations with respect to the \(\tilde{R}_n (z^{(l_1 \ldots l_k)})\), \(0 < l_1 < \cdots < l_k \leq m\), assuming that \(z \in O_\infty^1 \setminus \pi^{-1}(F)\). First of all we evaluate the determinant \(\| B_{j_1,\ldots,j_k} (z^{(l_1 \ldots l_k)}) \|\) of the matrix of this system. By the generalized Sylvester identity (for example, see [13], §2.7), which expresses the determinant of the matrix of prescribed-order minors containing a fixed corner block (or, as in our case, an element) in terms of the determinant of the original matrix, we have \(\det \| M_{0,j_1,\ldots,j_k}^{0,l_1,\ldots,l_k} (z) \| = (\det A)^{c_{m-1}}\). Therefore,
\[
\det \| B_{j_1,\ldots,j_k} (z^{(l_1 \ldots l_k)}) \| = \det \left\| \frac{M_{0,j_1,\ldots,j_k}^{0,l_1,\ldots,l_k} (z)}{M_w^{l_1,\ldots,l_k} (z)} \right\| = \frac{(\det A)^{c_{m-1}}}{\prod_{0 < l_1 < \cdots < l_k \leq m} M_w^{l_1,\ldots,l_k} (z)}. 
\tag{30}
\]
Since, as pointed out above, \(\det A \neq 0\) (because the original functions \(1, f_1, \ldots, f_m\) are linearly independent over \(\mathbb{C}(z)\)) and \(M_w^{l_1,\ldots,l_k} (z) \neq 0\) for all \(0 \leq l_1 < \cdots < l_k \leq m\),
(see the definition of the functions $M_{j_1,...,j_k}(\bar{z}^{(l_1,...,l_k)})$ in (19)), we also have $\det \|B_{j_1,...,j_k}(\bar{z}^{(l_1,...,l_k)})\| \neq 0$. Hence the matrix $\|B_{j_1,...,j_k}(\bar{z}^{(l_1,...,l_k)})\|$ is invertible. Moreover, since the columns of $\|B_{j_1,...,j_k}(\bar{z}^{(l_1,...,l_k)})\|$ contain the values of meromorphic functions on $O_{\infty}^1$ on sheets of this surface, we see that the rows of the inverse matrix $\|B_{j_1,...,j_k}(\bar{z}^{(l_1,...,l_k)})\|^{-1}$ also consist of the values on sheets of $O_{\infty}^1$ of meromorphic functions on this surface. (This property, which can be verified directly, was in fact established in the proof of Proposition 1 for the functions $A_{j_1,...,j_k}(\bar{z})$: here we just need to replace $R_{[k]}$ by $O_{\infty}^1$.) Thus the singularities of the matrix $\|B_{j_1,...,j_k}(\bar{z}^{(l_1,...,l_k)})\|^{-1}$ over $\infty$ are ‘finite’. More precisely, let $O \subset O_{\infty}$ be a neighbourhood of $\infty$ such that on $\pi^{-1}(O)$ the meromorphic functions in rows of the matrix $\|B_{j_1,...,j_k}(\bar{z}^{(l_1,...,l_k)})\|^{-1}$ can only have poles at points in $\pi^{-1}(\infty)$. Then there exists $p \in \mathbb{N} \cup \{0\}$ such that $z^p \|B_{j_1,...,j_k}(\bar{z}^{(l_1,...,l_k)})\|^{-1}$ (the matrix obtained from $\|B_{j_1,...,j_k}(\bar{z}^{(l_1,...,l_k)})\|^{-1}$ by multiplying each element of it by $z^p$) is bounded on $O$. Solving system (28) for $R_n(\bar{z}^{(l_1,...,l_k)})$ we arrive at the required result.

Remark 3. In the case when $\infty \notin \Sigma$ and the functions $A_{i_1,...,i_k}(\bar{z})$ have no poles at points in $\pi^{-1}(\infty)$, there always exist polynomials $\widetilde{P}_{n;i_1,...,i_k}(z)$ of degree at most $(m+1-k)n$ that satisfy (22) for $p = 0$. In fact, it is easy to see that in this case conditions (22), similarly to (1), form a system of $(m+1-k)(\binom{m}{k})$ homogeneous linear equations with respect to the $(n(m+1-k)+1)(\binom{m+1}{k}) = n(m+1)(\binom{m}{k})+(m+1)$ unknown coefficients of the polynomials $\widetilde{P}_{n;i_1,...,i_k}(z)$. Moreover, in the case when $\infty \notin \Sigma$ all germs $A_{i_1,...,i_k}(\bar{z}^{(l_1,...,l_k)})$, where $0 \leq i_1 < \cdots < i_k \leq m$ and $0 < l_1 < \cdots < l_k \leq m$, can be replaced by an arbitrary tuple of holomorphic germs, and there will still exist polynomials $\widetilde{P}_{n;i_1,...,i_k}(z)$ of degree at most $(m+1-k)n$ that satisfy (22) for $p = 0$. (However, they will have nothing to do with our original problem.)

However, in the case when $\infty \in \Sigma$ the situation is completely different. In this case we cannot a priori (without recourse to Theorem 3) assert that the polynomials $\widetilde{P}_{n;i_1,...,i_k}(z)$ exist, even if we replace the right-hand side of (22) by $O(z^{-1})$ and the functions $A_{i_1,...,i_k}(\bar{z})$ have no poles at the points in $\pi^{-1}(\infty)$. This is because the $A_{i_1,...,i_k}(\bar{z})$, regarded as functions of $z \in \mathbb{C}$, can have branch points in the set $\pi^{-1}(\infty)$.

The following corollary shows that under some assumption of ‘general position’ system (22) for $p = 0$ gives a new definition of the $k$th polynomials of the Hermite-Padé $m$-system, which is completely equivalent to the original definition. (All the notation used in this statement is inherited from the proof of Theorem 3.)

Corollary 2. Let $\infty \notin \Sigma$ and assume that the functions $B_{j_1,...,j_k}(\bar{z})$ and the functions in the rows of the matrix $\|B_{j_1,...,j_k}(\bar{z}^{(l_1,...,l_k)})\|^{-1}$, have no poles at points in $\pi^{-1}(\infty)$. Then the $k$th polynomials of the Hermite-Padé $m$-system (as defined in (1)) coincide with the polynomials of degree at most $(m+1-k)n$ that satisfy (22) for $p = 0$.

Proof. Let $\{P_{k;i_1,...,i_k}(z)\}$ be the $k$th polynomials of the Hermite-Padé $m$-system, that is, the solution of (1). Since the functions in the rows of the matrix $\|B_{j_1,...,j_k}(\bar{z}^{(l_1,...,l_k)})\|^{-1}$, have no poles at points in $\pi^{-1}(\infty)$, we can set $p = 0$ in the proof of Theorem 3.
Conversely, let $\tilde{P}_{n;i_1,\ldots,i_k}(z)$ be polynomials of degree at most $(m + 1 - k)n$ satisfying (22) for $p = 0$. Consider the meromorphic function
\[
\tilde{R}_n^*(\mathcal{Z}) := \sum_{0 \leq i_1 < \cdots < i_k \leq m} \tilde{P}_{n;i_1,\ldots,i_k}(z) A_{i_1,\ldots,i_k}(\mathcal{Z}) \tag{31}
\]
on $\tilde{R}_{[k]}$. In the same way as in the proof of Theorem 3, we express $\tilde{P}_{n;i_1,\ldots,i_k}(z)$ from (31):
\[
\tilde{P}_{n;i_1,\ldots,i_k}(z) = \sum_{0 \leq l_1 < \cdots < l_k \leq m} M_{i_1,\ldots,i_k}^{(l_1 \ldots l_k)}(\mathcal{Z}) \tilde{R}_n^*(\mathcal{Z}^{(l_1 \ldots l_k)}). \tag{32}
\]
Substituting these expressions for the $\tilde{P}_{n;i_1,\ldots,i_k}(z)$ into the left-hand side of (1), carrying out the same transformations as in (26)–(28), in the proof of Theorem 3, and using (29) we find that
\[
\tilde{P}_{n;j_1,\ldots,j_k}(z) + \sum_{s=1}^k (-1)^s \tilde{P}_{n;0,j_1,\ldots,j_{s-1},j_{s+1},\ldots,j_k}(z) f_{j_s,\infty}(z) = \sum_{0 < l_1 < \cdots < l_k \leq m} B_{j_1,\ldots,j_k}(\mathcal{Z}^{(l_1 \ldots l_k)}) \tilde{R}_n^*(\mathcal{Z}^{(l_1 \ldots l_k)}). \tag{33}
\]
Note that by assumption the $B_{j_1,\ldots,j_k}(\mathcal{Z})$ have no poles at points in $\mathcal{R}^{-1}(\infty)$. Furthermore, condition (22) for $p = 0$ is equivalent to the relations $\tilde{R}_n^*(\mathcal{Z}^{(l_1 \ldots l_m)}) = O(z^{-(nk+1)})$ as $z \to \infty$ for all $0 < l_1 < \cdots < l_k \leq m$. Hence from (33) we have
\[
\tilde{P}_{n;j_1,\ldots,j_k}(z) + \sum_{s=1}^k (-1)^s \tilde{P}_{n;0,j_1,\ldots,j_{s-1},j_{s+1},\ldots,j_k}(z) f_{j_s,\infty}(z) = O\left(\frac{1}{z^{nk+1}}\right) \tag{34}
\]
as $z \to \infty$, which is condition (1). The proof is complete.

**Proposition 2.** If the surface $\tilde{R}_{[k]}$ is connected, then for any set of indices $0 \leq j_1 < \cdots < j_k \leq m$ neither the function $M_{j_1,\ldots,j_k}(\mathcal{Z}) \equiv 0$ meromorphic on this surface is identically equal to 0, nor the function $M_{j_1,\ldots,j_k}(z)$ can vanish identically in any domain in $\tilde{C} \setminus F$.

**Proof.** Assume that for some indices $0 \leq j_1 < \cdots < j_k \leq m$ we have $M_{j_1,\ldots,j_k}(\mathcal{Z}) \equiv 0$ in some domain on $\tilde{R}_{[k]}$. Since $\tilde{R}_{[k]}$ is connected, this means that $M_{j_1,\ldots,j_k}(\mathcal{Z}) \equiv 0$ on the whole of $\tilde{R}_{[k]}$. Therefore, $\det ||M_{j_1,\ldots,j_k}(\mathcal{Z}^{(l_1 \ldots l_k)})|| \equiv 0$ for $z \in \tilde{C} \setminus F$. On the other hand, since $||M_{j_1,\ldots,j_k}(z)||$ is a matrix formed by all $k \times k$ minors of $A$, we have
\[
\det ||M_{j_1,\ldots,j_k}(\mathcal{Z}^{(l_1 \ldots l_k)})|| = \det \frac{M_{j_1,\ldots,j_k}^{(l_1 \ldots l_k)}(z)}{M_{\ell_1,\ldots,\ell_k}^{(l_1 \ldots l_k)}(z)} = \frac{(\det A(z))^{n-1}}{\prod_{0 \leq l_1 < \cdots < l_k \leq m} M_{\ell_1,\ldots,\ell_k}^{(l_1 \ldots l_k)}(z)}. \tag{35}
\]
Therefore, $\det A \equiv 0$ on $\tilde{C} \setminus F$. This contradicts the linear independence of the original functions $1, f_1, \ldots, f_m$ over $\mathbb{C}(z)$.

The second assertion of the proposition follows clearly from the fact that
\[
M_{j_1,\ldots,j_k}(z) = M_{j_1,\ldots,j_k}(\mathcal{Z}^{(01 \ldots k-1)}) M_{w_1,\ldots,k-1}^{(01 \ldots k-1)}(z) \tag{36}
\]
for $z \in \tilde{C} \setminus F$. The proof is complete.
§ 5. The proofs of Theorems 1 and 2

As we pointed out above, our proofs of Theorems 1 and 2 are close to the proofs of the analogous theorems in [8] for Hermite-Padé polynomials of type I. Roughly speaking, we replace the function $u(z)$ which defines the Nuttall partition of the surface $\mathcal{R}$ by the function $-\tilde{u}(\tilde{z})$ on $\mathcal{R}_{[k]}$ that we now construct from $u$, and we replace the remainder function for Hermite-Padé polynomials of type I by $\tilde{R}_n(\tilde{z})$ defined in (23).

5.1. Auxiliary results. Now, for $\tilde{z} \in \mathcal{R}_{[k]} \setminus \bar{\pi}^{-1}(F)$ we consider the function

$$\tilde{u}(\tilde{z}^{(l_1 \ldots l_k)}) := u_{l_1}(z) + \cdots + u_{l_k}(z),$$

where the $u_j(z)$ were introduced in (6).

