A New CNT-Oriented Shell Theory

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Abstract

A theory of linearly elastic orthotropic shells is presented, with potential application to the continuous modeling of Carbon NanoTubes. Two relevant features are: the selected type of orthotropic response, which should be suitable to capture differences in chirality; the possibility of accounting for thickness changes due to changes in inter-wall separation to be expected in multi-wall CNTs. A simpler version of the theory is also proposed, in which orthotropy is preserved but thickness changes are excluded, intended for possible application to single-wall CNTs. Another feature of both versions of the present theory is boundary-value problems of torsion, axial traction, uniform inner pressure, and rim flexure, can be solved explicitly in closed form. Various directions of ongoing further research are indicated.

Keywords: shell theory, single- and multi-wall carbon nanotubes, torsion, traction, pressure, and rim-flexure problems

1 Introduction

The application that motivated this work is the modeling of carbon nanotubes (CNTs). When CNTs are employed as nanodevice components, they are regarded as elastic beam-like or shell-like objects and their mechanical response is characterized in terms of an as-small-as-possible number of stiffness and inertia parameters. To define and evaluate these parameters is the common goal of all modelers; a way to achieve it is to try and bridge between the microscopic scale of molecular mechanics and the macroscopic scale of continuous structure mechanics, by way of a mesoscopic scale, at which concepts from discrete structure mechanics apply. At the onset of putting together a bottom-up model of this sort [1], we realized that ordinary shell theories, which presume an isotropic three-dimensional response of the material comprising the shell, could not possibly guarantee an accurate macroscopic account of the mesoscopic texture of single-wall carbon nanotubes (SWCNTs): a glance to armchair and zigzag CNTs (Figure 1) suggests instead an orthotropic response in planes orthogonal to radial directions (see Figure 2, where the three little cylinders suggest what probes one should cut out of a cylindrical shell-like body in order to determine its material
moduli). Moreover, when modeling multi-wall carbon nanotubes (MWCNTs), it seems to us important to allow for thickness distension: we conjecture that thickness changes are essentially due to changes in the inter-wall distances, as a consequence of the interplay of the applied loads with the van der Waals interactions between adjacent walls. A search of the literature convinced us that we better produced such a shell theory ourselves. This paper describes the results of our efforts, results that have been partly anticipated in [4] and that – so we believe – may find application also in contexts different from the mechanics of nanotubes.

In the next section, we expound the general lines of a theory of linearly elastic orthotropic shells of constant referential thickness $2\varepsilon$, whose geometry (Section 2.1) is dictated by a piecewise smooth referential model surface $\mathcal{S}$. Imitating the classic approach of Kirchhoff, we specify the admissible kinematics (Section 2.2) by choosing for the displacement field in the tubular region $\mathcal{G}(\mathcal{S}, \varepsilon)$ a representation parameterized by a few fields defined over $\mathcal{S}$. In the deformations we envisage, (i) thickness may change, if the applied loads require and the boundary conditions permit; (ii) material fibers orthogonal to $\mathcal{S}$ must remain orthogonal to the deformed shape of $\mathcal{S}$ itself, an internal constraint we refer to as unshearability. Both the balance and the constitutive equations of our shell theory (Sections 2.3 and 2.4, resp.) are inherently consistent with the corresponding equations of three-dimensional linear elasticity:
- the balance equations follow from a two-dimensional Principle of Virtual Powers that is a direct consequence of stating the corresponding three-dimensional Principle for all virtual velocity fields in the linear space to which the admissible displacement belong; they are expressed in terms of a pair of two-dimensional stress measures that are defined as weighted thickness averages of the three-dimensional stress field in $\mathcal{G}(\mathcal{S}, \varepsilon)$ (Section 2.5);
- the constitutive equations are arrived at when the three-dimensional constitu-
The equations for unshearable orthotropic materials are inserted in the definitions of the two-dimensional stress measures.

The remaining part of the paper is dedicated to cylindrical shells. We begin with shells whose thickness can change. In sections from 3.1 to 3.4, we parallel and specify the developments of Section 2 as to, respectively, geometry, kinematics, balance laws, and constitutive equations. Then, we confine attention to axisymmetric equilibrium problems, and solve explicitly and exactly those of torsion and axial traction (Sections 4.1 and 4.2) – the cases for which experimental tests and numerical simulations seem to be especially easy to set up for CNTs – as well as the problems of pressure and rim flexure (Sections 4.3 and 4.4). Finally, we lay down natural geometrical notions of thinness and slenderness, and we show how remarkably the analytical solutions derived in the previous section simplify for slender shells (Sections 5.1 and 5.2), and how effective contraction moduli and effective stiffnesses can be defined for a cylindrical shell, regarded as a traction or torsion probe (Sections 5.3 and 5.4).

Next, in Section 6, we take up orthotropic shells whose thickness is constitutively immutable, a class that we designate by the names of Kirchhoff and Love by analogy with the corresponding classic plate theory. We adapt to the simpler case of Kirchhoff-Love cylindrical shells all the formulas derived in Sections 4 and 5, both for whatever thinness and in the small thickness limit; in the latter case, we show how the four constitutive moduli characterizing the mechanical response could be determined on the basis of simple real or computer experiments.

In our final Section 7, we briefly recapitulate our main findings, and we indicate the directions of our future research, with special attention to the ap-
lication to CNTs of the concepts and methods developed in the present paper.

2 General Theory

2.1 Geometry

In this opening section we recapitulate some well known notions, with the main purpose of introducing our notation and terminology.

Following the approach to construct a shell theory proposed in [10], we let $S$ denote a compact, regular, orientable and oriented surface embedded in the three-dimensional Euclidean space $E$, and we let $x$ denote its typical point and $n(x)$ the value of its normal vector field at $x$, with $|n(x)| \equiv 1$. We choose an origin $o \in E$, and denote by $x := x - o$ the position vector of $x$ with respect to $o$. We assume that $S$ admits a tubular $\varepsilon$-neighborhood $G(S,\varepsilon)$ (see Section 2.2 of [3]) and a global parametrization

$$\mathbb{R}^2 \ni (z^1, z^2) \mapsto x(z^1, z^2) \in S \subset E$$

(here $S$ is an open set). A point $p \in G(S,\varepsilon)$ has position vector

$$p := p - o = x - o + \zeta n(x), \quad x \in S, \quad \zeta \in I := (-\varepsilon, +\varepsilon),$$

with respect to $o$; $|\zeta|$ is the distance of $p$ from $x$, the point where the straight line through $p$ perpendicular to $S$ intersects $S$ itself. The mapping

$$(z^1, z^2, \zeta) \mapsto p(z^1, z^2, \zeta) := x(z^1, z^2, \zeta) + \zeta n(x(z^1, z^2))$$

is a global parametrization of $G(S,\varepsilon)$, with $(z^1, z^2, \zeta)$ the triplet of normal curvilinear coordinates of $p$ (Figure 3). We term the region $G(S,\varepsilon)$ of $E$ a shell-shaped region, of model surface $S$ and constant thickness $2\varepsilon$. The thickness of $G(S,\varepsilon)$ can be visualized as the length, whatever $x \in S$ one picks, of the material fiber $\mathcal{F}(x)$ through $x$ perpendicular to $S$; clearly, $\mathcal{F}(x) := \{p \in G(S,\varepsilon) \mid p = x + \zeta n(x)\}$.

![Figure 3: Geometrical equipment of a typical shell-shaped region.](image)

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The chosen parametrization induces a system of coordinate curves on $\mathcal{S}$, described by the mappings $z^\alpha \mapsto x(z^1, z^2)$ ($\alpha = 1, 2$). The tangent space $T_x := \text{span} \{ e_\alpha(x) \}$ to $\mathcal{S}$ at $x \equiv (z^1, z^2)$ is spanned by the tangent vectors to the coordinate curves at that point:

$$e_\alpha(z^1, z^2) := x_{,\alpha}(z^1, z^2).$$

On taking the normal field to be $n(x) = \text{vers}(e_1(x) \times e_2(x))$, and on setting

$$\varsigma(x) := e_1(x) \times e_2(x) \cdot n(x) > 0,$$

the covariant and contravariant bases at $x$ are, respectively, $\{ e_1(x), e_2(x), n(x) \}$ and $\{ e^1(x), e^2(x), n(x) \}$, where

$$\varsigma(x)e^\alpha(x) := (-1)^\alpha n(x) \times e_{\alpha+1}(x) \pmod{2}, \alpha \text{ not summed}.$$

For the rest of this subsection we leave the indication of the typical point $x \in \mathcal{S}$ tacit. Accordingly, we write

$$P := e_i \otimes e^i = e_i \otimes e_i$$

for the metric tensor and

$$^*P := e_\alpha \otimes e^\alpha = e^\alpha \otimes e_\alpha$$

for the surface metric tensor.

A vector field $v$ defined over $\mathcal{S}$ can be represented both in the covariant basis and in the controvariant basis:

$$v = v_i e^i = v^j e_j, \quad \text{with} \quad v_i := v \cdot e_i, \quad v^j := v \cdot e^j, \quad (i, j = 1, 2, 3).$$

Analogously, a second-order tensor field $T$ can be represented as

$$T = T^{ij} e_i \otimes e_j = T^i_j e^i \otimes e^j = T^j_i e_i \otimes e^j = T^{ij} e^i \otimes e_j,$$

in terms of its covariant, contravariant, or mixed components $T^{ij} = T \cdot e^i \otimes e^j$, $T_{ij} = T \cdot e_i \otimes e_j$, or $T^i_j = T \cdot e^i \otimes e_j$ and $T^j_i = T \cdot e_i \otimes e^j$ ($i, j = 1, 2, 3$).

Remark. The physical dimensions of covariant and contravariant basis vectors, and hence of the corresponding components of vectors and tensors, may differ.

To circumvent this difficulty is easy, whenever it so happens that

$$e_1 \cdot e_2 = 0.$$

Simply, one introduces the so called physical basis at $x$, that is to say, the orthonormal basis

$$\{ e_{<1>}(x), e_{<2>}(x), n(x) \}, \quad e_{<\alpha>} := \frac{e_\alpha}{|e_\alpha|} = \frac{e^\alpha}{|e^\alpha|}. $$

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We shall be making use of physical bases and components (e.g., \( v \cdot e_{<k>} = v \cdot e_{<k>} \)) in Section 3, where we deal with shells whose model surface is a right circular cylinder.

At a point of \( G(S, \varepsilon) \), the covariant and contravariant basis vectors are, respectively,

\[ g_\alpha := p,\alpha = e_\alpha + \zeta n,\alpha, \quad g_3 := p,3 = n; \quad (1) \]

and

\[ \zeta g^\alpha := (-1)^\alpha n \times g_{\alpha+1} \quad (\text{modulo } 2, \alpha \text{ not summed}), \quad g^3 = n, \]

where

\[ \zeta := g_1 \times g_2 \cdot n. \]

The metric tensor \( G \) is defined to be:

\[ G := g^i \otimes g_i = g_\alpha \otimes g^\alpha, \]

with

\[ ^*G := g^\alpha \otimes g_\alpha \]

its surface part.

The covariant bases \( \{ e_i \} \) at \( x \in S \) and \( \{ g_i \} \) at \( x + \zeta n(x) = p \in G(S, \varepsilon) \) are related by the shift tensor \( A \) - briefly, the shifter \( A \):

\[ A(x, \zeta) = g_i \otimes e^i \quad \Leftrightarrow \quad A e_k = g_k; \]

it can be shown \[10\] that the ratio of the volume measures at \( p \) and \( x \) is equal to the determinant of \( A \):

\[ dvol(p) = \alpha(x, \zeta) dvol(x), \quad \alpha := \det A. \quad (2) \]

The shifter

\[ B(x, \zeta) := g^i \otimes e_i = ^*B + n \otimes n, \quad ^*B := g^\alpha \otimes e_\alpha, \]

maps the controvariant basis at \( x \) into the controvariant basis at \( p \). We have that

\[ B^T A = P, \quad AB^T = G. \]

Note that

\[ A(x, \zeta) = ^*A(x, \zeta) + n(x) \otimes n(x), \quad ^*A(x, \zeta) := g_\alpha \otimes e^\alpha = ^*P(x) - \zeta W(x), \quad (3) \]

where

\[ W := -^*\nabla n = -n,\alpha \otimes e^\alpha \]

is the curvature tensor of the oriented surface \( S \). We have here denoted by \(^*\nabla \) the operation of taking the surface gradient of a smooth vector field \( v \) over \( S \): \(^*\nabla v = v,\alpha \otimes e^\alpha \). Likewise, we denote by \( \nabla \) the gradient of a vector field \( v \) defined over \( G(S, \varepsilon) \): \( \nabla v = v, i \otimes g^i \). Two divergence operators are associated
with the gradient operators, $\text{Div} \ v := \nabla \cdot \mathbf{G}$ and $^*\text{Div} \ v := {^*\nabla} \cdot {^*\mathbf{P}}$, where the field $\mathbf{v}$ is defined, respectively, over $\mathcal{G}(S, \varepsilon)$ and over $S$. The surface divergence of a tensor field $\mathbf{T}$ over $S$ is defined as follows: $^*\text{Div}(\mathbf{T}^T \mathbf{v}) := ^*\text{Div} \mathbf{T} \cdot \mathbf{v}$, for all constant vectors $\mathbf{v}$.

Let $\bar{\mathbf{a}}$ the fiber-wise constant extension to $\mathcal{G}(S, \varepsilon)$ of a vector field $\mathbf{a}$ defined over $S$. When taking the gradient of $\bar{\mathbf{a}}$, we have that

$$\nabla \bar{\mathbf{a}} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{g}^\alpha = \bar{\mathbf{a}}_\alpha \otimes (^*\mathbf{b} \ e^\alpha) = (^*\nabla \mathbf{a})^*\mathbf{B}^T.$$  

In the following, we will not make any notational distinction between a field defined over $S$ and its fiber-wise constant extension to $\mathcal{G}(S, \varepsilon)$: e.g., we shall write:

$$\nabla \mathbf{a} = (^*\nabla \mathbf{a})^*\mathbf{B}^T. \quad (4)$$

### 2.2 Kinematics

The three-dimensional strain measure we use is the standard symmetrized gradient of the displacement field:

$$E(\mathbf{u}) = \text{sym} \nabla \mathbf{u} := \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T).$$

It follows from this definition that the covariant components of $E$ are:

$$2E_{ij} = 2E \cdot \mathbf{g}_i \otimes \mathbf{g}_j = (\mathbf{u}_k \otimes \mathbf{g}_k^k + \mathbf{g}_k \otimes \mathbf{u}_k) \cdot (\mathbf{g}_i \otimes \mathbf{g}_j)$$

$$= (\mathbf{u} \cdot \mathbf{g}_i)_{,j} + (\mathbf{u} \cdot \mathbf{g}_j)_{,i} - \mathbf{u} \cdot (\mathbf{g}_i_{,j} + \mathbf{g}_j_{,i}),$$

whence

$$E_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} - \mathbf{u} \cdot (g_{i,j} + g_{j,i}) \right). \quad (5)$$

We restrict attention to shell-shaped bodies $\mathcal{G}(S, \varepsilon)$ whose admissible deformations may induce thickness changes but must keep the material fibers orthogonal to the model surface, in the sense that, whatever $x \in S$, the deformed material fiber $\mathbf{u}(\mathcal{F}(x))$ must be found orthogonal to the deformed model surface $\mathbf{u}(S)$. This pointwise internal constraint — a restriction on admissible displacement fields that, as anticipated, we refer to as unshearability — can be expressed in terms of the linear strain measure $E$ in the following form:

$$E(\mathbf{u}) \mathbf{n} \cdot \mathbf{v} = 0 \quad \text{for all } \mathbf{v} \text{ such that } \mathbf{v} \cdot \mathbf{n} = 0, \quad (6)$$

or rather, equivalently, as

$$E(\mathbf{u}) \mathbf{n} \cdot \mathbf{g}_\alpha = 0 \quad \text{in } \mathcal{G}(S, \varepsilon); \quad (7)$$

with the use of (1) and (147), (7) becomes:

$$(\mathbf{u} \cdot \mathbf{g}_\alpha)_{,3} + (\mathbf{u} \cdot \mathbf{n})_{,\alpha} - 2 \mathbf{u} \cdot \mathbf{n}_{,\alpha} = 0, \quad (8)$$
with the same quantification. We look for solutions of this system of two PDEs having the following form:

\[
\mathbf{u}(x, \zeta) = \mathbf{u}^0(x) + \zeta \mathbf{u}^1(x);
\]  

(9)

note, in particular, that

\[
(1)^1 \mathbf{u}(x) \cdot \mathbf{n}(x) = \mathbf{E}(\mathbf{u}(x)) \cdot \mathbf{n}(x) \otimes \mathbf{n}(x)
\]

is the stretch of the material fiber \( \mathcal{F}(x) \), uniform all along it. Substituting (9) into (8), and exploiting the quantification with respect to the variable \( \zeta \), we find the following restrictions on the choice of the fields \( \mathbf{u}^0, \mathbf{u}^1 \) over \( \mathcal{S} \):

\[
(1)^1 \mathbf{u} \cdot \mathbf{e}_\alpha = -(0)^0 \mathbf{u} \cdot \mathbf{n}, \quad (1)^1 \mathbf{u} \cdot \mathbf{n} = 0. \tag{10}
\]

