# l-PROPERNESS OF MABUCHI’S K-ENERGY

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**Abstract.** Over the space of Kähler metrics associated to a fixed Kähler class, we first prove the lower bound of the energy functional \( \tilde{E}^β \), then we provide the criterions of the geodesics rays to detect the lower bound of \( \tilde{J}^β \)-functional. They are used to obtain the properness of Mabuchi’s K-energy. The criterions are examined under \( \| . \|_{L^1} \) by showing the convergence of the negative gradient flow of \( \tilde{J}^β \)-functional.

## Contents

1. Introduction
2. Variational structure of \( \tilde{J} \) and \( \tilde{E} \)
3. Geodesics in the space of Kähler potentials
4. A functional inequality of \( \tilde{J}^β \) and \( \tilde{E}^β \)
5. Proof of Theorem 1.1
6. Proof of Theorem 1.2
7. Proof of Theorem 1.3
8. Geodesic stability
9. Proof of Theorem 1.6
10. Proof of Theorem 1.4
11. Proof of Theorem 1.5
11.1. Lower bound of the 2nd derivatives
11.2. Upper bound of the 2nd derivatives
11.3. Zero order estimate
References

## 1. Introduction

Let \( M \) be a compact Kähler manifold and \( \Omega \) be an arbitrary Kähler class. We choose a Kähler metric \( \omega \in \Omega \) and denote the space of Kähler potentials associated to \( \Omega \) by

\[
\mathcal{H}_\Omega = \{ \varphi \in C^\infty(M, \mathbb{R}) \mid \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \}.
\]

Mabuchi’s K-energy \[18\] has the explicit formula (cf. \[5\], \[25\]) for any \( \varphi \in \mathcal{H}_\Omega \),

\[
\nu_\omega(\varphi) = E_\omega(\varphi) + \mathfrak{S} \cdot D_\omega(\varphi) + j_\omega(\text{Ric}(\omega), \varphi).
\]

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In which,

\[ E_\omega(\varphi) = \int_M \log \frac{\omega^n}{\omega^n_\varphi}, \]

\[ D_\omega(\varphi) = \frac{1}{V} \int_M \varphi \omega^n - J_\omega(\varphi), \]

\[ J_\omega(\varphi) = \frac{\sqrt{-1}}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_i \wedge \omega^n_{\varphi}^{n-1-i}. \]

and

\[ j_\omega(\text{Ric}(\omega), \varphi) \]

\[ = \frac{-1}{V} \sum_{i=0}^{n-1} \frac{n!}{(i+1)!(n-i-1)!} \int_M \varphi \cdot \text{Ric}(\omega) \wedge \omega^{n-1-i} \wedge (\sqrt{-1} \partial \bar{\partial} \varphi)^i. \]

We also recall Aubin’s \( I \)-function,

\[ I_\omega(\varphi) = \frac{1}{V} \int_M \varphi (\omega^n - \omega^n_\varphi) = \frac{\sqrt{-1}}{V} \sum_{i=0}^{n-1} \int_M \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_i \wedge \omega^n_{\varphi}^{n-1-i}. \]

The properness of the \( K \)-energy \( \nu_\omega(\varphi) \) is a kind of "coercive" condition in the variational theory. It was introduced in Tian \[19\], which states that there is a nonnegative, non-decreasing function \( \rho(t) \) satisfying \( \lim_{t \to \infty} \rho(t) = \infty \) such that \( \nu_\omega(\varphi) \geq \rho(I_\omega(\varphi)) \) for all \( \varphi \in \mathcal{H}_\Omega \). It is conjectured to be equivalent to the existence of the constant scalar curvature Kähler (cscK) metrics (Conjecture 7.12 in Tian \[25\]).

When \( \Omega = -C_1(M) \) or \( C_1(M) = 0 \), the function \( \rho \) is proved to be linear in Tian \[25\], Theorem 7.13, i.e. there are two positive constants \( A \) and \( B \) such that for all \( \varphi \in \mathcal{H}_\Omega \),

\[ (1.2) \quad \nu_\omega(\varphi) \geq AI_\omega(\varphi) - B. \]

In order to destine different notions of properness, in our paper, we say the \( K \)-energy is \( I \)-proper, if \( (1.2) \) holds.

When \( \Omega = C_1(M) > 0 \) and there is no holomorphic vector field on a Kähler-Einstein manifold \( M \), Phong-Song-Sturm-Weinkove \[21\] proved that Ding functional \( F_\omega(\varphi) \) (defined in Ding \[8\]) satisfies

\[ F_\omega(\varphi) \geq AI_\omega(\varphi) - B. \]

This inequality is a generalisation of the Moser-Trudinger inequalities on the sphere \[20\] \[19\] \[26\]. The \( I \)-properness of Ding functional also implies \( (1.2) \) by using the identity between \( \nu_\omega(\varphi) \) and \( F_\omega(\varphi) \) in Ding-Tian \[9\], we include the proof in Lemma \[10.2\] for readers’ convenience.

There are different notions of properness. In \[7\], Chen defined another properness of the \( K \)-energy regarding to the entropy \( E_\omega(\varphi) \). The equivalent relation between the \( I \)-properness and the \( E \)-properness is discussed in \[17\]. Chen also suggest another properness which means that the \( K \)-energy bounds the geodesic distance function. He furthermore conjectured that \( d \)-properness should be a necessary condition of the existence of the cscK or the general extremal Kähler metrics (see Conjecture/Question 2 in \[5\] and Conjecture/Question 6.1 in \[6\]).
Let $\chi$ be a closed $(1,1)$-form. The $J$-functional is defined to be the last two terms of the $K$-energy with $\text{Ric}(\omega)$ replaced by $\chi$,

$$J_{\omega,\chi}(\varphi) = G \cdot D_{\omega}(\varphi) + _J\omega(\chi, \varphi).$$

We introduce a new parameter $\beta$ within a range $0 \leq \beta < \frac{n}{n+1} - \alpha$.

We then define a new functional to be

$$\tilde{J}^\beta_{\omega,\chi}(\varphi) = J_{\omega,\chi}(\varphi) + \beta J_{\omega}(\varphi). \quad (1.3)$$

Now we return back to the formula of the $K$-energy. With the notations above it is split into

$$\nu_{\omega}(\varphi) = E_{\omega}(\varphi) - \beta J_{\omega}(\varphi) + \tilde{J}^\beta_{\omega,\text{Ric}(\omega)}(\varphi). \quad (1.4)$$

The lower bound of $E_{\omega}(\varphi)$ is $\alpha I_{\omega}(\varphi) - C$ in Lemma 10.1. Inserting it into the $K$-energy, we arrive at the lower bound

$$\nu_{\omega}(\varphi) \geq \alpha I_{\omega}(\varphi) - C - \beta J_{\omega}(\varphi) + \inf_{\varphi \in \mathfrak{H}_{\Omega}} \tilde{J}^\beta_{\omega,\text{Ric}(\omega)}(\varphi). \quad (1.5)$$

Note that $I$-functional is equivalent to the $J$-functional,

$$\frac{1}{n+1}I_{\omega}(\varphi) \leq J_{\omega}(\varphi) \leq \frac{n}{n+1}I_{\omega}(\varphi),$$

then we have

$$\nu_{\omega}(\varphi) \geq (\alpha - \frac{n\beta}{n+1})I_{\omega}(\varphi) - C + \inf_{\varphi \in \mathfrak{H}_{\Omega}} \tilde{J}^\beta_{\omega,\text{Ric}(\omega)}(\varphi). \quad (1.6)$$

From this inequality, we observe that in order to prove the $I$-properness of the $K$-energy, it suffices to obtain the lower bound of the functional $\tilde{J}^\beta_{\omega,\text{Ric}(\omega)}$.

The critical points of $\tilde{J}^\beta_{\omega,\chi}$ satisfy a new fully nonlinear equation in $\mathfrak{H}_{\Omega}$,

$$n \cdot \chi \wedge \omega_{\varphi}^{n-1} = c_{\beta} \cdot \omega_{\varphi}^n + \frac{\beta}{V} \omega^n. \quad (1.6)$$

The constant $c$ is a topological constant determined by

$$c_{\beta} = n \frac{[\chi] \cdot \Omega^{n-1}}{\Omega^n} - \frac{\beta}{V}.$$

We call such $\omega_{\varphi}$ a $\tilde{J}^\beta$-metric. We say that $\chi$ is semi-definite if it is negative semi-definite or positive semi-definite.

In these degenerate situation, (1.6) might have more than one solution. We first prove the lower bound the $\tilde{J}^\beta$-functional, when there is a $\tilde{J}^\beta$-metric in $\Omega$.