**Proposition 3.** The function $\tilde{u}(\tilde{z})$ extends to a harmonic function on $\mathcal{R}_{[k]} \setminus \bar{\pi}^{-1}(\infty)$ with the following logarithmic singularities at the points in $\bar{\pi}^{-1}(\infty)$:

$$\tilde{u}(z^{(0)}}) = -(m + 1 - k) \log |z| + O(1), \quad z \to \infty,$$

$$\tilde{u}(z^{1)} = k \log |z| + O(1), \quad z \to \infty, \quad l_1 \neq 0.$$  

**Proof.** As pointed out above, $F$ is a one-dimensional piecewise analytic set without isolated points (that is, $F$ is in fact the closure of a finite number of analytic arcs), hence $\bar{\pi}^{-1}(F)$ has the same properties. So it suffices to show that $\tilde{u}$ glues into a single harmonic function along the arcs in $\bar{\pi}^{-1}(F)$. Assume that in crossing such an (open) arc $\gamma \in \bar{\pi}^{-1}(F)$ we go from a sheet $\mathcal{R}_{[k]}^{(l_1 \ldots l_k)}$ to $\mathcal{R}_{[k]}^{(i_1 \ldots i_k)}$. Consider a small neighbourhood $O$ of the arc $\bar{\pi}(\gamma)$ such that $O \cap F = \bar{\pi}(\gamma)$. Over $O$ the Riemann surface $\mathcal{R}$ decomposes into a disjoint union of $m + 1$ neighbourhoods $O_{s}$, $s = 0, \ldots, m$. Set $u_s(z) := u|_{O_s}(z)$. Then on either side of $\bar{\pi}(\gamma)$ $u_j$ coincides with some function $u_s$. Moreover, if the set $O_{s} \cap \pi^{-1}(\bar{\pi}(\gamma))$ separates two sheets $\mathcal{R}^{(l)}$ and $\mathcal{R}^{(i)}$ locally (where the case when $l = i$ is possible), then the functions $u_l$ and $u_i$ glue into a single harmonic function $u_s(z)$ along $\bar{\pi}(\gamma)$. Hence, by the construction of $\mathcal{R}_{[k]}$, when we cross the arc $\bar{\pi}(\gamma)$, the functions $u_{l_1}(z) + \cdots + u_{l_k}(z)$ and $u_{i_1}(z) + \cdots + u_{i_k}(z)$ glue into a single harmonic function. This means that $\tilde{u}$ glues into a harmonic function along $\gamma$.

The behaviour of $\tilde{u}$ at points in $\bar{\pi}^{-1}(\infty)$ (see (38)) follows directly from the asymptotic behaviour of $u$ (see (5)) and the definition of the surface $\mathcal{R}_{[k]}$. The proof is complete.

Note that $\tilde{u}$, similarly to the function $u$ in (5) determining it, is defined up to an additive constant, which we specify below.

As noted above, the function $\tilde{R}_n$ defined by (23) is a meromorphic function on $\mathcal{R}_{[k]}$. By assumption $\mathcal{R}_{[k]}$ is connected, and therefore either $\tilde{R}_n \equiv 0$ on the whole of $\mathcal{R}_{[k]}$, or $\tilde{R}_n$ vanishes at a finite number of points. We show that the first case is impossible. Assume the converse: $\tilde{R}_n \equiv 0$. Then, since $\det \|A_{j_1 \ldots j_k}(\tilde{z}^{(l_1 \ldots l_k)})\| \neq 0$ (see (21)), it follows from (24) that all the polynomials $P_{n;i_1 \ldots i_k} \equiv 0$. This contradicts their definition (1). So $\tilde{R}_n \in \mathcal{M}(\mathcal{R}_{[k]})$ and $\tilde{R}_n \neq 0$. We find its divisor $(\tilde{R}_n)$. 


We denote the highest order of the poles of all the functions $A_{j_1,\ldots,j_k}$ in (20) at points in $\bar{\pi}^{-1}(\infty)$ by $p_1$. We have $\infty^{(0)} \notin \Sigma$, so in traversing $\tilde{R}_{[k]}$ near $\bar{\pi}^{-1}(\infty)$ we cannot go from a sheet $\tilde{R}_{[k]}^{(0)j_2\ldots j_k}$ to a sheet $\tilde{R}_{[k]}^{(l_1\ldots l_k)}$ where $l_1 > 0$. First assume that $\infty \notin \Sigma$. Then by Theorem 3, there are precisely $\binom{m}{k}$ points in the set $\bar{\pi}^{-1}(\infty)$ such that $\tilde{R}_{n}$ behaves like $O(z^{-(nk+1-p)})$ near each such point. (Such points lie either on sheets of the form $\tilde{R}_{[k]}^{(l_1\ldots l_k)}$, where $l_1 > 0$, or on the boundaries of such sheets in the case when $\infty \in F$.) Therefore, $\tilde{R}_{n}$ has a zero of order at least $nk + 1 - p$ at each of these points. By definition $\deg P_{n;j_1,\ldots,j_m} \leq (m+1-k)n$, hence $\tilde{R}_{n}$ has poles of order at most $(m+1-k)n + p_1$ at each of the remaining $\binom{m-1}{k-1}$ points in $\bar{\pi}^{-1}(\infty)$. Now assume that the case when $\infty \in \Sigma$ is allowed. We let $\tilde{\infty}_{1,\ldots,\infty}_{M_1}$ denote the points in $\bar{\pi}^{-1}(\infty)$ that lie on sheets of the form $\tilde{R}_{[k]}^{(l_1\ldots l_k)}$ with $l_1 > 0$ or on their boundaries. We also let $\tilde{\infty}_{M_1+1,\ldots,\infty}_{M_2}$ denote the other points in $\bar{\pi}^{-1}(\infty)$. Let $d_j = 1$ be the order of $\tilde{\infty}_{j}$ as a critical point of $\bar{\pi}$ ($d_j = 1$ if $\tilde{\infty}_{j}$ is not a critical point). By Theorem 3, the function $\tilde{R}_{n}$ has a zero of order at least $d_j (nk + 1 - p)$ at $\tilde{\infty}_{j}$ for $j = 1,\ldots,M_1$, and $\sum_{j=1}^{M_1} d_j = \binom{m}{k}$. We have $\deg P_{n;j_1,\ldots,j_m} \leq (m+1-k)n$, and therefore the function $\tilde{R}_{n}$ has a pole of order at most $d_j (m+1-k)n + p_1$ at $\tilde{\infty}_{j}$ for $j = M_1+1,\ldots,M_2$ and $\sum_{j=M_1+1}^{M_2} d_j = \binom{m-1}{k-1}$.

Let $\{\tilde{\alpha}_j(n)\}_{j=1}^{S_1(n)}$ be the zeros of $\tilde{R}_{n}$ over $\mathbb{C}$, repeated according to their multiplicities, and let $\{\tilde{\beta}_j(n)\}_{j=1}^{S_2(n)}$ be its poles, also taking their multiplicities into account. Then

$$\tilde{R}_{n} = \sum_{j=1}^{M_1} (d_j (nk + 1 - p) + r_j(n)) \tilde{\infty}_{j} - \sum_{j=M_1+1}^{M_2} (d_j (m+1-k)n + p_1 - r_j(n)) \tilde{\infty}_{j} + \sum_{j=1}^{S_1(n)} \tilde{\alpha}_j(n) - \sum_{j=1}^{S_2(n)} \tilde{\beta}_j(n),$$

(39)

where $r_j(n) \in \mathbb{N} \cup \{0\}$. We rewrite (39) in the form

$$\tilde{R}_{n} = n \left( k \sum_{j=1}^{M_1} d_j \tilde{\infty}_{j} - (m+1-k) \sum_{j=M_1+1}^{M_2} d_j \tilde{\infty}_{j} \right) + (\tilde{T}_{n}),$$

(40)

where

$$\tilde{T}_{n} := (1-p) \sum_{j=1}^{M_1} d_j \tilde{\infty}_{j} - p_1 \sum_{j=M_1+1}^{M_2} \tilde{\infty}_{j} + \sum_{j=1}^{S_1(n)} \tilde{\alpha}_j(n) - \sum_{j=1}^{S_2(n)} \tilde{\beta}_j(n).$$

(41)

Since $\sum_{j=1}^{M_1} d_j = \binom{m}{k}$ and $\sum_{j=M_1+1}^{M_2} d_j = \binom{m}{k-1}$, the identity $\deg(\tilde{R}_{n}) = 0$ is equivalent to $\deg(\tilde{T}_{n}) = 0$. We look at the points included with minus sign in the
divisor \((\widetilde{T}_n)\). First of all, these are the points \(\widetilde{\beta}_j(n)\). The function \(\widetilde{R}_n\) can have poles over \(\mathbb{C}\) only at points where some of the functions \(A_{j_1,\ldots,j_k}\) have poles, and the order of the pole of \(\widetilde{R}_n\) at such a point is no higher than the maximum order of the poles of the \(A_{j_1,\ldots,j_k}\) at this point. Hence \(\{\widetilde{\beta}_j(n)\}_{j=1}^{S_2(n)}\) is a subset (with regard for multiplicities) of the set of poles of the functions \(A_{j_1,\ldots,j_k}\over \mathbb{C}\). All the other points included in \((\widetilde{T}_n)\) with negative signs lie over \(\infty\) and do not depend on \(n\). Therefore, adding the missing points (with suitable signs) to \((\widetilde{T}_n)\) if necessary, we can assume that the negative terms in \((\widetilde{T}_n)\) are precisely the poles of all functions \(A_{j_1,\ldots,j_k}\) (with regard for multiplicities) and the points \(\infty_j\) with the multiplicities given in \((41)\). So now the points included in \((\widetilde{T}_n)\) with negative sign do not depend on \(n\). We denote these points (taking no account of signs) by \(\tilde{b}_j\), \(j = 1,\ldots,S\), where \(S\) is the total number of them (counting multiplicities). We have \(\deg(\widetilde{T}_n) = 0\), hence now \((\widetilde{T}_n)\) has exactly \(S\) unknown zeros, which we denote by \(\tilde{a}_j(n), j = 1,\ldots,S\). As a result, we have

\[
(\widetilde{T}_n) = \sum_{j=1}^{S} \tilde{a}_j(n) - \sum_{j=1}^{S} \tilde{b}_j. \tag{42}
\]

Substituting this expression into \((40)\) we obtain

\[
(\widetilde{R}_n) = n \left( k \sum_{j=1}^{M_1} d_j \infty_j - (m + 1 - k) \sum_{j=M_1+1}^{M_2} d_j \infty_j \right) + \sum_{j=1}^{S} \tilde{a}_j(n) - \sum_{j=1}^{S} \tilde{b}_j. \tag{43}
\]

For any two distinct points \(\tilde{q}, \tilde{p} \in \widetilde{R}_{[k]}\) let \(g(\tilde{q}, \tilde{p}; z)\) denote the standard bipolar Green’s function, that is, the harmonic function on \(\widetilde{R}_{[k]} \setminus \{\tilde{q}, \tilde{p}\}\) with logarithmic singularities at \(\tilde{q}\) and \(\tilde{p}\) that have the following form in the local coordinate \(\zeta\):

\[
g(\tilde{q}, \tilde{p}; z) = \log |\zeta(z) - \zeta(\tilde{q})| + O(1), \quad z \to \tilde{q},
g(\tilde{q}, \tilde{p}; z) = -\log |\zeta(z) - \zeta(\tilde{p})| + O(1), \quad z \to \tilde{p}. \tag{44}
\]

The fact that bipolar Green’s functions exist on any compact Riemann surface (which is well known) and all their properties we need were established in [8] (see also [5]). The functions \(g(\tilde{q}, \tilde{p}; z)\) are still defined up to a additive constant. We choose their normalization later on.

From \((43)\), \((38)\) and since \(\widetilde{R}_{[k]}\) is connected, it clearly follows that

\[
\log |\widetilde{R}_n(z)| = -n\tilde{u}(z) + \sum_{j=1}^{S} g(\tilde{a}_j(n), \tilde{b}_j; z) + c_n, \tag{45}
\]

where \(c_n\) is a real constant. Setting

\[
\psi_n(z) := \exp\left\{ \sum_{j=1}^{S} g(\tilde{a}_j(n), \tilde{b}_j; z) \right\}, \tag{46}
\]

we have

\[
|\widetilde{R}_n(z)| = C_n e^{-n\tilde{u}(z)} \psi_n(z), \tag{47}
\]

where \(C_n = e^{c_n}\) is a positive constant.
Now we fix normalizations of $g$, $\tilde{u}$, $\tilde{R}_n$ and the $k$th polynomials $P_{n; i_1, \ldots, i_k}$ of the Hermite-Padé $m$-system. The functions $g$ and $\tilde{u}$ are defined up to additive constants, hence the choice of a normalization for them is equivalent to choosing a multiplicative constant $C_n$ in (47). Since $\tilde{R}_n$ is expressed in terms of the $P_{n; i_1, \ldots, i_k}$ by (23), using multiplication of all the $P_{n; i_1, \ldots, i_k}$ by the same constant we can obtain any suitable normalization of $\tilde{R}_n$. In what follows we assume that $g$ and $\tilde{u}$ are spherically normalized on the sheet $\tilde{\mathcal{R}}^{(01\ldots k-1)}$, that is,

$$\int_{\tilde{\mathcal{C}} \setminus F_k} g(\tilde{q}, \tilde{p}; \tilde{z}^{(01\ldots k-1)}) \, d\sigma(z) = 0 \quad \text{and} \quad \int_{\tilde{\mathcal{C}} \setminus F_k} \tilde{u}(\tilde{z}^{(01\ldots k-1)}) \, d\sigma(z) = 0, \quad (48)$$

where

$$d\sigma := \frac{i}{2\pi} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}$$

is the normalized area form of the spherical metric on $\tilde{\mathcal{C}}$. To fix a normalization of $|\tilde{R}_n|$, set $c_n = 0$ in (45) (which is equivalent to taking $C_n = 1$ in (47)), that is, let

$$|\tilde{R}_n(\tilde{z})| = e^{-n \tilde{u}(\tilde{z})} \psi_n(\tilde{z}). \quad (49)$$

So $|\tilde{R}_n|$ is also spherically normalized on $\tilde{\mathcal{R}}^{(01\ldots k-1)}$. Thus we have fixed a normalization of the $k$th polynomials of the Hermite-Padé $m$-system. In what follows the $P_{n; i_1, \ldots, i_k}$ denote precisely those $k$th polynomials of the Hermite-Padé $m$-system for which the function $\log |\tilde{R}_n|$ is spherically normalized on the sheet $\tilde{\mathcal{R}}^{(01\ldots k-1)}$.

Note that, in general, the functions $\log |P_{n; i_1, \ldots, i_k}|$ are not spherically normalized. Therefore, along with the $P_{n; i_1, \ldots, i_k}$ we will also consider the polynomials $P^*_{n; i_1, \ldots, i_k} = c_{n; i_1, \ldots, i_k} P_{n; i_1, \ldots, i_k}$ such that

$$\int_{\tilde{\mathcal{C}}} \log |P^*_{n; i_1, \ldots, i_k}| \, d\sigma = 0, \quad (50)$$

where the $c_{n; i_1, \ldots, i_k}$ are positive constants. We emphasize that, in general, the $P^*_{n; i_1, \ldots, i_k}$ do not satisfy (1).

