Melnikov method approach to control of homoclinic/heteroclinic chaos by weak harmonic excitations

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Abstract

A review on the application of Melnikov’s method to control homoclinic and heteroclinic chaos in low-dimensional, non-autonomous and dissipative, oscillator systems by weak harmonic excitations is presented, including diverse applications such as chaotic escape from a potential well, chaotic solitons in Frenkel-Kontorova chains, and chaotic charged particles in the field of an electrostatic wave packet.

Keywords: chaos control; Melnikov method; homoclinic chaos; heteroclinic chaos; harmonic perturbation

1 Introduction

During the past 15 years or so, diverse techniques of non-feedback chaos control have been proposed (Chen & Dong 1998) that may be roughly classified into three types: (i) The parametric excitation of an experimentally adjustable parameter; (ii) Entrainment to the target dynamics; and (iii) The application of a coordinate-independent (or dipole) external periodic excitation. It is shown below that techniques (i) and (iii) may be unified in a general setting for the class of dissipative systems considered in this present review. There exists numerical, experimental, and theoretical evidence that the period of the most effective chaos-controlling excitations usually is a rational fraction of a certain period associated with the uncontrolled system, although the effectiveness of incommensurate excitations has also been demonstrated in some particular cases (Chacón & Martínez 2002). Indeed, resonances between the chaos-controlling excitation and (i) a periodic chaos-inducing excitation, (ii) an unstable periodic orbit embedded in the chaotic attractor, (iii) a natural period in a period doubling route to chaos, or (iv) a period associated with some peak in the power...
spectrum, have been considered in diverse successful chaos-controlling excitations. This is not really surprising since these types of resonances are closely related to each other. For instance, when a damped, harmonically forced oscillator exhibits a steady chaotic state, the power spectrum corresponding to a given state variable typically presents its main peaks at frequencies which are rational fractions of the chaos-inducing frequency for certain ranges of the chaos-inducing amplitude. The extensive literature concerning experimental, theoretical, and numerical studies of non-feedback methods is frankly unapproachable because of its volume in a review of the present type. Therefore, only pioneering key work (from the author’s viewpoint) is mentioned in the following. The effectiveness of periodic parametric excitations in suppressing chaos was shown by Alekseev & Loskutov (1987). Hübner & Lüscher (1989) discussed how a nonlinear oscillator can be driven toward a given target dynamics by means of resonant excitations. Braiman & Goldhirsch (1991) provided numerical evidence to show the possibility of inhibiting chaos by an additional periodic coordinate-independent excitation. Salerno (1991) showed, by the analysis of a phase-locked map, the possibility of suppressing chaos in long biharmonically driven Josephson junctions. Chacón & Díaz Bejarano (1993) discussed a new way to reduce or suppress steady chaotic states, by only altering the geometrical shape of weak periodic perturbations. Kivshar et al. (1994) showed analytically and numerically that the suppression of chaos may be effectively achieved by applying a high-frequency parametric force to a chaotic dynamical system. Experimental control of chaos by weak periodic excitations has been demonstrated in many diverse systems, including magnetoelastic systems (Ditto et al. 1990), ferromagnetic systems (Azevedo & Rezende 1991), electronic systems (Hunt 1991), laser systems (Roy et al. 1992; Meucci et al. 1994; Chizhevsky & Corbalán 1996; Uchida et al. 1998), chemical reactions (Petrov et al. 1993; Alonso et al. 2003), neurological systems (Schiff et al. 1994), and plasma systems (Ding et al. 1994).

This paper summarizes some main results concerning the application of Melnikov’s method (Melnikov 1963; Arnold 1964; Guckenheimer & Holmes 1983; Wiggings 1990) to the problem of control of chaos in low-dimensional, non-autonomous and dissipative, oscillator systems by small-amplitude harmonic perturbations. Specifically, the class of systems considered is described by the differential equation

\[ \ddot{x} + \frac{dU(x)}{dx} = -d(x, \dot{x}) + p_c(x, \dot{x})F_c(t) + p_s(x, \dot{x})F_s(t), \]

where \( U(x) \) is a nonlinear potential, \(-d(x, \dot{x})\) is a generic dissipative force which may include constant forces and time-delay terms, \( p_c(x, \dot{x})F_c(t) \) is a chaos-inducing excitation, and \( p_s(x, \dot{x})F_s(t) \) is an as yet undetermined suitable chaos-controlling excitation, with \( F_c(t) \), \( F_s(t) \) being harmonic functions of initial phases \( 0, \Theta \), and frequencies \( \omega, \Omega \), respectively. It is worth mentioning that Melnikov’s method imposes on (1.1) some additional limitations: the excitation, time-delay, and dissipation terms are weak perturbations of the underlying conservative system \( \ddot{x} + dU(x)/dx = 0 \) which has a separatrix. The original work
of Melnikov (1963) was generalized by Arnold (1964) to a particular instance of a time-periodic Hamiltonian perturbation of a two-degree-of-freedom integrable Hamiltonian system. Fifteen years later, Holmes (1979) was the first to apply Melnikov’s method (to a damped forced two-well Duffing oscillator) in the west. From then on the method began to be popular. Chow et al. (1980) rediscovered Melnikov’s results using alternative methods and emphasized that homoclinic and subharmonic bifurcations are closely related. Through the 1980s a great variety of extensions and generalizations of Melnikov’s approach were developed (Greenspan 1981; Holmes & Marsden 1982; Lerman & Umanski 1984; Greunderl 1985; Salam 1987; Schecter 1987; Wiggins 1987). The interested reader is referred to the books by Lichtenberg & Lieberman (1983), Guckenheimer & Holmes (1983), Wiggins (1988), and Arrowsmith & Place (1990) for more details and references. The work of Cai et al. (2002) provides the simplest extension of Melnikov’s method to include perturbational time-delay terms.

The application of Melnikov’s method to controlling chaos in low-dimensional systems by weak periodic perturbations began in about 1990. Indeed, Lima & Pettini (1990) provided a heuristic argument to extend the idea that parametric perturbations can modify the stability of hyperbolic or elliptic fixed points, in the phase space of linear systems, to the case of nonlinear systems, and hence that parametric perturbations could also provide a means to reduce or suppress chaos in nonlinear systems. They used for the first time the Melnikov method to analytically demonstrate this conjecture in the case of a damped driven two-well Duffing oscillator subjected to a chaos-suppressing parametric excitation. However, their insufficient analysis of the corresponding Melnikov function led them into gross errors in their final results and conclusions. Specifically, they failed both theoretically to demonstrate the sensitivity of the suppression scenario to the initial phase of the chaos-suppressing excitation and to find it numerically. They also failed theoretically to predict the suppression of chaos in the case of subharmonic resonances (between the chaos-inducing and chaos-suppressing excitations) higher than the main one. Although a part of their erroneous analysis of the Melnikov function originated from a mistake in its calculation (Cuadros & Chacón 1993; Lima & Pettini 1993), its main weakness was in not providing a correct necessary and sufficient condition for the Melnikov function to always have the same sign (i.e., for the frustration of homoclinic bifurcations). For the two-well Duffing oscillator that they considered, such a correct necessary and sufficient condition was first deduced for the general case of subharmonic resonances by Chacón (1995a), where the extremely important role of the initial phase (of the chaos-suppressing excitation) on the suppression scenario was demonstrated theoretically. Cicogna & Fronzoni (1990) studied the suppression of chaos in the Josephson-junction model

\[ \ddot{\phi} + [1 + \xi \cos(\Omega t + \theta)] \sin \phi = -\delta \dot{\phi} + \gamma \cos(\omega t), \]

