Global large solution for the tropical climate model with diffusion

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Abstract
This paper studies the d-dimensional (d=2,3) tropical climate model with only the dissipation of the first baroclinic model of the velocity (−ηΔv). By choosing a class of special initial data (u₀, v₀, θ₀) whose Hs(Rd) norm can be arbitrarily large, we obtain the global smooth solution of d-dimensional tropical climate model.

Key words: Tropical climate model, global smooth solution, large initial data.

MSC(2000): 35Q35, 76D03, 86A10.

1 Introduction
This paper focuses on the d-dimensional (d=2,3) tropical climate model (TCM) given by

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla)u + \nu u + \nabla p + \text{div}(v \otimes v) &= 0, \quad x \in \mathbb{R}^d, t > 0, \\
\partial_t v + (u \cdot \nabla) v + (v \cdot \nabla)u - \eta \Delta v + \nabla \theta &= 0, \\
\partial_t \theta + (u \cdot \nabla) \theta + \lambda \theta + \nabla \cdot v &= 0, \\
\nabla \cdot u &= 0, \\
u(x,0) &= u_0(x), v(x,0) = v_0(x), \theta(x,0) = \theta_0(x),
\end{aligned}
\tag{1.1}
\]

where vector fields u(x,t) and v(x,t) represent the barotropic model and the first baroclinic model of the velocity field, respectively. p = p(x,t) and θ = θ(x,t) stand for the scalar pressure and scalar temperature. The parameters \(\nu, \eta, \lambda\) are non-negative.

When \(\nu = \eta = \lambda = 0\), the system (1.1) reduces to the original tropical climate model derived by Frierson, Majda and Pauluis [4] by performing a Galerkin truncation to the hydrostatic Boussinesq. The issues on well-posedness, global regularity and blow-up criterion of TCM have attracted considerable attention. Let us briefly recall some progress. By introducing a new velocity variable, Li-Titi [6] constructed a unique global strong solution for the system (1.1) with \(-\Delta u, \nu = \lambda = 0, \eta = 1\) and \(d = 2\). Focused on the same version as the system in [6], Li-Zhai-Yin [9] established the global well-posedness in negative-order Besov spaces with small initial data. Ye [14] obtained the global regularity result of the 2-dimensional zero thermal diffusion TCM with

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fractional dissipation, given by $(-\Delta)^\alpha u$ and $(-\Delta)^\beta v$ where $\alpha + \beta \geq 2$ ($1 < \alpha < 2$). To further have a complete view of current studies on TCM, readers can see [2, 3, 10, 11, 12, 13, 15, 16, 17].

When $\theta = 0$, the system (1.1) reduces to the MHD-like equations. In [8], Li-Yang-Yu established a unique global large solution of the 2-dimensional MHD equations with only damping of velocity and magnetic. Later, Dai-Tan-Wu in [1] introduced an effective approach to construct a class of global large solutions to the $d$-dimensional ($d = 2, 3$) MHD equations with any fractional dissipation. Naturally, we consider that whether or not the system (1.1) with a class of special large initial data can generate a unique global solution. Motivated by [1], we established a class of global solutions of system (1.1) by choosing a class of special initial data $(u_0, v_0, \theta_0)$ whose $H^s(s > 1 + \frac{d}{2})$ norm can be arbitrarily large.

There are some differences between the 2-dimensional and 3-dimensional cases, so we split our results into two cases. Before presenting our main result of 3-dimensional case, we define a vector field $m_0 \in C_0^\infty(\mathbb{R}^3)$ and two scalar functions $n_0, r_0 \in C_0^\infty(\mathbb{R}^3)$ which are used to construct the large initial data of system (1.1)

\[
\begin{align*}
\hat{m}_0(\xi) &= \left(\varepsilon^{-1} \log \log \frac{1}{\xi}\right)\chi(\xi), & \xi &\in \mathbb{R}^3, \\
\hat{n}_0(\xi) &= \hat{r}_0(\xi) = \left(\varepsilon^{-1} \left(\log \log \frac{1}{\xi}\right)^\frac{1}{2}\right)\chi(\xi), & \xi &\in \mathbb{R}^3,
\end{align*}
\]  

where $0 < \varepsilon < 1$ is a small parameter depending on $\nu, \eta$ and $\lambda$. The smooth cutoff functions $\chi = (\chi_1, \chi_2, \chi_3)$ and $\chi$ satisfy

\[
\text{supp } \chi_i, \chi, \chi \subset [0, 1] \quad \text{and } \chi_i, \chi = 1 \text{ on } C_1 \quad (i = 1, 2, 3),
\]

where $C$ and $C_1$ denote the annuli,

\[
C := \left\{ \xi \in \mathbb{R}^3 \mid |\xi_i + \xi_j| \leq \varepsilon, i, j = 1, 2, 3, i \neq j, |\xi| \leq 2 \right\},
\]

\[
C_1 := \left\{ \xi \in \mathbb{R}^3 \mid |\xi_i + \xi_j| \leq \varepsilon, i, j = 1, 2, 3, i \neq j, \frac{4}{3} \leq |\xi| \leq \frac{5}{3} \right\}.
\]

Now we state our result for 3-dimensional case as follows.

