Global Wellposedness of the Primitive Equations with Nonlinear Equation of State in Critical Spaces

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Dedicated to Yoshihiro Shibata on the occasion of his 70th-Birthday

Abstract. This article investigates the primitive equations with nonlinear equations of state. A global, strong well-posedness result for this set of equations is established for initial data lying in critical spaces provided that the density, depending on temperature, salinity and pressure, satisfies certain regularity assumptions. These assumptions are in particular satisfied for the TEOS-80 formulation of the equation of state.

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1. Introduction

The primitive equations for the ocean and the atmosphere with linear equation of state are derived from the Navier–Stokes equations by assuming a hydrostatic balance for the pressure term in the vertical direction. The analysis of these equations was pioneered by Lions, Temam and Wang in their articles [12–14], where they proved the existence of a global, weak solution to these equations. Their uniqueness remains an open problem until today. A breakthrough result concerning global strong well-posedness of the primitive equations for initial data in $H^1$ was shown by Cao and Titi [2] using energy methods. Different approaches based on the theory of evolution equations and maximal regularity are due to Hieber and Kashiwabara [8], and Giga, Gries, Hieber, Hussein and Kashiwabara [5]. The vertical momentum equation of the fluid is represented due to the hydrostatic balance as

$$\partial_z \pi + g \rho = 0,$$

where $\pi$ denotes the pressure and $\rho$ the density of the fluid, respectively, and $g > 0$ is the acceleration of gravity. The special case of the linear equation of state

$$\rho = \rho_0 - \alpha (\tau - \bar{\tau}) + \beta (\sigma - \bar{\sigma})$$

was investigated in [2,7,11]. Here $\tau$ and $\sigma$ denote temperature and salinity, and $\rho_0$, $\bar{\tau}$ and $\bar{\sigma}$ given constants. In this article we consider the primitive equations with a general nonlinear equation of state that relates temperature $\tau$, salinity $\sigma$ and pressure $\pi$ to the density $\rho$. Following Section 1.8 of [18], the physical properties of seawater in thermodynamical equilibrium are described by an equation of state for $\rho$ of the form

$$\rho = R(\pi, \tau, \sigma).$$

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Samelson explains furthermore that “a theoretical derivation of an equation of state from a molecular description is not currently available” and “consequently, the function  must be determined empirically, which is generally done by fitting elementary functions in , , and to data of experimental measurements. The function  proves to be nonlinear. Measurement in the ocean of pressure, temperature, and salinity is much easier than direct measurement of density and, consequently, the equation of state is used regularly to determine the dynamically important density from ocean observations”. In Sect. 3 we give examples of equations of state, which are used in applied geophysical communities. Following [18] we assume that for the Boussinesq approximation, the pressure in the equation of state (1.1) equals the static pressure only. The primitive equations with a general equation of state are thus given by

\[
\begin{align*}
\partial_t v - \Delta v + (u \cdot \nabla) v + \nabla_H \pi &= f_v & \text{in } \Omega \times (0, T), \\
\text{div}(u) &= 0 & \text{in } \Omega \times (0, T), \\
\partial_t \tau - \Delta \tau + (u \cdot \nabla) \tau &= f_\tau & \text{in } \Omega \times (0, T), \\
\partial_t \sigma - \Delta \sigma + (u \cdot \nabla) \sigma &= f_\sigma & \text{in } \Omega \times (0, T), \\
\partial_z \pi(x, y, z) + g \rho(z, \tau, \sigma) &= 0 & \text{in } \Omega \times (0, T)
\end{align*}
\]

with initial values \( v(0) = v_0, \tau(0) = \tau_0 \) and \( \sigma(0) = \sigma_0 \) and subject to boundary condition described in detail in Sect. 2. Here \( \Omega = G \times (-h, 0) \) denotes a cylindrical domain with \( G = (0, 1)^2 \) and \( h > 0 \).

Very recently, Korn [10] investigated the primitive equations subject to certain specific equations of state, as the TEOS-10 or the UNESCO-80 equations of state, and extended the method of Cao and Titi [2] or the one by Kukavica [11] to the situation of these nonlinear equations of state. In particular, he proved global strong well-posedness for these equations for initial data belonging to \( H^1 \)-data.

In this article we consider general nonlinear equations of state and consider densities \( \rho \) belonging to the class

\[
\rho \in L^\infty_t((-h, 0), W^{2, \infty}(\mathbb{R}^2, \mathbb{R})).
\]

We then prove global strong well-posedness for the set of equations (1.2) subject to (1.3) for initial data lying in critical spaces, which in the given situation are the Besov spaces

\[
B^{\mu}_{pq} \text{ for } p, q \in (1, \infty) \text{ with } \frac{1}{p} + \frac{1}{q} \leq \mu \leq 1.
\]

The above spaces are critical spaces and they correspond in the situation of the Navier–Stokes equations to the function spaces \( B^{n/p-1}_{pq} \) introduced by Cannone [1]. Choosing in particular \( p = q = 2 \) and \( \mu = 1 \) and noting that \( B^{1}_{22} = H^1 \), we obtain the result by Korn for the above mentioned special equations of state. This paper is organized as follows. Section 2 presents preliminaries on the approach to the primitive equations by evolution equations; in Sect. 3 we discuss typical examples of densities arising in ocean dynamics. Our main result of strong, global well-posedness of the primitive equations subject to equations of state of the form (1.3) is presented in Sect. 4. In Sect. 5 we give estimates needed for the proof of the local existence, whereas in Sect. 6 we prove a priori estimates in the situation of equations of state satisfying (1.3). Local and global well-posedness for the system (1.2) are proved in Sect. 7. Finally, we summarize results on semilinear evolution equations needed for our proof in the Appendix A.

2. Preliminaries

Let \( \Omega = G \times (-h, 0) \) a cylindrical domain with \( G = (0, 1)^2 \) and \( h > 0 \). The velocity \( u \) of the fluid is given by \( u = (v, w) \) with \( v = (v_1, v_2) \), where \( v \) and \( w \) denote the horizontal and vertical components of \( u \), respectively. Furthermore, \( \tau, \sigma \) and \( \pi \) denote the temperature, the salinity and the pressure of the fluid, respectively. The primitive equations are given by
We assume the following Assumption (A) of the boundary JMFM Primitive Equations with Nonlinear Equation of State Page 3 of 18

\[\begin{align*}
\partial_t v - \Delta v + (u \cdot \nabla)v + \nabla_H \pi &= f_v \quad \text{in } \Omega \times (0, T), \\
\nabla(\nabla_h) &= 0 \quad \text{in } \Omega \times (0, T), \\
\partial_t \tau - \Delta \tau + (u \cdot \nabla)\tau &= f_\tau \quad \text{in } \Omega \times (0, T), \\
\partial_t \sigma - \Delta \sigma + (u \cdot \nabla)\sigma &= f_\sigma \quad \text{in } \Omega \times (0, T), \\
\partial_z \pi(x, y, z) + g\rho(z, \tau, \sigma) &= 0 \quad \text{in } \Omega \times (0, T),
\end{align*}\]

with initial values \(v(0) = v_0, \tau(0) = \tau_0\) and \(\sigma(0) = \sigma_0\). We equip this system with the boundary conditions

\[
\begin{align*}
\partial_z v &= 0, & w &= 0, & \partial_z \tau + a\tau &= 0, & \partial_z \sigma &= 0 \quad \text{on } \Gamma_u \times (0, T), \\
v &= 0, & w &= 0, & \partial_z \tau &= 0, & \partial_z \sigma &= 0 \quad \text{on } \Gamma_b \times (0, T), \\
u, \pi, \tau, \sigma \text{ are periodic} & \quad \text{on } \Gamma_1 \times (0, T),
\end{align*}
\]

where \(\Gamma_u := G \times \{0\}, \Gamma_b := G \times \{-h\}, \Gamma_1 := \partial G \times [-h, 0]\) denote the upper, bottom and lateral parts of the boundary \(\partial \Omega\), respectively. Here \(g \in \mathbb{R}\) and \(a > 0\) are constants and \(\rho\) is a function satisfying the following Assumption (A).

**Assumption.** (A) We assume

\[
\rho \in L^\infty_\Omega((-h, 0), W^{2, \infty}_\Omega(K_1, K_2) \times (L_1, L_2), \mathbb{R}))
\]

for some \(-\infty \leq K_1, K_2, L_1, L_2 \leq \infty\) provided \(\tau(t) \in (K_1, K_2)\) and \(\sigma(t) \in (L_1, L_2)\) for all \(t \in (0, T)\).

**Remark 2.1.** In the particular case of \(K_1, L_1 = -\infty\) and \(K_2, L_2 = +\infty\) we obtain Assumption (A) reads as \(\rho \in L^\infty_\Omega((-h, 0), W^{2, \infty}(\mathbb{R}^2, \mathbb{R}))\).

