Existence Theorems for Fractional Semilinear Integrodifferential Equations with Noninstantaneous Impulses and Delay

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In this paper, we consider a class of fractional semilinear integrodifferential equations with noninstantaneous impulses and delay. By the semigroup theory and fixed point theorems, we establish various theorems for the existence of mild solutions for the problem. An example involving partial differential equations with noninstantaneous impulses is given to show the application of our main results.

1. Introduction

Over the last couple of decades, fractional differential equations have been applied successfully to model many phenomena in physics, engineering, chemistry, financial, biology, etc. Consequently, the subject of fractional differential equations has attracted more and more attention worldwide and for more details, see for example [1–19] and the references therein.

Meanwhile, differential equations with impulsive effects have been used widely as mathematical models for the study of many phenomena in physical, biology, optimal control model of economics, etc. Much attention has been paid to the existence of solutions for the differential equations with impulses in abstract space. For details, see [7, 20–28].

In [26], the author initially studied the differential equations with noninstantaneous impulse effects as follows:

\[
\begin{align*}
&u'(t) = Au(t) + f(t, u(t)), \quad t \in (s_k, t_{k+1}], \quad k = 0, 1, \ldots, N, \\
&u(t) = g_k(t, u(t)), \quad t \in (t_k, s_k], \quad k = 1, 2, \ldots, N, \\
&u(0) = u_0, \\
&\Delta u_{|t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \ldots, m,
\end{align*}
\]

where \( A : D(A) \subset E \to E \) is the generator of a \( C_0 \)-semigroup of bounded operators \( \{T(t)\}_{t \geq 0} \) defined on a Banach space \( E \).

In [20], the author studied the following integer order integrodifferential equations with instantaneous impulses in a Banach space \( E \):

\[
\begin{align*}
&u'(t) = Au(t) + f\left(t, u(t_1(t)), u(t_2(t)), \ldots, u(t_n(t))\right) + \int_0^t h(t, s, u(t_{n+1}(s))) \, ds, \\
&t \in [0, b], \quad t \neq t_k, \quad k = 1, 2, \ldots, m, \\
&u(0) + g(u) = u_0, \\
&\Delta u_{|t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \ldots, m,
\end{align*}
\]
where for any $t \geq 0$, the linear operator $A$ is the infinitesimal generator of a compact, analytic semigroup, and the nonlinear term is Lipschitz continuous. The existence of mild solutions has been proved.

In this paper, we investigate the following fractional semilinear integrodifferential equations with noninstantaneous impulses and delay:

\begin{equation}
\begin{aligned}
\mathcal{D}_t^\beta\mathcal{E}(t)u(t) &= A(t)u(t) + f \left( t, u(\tau_1(t)), u(\tau_2(t)), \ldots, u(\tau_n(t)) \right), \\
&= \int_0^t h(t, s, u(\tau_{n+1}(s)))ds,
\end{aligned}
\end{equation}

where $\beta \in (0, 1]$, $\mathcal{D}_t^\beta$ is the Caputo’s fractional derivative of order $\beta$, $A(t)$ is a closed and linear operator with domain $D(A)$ defined on a Banach space $E$, and the fixed points $s_j$ and $t_i$ satisfying $0 = s_0 < t_1 \leq s_1 < t_2 \leq \ldots < t_N < s_N < t_{N+1} = T_0$ are prefixed numbers. $f_i, I_{s_j}, h, g (k = 1, 2, \ldots, N)$ and $\tau_j (i = 1, 2, \ldots, n + 1)$ are to be specified later.

Inspired by the results mentioned above, by the semigroup theory and fixed point theorems, we consider the existence of mild solutions for the fractional semilinear integrodifferential equations with noninstantaneous impulses and delay (3). In [7], the authors discussed the existence of solutions for the fractional ordinary differential equation with a generalized impulsive term. In [20–23], the authors discussed the integer or fractional differential equations with instantaneous impulses and the linear operator $A$ is independent of $t$. In [26, 29, 30], the authors discussed the integer-order differential equations with noninstantaneous impulses and the linear operator $A$ is independent of $t$. In [31–33], the authors discussed the fractional differential equations with noninstantaneous impulses and the linear operator $A$ is also independent of $t$. In this paper, we consider the fractional semilinear integrodifferential equations with noninstantaneous impulses and delay, and the linear operator $A(t)$ is assumed to be dependent on $t$. Therefore, the mentioned results above are special cases of the problem investigated in this paper. Our results improve and generalize the results in References [7, 20–23, 26, 29–33].

