Embedded solitons in the third-order nonlinear Schrödinger equation

Debabrata Pal, Sk Golam Ali and B Talukdar

Department of Physics, Visva-Bharati University, Santiniketan 731235, India
E-mail: binoy123@bsnl.in

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Abstract
We work with a \textit{sech} trial function with space-dependent soliton parameters and envisage a variational study for the nonlinear Schrödinger (NLS) equation in the presence of third-order dispersion. We demonstrate that the variational equations for pulse evolution in this NLS equation provide a natural basis to derive a potential model which can account for the existence of a continuous family of embedded solitons supported by the third-order NLS equation. Each member of the family is parameterized by the propagation velocity and co-efficient of the third-order dispersion.

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1. Introduction
The distortion of pulses propagating through waveguides arises due to dispersive spreading. However, if the waveguide is made using materials the refractive index of which depends nonlinearly on the intensity of the propagating pulses, it will be possible to compensate the effect of dispersion by a pulse-narrowing effect [1]. Here, the weak nonlinearity of the index of refraction arising due to the Kerr effect produces a self-modulation which in turn causes steepening of the wave. The dynamical interplay between the dispersive and narrowing effects produces the so-called optical solitons. For an isotropic medium with cubic nonlinearity, the dynamics of these solitons is governed by the nonlinear Schrödinger (NLS) equation

\begin{equation}
i\phi_t + \phi_{xx} + |\phi|^2\phi = 0,
\end{equation}

with $\phi = \phi(x,t)$, a complex wave field. Here, $t$ stands for time and $x$, the coordinate along the direction of propagation. The suffixes $x$ and $t$ of $\phi$ denote partial derivatives with respect to these variables. In particular, $\phi_{tt} = \partial^2\phi/\partial t^2$. Equation (1) is exactly integrable using the inverse scattering theory and can support two-parameter soliton solutions. These parameters are amplitude and velocity. Pulse width is just the reciprocal of the amplitude. The main reason for the robustness of the solitons described by (1) is that the wavenumbers of the solitons lie in a range that is forbidden for the linear dispersive waves [2]. As a result, linear wave cannot be in resonance with the soliton to receive energy from the latter. We then have a stable soliton.

The propagation of picosecond pulses through optical fibers is well described by the NLS equation in (1) which accounts for only the second-order dispersion and self-modulation. But, for femtosecond pulses another important physical effect namely, the third-order dispersion comes into play. In this case, the appropriate evolution equation for the pulse propagation is given by [3]

\begin{equation}
i\phi_t + \phi_{xx} + |\phi|^2\phi = i\beta\phi_{tt}.
\end{equation}

Understandably, the parameter $\beta$ is a measure of the perturbation caused by the third-order dispersion. Numerical integration of (2) for $\beta > 0$ indicates that the time evolution of its solitary wave solution exhibits a structure. For example, the solution is characterized by some low-energy pedestal consisting of a platform response on the pulse edge for $t < 0$ and an asymmetric oscillation on the trailing edge. Elgin et al [4] attributed the physical origin of this structure to the radiation field emitted by the NLS soliton due to its perturbative interaction with the third-order dispersive effect. In this context, Akhmediev and Karlsson [5] calculated the amount of radiation and identified the radiation mechanism as analogous to the well-known Cherenkov radiation process. However, an important implication of having a third-order dispersive term in (2) is that the solitary wave now possesses wavenumbers residing inside the linear spectrum of the system. As a result, the soliton could be in resonance with the perturbation to generate continuous wave radiation for subsequent decay. Despite that, there may be values of the solitons internal frequency in which the oscillation on the tail...
exactly vanishes and we have a stable exponentially localized wave embedded into the continuous spectrum of the radiation. Such a stable wave is called an embedded soliton [6], although (2) is not integrable. The stable embedded solitons are not isolated objects rather they occur as a continuous family. This point has been demonstrated by Yang [7] using the soliton perturbation theory. On the other hand, Yang and Akylas [8] took recourse to the use of a gauge transformation for \( \phi(x,t) \) in (2) to explicitly demonstrate the existence of such families. Further, they found that each member of the family is parameterized by the propagation velocity.

The object of the present paper is to derive a straightforward analytical model to demonstrate the existence of the above family and study some of its characteristic features. The equation governing the propagation of ultra-short pulses in optical fibers involves nonlinear dispersive terms like \( |\phi|^2 \phi_t \) and \( \phi^2 \phi_x \) [9], in addition to the term \( i \beta \phi_{3x} \) as written in the right-hand side of (2). The third-order linear term plays the most important role during usual operation of fibers in the spectral region near small values of the second-order dispersion [1]. In view of this, we shall use (2) to present our model. We shall work with a variational method involving trial functions in order to describe the main characteristics of the embedded solitons as determined by the third-order NLS equation in (2). Although the chosen trial function has a specific form, the shape parameters are allowed to evolve as the solitons propagate. With this assumption the evolution equation under consideration simplifies to a reduced Lagrangian problem [10]. We believe that the approach to be followed by us has distinct advantages to deal with physical problems because many unknown effects are then readily expressed and evaluated.