We fix some conformal metric $\rho$ on $\tilde{\mathcal{R}}^{[k]}$, and we denote distance with respect to it by $\text{dist}_\rho$ and the corresponding area form by $\sigma_\rho$. Since the spaces $L^p$ corresponding to the area forms of any two positive Riemannian metrics on a compact Riemannian surface coincide, we denote the space $L^p$ corresponding to the area form $\sigma_\rho$ by $L^p(\mathcal{R}^{[k]})$. For the Riemann sphere $\tilde{\mathcal{C}}$ and subsets of it we will consider the spaces $L^p$ that correspond to the normalized area form $d\sigma$ of the spherical metric $\rho_{sp}$, without indicating this explicitly. We denote distances on $\tilde{\mathcal{C}}$ with respect to the metric $\rho_{sp}$ by $\text{dist}$.

We recall the facts from potential theory on a compact Riemann surface $\mathcal{S}$ that we need. We will use this theory only on the Riemann sphere $\tilde{\mathcal{C}}$, so we can assume that $\mathcal{S} = \tilde{\mathcal{C}}$. Let $\Lambda^j(\mathcal{S})$, $j = 0, 2$, be the spaces of smooth $j$-forms on $\mathcal{S}$, with the topology of uniform convergence with all derivatives. In particular, $\Lambda^0(\mathcal{S}) = C^\infty(\mathcal{S})$ is the space of smooth functions on $\mathcal{S}$. Let $\Lambda^j(\mathcal{S}) := (\Lambda^{2-j}(\mathcal{S}))^*$ denote the dual space of $\Lambda^{2-j}(\mathcal{S})$, that is, the space of (de Rham) currents of degree $j$ with the weak* topology of the dual space; for example, see [3], [16], or [4]. In the general
case, on currents of degree 0 on $S$ the operator $\text{dd}^c : \Lambda^0(S) \to \Lambda^2(S)$ acts by $\text{dd}^c T(\tau) = T(\text{dd}^c \tau)$, where $T \in \Lambda^0(S)$ and $\tau \in \Lambda^0(S)$ is an arbitrary test function. (As pointed out in §2, on smooth functions $\varphi$, in the local coordinate $\zeta = x + iy$ the operator $\text{dd}^c$ acts as the Laplacian: $\text{dd}^c \varphi = (\varphi_{xx} + \varphi_{yy}) \, dx \, dy = \Delta \varphi \, dx \, dy$.) It is well known that the equation $\text{dd}^c T = \overline{T}$ on $S$ is solvable for currents $\overline{T} \in \Lambda^2(S)$ if and only if $\overline{T}(1) = 0$, and if $\overline{T}(1) = 0$, then $T$ is defined up to an additive constant (because the solutions of the equation $\text{dd}^c T = 0$ are harmonic functions on $S$ by Weyl’s lemma, and so they are constants because $S$ is compact). We are interested in solutions of the equation $\text{dd}^c T = \overline{T}$ in the case when $\overline{T} = 0$ is a neutral real-valued signed measure, that is, a signed measure satisfying $\int_S \nu = 0$.

We denote the space of such signed measures by $\text{Meas}_0(S)$. In this case $T$ is called the potential of $\nu$ and denoted by $\hat{\nu} := T = (\text{dd}^c)^{-1} \nu$ (the potential $\hat{\nu}$ is defined up to an additive constant). A function $\varphi$ on $S$ is said to be $\delta$-subharmonic if it can be represented locally as the difference of two subharmonic functions. Let $\delta\text{-sh}(S)$ denote the space of $\delta$-subharmonic functions on $S$. It is well known (see [5], for example) that $\delta\text{-sh}(S)$ consists precisely of the potentials of all neutral signed measures on $S$ and $\delta\text{-sh}(S) \in L^p(S)$ for each $p \in [1, \infty)$. To get rid of the ambiguity in the action of $(\text{dd}^c)^{-1}$ on the space of neutral signed measures which is related to the possible addition of a constant, authors fix a continuous linear functional $\phi$ on the space $L^1(S)$, for instance, satisfying the condition $\phi(1) \neq 0$ and, in place of $\delta\text{-sh}(S)$, consider the space

$$\text{Pot}_\phi(S) := \{ v \in \delta\text{-sh}(S) : \phi(v) = 0 \}.$$  

The operator $\text{dd}^c : \text{Pot}_\phi(S) \to \text{Meas}_0(S)$ is one-to-one on $\text{Pot}_\phi(S)$. For $\nu \in \text{Meas}_0(S)$ we denote its potential $\hat{\nu}$ in the space $\text{Pot}_\phi(S)$ by $\{ \hat{\nu} \}_\phi$. In what follows we deal with the Riemann sphere $\hat{\mathbb{C}}$ and use as $\phi$ the functional determined by the area form $d\sigma$ of the spherical metric $\rho_{sp}$:

$$\phi(v) = d\sigma(v) := \int_{\hat{\mathbb{C}}} v \, d\sigma,$$

where $v \in \delta\text{-sh}(\mathbb{C}) \subset L^1(\hat{\mathbb{C}})$.

### 5.2. The proof of Theorem 1.

Theorem 1 is the combination of Statements 1 and 2 below, which we prove in this subsection.

Recall that $S$ is the number of the unknown zeros of the divisor $(\tilde{R}_n)$ (see (43)). Let $\alpha_{j_{1}}\ldots j_{k}$ be the number of zeros of the function $M_{j_{1},\ldots,j_{k}}(z)$ in (19) on the sheet $\tilde{\mathbb{R}}_{[k]}^{01\ldots k-1}$, which is defined over $\hat{\mathbb{C}} \setminus F_k$. In what follows the notation $M_{j_{1},\ldots,j_{k}}$ means the meromorphic function $M_{j_{1},\ldots,j_{k}}(z)$ on $\tilde{\mathbb{R}}_{[k]}$ (see (19)).

**Statement 1.** Assume that the surface $\tilde{\mathbb{R}}_{[k]}$ constructed in terms of $\pi$ is connected. Then for $L_{j_{1}}\ldots j_{k} := S + \alpha_{j_{1}}\ldots j_{k}$, for any neighbourhood $V$ of the compact set $F_k$ there exists $N = N(V)$ such that for all $n > N$ at most $L_{j_{1}}\ldots j_{k}$ zeros (taken with multiplicities) of the polynomial $P_{n,j_{1}}\ldots j_{k}(z)$ lie outside $V$. 

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Proof. The meromorphic functions $M_{j_1,\ldots,j_k}(\bar{z})$ on $\widehat{\mathcal{R}}_{[k]}$ (see (19)) are independent of $n$, hence (considering a smaller neighbourhood $V$ if necessary) we can assume that they have neither zeros nor poles on the set $\pi^{-1}(V \setminus F_k)$. In the case when $\infty \notin F_k$ we also assume that $\infty \notin V$. We set
\[ \delta := \frac{\text{dist}(\partial V, F_k)}{2(S + 3)}, \]
where $\partial V$ is the boundary of $V$. Since $F$ is a piecewise analytic subset of $\widehat{\mathbb{C}}$ without isolated points (see [8], Appendix 1, Lemma 3), for each $n$ we can find a system of disjoint smooth contours $\Gamma_n$ bounding open sets $D_n$ such that $F_k \subset D_n \subset V$, and the following conditions hold: $\text{dist}(\Gamma_n, F_k) \geq \delta$, $\text{dist}(\Gamma_n, \partial V) \geq \delta$, $\text{dist}(\Gamma_n, b_i) \geq \delta$ and $\text{dist}(\Gamma_n, a_i(n)) \geq \delta$, $i = 1, \ldots, S$, where the $b_i = \pi(b_i)$ and $a_i(n) = \pi(\bar{a}_i)$ are the projections of poles and zeros of the remainder function $\bar{R}_n$ (see (43)).

We find an upper estimate for the number of zeros of $P_{n:j_1,\ldots,j_k}(z)$ in $\Omega_n := \widehat{\mathbb{C}} \setminus D_n$ (and therefore also in $\widehat{\mathbb{C}} \setminus V$). Since the functions $M_{j_1,\ldots,j_k}(\bar{z})$ and $\bar{R}_n(\bar{z})$ are meromorphic on $\widehat{\mathcal{R}}_{[k]}$, the expression (25) extends to the whole of $\widehat{\mathbb{C}}$ in the following way:
\[ P_{n:j_1,\ldots,j_k}(z) = \sum_{\bar{z} \in \pi^{-1}(z)} M_{j_1,\ldots,j_k}(\bar{z})\bar{R}_n(\bar{z}). \]
(54) As we pointed out in §4, the projection $\pi: \widehat{\mathcal{R}}_{[k]}^{(0)\ldots(k-1)} \rightarrow \widehat{\mathbb{C}} \setminus F_k$ is biholomorphic on the sheet $\widehat{\mathcal{R}}_{[k]}^{(0)\ldots(k-1)}$. Hence for $z \in \widehat{\mathbb{C}} \setminus F_k$, (54) is equivalent to
\[ P_{n:j_1,\ldots,j_k}(z) = M_{j_1,\ldots,j_k}(\bar{z}^{(0)\ldots(k-1)})\bar{R}_n(\bar{z}^{(0)\ldots(k-1)})(1 + h_{n:j_1,\ldots,j_k}(z)), \]
(55) where
\[ h_{n:j_1,\ldots,j_k}(z) := \sum_{\bar{z} \in \pi^{-1}(z) \setminus \bar{z}^{(0)\ldots(k-1)}} \frac{M_{j_1,\ldots,j_k}(\bar{z})}{M_{j_1,\ldots,j_k}(\bar{z}^{(0)\ldots(k-1)})} \cdot \frac{\bar{R}_n(\bar{z})}{\bar{R}_n(\bar{z}^{(0)\ldots(k-1)})}. \]
(56) First we show that $\lim_{n \rightarrow \infty} \max_{z \in \Gamma_n} |h_{n:j_1,\ldots,j_k}(z)| = 0$. By the definitions of the function $\bar{u}$ in (37) and the functions $u_k$ in (6), we have $\bar{u}(\bar{z}) - \bar{u}(\bar{z}^{(0)\ldots(k-1)}) \geq u_k(z) - u_{k-1}(z)$ for $\bar{z} \in \pi^{-1}(z) \setminus \bar{z}^{(0)\ldots(k-1)}$. Therefore, we conclude from (49) that for $z \in \widehat{\mathbb{C}} \setminus F_k$,
\[ |h_{n:j_1,\ldots,j_k}(z)| \leq \sum_{\bar{z} \in \pi^{-1}(z) \setminus \bar{z}^{(0)\ldots(k-1)}} \left| \frac{M_{j_1,\ldots,j_k}(\bar{z})}{M_{j_1,\ldots,j_k}(\bar{z}^{(0)\ldots(k-1)})} \right| \cdot \left| \frac{\psi_n(\bar{z})}{\psi_n(\bar{z}^{(0)\ldots(k-1)})} \right| e^{-n(u_k(z) - u_{k-1}(z))}. \]
(57) Let
\[ K := \left\{ z \in V : \text{dist}(z, F_k) \geq \frac{\delta}{2}, \text{dist}(z, \partial V) \geq \frac{\delta}{2} \right\}, \]
(58) where $\delta$ is defined by (53). Note that $\Gamma_n \subset K$ for all $n$. The meromorphic functions $M_{j_1,\ldots,j_k}(\bar{z})$ have neither zeros nor poles on $\pi^{-1}(V \setminus F_k)$, and so
\[ C_{j_1,\ldots,j_k} := \max_{\bar{z} \in \pi^{-1}(K) \setminus \bar{R}_{[k]}^{(0)\ldots(k-1)}} \left| \frac{M_{j_1,\ldots,j_k}(\bar{z})}{M_{j_1,\ldots,j_k}(\bar{z}^{(0)\ldots(k-1)})} \right| < \infty. \]
(59)
The functions $u_i$ are continuous in $\mathbb{C}$ (see [8], Appendix 1, Lemma 1) and the functions $u_k(z) - u_{k-1}(z)$ for $k > 1$ are continuous in a neighbourhood of $\infty$, while for $k = 1$ this function tends to $+\infty$ as $z \to \infty$. So, since $K$ is disjoint from $F_k$, we have

$$\varkappa := \min_{z \in K} (u_k(z) - u_{k-1}(z)) > 0. \quad (60)$$

In order to estimate $\psi_n(\bar{z})/\psi_n(\bar{z}(01...k))$, where $\psi_n(\bar{z}) = \exp\{\sum_{i=1}^{S} g(\bar{a}_i(n), \bar{b}_i; \bar{z})\}$ (see (46)), we derive an estimate for the functions $g(\bar{a}_i(n), \bar{b}_i; \bar{z})$ themselves. To do this we use Corollary 6 in [8], which states the following in our setting. Given a bipolar Green’s function $g(\bar{q}, \bar{p}; \bar{z})$ that is spherically normalized on the sheet $\tilde{\mathfrak{H}}_{[k]}(01...k-1)$ and an arbitrary $\delta' > 0$, there exists a constant $C = C(\delta')$ (independent of $\bar{q}$ and $\bar{p}$) such that for all $\bar{z} \in \tilde{\mathfrak{H}}_{[k]}$ satisfying $\text{dist}_\rho(\bar{z}, \bar{q}) \geq \delta'$ and $\text{dist}_\rho(\bar{z}, \bar{p}) \geq \delta' \Rightarrow |g(\bar{q}, \bar{p}; \bar{z})| < C$. The systems of contours $\Gamma_n$ are chosen so that $\text{dist}(\Gamma_n, a_i(n)) \geq \delta$ and $\text{dist}(\Gamma_n, b_i) \geq \delta$, hence (since for any conformal metric $\rho$ on $\mathfrak{H}$ there exists a constant $C_\rho > 0$ such that for arbitrary $\bar{z}_1, \bar{z}_2 \in \tilde{\mathfrak{H}}_{[k]}$ it follows that there exists a constant $C$ such that $|g(\bar{a}_i(n), \bar{b}_i; \bar{z})| \leq C$ for $\bar{z} \in \pi^{-1}(\Gamma_n)$. So for $\bar{z}, \bar{z}_2 \in \pi^{-1}(\Gamma_n)$ we have

$$\frac{\psi_n(\bar{z}_1)}{\psi_n(\bar{z}_2)} \leq e^{2S\bar{C}}. \quad (61)$$

Combining (59)–(61) and taking into account that $\tilde{\mathfrak{H}}_{[k]}$ has $(m+1)$ sheets, from (57) we obtain that for $z \in \Gamma_n$

$$|h_{n;j_1,...,j_k}(z)| \leq C_{m+1}^{\kappa} C_{j_1,...,j_k} e^{2S\bar{C}} e^{-n\varkappa}. \quad (62)$$

Since $\varkappa > 0$, we have $\lim_{n \to \infty} \max_{z \in \Gamma_n} |h_{n;j_1,...,j_k}(z)| = 0$.