The rest of this paper is organized as follows. In Section 2, we present the basic notation and preliminary results. In Section 3, we prove the existence of mild solutions. In Section 4, we give an illustrative example, followed by the conclusion of this paper in Section 5.

2. Preliminaries

Let $(E, \| \cdot \|)$ be a Banach space, $J = [0, T_0]$, $C(J, E) = \{ u : J \rightarrow E \text{ is continuous} \}$, $PC(J, E) = \{ u : J \rightarrow E : u \in C([s_k, t_{k+1}], E) \}$ and there exist $u(t_k)$ and $u(t_k^*)$ with $u(t_k) = u(t_k^*)$, $k = 1, 2, \ldots, N$ with the PC-norm $\| u \|_{PC} = \sup \{ \| u(t) \| : t \in J \}$.

\begin{equation}
\begin{aligned}
\mathcal{D}_t^\beta\mathcal{E}(t)u(t) &= A(t)u(t) + f \left( t, u(\tau_1(t)), u(\tau_2(t)), \ldots, u(\tau_n(t)) \right), \\
&= \int_0^t h(t, s, u(\tau_{n+1}(s)))ds,
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Lemma 5. (Sadovskii fixed point theorem). Let $B$ be a convex, closed, and bounded subset of a Banach space $E$ and $Q : B \rightarrow B$ be a condensing map. Then, $Q$ has one fixed point in $B$.

3. Main Results

For convenience in presentation, we give here the basic assumptions to be used later throughout the paper.

(H$_1$) The function $f : J \times E^{n+1} \rightarrow E$ is continuous, and for all $r > 0$, there exist nonnegative Lebesgue integrable functions $L_i \in L^1(J, R^+)$ ($i = 1, 2, \ldots, n + 1$) such that for any $t \in J$, $x, y, \overline{y} \in T_r = \{x \in E : \|x\| \leq r\}$ ($i = 1, 2, \ldots, n + 1$), we have

$$\|f(t, x_1, x_2, \ldots, x_{n+1}) - f(t, y_1, y_2, \ldots, y_{n+1})\| \leq \sum_{i=1}^{n+1} L_i(t) \|x_i - y_i\|.$$ (8)

(H$_2$) The function $f : J \times E \rightarrow E$ is continuous, and for all $r > 0$, there exists a nonnegative Lebesgue integrable function $L_h(t, s) \in L^1(J \times J, R^+)$ such that for any $t \in J, x, y \in T_r$, we have

$$\|h(t, s, x) - h(t, s, y)\| \leq L_h(t, s) \|x - y\|.$$ (9)

(H$_3$) $\tau : J \rightarrow J$ are continuous such that $0 \leq \tau_i(t) \leq t$ ($i = 1, 2, \ldots, n + 1$) for all $t \in J$.

(H$_4$) The functions $I_k : J \times E \rightarrow E$ and $g : PC(J, E) \rightarrow E$ are continuous, and for all $r > 0$, there exist nonnegative constants $L_{I_k}, L_g$ such that for any $t \in J, x, y \in T_r, u, v \in PC(J, E)$, we have

$$\|I_k(t, x) - I_k(t, y)\| \leq L_{I_k} \|x - y\|, \quad k = 1, 2, \ldots, N,$$

$$\|g(u) - g(v)\| \leq L_g \|u - v\|_{PC}.$$ (10)

$$r \geq \max \left\{\frac{\|I_k(t, 0)\|}{1 - L_{I_k}}, \frac{M}{1 - M} \left(\frac{\|g(0)\|_{PC} + \int_0^{t'} \|f(s, 0, 0, \ldots, 0)\| ds + \int_0^{t'} \|h(s, 0)\| ds}{\int_0^{t'} \sum_{i=1}^{n+1} L_i(s) ds + \int_0^{t'} \sum_{i=1}^{n+1} L_{I_k}(s) ds + \int_0^{t'} L_{I_k}(s) \|I_k(t, 0)\| ds} \right)\right\}, \quad k = 1, 2, \ldots, N\right\}. \quad (12)$$

Proof. Define an operator $Q : PC(J, E) \rightarrow PC(J, E)$ by

$$Q(t) = \left\{ \begin{array}{ll}
U(t_0)(u_0 - g(u)) + \int_0^{t'} U(t, s) f(s, u(\tau_1(s)), \ldots, u(\tau_n(s))), h(s, \sigma, u(\tau_{n+1}(\sigma))) ds, & t \in [0, t_1], \\
I_k(t, u(t)), & t \in (t_k, t_{k+1}], \quad k = 1, 2, \ldots, N, \\
U(t, s) I_k(s, u(s)) + \int_0^{t'} U(t, s) f(s, u(\tau_1(s)), \ldots, u(\tau_n(s))), h(s, \sigma, u(\tau_{n+1}(\sigma))) ds, & t \in (s_k, t_{k+1}] , \quad k = 1, 2, \ldots, N. \end{array} \right.$$ (13)

