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Hom-Lie structures on 3-dimensional skew symmetric algebras

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Abstract. We describe the dimension of the space of possible linear endomorphisms that turn skew-symmetric three-dimensional algebras into Hom-Lie algebras. We find a correspondence between the rank of a matrix containing the structure constants of the bilinear product and the dimension of the space of Hom-Lie structures. Examples from classical complex Lie algebras are given to demonstrate this correspondence.

1 Introduction

Hom-Lie algebras were first introduced by Hartwig, Larsson and Silvestrov in [1] by studying some examples of deformed Lie algebras which arise from twisted discretizations of vector fields. Hom-Lie algebras are therefore generalisations of Lie algebras by having an additional twist $\alpha$, a linear endomorphism. The space of such linear endomorphisms that turn a skew symmetric algebra into a Hom-Lie algebra forms a vector subspace, known as Hom-Lie structures. In [3], it is proved that every 3-dimensional skew-symmetric algebra can be turned into a Hom-Lie algebra. The authors in [2] give results where the dimension of the space of such linear endomorphisms is 6.

This work gives results for the remaining dimensions of the Hom-Lie structures in 3-dimension case. The first section gives preliminaries on constructing polynomial equations from the Hom-Jacobi identity which are linear in structure constants of the linear map and gives specific equations for the dimension 3 case. The second section of this work proceeds to give results showing correspondence between the dimension of the space of possible linear endomorphisms and the rank of a 3 by 3 matrix of all structure constants of the bilinear map, denoted by $C$. The last section demonstrates the results obtained using examples from classical Lie algebras. The Hom-Lie structures for Simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ have been studied previously, in for example, [7] and [8]. Hom-Lie structures for multiplicative Heisenberg Lie algebras have also been studied in [6].

2 Preliminaries

All algebras in this article are over an algebraically closed field, $\mathbb{K}$, of characteristic 0. We begin by giving the definition of a Hom-Lie algebra.
Definition 2.1. A Hom-Lie algebra is a triple $(V, [\cdot, \cdot], \alpha)$ consisting of a linear space $V$, bilinear map $[\cdot, \cdot] : V \times V \rightarrow V$ and a linear space homomorphism $\alpha : V \rightarrow V$ satisfying

\begin{align*}
[x, y] &= -[y, x] \quad \text{(skew-symmetry)} \quad (1) \\
[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] &= 0 \quad \text{(Hom - Jacobi identity)} \quad (2)
\end{align*}

for all $x, y, z \in V$.

Let the structure constants associated to both the bilinear product and linear map be given by $\{C^{r}_{ij}\}_{i<j}$ and $\{a_{it}\}$ respectively. We have the following equations involving the structure constants in general for $n$-dimensional case:

\begin{equation}
[e_{i}, e_{j}] = \sum_{s=1}^{n} C^{s}_{ij} e_{s} \quad \text{and} \quad \alpha(e_{i}) = \sum_{t=1}^{n} a_{it} e_{t} \quad (3)
\end{equation}

Replacing equations in (3) in the Hom-Jacobi identity and writing the equations as linear in $a_{it}$, we have the following:

\begin{equation}
\sum_{t=1}^{n} a_{it} \left( \sum_{s=1}^{n} C^{s}_{jk} C^{r}_{ls} \right) + a_{jt} \left( \sum_{s=1}^{n} C^{s}_{kl} C^{r}_{ts} \right) + a_{kt} \left( \sum_{s=1}^{n} C^{s}_{il} C^{r}_{ts} \right) = 0 \quad (4)
\end{equation}

with $1 \leq i < j < k \leq n$, $r = 1, 2, \ldots, n$.

We denote the matrix of linear transformation with coefficients involving structure constants of the bilinear map by $M$ and the column matrix involving the structure constants of linear map $\alpha$ by $a_{\alpha}$. In $n$-dimensional case the matrix $M$ has $n^{2}$ columns and $\binom{n}{3}$ rows. The elements of $M$ are the coefficients associated to $a_{it}, a_{jt}, a_{kt}$ given in (4). A general construction of $M$ is presented in [2].