The equation of state \(\partial_t \pi(x, y, z) + g\rho(z, \tau, \sigma) = 0\) yields

\[
\nabla_H \pi(\cdot, \cdot, z) = \nabla_H \pi(\cdot, \cdot, -h) + \nabla_H \int_{-h}^{z} \partial_z \pi(\cdot, \cdot, \xi) \, d\xi
\]

\[
= \nabla_H \pi_s - \nabla_H \int_{-h}^{z} g\rho(\xi, \tau(\cdot, \cdot, \xi), \sigma(\cdot, \cdot, \xi)) \, d\xi
\]

\[
= \nabla_H \pi_s - \Pi(\tau, \sigma),
\]

where \(\pi_s(x, y) := \pi(x, y, -h)\) and

\[
\Pi(\tau, \sigma) := \nabla_H \int_{-h}^{z} g\rho(\xi, \tau(\cdot, \cdot, \xi), \sigma(\cdot, \cdot, \xi)) \, d\xi.
\]

Using this and \(w = 0\) on \(\Gamma_b\) we rewrite (2.1) and (2.2) as

\[
\begin{align*}
\partial_t v - \Delta^\nu v + (v \cdot \nabla_H)v + v(\cdot, \cdot, \xi) \, d\xi & = f_v + \Pi(\tau, \sigma) \quad \text{in } \Omega \times (0, T), \\
\nabla(\nabla^\nu) &= 0 \quad \text{in } \Omega \times (0, T), \\
\partial_t \tau - \Delta^\tau \tau + (v \cdot \nabla_H)\tau + w(\cdot, \cdot, \xi) \, d\xi & = f_\tau \quad \text{in } \Omega \times (0, T), \\
\partial_t \sigma - \Delta^\sigma \sigma + (v \cdot \nabla_H)\sigma + w(\cdot, \cdot, \xi) \, d\xi & = f_\sigma \quad \text{in } \Omega \times (0, T), \\
\partial_z \pi & = 0 \quad \text{in } \Omega \times (0, T),
\end{align*}
\]

where

\[
\text{div}_H(v) := \partial_x v_1 + \partial_y v_2, \quad \bar{v} := \frac{1}{h} \int_{-h}^{0} v(\cdot, \cdot, \xi) \, d\xi, \quad w(v) := -\int_{-h}^{z} \text{div}_H(v(\cdot, \cdot, \xi)) \, d\xi,
\]

and

\[
\Delta^\nu v := \Delta v, \quad D(\Delta^\nu) := \{v \in W^{2,q}_\text{per}(\Omega): (\partial_z v)|\Gamma_u = 0, v|\Gamma_b = 0\},
\]

\[
\Delta^\tau \tau := \Delta \tau, \quad D(\Delta^\tau) := \{\tau \in W^{2,q}_\text{per}(\Omega): (\partial_z \tau + a\tau)|\Gamma_u = 0, (\partial_z \tau)|\Gamma_b = 0\},
\]

\[
\Delta^\sigma \sigma := \Delta \sigma, \quad D(\Delta^\sigma) := \{\sigma \in W^{2,q}_\text{per}(\Omega): (\partial_z \sigma)|\Gamma_u = 0, (\partial_z \sigma)|\Gamma_b = 0\}.
\]
Here horizontal periodicity is modelled by the closure
\[
W^2,q_{\text{per}}(\Omega) := \frac{C^\infty_{\text{per}}(\Omega)}{\| \cdot \|_{W^2,q(\Omega)}}
\]
of the functions spaces \(C^\infty_{\text{per}}(\Omega)\) and \(C^\infty_{\text{per}}(G)\) as defined in [8, Section 2], where periodicity is only with respect to \(x, y\) coordinates and not necessary in the \(z\) coordinate. To simplify our notation we use \(\zeta := (\tau, \sigma)\), \(\Delta^\zeta := \Delta^\tau \otimes \Delta^\sigma\), \(F(v, \psi) := (v \cdot \nabla_H)\psi + w(v) \partial_z \psi\) and \(f_\zeta := (f_r, f_\sigma)\) and obtain
\[
\begin{align*}
\partial_t v - \Delta^\tau v + F(v, \psi) + \nabla_H \pi_s &= f_v + \Pi(\zeta) \quad \text{in } \Omega \times (0, T), \\
\nabla_H(\bar{v}) &= 0 \quad \text{in } \Omega \times (0, T), \\
\partial_t \zeta - \Delta^\zeta \zeta + F(v, \bar{\zeta}) &= f_\zeta \quad \text{in } \Omega \times (0, T), \\
\partial_z \pi_s &= 0 \quad \text{in } \Omega \times (0, T).
\end{align*}
\tag{2.4}
\]

The hydrostatically solenoidal vector fields introduced in [8] are considered as a subspace of \(L^q(\Omega)^2\) for \(q \in (1, \infty)\) and are defined by
\[
L^q_s(\Omega) = \{ v \in C^\infty_{\text{per}}(\Omega) : \text{div}_H u = 0 \} \| \cdot \|_{L^q(\Omega)^2}.
\]

For \(q \in (1, \infty)\) and \(s \in [0, \infty)\) we define the Bessel potential spaces
\[
W^{s,q}_{\text{per}}(\Omega) := \frac{C^\infty_{\text{per}}(\Omega)}{\| \cdot \|_{W^{s,q}(\Omega)}}
\]
and \(W^{0,q}_{\text{per}} := L^q\), where \(W^{s,q}(\Omega)\) denotes the Bessel potential spaces, see e.g. [21, Definition 3.2.2]. It is well known that for \(s = m \in \mathbb{N}\) the Bessel spaces coincide with the classical Sobolev spaces of order \(m\). Further, for \(p, q \in (1, \infty)\) and \(s \in [0, \infty)\) we define the Besov spaces
\[
B^{s,q}_{qp,\text{per}}(\Omega) := \frac{C^\infty_{\text{per}}(\Omega)}{\| \cdot \|_{B^{s,q}_{qp}(\Omega)}}
\]
where \(B^{s,q}_{qp}(\Omega)\) denotes the Besov spaces, see e.g. [21, Definition 3.2.2]. The hydrostatic Helmholtz projection \(P_q\) is a continuous projection from \(L^q(\Omega)^2\) onto \(L^q_s(\Omega)\), see [8] or [5]. Note that it annihilates the pressure term \(\nabla_H \pi_s\). Applying it to the first line of (2.4) we obtain
\[
\begin{align*}
\partial_t v - A^v_q v + P_q F(v, v) &= P_q (f_v + \Pi(\zeta)) \quad \text{in } \Omega \times (0, T), \\
\partial_t \zeta - \Delta^\zeta \zeta + f_\zeta &= f_\zeta \quad \text{in } \Omega \times (0, T),
\end{align*}
\tag{2.5}
\]
on \(L^q_s(\Omega) \times L^q(\Omega)^2\). Here the hydrostatic Stokes operator \(A^v_q : D(A^v_q) \subset L^q_s(\Omega) \to L^q_s(\Omega)\) is given by
\[
A^v_q v := P_q \Delta^v_q v, \quad D(A^v_q) := D(\Delta^v_q) \cap L^q_s(\Omega).
\tag{2.6}
\]
The Eq. (2.5) can be thus rewritten as
\[
\partial_t \left( \begin{array}{c} v \\ \zeta \end{array} \right) - A_q \left( \begin{array}{c} v \\ \zeta \end{array} \right) + \left( \begin{array}{c} P_q F(v, v) \\ F(v, \zeta) \end{array} \right) = \left( \begin{array}{c} P_q (f_v + \Pi(\zeta)) \\ f_\zeta \end{array} \right) \quad \text{in } \Omega \times (0, T)
\tag{2.7}
\]
on \(L^q_s(\Omega) \times L^q(\Omega)^2\), where \(A_q : D(A_q) \subset L^q_s(\Omega) \times L^q(\Omega)^2 \to L^q_s(\Omega) \times L^q(\Omega)^2\) is given by
\[
A_q = \left( \begin{array}{cc} A^v_q & 0 \\ 0 & \Delta^\zeta_q \end{array} \right), \quad D(A_q) := D(A^v_q) \times D(\Delta^\zeta_q).
\tag{2.8}
\]
In [3] it is shown that the Laplacian subject to the boundary condition described above has the maximal \(L^p\)-regularity on \(X^\gamma_{\partial} := X^\gamma_{\partial} := L^q\). We set \(X^\gamma_0 := X^\gamma_0 \times X^\gamma_{\partial}\). Further, it was shown in in [7] that \(A^v_q\) has maximal \(L^p\)-regularity on \(X^\gamma_0 := L^q_s(\Omega)\). This means that
\[
\left( \begin{array}{c} \partial_t - A^v_q \\ \gamma_0 \end{array} \right) : E^\gamma_0((0, T) \to E^\gamma_0((0, T) \times X^{\gamma-\frac{1}{p},p})
\]
is an isomorphism, where \(\gamma_0 := v(0)\) denotes the trace operator. Here
\[
E^\gamma_0((0, T) := L^p((0, T) \times X^{\gamma-\frac{1}{p},p})
\]
and the space of maximal regularity is given by
\[ E_1^\eta(t, T) := W^{1,p}((0, T), X_0^\eta) \cap L^p((0, T), X_1^\eta). \]
for \( \eta \in \{v, \tau, \sigma, \zeta\} \), where \( X_1^\eta \) denotes the domain of the operator involved. Both spaces are Banach spaces equipped with its natural norms. Further, the trace space is given by \( X_{\partial, p}^\eta := (X_1^\eta, X_0^\eta)_{\theta, p} \), where \( (\cdot, \cdot)_{\theta, p} \) denotes the real interpolation functor for \( \theta \in (0, 1) \) and \( q \in (1, \infty) \) and \( \eta \in \{v, \tau, \sigma, \zeta\} \). Further, we denote the product space \( X_{\partial, p} := X_{\partial, p}^v \times X_{\partial, p}^\tau \). For more information on maximal regularity and the Navier–Stokes equations, we refer to [3,16,19]. Following [16, Theorem 3.5.4] the maximal regularity is equivalent within the real interpolation space
\[ t^{1-\mu}v \in L^p((0, T), T^0), \]
for \( \eta \in \{v, \tau, \sigma, \zeta\} \) and analogous for \( X_1^\eta \) instead of \( X_0^\eta \). Now, analogously to \( E_1^\eta(t, T) \) we define the space \( E_{1, \mu}^\eta(t, T) \) by
\[ E_{1, \mu}^\eta(t, T) := H^{1,p}((0, T), X_0^\eta) \cap L^p((0, T), X_1^\eta) \]
for \( \eta \in \{v, \tau, \sigma, \zeta\} \). Further, we define \( E_{0, \mu} := L_\mu^p((0, T), X_0^\eta) \) for \( \eta \in \{v, \tau, \sigma, \zeta\} \) and consider initial data within the real interpolation space
\[ X_{\gamma, \mu}^\eta := (X_0^\eta, X_1^\eta)_{\mu-1/p, p} \]
for \( \eta \in \{v, \tau, \sigma, \zeta\} \).

