The direct link model for polymer rings as a topological field theory and the second topological moment in dense systems

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Abstract

Polymer rings in solution are either permanently entangled or not. Permanent topological restrictions give rise to additional entropic interactions apart from the ones arising due to mere chain flexibility or excluded volume. Conversely, entangled polymer rings systems may be formed by closing randomly entangled flexible linear chains. The dependance of linking numbers between randomly entangled rings on the chain length, more specifically the second topological moment $\langle n^2 \rangle$, i.e. the average squared linking number, may be determined. In this paper, an approach recently discussed in mathematical physics and called abelian BF theory, is presented which allows to express the linking constraint in its simplest form, the Gauss integral, in terms of two gauge fields. The model of Brereton and Shah for a single ring entangled with many other surrounding rings is rederived. The latter model is finally used to calculate the second topological moment, in agreement with a recent result by Ferrari, Kleinert, and Lazzizzera obtained by using $n$-component $\phi^4$ theory in the limit $n \to 0$.

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1 Introduction

The topology of knots and links has attracted the interest of mathematicians and physicists alike (see for a general introduction e.g. [1, 2, 3, 4, 5, 6]). In polymer physics, the role of topological constraints has been discussed since its beginnings [7, 8]. On the level of a self-avoiding walk description for the polymer conformation, the Gauss invariant (defined below) was discussed by Edwards who also considered the role of higher order link invariants (HOLC). In [9], the abelian Chern-Simons theory (however not under the present term which is taken from the authors of [10]) was used to express the self-linking number of a single ring. In his discussion of HOLC, Edwards pointed to the role of a 3-vertex connecting 3 gauge fields, a remark anticipating the non-abelian Chern-Simons theory which was used by Witten [11] some 20 years later to obtain polynomial invariants for knots and links. These algebraic invariants which are usually defined by mathematical knot theorists in terms of recursion relations (skein relations) with respect to the crossings appearing in the 2-dimensional projections of knots and links have also been applied to polymers (for this important approach which will not explored here, the reader is referred to [6]). The conjecture by Edwards has been validated by the work of Cotta-Ramusino, Guadagnini, et al. (see [2, 3] and references therein) and by Axelrod et al., and by Alvarez and Labastida (see [13] and references therein), who show that the Gauss invariant and all HOLC appear in a perturbation series of the non-abelian Chern-Simons theory when the vacuum expectation value of products of Wilson loops is evaluated. This result suggests to treat the Gauss invariant as an approximate measure of the true topology.

The concept of the abelian variant of Chern-Simons field theory was reconsidered by many authors, among them [12, 14, 15, 16, 17, 18], also for many rings systems, as it arises naturally whenever the Gaussian linking number is enforced as a constraint in the partition function. Partial reviews of the subject are [4, 6, 2]. The non-abelian variant has been discussed only recently by some authors [12, 4, 15], however its implementation in terms of a well motivated physical model for entangled polymers is not so clear.

Whenever the abelian Chern-Simons theory is used in the context of the Gauss integral, an important problem arises in terms of self-linking integrals, where the double line integration is not taken with respect to two different ring contours but with respect to a single ring contour. These integrals attain a meaning only when using the so-called vertical framing, i.e. by replacing the single ring by a band. For details and the relevant literature we refer the reader to [2].

Following a suggestion of Thompson (see e.g. the review [20]), an alternative topological quantum field theory, termed BF theory [21], is used in the present paper. In contrast to Chern-Simons theory, it avoids the appearance of self-linking integrals, and was introduced, at least to the knowledge of the author, to the subject of polymer topology in [15]. Recently, BF theory was rediscovered in [18, 19] which is also concerned about avoiding self-linking integrals. In agreement with existing literature on topological quantum field theory as done in [20], the term BF theory will be used to underline the difference to Chern-Simons theory explained above.

