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A survey on fractional variational calculus

Abstract: Main results and techniques of the fractional calculus of variations are surveyed. We consider variational problems containing Caputo derivatives and study them using both indirect and direct methods. In particular, we provide necessary optimality conditions of Euler–Lagrange type for the fundamental, higher-order, and isoperimetric problems, and compute approximated solutions based on truncated Grünwald–Letnikov approximations of Caputo derivatives.

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1 Introduction

Calculus of Variations is a prolific field of mathematics, whose founders were Leonhard Euler (1707–1783) and Joseph Louis Lagrange (1736–1813). The name of the field was given by Euler in his *Elementa Calculi Variationum* and is related with the main technique used by Lagrange to derive first order necessary optimality conditions. Its development is in intimate connection with various problems occurring in biology, chemistry, control theory, dynamics, economics, engineering, physics, etc. The current development of the calculus of variations is based on modern analysis techniques [21, 22].

The Fractional Variational Calculus (FVC) is devoted to generalizations of the classic calculus of variations and its modern face, the theory of optimal control, to the case in which derivatives and integrals are understood as fractional operators of arbitrary order. The subject has been developed in the last two decades and provides nontrivial generalizations of the calculus of variations and optimal control, which open doors to new and interesting modern scientific problems. Indeed, the combination of the calculus of variations and FC is not artificial, coming naturally from more accurate descriptions of physical phenomena, in particular to better describe non-conservative systems in mechanics,
where the inclusion of non-conservatism in the theory is extremely important
[25, 26].

This chapter is organized as follows. In Section 2 we formulate the fundamental problem of the FVC. Then our survey proceed in two parts: indirect methods, in Section 3 and a direct method, in Section 4. In an indirect method, the Euler–Lagrange equations, and possible other optimality conditions as well, are used. Typically, the indirect approach leads to a boundary-value problem that should be solved to determine candidate optimal trajectories, called extremals. Direct methods, in contrast, begin by approximating the variational problem, reformulating it as a standard nonlinear optimization problem [27].

Here we restrict ourselves to scalar problems of Caputo type. However, the techniques presented are easily adapted to the vectorial case and other types of FDs and FIs. For a survey on the FVC with generalized fractional operators see [23]. For more on this fruitful area, we refer the readers to the four books on the subject and references therein: for a solid introduction to the FVC, see [19]; for computational aspects, begin with [4]; for advanced mathematical methods, including the question of existence of solutions, see [18]; for the variable order FVC and the state of the art see [5].

2 The fundamental problem

The calculus of variations deals with functional optimization problems involving an unknown function $x$ and its derivative $x'$. The general way of formulating the problem is as follows:

$$\min J(x) = \int_a^b L(t, x(t), x'(t)) \, dt$$

subject to the boundary conditions

$$x(a) = x_a \quad \text{and} \quad x(b) = x_b, \quad \text{with } x_a, x_b \in \mathbb{R}.$$  \tag{1}

One way to solve this problem consists to determine the solutions of the second order differential equation

$$\partial_2 L(t, x(t), x'(t)) = \frac{d}{dt} \partial_3 L(t, x(t), x'(t)), \quad t \in [a, b],$$  \tag{2}

together with the boundary conditions (1). Equation (2) was studied by Euler, and later by Lagrange, and is known as the Euler–Lagrange equation.
The fractional calculus of variations is a generalization of the ordinary variational calculus, where the integer-order derivative is replaced by a fractional derivative $D^\alpha x$:

$$\min_x J(x) = \int_a^b L(t, x(t), D^\alpha x(t)) \, dt.$$

As Riewe noted in [25], “traditional Lagrangian and Hamiltonian mechanics cannot be used with nonconservative forces such as friction”. FVC allows it in an elegant way.

