Effective Field Theory for Ultrasoft Momenta in NRQCD and NRQED
A. Pineda and J. Soto

We propose an effective field theory for heavy quark-antiquark bound states once the soft gluons have been integrated out. We also give new results for the matching between QCD (QED) and NRQCD (NRQED).

1. INTRODUCTION

Two particle non-relativistic bound states in QCD (QED) have, at least, three well separated scales: the mass $m$ (hard scale), the typical relative momentum $|\vec{p}|$ (soft scale) and the typical bound state energy $E$ (ultrasoft scale). This allows to introduce a hierarchy of effective field theories when sequentially integrating out each of these scales. After integrating out the hard scale $m$ QCD (QED) becomes a non-relativistic theory, the so called NRQCD (NRQED). This effective theory is local in space and time and it is still naturally written in terms of quark fields. Integrating out the soft scale leads to a so far elusive effective field theory which we call potential NRQCD (pNRQCD). We claim that this effective theory is local in time but it is non-local in space and it is naturally written in terms of wave function fields.

We present some original results for the above mentioned effective field theories. In section 2 we give the matching coefficients for the four-quark operators at one loop for NRQED and also for NRQCD in the case of different quark masses. In section 3 we put forward our proposal for pNRQCD and give some results for pNRQED.

2. FROM QCD TO NRQCD

The matching for NRQCD (and HQET) has been known at tree level since long. This can be obtained doing the matching at tree level of S-matrix elements or just performing a F-W transformation of the QCD lagrangian. Some results at one loop have also been known for HQET. Nevertheless, attempts to perform the matching beyond tree level using dimensional regularisation (DR) in NRQCD have not appeared until recently. The problem was that in NRQCD unlike in HQET the kinetic term was incorporated in the propagator. In DR the high modes are not explicitly suppressed by a cut-off $\mu$ ($\mu << m$) and give non-vanishing contributions which break the power counting rules.

Lately several people have addressed this problem and recently the situation has been clarified. There, it is claimed that the matching should be performed just like in HQET. Let us make some comments in favor of this approach. The key point is that when doing the matching it is not so important to know the power counting in the effective theory as to know that the scales of the effective theories are much lower than the mass. The power counting will tell us the relative importance between different operators but this would not change the value of the matching coefficients. That is, we only need

$$m >> |\vec{p}|, E, \Lambda_{QCD}$$

wherever the relation between $|\vec{p}|$, $E$ and $\Lambda_{QCD}$ is.

In ref. DR was used for both ultraviolet (UV) and infrared (IR) divergences in the full and the effective theory. In fact, it is not so important to know the way the UV divergences of the full theory are regulated since the comparison is done...
between S-matrix elements which are UV finite. Nevertheless, it is essential to regulate in the same way the IR divergences of both theories in order to cancel. This will happen since both theories have by construction the same IR behavior. It is also very important, from a practical point of view, to regulate the UV divergences of the effective theory using DR. In this way, in ref. the calculation in the effective theory becomes trivial, since there is no dimensionful parameter in the integrand, and the matching coefficients for the bilinear terms in fermions are calculated at one loop. Nevertheless, the four-quark operators were not taken into account. The way to deal with these operators is not obvious since we are faced with the computation of S-matrix elements of four-quarks in QCD and HQET. It is in these S-matrix elements, which are never calculated in traditional applications of HQET, where the distinct IR behavior of two heavy quark bound states becomes apparent. Power-like IR divergences appear in loops where a quark and an antiquark interact through a potential. We call these divergences Coulomb pole. They are naturally regulated once the kinetic term is introduced.

This IR behavior should appear in both the full and the effective theory. However, it is important to bear in mind that the matching coefficients are independent of the IR behavior. Therefore, we do not need to regulate the IR singularity with the introduction of the kinetic term and hence we can take advantage of a more convenient regularisation. For instance, a regularisation such that we could avoid the effort of computing this pole in both theories (which can be very painful).

The procedure consist of computing the matrix elements on-shell and at threshold (|\vec{p}| = 0). In this way there is no scale in the effective theory, and hence the diagrams in the effective theory are just trivially zero. Therefore, only the diagrams in QCD need to be computed in this peculiar kinematical situation. This precise kinematical situation produces IR divergences which get canceled by those of the effective theory. The computation is done in the \( \overline{MS} \) scheme.