The second of (10) tells us that, for (7) to have a solution of type (9), the fiber stretch must in fact be uniform all over \( \mathcal{G}(\mathcal{S}, \varepsilon) \); moreover, the tangential part of \( \mathbf{u} \) must be expressed as follows in terms of \( \mathbf{u}^1 \) and the curvature tensor of \( \mathcal{S} \):

\[
(1)^1 \mathbf{u} - (1)^1 \mathbf{u} \cdot \mathbf{n} = -\nabla (0)^0 \mathbf{u} \cdot \mathbf{n} - W(0)^0 \mathbf{u}.
\]

With the use of (3), we conclude that we should pick:

\[
\mathbf{u}_S(x, \zeta) := \mathbf{A}(x, \zeta) \mathbf{a}(x) + \mathbf{w}(x) \mathbf{n}(x) + \zeta (-\nabla \mathbf{w}(x) + \gamma \mathbf{n}(x)), \tag{11}
\]

where we have set:

\[
\mathbf{a} := (0)^0 \mathbf{u} - (0)^0 \mathbf{u} \cdot \mathbf{n}, \quad \mathbf{w} := (0)^0 \mathbf{u} \cdot \mathbf{n}, \quad \gamma := (1)^1 \mathbf{u} \cdot \mathbf{n}. \tag{12}
\]

Note that the displacement field (11) is parameterized by two fields over \( \mathcal{S} \), the tangential vector field \( \mathbf{a} \) and the scalar field \( \mathbf{w} \), and by the constant \( \gamma \)\footnote{A vector field \( \mathbf{a} \) (a tensor field \( T \)) defined over \( \mathcal{S} \) is termed \textit{tangential} if it so happens that \( \mathbf{a}(x) \cdot \mathbf{n}(x) \equiv 0 \) \( (T \mathbf{n} \equiv 0) \).}. Note also that, by taking \( \gamma = 0 \), we recover the kinematic Ansatz typical of the linear theory of Kirchoff-Love shells \[\text{[10]}\]; interestingly, the displacement field:

\[
\mathbf{u}_{KL}(x, \zeta) = \mathbf{A}(x, \zeta) \mathbf{a}(x) + \mathbf{w}(x) \mathbf{n}(x) - \zeta \nabla \mathbf{w}(x), \quad \mathbf{a}(x) \cdot \mathbf{n}(x) = 0, \tag{13}
\]

can be shown to be the general solution of the system of three PDEs that expresses the Kirchhoff-Love constraint, namely,

\[
\mathbf{E}(\mathbf{u}) \mathbf{n} = 0 \quad \text{in} \quad \mathcal{G}(\mathcal{S}, \varepsilon) \tag{14}
\]

(cf. \[\text{[7]}\]). \textbf{Remark.} That the fiber stretch \( \gamma \) defined by (12) be constant at any point of \( \mathcal{G}(\mathcal{S}, \varepsilon) \) to satisfy the second of (10) is a counterintuitive consequence of expressing the unshearability constraint in terms of the linear measure of
deformation $E(u)$. In fact, on adopting one of the standard exact deformation measures, namely, 
\[ D := \frac{1}{2}((\nabla f)^T(\nabla f) - I), \quad \text{where} \quad f(p) := p + u(p), \]
one finds that 
\[ D = E + \frac{1}{2}H^T H, \quad H := \nabla u. \]
Thus, the exact counterpart of (7) is:
\[ 0 = Dn \cdot \alpha = En \cdot \alpha + \frac{1}{2}Hn \cdot Hg_{\alpha}, \]
where the term quadratic in $|H|$ does not vanish in general. In particular, for $u$ of the form (9), we find that 
\[ Hn \cdot Hg_{\alpha} = (1)u \cdot n + (0)u \cdot (1)u, \alpha \]
hence, the second of (10) is replaced by 
\[ (1)u \cdot n + (1)u \cdot |u|^2, \alpha = 0, \]
and intuition rings no bell.

2.3 Balance Assumptions

Given a material body occupying a three-dimensional open and bounded region $\Omega$ and a velocity field $v$ over $\Omega$, the internal power expenditure over a part $\Pi$ of $\Omega$ associated with $v$ is:
\[ \int_{\Pi} S \cdot \nabla v \]
where $S$ denotes the stress field in $\Omega$; and, the external virtual power expenditure over $\Pi$ is:
\[ \int_{\Pi} d_o \cdot v + \int_{\partial \Pi} c_o \cdot v, \]
where $(d_o, c_o)$ denote, respectively, the distance force for unit volume and the contact force per unit area exerted on $\Pi$ by its own complement with respect to $\Omega$ and by the environment of the latter. These representations of power expenditures are those typical in the theory of the so-called simple material bodies. In that theory, a part is customarily a subset of non-null volume of $\Omega$ (which makes it for a part collection deemed sufficiently rich), and a virtual velocity field is a smooth vector field whose support is a part; the Principle of Virtual Powers is the stipulation that
\[ \int_{\Omega} S \cdot \nabla v = \int_{\Omega} d_o \cdot v + \int_{\partial \Omega} c_o \cdot v \quad (15) \]
for all virtual velocities in a collection modeled after a chosen collection of admissible motions, an alternative stipulation being

$$\int_{\Pi} S \cdot \nabla v = \int_{\Pi} d_o \cdot v + \int_{\beta\Pi} c_o \cdot v$$

(16)

for all parts $\Pi$ of $\Omega$ and for all velocity fields $v$ consistent with the admissible motions.

The virtual-power principle is interpreted as a balance statement for the internal and external fields $S$ and $d_o, c_o$. As exemplified in [10] for beams and plates, we use here below a restricted version of the three-dimensional statement to deduce a two-dimensional Principle of Virtual Powers and the associated balance laws for shells in terms of two-dimensional stress measures and applied loads. The first restriction we introduce has to do with the special shape of the three-dimensional bodies we consider: they must be shell-shaped in the sense of Section 2.1. The second restriction stems from the special class of admissible body parts we choose: they all must have the same thickness of the shell-like body. Thirdly and lastly, we pick a special class of virtual velocities, consistent with the representations of admissible displacements discussed in the previous section.

2.3.1 Internal power expenditure. Stress measures

The first two restrictions we listed imply that a typical part of $\Omega = G(S, \varepsilon)$ can be identified with the Cartesian product $\Pi = P \times I$ of $P$, an open subset of $S$, and the interval $I$. As to virtual velocities, we choose them of the form (9):

$$v(x, \zeta) = (0)v(x) + \zeta (1)v(x),$$

(17)

with $P$ the intersection of the supports of the vector fields $(0)v$ and $(1)v$. On recalling (4), we have that:

$$\nabla v = (\nabla (0)v + \zeta \nabla (1)v) s B^T + (1)v \otimes n.$$

Hence,

$$S \cdot \nabla v = (S^sB) \cdot (\nabla (0)v) + \zeta (S^sB) \cdot (\nabla (1)v) + Sn \cdot (1)v,$$

so that, having recourse to Fubini theorem and recalling (2), the internal power expenditure reads:

$$\int_{\Pi} S \cdot \nabla v = \int_{P} (sF \cdot (\nabla (0)v) + sM \cdot (\nabla (1)v) + f(3) \cdot (1)v).$$

Footnote:

3Choosing instead virtual velocities of the less general, constrained form (11) would preclude the appearance of reactive terms in the balance equations following from (16). We shall return on this important issue after completion of the generating procedure we are now about to start, in Remark 2.5 at the end of Section 2.5.
Here we have made use of the following definitions:

\[ sF(x) := \int \alpha(x, \zeta) S(x, \zeta) sB(x, \zeta) d\zeta = \left( \int \alpha(x, \zeta) S(x, \zeta) g^\alpha(x, \zeta) d\zeta \right) \otimes e_\alpha, \]

\[ sM(x) := \int \alpha(x, \zeta) \zeta S(x, \zeta) sB(x, \zeta) d\zeta = \left( \int \alpha(x, \zeta) \zeta S(x, \zeta) g^\alpha(x, \zeta) d\zeta \right) \otimes e_\alpha, \]

and

\[ f^{(3)}(x) := \int \alpha(x, \zeta) S(x, \zeta) n(x) d\zeta, \]

for the two-dimensional stress measures that we call, respectively, force tensor, moment tensor, and shear vector.

It is important to observe that the force and moment tensors are tangential. Now, for a tangential tensor field \( T \) over \( S \), it follows from the general identity

\[ T \cdot \nabla v = \text{Div}(T^T v) - s\text{Div} T \cdot v \]

and the standard divergence theorem that

\[ \int_P T \cdot \nabla v = \int_{\partial P} T m \cdot v - \int_P s\text{Div} T \cdot v, \]

where, if \( t \) is the tangential vector to the curve \( \partial P \) in the tangent plane to the surface \( S \), \( m = t \times n \) is the normal to \( \partial P \). Thus,

\[ \int_P S \nabla v = \int_P \left( -s\text{Div} sF \cdot v + (-s\text{Div} sM + f^{(3)}) \cdot v \right) + \int_{\partial P} \left( sF m \cdot v + sM m \cdot v \right). \]

### 2.3.2 External power expenditure. Applied loads

As to power expenditure of the applied forces, we find:

\[ \int_{\Pi} d_o \cdot v = \int_P \left( \left( \int \alpha d_o \right) \cdot (0) v + \left( \int \alpha \zeta d_o \right) \cdot (1) v \right) \]

for the distance force, and

\[ \int_{\partial \Pi} c_o \cdot v = \int_P \left( (\alpha^+ c_o^+ + \alpha^- c_o^-) \cdot (0) v + \varepsilon (\alpha^+ c_o^+ - \alpha^- c_o^-) \cdot (1) v \right) + \int_{\partial P} \left( \left( \int c_o \right) \cdot (0) v + \left( \int \zeta c_o \right) \cdot (1) v \right) \]

for the contact force, where we have set:

\[ \alpha^\pm(x) := \alpha(x, \pm \varepsilon), \quad c_o^\pm(x) := c_o(x, \pm \varepsilon), \quad x \in \mathcal{P}. \]

We are now in a position to define the load fields over \( S \) induced by the three-dimensional fields \((d_o, c_o)\). These are:

\[ \text{Here we have made use of the fact that } \partial \Pi = \{ \mathcal{P} \times \{ \pm \varepsilon \} \} \cup \{ \partial \mathcal{P} \times I \}. \]
• the distance force and distance couple per unit area

\[ q_o(x) := \int_\Omega \alpha(x, \zeta) d_o(x, \zeta) d\zeta + \alpha^+(x) c^+(x) + \alpha^-(x) c^-(x), \]

\[ r_o(x) := \int_\Omega \alpha(x, \zeta) d_o(x, \zeta) d\zeta + \varepsilon(\alpha^+(x) c^+(x) - \alpha^-(x) c^-(x)); \]

• the contact force and contact couple per unit length

\[ l_o(x) := \int_\eta c_o(x, \zeta) d\zeta, \quad m_o(x) := \int_\eta \zeta c_o(x, \zeta) d\zeta. \]

All in all, the external virtual power expenditure takes the following form:

\[ \int P d_o \cdot v + \int \partial P c_o \cdot v = \int P q_o \cdot (\eta) v + r_o \cdot (1) v + \int P l_o \cdot (0) v + m_o \cdot (1) v. \]

2.3.3 Principle of Virtual Powers. Field equations. Boundary equations

The two-dimensional Principle of Virtual Powers we arrive at is:

\[ 0 = \int P (-^s \text{Div}^s F - q_o) \cdot (0) v + (-^s \text{Div}^s M + f^{(3)} - r_o) \cdot (1) v + \int \partial P \left( (^s F m - l_o) \cdot (0) v + (^s M m - m_o) \cdot (1) v \right), \]

a statement to hold for every pair of vector fields \( (0) v \) and \( (1) v \) on \( S \) and for every part \( P \) of \( S \). Under the standard blanket assumptions of smoothness, and with the use of a standard localization lemma, (22) yields the (two-dimensional) field equations to hold at any interior point of \( S \):

\[ ^s \text{Div}^s F + q_o = 0, \]

\[ ^s \text{Div}^s M - f^{(3)} + r_o = 0. \]

Granted (24), what remains of (22) is:

\[ \int \partial P \left( (^s F m - l_o) \cdot (0) v + (^s M m - m_o) \cdot (1) v \right) = 0 \]

for all admissible variations \( (0) v \) and \( (1) v \). Localization of (21) yields different results according to where it is performed. At an interior point of \( S \), where it can be combined with arbitrariness in the choice of \( P \), we have that:

\[ ^s F m = l_o, \quad ^s M m = m_o, \]

two relations that parallel the classic relation between stress tensor and contact-force vector for three-dimensional Cauchy bodies.\footnote{See [15] for a discussion of this issue that covers second-gradient materials as well.}
Dirichlet boundary condition prevails – that is to say, where one or more components of the boundary trace of the displacement field are prescribed – the corresponding components of the admissible variations must vanish; accordingly, the complementing components of both vectors \((^\alpha F \mathbf{m} - l_\alpha)\) and \((^\alpha M \mathbf{m} - m_\alpha)\) must also vanish (yielding boundary equations of Neumann type), because localization of (24) leads to:

\[
(^\alpha F \mathbf{m} - l_\alpha) \cdot (\mathbf{0})\mathbf{v} = 0 \quad \text{and} \quad (^\alpha M \mathbf{m} - m_\alpha) \cdot (\mathbf{1})\mathbf{v} = 0,
\]

for all admissible choices of \((\mathbf{0})\mathbf{v}\) and \((\mathbf{1})\mathbf{v}\).

Remark. The shear vector \(f^{(3)}\), defined by (19) and entering the last of equations (32), is one of the force vectors:

\[
f^{(i)}(x) := \int_I \alpha(x, \zeta) \mathbf{S}(x, \zeta) g^i(x, \zeta) d\zeta;
\]

and

\[
m^{(i)}(x) := \int_I \alpha(x, \zeta) \zeta \mathbf{S}(x, \zeta) g^i(x, \zeta) d\zeta.
\]

With these definitions, we may further set:

\[
\mathbf{F} := f^{(i)} \otimes e_i = \mathbf{F} + f^{(3)} \otimes \mathbf{n}, \quad \mathbf{M} := m^{(i)} \otimes e_i,
\]

with

\[
^s\mathbf{F} = f^{(\alpha)} \otimes e_\alpha, \quad ^s\mathbf{M} = m^{(\alpha)} \otimes e_\alpha,
\]

and

\[
f^{(\alpha)} = ^s\mathbf{F} e_\alpha, \quad m^{(\alpha)} = ^s\mathbf{M} e_\alpha.
\]

In terms of force and moment vectors, equations (20) become:

\[
\begin{align*}
\left(f^{(\alpha)} - \gamma^{\alpha\beta} f^{(\beta)} \right)_{,\alpha} + q_\alpha &= 0, \\
\left(m^{(\alpha)} - \gamma^{\alpha\beta} m^{(\beta)} \right)_{,\alpha} - f^{(3)} + r_\alpha &= 0.
\end{align*}
\]

We call membrane forces the components \(F^{\alpha\beta} := \mathbf{F} \cdot e_\alpha \otimes e_\beta = f^{(\beta)} \cdot e_\alpha\) – respectively, normal \(\mathbf{F}\) for \(\alpha = \beta\) and shear \(\mathbf{F}\) for \(\alpha \neq \beta\); and we call \(M^{\alpha\beta} := \mathbf{M} \cdot e_\alpha \otimes e_\beta = m^{(\beta)} \cdot e_\alpha\) the bending moments and \(M^{\alpha\beta} := \mathbf{M} \cdot e_\alpha \otimes e_\beta = m^{(\alpha)} \cdot e_\beta\) the twisting moments. Finally, we call \(F^{3\alpha} = f^{(3)} \cdot e_\alpha\) the transverse shears, \(F^{33} = f^{(3)} = f^{(3)} \cdot e_3\) the thickness shear, and \(M^{3\alpha} = m^{(3)} \cdot e_\alpha\) the thickness moments. Component-wise, the general field equations (23) can be written in the following form:

\[
\begin{align*}
F^{3\alpha}_{,\alpha} - W_\alpha F^{3\alpha} + q^\delta_\alpha &= 0 \quad (\delta = 1, 2), \\
F^{3\alpha}_{,\beta} + W_\beta F^{3\alpha} + q^\delta_\beta &= 0; \\
M^{3\alpha}_{,\alpha} - F^{3\alpha} + r^\delta &= 0 \quad (\delta = 1, 2), \\
M^{3\alpha}_{,\beta} + W_\beta M^{3\alpha} - F^{33} + r^\delta &= 0,
\end{align*}
\]
where a vertical bar denotes covariant differentiation (given a tensor field $T$ over $S$, $T^{\delta\alpha}|_{\alpha} := T^{\delta\alpha} + \gamma^{\delta}_{\beta\alpha} T^{\beta\alpha} + \gamma^{\delta}_{\alpha\beta} T^{\beta\alpha}$, with $T^{\delta\alpha} := T \cdot e^\delta \otimes e^\alpha$) and where

$$\gamma^{\delta}_{\beta\alpha} := e^\delta \cdot e^\beta_{\alpha},$$

are the surface Christoffel symbols.

