On the area-preserving Willmore flow of small bubbles sliding on a domain’s boundary

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Abstract

We consider the area-preserving Willmore evolution of surfaces \( \phi \) that are close to a half-sphere with a small radius, sliding on the boundary \( S \) of a domain \( \Omega \) while meeting it orthogonally. We prove that the flow exists for all times and keeps a ‘half-spherical’ shape. Additionally, we investigate the asymptotic behaviour of the flow and prove that for large times the barycenter of the surfaces approximately follows an explicit ordinary differential equation. Imposing additional conditions on the mean curvature of \( S \), we then establish convergence of the flow.

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1 Introduction

Given an immersion \( \phi : S^2_+ \to \mathbb{R}^3 \) with mean curvature \( H \) the Willmore energy is defined as

\[ W(\phi) := \frac{1}{4} \int_{S^2_+} H^2 \, d\mu_\phi. \]

For a sufficiently smooth bounded domain \( \Omega \subset \mathbb{R}^3 \) with boundary \( S := \partial \Omega \) we consider the class

\[ \mathcal{M}(S) := \left\{ \phi \in C^4(S^2_+, \mathbb{R}^3) \text{ immersed } \mid \phi(S^2_+) \subset S, \frac{\partial \phi}{\partial \eta} = N^S \circ \phi \right\} \]

of immersed surfaces meeting \( S \) orthogonally along the boundary. Here \( \eta \) and \( N^S \) denote the interior unit normals of \( S^2_+ \) and \( \Omega \) along their respective boundaries. In this article we study the area preserving Willmore flow inside a suitable subclass of \( \mathcal{M}(S) \). That is, denoting the scalar Willmore operator by \( W \) and the inner normal of \( \phi \) by \( \nu \) the equation

\[
\begin{cases}
\langle \dot{\phi}(t), \nu \rangle = -W(\phi(t)) + \alpha(t)H(\phi(t)) \\
\frac{\partial H}{\partial \eta} + h^S(\nu, \nu)H = 0 \\
\dot{\phi}(t) \in \mathcal{M}(S)
\end{cases}
\]
Here $\alpha$ is a suitable Lagrange multiplier. The additional third-order Neumann-type boundary condition appearing in (1.1) arises naturally from the requirement of steepest energy descent (for details see Subsection 2.2).

Alessandroni and Kuwert studied the corresponding elliptic problem in [2] and constructed critical points inside $\mathcal{M}(S)$ that are (up to rotation and scaling) close to half-spheres attached to $S$. Introducing the class $\mathcal{S}^{4,\gamma}(\lambda, \theta)$ as the subset of $\mathcal{M}(S)$ consisting of almost half-spheres with area $2\pi\lambda^2$ where $\theta > 0$ measures the $C^{4,\gamma}$-deviation from the round half-sphere (for a detailed definition see Subsection 2.1) we first prove long-time existence of the flow:

**Theorem 1.1 (Long-time existence).**

Let $\Omega \in C^{17}$. Then there exist $\lambda_0 > 0$ and $\theta_1, \theta_0 > 0$ such that:

1. For $\lambda \in (0, \lambda_0]$ and $\phi_0 \in \mathcal{S}^{4,\gamma}(\lambda, \theta_0)$ there exists a classical solution $\phi : [0, \infty) \to C^{4,\gamma}(S^2_+) \cap \mathcal{M}(S)$ to the area preserving Willmore flow that stays in the class $\mathcal{S}^{4,\gamma}(\lambda, \theta_1)$ and satisfies $\phi(0) = \phi_0$.

 Moreover $\phi$ is unique up to reparameterizations.

2. Given $\phi(t)$ as in part 1, any sequence $t_n \to \infty$ contains a subsequence $(t_{n_k})$ such that $\phi(t_{n_k})$ converges to a critical point of the elliptic problem (2.9) in $C^{4,\beta}$ for any $\beta < \gamma$.

We prove Theorem 1.1 by writing $\phi(t)$ as a normal graph over a small half-sphere centered at the surface’s Riemannian barycenter $\xi(t) := C[\phi(t)] \in S$ and derive evolution equations for both the graph function $u(t)$ and $\xi(t)$. The proof then follows a two-step approach. First, an arbitrary curve $\xi(t)$ is chosen, rendering the evolution equation for $u$ as a gradient flow with the time-dependent constraint of prescribed barycenter curve. Existence of $u$ and suitably decay estimates can then be established by an application of the implicit function theorem as the additional constraint fixes the kernel of the elliptic operator appearing in the linearized evolution equation for $u$. Afterwards, an appropriate choice for $\xi$ is derived.

Due to the application of the implicit function theorem, Theorem 1.1 is of perturbative nature, which is reflected in the smallness assumption on $\theta_0$. In Riemannian manifolds, similar arguments have been used by Mattuschka [14] to study the area-preserving Willmore flow and by Alikakos and Freire [3] in the context of the normalised mean curvature flow. In [18] similar techniques are used to prove exponential convergence of the Willmore flow to a sphere if it is initialised sufficiently close to one. In [5] Bellettini and Fusco study the volume-preserving mean curvature flow of surfaces close to half-spheres that slide on the boundary of a domain $\Omega$ while meeting it orthogonally and prove results similar to the ones we establish here. It must, however, be noted that in their situation, the orthogonality boundary condition arises naturally while we impose it artificially. Our natural boundary condition is the third-order one appearing in (1.1). General gradient flows with time-dependent constraints have been studied in [6].

The general methodology of these papers is a form of Lyapunov-Schmitt reduction and originates from Ye [20]. It has also been employed in the study of related time-independent problems (see e.g. [2, 10, 12, 13, 15, 16]). An application to Willmore tori in Riemannian manifolds is considered in [9]. For a general overview, we refer to [8].
The present article establishes similar results to the ones in [5]. In doing so we face the difficulties of dealing with both a fourth-order and an initial boundary value problem. After an extensive search through the literature, it seems that such problems have not been studied using Ye’s methodology.

Inspired by Theorem 1.6 in [5], for $K > 0$ we introduce the subclass $S^-(\lambda, K) \subset S^{k,7}(\lambda, K\lambda)$ which imposes an additional smallness requirement on the antisymmetric part of the graph function (see Subsection 2.1 for a formal definition). The subclass $S^-(\lambda, K)$ arises naturally from the analysis of Alessandroni and Kuwert [2] as the critical points they construct belong to it (for a proof of this see Appendix B). We prove that all flow lines constructed in Theorem 1.1 eventually must enter and remain in $S^-(\lambda, K)$ for some $K > 0$ independent of $\lambda$ and use this observation to study the asymptotic behaviour of the associated Riemannian barycenter curves.

**Theorem 1.2** (Long-time behaviour of the flow).

1. Let $\Omega, \lambda_0, \theta_0$ be as in Theorem 1.1 and $\lambda \leq \lambda_0$. Then there exists a time $T_0 = T_0(\lambda, \theta_0) > 0$ and $K > 0$ independent of $\lambda$ and $\theta_0$ such that given $\phi_0 \in S(\lambda, \theta_0)$ the flow line $\phi(t)$ from Theorem 1.1 satisfies $\phi(t) \in S^-(\lambda, K)$ for all $t \geq T_0$.

2. Let $\Omega \in C^{21}, M > 0$ and denote the mean curvature of $S = \partial \Omega$ by $H^S$. There exists $\lambda_0(M) > 0$ such that for all $\lambda \leq \lambda_0$ and flow lines satisfying $\phi(t) \in S^-(\lambda, M)$ for all times $t \geq 0$ the barycenter curve $\xi(t)$ satisfies the estimate

$$\sup_{t \geq 0} \left| \dot{\xi}(t) - \frac{3}{2\lambda} \nabla H^S(\xi(t)) \right| \leq C(M).$$

In particular, when introducing the fast time $\tau := \lambda^{-1}t$, and sending $\lambda \to 0^+$ the flow collapses to a point that moves according to the ordinary differential equation

$$\frac{d}{d\tau} \xi(\tau) = \frac{3}{2} \nabla H^S(\xi(\tau)). \tag{1.2}$$

Let us motivate the structure of evolution equation (1.2). The critical points $\phi^{a,\lambda}$ Alessandroni and Kuwert construct in [2] are labelled by their ‘radius’ $\lambda$ and barycenter $a \in S$ (see Appendix B). In their paper they derive the expansion

$$W[\phi^{a,\lambda}] = 2\pi - \lambda \pi H^S(a) + O((\lambda^2). \tag{1.3}$$

Given a flow line $\phi(t) \in S^-(\lambda, M)$ with barycenter $\xi(t) := C[\phi(t)]$ let us assume that Equation (1.3) holds, at least for large times. Then formally differentiating with respect to time and substituting (1.2) we recover dissipation of energy:

$$\frac{d}{dt} W[\phi(t)] = -\frac{3\pi}{2} |\nabla H^S(\xi(t))|^2 + O(\lambda) \tag{1.4}$$

The structure of the barycenter equation is degenerate in the limit $\lambda \to 0^+$. This leads to additional technical difficulties when studying Equation (1.1) in the limit $\lambda \to 0^+$ that are absent.
in both the mean curvature and the Riemannian case (see [5,14]). This problem is overcome by altering the method followed in [14].

Finally, we investigate the convergence of the flow (1.1).

**Theorem 1.3** (Convergence).
Let \( \Omega \in C^{2,1} \) and assume that the mean curvature \( H^S \) of \( S \) only has finitely many critical points \( p_1, \ldots, p_m \) all of which are nondegenerate. Then for any flow line from Theorem 1.1 there exists a critical point \( \phi_\ast \in S^-((\lambda, K)) \) of the elliptic problem (2.9) such that \( \phi(t) \to \phi_\ast \) in \( C^{4,\beta} \) for \( t \to \infty \) and any \( \beta < \gamma \). \( \phi_\infty \) is of the form described in Appendix B.

The proof is based on an idea from [4]. Equation (1.4) implies that the barycenter curve \( \xi \) of a solution cannot stay far away from critical points for all times. As Equation (1.2) implies that \( H^S(\xi(t)) \) is increasing along the flow, the assumption of finitely many critical points then implies a stabilisation of \( \xi \) near a critical point of \( H^S \). Using a uniqueness Theorem from [2] (that requires the nondegeneracy assumption) then allows us to establish convergence.

### 2 Preliminaries

#### 2.1 Terminology

Let \( \tilde{g} \) be a metric on \( \mathbb{R}^3 \) close to the euclidean metric \( \delta \) in \( C^l \) for large enough \( l \in \mathbb{N} \), consider a suitably smooth immersion \( f : S^2_+ \to (\mathbb{R}^3, \tilde{g}) \) and put \( g := f^* \tilde{g} \). We denote the inner normal \((-\omega \text{ for the round half-sphere in euclidean space}) \) of \( f \) with respect to \( \tilde{g} \) by \( \tilde{\nu} \) and its inner conormal \((e_3 \text{ for the round half-sphere in euclidean space}) \) by \( \tilde{\eta} \). We define the mean curvature \( H[f, \tilde{g}] \) of \( f \) with the convention that for the round half-sphere in euclidean space \( H = 2 \). Let \( h^0 \) denote the traceless second fundamental form of \( f \) and \( \text{Ric}_{\tilde{g}} \) the Ricci tensor of \((\mathbb{R}^3, \tilde{g})\). The scalar Willmore gradient is then given by

\[
W[f, \tilde{g}] := \frac{1}{2} \left( \Delta g H + (|h^0|^2 + \text{Ric}_{\tilde{g}}(\tilde{\nu}, \tilde{\nu}))H \right).
\]

For a proof see Theorem 1 in [2] and note the following difference in conventions: We have included \( 1/2 \) in the definition of \( W \). Given a function \( F[f, \tilde{g}] \) that is invariant under reparameterizations (e.g. the area \( A \)) we denote the gradient along the normal bundle by \( \nabla F[f, \tilde{g}] \). That is, for \( \psi : S^2_+ \to \mathbb{R} \)

\[
\left. \frac{d}{ds} \right|_{s=0} F[f + s\psi \tilde{\nu}, \tilde{g}] = \int_{S^2_+} \nabla F[f, \tilde{g}]\psi d\mu_g.
\]

As an example, denoting the inclusion \( S^2_+ \to \mathbb{R}^3 \) by \( f_0 \) we have \( \nabla A[f_0, \delta] = -H[f_0, \delta] = -2 \).

