REGULARISATION OF SUPERSYMMETRIC THEORIES

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We discuss issues that arise in the regularisation of supersymmetric theories.

1 Beyond the tree approximation

In this chapter we consider issues of both practice and principle that arise when we take supersymmetric theories and calculate radiative corrections. It is usually the case that a symmetry of the Lagrangian is still a symmetry of the full quantum effective action; in which case we say that radiative corrections preserve the symmetry. There are important exceptions to this rule, however: for example conformal invariance is in general violated by radiative corrections, and massless quantum electrodynamics has a global $U_1$ axial symmetry which is violated at the one-loop level (in accordance with the famous Adler-Bardeen theorem). It is not a priori obvious, therefore, that supersymmetry is a symmetry of the full quantum theory in any particular case. Indeed it has been occasionally claimed that there exists a supersymmetry anomaly. In some cases these claims have been erroneous, and have occurred because it is difficult to distinguish between a genuine anomaly and an apparent violation of a supersymmetric Ward identity due to use of a regularisation method that itself violates supersymmetry. Contrariwise, a detailed formal renormalisation program has been pursued in a series of papers by Piguet and collaborators including one where a proof that supersymmetry is not anomalous was presented. There is no real reason to doubt this conclusion (although the treatment of infra-red singularities in the program is a possible weakness), for the class of theories considered; the evidence adduced so far points to supersymmetry being a symmetry of the full quantum theory. (Note, however, recent suggestions that there may indeed exist a supersymmetry anomaly in composite operators.)

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The existence of an anomaly is intimately related to the question of regularisation. Beyond the tree level, certain amplitudes in any given quantum field theory are not defined, due to divergences caused by the need to integrate over all momenta for particles in intermediate states. Regularisation is the process whereby the result of an ill-defined correction to a given amplitude is separated into a finite part (which is retained) and an “infinite” part (or more precisely, a part which tends to infinity in the limit that a certain parameter, or parameters, specific to the regularisation method is removed) which is removed from the theory (“subtracted”) by introducing a counter-term which precisely cancels it. If the regularised theory fails to respect any given symmetry then the finite amplitudes will fail to satisfy the Ward identities of the symmetry, giving rise to an apparent anomaly and the confusion alluded to above. When the anomaly really is specious, it is possible to restore invariance by modifying the counter-terms by finite amounts. (Obviously the counter-terms are ambiguous, in that their defining role is to cancel something which becomes infinite as the regulator is removed, so that adding a finite quantity to any counter-term leaves its raison d’être intact. If no modification of the counter-terms will restore the Ward identity then there is an anomaly.

From a formal point of view, the choice of regularisation scheme made in the implementation of a renormalisation is not of great significance; it is important only that it corresponds to addition of local counter-terms. For the extraction of physical predictions, however, the choice becomes a matter of considerable practical significance. It is convenient, for example, to use a regularisation method that preserves symmetries. It should be clear, in fact, from the above discussion that the existence of a regulator consistent with a given symmetry suffices to prove that symmetry to be anomaly-free. In this context the approach of West, consisting of higher derivative regularisation supplemented by Pauli-Villars at one loop, is worthy of consideration; but as the author himself remarks, the issue of a possible anomaly is not thereby fully resolved because of the infra-red difficulties already alluded to.

Dimensional regularisation (DREG) is an elegant and convenient way of dealing with the infinities that arise in quantum field theory beyond the tree approximation. It is well adapted to gauge theories because it preserves gauge invariance; it is less well-suited, however, for supersymmetry because invariance of an action with respect to supersymmetric transformations only holds

\[\text{sometimes an anomaly can be apparently removed only to reappear in another guise. This is the case with the Adler–Bardeen anomaly, which is a property of the fermion triangle with two vector and one axial–vector vertices. The anomaly may affect the axial current or the vector current, depending on how the theory is regularised.}

\[\text{Use of Pauli-Villars at one loop in supersymmetry was also advocated by Gaillard in 1995.}\]
in general for specific values of the space-time dimension \( d \). This is essentially due to the fact that a necessary condition for supersymmetry is equality of Bose and Fermi degrees of freedom. In non-gauge theories it is relatively easy to circumvent this problem, and DREG as usually employed is, in fact, a supersymmetric procedure. Gauge theories are a different matter, however, and the question as to whether there exists a completely satisfactory supersymmetric regulator for gauge theories remains controversial. This fact has been exploited recently to suggest that there may be supersymmetric anomalies.

An elegant attempt to modify DREG so as to render it compatible with supersymmetry was made by Siegel. The essential difference between Siegel’s method (DRED) and DREG is that the continuation from 4 to \( d \) dimensions is made by *compactification*, or *dimensional reduction*. Thus while the momentum (or space-time) integrals are \( d \)-dimensional in the usual way, the number of field components remains unchanged and consequently supersymmetry is undisturbed. (A pedagogical introduction to DRED was given by Capper et al.)

As pointed out by Siegel himself, there remain potential ambiguities with DRED associated with treatment of the Levi-Civita symbol, \( \epsilon^{\mu\nu\rho\sigma} \). We will address this difficulty and the related one involving \( \gamma^5 \) in Section 3.

