ALGEBRAIC GEOMETRY OVER BOOLEAN ALGEBRAS
IN THE LANGUAGE WITH CONSTANTS

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Abstract. We study equations over Boolean algebras with distinguished elements. We prove criteria for which a Boolean algebra is equationally Noetherian, weakly equationally Noetherian, \( q_\omega \)-compact, or \( u_\omega \)-compact. Also we solve the problem of geometric equivalence in the class of Boolean algebras with distinguished elements.

Introduction

For an arbitrary algebraic structure (algebra, for shortness) of a language \( L \) one can define the notion of an equation as an atomic formula of the language \( L \). Therefore, there is posed the classification problem of algebraic sets (i.e., sets defined by systems of equations) over an algebra \( A \). One can also define the useful notion of a coordinate algebra over an algebra \( A \). The coordinate algebra is an analog of a coordinate ring in commutative algebra, and it defines an algebraic set up to isomorphism. In papers by E. Daniyarova, A. Miasnikov, and V. Remeslennikov [1, 2], there were proved two so-called unifying theorems, which classify coordinate algebras over an algebra \( A \) with seven equivalent approaches.

The unique constraint in both unifying theorems is the Noetherian property of the algebra \( A \) (i.e., any system of equations over \( A \) is equivalent to its finite subsystem).

In [3], there were defined three classes of algebras that generalize the class of equationally Noetherian algebras. These classes are called weakly equationally Noetherian, \( u_\omega \)-compact, and \( q_\omega \)-compact algebras and denoted by \( N' \), \( U \), and \( Q \), respectively (see the definitions of such classes in Sec. 2).

Each of the classes \( N' \), \( U \), and \( Q \) inherits some properties of the class of equationally Noetherian algebras \( N \). For example, any algebraic set over an algebra \( A \in N' \) is defined by a finite system of equations. For every \( u_\omega \)-compact algebra \( A \), both unifying theorems remain true, whereas for a \( q_\omega \)-compact algebras only the first unifying theorem holds.

Thus, any application of unifying theorems means the preliminary solution of the next problem.

Problem. Find the classes \( N \), \( N' \), \( U \), or \( Q \) that contain the given algebra \( A \).

It is known that the classes \( N \), \( N' \), \( Q \), and \( U \) are pairwise distinct. In [7], there was constructed a \( q_\omega \)-compact but not equationally Noetherian group. In [4], there was shown that the classes \( N \), \( N' \), \( U \), and \( Q \) of algebras in the language \( L = \{ f_i^{(1)} \mid i \in \mathbb{N} \} \) (\( f_i^{(1)} \) is an unary function) are pairwise distinct. In [9], we constructed a series of semilattices in the language

\[ L = \{ \land \} \cup \{ c_i \mid i \in \mathbb{N} \} \]

with countable many constants. The obtained series shows that the classes \( N \), \( N' \), \( U \), and \( Q \) of semilattices in the language \( L \) are pairwise distinct.

An interesting result devoted to the problem above was obtained in [4] and it needs the next definition.

A system of equations \( S \) over an algebra \( A \) is called an \( E_k \)-system \( (k \in \mathbb{N}) \) if \( S \) has exactly \( k \) solutions in \( A \); however, the solution set of any finite subsystem \( S_0 \subseteq S \) is infinite.

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Theorem 6.3. Let \( A \) be a \( q_\omega \)-compact (\( u_\omega \)-compact) algebra. Then for any \( k \in \{0, 1\} \) \( (k \in \mathbb{N}) \) there does not exist any \( E_k \)-system over \( A \).

There are algebras where the conditions of the theorem above become sufficient for an algebra \( A \) to be \( q_\omega \)-compact (\( u_\omega \)-compact). In [8], this was proved for linearly ordered lattices in the language \( \{\land, \lor\} \cup \{c_i \mid i \in I\} \). In the current paper, we prove this for Boolean algebras in the language \( \mathcal{L} = \{\lor, \land, 0, 1\} \cup \{c_i \mid i \in I\} \) extended by an arbitrary set of constants (the functions \( \lor, \land, \) and \( \neg \) means the disjunction, conjunction, and negation, respectively).

