A GEOMETRY FOR SECOND-ORDER PDES AND THEIR INTEGRABILITY,
PART I

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Abstract. For the purpose of understanding second-order scalar PDEs and their hydrodynamic integrability, we introduce G-structures that are induced on hypersurfaces of the space of symmetric matrices (interpreted as the fiber of second-order jet space) and are defined by non-degenerate scalar second-order-only (Hessian) PDEs in any number of variables. The fiber group is a conformal orthogonal group that acts on the space of independent variables, and it is a subgroup of the conformal orthogonal group for a semi-Riemannian metric that exists on the PDE. These G-structures are automatically compatible with the definition of hydrodynamic integrability, so they allow contact-invariant analysis of integrability via moving frames and the Cartan–Kähler theorem. They directly generalize the GL(2)-structures that arise in the case of Hessian hyperbolic equations in three variables as well as several related geometries that appear in the literature on hydrodynamic integrability. Part I primarily discusses the motivation, the definition, and the solution to the equivalence problem, and Part II will discuss integrability in detail.

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INTRODUCTION

The primary motivation for this article and its sequel is a geometric classification of second-order scalar partial differential equations [PDEs] in any number of variables,

$$F \left( x^1, \ldots, x^n, z, \frac{\partial z}{\partial x^1}, \ldots, \frac{\partial z}{\partial x^n}, \frac{\partial^2 z}{\partial x^1 x^1}, \ldots, \frac{\partial^2 z}{\partial x^n x^n} \right) = 0.$$ 

Such a classification should describe the intrinsic structure of the PDEs, meaning that it should be invariant under any contact transformation of the PDE. Such a classification is especially interesting if it also highlights the integrable PDEs as those with certain explicit conditions on their defining invariants.

A classification of all scalar second-order PDEs under the entire pseudo-group of contact transformations is infinitely far beyond the scope of this article. Instead, this article focuses on the special case of Hessian PDEs, which are those of the form

$$F \left( \frac{\partial^2 z}{\partial x^1 x^1}, \ldots, \frac{\partial^2 z}{\partial x^n x^n} \right) = 0.$$ 

A Hessian PDE may be interpreted as a hypersurface $F^{-1}(0)$ in $\text{Sym}^2(\mathbb{R}^n)$. The class of Hessian PDEs is important for two reasons. First, this class of PDEs contains many interesting examples, such as the wave equation, the first flow of the dispersionless Kadomtsev–Petviashvili hierarchy, and the symplectic Monge–Ampère equations. Second, a recent theorem of Dennis The shows that any classification of Hessian PDEs with non-degenerate symbol up to the standard action of the conformal symplectic group yields a (somewhat coarse) classification of all second-order PDEs with non-degenerate symbol up to contact transformation [The10, Section 2.3]. This is because the contact transformations restrict on each fiber of second-order jet space to give the conformal symplectic transformations, and Hessian PDEs may be considered as the intersection of a general second-order PDE with any particular fiber of second-order jet space.

For functions $z$ over the real numbers, hyperbolic PDEs—those with leading symbol of signature $(n - 1, 1)$—are particularly interesting, since these have the least-degenerate characteristics and are thus most relevant to any reasonable notion of integrability. Aside from occasionally superfluous scaling factors, most results here also apply to the complex case with arbitrary non-degenerate symbol.

This article fits within a constellation of recent results on hyperbolic Hessian PDEs. In the case $n = 2$, a highly detailed classification of hyperbolic Hessian PDEs is given by Dennis The using the extrinsic geometry of surfaces embedded in $\text{Sym}^2(\mathbb{R}^2) \cong \mathbb{R}^3$ [The08, The10]. In the case $n = 3$, an intrinsic classification of integrable hyperbolic Hessian PDEs is given by the leaves of a singular foliation of $\mathbb{R}^9$, as found by the present author using intrinsic $GL(2)$ geometry [Smi10]. The project [Smi10] was inspired by earlier extrinsic work of Ferapontov et al. that links the Veronese cone to the notion of hydrodynamic integrability [FHK09]. In the case $n = 4$, Doubrov and Ferapontov classify the symplectic Monge–Ampère equations, which form an important subclass of integrable hyperbolic Hessian PDEs [DF09]. Another related study is [AABMP10], which uses the Veronese cone to investigate the Cauchy problem for a class of hyperbolic Hessian PDEs introduced by Goursat. Also, in the case of arbitrary $n$, quasilinear second-order PDEs (not Hessian) are analyzed in [BFT08] using an extrinsic $SL(n + 1)$ geometry that is related to the geometry seen here for Hessian PDEs by the sort of fiber-wise coarse classification mentioned above. In this project, the geometry of Hessian PDEs with non-degenerate symbol is considered for general $n$ in both the integrable and non-integrable cases. Thus, this project offers a unified view of the geometries that provide these recent results.
In Part I, we introduce a particular $G$-structure (see Definition 2.1) that is induced on non-degenerate Hessian PDEs, interpreted as hypersurfaces of $\text{Sym}^2(\mathbb{R}^n)$. These $G$-structures admit very simple global structure equations that have only finitely many local invariants (see Theorem 2.11). These structure equations can be readily computed for any $n$, but attention here is restricted to the cases $n = 2$, $n = 3$, and $n = 4$, which are most relevant for physical examples. This geometry is not strictly-speaking “new,” as it has been encountered in various guises and special cases in each of the references given above, but no previous work takes complete advantage of this geometry as a concept that is well-defined on any non-degenerate Hessian PDE in any number of variables and admits invariant analysis using the method of equivalence. The main theorem is a complete set of structure equations that indicate how to classify all such PDEs. In implementing the method of equivalence here, we attempt to keep the representation theory as elementary as possible, focusing only on a rough decomposition of vector spaces into submodules under the action of specific orthogonal groups. This approach has the advantage of keeping the structure equations in a form that can be easily entered into computer algebra systems without any understanding of spinor representations or Clebsch–Gordan decompositions. However, finer detail would result from application of such knowledge for a specific group $SO(n - k, k)$. In the sequel\(^1\), Part II, the relationship between these local invariants and the property of hydrodynamic integrability will be explored more deeply.

I wish to express my thanks to Niky Kamran, for his encouragement of my pursuit of this geometry, to Dennis The for many stimulating discussions of PDE and jet-space geometry, and to Francis Valiquette for expanding my perspective on the equivalence problem.

### Use of Indices

Because the geometry in this article uses a representation of $SO(n)$ other than the standard representation, the conventional up/down index summation notation is not used, as it would lead to confusion. Sums are indicated explicitly with $\sum$. Whether an index is up or down for a particular object depends only on aesthetics and convenience, and the reader must keep track of whether a particular object is co- or contra-variant. Moreover, throughout this article, juxtaposition without indices always means matrix multiplication in $\text{gl}(n)$. Thus, if $a$ and $b$ are differential forms valued in $\text{gl}(n)$, then $a \wedge b$ indicates the $\text{gl}(n)$-valued differential form with $(i, j)$ entry $(a \wedge b)_{ij} = \sum_k a_{ik} \wedge b_{kj}$. In particular, juxtaposition is never used to indicate composition of functions; instead, the notations $f \circ g$ and $f(g)$ are used as appropriate. The tensor, symmetric, and skew products are denoted by $\otimes$, $\odot$, and $\wedge$, respectively. The symbol $\nabla(x)$ always indicates the covariant derivative of $x$ with respect to a connection $\theta$ that acts on $x$ through a representation $\rho$, so $\nabla(x) = dx + \rho_\theta(x)$. Finally, $V \otimes W^*$ is identified with $\text{Hom}(W, V)$.

### 1. Background and Definitions

This section discusses the conformal symplectic group over $\text{Sym}^2(\mathbb{R}^n)$, which is the ambient structure that induces the intrinsic geometry on a Hessian PDE.

#### 1.1. The Conformal Symplectic Group

Let $V = \mathbb{R}^n$, considered as row vectors. Let $V \odot V = \text{Sym}^2(V)$ be identified with the space of $n \times n$ symmetric matrices. Consider the principal bundle $\mathcal{F}$ of $(V \odot V)$-valued co-frames over the manifold $\Lambda^o = \text{Sym}^2(V)$. Obviously, $\Lambda^o$ and $V \odot V$ are identical, but here $\Lambda^o$ refers to the set of symmetric matrices regarded as a manifold, which can be seen as a simply connected coordinate chart in the Lagrangian Grassmannian $\Lambda = LG(n, 2n)$, and $V \odot V$ refers to the set of symmetric matrices regarded as a vector space. The fiber over $U \in \Lambda^o$ is

$$\mathcal{F}_U = \left\{ a : T_U \Lambda^o \cong V \odot V \right\} \cong GL(n(n + 1)/2).$$

\(^1\)Part I and Part II will be separate as preprints on the arXiv, but it is likely that they will be unified for journal submission.
There is a distinguished reduction of this coframe bundle to a $GL(n)$-bundle, and after prolongation, this distinguished reduction has a total space isomorphic to the conformal symplectic group. There are three ways to see this structure, and all of them are important to understand.

First, the projective space $\mathbb{P}(V \odot V)$ contains a non-degenerate sub-variety, the Veronese variety, which is defined as the projective variety corresponding to the image of the Veronese map $\text{ver}_2 : V \to V \odot V$ by $\text{ver}_2(v) = v^T v$. In particular, the Veronese cone $\text{ver}_2(V)$, is the de-projectivized variety comprised of rank-one symmetric matrices, and the cone is in birational correspondence with $V$. The action on $\mathbb{P}(V \odot V)$ by symmetries of $\text{ver}_2(\mathbb{P}V)$ is a representation of $\text{PGL}(n)$ given by $[A] \mapsto [g^T A g]$ for any representative $g \in \text{PGL}(n) = GL(n)/(\mathbb{R}I)$. Here, this representation is called $\text{POd}(\text{PGL}(n))$. For the Veronese cone in the affine space $V \odot V$ over $\mathbb{R}$, an accurate description of the symmetry group $\text{Od}(\text{GL}(n))$ is a little more complicated; see Appendix \[A\]. One may consider an $\text{Od}(\text{GL}(n))$ reduction of $\mathcal{F}$. The reduced bundle, $\mathcal{F}_{\text{Od}(\text{GL}(n))}$, has a tautological semi-basic 1-form $\alpha = (\alpha_{ij}) = \alpha^\top$ that takes values in $V \odot V$, and the flat choice of section for the frame bundle, $a_{ij} = dU_{ij}$, defines a pseudo-connection $\beta = (\beta_{ij})$ valued in $\mathfrak{gl}(n)$. A study of the equivalence problem for this bundle (see [SS65]) shows that after one prolongation, the total space of the bundle has the structure equations of $\mathfrak{sp}(n)$, the symplectic algebra on $\mathbb{R}^{2n}$:

$$\mu = \begin{pmatrix} \beta & \gamma \\ \alpha & -\beta^\top \end{pmatrix}, \text{ for } \alpha = \alpha^\top, \gamma = \gamma^\top, \beta \in \mathfrak{gl}(n)$$

(1) $$0 = d\mu + \mu \wedge \mu = \begin{cases} d\alpha_{ij} + \sum_k \alpha_{ik} \wedge \beta_{kj} - \beta_{ki} \wedge \alpha_{kj}, \\ d\beta_{ij} + \sum_k \beta_{ik} \wedge \beta_{kj} + \gamma_{ik} \wedge \alpha_{kj}, \\ d\gamma_{ij} + \sum_k \beta_{ik} \wedge \gamma_{kj} - \gamma_{ik} \wedge \beta_{kj}. \end{cases}$$

Thus, the total space of the prolonged, flat $\text{Od}(\text{GL}(n))$-structure is the unique simply-connected Lie group that is the open set near the identity of $\text{Sp}(n)$. Over the reals, when the $\pm 1$ scaling is allowed, the group completes to become the conformal symplectic group, $\text{CSp}(n) \subset \text{GL}(2n, \mathbb{R})$,

$$\text{CSp}(n) = \left\{ \begin{pmatrix} B & C \\ A & D \end{pmatrix}, A^\top B - B^\top A = D^\top C - C^\top D = 0, D^\top B - C^\top A = cI_n \neq 0 \right\}. $$

(2) Second, consider the action on the space of symmetric bilinear forms, $V^* \odot V^*$, that is induced by the standard representation of $\text{GL}(n)$ on $V$. That is, consider how the coefficients $\Phi_{ij}$ in the equation $\sum_i \Phi_{ij} v_i w_j = 0$ vary when the coordinates $(v_1, \ldots, v_n)$ are transformed by $v \mapsto v g$ on $V$. This action is $\text{Od}^*(\text{GL}(n))$, given by $\Phi \mapsto \lambda g^{-1} \Phi g^{-1} \cdot \cdot \cdot$. Now, any such $\Phi \in V^* \odot V^*$ may be considered as an element of $(V \odot V)^*$, so it defines a hyperplane $\Phi^\perp = \{ a \in V \odot V : \text{tr}(\Phi a) = 0 \} \subset V \odot V$, which is unique up to scale. By the law of inertia, the action $\text{Od}_g$ is transitive on symmetric matrices of the same signature. Thus, $\text{Od}(\text{GL}(n))$ may also be characterized as the group that acts transitively on the space of non-degenerate hyperplanes in $V \odot V$ (preserving signature over the reals), so $\mathcal{F}_{\text{Od}(\text{GL}(n))}$ is also the bundle of frame changes that act transitively on the corresponding codimension-one distributions over $\Lambda^\circ$. This is essentially the perspective taken by Cartan in case "$\alpha$" of Theorems XIX and XX of [Car09] and subsequently clarified in [SS65, GQS66], and [Shm70].