We stress that the Coulomb pole does not appear at all doing the matching in this way. Let us explain what happens. In order to define some integrals we have to move to dimensions high enough for the IR Coulomb singularity to be regulated. When coming back to four dimensions we can trace back the IR Coulomb singularity as poles in dimensions different from four. The point is that we have not introduce the relative momentum and hence DR has no way to reproduce the Coulomb pole or any non-local behavior in the relative momentum. This fact has already been observed in ref. [3], where it was noticed that in HQET with a quark and antiquark moving exactly at the same velocity no imaginary anomalous dimensions occur using DR.

The important thing doing the matching is to take into account all the non-analytical behavior which cannot be obtained in the effective theory. We are taking into account all the non-analytical behavior due to the masses. The remaining non-analytical behavior is encoded in the effective theory.

Let us now give the results for the four-quark effective lagrangian (non-equal mass case)

\[
\delta \mathcal{L}_{NRQCD} = \frac{d_{ss}}{m_1 m_2} \psi_1^\dagger \psi_1 \chi_2^\dagger \chi_2 \\
+ \frac{d_{sv}}{m_1 m_2} \psi_1^\dagger \bar{\sigma} \psi_1 \chi_2^\dagger \bar{\sigma} \chi_2 \\
+ \frac{d_{vs}}{m_1 m_2} \psi_1^\dagger T^a \bar{\sigma} \psi_1 \chi_2^\dagger T^a \bar{\sigma} \chi_2 ,
\]

(2)

\[
d_{ss} = -\frac{N_c^2 - 1}{4 N_c^2} \frac{\alpha_s^2}{m_1^2 - m_2^2} \left\{ m_1^2 \left( \ln \frac{m_1^2}{\nu^2} + \frac{1}{3} \right) \\
- m_2^2 \left( \ln \frac{m_2^2}{\nu^2} + \frac{1}{3} \right) \right\}
\]

(3)

\[
d_{sv} = \frac{2 \alpha_s^2}{m_1^2 - m_2^2} \left\{ m_1^2 \left( \ln \frac{m_1^2}{\nu^2} + \frac{1}{3} \right) \\
- m_2^2 \left( \ln \frac{m_2^2}{\nu^2} + \frac{1}{3} \right) \right\}
\]

(5)
and replace $N$ these results. We just have to omit $d$. For QED, joining all the contributions we are allowed and they should be taken into account although local in time. The important point to be realized is that the appearance of a potential can be understood as the effect of integrating out soft gluons. Hence the potential can be calculated by matching NRQCD to pNRQCD.

The QED coefficients are easily obtained from these results. We just have to omit $d_{os}$ and $d_{vv}$ and replace $\frac{N^2 - 1}{2N_c}$ by 1.

In the equal mass case annihilation processes are allowed and they should be taken into account. For QED, joining all the contributions we get

$$d_{ss} = \frac{3\pi \alpha}{2} \left\{ 1 - \frac{2\alpha}{3\pi} \left( \ln \frac{m_1^2}{\nu^2} + \frac{23}{3} - \ln 2 + \frac{\pi}{2} \right) \right\} \quad (7)$$

$$d_{sv} = -\frac{\pi \alpha}{2} \left\{ 1 - \frac{2\alpha}{\pi} \left( \ln 2 - \frac{22}{9} + \frac{\pi}{2} \right) \right\} \quad (8)$$

These results are compatible with those found by Labelle et al. [10] except for a finite piece in (7). A more detailed explanation of the procedure and the full results at one loop for the equal and non-equal mass case in QCD will be given in ref. [6].

3. FROM NRQCD TO pNRQCD

In the last section we have integrated out the hard gluons. Here, we integrate out soft gluons, with energies of the order of the relative momentum.

Two point functions are insensitive to the relative momentum and hence the bilinear terms in fermions in the NRQCD lagrangian and in the pNRQCD at quark level lagrangian will read exactly the same. However one has to keep in mind that in the latter only gluons with ultrasoft momenta are kept. On the contrary, four point functions do know about relative momentum and generate non-trivial terms in the pNRQCD lagrangian. These terms are nothing but the potential piece of the new lagrangian. Due to the massless nature of the gluons the coefficients are going to be non-local in the relative space coordinate although local in time. The important point is that the appearance of a potential can be understood as the effect of integrating out soft gluons. Hence the potential can be calculated by matching NRQCD to pNRQCD.