Thus (4) can be written as $M a_{\alpha} = 0$ and the matrix $M$ represents a linear transformation $L : \mathbb{K}^{9} \rightarrow \mathbb{K}^{3}$. In order to realize a Hom-Lie algebra, it is then required that $a_{\alpha} \in \ker L$. We have that $6 \leq \dim \ker L \leq 9$. Hom-Lie structures, denoted by $\text{HomLie}(\mu)$, is the space of all linear endomorphisms that satisfy the Hom-Jacobi identity for some skew symmetric algebra $H$. That is the vector subspace

\[ \text{HomLie}(\mu) = \{ \alpha \in \text{End} V | \circ_{x,y,z} [\alpha(x), [y, z]] = 0 \} \]

where $\circ_{x,y,z} = [\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]]$.

This study concerns describing the dimensions of the space of such Hom-Lie structures in 3-dimensional case. The matrix $a_{\alpha}$ is given as $a_{\alpha} = (a_{11} a_{21} a_{31} a_{12} a_{22} a_{32} a_{13} a_{23} a_{33})$ and matrix $M$ associated to the bilinear product is given as

\[ M = \begin{pmatrix}
M_{1,1} & M_{1,2} & M_{1,3} & M_{1,4} & M_{1,5} & M_{1,6} & M_{1,7} & M_{1,8} & M_{1,9} \\
M_{2,1} & M_{2,2} & M_{2,3} & M_{2,4} & M_{2,5} & M_{2,6} & M_{2,7} & M_{2,8} & M_{2,9} \\
M_{3,1} & M_{3,2} & M_{3,3} & M_{3,4} & M_{3,5} & M_{3,6} & M_{3,7} & M_{3,8} & M_{3,9}
\end{pmatrix} \quad (5)
\]

where

\begin{align*}
M_{r.1} &= (C^{2}_{23} C^{r}_{12} + C^{3}_{23} C^{r}_{13}) \quad , \quad M_{r.2} = (-C^{2}_{12} C^{r}_{13} - C^{3}_{13} C^{r}_{13}) \quad , \quad M_{r.3} = (C^{2}_{12} C^{r}_{13} + C^{3}_{13} C^{r}_{13}) \\
M_{r.4} &= (C^{3}_{23} C^{r}_{23} - C^{2}_{23} C^{r}_{23}) \quad , \quad M_{r.5} = (C^{1}_{13} C^{r}_{12} - C^{3}_{13} C^{r}_{23}) \quad , \quad M_{r.6} = (C^{1}_{13} C^{r}_{12} - C^{1}_{13} C^{r}_{12}) \\
M_{r.7} &= (-C^{1}_{23} C^{r}_{13} - C^{2}_{23} C^{r}_{23}) \quad , \quad M_{r.8} = (C^{1}_{13} C^{r}_{13} + C^{2}_{13} C^{r}_{23}) \quad , \quad M_{r.9} = (-C^{1}_{13} C^{r}_{13} - C^{2}_{13} C^{r}_{23})
\end{align*}
\[ r = 1, 2, 3. \]

We denote by \( C \) the matrix of structure constants \( \{C_{ij}^k\}_{i<j} \) for \( i, j, k = 1, 2, 3 \) given as

\[
C = \begin{pmatrix}
C_{12}^1 & C_{12}^2 & C_{12}^3 \\
C_{13}^1 & C_{13}^2 & C_{13}^3 \\
C_{23}^1 & C_{23}^2 & C_{23}^3 
\end{pmatrix},
\]

### 3 Hom-Lie structure dimension

In this section, we show that the dimension of HomLie(\( \mu \)) can be described by the matrix \( C \).