Then, there exists a unique mild solution $u^*(t)$ of the problem (3) on $PC(J, E)$ and $\|u^*\|_{PC} \leq r$, where

Theorem 6. Assume $\rho = \max \{\Delta, M(\Delta + N_1 + N_2)\} < 1$ and conditions (H$_1$)-(H$_4$) hold, where $M = \max \{L_{I_k}(t), s, k = 1, 2, \ldots, N\}$,

$$N_1 = \max \left\{\int_0^{t_1} \sum_{i=1}^{n+1} L_i(s) ds, \int_0^{t_1} \sum_{i=1}^{n+1} L_{I_k}(s) ds, k = 1, 2, \ldots, N\right\},$$

$$N_2 = \max \left\{\int_0^{t_1} \sum_{i=1}^{n+1} L_i(s) ds, \int_0^{t_1} \sum_{i=1}^{n+1} L_{I_k}(s) ds, k = 1, 2, \ldots, N\right\}. \quad (11)$$

$$r \geq \max \left\{\frac{\|I_k(t, 0)\|}{1 - L_{I_k}}, \frac{M}{1 - M} \left(\frac{\|g(0)\|_{PC} + \int_0^{t'} \|f(s, 0, 0, \ldots, 0)\| ds + \int_0^{t'} \|h(s, 0)\| ds}{\int_0^{t'} \sum_{i=1}^{n+1} L_i(s) ds + \int_0^{t'} \sum_{i=1}^{n+1} L_{I_k}(s) ds + \int_0^{t'} L_{I_k}(s) \|I_k(t, 0)\| ds} \right)\right\}, \quad k = 1, 2, \ldots, N\right\}. \quad (12)$$

Proof. Define an operator $Q : PC(J, E) \rightarrow PC(J, E)$ by

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I_k(t, u(t)), & t \in (t_k, t_{k+1}], \quad k = 1, 2, \ldots, N, \\
U(t, s) I_k(s, u(s)) + \int_0^{t'} U(t, s) f(s, u(\tau_1(s)), \ldots, u(\tau_n(s))), h(s, \sigma, u(\tau_{n+1}(\sigma))) ds, & t \in (s_k, t_{k+1}] , \quad k = 1, 2, \ldots, N. \end{array} \right.$$ (13)
Let $B_r = \{ u \in \text{PC}(J, E) : \| u \|_{PC} \leq r \}$. We will prove $Q(B_r) \subset B_r$. Let $u \in B_r$, for $t \in [0, t_1]$, we have

$$
\|(Qu)(t)\| \leq M(L_g u) + M \left( \int_0^t \sum_{i=1}^n L_i(s) \|u(t_i, s)\| ds + M \int_0^t L_{n+1}(s) \right)
$$

\begin{align*}
&\cdot \|f(s, 0, 0, \cdots, 0)\| ds + M \int_0^t L_{n+1}(s) \\
&\cdot \left( \int_0^t L_h(s, \sigma) \|u(t_{r+1}(\sigma))\| d\sigma \right) ds \\
&+ \int_0^t \|h(s, \sigma, 0)\| d\sigma ds \leq M \left( \|g(0)\|_{PC} \right) \\
&+ \int_0^t \|f(s, 0, 0, \cdots, 0)\| ds + M \int_0^t \int_0^t \|h(s, \sigma, 0)\| ds \\
&\cdot \|d\sigma ds\| + M \left( \int_0^t \sum_{i=1}^n L_i(s) ds \right) \\
&+ \int_0^t L_{n+1}(s) \int_0^t L_h(s, \sigma) d\sigma ds r \leq r.
\end{align*}

(14)

