

Giovanni Gallavotti · Ian Jauslin

Kondo effect in the hierarchical \(s - d\) model

December 9, 2015

Abstract The \(s - d\) model describes a chain of spin-1/2 electrons interacting magnetically with a two-level impurity. It was introduced to study the Kondo effect, in which the magnetic susceptibility of the impurity remains finite in the 0-temperature limit as long as the interaction of the impurity with the electrons is anti-ferromagnetic. A variant of this model was introduced by Andrei, which he proved was exactly solvable via Bethe Ansatz. A hierarchical version of Andrei’s model was studied by Benfatto and the authors. In the present letter, that discussion is extended to a hierarchical version of the \(s - d\) model. The resulting analysis is very similar to the hierarchical Andrei model, though the result is slightly simpler.

Keywords Renormalization group · Non-perturbative renormalization · Kondo effect · Fermionic hierarchical model · Quantum field theory

The \(s - d\) model was introduced by Anderson [1] and used by Kondo [4] to study what would subsequently be called the Kondo effect. It describes a chain of electrons interacting with a fixed spin-1/2 magnetic impurity. One of the manifestations of the effect is that when the coupling is anti-ferromagnetic, the magnetic susceptibility of the impurity remains finite in the 0-temperature limit, whereas it diverges for ferromagnetic and for vanishing interactions.

A modified version of the \(s - d\) model was introduced by Andrei [2], which was shown to be exactly solvable by Bethe Ansatz. In [3], a hierarchical version of Andrei’s model was introduced and shown to exhibit a Kondo effect. In the present letter, we show how the argument can be adapted to the \(s - d\) model.

We will show that in the hierarchical \(s - d\) model, the computation of the susceptibility reduces to iterating an explicit map relating 6 running coupling constants (rcs), and that this map can be obtained by restricting the flow equation for the hierarchical Andrei model [3] to one of its invariant manifolds. The physics of both models are therefore very closely related, as had already been argued in [3]. This is particularly noteworthy since, at 0-field, the flow in the hierarchical Andrei model is relevant, whereas it is marginal in the hierarchical \(s - d\) model, which shows that the relevant direction carries little to no physical significance.
The s – d model [4] represents a chain of non-interacting spin-1/2 fermions, called electrons, which interact with an isolated spin-1/2 impurity located at site 0. The Hilbert space of the system is $\mathcal{F}_L \otimes \mathbb{C}^2$ in which $\mathcal{F}_L$ is the Fock space of a length-$L$ chain of spin-1/2 fermions (the electrons) and $\mathbb{C}^2$ is the state space for the two-level impurity. The Hamiltonian, in the presence of a magnetic field of amplitude $h$ in the direction $\omega \equiv (\omega_1, \omega_2, \omega_3)$, is

$$H_K = H_0 + V_0 + V_h \equiv H_0 + V$$

$$H_0 = \sum_{\alpha \in \{1, 2\}} \left( \sum_{x = -L/2}^{L/2 - 1} c^\dagger_\alpha(x) \left( -\frac{\Delta}{2} - 1 \right) c_\alpha(x) \right)$$

$$V_0 = -\lambda_0 \sum_{j=1, 2, 3, \sigma_1, \sigma_2} \tau^j \sigma_1^\dagger \sigma_2^\dagger \sigma_j \tau^j$$

$$V_h = -h \sum_{j=1, 2, 3} \omega_j \tau^j$$

where $\lambda_0$ is the interaction strength, $\Delta$ is the discrete Laplacian $c^\dagger_\alpha(x)$, $\alpha = \uparrow, \downarrow$ are creation and annihilation operators acting on electrons, and $\sigma_1^\dagger, \sigma_j, \tau^j$ are Pauli matrices. The operators $\tau^j$ act on the impurity, the boundary conditions are taken to be periodic.

In the Andrei model [2], the impurity is represented by a fermion instead of a two-level system, that is the Hilbert space is replaced by $\mathcal{F}_L \otimes \mathcal{F}_1$, and the Hamiltonian is defined by replacing $\tau^j$ in Eq.(1) by $d^+ \tau^j d^-$ in which $d^\dagger_\alpha(x)$, $\alpha = \uparrow, \downarrow$ are creation and annihilation operators acting on the impurity.

