Checking marking reachability with the state equation in Petri net subclasses

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Abstract. Although decidable, the marking reachability problem for Petri nets is well-known to be intractable in general, and a non-elementary lower bound has been recently uncovered. In order to alleviate this difficulty, various structural and behavioral restrictions have been considered, allowing to relate reachability to properties that are easier to check. For a given initial marking, the set of potentially reachable markings is described by the state equation solutions and over-approximates the set of reachable markings.

In this paper, we delineate several subclasses of weighted Petri nets in which the set of reachable markings equals the set of potentially reachable ones, a property we call the PR-R equality. When fulfilled, this property allows to use linear algebra to answer the reachability questions, avoiding a brute-force analysis of the state space. Notably, we provide conditions under which this equality holds in classes much more expressive than marked graphs, adding places with several ingoing and outgoing transitions, which allows to model real applications with shared buffers. To achieve it, we investigate the relationship between liveness, reversibility, boundedness and potential reachability in Petri nets. We also show that this equality does not hold in classes with close modeling capability when the conditions are relaxed.

Keywords: Weighted Petri net, State equation, Potential reachability, PR-R equality, Efficient analysis, Reverse net, Liveness, Reversibility, Augmented marked graph, Refinement, Place merging, T-net.

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1. Introduction

Petri nets, or equivalently vector addition systems (VASs), have proved useful to model numerous artificial and natural systems. Their weighted version allows weights (multiplicities) on arcs, making possible the bulk consumption or production of tokens, hence a more compact representation of the systems.

For many fundamental Petri net properties, the problem of their checking is decidable although intractable. Given a bounded Petri net, a naive analysis can be performed by constructing its finite reachability graph, whose size may be considerably larger than the net size. To avoid such a costly computation, subclasses are often considered, allowing to derive efficiently their behavior from their structure only. This approach has led to various polynomial-time checking methods dedicated to several subclasses, the latter being defined by structural restrictions in many cases [1, 2, 3, 4, 5, 6].

The reachability problem. Given a Petri net system $S$, the problem is to determine if a given marking is reachable in $S$. In weighted Petri nets, this question was known to be EXPSPACE-hard [7]; recently, a non-elementary lower bound has been obtained [8]. Reachability reduces to various well-known model-checking problems [8], which thus inherit this lower bound.

Relating reachability to the state equation. Consider a system $S = (N, M_0)$ with its incidence matrix $I$, where $N = (P, T, W)$ is the underlying net ($P$ being the set of places, $T$ the set of transitions and $W$ the weighting function).

The state equation associated to $S$ is expressed as $M = M_0 + I \cdot Y$, where the variable $M$ takes its value in the set of markings and the variable $Y$ ranges over the set of vectors whose components are non-negative integers.

The set of markings potentially reachable in $S$ is defined as $PR(S) = \{ M \in \mathbb{N}^{|P|} \mid \exists Y \in \mathbb{N}^{|T|}, M = M_0 + I \cdot Y \}$; this set is called the linearized reachability set of $S$ in [3]. Potential reachability is a necessary condition for reachability, but it is not sufficient in general; a Petri net satisfies the PR-R equality if its reachable markings are its potentially reachable ones. Thus, in the subclasses that are known to fulfill the PR-R equality, solving the reachability problem amounts to check the existence of a solution to an integer linear program (ILP) of polynomial size, trimming down its complexity to NP.

Petri net subclasses, applications and previous studies. In this work, we study conditions for the PR-R equality to hold in weighted Petri nets and several of their subclasses, notably:

– Weighted Marked Graphs with relaxed place constraint ($WMG_\leq$ for short), which force each place to have at most one input and one output, studied e.g. in [2, 10];
– Augmented Marked Graphs (AMG), which are unit-weighted and allow the addition of several shared places to a (unit-weighted) Marked Graph under some restrictions [11];
– H1S-WMG_\leq, i.e. homogeneous nets having at most one shared place, the deletion of which yields a WMG_\leq [10]; this class thus contains the WMG_\leq;

\footnote{A place is shared if it has at least two outgoing transitions.}

\footnote{Homogeneity means that, for each shared place $p$, all the output weights of $p$ are equal.}

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– Place-Composed Marked Graphs with relaxed place constraints (PCMG≤), obtained from a given undirected graph by replacing each edge with a marked graph and each vertex with a shared place, which is a kind of synthesis.

As far as we know, PCMG≤ have not been studied until now, while the previous works on AMG and H1S-WMG≤ did not focus on the PR-R equality. These net classes, although very restricted, can already model numerous real-world applications. Let us present some of them:

– WMG≤ generalize the Marked Graphs (MG)/Weighted Marked Graphs (WMG)/Weighted Event Graphs (WEG)/Weighted T-systems (WTS) [12] [13] [14] [1] [15], in which each place has exactly one input and one output. They are a special case of persistent systems [16], in which no transition firing can disable any other transition. They can model Synchronous DataFlow graphs [17], which have been fruitfully used to design and analyze many real-world systems such as embedded applications, notably Digital Signal Processing (DSP) applications [18] [19] [20]. Various analysis and synthesis methods have been developed for WMG≤ [15] [14] [21] [22] [9] [23] [24] [25] [26] and larger classes [2] [4] [10], dealing notably with reachability, liveness, boundedness and reversibility. In the same studies, some relationships between the behavior and the state equation are also provided.

– AMG, although unit-weighted, can model various manufacturing systems [11] [27], the dining philosophers problem [27], and several use-cases of the Model Checking Contest (MCC) such as the Swimming pool protocol, the Robot manipulation system, the Client and server protocol and the process-management method Kanban (under some conditions to be fulfilled by the initial marking). AMG benefit from results on their liveness, reversibility and reachability [11].

– H1S-WMG≤ extend both 1S-AMG (i.e. AMG with at most one shared place) and WMG≤, making more flexible the modeling of applications with WMG≤. The Swimming pool protocol can also be modeled with an H1S-WMG≤ [10].

– The class of PCMG≤, which is not included in the classes above nor contains them, allows to define a system first in terms of its topology of shared buffers, before refining the processes that connect and use these buffers. This model can be used for system synthesis under structural and behavioral constraints. Previous works, described in [28], have proposed conditions for merging sets of places into shared places while preserving various structural and behavioral properties, notably in unit-weighted asymmetric-choice Petri nets. However, the subclasses studied in [28] do not contain the PCMG≤.

Contributions. In this paper, we propose new sufficient conditions ensuring the PR-R equality: a general condition applying to all weighted Petri nets, and other ones dedicated to the subclasses mentioned above. When such conditions are known to be fulfilled, checking reachability then reduces to solving the state equation with linear algebra over the integers, trimming down the complexity to NP. So as to obtain these results, we exploit the next notions:

– directedness: a property stating the existence, for any two potentially reachable markings \( M_1 \) and \( M_2 \), of a marking reachable from both \( M_1 \) and \( M_2 \);

– initial directedness: a property stating the existence, for each potentially reachable marking \( M_1 \), of a marking reachable from both \( M_1 \) and the initial marking \( M_0 \);

– the reverse net: obtained by reversing all the arcs;

– liveness: a property stating the possibility, from each reachable marking, to fire some sequence con-

https://mcc.lip6.fr/models.php
taining all transitions;
− **boundedness**: a property stating the existence of an upper bound on the number of tokens of each place over all reachable markings;
− **reversibility**: a property stating the possibility to reach the initial marking from each reachable marking, meaning the strong connectedness of the reachability graph;
− the property \( \mathcal{R} \): we introduce this property for any Petri net system, stating reversibility of both the system and its reverse.

More precisely, we exploit these notions as follows.

First, we show that combining property \( \mathcal{R} \) with initial directedness is sufficient to ensure the PR-R equality in any weighted Petri net. This new condition is not necessary in general, but we show its tightness for live and bounded WMG\(_\leq\). We also present new results on liveness and deadlockability in WMG\(_\leq\), which help checking the precondition of liveness.

Then, we improve our understanding of the relationship between the state equation solutions (potential reachability), reachability, liveness and reversibility in the mentioned generalizations of marked graphs with shared places: in AMG, H1S-WMG\(_\leq\) and PCMG\(_\leq\), we provide new sufficient conditions ensuring that the PR-R equality is fulfilled. We highlight the sharpness of all conditions by providing counter-examples when only few assumptions are relaxed. We also propose methods to check the various conditions and give insight on their complexity in the subclasses mentioned and sometimes in larger ones. Notably, in a subclass of PCMG\(_\leq\), we provide a variant of Commoner’s theorem and of the Home Marking theorem which were developed for free-choice nets to characterize the live and reversible markings in polynomial-time \([1]\).

This work is the sequel to our previous paper \([10]\), in which we provided conditions for checking reachability, liveness and reversibility more efficiently in some subclasses. Since we often use liveness and reversibility as preconditions for the PR-R equality to hold, these previous results can be exploited to reduce their checking complexity.

**Organization of the paper.** In Section 2, we introduce general definitions, notations and properties. In Section 3, we define the main subclasses studied in this paper and compare their expressiveness.

In Section 4, we define the main notions related to directedness and recall related properties. We also recall known classes of the literature that fulfill directedness.

In Section 5, we give new properties of nets and their reverse; we provide notably a general sufficient condition for the PR-R equality, using reversibility, initial directedness and reverse nets. We apply this result to ensure the PR-R equality in the class of live homogeneous free-choice (HFC) nets, a weighted generalization of free-choice nets.

In Section 6, we show that live WMG\(_\leq\) fulfill the PR-R equality, we propose new characterizations of liveness and a new property about reachable deadlocks in this class, based on the state equation. We also discuss methods to check the behavioral properties of interest in WMG\(_\leq\).

In Section 7, we construct new examples of systems that do not fulfill the PR-R equality. They belong to the 2S-WMG\(_\leq\) subclass, i.e. the class of nets with at most 2 shared places, the deletion of which yields a WMG\(_\leq\). On these examples, we emphasize possible causes of roadblocks to the PR-R equality.

\(^4\)The different notion of *reversible computation* has been investigated in \([29, 30]\); contrarily to the global property of reversibility, reversible computation is a local mechanism that a system can use to undo some of the executed actions \([29, 30]\).
equality.

In Sections 8, 9 and 10 we study several subclasses of S-WMG$_\leq$, namely H1S-WMG$_\leq$, AMG and PCMG$_\leq$. We provide for them conditions that ensure the PR-R equality, exploiting the examples of Section 7. We also discuss methods for checking these conditions in the classes studied.

In Section 11 we further discuss related works.

Finally, Section 12 presents our conclusion with perspectives.

2. General Definitions, Notations and Properties

In the following, we define formally Petri nets, related notions and properties.

Petri nets, incidence matrices, pre- and post-sets, shared places. A (Petri) net is a tuple $N = (P, T, W)$ such that $P$ is a finite set of places, $T$ is a finite set of transitions, with $P \cap T = \emptyset$, and $W$ is a weight function $W: ((P \times T) \cup (T \times P)) \rightarrow \mathbb{N}_0$ setting the weights on the arcs. A marking of the net $N$ is a mapping from $P$ to $\mathbb{N}$, i.e. a member of $\mathbb{N}^P$, defining the number of tokens in each place of $N$.

A (Petri net) system is a tuple $S = (N, M_0)$ where $N$ is a net and $M_0$ is a marking, often called initial marking. The incidence matrix $I$ of $N$ (and $S$) is the integer place-transition matrix with components $I(p, t) = W(t, p) - W(p, t)$, for each place $p$ and each transition $t$.

The post-set $n^\bullet$ and pre-set $n^\bullet$ of a node $n \in P \cup T$ are defined as $n^\bullet = \{n' \in P \cup T \mid W(n, n') > 0\}$ and $n^\bullet = \{n' \in P \cup T \mid W(n', n) > 0\}$.

A place $p$ is shared if it has at least two outputs, i.e. $|p^\bullet| \geq 2$.

Some of these notions are illustrated in Figure 1.

Firings and reachability in Petri nets. Consider a system $S = (N, M_0)$ with $N = (P, T, W)$. A transition $t$ is enabled at $M_0$ (i.e. in $S$) if for each $p$ in $t^\bullet$, $M_0(p) \geq W(p, t)$, in which case $t$ is feasible or fireable from $M_0$. The firing of $t$ from $M_0$ leads to the marking $M = M_0 + I[P, t]$ where $I[P, t]$ is the column of $I$ associated to $t$: this is denoted by $M_0[t]M$.

A finite (firing) sequence $\sigma$ of length $n \geq 0$ on the set $T$, denoted by $\sigma = t_1 \ldots t_n$ with $t_1 \ldots t_n \in T$, is a mapping $\{1, \ldots, n\} \rightarrow T$. Infinite sequences are defined similarly as mappings $\mathbb{N} \setminus \{0\} \rightarrow T$. A sequence $\sigma$ of length $n$ is enabled (or feasible, fireable) in $S$ if the successive states obtained, $M_0[t_1]M_1 \ldots [t_n]M_n$, satisfy $M_{k-1}[t_k]M_k$, for each $k$ in $\{1, \ldots, n\}$, in which case $M_n$ is said to be reachable from $M_0$: we denote this by $M_0[\sigma]M_n$. If $n = 0$, $\sigma$ is the empty sequence $\epsilon$, implying $M_0[\epsilon]M_0$. The set of markings reachable from $M_0$ is denoted by $R(S)$ or $[S]$; when it is clear from the context, it is also denoted by $R(M_0)$ or $[M_0]$.

