The mean curvature flow for equifocal submanifolds

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14. The mean curvature flow for an isoparametric submanifold (Liu-Terng’s result)
The mean curvature flow for an isoparametric submanifold (Liu-Terng’s result)

The mean curvature flow for an isoparametric submanifold

\( M \): an \( n \)-dimensional manifold

\( f : M \hookrightarrow \mathbb{R}^{n+r} \): an embedding

We identify \( M \) with \( f(M) \).

**Definition**

\( M \): an isoparametric submanifold

\( \iff \)

- the normal holonomy group of \( M \) is trivial
- for any parallel normal vec. fd. \( v \) of \( M \),
  the principal curvatures for \( v_x \) are independent of \( x \in M \)
The mean curvature flow for an isoparametric submanifold (Liu-Terng’s result)

The mean curvature flow for an isoparametric submanifold

\( M \) : an isoparametric submanifold

\[ \text{Fix } x_0 \in M. \]

The focal set of \( M \) at \( x_0 \) consists of finite pieces of hyperplanes \( \{l_1, \cdots, l_k\} \) in \( T_{x_0}^\perp M \).

The reflections w.r.t. \( l_i \)'s generate a Weyl group. Fundamental domains of this group are called the \textbf{Weyl domain} of \( M \).
The mean curvature flow for an isoparametric submanifold (Liu-Terng’s result)

The mean curvature flow for an isoparametric submanifold

$T_{x_0}^\perp M$

Weyl domain
The mean curvature flow for an isoparametric submanifold (Liu-Terng’s result)

The mean curvature flow for an isoparametric submanifold

\[ M : \text{a compact isoparametric submanifold in } \mathbb{R}^{n+r} \]

\[ M_t (0 \leq t < T) : \text{the mean curvature flow for } M \]
The mean curvature flow for an isoparametric submanifold

**Theorem 14.1 (Liu-Terng (Duke M.J.-2009)).**

(i) $M_t (0 \leq t < T)$ are parallel submanifolds of $M$
(ii) $T < \infty$
(iii) $F := \lim_{{t \to T}} M_t$ is a focal submanifold of $M$
(iv) If the natural fibration $pr : M \to F$ is spherical, then $M_t (0 \leq t < T)$ is of type I singularity

\[
\begin{align*}
\left( \text{i.e., } \sup_{{t \in [0,T)}} \left( (T - t) \max_{{v \in S^\perp M_t}} ||A^t_v||^2 \right) < \infty \right) \\
\left( A^t : \text{the shape tensor of } M_t \right) \\
\left( S^\perp M_t : \text{the unit normal bd of } M_t \right)
\end{align*}
\]

**Remark**

\[ pr : M \to F \iff pr(f(x)) := \lim_{{t \to T}} f_t(x) \ (x \in M) \]
The mean curvature flow for an isoparametric submanifold (Liu-Terng’s result)

The mean curvature flow for an isoparametric submanifold

\[ \text{pr} : M \rightarrow F : \text{spherical} \]
The mean curvature flow for an isoparametric submanifold (Liu-Terng’s result)

The mean curvature flow for an isoparametric submanifold

\[ T_{x_0}^\perp M \]

\[ pr : M \to F \quad : \text{not spherical} \]
Question 2.

How does the m.c. flow for $F$ collapse in the case where $F$ is not minimal?
The mean curvature flow for an isoparametric submanifold (Liu-Terng’s result)

The mean curvature flow for an isoparametric submanifold

\[ \tilde{C} \subset T_{x_0}^\perp M \] : a Weyl domain
\[ C := \exp^\perp(\tilde{C}) (= x_0 + \tilde{C}) \]
\[ \sigma : \text{a simplex of } \partial C \ (\dim \sigma \geq 1) \]
\[ F : \text{a focal submanifold of } M \text{ through } \tilde{\sigma} \]
\[ F_t (0 \leq t < T) : \text{the mean curvature flow for } F \]
The mean curvature flow for an isoparametric submanifold (Liu-Terng’s result)

The mean curvature flow for an isoparametric submanifold

Theorem 14.2 (Liu-Terng (Duke M.J.-2009)).

