NON-AUTONOMOUS DYNAMICS IN $\mathbb{P}^k$

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Abstract. We study the dynamics of compositions of a sequence of holomorphic mappings in $\mathbb{P}^k$. We define ergodicity and mixing for non-autonomous dynamical systems, and we construct totally invariant measures for which our sequence satisfies these properties.

1. Introduction

Non-autonomous dynamics differs from standard dynamics in that instead of iterating a single map, we consider compositions of a sequence of maps. The main goal of non-autonomous dynamics is to generalize theorems that hold in the autonomous setting or to find counterexamples. Here, we will try to generalize theorems which state that for every complex mapping there exists a natural measure which is mixing and thus ergodic. This was first proved by Brolin for polynomials in the complex plane in [Br], by Bedford and Smillie for Hénon mappings in $\mathbb{C}^2$ in [BS], and for regular polynomial mappings of $\mathbb{C}^k$ by Fornaess and Sibony in [FS1]. It has also been shown for endomorphisms of $\mathbb{P}^k$, see for instance the articles by Briend and Duval [BD] or Guedj and Sibony [GS].

Non-autonomous systems of polynomials in the complex plane have been studied in the past years by many authors, see for instance the survey article by Comerford [Co], which has an extensive bibliography. It has turned out that a good setting in which to work is that of bounded sequences of monic polynomials of some fixed degree which were first considered in [FS2]. We will work with more general mappings, and our results imply the same results for such sequences.

Non-autonomous systems in higher complex dimensions have been studied only rarely. We will look at a compact sequence of holomorphic mappings on $\mathbb{P}^k$, which we will define more precisely in the next section. This setting has already been studied in [FW], but in a rather different way. There, the dynamics of all nearby mappings of a holomorphic mapping were studied at the same time, while we will study the dynamics of one fixed sequence.

Let $P_n$ be a compact sequence of holomorphic mappings, and let $\mu_n$ be the equilibrium measures for this sequence, which we will define later. The main results of this paper are the following two theorems:

Theorem 1. The system $\{(P_n, \mu_n)\}$ is randomly ergodic.

Theorem 2. The system $\{(P_n, \mu_n)\}$ is randomly mixing.

In Section 2 we will set our notation and give the precise definitions of randomly ergodic and randomly mixing, and in the Section 3 we will use pluripotential methods to introduce the equilibrium measures $\mu_n$. In Section 4 we will prove a series
of lemmas considering the convergence of preimages, and we will give the proofs of our two main theorems in the Section 5. In the last section we will prove that an autonomous system which is randomly ergodic is in fact weakly mixing.

2. Non-autonomous systems in $\mathbb{P}^k$

We will now introduce the setting for this paper. Let $\mathcal{P}$ be a compact family (in the coefficients topology) of holomorphic endomorphisms of $\mathbb{P}^k$ whose degrees are at least 2 and bounded from above, and let $P_0, P_1, \ldots$ be a sequence of polynomials in $\mathcal{P}$, where $P_n$ has degree $d_n$.

We define

$$P(n) = P_n \circ \ldots \circ P_1; \quad d(n) = d_n \cdot \ldots \cdot d_1$$

and for $n$ larger than $m$ we will write

$$P(m,n) = P_n \circ \ldots \circ P_{m+1}; \quad d(m,n) = d_n \cdot \ldots \cdot d_{m+1}$$

For a point $z = z_0$ in $\mathbb{P}^k$ we shall also write $z_n$ for $P(n)(z)$, which we shall say is a point at stage $n$. Thus $P_n$ is a mapping from stage $n-1$ to stage $n$.

Recall that a measure preserving automorphism $f$ of a space $X$ with probability measure $\mu$ is called ergodic if all totally invariant measurable subsets $A$ of $X$ either have full or empty measure, and that $f$ is called mixing if for all measurable sets $A$ and $B$ we have that

$$\mu(f^{-n}(A) \cap B) - \mu(A) \cdot \mu(B) \rightarrow 0.$$ 

Both definitions can also be studied in the autonomous setting, where a single map is iterated. Since continuous functions are dense in $L^2(\mu)$, we have that randomly mixing and mixing are equivalent. However, randomly ergodic is strictly stronger property than ergodic. It is easy to check that randomly ergodic implies ergodicity, but the only measure for which an automorphism is randomly ergodic is a point mass at a fixed point, which is certainly not the case for the classical definition. We note that a randomly mixing system is not necessarily randomly ergodic for the same reason.