where the parametric excitation \(\xi \cos(\Omega t + \theta) \sin \phi\) is the chaos-suppressing excitation, for the single case of the main resonance \(\Omega = \omega\) by using Melnikov’s method. Their insufficient analysis of the Melnikov function (in particular, that
of the role played by the initial phase $\theta$ led them also into gross mistakes in their final conclusions. On the contrary, it was demonstrated by Chacón (1995b) that the effect of the above parametric excitation in (1.2) for the general case of subharmonic resonances ($\Omega = p\omega$, $p$ an integer) is to suppress the chaotic behavior when a suitable initial phase is used and only for certain ranges of its amplitude. It was also shown (Chacón 1995b) for the first time that such suitable initial phases are compatible with the surviving natural symmetry under the parametric excitation. It was also conjectured (Chacón 1998) that such maximum survival of the symmetries of solutions from a broad and relevant class of systems, subjected both to primary chaos-inducing and chaos-suppressing excitations, corresponds to the optimal choice of the suppressory parameters; specifically, to particular values of the initial phase differences between the two types of excitations for which the amplitude range of the suppressory excitation is maximum. Rajasekar (1993) applied Melnikov’s method to study the suppression of chaos in the Duffing-van der Pol oscillator

$$\ddot{x} - \alpha^2 x + \beta x^3 = -p \left(1 - x^2\right) \dot{x} + f \cos(\omega t) + \eta \cos(\Omega t + \Omega\phi), \quad (1.3)$$

where the additional forcing $\eta \cos(\Omega t + \Omega\phi)$ is the chaos-suppressing excitation, for the single case of the main resonance $\Omega = \omega$. He pointed out the relevant role of the initial phase (of the chaos-suppressing excitation) on the suppression scenario for the first time, and he also deduced the analytical expression of the boundaries of the regions in the $\eta - \phi$ phase plane where homoclinic chaos is inhibited. A generalization of Rajasekar’s approach concerning the relative effectiveness of any two weak excitations in suppressing homoclinic/heteroclinic chaos is discussed in the work of Chacón (2002). There, general analytical expressions are derived from the analysis of generic Melnikov functions providing the boundaries of the regions as well as the enclosed area in the amplitude-initial phase plane of the chaos-suppressing excitation where homoclinic/heteroclinic chaos is inhibited. Also, a criterion based on the aforementioned area was deduced and shown to be useful in choosing the most suitable of the possible chaos-suppressing excitations. Cicogna & Fronzoni (1993) analyzed the Melnikov function associated with the family of systems

$$\ddot{x} = f(x) - \delta \dot{x} + \gamma \cos(\omega t) + \varepsilon g(x) \cos(\Omega t + \theta), \quad (1.4)$$

where $\varepsilon g(x) \cos(\Omega t + \theta)$ is the chaos-suppressing excitation, for the single case of the main resonance $\Omega = \omega$. They deduced both the suitable suppressory values of the initial phase $\theta$ and the associated chaotic threshold function $(\gamma/\delta)_{th}$ when the chaos-suppressing excitation acts on the system. General results (Chacón 1999) concerning suppression of homoclinic/heteroclinic chaos were derived on the basis of Melnikov’s for the family (1.1) for the general case of subharmonic resonance ($\Omega = p\omega$, $p$ an integer). There, a generic analytical expression was deduced for the maximum width of the intervals of the initial phase $\Theta$ for which homoclinic/heteroclinic bifurcations can be frustrated. It was also demonstrated that \{0, $\pi/2$, $\pi$, $3\pi/2$\} are, in general, the only optimal values of such initial phase, in the sense that they allow the widest amplitude ranges for the chaos-suppressing excitation. The work of Chacón (2001a)
presents general results concerning enhancement or maintenance of chaos for the family (1.1), where the connection with the results on chaos suppression was discussed in a general setting. It was also demonstrated that, in general, a second harmonic excitation can reliably play an enhancer or inhibitor role by solely adjusting its initial phase. The work of Chacón (2001b) provides a preliminary Melnikov-method-based approach concerning suppression of chaos by a chaos-suppressing excitation which satisfies an ultrasubharmonic resonance condition with the chaos-inducing excitation. This approach was further applied to the problem of the inhibition of chaotic escape from a potential well by incommensurate escape-suppressing excitations (Chacón & Martínez 2002).

2 Basic theoretical approach

To illustrate the theoretical approach with a paradigmatic example, consider a single Josephson junction subjected to a nonlinear dissipative term and driven by two harmonic excitations (Chacón et al. 2001)

\[ \ddot{x} + \sin x = -\alpha(1 + \gamma \cos x)\dot{x} + F\sin(\omega t) + \beta F\sin(\Omega t + \Theta), \]

(2.1)

where \(x\) and time are dimensionless variables, and \(\dot{x}\) is proportional to the difference of potential between the two superconductors. It is also assumed that the terms of dissipation and excitation are regarded as weak perturbations and \(\beta F\sin(\Omega t + \Theta)\) is the chaos-suppressing excitation. The nonlinear dissipative term appears in the study of a single Josephson junction when the conditions are such that the interference effects between the pair and quasiparticle currents should be taken into account (Barone & Paterno 1982). The application of the Melnikov method to (2.1) yields the Melnikov function

\[ M^{\pm}(t_0) = -C \pm A\sin(\omega t_0) \pm B\sin(\Omega t_0 + \Theta), \]

(2.2)

with

\[ C \equiv 8\alpha(1 + \gamma/3), \]

\[ A \equiv 2\pi F \text{sech} \left( \frac{\pi\omega}{2} \right), \]

\[ B \equiv 2\pi\beta \text{sech} \left( \frac{\pi\Omega}{2} \right). \]

(2.3)

Turning to the general case (1.1), let us assume that such a family of systems satisfies the requirements of the Melnikov method. Then, the application of the method to (1.1) provides the generic Melnikov function

\[ M_{h,h'}^{\pm}(\tau_0) = D \pm A\text{har}(\omega\tau_0) + B\text{har}'\left( \Omega\tau_0 + \Psi_{h,h'}^{\pm} \right), \]

(2.4)

where \(\text{har}(\tau)\) means indistinctly \(\sin(\tau)\) or \(\cos(\tau)\), and \(A\) is a non-negative function, while \(D, B\) can be non-negative or negative functions, depending upon
the respective parameters for each specific system. In particular, $D$ contains the effect of the damping, time-delay terms, and constant forces. In the absence of time-delay terms and constant drivings, $D < 0$, while one has the three cases $D \geq 0$ when a constant driving and a time-delay term act on the system. Also, $A$ and $B$ contain the effect of the chaos-inducing and chaos-controlling excitations, respectively. Note that changing the sign of $B$ is equivalent to having a fixed shift of the initial phase: $B \rightarrow -B \iff \Psi_{h, h'}^\pm \rightarrow \Psi_{h, h'}^\pm \pm \pi$, where the two signs before $\pi$ apply to each of the sign superscripts of $\Psi$. Therefore, $B$ will be considered (for example) as a positive function in the following. As phase and initial time $\tau_0$ are not fixed, one may study the simple zeros of $M_{h, h'}^\pm (\tau_0)$ by choosing quite freely the trigonometric functions in (2.4). Therefore, consider, for instant, the Josephson junction given by (2.1). It is worth noting that the Melnikov functions $M_{h, h'}^\pm (\tau_0)$, $M^\pm (t_0)$ (cf. (2.4) and (2.2), respectively) are connected by linear relationships which are known for each specific system (1.1):

$$\begin{align*}
\tau_0 &= \tau_0 (t_0, \omega), \\
\Psi_{h, h'}^\pm &= \Psi_{h, h'}^\pm (\Theta, \Omega, \omega).
\end{align*}$$

Therefore, the control theorems associated with any Melnikov function $M_{h, h'}^\pm (\tau_0)$ can be straightforwardly obtained from those associated with $M^\pm (t_0)$ (Chacón 1999).