\( \Delta \) always denotes the standard Laplacian on \( S^2 \).

For a bounded domain \( \Omega \in \mathbb{R}^3 \) of class \( C^n \) with large enough \( n \) put \( S := \partial \Omega \) and denote the inner normal (that is pointing into \( \Omega \)) of \( S \) by \( N^S \). Let \( p \in S \) and choose \( C^{n-1} \)-vector
fields \( b_1 \) and \( b_2 \) in a neighbourhood \( B_{r_0}(p) \cap S \) with \( r_0 \) independent of \( p \) such that \((b_1(q), b_2(q))\) provides an orthonormal basis of \( T_q S \) for each \( q \in B_{r_0}(p) \cap S \). Near \( p \) we have the local graph representation

\[
f[p, \cdot] : D_{r_1} \to \mathbb{R}^3, \quad f[p, x] := p + x^1 b_1(p) + x^2 b_2(p) + \varphi[p, x] N^S(p)
\]

of \( S \) defined on a disk \( D_{r_1} := \{ x \in \mathbb{R}^2 \mid |x| < r_1 \} \) with \( r_1 > 0 \) independent of \( p \). The map \( S \times D_{r_1} \ni (p, x) \mapsto \varphi[p, x] \in \mathbb{R} \) is of class \( C^{n-1} \), satisfies \( \varphi[p, 0] = 0 \), \( D_2 \varphi[p, 0] = 0 \) and the estimates

\[
|\varphi[p, x]| \leq C(\Omega)|x|^2, \quad |D_2 \varphi[p, x]| \leq C(\Omega)|x| \quad \text{and} \quad \|\varphi\|_{C^{\kappa-1}} \leq C(\Omega).
\]

After potentially shrinking \( r_1 \), we extend the graph representation to a diffeomorphism

\[
F[p, \cdot] : D_{r_1} \times (-r_1, r_1) \to \text{im}(F[p, \cdot]) \subset \mathbb{R}^3, \quad F[p, x, z] := f[p, x] + z N^S(p).
\]

For small \( \lambda \in (0, \lambda_0(\Omega)) \) we now introduce \( F^\lambda : S \times Z_2 \to \mathbb{R}^3 \) \((Z_2 := \bar{D}_2 \times [-2, 2])\) by mapping \( F^\lambda[p, x, z] := F[p, \lambda x, \lambda z] \) and set

\[
\tilde{g}^{p, \lambda} := \frac{1}{N^2} (F^\lambda[p, \cdot])^\ast \delta
\]

where \( \delta \) denotes the euclidean metric on \( \mathbb{R}^3 \). We abbreviate \( \partial_x \varphi[p, x] := \partial_{x^1} \varphi[p, x] \). A quick computation (that is carried out in [2]) gives the concrete formula

\[
\tilde{g}^{p, \lambda} = I_3 + \begin{bmatrix}
\partial_1 \varphi[p, \lambda x] \partial_1 \varphi[p, \lambda x] & \partial_1 \varphi[p, \lambda x] \partial_2 \varphi[p, \lambda x] & \partial_1 \varphi[p, \lambda x] \\
\partial_1 \varphi[p, \lambda x] \partial_2 \varphi[p, \lambda x] & \partial_2 \varphi[p, \lambda x] \partial_2 \varphi[p, \lambda x] & \partial_2 \varphi[p, \lambda x] \\
\partial_1 \varphi[p, \lambda x] & \partial_2 \varphi[p, \lambda x] & 0
\end{bmatrix},
\]

(2.2)

For \( n \geq 3 \), the map \((p, \lambda) \mapsto \tilde{g}^{p, \lambda} \in C^l(Z_2, M_3(\mathbb{R}))\) is of class \( C^{n-l-3} \) as \( \varphi \in C^{n-1} \). For regularity of such ‘meta maps’ we point to Appendix C.

**Functional spaces**

Let \( \gamma \in (0, 1) \) and \( k \in \mathbb{N} \). We consider the Hölder spaces \( C^{k, \gamma}(S^2_+) \), \( C^{k, \gamma}(\partial S^2_+) \) and denote their norms by \( \| \cdot \|_{C^{k, \gamma}} \) as long as the domain is clear from context.

Given \( T > 1 \) and denoting the floor function by \([ \cdot ]^−\), we also consider the parabolic Hölder spaces \( C^{k, [k/4]}^−(0, T] \times S^2_+ \) and \( C^{k, [k/4]}^−(0, T] \times \partial S^2_+ \) of functions that are \( k \)-times continuously differentiable in space, \([k/4]^-\)-times continuously differentiable in time and satisfy suitable Hölder conditions. A detailed definition is given in Appendix C. We denote the norms by \( \| \cdot \|_{C^{k, [k/4]}^-} \) as long as the spatial domain is clear from context.

For given \( U \subset \mathbb{R}^3 \), we consider the spaces \( C^{k, \tilde{T}}((0, T], U) \) and denote their norms by \( \| \cdot \|_{C_t^{1, \tilde{T}}} \) as \( U \) will always be clear from context.

If \( X \) is some functional space, \( x \in X \) and \( r > 0 \) we put \( X(x; r) := \{ \tilde{x} \in X \mid \|x - \tilde{x}\|_X \leq r \} \). We write \( X(r) := X(0; r) \).
**Summation convention**

We use the following summation convention. Every repeated index is summed over. If the index is Latin, it takes the values $i = 1, 2$ and if it is Greek, it takes the values $\mu = 1, 2, 3$ or $\mu = 0, 1, 2$ (which of the two will always be clear from context). In ambiguous cases the summation symbol is included.

**Almost half-spheres**

For $\Omega \subset \mathbb{R}^3$ and $S$ as described above and $\lambda > 0$ let

$$
\mathcal{M}^{4, \gamma}(S) := \{ \phi \in C^{4, \gamma}(S^2_+, \mathbb{R}^3) \mid \text{immersed, } \phi(\partial S^2_+) \subset S \text{ and } \phi \perp S \text{ along } \partial S^2_+ \},
$$

$$
\mathcal{M}^{4, \gamma}_\lambda(S) := \{ \phi \in \mathcal{M}^{4, \gamma}(S) \mid A[\phi] = 2\pi \lambda^2 \}.
$$

For $u \in C^{4, \gamma}(S^2_+)$ with small enough $C^{4, \gamma}$-norm, $\lambda > 0$ and $p \in S$ we consider

$$
f_u : S^2_+ \to \mathbb{R}^3, \quad f_u(\omega) := (1 + u(\omega))\omega \quad \text{and} \quad \phi^\lambda_{p,u}(\omega) := F^\lambda[p, f_u(\omega)].
$$

We say that an immersion $\phi \in \mathcal{M}^{4, \gamma}_\lambda(S)$ is an almost half-sphere of radius $\lambda$ if it can be written as $\phi^\lambda_{p,u}$ and, for given $\theta > 0$, put

$$
\mathcal{S}'(\lambda, \theta) := \{ \phi \in \mathcal{M}^{4, \gamma}_\lambda(S^2_+) \mid \phi \text{ is an almost half-sphere of radius } \lambda \text{ with } ||u||_{C^{4, \gamma}} < \theta \}.
$$

For small enough $\theta$ and $\lambda$ we prove in Appendix D that there exists a nonlinear projection $C$ that maps $\phi \in \mathcal{S}'(\lambda, \theta)$ to a point $C[\phi] \in S$ that we refer to as its (Riemannian) barycenter. For the round half-sphere attached to $\mathbb{R}^2$ this definition coincides with the origin. Similarly an analogue projection $C$ for immersions $f_u : S^2_+ \to (\mathbb{R}^3, \tilde{g})$ to $\mathbb{R}^2$ is constructed. These two projections satisfy

$$
C[F^\lambda[p, f_u]] = F^\lambda[p, C[f_u, \tilde{g}^p]]).
$$

The concept of the Riemannian barycenter is originally due to Karcher [11]. We use a slight variant of the local version introduced in [2]. Finally, we prove in Appendix D that for small enough $\lambda$ and $\theta_0$ each $\phi \in \mathcal{S}'(\lambda, \theta_0)$ may be parameterized over its barycenter. That is, there exists a parameterization of the form $\phi = F^\lambda[C[\phi], f_u]$. The parameterization depends on the orthonormal frame chosen at $p$ but is unique once a frame is fixed. We define

$$
\mathcal{S}(\lambda, \theta) := \{ \phi \in \mathcal{S}'(\lambda, \theta_0) \mid \phi \text{ is parameterized over its barycenter and } ||u||_{C^{4, \gamma}} < \theta \}.
$$

If we need to make $\gamma$ visible in the notation we write $\mathcal{S}^{4, \gamma}(\lambda, \theta)$. Given a functional $F$ (e.g. the area $A$ or the Willmore operator $W$ etc.) mapping an immersion $f_u$ and a metric $\tilde{g}$ to e.g. $\mathbb{R}$ we write $F[u, \tilde{g}] := F[f_u, \tilde{g}]$ and for $F$ invariant under reparameterizations $\nabla F[u, \tilde{g}] := \nabla F[f_u, \tilde{g}]$.

