SURGERY ON LINKS WITH UNKNOTTED COMPONENTS AND THREE-MANIFOLDS

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Abstract. It is shown that any closed three-manifold \( M \) obtained by integral surgery on a knot in the three-sphere can always be constructed from integral surgeries on a 3-component link \( L \) with each component being an unknot in the three-sphere. It is also interesting to notice that infinitely many different integral surgeries on the same link \( L \) could give the same three-manifold \( M \).

1. Introduction

It is well known that every closed, orientable, connected 3-manifold \( M \) can be obtained by integral surgery on a link in \( S^3 \). Moreover, one may always find a surgery presentation for \( M \) in which each component of the surgery link is an unknot (see [1]). For convenience, we use the word *simple* \( n \)-link to denote an \( n \)-component link with all its components being unknots in \( S^3 \). Then the minimal number \( \nu(M) \) of the components in all integral simple \( n \)-link surgery presentations for \( M \) is a topological invariant of \( M \), that is:

\[
\nu(M) := \min \{ n \mid L \text{ is a simple } n\text{-link in } S^3 \text{ and we can get } M \text{ by doing an integral surgery on } L \}
\]

For example: \( \nu(S^3) = 0 \) and \( \nu(L(p, 1)) = 1 \) where \( L(p, 1) \) is a lens space \((p \geq 2)\). However, it is not easy to compute \( \nu(M) \) in general. In particular, let \( S^3_K(m) \) denote the 3-manifold got from integral surgery on a knot \( K \subset S^3 \) with surgery index \( m \). Then it is easy to see that \( \nu(S^3_K(m)) \leq u(K) + 1 \), where \( u(K) \) is the unknotting number of \( K \). But in fact, We can prove the following:

**Theorem 1.1.** For any knot \( K \subset S^3 \) and any integer \( m \), \( \nu(S^3_K(m)) \leq 3 \), i.e. we can always construct \( S^3_K(m) \) by doing an integral surgery on a simple 3-link in \( S^3 \).

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Remark: In [2], D.Auckly defined a topological invariant called surgery number of a closed 3-manifolds. By his definition, the surgery number of $S^3_K(m)$ is 1 for any knot $K$. The $\nu(M)$ defined above can be considered as another type of surgery number which is more subtle than Auckly’s in the sense that $\nu(S^3_K(m))$ could be different for different knot $K$.

The geometric and topological properties of $S^3_K(m)$ have been studied intensively, which reveals much topological information of the knot $K$ itself. Theorem (1.1) ought to be useful to understand the geometry and topology of $S^3_K(m)$ and hence $K$ in the future.

2. Turn knot into simple 2-links

In this section, we will introduce some special operations on a knot diagram called skein-move. We will see that the skein-move along with plane isotopies and the Reidemeister moves can turn any knot diagram on a plane into the diagram of a simple 2-link.

First of all, for any knot $K \subset S^3$, we can use plane isotopy and Reidemeister moves to turn any diagram of $K$ into the form that all crossings in the diagram are on a short arc of $K$.

The idea is: starting from any diagram $D$ of $K$, we consider $D$ as the closure of a 1-tangle $T$. Then we label the crossings on $T$ according to their first appearance when we travel from the bottom end $A$ of $T$ to the top end $B$ of $T$, see figure [1] for example. Notice that we will meet each crossing of $T$ twice in the process, but when we meet a crossing for the second time, we will not relabel it or count it.

Next, extend the tangle horizontally via a line from $A$ to $A'$. See the figure [2] for example.

Denote the crossings by $z_1, \ldots, z_n$ according to their labels. Then, start from the crossing $z_1$, we can extend a small segment of the strand (overstrand or understrand) at $z_1$ down along the arc of $T$ that connects $z_1$ and the bottom end $A$, until it meet the line segment $AA'$. To be more precise, when we travel along the tangle starting from $A$ and meet the crossing $z_1$ at the first time, if we are standing on the understrand of $z_1$, we extend the overstrand of $z_1$ down via the process described. Otherwise, we extend the understrand of $z_1$ down (See the figure [2]). Obviously, this will reduce the number of crossings of the
Figure 1. Label the crossings of a tangle

Figure 2. tangle

tangle above the line segment $AA'$ by 1. Next, we do the same extension process to strands at $z_2, \ldots, z_n$ one by one according to their labeled order. When we finish this, all the crossings of the tangle will be moved to the segment $AA'$. Then connect $B$ and $A'$ via a simple arc far away from $T$, we get a diagram of $K$ in the required form. This form of knot diagram is called well-posed.
Figure 3. Use skein-move to turn a knot diagram into a simple 2-link

Remark: The Dowker notation (see [3]) of a well-posed knot diagram with $m$-crossings has the property that: in the two numbers associated to each crossing, one is $\leq m$, the other is $\geq m$.