It would be interesting to find a generalization of ergodicity that is useful for the study of the dynamics of a sequence of automorphisms.
3. Equilibrium Measures

The following construction of the equilibrium measures is fairly standard in holomorphic dynamics and can be found in [Si], and can also be found for non-autonomous systems in [FW].

Since \( \mathcal{P} \) is compact, we can extend all mappings \( P \in \mathcal{P} \) to homogeneous polynomial mappings \( \tilde{P} \) of \( \mathbb{C}^{k+1} \) in such a way that the coefficients of every \( \tilde{P} \) are bounded by some uniform constant \( M \), and such that the images of the unit sphere in \( \mathbb{C}^{k+1} \) are bounded away from the origin. In other words, there exists some constant \( t > 1 \) such that

\[
\frac{1}{t} \|z\| d^n < \|\tilde{P}_n(z)\| < t \|z\| d^n, \tag{1}
\]

holds for any nonzero \( z \) in \( \mathbb{C}^{k+1} \) and any \( n \).

For every \( i \in \mathbb{N} \) and \( n \geq i \), we define the function

\[
G_{n,i}(z) := \frac{1}{d(i,i+n)} \log \|\tilde{P}(i,i+n)(z)\|.
\]

**Lemma 5.** As \( n \to \infty \), the functions \( G_{n,i} \) converge uniformly on \( \mathbb{C}^{k+1} \) to a continuous and plurisubharmonic function \( G_i \).

**Proof.** Fix \( \epsilon > 0 \). It follows from (1) that for any \( z \) in \( \mathbb{C}^{k+1} \) we have

\[
|G_{n+1,i}(z) - G_{n,i}(z)| < \frac{\log(t)}{d(i,n+i+1)}.
\]

Therefore, we have for any \( m \geq n \) that

\[
|G_{m,i}(z) - G_{n,i}(z)| < \frac{\log(t)}{d(i,i+n)(d_{n+1} - 1)}.
\]

Since every \( d_n \) is at least 2 we can choose \( n \) large enough so that

\[
|G_{m,i}(z) - G_{n,i}(z)| < \epsilon,
\]

for any \( m \geq n \). It follows that the sequence \( G_{n,i} \) converges uniformly to a limit map \( G_i \), and since all the functions \( G_{n,i} \) are continuous and plurisubharmonic, the limit map is also continuous and plurisubharmonic. \( \square \)

It follows from (2) that \( G(z) = \log \|z\| + O(1) \). Also, since every \( \tilde{P}_n \) is homogeneous, we have that \( G_i(\lambda z) = \log(\lambda) + G_i(z) \). We get the equation

\[
\tilde{P}_n^* G_n = d_n G_{n-1}. \tag{3}
\]

Let \( \pi \) be the projection from \( \mathbb{C}^{k+1} \) to \( \mathbb{P}^k \). We can define \((1,1)\) currents \( T_i \) on \( \mathbb{P}^k \) which satisfy

\[
\pi^* T_i := dd^c G_i.
\]

\( T_i \) is a current of mass 1 on \( \mathbb{P}^k \), that does not depend on our choices for \( \tilde{P}_n \). It follows from equation (3) that

\[
P_n^* T_n = d_n T_n.
\]

Since \( G_n \) is continuous, it follows from [BT] that we can define \( \mu_n = (T_n)^k \). Since \( T_i \) has unit mass, we get that \( \mu_n \) is a probability measure and since \( G_n \) is locally bounded it follows from Proposition 4.6.4 in the book by Klimek [K] that \( \mu_n \) does not assign any mass to locally pluripolar sets.
We call $\mu_n$ the *equilibrium measure* at stage $n$ and we have that $P_n^*\mu_n = d_n^k\mu_{n-1}$, and that $P_n^*\mu_{n-1} = \mu_n$.