Remark. A shell-shaped body $G(S, \varepsilon)$ is in a membrane regime if it so happens that the following conditions:

$$f^{(\alpha)} \cdot n = 0, \quad f^{(3)} = 0, \quad m^{(\alpha)} = 0$$

hold identically in $G(S, \varepsilon)$. The first two of (33) imply that $F^{\beta\alpha} = 0$, the fourth that $M^{\alpha} = 0$. Consequently, equations (32) reduce to:

$$F^{\delta\alpha}|_{\alpha} + q^\delta_{\alpha} = 0,$$
$$W_{\beta\alpha} F^{\beta\alpha} + q^3_{\alpha} = 0,$$

(34)

together with the following compatibility condition on the data:

$$r_{\alpha} = 0.$$
Given a subgroup \( \mathcal{G} \) of the orthogonal group (or, for what it matters here, of the group of all rotations), one seeks a representation formula for all elasticity tensors \( \mathbb{C} \) such that \( \mathcal{G} \subset \mathcal{G} \), i.e., for all elasticity tensors sharing a given symmetry group. A linearly elastic material is called orthotropic when its stress response is insensitive to a rotation of \( \pi \) about a given axis \( c \), i.e., when the symmetry group of its elasticity tensor includes that rotation. We give here below a general representation formula for the elasticity tensors in question.

Let \( \{ e_i \ (i = 1, 2, 3) \} \) be an orthonormal basis of vectors. Consider the following orthonormal basis for the linear space \( \text{Sym} \) of all symmetric tensors:

\[
\begin{align*}
V_\alpha &= \frac{1}{\sqrt{2}}(c_\alpha \otimes c_3 + c_3 \otimes c_\alpha) \quad (\alpha = 1, 2), \\
V_3 &= c_3 \otimes c_3, \\
W_\alpha &= c_\alpha \otimes c_\alpha \quad (\alpha \text{ not summed}), \\
W_3 &= \frac{1}{\sqrt{2}}(c_1 \otimes c_2 + c_2 \otimes c_1).
\end{align*}
\]

With the use of this basis, any orthotropic elasticity tensor can be written in the following form:

\[
\mathbb{C} = C_{1111} W_1 \otimes W_1 + C_{2222} W_2 \otimes W_2 + C_{3333} V_3 \otimes V_3 + \\
+ C_{1212} W_3 \otimes W_3 + C_{3131} V_1 \otimes V_1 + C_{2323} V_2 \otimes V_2 + \\
+ C_{1122}(W_1 \otimes W_2 + W_2 \otimes W_1) + C_{1133}(W_1 \otimes V_3 + V_3 \otimes W_1) + \\
+ C_{2233}(W_2 \otimes V_3 + V_3 \otimes W_2).
\]

(cf. [6]); the orthotropic material class is then parameterized by the 9 elastic moduli \( C_{1111}, \ldots, C_{2233} \).

### 2.4.2 The elasticity tensor of unshearable orthotropic materials

As a direct consequence of the fact that the shell model we are after incorporates a kinematic constraint, the appropriate three-dimensional response is captured by an elasticity tensor somewhat simpler than (36). To see why, and to derive such an elasticity tensor, we apply to our present case a general representation result in constrained linear elasticity [14, 11].

We make the shell geometry agree with the geometry intrinsic to the material response, in the sense that, at any fixed point \( x \in S \), we identify \( c_3 \) with \( n(x) \). With this identification, the internal constraint [6] can be read as the requirement that all admissible strains be orthogonal to the following subspace of \( \text{Sym} \):

\[
\mathcal{G}^\perp := \text{span}(V_\alpha, \alpha = 1, 2).
\]

Accordingly, the space \( \text{Sym} \) is split into the direct sum of two orthogonal subspaces:

\[
\text{Sym} = \mathcal{G} \oplus \mathcal{G}^\perp,
\]

and the stress is split into reactive and active parts:

\[
S = S^{(R)} + S^{(A)},
\]
with
\[ \mathbf{S}^{(R)} = \psi^{(R)}_\alpha \mathbf{V}_\alpha, \quad \psi^{(R)}_\alpha \in \mathbb{R}, \]
\[ \mathbf{S}^{(A)} = \mathbf{\tilde{C}}[\mathbf{E}], \quad \mathbf{\tilde{C}} : \mathcal{D} \to \mathcal{D}; \]

here the coefficients \( \psi^{(R)}_\alpha \) are constitutively unspecified, and the constraint space \( \mathcal{D} \) implicitly defined by (37) and (38) can be identified as
\[ \mathcal{D} = \text{span}(\mathbf{V}_3; \mathbf{W}_i, i = 1, 2, 3). \]

On applying a general result proved in [14], a representation for the desired elasticity tensor \( \mathbf{\tilde{C}} \) can be deduced from the one given for \( \mathbf{C} \) in (36):
\[ \mathbf{\tilde{C}} = \mathbf{P}_\mathcal{D} \mathbf{C} |_{\mathcal{D}}, \quad \mathbf{P}_\mathcal{D} := \mathbf{I} - \mathbf{V}_\alpha \otimes \mathbf{V}_\alpha. \]

where \( \mathbf{P}_\mathcal{D} \) denotes the orthogonal projector of Sym on \( \mathcal{D} \). One finds the 7-parameter representation
\[ \mathbf{\tilde{C}} = C_{1111} \mathbf{W}_1 \otimes \mathbf{W}_1 + C_{2222} \mathbf{W}_2 \otimes \mathbf{W}_2 + C_{3333} \mathbf{V}_3 \otimes \mathbf{V}_3 + C_{1212} \mathbf{W}_3 \otimes \mathbf{W}_3 + C_{1122}(\mathbf{W}_1 \otimes \mathbf{W}_2 + \mathbf{W}_2 \otimes \mathbf{W}_1) + C_{1133}(\mathbf{W}_1 \otimes \mathbf{V}_3 + \mathbf{V}_3 \otimes \mathbf{W}_1) + C_{2233}(\mathbf{W}_2 \otimes \mathbf{V}_3 + \mathbf{V}_3 \otimes \mathbf{W}_2). \]

As is customary, we assume that \( \mathbf{\tilde{C}} \) is positive-definite, i.e., that
\[ \mathbf{E} \cdot \mathbf{\tilde{C}}[\mathbf{E}] > 0 \quad \text{for all } \mathbf{E} \in \mathcal{D} \setminus \{\mathbf{0}\}. \]

Remark. In case the internal constraint (6) is reinforced à la Kirchhoff-Love as specified by (14), so as to exclude thickness changes, the above procedure yields the elasticity tensor
\[ \mathbf{\hat{C}} = C_{1111} \mathbf{W}_1 \otimes \mathbf{W}_1 + C_{2222} \mathbf{W}_2 \otimes \mathbf{W}_2 + C_{1212} \mathbf{W}_3 \otimes \mathbf{W}_3 + C_{1122}(\mathbf{W}_1 \otimes \mathbf{W}_2 + \mathbf{W}_2 \otimes \mathbf{W}_1), \]

a linear transformation of the constraint space \( \mathcal{\hat{D}} = \text{span}(\mathbf{W}_i, i = 1, 2, 3) \) into itself.

2.4.3 The technical moduli

In technical applications of classic isotropic elasticity, the two Lamé constants are replaced by the technical moduli \( E \) and \( \nu \) of Young and Poisson, plus the shear modulus \( G \), under the condition that \( E = 2(1 + \nu)G \). Likewise, we here replace the seven Lamé-like constants \( C_{1111}, \ldots, C_{2233} \) in (42) by an equivalent list of ten technical moduli – three of them being Young-like, six Poisson-like, and one shear-like – that must satisfy three independent algebraic conditions. The technical moduli in question are precisely those that enter the following
representation of the compliance tensor $\tilde{C}^{-1}$:

$$
\tilde{C}^{-1} = \frac{1}{E_1} W_1 \otimes W_1 + \frac{1}{E_2} W_2 \otimes W_2 + \frac{1}{E_3} V_3 \otimes V_3 + \frac{1}{2G_{12}} W_3 \otimes W_3 + 
\frac{\nu_{12}}{E_1} (W_1 \otimes W_2 + W_2 \otimes W_1) - \frac{\nu_{13}}{E_1} (W_1 \otimes V_3 + V_3 \otimes W_1) + 
\frac{\nu_{23}}{E_2} (W_2 \otimes V_3 + V_3 \otimes W_2)
$$

(as is well known, the positivity assumption (43) guarantees invertibility of $\tilde{C}$).

To confirm that the three moduli $E_1, E_2$ and $E_3$ are Young-like, imagine to induce a state of uniaxial traction in the direction, say, $c_1$ in a specimen made of the material under examination, so that the stress is $S = \sigma W_1$ and the corresponding strain is

$$
E = \sigma \tilde{C}^{-1} [W_1] = \frac{\sigma}{E_1} (W_1 - \nu_{12} W_2 - \nu_{13} V_3).
$$

Then, we find that

$$
E_1 = \frac{S \cdot W_1}{E \cdot W_1},
$$

the ratio between the axial stress and the corresponding axial deformation. Moreover, we find that

$$
\nu_{12} = -\frac{E \cdot W_2}{E \cdot W_1}, \quad \nu_{13} = -\frac{E \cdot V_3}{E \cdot W_1},
$$

the Poisson-like negative ratios of the transverse deformations in the directions $c_2$ and $c_3$ and the axial deformation in the direction $c_1$. The defining formulae for $E_2, \nu_{21}, \nu_{23}$ and $E_3, \nu_{31}, \nu_{32}$ are completely analogous to (46) and (47). Due to the built-in symmetries of $\tilde{C}$, it turns out that

$$
\frac{E_1}{E_2} = \frac{\nu_{12}}{\nu_{21}}, \quad \frac{E_1}{E_3} = \frac{\nu_{13}}{\nu_{31}}, \quad \frac{E_2}{E_3} = \frac{\nu_{23}}{\nu_{32}}.
$$

Finally, the formula

$$
2G = \frac{S \cdot W_3}{E \cdot W_3},
$$

establishes the nature of $G$ as a shear modulus.

It is straightforward to see that $G = \tilde{C}_{1212}$. The other technical moduli can
be written as follows in terms of the components of \( \tilde{C} \):

\[
E_1 = \frac{C_{1111}C_{2222}C_{3333} - C_{3333}C_{1122} - C_{2222}C_{1133}^2 + 2C_{1122}C_{1133}C_{2233}}{C_{2222}C_{3333} - C_{2233}^2},
\]
\[
E_2 = \frac{C_{1111}C_{2222}C_{3333} - C_{3333}C_{2222}C_{1133}^2 - C_{2222}C_{1111}C_{2233} + 2C_{1122}C_{1133}C_{2233}}{C_{1111}C_{3333} - C_{1133}^2},
\]
\[
E_3 = \frac{C_{1111}C_{2222}C_{3333} - C_{3333}C_{1122}^2 - C_{2222}C_{1111}C_{2233} + 2C_{1122}C_{1133}C_{2233}}{C_{1111}C_{2222} - C_{1122}^2};
\]
\[
\nu_{12} = \frac{C_{3333}C_{1122} - C_{1133}C_{2222}}{C_{2222}C_{3333} - C_{2233}^2}, \quad \nu_{21} = \frac{C_{3333}C_{1122} - C_{1133}C_{2233}}{C_{1111}C_{3333} - C_{1133}^2},
\]
\[
\nu_{13} = \frac{C_{2222}C_{1133} - C_{1122}C_{2233}}{C_{2222}C_{3333} - C_{2233}^2}, \quad \nu_{31} = \frac{C_{2222}C_{1133} - C_{1122}C_{2233}}{C_{1111}C_{2222} - C_{1122}^2},
\]
\[
\nu_{23} = \frac{C_{1111}C_{2233} - C_{1122}C_{1133}}{C_{1111}C_{3333} - C_{1133}^2}, \quad \nu_{32} = \frac{C_{1111}C_{2233} - C_{1122}C_{1133}}{C_{1111}C_{2222} - C_{1122}^2}.
\]

To sum up, the 7-parameter representation (42) can be re-written in the form:

\[
\tilde{C} = \Delta^{-1} \left( E_1(1 - \nu_{23}\nu_{32}) W_1 \otimes W_1 + E_2(1 - \nu_{13}\nu_{31}) W_2 \otimes W_2 + E_3(1 - \nu_{12}\nu_{21}) V_3 \otimes V_3 + 2\Delta G W_3 \otimes W_3 + E_1(\nu_{21} + \nu_{23}\nu_{31})(W_1 \otimes W_2 + W_2 \otimes W_1) + E_1(\nu_{31} + \nu_{21}\nu_{32})(W_1 \otimes V_3 + V_3 \otimes W_1) + E_2(\nu_{32} + \nu_{31}\nu_{12})(W_2 \otimes V_3 + V_3 \otimes W_2) \right),
\]

where

\[
\Delta := 1 - \nu_{12}\nu_{21} - \nu_{13}\nu_{31} - \nu_{23}\nu_{32} - 2\nu_{12}\nu_{23}\nu_{31}.
\]

### 2.5 Force and Moment Vectors and Tensors

The final step in the assemblage of our shell theory – the posing of initial- and boundary-value problems – demands that the balance equations (42) are written in terms of the parameters involved in the general representation (11) for an admissible displacement field. All we have to do to construct the corresponding parametric representations for those components of the force and moment tensors that enter the balance equations is to insert in the definitions (27) and (28) the general parametric representation for the stress field in \( \mathcal{G}(\mathbf{S}, \varepsilon) \), and then make use of definitions (29). Now, the required stress representation is obtained by combining (30) – with \( \mathbf{c}_3 \equiv \mathbf{n}(x) \) – and (42) – with \( \mathbf{E} = \mathbf{E}(\mathbf{u}_S) \), and \( \mathbf{u}_S \) given by (11), one finds:

\[
\mathbf{S}(x, \zeta) = s^{(R)}(x, \zeta) \otimes \mathbf{n}(x) + \mathbf{n}(x) \otimes s^{(R)}(x, \zeta) + \tilde{C}[\mathbf{E}(\mathbf{u}_S(x, \zeta))],
\]
where the restriction to the fiber $F(x)$ of the reactive field $s^{(R)}$ is a vector field perpendicular to $n(x)$.

Consequently, the force and moment vectors have both reactive and active parts, namely,

$$
R_f^{(a)}(x) = \left( \int I(\alpha, \zeta) (s^{(R)}(x, \zeta) \cdot g^a(x, \zeta)) d\zeta \right) n(x),
$$

$$
R_f^{(3)}(x) = \int I(\alpha, \zeta) s^{(R)}(x, \zeta) d\zeta, \quad A_f^{(i)}(x) = \int I(\alpha, \zeta) \tilde{C}[E(u_S(x, \zeta))] g^i(x, \zeta) d\zeta.
$$

(53)

and

$$
R_m^{(a)}(x) = \left( \int I(\alpha, \zeta) (s^{(R)}(x, \zeta) \cdot g^a(x, \zeta)) d\zeta \right) n(x),
$$

$$
R_m^{(3)}(x) = \int I(\alpha, \zeta) s^{(R)}(x, \zeta) d\zeta, \quad A_m^{(i)}(x) = \int I(\alpha, \zeta) \tilde{C}[E(u_S(x, \zeta))] g^i(x, \zeta) d\zeta.
$$

(54)

It is not difficult to check that

$$
R_f^{(i)}(x) \cdot A_f^{(i)}(x) = 0, \quad R_m^{(i)}(x) \cdot A_m^{(i)}(x) = 0.
$$

(55)

It is also easy to see, in the light of (29), that the force and moment tensors $^aF$ and $^aM$ have active and reactive parts as well. Thus, the balance equations one arrives at are not pure, in the sense that, in addition to the parameter fields, they also include reactive terms. In fact, the transverse shears are reactive, and equations (32) relate them to the active bending moments:

$$
F^{3\delta} = M^{\delta a} |_{a} + r^3_\delta.
$$

(56)

The thickness moments are also reactive, and (33) relates their divergence to the active bending moments and the thickness shear:

$$
M^{3\alpha} |_{\alpha} = -W^{\beta\alpha} M^{\beta\alpha} + F^{3\delta} - r^3_\delta.
$$

(57)

These observations suggest a sequential strategy to solve a shell problem within our present theory, where the unknowns are (the fields $a, w$ and the constant $\gamma$ that parameterize) the displacement field $u_S$ and the reaction force and moment fields: firstly, by the use of the projection operator $P_{\varphi}$ defined in the Subsection 2.4.2, one derives a set of reaction-free consequences of the balance equations; secondly, one solves such ‘purified’ system of equations for $u_S$; thirdly, one returns to the full balance equations, where the active terms can now be computed explicitly, and solves them for the reactive fields.

---

6It follows from the first of (40) and the first two of (55) that $S^{(R)} = s^{(R)} \otimes c_3 + c_3 \otimes s^{(R)}$, with $s^{(R)} = \frac{1}{\sqrt{2}} \psi^{(R)}_{\alpha} e_\alpha$.

