Chirped periodic and localized waves in a weakly nonlocal media with cubic-quintic nonlinearity

Houria Triki\(^1\) and Vladimir I. Kruglov\(^2\)

\(^1\)Radiation Physics Laboratory, Department of Physics, Faculty of Sciences, Badji Mokhtar University, P. O. Box 12, 23000 Annaba, Algeria

\(^2\)Centre for Engineering Quantum Systems, School of Mathematics and Physics, The University of Queensland, Brisbane, Queensland 4072, Australia

We study the propagation of one-dimension optical beams in a weakly nonlocal medium exhibiting cubic-quintic nonlinearity. A nonlinear equation governing the evolution of the beam intensity in the nonlocal medium is derived thereby which allows us to examine whether the traveling-waves exist in such optical material. An efficient transformation is applied to obtain explicit solutions of the envelope model equation in the presence of all material parameters. We find that a variety of periodic waves accompanied with a nonlinear chirp do exist in the system in the presence of the weak nonlocality. Chirped localized intensity dips on a continuous-wave background as well as solitary waves of the bright and dark types are obtained in a long wave limit. A class of propagating chirped self-similar solitary beams is also identified in the material with the consideration of the inhomogeneities of media. The applications of the obtained self-similar structures are discussed by considering a periodic distributed amplification system.

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I. INTRODUCTION

Propagations of spatial optical solitons through nonlocal nonlinear media have drawn considerable attention because of their experimental observations in liquid crystals \cite{1} and lead glasses \cite{2} in addition to several important theoretical predictions \cite{3-5}. Nonlocal nonlinearity represents the fact that the refractive index change of a material at a particular location, is determined by the light intensity in a certain vicinity of this location \cite{3}. Compared with the local nonlinear medium for which the response at a given point is only dependent on the light intensity at that point \cite{6}, the response of the nonlocal nonlinear medium at a given point depends not only on the optical intensity at that point, but also on the intensity in its vicinity \cite{6-8}. It is noteworthy that the nonlocality plays a vital role for the very narrow beam propagation in the system and thus the nonlocal contribution to the refractive index change has to be taken into account in this case \cite{6}. It is worth mentioning that nonlinear media that feature the nonlocal nonlinearity also include nonlinear ion gas \cite{9}, thermal nonlinear liquid \cite{10}, quadratic nonlinear media \cite{11}, dipolar Bose-Einstein condensate \cite{12}, and photorefractive crystal \cite{13, 14}.

It has been demonstrated that the propagation dynamics of beams and their localization is significantly influenced by nonlocality \cite{15}. In this respect, important results have revealed that the nonlocal nonlinearity can suppress the modulational instability of plane waves \cite{3, 10} and support novel soliton states such as ring vortex solitons \cite{17, 18}, gap solitons \cite{19}, dipole solitons \cite{20}, spiraling solitons \cite{21, 22}, soliton clusters \cite{23}, and incoherent solitons \cite{24}. Although soliton structures have been extensively studied in nonlocal Kerr-type media \cite{6, 25-29}, their investigation in nonlocal non-Kerr systems has not been widespread. Some significant results have, however, been obtained, with previous theoretical studies considering nonlocality of nonlinear response and saturation \cite{30}. Specifically, Tsoy studied the soliton solutions in an implicit form which propagate through a weakly nonlocal medium with cubic-quintic nonlinearity, and derived explicit solutions in bright and dark solitons in particular case \cite{30}.

To the best of our knowledge, investigations discussing the formation and properties of periodic waves with nonlinear chirp in weakly nonlocal media exhibiting cubic-quintic nonlinearity are not available to date. Moreover, the control of chirped self-similar beams in a nonlocal nonlinear system with distributed diffraction, cubic-quintic nonlinearity, weak nonlocality, and gain or loss has been absent. The objective of the present work is to study the existence and propagation properties of periodic and localized waves with a nonlinear chirp in a weakly nonlocal cubic-quintic medium. Such chirping property is of practical interest in achieving effective beam compression or amplification. Additionally, the problem of self-similar light beam propagation through a weakly nonlocal medium in the presence of distributed cubic-quintic nonlinearity, diffraction, and gain (or loss) are investigated too.

The paper is organized as follows. In Sec. II, we present the cubic-quintic nonlinear Schrödinger equation (NLSE) with weak nonlocality describing optical beam propagation in a nonlocal medium with a saturation of the nonlinear response. We also present here the nonlinear equation that governs the evolution of the light beam intensity in the system and the general traveling-wave solutions of the mentioned equation. In Sec. III, we present results of novel chirped periodic wave solutions of the model equation and the nonlinear chirp accompanying these nonlinear wave-
forms. Considering the long-wave limit of the analytically determined periodic solutions, we find chirped solitary beam solutions which include gray, bright and dark solitons in Sec. IV. In Sec. V, the similarity transformation method is employed to construct exact self-similar periodic and localized wave solutions of the generalized NLS model with varied coefficients governing the beam evolution in presence of the inhomogeneities of nonlocal media. We also determine here the self-similar variables and constraints satisfied by the distributed coefficients in the inhomogeneous NLS model. We further investigate the propagation dynamics of the obtained self-similar localized waves (or “similaritons”) in a specified soliton control system. Finally, we give some conclusions in Sec. VI.

II. MODEL OF NLSE AND CHIRPED TRAVELING WAVES

The paraxial wave equation governing one-dimensional beam propagation in a nonlinear medium is given by [6]:

\[
\frac{\partial \psi}{\partial z} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + \Delta n(I) \psi = 0,
\]

(1)

where \( \psi(x, z) \) is the envelope of the electromagnetic field, \( x \) is transverse variable, and \( z \) is the longitudinal variable representing propagation distance. Here \( \Delta n \) and \( I(x, z) = |\psi(x, z)|^2 \) are the refractive index change and light intensity, respectively. In the case of nonlocal nonlinear media, \( \Delta n(I) \) can be written in general form as [3, 6, 30]

\[
\Delta n(I) = \int_{-\infty}^{+\infty} R(x - x') F(I(x', z)) \, dx',
\]

(2)

where \( R(x) \) is the response function of the nonlocal medium, which is a real symmetric function while \( F \) is the intensity-dependent function.

For weakly nonlocal media with cubic-quintic nonlinearity, one finds [30] \( \Delta n(I) = \gamma I + \sigma I^2 + \mu \partial^2 I \) with \( \mu \) being the nonlocality parameter which is defined through \( \mu = \frac{1}{2} \int_{-\infty}^{+\infty} x^2 R(x) \, dx \). The accordingly equation governing the beam propagation in such nonlinear media is given by the NLS equation with weak nonlocality presented in [30]. For our studies, this NLS equation model is expressed as

\[
\frac{i}{2} \frac{\partial \psi}{\partial z} + \frac{\beta}{2} \frac{\partial^2 \psi}{\partial x^2} + \gamma |\psi|^2 \psi + \sigma |\psi|^4 \psi + \mu \frac{\partial^2 |\psi|^2}{\partial x^2} = 0,
\]

(3)

where \( \gamma, \sigma \) and \( \mu \) are real parameters related to the cubic nonlinearity, quintic nonlinearity, and weak nonlocality, respectively, while the coefficient \( \beta \) accounts for the diffraction in the transverse plane.

