Bayesian approach for limited-aperture inverse acoustic scattering with total variation prior

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\textbf{ABSTRACT}
In this work, we apply the Bayesian approach for the acoustic scattering problem to reconstruct the shape of a sound-soft obstacle using the limited-aperture far-field measure data. A novel total variation prior scheme is developed for the obstacle shape parameterization. It is imposed on the Fourier coefficients of the parameterization not the parameterization itself. Using this prior, some less smooth objects can be reconstructed. We also investigate the well-posedness in the sense of the Hellinger distance, Wasserstein distance and Kullback–Leibler divergence. Extensive numerical tests are provided to illustrate the numerical performance.

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\section{1. Introduction}
This investigation is concerned with the problem of the acoustic obstacle reconstruction. The goal is to detect and identify the unknown object, provided the incident wave and far-field data are known. A variety of applications of the problem occur in many areas such as radar, medical imaging, geophysical prospection and remote sensing, nondestructive testing, etc. Over the past years, many satisfactory reconstructions have been developed [1–3]. One can refer to [2] for the linear sampling method, [3,4] for the factorization method, [5,6] for the direct sampling method and [7,8] for the Bayesian approach. As we can see, most of these methods focus on the reconstruction problem using the full-aperture data. Some methods for full-aperture data have been developed to process the limited-aperture case, e.g. [5,8–12]. In the real world, the limited-aperture case usually arises because of the difficulties to surround the whole region of interest by sensors. Due to the lack of full aperture measurements, the ill-posedness and nonlinearity of the inverse problem become more severe, which makes this problem more difficult to solve. The reconstruction is not as good as the full aperture case in general [13,14].

The present work aims at the application of the Bayesian statistical approach in the obstacle reconstruction problem, which is a further development of [8]. The Bayesian methodology presents the solution in the form of a posterior distribution by modeling the unknown as a random variable. This posterior distribution can then be used to compute useful estimates and quantify the uncertainty. Qualitative information about the solution may be formulated in the form of priors of the unknown parameters. In [7,8], the Gaussian prior is used in the shape parameterization. We propose a novel prior scheme, which is actually a modified version of total variation prior. We still call it the total variation (TV) prior. We give a brief overview of this proposed prior. The Gaussian random field prior

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for the shape parameterization discussed in [7,8] has the form of Karhunen–Loève (KL) expansion. Therefore by truncating the expansion, the infinite dimensional unknown is transformed to the finite dimensional expansion coefficients, a multi-dimensional Gaussian variable. Similar to [15], we transform the Gaussian to the TV variable. Meanwhile, the shape parameterization is regenerated with the TV prior parameter rather than the Gaussian in the KL expansion. Compared with the Gaussian random field prior, the TV prior can treat some less smooth scatters. In addition, since we only change the expansion coefficients, not the eigenvalues and eigenfunctions, in the KL expansion, the unknown shape parameterization remains smooth to some extent. This is different from the TV prior imposed on unknown parameterization itself [16]. The TV prior applied to the inverse scattering problem yields a non-Gaussian posterior distribution. Sampling it is challenging. Some classical MCMC techniques cannot be used directly. Due to the transformation between the TV and Gaussian prior, the normal sampling algorithms such as pCN-Metropolis–Hastings can be applied. With the prior, the well-posedness is discussed in the sense of Hellinger distance, Wasserstein distance, Kullback–Leibler divergence.

The rest of this paper is organized as follows: in Section 2, we introduce the inverse acoustic obstacle scattering problems and the Bayesian approach. In Section 3, a new TV prior is introduced. In Section 4, we investigate the well-posedness with respect to the Hellinger distance, Wasserstein distance and Kullback–Leibler divergence. In Section 5, we give some examples to illustrate the numerical effectiveness. Finally, the conclusion is stated.

2. Inverse scattering problems and Bayesian approach

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain with $C^2$ boundary $\partial \Omega$. Define $\mathbb{S} = \{ x \in \mathbb{R}^2, |x| = 1 \}$. Consider a time-harmonic plane wave propagating in the exterior of $\Omega$. The velocity potential is of this form

$$U(x, t) = \text{Re}\{ u^i(x) e^{-i\omega t} \} \quad \text{with} \quad u^i(x) := e^{i\kappa x \cdot d}, \quad x \in \mathbb{R}^2, \quad d \in \mathbb{S},$$

(1)

where $d$ is the propagating direction, $\omega > 0$ is the frequency, $\kappa = \omega / c > 0$ is the wavenumber and $c$ is the speed of sound. Hereafter, the term $\exp(-i\omega t)$ is factored out and only the space dependent part of all waves is considered. Let $\Omega$ be an impenetrable sound-soft obstacle. Clearly, the obstacle will 'scatter' the wave $u^i$ so that we may write the total field as $u = u^i + u^s$, where $u^s$ is the scattered wave. The classical physical problem is to explore the scattering phenomenon, i.e. find $u^s$. As we know, the scattered field $u^s$ is governed by an exterior boundary value problem for Helmholtz equation

$$\triangle u + \kappa^2 u = 0, \quad \text{in} \quad \mathbb{R}^2 \setminus \hat{\Omega},$$

(2a)

$$u = 0, \quad \text{on} \quad \partial \Omega,$$

(2b)

$$\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - i\kappa u^s \right) = 0,$$

(2c)

where Equation (2b) is the sound-soft boundary condition and (2c) is the Sommerfeld radiation condition.

