Combinatorics and Topology of partitions of spherical measures by 2 and 3 fans

Rade T. Živaljević
Mathematics Institute SANU, Belgrade

February 2002

Abstract

An arrangement of \( k \)-semilines in the Euclidean (projective) plane or on the 2-sphere is called a \( k \)-fan if all semilines start from the same point. A \( k \)-fan is an \( \alpha \)-partition for a probability measure \( \mu \) if \( \mu(\sigma_i) = \alpha_i \) for each \( i = 1, \ldots, k \) where \( \{\sigma_i\}_{i=1}^k \) are conical sectors associated with the \( k \)-fan and \( \alpha = (\alpha_1, \ldots, \alpha_k) \). The problem whether for a given collection of measures \( \mu_1, \ldots, \mu_m \) and given \( \alpha = (\alpha_1, \ldots, \alpha_k) \) there exists a simultaneous \( \alpha \)-partition by a \( k \)-fan was raised and studied in [3] in connection with some partition problems in Discrete and Computational Geometry. The set of all \( \alpha = (\alpha_1, \ldots, \alpha_m) \) such that for any collection of probability measures \( \mu_1, \ldots, \mu_m \) there exists a common \( \alpha \)-partition by a \( k \)-fan is denoted by \( A_{m,k} \). It was shown in [3] that the interesting cases of the problem are \( (m,k) = (3,2), (2,3), (2,4) \). We prove, as a central result of this paper, that \( A_{3,2} = \{(s,t) \in \mathbb{R}^2 \mid s+t = 1 \} \) and \( s, t > 0 \). The result follows from the fact that under mild conditions there does not exist a \( Q_{4n} \)-equivariant map \( f : S^3 \rightarrow V \backslash A(\alpha) \) where \( A(\alpha) \) is a \( Q_{4n} \)-invariant, linear subspace arrangement in a \( Q_{4n} \)-representation \( V \), where \( Q_{4n} \) is the generalized quaternion group. This fact is established by showing that an appropriate obstruction in the group \( \Omega_1(Q_{4n}) \) of \( Q_{4n} \)-bordisms does not vanish.

1 Introduction

A \( k \)-fan \( p = (x; l_1, l_2, \ldots, l_k) \) on the sphere \( S^2 \) consists of a point \( x \), called the center of the fan, and \( k \) great semicircles \( l_1, \ldots, l_k \) emanating from \( x \). It is always assumed that semicircles in a \( k \)-fan \( p = (x; l_1, \ldots, l_k) \) are enumerated counter clockwise, in an agreement with the standard orientation on the ambient 2-sphere. Sometimes it is more convenient to use the notation \( p = (x; \sigma_1, \sigma_2, \ldots, \sigma_k) \), where \( \sigma_i \) is the open angular sector between \( l_i \) and \( l_{i+1} \), \( i = 1, \ldots, k \). Here we adopt the circular order \( 1 < 2 < \ldots < k < 1 \) of indices and their addition “modulo \( k \)”, so for example \( \sigma_k \) is the open angular sector between \( l_k \) and \( l_1 = l_{k+1} \).

Let \( \mu_1, \mu_2, \ldots, \mu_m \) be Borel probability measures on \( S^2 \). We assume in this paper that all measures \( \mu_j \) are proper in the sense that \( \mu_j([a,b]) = 0 \) for any circular arc \( [a,b] \subset S^2 \) and that \( \mu_j(U) > 0 \) for each nonempty open set \( U \subset S^2 \). All the results, appropriately reformulated, can be extended by a standard limit argument to more general measures, including the counting measures of finite sets, see [3], [27], [28] for related examples.

Let \( (\alpha_1, \alpha_2, \ldots, \alpha_k) \) be a vector of positive real numbers where \( \alpha_1 + \alpha_2 + \ldots + \alpha_k = 1 \). Following [3], but taking into account our simplifying assumptions that we deal only with proper mea-
sures, we say that a $k$-fan $(x; l_1, \ldots, l_k)$ together with the associated conical sectors $\sigma_1, \ldots, \sigma_k$ is an $\alpha$-partition for the collection $\{\mu_j\}_{j=1}^m$ of measures if $\mu_j(\sigma_i) = \alpha_i$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, m$. Note that for more general measures, say for the weak limits of proper measures, instead of the equality $\nu(\sigma_i) = \alpha_i$ one could use a pair of inequalities $\nu(\sigma_i) \leq \alpha_i \leq \nu(\sigma_i)$.

**Definition 1.1.** A vector $\alpha \in \mathbb{R}^k$ is called admissible, or more precisely $(m, k)$-admissible, if for any collection of $m$ (proper) measures on $S^2$, there exists a simultaneous $\alpha$-partition. The collection of all $(m, k)$-admissible vectors is denoted by $A_{m,k}$.

**Problem ([3]):** Find a characterization of the set $A_{m,k}$ or at least describe some of its nontrivial properties. Equivalently, the problem is to decide for what combinations of $m, k$ and $\alpha \in \mathbb{R}^k$ one can guarantee that for any collection of measures $\mathcal{M} = \{\mu_1, \mu_2, \ldots, \mu_m\}$, there exist an $\alpha$-partition for $\mathcal{M}$.

The analysis given in [3] shows that the most interesting cases for the problem of the existence of $\alpha$-partitions are $(m, k) = (3, 2), (2, 3), (2, 4)$. In the case $(m, k) = (3, 2)$ it was shown that $\{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3})\} \subseteq A_{3,2}$.

In this paper we prove, Theorem 6.2, that $A_{3,2} = \{(s, t) \in \mathbb{R}^2 \mid s + t = 1 \text{ and } s, t > 0\}$.

## 2 The candidate space/test map paradigm

Imre Bárány and Jiří Matoušek demonstrated in [3] that the problem of $\alpha$-partitioning of $m$-measures on $S^2$ by a spherical $k$-fan can be in a very elegant way reduced to the problem of the existence of equivariant maps. Bárány and Matoušek introduced, for a given positive vector $\alpha = (\frac{a_1}{n}, \frac{a_2}{n}, \ldots, \frac{a_n}{n}) \in \frac{1}{n} \mathbb{Z}^k \subset \mathbb{Q}^k$, a $D_{2n}$-invariant, linear subspace arrangement $\mathcal{A} = A(\alpha) = A_{m,k}(\alpha)$ of $(W_n)^{\oplus (m-1)}$, where $W_n := \{x \in \mathbb{R}^n \mid x_1 + \ldots + x_n = 0\}$ and $D_{2n}$ is the dihedral group of order $2n$. They showed that an $\alpha$-partition exists if there does not exist a $D_{2n}$-equivariant map $f : V_2(\mathbb{R}^3) \to (W_n)^{\oplus (m-1)} \setminus \cup \mathcal{A}$, from the Stiefel manifold $V_2(\mathbb{R}^3) \cong SO(3)$ of all orthonormal 2-frames in $\mathbb{R}^3$ to the complement $M(\mathcal{A}) := (W_n)^{\oplus (m-1)} \setminus \cup \mathcal{A}$ of the arrangement $\mathcal{A}$.

Here is a brief outline of this construction. One starts with the definition of the candidate or configuration space $X_{\mu_1}$ which is constructed with the aid of the first measure $\mu_1$. By definition $q = (x; l_1, \ldots, l_n) \in X_{\mu_1}$ if and only if $\mu_1(\sigma_i) = \frac{1}{n}$ for each $i = 1, \ldots, n$. In other words $X_{\mu_1}$ is the collection of all $n$-fans on the sphere which form an equipartition of the first measure $\mu_1$. Note that $q = (x; l_1, \ldots, l_n) \in X_{\mu_1}$ is uniquely determined by the pair $(x, l_1)$ or equivalently the pair $(x, y)$, where $y$ is the unit tangent vector to $l_1$ at $x$. Hence, the space $X_{\mu_1}$ is homeomorphic to the Stiefel manifold $V_2(\mathbb{R}^3) \cong SO(3)$ of all orthonormal 2-frames, respectively all orthonormal, positive 3-frames in $\mathbb{R}^3$.