For all $t \in (s_k, s_{k+1}]$ $(k = 1, 2, \cdots, N)$, we have

$$
\|(Qu)(t)\| \leq M \left( \int_0^t \sum_{i=1}^n L_i(s) \right)
$$

\begin{align*}
&\cdot \|f(s, 0, 0, \cdots, 0)\| ds + M \int_0^t \sum_{i=1}^n L_i(s) ds \\
&\cdot \left( \int_0^t L_h(s, \sigma) d\sigma + \int_0^t \|h(s, \sigma, 0)\| d\sigma \right) ds \\
&\leq M \left( \left\| k(s_k, 0) \right\| + \int_0^t \|f(s, 0, 0, \cdots, 0)\| ds \right) \\
&+ \int_0^t \int_0^t \|h(s, \sigma, 0)\| d\sigma ds + M \left( \int_0^t \sum_{i=1}^n L_i(s) ds \right) \\
&+ \int_0^t \int_0^t L_n(s) ds + \int_0^t \int_0^t \|L_n(s)\| d\sigma ds \cdot r \leq r.
\end{align*}

(15)

For all $t \in (t_k, s_k]$(k = 1, 2, \cdots, N), we have

$$
\|(Qu)(t)\| \leq L_0 r + \|k(t, 0)\| \leq r.
$$

(16)

From the above inequalities, we have that $Qu \in B_r$. Next, we prove that $Q$ is a contraction map on $B_r$. For all $u, v \in \text{PC}(J, E)$ and $t \in [0, t_1]$, we have

$$
\| (Qu)(t) - (Qv)(t) \| \leq \| U^\alpha(t, 0)(g(u) - g(v)) \|
$$

\begin{align*}
&+ M \int_0^t \left\| f(s, u(t_1(s)), u(t_2(s)), \cdots) \right\| ds \\
&\cdot \|h(s, \sigma, u(t_{r+1}(\sigma))) d\sigma\| \\
&- f(s, v(t_1(s)), v(t_2(s)), \cdots) ds \\
&\cdot \|h(s, \sigma, v(t_{r+1}(\sigma))) d\sigma\| \\
&\leq ML_q \|u - v\|_{PC} + M \int_0^t \sum_{i=1}^n L_i(s) ds \\
&\cdot \|u(t_i(s)) - v(t_i(s))\| ds.
\end{align*}

(17)

For all $t \in (s_k, s_{k+1}]$ $(k = 1, 2, \cdots, N)$, we have

$$
\| (Qu)(t) - (Qv)(t) \| \leq \| U^\alpha(t, s_k)(k(s_k, u(s_k)) - k(s_k, v(s_k))) \|
$$

\begin{align*}
&+ M \int_0^t \left\| f(s, u(t_1(s)), u(t_2(s)), \cdots) \right\| ds \\
&\cdot \|h(s, \sigma, u(t_{r+1}(\sigma))) d\sigma\| \\
&- f(s, v(t_1(s)), v(t_2(s)), \cdots) ds \\
&\cdot \|h(s, \sigma, v(t_{r+1}(\sigma))) d\sigma\| \\
&\leq ML_q \|u - v\|_{PC} + M \int_0^t \sum_{i=1}^n L_i(s) ds \\
&\cdot \|u(t_i(s)) - v(t_i(s))\| ds.
\end{align*}

(18)
We introduce the decomposition

\[ M \int_{t_{n-1}}^{t} L_{n+1}(s) \left\| h(s, \sigma, u(\tau_{n+1}(\sigma))) \right\| ds \]

\[ - \int_{0}^{t} h(s, \sigma, v(\tau_{n+1}(\sigma))) ds \| ds \leq ML_{q} \| u - v \|_{PC} \]

\[ + M \int_{t_{n-1}}^{t} \sum_{i=1}^{n} L_{i}(s) ds \| u - v \|_{PC} + M \int_{t_{n-1}}^{t} L_{n+1}(s) \int_{0}^{t_{n}} L_{n}(s, \sigma) \]

\[ \cdot d\sigma ds \| u - v \|_{PC} \leq M (L_{q} + N_{1} + N_{2}) \| u - v \|_{PC}. \tag{18} \]

For all \( t \in (t_{k}, s_{k}) \), we have

\[ \| (Qu)(t) - (Qv)(t) \| = \| L_{k}(t, u(t)) - l_{k}(t, v(t)) \| \leq L_{q} \| u - v \|_{PC}. \tag{19} \]

From the above results, for all \( u, v \in PC(J, E) \), we have

\[ \| Qu - Qv \|_{PC} \leq \rho \| u - v \|_{PC}. \tag{20} \]

Therefore, \( Q : B_{r} \rightarrow B_{r} \) is a contraction map and there exists a unique mild solution \( u^{*} \) of the problem (3) in \( B_{r} \) and \( \| u^{*} \|_{PC} \leq r. \)