The trace spaces can be computed explicitly in terms of Besov spaces.

**Lemma 2.2.** (cf. [6, Lemma 2.1]) Let \( p, q \in (1, \infty) \). The trace spaces are given by

\[
X_{\partial, p}^\eta = \begin{cases} 
\{ v \in \text{B}_{q,p}^{2q}(\Omega) \cap L_1^q(\Omega) : \partial_z v|_{\Gamma_u} = 0, v|_{\Gamma_b} = 0 \}, & 1/2 + 1/2q < \theta < 1, \\
\text{B}_{q,p}^{2q}(\Omega), & 0 < \theta < 1/2q, \\
\{ \tau \in \text{B}_{q,p}^{2q}(\Omega) : (\partial_z \tau + a\tau)|_{\Gamma_u} = 0, \partial_z \tau|_{\Gamma_b} = 0 \}, & \tau \in \frac{1}{2} + \frac{1}{2q} < \theta < 1, \\
\text{B}_{q,p}^{2q}(\Omega), & 0 < \theta < \frac{1}{2} + \frac{1}{2q}.
\end{cases}
\]

\[
X_{\partial, p}^\sigma = \begin{cases} 
\{ \sigma \in \text{B}_{q,p}^{2q}(\Omega) : \partial_z \sigma|_{\Gamma_u} = 0, \partial_z \sigma|_{\Gamma_b} = 0 \}, & \tau \in \frac{1}{2} + \frac{1}{2q} < \theta < 1, \\
\text{B}_{q,p}^{2q}(\Omega), & 0 < \theta < \frac{1}{2} + \frac{1}{2q}.
\end{cases}
\]

### 3. Typical Ocean Densities

The set of equations (2.1) couples the velocity of the fluid to two advection-diffusion equations for the temperature and the salinity. It is closed by the equation of state expressing the density as a function of temperature, salinity and the pressure. In contrast to atmospheric dynamics where the equation of state can be expressed in the simplest case by the ideal gas law, the oceanic equation of state lacks a rigorous derivation. The latter needs to represent measurements of \( \tau, \sigma, \pi \) and should also be thermodynamically
consistent. In the TEOS formulation of the equation of state (see [20]), thermodynamical consistency is implemented by the introduction of a certain Gibbs-potential. We use thus the TEOS formulation of the equation of state and variants hereof as our main examples for the equations of state satisfying Assumption (A).

3.1. Linear Density

The density \( \rho \) considered in [7, 11] is linear and given by

\[
\rho(z, a, b) := 1 - \beta_r(a - 1) + \beta_r(b - 1)
\]

with \( \beta_r, \beta_s > 0 \). Obviously, we obtain \( \rho \in L^\infty_z((-h, 0), W^{2, \infty}(\mathbb{R} \times \mathbb{R})) \) and Assumption (A) is satisfied by Remark 2.1.

Remark 3.1. We point out that the density \( \rho \) above can be replaced by any smooth function.

3.2. Equation of State by TEOS-10

The equation of state TEOS-10 for the description of seawater by the Intergovernmental Oceanographic Commission is today the most accurate equation of state with respect to measurements and it rests on the Gibbs formalism of thermodynamics, see [20]. In this formulation, the density \( \rho \) is given by

\[
\rho(z, a, b) := \frac{1}{Q(z, a, b)} \quad \text{with} \quad Q(z, a, b) := \sum_{i,j,k=1}^{N,M,L} c_{ijk} z^i a^j b^k
\]

and such that there are constants \( c, C > 0 \) with

\[
c \leq |\rho(z, a, b)| \leq C
\]

for \( z \in (-h, 0), a \in [K_1, K_2] \) and \( b \in [L_1, L_2] \), see also [10, (20)]. Note that \( h: \mathbb{C} \setminus \{0\} \to \mathbb{C}, z \mapsto z^{-1} \) is smooth. Since \( \rho = h \circ Q \) is bounded on \((-h, 0) \times (K_1, K_2) \times (L_1, L_2)\), it follows that \( Q \) does not vanish on \((-h, 0) \times (K_1, K_2) \times (L_1, L_2)\). Further, since \( Q \) is a polynomial it is smooth and we obtain that \( \rho = h \circ Q \) is smooth. In particular \( \rho \in L^\infty_z((-h, 0), W^{2, \infty}((K_1, K_2) \times (L_1, L_2), \mathbb{R})) \) and Assumption (A) is satisfied.

Remark 3.2. The polynomial \( Q \) can be replaced by any \( L^\infty_z((-h, 0), W^{2, \infty}((K_1, K_2) \times (L_1, L_2), \mathbb{R})) \)-function which vanishes only on a Lebesgue-Null set \( U \subset (-h, 0) \times (K_1, K_2) \times (L_1, L_2) \).

3.3. Equation of State by McDongall–Jacket–Wright–Feistel

Another example of an equation of state of interest is the so-called McDongall–Jacket–Wright–Feistel equation, see [15]. In this case the density \( \rho \) is given by

\[
\rho(z, a, b) := \frac{Q_1(z, a, b)}{Q_2(z, a, b)},
\]

where \( Q_1, Q_2 \) are polynomials such that there are constants \( c, C > 0 \) with

\[
c \leq |\rho(z, a, b)| \leq C
\]

for \( z \in (-h, 0), a \in [K_1, K_2] \) and \( b \in [L_1, L_2] \), see also [10, (20)]. Denote again by \( h: \mathbb{C} \setminus \{0\} \to \mathbb{C} \) the inversion. As in the last subsection we conclude that \( Q_2 \) does not vanish on \((-h, 0) \times (K_1, K_2) \times (L_1, L_2)\). Since \( Q_1 \) and \( Q_2 \) are polynomials and hence smooth it follows that \( \rho = Q_1 \cdot (h \circ Q_2) \) is smooth. In particular, \( \rho \in L^\infty_z((-h, 0), W^{2, \infty}((K_1, K_2) \times (L_1, L_2), \mathbb{R})) \) and Assumption (A) is satisfied.
3.4. Equation of State by UNESCO-80

The UNESCO-80 equation of state is a classical equation of state, see [22]. The density $\rho$ is given in this case by

$$\rho(z, a, b) := \frac{Q(a, b)}{1 - \frac{cz}{K(z, a, b)}},$$

where $Q$ and $K$ are polynomials and such that there are constants $c, C > 0$ satisfying

$$c \leq |\rho(z, a, b)| \leq C$$

for $z \in (-h, 0)$, $a \in [K_1, K_2]$ and $b \in [L_1, L_2]$. The precise form of the polynomials are formulated e.g. in [10, (23)] and in [10, (25)]. Denote again by $h: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ the inversion. As in the last subsection we conclude that $1 - \frac{cz}{K(z, a, b)}$ does not vanish on $(-h, 0) \times (K_1, K_2) \times (L_1, L_2)$. Since $Q$ and $K$ are polynomials and hence smooth and the inversion $h$ is smooth it follows that $\rho$ is smooth. In particular, $\rho \in L^\infty((-h, 0), W^{2, \infty}((K_1, K_2) \times (L_1, L_2), \mathbb{R}))$ and Assumption (A) is satisfied.

4. Main Result

The following theorem is the main result of this article. We set $X_{\mu - \frac{1}{p}, p} := X_\mu^v \times X_\mu^\sigma$.

**Theorem 4.1.** Let $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$, $\mu \in [\frac{1}{p} + \frac{1}{q}, 1]$ and $T \in (0, \infty)$. Assume Assumption (A) and that

\[
\begin{align*}
(v_0, \tau_0, \sigma_0)^T &\in X_{\mu - \frac{1}{p}, p}, \\
f_v, f_\tau, f_\sigma &\in H^{1,p}_\mu((0, T), L^q(\Omega)) \cap H^{1,2}(\delta, T), L^2(\Omega))
\end{align*}
\]

for some $\delta > 0$ sufficiently small. Then there exists a unique, strong global solution

\[
(v, \tau, \sigma)^T \in H^{1,p}_\mu((0, T), L^q_\mu(\Omega) \times L^q(\Omega)^2) \cap L^p((0, T), D(A_q))
\]

to the primitive equations (2.1) subject to (2.2).

**Remark 4.2.** (a) As in [8, (6.2)] the surface pressure $\pi_s$ can be recovered from the velocity $v$ by

$$\nabla_H \pi_s := (1 - P_p)(f + \Pi(\zeta) - (\Delta v + v\nabla v + w\partial_z v)).$$

It follows $\pi_s \in L^p_\mu((0, T), W^{1,q}_{\text{per}, 0}(G))$, where $W^{1,q}_{\text{per}, 0}(G) = \{ \pi \in W^{1,q}_{\text{per}, 0}(G): \int_G \pi = 0 \}$.

(b) The above solution satisfies belongs to the regularity classes

\[
W^{1,p}_\mu((0, T), L^q_\mu(\Omega) \times L^q(\Omega)^2) \cap L^p_\mu((0, T), D(A_q)) \hookrightarrow C([0, T], X_{\mu - \frac{1}{p}, p}),
\]

\[
W^{1,p}_\mu((0, T), L^q_\mu(\Omega) \times L^q(\Omega)^2) \cap L^p_\mu((0, T), D(A_q)) \hookrightarrow C((\delta, T), X_{1 - \frac{1}{p}, p})
\]

for $\delta > 0$.

(c) For the existence of a local solution it is sufficient that $f \in L^p((0, T), L^q(\Omega, \mathbb{R}^3))$, see Proposition 7.1.

