Pfaff equation and Fourier analysis to phase extraction from an interferogram with carrier frequency

Francisco Lara-Cortez, Cruz Meneses-Fabian, and Gustavo Rodriguez-Zurita
Benemerita Universidad Autonoma de Puebla, Facultad de Ciencias Fisico-Matematicas, Av. San Claudio y 18 Sur, C. U. San Manuel, Puebla PUE72570, Mexico
E-mail: cmeneses@fcfm.buap.mx

Abstract. In the phase extraction techniques, one of the steps most used is to calculate the phase unwrapping from the wrapped phase, which is generally obtained via the inverse tangent function. With the idea to avoid this process, in the present manuscript a method based in the solution of the Pfaff equations is proposed. It is shown that the Pfaff equation is formed with the phase gradient and an auxiliary vector. The phase gradient is obtained from an interference patron with carrier frequency by applying the Fourier transform method and the partial derivatives. In the present manuscript, mathematical analysis, numerical simulation, and the phase extraction of some experimental interferograms are shown.

1. Introduction
Many techniques to extract the information phase from an interference pattern have been used amply. Some techniques are based in the unwrapping phase, from the wrap phase modulus of $2\pi$ radians [1-7]. Others, instead, have been used various mathematical tools in order to extract the information phase without to use conventional unwrapping phase, as for example, the Gabor transform [8], the functions of Green [9], the Helmholtz equation eigenfunctions [10] and the Poisson equation and genetic algorithmic [11], also the interpolation of Fourier transform technique [12], spiral spatial filtering techniques [13-14], elliptic equation partials [15], the differences of phase between adjacent points [16-18] or the phase gradient [15, 19-21]

In this paper, a quasi-homogenous differential equation is built by the scalar product of the phase gradient and an auxiliary vector. The general solution in order to extract of phase variations is obtained by means of solution methods of total differential equations or Pfaff equations. The phase gradient is calculated by applying Fourier-transform techniques and partial derivatives of the interference pattern.

2. Basic considerations
An interference pattern with a spatial carrier frequency in the horizontal direction is given by

$$I(x,y)=a(x,y)+b(x,y)\cos[\phi(x,y)+\frac{2\pi x}{\lambda}], \quad (1)$$

where $I(x,y)$ is the intensity laying in the plane of the camera CCD, $a(x,y)$ is the intensity of background, $b(x,y)$ is the intensity of the modulation, $\phi(x,y)$ is the phase variation in the test section,
and $\mu_0$ is the carrier frequency, which is chosen in such a way that the term $2\pi \mu_0 x$ is bigger than $a(x, y)$, $b(x, y)$, and $\phi(x, y)$.

The expression (1) can be written in a more convenient way

$$I(x, y) = a(x, y) + c(x, y)e^{i(2\pi \mu_0 x)} + c^*(x, y)e^{-i(2\pi \mu_0 x)},$$

where $i$ is the imaginary unit given by $\sqrt{-1}$ and the symbol $*$ denotes the conjugated complex of $c(x, y)$, which is given by

$$c(x, y) = \frac{1}{2}b(x, y)e^{i\phi(x, y)}.$$  

By Fourier-transform method introduced by Takeda et al.\,\cite{23-29} and by many works realized by many investigators,\,\cite{23-29} the three terms of the right hand side of Eq. (2) can be separated, in such a way that $c(x, y)$ can be calculated. Considering this fact, it is proposed to calculate the quotient

$$h(x, y) = \frac{c(x, y)}{c^*(x, y)} = e^{i2\phi(x, y)}.$$  

where it can be noted that term corresponding to the modulation intensity $b(x, y)$ has been eliminated, giving as result a complex exponential function with a relative exponent to double of the phase $\phi(x, y)$. For our objects, this can be advantageous, because the derivative of an exponential function always appears as a factor. Therefore, if the gradient of $h(x, y)$ is calculated from Eq. (4), then it is possible to obtain the phase gradient $\phi(x, y)$ as

$$\nabla \phi(x, y) = \frac{1}{i2h(x, y)}\nabla h(x, y),$$

where the phase gradient is denoted by $\nabla \phi(x, y)$, the operator $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j$ is given in Cartesian coordinates and $i, j$ denote unitary vectors on directions $x$ and $y$, respectively. By substituting Eq. (3) in Eq. (5), the phase gradient is given by

$$\nabla \phi(x, y) = \frac{1}{i2} \left[ \frac{\nabla c(x, y)}{c(x, y)} - \frac{\nabla c^*(x, y)}{c^*(x, y)} \right].$$

where $\nabla \phi(x, y)$ is now expressed in terms of $c(x, y)$ function, which is calculated of the interference pattern by applying the Fourier-transform method.

3. **Phase extraction by solution of Pfaff equation**

The problem is now how calculate $\phi(x, y)$ starting from Eq. (6). The theory of differential equations is adapted to interpret the gradient phase of Eq. (6) as part of a vectorial field. With this field and a suitable vector we build a Pfaff equation. The solution will give us a family of orthogonal surfaces to the vectorial field given. This family constituent the phase function mounted on a constant parameter. We start this task by building the following differential equation

$$\nabla \phi \cdot dr = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = P,$$

where $dr = dx + dy$ is a vector in the plane, the symbol $\cdot$ denotes the scalar product of two vectors, and $P = P(x, y)$ is a scalar function of two variables. For definition, the left part of Eq. (7) is an exact equation and therefore is a total differential equation. In this case, $P(x, y)$ is the differential of a scalar function $\zeta(x, y)$ (surface $S$), that is $P(x, y) = dz(x, y)$, and therefore Eq. (7) can be written as
\[ \nabla \phi \cdot \mathbf{dr} - dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy - dz = 0. \] (8)

