HAMiLTONIAN-MiNiMAL SUBMANIFOLDs IN KAeHLER MANIfOLDS WITH SYMMET RiES

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AbSTRACT. By making use of the symplectic reduction and the cohomogeneity method, we give a general method for constructing Hamiltonian minimal submanifolds in Kaehler manifolds with symmetries. As applications, we construct infinitely many nontrivial complete Hamiltonian minimal submanifolds in \( C^P \) and \( C^n \).

1. Introduction

Let \((M, \omega, g)\) be a symplectic manifold with a Riemannian metric \( g \) and let \( L \) be a Lagrangian submanifold in \( M \). A normal vector field \( V \) along \( L \) is called a Hamiltonian variation if the one form \( \alpha_V := \omega(V, \cdot) \) is exact. According to [O1,2], the Lagrangian submanifold \( L \) is called Hamiltonian minimal if it is a critical point of the volume functional with respect to all Hamiltonian variations along \( L \). In particular, this makes sense if \( M \) is a Kaehler manifold. A Hamiltonian minimal submanifold will be simply called \( H \)-minimal.

**Proposition 1.1.** ([O2]) Let \((M, \omega, g)\) be a Kaehler manifold. A Lagrangian submanifold \( L \subset M \) is \( H \)-minimal if and only if its mean curvature vector \( H \) satisfies

\[
\delta \alpha_H = 0
\]
on \( L \), where \( \delta \) is the Hodge-dual operator of \( d \) on \( L \).

\( H \)-minimal Lagrangian submanifolds offer a nice generalization of the minimal submanifold theory. It was Oh who first investigated these submanifolds (see [O1-2]). One motivation to study them is its similarity to some models in incompressible elasticity([Wo], [HR1]). In [O2], the author comments that \( H \)-minimal Lagrangian submanifolds seem to exist more often than minimal Lagrangian submanifolds do. In [CU], Castro and Urbano constructed some exotic Hamiltonian tori in \( C^2 \). Afterwards, Helein and Romon constructed \( H \)-minimal surfaces via integrable system method ([HR1,2]). Besides these explicit instances, Schoen and Wolfson [SW] established some important existence and regularity results for two-dimensional \( H \)-minimal surfaces. However, only a few non-trivial examples of \( H \)-minimal Lagrangian submanifolds of higher dimensions have been known so far.

The aim of this paper is to give some constructions of \( H \)-minimal Lagrangian submanifolds of higher dimensions. Note that the equation (1) is a third order P.D.E., which is
more complicated than the minimal submanifold equation. Even for the usual minimal submanifold, the existence is a difficult area of study, due to the nonlinearity of the equation. Recently, the symmetry reduction method leads to some important progress in explicit construction of special Lagrangian submanifolds by several authors (see [J1,2] and the references contained therein). In this paper, we will solve (1) by the same trick. Let $G$ be a compact connected Lie group of holomorphic isometries of a Kaehler manifold $M$ and let $\mu$ be the moment map of the $G$–action. First, we show that a $G$–invariant Lagrangian submanifold is $H$–minimal if and only if it is stationary with respect to any $G$–invariant Hamiltonian variation. From [J2], we know that a $G$–invariant Lagrangian submanifold is contained in a level set of $\mu$. The well-known Noether theorem tells us that the moment map $\mu$ is a conserved quantity for every $G$–invariant Hamiltonian deformation. This allows us to restrict the variational problem in a level set of $\mu$. By combining the symplectic reduction and the cohomogeneity method developed in [HsLa], we can reduce the equation (1) to a P.D.E. on the symplectic quotient with the Hsiang-Lawson metric. We have a very nice correspondence between the $G$–invariant $H$–minimal Lagrangian submanifolds in $M$ and the $H$–minimal Lagrangian submanifolds in the quotient space (see Corollary 2.8 and Theorem 2.9). The reduction procedure simplifies the original equation greatly. Actually, the reduced system becomes O.D.E. if the $G$–action is of cohomogeneity one. To show the procedure, we consider some concrete $G$–actions of cohomogeneity one on $\mathbb{C}P^n$ and $\mathbb{C}^n$ respectively. By solving the corresponding O.D.E., we construct infinitely many non-trivial closed $H$–minimal Lagrangian submanifolds and also non-trivial complete $H$–minimal Lagrangian submanifolds in $\mathbb{C}P^n$ and $\mathbb{C}^n$. Here the $H$–minimal Lagrangian submanifolds is called nontrivial, if they are not minimal in the usual sense.

### 2. Symmetry Reduction

Let $M$ be a connected manifold with a differentiable $G$–action, where $G$ is a compact, connected Lie group. For each $x \in M$ let $G_x$ be the isotropy subgroup of $x$, and $G(x) \approx G/G_x$ be the orbit of $x$ under $G$. Two orbits, $G(x)$ and $G(y)$, are said to be of the same type if $G_x$ and $G_y$ are conjugate in $G$. The conjugacy classes of the subgroup $\{G_x : x \in M\}$ are called the orbit types of the $G$–space $M$. The orbit types may be partially ordered as follows:

$$(H) > (K) \iff \exists g \in G \text{ s.t. } K \supseteq gHg^{-1}$$

where $(H)$ denotes the conjugacy class of $H$. We need the following important result ([MSY]):

**Proposition 2.1. (Principal orbit type)** Let $M$ be a connected manifold with a differentiable $G$–action. Then there exists a unique orbit type $(H)$ such that $(H) > (K)$ for all orbit types $(K)$ of the action. Moreover, the union of all orbits of type $(H)$, namely $M^* = \{x \in M : G_x \in (H)\}$, is an open, dense submanifold of $M$.

Following [MY] we call $(H)$ in Proposition 2.1 the principal orbit type of the $G$–space $M$. If $(H') \neq (H)$ but $\dim H' = \dim H$, then $(H')$ will be called an exceptional orbit type. All other orbit types will be called singular.

From now on we assume that $M$ is a Kaehler manifold with Kaehler form $\omega$ and complex structure $J$. Let $G$ be a compact, connected Lie group of holomorphic isometries of $M$. Let
Proposition 2.2. Suppose that \( G \) acts on the Kaehler manifold \((M, \omega, J)\) with moment map \( \mu \) and preserving \( J \). If \( c \in Z(g^*) \) then \( \mu^{-1}(c) \) is a stratified manifold which induces a stratified Kaehler space \( \mu^{-1}(c)/G \). In particular, \( \mu^*-1(c) \) is a manifold and the quotient space \( \mu^*-1(c)/G \) inherits a natural Kaehler structure \((\tilde{\omega}, \tilde{J})\) such that \( \pi^*\tilde{\omega} = \omega|_{\mu^*-1(c)} \) and \( \tilde{J}\tilde{X} = \pi_*(JX) \) for any \( \tilde{X} \in T_p(\mu^*-1(c)/G) \), where \( X \) is the horizontal lift of \( \tilde{X} \).

Remark 2.1. (i) Actually, \( \mu^*-1(c) \) is the stratum of \( \mu^{-1}(c) \) corresponding to the principal orbit type. It is easy to see that \( \pi : (\mu^*-1(c), ds^2) \to (\mu^*-1(c)/G, \tilde{ds}^2) \) is a Riemannian submersion, where \( ds^2 \) is the induced metric from the Kaehler metric of \( M \) and \( \tilde{ds}^2 \) is the metric determined by \( \tilde{\omega} \);

(ii) If \( c \in Z(g^*) \) is a regular value of \( \mu \) and the action of \( G \) is free, then \( \pi : \mu^{-1}(c) \to \mu^{-1}(c)/G \) is the well-known Marsden-Weinstein symplectic reduction. In this case, \( \mu^{-1}(c) \) and \( \mu^{-1}(c)/G \) are both smooth manifolds.

The following result shows that moment maps are a useful tool for studying Lagrangian submanifolds with symmetries.

Lemma 2.3. (cf. [J2]) If \( L \subset M \) is a connected \( G \)-invariant Lagrangian submanifold, then \( L \subset \mu^{-1}(c) \) for some \( c \in Z(g^*) \).

Proof. For \( \xi \in g \), we have the vector field \( \phi(\xi) \) on \( M \). Since \( \omega|_L \equiv 0 \) and \( \phi(\xi) \) is tangent to \( L \), we have \( d\mu|_{TL} \equiv 0 \) by (a). So \( \mu \) is constant on \( L \). By (b), we see that the constant \( \mu(L) \in Z(g^*) \). \( \square \)

Lemma 2.4. Let \( i : L \to M \) be a \( G \)-invariant Lagrangian submanifold of \( M \). Then \( L \) is \( H \)-minimal if and only if the volume of \( L \) is stationary w.r.t. all compactly supported, \( G \)-equivariant Hamiltonian variations.

