A simple and efficient algorithm for computing approximate Nash equilibria in sequential games with incomplete information

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Abstract
We present a simple primal-dual algorithm for computing approximate Nash equilibria in two-person zero-sum sequential games with incomplete information and perfect recall (like Texas Hold’em poker). Our algorithm only performs basic iterations (i.e matvec multiplications, clipping, etc., and no calls to external first-order oracles, no matrix inversions, etc.) and is applicable to a broad class of two-person zero-sum games including simultaneous games and sequential games with incomplete information and perfect recall. The applicability to the latter kind of games is thanks to the sequence-form representation [10] which allows us to encode any such game as a matrix game with convex polytopial profiles. We prove that the number of iterations needed to produce a Nash equilibrium with a given precision is inversely proportional to the precision. We also present experimental results on simulated and real games (Kuhn poker).

keywords: algorithmic-game theory; sequential game; incomplete information; perfect recall; approximate Nash equilibrium; primal-dual algorithm; convex-optimization

1 Introduction
A game-theoretic approach to playing games strategically optimally consists in computing Nash equilibria (in fact, approximations thereof) offline, and playing one’s part (an optimal behavioral strategy) of the equilibrium online. This technique is the driving-force behind solution concepts like CFR [20], CFR+ [18] and other variants, etc., which have recently had profound success in poker. However, solving games for equilibria remains a mathematical and computational challenge, especially in sequential games with imperfect information. This paper proposes a simple and efficient algorithm for solving for such equilibria approximately, in a sense which will be made clear shortly.

1.1 Notation and terminology
Given a set $X$, $2^X$ denotes the powerset of $X$, i.e the set of all subsets of $X$, or equivalently the set of all binary functions on $X$. Let $m$ and $n$ be positive integers. Given two vectors $z, w \in \mathbb{R}^n$, their inner product will be denoted $\langle z, w \rangle := \sum_j z_j w_j$. The components of $z$ will be denoted $z_0, z_1, ..., z_{n-1}$ (indexing begins from 0, not 1). $\mathbb{R}_n^+ := \{ z \in \mathbb{R}^n \mid z \geq 0 \}$ is the nonnegative $n$th orthant. The notation “$z \geq 0$” means that all the components of $z$ are nonnegative. $\|z\|$ denotes the 2-norm of $z$ defined by $\|z\| := \sqrt{\langle z, z \rangle}$. $(z)_+ := \max(0, z) \in \mathbb{R}_n^+$ is the point-wise maximum of $z$ with 0. For example, $((-2, \pi))_+ = (\max(-2, 0), \max(\pi, 0)) = (0, \pi)$. The
operator $(\cdot)_+$ is the well-known (multi-dimensional) ramp function. The $n$-simplex denoted $\Delta_n$, is defined by $\Delta_n := \{ z \in \mathbb{R}_n^+ | \sum_j z_j = 1 \}$. Given a matrix $A \in \mathbb{R}^{m \times n}$, its spectral norm, denoted $\|A\|$, is defined to be the largest singular value of $A$, i.e. the largest eigenvalue of $A^T A$ (or equivalently, of $AA^T$).

### 1.2 Nash-equilibrium concepts and statement of the problem

The sequence-form representation for two-person zero-sum games with incomplete information was introduced in [10], and the theory was further developed in [11, 17, 16] where it was established that for such games, there exist sparse matrices $A$ was introduced in [10], and the theory was further developed in [11, 17, 16] where it was established that for such games, there exist sparse matrices $A$ such that Nash equilibria correspond to pairs $(e_1, p)$ which equals 0

\[
\min_{(y,p)\in \mathbb{R}^n \times \mathbb{R}^l} (e_1, p) \quad \text{subject to: } y \geq 0, E_2 y = e_2, -A y + E_1^T p \geq 0.
\]

and the dual LCP

\[
\max_{(x,q)\in \mathbb{R}^n \times \mathbb{R}^l} -\langle e_2, q \rangle \quad \text{subject to: } x \geq 0, E_1 x = e_1, A^T x + E_2^T q \geq 0.
\]

The vectors \( p = (p_0, p_1, ..., p_{l-1}) \in \mathbb{R}^l \) and \( q = (q_0, q_1, ..., q_{l-1}) \in \mathbb{R}^l \) are dual variables. $A$ is the payoff matrix and each $E_k$ is a matrix whose entries are $-1$, 0 or 1, with exactly 1 entry per row which equals $-1$ except for the first whose whose first entry is 1 and all the others are 0. Each of the vectors $e_k$ is of the form $(1, 0, ..., 0)$.