**Theorem 1.1.** Assume that $\chi$ is negative semi-definite (positive semi-definite) and there is a $\tilde{J}^\beta$-metric in $\Omega$, then all $\tilde{J}^\beta$-metrics has the same critical value and $\tilde{J}^\beta$ has lower (resp. upper) bound.

There is another functional $\tilde{E}^\beta$ which is defined to be the square norm of the derivative of $\tilde{J}^\beta$,

$$\tilde{E}^\beta(\varphi) = \frac{1}{V} \int_M (c_{\beta} - \text{tr}_{\omega_{\varphi}} \chi + \frac{\beta}{V} \omega_{\varphi}^n)^2 \omega_{\varphi}^n. \quad (1.7)$$
The $\tilde{J}^\beta$-function and the $\tilde{E}^\beta$-functional play the roles as the $K$-energy and the Calabi energy in the study of extremal Kähler metrics. We next prove the lower bound of $\tilde{E}^\beta$.

When $\chi$ is semi-definite, according to the 2nd variation formula of $\tilde{J}^\beta$ in (2.1), it is convex or concave along a $C^{1,1}$ geodesic ray $\rho(t)$. Thus the limit of its first derivative along $\rho(t)$ exists

$$F^\beta(\rho) = \lim_{t \to \infty} \frac{1}{V} \int_M \frac{\partial c^\beta_{\rho}}{\partial t} (c^\beta_{\rho} - \text{tr}_\omega \chi + \frac{\beta}{V} \omega_{\rho}^n) \omega_{\rho}^n.$$  

(1.8)

We require the following notions of the geodesic ray in the space of Kähler potentials.

**Definition 1.1.** We say a $C^{1,1}$ geodesic ray is

- stable (semi-stable) if $\tilde{J}^\beta > 0$ ($\tilde{J}^\beta \geq 0$);
- destabilising (semi-destabilising) if $\tilde{J}^\beta < 0$ ($\tilde{J}^\beta \leq 0$);
- effective if $\lim \sup_{t \to \infty} \tilde{E}^\beta(\rho(t)) \cdot \frac{1}{t^2} = 0$.

**Theorem 1.2.** Assume that $\chi$ is negative semi-definite. The following inequality holds.

$$\inf_{\omega \in \Omega} \sqrt{\tilde{E}^\beta} \geq \sup_{\rho} (-\tilde{J}^\beta).$$

(1.9)

The supreme is taking over all $C^{1,1}$, effective, semi-destabilising geodesic $\rho$.

We remark that when $\beta = 0$ and $\chi$ and $\omega$ are both algebraic, the lower bound of $E^0$ was proved in Lejmi and Székelyhidi [15] in algebraic setting.

We then prove the lower bound of $\tilde{J}^\beta$ without the existence of $\tilde{E}^\beta$-metric.

**Theorem 1.3.** Suppose that $\chi$ is negative semi-definite. Assume that $\tilde{J}^\beta$ is bounded from below along a $C^{1,1}$ semi-destabilising geodesic ray and the infimum of the energy $\tilde{E}^\beta$ is zero along this ray. Then $\tilde{J}^\beta$ is uniformly bounded from below in the entire Kähler class $\Omega$.

The tool we use here to obtain these lower bounds is based on Chen [7][6]. The proof relies on the existence of the geodesic rays and the nonpositive curvature property of the infinite dimensional space $H_{\Omega}$. In general, it is difficult to examine the lower bound of functionals in an infinite dimensional space, however, Theorem 1.3 provides a method to examine it along only one geodesic ray.

Furthermore, we apply Theorem 1.3 to the $K$-energy. When $C_1(M) < 0$, according to Aubin-Yau’s solution of the Calabi conjecture [29][1], there exists a unique Kähler metric $\omega_0$ such that $\text{Ric}(\omega_0)$ represents the first Chern class. So let

$$\chi = \text{Ric}(\omega_0)$$

could be chosen to be $< 0$. We obtain the following criterion of the $I$-properness of the $K$-energy.

**Theorem 1.4.** Assume that there is a $C^{1,1}$ semi-destabilising geodesic ray $\rho(t)$ such that along $\rho(t)$

1. $\tilde{J}^\beta$ is bounded from below,
2. the infimum of the energy $\tilde{E}^\beta$ is zero.
Then the $K$-energy is $I$-proper.

When $\Omega$ admits a $\tilde{J}$-metric $\varphi$, the trivial geodesic ray $\rho(t) = \varphi, \forall t \geq 0$ provides such geodesic ray required in this theorem, since $\tilde{J} = 0$, the first condition follows from Theorem 1.1 and the second one follows from Theorem 1.2.

One way to obtain the critical metric of $J$-functional is its negative gradient flow. It was introduced in Chen [5] and also in Donaldson [10] from moment map picture. Theorem 1.1 in Song-Weinkove [22] showed that under the following condition of a Kähler class $\Omega$, that is, if there is a Kähler metric $\omega \in \Omega$ such that $-\chi > 0$ and $(-c_0 \cdot \omega + (n-1)\chi) \wedge \omega^{n-2} > 0$, the negative gradient flow of $\tilde{J}$-functional converges. Thus $I$-properness 1.2 holds when $\chi = \text{Ric}(\omega_0) \in C^1(M) < 0$ and $(-c_0 \cdot \omega + (n-1)\text{Ric}(\omega_0)) \wedge \omega^{n-2} > 0$. We extend their theorem to the negative gradient flow of $\tilde{J}$-functional

$$\frac{\partial \varphi}{\partial t} = - c_\beta + \frac{n \chi \wedge \omega^{n-1}}{\omega^V} - \frac{\beta \omega^n}{\omega^\beta},$$

and prove its convergence in Proposition 11.2 under the condition,

$$-\chi > 0 \text{ and } (-c_\beta \cdot \omega + (n-1)\chi) \wedge \omega^{n-2} > 0.$$

The extra term involving $\beta$ on the flow equation brings us trouble when we apply the second order estimate. In order to overcome this problem, we calculate a differential inequality by using the linear elliptic operator $L$ defined in (11.5) and apply the maximum principal.

We remark that 1.6 and its flow have been generalised in different directions [14][13][12][16]... which is far from a complete list.

Thus we verify the criterion in Theorem 1.4

**Theorem 1.5.** Assume that there is a $\omega \in \Omega$ such that

$$(-c_\beta \cdot \omega + (n-1)\text{Ric}(\omega_0)) \wedge \omega^{n-2} > 0.$$

Then from any Kähler potential $\varphi \in \mathcal{H}_\Omega$, there exists a $C^{1,1}$ semi-destabilising geodesic ray satisfying (1) and (2). Thus the $K$-energy is $I$-proper in $\Omega$.

Paralleling to Donaldson’s conjecture of existence of the cscK metrics (Conjecture/Question 12 in [11]), we propose a notion called geodesic stability w.r.t to the $\tilde{J}$-functional (see Definition 8.1). We at last link the existence of $\tilde{J}$-metric to this geodesic stability.

**Theorem 1.6.** Suppose that $\chi$ is negative semi-definite. Assume that $\Omega$ contains a $\tilde{J}$-metric $\varphi$, then $\Omega$ is geodesic semi-stable at $\varphi$ and moreover, it is weak geodesic semi-stable.

The criterion 8.1 means that along the geodesic ray, the first derivative of the $\tilde{J}$-functional is strictly increase. The question 8.1 suggests that there is no such geodesic ray satisfying 8.1 implies the existence of $\tilde{J}$-metric. Then from Theorem 1.1 and 1.3, the $K$-energy is $I$-proper. So according to Tian’s conjecture (Conjecture 7.12 in [25]), there exists cscK metrics. In this sense, the question 8.1 probably provides another possible point of view of Donaldson’s conjecture (Conjecture/Question 12 in [11]).
Lemma 2.1. The 1st variation of $\tilde{\beta}$-functional is
$$\delta \tilde{\beta} (\varphi) = \frac{1}{V} \int_M \varphi [c_\beta \cdot \omega^\varphi_n - n \cdot \chi \wedge \omega^\varphi_{n-1} + \frac{\beta}{V} \omega^n].$$

Proof. We compute
$$\delta \tilde{\beta} (\varphi) = \frac{1}{V} \int_M \varphi (c_\beta \cdot \omega^\varphi_n - n \cdot \chi \wedge \omega^\varphi_{n-1}) + \beta \int_M \varphi (\omega^n - \omega^\varphi_n)$$
$$= \frac{1}{V} \int_M \varphi [(c_\beta - \frac{\beta}{V}) \cdot \omega^\varphi_n - n \cdot \chi \wedge \omega^\varphi_{n-1} + \frac{\beta}{V} \omega^n].$$

Lemma 2.2. The 2nd variation of $\tilde{\beta}$-functional is
(2.1) $$\delta^2 \tilde{\beta} (\varphi, \dot{\varphi}) = \frac{1}{V} \int_M \dot{\varphi} [c_\beta - tr_\varphi \chi] \omega^\varphi_n - \frac{1}{V} \int_M \chi_{ij} \dot{\varphi} \beta_{ij} \omega^\varphi_n.$$
Therefore, when $\chi$ is strictly negative (positive), the $\tilde{J}^\beta$-metric is local minimum (maximum).