**Theorem 7.** Assume that the conditions \((H_{3}), (H_{4}), (H_{4}')\), \((H_{5})\), and \((H_{6})\) hold, \( U_{j}(t, s) \) is compact for \( t, s > 0 \) and

\[ q = \max \left\{ L_{q}, ML_{q}, M \left( L_{q} + \int_{0}^{t_{1}} \sum_{i=1}^{n} m_{j_{i}}(s) ds + \int_{t_{1}}^{t} m_{j_{i}}(s) ds \right), \right. \]

\[ \left. \int_{0}^{t_{1}} m_{j_{i}}(s) ds + \int_{t_{1}}^{t} m_{j_{i}}(s) ds \right\}, \]

\[ M_{k} + \int_{t_{1}}^{t} m_{j_{i}}(s) ds + \int_{t_{1}}^{t} m_{j_{i}}(s) ds, \]

\[ k = 1, 2, \ldots, N \] \( < 1. \)

Then, the problem (3) has a mild solution \( u^{*} \in PC(J, E) \) and \( \| u^{*} \|_{PC} \leq r, \) where

\[ r \geq \max \left\{ M_{1}, M_{2}, \sum_{i=1}^{n} M_{3}, \sum_{i=1}^{n} \int_{t_{1}}^{t} m_{j_{i}}(s) ds + \int_{0}^{t_{1}} \sum_{i=1}^{n} m_{j_{i}}(s) ds \right\}, \]

\[ k = 1, 2, \ldots, N. \] \( \tag{22} \]

**Proof.** We introduce the decomposition \( Q = Q_{1} + Q_{2} \), where \( Q_{j} : PC(J, E) \rightarrow PC(J, E) \) \( (j = 1, 2) \) are defined by

\[ (Q_{1} u)(t) = \]

\[ \begin{cases} U_{j}(t, t_{1})(u_{0} - g(u)), & t \in [0, t_{1}], \\ l_{k}(t, u(t)), & t \in (t_{k}, s_{k}), k = 1, 2, \ldots, N, \\ U_{j}(t, s_{k})l_{k}(s_{k}, u(s_{k})), & t \in (s_{k}, t_{k+1}), k = 1, 2, \ldots, N, 
\end{cases} \]

\[ \int_{0}^{t} U_{j}(t, s)f \left( s, u(\tau_{1}(s)), \ldots, u(\tau_{n}(s)) \right) \int_{0}^{t} h(s, \sigma, u(\tau_{n+1}(\sigma))) ds, & t \in [0, t_{1}], 
\end{cases} \]

\[ (Q_{2} u)(t) = \]

\[ \begin{cases} 0, & t \in (t_{k}, s_{k}), k = 1, 2, \ldots, N, \\ \int_{t_{k}}^{t} U_{j}(t, s)f \left( s, u(\tau_{1}(s)), \ldots, u(\tau_{n}(s)) \right) \int_{0}^{t} h(s, \sigma, u(\tau_{n+1}(\sigma))) ds, & t \in (s_{k}, t_{k+1}), k = 1, 2, \ldots, N. 
\end{cases} \] \( \tag{23} \]

Let \( B_{r} = \{ u \in PC(J, E) : \| u \|_{PC} \leq r \} \), we then prove that the operator \( Q = Q_{1} + Q_{2} \) is a condensing map on \( B_{r} \). It is easy to see that \( B_{r} \) is a closed, bounded, and convex subset of \( PC(J, E) \).
Step 1. We prove $Q(B_r) \subset B_r$. Let $u \in B_r$. For $t \in [0, t_1]$, we have
\[
\|Q(u)(t)\| \leq \|U_p(t, 0)(u - g(u))\| + \int_0^t \|U_p(u, s)\|
\]
\[
\times \|f(s, u(t(s)), u(t(s)), \cdots, u(t(s))) + L_g u_p\| ds + M\int_0^t \sum_{i=1}^n m_f(s) u(t(s)) ds
\]
\[
+ M\int_0^t m_{f_{i+1}}(s) \|h(s, \sigma, u(t_{n+1}(\sigma)))\| ds\leq M\|u_0\| + \|L_g u_p\|
\]
\[
+ M\int_0^t \sum_{i=1}^n m_f(s) u(t(s)) ds + M\int_0^t m_{f_{i+1}}(s) \|h(s, \sigma, u(t_{n+1}(\sigma)))\| ds\leq M\|u_0\| + \|L_g u_p\| + M\int_0^t \sum_{i=1}^n m_f(s) ds\|u_p\| PC
\]
\[
+ M\int_0^t m_{f_{i+1}}(s) \|h(s, \sigma, u(t_{n+1}(\sigma)))\| ds\leq M\|u_0\| + \|L_g u_p\| + M\int_0^t \sum_{i=1}^n m_f(s) ds\|u_p\| PC
\]
\[
+ M\int_0^t m_{f_{i+1}}(s) \|h(s, \sigma, u(t_{n+1}(\sigma)))\| ds\leq M\|u_0\| + \|L_g u_p\| + M\int_0^t \sum_{i=1}^n m_f(s) ds\|u_p\| PC.
\]
(24)