#### Proposition 3.1

Let \( H \) be a 3-dimensional skew symmetric algebra with structure constants \( \{C_{ij}^k\}_{i<j} \) and \( C \) be the matrix of structure constants. The dimension of the space of possible endomorphisms, HomLie(\( \mu \)), attains minimum dimension 6, if and only if \( \det C \) is non-zero.

From proposition 3.1, we see that if \( C \) is not of full rank then \( \dim H \mathcal{L}(\mu) \geq 7 \). For \( \dim H \mathcal{L}(\mu) = 7 \) we require that rank \( M = 2 \). Let us denote by \((m, n)_i\) any 2 by 2 sub-determinant of \( M \) involving columns \( m \) and \( n \) with \( 1 \leq m < n \leq 9 \) and \( i = 1, 2, 3 \). We proceed to list all 108 such 2 by 2 sub-determinants of \( M \).

For \( i = 1, 2, 3 \)

\[
\begin{align*}
(1, 2, i) &= A_i \cdot D_3 & (1, 3, i) &= A_i \cdot B_3 & (2, 3, i) &= A_i \cdot A_3 & (4, 5, i) &= B_i \cdot D_2 & (4, 6, i) &= B_i \cdot B_2 \\
(5, 6, i) &= A_i \cdot A_2 & (7, 8, i) &= D_1 \cdot D_1 & (7, 9, i) &= D_1 \cdot B_1 & (8, 9, i) &= D_1 \cdot A_1 \\
(1, 4, i) &= C_{13}^1 C_{23} C_3 A_i + C_{23}^2 C_{23} B_i + (C_{33}^2 D_i) & (3, 5, i) &= -C_{13}^1 C_{12}^1 A_i - C_{12}^2 C_{13}^1 B_i - C_{12}^3 C_{13}^1 D_i \\
(1, 5, i) &= -C_{13}^1 C_{23}^3 A_i - C_{23}^2 C_{23}^1 B_i - C_{23}^3 C_{23}^1 D_i & (3, 6, i) &= C_{12}^1 C_{13}^2 A_i + C_{12}^2 C_{13}^2 B_i + (C_{32}^1 D_i) \\
(1, 6, i) &= C_{23}^1 C_{12}^3 A_i + C_{23}^2 C_{23}^1 B_i + C_{23}^3 C_{23}^1 D_i & (3, 7, i) &= -C_{23}^1 C_{13} D_i - C_{23}^2 C_{13} D_i + C_{23}^3 C_{13} D_i \\
(1, 7, i) &= -C_{13}^1 C_{12}^3 A_i - (C_{23}^2 B_i - C_{23}^3 C_{23} D_i) & (3, 8, i) &= C_{12}^1 C_{13} D_i + C_{12}^2 C_{13} D_i + C_{12}^3 C_{13} D_i \\
(1, 8, i) &= C_{13}^1 C_{12}^2 A_i + C_{12}^2 C_{13} D_i + (C_{32}^1 D_i) & (3, 9, i) &= -C_{12}^2 C_{13} D_i - (C_{32}^3 D_i) - C_{12}^3 C_{13} D_i \\
(1, 9, i) &= -C_{12}^1 C_{23}^3 A_i - C_{23}^2 C_{23} D_i - C_{23}^3 C_{23} D_i & (4, 7, i) &= (C_{12}^3 D_i) A_i + C_{12}^3 C_{23} D_i + C_{12}^3 C_{23} D_i \\
(2, 4, i) &= -C_{13}^1 C_{23}^3 A_i - C_{23}^2 C_{13} B_i - C_{23}^3 C_{13} D_i & (4, 8, i) &= -C_{13}^1 C_{13}^2 D_i - C_{13}^2 C_{13} B_i - C_{13}^3 C_{13} D_i \\
(2, 5, i) &= C_{13}^1 C_{13}^2 A_i + C_{13}^2 C_{13} D_i + (C_{32}^3 D_i) & (4, 9, i) &= C_{12}^2 C_{13} D_i + C_{12}^2 C_{13} D_i + C_{12}^3 C_{13} D_i \\
(2, 6, i) &= -C_{12}^1 C_{13}^2 A_i - C_{23}^2 C_{13} B_i - C_{23}^3 C_{13} D_i & (5, 7, i) &= -C_{13}^1 C_{13}^2 A_i - C_{13}^2 C_{23} B_i - C_{13}^3 C_{23} D_i \\
(2, 7, i) &= C_{13}^1 C_{23}^3 A_i + C_{23}^2 C_{23} D_i + C_{23}^3 C_{13} D_i & (5, 8, i) &= (C_{13}^3 D_i) A_i + C_{13}^3 C_{23} D_i + C_{13}^3 C_{23} D_i \\
(2, 8, i) &= -C_{13}^1 C_{23}^2 A_i - (C_{23}^2 B_i - C_{23}^3 C_{23} D_i) & (5, 9, i) &= -C_{13}^2 C_{13} D_i - C_{13}^2 C_{23} B_i - C_{13}^2 C_{23} D_i \\
(2, 9, i) &= C_{13} C_{13} A_i + C_{13} C_{23} B_i + C_{23}^3 C_{13} D_i & (6, 7, i) &= C_{13} C_{13}^2 A_i + C_{13} C_{23} B_i + C_{23}^3 C_{13} D_i \\
(3, 4, i) &= C_{13} C_{13} A_i + C_{23}^2 C_{13} B_i + C_{23}^3 C_{13} D_i & (6, 8, i) &= -C_{13} C_{13} A_i - C_{13} C_{23} B_i - C_{13} C_{23} D_i \\
& & (6, 9, i) &= (C_{13}^2 D_i) A_i + C_{13} C_{13}^2 B_i + C_{13} C_{13} D_i \\
\end{align*}
\]