Since there are several definitions of fractional derivatives, one finds several works on the calculus of variations for these different operators. Here we consider the Caputo fractional derivative [11], but analogous results can be formulated for other types of fractional derivatives. Our choice of derivative is based on the following important fact: the Laplace transform of the Caputo fractional derivative depends on integer-order derivatives evaluated at the initial time $t = a$. Thus, we have an obvious physical interpretation for them. For other types of differentiation, this may not be so clear. For example, when dealing with the Riemann–Liouville fractional derivative, the initial conditions are fractional.

The main problem of the fractional calculus of variations, in the context of the Caputo fractional derivative, is stated as follows. Given a functional

$$J(x) = \int_a^b L(t, x(t), C D^\alpha_{a+} x(t)) \, dt, \quad x \in \Omega,$$

where $\Omega$ is a given class of functions, find a curve $x$ for which $J$ attains a minimum value. We suppose that $L : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable with respect to the second and third variables and, for every $x \in \Omega$, the function $C D^\alpha_{a+} x$ exists and is continuous on the interval $[a, b]$. On the set of admissible functions of the problem, the boundary conditions

$$x(a) = x_a \quad \text{and} \quad x(b) = x_b, \quad \text{with} \ x_a, x_b \in \mathbb{R},$$

may be imposed. We observe that a curve $x^* \in \Omega$ is a (local) minimizer of functional (3) if there exists some positive real $\delta$ such that, whenever $x \in \Omega$ and $\|x^* - x\| < \delta$, one has $J(x^*) - J(x) \leq 0$.

3 Indirect methods

The literature about the fractional calculus of variations is very vast [4, 5, 18, 19]. The usual procedure in the calculus of variations, to find a candidate to
minimizer, consists to take variations of the optimal curve \(x^\star\). Let \(\epsilon\) be a real number close to zero and \(\eta\) a function chosen so that \(x^\star + \epsilon \eta\) belongs to \(\Omega\). For example, if the functional is defined in a class of functions with the boundary conditions (4), then the curve \(\eta\) must satisfy the conditions \(\eta(a) = 0\) and \(\eta(b) = 0\).

The curve \(x^\star + \epsilon \eta\) is called a variation of the curve \(x^\star\). For simplicity of notation, we will use the operator \([\cdot]^{\alpha}\) defined by

\[
[x]^{\alpha}(t) := (t, x(t), C \, D_{a+}^{\alpha} x(t)).
\]

Suppose that \(x^\star\) minimizes \(\mathcal{J}\), and consider a variation \(x^\star + \epsilon \eta\). Depending on the existence of boundary conditions, some constraints over \(\eta(a)\) and \(\eta(b)\) may be imposed. Consider a function \(j\), defined on a neighborhood of zero, and given by

\[
j(\epsilon) := \mathcal{J}(x^\star + \epsilon \eta) = \int_{a}^{b} L(t, x^\star(t) + \epsilon \eta(t), C \, D_{a+}^{\alpha} x^\star(t) + \epsilon C \, D_{a+}^{\alpha} \eta(t)) \, dt.
\]

Since \(\mathcal{J}\) attains a minimum value at \(x^\star\), then we conclude that \(\epsilon \neq 0\) is a minimizer of \(j\), and thus \(j'(0) = 0\). Since

\[
j'(\epsilon) = \int_{a}^{b} \partial_2 L[x^\star + \epsilon \eta]^{\alpha}(t) \eta(t) + \partial_3 L[x^\star + \epsilon \eta]^{\alpha}(t) C \, D_{a+}^{\alpha} \eta(t) \, dt,
\]

we conclude that

\[
\int_{a}^{b} \partial_2 L[x^\star]^{\alpha}(t) \eta(t) + \partial_3 L[x^\star]^{\alpha}(t) C \, D_{a+}^{\alpha} \eta(t) \, dt = 0. \quad (5)
\]

Using now fractional integration by parts \([16]\), we obtain that

\[
\int_{a}^{b} \partial_3 L[x^\star]^{\alpha}(t) C \, D_{a+}^{\alpha} \eta(t) \, dt
\]

\[
= \int_{a}^{b} D_{a-}^{\alpha} (\partial_3 L[x^\star]^{\alpha}(t)) \eta(t) \, dt + \left[ I_{b-}^{1-\alpha} (\partial_3 L[x^\star]^{\alpha}(t)) \eta(t) \right]_{a}^{b}. \quad (6)
\]