We have carried out the matching to a given order in $1/m$ and $\alpha$ using HQET propagators and the Coulomb gauge. This produces a very strong simplification since the kinetic term can be treated perturbatively when computing the potential. Now, it is very easy to know how far we must go in the computation of the potential if we want to compute, for instance, the energy up to order $ma^n$. In this case, we must compute the matching up to order $(\frac{1}{m})^r \alpha^s$, with $s, r$ such that $s + r \le n - 1$. The lowest order just gives the standard Schrödinger equation.

In addition we use DR for both UV and IR divergences, and hence any loop in pNRQCD gives zero because there is no scale. The point is that on the one hand there is no way of reproducing the Coulomb pole since the computation is done with HQET propagators in both theories. On the other hand any loop with ultrasoft gluons in pNRQCD is zero since these are only sensible to the incoming energy and total momentum of the S-matrix element, which we set to zero (we remark that on-shell quarks in HQET have zero energy). Therefore, the calculation from NRQCD leads directly to the potential. Further terms in the pNRQCD lagrangian are obtained when matching four point functions with an arbitrary number of ultrasoft gluon legs.
In order to make explicit the distinction between soft and ultrasoft gluons it is most convenient to project NRQCD or pNRQCD to the two particle sector and promote the wave function \( \psi(\vec{x}_1, \vec{x}_2, t) \) to a field \( \psi(\vec{x}_1, \vec{x}_2, t) \) is a 3 x 3 matrix in color space and a 2 x 2 matrix in spin space). Then the relative coordinate \( \vec{x} = \vec{x}_1 - \vec{x}_2 \), whose typical size is the inverse of the soft scale, is explicit and can be consider a small scale compared with the typical wavelength of the ultrasoft gluon which is also of the order of the inverse of the energy \( E \). Gluon fields appear in this formalism at the points \( \vec{x}_1 \) and \( \vec{x}_2 \). Ultrasoft gluons are those which \( B_\mu(\vec{x}_1, t) \) can be expanded about the center of mass coordinate in a power series of derivatives and relative coordinates, the so-called multipole expansion (refs. [3] also deal with ultrasoft momenta through the multipole expansion). Then the most natural way of writing the pNRQCD lagrangian is as a functional of \( \psi(\vec{x}, \vec{X}, t) \) and \( B_\mu(\vec{X}, t) \), which is local in \( \vec{\nabla}_x, \vec{\nabla}_X \) but non-local in \( \vec{x} \). For definiteness we will give \( \mathcal{L}_{pNRQCD} \) at the leading order in the potential.

\[
\mathcal{L}_{pNRQCD} = \int d^3\vec{x}d^3\vec{X}dttr\left(\psi^\dagger(\vec{x}_1, \vec{x}_2, t)\psi(\vec{x}_1, \vec{x}_2, t)\right) \tag{9}
\]

Recall that the gluon fields in the covariant derivatives are ultrasoft and hence they must be multipole expanded which spoils the explicit gauge invariance. However, this can be recovered by writing \( \psi(\vec{x}, \vec{X}, t) \) in terms of the scalar \( S(\vec{x}, \vec{X}, t) \) and octet \( O(\vec{x}, \vec{X}, t) \) wave function fields, whose local gauge transformations depend on the center of mass coordinate only. Namely, \[\psi(\vec{x}_1, \vec{x}_2, t) = P twist\{\mathcal{F}\}S(\vec{x}, \vec{X}, t)\]

In this way \[\mathcal{L}_{pNRQCD}\] reads

\[
\mathcal{L}_{pNRQCD} = \int d^3\vec{x}d^3\vec{X}dttr\left\{ S^\dagger\left\{ i\partial_0 - \frac{\vec{p}^2}{m} + \frac{C_f\alpha}{|\vec{x}|}\right\}S + O^\dagger\left\{ iD_\mu - \frac{\vec{p}^2}{m} - \frac{1}{2N_c} \frac{\alpha}{|\vec{x}|}\right\}O + g\vec{x}O\vec{E}(\vec{X}, t)S + \frac{g}{2}\vec{x}O^\dagger\vec{E}(\vec{X}, t)\right\}. \tag{10}
\]

This lagrangian suffices to obtain the leading non-perturbative contributions to the two heavy quark bound states when \( E >> \Lambda_{QCD} \) (see also [3]).