Note that the LCPs above have the equivalent saddle point formulation

\[
\min_{y \in Q_2} \max_{x \in Q_1} \langle x, Ay \rangle,
\]

where the convex polytope (by which we mean a bounded convex polyhedron)

\[
Q_k := \{ z \in \mathbb{R}_n^+ | E_k z = e_k \} \subseteq [0,1]^n
\]

is identified with the strategy profile of player $k$ in the sequence-form representation. At a feasible point $(y, p, x, q)$ for the LCPs, the primal-dual gap $\tilde{G}(y, p, x, q)$ of these primal-dual pair is given by

\[
0 \leq \tilde{G}(y, p, x, q) := (e_1, p) - (-e_2, q) = (e_1, p) + (e_2, q) = G(x, y) := \max\{ \langle u, Ay \rangle - \langle x, Av \rangle | (u, v) \in Q_1 \times Q_2 \}.
\]

\[ ^{\text{The inequality being due to weak duality.}} \]

In (5), the quantity $G(x, y)$ is nothing but the primal-dual gap for the equivalent saddle point problem (3). It was shown (see Theorem 3.14 of [15]) that a pair $(x, y) \in Q_1 \times Q_2$ of realization plans is a solution to the LCPs (1) and (2) (i.e is a Nash equilibrium for the game) if and only if there exist vectors $p$ and $q$ such that

\[
-A y + E_1^T p \geq 0, \quad A^T x + E_2^T q \geq 0, \quad \langle x, -A y + E_1^T p \rangle = 0, \quad \langle y, A^T x + E_2^T q \rangle = 0.
\]

Moreover, at equilibria, strong duality holds and the value of the game equals $p_0 = -q_0$, i.e the primal-dual gap $\tilde{G}(y, p, x, q)$ defined in (5) vanishes at equilibria.

Solving the LCPs (1) and (2) exactly is impossible in practice (indeed, this system of problems is NP-hard [19]) and such a precision doesn’t have any fundamental practical advantage. Instead, it is customary compute “approximate” Nash equilibria. A popular notion of approximate equilibria is the following:

**Definition 1 (Nash $\epsilon$-equilibria).** Given $\epsilon > 0$, a Nash $\epsilon$-equilibrium is a pair $(x^*, y^*)$ of realization plans such that there exists dual vectors $p^*$ and $q^*$ for problems (1) and (2) such that the primal-dual gap at $(y^*, p^*, x^*, q^*)$ doesn’t exceed $\epsilon$. That is,
\[ 0 \leq \tilde{G}(y^*, p^*, x^*, q^*) \leq \epsilon. \quad (7) \]

A note about matrix games on simplexes. It should be noted that any matrix \( A \in \mathbb{R}^{n_1 \times n_2} \) specifies a matrix game with payoff matrix \( A \), for which each player’s strategy profile is a simplex; this simplex can be written in the form \((4)\) by taking \( E_k := (1, 1, \ldots, 1) \in \mathbb{R}^{1 \times n_k} \) and \( e_k = 1 \in \mathbb{R}^1 \). Thus every matrix game on simplexes can be seen as a sequential game. Thus the results presented in this manuscript can be trivially applied such games in particular. Here, the polytopial \( Q_k \) defined in \((4)\) reduce to simplexes \( \Delta_{n_k} \), and the primal-dual gap function \( G(x, y) \) writes

\[ G(x, y) = \max \{(u, Ay) - (x, Av) | (u, v) \in \Delta_{n_1} \times \Delta_{n_2}\} = \max_{0 \leq i < n_1} (Ay)_i - \min_{0 \leq j < n_2} (A^T x)_j. \quad (8) \]

1.3 Our contribution

One cannot directly attack the LCPs \((1)\) and \((2)\) via a traditional primal-dual algorithm (for example \([2, 3]\)) because computing the orthogonal projections \( \text{proj}_{Q_k} \) is very difficult. In fact, such subproblems would have to be solved iteratively\(^1\). Also, the primal-dual gap might explode even at points arbitrarily close to the set of feasible points, leaving the algorithm with no indication whatsoever, on whether progress is being made or not. So need to way to

1. avoid having to compute the projections \( \text{proj}_{Q_k} \),
2. have control over how far we are from the set of equilibria and avoid infinite (and thus non-informative) primal-dual gaps.

Developing on an alternative notion of approximate equilibria (see Definition \(4\)) homologous to that presented in Definition \(1\) we device a primal-dual algorithm (Algorithm \(1\)) for computing approximate Nash equilibria in sequential two-person zero-sum games with incomplete information and perfect recall, and which satisfies the above crucial requirements (1)–(2). We also prove (Theorem \(2\)) that –in an ergodic / Cesàro sense— the number of iterations required by the algorithm to produce an approximation equilibrium to a precision \( \epsilon \) is \( \mathcal{O}(1/\epsilon) \), with explicit values for the constants involved this worst-case cost. These contributions will be elaborated in section \(3\).