(d) Suppose the right hand sides of (2.1) satisfy $f_v \in W^{1,p}_\mu((0, T), L^q_\mu(\Omega)) \cap W^{1,2}(\delta, T), L^2(\Omega))$, $f_\tau \in W^{1,p}_\mu((0, T), L^q_\mu(\Omega)) \cap W^{1,2}(\delta, T), L^2(\Omega))$, $f_\sigma \in W^{1,p}_\mu((0, T), L^q_\mu(\Omega)) \cap W^{1,2}(\delta, T), L^2(\Omega))$ for sufficient small $\delta > 0$, the solution satisfies

\[
\begin{align*}
\begin{pmatrix} v \\ \tau \\ \sigma \end{pmatrix} \in W^{1,p}_\mu((0, T), L^q_\mu(\Omega) \times L^q_\mu(\Omega) \times L^q_\mu(\Omega)) \\
\cap L^p((0, T), D(A_q^v) \times D(\Delta_q^\tau) \times D(\Delta_q^\sigma)).
\end{align*}
\]
Lemma 5.1. Let $\Delta^v$ be replaced by $L^v$ given by

$$L^v v = \frac{1}{R e_1} \Delta_H v + \frac{1}{R e_2} \partial^2 z v + f \vec{k} \times v,$$

where $R e_1, R e_2 > 0$ denotes the horizontal and vertical Reynolds number, $f$ the Coriolis parameter and $\vec{k} \in \mathbb{R}^3$. Further, $\Delta^\tau$ and $\Delta^\sigma$ can be replaced by $L^\tau$ and $L^\sigma$ given by

$$L^\tau \tau = \frac{1}{R t_1} \Delta_H \tau + \frac{1}{R t_2} \partial^2 z \tau,$$

$$L^\sigma \sigma = \frac{1}{R s_1} \Delta_H \sigma + \frac{1}{R s_2} \partial^2 z \sigma,$$

where $R t_1, R t_2 > 0$ and $R s_1, R s_2 > 0$ denotes the horizontal and vertical mixing coefficients for the temperature and salinity.

Lemma 5.2. Let $q \in (1, \infty)$. Then the operator $-A_q + \lambda$ admits a bounded $H^\infty$-calculus on $L^q_0(\Omega) \times L^q(\Omega)^2$ with $\phi_\lambda^\infty = 0$.

**Proof.** By [5, Theorem 3.1 & Lemma 4.1] the hydrostatic Stokes operator $-A_q^v + \lambda$ has a bounded $H^\infty$-calculus for $\lambda > 0$ on $L^q_0(\Omega) \times L^q(\Omega)^2$ with $\phi_\lambda^\infty = 0$.

The boundary conditions for $v, \tau, \sigma$ on $\Gamma_u$ and $\Gamma_b$ can be replaced by Dirichlet, Neumann or Robin boundary conditions with positive coefficients. Note that by [5, Theorem 3.1 & Lemma 4.1] the corresponding operators $-A^v + \lambda, -\Delta^\tau + \lambda, -\Delta^\sigma + \lambda$ for $\lambda > 0$ have bounded $H^\infty$-calculus on $L^q_0(\Omega)$ and $L^q(\Omega)$, respectively. Further, the a priori bounds can be obtained as the bounds for $\tau$ in Lemma 6.1 and Lemma 6.3.

5. Estimates for the Local Existence

We start this section by showing that the operator $-A_q + \lambda$ admits a bounded $H^\infty$-calculus on $L^q_0(\Omega) \times L^q(\Omega)^2$.

**Lemma 5.1.** Let $q \in (1, \infty)$. Then the operator $-A_q + \lambda$ has a bounded $H^\infty$-calculus for $\lambda > 0$ on $L^q_0(\Omega) \times L^q(\Omega)^2$ with $\phi_\lambda^\infty = 0$.

**Proof.** By [5, Theorem 3.1] the hydrostatic Stokes operator $-A_q^v + \lambda$ has a bounded $H^\infty$-calculus for $\lambda > 0$ with $\phi_\lambda^\infty = 0$ on $L^q_0(\Omega)$. Furthermore, by [5, Lemma 4.1] the Laplacians $-\Delta_q^\tau + \lambda$ and $-\Delta_q^\sigma + \lambda$ for $\lambda > 0$ has bounded $H^\infty$-calculus with $\phi_\lambda^\infty = 0$ on $L^q(\Omega)$. We conclude that

$$-A_q + \lambda = \left( \begin{array}{ccc} -A_q^v + \lambda & 0 & 0 \\ 0 & -\Delta_q^\tau + \lambda & 0 \\ 0 & 0 & -\Delta_q^\sigma + \lambda \end{array} \right),$$

$$D(A_q) = D(A_q^v) \times D(\Delta_q^\tau) \times D(\Delta_q^\sigma)$$

for $\lambda > 0$ has a bounded $H^\infty$-calculus with on $L^q_0(\Omega) \times L^q(\Omega)^2$ with $\phi_\lambda^\infty = 0$.

We continue by providing properties of the nonlinearity $F$ and $\Pi$ given by

$$F(v, h) := v \cdot \nabla_H h + w(v) \partial_z h,$$

$$\Pi(\zeta) := \nabla_H \int_{-h}^z g \rho(\xi, \zeta(\cdot, \cdot, \xi)) d\xi.$$

The following lemma is crucial for our approach. It yields $L^q$-estimates and Lipschitz type estimates for the right hand side.

**Lemma 5.2.** Let $q \in (1, \infty)$ and assume Assumption (A). Set $\partial_2 := (\partial_{a_b})$ and $\partial_2^2 := \left( \begin{array}{c} \partial^2_{a_b} \\ \partial_{a_b} \partial_{b_a} \end{array} \right)$. Then

(i) there exists a constant $C > 0$, depending on $\|\partial_2 \rho\|_{L^\infty(\Omega)}$, such that

$$\|\Pi(\zeta)\|_{L^q(\Omega)} \leq C \cdot \|\nabla \zeta\|_{L^q(\Omega)}$$

for $\zeta \in W^{1,q}(\Omega)^2$. 
(ii) there exists a constant $C > 0$, depending on $\|\partial_2\rho\|_{L^\infty(\Omega)}$, $\|\partial_2^2\rho\|_{L^\infty(\Omega)}$, such that

$$\|\Pi(\zeta_1) - \Pi(\zeta_2)\|_{L^q(\Omega)} \leq C (1 + \|\zeta_1\|_{W^{1+\frac{4}{q}}(\Omega)} + \|\zeta_2\|_{W^{1+\frac{4}{q}}(\Omega)}) \cdot \|\zeta_1 - \zeta_2\|_{W^{1+\frac{4}{q}}(\Omega)}$$

for $\zeta_1, \zeta_2 \in W^{1+\frac{4}{q}}(\Omega)^2$.

**Proof.** The proof is similar to the proof of [7, Lemma 5.1]. We first note that

$$\Pi(\zeta) = \nabla_H \int_{-h}^z g\rho(\xi, \zeta(\cdot, \cdot, \xi)) d\xi = g \int_{-h}^z \nabla_H \rho(\xi, \zeta(\cdot, \cdot, \xi)) d\xi.$$ 

(i) Using Jensen’s inequality we obtain

$$\|\Pi(\zeta)\|_{L^q(\Omega)} \leq |g|^q \int_{\Omega} \left( \int_{-h}^z |\nabla_H \rho(\xi, \zeta(\cdot, \cdot, \xi))| d\xi \right)^q \leq |g|^q \cdot H^q \int_{\Omega} \int_{-h}^z |\nabla_H \rho(\xi, \zeta(\cdot, \cdot, \xi))|^q d\xi \leq |g|^q \cdot H^q \int_{\Omega} |\nabla_H \rho(\xi, \zeta(\cdot, \cdot, \xi))|^q = C \cdot \|((\partial_2\rho)(\xi, \zeta(\cdot, \cdot, \xi))) \cdot \nabla_H \zeta\|_{L^q(\Omega)}^q \leq C \cdot \|\partial_2\rho\|_{L^\infty(\Omega)} \cdot \|\nabla_H \zeta\|_{L^q(\Omega)}^q.$$ 

(ii) By the same arguments as in part (i) we obtain

$$\|\Pi(\zeta_1) - \Pi(\zeta_2)\|_{L^q(\Omega)}^q \leq C \cdot \|\nabla_H \rho(\xi, \zeta_1(\cdot, \cdot, \xi)) - \nabla_H \rho(\xi, \zeta_2(\cdot, \cdot, \xi))\|_{L^q(\Omega)}^q = C \cdot \|((\partial_2\rho)(\xi, \zeta_1(\cdot, \cdot, \xi))) \cdot \nabla_H \zeta_1 - (\partial_2\rho)(\xi, \zeta_2(\cdot, \cdot, \xi)) \cdot \nabla_H \zeta_2\|_{L^q(\Omega)}^q \leq C \cdot \|\partial_2\rho\|_{L^\infty(\Omega)} \cdot \|\nabla_H \zeta_1 - \nabla_H \zeta_2\|_{L^q(\Omega)}^q + C \cdot \|((\partial_2\rho)(\xi, \zeta_1(\cdot, \cdot, \xi))) \cdot \nabla_H \zeta_2\|_{L^q(\Omega)}^q \leq C \cdot \|\partial_2\rho\|_{L^\infty(\Omega)} \cdot \|\zeta_1 - \zeta_2\|_{W^{1,\frac{4}{q}}(\Omega)} + C \cdot \|((\partial_2\rho)(\xi, \zeta_1(\cdot, \cdot, \xi))) \cdot \nabla_H \zeta_2\|_{L^q(\Omega)}^q.$$ 

In order to bound the second term, let $r = \frac{3q}{2}$ and $p = 3q$ and note that

$$\frac{1}{p} + \frac{1}{r} = \frac{1}{3q} + \frac{2}{3q} = \frac{1}{q}.$$ 

Hence, Hölder’s inequality and the mean value theorem imply

$$\|((\partial_2\rho)(\xi, \zeta_1(\cdot, \cdot, \xi))) \cdot \nabla_H \zeta_2\|_{L^q(\Omega)} \leq \|((\partial_2\rho)(\xi, \zeta_1(\cdot, \cdot, \xi))) \cdot (\partial_2\rho)(\xi, \zeta_2(\cdot, \cdot, \xi))\|_{L^p(\Omega)} \cdot \|\nabla_H \zeta_2\|_{L^r(\Omega)} \leq \|\partial_2^2\rho\|_{L^\infty(\Omega)} \cdot \|\zeta_1 - \zeta_2\|_{L^p(\Omega)} \cdot \|\zeta_2\|_{W^{1,\frac{4}{q}}(\Omega)} \leq \|\partial_2^2\rho\|_{L^\infty(\Omega)} \cdot \|\zeta_1 - \zeta_2\|_{W^{1,\frac{4}{q}}(\Omega)} \cdot \|\zeta_2\|_{W^{1,\frac{4}{q}}(\Omega)}. $$