(i) \( F_t (0 \leq t < T) \) are focal submanifolds of \( M \) thr. \( \dot{\sigma} \)

(ii) \( T < \infty \)

(iii) \( F' := \lim_{t \to T} F_t \) is a focal submanifold of \( M \) thr. \( \partial \sigma \)

(iv) If the natural fibration \( \text{pr} : F \to F' \) is spherical

then \( F_t (0 \leq t < T) \) is of type I singularity.
The mean curvature flow for an isoparametric submanifold (Liu-Terng’s result)

The mean curvature flow for an isoparametric submanifold

\[ M_t \xrightarrow{(t \to T_1)} F^1 \]
\[ F^1_t \xrightarrow{(t \to T_2)} F^2 \]

\[ \vdots \]

\[ F^{k-1}_t \xrightarrow{(t \to T_k)} \{ \text{pt} \} \]

\[
\begin{pmatrix}
F^1 : \text{a focal submanifold of } M \\
F^i : \text{a focal submanifold of } F^{i-1} (i = 2, \cdots, k - 1)
\end{pmatrix}
\]
15. The outline of the proof of Liu-Terng’s result
$M$ : an isoparametric submanifold in $\mathbb{R}^{n+r}$

$M_t \ (0 \leq t < T) :$ the mean curvature flow for $M$
The outline of the proof of Liu-Terng’s result

$x_0 \in M$

$\tilde{C} (\subset T_{x_0}^\perp M)$ : the fundamental domain of the Weyl group of $M$ containing $x_0$

**Definition**

\[
\begin{align*}
X : & \text{ a tangent vector field on } \tilde{C} \\
X_w := & \left( H^w \right)_{x_0+w} (w \in \tilde{C}) \\
H^w : & \text{ the mean curvature vector of } \eta_{\tilde{w}}(M) \\
\eta_{\tilde{w}} : & \text{ the end-point map for a p. n. v. f. } \tilde{w} \text{ s.t. } \tilde{w}x_0 = w
\end{align*}
\]
The outline of the proof of Liu-Terng’s result

\[ \eta(w)(M) \]

\[ H^w \]

\[ x_0 + w \]

\[ X_w \]

\[ \tilde{C} \]
The outline of the proof of Liu-Terng’s result

\{ \psi_t \} : a local one-parameter transf. gr. of \( X \)

\( \xi(t) := \psi_t(0) \quad (0 : \text{the zero vector of } T_{x_0}^\perp M) \)

\( \widetilde{\xi(t)} : \text{the parallel n.v.f. of } M \text{ s.t. } \widetilde{\xi(t)}_{x_0} = \xi(t) \)

**Lemma 15.1.**

\[ M_t = \eta_{\widetilde{\xi(t)}}(M) \]

Thus the statement (i) of Theorem 14.1 is shown.
The outline of the proof of Liu-Terng’s result

\[ M \]

\[ \eta_{\xi(t)}(M) \]

\[ x_0(=0) \]

\[ \xi(t) \]

\[ \tilde{C} \]
Thus we suffice to analyze $X$ in order to analyze the mean curvature flow $M_t$. 

$$X \underbrace{\longrightarrow}_{\text{Lem 15.1}} \xi(t) \underbrace{\longrightarrow}_{\text{Lem 15.1}} M_t$$
The outline of the proof of Liu-Terng’s result

$A$: the shape tensor of $M$

$T_x M = \bigoplus_{i \in I_x} E^x_i \ (\text{common eigensp. decomp. of } A_v\text{'s})$

$(v \in T^\perp_x M))$

$\lambda^x_i : T^\perp_x M \to \mathbb{R} \overset{\text{def}}{\leftrightarrow} A_v|_{E^x_i} = \lambda^x_i(v) \text{id} \ (v \in T^\perp_x M)$

Fact 1.

$\lambda^x_i \in (T^\perp_x M)^*$
By ordering $E_i^x$’s ($x \in M$) suitably, we may assume that

\[ \forall i \in I(\equiv I_x), \]

\[ E_i : x \mapsto E_i^x (x \in M) : C^\infty \text{-distribution} \]

curvature distribution

\[ \lambda_i \in \Gamma((T^\perp M)^*) \iff (\lambda_i)_x := \lambda_i^x (x \in M) \]

principal curvature

\[ n_i \in \Gamma(T^\perp M) \iff \lambda_i = \langle n_i, \cdot \rangle \quad (i \in I) \]

curvature normal
Fact 2.

\[ \bigcup_{i \in I} (\lambda_i)_{x}^{-1}(1) = \text{"the focal set of } M \text{ at } x\text{"} \]

Fact 3.

\[ \tilde{C} = \{ w \in T_{x_0}^\perp M \mid (\lambda_i)_{x_0}(w) < 1 \ (i \in I) \} \]
The outline of the proof of Liu-Terng’s result

\[ m_i := \dim E_i \ (i \in I) \]

Lemma 15.2.

\[ X_w = \sum_{i \in I} \frac{m_i}{1 - (\lambda_i)x_0(w)(n_i)x_0} \]

Remark.