**Proposition 2.3.** When $\chi$ is strictly negative or strictly positive, the $\tilde{J}^\beta$-metric is unique up to a constant.

**Proof.** Assume $\varphi_1$ and $\varphi_2$ are two $\tilde{J}^\beta$-metrics. Then connecting them by the $C^{1,1}$ geodesic. Since all the computation above is well-defined along the $C^{1,1}$ geodesics, (2.1) implies that

$$\delta^2 \tilde{J}(\dot{\varphi}, \dot{\varphi}) = -\frac{1}{V} \int_M \chi_{ij}(\omega) \dot{\varphi}^i \dot{\varphi}^j \omega^n_\varphi.$$ 

Then integrating from 0 to 1, we have

$$\frac{1}{V} \int_0^1 \int_M \chi_{ij}(\omega) \dot{\varphi}^i \dot{\varphi}^j \omega^n_\varphi dt = \delta \tilde{J}(1) - \delta \tilde{J}(0) = 0.$$

Hence, $\dot{\varphi}$ is constant and $\varphi_1$ and $\varphi_2$ differ by a constant. $\Box$

We use the notion

$$\tilde{H} = \text{tr}_{\omega_\varphi} \chi - c^\beta - \beta \frac{\omega^n_\varphi}{\omega^n_{\tilde{\varphi}}}.$$

The $\tilde{J}^\beta$-metric is a Kähler metric satisfying

$$\tilde{H} = 0.$$

We define the energy $\tilde{E}^\beta$ as

$$\tilde{E}^\beta(\varphi) = \frac{1}{V} \int_M (\text{tr}_{\omega_\varphi} \chi - c^\beta - \beta \frac{\omega^n_\varphi}{\omega^n_{\tilde{\varphi}}})^2 \omega^n_\varphi.$$ 

Then we have

$$\delta \tilde{H}(\dot{\varphi}) = -\dot{\varphi}^i \chi_{ij} + \beta \frac{\omega^n_{\tilde{\varphi}}}{\omega^n_{\varphi}} \Delta_\varphi \dot{\varphi}.$$ 

**Lemma 2.4.** The 1st derivative of the modified energy $\tilde{E}$ is

$$\delta \tilde{E}^\beta(\dot{\varphi}) = \frac{2}{V} \int_M \tilde{H} \dot{\varphi}^i \chi_{ij} \omega^n_\varphi - \frac{2\beta}{V^2} \int_M \tilde{H} \dot{\varphi}^i \omega^n_\varphi.$$ 

**Proof.** We calculate that

$$\delta \tilde{E}^\beta(\dot{\varphi}) = \frac{2}{V} \int_M \tilde{H} (-\dot{\varphi}^j \chi_{ij} + \beta \frac{\omega^n_{\tilde{\varphi}}}{\omega^n_{\varphi}} \omega^n_{\varphi} + \frac{1}{V} \int_M \tilde{H} \Delta_\varphi \dot{\varphi} \omega^n_{\varphi}).$$

The first term is

$$\frac{2}{V} \int_M \tilde{H} \dot{\varphi}^i \chi_{ij} \omega^n_\varphi + \frac{2}{V} \int_M \tilde{H} \dot{\varphi}^i (\chi_{ij})^\tilde{\varphi} \omega^n_\varphi = \frac{2}{V} \int_M \tilde{H} \dot{\varphi}^i \chi_{ij} \omega^n_\varphi + \frac{2}{V} \int_M \tilde{H} \dot{\varphi}^i (\tilde{H} + \frac{\beta}{V} \frac{\omega^n_{\tilde{\varphi}}}{\omega^n_{\varphi}}) \omega^n_\varphi.$$

While, the second term is

$$\frac{2}{V} \int_M \tilde{H} \beta \frac{\omega^n_{\tilde{\varphi}}}{\omega^n_{\varphi}} \omega^n_\varphi = -\frac{2}{V} \int_M \tilde{H} \dot{\varphi}^i \chi_{ij} \omega^n_\varphi - \frac{2}{V} \int_M \tilde{H} \frac{\beta}{V} \frac{\omega^n_{\tilde{\varphi}}}{\omega^n_{\varphi}} \omega^n_\varphi.$$
and the third term is
\[-\frac{2}{V} \int_M \tilde{H} g^i_j \tilde{H}_j \phi \omega^n_{\phi}\]
which cancels the second component in the first term.

The critical points of $\tilde{E}$ satisfy that
\[ [\tilde{H}^i_i \chi_{ij} \omega^n_{\phi} - \frac{\beta}{V} \tilde{H}_i \omega^n_{i}] = 0. \]

**Lemma 2.5.** The 2nd derivative of the modified energy $\tilde{E}^\beta$ is
\[
\delta^2 \tilde{E}^\beta(u, v) = \frac{2}{V} \int_M (u^p \chi_{p ij})(u^q \chi_{q ij}) \omega^n_{\phi} + \frac{2\beta}{V^2} \int_M g^i_j (\Delta_v \omega^n_{\phi}) k g^i_j u_i \chi_{ij} \omega^n_{\phi} + \frac{2\beta}{V^2} \int_M g^i_j (\Delta_v \omega^n_{\phi}) u_i \chi_{ij} \omega^n_{\phi}.
\]

**Proof.** In the local coordinate, (2.3) is written as
\[
\delta \tilde{E}^\beta(u) = \frac{2}{V} \int_M g^i_j \tilde{H}_k g^i_j u_i \chi_{ij} \omega^n_{\phi} = \frac{2\beta}{V^2} \int_M g^i_j \tilde{H}_i u_i \omega^n_{\phi},
\]
we obtain that
\[
\delta^2 \tilde{E}^\beta(u, v) = -\frac{2}{V} \int_M v^k \tilde{H}_k g^i_j u_i \chi_{ij} \omega^n_{\phi} + \frac{2}{V} \int_M g^i_j u_i \chi_{ij} \omega^n_{\phi} + \frac{2\beta}{V^2} \int_M g^i_j \tilde{H}_i u_i \omega^n_{\phi} - \frac{2\beta}{V^2} \int_M \delta^2 u_i \chi_{ij} \omega^n_{\phi}.
\]
The second term is further reduced to,
\[
-\frac{2}{V} \int_M g^i_j (u^p \chi_{p ij}) k g^i_j u_i \chi_{ij} \omega^n_{\phi} = -\frac{2}{V} \int_M (u^p \chi_{p ij}) \tilde{H}_i u_i \chi_{ij} \omega^n_{\phi} = \frac{2}{V} \int_M (u^p \chi_{p ij}) u^i \chi_{ij} \omega^n_{\phi} + \frac{2}{V} \int_M (u^p \chi_{p ij}) u^i \tilde{H}_i \omega^n_{\phi}.
\]
Thus the lemmas holds by inserting this formula into (2.3).

When $\beta = 0$, the variational structure of $\tilde{E}^0$ is studied in Chen [4]. We denote
\[ H = tr \omega \chi - c_0. \]
The Kähler metric is called a $\tilde{J}$-metric if it satisfies $H = 0$. From (2.3), the 1st derivative of $\tilde{E}^0$-energy is
\[
\delta \tilde{E}^0(\phi) = \frac{2}{V} \int_M H^j_j \phi \chi_{ij} \omega^n_{\phi}.
\]
From this formula, the critical metrics satisfy the equation
\[
[H^j_j] = 0.
\]
The critical metrics of the modified energy include the $\tilde{J}$-metrics. (2.4) shows that, at the critical point of $\tilde{J}$,
\[ \delta^2 \tilde{E}^0(u,v) = 2 \sqrt{\int_M (u^{pq} \chi_{pq})(u^{ij} \chi_{ij}) \omega_{\varphi}^{n}}. \]
So the $\tilde{J}$-metric is local minimiser of $\tilde{E}^0$. However, it is not known whether all the critical metrics of the energy $\tilde{E}^0$ are minimisers. While, (2.6) suggests that when $\chi$ is strictly positive or negative, the critical metrics of the modified energy is the $\tilde{J}$-metric.