The embeddings $W^{1+\frac{4}{q}}(\Omega) \hookrightarrow L^{3q}(\Omega)$ and $W^{1+\frac{4}{q}}(\Omega) \hookrightarrow W^{1,\frac{4}{q}}(\Omega)$ yield

$$\|((\partial_2\rho)(\xi, \zeta_1(\cdot, \cdot, \xi))) - (\partial_2\rho)(\xi, \zeta_2(\cdot, \cdot, \xi))\|_{L^q(\Omega)} \leq \|\partial_2^2\rho\|_{L^\infty(\Omega)} \cdot \|\zeta_1 - \zeta_2\|_{W^{1+\frac{4}{q}}(\Omega)} \cdot \|\zeta_2\|_{W^{1+\frac{4}{q}}(\Omega)}.$$ 

Combining these estimates we obtain

$$\|\Pi(\zeta_1) - \Pi(\zeta_2)\|_{L^q(\Omega)} \leq C \cdot \|\partial_2^2\rho\|_{L^\infty(\Omega)} \cdot \|\zeta_1 - \zeta_2\|_{W^{1+\frac{4}{q}}(\Omega)} \cdot (1 + \|\zeta_2\|_{W^{1+\frac{4}{q}}(\Omega)})^\frac{1}{q}.$$
Lemma 6.1. \[ p \text{ we assume that } \]

6. A Priori Estimates

We start with the estimate for \( y \) yields

The results given in [8, Lemma 5.1], [6, Lemma 6.1] or [4, Lemma 5.1] imply then the following estimates.

Lemma 5.3. Let \( q \in (1, \infty) \) and \( F(v, \psi) := (v \cdot \nabla H) \psi + w(v) \partial_2 \psi \).

(i) Then \( F: W^{1+\frac{1}{q}}(\Omega) \times W^{1+\frac{1}{q}}(\Omega) \to L^q(\Omega) \) is a continuous bilinear map, i.e. there exists a constant \( C > 0 \), depending only on \( \Omega \) and \( q \), such that

\[
\| F(v, \psi) \|_{L^q(\Omega)} \leq C \cdot \| v \|_{W^{1+\frac{1}{q}}(\Omega)} \cdot \| \psi \|_{W^{1+\frac{1}{q}}(\Omega)}
\]

for \( v, \psi \in W^{1+\frac{1}{q}}(\Omega) \).

(ii) If \( \beta = \frac{1}{2}(1 + \frac{1}{q}) \), then there exists a constant \( C > 0 \) such that

\[
\| F(v, v) - F(v', v') \|_{L^q(\Omega)} \leq C \cdot (\| v \|_{X_\beta} + \| v' \|_{X_\beta}) \cdot \| v - v' \|_{X_\beta}.
\]

(iii) If \( v, \psi \in W^{2,q}(\Omega) \), then

\[
\| F(v, \psi) \|_{L^q(\Omega)} \leq C \cdot \| v \|_{W^{2,q}(\Omega)} \cdot \| \psi \|_{W^{1+\frac{1}{q}}(\Omega)} \cdot \| \psi \|_{W^{1,\frac{1}{2}}(\Omega)}.
\]

6. A Priori Estimates

In this section we prove a priori estimates in the space of maximal regularity. Throughout this section we assume that \( p = q = 2 \). We start with energy estimates.

Lemma 6.1. Assume Assumption (A) and let \( v_0 \in L^2(\Omega)^2, \zeta_0 \in L^2(\Omega)^2 \) and \( f_v, f_\zeta \in L^2((0, T), L^2(\Omega)) \).

Further, let \( (v, \zeta) \in E_1^\Omega(0, T) \times E_1^\Omega(0, T) \) be a solution of (2.5).

Then there exists constants \( C_\zeta, C_v > 0 \), depending on \( \| \zeta_0 \|_{L^2(\Omega)}, \| f_\zeta \|_{L^2((0, T), L^2(\Omega))} \) and \( \| v_0 \|_{L^2(\Omega)}, \| f_v \|_{L^2((0, T), L^2(\Omega))} \), respectively, such that

\[
\frac{1}{2} \| \zeta(t) \|_{L^2(\Omega)}^2 + \int_0^t \| \nabla \zeta(s) \|_{L^2(\Omega)}^2 \, ds + a \int_0^t \| \nabla \zeta(s) \|_{L^2(\Omega)}^2 \, ds \leq C_\zeta \cdot e^t =: B_\zeta^L(t),
\]

\[
\frac{1}{2} \| v(t) \|_{L^2(\Omega)}^2 + \int_0^t \| \nabla v(s) \|_{L^2(\Omega)}^2 \, ds \leq C_v \cdot e^t + C_\zeta \cdot (e^{2t} + e^t - 1) =: B_v^L(t).
\]

Proof. The proof is divided into two steps. It extends [7, Step 1 & 2, Proof of Lemma 6.1] to the present situation.

Step 1 [\( L^\infty((0, T), L^2(\Omega)) \) and \( L^2((0, T), W^{1,2}(\Omega)) \)-bounds of temperature and salinity]

We start with the estimate for \( \zeta \). Multiplying the second equation in (2.4) with \( \zeta \) and integrating over \( \Omega \) yields

\[
\frac{1}{2} \partial_t \| \zeta \|_{L^2(\Omega)}^2 + \| \nabla \zeta \|_{L^2(\Omega)}^2 + a \| \nabla \zeta \|_{L^2(\Omega)}^2 = \int_{\Omega} f_\zeta \cdot \zeta,
\]

since \( \int_{\Omega} (u \cdot \nabla) \zeta \cdot \zeta = 0 \). Integration in time and Young's inequality imply

\[
\frac{1}{2} \| \zeta \|_{L^2(\Omega)}^2 + \int_0^t \| \nabla \zeta \|_{L^2(\Omega)}^2 + a \int_0^t \| \nabla \zeta \|_{L^2(\Omega)}^2 = \| \zeta_0 \|_{L^2(\Omega)}^2 + \int_0^t \int_{\Omega} f_\zeta \cdot \zeta
\]

\[
\leq \| \zeta_0 \|_{L^2(\Omega)}^2 + \frac{1}{2} \| f_\zeta \|_{L^2((0, T), L^2(\Omega))}^2 + \frac{1}{2} \int_0^t \| \zeta \|_{L^2(\Omega)}^2.
\]
In particular,
\[ \|\zeta\|^2_{L^2(\Omega)} \leq 2\|\zeta_0\|^2_{L^2(\Omega)} + \|f_\zeta\|^2_{L^2((0,T),L^2(\Omega))} + \int_0^t \|\zeta\|^2_{L^2(\Omega)} . \]

Gronwall’s lemma yields
\[ \|\zeta\|^2_{L^2(\Omega)} \leq (2\|\zeta_0\|^2_{L^2(\Omega)} + \|f_\zeta\|^2_{L^2((0,T),L^2(\Omega))}) \cdot e^t =: B_1(t) . \]

From this estimate we conclude
\begin{align*}
\frac{1}{2} \|\zeta\|^2_{L^2(\Omega)} &+ \int_0^t \|\nabla\zeta\|^2_{L^2(\Omega)} + a \int_0^t \|\tau\|^2_{L^2(\Gamma^\ast)} \leq \|\zeta_0\|^2_{L^2(\Omega)} + \frac{1}{2} \|f_\zeta\|^2_{L^2((0,T),L^2(\Omega))} + \frac{1}{2} \int_0^t \|\zeta\|^2_{L^2(\Omega)} \\
&\quad \leq \|\zeta_0\|^2_{L^2(\Omega)} + \frac{1}{2} \|f_\zeta\|^2_{L^2((0,T),L^2(\Omega))} + \frac{1}{2} \int_0^t B_1(s) \, ds \\
&\quad = (2\|\zeta_0\|^2_{L^2(\Omega)} + \|f_\zeta\|^2_{L^2((0,T),L^2(\Omega))}) \cdot e^t \\
&\quad =: C_\zeta \cdot e^t =: B_{L^2}(t) .
\end{align*}

**Step 2** \([L^\infty((0,T), L^2(\Omega))] \text{ and } L^2((0,T), W^{1,2}(\Omega))-\text{bounds of the velocity}\]
For the estimate for \(v\) we multiply the first equations of (2.4) with \(v\) and integrate over \(\Omega\) to obtain
\[ \frac{1}{2} \partial_t \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} = \int_{\Omega} f_v \cdot v + \int_{\Omega} \Pi(\zeta) \cdot v , \]

since \(\int_{\Omega} (u \cdot \nabla)v \cdot v = 0\). Integration in time and Young’s inequality imply
\begin{align*}
\frac{1}{2} \|v\|^2_{L^2(\Omega)} &+ \int_0^t \|\nabla v\|^2_{L^2(\Omega)} \leq \|v_0\|^2_{L^2(\Omega)} + \|f_v\|^2_{L^2((0,T),L^2(\Omega))} + \int_0^t \|\Pi(\zeta)\|^2_{L^2(\Omega)} + \frac{1}{2} \int_0^t \|v\|^2_{L^2(\Omega)} .
\end{align*}