2 Related work

We now present a selection of algorithms that is representative of the efforts that have been made in the literature to compute Nash \( \epsilon \)-equilibria for two-person zero-sum games with incomplete information like Texas Hold’em poker, etc. First and foremost, let us note that for the class of games considered here (sequential games with incomplete information), the LCPs \((1)\) and \((2)\) are exceedingly larger than what state-of-the-art LCP and interior-point solvers can handle. See for example \([9, 7]\).

In \([9]\), a nested iterative procedure using the Excessive Gap Technique (EGT) \((14)\) was used to solve the equilibrium problem \((3)\). The authors reported a \( \mathcal{O}(1/\epsilon) \) convergence rate (which derives from the general EGT theory) for the outer-most iteration loop. \([7]\) proposed a modified version of the techniques in \([9]\) and proved a \( \mathcal{O}((\|A\|/\delta) \ln (1/\epsilon)) \) convergence rate in terms of the number of calls made to a first-order oracle. Here \( \delta = \delta(A, E_1, E_2, e_1, e_2) > 0 \) is a certain condition number for the game. The crux of their technique was to observe that \((3)\) can further be written as a minimization of the primal-dual gap function \( G(x, y) \) (defined in \((5)\)) for the game\(^2\), viz

\[ \minimize\{G(x, y) | (x, y) \in Q_1 \times Q_2\}, \quad (9) \]

\(^1\)An “exception to the rule” is the case where the \( Q_k \)s are simplexes, so that the projections can be computed exactly using \([6]\).

\(^2\)The minimizers of \( G \) are precisely the equilibria of the game.
and then show there exists a scalar \( \delta > 0 \) such that for any pair of realization plans \((x, y)\) \( \in Q_1 \times Q_2 \),

\[
\text{“distance between } (x, y) \text{ and the set of equilibria” } \leq G(x, y)/\delta.
\]

Their algorithm is then derived by iteratively applying Nesterov smoothing \([15]\) with a geometrically decreasing sequence of tolerance levels \( \epsilon_{n+1} = \epsilon_n/\gamma \) (with \( \gamma > 1 \)) \( G \). It should be noted however that

- The constant \( \delta > 0 \) can be arbitrarily small, and so the factor \( \|A\|/\delta \) in the \( O((\|A\|/\delta)\ln(1/\epsilon)) \) convergence rate can be arbitrarily large for ill-conditioned games.

- The reported linear convergence rate is not in terms of basic operations (addition, multiplication, matvec, clipping, etc.), but in terms of the number of calls to a first-order oracle. Most notably, the complicated projections \( \text{proj}_{Q_k} \) are applied at each iteration.

The primal-dual algorithm first developed in \([2]\), was proposed in \([3]\) as a way of solving matrix games on simplexes. It should be stressed that such matrix games on simplexes are considerably simpler than the games considered here. Indeed, the authors in \([3]\) used the fact that computing the orthogonal projection of a point onto a simplex can be done in linear time as in \([6]\). In contrast, no such efficient algorithm is known nor is likely to exist, for the polytopes \( Q_k \) defined in \([4]\). That notwithstanding, such projections can still be done iteratively using for example, the algorithm in proposition 4.2 of \([4]\) or the algorithms developed in \([19]\). Unfortunately, as with any nested iterative scheme, one would have to solve this sub-problem with finer and finer precision.

Sampling techniques like the CFR (CounterFactual Regret minimization) \([20]\ [13] [1]\) have also become state-of-the-art, and are particularly useful in many-player games, since convex-analytical methods cannot help much in such games.

3 Methods

3.1 The Generalized Saddle-point Problem (GSP) connection

In the next theorem, we show that the LCPs \((1)\) and \((2)\) can be conveniently written as a Generalized Saddle-point Problem (GSP) in the sense of \([8]\). The crux of idea is to remove the linear constraints in the definitions of the strategy polytopes \( Q_k \), by modifying the payoff matrix to yield an equivalent saddle-point problem.