Proof. Let \( H \) be the mean curvature vector of \( L \). Since \( H \) depends only on the immersion \( i \), and \( i \) is \( G \)-invariant, we have \( k_iH = H \) for any \( k \in G \). This implies that the one form \( \alpha_H := H \omega \) and thus its codifferential \( \delta \alpha_H \) are \( G \)-invariant. Let \( \varphi \) be any smooth, \( G \)-invariant, compactly supported function on \( L \). We define a variation \( i_t, -\varepsilon < t < \varepsilon \), of the immersion \( i \) by:

\[ i_t(x) = \exp_x(tV) \]
where $JV = \nabla (\varphi \delta \alpha_H)$, i.e., $\alpha_V = d(\varphi \delta \alpha_H)$. We choose $\varepsilon > 0$ small enough that each $i_t$ is an immersion. Observe that

$$
k \circ i_t(x) = k \circ \exp_x(tV) = \exp_{kx}(tkxV) = \exp_{kx}(tV_{kx}) = i_t \circ k(x).
$$

Hence each $i_t$ is equivariant. By the first variational formula of volume, we have

$$
\frac{d}{dt}|_{t=0} \text{Vol}(i_t(L)) = -\int_L <H, V> = -\int_L <\alpha_H, \alpha_V> = -\int_L |\delta \alpha_H|^2 \varphi.
$$

So, by assumption $\frac{d}{dt}|_{t=0} \text{Vol}(i_t(L)) = 0$ and the fact that $\varphi$ is arbitrary, we see that $\delta \alpha_H = 0$. □

We have the generalized Noether Theorem:

**Proposition 2.5.** (cf. [Si]) If $F$ is a $G$–invariant Hamiltonian, then the moment map $\mu$ is a conserved quantity for the Hamiltonian flow of $F$.

Lemma 2.3 shows that any connected $G$–invariant Lagrangian submanifold $L$ is contained in a level set $\mu^{-1}(c)$ for some $c \in Z(g^*)$. Proposition 2.5 implies that the deformation $L_t$ of $L$ by a $G$–invariant Hamiltonian flow is still contained in the same level set $\mu^{-1}(c)$. So we may restrict the equivariant Hamiltonian variational problem in a fixed level set.

For $c \in \text{Im}(\mu^*) \cap Z(g^*)$, we have a Riemannian submersion $\pi : (\mu^{-1}(c), ds^2) \to (\mu^{-1}(c)/G, \widetilde{\omega}^2)$, whose fibers are the principal orbits of the $G$–action. Obviously, $\mu^{-1}(c)$ and $\mu^{-1}(c)/G$ are open dense submanifolds of the stratified spaces $\mu^{-1}(c)$ and $\mu^{-1}(c)/G$ respectively. According to [HsLa], we define the volume function of the orbits as follows:

$$
V : \mu^{-1}(c)/G \to \mathbb{R}^+
$$

$$
x \mapsto \text{Vol}(\pi^{-1}(x)).
$$

(2)

Let $i : L \to \mu^{-1}(c) \subset M$ be a $G$–invariant Lagrangian submanifold, and for simplicity assume that $i(L) \cap \mu^{-1}(c) \neq \emptyset$. (There is no loss of generality in this assumption since $\mu^{-1}(c)$ may always be replaced by certain natural substrata for which the assumption holds and to which all subsequent arguments apply.) The cohomogeneity of $L$ is defined as the integer $\dim L - v$, where $v$ is the common dimension of the principal orbits. Obviously, if $L$ is of cohomogeneity $k$, then it project to a map $\widetilde{i} : L/G \to \mu^{-1}(c)/G$ such that $\widetilde{i}|L^*/G : L^*/G \to \mu^{-1}(c)/G$ is a $k$–dimensional Lagrangian submanifold of $(\mu^{-1}(c)/G, \widetilde{\omega})$. We will denote $L^*/G$ by $\widetilde{L}$. The Hsiang-Lawson metric on $\mu^{-1}(c)/G$ is defined as follows (cf. [HsLa]):

$$
\widetilde{g}_{HL} = V^{2/k} \widetilde{g},
$$

(3)

which goes continuously to zero at the singular boundary.
Theorem 2.6. Let $i : L \to M$ be a $G$–invariant Lagrangian submanifold of $M$ with $L \subset \mu^{-1}(c)$ for some $c \in \mathbb{Z}(g^*)$. Then $L$ is a $H$–minimal Lagrangian submanifold of $M$ if and only if $i : \tilde{L} \to \mu^*(c)/G$ is a $H$–minimal Lagrangian submanifold of $(\mu^*(c)/G, \tilde{\omega}, \tilde{g}_{HL})$. Furthermore, if $L$ is of cohomogeneity $k$, then $\tilde{L}$ is $H$–minimal if and only if

$$\delta_{HL}(V^{4/k} \hat{H} \tilde{\omega}) = 0,$$

where $\delta_{HL}$ is the codifferential operator w.r.t. the metric $\tilde{g}_{HL}$ and $\hat{H}$ is the mean curvature vector field of the Lagrangian submanifold $\tilde{L} \hookrightarrow (\mu^*(c)/G, \tilde{\omega}, \tilde{g}_{HL})$.

Proof. Denote by $H_G^\infty(T \mu^*(c))$ the set of $G$–invariant horizontal vector fields on $\mu^*(c)$ and $C^\infty(T(\mu^*(c)/G))$ the set of vector fields on $\mu^*(c)/G$. It is easy to see that $\pi_* : H_G^\infty(T \mu^*(c)) \to C^\infty(T(\mu^*(c)/G))$ is a bijective correspondence. For any $W, X \in H_G^\infty(T \mu^*(c))$, we have by Proposition 2.2 that

$$(W \lrcorner \omega)(X) = \omega(W, X) = (\pi^* \tilde{\omega})(W, X) = \tilde{\omega}(\tilde{W}, \tilde{X}) = (\tilde{W} \lrcorner \tilde{\omega})(\tilde{X})$$

where $\tilde{W} = \pi_* W$ and $\tilde{X} = \pi_* X$. Obviously, $W|_{\tilde{L}}$ is a Hamiltonian vector field along $L$ w.r.t. $\omega$ if and only if $\tilde{W}|_{\tilde{L}}$ is a Hamiltonian field along $\tilde{L}$ w.r.t. $\tilde{\omega}$. On the other hand, (3) implies that:

$$Vol(L, g) = Vol(\tilde{L}, \tilde{g}_{HL}).$$

Thus the first part of the Theorem follows immediately from Lemma 2.4.

Let $f$ be an arbitrary compactly supported function on $\tilde{L}$, which determines a Hamiltonian normal vector field $\tilde{W}$ along $\tilde{L}$, i.e., $\tilde{W} \lrcorner \tilde{\omega} = df$. By the first variational formula we have

$$\frac{dVol((\tilde{L}, \tilde{g}_{HL}))}{dt}|_{t=0} = - \int_L < \hat{H}, \tilde{W} \tilde{g}_{HL} > dVol_{HL}$$

$$= - \int_L < J\hat{H}, J\tilde{W} \tilde{g}_{HL} > dVol_{HL}$$

$$= - \int_L V^{4/k} < \hat{H} \tilde{\omega}, \tilde{W} \tilde{\omega} \tilde{g}_{HL} > dVol_{HL}$$

$$= - \int_L V^{4/k} < \hat{H} \tilde{\omega}, df \tilde{g}_{HL} > dVol_{HL}$$

$$= - \int_L \delta_{HL}(V^{4/k} \hat{H} \tilde{\omega}) f dVol_{HL}. $$

Then (4) follows immediately from (5). □
Corollary 2.7. Under the assumption of Theorem 2.6, if $\tilde{L}$ is minimal in $(\mu^{-1}(c)/G, \bar{g}_{HL})$ then $L$ is $H$–minimal in $M$.

Theorem 2.6 is interesting and particular simple when all the $G$–orbits are isometric. In this case, the volume function of orbits is constant, and thus the metrics $\bar{g}, \bar{g}_{HL}$ are equivalent.