### 2.1 Suppression of chaos

As is well-known, the Melnikov method provides estimates in parameter space for the appearance of homoclinic (and heteroclinic) bifurcations, and hence for transient chaos. This means that in most of cases only necessary conditions for steady chaos (strange chaotic attractor) are obtained from the method. Therefore, one may always get sufficient conditions for the inhibition of even transient chaos (frustration of homoclinic/heteroclinic bifurcation) and, a fortiori, for the inhibition of the steady chaos that ultimately arises from such a homoclinic/heteroclinic bifurcation. This is the principal foundation of the utility of Melnikov method in predicting the suppression of (steady) chaos when a homoclinic/heteroclinic bifurcation occurs prior to its emergence. For the Josephson junction (2.1) one has the following theorem (Chacón et al. 2001):

**Theorem 1** Let $\Omega = p\omega$, $p$ an integer, such that, for $M^+ (t_0)$ ($M^- (t_0)$), $p = \frac{4m-1-2\Theta/\pi}{4n-1}$ ($p = \frac{4m+1-2\Theta/\pi}{4n+1}$) is satisfied for some integers $m$ and $n$. Then $M^\pm (t_0)$ always has the same sign, specifically $M^\pm (t_0) < 0$, if and only if the
following condition is satisfied:

\[ \beta_{\text{min}} < \beta \leq \beta_{\text{max}}, \]

\[ \beta_{\text{min}} \equiv \left(1 - \frac{C}{A}\right)R, \]

\[ \beta_{\text{max}} \equiv \frac{R}{p^2}, \]

\[ R \equiv \frac{\cosh \left(\frac{\pi \Omega}{2}\right)}{\cosh \left(\frac{\pi \omega}{2}\right)}. \quad (2.6) \]

Now, the following remarks are in order.

First, one can test the suppression theorem theoretically by considering the
limiting Hamiltonian case \((\alpha = 0)\). Notice that, in the absence of dissipation,
\((2.6)\) must be rewritten as \(\beta_{\text{min}} \leq \beta \leq \beta_{\text{max}}, \beta_{\text{min}} \equiv R, \beta_{\text{max}} \equiv R/p^2,\) since
\(\beta_{\text{min}}\) cannot now be zero. Thus, one obtains (Chacón et al. 2001) \(\Omega = \omega, \Theta = \pi,\) and \(\beta = 1\) as a necessary and sufficient condition for suppressing stochasticity.
(This result can be trivially obtained, to first perturbative order, from \((2.2),\)
\((2.3)\) with \(\alpha = 0,\) i.e., having \(M^\pm (t_0) = 0\) for all \(t_0.\)

Second, the lower threshold for the chaos-suppressing amplitude, \(\beta_{\text{min}},\) takes
into account the strength of the initial chaotic state through the factor \(1 - C/A,\)
since one usually finds that the corresponding maximal Lyapunov exponent \(\lambda^+\)
increases as the ratio \(C/A\) decreases over a certain range of parameters. Therefore,
for fixed-chaos inducing and chaos-suppressing frequencies (and hence \(R\) fixed), one would expect that \(\beta_{\text{min}}\) will increase as \(\lambda^+\) is increased. Note that
the corresponding upper threshold, \(\beta_{\text{max}},\) does not verify this important property,
which is because \(\beta_{\text{max}}\) arises from a necessary condition for the necessary condition yielding \(\beta_{\text{min}}\) to be also a sufficient condition. This means that \(\beta_{\text{max}}\)
slightly underestimates the upper threshold for the chaos-suppressing amplitude, as is numerically and experimentally observed in different instances. It is
worth noting that this remark holds for any Melnikov function \((2.4)\).

Third, the asymmetry between the upper and lower homoclinic orbits (cf.
\((2.2),\) \((2.3)\)) gives rise to two distinct sets of optimal initial phases that are
suitable for suppressing chaos. The optimal suppressory values of \(\Theta\) (hereafter
denoted as \(\Theta_{\text{opt}}\)) are those values allowing the widest amplitude ranges for the
chaos-suppressing excitation (the use of the adjective is justified below in the
discussion of the suitable intervals of initial phase difference for taming chaos).
Indeed, Theorem 1 requires having \(\Theta = \Theta_{\text{opt}} \equiv \pi, \pi/2, 0, 3\pi/2, 3\pi/2, 0, \pi/2\)
for \(p = 4n - 3, 4n - 2, 4n - 1, 4n\) \((n = 1, 2, \ldots),\) respectively, in order to inhibit
chaos when one considers orbits initiated near the upper (lower) homoclinic
orbit. These distinct values are those compatible with the surviving natural
symmetry under the additional forcing. Indeed, the dissipative, harmonically
driven Josephson junction \((\beta = 0)\) is invariant under the transformation

\[ x \rightarrow -x, \]

\[ t \rightarrow t + \frac{(2n + 1)\pi}{\omega}, \quad (2.7) \]
where \( n \) is an integer, i.e., if \( [x(t), \dot{x}(t)] \) is a solution of (2.1) with \( \beta = 0 \), then so is \([-x(t + (2n + 1) \pi/\omega), -\dot{x}(t + (2n + 1) \pi/\omega)]\). This pair of solutions may be essentially the same in the sense that they may differ by an integer number of complete cycles, i.e.,

\[
x(t) = -x\left[t + (2n + 1) \pi/\omega\right] + 2\pi l,
\]

with \( l \) an integer, and they are termed symmetric. Otherwise, the time-shifting and sign reversal procedure yields a different solution, and the two solutions are termed broken-symmetric. When \( \beta > 0 \) and \( \Theta \) is arbitrary the aforementioned natural symmetry is generally broken. The reason for that breaking is

\[
\sin (\Omega t + \Theta) \neq \sin [\Omega t + \Theta + (2n + 1) \pi/\omega],
\]

for arbitrary \( \omega, \Omega, \) and \( \Theta \). Assuming a resonance condition \( \Omega = p\omega \), the survival of the above symmetry implies

\[
\sin (p\omega t + \Theta) = (-1)^{p+1} \sin (p\omega t + \Theta).
\]

Obviously, this is only the case for \( p \) an odd integer. For \( p \) an even integer, one has the new transformation \( [x \rightarrow -x, t \rightarrow t + (2n + 1) \pi/\omega, \Theta \rightarrow \Theta \pm \pi] \). In other words, if \([x(t), \dot{x}(t)]\) is a solution for a value \( \Theta \), then so is \([-x(t + (2n + 1) \pi/\omega), -\dot{x}(t + (2n + 1) \pi/\omega)]\) for \( \Theta \pm \pi \). Thus, this explains the origin of the differences between the corresponding (at the same resonance order) allowed \( \Theta_{\text{opt}} \) values for the two homoclinic orbits. Similar results have been found for the damped, driven pendulum mounted on a vertically oscillating point of suspension (Chacón 1995b). Therefore, this maximum symmetry principle appears to be the common background in the mechanism of regularization by the application of resonant excitations.

