Non-commutative world-volume interactions on D-brane and Dirac-Born-Infeld action

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ABSTRACT

By integrating the Seiberg-Witten differential equation in a special path, we write ordinary gauge fields in terms of their non-commutative counterparts up to three non-commutative gauge fields. We then use this change of variables to write ordinary abelian Dirac-Born-Infeld action in terms of non-commutative fields. The resulting action is then compared with various low energy contact terms of world-sheet perturbative string scattering amplitudes from non-commutative Dp-brane. We find completely agreement between the field theory and string theory results. Hence, it shows that perturbative string theory knows the solution of the Seiberg-Witten differential equation.

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1 Introduction

Recent years have seem dramatic progress in the understanding of non-perturbative aspects of string theory[1]. With these studies has come the realization that extended objects, other than just strings, play an essential role. An important tool in these investigations has been Dirichlet branes[2]. D-branes are non-perturbative states on which open string can live, and to which various closed strings including Ramond-Ramond states can couple.

Another interesting aspect of D-branes is that in the presence of background flux the world-volume of D-brane becomes non-commutative[3, 4, 5]. Hence, at low energy the D-brane dynamics may be described by non-commutative gauge theory. On the other hand, it is known that the D-brane is properly described by Dirac-Born-Infeld action with appropriate background flux (see e.g., [6]). Using this idea, Seiberg and Witten were able to find, among other things, an explicit differential equation that relates non-commutative gauge fields at different non-commutative parameter[7]. This Seiberg-Witten differential equation can be integrated to find a transformation that changes ordinary gauge fields into non-commutative fields.

The purpose of this paper is to show that the world-sheet perturbative string theory can capture above transformation. To this end, we integrate the differential equation in a special path to write ordinary gauge fields in terms of their non-commutative counterparts up to three non-commutative fields. The resulting transformation for abelian case contains two different multiplication rules. One is the familiar non-commutative $\ast$ multiplication that appears in the definition of non-commutative gauge field strength and the other one, we call $\ast'$ multiplication (see (13)), operates as commutative multiplication rule between two non-commutative field strengths. We use this transformation to rewrite the ordinary DBI action in terms of non-commutative fields. The resulting field theory action which contains various new interactions between ordinary closed string and non-commutative open string fields are then compared with appropriate low energy contact terms of perturbative string theory. Our results in string theory side are fully consistent with the new interactions in field theory side. Hence, it shows that perturbative string theory knows about the solution of the Seiberg-Witten differential equation.

One of the outcome of our calculations is that if one only replaces ordinary fields in the DBI action in terms of their non-commutative counterparts, the resulting action is not completely identical to the appropriate contact terms of string theory scattering amplitudes. To have an action that is fully compatible with string theory, one should transform ordinary multiplication rule between open string fields to the $\ast'$ rule as well.

The paper is organized as follows. In the following section we expand the DBI action to produce various interactions involving one closed and one or two ordinary open string fields. In Section 3, we integrate the Seiberg-Witten differential equation to transform the ordinary open string fields to their non-commutative counterparts. This transformation
leads us to propose that the ordinary multiplication rule between two open string fields in 
the expansion of DBI action should be replaced by the commutative $\ast'$ rule. In Section 
4, using the conformal field theory technique, we calculate various string theory ampli-
tudes describing scattering of closed and open string states from the non-commutative 
$D_p$-branes. Using these string theory amplitudes, we determine various low energy am-
plitudes and contact terms and compare them with the field theory results. We conclude 
with a brief discussion of our results in Section 5. Appendix contains our conventions and 
some useful comments on conformal field theory propagators and vertex operators used in 
our calculations.

2 Dirac-Born-Infeld Couplings

The world-volume theory of a single D-brane in type 0 theory includes a massless $U(1)$ 
vector $A_\alpha$ and a set of massless scalars $X^i$, describing the transverse oscillations of the brane 
$[9, 10]$. The leading order low-energy action for these fields corresponds to a dimensional 
reduction of a ten dimensional $U(1)$ Yang Mills theory. As usual in string theory, there are 
higher order $\alpha' = \ell_s^2$ corrections, where $\ell_s$ is the string length scale. As long as derivatives 
of the field strengths (and second derivatives of the scalars) are small compared to $\ell_s$, 
then the action takes a Dirac-Born-Infeld form $[11]$. To take into account the couplings of 
the open string states with closed strings, the DBI action may be extended naturally to 
include background closed string fields, in particular, the metric, dilaton, Kalb-Ramond 
and tachyon field $[12, 13]$. In this case one arrives at the following world-volume action:

$$S_{BI} = -T_p \int d^{p+1}\sigma \, g(T) e^{-\Phi} \sqrt{-\det(\tilde{G}_{ab} + \tilde{B}_{ab} + 2\pi \ell_s^2 F_{ab})}$$

(1)

where the tachyon function is $g(T) = 1 + T/4 + 3T^2/32 + \cdots$ $[12]$. Here, $F_{ab}$ is the abelian 
field strength of the world-volume ordinary gauge field, while the metric and antisymmetric 
tensors are the pull-backs of the bulk tensors to the D-brane world-volume, e.g.,

$$\tilde{G}_{ab} = G_{ab} + 2G_{i(a} \partial_b X^{i)} + G_{ij} \partial_a X^i \partial_b X^j.$$  

(2)

In general, the closed string fields are function of world-volume and transverse coordinates, 
i.e., $X^a$ and $X^i$ respectively, however, for simplicity we assume they are just function of $X^a$.

In order to find the interactions expected from the DBI action, we expand the action 
for fluctuations around $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = F^{ab} \eta_{\mu a} \eta_{\nu b}$, $\Phi = 0$. The fluctuations should 
be normalized as the conventional field theory modes which appear in the string vertex 
operators. As a first step, we recall that the graviton vertex operator corresponds to 
string frame metric. Hence, one should transform the Einstein frame metric $G_{\mu\nu}$ to the 
string frame metric $g_{\mu\nu}$ via $G_{\mu\nu} = e^{\Phi/2} g_{\mu\nu}$. Now with conventions of $[14]$, the string mode
fluctuations take the form
\[ g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} \]
\[ \Phi = \sqrt{2}\kappa \phi \]
\[ B_{\mu\nu} = \mathcal{F}^{a\nu} \eta_{a\mu} \eta_{b\nu} - 2\kappa b_{\mu\nu} \]
\[ T \equiv 2\kappa \tau \]
\[ A_a = \frac{1}{\sqrt{T_p^2 \pi l_s^2}} a_a \]
\[ X^i = \frac{1}{\sqrt{T_p}} \lambda^i . \]  

With these normalizations, the pull back of the Einstein frame metric becomes:
\[ \tilde{G}_{ab} = \eta_{ab} (1 + \kappa \sqrt{2} \phi) + 2\kappa \tilde{h}_{ab} + \frac{1}{T_p} (1 + \kappa \sqrt{2} \phi) \partial_a \lambda^i \partial_b \lambda^j + \cdots \]  

where the dots represents terms with two and more closed string fields.

Now it is straightforward, to expand eq. (1) using
\[ \sqrt{det(M_0 + M)} = \sqrt{det(M_0)(1 + \frac{1}{2} Tr(M_0^{-1} M) - \frac{1}{4} Tr(M_0^{-1} M M_0^{-1} M) + \frac{1}{8} (Tr(M_0^{-1} M))^2 + \frac{1}{6} Tr(M_0^{-1} M M_0^{-1} M) - \frac{1}{8} Tr(M_0^{-1} M) Tr(M_0^{-1} M M_0^{-1} M) + \frac{1}{48} (Tr(M_0^{-1} M))^3 + \cdots) \]

to produce a vast array of interactions. We are mostly interested in the interactions linear in the closed string fluctuations, and linear or quadratic in the open string fields.