4. Uniform Convergence of Preimages

Recall the following theorem, which was proved by H. Brolin [Br] for polynomials and by M. Lyubich [Ly] and independently by A. Freire, A. Lopes and R. Mañé [FLM] for rational functions:

**Theorem 6.** Let $R(z)$ be a rational function of degree $d \geq 2$, and let $R^n$ be its $n$-th iterate. Then for all $a \in \mathbb{P} \setminus \mathcal{E}_R$, $\text{card}(\mathcal{E}_R) \leq 2$,

$$\frac{1}{d^n}(R^n)^*\delta_x \to \mu.$$  

Here $\delta_x$ is the dirac mass at $x$. It follows from Theorem 1.2 of [RS] that this theorem can be generalized to our setting. However, to prove Theorems 1 and 2 we will need the uniform versions of this theorem which we will prove in this section. Our proofs will be similar to the method used by Lyubich to prove the above theorem, and which was later used by J. Briend and J. Duval in [BD] to prove similar results for endomorphisms of $\mathbb{P}^k$.

Define $\eta_{x,n,i}$ to be the probability measure with mass $\frac{1}{d(i,n+i)^k}$ at all the preimages $P(i, n + i)^{-1}(x)$ counting multiplicity. In other words,

$$\eta_{x,n,i} = \frac{P(i, n + i)^*\delta_x}{d(i, n + i)^k}.$$  

(For simplicity of notation, we shall write $\eta_{x,n}$ for $\eta_{x,n,0}$).

For two probability measures $\mu_1, \mu_2$ on $\mathbb{P}^k$ we define the distance

$$d(\mu_1, \mu_2) = \sup_{\phi} \left| \int \phi d\mu_1 - \int \phi d\mu_2 \right|,$$

where the supremum is taken over all $C^1(\mathbb{P}^k)$ functions $\phi$ for which $|\phi(z)|$ and $|\nabla \phi(z)|$ are bounded by 1. It is clear that the topology induced by this distance is weaker than the strong topology on probability measures. In fact a sequence of probability measures $\nu_n$ converges weakly to $\mu$ if and only if $d(\nu_n, \mu) \to 0$ since we are working in a compact space.

The following proposition shows that as $n$ gets large, the measures $\eta_{x,n}$ depend less and less on the point $x$.

**Proposition 7.** Let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ and subsets $X_n$ of $\mathbb{P}^k$ such that for every $n$ larger than $N$ we have that $\mu_n(X_n) < \epsilon$, and also

$$d(\eta_{x,n}, \eta_{y,n}) < \epsilon,$$

for every $x, y$ outside of $X_n$.

The proof is given below.

Fix $\epsilon > 0$, and let $l = l(\epsilon)$ be some large enough number that we will define later. For $n$ greater or equal to $l$, let $V_{l,n}$ be the set of critical values of the holomorphic mapping $P(n - l, n)$.

**Lemma 8.** There exists a $\delta$ such that the $\mu_n$ mass of the $\delta$-neighborhood of $V_{l,n}$ is less than $\epsilon$ for any $n$ larger than $l$.  

Proof. We have seen that the measures $\mu_n$ do not assign any mass to pluripolar sets. Therefore, there exists for each $n \geq l$ a $\delta_n$ such that the $\delta_n$-neighborhood of $V_{l,n}$ has $\mu_n$ mass less than $\epsilon$. Let $S$ be the set of sequences of polynomials of $\mathcal{P}$ with the product topology, so that $S$ is a compact set.

The maps $G_{n,i}$ depend continuously on the sequence in $S$, and since $G_{n,i}$ converges uniformly to the map $G_i$, we have that $G_i$ also depends continuously on $S$. Let $\{S^j\}$ be a sequence of sequences in $S$ that converges uniformly to $S \in S$. Write $G_i^j, G_i, \mu_i^j, \mu_i$ for the Green's functions and equilibrium measures corresponding to the sequences $S^j$ and $S$. Then we have that $G_i^j \to G_i$ uniformly on $\mathbb{C}^k$, and therefore it follows from [CLN] that $\mu_i^j$ converges weakly to $\mu_i$.