Combining equations (5) and (6), we prove that

\[
\int_{a}^{b} \left[ \partial_2 L[x^\star]^{\alpha}(t) + D_{b-}^{\alpha} (\partial_3 L[x^\star]^{\alpha}(t)) \right]^{\alpha} \eta(t) \, dt
\]

\[
+ \left[ I_{b-}^{1-\alpha} (\partial_3 L[x^\star]^{\alpha}(t)) \eta(t) \right]_{a}^{b} = 0. \quad (7)
\]
Assume that the set of admissible functions have to satisfy the boundary constraints \( (4) \). In this case, the variation function \( \eta \) must satisfy the conditions \( \eta(a) = 0 \) and \( \eta(b) = 0 \). Thus, equation \( (7) \) becomes

\[
\int_{a}^{b} \left[ \partial_2 L[x^*]^{\alpha}(t) + D_{b-}^{\alpha} (\partial_3 L[x^*]^{\alpha}(t)) \right] \eta(t) \, dt = 0.
\]

Assume now that the function \( t \mapsto D_{b-}^{\alpha} (\partial_3 L[x^*]^{\alpha}(t)) \) is continuous on \([a, b]\). Since \( \eta \) is an arbitrary function on the open interval \([a, b]\), we conclude, by the fundamental lemma of the calculus of variations \([14]\), that

\[
\partial_2 L[x^*]^{\alpha}(t) + D_{b-}^{\alpha} (\partial_3 L[x^*]^{\alpha}(t)) = 0
\]

for all \( t \in [a, b] \). Assume now that \( x(a) \) and \( x(b) \) are free. By the arbitrariness of \( \eta \), we can first consider variations such that \( \eta(a) = 0 \) and \( \eta(b) = 0 \), and in this case we prove equation \( (8) \). Therefore, replacing \( (8) \) in \( (7) \), we deduce that

\[
\left[ I_{b-}^{1-\alpha} (\partial_3 L[x^*]^{\alpha}(t)) \, \eta(t) \right]_{a}^{b} \, dt = 0.
\]

If \( \eta(a) = 0 \) and \( \eta(b) \neq 0 \), then

\[
I_{b-}^{1-\alpha} (\partial_3 L[x^*]^{\alpha}(t)) = 0 \quad \text{at} \quad t = b,
\]

and if \( \eta(a) \neq 0 \) and \( \eta(b) = 0 \), then

\[
I_{b-}^{1-\alpha} (\partial_3 L[x^*]^{\alpha}(t)) = 0 \quad \text{at} \quad t = a.
\]

In conclusion, we proved the following result, which provides a necessary optimality condition, known as the fractional Euler–Lagrange equation, and two subsidiary conditions, known as transversality conditions.

**Theorem 1.** Let \( x^* \) be an admissible function and suppose that \( J \) attains a minimum value at \( x^* \). If there exists and is continuous the function \( t \mapsto D_{b-}^{\alpha} (\partial_3 L[x^*]^{\alpha}(t)) \), then

\[
\partial_2 L[x^*]^{\alpha}(t) + D_{b-}^{\alpha} (\partial_3 L[x^*]^{\alpha}(t)) = 0
\]

for every \( t \in [a, b] \). Moreover, if \( x(a) \) is free, then

\[
I_{b-}^{1-\alpha} (\partial_3 L[x^*]^{\alpha}(t)) = 0 \quad \text{at} \quad t = a
\]

and, if \( x(b) \) is free, then

\[
I_{b-}^{1-\alpha} (\partial_3 L[x^*]^{\alpha}(t)) = 0 \quad \text{at} \quad t = b.
\]
As observed in [7], using the well–known relation
\[ D_0^\alpha f(t) = C D_0^\alpha f(t) + D_0^\alpha f(b) = C D_0^\alpha f(t) + \frac{f(b)}{\Gamma(1-\alpha)(b-t)^\alpha}, \]
between Riemann–Liouville and Caputo derivatives, one can easily rewrite the Euler–Lagrange equation (9) in the equivalent form
\[ \partial_2 L[x^*]^{\alpha}(t) + C \frac{D_b^\alpha (\partial_3 L[x^*]^{\alpha}(t))}{\Gamma(1-\alpha)(b-t)^\alpha} = 0. \]