In the absence of quintic nonlinearity (i.e., \( \sigma = 0 \)), Eq. (3) becomes the modified NLS equation which applies to the description of optical beam propagation in nonlocal nonlinear Kerr-type media [3]. Moreover, in the limit of vanishing weak nonlocality (i.e., \( \mu = 0 \)), Eq. (3) is reduced to the cubic-quintic NLS equation which governs the evolution of the optical beam in a local medium exhibiting third- and fifth-order nonlinearities. As previously mentioned, the soliton solutions in an implicit form of the model [3] with \( \beta = 1 \), which are expressed in terms of the elliptic integrals have been presented in Ref. [30]. But here we are concerned with explicit periodic wave and soliton solutions which are characterized by a nonlinear chirp. Our results introduce for the first time an efficient transformation which allows the derivation of periodic and localized solutions of Eq. (3) in an explicit form.

To obtain exact traveling wave solutions to the cubic-quintic NLS equation with weak nonlocality [3], we assume a solution given by the expression

\[
\psi(x, z) = u(\xi) \exp[i(\kappa z - \delta x) + i\phi(\xi)],
\]

(4)

where both \( u(\xi) \) and \( \phi(\xi) \) are real functions of the traveling coordinate \( \xi = x - vz \), with \( v \) being the transverse velocity of the wave. Also \( \kappa \) and \( \delta \) represent the propagation constant and frequency shift, respectively. Substitution of Eq. (4) into Eq. (3) and separation of the real and imaginary parts of the equation yields the following two coupled ordinary differential equations,

\[
\frac{\beta}{2} \left( u \frac{d^2 \phi}{d\xi^2} + 2 \frac{d\phi}{d\xi} \frac{du}{d\xi} \right) - (v + \beta \delta) \frac{du}{d\xi} = 0,
\]

(5)

\[
\frac{\beta}{2} \frac{d^2 u}{d\xi^2} - \left( \kappa - v \frac{d\phi}{d\xi} \right) u - \frac{\beta}{2} \left( \frac{d\phi}{d\xi} - \delta \right)^2 u + \gamma u^3 + \sigma u^5 + 2\mu \left[ u \left( \frac{du}{d\xi} \right)^2 + u^2 \frac{d^2 u}{d\xi^2} \right] = 0.
\]

(6)
The multiplication of Eq. (5) by the function $u(\xi)$ and integration of the resulting equation leads to the following equation,

$$\beta u^2 \frac{d\phi}{d\xi} - (v + \beta \delta)u^2 = J,$$

(7)

where $J$ is the integration constant. Then Eq. (7) yields the following expression,

$$\frac{d\phi}{d\xi} = \delta + \frac{v}{\beta} + \frac{J}{\beta u^2(\xi)}.$$

(8)

The accompanying chirp $\Delta \omega$ defined as $\Delta \omega = -\partial [\kappa z - \delta x + \phi(x)] / \partial x$ is given by

$$\Delta \omega = -\frac{v}{\beta} - \frac{J}{\beta u^2(\xi)}.$$

(9)

Further insertion of the result (8) into (6) gives to the following nonlinear ordinary differential equation,

$$\frac{d^2 u}{d\xi^2} + a \left[ u \left( \frac{du}{d\xi} \right)^2 + u^2 \frac{d^2 u}{d\xi^2} \right] + bu + cu^3 + du^5 - \frac{J^2}{\beta^2 u^2} = 0,$$

(10)

where the parameters $a$, $b$, $c$ and $d$ are defined by

$$a = \frac{4\mu}{\beta}, \quad b = \frac{v^2 + 2\beta (\delta v - \kappa)}{\beta^2}, \quad c = \frac{2\gamma}{\beta}, \quad d = \frac{2\sigma}{\beta}.$$

(11)

Multiplying Eq. (10) by $du/d\xi$ and integrating the resultant equation, we obtain

$$(1 + au^2) \left( \frac{du}{d\xi} \right)^2 + bu^2 + \frac{1}{2} cu^4 + \frac{1}{3} du^6 + \frac{J^2}{\beta^2 u^2} + C = 0,$$

(12)

where $C$ is another integration constant. We define new function $f(\xi) = u^2(\xi)$ which transforms Eq. (12) to the following ordinary differential equation,

$$(1 + af) \left( \frac{df}{d\xi} \right)^2 = \nu_0 + \nu_1 f + \nu_2 f^2 + \nu_3 f^3 + \nu_4 f^4,$$

(13)

where

$$\nu_0 = -\frac{4J^2}{\beta^2}, \quad \nu_1 = -4C, \quad \nu_2 = -4b, \quad \nu_3 = -2c, \quad \nu_4 = -\frac{4d}{3}.$$

(14)

Equation (13) presents one of the main results of our analysis, describing the evolution of beam intensity in a weakly nonlocal medium with cubic-quintic nonlinearity. This equation allow us to know whether the traveling-wave solutions exist in the nonlocal medium and in what parametric conditions they are formed. In general, this nonlinear differential equation with coexisting $f(\xi)$ and $f^4$ terms is difficult to handle analytically. However, by introducing a special transformation in this paper, Eq. (13) is solved analytically to obtain a rich variety of nonlinear waveforms for the model (3). Such a transformation, to our knowledge not used before testifies about the novelty of the solutions obtained.

Incorporating these results back into Eq. (11), we find that general form of traveling wave solutions to the cubic-quintic NLS equation with weak nonlocality (9) is

$$\psi(x, z) = \pm \sqrt{f(\xi)} \exp[i(\kappa z - \delta x + \phi(\xi))],$$

(15)

where $f(\xi)$ satisfies Eq. (13) while $\phi(\xi)$ can be evaluated explicitly using Eq. (8) as

$$\phi(\xi) = \left( \frac{\delta + \nu}{\beta} \right) (\xi - \eta) + \frac{J}{\beta} \int_{\eta}^{\xi} \frac{1}{u^{2}(\xi)} d\xi + \phi_0,$$

(16)

with $\phi_0$ being the initial phase and $\eta$ is an arbitrary constant.

This result shows that the phase modification $\phi(\xi)$ involves a nonlinear contribution that is inversely proportional to light beam intensity $|\psi(x, z)|^2 = |u(\xi)|^2$. Interestingly, the nontrivial nature of the phase leads to the formation of chirped beams in the system. In particular, when $J = 0$, the phase $\phi(\xi)$ in (13) can be reduced to a simple linear form as $\phi(\xi) = \left( \frac{\delta + \nu\beta^{-1}}{\beta} \right) (\xi - \eta) + \phi_0$. In what follows, we are interested in periodic and solitary pulse solutions to Eq. (13) in the most general case when $J \neq 0$, which describe nonlinearly chirped structures to the model (3).
III. PERIODIC WAVE SOLUTIONS

As previously noted, the general solutions to Eq. (3) which are implicit and are expressed in terms of the elliptic integrals have been found in [30]. In this section, we introduce an efficient transformation that enables one to obtain explicit solutions of the full underlying cubic-quintic NLS equation with weak nonlocality (3). Interestingly, exact chirped periodic solutions are found in the presence of all material parameters for the first time.