\[ X_w = 0 \iff \eta_{\tilde{w}}(M) \text{ : minimal} \]
Proof of (ii) of Theorem 14.1

\[ \xi_0 \in \bigcap_{i \in I} (\lambda_i)^{-1}(x_0)(1) \]

From Lemma 15.2, we have

\[ \frac{d}{dt} ||\xi(t) - \xi_0||^2 = 2\langle \xi'(t), \xi(t) - \xi_0 \rangle \]
\[ = 2\langle X_{\xi(t)}, \xi(t) - \xi_0 \rangle = -2n \quad (n := \dim M) \]

On the other hand, we can show the following fact:

∃Φ : a polynomial map of \( T_{x_0}^\perp M \) onto \( \mathbb{R}^r \) (r := codim M)

s.t. \( \Phi|_{\widetilde{C}} : (\widetilde{C} \to \mathbb{R}^r) : \text{into homeomorphism} \)

\( \Phi_*X : \text{a polynomial vec. fd.} \)
The outline of the proof of Liu-Terng’s result

$\Phi(T_{x_0}^\perp M)$

$(\Phi|_{\overline{C}})^{-1}$

Flows of $X|_{\overline{C}}$

$Y$

(The extension of $\Phi_*(X)$)
From these facts, it is shown that

$$\xi(t)$$ converges to a point $w_1$ of $\partial \tilde{C}$
as $t \to T(< \infty)$.

Since $M_t = \eta_{\tilde{\xi}(t)}(M)$ by Lemma 15.1,

$M_t$ collapses to the focal submanifold $\eta_{\tilde{w}_1}(M)$
as $t \to T(< \infty)$.

q.e.d.
16. The mean curvature flow for an equifocal submanifold
The mean curvature flow for an equifocal submanifold

\[(N, \langle , \rangle) : \text{a Riemannian manifold}\]

\[M : \text{an embedded submanifold in } N\]

\[\exp^\perp : \text{the normal exponential map of } M\]

\[v_0 \in T^\perp_{x_0} M\]

**Definition**

\[s_0 : \text{a focal radius along } \gamma_{v_0}\]

\[\overset{\text{def}}{\iff} \gamma_{v_0}(s_0) : \text{a focal point of } M \text{ along } \gamma_{v_0}\]

\[\overset{\iff}{(\text{Ker } \exp^\perp_{s_0 v}) \cap (T_{s_0 v}(T^\perp M) \setminus V_{s_0 v}) \neq \{0\})\]

\[(V_{s_0 v} : \text{the vertical space of } T^\perp M \text{ at } s_0 v)\]
The mean curvature flow for an equifocal submanifold

\[ G/K : \text{a symmetric space of compact type} \]
\[ M : \text{an embedded submanifold in } G/K \]

**Definition (Terng-Thorbergsson (JDG-1995))**

\[ M : \text{an equifocal submanifold} \]

\[ \begin{align*}
&\exists \text{ M is compact} \\
&\exists \text{ the normal holonomy group of } M \text{ is trivial} \\
&\exists \text{ M has flat section} \\
&\exists \text{ for any parallel normal vec. fd. } v \text{ of } M, \\
&\text{the focal radii along } \gamma_{vx} \text{ are indep. of } x \in M
\end{align*} \]
$M$ has flat section

\[ \iff \text{def} \quad \forall x \in M, \quad \Sigma_x := \exp_{x}^{-1}(T_{x}^\perp M) \text{ is totally geodesic and flat.} \]
\( M \): an equifocal submanifold in \( G/K \)

\( x_0 \in M \)

The focal set of \( M \) at \( x_0 \) consists of the images of finite pieces of infinite parallel families of hyperplanes \((\mathcal{L}_a := \{l_{ai} \mid i \in \mathbb{Z}\} \ (a = 1, \cdots, k))\) in \( T_{x_0} M \) by the normal exponential map.

The reflections w.r.t. \( l_{ai} \)'s generate a discrete group, that is, a Coxeter group.

This group is called the Coxeter group of \( M \).
The mean curvature flow for an equifocal submanifold
The mean curvature flow for an equifocal submanifold

$G/K$ : a symmetric space of compact type

$M$ : a non-minimal equifocal submanifold in $G/K$

$M_t (0 \leq t < T)$ : the mean curvature flow for $M$
The mean curvature flow for an equifocal submanifold

The mean curvature flow for an equifocal submanifold

Theorem 16.1(K. (Asian J.M.-2011)).