Since the sets of critical values $V_{l,n}$ also vary continuously as a function in $S$, we have that $\delta_n$ is also sufficient for an open neighborhood of our sequence $S$. Since $S$ is compact, this means that we can choose one $\delta$ that suffices for all sequences, in particular for the sequences $P_j, P_{j+1}, \ldots$, which completes the proof. \qed

Fix $\delta$ as in the above lemma, and we now fix $l$ such that $4\tau 2^{-l} < \epsilon$, where $\tau$ is the maximum possible algebraic degree of the sets $V_n$, the critical values of $P_n$. Let $\gamma$ be the maximum possible degrees of the algebraic sets $V_{l,n}$. We can choose $\tau$ and $\gamma$ since the degrees of the polynomials $P_n$ are bounded from above, which follows from the compactness of $\mathcal{P}$.

We shall call a holomorphic disc $\Delta$ in a complex line $L$ $\delta$-extendable if the $\delta$-neighborhood of $\Delta$ in $L$ is simply connected.

**Lemma 9.** There exists a constant $c \in \mathbb{R}$ such that for every $n$ large enough, every complex line $L$ and every $\frac{1}{4\gamma}$-extendable holomorphic disc $\Delta \subset L$ that does not intersect a $\frac{1}{4\gamma}$-neighborhood of $L \cap V_{l,n}$, there exist at least $(1-\epsilon)d(n)^k$ inverse branches of $P(n)$ on $\Delta$ for which the preimages $\Delta_i = P(n)^{-1}(\Delta)$ satisfy

$$\text{diam}(\Delta_i) < cd(n)^{k/2}.$$  

**Proof.** We can exactly follow the proof of the lemma in [BD] to get that for every such disc $\Delta$, there exists a constant $c$ such that there are at least $(1-\epsilon)d(n)^k$ preimages $\Delta_i$ of diameter less than $cd(n)^{k/2}$. To see that we can choose $c$ independently of $\Delta$, note that we can take the larger disc $\Delta$ in that proof as the $\frac{1}{4\gamma}$-neighborhood of $\Delta$ in $L$. It follows that $\text{Mod}(\Delta - \Delta)$ is bounded from below by some strictly positive constant, and this gives a bound on $c$ which completes the proof of the lemma. \qed

Note that for every line $L$ that intersects $V(l,n)$ in a finite number of points and every $x, y$ in the complement of the $\delta/(4\gamma)$-neighborhood of $V(l,n)$ in $L$, we can choose a $\delta/(4\gamma)$-extendable holomorphic disc outside of the $\delta/(2\gamma)$-neighborhood of $V(l,n)$. Indeed, we can take the shortest curve in $L$ from $x$ to $y$ that avoids the $3\delta/(4\gamma)$-neighborhood of $V(l,n)$ and take the $\delta/(4\gamma)$-neighborhood of the curve as our extendable disc.

**Proof of Proposition 9** Let $X_n$ be the $\delta$-neighborhood of $V_{l,n}$. We have that $\mu_n(X_n) < \epsilon$ for any $n \in \mathbb{N}$. Let $x, y$ be points outside of $X_n$. We can choose $z$ outside of $X_n$ such that the lines $L_1$ and $L_2$ through respectively $x, z$ and $y, z$ intersect $V_{l,n}$ in at most $\gamma$ points. This means that there exist $\delta/(4\gamma)$-extendable holomorphic discs $\Delta_1 \subset L_1$ and $\Delta_2 \subset L_2$ such that $x, z \in \Delta_1$ and $y, z \in \Delta_2$, and
such that $\Delta_1$ and $\Delta_2$ avoid the $\frac{\epsilon}{2\gamma}$ neighborhood of $V_{l,n}$. Now it follows from the lemma that there are at least $(1 - \epsilon)d^n$ preimages $x_j^{-n}$, $y_j^{-n}$ and $z_j^{-n}$ such that

$$\text{dist}(x_j^{-n}, y_j^{-n}) \leq \text{dist}(x_j^{-n}, z_j^{-n}) + \text{dist}(y_j^{-n}, z_j^{-n}) \leq 2\frac{c}{d(n^{k/2})}.$$