Fourth, the width of the allowed interval \([\beta_{\text{min}}, \beta_{\text{max}}]\) for regularization is

\[
\Delta \beta \equiv \beta_{\text{max}} - \beta_{\text{min}} = \left[\frac{C}{A} - \frac{p^2 - 1}{p^2}\right] R,
\]

with \( R \) given by (2.6). Since \( R \) is a positive function, there always exists a maximum resonance order \( p_{\text{max}} \) for suppression of homoclinic (and heteroclinic) chaos, for each fixed initial chaotic state (i.e., \( C/A \) fixed), which is

\[
p_{\text{max}} \equiv \left[\left(1 - \frac{C}{A}\right)^{-1/2}\right],
\]

where the brackets indicate integer part. From (2.12), one sees that \( p_{\text{max}} \) increases as the ratio \( C/A \) is increased, which would associated with the decrease of the corresponding maximal Lyapunov exponent over a certain range of parameters. For a given set of parameters satisfying the above theorem’s hypothesis, as the resonance order \( p \) is increased the allowed interval \([\beta_{\text{min}}, \beta_{\text{max}}]\) shrinks quickly for low frequencies. This means that initial chaotic states cannot
necessarily be regularized to periodic attractors of arbitrary long period, since numerical experiments indicate that the regularized responses are typically a period-1 attractor for \( p = 1 \) and a period-2 attractor for \( p = 2 \). On the other hand, the asymptotic behavior \( \Delta \beta (\omega \to \infty) = \infty \) (the remaining parameters being held constant) means that chaotic motion is not possible in this limit, as expected.

Fifth, to establish the suppression theorem corresponding to any Melnikov function (2.4), it is enough to transform \( M_{h,h'}(\tau_0) \) into the form given by (2.2). Therefore, taking into account (2.5) and the aforementioned \( \Theta_{\text{opt}} \) values, one finds that in general there exist at most four suitable optimal values for the suppressory initial phase difference between the two (commensurate: \( \Omega = p\omega \)) excitations: 0, \( \pi/2 \), \( \pi \), \( 3\pi/2 \).

Sixth, It has been stated above that the suitable values of the initial phase difference (between the two excitations involved) given by Theorem 1 are optimal, in the sense that they allow the widest amplitude ranges for the chaos-suppressing excitation. One therefore could expect reliable control of the dynamics over certain suitable phase difference intervals, which would be centered on such optimal values, although this would imply a reduction of the respective amplitude ranges. It has been deduced (Chacón 1999; Chacón et al. 2001) that there always exists a maximum-range interval

\[
\left[ \Theta_{\text{opt}} - \Delta \Theta_{\text{max}}, \Theta_{\text{opt}} + \Delta \Theta_{\text{max}} \right],
\]

(2.13)

of permitted initial phase differences for homoclinic/heteroclinic chaos inhibition, where

\[
\Delta \Theta_{\text{max}} \equiv \arcsin \left( \frac{C}{A} \right).
\]

(2.14)

For each value of \( \Theta \) belonging to this interval there exists a reduced interval (with regard to the limiting case where the only suitable values of \( \Theta \) are \( \Theta_{\text{opt}} \)) of amplitudes of the chaos-suppressing excitation which is

\[
\beta_{\min} (\Theta = \Theta_{\text{opt}} \pm \Delta \Theta) < \beta \leq \beta_{\max} (\Theta = \Theta_{\text{opt}} \pm \Delta \Theta),
\]

\[
\beta_{\min} (\Theta = \Theta_{\text{opt}} \pm \Delta \Theta) \equiv \left( 1 - \frac{C}{A} \right) R \sec (\Delta \Theta),
\]

\[
\beta_{\max} (\Theta = \Theta_{\text{opt}} \pm \Delta \Theta) \equiv \frac{R \cos (\Delta \Theta)}{p^2},
\]

(2.15)

where \( R \) is given by (2.6) and \( 0 \leq \Delta \Theta \leq \Delta \Theta_{\text{max}} \). Thus, the width of the range for the chaos-suppressing amplitude is

\[
\Delta \beta (\Theta = \Theta_{\text{opt}} \pm \Delta \Theta) = \left\{ \frac{\cos (\Delta \Theta)}{p^2} - \frac{1 - C/A}{\cos (\Delta \Theta)} \right\} R,
\]

(2.16)

i.e., for fixed \( C, A \) and \( \Delta \Theta \) there always exists a maximum resonance order \( p_{\text{max}} \) for homoclinic chaos suppression which is

\[
p_{\text{max}} (\Theta = \Theta_{\text{opt}} \pm \Delta \Theta) = \left[ \frac{\cos (\Delta \Theta)}{\sqrt{1 - \sin (\Delta \Theta_{\text{max}})}} \right],
\]

(2.17)
where the brackets indicate integer part. Also, one can put

\[ \Delta \Theta_{\text{max}} \simeq \frac{C}{A} + O \left[ \left( \frac{C}{A} \right)^3 \right]. \]  

(2.18)

Thus, one can use a linear approximation for \( \Delta \Theta_{\text{max}} \) suitable for chaotic motions arising away from the limiting case of tangency between the stable and unstable manifolds \( (C/A \ll 1) \). It is worth mentioning that the last inequality is usually associated with the observation of steady chaos (strange chaotic attractor).

### 2.2 Enhancement of chaos

It has been mentioned above that the mechanism for suppressing homoclinic (and heteroclinic) chaos is the frustration of a homoclinic/heteroclinic bifurcation, which prevents the appearance of horseshoes in the dynamics. Chacón (2001a) showed that the enhancement of the initial chaos is achieved by moving the system from the homoclinic tangency condition even more than in the initial situation with no second periodic excitation. Thus, enhancement of chaos can mean increasing the duration of a chaotic transient, passing from transient to steady chaos, or increasing the maximal Lyapunov exponent. Consider again that the family of systems modeled by (1.1) satisfies the requirements of the Melnikov method. Similarly to the preceding discussion of the suppression of chaos, one can assume any particular form of the Melnikov function (2.4) to discuss the enhancement of chaos. Therefore, consider, for instance, the following nonlinearly damped, biharmonically driven, two-well Duffing oscillator:

\[
\dot{x} - x + \beta x^3 = -\delta x |x|^{n-1} + F \cos(\omega t) - \eta \beta x^3 \cos(\Omega t + \Theta),
\]  

(2.19)

where \( \eta, \Omega, \) and \( \Theta \) are the normalized amplitude factor, frequency, and initial phase, respectively, of the chaos-controlling parametric excitation \( (0 < \eta \ll 1) \), and \( \beta, \delta, n, F, \) and \( \omega \) are the normalized parameters of the potential coefficient, damping coefficient, damping exponent, chaos-inducing amplitude, and chaos-inducing frequency, respectively \( (0 < \delta, F \ll 1, \beta > 0, n = 1, 2, ...) \). The application of the Melnikov method to (2.19) yields the Melnikov function

\[
M^\pm(t_0) = D \pm A \sin(\omega t_0) - C \sin(\Omega t + \Theta),
\]  

(2.20)

with

\[
D \equiv -\delta \left( \frac{2}{\beta} \right)^{(n+1)/2} B \left( \frac{n+2}{2}, \frac{n+1}{2} \right),
\]

\[
A \equiv \left( \frac{2}{\beta} \right)^{1/2} \pi \omega F \text{sech} \left( \frac{\pi \omega}{2} \right),
\]

\[
C \equiv \frac{\pi \eta}{6 \beta} (\Omega^4 + 4\Omega^2) \text{csch} \left( \frac{\pi \Omega}{2} \right),
\]  

