A GENERALIZATION OF A POWER-CONJUGACY PROBLEM IN TORSION-FREE NEGATIVELY CURVED GROUPS

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Abstract. Let \( H \) and \( K \) be quasiconvex subgroups of a negatively curved torsion-free group \( G \). We give an algorithm which decides whether an element of \( H \) is conjugate in \( G \) to an element of \( K \).

1. Introduction

In 1911 Max Dehn introduced in [10] three basic algorithmic problems in group theory: the word problem, the conjugacy problem, and the isomorphism problem. Let a group \( G \) be given by a presentation \( G = \langle X | R \rangle \). The word problem asks if there exists an algorithm to decide if any word in the alphabet \( X \) represents the trivial element of \( G \). The word problem was shown to be undecidable, in general, by Novikov, [24], and independently, by Boone, [4]. The conjugacy problem asks if there exists an algorithm which for any pair of words in the alphabet \( X \) decides whether they represent conjugate elements in \( G \). A special case of the conjugacy problem, namely the existence of an algorithm deciding if a given word in the alphabet \( X \) represents an element of \( G \) conjugate to the identity of \( G \), is the word problem. Hence the conjugacy problem is also undecidable, in general. The isomorphism problem asks if for any pair of presentations there exists an algorithm to decide if they define isomorphic groups. The isomorphism problem was shown to be undecidable, in general, by Adian, [1], and independently by Rabin, [26]. The membership problem for a subgroup \( H \) of a group \( G \) asks if there exists an algorithm deciding if any element of \( G \) belongs to \( H \). As the word problem, in general, is undecidable, it follows that the membership problem is, in general, undecidable. The power-conjugacy problem for a group \( G \) asks if for any two elements of \( G \) there exists an algorithm to decide if one of them is conjugate to some power of the other. A special case of the power-conjugacy problem, namely the existence of an algorithm deciding if any element of \( G \) is conjugate to some power of the identity element in \( G \), is the word problem. Hence the power-conjugacy problem is undecidable, in general. For more detailed information about the aforementioned algorithmic problems see, for example, survey articles [14], [15], and [23].

Even though the aforementioned algorithmic problems are undecidable in general, they are decidable in negatively curved groups. The solution of the word problem in negatively curved groups follows from the work of Greendlinger, [16]. The solution of the conjugacy problem for negatively curved groups was given by Gromov in [17] p.199. The solution of the isomorphism problem for negatively curved
The solution of the membership problem for quasiconvex subgroups of negatively curved groups was given by Dahmani and Guirardel in [9]. The solution of the membership problem for quasiconvex subgroups of negatively curved groups was given by the author, [12], [13], and [14], and independently, by Farb, [11], I. Kapovich, [19], and Kharlampovich, Miasnikov, and Weil, [20]. The power-conjugacy problem was shown to be decidable when $G$ is negatively curved by Lysenok, [21].

The power-conjugacy problem has been the subject of extensive research and was solved for several additional classes of groups, see for example, [2], [3], [5], [6], [7], [8], [22], and [25]. In this paper we prove a generalized version of the power-conjugacy problem for torsion-free negatively curved groups.

**Theorem 1.** Let $H$ and $K$ be $\mu$-quasiconvex subgroups of a $\delta$-negatively curved torsion-free group $G$. There exists an algorithm to decide if an element of $H$ is conjugate in $G$ to an element of $K$.

**Corollary 1.** Let $K$ be a quasiconvex subgroup of a torsion-free negatively curved group $G$, and let $u$ be a non-trivial element of $G$. There exists an algorithm to decide whether some power of $u$ is conjugate in $G$ to an element of $K$.

**Proof.** As a cyclic subgroup in a negatively curved group is quasiconvex, [17] p. 210, we can apply Theorem 1 with $H$ being the cyclic subgroup generated by $u$. \hfill \Box

**Corollary 2.** Let $K$ be a quasiconvex subgroup of a torsion-free negatively curved group $G$ and let $u$ be a non-trivial element of $G$. There exists an algorithm to decide whether $u$ is conjugate in $G$ to an element of $K$.