**Parity**

We may write points $\omega \in S^2_+$ as $(\bar{\omega}, \omega^3)$ with $\bar{\omega} \in \mathbb{R}^2$ and $\omega^3 \geq 0$ and introduce the reflection $r : S^2_+ \to S^2_+,(\bar{\omega}, \omega^3) \mapsto (-\bar{\omega}, \omega^3)$. A map $u : S^2_+ \to \mathbb{R}$ is said to be even if $u \circ r = u$ and is called odd if $u \circ r = -u$. It is readily checked that any function $u : S^2_+ \to \mathbb{R}$ possesses a unique decomposition $u = u^+ + u^-$ into an even part $u^+$ and an odd part $u^-$. For $\lambda, K > 0$ we define

$$
\mathcal{S}^-(\lambda, K) := \{ \phi \in \mathcal{S}(\lambda, K\lambda) \mid ||u^-||_{C^{4, \gamma}} \leq K\lambda^2 \}.
$$
2.2 The area preserving Willmore flow

Given \( \lambda > 0 \) and \( \phi_0 \in \mathcal{M}^{\lambda}(S) \) we say that \( \phi : [0, \infty) \to \mathcal{M}^{\lambda}(S) \) solves the area preserving Willmore flow with initial value \( \phi_0 \) if

\[
\begin{align*}
\langle \dot{\phi}(t), \nu \rangle &= -P_H^\perp [W(\phi(t))] \\
\frac{\partial \phi}{\partial \eta} &= N^S \circ \phi & \text{on } [0, \infty) \times \partial S^2_+, \\
\frac{\partial H}{\partial \eta} + h^S(\nu, \nu)H &= 0 & \text{on } [0, \infty) \times \partial S^2_+,
\end{align*}
\tag{2.4}
\]

Here \( \nu \) and \( \eta \) denote the normal and conormal of \( \phi \) respectively. Also, we have introduced the projection operator \( P_H^\perp[W(\phi)] \), abbreviated by \( P^\perp[H(\phi)] \), defined by

\[
P^\perp[H(\phi)] = W[\phi] - \frac{\int_{S^2_+} W[\phi]H[\phi]d\mu_{\phi}^{s'}}{\int_{S^2_+} H[\phi]^{1+s'}d\mu_{\phi}^{s'}}H[\phi].
\tag{2.5}
\]

This is precisely the structure appearing in (1.1) with \( \alpha \) chosen to ensure constant \( A[\phi(t)] \). The first order boundary condition is a rephrasing of \( \phi \) meeting \( S \) orthogonal along \( \partial S^2_+ \). The third order boundary condition is natural for two reasons. First it is automatically satisfied for critical points of \( W \) (see [2]). Second it is crucial to establish optimal dissipation of Willmore energy. Indeed, let \( t \geq 0 \) and consider the variation \( \epsilon \to \phi(t + \epsilon) \) of \( \phi(t) \). We may decompose the variational vector field as \( \vec{V} = \varphi \nu + D\phi \xi \) where \( \varphi \) is a scalar function \( \xi \) is a tangential vector field. As \( \phi(t + \epsilon) \in \mathcal{M}^{\lambda}(S) \) for all \( \epsilon > 0 \) this variation is admissible in the sense described in [2] (see Equation (1.17) and the proceeding analysis) and must therefore satisfy

\[
\langle \phi^* \delta \rangle(\xi, \eta) = 0 \quad \text{and} \quad \frac{\partial \varphi}{\partial \eta} + \varphi h^S(\nu, \nu) = 0 \quad \text{on } \partial S^2_+.
\tag{2.6}
\]

Applying Theorem 1 from [2] now yields

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} W[\phi(t + \epsilon)] = \int_{S^2_+} W[\phi(t)]\varphi d\mu_{\phi} + \frac{1}{2} \int_{S^2_+} \varphi \left( \frac{\partial H}{\partial \eta} - \frac{\partial \varphi}{\partial \eta}H - \frac{1}{2}H^2g(\xi, \eta) \right) dS_{\phi}
\]

\[
\tag{2.4}
\tag{2.6}
\]

\[
\equiv - \int_{S^2_+} W[\phi(t)]P_H^\perp[W(\phi(t))]d\mu_{\phi} + \frac{1}{2} \int_{S^2_+} \varphi \left( \frac{\partial H}{\partial \eta} + Hh^S(\nu, \nu) \right) dS_{\phi}
\]

\[
\equiv - \int_{S^2_+} |P_H^\perp[W(\phi(t))]|^2 d\mu_{\phi}.
\tag{2.4}
\]

Given a suitable \( \phi_0 \in S(\lambda, \theta) \) we construct a solution \( \phi : [0, \infty) \to S(\lambda, \theta) \) to (2.4). Once such a solution is constructed it is standard to construct \( \phi_*(t) \) that solves the same boundary and initial conditions as well as the evolution equation

\[
\dot{\phi}_*(t) = -P_H^\perp[W(\phi_*(t))\nu].
\tag{2.7}
\]
Indeed, once a solution \( \phi \) to (2.4) is constructed one can follow the analysis that is e.g. presented in [19] to deduce the existence of \( \phi_0 \) by considering a reparameterization \( \phi_x(t, \omega) := \phi(t, \psi(t, \omega)) \). Inserting this ansatz into the evolution equation (2.7) gives the following ordinary differential equation (see e.g. [19]) for \( \psi \):

\[
\begin{align*}
\frac{\partial \psi(t, \omega)}{\partial t} &= - \left( D_2 \phi(t, \psi(t, \omega)) \right)^{-1} \left( D_1 \psi(t, \psi(t, \omega)) \right)^T, \\
\psi(0, \omega) &= \omega.
\end{align*}
\]

Here \( T \) denotes component of the velocity vector tangential to \( \phi \).

### The elliptic problem

We also state the formulation of the corresponding elliptic problem studied by Alessandroni and Kuwert in [2] as it is needed for the formulation of Theorem 1.3.

\[
\begin{align*}
&\left. \left[ P^i_{H} \right] W(\phi_0) \right|_{S^2_+} = 0 & \text{in } S^2_+ \\
&\frac{\partial \phi_0}{\partial \eta} = N^S \circ \phi_0 & \text{on } \partial S^2_+ \\
&\frac{\partial H}{\partial \eta} + h^S(\nu, \nu) H = 0 & \text{on } \partial S^2_+.
\end{align*}
\]

### 2.3 Flow equation in a moving reference frame

Suppose that we have a solution \( \phi : [0, \infty) \to S(\lambda, \theta) \) to (2.4) with initial barycenter \( C[\phi(0)] = p_0 \in S \) and have chosen an orthonormal frame \( b_i \) in a neighbourhood of \( p_0 \). Putting \( \xi(t) := C[\phi(t)] \) to be the Riemannian barycenter of \( \phi(t) \), the expression \( b_i(\xi(t)) \) is then well well defined for small times and we may write

\[
\phi(t) = F^\lambda[\xi(t), (1 + u(t, \omega))\omega].
\]

Inserting this ansatz into (2.4) must then give equations for the graph function \( u \) and the curve \( \xi \). We abbreviate the time-dependent metric \( \tilde{g}^{\xi(t), \lambda} \) by \( \bar{g}^{\xi, \lambda} \). Recall that we denote the inner normal of \( f_u \) with respect to \( \tilde{g} \) by \( \tilde{\nu} \) and its conormal by \( \bar{\eta} \) to distinguish them from the corresponding quantities for \( \phi \) which do not carry the tilde.

### The graph function equation

Inserting (2.10) into the evolution equation from (2.4) gives

\[
- P^i_{H[\phi(t)]} W[\phi(t)] = \langle \dot{\phi}(t), \nu \rangle = \langle D_1 F^\lambda[\xi(t), f_u(t)]\dot{\xi}(t), \nu \rangle + \langle D_2 F^\lambda[\xi(t), f_u(t)]f_u(t), \nu \rangle.
\]

We use that \( W \) and \( H \) are invariant under diffeomorphisms to get \( W[\phi(t)] = W[\lambda f_u, F[\xi(t), \cdot]^* \delta], H[\phi(t)] = H[\lambda f_u, F[\xi(t), \cdot]^* \delta] \) and substitute the scaling behaviour of \( W, H \) to find

\[
- \frac{1}{\lambda^2} P^i_{H[f_u, \bar{g}^{\xi, \lambda}]} W[f_u, \bar{g}^{\xi, \lambda}] = \langle D_1 F^\lambda[\xi(t), f_u(t)]\dot{\xi}(t), \nu \rangle + \langle D_2 F^\lambda[\xi(t), f_u(t)]f_u(t), \nu \rangle. \tag{2.11}
\]
Here we have introduced the $L^2(f_u^*\tilde{g}^{\xi,\lambda})$-orthogonal projection onto $H[f_u, \tilde{g}^{\xi,\lambda}]$ that is defined as in Equation (2.5). Next we rewrite the right hand side of this equation by exploiting the formula

$$\tilde{\nu} = \lambda \left( D_2 F^{\lambda}[\xi(t), f_u(t)] \right)^{-1} \nu.$$  

The factor of $\lambda$ is due to the scaling in the definition of $\tilde{g}^{\xi,\lambda}$ in Equation (2.1). This formula implies the identities

$$\langle D_2 F^{\lambda}[\xi(t), f_u(t)] \tilde{f}_u(t), \nu \rangle = \lambda \tilde{g}^{\xi,\lambda} [\tilde{f}_u, \tilde{\nu}]$$  

$$\langle D_1 F^{\lambda}[\xi(t), f_u(t)] \tilde{\xi}(t), \nu \rangle = \lambda \tilde{g}^{\xi,\lambda} \left[ \left( D_2 F^{\lambda}[\xi(t), f_u(t)] \right)^{-1} D_1 F^{\lambda}[\xi(t), f_u(t)] \right] \tilde{\xi}(t), \tilde{\nu} \right].  \tag{2.12}$$

Inserting Equations (2.12) and (2.13) into (2.11) we find

$$\tilde{g}^{\xi,\lambda} [\tilde{f}_u, \tilde{\nu}] = -\frac{1}{\lambda^2} P^\perp_{\tilde{g}^{\xi,\lambda}} W [f_u, \tilde{g}^{\xi,\lambda}]$$

$$-\tilde{g}^{\xi,\lambda} \left[ \left( D_2 F^{\lambda}[\xi(t), f_u(t)] \right)^{-1} D_1 F^{\lambda}[\xi(t), f_u(t)] \right] \tilde{\xi}(t), \tilde{\nu} \right].  \tag{2.14}$$

The identities $A[\phi(t)] = 2\pi \lambda^2$ and $C[\phi(t)] = \xi(t)$ imply $A[f_u, \tilde{g}^{\xi,\lambda}] = 2\pi$ and $C[f_u, \tilde{g}^{\xi,\lambda}] = 0$ (see Equation (2.3)). We put $K[u, \tilde{g}^{\xi,\lambda}] := \operatorname{span}_{\mathbb{R}} (\nabla A[f_u, \tilde{g}^{\xi,\lambda}], \nabla C^i[f_u, \tilde{g}^{\xi,\lambda}])$ where $i = 1, 2$. In Appendix E we examine this space in more detail. In particular we show that for small enough $\|u(t)\|_{C_{4,\gamma}}$ and $\lambda$ we may consider the $L^2(\tilde{g}^{\xi,\lambda})$-projection operators $P_{K[u, \tilde{g}^{\xi,\lambda}]}^\perp$ and $P_{K[u, \tilde{g}^{\xi,\lambda}]}^\perp$ onto $K[u, \tilde{g}^{\xi,\lambda}]$ and its $L^2(\tilde{g}^{\xi,\lambda})$-orthogonal complement $K[u, \tilde{g}^{\xi,\lambda}]^\perp$. Differentiating $A[f_u, \tilde{g}^{\xi,\lambda}] = 2\pi$ and $C^i[f_u, \tilde{g}^{\xi,\lambda}] = 0$ with respect to time gives

$$0 = \langle H[f_u, \tilde{g}^{\xi,\lambda}], \tilde{g}^{\xi,\lambda} (\tilde{f}_u, \tilde{\nu}) \rangle_{L^2(\tilde{g}^{\xi,\lambda})} + D_2 A[f_u, \tilde{g}^{\xi,\lambda}] \tilde{g}^{\xi,\lambda},$$

$$0 = \langle \nabla C^i[f_u, \tilde{g}^{\xi,\lambda}], \tilde{g}^{\xi,\lambda} (\tilde{f}_u, \tilde{\nu}) \rangle_{L^2(\tilde{g}^{\xi,\lambda})} + D_2 C^i[f_u, \tilde{g}^{\xi,\lambda}] \tilde{g}^{\xi,\lambda}.$$

Putting $C^0 := A$ and $\psi := ||\nabla C^\mu[f_u, \tilde{g}^{\xi,\lambda}]||_{L^2(\tilde{g}^{\xi,\lambda})}^{-1} \nabla C^\mu[f_u, \tilde{g}^{\xi,\lambda}]$ for $\mu = 0, 1, 2$ we may rewrite these equations as

$$\langle \psi, [f_u, \tilde{g}^{\xi,\lambda}] \rangle_{L^2(\tilde{g}^{\xi,\lambda})} (\tilde{f}_u, \tilde{\nu}) = -\frac{D_2 C^\mu[f_u, \tilde{g}^{\xi,\lambda}] \tilde{g}^{\xi,\lambda}}{||\nabla C^\mu[f_u, \tilde{g}^{\xi,\lambda}]||_{L^2(\tilde{g}^{\xi,\lambda})}}$$

for $\mu = 0, 1, 2$.

