On the construction of Fermi-Walker transported frames

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Abstract

We consider tetrad fields as reference frames adapted to observers that move along arbitrary timelike trajectories in spacetime. By means of a local Lorentz transformation we can transform these frames into Fermi-Walker transported frames, which define a standard of non-rotation for accelerated observers. Here we present a simple prescription for the construction of Fermi-Walker transported frames out of an arbitrary set of tetrad fields.

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1 Introduction

A large class of reference frames in spacetime experience inertial forces, and therefore constitute noninertial frames. For instance, the frame adapted to an observer “at rest” on the surface of the (rotating) Earth is noninertial. Observers that follow arbitrary timelike trajectories in spacetime will regard as natural a reference frame in which they are at rest, and their spatial axes do not rotate. These observers will carry with them a set of tetrad fields such that the timelike component is always tangent to the trajectory $C$, and the three spacelike components are normal to the observers’ worldlines. However, if we consider a single timelike worldline $C$, in general the standard parallel transport of a tangent $A^\mu$ at a point $x^\mu$ does not lead to a tangent vector at the point $x^\mu + dx^\mu$ on $C$. The trajectory $C$ in general is not geodesic. The transport of vectors on $C$ that lead tangent vectors into tangent vectors is realized by the Fermi-Walker transport [1]. The latter describes the appropriate evolution in spacetime of noninertial frames, since the timelike component of the tetrad field is transported into itself, and is always tangent to the worldline $C$ of the observer. A Fermi-Walker transported set of tetrad fields is the best approximation to a nonrotating reference frame in the sense of Newtonian mechanics. It is physically realized by a system of gyroscopes.

Fermi-Walker transported frames are important in several investigations. A frame that undergoes linear and rotational acceleration may be described by the Frenet-Serret frame. The relative rotational acceleration of a Frenet-Serret frame with respect to a Fermi-Walker transported frame is taken to characterize important phenomena, like the gyroscopic precession [2]. Noninertial reference frames in Minkowski spacetime that undergo Fermi-Walker transport are useful, for instance, in the analysis of the inertial effects on a Dirac particle [3]. Fermi-Walker frames have been used in ref. [4] in the study of the geodetic and Lense-Thirring precessions, and in the analysis of gravitational wave resonant detectors.

A procedure for obtaining Fermi-Walker frames in the Kerr spacetime has been worked out in ref. [5]. The method is based on the property of separability of the Hamilton-Jacobi equation for the geodesics of vacuum solutions of Petrov type D (including the Kerr solution), and depends on the existence of a Killing-Yano tensor that satisfies some specific properties [6].

In this paper we present a simple prescription for obtaining a Fermi-Walker transported frame out of any set of tetrad fields, for an arbitrary
spacetime, assuming that the frame is transported along a timelike trajectory. The mechanism consists in finding suitable coefficients for a local Lorentz transformation of the spatial sector of the tetrad field. Although the resulting equation for these coefficients is simple, in practice it may not be straightforward to solve it in the general case.

This paper is organized as follows. In section 2 we present the Frenet-Serret equations and the definition of the Fermi-Walker transport. In section 3 we recall the interpretation of tetrad fields as reference frames adapted to a field of observers in spacetime, and define the acceleration tensor. This tensor determines the inertial forces that act on the frame, and therefore may be taken to characterize the latter. In section 4 we show that the vanishing of certain components of the acceleration tensor implies that the frame is Fermi-Walker transported. The vanishing of these components is achieved by means of a local Lorentz transformation. We find the equation that the coefficients of the local Lorentz transformation must satisfy in order to obtain the Fermi-Walker transported frame. In section 5 we make a simple application of the results of section 4 to the determination of Fermi-Walker transported frames in Kerr spacetime.

Notation: spacetime indices $\mu, \nu, ...$ and $\text{SO}(3,1)$ indices $a, b, ...$ run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i,\ a = (0), (i)$. The tetrad field is denoted by $e^a_\mu$, and the object of anholonomy reads $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$. The flat, Minkowski spacetime metric tensor raises and lowers tetrad indices and is fixed by $\eta_{ab} = e^a_\mu e^b_\nu g^{\mu\nu} = (-+++)$.