Finally, here is the tautological description of jet space that appears in [Yam83]. In this description, zeroth-order jet space is $J^0$, a trivial bundle over $\mathbb{R}^n$ with fiber $\mathbb{R}$. The first-order jet space is $J^1 = \text{Gr}_n(T^* J^0) = T^* J^0$, which admits a tautological 1-form, $\gamma_E = E^\perp \circ \pi_0^1$, where $E^\perp$ is the annihilator of $E \subset \text{Gr}(T^* J^0)$ and $\pi_0^1$ is the projection from $J^1$ to $J^0$. The Pfaff theorem holds on the differential system generated by $\gamma$, so the space of maximal integral elements $\Lambda_{\text{max}}(\gamma) \subset \text{Gr}_n(T^* J^1)$ is a smooth bundle over $J^1$. The total space of this bundle is $J^2$, and each fiber of the projection $\pi_0^2 : J^2 \to J^1$ is isomorphic to the Lagrangian Grassmannian $\Lambda = \{ E \in \text{Gr}_n(2n) : \sigma|_E = 0 \}$ that is associated to the standard symplectic form $\sigma$ on $\mathbb{R}^{2n}$. Given local coordinates $(x^i, z, p_i, U_{ij})$ with
\[ U_{ij} = U_{ji} \] on \( J^2 \) and an independence condition \( dx_1 \wedge \cdots \wedge dx^n \neq 0 \), the canonical system generated by \( \Upsilon \) may be written as

\[
\Upsilon = dz - \sum_i p_i dx^i, \quad \text{and} \quad d\Upsilon = \sum_i dp_i \wedge dx^i, \quad \text{so} \quad 0 = dp_i - \sum_j U_{ij} dx^j.
\]

(3)

A contact transformation is an isomorphism of the bundle \( J^2 \) that preserves the canonical system generated by \( \Upsilon \) up to scale. The contact transformation on \( J^2 \) restricts to the fiber as an action of the conformal symplectic group on \( LG(n, 2n) \). In local coordinates \( (U_{ij}) \) for the fiber, this appears as the \( Od(GL(n)) \) action on the matrix \( U \in \Lambda^o \).

Thus, the conformal symplectic group, which is the prolongation of the flat \( Od(GL(n)) \) frame bundle on \( \Lambda^o \), is the appropriate setting to study the properties of Hessian PDEs that are invariant under the family of contact transformations that preserve a specific fiber of \( J^2 \). For more detail regarding this fiber-wise action and the associated notion of constant symplectic invariant, consult Section 2.3 of \([\text{The10}]\).

1.2. Hypersurfaces and Hyperbolicity. Suppose that \( F^{-1}(0) \) is a hypersurface in \( \Lambda^o \), and only consider the hypersurface near points where \( dF \neq 0 \) so that the implicit function theorem applies. Let \( a = (a_{ij}) \) be a \( (V \otimes V) \)-valued moving frame on \( \Lambda^o \) that is \( Od(GL(n)) \)-equivalent to the flat section, \( dU_{ij} \), so \( a \) is a section of \( F_{Od(GL(n))} \). Then \( dF_U = \sum_{ij} \Phi_{ij}(U)a_{ij}(U) \) for some \( \Phi : \Lambda^o \rightarrow (V \otimes V)^* = V^* \otimes V^* \). When \( \Phi \) is interpreted as a symmetric bilinear form on \( V \), namely \( (v, w) \mapsto v\Phi w^T \), it is precisely the leading symbol of the PDE \( F \) as written in the flat coordinates determined by the co-frame \( a \). Under a coordinate change \( g : v \mapsto vg \) for \( g \in GL(n) \), the symmetric bilinear form changes as \( \Phi \mapsto g^{-1}\Phi g^T \). This corresponds to the \( Od_g \) action on the co-frame \( a \).

At each point \( U \in F^{-1}(0) \), the intersection \( a(\ker dF|_U) \cap \text{ver}(V) \) gives the equation of a quadric in \( \mathbb{P}V \), \( 0 = \sum_{ij} \Phi_{ij}(U)v_iv_j \). This quadric is non-degenerate if and only if the matrix \( \Phi(U) \) is non-singular. In the real case, the most interesting case is the maximal intersection, which occurs when \( \Phi(U) \) has signature \((n-1, 1)\). In this case, the hypersurface is called hyperbolic at \( U \). Given the discussion \( J^2 \) in the previous section, this matches the traditional notion of hyperbolicity for PDEs. Because the signature of \( \Phi(U) \) is preserved by the \( Od_g(GL(n)) \) action, the signature of non-degenerate \( \Phi(U) \) is an example of a “constant symplectic invariant” in the terminology of \([\text{The10}]\), so this is an easy way to see that hyperbolicity (or any other non-degenerate signature) is a contact-invariant property of a real second-order PDE near generic points in jet space where the highest-order terms of \( F \) have maximal rank.

Now, consider the subgroup of \( Od(GL(n)) \) that preserves \( \ker dF|_U \). This subgroup must preserve the bilinear form \( \Phi(U) \) up to scale, so it contains \( O(n, \Phi(U)) = \{ g \in GL(n) : g\Phi(U)g^T = \Phi(U) \} \). The representation theory of this group is central to the main result, so it is explored in the next section before proceeding to the local geometry.

1.3. Infinitesimal Geometry. Consider \( V = \mathbb{R}^n \) as the vector space of row vectors. Fix a non-degenerate symmetric bilinear form \( \Phi \) on \( V \). Denote the \( \Phi \)-null cone in \( V \) by \( N = \{ v \in V : v\Phi v^T = 0 \} \). Let \( O(n, \Phi) = \{ g \in GL(n) : g\Phi g^T = \Phi \} \) which has Lie algebra \( \mathfrak{so}(n, \Phi) = \{ X \in \mathfrak{gl}(n) : X\Phi + \Phi X^T = 0 \} \). If \( \Phi \) has signature \((n-1, 1)\), then the Lie group \( O(n, \Phi) \) is isomorphic to \( O(n-1, 1) \), the Lorentz group. Let \( G \) denote the conformal Lorentz group, \( G = CO(n, \Phi) = \{ g \in GL(n) : g\Phi g^T = \lambda \Phi, \quad \lambda \neq 0 \} \), which has Lie algebra \( \mathfrak{g} = \mathfrak{so}(n, \Phi) + \mathbb{R}I_n \). Under
the standard representation \( v \mapsto vg \), the group \( G \) has three orbits on \( \mathbb{R}^n \): the light cone \( N \), the time-like region \( \{ v : v \Phi v^\top < 0 \} \), and the space-like region \( \{ v : v \Phi v^\top > 0 \} \).

For any \( A \in \mathfrak{gl}(n) \), define the \( \Phi \)-trace of \( A \) as \( \text{tr}_\Phi(A) = \sum_{ij} \Phi_{ij} A_{ji} = \text{tr}(\Phi A) \). Let \( S \) denote the vector space of \( \Phi \)-traceless symmetric matrices, so

\[
S = \{ A \in \text{Sym}^2(V) : \text{tr}_\Phi(A) = 0 \}.
\]

Note that \( \text{ver}_2(N) = S \cap \text{ver}_2(V) \) is the space of rank-one, symmetric, \( \Phi \)-traceless matrices.

Consider the “orthogonal adjoint” representation of \( O(n, \Phi) \) on the vector space \( \mathfrak{gl}(n) \) given as

\[
\text{od}(g) = g^\top X g, \ X \in \mathfrak{gl}(n)
\]

Let \( Od(G) \) denote the subgroup of \( Od(GL(n)) \) defined as \( Od(CO(n, \Phi)) \) in Appendix A so

\[
Od(G) = \{ A \mapsto \lambda g^\top Ag, \ g \in O(n, \Phi), \lambda \in \mathbb{R}^x \}.
\]

The Lie algebra of this group yields a faithful representation of \( g \), namely

\[
\text{od}(g) = \{ A \mapsto X^\top A + AX, \ X \in g \}.
\]

**Lemma 1.1.** The group \( Od(G) \) preserves \( S \), \( \text{ver}_2(V) \), and \( \text{ver}_2(N) \) as varieties in \( V \circ V \). Moreover, \( S \) is an irreducible \( Od(G) \)-module.

**Proof.** For any \( A \in S \), then \( \sum_{ij} \Phi_{ij} \text{od}_g(A)_{ij} = \sum_{ijkl} \Phi_{ijkl} g_{ki} A_{kl} g_{lj} = \sum_{kl} \left( \sum_{ij} g_{ij} \Phi_{ijkl} \right) A_{kl} = \lambda \sum_{kl} \Phi_{kl} A_{kl} = 0 \). For any \( A \in \text{ver}_2(V) \), there exists \( v \in V \) such that \( A = v^\top v \). Therefore \( \text{od}_g(A) = g^\top v^\top v g = (vg)^\top (vg) \in \text{ver}_2(V) \). Since \( \text{ver}_2(N) = \text{ver}_2(V) \cap S \), it is also preserved. The action \( \text{od}_g \) is transitive on the non-zero elements of \( \text{ver}_2(V) \) because \( \text{ver}_2 \) is a bijection onto its image and \( G \) is transitive on \( PV \).

Over \( \mathbb{C} \), \( N \) is a non-degenerate affine variety. Since \( \text{ver}_2(N) \) is in bi-rational correspondence with \( N \), the action of \( Od(G) \) is irreducible on the non-zero elements of \( \text{ver}_2(N) \).

In the language of Cartan \[\text{car81}], \text{ver}_2(N) \) is an irreducible Euclidean representation for \( Od(G) \) spanning the vector space \( S \). In particular, this implies that \( S \) is an irreducible \( Od(G) \)-module. \( \square \)

As an \( Od(G) \)-module, \( V \circ V \) decomposes into irreducible submodules as \( S + \mathbb{R} \Phi^{-1} \). The projection onto the second component is \( A \mapsto \frac{1}{n} \text{tr}(\Phi A) \Phi^{-1} \).

Of course, \( \mathfrak{gl}(n) \) also admits another action of \( O(n, \Phi) \), the well-known (right) adjoint representation.

\[
\text{Ad}_g(A) = g^{-1}Ag.
\]

Augmenting this action with scalings as in Appendix A consider the group \( \text{Ad}(G) = \{ A \mapsto \lambda g^{-1}Ag, \ g \in O(n, \Phi), \lambda \neq 0 \} \). Let \( \mathcal{R} = \{ a : a^\top = \Phi^{-1}a\Phi, \text{tr} a = 0 \} \), which is the space of traceless Ricci tensors for the nondegenerate symmetric bilinear form \( \Phi \). Then \( \mathfrak{gl}(n) \) decomposes into the \( \text{Ad}(G) \)-submodules \( \mathbb{R} I + \mathfrak{so}(n, \Phi) + \mathcal{R} \). The first two summands together are \( g \), the Lie algebra of \( G \). The projections are given by

\[
\begin{align*}
\pi_{\mathbb{R}I} : a &\mapsto \frac{1}{n} \text{tr}(a) I, \\
\pi_{\mathfrak{so}(n, \Phi)} : a &\mapsto \frac{1}{2} (a - \Phi a^\top \Phi^{-1}), \\
\pi_{\mathcal{R}} : a &\mapsto \frac{1}{2} (a + \Phi a^\top \Phi^{-1}) - \frac{1}{n} \text{tr}(a) I, \\
\pi_{g} : a &\mapsto \pi_{\mathbb{R}I}(a) + \pi_{\mathfrak{so}(n, \Phi)}(a).
\end{align*}
\]

Moreover, the \( \text{Ad}(G) \)-module \( \mathcal{R} \) is isomorphic to the (irreducible) \( \text{Ad}(G) \)-module \( S \), as seen in Figure B. Generally, the \( \text{Ad}(G) \)-module \( \mathfrak{so}(n, \Phi) \) is not irreducible. The corresponding decomposition for \( Od \) is \( \mathfrak{gl}(n) = \mathbb{R} \Phi^{-1} + \mathfrak{so}(n, \Phi) + S \) with the obvious projections.
Lemma 1.2. The bilinear pairing $\langle \cdot , \cdot \rangle : S^* \otimes S^* \rightarrow \mathbb{R}$ defined by $\langle A, B \rangle = \frac{1}{n} \text{tr}(\Phi A \Phi B)$ is non-singular and is $\text{Od}(G)$-invariant, up to scale. Therefore, $\text{Od}(G)$ is a subgroup of the conformal group $\text{CO}(m, \langle \cdot , \cdot \rangle)$.

Proof. Trace is cyclic, so the pairing is symmetric. Because $\Phi$ is non-singular, $\langle \cdot , \cdot \rangle$ is also non-singular. To check invariance up to scale, it suffices to check over $\text{Od}(O(n, \Phi))$. For any $g \in \text{Od}(n, \Phi)$,

$$\langle \text{Od}_g(A), \text{Od}_g(B) \rangle = \text{tr}(\Phi g^\top A g \Phi g^\top B g) = \text{tr}(g \Phi g^\top A g \Phi g^\top B) = \langle A, B \rangle.$$  

\[\square\]

Though the pairing is nonsingular, it does have null directions if $\Phi$ does. In particular, any element of $\text{ver}(\mathcal{N})$ is null. The trilinear form $(A^1, A^2, A^3) = \text{tr}(\Phi A^1 \Phi A^2 \Phi A^3)$ is also fully symmetric. Under the isomorphism from $\text{Od}(G)$-modules to $\text{Ad}(G)$-modules, it is seen that the pairing $\langle \cdot , \cdot \rangle$ for $S$ corresponds to the “trace form” pairing $(R, S) \mapsto \text{tr}(RS)$ for $R, S \in \mathfrak{gl}(n)$ using the $\text{Ad}$ action.