(i) $M_t (0 \leq t < T)$ are parallel submanifolds of $M$
(ii) $T < \infty$
(iii) $F := \lim_{t \to T} M_t$ is a focal submanifold of $M$
(iv) If $M$ is irreducible, if $\text{codim } M \geq 2$, and if the natural fibration $\text{pr} : M \to F$ is spherical, then $M_t (0 \leq t < T)$ is of type I singularity.
The mean curvature flow for an equifocal submanifold.

Question.

How does the mean curvature flow for $F$ collapse?
The mean curvature flow for an equifocal submanifold

\( \tilde{C}(\subset T^\perp_{x_0} M) \) : a fundamental domain (s.t. \( 0 \in \tilde{C} \)) of the Coxeter group of \( M \)

\( C := \exp^\perp(\tilde{C}) \)

\( \sigma \) : a stratum of \( \partial C \) (\( \dim \sigma \geq 1 \))

\( F \) : a non-minimal focal submanifold through \( \tilde{\sigma} \)

\( F_t (0 \leq t < T) \) : the mean curvature flow for \( F \)
The mean curvature flow for an equifocal submanifold

Theorem 16.2 (K. (Asian J.M.-2011)).

(i) $F_t (0 \leq t < T)$ are focal submanifolds of $M$ through $\sigma$
(ii) $T < \infty$
(iii) $F' := \lim_{t \to T} F_t$ is a focal submanifold of $M$ through $\partial \sigma$
(iv) If $M$ is irreducible, if codim $M \geq 2$
and if the natural fibration $\text{pr} : F \to F'$ is spherical,
then $F_t (0 \leq t < T)$ is of type I singularity.
The mean curvature flow for an equifocal submanifold

\[ M_t \underset{(t \rightarrow T_1)}{\longrightarrow} F^1 \text{ non-min.} \]
\[ F^1_t \underset{(t \rightarrow T_2)}{\longrightarrow} F^2 \text{ non-min.} \]
\[ \cdots \]
\[ F^{k-1}_t \underset{(t \rightarrow T_k)}{\longrightarrow} F^k \text{ min.} \]

\( F^1 : \text{a focal submanifold of } M \)

\( F^i : \text{a focal submanifold of } F^{i-1} \) \((i = 2, \cdots, k)\)
The mean curvature flow for an equifocal submanifold

\[ \tilde{M} := (\pi \circ \phi)^{-1}(M) \hookrightarrow H^0([0, 1], g) \]

\[ \downarrow \phi \]

\[ \downarrow G \]

\[ \downarrow \pi \]

\[ M \leftrightarrow G/K \]

\[ M : \text{equifocal} \iff \tilde{M} : \text{isoparametric} \]
17. Isoparametric submanifolds in a Hilbert space
Isoparametric submanifolds in a Hilbert space

\[ V : \text{an } \infty\text{-dimensional (separable) Hilbert space} \]

\[ f : M \hookrightarrow V : \text{an immersion of finite codimension} \]

**Definition (Terng (JDG-1989)).**

\[ f : M \hookrightarrow V : \text{a proper Fredholm submanifold} \]

\[ \iff \quad \begin{cases} 
\exp_{\perp}|_{B_{-1}(M)} : \text{proper map} \\
\exp_{\ast v} : \text{Fredholm op. } (\forall v \in T_{-1}^\perp M) \\
\exp_{\perp} : \text{the normal exponential map of } M \\
B_{-1}(M) : \text{the unit normal bundle map of } M 
\end{cases} \]
Fact

The shape operator of a proper Fredholm submanifold is compact operator.

Definition (Terng (JDG-1989)).

\( f : M \leftrightarrow V \) : an isoparametric submanifold

\[\begin{align*}
\text{def} & \quad \text{the normal holonomy group of } M \text{ is trivial} \\
& \quad \text{For any parallel normal vec. fd. } v \text{ of } M, \\
& \quad \text{the principal curvature}'s for } v_x \\
& \quad \text{are independent of } x \in M
\end{align*}\]
$f : M \hookrightarrow V$ : an isoparametric submanifold

$x_0 \in M$

The focal set of $M$ at $x_0$ consists of finite pieces of infinite parallel families of hyperplanes in $T_{x_0} M$.