The well-posedness of the direct scattering problem (2a)–(2c) has been established, see [17]. It is well known that the unique solution $u^s$ of (2) admits an asymptotic expansion [17]

$$u^s(x, d) = \frac{e^{i\pi}}{\sqrt{8\kappa \pi}} \frac{e^{i\kappa r}}{\sqrt{r}} \left\{ u^\infty(\hat{x}, d) + O\left( \frac{1}{r} \right) \right\} \quad \text{as} \quad r := |x| \to \infty$$

(3)

uniformly in all directions $\hat{x} = x/|x|$. The function $u^\infty(\hat{x}, d)$ is called the far-field pattern. The inverse scattering problem in this paper is to find the shape of obstacle $\Omega$ from some observed far-field data.
Let \( \gamma^o \subseteq S \) be the observation aperture and \( \gamma^i \subseteq S \) be the incident direction. The direct scattering problem can be formulated as

\[
u^\infty(\hat{x}, d) = \mathcal{F}(\Omega), \quad (\hat{x}, d) \in \gamma^o \times \gamma^i,
\]

(4)

where \( \mathcal{F} \) is the shape-to-measurement operator. If \( \gamma^o \subseteq S \), i.e. \( \gamma^o \) is a proper subset of \( S \), the case is referred to as the limited-aperture problem. If \( \gamma^o = S \), it is regarded as the full-aperture case.

We consider a starlike obstacle \( \Omega \) centered at \( x_c \) with boundary being parametrized as

\[
\partial \Omega := x_c + r(\theta)(\cos \theta, \sin \theta) = x_c + \exp(q(\theta))(\cos \theta, \sin \theta),
\]

(5)

where \( q(\theta) = \log r(\theta), \ 0 < r(\theta) < r_{\text{max}}, \ \theta \in [0, 2\pi) \). With the parameterization, we take the noise in measurements into account and rewrite (4) as a statistical inference model

\[
y^\dagger = \mathcal{F}(q) + \eta,
\]

(6)

where \( q \in X \) and \( y = u^\infty(\hat{x}, d) \in Y \) for some suitable Banach spaces \( X \) and \( Y \), respectively. In particular, \( y^\dagger \) is the noisy observations of \( u^\infty(\hat{x}, d) \) and \( \eta(\hat{x}, d) \) is the noise.

Apart from mere estimation purposes, a major goal of statistical inference is to find the posterior distribution \( \mu(q \mid y^\dagger) \). To achieve this, the Bayesian approach proceeds as follows. First, the parameter \( q \) is modeled as a random variable, \( q \sim \mu_{\text{prior}} \). The distribution \( \mu_{\text{prior}} \) is the so-called prior distribution and characterizes the uncertainty in \( q \). Moreover, we assume that \( q, \eta \) are independent random variables defined on an underlying probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \). In this setting, the inverse scattering problem is an event

\[
\{ y^\dagger = \mathcal{F}(q) + \eta \} \in \mathcal{A},
\]

where the data \( y^\dagger \) is a realization of the random variable \( \mathcal{F}(q^\dagger) + \eta \). The posterior distribution is then given

\[
\mu(q \mid y^\dagger) := \mathbb{P}(q \in \cdot \mid \mathcal{F}(q) + \eta = y^\dagger).
\]

(7)

In this process, \( y := \mathcal{F}(q) + \eta \) is regarded as a random variable reflecting the uncertainty of the data, given parameter \( q \). The distribution for noise \( \eta \) and the forward model induce the conditional measure of the data \( y \) given \( q = q' \)

\[
\mu_L(y \mid q') := \mathbb{P}(y \in \cdot \mid q = q') = \mu_{\text{noise}}(\cdot \mid \mathcal{F}(q')).
\]

(8)

Assume that there are \( \sigma \)-finite measure spaces \( (X, \mathcal{B}_X, \nu_X) \) and \( (Y, \mathcal{B}_Y, \nu_Y) \), where \( \mu_{\text{prior}} \ll \nu_X \) and \( \mu_L(\cdot \mid q') \ll \nu_Y (q' \in X, \mu_{\text{prior}}\text{-almost surely}) \). Here \( \mu \ll \nu \) means that \( \mu \) is absolutely continuous with respect to \( \nu \). The Radon–Nikodym theorem yields that their probability density functions (pdfs) exist:

\[
\frac{d\mu_L}{d\nu_Y}(y^\dagger) = L(y^\dagger \mid q'), \quad \frac{d\mu_{\text{prior}}}{d\nu_X}(q) =: \pi_{\text{prior}}(q).
\]