The reachability graph of $S$, denoted by $RG(S)$, is the rooted directed graph $(V, A, \epsilon)$ where $V$ represents the set of vertices labeled bijectively with the markings $[M_0]$, $A$ is the set of arcs labeled with transitions of $T$ such that the arc $M \xrightarrow{t} M'$ belongs to $A$ if and only if $M[t]M'$ and $M \in [M_0]$, and $\epsilon$ is the root, labeled with $M_0$.

In Figure 1 a weighted system is pictured on the left. Its reachability graph is pictured on the right, where $v^T$ denotes the transpose of vector $v$. 
Subnets and subsystems. Let \( N = (P,T,W) \) and \( N' = (P',T',W') \) be two nets. \( N' \) is a subnet of \( N \) if \( P' \) is a subset of \( P \), \( T' \) is a subset of \( T \), and \( W' \) is the restriction of \( W \) to \( (P' \times T') \cup (T' \times P') \). \( S' = (N',M'_0) \) is a subsystem of \( S = (N,M_0) \) if \( N' \) is a subnet of \( N \) and its initial marking \( M'_0 \) is the restriction of \( M_0 \) to \( P' \), denoted by \( M'_0 = M_{0\mid P'} \).

\( N' \) is a P-subnet of \( N \) if \( N' \) is a subnet of \( N \) and \( T' = \bullet P' \cup P'\bullet \), the pre- and post-sets being taken in \( N \). \( S' = (N',M'_0) \) is a P-subsystem of \( S = (N,M_0) \) if \( N' \) is a P-subnet of \( N \) and \( S' \) is a subsystem of \( S \). We say that \( N' \) and \( S' \) are induced by the subset \( P' \).

Similarly, \( N' \) is a T-subnet of \( N \) if \( N' \) is a subnet of \( N \) and \( P' = \bullet T' \cup T'\bullet \), the pre- and post-sets being taken in \( N \). \( S' = (N',M'_0) \) is a T-subsystem of \( S = (N,M_0) \) if \( N' \) is a T-subnet of \( N \) and \( S' \) is a subsystem of \( S \). We say that \( N' \) and \( S' \) are induced by the subset \( T' \).

Subsystems play a fundamental role in the analysis of Petri nets, typically leading to characterizations relating the system’s behavior to properties of its subsystems; this approach yielded polynomial-time checking methods in various subclasses, e.g. [14, 4, 6]. We exploit such subsystems in this paper to obtain some of our new results on reachability.

Examples are given in Figure 2

Siphons and traps. Consider a net \( N = (P,T,W) \). A subset \( D \subseteq P \) of places is a siphon (sometimes also called a deadlock) if \( \bullet D \subseteq D\bullet \). A subset \( Q \subseteq P \) of places is a trap if \( \overline{Q} \subseteq Q \). Siphons and traps are most often assumed to be non-empty; for the sake of conciseness, we allow emptiness explicitly when it is needed.

There exist various studies relating the structure to the behavior with the help of siphons and traps. Intuitively, insufficiently marked siphons induce P-subsystems that cannot receive new tokens and thus block some transitions irremediably, while marked traps always keep some token and favor the enabledness of outgoing transitions, at least in some ordinary subclasses; see e.g. [11, 2].

A siphon (respectively trap) is minimal if it does not contain any proper siphon (respectively trap), i.e. there is no subset of the same type with smaller cardinality.

In Figure 2 on the left, \( \{p_1,p_2,p_3,p_4\} \) is both a siphon and a trap, and includes smaller ones,
namely \( \{p_1, p_3\} \) and \( \{p_2, p_4\} \), while \( \{p_1\} \) is neither a siphon nor a trap; in the middle, \( \{p_3\} \) is a minimal siphon and is not a trap, while \( \{p_1\} \) is a minimal trap and is not a siphon.

**Vectors, semiflows, conservativeness and consistency.** The support of a vector is the set of the indices of its non-null components. Consider any net \( N = (P, T, W) \) with its incidence matrix \( I \).

A T-vector (respectively P-vector) is an element of \( \mathbb{N}^T \) (respectively \( \mathbb{N}^P \)); it is called prime if the greatest common divisor of its components is one (i.e. its components do not have a common non-unit factor). It is called minimal when it is prime and its support is not a proper superset of the support of any other T-vector. The cardinality of a T-vector is the sum of its components; for instance, the cardinality of \((1, 0, 2, 5)\) is 8.

The Parikh vector \( P(\sigma) \) of a finite sequence \( \sigma \) of transitions is the T-vector counting the number of occurrences of each transition in \( \sigma \), and the support of \( \sigma \) is the support of its Parikh vector, i.e. \( \text{supp}(\sigma) = \text{supp}(P(\sigma)) = \{ t \in T \mid P(\sigma)(t) > 0 \} \).

We denote by \( 0^n \) (respectively \( 1^n \)) the column vector of size \( n \) whose components are all equal to 0 (respectively 1). The exponent \( n \) may be omitted when it is clear from the context.

A T-semiflow (respectively P-semiflow) \( Y \) of the net is a non-null T-vector (respectively P-vector) whose components are only non-negative integers (i.e. \( Y \geq 0 \)) and such that \( I \cdot Y = 0 \) (respectively \( Y^T \cdot I = 0 \)).

\( N \) is conservative, or invariant, if a P-semiflow \( X \in \mathbb{N}^P \) exists for \( I \) such that \( X \geq 1^P \), in which case \( X \) is called a conservativeness vector. In case such a P-vector \( X \) exists and, in addition, \( X = 1^P \), \( N \) is called 1-conservative, or 1-invariant.

\( N \) is consistent if a T-semiflow \( Y \in \mathbb{N}^T \) exists for \( I \) such that \( Y \geq 1^T \), in which case \( Y \) is called a consistency vector.

Such vectors are frequently exploited in the structural and behavioral analysis of Petri nets, see e.g. [3].

**State equation, potential reachability and the PR-R equality.** Consider any system \( S = (N, M_0) \) with incidence matrix \( I \). The state equation associated to \( S \) is expressed as \( M = M_0 + I \cdot Y \), whose solutions are described by the variables \( M \) and \( Y \), denoting respectively markings and T-vectors. The set of markings potentially reachable in \( S \) is defined as \( PR(S) = \{ M \in \mathbb{N}^P \mid \exists Y \in \mathbb{N}^T, M = \ldots \} \).
$M_0 + I \cdot Y$. We denote by $PRG(S)$ the potential reachability graph of $S$, defined as the rooted directed graph $(V, A, t)$ where $V$ represents the set of vertices $PR(S)$, $A$ is the set of arcs labeled with transitions of $S$ such that, for each transition $t$, the arc $M \xrightarrow{t} M'$ belongs to $A$ if and only if $M(t)M'$ and $M \in PR(S)$, and $t = M_0$ is the root.

A Petri net system $S$ fulfills the PR-R equality if $R(S) = PR(S)$.

**Deadlockability, liveness, boundedness and reversibility.** Consider any system $S = (N, M_0)$. A transition $t$ is *dead* in $S$ if no marking of $[M_0]$ enables $t$. A deadlock, or *dead marking*, is a marking enabling no transition. $S$ is deadlock-free if no deadlock belongs to $[M_0]$; otherwise it is deadlockable.

A transition $t$ is *live* in $S$ if for every marking $M$ in $[M_0]$, there is a marking $M' \in [M]$ enabling $t$. $S$ is *live* if every transition is live in $S$. $N$ is *structurally live* if a marking $M$ exists such that $(N, M)$ is live.

A marking $M$ is a *home state* of $S$ if it can be reached from every marking in $[M_0]$. $S$ is *reversible* if its initial marking is a home state, meaning that $RG(S)$ is strongly connected.

$S$ is *$k$-bounded* (or *$k$-safe*) if an integer $k$ exists such that: for each $M$ in $[M_0]$, for each place $p$, $M(p) \leq k$. It is *bounded* if an integer $k$ exists such that $S$ is $k$-bounded. $N$ is *structurally bounded* if $(N, M)$ is bounded for each $M$.

$N$ is *well-formed* if it is structurally bounded and structurally live.

The underlying net $N$ in Figure 1 is structurally live and bounded, hence well-formed. In the same figure, the system $S = (N, M_0)$ is live, $4$-bounded and reversible, thus non-deadlockable, which can be checked on its finite reachability graph.

## 3. Petri net subclasses

In this section, we define the subclasses of Petri nets studied in this paper.

### 3.1. Classical restrictions on the structure

Let us define subclasses from restrictions on the structure of any net $N = (P, T, W)$.

− Subclasses defined by restrictions on the weights. $N$ is *ordinary* (or *plain, unit-weighted*) if no arc weight exceeds $1$; $N$ is *homogeneous* if for each place $p$, all outgoing weights of $p$ are equal. In particular, ordinary nets are homogeneous. In this paper, for any class of nets $C$, we denote by HC the homogeneous subclass of $C$. Examples are pictured in Figures 3 and 4.

− Subclasses without shared places. $N$ is *choice-free* (CF, also called place-output-nonbranching) if each place has at most one output, i.e. \( \forall p \in P, |p^*| \leq 1 \); it is a *weighted marked graph with relaxed place constraints* (WMG<sub>$<$</sub>) if it is choice-free and, in addition, each place has at most one input, i.e. \( \forall p \in P, |p| \leq 1 \) and $|p^*| \leq 1$. W MG<sub>$<$</sub> contain the weighted T-systems (WTS) of [15], also known as *weighted event graphs* (WEG) in [14] and *weighted marked graphs* (WMG), in which \( \forall p \in P, |p^*| = 1 \) and $|p^*| = 1$. The nets of Figure 2 are W MG<sub>$<$</sub>. We denote by MG<sub>$<$</sub> the unit-weighted
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Figure 3. The net on the left is ordinary (i.e. unit-weighted, plain), thus in particular it is homogeneous. The net on the right is homogeneous and is not ordinary. Both nets have a single shared place, which is $p_2$.

WMG$_\leq$. Well-studied ordinary subclasses are marked graphs [12], also known as T-nets [1], which fulfill $|p^*| = 1$ and $|p| = 1$ for each place $p$.

– Subclasses with shared places. $N$ is asymmetric-choice (AC) if it satisfies the following condition for any two input places $p_1, p_2$ of each synchronization $t$, $p_1^* \subseteq p_2^*$ or $p_2^* \subseteq p_1^*$. It is free-choice (FC) if for any two input places $p_1, p_2$ of each synchronization $t$, $p_1^* = p_2^*$. Thus, FC nets form a subclass of AC nets. It is a state machine if it is ordinary and each transition has exactly one input and one output.

Figure 4. The net on the left is HFC, the second one is HAC. The third net is homogeneous, non-AC since $t_1^* = \{p_1, p_2\}$, while $p_1^* \not\subseteq p_2^*$ and $p_2^* \not\subseteq p_1^*$. The fourth net is a state machine. None of these nets is CF.

3.2. S-WMG$_\leq$

We introduce the new classes of $k$S-WMG$_\leq$ and S-WMG$_\leq$.

Definition 3.1. ($k$S-WMG$_\leq$ and S-WMG$_\leq$)

A net (or system) is a $k$S-WMG$_\leq$ if it has at most $k$ shared places and if the deletion of all its shared places yields a WMG$_\leq$. A net (or system) is a S-WMG$_\leq$ if it is a $k$S-WMG$_\leq$ for some positive integer $k$.

Figure 5 pictures an homogeneous 1S-WMG$_\leq$ (H1S-WMG$_\leq$) on the left.