In the present paper, the direct link model for a system of \( n_p \) rings of [22] is used. It singles out a single ring, say \( C_1 \) and enforces a linking number \( n \) of this
test ring with all other rings $C_\beta \in \{C_2, \ldots, C_{n_p}\}$ in the system. The Gauss invariant expressing the linking number as a double line integral is then represented in terms of an abelian BF theory. Next, the conformational degrees of freedom of all rings but $C_1$ are summed over, giving rise to a “mass term” to be added to the BF theory. Then, one gauge field is integrated over, and the field theory of Brereton and Shah [22] is obtained, expressed in terms of one remaining gauge field and the conformational field $r^1(s)$, where $s$ parametrizes the polymer contour. This theory is then used to evaluate what has recently been called the second topological moment [18], $\langle n^2 \rangle$. Essentially, the purpose of this paper is two-fold: one, to inform polymer physicists of a new approach (BF theory) to tackle entangled polymers, and two, to present a very simple derivation of the second topological moment in dense systems.

The outline of the paper is as follows. In section 2, the direct link model is presented and expressed as an abelian BF theory. The model of Brereton and Shah is derived. In section 3, the second topological moment is evaluated. Finally, in the concluding section the relation to similar work is discussed and a simple scaling argument is evoked to interpret the result.

2 The direct link model as a BF theory

Let the conformation of the single test ring and all other rings be given by their coordinates $\{r^1(s)\}$ and $\{r^\beta(s)\}$ where $\beta = 2, \ldots, n_p$ respectively. Now let $n$ be the linking number of the ring $C_1$ with all other rings $C_\beta$

\[
 n = \sum_{\beta \neq 1}^{n_p} \Phi(C_1, C_\beta). \tag{1}
\]

where

\[
 \Phi(C_\alpha, C_\beta) = \frac{1}{4\pi} \oint_{C_\alpha} \oint_{C_\beta} dr^\alpha \wedge dr^\beta \cdot \frac{r^\alpha - r^\beta}{|r^\alpha - r^\beta|^3} \tag{2}
\]

is the Gaussian linking number. As discussed in the literature [4] [5], the double line integral on the r.h.s. is a topological invariant, i.e. is invariant with respect to all deformations of the rings given by their coordinates $\{r^\alpha\}$ and $\{r^\beta\}$ which do not alter the topological state. However, this invariant is not unique. A counterexample is the Whitehead link [6] which represents a linked state between two rings and has linking number $\Phi(C_\alpha, C_\beta) = 0$, the same value which results for a pair of unlinked rings. However, the Whitehead is maintained by self-interactions of one ring with itself. For random walk rings, these self-interactions effectively disappear, and thus the Whitehead link is unknotted to a pair of unlinked rings.

The partition function of the system with a given linking number $n$ reads as:

\[
 Z(n) = Z_0 \left\langle \delta(n, \sum_{\beta \neq 1}^{n_p} \Phi(C_1, C_\beta)) \right\rangle \tag{3}
\]

The Kronecker delta appearing in Eq.(3) may be expressed in terms of the integral

\[
 \delta(n, f) = \int_{-\pi}^{\pi} \frac{dg}{2\pi} e^{ign - igf} \tag{4}
\]
giving rise to a topological “charge” $g$ conjugate to $n$. One obtains the corresponding conjugate partition function $Z(g)$:

$$Z(g) = \left\langle \exp \left( ig \sum_{\beta \neq 1} \Phi(C_1, C_{\beta}) \right) \right\rangle \{r^1(s), \{r^\beta(s)\}\}$$  \hspace{1cm} (5)

The parameter $g$ may therefore be considered as chemical potential for linking numbers. In what follows, the conformational model for the rings is restricted to closed random walk chains. The excluded volume effect is neglected. As discussed in [15], this simplification restores the uniqueness of the Gauss invariant for links.

The argument inside the average in Eq.(5) may now be expressed (in the spirit of a Hubbard-Stratonovich transformation) in terms of an abelian BF theory as follows [20]:

$$Z(g) = N \int \mathcal{D}A \int \mathcal{D}B \exp \left( i \int B \wedge dA \right) \left\langle \exp \left( ig \oint_{C_1} dr^1 \cdot A \right) \right\rangle \{r^1(s)\} \left\langle \exp \left( i \int B \cdot j \right) \right\rangle \{r^\beta(s)\}$$  \hspace{1cm} (6)