where

\[
A_1 = \begin{pmatrix} C_{12}^1 C_{13} & C_{12}^2 C_{13} \\ C_{13} & C_{13} \end{pmatrix}, \quad A_2 = \begin{pmatrix} C_{12}^1 C_{13} & C_{12}^2 C_{13} \\ C_{13} & C_{13} \end{pmatrix}, \quad A_3 = \begin{pmatrix} C_{12}^2 C_{13} & C_{12}^3 C_{13} \\ C_{13} & C_{13} \end{pmatrix}
\]
\[
B_1 = \begin{vmatrix} C_{12}^1 & C_{12}^2 \\ C_{23}^1 & C_{23}^2 \end{vmatrix}, \quad B_2 = \begin{vmatrix} C_{12}^3 & C_{12}^2 \\ C_{23}^3 & C_{23}^2 \end{vmatrix}, \quad B_3 = \begin{vmatrix} C_{12}^3 & C_{12}^1 \\ C_{23}^3 & C_{23}^1 \end{vmatrix}
\]
\[
D_1 = \begin{vmatrix} C_{13}^1 & C_{13}^2 \\ C_{23}^1 & C_{23}^2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} C_{13}^3 & C_{13}^2 \\ C_{23}^3 & C_{23}^2 \end{vmatrix}, \quad D_3 = \begin{vmatrix} C_{13}^3 & C_{13}^1 \\ C_{23}^3 & C_{23}^1 \end{vmatrix}
\]

Thus from the computations, we observe that all 2 by 2 minors of \( M \) are either factored into 2 by 2 minors of \( C \) or can be written as sums involving such minors of \( C \). We give the following proposition.

**Proposition 3.2.** Let \( H \) be a 3-dimensional skew symmetric algebra with structure constants \( \{C_{ij}^k\}_{i < j} \) and \( C \) be the matrix of structure constants. If rank \( C \) is 2 then the dimension of the space of possible endomorphisms, \( \text{HomLie}(\mu) \), is 7.