We must also address problems which arise only when DRED is applied to non-supersymmetric theories. That DRED is a viable alternative to DREG in the non-supersymmetric case was claimed early on. Subsequently it has been adopted occasionally, motivated usually by the fact that Dirac matrix algebra is easier in four dimensions—and in particular by the desire to use Fierz identities. One must, however, be very careful in applying DRED to non-supersymmetric theories because of the existence of *evanescent couplings*. These were first described in 1979, and independently discovered by van Damme and ‘t Hooft. They argued, in fact, that while DRED is a satisfactory procedure for supersymmetric theories (modulo the subtleties alluded to above) it leads to a catastrophic loss of unitarity in the non-supersymmetric case. Evidently there is an important issue to be resolved here—is use of DRED in fact forbidden (except in the supersymmetric case) in spite of its apparent convenience? It has been conclusively demonstrated that if DRED is employed in the manner envisaged by Capper et al, (which as we shall see differs in an important way from the definition of DRED primarily used by ‘t Hooft and van Damme) then there is no problem with unitarity. There exist a set of transformations whereby the \( \beta \)-functions of a particular theory (calculated using DRED) may be related to the \( \beta \)-functions of the same theory (calculated using DREG) by means of coupling constant reparametrisation. The key is that a correct description of any non-supersymmetric theory impels
us to a recognition of the fundamental fact that in general the evanescent couplings renormalise in a manner different from the "real" couplings with which we may be tempted to associate them. This means that care must be taken as we go beyond one loop; nevertheless it is still possible to exploit the simplifications in the Dirac algebra which have motivated the use of DRED. We will return to this point later.

At this point the reader may wonder why, in a book about supersymmetry, we should worry about renormalising non-supersymmetric theories at all. The main practical reason is that the supersymmetric standard model is an effective theory in which supersymmetry is \textit{explicitly} broken, albeit by terms with non-zero dimension of mass.

The reader may also feel that, given the problems with DRED, we should explore other regulators. For example, there has been some recent work on a new approach known as differential regularisation\textsuperscript{18}. The fact is, however, that the convenience of DREG for calculations beyond one loop makes the use of some variant of it very desirable. Use of other proposed regulators is rarely pursued beyond verification of some already known (and usually one-loop) results.

2 Introduction to DRED

As a concrete example, let us consider a non-abelian gauge theory with fermions but no elementary scalars. The theory to be studied consists of a Yang-Mills multiplet $W^a_\mu(x)$ with a multiplet of spin $\frac{1}{2}$ fields $\psi^\alpha(x)$ transforming according to an irreducible representation $R$ of the gauge group $G$. Of course if $\psi$ is Majorana, then $R$ must be a real representation, since the Majorana condition is not preserved by a unitary transformation.

The Lagrangian density (in terms of bare fields) is

$$L_B = \frac{-1}{4} G^2_{\mu\nu} + \frac{1}{2\alpha} (\partial^\mu W^a_{\mu})^2 + C^a \partial^\mu D^a_\mu + i \overline{\psi} \gamma^\mu D^{a\beta}_\mu \psi^\beta$$

where

$$G^a_{\mu\nu} = \partial_\mu W^a_{\nu} - \partial_\nu W^a_{\mu} + gf^{abc} W^b_{\mu} W^c_{\nu}$$

and

$$D^{a\beta}_\mu = \delta^{a\beta} \partial_\mu - ig (R^a)^{a\beta} W^a_{\mu}$$

and the usual covariant gauge fixing and ghost terms have been introduced.\textsuperscript{d}

\textsuperscript{d}which may be Dirac or Majorana at this stage
The process of dimensional reduction consists of imposing that all field variables depend only on a subset of the total number of space-time dimensions—in this case \( d \) out of 4 where \( d = 4 - \epsilon \). We will use \( \mu, \nu \cdots \) to denote 4-dimensional indices and \( i, j \) to denote \( d \)-dimensional ones, with corresponding metric tensors \( g_{\mu\nu} \) and \( g_{ij} \). It is also convenient to introduce “hatted” quantities (such as \( \hat{g}_{\mu\nu} \) and \( \hat{\gamma}^\mu \)) which are identical to the corresponding \( d \)-dimensional quantities \( (g_{ij}, \gamma_i \cdots) \) within the \( d \)-dimensional subspace, but whose remaining components are zero. Momenta \( p_\mu \) exist only in the \( d \) dimensional subspace so we do not bother to “hat” them. Thus we have for example

\[
\hat{p} = p_\mu \hat{\gamma}^\mu = p_\mu \hat{\gamma}^\mu
\]

and

\[
g^{\mu\nu} g_{\mu\nu} = 4, \quad \hat{g}^{\mu\nu} \hat{g}_{\mu\nu} = \hat{g}^{ij} \hat{g}_{ij} = d.
\]

In particular, we have that

\[
\hat{g}^{\mu\nu} g_{\nu \lambda} = \hat{g}^{\mu\lambda} \quad \text{and} \quad \hat{g}^{\mu\nu} \gamma_\nu = \hat{\gamma}^\mu.
\]

These apparently innocuous relations will cause us trouble in the next section.

