On the similarity between Thomson scattering from high-intensity circularly polarised lasers and synchrotron radiation

Abstract
The mathematical similarity between Thomson Scattering (TS) and Synchrotron Radiation (SR) phenomena is elucidated, and some physical consequences are extracted from it. Analytical computations are reported in the frequency- and the time-domains so as to make this relationship more compelling. Algebraic equations for the relevant parameters implementing that similarity are obtained in the time-domain in their final form. Subsequently, those algebraic equations are numerically solved so as to make the connection available for a wide range of parameters in both phenomena. Prospects for a compact vacuum ultraviolet (VUV) or even x-ray source based on this analogy and using the interaction of high-intensity circularly polarized laser radiation with essentially monoenergetic electron beams are discussed.

1. Introduction and motivation
The scattering of electromagnetic radiation by free electrons, i.e. Thomson Scattering (TS), is at the basis of one of the most powerful diagnostics in fusion plasmas, being routinely used to obtain electron temperature and density with both spatial and temporal resolution [1–3]. TS is also a key ingredient in new (W7X, with first plasmas in 2015) and future (ITER) plasma devices. In particular, electron temperatures in excess of 40keV are expected in ITER when fully operational, and that will pose substantial experimental challenges due (in part, but not only) to the large contribution of relativistic effects, see for example [4].

Linear, incoherent TS is well documented, see [5–10] and references therein. By linear it is meant that the electron trajectory is very weakly perturbed by the laser, while incoherent TS means the electrons scatter the electromagnetic radiation independently of one another, so collective effects of electron motion in the computation of the scattered spectrum can be disregarded.

However, if radiation from ultra-high intensity lasers impinges upon the electron(s), the scattering becomes nonlinear and interesting new effects (harmonic generation, for instance) appear that are beyond conventional treatments of this effect. The computation of nonlinear TS (and, in general, of nonlinear radiation emission) from accelerated electrons has been addressed by several authors, using a variety of methods: analytic or semianalytic computations, Particle-in-Cell codes (PIC), or by following electron trajectories and subsequent computation of the Liénard–Wiechert radiated fields, for applications ranging from the physics and diagnosis of relativistic beams to high-energy astrophysical phenomena. These developments point to the lively and timely interest of this problem, in the context of new and powerful radiation sources (ultra-intense lasers) and their interaction with relativistic electrons. A substantial body of literature has accumulated on these subjects, see for example [11–23].

In previous papers the authors have developed Monte Carlo and analytical methods for the ab-initio computation (in a classical framework, disregarding quantum effects) of TS spectra under very general conditions regarding the electron distribution function, as well as the intensity and the polarization of the incoming laser radiation, for plasma fusion diagnostic applications [24–28]. In the course of these investigations,
the close similarity of the temporal behaviour of radiated fields coming from TS from a circularly polarized laser (TSC) and Synchrotron Radiation (SR) has emerged. The present paper is devoted to a detailed investigation of this relationship.

The similarity to be investigated below employs features of the SR phenomenon that are not quite common in modern laboratory applications of this effect, namely, out-of-the-orbital plane radiated fields in general and, perhaps more important, electron energies ranging from hundreds of keV up to some tens of MeV have to be used. This is to be compared with modern SR facilities, where energies in excess of several GeV are now standard. Also, the mathematical treatment of the SR below has included the presence of a component of the velocity/momentum along the direction of the externally applied magnetic field \(\mathbf{B}_0\), so that the general trajectory of the charged particle is a helix with axis parallel to \(\mathbf{B}_0\), hence we shall refer to it as Helical Synchrotron Radiation (HSR). Various physical features of SR are well documented in the scientific literature (including in some cases helical trajectories): see \[39–32\], among others.

The nonlinear interaction studies of ultra-intense lasers with energetic electrons have experienced a rapid increase over the last decade, driven by the availability of the former ones, which are now much more accessible and less bulky than a few years ago. In this connection, \[33, 34\] represent two recent experimental milestones in this area of research, which are relevant for the problems dealt with in this paper, in particular \[34\], where very high order harmonics of the laser radiation are experimentally produced and studied. In these publications a more or less close connection between TS and SR is mentioned, without entering into a detailed comparison. Both phenomena deal with radiation from accelerated charges at relativistic energies, a major reason for the similarities. What is new in the present work (to the best of our knowledge) is the statement that if TS is performed with circularly polarized lasers, then, under some suitable conditions, the radiated fields for TSC and HSR in the time-domain have exactly the same form. Practical consequences can also be derived from this connection between the two phenomena: for example, a high-intensity, circularly polarized laser radiation interacting with a suitable electron population can be made a compact, tunable source of VUV or even x-ray radiation source, with spectral and polarization properties that are precisely those of Helical Synchrotron Radiation.

Even if originality is not claimed in the derivation of the formulae for the emitted fields for HSR, however we do claim it regarding the detailed comparisons (both analytical and numerical) of the HSR formulae versus TSC ones, all of which require an elaborate and complex work, as we shall see.

The paper is organized as follows: sections 2 and 3 present the bulk of the analytical work regarding to the analogy between TSC and HSR both in frequency and time domains. In section 3.5, algebraic equations are obtained that connect both phenomena in the time domain: given the relevant parameters in TSC, the algebraic equations provide the corresponding ones in HSR making the radiated fields proportional to each other up to a dilation in time and an overall phase, and conversely. Section 4 deals with the numerical solution of those algebraic equations, and the results are illustrated with some examples. Section 5 mentions the physical applications of the TSC-HSR similarity, ranging from the design of synchrotron-like light sources based on lasers to the study of energetic astrophysical phenomena under laboratory conditions. Section 6 summarizes the conclusions.

2. Towards the TSC-HSR similarity: frequency-domain analytical computations

In this section the mathematical formalism used to compare both phenomena in the frequency-domain is presented; it will provide the basis for the comparison in the time-domain in section 3. The steps to be taken closely follow the results for TS in \[24–27\]. Excellent monographs on classical electrodynamics and SR exist, which have been instrumental at the starting level of the present research: see, for example, \[29, 30, 35, 36\].

2.1. Helical electron trajectory for HSR

In the computation of the radiated magnetic field for HSR, the following assumptions will be made. The electron trajectory is helical, and its projection in a plane perpendicular to the applied constant magnetic field \(\mathbf{B}_0\) is a circle of radius \(r_0\). The helix axis is defined by the unit vector

\[
\mathbf{\kappa}_0 = \sin \psi_0 \cos \phi_0 \hat{\mathbf{i}} + \sin \psi_0 \sin \phi_0 \hat{\mathbf{j}} + \cos \psi_0 \hat{\mathbf{k}}
\]

\(\psi_0\) and \(\phi_0\) being constant polar and azimuthal angles, \(\hat{\mathbf{i}}, \hat{\mathbf{j}}\) and \(\hat{\mathbf{k}}\) are the standard orthonormal unit vectors along the X, Y and Z axis, respectively. The charged particle moves with constant velocity \(v_0 = c \beta_0 (-1 < \beta_0 < 1)\) parallel to the helix’s axis and its velocity along the circle has constant magnitude \(\beta_0 = v_0 / c\), \((0 < \beta_0 < 1)\), the corresponding angular velocity being denoted as \(\Omega_0 = c \beta_0 / r_0\). The geometry just described is general enough to capture the essentials of the HSR phenomenon and its relationship to the TSC one.
Three important mutually perpendicular unit vectors are $\mathbf{e}_0$ and

\[
\mathbf{e}_0 = \cos \phi_0 \mathbf{i} + \cos \psi_0 \mathbf{j} - \sin \psi_0 \mathbf{k}
\]

(2)

\[
\phi_0 = -\sin \phi_0 \mathbf{i} + \cos \phi_0 \mathbf{j}
\]

(3)

For HSR, $t'$ will denote in what follows the 'particle time' (to be clearly distinguished from the 'detector time' or 'retarded time at detector' denoted by $t$ when needed). The electron trajectory and its velocity are given by:

\[
\mathbf{x}(t')_{SR} = c\beta_p t' \mathbf{e}_0 + r_0 (\cos (\Omega_0 t' + \delta_0) \psi_0 + \sin (\Omega_0 t' + \delta_0) \phi_0).
\]

(4)

\[
\mathbf{v}(t')_{SR} = c\beta_p \mathbf{e}_0 + \beta_0 (-\sin (\Omega_0 t' + \delta_0) \psi_0 + \cos (\Omega_0 t' + \delta_0) \phi_0),
\]

(5)

\[
\beta(t')_{SR} = \mathbf{v}(t')_{SR}/c = \beta_x(t') \mathbf{i} + \beta_y(t') \mathbf{j} + \beta_z(t') \mathbf{k}
\]

(6)

The components $\beta_x(t')$, $\beta_y(t')$ and $\beta_z(t')$ of the normalized velocity are given in appendix A. The modulus of velocity is constant in time, and, in agreement with Special Relativity $\beta_p$ and $\beta_0$ fulfill $0 \leq \beta_p^2 + \beta_0^2 < 1$. In figure 1, the trajectory and scattering geometry considered in this paper for HSR are shown in sketch.

2.2. Frequency-domain radiated fields at detector position for HSR

The radiated field is computed far away from the electron orbit, at a point/detector located at distance $D_0 \gg r_0$ from the origin, and lying in the X-Z plane. The scattering angle is denoted $\theta_0$, hence $\mathbf{n}_0 = \sin \theta_0 \mathbf{i} + \cos \theta_0 \mathbf{k}$ is the scattering vector, assumed to be constant under the above general assumptions, and $\mathbf{y}_0 = D_0 \mathbf{n}_0$. Only the radiation part of the Liénard-Wiechert emitted fields is computed according to ($\epsilon$ and $\epsilon_0$ are the electron charge and the dielectric permittivity of vacuum, respectively):

\[
\mathbf{B}(\mathbf{y}_0, \omega)_{SR} = \frac{i\omega \mathbf{e}_0 \exp(i\omega D_0/c)}{4\pi \epsilon_0 c^2 D_0} \int_{-\infty}^{+\infty} \frac{dt'}{(2\pi)^2} \left[ \frac{\mathbf{n}_0 \times \mathbf{v}(t')_{SR}}{c} \right] \exp \left[ i\omega(t' - c^{-1} \mathbf{n}_0 \cdot \mathbf{x}(t')_{SR}) \right],
\]

(7)

\[
A \text{ priori, there could be one subtlety, namely, one could argue that the contribution of } c\beta_p t'/x(t')_{SR} \text{ (in the exponential inside the integral in equation (7)) could be larger than } D_0, \text{ for suitably large } |t'|, \text{ so that the reliability of equation (7) would be in question. However, as } 1 - \beta_p (\mathbf{n}_0 \cdot \mathbf{e}_0) = 0, \text{ the contributions due to } \left[ \int_{-\infty}^{-T} + \int_{T}^{+\infty} \right]dt' \exp[i\omega t'(1 - \beta_p (\mathbf{n}_0 \cdot \mathbf{e}_0))] \text{ for large positive } T, \text{ due to cancellations arising from oscillations in}
\]
the integrations over \( t' \), can be regarded as negligible compared to other contributions, thereby supporting \textit{a posteriori} the physical validity of equation (7).