In section 2, we convert the initial boundary value problem (2) into a variational problem and introduce a sech-type trial function with a view to construct an expression for the effective Lagrangian. In section 3, we use this effective Lagrangian to study evolution of the characteristic pulse parameters of our trial function as the solitons propagate. In section 4, we work out the model of our interest. In particular, we construct a potential function formulation for the embedded soliton dynamics. We show that the results of soliton the perturbation theory or gauge theory can be obtained from a rather elementary consideration. The present work should, therefore, be regarded as a very useful supplement for the more detailed investigation in [7, 8]. Finally, in section 5, we make some concluding remarks.

2. Variational description of the third-order NLS equation

For the NLS equation in (1) there exists a well-defined spectral problem [11], such that one can write a closed form analytical solution of it in terms of sech functions. As opposed to this, the third-order NLS equation is nonintegrable and cannot be solved analytically. To solve (2) one would, therefore, take a recourse to the use of either perturbation techniques [4, 7] or variational methods [10]. We are interested in a variational formulation of our third-order NLS to find an accurate approximation solutions. The variational method can sometimes give rise to false instabilities when applied to study the dynamics of solitary waves propagating in one-spatial dimension [12]. Fortunately, Kaup and Lakoba [13] proved that such instabilities do not occur for NLS-type equation as that of ours. In an excellent review article, Malomed [14] expounded variational methods for application in nonlinear fiber optics and related fields.

For our treatment, we will essentially rely on a Ritz optimization procedure [15] applied to the Lagrangian function for the NLS equation in (2). We have found that the action functional

\[
W = \int \int \mathcal{L}(\phi, \phi_t, \phi_x, \phi_y, \phi_{tt}, \phi_{xx}, \phi_{yy}) \, dx \, dt
\]

with the Lagrangian density

\[
\mathcal{L} = \frac{i}{2} (\phi \phi_{xx}^\star - \phi^\star \phi_{xx}) - \frac{1}{2} \phi^2 \phi_{xx}^2 + \phi \phi_{xx} + \frac{i}{2} \beta (\phi \phi_{xx}^\star - \phi^\star \phi_{xx})
\]

reproduces the third-order NLS equation via the Hamilton’s variational principle.

In the absence of third-order term in (2), the soliton parameters remain fixed and we have an ideal soliton. Peleg and Chung [16] used the singular perturbation theory around the ideal soliton to calculate the solution of the NLS equation in the presence of nonzero third-order dispersion. We postulate that in the presence of third-order dispersion the soliton parameters become \( x \)-dependent. Thus, in analogy with the well-known solution of (1) we introduce a sech-type trial function

\[
\phi(x,t) = \eta(x) \text{sech} \left( \frac{t - y(x)}{a(x)} \right)
\]

\[
\times \exp \left[ V(x)(t - y(x)) + \frac{b(x)}{2a(x)} (t - y(x))^2 + \sigma(x) \right].
\]

Here, the parameters \( \eta, y \) and \( a \) are related to the three lowest-order moments of the \( \phi \) envelope and represent, respectively, its amplitude, central position and width, respectively. The other parameters \( \sigma, V \) and \( b \) stand for the phase, velocity (center of the soliton) and frequency chirp. Understandably, these parameters will all vary with the distance of propagation. Using (5) in (4) we get

\[
\mathcal{L}_i = \sum_{i=1}^{4} \mathcal{L}_i^{(i)}
\]

where

\[
\mathcal{L}_i^{(1)} = \eta^2 \left[ a \frac{dV}{dx} \left( \frac{t - y}{a} \right) - \frac{dy}{dx} + \frac{1}{2} \left( a \frac{dh}{dx} - b \frac{da}{dx} \right) \left( \frac{t - y}{a} \right)^2 \right. \\
\left. - b \frac{dy}{dx} \left( \frac{t - y}{a} \right) + \frac{da}{dx} \right] \text{sech}^2 \left( \frac{t - y}{a} \right),
\]

\[
\mathcal{L}_i^{(2)} = -\frac{1}{2} \eta^2 \text{sech}^4 \left( \frac{t - y}{a} \right),
\]

\[
\mathcal{L}_i^{(3)} = \eta^2 \left[ \frac{1}{a^2} \tanh^2 \left( \frac{t - y}{a} \right) + \left( V + \frac{b(t - y)}{a} \right) \right] \times \text{sech}^2 \left( \frac{t - y}{a} \right)
\]
and
\[ L_s^{(4)} = -\beta \eta^2 \left\{ \frac{b}{a^2} \tanh \left( \frac{t - y}{a} \right) \right. \]
\[- \frac{1}{a^2} \left\{ V + \frac{b(t - y)}{a} \right\} \tanh^2 \left( \frac{t - y}{a} \right) \right.
\[- \frac{1}{a^2} \left\{ V + \frac{b(t - y)}{a} \right\} \text{sech}^2 \left( \frac{t - y}{a} \right) \right.
\left. - \left\{ V + \frac{b(t - y)}{a} \right\}^3 \text{sech}^2 \left( \frac{t - y}{a} \right) \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right.
We now introduce a normalized pulse width \( z(x) = \frac{a(x)}{\Delta t} \) to rewrite (15) as