Put $A_{\mu\nu} := \langle \psi, \psi \rangle_{L^2(\tilde{g}^{\xi,\lambda})}$. In Appendix E we argue that we may invert the matrix $A_{\mu\nu}$ if $\lambda$ and $||u(t)||_{C_{4,\gamma}}$ are small. We denote the inverse matrix by $A^{\mu\nu}$. Multiplying the last equation with $A^{\mu\nu} \psi$, summing over $\mu, \nu$ and using Equation (E.3) we get

$$P_{K[u, \tilde{g}^{\xi,\lambda}]} (\tilde{f}_u, \tilde{\nu}) = -\sum_{\mu, \nu = 0}^2 A_{\mu\nu} \tilde{g}^{\xi,\lambda}.  \tag{2.15}$$

Equation (2.15) gives us the component of $\tilde{g}^{\xi,\lambda}(\tilde{f}_u, \tilde{\nu})$ inside the space $K[u, \tilde{g}^{\xi,\lambda}]$. The component inside $K[u, \tilde{g}^{\xi,\lambda}]^\perp$ can be derived by applying the projection $P_{K[u, \tilde{g}^{\xi,\lambda}]}^\perp$ to Equation (2.14).
Combining these two components we find

\[
g^\xi \lambda(f_u, \bar{\nu}) = \tau[u, g^\xi \lambda] - P_{K[u, g^\xi \lambda]}^\perp \left[ \frac{1}{\lambda^4} W[f_u, \bar{g}^\xi \lambda] \right] - \left[ g^\xi \lambda \left[ \left(D_2 F^\lambda[\xi(t), f_u(t)]\right)^{-1} D_1 F^\lambda[\xi(t), f_u(t)] \dot{\xi}(t), \bar{\nu} \right] \right]
\]

where we have introduced

\[
\tau[u, g^\xi \lambda] := -2 \sum_{\mu, \nu=0} A^{\mu \nu}[u, g^\xi \lambda] \frac{D_2 C^{\mu}[u, g^\xi \lambda] g^\xi \lambda}{\|\nabla C^{\mu}[u, g^\xi \lambda]\|_{L^2(f_u g^{\xi \lambda})}} \psi_{\nu}.
\]

**The barycenter equation**

Next, we derive an equation for the barycenter curve \( \xi(t) := C[\phi(t)] \). For that purpose we consider small \( \epsilon > 0 \) and express \( \xi(t + \epsilon) \) in the chart \( f[\xi(t), \cdot] \) centered at \( \xi(t) \). That is,

\[
\xi(t + \epsilon) = f[\xi(t), (\xi^1(t + \epsilon), \xi^2(t + \epsilon))] = \xi(t) + \sum_{i=1}^2 \xi^i(t + \epsilon) b_i(\xi(t)) + \varphi[\xi(t), (\xi^1(t + \epsilon), \xi^2(t + \epsilon))] N^S(\xi(t)).
\]

Clearly \( \xi^i(t) = 0 \). Taking the scalar product with \( b_i(\xi(t)) \) and differentiating with respect to \( \epsilon \) at \( \epsilon = 0 \) we learn

\[
\dot{\xi}^i(t) = \langle \xi(t), b_i(\xi(t)) \rangle.
\]

Recall \( \phi(t) = F^\lambda[\xi(t), f_u(t)] \) and write \( \phi(t + \epsilon) = F^\lambda[\xi(t), f_v(t, \epsilon)] \). Then clearly \( v(t, 0) = u(t) \).

We wish to express the equation \( \xi(t + \epsilon) = C[\phi(t + \epsilon)] \) using the local diffeomorphism \( F^\lambda[\xi(t), \cdot] \) centered at \( \xi(t) \). To do this note \( \xi(t + \epsilon) = f[\xi(t), (\xi^1(t + \epsilon), \xi^2(t + \epsilon))] = F^\lambda[\xi(t), \lambda^{-1}(\xi^1(t + \epsilon), \xi^2(t + \epsilon)), 0]. \) Using Identity (2.3) we get

\[
\lambda^{-1} \xi^i(t + \epsilon) = C^i[f_v(t, \epsilon), \bar{g}^\xi \lambda].
\]

Differentiating the left hand side at \( \epsilon = 0 \) gives \( \lambda^{-1} \dot{\xi}^i(t) \). We now differentiate the right hand side. To do so we investigate \( \partial_\epsilon v(t, \epsilon) \) at \( \epsilon = 0 \). Following the derivation for the graph function from the previous paragraph we can use Equation (2.14) but must drop the last term as it derived from differentiating the time-dependent chart which we do not have here. This gives

\[
g^\lambda(\xi(t)) \left( \frac{\partial f_v}{\partial \epsilon} \bigg|_{\epsilon=0}, \bar{\nu} \right) = \frac{1}{\lambda^3} P_{H[f_u(t), \bar{g}^\xi \lambda]} \left( W[f_u(t), \bar{g}^\xi \lambda] \right).
\]

Here we have used that \( v(t, 0) = u(t) \). Differentiating Equation (2.19) with respect to \( \epsilon \) at \( \epsilon = 0 \) and substituting Equations (2.18) and (2.20) we get the evolution equation for \( \xi \):

\[
\dot{\xi}^i(t) = \langle \xi(t), b_i(\xi(t)) \rangle = \frac{1}{\lambda^3} (\nabla C^i[f_u(t), \bar{g}^\xi \lambda], P_{H[f_u(t), \bar{g}^\xi \lambda]} \left( W[f_u(t), \bar{g}^\xi \lambda] \right)) \|_{L^2(f_u \bar{g}^{\xi \lambda})}.
\]
Equivalence to the Flow
We now prove that \( \phi(t) := F^\lambda[\xi(t), f_u(t)] \) solves the evolution equation (2.4) if (2.16) and (2.21) are satisfied. Using similar manipulations as in the derivation of (2.14) we find

\[
\langle \phi(t), \nu \rangle = \lambda \bar{\phi}^{\xi, \lambda}[\tilde{f}_u, \tilde{v}] + \lambda \bar{\phi}^{\xi, \lambda} \left[ \left( D_2 F^\lambda[\xi(t), f_u(t)] \right)^{-1} D_1 F^\lambda[\xi(t), f_u(t)] \tilde{\xi}(t), \tilde{v} \right]
\]

\[(2.16) = \lambda \tau[u, \tilde{\phi}^{\xi, \lambda}] - \frac{1}{\lambda^3} P_{\tilde{K}[u, \tilde{\phi}^{\xi, \lambda}]}(W[f_u, \tilde{\phi}^{\xi, \lambda}]) + \lambda P_{\tilde{K}[u, \tilde{\phi}^{\xi, \lambda}]}(X). \quad (2.22)
\]

Next we rewrite the definition of \( \tau \) in Equation (2.17) by computing \( D_2 C^\mu[u, \tilde{\phi}^{\xi, \lambda}] \tilde{\phi}^{\xi, \lambda} \). This computation is moved to Appendix F as it is quite lengthy. There we show

\[
\tau[u, \tilde{\phi}^{\xi, \lambda}] = -P_{\tilde{K}[u, \tilde{\phi}^{\xi, \lambda}]}(X) + \sum_{\nu=0}^{2} \sum_{i=1}^{2} \frac{A^\mu[u, \tilde{\phi}^{\xi, \lambda}] \bar{\xi}^i}{\lambda^3 \| \nabla C^i[f_u, \tilde{\phi}^{\xi, \lambda}] \|_{L^2(f_u \tilde{\phi}^{\xi, \lambda})}} \| \nabla C^i[f_u, \tilde{\phi}^{\xi, \lambda}] \|_{L^2(f_u \tilde{\phi}^{\xi, \lambda})} \psi_\nu[u, \tilde{\phi}^{\xi, \lambda}].
\]

(2.23)

Inserting this formula into (2.22) and substituting Equation (2.21) for \( \tilde{\xi}(t) \) we find

\[
\langle \dot{\phi}(t), \nu \rangle = \frac{1}{\lambda^3} P_{\tilde{K}[u, \tilde{\phi}^{\xi, \lambda}]}(W[f_u, \tilde{\phi}^{\xi, \lambda}]) - \frac{1}{\lambda^3} P_{\tilde{K}[u, \tilde{\phi}^{\xi, \lambda}]}(P_{\tilde{H}[u, \tilde{\phi}^{\xi, \lambda}]} W[f_u, \tilde{\phi}^{\xi, \lambda}])
\]

\[
= -\frac{1}{\lambda^3} P_{\tilde{H}[u, \tilde{\phi}^{\xi, \lambda}]}(W[f_u, \tilde{\phi}^{\xi, \lambda}]).
\]

Here we have used \( H[f_u, \tilde{\phi}^{\xi, \lambda}] = \nabla C^0[f_u, \tilde{\phi}^{\xi, \lambda}] \in K[u, \tilde{\phi}^{\xi, \lambda}] \) and hence \( P_{\tilde{K}}(H[f_u, \tilde{\phi}^{\xi, \lambda}]) = 0 \) to argue \( P_{\tilde{K}} = P_{\tilde{K}} \circ P_{\tilde{H}} \). Finally, we use the diffeomorphism invariance and scaling of \( W \) and \( H \) to deduce

\[
\langle \dot{\phi}(t), \nu \rangle = -P_{\tilde{H}[\phi(t)]}(W[\phi(t)]).
\]

Rewriting the system
In principal equations (2.16) and (2.21) constitute the system we must examine. We may however rewrite it as follows: Consider a pair \((u, \xi)\) that solves (2.16) and (2.21). First we write \( \tilde{\xi}(t) = \sum_{i=1}^{2} (\xi(t), b_i(\xi(t)))b_i(\xi(t)) \). Now insert the evolution equation for \( (\xi(t), b_i(\xi(t))) \) and use this formula to eliminate \( \xi(t) \) from the last term in Equation (2.16). Putting

\[
\tilde{I}[u, \tilde{\phi}^{\xi, \lambda}] := \tilde{\phi}^{\xi, \lambda} \left[ \left( D_2 F^\lambda[\xi(t), f_u(t)] \right)^{-1} D_1 F^\lambda[\xi(t), f_u(t)] \tilde{\xi}(t), \tilde{v} \right]
\]

\[
I^i[u, \tilde{\phi}^{\xi, \lambda}] := \langle W[u, \tilde{\phi}^{\xi, \lambda}], P_{\tilde{H}[u, \tilde{\phi}^{\xi, \lambda}]} \nabla C^i[u, \tilde{\phi}^{\xi, \lambda}] \rangle_{L^2(f_u \tilde{\phi}^{\xi, \lambda})}
\]
for \( i = 1, 2 \), we derive the following system (note the summation convention):

\[
\tilde{g}^{\xi, \lambda}(f_u, \tilde{\nu}) = \tau[u, \tilde{g}^{\xi, \lambda}] - P_{K[u, \tilde{g}^{\xi, \lambda}]} \left[ \frac{1}{\lambda^2} W[u, \tilde{g}^{\xi, \lambda}] \right] = \frac{1}{\lambda^2} I^r[u, \tilde{g}^{\xi, \lambda}] P_{K[u, \tilde{g}^{\xi, \lambda}]}[\tilde{J}_i[u, \xi, \lambda]]
\]

\[
\langle \xi(t), b_i(\xi(t)) \rangle = - \frac{1}{\lambda^2} I^r[u, \tilde{g}^{\xi, \lambda}]
\]