Corollary 2.8. If all $G$–orbits in $\mu^{-1}(c)$ are mutually isometric, then $L \subset \mu^{-1}(c)$ is a $G$–invariant $H$–minimal submanifold in $M$ if and only if $\pi(L)$ is a $H$–minimal submanifold in $(\mu^{-1}(c)/G, \bar{w}, \bar{g})$.

Let us consider an important special case of Corollary 2.8. Define a $S^1$–action on $C^n$ by

$$e^{i\theta}(z_1, \ldots, z_n) = (e^{i\theta}z_1, \ldots, e^{i\theta}z_n).$$

This is a Hamiltonian action with moment map

$$\mu(z) = -\frac{i}{2}|z|^2.$$

Its level set at a value $-\frac{i}{2}t$ is $S^{2n-1}(\sqrt{t})$. The symplectic reduction at a regular value $-\frac{i}{2}t$ ($t > 0$) gives a fibration $\mu^{-1}(\frac{i}{2}t) \to \mu^{-1}(\frac{i}{2}t)/S^1$. In particular, we have the well-known Hopf-fibration $\pi : S^{2n-1} \to CP^{n-1}$ by taking $t = 1$. Since all $S^1$–orbits in $\mu^{-1}(\frac{i}{2}) = S^{2n-1}$ are isometric, we obtain immediately from Corollary 2.8 the following result:

**Theorem 2.9.** Let $\pi : S^{2n-1} \to CP^{n-1}$ be the Hopf fibration. Let $\tilde{L}^{n-1} \hookrightarrow CP^{n-1}$ be a Lagrangian submanifold and $L^n = \pi^{-1}(\tilde{L}^{n-1})$ the inverse image of $\tilde{L}^{n-1}$ by the Hopf projection. Then $L^n$ is a $H$–minimal Lagrangian submanifold in $C^n$ if and only if $\tilde{L}^{n-1}$ is a $H$–minimal Lagrangian submanifold in $CP^{n-1}$.

**Remark 2.2.** (i) In [Oh2], it was proved that the inverse image $\pi^{-1}(\tilde{L}^{n-1})$ is a $H$–minimal Lagrangian submanifold in $C^n$ provided that $\tilde{L}^{n-1}$ is a usual minimal Lagrangian submanifold in $CP^{n-1}$. So Theorem 2.9 generalizes Oh’s result; (ii) In [HR2], the authors constructed $H$–minimal Lagrangian tori in $CP^2$, which are not minimal, by integrable system method. By applying Theorem 2.9 to their examples, we can get a large number of non-trivial $H$–minimal $T^d$ in $C^3$; (iii) Corollary 2.7 may also be regarded as a generalization of Oh’s result in another direction.

If the Lagrangian submanifold is of cohomogeneity one, we may simplify the equation (4) as follows:

**Corollary 2.10.** Let $L \hookrightarrow \mu^{-1}(c) \subset M$ be a $G$–invariant Lagrangian submanifold of cohomogeneity one, then $L$ is $H$–minimal if and only if

$$V^2k_L = K,$$

where $k_L$ is the mean curvature of the curve $\tilde{L}$ in $(\mu^*^{-1}(c)/G, \bar{g}_{HL})$ and where $K$ is any constant.
Proof. Let $\hat{e}_1$ be the unit tangent vector field of the curve $\tilde{L}$ with respect to the Hsiang-Lawson metric. From (4), we have

$$0 = \hat{e}_1(V^2\tilde{g}_{HL}(\tilde{J}\tilde{H}, \hat{e}_1))$$

$$= -\hat{e}_1(V^2\tilde{g}_{HL}(\tilde{H}, J\hat{e}_1))$$

$$= \hat{e}_1(V^2k_\tilde{L}),$$

i.e., $V^2k_\tilde{L} = \text{const.}$ \qed

Remark 2.3. Corollary 2.10 reduces the third order P.D.E (1) to a second order O.D.E. (8) with a constant $K$.

In the remaining two sections, we will use Corollary 2.10 to construct Hamiltonian minimal submanifolds of cohomogeneity one in $CP^n$ and $C^n$ respectively.

3. Hamiltonian minimal Lagrangian submanifolds in $CP^n$

3.1. $SO_n$–invariant $H$–minimal Lagrangian submanifolds.

Let $G = SO(n)$, that can be regarded as a subgroup of $PU(n + 1) = U(n + 1)/S^1$ in the natural way. The group $G$ acts on $(CP^n, \omega_F S)$ by

$$A \cdot [z] = [z_0 : \tilde{z}_1 : \cdots : \tilde{z}_n]$$

where $z = (z_0, z_1, ..., z_n)$ and $(\tilde{z}_1, ..., \tilde{z}_n)^t = A(z_1, ..., z_n)^t$. This is a Hamiltonian action on $CP^n$, whose moment map is given by

$$\mu([z]) = \frac{1}{|z|^2}(Im(z_1\bar{z}_2), ..., Im(z_1\bar{z}_n), Im(z_2\bar{z}_3), ..., Im(z_2\bar{z}_n), ..., Im(z_{n-1}\bar{z}_n)).$$

As $Z(g^*) = \{0\}$, any $G$–invariant connected Lagrangian submanifold lies in $\mu^{-1}(0)$. All points in $\mu^{-1}(0)$ may be written as

$$[x_0 : \lambda x_1 : \lambda x_2 : \cdots : \lambda x_n]$$

where $\lambda \in C$ and $x_0, x_1, ..., x_n$ are real, and normalized so that $\sum_{\alpha=1}^{n} x_{\alpha}^2 = 1$ and $x_{0}^2 + |\lambda|^2 = 1$. Therefore the orbits of $G$ in $\mu^{-1}(0)$ are $O_{\lambda}$ for $\lambda \in C$, where

$$O_{\lambda} = \{[x_0 : \lambda x_1 : \lambda x_2 : \cdots : \lambda x_n] : x_\alpha \in R, x_{0}^2 + |\lambda|^2 = 1, \sum_{i=1}^{n} x_{i}^2 = 1\}.$$ 

The orbit space $\mu^{-1}(0)/G$ may be parameterized as

$$\mu^{-1}(0)/G = \{[\sqrt{1-r^2}, re^{i\theta}, 0, ..., 0] : 0 \leq r \leq 1, \theta \in [0, 2\pi)\}.$$ 

Note that $r = 0$ and $r = 1$ correspond to a singular orbit and an exceptional orbit respectively. Each orbit in $\mu^{*^{-1}}(0)/G$ has the following unique representative element in $S^{2n+1}(1)$

$$F(r, \theta) = (\sqrt{1-r^2}, 0, r \cos \theta, r \sin \theta, 0, ..., 0) \in S^{2n+1}(1),$$

$$0 < r < 1, \quad 0 \leq \theta < 2\pi.$$
At $F(r, \theta)$, the unit vertical vector $\eta$ of the Hopf fibration $\pi_H : S^{2n+1}(1) \to CP^n$ is given by

$$\eta(r, \theta) = (0, \sqrt{1-r^2}, -r \sin \theta, r \cos \theta, 0, \ldots, 0).$$

To determine the tangent space of the $G$–orbit $O_\lambda$, we consider any tangent vector $X = (0, v_2, \ldots, v_n) \in T_pS^{2n-1}(1)$ at $p = (1, 0, \ldots, 0) \in S^{2n-1}(1)$. Set

$$\xi_X = (0, 0, 0, 0, rv_2 \cos \theta, rv_2 \sin \theta, \ldots, rv_n \cos \theta, rv_n \sin \theta) \in R^{2n+2} = C^{n+1}.$$

Obviously $< \xi_X, \eta > = 0$, and thus $\text{Span}\{(\pi_H)_*\xi_X : X \in T_pS^{2n-1}(1)\}$ is just the tangent space of the $G$–orbit at the corresponding point. From (11), we have

$$dF\left(\frac{\partial}{\partial r}\right) = \left(\frac{-r}{\sqrt{1-r^2}}, 0, \cos \theta, \sin \theta, 0, \ldots, 0\right)$$

and

$$dF\left(\frac{\partial}{\partial \theta}\right) = (0, 0, -r \sin \theta, r \cos \theta, 0, \ldots, 0).$$

Obviously $< dF\left(\frac{\partial}{\partial r}\right), \eta > = < dF\left(\frac{\partial}{\partial r}\right), \xi_X > = 0$. So,

$$|\pi_*\pi_H*(dF\left(\frac{\partial}{\partial r}\right))|^2 = \frac{1}{1-r^2}.$$