In order to check how far is a $n$-fan $q \in X_{\mu_1}$ from being an equipartition for the measure $\mu_j$, $j \geq 2$, one introduces a test map $f_j : X_{\mu_1} \to \mathbb{R}^n$ by

$$f_j(q) = (\mu_j(\sigma_1) - \frac{1}{n}, \ldots, \mu_j(\sigma_n) - \frac{1}{n}),$$

where as before $\sigma_i$ is the sector on the sphere bounded by $l_i$ and $l_{i+1}$. Taken together, the maps $\{f_j\}_{j=2}^m$ define a test map $F : X_{\mu_1} \to (\mathbb{R}^n)^{\oplus (m-1)}$ where

$$F(q) := (f_2(q), \ldots, f_m(q)) \in (\mathbb{R}^n)^{\oplus (m-1)}.$$
Sometimes it is convenient to interpret the target space \((\mathbb{R}^n)^{\oplus (m-1)}\) as the space \(\text{Mat}_{(m-1)\times n}(\mathbb{R})\) of all \((m-1)\) by \(n\) matrices. Note also that \(\text{Im}(f_j) \subset W_n := \{x \in \mathbb{R}^n \mid x_1 + \ldots + x_n = 0\}\) which implies that the matrix \(F(q)\) has an additional property that the entries in each row add up to zero. The column decomposition \(\text{Mat}_{(m-1)\times n}(\mathbb{R}) \cong (\mathbb{R}^{(m-1)})^{\oplus n} \cong L_1 \oplus \ldots \oplus L_n, x = x_1 + \ldots + x_n = (x_1, \ldots, x_n), \) where \(L_i \cong \mathbb{R}^{m-1}\) is also useful. All these target spaces are identified, so the actual test space \(W = W_n^{\oplus (m-1)}\) is seen as a vector subspace in each of spaces

\[
\text{Mat} := \text{Mat}_{(m-1)\times n}(\mathbb{R}) \cong (\mathbb{R}^{(m-1)})^{\oplus n} \cong (\mathbb{R}^n)^{\oplus (m-1)}.
\]

The dihedral group \(D_{2n}\), defined as the group generated by \(E\) and \(J\) subject to the relations

\[
E^n = 1 \quad JEJ = E^{n-1},
\]  
(1)

acts on \(\mathbb{R}^n, W_n\) and similarly on linear spaces \(\text{Mat}_{(m-1)\times n}(\mathbb{R})\) and \(W_n^{\oplus (m-1)}\) by

\[
E(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n, x_1), \quad J(x_1, x_2, \ldots, x_n) = (x_n, x_{n-1}, \ldots, x_1).
\]

The group \(D_{2n}\) acts also on the configuration space \(X_{\mu_1}\) by

\[
E(x; l_1, l_2, \ldots, l_n) = (x; l_2, \ldots, l_n, l_1), \quad J(x; l_1, l_2, \ldots, l_n) = (-x; l_1, l_n, \ldots, l_2).
\]

Perhaps the action of \(J\) appears more natural if expressed as

\[
J(x; \sigma_1, \sigma_2, \ldots, \sigma_n) = (-x; \sigma_n, \sigma_{n-1}, \ldots, \sigma_1)
\]

where \(\sigma_i\) are the sectors on \(S^2\) associated to the \(n\)-fan \((x; l_1, \ldots, l_n)\). The following proposition doesn’t require a proof.

**Proposition 2.1.** The test map \(F : X_{\mu_1} \to W_n^{\oplus (m-1)}\) is \(D_{2n}\)-equivariant.

Let us associate to each vector \(\alpha = (a_1n, \ldots, a_kn)\) a space \(L = L(\alpha) \subset (W_n)^{\oplus (m-1)} \subset \text{Mat},\)

\[
L = L(\alpha) := \{x \in (W_n)^{\oplus (m-1)} \mid z_1(x) = z_2(x) = \ldots = z_k(x) = 0\},
\]  
(2)

where \(z_i : \text{Mat} \to W_n\) are the linear maps defined by

\[
\begin{align*}
z_1 &= x_1 + \ldots + x_{a_1} \\
z_2 &= x_{a_1+1} + \ldots + x_{a_1+a_2} \\
z_3 &= x_{a_1+a_2+1} + \ldots + x_{a_1+a_2+a_3} \\
\vdots \\
z_k &= x_{a_1+\ldots+a_{k-1}+1} + \ldots + x_n
\end{align*}
\]

(3)

and \(x_j : \text{Mat} \to L_j\) are projections on the corresponding column spaces.

Let \(\mathcal{A} = \mathcal{A}(\alpha)\) be the smallest \(D_{2n}\)-invariant, linear subspace arrangement in \((W_n)^{\oplus (m-1)}\) which contains \(L(\alpha)\). Hence, \(\mathcal{A}(\alpha)\) is the arrangement which contains all subspaces of the form \(L_g(\alpha) := g(L(\alpha)), g \in D_{2n},\) together with their intersections.

**Proposition 2.2.** \((\square)\) Let \(\alpha = \left(\frac{a_1}{n}, \frac{a_2}{n}, \ldots, \frac{a_k}{n}\right) \in \mathbb{R}^k\) be a vector such that all \(a_i\) are positive integers and \(a_1 + \ldots + a_k = n\). Let us suppose that there does not exist a \(D_{2n}\)-equivariant map \(F : V_2(\mathbb{R}^3) \to M(\alpha),\) where \(M(\alpha) := (W_n)^{\oplus (m-1)} \setminus \cup \mathcal{A}(\alpha)\). Then, \(\alpha \in \mathcal{A}_{m,k}\), or in other words for any collection \(\{\mu_j\}_{j=1}^m\) of (proper) measures on \(S^2\), there always exists a \(k\)-fan which \(\alpha\)-partitions each of the measures \(\mu_i\).
Remark 2.3. The “configuration space/test map” scheme, used by Bárány and Matoušek in the α-partition problem, has been successfully applied on numerous problems in discrete geometry and combinatorics. The review papers [9, 10, 12, 27, 29, 30] provide more information and give references to the original papers where these ideas were introduced and developed. This method continues to serve as a central guiding principle for applying topological methods in discrete problems.

3 The change of the group and equivariant maps

If $G$ acts on a space $X$, say on the left, and $H \triangleleft G$ is a normal subgroup, then the quotient group $Q := G/H$ acts on the orbit space $X/H$ by $gH(Hx) := H(gx)$. For example if the cyclic group $G = \mathbb{Z}/2n \cong \{1, \omega, \ldots, \omega^{2n-1}\}$ acts freely on $S^3$, then there is an induced action of $\mathbb{Z}/n \cong G/H = (\mathbb{Z}/2n)/(\mathbb{Z}/2)$ on $S^3/(\mathbb{Z}/2) \cong RP^3$, where $H \cong \mathbb{Z}/2 = \{1, \omega^n\} \subset G$.

A $Q$-space $Z$ is also seen as a $G$-space via the obvious homomorphism $\rho : G \to Q$ where for $g \in G$ and $z \in Z$, $g \cdot z := \rho(g)z$.

The following proposition is a topological analogue of the well known “extension of scalars” equivalence from homological algebra, [27, Section III.3].

Proposition 3.1. Suppose that $X$ and $Z$ are $G$-spaces and $H$ is a normal subgroup of $G$ which acts trivially on $Z$. There exists a $G$-equivariant map $\alpha : X \to Z$ if and only if there exists a $Q$-equivariant map $\beta : X/H \to Z$, where $Q := G/H$ and $X/G$ and $Z/H = Z$ are interpreted as $Q$-spaces.

Proof: The obvious quotient map $p : X \to X/H$ is $G$-equivariant. As before the $Q$-space $X/H$ is automatically a $G$-space via the homomorphism $\rho : G \to Q$. If $\beta : X/H \to Z$ is $Q$-equivariant, it is also $G$-equivariant and the composition $\alpha := p \circ \beta : X \to Z$ is $G$-equivariant. Conversely, if $\alpha : X \to Z$ is a $G$-equivariant map then, in light of the fact that $H$-acts trivially on $Z$, there is a factorization $\alpha = p \circ \beta$ for some $\beta : X/H \to Z$, i.e. $\beta(Hx) = \alpha(x)$ for each $x \in X$. It is easily checked that $\beta$ is $Q$-equivariant,

$$\beta(gH \cdot Hx) := \alpha(gx) = g\alpha(x) = g\beta(Hx) = (gH)\beta(Hx).$$

In this paper we pay a special attention to the class of generalized quaternion groups, [8] p. 253, which are also known as binary dihedral groups. Let $S^3 = S(\mathbb{H}) = Sp(1)$ be the group of all unit quaternions. Let $e = \epsilon_2n = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n} \in S(\mathbb{H})$ be a root of unity generating a subgroup of $S(\mathbb{H})$ of order $2n$. Then

$$Q_{4n} = \{1, \epsilon, \ldots, \epsilon^{2n-1}, j, \epsilon j, \ldots, \epsilon^{2n-1}j\}$$

is a subgroup of $S^3$ of order $4n$ called the generalized quaternion group. It is easily checked that $Q_{4n}$ is isomorphic to the group freely generated by $\epsilon$ and $j$ subject to the relations

$$\epsilon^n = j^2 \quad \epsilon j = j.
$$

Let $H = \{1, e^n\} = \{1, -1\} \subset Q_{4n}$ and let $D_{2n}$ be the associated quotient group $D_{2n} = Q_{4n}/H$. Then $D_{2n}$ is easily found to be isomorphic to the dihedral group of order $2n$. Moreover, since $Q_{4n}/H \subset S^3/H \cong SO(3)$, we identify $D_{2n}$ with a subgroup $D^{'2n}$ of $SO(3)$. The
following well known description of the homomorphism \( \theta : S^3 \to SO(3) \) will help us give a more transparent picture of the group \( D_{2n}' \). Given \( q \in S^3 \), the isometry \( \theta(q) \in SO(3) \) is described by \( \theta(q)x = qxq \) where \( x \in \mathbb{R}^3 \cong \text{Im}(\mathbb{H}) \). If \( q = \cos \alpha + u \sin \alpha \), where \( 0 \leq \alpha \leq \pi \), and \( u \in \text{Im}(\mathbb{H}) \), then \( \theta(q) \) has a geometric description as the positive rotation \( R_u(2\alpha) \) around the oriented axes determined by \( u \), through the angle \( 2\alpha \).