Let $\text{Od}^i$ denote the adjoint for the pairing, $(\text{Od}_g^i(A), B) = \langle A, \text{Od}_g(B) \rangle$, so $\text{Od}_g^i(A) = g^\top A g^{-1} = \text{Od}_{g^{-1}}(A)$. A dual group $\text{Od}^*(G)$ acts on $S^* = \text{Hom}(S, \mathbb{R})$ with the rule $F(\text{Od}_g(A)) = \text{Od}_g^{-1}(F)(A)$, so

$$\sum_{ij} F_{ij}^i (g^\top A)_{ij} = \sum_{ijkl} F_{ij}^k g_{ki} A_{kl} g_{lj} = \sum_{kl} (gC^k g^\top)_{kl} A_{kl} = \sum_{kl} (gC g^\top)_{kl} A_{kl}.$$  

Therefore, $S^*$ embeds in the symmetric matrices as $\{ \Phi A \Phi : A \in S \}$ with the action $\text{Od}_g^*$ acting like $\text{Od}_{g^{-1}}$.

Using the $\text{Od}^*$ and $\text{Od}^i$ identifications of $S$ with $S^*$, the space $\text{Hom}(S, S) = S \otimes S^*$ is identified with $\text{Hom}(S, S^*) = S^* \otimes S^*$ and with $\text{Hom}(S \otimes S, \mathbb{R}) = (S \otimes S)^*$ using the bilinear form $\langle \cdot , \cdot \rangle$ as seen here:

$$\{ A \mapsto Q(A) \} \leftrightarrow \{ A \mapsto \Phi Q(A) \Phi \} \leftrightarrow \{ A \otimes B \mapsto \langle Q(A), B \rangle \}.$$  

These have some important submodules that arise frequently here

$$\mathcal{Q}_0 = \{ A \mapsto Q(A) = c A, \ c \in \mathbb{R} \}$$

$$\leftrightarrow \{ A \otimes B \mapsto c \langle A, B \rangle, \ c \in \mathbb{R} \} \cong \mathbb{R}$$

$$\mathcal{Q}_1 = \{ A \mapsto Q(A) = \frac{1}{2} (A \Phi C + C \Phi A) - \langle A, B \rangle \Phi^{-1}, \ C \in S \}$$

$$\leftrightarrow \{ A \otimes B \mapsto \langle A, B, C \rangle, \ C \in S \} \cong S$$

$$\mathcal{Q}_- = \{ A \mapsto Q(A), \langle Q(A), B \rangle = - \langle A, Q(B) \rangle \} \cong S^* \wedge S^*,$$

$$\mathcal{Q}_+ = \{ A \mapsto Q(A), \langle Q(A), B \rangle = \langle A, Q(B) \rangle \} \cong S^* \circ S^*.$$  

Of course, $\mathcal{Q}_-$ is the set of endomorphisms of $S$ that are anti-self-adjoint for $\langle \cdot , \cdot \rangle$, and $\mathcal{Q}_+$ is the set of endomorphisms of $S$ that are self-adjoint for $\langle \cdot , \cdot \rangle$. When considered as elements of $S^* \otimes S^*$ using the pairing, they describe $S^* \wedge S^*$ and $S^* \circ S^*$, respectively. Both $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are submodules of $\mathcal{Q}_+$, and $\mathcal{Q}_0 + \mathcal{Q}_1$ is identified with the space of symmetric matrices by mapping the parameters $c$ and $C$ to $c \Phi + C \in \mathbb{R} \Phi^{-1} + S$. There are preferred projections onto these two components, given again by the identification of $S$ and $S^*$,

$$\Pi_0 : A^* \otimes B^* \mapsto \langle A, B \rangle \Phi^{-1}$$

$$\Pi_1 : A^* \otimes B^* \mapsto \frac{1}{2} (A \Phi B + (A \Phi B)^\top) - \langle A, B \rangle \Phi^{-1} \in S.$$  

Writing $\Pi = (\Pi_0 + \Pi_1) : S^* \otimes S^* \mapsto (\mathbb{R} \Phi^{-1} + S)$, set $\mathcal{Q}_2 = \ker \Pi \subset \mathcal{Q}_+$, so $\mathcal{Q}_2 \cong (S^* \circ S^*)/(\mathbb{R} \Phi^{-1} + S)$ and $\mathcal{Q}_+ = \mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_2$. Overall, $S^* \otimes S^* = \mathcal{Q}_- + \mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_2$. 

Figure 1. The \((\Phi \text{ or } I)\)-traceless and \((I \text{ or } \Phi)\)-symmetric \(G\)-modules and their duals.

2. \(Od(G)\)-Structures

Let \(M\) denote a smooth manifold of dimension \(m = \frac{3}{2}n(n + 1) - 1\), and let \(\mathcal{F}(M)\) denote the \(\mathcal{S}\)-valued co-frame bundle over \(M\), meaning that elements of \(\mathcal{F}_p(M)\) are linear isomorphisms \(\eta_p : T_pM \rightarrow \mathcal{S}\). Conventionally \(\mathcal{F}(M)\) is a principal right \(GL(\mathcal{S})\)-bundle.

**Definition 2.1.** \(^\text{2}\) For \(M\) of dimension \(m = \frac{3}{2}n(n + 1) - 1\), an \(\text{“}Od(G)\text{”-structure with respect to } \Phi\) is a reduction of the \(\mathcal{S}\)-valued co-frame bundle by the action of \(Od(G)\). In particular, an \(Od(G)\)-structure is a principal right \(G\)-bundle \(\mathcal{B} \rightarrow M\).

Let \(\omega\) denote the tautological \(\mathcal{S}\)-valued one-form of an \(Od(G)\)-structure \(\mathcal{B} \rightarrow M\), so \(\omega_b(X) = b \circ \pi(X)\) for all \(X \in T_b\mathcal{B}\). Let \(\eta\) be a local section, \(\eta : M \rightarrow \mathcal{B}\). Define a local trivialization \(H : M \times G \rightarrow \mathcal{B}\) by \(H(p, g) = Od_g(\eta_p)\). Then \(\eta^*(\omega) = \eta\) and \(H^*(\omega) = Od_g \circ \eta\). Then

\[
H^*(d\omega) = d(H^*(\omega))
\]

\[
= d(g^\top \eta g) = dg(g^\top)^{-1}g^\top \eta g + Od_g \circ d\eta - g^\top \eta g g^{-1}dg
\]

\[
= (gdg)^\top \wedge H^*(\omega) - H^*(\omega) \wedge (g^{-1}dg) + Od_g \circ d\eta
\]

Notice that \(\theta = (H^*)^{-1}(g^{-1}dg)\) is a \(\mathfrak{g}\)-valued pseudo-connection on \(\mathcal{B}\) with apparent torsion \(T(\omega \wedge \omega) = (H^*)^{-1}(Od_g \circ d\eta)\). The apparent torsion \(T\) is a function on \(\mathcal{B}\) valued in \(\mathcal{S} \otimes (\mathcal{S}^* \wedge \mathcal{S}^*)\). Cartan’s first structure equation for \(Od(G)\)-structures is thus

\[
d\omega_{ij} = \sum_k (\theta_{ki} \wedge \omega_{kj} - \omega_{ik} \wedge \theta_{kj}) + T_{ij}(\omega \wedge \omega).
\]

**Definition 2.2.** An \(Od(G)\)-structure \(\mathcal{B} \rightarrow M\) is said to be “embeddable” if there exists a bundle embedding into \(CSp(N)^0 \rightarrow \Lambda^0\). An \(Od(G)\)-structure is said to be “locally embeddable near \(b\)” if there is an open neighborhood \(U\) of \(b \in \mathcal{B}\) such that \(U\) is embeddable.

The discussion in Section III and the fundamental lemma of Lie groups immediately provide a simple characterization of locally embeddable structures.

**Lemma 2.3.** The following are equivalent for an \(Od(G)\)-structure \(\mathcal{B} \rightarrow M\) with \(p \in M\).

1. \(\mathcal{B}\) is locally embeddable near \(b\) for some \(b \in \mathcal{B}_p\);

\(^{2}\)In an earlier preprint, these were called “Veronese structures,” but in retrospect, that name is misleading in this context and is better used for a different but related structure that will appear elsewhere.
(2) there exists a local inclusion $i : M \to \Lambda^a$ near $p$ such that $i(M) = F^{-1}(0)$ for a Hessian PDE $F = 0$;
(3) there is an $\mathfrak{sp}(n)$-valued 1-form $\mu$ defined in a neighborhood of $b \in B_p$ such that

$$
\mu = \begin{pmatrix} \beta & \gamma \\ \alpha & -\beta^\top \end{pmatrix}
$$

with $\alpha = \alpha^\top$, $\gamma = \gamma^\top$, $d\mu + \mu \wedge \mu = 0$, and such that $\alpha$ is semi-basic and of maximum rank $m$ on $B$.

Henceforth, only (locally) embeddable structures are considered.

### 2.1. Embeddable Torsion and The First Fundamental Lemma.

In this section, we apply the first step of Cartan’s method of equivalence to normalize the first-order structure equations and find global forms of the connection and torsion for embeddable $OdG$-structures [IL03].

Before beginning the method, note an important algebraic curiosity that is inherent to these structures. For an arbitrary Lie group $H \subset GL(S)$, consider a $H$-structure $B$ over a manifold $M$ with tangent space $T_p M \cong \mathbb{R}^m = S$. In Cartan’s tradition, the local equivalence of $H$-structures can be understood by determining how much apparent torsion $T : B \to S \otimes (S^* \otimes S^*)$ can be absorbed by making an alteration of the $h$-valued pseudo-connection of the form $\theta \mapsto \theta + P(\omega)$ for $P : B \to h \otimes S^*$. Thus, the solution to the equivalence problem of $H$-structures involves the computation of the skewing map $\delta$ that defines the exact sequence

$$
0 \to h^{(1)} \to h \otimes S^* \xrightarrow{\delta} S \otimes (S^* \wedge S^*) \to H^{0,2}(h) \to 0
$$

Informally, $H^{0,2}(h)$ is the space where invariant torsion is valued, and $h^{(1)}$ controls the uniqueness of global connections with a given essential torsion. In the most historically important equivalence problems, the action by the Lie group $H$ on the co-frames of the manifold $M^m$ is defined by the standard representation of $H$ as embedded in $GL(S)$, so the inclusion of $h$ into $S \otimes S^*$ is the identity embedding, and the map $\delta$ is given by the composition of the maps

$$
h \otimes S^* \xrightarrow{\mathbb{1}} (S \otimes S^*) \otimes S^* \xrightarrow{(1,\wedge)} S \otimes (S^* \wedge S^*).$$

However, for $OdG$-structures, the action of $g$ on $S$ is defined by a different representation, namely Equation (7). Thus, for any $P \in g \otimes S^*$, the image of $P$ in $S \otimes S^* \otimes S^*$ is the map $(A,B) \mapsto P(A)^\top B + BP(A)$, so $\delta(P)(A,B) = \frac{1}{2} \left( P(A)^\top B + BP(A) - P(B)^\top A - AP(B) \right)$. As an identification of two-forms valued in $S$, this is written as $\delta(P)(\omega \wedge \omega) = P(\omega)^\top \wedge \omega - \omega \wedge P(\omega)$. Note the apparent sign change, which is really just a consequence of the rule $(\alpha \wedge \beta)^\top = (-1)^{pq} \beta^\top \wedge \alpha^\top$ for matrix-valued $p$- and $q$-forms $\alpha$ and $\beta$. Conceptually, the computation of the kernel and co-kernel of $\delta$ is still the appropriate approach, but the computation relies on this matrix arithmetic.

We now proceed to study $\delta$ by first examining a related linear map $\bar{\Delta}$ on a larger domain. Consider the space

$$
\text{gl}(N) \otimes S^* = \left\{ \left( Y_{ij}^{kl} : Y_{ij}^{kl} = Y_{ij}^{lk}, \sum_{kl} Y_{ij}^{kl} (\Phi^{-1})_{kl} = 0 \right) \right\}.
$$

Let $f$ denote the map $\text{gl}(n) \otimes S^* \to \text{gl}(n) \otimes (S^* \otimes S^*)$ defined by $f(Y)(A,B) = Y(A)^\top B$ for any $Y \in \text{gl}(n) \otimes S^*$. Let $\bar{\Delta}$ denote the skew of $f$, so map $\bar{\Delta}(Y)(A,B) = f(Y)(A,B) - f(Y)(B,A) = Y(A)^\top B - Y(B)^\top A$.

### Lemma 2.4

The map $\bar{\Delta}$ is injective.
Proof. To study $\Delta$, it is expedient to evaluate the map on 1-forms, which is anyway the situation that is always needed for local geometry. Fix arbitrary $Y \in \ker \Delta$, and let $\psi = Y(\omega)$. Aside from symmetry, the 1-forms $\omega_{ij} = \omega_{ji}$ have only one linear relation, namely that $\text{tr}(\Phi \omega) = 0$. Because $\Phi$ is assumed to be non-degenerate, the entries of each row (or column) of $\omega$ are independent 1-forms.