The horizontal component (w.r.t. $\pi_H$) of $dF\left(\frac{\partial}{\partial \theta}\right)$ is given by:

$$F_\theta := dF\left(\frac{\partial}{\partial \theta}\right) - r^2 \eta.$$

From (12), (13) and (14), we see that $< F_\theta, \xi_X > = 0$. Then

$$|\pi_*\pi_H*(F_\theta)|^2 = r^2 - r^4.$$

Also $< dF\left(\frac{\partial}{\partial r}\right), F_\theta > = 0$. Hence the induced metric on the orbit space $\mu^{-1}(0)/G$ is given by

$$\tilde{g} = \frac{1}{1-r^2} dr^2 + r^2 (1-r^2)d\theta^2.$$

Up to a constant the volume function of the orbits is $r^{n-1}$. Therefore the Hsiang-Lawson metric on $\mu^{-1}(0)/G$ is given by

$$\tilde{g}_{HL} = r^{2n-2} \left[\frac{1}{1-r^2} dr^2 + r^2 (1-r^2)d\theta^2\right]$$

If we set $r = \sin \varphi$, $\tilde{g}_{HL}$ can be expressed as

$$\tilde{g}_{HL} = \sin^{2n-2} \varphi [d\varphi^2 + \sin^2 \varphi \cos^2 \varphi d\theta^2],$$

$$0 \leq \varphi \leq \pi/2, \ 0 \leq \theta < 2\pi.$$
Here $\theta$ is the rotational parameter, and $\varphi$ is the radial parameter. Note that $\varphi = 0$ and $\varphi = \frac{\pi}{2}$ correspond to singular points on $\mu^{-1}(0)/G$.

Observe that the metric is invariant under the rotation in $\theta$ and the reflection $\theta_0 + \theta \to \theta_0 - \theta$ for any $\theta_0$. So $\theta \equiv \text{const.}$ are all geodesics on $(\mu^{-1}(0)/G)$, whose inverse images are mutually congruent in $CP^n$. The congruence class corresponds to the totally geodesic Lagrangian immersion $S^n \to RP^n \subset CP^n$. We are not interested in this case.

Let $\varphi(\theta)$ be any curve in $\mu^*(0)/G$, where $\theta$ is now allowed to vary over all real numbers. The unit tangent vector field and the normal vector field of $(\theta, \varphi(\theta))$ are given respectively by

$$e = \frac{1}{\sin^{n-1} \varphi \sqrt{(\varphi')^2 + \sin^2 \varphi \cos^2 \varphi}}(\varphi' \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \theta})$$

and

$$n = \frac{1}{\sin^n \varphi \cos \varphi \sqrt{((\varphi')^2 + \sin^2 \varphi \cos^2 \varphi}}(-\sin^2 \varphi \cos^2 \varphi \frac{\partial}{\partial \varphi} + \varphi' \frac{\partial}{\partial \theta}).$$

For any variation $\varphi + s\eta$ of $\varphi$, we have the corresponding variation vector field

$$\xi = \eta \frac{\partial}{\partial \varphi}.$$ 

So we get

$$<\xi, n> = -\frac{\eta \sin^n \varphi \cos \varphi}{\sqrt{(\varphi')^2 + \sin^2 \varphi \cos^2 \varphi}}$$

By definition, $k_L = <\nabla e^H, n>_H$, where $\nabla^H$ denotes the Levi-Civita connection of the metric $\tilde{g}_H$. From Corollary 2.10, we know that the $H$–minimal equation for $\tilde{L}$ is

$$k_L \sin^{2n-2} \varphi = K.$$ 

It is easy to see from the first variation formula of the arc length w.r.t. the variation (17) that (19) is the Euler-Lagrange equation for the following functional:

$$J = \int \left[\sin^{n-1} \varphi \sqrt{\sin^2 \varphi \cos^2 \varphi + (\varphi')^2} - \frac{K}{2} \sin^2 \varphi\right]d\theta.$$ 

By a direct computation from (20), we get the E-L equation for $\varphi(\theta)$:

$$\frac{\sin^{n-1} \varphi}{(\sqrt{\sin^2 \varphi \cos^2 \varphi + (\varphi')^2})^3}\{-\varphi'' \sin \varphi \cos \varphi + [(n + 1) \cos^2 \varphi - 2 \sin^2 \varphi](\varphi')^2$$

$$+ \sin^2 \varphi \cos^2 \varphi[n \cos^2 \varphi - \sin^2 \varphi]\} = K,$$

where $K$ is a constant. We will assume $K \neq 0$, because this condition keeps the $H$–minimal submanifolds from being minimal submanifolds. Set

$$L(\theta, \varphi, \varphi') = \sin^{n-1} \varphi \sqrt{\sin^2 \varphi \cos^2 \varphi + (\varphi')^2} - \frac{K}{2} \sin^2 \varphi.$$
We perform a Legendre transformation
\[ p = L\varphi' = \frac{\varphi' \sin^{n-1} \varphi}{\sqrt{\sin^2 \varphi \cos^2 \varphi + (\varphi')^2}}. \]

The Hamiltonian \( H \) of the equation (21) is defined as
\[
H(\theta, \varphi, p) = \varphi' p - L = -\sin \varphi \cos \varphi \sqrt{\sin^{2n-2} \varphi - p^2} + \frac{K}{2} \sin^2 \varphi.
\]

Note that \( H \) does not depend explicitly on the variable \( \theta \). So \( H \) is a constant of motion, i.e., constant along any solution of the equation (cf. [JL]). It follows from (22) that
\[
\frac{\sin^{n+1} \varphi \cos^2 \varphi}{\sqrt{\sin^2 \varphi \cos^2 \varphi + (\varphi')^2}} = \lambda + \frac{K}{2} \sin^2 \varphi,
\]
where \( \lambda \) and \( K \) are constants.

We will solve the ODE (21) by considering the following initial conditions
\[
(24) \quad \begin{align*}
\varphi(0) &= a \in (0, \frac{\pi}{2}), \\
\varphi'(0) &= b.
\end{align*}
\]

So \( \lambda \) is determined by the initial conditions from (23) as follows
\[
(25) \quad \lambda = \frac{\sin^{n+1} a \cos^2 a}{\sqrt{\sin^2 a \cos^2 a + b^2}} - \frac{K}{2} \sin^2 a.
\]

At least the ODE (23) with the initial values (24) can be solved locally. Any such a solution gives a (local) \( H \)-minimal Lagrangian submanifold in \( CP^n \). We now give the initial values \((a, b)\) to ensure the global existence of the solution.

**Lemma 3.1.** For any initial values \((a, b)\) with \( \lambda \notin [0, -\frac{K}{2}] \) or \([-\frac{K}{2}, 0]\) according to \( K < 0 \) or \( K > 0 \), there is a unique global solution of (21) satisfying the initial conditions (24).

**Proof.** Set \( \psi = \varphi' \). Then we may rewrite the ODE (21) as an ODE system of first order for \((\theta, \varphi, \psi)\) on the domain \((-\infty, \infty) \times (0, \frac{\pi}{2}) \times (-\infty, \infty)\). Under the hypothesis in the Lemma, we see from (23) that there exists no finite value \( \theta_0 \in \mathbb{R} \) such that \( \varphi(\theta) \to 0 \), \( \frac{\pi}{2} \) or \( \psi(\theta) \to \infty \) as \( \theta \to \theta_0 \). Hence the local solution may be extended to a global solution. \( \square \)

Now we hope to determine the initial values for which the corresponding \( \lambda \) satisfies the condition of Lemma 3.1. First, if \( K < 0 \), it is easy to see from (25) that \( \lambda \notin [0, -\frac{K}{2}] \) is equivalent to
\[
(26) \quad \frac{\sin^{n+1} a}{\sqrt{\sin^2 a \cos^2 a + b^2}} > -\frac{K}{2}.
\]
Similarly, if $K > 0$, the condition $\lambda \notin [-\frac{K}{2}, 0]$ is equivalent to

\begin{equation}
\frac{\sin^{n-1} a \cos^2 a}{\sqrt{\sin^2 a \cos^2 a + b^2}} > \frac{K}{2}.
\end{equation}

Obviously we can always find initial values $(a, b)$ such that (26) or (27) is satisfied, provided that $|K|$ is small enough. Hence the ODE (21) has a global solution for any initial values (24) which satisfy (26) or (27) according to $K < 0$ or $K > 0$. From the proof of Lemma 3.1, we see that $\inf_{(-\infty, \infty)} \sin^2 \varphi(\theta) \cos^2 \varphi(\theta) = B > 0$ for the global solution. The length of $\varphi(\theta)$ with respect to $\tilde{g}$ given by (15) is

$$L(\varphi) = \int_{-\infty}^{\infty} \sqrt{(\varphi')^2 + \sin^2 \varphi \cos^2 \varphi} d\theta$$

$$\geq \sqrt{B} \int_{-\infty}^{\infty} d\theta$$

$$= \infty,$$

i.e., the solution curve has infinite length. As a Riemannian manifold of one dimension, the solution curve is complete. Since the fibres of the projection $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ are compact, it is easy to prove that the corresponding $H$–minimal submanifold $\pi^{-1}(\tilde{L})$ is complete as a metric space. So it is complete via Hopf-Rinow Theorem. We have proved

**Theorem 3.2.** There exist infinitely many non-trivial complete $H$–minimal Lagrangian immersions of $R^1 \times S^{n-1}$ into $CP^n$.