**Proof.** Lemma 1, stated below, shows that if $H$ is the cyclic subgroup generated by $u$ and $u$ is conjugate to an element of $K$, then there exists $g \in G$ with $|g| < C$ such that $gug^{-1} \in K$, ($C$ is defined in the statement of Lemma 1). As $G$ is finitely generated, there are only finitely many elements shorter than $C$ in $G$. Hence we need to check if one of finitely many elements of the form $gug^{-1}$ with $|g| < C$ is in $K$, which we can do because the membership problem for $K$ in $G$ is decidable. \hfill \Box

**Corollary 3.** The power-conjugacy problem is decidable for torsion-free negatively curved groups.

**Proof.** Let $u$ be a non-trivial element of $G$ and let $v$ be any element of $G$. Corollary 2 implies that there is an algorithm to decide whether $u$ is conjugate in $G$ to an element of a cyclic group generated by $v$, which is the power-conjugacy problem. \hfill \Box

Theorem 1 follows from two technical results stated below.

**Lemma 1.** Let $H$ and $K$ be $\mu$-quasiconvex subgroups of a $\delta$-negatively curved torsion-free group $G$, and let $g \in G$ be a shortest representative of the double coset $KgH$ such that $gug^{-1}$ is in $K$ for some non-trivial element $h$ of $H$. Then $g$ is shorter than $C = 4\delta + 2\mu + (m^2 + 1) \cdot L$, where $L$ is the number of words in $G$ with length less than $8\delta + \mu$, and $m$ is the number of elements in $G$ with length not greater than $42\delta + 12\mu$.

**Lemma 2.** Let $H$ and $K$ be $\mu$-quasiconvex subgroups of a $\delta$-negatively curved torsion-free group $G$ and let $h$ be a shortest non-trivial element of $H$ such that $ggh^{-1}$ is in $K$ for some $g \in G$ with $|g| < C$. Then $h$ is shorter than $C' = (L' + 2)2\mu + 8\delta$, where $L'$ is the number of words in $G$ shorter than $(2\delta + 2\mu)$. 
Remark 1. Note that if there exist \( h \in H \) and \( g \in G \) such that \( ghg^{-1} \in K \), then for any \( h_0 \in H \) and \( k_0 \in K \), \( (k_0gh_0)(h_0^{-1}hh_0)(h_0^{-1}g^{-1}k_0^{-1}) \in K \). So if \( g \in G \) conjugates an element of \( H \) to an element of \( K \), then any \( g_0 \) in the double coset \( KgH \) has the same property.

Proof of Theorem 1.

Assume that there exists a non-trivial element \( h \in H \) and an element \( g \in G \) such that \( ghg^{-1} \in K \). Let \( g_1 \) be a shortest element in the double coset \( KgH \). Lemma 1 states that \( |g_1| < C \). Remark 1 implies that there exists an element \( h' \in H \) such that \( g_1h'g_1^{-1} \in K \). Let \( h_1 \in H \) be a shortest non-trivial element such that \( g_1h_1g_1^{-1} \in K \). Lemma 2 states that \( |h_1| < C' \). As \( G \) is finitely generated, there are finitely many possible \( g_1 \) and \( h_1 \). Hence we need to form finitely many products \( g_1h_1g_1^{-1} \) and to check if they belong to \( K \), which we can verify because the generalized word problem is solvable for quasiconvex subgroups of negatively curved groups.

2. Preliminaries

Let \( X \) be a set, let \( X^* = \{ x, x^{-1} | x \in X \} \), and for \( x \in X \) define \( (x^{-1})^{-1} = x \). A word in \( X \) is any finite sequence of elements of \( X^* \). Denote the set of all words in \( X \) by \( W(X) \), and denote the equality of two words by “\( \equiv \)”. Recall that the Cayley graph of \( G = \langle X|R \rangle \), denoted \( Cayley(G) \), is an oriented graph whose set of vertices is \( G \) and the set of edges is \( G \times X^* \), such that an edge \( (g, x) \) begins at the vertex \( g \) and ends at the vertex \( gx \). Since the Cayley graph depends on the generating set of the group, we work with a fixed generating set.

A geodesic in the Cayley graph is a shortest path joining two vertices. A geodesic triangle in the Cayley graph is a closed path \( p = p_1p_2p_3 \), where each \( p_i \) is a geodesic. A group \( G = \langle X|R \rangle \) is \( \delta \)-negatively curved if any side of any geodesic triangle in the Cayley graph of \( G = \langle X|R \rangle \) belongs to the \( \delta \)-neighborhood of the union of the other two sides.