As a consequence we obtain that \( \theta(\epsilon) = R_i(\frac{2\pi}{n}), \theta(j) = R_j(\pi) \) and \( D_{2n}' \) is identified as a dihedral group of order \( 2n \). Note that an abstract dihedral group is generated by two generators \( E \) and \( J \) subject to relations described in (1). We often identify all these dihedral groups so in particular

\[
E = R_i(\frac{2\pi}{n}), \quad J = R_j(\pi), \quad \theta(\epsilon) = E, \quad \theta(j) = J. \tag{4}
\]

The following simple and useful result allows us to change or modify the action of the group \( G \) on the domain \( S^3 \).

**Proposition 3.2.** Suppose that \( \gamma_i : G \times S^3 \to S^3, \gamma_i(g, x) = g \cdot_i x \), \( i = 1, 2 \) are two actions of a finite group \( G \) on the 3-sphere \( S^3 \). Suppose that the action \( \gamma_1 \) is free. Then there exists a \( G \)-equivariant map \( f : S^3 \to S^3 \) between these two actions, that is a map such that for each \( g \in G \) and \( x \in S^3 \),

\[
f(g \cdot_i x) = g \cdot f(x)
\]

**Proof:** The proof of this fact is routine and relies on the fact that \( S^3 \) is a 2-connected, \( \gamma_1 \)-free, \( CW \)-complex so there are no obstructions to extend equivariantly a map defined on 0-skeleton of \( S^3 \). \( \square \)

**Corollary 3.3.** Suppose that \( \gamma_i : G \times S^3 \to S^3, \gamma_i(g, x) = g \cdot_i x \), \( i = 1, 2 \) are two free actions of a finite group \( G \) on the 3-sphere \( S^3 \) and let \( Z \) be an arbitrary \( G \)-space. Then there exists a \( \gamma_1 \)-equivariant map \( f : S^3 \to Z \) if and only if such a map exists for the \( \gamma_2 \) action.

As an illustration of the use of Proposition 3.2 and its Corollary 3.3 we prove the following proposition which is needed in Section 6. Let us start with a definition.

**Definition 3.4.** Suppose that the linear subspace arrangements \( A \) and \( B \) in \( \mathbb{R}^n \) are both \( G \)-invariant for some \( G \subset GL(n, \mathbb{R}) \). \( A \) and \( B \) are isomorphic if there exists a non singular linear map \( C : \mathbb{R}^n \to \mathbb{R}^n \) such that \( K \in A \iff C(K) \in B \). If \( C : \mathbb{R}^n \to \mathbb{R}^n \) is \( G \)-equivariant, i.e. if \( Cg = gC \) for each \( g \in G \), we say that \( A \) and \( B \) are \( G \)-isomorphic. Finally \( A \) and \( B \) are weakly \( G \)-isomorphic or \( wG \)-isomorphic if there exists an automorphism \( \theta : G \to G \) of the group \( G \), such that \( C(g \cdot x) = \theta(g) \cdot C(x) \).

**Proposition 3.5.** If two arrangements \( A \) and \( B \) are \( wG \)-isomorphic, then there exists a \( G \)-equivariant map \( f : S^3 \to M(A) \) if and only if there exists a \( G \)-equivariant map \( g : S^3 \to M(B) \).

Here is an example of two \( wG \)-isomorphic arrangements.

**Proposition 3.6.** Let \( \alpha = (\frac{1}{n}, \frac{n-1}{n}) \) and \( \beta = (\frac{2}{n}, \frac{n-2}{n}) \), where \( p \) is an integer and \( 1 \leq p \leq n - 1 \). Then the arrangements \( A(\alpha) \) and \( A(\beta) \), interpreted as arrangements both in \( \text{Mat}_{2 \times n} \cong (\mathbb{R}^2)^{\otimes n} \) and its subspace \( W_{n^2} \), are \( wD_{2n} \)-isomorphic.
Proof: Let $x_i : (\mathbb{R}^2)^\oplus_n \to \mathbb{R}^2$, $i = 1, \ldots, n$ be the projection on the $i$-th factor. Let $y_j : (\mathbb{R}^2)^\oplus_n \to \mathbb{R}^2$, $j = 1, \ldots, n$ be defined by $y_j = x_{j+1} + \ldots + x_{j+p}$. The arrangement $A(\alpha)$ is generated by subspaces $\text{Ker}(x_i)$ and similarly, $A(\beta)$ is generated by $\text{Ker}(y_j)$. Then

$$x_i \circ E = x_{i+1}, \quad y_i \circ E = y_{i+1}, \quad x_i \circ J = x_{n-i+1}, \quad y_i \circ J = y_{n-i-p}$$

and if $C : (\mathbb{R}^2)^\oplus_n \to (\mathbb{R}^2)^\oplus_n$ is the linear map such that $y_i = x_i \circ C$ then

$$EC = CE \quad \text{and} \quad CJ = JE^{-p-1}C.$$  

Hence, $A(\alpha)$ and $B(\beta)$ are $wD_{2n}$-isomorphic with the associated automorphism $\theta : D_{2n} \to D_{2n}$ defined by $\theta(E) = E$ and $\theta(J) = JE^{-p-1}$. Note that if $p = 2s - 1$ is an odd integer then $JE^{-p-1} = E^{s}JE^{-s}$ and $\theta$ is an inner automorphism of $D_{2n}$. □

Corollary 3.7. There exists a $D_{2n}$-equivariant map $f : V_2(\mathbb{R}^3) \to W_n^{\oplus 2} \setminus \cup A(\alpha)$ if and only if there exists an equivariant map $g : V_2(\mathbb{R}^3) \to W_n^{\oplus 2} \setminus \cup A(\beta)$

4 Evaluation of the group $\Omega_1(Q_{4n})$

In this section we collect some elementary facts about the group $\Omega_1(Q_{4n})$ of $Q_{4n}$-bordism classes of oriented, 1-dimensional $Q_{4n}$-manifolds. In light of natural isomorphisms,

$$\Omega_1(G) \cong \Omega_1(BG) \cong H_1(G, \mathbb{Z}), \quad (5)$$

see [1] or [12] Lemma 2, the computation of $\Omega_1(G)$ is reduced to evaluation of group homology. On the other hand the Abelianization functor $G \mapsto \text{Ab}(G) := G/[G, G]$ allows us to compute $H_1(X, \mathbb{Z})$, at least for connected $X$, as the group $\text{Ab}(\pi_1(X))$. In particular we have isomorphisms

$$H_1(G, \mathbb{Z}) = H_1(BG, \mathbb{Z}) \cong \text{Ab}(\pi_1(BG)) \cong \text{Ab}(G).$$

![Figure 1: The class $[\mathbb{Z}/4 \times_{\mathbb{Z}/2} S^1]$ in $\Omega_1(\mathbb{Z}/4)$.]

Proposition 4.1. Suppose that $Q_{4n} = \{1, \epsilon, \ldots, \epsilon^{2n-1}, j, \epsilon j, \ldots, \epsilon^{2n-1}j\}$ is a generalized quaternion group of order $4n$ where $n$ is an odd integer. Let $H = \{1, j, j^2, j^3\}$ be the subgroup generated by $j$. Then the map, $\Omega_1(H) \xrightarrow{i} \Omega_1(Q_{4n})$, induced by the inclusion $H \to G$, is an isomorphism. Hence,

$$\Omega_1(Q_{4n}) \cong \Omega_1(\mathbb{Z}/4) \cong \mathbb{Z}/4.$$
**Proof:** The commutativity of the following diagram, cf. [1], Section III.20,

\[
\begin{array}{ccc}
\Omega_1(H) & \to & \Omega_1(Q_{4n}) \\
\downarrow & & \downarrow \\
H_1(H, \mathbb{Z}) & \to & H_1(Q_{4n}, \mathbb{Z})
\end{array}
\]

(6)

where the vertical arrows are isomorphisms, allows us to prove the result on the level of homology. Let \( A := \{\epsilon_j \} \) be the subgroup of \( Q_{4n} \) generated by \( \epsilon^j \). Since \( A = [Q_{4n}, Q_{4n}] \) is the commutator group, it is a normal subgroup of \( Q_{4n} \). Moreover, since \( n \) is an odd number, \( A \cap H = \{1\} \). As a consequence one obtains that the following composition of homomorphisms

\[
H \to Q_{4n} \to Q_{4n}/A \cong H
\]

is an isomorphism. By applying the *Abelization* functor on this sequence, we obtain that the horizontal, as well as the vertical arrows in the diagram

\[
\begin{array}{ccc}
H & \to & \text{Ab}(Q_{4n}) \\
\downarrow & & \downarrow \\
H_1(H, \mathbb{Z}) & \to & H_1(Q_{4n}, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_1(H, \mathbb{Z}) & \to & H_1(H, \mathbb{Z})
\end{array}
\]

(7)

are also isomorphisms. Hence, the map \( H_1(H, \mathbb{Z}) \to H_1(Q_{4n}) \) is an isomorphism. \( \square \)

The following proposition is obtained by a similar proof.

