A 2-Dimensional Binary Search for Integer Pareto Frontiers

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Abstract

For finite integer squares, we consider the problem of learning a classification \( I \) that respects Pareto domination. The setup is natural in dynamic programming settings. We show that a generalization of the binary search algorithm achieves an optimal \( \theta(n) \) worst-case run time.

1 Problem description and preliminaries

We consider a square finite integer grid \( G \) parameterized by \( n \) with
\[
G = \{(x, y) \mid x \in [n], y \in [n]\}
\]
where \([n] = \{1, \ldots, n\}\). There is a classification function \( I : G \to \{0, 1\} \) that respects Pareto domination. In

**Definition 1.** Pareto domination We say a tuple \((a_1, a_2)\) Pareto-dominates a tuple \((b_1, b_2)\) if \(\forall i, a_i \geq b_i \land \exists i : a_i > b_i\).

Pareto frontier Given a set \( S \) with tuple elements of the form \((x, y)\), we say \( F(S) \) is the Pareto frontier of \( S \) if every element in \( S \) is Pareto-dominated by an element of \( F(S) \) and no element of \( F(S) \) is Pareto-dominated by an element of \( F(S) \).

Since \( I \) respects Pareto domination, it is uniquely defined by a Pareto frontier of some set \( S \subseteq G \).

**Lemma 1.** There is a 1-1 mapping between classification functions \( I \) and Pareto frontiers in \( G \).

**Proof.** Every function \( I \) has a 1-1 mapping to the set \( S = \{e \mid I(e) = 1\} \subseteq G \). The set \( S \) has a 1-1 mapping to its Pareto frontier \( F(S) \). Composing the two we get the result.

We wish to learn the function \( I \). Our query model assumes that we can at each step choose some element \( e \) of \( G \) and observe \( I(e) \). We show two simple upper bounds over learning \( I \):

**Lemma 2.** Learning \( I \) is \( O(n^2) \)

**Proof.** We query each element in \( G \), and \( |G| = n^2 \).
Lemma 3. Learning $I$ is $O(n \log_2 n)$

For this lemma we consider the notion of a clash.

Definition 2. For a row $R = \{(x, y)\}_{a \leq y \leq b}$ for some fixed $a, b, x$, a clash is some integer value $a \leq y_c \leq b + 1$ so that $\forall y < y_c, I((x, y)) = 1$ and $\forall y \geq y_c, I((x, y)) = 0$.

Claim 1. We can find a clash in $O(\log_2 (b - a))$

Proof. We implement a binary search over the query model: If we query some tuple $(x, y)$ and get 1, we go “up”, and if we get 0 we go “down”. We stop when we get 0 and we know $(x, y - 1)$ returned 1 (or $y = a$), or we get 1 and know $(x, y + 1)$ returned 0 (or $y = b$).

Claim 2. Given a clash point $y_c$ for a row $R$ we know $I$ values for any $e \in R$.

Proof. Immediate from Pareto domination.

We can now prove Lemma 3.

Proof. For every row $i \in [n]$, we find a clash point. We thus know $I$ values for all elements in the row. Since we do it for all rows, we know $I$. The run time for each row is $O(\log_2 n)$ and over $n$ rows this is $O(n \log_2 n)$.

2 Tight bounds

The above simple algorithms do not establish a tight upper bound for the problem.

Theorem 1. Learning $I$ is $\Omega(n)$.

Proof. We use the 1-1 mapping of Lemma 1. Since we learn at most 1 bit of information with each query, log of the number of Pareto frontiers for $G$ is a lower bound for any algorithm. We show an injection from permutations over $n$ values of 1 and $n$ values of 0 (overall $2^n$ values) into the Pareto frontiers of $G$. For example, $\{(0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0)\}$ are all the permutations for $n = 2$.

Claim 3. There is an injection from permutations over $n$ values of 1 and $n$ values of 0 (which is of size $\binom{2n}{n}$) into the Pareto frontiers in $G$.

Proof. Let $s$ be some permutation as described. We inject it into a series of clash points of length $n$ (one for each row of $G$). Let $k_i$ be a running index over the permutation, which we initialize with $k_1 = 0$. Let $y_1 = n + 1$, i.e., for any element in the first row $I$ value is 1. Let $n_0^i$ be the number of 0 values in the permutation $s$ starting from $k_i$. Let $y_i = y_{i-1} - n_0^i, k_i = k_{i-1} + n_0^i + 1$. Since there are $n$ 1 values in $s$, and each $k_i$ is immediately after the $i - 1$ such 1 value in the permutation, we have that $k_{n+1} = 2n$, i.e., it marks the end of the permutation. Thus $y_n = 1$, as there are $n$ 0 values in the permutation and all of them are covered by $n_0^n$.