The constant $N$ is a normalization. The wedge product $\wedge$ is an exterior product which generates the algebra of differential forms over $\mathbb{R}^3$, $\Omega^*(\mathbb{R}^3)$ [23]. In this language $A$ and $B$ are 1-forms, and the invariant expression $B \wedge dA$ is a 3-form. In terms of local coordinates it reads as $\epsilon_{\lambda\mu\nu} B_\lambda \partial_\mu A_\nu dx^1 \wedge dx^2 \wedge dx^3$. Therefore, no volume element appears in the integration. Line integrations are specified explicitly. The notation given above is widely used in the field theory literature [20], and is used here to make manifest the coordinate invariance of the field theory. Later on, when evaluating 2-point functions, we return to explicit coordinate representations of the fields. The variable $j(x)$ is the tangent vector density, simply called the “current”, of the chains $\beta = 2..n_p$

$$j(x) = \sum_\beta \oint_0^N ds \dot{r}^\beta(s) \delta(x - r^\beta(s)).$$  \hspace{1cm} (7)

The conformational averages with respect to the test ring $C_1$ and with respect to the other rings factorize for random walk chains. The conformation of the surrounding rings are represented in terms of the current $j$. In fact, they have fused to a single effective ring. As a reminder, the appearance of self-linking numbers has been avoided.

Next, the conformational coordinates for the surrounding rings $C_1$ for $\beta \neq 1$ are summed over. Then, the functional integral with respect to the gauge field $B$ is performed. First, the average with respect to the ring chains $C_{\beta}$ is carried out. For random walk rings one obtains:

$$\left\langle \exp \left( i \int B \cdot j \right) \right\rangle \{r^\beta(s)\} = \exp \left( -\frac{1}{2} \int dx \int dx' B_{\mu}(x)B_{\nu}(x') \left\langle j_{\mu}(x)j_{\nu}(x') \right\rangle \right).$$  \hspace{1cm} (8)

On the r.h.s. of this equation the chain index has been omitted. The average of the correlation function for the currents at points $x$ and $x'$ is carried out in the
limit of very long rings. Let us denote the number of segments per ring by $N$ and the average segment length by $l$. Neglecting $1/N$ terms arising due to the closure constraint \cite{27} for the surrounding rings, the current-current correlation function is given by the following expression (the same approximation is made in \cite{22}):

$$\langle j_\mu(x)j_\nu(x') \rangle = \delta_{\mu\nu} \frac{\rho l^2}{d} \delta(x - x')$$

where $\rho = (n_p N)/V$ is the average segment density. Eq.(9) is an approximation valid for sufficiently concentrated systems. The question may be raised for these systems whether the Gauss invariant is still valid. Certainly the probability for more complicated links, such as the Borromean rings (a 3 ring link which falls apart when cutting one ring), which are only detected by HOLC, is higher but remains low compared to the Gauss invariant measuring the pairwise entanglement of rings. In order to determine the second topological moment given in the next section, these considerations are irrelevant as the Gauss invariant is simply an observable of the system (see below).

Let $G = \rho l^2/3$, then the partition function reads as

$$Z(g) = \mathcal{N} \int \mathcal{D}A \int \mathcal{D}B \exp \left( i \int B \wedge dA - \frac{G}{2} \int B \cdot B \right) \left\langle \exp \left( ig \oint_{C_1} dr^1 \cdot A \right) \right\rangle_{\{r^1(s)\}}$$

where $\mathcal{N}$ is an adjusted normalization factor. Now the integration with respect to the gauge fields $B$ may be carried out, which yields the effective model of Brereton und Shah \cite{22}:

$$Z(g) = \mathcal{N} \int \mathcal{D}A \exp \left( - \frac{1}{2G} \int (\nabla \wedge A)^2 \right) \left\langle \exp \left( ig \oint_{C_1} dr^1 \cdot A \right) \right\rangle_{\{r^1(s)\}}$$

Leaving aside for the moment the average with respect to the ring $C_1$ and using the language of quantum field theory, one is left with a so-called Wilson loop that is averaged with respect to a euclidean Yang-Mills theory in 2 + 1 dimensions. Wilson loops are considered in quantum field theory to study the confinement problem. An analogy to this problem has been used in \cite{24} to study the collapse transition of randomly entangled polymer rings in 2D (for earlier studies see \cite{23,26}).