The partition function $Z = \text{Tr} e^{-\beta H_K}$ can be expressed formally as a functional integral:

$$Z = \text{Tr} \int P(d\psi) \sum_{n=0}^{\infty} (-1)^n \int dt_1 \cdots dt_n V(t_1) \cdots V(t_n)$$

in which $V(t)$ is obtained from $V$ by replacing $c^\dagger_\alpha(0)$ in Eq.(1) by a Grassmann field $\psi_\alpha(0, t)$. $P(d\psi)$ is a Gaussian Grassmann measure over the fields $\{\psi_\alpha(0, t)\}_{1, \alpha}$ whose propagator (i.e. covariance) is, in the $L \to \infty$ limit,

$$g(t, t') = \frac{1}{(2\pi)^2} \int dk_0 e^{ik_0(t-t')}$$

and the trace is over the state-space of the spin-1/2 impurity, that is a trace over $\mathbb{C}^2$.

We will consider a hierarchical version of the s – d model. The hierarchical model defined below is inspired by the s – d model in the same way as the hierarchical model defined in [3] was inspired by the Andrei model. We will not give any details on the justification of the definition, as such considerations are entirely analogous to the discussion in [3].

The model is defined by introducing a family of hierarchical fields and specifying a propagator for each pair of fields. The average of any monomial of fields is then computed using the Wick rule.

Assuming $\beta = 2^{N_\beta}$ with $N_\beta = \log_2 \beta \in \mathbb{N}$, the time axis $[0, \beta)$ is paved with boxes (i.e. intervals) of size $2^{-m}$ for every $m \in \{0, -1, \ldots, -N_\beta\}$; let

$$Q_m \equiv \left\{ [i2^{m}]^+, (i+1)2^{m}] \right\}_{i=0, 1, \ldots, 2^{-N_\beta} - |m| + 1}$$

Given a box $\Delta \in Q_m$, let $t_\Delta$ denote the center of $\Delta$, and given a point $t \in R$, let $\Delta^{[m]}(t)$ be the (unique) box on scale $m$ that contains $t$. We further decompose each box $\Delta \in Q_m$ into two half boxes: for $\eta \in \{-, +\}$, let

$$\Delta^n \equiv \Delta^{[m+1]}(t_\Delta + \eta 2^{-m-2})$$

for $m \leq 0$. Thus $\Delta_-$ can be called the “lower half” of $\Delta$ and $\Delta_+$ the “upper half”.

The elementary fields used to define the hierarchical s – d model will be constant on each half-box and will be denoted by $\psi_\alpha^{[m]+}(\Delta_\eta)$ for $m \in \{0, -1, \cdots, -N_\beta\}$, $\Delta \in Q_m$, $\eta \in \{-, +\}$, $\alpha \in \{\uparrow, \downarrow\}$. 

The propagator of the hierarchical \( s - d \) model is defined as
\[
\left\langle \psi_{\alpha}^{[m]}(\Delta, \eta) \psi_{\alpha}^{[m]+}(\Delta, \eta) \right\rangle \overset{def}{=} \eta
\]
for \( m \in \{0, -1, \ldots, -N_\beta\} \), \( \Delta \in \mathbb{Q}_m \), \( \eta \in \{-, +\} \), \( \alpha \in \{\uparrow, \downarrow\} \). The propagator of any other pair of fields is set to 0.

Finally, we define
\[
\psi_{\alpha}^{\pm}(t) \overset{def}{=} \sum_{m=0}^{-N_\beta} 2^{2m+1} \psi_{\alpha}^{[m] \pm} (\Delta^{m+1}(t)).
\]
The partition function for the hierarchical \( s - d \) model is
\[
Z = \text{Tr} \left\{ \sum_{n=0}^{\infty} (-1)^n \int_0 dt_1 \cdots dt_n \mathcal{V}(t_1) \cdots \mathcal{V}(t_n) \right\}
\]
in which the \( \psi_\alpha^{\pm}(0, t) \) in \( \mathcal{V}(t) \) have been replaced by the \( \psi_\alpha^{\pm}(t) \) defined in Eq.(6):
\[
\mathcal{V}(t) \overset{def}{=} -\lambda_0 \sum_{j=1,2,3} \psi_{\alpha_1}^{\pm}(t) \sigma_{\alpha_1, \alpha_2} \psi_{\alpha_2}^{\pm}(t) + \hbar \sum_{j=1,2,3} \omega_j \tau^j.
\]

This concludes the definition of the hierarchical \( s - d \) model.