**Theorem 1.1.** Assume $m_0, n_0, r_0$ are defined by (1.2) and (1.3). Consider the initial data in the TCM system (1.1) which fulfills $\text{div} v_0 = 0$ and

\[
\begin{align*}
u_0 &= U_0 + w_0, & \nu_0 &= V_0 + z_0 \quad \text{and } \quad \Theta_0 = \Theta_0 + \psi_0,
\end{align*}
\]

where

\[
\begin{align*}
U_0 &= \nabla \times m_0, & V_0 &= (n_0, n_0, n_0) \quad \text{and } \quad \Theta_0 = r_0.
\end{align*}
\]

Let $s > \frac{5}{2}$, there exists a sufficiently small parameter $\delta$ such that, if $(w_0, z_0, \psi_0)$ satisfies

\[
\left[ \|(w_0, z_0, \psi_0)\|_{H^s} + C\left( \varepsilon \left(\|(m_0, n_0, r_0)\|_{H^{s+2}}^2 + \|(n_0, r_0)\|_{H^{s+2}}^2\right) \right) \right] 
\times e^{C\left(\|(m_0, n_0, r_0)\|_{H^{s+2}} + \|(n_0, r_0)\|_{H^{s+2}}^2\right)} \leq \delta \min\{\nu, \eta, \lambda\},
\]

then the system (1.1) has a unique global solution.
Remark 1.1. The initial data \((u_0, v_0, \theta_0)\) in Theorem 1.1 is not small. In fact,

\[
\|u_0\|_{H^s} \geq \|U_0\|_{H^s} - \|w_0\|_{H^s}
\]

where

\[
\|\mathcal{E}\|_{L^1} \geq \|\mathcal{E}_0\|_{L^1} - \|\mathcal{E}_0\|_{L^1}
\]

which will be really large when \(\varepsilon\) is taken to be small. By applying the same argument, we can show that \(\|\nu\|_{H^s}\) and \(\|\theta_0\|_{H^s}\) are not small.

When \(d = 2\), we can also obtain a unique global solution for the TCM. Likewise, we define three scalar functions \(\hat{m}_0, \hat{n}_0, \hat{\gamma}_0 \in C_0^\infty(\mathbb{R}^2)\) with their Fourier transform satisfying

\[
\hat{m}_0 = \left(\varepsilon^{-\frac{1}{2}} \log \frac{1}{\varepsilon}\right) \chi^*(\xi), \quad \xi \in \mathbb{R}^2, \tag{1.5}
\]

\[
\hat{n}_0 = \left(\varepsilon^{-\frac{1}{2}} \left( \log \frac{1}{\varepsilon} \right)^\frac{1}{4}\right) \chi^*(\xi), \quad \xi \in \mathbb{R}^2, \tag{1.6}
\]

\[
\hat{\gamma}_0 = \left(\varepsilon^{-1} \left( \log \frac{1}{\varepsilon} \right)^\frac{1}{4}\right) \chi^*(\xi), \quad \xi \in \mathbb{R}^2 \tag{1.7}
\]

the smooth cutoff function \(\chi^*(\xi)\) satisfies

\[
\text{supp } \chi^* \subset \mathcal{D}, \quad \chi^* \in [0, 1] \quad \text{and} \quad \chi^* = 1 \quad \text{on} \quad \mathcal{D}_1, \\
\text{supp } \chi^* \subset \mathcal{E}, \quad \chi^* \in [0, 1] \quad \text{and} \quad \chi^* = 1 \quad \text{on} \quad \mathcal{E}_1,
\]

where \(\mathcal{D}\) and \(\mathcal{D}_1\) denote the annuli,

\[
\mathcal{D} := \left\{ \xi \in \mathbb{R}^2 \mid |\xi_1 + \xi_2| \leq \varepsilon, 1 \leq |\xi| \leq 2 \right\},
\]

\[
\mathcal{D}_1 := \left\{ \xi \in \mathbb{R}^2 \mid |\xi_1 + \xi_2| \leq \varepsilon, \frac{4}{3} \leq |\xi| \leq \frac{5}{3} \right\},
\]

\[
\mathcal{E} := \left\{ \xi \in \mathbb{R}^2 \mid |\xi_1 + \xi_2| \leq \varepsilon, \varepsilon \leq |\xi| \leq 2\varepsilon \right\},
\]

\[
\mathcal{E}_1 := \left\{ \xi \in \mathbb{R}^2 \mid |\xi_1 + \xi_2| \leq \varepsilon, \frac{4}{3}\varepsilon \leq |\xi| \leq \frac{5}{3}\varepsilon \right\}.
\]

The result for 2-dimensional case can be stated as follows.

**Theorem 1.2.** Assume \(\overline{m}_0, \overline{n}_0, \overline{\gamma}_0\) are defined by (1.5), (1.6) and (1.7). Consider the initial data in the TCM system (1.1) fulfills \(\text{div}u_0 = 0\) and

\[
u_0 = \overline{U}_0 + w_0, \quad v_0 = \overline{V}_0 + z_0 \quad \text{and} \quad \theta_0 = \overline{\theta}_0 + \psi_0,
\]

where

\[
\overline{U}_0 = \nabla^2 m_0, \quad \overline{V}_0 = (\overline{m}_0, \overline{n}_0) \quad \text{and} \quad \overline{\theta}_0 = \overline{\gamma}_0. \tag{1.8}
\]

Let \(s > 2\), there exists a sufficiently small parameter \(\delta\) such that, if \((w_0, z_0, \psi_0)\) satisfies

\[
\|w_0, z_0, \psi_0\|_{H^s} + C\left(\varepsilon (\|(m_0, n_0, \gamma_0)\|_{H^{s+2}}^2 + \|(m_0, n_0, \gamma_0)\|_{H^{s+2}}^2)\right)
\]

\[
\leq \delta \min\{\nu, \eta, \lambda\},
\]

then the system (1.1) has a unique global solution.
Remark 1.2. Li-Yu [7] studied the 2-dimensional TCM in (1.1) with \( \lambda = 0 \) and constructed a class of global solutions which permit the initial data \((u_0, v_0)\) large in \(L^\infty(\mathbb{R}^2)\), but the initial data \(\theta_0\) is small in \(L^\infty(\mathbb{R}^2)\). In Theorem 1.2, we set the initial data \((u_0, v_0, \theta_0)\) can be large in \(H^s(\mathbb{R}^2)\) for \(s > 2\), which further improve the result in [7].