The spectral amplitude is clearly perpendicular to the \( \mathbf{n}_0 \) vector. The vector product \( \mathbf{n}_0 \times \mathbf{v}(t')_{\text{SR}} \) is easily computed as

\[
\frac{\mathbf{n}_0}{c} \times \mathbf{v}(t')_{\text{SR}} = \mathbf{n}_0 \times \mathbf{v}(t')_{\text{SR}} = - \cos \theta'_0 \beta'_0 \mathbf{i} + (\cos \theta'_0 \beta'_0 - \sin \theta'_0 \beta'_0) \mathbf{j} + \sin \theta'_0 \beta'_0 \mathbf{k}
\]

On the other hand, by using equations (1), (2), (3) and (4), we have:

\[
\omega [t' - c^{-1} \mathbf{n}_0 \cdot \mathbf{x}(t')_{\text{SR}}] = \omega \left[ \frac{t' \omega_0}{\Omega_0} + a_{\text{SR,}c} \cos(\Omega_0 t') + a_{\text{SR,}s} \sin(\Omega_0 t') \right]
\]

\[
\omega \left[ \frac{t' \omega_0}{\Omega_0} + a_{\text{SR,}c} \sin(\Omega_0 t' - \varphi_{\omega}) \right]
\]

where we have introduced the new (Doppler) frequency \( \Omega_0 \) for HSR:

\[
\Omega_0 = \frac{\Omega_0}{1 - \beta_p (\mathbf{n}_0 \cdot \mathbf{n}_0)}
\]

and \( a_{\text{SR,}} \), \( a_{\text{SR,c}} \), \( a_{\text{SR,s}} \) and the angle \( \varphi_{\omega} \):

\[
a_{\text{SR,c}} = - \frac{n_0}{c} [\cos \delta_0 (\mathbf{\psi}_0 \cdot \mathbf{n}_0) + \sin \delta_0 (\mathbf{\phi}_0 \cdot \mathbf{n}_0)] = - a_{\text{SR}} \sin \varphi_{\omega}
\]

\[
a_{\text{SR,s}} = \frac{n_0}{c} [\sin \delta_0 (\mathbf{\psi}_0 \cdot \mathbf{n}_0) - \cos \delta_0 (\mathbf{\phi}_0 \cdot \mathbf{n}_0)] = a_{\text{SR}} \cos \varphi_{\omega}
\]

Alternative expressions for \( \Omega_0 a_{\text{SR,c}} \) and \( a_{\text{SR,s}} \) and the modulus of \( \mathbf{B}_0 \) are given in appendix A. Let us introduce the unit vector:

\[
\mathbf{u}_0 = - \cos \theta'_0 i + \sin \theta'_0 k
\]

\( \mathbf{u}_0 \) lies in the X-Z plane and is perpendicular to \( \mathbf{j} \) and to the scattering vector \( \mathbf{n}_0 \). Since the radiated magnetic field is perpendicular to \( \mathbf{n}_0 \), it can be projected onto two orthogonal quadratures along (among other choices) the unit orthogonal vectors \( \mathbf{j} \) and \( \mathbf{u}_0 \). Accordingly:

\[
\mathbf{j} \cdot \mathbf{B}(\mathbf{y}, \mathbf{\omega})_{\text{SR}} = \mathbf{i} e^{i \Omega_0 (\omega/\Omega_0)} \exp[i (\omega/\Omega_0) (D_0\Omega_0/\omega_0)] \int_{-\infty}^{+\infty} \frac{\Omega_0 dt'}{(2\pi)^2} \mathbb{A}_{\text{SR,}c}\]

\[
\mathbf{u}_0 \cdot \mathbf{B}(\mathbf{y}, \mathbf{\omega})_{\text{SR}} = \mathbf{i} e^{i \Omega_0 (\omega/\Omega_0)} \exp[i (\omega/\Omega_0) (D_0\Omega_0/\omega_0)] \int_{-\infty}^{+\infty} \frac{\Omega_0 dt'}{(2\pi)^2} \mathbb{A}_{\text{SR,}u}\]

\( t' \Omega_0 \) is dimensionless and, for the sake of the HSR-TSC comparison, we have scaled all \( \omega \)- dependent terms, so that they depend on \( \omega/\Omega_0 \) and, consequently, on \( D_0\Omega_0/\omega_0 \). The \( t' \)- independent dimensionless amplitudes \( \mathbb{A}_{\text{SR,}c}, \mathbb{A}_{\text{SR,}u}, \mathbb{A}_{\text{SR,}s}, \mathbb{A}_{\text{SR,}c}, \mathbb{A}_{\text{SR,}u}, \mathbb{A}_{\text{SR,}sc}, \mathbb{A}_{\text{SR,}uc} \) and \( \mathbb{A}_{\text{SR,se}} \) are given in appendix B.

### 2.3. Frequency-domain radiated fields at detector position for TSC

Let an incoming circularly polarized monochromatic plane wave propagate along \( \mathbf{k} \) (with frequency \( \omega_0 \) and free relativistic electron, giving rise to Thomson Scattering. The scattered radiated (magnetic) TSC field is:

\[
\mathbf{B}(\mathbf{y}, \mathbf{\omega}_{\text{TS}}) = \mathbf{i} e^{i \omega (R/c)} \frac{c}{4\pi e_0 c^2 \gamma_1} \mathbf{n} \times \int_{-\infty}^{+\infty} \frac{d\xi}{(2\pi)^2} \mathbf{p}(\xi) \exp[i \omega (\Lambda(\xi)/c)]
\]

The scattering unit vector \( \mathbf{n} \) indicates the direction at which a detector is located at \( \mathbf{y} : \mathbf{n} = - \sin \theta_0 \mathbf{i} + \cos \theta_0 \mathbf{k} \) and \( \mathbf{n} = R^{-1} \mathbf{y} \). In turn:

\[
\Lambda(\xi)/c = \xi + \frac{1 - \cos \theta_0 x_3(\xi) - \sin \theta_0 x_3(\xi)}{c}
\]

\( x(\xi) = x_1(\xi) i + x_2(\xi) j + x_3(\xi) k \) is the solution of the dynamical equations of motion for the incoming circularly polarized monochromatic plane wave and \( \xi = t' - x_3(\xi)/c, t' \) being the ‘radiation’ time for the electron; \( \mathbf{p}(\xi) \) is the momentum of the radiating electron, also following from those dynamical equations [24, 25]. \( x_1(\xi), x_2(\xi), x_3(\xi) \) and \( \mathbf{p}(\xi) \) are collected in appendix C. The quantities \( \gamma_1, f_1 \) and \( f_2 \), employed below, are
integration constants of the dynamical equations of motion \cite{24, 25}. $\Lambda (\xi)$ can be recast as:

$$\Lambda (\xi) = \frac{1}{\omega_0} (g_{0,C} + g_{1,C} \omega_0 \xi + g_{2,C} \sin(\omega_0 \xi - \varphi_C))$$  \hfill (18)$$

\begin{align*}
g_{0,C} &= -\sin \theta_0 \left[ \frac{\omega_0 x_{0,1}}{c} + \frac{\epsilon E_0}{\gamma_1 \omega_0} \right] (1 - \cos \theta_0) \left[ \frac{\omega_0 x_{3,0}}{c} + \frac{\epsilon E_0 f_1}{\gamma_1^2 \omega_0} \right] \\
g_{1,C} &= 1 - f_1 \sin \theta_0 + (1 - \cos \theta_0) \left[ \frac{2}{c} + \frac{\epsilon E_0}{\gamma_1^2 \omega_0} \left( f_2 + \frac{\epsilon E_0}{\omega_0} \right) \right] \\
g_{2,C} \cos \varphi &= -\left( 1 - \cos \theta_0 \right) \frac{\epsilon E_0 (f_2 + \epsilon E_0 / \omega_0)}{\gamma_1^2 \omega_0} \\
g_{2,C} \sin \varphi &= -\sin \theta_0 \frac{\epsilon E_0}{\gamma_1 \omega_0} + (1 - \cos \theta_0) \frac{\epsilon E_0 f_1}{\gamma_1^2 \omega_0} 
\end{align*}$$  \hfill (19)$$

One has:

$$\mathbf{n} \times \mathbf{p}(\xi) = -\cos \theta_0 p_z(\xi) \mathbf{i} + (\cos \theta_0 p_x(\xi) - \sin \theta_0 p_y(\xi)) \mathbf{j} + \sin \theta_0 p_z(\xi) \mathbf{k}$$  \hfill (23)$$

Like for HSR, we introduce the unit vector in the X-Z plane, perpendicular to both $\mathbf{j}$ and $\mathbf{n}$:

$$\mathbf{u} = -\cos \theta \mathbf{i} + \sin \theta \mathbf{k}$$  \hfill (24)$$

We shall project the spectral amplitude $\mathbf{B}(\mathbf{y}, \omega)_{TS}$ on the orthonormal vectors $\mathbf{j}$ and $\mathbf{u}$:

$$\begin{align*}
\mathbf{j} \cdot \mathbf{B}(\mathbf{y}, \omega)_{TS} &= \frac{i(\omega g_{0,C}/\omega_0)(\omega_0 / g_{0,C}) e \exp[i(\omega g_{0,C}/\omega_0)(R \omega_0 / \epsilon g_{0,C})]}{4 \pi \epsilon c^2 \gamma_1 R \omega_0} \\
& \times \int_{-\infty}^{+\infty} \frac{\omega_0 d\xi}{(2\pi)^{1/2}} \left[ \Lambda_{TS,\xi,0} + \Lambda_{TS,\xi,1} \cos(\omega_0 \xi) + \Lambda_{TS,\xi,2} \sin(\omega_0 \xi) \right] \\
& \times \exp \left[ i \left( \frac{\omega g_{0,C}}{\omega_0} \right) \omega_0 \xi \right] \exp \left[ i \left( \frac{\omega g_{0,C}}{\omega_0} \right) \frac{\epsilon E_0 (g_{2,C} / g_{0,C}) \sin(\omega_0 \xi - \varphi)}{\gamma_1^2 \omega_0} \right] = \tilde{B}_j(\omega)_{TS}
\end{align*}$$  \hfill (25)$$

$$\begin{align*}
\mathbf{u} \cdot \mathbf{B}(\mathbf{y}, \omega)_{TS} &= \frac{i(\omega g_{0,C}/\omega_0)(\omega_0 / g_{0,C}) e \exp[i(\omega g_{0,C}/\omega_0)(R \omega_0 / \epsilon g_{0,C})]}{4 \pi \epsilon c^2 \gamma_1 R \omega_0} \\
& \times \int_{-\infty}^{+\infty} \frac{\omega_0 d\xi}{(2\pi)^{1/2}} \left[ \Lambda_{TS,\xi,0} + \Lambda_{TS,\xi,1} \cos(\omega_0 \xi) \right] \\
& \times \exp \left[ i \left( \frac{\omega g_{0,C}}{\omega_0} \right) \omega_0 \xi \right] \exp \left[ i \left( \frac{\omega g_{0,C}}{\omega_0} \right) \frac{g_{2,C} / g_{0,C} \sin(\omega_0 \xi - \varphi)}{\gamma_1^2 \omega_0} \right] = \tilde{B}_u(\omega)_{TS}
\end{align*}$$  \hfill (26)$$

Notice that $\omega_0 \xi$ is dimensionless and that we have approximated $\exp[i(\omega_0 R / c) + (g_{0,C}/\omega_0)] \simeq \exp[i(\omega_0 R / c)]$ in the quadratures given in equations (25) and (26), since $(R / c) \gg (g_{0,C} / \omega_0)$. We have scaled all $\omega$-dependent terms (for the sake of the HSR-TSC comparison), so that they depend on $\omega g_{0,C}/\omega_0$ and, consequently, on $R \omega_0 / \epsilon g_{0,C}$. The $\xi$-dependent amplitudes $\Lambda_{TS,\xi,0}$, $\Lambda_{TS,\xi,1}$, $\Lambda_{TS,\xi,2}$, $\Lambda_{TS,\xi,0}$, and $\Lambda_{TS,\xi,1}$ are given in appendix D.