(9)

The conditional density \( L(\cdot \mid q') \) is called data likelihood. Bayes’ theorem establishes a connection of \( \mu_{\text{prior}}, \mu(q \mid y), \) and \( \mu_L \) in terms of pdfs

\[
\pi(q' \mid y^\dagger) = \frac{L(y^\dagger \mid q')\pi_{\text{prior}}(q')}{Z(y^\dagger)},
\]

(10)

where \( Z(y^\dagger) := \int L(y^\dagger \mid q) \, d\mu_{\text{prior}}(q) \) is the normalized constant. To simplify this discussion, we assume that the space \( Y \) is finite dimensional, i.e. \( Y = \mathbb{C}^N \) and \( \nu_{\text{noise}} \) is a finite dimensional Gaussian distribution \( \mathcal{N}(0, \Sigma) \). The reference measure \( \nu_Y \) is the Lebesgue measure in \( \mathbb{C}^N \). Thereby, the
likelihood admits by (6) and (9)

\[ L(y^+ | q') = \det(2\pi \Sigma)^{-1/2} \exp \left( -\frac{\| \Sigma^{-1/2}(\mathcal{F}(q') - y^+) \|_Y^2}{2} \right) \]

\[ := \det(2\pi \Sigma)^{-1/2} \exp \left( -\Phi(q', y^+) \right), \tag{11} \]

where \( \Phi(q', y) = \| \Sigma^{-1/2}(\mathcal{F}(q') - y^+) \|_Y^2 \) is the negative log likelihood function. It is obvious that the likelihood function \( 0 < L(y^+ | q') < (\det(2\pi \Sigma))^{-1/2} \) and therefore is in \( L^1(\mathcal{X}, p_{\text{prior}}) \). Consequently, Bayes’ theorem shows that the posterior distribution \( \pi(q | y^+) \) with density (10) is well-defined (see Theorem 2.5 in [18]).

The Bayesian approach answers richer questions than the traditional regularization-based methodologies. As we can see, the solution it produces is not limited to individual values but consists of probability distributions. Therefore, it provides an effective tool to analyze the relative probabilities of different approximate solutions or the probability of a solution to lie in a subset of the solution space for decision makers.

3. Total variation prior

The prior is key in Bayesian inversion and usually plays a defining role in Bayesian statistical inference. There is a vast literature on the related discussion [15,19,20]. A rich class of prior distributions can be derived from the theory of Markov random fields (MRF) [15,19,21]. In this paper, the prior for \( q \) is taken as the Gauss MRF \( N(0, \Gamma) \) with covariance operator \( \Gamma = Q^{-1} \). Here \( Q \) is the precision operator defined by the fractional diffusion operator \(-\left( \frac{d^2}{d\tau^2} \right)^{\frac{s}{2}} \) with periodic boundary condition.

In the numerical implementation, we need the discrete form of a random variable. There are two common ways to transform the infinite dimensional random variable into its finite pattern. In the first way, we take a mesh partition of \( \{\theta_1, \theta_2, \ldots, \theta_m\} \subset [0, 2\pi) \). Denote the discretized version of precision operator \( Q \) by \( Q \). The discretize can be done by the finite difference, finite element, etc. The joint density function for \( q := [q(\theta_1), q(\theta_2), \ldots, q(\theta_m)]^\top \) is Gaussian and admits the form

\[ \pi_{\text{prior}}(q) = (2\pi)^{-m/2} | Q |^{1/2} \exp \left( -\frac{1}{2} q^\top Q q \right). \tag{12} \]

In this way, the finite dimensional samples of \( q \) can be generated by the probability distribution (12). Denote eigenvalues and eigenvectors of precision matrix \( Q \) by \( \lambda_Q \) and \( \psi_Q \). In the second way, we use the Karhunen–Loève expansion to represent the prior random variate. The precision operator \( Q \) admits eigenvalues \( \lambda_k \) and eigenvectors \( \psi_k \), \( k = 1, 2, \ldots \), i.e.

\[ Q\psi_k = \lambda_k \psi_k, \quad k = 1, 2, \ldots \tag{13} \]

According to Karhunen–Loève expansion of \( q \), we have

\[ q(\theta) = \sum_{k=1}^{\infty} \frac{B_k}{\lambda_k} \psi_k(\theta) \approx \sum_{k=1}^{m} \frac{B_k}{\lambda_k} \psi_k(\theta) := \tilde{q}(\theta), \tag{14} \]

where \( B_k \)s are independent identity distribution normal Gaussian. The truncated form \( \tilde{q}(\theta) \) is used in the numerical implement process.