Using Lemma 5.2(i) and Step 1 we conclude
\begin{align*}
\frac{1}{2} \|v\|^2_{L^2(\Omega)} &+ \int_0^t \|\nabla v\|^2_{L^2(\Omega)} \leq \|v_0\|^2_{L^2(\Omega)} + \|f_v\|^2_{L^2((0,T),L^2(\Omega))} + C \cdot \int_0^t \|\nabla \zeta\|^2_{L^2(\Omega)} + \frac{1}{2} \int_0^t \|v\|^2_{L^2(\Omega)} \\
&\quad \leq \|v_0\|^2_{L^2(\Omega)} + \|f_v\|^2_{L^2((0,T),L^2(\Omega))} + C \cdot B_{L^2}(t) + \frac{1}{2} \int_0^t \|v\|^2_{L^2(\Omega)} .
\end{align*}

In particular,
\[ \|v\|^2_{L^2(\Omega)} \leq 2 \cdot (\|v_0\|^2_{L^2(\Omega)} + \|f_v\|^2_{L^2((0,T),L^2(\Omega))} + C \cdot B_{L^2}(t)) + \int_0^t \|v\|^2_{L^2(\Omega)} . \]

Gronwall’s lemma implies
\begin{align*}
\|v\|^2_{L^2(\Omega)} &\leq 2 \cdot (\|v_0\|^2_{L^2(\Omega)} + \|f_v\|^2_{L^2((0,T),L^2(\Omega))} + C \cdot B_{L^2}(t)) \cdot e^t \\
&= 2 \cdot (\|v_0\|^2_{L^2(\Omega)} + \|f_v\|^2_{L^2((0,T),L^2(\Omega))}) \cdot e^t + 2C \cdot (2\|\zeta_0\|^2_{L^2(\Omega)} + \|f_\zeta\|^2_{L^2((0,T),L^2(\Omega))}) \cdot e^{2t} =: B_2(t) .
\end{align*}

Finally, we conclude
\begin{align*}
\frac{1}{2} \|v\|^2_{L^2(\Omega)} &+ \int_0^t \|\nabla v\|^2_{L^2(\Omega)} \leq \|v_0\|^2_{L^2(\Omega)} + \|f_v\|^2_{L^2((0,T),L^2(\Omega))} + C \cdot B_{L^2}(t) + \frac{1}{2} \int_0^t B_2(s) \, ds \\
&\quad \leq (\|v_0\|^2_{L^2(\Omega)} + \|f_v\|^2_{L^2((0,T),L^2(\Omega))}) \cdot e^t \\
&\quad + C \cdot (2\|\zeta_0\|^2_{L^2(\Omega)} + \|f_\zeta\|^2_{L^2((0,T),L^2(\Omega))}) \cdot (e^{2t} + e^t - 1) \\
&\quad =: C_v \cdot e^t + C_\zeta \cdot (e^{2t} + e^t - 1) =: B_{L^2}^v .
\end{align*}
Remark 6.2. By Young’s inequality we see that the long time behaviour of the right hand side can be improved to
\[
\frac{1}{2} \| \zeta(t) \|_{L^2(\Omega)}^2 + \int_0^t \| \nabla \zeta(s) \|_{L^2(\Omega)}^2 \, ds + a \int_0^t \| \tau \|_{L^2(\Gamma_u)}^2 \leq C \zeta \cdot (e^{\varepsilon t} + 1),
\]
\[
\frac{1}{2} \| v(t) \|_{L^2(\Omega)}^2 + \int_0^t \| \nabla v(s) \|_{L^2(\Omega)}^2 \, ds \leq C_v \cdot e^{\varepsilon t} + C \zeta \cdot (e^{2\varepsilon t} + e^{\varepsilon t} + 1)
\]
for every \( \varepsilon > 0 \).

Next we show \( L^\infty((0, T), W^{1,2}(\Omega)) \) and \( L^2((0, T), W^{2,2}(\Omega)) \)-priori bounds.

Lemma 6.3. Assume Assumption (A) and let \( v_0 \in \{ v \in W^{1,2}(\Omega)^2 \cap L^2_s(\Omega) : v|_{\Gamma_b} = 0 \}, \zeta_0 \in W^{1,2}(\Omega)^2 \) and \( f_v, f_\zeta \in L^2((0, T), L^2(\Omega)) \). Furthermore, let \( (v, \zeta) \in E^*_1(0, T) \times E^*_1(0, T) \) be a solution of \( (2.5) \).

Then there exists continuous functions \( B^v_{W^{1,2}} \) and \( B^\zeta_{W^{1,2}} \) on \([0, T]\), depending on \( \| v_0 \|_{W^{1,2}(\Omega)^2}, \| \zeta_0 \|_{W^{1,2}(\Omega)^2}, \| f_v \|_{L^2((0, T), L^2(\Omega))}, \| f_\zeta \|_{L^2((0, T), L^2(\Omega))} \), such that
\[
\| \nabla v(t) \|_{L^2(\Omega)}^2 + \int_0^t \| \Delta v(s) \|_{L^2(\Omega)}^2 \, ds \leq B^v_{W^{1,2}}(t),
\]
\[
\| \nabla \zeta(t) \|_{L^2(\Omega)}^2 + \int_0^t \| \Delta \zeta(s) \|_{L^2(\Omega)}^2 \, ds \leq B^\zeta_{W^{1,2}}(t).
\]

Proof. The proof is divided into two steps.

Step 1 \([L^\infty((0, T), W^{1,2}(\Omega)) \) and \( L^2((0, T), W^{2,2}(\Omega))\)-bounds of the velocity\]

Lemma 5.2(i) and Lemma 6.1 imply that \( \Pi(\zeta) \in L^2((0, T), L^2(\Omega)) \). Since \( f_v \in L^2((0, T), L^2(\Omega)) \) it follows that the right hand side of the first equation in \((2.4)\) is \( f_v + \Pi(\zeta) \in L^2((0, T), L^2(\Omega)) \). Now [4, equation (4.1)] implies
\[
\| \nabla v(t) \|_{L^2(\Omega)}^2 + \int_0^t \| \Delta v(s) \|_{L^2(\Omega)}^2 \, ds \leq B^v_{W^{1,2}} \left( \| v_0 \|_{W^{1,2}(\Omega)}, \int_0^t \| f_v \|_{L^2(\Omega)}, \int_0^t \| \nabla \zeta \|_{L^2(\Omega)} \right).
\]

Using Lemma 6.1 again, we obtain
\[
\int_0^t \| \nabla \zeta \|_{L^2(\Omega)} \leq B^\zeta_{L^2}(t)
\]
where \( B^\zeta_{L^2}(t) \) depends on \( \| \zeta_0 \|_{L^2(\Omega)} \) and \( \| f_\zeta \|_{L^2((0, T), L^2(\Omega))} \). Now the claim follows since \( \int_0^t \| f \|_{L^2(\Omega)} \leq \| f \|_{L^2((0, T), L^2(\Omega))} \).

Step 2 \([L^\infty((0, T), W^{1,2}(\Omega)) \) and \( L^2((0, T), W^{2,2}(\Omega))\)-bounds of temperature and salinity\]

We multiply the second equation in \((2.4)\) by \( -\Delta \zeta \) and integrate over \( \Omega \) and obtain
\[
\frac{1}{2} \frac{\partial}{\partial t} \left( \| \nabla \zeta \|_{L^2(\Omega)}^2 + a \| \tau \|_{L^2(\Gamma_u)}^2 \right) + \| \Delta \zeta \|_{L^2(\Omega)}^2 = \int_{\Omega} v \nabla H \zeta \cdot \Delta \zeta + w \partial_2 \zeta \cdot \Delta \zeta - \int_{\Omega} g \cdot \Delta \zeta.
\]

For the terms on the right hand side we obtain the following estimates.

(i) Since \( W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega) \) Hölder’s and Young’s inequality yield
\[
\int_{\Omega} |v \nabla H \zeta \cdot \Delta \zeta| \leq \| v \|_{L^\infty(\Omega)} \cdot \| \nabla H \zeta \|_{L^2(\Omega)} \cdot \| \Delta \zeta \|_{L^2(\Omega)}
\]
\[
\leq C \cdot \| v \|_{W^{2,2}(\Omega)} \cdot \| \nabla \zeta \|_{L^2(\Omega)} \cdot \| \Delta \zeta \|_{L^2(\Omega)}
\]
\[
\leq C \cdot \| v \|_{W^{2,2}(\Omega)} \cdot \| \nabla \zeta \|_{L^2(\Omega)}^2 + \frac{1}{6} \cdot \| \Delta \zeta \|_{L^2(\Omega)}^2.
\]