In order to fully appreciate the consequences of DRED for \( L_B \) we must make the decomposition

\[
W^a_\mu(x^j) = \{W^a_i(x^j), W^a_\sigma(x^j)\}
\]

where

\[
\delta^i_i = \delta^j_j = d \quad \text{and} \quad \delta^\sigma_\sigma = \epsilon.
\]

It is then easy to show that

\[
L_B = L_B^d + L_B^\epsilon
\]

where

\[
L_B^d = -\frac{1}{4} G_{ij}^2 - \frac{1}{2\kappa} (\partial^i W_i)^2 + C^{\alpha} D_i^{ab} C^b + \bar{\psi}^a \gamma^i D^{\alpha\beta} \psi^\beta \]

and

\[
L_B^\epsilon = \frac{1}{2} (D_\sigma^{ab} W^{\sigma b})^2 - g \bar{\psi} \gamma_\sigma R^a \psi W^{\sigma a} - \frac{1}{4} g^2 f^{abc} f^{def} W^b_\sigma W^c_\sigma W^d_\sigma W^e_\sigma.
\]

Conventional dimensional regularisation (DREG) amounts to using Eq. (10) and discarding Eq. (11). For DRED, on the other hand we include both.

\[\text{The additional contributions from } L_B^\epsilon \text{ are precisely what is required to restore the supersymmetric Ward identities at one loop in supersymmetric theories, as described in section 4.}\]
simple applications it is in general more convenient to eschew the separation performed above and calculate with 4-dimensional and \(d\)-dimensional indices rather than \(d\)-dimensional and \(\epsilon\)-dimensional ones. As a simple illustration, consider the following typical calculation:

\[
\gamma^\mu p_\gamma = p_\nu \gamma^\nu \gamma_\mu = -2p_\nu \gamma^\nu = -2\dot{p}
\]

or equivalently

\[
\gamma^\mu p_\gamma = \gamma^i p_\gamma + \gamma^\sigma p_\gamma = (2 - d)\dot{p} + (d - 4)\dot{p} = -2\dot{p}.
\]

From the dimensionally reduced form of the gauge transformations:

\[
\begin{align*}
\delta W^a_i &= \partial_i \Lambda^a + g f^{abc} W^b_i \Lambda^c \\
\delta W^a_\sigma &= g f^{abc} W^b_\sigma \Lambda^c \\
\delta \psi^i &= ig (R^a)^{\alpha \beta} \psi^\beta \Lambda^a
\end{align*}
\]

we see that each term in Eq. \([11]\) is separately invariant under gauge transformations. The \(W_\sigma\)-fields behave exactly like scalar fields, and are hence known as \(\epsilon\)-scalars. The significance of this is that gauge invariance \textit{per se} provides no reason to expect the \(\bar{\psi}\psi W_\sigma\) vertex to renormalise in the same way as the \(\bar{\psi}\psi W_i\) vertex. In the case of the quartic \(\epsilon\)-scalar coupling the situation is more complex since in general of course more than one such coupling is permitted by Eq. \([14]\). In other words, we cannot in general expect the \(f-f\) tensor structure present in Eq. \([11]\) to be preserved under renormalisation. This is clear from the abelian case, where there is no such quartic interaction in \(L_B\) but there is a divergent contribution at one loop from a fermion loop.

In the case of supersymmetric theories, however, these difficulties do not arise. If \(\psi\) above is a Majorana fermion in the adjoint representation, then \(L_B\) is supersymmetric. This links \(W_i\) and \(W_\sigma\) in a way that is not severed by the dimensional reduction. Thus the \(\bar{\psi}\psi W_\sigma\) and \(\bar{\psi}\psi W_i\) vertices (both equal to \(g\) at the tree level) remain equal under renormalisation. We will return in section 6 to the application of DRED to non-supersymmetric theories.

### 3 DRED ambiguities

With DRED it would seem that necessarily \(d < 4\), since the regulated action is, after all, defined by dimensional \textit{reduction}. Then, given \(d < 4\), one can define an object \(\hat{\epsilon}^{\mu\nu\rho\sigma}\) as follows:

\[
\hat{\epsilon}^{\mu\nu\rho\sigma} = \hat{g}^{\mu\alpha} \hat{\gamma}^{\nu\beta} \hat{g}^{\rho\gamma} \hat{g}^{\sigma\delta} \epsilon_{\alpha\beta\gamma\delta}
\]
where $\epsilon_{\alpha\beta\gamma\delta}$ is the usual 4-dimensional tensor. Unfortunately it is now possible to show that algebraic inconsistencies result unless $d = 4$. Let us illustrate these problems in the two dimensional case. The alternating tensor $\epsilon^{\mu\nu}$ satisfies (in two Euclidean dimensions) the relation

\[ \epsilon^{\mu\nu} \epsilon^{\rho\sigma} = g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}. \quad (16) \]

Using Eq. 6 it is easy to show that

\[ \tilde{\epsilon}^{\mu\nu} \tilde{\epsilon}^{\rho\sigma} = \tilde{g}^{\mu\rho} \tilde{g}^{\nu\sigma} - \tilde{g}^{\mu\sigma} \tilde{g}^{\nu\rho}. \quad (17) \]

where $\tilde{\epsilon}^{\mu\nu}$ is defined similarly to Eq. 15.