In fact, it can be verified that \( \lambda_Q \) behaves like \( \lambda_k \) and \( q_Q \)s are vectors of values of \( \psi_k \) at discrete points \( \{\theta_1, \theta_2, \ldots, \theta_m\} \). Based on this reason, in our numerical test, we do not distinguish them any more. And in the subsequent parts, we only use the KL expansion to represent the prior. Therefore, the prior for the unknown \( q \) is determined by its expansion coefficients \( B_k \)s.
Remark 3.1: In the KL expansion prior condition, we actually assume the function $q(\theta)$ admits Fourier series representation

$$q(\theta) = \sum_{k=1}^{\infty} \frac{B_k}{\lambda_k} \psi_k(\theta).$$  \hspace{1cm} (15)

And the prior for $B_k$ is given by $N(0, I)$ from (14).

For a Gaussian random vector with mutually independent components, we may transform it to an $l_1$-type prior according to [15]. For this purpose, an invertible mapping function $g : \mathbb{R} \rightarrow \mathbb{R}$ is introduced

$$g(B_k) = \mathcal{L}^{-1}(\mathcal{G}(B_k)) = -\frac{1}{\lambda} \text{sign}(B_k) \log \left(1 - |2\mathcal{G}(B_k) - 1|\right),$$ \hspace{1cm} (16)

where $\mathcal{L}$ is the cumulative distribution function (cdf) of the Laplace distribution and $\mathcal{G}$ is the cdf of the standard Gaussian distribution. This function $g$ relates a Gaussian reference random variable $B_k \in \mathbb{R}$ to the Laplace-distributed parameter $A_k \in \mathbb{R}$, such that $A_k = g(B_k)$. Then we apply the transformation to each component of $B$

$$A(B) := [g(B_1), g(B_2), \ldots, g(B_m)]^T.$$ \hspace{1cm} (17)

A prior transformation for the $l_1$-type prior is $Z_q := D^{-1}A(B)$, i.e.

$$\pi_{\text{prior}}(Z_q) \propto \exp\left(-\lambda |DZ_q|\right) = \exp\left(-\lambda \sum_{i=1}^{m} |DZ_q|_i\right),$$ \hspace{1cm} (18)

where $D$ is an invertible matrix. For total variation prior, the invertible matrix $D$ is taken as [15]

$$D_0 = \begin{bmatrix}1 & 1 \\ -1 & 1 \\ \vdots & \vdots \\ -1 & 1 \end{bmatrix}_{m \times m}. \hspace{1cm} (19)$$

We use a modified version of $D_0$ as $D = I + \alpha D_0$ with a small constant $\alpha_0 > \alpha > 0$. By this, we actually take $|DZ_q|$ in (18) as

$$|DZ_q| = |(1 + \alpha)z_1 + \alpha z_m| + \sum_{i=2}^{m} |z_i + \alpha(z_i - z_{i-1})|,$$ \hspace{1cm} (20)

where $z_i$ is the $i$th component of $Z_q$. Since $D$ is a positive definite matrix, (20) defines a norm of vector $Z_q$. Therefore, it follows that the norm in (20) is equivalent to the Euclidian norm.

Remark 3.2: The total variation strengthens the correlations of the Fourier coefficients. In (14), the Fourier coefficients are required to be independent. In TV prior, the coefficients are not completely uncorrelated.

Figure 1 plots some shape samples using $\lambda = 0.3, \alpha = 0.1$ from TV prior. The corresponding Gaussian prior samples are also plotted in Figure 1. These TV samples are intuitively less smooth compared with the Gaussian prior samples.
4. Well-posedness of the posterior distribution

In this section, we study the well-posedness of the posterior distribution. One can refer to [22] for the general framework about the well-posedness of Bayesian approach. It is further extended to inverse scattering problems [7,8]. In [18], the author discusses the well-posedness of Bayesian inverse
problems under some weaker assumptions. We apply the theories of [18] to give the well-posedness of our problem.

The existence and uniqueness of the posterior distribution have been established by Bayes’ theorem. Now we focus on the stability using different deviation measures between two measures \( \mu, \mu' \), such as Hellinger distance, Wasserstein distance and Kullback–Leibler divergence.

- Hellinger distance: Assume that \( \mu \ll \tilde{\mu} \) and \( \mu' \ll \tilde{\mu} \). The Hellinger distance between \( \mu \) and \( \mu' \) is defined by

\[
    d_{\text{Hell}}(\mu, \mu') = \left( \frac{1}{2} \int \left( \sqrt{\frac{d\mu}{d\tilde{\mu}}} - \sqrt{\frac{d\mu'}{d\tilde{\mu}}} \right)^2 d\tilde{\mu} \right)^{\frac{1}{2}}.
\]

- Wasserstein distance: The \( p \)-Wasserstein distance between \( \mu \) and \( \mu' \) is defined by

\[
    W_p(\mu, \mu') = \left( \inf_{C(\mu, \mu')} \int_{X \times X} \|q - q'\|_X^p d\tilde{\mu}(q, q') \right)^{\frac{1}{p}}, \quad p \geq 1,
\]

where \( C(\mu, \mu') \) is the set of probability measures \( \tilde{\mu} \) on \( X \times X \) such that \( \tilde{\mu}(A \times X) = \mu(A) \) and \( \tilde{\mu}(X \times B) = \mu'(B) \) for all measurable \( A, B \subseteq X \). The measure \( \tilde{\mu} \) with marginals \( \mu \) and \( \mu' \) on the two components of \( X \times X \) is called a coupling of \( \mu \) and \( \mu' \).