Clearly any solution to this pair of equations also solves (2.16) and (2.21). Finally, we rescale the evolution equations by introducing \( u'(t, \omega) := u(\lambda^4 t, \omega) \) and \( \xi'(t) := \xi(\lambda^4 t) \). We do not write \( u' \) and \( \xi' \) however. From now on and \( \xi \) will represent the rescaled quantities. They satisfy

\[
\begin{aligned}
\tilde{g}^{\xi, \lambda}(f_u, \tilde{\nu}) &= \tau[u, \tilde{g}^{\xi, \lambda}] - P_{K[u, \tilde{g}^{\xi, \lambda}]} \left[ \frac{1}{\lambda^2} W[u, \tilde{g}^{\xi, \lambda}] \right] + \lambda I^r[u, \tilde{g}^{\xi, \lambda}] P_{K[u, \tilde{g}^{\xi, \lambda}]}[\tilde{J}_i[u, \xi, \lambda]] \\
\langle \xi(t), b_i(\xi(t)) \rangle &= - \lambda I^r[u, \tilde{g}^{\xi, \lambda}].
\end{aligned}
\]

### Boundary conditions

Next we rewrite the boundary conditions in terms of \( u \) and \( \xi \). As \( \tilde{g}^{\xi, \lambda} = \lambda^{-2} F^\lambda[\xi(t), \cdot]^* \delta \) we have

\[
\frac{\partial}{\partial \eta} = N^S \circ \phi \iff \frac{\partial f_u}{\partial \eta} = \tilde{\nu}_{R^2} \circ f_u.
\]

\[
\frac{\partial H}{\partial \eta} + H h^S(\nu, \nu) = 0 \iff \frac{\partial H}{\partial \eta} + H \tilde{h}_{R^2}(\tilde{\nu}, \tilde{\nu}) = 0.
\]

Here \( \tilde{\nu}_{R^2} \) denotes the normal of \( R^2 \times \{0\} \) with respect to \( \tilde{g}^{\xi, \lambda} \) that coincides with \( e_3 \) in the euclidean case. We now reformulate the first condition. Let \( \tilde{\tau} \) denote (one of) the unit tangential vector fields along \( \partial S^2_+ \). Then of course \( Df_u \tilde{\tau}, Df_u \tilde{\eta} \) and \( \tilde{\nu} \) are defined at each point on the boundary of \( f_u \) and constitute a \( \tilde{g}^{\xi, \lambda} \)-orthogonal basis for \( R^3 \). We now have

\[
Df_u \tilde{\eta} = \tilde{\nu}_{R^2} \iff Df_u \tilde{\eta} \perp_{\tilde{g}^{\xi, \lambda}} R^2 \iff \tilde{\nu} \in R^2 \iff \tilde{g}^{\xi, \lambda}(\tilde{\nu}, \tilde{\nu}_{R^2}) = 0.
\]

In the step marked by (!) we used \( \langle Df \tilde{\eta} \rangle^\perp = \text{span}_R \{ Df_u \tilde{\tau}, \tilde{\nu} \} \). We summarize these boundary conditions as \( B[u, \tilde{g}] = 0 \) where \( \tilde{g} = \tilde{g}^{\xi, \lambda} \) and

\[
B[u, \tilde{g}] := \left( \tilde{g}(\tilde{\nu}, \tilde{\nu}_{R^2}), \frac{\partial H}{\partial \eta} + H \tilde{h}_{R^2}(\tilde{\nu}, \tilde{\nu}) \right).
\]

### Initial values

Finally, we discuss possible initial values \( u_0 \) and \( p := \xi(0) \) for the evolution equations (2.24). We say that a pair \( (u_0, p) \in C^4, \gamma(S^2_+) \times S \) is admissible if

(A1) it is compatible with the boundary conditions. That is, \( B[u_0, \tilde{g}^{p, \lambda}] = 0 \),

(A2) it satisfies \( A[u_0, \tilde{g}^{p, \lambda}] = 2\pi \) and \( C[u_0, \tilde{g}^{p, \lambda}] = 0 \).
Let the following system of equations:
\[ g(\tilde{f}, \tilde{v}) = \tau[u, \tilde{g}] - P_{K[u,\tilde{g}]}[W[u, \tilde{g}]] + \lambda I^l[u, \tilde{g}]P_{K[u,\tilde{g}]}[\tilde{f}, u, \xi, \lambda,], \]
\[ [\tilde{\xi}(t), b_i(\tilde{\xi}(t))] = -\lambda I^l[u, \tilde{g}], \]
\[ B[u, \tilde{g}] = 0 \]
\[ u(0) = u_0 \]
\[ \xi(0) = p \]

(2.26)

3 An abstract perturbation problem

Let \(T > 1\) and denote the inner conormal of \(S_+^2 \subset (\mathbb{R}^3, \delta)\) by \(\eta\). We define the spaces
\[ X_T := \left\{ w \in C^{4,\gamma}([0, T] \times S_+^2) \mid \frac{\partial w}{\partial \eta} = \frac{\partial \Delta w}{\partial \eta} = 0 \right\}, \]
\[ Y_T := \left\{ \tilde{w} \in C^{4,\gamma}([0, T] \times S_+^2) \mid \forall t \geq 0 : \int_{S_+^2} \tilde{w}(t)d\mu_{S_+^2} = 0, \exists c_1 \in C^{0,\gamma}([0, T], \mathbb{R}), c_0 \in \mathbb{R} : \tilde{w} + \Delta^2 \tilde{w} = c_1, \Delta^2 \tilde{w}(0, \cdot) = c_0 \right\}, \]
as well as the following elliptic analogues:
\[ X_0 := \left\{ w_0 \in C^{4,\gamma}(S_+^2) \mid \frac{\partial w_0}{\partial \eta} = \frac{\partial \Delta w_0}{\partial \eta} = 0 \right\}, \]
\[ Y_0 := \left\{ \tilde{w}_0 \in C^{4,\gamma}(S_+^2) \mid \int_{S_+^2} \tilde{w}_0d\mu_{S_+^2} = 0, \exists c_0 \in \mathbb{R} : \Delta^2 \tilde{w}_0 = c_0 \right\} \]

**Lemma 3.1** (A direct decomposition of \(C^{4,\gamma}([0, T] \times S_+^2)\) and \(C^{4,\gamma}(S_+^2)\)).

1. \(C^{4,\gamma}(S_+^2) = X_0 \oplus Y_0\). The linear projections \(\pi_{X_0} : C^{4,\gamma}(S_+^2) \rightarrow X_0\) and \(\pi_{Y_0} : C^{4,\gamma}(S_+^2) \rightarrow Y_0\) are continuous.

2. \(C^{4,\gamma}([0, T] \times S_+^2) = X_T \oplus Y_T\). The linear projections \(\pi_{X_T} : C^{4,\gamma}([0, T] \times S_+^2) \rightarrow X_T\) and \(\pi_{Y_T} : C^{4,\gamma}([0, T] \times S_+^2) \rightarrow Y_T\) are continuous.
Proof. We first prove the time-independent case. It is easy to check that $X_0 \cap Y_0 = \{0\}$. Now let $u_0 \in C^{4,\gamma}(S^2_+)$. We consider the following problem:

$$
\begin{align*}
\Delta^2 \tilde{w}_0 &= -\int_{\partial S^2_+} \frac{\partial \Delta u}{\partial n} dS & \text{in } S^2_+, \\
\frac{\partial}{\partial n} \tilde{w}_0 &= \frac{\partial}{\partial n} u_0 & \text{on } \partial S^2_+, \\
\frac{\partial}{\partial n} \Delta \tilde{w}_0 &= \frac{\partial}{\partial n} \Delta u_0 & \text{on } \partial S^2_+, \\
\int_{S^2_+} \tilde{w}_0 d\mu_{S^2} &= 0.
\end{align*}
$$

(3.1)

This problem has a unique solution $\tilde{w}_0 \in C^{4,\gamma}(S^2_+)$ (see e.g. [1]). We set $w_0 := u_0 - \tilde{w}_0$ and claim that this gives the required decomposition. Considering (3.1) it is clear that $\tilde{w}_0 \in Y_0$ and it is readily shown that $w_0 \in X_0$. All that is left to do is establishing the continuity of the projections. Schauder theory (e.g. [1]) implies

$$\|\tilde{w}_0\|_{C^{4,\gamma}} \leq C\|u_0\|_{C^{4,\gamma}}.$$ (3.2)

Using the definition of $w_0$ and (3.2) the estimate $\|w_0\|_{C^{4,\gamma}} \leq C(S^2_+,\gamma)\|u_0\|_{C^{4,\gamma}}$ follows.

Next, we show the time-dependent case. First suppose that $u \in X_T \cap Y_T$. It is easy to see that such $u$ solves

$$
\begin{align*}
\dot{u} + \Delta^2 u &= 0 & \text{in } [0,T] \times S^2_+, \\
\frac{\partial}{\partial n} u &= 0 & \text{on } [0,T] \times \partial S^2_+, \\
\frac{\partial}{\partial n} \Delta u &= 0 & \text{on } [0,T] \times \partial S^2_+, \\
u(0,\cdot) &= 0 & \text{on } \{0\} \times S^2_+.
\end{align*}
$$

and is therefore 0 by uniqueness of this problem. For $u \in C^{4,1,\gamma}([0,T] \times S^2_+)$ set $u_0 := u(0,\cdot) \in C^{4,\gamma}(S^2_+)$. By the time-independent case there exist unique $w_0 \in X_0$ and $\tilde{w}_0 \in Y_0$ such that $u_0 = w_0 + \tilde{w}_0$. Now consider the problem

$$
\begin{align*}
\dot{\tilde{w}} + \Delta^2 \tilde{w} &= -\int_{\partial S^2_+} \frac{\partial \Delta u}{\partial n} dS & \text{in } [0,T] \times S^2_+, \\
\frac{\partial}{\partial n} \tilde{w} &= \frac{\partial}{\partial n} u & \text{on } [0,T] \times \partial S^2_+, \\
\frac{\partial}{\partial n} \Delta \tilde{w} &= \frac{\partial}{\partial n} \Delta u & \text{on } [0,T] \times \partial S^2_+, \\
\tilde{w}(0,\cdot) &= w_0 & \text{on } \{0\} \times S^2_+.
\end{align*}
$$