Notice that each Petri net system with $k$ places can be transformed into an S-WMG$_\leq$ by inserting, for each place $p$, a new transition $t_p$ with only input $p$ and only output $p$. This transformation preserves numerous behavioral properties of the original system (e.g. liveness, boundedness, reversibility, the reachable markings), hence there is no hope of reducing the checking complexity of such properties in S-WMG$_\leq$. However, as we will highlight, several intractable problems can be alleviated when $k = 1$ (in the H1S-WMG$_\leq$ class). We will show that the same methods do not work anymore when $k = 2$. 
3.3. Augmented Marked Graphs (AMG)

Augmented Marked Graphs (AMG) extend marked graphs with shared places having the same number of inputs and outputs, in addition to other constraints such as the existence of elementary paths connecting outputs to inputs in the underlying marked graph and restrictions on the initial marking. They form a proper subclass of S-WMG≤. We recall their most general definition next, as introduced in [11]:

Definition 3.2. (AMG [11])

An augmented marked graph is an ordinary Petri net system composed of two distinct sets \( P \) and \( R \) of places (\( R \) for resources, denoting the set of shared places; \( P \) denoting the other places) and a set \( T \) of transitions satisfying the following conditions:

(H1) The net \( G \) obtained by removing the places of \( R \) is a marked graph;

(H2) For each place \( r \in R \), there exist an integer \( k \geq 2 \) and \( k \) pairs of transitions described by the set \( D_r = \{ (a_{r_1}, b_{r_1}), \ldots, (a_{r_k}, b_{r_k}) \} \) such that \( \bullet_r = \{ a_{r_1}, \ldots, a_{r_k} \}, \circ_r = \{ b_{r_1}, \ldots, b_{r_k} \}, a_{r_i} \neq a_{r_j}, b_{r_i} \neq b_{r_j}, \forall i \neq j \in \{1, \ldots, k\} \), and for each pair \( (a_{r_i}, b_{r_i}) \in D_r \) such that \( a_{r_i} \neq b_{r_i} \), there exists an elementary path in \( G \) from \( a_{r_i} \) to \( b_{r_i} \);

(H3) Each elementary circuit in \( G \) is marked by \( M_0 \);

(H4) Each place in \( R \) is marked by \( M_0 \), and for each pair \( (a_{r_i}, b_{r_i}) \in D_r \) such that \( a_{r_i} \neq b_{r_i} \), there is an elementary path \( O_{r_i} \) in \( G \) from \( a_{r_i} \) to \( b_{r_i} \) that is unmarked by \( M_0 \).

This definition allows the existence of marked elementary paths in \( G \) from \( a_{r_i} \) to \( b_{r_i} \) for any \( r_i \), as well as the existence of other sets of pairs \( D^{r_i} \) containing a pair \( (a'_{r_i}, b'_{r_i}) \) such that no unmarked elementary path exists in \( G \) from \( a'_{r_i} \) to \( b'_{r_i} \). Examples are provided in Figure 6 with only one resource place (i.e. a shared place), but an arbitrary number of resource places is allowed in general (contrarily to H1-S-WMG≤). More restricted definitions of AMG exist, adding notably the constraint of 1-boundedness (also called 1-safeness, or safeness) [31].

Figure 5. Deleting place \( p \) in the H1S-WMG≤ on the left yields the WMG≤ on the right.
Figure 6. The system on the left is an AMG, with set of resource places \( R = \{ p_5 \} \). Let us choose \( D^r = \{(t_1, t_3), (t_2, t_4)\} \), \( O_r \), the unmarked elementary path \( t_1p_3t_3 \) and \( O_r \), the unmarked elementary path \( t_2p_6t_4 \).

Notice that choosing \( D'^r = \{(t_1, t_4), (t_2, t_3)\} \) does not permit to find an unmarked elementary path from \( t_2 \) to \( t_3 \) in \( G \) since each path must visit \( p_2 \), which is marked. Its underlying marked graph \( G \) is pictured in the middle. The system on the right is not an AMG. None of these systems is 1-bounded.

### 3.4. Place-Composed Marked Graphs with relaxed place constraints (PCMG\( _\leq \))

We introduce the new class of PCMG\( _\leq \) and a dedicated notion of well-structuredness. For that purpose, we need to define a place-merging operation on Petri nets.

**Definition 3.3. (Place-merging)**

Consider a net \( N = (P, T, W) \) and a subset \( A \) of \( 2^P \) whose elements are mutually disjoint. The net \( N' = (P', T', W') \) obtained by place-merging \( A \) is defined as follows:
- \( T' = T \);
- for each element \( x = \{ p_1, \ldots, p_k \} \) in \( A \), a place \( p_x \) belongs to \( P' \) such that, for each transition \( t \), \( W'(p_x, t) = \sum_{p \in x} W(p, t) \) and \( W'(t, p_x) = \sum_{p \in x} W(t, p) \);
- denoting by \( P'' \) the set of places in \( P \) that do not appear in \( A \), \( P' = \bigcup_{x \in A} p_x \cup P'' \), and for each place \( p \) in \( P'' \) and each transition \( t \), \( W''(p, t) = W(p, t) \) and \( W''(t, p) = W(t, p) \).

**Definition 3.4. (PCMG\( _\leq \))**

Consider any connected, undirected graph \( G = (V, E) \), where \( V = \{ v_1, \ldots, v_x \} \) is a finite set of vertices and \( E = \{ e_1, \ldots, e_y \} \) is a finite set of edges connecting distinct vertices of \( V \). A place-composed marked graph with relaxed place constraints (PCMG\( _\leq \)) \( N = (P, T, W) \) is obtained from \( G \) by refining vertices with places and edges with MG\( _\leq \) as follows:

- first, define a MG\( _\leq \) \( N' = (P', T', W') \) containing \( y \) maximal connected components \( C_1, \ldots, C_y \), each of which contains at least two places; denote by \( C \) the set of these components;

- then, define a bijective mapping \( \beta : E \mapsto C \) that associates to each edge a component, and a mapping \( \gamma : E \mapsto P' \times P' \) that associates to each edge \( e_i = \{ v_a, v_b \} \), \( a < b \), a pair of distinct places \( (p_{i,a}, p_{i,b}) \) in the component \( \beta(e_i) \);
• finally, for each vertex \( v_j \) of \( V \), denote by \( A_j \) the set of all the places associated to \( v_j \) through \( \gamma \); denote by \( A \) the set \( \{A_j \mid v_j \in V\} \);

• \( N \) is obtained by place-merging \( A \).

Examples are pictured in Figure 7.

![Figure 7](image)

Figure 7. In the middle, a simple undirected graph \( G = (V, E) \) is pictured, where \( V = \{v_0, v_1\} \) and \( E = \{e_1\} \). The MG\(_{\leq}\) on the left is obtained from \( G \) by identifying place \( p_0 \) to vertex \( v_0 \) and place \( p_1 \) to vertex \( v_1 \), i.e. \( \beta(e_1) = C_1, \gamma(e_1) = (p_0, p_1) \). On the right, \( e_1 \) is replaced by a component \( C_1 \) formed of a single transition \( t_0 \) synchronizing its input places \( p_0, p_1 \) and \( p_2 \) and writing in \( p_2 \).

In the design phase, PCMG\(_{\leq}\) allow to define first the topology of shared places, i.e. the communication links between the buffers of the system. The processes reading and writing the buffers can then be defined through refinement.

In the sequel, we focus mainly on well-formed MG subnets, i.e. MG subnets that are structurally live (meaning that a live marking exists) and structurally bounded (i.e. for each initial marking, the system is bounded). We define next a notion of well-structuredness for PCMG\(_{\leq}\).

**Definition 3.5. (Well-structured PCMG\(_{\leq}\))**

A PCMG\(_{\leq}\) \( S = (N, M_0) \), where \( N = (P, T, W) \), obtained from a graph \( G = (V, E) \), is well-structured if each component of \( C \) is a strongly connected and well-formed MG.

The Petri net obtained on the left of Figure 7 is a well-structured PCMG\(_{\leq}\), while the net on the right is not, since the unique MG component is not structurally live. Another well-structured PCMG\(_{\leq}\), with shared places, is given in the middle of Figure 8.

In the definition of PCMG\(_{\leq}\), the undirected graph representing the topology stems from the fundamental behavioral properties fulfilled by this class. Indeed, we provide in Section 10 a characterization of reversibility, together with a sufficient condition of PR-R equality, and show they are no more valid when the underlying undirected graph topology of PCMG\(_{\leq}\) is relaxed. Thus, PCMG\(_{\leq}\) are defined so as to benefit from stronger conditions ensuring liveness, boundedness, reversibility and to fulfill the PR-R equality, while extending the expressiveness of marked graphs and state machines.

### 3.5. Expressiveness comparison

The AMG class contains all the MG but not all WMG (nor the WMG\(_{\leq}\)), since the former are unit-weighted and the latter have arbitrary weights. Since AMG allow shared places, WMG\(_{\leq}\) do not contain
them all either. AMG and PCMG\_\leq are incomparable (see Figure 8). H1S-WMG\_\leq are not included in PCMG\_\leq nor in AMG.

Figure 8. The system in the middle is a PCMG\_\leq but not an AMG: in the underlying marked graph \( G \), there is no path connecting the transitions. The system on the right is an AMG but not a PCMG\_\leq: \( t_0 \) connects 3 shared places, thus the system cannot be obtained from an undirected graph as in the definition of PCMG\_\leq.

Deterministically synchronized sequential processes (DSSP), introduced in \[32\], aim at modelling several agents that cooperate through asynchronous message passing, in a modular way, each module representing an agent. We do not study DSSP in this work, whose formal definition, together with examples, can be found in \[32\]. This unit-weighted class does not contain all state machines even with a single shared place, nor all the AMG, the H1S-WMG\_\leq and the PCMG\_\leq. However, it is worth mentioning that each HFC system can be transformed into a DSSP system with the same set of feasible sequences, as shown in \[32\].

Examples are pictured in Figure 9 and the inclusion relation between the main subclasses studied in this paper is depicted in Figure 10.

Figure 9. The net on the left is an H1S-WMG\_\leq, the one in the middle is an HFC and simulated by the one on the right, which is a DSSP.

In the following sections, we develop new conditions ensuring the PR-R equality in weighted Petri nets and some of their subclasses mentioned above, namely H1S-WMG\_\leq, AMG and PCMG\_\leq; we also consider the HFC class to a smaller extent. To achieve it, we introduce the notion of \textit{directedness} of the potential reachability graph, together with variants, in the next section.
Figure 10. Inclusion of several classes mentioned in this paper. We focus on the boxed classes, for which we develop new results; the HFC class is also studied to a smaller extent (dashed box). Straight arrows represent the inclusion relation: MG are included in AMG, PCMG≤, WMG≤ and DSSP, WMG≤ form a subclass of H1S-WMG≤ and HFC nets, while S-WMG≤, HFC and DSSP nets are incomparable, and AMG, PCMG≤ and H1S-WMG≤ are also incomparable. However, the wavy arrow represents the possibility of transforming each HFC into a DSSP with the same set of transitions and feasible sequences. We do not depict the transitive closure of the inclusion relation for the sake of readability.

Previous works exist that study properties related to the state equation in other classes, which do not contain our classes or do not tackle the PR-R equality problem, as summarized in the related work at the end of this paper.

4. Directedness

In this section, we first introduce the notion of directedness of the potential reachability graph, with variants, extracted from [2, 32, 10]. Then, we present an overview of the classes from the literature that benefit from directedness, including the persistent class for which a stronger form of directedness exists, embodied by Keller’s theorem.

4.1. Directedness and variants

**Definition 4.1. (Directedness of the potential reachability graph)**

Let us consider any system $S = (N, M_0)$ and its potential reachability graph $PRG(S)$:

- $PRG(S)$ is **directed** if every two potentially reachable markings have a common reachable marking.

More formally: $\forall M_1, M_2 \in PR(S): R((N, M_1)) \cap R((N, M_2)) \neq \emptyset$.

- $PRG(S)$ is **initially directed** if $\forall M_1 \in PR(S): R(S) \cap R((N, M_1)) \neq \emptyset$.

The directedness of $PRG(S)$ is called **structural directedness** in [33].

We shall also consider the particular case of directedness restricted to the reachability graph, i.e. when every two reachable markings have a common reachable marking.

Figure 11 illustrates these properties.
4.2. Initial directedness and strong liveness

A system \((N, M_0)\) is strongly live if, for each potentially reachable marking \(M\), \((N, M)\) is live. We recall next lemma.

**Lemma 4.2. (Strong liveness)**

Consider a live system \(S\). If \(PRG(S)\) is initially directed, then \(S\) is strongly live.

We know that the converse of Lemma 4.2 does not hold, even in the class of ordinary asymmetric-choice Petri nets [10].

4.3. Known classes with directed reachability graph

The potential reachability graph of live HFC systems is known to be directed (Theorem 12 in [2]); the reachability graph of live DSSP is directed as well (Theorem 4 in [32]).

Persistent systems have a strong restriction on their behavior: no transition firing can disable any other transition. They do not include all HFC nor all DSSP systems, and their reachability graph is directed in a stronger form, as expressed by Keller's theorem below. We first need to recall the notion of residues, on which this theorem is based.
Definition 4.3. ((Left) Residue)
Let $T$ be a set of labels (typically, transitions) and $\tau, \sigma \in T^*$ two sequences over this set. The (left) residue of $\tau$ with respect to $\sigma$, denoted by $\tau^\bullet \sigma$, arises from cancelling successively in $\tau$ the leftmost occurrences of all symbols from $\sigma$, read from left to right. Inductively:

$$\tau^\bullet \varepsilon = \tau; \quad \tau^\bullet t = \tau$$

if $t \not\in \text{supp}(\tau)$; $\tau^\bullet t$ is the sequence obtained by erasing the leftmost $t$ in $\tau$ if $t \in \text{supp}(\tau)$; and

$$\tau^\bullet (t\sigma) = (\tau^\bullet t)^\bullet \sigma.$$

For example, $acbcacbc^\bullet abbcb = cacc$ and $abbcb^\bullet acbcacbc = b$.