Applying the transformation (A1) to Eq. (13), one obtains a modified nonlinear differential equation of the form [see Appendix A]:

\[
\left( \frac{df}{d\xi} \right)^2 = \alpha_0 + \alpha_1 f + \alpha_2 f^2 + \alpha_3 f^3, \tag{17}
\]

with the new coefficients \(\alpha_i\) \((i = 1, 2, 3)\) that are found in the Appendix A as

\[
\alpha_0 = -\frac{4J^2}{\beta^2}, \quad \alpha_1 = -\frac{4b}{a} + \frac{2c}{a^2} - \frac{4d}{3a^3}, \tag{18}
\]

\[
\alpha_2 = -\frac{2c}{a} + \frac{4d}{3a^2}, \quad \alpha_3 = -\frac{4d}{3a}. \tag{19}
\]

Thus the coefficient \(\alpha_0\) in Eq. (17) is a free parameter because \(J\) is the integration constant. Note that in Eq. (13) there are two free coefficients as \(\nu_0\) and \(\nu_1\) because \(J\) and \(C\) are two independent integration constants. However, Eq. (17) has only one free parameter \(\alpha_0\). Thus one can use different parameter \(\alpha_0\) for different solutions of Eq. (17).

We now introduce a new function \(y(\xi)\) as

\[
y(\xi) = -\alpha_3 f(\xi), \quad f(\xi) = u^2(\xi). \tag{20}
\]

Thus the equation for function \(y(\xi)\) is

\[
\left( \frac{dy}{d\xi} \right)^2 = c_0 + c_1 y + c_2 y^2 - y^3, \tag{21}
\]

where the coefficients \(c_n\) are given by

\[
c_0 = \alpha_0 \alpha_3^2, \quad c_1 = -\alpha_1 \alpha_3, \quad c_2 = \alpha_2. \tag{22}
\]

We also introduce the polynomial \(P(y) = c_0 + c_1 y + c_2 y^2 - y^3\) which is given by the right side of Eq. (21). The roots of polynomial \(P(y)\) are given by equation,

\[
y^3 - \alpha_2 y^2 + \alpha_1 \alpha_3 y - c_0 = 0, \tag{23}
\]

where the coefficient \(c_0 = \alpha_0 \alpha_3^2\) is a free parameter because \(\alpha_0 = -4J^2/\beta^2\) and \(J\) is integration constant.

The periodic bounded solution of Eq. (21) defined in the interval \(y_2 \leq y(\xi) \leq y_3\) is

\[
y(\xi) = y_2 + (y_3 - y_2) \text{cn}^2(w(\xi - \eta), k), \tag{24}
\]

where the roots are real and ordered \((y_1 < y_2 < y_3)\), and \(\text{cn}(z, k)\) is elliptic Jacobi function. The parameters \(w\) and \(k\) in this solution are

\[
w = \frac{1}{2} \sqrt{y_3 - y_1}, \quad k = \sqrt{\frac{y_3 - y_2}{y_3 - y_1}}, \tag{25}
\]

where \(0 < k < 1\). It is shown in the Appendix B that ordered real roots are

\[
y_1 = \frac{\alpha_2}{3} + \frac{4(k^2 - 2)w^2}{3}, \quad y_2 = \frac{\alpha_2}{3} + \frac{4(1 - 2k^2)w^2}{3}, \tag{26}
\]

\[
y_3 = \frac{\alpha_2}{3} + \frac{4(1 + k^2)w^2}{3}. \tag{27}
\]
The parameter \( w \) in these equations is given by
\[
\alpha = \frac{1}{2} \left( \frac{\alpha_2}{k^4 - k^2 + 1} \right)^{1/4},
\] (28)
where it is assumed that \( \alpha_2 - 3\alpha_1\alpha_3 > 0 \). It is shown in the Appendix B that the integration constant \( J \) is fixed by relation,
\[
J^2 = -\frac{\beta^2 y_1 y_2 y_3}{4\alpha_3}.
\] (29)
Hence, in general case the roots \( y_n \) must satisfy the condition \( y_1 y_2 y_3 \leq 0 \).

1. Family of chirped periodic bounded waves with \( J \neq 0 \).

The amplitude in Eq. (28) is \( u(\xi) = \pm \sqrt{-y(\xi)/\alpha_3} \) which lead to a family of periodic bounded solutions of Eq. (5) (with \( 0 < k < 1 \)) as
\[
\psi(x, z) = \pm [A + Bcn^2(w(\xi - \eta), k)]^{1/2} \exp[i(\kappa z - \delta x) + i\phi(\xi)],
\] (30)
where the parameters \( A = -y_2/\alpha_3 \) and \( B = -(y_3 - y_2/\alpha_3 \) are
\[
A = \frac{1}{3\alpha_3}[-\alpha_2 + 4(2k^2 - 1)w^2], \quad B = -\frac{4k^2w^2}{\alpha_3}.
\] (31)
The parameter \( w \) in this solution is given in Eq. (28). It follows from this solution that the parameters \( A \) and \( B \) should satisfy the conditions \( A > 0 \) and \( A + B \geq 0 \). Moreover, the conditions \( \alpha_2 - 3\alpha_1\alpha_3 > 0 \) and \( y_1 y_2 y_3 < 0 \) are also necessary for this family of periodic solutions with integration constant \( J \neq 0 \).

We note that parameter \( w \) is given by Eq. (28), however one can consider \( w \) as independent variable parameter in Eqs. (30) and (31) because the parameter \( \alpha_1 \) depends on \( b \) which is variable parameter (see Eqs. (11) and (18)). In this case using Eq. (28) and relation \( \alpha_1 = -4b/a - \alpha_2/a \) we obtain the equation for the wave number \( \kappa \) as
\[
\kappa = v\delta + \frac{v^2}{2\beta} + \frac{\beta\alpha_2}{8} + \frac{\beta\alpha_3^2(1 - k^2)}{8\alpha_3(2 - k^2)^2} - \frac{2\alpha_3 w^4}{3\alpha_3(k^4 - k^2 + 1)}.
\] (32)
This equation means that the variable parameters \( w, v, \delta \) and \( \kappa \) are connected by this relation for fixed parameter \( k \) of Jacobi elliptic function \( cn(\zeta, k) \). We present below three particular cases of periodic solutions with the integration constant \( J = 0 \).

2. Bounded periodic dn-waves for the condition \( y_1 = 0 \).

The solution in Eq. (30) for \( y_1 = 0 \) (\( J = 0 \)) and \( 0 < k < 1 \) reduces to the periodic waves,
\[
\psi(x, z) = \pm \Lambda dn(w(\xi - \eta), k) \exp(i\theta(x, z)),
\] (33)
where the parameters \( \Lambda \) and \( w \) are
\[
\Lambda = \sqrt{-\frac{\alpha_2}{\alpha_3(2 - k^2)}}, \quad w = \frac{1}{2} \sqrt{\frac{\alpha_2}{2 - k^2}}.
\] (34)
The necessary conditions for this solution are \( \alpha_2 > 0 \) and \( \alpha_3 < 0 \). The phase \( \theta(x, z) \) in this periodic solution is
\[
\theta(x, z) = \left( \kappa - v\delta - \frac{v^2}{\beta} \right) z + \frac{v}{\beta} x + \theta_0,
\] (35)
where \( \theta_0 = \phi_0 - \eta(\delta + v/\beta) \). In this case (\( y_1 = 0 \)) the parameter \( w \) is fixed by Eq. (34) and hence Eq. (32) has the form,
\[
\kappa = v\delta + \frac{v^2}{2\beta} + \frac{\beta\alpha_2}{8} + \frac{\beta\alpha_3^2(1 - k^2)}{8\alpha_3(2 - k^2)^2}.
\] (36)