**Proof.** From previous result in proposition 3.1, \( M \) is of full rank if and only if \( \det C = 0 \). This implies that if rank \( C \) is 2 then \( M \) is not of full rank. We show that if rank \( C \) is 2 then there exists some non-zero 2 by 2 minor of \( M \). With rank \( C = 2 \) then it means that one of the rows can be written as a linear combination of the other two rows. That is Row \( i = \alpha \) Row \( j + \beta \) Row \( k \) for rows \( i, j, k \) and \( \alpha, \beta \in \mathbb{K} \). We consider all the possible cases.

**Case 1:** \( \alpha, \beta \neq 0 \).

We begin with the case where the first two rows are linearly independent.

If Row 3 = \( \alpha \) Row 1 + \( \beta \) Row 2 then

\[
D_i = -\alpha A_i \quad \text{and} \quad B_i = \beta A_i
\]  

(6)

At least one of the 2 by 2 minors is non-zero since \( C \) is of rank 2. Thus since \( A_i \) is a common factor for both \( B_i \) and \( D_i \) for \( i = 1, 2, 3 \), it is enough to consider only \( A_i \). So at least one \( A_i \) must be non-zero. We consider all such possible cases.

(a) \( A_1 \neq 0 \implies (2,3)_1 = A_1 \cdot A_3 \neq 0 \).

(b) \( A_1 \neq 0, A_2 \neq 0, A_3 = 0 \implies (5,6)_1 = -B_1 \cdot A_2 \neq 0 \) since \( B_1 \neq 0 \).

(c) \( A_1 \neq 0, A_2 = 0, A_3 \neq 0 \implies (1,3)_3 = A_3 \cdot B_3 \neq 0 \) since \( B_3 \neq 0 \).

(d) \( A_1 = 0, A_2 \neq 0, A_3 \neq 0 \implies (2,3)_3 = A_3 \cdot A_3 \neq 0 \).

(e) \( A_1 = 0, A_2 = 0, A_3 \neq 0 \implies (2,3)_3 = A_3 \cdot A_3 \neq 0 \).

(f) \( A_1 = 0, A_2 \neq 0, A_3 = 0 \implies (5,6)_2 = A_2 \cdot A_2 \neq 0 \).

(g) \( A_1 \neq 0, A_2 = 0, A_3 = 0 \implies (8,9)_1 = D_1 \cdot A_1 \neq 0 \) since \( D_1 \neq 0 \).

If Row 2 = \( \alpha \) Row 1 + \( \beta \) Row 3 then

\[
D_i = \alpha B_i \quad \text{and} \quad A_i = \beta B_i \implies D_i = \alpha' A_i \quad \text{and} \quad B_i = \beta' A_i
\]

which is the same case as in (6) since \( \alpha', \beta' \neq 0 \).

Similarly, if Row 1 = \( \alpha \) Row 2 + \( \beta \) Row 3 then

\[
A_i = -\beta D_i \quad \text{and} \quad B_i = \alpha D_i \implies D_i = \beta' A_i \quad \text{and} \quad B_i = \alpha' A_i
\]

which is the same case shown from (6) since \( \beta', \alpha' \neq 0 \).

**Case 2:** Either \( \alpha = 0 \) or \( \beta = 0 \) but not both.
In this case, it implies we check the cases where one of the rows can be written as a scalar product of another row.

**Row 2 = k Row 1**

We have

\[ D_i = kB_i \quad \text{and} \quad A_i = 0 \]  \hspace{1cm} (7)

So at least one \( B_i \) must be non-zero. We consider all such possible cases.