Residues naturally extend to T-vectors as follows: for any sequence $\sigma$ and T-vector $Y$, $\sigma^\bullet Y$ is $\sigma$ in which, for each transition $t$ in $\text{supp}(Y)$, the $\min\{P(\sigma)(t), Y(t)\}$ leftmost occurrences of $t$ have been removed.

Theorem 4.4. (Keller [34])
Let $S$ be a persistent system. Let $\tau$ and $\sigma$ be two sequences feasible in $S$. Then $\tau(\sigma^\bullet \tau)$ and $\sigma(\tau^\bullet \sigma)$ are both feasible in $S$ and lead to the same marking.

Keller’s theorem applies to WMG≤ and the larger class of CF nets, since they are structurally persistent (each place having at most one output).

In the next section, we exploit directedness to develop our first general condition ensuring the PR-R equality in weighted Petri nets.

5. Reverse nets, properties and the PR-R equality

In order to study the relationship between reachability and potential reachability, we introduce the notion of reverse nets and sequences. We also introduce related notation and behavioral properties, and develop new relations between these properties.

Then, we relate reversibility of a system and of its reverse to initial directedness, yielding a new general sufficient condition of PR-R equality for weighted Petri nets. We recall methods checking its reversibility assumption in weighted subclasses of Petri nets.

We deduce a sufficient condition of PR-R equality for the live HFC subclass and a polynomial-time variant of it. Finally, we recall a liveness characterization for CF nets, which will prove useful in the study of WMG≤.

5.1. Reverse nets and properties

Definition 5.1. (Reverse nets, systems and sequences)
The reverse of a net $N$, denoted by $-N$, is obtained from $N$ by reversing all the arcs while keeping the weights. The reverse of a system $S = (N, M_0)$, denoted by $-S$, is the system $(-N, M_0)$. We denote by $\sigma^\bullet$ the sequence $\sigma$ followed in reverse order, called its reverse. For example, if $\sigma = t_1t_2t_2t_3$, then $\sigma^\bullet = t_3t_2t_2t_1$.

The notation $-N$ stems from the fact that the incidence matrix of the reverse of $N$ is the opposite $-I$ of the incidence matrix $I$ of $N$, so that $-I + I$ is null.
Definition 5.2. (Properties $L$, $R$ and $B$)

A system $S$ fulfills property $L$ if $S$ and $-S$ are live; it fulfills property $R$ if $S$ and $-S$ are reversible; it fulfills property $B$ if $S$ and $-S$ are bounded.

We assume that each Petri net has at least one transition. Next lemmas relate properties of a system to the same properties in its reverse, and will prove useful in the study of subclasses.

Lemma 5.3. (Properties $R$ and $B$)

Let us suppose that a system $S$ fulfills property $R$. Then:

$S$ is bounded iff $-S$ is bounded.

Proof:

If $-S$ is reversible and unbounded, consider an unbounded place $p$ in $-A$: for each $k$, there exists a sequence $\sigma_k$ in $-A$ that visits a marking $M_k$ such that $M_k(p) \geq k$ and comes back to $M_0$, thus $\sigma_k$ is feasible in $A$, visiting the same marking $M_k$. Thus, $-A$ is bounded.

We obtain next result.

Lemma 5.4. ($LR$ PR markings and property $R$)

Consider a system $S$. Suppose that every potentially reachable marking of $S$ is live and reversible ($LR$). Then $-S$ is live and reversible.

Proof:

Denote by $I$ the incidence matrix of $S$. Suppose that $-S = (-N, M_0)$ is not reversible; consider a marking $M$ reachable in $-S$ with some sequence $\sigma^d$ such that $M_0$ is not reachable in $(-N, M)$. Since $S$ is live and reversible, a sequence $\alpha$ is feasible in $S$ that contains all transitions and leads back to $M_0$, hence there exists some positive integer $k$ such that $M = M_0 + I \cdot Y$ with $Y = k \cdot P(\alpha) - P(\sigma) \geq 0$. Thus, $M$ is potentially reachable in $S$, so that $(N, M)$ is live and reversible. Consequently, since $M_0$ is reached from $(N, M)$ by firing $\sigma$, a sequence $\tau$ is feasible in $(N, M_0)$ that leads to $M$. We deduce that the sequence $\tau^d$ is feasible in $(-N, M)$ and leads to $(-N, M_0)$, contradiction. Thus $-S$ is reversible. Moreover, $\alpha^d$ is feasible in $-S$, which is consequently live. Hence the claim.

5.2. Ensuring the PR-R equality from reversibility and initial directedness

We obtain the next sufficient condition of reachability for the markings in $PR(S)$. Its proof is illustrated in Figure 13. This new result will be exploited in the sequel to ensure the PR-R equality in Petri net subclasses.

Theorem 5.5. (Combining initial directedness with property $R$)

Consider a Petri net system $S = (N, M_0)$ satisfying property $R$ and such that $PRG(S)$ is initially directed. Then $R(S) = PR(S)$.

Proof:

The proof is illustrated in Figure 13. Consider any marking $M$ potentially reachable from $M_0$. By initial directedness, there exists $M' \in R((N, M_0)) \cap R((N, M))$, with feasible sequences $M_0 \xrightarrow{\sigma_0} M'$ and $M \xrightarrow{\sigma_1} M'$. The marking $M_0$ is reachable from $M'$ with some sequence $\sigma_2$ since the system is
reversible. Now, let us consider the reverse of these sequences. In particular, the sequence $\sigma_2 \sigma_1$ leads to $M$ in the reverse system $-S$; since this system is also reversible, a sequence $\sigma_3$ exists that leads to the initial marking. In $S$, $\sigma_3$ leads to $M$, which is thus reachable. 

\[ M' \in R(M_0) \cap R(M) \]

Figure 13. Illustration of the proof of Theorem 5.5. Part of the reachability graph of $S$ is depicted on the left. On the right, sequences in $-S$ are considered.

5.3. Checking reversibility

The reversibility checking problem is PSPACE-hard \[35\]. However, under the liveness assumption, characterizations of reversibility exist for HFC nets that often avoid to explore the reachability graph exhaustively. Polynomial-time sufficient conditions of liveness and reversibility also exist for well-formed HFC nets and join-free (JF) nets (i.e. without synchronizations) \[14, 4, 6\].

We recall the notion of a T-sequence and its importance for reversibility.

**Definition 5.6. (T-sequence \[36, 4\])**

Consider a system $S$ whose set of transitions is $T$ and denote by $I$ its incidence matrix. A firing sequence $\sigma$ of $S$ is a T-sequence if it contains all transitions of $T$ (i.e. $\text{supp}(\sigma) = T$) and $I \cdot P(\sigma) = 0$ (i.e. $P(\sigma)$ is a consistency vector).

In all weighted Petri nets, the existence of a feasible T-sequence is a known necessary condition of liveness and reversibility, taken together \[36\]. It has also been proven sufficient for reversibility in live HFC systems, also called Equal-Conflict systems \[4\], in a proper subclass of the live join-free systems (with an additional constraint on the reachable markings) \[6\] and in live H1S systems \[10\].

5.4. Directedness and PR-R equality in HFC systems

We now consider the special case of the HFC subclass.

**Proposition 5.7. (Directedness of live HFC systems (Theorem 12 in \[2\]))**

Consider any HFC system $S$. If $S$ is live, then $PRG(S)$ is directed.

Applying Theorem 5.5, we deduce next result.
Corollary 5.8. (PR-R equality in live HFC systems)
Consider a system $S$ satisfying property $\mathcal{R}$. If $S$ is live and HFC, then $R(S) = PR(S)$.

Consider any well-formed HFC system $S$ whose reverse is also a well-formed HFC. Theorem 28 in [2] provides a polynomial-time characterization of well-formedness for HFC nets. A wide-ranging linear-time sufficient condition of liveness and reversibility in well-formed HFC systems is given by Theorem 6.6 in [4]. Thus, in this subclass, we deduce a polynomial-time sufficient condition of PR-R equality.

5.5. Liveness of CF systems

We recall the next characterization of liveness for weighted choice-free systems (which form a subclass of the HFC systems) given as Corollary 4 in [37].

Proposition 5.9. (Liveness of choice-free systems [37])
Let $(N, M_0)$ be a choice-free system with incidence matrix $I$. It is live iff there exist a marking $M \in R((N, M_0))$ and a firing sequence $\sigma \in L(N, M)$ such that $P(\sigma) \geq 1$ and $I \cdot P(\sigma) \geq 0$.

This result will prove useful in the study of WMG$_{\leq}$ in the next section.

6. Reachability properties of WMG$_{\leq}$

Live Weighted T-Systems (WTS), in which each place has exactly one input and one output, fulfill the PR-R equality [15]. In this section, we extend this result to the live WMG$_{\leq}$, which allow places without inputs and places without output. With the aim of checking the liveness of a WMG$_{\leq}$ (as a precondition), we also provide new characterizations of liveness for WMG$_{\leq}$ and their circuit subclass, as well as properties on their deadlocks. Finally, we recall several other ways of checking liveness, as well as reversibility and boundedness in WMG$_{\leq}$.

6.1. Liveness, deadlockability and PR-R equality in WMG$_{\leq}$

To obtain the PR-R equality result, we need the following proposition, which recalls Corollary 1 of [9].

Proposition 6.1. (Fireable T-vectors in WMG$_{\leq}$ [9])
Let $N = (P, T, W)$ be a WMG$_{\leq}$ with incidence matrix $I$. Let $M_0$ be any marking and $Y \in \mathbb{N}^T$ be a T-vector such that $M = M_0 + C \cdot Y \geq 0$. Let $\sigma$ be a transition sequence such that $Y \leq P(\sigma)$. Then, if $M_0[\sigma]$, there is a firing sequence $M_0[\sigma']M$ such that $P(\sigma') = Y$.

We deduce that liveness is sufficient for ensuring the PR-R equality in this class.

Corollary 6.2. Every live WMG$_{\leq}$ fulfills the PR-R equality.
Proof:
For each solution \((M, Y)\) of the state equation, since the system is live, there exists a feasible sequence \(\sigma\) whose Parikh vector is greater than or equal to \(Y\), hence Proposition 6.1 applies and \(M\) is reachable.

Notice that the live \(\text{WMG}_{\leq}\) form a subclass of live HFC systems, hence their potential reachability graph is directed (as recalled for the HFC class in Section 4). Moreover, every live and bounded \(\text{WMG}_{\leq}\) is reversible and fulfills liveness, boundedness and reversibility in its reverse \([15]\); when boundedness is dropped, \(\text{WMG}_{\leq}\) may be live without being reversible, and their reverse are not always live, so that Theorem 5.3 cannot be applied to derive the PR-R equality. When the liveness assumption is dropped, examples not fulfilling the PR-R equality are easily built, as the one in Figure 14.

![Figure 14. A deadlocked MG\(_{\leq}\)](image)

Figure 14. A deadlocked MG\(_{\leq}\) (i.e. a unit-weighted WMG\(_{\leq}\)) with marking \((0, 0)\). For each integer \(k > 0\), the marking \((0, k)\) is potentially reachable but not reachable.

So as to check liveness, we introduce several characterizations. The next characterization of liveness, expressed in terms of the liveness of circuit subsystems, is extracted from \([15]\).

Proposition 6.3. (Liveness of WTS \([15]\))
A WTS \(S = (N, M_0)\) is live iff every elementary circuit P-subsystem \(C\) of \(S\) is live.

This proposition readily extends to \(\text{WMG}_{\leq}\) without source places (i.e. places with no input):

Corollary 6.4. (Extension of Theorem 4.12 in \([15]\))
A \(\text{WMG}_{\leq}\) without source places \(S = (N, M_0)\) is live iff every elementary circuit P-subsystem \(C\) of \(S\) is live.

We now introduce property \(E\), which we use to obtain a variant of Corollary 6.4.

Definition 6.5. (Property \(E\))
A Petri net system \(S = (N, M_0)\) has the property \(E\) if, for each solution \((M, Y)\) of its state equation, \(M\) enables at least one transition.

Theorem 6.6. A \(\text{WMG}_{\leq}\) without source places is live iff property \(E\) is true in each elementary circuit P-subsystem.

Proof:
We show the first direction (\(\Rightarrow\)). Consider any elementary circuit P-subsystem \(C = (N_{P^\prime}, M_0_{P^\prime})\) of the live \(\text{WMG}_{\leq}\), where \(P^\prime\) is the set of places of \(C\): applying Corollary 6.4 \(C\) is live. Denote by \(I_C\) the incidence matrix of the circuit \(C\). By Proposition 6.1 each solution \((M_c, Y)\) of the state equation
Since \(\text{WMG}\) Proof: 
that deadlock, denoted by \(M\) by there might be several ones with minimal length, but we show below that there is only one. Denote by \(\text{Theorem 6.7.}\) matrix \(I\) and solution enables at least one transition of \(C\). Thus, each circuit P-subsystem satisfies \(E\).