(ii) The embedding \( W^{1,2}(-h, 0) \hookrightarrow L^\infty(-h, 0) \) as well as the inequality \( \| v \|_{L^4(G)} \leq C \| v \|_{L^2(G)}^{\frac{1}{2}} \| \nabla H v \|_{L^2(G)}^{\frac{1}{2}} \) and Hölder’s and Young’s inequality yield
\[
\int_{\Omega} |w \partial_2 \zeta \cdot \Delta \zeta| \leq \| w \partial_2 \zeta \|_{L^2(\Omega)} \cdot \| \Delta \zeta \|_{L^2(\Omega)}
\]
By Hölder’s and Young’s inequality we obtain
\[
\int_\Omega |g \cdot \Delta| \leq \|g\|_{L^2(\Omega)} \cdot \|\Delta \zeta\|_{L^2(\Omega)} \leq C \cdot \|g\|^2_{L^2(\Omega)} + \frac{1}{6} \cdot \|\Delta \zeta\|^2_{L^2(\Omega)}.
\]
Combining these inequalities we obtain by absorbing the \(\|\Delta \zeta\|^2_{L^2(\Omega)}\)-term
\[
\partial_t \left( \|\nabla \zeta\|^2_{L^2(\Omega)} + a^2 \|\tau\|^2_{L^2(\Gamma_u)} \right) + \|\Delta \zeta\|^2_{L^2(\Omega)} \\
\leq C \cdot \|g\|^2_{L^2(\Omega)} + C \cdot \left( \|v\|^2_{W^{1,2}(\Omega)} + \|v\|^2_{W^{1,2}(\Omega)} \cdot \|v\|^2_{W^{2,2}(\Omega)} \right) \cdot \|\nabla \zeta\|^2_{L^2(\Omega)}.
\]
Now adding the term \(a \cdot C \cdot \left( \|v\|^2_{W^{1,2}(\Omega)} + \|v\|^2_{W^{1,2}(\Omega)} \cdot \|v\|^2_{W^{2,2}(\Omega)} \right) \cdot \|\tau\|^2_{L^2(\Gamma_u)} > 0 \) it follows
\[
\partial_t \left( \|\nabla \zeta\|^2_{L^2(\Omega)} + a \|\tau\|^2_{L^2(\Gamma_u)} \right) + \|\Delta \zeta\|^2_{L^2(\Omega)} \\
\leq C \cdot \|g\|^2_{L^2(\Omega)} + C \cdot \left( \|v\|^2_{W^{1,2}(\Omega)} + \|v\|^2_{W^{1,2}(\Omega)} \cdot \|v\|^2_{W^{2,2}(\Omega)} \right) \cdot \left( \|\nabla \zeta\|^2_{L^2(\Omega)} + a \|\tau\|^2_{L^2(\Gamma_u)} \right).
\]
Integration in time yields
\[
\left( \|\nabla \zeta(t)\|^2_{L^2(\Omega)} + a^2 \|\tau(t)\|^2_{L^2(\Gamma_u)} \right) + \int_0^t \|\Delta \zeta(s)\|^2_{L^2(\Omega)} \, ds \\
\leq C \cdot \|g\|^2_{L^2((0,T),L^2(\Omega))} + \|\nabla \zeta_0\|^2_{L^2(\Omega)} + a^2 \|\tau_0\|^2_{L^2(\Gamma_u)} \\
+ C \cdot \int_0^t \left( \|v(s)\|^2_{W^{1,2}(\Omega)} + \|v(s)\|^2_{W^{1,2}(\Omega)} \cdot \|v(s)\|^2_{W^{2,2}(\Omega)} \right) \cdot \left( \|\nabla \zeta(s)\|^2_{L^2(\Omega)} + a \|\tau(s)\|^2_{L^2(\Gamma_u)} \right) \, ds.
\]
Gronwall’s Lemma implies
\[
\left( \|\nabla \zeta(t)\|^2_{L^2(\Omega)} + a^2 \|\tau(t)\|^2_{L^2(\Gamma_u)} \right) + \int_0^t \|\Delta \zeta(s)\|^2_{L^2(\Omega)} \, ds \\
\leq \left( C \cdot \|g\|^2_{L^2((0,T),L^2(\Omega))} + \|\nabla \zeta_0\|^2_{L^2(\Omega)} + a^2 \|\tau_0\|^2_{L^2(\Gamma_u)} \right) \cdot e^{C \int_0^t \left( \|v(s)\|^2_{W^{1,2}(\Omega)} + \|v(s)\|^2_{W^{1,2}(\Omega)} \cdot \|v(s)\|^2_{W^{2,2}(\Omega)} \right) \, ds}
\]
Finally, Step 1 and Hölder’s inequality imply
\[
\int_0^t \|v(s)\|^2_{W^{1,2}(\Omega)} \cdot \|v(s)\|^2_{W^{2,2}(\Omega)} \, ds \leq \|v\|^2_{L^\infty((0,T),W^{1,2}(\Omega))} \cdot \|v\|^2_{L^2((0,T),W^{2,2}(\Omega))} \\
\leq (B_{W^{1,2}}^2(t))^4
\]
Setting
\[
B_{W^{1,2}}^2(t) := \left( C \cdot \|g\|^2_{L^2((0,T),L^2(\Omega))} + \|\nabla \zeta_0\|^2_{L^2(\Omega)} + a^2 \|\tau_0\|^2_{L^2(\Gamma_u)} \right) \cdot e^{(B_{W^{1,2}}^2(t))^2 + (B_{W^{1,2}}^2(t))^4},
\]
we obtain
\[
\left(\|\nabla \zeta(t)\|^2_{L^2(\Omega)} + a \|\tau(t)\|^2_{L^2(\Gamma_u)}\right) + \int_0^t \|\Delta \zeta(s)\|^2_{L^2(\Omega)} \, ds \leq B_{W^{1,2}}^\zeta(t).
\]

We conclude the following a priori bounds in maximal regularity spaces.

**Proposition 6.4.** Assume Assumption (A), \(0 < T < \infty\), and let \(v_0 \in \{v \in W^{1,2}(\Omega)^2 \cap L^2_\sigma(\Omega) : v|_{\Gamma_k} = 0\}\) \(\zeta_0 \in W^{1,2}(\Omega)^2\) and \(f_v, f_\zeta \in L^2((0,T),L^2(\Omega)^2)\). Furthermore, let \((v, \zeta) \in E^v_T(0,T) \times E^\zeta_T(0,T)\) be a solution of (2.5).

Then there exists a continuous function \(B\) on \([0,T]\), depending on \(v_0\) \(\|
m_0\|_{w^{1,2}(\Omega)^2},\) \(\|\zeta_0\|_{w^{1,2}(\Omega)^2},\) \(\|f_v\|_{L^2((0,T),L^2(\Omega))},\) \(\|f_\zeta\|_{L^2((0,T),L^2(\Omega))}\), \(T\), such that
\[
\|v\|_{E^v_T} + \|\zeta\|_{E^\zeta_T} \leq B.
\]

**Proof.** The proof extends [6, Step 4, Proof of Theorem 6.9] to the present situation. Lemma 5.1 implies that \(A_2 - \lambda\) has maximal \(L^2\)-regularity for \(\lambda > 0\) on \(L^2_\sigma(\Omega) \times L^2(\Omega)\). Hence, there exists a constant \(C > 0\) such that
\[
\|v\|_{E^v_T} + \|\zeta\|_{E^\zeta_T} \leq C \cdot \left(\|\partial_t - (A_2 - \lambda)\| E^v_T \times E^\zeta_T\right) v \zeta
\leq C \cdot \left(\|P_2(F(v, v) + \Pi(\zeta) + f_v - \lambda v)\| E^v_T \times E^\zeta_T\right)
\leq C \cdot (\|F(v, v)\|_{E^v_T} + \|\Pi(\zeta)\|_{E^\zeta_T} + \|f_v\|_{E^v_T} + \|\zeta\|_{E^\zeta_T} + \lambda \|v\|_{E^v_T} + \|f_\zeta\|_{E^\zeta_T} + \lambda \|\zeta\|_{E^\zeta_T})
\]
where we have used the boundedness of \(P_2\) in the last line. Using Lemma 5.3(iii) and Hölder’s and Young’s inequality we obtain
\[
\|F(v, v)\|_{E^v_T} \leq C \cdot \|v(s)\|_{L^2((0,T),w^{2,2}(\Omega))} \cdot \|v(s)\|_{L^\infty((0,T),w^{1,2}(\Omega))}
\]
as well as
\[
\|F(v, \zeta)\|_{E^\zeta_T} \leq C \cdot (\|v(s)\|_{L^2((0,T),w^{2,2}(\Omega))} + \|\zeta(s)\|_{L^2((0,T),w^{2,2}(\Omega))})
\cdot (\|v(s)\|_{L^\infty((0,T),w^{1,2}(\Omega))} + \|\zeta(s)\|_{L^\infty((0,T),w^{1,2}(\Omega))}).
\]
Furthermore, Lemma 5.2(i) implies
\[
\|\Pi(\zeta)\|_{E^\zeta_T} \leq C \int_0^T \|\Pi(\zeta(s))\|_{L^2(\Omega)}^2 \, ds \leq C \int_0^T \|\nabla \zeta(s)\|_{L^2(\Omega)}^2 \, ds
\leq C \cdot \|\zeta\|_{L^2((0,T),w^{1,2}(\Omega))}^2.
\]
Finally, Lemmas 6.1 and 6.3 imply
\[
\|v\|_{E^v_T} + \|\zeta\|_{E^\zeta_T} \leq C \cdot ((B_{W^{1,2}}^v + B_{W^{1,2}}^\zeta + f_v)_{E^v_T} + \lambda B_{E^v_T} + (B_{W^{1,2}}^v + B_{W^{1,2}}^\zeta + f_\zeta)_{E^\zeta_T} + \lambda B_{E^\zeta_T}) =: B.
\]

The following elementary lemma shows that controlling the \(E_{1,\mu}\)-norm of local solutions for all times uniformly suffices to exclude blow ups.

**Lemma 6.5.** ([6, Lemma 6.6]) Let \(h \in E_{1,\mu}(0,T')\) for any \(0 < T' < T\) and \(\sup_{0 < T' < T} \|h\|_{E_{1,\mu}(0,T')} \leq C\) for some constant \(C > 0\). Then \(h \in E_{1,\mu}(0,T)\).
7. Proof of Theorem 4.1

7.1. Local Wellposedness

We start with the proof of local well-posedness.

**Proposition 7.1.** Let $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$, $\mu \in \left[\frac{1}{p} + \frac{1}{q}, 1\right]$ and $T \in (0, \infty)$. Assume Assumption (A) and that

$$(v_0, \tau_0, \sigma_0)^T \in X_{\mu - \frac{1}{p}, p} \quad \text{and} \quad f_v, f_r, f_\sigma \in L^p((0, T), L^q(\Omega)).$$

Then there exists $T' = T'(v_0, \tau_0, \sigma_0, f)$ with $0 < T' \leq T$ and a unique strong solution

$$(v, \tau, \sigma)^T \in H^1(\mu, (0, T'), L^2(\Omega))^2 \cap L^\infty((0, T'), D(A_v^q) \times D(A_\sigma^q))$$

to (2.3) on $(0, T')$.