However it is trivial to demonstrate that the result of applying Eq. 17 to the tensor

\[ A^{\mu\nu} = \tilde{\epsilon}^{\mu\nu} \tilde{\epsilon}^{\rho\sigma} \tilde{\epsilon}^{\rho\sigma} \quad (18) \]

is ambiguous inasmuch that it differs according to which pair of $\tilde{\epsilon}$-tensors are selected: the result is the identity

\[ (d + 1)(d - 2) \tilde{\epsilon}^{\mu\nu} = 0. \quad (19) \]

A related problem (of course) is the fact that the only mathematically consistent treatment of $\gamma^5$ within DREG is predicated on having $d > 4$. Given Eq. 6 and the usual relation

\[ \{ \gamma_{\mu}, \gamma^5 \} = 0. \quad (20) \]

it follows that

\[ \{ \tilde{\gamma}_{\mu}, \gamma^5 \} = 0. \quad (21) \]

and hence that

\[ (d - 4) \text{Tr} \left[ \gamma^5 \tilde{\gamma}_{\mu} \tilde{\gamma}_{\nu} \tilde{\gamma}_{\rho} \tilde{\gamma}_{\sigma} \right] = 0. \quad (22) \]

This is unfortunate since it renders problematic the discussion of the axial anomaly.

For $d > 4$, however, Eq. 6 does not hold and so Eq. 21 no longer follows. Instead we impose

\[ [\gamma_{\sigma}, \gamma^5] = 0, \quad \text{for} \quad 4 < \sigma < d. \quad (23) \]

and this leads to a straightforward and unambiguous derivation of the axial anomaly. It has been verified that this prescription correctly reproduces the Adler-Bardeen theorem at the two-loop level using DREG.
Returning to the DRED prescription, there are a number of possible “fixes” at one loop, at two loops, it was shown that the Adler-Bardeen theorem could indeed still be satisfied if relations like
\[ \gamma^i \gamma^j = (d - 8) \gamma^5 \] (24)
which follow in the \( d > 4 \) case, are used in conjunction with DRED.

A possible point of view concerning all this is that DRED is terminally inconsistent and should not be used. We believe, however, that the difficulties are essentially technical and can be evaded. For example, one well-defined procedure would be to write
\[ \gamma^5 = \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma_{\sigma} \] (25)
and factor all out \( \epsilon \)-tensors. Renormalised amplitudes may then be calculated, which, being finite as \( d \to 4 \), are unambiguous when the \( \epsilon \)-tensors are contracted in. We would claim also that other modes of procedure which would give different answers because of the ambiguities detailed above, correspond nevertheless to the same physical results. This assertion has in fact been verified in one particular case where a prescription first suggested by Hull and Townsend was used. This amounted to employing as \( \epsilon^{\mu\nu} \) not the usual alternating tensor but instead a structure satisfying
\[ \tilde{\epsilon}^{\mu\nu} \tilde{\epsilon}^{\nu\rho} = (1 + c \epsilon) \tilde{g}^{\mu\rho} \] (26)
(where here \( \epsilon = 2 - d \)). It turns out that the dependence of the results on the parameter \( c \) can be absorbed into redefinitions of the renormalised metric and torsion tensors. In the special case \( c = 0 \), \( \tilde{\epsilon}^{\mu\nu} \) is an almost complex structure.

There have been a considerable number of papers discussing the interpretation of \( \gamma^5 \) in both DREG and DRED, and the reader may consult them for further enlightenment. We turn in the next section to another (but again related) problem with DRED, arising from the fact that in spite of the correct counting of degrees of freedom, there are still ambiguities associated with establishing invariance of the action: we will look at this in more detail below in the context of the supersymmetry Ward identity.

4 The supersymmetry Ward identity

The first concrete illustration of the different results provided by DRED and DREG for a supersymmetric theory was as follows. Consider the basic su-
persymmetric gauge theory in the Wess-Zumino gauge as defined by the Lagrangian 

$$L_S = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + i\frac{1}{2}\chi^\alpha \gamma_\mu D_\mu^\alpha \chi^\beta + \frac{1}{2}D^2.$$  

(27)

In \(d=4\), \(L_S\) is invariant (up to a total derivative) under the transformations 

$$\delta W_a^\mu = i\epsilon_{\mu\lambda}^a, \quad \delta \lambda^a = \frac{1}{2}G_{\mu\nu}^a \gamma^\mu \gamma^\nu \epsilon - iD^a \gamma^5 \epsilon$$

$$\delta D^a = -\tau_{\gamma \mu} \gamma^5 (D^\mu \lambda)^a.$$  

(28)

It is an excellent exercise in spinor algebra to verify this invariance. Note the presence in Eq. 28 of \(\gamma^5\)-terms; to obtain invariance one must assume that \(\gamma^5\) is totally anti-commuting. Of course in this particular case we could set \(D^a = 0\), and still have an invariance (not involving \(\gamma^5\)). However this does not escape the Siegel ambiguity, as we shall now show. With due care, one obtains (up to total derivatives)

$$\delta L_S = g\frac{1}{2} f^{abc} \tau_{\gamma \mu} \lambda^a \lambda^b \gamma_\mu \lambda^c.$$  

(29)

This is identically zero for \(d=4\), though this is not obvious even if we rewrite in two–component formalism; a Fierz re-ordering is required. For \(d \neq 4\), \(\delta L_S\) is not zero; and the key to the distinction between DRED and DREG lies in the \(\gamma_\mu \otimes \gamma_\mu\) contraction, which is \(d\)-dimensional for DREG and four-dimensional for DRED. There are important consequences for the regularisation of supersymmetric theories, as we shall now see. Let us add to \(L_S\) gauge fixing and ghost terms:

$$L_S \rightarrow L_T = L_S + L_G$$  

(30)

where

$$L_G = -\frac{1}{2\alpha} (\partial_\mu W_\mu)^2 + C^{a*} \partial_\mu D_\mu^{ab} C^b.$$  