(2.21)
where the positive (negative) sign refers to the right (left) homoclinic orbit of the underlying integrable two-well Duffing oscillator \((\delta = F = \eta = 0)\), and where \(B(m, n)\) is the Euler beta function. It has been demonstrated (Chacón 2001a) that, in general, a second harmonic excitation can reliably play an enhancer or inhibitor role solely from adjusting its initial phase. The Melnikov function \(M^+(t_0)\) will be used here to illustrate the approach to the enhancement of chaos. Indeed, consider that, in the absence of any second parametric excitation \((C = 0)\), the associated Melnikov function \(M^+_0(t_0) = -|D| + A \sin(\omega t_0)\) changes sign at some \(t_0\), i.e., \(|D| \leq A\). If one now lets the second excitation act on the system such that \(C \leq A - |D|\), this relationship represents a sufficient condition for \(M^+(t_0)\) to change sign at some \(t_0\). Thus, a necessary condition for \(M^+(t_0)\) to always have the same sign \((M^+(t_0) < 0)\) is \(C > A - |D| = C_{\text{min}}\). It was above mentioned (Chacón 1999) that a sufficient condition for \(C > C_{\text{min}}\) to also be a sufficient condition for inhibiting chaos is \(\Omega = p\omega\) (subharmonic resonance condition), \(C \leq C_{\text{max}} = A/p^2\), \(p\) an integer, and that \(M^+_0(t_0)\) and \(-C_{\text{min, max}} \sin(\Omega_0 + \Theta)\) are in opposition. This condition yields the optimal suppressory values \(\Theta_{\text{opt}}^\text{sup} = \Theta_{\text{opt}}\). It was demonstrated (Chacón 2001a) that imposing \(M^+_0(t_0)\) to be in phase with \(-C_{\text{min, max}} \sin(\Omega_0 + \Theta)\) is a sufficient condition for \(M^+(t_0)\) change sign at some \(t_0\). This condition provides the optimal enhancer values of the initial phase, \(\Theta_{\text{enh}}^\text{opt}\), in the sense that \(M^+(t_0)\) presents its highest maximum at \(\Theta_{\text{enh}}^\text{opt}\), i.e., one obtains the maximum gap from the homoclinic tangency condition. Now, the following remarks are in order.

First, for a given homoclinic orbit forming (part of) a separatrix, one has in general (i.e., for any Melnikov function (2.4)) that
\[
|\Theta_{\text{opt}}^\text{sup} - \Theta_{\text{opt}}^\text{enh}| = \pi, \tag{2.22}
\]
for each resonance order.

Second, for \(C = C_{\text{min}}\) there always exists a maximum-range interval
\[
[\Theta_{\text{opt}}^\text{enh} - \Delta\Theta_{\max}^\text{enh} (C = C_{\text{min}}), \Theta_{\text{opt}}^\text{enh} + \Delta\Theta_{\max}^\text{enh} (C = C_{\text{min}})] \tag{2.23}
\]
of permitted initial phases for enhancement of chaos in the sense that, for values of \(\Theta\) belonging to that interval, the maxima of \(M^+(t_0)\) are higher than those of \(M^+_0(t_0)\). Similarly, for \(C = C_{\text{max}}\) there always exists a different maximum-range interval
\[
[\Theta_{\text{opt}}^\text{enh} - \Delta\Theta_{\max}^\text{enh} (C = C_{\text{max}}), \Theta_{\text{opt}}^\text{enh} + \Delta\Theta_{\max}^\text{enh} (C = C_{\text{max}})] \tag{2.24}
\]
of allowed initial phases for enhancement of chaos, and also
\[
\Delta\Theta_{\max}^\text{enh} (C = C_{\text{max}}) \geq \Delta\Theta_{\max}^\text{enh} (C = C_{\text{min}}), \tag{2.25}
\]
which is a consequence of the dissipation. It must be emphasized that the definition of \(\Theta_{\text{opt}}^\text{enh}\) is general; i.e., it refers to any resonance and any Melnikov function (2.4).

Third, for general separatrices, i.e., those formed by several homoclinic and (or) heteroclinic loops, the above scenario of control of chaos holds for each
homoclinic (heteroclinic) orbit. However, it is common to find that the different homoclinic (heteroclinic) orbits of a given separatrix yield distinct $\Theta_{\text{enh}}$ values. This is a consequence of the survival of the symmetries existing in the absence of the second excitation. Thus, the actual scenario is usually more complicated. For instance, let $\Theta_{\text{opt},r}^{\text{sup}}, \Theta_{\text{opt},l}^{\text{sup}}$ be the optimal values associated with the right and left homoclinic orbits, respectively, of a typical separatrix with a “figure-of-eight” loop, as in the two-well Duffing oscillator (2.19). One then obtains that the best chance for enhancing chaos should now be at $\Theta_{\text{opt}}^{\text{enh}} \sim (\Theta_{\text{opt},r}^{\text{enh}} - \Theta_{\text{opt},l}^{\text{enh}})/2 \mod (2\pi)$. See Chacón (2001a) for more details.

2.3 Further developments

The case of subharmonic resonance between the chaos-inducing and chaos-controlling frequencies has been briefly discussed above. However, a number of theoretical (Salerno 1991; Salerno & Samuelsen 1994), numerical (Braiman & Goldhirsch 1991), and experimental (Uchida et al. 1998) studies show that chaos can be reliably controlled by other non-subharmonic resonances. The work of Chacón (2001b) presents a Melnikov-method-based approach concerning reduction of homoclinic and heteroclinic instabilities for the family of systems (1.1) where the harmonic excitations verify an ultrasubharmonic resonance condition: $\Omega/\omega = p/q, q > 1 (p \neq q), p, q$ positive integers and $\Omega (\omega)$ the chaos-suppressing (inducing) frequency. Such general results can be used to approach the case of incommensurate chaos-suppressing excitations by means of a series of ever better rational approximations, which are the successive convergents of the infinite continued fraction associated with the irrational ratio $\Omega/\omega$. This procedure has been much employed in characterizing strange non-chaotic attractors in quasiperiodically forced systems as well as in studying phase-locking phenomena in both Hamiltonian and dissipative systems. To illustrate the method one intentionally chooses the golden section $\Omega/\omega = \Phi = (\sqrt{5} - 1)/2$, since it is the irrational number which is the worst approximated by rational numbers (in the sense of the size of the denominator). As is well-known, the golden section can be approximated by the sequence of rational numbers $\{(\Omega/\omega)_i = F_{i-1}/F_i\}$ where $F_i = 1, 1, 2, 3, 5, 8, \ldots$, are the Fibonacci numbers such that $\lim_{i \to \infty} (\Omega/\omega)_i = \Phi$. For each $(\Omega/\omega)_i$ one replaces each quasiperiodically excited system

$$\dot{x} + \frac{dU(x)}{dx} = -d(x, \dot{x}) + p_c(x, \dot{x})\text{har}(\omega t) + p_s(x, \dot{x})\text{har}'(\Psi_{\omega t} + \Psi_{h,h'}) \quad (2.26)$$

by the respective periodically excited system

$$\dot{x} + \frac{dU(x)}{dx} = -d(x, \dot{x}) + p_c(x, \dot{x})\text{har}(\omega t) + p_s(x, \dot{x})\text{har}'\left(\frac{F_{i-1}}{F_i}\omega t + \Psi_{h,h'}\right) \quad (2.27)$$

giving a sequence of periodically excited systems whose associated frequencies satisfy an ultrasubharmonic resonance condition. The work of Chacón & Martínez (2002) applied this approach to the problem of the reduction of chaotic
escape from a potential well using the simple model

\[
\ddot{x} = x - \beta x^2 - \delta \dot{x} + \gamma \sin(\omega t) - \beta \eta x^2 \sin(\Omega t + \Theta),
\]  

(2.28)

where \(\beta \eta x^2 \sin(\Omega t + \Theta)\) is the escape-suppressing excitation. They found that, for irrational escape-suppressing frequencies, the effective escape-reducing initial phases are found to lie close to the accumulation points of the set of suitable initial phases that are associated with the complete series of convergents up to the convergent giving the chosen rational approximation.