**Proposition 4.2.** The map \( \Omega_1(\mathbb{Z}/2) \to \Omega_1(\mathbb{Z}/4) \), induced by the map \( \omega \mapsto j^2 \), where \( \mathbb{Z}/2 = \{1, \omega\} \) and \( \mathbb{Z}/4 = \{1, j, j^2, j^3\} \), is a monomorphism. Moreover, the image of the generator of \( \Omega(\mathbb{Z}/2) \cong H_1(\mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2 \) in \( \Omega_1(\mathbb{Z}/4) \cong \mathbb{Z}/4 \) is the 1-dimensional, \( \mathbb{Z}/4 \)-manifold \( \mathbb{Z}/4 \times_{\mathbb{Z}/2} S^1 \), where \( S^1 \) is seen as a \( \mathbb{Z}/2 \)-manifold.

Figure [ ] gives a pictorial proof of Proposition [4.2]. It shows that the union of circles \( C_1 = \text{“ABcdEFgh”} \) and \( C_2 = \text{“abCDefHI”} \) is \( \mathbb{Z}/4 \)-bordant to the union of two (positive) \( \mathbb{Z}/4 \)-circles, hence it represents the element \( 2 \in \mathbb{Z}/4 \cong \Omega_1(\mathbb{Z}/4) \).

**Corollary 4.3.** Let \( \Delta := Q_{4n} \times_{\mathbb{Z}/2} S^1 = Q_{4n} \times_{\mathbb{Z}/2} M^1 \) where \( \mathbb{Z}/4 \) and \( \mathbb{Z}/2 \) are respectively the subgroups of \( Q_{4n} \) generated by \( j \) and \( j^2 \), the action of \( j^2 \) on \( S^1 \) is antipodal and \( M^1 := \mathbb{Z}/4 \times_{\mathbb{Z}/2} S^1 \) is the union of two circles \( C_1 \) and \( C_2 \). Then \( [\Delta] \in \Omega_1(Q_{4n}) \) is a nontrivial element.

**Proof:** All we need, aside from Propositions [3] and [4.2], is the well known fact, cf. Section III.20, that the transfer map \( t : \Omega_1(H) \to \Omega_1(G) \) is defined by \( t([M]) = [G \times_H M] \). \( \square \)

## 5 Transversality, singular sets and G-bordism

The concept of *transversality* is central for the whole of Differential Topology. The basic facts needed in this paper can be extracted from [7], Chapter II, [8] Chapter 18, or other standard references. As usual, the relation of transversality is denoted by \( f \pitchfork Z \). Here is a list of basic facts.

**F1** If \( f : M \to N \) is transverse to a submanifold \( Z \subset N \), then \( f^{-1}(Z) \) is a submanifold of \( M \) such that \( \text{codim}_M(f^{-1}(Z)) = \text{codim}_N(Z) \).
\textbf{F}_2 Each smooth map \( f : M \to N \) can be perturbed by a “small” homotopy to a map \( f' : M \to N \), transverse to \( Z \subset N \). Moreover, if \( f \) is already transverse to \( Z \) over an open set \( U \subset M \) and \( K \subset U \) is a compact set, then \( f' \) can be chosen to coincide with \( f \) over \( K \).

\textbf{F}_3 A “small” perturbation \( f' \) of a map \( f : M \to N \) which is transverse to \( Z \) is also transverse to \( Z \).

\textbf{F}_4 Facts \( F_1 - F_3 \) together imply that for a given “arrangement” \( A = \{ Z_1, \ldots, Z_k \} \) of submanifolds of \( N \), there always exists a map \( f : M \to N \) such that \( f \cap Z_i \) for all \( i = 1, \ldots, k \). In this case we write \( f \cap A \).

\textbf{F}_5 All results remain valid for sections of smooth bundles. In particular they remain true if \( M \) is a free \( G \)-manifold, where \( G \) is a finite group acting on both \( M \) and \( N \) as a group of diffeomorphism and \( Z \) is a submanifold of \( N \), while all maps are assumed to be \( G \)-equivariant. This follows from the fact that \( G \)-equivariant maps \( f : M \to N \) are in \( 1 - 1 \)-correspondence with the sections of the bundle \( M \times_G N \to M/G \).

Suppose that a finite group \( G \) acts freely on \( S^3 \) as a group of diffeomorphisms. Suppose that \( V \) is a real \( G \)-representation and \( A \) a \( G \)-invariant, linear subspace arrangement of \( V \). As usual, let \( D(A) := \bigcup A \) be the link and \( M(A) := V \setminus D(A) \) the complement of the arrangement. Also, let \( \text{Max}(A) = \{ P_1, \ldots, P_s \} \) be the collection of linear subspaces in \( A \) of maximum dimension. Recall that the intersection poset \( P = P_A \) is an abstract poset that records the containment relation in \( A \), i.e. \( (P_A, \leq) \cong (A, \subseteq) \). In some papers the opposite poset \( P_A^{op} \) is called the intersection poset of \( A \). Our choice has the merit that the dimension function \( d : P_A \to N, P \mapsto \dim(P) \) is monotone. Keeping in mind the intended applications in this paper, we introduce the following, rather restrictive assumption on the arrangement \( A \).

\textbf{A}_1 All maximal elements in \( A \cong P_A \) have dimension \( n - 2 \) where \( n := \dim V \). Moreover, for any 2 maximal elements \( P, Q \in P_A \), if \( P \neq Q \) then \( \dim(P \cap Q) \leq n - 4 \).

Given a \( G \)-equivariant map \( f : S^3 \to V \), the singular set \( \Delta(f) \) is defined as \( \Delta(f) := f^{-1}(D(A)) \). The singular set \( \Delta(f) \) is clearly \( G \)-invariant. If \( f \) is transverse to all subspaces in the arrangement \( A \) then, as a consequence of \( F_1 \), \( f^{-1}(C) = \emptyset \) unless \( \dim(C) = n - 2 \) i.e. unless \( C \) is a maximal subspace in \( A \). It follows that \( \Delta(f) = \bigcup \{ f^{-1}(C) \mid \dim(C) = n - 2 \} \) is a 1-dimensional, \( G \)-submanifold of \( S^3 \) which has at least \( s \) connected components where \( s \) is the cardinality of \( \text{Max}(A) \).

Here are more assumptions on the action of \( G \).

\textbf{A}_2 The group \( G \) preserves the orientation of both \( S^3 \) and \( V \).

\textbf{A}_3 Each \( P \in \text{Max}(A) \) is assigned an orientation in such a way that all these orientations are compatible with the \( G \)-action in the sense that the orientation assigned to \( g(P) \) agrees with the orientation induced on \( g(P) \) from \( P \in \text{Max}(A) \) by the map \( g : P \to g(P) \).

\textbf{Remark 5.1.} Let \( \text{Stab}(P) = \{ g \in G \mid g(P) = P \} \) be the stabilizer of \( P \in \text{Max}(A) \). Then \( \text{Stab}(P) \) acts on \( P \) and it is easy to see that all \( Q \) in the orbit \( \{ g(P) \}_{g \in \text{Stab}(P)} \) of \( P \) can be assigned orientations such that the condition \( A_3 \) is satisfied if and only if \( \text{Stab}(P) \) preserves the orientation on \( P \).
Under assumptions $A_1-A_3$, $\Delta(f)$ is an oriented, 1-dimensional, free $G$-manifold which therefore defines an element $[\Delta(f)]$ in the group $\Omega_1(G)$ of oriented $G$-bordisms, $\mathfrak{I}$.

**Proposition 5.2.** Under assumptions $A_1-A_3$, the element $[\Delta(f)] \in \Omega_1(G)$ does not depend on the smooth map $f$. As a consequence if $\Delta = \Delta(f)$ is nontrivial for one smooth function $f \pitchfork A$, then for any continuous function $g : S^3 \to V$, the singular set $\Delta(g)$ is nonempty.