Following similar ideas, we can study the case for several dependent variables \((x_1(t), \ldots, x_m(t))\), with \(m \in \mathbb{N}\). In this case, we obtain \(m\) fractional differential equations, one for each function \(x_i, i = 1, \ldots, m\).

**Theorem 2.** Let \(J\) be the functional
\[ J(x_1, \ldots, x_m) := \int_a^b L[x]^{\alpha}_m(t) \, dt \]
with
\[ [x]^{\alpha}_m(t) := (t, x_1(t), \ldots, x_m(t), C D_{a+1}^{\alpha_1} x_1(t), \ldots, C D_{a+1}^{\alpha_m} x_m(t)), \]
where \(\alpha_1, \ldots, \alpha_m \in ]0,1[\) and \(m \in \mathbb{N}\), defined on the set of functions
\[ \{ x \in C^1([a,b], \mathbb{R}^m) : x(a) = x_a \land x(b) = x_b \}, \]
where \(x_a, x_b\) are two fixed vectors in \(\mathbb{R}^m\). Suppose also that \(L : [a,b] \times \mathbb{R}^{2m} \to \mathbb{R}\) is continuously differentiable with respect to its \(i\)th variable, \(i = 2, \ldots, 2m + 1\). Let \((x_1^*, \ldots, x_m^*)\) be a minimizer of \(J\), and suppose that the functions \(t \mapsto D_{b-}^{\alpha_i} (\partial_{i+1+m} L[x^*_m]^{\alpha_i}(t))\) exist and are continuous on \([a,b], i = 1, \ldots, m\). Then,
\[ \partial_{i+1} L[x^*_m]^{\alpha_i}(t) + D_{b-}^{\alpha_i} (\partial_{i+1+m} L[x^*_m]^{\alpha_i}(t)) = 0 \]
for all \(i = 1, \ldots, m\) and for all \(t \in [a,b]\).

In the next result, the lower bound of the cost functional is a real \(A\), where \(A > a\), that is, \(A\) is greater than the lower bound of the fractional derivative. For simplicity of presentation, we restrict ourselves to the case \(m = 1\).

**Theorem 3.** Define the functional \(A\) as
\[ J(x) := \int_A^b L[x]^{\alpha}(t) \, dt, \]
where $A > a$. If $J$ achieves a minimum value at $x^*$, and if the maps $t \mapsto D^\alpha_{A-} (\partial_3 L[x^*]^{\alpha}(t))$, for $t \in [a, A]$, and $t \mapsto D^\alpha_{b-} (\partial_3 L[x^*]^{\alpha}(t))$, for $[a, b]$, exist and are continuous, then

$$D^\alpha_{b-} (\partial_3 L[x^*]^{\alpha}(t)) - D^\alpha_{A-} (\partial_3 L[x^*]^{\alpha}(t)) = 0, \quad t \in [a, A] ,$$

and

$$\partial_2 L[x^*]^{\alpha}(t) + D^\alpha_{b-} (\partial_3 L[x^*]^{\alpha}(t)) = 0, \quad t \in [A, b] .$$

Moreover, the following transversality conditions are fulfilled:

\[
\begin{cases}
I^{1-\alpha}_{b-} (\partial_3 L[x^*]^{\alpha}(t)) - I^{1-\alpha}_{A-} (\partial_3 L[x^*]^{\alpha}(t)) = 0 \quad \text{at } t = a, \text{ if } x(a) \text{ is free}; \\
I^{1-\alpha}_{A-} (\partial_3 L[x^*]^{\alpha}(t)) = 0 \quad \text{at } t = A, \text{ if } x(A) \text{ is free}; \\
I^{1-\alpha}_{b-} (\partial_3 L[x^*]^{\alpha}(t)) = 0 \quad \text{at } t = b, \text{ if } x(b) \text{ is free}.
\end{cases}
\]