Let us now consider the other direction \((\Leftarrow)\). If \(E\) is true in each elementary circuit P-subsystem \(C\), then in particular each marking reachable in \(C\) enables some transition, \(C\) is thus deadlock-free, hence live. By Corollary 6.4, we deduce the \(\text{WMG_{\leq}}\) to be live.

Let us now characterize the set of reachable markings of the non-live, connected \(\text{WMG_{\leq}}\) without source places.

**Theorem 6.7.** Consider a connected \(\text{WMG_{\leq}}\) without source places \(S = (N, M_0)\) with incidence matrix \(I\). If it is non-live, there exists a unique \(T\)-vector \(Y_d\) of smallest cardinality such that \(M_d = M_0 + I \cdot Y_d\) is a deadlock. Moreover, a sequence \(\sigma_d\) exists that is feasible in \(S\) and leads to \(M_d\) such that \(P(\sigma_d) = Y_d\).

**Proof:**
Since \(\text{WMG_{\leq}}\) are persistent, Keller’s theorem applies (Theorem 4.4) and there is only one reachable deadlock, denoted by \(M_d\). Denote by \(\sigma_d\) one of the sequences leading to \(M_d\) with minimal length: there might be several ones with minimal length, but we show below that there is only one. Denote by \(Y_d\) the Parikh vector of \(\sigma_d\).

Suppose there is some \(T\)-vector \(Y_d'\) defining a potentially reachable deadlock \(M_d'\) that is not reachable, such that \(Y_d' \not\geq Y_d\), i.e. either \(Y_d' \leq Y_d\), or both are incomparable.

Let us prove by induction on the length \(n\) of \(\sigma_d\) that \(Y_d' = Y_d\). If \(n = 0\), it is clear since we assumed \(Y_d' \not\geq Y_d\). If \(n > 0\), let us write \(\sigma_d = t_0\tau\), where \(t_0\) leads to a marking \(M\). We have two cases: \(Y_d'(t_0) = 0\) and \(Y_d'(t_0) > 0\). In the first case, by structural persistence, \(t_0\) is still enabled at \(M_d'\), a contradiction. In the second case, given that \(M_d' = M + I \cdot (Y_d' - \mathbb{1}_{t_0})\) (where \(\mathbb{1}_{t_0}\) denotes the \(T\)-vector whose only non-null component equals 1 and has index \(t_0\)) applying the induction hypothesis to \(M\), \((Y_d' - \mathbb{1}_{t_0})\) and \(\tau\) (whose length is \(n - 1\) and Parikh vector is \(Y_d - \mathbb{1}_{t_0}\)) yields \((Y_d' - \mathbb{1}_{t_0}) = (Y_d' - \mathbb{1}_{t_0})\), thus \(Y_d = Y_d'\).

Hence, there is a unique minimal \(Y_d\), and for each sequence \(\sigma_d\) feasible in \(S\) that leads to \(M_d\), we have \(P(\sigma_d) = Y_d\).

We denote by \(\text{DEAD}\) the predicate on nets and markings such that \(\text{DEAD}(N, M) = \text{true}\) iff the marking \(M\) is a deadlock for the net \(N = (P, T, W)\).

We obtain next theorem for checking liveness in a weighted circuit, relaxing the non-negativity constraint on the components of the potentially reachable markings.

**Theorem 6.8.** (Checking liveness of weighted circuits)
A circuit system \(S = (N, M_0)\) with incidence matrix \(I\) is live iff the following system has no solution \((M_d, Y) \in \mathbb{Z}^{|P|} \times \mathbb{N}^{|T|}\):

\[
\begin{cases}
M_d = M_0 + I \cdot Y \\
\text{DEAD}(M_d)
\end{cases}
\]
Proof:
(⇐) If the system has no such solution, then in particular it has no solution \((M_d, Y) \in \mathbb{N}^{|P|} \times \mathbb{N}^{|T|} \), hence no feasible sequence leads to a deadlock, thus \(S\) is live.

(⇒) Suppose that \(S\) is live and that some solution \((M_d, Y) \in \mathbb{Z}^{|P|} \times \mathbb{Z}^{|T|}\) to the system exists. Denote by \(\sigma'\) the sequence of maximal length that is feasible in \(S\) and such that \(P(\sigma') = Y' \leq Y\), leading to \(M' = M_0 + I \cdot Y'\). Let us define \(Y'' = Y - Y'\), which has a non-empty support since \(Y'' \leq Y\). By definition of \(Y''\), \(M'\) does not enable any transition in the support of \(Y''\). Since \(S\) is live, \(M'\) enables some transition \(t\) not in \(\text{supp}(Y'')\) with unique input place \(p\). Since \(M_d\) is a deadlock, \(M_d(p) < W(p, t)\). By definition of \(t\) and \(Y''\), \(M'(p) = M_d(p) < W(p, t)\), thus \(t\) is not enabled at \(M'\), a contradiction.

We deduce the claim. \(\square\)

We deduce next corollary when conservativeness is assumed, allowing to relax the non-negativity constraint not only on the components of potentially reachable markings, but also on the components of the \(T\)-vectors.

**Corollary 6.9. (Liveness of conservative weighted circuits)**
A conservative circuit system \(S = (N, M_0)\) with incidence matrix \(I\) is live iff the following system has no solution \((M_d, Y) \in \mathbb{Z}^{|P|} \times \mathbb{Z}^{|T|}\):

\[
\begin{align*}
M_d &= M_0 + I \cdot Y \\
\text{DEAD}(M_d)
\end{align*}
\]

**Proof:**
(⇐) This direction is obtained as in the proof of Theorem 6.8.

(⇒) Suppose that \(S\) is live and that some solution \((M_d, Y) \in \mathbb{Z}^{|P|} \times \mathbb{Z}^{|T|}\) to the system exists. If \((M_d, Y) \in \mathbb{N}^{|P|} \times \mathbb{N}^{|T|}\), applying Proposition 6.1, \(M_d\) is reachable, contradicting liveness. Thus, let us suppose that negative components appear in \(M_d\) or \(Y\).

Since \(S\) is live, conservative (hence bounded) and strongly connected, it has a minimal \(T\)-semiflow \(Y\) with support \(T\) (by consistency and Theorem 8 in [37]). If \(Y\) has some negative component, then there exists a positive integer \(k\) such that \(Y_k = Y + k \cdot Y\) has only positive components and such that \(M_d = M_0 + I \cdot Y_k\). Hence we suppose without loss of generality that \((M_d, Y) \in \mathbb{Z}^{|P|} \times \mathbb{N}^{|T|}\), where \(M_d\) has at least one negative component and \(Y\) is a \(T\)-vector. The rest of the proof is the same as in the proof of Theorem 6.8 \(\square\)

6.2. Checking Properties of WMG≤

Structural boundedness means boundedness for each marking, while bounded systems are not always structurally bounded, even when they are live [3]. If a system is unbounded, the underlying net is not structurally bounded. We recall the next characterization for this property, which appears in various studies, e.g. in [3].
Proposition 6.10. (Corollary 16 in [3])

A net with incidence matrix $I$ is not structurally bounded iff there exists a $T$-vector $Y$ such that $I \cdot Y \geq 0$.

Consequently, structural boundedness can be checked in polynomial time with linear programming (over the rationals, obtaining an integer-valued solution from a rational-valued one) for any weighted Petri net, contrarily to boundedness, whose checking problem is EXPSPACE-complete.

To check liveness of $WMG_{\leq}$, either Proposition 5.9 can be used directly (since it is a liveness characterization for CF nets, which contain the $WMG_{\leq}$), or we use the results of the previous subsection; since using these conditions in a naive way is generally costly, one can try first the polynomial-time sufficient (and non-necessary) conditions of [14].

To check boundedness of $WMG_{\leq}$ using liveness: if it is live, it is bounded iff it is structurally bounded [15, 37, 2], which can be checked in polynomial time via Proposition 6.10. Otherwise, suppose it is connected, without source places and non-live: then, by Theorem 6.7 no infinite sequence is feasible and all feasible sequences of maximal length lead to the same deadlock, thus the $WMG_{\leq}$ is necessarily bounded. However, $WMG_{\leq}$ exist that are bounded, and not live nor structurally bounded (e.g. Figure 14).

To check reversibility using liveness: suppose it is live. Then if it is bounded, it is reversible [15]. Otherwise, it is unbounded, and it is reversible iff a $T$-sequence (i.e. a sequence containing all transitions and getting back to the same marking) is feasible from the initial marking; thus consistency is a necessary condition for reversibility under the liveness assumption. More precisely, in live and connected $WMG_{\leq}$, it is necessary and sufficient to check the existence of a $T$-sequence whose Parikh vector equals the unique minimal $T$-semiflow of the incidence matrix [15, 37]. By Theorem 4.10 in [15], if $S$ is a consistent, non-deadlocked WTS, then reversibility and liveness of $S$ are equivalent; this extends readily to $WMG_{\leq}$, since consistency implies that the $WMG_{\leq}$ is a WTS.

7. Systems with shared places not fulfilling the PR-R equality

In this section, we provide various examples of systems belonging to the subclasses studied in this paper and that do not fulfill the PR-R equality, while several other structural and behavioral properties are ensured. These examples will prove useful to obtain our sufficient conditions of PR-R equality in the subsequent sections and to show the sharpness of their assumptions. Notably, they permit to highlight the importance of siphon properties.

Examples of AMG and $H1S-WMG_{\leq}$. In Figure 15, we provide three examples of AMG satisfying specific conditions, together with two of their P-subsystems:

- The AMG on the left, which is also a $H1S-WMG_{\leq}$, shows that being live, reversible and bounded ($LRB$) with only one shared place, while not having a reversible reverse, is not sufficient for ensuring the PR-R equality.

- The other two AMG in the middle have two shared places (hence are $H2S-WMG_{\leq}$): each one is the reverse of the other one, both are $LRB$, thus in particular fulfill property $R$, but do not satisfy the PR-R equality.
On the right, two live, unbounded P-subsystems of the AMG with two shared places, induced by a minimal siphon, are pictured, and do not fulfill the PR-R equality.

![Diagram of AMG systems](image)

Figure 15. On the left, the AMG system $S = (N, M_0)$ is live, reversible and bounded ($LRB$), with only one shared place. Its reverse is $\overline{LRB}$ (where $\overline{Q}$ denotes the negation of property $Q$), hence does not fulfill property $R$. Indeed, in the reverse $-S$, $t_2$ can be fired two times, leading to the deadlock $M_D = (0, 0, 2, 0, 0, 1, 0)$. This marking is potentially reachable in $S'$ through the Parikh vector $Y = (2, 0, 2, 2, 2)$, although not reachable. On the top middle, the AMG $S'$ satisfies property $R$: it is $LRB$ and its reverse $-S'$, pictured on the bottom middle, is also $LRB$. $S'$ contains the minimal siphon $D' = \{p_3, p_4\}$ which induces the strongly connected, $LRB$ subsystem $S_{D'}$ on the top right. On the bottom right, $-S_{D'}$ is $LRB$. In $S'$, the marking $(1, 1, 0, 0)$ is a potentially reachable deadlock obtained with the Parikh vector $Y = (1, 1, 0, 0)$ that is not reachable.

On the left of Figure 16 we provide an example of an AMG that is H2S-WMG$_\leq$, live and fulfills property $R$, whose minimal siphons have at most one shared place, but does not fulfill the PR-R equality.

**Importance of the structure of P-subsystems induced by minimal siphons.** In the middle of Figure 16 we exhibit a minimal siphon that induces an ill-formed (i.e. non well-formed) P-subsystem of the live AMG system on the left, explaining the non-reachability of some potentially reachable marking that empties the siphon. Indeed, no siphon can become unmarked at any reachable marking in live AMG [11].

Going back to the examples in the middle of Figure 15 the unique minimal siphon with shared places contains 2 shared places and induces a live, strongly connected P-subsystem fulfilling property $R$. The two other minimal siphons induce two strongly connected, (1-)safe, live and reversible state machines with one shared place, their reverse having the same properties. Notice also that the set of siphons of $S'$ is the same as in $-S'$ in this example, although it is not the case in general. Thus, assume each minimal siphon of $S'$ and $-S'$ to induce a strongly connected, live and reversible P-subsystem is not sufficient for ensuring the PR-R equality in AMG, even with only two shared places (resource places), hence in 2S-WMG$_\leq$.

Besides, we show in Figure 17 a non-homogeneous 4S-WMG$_\leq$ in which each minimal siphon
induces a well-formed, live and reversible P-subsystem and that does not satisfy the PR-R equality.
assumptions on siphons. Later on, in Section 9, we take inspiration from these examples to derive a sufficient condition of PR-R equality in AMG with an arbitrary number of shared places, using properties of the minimal siphons. We investigate reachability for the new class of PCMG\(_\leq\) in Section 10.