(31)

Then we introduce the functional \(Z(J, j, j_D)\) where

$$Z = \int d\{W_\mu\} d\{\lambda\} d\{D\} e^{i \int d^d x \left[ L_T + J^\mu W_\mu + \bar{\lambda} j_D + j_\mu \right]}$$  

(32)

and use of Eq. 28 leads to the following Ward identity:

$$0 = < \int d^d x \left[ J^\mu \delta W_\mu + \bar{\lambda} \delta \lambda + j_D \delta D + \delta L_S + \delta L_G \right] >$$  

(33)

where

$$\delta L_G = -\frac{1}{\alpha} \partial_\mu W_\mu \partial^\mu \delta W_\mu + gf^{abc} C^{a*} \partial^\mu \delta W_\mu^{c} C^b$$  

(34)
and

\[ < X > = \int d\{W_µ\}d\{\lambda\}d\{D\}Xe^{i{\int d^dx[L_T + j^{\mu}W_\mu + \bar{\lambda}j_\mu D + J_\mu]}}. \]  

(35)

Notice we have included the term \(\delta L_S\) from Eq. 29 to allow for the fact that this may not be zero away from \(d = 4\). When this Ward identity was investigated at one loop, it was found to be true with DRED and false with DREG; which conclusion was arrived at because the contribution from \(\delta L_S\) was ignored. It is easy to show that this contribution is zero for DRED, and in the case of DREG serves precisely to restore Eq. 34. The distinction between DRED and DREG is manifest in the fact that the contribution of \(\delta L_S\) is zero in the former case. It is in this sense that DRED is more consistent with supersymmetry. In terms of this Ward identity the difference may not seem crucial, but the fact that \(\delta L_S\) is effectively non-zero (with DREG) means that if we employ DREG in a supersymmetric theory then great care must be taken with the formulation of physical predictions. This becomes particularly clear when we generalise to include matter fields; in supersymmetric QCD, for example, the fact that the gluino-quark-squark coupling is equal to the gauge coupling \(g\) is a consequence of supersymmetry and will not be preserved under renormalisation if DREG is employed. In fact, this is very similar to the problems that occur when we want to apply DRED to non–supersymmetric theories; once again there are coupling constant relations that are not preserved by radiative corrections.

While DRED was successful in the above application, it does not follow that an insertion of \(\delta L_S\) in a diagram of arbitrary complexity gives zero. It can be shown\(^2\) that such an insertion depends on the quantity \(\Delta\) where

\[ \Delta = \text{Tr}(A\gamma^{\mu}B\gamma_{\mu}) + \text{Tr}(A\gamma^{\mu})\text{Tr}(B\gamma_{\mu}) - (-1)^k\text{Tr}(A\gamma^{\mu}B^R\gamma_{\mu}). \]  

(36)

Here \(A\) and \(B\) are products of Dirac \(\gamma\)-matrices; \(k\) is the number of such matrices in \(B\), and \(B^R\) consists of the the same set of matrices as \(B\) but written down in reverse order. In strict \(d = 4\), \(\Delta\) is zero; but because \(A, B\) may contain \(d\)-dimensional \(\gamma\)-matrices (due to contraction with momenta) \(\Delta\) is non-zero in general. If we set

\[ A = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_5} \quad \text{and} \quad B = \gamma_{\nu_1}\gamma_{\nu_2}\cdots\gamma_{\nu_5} \]  

(37)

then

\[ \Delta = 48\delta^{[\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_5}_{\nu_5]} \]  

(38)

where the brackets \([\cdots]\) denote antisymmetrisation. This is clearly zero for integer \(d \leq 4\) but if the various indices are \(d\)-dimensional then it is not. For instance if we calculate the trace by contracting \(\Delta\) with \(\delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_5}_{\nu_5}\) then we obtain \(\text{Tr}\Delta = 48d(d-1)(d-2)(d-3)(d-4)\). A diagram with at least four
loops is required so that we get enough \( \gamma \)-matrices to activate this problem in the propagator Ward identity. This is clearly the same ambiguity at bottom addressed by Siegel.

One might hope that this problem is somehow resolved by use of superfields. Using superfield perturbation theory Feynman rules maintains supersymmetry in a manifest way; but a crucial part of the calculational procedure relies on the reduction of products of supercovariant \( D_\alpha \) and \( \bar{D}_\dot{\alpha} \) derivatives to products of four or less, and this is possible only when the fact that the \( \alpha, \dot{\alpha} \) indices are two-valued is used. Thus the same ambiguity must be present, albeit in a somewhat different form. However as we argued in the previous section, the ambiguity will not affect physical results since it is equivalent to the freedom available in choice of regularisation scheme, as long as a systematic procedure is adopted. Despite all difficulties, DRED remains the regulator of choice for supersymmetric theories, and has survived many practical tests.

5 \( N = 2 \) and \( N = 4 \) supersymmetry

\( N = 2 \) supersymmetry corresponds, in the language of \( N = 1 \) superfields, to the special case of a superpotential taking the form:

\[
W = \sqrt{2} g^{\alpha \dot{\alpha}} \xi^T S_\alpha \chi, \tag{39}
\]

where \( \xi, \chi, \phi \) are multiplets transforming under the \( S^*, S \) and adjoint representations of the gauge group \( G \) respectively. In the special case that \( S \) is the adjoint representation we have \( N = 4 \) supersymmetry. \( N = 2 \) theories are extraordinary in that they have only one-loop divergences. This means that for \( N = 2 \), \( \beta_g \) vanishes beyond one-loop if computed using DRED; crucial here is the fact that DRED incorporates minimal subtraction. In the \( N = 4 \) case the one-loop contribution also vanishes, so \( N = 4 \) theories are ultra-violet finite to all orders of perturbation theory.