For our next result, besides the boundary conditions, an integral constraint is imposed on the set of admissible functions. Such type of problems are known as isoperimetric problems, and the first example goes back to Dido, Queen of Cartage in Africa. She was interested if finding the shape of a curve with fixed perimeter, of maximum possible area. As we know, the solution is given by a circle. The calculus of variations presents a solution to such problem, with an integral constraint of type

$$\int^b_a \sqrt{1 + (x'(t))^2} \, dt = \text{constant}.$$ 

In our case, we consider that the new constraint depends also on a fractional derivative, and it is of form

$$G(x) := \int^b_a M \left( t, x(t), C D^\alpha_{a+} x(t) \right) \, dt = K, \quad K \in \mathbb{R}, \quad (10)$$

where $M : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is a constant, such that there exist and are continuous the functions $\partial_2 M$ and $\partial_3 M$.

**Theorem 4.** Suppose that $J$ (3), subject to the constraints (1) and (10), attains a minimum value at $x^*$. If $x^*$ is not a solution for

$$\partial_2 M[x]^{\alpha}(t) + D^\alpha_{b-} (\partial_3 M[x]^{\alpha}(t)) = 0, \quad \forall t \in [a, b], \quad (11)$$

and if there exist and are continuous the functions $t \mapsto D^\alpha_{b-} (\partial_3 M[x^*]^{\alpha}(t))$ and $t \mapsto D^\alpha_{b-} (\partial_3 M[x^*]^{\alpha}(t))$ on $[a, b]$, then there exists $\lambda \in \mathbb{R}$ such that $x^*$ satisfies

$$\partial_2 F[x]^{\alpha}(t) + D^\alpha_{b-} (\partial_3 F[x]^{\alpha}(t)) = 0, \quad \forall t \in [a, b],$$

where we define the function $F$ as $F := L + \lambda M$. 
The case when \( x^* \) is a solution of (11) can be easily included:

**Theorem 5.** Suppose that \( J (3) \), subject to the constraints (4) and (10), attains a minimum value at \( x^* \). If there exist and are continuous the functions \( t \mapsto D_\alpha^\omega (\partial_3 L[x^*]^\alpha(t)) \) and \( t \mapsto D_\alpha^\omega (\partial_3 M[x^*]^\alpha(t)) \) on \([a, b] \), then there exist \( \lambda_0, \lambda \in \mathbb{R} \), not both zero, such that \( x^* \) satisfies the equation

\[
\partial_2 F[x]^\alpha(t) + D_\alpha^\omega (\partial_3 F[x]^\alpha(t)) = 0, \quad \forall t \in [a, b],
\]

where we define the function \( F \) as \( F := \lambda_0 L + \lambda M \).

Next, we present another constrained type problem, but now the restriction is given by

\[
g(t, x(t)) = 0, \quad \forall t \in [a, b], \tag{12}
\]

where \( x = (x_1, x_2) \) is a vector and \( g : [a, b] \times \mathbb{R}^2 \to \mathbb{R} \) is differentiable with respect to \( x_1 \) and \( x_2 \). Also, we have the boundary conditions

\[
x(a) = x_a \quad \text{and} \quad x(b) = x_b, \quad x_a, x_b \in \mathbb{R}^2. \tag{13}
\]