2 Preliminaries

To prove Theorem 1.1, we renormalize the system (1.1). Let \((m, n, r)\) be the solution of the following system

\[
\begin{aligned}
&\partial_t m + \nu m = 0, \\
&\partial_t n - \eta \Delta n + \partial_1 r + \partial_2 r + \partial_3 r = 0, \\
&\partial_t r + \lambda r = 0, \\
&m(x, 0) = m_0(x), \quad n(x, 0) = n_0(x), \quad r(x, 0) = r_0(x),
\end{aligned}
\]

which implies

\[
m(t) = e^{-\nu t} m_0, \quad n(t) = e^{\eta \Delta t} n_0 - \int_0^t e^{\eta \Delta (t-\tau)} e^{-\lambda \tau} (\partial_1 + \partial_2 + \partial_3) r_0 d\tau, \quad r(t) = e^{-\lambda t} r_0.
\]

Defining

\[U = \nabla \times m, \quad V = (n, n, n) \quad \text{and} \quad \Theta = r,
\]

we can deduce from (2.1) that

\[
\begin{aligned}
&\partial_t U + \nu U = 0, \\
&\partial_t V - \eta \Delta V + \nabla \Theta + \lambda \Theta = 0, \\
&\partial_t \Theta + \lambda \Theta = 0, \\
&U(x, 0) = U_0(x), \quad V(x, 0) = V_0(x), \quad \Theta(x, 0) = \Theta_0(x),
\end{aligned}
\]

where operator \(A = (\partial_1 + \partial_3, \partial_1 + \partial_3, \partial_1 + \partial_3)\).

By (2.2) and the definition of \(U_0, V_0, \Theta_0\) in Theorem 1.1, \(U, V, \Theta\) can be written as

\[
\begin{aligned}
U(t) &= e^{-\nu t} U_0 = e^{-\nu t} \nabla \times m_0, \\
V(t) &= e^{\eta \Delta t} V_0 - \int_0^t e^{\eta \Delta (t-\tau)} (\nabla \Theta + \lambda \Theta) d\tau \\
&= e^{\eta \Delta t} (n_0, n_0, n_0) - \int_0^t e^{\eta \Delta (t-\tau)} e^{-\lambda \tau} (\nabla r_0 + \lambda r_0) d\tau, \\
\Theta(t) &= e^{-\lambda t} \Theta_0 = e^{-\lambda t} r_0.
\end{aligned}
\]

Denoting \(w = u - U, z = v - V\) and \(\psi = \theta - \Theta\), then \((w, z, \psi)\) is the solution of the following equations

\[
\begin{aligned}
&\partial_t w + (w \cdot \nabla) w + (z \cdot \nabla) z + z(\nabla \cdot z) + v w + \nabla p \\
&\quad = f - (U \cdot \nabla) w - (w \cdot \nabla) U - (V \cdot \nabla) z - (z \cdot \nabla) V - z(\nabla \cdot V) - V(\nabla \cdot z), \\
&\partial_t z + (w \cdot \nabla) z + z(\nabla \cdot z) + w \cdot \nabla \psi \\
&\quad = g - (U \cdot \nabla) z - (w \cdot \nabla) V - (V \cdot \nabla) w - (z \cdot \nabla) U, \\
&\partial_t \psi + (w \cdot \nabla) \psi + \lambda \psi + \nabla \cdot z = h - (w \cdot \nabla) \Theta - (U \cdot \nabla) \psi, \\
&\text{div} w = 0, \\
&w(x, 0) = w_0(x), \quad z(x, 0) = z_0(x), \quad \psi(x, 0) = \psi_0(x),
\end{aligned}
\]
where

\[ f = - (U \cdot \nabla)U - (V \cdot \nabla)V - V(\nabla \cdot V), \]
\[ g = - (U \cdot \nabla)V - (V \cdot \nabla)U + A\Theta, \]
\[ h = - \nabla \cdot V - (U \cdot V)\Theta. \]

Next, we present two lemmas which will be used in the proof of Theorem 1.1. Lemma 2.1 offers upper bounds on \((U, V, \Theta)\) and \((f, g, h)\).

**Lemma 2.1.** Assume \(m_0, n_0\) and \(r_0\) are defined by (1.2) and (1.3), \(U(t), V(t)\) and \(\Theta(t)\) by (2.3), and \(f, g\) and \(h\) by (2.5). Then for any \(s > \frac{5}{2}\), the following estimates hold

\[
\begin{align*}
\|U\|_{L^1_t H^{s+1}} &\leq C\|m_0\|_{H^{s+2}}, \\
\|V\|_{L^1_t H^{s+1}} &\leq C(\|n_0\|_{H^{s+2}} + \|r_0\|_{H^{s+2}}), \\
\|\Theta\|_{L^1_t H^{s+1}} &\leq C\|r_0\|_{H^{s+2}}, \\
\|f\|_{L^1_t H^{s}} &\leq C\|m_0\|_{H^{s+2}}^2 + \|n_0\|_{H^{s+2}}^2 + \|r_0\|_{H^{s+2}}^2, \\
\|g\|_{L^1_t H^{s}} &\leq C\|m_0\|_{H^{s+2}} + \|n_0\|_{H^{s+2}} + \|r_0\|_{H^{s+2}} + C\|r_0\|_{H^{s}} \\
\|h\|_{L^1_t H^{s}} &\leq C\|m_0\|_{H^{s}} + \|r_0\|_{H^{s+1}} + \|m_0\|_{H^{s+1}}^2 + \|r_0\|_{H^{s+1}}^2.
\end{align*}
\]  

**Proof.** It is easy to prove the first three terms of (2.6) by (2.3) and the definition of \((m_0, n_0, r_0)\), we shall omit the detailed steps. Now, we prove the estimates of \(f, g\) and \(h\). Our main idea is to rewrite the terms in \(f, g\) and \(h\) so that each term contains \(\partial_i + \partial_j (i, j = 1, 2, 3, i \neq j)\).