A subgroup \( H \) of a group \( G = \langle X|R \rangle \) is \( \mu \)-quasiconvex in \( G = \langle X|R \rangle \) if any geodesic in the Cayley graph of \( G = \langle X|R \rangle \) with endpoints in \( H \) belongs to the \( \mu \)-neighborhood of \( H \). A subgroup is quasiconvex in \( G = \langle X|R \rangle \) if it is \( \mu \)-quasiconvex in \( G = \langle X|R \rangle \) for some \( \mu \). As usual, we assume that all negatively curved groups are finitely generated.

The label of a path \( p = (g, x_1)(g \cdot x_1, x_2) \cdots (g \cdot x_1 \cdots x_{n-1}, x_n) \) in \( Cayley(G) \) is the function \( Lab(p) \equiv x_1x_2 \cdots x_n \in W(X) \). As usual, we identify the word \( Lab(p) \) with the corresponding element in \( G \).

The following result has been proven in [15].

Theorem GMRS

Let \( H \) be a \( \mu \)-quasiconvex subgroup of a \( \delta \)-negatively curved torsion-free group \( G \), and let \( m \) be the number of elements in \( G \) with length not greater than \( 42\delta + 12\mu \). Let \( S = \{ q_i^{-1}Hq_i | 1 \leq i \leq n \} \) be a collection of essentially distinct conjugates of \( H \), where the conjugates \( q_i^{-1}Hq_i \) and \( q_j^{-1}Hq_j \) are called essentially distinct if \( Hq_i \neq Hq_j \) for \( i \neq j \). If \( n > m^2 \), then the intersection of some pair of elements of \( S \) is trivial.
3. PROOFS OF THE RESULTS

Let $g$ be a shortest element in the double coset $KgH$ such that $ghg^{-1} = k$ is in $K$ for some non-trivial element $h$ of $H$.

Let $p, p_k$ and $p'$ be geodesics in $Cayley(G)$ such that $\text{Lab}(p) \equiv \text{Lab}(p') \equiv g$, $\text{Lab}(p_k) = h$, $p$ begins at 1 and ends at $g$, $p'$ begins at $gh^{-1}$ and ends at $gh$, and $p_k$ begins at $g$ which is the endpoint of $p$ and ends at $gh$ which is the endpoint of $p'$.

Denote the vertices of $p$ in their linear order by $1 = v_0, v_1, \ldots, v_n = g$ and denote the vertices of $p'$ in their linear order by $gh^{-1} = v_0', v_1', \ldots, v_n' = gh$. Note that $|g| = |p| = |p'| = n$.

Let $p_k$ be a geodesic in $Cayley(G)$ joining $v_0 = 1$ and $v_0' = ghg^{-1}$. Then the paths $p, p_k, p'$ and $p_k$ form a geodesic 4-gon which is $2\delta$-thin in $Cayley(G)$, because $G$ is $\delta$-negatively curved.

**Lemma 3.** For any index $i$ such that $2\delta + \mu \leq i \leq n - 2\delta - \mu$ the distance $d(v_i, v_i')$ is less than $8\delta + \mu$.

**Proof.** Let $l$ be the biggest index such that the vertex $v_i$ belongs to the $2\delta$-neighborhood of $p_k$, let $w_l$ be a vertex in $p_k$ closest to $v_l$, and let $r$ be a geodesic joining $w_l$ to $v_i$. By construction, $\text{Lab}(p_k) = k \in K$. As $K$ is $\mu$-quasiconvex, $p_k$ belongs to the $\mu$-neighborhood of $K$ in $Cayley(G)$, hence there exists a vertex $u_l \in K$ such that $d(w_l, u_l) < \mu$. Let $r'$ be a geodesic joining $u_l$ to $w_l$. Let $s_l$ be a subpath of $p$ joining $v_0$ to $v_l$, let $\bar{s}_l$ be the inverse of the path $s_l$, and let $t_l$ be a subpath of $p$ joining $v_l$ to $v_n$. Note that $\text{Lab}(r't_l) = \text{Lab}(r's_l) = \text{Lab}(r'\bar{s}_l)g \in Kg$. As $g$ is a shortest representative of $KgH$, it follows that $|g| = |p| = |s_l| + |t_l| \leq |r't_l| < 2\delta + \mu + |t_l|$, so $|s_l| = d(v_0, v_l) = l < 2\delta + \mu$. Hence if $i \geq \mu + 2\delta$, then $d(v_i, p_k) > 2\delta$

Let $i$ be the smallest index such that the vertex $v_i$ belongs to the $2\delta$-neighborhood of $p_k$. An argument, similar to the above, shows that for any $j \leq n - 2\delta - \mu$, $d(v_j, p_k) > 2\delta$.