We begin with the linear couplings of the closed strings to the D-brane source itself
\[ \mathcal{L}_{0,1} = -T_p \kappa c \left( \frac{1}{2} \tau + V^{ab}(h_{ba} - b_{ba}) + \frac{1}{2\sqrt{2}} (Tr(V) - 4) \phi \right) \]  

where we defined the overall square root of the metric as \( \sqrt{-det(\eta_{ab} + \mathcal{F}_{ab})} \equiv c \), and matrix \( V^{ab} \) as the dual of the metric, that is
\[ V^{ab} \equiv ((\eta + \mathcal{F})^{-1})^{ab} . \]  

Next there are interactions involving one closed string mode and one open string mode, that is
\[ \mathcal{L}_{1,1} = -\sqrt{T_p \kappa c} \left( \frac{1}{2} V^{ab} f_{ba} (\frac{1}{2} \tau + V^{ab}(h_{ba} - b_{ba}) + \frac{1}{2\sqrt{2}} (Tr(V) - 4) \phi) - V^{ab}(h_{bc} - b_{bc} + \frac{1}{2\sqrt{2}} \phi \eta_{bc}) V^{cd} f_{da} + 2V^{ab}(h_{i}(\partial_a \lambda^i - b_{i[a} \partial_a \lambda^{i]}) \right) \]  

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where $f_{ab} = \partial_a \alpha_b - \partial_b \alpha_a$. We will need also to compare our results with the DBI terms that have one closed and two open string states,

$$\mathcal{L}_{2,1} = -\kappa C \left( \frac{1}{2} \tau + V^{ab}(h_{ba} - b_{ba}) + \frac{1}{2\sqrt{2}}(\text{Tr}(V) - 4\phi) \right)$$

$$\times \left( \frac{1}{4} V^{ab} \partial_a \lambda^i \partial_b \lambda_i - \frac{1}{4} V^{ab} f_{bc} V^{cd} f_{da} + \frac{1}{8} (V^{ab} f_{ba})^2 \right)$$

$$- V^{ab} (h_{bc} - b_{bc} + \frac{1}{2\sqrt{2}} \phi \eta_{bc}) (V^{cd} \partial_d \lambda^i \partial_a \lambda_i - V^{cd} f_{dc} V^{ef} f_{fa} + \frac{1}{2} V^{cd} f_{da} V^{ef} f_{fe})$$

$$+ \frac{1}{2\sqrt{2}} \phi V^{ab} \partial_a \lambda^i \partial_b \lambda_i + V^{ab} (h_{ij} - b_{ij}) \partial_b \lambda^i \partial_a \lambda^j$$

$$+ V^{ab} (h_{i(l} \partial_{a)} \lambda^i - b_{i[l} \partial_{a]} \lambda^i) V^{cd} f_{dc} - 2 V^{ab} (h_{i(l} \partial_{c]} \lambda^i - b_{i[l} \partial_{c]} \lambda^i) V^{cd} f_{da} f r a c) . \ (8)$$

Finally, to compare the couplings of three open string states and massless poles of string amplitudes with the corresponding terms in the DBI theory, we will need also the following action:

$$\mathcal{L}_{2,0} = -c \left( \frac{1}{2} V^{ab} \partial_a \lambda^i \partial_b \lambda_i - \frac{1}{4} V^{ab} f_{bc} V^{cd} f_{da} + \frac{1}{8} (V^{ab} f_{ba})^2 \right)$$

$$= -c \left( \frac{1}{2} (V_S)^{ab} \partial_a \lambda_i \partial_b \lambda^i - \frac{1}{4} (V_S)^{ab} f_{bc} (V_S)^{cd} f_{da} \right) \ (9)$$

where we have drop some total derivative terms in the second line above.

### 3 From commutative to non-commutative variables

Taking into account that the open string vertex operators correspond to non-commutative gauge fields, one should write the ordinary open string fields in terms of their non-commutative counterparts. In the differential equation for non-commutative gauge field was found to be

$$\delta \hat{F}_{ab}(\theta) = \frac{1}{4} \delta \theta^{cd} (2 \hat{F}_{ac} * \hat{F}_{bd} + 2 \hat{F}_{bc} * \hat{F}_{ad} - \hat{A}_c * (\hat{D}_d \hat{F}_{ab} + \partial_d \hat{F}_{ab}) - (\hat{D}_d \hat{F}_{ab} + \partial_d \hat{F}_{ab}) * \hat{A}_c) + O(\hat{F}^3) \ (10)$$

where the field strength and * product were defined to be

$$\hat{F}_{ab} = \partial_a \hat{A}_b - \partial_b \hat{A}_a - i \hat{A}_a \hat{A}_b + i \hat{A}_b \hat{A}_a$$

$$\hat{F}(x) * \hat{g}(x) = e^{\frac{i}{\theta^{ab}} \partial_x^a \partial_{x'}^b} \hat{f}(x') \hat{g}(x'')|_{x'=x''=x} . \ (11)$$

Scattering amplitudes in the next section reproduce different couplings for finite $\theta$. Therefore, to compare expected coupling of DBI action and string amplitude we should integrate the differential equation to find relation between ordinary field strength,
\[ \dot{F}_{ab}(\theta = 0), \text{and its non-commutative counterpart, i.e., } \dot{F}_{ab}(\theta). \text{ We take the integral in the special path that } \theta_{ab} \text{ is proportional to a scalar, i.e., } \theta_{ab} = \alpha \theta_{ab}, \text{ and take integral over } \alpha \text{ from } \alpha = 0 \text{ to } \alpha = 1. \text{ The result is} \]

\[ F_{ab} = \dot{F}_{ab} - \frac{1}{2} \theta^{cd} \left( \dot{F}_{ac} \ast'' \dot{F}_{bd} + \dot{F}_{bd} \ast'' \dot{F}_{ac} - \dot{A}_c \ast'' \partial_d \dot{F}_{ab} - \partial_d \dot{F}_{ab} \ast'' \dot{A}_c \right) + O(\dot{A}^3) \]

where now the non-commutative \ast'' product is defined to be

\[ \dot{f}(x) \ast'' \dot{g}(x) = \left. \frac{e^{\frac{i}{2} \theta_{ab} \partial_{a'} \partial_{b'}} - 1}{\frac{1}{2} \theta_{ab} \partial_{a'} \partial_{b'}} \dot{f}(x') \dot{g}(x'') \right|_{x' = x'' = x} \]

to check the result, one may differentiate it to get equation (10) up to order \( O(\dot{A}^3) \). For abelian case that we are interested in, the transformation becomes

\[ F_{ab} = \dot{F}_{ab} - \theta^{cd} \left( \dot{F}_{ac} \ast' \dot{F}_{bd} - \dot{A}_c \ast' \partial_d \dot{F}_{ab} \right) + O(\dot{A}^3) \]

where the commutative \ast' operates as

\[ \dot{f}(x) \ast' \dot{g}(x) = \left. \frac{\sin(\frac{1}{2} \theta_{ab} \partial_{a'} \partial_{b'})}{\frac{1}{2} \theta_{ab} \partial_{a'} \partial_{b'}} \dot{f}(x') \dot{g}(x'') \right|_{x' = x'' = x} \]

Now if one compare equation (12) for infinitesimal \( \delta \theta \) and finite \( \theta \), one may conclude that to go from ordinary product of two open string fields at \( \theta = 0 \) to finite \( \theta \) one should use the following transformation as well:

\[ fg|_{\theta = 0} \rightarrow f \ast' g|_{\theta \neq 0} \]

where \( f \) and \( g \) are any arbitrary open string fields. We will see in the next section that this multiplication rule is consistent with string theory scattering amplitudes of two open and one closed string states.