For all $t \in (t_k, s_k)$ $(k = 1, 2, \cdots, N)$, we have
\[
\|Q(u)(t)\| = \|Q(u)(t)\| = \|Q(t, u(t))\| \leq M_s \leq r.
\]
(25)

For all $t \in (s_k, t_{k+1})$, $(k = 1, 2, \cdots, N)$, we have
\[
\|Q(u)(t)\| \leq \|U_p(t, 0)(u - g(u))\| + \int_0^t \|U_p(u, s)\|
\]
\[
\times \|f(s, u(t(s)), u(t(s)), \cdots, u(t(s))) + L_g u_p\| ds + M\int_0^t \sum_{i=1}^n m_f(s) u(t(s)) ds
\]
\[
+ M\int_0^t m_{f_{i+1}}(s) \|h(s, \sigma, u(t_{n+1}(\sigma)))\| ds\leq M\|u_0\| + \|L_g u_p\|
\]
\[
+ M\int_0^t \sum_{i=1}^n m_f(s) u(t(s)) ds + M\int_0^t m_{f_{i+1}}(s) \|h(s, \sigma, u(t_{n+1}(\sigma)))\| ds\leq M\|u_0\| + \|L_g u_p\| + M\int_0^t \sum_{i=1}^n m_f(s) ds\|u_p\| PC
\]
\[
+ M\int_0^t m_{f_{i+1}}(s) \|h(s, \sigma, u(t_{n+1}(\sigma)))\| ds\leq M\|u_0\| + \|L_g u_p\| + M\int_0^t \sum_{i=1}^n m_f(s) ds\|u_p\| PC.
\]
(26)

Hence, $Q(B_r) \subset B_r$.

Step 2. We prove that $Q_1$ is a contraction on $B_r$. For any $u, v \in B_r$ and $t \in [0, t_1]$, we have
\[
\|Q_1(u)(t) - Q_1(v)(t)\| \leq \|U_p(t, 0)(g(u) - g(v))\| \leq ML_g \|u - v\| PC.
\]
(27)

For all $t \in (t_k, s_k)$ $(k = 1, 2, \cdots, N)$, we have
\[
\|Q_1(u)(t) - Q_1(v)(t)\| \leq L_k \|u - v\| PC.
\]
(28)

For all $t \in (s_k, t_{k+1})$, $(k = 1, 2, \cdots, N)$, we have
\[
\|Q_1(u)(t) - Q_1(v)(t)\| \leq \|U_p(t, s_k)(I_k(s_k, u(s_k))) - I_k(s_k, v(s_k))\| \leq ML_k \|u - v\| PC.
\]
(29)

Thus, since $q < 1$, we get that $Q_1$ is a contraction on $B_r$.

Step 3. We prove that $Q_1$ is completely continuous on $B_r$. Firstly, we prove that the operator $Q_1$ is continuous. Let $\{u_m\}_{m=1}^{\infty} \subset B_r$, and $u_m \to u \in B_r$. For any $t \in [0, t_1]$, by $(H_3)$, we have
\[
\|u_m(t_{n+1}) - u(t_{n+1})\| \leq \|u_m - u\| PC \to 0 \quad (m \to \infty),
\]
\[
i = 1, 2, \cdots, n + 1.
\]
(30)

By $(H_3)$,
\[
\|f\left(t, u_m(t_{n+1}), u_m(t_{n+1}), \cdots, u_m(t_{n+1}), h(s, u_m(t_{n+1}(\sigma)))\right)\| \leq f\left(t, u(t_{n+1}), u(t_{n+1}), \cdots, u(t_{n+1}), h(s, u(t_{n+1}(\sigma)))\right),
\]
\[
a.e. \; t \in [0, t_1].
\]
(31)

By the Lebesgue dominated convergence theorem, we have
\[\| (Q_2 u_m) (t) - (Q_2 u) (t) \| \leq M \int_0^t \| f (s, u_m (t_1 (s)), \cdots, u_m (t_n (s)), h (s, \sigma, u_m (t_{n+1} (\sigma))) d \sigma \| ds \to 0,\]
\[(32)\]

as \( m \to \infty \). Therefore, \( \| Q_2 u_m - Q_2 u \|_{PC} \to 0 \), as \( m \to \infty \).