If $\varphi = \varphi(\theta)$ corresponds to a closed curve, then there is some point, which we may assume is $\theta = 0$, at which $\varphi$ assumes a maximum or minimum. Hence we consider the following initial conditions

\begin{equation}
\varphi(0) = a \\
\varphi'(0) = 0
\end{equation}

for $a \in (0, \frac{\pi}{2})$.

According to Lemma 3.1, we will choose the initial value $a$ such that

$$a \in I_K := \{ x \in (0, \frac{\pi}{2}) | \sin^2 x \sec x > -\frac{K}{2} \text{ for } K < 0 \}
\text{ or } \sin^{n-2} x \cos x > \frac{K}{2} \text{ for } K > 0 \}.$$ 

So we get a global solution $\varphi$ for such initial values. If $\varphi''(0) = 0$, then $\varphi \equiv \text{const.}$ and the constant solution can be determined from (21).

Without lose of generality, we assume that $\varphi''(0) < 0$. From (23), we have

\begin{equation}
\frac{d\varphi}{d\theta} = \pm \frac{\sin \varphi \cos \varphi \sqrt{\sin^{2n} \varphi \cos^2 \varphi - (\lambda + \frac{K}{2} \sin^2 \varphi)^2}}{\lambda + \frac{K}{2} \sin^2 \varphi}
\end{equation}
and thus

\begin{equation}
\theta = \pm \int_\varphi^{\varphi(a)} \frac{(\lambda + \frac{K}{2} \sin^2 \varphi) d\varphi}{\sin \varphi \cos \varphi \sqrt{\sin^2 \varphi \cos^2 \varphi - (\lambda + \frac{K}{2} \sin^2 \varphi)^2}}.
\end{equation}

Set $f(x) := \sin^{2n} x \cos^2 x - (\lambda + \frac{K}{2} \sin^2 x)^2$. We get from (28) and (29) that $f(a) = 0$ and

\[ \frac{d^2 \varphi}{d\theta^2}(0) = \frac{f'(a)}{2 \sin^{2n-2} a} < 0 \]

i.e., $f'(a) < 0$. So $f(a - \varepsilon) > 0$ for small $\varepsilon > 0$. On the other hand, $f(0) = -\lambda^2 < 0$. Thus there is $b \in (0, a)$ such that $f(b) = 0$. Set $\hat{b} = \max\{b : f(b) = 0, 0 < b < a\}$ and

\begin{equation}
\Omega_a = \min\{\theta | \varphi(\theta) = \hat{b}, \theta \in (0, +\infty)\}.
\end{equation}

Then $\varphi$ is a decreasing function on $[0, \Omega_a]$ and $\varphi(\Omega_a) = \hat{b}$. By (30), $\Omega_a$ is given by

\begin{equation}
\Omega_a = -\int_\hat{b}^{b} \frac{(\lambda + \frac{K}{2} \sin^2 \varphi) d\varphi}{\sin \varphi \cos \varphi \sqrt{\sin^2 \varphi \cos^2 \varphi - (\lambda + \frac{K}{2} \sin^2 \varphi)^2}}.
\end{equation}

We also note that the solution of (21) is invariant under the reflection

\[ \theta_0 + \theta \rightarrow \theta_0 - \theta \]

for any $\theta_0$. By reflection at points $\{0, \pm n\Omega_a ; n = 1, 2, \ldots\}$, we get a global solution $\bar{\varphi}(\theta)$ on $(-\infty, \infty)$ with period $\Omega_a$. By uniqueness Theorem of ODE, $\varphi \equiv \bar{\varphi}$. Obviously, the solution curve $\bar{L}$ is closed if and only if $\Omega_a$ is a rational multiple of $\pi$. Since $\Omega_a$ is a non-constant continuous function of $a$, we may obtain countable many such closed curves. Set

\[ A_n(K) = \{a \in I_K | \Omega_a/\pi \text{ is rational}\}. \]

Let $L_a$ denote the inverse image in $\mathbb{C}P^n$ of the closed solution curve $\varphi(\theta)$ with the initial value $a \in A_n(K)$. Then we have

**Theorem 3.3.** There exist countable infinite non-trivial closed $H$–minimal Lagrangian submanifolds $\{L_a\}_{a \in A_n(K)}$ in $\mathbb{C}P^n$, which are invariant under the $SO(n)$ – action.

**Remark 3.1.** These submanifolds $\{L_a\}$ are immersions of $S^1 \times S^{n-1}$ in $\mathbb{C}P^n$.

**3.2. $T^{n-1}$–invariant $H$–minimal Lagrangian submanifolds.**

We only consider the following simple $T^{n-1}$ – action on $\mathbb{C}P^n$:

\begin{equation}
(e^{i\theta_1}, \ldots, e^{i\theta_{n-1}}) \cdot [z] = [z_0 : e^{i\theta_1} z_1 : \cdots : e^{i\theta_{n-1}} z_{n-1} : z_n]
\end{equation}

whose moment map is

\[ \mu([z]) = -\frac{i}{2|z|^2}(|z_1|^2, |z_2|^2, \ldots, |z_{n-1}|^2). \]
As $G = T^{n-1}$ is Abelian, $Z(g^*) = g^*$. Choose $c_i \in R$ such that $c_j > 0$ ($j = 1, ..., n-1$) and $\sum_{j=1}^{n-1} c_j < 1$. Set $c = -\frac{i}{2}(c_1, c_2, ..., c_{n-1})$. Then we have the level set

$$
\mu^{-1}(c) = \{ [z] \in CP^n : z \in S^{2n+1}, |z_j|^2 = c_j, 1 \leq j \leq n-1 \}.
$$

We may parametrize the orbit space $\mu^{-1}(c)/G$ as follows:

$$
(r, \theta) \xrightarrow{F} \left( \sqrt{1 - \sum_{j=1}^{n-1} c_j - r^2}, 0, \sqrt{c_1}, 0, \sqrt{c_2}, 0, ..., \sqrt{c_{n-1}}, 0, r \cos \theta, r \sin \theta \right).
$$

From (35), we have

$$
\begin{align*}
\frac{\partial F}{\partial r} &= \left( -\frac{r}{\sqrt{1 - \sum_{j=1}^{n-1} c_j - r^2}}, 0, ..., 0, \cos \theta, \sin \theta \right), \\
\frac{\partial F}{\partial \theta} &= (0, ..., 0, -r \sin \theta, r \cos \theta).
\end{align*}
$$

Set

$$
\xi_j = (0, ..., 0, \sqrt{c_j}, 0, ..., 0), 1 \leq j \leq n-1.
$$

Then $d\pi_H(\xi_j)$ ($j = 1, ..., n-1$) span the tangent space of the $G$–orbit at the corresponding point, where $\pi_H : S^{2n+1} \to CP^n$ is the Hopf fibration. It is easy to see from (36), (37) that the volume function of the orbits is constant. Using the similar method as in section 3.1, we can also get from (36), (37) the Hsiang-Lawson metric on $\mu^{-1}(c)/G$ (up to a constant):

$$
\tilde{g}_{HL} = \frac{1}{\delta - r^2} dr^2 + \frac{r^2(\delta - r^2)}{\delta^2} d\theta^2,
$$

where $\delta = 1 - \sum_{j=1}^{n-1} c_j$. If we introduce $r = \sqrt{\delta} \sin \varphi$, then

$$
\tilde{g}_{HL} = d\varphi^2 + \sin^2 \varphi \cos^2 \varphi d\theta^2,
$$

where $0 \leq \varphi < \frac{\pi}{2}$ and $0 \leq \theta < 2\pi$. From Corollary 2.10, we know that $H$–minimal equation for $\tilde{L}$ is

$$
k_{\tilde{L}} = K,
$$

where $K$ is a constant.