Then $0 = \sum_k \psi_{ik} \wedge \omega_{kj} = \sum_k \psi_{ik} \wedge \omega_{jk}$ for all $i, j$. In the case $j = 1$, the 1-forms $\omega_{11}, \ldots, \omega_{1n}$ are independent, so Cartan’s lemma implies that $\psi_{ik} = \sum_l C_{ilk}^l \omega_{1l}$ for some functions $C_{ilk}^l = C_{ilk}$. The case $j = 2$ similarly implies that $\psi_{ik} = \sum_l C_{ilk}^l \omega_{2l}$ for some functions $C_{ilk}^l = C_{ilk}$. The case $j = n$ similarly implies that $\psi_{ik} = \sum_l C_{ilk}^l \omega_{nl}$ for some functions $C_{ilk}^l = C_{ilk}$. Comparing cases $j = 1$ and $j = 2$, it must be that $\psi_{ik} \equiv 0$ modulo $\omega_{12}$. Comparing cases $j = 1$ and $j = n$, it must be that $\psi_{ik} \equiv 0$ modulo $\omega_{1n}$. So, $\psi_{ik} = 0$, and $Y = 0$. \hfill \Box

Let $\bar{\delta}$ denote the map $\mathfrak{gl}(N) \otimes S^* \to (V \otimes V) \otimes (S^* \wedge S^*)$ that is defined by

$$\bar{\delta}(Y)(A, B) = \frac{1}{2}(\text{od}_{Y(A)}(B) - \text{od}_{Y(B)}(A))$$

(18)

$$= \frac{1}{2} \left( (Y(A)^T B + (Y(A)^T B)^T - Y(B)^T A - (Y(B)^T A)^T \right)$$

$$= \Delta(Y)(A, B) + (\Delta(Y)(A, B))^T.$$  

If $Y$ happens to be a change of connection valued in $\mathfrak{g} \otimes S^*$, then $\bar{\delta}(Y) \in S \otimes (S^* \wedge S^*)$ is the resulting change of torsion. Thus the skewing map is computed as $\delta = \bar{\delta}|_{\mathfrak{g} \otimes S^*}$.  

Lemma 2.5.

$$\ker \bar{\delta} = \{A \mapsto c\Phi A, c \in \mathbb{R}\} \cup \{A \mapsto \Phi C \Phi A, C \in S\}.$$  

Moreover, $\mathfrak{g}^{(1)} = \ker \delta = \ker \bar{\delta} \cap (\mathfrak{g} \otimes S^*) = 0$. Therefore, for any choice of co-kernel of $\delta$, every $\text{Od}(G)$-structure admits a unique and global connection such that the torsion map $T$ takes values in that co-kernel.  

Proof. Suppose $Y \in \ker \bar{\delta} \subset \mathfrak{gl}(n) \otimes S^*$. Let $\tau_{ij} = Y_{ij}^{(kl)} \omega_{kl}$. Then

$$0 = \sum_a \tau_{ai} \wedge \omega_{aj} - \omega_{ia} \wedge \tau_{aj}, \quad \forall i, j.$$  

(19)

In the case $i=j$, this implies $0 = \sum_a \tau_{ai} \wedge \omega_{ai}$. For fixed $i$, the collection $\{\omega_{1i}, \ldots, \omega_{ni}\}$ is linearly independent, so the Cartan Lemma implies that there exist functions $C(i)^b_a = C(i)^b_a$ such that $\tau_{ia} = \sum_b C(i)^b_a \omega_{ib}$ for all $a, i$. Then, for any $i \neq j$,

$$0 = \sum_a (\tau_{ai} \wedge \omega_{aj} + \tau_{aj} \wedge \omega_{ai})$$

$$= \sum_{a,b} (C(i)^b_a \omega_{bi} \wedge \omega_{aj} + C(j)^b_a \omega_{bj} \wedge \omega_{ai})$$

$$= \sum_a (C(i)^a_a \omega_{ai} \wedge \omega_{aj} + C(j)^a_a \omega_{aj} \wedge \omega_{ai})$$

$$+ \sum_{a < b} ((C(i)^b_a - C(j)^b_a) \omega_{bi} \wedge \omega_{aj} + (C(j)^b_a - C(i)^b_a) \omega_{bj} \wedge \omega_{ai})) .$$

Each term in the previous expression is linearly independent, so $C(i)^b_a = C(j)^b_a = C^a_b = C^b_a$ for all $i, j, a, b$. Thus, the kernel of $\bar{\delta}$ is isomorphic to $V \otimes V$ as determined by these $\frac{1}{2}n(n+1)$ constants. It is now easy to check that $\ker \bar{\delta}$ intersects trivially with $\mathfrak{g} \otimes S^*$. \hfill \Box
Note that the output of $\delta(P)$ is a symmetric matrix, but it is not $\Phi$-traceless for general $P \in \mathfrak{gl}(n) \otimes S^*$, so general $\delta(P)$ cannot represent an apparent torsion in $S \otimes (S^* \wedge S^*)$. Consider the subspace

$$\mathcal{P} = \delta^{-1}(S \otimes (S^* \wedge S^*)) = \{P \in \mathfrak{gl}(n) \otimes S^* : (\text{tr}_\Phi \otimes 1)(\delta P) = 0\},$$

along with its image, $\mathcal{E} = \delta(\mathcal{P}) \subset S \otimes (S^* \wedge S^*)$. Of course, $\mathfrak{g} \otimes S^* \subset \mathcal{P}$, and $\delta(\mathfrak{g} \otimes S^*) \subset \mathcal{E}$. As justified by Lemma [2.6], $\mathcal{E}$ is called the space of “embeddable torsion.”

**Lemma 2.6.** If $\mathcal{B} \to \mathcal{M}$ is an embeddable $\text{Od}(G)$-structure, then for any (local) pseudo-connection $\theta$, the associated apparent torsion is a map $T : \mathcal{B} \to \mathcal{E}$. In particular, $\mathcal{B}$ admits a function $P : \mathcal{B} \to \mathcal{P}$, unique up to $\ker \delta$, such that

$$d\omega = (\theta + P(\omega))^\top \wedge \omega - \omega \wedge (\theta + P(\omega)).$$

**Proof.** Fix a pseudo-connection $\theta$ on $\mathcal{B}$ with apparent torsion $T(\omega \wedge \omega)$. Let $\alpha$, $\beta$, and $\gamma$ denote the blocks of the Maurer–Cartan form of $\text{Sp}(n)$, as in Equation [1]

If $\mathcal{B}$ is locally embeddable via a bundle embedding $h$, then the $\frac{1}{2} n(n+1)$ semi-basic one-forms $h^*(\alpha_{jk})$, $j \leq k$ must have a single linear relation, namely that $\text{tr}(\Phi h^*(\alpha)) = 0$. Thus, the components of the $\mathcal{S}$-valued tautological form $\omega$ on $\mathcal{B}$ are given by $\omega_{jk} = h^*(\alpha_{jk})$. The $\mathfrak{gl}(n)$-valued one-form $h^*(\beta)$ may have both vertical and semi-basic components, so write $h^*(\beta_{kl}) = \theta' + \tau$ where $\tau \equiv 0 \mod \{\omega_{kl}\}$ and $\theta' \equiv 0 \mod \{\theta_{kl}\}$. Then

$$0 = d\omega - h^*(d\alpha) = (\theta - h^*(\beta))^\top \wedge \omega - \omega \wedge (\theta - h^*(\beta)) + T(\omega \wedge \omega)$$

$$= \begin{cases} (\theta - \theta')^\top \wedge \omega - \omega \wedge (\theta - \theta') \\ T(\omega \wedge \omega) - (\tau^\top \wedge \omega - \omega \wedge \tau). \end{cases}$$

By Lemma [A.1] $\theta' = \theta$. Also, since $\tau$ is semi-basic, one may write $\tau = P(\omega)$ for some $P : \mathcal{B} \to \mathfrak{gl}(N) \otimes S^*$. Then $T = \delta(P)$. Because $T$ is a priori valued in $S \otimes (S^* \wedge S^*)$, it must be that $P \in \mathcal{P}$ and $T \in \mathcal{E}$. \hfill $\Box$

The next task is to write $\theta$ and $T$ in a preferred way so that the structure equations of an embeddable $\text{Od}(G)$-structure are global. Consider Figure [2] with $\mathcal{T} = \mathcal{E}/\delta \subset H^{0,2}(\mathfrak{g})$. Since $\ker \delta$ and $\mathfrak{g} \otimes S^*$ are naturally subspaces of $\mathcal{P}$, a preferred representative of $[T]$ can be given by specifying any section $\sigma : \mathcal{T} \to \mathcal{P}$, which then yields a corresponding decomposition $\mathcal{P} = \ker \delta + (\mathfrak{g} \otimes S^*) + \sigma(\mathcal{T})$.

The next few lemmas specify a preferred cokernel $\mathcal{T}' = \sigma(\mathcal{T})$ by analyzing the sub-modules of $\mathfrak{gl}(n) \otimes S^*$ under the $\text{Ad}(G) \otimes \text{Od}^*(G)$ action.

**Lemma 2.7.** Let $\mathcal{P}_R = \{P \in \mathcal{P} : \pi_\mathcal{g}(P(A)) = 0 \text{ for all } A \in \mathcal{S}\} = (\pi_\mathcal{R} \otimes 1)(\mathcal{P})$ Then $\mathcal{P}_R$ is isomorphic to $\mathcal{Q}_+$, so $\mathcal{P} = \mathcal{Q}_+ \oplus (\mathfrak{g} \otimes S^*)$.

**Proof.** It suffices to define a map $e$ such that the sequence

$$0 \to \mathfrak{g} \otimes S^* \to \mathcal{P} \xrightarrow{\delta} \mathcal{Q}_+ \to 0$$

is exact. For any $P : \mathcal{S} \to \mathfrak{gl}(n)$, define the map $e(P) : \mathcal{S} \to \mathcal{S}$ by $e(P)(A) = \Phi^{-1}\pi_\mathcal{R}(P(A))$. Let $E(P) \in S^* \otimes S^*$ denote the bilinear pairing associated to $e(P)$ using the identifications from
The torsions of an embeddable $Od(G)$-structure. The quantity $P$ depends on the choice of the section $\sigma$.

Section 1.3 So,

$$E(P)(A, B) = \langle e(P)(A), B \rangle$$

$$= \left\langle \Phi^{-1} \frac{1}{2} (P(A) + \Phi P(A)^\top \Phi^{-1}) - \frac{1}{n} \text{tr}(P(A))\Phi^{-1}, B \right\rangle$$

$$= \frac{1}{2n} (\text{tr}(P(A)\Phi B) + \text{tr}(B\Phi P(A)^\top))$$

$$= \frac{1}{n} \text{tr}(P(A)\Phi B).$$

Note that $\text{tr}_\Phi(\bar{\delta}(P)(A, B)) = \text{tr}(P(A)\Phi B) - \text{tr}(P(B)\Phi A)$, so $P \in \mathcal{P}$ if and only if $E(P) \in \mathcal{S}^* \otimes \mathcal{S}^*$, which is true if and only if $e(P) \in \mathcal{Q}_+$. Moreover, $e(P) = 0$ for all $P \in \mathfrak{g} \otimes \mathcal{S}^*$.

To prove that $\mathfrak{g} \otimes \mathcal{S}^*$ is the entire kernel of $e$, it suffices to prove that dimension of $\mathcal{P}$ is $\frac{1}{2}m(m + 1) + \dim(\mathfrak{g} \otimes \mathcal{S}^*)$ or equivalently that the co-dimension of $\mathcal{P}$ in $\mathfrak{gl}(n) \otimes \mathcal{S}^*$ is $\frac{1}{2}m(m - 1)$. To do this, it suffices to prove that the map $(\text{tr}_\Phi \otimes 1) \circ \bar{\delta} : \mathfrak{gl}(n) \otimes \mathcal{S}^* \to \mathbb{R} \otimes (\mathcal{S}^* \wedge \mathcal{S}^*)$ is a surjection.

Let $\check{P}(A) = P(A)\Phi$; this changes the action on the image from $Ad$ to $Od^*$, which effectively reveals the isomorphism between $\mathcal{Q}_-$ and $\mathcal{S}^* \wedge \mathcal{S}^*$. Any $\check{P} \in \mathfrak{gl}(n) \otimes \mathcal{S}^*$ may be written as $\check{P}(A)_{ij} = \frac{1}{2} \sum_{ab} C_{ij}^{ab} A_{ab}$ such that $C_{ij}^{ab} = C_{ij}^{ba}$. Then

$$\text{tr}(P(A)\Phi B - P(B)\Phi A) = \text{tr}(\check{P}(A)B - \check{P}(B)A)$$

$$= \sum_{ij} \check{P}(A)_{ij} B_{ij} - \check{P}(B)_{ij} A_{ij}$$

$$= \frac{1}{2} \sum_{ijab} C_{ij}^{ab} A_{ab} B_{ij} - C_{ij}^{ab} B_{ab} A_{ij}$$

$$= \sum_{ijab} C_{ij}^{ab} (A_{ij} \wedge B_{ab}).$$

Hence, any element of $\mathcal{S}^* \wedge \mathcal{S}^*$ may be obtained by choosing appropriate $C_{ij}^{ab}$.