2 The Frenet-Serret equations and the Fermi-Walker transport

In this section we will adopt the notation of ref. [1] for the vector quantities. The absolute derivative of a vector $V^\mu$ along a worldline $C$ is written as

$$\frac{DV^\mu}{ds} = \frac{dV^\mu}{ds} + \Gamma^\mu_{\alpha\beta} V^\alpha \frac{dx^\beta}{ds},$$

where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols. Let us consider four vectors, $A^\mu, B^\mu, C^\mu$ and $D^\mu$ that satisfy the following equations and properties:
\[
\frac{DA^\mu}{ds} = bB^\mu, \\
\frac{DB^\mu}{ds} = cC^\mu + bA^\mu, \\
\frac{DC^\mu}{ds} = dD^\mu - cB^\mu, \\
\frac{DD^\mu}{ds} = -dC^\mu,
\]

(2) \hspace{1cm} (3) \hspace{1cm} (4) \hspace{1cm} (5)

where \( A^\mu A_\mu = -1 \) and \( B^\mu B_\mu = C^\mu C_\mu = D^\mu D_\mu = 1 \), and \( b, c, d \) are nonnegative coefficients. Given the vector \( A^\mu \), eq. (2) defines \( B^\mu \), eq. (3) defines \( C^\mu \) and eq. (4) defines \( D^\mu \). Equation (5) is verified in view of eqs. (2) to (4). It is not difficult to verify that eqs. (2) to (5) imply that \( A^\mu, B^\mu, C^\mu \) and \( D^\mu \) form an orthonormal set of vectors, i.e., \( A^\mu B_\mu = 0 \), etc.

We identify \( A^\mu \) as the unit vector tangent to the trajectory \( C \), \( A^\mu = dx^\mu/ds \). In this case \( A^\mu, B^\mu, C^\mu \) and \( D^\mu \) establish the Frenet-Serret frame, and eqs. (2-5) are called the Frenet-Serret equations. \( B^\mu, C^\mu \) and \( D^\mu \) are the first, second and third normals to \( C \), and \( b, c, d \) are the first, second and third curvatures of \( C \), respectively [1]. The Frenet-Serret orthonormal basis is suitably adapted to special curves in spacetime. For instance, if \( b = c = d = 0 \), the curve \( C \) is a geodesic; if \( b = \text{constant} \) and \( c = d = 0 \), \( C \) represents a hyperbola, and if \( b = \text{constant}, c = \text{constant} \) and \( d = 0 \), then \( C \) is a helix [1].

Let us consider a vector \( F^\mu(x) \) defined on the timelike trajectory \( C \), in a spacetime determined by the metric tensor \( g_{\mu\nu} \). The Fermi-Walker transport of \( F^\mu \) on \( C \) is defined by [1]

\[
\frac{DF^\mu}{ds} = bF_\alpha(A^\mu B^\alpha - A^\alpha B^\mu).
\]

(6)

Given the value \( F^\mu(s_0) \) at a certain initial position \( s_0 \), eq. (6) formally determines \( F^\mu \) along the curve \( C \) determined by \( x^\mu = x^\mu(s) \). The Fermi-Walker transport of a second rank tensor along \( C \) is defined by

\[
\frac{DT^{\mu\nu}}{ds} = bT_\alpha^\nu(A^\mu B^\alpha - A^\alpha B^\mu) + bT_\nu^\mu(A^\nu B^\alpha - A^\alpha B^\nu).
\]

(7)

It follows from the equation above that
\[
\frac{Dg^{\mu\nu}}{ds} = 0 = \frac{D\delta^\mu_\nu}{ds}.
\] (8)

The unit velocity vector \( A^\mu = dx^\mu/\,ds \) naturally undergoes Fermi-Walker transport. Application of eq. (6) to \( A^\mu \) leads to eq. (2). It is also easy to show that the scalar product of two vectors is preserved under the Fermi-Walker transport. Let \( \phi \) represent the scalar product of the vectors \( \Sigma^\mu \) and \( \Lambda_\mu \). Along \( C \) we have

\[
\phi(s + ds) - \phi(s) = \Sigma^\mu(s + ds)\Lambda_\mu(s + ds) - \Sigma^\mu(s)\Lambda_\mu(s)
= \Lambda_\mu(\delta^{FW}\Sigma^\mu) + \Sigma^\mu(\delta^{FW}\Lambda_\mu),
\] (9)

where

\[
\delta^{FW}\Sigma^\mu = -\Gamma^\mu_{\alpha\beta}\Sigma^\beta dx^\alpha + b\Sigma_\alpha(A^\mu B^\alpha - A^\alpha B^\mu)ds,
\]
\[
\delta^{FW}\Lambda_\mu = \Gamma^\mu_{\alpha\mu}\Lambda_\lambda dx^\alpha + b\Lambda_\alpha(A_\mu B^\alpha - A^\alpha B_\mu)ds.
\] (10)

Equations (9) and (10) imply that \( \phi(s + ds) - \phi(s) = 0 \).