7In fact, $F^{3\delta} = f^{(3)} \cdot \varepsilon^\delta = R^{(3)} \cdot \varepsilon^\delta$; and, $M^{\delta a} = m^{(3)} \cdot e^a = A^{(3)} \cdot e^a$.

8The annihilation procedure of the reactive terms occurring in the three-dimensional balance equations of constrained linear elasticity is discussed in [11], Section 17.2.
Remark. As anticipated in footnote 2, reactive forces and moments are found in the balance equations (32), because the class of variations (17) we used to derive those equations from the Principle of Virtual Powers (16) is visibly larger than the class (11) of admissible displacements (this will not be the case in the next section). Although we here do not pursue this issue any further, we recall that having at one’s disposal the reactive dynamical descriptors associated with the kinematical Ansatz adopted to construct a lower-dimensional structure theory can be proved beneficial to improve the pointwise approximation of the relative three-dimensional stress field [9, 7, 5, 13].
3 Cylindrical Shells: Generalities

3.1 Geometry

We now restrict our attention to shells whose model surface $S$ is a portion of a right circular cylinder, that we parameterize as usual by means of cylindrical coordinates:

$$z^1 = x_1, \quad z^2 = \vartheta$$

(Figure 4). For $\rho_o$ the radius of the directrix of $S$ and $n$ the normal to $S$, and for $\varepsilon < \rho_o$, the thickness of the shell-shaped region $\mathcal{G}(S, \varepsilon)$, the position vectors with respect to the origin $o$ of two typical points $p \in \mathcal{G}(S, \varepsilon)$ and $x \in S$ are, respectively,

$$p - o = p(x_1, \vartheta, \zeta) = x(x_1, \vartheta) + \zeta n(\vartheta) = x_1 c_1 + \rho_o \left(1 + \frac{\zeta}{\rho_o}\right) n(\vartheta).$$

and

$$x - o = x(x_1, \vartheta) = x_1 c_2 + \rho_o n(\vartheta), \quad n(\vartheta) = \sin \vartheta c_1 + \cos \vartheta c_3.$$

The relative covariant and contravariant bases can both be represented in terms of the orthonormal basis $\{e_1, n'(\vartheta), n(\vartheta)\}$, that serves as physical basis (recall Remark 2.1):

$$g_1(p) = e_1(x), \quad e_1(x) = c_1,$$

$$g_2(p) = \left(1 + \frac{\zeta}{\rho_o}\right) e_2(x), \quad e_2(x) = \rho_o n'(\vartheta), \quad (58)$$

$$g_3(p) = e_3(x) = n(\vartheta).$$

and

$$g^1(p) = e^1(x), \quad e^1(x) = c_1,$$

$$g^2(p) = \left(1 + \frac{\zeta}{\rho_o}\right)^{-1} e^2(x), \quad e^2(x) = \rho_o^{-1} n'(\vartheta), \quad (59)$$

$$g^3(p) = e^3(x) = n(\vartheta).$$
The surface shifter defined by \( \mathcal{A}_2 \) turns out to be:

\[
\mathcal{A}(\vartheta, \zeta) = c_1 \otimes c_1 + \left( 1 + \frac{\zeta}{\rho_o} \right) n'(\vartheta) \otimes n'(\vartheta),
\]

whence

\[
\alpha(\zeta) = 1 + \frac{\zeta}{\rho_o}.
\]

Finally, the only nonnull Christoffel symbols on \( S \) are:

\[
\gamma_{22}^3 = -\rho_o, \quad \gamma_{23}^2 = \gamma_{32}^2 = \rho_o^{-1}.
\]

### 3.2 Kinematics

The displacement field (11) now reads:

\[
u_C(x_1, \vartheta, \zeta) = \nu_{<1>}(x_1, \vartheta, \zeta) c_1 + \nu_{<2>}(x_1, \vartheta, \zeta) n'(\vartheta) + \nu_{<3>}(x_1, \vartheta, \zeta) n(n'(\vartheta)),
\]

with components:

\[
u_{<1>} = a_{<1>} - \zeta w_{,1},
\]

\[
u_{<2>} = \left( 1 + \frac{\zeta}{\rho_o} \right) a_{<2>} - \frac{\zeta}{\rho_o} w_{,2},
\]

\[
u_{<3>} = w + \zeta \gamma.
\]

Since

\[
\nabla u_C = u_i \otimes g^i = u_{,1} \otimes c_1 + \frac{1}{\rho_o} \left( 1 + \frac{\zeta}{\rho_o} \right)^{-1} u_{,2} \otimes n' + u_{,3} \otimes n,
\]
the nonnull physical components of the strain tensor are:

\[ E_{11} = \nabla u \cdot c_1 \otimes c_1 = u_{11} = a_{11} - \zeta w_{11}, \]

\[ 2E_{12} = 2E_{21} = \nabla u \cdot (c_1 \otimes n' + n' \otimes c_1) = u_{21} + \rho_0 \left(1 + \frac{\zeta}{\rho_o}\right)^{-1} (a_{12} - \zeta w_{12}), \]

\[ E_{22} = \nabla u \cdot n' \otimes n' = \rho_0 \left(1 + \frac{\zeta}{\rho_o}\right)^{-1} (u_{22} + u_{33}), \]

\[ E_{33} = \nabla u \cdot n \otimes n = u_{33} = \gamma. \]

(63)

### 3.3 Balance Assumptions

We exploit the virtual-power procedure detailed in Section 2.3, with the difference anticipated in Remark 2.5: the variations we now employ have the same structure as the admissible displacements (61)-(62); hence, no reactive contributions are going to enter the field and boundary equations that the procedure delivers. Precisely, the variations in question have the following form:

\[ \begin{align*}
{v}^{(0)} &= v_1 c_1 + v_2 n' + v_3 n, \\
{v}^{(1)} &= -v_{3,1} c_1 + \rho_o^{-1}(v_2 - v_{3,2}) n' + v_4 n,
\end{align*} \]

(64)

with the scalar fields \( v_i = v_i(x_1, \vartheta) \) \((i = 1, 2, 3)\) compactly supported in \( S \) and with \( v_4 \) a constant.

#### 3.3.1 Field equations

With a view toward deriving the field equations to which the general balances reduce in the present situation, we firstly take \( v_4 = 0 \) in (64). In this instance, on setting:

\[ \begin{align*}
f^{(0)} := -(^s \text{Div}^s F + q_o), \quad m^{(0)} := -(^s \text{Div}^s M - f^{(3)} + r_o),
\end{align*} \]

the formulation (22) of the Principle of Virtual Powers reads:

\[ 0 = \int_{\mathcal{P}} (f^{(0)} \cdot v^{(0)} + m^{(0)} \cdot v^{(1)}) = \int_{\mathcal{P}} (v_1 (f^{(0)} \cdot c_1) + v_2 (f^{(0)} + \rho_o^{-1} m^{(0)}) \cdot n' + v_3 (f + (m \cdot c_1)_{11} + \rho_o^{-1} (m \cdot n')_{22}) \cdot n). \]
Hence, three field equations must hold at each interior point of \( \mathcal{S} \), namely,

\[
\begin{align*}
(0) & \quad f \cdot c_1 = 0, \\
(0) & \quad (f + \rho_o^{-1} m) \cdot n' = 0, \\
(0) & \quad (f + (m \cdot c_1) + \rho_o^{-1} (m \cdot n') \cdot n = 0.
\end{align*}
\]

To find the balance law that follows from testing the equality of internal and external powers by way of virtual velocity fields of the form:

\[
v = \zeta (1) v, \quad (1) v = v^4 n,
\]

with \( v_4 \) an arbitrary real number, we turn to the formulation (15) of the Principle: indeed, \( v_4 \) must be taken constant over the whole of \( \Omega \), because it is meant to be a variation of the constant thickness strain \( E_{<33>} = \gamma \) in the representation (62) of the admissible displacement. Given that, for any \( v \) as in (66),

\[
\nabla v = v_4 (n \otimes n + \frac{\zeta}{\rho_o} n' \otimes n'),
\]

and that \( v_4 \) can be chosen arbitrarily, (15) and (66) yield:

\[
\int_{-l}^{+l} \int_0^{2\pi} \left( \int_I \alpha (S \cdot n \otimes n + \frac{\zeta}{\rho_o} S \cdot n' \otimes n' - \zeta d_o \cdot n) d\zeta \right) dx_1 d\theta = 0. \quad (67)
\]

In terms of physical components of the force and moment tensors, the system of equations (65) and (67) can be written as:

\[
\begin{align*}
F_{<11>,1} + \rho_o^{-1} F_{<12>,2} + q_o <1> & = 0, \\
(F_{<21>} + \rho_o^{-1} M_{<21>,1} + \rho_o^{-1} (F_{<22>} + \rho_o^{-1} M_{<22>}) )_{12} + q_o <2> + \rho_o^{-1} r_o <2> & = 0, \\
M_{<11>,1} + \rho_o^{-1} (M_{<12>} + M_{<21>})_{12} + \frac{1}{\rho_o} M_{<22>,22} & = 0, \\
- \rho_o^{-1} F_{<22>} + q_o <3> + r_o <1>,1 + \rho_o^{-1} r_o <2>,2 & = 0, \\
\int_{-l}^{+l} \int_0^{2\pi} \left( \rho_o^{-1} M_{<22>} + F_{<33>} - r_o <3> \right) dx_1 d\theta & = 0;
\end{align*}
\]

(68)

\[9\text{More generally, we point out that, whenever the admissible motions are such as to suggest the use of test fields of the form } v = \nu_o w, \text{ with } \nu_o \text{ an arbitrary constant and } w \text{ a given vector field, then (15) is the appropriate formulation of the Principle of Virtual Powers, and the associated balance information is an integral relation of the form:}

\[
\int_{\Omega} S \cdot \nabla w = \int_{\Omega} d_o \cdot w + \int_{\partial \Omega} c_o \cdot w.
\]

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the following version in terms of contravariant components may ease a comparison with (32):

\[
F_{11}^{1} + F_{12}^{2} + q_1^{0} = 0,
\]

\[
F_{21}^{1} + F_{22}^{2} + \rho_0^{-1}(M_{21}^{1} + M_{22}^{2} + r_0^2) + q_0 = 0,
\]

\[
M_{11}^{1} + M_{22}^{2} + (M_{21}^{1} + M_{12}^{2})_{,12} - \rho_0 F_{22}^{2} + q_3^{0} + r_{1,1}^{0} + r_{0,2}^{0} = 0,
\]

\[
\int_{-l}^{+l} \int_{0}^{2\pi} (\rho_0 M_{22}^{2} + F_{33}^{3} - r_0^{3}) \, dx_1 d\vartheta = 0.
\]

### 3.3.2 Boundary equations

We start from the general weak statement (24), repeated here for the reader’s convenience:

\[
\int_{\partial \mathcal{P}} \left( (\textbf{m} \cdot (\textbf{v}) + (\textbf{m} \cdot (\textbf{v})) \right) = 0,
\]

where we now insert variations of type (64). For simplicity, we restrict attention to parts of the model surface whose boundary consists of the union of two identical parts \(d_0, d_1\) of directrices at different axial abscissae and the relative two segments of generatrices \(g_0, g_1\), that is to say, with reference to Figure 5, parts \(\mathcal{P}\) such that

![Figure 5](image-url)

Figure 5: A typical part of the model surface of a cylindrical shell, bounded by generatrices and directrices.

\[
\partial \mathcal{P} = \bigcup_{a} d_a \cup g_a \ (a = 0, 1),
\]

with

\[
d_a := \{ x \in \mathcal{S} \mid x \equiv (l_a, \vartheta), \ \vartheta \in (\vartheta_a, \vartheta_{a1}) \}, \ g_a := \{ x \in \mathcal{S} \mid x \equiv (x_1, \vartheta_a), \ x_1 \in (l_0, l_1) \}.
\]
Over \( d_1 \), where \( m = c_1 \), the integral condition (24) takes the form:

\[
0 = \int_{\partial_0}^{\partial_1} \left( (F \cdot c_1 - l_o) \cdot (v_1 c_1 + v_2 n' + v_3 n) + \right.
\]
\[
+ \left( (M \cdot c_1 - m_o) \cdot (-v_{3,1} c_1 + \rho_o^{-1} (v_2 - v_{3,2}) n' + v_4 n) \right) \bigg|_{x_1 = t_1} d\theta =
\]
\[
= \int_{\partial_0}^{\partial_1} \left( v_1 (F_{<12> - l_o<1>}) + v_2 (F_{<22> + \rho_o^{-1} M_{<22>} - l_o<2> - \rho_o^{-1} m_o<2>) + 
\right.
\]
\[
+ \left. v_3 (F_{<32> - \rho_o^{-1} M_{<32>,2} - l_o<3> - \rho_o^{-1} m_o<2,2>) + 
\right.
\]
\[
- v_{3,1} (M_{<11> - m_o<1>} + v_4 (M_{<31> - m_o<3>}) \bigg|_{x_1 = t_1} \right. d\theta +
\]
\[
+ \left. \left( v_3 M_{<21> - \rho_o^{-1} m_o<2>} \right) \bigg|_{(t_1, \theta_1)} \right]
\]

(69)

over \( g_1 \), where \( m = n'(\theta) \), (24) becomes:

\[
0 = \int_{l_o}^{l_1} \left( (F \cdot n' - l_o) \cdot (v_1 c_1 + v_2 n' + v_3 n) + \right.
\]
\[
+ \left( (M \cdot n' - m_o) \cdot (-v_{3,1} c_1 + \rho_o^{-1} (v_2 - v_{3,2}) n' + v_4 n) \right) \bigg|_{\theta = \theta_1} dx_1 =
\]
\[
= \int_{l_o}^{l_1} \left( v_1 (F_{<12>} - l_o<1>) + v_2 (F_{<22> + \rho_o^{-1} M_{<22>} - l_o<2> - \rho_o^{-1} m_o<2>) + 
\right.
\]
\[
+ \left. v_3 (F_{<32>} - \rho_o^{-1} M_{<32>,1} - l_o<3> - \rho_o^{-1} m_o<2,1>) + 
\right.
\]
\[
- v_{3,2} (M_{<22> - m_o<1>} + v_4 (M_{<32> - m_o<3>}) \bigg|_{\theta = \theta_1} dx_1 +
\]
\[
+ \left. \left( v_3 M_{<12> - \rho_o^{-1} m_o<2>} \right) \bigg|_{(l_0, \theta_1)} \right)
\]

(70)

needless to say, relations completely similar to (69) and (70) hold, respectively, at \( d_0 \) and \( g_0 \).

As mentioned in Section 5, if \( u_3^{(0)} \) or anyone of the components of \( u^{(0)} \) and \( \nabla u_3^{(0)} \) is assigned at a boundary point, then \( v_4 \) or the corresponding component of \( v^{(0)} \) and \( \nabla v_3^{(0)} \) must vanish at that point; on the other hand, the arbitrary variation of each of the remaining parameters in (64) induces at the same point a boundary equation of Neumann type: for example, inspection of (69) shows that, if \( u \cdot c_1 = a_{<1>} \) is assigned over \( d_a \), then \( v_1 \) must be taken there identically null, while the Neumann boundary equations

\[ F_{<21>} + \rho_o^{-1} M_{<21>} = l_o<2> + \rho_o^{-1} m_o<2> \]

etc.

must hold; in other words, at a point of \( d_a \) admissible boundary conditions must consist of a list of mutually exclusive assignments of the one or the other element
of the following five power-conjugate pairs:

\[
(F_{<11>}, a_{<1>}), \ (F_{<21>} + \rho_o^{-1} M_{<21>}, a_{<2>}), \ (F_{<31>} - \rho_o^{-1} M_{<21>}, w), \ (M_{<11>}, w_{1}), \ (M_{<31>}, \gamma). \tag{71}
\]

Likewise, from (70) we deduce that the boundary conditions at a point of \( g_a \) should consist of mutually exclusive assignments of the pairs:

\[
(F_{<12>}, a_{<1>}), \ (F_{<22>} + \rho_o^{-1} M_{<22>}, a_{<2>}), \ (F_{<32>} - \rho_o^{-1} M_{<12>}, w), \ (M_{<22>}, w_{2}), \ (M_{<32>}, \gamma). \tag{72}
\]