3. Geodesics in the space of Kähler potentials

We recall the necessary progress of constructing the geodesic ray in this section for the next several sections. the existence of the $C^{1,1}$ geodesic segment is proved in Chen [7]. In Calamai-Zheng [3], we improve the following existence of the geodesic segment with slightly weaker boundary geometric conditions. Now we specify the geometric conditions on the boundary metrics.

Definition 3.1. We label as $H_C \subset H_\Omega$ one of the following spaces;

$J_1 = \{ \varphi \in H_\Omega \text{ such that } \sup \text{Ric}(\omega_{\varphi}) \leq C \}$;

$J_2 = \{ \varphi \in H_\Omega \text{ such that } \inf \text{Ric}(\omega_{\varphi}) \geq C \}$.

Theorem 3.1. (Calamai-Zheng [3]) Any two Kähler metrics in $H_C$ are connected by a unique $C^{1,1}$ geodesic. More precisely, it is the limit under the $C^{1,1}$-norm by a sequence of $C^\infty$ approximate geodesics.

Due to Calabi-Chen [2], $H$ has positive semi-definite curvature in the sense of Aleksandrov. Two geodesic ray $\rho_i$ are called paralleling if the geodesic distance between $\rho_i(t)$ and $\rho_2(t)$ is uniformly bounded.

Lemma 3.2. Given a geodesic ray $\rho(t)$ in $H_C$ and a Kähler potential $\varphi_0$ which is not in $\rho(t)$. There is a $C^{1,1}$ geodesic ray starting from $\varphi_0$ and paralleling to $\rho(t)$.

Proof. According to Theorem 3.1 we could connect $\varphi_0$ and $\rho(t)$ by a $C^{1,1}$ geodesic segment $\gamma_t(s)$ which have uniform $C^{1,1}$ norm. Thus after taking limit of the parameter $t$, we obtain a limit geodesic ray in $W^{2,p}, \forall p \geq 1$ and $C^{1,\alpha}, \forall \alpha < 1$,
\[ \gamma(s) = \lim_{t \to \infty} \gamma_t(s). \]

Remark 3.1. The condition of $\rho(t)$ could be weakened to be the tamed condition in Chen [7]. We only require that there is a $\tilde{\rho}(t) \in H_C$ and $\tilde{\rho}(t) - \rho(t)$ is uniformly bounded.

4. A functional inequality of $\tilde{J}^\beta$ and $\tilde{E}^\beta$

We first prove a functional inequality.

Proposition 4.1. Let $\varphi_0$ and $\varphi_1$ be two Kähler potentials then the following inequality holds.
\[ \tilde{J}^\beta(\varphi_1) - \tilde{J}^\beta(\varphi_0) \leq d(\varphi_0, \varphi_1) \cdot \sqrt{\tilde{E}^\beta(\varphi_1)}. \]
Proof. The functional inequality is proved by direct computation. Let \( \rho(t) \) be a \( C^{1,1} \) geodesic segment connecting \( \varphi_0 \) and \( \varphi_1 \).

\[
\tilde{J}^\beta(\varphi_1) - \tilde{J}^\beta(\varphi_0) \leq \int_0^1 d\tilde{J}^\beta \left( \frac{\partial \rho}{\partial t} \right)_{\varphi_1} dt \\
\leq \sqrt{\frac{1}{V}} \int_M H^2 \omega^\beta_{\varphi_1} \cdot \sqrt{\int_0^1 \int_M \left( \frac{\partial \rho}{\partial t} \right)^2 \omega^\beta_{\varphi_1} dt}.
\]

Thus the resulting inequality follows from the Hölder inequality. \( \square \)

5. PROOF OF THEOREM 1.1

Proof. Let \( \varphi_1 \) be any Kähler potential in \( \mathcal{H}_\Omega \) and \( \varphi_0 \) be a \( \tilde{J}^\beta \)-metric. Connecting \( \varphi_1 \) and \( \varphi_0 \) by a \( C^{1,1} \) geodesic segment \( \gamma(t) \) and computing the expansion formula along \( \gamma(t) \)

\[
\tilde{J}^\beta(1) - \tilde{J}^\beta(0) = \int_0^1 \frac{\partial \tilde{J}^\beta}{\partial t} dt \\
= \int_0^1 \frac{\partial \tilde{J}^\beta}{\partial t} (t) - \frac{\partial \tilde{J}^\beta}{\partial t} (0) dt \\
= \int_0^1 \int_0^t \frac{\partial^2 \tilde{J}^\beta}{\partial t^2} ds dt.
\]

In the second identify we use the assumption that \( \varphi_0 \) is a \( \tilde{J}^\beta \)-metric, so

\[
\frac{\partial \tilde{J}^\beta}{\partial t} (0) = 0.
\]

Applying the 2nd formula of the \( \tilde{J}^\beta \), Lemma 2.1, we see that

\[
(\tilde{J}^\beta)'' \geq 0
\]

along \( \gamma(t) \). As a result, we obtain that

\[
\tilde{J}^\beta(1) \geq \tilde{J}^\beta(0).
\]

Furthermore, assume that \( \varphi_1 \) is another \( \tilde{J}^\beta \)-metric when the solution is not unique, then we have

\[
\tilde{J}^\beta(1) \geq \tilde{J}^\beta(0).
\]

Switching the positions of \( \varphi_0 \) and \( \varphi_1 \), we see that all \( \tilde{J}^\beta \)-metrics has the same critical value of \( \tilde{J}^\beta \). \( \square \)

6. PROOF OF THEOREM 1.2

Proof. Let \( \rho(t) \) be a geodesic ray parameterized by the arc length and satisfy the assumption in the theorem. Let \( \varphi_0 \) be a Kähler potential outside \( \rho(t) \) and connecting \( \varphi_0 \) and \( \rho(t) \) by a \( C^{1,1} \) geodesic \( \gamma(s) \) which is also parameterized by the arc length. Let \( \theta \) be the angle expanding by \( \overrightarrow{\rho(t)\rho(0)} \) and \( \overrightarrow{\rho(t)\varphi(0)} \).

Since \( \mathcal{H}_\Omega \) is nonpositive curve, we obtain

\[
d(\varphi_0, \rho(0)) \geq d
\]