We identify the velocity vector on \( C \) with the timelike component of the tetrad field \( e^{(0)}_\mu \),

\[
A^\mu = \frac{dx^\mu}{ds} = e^{(0)}_\mu.
\] (11)

The Fermi-Walker transport of \( e^{(0)}_\mu \) along \( C \) guarantees that \( e^{(0)}_\mu \) will always be tangent to \( C \). The spacelike components \( e^{(k)}_\mu \) are everywhere orthogonal to \( e^{(0)}_\mu \). Along \( C \), \( e^{(k)}_\mu \) also undergoes Fermi-Walker transport. Since \( e^{(k)}_\mu \) and \( e^{(0)}_\mu \) are orthogonal, we have

\[
\frac{De^{(k)}_\mu}{ds} = be^{(0)}_\mu e^{(k)}_\lambda B_\lambda.
\] (12)

In view of eq. (2) the equation above may be rewritten as

\[
\frac{De^{(k)}_\mu}{ds} = e^{(0)}_\mu e^{(k)}_\lambda \frac{De^{(0)}_\lambda}{ds}.
\] (13)

This equation determines the transport of the orthonormal basis \( e_a^\mu \) along an arbitrary timelike curve \( C \), such that \( e^{(0)}_\mu \) is always tangent to \( C \).
3 The tetrad field as a reference frame and the acceleration tensor

In this section we will recall the discussion presented in ref. [7] regarding the characterization of tetrad fields as reference frames in spacetime. A frame may be characterized in a coordinate invariant way by its inertial accelerations, represented by the acceleration tensor.

The notation will be slightly different from the previous section. The vector $A^\mu = dx^\mu/ds$ on a curve $C$ will be denoted here by the standard notation $u^\mu$. Thus the velocity vector of an observer on $C$ reads $u^\mu = dx^\mu/ds$. We identify the observer’s velocity with the $a = (0)$ component of $e_a^\mu$: $u^\mu(s) = e_{(0)}^\mu$. The observer’s acceleration $a^\mu$ is given by the absolute derivative of $u^\mu$ along $C$ [8],

$$a^\mu = \frac{Du^\mu}{ds} = \frac{De_{(0)}^\mu}{ds} = u^\alpha \nabla_\alpha e_{(0)}^\mu,$$

(14)

where the covariant derivative is constructed out of the Christoffel symbols. Thus $e_a^\mu$ determines the velocity and acceleration along the worldline of an observer adapted to the frame. The set of tetrad fields for which $e_{(0)}^\mu$ describes a congruence of timelike curves is adapted to a class of observers characterized by the velocity field $u^\mu = e_{(0)}^\mu$ and by the acceleration $a^\mu$.

We may consider not only the acceleration of observers along trajectories whose tangent vectors are given by $e_{(0)}^\mu$, but the acceleration of the whole frame along $C$. The acceleration of the frame is determined by the absolute derivative of $e_a^\mu$ along the path $x^\mu(s)$. Thus, assuming that the observer carries an orthonormal tetrad frame $e_a^\mu$, the acceleration of the latter along the path is given by [9]

$$\frac{De_a^\mu}{ds} = \phi_{ab}^\mu e_b^\mu,$$

(15)

where $\phi_{ab}$ is the antisymmetric acceleration tensor. According to ref. [9], in analogy with the Faraday tensor we can identify $\phi_{ab} \rightarrow (a, \Omega)$, where $a$ is the translational acceleration ($a_{(0)(i)} = a_{(i)}$) and $\Omega$ is the angular velocity of the local spatial frame with respect to a nonrotating (Fermi-Walker transported) frame. It follows from Eq. (11) that
\[ \phi_{a}^{b} = e_{\mu}^{b} \frac{De_{a}^{\mu}}{ds} = e_{\mu}^{b} u^{\lambda} \nabla_{\lambda} e_{a}^{\mu}. \]  

(16)

Therefore given any set of tetrad fields for an arbitrary gravitational field configuration, its geometrical interpretation may be obtained by suitably interpreting the velocity field \( u^\mu = e_{(0)}^\mu \) and the acceleration tensor \( \phi_{ab} \). The acceleration vector \( a^\mu \) defined by Eq. (14) may be projected on a frame in order to yield

\[ a^{b} = e_{\mu}^{b} a^\mu = e_{\mu}^{b} u^{\alpha} \nabla_{\alpha} e_{(0)}^{\mu} = \phi_{(0)}^{b}. \]  

(17)

Thus \( a^\mu \) and \( \phi_{(0)i} \) are not different accelerations of the frame.