8. Ensuring the PR-R Equality in H\(_1\)S-WMG\(_\leq\)

In this section, we first recall a result on the potential reachability graph of H\(_1\)S-WMG\(_\leq\), and deduce a sufficient condition of PR-R equality in this class. Then, we recall ways of checking its assumptions of liveness and reversibility in the larger class of H1S systems.

8.1. Potential reachability in H\(_1\)S-WMG\(_\leq\)

We recall next theorem, illustrated in Figure 18, which applies to H\(_1\)S-WMG\(_\leq\) and exploits liveness, but does not need the reversibility nor boundedness assumptions.

Theorem 8.1. (Properties of the potential reachability graph in H\(_1\)S-WMG\(_\leq\))

Consider a live H\(_1\)S-WMG\(_\leq\) \(S = (N, M_0)\). For any Parikh vector \(Y\) and marking \(M\) such that \(M_0 + I \cdot Y = M\), there exists a firing sequence \(M_0 \xrightarrow{\sigma} M'\) such that \(M'\) is also reached by firing \(\sigma \cdot Y\) from \(M\), where \(P(\sigma) \geq Y\). Consequently, \(PRG(S)\) is initially directed and \((N, M)\) is live.

![Figure 18. Illustration of the claim of Theorem 8.1](image)

We deduce next result.

Corollary 8.2. (Sufficient condition for PR-R equality in H\(_1\)S-WMG\(_\leq\))

Let \(S = (N, M_0)\) be a live, H\(_1\)S-WMG\(_\leq\) satisfying property \(\mathcal{R}\). Then \(R(S) = PR(S)\).

Proof:

Since \(S\) fulfills the conditions of Theorem 8.1, \(PRG(S)\) is initially directed. Moreover, \(S\) satisfies property \(\mathcal{R}\). Hence, Theorem 5.5 applies.

□

This corollary is not true in the class of live, H\(_2\)S-WMG\(_\leq\) satisfying property \(\mathcal{R}\), as exemplified in the middle of Figure 15.
8.2. Checking liveness and reversibility in H1S systems

Let us recall characterizations of liveness and reversibility for the H1S systems, which contain the H1S-WMG \( \leq \).

A siphon \( D \) of a system \( S = (N, M_0) \) is said to be *deadlocked* if, for each place \( p \) in \( D \), for each \( t \in p^* \), \( M_0(p) < W(p, t) \).

**Theorem 8.3. (Liveness of H1S systems [10])**

Consider an H1S system. It is live iff no minimal siphon is deadlocked at any reachable marking.

**Theorem 8.4. (Reversibility of live H1S systems [10])**

Consider a live H1S system \( S = (N, M_0) \). Then \( S \) is reversible iff \( S \) enables a T-sequence.

This theorem is no more true in the case of two shared places, as shown in Figure 19.

![Diagram](image)

**Figure 19.** On the left, a unit-weighted, live, structurally bounded system with only two shared places, namely \( p_1 \) and \( p_2 \). The system enables the T-sequence \( t_0 \ t_3 \ t_2 \ t_1 \) but is not reversible. On the right, its non strongly connected reachability graph is pictured, with initial state \( s_0 \).

Once such a system, with one shared place, is known to be live, checking reversibility thus amounts to checking the existence of a feasible T-sequence. It applies in particular to H1S-WMG \( \leq \) systems.

However, these characterizations of liveness and reversibility do not provide checking algorithms. We leave their design as future work.

In the following sections, we study other classes of systems with shared places, namely AMG and PCMG \( \leq \). We investigate conditions inducing the PR-R equality in these systems.

9. Ensuring the PR-R Equality in Augmented Marked Graphs

In this section, we first recall reachability properties of AMG developed in [11]. Then, we develop results leading to a sufficient condition of PR-R equality. We also discuss methods for checking the latter condition.
9.1. Previous results on AMG

Let us recall some of the known results about AMG.

**Proposition 9.1. (Invariant number of tokens, Property 24 in [11])**
Each resource (i.e. shared) place \( r \in R \) together with the places in paths \( O_{r_i} \) induces a P-semiflow and for all \( M \in R(M_0), M(r) + \sum_{i \in N_r} \sum_{p \in O_{r_i}} M(p) = M_0(r) + \sum_{i \in N_r} \sum_{p \in O_{r_i}} M_0(p). \)

**Proposition 9.2. (Liveness implies reversibility, Property 25 in [11])**
An augmented marked graph is reversible if it is live.

**Proposition 9.3. (Property 26 in [11])**
Let \((N, M_0)\) be a live Petri net system with incidence matrix \( I \) and satisfying the assumptions \( H_1, H_2 \) and \( H_3 \) of the AMG. Let \( M^* \geq 0 \) be a marking satisfying \((C1): \exists Y \in \mathbb{N}^{|T|} \) such that \( M^* = M_0 + I \cdot Y \), and \((C2): \) no place in paths \( O_{r_i} \) is marked by \( M^* \). Then \( M^* \) is reachable from \( M_0 \) and \( M^*(r) > 0, \forall r \in R \).

**Proposition 9.4. (Liveness and siphons, Property 27 in [11])**
An augmented marked graph is live iff it cannot reach any marking at which some siphon is unmarked.

**Proposition 9.5. (Liveness, siphons and home states, Property 29 in [11])**
Let \((N, M_0)\) be a Petri net satisfying assumptions \( H_1, H_2 \) and \( H_3 \). If there exists a marking \( M^* \) satisfying conditions \( C1 \) and \( C2 \) of Proposition 9.3 then \((N, M_0)\) is live iff no siphon is unmarked at any reachable marking. Furthermore, \( M^* \) is a home state.

9.2. New results on directedness, strong liveness and the PR-R equality in AMG

We obtain next lemma, which proves the converse of Lemma 4.2 in the live AMG class, and even a stronger version since we get directedness instead of initial directedness.

**Lemma 9.6. (Directedness in strongly live AMG)**
If an AMG \( S \) is strongly live, then \( PRG(S) \) is directed.

**Proof:**
The proof is illustrated in Figure 20. By Proposition 9.2 \( S = (N, M_0) \) is reversible. Consider any solution \((M, Y)\) of the state equation of \( S \). By liveness and reversibility, there exists a sequence \( \sigma \) feasible in \( S \) such that \( P(\sigma) \geq Y \) and \( \sigma \) leads to the initial marking \( M_0 \). By assumption, \((N, M)\) is live, and from the above we deduce \( M_0 = M + I \cdot (P(\sigma) - Y) \), i.e. \( M_0 \) is potentially reachable from \( M \) with the vector \( P(\sigma) - Y \). Now, Proposition 9.3 applies by renaming \( M_0 \) as \( M^* \) and \( M \) as \( M_0 \); indeed, \( S \) is supposed to be an AMG, hence fulfills all the conditions \( H_1 \) to \( H_4 \), and \((N, M)\) is live and fulfills the conditions \( H_1 \) to \( H_3 \). Condition \( C1 \) is fulfilled by \( S \) since \( M_0 = M + I \cdot (P(\sigma) - Y) \), and condition \( C2 \) is fulfilled by \( S \) too since it is an AMG. We deduce that \( M_0 \) is reachable from \( M \) with a firing sequence \( \tau \). Consequently, for all pairs of potentially reachable markings \((M, M')\), \( M_0 \) is a marking reachable from both \( M \) and \( M' \), implying that \( PRG(S) \) is directed. \( \square \)
Checking P Petri Nets Properties Using the State Equation

Figure 20. Illustration of the proof of Lemma 9.6 for any solution \((M, Y)\) to the state equation associated to \(S = (N, M_0)\), some sequence \(\sigma\) is feasible at \(M_0\) and leads to \(M_0\) such that \(P(\sigma) \geq Y\); using \(\sigma\) and \(Y\), we deduce that some sequence \(\tau\) is feasible at \(M\) and leads to \(M_0\). Thus, \(M_0\) is reachable from each potentially reachable marking, hence the directedness of \(PRG(S)\), illustrated for two solutions \((M, Y)\) and \((M', Y')\) to the state equation.

**Lemma 9.7.** Consider a live AMG \(S = (N, M_0)\) in which each minimal siphon induces a conservative P-subnet, and consider any potentially reachable marking \(M\). Then, \((N, M)\) is live, thus \(S\) is strongly live.

**Proof:**
Since \(S\) is live, each minimal siphon is initially marked. Besides, it is reversible. By conservativeness, each marking that is potentially reachable in the P-subsystem induced by any minimal siphon necessarily marks this siphon, hence also each non-minimal siphon that contains it. In particular, each marking that is potentially reachable in \(S\) marks each siphon. By liveness and reversibility, \(M_0\) is potentially reachable from \(M\). Now, in Proposition 9.5 let us rename \(M_0\) as \(M\) and \(M^*\) as \(M_0\). Applying it, since no marking reachable from \((N, M)\) empties any siphon, \((N, M)\) is live. By Lemma 9.6, \(PRG(S)\) is directed. Lemma 4.2 applies and \(S\) is strongly live.

We deduce next theorem.

**Theorem 9.8.** (Potential reachability in AMG)
Let \(S\) be a live AMG system satisfying property \(R\), in which each minimal siphon induces a conservative P-subnet. Then \(PR(S) = R(S)\).

**Proof:**
Applying Lemma 9.7 for each potentially reachable marking \(M\), \((N, M)\) is live. Now, Lemma 9.6 applies: \(PRG(S)\) is directed, thus also initially directed. By Theorem 5.5 \(PR(S) = R(S)\).

Figure 16 provides a counter-example when a minimal siphon induces a non-conservative P-subsystem, which is the only assumption relaxed.

9.3. Checking Properties of Augmented Marked Graphs

Proposition 9.4 states that an AMG \(S\) is live iff it cannot reach any marking at which some siphon is unmarked. Now, if each minimal siphon of \(S\) and of \(\neg S\) induces a conservative P-subsystem, \(S\) fulfills
property $L$, since both $S$ and $-S$ are AMG and each siphon is initially marked. Applying Proposition $9.2$, $S$ also fulfills property $R$. Then, Theorem $9.8$ can be exploited.

In addition to these remarks, we obtain below a result relating the behavior of an AMG to the behavior of its reverse.

**Lemma 9.9. (Properties of the reverse AMG)**

Let us suppose that the AMG $A$ is live, bounded and reversible ($LBR$). Then:

1. If $-A$ is live, then $-A$ is $LBR$.
2. If the underlying MG $G$ of $A$ is bounded, then $-A$ is bounded.

**Proof:**

(1.) Applying Proposition $9.2$ and noticing that the reverse of an AMG is also an AMG, $-A$ is reversible. By Lemma $5.3$, $-A$ is bounded, hence it is $LBR$.

(2.) Since $A$ is live, $G$ is live $[15, 37]$. Since $G$ is also bounded, it is conservative and structurally bounded $[15, 37]$. By Proposition $9.1$, each resource place of $-A$ (which is also an AMG) is bounded. Since the other places belong to $-G$, which is structurally bounded since its reverse is a conservative MG, each place of $-A$ is bounded, hence the claim. 

\[ \square \]

10. Reachability in PCMG\(_{\leq}\)

In this section, we focus on well-structured PCMG\(_{\leq}\). First, in this class, we exhibit the structure of minimal siphons when the undirected graph $G$ is acyclic; under the same constraint, we develop a characterization of liveness in terms of marked siphons. Then, we develop a characterization of reversibility under the liveness assumption, without the acyclicity constraint. Finally, assuming that a live PCMG\(_{\leq}\) system $S$ is obtained from an acyclic undirected graph $G$, we show that $S$ is reversible and fulfills the PR-R equality.

10.1. Structure of siphon-induced P-subnets in well-structured PCMG\(_{\leq}\) obtained from an acyclic graph $G$

Next theorem highlights the state machine structure of siphon-induced P-subnets when $G$ is acyclic.

**Theorem 10.1. (Structure of minimal siphons)**

Consider a well-structured PCMG\(_{\leq}\) $N$ obtained from an acyclic, connected, undirected graph $G$. Then each place belongs to a minimal siphon and a minimal trap of $N$, and each minimal siphon, each minimal trap, induces a (strongly) connected state machine P-subnet of $N$.

**Proof:**

Consider any P-subnet $N_D$ induced by a minimal siphon $D$, then $N_D$ is connected, since otherwise $D$
would not be minimal. We prove the claim by strong induction on the number \( n \) of shared places in \( N \), for \( n \geq 0 \).

- **Base case:** \( n = 0 \), \( N \) is either empty, or an isolated place, or a strongly connected and well-formed marked graph. In the first two cases, the claim is trivially true. In the third case, each place belongs to an elementary circuit P-subnet, hence the claim.

- **Inductive case:** \( n \geq 1 \), \( N \) has some shared place. If \( D \) does not contain any shared place, it induces an elementary circuit P-subnet of \( N \). Otherwise, \( D \) contains at least one shared place \( p \) corresponding to a vertex \( v \) of \( G \). Consider any edge \( e \) having \( v \) as an extremity.