For \( t \in (s_k, t_{k+1}] \) \((k = 1, 2, \cdots, N)\), the proof is similar to that for \( t \in [0, t_1] \). Hence, \( Q_2 \) is continuous. By the proof of Step 1, for any \( t \in J \), it is easy to see that \( Q_2 \) maps a bounded set into a bounded set in \( B_r \).

Secondly, we prove that \( Q_2 (B_r) \) is equicontinuous.

\textbf{Case 1.} For \([0, t_1], 0 \leq e_1 < e_2 \leq t_1, u \in B_r,\]
\[\| (Q_2 u) (e_2) - (Q_2 u) (e_1) \| \leq \left\| \int_0^{e_1} \left( U_{\beta} (e_2, s) - U_{\beta} (e_1, s) \right) \cdot f (s, u (t_1 (s)), \cdots, u (t_n (s))) \cdot h (s, \sigma, u (t_{n+1} (\sigma))) d \sigma d s \right\| + \left\| \int_{e_1}^{e_2} U_{\beta} (e_2, s) \cdot f (s, u (t_1 (s)), \cdots, u (t_n (s))) \cdot h (s, \sigma, u (t_{n+1} (\sigma))) d \sigma d s \right\| \]
\[= I_1 + I_2. \quad (33)\]

By \( (H_3) \), we have
\[I_1 \leq \left( \int_0^{e_1} \sum_{i=1}^{n} m_f (s) d s + \int_0^{e_1} m_{f_{n+1}} (s) \sum_{i=1}^{n} m_h (s, \sigma) d \sigma d s \right) \cdot r \sup_{s \in [0, t_1]} \| U_{\beta} (e_2, s) - U_{\beta} (e_1, s) \|, \]
\[I_2 \leq M \left( \int_{e_1}^{e_2} \sum_{i=1}^{n} m_f (s) d s + \int_{e_1}^{e_2} m_{f_{n+1}} (s) \int_0^{e_1} m_h (s, \sigma) d \sigma d s \right) r. \quad (34)\]

Further, the compactness of \( U_{\beta} (t, s) \) for \( t, s > 0 \) implies that the \( U_{\beta} (t, s) \) for \( t, s > 0 \) is continuous in the sense of uniformly operator topology. Thus, \( I_1 \to 0 \) as \( e_2 \to e_1 \). It is easy to see that \( I_2 \to 0 \) as \( e_2 \to e_1 \) since the functions \( m_f \in L^1 (J, R^+) \) \((i = 1, 2, \cdots, n + 1)\) and \( m_h (t, s) \in L^1 (J \times J, R^+) \). Therefore, \( \| (Q_2 u) (e_2) - (Q_2 u) (e_1) \| \to 0 \) as \( e_2 \to e_1 \to 0 \).

\textbf{Case 2.} For \((s_k, t_{k+1}] \) \((k = 1, 2, \cdots, N)\), \( e_1 \leq e_2 \leq t_{k+1}, u \in B_r,\]
\[\| (Q_2 u) (e_2) - (Q_2 u) (e_1) \| \leq \left\| \int_{s_k}^{t_{k+1}} \left( U_{\beta} (e_2, s) - U_{\beta} (e_1, s) \right) \cdot f (s, u (t_1 (s)), \cdots, u (t_n (s))) \cdot h (s, \sigma, u (t_{n+1} (\sigma))) d \sigma d s \right\| \]
\[= I_3 + I_4. \quad (35)\]

Similar to the proof for Case 1, we get \( \| (Q_2 u) (e_2) - (Q_2 u) (e_1) \| \to 0 \) as \( e_2 \to e_1 \). Therefore, \( Q_2 (B_r) \) is equicontinuous.

Finally, we prove that \( Q_2 (B_r) \) is precompact. For every fixed \( t \) \((0 < t \leq t_1)\) and \( 0 < \varepsilon < t \), let \( u \in B_r \) and define
\[Q_2 (u) (t) = \int_0^{t-\varepsilon} U_{\beta} (t, s) \cdot f (s, u (t_1 (s)), \cdots, u (t_n (s))) \cdot h (s, \sigma, u (t_{n+1} (\sigma))) d \sigma d s. \quad (36)\]

Due to the compactness of \( U (t, s) \) for \( t, s > 0 \), the set \( Y_\varepsilon \)
\[\{ Q_2 (u) (t) : u \in B_r \} \text{ is relatively compact in } E \text{ for any } \varepsilon \text{ } (0 < \varepsilon < t). \]