3 The second topological moment

The second topological moment $\langle n^2 \rangle$ for a single test ring entangled with many other rings given the restriction that all conformational averages are taken with respect to random walks may now be determined rather simply. Following \cite{18,19} $\langle n^2 \rangle$ may be determined from:

$$\langle n^2 \rangle = - \frac{\partial^2}{\partial g^2} Z(g) \big|_{g=0}$$

5
More generally \( Z(g) \) is the generating function for all topological moments. It is formulated for the present purpose as follows:

\[
Z(g) = \left\langle \exp \left( ig \oint_{C_o} \, d\sigma \, \mathbf{A} \right) \right\rangle_{\{\mathbf{A}\}, \{\mathbf{r}(s)\}}
\]

(13)

The normalization factor has been absorbed into the average with respect to \( \mathbf{A} \). To separate the gauge fields and the conformational coordinates, the line integral in Eq.(13) is expressed as follows:

\[
\oint_C \, d\mathbf{r} \cdot \mathbf{A} = \int_k A_\mu(k) \oint_C \, ds \, \dot{r}_\mu(s) e^{ik \cdot \mathbf{r}(s)}
\]

(14)

The summation convention is understood. The abbreviation \( \int_k \) represents \( \int d^3k/(2\pi)^3 \). In order to perform the functional integral with respect to \( \mathbf{A} \), a gauge has to be used, which in the present case is \( \nabla \cdot \mathbf{A} = 0 \). In this case the 2-point correlation function for \( \mathbf{A} \) in Fourier space is given by:

\[
\left\langle A_\mu(k) A_\nu(q) \right\rangle_{\{\mathbf{A}\}} = \delta(k+q) \frac{G}{k^2} \left( \delta_{\mu\nu} - \hat{k}_\mu \hat{k}_\nu \right)
\]

(15)

where \( \hat{k}_\mu = k_\mu/|k| \) are the components of a unit vector. In order to perform the conformational average, the only expression needed is

\[
\Phi_{\mu\nu}(s_1, s_2; k) = \left\langle \dot{r}_\mu(s_1) \dot{r}_\nu(s_2) e^{ik \cdot (\mathbf{r}(s_1) - \mathbf{r}(s_2))} \right\rangle
\]

(16)

This average has been evaluated for random walks rings [27] and reads as follows:

\[
\Phi_{\mu\nu}(s_1, s_2; k) = \left[ \frac{l^2}{3} \delta_{\mu\nu} \left( \delta(s_1 - s_2) - \frac{1}{N} \right) + \frac{l^4}{9N} k_\mu k_\nu |s_1 - s_2| \left( 1 - \frac{|s_1 - s_2|}{N} \right) \right] \\
\times \exp \left[ -\frac{l^2 k^2}{6} |s_1 - s_2| \left( 1 - \frac{|s_1 - s_2|}{N} \right) \right]
\]

(17)

The \( k \)-dependent terms in the first factor on the RHS of this equation do not give any contributions due to gauge invariance. This is most obvious in the Landau gauge, \( k \cdot \mathbf{A}(k) = 0 \).

Now, the second topological moment is given by (using a summation convention for repeated indices):

\[
\left\langle n^2 \right\rangle = -\left\langle \int_k \int_q A_\mu(k) A_\nu(q) \oint ds_1 \oint ds_2 \dot{r}_\mu(s_1) \dot{r}_\nu(s_2) e^{ik \cdot \mathbf{r}(s_1) - i q \cdot \mathbf{r}(s_2)} \right\rangle_{\{\mathbf{A}\}, \{\mathbf{r}(s)\}}
\]

(18)

Using the correlators (13) and (17), one obtains the expression:

\[
\left\langle n^2 \right\rangle = \frac{1}{k^2} \int ds_1 \int ds_2 \left[ \frac{l^2}{3} \left( \delta_\mu_\nu - 1 \right) \left( \delta(s_1 - s_2) - \frac{1}{N} \right) \right] \\
\times \exp \left[ -\frac{l^2 k^2}{6} |s_1 - s_2| \left( 1 - \frac{|s_1 - s_2|}{N} \right) \right]
\]

(19)