The acceleration \( a^\mu \) given by Eq. (14) may be rewritten as

\[ a^\mu = u^{\alpha} \nabla_{\alpha} e_{(0)}^{\mu} = u^{\alpha} \nabla_{\alpha} u^\mu = \frac{dx^\alpha}{ds} \left( \frac{\partial u^\mu}{\partial x^\alpha} + \Gamma^{\mu}_{\alpha\beta} u^\beta \right) \]

\[ = \frac{d^2 x^{\mu}}{ds^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}, \]  

(18)

where \( \Gamma^{\mu}_{\alpha\beta} \) are the Christoffel symbols. Thus if \( u^\mu = e_{(0)}^\mu \) represents a geodesic trajectory, then the frame is in free fall and \( a^\mu = 0 = \phi_{(0)i} \). Therefore we conclude that nonvanishing values of \( \phi_{(0)i} \) represent inertial accelerations of the frame.

Following ref. [7], we take into account the orthogonality of the tetrads and write Eq. (16) as \( \phi_{a}^{b} = -u^{\lambda} e_{a}^{\mu} \nabla_{\lambda} e_{b}^{\mu} \), where \( \nabla_{\lambda} e_{b}^{\mu} = \partial_{\lambda} e_{b}^{\mu} - \Gamma^{\sigma}_{\lambda\mu} e_{b}^{\sigma} \). Now we consider the identity \( \partial_{\lambda} e_{b}^{\mu} - \Gamma^{\sigma}_{\lambda\mu} e_{b}^{\sigma} + {}^{0} \omega_{b}^{c} e_{c}^{\mu} = 0 \), where \( {}^{0} \omega_{b}^{c} e_{c}^{\mu} \) is the metric compatible, torsion-free Levi-Civita connection, and express \( \phi_{a}^{b} \) according to

\[ \phi_{a}^{b} = e_{(0)}^{\mu} ( {}^{0} \omega_{\mu}^{b} a ) . \]  

(19)

Finally we take into account the identity \( {}^{0} \omega_{\mu}^{a} b = -K_{\mu}^{a} b \), where \( -K_{\mu}^{a} b \) are the Ricci rotation coefficients defined by

\[ K_{\mu ab} = \frac{1}{2} e_{\alpha}^{c} e_{b}^{\nu} ( T_{\lambda\mu\nu} + T_{\nu\lambda\mu} + T_{\mu\lambda\nu} ) , \]  

(20)

and \( T_{\lambda\mu\nu} = e_{\lambda}^{a} T_{a\mu\nu} \). After simple manipulations we arrive at
\[ \phi_{ab} = \frac{1}{2} [T_{(0)ab} + T_{a(0)b} - T_{b(0)a}] . \]  \hfill (21)

The expression above is not invariant under local SO(3,1) transformations, and for this reason the values of \( \phi_{ab} \) may characterize the frame. However, eq. (21) is invariant under coordinate transformations. We interpret \( \phi_{ab} \) as the inertial accelerations along the trajectory \( C \).

4 Construction of Fermi-Walker frames

Let us consider the expression for the Fermi-Walker transport of tetrad fields given by eq. (13). Taking into account eq. (14) we may rewrite (13) as

\[ \frac{De^{(k)}_{\mu}}{ds} = u^\mu e^{(k)}_{\lambda} a^\lambda. \]  \hfill (22)

In view of (17) we have \( e^{(k)}_{\lambda} a^\lambda = a^{(k)} = \phi^{(0)(k)} \). Therefore the Fermi-Walker transport of a frame may be written as

\[ \frac{De^{(k)}_{\mu}}{ds} = u^\mu \phi^{(0)(k)} . \]  \hfill (23)

On the other hand, it is easy to verify that the total acceleration of the frame components \( e^{(k)}_{\mu} \) given by eq. (15) may be expressed in terms of \( \phi^{(0)(k)} \) and \( \phi^{(j)(k)} \) as

\[ \frac{De^{(k)}_{\mu}}{ds} = u^\mu \phi^{(0)(k)} + \phi^{(k)}_{(j)} e^{(j)}_{\mu} . \]  \hfill (24)

Therefore if \( \phi^{(j)(k)} = 0 \), the frame is Fermi-Walker transported, in agreement with the discussion after eq. (15). It turns out that we can easily require \( \phi^{(j)(k)} = 0 \), at least formally.