Next, we orient the knot from $A'$ to $A$. The general picture of a well-posed knot diagram is like figure 3. Notice that, we can always use the skein move defined in figure 3 to turn a well-posed knot diagram into a two-component link $\mathcal{L}$. And it is easy to see that each component in $\mathcal{L}$ is a diagram of the unknot, i.e. the link $\mathcal{L}$ is a simple 2-link (see figure 4 for an example).

Conversely, given a diagram of simple 2-link $\mathcal{L}$, we can use Reidemeister moves and the skein move to turn it into a knot diagram.

Remark: The well-posed diagram for a knot $K$ is not unique, nor is the corresponding simple 2-link.
3. 3-MANIFOLDS FROM INTEGRAL SURGERY ON A KNOT

Suppose $K$ is a knot in $S^3$, let $N(K) \subset S^3$ be a small tubular neighborhood of $K$ and $E(K) := S^3 - N(K)$. Up to isotopy, $\partial E(K)$ has a canonical longitude $l$ which is homologically trivial in $S^3 - K$. And let $m$ be a meridian of $\partial E(K)$ which bounds a disk in $N(K)$. Then doing $(p, q)$-surgery on $K$ is first removing $N(K)$ from $S^3$ and then glue back a standard solid torus $S^1 \times D^2$ via a homeomorphism of $h : \partial D^2 \times S^1 \rightarrow \partial E(K)$ where $h$ maps the $\partial D^2 \times 0$ to a curve on $\partial E(K)$ which is isotopic to $p \cdot m + q \cdot l$ on $\partial E(K)$. The 3-manifold we get is denoted by $S^3_K(p, q)$. A $(p, q)$-surgery is called integral if $q = \pm 1$. Moreover, $S^3_K(p, q)$ is always an orientable 3-manifold.

Remark: We do not need to orient the knot $K$ in the surgery since the topological type of $S^3_K(p, q)$ depends only on the knot $K$.

Moreover, we can similarly define surgery on any link $L \subset S^3$. The surgery is called integral if the surgery on each component of $L$ is integral.

**Theorem 3.1** (Lickorish\textsuperscript{[4]} and Wallace\textsuperscript{[5]}). Every closed orientable 3-manifold can be obtained from $S^3$ by an integral surgery on a link in $S^3$. Moreover, each component of the link can be required to be an unknot in $S^3$.

Integral surgery on a link $L = L_1 \cup \cdots \cup L_m$ decides an integer $n_i$ for each component $L_i$ in $L$, which is called a framing of $L$. A link $L$ with a fixed framing will be called framed link. So we can also say that any closed orientable 3-manifolds can be got from a surgery on a framed link in $S^3$. 
Surgery on different framed links may give the same 3-manifold. Following are two elementary operations on a framed link $L$ called Kirby moves (see [6]) which do not change the corresponding 3-manifold.

**K1 Move**: Add or delete an unknotted circle with framing $\pm 1$ which belongs to a 3-ball that does not intersect the other components on $L$.

**K2 Move**: Slide one component $L_1$ onto another component $L_2$. Namely, let $L_2^*$ be a longitude of the tubular neighborhood of $L_2$ whose linking number with $L_2$ is the framing index $n_2$ of $L_2$. Now replace $L_1$ by $L_1' = L_1 \# b L_2^*$ where $b$ is any band connecting $L_1$ to $L_2^*$ and disjoint from the other components of $L$. The framing of $L_1'$ is $n_1 + n_2 + 2 \text{lk}(L_1, L_2)$ where $\text{lk}(L_1, L_2)$ is the linking number of $L_1$ and $L_2$ in $S^3$ with respect to some orientations of them. The rest of the framed link $L$ remains unchanged. To compute $\text{lk}(L_1, L_2)$, we orient $L_1$ and $L_2$ in such a way that together they define an orientation on $L_1'$. So different orientations of $L_1$ and $L_2$ may end up with different framed links (see [7]).