We will now show how to compute the partition function Eq.(7) using a renormalization group iteration. We first rewrite
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \mathcal{V}(t_{\Delta}[0])^n = C \left( 1 + \sum_{p} \ell_p^{[0]} O^{[0]}_{p, \eta}(\Delta^{[0]}) \right)
\]
and find that
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \mathcal{V}(t_{\Delta^{[0]}})^n = C \left( 1 + \sum_{p} \ell_p^{[0]} O^{[0]}_{p, \eta}(\Delta^{[0]}) \right)
\]
with
\[
O_{\eta, \eta}^{[0]}(\Delta) \overset{def}{=} \frac{1}{2} A_{\eta, \eta}^{[0]}(\Delta) \cdot \tau,
\]
\[
O_{\eta, \eta}^{[0]}(\Delta) \overset{def}{=} \frac{1}{2} A_{\eta, \eta}^{[0]}(\Delta) \cdot \omega,
\]
\[
O_{\eta, \eta}^{[0]}(\Delta) \overset{def}{=} \frac{1}{2} A_{\eta, \eta}^{[0]}(\Delta) \cdot (\tau \cdot \omega),
\]

(11)

(the numbering is meant to recall that in [3]) in which \( \mathbf{A}_{\eta, \eta}^{[0]}(\Delta) \) is a vector of polynomials in the fields whose \( j \)-th component for \( j \in \{1, 2, 3\} \) is
\[
A_{\eta, \eta}^{[0]}(\Delta) \overset{def}{=} \sum_{(\alpha, \alpha') \in \{\uparrow, \downarrow\}^2} \psi_{\alpha, \alpha'}^{[0]+}(\Delta) \sigma_{\alpha, \alpha'} \psi_{\alpha, \alpha'}^{[0] -}(\Delta) \psi_{\alpha, \alpha'}^{[0] -}(\Delta)
\]
(12)

\[
\psi_{\alpha}^{[0]} \overset{def}{=} \sum_{m=0}^{\infty} 2^{2m+1} \psi_{\alpha}^{[m] \pm},
\]
and
\[
C = \cosh(\hbar), \quad \ell_0^{[0]} = \frac{1}{C} \frac{\lambda_0}{\hbar} \sinh(\hbar)
\]
\[
\ell_1^{[0]} = \frac{1}{C} \frac{\lambda_0^2}{12\hbar} (\hbar \cosh(\hbar) + 2 \sinh(\hbar))
\]
\[
\ell_2^{[0]} = \frac{1}{C} \frac{\lambda_0}{\hbar} \sinh(\hbar), \quad \ell_3^{[0]} = \frac{2}{C} \sinh(\hbar)
\]
\[
\ell_4^{[0]} = \frac{1}{C} \frac{\lambda_0}{\hbar} (\hbar \cosh(\hbar) - \sinh(\hbar))
\]
\[
\ell_5^{[0]} = \frac{1}{C} \frac{\lambda_0^2}{12\hbar^2} (\hbar^2 \sinh(\hbar) + 2 \hbar \cosh(\hbar) - 2 \sinh(\hbar))
\]
(13)
in which $\hat{h} := h/2$.