by comparing the cosine formulae in the Euclidean space

\[
d^2 = d^2(\varphi_0, \rho(t)) + d^2(\rho(0), \rho(t)) - 2d(\varphi_0, \rho(t))d(\rho(0), \rho(t)) \cos \theta.
\]
Then knowing that
\[ d(\rho(0), \rho(t)) = t, \]
and letting \( d_t = d(\varphi_0, \rho(t)) \) be the distance between \( \varphi_0 \) and \( \rho(t) \), we have
\[
\begin{align*}
    d_0^2 &\geq d_t^2 + t^2 - 2d_t \cdot t \cdot \cos \theta \\
    &= d_t^2 + t^2 - 2d_t \cdot t + 2d_t \cdot t - 2d_t \cdot t \cdot \cos \theta \\
    &\geq 2d_t \cdot t \cdot (1 - \cos \theta).
\end{align*}
\]
Thus the cosine formula implies
\[ 2(1 - \cos \theta) \leq \frac{d_0^2}{t \cdot d_t}. \]
While, the triangle inequality implies that
\[ t - d_0 \leq d_t \leq t + d_0. \]
When \( t \) is sufficient large, we further have
\[ d_0 \leq \frac{t}{2}. \]
Thus
\[
0 \leq 2(1 - (\partial \rho / \partial t, \partial \gamma / \partial s))_{\rho(t)} \\
= 2(1 - \cos \theta) \\n\leq \frac{d_0^2}{t \cdot d_t} \\
\leq \frac{d_0^2}{t \cdot (t - d_0)} \\
\leq \frac{2d_0^2}{t^2}.
\]
Applying the H"{o}lder inequality to
\[
d\tilde{J}^\beta(\partial \gamma / \partial s)_{\rho(t)} \leq d\tilde{J}^\beta(\partial \gamma / \partial s - \partial \rho / \partial t)_{\rho(t)} + d\tilde{J}^\beta(\partial \rho / \partial t)_{\rho(t)},
\]
then using (6.1), we obtain
\[
d\tilde{J}^\beta(\partial \gamma / \partial s)_{\rho(t)} \leq \sqrt{\tilde{E}^\beta(\rho(t))} \sqrt{2 - 2(\partial \gamma / \partial s - \partial \rho / \partial t)_{\rho(t)} + d\tilde{J}^\beta(\partial \rho / \partial t)_{\rho(t)}} \\
\leq \sqrt{\tilde{E}^\beta(\rho(t))} \frac{\sqrt{2} \cdot d_0}{t} + d\tilde{J}^\beta(\partial \rho / \partial t)_{\rho(t)}.
\]
Since \( \rho(t) \) is effective
\[ \tilde{E}^\beta(\rho(t)) = o(t) t^2, \]
the first term becomes \( o(t) \). Then
\[ \tilde{J}^\beta(\partial \gamma / \partial s)_{\rho(t)} \leq o(t) + d\tilde{J}^\beta(\partial \rho / \partial t)_{\rho(t)}. \]
(6.3)
On the other hand, note that \( (\tilde{J}^\beta)’ \) and \( (\tilde{J}^\beta)” \) are well-defined along \( C^{1,1} \) geodesic. When \( \chi \) is negative semi-definite, from Lemma 2.1
\[ (\tilde{J}^\beta)”(\gamma(s)) \geq 0. \]
So
\[ d\tilde{\beta}(\frac{\partial\gamma}{\partial s})\varphi(0) \leq d\tilde{\beta}(\frac{\partial\rho}{\partial t})\rho(t). \]
Thus combining (6.3), we have
\[ d\tilde{\beta}(\frac{\partial\gamma}{\partial s})\varphi(0) \leq o(t) + d\tilde{\beta}(\frac{\partial\rho}{\partial t})\rho(t). \]
Inverting this inequality,
\[ -o(t) - d\tilde{\beta}(\frac{\partial\rho}{\partial t})\rho(t) \leq -d\tilde{\beta}(\frac{\partial\gamma}{\partial s})\varphi(0). \]
The right hand side is controlled by the Hölder inequality again
\[ \sqrt{\tilde{E}_\beta(\varphi_0)} \cdot (\int_M (\frac{\partial\gamma}{\partial s})^2|_{s=0}\omega^n_0) \frac{1}{2} = \sqrt{\tilde{E}_\beta(\varphi)} . \]
The inequality follows from choosing the unit arc-length of \( \gamma \). Taking \( t \to \infty \) on both sides of (6.4),
\[ -\tilde{\beta}(\rho) \leq \sqrt{\tilde{E}_\beta(\varphi_0)} . \]
Thus the theorem follows.

7. PROOF OF THEOREM 1.3

Proof. Since when \( \chi \) is negative semi-definite, \( (\tilde{\beta})'' \geq 0 \) along geodesic ray \( \gamma_t(s) \), \( \frac{\partial \tilde{\beta}}{\partial s} \) is non-decreasing. Then letting \( \tau(t) \) be the length of the \( \gamma_t(s) \), we have
\[ \tilde{\beta}(\rho(t)) - \tilde{\beta}(\varphi_0) = \int_0^{\tau(t)} d\tilde{\beta}(\frac{\partial\gamma}{\partial s})ds \]
\[ \leq \int_0^{\tau(t)} d\tilde{\beta}(\frac{\partial\gamma}{\partial s})\rho(t)ds . \]
From (6.2) in the proof above, we obtain that
\[ d\tilde{\beta}(\frac{\partial\gamma}{\partial s})\rho(t) \leq \sqrt{\tilde{E}_\beta(\rho(t))} \sqrt{2 - 2(\frac{\partial\gamma}{\partial s} \cdot \frac{\partial\rho}{\partial t})\rho(t) + d\tilde{\beta}(\frac{\partial\rho}{\partial t})\rho(t) . \]
(7.1)
From the assumption that \( \rho(t) \) is semi-destabilising, so
\[ d\tilde{\beta}(\frac{\partial\rho}{\partial t})\rho(t) \leq 0 . \]
Putting the inequalities above together, we arrive at
\[ \tilde{\beta}(\rho(t)) - \tilde{\beta}(\varphi_0) \leq \sqrt{\tilde{E}_\beta(\rho(t))} \frac{C \cdot d(\varphi_0, \rho(0))}{t} \tau(t) . \]
Taking limit of \( t \), since
\[ \tau(t) = O(t) \]
and from assumption in Theorem 1.3 along \( \rho(t) \),
\[ \lim_{t \to \infty} \sqrt{\tilde{E}_\beta(\rho(t))} = 0 , \]
we have
\[ \tilde{\beta}(\varphi_0) \geq \lim_{t \to \infty} \tilde{\beta}(\rho(t)) . \]
Thus the theorem follows from the assumption that $\tilde{3}^\beta$ is bounded below along $\rho(t)$. \hfill \Box

8. Geodesic stability

Inspired from the geodesic conjecture of the extremal metrics in Donaldson [11], we proposal a counterpart of $\tilde{3}^\beta$-metric.

Conjecture/Question 8.1. The following are equivalent:
(1) There is no $\tilde{3}^\beta$-metric in $\mathcal{K}_\Omega$.
(2) There is infinite geodesic ray $\varphi(t)$, $t \in [0, \infty)$, in $\mathcal{K}_\Omega$ such that
\begin{equation}
\frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} (c_\beta - \text{tr}_\omega \chi + \frac{\beta}{\sqrt{\omega}} \omega^n) > 0
\end{equation}
for all $t \in [0, \infty)$.
(3) For any point $\varphi \in \mathcal{K}_\Omega$, there is a geodesic ray in (2) starting at $\varphi$.

We need some definitions.

Definition 8.1. A Kähler class is called
- geodesic semi-stable at a point $\varphi_0$ if every non-trivial $C^{1,1}$ geodesic ray starting from $\varphi_0$ is semi-stable.
- geodesic semi-stable if every non-trivial $C^{1,1}$ geodesic ray is semi-stable.
- weak geodesic semi-stable if every non-trivial geodesic ray with uniform $C^{1,1}$ bound is semi-stable.

We say a $C^{1,1}$ geodesic ray is trivial if it is just a point.

Proposition 8.2. Suppose that $\chi$ is negative semi-definite. We assume that there is a $C^{1,1}$ geodesic ray $\rho(t)$ staying in $\mathcal{K}_C$ and the $\tilde{3}^\beta$-functional is non-increasing along $\rho(t)$. If there is a $\tilde{3}^\beta$-metric, then $\rho(t)$ converges to the $\tilde{3}^\beta$-metric.

Proof. Let $\varphi_0$ be a $\tilde{3}^\beta$-metric. We first connect $\varphi_0$ and $\rho(t)$ by a $C^{1,1}$ geodesic segment $\gamma(s)$, this follows from Theorem [11]. Moreover, since the $C^{1,1}$ norm is uniform, after taking limit on $t$, we obtain a $C^{1,1}$ geodesic ray $\gamma(s)$ starting at $\varphi_0$. Thus, $\tilde{3}^\beta$ strongly converges and is well-defined along $\gamma(s)$.

Since the $\tilde{3}^\beta$ is non-increasing along $\rho(t)$, so $\tilde{3}^\beta$ has upper bound along $\gamma(s)$. While, Theorem [11] implies that when $\Omega$ has a $\tilde{3}^\beta$-metric, then $\tilde{3}^\beta$ has lower bound.

Meanwhile, when $\chi$ is negative semi-definite, from Lemma [23], $\tilde{3}^\beta$ is convex along the geodesic ray $\gamma(s)$. Moreover, $\tilde{3}^\beta$ obtains its lower bound at $s = 0$. So, we claim that $\tilde{3}^\beta(s) \equiv \min \tilde{3}^\beta$ along $\gamma(s)$. I.e. $\gamma(s)$ are constituted of $\tilde{3}^\beta$-metrics.

We prove this claim by the contradiction method. Since along $\gamma(s)$, the first derivative $(\tilde{3}^\beta)'$ is non-negative, we assume that $s_0$ is the first finite time such that $(\tilde{3}^\beta)'(s_0)$ is strictly positive, otherwise, the claim is proved. Since along $\gamma(s)$, $(\tilde{3}^\beta)''$ is also non-negative, so $(\tilde{3}^\beta)'$ is strictly positive for any $s \geq s_0$. This is a contradiction to $\lim_{s \to \infty} (\tilde{3}^\beta)'(s) = 0$ which follows from that $\tilde{3}^\beta$ is bounded and monotonic. \hfill \Box

Remark 8.1. When $\chi$ is strictly negative, using Lemma [23] again, we see that
\begin{equation}
\frac{1}{V} \int_M \chi_{ij} \tilde{\gamma}_{ij} \omega^n = 0.
\end{equation}
This implies $\gamma(s)$ is just a point which coincides with $\varphi_0$. Therefore $\rho(t)$ will converges to $\varphi_0$.