Let us formulate the main results of the current paper.

Theorem 6.3. A Boolean \( C \)-algebra \( B \) is \( q_\omega \)-compact if and only if there are no \( E_0 \)- or \( E_1 \)-systems over \( B \).

Theorem 6.4. A Boolean \( C \)-algebra \( B \) is \( u_\omega \)-compact if and only if there are no \( E_k \)-systems over \( B \) for any \( k \in \mathbb{N} \).

Theorem 4.1. A Boolean \( C \)-algebra \( B \) is equationally Noetherian if and only if the subalgebra \( C \) generated by the constants of \( \mathcal{L} \) is finite.

Theorem 5.1. A Boolean \( C \)-algebra \( B \) is weakly equationally Noetherian if and only if the algebra \( C \) of constants is complete in \( B \), i.e., any set of elements \( \{c_j \mid j \in J\} \subseteq C \) has an infimum in \( B \) and the infimum belongs to \( C \).

Note that the analog of Theorem 4.1 holds for semilattices of the language \( \{\land\} \cup \{c_i \mid i \in I\} \) extended by an infinite set of constants (see [9]).

Let us consider another generalization of the Noetherian property. An algebra \( A \) is consistently Noetherian if any consistent system of equations over \( A \) is equivalent to its finite subsystem. This definition is not equivalent to the Noetherian property in general, since there exists a semilattice of the language \( \{\land\} \cup \{c_i \mid i \in I\} \) that is consistently Noetherian, but there is an inconsistent system of equations \( S \) all of whose finite subsystems are consistent (see [9]).

However, for Boolean algebras of the language \( \mathcal{L} \) we have the following result.

Theorem 4.2. If a Boolean \( C \)-algebra \( B \) is consistently Noetherian, then it is equationally Noetherian.

Section 7 is devoted to the problem of geometric equivalence of Boolean algebras in the language \( \mathcal{L} \). By definition, Boolean algebras \( B_1 \) and \( B_2 \) of the language \( \mathcal{L} \) are geometrically equivalent if for any system of equations \( S \) the coordinate algebras \( \Gamma_{B_1}(S) \) and \( \Gamma_{B_2}(S) \) are isomorphic to each other. This means that the description of coordinate algebras over an algebra \( B_1 \) automatically implies the corresponding description over any algebra \( B_2 \) that is geometrically equivalent to \( B_1 \).

The problem of geometric equivalence was posed in [7]. In [6], this problem was solved for equationally Noetherian groups. Theorem 7.2 of the current paper contains a criterion for a pair of Boolean algebras of the language \( \mathcal{L} \) to be geometrically equivalent.

As follows from [6], the geometric equivalence of algebras \( B_1 \) and \( B_2 \) is highly connected to the universal classes generated by the algebras \( B_1 \) and \( B_2 \). For instance, the equality of the pre-varieties generated by \( B_1 \) and \( B_2 \) is equivalent to their geometric equivalence. Thus, Theorem 7.3 contains statements about the universal classes generated by Boolean algebras of the language \( \mathcal{L} \).

1. Boolean Algebras

Following [5], let us give the main properties of Boolean algebras.

Let \( \mathcal{L}_0 = \{\lor^{(2)}, \land^{(2)}, \neg^{(1)}, 0, 1\} \) be a language with two binary functions \( \lor \) and \( \land \), one unary function \( \neg \), and constants 0 and 1. The functions \( \lor \), \( \land \), and \( \neg \) are called disjunction, conjunction, and negation, respectively. We also use the symbol \( \land \) for the conjunction. For example, the denotation \( \land_{i \in I} b_i \) means the conjunction of the elements \( b_i \) with indices from the set \( I \).

A Boolean algebra \( B \) is an algebraic structure of the language \( \mathcal{L}_0 \) that satisfies the following axioms:

(1) \( a \lor b = b \lor a, \ ab = ba \);