In the subgraph of \( G \) obtained by deleting all the edges adjacent to \( v \) except \( e \), denote by \( G' \) the maximal connected component containing \( e \). The associated PCMG\(_{\leq} \) \( N|_{G'} \) is strongly connected, well-structured and has a strictly smaller number of shared places. The subset \( D' \) of \( D \) obtained by projection on \( N|_{G'} \) is a siphon, and it is minimal in \( N|_{G'} \) since otherwise \( D \) would not be minimal in \( N \).

Hence, the inductive hypothesis applies to \( N|_{G'} \): the siphon \( D' \) induces a strongly connected state machine P-subnet \( N_{D'} \) of \( N|_{G'} \).

Now, we apply the same reasoning to every other edge adjacent to \( v \). We deduce that \( D \) induces a strongly connected state machine P-subnet of \( N \).

We proved the base case and the inductive case: the property is true for each \( n \geq 0 \). Hence the claim for the siphon case.

Using the same reasoning for traps or noticing that the reverse of such a well-structured PCMG\(_{\leq} \) is also a well-structured PCMG\(_{\leq} \) obtained from the same undirected, acyclic graph \( G \), we deduce the claim for the trap case.

\[\square\]

If the acyclicity assumption is dropped, this result is no more true, as exemplified in Figure 21.

---

**Figure 21.** An undirected graph \( G \) with place labels on the left, from which the PCMG\(_{\leq} \) in the middle is derived by refinement. This PCMG\(_{\leq} \) is well-structured and not structurally live. It contains the minimal siphon \( \{p_0, p_2, p_4\} \) inducing the P-subnet depicted on the right, which is not a state machine. In the reverse net, the minimal siphon becomes a minimal trap.
10.2. Liveness of well-structured $\text{PCMG}_{\leq}$ obtained from an acyclic graph $G$, in PTIME

Next theorem provides a characterization of liveness when $G$ is acyclic. Noticing that each minimal siphon of the nets considered is a trap (by Theorem [10.1]), the result can be seen as a variant of Commoner's theorem and the Home Marking theorem developed for free-choice nets (see e.g. [1]). We derive its polynomial-time complexity as a corollary.

**Theorem 10.2. (Liveness of well-structured $\text{PCMG}_{\leq}$ with acyclic graph)**

Consider a well-structured $\text{PCMG}_{\leq}$ system $S = (N, M_0)$ obtained from an acyclic undirected graph $G$ and having at least one transition. Then $S$ is live iff each minimal siphon—equivalently each minimal trap—of $N$ is marked by $M_0$.

**Proof:**

By Theorem [10.1] each minimal siphon $D$ induces a strongly connected state machine $\text{P-subnet}$ $N_D$, and each place belongs to some minimal siphon. If a siphon is not initially marked, then $S$ cannot be live. Let us prove the other direction.

We prove the claim by strong induction on the number $n$ of edges, $n \geq 1$ since we assumed $N$ to have at least one transition.

Base cases: $n = 1$, $N$ is a well-formed marked graph with at least one transition, in which each elementary circuit $\text{P-subnet}$ is initially marked, from which liveness is derived (by Corollary 6.4).

Inductive case: $n > 1$. We suppose the claim to be true for each $n' < n$.

Assume that each $D$ is initially marked and that a transition $t$ is dead at a reachable marking $M_t$. Denote by $N_t = (P_t, T_e, W_t)$ the well-formed $\text{MG T-subnet}$ containing $t$ and induced by the associated edge $e_t$. Since $t$ is dead at $M_t$, $(N_t, M_t|_{P_t})$ is deadlockable, so that at least one (elementary) circuit $\text{P-subnet}$ $C_t$ of $N_t$ is unmarked by $M_t$. Since each minimal siphon of $N$ is initially marked, $C_t$ contains one or two shared places of $N$.

If $C_t$ contains exactly one shared place $p$, associated to vertex $v$ in $G$, consider the subgraph of $G$ obtained by deleting $e_t$ and denote by $G'$ the maximal connected component containing $v$. Let $D$ be a minimal siphon of $N$ containing the places $P_C$ of $C$, then $D \setminus (P_C \setminus \{p\})$ is a minimal siphon of $N|_{G'}$ that is marked by $M_t$. Consider any minimal siphon $D'$ of $N|_{G'}$ containing $p$: $D'$ is marked by $M_t$, since otherwise $D' \cup P_C$ is a minimal siphon of $N$ unmarked by $M_t$, which is impossible (by Theorem [10.1]). Each other minimal siphon of $N|_{G'}$ not containing $p$ is also a minimal siphon of $N$ and is marked by $M_t$. Applying the inductive hypothesis, the $\text{T-subsystem}$ $(N|_{G'}, M_t|_{G'})$ of $(N, M_t)$ is live and enables a sequence sending a token to $p$.

This reasoning applies symmetrically to the other extremity of $e_t$.

Now, if $C_t$ contains exactly two shared places $p$ and $p'$, associated to $v$ and $v'$ in $G$, consider the subgraph of $G$ obtained by deleting $e_t$ and denote by $G_v$ the maximal connected component containing $v$, by $G_{v'}$ the maximal connected component containing $v'$. If $D_v$ is a minimal siphon of $G_v$ unmarked by $M_t$ and containing $v$, and if $D_{v'}$ is a minimal siphon of $G_{v'}$ unmarked by $M_t$ and containing $v'$ (they exist by Theorem [10.1]), then $D_v \cup D_{v'} \cup P_C$ is a minimal siphon of $N$ unmarked by $M_t$, which is impossible. Consequently, $D_v$ of $D_{v'}$ is marked by $M_t$, and $(N|_{G_v}, M_t|_{G_v})$ or $(N|_{G_{v'}}, M_t|_{G_{v'}})$ is a live $\text{T-subsystem}$ of $(N, M_t)$ in which a token can be sent to $p$ or $p'$.
We deduce that a marking $M'_t$ is reachable from $M_t$ such that $(N_t, M'_t | P_t)$ is live. This contradicts the fact that $t$ is dead at $M_t$. Thus, $S$ is live.

We proved the base cases and the inductive case, so that the claim is true for each $n$. Hence the result for the siphon case.

The trap case is derived directly from the above, since, in the class of nets considered, each minimal siphon is a minimal trap and reciprocally, by Theorem 10.1.

**Corollary 10.3. (Polynomial-time complexity of Theorem 10.2)**

Checking the liveness of a well-structured, connected PCMG$_\leq$ obtained from an acyclic graph, is a polynomial-time problem.

**Proof:**

Denote by $Q$ the set of all the places that are not marked by the initial marking $M_0$. Computing the unique maximal trap or siphon $Q_{\text{max}}$ included in $Q$ is done in polynomial-time, as detailed in the proof of Theorem 8.12 in [1]. Using Theorem 10.2 we have to check that each minimal siphon and trap is initially marked. If $Q_{\text{max}}$ is empty, then each of them is marked and the system is live. Otherwise, either $Q_{\text{max}}$ is minimal and the system is not live, or it is not minimal, meaning that it contains a proper, minimal, non-empty and unmarked siphon or trap, implying the system is not live: the result is the same in both cases, so that we do not have to compute any minimal siphon or trap.

$\blacksquare$

### 10.3. Live and well-structured PCMG$_\leq$ are not always reversible

The existence of a feasible T-sequence is yet not known to be sufficient for reversibility in the class of live and well-structured PCMG$_\leq$, which are not included in the class of HFC nets. Also, the class of AMG benefits from conditions ensuring liveness, boundedness and reversibility [38], but does not contain the PCMG$_\leq$, hence the answer cannot be deduced directly from them.

In the class of well-structured PCMG$_\leq$, reversibility is not necessarily deduced from liveness, as shown in Figure 22.

![Figure 22. An undirected graph $G$ on the left from which the PCMG$_\leq$ system on the right is derived by refinement. The latter is well-structured, live and non-reversible. It is not an AMG since $p_1$, $p_3$ and $p_5$ are initially marked, hence condition ($H_4$) is not fulfilled.]
10.4. A characterization of reversibility for well-structured and live $\text{PCMG}_{\leq}$

We provide the characterization through the next theorem and the subsequent corollary. The theorem studies the reversing of a single transition firing, and the corollary generalizes the result to finite sequences of arbitrary length.

**Theorem 10.4. (Reversing the action of a single transition)**

Consider a live and well-structured $\text{PCMG}_{\leq}$ system $S = (N, M_0)$ in which a T-sequence is feasible. After the firing of any single transition in $S$, a feasible sequence exists that leads to $M_0$.

**Proof:**

The proof is illustrated in Figure 23. Denote by $G$ the undirected graph from which $S$ is obtained. Denote by $\alpha$ some T-sequence feasible in $S$. Consider the firing of some transition $t$ from $M_0$ leading to the marking $M_t$.

If no input place $p$ of $t$ is a shared place, meaning $p^* = \{t\}$, then $\alpha \bullet t$ is feasible from $M_t$ and reaches $M_0$, in which case we deduce the claim. Otherwise, some place $p$ in $\bullet t$ is a shared place, in which case we prove in the following that a sequence leading to $M_0$ from $M_t$ also exists.

The transition $t$ belongs necessarily to a marked graph T-subsystem $S_t$ of $S$ corresponding to an edge $e_t = (v, v')$ of $G$ (i.e. the component $\beta(e_t)$), where $v$ is associated to $p$ and $v'$ to $p'$ (i.e. $\gamma(e_t) = (p, p')$). If $S_t$ is reversible, then the initial marking can be reached trivially. Otherwise, $S_t$ deadlocks at some marking $M_{dt}$ reachable from $M_t$ (since $S_t$ is a strongly connected and well-formed MG, non-reversibility implies deadlockability). In this case, $S_t$ necessarily contains at least one synchronization and at least one unmarked elementary circuit in which $p'$ occurs (by Proposition 6.3). Since $S$ is live, $p'$ is a shared place. The following algorithm builds a tree $G'$ that is a subgraph of $G$.

Initially, $G' = (V', E')$ contains only the edge $e_t = (v, v')$, i.e. $V' = \{v, v'\}$ and $E' = e_t$. Until some edge $e_r = (v_r, v'_r)$ exists such that $v_r$ belongs to $V' \setminus \{v\}$ and the marked graph T-subsystem $S_{e_r}$ of $(N, M_{dt})$ associated to $e_r$ enables a sequence sending a token to $v_r$, we iterate the following: add to $G'$ each non-visited edge $e = (v_1, v_2)$ such that $v_1$ belongs to $V' \setminus \{v\}$, but $v_2$ does not, the MG T-subsystem $S_e$ of $(N, M_{dt})$ induced by $e$ is not live and becomes live if a token is added to $v_2$.

The loop terminates since each edge is visited at most once, and the non-existence of the edge $e_r$ above means that the T-subsystem of $S$ corresponding to $G'$ can deadlock as a T-subsystem and cannot receive tokens from other transitions of the T-subsystem associated to $G \setminus G'$, contradicting liveness.

Hence, a token can be sent to $v_r$ by firing only in $S_{e_r}$, leading to a new marking $M$ through some sequence $\sigma_r$. Some elementary sequence of edges $\mu = e_1 \ldots e_k$ exists in $G'$ from $v'$ to $v_r$ and represents a live and reversible T-subsystem $S'_\mu$ of $(N, M)$, by construction.

Walking along this path backwards, i.e. firing successively in the marked graph T-subsystems associated to $\beta(e_k) \ldots \beta(e_1)$, a token is sent to $v'$ through some sequence $\sigma$, leading to $M_\sigma$. The first firing of $t$ in $S_t$ can thus be canceled, leading to $M'$ such that for each place $p'' \neq v, v'$ of $P_{e_t}$, $M'(p'') = M_0(p'')$. The T-subsystem $S'_\mu$ associated to $\mu$ and marked by $M'$ is live and reversible. Thus $\sigma$ can be canceled in $S'_\mu$, leading to $M''$. At $M''$, only places of the marked graph $N_r$ associated to $e_r$ (i.e. $N_r = \beta(e_r)$) might be marked differently from $M_0$. Either $(N_r, M_t|_{P_{e_r}})$ is live and reversible, in which case it is also live and reversible at $M''$ and $M_0$ is reachable from $M''$, or $(N_r, M_t|_{P_{e_r}})$ deadlocks, thus deadlocks also from $M''$, meaning that the sequence $\sigma_r$ that sent tokens to $v_r$ using
only transitions of $N_r$ did not use any token from any shared place: consequently, the initial marking $M_0$ can be reached from $M''$ by firing $\alpha^k \cdot \sigma_r$ for the smallest positive integer $k$ such that $P(\alpha^k) \geq P(\sigma_r)$. \hfill \Box

Figure 23. Illustration of the proof of Theorem 10.4. The graph $G$ is pictured on the left, with $\mu = e_1 e_2$. The PCMG $S$ obtained from $G$ is pictured on the right and contains cycles.

Figure 24. Another illustration of the proof of Theorem 10.4 with $\mu = e_1 e_2$, where $e_2, e_1$ and $e_r$ share a node.