The expression of \( \phi^{(i)(j)} \) is given by

\[ \phi^{(i)(j)} = \frac{1}{2} \left[ e^{(i)}_{\mu} e^{(j)}_{\nu} T_{(0)\mu\nu} + e^{(0)}_{\mu} e^{(j)}_{\nu} T_{(i)\mu\nu} - e^{(0)}_{\mu} e^{(i)}_{\nu} T_{(j)\mu\nu} \right] . \]  \hfill (25)

We perform a local Lorentz rotation,

\[ \tilde{e}^{(i)}_{\mu} = \Lambda^{(i)}_{(k)} e^{(k)}_{\mu} , \]  \hfill (26)
that yields

\[
\tilde{T}_{(i)\mu\nu} = \partial_\mu \tilde{e}_{(i)\nu} - \partial_\nu \tilde{e}_{(i)\mu} = \Lambda_{(i)}^{(k)} T_{(k)\mu\nu} + [\partial_\mu \Lambda_{(i)}^{(k)}] e_{(k)\nu} - [\partial_\nu \Lambda_{(i)}^{(k)}] e_{(k)\mu}.
\]  

(27)

The local Lorentz coefficients \(\{\Lambda_{(i)}^{(j)}\}\) will be fixed such that \(\tilde{\phi}_{(i)(j)} = 0\).

Equations (26) and (27) imply

\[
\tilde{\phi}_{(i)(j)} = \frac{1}{2} \left\{ e^{(0)\mu} \Lambda_{(j)}^{(k)} \right\} \left[ e^{\nu(k)} e_{(i)(l)} - e^{\nu(l)} e_{(i)(k)} \right] + \Lambda_{(i)}^{(k)} \phi_{(k)(l)}^{(j)}
\]  

\[
+ \frac{1}{2} \left\{ e^{(0)\mu} \Lambda_{(i)}^{(k)} \right\} \left[ e^{\nu(k)} e_{(i)(l)} - e^{\nu(l)} e_{(i)(k)} \right] \right\}
\]  

(28)

The equation above may be written in a more convenient way as

\[
\tilde{\phi}_{(i)(j)} = \frac{1}{2} \left[ \Lambda_{(j)}^{(m)} \left( \Lambda_{(i)}^{(k)} \phi_{(k)(m)} + e^{(0)\mu} \partial_\mu \Lambda_{(i)(m)} \right) \right.
\]

\[
- \Lambda_{(i)}^{(m)} \left( \Lambda_{(j)}^{(k)} \phi_{(k)(m)} + e^{(0)\mu} \partial_\mu \Lambda_{(j)(m)} \right) \right].
\]

(29)

Therefore we obtain \(\tilde{\phi}_{(i)(j)} = 0\) if

\[
\Lambda_{(i)}^{(k)} \phi_{(k)(m)} + e^{(0)\mu} \partial_\mu \Lambda_{(i)(m)} = 0,
\]

(30)

or, equivalently

\[
e^{(0)\mu} \Lambda_{(j)}^{(m)} \partial_\mu \Lambda_{(j)(k)} - \phi_{(k)(m)} = 0.
\]

(31)

The equation above is the main result of the paper: given an arbitrary frame transported along \(e^{(0)\mu}\), we calculate the angular velocity \(\phi_{(k)(m)}\), and by means of eq. (31) we determine the coefficients \(\{\Lambda_{(i)}^{(j)}\}\). Equation (31)
ensures that the frame obtained according to eq. (26) is Fermi-Walker transported. We note that the local Lorentz rotation does not affect the timelike component $e_{(0)}^\mu$.