$(\mathcal{L}_a := \{l_{ai} \mid i \in \mathbb{Z}\} \ (a = 1, \cdots, k))$

The reflections w.r.t. $l_{ai}$’s generate a discrete group, that is, a Coxeter group.
This group is called the Coxeter group of $M$. 
Isoparametric submanifolds in a Hilbert space
Isoparametric submanifolds in a Hilbert space

\[ f : M \hookrightarrow V : \text{a proper Fredholm submanifold} \]

**Definition (Heintze-Liu-Olmos (2006)).**

\[ f : M \hookrightarrow V : \text{a regularizable submanifold} \]

\[ \left\{ \begin{array}{l}
\forall v \in T^\perp M, \\
\exists \text{Tr}_r A_v (< \infty), \quad \exists \text{Tr}(A_v^2) (< \infty) \\
\text{Tr}_r A_v := \sum_{i=1}^{\infty} (\lambda_i + \mu_i) \\
(Spec A_v = \{ \mu_1 \leq \mu_2 \leq \cdots \leq 0 \leq \cdots \leq \lambda_2 \leq \lambda_1 \}) \\
\text{Tr}(A_v^2) := \sum_{i=1}^{\infty} \nu_i \\
(Spec A_v^2 = \{ \nu_1 \geq \nu_2 \geq \cdots > 0 \})
\end{array} \right. \]
18. The mean curvature flows for a regularizable submanifold
The mean curvature flows for a regularizable submanifold

$V : \text{an } \infty\text{-dimensional (separable) Hilbert space}$

$f : M \hookrightarrow V : \text{a regularizable submanifold}$

**Definition (Heintze-Liu-Olmos (2006)).**

$H \overset{\text{def}}{\leftrightarrow} \langle H, v \rangle = \text{Tr}_r A_v \quad (\forall v \in T^\perp M)$

This normal vector field $H$ is called

a regularized mean curvature vector.
$f_t : M \hookrightarrow V \ (0 \leq t < T) : \text{a } C^\infty\text{-family of regularizable submanifolds}$

$\tilde{f} : M \times [0, T) \to V$

$\overset{\text{def}}{\iff} \tilde{f}(x, t) := f_t(x) \ ((x, t) \in M \times [0, T))$
The mean curvature flows for a regularizable submanifold

Definition (K. (Asian J.M.-2011))

\[ f_t \quad (0 \leq t < T) \quad : \text{the (regularized) mean curvature flow} \]

\[ \frac{\partial f}{\partial t} \quad \overset{\text{def}}{=} \quad H_t \quad (0 \leq t < T) \]

\[ (H_t \quad : \text{the regularized mean curv. vec. of } f_t) \]

Question.

For any regularizable submanifold \( f \), does the mean curvature flow for \( f \) uniquely exist in short time?
In order to solve this question affirmatively, we must show the Hilbert vector bundle version of the Hamilton’s theorem for the evolution of a section of a (finite dim.) vector bundle. However, since a regularizable submanifold can be not compact, we must assume a certain kind of compactness for the submanifold.
The mean curvature flows for a regularizable submanifold

\( G/K \) : a symmetric space of compact type

\( M \) : a compact submanifold in \( G/K \)

\( \phi : H^0([0, 1], g) \rightarrow G \) : the parallel transport map for \( G \)

\[
\begin{align*}
\phi(u) &:= g_u(1) \quad (u \in H^0([0, 1], g)), \\
\text{where } g_u &\text{ is the element of } H^1([0, 1], G) \text{ s.t.} \\
g_u(0) &= e \text{ and } (R_{g_u(t)})^{-1}(g_u'(t)) = u(t) \quad (\forall t \in [0, 1])
\end{align*}
\]

\pi : G \rightarrow G/K : the natural projection

Set \( \tilde{\phi} := \pi \circ \phi \).

\( \tilde{M} := \tilde{\phi}^{-1}(M) \) (\( \leftarrow \) \( H^0([0, 1], g) \))
The mean curvature flows for a regularizable submanifold

Fact.

- \( \widetilde{M} \) is a regularizable submanifold.
- There uniquely exists the mean curvature flow \( \widetilde{M}_t \) for \( \widetilde{M} \) in short time.
19. The outline of the proof of Theorems 16.1 and 16.2
$M$ : a non-minimal equifocal submanifold in $G/K$

$M_t \ (0 \leq t < T) :$ the mean curvature flow for $M$
Theorem 16.1(K. (Asian J.M.-2011)).