**Theorem 2.** Define two proper closed convex functions

\[
g_1 : \mathbb{R}^{n_2} \times \mathbb{R}^{l_1} \rightarrow (-\infty, +\infty], \quad g_1(y, p) := i_{y \geq 0} + \langle e_1, p \rangle
\]

\[
g_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{l_2} \rightarrow (-\infty, +\infty], \quad g_2(x, q) := i_{x \geq 0} + \langle e_2, q \rangle
\]

Also define two bilinear forms \( \Psi_1, \Psi_2 : \mathbb{R}^{n_2} \times \mathbb{R}^{l_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{l_2} \rightarrow \mathbb{R} \) with \( \Psi_2 = -\Psi_1 \) by letting

\[
K := \begin{bmatrix}
A & -E_1^T \\
E_2 & 0
\end{bmatrix}, \quad \Psi_1(y, p, x, q) := \begin{bmatrix} x \\ q \end{bmatrix}^T K \begin{bmatrix} y \\ p \end{bmatrix} = \langle x, Ay \rangle - \langle x, E_1^T p \rangle + \langle q, E_2 y \rangle,
\]

and define the functions \( \hat{\Psi}_1, \hat{\Psi}_2 : \mathbb{R}^{n_2} \times \mathbb{R}^{l_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{l_2} \rightarrow (-\infty, +\infty] \) by

\[
\hat{\Psi}_1(y, p, x, q) := \begin{cases} \Psi_1(y, p, x, q) + g_1(y, p) & \text{if } y \geq 0, \\
\infty & \text{otherwise}
\end{cases}
\]

\[
\hat{\Psi}_2(y, p, x, q) := \begin{cases} \Psi_2(y, p, x, q) + g_2(x, q) & \text{if } x \geq 0, \\
\infty & \text{otherwise}
\end{cases}.
\]

Finally, define the sets \( S_1 := \mathbb{R}^{n_2} \times \mathbb{R}^{l_1} \) and \( S_2 := \mathbb{R}^{n_1} \times \mathbb{R}^{l_2} \) and consider the GSP\((\Psi_1, \Psi_2, g_1, g_2)\): Find a quadruplet \((y^*, p^*, x^*, q^*)\) \( \in S_1 \times S_2 \) such that \( \forall (y, p, x, q) \in S_1 \times S_2 \), it holds that

\[
\hat{\Psi}_1(y^*, p^*, x^*, q^*) \leq \hat{\Psi}_1(y, p, x^*, q^*) \text{ and } \hat{\Psi}_2(y^*, p^*, x^*, q^*) \leq \hat{\Psi}_2(y^*, p^*, x, q).
\]
Indeed, the unconstrained objective in (1), say the primal LCP (1) and the dual LCP (2) equals the primal-dual gap of GSP(Ψ1, Ψ2, g1, g2).

Proof. It suffices to show that at any point \((y, p, x, q) \in S1 \times S2\), the primal-dual gap between the primal LCP (1) and the dual LCP (2) equals the primal-dual gap of GSP(Ψ1, Ψ2, g1, g2). Indeed, the unconstrained objective in (1), say \(a(x, y)\), can be computed as

\[
a(y, p) = \langle e_1, p \rangle + i_{y \geq 0} + i_{A_gey + E_T^2p > 0} + i_{E_2y = e_2} = g_1(y, p) + \max_{x \geq 0} \langle x', Ay - E_1^Tp \rangle + \max_{q} \langle q', E_2y - e_2 \rangle
\]

Similarly, the unconstrained objective, say \(b(x, q)\), in the dual LCP (2) writes

\[
b(x, q) = -\langle q, e_2 \rangle - i_{x \geq 0} - i_{A_xT + E_T^2q > 0} - i_{E_1x = e_1}
\]

Thus, noting that \(-\infty \leq \phi_1(x, q), \phi_2(y, p), < +\infty\) (so that all the operations below are valid), one computes the primal-dual gap between the primal LCP (1) and dual the LCP (2) at \((y, p, x, q)\) as

\[
a(y, p) - b(x, q) = g_1(y, p) - \phi_2(y, p) + g_2(x, q) - \phi_1(x, q)
\]

3.2 Interlude: Proximal calculus

Let us introduce some basic but powerful modern convex-analytical notions which will be essential in the sequel. Given a subset \(C\) of \(\mathbb{R}^n\), \(i_C\) denotes its indicator function defined by

\[
i_C(x) = 0 \text{ if } x \in C \text{ and } +\infty \text{ otherwise.} \tag{15}
\]