For simplicity, here we take the first component of each term in \(f\) as an example, the rest can be obtained in the same way.

\[ U = \nabla \times m = (\partial_2 m_3 - \partial_3 m_2, \partial_3 m_1 - \partial_1 m_3, \partial_1 m_2 - \partial_2 m_1), \]
\[ V = (n, n, n). \]

Direct calculations show that

\[ U \cdot \nabla U = \partial_3 m_1 \partial_1 m_3 - \partial_1 m_1 \partial_3 m_3 + \partial_1 m_2 \partial_3 m_2 - \partial_2 m_2 \partial_3 m_2 + \partial_1 m_1 \partial_3 m_3 - \partial_3 m_3 \partial_1 m_3 - \partial_3 m_2 \partial_1 m_2 \]
\[ + \partial_3 m_1 \partial_2 m_3 - \partial_2 m_1 \partial_3 m_2 + \partial_2 m_1 \partial_2 m_3 - \partial_3 m_1 \partial_2 m_2 + \partial_3 m_2 \partial_2 m_1 \]
\[ + \partial_3 m_2 \partial_1 m_3 - \partial_2 m_3 \partial_1 m_3 - \partial_2 m_1 \partial_3 m_3 + \partial_2 m_1 \partial_3 m_2 - \partial_3 m_1 \partial_3 m_2 + \partial_3 m_2 \partial_3 m_1 \]
\[ + \partial_3 m_2 \partial_2 m_3 - \partial_2 m_3 \partial_2 m_3 - \partial_2 m_1 \partial_2 m_2 + \partial_2 m_1 \partial_3 m_2 - \partial_2 m_1 \partial_3 m_2 + \partial_2 m_2 \partial_3 m_1 \]
\[ + \partial_2 m_2 \partial_2 m_1 - \partial_2 m_1 \partial_2 m_1 - \partial_2 m_2 \partial_2 m_1 + \partial_2 m_2 \partial_2 m_2 \]
\[ \leq C\|m_0\|_{H^{s+2}}^2 + \|n_0\|_{H^{s+2}} + \|r_0\|_{H^{s+2}} + C\|r_0\|_{H^{s}}. \]

Taking the \(H^s\) norm yields

\[
\|U \cdot \nabla U\|_{H^s} \leq C(\|\partial_2 + \partial_3\|_{H^s} \|m\|_{H^{s+2}} + \|m\|_{H^{s+2}} \|\partial_3 + \partial_1\|_{H^{s+1}} + \|\partial_1 + \partial_2\|_{H^s} \|m\|_{H^{s+2}} + \|m\|_{H^{s+2}} \|\partial_2 + \partial_3\|_{H^{s+1}} + \|\partial_2 + \partial_3\|_{H^s} \|m\|_{H^{s+2}} + \|m\|_{H^{s+2}} \|\partial_3 + \partial_1\|_{H^{s+1}} + \|\partial_3 + \partial_2\|_{H^s} \|m\|_{H^{s+2}} + \|m\|_{H^{s+2}} \|\partial_1 + \partial_3\|_{H^{s+1}} \|\partial_2 + \partial_3\|_{H^{s+1}} + \|\partial_2 + \partial_3\|_{H^s} \|m\|_{H^{s+2}} + \|m\|_{H^{s+2}} \|\partial_1 + \partial_3\|_{H^{s+1}}). \]

\]  

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where $i, j$ in the last line are summed over $i, j = 1, 2, 3$ and $i \neq j$. Similarly,

$$V \cdot \nabla V = n\partial_1 n + n\partial_2 n + n\partial_3 n = \frac{\partial_1 + \partial_2 + \partial_3}{2} n + n\frac{\partial_2 + \partial_3}{2} n + n\frac{\partial_3 + \partial_1}{2} n = V^1(\nabla \cdot V).$$

Taking the $H^s$ norm yields

$$\|V \cdot \nabla V\|_{H^s} = \|V^1(\nabla \cdot V)\|_{H^s} \leq C(\|\partial_1 + \partial_2 + \partial_3\|_{H^s} + \|\partial_2 + \partial_3\|_{H^s} + \|\partial_3 + \partial_1\|_{H^s}),$$

where $*$ is convolution operator and $(e^{-C_s(t)} * e^{-C_0(t)})(t) = \int_0^t e^{-C_s(t-\tau)} e^{-C_0 \tau} d\tau$. Similiar argument is used to the rest components, it holds that

$$\|f\|_{H^s} \leq C(\|\partial_1 + \partial_2 + \partial_3\|_{H^s} + \|\partial_2 + \partial_3\|_{H^s} + \|\partial_3 + \partial_1\|_{H^s}),$$

Integrating both sides of the above inequality over $[0, t]$, by convolution Young's inequality, we deduce that

$$\|f\|_{L^t_{1} H^s} \leq C\|m_0\|_{H^{s+2}} \int_0^t e^{-2C_0 \tau} d\tau + C\|n_0\|_{H^{s+2}} \int_0^t e^{-2C_1 \tau} d\tau + C\|r_0\|_{H^{s+2}} \int_0^t e^{-C_1 \tau} d\tau \leq C\|m_0\|_{H^{s+2}} + C\|n_0\|_{H^{s+2}} + C\|r_0\|_{H^{s+2}}.$$