(i) $M_t (0 \leq t < T)$ are parallel submanifolds of $M$
(ii) $T < \infty$
(iii) $F := \lim_{t \to T} M_t$ is a focal submanifold of $M$
(iv) If $M$ is irreducible, if $\operatorname{codim} M \geq 2$, and if the natural fibration $\operatorname{pr} : M \to F$ is spherical, then $M_t (0 \leq t < T)$ is of type I singularity.
\[ \tilde{M} := (\pi \circ \phi)^{-1}(M) \hookrightarrow H^0([0, 1], g) \]
\[ \downarrow \phi \]
\[ \downarrow G \]
\[ \downarrow \pi \]
\[ M \hookrightarrow G/K \]

\[ M : \text{equifocal} \quad \rightarrow \quad \tilde{M} : \text{regularizable isoparametric} \]

\[ M_t : \text{the mean curvature flow for } M \]
\[ \tilde{M}_t : \text{the mean curvature flow for } \tilde{M} \]
Lemma 19.1.

\[ \widetilde{M}_t = (\pi \circ \phi)^{-1}(M_t) \]

According to this fact, the investigation of the flow \( M_t \) is reduced to that of the flow \( \widetilde{M}_t \).
The outline of the proof of Theorems 16.1 and 16.2

\[ x_0 \in M \]

\[ u_0 \in (\pi \circ \phi)^{-1}(x_0) \subset \widetilde{M} \]

\[ \tilde{C} \subset T_{u_0} \widetilde{M} : \text{the fund. domain of the Coxeter group of } \widetilde{M} \text{ containing } u_0 \]

**Definition**

\[ X : \text{a vector field on } \tilde{C} \]

\[
\begin{align*}
X_w := (\tilde{H}^w)_{u_0 + w} (w \in \tilde{C}) \\
\quad \tilde{H}^w : \text{the reg. mean curv. vec. of } \eta_{\tilde{w}}(\widetilde{M}) \\
\quad \eta_{\tilde{w}} : \text{the end – point map for a p. n. v. f. } \tilde{w} \text{ s.t. } \tilde{w}_{u_0} = w
\end{align*}
\]

\[ \iff \text{def} \]

\[ \begin{cases} 
X_w := (\tilde{H}^w)_{u_0 + w} (w \in \tilde{C}) \\
\quad \tilde{H}^w : \text{the reg. mean curv. vec. of } \eta_{\tilde{w}}(\widetilde{M}) \\
\quad \eta_{\tilde{w}} : \text{the end – point map for a p. n. v. f. } \tilde{w} \text{ s.t. } \tilde{w}_{u_0} = w
\end{cases} \]
The outline of the proof of Theorems 16.1 and 16.2

\[ u_0 + w \]

\[ \eta_{\tilde{w}}(\tilde{M}) \quad \tilde{M} \]

\[ \tilde{H}^w \]

\[ T_{u_0} \tilde{M} \]
The outline of the proof of Theorems 16.1 and 16.2

\{\psi_t\} : a local one-parameter transformation gr. of X

\xi(t) \equiv \psi_t(0) \quad (0 : the zero vector of T_{u_0} \overline{M})

\overline{\xi(t)} : the parallel normal vec. fd. of \overline{M} s.t. \overline{\xi(t)}_{u_0} = \xi(t)

Lemma 19.2.

\overline{M_t} = \eta_{\overline{\xi(t)}}(\overline{M})

Proof of (i) of Theorem 16.1.

\[ M_t = (\pi \circ \phi)(\overline{M_t}) = (\pi \circ \phi)(\eta_{\overline{\xi(t)}}(\overline{M})) \]

\[ = \eta_{(\pi \circ \phi)_*(\overline{\xi(t)}}}(M) \]

q.e.d.
The outline of the proof of Theorems 16.1 and 16.2

\[ \eta_{\xi(t)}(\widetilde{M}) \]

\[ T_{u_0} \widetilde{M} \]
The outline of the proof of Theorems 16.1 and 16.2

\[ X \rightarrow \xi \rightarrow \tilde{M}_t \rightarrow M_t \]

\( M : \text{equifocal} \rightarrow \tilde{M} : \text{reg. isoparametric} \)
\( \tilde{A} : \text{the shape tensor of } \tilde{M} \)
\[ T_u \tilde{M} = \bigoplus_{i \in I_u} E_i^u \quad (\text{common eigensp. decomp. of } \tilde{A}_v \text{'s}) \]
\[ (v \in T_u^{\perp} \tilde{M}) \]
\[ \lambda_i^u : T_u^{\perp} \tilde{M} \rightarrow \mathbb{R} \quad \overset{\text{def}}{\leftrightarrow} \quad \tilde{A}_v | E_i^u = \lambda_i^u(v) \text{id} \quad (v \in T_u^{\perp} \tilde{M}) \]

Fact.
\[ \lambda_i^u \in (T_u^{\perp} \tilde{M})^* \]
By choosing $E_i^u$'s ($u \in \tilde{M}$) suitably, we may assume that

$$\forall i \in I(:= I_u),$$

$$E_i : u \mapsto E_i^u (u \in \tilde{M}) : C^\infty \text{-distribution}$$

curvature distribution

$$\lambda_i \in \Gamma((T^\perp \tilde{M})^*) \iff (\lambda_i)_u := \lambda_i^u (u \in \tilde{M})$$

principal curvature

$$n_i \in \Gamma(T^\perp \tilde{M}) \iff \lambda_i = \langle n_i, \cdot \rangle \quad (i \in I)$$

curvature normal
The set of all principal curvatures of $f_M$.