By a straightforward induction, we find that the partition function Eq.(7) can be computed by defining

$$C^{[m]}\mathcal{W}^{[m-1]}(\Delta^{[m]}) \overset{def}{=} \left( \prod_{\eta} \mathcal{W}^{[m]}(\Delta^{[m]}_\eta) \right)_m$$

in which $\langle \cdot \rangle_m$ denotes the average over $\psi^{[m]}$, $C^{[m]} > 0$ and

$$\mathcal{W}^{[m-1]}(\Delta^{[m]}) = 1 + \sum_p \ell_p^{[m]} O^{\leq m}_p(\Delta^{[m]})$$

in terms of which

$$Z = C^{2|\mathcal{Q}_0|} \prod_{m = -N(\beta) + 1}^0 (C^{[m]}|\mathcal{Q}^{m-1}|)$$

in which $|\mathcal{Q}_m| = 2^{N(\beta)-|m|}$ is the cardinality of $\mathcal{Q}_m$. In addition, similarly to [3], the map relating $\ell^{[m]}_p$ to $\ell^{[m-1]}_p$ and $C^{[m]}$ can be computed explicitly from Eq.(14):

$$C^{[m]} = 1 + \frac{3}{2} \ell_0^2 + \ell_0 \ell_6 + 9 \ell_2^2 + \frac{\ell_2^4}{2} + \frac{\ell_2^6}{4} + \frac{\ell_6^2}{2} + 9 \ell_2^2$$

$$\ell^{[m-1]}_0 = \frac{1}{C} (-\ell_0 - \ell_6^2 + 3 \ell_0 \ell_1 - \ell_0 \ell_6)$$

$$\ell^{[m-1]}_2 = \frac{1}{C} (-\ell_2^2 + \ell_2^4 + \ell_0 \ell_6 + \ell_0 \ell_2 + \ell_2^2 + \ell_2^4)$$

$$\ell^{[m-1]}_4 = \frac{1}{C} (\ell_4 + \ell_6 \ell_5 + 3 \ell_0 \ell_7 + 3 \ell_1 \ell_4 + \ell_5 \ell_7 + 3 \ell_6 \ell_7)$$

$$\ell^{[m-1]}_5 = \frac{1}{C} (2 \ell_5 + 2 \ell_6 \ell_4 + 3 \ell_1 \ell_7 + 2 \ell_4 \ell_6)$$

$$\ell^{[m-1]}_6 = \frac{1}{C} (\ell_6 + \ell_0 \ell_6 + 3 \ell_1 \ell_6 + \ell_4 \ell_5 + 3 \ell_4 \ell_7)$$

$$\ell^{[m-1]}_7 = \frac{1}{C} (\ell_7 + \ell_0 \ell_4 + \ell_1 \ell_5 + \ell_4 \ell_6 + \ell_5 \ell_7)$$

in which the $|m|$ have been dropped from the right hand side.

The flow equation Eq.(17) can be recovered from that of the hierarchical Andrei model studied in [3] (see in particular [3, Eq.(C1)]) by restricting the flow to the invariant submanifold defined by

$$\ell^{[m]}_2 = \frac{1}{3}, \quad \ell^{[m]}_3 = \frac{1}{6} \ell^{[m]}_1, \quad \ell^{[m]}_4 = \frac{1}{6} \ell^{[m]}_4.$$  \hspace{3em} (18)

This is of particular interest since $\ell^{[m]}_2$ is a relevant coupling and the fact that it plays no role in the $s-d$ model indicates that it has little to no physical relevance.

The qualitative behavior of the flow is therefore the same as that described in [3] for the hierarchical Andrei model. In particular the susceptibility, which can be computed by deriving $-\beta^{-1} \log Z$ with respect to $\hbar$, remains finite in the 0-temperature limit as long as $\lambda_0 < 0$, that is as long as the interaction is anti-ferromagnetic.

**Acknowledgements** We are grateful to G. Benfatto for many enlightening discussions on the $s-d$ and Andrei’s models.

**References**

1. Anderson, P.: Local magnetized states in metals. Physical Review **124**, 41–53 (1961)
2. Andrei, N.: Diagonalization of the Kondo Hamiltonian. Physical Review Letters **45**, 379–382 (1980)
3. Benfatto, G., Gallavotti, G., Jauslin, I.: Kondo effect in a fermionic hierarchical model. arXiv: 1506.04381 (2015)
4. Kondo, J.: Resistance Minimum in Dilute Magnetic Alloys. Progress of Theoretical Physics **32**, 37–49 (1964)