which is known as a Pfaff equation. If we define a vectorial field 
\[ \mathbf{F} = \mathbf{k} \cdot \left( \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} - \mathbf{k} \right), \]

it is easy to prove that \( \text{rot} \mathbf{F} = 0 = (0,0,0) \), so it turns out to hold what vectorial field to admit to attain by \( \mathbf{F} = \nabla U \), where \( U = U(x,y,z) = C \) is a potential function, with \( C \) constant. Then as it is known, \( \mathbf{F} \) is orthogonal to the potential \( U \) and Eq. (8) can be rewritten as

\[ \mathbf{F} \cdot \mathbf{dr} = \nabla U \cdot \mathbf{dr} = dU = 0. \] (9)

where an arbitrary vector \( \mathbf{dr} = dx + j dy - k dz \) lying in the tangent plane to the surface \( S \) in whichever point has been defined. As it is well known in theory of differential equations, the solution of Eq. (9) consists of a family of orthogonal surfaces to the vectorial field \( \mathbf{F} \) and each solution is of form \( U(x,y,z) = C \) with \( C \) a constant. More clearly \( U \) is obtained from

\[ dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy - \frac{\partial U}{\partial z} dz = 0. \] (10)

Relating Eq. (8) to Eq. (10), the potential function can be written as \( U(x,y,z) = \phi(x,y) - z \) and the function \( z = z(x,y) \) can be described by

\[ z(x,y) = \phi(x,y) - C, \] (11)

so, the solutions viewed with respect to \( z(x,y) \) consist of an infinity family the surfaces given per a constant \( C \), which is consistent for any value of \( C \) with the interferometric techniques when the retrieval phase from an interferometry pattern is obtained, due to in interferometry is only observed phase variations between two waves.

**Figure 1.** Simulation of computer, (a) Phase function, (b) Interferogram pattern (Eq. 1) evaluated with a spatial carry frequency and phase function given by (a).

Eq. (11) is the solution of Pfaff equation formed by using the phase gradient calculated by Fourier-transform method applied to pattern interferometry. The solution of Pfaff equation is well known, we here show a solution based by integration through of a curvilinear trajectory, which is invariant to the integration trajectory used. For example, if it is integrated through the parallel trajectories to the axis, initiating at the point \( (x_0, y_0) \) and ending at the point \( (x, y) \), the general solution is obtained as
\[
\int_{(x_0, y_0)}^{(x, y)} \frac{\partial \phi}{\partial x} \, dx + \int_{(x_0, y_0)}^{(x, y)} \frac{\partial \phi}{\partial y} \, dy = \int_{(x_0, y_0)}^{(x, y)} \frac{\partial \phi}{\partial y} \, dx + \int_{(x_0, y_0)}^{(x, y)} \frac{\partial \phi}{\partial x} \, dy = \phi(x, y) - \phi(x_0, y_0) = z(x, y),
\]

where \((x_0, y_0)\) is first point and \((x, y)\) is the last point in the interferogram in order to obtain the phase of the all possible points. The solution \(z(x, y)\) is a family surfaces that dependent of the initial boundary value \(\phi(x_0, y_0)\), which do not represent an important problem, because in analysis fringes is more important to observe variation of phase, missing of constant terms, therefore, in practical terms the solution \(z(x, y)\) can be considered as the phase desired \(\phi(x, y)\). Note that the constant \(C\) is to be equal to \(\phi(x_0, y_0)\).

**Figure 2.** Image corresponding to Eq. (3). (a) Real and (b) Imaginary part obtained by Fourier-transform method.

**Figure 3.** Real part of the phase gradient: (a) Partial derivative with respect to \(x\) and (b) Partial derivative with respect to \(y\).

### 4. Numerical simulations

For numerical simulations, Fig. 1a depicted arbitrary phase variations. Fig. 1b shows a graphic of the interference pattern corresponding to phase of Fig. 1a, where a carrier frequency has been added, by simplicity it has been chosen \(a(x, y) = b(x, y) = 1\). Proceeding as the theory has been exposed above, the Fig. 2 shows the function \(c(x, y)\) (Eq. 4) obtained by applying Fourier-transform method, the real part is shown in Fig. 2a and the imaginary part is shown in Fig. 2b. In order to show as the phase gradient is numerically calculated with a good approximation, we first calculate Eq. (4) due to the modulation illumination is missed by division pixel to pixel and this way there is no effect in the calculate of phase gradient, which is made by numerically implementing Eq. (5). Moreover, Fig. 3 shows the phase gradient, where Fig. 3a and Fig. 3b depict the real part of the partial derivatives with respect \(x\) and respect \(y\), respectively, which were calculated by Eq. (6). In Fig. 4a is shown the results of the
evaluation of a curvilinear integral applied to the data shown in the Fig. 3 by using Eq. (12). This image is the information of desired phase function, with the constant \( C = \phi(x_0, y_0) = 0 \). This computation has an error of 0.2 % approximately (Fig. 4b) with possibility to be reduced or minimize.

![Figure 4](image)

**Figure 4.** (a) Phase extraction by curvilinear integration as indicated in Eq. (12) using the data in Fig. 3 and (b) Show the error obtained by rest of the data in Fig. 1a and data in Fig. 4a.

5. Conclusion

We have demonstrated a way to see analytically the phase retrieval from an interference pattern with a carrier frequency. We think that also can be possible to adapt it method to phase-shifting interferometry. In the present technique, there is no ambiguity of phase sign, wrapped phase has been avoided by calculating phase gradient, and desired phase has been measured by solving Pfaff equation. In this technique the Pfaff equation is built by using phase gradient and auxiliary vector. This way, phase unwrapping has been avoided. This method is achievable and also its numerical implementation is practicable.

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