A Melnikov-method-based approach (Chacón 2002) was presented concerning the relative effectiveness of harmonic excitations in suppressing homoclinic (and heteroclinic) chaos of the family (1.1) for the main resonance between the chaos-inducing and chaos-suppressing excitations. A criterion based on the area in the suppressory amplitude/initial phase parameter plane, where suppression of homoclinic chaos is guaranteed, was deduced and shown to be useful in choosing the most suitable of the possible chaos-suppressing excitations. Additionally, the choice of the most suitable chaos-suppressing excitation was shown to exhibit sensitivity to the particular initial chaotic state.

The work of Chacón et al. (2003) presents general findings concerning control of chaos for the family

\[
\dot{x} + \frac{dU(x)}{dx} = -d(x, \dot{x}) + \sum_{i=1}^{N} h_{ch,i}(x, \dot{x}) F_{ch,i}(t) + \sum_{j=1}^{M} h_{co,j}(x, \dot{x}) F_{co,j}(t),
\]  

(2.29)

where \(U(x)\) is a general potential, \(-d(x, \dot{x})\) represents a generic dissipative force, \(\sum_{i=1}^{N} h_{ch,i}(x, \dot{x}) F_{ch,i}(t)\) is a general multiple chaos-inducing excitation, and \(\sum_{j=1}^{M} h_{co,j}(x, \dot{x}) F_{co,j}(t)\) is an as yet undetermined suitable multiple chaos-controlling excitation, with \(F_{ch,i}(t), F_{co,j}(t)\) being harmonic functions of common frequency \(\omega\) and initial phases \(\phi_i (i = 1, \ldots, N), \phi_j (j = 1, \ldots, M)\). The effectiveness of this approach in suppressing spatio-temporal chaos of chains of identical chaotic coupled oscillators was demonstrated through the example of coupled Duffing oscillators, where coherent oscillations were achieved under localized control.

The work of Chacón et al. (2002) studied the robustness of the suppression of bidirectional chaotic escape of a harmonically driven oscillator from a quartic potential well by the application of weak parametric excitations. It was numerically shown that Melnikov-method-based theoretical predictions also work for harmonic escape-inducing excitations in the presence of external noise, and for chaotic-escape-inducing excitations having a sharp Fourier component with a sufficiently high power.

The method proposed in the work of Lenci & Rega (2003) consists of choosing the shape of external and/or parametric periodic excitations, which permits one avoid, in an optimal manner, a homoclinic bifurcation. They numerically investigated the effectiveness of the control method with respect to the basin erosion and escape phenomena of a perturbed Helmholtz oscillator.
3 Some applications

3.1 Taming chaotic escape from a potential well

The work of Chacón et al. (1996, 1997, 2001) and Balibrea et al. (1998) applies the above Melnikov-method-based approach to the problem of chaotic escape from a potential well. This is a general and ubiquitous phenomenon in science. Indeed, one finds it in very distinct contexts: the capsizing of a boat subjected to trains of regular waves (Thompson 1989), the stochastic escape of a trapped ion induced by a resonant laser field (Chacón & Cirac, 1995), and the escape of stars from a stellar system (Contopoulos et al. 1993) are some important examples. Remarkably, such complex escape phenomena can often be well described by a low-dimensional system of differential equations. The case considered by Chacón and coworkers is that where escape is induced by an external periodic excitation added to the model system, so that, before escape, chaotic transients of unpredictable duration (due to the fractal character of the basin boundary) are usually observed for orbits starting from chaotic generic phase space regions (such as those surrounding separatrices), in both dissipative and Hamiltonian systems. In particular, Chacón et al. (1996) studied the simplest model for a universal chaotic escape situation:

\begin{equation}
\dot{x} - x + \beta x^2 = -\delta \dot{x} + \gamma \sin(\omega t) + \left(\frac{-\beta \eta x^2 \sin(\Omega t + \Theta)}{\beta \eta x \sin(\Omega t + \Theta)}\right),
\end{equation}

where \(\beta \eta x^2 \sin(\Omega t + \Theta)\) and \(\beta \eta x \sin(\Omega t + \Theta)\) are the (independently considered) escape-suppressing parametric excitations. It was demonstrated that the parametric excitation of the linear (quadratic) term suppress chaotic escape more efficiently than that of the quadratic (linear) term for small (large) driving periods of the primary chaos-inducing excitation. Chacón et al. (1997) studied the inhibition of chaotic escape of a driven oscillator from the cubic potential well that typically models a metastable system close to a fold:

\begin{equation}
\dot{x} + x - \beta x^2 = -\left(\frac{\delta_1 \dot{x}}{\delta_2 x^2 + \delta_3 x^4}\right) + \gamma \cos(\omega t) - \eta x \cos(\Omega t + \Theta),
\end{equation}

where \(\delta_1 \dot{x}\) and \((\delta_2 x^2 + \delta_3 x^4) \dot{x}\) are the (independently considered) linear and nonlinear damping terms, respectively. They demonstrated that the effectiveness of a parametric excitation at suppressing chaotic escape from such a cubic potential well diminishes as the system approaches a period-1 parametric resonance, and that, for linear damping, the parametric excitation inhibits chaotic escape more efficiently than for nonlinear damping. The role of a nonlinear damping term, proportional to the \(n\)th power of the velocity, on the escape-inhibition scenario is considered in the work of Chacón et al. (2001):

\begin{equation}
\dot{x} + x - x^2 = -\delta \dot{x} |\dot{x}|^{n-1} + \gamma \cos(\omega t) + \eta x^2 \cos(\Omega t + \Theta),
\end{equation}

where \(\eta x^2 \cos(\Omega t + \Theta)\) is the escape-suppressing parametric excitation. In this case, the effectiveness of the parametric excitation of the quadratic potential
well at inhibiting chaotic escape diminishes as the system approaches either a period-1 or a period-2 parametric resonance. Also, the effectiveness of the parametric excitation in the presence of the nonlinear dissipative force is less than for a linear dissipative force.

3.2 Taming chaotic solitons in Frenkel-Kontorova chains

Control of chaos in spatially extended systems is one of the most important and challenging problems in the field of nonlinear dynamics. Instances of possible applications include the stabilization of superconducting Josephson-junction arrays (Barone & Paterno 1982), periodic patterns in optical turbulence, and semiconductor laser arrays (Schöll 2001), to cite just a few. Martínez & Chacón (2004) presented a Melnikov-method-based general theoretical approach to control chaotic solitons in damped, noisy and driven Frenkel-Kontorova chains. Specifically, they studied the model

$$\ddot{x}_j + \frac{K}{2\pi} \sin(2\pi x_j) = x_{j+1} - 2x_j + x_{j-1} - \alpha \dot{x}_j + F \cos(\omega t) + \beta F \cos(\Omega t + \varphi) + \xi(t),$$  \hspace{1cm} (3.4)

where $\beta F \cos(\Omega t + \varphi)$ is the chaos-suppressing excitation, and $\xi(t)$ is a bounded noise term. They obtained an effective equation of motion governing the dynamics of the soliton center of mass for which they deduced Melnikov-method-based predictions concerning the regions in the control parameter space where homoclinic bifurcations are frustrated. Numerical simulations indicated that such theoretical predictions can be reliably applied to the original Frenkel-Kontorova chains, even for the case of localized application of the soliton-taming excitations. It is worth mentioning that the same effectiveness of such a localized control in suppressing spatio-temporal chaos of chains of identical chaotic coupled oscillators was demonstrated through the example of coupled Duffing oscillators (Chacón et al. 2003).