- Kullback–Leibler divergence: Kullback–Leibler divergence is defined by

\[
    D_{\text{KL}}(\mu' || \mu) = \int_X \log \left( \frac{d\mu'}{d\mu} \right) d\mu', \quad \mu' \ll \mu.
\]

Some assumptions are presented to derive the well-posedness of Bayesian inverse problems [18]. We list them as follows:

**Assumption:** Let the following assumptions hold for \( \mu_{\text{prior}} \)-almost every \( q' \in X \) and every \( y^\dagger \in Y \).

(A1) \( L(\cdot \mid q') \) is a strictly positive probability density function

(A2) \( L(y^\dagger \mid \cdot) \in L^1(X, \mu_{\text{prior}}) \)

(A3) \( g \in L^1(X, \mu_{\text{prior}}) \) exists such that \( L(\tilde{y} \mid \cdot) \leq g \) for all \( \tilde{y} \in Y \)

(A4) \( L(\cdot \mid q') \) is continuous

(A5) \( \hat{g} \in L^1(X, \mu_{\text{prior}}) \) exists such that \( \|q'\|_X^p \cdot L(y^\dagger \mid q') \leq \hat{g}(q') \)

(A6) There is a \( \delta > 0 \) and a function \( h(\cdot, y^\dagger) \in L^1(X, \mu(\cdot \mid y^\dagger)) \) such that

\[
    |\log L(\tilde{y} \mid \cdot)| \leq h(\cdot, y^\dagger), \quad (\tilde{y} \in Y: \|y^\dagger - \tilde{y}\|_Y \leq \delta).
\]

For our inverse scattering problem, \( 0 < L(\tilde{y} \mid \cdot) < (\det(2\pi \Sigma))^{-1/2} \) implies that (A1)–(A3) hold. And (A4) is obvious for the likelihood function (11). As a result, we conclude that the Bayesian approach is well-posed in the sense of Hellinger metric according to [18] (see Theorem 3.6).

**Theorem 4.1 (Hellinger stability):** The posterior distribution \( \mu(q \mid \cdot) \) in (10) is locally Lipschitz continuous in the sense of Hellinger distance, i.e. there exists \( M > 0 \), such that for all \( y^\dagger, \tilde{y} \in .Y \subseteq Y \), \( y^\dagger, \tilde{y} \in Y \),

\[
    d_{\text{Hell}}(\mu(q \mid y^\dagger), \mu(q \mid \tilde{y})) \leq M\|y^\dagger - \tilde{y}\|_Y, \quad q \in X.
\]

Next let us check assumption (A5) for \( p \in [1, \infty) \). It holds for \( \mu_{\text{prior}} = N(0, \Gamma) \) that

\[
    \int_X \|q\|_X^p d\mu_{\text{prior}}(q) < +\infty.
\]

Set \( \hat{g}(q') = (\det(2\pi \Sigma))^{-1/2} \|q'\|_X^p \) and then we get that (A5) is satisfied according to [18] (see Proposition 3.11 in [18]). With (A1)–(A5) holding, we obtain the following Wasserstein stability [18]:
Theorem 4.2 (Wasserstein stability): Let $p \in [1, \infty)$, the posterior $\mu(q | \cdot)$ with density (10) is Wasserstein stable.

Finally, we assume that $X = C^{2,\tau}(0, 2\pi)$, $\tau \in (0, 1]$. The following estimation holds.

**Lemma 4.3 ([8]):** For fixed $\hat{x}$, $d$ and every $\epsilon > 0$, there exists $M = M(\epsilon)$ such that

$$\|\Sigma^{-1/2}F(q)\|_Y \leq \exp(\epsilon\|q\|^2_X + M)$$

for all $q \in X$.

Then we give the Kullback–Leibler stability.

**Theorem 4.4 (Kullback–Leibler stability):** Let $X = C^{2,\tau}(0, 2\pi)$. Then for all $y^\dagger \in Y$ and $\epsilon > 0$, there is $\delta(\epsilon) > 0$, such that

$$D_{KL}(\mu(q | y^\dagger) || \mu(q | \hat{y})) \leq \epsilon, \quad (\hat{y} \in Y : \|y^\dagger - \hat{y}\|_Y \leq \delta(\epsilon)).$$

**Proof:** The stability can be obtained when (A1)–(A4) and (A6) are satisfied [18], (A1)–(A4) hold. We only need to verify assumption (A6), i.e. find a function $h(\cdot, y^\dagger) \in L^1(X, \mu(\cdot | y^\dagger))$ such that

$$\Phi(\cdot, \hat{y}) = |\log L(\hat{y} | \cdot)| \leq h(\cdot, y^\dagger), \quad (\hat{y} \in Y : \|y^\dagger - \hat{y}\|_Y \leq \delta).$$