3. Bounded periodic cn-waves for the condition \( y_2 = 0 \).

The solution in Eq. (30) for \( y_2 = 0 \) (\( J = 0 \)) and \( 0 < k < 1 \) reduces to the periodic waves,
\[
\psi(x, z) = \pm \Lambda cn(w(\xi - \eta), k) \exp(i\theta(x, z)),
\] (37)
where the parameters $\Lambda$ and $w$ are

$$\Lambda = \sqrt{\frac{-\alpha_2 k^2}{\alpha_3(2k^2 - 1)}}, \quad w = \frac{1}{2} \sqrt{\frac{-\alpha_2}{2k^2 - 1}}.$$ (38)

The necessary conditions for this solution are $\alpha_2 > 0$ and $\alpha_3 < 0$ for $1/\sqrt{2} < k < 1$; and $\alpha_2 < 0$ and $\alpha_3 < 0$ for $0 < k < 1/\sqrt{2}$. The phase $\theta(x, z)$ in this solution is given by Eq. (35). In this case ($y_2 = 0$) the parameter $w$ is fixed by Eq. (38) and hence Eq. (32) has the form,

$$\kappa = v\delta + \frac{v^2}{2\beta} + \frac{\beta\alpha_2}{8} - \frac{a\beta\alpha_2^2 k^2(1 - k^2)}{8\alpha_3(2k^2 - 1)^2}.$$ (39)

4. Bounded periodic sn-waves for the condition $y_3 = 0$.

The solution in Eq. (30) for $y_3 = 0$ ($J = 0$) and $0 < k < 1$ reduces to the periodic waves,

$$\psi(x, z) = \pm \Lambda \text{sn}(w(x - \eta), k) \exp(i\theta(x, z)),$$ (40)

where the parameters $\Lambda$ and $w$ are

$$\Lambda = \sqrt{\frac{-\alpha_2 k^2}{\alpha_3(1 + k^2)}}, \quad w = \frac{1}{2} \sqrt{\frac{-\alpha_2}{1 + k^2}}.$$ (41)

The necessary conditions for this solution are $\alpha_2 < 0$ and $\alpha_3 > 0$. The phase $\theta(x, z)$ in this solution is given by Eq. (35). In this case ($y_3 = 0$) the parameter $w$ is fixed by Eq. (41) and hence Eq. (32) has the form,

$$\kappa = v\delta + \frac{v^2}{2\beta} + \frac{\beta\alpha_2}{8} + \frac{a\beta\alpha_2^2 k^2}{8\alpha_3(1 + k^2)^2}.$$ (42)

We emphasize that in Eqs. (36), (39) and (42) there are three variable parameters: $\delta$, $v$ and $\kappa$ for fixed parameter $k$. This means that one can fix any two of these variable parameters and then the third variable parameter follows from Eqs. (36), (39) and (42) respectively. For an example, if the variable parameters $\delta$ and $\kappa$ are fixed then the above equations yield the parameter $v$.

IV. SOLITARY WAVE SOLUTIONS

In this section, we present various nonlinearly chirped solitary wave solutions of the cubic-quintic NLS equation with weak nonlocality (3). At first we consider the solution in Eq. (30) in the limiting case $k = 1$ for integration constant $J \neq 0$. In this case the periodic bounded solution in Eq. (30) reduces to solitary wave as

$$\psi(x, z) = \pm [A + B \text{sech}^2(w_0(x - \eta))]^{1/2} \exp[i(\kappa z - \delta x) + i\phi(\xi)],$$ (43)

where the inverse width is $w_0 = \frac{1}{2}(\alpha_2^2 - 3\alpha_1\alpha_3)^{1/4}$. It is assumed here that the condition $\alpha_2^2 - 3\alpha_1\alpha_3 > 0$ is satisfied. The parameters $A$ and $B$ are

$$A = \frac{1}{3\alpha_3}(4w_0^2 - \alpha_2), \quad B = -\frac{4w_0^2}{\alpha_3}.$$ (44)

Note that in the case with $k = 1$ the roots are

$$y_1 = \frac{\alpha_2}{3} - \frac{4w_0^2}{3}, \quad y_2 = \frac{\alpha_2}{3} - \frac{4w_0^2}{3}, \quad y_3 = \frac{\alpha_2}{3} + \frac{8w_0^2}{3},$$ (45)

where $y_1 = y_2$. The equation for variable parameters $w_0$, $v$, $\delta$ and $\kappa$ for this case ($k = 1$) are connected by relation,

$$\kappa = v\delta + \frac{v^2}{2\beta} + \frac{\beta\alpha_2}{8} + \frac{a\beta\alpha_2^2}{24\alpha_3} - \frac{2a\beta w_0^4}{3\alpha_3}.$$ (46)

We consider here the case with integration constant $J \neq 0$, and hence Eq. (29) yields the condition $y_3 < 0$ because $y_1 = y_2$ for $k = 1$. It follows from Eq. (45) that the condition $y_3 < 0$ is satisfied for $\alpha_2 < -8w_0^2$, and hence we have
\( \alpha_2 < 0 \). Moreover, in this case \((J \neq 0)\) we have \( y_n \neq 0 \), and hence \( 4w_0^2 - \alpha_2 \neq 0 \) and \( A \neq 0 \). The parameter \( A \) in solution (33) should be positive which is possible only in the case with \( \alpha_2 > 0 \) because we have \( \alpha_2 < 0 \) (see Eq. (14)). It also follows from Eq. (14) that \( A + B = -(8w_0^2 + \alpha_2)/3\alpha_3 > 0 \) because \( y_3 < 0 \) and \( \alpha_3 > 0 \). Thus we have found that the conditions \( A > 0, B < 0 \) and \( A + B > 0 \) are satisfied for the solution given in Eq. (33). These conditions lead to gray soliton solution which is a dark soliton with nonzero minimum intensity.

5. Chirped gray soliton solution.

The solution given in Eq. (33) with \( J \neq 0 \) can also be written in the equivalent form as

\[
\psi(x, z) = \pm [A + D \tanh^2(w_0(\xi - \eta))]^{1/2} \exp[i(\kappa z - \delta x) + i\phi(\xi)],
\]

(47)

where the inverse width of the gray soliton is \( w_0 = \frac{1}{2}(\alpha_2^2 - 3\alpha_1\alpha_3)^{1/4} \) with the condition \( \alpha_2^2 > 3\alpha_1\alpha_3 \). The parameters \( \Lambda = A + B \) and \( D = -B \) are given by

\[
\Lambda = -\frac{1}{3\alpha_3}(\alpha_2^2 + 8w_0^2), \quad D = \frac{4w_0^2}{\alpha_3}.
\]

(48)

We have shown that this solution occur only in the case when \( \alpha_2 < -8w_0^2 \) and \( \alpha_3 > 0 \). These conditions also lead to inequalities \( \Lambda > 0 \) and \( D > 0 \). Hence, the minimum intensity of the pulse is \( I = \Lambda \neq 0 \) which typically for gray solitons. The equation for variable parameters \( w_0, v, \delta \) and \( \kappa \) for this case \((k = 1)\) are connected by Eq. (40).