There is no contribution accompanying the $\sin(\omega_0 \xi)$ term in $\mathbf{u} \cdot \mathbf{B}(\mathbf{y}, \omega)_{TS}$. In order to overcome the latter limitation, we could introduce two new orthonormal unit vectors $\mathbf{j}'$ and $\mathbf{u}'$ which are orthonormal to each other and to $\mathbf{n}$. We shall omit the specific expressions for $\mathbf{j}'$ and $\mathbf{u}'$, which would require a suitable rotation angle in the $\mathbf{j}-\mathbf{u}$ plane. As before, we could project the spectral amplitude $\mathbf{B}(\mathbf{y}, \omega)_{TS}$ on the orthonormal vectors $\mathbf{j}'$ and $\mathbf{u}'$:

$$\begin{align*}
\mathbf{j}' \cdot \mathbf{B}(\mathbf{y}, \omega)_{TS} &= \frac{i(\omega g_{0,C}/\omega_0)(\omega_0 / g_{0,C}) e \exp[i(\omega g_{0,C}/\omega_0)(R \omega_0 / \epsilon g_{0,C})]}{4 \pi \epsilon c^2 \gamma_1 R \omega_0} \\
& \times \int_{-\infty}^{+\infty} \frac{\omega_0 d\xi}{(2\pi)^{1/2}} \left[ \Lambda_{TS,\xi,0} + \Lambda_{TS,\xi,1} \cos(\omega_0 \xi) \right] \\
& \times \exp \left[ i \left( \frac{\omega g_{0,C}}{\omega_0} \right) \omega_0 \xi \right] \exp \left[ i \left( \frac{\omega g_{0,C}}{\omega_0} \right) \frac{g_{2,C} / g_{0,C} \sin(\omega_0 \xi - \varphi)}{\gamma_1^2 \omega_0} \right] = \tilde{B}_j'(\omega)_{TS}
\end{align*}$$  \hfill (27)$$
\[ u' \cdot \hat{B}(y, \omega)_{TS} = \frac{i(\omega g_{2,C}/\omega_0)(\omega_0/g_{1,C}) e \exp[i(\omega g_{2,C}/\omega_0)(R\omega_0/\epsilon g_{1,C})]}{4\pi \epsilon_0 c^2 \gamma R\omega_0} \]

\[ \times \int_{-\infty}^{+\infty} \frac{\omega_0 d\xi}{(2\pi)^{1/2}} \left[A_{TS,u',\omega} + A_{TS,u,\omega} \cos(\omega_0 \xi) + A_{TS,u,\omega} \sin(\omega_0 \xi)\right] \]

\[ \times \exp\left[i\left(\frac{\omega g_{2,C}}{\omega_0}\right)\omega_0 \xi\right] \exp\left[i\left(\frac{\omega g_{2,C}}{\omega_0}\right)(g_{2,C}/g_{1,C}) \sin(\omega_0 \xi - \varphi_l)\right] \equiv \hat{B}_{u'}(\omega)_{TS} \]

where one now finds a contribution accompanying the \(\sin(\omega_0 \xi)\) term in \(u' \cdot \hat{B}(y, \omega)_{TS}\). We shall omit for brevity the specific expressions for the \(\xi\)-independent amplitudes \(A_{TS,u,\omega}^\prime\), \(A_{TS,u',\omega}^\prime\), \(A_{TS,u,\omega}^\prime\), \(A_{TS,u',\omega}^\prime\), and \(A_{TS,u,\omega}^\prime\). Of course, if \(j' = j\) and \(u' = u\), then \(A_{TS,u,\omega} = 0\) and the resulting \(A_{TS,u,\omega}^\prime\), \(A_{TS,u',\omega}^\prime\), \(A_{TS,u,\omega}^\prime\), \(A_{TS,u',\omega}^\prime\), and \(A_{TS,u,\omega}^\prime\) are given in appendix D. In the following developments, we shall make use of \(\hat{B}_{u'}(\omega)_{TS}\) and \(\hat{B}_{u}(\omega)_{TS}\), as they are more general. However, upon proceeding to numerical computations (in the time-domain), we shall restrict to \(j' = j\) and \(u' = u\), hence dealing with \(\hat{B}_{u}(\omega)_{TS}\) and \(\hat{B}_{u}(\omega)_{TS}\).

### 3. The mathematical basis of the TSC-HSR similarity: time-domain analytical computations and comparison

In this section the mathematical formalism used to compare both phenomena in the time-domain is presented. Time-domain analytical computations of fields for HSR and TSC have been operational to first suggest the connection investigated in this paper.

Using the standard expansion [37]

\[ \exp(ix \sin \Phi) = \sum_{l=-\infty}^{\infty} \exp(i\Phi) J_l(x), \]  

and applying it successively to \(\exp(i(\omega/\Omega_D)(\Omega_D a_{SR}\sin(\Omega_D t' - \varphi_\omega)))\) inside the integrals in \(\hat{B}_{u}(\omega)_{SR}\) and \(\hat{B}_{u}(\omega)_{SR}\) and to the factor \(\exp[i\left(\frac{\omega}{\Omega_D}\right)\omega_0 \xi \times \exp[i\left(\frac{\omega}{\Omega_D}\right)(g_{2,C}/g_{1,C}) \sin(\omega_0 \xi - \varphi_l)]]\) inside the integrals in \(\hat{B}_{u}(\omega)_{TS}\) and \(\hat{B}_{u}(\omega)_{TS}\), one can express \(\hat{B}_{u}(\omega)_{SR}, \hat{B}_{u}(\omega)_{SR}, \hat{B}_{u}(\omega)_{TS}\) and \(\hat{B}_{u}(\omega)_{TS}\) as series in terms of the Bessel Functions of the first kind \((J_l(x), l = 0, \pm 1, \pm 2, \pm 3, \ldots)\). In turn, in those series we use

\[ \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi}} \exp[i(\Omega + l)\tau] = \sqrt{2\pi} \delta(\Omega + l), \delta(\Omega)\text{ being the Dirac delta function, and related ones. Those integrals enable to carry out the integrations over }\Omega_D t' = \Omega_D t\text{ in }\hat{B}_{u}(\omega)_{SR}\text{ and }\hat{B}_{u}(\omega)_{SR}\text{ and over }\omega_0 \xi = \Omega_D \text{ in }\hat{B}_{u}(\omega)_{TS}\text{ and }\hat{B}_{u}(\omega)_{TS}.\]

We shall omit the lengthy intermediate computations and the final expansions for \(\hat{B}_{u}(\omega)_{SR}, \hat{B}_{u}(\omega)_{SR}, \hat{B}_{u}(\omega)_{TS}\) and \(\hat{B}_{u}(\omega)_{TS}\) as series in terms of the Bessel functions of the first kind \((J_l(x))\) and Dirac delta functions. We emphasize the important simplifying role played by the properties of the \(J_l(x)\)’s summarized in appendix E: the first three allow to reduce \(\sum_{l=-\infty}^{\infty} \) to \(\sum_{l=1}^{\infty} \) while the fourth one allows to replace \(J_{l+1}(x) + J_{l-1}(x)\) by \((2\lambda/x)J_l(x)\).

Next, we shall apply the inverse Fourier transform so as to proceed to the time-domain, namely, \(\int_{-\infty}^{+\infty} \frac{dx}{(2\pi)^{1/2}} \exp(-ix\zeta)\) \((\tau\text{ being the time variable})\) applied to \(\hat{B}_{u}(\omega)_{SR}, \hat{B}_{u}(\omega)_{SR}, \hat{B}_{u}(\omega)_{TS}\) and \(\hat{B}_{u}(\omega)_{TS}\). The resulting formulae for HSR and for TSC in the time-domain are given in the following two subsections, omitting intermediate computations.

#### 3.1. Time-domain radiated fields at detector position for HSR

One obtains the following explicitly real expansions into harmonics \((\zeta = \Omega_D a_{SR}, \varphi'_\omega = \varphi_\omega + D_0 \Omega_D/c)\):

\[ \int_{-\infty}^{+\infty} \frac{d\zeta}{(2\pi)^{1/2}} \hat{B}_{u}(\omega)_{SR} = \frac{e(\Omega_D)^2}{4\pi \epsilon_0 c^2 D_0 \Omega_D} \sum_{l=1}^{\infty} (-1)^l [J_l(l\zeta)] \]

\[ \times (-2A_{SR,\omega} + \frac{2}{\zeta} [\sin \varphi_\omega A_{SR,\omega} + \cos \varphi_\omega A_{SR,\omega}]) \sin l(\varphi'_\omega - \Omega_D t) \]

\[ - (J_{l+1}(l\zeta) - J_{l-1}(l\zeta)) (\cos \varphi_\omega A_{SR,\omega} - \sin \varphi_\omega A_{SR,\omega}) \cos l(\varphi'_\omega - \Omega_D t) \]

\[ \int_{-\infty}^{+\infty} \frac{d\zeta}{(2\pi)^{1/2}} \hat{B}_{u}(\omega)_{SR} = \frac{e(\Omega_D)^2}{4\pi \epsilon_0 c^2 D_0 \Omega_D} \sum_{l=1}^{\infty} (-1)^l [J_l(l\zeta)] \]

\[ \times (-2A_{SR,\omega} + \frac{2}{\zeta} [\sin \varphi_\omega A_{SR,\omega} + \cos \varphi_\omega A_{SR,\omega}]) \sin l(\varphi'_\omega - \Omega_D t) \]

\[ - (J_{l+1}(l\zeta) - J_{l-1}(l\zeta)) (\cos \varphi_\omega A_{SR,\omega} - \sin \varphi_\omega A_{SR,\omega}) \cos l(\varphi'_\omega - \Omega_D t) \]
3.2. Time-domain radiated fields at detector position for TSC