We have by the triangle inequality

$$\Phi(q, \hat{y}) \leq \frac{1}{2}\left(\|\Sigma^{-1/2}(F(q) - y^\dagger)\|^2_Y + \|\Sigma^{-1/2}(y^\dagger - \hat{y})\|^2_Y\right)$$

$$\leq \frac{1}{2}\left(\|\Sigma^{-1/2}(F(q) - y^\dagger)\|^2_Y + \|\Sigma^{-1/2}\delta^2\right) =: h(q, y^\dagger).$$

Next we prove $h(q, y^\dagger) \in L^1(X, \mu(q | y^\dagger))$. To get this, we only need show that $\|\Sigma^{-1/2}(F(q) - y^\dagger)\|_Y^2$ is $\mu(q | y^\dagger)$-integrable. In fact, according to Lemma 4.3, we can get

$$\int_X \|\Sigma^{-1/2}(F(q) - y^\dagger)\|_Y^2 d\mu(q | y^\dagger)$$

$$\leq \int_X 2\exp(2\epsilon\|q\|^2_X + 2M) + 2\|\Sigma^{-1/2}y^\dagger\|_Y^2 d\mu(q | y^\dagger)$$

$$\leq \frac{2(\det(2\pi \Sigma))^{1/2}}{Z(y^\dagger)} \int_X \left(\exp(2\epsilon\|q\|^2_X + 2M) + \|\Sigma^{-1/2}y^\dagger\|_Y^2\right) d\mu_{\text{prior}}.$$

Table 1. The boundary parameterizations of the object obstacles.

| Object         | Parameterization                                      |
|----------------|-------------------------------------------------------|
| Kite           | $(x_1, x_2) = (\cos \theta + 0.65 \cos 2\theta - 0.65, 1.5 \sin \theta)$ |
| Roundrect      | $r(\theta) = (\cos^4 \theta + (2/3 \sin \theta)^4)^{-1/4}$ |
| Acorn          | $r(\theta) = \frac{1}{3} \sqrt{\frac{12}{\theta} + 2 \cos 3\theta}$ |
| Pear           | $r(\theta) = \frac{1}{6} \cos 3\theta$ |
| Bean           | $r(\theta) = 0.4 \sqrt{\cos^2 \theta + \sin^2 \theta}$ |
| Threeolobes    | $r(\theta) = 0.5 + 0.25 \exp(- \sin 3\theta) - 0.1 \sin \theta$ |
| Star           | $r(\theta) = 1 + 0.3 \sin 5\theta$ |
| Cloverleaf     | $r(\theta) = 1 + 0.3 \cos 4\theta$ |
| Peanut         | $r(\theta) = 0.4 \sqrt{\cos^2 \theta + \sin^2 \theta}$ |
| Drop           | $(x_1, x_2) = (-1 + 2 \sin(\theta/2), - \sin \theta)$ |
Using Fernique’s theorem, we get for Gaussian prior $\mu_{\text{prior}} = N(0, \Gamma)$ that

$$\int_X \exp(2\epsilon \|q\|^2_X) d\mu_{\text{prior}} < +\infty.$$  \hfill (25)

**Figure 2.** Reconstructions for different shapes using $\varphi^\infty(\hat{x}, d), (\hat{x}, d) \in \gamma_1 \times \gamma_2$ with $\eta_1 = 1\%$, $\eta_2 = 1\%$. (a) Kite; (b) Roundrect; (c) Pear; (d) Acorn; (e) Bean; (f) Threelobes; (g) Star; (h) Cloverleaf; (i) Peanut; (j) Drop.
Thus \( \| \Sigma^{-1/2}(F(q) - y^\dagger) \|_2^2 \) is \( \mu(q \mid y^\dagger) \)-integrable. As a result, we get \( h(\cdot, y^\dagger) \in L^1(X, \mu(q, y^\dagger)) \). Therefore (A6) holds for \( X = C^{2,\tau}(0, 2\pi) \). We finish the proof. 

When the prior on \( q \) is transformed to the prior on the Fourier coefficients, the forward model actually is of the form

\[
\tilde{G}(B) := \mathcal{F}(q(\theta; B)) = y^\dagger,
\]

where the Fourier coefficients \( B \) are the unknowns. The posterior density is then

\[
\pi(B \mid y^\dagger) = \frac{L(y^\dagger \mid q(\theta; B)) \pi_{\text{prior}}(B)}{Z(y^\dagger)}.
\]

For the posterior density (27), the Hellinger stability and Wasserstein stability still hold since \( G : \mathbb{R}^m \to \mathbb{C}^N \) (see Corollary 5.1 in [18]). Specially, for the proposed TV prior, we transform the Gaussian \( B \) to the TV \( D^{-1}A(B) \) in Section 3. Then the forward model is

\[
\tilde{G}(B) := \mathcal{F}(q(\theta; D^{-1}A(B))) = y^\dagger,
\]

where \( \tilde{G} : \mathbb{R}^m \to \mathbb{C}^N \). This implies that the Hellinger stability and Wasserstein stability hold.