Since $A = (\partial_2 + \partial_3, \partial_1 + \partial_3, \partial_1 + \partial_3)$, we have

$$\|Ar\|_{H^s} \leq \|\partial_2 + \partial_3\|_{H^s} + \|\partial_1 + \partial_3\|_{H^s} \leq C\|e^{-C_0 t} r\|_{H^s}.$$

Then, by the similarly arguments that used in evaluating $\|f\|_{L^1_{1} H^s}$, $\|g\|_{L^1_{1} H^s}$ is estimated as

$$\|g\|_{L^1_{1} H^s} \leq C\|m_0\|_{H^{s+2}} + C\|n_0\|_{H^{s+2}} + C\|r_0\|_{H^s}.$$

For $h$, since

$$\|\nabla \cdot V\|_{H^s} = \|\partial_1 + \partial_2 + \partial_3\|_{H^s},$$

and

$$\|(U \cdot \nabla) \Theta\|_{H^s} \leq \|\partial_3 m_3 - \partial_2 m_3\|_{H^s} + \|\partial_3 m_2 - \partial_2 m_2\|_{H^s} + \|\partial_3 m_1 - \partial_1 m_1\|_{H^s} + \|\partial_3 m_2 - \partial_2 m_2\|_{H^s} \leq C\|e^{-C_1 t} n_0\|_{H^s} + C\|e^{-C_0 t} (t)\|_{H^{s+1}},$$

and

$$\|(U \cdot \nabla) \Theta\|_{H^s} \leq C\|\partial_3 m_3 - \partial_2 m_3\|_{H^s} + \|\partial_3 m_2 - \partial_2 m_2\|_{H^s} + \|\partial_3 m_1 - \partial_1 m_1\|_{H^s} + \|\partial_3 m_2 - \partial_2 m_2\|_{H^s} \leq C\|e^{-2C_0 t} (t)\|_{H^{s+1}}.$$
we have
\[ \|h\|_{L^1_t H^s} \leq C\varepsilon\|n_0\|_{H^s} \int_0^t e^{-C_1\tau}d\tau + C\varepsilon\|r_0\|_{H^{s+1}} \int_0^t e^{-C_0\tau}d\tau \int_0^t e^{-C_1\tau}d\tau \]
\[ + C\varepsilon(\|m_0\|_{H^{s+1}}^2 + \|r_0\|_{H^{s+1}}^2) \int_0^t e^{-2C_0\tau}d\tau \]
\[ \leq C\varepsilon(\|n_0\|_{H^s} + \|r_0\|_{H^{s+1}} + \|m_0\|_{H^{s+1}}^2 + \|r_0\|_{H^{s+1}}^2). \]
This proves (2.6).

Next, we recall the following commutator and bilinear estimates, the detailed proof can be found in [5].

**Lemma 2.2.** Let \( s > 0 \). Let \( p, p_1, p_3 \in (1, \infty) \) and \( p_2, p_4 \in [1, \infty] \) satisfies
\[ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \]
Then there exists constants \( C \) such that
\[ \| [J^s, F]G \|_{L^p} \leq C(\| J^s F \|_{L^{p_1}} \| G \|_{L^{p_2}} + \| J^{s-1} G \|_{L^{p_3}} \| \nabla F \|_{L^{p_4}}), \]
\[ \| J^s(FG) \|_{L^p} \leq C(\| J^s F \|_{L^{p_1}} \| G \|_{L^{p_2}} + \| J^s G \|_{L^{p_3}} \| F \|_{L^{p_4}}), \]
where \( J = (I - \Delta)^{\frac{1}{2}} \) and the commutator \([ J^s, F]G = J^s(FG) - F(J^sG) \). The operator \( J \) is defined via the Fourier transform
\[ \hat{Jf}(\xi) = (1 + |\xi|^2)^{\frac{1}{2}} \hat{f}(\xi), \]
and
\[ \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}f(x)dx. \]

### 3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We derive the \( H^s \) estimates of \((w, z, \psi)\) and apply the bootstrap argument to complete the proof.

**Proof.** Applying \( J^s \) to (2.4)_{1,2,3} and dotting with \((J^s w, J^s z, J^s \psi)\) yields
\[ \frac{1}{2} \frac{d}{dt}\|(w, z, \psi)\|_{H^s}^2 + \nu \|w\|_{H^s}^2 + \eta \|z\|_{H^{s+1}}^2 + \lambda \|\psi\|_{H^s}^2 = \sum_{i=1}^7 I_i, \] (3.1)
with
\begin{align*}
I_1 &= - \int_{\mathbb{R}^3} [J^s, w \cdot \nabla] w \cdot J^s w dx - \int_{\mathbb{R}^3} [J^s, w \cdot \nabla] z \cdot J^s z dx - \int_{\mathbb{R}^3} [J^s, w \cdot \nabla] \psi \cdot J^s \psi dx, \\
I_2 &= - \int_{\mathbb{R}^3} [J^s, U \cdot \nabla] w \cdot J^s w dx - \int_{\mathbb{R}^3} [J^s, U \cdot \nabla] z \cdot J^s z dx - \int_{\mathbb{R}^3} [J^s, U \cdot \nabla] \psi \cdot J^s \psi dx, \\
I_3 &= - \int_{\mathbb{R}^3} J^s (z \cdot \nabla z) \cdot J^s w dx - \int_{\mathbb{R}^3} J^s (z(\nabla \cdot z)) \cdot J^s z dx - \int_{\mathbb{R}^3} J^s (z \cdot \nabla w) \cdot J^s z dx, \\
I_4 &= - \int_{\mathbb{R}^3} J^s (w \cdot \nabla U) \cdot J^s w dx - \int_{\mathbb{R}^3} J^s (z \cdot \nabla U) \cdot J^s z dx - \int_{\mathbb{R}^3} J^s (w \cdot \nabla \Theta) \cdot J^s \psi dx, \\
I_5 &= - \int_{\mathbb{R}^3} J^s (V \cdot \nabla z) \cdot J^s w dx - \int_{\mathbb{R}^3} J^s (V(\nabla \cdot z)) \cdot J^s w dx - \int_{\mathbb{R}^3} J^s (V \cdot \nabla w) \cdot J^s z dx, \\
I_6 &= - \int_{\mathbb{R}^3} J^s (z \cdot \nabla V) \cdot J^s w dx - \int_{\mathbb{R}^3} J^s (z(\nabla \cdot V)) \cdot J^s w dx - \int_{\mathbb{R}^3} J^s (w \cdot \nabla V) \cdot J^s z dx, \\
I_7 &= \int_{\mathbb{R}^3} J^s f \cdot J^s w dx + \int_{\mathbb{R}^3} J^s g \cdot J^s z dx \quad \text{and} \quad \int_{\mathbb{R}^3} J^s h \cdot J^s \psi dx,
\end{align*}