Now with the help of equation (12) and (14), one can write the DBI coupling (7), (8) and (9) at \( \theta = 0 \) in terms of non-commutative fields at \( \theta \neq 0 \) corresponding to open string vertex operator. In doing so, one should first using (14) replace ordinary multiplication of two open string fields by the \ast' multiplication. Then, using (12), the ordinary fields are shifted to their non-commutative counterparts. For example, transformation of equation (8) up to three open string states becomes

\[ \dot{\mathcal{L}}_{2,1} = -\kappa_C \left( \left( \frac{1}{2} \tau + V^{ab}(h_{ba} - b_{ba}) + \frac{1}{2\sqrt{2}}(\text{Tr}(V) - 4)\phi \right) \right. \]

\[ \times \left. \left( \frac{1}{2} V^{ab} \partial_a \dot{\lambda}^i \ast' \partial_b \dot{\lambda}_i - \frac{1}{4} V^{ab} \dot{f}_{bc} \ast' V^{cd} \dot{f}_{da} + \frac{1}{8} V^{ab} \dot{f}_{ba} \ast' V^{cd} \dot{f}_{dc} \right) \right. \]

\[ - V^{ab}(h_{bc} - b_{bc} + \frac{\phi^{bc}}{2\sqrt{2}})(V^{cd} \partial_d \dot{\lambda}^i \ast' \partial_a \dot{\lambda}_i - V^{cd} \dot{f}_{de} \ast' V^{ef} \dot{f}_{fa} + \frac{1}{2} V^{cd} \dot{f}_{da} \ast' V^{ef} \dot{f}_{fe}) \]

\[ + \frac{1}{2\sqrt{2}} \phi V^{ab} \partial_a \dot{\lambda}^i \ast' \partial_b \dot{\lambda}_i + V^{ab}(h_{ij} - b_{ij}) \partial_b \dot{\lambda}_i \ast' \partial_a \dot{\lambda}_j \]

\[ + V^{ab}(h_{i(b} \partial_{a)} \dot{\lambda}^i - b_{i[b} \partial_{a]} \dot{\lambda}^i) \ast' V^{cd} \dot{f}_{dc} - 2V^{ab}(h_{i(b} \partial_{c)} \dot{\lambda}^i - b_{i[b} \partial_{c]} \dot{\lambda}^i) \ast' V^{cd} \dot{f}_{da} \right) \]  \ (15)
We now turn to string theory side and evaluate these couplings using the conformal field theory technique.

4 Scattering Calculations

In this section, we calculate various string scattering amplitudes. The amplitude describing scattering of two closed strings from D-brane with a background magnetic flux was calculated in \[8\]. There by analyzing the t-channel of the amplitude, we were able to find the linear coupling of closed string fields to the D-brane and to show that they are consistent with the coupling in \[5\].

4.1 Closed-Open couplings

Here, we wish to compare the coupling of one massless non-commutative open string field and one closed string field on the D-brane to the results of the appropriate string couplings. In field theory, this coupling can be read from eq. (7) and the transformation (12). In string theory side, on the other hand, this coupling is given by the string scattering amplitude of one open and one closed string state from the D-brane, that is,

\[ A_{NS,NS-NS} \sim \int dx_1 d^2 z_2 <V_{NS}(k_1, \zeta_1, x_1)V_{NS-NS}(p_2, \varepsilon_2, z_2, \bar{z}_2) >. \]  

(16)

The details of the vertex operators appear in the Appendix. We already assumed in Sec. 2 that the closed string fields in the DBI action (1) are independent of transverse coordinates. In string theory side, it means that the momentum of closed string vertex operators have component only in the world-volume directions, \(i.e., p_i = 0\). The techniques in calculating the above string scattering amplitude may be found in refs. [13, 14, 16]. The final result is

\[ A_{NS,NS-NS} = \frac{\sqrt{T_p \kappa C}}{2} \left( 2k_{1a}(\mathcal{G} \cdot \varepsilon_2 \cdot D \cdot \mathcal{G}^T)^{\mu a} \zeta_{1\mu} - 2k_{1a}(\mathcal{G} \cdot \varepsilon_2 \cdot D \cdot \mathcal{G}^T)^{a\mu} \zeta_{1\mu} - p_{2\mu}(D \cdot \mathcal{G}^T)^{\mu \nu} \zeta_{1\nu} Tr(\varepsilon_2 \cdot D) \right) \]

\[ A^{NS,\tau} = \frac{\sqrt{T_p \kappa C}}{2} \zeta_1 \cdot \mathcal{G} \cdot p_2 \]

where \(D^{\mu \nu}\) and \(\mathcal{G}^{\mu \nu}\) matrices coming from closed and open string vertices, respectively (see Appendix). We have also normalized the amplitudes at this point by \(-i \sqrt{T_p \kappa C}/2\), where \(1/\sqrt{T_p}, \kappa\) and \(T_p C\) are open string, closed string and D-brane coupling constants, respectively. Substituting the appropriate polarizations for the open and closed string fields from
the Appendix, one finds
\[ A(\lambda, h) = \sqrt{T_p} \kappa_c \left( \zeta_1 N \varepsilon_2 V^T k_1 + k_1 V^T \varepsilon_2 N \zeta_1 \right) \]
\[ A(a, h) = -\sqrt{T_p} \kappa_c \left( \zeta_1 V \varepsilon_2 V k_1 - k_1 V \varepsilon_2 V \zeta_1 + k_1 V \zeta_1 \text{Tr}(\varepsilon_2 V^T) \right) \]
\[ A(a, \phi) = -\sqrt{T_p} \kappa_c \left( \zeta_1 V V k_1 - k_1 V V \zeta_1 + k_1 V \zeta_1 (\text{Tr}(V) - 4) \right) \]
\[ A(a, \tau) = -\frac{\sqrt{T_p} \kappa_c}{2} k_1 V A \zeta_1 \]
where here and in the scattering amplitudes in subsequent sections \( h \) stands for both graviton and Kalb-Ramond fields. In writing above equations, we have used the on-shell condition \( k_1 \cdot V T \cdot F \cdot V \cdot k_2 = 0 \) (see Appendix) and momentum conservation \( k_1 + p_2 = 0 \). These terms are reproduced by the following action:
\[ \hat{L}_{1,1} = -\sqrt{T_p} \kappa_c \left( \frac{1}{2} V^{ab} \hat{f}_{ba} \left( \frac{1}{2} \tau + V^{ab}(h_{ba} - b_{ba}) \right) + \frac{1}{2\sqrt{2}} (\text{Tr}(V) - 4) \phi \right) \]
\[ -V^{ab}(h_{bc} - b_{bc} + \frac{1}{2\sqrt{2}} \phi \eta_{bc}) V^{cd} \hat{f}_{da} + 2 V^{ab}(h_{i[a} \partial_{a} \hat{\lambda}^i - b_{[i} \partial_{a]} \hat{\lambda}^i) \right) \] \( . \) \[ (17) \]
This is consistent with the DBI interaction (7) and the transformation (12) up to two open string fields. Note that transformation of \( \partial_{a} \lambda \) can be read from dimensional reduction of (12).

### 4.2 Open-Open-Open couplings

Next, we turn to the coupling of three open string states. In string theory side this coupling is given by
\[ A^{NS,NS,NS} \sim \int dx_1 dx_2 dx_3 < V^{NS}(\zeta_1, k_1, x_1) V^{NS}(\zeta_2, k_2, x_2) V^{NS}(\zeta_3, k_3, x_3) > \]
where the appropriate vertex operators are given in the Appendix. Using the world-sheet conformal field theory, it is not difficult to perform the correlators above and show that the integrand is invariant under \( SL(2, R) \). Fixing this symmetry, one finds
\[ A = \frac{c \sin(\pi l)}{\pi \sqrt{T_p}} \left( k_1 \mathcal{G} \mathcal{G}^T \zeta_3 \zeta_1 \mathcal{G} \mathcal{G}^T \zeta_2 + k_2 \mathcal{G} \mathcal{G}^T \zeta_2 \zeta_1 \mathcal{G} \mathcal{G}^T \zeta_3 - k_3 \mathcal{G} \mathcal{G}^T \zeta_1 \zeta_2 \mathcal{G} \mathcal{G}^T \zeta_3 \right) \]
where we have defined \( l \equiv -2k_1 V^T \mathcal{F} \cdot V \cdot k_2 = 2k_1 V_A k_2 \) and \( V_A \) is antisymmetric part of the \( V \) matrix (6). We have also normalized the amplitude by the appropriate coupling factor \( -c/2\pi \sqrt{T_p} \). The \( \sin(\pi l) \) factor above arises basically from two different phase factors.
corresponding to two distinct cyclic orderings of the vertex operators. Each phase factor steam from the second terms of the world-sheet propagator (18). Using polarization for scalar and gauge field, one finds the following non-vanishing terms:

\[
A(\lambda, \lambda, a) = \frac{c \sin(\pi l)}{\pi \sqrt{T_p}} \zeta_1 N \zeta_2 k_1 V_S \zeta_3
\]

\[
A(a, a, a) = \frac{c \sin(\pi l)}{\pi \sqrt{T_p}} (\zeta_1 V_S \zeta_2 \zeta_3 V_S k_1 + \zeta_2 V_S \zeta_3 \zeta_1 V_S k_2 + \zeta_1 V_S \zeta_3 \zeta_2 V_S k_3)
\]  