Moreover, it is shown in [6] that any two framed links which give the same 3-manifolds can always be transformed into each other via a finite number of Kirby moves. We can use this to show the following lemma.

**Lemma 3.2** (proposition 3.3 [7]). *If in a framed link $L$ a component $L_0$ is an unknot with framing zero which links only one other component $L_1$ geometrically once, then $L_0 \cup L_1$ may be moved away from the link $L$ without changing the resulting 3-manifold and framings of other components, and cancelled (See the following figure 7).*
From the proof of theorem 1.1 we can see the following:

1. The diagrams for $L_1, L_2$ have no self crossings and the geometric intersection number of $L_1$ or $L_2$ with a 2-disk bounded by
We can fix the framing on one of the $L_1, L_2$ to be 1 (or $-1$) in the simple 3-link.

(3) There are infinite different framings on a fixed simple 3-link that can give the same 3-manifold $S^3_K(m)$!

(4) We can require the linking number $lk(L_1, L_2) = 0$ in the simple 3-link by doing second Kirby moves to $L_1$ and $L_0$ in the figure.

Remark: Obviously, integral surgeries on simple 3-links will give lots of 3-manifolds other than $S^3_K(m)$. We can change the way how $L_0$ is linked to $L_1, L_2$ and the surgery index of $L_0$. So theorem may be useful for us to construct some interesting examples like integral homology 3-spheres other than $S^3_K(1)$.

Corollary 3.3. Suppose $M$ is constructed from integral surgery on a $n$-component link $\mathcal{L}$ in $S^3$, then $\nu(M) \leq 3n$.

Proof. Apply the argument in the proof of theorem to each component of $\mathcal{L}$. 

Obviously, if $\nu(M^3) = 1$, $M^3$ must be lens space. But it is not clear how to classify closed 3-manifolds $M^3$ with $\nu(M^3) = 2$. In particular, we can ask the following question.

Question 1: For what knot $K$ and integer $m$, $\nu(S^3_K(m)) \leq 2$?

There are some obvious candidates for the question. For example: if the unknotting number of $K$ is 1, $\nu(S^3_K(m)) \leq 2$ for any $m$. But it is not clear how to give a complete answer to this question. In particular, it is interesting to know whether $\nu(S^3_K(m)) \leq 2$ for all knot $K$ and $m \in \mathbb{Z}$.

Also, it is natural to consider $\nu(S^3_K(p, q))$ for $p/q \notin \mathbb{Z}$. For example when $K$ is the unknot, $S^3_K(p, q)$ is the lens space $L(p, q)$. Suppose the continued fraction decomposition of $p/q$ is $[x_1, \ldots, x_n]$, where

$$[x_1, \ldots, x_n] = x_1 - \frac{1}{x_2 - \frac{1}{\cdots - \frac{1}{x_n}}}$$
then $L(p, q)$ has a surgery presentation as shown in the figure 8. So $
u(L(p, q)) \leq n$. Notice that there are examples for $p/q = [x_1, \ldots, x_n]$ with $n > 3$ but $\nu(L(p, q)) \leq 3$. In fact, in [8], it is shown that $L(23, 7)$ could be obtained by $-23$-surgery on the $(11, 2)$-cable knot about the trefoil knot, so $\nu(L(23, 7)) \leq 3$ while $23/7 = [4, 2, 2, 3]$. More examples of getting lens space via integral surgeries on knots in $S^3$ can be found in [8, 9, 10, 11].

**Question 2:** Does there exist an integer $C$ such that $\nu(L(p, q)) \leq C$ for all $p, q \in \mathbb{Z}$?

**Remark:** [12] also gave a way of presenting $S^3_K(p, q)$ by an integral surgery on some link. But we will not get any universal bounds of $\nu(S^3_K(p, q))$ for all $K$ and $(p, q)$ via the method in [12].

Theorem [11] provides an interesting way to see $S^3_K(m)$ via surgery diagrams. From the proof of theorem [11] we can see that the topological information of $S^3_K(m)$ is completely encoded in how $L_1$ and $L_2$ are linked together and the surgery index $m$. Notice all the crossings in the diagrams of $L_1 \cup L_2$ are between $L_1$, $L_2$. So similar to Dowker notation for knots, we can use a sequence of numbers to represent $L_1 \cup L_2$. This could be interesting in its own sense.

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