□
Lemma 2.8. The space of essential torsion, $\mathcal{T}$, is isomorphic to $Q_2 = \ker \Pi$. In particular, identifying $\mathcal{T}' = e^{-1}(Q_2)$ as a co-kernel of $\delta$ allows the splittings $P_R = (\pi_R \otimes 1)(\ker \delta) + \mathcal{T}'$ and $\mathcal{P} = (g \otimes S^*) + \ker(\delta) + \mathcal{T}'$.

Proof. Consider the pair of exact sequences

\begin{equation}
0 \longrightarrow (\pi_R \otimes 1)(\ker \delta) \longrightarrow P_R \xrightarrow{\delta} \mathcal{T} \longrightarrow 0
\end{equation}

(26)

Recall that $e \circ (\pi_R \otimes 1) = e$. Because both sequences are exact, it suffices to prove that the map $\Pi \circ e$ is an isomorphism from $\ker \delta$ to $Q_0 + Q_1$. Suppose $K \in \ker \delta$, so $K(A) = c\Phi A + \Phi C \Phi A$ for arbitrary $c \in \mathbb{R}$ and $C \in S$. Then

\begin{align}
E(K)(A,B) &= \left(\Phi^{-1} \frac{1}{2} (K(A) + \Phi K(A)^\top \Phi^{-1}) - \frac{1}{n} \text{tr}(K(A)) \Phi^{-1}, B\right) \\
&= \frac{1}{2n} \text{tr}(c\Phi A \Phi B + \Phi C \Phi A \Phi B + c\Phi A \Phi B + \Phi A \Phi C \Phi B) \\
&= c \langle A, B \rangle + \langle A, B, C \rangle.
\end{align}

(27)

Lemma 2.9 (The First Fundamental Lemma). An embeddable $O(d)$-structure $\mathcal{B}$ admits a unique function $P : B \rightarrow \mathcal{T}' \cong Q_2$ and a unique connection $\theta$ such that $B$ has first structure equations:

$$d\omega = (\theta + P(\omega))^\top \wedge \omega - \omega \wedge (\theta + P(\omega)).$$

Moreover, $\theta$ decomposes as $\theta = \varphi - \frac{1}{2} \lambda I$ for $-\frac{1}{2} \lambda = \frac{1}{n} \text{tr}(\theta)$ valued in $\mathbb{R}$ and $\varphi$ valued in $\mathfrak{so}(n, \Phi)$.

Proof. Fix an embeddable $O(d)$-structure $\mathcal{B}$ with (local) pseudo-connection $\hat{\theta}$ and apparent torsion $T$ and structure equation

\begin{equation}
\omega = \theta^\top \wedge \omega - \omega \wedge \theta + T(\omega \wedge \omega).
\end{equation}

(28)

The torsion $T$ may be written as $\hat{\delta}(\hat{P})$ for some $\hat{P} : B \rightarrow P$ that is unique up to the addition of any $K : \mathcal{B} \rightarrow \ker \delta$. Lemma 2.5 implies that, among all such $\hat{P}$, there is a unique one such that $\Pi(E(\pi_R \circ \hat{P})) = 0$.

Set $\theta = \hat{\theta} + \pi_q(\hat{P}(\omega))$, and write $P(\omega) = \pi_R(\hat{P}(\omega))$. The decomposition of $\theta$ follows from the definition of the scaling action in Appendix A. The given structure equation now holds. Because $\Pi(e(\hat{P}_R)) = 0$, Diagram 2.3 shows that $\hat{P}_S = e^{-1}(Y) + \ker Y \in Q_2 = \ker \Pi$. Therefore, $P \in \mathcal{T}'$.

Henceforth, the words “connection” and “torsion 1-form” refer to the symbols $\theta = \varphi - \frac{1}{2} \lambda I$ and $\tau = P(\omega)$ normalized in this way.

2.2. Embeddable Curvature and the Structure Theorem. We now consider similar restrictions on the curvature,

$$R(\omega \wedge \omega) = d\theta + \theta \wedge \theta = d\varphi + \varphi \wedge \varphi - \frac{1}{2} d\lambda I.$$

(A priori, the curvature $R$ may live in $g \otimes (S^* \wedge S^*)$; however, the condition that $\mathcal{B}$ is embeddable imposes conditions determining which submodules may actually appear.)
Lemma 2.10 (The Second Fundamental Lemma). Let \( \mathcal{B} \) be an embeddable \( Od(G) \)-structure with connection 1-form \( \theta \) and torsion 1-form \( \tau \). Then there exists a function \( C \) valued in \((R\Phi + S^*) \otimes S^* \) such that
\[
(29) \quad R(\omega \wedge \omega) + \tau \wedge \tau + \nabla(\tau) = -C(\omega) \wedge \omega.
\]

Proof. If \( \mathcal{B} \) is embeddable, then there exists a (local) map \( h : B \to Sp(n) \) such that \( h^*(\alpha) = \omega \), \( h^*(\beta) = \theta + \tau \), and \( h^*(d\beta + \beta \wedge \beta + \gamma \wedge \alpha) = 0 \).

Note that \( h^*(d\beta + \beta \wedge \beta) \) is a \( \mathfrak{gl}(n) \)-valued 2-form on \( B \), but it decomposes into the two terms \( R(\omega \wedge \omega) + \tau \wedge \tau \), which are semi-basic and valued in \( \mathfrak{g} \), and \( \nabla(\tau) \), which is semi-basic and valued in \( \mathcal{R} \). Thus, \( h^*(\gamma) \wedge \omega \) must be semi-basic, so \( h^*(\gamma) \equiv 0 \mod \{\omega_{jk}\} \).

Since \( \gamma = \gamma^\top \), \( h^*(\gamma) \) must take values in the symmetric matrices, which decompose into either \( R\Phi + S^* \) or \( R\Phi^{-1} + S \) depending on the action of \( G \). The \( Ad(G) \) action on \( h^*(d\beta + \beta \wedge \beta) \) shows that the appropriate action on \( h^*(\gamma) \) is \( Od^*(G) \). Thus \( h^*(\gamma) \) may take values in the sum of the two irreducible \( Od^*(G) \)-modules \( S^* \) and \( R\Phi \).

Theorem 2.11 (The Structure Theorem). Suppose \( \mathcal{B} \to M \) is an embeddable \( Od(G) \)-structure. Then there are unique \( G \)-equivariant functions \( P : \mathcal{B} \to T^* \), \( Q_{12} : \mathcal{B} \to (Q_1 + Q_2) \), \( Q_- : \mathcal{B} \to Q_- \), \( r : \mathcal{B} \to \mathbb{R} \), and \( s : \mathcal{B} \to S^* \) such that \( \mathcal{B} \) has structure equations
\[
d\omega = -\lambda I \wedge \omega + (\varphi + P(\omega))^\top \wedge \omega - \omega \wedge (\varphi + P(\omega)),
\]
action by \( Od \)
\[
d(P(\omega)) = -\theta^\top \wedge P(\omega) - P(\omega) \wedge \theta - s(\omega)\Phi \wedge \omega - \pi_R ([Q_{12}(\omega) + Q_-(\omega)] \wedge \omega),
\]
action by \( Ad \)
\[
d\varphi = -\varphi \wedge \varphi - P(\omega) \wedge P(\omega) + \frac{1}{n} tr(P(\omega) \wedge P(\omega))I - r\Phi \omega \wedge \Phi \omega - \pi_{so(n,\Phi)} ([Q_{12}(\omega) + Q_-(\omega)] \wedge \omega)
\]
action by \( Ad \)
\[
d\lambda = 2 \tr(P(\omega) \wedge P(\omega)) + 2\pi_{RI} (Q_-(\omega) \wedge \omega)
\]
trivial action

In the general case, all of the projections are injections on the shown representations. In particular, \( \nabla(\theta) = R(\omega \wedge \omega) \) depends only on \( P \), \( \nabla(P) \), and a single scalar curvature, \( r \).

Note! Although the structure equations in Theorem 2.11 are true for any non-degenerate \( \Phi \), their precise formulation depends on \( \Phi \). The group \( Od(G) \) depends on the initial choice of \( \Phi \). This group determines the “shape” of the matrices \( \theta \) and \( \omega \) as well as the projections that define the components of \( P \), \( Q_{12} \), \( Q_- \), \( r \), and \( s \). However, by the law of inertia, changing \( \Phi \) for another non-degenerate symmetric bilinear form of the same signature amounts only to re-indexing these equations. Under such a change, the spaces of invariants will be represented by different (but isomorphic) submodules of \( \mathfrak{gl}(n) \otimes \mathfrak{gl}(n) \).

Proof. Consider the second-order embeddable skewing map \( \Delta : (S^* \otimes R\Phi) \otimes S^* \to \mathfrak{gl}(n) \otimes (S^* \wedge S^*) \) defined by \( \Delta(C)(\omega \wedge \omega) = C(\omega) \wedge \omega. \) Aside from the restricted domain, this is the same as \( \Delta \) that appears in Lemma 2.4.1 so it is injective. By the identifications in Section 1.3 any \( C \) decomposes as \( C(\omega) = s(\omega)\Phi + r\Phi \omega + Q_1(\omega) + Q_2(\omega) + Q_-(\omega) \) according to \((R\Phi + S^*) \otimes S^* = R\Phi \otimes S^* + Q_0 + Q_1 + Q_2 + Q_- \). Thus, it is known that this space provides all the second-order invariants of \( \mathcal{B} \). The only question is where the various components appear in the structure equations.

Equation (29) is an equality of matrices in the \( Ad(G) \)-module \( \mathfrak{gl}(n) \), so it can be interpreted as the three distinct equations by projecting onto the \( \mathcal{R}, \mathbb{R}I, \) and \( so(n,\Phi) \) submodules. In particular,
the second-order invariants of $\mathcal{B}$ will appear in the equations

\begin{equation}
\pi_R(\Delta(C)(\omega \wedge)) = -\nabla(\tau)
\end{equation}

\begin{equation}
\pi_R(\Delta(C)(\omega \wedge \omega)) = \frac{1}{2}d\lambda - \frac{1}{n}\text{tr}(\tau \wedge \tau)
\end{equation}

\begin{equation}
\pi_{so(n, \Phi)}(\Delta(C)(\omega \wedge \omega)) = -d\varphi - \varphi \wedge \varphi - \tau \wedge \tau + \frac{1}{n}\text{tr}(\tau \wedge \tau).
\end{equation}

Suppose $V \subset \mathfrak{gl}(n) \otimes (S^* \wedge S^*)$ is an irreducible component of the image of $\Delta$. If $(\pi_R \otimes 1)(V) \neq 0$ and $(\pi_g \otimes 1)(V) \neq 0$, then there is an isomorphism between these two images. In this case, the irreducible component of $R(\omega \wedge \omega) + \tau \wedge \tau$ appearing as $(\pi_g \otimes 1)(V)$ may be expressed as a multiple of the irreducible component of $\nabla(\tau)$ that appears as $(\pi_R \otimes 1)(V)$. If $(\pi_R \otimes 1)(V) = 0$, then $V = (\pi_g \otimes 1)(V)$, so $V$ is an irreducible component of $R(\omega \wedge \omega) + \tau \wedge \tau$ that is independent of $\nabla(\tau)$. Finally, if $(\pi_g \otimes 1)(V) = 0$, then $V = (\pi_R \otimes 1)(V)$, so $V$ is an irreducible component of $\nabla(\tau)$ that is independent of $R(\omega \wedge \omega) + \tau \wedge \tau$. Similarly, the image on $\mathfrak{g}$ can be projected onto the $\mathbb{R} I$ and $\mathfrak{so}(n, \Phi)$ submodules. Thus, the form of Equation (30) relies only on the decomposition of the images of the projections of $\Delta(C)$. The proof is completed by Lemmas 2.15, 2.16, and 2.17 below. □

Note that $r$ and $s$ take values in irreducible $(Od^*(G) \otimes Od^*(G))$-modules, but $P$ and $Q$ are not irreducible for general $n$. To see the syzygies of the invariants $P$, $Q$, $r$, and $s$, it is necessary to differentiate once more.