**Proof:** Suppose that $f$ and $g$ are two maps which are both transverse to the arrangement $A$. Then $\Delta(f)$ and $\Delta(g)$ are both oriented, $G$-invariant 1-manifolds and we are supposed to show that there exists an oriented $G$-bordism between them. Let $F : S^3 \times I \to V$ be a smooth homotopy between $f = F(\cdot, 0)$ and $g = F(\cdot, 1)$, for example $F$ can be obtained by smoothing the linear homotopy $G(x, t) := (1-t)f(x) + tg(x)$. One can assume that $F$ is transverse to $A$. Indeed, by assumption on $f$ and $g$, it is already transverse to $A$ in a neighborhood $U$ of $S^3 \times \{0, 1\}$. By property F$_2$, $F$ can be made transverse to $A$ by a small perturbation outside a compact set $V \subset U$ where $V$ is a neighborhood of $S^3 \times \{0, 1\}$. Then,

$$\Delta(F) = F^{-1}(D(A)) = \cup_{P \in A} F^{-1}(P) \subset S^3 \times I$$

is an oriented, $G$-invariant, 2-manifold, which may have singularities of a very special form. Let us show that these singularities can be removed and that the desingularized manifold $\Delta'(F)$ provides the desired $G$-bordism between $\Delta(f)$ and $\Delta(g)$. Let us note that the singular set $S(F)$ has the following description

$$S(F) = \cup \{F^{-1}(C) \mid \dim(C) = n - 4\}.$$ 

Note that $S(F)$ is a $G$-invariant subset of $S^3 \times I$. Also, by the property F$_1$, the singular set $S(F)$ is 0-dimensional. Each $C \in A$ of dimension $n - 4$ is the intersection of 2 or more maximal subspaces in $A$ so let $A^C := \{P \in \operatorname{Max}(A) \mid C \subset P\}$. It follows that if $x \in F^{-1}(C)$ is a singular point associated to $C$, then $x$ is a point where manifolds $F^{-1}(P)$, $P \in A^C$ intersect transversally. Let $D_x$ be a very small disc in $F^{-1}(P)$ around $x$. Let us remove the interior $D_x$ of this disc from $\Delta(F)$ as well as the interiors of all discs of the form $g(D_x)$ for some $g \in G$. Note that the freeness of the action guarantees that $D_x \cap g(D_x) = \emptyset$ for each $g \neq 1$. This means that we enlarged the boundary of $\Delta(F)$ by an oriented 1-dimensional, $G$-manifold diffeomorphic to the manifold $S^1 \times G$ with the obvious $G$-action. The process of removing discs around singular point can be repeated until we obtain a 2-dimensional, free $G$-manifold $\Delta_1(F)$ with the boundary consisting of $\Delta(f)$, $\Delta(g)$ and possibly several copies of $G \times S^1$. Since $G \times S^1$ is clearly $G$-cobordant to zero, we can fill the holes with manifolds of the form $G \times D^2$, $(S^1 = \partial(D^2))$ and eventually arrive at the desired $G$-cobordism $\Delta'(F)$ between $\Delta(f)$ and $\Delta(g)$. This shows that $\Delta = [\Delta(f)]$ is a well defined element in $\Omega_1(G)$, independent of $f \pitchfork A$. Suppose that $\Delta \neq 0$ and $\Delta(g) = \emptyset$ for some continuous, $G$-equivariant map $g : S^3 \to V$. Then there exists a smooth $\epsilon$-approximation $h$ of $g$ such that $h \pitchfork A$. If $\epsilon$ is small enough then $\Delta(h) = \emptyset$ which is in contradiction with $\Delta = [\Delta(h)] \neq 0$. 

\[\square\]

### 6 Partitions by 2-fans

**Theorem 6.1.** Suppose that $n$ and $p$ are two positive integers such that $n = 2k - 1$ for some $k$ and $1 \leq p \leq n - 1$. Let $\alpha = (\frac{p}{n}, \frac{q}{n})$ where $p + q = n$. Then there does not exist a $Q_{4n}$-equivariant map

$$F : S^3 \to (W_n)^{\otimes 2} \cup \mathcal{A}(\alpha).$$
**Proof:** According to Propositions 3.3 and 3.6 it is sufficient to prove the theorem in the case \( p = 1 \). By identifying the space \( W := (W_n)^{\oplus 2} \) with an appropriate linear subspace of the vector space \( \text{Mat} := \text{Mat}_{2 \times n}(\mathbb{R}) \) of all \( 2 \) by \( n \) matrices, we may view \( \mathcal{A}(\alpha) \) as an arrangement in the latter space. Note that \( W \) is \( D_{2n} \)-invariant, hence \( Q_{4n} \)-invariant subspace of \( \text{Mat} \), actually \( W \) is precisely the orthogonal complement to the linear space of all \( D_{2n} \)-fixed points in \( \text{Mat} \). The action of \( D_{2n} \) on \( \text{Mat} \) is orientation preserving. Indeed, if \( \text{Mat} \cong \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \ldots \oplus \mathbb{R}^2 = \oplus_{i=1}^{n} L_i \) is the column decomposition of \( \text{Mat} \), then \( D_{2n} \) acts essentially by permuting \( 2 \)-dimensional subspaces \( L_i \). This shows that the condition \( A_2 \) from Section 3 is satisfied. Let \( \pi_i : \text{Mat} \rightarrow \mathbb{R}^2 \) be an orthogonal projection on \( L_i \cong \mathbb{R}^2 \) and let \( S_i := \pi_i^{-1}(0) \subset \text{Mat} \). Then, since \( \alpha = (\frac{2}{n}, -\frac{2}{n}) = (\frac{1}{n}, \frac{n-1}{n}) \), the subspaces in \( \mathcal{A}(\alpha) \) of maximum dimension are the spaces \( R_i := W \cap S_i \), \( i = 1, \ldots , n \). It follows that the condition \( A_3 \) is also satisfied. Finally, since \( \text{Stab}(R_1) = \{ I, JE \} \cong \mathbb{Z}/2 \) and \( JE \) fixes all vectors in \( R_1 \), the condition \( A_3 \) in Section 3 and Proposition 5.2 is also fulfilled.

In order to apply Proposition 5.2, one is supposed to select carefully a \( Q_{4n} \)-equivariant map \( f : S^3 \rightarrow W \) transverse to the arrangement \( \mathcal{A}(\alpha) \), compute the associated singular set \( \Delta(f) \) and show that the corresponding element \( [\Delta(f)] \in \Omega_1(Q_{4n}) \) is nontrivial.

In the construction of the map \( f \), we will use both the smooth and a simplicial model for the sphere \( S^3 \). As a complex vector space, the quaternions have the decomposition \( \mathbb{H} \cong \mathbb{C}(1) \oplus \mathbb{C}(2) \) where \( 1, i \in \mathbb{C}(1) \) and \( j, k \in \mathbb{C}(2) \). Each unit quaternion \( q \in S^3 \) has a unique decomposition of the form \( q = (\cos \alpha)q_1 + (\sin \alpha)q_2 \) where \( 0 \leq \alpha \leq \frac{\pi}{2} \) and \( q_1 \) and \( q_2 \) are unit quaternions in \( \mathbb{C}(1) \) and \( \mathbb{C}(2) \) respectively. This simple observation is at the root of the well known join decomposition \( S^3 \cong S^1_{(1)} \ast S^1_{(2)} \) where \( S^1_{(1)} \) and \( S^1_{(2)} \) are respectively the unit circles in \( \mathbb{C}(1) \) and \( \mathbb{C}(2) \). Let \( P_{(1)} \) and \( P_{(2)} \) be regular polygons in \( \mathbb{C}(1) \) and \( \mathbb{C}(2) \) with respective vertices \( a_p := \cos \frac{\pi}{n} + i \sin \frac{\pi}{n} \) and \( b_q := j \cos \frac{\pi}{n} - k \sin \frac{\pi}{n} \) \( k \) for \( p, q = 0, \ldots , 2n - 1 \). This enumeration of vertices of the polygon \( P_{(2)} = \text{conv}\{b_q\}_{q=1}^{n} \) is chosen because we work with left actions and we want to have \( b_q = ja_q \). The (simplicial) join \( P := P_{(1)} \ast P_{(2)} \) is a triangulation of the sphere \( S^3 \). We shall construct a simplicial map \( f : P \rightarrow W \) which is \( Q_{4n} \)-equivariant. Although strictly speaking \( f \) is not a smooth map, \( \Delta(f) \) still provides the information needed for an application of Proposition 5.2.

Indeed, one can from the start interpret \( P \) as a smooth triangulation of \( S^3 \) or alternatively recall that there is no essential difference between the smooth and \( PL \) category in dimension 3 so in particular all results in Section 3 remain valid for \( PL \)-manifolds.