**Theorem 6.** Consider the functional

\[
J(x) = \int_a^b L[x]^\alpha_2(t) \, dt,
\]

where

\[
[x]^\alpha_2(t) := (t, x_1(t), x_2(t), C D_\alpha^\omega + x_1(t), C D_\alpha^\omega + x_2(t)),
\]

defined on \( C^1[a, b] \times C^1[a, b] \), subject to the constraints (12) and (13). If \( J \) attains an extremum at \( x^* = (x_1^*, x_2^*) \), the maps \( t \mapsto D_\alpha^\omega (\partial_{i+3} L[x^*]^\alpha_2(t)) \), \( i = 1, 2 \), are continuous, and \( \partial_3 g(t, x(t)) \neq 0 \) for all \( t \in [a, b] \), then there exists a continuous function \( \lambda : [a, b] \to \mathbb{R} \) such that

\[
\partial_{i+1} L[x^*]^\alpha_2(t) + C D_\alpha^\omega (\partial_{i+3} L[x^*]^\alpha_2(t)) + \lambda \partial_{i+1} g(t, x(t)) = 0
\]

for all \( t \in [a, b] \) and \( i = 1, 2 \).

Infinite horizon problems are an important field of research, that deal with phenomena that spread along time \([1, 13, 20] \). In this case, the cost functional is evaluated in an infinite interval \([a, \infty] \):

\[
J(x) := \int_a^\infty L[x]^\alpha \, dt, \tag{14}
\]
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defined on a set of functions with fixed initial conditions: \( x(a) = x_a \). Since we are dealing with improper integrals, some attention to what is a minimizer is needed. We say that \( x^* \) is a local minimizer for \( J \) as in (14) if there exists some \( \epsilon > 0 \) such that, for all \( x \), if \( \|x^* - x\| < \epsilon \), then

\[
\lim_{T \to \infty} \inf_{b \geq T} \int_a^b [L[x^*]^{\alpha}(t) - L[x]^{\alpha}(t)] dt \leq 0.
\]

Also, let us define the functions

\[
A(\epsilon, b) := \int_a^b \frac{L[x^* + \epsilon v]^{\alpha}(t) - L[x^*]^{\alpha}(t)}{\epsilon} dt;
\]

\[
V(\epsilon, T) := \inf_{b \geq T} \int_a^b [L[x^* + \epsilon v]^{\alpha}(t) - L[x^*]^{\alpha}(t)] dt;
\]

\[
W(\epsilon) := \lim_{T \to \infty} V(\epsilon, T),
\]

where \( v \in C^1[a, \infty] \) is a function and \( \epsilon \) a real.

**Theorem 7.** Let \( x^* \) be a local minimal for \( J \) as in (14). Suppose that:

1. \( \lim_{\epsilon \to 0} V(\epsilon, T) \) exists for all \( T \);
2. \( \lim_{T \to \infty} V(\epsilon, T) \) exists uniformly for all \( \epsilon \);
3. For every \( T > a \) and \( \epsilon \neq 0 \), there exists a sequence \( (A(\epsilon, b_n))_{n \in \mathbb{N}} \) such that

\[
\lim_{n \to \infty} A(\epsilon, b_n) = \inf_{b \geq T} A(\epsilon, b)
\]

uniformly for \( \epsilon \).

If there exists and is continuous the function \( t \mapsto D_{b-}^{\alpha} (\partial_3 L[x^*]^{\alpha}(t)) \) on \([a, b]\), for all \( b > a \), then

\[
\partial_2 L[x^*]^{\alpha}(t) + D_{b-}^{\alpha} (\partial_3 L[x^*]^{\alpha}(t)) = 0,
\]

for all \( b > a \). Also, we have

\[
\lim_{T \to \infty} \inf_{b \geq T} I_{b-}^{1-\alpha} (\partial_3 L[x^*]^{\alpha}(t)) = 0 \quad \text{at} \quad t = b.
\]

So far, we considered variational problems with order \( \alpha \in [0, 1] \). Now we proceed by extending them to functionals depending on higher-order derivatives. To that
purpose, consider the functional

\[ J(x) := \int_a^b L \left( t, x(t), C D_{a+1}^\alpha x(t), \ldots, C D_{a+m}^\alpha x(t) \right) \, dt, \tag{15} \]

defined on \( C^m[a, b] \), such that \( x^{(i)}(a) \) and \( x^{(i)}(b) \) are fixed reals for \( i \in \{0, 1, \ldots, m-1\} \). Here, \( m \) is a positive integer, \( \alpha_i \in (i-1, i) \), for all \( i \in \{1, \ldots, m\} \), and \( L : [a, b] \times \mathbb{R}^{m+1} \to \mathbb{R} \) is differentiable with respect to the \( i \)th variable, for \( i \in \{2, 3, \ldots, m+1\} \).