(18)

where \(V_S\) is the symmetric part of the \(V\) matrix (6). These couplings may be reproduced by

\[
\hat{L}_{3,0} = \frac{i c}{8 \pi \sqrt{T_p}} \left( 2 V^{ab}_{S} \partial_a \hat{\lambda}_i \ast' \partial_b \hat{\lambda}_i \right) M - V^{ab}_{S} \hat{f}_{bc} \ast' V^{cd}_{S} \left[ \hat{a}_d, \hat{a}_a \right] M
\]

(19)

where the Moyal bracket is defined in (11) and the \(\ast'\) operates on the whole Moyal bracket. This fix the relation between the non-commutative two-form in (11) and the background metric and flux to be

\[
\hat{\theta}^{ab} = 4 \pi V^{ab}_{A}.
\]

(20)

The DBI interaction (8) and the transformations (14) and (12), reproduce various terms having two, three and more open string fields. Terms which have three fields contain two different parts. One part is just above action and the other part which have three momentum may be verified to be zero. In action (19), one may use, instead of \(\ast'\), another multiplication rule, e.g., ordinary multiplication [17] or \(\ast\) multiplication [7], all produce the same momentum space couplings (18). Hence, although above calculations of three open string couplings can fix the non-commutative multiplication rule in the definition of field strength, i.e., \(\ast\), it can not however uniquely fix the multiplication rule between two open string field strengths.

If one multiplies action (19) with a closed string field, say tachyon, then the ordinary, \(\ast\) and \(\ast'\) and any other multiplication rule produce different momentum space couplings. In this case, string theory calculations can be used to fix uniquely the multiplication rule. To fix the multiplication rule between two open string fields, we calculate string theory couplings of two open and one closed string states in the momentum space. These can be extracted from string scattering amplitudes of two open and one closed string states from the non-commutative D-brane which we now turn to calculate them.

### 4.3 Closed-Open-Open amplitudes

Scattering amplitude of one closed and two open string states can be related to the appropriate amplitude of four open string states in type I theory [14, 15]. However, type I theory
does not have open string tachyon, so the scattering amplitude describing the decay of two massless open string to one closed string tachyon in type 0 is not related to the known amplitude in type I theory. So we explicitly calculate the tachyon amplitude, while using the idea in \[14, 15\] we find the massless closed string amplitude from the known amplitude of type I theory.

4.3.1 Tachyon amplitudes

The amplitudes describing interaction of one closed string tachyon and two massless open strings is given by

$$A_{NS,NS,\tau} \sim \int dx_1 dx_2 d^2 z < V^{NS}(\zeta_1, k_1, x_1) V^{NS}(\zeta_2, k_2, x_2) V^{\tau}(p_3, z_3)>$$

where the closed and open string vertex operators are given in the Appendix. Here again using appropriate world-sheet propagators from \[14\], one can evaluate the correlations above and show that the integrand in $SL(2, R)$ invariant. Gauging this symmetry by fixing $z_3 = i$ and $x_2 = \infty$, one arrives at

$$A \sim 2^{-2s-2} \int dx_1 \left( (2s + 1) \zeta_1 G^T \zeta_2 - \frac{2i \zeta_1 G \cdot D^T p_3 \zeta_2 G p_3}{x_1 - i} + \frac{2i \zeta_1 G p_3 \zeta_2 G \cdot D^T p_3}{x_1 + i} \right) \times (x_1 - i)^{s-l} (x_1 + i)^{s+l}$$

where the integral is taken from $-\infty$ to $+\infty$, and $s = -(p_3 V^T)^2 = -2k_1 V_s k_2$. This integral is doable and the result is

$$A = -\frac{ikc}{2} (a_1(s + l) - a_2(s - l)) \frac{\Gamma(-2s)}{\Gamma(1 - s - l)\Gamma(1 - s + l)}$$

(21)

where $a_1$ and $a_2$ are two kinematic factors depending only on the space time momentum and polarization vectors

$$a_1 = -\zeta_1 G \cdot D^T p_3 \zeta_2 G p_3$$

$$a_2 = (s + l) \zeta_1 G^T \zeta_2 + \zeta_1 G p_3 \zeta_2 G \cdot D^T p_3 .$$

We have also normalized the amplitude (21) at this point by the coupling factor $-ikc/2\pi$. A check of our calculations is that the amplitude (21) satisfies the Ward identity associated with the gauge invariance of the open string states, i.e., the amplitude vanishes upon substituting $\zeta_{ia} \rightarrow k_{ia}$. This amplitude has the pole structure at $m^2_{open} = n/\alpha'$.

\[1\]We explicitly restore $\alpha'$ here. Otherwise our conventions set $\alpha' = 2$
4.3.2 NS-NS amplitudes

Next, we evaluate the amplitude describing the decay of two massless open NS strings into one massless closed NSNS state. Using the idea in \[14, 15\], we relate this amplitude to amplitude of four massless open NS strings. Hence, we begins with the closed string amplitude which is given by

\[
A \sim \int dx_1 dx_2 d^2 z_3 < V^{NS}(k_1, \zeta_1, x_1) V^{NS}(k_2, \zeta_2, x_2) V^{NSNS}(p_3, \varepsilon_3, z_3, \tilde{z}_3) > .
\] (22)

If one evaluated the above correlators, one would find that the integrand is \(SL(2, R)\) invariant. Similar to the tachyon amplitude, the appropriate way to fix this gauge would be to fix operators at \(\{\varepsilon_3, x_1, z_3, x_2\} = \{-i, y, i, \infty\}\).

To relate the calculation here to that of a four point amplitude of open superstrings\[18\], we write the latter in \(SL(2, R)\) invariant form, that is

\[
A' \sim \int dx_1 dx_2 dx_3 dx_4 x_1^{4k_1} x_2^{4k_2} \frac{a_1'}{x_1 x_2 x_3 x_4} - \frac{a_2'}{x_3 x_4 x_5 x_6}
\] (23)

where the kinematic factors \(a_1'\) and \(a_2'\) are

\[
a_1' = 4 \{ \zeta_1 \zeta_2 \zeta_3 \zeta_4 k_2 k_3 + \zeta_1 \zeta_4 k_1 \zeta_2 k_4 \zeta_3 + \zeta_2 \zeta_3 k_2 \zeta_1 k_5 \zeta_4
\]

\[
+ \zeta_3 \zeta_4 k_1 k_2 \zeta_4 \zeta_1 + \zeta_1 \zeta_2 k_2 \zeta_3 k_1 \zeta_4 - \zeta_1 \zeta_3 k_1 k_2 \zeta_3 \zeta_4
\]

\[
- \zeta_2 \zeta_4 k_1 k_2 \zeta_3 \zeta_4 - \zeta_3 \zeta_4 k_2 k_3 \zeta_1 - \zeta_4 k_1 k_3 \zeta_4 \zeta_1 \}
\] (24)

\[
a_2' = 4 \{ k_2 k_4 k_3 k_1 k_3 k_2 k_3 + \zeta_2 k_4 k_3 \zeta_3 \zeta_2 k_4 \zeta_3 - \zeta_1 \zeta_2 \zeta_3 k_4 \zeta_1 k_3
\]