Each \( L_i \cong \mathbb{R}^2 \) in the column decomposition \( \text{Mat} = \oplus_{i=1}^{n} L_i \) is identified with the complex plane \( \mathbb{C} = \mathbb{C}(i) \). Choose a nonzero vector \( v_0 \in L_1 = \mathbb{C}(1) \). Then \( v_i := e^{2i}v_0 \) are vertices of a regular \( n \)-gon \( E_n \) in \( L_1 = \mathbb{C}(1) \).

**Caveat:** As before we allow the indices \( p, q \) etc. of vertices \( a_p, b_q \) of the complex \( P \) to range over all integers and assume that \( x_p = x_{p'} \) whenever \( p = 2n \) \( p' \) are congruent modulo \( 2n \). Similarly, the indices \( j \) of vertices \( v_j \) also range over \( \mathbb{Z} \) with the convention that \( v_i = v_j \) if \( i = n \).

We want to have \( f(\varepsilon_0a_0) = Ef(a_0) = e^2 \cdot f(a_0) \) so \( f(a_0) \) is defined by

\[
f(a_0) := v_0 \oplus e^2v_0 \oplus \ldots \oplus e^{2n-2}v_0 = v_0 \oplus v_1 \oplus \ldots \oplus v_{n-1} \in \oplus_{i=1}^{n} L_i.
\]

The condition \( f(g \cdot a_0) = g \cdot f(a_0) \) forces us to define \( f(a_p) = f(e^p a_0) \) and \( f(b_q) = f(j \cdot a_q) \) by

\[
f(a_p) = e^{2p}v_0 \oplus e^{2p+2}v_0 \oplus \ldots \oplus e^{2p+2n-2}v_0 = v_p \oplus v_{p+1} \oplus \ldots \oplus v_{p+n-1}
\]

\[8\]

\[
f(b_q) = JE^q(v_0 \oplus v_1 \oplus \ldots \oplus v_{n-1}) = v_{q+n-1} \oplus v_{q+n-2} \oplus \ldots \oplus v_q
\]

\[9\]
so for example,
\[ f(b_0) = f(j \cdot a_0) = v_{n-1} \oplus v_{n-2} \oplus \ldots \oplus v_0. \]

The map \( f \), already defined as a \( Q_{4n} \)-equivariant map on the set of vertices of \( P = P_{(1)} \ast P_{(2)} \), admits a unique simplicial extension on the whole of \( P \). Let us show that \( f \) is \( Q_{4n} \)-equivariant. This will follow once we convince ourselves that \( P = P_{(1)} \ast P_{(2)} \) is a free \( Q_{4n} \)-complex. Note that a 3-simplex \( \sigma = a_p a_{p+1} b_q b_{q+1} \in P \) is uniquely reconstructed from its label \( l(\sigma) = (p, q) \in \mathbb{Z}/2n \times \mathbb{Z}/2n \). The induced action of \( Q_{4n} \) on the set \( \mathbb{Z}/2n \times \mathbb{Z}/2n \) of labels of all 3-simplices is obviously free since \( \epsilon(p, q) = (p + 1, q - 1) \) and \( j(p, q) = (q + n, p) \). Indeed if for example \( j^m(p, q) = (p, q) \), then \( (q - m + n, p + m) = (p, q) \) which implies \( p = 2n, p + n, \) a contradiction.

Let us determine the singular set \( \Delta(f) \subset P \) of \( f \), proving along the way that \( f \) is transverse to the arrangement \( \mathcal{A} \). The maximal subspaces in the arrangement \( \mathcal{A} \) are \( R_i = \text{Ker}(\pi_i) \cap W \) where \( \pi_i : \text{Mat} \to L_i \) is the projection. Then,

\[ \Delta(f) = \bigcup_{i=1}^{n} f^{-1}(R_i) = \bigcup_{i=1}^{n} (\pi_i \circ f)^{-1}(0). \tag{10} \]

It is sufficient to determine \( O := (\pi_1 \circ f)^{-1}(0) \) as a (left) \( \mathbb{Z}/4 \)-manifold. Indeed, the stabilizer of \( R_1 \) in \( Q_{4n} \) is the group

\[ \text{Stab}(R_1) = \{ 1, j, -1, -j \} \cong \mathbb{Z}/4. \]

Then \( O \) is a \( \text{Stab}(R_1) \)-manifold and because of \([11]\), \( \Delta(f) \cong Q_{4n} \times \mathbb{Z}/4 O \) as a (left) \( Q_{4n} \)-manifold. Note that \( \pi_i f(a_p) = v_{p+i-1} \) and \( \pi_i f(b_q) = v_{q+n-i} \) for all \( i = 1, \ldots, n \). Hence, for \( i = 1 \)

\[ \pi_1 f(a_p) = v_p \quad \text{and} \quad \pi_1 f(b_q) = v_{q+n-1}. \]

A triangle \( a_p a_{p+1} b_q \in P = P_{(1)} \ast P_{(2)} \) is called good if it intersects \( O := (\pi_1 \circ f)^{-1}(0) \) i.e. if and only if

\[ 0 \in \text{conv}\{\pi_1 f(a_p), \pi_1 f(a_{p+1}), \pi_1 f(b_q)\} = \text{conv}\{v_p, v_{p+1}, v_{q+n-1}\}. \tag{11} \]

Similarly, a triangle \( a_p b_q b_{q+1} \in P \) is good if it intersects \( O := (\pi_1 \circ f)^{-1}(0) \) which happens if and only if

\[ 0 \in \text{conv}\{\pi_1 f(a_p), \pi_1 f(b_q), \pi_1 f(b_{q+1})\} = \text{conv}\{v_p, v_{q+n-1}, v_{q+n}\}. \tag{12} \]

Note that if \( a_p a_{p+1} b_q \) is a good triangle, then the same property have the triangles

\[ \ldots \ a_p b_{q-1} b_q \ a_p a_{p+1} b_q \ a_p a_{p+1} b_{q+1} \ a_{p+1} a_{p+2} b_{q+1} \ a_{p+1} a_{p+2} b_{q+2} \ a_{p+2} b_{q+1} b_{q+2} \ \ldots \tag{13} \]

Let us show that the 1-manifold \( O \) is actually the union of two circles, \( O = C_1 \cup C_1 \). The triangle \( a_0 a_1 b_k \), where \( n = 2k - 1 \), intersects one of these circles, say \( C_1 \) while the triangle \( a_0 a_1 b_{k-1} = a_0 a_1 b_{n+k} \) intersects the other. Actually the circles, connected components of \( O \), are in 1–1 correspondence with chains of consecutive triangles of the form \([13]\). It is than easy to check that there are precisely two such chains. Alternatively, one can show that both \( a_p a_{p+1} b_q \) and \( a_{p+1} b_q b_{q+1} \) are good triangles if and only if either \( q - p = k \) or \( q - p = n + k = 3k - 1 \).
Then, if both \( a_p a_{p+1} b_q \) and \( a_{p+1} b_q b_{q+1} \) are labelled by \((p, q) \in \mathbb{Z}/2n \times \mathbb{Z}/2n\), then the set \( L_\Delta \) of labels of all good triangles is described as

\[
L_\Delta = \theta^{-1}(k) \cup \theta^{-1}(3k - 1)
\]

where \( \theta : \mathbb{Z}/2n \times \mathbb{Z}/2n \to \mathbb{Z}/2n \) is the map defined by \( \theta(p, q) = q - p \). In this representation the sets of labels \( \theta^{-1}(k) \) and \( \theta^{-1}(3k - 1) \) correspond to circles \( C_1 \) and \( C_2 \) respectively. The meaning of labels becomes even more transparent if we observe that the triangles \( a_p a_{p+1} b_q \) and \( a_{p+1} b_q b_{q+1} \) are faces of a common 3-simplex \( \sigma_{(p, q)} = a_p a_{p+1} b_q b_{q+1} \in P \). Note also that if \( \gamma : \mathbb{Z}/2n \times \mathbb{Z}/2n \to \mathbb{Z}/n \) is the map defined by \( \gamma(p, q) = [\theta(p, q)]_{\text{mod } n} \), then the decomposition \( \Delta(f) = \bigcup_{i=1}^{n^3} \gamma^{-1}(i) \) corresponds to the decomposition \( [\mathbb{1}] \) of the singular set \( \Delta(f) \).