**Theorem 8.** Let \( x^\star \) be a minimizer of \( J \) (15), and suppose that for all \( i \in \{1, \ldots, m\} \), there exist and are continuous the functions \( t \mapsto D_{b-}^\alpha \left( \partial_{i+2} L[x^\star]_m^\alpha (t) \right) \) on \([a, b]\). Then,

\[ \partial_2 L[x^\star]_m^\alpha (t) + \sum_{i=1}^m D_{b-}^\alpha \left( \partial_{i+2} L[x^\star]_m^\alpha (t) \right) = 0 \]

for all \( t \in [a, b] \), where \( [x^\star]_m^\alpha (t) := (t, x(t), C D_{a+1}^\alpha x^\star (t), \ldots, C D_{a+m}^\alpha x^\star (t)) \).

After solving the presented Euler–Lagrange equations, we still need to verify if we are in presence of a minimizer of the functional or not. We recall that the Euler–Lagrange equation is just a necessary optimality condition of first order, and thus its solutions may not be a solution for the problem. For the question of existence of solutions, we refer to [8, 9]. One possible way to check if we have a candidate for minimizer or maximizer is to apply the Legendre condition, which is a second order necessary optimality condition [3, 17]. Under suitable convexity of the Lagrangian, sufficient conditions for global minimizer hold [6].

For the Legendre condition, we have to assume that the Lagrange function \( L \) is such that its second order partial derivatives \( \partial^2_{ij} L \), with \( i, j \in \{2, 3\} \), exist and are continuous.

**Theorem 9.** Suppose that \( x^\star \) is a local minimum for \( J \) as in (3). Then,

\[ \partial^2_{33} L[x^\star]_m^\alpha (t) \geq 0, \quad \forall t \in [a, b]. \]

For our next result, we recall that a function \( L : [a, b] \times \mathbb{R}^2 \to \mathbb{R} \) is convex with respect to the second and third variables if

\[ L(t, x + v, y + w) - L(t, x, y) \geq \partial_2 L(t, x, y)v + \partial_3 L(t, x, y)w \]

for all \( t \in [a, b] \) and \( x, y, v, w \in \mathbb{R} \).

**Theorem 10.** If \( L \) is convex and if \( x^\star \) satisfies the Euler–Lagrange equation (9), then \( x^\star \) minimizes \( J \) (3) when restricted to the boundary conditions (4).
4 Direct methods

There are several different numerical approaches and methods to solve fractional differential equations \([2, 10, 12]\). Different discretizations of fractional derivatives are possible, but many of them do not preserve the fundamental properties of the systems, such as stability \([15, 29]\). For numerical calculations, a simple and powerful method preserving stability is obtained from Grünwald–Letnikov discretizations. For a detailed numerical treatment of fractional differential equations, based on Grünwald–Letnikov fractional derivatives, we refer the interested reader to \([24, 28]\). Here we just mention that both explicit and implicit methods are possible: Theorem 5.1 of \([28]\) shows that the explicit and implicit Grünwald–Letnikov methods are asymptotically stable; while Theorem 5.2 of \([28]\) gives conditions on the step size for the explicit method to be absolute stable, asserting that the implicit method is always absolute stable, without any step size restriction. It is also worth to underline the good convergence properties and error behavior of Grünwald–Letnikov methods \([28]\).