\( N = 2 \) and \( N = 4 \) theories, although obviously of great interest, possess a property that is unfortunate if we want to try and incorporate them into a realistic theory. This property is that since the chiral superfields are either adjoint or in \( S, S^* \) pairs, gauge invariant mass terms are possible for the fermionic components of the multiplets. It is difficult, therefore, to arrange for fermion masses (such as the electron mass) to be much less than the scale of supersymmetry–breaking, at least. Nevertheless there have been occasional attempts to construct phenomenologically viable models and explore their consequences.
6  Non–supersymmetric theories

We saw in section 2 that under renormalisation the $\epsilon$-scalars behave differently from the gauge fields, except in supersymmetric theories. On might be tempted to assert that it doesn't matter if Green’s functions with external $\epsilon$-scalars are divergent (since they are anyway unphysical) and introduce a common wave function subtraction for $W_i$ and $W_\sigma$, a wave function subtraction for $\psi$ and a coupling constant subtraction for $g$, these being determined (as usual) by the requirement that Green's functions with real particles be rendered finite. This was the procedure adopted in the main by van Damme and ’t Hooft. On the other hand we could insist on all Green’s functions (including those with external $\epsilon$-scalars) being finite, leading to the introduction of a plethora of new subtractions or equivalently coupling constants. We have shown that it is only the latter procedure which leads to a consistent theory; the former manifestly breaks unitarity.

Now in a supersymmetric theory the complications described above can be safely ignored. The wave function renormalisations of $W_\sigma$ and $W_i$ are equal because of supersymmetry, and the evanescent couplings remain equal to their “natural” values after renormalisation. At first sight, this conclusion also appears to obtain when supersymmetry is softly broken, since the dimensionless couplings renormalise exactly as in the fully supersymmetric theory. This is not quite true, however, since there is nothing to protect the $\epsilon$–scalars acquiring a mass through interacting with the genuine fields; and, indeed, precisely this happens. In other words, the $\beta$–function for the $\epsilon$-scalar mass $\tilde{m}$ is inhomogeneous with respect to $\tilde{m}$:

$$\beta_{\tilde{m}^2} = A(g,Y)\tilde{m}^2 + \sum_i B_i(g,Y)m_i^2 + \cdots ,$$

(40)

where the $m_i^2$ are the genuine scalar masses, $Y$ represents the Yukawa couplings and the $+\cdots$ denotes terms involving the gaugino mass(es) and the $A$-parameter(s). Moreover, the two–loop $\beta$–functions for the genuine scalar masses depend explicitly on the $\epsilon$–scalar masses, when calculated using DRED. This fact would annoyingly complicate an extension to two loops of the standard running analysis relating the low energy values of the various soft parameters to the corresponding values at gauge unification. Fortunately, however, there exists a hybrid scheme which decouples this $\epsilon$–scalar dependence both from the $\beta$–functions and from the threshold corrections to the physical masses. At leading order, this scheme is arrived at from DRED by redefining the masses $m_i^2$ as follows:

$$m_i^2|_{\text{DRED}^\prime} = m_i^2|_{\text{DRED}} - C_i(g)\tilde{m}^2$$

(41)
where $C_i(g)$ is easily calculated\[31\]. The resulting $\beta_{m^2}$ is independent of $\tilde{m}^2$ through two loops; and conveniently the same transformation removes the $\tilde{m}^2$ term from the one-loop relationship between the renormalised and physical scalar masses $m^2$. So in the DRED’ scheme, although $\tilde{m}$ evolves under the renormalisation group, it is decoupled from physical quantities and so can be safely ignored. To sum up, the DRED’ procedure for the standard running analysis (at two loops) and extraction of predictions for physical masses is:

1. Use the DRED’ $\beta$-functions\[30,35\] to do the running analysis.
2. Calculate the one loop corrections to convert the renormalised masses to the physical (pole) masses using DRED but with $\tilde{m}^2 = 0$. The top quark mass would not in any case have $\tilde{m}$ dependence, but note that the result for the one–loop gluon contribution to it is

\begin{align}
m^\text{pole}_t &= m_t(\mu) \left[ 1 + \frac{\alpha_3(\mu)}{3 \pi} \left( 5 - 3 \ln \frac{m^2_t}{\mu^2} \right) \right] \quad \text{(42)} \\

\text{not} \quad m^\text{pole}_t &= m_t(\mu) \left[ 1 + \frac{\alpha_3(\mu)}{3 \pi} \left( 4 - 3 \ln \frac{m^2_t}{\mu^2} \right) \right] \quad \text{(43)}
\end{align}

as in DREG.