\[
- \zeta_1 \zeta_2 \zeta_3 k_1 k_2 \zeta_3 k_3 - \zeta_2 \zeta_3 k_1 k_2 \zeta_3 k_4 - \zeta_3 \zeta_4 k_3 \zeta_2 \zeta_4 k_3
\]

\[
- \zeta_1 \zeta_2 k_2 k_3 \zeta_3 k_1 k_4 + \zeta_2 k_3 k_1 k_4 k_2 k_3 + \zeta_1 \zeta_4 k_4 k_2 k_1 k_3
\]

\[
+ \zeta_2 k_4 k_3 k_2 k_3 + \zeta_1 \zeta_3 k_3 k_2 k_4 \zeta_1 \}
\] (25)

where \(\zeta_i\)’s are the polarization of external states. Since we are interested in this amplitude for transforming it to scattering amplitude of one closed and two open string states, we do not consider the phase factor associated with the second term of propagator \[18\]. If one fixes the \(SL(2, R)\) symmetry by fixing the operators at \(\{x_1, x_2, x_3, x_4\} = \{-1, x, 1, \infty\}\), one finds

\[
A' \sim \frac{1}{1 + x} \int_{-1}^{+1} (1 + x)^{4k_1} (1 - x)^{4k_2} x_{12} x_{13} x_{24} x_{34} \cdot \frac{a_1'}{1 + x} - \frac{a_2'}{1 - x}.
\] (26)

Now scattering amplitude \(22\) can be read from amplitude \(24\) by replacing

\[
2k_1 \rightarrow p_3 \cdot D , k_2 \rightarrow k_1 \cdot G , 2k_3 \rightarrow p_3 , k_4 \rightarrow k_2 \cdot G , x \rightarrow iy
\]

\[
\zeta_{\mu \nu} \rightarrow (\varepsilon \cdot D)_{\mu \nu} , \zeta_{2 \mu} \rightarrow \zeta_1 G_{\mu} , \zeta_{4 \mu} \rightarrow \zeta_2 G_{\mu}.
\] (27)
Under these transformations the amplitude (26) transforms to

\[ A \sim 2^{-s-1} \int_{-\infty}^{+\infty} (1 + iy)^{s-l}(1 - iy)^{s+l} \times \left( \frac{a_1}{1 + iy} - \frac{a_2}{1 - iy} \right) \]

and the kinematic factors (24) and (25) become

\[ a_1 = (s + l)\zeta_2 \varepsilon_3 D \varepsilon_3 T \zeta_1 + 2k_2 \varepsilon_3 D \varepsilon_3 T \zeta_2 p_3 D \varepsilon_3 T \zeta_1 \]
\[ + 2\zeta_1 \varepsilon_3 D \varepsilon_3 T k_1 p_3 D \varepsilon_3 T \zeta_2 + 2\zeta_2 \varepsilon_3 D \varepsilon_3 T k_2 p_3 D \varepsilon_3 T \zeta_1 \]
\[ + 2k_1 \varepsilon_3 D \varepsilon_3 T \zeta_1 p_3 D \varepsilon_3 T \zeta_2 - \text{Tr}(\varepsilon_3 D) p_3 D \varepsilon_3 T \zeta_1 p_3 D \varepsilon_3 T \zeta_2 \]
\[ - 4k_2 \varepsilon_3 D \varepsilon_3 T k_1 \zeta_2 D \varepsilon_3 T \zeta_1 - 2\zeta_2 \varepsilon_3 D p_3 k_2 D \varepsilon_3 T \zeta_1 \]
\[ - 2p_3 D \varepsilon_3 D \varepsilon_3 T \zeta_1 k_1 D \varepsilon_3 T \zeta_2 \]
\[ a_2 = -2s \zeta_1 \varepsilon_3 D \varepsilon_3 T \zeta_2 + (s + l)\text{Tr}(\varepsilon_3 D) \zeta_1 \varepsilon_3 D \varepsilon_3 T \zeta_2 - (s + l)\zeta_2 \varepsilon_3 D \varepsilon_3 T \zeta_1 \]
\[ - 2k_2 \varepsilon_3 D \varepsilon_3 T \zeta_2 p_3 D \varepsilon_3 T \zeta_1 - 2\zeta_1 \varepsilon_3 D \varepsilon_3 T k_1 p_3 D \varepsilon_3 T \zeta_2 \]
\[ - 2\zeta_2 \varepsilon_3 D \varepsilon_3 T k_2 p_3 D \varepsilon_3 T \zeta_1 - 2k_1 \varepsilon_3 D \varepsilon_3 T \zeta_1 p_3 D \varepsilon_3 T \zeta_2 \]
\[ + 2\zeta_1 \varepsilon_3 D p_3 k_1 D \varepsilon_3 T \zeta_2 + 2p_3 D \varepsilon_3 D \varepsilon_3 T \zeta_2 k_2 D \varepsilon_3 T \zeta_1 \]
\[ + 2k_1 \varepsilon_3 D \varepsilon_3 T k_2 \zeta_1 D \varepsilon_3 T \zeta_2 + \text{Tr}(\varepsilon_3 D) p_3 D \varepsilon_3 T \zeta_1 p_3 D \varepsilon_3 T \zeta_2 . \]

The integral (28) is doable and the result is the same as equation (21) with above kinematic factors. As a check of our calculations, we have inserted the dilaton polarization (49) into the kinematic factor and found that it is independent of the auxiliary vector \( \ell^\mu \). Another check is that the amplitude satisfies the Ward identity associated with the gauge invariance of the open string states.

### 4.3.3 Massless poles

Given the general form of the string amplitude in eq. (21), one can expand this amplitude as an infinite sum of terms reflecting the infinite tower of open string states that propagate on the world-Volume of D-brane. In the low energy domain, i.e., \( \alpha'^{\text{open}} \ll 1 \), the first term representing the exchange of massless string states dominates. In this case the scattering amplitude (21) reduces to

\[ A = \frac{i\kappa c \sin(\pi l)}{4\pi s} (a_1 + a_2) + \cdots \]

where dots represent contact terms and the infinite massive poles. Making the appropriate explicit choices of polarizations, we find

\[ A_s(\lambda, \lambda, \tau) = \frac{i\kappa c \sin(\pi l)}{8\pi s} (\kappa_1\zeta_2 + 1 \leftrightarrow 2) \]
Replacing above propagator and vertices into (32), one finds exactly the string massless string amplitude (21) should be gauge invariant. We turn now to evaluate these low energy contact terms of be gauge invariant either. However, combination of the massless pole and contact terms explicitly. So one expects that the low energy contact terms of string amplitude not to

In writing explicitly the above massless poles, one finds some terms which is proportional to s as well. We will add these terms which have no contribution to the massless poles of field theory to the contact terms in (33). These amplitudes should be reproduced in s-channel of field theory. We present the calculation explicitly for the decay of two gauge fields into tachyon. This amplitude can be evaluated in field theory as

\[ A'_s(a, a, \tau) = (\tilde{V}_{ra})^a (\tilde{G}_a)_{ab} (\tilde{V}_{aaa})^b \]  

where the propagator and the vertices can be read from (9), (17) and (19). They are

\[ (\tilde{G}_a)^{ab} = \frac{i}{c} \frac{(V_S^{-1})^{ab}}{s} \]

\[ (\tilde{V}_{ra})^a = \frac{\sqrt{T_p}K_C}{2} \frac{p_3 V_A}{p_3 V_A} \]

\[ (\tilde{V}_{aaa})^a = \frac{c \sin(\pi l)}{2\pi \sqrt{T_p}} (\zeta_1 V_S \zeta_2 k_1 - 2k_1 V_S \zeta_2 \zeta_1) V_S^a + 1 \leftrightarrow 2 \]  

In writing the above propagator from (9), we have used the covariant gauge \( V_S^{ab} \partial_a \dot{A}_b = 0 \). Replacing above propagator and vertices into (32), one finds exactly the string massless pole \( A_s(a, a, \tau) \). Similar calculations for the other open string modes reproduces exactly the corresponding massless poles of the string amplitudes (31).