For an arbitrary set of tetrad fields it may not be straightforward to solve eq. (31) for $\{\Lambda_{(i)}^{(j)}\}$. However, in the weak field approximation, or in case the spacetime is asymptotically flat and we have

$$e^a_{\mu} \approx \delta^a_{\mu} + \frac{1}{2} h^a_{\mu},$$

we may find the approximate expression of $\Lambda_{(i)}^{(j)}$. In this case we may write

$$\Lambda_{(j)(k)} \approx \eta_{(j)(k)} + \varepsilon_{(j)(k)}.$$  

We assume that both $h^a_{\mu}$ and $\varepsilon_{(j)(k)}$ are of order $\epsilon$, where $\epsilon << 1$. Then $\phi_{(j)(k)}$ is also of order $\epsilon$. In this approximation we have $e_{(0)}^\mu \partial_{\mu} \approx \partial_0 = \partial/\partial t$. Under these conditions we may solve eq. (31) and obtain

$$\Lambda_{(j)(k)} = \begin{pmatrix} 1 & \varepsilon_{(1)(2)} & \varepsilon_{(1)(3)} \\ -\varepsilon_{(1)(2)} & 1 & \varepsilon_{(2)(3)} \\ -\varepsilon_{(1)(3)} & -\varepsilon_{(2)(3)} & 1 \end{pmatrix},$$

where

$$\dot{\varepsilon}_{(j)(k)} = \dot{\phi}_{(k)(j)}.$$  

The dot denotes time derivative.

5 Fermi-Walker frames in the Kerr spacetime

Although the solution of eq. (31) in the general case (for an arbitrary frame) is not always feasible, in certain situations we obtain simple and interesting results. We will consider here the frame addressed in ref. [7] that describes static observers in the Kerr spacetime, and construct Fermi-Walker transported frames in the equatorial plane $\theta = \pi/2$.

The Kerr spacetime is determined by the line element

$$ds^2 = -\frac{\psi^2}{\rho^2} dt^2 - \frac{2 \chi \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} d\tau^2$$
\[ + \rho^2 d\theta^2 + \frac{\Sigma^2 \sin^2 \theta}{\rho^2} d\phi^2, \]  
\[ (36) \]

with the following definitions:

\[
\begin{align*}
\Delta &= r^2 + a^2 - 2mr, \\
\rho^2 &= r^2 + a^2 \cos^2 \theta, \\
\Sigma^2 &= (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta, \\
\psi^2 &= \Delta - a^2 \sin^2 \theta, \\
\chi &= 2amr. 
\end{align*}
\]
\[ (37) \]

A static reference frame in Kerr’s spacetime is defined by the congruence of timelike curves \( u^\mu(s) \) such that \( u^i = 0 \), i.e., the spatial velocity of the observers is zero with respect to static observers at spacelike infinity. Since we identify \( u^i = e_{(0)}^i \), a static reference frame is established by the condition

\[ e_{(0)}^i = 0. \]
\[ (38) \]

In view of the orthogonality of the tetrads, the equation above implies \( e^{(k)}_0 = 0 \). This latter equation remains satisfied even after a local rotation of the frame, \( \tilde{e}^{(k)}_0 = \Lambda^{(k)}_{(j)} e^{(j)}_0 = 0 \). Therefore condition (38) determines the static character of the frame, up to an orientation of the frame in the three-dimensional space.

A simple form for the tetrad field that satisfies Eq. (38) reads

\[
e_{\alpha\mu} = 
\begin{pmatrix}
-A & 0 & 0 & -B \\
0 & C \sin \theta \cos \phi & \rho \cos \theta \cos \phi & -D \sin \theta \sin \phi \\
0 & C \sin \theta \sin \phi & \rho \cos \theta \sin \phi & D \sin \theta \cos \phi \\
0 & C \cos \theta & -\rho \sin \theta & 0
\end{pmatrix},
\]
\[ (39) \]

with the definitions

\[
A = \frac{\psi}{\rho}, \\
B = \frac{\chi \sin^2 \theta}{\rho \psi},
\]
\[ C = \frac{\rho}{\sqrt{\Delta}}, \quad D = \frac{\Lambda}{\rho \psi}, \] (40)

and \( \Lambda = (\psi^2 \Sigma^2 + \chi^2 \sin^2 \theta)^{1/2} \) (\( a \) and \( \mu \) represent lines and columns, respectively). The frame (39) is additionally fixed by the condition that the vector \( e_{(3)}^\mu \) is oriented along the \( z \) axis [7].