Therefore, for any index $i$ such that $2\delta + \mu \leq i \leq n - 2\delta - \mu$, the vertex $v_i$ belongs to the $2\delta$-neighborhood of $p'$. Similarly, for any index $i$ such that $2\delta + \mu \leq i \leq n - 2\delta - \mu$ the vertex $v_i'$ belongs to the $2\delta$-neighborhood of $p$.

Let $b = n - 2\delta - \mu$. We claim that $d(v_b, v'_b) < 4\delta + \mu$. Indeed, let $j(b) \leq b$ be an index such that $d(v_b, v'_j) < 2\delta$. Let $t_b$ be the subpath of $p$ joining $v_b$ and $v_b$, let $t'_j(b)$ be the subpath of $p'$ joining $v'_j(b)$ to $v'_n$, and let $\gamma$ be a geodesic joining $v_b$ and $v'_j(b)$. Consider the geodesic 4-gon formed by $t_b, p_h, t'_j(b)$ and $\gamma$.

As $b \leq n - 2\delta - \mu$, it follows that $d(v'_i, p_h) > 2\delta$. If $d(v'_i, \gamma) < 2\delta$, then $d(v_b, v'_i) \leq |\gamma| + d(v'_i, \gamma) < 4\delta$. If $d(v'_i, t_b) < 2\delta$, then $d(v_b, v'_i) \leq |t_b| + d(v'_i, t_b) < 4\delta + \mu$.

Now consider $2\delta + \mu \leq i \leq n - 2\delta - \mu$. Let $j(i)$ be an index such that $d(v_i, v'_j(i)) < 2\delta$. By interchanging $v_i$ and $v'_j(i)$, if needed, we can assume that $j(i) \geq i$. As $p$ is a geodesic, $d(v_i, v_b) = b - i \leq d(v_i, v'_j(i)) + d(v'_j(i), v'_b) + d(v_b, v'_b) < 2\delta + (b - j(i)) + 4\delta + \mu$, hence $0 \leq j(i) - i < 6\delta + \mu$. But then $d(v_i, v'_i) \leq d(v_i, v'_j(i)) + d(v'_j(i), v'_i) < 2\delta + (j(i) - i) < 8\delta + \mu$, proving Lemma 3.

**Proof of Lemma 1.**

Assume that $|g| = n \geq C$, where $C$ is defined in the statement of Lemma 1. It follows that $(n - 2\delta - \mu) - (2\delta + \mu) \geq C - 4\delta - 2\mu = L \cdot (m^2 + 1)$. Therefore Lemma 3 implies that there exists a set of distinct indexes $\{i_j | 1 \leq j \leq m^2 + 1\}$ such that:

\begin{align*}
(1) &\ n - 2\delta - \mu \geq i_j \geq 2\delta + \mu,
\end{align*}
(2) the paths connecting \( v_{i_j} \) to \( v'_{i_j} \) have the same label, say \( a \), for all \( i_j \).

Let \( s_{i_j} \) be the initial subpath of \( p \) connecting \( v_0 \) and \( v_{i_j} \) and let \( s'_{i_j} \) be the initial subpath of \( p' \) connecting \( v'_0 \) and \( v'_{i_j} \). If \( a = 1 \), then \( v_{i_j} = v'_{i_j} \). It follows that 
\[
\text{Lab}(s_{i_j})^{-1}\text{Lab}(s'_{i_j}) = 1,
\]
\( k = h = 1 \), contradicting the choice of \( h \).