Although the whole string scattering amplitude (21) is gauge invariant, vanishing upon substituting \( \zeta_{ia} \rightarrow k_{ia} \), its massless pole (31) is not gauge invariant which can be checked explicitly. So one expects that the low energy contact terms of string amplitude not to be gauge invariant either. However, combination of the massless pole and contact terms should be gauge invariant. We turn now to evaluate these low energy contact terms of string amplitude (21).
4.3.4 Contact terms

Having examined in detail the massless poles of string amplitudes, we now extract the low energy contact terms of the string amplitude \((21)\). Expanding the gamma function appearing in this amplitude, one will find

\[
A = \frac{ic}{2} \left( \frac{(a_1 + a_2)}{2\pi} \sin(\pi l) s + \frac{(a_1 - a_2)}{\pi l} \right) + (a_1 + a_2) \frac{\sin(\pi l)}{\pi l} \sum_{n=1}^{\infty} \zeta(2n+1) l^{(2n+1)} + k^2 O(s, l). \tag{34}
\]

The factor \(\sin(\pi l) / (\pi l)\) appears for all the contact terms. This indicates that multiplication rule between any two open strings is \(*\). However, as we will see in a moment, some of the terms in the kinematic factors \(a_1\) and \(a_2\) are proportional to \(l\). Hence, these terms have overall factor of \(\sin(\pi l)\) which produce the non-commutative multiplication rule \(\ast\) instead of \(*\) in field theory. It is important to note that this higher order derivative terms associated with \(*\) or \(\mathcal{O}(s, l)\) are not the ones that steams from massive pole of string amplitude. This factors appear for both the low energy contact terms and the contact terms corresponding to massive poles. Terms in the first line of \((34)\) are the low energy massless pole and contact terms, whereas the terms in the second line are effect of the massive poles of the string amplitude \((21)\).

As anticipated above, not all the low energy contact terms are gauge invariant. Therefore, we separate the contact terms into gauge invariant and gauge non-invariant terms. Moreover, we divide the gauge non-invariant terms into two parts, terms which have, apart from the overall factor, no momentum and two momenta. That is

\[
\frac{ic \sin(\pi l)}{4\pi l} (a_1 - a_2) \equiv A_c^M + A_c^{ng} + A_c^g \tag{35}
\]

where \(A_c^M\), \(A_c^{ng}\) and \(A_c^g\) are gauge non-invariant terms which have no momentum, two momentum and gauge invariant terms, respectively. The \(A_c^M\) terms are:

\[
A_c^M(a, a, \tau) = -\frac{ic \sin(\pi l)}{4\pi} \zeta_1 V_A \zeta_2
\]

\[
A_c^M(a, a, \phi) = \frac{ic \sin(\pi l)}{4\pi \sqrt{2}} \left( \zeta_1 V_A \zeta_2 - \zeta_2 V_A \zeta_1 - (\text{Tr}(D) + 2) \zeta_1 V_A \zeta_2 \right)
\]

\[
A_c^M(\lambda, a, h) = -\frac{ic \sin(\pi l)}{2\pi} \left( \zeta_1 V_A \zeta_2 + \zeta_2 V_A \zeta_1 \right)
\]

\[
A_c^M(a, a, h) = \frac{ic \sin(\pi l)}{2\pi} \left( \zeta_2 V_A \zeta_1 - \zeta_1 V_A \zeta_2 - \frac{1}{2} \text{Tr}(\varepsilon_3 D) \zeta_1 V_A \zeta_2 \right)
\]

In all above terms, the factor \(\sin(\pi l) / (\pi l)\) reduces to \(\sin(\pi l)\) which produces \(*\) operator between two open string fields. In fact it is not difficult to see that above terms exactly
reproduce by the following action:

\[
\hat{\mathcal{L}}_{2,1}^{M} = \frac{i\kappa c}{4\pi} \text{Tr} \left( \frac{1}{2} V^{ab} \hat{a}_{b} \hat{a}_{a} [\hat{M}_{x}, \hat{M}_{y}] \frac{1}{2} \tau + V^{ab}(h_{ba} - b_{ba}) + \frac{1}{2\sqrt{2}}(\text{Tr}(V) - 4)\phi \right) - \frac{V^{ab}(h_{bc} - b_{bc} + \frac{1}{2\sqrt{2}}(\phi\eta_{bc})V^{cd}[\hat{a}_{d}, \hat{a}_{a} [\hat{M}_{x}, \hat{M}_{y}] M + 2V^{ab}(h_{i[b[\hat{a}_{a}, \hat{M}^{i} - b_{i[b[\hat{a}_{a}, \hat{M}^{i} ] M})] M} \right)
\]

where the non-commutative parameter of the Moyal bracket is the one appearing in (20). Appearance of the Moyal bracket in (36) is consistent with the transformation (12).

The gauge non-invariant terms \( A_{c}^{ng} \) are:

\[
A_{c}^{ng}(a, a, \tau) = \frac{i\kappa c \sin(\pi l)}{2\pi l} (\zeta_{i1} V_{A} k_{1} \zeta_{2} V_{A} k_{1}) + 1 \leftrightarrow 2
\]

\[
A_{c}^{ng}(a, a, \phi) = \frac{i\kappa c \sin(\pi l)}{4\pi \sqrt{2} l} ((\text{Tr}(D) + 2) k_{1} V_{A} \zeta_{1} k_{1} V_{A} \zeta_{2} - 2k_{1} V_{A} \zeta_{2} (k_{1} \zeta_{1} - \zeta_{1} \zeta_{3} V_{k_{1}})) + 1 \leftrightarrow 2
\]

\[
A_{c}^{ng}(\lambda, a, h) = \frac{i\kappa c \sin(\pi l)}{2\pi l} (2\zeta_{1} \nu_{3} V_{3}^{T} k_{1} + 2k_{1} V_{3}^{T} \nu_{3} \zeta_{1}) \zeta_{2} V_{A} k_{1}
\]

\[
A_{c}^{ng}(a, a, h) = \frac{i\kappa c \sin(\pi l)}{2\pi l} (\text{Tr}(\varepsilon_{3} D) k_{1} V_{A} \zeta_{1} k_{1} V_{A} \zeta_{2} - 2k_{1} V_{A} \zeta_{2} (k_{1} \varepsilon_{3} V_{k_{1}} - \varepsilon_{3} V_{k_{1}})) + 1 \leftrightarrow 2.
\]

It is easy to check that, as expected, under replacing \( \zeta_{ia} \rightarrow k_{ia} \) the non-zero terms of \( A_{c}^{M} + A_{c}^{ng} \) cancels exactly the non-zero terms of the massless poles in (31). The factor \( \sin(\pi l)/(\pi l) \) reproduces the \( \ast' \) operator between two open string fields. Above momentum space couplings are reproduced by the following action:

\[
\hat{\mathcal{L}}_{2,1}^{ng} = -\kappa c \text{Tr} \left( (V_{A})^{ab} \hat{a}_{a} \ast' \left( \frac{1}{2} V^{cd} \partial_{h_{de}} \frac{1}{2} \tau + V^{ef}(h_{fe} - b_{fe}) + \frac{1}{2\sqrt{2}}(\text{Tr}(V) - 4)\phi 
\right) - V^{cd}(h_{de} - b_{de} + \frac{1}{2\sqrt{2}}(\phi\eta_{de})V^{ef}\partial_{h_{fe}} \hat{f}_{dc} + 2V^{cd}(h_{i[d[\partial_{c} \partial_{h} \hat{M}^{i} - b_{i[d[\partial_{c} \partial_{h} \hat{M}^{i} ] M} \right)
\right)
\]

Here also appearance of the non-commutative gauge field and derivative of field strength is consistent with the transformation (12).