**Corollary 10.5. (A characterization of reversibility for well-structured, live PCMG$_\prec$)**
Consider a live and well-structured PCMG$_\prec$ system $S = (N, M_0)$. It is reversible iff it enables a T-sequence.
**Proof:**
As recalled earlier, the left to right direction ($\Rightarrow$) is clear. To prove the other direction ($\Leftarrow$), we reason by induction on the length $n$ of any feasible sequence $\sigma$, as follows.

**Base case:** $n = 0$, $\sigma$ is the empty sequence, hence $M_0$ is trivially reachable.

**Inductive case:** $n > 0$ and we suppose the claim to be true for $n - 1$. The sequence is of the form $\sigma = t\sigma_0$ and leads to a marking $M$. The firing of $t$ leads to the system $(N, M_t)$, which is live and enables a $T$-sequence by Theorem 10.4. By the induction hypothesis, some sequence $\sigma'_0$ is feasible at $M$ that leads to $M_t$. Thus, $M_0$ can be reached from $M$. We proved the base and inductive cases, hence the claim is true for each length $n$, thus any feasible sequence. $\blacksquare$

Figure 25 pictures a well-formed, non-reversible system allowing a $T$-sequence, obtained by merging three pairs of places in a well-formed marked graph: the system obtained is not a PCMG$_\leq$, since there does not exist any undirected graph describing its topology. It has three shared places, whereas in each PCMG$_\leq$ each $N_i$ contains at most two shared places. Since a PCMG$_\leq$ can be seen as the result of place-merging subsets of places in a non-connected marked graph, with additional constraints reducing the set of mergeable subsets, this example can be considered as close to a PCMG$_\leq$. Thus, relaxing the definition of PCMG$_\leq$ easily leads to examples invalidating Corollary 10.5. Another similar counter-example, with an isomorphic reachability graph and only two shared places, is pictured in Figure 26. Notice that both figures use the same MG, with a different layout.

**10.5. Reversibility and the PR-R equality in the acyclic case**

We obtain next theorem.

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Footnote: The system is inspired from Figure 21 in [4].
Figure 26. On the left, a unit-weighted, well-formed MG. Place-merging its subsets \{p_1, p'_1\} and \{p_2, p'_2\} leads to the system in the middle, which is unit-weighted, live, structurally bounded with only two shared places, namely \(p_1\) and \(p_2\). The system obtained, with the initial marking pictured, enables the T-sequence \(t_0 t_3 t_2 t_1\) but is not reversible. On the right, its non strongly connected reachability graph is given, with initial state \(s_0\).

**Theorem 10.6. (Directedness and reversibility in a subclass of PCMG\(\leq\))**

Consider a well-structured and live PCMG\(\leq\) \(S = (N, M_0)\) obtained from an acyclic, undirected and connected graph \(G\). Then \(S\) is reversible and fulfills the PR-R equality.

**Proof:**
The proof is illustrated in Figure [27]. We reason by induction on the number \(n\) of shared places.
- **Base case:** \(n = 0\), \(S\) is a live and bounded marked graph, and Proposition [6.1] applies.
- **Inductive case:** \(n > 0\). We suppose the claim to be true for \(n - 1\). Since \(G\) is acyclic, consider an edge \(e\) of \(G\) having only one shared extremity; denote by \(p\) the shared place of \(S\) associated to this extremity. Denote by \(S_e\) the T-subsystem \((N_e, M_0|P_e)\) of \(S\), where \(N_e = (P_e, T_e, W_e)\) is the marked graph associated to \(e\), \(N_e = \beta(e)\), and denote by \(Y\) a \(T\)-vector such that \(M = M_0 + I \cdot Y\). The transition set \(T_e\) of \(N_e\) is disjoint from the one associated to \(G - e\), by definition of PCMG\(\leq\). In the following, for any PCMG\(\leq\) system \(S\) obtained from a graph \(G\), for any subgraph \(G'\) of \(G\), we denote by \(S|G'\) the T-subsystem of \(S\) corresponding to \(G'\) and by \(Y|G'\) the projection of the \(T\)-vector \(Y\) on the set of transitions associated to \(G'\).

Fire a finite sequence \(\sigma_e\) in \(S_e\) leading to a marking \(M'\) in \(R(S)\) that maximizes the amount of tokens in \(p\), i.e. such that \(p\) is \(M'(p)\)-bounded in \(S_e\). Since \((N_e, M'|P_e)\) cannot produce additional tokens in \(p\), since \(S\) is live and since \((N, M')|_{G-e}\) is well-structured, Theorem [10.2] applies and we deduce that \((N, M')|_{G-e}\) is a live, well-structured PCMG\(\leq\). Thus, the inductive hypothesis applies to the latter: \((N, M')|_{G-e}\) is also reversible and every marking that is potentially reachable in \((N, M')|_{G-e}\) is reachable in it.

We show first that some T-sequence is feasible in \(S\), allowing to apply Theorem [10.6]. At each marking \(M''\) reachable in \(S\), if \(M''(p) \geq 1\) then \((N_e, M''|P_e)\) is live and reversible. By liveness
and reversibility of \((N, M')|_{G-e}\), the latter enables a sequence \(\tau\) that visits all transitions in \(N|_{G-e}\) and reaches a marking \(M_p\) such that \(M_p(p) \geq 1\); from the above, the marked graph T-subsystem \((N_e, M_p|_{P_e})\) enables a \(T_e\)-sequence \(\tau_e\), leading back to \(M_p\); then a sequence \(\tau'\) leads back to \(M'\). We deduce that \(\alpha = \tau \tau_e \tau'\) is a T-sequence feasible in \((N, M')\). Now, either \(M'(p) > 0\) so that \(S_e\) is reversible and \(M_0\) can be reached trivially from \(M'\), or \(M'(p) = 0\) so that \(\alpha\) is also feasible in \(S\), the intermediate marking \(M'\) being replaced by \(M_0\) in the reasoning above. In both cases, applying Theorem[10.6] \(S\) is reversible.

Now, let us show that \(M\) is reachable in \(S\). Since \(M\) is a marking, we have \(M(p) \geq 0\). By definition of \(M', M'(p) \geq M(p)\). Let us denote by \(T_{G-e}\) the set of transitions of \(N|_{G-e}\). We define the T-vector \(Y'\) as follows: for each transition \(t\), if \(t\) belongs to \(T_{G-e}\) then \(Y'(t) = Y(t)\), otherwise \(Y'(t) = 0\). Then, \(M_{Y'} = M' + I \cdot Y'\) is a marking potentially reachable in \((N, M')\), and the marking \(M_{Y'}|_{G-e}\) is potentially reachable in \((N, M')|_{G-e}\), thus is reachable in the latter.

We have two cases: either \(M_{Y'}(p) > 0\) or \(M_{Y'}(p) = 0\). Let us define the T-vector \(Z = k \cdot 1_{T_e} - P(\sigma_e) + Y|_e\), where \(k\) is the smallest positive integer \(k\) such that \(k \cdot 1_{T_e} \geq P(\sigma_e)\).

In the first case, \((N_e, M_{Y'}|_e)\) is a live, well-formed MG T-subsystem of \((N, M_{Y'})\), thus Proposition[6.1] applies: since \(M = M' + I \cdot Z\), some sequence with Parikh vector \(Z\) is feasible in \((N_e, M_{Y'}|_e)\), thus also in \((N, M')\) and leads to \(M\), hence the claim.

In the second case, a sequence \(\sigma_r\) is feasible in \((N, M_{Y'}),\) leading to a marking \(M_r\) such that \(M_r(p) = 1\). Since \((N, M_{Y'})|_{G-e}\) is reversible, a sequence \(\sigma'_r\) is feasible in \((N, M_r)|_{G-e}\), thus also in \((N, M_r)\), that leads back to \(M_{Y'}\). Let us define \(M'_r = M_r + I \cdot Z\), then \((N_e, M'_r|_e)\) is live, Proposition[6.1] applies and \(M'_r|_e\) is reachable in the latter T-subsystem, hence \(M'_r\) is reachable from \((N, M_r)\). Moreover, \(M'_r(p) \geq 1\) since otherwise we would have \(M(p) < 0\), which is impossible. Consequently, since the sequence \(\sigma'_r\) is feasible in \((N, M_r)\) and since \(M'_r|_{G-e} \geq M_r|_{G-e}\), it is also feasible in \((N, M'_r)\) and leads to \(M\), which is thus reachable from \(M_0\).

We proved the property to be true for every number \(n\) of shared places. We get the claim. \(\Box\)

![Figure 27](image-url) Illustration of the proof of Theorem[10.6] The graph \(G\) on the left is acyclic and is labeled with place names. The \(\text{PCMG}_{\leq}\) on the right is obtained from \(G\).
This theorem is no more true if homogeneous weights are allowed, as shown in Figure 28.

![Figure 28](image)

Figure 28. On the right, a weighted, homogeneous state machine that could be seen as a weighted, live and well-structured PCMG\(_{\leq}\), obtained from the graph on the left. It is live, but non reversible; from some reachable marking (e.g. after a single firing of \(t_2\)), the initial marking is potentially reachable but not reachable.

We are now able to derive next corollary.

**Corollary 10.7. (Property \(\mathcal{R}\) and PR-R equality in acyclic, live and well-structured PCMG\(_{\leq}\))**

Consider a live and well-structured PCMG\(_{\leq}\) \(S\) obtained from an acyclic undirected graph. Then \(S\) fulfills \(\mathcal{R}\) and the PR-R equality.

**Proof:**

We get the claim by combining Theorem 10.6 with Lemma 5.4. \(\Box\)

By Corollary 10.3, the liveness of a well-structured PCMG\(_{\leq}\) obtained from an acyclic graph can be checked in polynomial-time. Then, Corollary 10.7 derives property \(\mathcal{R}\) and the PR-R equality.

11. Related work

Results connected to potential reachability in WMG, WMG\(_{\leq}\) and larger classes, together with the behavioral properties investigated in this paper, have been developed notably in [15, 2, 37, 9, 23].

As far as we know, the only work dedicated to the H1S class is [10].

AMG and their properties, such as liveness and reversibility, together with compositional methods, have been investigated in [11, 27, 31].

Concerning refinement, synthesis, composition (node merging, fusion...) and abstraction techniques, the previous works closest to our notion introduced for PCMG\(_{\leq}\) are [39, 40, 31, 28, 41, 42, 43, 44]. Numerous other studies provide algorithms for synthetizing, from a given labeled transition system, a Petri net with isomorphic reachability graph, see e.g. [9] for the synthesis of WMG\(_{\leq}\); such techniques are out of scope for this paper.

Other classes with shared places, such as \(S^1\)PR, PC\(^2\)R and L-S\(^3\)PR, which do not contain the H1S-WMG\(_{\leq}\), the AMG nor the PCMG\(_{\leq}\), have also been extensively studied in [40, 33, 45] and benefit from strong properties related to reachability, notably to directedness, deadlockability, liveness and reversibility. In [32, 46], DSSP and \{SC\}\(^*\)ECS systems, which allow weights and shared places in a
restricted fashion, benefit from structural and behavioral properties. Generalizations of AC nets have been studied in [47].

12. Conclusions and Perspectives

The reachability problem has attracted a lot of attention since the introduction of Petri nets in the 60’s by Carl Adam Petri. It is indeed a central model-checking problem that reduces to numerous other fundamental ones. Bounds on its complexity have been obtained and refined over the years. Recently, a non-elementary lower bound has been uncovered.

In this paper, for weighted Petri nets, we provided several sufficient conditions ensuring the PR-R equality, i.e. the equality of the set of reachable markings and the set of potentially reachable ones, the latter being described by the solutions of the state equation. When this equality is fulfilled, asking for the reachability of a marking reduces to solving the state equation for this marking, allowing to use integer linear programming. This avoids a brute-force analysis of the state space and trims down the complexity to NP. The main contributions of this paper are summarized as follows.

First, we developed results connecting some behavioral properties of a system to the same properties in the reverse system. Using the notion of directedness, we provided a general condition ensuring the PR-R equality in weighted Petri nets, based on liveness, reversibility in the system and its reverse. We deduced a sufficient condition of PR-R equality for homogeneous free-choice nets.

Then, we delineated several subclasses of weighted Petri nets generalizing the marked graphs, which have been extensively studied and fruitfully used in the modeling and analysis of various real-world applications. We recalled some use-cases of these larger classes, extracted from previous studies and the Petri Net Model Checking Contest database. In these classes, we proposed several sufficient conditions for PR-R equality, based on the reversibility of the systems and their reverse, and on liveness.

We also recalled and gathered methods for checking the sufficient conditions, notably relating the structure to liveness and reversibility, and highlighted cases in which these methods are more efficient.

As a perspective, we believe that our methods may be extended to other classes of Petri nets, notably modular ones. Also, more efficient methods checking the assumptions of the conditions, such as liveness, reversibility and properties of the siphons, might be uncovered for the classes of our study and more expressive ones.

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