This problem has a unique solution $\tilde{w} \in C^{4,1,\gamma}([0,T] \times S^2_+)$ (see e.g. [7,17]) as the necessary compatibility conditions are satisfied. Let $w := u - \tilde{w}$. We claim that this provides the desired decomposition. First we use parabolic Schauder theory to get

$$\|\tilde{w}\|_{C^{4,1,\gamma}} \leq C(T)(\|u\|_{C^{4,1,\gamma}} + \|\tilde{w}_0\|_{C^{4,\gamma}}) \leq C(T)\|u\|_{C^{4,1,\gamma}}.$$ (3.3)

Note that $\tilde{w}$ also satisfies

$$\int_{S^2_+} \tilde{w}(0) d\mu_{S^2} = 0 \quad \text{and} \quad \frac{d}{dt} \int_{S^2_+} \tilde{w}(t) d\mu_{S^2} = -2\pi \int_{\partial S^2_+} \frac{\partial \Delta u}{\partial n} dS - \int_{S^2_+} \Delta^2 u d\mu_{S^2} = 0.$$

So $\tilde{w} \in Y_T$. It is easy to see that $w \in X_T$. Finally, we combine the definition of $w$ and Equation (3.3) to get $\|w\|_{C^{4,1,\gamma}} \leq C(T)\|u\|_{C^{4,1,\gamma}}$. \hfill \qed
For $l \geq 0$ and $T > 1$ we introduce

$$G_0^l := C^l(Z_2, M_3^{\text{sym}}(\mathbb{R})) \quad \text{and} \quad \| \cdot \|_{G_0^l} := \| \cdot \|_{C^l(Z_2, M_3^{\text{sym}}(\mathbb{R}))},$$

$$G_T^l := C^1(\mathbb{R}, C^l(Z_2, M_3^{\text{sym}}(\mathbb{R}))), \quad \text{and} \quad \| \cdot \|_{G_T^l} := \| \cdot \|_{C^1(\mathbb{R}, C^l(Z_2, M_3^{\text{sym}}(\mathbb{R}))}.'

Let $l \geq 7$. We consider the operator

$$B_0 : C^{4,\gamma}(\mathbb{S}_+^2) \times G_0^l \to C^{3,\gamma}(\partial \mathbb{S}_+^2) \times C^{1,\gamma}(\partial \mathbb{S}_+^2)$$

$$(u_0, g_0) \mapsto \left( \bar{g}_0(\bar{v}, \bar{v}_{\mathbb{R}^2}), \frac{\partial H}{\partial \eta} + H \bar{h}^2(\bar{v}, \bar{v}) \right).$$

Here $\bar{v}$ and $\bar{\eta}$ are defined with respect to the metric $\bar{g}_0$. $B_0$ is well defined on the set $\|u_0\|_{C^{4,\gamma}} + \|\bar{g}_0 - \delta\|_{G_0^l} < \epsilon$ and is of class $C^{l-7}$. Next we define the analogue time-dependent operator. For $T > 1$ let

$$B_T : C^{4,\gamma}([0, T] \times \mathbb{S}_+^2) \times G_T^l \to C^{3,\gamma}([0, T] \times \partial \mathbb{S}_+^2) \times C^{1,\gamma}([0, T] \times \partial \mathbb{S}_+^2)$$

$$B_T[u, \bar{g}] := \left( \bar{g}(\bar{v}, \bar{v}_{\mathbb{R}^2}), \frac{\partial H}{\partial \eta} + H \bar{h}^2(\bar{v}, \bar{v}) \right).$$

Now $\bar{v}$ and $\bar{\eta}$ are defined with respect to the metric $\bar{g}$. $B_T$ is well defined on the set $\|u\|_{C^{4,\gamma}} + \|\bar{g} - \delta\|_{G_T^l} < \epsilon$ and is of class $C^{l-7}$.

**Lemma 3.2 (Boundary value Lemma).** Let $l \geq 8$.

1. There exist $\sigma_0, \theta_0, \delta_0 > 0$ and a $C^{l-7}$-map $\psi_0 : X_0(\theta_0) \times G_0^l(\delta; \sigma_0) \to Y_0(\delta_0)$ such that within the respective neighbourhoods

   $$B_0[w + \bar{w}_0, \bar{g}_0] = 0 \iff \bar{w}_0 = \psi_0[w_0, \bar{g}_0].$$

   The map $\psi_0$ satisfies $D_1 \psi_0[0, \delta] = 0$.

2. There exist $\sigma(T), \theta(T), \delta(T) > 0$ and a $C^{l-7}$-map $\psi_T : X_T(\theta(T)) \times G_T^l(\delta; \sigma(T)) \to Y_T(\delta(T))$ such that within the respective neighbourhoods

   $$B_T[w + \bar{w}, \bar{g}] = 0 \iff \bar{w} = \psi_T[w, \bar{g}].$$

   The map $\psi_T$ satisfies $D_1 \psi_T[0, \delta] = 0$.

**Proof.** We first prove the time-independent case. $B_0$ is of class $C^{l-7}$ and in Lemma A.1 we show

$$D_1 B_0[0, \delta] \varphi = \left( \frac{\partial \varphi}{\partial \eta}, -\frac{\partial}{\partial \eta}(\Delta + 2) \varphi \right).$$

Note that $X_0 = \ker(D_1 B_0[0, \delta])$. On a neighbourhood of $(0, 0, \delta) \in X_0 \times Y_0 \times G_0^l$ we define $B_0 : X_0 \times Y_0 \times G_0^l \to C^{3,\gamma}(\partial \mathbb{S}_+^2) \times C^{1,\gamma}(\partial \mathbb{S}_+^2)$ by mapping $B_0[w_0, \bar{w}_0, \bar{g}_0] := B_0[w_0 + \bar{w}_0, \bar{g}_0]$. By
construction \( D_2 \bar{B}_0[0, 0, \delta] \) is injective. It is also onto as for any \((\alpha_0, \beta_0) \in C^{3, \gamma}(\partial S^3_+) \times C^{1, \gamma}(\partial S^2_+)\) we can obtain \( \tilde{w}_0 \in Y_0 \) satisfying \( D_2 \bar{B}_0[0, 0, \delta] = (\alpha_0, \beta_0) \) by solving

\[
\left\{ \begin{array}{ll}
\Delta^2 \varphi_0 &= f_{\partial S^2_+} \beta_0 + 2\alpha_0 \quad \text{in } S^2_+ \\
\frac{\partial \varphi_0}{\partial n} &= \alpha_0 \quad \text{on } \partial S^2_+ \\
\frac{\partial \varphi_0}{\partial n} &= -\beta_0 - 2\alpha_0 \quad \text{on } \partial S^2_+ \\
\int \varphi_0 d\mu &= 0.
\end{array} \right.
\]

This problem has a unique solution \( \varphi_0 \in C^{4, \gamma}(S^2_+) \) (see e.g. [1]) which clearly lies in \( Y_0 \). The Schauder estimate

\[ ||\varphi_0||_{C^{4, \gamma}} \leq C(||\alpha_0||_{C^{4, \gamma}} + ||\beta_0||_{C^{1, \gamma}}) \tag{3.4} \]

guarantees \( D_2 \bar{B}_0[0, 0, \delta] \) to have a bounded inverse. The implicit function theorem yields the neighbourhoods as well as the \( C^{4-7} \) diffeomorphism. Finally, we prove the formula. For that let \( w_0 \in X_0 \) and compute

\[ 0 = \frac{d}{dt} B_t[tw_0 + \psi_0[tw_0, \delta], \delta] = D_1 B_0[0, \delta] w_0 + D_1 B_0[0, \delta] D_1 \psi_0[0, \delta] w_0. \]

Note \( w_0 \in X_0 = \ker D_1 B_0[0, \delta] \). Hence the first term vanishes and \( D_1 \psi_0[0, \delta] w_0 \in \ker D_1 B_0[0, \delta] = X_0. \) So \( D_1 \psi_0[0, \delta] w_0 \in X_0 \cap Y_0 = \{ 0 \} \) for all \( w_0 \in X_0 \) and therefore \( D_1 \psi_0[0, \delta] = 0 \).

For the time-dependent case we essentially repeat the same argument. First, on a neighbourhood of \((0, 0, \delta) \in X_T \times Y_T \times G_T^d\), we define \( \bar{B}_T : X_T \times Y_T \times G_T^d \rightarrow C^{3,0, \gamma}([0, T] \times S^2_+) \times C^{1,0, \gamma}([0, T] \times S^2_+) \)
by mapping \( \bar{B}_T[w, \tilde{w}, \tilde{g}] := B_T[w + \tilde{w}, \tilde{g}] \) and compute

\[ D_2 \bar{B}_T[0, 0, \delta] \varphi = \left( \frac{\partial \varphi}{\partial \eta}, -\frac{\partial (\Delta + 2) \varphi}{\partial \eta} \right). \]

\( D_2 \bar{B}_0[0, 0, \delta] \) is injective as any element \( \varphi \in \ker D_2 \bar{B}_T[0, 0, \delta] \) is in \( X_T \cap Y_T \) and thus 0. Next \( D_2 \bar{B}_T[0, 0, \delta] \) is onto. For \((\alpha, \beta) \in C^{3,0, \gamma}([0, T] \times S^2_+) \times C^{1,0, \gamma}([0, T] \times S^2_+) \) let \( \alpha_0 := \alpha(0, \cdot) \) and \( \beta_0 := \beta(0, \cdot) \). Then we have seen in the time-independent case that there exists a unique \( \varphi \in Y_0 \) such that \( D_2 \bar{B}_0[0, 0, \delta] \varphi = (\alpha_0, \beta_0) \). Now consider the problem

\[
\left\{ \begin{array}{ll}
\varphi + \Delta^2 \varphi &= f_{\partial S^2_+} \beta + 2\alpha \quad \text{in } [0, T] \times S^2_+ \\
\frac{\partial \varphi}{\partial n} &= \alpha \quad \text{on } [0, T] \times \partial S^2_+ \\
\frac{\partial \varphi}{\partial n} &= -\beta - 2\alpha \quad \text{on } [0, T] \times \partial S^2_+, \\
\varphi(\cdot, 0) &= \varphi_0 \quad \text{on } \{ 0 \} \times S^2_+.
\end{array} \right.
\]

By construction of \( \varphi_0 \) the necessary compatibility conditions are satisfied and we deduce that there exists a unique function \( \varphi \in C^{4,1, \gamma}([0, T] \times S^2_+) \) that is a solution of this problem (see e.g. [7, 17]). Proving that this solution lies in \( Y_T \) is done as in the proof of Lemma 3.1. Due to Schauder estimates and estimate (3.4) we get

\[ ||\varphi||_{C^{4,1, \gamma}_T} \leq C(T) \left( ||\varphi_0||_{C^{4, \gamma}_T} + ||\alpha||_{C^{3,0, \gamma}_T} + ||\beta||_{C^{1,0, \gamma}_T} \right) \tag{3.4} \]

\[ \leq C(T) \left( ||\alpha||_{C^{3,0, \gamma}_T} + ||\beta||_{C^{1,0, \gamma}_T} \right) \]
and hence learn that the inverse map is continuous. The implicit function theorem guarantees the existence of the neighbourhoods and of the function $\psi_T$. The formula follows as in the time-independent case.

\[ \square \]