5. Numerical examples

We use the pCN Markov chain Monte Carlo algorithm [22] listed in Algorithm 1 to implement the sampling process.

**Algorithm 1** pCN MCMC algorithm

1. Initialization: Pick \( B^0 \). Transform \( B^0 \) to \( Z_q^0 \) according to (17)-(19) and compute \( \tilde{q}(Z_q^0) \) by (14). Set \( j = 0 \).
2. While \( j < N \),
   - Draw a proposal \( \hat{B} \) according to
     \[
     \hat{B} = \sqrt{1 - \beta^2} B^j + \beta B_{pr},
     \]
     where \( B_{pr} \sim N(0, I) \) and \( \beta \in (0, 1) \) is a constant. Transform \( \hat{B} \) to \( \hat{Z}_q \) according to (17)-(19).
     Generate \( \tilde{q}(\hat{Z}_q) \) using (14) with coefficients \( \hat{Z}_q \).
   - Compute the acceptance probability:
     \[
     \alpha = \min \left\{ 1, \frac{\exp(-\Phi(\tilde{q}(\hat{Z}_q), g))}{\exp(-\Phi(\tilde{q}(Z_q^j), g))} \right\}.
     \]
   - Flip an \( \alpha \)-coin: Draw \( \xi \sim \text{Uniform}(0, 1) \):
     - If \( \xi \leq \alpha \), accept the proposal, setting \( B^{j+1} = \hat{B} \),
     - else, reject the move and stay put \( B^{j+1} = B^j \).
   - Increase \( j \leftarrow j + 1 \).

We take several benchmark examples in acoustic obstacle scattering to verify the numerical effectiveness (\( \theta \in (0, 2\pi) \)), see Table 1.
The true data is generated by solving the forward problem by boundary integral equation method [17]. Let \( \phi \) be the observation angle such that \( \hat{x} := (\cos \phi, \sin \phi) \). We use the observation/measurement apertures as follows:

\[
\gamma_1^o = \{ (\cos \phi, \sin \phi) \mid \phi \in [0, 2\pi] \},
\gamma_2^o = \{ (\cos \phi, \sin \phi) \mid \phi \in [0, \pi] \},
\gamma_3^o = \{ (\cos \phi, \sin \phi) \mid \phi \in [0, \pi/2] \}.
\]

The incident apertures are

\[
\gamma_1^i = \{(1,0)\},
\gamma_2^i = \{(\cos \theta, \sin \theta) \mid \theta = \{\pi/2, 3\pi/2\}\}.
\]

The relative noise is added

\[
y^\dagger = u^\infty(\hat{x}, d) + (\eta_1 + \eta_2 i)\|u^\infty(\hat{x}, d)\|\infty.
\]  

(29)

We first plot the numerical results in Figure 2 using different parameters \( \alpha = 0, 0.1, 0.5, 1, 2 \) by setting \( \lambda = 0.2 \) for \( \gamma^o \times \gamma^i = \gamma_1^o \times \gamma_2^i \). For all plots in Figure 2, the wave number is set to \( \kappa = 1 \), the Fourier expansion of \( q \) is truncated to the first 27 terms, the true center position \( x_c \) is fixed at \((0,0)\). And we take the samples after 1000 iteration and use the mean estimation as the approximation of the unknown parameters. The parameter \( \beta \) in Algorithm 1 is set to 0.002. We generate \( 5 \times 10^5 \) samples for Star and Cloverleaf shapes and \( 2 \times 10^4 \) samples for other shapes. Meanwhile, the numerical results are used to compare with Gaussian prior and the TV with \( D = I + \alpha D_0 \) replaced by \( D_0 \). The results show that the reconstructions are better for smaller \( \alpha \) for this proposed method. When \( \alpha = 2 \), the results become worse, so do for \( D = D_0 \). We also find that the reconstructions with the proposed prior is superior to the Gaussian prior and the case \( D = D_0 \). Especially for Kite and Star shapes, the reconstructions are much better with the TV prior than with the other two priors.

In Figure 3, we check the numerical performance for the proposed method when the Gaussian prior parameter \( s \) varies for fixed \( \alpha = 0.1 \) and \( \lambda = 0.8 \). We generate samples for four shapes: Peanut,
Roundrect, Cloverleaf, Acorn, for different $s$. From Figure 3, we can see that the value of $s$ affects the smoothness of the reconstructions. This point can be verified from the form of Gaussian prior [22].

It is better to choose $\lambda = 0.2$ by comparing the results of Cloverleaf in Figure 2 with Figure 3. And for other shapes, there is no obvious difference for $\lambda = 0.2$ and $\lambda = 0.8$.