At times, we will write \(i_{x \in C}\) for \(i_C(x)\) (to ease notation, etc.). For example, we will write \(i_{z \geq 0}\) for \(i_{\mathbb{R}^n_+}(z)\), etc. Let \(f : \mathbb{R}^n \to (-\infty, +\infty]\) be a convex function. The orthogonal projector onto \(C\), is the ”closes-point function”

\[
\text{proj}_C : \mathbb{R}^n \to C, x \mapsto \arg\min_{z \in C} \frac{1}{2} ||z - x||^2. \tag{16}
\]
The effective domain of \( f \), denoted \( \text{dom}(f) \), is defined as
\[
\text{dom}(f) := \{ x \in \mathbb{R}^n | f(x) < +\infty \}. \tag{17}
\]
If \( \text{dom}(f) \neq \emptyset \) then we say \( f \) is proper. The subgradient of \( f \) is the set-valued function
\[
\partial f : \mathbb{R}^n \to 2^{\mathbb{R}^n}, \quad x \mapsto \{ s \in \mathbb{R}^n | f(z) \geq f(x) + \langle s, z - x \rangle, \forall z \in \mathbb{R}^n \}. \tag{18}
\]
Of course \( \partial f(x) \) reduces to the singleton \( \{ \nabla f(x) \} \) in case \( f \) is differentiable at \( x \). If \( f \) is convex, its proximal operator is the function \( \text{prox}_f : \mathbb{R}^n \to \mathbb{R}^n \) defined by
\[
\text{prox}_f(x) := \min_{z \in \mathbb{R}^n} \frac{1}{2} \| z - x \|^2 + f(z). \tag{19}
\]
For example, if \( C \) is a closed convex subset of \( \mathbb{R}^n \), then \( \text{prox}_C = \text{proj}_C \), the orthogonal projector onto \( C \). Thus proximal operators generalize orthogonal projectors. For example \( \text{prox}_{\epsilon f}(z) \equiv \text{proj}_{\mathbb{R}_+^n}(z) \equiv (z)_+ \). One also has the useful characterization
\[
p = \text{prox}_f(x) \iff x - p \in \partial f(p).
\]
Thus given \( \gamma > 0 \), the operator \( \text{prox}_{\gamma f} \) can (and should) be thought of as performing an implicit gradient step of size \( \gamma \).

The interested reader should refer to [5] and the references therewithin, for a more elaborate exposition on proximal calculus and its use in modern convex-optimization.

### 3.3 A more comfortable notion of approximate Nash equilibrium

**Definition 3.** Given a scalar \( \epsilon > 0 \) and a function \( f : \mathbb{R}^n \to [-\infty, +\infty] \), the \( \epsilon \)-enlarged subgradient (or \( \epsilon \)-subgradient, for short) of \( f \) is the set-valued function
\[
\partial_{\epsilon} f : \mathbb{R}^n \to 2^{\mathbb{R}^n}, \quad x \mapsto \{ s \in \mathbb{R}^n | f(z) \geq f(x) + \langle s, z - x \rangle - \epsilon, \forall z \in \mathbb{R}^n \}. \tag{21}
\]
The idea behind \( \epsilon \)-subgradients is the following. Say we wish to minimize the function \( f \). Replace the usual necessary condition “\( 0 \in \partial f(x) \)” for the optimality of \( x \) with the weaker condition “\( \partial_{\epsilon} f(x) \)” contains a sufficiently small vector \( v \). In fact, it is easy to see that, for each point \( x \in \mathbb{R}^n \), we have \( \lim_{\epsilon \to 0^+} \partial_{\epsilon} f(x) = \partial f(x) \).

This approximation concept for subgradients yields the following concept of approximate Nash equilibria (adapted from [8]).

**Definition 4 (Nash \((\epsilon_1, \epsilon_2)\)-equilibria).** Given tolerance levels \( \epsilon_1, \epsilon_2 > 0 \), a Nash \((\epsilon_1, \epsilon_2)\)-equilibrium for the GSP \([14]\) is any quadruplet \((x^*, y^*, x^*, q^*)\) for which there exists a perturbation vector \( v^* \) such that
\[
\| v^* \| \leq \epsilon_1 \text{ and } v^* \in \partial_{\epsilon_2}[\hat{\psi}_1(.,.,x^*,q^*) + \hat{\psi}_2(y^*,p^*,.,)](y^*,p^*,x^*,q^*). \tag{22}
\]
Such a vector \( v^* \) is called a Nash \((\epsilon_1, \epsilon_2)\)-residual at \((x^*, y^*, x^*, q^*)\).

The above definition is a generalization of the notion of Nash equilibria since:

- Exact Nash equilibria correspond to Nash \((0,0)\)-equilibria.
- Nash \(\epsilon\)-equilibria (in the sense of Definition 1) correspond to Nash \((0, \epsilon)\)-equilibria.

### 3.4 The proposed algorithm

We now derive the algorithm (Algorithm 1) which is the main object of this manuscript, and establish the theoretical properties.