We shall present the time-domain radiated field, for the general choice \( \hat{j}' = \hat{j} \) and \( u' = u \), as a spin-off to the calculations in the frequency domain. One gets the following explicitly real expansions into harmonics 
\[
\psi_c' = \phi_c + \left. \frac{\rho_0}{\omega_0} \right|_{r_C} \text{ and } \omega_D = \omega_0 / B_{1,C} \text{ being the Doppler frequency for TSC}:
\]
\[
\int_{-\infty}^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} B^c(\omega)_{TS} = \frac{\omega_0}{4\pi \epsilon_0 c^2 R_1 B_{1,C}} \sum_{l=1}^{+\infty} (-1)^l [\gamma_j(l\zeta')] \\
\times (-2A_{TS,j',\omega} + \frac{2}{\zeta}[\sin \phi_j A_{TS,j',\omega} + \cos \phi_j A_{TS,j',\omega}]) \sin \{l\phi_c' - \omega_D t\} \\
- (\Omega_{j+1}(l\zeta') - \Omega_{j-1}(l\zeta'))(\cos \phi_j A_{TS,j',\omega} - \sin \phi_j A_{TS,j',\omega}) \cos \{l\phi_c' - \omega_D t\} \\
\int_{-\infty}^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} B^c(\omega)_{TS} = \frac{\omega_0}{4\pi \epsilon_0 c^2 R_1 B_{1,C}} \sum_{l=1}^{+\infty} (-1)^l [\gamma_j(l\zeta')] \\
\times (-2A_{TS,u',\omega} + \frac{2}{\zeta}[\sin \phi_j A_{TS,u',\omega} + \cos \phi_j A_{TS,u',\omega}]) \sin \{l\phi_c' - \omega_D t\} \\
- (\Omega_{j+1}(l\zeta') - \Omega_{j-1}(l\zeta'))(\cos \phi_j A_{TS,u',\omega} - \sin \phi_j A_{TS,u',\omega}) \cos \{l\phi_c' - \omega_D t\}
\]

3.3. Time-domain: comparison of radiated fields for HSR and TSC in the general case (\( \hat{j}' = \hat{j} \) and \( u' = u \))

We shall now impose that the above projections of the spectral HSR and TSC fields have entirely similar functional forms and variations in the time-domain and we shall find the conditions that warrant those similarities. Specifically, we shall impose that the time-domain radiated field for HSR, as a function of \( t \frac{\omega_0}{B_{1,C}} \) and the one for TSC, as a function of \( t \frac{\omega_0}{B_{1,C}} \), have similar functional structures, after having performed suitable changes in the time scales for both HSR and TSC (say, suitable scalings \( t \Omega_D \rightarrow \tau \) for HSR and \( t \frac{\omega_0}{B_{1,C}} \rightarrow \tau \) for TSC).

An inspection of the arguments of the various Bessel functions leads to the necessary condition (equivalent to \( \zeta = \zeta' \)):
\[
\frac{1}{\Omega_D} = \frac{\zeta_D}{\zeta}
\]

The two phase angles \( \phi_{j,\omega} \) and \( \phi_{j,\omega} \) differ from each other (\( \phi_{j,\omega} \neq \phi_{j,\omega} \)) in general, although in special cases it could well be that \( \phi_{j,\omega} = \phi_{j,\omega} \). Similarly, in the actual time-domain comparison, \( D_0 \Omega_D \) and \( R_0 \omega_0 / B_{1,C} \) are not necessarily related to each other, either. Rather, it appears more adequate to regard \( \phi_{j,\omega} + \frac{D_0 \Omega_D}{\zeta} \) and \( \phi_{j,\omega} + \frac{R_0 \omega_0}{B_{1,C}} \) just as constant phase shifts which do not prevent the HSR-TSC relationship. Then, no restriction between \( \phi_{j,\omega} + \frac{D_0 \Omega_D}{\zeta} \) and \( \phi_{j,\omega} + \frac{R_0 \omega_0}{B_{1,C}} \) needs be imposed for the relationship between HSR and TSC to hold.

Next, we shall combine the HSR formulae with the general set of expressions for the time-domain TSC radiated fields (namely, for \( \hat{j}' = \hat{j} \) and \( u' = u \)). Additional conditions, warranting that the projections of HSR and TSC fields have entirely similar functional forms and variations in the time-domain, are:
\[
\begin{align*}
- \frac{A_{SR,j,\omega}}{\zeta_{SR,j,\omega}} + \frac{1}{\zeta}(\sin \phi_{j,\omega} A_{SR,j,\omega} + \cos \phi_{j,\omega} A_{SR,j,\omega}) \\
- \frac{A_{TS,j',\omega}}{\zeta_{TS,j',\omega}} + \frac{1}{\zeta}(\sin \phi_{j,\omega} A_{TS,j',\omega} + \cos \phi_{j,\omega} A_{TS,j',\omega}) \\
= \cos \phi_{j,\omega} A_{SR,j,\omega} - \sin \phi_{j,\omega} A_{SR,j,\omega} \\
= \cos \phi_{j,\omega} A_{TS,j',\omega} - \sin \phi_{j,\omega} A_{TS,j',\omega}
\end{align*}
\]

\[
\begin{align*}
- \frac{A_{SR,j,\omega}}{\zeta_{SR,j,\omega}} + \frac{1}{\zeta}(\sin \phi_{j,\omega} A_{SR,j,\omega} + \cos \phi_{j,\omega} A_{SR,j,\omega}) \\
- \frac{A_{TS,u',\omega}}{\zeta_{TS,u',\omega}} + \frac{1}{\zeta}(\sin \phi_{j,\omega} A_{TS,u',\omega} + \cos \phi_{j,\omega} A_{TS,u',\omega}) \\
= \cos \phi_{j,\omega} A_{SR,j,\omega} - \sin \phi_{j,\omega} A_{SR,j,\omega} \\
= \cos \phi_{j,\omega} A_{TS,u',\omega} - \sin \phi_{j,\omega} A_{TS,u',\omega}
\end{align*}
\]

We emphasize the role of the properties, summarized in appendix E, of the Bessel functions: they have led to the structures in the various equations above and, in particular, to the fact that the conditions warranting the TSC-HSR connection be independent on the \( f_0(x) \)'s and hold for any harmonic \( h \).

We shall also impose that the ratio in equation (34) has the same value as the ratio in equation (35) for the general case \( \hat{j}' = \hat{j} \) and \( u' = u \), as it is physically natural that those ratios be the same for both projections upon formulating the TSC-HSR relationship. And, once the later condition has been imposed, it will be useful to compute the common value of those ratios, which will give rise to another equation. All in all, we are left with a total of five equations as the basic framework for the TSC-HSR relationship.
In section 3.5, we shall treat (in addition to equation (33)) equations (34) and (35) for \( \hat{\mathbf{J}} = \mathbf{j} \) and \( \mathbf{u}' = \mathbf{u} \). And, in so doing, we shall also write explicitly the condition that the ratio in equation (34) has the same value as the ratio in equation (35) as well as the equation which determines the latter ratio (denoted below as \( 1/\tilde{g}_2 \)). See equations (36) and (41).

3.4. Comparison of frequency-domain radiated fields for HSR and TSC: a short discussion

Let the conditions that warrant the similarities of the projections of the spectral HSR and TSC fields in the time-domain be imposed, namely, equations (33)–(35) and the additional equation expressing the equality between (34) and (35). Then, one can apply the Fourier transform to proceed back from the time-domain to the frequency one (namely, \( \int_{-\infty}^{+\infty} \frac{\exp(i\omega t)}{2\pi} dt \)), that is, to equations (29)–(32). Then, \( \mathbf{j} \cdot \mathbf{B}(\gamma_0, \omega)_{\text{SR}} \) as a function of \( \omega/\Omega_B \) has the same functional structure as \( \mathbf{j}' \cdot \mathbf{B}(\gamma, \omega)_{\text{TS}} \) has as a function of \( \omega_{\text{GC}}/\omega_0 \), and the same holds for \( \mathbf{u}_0 \cdot \mathbf{B}(\gamma_0, \omega)_{\text{SR}} \) and \( \mathbf{u}' \cdot \mathbf{B}(\gamma, \omega)_{\text{TS}} \).

We emphasize here the difficulties involved in trying to arrive at the above conditions (33)–(35) and so on) directly in the frequency-domain. In fact, inspections of \( \mathbf{j} \cdot \mathbf{B}(\gamma_0, \omega)_{\text{SR}}, \mathbf{j}' \cdot \mathbf{B}(\gamma, \omega)_{\text{TS}}, \mathbf{u}_0 \cdot \mathbf{B}(\gamma_0, \omega)_{\text{SR}} \) and \( \mathbf{u}' \cdot \mathbf{B}(\gamma, \omega)_{\text{TS}} \) in the frequency domain do not appear to lead in a direct or transparent way to equations (33)–(35) and so on. This justifies, a posteriori, the convenience of having proceeded first to the time-domain.

3.5. Algebraic equations connecting TSC and HSR for \( \hat{\mathbf{j}} = \mathbf{j} \) and \( \mathbf{u}' = \mathbf{u} \): explicit dimensionless form