Write $\tilde{L}$ as $\varphi = \varphi(\theta)$. Similar to the discussion in section 3.1, we see that (40) is just the Euler-Lagrange equation of the following functional

$$
J = \int_{\theta_1}^{\theta_2} (\sqrt{(\varphi')^2 + \sin^2 \varphi \cos^2 \varphi} - \frac{K}{2} \sin^2 \varphi) d\theta
$$

Since (41) is only a special case of (20), we obtain the following:

**Theorem 3.4.**

(i) There exist infinitely many non-trivial complete $H$–minimal Lagrangian immersions of $R^1 \times T^{n-1}$ into $CP^n$;

(ii) There exist countable infinite non-trivial $H$–minimal Lagrangian immersions of $T^n$ into $CP^n$. 

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4. Hamiltonian minimal Lagrangian submanifolds in $C^{n+1}$

4.1. Inverse images of the Hopf map.

From Theorem 2.9, Theorem 3.3 and Theorem 3.4, we immediately have the following:

**Theorem 4.1.** There exist infinitely many non-trivial $H$–minimal Lagrangian immersions of $R^1 \times S^1 \times S^{n-1}$, $S^1 \times S^1 \times S^{n-1}$, $R^1 \times T^n$ and $T^{n+1}$ into $C^{n+1}$.

In the rest of this paper, we will consider two Hamiltonian actions on $C^{n+1}$, which were used to construct special Lagrangian submanifolds in $C^{n+1}$ by Havey and Lawson [HL](cf. also [J2]). We now use them to construct some new non-trivial complete $H$–minimal Lagrangian submanifolds in $C^{n+1}$.

4.2. $SO_{n+1}$–invariant $H$–minimal Lagrangian submanifolds.

Let $G = SO(n + 1) \subset SU(n + 1)$, which acts on $C^{n+1}$ ($n > 1$) in the following way:

$$\gamma \cdot z = (\gamma x, \gamma y), z \in C^{n+1}, \gamma \in SO(n + 1)$$

where we write $z = x + iy$. Then the moment map of the action is

$$\mu(z_1, ..., z_{n+1}) = (Im(z_1\overline{z}_2), ..., Im(z_1\overline{z}_{n+1}), Im(z_2\overline{z}_3), ..., Im(z_2\overline{z}_{n+1}), ..., Im(z_n\overline{z}_{n+1}))$$

As $Z(g^*) = \{0\}$, any $G$–invariant connected Lagrangian submanifold is contained in $\mu^{-1}(0)$.

Obviously any point of $\mu^{-1}(0)$ may be written as $(\lambda x_1, ..., \lambda x_{n+1})$ with $\lambda \in C$, $x_1, ..., x_{n+1} \in R$ and $x_1^2 + \cdots + x_{n+1}^2 = 1$. So a $G$–orbit in $\mu^{-1}(0)$ is

$$O_0 = \{(\lambda x_1, \cdots, \lambda x_{n+1}) : x_j \in R, x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

Clearly $O_0$ is a point, and $O_\lambda = O_{-\lambda} \cong S^n$ if $\lambda \neq 0$. So the orbit space $\mu^{-1}(0)/G$ is

$$\mu^{-1}(0)/G = \{[(\lambda, 0, ..., 0)] : \lambda \in C\}.$$

which has the following parametrization

$$(r, \theta) \rightarrow [(r \cos \theta, r \sin \theta, 0, ..., 0)].$$

It is easy to see that the induced metric $\tilde{g}$ on the orbit space is flat, i.e.,

$$\tilde{g} = dr^2 + r^2 d\theta^2.$$  

(43) Here $V = r^n$.

The volume of the orbit space at $\lambda$ is (up to a constant):

$$V = r^n.$$ 

So, the Hsiang-Lawson metric is given by

$$\tilde{g}_{HL} = r^{2n}(dr^2 + r^2 d\theta^2).$$
Obviously, \( \theta \equiv \text{const.} \) is a geodesic (\( \mu^{-1}(0)/G, \tilde{g}_{H\ell} \)), whose inverse image in \( C^{n+1} \) is a \((n + 1)-\text{dim Lagrangian plane passing through the origin of } C^{n+1}\). Now we allow \( \theta \) to vary over all real number and write the curve on the orbit space as \( r = r(\theta) \). Similar to the above discussion, we see that the Hamiltonian-minimal equation for \( r(\theta) \) is

\[
\left( r^2 + r^2 \right)^{3/2} \left( -r \ddot{r} + 2 \dot{r}^2 + r^2 \right) + \frac{n}{\sqrt{r^2 + r^2}} = \frac{K}{r^n},
\]

which is a critical point of the following functional

\[
J = \int \left( r^n \sqrt{r^2 + \dot{r}^2} - \frac{K}{2} r^2 \right) d\theta,
\]

where \( K \) is a nonzero constant. Since \( r = 0 \) corresponds to a singular point of the orbit space, we will solve the equation (44) on \( (\mu^{-1}(0)/G) - \{ r = 0 \} \). Set

\[
L(\theta, r, \dot{r}) = r^n \sqrt{r^2 + \dot{r}^2} - \frac{K}{2} r^2
\]

and

\[
p = L_r = \frac{r^n \dot{r}}{\sqrt{r^2 + \dot{r}^2}}.
\]

We get the Hamiltonian for the equation:

\[
H(\theta, r, p) = -r \sqrt{r^2 n - p^2} + \frac{K}{2} r^2.
\]

Since \( H(\theta, r, p) \) does not depend on \( \theta \) explicitly, it must be a constant of the motion. It follows from (46) and (47) that

\[
\frac{r^{n+2}}{\sqrt{r^2 + \dot{r}^2}} = \lambda + \frac{K}{2} r^2,
\]

where \( \lambda, K \) are constants. For initial values

\[
r(0) = a > 0,
\]

\[
r'(0) = b,
\]

we have

\[
\lambda = \frac{a^{n+2}}{\sqrt{a^2 + b^2}} - \frac{K}{2} a^2.
\]

From (48), we see that there exists no point \( \theta_0 \) such that \( r(\theta) \to 0 \) as \( \theta \to \theta_0 \), provided that \( \lambda \neq 0 \). Any local solution of (44) with the initial values (49) corresponds to a local \( H \)-minimal submanifolds in \( C^{n+1} \).

In following, we always consider the initial values with \( \lambda \neq 0 \). From (48), we have

\[
\frac{dr}{d\theta} = \pm \frac{r^2 (n+1) - \left( \lambda + \frac{K}{2} r^2 \right)^2}{\lambda + \frac{K}{2} r^2}.
\]
Lemma 4.2. If the solution curve $r = r(\theta)$ of (44) is closed, then $r \equiv \left(\frac{K}{n+1}\right)^{1/(n-1)}$ ($K > 0$).

Proof. Since $r = r(\theta)$ is closed, then there are two points $\theta_1$ and $\theta_2$, at which $r$ assumes the maximum and minimum respectively. Set $A_i = r(\theta_i), i = 1, 2$. So, we have

$$
\begin{align*}
& r'(\theta_i) = 0, \ i = 1, 2, \\
& r''(\theta_1) \leq 0, \ r''(\theta_2) \geq 0.
\end{align*}
$$

(52)

Obviously, $\lambda = A_i^{n+1} - \frac{K}{2} A_i^2$. From (51), and the first equation of (52), we get

$$
\frac{d^2 r}{d\theta^2}(\theta_i) = \frac{1}{A_i^{n-2}}((n + 1)A_i^{n-1} - K).
$$

Also $A_1 \leq \left(\frac{K}{n+1}\right)^{1/n+1}$ and $A_2 \geq \left(\frac{K}{n+1}\right)^{1/(n-1)}$ by (52). Thus $r \equiv \left(\frac{K}{n+1}\right)^{1/(n-1)}$. □

We assume now that $K > 0$ and consider the following initial values

$$
\begin{align*}
& r(0) = a > 0, \\
& r'(0) = 0.
\end{align*}
$$

(53)

So we get

$$
\lambda = a^{n+1} - \frac{K}{2} a^2.
$$

(54)