Fact: \[ \bigcup_{\lambda \in \Lambda} \lambda_u^{-1}(1) = "\text{the focal set of } \tilde{M} \text{ at } u" \]

The focal set of $\tilde{M}$ at $u$ consists of finite pieces of infinite families of parallel hyperplanes in $T_u \perp \tilde{M}$. 
The outline of the proof of Theorems 16.1 and 16.2

Fact

The set $\Lambda$ is described as

$$\Lambda = \bigcup_{a=1}^{\bar{r}} \left\{ \frac{\lambda_a}{1 + b_a j} \mid j \in \mathbb{Z} \right\}$$

for some $\lambda_a \in \Gamma((T^\perp \widetilde{M})^*)$ and some constant $b_a > 1$ ($a = 1, \ldots, \bar{r}$).

Fact

$$\tilde{C} = \{ w \in T_{u_0}^\perp \tilde{M} \mid (\lambda_a)_{u_0}(w) < 1 \ (a = 1, \ldots, \bar{r}) \}$$

$$E_{a,j} := \text{Ker} \left( \tilde{A} - \frac{\lambda_a(\cdot)}{1 + b_a j} \text{id} \right)$$

$$m^e_a := \dim E_{a,2j}, \quad m^o_a := \dim E_{a,2j+1}$$
The outline of the proof of Theorems 16.1 and 16.2

\[(a)^1 u_0 (1 + b a)\]
The outline of the proof of Theorems 16.1 and 16.2

Lemma 19.3.

\[ X_w = \sum_{a=1}^{\bar{r}} \left( m_a^e \cot \frac{\pi}{b_a} (1 - (\lambda_a)u_0(w)) \right) \left( -m_a^o \tan \frac{\pi}{b_a} (1 - (\lambda_a)u_0(w)) \right) \frac{\pi}{2b_a} (n_a)u_0 \]

\[ \left( n_a \overset{\text{def}}{\equiv} \langle n_a, \cdot \rangle = \lambda_a(\cdot) \right) \]

Remark.

\[ X_w = 0 \iff \eta_{\bar{w}}(\bar{M}) : \text{minimal} \]
The outline of the proof of Theorems 16.1 and 16.2

Proof of (ii) and (iii) of Theorem 16.1

\[ \rho \in C^\infty(\tilde{C}) \]

\[ \rho(w) := - \sum_{a=1}^{\tilde{r}} \left( m_a^e \log \sin \frac{\pi}{b_a} (1 - (\lambda_a)_{u_0}(w)) \right) + m_a^o \log \cos \frac{\pi}{b_a} (1 - (\lambda_a)_{u_0}(w)) \quad (w \in \tilde{C}) \]

Then we have

\[ \text{grad} \rho = X \quad \text{and} \quad \rho : \text{downward convex} \]

Also we have

\[ \rho(w) \to \infty \quad (w \to \partial\tilde{C}) \]
The outline of the proof of Theorems 16.1 and 16.2
Hence we see that
\[ \rho \text{ has the only minimal point.} \]

Denote by \( w_0 \) this point. Clearly we have \( X_{w_0} = 0 \)

On the other hand, we can show the following fact:

\[ \exists \Phi : \text{a } C^\infty \text{ map of } T_{u_0}^\perp \tilde{M} \text{ onto } \mathbb{R}^r \quad (r := \text{codim } M) \]

s.t. \( \left\{ \Phi|_{\tilde{C}} : \tilde{C} \to \mathbb{R}^r \right\} : \text{into homeomorphism} \)

\[ \Phi_* X : \text{a } C^\infty \text{ vec. fd.} \]
From these facts, we see that

- the flow of $X$ starting from a point other than $w_0$ converges to a point of $\partial \tilde{C}$ in finite time.

Since $\tilde{M}$ is not minimal, we can show

- $0 \neq w_0$ and the flow $\xi(t)$ of $X$ starting from $0$ converges to a point $w_1$ of $\partial \tilde{C}$ in finite time $T$.

Since $M_t = \eta_{(\pi \circ \phi)_* (\xi(t))}(M)$,

$M_t$ collapses to the focal submanifold $\eta_{(\pi \circ \phi)_* (\tilde{w}_1)}(M)$ in the time $T$. 

q.e.d.
The outline of the proof of Theorems 16.1 and 16.2

\[ F := \eta(\pi \circ \phi)_*(\tilde{w}_1)(M) \]

(iv) of Theorem 16.1.