where we have used the simple fact
\[
\int_{\mathbb{R}^3} J^s \nabla \psi \cdot J^s z dx + \int_{\mathbb{R}^3} J^s \nabla \cdot J^s \psi dx = 0.
\]
We now estimate the terms on the right hand side of (3.1). By Hölder’s inequality, Lemma 2.2 and Young inequality, $I_1$ can be bounded by
\[
|I_1| \leq C \|J^s, w \cdot \nabla w\|_{L^2} \|w\|_{H^s} + C \|J^s, w \cdot \nabla z\|_{L^2} \|z\|_{H^s} + C \|J^s, w \cdot \nabla \psi\|_{L^2} \|\psi\|_{H^s} \\
\leq C \|\nabla w\|_{L^\infty} \|w\|_{H^s} + C \|\nabla z\|_{L^\infty} \|z\|_{H^s} + C \|\nabla \psi\|_{L^\infty} \|\psi\|_{H^s} \\
+ C \|\nabla z\|_{L^\infty} \|w\|_{H^s} + C \|\nabla \psi\|_{L^\infty} \|\psi\|_{H^s} \\
\leq C \|w\|_{H^s} + C \|z\|_{H^{s+1}} \|w\|_{H^s} + C \|w\|_{H^s} \|\psi\|_{H^s}^2 \\
\leq C \|w, z, \psi\|_{H^s} \|w\|_{H^s}^2 + \|z\|_{H^{s+1}}^2 + \|\psi\|_{H^s}^2,
\]
where we have used the following facts, for $s > \frac{\tilde{\alpha}}{2}$
\[
\|\nabla w\|_{L^\infty} \leq C \|w\|_{H^s}, \quad \|\nabla w\|_{L^\infty} \leq C \|w\|_{H^s} \quad \text{and} \quad \|\nabla w\|_{L^\infty} \leq C \|z\|_{H^s} \leq C \|z\|_{H^{s+1}}.
\]

For $I_2$, by the estimates of $U$ and $\nabla U$ in Lemma 2.1, we deduce
\[
|I_2| \leq C \|U\|_{H^s} (\|\nabla w\|_{L^\infty} \|w\|_{H^s} + \|\nabla z\|_{L^\infty} \|z\|_{H^s} + \|\nabla \psi\|_{L^\infty} \|\psi\|_{H^s}) \\
+ C \|\nabla U\|_{L^\infty} \|(w, z, \psi)\|_{H^s}^2 \\
\leq C \|U\|_{H^s} \|(w, z, \psi)\|_{H^s}^2.
\]

To evaluate $I_3$, we use the fact $z \cdot \nabla w = \nabla \cdot (w \otimes z) - w(\nabla \cdot z)$, then it holds that
\[
|I_3| \leq C \|z \cdot \nabla z\|_{H^s} \|w\|_{H^s} + C \|z(\nabla \cdot z)\|_{H^s} \|w\|_{H^s} + C \|w \cdot \nabla \otimes z\|_{H^s} \|\psi\|_{H^s} \\
+ C \|w(\nabla \cdot z)\|_{H^s} \|w\|_{H^s} \\
\leq C \|z\|_{H^{s+1}} \|z\|_{H^s} \|w\|_{H^s} + C \|w\|_{H^s} \|z\|_{H^s} \|z\|_{H^{s+1}} \\
\leq C \|z\|_{H^s} \|z\|_{H^{s+1}} \|w\|_{H^s} + C \|w\|_{H^s} \|w\|_{H^s} \|z\|_{H^{s+1}}.
\]

For $I_4$
\[
|I_4| \leq C \|w \cdot \nabla U\|_{H^s} \|w\|_{H^s} + C \|z \cdot \nabla U\|_{H^s} \|z\|_{H^s} + \|w \cdot \nabla \Theta\|_{H^s} \|\psi\|_{H^s} \\
\leq C \|w\|_{H^s} \|\nabla U\|_{H^s} \|w\|_{H^s} + C \|z\|_{H^s} \|\nabla U\|_{H^s} \|z\|_{H^s} + C \|w\|_{H^s} \|\nabla \Theta\|_{H^s} \|\psi\|_{H^s} \\
\leq C \|U\|_{H^{s+1}} + \|\Theta\|_{H^{s+1}} \|(w, z, \psi)\|_{H^s}^2.
\]
$I_5$ can be bounded by the similar idea of $I_4$

$$
|I_5| \leq C \||V \cdot \nabla z||_{H^s}||w||_{H^{s+1}} + C\|V (\nabla \cdot z)||_{H^s}||w||_{H^{s}} + C\|w \odot V||_{H^s} \|\nabla z||_{H^s} \\
+ C\|w(\nabla \cdot V)||_{H^s} \|z||_{H^s} \\
\leq C(||V||_{H^{s+1}}^2 + \|V||_{H^{s+1}}^2)\|w||_{H^{s}}^2 + \frac{\eta}{2}||z||_{H^{s+1}}^2.
$$