For any \( u \in B_r \), it follows from
\[\| (Q_2 u) (t) - (Q_2 u) (t') \| \leq \left\| \int_{t-\varepsilon}^{t'} U_{\beta} (t, s) \cdot f (s, u (t_1 (s)), \cdots, u (t_n (s))) \cdot h (s, \sigma, u (t_{n+1} (\sigma))) d \sigma d s \right\| \]
\[= M \left( \int_{t-\varepsilon}^{t'} \sum_{i=1}^{n} m_f (s) d s + \int_{t-\varepsilon}^{t'} m_{f_{n+1}} (s) \int_0^{t-\varepsilon} m_h (s, \sigma) d \sigma d s \right) r. \quad (37)\]

that \( Y (t) = \{ (Q_2 u) (t) : u \in B_r \} \text{ is totally bounded. Thus, } \)
\( Y (t) \text{ is relatively compact in } E. \) Similarly, we can prove that \( Q_2 (B_r) \) \( \subset E \text{ for any } \varepsilon \) \((0 < \varepsilon < t). \)

Using the Arzelà-Ascoli theorem, we have that \( Q_2 : PC [J, E] \to PC [J, E] \) is completely continuous. Hence, the operator \( Q = Q_1 + Q_2 \) is a condensing operator on \( B_r \). By Lemma 5, \( Q \) has a fixed point \( u^* \) in \( B_r \), which is a mild solution of the problem (3).
Remark 8. In this paper, we consider the fractional semilinear integrodifferential equations with noninstantaneous impulses and delay, and the linear operator \( A(t) \) is assumed to be dependent on \( t \). In conditions \((H_1)\) and \((H_2)\), \( I_i \in L^1(J, R^+) \) \((i = 1, 2, \cdots, n + 1)\) and \( L_i(t, \cdot) \in L^1(J \times J, R^+) \) are Lebesgue integrable functions, not constants. The mentioned results above are special cases of the problem investigated in this paper.

4. An Application

In order to show the application of the main results of this paper, we consider the following problem:

\[
\begin{aligned}
D^\alpha_t u(x, t) &= \frac{\partial^2}{\partial x^2} u(x, t) + \frac{t}{1 + t^2} u(x, \sin t) + \frac{1}{1 + t^2} \left( \int_0^t u(x, s) ds \right)^{1/2}, \\
(t, x) &\in \bigcup_{i=1}^n [s_i, s_{i+1}] \times [0, \pi], \\
u(x, t) &= G_i(t, u(x, t)), \quad x \in [0, \pi], \ t \in (t_i, s_i], \\
u(0, t) &= u(\pi, t) = 0, \ 0 \leq t < 1, \\
u(x, 0) &= u_0(x), \quad 0 \leq x \leq \pi.
\end{aligned}
\]

(38)

where \( 0 = s_0 < t_1 < s_1 < \cdots < s_{n-1} < t_n+1 = 1, 0 < \alpha \leq 1, u_0(x) \in E = L^2[0, \pi] \) and the operator \( A : D(A) \subset E \rightarrow E \) is defined by \( A(t)(z) = t(\partial^2 u/\partial x^2) \) with

\[
D(A) = \left\{ u \in E : u'' \in E, u(0) = u(\pi) = 0 \right\}. \quad (39)
\]

It is well known that \( A(t) \) generates a \( \beta \)-resolvent family on \( E \).

Let

\[
\begin{align*}
u(t) &= u(\cdot, t), \\
f(t, x, y) &= \frac{t}{1 + t^2} x \sin t + \frac{1}{1 + t^2} y^{1/2}, \\
l_i(t, u) &= G_i(t, u(\cdot, t)), \\
g(u) &= \frac{1}{3} u \cos u.
\end{align*}
\]

Then, (38) takes the abstract form (3). Assume that all the conditions of Theorems 6 and 7 are satisfied, then the problem (38) has a mild solution on \([0, 1]\).

5. Conclusions

This paper investigates the existence of mild solutions for a class of fractional semilinear integrodifferential equations with noninstantaneous impulses and delay by the fixed point theorem and the semigroup theory. In [7, 20–23, 26, 29–33], the linear operator \( A \) is independent of \( t \), while in this paper, the linear operator \( A(t) \) is allowed to be dependent on \( t \). Therefore, the differential equations in [7, 20–23, 26, 29–33] are special cases of the problem investigated in this paper.

Hence, our results improve and generalize the results of References [7, 20–23, 26, 29–33].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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