**Theorem 5 (Ergodic / Cesário O(1/\epsilon) convergence).** Let \( d_0 \) be the orthogonal distance between the starting point \((y^{(0)}, p^{(0)}, x^{(0)}, q^{(0)})\) of Algorithm 1 and the set of equilibria for the GSP \([14]\). Then given any \( \epsilon > 0 \), there exists an index \( k_0 \leq \frac{2d_0\|K\|}{\epsilon} \) such that after \( k_0 \) iterations the algorithm produces the following a quadruplet \((y^{(k_0)}, p^{(k_0)}, x^{(k_0)}, q^{(k_0)})\) and a vector \( v^{(k_0)} \) such that
\[
\| v^{(k_0)} \| \leq \epsilon \text{ and } v^{(k_0)} \in \partial_{\epsilon}[\hat{\psi}_1(.,.,x^{(k_0)},q^{(k_0)}) + \hat{\psi}_2(y^{(k_0)},p^{(k_0)},.,)](y^{(k_0)},p^{(k_0)},x^{(k_0)},q^{(k_0)}), \tag{23}
\]
where \( v^{(k_0)} := \frac{1}{k_0} v^{(k_0)} \). Thus Algorithm 1 outputs an \((\epsilon, 0)\)-Nash equilibrium for the GSP \([14]\) in at most \( \frac{2d_0\|K\|}{\epsilon} \) iterations.
Algorithm 1 Primal-dual algorithm for computing approximate Nash Equilibria in two-person zero-sum games with incomplete information and perfect recall

Require: $\epsilon > 0$; $(y^{(0)}, p^{(0)}, x^{(0)}, q^{(0)}) \in \mathbb{R}^{m_2} \times \mathbb{R}^{r_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{l_2}$.
Ensure: A Nash $(\epsilon, 0)$-equilibrium $(y^*, p^*, x^*, q^*) \in S_1 \times S_2$ for the GSP (14).

1: $\lambda \leftarrow 1/\|K\|$, $v^{(0)} \leftarrow 0$, $k \leftarrow 0$
2: while $k = 0$ or $\frac{1}{k} \|v^{(k)}\| \geq \epsilon$ do
3: $y^{(k+1)} \leftarrow (y^{(k)} - \lambda(A^T x^{(k)} + E_2 \Delta x^{(k)}))_+$,
4: $x^{(k+1)} \leftarrow (x^{(k)} + \lambda(y^{(k+1)} - E_1 \Delta y^{(k+1)})_+)$,
5: $\Delta x^{(k+1)} \leftarrow x^{(k+1)} - x^{(k)}$
6: $y^{(k+1)} \leftarrow (y^{(k+1)} - \lambda(A^T \Delta x^{(k+1)} + E_2 \Delta q^{(k+1)}))_+$,
7: $\Delta q^{(k+1)} \leftarrow q^{(k+1)} - q^{(k)}$
8: $p^{(k+1)} \leftarrow (p^{(k+1)} + \lambda y^{(k+1)} \Delta x^{(k+1)})_+$,
9: $\Delta p^{(k+1)} \leftarrow p^{(k+1)} - p^{(k)}$
10: $k \leftarrow k + 1$
end while

Proof. It is clear to see that the quadruplet $(\Psi_1, \Psi_2, g_1, g_2)$ satisfies assumptions B.1, B.2, B.3, B.5, and B.6 of [8] with $L_{xy} = L_{yx} = 0$ and $L_{yx} = L_{yx} = \|K\|$. Now, one easily computes the proximal operator of $g_j$ in closed-form as $\text{prox}_{\lambda g_j}(a, b) \equiv ((a)_+ - \lambda j, b - \lambda e_j)$. With all these ingredients in place, Algorithm 1 is then obtained from [8] Algorithm T-BD, applied on the GSP (14) with the choice of parameters: $\sigma_1 \in (0, 1)$, $\sigma_x = \sigma_y = 0 \in [0, \sigma_1]$, $\lambda_{xy} := \frac{1}{\lambda_{xy}} \sqrt{\sigma_x^2 - \sigma_y^2}$, $\lambda = \lambda_{xy} \in (0, \lambda_{xy}]$. The convergence result follows immediately from [8] Theorem 4.2. $\square$

3.5 Practical considerations

(a) Efficient computation of $A y$ and $A^T x$. In Algorithms 1 most of the time is spent pre-multiplying vectors by $A$ and $A^T$ (precisely 3 such operations are done per iteration). For flop-type poker games like Texas Hold’em and Rhode Island Hold’em, $A$ (and thus $A^T$ too) is very big (up to $10^{14}$ rows and columns!) but has a rich block-diagonal structure which can be carefully exploited, as was done in [9]. For the purpose of completeness, we explain the details of the tricks used in [9] for speeding up the computation of these matvec products.