3.3.3 Inertial interactions. Evolution equations

The time evolution of an unshearable orthotropic shell is ruled by the partial differential equations that follow from (23) when the inertial force is separated from the rest of the distance force per unit volume of \( G(S, \varepsilon) \). To this effect, we set:

\[ d_{\text{o}}^{\text{in}} := -\delta_o \ddot{u} \quad \text{and} \quad d_{\text{n}}^{\text{o}} := d_{\text{o}} - d_{\text{o}}^{\text{in}}, \]

for, respectively, the inertial and noninertial distance forces (here \( \delta_o \) denotes the mass density per unit reference volume and a superposed dot signifies time differentiation). We then set:

\[
q_{\text{o}}^{\text{in}} := -\int_I \alpha(x, \zeta) \delta_o(x, \zeta) \ddot{u}(x, \zeta; t) d\zeta \quad \text{and} \quad r_{\text{o}}^{\text{in}} := -\int_I \alpha(x, \zeta) \zeta \delta_o(x, \zeta) \ddot{u}(x, \zeta; t) d\zeta, \tag{73}
\]

for the inertial force and the inertial couple per unit area of \( S \). Finally, we return to definitions (21) and set:

\[ q_{\text{o}}^{\text{n}} := q_{\text{o}} - q_{\text{o}}^{\text{in}} \quad \text{and} \quad r_{\text{o}}^{\text{n}} := r_{\text{o}} - r_{\text{o}}^{\text{in}}, \]

for the relative noninertial loadings. In conclusion, the balance equations (23) take the evolutionary form:

\[
q_{\text{o}}^{\text{in}} = ^a\text{Div} \cdot ^aF + q_{\text{o}}^{\text{n}}, \quad r_{\text{o}}^{\text{in}} = ^a\text{Div} \cdot ^aM - f^{(3)} + r_{\text{o}}^{\text{n}}. \tag{74}
\]
In the case of cylindrical shells whose mass distribution is uniform, definitions (73) yield:

\[
q_o^{\text{in}1} = -\delta_o \left( \ddot{a}^{\text{<1>}} - \frac{1}{3} \varepsilon^2 \rho_o^{-1} \ddot{\mathbf{w}}^{1} \right),
\]

\[
q_o^{\text{in}2} = -\delta_o \left( 1 + \frac{1}{3} \varepsilon^2 \rho_o^{-1} \ddot{a}^{\text{<2>}} - \frac{1}{3} \varepsilon^2 \rho_o^{-1} \ddot{\mathbf{w}}^{2} \right),
\]

\[
q_o^{\text{in}3} = -\delta_o \left( \ddot{w} + \frac{1}{3} \varepsilon^2 \rho_o^{-1} \ddot{\gamma} \right),
\]

\[
r_o^{\text{in}1} = -\delta_o \left( \ddot{a}^{\text{<1>}} - \ddot{\mathbf{w}}^{1} \right),
\]

\[
r_o^{\text{in}2} = -\delta_o \left( \frac{1}{3} \varepsilon^2 \rho_o^{-1} ( \ddot{a}^{\text{<2>}} - \ddot{\mathbf{w}}^{2} ) \right),
\]

\[
r_o^{\text{in}3} = -\delta_o \left( \ddot{w} + \frac{1}{3} \varepsilon^2 \rho_o^{-1} \ddot{\gamma} \right),
\]

where \(\delta_o := (2\varepsilon)\delta_o\) is the uniform mass density per unit area of the model surface \(S\). Hence, the evolution equations corresponding to the balance equations (68) are:

\[
\delta_o \left( \ddot{a}^{\text{<1>}} - \frac{1}{3} \varepsilon^2 \rho_o^{-1} \ddot{w}^{1} \right) = F^{\text{<11>,1}} + \rho_o^{-1} F^{\text{<12>,2}} + q_o^{\text{in}1} <1>,
\]

\[
\delta_o \left( 1 + \frac{2}{3} \varepsilon^2 \rho_o^{-1} \ddot{a}^{\text{<2>}} - \frac{2}{3} \varepsilon^2 \rho_o^{-1} \ddot{w}^{2} \right) =
\]

\[
= (F^{\text{<21>}} + \rho_o^{-1} M^{\text{<21>}},1) + \rho_o^{-1} (F^{\text{<22>}} + \rho_o^{-1} M^{\text{<22>}},2) + q_o^{\text{in}2} + \rho_o^{-1} r_o^{\text{ni}2},2>,
\]

\[
\delta_o \left( \ddot{w} + \frac{1}{3} \varepsilon^2 \rho_o^{-1} \ddot{\gamma} \right) =
\]

\[
= M^{\text{<11>,11}} + \rho_o^{-1} (M^{\text{<12>}} + M^{\text{<21>}},2) + \frac{1}{\rho_o^2} M^{\text{<22>}},22 - \rho_o^{-1} F^{\text{<22>}} +
\]

\[
+ q_o^{\text{in}3} + r_o^{\text{ni}3} <1>,1 + \rho_o^{-1} r_o^{\text{ni}2},2,
\]

\[
\delta_o \left( \ddot{w} + \frac{1}{3} \varepsilon^2 \rho_o^{-1} \ddot{\gamma} \right) = \rho_o^{-1} M^{\text{<22>}} + F^{\text{<33>}} - r_o^{\text{ni}3} <3>.
\]

Needless to say, these equations are to be equipped with a set of initial conditions for the unknown fields \(a^{\text{<1>}}, a^{\text{<2>}}, w,\) and \(\gamma\), and for their time rates.

### 3.4 Constitutive Assumptions

The components of force and moment tensors that appear in (68) depend on the active part of the three-dimensional stress field. The latter has the following expression in terms of the constitutive law (50) and the strain tensor (63):

\[
S(x, \zeta) = \mathbb{C}[E(u_C(x, \zeta))];
\]

(77)
in components, this equation reads:

\[ S_{11} = \frac{E_1}{\Delta} (1 - \nu_{23}\nu_{32}) E_{11} + \frac{E_2}{\Delta} \left( \nu_{21} + \nu_{23}\nu_{31} \right) E_{22} + \frac{E_1}{\Delta} (\nu_{31} + \nu_{21}\nu_{32}) E_{33}, \]

\[ S_{22} = \frac{E_2}{\Delta} (1 - \nu_{13}\nu_{31}) E_{22} + \frac{E_1}{\Delta} \left( \nu_{21} + \nu_{23}\nu_{31} \right) E_{11} + \frac{E_2}{\Delta} (\nu_{32} + \nu_{31}\nu_{12}) E_{33}, \]

\[ S_{33} = \frac{E_3}{\Delta} (1 - \nu_{12}\nu_{21}) E_{33} + \frac{E_1}{\Delta} (\nu_{31} + \nu_{21}\nu_{32}) E_{11} + \frac{E_3}{\Delta} (\nu_{32} + \nu_{31}\nu_{12}) E_{22}, \]

\[ S_{12} = S_{21} = 2G E_{12}, \]  

(78)

where the constitutive modulus \( \Delta \) is defined as in (51), the components \( E_{\alpha\beta} \) as in (83). On inserting these relations in the appropriate consequences of definitions (29), we find:

\[ F_{11} = \int_I \left( 1 + \frac{\zeta}{\rho_o} \right) S_{11} = \frac{2}{\rho_o} \left[ \frac{E_1}{\Delta} (1 - \nu_{23}\nu_{32}) \left( \rho_o a_{<1>,1} - \frac{1}{3} \varepsilon^2 w_{11} \right) \right. \]

\[ + \left( \nu_{21} + \nu_{23}\nu_{32} \right) (a_{<2>,2} + w) + \left( \nu_{31} + \nu_{21}\nu_{32} \right) \rho_o \gamma \right], \]

(79)

\[ F_{22} = \int_I \left( 1 + \frac{\zeta}{\rho_o} \right) S_{22} = \frac{2}{\rho_o} \left[ \frac{E_2}{\Delta} (1 - \nu_{13}\nu_{31}) \left( a_{<2>,2} - \left( 1 - \frac{1}{2} \varepsilon^2 \rho_o \gamma \right) \right) \right. \]

\[ + \left( \nu_{21} + \nu_{23}\nu_{32} \right) (a_{<2>,2} + w) + \frac{1}{2} \log \left( \frac{1 + \frac{\varepsilon^2}{\rho_o}}{1 - \frac{\varepsilon^2}{\rho_o}} \right) \rho_o \gamma \right], \]

(80)

\[ F_{33} = \int_I \left( 1 + \frac{\zeta}{\rho_o} \right) S_{33} = \frac{2}{\rho_o} \left[ \frac{E_3}{\Delta} (1 - \nu_{12}\nu_{21}) \rho_o \gamma + \right. \]

\[ + \left( \nu_{31} + \nu_{21}\nu_{32} \right) \left( \rho_o a_{<1>,1} - \frac{1}{3} \varepsilon^2 w_{11} \right) + \frac{\nu_{32} - \nu_{31}\nu_{12}}{\mu} \left( a_{<2>,2} + w \right) \right], \]

(81)

\[ F_{21} = \int_I \left( 1 + \frac{\zeta}{\rho_o} \right) S_{21} = 2\varepsilon G \left[ \left( 1 + \frac{1}{3} \varepsilon^2 \rho_o \gamma \right) a_{<2>,1} - \frac{1}{3} \varepsilon^2 w_{21} + \frac{1}{\rho_o} a_{<1>,2} \right], \]

(82)

and

\[ M_{11} = \int_I \left( 1 + \frac{\zeta}{\rho_o} \right) \zeta S_{11} = \frac{2}{\rho_o} \left[ \frac{E_1}{\Delta} (1 - \nu_{23}\nu_{32}) (a_{<1>,1} - \rho_o w_{11}) + \right. \]

\[ + \left( \nu_{21} + \nu_{23}\nu_{31} \right) (a_{<2>,2} - w_{21}) + \left( \nu_{31} + \nu_{21}\nu_{32} \right) \gamma \right], \]

(83)

\[ M_{21} = \int_I \left( 1 + \frac{\zeta}{\rho_o} \right) \zeta S_{21} = \frac{4}{3} \varepsilon^4 G (a_{<2>,1} - w_{12}), \]  

(84)
\[ M_{<12>} = \int \zeta S_{<12>} = \frac{2}{3} \varepsilon^3 \frac{G}{\rho_o} \left[ a_{<2>,1} - w_{,21} + 3 \frac{\rho_o}{\varepsilon} \left( 1 - \frac{1}{2} \log \frac{1 + \frac{\rho_o}{\rho_o}}{1 - \frac{\rho_o}{\rho_o}} \right) a_{<1>,2} + \right. \\
\left. - 3 \frac{\varepsilon}{\rho_o} \left( 1 - \log \frac{1 + \frac{\rho_o}{\rho_o}}{1 - \frac{\rho_o}{\rho_o}} \right) w_{,12} \right], \tag{85} \]

\[ M_{<22>} = \int \zeta S_{<22>} = 2 \varepsilon E_2 \Delta \left[ (1 - \nu_{13} \nu_{31}) \left( 1 - \frac{1}{2} \frac{\rho_o}{\rho_o} \log \frac{1 + \frac{\rho_o}{\rho_o}}{1 - \frac{\rho_o}{\rho_o}} \right) \left( w - w_{,22} + \rho_o \gamma \right) + \\
- \frac{1}{3} \varepsilon^2 \frac{\nu_{12} + \nu_{23} \nu_{31}}{\eta} w_{,11} \right], \tag{86} \]

where
\[ \frac{E_1}{E_2} = \frac{\nu_{12}}{\nu_{21}} =: \frac{1}{\eta}, \quad \frac{E_1}{E_3} = \frac{\nu_{13}}{\nu_{31}} =: \frac{1}{\lambda}, \quad \frac{E_2}{E_3} = \frac{\nu_{23}}{\nu_{32}} =: \frac{1}{\mu}. \tag{87} \]

The equations governing the equilibria of unshearable cylindrical shells are arrived at when the above constitutive equations are inserted into the balances \[ \text{[85]}. \] In their general form, those equations are complicated to solve analytically; we choose not to list them here. However, certain highly symmetric problems admit simple and explicit solutions. We deal with such problems in the next section, while the noticeable simplifications obtained when the approximations judged appropriate for thin and slender shells will be introduced and exemplified in Section 5.

## 4 Cylindrical Shells: Axisymmetric Boundary-Value Problems

A boundary-value problem for a cylindrical shell is \textit{axisymmetric} if the load and confinement data induce solution displacement fields whose physical components \( u_{<1>}, u_{<2>}, u_{<3>} \) are all independent of the circumferential coordinate \( \vartheta \), that is to say, in view of \( \text{[62]} \), if

\[ u_{<1>} = a_{<1>} - \zeta w', \quad u_{<2>} = \left( 1 + \frac{\zeta}{\rho_o} \right) a_{<2>}, \quad u_{<3>} = w + \zeta \gamma, \tag{88} \]

where a prime denotes differentiation with respect to \( x_1 \), the only space variable from which all of the parameter fields \( a_{<1>}, a_{<2>}, \) and \( w \), may depend. When the displacement field has the form \( \text{[88]} \),
(i) the strain components (63) take the simpler form:

\[ E_{<11>} = a_{<1>}' - \zeta w'', \]
\[ E_{<12>} = E_{<21>} = \frac{1}{2} \left(1 + \frac{\zeta}{\rho_o}\right) a_{<2>}', \]
\[ E_{<22>} = \left(\rho_o \left(1 + \frac{\zeta}{\rho_o}\right)^{-1}\right) (w + \zeta \gamma), \]
\[ E_{<33>} = \gamma; \]  

(89)

(ii) the constitutive equations (82)-(86) for the force and moment components become:

\[ F_{<11>} = 2 \frac{\varepsilon}{\rho_o} E_1 \left[ (1 - \nu_{23}\nu_{32}) \left(\rho_o a_{<1>}' - \frac{1}{3} \varepsilon w''\right) + (\nu_{21} + \nu_{23}\nu_{31}) w + (\nu_{31} + \nu_{21}\nu_{32}) \rho_o \gamma \right], \]
\[ F_{<22>} = 2 \frac{\varepsilon}{\rho_o} E_2 \left[ (1 - \nu_{13}\nu_{31}) \left(\frac{1 - \frac{1}{2} \varepsilon}{\rho_o} \log \frac{1 + \frac{\rho_o}{\rho_o}}{1 - \frac{\rho_o}{\rho_o}} \right) \rho_o \gamma + \frac{1}{2} \frac{\varepsilon^2}{\rho_o} log \frac{1 + \frac{\rho_o}{\rho_o}}{1 - \frac{\rho_o}{\rho_o}} w \right] + \rho_o \frac{\nu_{21} + \nu_{23}\nu_{31}}{\eta} a_{<1>}' + (\nu_{32} + \nu_{31}\nu_{12}) \rho_o \gamma \]
\[ F_{<33>} = 2 \varepsilon G \left[ 1 + \frac{1}{3} \frac{\varepsilon^2}{\rho_o} \right] a_{<2>'}, \]
\[ F_{<33>} = 2 \frac{\varepsilon}{\rho_o} E_3 \left[ (1 - \nu_{12}\nu_{21}) \rho_o \gamma + \frac{\nu_{31} + \nu_{21}\nu_{32}}{\lambda} \left(\rho_o a_{<1>}' - \frac{1}{3} \varepsilon w''\right) + \frac{\nu_{32} + \nu_{31}\nu_{12}}{\mu} w \right], \]  

(90)

and

\[ M_{<11>}' = -2 \frac{\varepsilon^3}{3 \rho_o} E_1 \left[ (1 - \nu_{23}\nu_{32})(\rho_o w'' - a_{<1>}'') - (\nu_{31} + \nu_{21}\nu_{32}) \right], \]
\[ M_{<21>} = \frac{4 \varepsilon^3}{3 \rho_o} G a_{<2>'}, \quad M_{<12>} = \frac{2 \varepsilon^3}{3 \rho_o} G a_{<2>'}, \]
\[ M_{<22>} = 2 \frac{\varepsilon}{\rho_o} F_2 \Delta \left[ (1 - \nu_{13}\nu_{31}) \left(1 - \frac{1}{2} \frac{\varepsilon}{\rho_o} \log \frac{1 + \frac{\rho_o}{\rho_o}}{1 - \frac{\rho_o}{\rho_o}} \right) (w + \rho_o \gamma) - \frac{1}{3} \frac{\varepsilon^2}{\rho_o} \log \left[1 + \frac{\rho_o}{\rho_o} \right] w'' \right] \]  

(91)

(iii) the reaction-free equations (68) reduce to the field equations:

\[ F_{<11>}' + q_{o<1>'} = 0, \]
\[ (F_{<21>} + \rho_o^{-1} M_{<21>'})' + q_{o<2>} + \rho_o^{-1} r_{o<2>} = 0, \]
\[ M_{<11>''} - \rho_o^{-1} F_{<22>} + q_{o<3>} + r_{o<1>}' = 0, \]  

(92)
holding in the interval \((-l, +l)\), plus the integral relation:

\[
\int_{-l}^{+l} \left( \rho_o^{-1} M_{<22>} + F_{<33>} - r_o <3> \right) \, dx_1 = 0. \tag{93}
\]

(iv) the reaction-free boundary conditions consist in specifications at \(\pm l\) of one of the elements in each of the pairs \(\mathcal{J}_1\), \(\mathcal{J}_2\), and \(\mathcal{J}_4\):

\[
(F_{<11>}, a_{<1>}), \quad (F_{<21>} + \rho_o^{-1} M_{<21>}, a_{<2>}), \quad (M_{<11>}, w'); \tag{94}
\]

Insertion of (90) and (91) into (92) and (93) yields the system of equations ruling axisymmetric boundary-value problems in our theory. We are going to solve this system for assignments of Neumann data corresponding, respectively, to problems of torsion, axial traction, pressure, and rim flexure. Interestingly, as we shall quickly demonstrate, this system splits into one equation for the circumferential displacement \(a_{<2>}\) plus a system of three equations for the axial and radial displacements \(a_{<1>}\) and \(w\) and the thickness stretch \(\gamma\).