We shall concentrate here on equations (34) and (35) for the important case \( \hat{\mathbf{j}} = \mathbf{j} \) and \( \mathbf{u}' = \mathbf{u} \). By using all the definitions given in appendix D, with \( \zeta = \Omega_B \omega_{\text{SR}} = \zeta' = \omega_{\text{GC}}/\Omega_B \), introducing the new angle \( \delta''_0 = \delta_0 + \phi_0 \), the normalised variables \( p_0 = (p_0/m) \), \( p_y = (p_0/m) \), \( p_{\text{sr}} = (p_0/m) \), \( \gamma_0 = \gamma_{\text{GC}}/m \), \( \gamma = \gamma_{\text{GC}}/m \), \( p_0 = \text{modulus of the initial momentum} \), \( \p(\xi = 0) \) of the electron, from appendix C), and after a little algebra equations (34) and (35) can be recast into the following explicit dimensionless equations that connect TSC and HSR:

\[
\frac{F_{\text{SR},j,s}}{F_{\text{TS},j,s}} = \frac{F_{\text{SR},j,c}}{F_{\text{TS},j,c}} = \frac{1}{g_2} \tag{36}
\]

\[
F_{\text{SR},j,s} = \beta_\varphi (\cos \theta_0' \sin \psi_0 \cos \phi_0 - \sin \theta_0' \cos \psi_0) \\
+ \frac{\beta_0}{\zeta} (\cos \theta_0' [\cos \psi_0 \cos \phi_0 \sin \delta''_0 + \sin \phi_0 \cos \delta''_0] + \sin \theta_0' \sin \psi_0 \sin \delta''_0) \tag{37}
\]

\[
F_{\text{SR},j,c} = \beta_\varphi (\cos \theta_0' \sin \psi_0 \cos \phi_0 \cos \delta''_0 - \sin \phi_0 \sin \delta''_0 + \sin \theta_0' \sin \psi_0 \cos \delta''_0) \tag{38}
\]

\[
F_{\text{TS},j,s} = \cos \theta_0 p_{0x} - \sin \theta_0 \left( \frac{p_{0y}^2 + \left( p_{0y} + \alpha/2\pi \right)^2 + \alpha/2\pi^2 + 1 - \gamma_1^2 }{2\gamma_1} \right) \\
- \frac{1}{2\alpha/\pi} \cos \theta_0 \sin \varphi \sin \frac{\varphi}{\gamma_1} \left[ \left( \frac{P_{0y} + \alpha / 2\pi}{2\pi} \right) \cos \varphi_i P_{0x} \sin \varphi_i \right] \tag{39}
\]

\[
F_{\text{TS},j,c} = \frac{\alpha}{2\pi} \left( \frac{P_{0y} + \alpha / 2\pi}{2\pi} \right) \cos \varphi_i P_{0x} \sin \varphi_i - \left[ \cos \theta_0 - \sin \theta_0 \varphi_i P_{0x} \cos \varphi_i \right] \tag{40}
\]

\[
\frac{F_{\text{SR},u,s}}{F_{\text{TS},u,s}} = \frac{F_{\text{SR},u,c}}{F_{\text{TS},u,c}} = \frac{1}{g_2} \tag{41}
\]

\[
F_{\text{SR},u,s} = \beta_\varphi \sin \psi_0 \cos \phi_0 + \beta_0 \cos \psi_0 \sin \phi_0 \sin \delta''_0 - \cos \phi_0 \cos \delta''_0 / \zeta \tag{42}
\]

\[
F_{\text{SR},u,c} = \beta_\varphi \cos \psi_0 \sin \phi_0 \cos \delta''_0 + \cos \phi_0 \sin \delta''_0 \tag{43}
\]

\[
F_{\text{TS},u,s} = \left( P_{0y} + \frac{\alpha}{2\pi} \right) \cos \varphi_i / \zeta \tag{44}
\]

\[
F_{\text{TS},u,c} = -\frac{\alpha}{2\pi} \sin \varphi_i \tag{45}
\]

\( \alpha \) is the so-called laser parameter for TSC, given in appendix D, together with \( \theta \) and \( \phi \) (for the initial momentum \( \p(\xi = 0) \) of the electron). Notice that we have imposed directly that the ratio in equation (36) has the same value (namely, \( 1/g_2 \)) as the ratio in equation (41), for the actual case \( \hat{\mathbf{j}} = \mathbf{j} \) and \( \mathbf{u}' = \mathbf{u} \). The numerical computations will be based upon equations (33), (36) and (41). By starting out from the parameters characterizing one phenomenon, these five equations enable, in principle, to find the corresponding parameters in the other phenomenon giving a similar pattern of radiated fields (in the sense previously discussed), and conversely.
Table 1. HSR parameters fitting a given TSC run, see main text for details.

| $E_{\text{kin,TS}}$ (keV) | $\alpha$ | $\lambda_i$ (nm) | $\theta$ (deg) | $\phi$ (deg) | $\delta_0$ (deg) |
|--------------------------|--------|------------------|--------------|-------------|---------------|
| 100.0                    | $\pi$  | 1.064.0          | 121.713      | 292.508     | 99.0          |
| $E_{\text{kin,HSR}}$ (keV) | Frac  | $\psi_0$ (deg)  | $\phi_0$ (deg) | $\delta_0$ (deg) | $\delta_0'$ (deg) |
| 250.15                   | $-1.968\ 45$ | 117.849       | 179.383       | 120.93      | 5.734.89      |

4. Numerical computations backing up the analytical results

In this section the approach to the numerical solution of the algebraic equations in section 3.5 is presented. Most runs have as a target a TSC case (i.e., a set-up consisting of an electron of given momentum interacting with a large intensity laser), and then find the corresponding parameters in HSR giving the same pattern of radiated fields. The equations work in either direction however, and it is possible to begin with a set of HSR parameters to find the corresponding ones in TSC; some examples will be provided in section 5. Hundreds of different runs have been performed, varying the initial kinetic energy of the electron, the orientation of its momentum with respect to the laser propagation direction, and the laser parameter over wide ranges. The conclusion is that those algebraic equations have indeed a solution that can be efficiently computed to a high degree of accuracy. In some cases, the numerical computation has suggested an Ansatz for exact analytical solutions to the equations. We will also address here this aspect of the problem.

Before presenting a selection of cases backing up the connection between TSC-HSR, let us point out that if we just solve equations (33), (36) and (41) (five equations in total) one can find multiple solutions to them, differing (among others) in the relative Doppler frequencies at detector, namely, in $\Omega_D/\Omega_0$ for HSR and $\omega_D/\omega_0$ for TSC. This freedom/indeterminacy of solutions will allow to extend the potential physical applications, because it would be possible to connect both phenomena over largely different frequency regimes. Those aspects will also be addressed in section 5, and summarized in the Conclusions.

The algebraic equations have been numerically solved using the Mathematica® package, in particular the intrinsic function/procedure Minimizem[38]. Hence, the problem of finding a numerical solution to the algebraic equations is transformed into a minimization problem, namely, finding the minimum of a sum of square terms, each of them of the form $w_i(g_i, N_i(\zeta_1, \zeta_2, \ldots, \zeta_n) - \lambda_i)^2$. $\lambda_i$ are the target values (the denominators of equations (36) and (41)) and $N_i(\zeta_1, \zeta_2, \ldots, \zeta_n)$ are the numerators of the corresponding equations as functions of the parameters to be found ($n_p$ denoting the number of parameters). In the case of TSC target values, the HSR parameters we are looking for would be the parallel and perpendicular velocities $\beta_p, \beta_0$, the angles $\psi_0, \phi_0$, the orientation of helix axis (magnetic field) and the initial position of electron with respect to it (that is, $r_0$ and $\delta_0$), and the scattering angle $\delta_0'$. An equivalent set of HSR parameters turns out to be the kinetic energy $E_{\text{kin,HSR}}$ of the electron, the fraction of parallel to perpendicular velocities, $\text{Frac} = \beta_p/\beta_0$, and the set of angles, together with $r_0$ mentioned above. The ‘extra’ parameter $g_2$ in the minimization is always found to be numerically equal to the initial Lorentz $\gamma$ factor of the electron (namely, $|1 - (\beta_0/t = 0)_{\text{SR}}^2|^{1/2}$) and hence is not an independent one.

The problem is reduced to finding the minimum of $\sum_{i=1}^{n_p} w_i(g_i, N_i(\zeta_1, \zeta_2, \ldots, \zeta_n) - \lambda_i)^2$ as parameters $\zeta_1, \zeta_2, \ldots, \zeta_n$ vary over given ranges. $w_i$ are weights that control the relative importance of the different terms in the minimization process. The weights associated to equations (33), (36) and (41) (five equations in total) have always been set to 1.0, while the weight associated to the relative Doppler frequency at detector has been set either to 0.0 or 1.0 (a value of 0.0 for a weight means of course that this particular condition is not included in the minimization for that particular run). In the large majority of cases that have been tested, the final value for the weighed sum is found to be $\approx 10^{-15}$ or even less, meaning that minimization conditions have simultaneously been met to better than seven significant digits.

Case-study number 1: The general conditions for the comparison TSC-HSR are summarised in table 1. The value for the weighed sum of squares after minimization is $6.8 \times 10^{-16}$. In this run the equality of normalised Doppler frequencies at detector has been imposed, i.e., the corresponding weight has been set to 1.0. Figure 2 gives the corresponding radiated fields in (normalised) time domain, and the plot of $j$-quadrature versus $u$-quadrature. Taking into account that ‘standard’ synchrotron radiation set-up (with $\beta_p = 0.0$) gives necessarily $(\Omega_D/\Omega_0)_{\text{SR}} = 1.0$, a substantial $\beta_p$ is needed to meet a target value $1/\Theta_{1C} = (\omega_D/\omega_0)_{\text{TS}} = 1.576.04$ in this case. The match is excellent; 16 harmonics have been added to compute the quadratures in time domain.

Case-study number 2: Target conditions are summarised in table 2. Final value for the weighed sum of squares after minimization is $1.25 \times 10^{-15}$. In this run, that has $1/\Theta_{1C} = (\omega_D/\omega_0)_{\text{TS}} = 0.402.414$ (a redshifted electromagnetic field with respect to laser frequency), we have forced two different values for $(\Omega_D/\Omega_0)_{\text{SR}}$, namely $(\Omega_D/\Omega_0)_{\text{SR}} = 1.0$ and $(\Omega_D/\Omega_0)_{\text{SR}} = 2.0$. Figures 3, 4 give the corresponding radiated fields in time...
domain, and the plot of \( j \)-quadrature versus \( u \)-quadrature. This example shows that imposing different forcing different values for the DS\( R_0\) WW(\( )\) is possible by a simultaneous change of electron energy, helical trajectory parameters, share of parallel to perpendicular velocity and scattering angle, and this large freedom in fitting different observed Doppler frequencies opens the possibility of modeling some phenomena associated with synchrotron radiation at other frequency range (s) using the connection TSC-HSR. Also in this case, the match between both phenomena \( \text{figures } 3, 4 \) is excellent; 16 harmonics have been added to compute the quadratures.