(a) \( B_1 \neq 0 \implies (4,6)_1 = B_1 \cdot B_2 \neq 0 \).
(b) \( B_1 \neq 0, B_2 \neq 0, B_3 = 0 \implies (4,6)_1 = B_1 \cdot B_2 \neq 0 \).
(c) \( B_1 \neq 0, B_2 = 0, B_3 \neq 0 \implies (7,8)_1 = D_1 \cdot D_1 \neq 0 \) since \( D_1 \neq 0 \).
(d) \( B_1 = 0, B_2 \neq 0, B_3 = 0 \implies (4,5)_2 = -B_2 \cdot D_2 \neq 0 \) since \( D_2 \neq 0 \).
(e) \( B_1 = 0, B_2 = 0, B_3 \neq 0 \implies \) Either \((3,4)_3\) or \((1,6)_3\) is non-zero.
(f) \( B_1 = 0, B_2 \neq 0, B_3 = 0 \implies (4,5)_2 = -B_2 \cdot D_2 \neq 0 \) since \( D_2 \neq 0 \).
(g) \( B_1 \neq 0, B_2 = 0, B_3 = 0 \implies (7,8)_1 = D_1 \cdot D_1 \neq 0 \) since \( D_1 \neq 0 \).

**Row 3 = k Row 1**

We have

\[ D_i = kA_i \quad \text{and} \quad B_i = 0 \]  \hspace{1cm} (8)

So at least one \( A_i \) must be non-zero. We consider all such possible cases.

(a) \( A_i \neq 0 \implies (2,3)_1 = A_1 \cdot A_3 \neq 0 \).
(b) \( A_1 \neq 0, A_2 \neq 0, A_3 = 0 \implies (8,9)_1 = D_1 \cdot A_1 \neq 0 \) since \( D_1 \neq 0 \).
(c) \( A_1 \neq 0, A_2 = 0, A_3 \neq 0 \implies (8,9)_1 = D_1 \cdot A_1 \neq 0 \) since \( D_1 \neq 0 \).
(d) \( A_1 = 0, A_2 \neq 0, A_3 \neq 0 \implies (2,3)_2 = A_2 \cdot A_3 \neq 0 \).
(e) \( A_1 = 0, A_2 = 0, A_3 \neq 0 \implies (2,3)_3 = A_3 \cdot A_3 \neq 0 \).
(f) \( A_1 = 0, A_2 \neq 0, A_3 = 0 \implies \) Either \((3,5)_2\) or \((2,6)_2\) is non-zero.
(g) \( A_1 \neq 0, A_2 = 0, A_3 = 0 \implies (8,9)_1 = D_1 \cdot A_1 \neq 0 \) since \( D_1 \neq 0 \).

**Row 3 = k Row 2**

We have

\[ B_i = kA_i \quad \text{and} \quad D_i = 0 \]  \hspace{1cm} (9)

So at least one \( A_i \) must be non-zero. We consider all such possible cases.

(a) \( A_i \neq 0 \implies (2,3)_1 = A_1 \cdot A_3 \neq 0 \).
(b) \( A_1 \neq 0, A_2 \neq 0, A_3 = 0 \implies (5,6)_1 = -B_1 \cdot A_2 \neq 0 \) since \( B_1 \neq 0 \).
(c) \( A_1 \neq 0, A_2 = 0, A_3 \neq 0 \implies (1,3)_1 = -A_1 \cdot B_3 \neq 0 \) since \( B_3 \neq 0 \).
(d) \( A_1 = 0, A_2 \neq 0, A_3 \neq 0 \implies (2,3)_3 = A_3 \cdot A_3 \neq 0 \).
(e) \( A_1 = 0, A_2 = 0, A_3 \neq 0 \implies (2,3)_3 = A_3 \cdot A_3 \neq 0 \).
(f) \( A_1 = 0, A_2 \neq 0, A_3 = 0 \implies (5,6)_2 = -B_2 \cdot A_2 \neq 0 \) since \( B_2 \neq 0 \).
(g) \( A_1 \neq 0, A_2 = 0, A_3 = 0 \implies \) Either \((3,8)_1\) or \((2,9)_1\) is non-zero.

**Case 3: \( \alpha, \beta = 0 \)**

In this case, it implies we check the cases where one of the rows is zero while the other two are linearly independent.

**Row 3 = 0**
We have
\[ B_i = D_i = 0 \text{ and } A_i \neq 0 \] (10)

So at least one \( A_i \) must be non-zero. We consider all such possible cases.