3.3 Taming chaotic charged particles in the field of an electrostatic wave packet

The interaction of charged particles with an electrostatic wave packet is a basic and challenging problem appearing in diverse fundamental fields such as astrophysics, plasma physics, and condensed matter physics. While the Hamiltonian approach to this problem is suitable in many physical contexts, the consideration of dissipative forces seems appropriate in diverse phenomena such as the stochastic heating in the dynamics of charged particles interacting with plasma oscillations. In any case, stochastic (chaotic) dynamics already appears (can appear) when the wave packet solely consists of two electrostatic plane waves. Such a non-regular behavior of the charged particles may yield undesirable effects on a number of technological devices such as the destruction of magnetic surfaces in tokamaks. Thus, apart from its general intrinsic interest, the problem
of regularization of the dissipative dynamics of charged particles in an electro-
static wave packet by a small-amplitude uncorrelated wave (which is added to
the initial wave packet) is especially relevant in plasma physics. Chacón (2004)
considered the simplest model equation to examine this problem:

\[
\ddot{x} + \delta \dot{x} = -\frac{e}{m} \left[ E_0 \sin (k_0 x - \omega_0 t) + E_c \sin (k_c x - \omega_c t) \right] - \frac{e}{m} E_s \sin (k_s x - \omega_s t),
\]

where \(E_c \sin (k_c x - \omega_c t)\) and \(E_s \sin (k_s x - \omega_s t)\) are the chaos-inducing and chaos-
suppressing waves, respectively. In a reference frame moving along the main
wave \(E_0 \sin (k_0 x - \omega_0 t)\), (3.5) transforms into a perturbed pendulum equation
which is capable of being studied by means of Melnikov’s method. Two sup-
pressory mechanisms were identified: One mechanism requires chaos-inducing
and chaos-suppressing waves to have both commensurate wavelengths and com-
mensurate relative phase velocities, while the other allows chaos to be tamed
when these quantities are incommensurate.

4 Conclusions and open problems

The present review summarizes some of the main results and applications of a
preliminary theoretical approach to control chaos in dissipative, non-autonomous
dynamical systems, capable of being studied by Melnikov’s method, by means of
periodic excitations. Diverse extensions and applications of the current theory
remain to be developed. Among them:

(i) To obtain the boundaries of the regularization regions in the control
parameter space for the case of a general resonance (not just the main) between
the involved excitations.

(ii) To extend the theoretical approach to (some family of) multidimensional
systems capable of being studied by (some generalized version of) Melnikov’s
method.

(iii) To develop a multiharmonic control theory beyond the main resonance
case.

(iv) To extend the theoretical approach for the case of periodic excitations
to the case of random excitations.

(v) To obtain analytical approximations of the regularized responses for the
deterministic case of a general resonance between the chaos-inducing and chaos-
suppressing excitations.

(vi) To extend the current theory described for harmonic excitations to the
case of general periodic excitations (both chaos-inducing and chaos-controlling).
In particular, the waveform effect should be taken into account in the control
scenario.

(vii) To extend the current theory to the case where the chaos-controlling
excitation is a parametric excitation of the amplitude of the chaos-inducing
excitation, as well as to the case where it is a parametric excitation of the
frequency of the chaos-inducing excitation.
(viii) To apply the current theory to ratchet systems to improve the directed energy transport.
(ix) To apply the current theory to control chaotic population oscillations between two coupled Bose-Einstein condensates with time-dependent asymmetric potential and damping.

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References
[1] Alekseev, V. V. & Loskutov, A. Y. 1987 Control of a system with a strange attractor through periodic parametric action. *Sov. Phys. Dokl.* **32**, 1346-1348.
[2] Alonso, S., Sagüés, F. & Mikhailov, A. S. 2003 Taming Winfree turbulence of scroll waves in excitable media. *Science* **299**, 1722-1725.
[3] Arnold, V. I. 1964 Instability of dynamical systems with many degrees of freedom. *Sov. Math. Dokl.* **5**, 581-585.
[4] Arrowsmith, D. K. & Place, C. M. 1990 *An introduction to dynamical systems*. Springer.
[5] Azevedo, A. & Rezende, S. M. 1991 Controlling chaos in spin-wave instabilities. *Phys. Rev. Lett.* **66**, 1342-1345.
[6] Balibrea, F., Chacón, R. & López, M. A. 1998 Inhibition of chaotic escape by an additional driven term. *Int. J. Bifurcation Chaos* **8**, 1719-1723.
[7] Barone, A. & Paterno, G. 1982 *Physics and applications of the Josephson effect*. Wiley.
[8] Braiman, Y. & Goldhirsh, I. 1991 Taming chaotic dynamics with weak periodic perturbations. *Phys. Rev. Lett.* **66**, 2545-2548.
[9] Cai, C., Xu, Z. & Xu, W. 2002 Melnikov’s analysis of time-delayed feedback control in chaotic dynamics. *IEEE Trans. Circuits Syst. I* **49**, 1724-1728.
[10] Chacón, R. 1995a Suppression of chaos by selective resonant parametric perturbations. *Phys. Rev. E* **51**, 761-764.
[11] Chacón, R. 1995b Natural symmetries and regularization by means of weak parametric modulations in the forced pendulum. *Phys. Rev. E* **52**, 2330-2337.
[12] Chacón, R. 1998 Comparison between parametric excitation and additional forcing terms as chaos-suppressing perturbations. *Phys. Lett.* **249 A**, 431-436.
[13] Chacón, R. 1999 General results on chaos suppression for biharmonically driven dissipative systems. Phys. Lett. **257 A**, 293-300.

[14] Chacón, R. 2001a Maintenance and suppression of chaos by weak harmonic perturbations: a unified view. Phys. Rev. Lett. **86**, 1737-1740.

[15] Chacón, R. 2001b Role of ultrasubharmonic resonances in taming chaos by weak harmonic perturbations. Europhys. Lett. **54**, 148-153.

[16] Chacón, R. 2002 Relative effectiveness of weak periodic excitations in suppressing homoclinic/heteroclinic chaos. Eur. Phys. J. B **30**, 207-210.

[17] Chacón, R. 2004 Dissipative dynamics of a charged particle in the field of three plane waves: chaos and control. [arXiv:lin.CD/0406060](http://arXiv:lin.CD/0406060).

[18] Chacón, R., Balibrea, F. & López, M. A. 1996 Inhibition of chaotic escape from a potential well using small parametric modulations. J. Math. Phys. **37**, 5518-523.

[19] Chacón, R., Balibrea, F. & López, M. A. 1997 Role of parametric resonance in the inhibition of chaotic escape from a potential well. Phys. Lett. **235 A**, 153-158.

[20] Chacón, R., Balibrea, F. & López, M. A. 2001 Role of nonlinear dissipation in the suppression of chaotic escape from a potential well. Phys. Lett. **279 A**, 38-46.

[21] Chacón, R. & Cirac, J. I. 1995 Chaotic and regular behavior of a trapped ion interacting with a laser field. Phys. Rev. A **51**, 4900-4906.