**Case-study number 3:** Let us now present two examples of numerical computations which lead to obtain much simpler solutions to the algebraic equations. This is interesting at least for two reasons: on the one hand an explicit analytical solution to the full TSC-HSR connection can be offered at least for some combinations of parameters, and on the other hand, the solutions obtained can benchmark the numerical techniques used, increasing our confidence on their correctness under more challenging conditions. The cases to be studied now require that \( f = \pi/2 \), and a sufficiently high initial kinetic energy such that the condition \( p_{0y} + \alpha/2\pi = 0 \) (equivalent to \( \varphi_s = -\pi/2 \)) is fulfilled. In such a case, a HSR solution exists at exactly the same energy and scattering angle, and with \( \psi_0 = \phi_0 = \delta_0 = 0 \). The additional condition that the arguments inside Bessel functions be the same in TSC and HSR, fully solves the problem by providing the fraction of parallel to

![Figure 2. Comparison between the radiated TSC fields (a) and SR fields (b). The temporal shape of both quadratures in HSR is exactly the same as in TSC, once the particle velocity/energy, helix and scattering angle, etc are chosen according to the solution of the algebraic equations connecting both phenomena. Panel (c) shows \( B_u \) versus \( B_j \) for both processes; matching is excellent. In figures 2–11, B-field quadrature(s) along j and u unit vectors (time evolution signals) are always depicted in red and green, respectively. When plotting \( B_u \) versus \( B_j \), TSC data are always plotted in green, and HSR are always plotted in black (except in figure 8: see its caption for details); ‘adim’ means dimensionless.](image)

| TSC/HSR parameters (target/found) |
|-----------------------------------|
| \( E_{TSC,\text{kin}} \) (keV) | \( \alpha \) | \( \lambda_0 \) (nm) | \( \theta \) (deg) | \( \phi \) (deg) | \( \delta_0 \) (deg) |
| 100.0 | \( \pi \) | 1 064.0 | 21.897 | 6 | 259.77 | 90.0 |
| 131.476 | \( \text{Frac} \) | \( \psi_0 \) (deg) | \( \phi_0 \) (deg) | \( \delta_0 \) (deg) | \( \theta_0 \) (deg) |
| 1.150 28 | 33.433 8 | 223.647 | 345.831 | 64.464 2 |
| 204.444 | \(-1.685 \) | 16.592 6 | 312.142 | 4.920 17 | 159.51 |
perpendicular velocity. In general, the solution so obtained differs in normalised Doppler frequency at the detector, i.e., \( \frac{1}{\gamma_1} = \frac{\omega_D}{\omega_0} \) for TSC and \( \frac{\Omega_D}{\Omega_0} \) for SR. It is also possible to prove that in those cases, a scattering angle exists \( \theta^* \), given by the condition \( \tan \theta^* = \frac{\alpha}{(2\pi\zeta(\theta^*)p_{\text{in}})} \), at which the \( j \)-quadratures of the emitted \( B \)-field exactly vanishes, and then the radiated field is fully linearly polarised as is standard SR when measured in the plane of the electron(s) orbit. See appendix F for details. The target parameters for TSC illustrating the above mentioned results are given in table 3; figures 5, 6 show the field quadratures in time domain when 32 harmonics are added. It is perhaps interesting to mention that if \( \phi \neq 3\pi/2 \), but the condition \( \varphi_c = -\pi/2 \) can still be met,

**Figure 3.** Comparison between the radiated TS fields (a) and SR fields (b). Apart from a dilation/contraction in time axis, readily accounted for through the corresponding Doppler effects, the temporal shape of both quadratures in HSR is exactly the same as in TSC. Panel (c) shows \( B_x \) versus \( B_y \) for both processes; matching is excellent.

**Figure 4.** Comparison between the radiated TS fields (a) and SR fields (b). Apart from a dilation/contraction in time axis, readily accounted for through the corresponding Doppler effects, the temporal shape of both quadratures in HSR is exactly the same as in TSC.
numerical evidence suggests that another exact solution exists fulfilling $\psi_0 = 0$, $\phi_0 + \delta_0 = k \pi$, but in general $E_{TS,kin} \neq E_{SR,kin}$, $\theta_0 = \theta_0'$. The latter case is presently under study.

Case-study number 4: To end this section, let us present some numerical computations that are inspired in, and connect with [34]. In the latter work, the authors report on high-order nonlinear TS for a set-up in which high energy electrons (up to 200MeV) counterpropagate against an ultrahigh-intensity laser, and the corresponding scattering is also measured in a (basically) backscattering geometry. We shall show here 3 examples where the electron is counterpropagating against the laser (hence $\theta = \pi$, $\phi = 0$), with scattering angle $\theta_0 = \pi$, laser parameter $\alpha = 4\pi$ and initial kinetic energies for the electron successively equal to 250.0 keV, 2.5 MeV and 25.0 MeV. The results are summarised in Table 4 and Figure 7.
5. Physical applications suggested by the relationship between TSC-HSR

The relationship between nonlinear TS from circularly polarised high-intensity lasers and HSR suggests some physical applications that we shall briefly mention below. It is not the aim of this paper to fully develop them, but the tools needed to do so, both in frequency and time domains, are at hand. For completeness, a brief comment on the interaction of high-intensity circularly polarized lasers with fusion plasmas is included, although in that case the emphasis is not placed specifically on the TSC-HSR relationship.

(a) The laser synchrotron light source. When the condition \( p_0 \gamma + \alpha / 2\pi = 0 \) is met (equivalent to \( \phi_0 = -\pi/2 \)), the field quadratures radiated by TSC have the expected symmetry of SR for an electron orbit lying in the X-Y plane, the radiation being measured at a point out of the orbital plane in general. Usually \( F \equiv (\beta_0/\beta_0) \approx 0 \) is needed to obtain quadrature matching, but the analogy can be made closer by explicitly looking for solutions of the algebraic equations having \( F = 0 \), see table 5 and figure 8 for a typical example. If, in addition, \( \phi = 3\pi/2 \) holds then an angle \( \theta^* \) exists, given by the condition \( \tan 2\theta^* = \alpha / (2\pi \zeta (\theta^*) \beta_0) \), at which the \( j \)-quadratures of the emitted B-field exactly vanishes \( ((p_0 + \alpha / 2\pi) = 0 \) can be compatible with \( \phi = 3\pi/2 \) for a certain range of \( \phi \) angles, but the condition to obtain the vanishing of the \( j \)-quadrature is more complicated). The radiated field is then fully linearly polarized as is the standard SR measured in the plane of the electron(s) orbit. The spectral content of the emitted fields can be tuned by a combination of laser wavelength/intensity tuning and/or electron energy tuning; see, for instance, the case reported in table 6 and figure 9. In summary, varying \( E_{SR,kin}, \alpha, \beta, \phi, \beta_0 \), a tunable radiation source is feasible, based on circularly polarised lasers interacting with (essentially) monoenergetic electrons, having the same properties as regards to its spectrum, spatial distribution of power, polarization, etc., as SR, provided that the magnitude and orientation of the initial momentum be adequately chosen.

(b) Learning the basics of SR from table-top laser experiments. A compact laser/electron beam set-up can be used to learn the basic physics of SR. Synchrotrons radiation properties can be (partly) modeled and characterized in more comfortable optical/UV range(s), provided that the laser wavelength, its intensity, and the energy of the electron beam be suitably chosen. Of course, that set-up would not be an alternative to those critical and very important modern applications of synchrotrons making use of very energetic electrons (in the range of a few GeV to \( \approx 10 \) GeV).

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**Table 4. HSR parameters fitting a given TSC run, see main text for details.**

| \( E_{TSC,kin} \) (keV) | \( \alpha \) | \( \lambda_0 \) (nm) | \( \theta \) (deg) | \( \phi \) (deg) | \( \theta_0 \) (deg) |
|-----------------------|------------|-----------------|----------------|---------------|----------------|
| 250.0                 | 4\pi       | 1 064.0         | 180.0          | 0.0           | 180            |
| 2 500.0               | 4\pi       | 1 064.0         | 180.0          | 0.0           | 180            |
| 25 000.0              | 4\pi       | 1 064.0         | 180.0          | 0.0           | 180            |

| \( E_{SR,kin} \) (keV) | \( Frac \) | \( \psi_0 \) (deg) | \( \phi_0 \) (deg) | \( \delta_0 \) (deg) | \( \theta_0' \) (deg) |
|-----------------------|-----------|-------------------|-------------------|--------------------|---------------------|
| 693.761               | 0.373     | 114.237           | 266.86            | 224.331            | 6.936 64           |

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**Figure 6.** Comparison between the radiated TS fields (a) and SR fields (b). Apart from a dilation/contraction in time axis, readily accounted for through the corresponding Doppler effects, the temporal shape of both quadratures in HSR is exactly the same as in TSC.
Gyrotrons as model systems in the TSC-HSR comparison. Gyrotrons are high power microwave sources that use moderately relativistic electrons (usually from 70 to 100 keV) spiraling in a strong magnetic field. By design, electron trajectories are helical with a substantial component of velocity parallel to the magnetic field.

(c) Gyrotrons as model systems in the TSC-HSR comparison. Gyrotrons are high power microwave sources that use moderately relativistic electrons (usually from 70 to 100 keV) spiraling in a strong magnetic field. By design, electron trajectories are helical with a substantial component of velocity parallel to the magnetic field.

**Figure 7.** Matching three TSC cases with the electron counterpropagating with respect to a laser beam of $\alpha = 4\pi$ (backscattering geometry). Electron energies of 250.0 keV, 2.5 MeV and 25.0 MeV have been considered (panels (a), (b), (c) respectively). Quadrature matching is shown in panels (d), (e), (f) where $B_x$ versus $B_y$ for both processes (TSC quadratures plotted with open circles, HSR ones plotted with crosses) is plotted.

**Table 5.** HSR parameters fitting a given TSC run, see main text for details.

| TSC/HSR parameters (target/found) | $E_{TS,kin}$ (keV) | $\alpha$ | $\lambda_0$ (nm) | $\theta$ (deg) | $\phi$ (deg) | $\theta_0$ (deg) |
|----------------------------------|-------------------|----------|-----------------|---------------|-------------|--------------|
| $E_{TS,kin}$ (keV)              | $\alpha$          | $\lambda_0$ (nm) | $\theta$ (deg) | $\phi$ (deg) | $\theta_0$ (deg) |
| 500.0                           | 3$\pi$            | 1 064.0   | 61.480 7        | 270.0         | 90.0        |
| $E_{TS,kin}$ (keV)              | $\rho_{\mu c}$    | $\psi_0$ (deg) | $\phi_0$ (deg) | $\ell_0$ (deg) | $\theta_0'$ (deg) |
| 480.093                         | 0.476 905         | 0.0       | 107.524         | 213.227       | 132.673     |
| 410.217                         | 0.0               | 0.0       | 131.301         | 238.000       | 114.329     |

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field, and the theory developed in this paper should apply as to the computation and basic properties of the emitted radiation. Other subtle effects in gyrotron physics, as the role of the resonant cavity, or the coordinated movement of charges, which gives rise to the output coherent radiation, lie beyond our scope here.

Figures 10, 11 give an example of electrons of 75 keV kinetic energy, with \( F(\equiv \beta_p/\beta_0) = 0.5 \), and moving in a magnetic field \( B_0 = 1 \) T. Larmor radius and Larmor frequency are computed according to the well-known formulae (16):

\[
\begin{align*}
    r_L &= \frac{mc}{q} \frac{\gamma_0 \beta_0}{B_0} \\
    \omega_L &= \frac{q}{m} \frac{B_0}{\gamma_0}
\end{align*}
\]  

In this particular case, and for \( \theta'_0 = \pi/2 \), two TSC configurations giving rise to quadrature matching are given. The difference among the two cases is that in figure 10, the relative Doppler frequencies are matched (to

| Table 6. HSR parameters fitting a given TSC run. See main text for details. |
|---------------------------------------------------------------|
| TSC/HSR parameters (target/found)                             |
| \( E_{\text{TSC}} \) (keV) | \( \alpha \) | \( \lambda_0 \) (nm) | \( \theta \) (deg) | \( \phi \) (deg) | \( \theta_0 \) (deg) |
| 500.0     | 3\pi        | 1 064.0     | 61.480        | 270.0           | 65.670        |
| 1 000.0   | 15.0        | 1 064.0     | 59.082        | 270.0           | 61.083        |
| \( E_{\text{SR,kin}} \) (keV) | Frac | \( \phi_0 \) (deg) | \( \phi_0 \) (deg) | \( \delta_0 \) (deg) | \( \theta'_0 \) (deg) |
| 410.217   | 0.0         | 213.104     | 337.322       | 90.0            |
| 811.621   | 0.0         | 168.516     | 311.769       | 90.0            |
exactly one), while in figure 11, the TSC relative Doppler frequency has been selected to be \( \omega_0/\omega_0 \)TSC = 2/3. The interacting configuration of electron beam/laser in this case could be called an ‘optical gyrotron’ since the quadratures follow the same pattern as in its microwave counterpart. Obviously, the frequency ranges in which energy is emitted are quite different in one case from the other, but due to the freedom in choosing the matching relative frequency, the output frequency of the optical gyrotron can be (widely) tuned, an interesting feature to explore in more detail. The capability of tuning the gyrotron frequency shot to shot or in real time would be a possible break-through in electron cyclotron resonance heating in fusion plasmas.