By an argument similar to that used in evaluating $I_4$, it follows that

$$
|I_6| \leq C\|V||_{H^{s+1}}(||w||_{H^{s}}^2 + ||z||_{H^{s}}^2).
$$

Thanks to Lemma 2.1, the term $I_7$ can be bounded by

$$
|I_7| \leq \|f||_{H^s}||w||_{H^{s}} + \|g||_{H^{s}}||z||_{H^{s}} + \|h||_{H^{s}}||\psi||_{H^{s}}.
$$

Inserting all the estimates above for $I_1$ through $I_7$ in (3.1) yields

$$
\frac{d}{dt}||w(z, \psi)||_{H^{s}}^2 + \nu||w||_{H^{s}}^2 + \frac{\eta}{2}||z||_{H^{s+1}}^2 + \lambda||\psi||_{H^{s}}^2 \\
\leq 2C_2(||w||_{H^{s}}^2 + ||z||_{H^{s+1}}^2 + ||\psi||_{H^{s}}^2) \\
+ C_3(||U||_{H^{s+1}} + \|V||_{H^{s+1}} + ||\Theta||_{H^{s+1}} + \|V||_{H^{s+1}}^2)||w(z, \psi)||_{H^{s}}^2 \\
+ C_4(||f||_{H^{s}} + ||g||_{H^{s}} + \|h||_{H^{s}})||w(z, \psi)||_{H^{s}}.
$$

Then we deduce

$$
\frac{d}{dt}||w(z, \psi)||_{H^{s}}^2 + (\nu - 2C_2||w(z, \psi)||_{H^{s}})||w||_{H^{s}}^2 \\
+ (\eta - 2C_2||w(z, \psi)||_{H^{s}})||z||_{H^{s+1}}^2 + (\lambda - 2C_2||w(z, \psi)||_{H^{s}})||\psi||_{H^{s}}^2 \\
\leq 2C_3(||U||_{H^{s+1}} + \|V||_{H^{s+1}} + ||\Theta||_{H^{s+1}} + \|V||_{H^{s+1}}^2)||w(z, \psi)||_{H^{s}}^2 \\
+ 2C_4(||f||_{H^{s}} + ||g||_{H^{s}} + \|h||_{H^{s}})||w(z, \psi)||_{H^{s}}. \tag{3.2}
$$

Assume that $(w, z, \psi)||_{H^{s}}$ is bounded, we prove $(w, z, \psi)||_{H^{s}}$ actually admits a smaller bound when $(w_0, z_0, \psi_0)||_{H^{s}}$ is taken to be sufficiently small. To apply the bootstrap argument to (3.2), we make the ansatz

$$
||(w, z, \psi)||_{H^{s}} \leq M := \frac{1}{2C_2} \min{\nu, \eta, \lambda}. \tag{3.3}
$$

Obviously, (3.3) implies

$$
\nu - 2C_2||w(z, \psi)||_{H^{s}} \geq 0, \quad \eta - 2C_2||w(z, \psi)||_{H^{s}} \geq 0, \quad \lambda - 2C_2||w(z, \psi)||_{H^{s}} \geq 0.
$$

Thus we observe that (3.2) leads to

$$
\frac{d}{dt}||w(z, \psi)||_{H^{s}} \\
\leq 2C_3(||U||_{H^{s+1}} + \|V||_{H^{s+1}} + ||\Theta||_{H^{s+1}} + \|V||_{H^{s+1}}^2)||w(z, \psi)||_{H^{s}} \\
+ 2C_4(||f||_{H^{s}} + ||g||_{H^{s}} + \|h||_{H^{s}}).$$
By Grönwall inequality, we obtain
\[
\| (w, z, \psi) \|_{H^s} \leq e^{2C_3 \int_0^t (\| U \|_{H^{s+1}} + \| V \|_{H^{s+1}} + \| \Theta \|_{H^{s+1}} + \| V \|_{H^{s+1}}^2) d\tau} \left[ \| (w_0, z_0, \psi_0) \|_{H^s} + 2C_4 \int_0^t (\| f \|_{H^{s+1}} + \| g \|_{H^{s+1}} + \| h \|_{H^{s+1}}) d\tau \right] \\
\leq e^C \left( \| (m_0, n_0, r_0) \|_{H^{s+2}} + \| (n_0, r_0) \|_{H^{s+2}}^2 \right) \left[ \| (w_0, z_0, \psi_0) \|_{H^s} + C_5 \left( \| (m_0, n_0, r_0) \|_{H^{s+2}} + \| (n_0, r_0) \|_{H^{s+2}} \right) \right] \\
\leq \delta \min \{ \nu, \eta, \lambda \}. \quad (3.4)
\]
Here we take
\[
\delta = \frac{1}{4C_3},
\]
\[
\| (w, z, \psi) \|_{H^s} \text{ admits a smaller upper bound} \quad \| (w, z, \psi) \|_{H^s} \leq M, \]
the bootstrap argument then implies that
\[
\| (w, z, \psi) \|_{H^s} \leq \frac{1}{4C_3} \min \{ \nu, \eta, \lambda \} \quad \text{for } 0 < t < \infty.
\]
This completes the proof of Theorem 1.1. \qed

4 Proof of Theorem 1.2

In this section, we shall just provide the preparations of the proof of Theorem 1.2, the main proof follows the same argument of that for Theorem 1.1.