(a) \( A_1 \neq 0 \implies (2,3)_1 = A_1 \cdot A_3 \neq 0 \).
(b) \( A_1 \neq 0, A_2 \neq 0, A_3 = 0 \implies \) Either (6,9)_2 or (5,8)_2 is non-zero.
(c) \( A_1 \neq 0, A_2 = 0, A_3 \neq 0 \implies (2,3)_1 = A_3 \cdot A_3 \neq 0 \).
(d) \( A_1 = 0, A_2 \neq 0, A_3 \neq 0 \implies (2,3)_3 = A_3 \cdot A_3 \neq 0 \).
(e) \( A_1 = 0, A_2 = 0, A_3 \neq 0 \implies (2,3)_3 = A_3 \cdot A_3 \neq 0 \).
(f) \( A_1 = 0, A_2 \neq 0, A_3 = 0 \implies \) Either (6,9)_2 or (5,8)_2 is non-zero.
(g) \( A_1 \neq 0, A_2 = 0, A_3 = 0 \implies \) Either (3,8)_1 or (2,9)_1 is non-zero.

Row 2 = 0

We have
\[ A_i = D_i = 0 \text{ and } B_i \neq 0 \] (11)

So at least one \( B_i \) must be non-zero. We consider all such possible cases.

(a) \( B_1 \neq 0 \implies (4,6)_1 = B_1 \cdot B_2 \neq 0 \).
(b) \( B_1 \neq 0, B_2 \neq 0, B_3 = 0 \implies (4,6)_1 = B_1 \cdot B_2 \neq 0 \).
(c) \( B_1 \neq 0, B_2 = 0, B_3 \neq 0 \implies \) Either (3,9)_3 or (1,7)_3 is non-zero.
(d) \( B_1 = 0, B_2 \neq 0, B_3 \neq 0 \implies (4,6)_3 = B_3 \cdot B_2 \neq 0 \).
(e) \( B_1 = 0, B_2 = 0, B_3 \neq 0 \implies \) Either (3,9)_3 or (1,7)_3 is non-zero.
(f) \( B_1 = 0, B_2 \neq 0, B_3 = 0 \implies (4,6)_2 = B_2 \cdot B_2 \neq 0 \).
(g) \( B_1 \neq 0, B_2 = 0, B_3 = 0 \implies \) Either (3,9)_1 or (1,7)_1 is non-zero.

Row 1 = 0

We have
\[ A_i = B_i = 0 \text{ and } D_i \neq 0 \] (12)

So at least one \( D_i \) must be non-zero. We consider all such possible cases.

(a) \( D_1 \neq 0 \implies (7,8)_1 = D_1 \cdot D_1 \neq 0 \).
(b) \( D_1 \neq 0, D_2 \neq 0, D_3 = 0 \implies (7,8)_2 = D_2 \cdot D_1 \neq 0 \).
(c) \( D_1 \neq 0, D_2 = 0, D_3 \neq 0 \implies (7,8)_3 = D_3 \cdot D_1 \neq 0 \).
(d) \( D_1 = 0, D_2 \neq 0, D_3 \neq 0 \implies \) Either (4,8)_2 or (5,7)_2 is non-zero.
(e) \( D_1 = 0, D_2 = 0, D_3 \neq 0 \implies \) Either (1,8)_3 or (2,6)_3 is non-zero.
(f) \( D_1 = 0, D_2 \neq 0, D_3 = 0 \implies \) Either (4,8)_2 or (5,7)_2 is non-zero.
(g) \( D_1 \neq 0, D_2 = 0, D_3 = 0 \implies (7,8)_1 = D_1 \cdot D_1 \neq 0 \).

Thus we have shown that there always exists a non-zero \( 2 \times 2 \) sub-determinant of \( M \) whenever rank \( C \) is 2. This implies that dimension of HomLie(\( \mu \)) is 7.