We note that the condition \( \Lambda D > 0 \) is satisfied for gray soliton solution. In this case the phase \( \phi(\xi) \) in Eq. (47) follows by Eq. (10) as

\[
\phi(\xi) = -\frac{J D}{2\beta R w_0} \arcsin \left( \frac{(\Lambda + D) - (\Lambda - D) \cosh(2w_0(\xi - \eta))}{(\Lambda - D) + (\Lambda + D) \cosh(2w_0(\xi - \eta))} \right)
+ \left( \delta + \frac{v}{\beta} + \frac{J}{\beta(\Lambda + D)} \right)(\xi - \eta) + \phi_0,
\]

(49)

where \( R = \sqrt{\Lambda D(\Lambda + D)} \).

Typical intensity profiles of the chirped solitary wave (47) at \( z = 0 \) are shown in Fig. 1 for different degrees of nonlocality \( \mu \) as: \( \mu = 0.25, \mu = 0.27, \mu = 0.30 \). The parameter values used are \( \beta = 1, \sigma = -1, \gamma = 1, v = 0.1, \delta = 2.81, \kappa = 0.12, \eta = 0 \). An interesting observation in this figure is that the minimum intensity of chirped gray solitary wave decreases continuously with the increasing of the degree of nonlocality. We also find that the background intensity of the chirped gray solitary waves remains the same for different values of the nonlocality parameter \( \mu \).

6. Bright soliton solution.
The solutions in Eqs. (33) and (37) for limiting case \( k = 1 \) and \( J = 0 \) reduce to chirped bright soliton solution,

\[
\psi(x, z) = \pm \sqrt{\frac{\alpha_2}{\alpha_3}} \text{sech}\left(\sqrt{\frac{\alpha_2}{2}}(\xi - \eta)\right) \exp(i\theta(x, z)),
\]

where the necessary conditions for this solution are \( \alpha_2 > 0 \) and \( \alpha_3 < 0 \). In this bright soliton solution the phase \( \theta(x, z) \) is given by Eq. (35). Note that we have \( c_0 = y_2^2 y_3 \) which leads to relation \( c_0 = \alpha_0 \alpha_2^2 = 0 \) because \( y_2 = 0 \). This yields \( \alpha_0 = 0 \) and integration constant \( J = 0 \). Moreover, Eqs. (36) and (39) lead to equation for variable parameters \( \delta \) and \( \kappa \) as

\[
\kappa = v\delta + \frac{v^2}{2\beta} + \frac{\beta\alpha_2}{8}.
\]

Note that in this case the inverse width is fixed as \( w_0 = \sqrt{-\alpha_2/2} \).

Figure 2 depicts the intensity profiles of the chirped bright solitary wave solution (50) for different values of the nonlocality parameter \( \mu \). It can be observed that with the increasing of nonlocality, the intensity of the solitary wave gradually decreases, while the width increases leading to broadening of the light beam if the parameter \( \mu \) is further increased.

7. Dark soliton solution.

The solution in Eq. (40) for limiting case \( k = 1 \) and \( J = 0 \) reduces to chirped dark soliton solution,

\[
\psi(x, z) = \pm \sqrt{\frac{\alpha_2}{\alpha_3}} \tanh\left(\sqrt{\frac{\alpha_2}{8}}(\xi - \eta)\right) \exp(i\theta(x, z)),
\]

where the necessary conditions for this solution are \( \alpha_2 < 0 \) and \( \alpha_3 > 0 \). In this dark soliton solution the phase \( \theta(x, z) \) is given by Eq. (35). Note that in this case we have \( c_0 = \frac{y_2^2}{2} y_3 \) which leads to relation \( c_0 = \alpha_0 \alpha_3^2 = 0 \) because \( y_2 = 0 \). This yields \( \alpha_0 = 0 \) and integration constant \( J = 0 \). Moreover, Eq. (42) leads to equation for variable parameters \( \delta \) and \( \kappa \) and \( v \) as

\[
\kappa = v\delta + \frac{v^2}{2\beta} + \frac{\beta\alpha_2}{8} + \frac{a\beta\alpha_2^2}{32\alpha_3}.
\]

In this case the inverse width is fixed as \( w_0 = \sqrt{-\alpha_2/8} \). We note that in Eqs. (51) and (53) there are three variable parameters: \( \delta \), \( v \) and \( \kappa \). This means that one can fix any two of these variable parameters and then the third parameter follows from above equations respectively.

The intensity profile of the chirped dark solitary wave (52) for different values of \( \mu \) is displayed in Fig. 3. One can see that unlike chirped gray solitary waves, the background intensity of the chirped dark solitary waves increase as the degree of nonlocality grows.
FIG. 3: Intensity profiles of the chirped dark solitary wave solution for different values of the nonlocality parameter $\mu$: $\mu = 0.10$, $\mu = 0.15$, $\mu = 0.20$. Other parameters are $\beta = -1$, $\gamma = 1$, $\sigma = -0.3$, $\nu = 0.1$, $\delta = 2.81$, $\kappa = 0.12$, $\eta = 0$.

V. SIMILARITY TRANSFORMATION OF GENERALIZED CQNLS EQUATION

In a real optical material including nonlocal nonlinear media, the physical parameters vary along with the propagation of light beams due to the presence of the inhomogeneities of media. The inclusion of the distributed coefficients into the NLS equations is currently an effective way to reflect the inhomogeneous effects of the optical beams [31]. In what follows, we analyze the beam propagation phenomena in a realistic weakly nonlocal cubic-quintic medium exhibiting varied physical parameters. An accurate description of the beam evolution in such inhomogeneous system can be achieved by means of variation with respect to the propagation distance of all the material parameters in Eq. (3), resulting in the generalized NLS model with distributed coefficients:

$$i\frac{\partial \Phi}{\partial s} + D(s) \frac{\partial^2 \Phi}{\partial \chi^2} + R_1(s) |\Phi|^2 \Phi + R_2(s) |\Phi|^4 \Phi + N(s) \Phi \frac{\partial^2 (|\Phi|^2)}{\partial \chi^2} = iG(s)\Phi,$$

(54)

where $D(s)$, $R_1(s)$ and $R_2(s)$ represent the variable diffraction, cubic and quintic nonlinearity coefficients, respectively. Parameter $N(s)$ denotes the weak nonlocality coefficient, while $G(s)$ represents the loss ($G(s) < 0$) or gain ($G(s) > 0$) coefficient.