Hence, for any continuous function $\phi$ of norm 1 we have that

$$| \int \phi d\eta_{x,n} - \int \phi d\eta_{y,n} | \leq 2\epsilon + \frac{1}{d^n} \sum_j \left| \phi(y_j^{-n}) - \phi(x_j^{-n}) \right| \leq 2\epsilon + 2\frac{c}{d(n^{k/2})}.$$

For $n$ large enough, this is smaller than $3\epsilon$, which completes the proof. □

Now, for some fixed small $\epsilon > 0$, let $\epsilon_1, \epsilon_2, \ldots$ be a monotone decreasing sequence such that the sum over all $\epsilon_j$ is smaller than $\epsilon$. For every $j$, define a set $X_{n,j}$ as in Proposition [1] and $N_j$ in $\mathbb{N}$ such that $\mu(X_{n,j}) < \epsilon_j$ and $d(\eta_{n,x}, \eta_{n,y}) < \epsilon_j$ for any $n$ larger than $N_j$ and $x, y$ outside of $X_{n,j}$. Now set

$$U_n := \mathbb{P}^k - \bigcup_{N_j \leq n} X_{n,j}.$$

We see in particular that $\mu_n(U_n)$ is larger than $1 - \epsilon$ for every $n$. Fixing a sequence $x_1, x_2, \ldots$ such that $x_n$ is an element of $U_n$, we get the following uniform version of Theorem [2].

**Lemma 10.** For every $\epsilon > 0$ there exists an $N$ so that for every $m$ and every $n \geq N$ we have that

$$d(\eta_{x_n,n-m}, \mu_m) < \epsilon.$$

**Proof.** We have that

$$\mu_m = \int \delta_y d\mu_m(y),$$

and therefore we have

$$\mu_m = \frac{P(m, n + m) \ast \mu_{n+m}}{d(m, n + m)^k} = \int \eta_{y,n,m} d\mu_{n+m}(y).$$

It follows that

$$\mu_m - \eta_{x_{n+m},n,m} = \int (\eta_{y,n,m} - \eta_{x_{n,m}}) d\mu_m(y).$$

We can choose a $j$ such that $2\epsilon_j < \epsilon$, and by our construction of $X_{n,j}$ and $U_n$, it follows that for $n \geq N_j$ we have $d(\eta_{y,n,m}, \eta_{x_{n+m}}) < \epsilon_j$ for any $y$ outside of $X_{n,j}$, while $\mu_{n+m}(X_{n+m,j}) < \epsilon_j$. Therefore, $d(\mu_m, \eta_{x_{n,m}}) < 2\epsilon_j$, which completes the proof. □

In the autonomous setting it is known that the equilibrium measure is the only totally invariant measure that doesn’t charge the exceptional set [BD]. We can’t expect such a result to hold here. Consider for instance the map $z \mapsto z^2$ in $\mathbb{P}^1$. The equilibrium measures $\mu_n$ are all equal to the normalized Lebesgue measure on the unit circle. However, let $\nu_n$ be the normalized Lebesgue measure on the disc of radius $1/2^n$. Then $\{\nu_n\}$ is totally invariant and doesn’t charge the exceptional set $\{0, \infty\}$.

We do have the following related uniqueness result:
Corollary 11. Let $\nu$ be a probability measure on $\mathbb{P}^k$ that doesn’t charge locally pluripolar sets. Then we have that
\[
P(m, n + m)^* \frac{d(m, n + m)^k \nu}{\nu \rightarrow \mu}_m,
\] weakly.

The corollary follows from Proposition 7 as in the proof of lemma 10.