A consequence of this analysis is that \( \Delta(f) \) is, as an oriented \( Q_{4n} \)-manifold, isomorphic to \( Q_{4n} \times_{\mathbb{Z}/2} S^1 \). Hence, by Corollary \ref{cor:nontrivial}, \([\Delta(f)]\) is a nontrivial element \( \Delta \) of \( \Omega_1(Q_{4n}) \). By Proposition \ref{prop:nonempty}, \( \Delta(g) \) is nonempty for each continuous, \( Q_{4n} \)-equivariant map \( g : S^3 \to (W_n)^{\oplus 2} \), i.e. there does not exist a \( Q_{4n} \)-equivariant map \( g : S^3 \to (W_n)^{\oplus 2} \setminus A(\alpha) \). This completes the proof of the theorem. \hfill \( \Box \)

**Theorem 6.2.** Suppose that \( \alpha = (s, t) \) is a vector in \( \mathbb{R}^2 \) such that \( s + t = 1 \) and \( s, t > 0 \). Then any collection of three proper measures \( \mu_1, \mu_2, \mu_3 \) on the sphere \( S^2 \) admits an \( \alpha \)-partition by a 2-fan \( p = (x, l_1, l_2) \). In other words,

\[
A_{3,2} = \{(s, t) \in \mathbb{R}^2 \mid s + t = 1 \text{ and } s, t > 0\}.
\]

**Proof:** A simple limit and compactness argument shows that \( A_{3,2} \) is a closed subspace of \( T := \{(s, t) \in \mathbb{R}^2 \mid s + t = 1 \text{ and } s, t > 0\} \). Hence it is sufficient to show that \( S \subset A_{3,2} \) for a dense subset \( S \subset T \). Let \( (s, t) = \left( \frac{p}{n}, \frac{q}{n} \right) \), where \( n \) is an odd number. In order to show that \( \left( \frac{p}{n}, \frac{q}{n} \right) \in A_{3,2} \) it is sufficient, according to Proposition \ref{prop:nontrivial}, to show that there does not exist a \( D_{2n} \)-equivariant map \( f : V_2(\mathbb{R}^3) \to (W_n)^{\oplus 2} \setminus A(\alpha) \). In light of Proposition \ref{prop:3-fan} applied on \( X = S^3, G = Q_{4n} \) and \( H = \mathbb{Z}/2 \), this is precisely the statement of Theorem \ref{thm:3-fan}. \hfill \( \Box \)

### 7 Partitions by 3-fans

The central part of the proof of Theorem \ref{thm:3-fan} was the construction of a special, \( Q_{4n} \)-equivariant map \( f : S^3 \to W_n \) and the evaluation of its singular set \( \Delta(f) \). A similar strategy is applied in the case of partitions by 3-fans. Unfortunately, the invariant \( \Delta = [\Delta(f)] \in \Omega_1(Q_{4n}) \) turns out to be zero in this case, Proposition \ref{prop:3-fan}. We nevertheless reproduce here this computation for two reasons. The combinatorial details appear to be interesting in themselves and, since the first obstruction to the existence of an equivariant map vanishes, we expect that they may be useful in the computation of the secondary obstruction. Secondly, the evaluation of \( \Delta \) in the case of 3-partitions provides a neat example what in principle can “go wrong” with this approach.

We keep the same notation as before so in particular \( \mathbb{H} \cong C_1 \oplus C_2 \) and \( S^3 = S^1_{(1)} \ast S^1_{(2)} \), while \( P = P_{(1)} \ast P_{(2)} \) is the triangulation of the 3-sphere obtained as the join of regular \( (2n) \)-gons \( P_{(1)} \) and \( P_{(2)} \) in \( C_1 \) and \( C_2 \) respectively. The indexing of vertices \( \{a_p\}_{p=1}^{2n} \) of \( P_{(1)} \) is also unchanged so as before \( a_p = e^p a_0 = \cos \frac{pe}{n} + i \sin \frac{pe}{n} \). However, the enumeration of vertices \( \{b_q\}_{q=1}^{2n} \) of \( P_{(2)} = j(P_{(1)}) \) is different. In order to make the calculations more transparent we assume that \( ja_p = b_q \) is equivalent to \( p + q = n + 1 \) (mod \( 2n \)) or in other words \( ja_p = b_{n-p+1} \).
for each \(p = 0, 1, \ldots, 2n - 1\). From here one easily deduces the following equalities

\[
\begin{align*}
ja_p &= b_{n-p+1} \\
\epsilon a_p &= a_{p+1} \\
jb_q &= a_{q+1} \\
\epsilon b_q &= b_{q+1}.
\end{align*}
\] (15)

Let \(v_i = e_i - (1/n) \sum_{j=1}^n e_j, i = 1, \ldots, n\) be the vertices of the regular, \(\mathbb{D}_{2n}\)-invariant simplex in \(W_n\). Let \(f' : P^{(0)} \to W_n\) be the map defined on the 0-skeleton \(P^{(0)} = \{a_p\}_{p=1}^{2n} \cup \{b_q\}_{q=1}^{2n}\) of \(P\) by

\[
f'(a_p) = v_p \quad \text{and} \quad f'(b_q) = v_q.
\] (16)

Let \(f : P \to W_n\) be the simplicial extension of \(f'\) on the complex \(P = P(1) \ast P(2) \cong S^3\). Then,

\[
\begin{align*}
f(ja_p) &= f(b_{n-p+1}) = v_{n-p+1} = Jv_p = Jf(a_p) \\
f(\epsilon a_p) &= f(a_{p+1}) = v_{p+1} = Ev_p = Ef(a_p)
\end{align*}
\] (17)

which means that \(f'\) is \(\mathbb{Q}_{4n}\)-equivariant. As in the proof of Theorem 12, \(f'\) is extended to a unique \(\mathbb{Q}_{4n}\)-equivariant, simplicial map

\[
f : P(1) \ast P(2) \to W_n.
\]

The next step is to determine the associated singular set

\[
\Delta(f) = f^{-1}(\cup A(\alpha)) = \bigcup_{g \in \mathbb{D}_{2n}} f^{-1}(gL(\alpha)) = \bigcup_{h \in \mathbb{Q}_{4n}} hf^{-1}(L(\alpha)).
\] (18)

Let us write the defining equations (12) and (13) for \(L(\alpha)\) in the special case \(\alpha = (\frac{p}{n}, \frac{q}{n}, \frac{r}{n})\). If \(P := p\) and \(Q := p + q\) then the equations \(z_1 = z_2 = z_3 = 0\) can be rewritten as

\[
x_1 + \ldots + x_P = x_{P+1} + \ldots + x_Q = x_{Q+1} + \ldots + x_n = 0.
\] (19)

In light of the condition \(z_1 + z_2 + z_3 = 0\), one of the equations in (13) is redundant so \(L(\alpha)\) is a subspace of \(W_n\) of codimension 2. Clearly \(\Delta(f) \cong \mathbb{Q}_{4n} \times H\), where \(O := f^{-1}(L(\alpha))\) and \(H := \text{Stab}(L(\alpha)) \subset \mathbb{Q}_{4n}\) is the stabilizer of \(L(\alpha)\). It is clear that the group \(H = H_\alpha\) depends on the vector \(\alpha = (\frac{p}{n}, \frac{q}{n}, \frac{r}{n})\). More precisely, \(H_\alpha = \{−1, 1\}\) if all three integers \(p, q, r\) are pairwise distinct, \(H_\alpha = \{1, j\epsilon^q, −1, −j\epsilon^q\} \cong \mathbb{Z}/4\) if \(\alpha = (\frac{p}{n}, \frac{q}{n}, \frac{r}{n})\) and \(p \neq q\), and \(H_\alpha \cong S_3\) if \(p = q = r\).

It turns out that the \(H\)-manifold \(O\) is always the union of 2 circles, \(O = O_1 \cup O_2\). This assertion is proved by a careful analysis of which simplices \(a_pa_{p+1}bqb_{q+1}\) intersect \(\Delta(f)\) and \(O\). The triangles \(a_pa_{p+1}b_q\) and \(a_pbqb_{q+1}\) are called \textit{good} if they intersect \(O = f^{-1}(L(\alpha))\). Let \([1, 2n] = \mathcal{P} \cup \mathcal{Q} \cup \mathcal{R}\) be the partition of the index set \([1, 2n] = \{1, 2, \ldots, 2n\}\) defined by \(\mathcal{P} = [1, P] \cup [n, P + n], \mathcal{Q} = [P + 1, Q] \cup [P + n + 1, Q + n], \mathcal{R} = [Q + 1, n] \cup [Q + n + 1, 2n]\).

**Claim 1:** A triangle \(a_pa_{p+1}b_q\) is \textit{good} if and only if \(\{p, p + 1, q\}\) intersects each of the sets \(\mathcal{P}, \mathcal{Q}\) and \(\mathcal{R}\). Similarly, a triangle \(a_pbqb_{q+1}\) is \textit{good} if and only if \(\{p, q, q + 1\}\) has a nonempty intersection with each of the sets \(\mathcal{P}, \mathcal{Q}\) and \(\mathcal{R}\).