From the list of all \( 2 \times 2 \) minors of \( M \) given, it follows that if rank \( C \) is 1 then all such minors are zero and hence the dimension of HomLie(\( \mu \)) > 7. So far, we have seen there is a correspondence between the rank of \( C \) and \( M \), for rank \( C = 2 \) and 3. However, if rank \( C = 1 \) this dimension can either be 8 or 9. It is obvious that if rank \( C = 0 \) then \( M \) is of rank 0 and consequently dim HomLie(\( \mu \)) = 9. We give possible cases for rank \( C = 1 \) with dim HomLie(\( \mu \)) = 9.
Proposition 3.3. If two skew-symmetric algebras \( \mathcal{A} = \mathcal{B} = \mathcal{A}(\alpha, \beta, \phi) \) of classical complex Lie algebras are isomorphic, then \( \mathcal{A} \rightarrow \mathcal{B} \) is isomorphic to \( \mathcal{B} \rightarrow \mathcal{A} \).

Proof. Let \( \phi : \mathcal{A} \rightarrow \mathcal{B} \) be the isomorphism of the algebras. Let \( \alpha \in \text{HomLie}(\mathcal{A}) \) and \( \beta \in \text{HomLie}(\mathcal{B}) \). The isomorphism between the Hom-Lie structures \( \varphi : \text{HomLie}(\mathcal{A}) \rightarrow \text{HomLie}(\mathcal{B}) \) is defined by \( \beta = \varphi(\alpha) := \phi \circ \alpha \circ \phi^{-1} \). Given \( x, y, z \in \mathcal{A} \), then there exists \( x', y', z' \in \mathcal{B} \) as images of \( x, y, z \) respectively under the isomorphism \( \phi \). That is, \( \phi(x) = x', \phi(y) = y' \) and \( \phi(z) = z' \). \( \beta = \phi \circ \alpha \circ \phi^{-1} \) implies \( \beta(x') = \phi(\alpha(x)), \beta(y') = \phi(\alpha(y)) \) and \( \beta(z') = \phi(\alpha(z)) \). We show that if \( \alpha \in \text{HomLie}(\mathcal{A}) \) then \( \beta \in \text{HomLie}(\mathcal{B}) \).

\[
\mu'(\beta(x'), \mu'(y', z')) + \mu'(\beta(y'), \mu'(z', x')) + \mu'(\beta(z'), \mu'(x', y'))
= \mu'(\alpha(\alpha(x)), \mu'(\phi(y'), \phi(z)) + \mu'(\phi(\alpha(y)), \mu(\phi(\alpha(z)), \mu(\phi(z), \phi(x))) + \mu'(\phi(\alpha(z)), \mu'(\phi(x), \phi(y)))
= \phi(\mu((\alpha(x), \mu(y, z)) + \mu((\alpha(y)), \mu(z, x)) + \mu((\alpha(z)), \mu(x, y)))
= 0
\]

Thus using proposition (3.3) we can describe dimension of all skew-symmetric algebras including Lie algebras by using representatives of such algebras.

4 3-dimensional Lie algebras
In this section, we use our results to deduce dim \( \text{HomLie}(\mathcal{A}) \) of classical complex Lie algebras. Such non-isomorphic representatives of all complex 3-dimensional Lie algebras can be seen in [4] and [5]. Let \( \{e_1, e_2, e_3\} \) be the basis of the 3-dimensional Lie algebras.
In [7] (proposition 7.1) the matrix of linear endomorphism $\alpha$ for all Hom-Lie algebras of $\mathfrak{sl}(2)$ type is in the form\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
2a_{13} & a_{22} & a_{31} \\
2a_{12} & a_{32} & a_{22}
\end{pmatrix},
\]
with 6 parameters $a_{11}, a_{12}, a_{13}, a_{13}, a_{22}, a_{32} \in \mathbb{K}$. Thus the space of HomLie($\mu$) for this case is of dimension 6.

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