**Remark.** As mentioned in closing Section 2.5, once the equilibrium displacement field has been found, the axial distributions of reactive stress measures \(F_{<31>}\) and \(M_{<31>}\) consistent with boundary conditions compatible with \(\mathcal{J}_1\) and \(\mathcal{J}_5\) can be determined by a use of, respectively, the balance equation (57), keeping into account (93). The situations of our present interest occur when homogeneous Neumann data are prescribed at the boundary, so that the problems to solve are:

\[
F_{<31>} = M_{<11>}' + r_o <1> \quad \text{in} \quad (-l, +l), \quad F_{<31>} (\pm l) = 0, \tag{95}
\]

and

\[
M_{<31>} = 0 \quad \text{in} \quad (-l, +l), \quad M_{<31>} (\pm l) = 0 \quad \iff \quad M_{<31>} \equiv 0 \quad \text{in} \quad [-l, +l]. \tag{96}
\]

Note that equation (95) has the familiar structure of the moment balance in Bernoulli-Navier rod theory, with \(F_{<31>}\) playing the role of the shear resultant, \(M_{<11>}\) of the bending moment, and \(r_o <1>\) of the diffused applied couples; both \(F_{<31>}\) here and the shear resultant in that classic rod theory have reactive nature, as a consequence of one and the same unshearability constraint.

### 4.1 Torsion

From (90) and (91), we have that

\[
F_{<21>} + \rho_o^{-1} M_{<21>} = 2\varepsilon \left( 1 + \frac{\varepsilon^2}{\rho_o^2} \right) G a_{<2>}' \tag{97}
\]

with this, equation (92) takes the form of a second-order equation for \(a_{<2>}\):

\[
2\varepsilon \left( 1 + \frac{\varepsilon^2}{\rho_o^2} \right) G a_{<2>>}'' + q_{<2>} + \rho_o^{-1} r_o <2> = 0. \tag{98}
\]
This equation accounts for whatever twisting about its axis an unshearable cylindrical shell may have; it can be associated with boundary conditions specifying the values at \(x_1 = \pm l\) of either \(a<2>_+\) or \(a<2>_'\).

When the only applied load is a distribution of end tractions statically equivalent to two mutually balancing torques of magnitude
\[
T = (2\pi \rho_o^2)t, \quad \text{with } t = O(\varepsilon),
\]
the thickness stretch \(\gamma\) and the axial displacement \(a<2>_+\) vanish, and \(\rho_o^{-1} a<2>_+ = \text{constant}\), whose value is determined by the boundary condition:
\[
F<21>_+ + \rho_o^{-1} M<21>_+ = t = r_T a<2>_+', \quad r_T := 2\varepsilon \left(1 + \frac{\varepsilon^2}{\rho_o^2}\right) G; \quad (100)
\]
it follows from \((100)\) that a twisted shell of the type we study undergoes a rotation per unit length
\[
\Theta := \rho_o^{-1} a<2>_+'
\]
proportional to \((\rho_o r_T)^{-1}\). Moreover, given that the function \(a<2>\) must be odd,
\[
a<2>(x_1) = \frac{1}{1 + \frac{\varepsilon^2}{\rho_o^2}} \frac{\varepsilon^{-1}_1}{2G} x_1.
\]

**Remark.** In line with \((75)\), set
\[
q_o<2>_+ = q_o <2>_+ = -\delta_o \left(1 + \frac{\varepsilon^2}{3 \rho_o^2}\right) \tilde{a}<2>_+
\]
in equation \((98)\), so that it takes the form of the classical wave equation:
\[
\frac{\partial^2 a<2>(x_1,t)}{\partial t^2} - c^2 \frac{\partial^2 a<2>(x_1,t)}{\partial x_1^2} = 0, \quad \text{with } c^2 = \frac{G}{\delta_o} \frac{1 + \frac{\varepsilon^2}{\rho_o^2}}{1 + \frac{2\varepsilon^2}{3 \rho_o^2}}.
\]
Then,
\[
a<2>(x_1,t) = \varphi(x_1 + ct) + \psi(x_1 - ct),
\]
and two *twist waves* propagate along the axis with speed \(|c|\), the one in the positive direction the other in the negative direction.

**Remark.** In all boundary-value problems we shall solve next – we recall, axial traction, uniform pressure, and rim flexure – the twisting loads are null. As a consequence, in all three cases, the second equation of system \((92)\) and the boundary equations:
\[
(F<21>_+ + \rho_o^{-1} M<21>_+)(\pm l) = 0
\]
together imply that the construct \((F<21>_+ + \rho_o^{-1} M<21>_+)\) is identically null in \([-l, +l]\). Hence, by the constitutive relations \((90)_3\) and \((91)_2\), the circumferential
displacement $a_{<2>}$ has to have constant value, as is the case for a rotation about the $x_1$-axis; we note that this, because of (91)_2, implies that

$$M_{<21>}(-l) = 0,$$

and we take $a_{<2>} = 0$. In fact, as is customary in elasticity with Neumann data, in all three cases we expect to arrive at a displacement solution being unique to within an ignorable rigid motion. Moreover, given the common built-in symmetries, there will be no loss of generality in searching for solutions with $a_{<1>}$ an odd function of $x_1 \in [-l, +l]$, and $w$ even.

4.2 Traction

Let us take all distance forces and couples null and all boundary conditions of Neumann type and homogeneous, except for a distribution of tractions at the ends equivalent to two mutually balancing axial forces of magnitude

$$P = (2\pi \rho_o)p, \quad p = O(\varepsilon). \quad (104)$$

Two of the balance equations (102) are in force:

$$F_{<11>}' = 0, \quad M_{<11>}' - \rho_o^{-1}F_{<22>} = 0; \quad (105)$$

the accompanying boundary equations are:

$$F_{<11>}(-l) = p, \quad M_{<11>}(-l) = 0, \quad M_{<11>}'(-l) = 0 \quad (106)$$

the last one following from (103). Now, (105) and (106) imply that the membrane force $F_{<11>}$ is constant and equal to $p$ in the closed interval $[-l, +l]$; then, making use of (90), we obtain that the differential relation:

$$(1 - \nu_{23}\nu_{32}) \left( \rho_o a_{<1>}' - \frac{1}{3} \varepsilon^2 w'' \right) + (\nu_{21} + \nu_{23}\nu_{31})w +$$

$$+ (\nu_{31} + \nu_{21}\nu_{32})\rho_o \gamma = \rho_o \Delta \varepsilon^{-1} p \quad (107)$$

must hold in $[-l, +l]$. Moreover, with the use of (91) and under the parity assumptions we made for $a_{<1>}$ and $w$, we see that the boundary conditions (106) take the common form:

$$(1 - \nu_{23}\nu_{32}) \left( a_{<1>}'(l) - \rho_o w''(l) \right) + (\nu_{31} + \nu_{21}\nu_{32})\gamma = 0. \quad (108)$$

Finally, with the use of (91) again and of (90)_2, and under the provisional assumption that the traction problem admits a solution with $\gamma$ a constant,
\[ \frac{1}{3} \varepsilon^2 \left( 1 - \nu_{23} \nu_{32} \right) \left( \rho_o^2 w'' - \rho_o a_{<1>'} \right)'' + 
\]
\[ + \left( 1 - \nu_{13} \nu_{31} \right) \left( 1 - \frac{1}{2 \rho_o} \log \frac{1 + \frac{\rho_o}{\rho_o}}{1 - \frac{\rho_o}{\rho_o}} \right) \rho_o \gamma + \frac{1}{2 \rho_o} \log \frac{1 + \frac{\rho_o}{\rho_o}}{1 - \frac{\rho_o}{\rho_o}} w \] + (109)

On eliminating \( a_{<1>'} \) by means of (107), (109) yields:

\[ \varepsilon^2 \rho_o^2 \left( 1 - \frac{\varepsilon^2}{3 \rho_o^2} \right) w''' + \varepsilon^2 a w'' + b w + c \rho_o \Delta \varepsilon^{-1} p \frac{\varepsilon^{-1} p}{2 E_1} + d \rho_o \gamma = 0, \] (110)

where

\[ a := \frac{2 (\nu_{21} + \nu_{23} \nu_{32})}{1 - \nu_{23} \nu_{32}}, \]
\[ b := \frac{3}{1 - \nu_{23} \nu_{32}} \left( \eta (1 - \nu_{13} \nu_{31}) \frac{1}{2 \rho_o} \log \frac{1 + \frac{\rho_o}{\rho_o}}{1 - \frac{\rho_o}{\rho_o}} - \frac{(\nu_{21} + \nu_{23} \nu_{32})^2}{1 - \nu_{23} \nu_{32}} \right), \]
\[ c := \frac{3 (\nu_{21} + \nu_{23} \nu_{31})}{(1 - \nu_{23} \nu_{32})^2}, \] (111)
\[ d := \frac{3}{1 - \nu_{23} \nu_{32}} \left( \eta (1 - \nu_{13} \nu_{31}) \left( 1 - \frac{1}{2 \rho_o} \log \frac{1 + \frac{\rho_o}{\rho_o}}{1 - \frac{\rho_o}{\rho_o}} \right) - \right. \]
\[ - \frac{(\nu_{21} + \nu_{23} \nu_{32}) (\nu_{21} + \nu_{23} \nu_{31})}{1 - \nu_{23} \nu_{32}} \eta \nu_{31} + \nu_{31} \nu_{12} \right) \eta (\nu_{32} + \nu_{31} \nu_{12}). \]

The general solution of the homogeneous equation associated with (110) has the form:

\[ w_h(x_1) = c_1 \left( \exp(\alpha_1 x_1) + \exp(-\alpha_1 x_1) \right) + c_2 \left( \exp(\alpha_2 x_1) + \exp(-\alpha_2 x_1) \right), \] (112)

where

\[ \alpha_1^2 := \frac{-a + \sqrt{a^2 - 4 b \varepsilon^2 \left( 1 - \frac{\varepsilon^2}{3 \rho_o^2} \right)}}{2 \rho_o^2 \left( 1 - \frac{\varepsilon^2}{3 \rho_o^2} \right)}, \quad \alpha_2^2 := \frac{-a - \sqrt{a^2 - 4 b \varepsilon^2 \left( 1 - \frac{\varepsilon^2}{3 \rho_o^2} \right)}}{2 \rho_o^2 \left( 1 - \frac{\varepsilon^2}{3 \rho_o^2} \right)}. \] (113)

With this, we write:

\[ w(x_1) = w_h(x_1) + w_p, \quad w_p := -\rho_o \left( \frac{c}{b} \Delta \frac{\varepsilon^{-1} p}{2 E_1} + \frac{d}{b} \gamma \right), \] (114)

with \( w_p \) the constant solution of (110).
With a view to determining the coefficients $c_1$ and $c_2$, we firstly return to the boundary condition (106), that, when combined with (91), reads:

$$\rho_0 w''(l) - a_{<1>''}(l) = 0;$$

in addition, by differentiating (107) and invoking continuity up to the boundary of the resultant expression, we obtain that

$$\rho_0 \alpha_{<1>''}(l) = \frac{1}{3} \frac{\varepsilon^2}{\rho_o^2} w''(l) - \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}} w'(l);$$

the last two relations together imply that

$$\left(1 - \frac{1}{3} \frac{\varepsilon^2}{\rho_o^2}\right) \rho_0 w''(l) + \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}} \rho_0 w'(l) = 0. \quad (115)$$

Secondly, on eliminating $a_{<1>}'(l)$ in (108) by means of (107), we obtain:

$$\left(1 - \frac{1}{3} \frac{\varepsilon^2}{\rho_o^2}\right) \rho_0 w''(l) + \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}} \rho_0 w(l) - \frac{1}{(1 - \nu_{23} \nu_{32})} \frac{\Delta \varepsilon^{-1} P}{2E_1} = 0. \quad (116)$$

On taking (114) into account, the system of equations (115) and (116) determines the coefficients $c_\alpha$ in (112):

$$c_1 = \frac{\kappa_1}{\alpha_1 (\exp(2\alpha_1 l) - 1)(\exp(2\alpha_2 l) - 1) - \alpha_2 (\exp(2\alpha_1 l) + 1)(\exp(2\alpha_2 l) - 1),}$$

$$c_2 = \frac{\kappa_2}{\alpha_2 (\exp(2\alpha_2 l) - 1)(\exp(2\alpha_1 l) + 1) - \alpha_1 (\exp(2\alpha_2 l) + 1)(\exp(2\alpha_1 l) - 1)},$$

with

$$\kappa_1 := -\alpha_2 \left(\frac{\varepsilon}{\rho_o} \left(1 - \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}}\right) \frac{\Delta \varepsilon^{-1} P}{2E_1} + \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}} \frac{d}{b} \right),$$

$$\kappa_2 := -\alpha_1 \left(\frac{\varepsilon}{\rho_o} \left(1 - \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}}\right) \frac{\Delta \varepsilon^{-1} P}{2E_1} + \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}} \frac{d}{b} \right); \quad (118)$$

note that both the coefficients $c_\alpha$ depend on $\gamma$.

Having found the form of the radial displacement $w$ in $[-l, +l]$, we revert to equation (107) to find the axial displacement $a_{<1>}$.

A simply calculation yields:

$$a_{<1>}(x_1) = \left[\left(1 + \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}} \frac{c}{b}\right) \frac{\varepsilon^{-1} P}{2E_1} + \left(1 - \nu_{23} \nu_{32}\right)^{-1} \left(\frac{d}{b} (\nu_{21} + \nu_{23} \nu_{31}) - (\nu_{31} + \nu_{21} \nu_{32})\right) \gamma\right] x_1 +$$

$$+ \frac{1}{3\rho_o \alpha_1 \alpha_2} \left[\alpha_2 c_1 \left(\varepsilon^2 - \frac{3}{2} \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}}\right) (\exp(\alpha_1 x_1) - \exp(-\alpha_1 x_1)) + \right.$$\n
$$+ \alpha_1 c_2 \left(\varepsilon^2 - \frac{3}{2} \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}}\right) (\exp(\alpha_2 x_1) - \exp(-\alpha_2 x_1)) \right]$$

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(needless to say, the parity condition $a_{<1>}(0) = 0$ is satisfied). The one task remaining is to find the constant $\gamma$. This we do by a sequence of manipulations of the integral balance equation (93):

(i) using (90) and (91), we give equation (93) the form:

$$
\int_{-l}^{+l} \left( 1 - \nu_{13}\nu_{31} \right) \left( 1 - \frac{1}{2\rho_o} \log \frac{1 + \frac{\alpha}{\rho_o}}{1 - \frac{\rho}{\rho_o}} \right) (w + \rho_o\gamma) - \frac{1}{3} \varepsilon^2 \frac{\nu_{12} + \nu_{23}\nu_{31}}{\bar{\eta}} w'' + \\
+ \mu(1 - \nu_{12}\nu_{21}) \rho_o\gamma + \frac{\nu_{31} + \nu_{21}\nu_{32}}{\eta} \left( \rho_o a_{<1>}' - \frac{1}{3} \varepsilon^2 w'' \right) + (\nu_{32} - \nu_{31}\nu_{12})w = 0;
$$

(119)

(ii) on expunging the construct $(\rho_o a_{<1>}' - \frac{1}{3} \varepsilon^2 w'')$ by means of (107), we have:

$$
2\varepsilon^2 a_0 w'_h(l) + a_1 \int_{-l}^{+l} w_h = 2l \left( \rho_o \left( \frac{\varepsilon^{-1}p}{2E_1} + a_2 \gamma \right) - a_1 w_p \right),
$$

where

$$a_0 := \frac{1}{3} \left( \nu_{31} + \nu_{21}\nu_{32} \right) \left( \nu_{12} + \nu_{23}\nu_{31} \right),$$

$$a_1 := \eta \frac{1 - \nu_{23}\nu_{32}}{\nu_{31} + \nu_{21}\nu_{32}} \left( \nu_{21} + \nu_{32}\nu_{31} - \nu_{23} + \nu_{31}\nu_{12} - (1 - \nu_{13}\nu_{31}) \left( 1 - \frac{1}{2\rho_o} \log \frac{1 + \frac{\alpha}{\rho_o}}{1 - \frac{\rho}{\rho_o}} \right) \right),$$

$$a_2 := \eta \frac{1 - \nu_{23}\nu_{32}}{\nu_{31} + \nu_{21}\nu_{32}} \left( 1 - \nu_{13}\nu_{31} \right) \left( 1 - \frac{1}{2\rho_o} \log \frac{1 + \frac{\rho}{\rho_o}}{1 - \frac{\rho}{\rho_o}} \right) + \mu(1 - \nu_{12}\nu_{21}) + \right.$$

$$- (\nu_{31} + \nu_{21}\nu_{32});$$

(iii) in view of (114), we end up with:

$$
\left( a_1 \frac{d}{b} + a_2 \right) \gamma = - \left( 1 + c \frac{a_1}{b} \right) \Delta \frac{\varepsilon^{-1}p}{2E_1} + \\
+ \frac{2}{l\rho_o} \left( \alpha_1^{-1} c_1 (a_1 + \varepsilon^2 a_o a_1^2) \sinh(\alpha_1 l) + \alpha_2^{-1} c_2 (a_1 + \varepsilon^2 a_o a_2^2) \sinh(\alpha_2 l) \right),
$$

(120)

an implicit equation for $\gamma$. Figures [38] help visualizing some relevant features of the analytic solution we just constructed.
Figure 6: Axial Traction Problem: qualitative diagrams of bending moment $M_{<11>}$ and shear force $F_{<31>}$. 