If \( a \neq 1 \), consider the set 
\[
S = \{ \text{Lab}(s_{i_j})k\text{Lab}(s_{i_j}), 1 \leq i_j \leq m^2 + 1 \}.
\]
As 
\[
\text{Lab}(s_{i_j})^{-1}\text{Lab}(s_{i_j}) = a \neq 1 \quad \text{for all} \quad 1 \leq i_j \leq m^2 + 1,
\]
and as \( G \) is torsion-free, it follows that the intersection of any pair of elements of \( S \) is infinite.

However, the elements of \( S \) are essentially distinct. Indeed, assume that there exists \( k_0 \in K \) such that 
\[
k_0\text{Lab}(s_{i_j}) = \text{Lab}(s_{i_j}).
\]
Without loss of generality, \( i_i > i_j \). Let \( t_{i_j} \) be the subpath of \( p \) joining \( v_{i_j} \) to \( v_n \).
Then 
\[
g = \text{Lab}(s_{i_j})\text{Lab}(t_{i_j}) = k_0\text{Lab}(s_{i_j})\text{Lab}(t_{i_j}).
\]
Hence the element \( \text{Lab}(s_{i_j})\text{Lab}(t_{i_j}) \) belongs to \( Kg \) and 
\[
|\text{Lab}(s_{i_j})\text{Lab}(t_{i_j})| \leq |s_{i_j}| + |t_{i_j}| < |s_{i_j}| + |t_{i_j}| = |g|,
\]
contradicting the choice of \( g \) as a shortest representative of the double coset \( KgH \). So \( S \) is a collection of \( m^2 + 1 \) distinct conjugates of \( K \) such that any two elements of \( S \) have infinite intersection, contradicting Theorem GMRS.

Hence \( |g| < C \), proving Lemma 1.

**Remark 2.** By increasing the quasiconvexity constant \( \mu \) if needed, we can assume that \( \mu \) is a positive integer.

**Lemma 4.** Let \( g \) be an element shorter than \( 4\delta + 2\mu \) such that \( ghg^{-1} \in K \) for a non-trivial \( h \in H \). If \( h \) is longer than \( (L' + 2)2\mu + 8\delta \), where \( L' \) is the number of words in \( G \) shorter than \( 2\delta + 2\mu \), then there exist a non-trivial \( h_0 \in H \) with 
\[
|h_0| \leq 2\mu(L' + 2) \quad \text{and} \quad g_0 \in G \quad \text{with} \quad |g_0| < 2\delta + 2\mu \quad \text{such that} \quad g_0h_0g_0^{-1} \in K.
\]

**Proof.** Let \( p, p_k, p' \) and \( p_h \) be a geodesic 4-gon, as in the proof of Lemma 3. Denote the vertices of \( p_h \) in their linear order by \( g = v_0, v_1, \ldots, v_f = gh \).

Let \( q' \) be the maximal initial subpath of \( p_h \) which belongs to the \( 2\delta \)-neighborhood of \( p \). Note that the length of \( q' \) is at most \( 4\delta + \mu \). Indeed, let \( v_q' \) be the terminal vertex of \( q' \). Let \( v_q' \) be a vertex of \( p \) such that \( d(v_q', q') \leq 2\delta \). Let \( \alpha \) be a geodesic in \( Cayley(G) \) which begins at \( v_q' \) and ends at \( v_q' \).

Let \( s_q \) be the initial subpath of \( p \) joining \( v_0 \) to \( v_q' \) and let \( t_q' \) be the terminal subpath of \( p \) joining \( v_q' \) to \( v_n = g \).
As \( H \) is \( \mu \)-quasiconvex in \( G \), there exists a vertex \( x_q' \) in \( Cayley(G) \) and a geodesic \( \alpha' \) joining \( v_q' \) to \( x_q' \) such that \( \text{Lab}(q_0\alpha') \in H \) and \( |\alpha'| < \mu \).

As \( g \) is a shortest element in the double coset \( KgH \), it follows that 
\[
|g| = |s_q' + t_q'| \leq |s_q'| + |\alpha| + |\alpha'| \leq |s_q'| + 2\delta + \mu.
\]
Hence \( |t_q'| \leq 2\delta + \mu \). It follows that 
\[
|q'| \leq |t_q'| + |\alpha| \leq 4 \delta + \mu.
\]

Similarly, the length of the maximal subpath of \( p_h \) which belongs to the \( 2\delta \)-neighborhood of \( p' \) is at most \( 4 \delta + \mu \).