The series of clash points $y_1, \ldots, y_n$ is weakly monotone decreasing and valid in the sense that each clash point is between 1 and $n + 1$. It can be shown that every such series of clash points defines a unique Pareto frontier in $G$. $\blacksquare$
Using this injection, we are now able to prove the lower bound. Asymptotically, the central binomial coefficient \( \binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}} \) using Stirling’s approximation [2]. Taking \( \log_2 \), we get 
\[ 2n - \frac{1}{2} \log_2 \pi n \in \Omega(n), \] as required.

We now present an asymptotically optimal algorithm that finds a series of clashes \( y_1, \ldots, y_n \) in \( O(n) \), establishing a tight bound. As we know the series of clashes uniquely defines a Pareto frontier in \( G \), which defines a function \( I \).

**Theorem 2.** There is an \( O(n) \) algorithm to find a Pareto frontier in \( G \).

**Proof.** Assume for simplicity that \( \log_2 n \) is an integer number.

**Algorithm 1: RecursiveClashFinder**

**Input:** Recursion depth \( d \), Row number \( k \), lower border \( a \), upper border \( b \)

**Output:** A series \( y_1, \ldots, y_n \) of all clashes in \( G \)

1. Find a clash \( y_k \) in row \( k \) with borders \( a, b \).
2. if \( d \leq \log_2 n \) then
3. \( \text{RecursiveClashFinder}(d = d + 1, k = k + \frac{n}{2d}, a = y_k, b = b) \)
4. \( \text{RecursiveClashFinder}(d = d + 1, k = k - \frac{n}{2d}, a = a, b = y_k) \)

The algorithm runs over all rows, and thus finds all clashes and the Pareto frontier.

We examine the aggregate run time for all calls of the algorithm with the same depth \( d \). There are \( 2^{d-1} \) such calls. Notice that the calls are bounded to subsequent intervals that together cover \([1, n]\). Each call finds a clash in the interval, and so is \( O(\log_2(b-a)) \). By Jensen inequality \[ \frac{1}{2d-1} \sum_{i=1}^{2d-1} \log_2(b_i-a_i) \leq 2^{d-1} \log_2 \frac{n}{2d-1} \leq \log_2 \frac{n}{2d-1} = 2^{d-1}(\log_2 n - (d-1)). \]

Summing over all possible depths, we get

\[
\sum_{d=1}^{\log_2 n+1} 2^{d-1}(\log_2 n - (d-1)) =
\sum_{d=0}^{\log_2 n} 2^d(\log_2 n - d)
\]

We show by induction over \( \log_2 n \) that this sum is exactly \( 2n - \log_2 n - 2 \), and thus establish an \( O(n) \) run time overall for the algorithm.

**Claim 4.**

\[
\sum_{d=0}^{\log_2 n} 2^d(\log_2 n - d) = 2n - \log_2 n - 2
\]

**Proof.** For \( n = 1, \log_2 n = 0 \) both sides are 0. Now,
\[
\begin{align*}
\log_2 n + 1 \\
\sum_{d=0}^{\log_2 n} 2^d (\log_2 n + 1 - d) &= \\
\sum_{d=0}^{\log_2 n} 2^d (\log_2 n + 1 - d) &= \\
\sum_{d=0}^{\log_2 n} 2^d (\log_2 n - d) + \sum_{d=0}^{\log_2 n} 2^d = \\
2n - \log_2 n - 2 + \sum_{d=0}^{\log_2 n} 2^d &= \\
2n - \log_2 n - 2 + 2^{\log_2 n + 1} - 1 &= \\
4n - (\log_2 n + 1) - 2,
\end{align*}
\]

which is exactly the expression we expect for \(\log_2 n + 1\) (i.e., \(2n\)).

3 Discussion

The motivation of the problem is for dynamic programming. Say that there is some recursive rule over tuples \((x, y)\) that defines, given a current valid set \(S_i\), the next valid set \(S_i \subseteq S_{i+1}\). We also know that the recursive rule respects Pareto domination. In particular, the problem of finding an optimal deterministic 1-realizable online learning algorithm [4] for label set \\{0, 1\} takes this form, but it should be pretty useful in dynamic programming and game theory in general.

An obvious generalization would be to \(d\)-tuples, where much of the discussion extends naturally. The interesting question would be the run time of the algorithm. Since for \(d = 1\) we get the binary search in \(O(\log n)\), and for \(d = 2\) we get the algorithm we presented in \(O(n)\), it is not clear what should be the formula for general \(d\). We are however guaranteed an upper bound of \(O(n^{d-1})\) by simply iterating over all possible values for the first \(d - 2\) coordinates and applying the \(O(n)\) two dimensional result.

References

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