We turn now to the gauge invariant terms. The contact terms of two open string scalars and one closed string are:

\[
A_{c}^{g}(\lambda, \lambda, \tau) = \frac{i\kappa c \sin(\pi l)}{4\pi l} (-s \zeta_{1} \nu_{2})
\]

\[
A_{c}^{g}(\lambda, \lambda, \phi) = \frac{i\kappa c \sin(\pi l)}{8\pi \sqrt{2} l} \left( (\text{Tr}(D) + 2)(-s \zeta_{1} \nu_{2}) - 4(s + k_{1} V_{k_{2}} + k_{2} V_{k_{1}}) \zeta_{1} \nu_{2} \right)
\]

(39)
\[ A^\varphi_c(\lambda,\lambda,\hbar) = \frac{i\kappa c \sin(\pi l)}{4\pi l} \left( \frac{1}{2} \text{Tr}(\varepsilon_3 D)(-s\zeta_1 N\zeta_2) - 4k_1 V\varepsilon_3^T V k_2 \zeta_1 N\zeta_2 - 2(s - l)\zeta_1 N\varepsilon_3 N\zeta_2 \right) + 1 \leftrightarrow 2. \]

These terms reproduce exactly by appropriate terms in \((\ref{13})\). This confirms the conjectured multiplication rule \((\ref{14})\) between two open string fields. Now the other gauge invariant contact terms are

\[ A^\varphi_c(\lambda,\lambda,\hbar) = \frac{i\kappa c \sin(\pi l)}{2\pi l} \left( 2k_1 V\varepsilon_3^T V k_2 - k_2 V\varepsilon_3^T V k_2 \right) + s(\zeta_1 N\varepsilon_3 V T \zeta_2 - \zeta_2 V T \varepsilon_3 N\zeta_1) - 2k_2 V_A \zeta_2(k_1 V\varepsilon_3 N\zeta_1 + \zeta_1 N\varepsilon_3 V k_1) \]

\[ A^\varphi_c(\varphi,\varphi,\tau) = \frac{i\kappa c \sin(\pi l)}{4\pi l} \left( \zeta_1 V_A k_1 \zeta_2 V_A k_2 \right) + \frac{l}{2} \zeta_1 V_A \zeta_2 - \frac{s}{2} \zeta_1 V_S \zeta_2 + \zeta_1 V_A k_2 \zeta_2 V_A k_1 - \zeta_1 V_S k_2 \zeta_2 V_S k_1 \]

\[ A^\varphi_c(\varphi,\varphi,\phi) = \frac{i\kappa c \sin(\pi l)}{8\pi \sqrt{2}} \left( (\text{Tr}(D) + 2)\zeta_1 V_A k_1 \zeta_2 V_A k_2 - 4k_1 V_A \zeta_1(k_2 V V k_2 - \zeta_2 V V k_2) \right) + \text{Tr}(\varepsilon_3 D) \left( \frac{1}{2} \zeta_1 V_A \zeta_2 - \frac{s}{2} \zeta_1 V_S \zeta_2 + \zeta_1 V_A k_2 \zeta_2 V_A k_1 - \zeta_1 V_S k_2 \zeta_2 V_S k_1 \right) + 2s \zeta_1 V_S \zeta_2 k_1 V V k_2 + 4k_2 V_S \zeta_1(k_1 V V \zeta_2 + \zeta_2 V V \zeta_2) \right) \]

\[ A^\varphi_c(\varphi,\varphi,\phi) = \frac{i\kappa c \sin(\pi l)}{4\pi l} \left( \text{Tr}(\varepsilon_3 D) \zeta_1 V_A k_1 \zeta_2 V_A k_2 - 4k_1 V_A \zeta_1(k_2 V_x \varepsilon_3^T V k_2 - \zeta_2 V_x \varepsilon_3^T V k_2) \right) + \text{Tr}(\varepsilon_3 D) \left( \frac{1}{2} \zeta_1 V_A \zeta_2 - \frac{s}{2} \zeta_1 V_S \zeta_2 + \zeta_1 V_A k_2 \zeta_2 V_A k_1 - \zeta_1 V_S k_2 \zeta_2 V_S k_1 \right) + 2s \zeta_1 V_S \varepsilon_3^T V k_2 - 4\zeta_1 V_S \zeta_2 k_1 V x \varepsilon_3^T V k_2 + 4k_2 V_S \zeta_1(k_1 V x \varepsilon_3^T V \zeta_2 + \zeta_2 V x \varepsilon_3^T V k_1) \right) \]

plus \((1 \leftrightarrow 2)\) for equations that have two gauge fields. These gauge invariant terms are not fully consistent with the DBI terms in \((\ref{13})\). However, adding the following terms to \((\ref{13})\), the resulting action reproduces all the contact terms in \((\ref{10})\),

\[ \hat{L}^g_{2,1} = -\kappa c \text{Tr} \left( \frac{1}{2} V^{ab} \hat{f}_{bc} \ast' (V_A)^{cd} \hat{f}_{da} \left( \frac{1}{2} + V^{ab} (h_{ba} - b_{ab}) + \frac{1}{2\sqrt{2}} (\text{Tr}(V) - 4) \phi \right) \right) - V^{ab} (h_{bc} - b_{bc}) + \frac{1}{2\sqrt{2}} \phi h_{bc} (V^{cd} \hat{f}_{de} \ast' (V_A)^{ef} \hat{f}_{fa}) + V^{ab} (h_{ib} - b_{ib}) \partial_i \hat{\lambda}^{i'} (V_A)^{cd} \hat{f}_{da} + (V_A)^{ab} (h_{ic} + b_{ic}) \partial_i \hat{\lambda}^{i'} V^{cd} \hat{f}_{da}. \]

Now the string theory contact terms in equations \((\ref{17}), (\ref{36}), (\ref{38})\) and \((\ref{11})\) are exactly the DBI interactions \((\ref{7})\) in which using the transformation \((\ref{12})\) with the normalization \((\ref{3})\) its ordinary fields are written in terms of their non-commutative fields up to three open string fields. This ends our illustration of consistency between string theory scattering amplitudes and ordinary DBI action in which using transformations \((\ref{12})\) the ordinary open string fields are written in terms of non-commutative fields.
5 Discussion

Having integrated the Seiberg-Witten differential equation in a special path, we write ordinary gauge fields in terms of non-commutative fields for finite non-commutative two-form parameter. We then use this change of variables to express the ordinary DBI action in terms of non-commutative fields. We have also proposed a transformation for multiplication of two arbitrary open string fields when one write DBI action in terms of non-commutative variables. The resulting action was then compared with various world-sheet perturbative string theory scattering amplitudes. We find completely agreement between the field theory and string theory results. This indicates that the perturbative string theory knows about the Seiberg-Witten differential equation.

Our calculations of string scattering amplitude of two massless open and one closed string states confirmed our proposed commutative multiplication rule (14) between two open string fields. It would be interesting to perform the calculation of one closed and three open string states to find transformation of multiplication rules between three open string states.

The scattering amplitudes considered in this paper fixed the relation between ordinary and non-commutative fields up to three non-commutative fields. In principle, the perturbative string theory knows about all the terms in this change of variable. So it would be interesting to extend our method to higher point functions to find other terms of the relation.

In section 4.2 we reach to the conclusion that string scattering amplitude (18) reproduce the action (19) in field theory. In [7], Seiberg and Witten conclude different action, i.e., similar to (19) with \( \ast \) instead of \( \ast' \) operator. These two actions are identical up to some total derivative terms. In fact, using the antisymmetric property of the non-commutative parameter, one finds

\[
V_S^{ab} \hat{F}_{bc} \ast' V_S^{cd} \hat{F}_{da} = V_S^{ab} \hat{F}_{bc} \ast V_S^{cd} \hat{F}_{da}
\]

up to some total derivative terms.