The last equation gives rise to 2 terms. The first stems from the delta function \( \delta(s_1 - s_2) \) and is readily evaluated. The second needs further analysis: if \( s_1 = s_2 \),
the exponential equals 1, and one needs to evaluate an integral in k space which depends on a cutoff $\Lambda \sim \xi^{-1}$; if $s_1 \neq s_2$, the integration with respect to $s_1$, $s_2$ gives a non-trivial contribution to the integration in k-space, the details of which are discussed in the appendix. The result is when inserting for $G$

$$\langle n^2 \rangle = \rho l^3 \left( \frac{NI\Lambda}{9\pi^2} - \frac{\sqrt{N}}{\sqrt{2}(3\pi)^3} - \frac{l\Lambda}{9\pi^2} \right) \tag{20}$$

where $\Lambda = \xi^{-1}$ is an inverse cutoff length. As the result has been obtained in the limit $N \gg 1$, the last term which is independent of $N$ may be dropped, so the final result in the large $N$ limit reads as:

$$\langle n^2 \rangle \simeq \rho l^3 \left( \frac{NI\xi^{-1}}{6\pi^2} - \frac{\sqrt{N}}{\sqrt{3}(2\pi)^3} \right) \tag{21}$$

Apart from the numerical prefactors, the result is identical to the one derived from an n-component field theory in the limit $n \to 0$ in [18]. The leading scaling term in Eq.(21) was also found by Brereton and Shah [22] who calculated the generating function $Z(g)$ directly in the conformational space without the use of gauge fields, using a pre-averaging procedure as a further approximation. Their result contains however the square root of the ratio $l/\xi$ in contrast to the present result which is a minor difference and might be due to the approximation used in their work.

4 Discussion

The topological constraint of fixed linking number of a test ring entangled with $n_p$ surrounding rings has been implemented within the framework of a random walk model for the polymer conformation, by using the simplest link invariant, the Gauss integral. The latter one has been reformulated in terms of a topological quantum field theory, the so-called abelian BF theory. Two results have been obtained: first, the model of Brereton and Shah [22] has been derived, second, the second topological moment in the limit of large segment numbers $N \gg 1$ has been calculated, in agreement with the previous result by Ferrari, Kleinert, and Lazzizzera [18, 19]. The same result may also be obtained from directly averaging the squared Gauss integral with respect to a random walk conformation, a work which will be presented elsewhere. The gauge field approach, however, is conceptually very appealing as it separates topological interactions from conformational entropy.

The present method avoids the complicated n-component field theory used in [18]. The latter approach is, however, better suited to treat the full problem including excluded volume interactions. On the other hand, the Gauss invariant is ambiguous for self-avoiding walk rings. Considering work by Moroz and Kamien [30] it remains an open problem whether a topological field coupled to the n-component field for the polymer conformation changes the random walk result significantly. The authors consider self-avoiding walks with writhe. In fact, a chemical potential for writhe is introduced which gives the coupling constant for the interaction between a topological gauge field and conformational n-component field, in a very similar way as in [18]. The scaling behavior of the radius of gyration and the first two moments
of the writhe are calculated. As to the first question, the effect of writhe is found to be irrelevant (to one-loop), i.e. the self-avoiding walk fix point corresponding to the exponent $\nu \simeq 0.588$ remains unchanged. Essentially this result is due to the fact that a topological field theory lacks a scale. Concerning moments of the writhe (which is similar to calculating moments of $n$ as done above), no scaling dependence on the number of polymer segments is found which is a surprising result. A clarification of this issue is certainly necessary. Let us note that Kholodenko and Vilgis \cite{31} calculated the writhe of semiflexible polymers which is found to scale as $N^{1/2}$.