**Theorem 2.12** (The Structure Theorem, cont’d). In the setting of Theorem 2.11, the following equations also hold for $\tau = P(\omega)$ and $\rho = r\Phi \wedge \Phi + Q_{12}(\omega) + Q_-(\omega)$.

\begin{equation}
d(s(\omega)) = \lambda \wedge s(\omega) - 2\text{tr}(\Phi^{-1}\rho \wedge P(\omega)),
\end{equation}

\begin{equation}
\nabla(\rho) = s(\omega) \wedge \left(\varphi\Phi + \Phi\varphi^\top + \tau\Phi + \Phi\tau^\top\right) - \tau \wedge \rho + \rho \wedge \tau^\top + \frac{2}{n}\text{tr}(\Phi^{-1}\rho \wedge \tau)\Phi.
\end{equation}

**Proof.** Write $C(\omega) = s(\omega)\Phi + \rho$ for $\rho = r\Phi \omega \Phi + Q_{12}(\omega) + Q_-(\omega)$ a semi-basic 1-form valued in $S^*$. By pulling back the final part of Equation 1, it must be that

\begin{equation}
0 = d(s(\omega)\Phi + \rho) + \left(\varphi - \frac{1}{2}\lambda I + \tau\right) \wedge (s(\omega)\Phi + \rho) - (s(\omega)\Phi + \rho) \wedge \left(\varphi - \frac{1}{2}\lambda I + \tau\right)^\top.
\end{equation}

So, $ds(\omega)$ is given by the $\Phi^{-1}$-trace of this equation, and the components of $d\rho$ are given by the projection of this equation onto $S^*$ by subtracting the $\Phi^{-1}$-trace part. The terms $(\varphi + \tau) \wedge s(\omega)\Phi + \varphi \wedge \rho - \frac{1}{2}\lambda \wedge \rho$ and their transposes are $\Phi^{-1}$-traceless. The component $-\frac{1}{2}\lambda \wedge s(\omega)\Phi$ and $\tau \wedge \rho$ and their transposes have non-trivial $\Phi^{-1}$-trace. □

**Corollary 2.13.** $P = 0$ locally if and only if $P = Q_{12} = Q_- = s = 0$ at a point. In this case, $dr = 2r\lambda$. 
Proof. If $P = 0$ locally, then $d(P(\omega)) = 0$, but the kernel of $\pi_R \circ \Delta$ is $Q_0$. Therefore, $s = Q_{12} = Q_\perp = 0$. Write $\omega = \Phi \omega \Phi$.

\[
0 = d(r\omega) + r\theta \wedge \omega - r\omega \wedge \theta^T \\
= dr \wedge \omega + r\Phi (-\lambda \wedge \omega + \varphi^T \wedge \omega - \omega \wedge \varphi) \Phi + r\theta \wedge \omega - r\omega \wedge \theta^T \\
(34) = dr \wedge \omega - r\lambda \wedge \omega - r\varphi \Phi \wedge \omega \Phi + r\Phi \omega \wedge \Phi \varphi^T + -r\varphi \wedge \omega - r\lambda \wedge \omega - r\varphi \wedge \omega \\
= (dr - 2r\lambda) \wedge \omega.
\]

Therefore, $dr - 2r\lambda \equiv 0$ modulo $\omega_{ij}$ for all $i, j$. \hfill \Box

This helps us see that the invariant $r$ is somewhat spurious, in the sense that it represents the scaling implicit in the identification $S \leftrightarrow S^*$ or equivalently in the pairing $\langle \cdot, \cdot \rangle$. Since this pairing gives pseudo-Riemannian metric over $M$ that is only defined up to a conformal factor, the scalar curvature can be varied freely.

**Corollary 2.14.** Other than signature, there are no intrinsic invariants of Hessian PDEs in two variables. All invariants arise from the particular embedding.

Proof. Consider an $\text{Od}(G)$-structure determined by $\Phi$ on a surface. Then $S \cong \mathbb{R}^2$, so $S \otimes S \cong \mathfrak{so}(2, \Phi) + \mathbb{R}I + \mathcal{R}$, and $\mathcal{T}' = 0$. In this case, $Q_{12}, Q_\perp$ and $s$ must also be identically zero, as they are derivatives of $P = 0$. The scalar function $r$ is still present, but Riemannian and semi-Riemannian surfaces are locally conformally flat, so $r$ can be normalized away. Thus, there are no intrinsic invariants of hyperbolic Hessian PDEs in two variables. To glean any information in this case, one must examine the extrinsic invariants of the hypersurface $F^{-1}(0)$, as done in [The10]. \hfill \Box

Here are the lemmas that provide the decompositions in the structure theorem.

**Lemma 2.15.** The kernel of $(\pi_{RI} \otimes 1) \circ \Delta$ is $(\mathbb{R}\Phi \otimes S^*) + Q_+ \subset (\mathbb{R}\Phi \oplus S^*) \otimes S^*$.

Proof. Suppose that $0 = \pi_{RI}(\Delta(C)(A, B)) = \frac{1}{n} \text{tr}(C(A)B - C(B)A)$ for all $A, B$ in $S$. Write $C(A) = C_0 + C_1$ where $C_0 \in \mathbb{R}\Phi^{-1} \otimes S^*$ and $C_1 \in S^* \otimes S^*$. Clearly, any $C = C_0$ is in the kernel, since $\Phi$-tracelessness defines $S$. So, $C = C_0 + C_1$ is in the kernel if and only if $C_1$ is in the kernel, meaning $0 = \frac{1}{n} \text{tr}(C_1(A)B - C_1(B)A)$ for all $A, B \in S$. By writing $C_1 = \Phi C_1^* \Phi$, one sees that this is precisely the condition that $C_1 \in Q_+$. \hfill \Box

**Lemma 2.16.** $\ker(\pi_R \otimes 1) \circ \Delta$ is isomorphic to the 1-dimensional submodule $Q_0$ of $S^* \otimes S^*$ that is given by the scalar form $\langle \cdot, \cdot \rangle$.

Proof. Write $C = Y \Phi$ for $Y \in (\mathbb{R}I + \mathcal{R}) \otimes S^*$. Suppose $C$ is in the kernel, so for all $A, B \in S$,

\[
0 = \pi_R (\Delta(C)(A, B)) \\
= \frac{1}{2} (C(A)B - C(B)A + \Phi(C(A)B - C(B)A)^T \Phi^{-1}) - \frac{1}{n} \text{tr}(C(A)B - C(B)A)I \\
= \frac{1}{2} (\Phi Y(A)B - \Phi Y(B)A + \Phi BY(A) - \Phi AY(B)) - \frac{1}{n} \text{tr}(\Phi Y(A)B - \Phi Y(B)A)I \\
(35) = \Phi \left( \frac{1}{2} (Y(A)B - Y(B)A + BY(A) - AY(B)) - \frac{1}{n} \text{tr}(\Phi Y(A)B - \Phi Y(B)A)\Phi^{-1} \right) \\
= \Phi (\delta(Y)(A, B) - \frac{1}{n} \text{tr}(\Phi \delta(Y)(A, B))\Phi^{-1}) \\
= \Phi \pi_S(\delta(Y)(A, B)).
\]
So, either \( Y \in \ker \delta \) or \( \delta(Y)(A,B) \) is a multiple of \( \Phi^{-1} \). By Lemma 2.18, the former case implies that \( C(A) = c\Phi.A\Phi \). By the proof of Lemma 2.7, the latter case means that \( e(Y) = \Phi^{-1}C \) projects non-trivially to \( Q_- \); however, the actual pre-image via \( \delta \) of \( \Phi^{-1} \) also projects non-trivially to \( \mathfrak{so}(n) \otimes S^* \), so it is not contained in the domain of \( \Delta \). \( \square \)

**Lemma 2.17.** \( \ker((\pi_{\mathfrak{so}(n,\Phi)} \otimes 1) \circ \Delta) = \ker(\Delta - \Delta^\top) \) is isomorphic to the \( m \)-dimensional irreducible representation \( \mathbb{R}\Phi \otimes S^* \).

The proof of this lemma is a repeated use of Cartan’s lemma, analogous to that of Lemmas 2.4 and 2.5 but more tedious.

### 2.3. A Classifying Space for Second-Order PDEs.

Because the Theorems 2.11 and 2.12 provide structure equations that are closed under exterior derivative and have finitely many structure coefficients, they fit into the framework of Cartan’s generalization of Lie’s third fundamental theorem, which concerns the existence and uniqueness of pseudo-groups with putative structure equations \([\text{Car04}]\). For a summary of the case needed here, see Appendix A of [Bry01]. A modern interpretation of this theorem arises in the theory of groupoids and Lie algebroids \([\text{Mac05}]\). All of the results here are standard consequences of the fact that the space of differential invariants is finite-dimensional \([Olv95]\). It is worthwhile to compare this section to Sections 4 and 5 of [Smi10], which only applies to integrable hyperbolic Hessian PDEs. The presentation here intentionally mirrors that one closely, but the theorems here apply more generally to non-degenerate Hessian PDEs in any number of variables.

**Definition 2.18.** The notation \((B, M, p)_\Phi\) denotes a smooth, embeddable \( \text{Od}(G) \)-structure over \( M \) such that \( M \) is connected and such that \( p \in M \).

Connectedness is very important for these corollaries.

**Definition 2.19.** Let \( K_{\Phi} = T' \oplus (Q_1 \oplus Q_2) \oplus Q_- \oplus Q_0 \oplus S^* \), a vector space of dimension \( m^2 + \frac{1}{2}m(m + 1) - 1 \). This space is called the “classifying space” for \( \text{Od}(G) \)-structures. For an \( \text{Od}(G) \)-structure \( B \), let \( \kappa : B \to K_{\Phi} \) denote the function \( \kappa(b) = (P(b), Q_{12}(b), Q_-(b), r(b), s(b)) \).

Note that \( K_{\Phi} \) is an \( O(n, \Phi) \)-module by the appropriate actions on each component. Suppose that \( T' \cong Q_2 \) decomposes into irreducible \( (\text{Ad}(G) \otimes \text{Od}^*(G)) \)-modules or \( (\text{Od}^*(G) \otimes \text{Od}^*(G)) \)-modules as \( Q_{2,1}, \ldots, Q_{2,q} \), and that \( Q_- \) decomposes into irreducible modules \( Q_{-1}, \ldots, Q_{-w} \). The component \( S^* \) is an irreducible \( \text{Od}^*(G) \)-module, and the component \( \mathbb{R} \) is trivial. Then

\[
K_{\Phi} \cong (Q_{2,1} \oplus \cdots \oplus Q_{2,q}) \oplus Q_1 \oplus (Q_{2,1} \oplus \cdots \oplus Q_{2,q}) \oplus (Q_{-1} \oplus \cdots \oplus Q_{-w}) \oplus Q_0 \oplus Q_1
\]

is a decomposition of classifying space into irreducible \( O(n, \Phi) \)-modules. The infinitesimal scaling action acts on each component of \( K_{\Phi} \) as well, so there is a group action corresponding to \( \mathfrak{g} \), constructed analogously to Appendix A. If the scaling action is removed by considering the projective group, then each component becomes the corresponding projective space, in which case the component \( Q_0 \cong \mathbb{R} \) (where \( r \) lives) vanishes to a point.

Implicit in Theorem 2.12 is a set of equations for \( d\kappa \) that schematically looks like

\[
d\kappa = \begin{pmatrix} P \\ Q_{12} \\ Q_- \\ r \\ s \end{pmatrix} = \begin{pmatrix} \text{linear in } Q_{12}, Q_-, s & \text{quadratic in } P \\ \text{quadratic in } P, Q_{12}, Q_-, r, s & \text{linear in } Q_{12}, Q_-, r, s \\ \text{quadratic in } P, Q_{12}, Q_-, r, s & \text{linear in } Q_{12}, Q_-, r, s \\ \text{quadratic in } P, Q_{12}, Q_-, r, s & \text{linear in } Q_{12}, Q_-, r, s \\ \text{quadratic in } P, Q_{12}, Q_-, r, s & \text{linear in } s \end{pmatrix} \left( \begin{array}{c} \omega_{ij} \\ \theta_{ij} \end{array} \right) = J(\kappa(b)) \left( \begin{array}{c} \omega \\ \theta \end{array} \right)
\]

The matrix \( J(K) \) given by these formulas is well-defined for any \( K \in K_{\Phi} \). The matrix has \( \dim B = n^2 \) columns and \( \dim K_{\Phi} = m^2 + \frac{1}{2}m(m + 1) - 1 \) rows, and its entries are algebraic
functions of \( K \in K_\Phi \) as given by a complete expansion of Theorem 2.12. Because the equations in Theorems 2.11 and 2.12 are closed under exterior derivative, the matrix \( J \) defines the anchor map of a Lie algebroid over \( K_\Phi \). This Lie algebroid can be integrated to a smooth groupoid over \( K_\Phi \) \cite{ste80, mac05}.

**Lemma 2.20.** The singular distribution on \( K_\Phi \) defined by the columns of the matrix \( J \) is integrable, providing a singular foliation of \( K_\Phi \) into leaves that are submanifolds. That is, for any \( K \in K_\Phi \), there exists a unique submanifold \( O_J(K) \) such that \( T_K O_J(K) = \text{range } J(K) \).

Through each point \( K \in K_\Phi \), there passes an orbit, \( O_J(K) \), and each orbit is a submanifold of the base. These orbits may be regarded as the leaves of a singular foliation of \( K_\Phi \).

**Corollary 2.21** (Existence). For any non-degenerate \( \Phi \) and for any choice of \( K \in K_\Phi \), there exists \((B, M, p)_\Phi\) such that \( \kappa(b) = K \) for some \( b \in B_p \). Moreover, this structure is analytic. The map \( \kappa \) is a submersion to the leaf \( O(J) \).

**Corollary 2.22** (Uniqueness). Suppose that \((B, M, p)_\Phi\) and \((B', M', p')_\Phi\) are two Od(G)-structures such that \( \kappa(b) = \kappa'(b') \) for some \( b \in B_p \) and \( b' \in B'_{p'} \). Then there exist neighborhoods \( U \) of \( p \) and \( U' \) of \( p' \) such that \( B_U \) and \( B'_{U'} \) are isomorphic as Od(G)-structures.

Note that, *a priori*, the local isomorphism only holds when comparing two structures using the same \( \Phi \); however, the law of inertia allows one to construct a bundle isomorphism between \((B, M, p)_\Phi\) and \((B', M', p')_\Phi\) as long as \( \Phi \) and \( \Phi' \) have the same signature.