Remark 8.2. If a $C^{1,1}$ geodesic ray $\gamma(t)$ is destabilizing, then the $\tilde{3}^\beta$-functional is non-increasing when $t$ is large enough.
9. Proof of Theorem 1.6

Proof. Due to Theorem 1.1, \( \varphi_0 \) is a global minimiser. So \( \tilde{J}^\beta_{\varphi} \) is non-decreasing along any \( C^{1,1} \) geodesic ray \( \rho(t) \). So the first statement holds. For the second statement, we consider the sign of \( F^\beta_{\varphi} \) and prove by contradiction method. Assume that \( \rho(t) \) is a geodesic ray with uniform \( C^{1,1} \) bundle and \( \tilde{J}^\beta_{\varphi} \) is strictly negative along it. So according to the definition of \( \tilde{J}^\beta_{\varphi} \) \([13]\), when \( t \) is large enough,

\[
\tilde{J}^\beta_{\varphi}(\frac{\partial \rho}{\partial t})_\rho(t) < 0.
\]

According to Proposition 8.2, \( \rho(t) \) will converge to a \( \tilde{J}^\beta_{\varphi} \)-metric and \( F^\beta_{\varphi} = 0 \). Contradiction! So the theorem follows. \( \square \)

10. Proof of Theorem 1.4

Recall the entropy

\[
E^\omega_{\varphi}(\varphi) = \frac{1}{V} \int_M \log \frac{\omega^\varphi_n}{\omega^\omega_n}.\]

The proof of Theorem 1.4 follows from the following lemma and (1.4).

Lemma 10.1. (Tian \([25]\)) There is a uniform constant \( C = C(\omega) > 0 \),

\[
E^\omega_{\varphi}(\varphi) \geq \alpha I^\omega_{\varphi}(\varphi) - C, \forall \varphi \in \mathcal{K}.
\]

(10.1)

Proof. The \( \alpha \)-invariant was introduced by Tian \([23]\):

\[
\alpha([\omega]) = \sup\{\alpha > 0| \exists C > 0, \text{ s.t. } \int_M e^{-\alpha(e^{-\sup M} \varphi)} \omega^\omega_n \leq C \}
\]

holds for all \( \varphi \in \mathcal{K} \} > 0 \).

From the definition of the \( \alpha \)-invariant

\[
\int_M e^{-\alpha(\varphi - \frac{1}{V} \int_M \varphi \omega^n)} - h \omega^\varphi_n = \int_M e^{-\alpha(\varphi - \frac{1}{V} \int_M \varphi \omega^n)} \omega^\omega_n
\]

\[
\leq \int_M e^{-\alpha(\varphi - \sup M \varphi)} \omega^\omega_n
\]

and then the Jensen inequality

\[
\int_M \alpha(-\varphi + \frac{1}{V} \int_M \varphi \omega^n) - \log \frac{\omega^\omega_n}{\omega^\omega_n} \omega^\varphi_n \leq C,
\]

we obtain the lower bound of the entropy. \( \square \)

Lemma 10.2. I-properness of Ding functional implies I-properness of Mabuchi K-energy.

Proof. From assumption, in \( \Omega = C_1(M) \), there are two positive constants \( A_3 \) and \( A_4 \) such that for all \( \varphi \in \mathcal{K}_\Omega \),

\[
F^\omega_{\varphi}(\varphi) \geq A_3 I^\omega_{\varphi}(\varphi) - A_4.
\]

(10.2)

Let \( f \) be the scalar potential which is defined to be the solution of the equation

\[
\Delta \varphi f = S - S
\]

with the normalisation condition

\[
\int_M e^f \omega^\varphi_n = V.
\]
Ding-Tian \[9\] introduced the following energy functional

\[ A(\varphi) = \frac{1}{V} \int_M f \omega^n. \]

Let \( \mathcal{H}_0 \) be the space of Kähler potential \( \varphi \) under the normalization condition

\[ \int_M e^{-\varphi + h} \omega^n = V. \]

In \( \mathcal{H}_0 \), the relation between Mabuchi K-energy and Ding F-functional is

\[ F_\omega(\varphi) = \nu_\omega(\varphi) + A(\varphi) - A(0). \]

Applying the Jensen inequality to the normalization condition of \( f \), we have

\[ A(\varphi) \leq 0. \]

Thus the I-properness of Mabuchi K-energy is achieved by another positive constant \( A_5 \) from (10.2),

\[ \nu_\omega(\varphi) \geq A_3 I_\omega(\varphi) - A_5. \]

\[ \square \]

11. Proof of Theorem 1.5

We construct the required geodesic ray by using the \( \tilde{\beta} \)-flow.

**Proposition 11.1.** Assume that the \( \tilde{\beta} \)-flow converges to a \( \tilde{\beta} \)-metric. From any Kähler potential \( \psi \), there exists a semi-destabilising \( C^{1,1} \)-geodesic ray such that

1. \( \tilde{\beta} \) is bounded from below,
2. the infimum of the energy \( \tilde{E}^\beta \) is zero.

**Proof.** We connect \( \psi \) to the \( \tilde{\beta} \)-flow \( \varphi(t) \) with the \( C^{1,1} \)-geodesic \( \varphi_t(s) \). Then we define \( \rho(s) = \lim_{t \to \infty} \varphi_t(s) \). Since the \( \tilde{\beta} \)-flow \( \varphi(t) \) satisfies two conclusions in this proposition and the end-points of each \( \rho_t(s) \) are all in \( \varphi(t) \), so \( \rho(s) \) also satisfies these two conclusion automatically. The semi-destabilising is proved as following.

\[ \tilde{\beta}(\rho) = \lim_{s \to \infty} \delta \tilde{\beta}(\frac{\partial}{\partial s})_{\rho(s)} \]

\[ \leq \lim_{s \to \infty} \lim_{t \to \infty} d\tilde{\beta}(\frac{\partial}{\partial s} - \frac{\partial \varphi}{\partial t})_{\rho_t(s)} + d\tilde{\beta}(\frac{\partial \varphi}{\partial t})_{\rho_t(s)} \]

\[ = \lim_{s \to \infty} \lim_{t \to \infty} d\tilde{\beta}(\frac{\partial}{\partial s})_{\varphi(t)} + d\tilde{\beta}(\frac{\partial \varphi}{\partial t})_{\varphi(t)} \]

\[ \leq \lim_{s \to \infty} \lim_{t \to \infty} d\tilde{\beta}(\frac{\partial}{\partial s} - \frac{\partial \varphi}{\partial t})_{\varphi(t)}. \]

From (6.1), we further have the right hand side is bounded by

\[ \leq \lim_{s \to \infty} \lim_{t \to \infty} \sqrt{\tilde{E}^\beta(\varphi(t))} \sqrt{2 - 2(\frac{\partial \varphi}{\partial s}, \frac{\partial \varphi}{\partial t})_{\varphi(t)}} \]

\[ \leq \lim_{s \to \infty} \lim_{t \to \infty} \sqrt{\tilde{E}^\beta(\varphi(t))} C \cdot d(\varphi_0, \rho(0)) \]

\[ = 0. \]

Thus, the proposition holds. \[ \square \]
Now we prove the convergence of the negative gradient flow $\tilde{J}_\beta$-functional. Assume that there is a $\omega \in \Omega$ such that
\[(11.1)\quad (\omega \chi - (n-1)\omega) \cdot \omega + (\omega.n)^2 > 0,\]
and
\[(11.2)\quad -\chi > 0.\]

**Proposition 11.2.** The conditions (11.1) and (11.2) is equivalent to convergence of the $\tilde{J}_\beta$-flow to a $\tilde{J}_\beta$-metric.

The shot tome existence from the fact that the linearisation operator $L$ is elliptic. In the following, we prove the a priori estimates. As long as we have the second order estimate and the zero estimate, the $C^2$ estimate follows from the Evans-Krylov estimate. The higher order estimates is obtained by the bootstrap method.