Finally let us interpret the result Eq.\((21)\) in terms of a simple picture. Obviously the second topological moment basically scales as $\langle n^2 \rangle \sim \rho l^3 N$ as $N$ becomes large. A simple argument, also discussed in \cite{29}, gives a similar result. Let the average segment density $\rho$ inside the test ring be given by the number of rings crossing the interior of the test ring, which is equal to the average linking number per ring $\bar{n}$, times the number of segments $N$, divided by the volume of the test ring $R^3$. Then the density reads as

$$\rho = \frac{\bar{n}N}{R^3} \quad (22)$$

Assuming that the segment density in the interior of the test ring is equal to the segment density elsewhere in the system, and employing $R^3 \sim l^3 N^{3/2}$, i.e. using the Gaussian result for the ring volume, one immediately obtains

$$\bar{n} = \rho l^3 \sqrt[3]{N} \quad (23)$$

Both $\bar{n}$ and $\sqrt{\langle n^2 \rangle}$ scale as $\sqrt[3]{N}$, so the theoretical calculation presented above reproduces the characteristic scaling with chain length derived from the simple argument. The difference between density-dependent prefactors appears to be superficial. In fact in the case of dense melts where $\rho \sim l^{-3}$ and which is the regime where the above argument applies, the prefactors coincide, and $\bar{n} = \sqrt{\langle n^2 \rangle}$.

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Appendix

In the following, the calculation of the r.h.s. of Eq.\((19)\)

$$\langle n^2 \rangle = \frac{2}{3}G \int \frac{1}{k^2} \int ds_1 \int ds_2 \left[ l^2 \left( \delta(s_1 - s_2) - \frac{1}{N} \right) \right]$$

\begin{equation}
\times \exp \left[ -\frac{l^2 k^2}{6} |s_1 - s_2| \left( 1 - \frac{|s_1 - s_2|}{N} \right) \right] \quad (24)
\end{equation}

is given in more detail. The delta function in the integrand gives rise to the integral

$$I_1 = \frac{2}{3}GNl^2 \int \frac{1}{k^2} = \frac{2}{3}GNl^2 \frac{4\pi}{(2\pi)^3} \int_0^\Lambda dk$$

$$= GNl^2 \frac{\Lambda}{3\pi^2} \quad (25)$$
where the $k$ integration has been carried out up to the cutoff parameter $\Lambda \sim \xi^{-1}$.

The $1/N$ term in the integrand on the r.h.s. of Eq. (24) is given by

$$I_2 = \frac{2}{3} G \frac{l^2}{N} \int \frac{1}{k^2} \int d s_1 \int d s_2 \exp \left[ -\frac{l^2 k^2}{6} |s_1 - s_2| \left( 1 - \frac{|s_1 - s_2|}{N} \right) \right]$$

Concerning the integration with respect to $s_1$ and $s_2$ in the last equation, two cases need to be considered: $s_1 = s_2$ and $s_1 \neq s_2$. In the first case, the exponential function gives 1 and the integration gives a factor of $N$, the result being

$$I_{2a} = -G l^2 \frac{\Lambda}{3\pi^2}$$

The case $s_1 \neq s_2$ is slightly more involved. First, let us integrate with respect to $k$, which gives:

$$I_{2b} = -G \frac{l^2}{N} \frac{4\pi}{(2\pi)^3} \int d s_1 \int d s_2 \sqrt{2\pi} \left[ \frac{l^2}{3} |s_1 - s_2| \left( 1 - \frac{|s_1 - s_2|}{N} \right) \right]^{-1/2}$$

Now the domain of integration with respect to $s_1$ and $s_2$ may be restricted to two times one half the original one giving:

$$I_{2b} = -\frac{2}{3} G \frac{l^2}{N} \frac{4\pi}{(2\pi)^3} 2 \int_0^N ds_1 \int_0^{s_1} ds_2 \sqrt{2\pi} \left[ \frac{l^2}{3} (s_1 - s_2) \left( 1 - \frac{(s_1 - s_2)}{N} \right) \right]^{-1/2}$$

A shift of variables $u = (s_1 - s_2)/N$, $v = (s_1 + s_2)/N$ is easily performed and leads to

$$I_{2b} = -\frac{2}{3} G \frac{l^2}{N} \frac{4\pi}{(2\pi)^3} \sqrt{2\pi N^2} \int_0^1 du \int_u^{2u} dv \left[ \frac{l^2}{3} N u (1 - u) \right]^{-1/2}$$

From now on the integration is elementary, and one obtains

$$I_{2b} = -G l N^{1/2} \frac{1}{\sqrt{6\pi^3}}$$

Adding $I_1$, $I_{2a}$, and $I_{2b}$ and inserting for $G = \rho l^3/d$ for $d = 3$ gives the r.h.s. of Eq. (19) in the main text.
References

[1] L.H. Kauffman, *Knots and physics*, World Scientific, Singapore 1993.