Figure 4. Reconstructions for different shapes using $u^\infty(\hat{x}, d), (\hat{x}, d) \in \gamma_1^u \times \gamma_1$ with $\eta_1 = 1\%, \eta_2 = 1\%$: (a) Kite; (b) Roundrect; (c) Pear; (d) Acorn; (e) Bean; (f) Threelobes; (g) Star; (h) Cloverleaf; (i) Peanut; (j) Drop.
Next, we consider the case of one incident wave, i.e. $\gamma^i = \gamma^i_1$, with full observation aperture. Fix $\alpha = 0.1$ and set all other parameters as that in Figure 2. Taking one observation aperture with $\gamma^o_1$, i.e. $\gamma^o \times \gamma^i = \gamma^o_1 \times \gamma^i_1$, we display the numerical reconstructions in Figure 4 using noise level $\eta_1 = 1\%$, $\eta_2 = 1\%$. These results show that the single incident wave performs satisfactory reconstructions.

The limited aperture cases are considered with the same noise and parameter configurations in Figure 4. We show the numerical reconstructions using observation aperture $\gamma^o = \gamma^o_2$ with two incident waves, i.e. $\gamma^o \times \gamma^i = \gamma^o_2 \times \gamma^i_2$ in Figure 5. With smaller observation aperture $\gamma^o = \gamma^o_3$, Figure 6 shows the results for two incident waves $\gamma^i = \gamma^i_2$. These results show that the reconstructions are better for larger observation aperture for two incident waves.

![Figure 5. Reconstructions for different shapes using $u^\infty(\hat{x}, d), (\hat{x}, d) \in \gamma^o_2 \times \gamma^i_2$ with $\eta_1 = 1\%, \eta_2 = 1\%$: (a) Kite; (b) Roundrect; (c) Pear; (d) Acorn; (e) Bean; (f) Threelobes; (g) Star; (h) Cloverleaf; (i) Peanut; (j) Drop.](image-url)
We list the acceptance rates for the TV prior in Table 2 with $\alpha = 0.1$ for $\beta = 0.002$ for the test examples in Figures 2, 4, 5, 6. These results show that the acceptance rates are in the interval [20\%, 50\%].

**Figure 6.** Reconstructions for different shapes using $u^\infty(\hat{x}, d)$, $(\hat{x}, d) \in \gamma^{d}_1 \times \gamma^{\infty}_1$ with $\eta_1 = 1\%$, $\eta_2 = 1\%$: (a) Kite; (b) Roundrect; (c) Pear; (d) Acorn; (e) Bean; (f) Threelobes; (g) Star; (h) Cloverleaf; (i) Peanut; (j) Drop.
Table 2. Acceptance rates.

| Shape   | $\gamma_o \times \gamma_1$ | $\gamma_o \times \gamma_2$ | $\gamma_o \times \gamma_3$ | $\gamma_o \times \gamma_4$ |
|---------|----------------------------|----------------------------|----------------------------|----------------------------|
| Kite    | 28%                        | 23%                        | 21%                        | 27%                        |
| Roundrect | 36%                        | 22%                        | 20%                        | 24%                        |
| Acorn   | 35%                        | 23%                        | 22%                        | 26%                        |
| Pear    | 47%                        | 31%                        | 30%                        | 44%                        |
| Bean    | 50%                        | 29%                        | 26%                        | 42%                        |
| Threelobes | 46%                        | 29%                        | 28%                        | 44%                        |
| Star    | 40%                        | 25%                        | 23%                        | 27%                        |
| Cloverleaf | 41%                        | 25%                        | 23%                        | 28%                        |
| Peanut  | 50%                        | 38%                        | 37%                        | 39%                        |
| Drop    | 50%                        | 29%                        | 27%                        | 29%                        |

Figure 7. Reconstructions for shapes: Kite, Star, Cloverleaf, Pear, using nine different centers with $(\tilde{x}, d) \in \gamma_o \times \gamma_2, \eta_1 = 5\%, \eta_2 = 5\%$.

We also check the numerical accuracy using inaccurate center positions $x_c$ in the reconstruction process. We use the following nine different centers:

$$(0.2, \pm 0.2), \ (-0.2, \pm 0.2), \ (0, \pm 0.2), \ (\pm 0.2, 0), \ (0, 0),$$

to reconstruct shapes: Kite, Cloverleaf, Pear and Star using $\eta_1 = 5\%, \eta_2 = 5\%$. The results are displayed in Figure 7. We conclude that even the center is not exact, the reconstruction has not been seriously affected. This means that if we can use some method, e.g. the extended sampling method [8], to find the approximated center first, the proposed method provides satisfactory reconstruction.

6. Conclusion

We discuss the application of the Bayesian method for the limited aperture inverse scattering problem. A novel total variation prior is proposed for the shape parameterization representation. Numerical examples show that the proposed method can yield satisfactory reconstructions. The reconstructions with multi-incident wave are better than with single incident wave. The observation aperture affects the numerical performance. If the observation aperture is not too small, the reconstructions are satisfactory. In addition, since the TV prior is imposed on Fourier coefficients, the prior curves preserve...
smoothness to a certain extent due to the KL expansion. Though the prior curve shows some weak smoothness, it is still difficult to deal with objects with sharp corners.

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