In order to connect solutions of Eq. (54) with those of Eq. (3), we will use the transformation [32–34] as

$$\Phi(s, \chi) = \rho(s) \psi [x(s, \chi), z(s)] \exp [i\varphi(s, \chi)],$$

(55)

where $\rho(s)$, $z(s)$, $x(s, \chi)$, and $\varphi(s, \chi)$ are real functions to be determined. Substituting Eq. (55) into Eq. (54) leads to Eq. (3), but we must have the following set of parametric equations:

$$2\rho_s + D\rho \varphi_{\chi\chi} - 2G\rho = 0,$$

$$x_x + D x \varphi_x = 0,$$

(56)

$$2\varphi_s + D (\varphi_\chi)^2 = 0,$$

$$D (x_\chi)^2 - \beta z_s = 0,$$

(57)

$$N\rho^2 (x_\chi)^2 - \mu z_s = 0,$$

$$x_{\chi\chi} = 0,$$

(58)

$$R_1\rho^2 - \gamma z_s = 0,$$

$$R_2\rho^4 - \sigma z_s = 0,$$

(59)
where subscripts denote partial differentiation. These equations can be solved self-consistently to obtain the self-similar wave amplitude \( \rho(s) \) and phase \( \varphi(s, \chi) \):

\[
\rho(s) = \frac{\sqrt[\gamma]{D(s)}}{\sqrt[\beta]{R_1(s)W(s)}}, \\
\varphi(s, \chi) = \frac{c_0 \Gamma(s)}{2} \chi^2 - b_0 \Gamma(s) \chi - \frac{b_0^2}{2} \Gamma(s)d(s),
\]

(60)

(61)

together with the similarity variable \( x(s, \chi) \) and effective propagation distance \( z(s) \):

\[
x(s, \chi) = \frac{\chi - \chi_c(s)}{W(s)}, \\
z(s) = \frac{d(s) \Gamma(s)}{\beta W_0^2},
\]

(62)

(63)

where the beam width \( W(s) \) and center position \( \chi_c(s) \) are given by

\[
W(s) = W_0/\Gamma(s), \quad \chi_c(s) = \chi_0 - (b_0 + c_0 \chi_0) d(s).
\]

(64)

Meanwhile, the accumulated diffraction \( d(s) \) and the parameter related to the phase-front curvature wave \( \Gamma(s) \) are given by

\[
d(s) = \int_0^s D(s')ds', \quad \Gamma(s) = [1 - c_0d(s)]^{-1}.
\]

(65)

Here the parameters \( \chi_0, W_0, c_0 \) and \( b_0 \) are constants representing the initial values of central position of beam, width, chirp, and position of the wavefront, respectively. Further, the constraint conditions on the management parameters depicting weak nonlocality, quintic nonlinearity, and gain (or loss) are given by

\[
N(s) = \frac{\mu R_1(s)W^2(s)}{\gamma}, \\
R_2(s) = \frac{\beta \sigma R_1^2(s)W^2(s)}{\gamma^2 D(s)}, \\
G(s) = \frac{1}{2} \left\{ \frac{W[R_1(s),D(s)]}{R_1(s)D(s)} - \frac{c_0 W_0 D(s)}{W(s)} \right\},
\]

(66)

(67)

(68)

with the notation for the Wronskian \( W[R_1(s),D(s)] = R_1 D_s - D R_1 s \).

We notice that the accumulated diffraction \( d(s) \) influences not only the characteristics of the self-similar pulse such as the amplitude, width, center position, and phase but also the effective propagation distance. Hence, a similarity transformation between Eq. (54) and Eq. (8) can be obtained as

\[
\Phi(s, \chi) = \frac{\sqrt[\gamma]{\sqrt[\beta]{R_1(s)W(s)}}}{\sqrt[\beta]{R_1(s)W(s)}} \psi [x(s, \chi), z(s)] e^{i\varphi(s, \chi)},
\]

(69)

where the phase \( \varphi(s, \chi) \) is given by Eq. (61) and \( \psi(x, z) \) satisfies Eq. (8).

With this transformation [Eq. (69)], one can construct exact self-similar solutions of the generalized NLS equation model (54) by using the above analytic solutions of the constant-coefficient NLS model (3). Let us first construct exact self-similar periodic wave solutions of Eq. (54). Using the transformation (69) with Eqs. (66)–(68) and periodic bounded solution (30) of Eq. (3), we obtain a self-similar cnoidal wave solution of the generalized NLS equation (54) in the form

\[
\Phi(s, \chi) = \pm \frac{\sqrt[\gamma]{D(s)}}{\sqrt[\beta]{R_1(s)W(s)}} [A + B \cosh^2(w \zeta, k)]^{1/2} \exp [i\Theta(s, \chi)],
\]

(70)

where the traveling coordinate \( \zeta \) is given by

\[
\zeta(s, \chi) = \frac{\Gamma(s) \{ \chi + (b_0 + c_0 \chi_0) d(s) - \chi_0 \}}{W_0} - \frac{v}{\beta W_0^2} d(s) \Gamma(s) - \eta.
\]

(71)
and the phase of field has the form,
\[ \Theta(s, \chi) = -\frac{c_0 \Gamma(s)}{2} \chi^2 - \left( b_0 + \frac{\delta}{W_0} \right) \Gamma(s) \chi - \left\{ \frac{b_0^2}{2} + \frac{\delta (b_0 + c_0 \chi_0) \kappa}{W_0^2} - \frac{\kappa}{\beta W_0^2} \right\} \Gamma(s) d(s) + \frac{\delta \chi_0}{W_0} \Gamma(s) + \phi(\xi). \] (72)

Note that the parameters \( A, B, \) and \( \kappa \) in the family of self-similar cnoidal wave solutions (70) are defined by Eqs. (31) and (32). While the phase structure for self-similar waves propagating in cubic-quintic nonlinear media seem to be quadratic [see Refs. 35, 36], the phase of self-similar light beams in the presence of weak nonlocality takes a more complicated form [Eq. (72)], which involves an extra intensity-dependent phase term \( \phi(\xi) \) [Eq. (10)]. This implies that the derived self-similar solutions are chirped nonlinearly which would find potential applications in light compression or amplification.

A second family of exact self-similar periodic wave solution of the generalized NLS equation (54) can be obtained by inserting the solution (44) into the transformation (69) as
\[ \Phi(s, \chi) = \pm \frac{\Lambda \sqrt{\gamma D(s)}}{\sqrt{\beta R_1(s) W(s)}} \text{dn}(w \zeta, k) \exp \left[ i \Theta(s, \chi) \right], \] (73)
where \( \zeta \) is the same as the one given by Eq. (71) while the phase \( \Theta(s, \chi) \) takes the form,
\[ \Theta(s, \chi) = -\frac{c_0 \Gamma(s)}{2} \chi^2 - \left( b_0 - \frac{v}{\beta W_0} \right) \Gamma(s) \chi - \left\{ \frac{b_0^2}{2} - \frac{v (b_0 + c_0 \chi_0)}{\beta W_0} - \frac{\beta (\kappa - \nu \delta - v^2)}{\beta^2 W_0^2} \right\} \Gamma(s) d(s) - \frac{v \chi_0}{\beta W_0} \Gamma(s) + \theta_0. \] (74)

Also \( \Lambda \) and \( w \) satisfy Eq. (38) and \( \kappa \) is given by Eq. (43).

A third family of exact self-similar periodic wave solution of the model (54) can be found by inserting Eq. (71) into the transformation (69) as
\[ \Phi(s, \chi) = \pm \frac{\Lambda \sqrt{\gamma D(s)}}{\sqrt{\beta R_1(s) W(s)}} \text{cn}(w \zeta, k) \exp \left[ i \Theta(s, \chi) \right], \] (75)
where the parameters \( \Lambda \) and \( w \) are given by Eq. (38) and \( \kappa \) is shown in Eq. (39). Meanwhile, the variable \( \zeta \) and phase \( \Theta(s, \chi) \) take the same form as Eqs. (71) and (74), respectively.

Another class of exact self-similar periodic wave solution of Eq. (54) can be determined by substituting Eq. (40) into the transformation (69) as
\[ \Phi(s, \chi) = \pm \frac{\Lambda \sqrt{\gamma D(s)}}{\sqrt{\beta R_1(s) W(s)}} \text{sn}(w \zeta, k) \exp \left[ i \Theta(s, \chi) \right], \] (76)
where \( \zeta \) and \( \Theta(s, \chi) \) are same as the ones given by Eqs. (71) and (74), respectively. Moreover, \( \Lambda \) and \( w \) are given by Eq. (41) while \( \kappa \) is determined by Eq. (42).