The projection $\bar{\pi}: \tilde{\mathfrak{H}}_{[k]}(01...k-1) \to \bar{\mathcal{C}} \setminus F_k$ is biholomorphic on the sheet $\tilde{\mathfrak{H}}_{[k]}(01...k-1)$, so we can consider the functions $\tilde{R}_n(\bar{z}(01...k-1))$ and $M_{j_1,...,j_k}(\bar{z}(01...k-1))$ in (55) as meromorphic functions of $z \in \tilde{\mathcal{C}} \setminus F_k$, that is, for example, $\tilde{R}_n(\bar{z}(01...k-1)) = \tilde{R}_n \circ (\bar{\pi}|_{\tilde{\mathfrak{H}}_{[k]}(01...k-1)})^{-1}(z)$. So all the functions in (55), apart from $1 + h_{n;j_1,...,j_k}$, are meromorphic in $\tilde{\mathcal{C}} \setminus F_k$, and therefore we also have $1 + h_{n;j_1,...,j_k}(z) \in \mathcal{M}(\tilde{\mathcal{C}} \setminus F_k)$. Since $\lim_{n \to \infty} \max_{z \in \Gamma_n} |h_{n;j_1,...,j_k}(z)| = 0$, there exists $N = N(V)$ such that for all $n > N$ we have $|h_{n;j_1,...,j_k}(z)| < 1/2$ for $z \in \Gamma_n$. Below we assume that $n > N$. Then $1 + h_{n;j_1,...,j_k}(z)$ has no zeros on $\Gamma_n$. Furthermore, by the choice of the system of contours $\Gamma_n$ the functions $\tilde{R}_n(\bar{z}(01...k-1))$ and $M_{j_1,...,j_k}(\bar{z}(01...k-1))$ have neither zeros nor poles on $\Gamma_n$. Hence all functions on the right-hand side of (55) are meromorphic in $\Omega_n$ and have no zeros on $\Gamma_n = \partial \Omega_n$. Therefore, the number of zeros of $P_{n;j_1,...,j_k}$ in $\Omega_n$ can be evaluated using the principle of the argument. We choose the orientation of each contour in $\Gamma_n$ so that $\Gamma_n$ is positively orientated with respect to $\Omega_n$.

First assume that $\infty \notin F_k$. (Recall that in this case $\infty \notin V$, and therefore $\infty \in \Omega_n$.) Then the number of zeros of a polynomial $P_{n;j_1,...,j_k}$ in $\Omega_n$ is equal to

$$\deg P_{n;j_1,...,j_k} + \frac{1}{2\pi} \sum_{z \in \Gamma_n} \Delta \arg P_{n;j_1,...,j_k}(z). \quad (63)$$
Since \(|h_{n;1,...,j_k}(z)| < 1/2\) on \(\Gamma_n\), we have \(\Delta_{z \in \Gamma_n} \arg(1 + h_{n;1,...,j_k}(z)) = 0\). Therefore, from (55) we obtain

\[
\Delta_{z \in \Gamma_n} \arg P_{n;1,...,j_k}(z) = \Delta_{z \in \Gamma_n} \arg \tilde{R}_n(z^{(01)...(k-1)}) + \Delta_{z \in \Gamma_n} \arg M_{j_1,...,j_k}(z^{(01)...(k-1)}),
\]

By the principle of the argument \((2\pi)^{-1} \Delta_{z \in \Gamma_n} \arg M_{j_1,...,j_k}(z^{(01)...(k-1)})\) is equal to the difference between the number of zeros and the number of poles (counting multiplicities) of the function \(M_{j_1,...,j_k}(z^{(01)...(k-1)})\) for \(z \in \Omega_n\). Hence

\[
\frac{1}{2\pi} \Delta_{z \in \Gamma_n} \arg M_{j_1,...,j_k}(z^{(01)...(k-1)}) \leq \alpha_{j_1,...,j_k}.
\] (64)

Since \(\infty \notin F_k\), the point \(\infty^{(01)...(k-1)}\) is not critical for \(\tilde{\pi}\). Hence, from the form of the divisor of \(\tilde{R}_n\) (see (43)) we can conclude that (apart from the poles \(b_j\) and free zeros \(\tilde{a}_j(n)\), both in the amount of \(S\)) \(\tilde{R}_n(z^{(01)...(k-1)})\) has a pole of order \((m+1-k)n\) at \(\infty^{(01)...(k-1)}\). Consequently,

\[
\frac{1}{2\pi} \Delta_{z \in \Gamma_n} \arg \tilde{R}_n(z^{(01)...(k-1)}) \leq -(m+1-k)n + S.
\] (66)

We have \(\deg P_{n;1,...,j_k} \leq (m+1-k)n\), and therefore, combining (63), (64) and (66) we see that the number of zeros of \(P_{n;1,...,j_k}\) in \(\Omega_n\) (for \(n > N\)) is at most \(L_{j_1,...,j_k} := S + \alpha_{j_1,...,j_k}\).

If \(\infty \in F_k\), then we always have \(\infty \notin \Omega_n\). Therefore, \(P_{n;1,...,j_k}\) has \((2\pi)^{-1} \Delta_{z \in \Gamma_n} \arg P_{n;1,...,j_k}(z)\) zeros in \(\Omega_n\). Proceeding as in the case when \(\infty \notin F\) we conclude that estimate (65) again holds for \(\Delta_{z \in \Gamma_n} \arg M_{j_1,...,j_k}(z^{(01)...(k-1)})\), and from (43) we obtain \((2\pi)^{-1} \Delta_{z \in \Gamma_n} \arg \tilde{R}_n(z^{(01)...(k-1)}) \leq S\). As a result, we see that the number of zeros of a polynomial \(P_{n;1,...,j_k}\) in \(\Omega_n\) (for \(n > N\)) is at most \(L_{j_1,...,j_k} := S + \alpha_{j_1,...,j_k}\). Statement 1 is proved.

Recall that the functions \(h_{n;1,...,j_k}(z)\) were defined in (56).

**Lemma 1.** Assume that the surface \(\tilde{\mathcal{R}}_{[k]}\) constructed in terms of \(\pi\) is connected. Then for any neighbourhood \(V\) of \(F_k\) there exists \(N = N(V)\) such that for all \(n > N\) the function \(1 + h_{n;1,...,j_k}(z)\) has at most \(L_{j_1,...,j_k}\) zeros and at most \(L_{j_1,...,j_k}\) poles (counting multiplicities) in \(\hat{\mathbb{C}} \setminus V\).

**Proof.** By definition (56)

\[
1 + h_{n;1,...,j_k}(z) = \frac{P_{n;1,...,j_k}(z)}{\tilde{R}_n(z^{(01)...(k-1)})M_{j_1,...,j_k}(z^{(01)...(k-1)})},
\] (67)

where, similarly to the proof of Statement 1, we regard \(\tilde{R}_n(z^{(01)...(k-1)})\) and \(M_{j_1,...,j_k}(z^{(01)...(k-1)})\) as meromorphic functions of \(z \in \hat{\mathbb{C}} \setminus F_k\). Hence \(h_{n;1,...,j_k}(z) \in \mathcal{M}(\mathbb{C} \setminus F_k)\). Taking account of the form (43) of the divisor \((\tilde{R}_n)\) and since we have \(\deg P_{n;1,...,j_k} \leq (m+1-k)n\), we conclude that the number of poles of the function \(1 + h_{n;1,...,j_k}(z)\) in \(\hat{\mathbb{C}} \setminus F_k\) is at most \(S + \alpha_{j_1,...,j_k} = L_{j_1,...,j_k}\). Consequently, the number of poles in \(\hat{\mathbb{C}} \setminus V\) is also at most \(L_{j_1,...,j_k}\).
For each \( n \) we choose the same system of contours \( \Gamma_n \) as in the proof of Statement 1. In particular, \( \Gamma_n \) bounds an open set \( D_n \) such that \( F_k \subset D_n \subset V \). In the proof of Statement 1 we showed that \( \lim_{n \to \infty} \max_{z \in \Gamma_n} |h_{n;j_1,\ldots,j_k}(z)| = 0 \). We choose \( N \) such that for each \( n > N \) we have \( |h_{n;j_1,\ldots,j_k}(z)| < 1/2 \) for \( z \in \Gamma_n \). Below we assume that \( n > N \). Since \( |h_{n;j_1,\ldots,j_k}(z)| < 1/2 \) on \( \Gamma_n \), by the principle of the argument the number of zeros of \( 1 + h_{n;j_1,\ldots,j_k}(z) \) in the open set \( \Omega_n := \widehat{C} \setminus \overline{D}_n \) is equal to the number of its poles there. We have \( \widehat{C} \setminus V \subset \Omega_n \), hence the function \( 1 + h_{n;j_1,\ldots,j_k}(z) \) has no more zeros for \( z \in \widehat{C} \setminus V \) than it has poles in \( \widehat{C} \setminus F_k \), that is, no more than \( L_{j_1,\ldots,j_k} \). The proof is complete.

Recall that the functions \( u_i \) defining the Nuttall partition (6) are uniquely defined, because we fixed a normalization of the function \( \tilde{u} \) from (37) in (48):

\[
\int_{\widehat{C}} \sum_{s=0}^{k-1} u_s(z) \, d\sigma(z) \equiv \int_{\widehat{C} \setminus F_k} \tilde{u}(z^{(01\ldots k-1)}) \, d\sigma(z) = 0. \tag{68}
\]

**Statement 2.** Assume that the surface \( \tilde{\mathcal{R}}_{[k]} \) constructed in terms of \( \pi \) is connected. Then for any \( p \in [1, \infty) \),

\[
\frac{1}{n} \log |P^*_{n;j_1,\ldots,j_k}(z)| \to - \sum_{s=0}^{k-1} u_s(z) \text{ in } L^p(\widehat{C}) \tag{69}
\]

and

\[
\frac{1}{n} \, \text{d}^{\mathcal{C}} \log |P_{n;j_1,\ldots,j_k}(z)| \to \psi_n(\widehat{z}^{(01\ldots k-1)}) - \sum_{s=0}^{k-1} u_s(z) \text{ in } C(\widehat{C})^* \tag{70}
\]

as \( n \to \infty \).

**Proof.** We fix a set of indices \( j_1, \ldots, j_k \). Using (55) and taking (49) into account, we see that for \( z \in \widehat{C} \setminus F_k \),

\[
\frac{1}{n} \log |P_{n;j_1,\ldots,j_k}(z)| = - \sum_{s=0}^{k-1} u_s(z)
+ \frac{1}{n} \log \left\{ \psi_n(\widehat{z}^{(01\ldots k-1)}) |M_{j_1,\ldots,j_k}(\widehat{z}^{(01\ldots k-1)})| |1 + h_{n;j_1,\ldots,j_k}(z)| \right\}, \tag{71}
\]

where \( h_{n;j_1,\ldots,j_k} \) was defined in (56). (Here \( \psi_n(\widehat{z}^{(01\ldots k-1)}) \) and \( M_{j_1,\ldots,j_k}(\widehat{z}^{(01\ldots k-1)}) \) are again understood as meromorphic functions of \( z \in \widehat{C} \setminus F_k \).) Since \( F_k \) is a piecewise analytic subset of \( \widehat{C} \) (see [8], Appendix 1, Lemma 3), we have \( \sigma(F_k) = 0 \). Hence (71) can be understood as the equality of two elements of \( L^p(\widehat{C}) \). In what follows we assume that \( p \in [1, \infty) \) is fixed.