The condition $\lambda \neq 0$ is equivalent to the following condition

$$
a \neq \left(\frac{K}{2}\right)^{\frac{1}{n-1}}.
$$

Set $f(x) = x^{n+1} - (\lambda + \frac{K}{2} x^2)$ for $x \in (0, \infty)$. It is easy to see that $f$ is increasing strictly on $(a, \infty)$, provided that

$$
a > \left(\frac{K}{n+1}\right)^{\frac{1}{n-1}}.
$$

(55)

Under the condition (55), we get

$$
\frac{d^2 r}{d\theta^2}(0) = \frac{1}{a^{n-2}} ((n + 1)a^{n-1} - K) > 0.
$$

It follows that $\frac{dr}{d\theta} > 0$ for $\theta \in (0, \varepsilon)$, if $\varepsilon$ is small enough. So we have from (51) that

$$
\frac{dr}{d\theta} = \frac{r \sqrt{r^{2(n+1)} - (\lambda + \frac{K}{2} r^2)^2}}{\frac{\lambda + \frac{K}{2} r^2}{16}},
$$

(56)
for \( \theta \in (0, \varepsilon) \). Since \( f \) is increasing, it is easy to see from (55) that \( \frac{dr}{d\theta} > 0 \) for \( \theta > 0 \). Let \( \theta_{\text{max}} \) be the maximal value such that the solution exits on \([0, \theta_{\text{max}}]\). From (56) we get

\[
\theta_{\text{max}} = \int_a^{r(\theta_{\text{max}})} \frac{\lambda + \frac{K}{2} \rho^2}{\rho \sqrt{\rho^{2(n+1)} - (\lambda + \frac{K}{2} \rho^2)^2}} d\rho.
\]

When \( n > 1 \), the integral (57) converges. So \( \theta_{\text{max}} \) is finite and \( r \) is an increasing function on \([0, \theta_{\text{max}}]\). By ODE theory, there are two possibilities: (i) \( \lim_{\theta \to \theta_{\text{max}}} r(\theta) = +\infty \); (ii) \( \lim_{\theta \to \theta_{\text{max}}} r'(\theta) = \infty \).

We assert that the case (ii) will not occur. For this case we will have by (48) and (54) that

\[
a^{n+1} - \frac{K}{2} a^2 + \frac{K}{2} r^2(\theta_{\text{max}}) = 0.
\]

But this is impossible, because \( r^2(\theta_{\text{max}}) > a^2 \). Hence we only have

\[
\lim_{\theta \to \theta_{\text{max}}} r(\theta) = +\infty.
\]

We note that the solution of (44) is invariant under the reflection

\[
\theta \to -\theta.
\]

By the reflection, we can get a positive solution \( r \) on \((-\theta_{\text{max}}, \theta_{\text{max}})\). Two lines \( \theta = \pm \theta_{\text{max}} \) are asymptotic lines of the solution curve. It is easy to see that the solution curve has infinite length with respect to the metric \( \tilde{g} \) given by (43). In conclusion, we have

**Theorem 4.3.**

(i) The only closed \( H \)-minimal Lagrangian submanifold invariant under the action (42) is the Lagrangian submanifold corresponding \( r \equiv \text{const.} \), which is given by

\[
S^1 \times S^n \to C^{n+1}
\]

\[
(e^{i\theta}, x_1, \ldots, x_{n+1}) \mapsto (\frac{K}{n+1})^{1/(n-1)} e^{i\theta}(x_1, \ldots, x_{n+1}),
\]

where \( x_1^2 + \cdots + x_{n+1}^2 = 1 \).

(ii) There are infinitely many non-trivial complete \( H \)-minimal Lagrangian immersions of \( R \times S^n \) into \( C^{n+1} \) which are invariant under the action (42).

**4.3. \( T^n \)-invariant \( H \)-minimal Lagrangian submanifolds.**

Let \( G \cong T^n \) be the group of diagonal matrices in \( SU(n+1) \), so that each \( \gamma \in G \) acts on \( C^{n+1} \) \((n > 1)\) by

\[
\gamma : (z_1, z_2, \ldots, z_{n+1}) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2, \ldots, e^{i\theta_{n+1}} z_{n+1})
\]

for some \( \theta_1, \ldots, \theta_{n+1} \in R \) with \( \theta_1 + \cdots + \theta_{n+1} = 0 \). The moment map of \( G \) is

\[
(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{n+1}) \mapsto -\frac{i}{2}(|\tilde{z}_1|^2 - |z_{n+1}|^2, |\tilde{z}_2|^2 - |z_{n+1}|^2, \ldots, |\tilde{z}_n|^2 - |z_{n+1}|^2).
\]
As $G$ is abelian, $Z(g^*) = g^*$. Let $c_i \in R$ such that $c_1 \cdots c_n \neq 0$ and set $c = -\frac{i}{2}(c_1, c_2, ..., c_n)$. The level set $\mu^{-1}(c)$ is given by

$$
\mu^{-1}(c) = \{(z_1, ..., z_{n+1}) : |z_1|^2 - |z_{n+1}|^2 = c_1, ..., |z_n|^2 - |z_{n+1}|^2 = c_n\}.
$$

So we can introduce the following parametrization of $\mu^{-1}(c)/G$:

$$(r, \theta) \mapsto F(\sqrt{r^2 + c_1}, \sqrt{r^2 + c_2}, ..., \sqrt{r^2 + c_n}, r \cos \theta + \sqrt{-1}r \sin \theta),$$

where $\sigma = \max_{1 \leq i \leq n} \{-c_i, 0\}$. Now we hope to derive the metric on $\mu^{-1}(c)/G$. Let $\xi_i, i = 1, ..., n$, be the standard basis of the Lie algebra of $T^n$. Then the tangent space of the orbit at $F(r, \theta)$ is spanned by

$$(59) \quad \phi(\xi_1) = (iz_1, 0, ..., 0, -iz_{n+1}) \longleftrightarrow (0, \sqrt{r^2 + c_1}, ..., r \sin \theta, -r \cos \theta)$$

$$(60) \quad \phi(\xi_n) = (0, 0, ..., iz_n, -iz_{n+1}) \longleftrightarrow (0, 0, ..., 0, \sqrt{r^2 + c_n}, r \sin \theta, -r \cos \theta)$$

Set $\Phi_{ij} = \langle \phi(\xi_i), \phi(\xi_i) \rangle$, $i, j = 1, ..., n$. From (59), we get

$$(61) \quad \Phi_{ii} = 2r^2 + c_i, \quad \Phi_{ij} = r^2 \quad \text{for} \ i \neq j.$$ 

The induce metric on each orbit is given by

$$(62) \quad ds_{T^n}^2 = \Phi_{ij} d\theta^i d\theta^j.$$ 

The volume of the orbit corresponding to the point $(r, \theta)$ is (up to a constant)

$$(63) \quad V(r, \theta) = \sqrt{\det(\Phi_{ij})}.$$ 

By a direct computation, we have

$$(64) \quad \det \Phi_{ij} = \sum_{j=1}^n r^2(r^2 + c_1) \cdots (r^2 + c_j) \cdots (r^2 + c_n) + \prod_{k=1}^n (r^2 + c_k).$$

Obviously $dF(\frac{\partial}{\partial r}) \perp \text{span}\{\phi(\xi_1), ..., \phi(\xi_n)\}$ and

$$(65) \quad \langle dF(\frac{\partial}{\partial r}), dF(\frac{\partial}{\partial r}) \rangle = \frac{\det(\Phi_{ij})}{\prod_{j=1}^n (r^2 + c_j)}.$$
Note that \( dF(\frac{\partial}{\partial \theta}) \) is not horizontal w.r.t. \( \pi : \mu^{-1}(c) \to \mu^{-1}(c)/G \). Denote by \( P_{jH} \) the projection on the horizontal space of the fibration \( \pi \). By an elementary computation, we may get

\[
|P_{jH}dF(\frac{\partial}{\partial \theta})|^2 = \frac{r^2 \prod_{k=1}^{n} (r^2 + c_k)}{\det(\Phi_{ij})}.
\]

Also \( \langle dF(\frac{\partial}{\partial r}), dF(\frac{\partial}{\partial \theta}) \rangle = 0 \). So we get from (65) and (66) the induced Kaehler metric on the orbit space as follows

\[
\tilde{g} = \frac{\det(\Phi_{ij})}{\prod_{j=1}^{n} (r^2 + c_j)} dr^2 + \frac{r^2 \prod_{k=1}^{n} (r^2 + c_k)}{\det(\Phi_{ij})} d\theta^2.
\]

Thus the Hsiang-Lawson metric on \( \mu^{-1}(c)/G \) is given by (up to a constant)

\[
\tilde{g}_{HL} = \frac{[\det(\Phi_{ij})]^2}{\prod_{k=1}^{n} (r^2 + c_k)} dr^2 + r^2 \prod_{k=1}^{n} (r^2 + c_k) d\theta^2.
\]

The \( H \)-minimal equation is

\[
V^2 k_L = K,
\]

where \( K \) is a constant. If there exists a point \( p \in \tilde{L} \) such that \( k_L(p) = 0 \), then we see that \( k_L \equiv 0 \). This corresponds the special Lagrangian submanifold found by Harvey-Lawson [HL](see Remark 4.1). The points with \( \det(\Phi_{ij}) = 0 \) correspond to the singular points on \( \mu^{-1}(c)/G \).