Definition 6. (Kronecker product) Let $F \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{r \times s}$ be matrices. The Kronecker product of $F$ and $B$ is defined by

\[
F \otimes B := \begin{bmatrix}
F_{0,0}B & F_{0,1}B & \cdots & F_{0,n-1}B \\
F_{1,0}B & F_{1,1}B & \cdots & F_{1,n-1}B \\
\vdots & \vdots & & \vdots \\
F_{m-1,0}B & F_{m-1,1}B & \cdots & F_{m-1,n-1}B
\end{bmatrix} \in \mathbb{R}^{mr \times ns}
\tag{24}
\]

One notes that $(F \otimes B)^T = F^T \otimes B^T$. $F \otimes B$ can be very much larger than both $F$ and $B$ in dimensions. For example, if $r = n = r = s = 1000$, so that $F$ and $B$ are $10^3$-by-$10^3$ matrices, then $F \otimes B$ is a $10^6$-by-$10^6$ matrix! Fortunately, given a vector $x \in \mathbb{R}^s$ one can compute the matvec product $(F \otimes B)x$ without forming $F \otimes B$. Indeed, let $y := (F \otimes B)x$. We rewrite the row vector $x$ as an $s$-by-$r$ matrix $X = [X_0, X_1, \ldots, X_{r-1}]$, where each column vector $X_i$ is the $i$th block of $s$ elements of $x$ read from left to right. Similarly, rewrite $y$ as an $n$-by-$m$ matrix $Y = [Y_0, Y_1, \ldots, Y_{m-1}]$. Then one easily verifies that $Y$ is the matrix product of the “small” matrices $B$, $X$, and $F^T$, i.e

\[
Y = BXF^T
\tag{25}
\]

Now, in Texas Hold’em for example, the payoff matrix $A$ can be written as block-diagonal matrix whose blocks are sums of Kronecker products of much smaller sparse matrices as follows
The pair of vectors \((0, A)\) curves are shown in Fig 2. One easy checks that this equilibrium is feasible. Indeed, one computes

\[
A = \begin{bmatrix}
F_1 \otimes B_1 \\
F_2 \otimes B_2 \\
F_3 \otimes B_3 \\
F_4 \otimes B_4 + S
\end{bmatrix}
\]

See [9]. The matrices \(F_i\) correspond to sequences of moves in round \(i\) which end in a fold action, and \(S\) to the sequences which end in a showdown. \(B_i\) encodes the betting structure of round \(i\), while \(W\) encodes the wind/lose/draw information determined by ranking the players’ hands at showdown. The component matrices \(F_i, B_i, S,\) and \(W\) are small enough to be explicitly represented whereas it is infeasible to explicitly represent \(A\). Furthermore, the matrices \(F_i, B_i, S,\) and \(W\) are themselves sparse, which allows one to use the compressed row storage data structure that only stores nonzero entries (for example scipy.sparse.csr_matrix, in the Python programming language).

Such a representation of the payoff matrix \(A\) trivializes matvec operations involving \(A\) or \(A^T\) (thanks to formula (25) above, applied to their diagonal blocks and exploiting the sparsity of these blocks themselves). Of course, one can always write the payoff matrix \(A\) of a flop-type poker game in a form similar to (26) by applying appropriate permutations to the enumeration of the players’ sequences.

(b) Computing \(\|K\|\). Also the 2-norm \(\|K\|\) of the linear operator \(K\), can be efficiently computed using the power iteration (Perron-Frobenius).

4 Experimental results

To access the practical quality of the proposed algorithm, we tested it on simulated and real games.

(a) Experiments on sequential games with incomplete information: Kuhn 3-card poker.

This game is a simplified form of poker developed by Harold W. Kuhn. It is a two-person zero-sum game which is simple enough to serve as a proof-of-concept example but contains all the complexity traits of an incomplete information sequential game. The deck includes only three playing cards: a King, Queen, and Jack. One card is dealt to each player, then the first player must bet or pass, then the second player may bet or pass. If any player chooses to bet the opposing player must bet as well (“call”) in order to stay in the round. After both players pass or bet the player with the highest card wins the pot. The sequence-form representation of the game has \(n_1 = n_2 = 13\) sequences and \(l_1 = l_2 = 7\) information sets per player. The sequence-form of the game is given by