The linear acceleration and angular velocity that are necessary to cancel the gravitational forces on the frame, and maintain it static in spacetime, are given by \( \phi_{ab} \). The acceleration tensor was calculated in ref. [7]. We define

\[ \textbf{a} = (\phi_{01}, \phi_{02}, \phi_{03}), \quad (41) \]

\[ \Omega = (\phi_{23}, \phi_{31}, \phi_{12}). \quad (42) \]

The expressions for \( \textbf{a} \) and \( \Omega \) are given by

\[ \textbf{a} = \frac{m}{\psi^2} \left[ \frac{\sqrt{\Delta}}{\rho} \left( \frac{2r^2}{\rho^2} - 1 \right) \hat{r} + \frac{2ra}{\rho^3} \sin \theta \cos \theta \hat{\theta} \right], \quad (43) \]

\[ \Omega = -\frac{\chi}{\Lambda \rho} \cos \theta \hat{r} + \frac{\psi^2 \sqrt{\Delta}}{2\Lambda \rho} \sin \theta \partial_r \left( \frac{X}{\psi^2} \right) \hat{\theta} - \frac{\psi^2}{2\Lambda \rho} \sin \theta \partial_\theta \left( \frac{X}{\psi^2} \right) \hat{r}, \quad (44) \]

where

\[ \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}, \]

\[ \hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}. \quad (45) \]

Restricting the analysis to the plane \( \theta = \pi/2 \), it is easy to verify that \( \Omega \) reduces to

\[ \Omega = \phi_{(1)(2)} \hat{z} = -\frac{\psi^2 \sqrt{\Delta}}{2\Lambda \rho} \partial_r \left( \frac{X}{\psi^2} \right) \hat{z}. \quad (46) \]

Returning now to eq. (31), we observe that the general structure of the matrix \( \Lambda_{(i)(j)} \) may be given in terms of Euler angles. There are several possible ways of writing a rotation matrix as function of Euler angles. For
our purposes, a convenient way is given in ref. [10] (appendix A, eq. (A.3y)). It reads

\[ \Lambda_{(i,j)} = \]

\[
\begin{pmatrix}
-\sin \alpha \sin \beta + \cos \alpha \cos \beta \cos \gamma & \cos \alpha \sin \beta + \sin \alpha \cos \beta \cos \gamma & -\cos \beta \sin \gamma \\
-\sin \alpha \cos \beta - \cos \alpha \sin \beta \cos \gamma & \cos \alpha \cos \beta - \sin \alpha \sin \beta \cos \gamma & \sin \beta \sin \gamma \\
\cos \alpha \sin \gamma & \sin \alpha \sin \gamma & \cos \gamma \\
\end{pmatrix}.
\]

(47)

We assume that \( \alpha, \beta \) and \( \gamma \) are functions (to be determined) of the time parameter \( t \).

The reason for choosing the expression above is the following. In eq. (31) we have \( e^{(0)} \mu \partial_\mu = e^{(0)} 0 \partial_0 \). We evaluate the quantities below,

\[
\Lambda^{(k)} (3) \partial_0 \Lambda^{(k)} (2) = -\dot{\gamma} \sin \alpha + \dot{\beta} \cos \alpha \sin \gamma = -\omega_x,
\]

\[
\Lambda^{(k)} (1) \partial_0 \Lambda^{(k)} (3) = -\dot{\gamma} \cos \alpha - \dot{\beta} \sin \alpha \sin \gamma = -\omega_y,
\]

\[
\Lambda^{(k)} (2) \partial_0 \Lambda^{(k)} (1) = -\dot{\beta} \cos \gamma - \dot{\alpha} = -\omega_z,
\]

(48)

where the dot represents time derivative, and note that the right hand side of the expressions in eq. (48) are, except for the sign, the angular velocities along the space axes \( (\omega_x, \omega_y, \omega_z) \) in eq. (A.8y) of ref. [10]).

The general solution of eq. (31) amounts to determining \( \alpha, \beta \) and \( \gamma \) where

\[
e^{(0)} 0 ( -\dot{\gamma} \sin \alpha + \dot{\beta} \cos \alpha \sin \gamma ) = \phi^{(2)} (3),
\]

\[
e^{(0)} 0 ( -\dot{\gamma} \cos \alpha - \dot{\beta} \sin \alpha \sin \gamma ) = \phi^{(3)} (1),
\]

\[
e^{(0)} 0 ( -\dot{\beta} \cos \gamma - \dot{\alpha} ) = \phi^{(1)} (2).
\]

(49)