Figure 7: Axial Traction Problem: radial displacement $w$. 

Figure 8: Axial Traction Problem: cartoon visualization of deformed and undeformed shapes.
4.3 Pressure

We now let the cylindrical shell we study be subject to a uniform pressure \( q_o \approx \varpi = O(\varepsilon) \), all the other applied loads being null. Accordingly, the field and boundary equations \((105)\) and \((106)\) are replaced by, respectively,

\[
F_{<11>'} = 0, \quad M_{<11>''} - \rho_o^{-1} F_{<22>} + \varpi = 0;
\]

and

\[
F_{<11>} (\pm l) = 0, \quad M_{<11>} (\pm l) = 0, \quad M_{<11>}' (\pm l) = 0.
\]

Equations \((121)_1\) and \((122)_1\) imply that \( F_{<11>} \) is identically null, whence, in view of \((90)_1\), that

\[
\rho_o a_{<1>'} - \frac{1}{3} \varepsilon^2 w'' = -\frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}} w - \frac{\nu_{31} + \nu_{21} \nu_{32}}{1 - \nu_{23} \nu_{32}} \rho_o \gamma \quad \text{over } [-l, +l].
\]

With this, \((91)_1\) and \((90)_2\), equation \((121)_2\) takes the form:

\[
\varepsilon^2 \rho_o \left( 1 - \frac{1}{3} \varepsilon^2 \right) w'''' + \varepsilon^2 a \ w'' + b \ w - \tilde{c} \rho_o \Delta \frac{\varepsilon^{-1} \varpi}{2E_1} + d \rho_o \gamma = 0,
\]

where

\[
\tilde{c} := \frac{3}{1 - \nu_{23} \nu_{32}},
\]

the other constants being defined by \((111)_{1,2,4}'\). The general solution of the homogeneous equation associated to \((124)\) has the form \((112)\),

\[
w_h (x_1) = \hat{c}_1 \left( \exp (\alpha_1 x_1) + \exp (-\alpha_1 x_1) \right) + \hat{c}_2 \left( \exp (\alpha_2 x_1) + \exp (-\alpha_2 x_1) \right),
\]

with the coefficients \(\alpha_\delta\) given by \((113)\); then,

\[
w (x_1) = w_h (x_1) + w_p, \quad w_p := \rho_o \left( \frac{\tilde{c}}{b} \rho_o \Delta \frac{\varepsilon^{-1} \varpi}{2E_1} - \frac{d}{b} \gamma \right),
\]

with \( w_p \) the constant solution of \((124)\). The boundary conditions \((122)_{2,3}\) become:

\[
\left( 1 - \frac{1}{3} \varepsilon^2 \right) \rho_o w'' (l) + \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}} \frac{1}{\rho_o} w (l) = 0,
\]

and are expedient to determine the constants \(\hat{c}_1\) and \(\hat{c}_2\):

\[
\hat{c}_1 = \tilde{\kappa}_1 \frac{\exp (2 \alpha_1 l) - 1}{\alpha_1 \left( \exp (2 \alpha_1 l) - 1 \right) \left( \exp (2 \alpha_2 l) + 1 \right) - \alpha_2 \left( \exp (2 \alpha_1 l) + 1 \right) \left( \exp (2 \alpha_2 l) - 1 \right)},
\]

\[
\hat{c}_2 = \tilde{\kappa}_2 \frac{\exp (2 \alpha_2 l) - 1}{\alpha_2 \left( \exp (2 \alpha_2 l) - 1 \right) \left( \exp (2 \alpha_1 l) + 1 \right) - \alpha_1 \left( \exp (2 \alpha_2 l) + 1 \right) \left( \exp (2 \alpha_1 l) - 1 \right)},
\]

with

\[
\tilde{\kappa}_1, \tilde{\kappa}_2.
\]
with
\[ \tilde{\kappa}_1 := \alpha_2 \left( \frac{\tilde{c}_b}{2} \left( 1 - \frac{\mu_1 \mu_2}{\mu_2 \mu_3} \right) \rho_o \frac{\Delta \varphi}{2E_1} - \frac{\mu_1 \mu_2}{\mu_2 \mu_3} \frac{d}{2} \right) \]  
\[ \tilde{\kappa}_2 := \alpha_1 \left( \frac{\tilde{c}_b}{2} \left( 1 - \frac{\mu_1 \mu_2}{\mu_2 \mu_3} \right) \rho_o \frac{\Delta \varphi}{2E_1} - \frac{\mu_1 \mu_2}{\mu_2 \mu_3} \frac{d}{2} \right) \]  

Finally, by a sequence of steps completely analogous to the one leading to (120), the integral balance (93) yields an implicit equation for the constant \( \gamma \):

\[ \left( a_1 \frac{d}{b} + a_2 \right) \gamma = \frac{\tilde{c}_b \rho_o \Delta \varphi}{2E_1} + \frac{2}{l \rho_o} \left( \alpha_1 c_1 (a_1 + \varepsilon^2 a_o \alpha_1^2) \sinh(\alpha_1 l) + \alpha_2 c_2 (a_1 + \varepsilon^2 a_o \alpha_2^2) \sinh(\alpha_2 l) \right) \]  

(131)

Figure 9: Pressure Problem: qualitative diagrams of bending moment \( M_{<11>} \) and shear force \( F_{<31>} \).

Figure 10: Pressure Problem: radial displacement \( w \).
4.4 Rim flexure

Lastly, we consider the case when a uniform distribution of bending couples $m = O(\varepsilon)$ per unit length is applied at both rims of the cylinder. This time, the field equations we have to satisfy are:

$$F_{<11>'} = 0,$$
$$M_{<11>''} - \rho_0^{-1} F_{<22>} = 0,$$

(132)

with the boundary conditions:

$$F_{<11>}(\pm l) = 0, \quad M_{<11>}(\pm l) = m, \quad M'_{<11>}(\pm l) = 0.$$  (133)

Just as in the pressure case, equations (132) and (133) imply that (123) holds; moreover, with the use of (111), (90), and (91), equation (132) takes the form:

$$\varepsilon^2 \rho_0^2 \left(1 - \frac{\varepsilon^2}{3 \rho_0^2}\right) w''' + \varepsilon^2 a w'' + b w + d \rho_o \gamma = 0$$

(134)

(cf. (124)), where the constants are defined in (111). We set:

$$w(x_1) = -\frac{d}{b} \rho_o \gamma + w_h(x_1),$$

with

$$w_h(x_1) = \hat{c}_1 \left( \exp(\alpha_1 x_1) + \exp(-\alpha_1 x_1) \right) + \hat{c}_2 \left( \exp(\alpha_2 x_1) + \exp(-\alpha_2 x_1) \right).$$

(135)

the constants $\alpha_1, \alpha_2$ being given by (113). On writing the boundary conditions (133) in terms of displacements:

$$\left(1 - \frac{\varepsilon^2}{3 \rho_0^2}\right) \rho_o w''(l) + \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}} \frac{1}{\rho_o} w(l) = -\frac{3}{2} \Delta \frac{m \rho_o^2}{\varepsilon^3 E_1},$$

(136)

$$\left(1 - \frac{\varepsilon^2}{3 \rho_0^2}\right) \rho_o w'''(l) + \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}} \frac{1}{\rho_o} w''(l) = 0,$$

(137)

we determine the constants $\hat{c}_1$ and $\hat{c}_2$:

$$\hat{c}_1 = \hat{k}_1^{-1} \left(3 \Delta \frac{m \rho_o^2}{\varepsilon^3 E_1} - \frac{d}{b} \gamma\right) \frac{\alpha_2 \left( \exp(2 \alpha_2 l) - 1 \right) \exp(\alpha_1 l)}{\alpha_1 \left( \exp(2 \alpha_2 l) + 1 \right) \exp(\alpha_1 l) - \alpha_2 \left( \exp(2 \alpha_2 l) + 1 \right) \left( \exp(2 \alpha_2 l) - 1 \right)}$$

$$\hat{c}_2 = \hat{k}_2^{-1} \left(3 \Delta \frac{m \rho_o^2}{\varepsilon^3 E_1} - \frac{d}{b} \gamma\right) \frac{\alpha_1 \left( \exp(2 \alpha_1 l) + 1 \right) \exp(\alpha_2 l)}{\alpha_2 \left( \exp(2 \alpha_1 l) + 1 \right) \exp(\alpha_2 l) - \alpha_1 \left( \exp(2 \alpha_2 l) - 1 \right) \left( \exp(2 \alpha_1 l) - 1 \right)}$$

where

$$\hat{k}_1 = \left(1 - \frac{\varepsilon^2}{3 \rho_0^2}\right) \rho_o \alpha_1^2 + \rho_o^{-1} \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}},$$

$$\hat{k}_2 = \left(1 - \frac{\varepsilon^2}{3 \rho_0^2}\right) \rho_o \alpha_2^2 + \rho_o^{-1} \frac{\nu_{21} + \nu_{23} \nu_{31}}{1 - \nu_{23} \nu_{32}}.$$
Once again, the integral condition balance \((93)\) yields an implicit equation for \(\gamma\):

\[
\left( a_1 \frac{d}{b} + a_2 \right) \gamma = \frac{2}{l \rho_o} \left( \alpha_1^{-1} c_1 (a_1 + \varepsilon a_o \alpha_1^2) \sinh(\alpha_1 l) + \alpha_2^{-1} c_2 (a_1 + \varepsilon a_o \alpha_2^2) \sinh(\alpha_2 l) \right).
\]

\[(138)\]
Figure 11: Rim Flexure Problem: qualitative diagrams of bending moment $M_{11}$ and shear force $F_{31}$.

Figure 12: Rim Flexure Problem: radial displacement $w$.

Figure 13: Rim Flexure Problem: cartoon visualization of deformed and undeformed shapes.
5 Cylindrical Shells: Thinness, Slenderness, Contraction Moduli, and Stiffnesses

As all figures from (6) to (13) make evident, a phenomenon of boundary localization takes place, whatever the shell’s thickness, in the boundary-value problems of traction, pressure and rim-flexure we have solved analytically. This phenomenon is more and more pronounced as the shell’s length grows ceteris paribus. In our opinion, this fact amply justifies the use of the much simpler formulas for slender shells that we derive in this section.

5.1 Thin shells and slender shells

We term thin a cylindrical shell of diameter $2(\rho_o + \varepsilon)$ and length $2l$ if $\varepsilon/\rho_o \ll 1$, slender if $\rho_o/l \ll 1$; most of times, $\rho_o$ is indeed smaller than $l$, so that a thin shell is slender as well (and $\varepsilon/l \ll 1$). For $\rho_o/l$ fixed, a thinner and thinner shell loses its bending and twisting stiffness more and more; in the limit for $\varepsilon/\rho_o \to 0$, it responds to loads as a tubular membrane. On the other hand, for both $\varepsilon$ and $\rho_o$ fixed, longer and longer shells become slender and slender, without losing their shell-like response.

To exemplify the simplifications ensuing from taking large-thinness limits, we take the integral balance (93), that has the same displacement form (119) in all three boundary-value problems of traction, pressure, and rim flexure. Since

$$\lim_{\varepsilon/\rho_o \to 0} \frac{1}{2 \rho_o} \log \left( \frac{1 + \frac{\varepsilon}{\rho_o}}{1 - \frac{\varepsilon}{\rho_o}} \right) = 1,$$

we see that (119) reduces to

$$\mu (1 - \nu_{12} \nu_{21}) \rho_o \gamma + \eta^{-1} (\nu_{31} + \nu_{21} \nu_{32}) \frac{\rho_o}{l} \int_{-l}^{+l} w = 0,$$

where of course the values of both $a_{1>(l)}$ and $\int_{-l}^{+l} w$ are problem-dependent.

Remark. CNTs, no matter if single- or multi-wall, are as a rule slender. However, they may be thin or not, in the sense of the above definition [5]. Of course, all shell theories concern thin objects, but the thinness notions they are constructed upon may differ (see the discussions in [12] and [13]); in particular, those notions need not be expressed in terms on one purely geometrical aspect ratio. Our shell theory works whatever that ratio, because its subtler and more complex notion of thinness is the one typical of the method of internal constraints, a method to derive the mathematical models of linear structure mechanics firstly sketched in [9].

5.2 Axisymmetric equilibria of slender shells

Hereafter, we display the slenderness approximations of the solutions to the fundamental problems analyzed in the previous section, except for the torsion
problem, where no such approximation is in order, because the solution does not depend on the cylinder’s length.

(i) **Traction.** Looking at (117), it is easy to conclude that
\[
\lim_{l \to \infty} c_1 = \lim_{l \to \infty} c_2 = 0;
\]
then, (112) takes the form:
\[
w_h(x_1) \equiv 0.
\]
Moreover, (120) yields that:
\[
\gamma = -\tilde{\gamma} \frac{\varepsilon^{-1} p}{2E_1}, \quad \tilde{\gamma} := \frac{1 + a_1 \tilde{c}}{a_2 + a_1 \frac{\tilde{c}}{\rho}} \Delta.
\]
With this, one finds that
\[
w(x_1) \equiv -\bar{w}_p \frac{\varepsilon^{-1} p}{2E_1}, \quad \bar{w}_p := \rho_o \left( \frac{c}{\tilde{c}} + \frac{1 + a_1 \tilde{c}}{a_2 + a_1 \frac{\tilde{c}}{\rho}} \right) \Delta.
\]
To arrive to a large-slenderness approximation for the function \(x_1 \mapsto a_{<1>}(x_1)\), the remaining unknown of our problem, we turn to (107). Under the present circumstances, that equation yields the value of the axial strain in a slender shell:
\[
\Lambda := a_{<1>}'(x_1) \equiv \frac{\Delta + \rho_o^{-1} \bar{w}_p (\nu_{21} + \nu_{23}\nu_{31}) + \tilde{\gamma} (\nu_{31} + \nu_{21}\nu_{32}) \varepsilon^{-1} p}{1 - \nu_{23}\nu_{32}} \frac{\varepsilon^{-1} p}{2E_1}.
\]

(ii) **Pressure.** This time, we look at (129) to conclude that, in limit for \(l \to \infty\), both constants \(\tilde{c}_o\) tend to zero. Thus, once again, \(w(x_1) \equiv w_p\), and we find that
\[
\gamma = \tilde{\gamma} \rho_o \varepsilon^{-1} \omega, \quad \tilde{\gamma} := \frac{a_1 \tilde{c}}{a_2 + a_1 \frac{\tilde{c}}{\rho}} \Delta,
\]
and
\[
w(x_1) \equiv \bar{w}_p \rho_o \varepsilon^{-1} \omega, \quad \bar{w}_p := \rho_o \frac{a_2 \tilde{c}}{a_2 + a_1 \frac{\tilde{c}}{\rho}} \Delta.
\]
Finally, a use of (123) yields:
\[
a_{<1>}'(x_1) \equiv -\rho_o^{-1} \bar{w}_p (\nu_{21} + \nu_{23}\nu_{31}) + \tilde{\gamma} (\nu_{31} + \nu_{21}\nu_{32}) \rho_o \varepsilon^{-1} \omega \frac{\varepsilon^{-1} p}{1 - \nu_{23}\nu_{32}} \frac{\varepsilon^{-1} p}{2E_1}.
\]

(iii) **Rim Flexure.** Given that \(\lim_{l \to \infty} \tilde{c}_1 = \lim_{l \to \infty} \tilde{c}_2 = 0\), it is easy to show that the displacement field is everywhere null, in the large-slenderness limit.
5.3 Effective contraction moduli

Let us now regard a slender cylindrical shell as a three-dimensional rod-like body, in short, a probe. On defining the cross-section strain measure:

$$E(x_1) := \frac{1}{2\varepsilon} \int_{-\varepsilon}^{+\varepsilon} \alpha(\zeta) E(x_1, \zeta),$$  \hspace{1cm} (146)

we have from (89) that

$$E_{<11>} = a_{<1>'}, \quad E_{<12>} = E_{<21>} = \frac{1}{2} a_{<2>'}, \quad E_{<22>} = \rho_o^{-1} w, \quad E_{<33>} = \gamma.$$  \hspace{1cm} (147)

Moreover, (88) implies that the deformed external radius of a shell of undeformed external radius $$r_{0\text{ext}} = \rho_0 + \varepsilon$$ is $$r_{\text{ext}} = \rho_0 + w + \varepsilon\gamma$$, so that

$$\Gamma := \frac{r_{\text{ext}} - r_{0\text{ext}}}{r_{0\text{ext}}} = E_{<22>} + O(\varepsilon).$$

Motivated by this observation, by an effective contraction modulus we mean:

- in case of axial traction,

$$\nu_{CA} := \frac{-E_{<22>}}{E_{<11>}} = \frac{\rho_o^{-1} \tilde{w}_p (1 - \nu_{23}\nu_{32})}{\Delta + \rho_o^{-1} \tilde{w}_p (\nu_{21} + \nu_{23}\nu_{31}) + \tilde{\gamma}(\nu_{31} + \nu_{21}\nu_{32})},$$ \hspace{1cm} (148)

where $$\tilde{w}_p$$ and $$\tilde{\gamma}$$ are given by, respectively, (141) and (140) (note that this definition is directly reminiscent of the laboratory procedure followed to measure Poisson’s modulus for isotropic materials);

- in case of uniform inner pressure,

$$\nu_{CP} := \frac{-E_{<11>}}{E_{<22>}} = \frac{\rho_o^{-1} \tilde{w}_p (\nu_{21} + \nu_{23}\nu_{31}) + \tilde{\gamma}(\nu_{31} + \nu_{21}\nu_{32})}{\rho_o^{-1} \tilde{w}_p (1 - \nu_{23}\nu_{32})},$$ \hspace{1cm} (149)

where $$\tilde{w}_p$$ and $$\tilde{\gamma}$$ are given by, respectively, (144) and (143).