If we leave aside non-perturbative effects (~\( \Lambda_{QCD} \)), each term in the lagrangian above has a well defined size, unlike in the NRQCD lagrangian. Relative coordinates \( \vec{x} \) and its associated momentum \( \vec{\nabla}_x \) must be counted as soft scales (~\( m\alpha \)). Then if we wish to calculate a given observable to a given order in \( \alpha \) we know immediately which terms are to be kept in the pNRQCD lagrangian. As an example, let us present \( \mathcal{L}_{pNRQED} \) from which one can obtain next to next to leading corrections to the energy (~\( m\alpha^5 \)).

\[
\mathcal{L}_{pNRQED} = \int d^3\vec{x}d^3\vec{X}dtS^\dagger(\vec{x}, \vec{X}, t)\]

\[
\left\{ i\partial_0 - \frac{\vec{p}^2}{m} + \frac{\alpha}{4m^2} \right\} + \left\{ \frac{\delta^{(3)}(\vec{x})}{m^2} \pi\alpha (e_D - 2e_F) + d_{ss} + 3d_{sv} \right\} + \frac{\lambda}{2m^2} \frac{1}{\vec{x}_2} \left( \vec{\nabla}^2 + \frac{1}{\vec{x}_2^2} \vec{E}(\vec{x}) \vec{p} \right) - \frac{\delta^{(3)}(\vec{x})}{m^2} \frac{1}{\vec{x}_2} \left( \pi\alpha \frac{4}{3} e_F - 2d_{sv} \right) - \frac{\alpha}{4m^2} \frac{1}{|\vec{x}|^3} L \cdot \vec{S} (2c_S + 4c_F) \right\}
\]
The coefficients $c_i$ are given in $\delta V$ and $d_{ss}$, $d_{sv}$ in $\beta(x)$ and $\delta V(x)$. $c_i = 1 + O(\alpha)$ and $d_{ij} = O(\alpha) + O(\alpha^2)$ are obtained from the one loop matching between QED and NRQED. The last term corresponds to the ultrasoft photons which contribute at this order. The potential terms are obtained upon matching NRQED to pNRQED up to one loop. $\delta V$ encodes the computation of the potential at one loop. It reads

$$
\delta V(x) = -\frac{\alpha^2}{m^2} \left( \frac{\alpha}{2\pi} \ln \nu^2 + 1 \frac{\ln \nu}{|x|^3} \right) - \frac{4\alpha^2}{3m^2} \left( \frac{\alpha}{2\pi} \ln \nu^2 + 1 \frac{\ln \nu}{|x|^3} \right) - \frac{\alpha^2}{m^2} \delta(x) C
$$

with unknown $C$. The first ln $\nu$ (which was produced by an UV divergence in the potential) gets canceled by the ln $\nu$ in $d_{ss}$. The second ln $\nu$ (with IR origin) cancels with a piece of the UV divergent contribution coming from the ultrasoft photons. The remaining contribution from the ultrasoft photons cancels the contribution coming from $c_D$. The net result is the total energy to be scale independent as it must be.

With $\delta V$ the binding energy at order $ma^5$ for arbitrary $n, l$ states can be calculated. We find agreement with $\alpha^5$ for the hyperfine splittings. The $ma^5 \ln \alpha$ correction has also been calculated finding agreement with known results. We have also used these techniques with DR to reproduce the Lamb shift in the simpler case of a hydrogen-like atom.

We remark that in this section we are assuming $|\vec{p}| >> E$ which was not needed in the previous section. As far as we are making the matching to some order in $\alpha$ (we are computing the potential perturbatively) we also assume $|\vec{p}| >> \Lambda_{QCD}$. Notice that the relative size between $E$ and $\Lambda_{QCD}$ is left arbitrary. We can further distinguish between two situations: (i) $E >> \Lambda_{QCD}$ and (ii) $\Lambda_{QCD} \geq E$. For the situation (i), as we mention before, we can calculate perturbatively from pNRQCD and parametrise the non-perturbative contributions by means of local condensates. For the situation (ii) the calculations in pNRQCD cannot be carried out perturbatively anymore.

If $\Lambda_{QCD} \geq |\vec{p}|$ the matching to pNRQCD cannot be carried out perturbatively. Even in this situation NRQCD is extremely useful to parametrise non-perturbative contributions in many processes.

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