7 The NSVZ $\beta$-function

In this section we examine the $\beta$-functions for an $N = 1$ supersymmetric theory defined by the superpotential

\begin{equation}
W = \frac{1}{6} Y^{ijk} \Phi_i \Phi_j \Phi_k + \frac{1}{2} \mu^{ij} \Phi_i \Phi_j.
\end{equation}

The multiplet of chiral superfields $\Phi_i$ transforms as a representation $R$ of the gauge group $G$, which has structure constants $f_{abc}$. In accordance with the non-renormalisation theorem\[36\] the $\beta$-functions for the Yukawa couplings $\beta^Y_{ijk}$ are given by

\begin{equation}
\beta^Y_{ijk} = Y^{p(ij)k}_p = Y^{ijk}_p \gamma_p + (k \leftrightarrow i) + (k \leftrightarrow j),
\end{equation}

where $\gamma(g, Y)$ is the anomalous dimension for $\Phi$. There exists an all-orders relation between the gauge $\beta$-function $\beta_g(g, Y)$ and $\gamma$ which was first derived using instanton calculus\[37\]:

\begin{equation}
\beta^{\text{NSVZ}}_g = \frac{g^3}{16 \pi^2} \left[ \frac{Q - 2 \pi^{-1} \text{Tr} \left[ \gamma^{\text{NSVZ}} C(R) \right]}{1 - 2C(G)g^2(16 \pi^2)^{-1}} \right].
\end{equation}
Here $Q = T(R) - 3C(G)$, $T(R)\delta_{ab} = \text{Tr}(R_a R_b)$, $C(G)\delta_{ab} = f_{acd}f_{bde}$, $r = \delta_{aa}$ and $C(R)^j_j = (R_a R_a)^j_j$.

We have added a “NSVZ” label to both $\beta_g$ and $\gamma$ in Eq. 46 because of scheme dependence issues which we will discuss shortly. In the special case $Y = 0$, the fixed point $g^* = 0$, defined by

$$Q = \frac{2}{r} \text{Tr} [\gamma^{\text{NSVZ}} C(R)] \quad (47)$$

is important for duality, in the context of the conformal window identified by Seiberg. We will return to this fixed point in the context of large-$N$ expansions.

It turns out that if $\beta_g$ and $\gamma$ are calculated using DRED, then they begin to deviate from Eq. 46 at three loops. The relationship between $\beta_g^{\text{NSVZ}}$ and $\beta_g^{\text{DRED}}$ has been explored recently, with the conclusion that there exists an analytic redefinition of $g$, $g \rightarrow g'(g, Y)$ which connects them. We emphasise that it is quite non-trivial that the redefinition exists at all; in the abelian case for example, the redefinition consists of a single term, but it affects four distinct terms (with different tensor structure) in the $\beta$-functions. By exploiting the fact that $N = 2$ theories are finite beyond one loop, it was possible to determine $\beta_g^{\text{DRED}}$ at three loops by a comparatively simple calculation, and at four loops in the general case except for one undetermined parameter. What one learns from this is that it is highly non-trivial that the DRED and NSVZ results correspond to schemes which can be related in this manner. Use of what regularisation scheme would lead to the NSVZ result, which is associated with the holomorphic nature of the Wilsonian action? Presumably, for example, a combination of Pauli-Villars and higher derivatives. Notwithstanding the existence of the exact NSVZ result, however, it is still important to have $\beta_g^{\text{DRED}}$ as accurately as possible, because in calculating physical predictions DRED (or more accurately DRED') is the scheme most often used. The three loop results for $\beta_g^{\text{DRED}}$ and $\gamma^{\text{DRED}}$ have found phenomenological applications. The results for $\beta_g$ in supersymmetric QCD (SQCD) with $N_f$ flavours and $N_c$ colours are:
\begin{align*}
16\pi^2 \beta^{(1)}_g &= (N_f - 3N_c) g^3, \\
(16\pi^2)^2 \beta^{(2)}_g &= \left( \left[ 4N_c - \frac{4}{N_c^2} \right] N_f - 6N_c^2 \right) g^5, \\
(16\pi^2)^3 \beta^{(3)}_g &= \left( \left[ \frac{3}{N_c} - 4N_c \right] N_f^2 + \left[ 21N_c^2 - \frac{2}{N_c} - 9 \right] N_f - 21N_c^3 \right) g^7.
\end{align*}

For \( \beta^{(4)}_g \) we have only a partial result:

\begin{align*}
(16\pi^2)^4 \beta^{(4)}_g &= \left( -\frac{2}{3N_c} N_f^3 + \left[ \frac{100}{3} + 4\alpha + \frac{6\alpha - 20}{3N_c^2} - \left( \frac{6\alpha}{3} + 2\kappa + 8\alpha \right) N_c^2 \right] N_f^2 \\
&\quad + \left[ 36(1 + \alpha)N_c^3 - (34 + 12\alpha)N_c - \frac{8}{N_c} - \frac{1}{N_c^2} \right] N_f \\
&\quad - (6 + 36\alpha)N_c^4 \right) g^9
\end{align*}

where \( \alpha \) is an as yet undetermined parameter, and where \( \kappa = 6\zeta(3) \). (A recent application of the method of asymptotic Padé approximants suggests that \( \alpha \approx 2.4 \).)

It is very interesting that the higher order group theory invariants found by van Ritbergen et al. in the corresponding calculation for QCD do not appear here. Of course the QCD calculation was done with DREG rather than DRED; but since these group structures first appear at four loops we would expect, for these particular terms, that DRED and DREG should give the same result at this order. It is an excellent check on both calculations, therefore, that when in the QCD case we go to the special case of \( N = 1 \) supersymmetry (by setting \( N_f = \frac{1}{2} \) and putting the fermions in the adjoint representation) the new invariants cancel. It is also interesting to note that in the pure gauge theory, these invariants signalled the first contribution from non-planar structures to \( \beta_g \) in QCD; for SQCD, it remains possible that \( \beta^{DRED}_g \) is free of such structures to all orders (this is manifestly so for \( \beta^{NSVZ}_g \) in the absence of chiral superfields, of course).