Since (see (48)) we have normalized all bipolar Green’s functions \( g(\tilde{q}, \tilde{p}; \widehat{z}) \) spherically on the sheet \( \tilde{\mathcal{R}}^{(01\ldots k-1)} \) of the compact Riemann surface \( \tilde{\mathcal{R}}_{[k]} \), which is connected by assumption, we obtain (see [8], Appendix 2, Corollary 5) that their norms in the space \( L_p(\tilde{\mathcal{R}}_{[k]}) \) are uniformly bounded by some constant \( C_1 \). (Note that in [8] this corollary was only formulated for \( p \in (1, \infty) \), but the proof also holds for \( p = 1 \). Moreover, this is immaterial for us, because on any compact set convergence in \( L^p \)
implies convergence in all $L^q$ for $1 \leq q < p$. Since the surface $\tilde{R}_k$ is compact, for any function $f \in L^p(\tilde{R}_k)$ we have $\|f(\tilde{z}^{(01...k-1)})\|_{L^p(\tilde{C})} \leq C_2\|f\|_{L^p(\tilde{R}_k)}$, where $C_2 := \max_{\tilde{z} \in \tilde{R}} (d\sigma(z)/d\sigma_{p}(\tilde{z}))^{1/p} < \infty$. Therefore from definition (46) we obtain

$$\| \log \psi_n(\tilde{z}^{(01...k-1)}) \|_{L^p(\tilde{C})} \leq C_2 \|\log \psi_n(\tilde{z})\|_{L^p(\tilde{R}_k)} \leq C_2 C_1 S. \tag{72}$$

Consequently, $n^{-1}\log \psi_n(\tilde{z}^{(01...k-1)}) \to 0$ in $L^p(\tilde{C})$. Since $M_{j_1,...,j_k}(\tilde{z}) \in M(\tilde{R}_k)$, we have $\log |M_{j_1,...,j_k}(\tilde{z})| \in L^p(\tilde{R}_k)$. This implies that $\log |M_{j_1,...,j_k}(\tilde{z}^{(01...k-1)})| \in L^p(\tilde{R}_k)$. Therefore, $n^{-1}\log |M_{j_1,...,j_k}(\tilde{z}^{(01...k-1)})| \to 0$ in $L^p(\tilde{C})$. Now we show that $n^{-1}\log |1 + h_{n;j_1,...,j_k}(z)| \to 0$ in $L^p_{\text{loc}}(\tilde{C} \setminus F_k)$. In view of the above, it follows from (71) that $n^{-1}\log |P_{n;j_1,...,j_k}| \to -\sum_{s=0}^{k-1} u_s(z)$ in $L^p_{\text{loc}}(\tilde{C} \setminus F_k)$.

Now fix a neighbourhood $V$ of $F_k$. We show that $n^{-1}\log |1 + h_{n;j_1,...,j_k}| \to 0$ in $L^p(\tilde{C} \setminus V)$. Passing to a smaller neighbourhood if necessary, we assume that the function $M_{j_1,...,j_k}(\tilde{z})$ has neither zeros nor poles on the set $\tilde{\pi}^{-1}(V \setminus F_k)$, and if $\infty \notin F_k$, then we assume that $\infty \notin V$. We set

$$\delta := \frac{\text{dist}(\partial V,F_k)}{2(2S + 2L_{j_1,...,j_k} + 3)}, \tag{73}$$

where $L_{j_1,...,j_k} := S + \alpha_{j_1,...,j_k}$, $S$ is the number of unknown zeros in the divisor $(\tilde{R}_n)$ from (43), and $\alpha_{j_1,...,j_k}$ is the number of zeros of the function $M_{j_1,...,j_k}(\tilde{z})$ on the sheet $\tilde{R}_k^{(01...k-1)}$. Let $V_\delta := \{z \in \tilde{C}: \text{dist}(z,F_k) < \delta\}$. By Lemma 1 there exists $N$ such that for all $n > N$ the function $1 + h_{n;j_1,...,j_k}(z)$ has at most $L_{j_1,...,j_k}$ zeros and at most $L_{j_1,...,j_k}$ poles in $\tilde{C} \setminus V_\delta$. Now assume that $n > N$. Let $q_1(n),...,q_{\ell(n)}(n)$ be the zeros (taken with multiplicities) of the function $(1 + h_{n;j_1,...,j_k})$ in $\tilde{C} \setminus V_\delta$ and $p_1(n),...,p_{\ell'(n)}(n)$ be its poles (also taken with multiplicities). Then $l(n),l'(n) \leq L_{j_1,...,j_k}$. We set $\tilde{q}_s(n) := \tilde{\pi}^{-1}(q_s(n)) \cap \tilde{R}_k^{(01...k-1)}$ and $\tilde{p}_s(n) := \tilde{\pi}^{-1}(p_s(n)) \cap \tilde{R}_k^{(01...k-1)}$. We fix a point $\tilde{z}^* \in \partial \tilde{R}_k^{(01...k-1)}$ and for $\tilde{z} \in \tilde{R}_k$ consider the function

$$\psi_{n;j_1,...,j_k}(\tilde{z}) := \exp\left\{ \sum_{s=1}^{L_{j_1,...,j_k}} g(\tilde{q}_s(n),\tilde{z}^*;\tilde{z}) + \sum_{s=1}^{L_{j_1,...,j_k}} g(\tilde{z}^*,\tilde{p}_s(n);\tilde{z}) \right\}, \tag{74}$$

where the bipolar Green’s functions $g(\tilde{q},\tilde{p};\tilde{z})$ in (44) are spherically normalized on the sheet $\tilde{R}_k^{(01...k-1)}$ (see (48)). In (74) we assume that $l(n) = l'(n) = L_{j_1,...,j_k}$, complementing the sets $\{\tilde{q}_s(n)\}_{s=1}^{l(n)}$ and $\{\tilde{p}_s(n)\}_{s=1}^{l'(n)}$ with copies of $\tilde{z}^*$ (taken $L_{j_1,...,j_k} - l(n)$ and $L_{j_1,...,j_k} - l'(n)$ times, respectively) if necessary; we also complement $\{q_s(n)\}$ and $\{p_s(n)\}$ with the same number of copies of $z^* = \tilde{\pi}(\tilde{z}^*)$. Proceeding as in deducing a uniform estimate for the function $\psi_n$ in (72) we find that

$$\| \log \psi_{n;j_1,...,j_k}(\tilde{z}^{(01...k-1)}) \|_{L^p(\tilde{C})} \leq 2C_2C_1L_{j_1,...,j_k}.$$ 

Therefore, $n^{-1}\log \psi_{n;j_1,...,j_k}(\tilde{z}^{(01...k-1)}) \to 0$ in $L^p(\tilde{C})$. It remains to verify that

$$\frac{1}{n} \log \psi_{n;j_1,...,j_k}(\tilde{z}^{(01...k-1)}) \to 0 \quad \text{in} \quad L^p(\tilde{C} \setminus V). \tag{75}$$
We prove that the functions

\[
\log \frac{|1 + h_{n;j_1,\ldots,j_k}(z)|}{\psi_{n;j_1,\ldots,j_k}(\mathbf{z}^{(01\ldots k-1)})}
\]

are uniformly bounded on the set \( \hat{\mathcal{C}} \setminus V \). Of course, this will imply (75). As pointed out above (see (67)), the function \( 1 + h_{n;j_1,\ldots,j_k} \) is meromorphic in \( \hat{\mathcal{C}} \setminus F_k \). Therefore, \( \log |1 + h_{n;j_1,\ldots,j_k}| \) is a harmonic function in \( \hat{\mathcal{C}} \setminus V_\delta \) outside the points \( q_1(n), \ldots, q_l(n) \) and \( p_1(n), \ldots, p_{l'}(n) \), where it has the corresponding logarithmic singularities. By construction (see (74)) the function \( \log \psi_{n;j_1,\ldots,j_k} \) is also harmonic in this domain and has the same singularities as \( \log |1 + h_{n;j_1,\ldots,j_k}| \) at the points \( q_1(n), \ldots, q_l(n) \) and \( p_1(n), \ldots, p_{l'}(n) \); hence the function \( \log(|1 + h_{n;j_1,\ldots,j_k}(z)|/\psi_{n;j_1,\ldots,j_k}(\mathbf{z}^{(01\ldots k-1)})) \) is harmonic in \( \hat{\mathcal{C}} \setminus V_\delta \). By the choice of \( \delta \) (see (73)), for each \( n \) we can find a system of disjoint smooth contours \( \Gamma_n \) bounding an open set \( D_n \) such that \( F_k \subset D_n \subset V \), and the following conditions hold true: \( \text{dist}(\Gamma_n, F_k) \geq \delta, \text{dist}(\Gamma_n, \partial V) \geq \delta, \text{dist}(\Gamma_n, q_s(n)) \geq \delta \) and \( \text{dist}(\Gamma_n, p_s(n)) \geq \delta, s = 1, \ldots, L_{j_1,\ldots,j_k} \), \( \text{dist}(\Gamma_n, a_i) \geq \delta \) and \( \text{dist}(\Gamma_n, b_i(n)) \geq \delta, i = 1, \ldots, S \), where the \( a_i(n) = \pi(a_i(n)) \) and \( b_i = \pi(b_i) \) are the projections of the zeros and poles of the remainder function \( \tilde{R}_n \) (see (43)). In particular, we have \( \text{dist}(\Gamma_n, F_k) \geq \delta, \text{dist}(\Gamma_n, \partial V) \geq \delta, \text{dist}(\Gamma_n, b_i) \geq \delta, \text{dist}(\Gamma_n, a_i(n)) \geq \delta \), and so \( \lim_{n \to \infty} \max_{z \in \Gamma_n} |h_{n;j_1,\ldots,j_k}(z)| = 0 \) (see the derivation of the analogous property in the proof of Statement 1). Therefore, there exists \( N' \) such that \( |h_{n;j_1,\ldots,j_k}(z)| < 1/2 \) on \( \Gamma_n \) for all \( n > N' \). Below we assume that \( n > N' > N \). Hence for \( z \in \Gamma_n \),

\[
\frac{1}{2} < |1 + h_{n;j_1,\ldots,j_k}(z)| < \frac{3}{2}.
\]

We have \( \text{dist}(\Gamma_n, \tilde{q}_s(n)) \geq \delta, \text{dist}(\Gamma_n, \tilde{q}_s(n)) \geq \delta \) and \( \text{dist}(\Gamma_n, z^*) \geq \delta \), so proceeding as in deducing (61) in the proof of Statement 1, we obtain that there exists a constant \( \tilde{C} \) such that \( |g(\tilde{q}_s(n), \tilde{z}^*; \tilde{z})| \leq \tilde{C} \) and \( |g(\tilde{z}^*, \tilde{p}_s(n); \tilde{z})| \leq \tilde{C} \) for \( \tilde{z} \in \pi^{-1}(\Gamma_n) \). Hence for \( z \in \Gamma_n \), from (74) we obtain

\[
|\log \psi_{n;j_1,\ldots,j_k}(\mathbf{z}^{(01\ldots k-1)})| \leq 2L_{j_1,\ldots,j_k} \tilde{C}.
\]

Combining (76) and (77), for \( z \in \Gamma_n \) we find that

\[
|\log \frac{|1 + h_{n;j_1,\ldots,j_k}(z)|}{\psi_{n;j_1,\ldots,j_k}(\mathbf{z}^{(01\ldots k-1)})} | \leq 2L_{j_1,\ldots,j_k} \tilde{C} + 1.
\]

We set \( \Omega_n := \hat{\mathcal{C}} \setminus \overline{D}_n \). Since the function \( \log(|1 + h_{n;j_1,\ldots,j_k}(z)|/\psi_{n;j_1,\ldots,j_k}(\mathbf{z}^{(01\ldots k-1)})) \) is harmonic in \( \hat{\mathcal{C}} \setminus V_\delta \supset \Omega_n \) and \( \Gamma_n = \partial \Omega_n \), it follows from the maximum principle that (78) holds in the whole of \( \Omega_n \), and therefore also in \( \hat{\mathcal{C}} \setminus V \subset \Omega_n \). So we have shown that \( n^{-1} \log |P_{n;j_1,\ldots,j_k}| \to - \sum_{s=0}^{k-1} u_s(z) \) in \( L^p_{\text{loc}}(\hat{\mathcal{C}} \setminus F_k) \).

To prove the statement we show that from any sequence \( \{P_{n;j_1,\ldots,j_k}^*\}, n \in \Lambda \), of polynomials \( P_{n;j_1,\ldots,j_k}^* \) we can extract a subsequence \( \{P_{n;j_1,\ldots,j_k}^*\}, n \in \Lambda' \subset \Lambda \), satisfying (69) and (70). By the Poincaré-Lelong formula (see [3], for example)
we have
\[
\mu_{n;j_1,...,j_k} := dd^c \log |P_{n;j_1,...,j_k}^*| = 2\pi \left( \sum_{z : \, P_{n;j_1,...,j_k}^*(z) = 0} \delta_z - \deg P_{n;j_1,...,j_k}^* \cdot \delta_\infty \right),
\]
where \(\delta_x\) is the delta measure at the point \(x \in \hat{\mathbb{C}}\). Since \(\deg P_{n;j_1,...,j_k}^* \leq (m+1-k)n\), we have \(\|n^{-1} \mu_{n;j_1,...,j_k} \|_{C(\hat{\mathbb{C}}^*)} \leq 4\pi (m+1-k)\). By the Banach-Alaoerglu theorem (on the compactness of a norm-closed ball in a dual space in the weak* topology), the sequence \(\{n^{-1} \mu_{n;j_1,...,j_k}\}, \, n \in \Lambda\), has a subsequence \(\{n^{-1} \mu_{n;j_1,...,j_k}\}, \, n \in \Lambda' \subset \Lambda\), that weak* converges to a signed measure \(\mu_{j_1,...,j_k} \in C(\hat{\mathbb{C}})^*\). We show that \(\Lambda'\) satisfies (69) and (70). Consider the space of potentials \(\text{Pot}_\phi(\hat{\mathbb{C}})\) (see (51)), where the functional \(\phi\) is defined in terms of the area form \(d\sigma\) (see (52)). The measure \(d\sigma\) has a smooth density with respect to Lebesgue measure in each coordinate neighbourhood, so its local potentials are continuous. So we can use the corollary to Lemma 2.3 in [5], which states that if the measure defining the functional \(\phi\) has continuous local potentials, then the weak convergence of measures implies the convergence of their potentials in \(\text{Pot}_\phi\) in all spaces \(L^p\), \(p \in [1, \infty)\). Hence
\[
n^{-1}(\mu_{n;j_1,...,j_k})_\phi \to (\mu_{j_1,...,j_k})_\phi \quad \text{for } n \in \Lambda' \text{ in } L^p(\hat{\mathbb{C}}) \text{ for all } p \in [1, \infty).
\]
Since the polynomials \(P_{n;j_1,...,j_k}^*\) are spherically normalized (see (50)), we have \(P_{n;j_1,...,j_k}^* = (\mu_{n;j_1,...,j_k})_\phi\). Consequently, as \(n \to \infty, \, n \in \Lambda'\),
\[
\frac{1}{n} \log |P_{n;j_1,...,j_k}^*| \to (\mu_{j_1,...,j_k})_\phi \quad \text{in } L^p(\hat{\mathbb{C}})
\]
for each \(p \in [1, \infty)\).