Next we construct the exact self-similar localized solutions of the generalized NLS equation with distributed coefficients (54). Substitution of the solution (43) into the transformation (69) yields an exact self-similar solitary wave solution of Eq. (54) of the form,
\[ \Phi(s, \chi) = \pm \frac{\sqrt{\gamma D(s)}}{\sqrt{\beta R_1(s) W(s)}} \left[ A + B \text{sech}^2(w_0 \zeta) \right]^{1/2} \exp \left[ i \Theta(s, \chi) \right], \] (77)
where \( A \) and \( B \) are defined by Eq. (44), \( \zeta \) and \( \Theta(s, \chi) \) are shown in Eqs. (71) and (72), respectively, with \( \kappa \) given by Eq. (46).

Moreover, substitution of the solution (47) into the transformation (69) leads to an exact chirped self-similar gray soliton solution of Eq. (54) of the form,
\[ \Phi(s, \chi) = \pm \frac{\sqrt{\gamma D(s)}}{\sqrt{\beta R_1(s) W(s)}} \left[ A + D \text{tanh}^2(w_0 \zeta) \right]^{1/2} \exp \left[ i \Theta(s, \chi) \right], \] (78)
where \( A \) and \( D \) are shown in Eq. (48) and \( \zeta \) and \( \Theta(s, \chi) \) are given by Eqs. (71) and (72) respectively, with the phase shift \( \phi(\xi) \) given by the relation (49).

It is interesting that we can construct a chirped self-similar bright-type soliton solution for the generalized NLS equation (54) by combining Eqs. (31) and (69) as
\[ \Phi(s, \chi) = \pm \frac{\sqrt{\gamma D(s)}}{\sqrt{\beta R_1(s) W(s)}} \sqrt{\frac{\alpha_2}{\alpha_3}} \text{sech}\left(\frac{\alpha_3}{2} \zeta\right) \exp \left[ i \Theta(s, \chi) \right], \] (79)
in the case where $\alpha_3 < 0$ and $\alpha_2 > 0$. It should be noted here that $\zeta$ and $\Theta(s, \chi)$ take the same expressions stated in (71) and (74) respectively and $\kappa$ is given by Eq. (51).

A chirped self-similar dark-type soliton solution can be also obtained for Eq. (54) by using Eqs. (52) and (69) as

$$\Phi(s, \chi) = \pm \sqrt[2]{\frac{\gamma D(s)}{\beta R_1(s) W(s)}} \sqrt{-\frac{\alpha_2}{2\alpha_3}} \tanh\left(\sqrt{-\frac{\alpha_2}{8}} \zeta\right) \exp\left[i \Theta(s, \chi)\right],$$

(80)

when $\alpha_2 < 0$ and $\alpha_3 > 0$. Here the variable $\zeta$ and phase $\Theta(s, \chi)$ are same as the ones given by Eqs. (71) and (74) while the wave number $\kappa$ satisfies Eq. (53).

Now we discuss how to realize the control for chirped self-similar beams presented above. To demonstrate the controllable self-similar structures, here we take the chirped self-similar localized solutions (78) and (79) as examples to discuss the dynamical behaviors of the self-similar gray and bright solitary waves in the weakly nonlocal medium. In this situation, we take a periodic distributed amplification system with varying diffraction and nonlinear parameters of the form [36–38]:

$$D(s) = D_0 \cos(gs) , \quad R_1(s) = R_0 \exp(-rs) \cos(gs),$$

(81)

where $D_0$ and $g$ are the parameters used to describe the diffraction, while $R_0$ and $r$ are related to the nonlinearity. The corresponding weak nonlocality, quintic nonlinearity, and gain (or loss) of the optical medium defined by Eqs. (66), (67) and (68) read

$$N(s) = \frac{\mu R_0 W_0^2 \cos(gs) \exp(-rs)}{\gamma} [1 - \epsilon \sin(gs)]^2,$$

(82)

$$R_2(s) = \frac{\sigma \beta R_0^2 W_0^2 \cos(gs) \exp(-2rs)}{\gamma^2 D_0} [1 - \epsilon \sin(gs)]^2,$$

(83)

$$G(s) = \frac{r}{2} - \frac{c_0 D_0 \cos(gs)}{2 [1 - \epsilon \sin(gs)]},$$

(84)

where we introduced for brevity the parameter $\epsilon = c_0 D_0 / g$.

First, we analyze the evolutional behavior of self-similar solitary beams in the case when the initial chirp $c_0 = 0$. In this situation, the gain (loss) function (84) takes a constant form $G(s) = r/2$, which corresponds to the diffraction decreasing nonlocal medium for $r < 0$. Here, without loss of generality, we use the parameters $D_0 = 1$, $g = 1$, and
$R_0 = 1$ and choose the value of the nonlocality parameter as $\mu = 0.125$. Figure 4 shows the evolution and contour plots of the chirped self-similar solitary wave solutions (78) and (79) with $D(s)$ and $R_1(s)$ given by Eq. (81) for the parameters $b_0 = 1$, $r = 0$, $v = 0.9$, $\delta = 1.2$, $\kappa = 1.807$, $W_0 = 1$, and $\chi_0 = \eta = 0$. From it, one observes a snake-like behavior of the gray and bright solitary beams as they propagate through the weakly nonlocal medium. For such snaking-like trajectory, the profile of the self-similar solitary wave maintains its structural integrity in propagating along the system although its position varies periodically. Notice that for this case, the central position as given by Eq. (64) reads $\chi_c(s) = \chi_0 - [(b_0 + c_0\chi_0) D_0/\bar{g}] \sin(gs)$, thus indicating that it oscillates periodically along distance even if the value of initial chirp is zero (i.e., $c_0 = 0$). However, for the same case, the width given by Eq. (64) which
obey the relation $W(s) = W_0 [1 - \epsilon \sin (qs)]$, becomes a constant $W(s) = W_0$ when $c_0 = 0$. One should note here that the snakelike evolitional behavior of the gray and bright structures takes place due to the presence of the parameters $b_0$ and $v$ in the traveling coordinate $\zeta$ [Eq. (7)].

Next, we discuss the dynamical behavior of self-similar waves in the case of practical interest $c_0 \neq 0$. Considering the initial chirp parameter $c_0 = 0.2$ and using the same values of $b_0$ and $v$ as in Fig. 4, one can see that the chirped self-similar gray and bright solitary beams show a periodical change in their intensity but their profile remain unchanged in propagation along the nonlocal medium [Figs. 5(a)-(b) and 5(c)-(d)]. We can conclude that the oscillation of the self-similar beam intensity here results from the term $c_0 d(s)$ in the expression for the amplitude given by Eq. (60). Taking the same value of the initial chirp $c_0 = 0.2$ and for $b_0 = 0$ and $v = 0$, we see that that the chirping leads to localization and yields the periodic emergence of solitary waves [Figs. 6(a)-(b) and 6(c)-(d)].