(d) The interaction of high-intensity circularly polarised laser radiation with fusion plasmas. Fusion plasmas are currently probed by a host of active and passive diagnostics in order to ascertain their properties. One of the most important diagnostics to measure electron temperature and density is TS from powerful lasers, probing the electron distribution function through the measurement of the scattered spectrum. To the best of our knowledge, no attempt has been made to probe fusion plasmas with ultra-intense circularly polarised lasers, either with diagnostic purposes, or to obtain from them a light source with the special properties predicted from the nonlinear effects. Numerical work presented by the authors in [24, 25, 27] suggest that using an ultra-intense laser could shift the emitted spectrum to the red, and hence spectral zones that are normally unaccessible due to practical considerations, like the one around the laser wavelength, could now be directly observed, increasing the diagnostic capability. The predicted harmonics to the (bulk) scattered spectrum would also be of potential diagnostic use. Experiments like the one reported in [34] make fusion plasmas a natural candidate for further study.

(e) Geophysical/astrophysical applications. Short portions of the trajectories of charged particles on the Earth or planetary magnetospheres are well modeled by the helicoidal trajectories considered in this paper. For suitable energy ranges of the corresponding particles and local magnetic fields in which they are moving, it could be possible to find the corresponding parameters of a TSC laser experiment that would model that behaviour, and predict their electromagnetic emission.

Also, the emission from high-energy electrons in pulsars having magnetic fields substantially larger than those achieved under laboratory conditions seems a good candidate for study. In fact, SR from charges spiraling in the magnetic fields of pulsars has been considered as a possible mechanism explaining (part of) their rich phenomenology [31, 32, 39]. For those objects, the far-field approximation made in this paper for the emitted radiation would be a very good one, even for particles that travel along the magnetic field for substantial
distances. Depending on the share of parallel to perpendicular velocity of charged particles in the pulsar magnetosphere, some of the most energetic radiation processes taking part in them could be modeled by TSC experiments under conditions that could be achieved with current laser technology. The theoretical analogy developed in this paper can be easily extended to other charged particles: it could be applied to the emission of, say, ultra-energetic protons in pulsar magnetospheres, and one could speculate about a corresponding TSC experiment capturing the characteristics of such emission.

6. Conclusions

It is proven that Thomson Scattering from ultrahigh-intensity circularly polarised lasers (TSC) interacting with (essentially) monoenergetic electrons, and helical synchrotron radiation (HSR) have the same mathematical form. To achieve this connection, the circular polarization of the laser is essential; for other states of polarization (linear, say) some similarities still remain, but no longer an exact one.
The TSC-HSR similarity is not due to the ‘trivial’ reason that both HSR and TSC trajectories be helices in real space-time \((x - t')\), because that is not true in a strict sense. Although the HSR trajectories are indeed helices in \(x - t'\), the TSC ones are not, as shown by extensive numerical computation of the implicit equations 
\[ \xi = t' - x_3(\xi)/c, \] etc in appendix C. TSC trajectories in \(x - t'\) show in general highly non-harmonic (although periodic) velocity components in all co-ordinate axis, and hence they substantially differ from the standard helix considered for HSR. It is after the change of variables \(t' \rightarrow \xi = t' - x_3(\xi)/c\) that the mathematical similarity between TSC-HSR is displayed in full generality, and so it plays an absolutely central role in providing a basis for the TSC-HSR connection. The kinematics of TSC trajectories, and its role for a deeper physical understanding of the TSC-HSR similarity, would deserve further investigation.

The TSC-HSR similarity has been studied both in the frequency and time domains. As a by-product, exact and general expressions in the time domain giving the radiated magnetic fields for general helical trajectories and scattering geometries are obtained.

Figure 11. Comparison of first-harmonic and full-field quadratures for HSR (top panels) and TSC (middle panels). Bottom panel shows \(B_u\) versus \(B_i\) for both processes (TSC and HSR). As it is apparent, quadrature matching is excellent. See main text for details.
The conditions leading to this similarity are fully clarified, and the problem is reduced to solving a set of nonlinear algebraic equations connecting both phenomena. Those equations have been numerically solved to provide compelling examples of the above mentioned connection. In some specific cases (Appendix F), they are also amenable to large analytical simplifications. In general the solution to the set of algebraic equations is not unique, but can be made so if further conditions are imposed, as for example a definite value for the target normalised frequency. This opens prospects for the study of synchrotron radiation processes in spectral ranges that can perhaps be more readily accessible from an experimental point of view (visible or UV, for example) using current ultrahigh laser technology.

Physical applications derived from the mathematical analogy, ranging from the all-optical production of radiation having exactly the same properties as regards to polarization, harmonic content, etc as synchrotron radiation, to the experimental simulation of geophysical or astrophysical processes under laboratory conditions exploiting current laser technology, have been discussed.

This paper has developed what could be called a ‘dictionary’ in which every process in TSC has a precise translation (although possibly not a single one) into a HSR one, and vice versa. Whenever a mathematical connection between different physical phenomena has been established, the corresponding physical realms have benefited from insights coming from each other. In this sense, it is hoped that our work provides a connection that allows to see TS and synchrotron processes from an unified point of view.

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Appendix A. HSR: various formulae

The components\( \beta_x(t'), \beta_y(t') \) and \( \beta_z(t') \) are (with \( \frac{B_0}{c} = \beta_0 \):

\[
\beta_x(t') = \beta_p \sin \psi_0 \cos \phi_0 + \beta_0 (-\cos \psi_0 \cos \phi_0 \sin \delta_0 - \sin \phi_0 \cos \delta_0) \cos(\Omega_0 t') + \beta_0 (-\cos \psi_0 \cos \phi_0 \cos \delta_0 + \sin \phi_0 \sin \delta_0) \sin(\Omega_0 t') \tag{A.1}
\]

\[
\beta_y(t') = \beta_p \sin \psi_0 \cos \phi_0 + \beta_0 (-\cos \psi_0 \sin \phi_0 \cos \delta_0 + \sin \phi_0 \cos \delta_0) \cos(\Omega_0 t') - \beta_0 (-\cos \psi_0 \sin \phi_0 \sin \delta_0 + \cos \phi_0 \cos \delta_0) \sin(\Omega_0 t') \tag{A.2}
\]

\[
\beta_z(t') = \beta_p \cos \psi_0 + \beta_0 \sin \psi_0 \sin \delta_0 \cos(\Omega_0 t') + \beta_0 \sin \psi_0 \cos \delta_0 \sin(\Omega_0 t') \tag{A.3}
\]

The expressions for \( \Omega_0, a_{SRc}, \) and \( a_{SRs} \), are:

\[
\Omega_0 = \frac{\Omega_0}{1 - \beta_p (\sin \theta'_0 \sin \psi_0 \cos \phi_0 + \cos \theta'_0 \cos \psi_0)} \tag{A.4}
\]

and:

\[
a_{SRc} = -\frac{n}{c} (\cos \delta_0 [\sin \theta'_0 \cos \psi_0 \cos \phi_0 - \cos \theta'_0 \sin \psi_0] - \sin \delta_0 \sin \theta'_0 \cos \phi_0) \tag{A.5}
\]

\[
a_{SRs} = \frac{n}{c} (\sin \delta_0 [\sin \theta'_0 \sin \psi_0 \cos \phi_0 - \cos \theta'_0 \cos \psi_0] + \cos \delta_0 \sin \theta'_0 \sin \phi_0) \tag{A.6}
\]

The modulus of the applied magnetic field \( B_0 \) is (compare with equation (46)): \( B_0 = [(mc \beta_0)/(\epsilon_0 \epsilon_r) (1 - \beta_0^2)^{1/2}] \).

Appendix B. The dimensionless amplitudes \( A_{SR,jx}, A_{SR,jy}, A_{SR,jz}, A_{SR,kx}, A_{SR,ky}, A_{SR,kz} \) and \( A_{SR,uu} \)

We give the \( t' \)-independent dimensionless amplitudes \( A_{SR,jx}, A_{SR,jy}, A_{SR,jz}, A_{SR,kx}, A_{SR,ky}, A_{SR,kz} \) and \( A_{SR,uu} \) in terms of \( \beta_p, \beta_0 \) and the angles characterising the helical trajectory and the scattering angle (\( \psi_0, \phi_0, \delta_0, \theta_0 \)), namely:
Appendix C. TSC: position and momentum of the radiating electron

The solution of the dynamical equations of motion for the incoming circularly polarized monochromatic plane wave [24, 25, 35, 36] is:

\[
\mathbf{x}(\xi) = x_1(\xi) \mathbf{i} + x_2(\xi) \mathbf{j} + x_3(\xi) \mathbf{k}
\]

\[
x_1(\xi) = x_{1,0} + \frac{\dot{\mathbf{f}} \xi}{\gamma_1} + \frac{c e E_0 (1 - \cos \omega_0 \xi)}{\gamma_1 \omega_0^2}
\]

\[
x_2(\xi) = x_{2,0} + \left[ \frac{\dot{\mathbf{f}} \xi}{\gamma_1} + \frac{c e E_0}{\gamma_1 \omega_0} \right] - \frac{c e E_0 \sin \omega_0 \xi}{\gamma_1 \omega_0^2}
\]

\[
x_3(\xi) = x_{3,0} + \left[ s \frac{c e E_0 [f_1 + (e E_0)/\omega_0]}{\gamma_1 \omega_0} - \frac{c e E_0}{\gamma_1 \omega_0^2} \right]
\]

\[
x = \frac{m c^2 + \dot{\mathbf{f}}^2}{2 \gamma_1} - \frac{c}{2}
\]

\[\xi = t' - x_0(\xi)/c, t' \text{ being the 'radiation' time for the electron, and } \dot{\mathbf{f}}^2 = f_1^2 + f_2^2.\]

The momentum of the radiating electron is (from the dynamical equations) [24, 25]:

\[
\mathbf{p}(\xi) = p_1(\xi) \mathbf{i} + p_2(\xi) \mathbf{j} + p_3(\xi) \mathbf{k}
\]

\[
p_1(\xi) = f_1 + \frac{e E_0 \sin(\omega_0 \xi)}{\omega_0}
\]

\[
p_2(\xi) = f_2 + \frac{e E_0 (1 - \cos(\omega_0 \xi))}{\omega_0}
\]

\[
p_3(\xi) = -\frac{\gamma_1}{2} + \frac{m c^2}{2 \gamma_1} + \frac{f_1^2 + (f_2 + (e E_0/\omega_0)^2 + (e E_0/\omega_0)^2)}{2 \gamma_1}
\]

\[
+ \frac{2 f_1 (e E_0/\omega_0) \sin(\omega_0 \xi)}{2 \gamma_1} - 2 (f_2 + (e E_0/\omega_0) (e E_0/\omega_0) \cos(\omega_0 \xi)}
\]

\[x_{0,1}, x_{0,2}, x_{0,3}, \gamma_1, f_1, f_2 \text{ are the six integration constants of the dynamical equations of motion for } \xi = 0.\]

In the case of interest here, one can safely choose \(x_{0,1} = x_{0,2} = x_{0,3} = 0\). The Doppler frequency for TSC is: \(\omega_D = \omega_0/g_1, c\).