Using (80) and since \(n^{-1} dd^c \log |P_{n;j_1,...,j_k}^*| \to \mu_{j_1,...,j_k}\) for \(n \in \Lambda'\) by construction, it remains to show that \((\mu_{j_1,...,j_k})_\phi = -\sum_{s=0}^{k-1} u_s\) in \(L^p(\hat{\mathbb{C}})\). In fact, we have
\[
n^{-1} \log |P_{n;j_1,...,j_k}| \to -\sum_{s=0}^{k-1} u_s \text{ in } L^p_{\text{loc}}(\hat{\mathbb{C}} \setminus F_k) \quad \text{and} \quad P_{n;j_1,...,j_k} = c_{n;j_1,...,j_k} P_{n;j_1,...,j_k},
\]
where the \(c_{n;j_1,...,j_k}\) are some positive constants. Hence from (80) we find that
\[
\frac{1}{n} \log c_{n;j_1,...,j_k} \to (\mu_{j_1,...,j_k})_\phi + \sum_{s=0}^{k-1} u_s \quad \text{in } L^p_{\text{loc}}(\hat{\mathbb{C}} \setminus F_k)
\]
as \(n \to \infty, \, n \in \Lambda'\). Since the \(c_{n;j_1,...,j_k}\) are constants, they can only converge to a constant. Therefore, \((\mu_{j_1,...,j_k})_\phi = -\sum_{s=0}^{k-1} u_s + \text{const in } L^p(\hat{\mathbb{C}})\). On the other hand the function \(-\sum_{s=0}^{k-1} u_s\) is spherically normalized (see (68)), and so
\[
\phi((\mu_{j_1,...,j_k})_\phi) = \phi\left(-\sum_{s=0}^{k-1} u_s\right) = 0.
\]
Thus, \(\text{const} = 0\), that is, \((\mu_{j_1,...,j_k})_\phi = -\sum_{s=0}^{k-1} u_s\). Statement 2 is proved.

5.3. Proof of Theorem 2. Theorem 2 is precisely Corollary 3 below (see also Remark 4), which is proved in this subsection.

Let \(\tilde{w}_1, \ldots, \tilde{w}_W\) be the zeros and poles (taking no account of multiplicities) of all functions \(M_{j_1,...,j_k}(\tilde{z}), \, 0 \leq j_1 < \cdots < j_k \leq m\), and let \(w_i = \tilde{\pi}(\tilde{w}_i), \, i = 1, \ldots, W,\)
be their projections. Recall that \( a_i(n) = \tilde{\pi}(\tilde{a}_i(n)) \) and \( b_i = \tilde{\pi}(\tilde{b}_i), \) \( i = 1, \ldots, S, \) are the projections of the zeros and poles of the remainder function \( R_n \) (see (43)). For any \( \varepsilon > 0 \) and any point \( z^* \in \hat{\mathbb{C}} \) we denote the disc in the spherical metric with centre \( z^* \) and radius \( \varepsilon \) by \( O_{z^*}^\varepsilon := \{ z : \text{dist}(z, z^*) < \varepsilon \} \). For each compact set \( K \subset \hat{\mathbb{C}} \) and \( \varepsilon > 0, \) set
\[
K^\varepsilon(n) := K \setminus \left( \bigcup_{i=1}^{W} O_{w_i}^\varepsilon \cup \bigcup_{i=1}^{S} O_{a_i(n)}^\varepsilon \cup \bigcup_{i=1}^{S} O_{b_i}^\varepsilon \right).
\]

**Statement 3.** Assume that the surface \( \tilde{\mathfrak{R}}_{[k]} \) constructed in terms of \( \pi \) is connected. Then for any compact set \( K \subset \hat{\mathbb{C}} \setminus F_k \) and an arbitrary \( \varepsilon > 0, \)
\[
\lim_{n \to \infty} \max_{z \in K^\varepsilon(n)} \left| \frac{P_{n:i_1,\ldots,i_k}(z)}{P_{n;i_1,\ldots,i_k}(z)} - \frac{M_{j_1,\ldots,j_k}(\tilde{z}(01\ldots k-1))}{M_{i_1,\ldots,i_k}(\tilde{z}(01\ldots k-1))} \right| = 0.
\]
Moreover, the following estimate holds for the rate of convergence:
\[
\lim_{n \to \infty} \max_{z \in K^\varepsilon(n)} \left( \left| \frac{P_{n:i_1,\ldots,i_k}(z)}{P_{n;i_1,\ldots,i_k}(z)} - \frac{M_{j_1,\ldots,j_k}(\tilde{z}(01\ldots k-1))}{M_{i_1,\ldots,i_k}(\tilde{z}(01\ldots k-1))} \right|^{1/n} e^{u_k(z)-u_{k-1}(z)} \right) \leq 1.
\]

**Proof.** We fix a compact set \( K \subset \hat{\mathbb{C}} \setminus F_k \) and \( \varepsilon > 0. \) From the representation for \( P_{n:j_1,\ldots,j_k} \) in (55) we see that for \( z \in K \)
\[
\frac{P_{n:j_1,\ldots,j_k}(z)}{P_{n;i_1,\ldots,i_k}(z)} = \frac{M_{j_1,\ldots,j_k}(\tilde{z}(01\ldots k-1))}{M_{i_1,\ldots,i_k}(\tilde{z}(01\ldots k-1))} \frac{1+h_{n:j_1,\ldots,j_k}(z)}{1+h_{n:i_1,\ldots,i_k}(z)},
\]
where the \( h_{n:j_1,\ldots,j_k} \) were defined in (56). We show that
\[
\lim_{n \to \infty} \max_{z \in K^\varepsilon(n)} |h_{n:j_1,\ldots,j_k}(z)| = 0.
\]
Of course, this will imply (82). We proceed as in deriving the property
\[
\lim_{n \to \infty} \max_{z \in \Gamma_n} |h_{n:j_1,\ldots,j_k}(z)| = 0
\]
in the proof of Statement 1. So, for \( z \in K \) we use estimate (57) for \( h_{n:j_1,\ldots,j_k}: \)
\[
|h_{n:j_1,\ldots,j_k}(z)| \leq \sum_{\tilde{z} \in \pi^{-1}(z) \setminus \tilde{z}(01\ldots k-1)} \left| \frac{M_{j_1,\ldots,j_k}(\tilde{z})}{M_{j_1,\ldots,j_k}(\tilde{z}(01\ldots k-1))} \right| e^{-n(u_k(z)-u_{k-1}(z))}.
\]
Let \( K_1 := K \setminus \bigcup_{k=1}^{W} O_{w_k}^\varepsilon. \) (In particular, \( K^\varepsilon(n) \subset K_1. \)) Then
\[
C_{j_1,\ldots,j_k} := \max_{\tilde{z} \in \pi^{-1}(K_1)} \left| \frac{M_{j_1,\ldots,j_k}(\tilde{z})}{M_{j_1,\ldots,j_k}(\tilde{z}(01\ldots k-1))} \right| < \infty.
\]
We have \( \text{dist}(K^\varepsilon(n), a_i(n)) \geq \varepsilon \) and \( \text{dist}(K^\varepsilon(n), b_i) \geq \varepsilon, \) \( i = 1, \ldots, S, \) so proceeding as in the derivation of (61) in the proof of Statement 1 we show that there exists a constant \( \tilde{C}_1 = \tilde{C}_1(\varepsilon) \) such that, for \( \tilde{z}_1, \tilde{z}_2 \in \pi^{-1}(K^\varepsilon(n)), \)
\[
\frac{\psi_n(\tilde{z}_1)}{\psi_n(\tilde{z}_2)} \leq e^{2S\tilde{C}_1}.
\]
Consequently,
\[
|h_{n;j_1,\ldots,j_k}(z)| \leq C_{m+1}^k C_{j_1,\ldots,j_k} e^{2S\tilde{C}_1} e^{-n(u_k(z)-u_{k-1}(z))}. \tag{85}
\]

The functions \( u_i \) are continuous in \( \mathbb{C} \) (see [8], Appendix 1, Lemma 1) and the function \( u_k(z) - u_{k-1}(z) \) for \( k > 1 \) is continuous in a neighbourhood of \( \infty \), while for \( k = 1 \) it converges to \( +\infty \) as \( z \to \infty \). Hence, as the compact set \( K \) is disjoint from \( F_k \), we have \( \varepsilon := \min_{z\in K} (u_k(z)-u_{k-1}(z)) > 0 \). Thus
\[
|h_{n;j_1,\ldots,j_k}(z)| \leq C_{m+1}^k C_{j_1,\ldots,j_k} e^{2S\tilde{C}_1} e^{-n\varepsilon}.
\]

Since \( \varepsilon > 0 \), we have \( \lim_{n\to\infty} \max_{z\in K^\varepsilon(n)} |h_{n;j_1,\ldots,j_k}(z)| = 0. \)

Now we prove (83). From (84), for \( z \in K \) we obtain
\[
\frac{P_{n;i_1,\ldots,i_k}(z)}{P_{n;j_1,\ldots,j_k}(z)} - \frac{M_{j_1,\ldots,j_k}(z^{(01\ldots k-1)})}{M_{i_1,\ldots,i_k}(z^{(01\ldots k-1)})} = \frac{M_{j_1,\ldots,j_k}(z^{(01\ldots k-1)})}{M_{i_1,\ldots,i_k}(z^{(01\ldots k-1)})} \left( \frac{|h_{n;j_1,\ldots,j_k}(z) - h_{n;i_1,\ldots,i_k}(z)|}{1 + h_{n;i_1,\ldots,i_k}(z)} \right).
\]

We have \( K^\varepsilon(n) \subset K_1 \), and hence
\[
\max_{z\in K^\varepsilon(n)} \left| \frac{M_{j_1,\ldots,j_k}(z^{(01\ldots k-1)})}{M_{i_1,\ldots,i_k}(z^{(01\ldots k-1)})} \right| \leq \max_{z\in K_1} \left| \frac{M_{j_1,\ldots,j_k}(z^{(01\ldots k-1)})}{M_{i_1,\ldots,i_k}(z^{(01\ldots k-1)})} \right| =: C_{j_1,\ldots,j_k}^{i_1,\ldots,i_k} < \infty. \tag{87}
\]

Since we have already shown that \( \lim_{n\to\infty} \max_{z\in K^\varepsilon(n)} |h_{n;i_1,\ldots,i_k}(z)| = 0 \), there exists \( N \) such that \( |1 + h_{n;i_1,\ldots,i_k}(z)| > 1/2 \) for all \( n > N \) and \( z \in K^\varepsilon(n) \). So, using (87) and (85), from (86), for \( n > N \) and \( z \in K^\varepsilon(n) \) we have
\[
\frac{P_{n;j_1,\ldots,j_k}(z)}{P_{n;i_1,\ldots,i_k}(z)} - \frac{M_{j_1,\ldots,j_k}(z^{(01\ldots k-1)})}{M_{i_1,\ldots,i_k}(z^{(01\ldots k-1)})} \leq \tilde{C}_0 e^{-n(u_k(z)-u_{k-1}(z))}, \tag{88}
\]

where \( \tilde{C}_0 = 2^{(m+1)} C_{j_1,\ldots,j_k}^{i_1,\ldots,i_k} (C_{j_1,\ldots,j_k} + C_{i_1,\ldots,i_k}) e^{2S\tilde{C}_1} \) is a constant. Estimate (83) evidently follows from (88). Statement 3 is proved.

**Corollary 3.** Assume that the surface \( \tilde{R}_{[k]} \) constructed in terms of \( \pi \) is connected. Then for any compact set \( K \subset \mathbb{C} \setminus F_k \),
\[
\frac{P_{n;j_1,\ldots,j_k}(z)}{P_{n;i_1,\ldots,i_k}(z)} \xrightarrow{\text{cap}} \frac{M_{j_1,\ldots,j_k}(z)}{M_{i_1,\ldots,i_k}(z)}, \quad z \in K, \tag{89}
\]
as \( n \to \infty \). Moreover, for an arbitrary \( \varepsilon' > 0 \)
\[
\text{cap}\left\{ z \in K : \left| \frac{P_{n;j_1,\ldots,j_k}(z)}{P_{n;i_1,\ldots,i_k}(z)} - \frac{M_{j_1,\ldots,j_k}(z)}{M_{i_1,\ldots,i_k}(z)} \right|^{1/n} e^{u_k(z)-u_{k-1}(z)} \geq 1 + \varepsilon' \right\} \to 0. \tag{90}
\]
Proof. To prove the corollary, we show roughly speaking that in Statement 3, in the case when \( \infty \notin K \), instead of discs in the spherical metric, we can remove small Euclidean discs with the same centres from the compact set \( K \), and we do not take account of those discs with centres ‘far’ from \( K \). So let \( B^e_{z^*} := \{ z : |z - z^*| < \varepsilon \} \) denote the disc of radius \( \varepsilon \) with centre \( z^* \) in the standard Euclidean metric in \( \mathbb{C} \). We set

\[
r = \max_{\zeta \in K} \text{dist}(0, \zeta) + \frac{1}{3} \text{dist}(K, \infty) \quad \text{and} \quad R = \max_{\zeta \in K} \text{dist}(0, \zeta) + \frac{2}{3} \text{dist}(K, \infty).
\]

Let \( \{v_i(n)\}_{i=1}^{l(n)} \), \( l(n) \leq 2S + W \) denote those points among the \( a_i(n), b_i \) and \( w_i \) that lie in \( \overline{O_0^e} \) and set \( \tilde{K}^e(n) := K \setminus \bigcup_{i=1}^{l(n)} B^e_{v_i(n)} \). Since the spherical metric and the Euclidean metric on \( \overline{O_0^e} \) are equivalent, there exists a constant \( C \) such that \( |z_1 - z_2| \leq C \text{dist}(z_1, z_2) \) for \( z_1, z_2 \in \overline{O_0^e} \). Now let \( \varepsilon < C \text{dist}(K, \infty)/3 \). Then on the one hand, for \( z^* \in \overline{O_0^e} \) we have \( O^e_{z^*/C} \subset B^e_{z^*} \), and on the other hand, for \( z^* \in \mathbb{C} \setminus \overline{O_0^e} \) we have \( O^e_{z^*/C} \cap K = \emptyset \). Consequently, \( \tilde{K}^e(n) \subset K^{e/C}(n) \). Hence, in the case when \( \infty \notin K \) the compact sets \( K^e(n) \) can be replaced by the \( \tilde{K}^e(n) \) and the conclusion of Statement 3 remains valid. Since \( K \setminus \tilde{K}^e(n) \) is contained in a union of at most \( 2S + W \) Euclidean discs with centres on \( \overline{O_0^e} \) and radius \( \varepsilon \), we have \( \text{cap}(K \setminus \tilde{K}^e(n)) \leq \text{const} \varepsilon^{1/(2S+W)} \) (this follows from the standard estimate for the capacity of a union of sets: see [15], Theorem 5.1.4, for example). Therefore, \( \text{cap}(K \setminus \tilde{K}^e(n)) \to 0 \) as \( n \to \infty \). Now the required result clearly follows from Statement 3 (with \( K^e(n) \) replaced by \( \tilde{K}^e(n) \)).