**Lemma 2.23** shows that the existence and uniqueness theorems apply to local non-degenerate Hessian PDEs, too:

**Corollary 2.23.** For a fixed \( \Phi \) and for any choice of \( K \in K_\Phi \), there exists an analytic function \( F \) defined in a neighborhood of the origin of \( \Lambda^p \) such that the Od(G)-structure induced on \( F^{-1}(0) \) takes the chosen value of \( K \) in the fiber over the origin. The Taylor series of this function at the origin is uniquely determined by the G-orbit of \( K \).

This construction is essentially what is done in Section 6 of \cite{smi10} for several examples in the integrable case for \( n = 3 \). For any \( n \), the flat structure yields the PDE \( 0 = \sum_{ij} \Phi_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \).

**Definition 2.24.** \((B, M, p)_\Phi\) is said to “represent \( K \)” if \( K = \kappa(b) \) for some \( b \in B_p \).

**Definition 2.25** (Leaf-equivalence). \((B_0, M_0, p_0)_\Phi\) and \((B_k, M_k, p_k)_\Phi\) are said to be leaf-equivalent if there exist finite sequences \( \{(B_i, M_i, p_i)_\Phi\} \) and \( \{K_i\} \) with \( 1 \leq i \leq k-1 \) such that \((B_i, M_i, p_i)_\Phi\) and \((B_{i+1}, M_{i+1}, p_{i+1})_\Phi\) both represent \( K_i \) for \( 0 \leq i \leq k-1 \).

The term “leaf-equivalence” arises from the leaves of the singular foliation of \( K_\Phi \) from Lemma 2.20. Again, a standard argument for Lie pseudo-groups shows that these leaves separate all possible \((B, M, p)_\Phi\)'s into equivalence classes by the value of \( K \).

**Theorem 2.26** (Leaf-equivalence). \((B, M, p)_\Phi\) and \((\hat{B}, \hat{M}, \hat{p})_\Phi\) are leaf-equivalent if and only if \( O_J(B) = O_J(\hat{B}) \). Moreover, \( \kappa : B \to O_J(B) \) is a submersion.

Because \( \kappa \) is a submersion, the leaves have dimension at most \( \dim B = n^2 \), which is much smaller than \( \dim K_\Phi \). Identifying these leaves explicitly is potentially an extremely challenging task, but it is ultimately the way towards a thorough understanding of non-degenerate Hessian PDEs. For the special case of integrable hyperbolic Hessian PDEs in three variables, this was accomplished due to a small miracle (Lemma 5.1 of \cite{smi10}).
2.4. The Case of Three Variables. Consider the hyperbolic signature (2, 1) for \( n = 3 \). The base manifold \( M \) has dimension 5 and \( \Lambda^o \) has dimension 6. If \( \Phi \) is chosen to be
\[
\Phi = \begin{pmatrix}
0 & 0 & -\frac{1}{2} \\
0 & 1 & 0 \\
-\frac{1}{2} & 0 & 0
\end{pmatrix},
\]
then \( S \) is the vector space of matrices of the form
\[
\begin{pmatrix}
\alpha_{-4} & \alpha_{-2} & \alpha_0 \\
\alpha_{-2} & \alpha_0 & \alpha_2 \\
\alpha_0 & \alpha_2 & \alpha_4
\end{pmatrix}, \alpha_{-4}, \alpha_{-2}, \alpha_0, \alpha_2, \alpha_4 \in \mathbb{R}.
\]

The Lie algebra \( \mathfrak{so}(n, \Phi) \) is isomorphic to \( \mathfrak{sl}_2(\mathbb{R}) \). The finite-dimensional irreducible representations of \( \mathfrak{sl}_2(\mathbb{R}) \) are given by the action of \(-x \frac{\partial}{\partial y}, y \frac{\partial}{\partial x}\) and \(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\) on \( V_r \), the vector space of degree-\( r \) homogeneous polynomials in \( x \) and \( y \). These and the scaling action correspond to the generators of the Lie algebra \( \mathfrak{od}(g) \).

The intersection of the Veronese cone with \( S \) is all matrices of the form
\[
\begin{pmatrix}
s^{4} & s^{3}t & s^{2}t^{2} \\
s^{3}t & t^{2}s^{2} & st^{3} \\
s^{2}t^{2} & st^{3} & t^{4}
\end{pmatrix}, \ s, t \in \mathbb{R}.
\]

It is easy to see that if \( S \) is identified with \( V_4 = \{a_{-4}x^4 + a_{-2}4x^3y + a_06x^2y^2 + a_24xy^3 + a_4y^4\} \), then the intersection of the Veronese cone with \( S \) is the rational normal cone \( \{(sx + ty)^4 : s, t \in \mathbb{R}\} \), which has symmetry group \( GL(2) \). Hence, a hyperbolic \( Od(G) \)-structure in \( n = 3 \) variables corresponds to a \( GL(2) \)-structure of degree 4, as seen in [FHK09] and [Smi10]. The identification of \( S = V_4 \) above is precisely the identification that was used in [Smi10] to reconstruct integrable Hessian PDEs from the structure equations for 2,3-integrable \( GL(2) \)-structures.

So, examining the structure theorem, we arrive at the following irreducible decompositions:
\[
\mathfrak{g} \cong V_0 \oplus V_2 \\
Q_0 \cong V_0 \\
Q_1 \cong V_4 \\
Q_2 \cong V_8 \\
Q_- \cong V_2 \oplus V_6 \\
K_0 \cong (V_8) \oplus (V_4 \oplus V_8) \oplus (V_2 \oplus V_6) \oplus V_0 \oplus V_4 \cong \mathbb{R}^{30}
\]

**Theorem 2.27.** For a hyperbolic \( Od(G) \)-structures in \( n = 3 \) variables or equivalently for \( GL(2) \)-structures of degree 4, the condition of 2-integrability (see Section 3) is equivalent to the condition of embeddability.

**Proof.** In [Smi10], it is shown that a generic \( GL(2) \)-structure of degree 4 is 2-integrable if and only if its torsion only takes values in the irreducible representation \( V_8 \). Compare to Lemma 2.8. \( \square \)

Thus, the theory of embeddable hyperbolic \( Od(G) \)-structures in \( n = 3 \) variables is equivalent to the study of 2-integrable \( GL(2) \)-structures of degree 4. See Corollary 3.2 of [Smi10], where structure equations appear that are equivalent to Theorem 2.11 above, albeit with a different collection of projections. To see the relationship between the projections, note that the decomposition \( V \otimes V = \mathbb{R} \Phi^{-1} + \mathfrak{so}(3) + S \) corresponds to the decomposition \( V_2 \otimes V_2 = V_0 \oplus V_2 \oplus V_4 \). The coefficients

of the Clebsch–Gordan pairings for $SL(2)$ $\langle v, w \rangle_0$, $\langle v, w \rangle_1$, and $\langle v, w \rangle_2$ are scalar multiples of the coefficients of $\pi_\mathcal{R}(v^\top w)$, $\pi_{\mathfrak{so}(3)}(v^\top w)$ and $\pi_{\mathfrak{so}(3)}(v^\top w)$, respectively.

The leaves in $\mathbf{K}_\Phi$ can have dimension no greater than $3^2 = 9$. In the 2,3-integrable case, all of the second-order invariants become functions of $P$, and there are indeed leaves in $\mathcal{V}_8 = \mathcal{Q}_2$ of maximum dimension nine.

2.5. The Case of Four Variables. The Lie algebra $\mathfrak{so}(3,1)$ is isomorphic to $\mathfrak{so}_2(\mathbb{R}) \times \mathfrak{so}_2(\mathbb{R})$, and the finite-dimensional representations are given by products of homogeneous polynomials $\mathcal{V}_{p,q} = \{ f(x,y)g(x',y'), \text{deg } f = p, \text{deg } g = q \}$, which has dimension $(p+1)(q+1)$. The standard representation on $V$ is denoted by $\mathcal{V}_{1,1}$, and the $Od$ representation on $\mathcal{S}$ is $\mathcal{V}_{2,2}$. Using the well-known decomposition for these representations (see [Car81, GMS58]) we can compute

$$\mathfrak{g} \cong \mathcal{V}_{0,0} \oplus \mathcal{V}_{1,0} \oplus \mathcal{V}_{2,0}$$
$$\mathcal{Q}_0 \cong \mathcal{V}_{0,0}$$
$$\mathcal{Q}_1 \cong \mathcal{V}_{2,2}$$
$$\mathcal{Q}_2 \cong \mathcal{V}_{0,4} \oplus \mathcal{V}_{4,0} \oplus \mathcal{V}_{4,4}$$
$$\mathcal{Q}_- \cong \mathcal{V}_{2,0} \oplus \mathcal{V}_{0,2} \oplus \mathcal{V}_{2,4} \oplus \mathcal{V}_{4,2}$$
$$\mathbf{K}_\Phi \cong \mathbb{R}^{90}. $$

The leaves in $\mathbf{K}_\Phi$ can have dimension no greater than $4^2 = 16$, but $P$ takes values in a sum of irreducible representations of dimensions five, five, and 25.

3. Secant submanifolds and Hydrodynamic Integrability

When studying PDEs, one often considers the question of integrability; that is, when can one construct "many" exact solutions making clever use of characteristics or conservation laws? This section is a summary of the approach that has been championed by Tsarev, Ferapontov and their many collaborators over the past two decades [Tsa90, Tsa93, Tsa00, FKS02, FK04a, FK04b, BFT08, FHK09, DF09]. (In fact, this article is the result of an effort to understand their approach and is a happy accident that this effort led to broader statements about general, non-integrable PDEs.) This section exists simply to demonstrate that $Od(G)$-structures provide a convenient geometric framework to investigate integrability; Part II will be dedicated to that investigation.

For scalar PDEs in three or more variables, a popular notion of integrability seems to be tied to the existence of hydrodynamic reductions.

**Definition 3.1** (Hydrodynamic Reduction). Consider a PDE $F^{-1}(0) \subseteq J^2(\mathbb{R}^n, \mathbb{R})$. A $k$-parameter hydrodynamic reduction for $F^{-1}(0)$ is a pair of maps $(R, Z)$ of the form

$$\mathbb{R}^n \xrightarrow{R} \mathbb{R}^k \xrightarrow{Z} J^2(\mathbb{R}^n, \mathbb{R})$$

such that

1. $R$ is a submersion, and $Z$ is an immersion;
2. $N = Z(\mathbb{R}^k)$ is a $k$-dimensional submanifold of $F^{-1}(0)$ and is an integral of the contact system on $J^2$;
3. for $l = 1, \ldots, k$ there exist functions $\lambda^l : \mathbb{R}^k \to \mathbb{R}^n$ such that $\frac{\partial}{\partial x^i} R^l = \lambda^l_i(R) \frac{\partial}{\partial x^i} R^l$;
4. there exist $\Gamma_b^l : \mathbb{R}^n \to \mathbb{R}$ such that $\frac{\partial}{\partial x^i} \lambda^l_i = (\lambda^l_i - \lambda^l_b) \Gamma^l_b$.

This definition is built upon the notion of constructing systems of conservation laws that foliate the hypersurface $F^{-1}(0) \subseteq J^2$. Condition (1) is a simple non-degeneracy assumption, for if $R$ were not a submersion, then one would simply reduce the dimension $k$ to match the image of $R$. 
Condition (2) essentially means that $N$ can be treated as an intermediate solution of $F$. Conditions (3) and (4) may seem cumbersome, but they are perfect for reducing $F = 0$ to a system of coupled first-order PDEs in the $\lambda$’s. In fact, conditions (3) and (4) are familiar from the definition of systems of conservation laws in $(1+1)$ variables that are rich or semi-Hamiltonian. The vectors $\{\lambda^1, \ldots \lambda^k\}$ can be interpreted as the characteristic speeds of a traveling wave within $F^{-1}(0)$. When these reductions exist, the reduced systems can be used to construct a solution to the original equation using the generalized hodograph method \cite{Tsa90}. If this can be done in “many” ways, then the PDE is called integrable.

Definition 3.2 (Integrability for PDEs). A PDE $F = 0$ in $n \geq 3$ independent variables is integrable if, for all $k = 1, \ldots, n$, there are infinitely many $k$-parameter hydrodynamic reductions of $F = 0$, and this collection is parametrized by $k(n - 2)$ functions of one variable.

In the references cited above, the number $k(n - 2)$ is expected to be the maximal possible Cartan character in the generic case. This parametrization will be discussed in greater detail in Part II; a more geometric definition is provided below.

Of course, these definitions can be extended in sensible ways for PDEs of higher order or in more dependent variables, but we focus on Hessian scalar PDEs here. In this case, the map $Z$ of a hydrodynamic reduction takes values in $N = Z(\mathbb{R}^k) \subset F^{-1}(0) \subset \Lambda^o$. Recall that $dU$ is a flat coframing on $\Lambda^o$ from Section \ref{sec:geometry}

Lemma 3.3 (e.g., \cite{FHK09}). Suppose $F = 0$ is a Hessian PDE in $n \geq 3$ variables such that $dF = \sum_{ij} \Phi(U)_{ij} a(U)_{ij}$ for a flat $(V \odot V)$-valued coframe $a = a^\top$ on $\Lambda^0$. Let $(R, Z)$ be a $k$-parameter hydrodynamic reduction of $F = 0$ with $N = Z(\mathbb{R}^k)$. Then $T \mathcal{N}$ is everywhere spanned by $k$ tangent vectors $\{A^1, \ldots, A^k\}$ such that $a(A^l)$ lies in the intersection of the hyperplane $\Phi(U)^\perp \subset V \odot V$ with the Veronese cone in $V \odot V$.