\[
\frac{1}{2} \left( \frac{dz}{dx} \right)^2 + \Pi(z) = 0
\]

with

\[
\Pi(z) = (1 + 3\beta V) \left( \frac{1}{z^2} - 1 \right) \left( \mu (1 + 3\beta V) \left( \frac{1}{z^2} + 1 \right) + \nu \right)
\]

and

\[
\mu = \frac{8}{\pi^2 q_0^4}, \quad \nu = -\frac{8E_0}{\pi^2 q_0^2}.
\]

The potential function \( \Pi(z) \) vanishes for two values of \( z \), namely,

\[
z_1 = 1 \quad \text{and} \quad z_2 = -\frac{\mu (1 + 3\beta V)}{\nu + \mu (1 + 3\beta V)}
\]

implying that there is a minimum in between these points. The point at which \( \Pi(z) \) is minimum can be obtained from \( \frac{d\Pi(z)}{dz} = 0 \) as

\[
z_m = -\frac{2\mu (1 + 3\beta V)}{\nu}.
\]

In the limit, when \( \frac{\nu}{\mu} = -2(1 + 3\beta V) \), we have \( z_2 = 1 = z_m \) and \( \frac{d\Pi(z)}{dz} \bigg|_{z=1} = \Pi(z_m) = 0 \). In this case, the potential well degenerates into a single point such that a particle released at that point will stay there. In the context of the present work, this signifies that a wave pulse for which \( \frac{\nu}{\mu} = -2(1 + 3\beta V) \) propagates with unchanged shape.

In figure 1, we plot the potential function \( \Pi(z)/\mu \) as a function of \( z \) for \( \beta V = -0.2, -0.1, 0, 0.1 \) and 0.2. Each of these potentials has \( z_1 = z_2 = z_m = 1 \) and is, therefore, associated with an embedded soliton. The solid curve for \( \beta V = 0 \) is the potential function corresponding to the usual second-order NLS equation while the dotted curves represent the potential functions of the embedded solitons as found in the solution of the third-order NLS equation. For \( \beta V > 0 \), the potential curves for the embedded solitons lie above the solid curve for \( z > 1 \). For \( \beta V < 0 \), the dotted curves fall below the solid curve. We have drawn only a few curves for discrete values of \( \beta V \). A dense set of curves corresponding to a continuous family of embedded solitons can be drawn by varying \( \beta V \) continuously. In the neighborhood of each embedded soliton there is another one having a slightly higher or lower energy. A particular embedded soliton when perturbed may always relax at the adjacent one. This fact has often been regarded as the reason for stability of an embedded soliton [7].

We have numerically solved the coupled set of equations (11f), (12d) and (13), and an equation resulting from (12a) to determine the soliton parameters in (5). We have chosen to work with \( V = 1 \) and \( \beta = 0.2, \beta = 0.0 \) and \( \beta = -0.2 \). Clearly, the potential curves corresponding to these values are given in figure 1. In figure 2, we plot \(|\phi(x, 0)|\) for these \( \beta \) values as a function of \( x \). The solid curve for \( \beta = 0 \) gives the exponentially localized NLS soliton. It is of interest to note that the other two curves for the embedded solitons are also exponentially localized. This conclusion is true for every member of this family. Our curves in figure 2 are not discernible from those obtained by numerical simulation which utilizes the split-step method with periodic boundary conditions [16].

5. Conclusions

The existence of embedded solitons has been predicted in various physical systems governed by the fifth-order KdV equation [17], coupled KdV equations [18], second-harmonic generation model [6] and many others. A common feature of all these embedded solitons is that they exist at isolated parameter points. The isolated solitons can at most be semistable. This implies that the perturbed soliton persists or decays depending on whether the initial pulse energy is higher or lower than that of the embedded soliton. This observation is supported by both analytical calculation and numerical simulation. There was, however, an open question: could embedded solitons exist as continuous families? Admittedly, if such families are there, a soliton when perturbed will, in principle, shed some energy and approach nearby embedded solitons.

In two remarkable works, Yang [7] and Yang and Akylas [8] made use of the machineries for stability analysis of dynamical systems to demonstrate the existence of such
families. We deal with a variational approach to the problem and derive a potential model to visualize the embedded soliton family. The merit of the method used by us is that it is, on the one hand, physically transparent and, on the other hand, mathematically uncomplicated. A problem of considerable current interest is to study the effect of third-order dispersion on the collision between the two soliton pulses. In respect of this Malomed [19] and Peleg et al. [20] have carried out detailed analytical and numerical studies to show that the soliton–soliton interactions can be regarded as an inelastic collision in which energy is lost to continuous radiation due to small but finite third-order dispersion.

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