As a preparation of the proof, we first renormalize the system (1.1) in the 2D case and provide several global upper bounds. To this end, we assume that \((\overline{m}, \overline{n}, \overline{r})\) is the solution of the following system
\[
\begin{align*}
\partial_t \overline{m} + \nu \overline{m} &= 0, \\
\partial_t \overline{n} - \eta \Delta \overline{n} + \partial_1 \overline{r} + \partial_2 \overline{r} &= 0, \\
\partial_t \overline{r} + \lambda \overline{r} &= 0.
\end{align*}
\]
(4.1)

Defining
\[
\overline{U} = \nabla \cdot \overline{m}, \quad \overline{V} = (\overline{n}, \overline{n}), \quad \overline{\Theta} = \overline{r}.
\]
(4.2)

Taking the differences \(w = u - \overline{U}, z = v - \overline{V}, \psi = \theta - \overline{\Theta}\), it is easy to see that \((w, z, \psi)\) satisfy (2.4) with \(U, V, \Theta\) and \(f, g, h\) replaced by \(\overline{U}, \overline{V}, \overline{\Theta}\) and \(\overline{f}, \overline{g}, \overline{h}\), respectively. Where \(\overline{f}, \overline{g}, \overline{h}\) are defined as follows
\[
\begin{align*}
\overline{f} &= - (\overline{U} \cdot \nabla) \overline{U} - (\overline{V} \cdot \nabla) \overline{V} - (\nabla \cdot \overline{V}), \\
\overline{g} &= - (\overline{U} \cdot \nabla) \overline{V} - (\nabla \cdot \overline{V}) \overline{U} + \overline{k} \overline{\Theta}, \\
\overline{h} &= \nabla \cdot \overline{V} - (\overline{U} \cdot \nabla) \overline{\Theta},
\end{align*}
\]
(4.3)

with \(\overline{k} = (\partial_2, \partial_1)\).

Now we are in a position to prove upper bounds for \(\overline{U}, \overline{V}, \overline{\Theta}\) and \(\overline{f}, \overline{g}, \overline{h}\).
Lemma 4.1. Assume \( \mathbf{m}_0, \mathbf{n}_0 \) and \( \mathbf{r}_0 \) are defined by (1.5), (1.6) and (1.7), \( \mathbf{U}(t), \mathbf{V}(t) \) and \( \mathbf{G}(t) \) by (4.2), and \( \mathbf{f}, \mathbf{g} \) and \( \mathbf{h} \) by (4.3). Then for any \( s > 2 \), the following estimates hold

\[
\begin{align*}
\|\mathbf{U}\|_{L^1_t H^{s+1}} & \leq C\|\mathbf{m}_0\|_{H^{s+2}}, \\
\|\mathbf{V}\|_{L^1_t H^{s+1}} & \leq C(\|\mathbf{m}_0\|_{H^{s+2}} + \|\mathbf{r}_0\|_{H^{s+2}}), \\
\|\mathbf{G}\|_{L^1_t H^{s+1}} & \leq C\|\mathbf{r}_0\|_{H^{s+1}},
\end{align*}
\]

(4.4)

Proof. It is easy to prove the first three terms of (4.4) by (4.2) and the definition of \( \langle \mathbf{m}_0, \mathbf{n}_0, \mathbf{r}_0 \rangle \).

To prove the upper bound for \( \|f\|_{H^s} \), we rewrite the first component of \( f \) as

\[
\mathbf{f}_1 = -\partial_2 \mathbf{m} \partial_1 \mathbf{m}_1 + \partial_1 \mathbf{m} \partial_2 \mathbf{m}_2 - 2 \partial_1 \mathbf{m} \partial_1 \mathbf{m}_1 + 2 \partial_2 \mathbf{m} \partial_2 \mathbf{m}_2
\]

\[
= -(\partial_1 + \partial_2) \mathbf{m} \partial_2 \mathbf{m}_1 + \partial_1 \mathbf{m} (\partial_1 + \partial_2) \partial_2 \mathbf{m} - 2 (\partial_1 + \partial_2) \mathbf{m} \partial_1 \mathbf{m}_1 + 2 \partial_2 \mathbf{m} (\partial_1 + \partial_2) \partial_1 \mathbf{m}_2.
\]

Taking the \( H^s \) norm, by Hölder inequality and Sobolev embedding, we deduce that

\[
\|\mathbf{f}_1\|_{H^s} \leq C((\|\partial_1 + \partial_2\|_H^s)\|\mathbf{m}\|_{H^{s+2}} + (\|\partial_1 + \partial_2\|_H^s)\|\mathbf{m}\|_{H^{s+1}}\|\mathbf{m}\|_{H^{s+1}}
\]

\[
+ ((\|\partial_1 + \partial_2\|_H^s)\|\mathbf{m}\|_{H^{s+2}} + (\|\partial_1 + \partial_2\|_H^s)\|\mathbf{m}\|_{H^{s+1}}\|\mathbf{m}\|_{H^{s+1}})
\]

\[
\leq C\varepsilon^2 + C\varepsilon e^{-2C_0 t} \|\mathbf{m}_0\|_{H^{s+2}} + C\varepsilon e^{-2C_0 t} \|\mathbf{m}_0\|_{H^{s+2}} + C\varepsilon (e^{-2C_0 t} + e^{-C_0 t})(t)\|\mathbf{r}_0\|_{H^{s+2}}.
\]

In addition, \( \mathbf{f}_2 \) admits the same bound, which implies that

\[
\|\mathbf{f}\|_{L^1_t H^s} \leq C\varepsilon(\|\mathbf{m}_0\|_{H^{s+2}} + \|\mathbf{n}_0\|_{H^{s+2}} + \|\mathbf{r}_0\|_{H^{s+2}}).
\]

The upper bounds for \( \|f\|_{L^1_t H^s}, \|\mathbf{h}\|_{L^1_t H^s} \) can be similarly obtained. The proof of Lemma 4.1 is thus complete.

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