Remark 4. Since in Statement 3 and Corollary 3 we consider the ratios of \( k \)th polynomials of the Hermite-Padé \( m \)-system, these results also hold true for arbitrary \( k \)th polynomials of the Hermite-Padé \( m \)-system that satisfy (1) (and not only for those for which \( \log |\tilde{R}_n| \) is spherically normalized on the sheet \( \tilde{R}_n^{(01\ldots k-1)} \) as in (49)).

§ 6. The condition that the Riemann surface \( \tilde{R}_n^{[k]} \) be connected

In this section we discuss the condition in Theorems 1 and 2 that the surface \( \tilde{R}_n^{[k]} \) is connected. First we explain why this condition is necessary to the proofs of Theorems 1 and 2 (using our methods). The key point for us was deriving expression (43) for the divisor \( (\tilde{R}_n) \) and deducing the representation (47) for the function \( |\tilde{R}_n| \) in (43):

\[
|\tilde{R}_n(\tilde{z})| = C_n e^{-n\tilde{u}(\tilde{z})} \psi_n(\tilde{z}),
\]

where \( C_n \) is a positive constant. After that we fixed the normalizations of the functions \( \tilde{u} \) and \( \log \psi_n \), which were spherical on the sheet \( \tilde{R}_n^{(01\ldots k-1)} \) (see (48)). Now, since \( \tilde{R}_n \) is expressed in terms of the \( P_{n,i_1,\ldots, i_k} \) via (23), multiplying all the \( P_{n,i_1,\ldots, i_k} \) by the same constant, we were able to take an arbitrary (convenient for us) value of \( C_n \) (we set \( C_n = 1 \); see (49)). In the case when the surface \( \tilde{R}_n^{[k]} \) is disconnected, it follows from (43) that the representation (47) holds on each connected component of \( \tilde{R}_n^{[k]} \). Therefore, the constant \( C_n \) may depend on this connected component, while we can only multiply all the \( P_{n,i_1,\ldots, i_k} \) (and therefore also \( \tilde{R}_n \)) by the same constant,
so that we can only multiply all the constants \( C_n \) by the same factor simultaneously. So we cannot take an arbitrary convenient normalization of the function \( \check{R}_n \) in this case. Moreover, if \( \check{\mathcal{R}}_{[k]} \) is disconnected, then the condition \( \check{R}_n \neq 0 \) only implies that there exists at least one connected component on which \( \check{R}_n \neq 0 \), but \( \check{R}_n \) can vanish identically on some other components. We can show that in this case \( |\check{R}_n| \) has a representation (47) on each connected component of \( \check{\mathcal{R}}_{[k]} \) with its own constant \( C_n \geq 0 \), that is, \( C_n \) may vanish on some components. Moreover, similarly to \( \check{R}_n \), the functions \( M_{j_1, \ldots, j_k}(\check{z}) \) in (19) may also vanish identically on some connected components of \( \check{\mathcal{R}}_{[k]} \). Therefore, if \( \check{\mathcal{R}}_{[k]} \) is disconnected, then even if we assume that the problem of normalizing \( \check{R}_n \) is solved somehow, the leading asymptotic term in the representation (54) for \( P_{n; i_1, \ldots, i_k} \) in terms of \( \check{R}_n \) ‘need not correspond’ to the sheet \( \check{\mathcal{R}}_{[k]}^{(01 \ldots k-1)} \). In particular, the function \( M_{j_1, \ldots, j_k}(\check{z}^{(01 \ldots k-1)}) \) in (55) may vanish identically in some domain. Moreover, if \( \check{R}_n(\check{z}^{(01 \ldots k-1)}) \) vanishes identically in some domain, then the representation (55) is meaningless because in this case the function \( h_{n; j_1, \ldots, j_k} \) in (56) is not defined.

Now we discuss the condition that the surface \( \check{\mathcal{R}}_{[k]} \) be connected. Recall that we have fixed a compact Riemann surface \( \mathcal{R} \) and an \((m + 1)\)-sheeted branched cover \( \pi: \mathcal{R} \to \hat{\mathbb{C}} \). So we can consider \( \mathcal{R} \) as the standard compactification of the Riemann surface \( \mathcal{R}' \) of the \((m + 1)\)-valued global analytic function (GAF) \( w(\cdot) := \pi^{-1}(\cdot) \) in the domain \( \hat{\mathbb{C}} \setminus \Sigma \) (where \( \Sigma \) is the set of critical values of \( \pi \)). Points in \( \mathcal{R}' \) are pairs \((z, w^z)\), where \( z \in \hat{\mathbb{C}} \setminus \Sigma \) and \( w^z \) is the germ of the GAF \( w \) at the point \( z \). The surface \( \check{\mathcal{R}}_{[k]} \) is defined to be the standard compactification of the Riemann surface \( \check{\mathcal{R}}_{[k]}' \) of all possible unordered tuples of \( k \) distinct germs of \( w \) at the same point \( z \in \hat{\mathbb{C}} \setminus \Sigma \) (for more details, see §4). So \( \check{\mathcal{R}}_{[k]} \) is connected if and only if \( \check{\mathcal{R}}_{[k]}' \) is. In our argument we prove exactly the connectedness/disconnectedness of the surface \( \check{\mathcal{R}}_{[k]}' \), without indicating this in formulations. As before, points on \( \check{\mathcal{R}}_{[k]}' \) are denoted by \((z, \{w^z_1, \ldots, w^z_k\})\), where \( z \in \hat{\mathbb{C}} \setminus \Sigma \) and \( \{w^z_1, \ldots, w^z_k\} \) is an unordered tuple of \( k \) distinct germs of the function \( w(\cdot) \) at the point \( z \).

We first note that, by construction, for \( k = 1 \) the surface \( \check{\mathcal{R}}_{[1]}' \) coincides precisely with \( \mathcal{R}' = \mathcal{R} \setminus \pi^{-1}(\Sigma) \). It is easy to see that for \( k = m \) the surface \( \check{\mathcal{R}}_{[m]}' \) is isomorphic to \( \mathcal{R}' \). In fact, let \((z, \{w^z_1, \ldots, w^z_m\}) \in \check{\mathcal{R}}_{[m]}' \). We set \( w^z_{m+1} \) to be the only germ of the GAF \( w \) at \( z \) that is not contained in \( \{w^z_1, \ldots, w^z_m\} \). The required isomorphism takes \((z, \{w^z_1, \ldots, w^z_m\}) \) to \((z, w^z_{m+1})\). Thus the surfaces \( \check{\mathcal{R}}_{[1]}' \) and \( \check{\mathcal{R}}_{[m]}' \) are always connected.

Now we show that for the following class of projections \( \pi \) all surfaces \( \check{\mathcal{R}}_{[k]} \), \( k = 1, \ldots, m \), are connected.

**Statement 4.** Assume that all critical points of the projection \( \pi: \mathcal{R} \to \hat{\mathbb{C}} \) are of the first order and there is at most one critical point of \( \pi \) over each \( z \in \hat{\mathbb{C}} \). Then all the surfaces \( \check{\mathcal{R}}_{[k]} \), \( k = 1, \ldots, m \), are connected.

**Proof.** Since, as we mentioned above, the surfaces \( \check{\mathcal{R}}_{[1]}' \) and \( \check{\mathcal{R}}_{[m]}' \) are always connected, we assume in what follows that \( m \geq 3 \) and fix some \( k = 2, \ldots, m - 1 \). We show that the surface \( \check{\mathcal{R}}_{[k]}' \) is connected.
Let the set $\Sigma$ of branch points of the GAF $w := \pi^{-1}$ consist of points $a_1, \ldots, a_J$. The assumptions of the theorem mean that there is precisely one ramification point of $w$ over each point $a_j$, and its order is 2. Since the set $\Sigma$ is finite, there exists $a \in \mathbb{C}$ such that the line segments $[a, a_j]$, $j = 1, \ldots, J$, only intersect in $a$ (in the case when $\infty \in \Sigma$, we mean by $[a, \infty]$ any ray from $a$ to $\infty$ not containing other points $a_j$). Set $S := \bigcup_{j=1}^{J} [a, a_j]$. Then $\mathbb{C} \setminus S$ is connected. It is clear that over $\mathbb{C} \setminus S$ the surface $\mathcal{R}$ decomposes into a union of $m+1$ disjoint connected sheets, and $\pi$ is biholomorphic on each of them. In terms of $w$ this means that in $\mathbb{C} \setminus S$ the GAF $w$ splits into $m+1$ distinct holomorphic functions (the branches of $w$), which we denote by $w_1(\cdot), \ldots, w_{m+1}(\cdot)$. In what follows we assume that for $z \in \mathbb{C} \setminus S$ $w^{z}$ is the germ of exactly the function $w_i(\cdot)$ at the point $z$, $i = 1, \ldots, m+1$. Fix $z^* \in \mathbb{C} \setminus S$. We show that for any tuple of $k+1$ distinct indices $i_1, \ldots, i_{k+1}$ such that $1 \leq i_s \leq m+1$ there exists a path $\gamma \subset \mathbb{C} \setminus \Sigma$ beginning and ending at $z^*$ such that in continuing each germ $w^{z^*}_{i_s}$, $s = 1, \ldots, k+1$, along this path we obtain $w^{z^*}_{i_s}$ again, and in continuing $w^{z^*}_{i_k}$ we obtain $w^{z^*}_{i_k+1}$. So the lifting of this path to $\mathcal{R}'_{[k]}$ connects the point $(z^*, \{w^{z^*}_{i_1}, \ldots, w^{z^*}_{i_{k-1}}, w^{z^*}_{i_k}\})$ with $(z^*, \{w^{z^*}_{i_1}, \ldots, w^{z^*}_{i_{k+1}}\})$. It is clear that taking a composition of several paths similar to $\gamma$, we can construct a path connecting $(z^*, \{w^{z^*}_{i_1}, \ldots, w^{z^*}_{i_k}\})$ with an arbitrary prescribed point $(z^*, \{w^{z^*}_{j_1}, \ldots, w^{z^*}_{j_k}\}) \in \mathcal{R}'_{[k]}$. Since $\mathbb{C} \setminus S$ is connected, this means that $\mathcal{R}'_{[k]}$ is too.

Now we construct the path $\gamma$. By assumption the Riemann surface $\mathcal{R}'$ of the GAF $w$ is connected, hence there exists a path $\gamma' \subset \mathbb{C} \setminus \Sigma$ such that continuing $w^{z^*}_{i_k}$ along this path we obtain $w^{z^*}_{i_{k+1}}$. For each $j = 1, \ldots, J-1$ we fix an (oriented) loop $\alpha_j$ around $a_j$ with endpoints at $z^*$ such that $\alpha_j$ intersects $S$ in a unique point, which lies on the open segment $(a, a_j)$. Then $\gamma'$, as a path in $\mathbb{C} \setminus \Sigma$, is homotopic to the path $\gamma''$ that is the composition of some paths $\alpha_j$ and some paths $\alpha_j^{-1}$, where $\alpha_j^{-1}$ is the loop $\alpha_j$ traversed in the opposite direction. Consequently, continuing any germ of $w$ along the path $\gamma'$ or $\gamma''$ gives the same result; in particular, $w^{z^*}_{i_k}$ is continued to $w^{z^*}_{i_{k+1}}$ along $\gamma''$. Let us see what happens when germs of $w$ are continued along the loops $\alpha_j$. Since all the ramification points of $w$ are of the second order, the results of continuing germs of $w$ at $z^*$ along the loops $\alpha_j$ and $\alpha_j^{-1}$ coincide. Thus we assume below that the paths $\gamma''$ consists only of loops $\alpha_j$. Since there is precisely one ramification point of the second order over each point $a_j$, we see that when we traverse the loop $\alpha_j$, only two germs in the tuple $w^{z^*}_{1}, \ldots, w^{z^*}_{m+1}$ get interchanged, while the others remain unchanged. Suppose that, in traversing $\alpha_j$, a germ $w^{z^*}_{s}$ is transformed into $w^{z^*}_{t}$ (possibly coinciding with $w^{z^*}_{s}$). Then we write $w^{z^*}_{s} \xrightarrow{\alpha_j} w^{z^*}_{t}$. So assume that the path $\gamma''$ consist of $B$ loops $\alpha_j$:

$$\gamma'' = \alpha_{j_1} \circ \cdots \circ \alpha_{j_s},$$

where $j_s \subset \{1, \ldots, J-1\}$. Set $w^{z^*}_{r_1} := w^{z^*}_{i_k}$ and $w^{z^*}_{r_{B+1}} := w^{z^*}_{i_{k+1}}$. Let $w^{z^*}_{r_l} \xrightarrow{\alpha_j} w^{z^*}_{r_{l+1}}$, $l = 1, \ldots, B$, that is,

$$w^{z^*}_{i_k} =: w^{z^*}_{r_1} \xrightarrow{\alpha_{j_1}} w^{z^*}_{r_2} \xrightarrow{\alpha_{j_2}} \cdots \xrightarrow{\alpha_{j_{B-1}}} w^{z^*}_{r_{B}} \xrightarrow{\alpha_{j_B}} w^{z^*}_{r_{B+1}} := w^{z^*}_{i_{k+1}}. \tag{93}$$

Suppose that $w^{z^*}_{r_l} = w^{z^*}_{r_{l'}}$ for some $1 \leq l < l' \leq B + 1$. Then removing the piece $\alpha_{j_{l'}} \circ \cdots \circ \alpha_{j_l}$ from the representation (92) for $\gamma''$ we also obtain a path
transforming $w_{ik}^*$ into $w_{ik+1}^*$. Hence we assume in what follows that $\gamma''$ is a path such that all the $w_{ir}^*$ in (93) are distinct. We show that in this case the path

$$\gamma := \alpha_{j_1} \circ \cdots \circ \alpha_{j_{B-1}} \circ \alpha_j \circ \alpha_{j_{B-1}} \circ \cdots \circ \alpha_{j_1}$$

(94)
is the required one.