**Remarks**

1. It is easy to check that $\psi_T[w, \bar{g}](0) = \psi_0[w(0), \bar{g}(0)]$.

2. $\psi_0, \psi_T \in C^{2}$ for $l \geq 9$. Using $D_1 \psi_0[0, \delta] = 0$ and $D_1 \psi_T[0, \delta] = 0$ it is then readily shown that after potentially shrinking the neighbourhoods in Lemma 3.2 the following hold: Given $w_0 \in X_0(\theta_0)$ and $\bar{g}_0 \in G^l(\sigma_0)$ or $w \in X_T(\theta(T))$ and $\bar{g} \in G^l(\sigma(T))$ we have

\[
\|\psi_0[w_0, \bar{g}_0]\|_{C^{4, \gamma}} \leq C(\|\bar{g}_0 - \delta\|_{C^l} + \|w_0\|^2_{C^{4, \gamma}}),
\]

\[
\|\psi_T[w, \bar{g}]\|_{C^{4,1,\gamma}} \leq C(T)\left(\|\bar{g} - \delta\|_{C^l_T} + \|w\|^2_{C^{4,1,\gamma}}\right).
\]

We now wish to study a prototype of the flow equations (2.26). For $\bar{g} \in G^l_T$ and $u \in C^{4,1,\gamma}([0, T] \times S^1_\sigma)$ we put

\[
\tau[u, \bar{g}] := -\sum_{\mu, \nu=0}^2 A^{\mu\nu}[u, \bar{g}] \frac{D_2 C^\mu[u, \bar{g}] \tilde{g}}{\|\nabla C^\mu[u, \bar{g}]\|_{L^2(f_T^2 \tilde{g})}} \frac{\nabla C^\nu[u, \bar{g}]}{\|\nabla C^\nu[u, \bar{g}]\|_{L^2(f_T^2 \tilde{g})}}
\]

and define the operator

\[
Q_T : \mathcal{C}^{4,1,\gamma}([0, T] \times S^1_\sigma) \times \mathcal{C}^{4,\gamma}(S^1_\sigma) \times G^l_T \times C^{0,0,0,\gamma}([0, T] \times S^1_\sigma) \to C^{0,0,\gamma}([0, T] \times S^1_\sigma)
\]

\[
Q_T[u, w_0, \bar{g}, f] := \tilde{g}(\hat{f}_u, \tilde{\nu}[u, \bar{g}]) - \tau[u, \bar{g}] + P^\perp_{K[u, \bar{g}]}[W[u, \bar{g}] - f].
\]

For $l \geq 8$ the operator $Q_T$ is well defined on a small neighbourhood of $(0, 0, \delta, 0)$ and is of class $\mathcal{C}^{l-8}$. We also introduce the operator $Q_T : X_T \times X_0 \times G^l_T \times C^{0,0,0,\gamma}([0, T] \times S^1_\sigma) \to C^{0,0,0,\gamma}([0, T] \times S^1_\sigma)$ that maps

\[
Q_T[w, w_0, \bar{g}, f] := (Q_T[w + \psi_T[w, \bar{g}], w_0 + \psi_0[w_0, \bar{g}(0)], \bar{g}, f], w(0, \cdot) - w_0).
\]

By the standard rules for composition of operators we see that $Q_T$ is also of class $\mathcal{C}^{l-8}$ for $l \geq 8$.

**Theorem 3.3.** Let $T > 1$ and suppose $l \geq 9$. There exists $\theta(T) > 0$, $\theta'(T) > 0$, $\sigma(T) > 0$ and $\delta(T) > 0$ and a $\mathcal{C}^{l-8}$-map $w : X_0(\theta(T)) \times G^l_T(\delta; \sigma(T)) \times C^{0,0,0,\gamma}(\delta(T)) \to X_T(\theta(T))$ such that

\[
Q_T[w, w_0, \bar{g}, f] = (0, 0) \Leftrightarrow w = w[w_0, \bar{g}, f].
\]

If additionally $l \geq 10$, $A[w_0 + \psi_0[w_0, \bar{g}(0)], \bar{g}(0)] = 2\pi$ and $C[w_0 + \psi_0[w_0, \bar{g}(0)], \bar{g}(0)] = 0$ the following estimates hold:

\[
\|w\|_{C^{4,1,\gamma}} \leq C(T)\left[\|w_0\|_{C^{4,\gamma}} + \|\bar{g} - \delta\|_{C^l_T} + ||P^\perp_{K[0, \delta]}[f||_{C^{0,0,\gamma}} + \|f\|^2_{C^{0,0,\gamma}}\right],
\]

\[
\|w(T)\|_{C^{4,\gamma}} \leq Ce^{-6T}\|w_0\|_{C^{4,\gamma}} + C(T)\left[\|\bar{g} - \delta\|_{C^l_T} + ||P^\perp_{K[0, \delta]}[f||_{C^{0,0,\gamma}} + \|f\|^2_{C^{0,0,\gamma}}\right].
\]

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Proof. For $l \geq 9$ both $Q_T$ and $\bar{Q}_T$ are of class $C^{l-8} \subset C^1$. Using $D_1 \psi_T[0, \delta] = 0$, $D_1 \psi_0[0, \delta] = 0$ and

$$D_1 Q_T[0, 0, \delta, 0] \varphi = -\left( \dot{\varphi} + \frac{1}{2} \Delta (\Delta + 2) \varphi \right)$$ (3.7)

we learn that $D_1 \bar{Q}_T[0, 0, \delta, 0]$ is an isomorphism with bounded inverse. Indeed $\bar{Q}_T[0, 0, \delta, 0] \varphi = (\psi, \varphi_0)$ for $\varphi \in X_T$ if and only if

$$\begin{cases}
\dot{\varphi} + \frac{1}{2} \Delta (\Delta + 2) \varphi = -\psi & \text{on } [0, T] \times S^2_+
\frac{\partial \varphi}{\partial \eta} = 0 & \text{on } [0, T] \times \partial S^2_+
\frac{\partial \Delta \varphi}{\partial \eta} = 0 & \text{on } [0, T] \times \partial S^2_+
\varphi(0, \cdot) = \varphi_0 & \text{on } \{0\} \times S^2_+.
\end{cases}$$

Schauder Theory (see e.g. [7, 17]) implies the existence and uniqueness of a solution $\varphi \in C^{4,1,\gamma}([0, T] \times S^2_+)$ that must then be an element in $X_T$. Additionally the Schauder estimate

$$\|\varphi\|_{C^{4,1,\gamma}} \leq C(T) \left( \|\varphi_0\|_{C^{4,\gamma}} + \|\psi\|_{C_T^{0,\gamma}} \right)$$

implies that $D_1 \bar{Q}_T[0, 0, \delta, 0]$ is an isomorphism with bounded inverse. Hence the first part of the theorem follows. To prove the estimates we first note that after potentially shrinking all neighbourhoods in the Theorem the implicit function theorem implies the estimate

$$\|w\|_{C_T^{4,1,\gamma}} \leq C(T, \gamma) (\|\bar{g} - \delta\|_{C_T^4} + \|f\|_{C_T^{0,0,\gamma}} + \|w_0\|_{C^{4,\gamma}}).$$ (3.8)

Expanding $\bar{Q}_T[w, w_0, \bar{g}, f] \text{ around } (0, 0, \delta, 0)$ yields the equation

$$\begin{cases}
\ddot{w} + \frac{1}{2} \Delta (\Delta + 2) w = \sum_{\mu=0}^{2} \frac{D_2 C^\mu [0, \delta] \bar{g}}{\|\nabla C^\mu [0, \delta]\|_{L^2(S^2_+)}^2} \nabla C^\mu [0, \delta]
+ P_K^0 [0, \delta] [D_2 W [0, \delta] (\bar{g} - \delta) - f] + R,
\end{cases}$$ (3.9)

where the remainder $R$ satisfies the estimate $\|R\|_{C_T^{0,0,\gamma}} \leq C(T) \left( \|w\|_{C_T^{4,1,\gamma}}^2 + \|\bar{g} - \delta\|^2_{C_T^4} + \|f\|^2_{C_T^{0,0,\gamma}} \right)$ as due to $l \geq 10$ we know $\bar{Q}_T \in C^2$. Inserting Estimate (3.8) for $\|w\|_{C_T^{4,1,\gamma}}$ yields $\|R\|_{C_T^{0,0,\gamma}} \leq C(T) \left( \|w_0\|^2_{C_T^{4,1,\gamma}} + \|\bar{g} - \delta\|^2_{C_T^4} + \|f\|^2_{C_T^{0,0,\gamma}} \right)$. Applying the Schauder Theory from e.g. [7, 17] to problem (3.9) and inserting this estimate for $R$ we immediately find the estimate

$$\|w\|_{C_T^{4,1,\gamma}} \leq C(T, \gamma) (\|\bar{g} - \delta\|_{C_T^4} + \|P_K^0 [0, \delta] f\|_{C_T^{0,0,\gamma}} + \|w_0\|_{C^{4,\gamma}} + \|f\|^2_{C_T^{0,0,\gamma}}).$$ (3.10)
This is the first estimate claimed in the Theorem. To deduce the decay estimate we decompose \( w = w^1 + w^2 \) where \( w^1(t) \in \text{span}_R \{ 1, \omega^1, \omega^2 \} \) for all \( t \in [0, T] \) and \( w^2(t) \perp L^2(S^2) \) span \( R \{ 1, \omega^1, \omega^2 \} \). Splitting the remainder \( R \) and the initial value \( w_0 \) similarly into \( R^1 + R^2 \) and \( w_0^1 + w_0^2 \) we now examine separate problems for \( w^1 \) and \( w^2 \). \( w^1 \) satisfies

\[
\begin{aligned}
& \dot{w}^1 + \frac{1}{2} \Delta (\Delta + 2) w^1 = \sum_{\mu = 0}^{2} \frac{D_2 C^\mu[0, \delta] \hat{g}}{\| \nabla C^\mu[0, \delta] \|^2_{L^2(S^2)}} \nabla C^\mu[0, \delta] + R^1, \\
& \frac{\partial w^1}{\partial \eta} = \frac{\partial \Delta w^1}{\partial \eta} = 0, \\
& w^1(0) = w_0^1.
\end{aligned}
\]

We apply the Schauder Theory from e.g. [7, 17] to derive an estimate for \( w^1 \). Inserting the estimate for the remainder from above we find

\[
\| w^1 \|_{C^{1, \gamma}_T} \leq C(T, \gamma)(\| \hat{g} - \delta \|_{C^1_T} + \| w_0^1 \|_{C^{4, \gamma}} + \| w_0 \|_{C^{4, \gamma}} + \| f \|_{C^{0,0, \gamma}_T}). \tag{3.11}
\]

Next we establish an estimate for \( w_0^1 \) by expanding the equation \( A[w_0 + \psi_0[w_0, \tilde{g}_0, \hat{g}_0]] = 2\pi = A[0, \delta] \). Using \( A \in C^{4-\delta} \subset C^2 \), \( \psi_0 \in C^{4-\gamma} \subset C^2 \), \( \nabla A[0, \delta] = -2 \) and \( D_1 \psi_0[0, \delta] = 0 \) we deduce

\[
2 \int_{S^2} w_0 \, d\mu_2 = \| \nabla A[0, \delta], w_0 \|_{L^2(S^2)} \leq C(\| w_0 \|_{C^{4, \gamma}} + \| \hat{g} - \delta \|_{C^1_T}). \tag{3.12}
\]