5. PROOF OF THEOREMS 1 AND 2

Proof of Theorem 1: Let $A_0, A_1, \ldots$ be a sequence of measurable subsets of $\mathbb{P}^k$ such that $P_n^{-1}(A_n) = A_{n-1}$ for all $n$, and assume that $\mu_0(A_0)$ is not equal to 0. We need to show that $A_0$ has full measure. Define the measures $\nu_n$ by
\[
\nu_n(X) = \frac{\mu_n(X \cap A_n)}{\mu_n(A_n)}.
\]
Clearly, every $\nu_n$ is a probability measure. We see that
\[
P_n \nu_n = \nu_{n-1}(P_n^{-1}(X)) = \mu_{n-1}(P_n^{-1}(X) \cap A_{n-1}) / \mu_{n-1}(A_{n-1}) = \mu_n(X \cap A_n) / \mu_n(A_n) = \nu(X).
\]
Similarly, it follows from the total invariance of the sets $A_n$ and the measures $\mu_n$ that
\[
P_n \nu_n = \nu_{n-1}.
\]
As we have seen before in the proof of Lemma 10 we have the equation
\[
\mu_0 = \int \eta_{x,n} d\mu_n(x),
\]
and similarly,
\[
\nu_0 = \int \eta_{y,n} d\nu_n(y).
\]
Therefore we see that
\[
\mu_0 - \nu_0 = \int (\eta_{x,n} - \eta_{y,n}) d\mu_n(x) \otimes d\nu_n(y).
\]
It now follows from Proposition 1 that for any $\epsilon > 0$, we have
\[
\|
\mu_0 - \nu_0
\| < 3\epsilon.
\]
Thus $\nu_0 = \mu_0$ and $\mu_0(A_0)$ must equal 1, which completes the theorem.

The argument of the proof of Theorem 2 is similar to that of Theorem 17.1 in [Br].

Proof of Theorem 2: Let $\phi, \psi$ be test functions of norm at most 1, and let $\epsilon > 0$. Construct sets $U_n$ as we did for Lemma 10 such that $\mu_n(U_n) > 1 - \epsilon$ for each $n$. It follows from Lemma 10 that we can fix $n$ so large that $\| \eta_{\zeta,n} - \mu_0 \| < \epsilon$ for any $\zeta \in U_n$. 

Let $m$ be large enough so that

$$\int (\phi \circ P(n)) \cdot \psi \, d\mu_0 = \int (\phi \circ P(n)) \circ \psi \, d\eta_{x_{m+n},-(m+n)} + \epsilon_1,$$

where $|\epsilon_1| < \epsilon$. It follows from the definition of $\eta_{x_{m+n},-(m+n)}$ that the right hand side is equal to

$$\sum_{\nu} \phi(P(n)(\zeta_{m+n,-(m+n)})) \psi((\zeta_{m+n,-(m+n)})d(m+n)^{-k} + \epsilon_1$$

$$= \sum_{\nu} \phi(\zeta_{m+n,-m})d(n,n+m)^{-k} \sum_{\zeta_{m+n,-m}^\text{fixed}} \psi(\zeta_{m+n,-m})d(n)^{-k} + \epsilon_1.$$

Counting multiplicity, there are $d(n,n+m)^k$ preimages $\zeta_{m+n,-m}$, and since $\mu_n(U_n) > 1 - \epsilon$, we can increase $m$ if necessary so that at least $(1 - \epsilon)d(n,n+m)^k$ of the $\zeta_{m+n,-m}$ are in $U_n$. It follows that the above right hand side is equal to

$$\sum \phi(\zeta_{m+n,-m})d(m,n+m)^{-k} \left( \int \psi \, d\mu_0 + \epsilon_3 \right) + \epsilon_1 + \epsilon_2,$$

where $\epsilon_3$, which depends on $\nu$, and $\epsilon_2$ all have absolute value less than $\epsilon$. We can rewrite this as

$$\left( \int \psi \, d\mu_0 + \epsilon_3 \right) \sum \phi(\zeta_{m+n,-m})d(n,n+m)^{-k} + \epsilon_1 + \epsilon_2,$$

where $\epsilon_3$ no longer depends on $m$. By increasing $m$ if necessary we get

$$\left( \int \psi \, d\mu_0 + \epsilon_3 \right) \left( \int \phi \, d\mu_n + \epsilon_4 \right) + \epsilon_1 + \epsilon_2,$$

and so

$$\left| \int (\phi \circ P(n)) \cdot \psi \, d\mu_0 - \int \phi \, d\mu_n \int \psi \, d\mu_0 \right| < 4\epsilon.$$