Assume that \( h \) is longer than \( (L' + 2)2\mu + 8\delta \). Then there exists a subpath \( q \) of \( p_h \) of length at least \( (L' + 1)2\mu \) which belongs to the \( 2\delta \)-neighborhood of \( p_h \).

By construction, \( q \) begins at the vertex \( v_q' \). By definition of the path \( q \), for any vertex \( v_q' \) of \( q \) there exists a vertex \( w(v_q') \) in \( p_k \) such that \( d(v_q', w(v_q')) < 2\delta \). As \( H \) is \( \mu \)-quasiconvex in \( G \), for any vertex \( v_q' \) of \( q \) there exists a vertex \( x_i \) such that \( d(v_q', x_i) < \mu \), and the element \( x_i \) belongs to the coset \( gH \). Similarly, there exists a vertex \( k(v_q') \) such that \( d(w(v_q'), k(v_q')) < \mu \) and the element \( k(v_q') \) belongs to \( K \).

Let \( \beta_i \) be a geodesic joining \( k(v_q') \) and \( x_i \). Then \( |\beta_i| < 2\mu + 2\delta \).

Consider a subset of vertices of \( q \) with indexes \( v_{q'+2\mu}, v_{q'+2\mu+2\mu}, \ldots, v_{q'+2\mu+L'2\mu} \). The distance between two consecutive vertices in this subset is \( 2\mu \), hence \( x_{q' + i} \neq x_{q' + j} \) for \( i \neq j \).
By definition of the constant $L'$, there exist indexes $i \neq j$ such that $\text{Lab}(\beta_{q'+i:2\mu}) = \text{Lab}(\beta_{q+j:2\mu})$. By construction, $d(v_{q'+i:2\mu}, v_{q'+j:2\mu}) \leq 2\mu(L' + 1)$, so $d(x_{q'+i:2\mu}, x_{q'+j:2\mu}) < 2\mu + 2\mu(L' + 1) = 2\mu(L' + 2)$.

By construction, if $\nu$ is a geodesic joining $x_{q'+i:2\mu}$ and $x_{q'+j:2\mu}$, then $\text{Lab}(\nu) \in H$. Similarly, if $\nu'$ is a geodesic joining $k(v_{q'+i:2\mu})$ and $k(v_{q'+j:2\mu})$, then $\text{Lab}(\nu') \in K$. So take $g_0 = \text{Lab}(\beta(v_{q'+i:2\mu})$ and $h_0 = \text{Lab}(\nu)$, proving Lemma 4.

**Proof of Lemma 2.**

Let $h$ be a non-trivial element of $H$ such that $ghg^{-1} \in K$ for some $g \in G$ with $|g| < C$, where $C$ is defined in the statement of Lemma 1. We want to find $h_0 \in H$ with $|h_0| < C'$, where $C'$ is defined in the statement of Lemma 2, and $g_0 \in G$, which might be different from $g$, with $|g_0| < C$ such that $g_0h_0g_0^{-1} \in K$.

Consider three cases.

1. If $|g| < 4\delta + 2\mu$ and $|h| \leq (L' + 2)2\mu + 8\delta$, take $g_0 = g$ and $h_0 = h$.
2. If $|g| < 4\delta + 2\mu$ and $|h| > (L' + 2)2\mu + 8\delta$, then Lemma 4 states that there exist a non-trivial $h_0 \in H$ with $|h_0| \leq 2\mu(L' + 2)$ and $g_0 \in G$ with $|g_0| < 2\delta + 2\mu$ such that $g_0h_0g_0^{-1} \in K$.
3. If $C > |g| \geq 4\delta + 2\mu$, let $p, p', v_b, v'_b$ and $p_h$ be as in the proof of Lemma 3. It is shown in Lemma 3 that $d(v_b, v'_b) < 4\delta + \mu$. Then $|h| = |p_h| \leq d(v_b, v_h) + d(v_h, v'_b) + d(v'_b, p_h) < (\mu + 2\delta) + (\mu + 2\delta) + (\mu + 2\delta) < (3\mu + 8\delta)$. Hence we can take $g_0 = g$ and $h_0 = h$, proving Lemma 2.

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