Recall the $\tilde{J}_\beta$-flow,
\[
\dot{\phi} = -c_\beta + \frac{\omega^{n-1}}{\omega^\beta} \omega \cdot \nabla \phi - \frac{\beta \omega^n}{V} \omega_\phi^\beta.
\]

We take derivative $\partial_t$ on the both sides,
\[
\ddot{\phi} = -\dot{\phi} \omega_i \omega_j + \frac{\beta \omega^n}{V} \omega_\phi^\beta \omega_i \omega_j \\
= \omega_{ij} [-g_{ij} \omega_i \omega_j + \frac{\beta \omega^n}{V} \omega_\phi^\beta].
\]

We denote
\[
L = [-g_{ij} \omega_i \omega_j + \frac{\beta \omega^n}{V} \omega_\phi^\beta] \partial_i \partial_j.
\]

From (11.2), we see that on the short time interval, $L$ is an elliptic operator, i.e.
\[(11.6)\quad -\chi + \frac{\beta \omega^n}{V} \omega_\phi^\beta > 0.
\]

From the maximum principle, we have
\[(11.7)\quad \min_M \dot{\phi}(0) \leq \dot{\phi}(t) \leq \max_M \dot{\phi}(0).
\]

11.1. **Lower bound of the 2nd derivatives.** Using the flow equation, we have
\[
\min_M \dot{\phi}(0) \leq \dot{\phi}(t) = -c_\beta + \frac{\omega^{n-1}}{\omega^\beta} \omega \cdot \nabla \phi - \frac{\beta \omega^n}{V} \omega_\phi^\beta \\
= -c_\beta + g_i^j \omega_i \omega_j - \frac{\beta \omega^n}{V} \omega_\phi^\beta \\
\leq -c_\beta + g_i^j \omega_i \omega_j.
\]

In the following, we always use the normal coordinate diagonalize $\omega$ and $\omega_\phi$ such that their eigenvalues are 1 and $\lambda_i$ for $1 \leq i \leq n$ respectively. Denote the diagonal of $\chi$ by $\mu_i$.

Thus for any $1 \leq i \leq n$,
\[
\frac{-\mu_i}{\lambda_i} \leq \min_M \dot{\phi}(0) - c_\beta,
\]
or
\[
\lambda_i \geq \frac{-\mu_i}{\min_M \phi(0) - c_\beta}.
\]

11.2. Upper bound of the 2nd derivatives. Let
\[
A = \chi^{ij} g_{\phi ij}.
\]

When we work on the second order estimate, the extra term in the equation causes the trouble, we overcome it by using the linearisation operator \(L\) as the elliptic operator. Then we compute
\[
(\partial_t - L)(\log A - C\phi).
\]
Let
\[
B = g_{\phi ij} \chi_{pq}.
\]
We have
\[
B_{ij} = [g_{\phi ij} \chi_{pq}]_{ij} = -\left( g_{\phi}^{pq} g_{\phi}(g_{\phi rs})_{ij} \right)_{ij} \chi_{pq} - g_{\phi}^{pq} R_{\phi ij}^{pq}(\chi)
= \left[ -g_{\phi}^{pq} g_{\phi}(g_{\phi rs})_{ij} \right]_{ij} + g_{\phi}^{pq} \phi_{ij} (g_{\phi rs})_{ij}
+ g_{\phi}^{pq} \phi_{ij} (g_{\phi rs})_{ij} \chi_{pq} - g_{\phi}^{pq} R_{\phi ij}^{pq}(\chi).
\]
So using the flow equation,
\[
\begin{align*}
\partial_t A &= \chi^{ij} \phi_{ij} \\
&= \chi^{ij} [c_\beta + g_{\phi}^{pq} \chi_{pq} - \frac{\beta}{\omega^n} \omega^{pq} \chi_{pq}]_{ij}
= \chi^{ij} [c_\beta + g_{\phi}^{pq} \chi_{pq} - \frac{\beta}{\omega^n} \omega^{pq} \chi_{pq}]_{ij}
+ g_{\phi}^{pq} \phi_{ij} (g_{\phi rs})_{ij} \chi_{pq} - g_{\phi}^{pq} R_{\phi ij}^{pq}(\chi) - \frac{\beta}{\omega^n} \omega^{pq} \chi_{pq}.
\end{align*}
\]
Then computing under normal coordinate of \(\omega\),
\[
\begin{align*}
\frac{\omega^n}{\omega^n} &= [g^{kl}(g_{kl})]_{ij} \omega^n(\omega^n)^{-1} - \omega^n(\omega^n)^{-1} g^{kl}(g_{kl})_{ij}
= -g^{kl} R_{klij}(\omega) \omega^n(\omega^n)^{-1} + \omega^n(\omega^n)^{-1} g^{pq} (g_{pq ij})_{ij} g^{kl}(g_{kl})_{ij}
+ \omega^n(\omega^n)^{-1} g^{pq} (g_{pq ij})_{ij} (g_{kl})_{ij} - \omega^n(\omega^n)^{-1} g^{kl}(g_{kl})_{ij}.
\end{align*}
\]
Again,
\[
A_{kij} = [\chi^{pq} g_{pq ij}]_{kij}
= R^{pq}_{kij}(\chi) g_{pq ij} + \chi^{pq} (g_{pq ij})_{kij}.
\]
Furthermore, from the flow equation,
\[
(\partial_t - L)\phi
= -c_\beta + [g_{\phi}^{ij} \chi_{ij} + \frac{\beta}{\omega^n} \omega^n]_{ij} + \frac{\beta}{\omega^n} \omega^{pq} g^{pq} g^{kl} \phi_{kij}
= -c_\beta + 2g_{\phi}^{ij} \chi_{ij} - g_{\phi}^{kl} (g_{\phi ij})_{ij} g_{kl} - \frac{\beta}{\omega^n} \omega^n(n + 1) + \frac{\beta}{\omega^n} \omega^n g^{kl} g_{kl}.
\]
Putting them together, we obtain

\[(\partial_t - L)[\log A - C\varphi]\]

\[= \frac{1}{A} \partial_t A + g^k\_p g^\_q \chi_{ij}(A_k\_l - A_k\_l) - \frac{\beta \omega^n g^k\_p}{V \omega^p}(A_k\_l - A_k\_l)\]

\[- C[(\partial_t - L)\varphi]\]

\[= \frac{\partial_t A + g^k\_p g^\_q \chi_{ij} A_k\_l}{A} \cdot \frac{g^k\_p g^\_q \chi_{ij} A_k\_l}{A^2}\]

\[- \frac{\beta \omega_n g^k\_p}{V \omega^p} A_k\_l + \frac{\beta \omega_n g^k\_p}{V \omega^p} A_k\_l \]

\[- C[-c_\beta + 2g^k\_p \chi_{ij} - g^k\_p g^\_q \chi_{ij} g_{kl} - \frac{\beta \omega_n}{V \omega^p}(n + 1) + \frac{\beta \omega_n}{V \omega^p} g^k\_p g^k\_l].\]

The first line in the last identity is,

\[\frac{\partial_t A + g^k\_p g^\_q \chi_{ij} A_k\_l}{A} - \frac{g^k\_p g^\_q \chi_{ij} A_k\_l}{A^2}\]

\[= \frac{1}{A} \{ - \chi^{ij} g^\_p g^\_q (g_{\varphi^r s})_{ij} \chi_{pq} + \chi^{ij} g^\_p g^\_q g^\_s (g_{\varphi^p q})(g_{\varphi^r s})_i \chi_{pq} + \chi^{ij} g^\_p g^\_q g^\_s (g_{\varphi^p q})(g_{\varphi^r s})_i \chi_{pq} - g^\_p R_{pq}(\chi) - \frac{\beta \omega_n}{V \omega^p} \chi^{ij}\}\]

\[+ \frac{1}{A} \chi^{ij} g^\_p g^\_q [R^{pq \_kl}(\chi) g_{\varphi pq} + \chi^{p q} (g_{\varphi pq} k)] - \frac{g^k\_p g^\_q \chi_{ij} A_k\_l}{A^2}\]

\[= \frac{1}{A} \chi^{ij} g^\_p g^\_q g^\_s (g_{\varphi^p q})(g_{\varphi^r s})_i \chi_{pq} + \chi^{ij} g^\_p g^\_q g^\_s (g_{\varphi^p q})(g_{\varphi^r s})_i \chi_{pq} - g^\_p R_{pq}(\chi) - \frac{\beta \omega_n}{V \omega^p} \chi^{ij}\]

\[+ \frac{1}{A} \chi^{ij} g^\_p g^\_q [R^{pq \_kl}(\chi) g_{\varphi pq} + \chi^{p q} (g_{\varphi pq} k)] - \frac{g^k\_p g^\_q \chi_{ij} A_k\_l}{A^2}\]

Here we use the identity to cancel the first term in the 2nd line and the second term in the 4th line,

\[(g_{\varphi pq})_{kl} = R_{pq \_kl} + \frac{\partial^4}{\partial z^p \partial z^q \partial z^r \partial z^s} \varphi = R_{kl \_pq} + \frac{\partial^4}{\partial z^p \partial z^q \partial z^r \partial z^s} \varphi = (g_{\varphi kl})_{pq}.\]