This proves that

$$\int (\phi \circ P(n)) \cdot \psi \, d\mu_0 - \int \phi \, d\mu_n \int \psi \, d\mu_0 \to 0$$

for all test functions $\phi$ and $\psi$. The theorem follows since we can uniformly approximate any continuous function by test functions. \[\square\]

Remark 12. It is not clear if the theorem holds if we allow $\phi$ in the definition of randomly mixing to be in the intersection of all $L^2(\mu_n)$, since in general we will not be able to approximate these functions by continuous functions that are close in every $L^2(\mu_n)$ norm at the same time. The theorem does however hold for $\psi$ in $L^2(\mu_0)$.

6. Random ergodicity in the autonomous setting

We have already seen that random ergodicity is not equivalent to ergodicity in the classical case. Indeed, an automorphism can never have interesting measures that are randomly ergodic, so we can not expect randomly ergodic to be equivalent to any known condition from ergodic theory. We shall see in this section that random ergodicity implies a condition that is stronger than ergodicity, namely
weakly mixing. Recall that a measure preserving transformation \((P, \mu)\) is weakly mixing if for all \(\phi, \psi \in L^2(\mu)\) we have that
\[
\frac{1}{n} \sum_{k=1}^{n} \left( \int (\phi \circ P^k) \cdot \psi \, d\mu - \int \phi \, d\mu \int \psi \, d\mu \right)^2,
\]
converges to 0 as \(n \to \infty\). Weakly mixing implies ergodicity, see for example [CFS].

Let \(U_P\) be the adjoint operator working on \(L^2(\mu)\), that is, \(U_P(\phi) = \phi \circ P\).

The following result is well known and can be found in [CFS].

**Theorem 13.** A measure preserving transformation \(P\) is weakly mixing if and only if every eigenfunction of \(U_P\) is constant almost everywhere.

We can similarly express random ergodicity in terms of the operators \(U_{P_n}\).

**Lemma 14.** \(\{P_n, \mu_n\}\) is randomly ergodic if and only if we have the following property:

For every sequence \(f_0, f_1, \ldots\) with \(f_n \in L^2(\mu_n)\) for which \(f_n \circ P_n = f_{n-1}\) holds for every \(n\), we have that \(f_0\) is constant a.e..

**Proof.** First, assume that the system is randomly ergodic. Fix a totally invariant sequence of maps \(f_0, f_1, \ldots\) as above. For some \(r \in \mathbb{R}\), define \(A_k = \{z | f_k(z) > r\}\). Then \(x \in A_{k-1}\) if and only if \(f_k(P_k(x)) > r\), and thus if and only if \(P_k(x) \in A_k\), i.e. \(P_{k-1}^{-1}A_k = A_{k-1}\). This means that \(\mu(A_0) = 0\) or 1, and this holds for every \(r \in \mathbb{R}\) and thus we see that the function \(f_0\) is constant.

For the converse, let \(A_0, \ldots\) be such that \(P_k^{-1}(A_k) = A_{k-1}\) and define \(f_k = 1_{A_k}\). It follows that \(f_0\) is constant, and therefore that \(A_0\) is has mass 0 or 1. \(\square\)

**Proposition 15.** Any randomly ergodic measure preserving transformation \((P, \mu)\) is weakly mixing.

**Proof.** Suppose that \(U_P\) has an eigenfunction \(f\), say \(f \circ P = \sigma f\). Then define \(f_0 = f, f_1 = \sigma^{-1}f\), and so forth. Clearly this sequence satisfies \(f_k \circ P = f_{k-1}\), and therefore \(f_0 = f\) is constant a.e.. \(\square\)

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