**Proof.** We verify the Assumptions A.1 for (2.7). In fact, assumptions (H1) and (S) are satisfied since $L^q_\mu(\Omega) \times L^q(\Omega)^2$ is a UMD space and by Lemma 5.1 $-A_q + \lambda$ for some $\lambda \geq 0$ has a bounded $H^\infty$-calculus on $L^q_\mu(\Omega) \times L^q(\Omega)^2$ for some $\lambda \geq 0$.

Assumption (H2) follows by Lemma 5.2(ii), Lemma 5.3(ii) and (H3) by choosing $\beta = \frac{1}{2} \left(1 + \frac{1}{q}\right)$. The assertion follows then from Theorem A.2. \hfill $\square$

7.2. Global Wellposedness

We consider first the case $p = q = 2$.

**Proposition 7.2.** Let $T \in (0, \infty)$ and $v_0 \in \{v \in H^{1,2}(\Omega)^2 \cap L^2_\mu(\Omega) : v|_{\Gamma_0} = 0\}$, $\zeta_0 \in H^{1,2}(\Omega)^2$. Assume Assumption (A). If $f_v, f_r, f_\sigma \in L^2((0, T), L^2(\Omega))$, then there exists a unique, strong solution

$$(v, \tau, \sigma)^T \in H^{1,2}((0, T), L^2(\Omega)^2) \cap L^\infty((0, T), D(A_2))$$

to the primitive equations (2.1) subject to (2.2).

**Proof.** The proof follows essentially the lines of [6, Proof of Proposition 3.2]. Throughout the proof we assume that $p = q = 2$.

Consider

$$t_+(v_0, \zeta_0) := \sup\{T' > 0 : (2.4) \text{ has a solution in } E_{1,1}(0, T')\}.$$  

By Proposition 7.1, $t_1(v_0, \zeta_0) > 0$. Assuming that there exist two solutions $(v, \zeta, (v', \zeta')) \in E_{1,1}(0, T') := E_{1,1}^v(0, T') \times E_{1,1}^\zeta(0, T')$ to $(v_0, \zeta_0)$, we set

$$t_1(v_0, \zeta_0) := \sup\left\{s > 0 : \left(\left(\frac{\partial}{\partial s}\right) - (v(s), \zeta(s))\right)(s)\right\}_{X_{\gamma,1}} = 0\}.$$  

and see that $t_1(v_0, \zeta_0) > 0$ by Proposition 7.1. By continuity, $E_{1,1}(0, T') \hookrightarrow C([0, T'], X_{\gamma,1})$ and thus the above supremum is attained. Assume now that $t_1(v_0, \zeta) < T'$. By Proposition 7.1 there exists a unique solution with initial value $t_1(v_0, \zeta_0)$ on some interval, which contradicts $t_1(v_0, \zeta_0) < T'$.

Thus $t_1(v_0, \zeta_0) = T'$.

Assume now that $t_+(v_0, \zeta_0) < T$. By Proposition 6.5 we obtain

$$\|v\|_{E_{1,1}^v(0, T')} + \|\zeta\|_{E_{1,1}^\zeta(0, T')} \leq B$$

for the function $B$ on $[0, T']$, depending on $t_+(v_0, \zeta_0)$, $\|v\|_{H^{1,2}(\Omega)^2}$, $\|v_0\|_{H^{1,2}(\Omega)^2}$, $\|f_v\|_{L^2((0, T), L^2(\Omega))}$, $\|f_\zeta\|_{L^2((0, T), L^2(\Omega))}^2$ for all $0 < T' < t_+(v_0, \zeta_0)$. Lemma 6.5 implies $(v, \zeta)^T \in E_{1,1}(0, t_+(v_0))$ and $t_+(v_0, \zeta_0)) \in X_{\gamma,1}$ and $t_+(v_0, \zeta_0))$ can be taken as new initial value, hereby extending
the solution beyond $t_+(v_0, \zeta_0)$, which contradicts the maximality of $t_+(v_0, \zeta_0)$. Hence, $t_+(v_0, \zeta_0) = T$ and thus there exists a global solution. Moreover, by Proposition 6.4 and Lemma 6.5 we conclude that $(v, \zeta) \in E_{1,1}(0, T)$. 

We are now prepared to prove our main theorem.

**Proof of Theorem 4.1.** The proof is similar to the one of [6, Proof of Theorem 3.1]. By Proposition 7.1 there exists a local solution. The smoothing property of parabolic equations implies the existence of $0 < \delta \leq T' < T$ such that $(v, \zeta) \in H^{1,p}((\delta, T), D(A_0^* \times D(\Delta_q^r) \times D(\Delta_q^r)) \leftrightarrow C((\delta, T), D(A_0^* \times D(\Delta_q^r)) \times D(\Delta_q^r) \times D(\Delta_q^r)))$. Taking $(v(T'), \zeta(T')) \in D(A_p) \leftrightarrow (L^2_\sigma(\Omega), D(A_2))\frac{1}{2}, p$ for $p \in \left[\frac{6}{5}, \infty\right]$ and by assumption $f_v, f_\tau, f_\sigma \in H^{1,2}((\delta, T), L^2(\Omega))$, we obtain that $(v, \zeta)$ is a $L^2$ solution at least for $\delta > 0$. Using the bootstrapping argument from [7, Section 6.2] we see that this property holds also for $p \in \left(\frac{6}{5}, \infty\right]$ and by duality for $p \in (1, \frac{6}{5})$. By Proposition 7.2 the solution $(v, \zeta) \in C_b((\delta, T), D(A_2))$ is global and by classical embeddings we obtain

$$D(A_2) \hookrightarrow X_{\mu,p} \quad \text{for } 0 \leq \tilde{\mu} - \mu < 2 - \frac{2}{q},$$

and compactness of the embedding $X_{\mu,p} \hookrightarrow X_{\mu,\tilde{\mu}}$ for $1/q < \mu < \tilde{\mu} < 1$. Furthermore,

$$\|(v, \zeta)\|_{C((\delta, T), X_{\mu,p})} \leq C \cdot \|(v, \zeta)\|_{C((\delta, T), D(A_2))}.$$ Finally, the the claim follows from Theorem A.4. \qed

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**Declarations**

**Conflict of interest** The authors declare that they have no conflicts of interest.

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**Appendix A. Semilinear Evolution Equations and Maximal $L^r$-Regularity**

In this section we briefly recall and summarize results on local and global wellposedness for semilinear evolution equations, which we apply to the primitive equations. These results are mainly due to Prüss and Wilke [17]. To this end, let $X_0, X_1$ be Banach spaces such that $X_1$ is densely embedded into $X_0$ and let $A : X_1 \rightarrow X_0$ be bounded. For $0 < T \leq \infty$ we consider the equation

$$\begin{cases}
  u' + Au = F(u) + f \text{ on } (0, T) \\
  u(0) = u_0
\end{cases} \quad (A.1)$$

and make following assumptions. Here, for $\beta \in [0, 1]$ define $X_\beta$ to be the complex interpolation space $[X_0, X_1]_\beta$.

**Assumption A.1.** (H1) $A$ has maximal $L^r$-regularity for $r \in (1, \infty)$. 

(H2) $F: X_\beta \to X_0$ satisfies the estimate
\[
\|F(u_1) - F(u_2)\|_{X_0} \leq C(1 + \|u_1\|_{X_\beta} + \|u_2\|_{X_\beta}) \cdot \|u_1 - u_2\|_{X_\beta}
\]
for some constant $C > 0$ independent of $u_1$ and $u_2$.

(H3) $\beta - (\mu - \frac{1}{2}) \leq \frac{1}{2} (1 - (\mu - \frac{1}{2}))$, that is $2\beta + \frac{1}{r} \leq 1 + \mu$.

(S) $X_0$ is of class UMD, and the embedding
\[
W^{1, r}(\mathbb{R}, X_0) \cap L^r(\mathbb{R}, X_1) \hookrightarrow W^{1-\beta, r}(\mathbb{R}, X_\beta)
\]
is valid for each $\beta \in (0, 1)$ and $r \in (1, \infty)$.

The following result guarantees unique existence of local solution.

**Theorem A.2.** (\cite{16, Theorem 2.1}) Assume that Assumption A.1 holds and that
\[
u_0 \in X_{\gamma, \mu} \quad \text{and} \quad f \in L^r((0, T), X_0).
\]
Then there exists $T' = T'(u_0, f)$ with $0 < T' \leq T$ such that Eq. (A.1) admits a unique solution
\[
u \in W^{1, r}_\mu((0, T'), X_0) \cap L^r_\mu((0, T'), X_1).
\]
Furthermore, the solution depends continuously on the data.

**Remark A.3.**
(i) Condition (S) holds true whenever $X_0$ is of class UMD and there is an operator
$A_\# \in \mathcal{H}_\infty(X_0)$ with domain $D(A_\#) = X_1$ satisfying $\phi_{A_\#}^\infty < \frac{\pi}{2}$, see \cite{17, Remark 1.1} and \cite[Remark 5.2(a)]{6}.

(ii) Due to the embeddings
\[
E_1(0, T') \hookrightarrow C([0, T'], X_{\gamma, \mu}) \quad \text{and} \quad E_1(\delta, T') \hookrightarrow C([\delta, T'], X_{\gamma, \mu}), \quad \text{for} \ \delta > 0,
\]
there is an instantaneous smoothing effect typical for parabolic equations, compare e.g. \cite[Section 3.5.2]{16}.

Finally, we obtain the following abstract criterion for global existence.

**Theorem A.4.** (\cite[Theorem 5.7.1]{16}) Assume additional to Assumption A.1 that for $\mu < \bar{\mu} \leq 1$ the embedding
\[
X_{\gamma, \bar{\mu}} \hookrightarrow X_{\gamma, \mu}
\]
is compact, and that for some $\bar{T} \in (0, T_+(u_0))$ the solution of (A.1) satisfies
\[
u \in C_b([\bar{T}, T_+(u_0)), X_{\gamma, \bar{\mu}}).
\]
Then the solution is global, i.e. $T_+(u_0) = T$.

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