[2] E. Guadagnini, *The link invariants of the Chern-Simons field theory*, Walter de Gruyter, Berlin 1993.

[3] N.D. Gilbert, T. Porter, *Knots and surfaces*, Oxford University Press, Oxford 1994.

[4] H. Kleinert, *Path integrals in quantum mechanics, statistics, and polymer physics*, 2nd ed., World Scientific, Singapore 1995.

[5] S.K. Nechaev, *Statistics of knots and entangled random walks*, World Scientific, Singapore 1996.

[6] A.L. Kholodenko, T.A. Vilgis, Phys. Rep. 298, 251 (1998).

[7] E. Wasserman, J. Am. Chem. Soc. 82, 4433 (1960).

[8] H.L. Frisch, E. Wasserman, J. Am. Chem. Soc. 83, 3789 (1962).

[9] S.F. Edwards, J. Phys. A 1, 15 (1968).

[10] S.S. Chern, J. Simons, Ann. Math. 99, 48 (1974).

[11] E. Witten, Commun. Math. Phys. 121, 351 (1989).

[12] A.L. Kholodenko, T.A. Vilgis, J. Phys. (Paris) 4, 843 (1994).

[13] M. Alvarez, J.M.F. Labastida, Nucl. Phys. 395, 198 (1993); *ibid*. 433, 555 (1995).

[14] M.G. Brereton, T.A. Vilgis, J. Phys. A: Math. Gen. 28, 1149 (1995).

[15] M. Otto, “Die statistische Mechanik flexibler, verschlaufter und unverschlafter Polymerringe”, Ph.D. thesis, University of Mainz, 1996. Available at http://www.Theorie.Physik.UNI-Goettingen.DE/~otto/pubs.html.

[16] M. Otto, T.A. Vilgis, Phys.Rev.Lett 80, 881 (1998).

[17] F. Ferrari, I. Lazzizzera, Nucl. Phys. B539, 673 (1999); Phys. Lett. B444, 167 (1998).

[18] F. Ferrari, H. Kleinert, I. Lazzizzera, Phys. Lett. A 276, 31 (2000), cond-mat/0002049; *Calculation of second topological moment $\langle m^2 \rangle$ of two entangled polymers*. Preprint, cond-mat/0003353, available at http://xxx.lanl.gov.

[19] F. Ferrari, H. Kleinert, I. Lazzizzera, *Field theory of N entangled polymers*. Preprint, cond-mat/0005300, available at http://xxx.lanl.gov.

[20] M. Blau, G. Thompson, Ann. Phys. (N.Y.), 205, 130 (1991).
The term “BF” originates from its non-abelian form whose general action is given by (wedge) product of a field $\mathbf{B}$ and field tensor $\mathbf{F}_A$ (corresponding to another field $\mathbf{A}$), $S = \int Tr(\mathbf{B} \wedge \mathbf{F}_A)$. In the abelian case, $\mathbf{F}_A = d\mathbf{A}$.

[22] M.G. Brereton, S. Shah, J. Phys. A: Math. Gen. 13, 2751 (1980).

[23] M. Nakahara, Geometry, topology and physics, Adam Hilger, Bristol 1990.

[24] M. Otto, T.A. Vilgis, J. Phys. A: Math. Gen. 29, 3893 (1996).

[25] S.K. Nechaev, V.G. Rostiashvili, J. Phys. II France 3, 91 (1993).

[26] V.G. Rostiashvili, S.K. Nechaev, T.A. Vilgis, Phys. Rev. E 48, 3314 (1993).

[27] M.G. Brereton, T.A. Vilgis, Phys. Rev. A 45, 7413 (1992).

[28] M.G. Brereton, S. Shah, J. Phys. A: Math. Gen. 15, 985 (1982).

[29] T.A. Vilgis, M.Otto, Phys.Rev. E Rap.Comm. 56 R1314 (1997).

[30] J.D. Moroz, K.D. Kamien, Nucl.Phys. B506 695 (1997).

[31] A.L. Kholodenko, T.A. Vilgis, J. Phys. A: Math. Gen.29, 939 (1996).