Both $$\nu_{CA}$$ and $$\nu_{CP}$$ depend on thickness, through, respectively, $$\tilde{w}_p$$, $$\tilde{\gamma}$$ and $$\tilde{w}_p$$, $$\tilde{\gamma}$$.

5.4 Effective stiffnesses

It remains for us to introduce suitable notions of effective traction and torsion stiffnesses for a slender cylindrical shell regarded as a probe. This we do by mimicking the relative familiar formulas from one-dimensional rod theory.

As to effective traction stiffness, given that the traction stiffness of a rod is defined to be (axial load)/(axial strain), we set:

$$s_A := \frac{P}{\Lambda},$$ \hspace{1cm} (150)
whence
\[
s_A = \frac{(1 - \nu_{23} \nu_{32})}{\Delta + \rho_o^{-1} \bar{w}_p (\nu_{21} + \nu_{23} \nu_{31}) + \bar{\gamma} (\nu_{31} + \nu_{21} \nu_{32})} E_1 A(\varepsilon), \quad A(\varepsilon) := 4\pi \rho_o \varepsilon,
\]
(151)
where we have made use of (104), (142), and (147), and where \( A(\varepsilon) \) is the area of the shell’s cross-section (needless to say, \( s_A \) depends on \( \varepsilon \) also through \( \bar{w}_p \) and \( \bar{\gamma} \)).

As to effective torsion stiffnesses, we recall that, for a twisted rod, one takes it to be \( \text{torison moment}/\text{twist per unit length} \); accordingly, we set:

\[
s_T := \frac{T}{\Theta},
\]
(152)
so that, on recalling (99)-(101), we have that

\[
s_T = \frac{GJ(\varepsilon)}{\chi(\varepsilon)}, \quad J(\varepsilon) := 4\pi \rho_o^3 \varepsilon, \quad \chi(\varepsilon) := \left(1 + \frac{\varepsilon^2}{\rho_o^2}\right)^{-1},
\]
(153)
where \( J(\varepsilon) \) approximates to within \( O(\varepsilon^2) \) terms the polar inertia moment of the cross section, and where \( \chi(\varepsilon) = 1 + O(\varepsilon^2) \) is a sort of torsion factor.\(^{10}\)

### 6 Cylindrical Kirchhoff–Love Shells

Recall from Section 2.2 the Kirchhoff-Love representation (14) for the displacement field:

\[
u_{KL}(x, \zeta) = A(x, \zeta) a(x) + w(x) n(x) - \zeta \nabla w(x), \quad a(x) \cdot n(x) = 0,
\]
and compare it with the more general representation (11): in both cases the unshearability constraint is imposed implicitly, together with, but only in the first case, inextensibility of the fibers orthogonal to the middle surface. Thus, the thickness of these shells does not change whatever the applied loads, making the simpler Kirchhoff-Love theory suitable, in our opinion, for application to single-wall CNTs.

Formally, to recover the representation (14), one only has to take \( \gamma = 0 \) in (11). To adjourn the developments of Sections 3, 4, and 5, this measure has to be accompanied by a few other adjustments that we now categorize and detail.

(i) (Balance Assumptions) The variations entering the Principle of Virtual Powers must have the following form, to be compared with (64):

\[
\begin{align*}
^{(0)}v &= v_1 c_1 + v_2 n' + v_3 n, \\
^{(1)}v &= -v_{3;1} c_1 + \rho_o^{-1}(v_2 - v_{3;2}) n',
\end{align*}
\]
(154)
Consequently, the integral balance equations (68) disappears, and the same happens with the last power-conjugate pair in each of the boundary conditions (71) and (94).

\(^{10}\)Note that (153) holds whatever the slenderness of the shell under consideration.
(ii) (Constitutive Assumptions) The constraint space (41) reduces to
\[ \mathcal{D} = \text{span}(W_i, i = 1, 2, 3); \]
\[ S_{<33>} \] is then reactive and only four constitutive moduli survive. It is not difficult to show that (50) and (51) must be replaced, respectively, by
\[ \tilde{C} = \Delta^{-1}\left(E_1 W_1 \otimes W_1 + E_2 W_2 \otimes W_2 + 2\Delta G W_3 \otimes W_3 + E_1\nu_21(W_1 \otimes W_2 + W_2 \otimes W_1)\right), \] (155)
and
\[ \Delta := 1 - \nu_{12}\nu_{21}. \] (156)
This result is achieved by the first of various applications to follow of a procedure that consists in taking the limits for \( E_3, \lambda, \) and \( \mu, \) tending to infinity, and in setting \( \nu_{3\alpha} = \nu_{\alpha 3} = 0 \) (\( \alpha = 1, 2 \)). It is important to realize that, due to the first of (87) that we here repeat for the reader’s convenience:
\[ \frac{E_1}{E_2} = \frac{\nu_{12}}{\nu_{21}}, \] (157)
(155) and (156) integrate a 4-parameter constitutive representation.

(iii) (Torsion Problem) The solution given in Section 4.1 does not change, because it does not involve any of the constitutive moduli that take singular values in the case of Kirchhoff-Love shells.

(iv) (Traction Problem) The two governing equations are, mutatis mutandis, (107) and (110); specifically, the equation corresponding to (107) is:
\[ \rho_o a \epsilon_{11}^{<1>} - \frac{1}{3} \varepsilon^2 w'' + \nu_{21} w = \rho_o \Delta \frac{\epsilon^{-1} p}{2E_1}, \] (158)
while the one corresponding to (110) reads:
\[ \varepsilon^2 \rho_o^2 \left(1 - \frac{1}{3} \varepsilon^2 \right) w''' + \varepsilon^2 a w'' + b w + c \rho_o \Delta \frac{\epsilon^{-1} p}{2E_1} = 0, \] (159)
where the constants have the following expressions, that can be recovered from (111):
\[ a := 2\nu_{21}, \quad b := 3\eta \left(\frac{1}{2\rho_o} \log \frac{1 + \frac{\varepsilon}{\rho_o}}{1 - \frac{\varepsilon}{\rho_o}} - \nu_{12}\nu_{21}\right), \quad c := 3\nu_{21}. \] (160)
As to equation (114), it is replaced by
\[ w(x_1) = w_h(x_1) + w_{pA}, \quad w_{pA} = -\rho_o \nu_{12} \delta(\varepsilon) \frac{\epsilon^{-1} p}{2E_1}, \] (161)
where
\[ \delta(\varepsilon) := \frac{1 - \nu_{12}\nu_{21}}{\frac{1}{2} \rho_0 \log \frac{1 + \frac{\varepsilon}{\rho_0}}{1 - \frac{\varepsilon}{\rho_0}} - \nu_{12}\nu_{21}} \] (note for later use that \( \lim_{\varepsilon/\rho_0 \to 0} \delta(\varepsilon) = 1 \)). \hspace{1cm} (162)

For slender shells, \( w(x_1) = w_p \) and (158) allows to conclude that
\[ a<1>(x_1) = \left(1 - \nu_{12}\nu_{21} \left(1 - \delta(\varepsilon)\right)\right) \frac{\varepsilon^{-1} p}{2E_1} x_1. \] (163)

(v) (Pressure Problem) The two relevant equations, (123) and (124), become, respectively:
\[ \rho_0 a_{<1><'} - \frac{1}{3} \varepsilon^2 w''' = -\nu_{21} w, \] (164)
and
\[ \varepsilon^2 \rho_0^2 \left(1 - \frac{1}{3} \varepsilon^2 \right) w''' + \varepsilon^2 a w'' + b w - 3\rho_0^2 \Delta \frac{\varepsilon^{-1} w}{2E_1} = 0, \] (165)
where the constants \( a \) and \( b \) are the same as in equation (160). Equation (126) is replaced by
\[ w(x_1) = w_h(x_1) + w_p p, \quad w_p p = \rho_0^2 \delta(\varepsilon) \frac{\varepsilon^{-1} w}{2E_2}. \] (166)

By using (164), we conclude that, for slender shells,
\[ a<1> = -\nu_{21} \delta(\varepsilon) \frac{\rho_0 \varepsilon^{-1} w}{2E_2} x_1. \] (167)

(vi) (Effective Contraction Moduli. Effective Stiffnesses) As to contraction moduli, formulas (148) and (149) become:
\[ \nu_{CA} = \frac{\delta(\varepsilon)}{1 - \nu_{12}\nu_{21} \left(1 - \delta(\varepsilon)\right)} \nu_{12} \quad \text{and} \quad \nu_{CP} = \nu_{21}. \] (168)

As to traction stiffness, (151) reduces to
\[ s_A = \frac{1}{1 - \nu_{12}\nu_{21} \left(1 - \delta(\varepsilon)\right)} E_1 A(\varepsilon); \] (169)
torsion stiffness continues to be given by formula (153).

(vii) (Determination of Effective Constitutive Moduli) In principle, a set of simple torsion, traction, and pressure, experiments or simulations allows to deduce the values of \( s_T, s_A, \nu_{CA}, \) and \( \nu_{CP} \), from the prescribed values of applied torque, axial load, and inner pressure and the measured or
computed values of \( \Theta, \Lambda, \) and \( \Gamma \). In the large-thinness limit, one would have:

\[
GJ(\epsilon) = s_T, \quad E_1 A(\epsilon) = s_A, \quad \nu_{12} = \nu_{CA}, \quad \nu_{21} = \nu_{CP},
\]

so that the values of the four constitutive moduli \( G, E_1, \nu_{12}, \) and \( \nu_{21} \) would follow, provided one could measure or evaluate the values of the geometrical parameters \( \rho_o \) and \( \epsilon \). Now, given the well-known difficulties, commonly referred to as the Yakobson’s Paradox \(^{16}\), in choosing a representative value for the wall thickness of a SWCNT, we think it best to characterize the mechanical response of a Kirchhoff-Love shell by the contraction moduli \( \nu_{12}, \nu_{21} \) and the effective constitutive moduli \( \tilde{G} := G \epsilon, \tilde{E}_\alpha := E_\alpha \epsilon, (\alpha = 1, 2) \).

**Remark.** We find it appropriate to expand a little on this last point. For slender and thin cylindrical shells, we find that:

- when subject to end traction,

\[
w_T = -\rho_o \nu_{12} \frac{p}{2E_1}, \quad a<1>_T = \frac{p}{2E_1}, \tag{170}
\]

by way of equations \((161)\) and \((163)\);

- when subject to inner pressure,

\[
w_P = \rho_o^2 \frac{\omega}{2E_2}, \quad a<1>_P = -\nu_{21} \rho_o \frac{\omega}{2E_2}, \tag{171}
\]

by way of equations \((166)\) and \((167)\).

Equations \((170)\) and \((171)\) can be regarded as a system of four equations in the unknowns \( \tilde{E}_1, \tilde{E}_2, \nu_{12}, \nu_{21} \), whose solution is:

\[
\tilde{E}_1 = \frac{p}{2a<1>_T}, \quad \tilde{E}_2 = \rho_o^2 \frac{\omega}{2w_P}, \quad \nu_{12} = -\frac{w_T}{\rho_o a<1>_T}, \quad \nu_{21} = -\rho_o \frac{a<1>_P}{w_P}. \tag{172}
\]

Interestingly, the consistency condition \((87)_1\), which can now be written as

\[
\tilde{E}_1 \frac{\nu_{12}}{\nu_{21}},
\]

implies that

\[
\frac{p}{w_T} = \frac{\omega}{a<1>_P},
\]

a relation that can be used to check whether the present simplified form of our shell theory is applicable.
7 Conclusions and further developments

We have given a detailed presentation of a theory of linearly elastic orthotropic shells with potential application to the continuous modeling of carbon nanotubes. The novelty of this theory resides in two features: (1) the type of orthotropic response we have selected seems suitable, with minimal tuning, to capture chirality, not only in the extreme cases of zig-zag and armchair SWCNTs, but also when it varies in an essentially undetectable manner from wall to wall of a MWCNT; (2) the possibility of accounting for overall thickness changes, that should be almost exclusively due to changes in inter-wall separation. As a matter of fact, the referential thickness of an ideal MWCNT is dictated by inter-wall forces of van der Waals type, whose action may be modeled essentially in two ways: either they can be regarded small with respect to applied loads, and ignored altogether; or they can be thought of as inducing a referential equilibrated stress state that should be taken into account when studying the effects of boundary conditions, no matter if hard or soft: our sounding of this latter approach is encouraging.

In addition, we have proposed a simpler version of the theory, in which orthotropy is preserved but thickness changes are excluded in all admissible deformational vicissitudes; we believe this simpler theory to fit SWCNTs, whose effective thickness when regarded as cylindrical shells should not change appreciably when loaded no matter how evaluated.

Presuming a rather complex material response requires specification of a number of constitutive parameters – seven when thickness changes are allowed, four when they are not. Luckily, another feature of our present theory is that, in both its versions, it leads to a number of significant boundary-value problems that can be solved explicitly in closed form. These problems are: torsion, axial traction, uniform inner pressure, and rim flexure; were their solutions coupled with the corresponding measurements and/or simulation results, applicability of our theory could be unequivocally assessed and all constitutive parameters in it uniquely determined. It is not difficult to paste explicit solutions of the type we here derived for two or more coaxial shells, both in statical and dynamical situations; we are currently developing this line of research with a view to a better understanding of van der Waals forces and the role and evolution of defects.

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References
[1] C. Bajaj, A. Favata, P. Podio–Guidugli, On a Scale-Bridging Mechanical Model of Carbon Nanotubes, Forthcoming.
[2] A. Di Carlo, A. Favata, P. Podio-Guidugli, Modeling Multi-Wall Carbon Nanotubes as Elastic Multi-Shells, 2011 (Forthcoming).

[3] M.P. Do Carmo, Differential Geometry of Curves and Surfaces. Prentice Hall, New Jersey (1976).

[4] A. Favata, P. Podio-Guidugli, What shell theory fits carbon nanotubes? To appear in Proceedings of Euromech 527, Lutherstadt Wittenberg, Aug 22-26 2011.

[5] R.V. Goldstein, V.A. Gorodtsov, A.V. Chentsov, S.V. Starikov, V.V. Stegailov, G.E. Norman, To description of mechanical properties of nanotubes. Tube wall thickness problem. Size effect, Russian Academy of Sciences, A.Yu. Ishlinsky Institute for Problems in Mechanics, Preprint 937 (2010).

[6] M.E. Gurtin, The Linear Theory of Elasticity. Pp. 1-295 of *Handbuch der Physik VIa/2*, Springer (1972).

[7] M. Lembo, P. Podio-Guidugli, Internal constraints, reactive stresses, and the Timoshenko beam theory. J. Elasticity 65 (2001) 131-148.

[8] M. Lembo, P. Podio-Guidugli, How to use reactive stresses to improve plate-theory approximations of the stress field in a linearly elastic plate-like body. Int. J. Sol. Structures 44 (2007) 1337-1369.

[9] P. Podio-Guidugli, An exact derivation of the thin plate equation. J. Elasticity 22 (1989) 121-133.

[10] P. Podio–Guidugli, *Lezioni sulla teoria lineare dei gusci elastici sottili*, Masson, Milano (1991).

[11] P. Podio–Guidugli, A Primer in Elasticity, Kluwer (2000).

[12] P. Podio–Guidugli, On structure thinness, mechanical and variational, pp. 227-242 of *Variational Formulations in Mechanics: Theory and Applications*, E. Taroco, E.A. de Souza Neto, and A.A. Novotny (Eds.), CIMNE, 2007

[13] P. Podio–Guidugli, Concepts in the mechanics of thin structures. Pp. 77-110 of CISM Vol. 503, A. Morassi and R. Paroni (Eds.), Springer (2008).

[14] P. Podio–Guidugli, M. Vianello, The representation problem of constrained linear elasticity. J. Elasticity 28 (1992) 271-276.

[15] P. Podio–Guidugli, M. Vianello, Hypertractions and hyperstresses convey the same mechanical information, Cont. Mech. Thermodyn. 22 (2010) 163-176.

[16] O.A. Shenderova, V.V. Zhirnov, D.W. Brenner, Carbon Nanostructures. Crit. Rev. Solid State Mater. Sci. 27, 227356 (2002).