\[
E_i \in \mathbb{R}^{7 \times 13} \text{ with } E_1(0, 0) = E_1(1, 9) = E_1(1, 12) = E_1(2, 1) = E_1(2, 4) = E_1(3, 5) = E_1(3, 8) = E_1(4, 2) = E_1(4, 3) = E_1(5, 6) = E_1(5, 7) = E_1(6, 10) = E_1(6, 11) = 1, \\
E_1(1, 0) = E_1(2, 0) = E_1(3, 0) = E_1(4, 1) = E_1(5, 5) = E_1(6, 9) = -1; \\
E_2(0, 0) = E_2(1, 7) = E_2(1, 8) = E_2(2, 9) = E_2(2, 10) = E_2(3, 5) = \\
E_2(3, 6) = E_2(4, 11) = E_2(4, 12) = E_2(5, 1) = E_2(5, 2) = E_2(6, 3) = E_2(6, 4) = 1, \\
E_2(1, 0) = E_2(2, 0) = E_2(3, 0) = E_2(4, 0) = E_2(5, 0) = E_2(6, 0) = -1; \text{ and } A \in \mathbb{R}^{13 \times 13} \\
\text{with } A(3, 8) = A(3, 12) = A(4, 6) = A(4, 10) = A(7, 12) = A(8, 10) = -0.333333, A(1, 7) = A(1, 11) = A(2, 8) = A(2, 12) = A(5, 11) = A(6, 4) = A(6, 12) = A(10, 4) = A(10, 8) = -0.166667, A(7, 4) = A(8, 2) = A(11, 4) = A(11, 8) = A(12, 2) = A(12, 6) = 0.333333, \\
A(4, 5) = A(4, 9) = A(5, 3) = A(8, 1) = A(8, 9) = A(9, 3) = A(9, 7) = A(12, 1) = A(12, 5) = 0.166667.
\]

The pair of vectors \((x^*, y^*)\) \(\in \mathbb{R}^{13+13}\) plans given by

\[
x^* = [1, 0.759, 0.759, 0, 0.241, 1, 0.425, 0.575, 0, 0.275, 0, 0.275, 0.725]^T,
\]

\[
y^* = [1, 1, 0, 0.667, 0.333, 0.667, 0.333, 1, 0, 0, 1, 0, 1]^T
\]

is a Nash \((10^{-4}, 0)\)-equilibrium computed in 1500 iterations of Algorithm [3]. The convergence curves are shown in Fig[2]. One easy checks that this equilibrium is feasible. Indeed, one computes
Finally, one checks that $x^* y^* = -0.05555$, which agrees to 5 d.p with the value of $-1/18$ computed analytically by H. W. Kuhn in his 1950 paper [12]. The evolution of the dual gap and the expected value of the game across iterations are shown in Figure 2. We have not benchmarked this against the algorithms proposed in [15] and Gilpin’s et al. [7] because implementing them from scratch for such games would require us to compute the complicated projections $\text{prox}_{Q_k}$. We recall that avoiding these projections was one of the goals of the manuscript.

(b) Basic test-bed: Matrix games on simplexes. As in [15, 3], we generate a $1000 \times 1000$ random matrix whose entries are uniformly identically distributed in the closed interval $[-1, 1]$. We compare our proposed Algorithm 1 with Nesterov’s [15] and Gilpin’s et al. [7]. The results of the benchmarks are shown in Figure 2 (a).

Figure 2: Convergence curves of Algorithm 1. In (a), the duality gaps are computed according to formula (8). One can see the linear (i.e exponential) behavior of the algorithm in [7], inbetween consecutive breakpoints on the $c$ grid (though the rate of exponential growth seems to by quite close to 1 here). The first-order smoothing algorithm from [15] jitters around as the iterations go on because even the smoothed problem becomes heavily ill-conditioned near solutions. On the other hand, our proposed algorithm beats both of the aforementioned algorithms, and its proven $O(1/k)$ convergence rate is clearly verified empirically.
5 Concluding remarks

Making use of the sequence-form representation \[10, 17, 16\], we have devised a simple and efficient primal-dual algorithm for computing Nash equilibria in two-person zero-sum sequential games with incomplete information (like Texas Hold’em, etc.). Our algorithm is simple to implement, with a low constant cost per iteration, and enjoys a rigorous convergence theory with a proven $O(1/\varepsilon)$ convergence in terms of basic operations (matvec products, clipping, etc.), to a Nash $(\varepsilon, 0)$-equilibrium of the game.

Equilibrium problems are saddle-point convex-concave problems, and as such a natural choice for algorithms for solving them would be in the family of primal-dual algorithms. We believe such primal-dual schemes will receive more attention in the algorithmic game theory community in future.

Software. The authors’ implementation of the proposed algorithm is available upon request.

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