For the numerical computations, we consider the standard polar and azimuthal angles \(\theta\) and \(\phi\) and the kinetic energy \(E_{TS,kin}\) characterizing the initial momentum of the electron (namely, \(\mathbf{p}(\xi = 0)\)), as displayed in the various tables.

Appendix D. The amplitudes \(A_{TS,j,0}, A_{TS,j,c}, A_{TS,j,s}, A_{TS,u,0} \text{ and } A_{TS,u,c}\) for \(j' = j\) and \(u' = u\)

The \(\xi\)-independent amplitudes \(A_{TS,j,0}, A_{TS,j,c}, A_{TS,j,s}, A_{TS,u,0} \text{ and } A_{TS,u,c}\) are:

\[
A_{TS,j,0} = \cos \theta_0 f_1 - \sin \theta_0 \left[ -\frac{\gamma_1}{2} + \frac{m c^2}{2 \gamma_1} + \frac{f_1^2 + (f_2 + (e E_0/\omega_0))^2 + (e E_0/\omega_0)^2}{2 \gamma_1} \right]
\]

\[
A_{TS,j,c} = \sin \theta_0 \left( f_2 + (e E_0/\omega_0) (e E_0/\omega_0) \right) \frac{1}{\gamma_1}
\]

\[
A_{TS,j,s} = \cos \theta_0 (e E_0/\omega_0) - \sin \theta_0 (e E_0/\omega_0) \frac{f_1}{\gamma_1}
\]
\[ A_{\text{TS,acc}} = f_2 + \frac{eE_0}{\omega_0} \]
\[ A_{\text{TS,acc}} = -\frac{eE_0}{\omega_0} \]  

(D.4)  
(D.5)

In order to connect TSC and HSR through dimensionless equations, we shall define the normalised (dimensionless) quantities \( a_{\text{TS},...} = A_{\text{TS},...}/mc \), \( p_{0x} = f_1/mc \) and so on. Then, the equations expressing the TSC-HSR similarity read:

\[
a_{\text{TS}0} = \cos \theta_0 p_{0x} - \sin \theta_0 \left( \frac{p_{0x}^2 + (p_{0y} + (\alpha/2\pi))^2 + (\alpha/2\pi)^2 + 1 - \gamma_1^2}{2\gamma_1} \right) \]
\[
a_{\text{TS}y} = \frac{\alpha}{2\pi} (\cos \theta_0 - \sin \theta_0 \frac{p_{0x}}{\gamma_1}) \]
\[
a_{\text{TS}z} = \frac{\alpha}{2\pi} \sin \theta_0 \left( p_{0y} + \alpha/2\pi \right) \]
\[
a_{\text{TS}su} = p_{0y} + \alpha/2\pi \]
\[
a_{\text{TS}su} = 0 \]
\[
a_{\text{TS}sc} = -\alpha/2\pi. \]

\( \alpha = (|e|E_0 \lambda_0)/(mc^2) \) is the so-called laser parameter. \( \alpha < 0 \) and \( m \) are the electron charge and mass, respectively, and \( \lambda_0 = (2\pi c)/\omega_0 \) is the incoming wavelength.

**Appendix E. Useful properties of the Bessel functions of the first kind \( (f_l(x), l = 0, \pm 1, \pm 2, \pm 3, \ldots) \)**

The following properties of \( f_l(x) \) [37] are made use upon deriving the time-domain fields for HSR and TSC:

\( f_l(-x) = (-1)^l f_l(x) \)
\( f_{l+1}(x) + f_{l-1}(x) = \frac{2l}{x} f_l(x) \)

(E.1)  
(E.2)  
(E.3)  
(E.4)

**Appendix F. Special configurations for HSR and TSC (for \( j' = \hat{j} \) and \( u' = \hat{u} \))**

We shall consider equations (36), (41) and (33), in certain specific configurations for HSR and TSC.

For HSR, we shall suppose that \( \psi_0 = \phi_0 = \delta_0 = 0 \) and \( \varphi_{st} = \pi/2 \). Then, from section 3.5, it follows that:

\[ F_{\text{SR},t,s} = -\beta_0 \sin \theta_0' + \frac{\beta_0}{\zeta} \cos \theta_0' \]  
\[ F_{\text{SR},x,s} = 0 \]  
\[ F_{\text{SR},u,c} = \beta_0 \]  
\[ F_{\text{SR},u,s} = 0 \]  

(F.1)  
(F.2)  
(F.3)  
(F.4)

The last four equations imply that in \( \int_{-\infty}^{+\infty} \frac{db}{(2\pi)^{3/2}} B_j(\omega)s_{\text{SR}}(\omega) \) only the terms \( \sin l(\varphi_{st}' - \Omega_{0t}) \) contribute and that in \( \int_{-\infty}^{+\infty} \frac{db}{(2\pi)^{3/2}} B_j(\omega)s_{\text{SR}}(\omega) \) only the terms \( \cos l(\varphi_{st}' - \Omega_{0t}) \) are not vanishing.

For TSC, we shall suppose that \( p_{0y} + \frac{\alpha}{2\pi} = 0 \) (or, equivalently, that \( \cos \varphi_c = 0 \), with the specific determination \( \varphi_c = -\pi/2 \), as equation (21) indicates) and \( p_{0x} = 0 \) (consistent with \( \phi = 3\pi/2 \)). Then, from section 3.5, it follows that:

\[ F_{\text{TS},t,s} = -\sin \theta_0 \left( \frac{(\alpha/2\pi)^2 + 1 - \gamma_1^2}{2\gamma_1} \right) + \frac{\alpha}{\zeta 2\pi} \cos \theta_0 \]  
\[ F_{\text{TS},x,s} = 0 \]  
\[ F_{\text{TS},u,c} = \frac{\alpha}{2\pi} \]  
\[ F_{\text{TS},u,s} = 0 \]  

(F.5)  
(F.6)  
(F.7)  
(F.8)
The last four equations imply that in \( \int_{-\infty}^{+\infty} \frac{d\omega}{(2\pi)^{2}}e^{-i\omega t}B_{1}(\omega)_{TS} \) only the terms \( \sin(\varphi_{1}' - \Omega_{D}t) \) contribute and that in \( \int_{-\infty}^{+\infty} \frac{d\omega}{(2\pi)^{2}}e^{-i\omega t}B_{1}(\omega)_{SB} \) only the terms \( \cos(\varphi_{1}' - \Omega_{D}t) \) are not vanishing.

One has:

\[
\frac{F_{SR,i,s}}{F_{TS,i,s}} = \frac{1}{g_{2}} \tag{F.9}
\]

\[
\frac{F_{SR,u,c}}{F_{TS,u,c}} = \frac{1}{g_{2}} \tag{F.10}
\]

The interesting outcome, so far, in this appendix is that for the special configurations for HSR and TSC considered above the four equations (36) and (41) have reduced to the two equations (F.9) and (F.10) or, stated into another form, to the single equation:

\[
\frac{F_{SR,i,s}}{F_{TS,i,s}} = \frac{F_{SR,u,c}}{F_{TS,u,c}} \tag{F.11}
\]

On the other hand, equation (33) becomes:

\[
\frac{\Omega_{D}}{\Omega_{0}}\sin \theta_{0}' = \frac{\omega_{0}c}{\omega_{0}2\pi \gamma_{1}} \sin \theta_{0} \tag{F.12}
\]

Equations (F.11) and (F.12) embody the conditions on the remaining parameters to be fulfilled, for the HSR-TSC similarity to hold.

We are now ready to consider the following interesting solution of both equations (F.11) and (F.12), based upon certain specific choices of \( \theta_{0}' \) and \( \theta_{0} \). Let \( \theta_{0}' = \theta_{0}^{*} \) be such that for TSC:

\[
F_{TS,i,s} = 0 \tag{F.13}
\]

After a little algebra this last equation is shown to be equivalent to \( (\gamma_{1} = \gamma_{1}/mc) \)

\[
\tan \theta_{0}^{*} \sin \theta_{0}^{*} = \frac{2\gamma_{1}^{2}(1 + (\alpha_{1}/2\pi)^{2} - \gamma_{1}^{2})(1 - \cos \theta_{0}^{*})}{\gamma_{1}^{2} \sin \theta_{0}} \tag{F.14}
\]

and since the left-hand side takes all values from +\( \infty \) to \(-\infty \) when \( \theta_{0}^{*} \) runs from 0 to \( \pi \) while the right-hand side remains finite, there is always at least one value of \( \theta_{0}^{*} \) fulfilling it. The particular choice made above implies that \( \int_{-\infty}^{+\infty} \frac{d\omega}{(2\pi)^{2}}e^{-i\omega t}B_{1}(\omega)_{TS} = 0 \) (while \( \int_{-\infty}^{+\infty} \frac{d\omega}{(2\pi)^{2}}e^{-i\omega t}B_{1}(\omega)_{SB} \) does not vanish). Since \( F_{SR,u,c}/F_{TS,u,c} \) does not vanish (when (F.13) holds), equation (F.11) implies that there is an angle \( \theta_{0}' = \theta_{0}^{*} \) for HSR such that:

\[
F_{SR,i,s} = 0 \tag{F.15}
\]

In turn, this choice implies that \( \int_{-\infty}^{+\infty} \frac{d\omega}{(2\pi)^{2}}e^{-i\omega t}B_{1}(\omega)_{SB} = 0 \) (while \( \int_{-\infty}^{+\infty} \frac{d\omega}{(2\pi)^{2}}e^{-i\omega t}B_{1}(\omega)_{SB} \) does not vanish). We have extensive numerical evidence showing that equation (F.12) is also satisfied by the two angles \( \theta_{0}' = \theta_{0}^{*} \) and \( \theta_{0} = \theta_{0}^{*} \).

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