Let \( \tilde{L} \) be given by \( r = r(\theta) \) on \( \mu^{-1}(c)/G - \{ \det(\Phi_{ij}) = 0 \} \). By a similar method as in previous sections, we may show that (69) is the Euler-Lagrange equation of the following functional:

\[
J = \int_{\theta_1}^{\theta_2} \sqrt{\frac{[\det(\Phi_{ij})]^2}{\prod_{k=1}^{n} (r^2 + c_k)} (r')^2 + r^2 \prod_{k=1}^{n} (r^2 + c_k) - \frac{K}{2} r^2} d\theta.
\]

So the Hamiltonian for the Euler-Lagrange equation of the functional (70) is given by

\[
H(\theta, r, p) = -\frac{r \prod_{k=1}^{n} (r^2 + c_k)}{\det(\Phi_{ij})} \sqrt{\frac{[\det(\Phi_{ij})]^2}{\prod_{k=1}^{n} (r^2 + c_k)} - p^2 + \frac{K}{2} r^2},
\]

where

\[
p = L_{r'} = \frac{[\det(\Phi_{ij})]^2 r'}{\prod_{k=1}^{n} (r^2 + c_k) \sqrt{\frac{[\det(\Phi_{ij})]^2}{\prod_{k=1}^{n} (r^2 + c_k)} (r')^2 + r^2 \prod_{k=1}^{n} (r^2 + c_k)}}.
\]

Since \( H(\theta, r, p) \) doesn’t depend on \( \theta \) explicitly, it must be a constant of the motion. Thus we get from (71) and (72) that

\[
\frac{r^2 \prod_{k=1}^{n} (r^2 + c_k)}{\sqrt{\frac{[\det(\Phi_{ij})]^2}{\prod_{k=1}^{n} (r^2 + c_k)} (r')^2 + r^2 \prod_{k=1}^{n} (r^2 + c_k)}} = \lambda + \frac{K}{2} r^2.
\]
From (73), we see that there exists no point $\theta_0$ such that
\[
\lim_{\theta \to \theta_0} [r^2(\theta) \prod_{k=1}^{n} (r^2(\theta) + c_k)] = 0
\]
as $\theta \to \theta_0$, provided that $\lambda \neq 0, -\frac{K}{2}c_i$ ($i = 1, ..., n$).

To simplified the discussion, we assume now that $K > 0$ and consider the following initial values
\[
(74) \quad r(0) = a > \sqrt{\sigma}, \\
r'(0) = 0.
\]
From (73) and (74), we get
\[
(75) \quad \lambda = a \sqrt{\prod_{k=1}^{n} (a^2 + c_k) - \frac{K}{2}a^2}.
\]
In following, we always choose initial values such that $\lambda \neq 0, -\frac{K}{2}c_i$.

If $r''(0) = 0$, then the solution $r(\theta) \equiv a$ by the uniqueness Theorem of ODE. From (73), we have
\[
(76) \quad \frac{dr}{d\theta} = \pm \frac{r \prod_{k=1}^{n} (r^2 + c_k) \sqrt{r^2 \prod_{k=1}^{n} (r^2 + c_k) - (\lambda + \frac{K}{2}r^2)^2}}{\det(\Phi_{ij})(\lambda + \frac{K}{2}r^2)}.
\]
Without loss of generality, we assume that $r''(0) > 0$. By taking derivative of (76) and using (74), we see that this assumption is equivalent to
\[
(77) \quad \frac{\det(\Phi_{ij}(a))}{a \sqrt{\prod_{k=1}^{n} (a^2 + c_k)}} > K.
\]
Set $F(x) = \sqrt{x \prod_{k=1}^{n} (x + c_k) - (\lambda + \frac{K}{2}x)}$ on $(\sigma, +\infty)$. Obviously, we may choose a large enough so that (77) is satisfied and $F$ is increasing strictly on $(a^2, +\infty)$. Let $\theta_{\text{max}}$ be the maximal value such that the solution $r(\theta)$ exists on $[0, \theta_{\text{max}})$. From (76), we get
\[
(78) \quad \theta_{\text{max}} = \int_{a}^{r(\theta_{\text{max}})} \frac{\det(\Phi_{ij})(\lambda + \frac{K}{2}r^2)dr}{r \prod_{k=1}^{n} (r^2 + c_k) \sqrt{r^2 \prod_{k=1}^{n} (r^2 + c_k) - (\lambda + \frac{K}{2}r^2)^2}}.
\]
Since the integral (79) exists, $\theta_{\text{max}}$ is finite. Similar to the previous discussion in section 4.2, we see that $r(\theta)$ is a strictly increasing solution on $[0, \theta_{\text{max}})$ such that
\[
(79) \quad \lim_{\theta \to \theta_{\text{max}}} r(\theta) = +\infty.
\]
Note also that the solution of (69) (or see (73)) is invariant under the reflection $\theta \to -\theta$. 20
By the reflection, we can get a positive solution \( r \) on \((-\theta_{\text{max}}, \theta_{\text{max}})\) so that

\[
\lim_{\theta \to \pm \theta_{\text{max}}} r(\theta) = +\infty.
\]

Now we show that the solution curves \( r = r(\theta) \) are complete with respect to the metric \( \tilde{g} \) given by (67). For the solution \( r = r(\theta) \) on \((-\theta_{\text{max}}, \theta_{\text{max}})\), we have from (67) and (64) that

\[
L_{\tilde{g}}(r) = 2 \int_{0}^{\theta_{\text{max}}} \sqrt{\det(\Phi_{ij})} \frac{(r')^2 + \frac{2 \prod_{k=1}^{n} (r^2 + c_k)}{\det(\Phi_{ij})}}{\prod_{j=1}^{n} (r^2 + c_j)} d\theta
\]

\[
\geq 2 \int_{0}^{\theta_{\text{max}}} \frac{\det(\Phi_{ij})}{\prod_{j=1}^{n} (r^2 + c_j)} (r')^2 d\theta
\]

\[
= 2 \int_{0}^{\theta_{\text{max}}} r' d\theta
\]

\[
= 2 (r(\theta_{\text{max}}) - a)
\]

\[
= +\infty.
\]

In conclusion, we have proved that

**Theorem 4.4.** There are infinitely many non-trivial complete \( H \)-minimal immersions of \( R \times T^n \) into \( C^{n+1} \), which are invariant under the action (58).

**Remark 4.1.**

1. Note that the \( H \)-minimal immersions of \( R \times T^n \) in Theorem 4.4 are different from those in Theorem 4.1, because their actions of groups are different;
2. If we set \( R = \sqrt{r^2 \prod_{j=1}^{n} (r^2 + c_j)} \), then the Hsiang-Lawson metric becomes

\[
\tilde{g}_{HL} = dR^2 + R^2 d\theta^2
\]

which is actually a flat metric. Using the parametrization here, the \( T^n \)-invariant special Lagrangian \((n + 1)\)-folds in \( C^{n+1} \) of Harvey-Lawson [HL] correspond to the straight lines \( R \sin \theta \equiv \text{const.} \) or \( R \cos \theta \equiv \text{const.} \) on the orbit space \( \mu^{-1}(c)/G \). Up to a \( SO(2) \)-motion, the different straight lines in the \((x = R \cos \theta, y = R \sin \theta)\)-plane corresponds to special Lagrangian \((n + 1)\)-folds in \( C^{n+1} \) with different phases. So we see that the complete \( H \)-minimal submanifolds in Theorem 4.4 are asymptotic to two singular \( T^n \)-invariant \( \pm \theta_0 \)-special Lagrangian \((n + 1)\)-folds.

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