An example of a good triangle is \(a_2a_1b_{p+1}\) which intersects \(O\) precisely in the point \(c = (r/n)a_{2n} + (p/n)a_1 + (q/n)b_{p+1} \in \mathcal{P}\). More generally, it is easily verified that if \(x_iy_jz_k\) is a good triangle such that \(i \in \mathcal{P}, j \in \mathcal{Q}\) and \(k \in \mathcal{R}\), then \(c = (r/n)x_i + (q/n)y_j + (r/n)z_k\) is the unique point in \(x_iy_jz_k\) which intersects \(O\). This observation is essentially all we need for the proof of Claim 1 and at the same time it shows that the map \(f\) is transverse to \(L(\alpha)\). Two good triangles are called \textit{adjacent} if they are faces of a common 3-simplex in \(P\). They are in the same component if they can be connected by a chain of adjacent triangles. It turns out that
as a D satisfies. The analysis of the singular set $\Delta(\omega)$ means that as a $Q$ implies that as a $D$.

**Proof:** The assumptions on $Q$ first obstruction for the existence of a set of the map $\Delta = \{\Delta(p) \mid p \in \Omega\}$.

Suppose that

Proposition 7.1. Suppose that $n$ is odd, $\alpha = (\frac{q}{n}, \frac{p}{n}, \frac{r}{n})$ and the integers $p, q, r$ are pairwise distinct. Then the obstruction element $\Delta = [\Delta(f)] \in \Omega_1(Q_{4n})$ is zero, where $\Delta(f)$ is the singular set of the map $f : S^3 \to W_n$ described by (16) and (17). Moreover, under these conditions the first obstruction for the existence of a $Q_{4n}$-equivariant map $f : S^3 \to W_n \cup A(\alpha)$ also vanishes.

**Proof:** The assumptions on $n$ and $\alpha$ guarantee that the conditions $A_1 - A_3$ of Proposition 5.2 are satisfied. The analysis of the singular set $\Delta(f)$ reveals that $\Delta(f) \cong Q_{4n} \times S^1$ as a $Q_{4n}$-set, hence $\Delta = [\Delta(f)] \in \Omega_1(Q_{4n})$ is a trivial element.

The first obstruction to the existence of an equivariant map $g : S^3 \to W_n \setminus A(\alpha)$ is an element $\omega \in H^2_{Q_{4n}}(S^3, \pi_1(W_n \setminus A(\alpha)))$. Condition $A_1$ implies that

$$A := \pi_1(W_n \setminus A(\alpha)) \cong H_1(W_n \setminus A(\alpha); \mathbb{Z}) \cong \mathbb{Z}^{2n}.$$

As a $D_{2n}$-module, $A \cong D_{2n} \times \mathbb{Z}$ is isomorphic to the regular representation of $D_{2n}$ over $\mathbb{Z}$. This means that as a $Q_{4n}$-module, $A \cong Q_{4n} \times_{\mathbb{Z}/2} \mathbb{Z}$ for an appropriate subgroup $\mathbb{Z}/2 \subset Q_{4n}$. It follows that $H^2_{Q_{4n}}(S^3, A) \cong H^2_{\mathbb{Z}/2}(S^3, \mathbb{Z})$ by the “extensions of scalars” isomorphism. By equivariant
Poincaré duality, \cite{26}, $H^2_{\mathbb{Z}/2}(S^3, \mathbb{Z}^2) \cong H^{\mathbb{Z}/2}_{1}(S^3, \mathbb{Z})$ and the obstruction class $\omega$ corresponds to the class $\hat{\omega} \in H^{\mathbb{Z}/2}_{1}(S^3, \mathbb{Z}^2)$ represented by two circles. Finally, the triviality of $\hat{\omega}$ follows from the geometric fact that two circles represent a trivial element in

$$H^2_{\mathbb{Z}/2}(S^3, \mathbb{Z}) \cong H^{\mathbb{Z}/2}_{1}(S^3, \mathbb{Z}) \cong \Omega_1(\mathbb{Z}/2).$$
References

[1] J. Akiyama, A. Kaneko, M. Kano, G. Nakamura, E. Rivera-Campo, S. Tokunaga, and J. Urutia. Radial perfect partitions of convex sets in the plane. In Discrete and Computational Geometry (J. Akiyama et al. eds.), Lect. Notes Comput. Sci. 1763, pp. 1–13. Springer, Berlin 2000.

[2] I. Bárány, Geometric and combinatorial applications of Borsuk’s theorem, New trends in Discrete and Computational Geometry, János Pach, ed., Algorithms and Combinatorics 10, Springer-Verlag, Berlin, 1993.

[3] I. Bárány, J. Matoušek. Simultaneous partitions of measures by $k$-fans, Discrete Comp. Geometry, 25 (2001), 317–334.

[4] I. Bárány, J. Matoušek. Equipartitions of two measures by a 4-fan, (preprint).

[5] S. Bespamyatnikh, D. Kirkpatrick, and J. Snoeyink. Generalizing ham sandwich cuts to equitable subdivisions. Discrete Comput. Geom., 24:605–622, 2000.

[6] A. Björner. Topological methods, In R. Graham, M. Grötschel, and L. Lovász, editors, Handbook of Combinatorics. North-Holland, Amsterdam, 1995.

[7] K.S. Brown. Cohomology of groups, Springer-Verlag, New York, Berlin, 1982.

[8] H. Cartan and S. Eilenberg. Homological Algebra Princeton University Press, 1956.

[9] P.E. Conner and E.E. Floyd. Differentiable periodic maps, Springer-Verlag, Berlin 1964.

[10] T. tom Dieck, Transformation groups, de Gruyter Studies in Math. 8, Berlin, 1987.

[11] M. Golubitsky, V. Guillemin. Stable Mappings and Their Singularities. Graduate Texts in Mathematics 14, Springer–Verlag 1973.

[12] C.M. Gordon. The $G$-signature theorem in dimension 4, in A la recherche de la topologie perdue, L. Guillou, A. Marin (eds.), Birkhäuser, 1986.

[13] H. Ito, H. Uehara, and M. Yokoyama. 2-dimension ham-sandwich theorem for partitioning into three convex pieces. In Discrete and Computational Geometry (J. Akiyama et. al eds.), Lect. Notes Comput. Sci. 1763, pp. 129–157. Springer, Berlin 2000.

[14] A. Kaneko, M. Kano. Balanced partitions of two sets of points in the plane. Comput. Geom. Theor. Appl., 13(4), 253–261, 1999.

[15] P. Mani-Levitzka, S. Vrečica, R. Živaljević, Combinatorics and topology of partitions of masses by hyperplanes, (in preparation).

[16] J. Matoušek, Topological methods in Combinatorics and Geometry, Lecture notes, Prague 1994. (updated version, February 2002, www.ms.mff.cuni.cz/ matousek/lecturenotes.html).

[17] J.W. Milnor, J.D. Stasheff. Characteristic Classes. Annals of Mathematics Studies 76, Princeton University Press 1974.

[18] P. Orlik, H. Terao. Arrangements of Hyperplanes. Grundlehren der mathematischen Wissenschaften 300, Springer-Verlag 1992.
[19] J. Pach (Ed.), *New Trends in Discrete and Computational Geometry*, Algorithms and Combinatorics 10, Springer 1993.

[20] E. Ramos, Equipartitions of mass distributions, by hyperplanes. *Discrete Comput. Geom.*, 15 : 147–167, 1996.

[21] T. Sakai. Radial partitions pf point sets in $R^2$. Manuscript, Tokoha Gakuen University, 1998.

[22] H. Tverberg, S. Vrečica, On generalizations of Radon’s theorem and the ham sandwich theorem, *Europ. J. Combinatorics* 14, 1993, pp. 259–264.

[23] S. Vrečica, R. Živaljević, The ham sandwich theorem revisited, *Israel J. Math.* 78, 1992, pp. 21–32.

[24] S. Vrečica, R. Živaljević. Conical equipartitions of massdistributions. *Discrete Comput. Geom.*, 225:335–350, 2001.

[25] S. Vrečica, R. Živaljević. Arrangements, equivariant maps and partitions of measures by 4-fans, (preprint).

[26] C.T.C. Wall. *Surgery on Compact Manifolds*. Academic Press, 1970.

[27] R.T. Živaljević, Topological methods, in *CRC Handbook of Discrete and Computational Geometry*, J.E. Goodman, J. O’Rourke, eds. CRC Press, Boca Raton 1997.

[28] R.T. Živaljević. The Tverberg–Vrečica problem and the combinatorial geometry on vector bundles, *Israel J. Math.* 111 (1999), 53–76.

[29] R.T. Živaljević, User’s guide to equivariant methods in combinatorics, *Publ. Inst. Math. Belgrade*, 59(73), 1996, 114–130.

[30] R.T. Živaljević, User’s guide to equivariant methods in combinatorics II, *Publ. Inst. Math. Belgrade*, 64(78), 1998, 107–132.

[31] R. Živaljević, S. Vrečica, An extension of the ham sandwich theorem, *Bull. London Math. Soc.* 22, 1990, pp. 183–186.