Our calculations of string scattering amplitude of two open and one closed string states from non-commutative D-brane are also original. Using the two-dimensional conformal field theory, we performed calculations for scattering amplitude of two massless open string states and one closed string tachyon explicitly. Whereas, using the idea that scattering amplitude of open and closed string states can be read from appropriate amplitude of only open string states[14, 15], we were able to find an expression for scattering amplitude of two open and one massless closed string states from non-commutative D-branes.
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When I was finalizing this paper, the paper [20] came out which has some overlap with the results in section 4.
A Perturbative string theory with background field

In perturbative superstring theories, to study scattering amplitude of some external string states in conformal field theory frame, one usually evaluate correlation function of their corresponding vertex operators with use of some standard conformal field theory propagators [19]. In trivial flat background one uses an appropriate linear $\sigma$-model to derive the propagators and define the vertex operators. In nontrivial D-brane background the vertex operator remain unchanged while the standard propagators need some modification. Alternatively, one may use a doubling trick to convert the propagators to standard form and give the modification to the vertex operators[14]. In this appendix we would like to consider a D-brane with constant gauge field strength / or antisymmetric Kalb-Ramond field in all directions of the D-brane. The modifications arising from the appropriate linear $\sigma$-model appear in the following boundary conditions [9]

\[
\partial_y X^a - i F^a_b \partial_x X^b = 0 \quad \text{for} \quad a, b = 0, 1, \ldots, p
\]

\[
X^i = 0 \quad \text{for} \quad i = p + 1, \ldots, 9
\]

where $F_{ab}$ are the constant background fields, and these equations are imposed at $y = 0$. The world-volume (orthogonal subspace) indices are raised and lowered by $\eta^{ab}(N^i)$ and $\eta_{ab}(N_{ij})$, respectively. Now we have to understand the modification of the conformal field theory propagators arising from these mixed boundary conditions. To this end consider the following general expression for propagator of $X^\mu(z, \bar{z})$ fields:

\[
< X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) > = -\eta^{\mu\nu} \log(z - w) - \eta^{\mu\nu} \log(\bar{z} - \bar{w}) - D^{\mu\nu} \log(z - \bar{w}) - D^{\nu\mu}(\bar{z} - w)
\]

where $D^{\mu\nu}$ is a constant matrix. To find this matrix, we impose the boundary condition (42) on the propagator (43), which yields

\[
\eta^{ab} - D^{ba} - F^{ab} - \mathcal{F}_{ac} D^{bc} = 0
\]

for the world-volume directions, $D^{ij} = -N^{ij}$ for the orthogonal directions, and $D^{ia} = 0$ otherwise. Now equation (44) can be solved for $D^{ab}$, that is

\[
D_{ab} = 2(\eta - F_{ab})^{-1} - \eta_{ab}
\]

\[
= 2V_{ba} - \eta_{ab}
\]

2 Our notation and conventions follow those established in [14]. So we are working on the upper-half plane with boundary at $y = 0$ which means $\partial_y$ is normal derivative and $\partial_x$ is tangent derivative. And our index conventions are that lowercase Greek indices take values in the entire ten-dimensional space-time, e.g., $\mu, \nu = 0, 1, \ldots, 9$; early Latin indices take values in the world-volume, e.g., $a, b, c = 0, 1, \ldots, p$; and middle Latin indices take values in the transverse space, e.g., $i, j = p + 1, \ldots, 8, 9$. Finally, our conventions set $\ell_s^2 = \alpha' = 2$. 

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where matrix $V$ is the dual metric that appears in the expansion of DBI action (1). Note that the $D^{\mu\nu}$ is orthogonal matrix, i.e., $D^\mu_{\alpha}D^{\nu\alpha} = \eta^{\mu\nu}$.

Using two dimensional equation of motion, one can write the world-sheet fields in terms of right- and left-moving components. In terms of these chiral fields, closed NSNS and open NS vertex operators are

\[
V^{\text{NSNS}} = \text{op}(X(z), \psi(z), \phi(z), p) \quad \text{and} \quad V^{\text{NS}} = \text{op}(X(x) + \tilde{X}(x), \psi(x) + \tilde{\psi}(x), \phi(x) + \tilde{\phi}(x), k).
\]

where $\psi^\mu$ is super partner of world-sheet field $X^\mu$ and $\phi$ is world-sheet superghost field. The indices $n, m$ refer to the superghost charge of vertex operators, and $p$ and $k$ are closed and open string momentum, respectively. In order to work with only right-moving fields, we use the following doubling trick:

\[
X^\mu(\bar{z}) \rightarrow D^\mu_{\nu}X^\nu(\bar{z}) \quad \tilde{\psi}^\mu(\bar{z}) \rightarrow D^\mu_{\nu}\psi^\nu(\bar{z}) \quad \tilde{\phi}(\bar{z}) \rightarrow \phi(\bar{z}).
\]  

These replacements in effect extend the right-moving fields to the entire complex plane and shift modification arising from mixed boundary condition from propagators to vertex operators. Under these replacement, world-sheet propagator between all right-moving fields take the standard form [8] except the following boundary propagator:

\[
< X^\mu(x_1) X^\nu(x_2) > = -\eta^{\mu\nu} \log(x_1 - x_2) + \frac{i\pi}{2} \mathcal{F}^{\mu\nu} \Theta(x_1 - x_2)
\]  

where $\Theta(x_1 - x_2) = 1(-1)$ if $x_1 > x_2(x_1 < x_2)$. Note that the orthogonal property of the $D$ matrix is an important ingredient for writing the propagators in the standard form. The vertex operators under transformation (47) becomes

\[
V^{\text{NSNS}} = \text{op}(X(z), \psi(z), \phi(z), p) \quad \quad V^{\text{NS}} = \text{op}(X(x) + D X(x), \psi(x) + D \psi(x), 2\phi(x), k).
\]

The vertex operator for closed string tachyon, massless NSNS and massless NS states are

\[
\begin{align*}
V^r &= \text{op}(p, z) \quad \text{op}(pD, \bar{z}) \\
V^{\text{NSNS}} &= (\bar{z}D)_{\mu\nu} : V^\mu_n(p, z) : V^\nu_m(pD, \bar{z}) : \\
V^{\text{NS}} &= (\zeta G)_\mu : V^\mu_n(2k V^T, x) :
\end{align*}
\]

where $G^{ab} = (\eta^{ab} + D^{ab})/2 = V^{ba}$ for gauge field, $G^{ij} = (\eta^{ij} - D^{ij})/2 = N^{ij}$ for scalar field and $G^{ai} = 0$ otherwise. The open string vertex operators in (0) and (-1) pictures are

\[
\begin{align*}
V^\mu_0(k, x) &= (\partial X^\mu(x) + ik\psi(x) \psi^\mu(x)) e^{ik \cdot X(x)} \\
V^\mu_{-1}(k, x) &= e^{-\phi(x)} \psi^\mu(x) e^{ik \cdot X(x)} \\
V_0_0(k, x) &= ik\psi(x) e^{ik \cdot X(x)} \\
V_{-1}(k, x) &= e^{-\phi(x)} e^{ik \cdot X(x)}.
\end{align*}
\]
The physical conditions for the massless open string are

\[ kV_S k = 0 \quad , \quad kV_S \zeta = 0 \]

and for massless closed string are \( p^2 = 0 \) and \( p_\mu \varepsilon^\mu\nu = 0 \) where \( \varepsilon \) is the closed string polarization which is traceless and symmetric(antisymmetric) for graviton(Kalb-Ramond) and

\[ \varepsilon^{\mu\nu} = \frac{1}{\sqrt{8}} (\eta^{\mu\nu} - \ell^\mu p^\nu - \ell^\nu p^\mu) \quad , \quad \ell p = 1 \]  \hspace{1cm} (49)

for the dilaton. Using the fact that \( D^{\mu\nu} \) is orthogonal matrix, one finds the following identities:

\[ \mathcal{G} \mathcal{G}^T = \mathcal{G}^S \]  , \ \( (D\mathcal{G}^T)^{ab} = \mathcal{G}^{ab} \)  , \ \( (D\mathcal{G}^T)^{ij} = -N^{ij} \)

where the \( \mathcal{G}^S \) is symmetric part of the \( \mathcal{G} \) matrix.
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