Proof. Following \cite{FHK09}, we observe that (if $Z_{ij}$ is to be the Hessian matrix of a smooth function $z$)

$$
\frac{\partial Z_{ij}}{\partial x^k} = \frac{\partial Z_{ik}}{\partial x^j} = \frac{\partial Z_{jk}}{\partial x^i} 
$$

which implies that

$$0 = \frac{\partial Z_{ij}}{\partial x^1} - \frac{\partial Z_{ij}}{\partial x^2} = \sum_l \frac{\partial Z_{ij}}{\partial R^l} \frac{\partial R^l}{\partial x^1} - \sum_l \frac{\partial Z_{ij}}{\partial R^l} \frac{\partial R^l}{\partial x^2} = \sum_l \left( \frac{\partial Z_{ij}}{\partial R^l} - \frac{\partial Z_{ij}}{\partial R^l} \lambda^l_j \right) \frac{\partial R^l}{\partial x^1}.
$$

Therefore, because $R(x)$ is not constant, we have

$$
\frac{\partial}{\partial R^l} a dZ_{ij} = \frac{\partial Z_{ij}}{\partial R^l} = \frac{\partial Z_{ij}}{\partial R^l} \lambda^l_i \lambda^l_j.
$$

Thus, the image of $dZ$ is a rank-one symmetric matrix in a flat coframe.

Let $v^l = \lambda^l \circ Z^{-1}$ for $l = 1, \ldots, k$ and $A^l = (v^l)^\top v^l$.

Thus, hydrodynamic reductions and integrability are intimately tied to the Veronese variety and its intersection with $T(F^{-1}(0)) = \ker dF$. With this observation in mind, there are obvious analogous notions for $Od(G)$-structures. These definitions are designed to admit analysis using the Cartan–Kähler theorem.

Definition 3.4 (Secants). Suppose $\mathcal{B} \to M$ is an $Od(G)$-structure (not necessarily embeddable). A $k$-dimensional subspace $E^k \subset \mathcal{T}_p M$ is called $k$-secant if there exists $b \in \mathcal{B}_p$ such that $b(E) \subset \mathcal{S}$ is the span of $\{A^1, \ldots, A^k\}$ such that each $A^l$ is a symmetric rank-one matrix. That is, $A^l = \text{ver}_2(v^l) = (v^l)^\top (v^l)$ and $v^l \Phi(v^l)^\top = 0$. A $k$-dimensional submanifold $N \subset M$ is called $k$-secant if the sub-tangent space $\mathcal{T}_p N$ is $k$-secant for all $p \in N$.
Since the null vectors \( v^1, \ldots, v^k \) are independent, any \( k \)-secant subspace \( E \) contains exactly \( k \) distinct \( (k-1) \)-secant subspaces.

**Definition 3.5** (Integrability for Structures). An \( Od(G) \)-structure \( B \to M \) is called \( k \)-integrable if, for any \( k \)-secant subspace \( E \subset T_pM \), there exists a \( k \)-secant submanifold \( N \) with \( T_pN = E \). If \( B \to M \) is an example, 2-integrable and 3-integrable, this property is abbreviated as 2,3-integrable. “Integrable” is shorthand for “2, \ldots, \( n \)-integrable.”

Note that 1-integrability always holds, as it describes the existence of the flow of a vector field.

**Theorem 3.6** (2-Integrability). For any \( n \), every hyperbolic, embeddable \( Od(G) \)-structure \( B \to M \) is 2-integrable. In this case, The solution 2-secant submanifolds are locally parametrized by \( 2(\text{functions of one variable}) \). This is true in the smooth category.

**Proof.** Consider the hyperbolic bilinear form \( x_1x_n = \sum_{k=2}^{n-1}(x_k)^2 \), which corresponds to the symmetric matrix
\[
\Phi = \begin{pmatrix} 0 & \cdots & -1/2 \\ \vdots & I_{n-2} & \\ -1/2 & \cdots & 0 \end{pmatrix}
\]

Suppose \( E \subset T_pM \) is a bi-secant plane. In a neighborhood of \( p \), choose a section \( b : M \to B \). It must be that \( b(E) \) is spanned by two rank-one symmetric matrices that are in the Veronese image of null vectors. For any two null vectors in \( \tilde{V} \), there exists an element of \( G \) that moves these null vectors to the null vectors \( y^1 = (1, 0, \ldots, 0) \) and \( y^2 = (0, \ldots, 0, 1) \). So, one may use an \( Od(G) \) frame adaptation to assume that \( b(E) \) is the span of \( Y^1 = \text{ver}2(1, 0, \ldots, 0) \) and \( Y^2 = \text{ver}2(0, \ldots, 0, 1) \).

To prove the theorem, one must find the conditions on \( B \) that allow an arbitrary bi-secant plane \( E \subset Gr_2(T_pM) \) to be extended to a bi-secant surface \( N \subset M \). Let \( \tilde{E} = b_*(E) \) denote the “lifted” image of \( E \) in \( T_{b(p)}B \). Then \( \omega_1 \wedge \omega_n|_E \neq 0 \) and the annihilator of \( \tilde{E} \) is \( \{ \omega_{ij}, (i, j) \neq (1, 1), (n, n) \} \). Let \( \mathcal{I} \) denote the differential ideal generated by these 1-forms with the independence condition \( \omega_1 \wedge \omega_{nn} \). It suffices to prove the existence of integral manifolds of this differential ideal.

Recall that \( \omega_{ij} = \omega_{ji} \) and \( \omega_{1n} = \sum_{k=2}^{n-1} \omega_{kk} \). By the first-order structure equations, \( d\omega = (\theta + \tau)^\top \wedge \omega - \omega \wedge (\theta + \tau) \), it is clear that \( d\omega_{ij} \equiv 0 \) unless \( i \) or \( j \) equals 1 or \( n \). Also, \( \omega_{1n} = \omega_{22} + \cdots + \omega_{n-1,n-1} \) implies that \( d\omega_{1n} \equiv 0 \). Thus, the differential generators are
\[
d\left( \frac{\omega_1}{\omega_{in}} \right) \equiv \begin{pmatrix} \theta_{1i} + \tau_{1i} & 0 \\ 0 & \theta_{ni} + \tau_{ni} \end{pmatrix} \wedge \begin{pmatrix} \omega_{11} \\ \omega_{nn} \end{pmatrix}, \quad i = 2, \ldots, n-1.
\]

The \( \theta \)'s that appear here are independent, so Cartan’s test shows that the system is involutive with solutions depending on \( s_1 = 2(n-2) \) functions of one variable. Moreover, this is a hyperbolic linear Pfaffian system in the sense of [Yan87], so the Cartan–Kähler theorem applies in the \( C^\infty \) category.

**Theorem 3.6** is already well-known from the perspective of PDEs, but its proof provides a model for how to approach \( k \)-integrability in general. Either of Corollary 2.14 or Theorem 3.6 shows why hydrodynamic integrability is a trivial concept for PDEs in \( n = 2 \) variables. The case of integrability for hyperbolic Hessian PDEs in \( n = 3 \) variables is detailed in [DF09] and [Smi10]. The case of integrability (equivalently, 2,3-integrability) for symplectic Monge–Ampère equations in \( n = 4 \) variables is detailed in [DF09]. In both cases, the geometry induced from the symmetries of the Veronese cone is used to classify the integrable equations. Thus, the extrinsic geometry is tied to the \( Od(GL(n)) \)-structure over \( \Lambda^n \) and the intrinsic geometry is tied to the induced \( Od(G) \)-structure on \( F^{-1}(0) \). The mostly-open case of \( k \)-integrability for \( k \geq 3 \) and \( n \geq 3 \) will be studied in Part II.
4. Conclusion

A Hessian partial differential equation \( M = F^{-1}(0) \subset \text{Sym}^2(\mathbb{R}^n) \) of any fixed non-degenerate signature in any number, \( n \), of variables admits a geometry, called an embeddable \( Od(G) \)-structure, with structure equations that are complete at second-order. The natural notion of integrability from PDE theory translates to a natural notion of geometric integrability for these structures. Up to conformal factors, the fiber group of this \( Od(G) \)-structure is a subgroup of the orthogonal group \( O(n(n+1)/2-1, \langle \cdot, \cdot \rangle) \) for a pseudo-Riemannian structure on \( M \), and it is a representation of the orthogonal group \( O(n, dF) \). The structure equations are easy to write down in any dimension, and the structure functions take values in a finite-dimensional classifying space \( K_{\Phi} \) that is given by the 1-jet of a first-order invariant \( P \in \mathcal{Q}_2 \) along with a scalar curvature \( r \) that reflects the conformal factor of the pseudo-Riemannian structure.

Using the standard theory of Lie pseudo-groups of finite type, several conclusions can be drawn immediately. To each point in \( K_{\Phi} \) there is an associated Hessian PDE, which is locally unique. There is a singular foliation of \( K_{\Phi} \) that separates connected embeddable \( Od(G) \)-structures into equivalence classes, and the corresponding moduli space depends only on the signature of the leading symbol of \( F \). Because the leaves of \( K_{\Phi} \) have high co-dimension in the case \( n \geq 3 \), there are infinitely-many such equivalence classes.

Moreover, because any \( CSp(n) \)-invariant classification of non-degenerate Hessian PDEs induces a contact-invariant classification of second-order PDEs that have locally non-degenerate leading symbol, the moduli space of Hessian PDEs defined by the singular foliation of \( K_{\Phi} \) also provides a classification of all such second-order PDEs; however, the full contact-invariant classification of second-order PDEs will be much finer in general.

While these are standard results from the theory of Lie pseudo-groups of finite type, the explicit description of these structures should aid the investigation of integrability and help to explain the increasing complexity of PDEs in high dimensions and help identify special sub-classes of PDEs.
Appendix A. Scaling and the $O_d$ action

Let $V$ be a vector space of dimension $n$ over $\mathbb{R}$ or $\mathbb{C}$ with the standard basis. Let $V \odot V$ be identified with the vector space of symmetric $n \times n$ matrices.

For any subgroup $PH$ of $PGL(n)$, define the representation of $P\!Od$ on $\mathbb{P}(V \odot V)$ by $P\!Od_h(A) = [h^\top Ah]$. This representation is faithful.

For $g \in GL(n)$ and $A$ a symmetric matrix, consider the action $O_d(A) = g^\top Ag$. This action does not describe a faithful representation, as $O_d = O_d^{-1}$; however, the infinitesimal action describes a faithful representation of $gl(n)$:

**Lemma A.1.** Define $od : gl(n) \to gl(V \odot V)$ by $od_X(A) = X^\top A + AX$. Then $\ker od = 0$.

*Proof.* Suppose that $X \in gl(n)$ is such that $0 = X^\top A + AX$ for all symmetric matrices $A$. If $A$ is invertible, then $A$ represents a non-degenerate symmetric bilinear form, and $X^\top$ is a matrix that is skew with respect to $A$. Therefore, $X^\top$ lies in the intersection over all Lie algebras of the form $so(n, A) = A \cdot so(n)$. \qed

Over $\mathbb{C}$, the $O_d$ action for $g \in GL(n)$ is transitive on the Veronese cone in $\text{Sym}^2(\mathbb{C}^n)$. However, over $\mathbb{R}$ the scaling by $-1$ is never possible through the $O_d$ action, as $O_d(A) = \lambda^2 A$. This annoyance can be dealt with in three ways.

First, one could consider only the action of $[g] \in PGL(n)$, and then the target $[g^\top Ag]$ is a representative in $PGL(n(n+1)/2)$ without ambiguity. This representation, called $P\!Od(PGL(n))$, is faithful; however, it disguises the structure equations that arise in the equivalence problem.

Alternatively, observe that the infinitesimal action $od_X$ actually does allow for arbitrary scalings, as $X = -\frac{1}{2} \lambda I$ shows. So, define the group $O_d(GL(n))$ as the collection of actions $\{A \mapsto \lambda g^\top Ag, g \in GL(n), \lambda = \pm 1\}$. This group $O_d(GL(n))$ is not really a representation of $GL(n)$, rather it can be described as the semi-direct product of the faithful representation $P\!Od(PGL(n))$ and the one-dimensional scaling group $\mathbb{R}^\times$. Note that the Lie algebra of $O_d(GL(n))$ is isomorphic to the Lie algebra of $gl(n)$, though the scaling action is halved.

Another group of particular interest in this article is $G = CO(n, \Phi)$ for a non-degenerate symmetric bilinear form $\Phi$. The group $O_d(G)$ is defined analogously as $\{A \mapsto \lambda g^\top Ag, g \in CO(n, \Phi), \lambda = \pm 1\}$. The Lie algebra of this group is $\{A \mapsto X^\top A + AX + \lambda A, X \in so(n, \Phi), \lambda \in \mathbb{R}\}$. Of course, this is isomorphic to $g = so(n, \Phi) + \mathbb{R}I$. Thus, when considering irreducible $O_d(G)$-modules, one needs to examine the irreducible representations of $so(n, \Phi)$. Finally, if both of these approaches are distasteful, then one could simply pretend that this is a study of non-degenerate Hessian PDEs over the complex numbers and forget this detail entirely.
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