If \( M \) is irreducible, if \( \text{codim} \ M \geq 2 \), and if the natural fibration \( \text{pr} : M \to F \) is spherical, then \( M_t \ (0 \leq t < T) \) is of type I singularity.

\[
\left( \begin{array}{c} 
\text{i.e., } \sup_{t \in [0,T)} \left( (T - t) \max_{v \in S^\perp M_t} \|A^t_v\|^2 \right) < \infty \\
A^t : \text{the shape tensor of } M_t \\
S^\perp M_t : \text{the unit normal bd of } M_t 
\end{array} \right)
\]
Proof of (iv) of Theorem 16.1.

\[ \tilde{F} = \eta \tilde{w}_1(\tilde{M}) \]

\[ A^t \text{ (resp. } \tilde{A}^t \text{): the shape tensor of } M_t \text{ (resp. } \tilde{M}_t \text{)} \]

Since \( pr : M \to F \) is spherical,

\[ \exists 1 \ a_0 \in \{1, \ldots, r\} \text{ s.t. } w_1 \in ((\lambda_{a_0})_{u_0}^{-1}(1) \cap \partial \tilde{C})^\circ \]

Hence

\[ \lim_{t \to T^0} \| \tilde{A}^t_v \|_\infty^2 (T - t) = \lim_{t \to T^0} \frac{(\lambda_{a_0})_{u_0}(v)^2}{(1 - (\lambda_{a_0})_{u_0}(\xi(t)))^2} (T - t) \]

\[ = \frac{(\lambda_{a_0})_{u_0}(v)^2}{2m_{a_0} \|(n_{a_0})_{u_0}\|^2} \cdots \cdots \ (1) \]
The outline of the proof of Theorems 16.1 and 16.2

\[ M : \text{irr.} \& \text{codim } M \geq 2 \]

\[ \rightarrow M : \text{curvature-adapted} \]

\[ \lim_{t \to T - 0} \| \tilde{A}_v^t \|^2_{\infty} (T - t) = \lim_{t \to T - 0} \| A^t_{(\pi\circ\phi)^*} (v) \|^2_{\infty} (T - t) \]

\[ \cdots \cdots \text{(2)} \]

From (1) and (2), we have

\[ \lim_{t \to T - 0} \max_{v \in S^\perp_{\exp\perp (\xi(t))} M_t} \| A^t_{(\pi\circ\phi)^*} (v) \|^2_{\infty} (T - t) = \frac{1}{2m_{a_0}^e} < \infty \]

Thus \( M_t \) is of type I singularity.

q.e.d.
\[ \sigma : \text{a stratum of } \partial C \text{ s.t. } \dim \sigma \geq 1 \]

\[ F : \text{a non-minimal focal submanifold of } M \text{ thr. } \overset{\circ}{\sigma} \]

\[ F_t : \text{the mean curvature flow for } F \]
Theorem 16.2.

(i) $F_t (0 \leq t < T)$ are focal submanifolds of $M$ through $\partial \sigma$

(ii) $T < \infty$

(iii) $F' := \lim_{t \to T} F_t$ is a focal submanifold of $M$ through $\partial \sigma$

(iv) If $M$ is irreducible, if $\text{codim } M \geq 2$

and if the natural fibration $\text{pr} : F \to F'$ is spherical,
then $F_t (0 \leq t < T)$ is of type I singularity.
The outline of the proof of Theorems 16.1 and 16.2

\[ \tilde{\sigma} : \text{the simplex of } \partial \tilde{C} \text{ s.t. } \exp_{\tilde{\sigma}}(\tilde{\sigma}) = \sigma \]

\[ w \in (\tilde{\sigma})^\circ \]

\[ \tilde{F}_w : \text{the focal submanifold of } \tilde{M} \text{ through } w \]

\[ (\text{i.e., } \tilde{F}_w := \eta_{\tilde{w}}(\tilde{M})) \]

\[ \tilde{H}^w : \text{the mean curvature vector of } \tilde{F}_w \]

**Fact.**

\[ (\tilde{H}^w)_{u_0+w} : \text{tangent to } \tilde{\sigma} \]
The outline of the proof of Theorems 16.1 and 16.2

**Definition**

\[ X_{\sigma} \text{ : a tangent vector field on } \sigma \]

\[ X_{w}^{\sigma} := (H^w_{u0})_u + w (w \in \sigma) \]

By analyzing \( X^{\sigma} \), we can show the statements of Theorem 16.2.