VI. CONCLUSION

To conclude, we studied the formation and properties of spatial nonlinear waves in a weakly nonlocal media with cubic-quintic nonlinearity. A wide variety of periodic waves which are characterized by a nonlinear chirp have been found in the system for the first time. The chirping property of such nonlinear period waves makes them of practical importance for achieving effective beam compression or amplification. It is remarked that these chirped periodic structures do exist in nonlocal media in the presence of all material parameters. The explicit chirped gray optical dip solutions as well as chirped bright and dark soliton solutions are obtained in the long-wave limit of the derived periodic waveforms. In addition, we have investigated the self-similar propagation of optical beams in an inhomogeneous weakly nonlocal media wherein the light propagation is described by the generalized NLS equation with distributed weak nonlocality, diffraction, cubic-quintic nonlinearities and gain or loss. Interestingly, the wider class of periodic and localized solutions presented here possess attractive features such as nonlinearity in beam chirp and self-similarity in beam shape. We also discussed the applications of the obtained chirped self-similar localized beams by considering a periodic distributed amplification system. The results showed that the dynamical behaviors of those nonlinearly chirped self-similar beams could be controlled by choosing appropriate diffraction and quintic nonlinearity parameters. Future research includes a systematic study of the stability of such nonlinearly chirped solitary waves with respect to finite perturbations such as amplitude perturbation, the slight violation of the parametric conditions, and random noises.

Appendix A: Modified nonlinear differential equation

In this Appendix we consider the transformation of Eq. (13) to modified Eq. (17). Note that such transformation is possible because the coefficients $\nu_0$ and $\nu_1$ in Eq. (13) are free parameters. These coefficients are free because $\nu_0$ and $\nu_1$ depend on integration constants $J$ and $C$. We define the coefficients $\alpha_n$ by equation,

$$\sum_{n=0}^{4} \nu_n f^n(\xi) = (1 + af(\xi)) \sum_{n=0}^{3} \alpha_n f^n(\xi).$$  \hspace{1cm} (A1)

Equation (A1) yields the system of algebraic equations as

$$\alpha_0 = \nu_0, \quad \alpha_1 + a\alpha_0 = \nu_1, \quad \alpha_2 + a\alpha_1 = \nu_2, \quad \alpha_3 + a\alpha_2 = \nu_3,$$

where the last equation is

$$a\alpha_3 = \nu_4.$$ \hspace{1cm} (A3)

Relations (A2) lead to the coefficients as

$$\alpha_0 = \nu_0, \quad \alpha_1 = \nu_1 - a\nu_0, \quad \alpha_2 = \nu_2 - a\nu_1 + a^2\nu_0,$$

$$\alpha_3 = \nu_3 - a\nu_2 + a^2\nu_1 - a^3\nu_0.$$ \hspace{1cm} (A5)
We emphasize that Eqs. (A3) and (A5) yield the constraint for coefficients $\nu_n$ as
\[ \nu_4 = a \nu_3 - a^2 \nu_2 + a^3 \nu_1 - a^4 \nu_0. \] (A6)
Equations (A4), (A5) and the constraint (A6) lead to the following coefficients,
\[ \alpha_0 = \nu_0, \quad \alpha_1 = \frac{\nu_3 - \nu_2}{a^2} + \frac{\nu_4}{a^3}, \quad \alpha_2 = \frac{\nu_3}{a^2} - \frac{\nu_1}{a^3}, \quad \alpha_3 = \frac{\nu_4}{a}. \] (A7)
Note that one can assume that the constraint (A6) is satisfied because the coefficients $\nu_0$ and $\nu_1$ are free parameters. These coefficients are free because they depend on integration constants $J$ and $C$ (see Eq. (13)). Moreover, one can consider the coefficient $\nu_0$ as a free parameter and $\nu_1$ is fixed by (A6) for an arbitrary given $\nu_0$. The substitution of Eq. (A1) to Eq. (12) leads to modified nonlinear differential equation,
\[ \left( \frac{df}{d\xi} \right)^2 = \alpha_0 + \alpha_1 f + \alpha_2 f^2 + \alpha_3 f^3, \] (A8)
where the coefficients $\alpha_n$ are given by Eq. (A7). In this equation the coefficient $\alpha_0$ is a free parameter because $\alpha_0 = \nu_0$. We emphasize that in Eq. (13) there are two free coefficients as $\nu_0$ and $\nu_1$. However, the modified Eq. (A8) has one free coefficient $\alpha_0$.

**Appendix B: Ordered real roots of polynomial**

In this Appendix we present three ordered real and different roots $y_1 < y_2 < y_3$ in Eq. (23). The polynomial $P(y) = c_0 + c_1 y + c_2 y^2 - y^3$ can be written as
\[ c_0 + c_1 y + c_2 y^2 - y^3 = -\prod_{n=1}^{3} (y - y_n), \] (B1)
where $y_n$ are the roots in Eq. (23). Eq. (B1) yields the relations,
\[ c_1 = -y_2 y_3 - y_1 y_2 - y_1 y_3, \quad c_2 = y_1 + y_2 + y_3, \] (B2)
\[ c_0 = y_1 y_2 y_3, \quad c_0 = \alpha_0 \alpha_3^2, \] (B3)
where $c_1 = -\alpha_1 \alpha_3$ and $c_2 = \alpha_2$. Moreover, Eq. (B3) yields the relation,
\[ y_2 = k^2 y_1 + (1 - k^2) y_3. \] (B4)
Eqs. (B2) and (B3) lead to the following ordered real and different roots $y_n$ of the polynomial $P(y)$,
\[ y_1 = \frac{\alpha_2}{3} + \frac{(k^2 - 2)}{3} \sqrt{\frac{\alpha_2^2 - 3 \alpha_1 \alpha_3}{k^4 - k^2 + 1}}, \] (B5)
\[ y_2 = \frac{\alpha_2}{3} + \frac{(1 - 2 k^2)}{3} \sqrt{\frac{\alpha_2^2 - 3 \alpha_1 \alpha_3}{k^4 - k^2 + 1}}, \] (B6)
\[ y_3 = \frac{\alpha_2}{3} + \frac{(1 + k^2)}{3} \sqrt{\frac{\alpha_2^2 - 3 \alpha_1 \alpha_3}{k^4 - k^2 + 1}}. \] (B7)
Note that $k^4 - k^2 + 1 > 0$ for the arbitrary values of parameter $k$. One can show that under conditions $\alpha_2^2 - 3 \alpha_1 \alpha_3 \geq 0$ and $0 < k < 1$ these three roots are real, different and ordered as $y_1 < y_2 < y_3$.

It follows from Eq. (25) that the parameter $w$ is
\[ w = \frac{1}{2} \sqrt{y_3 - y_1} = \frac{1}{2} \left( \frac{\alpha_2^2 - 3 \alpha_1 \alpha_3}{k^4 - k^2 + 1} \right)^{1/4}. \] (B8)
Thus the parameters $A$ and $B$ in Eq. (29) are
\[ A = \frac{1}{3\alpha_3}[-\alpha_2 + 4(2k^2 - 1)w^2], \quad B = -\frac{4k^2w^2}{\alpha_3}. \] (B9)

We emphasize that the constraint in Eq. (B3) has the form,
\[ J^2 = -\frac{\beta^2 y_1 y_2 y_3}{4\alpha_3}, \] (B10)

where $J$ is integration constant. Hence, this constraint is satisfied for the integration constant $J$ given in Eq. (B10). However, in this case the roots $y_n$ must satisfy to the inequality $y_1 y_2 y_3 \leq 0$.

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