[22] Chacón, R. & Díaz Bejarano, J. 1993 Routes to suppressing chaos by weak periodic perturbations. Phys. Rev. Lett. **71**, 3103-3106.

[23] Chacón, R. & Martínez, J. A. 2002 Inhibition of chaotic escape from a potential well by incommensurate escape-suppressing excitations. Phys. Rev. E **65**, 036213/1-7.

[24] Chacón, R., Palmero, F. & Balibrea, F. 2001 Taming chaos in a driven Josephson junction. Int. J. Bifurcation Chaos **11**, 1897-1909.

[25] Chacón, R., Sánchez-Bajo, F. & Martínez, J. A. 2002 Robustness in the suppression of bidirectional chaotic escape from a potential well by weak parametric excitations. Phys. Lett. **303 A**, 190-196.

[26] Chen, G. & Dong, X. 1998 *From chaos to order: Perspectives, Methodologies and Applications*. World Scientific.

[27] Chizhevsky, V. N. & Corbalán, R. 1996 Experimental observation of perturbation-induced intermittency in the dynamics of a loss-modulated CO₂ Laser. Phys. Rev. E **54**, 4576-4579.
[28] Chow, S. N., Hale, J. K. & Mallet-Paret, J. 1980 An example of bifurcation to homoclinic orbits. *J. Diff. Eqns.* **37**, 351-373.

[29] Cicogna, G. & Fronzoni, L. 1990 Effects of parametric perturbations on the onset of chaos in the Josephson-junction model: Theory and analog experiments. *Phys. Rev. A* **42**, 1901-1906.

[30] Cicogna, G. & Fronzoni, L. 1993 Modifying the onset of homoclinic chaos: application to a bistable potential. *Phys. Rev. E* **47**, 4585-4588.

[31] Contopoulos, G., Kandrup, H. & Kaufmann, D. 1993 Fractal properties of escape from a two-dimensional potential. *Physica* **64D**, 310-323.

[32] Cuadros, F. & Chacón, R. 1993 Comment on “Suppression of chaos by resonant parametric perturbations.” *Phys. Rev. E* **47**, 4628-4629.

[33] Ding, W. X., She, H. Q., Huang, W. & Yu, C. X. 1994 Controlling chaos in a discharge plasma. *Phys. Rev. Lett.* **72**, 96-99.

[34] Ditto, W. L., Rauseo, S. N. & Spano, M. L. 1990 Experimental control of chaos. *Phys. Rev. Lett.* **65**, 3211-3214.

[35] Greenspan, B. D. 1981 *Bifurcations in periodically forced oscillations: Subharmonic and homoclinic orbits*. Ph.D. thesis, Cornell University.

[36] Greundler, J. 1985 The existence of homoclinic orbits and the method of Melnikov for systems in $\mathbb{R}^n$. *SIAM J. Math. Anal.* **16**, 907-931.

[37] Guckenheimer, J. & Holmes, P. 1983 *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*. Springer.

[38] Holmes, P. J. 1979 A nonlinear oscillator with a strange attractor. *Phil. Trans. Roy. Soc. A* **292**, 419-448.

[39] Holmes, P. J. & Marsden, J. E. 1982 Horseshoes in perturbations of Hamiltonian systems with two degrees of freedom. *Comm. Math. Phys.* **82**, 523-544.

[40] Hübler, A. W. & Lüscher, E. 1989 Resonant stimulation and control of nonlinear oscillators. *Naturwissenschaft* **76**, 67-69.

[41] Hunt, E. R. 1991 Stabilizing high-period orbits in a chaotic system: The diode resonator. *Phys. Rev. Lett.* **68**, 1953-1955.

[42] Kivshar, Y. S., Rödelsperger, F. & Benner, H. 1994 Suppression of chaos by nonresonant parametric perturbations. *Phys. Rev. E* **49**, 319-324.

[43] Lenci, S. & Rega, G. 2003 Optimal control of homoclinic bifurcation: Theoretical treatment and practical reduction of safe basin erosion in the Helmholtz oscillator. *Journal of Vibration and Control* **9**, 281-315
Lerman, L. M. & Umanski, Ia. L. 1984 On the existence of separatix loops in four dimensional systems similar to integrable Hamiltonian systems. *PMM U.S.S.R.* **47**, 335-340.

Lichtenberg, A. J. & Lieberman, M. A. 1983 *Regular and stochastic motion*. Springer.

Lima, R. & Pettini, M. 1990 Suppression of chaos by resonant parametric perturbations. *Phys. Rev. A* **41**, 726-733.

Lima, R. & Pettini, M. 1993 Reply to ‘Comment on “Suppression of chaos by resonant parametric perturbations”.’ *Phys. Rev. E* **47**, 4630-4631.

Martínez, P. J. & Chacón, R. 2004 Taming chaotic solitons in Frenkel-Kontorova chains by weak periodic excitations. *Phys. Rev. Lett.* (in press).

Melnikov, V. K. 1963 On the stability of the center for time periodic perturbations. *Trans. Moscow Math. Soc.* **12**, 1-57.

Meucci, R., Gadomski, W., Ciofini, M. & Arecchi, F. T. 1994 Experimental control of chaos by means of weak parametric perturbations. *Phys. Rev. E* **49**, R2528-R2531.

Petrov, V., Gaspar, V., Masere, J. & Showalter, K. 1993 Controlling chaos in the Belousov-Zhabotinsky reaction. *Nature* (London) **361**, 240-243.

Rajasekar, S. 1993 Controlling of chaos by weak periodic perturbations in Duffing-van der Pol oscillator. *Pramana J. Phys.* **41**, 295-309.

Roy, R., Murphy, T. W., Maier, T. D., Gills, Z. & Hunt, E. R. 1992 Dynamical control of a chaotic laser: Experimental stabilization of a globally coupled system. *Phys. Rev. Lett.* **68**, 1259-1262.

Salam, F. M. A. 1987 The Melnikov technique for highly dissipative systems. *SIAM J. App. Math.* **47**, 232-243.

Salerno, M. 1991 Suppression of phase-locking chaos in long Josephson junctions by biharmonic microwave fields. *Phys. Rev. B* **44**, 2720-2726.

Salerno, M. & Samuelsen, M. R. 1994 Stabilization of chaotic phase locked dynamics in long Josephson junctions. *Phys. Lett.* **190 A**, 177-181

Schechter, S. 1987 Melnikov’s method at a saddle-node and the dynamics of the forced Josephson junction. *SIAM J. Math. Anal.* **18**, 1699-1715.

Schiff, S. J., Jerger, K., Duong, D. H., Chang, T., Spano, M. L. & Ditto, W. L. 1994 Controlling chaos in the brain. *Nature* **370**, 615-620.

Schöll, E 2001 *Nonlinear spatio-temporal dynamics and chaos in semiconductors*. Cambridge University Press.
[60] Thompson, J. M. T. 1989 Chaotic phenomena triggering the escape from a potential well. *Proc. R. Soc. London A* **421**, 195-225.

[61] Uchida, A., Sato, T., Ogawa, T. & Kannari, F. 1998 Nonfeedback control of chaos in a microchip solid-state laser by internal frequency resonance. *Phys. Rev. E* **58**, 7249-7255.

[62] Wiggins, S. 1987 Chaos in the quasiperiodically forced Duffing oscillator. *Phys. Lett.** 124 A*, 138-142.

[63] Wiggins, S. 1988 *Global bifurcations and chaos*. Springer.

[64] Wiggins, S. 1990 *Introduction to applied nonlinear dynamical systems and chaos*. Springer.