First we show that continuing $w_{ik}^*$ along $\gamma$ gives $w_{ik+1}^*$. By construction all $w_{ir}^*$ in (93) are distinct, so in going along the loop $\alpha_{j_l}$, $l = 1, \ldots, B$, the germs $w_{ir}^*$ and $w_{ir+1}^*$ get interchanged, while all other germs of $w$ at $z^*$ remain unchanged. So, when we traverse the loops $\alpha_{j_l}$, $l = 1, \ldots, B - 1$, the germ $w_{ik+1}^*$ does not change.

The continuation of $w_{ik}^*$ along $\gamma''$ in (92) gives $w_{ik+1}^*$, hence taking the form (94) of $\gamma$ into account we conclude that the continuation of $w_{ik}^*$ along $\gamma$ also gives $w_{ik+1}^*$. Now we show that for all $s = 1, \ldots, k - 1$ the continuation of $w_{is}^*$ along $\gamma$ gives the same germ $w_{is}^*$. We fix some $s$ in this set. Let the continuation of $w_{is}^*$ along the path $\alpha_{j_{B-1}} \circ \cdots \circ \alpha_{j_1}$ produce a germ $w_{ib}^*$. We show that traversing the loop $\alpha_{j_B}$ does not change $w_{ib}^*$.

Assume the contrary. Then, since all the germs $w_{ir}^*$ in (93) are distinct, we have $w_{ib}^* = w_{iB+1}^*$. Hence continuing $w_{is}^*$ along $\alpha_{j_{B-1}} \circ \cdots \circ \alpha_{j_1}$ we obtain $w_{iB+1}^* = w_{ik+1}^*$. On the other hand, going along $\alpha_{j_l}$, $l = 1, \ldots, B$, interchanges the germs $w_{ir}^*$ and $w_{ir+1}^*$, while the other germs remain unchanged. Consequently (since $w_{ik+1}^* \neq w_{is}^*$), among the germs $w_{ir}^*$, $l = 1, \ldots, B - 1$, there is a germ $w_{ik+1}^* = w_{iB+1}^*$. But this contradicts the assumption that all germs $w_{ir}^*$ in (93) are distinct. Hence the continuation of the germ $w_{is}^*$ along $\gamma$ in (94) is equivalent to its continuation along the path

$$\alpha_{j_1} \circ \cdots \circ \alpha_{j_{B-1}} \circ \alpha_j \circ \alpha_{j_{B-1}} \circ \cdots \circ \alpha_{j_1} \equiv \text{Id},$$

(95)

that is, this germ does not change after continuation along $\gamma$. Statement 4 is proved.

Although Statement 4 shows that the class of projections $\pi$ for which the surfaces $\mathfrak{R}_{[k]}$ are connected is quite wide, the following result, Proposition 4, provides a natural class of projections $\pi$ for which the surfaces $\mathfrak{R}_{[k]}$ are disconnected for all $k = 2, \ldots, m - 1$.

Since $\mathfrak{R}$ is a compact Riemann surface and $\pi : \mathfrak{R} \to \hat{\mathbb{C}}$ is a $(m + 1)$-sheeted branched covering of $\hat{\mathbb{C}}$, $\pi$ is isomorphic to the covering of $\hat{\mathbb{C}}$ by a complex algebraic curve, defined in the affine part of $\hat{\mathbb{C}}_z \times \hat{\mathbb{C}}_w$ by an algebraic equation $P(z, w) = 0$, where $P$ is an irreducible polynomial of degree $(m + 1)$ with respect to $w$, by means of the natural projection $(z, w) \mapsto z$. Correspondingly, the surface $\mathfrak{R}'$ is isomorphic to the Riemann surface of the GAF $w$ in $\hat{\mathbb{C}} \setminus \Sigma$.

**Proposition 4.** Let $\mathfrak{R}$ be the Riemann surface defined by the algebraic equation $w^{m+1} = R(z)$, where $m \geq 3$ and $R(z) = P(z)/Q(z)$, where $P$ and $Q$ are polynomials and the polynomial $w^{m+1}Q(z) - P(z)$ is irreducible. Let $\pi : (z, w) \mapsto z$. Then the surfaces $\mathfrak{R}_{[k]}$ are disconnected for all $k = 2, \ldots, m - 1$. 

Proof. The surfaces \( \tilde{\mathfrak{R}}_{[k]} \) and \( \tilde{\mathfrak{R}}_{[m+1-k]} \) are isomorphic (the corresponding isomorphism sends a point \((z, \{w^z_1, \ldots, w^z_k\})\) to \((z, \{w^z_{k+1}, \ldots, w^z_{m+1}\})\), where \(w^z_{k+1}, \ldots, w^z_{m+1}\) are the germs of \(w\) at \(z\) that do not belong to the tuple \(\{w^z_1, \ldots, w^z_k\}\). So we can assume that \(k \leq (m+1)/2\).

The branch points of the GAF \(w(z) = \frac{m+1}{\sqrt{R(z)}}\) belong to the set of zeros and poles of the function \(R(z)\). Let \(a\) be a zero or a pole of order \(l\) of \(R(z)\), and let \(d\) be the greatest common divisor of \(m+1 \) and \(l\). Then \(W\) has \(d\) ramification points of order \((m+1)/d\) over \(a\). Therefore, when we go around \(a\), the ratio of two arbitrary germs of \(w\) is preserved. Hence if \(w^z_r\) and \(w^z_t\) are two germs of \(w(z)\) at a point \(z^* \in \hat{C} \setminus \Sigma\), then the ratio \(w^z_r/w^z_t\) is preserved under continuation along any path \(\gamma\) in \(\hat{C} \setminus \Sigma\) with endpoints at \(z^*\).

We fix a point \(z^* \in C\) that is neither a zero nor a pole of \(R(z)\). Let \(w^z_i\) be some germ of \(w\) at \(z^*\). For each \(j = 2, \ldots, m+1\) let \(w^z_j\) denote the germ of \(w\) at \(z^*\) such that

\[
w^z_j(z^*) = \exp\left(\frac{2\pi i (j-1)}{m+1}\right) w^z_i(z^*).
\]

Recall that \(k \leq (m+1)/2\) by assumption. We show that there is no path \(\gamma \subset \hat{C} \setminus \Sigma\) with endpoints at \(z^*\) such that continuation along this path transforms the unordered tuple of germs \(\{w^z_1, w^z_2, \ldots, w^z_k\}\) into the tuple \(\{w^z_1, w^z_2, \ldots, w^z_k, w^z_s\}\), where \(s = (m+1)/2 + 1\) for odd \(m\) and \(s = m/2 + 1\) for even \(m\). In fact, we have \(\arg(w^z_r(z^*)/w^z_i(z^*)) = 2\pi(s-1)/(m+1)\), and the arguments of all possible ratios \(w^z_r(z^*)/w^z_i(z^*)\), where \(r, t \leq k \leq (m+1)/2\), are no greater than \(2\pi(s-2)/(m+1)\) (modulo \(2\pi\)). Therefore, among \(w^z_1, w^z_2, \ldots, w^z_k\) there are no two germs whose ratio at \(z^*\) is \(w^z_r(z^*)/w^z_i(z^*)\). Since, in accordance with the above, the ratio of two arbitrary germs of \(w\) is preserved under continuation along any closed path, it follows that the required \(\gamma\) does not exist and the proof is complete.

At the end of this section we consider the simplest case when \(m = 3\) and therefore \(k = 2\) in greater detail. Since in this case the covering \(\pi: \tilde{\mathfrak{R}}_{[2]} \to \hat{C}\) is 4-sheeted, it follows that the covering \(\tilde{\pi}: \tilde{\mathfrak{R}}_{[2]} \to \hat{C}\) is 6-sheeted, and it is easy to see that any connected component of the latter covering is at least 2-sheeted. So three cases are possible: 1) \(\tilde{\pi}: \tilde{\mathfrak{R}}_{[2]} \to \hat{C}\) consists of three connected components of two sheets; 2) \(\tilde{\pi}: \tilde{\mathfrak{R}}_{[2]} \to \hat{C}\) consists of a component of two sheets and another of four sheets; 3) \(\tilde{\pi}: \tilde{\mathfrak{R}}_{[2]} \to \hat{C}\) is a connected 6-sheeted covering. We show that all these possibilities are realized by presenting explicit examples. All the corresponding surfaces \(\mathfrak{R}\) are topological spheres, that is, they have genus zero.

Example 1. Let \(\mathfrak{R}\) be the compactification of the Riemann surface of the GAF \(w(z) = \sqrt{z} + \sqrt{z-1}\) and let \(\pi: (z, w) \mapsto z\). Then the covering \(\tilde{\pi}: \tilde{\mathfrak{R}}_{[2]} \to \hat{C}\) is easily seen to be isomorphic (under a base-preserving isomorphism) to the disjoint union of the coverings of \(\hat{C}\) by means of the GAFs \(w_{(1)}(z) = \sqrt{z}\), \(w_{(2)}(z) = \sqrt{z-1}\) and \(w_{(3)}(z) = \sqrt{z}/(z-1)\).

Example 2. Let \(\mathfrak{R}\) be the compactification of the Riemann surface of the GAF \(w(z) = \sqrt[3]{z}\) and let \(\pi: (z, w) \mapsto z\). Then the covering \(\tilde{\pi}: \tilde{\mathfrak{R}}_{[2]} \to \hat{C}\) is easily seen to be isomorphic (under a base-preserving isomorphism) to the disjoint union of the coverings of \(\hat{C}\) by means of the GAFs \(w_{(1)}(z) = \sqrt[3]{z}\) and \(w_{(2)}(z) = \sqrt[3]{z}\).
Examples 1 and 2 show that, for coverings $\pi$ explicitly defined by means of a GAF $w(z) = \pi^{-1}(z)$ expressed in radicals in terms of $z$, we can hardly expect that the surface $\mathfrak{R}_{[2]}$ will be connected. Moreover, it is easy to show that if $w$ satisfies a polynomial equation $P(z, w) = 0$ that is biquadratic in $w$, then the corresponding surface $\mathfrak{R}_{[2]}$ is always disconnected.

Statement 4 provides a wide class of examples when the surface $\mathfrak{R}_{[2]}$ is connected. However, it does not itself explicitly define such surfaces, as solutions of some equations, for example. Let $w$ be a GAF defined an algebraic equation $z = R(w) := P_4(w)/Q_2(w)$, where $P_4(w)$ and $Q_2(w) = (w - a)(w - b)$ are polynomials of degrees 4 and 2, respectively, that have no common zeros, and let $a \neq b$. We also assume that the derivative $R'(w)$ has no multiple zeros (that is, all the zeros $w_1, \ldots, w_5$ of the polynomial $P_4Q_2 - P_4Q_2'$ are distinct). Let $\mathfrak{R}$ be the compactification of the Riemann surface of this GAF $w$ and let $\pi: (z, w) \mapsto z$. Then $\mathfrak{R}$ is a topological sphere (since $w$ is a global coordinate on it) and the GAF $w = \pi^{-1}$ has precisely five branch points of the second order $R(w_1), \ldots, R(w_5)$ in the affine part and one branch point of the second order at $\infty$. Therefore, in order that this $\pi$ satisfies the assumptions of Statement 4 and thus $\mathfrak{R}_{[2]}$ is connected, it is necessary that $R(w_j) \neq R(w_l)$ for $l \neq j$. Here is an explicit example of such a surface.

Example 3. Let $\mathfrak{R}$ be the Riemann surface of the algebraic function $w$ that is the solution of the equation

$$z = \frac{w^4 - (1 + i)w^3 + 3iw^2}{w^2 + (1 + i)w/3 + i/3}$$

(96)

and let $\pi: (z, w) \mapsto z$. Then the surface $\widetilde{\mathfrak{R}}_{[2]}$ is connected.

In fact, denote the right-hand side of (96) by $R(w)$, its numerator by $P_4(w)$ and its denominator by $Q_2(w)$. Then

$$P_4'(w)Q_2(w) - P_4(w)Q_2'(w) = 2w(w^4 - 1).$$

Consequently, 0, ±1 and ±i are the zeros of $R(w)$. In view of the above we should check that $\hat{R}$ takes distinct values at these points. In fact, we have $R(0) = 0$, $R(1) = (3 + 6i)/5$, $R(-1) = 3 + 6i$, $R(i) = (-3 + 6i)/5$ and $R(-i) = -3 + 6i$.

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