The second line in the last identity in (11.13) is

\[= - \frac{\beta \omega_n g^k\_p [R^{pq \_kl}(\chi) g_{\varphi pq} + \chi^{p q} (g_{\varphi pq} k)]}{V \omega^p} + \frac{\beta \omega_n g^k\_p A_k\_l}{V \omega^p} .\]

In order to annihilate the 2nd term with 2nd term in (11.15) and 2nd term in (11.16) with (11.14), we need the lemma,
Lemma 11.3. The following lemma holds.
\[
[\chi^i g^k g^l (g_{\varphi\check{p}q})_i (g_{\varphi\check{k}l})_1] A \geq g^k A_k A_l,
\]
\[
[\chi^i g^k g^l (g_{\varphi\check{p}q})_i (g_{\varphi\check{k}l})_1] A \geq g^k g^l \chi^i A_k A_l.
\]

Proof. Under the normal chordate of \( \chi \) which is negative-defined, and \( \omega_\chi \) is diagonalized, the first inequality becomes,
\[
[g^k g^l \sum_i (g_{\varphi\check{p}q})_i (g_{\varphi\check{k}l})_1] \sum_i g_{\varphi\check{i}i} \geq g^k \sum_i g_{\varphi\check{i}i} g^l \sum_i g_{\varphi\check{i}i}.
\]
This follows from the Hölder’s inequality. The second inequality is proved in Lemma 3.2 in [27].

Thus (11.13) becomes
\[
(\partial_t - L) [\log A - C_\varphi]
= \frac{1}{A} [-g^p_\varphi R_{pq} (\chi) + \frac{\beta}{V} \chi^i g^k R_{k\check{i}j} (\omega) \frac{\omega^n}{\omega_\varphi^n}]
+ \frac{1}{A} [g^k g^l \chi_{ij} R_{pq} (\chi) g_{\varphi\check{p}q} - \frac{\beta}{V} \omega^n g^k R_{p\check{i}q} (\chi) g_{\varphi\check{q}p}]
- C[-c_\beta + 2g^i_\varphi \chi_{ij} - g^k g^l \chi_{ij} g_{\varphi\check{k}l} - \frac{\beta}{V} \omega^n (n + 1) + \frac{\beta}{V} \omega_\varphi^n g^k g^l].
\]

Since \( \omega_\varphi \) has lower bound from Subsection 11.1, the first four terms and the 4th term in the last line are bounded above by constant \( C_1 \), thus at the maximum point \( p \) of \( \log A - C_\varphi \),
\[
0 \leq C_1 - C[-c_\beta + 2g^i_\varphi \chi_{ij} - g^k g^l \chi_{ij} g_{\varphi\check{k}l}].
\]
Written in the normal co-ordinate where \( \chi \) has negative diagonal \( \mu_i \), it becomes
\[
0 \leq C_1 - C[-c_\beta + 2\sum_{i=1}^n \frac{\mu_i}{\lambda_i} - \sum_{i=1}^n \frac{\mu_i}{\lambda_i^2}].
\]
From the condition,
\[
(-nc_\beta \cdot \omega + (n - 1)\chi) \wedge \omega^{n-2} > 0,
\]
We have there exists a positive constant \( \delta \) such that
\[
(-nc_\beta \cdot \omega + (n - 1)\chi) \wedge \omega^{n-2} \geq \delta \omega^{n-1},
\]
then
\[
-c_\beta + \sum_{i=1,i \neq k}^n \mu_i \geq \delta.
\]
From (11.18), we have for large \( C \),
\[
-c_\beta + 2\sum_{i=1}^n \frac{\mu_i}{\lambda_i} - \sum_{i=1}^n \frac{\mu_i}{\lambda_i^2} \leq \frac{C_1}{C} \leq 0.5\delta.
\]
We choose \( 1 \leq k \leq n \) and consider,

\[
0 \geq \sum_{i=1, i \neq k}^{n} \mu_i \left( \frac{1}{\lambda_i} - 1 \right)^2 + \frac{\mu_k}{\lambda_k^2}
\]

\[
= c_\beta - 2 \sum_{i=1}^{n} \frac{\mu_i}{\lambda_i} + \sum_{i=1}^{n} \frac{\mu_i}{\lambda_i^2} - \left[ c_\beta - \sum_{i=1, i \neq k}^{n} \mu_i - 2 \frac{\mu_k}{\lambda_k} \right]
\]

\[
\geq -0.5\delta + \delta + 2 \frac{\mu_k}{\lambda_k}.
\]

Thus,

\[
\lambda_k \leq -\frac{4\mu_k}{\delta},
\]

or at \( p \),

\[
\omega \varphi \leq -\frac{4}{\delta} \chi.
\]

Therefore, we obtain that at any \( x \in M \)

\[
\log A(x) - C \varphi(x) \leq \log A(p) - C \varphi(p),
\]

then,

\[
\log A(x) \leq \log \frac{4n}{\delta} - C \cdot (\varphi - \inf \varphi).
\]

Therefore, there is constant \( C \) such that

\[
(11.19) \quad \omega \varphi \leq e^{C_1 \cdot (\varphi - \inf \varphi)}.
\]

11.3. **Zero order estimate.** It suffices to obtain the iteration formula. Letting

\[
C_2 = \max\{1, -\varphi - c_\beta + 1\}
\]

from (11.3), we have

\[
\omega_n \varphi \leq (\varphi + c_\beta + C_2) \omega_n \varphi = n \omega_n \varphi^{-1} \wedge \chi - \frac{\beta}{V} \omega^n + C_2 \omega_n \varphi.
\]

We compute that

\[
(11.20) \quad \omega_n \varphi - \omega_n \varphi^{-1} \wedge \omega
\]

\[
\leq (\varphi + c_\beta + C_2) \omega_n \varphi - \omega_n \varphi^{-1} \wedge \omega
\]

\[
= n \omega_n \varphi^{-1} \wedge \chi - \frac{\beta}{V} \omega^n + C_2 \omega_n \varphi - \omega_n \varphi^{-1} \wedge \omega.
\]
Then we let $\phi = \varphi - \inf \varphi$ and $u = e^{-C_3 \phi}$, we multiply (11.20) with $u$ and integrate over $M$. The right hand side becomes,

$$
\int_M u [\omega^n - \omega^{n-1}_\varphi \wedge \omega] = \int_M e^{-C_3 \phi} [\omega^n - \omega^{n-1}_\varphi \wedge \omega]
$$

$$
= C_3 \int_M e^{-C_3 \phi} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^{n-1}_\varphi
$$

$$
= C_3 \int_M e^{-C_3 \phi} \partial \varphi \wedge e^{-C_3 \phi} \bar{\partial} \varphi \wedge \omega^{n-1}_\varphi
$$

$$
= \frac{4}{C_3} \int_M \partial u^+ \wedge \bar{\partial} u^+ \wedge \omega^{n-1}_\varphi
$$

$$
\geq \frac{C_4}{C_3} \int_M \partial u^+^2 \omega^n.
$$

In the last inequality we used the lower bound of $\omega_\varphi$. While, the right hand side is

$$
\int_M u [n \omega^{n-1}_\varphi \wedge \chi - \frac{\beta}{V} \omega^n + C_2 \omega^n - \omega^{n-1}_\varphi \wedge \omega]
$$

$$
\leq C_2 \int_M u \omega^n
$$

$$
\leq C_2 \int_M e^{-C_3 \phi} e^{C_1 (\varphi - \inf \varphi)} \omega^n
$$

$$
\leq C_2 \int_M e^{-C_3 \phi} e^{C_1 \phi} e^{-C_1 \inf \varphi \omega^n}
$$

$$
\leq C_2 ||u||_0 \int_M e^{C_3 (-1+\frac{C_1}{C_5}) \phi} \omega^n.
$$

We apply (11.19) in the second inequality. Let $v = e^{-C_3 \phi}$. We choose $C_3 = pC_5$ and $\frac{C_1}{C_5} = 1 - \delta$, we thus obtain

$$
\int_M |\partial v^+|^2 \omega^n \leq pC_6 ||v||_0^{1-\delta} \int_M e^{C_5 (-p+1-\delta) \phi} \omega^n
$$

$$
\leq pC_6 ||v||_0^{1-\delta} \int_M e^{C_5 (p-1+\delta) \phi} \omega^n.
$$

Thus the zero order estimate follows from the iteration Lemma 3.3 in [28].

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