Clearly there is no simple solution to \( \alpha, \beta \) and \( \gamma \). However, if we restrict the analysis to the equatorial plane defined by \( \theta = \pi/2 \), the problem is greatly simplified. We recall that in the equatorial plane \( \Omega \) given by eq. (46) is directed along the \( z \) axis. Thus eq. (47) must describe a rotation along the \( z \) axis. Making \( \beta = \gamma = 0 \) we have
\[
\Lambda_{(i)(j)} = \begin{pmatrix}
\cos \alpha & \sin \alpha & 0 \\
-\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
(50)

and as a consequence,

\[
\Lambda^{(k)} \partial_0 \Lambda_{(k)(1)} = -\dot{\alpha}.
\]
(51)

Taking into account eq. (46) we write

\[
-e^{(0)}_0 \dot{\alpha} = \phi_{(1)(2)} = -\frac{\psi^2 \sqrt{\Delta}}{2 \rho \Lambda} \partial_r \left( \frac{\chi}{\psi^2} \right),
\]
(52)

from what we obtain

\[
\alpha = \frac{\psi^2 \sqrt{\Delta}}{2 \rho^2 \Lambda} \partial_r \left( \frac{\chi}{\psi^2} \right) t.
\]
(53)

Equation (50) acts only on the spatial sector of the tetrad field, and allows obtaining the transformed tetrads according to eq. (26). Dropping the tilde, the transformed tetrad field \(e_{a\mu}(t, r, \theta = \pi/2, \phi)\) is given by

\[
e_{(0)\mu} = \left(-\frac{\psi}{\rho}, 0, 0, -\frac{\chi}{\rho \psi}\right)
\]

\[
e_{(1)\mu} = \left(0, \frac{\rho}{\sqrt{\Delta}} \left(\cos \phi \cos \alpha + \sin \phi \sin \alpha\right), 0, \frac{\Lambda}{\rho \psi} \left(-\sin \phi \cos \alpha + \cos \phi \sin \alpha\right)\right)
\]

\[
e_{(2)\mu} = \left(0, \frac{\rho}{\sqrt{\Delta}} \left(-\cos \phi \sin \alpha + \sin \phi \cos \alpha\right), 0, \frac{\Lambda}{\rho \psi} \left(\sin \phi \sin \alpha + \cos \phi \cos \alpha\right)\right)
\]

\[
e_{(3)\mu} = \left(0, 0, -\rho, 0\right).
\]
(54)

The frame above is Fermi-Walker transported, and is adapted to observers located at the equatorial plane. It is possible to check by direct calculations that \(\phi_{(i)(j)} = 0\) if the latter is evaluated out of the tetrads above.

Another simple construction of Fermi-Walker transported tetrad fields is the frame for observers located on the \(z\) axis, namely, for \(\theta = 0\). Equation (44) reduces to

\[
\Omega = -\frac{\chi}{\Lambda \rho} \dot{z}.
\]
(55)
The rotation matrix is again given by eq. (50). Similar to eq. (52) we find

\[-e_{(0)}^{\mu} \dot{\alpha} = \phi_{(1)(2)} = -\frac{\chi}{\Lambda \rho}, \]  

and thus we obtain

\[\alpha = \frac{\chi \psi \Lambda \rho}{t^{2}}.\]  

We remark that the frame presented in ref. [5] for the equatorial plane of the Kerr spacetime does not have a simple form as eq. (54), and is not adapted to static observers (as eq. (39)) because it satisfies \(e_{(1)}^{0} \neq 0 \neq e_{(3)}^{0}\), and consequently \(e_{(0)}^{i} \neq 0\).

6 Conclusions

We have found a mechanism for constructing Fermi-Walker frames out of an arbitrary frame transported along a curve \(C\), whose tangent vector is given by the observer’s velocity \(e_{(0)}^{\mu} = dx^{\mu}/ds\). The idea consists in performing a local Lorentz rotation and solving eq. (31) for the coefficients of the local Lorentz transformation. Although in principle it is difficult to obtain the general solution of eq. (31), in some particular situations the solution can be easily achieved. We have obtained Fermi-Walker transported frames for observers located in the equatorial plane and in the symmetry axis (the \(z\) axis) in the Kerr spacetime. It is formally possible to establish Fermi-Walker frames for observers at rest on the surface of the rotating Earth (the Frenet-Serret equations would determine the trajectory of the observer in spacetime). These frames are the best realization of reference frames that take into account the effects of the rotation of the Earth.

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