Recently, some exact results for soft supersymmetry-breaking masses and couplings have been derived by Hisano and Shifman using the holomorphy of the Wilsonian action. Soft breaking terms may be accommodated within the superfield formalism by the introduction of an external “spurion” field \( \eta \equiv \theta^2 \). The renormalisation-group functions for soft breaking parameters
may all be derived from the anomalous dimension $\gamma^n$ of the chiral fields in the presence of the spurion $\eta$. $\gamma^n$ may be expanded as

$$\gamma^n = \gamma + \gamma^{[1]} \eta + \gamma^{[1]} \bar{\eta} + \gamma^{[2]} \eta \bar{\eta},$$

where $\gamma$ is the conventional anomalous dimension for $\Phi$, in the absence of the spurion. Furthermore, simple rules may be derived for obtaining $\gamma^n$ directly from $\gamma$. It is possible to derive from the exact results of Hisano and Shifman an elegant formula for the $\beta$-function for the gaugino mass $M$, namely

$$\beta_M^{\text{NSVZ}} = \frac{2}{g^2} \left[ \frac{M \beta_g^{\text{NSVZ}} - 2g^3(16\pi^2 r)^{-1} \text{Tr} [\gamma^{[1]} \text{NSVZ} C(R)]}{1 - 2C(G)g^2(16\pi^2)^{-1}} \right],$$

where $\gamma^{[1]}$ is as defined in Eq. 50. This result is strikingly similar in form to the primordial NSVZ result for $\beta_g$ in Eq. 46. The other soft breaking $\beta$-functions are also simply related to $\gamma^{[1]}$, as follows (from now on we suppress the “NSVZ” label):

$$\beta^{ijk}_h = \gamma^{[i]} [h^{jk}] - 2\gamma^{[1]} [Y^{jk}] \gamma^{[i]}$$

and

$$\beta^{ij}_b = \gamma^{[i]} [b^{j}] - 2\gamma^{[1]} [\mu^{j}] \gamma^{[i]}$$

where $h^{ijk}$ and $b^{ij}$ are the soft $\phi^3$ and $\phi^2$ interactions, respectively.

8 Large-$N_f$ supersymmetric gauge theories

The large-$N$ expansion is an alternative to conventional perturbation theory. In both QCD and SQCD, the large $N_c$ expansion is of particular interest; more tractable, however, is the large $N_f$ expansion. Recently the leading and $O(1/N_f)$ terms in $\beta_g$ and $\gamma$ have been calculated (using DRED) for a number of supersymmetric theories. We give below the results for SQCD (noting that we have rescaled the gauge coupling, $g \rightarrow g/\sqrt{N_f}$):

$$\gamma = -\frac{(N_f^2 - 1)}{N_f N_c} \hat{K} G(\hat{K}),$$

and

$$\beta_g = g \hat{K} - \frac{3N_f}{N_f^2} g \hat{K} + 4g \hat{K} \frac{N_f}{N_f} \int_0^{\hat{K}} (1 - x) G(x) \, dx$$

$$- \frac{2g \hat{K}}{N_c N_f} \int_0^{\hat{K}} (1 - 2x) G(x) \, dx.$$
where \( \hat{K} = g^2/(16\pi^2) \), and
\[
G(x) = \frac{\Gamma(2 - 2x)}{\Gamma(2 - x)\Gamma(1 - x)^2\Gamma(1 + x)},
\]
(56)

These results do not satisfy Eq. 46, because they were calculated using DRED. It is quite remarkable that the \( O(1/N_f) \) corrections to the SQCD \( \beta \)-function depend only on simple integrals involving \( G(x) \). \( G \) has a simple pole at \( x = 3/2 \) and consequently \( \beta_g \) has a logarithmic singularity at \( g^2 = 24\pi^2 \) and a finite radius of convergence in \( g \). Using Eq. 55 for \( N_f = 6 \) and \( N_c = 3 \), values which lie in the conformal window \( 3N_c/2 < N_f < 3N_c \), we indeed find an infra-red fixed point in the gauge coupling evolution, corresponding to \( g^* \approx 8 \). It is interesting that the range of \( N_f \) such that \( g^* < 24\pi^2 \) is given by
\[
aN_c < N_f < 3N_c,
\]
where \( a \approx 1.7 \) depends weakly on \( N_c \). This is remarkably close to the exact conformal window.

Of course the result for \( g^* \) is scheme dependent. In the NSVZ scheme it is possible to show that the \( O(1/N_f) \) contribution to \( \gamma \) is in fact the same as in DRED, with the corresponding result for \( \beta_g \) being easily calculated from Eq. 46. One then finds that the fixed point (for \( N_c = 3 \)) corresponds to \( g^* \approx 7 \).

Of course it is not clear that this regime is within the region of validity of our approximation: do we believe that the appropriate expansion parameter is \( N_c/N_f \) or \( 3N_c/N_f \)? It would obviously be interesting if we could calculate more terms in the \( 1/N_f \) expansion. Even the \( O(1/N_f^2) \) contribution presents considerable technical problems, however.

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