We start this section by recalling the (left) Grünwald–Letnikov fractional derivative of a function \(x\). Let \(\alpha > 0\) be a real. The Grünwald–Letnikov fractional derivative of order \(\alpha\) is defined by

\[
^G \! \text{L} D^\alpha_{x} x(t) := \lim_{h \to 0^+} \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} x(t - kh),
\]

where \(\binom{\alpha}{k}\) stands for the generalization of binomial coefficients to real numbers. As usual, we will adopt the notation

\[
(w^\alpha_k) := (-1)^k \binom{\alpha}{k}.
\]

This fractional derivative is particularly useful to approximate the Caputo derivative. The method is now briefly explained.

Given an interval \([a, b]\) and a fixed integer \(N\), let \(t_j := a + jh, j = 0, 1, \ldots, N\) and \(h > 0\), be a partition of the interval \([a, b]\). Then,

\[
^C D^\alpha_{a+} x(t_j) = \frac{1}{h^\alpha} \sum_{k=0}^{j} (w^\alpha_k) x(t_{j-k}) - \frac{x(a)}{\Gamma(1-\alpha)} (t_j - a)^{-\alpha} + O(h).
\]

Thus, truncated Grünwald–Letnikov fractional derivatives are first-order approximations of the Caputo fractional derivatives. The approximation used for the Caputo derivative is then

\[
^C D^\alpha_{a+} x(t_j) \approx \frac{1}{h^\alpha} \sum_{k=0}^{j} (w^\alpha_k) x(t_{j-k}) - \frac{x(a)}{\Gamma(1-\alpha)} (t_j - a)^{-\alpha} := \hat{D} x(t_j).
\]
The next step is to discretize the functional $\mathcal{J}$. For simplicity, let $h = (b - a)/N$ and let us consider the grid $t_j = a + jh$, $j = 0, 1, \ldots, N$. Then,

$$
\mathcal{J}(x) = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} L(t, x(t), C D_{a+}^{\alpha} x(t)) \, dt \\
\approx \sum_{k=1}^{N} h L(t_k, x(t_k), C D_{a+}^{\alpha} x(t_k)) \\
\approx \sum_{k=1}^{N} h L(t_k, x(t_k), \tilde{D} x(t_k)) \, dt.
$$

(16)

The right hand side of (16) can be regarded as a function $\Psi$ of $N - 1$ unknowns:

$$
\Psi(x_1, x_2, \ldots, x_{N-1}) = \sum_{k=1}^{N} h L(t_k, x(t_k), \tilde{D} x(t_k)).
$$

To find an extremum for $\Psi$, one has to solve the following system of algebraic equations:

$$
\frac{\partial \Psi}{\partial x_i} = 0, \quad i = 1, \ldots, N - 1.
$$

Suppose that $h$ goes to zero. The solution obtained by this method converges to a function $x^*$. Then, $x^*$ is a solution of the Euler–Lagrange equation (9) (for details, see [4, Theorem 8.1]).

**Example 1.** Consider the functional

$$
\mathcal{J}(x) = \int_0^{10} \left( C D_{0+}^{0.5} x(t) - \frac{2}{\Gamma(3/2)} t^{1.5} \right)^2 \, dt
$$

subject to the boundary conditions $x(0) = 0$ and $x(10) = 100$. Since $C D_{0+}^{0.5} t^2 = \frac{2}{\Gamma(3/2)} t^{1.5}$, and $\mathcal{J}$ is nonnegative, we conclude that the function $x(t) = t^2$ is a minimizer of the functional. If we discretize the functional, for different values of $n$, we obtain several numerical approximations of $x$. In Table 1 we present the error of each $n$, where the error is given by the maximum of the absolute value of the difference between $x$ and the numerical approximation.

**Example 2.** For our second example, we consider a problem where we do not know its exact solution. Let

$$
\mathcal{J}(x) = \int_0^1 \left( x(t) \left( C D_{0+}^{0.5} x(t) \right)^2 - \sin(x(t)) \right)^2 \, dt
$$
subject to the boundary conditions $x(0) = 0$ and $x(1) = 1$. In Figure 1, we present the results for $n = 100$.

![Plot of the numerical solution of problem of Example 2]

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