By applying the same reasoning to the barycenter \( C^i \) we may derive a similar bound for \( \int_{S^2} w_0 \, d\mu_2 \) as \( \nabla C^i[0, \delta] = -\frac{3}{2\pi} \omega^i \). Hence \( \| w_0^1 \|_{C^{4, \gamma}} \leq C(\| w_0 \|_{C^{4, \gamma}} + \| \hat{g} - \delta \|_{C^1_T}) \). Combining this estimate with (3.11) gives

\[
\| w^1 \|_{C^{2,1, \gamma}_T} \leq C(T)(\| \hat{g} - \delta \|_{C^1_T} + \| P_{K[0, \delta]} f \|_{C^{0,0, \gamma}_T} + \| w_0 \|_{C^{4, \gamma}} + \| f \|_{C^{0,0, \gamma}_T}). \tag{3.13}
\]

Next, we examine the problem that is solved by \( w^2 \).

\[
\begin{aligned}
& \dot{w}^2 + \frac{1}{2} \Delta (\Delta + 2) w^2 = P_{K[0, \delta]}^T[0, \delta](\hat{g} - \delta) - f + R^2, \\
& \frac{\partial w^2}{\partial \eta} = \frac{\partial \Delta w^2}{\partial \eta} = 0, \\
& w^2(0) = w_0^2.
\end{aligned}
\]

As \( w^2(t) \perp L^2(S^2) \) span \( R \{ 1, \omega^1, \omega^2 \} \) for all \( t \geq 0 \) we may use Appendix C to derive the improved Schauder estimate.

\[
\begin{aligned}
& \| w^2(T) \|_{C^{4, \gamma}} \leq C e^{-6T} \| w_0^2 \|_{C^{4, \gamma}} + C(T) \| \hat{g} - \delta \|_{C^1_T} + \| P_{K[0, \delta]}^T f \|_{C^{0,0, \gamma}_T} + \| R^2 \|_{C^{0,0, \gamma}_T}) \\
& \leq C e^{-6T} \| w_0^2 \|_{C^{4, \gamma}} + C(T) \| \hat{g} - \delta \|_{C^1_T} + \| P_{K[0, \delta]}^T f \|_{C^{0,0, \gamma}_T} + \| w_0 \|_{C^{4, \gamma}} + \| f \|_{C^{0,0, \gamma}_T}). \tag{3.14}
\end{aligned}
\]

In the second step, we inserted the estimate for \( R \) from above. The decay estimate follows by combining (3.13), (3.14), the estimate for \( \| w_0^1 \|_{C^{4, \gamma}} \) from above and shrinking the neighbourhood from which \( w_0 \) is taken so that \( C(T) \| w_0 \|_{C^{4, \gamma}} \leq C e^{-6T} \).
4 Willmore flow with prescribed barycenter curve

We now apply the result from the previous section to derive the existence of a solution to the Willmore flow with \textit{prescribed barycenter curve}. That is, given a curve \( \xi : [0, T] \rightarrow S \) with initial condition \( \xi(0) = p \) for some \( p \in S \) we investigate the following system that only contains the evolution equation for the graph function form \((4.1)\) with \( \dot{J}_t \) written out explicitly.

\[
\begin{cases}
\bar{g}(\dot{f}, \dot{\nu}) = \tau[u, \bar{g}] - P^L_{K[u, \beta]}[W[u, \bar{g}]] \\
+ P^L_{K[u, \beta]} \left[ I^u[u, \bar{g}] \bar{g} \left( \lambda(D_2 F^\lambda[\xi(t), f_u(t)])^{-1} D_1 F^\lambda[\xi(t), f_u(t)] b_i(\xi(t)), \dot{\nu} \right) \right], \\
B_T[u, \bar{g}] = 0, \\
u(0) = u(0).
\end{cases}
\]

Ultimately we want to put \( \bar{g} := \bar{g}^{\varepsilon, \lambda} \) but will keep \( \bar{g} \) abstract for now. To parameterize the curves \( \xi \) near \( p \) we use the chosen orthonormal frame \((b_1, b_2)\) in a neighbourhood of \( p \). Given a vector field \( \bar{\beta} : [0, T] \rightarrow D_{r_1} \) with \( r_1 \) as in Subsection 2.1 we can consider the curve

\[
\xi_{p, \bar{\beta}} : [0, T] \rightarrow S, \quad \xi_{p, \bar{\beta}}(t) := f[p, \bar{\beta}(t)] = p + \beta'(t) b_i(p) + \varphi[p, \bar{\beta}(t)] N^S(p).
\]

It is easy to verify that this accounts for all curves close to \( p \) for \( t \in [0, T] \). For \( \Omega \in C^n \) and \( n \geq 4 \) the map \( C^1 \mathbb{T}([0, T], D_{r_1}) \ni \bar{\beta} \rightarrow \xi_{p, \bar{\beta}} \in C^1 \mathbb{T}([0, T], S) \) is of class \( C^{n-4} \). On a small neighbourhood of \((0, 0, 0)\), we may define the map

\[
J_i : (-1, 1) \times C^{4,1}_T([0, T] \times S^2_+) \times C^1 \mathbb{T}([0, T], \mathbb{R}^2) \rightarrow C^{0,0}_T([0, T] \times S^2_+),
\]

\[
(\lambda, u, \bar{\beta}) \mapsto \lambda \left( D_2 F^\lambda[\xi_{p, \bar{\beta}}, f_u] \right)^{-1} D_1 F^\lambda[\xi_{p, \bar{\beta}}, f_u] b_i(\xi_{p, \bar{\beta}}).
\]

Let \( \Gamma_{ij}^k := \langle b_k, \nabla_{b_i} b_j \rangle \). Then we may follow the analysis from [2] (Proof of Theorem 2) where the following explicit coordinate expression for \( J_i \) is derived:

\[
\lambda \left( D_2 F^\lambda[a, x, z] \right)^{-1} D_1 F^\lambda[a, x, z] b_i(a) = e_i + \left[ (\lambda x^k \Gamma_{ij}^k(a) - h^S_{ik}(a)(\varphi[a, \lambda x] + \lambda z)) e_k \right]
\]

\[
\lambda z h^S_{ik}(a) \partial_k \varphi[a, \lambda x] \]

Using this formula it is readily checked that \( J_i \) is of class \( C^{n-4} \). Formula \((4.2)\) also implies that for \( \|\beta\|_{C^1} \leq 1 \) and \( \|u\|_{C^{4,1}_T} \leq 1 \) we have the estimate

\[
\|J_i[\lambda, u, \bar{\beta}] - e_i\|_{C^{0,0}_T} \leq C(\Omega, \gamma, T) |\lambda|.
\]

Comparing the evolution equation \((4.1)\) to the abstract perturbation problem we studied in Theorem 3.3 we see

\[
I^u[u, \bar{g}] P^L_{K[u, \beta]} \left[ \bar{g} \left( \lambda \left( D_2 F^\lambda[\xi, f_u] \right)^{-1} D_1 F^\lambda[\xi, f_u] b_i(\xi), \dot{\nu} \right) \right] \leftrightarrow f.
\]
Suppose that $l \geq 9$ and $n \geq l$ are integers. With these choices $n - 4 \geq l - 8 \geq 1$. On a small neighbourhood of $(0, 0, \delta, 0, 0)$ we introduce the operator

$$\hat{Q}_T : X_T \times X_0 \times C^1_T \times (-1, 1) \times C^1 \hat{\tau}([0, T], \mathbb{R}^2) \to C^{0,0,0,\gamma}([0, T] \times S^2_+).$$

$$\hat{Q}_T[w, w_0, \tilde{g}, \lambda, \tilde{\beta}] := \hat{Q}_T[w, w_0, \tilde{g}, 0] + \left( P_{K,\{u, \delta\}}^\perp \left[ I^\perp[u, \tilde{g}][\gamma(J_i[\lambda, u, \tilde{\beta}], \tilde{\nu}[u, \tilde{g}])] \right] \right)_{|u=w+\psi_T[w, \tilde{g}]}.$$ Recalling that we use the reference point $p \in S$ to define the curve $\xi_{p,\tilde{\beta}}$. We already know that $\hat{Q}_T$ is of class $C^{l-8}$. It is also not difficult to check that $I^i \in C^{l-8}$, $\tilde{\nu} \in C^{l-8}$ and $J^i \in C^{n-4} \subset C^{l-8}$ as maps into $C^{0,0,0,\gamma}([0, T] \times S^2_+)$. Hence $\hat{Q}_T \in C^{l-8} \subset C^1$. As with the abstract perturbation problem we now prove that there exist a unique solution $w = w[w_0, \tilde{g}, \lambda, \tilde{\beta}]$ for small enough data.

**Theorem 4.1** (Short time existence with prescribed barycenter curve). Let $T > 1$, $n \geq l$ and $l \geq 9$. There exist $\theta(T) > 0$, $\theta'(T) > 0$, $\sigma(T) > 0$, $\rho(T) > 0$, $\lambda_0(T) > 0$ and a $C^{l-8}$-map $w : X_0(\theta(T)) \times C^1_T(\delta; \sigma(T)) \times [-\lambda_0(T), \lambda_0(T)] \times C^1_\hat{T} \om(T) \to X_T(\theta'(T))$ such that within the respective neighbourhoods

$$\hat{Q}_T[w, w_0, \tilde{g}, \lambda, \tilde{\beta}] = (0, 0) \Leftrightarrow w = w[w_0, \tilde{g}, \lambda, \tilde{\beta}].$$

For $l \geq 10$ the solution $w$ satisfies

$$\|w\|_{C^{l,1,\gamma}} \leq C(T) \left[ \|w_0\|_{C^{4,\gamma}} + \|\tilde{g} - \delta\|_{G^{l,\gamma}} + \lambda^2 \right]$$

$$\|w(T)\|_{C^{4,\gamma}} \leq C e^{-\rho T} \|w_0\|_{C^{4,\gamma}} + C(T) \left[ \|\tilde{g} - \delta\|_{G^{l,\gamma}} + \lambda^2 \right].$$

**Proof.** First we prove existence. As deduced above $\hat{Q}_T \in C^1$. Next note that $\hat{Q}_T[0, 0, \delta, 0, 0] = (0, 0)$ and $I^i[\delta, 0] = 0$ and thus $\hat{Q}_T[0, 0, \delta, 0, 0] = (0, 0)$. Next we compute the differential with respect to the first argument. To do this we note that $I^i[0, \delta] = 0$ and that $J_i[0, 0, 0] = \epsilon_i$. Hence we learn that the map

$$\mathcal{I} : X_T \times X_0 \times C^1_T \times (-1, 1) \times C^1 \hat{\tau}([0, T], \mathbb{R}^2) \to C^{0,0,0,\gamma}([0, T] \times S^2_+)$$

$$\mathcal{I}[w, w_0, \tilde{g}, \lambda, \tilde{\beta}] := P_{K,\{u, \delta\}}^\perp \left[ I^\perp[u, \tilde{g}][\gamma(J_i[\lambda, u, \tilde{\beta}], \tilde{\nu}[u, \tilde{g}])] \right]_{|u=w+\psi_T[w, \tilde{g}]}$$

satisfies

$$D_1 \mathcal{I}[0, 0, \delta, 0, 0] = P_{K,\{u, \delta\}}^\perp(-\omega^i) D_1 I^i[0, \delta] = 0.$$ Since $D_1 \psi_T[0, \delta] = 0$ we get $D_1 \hat{Q}_T[0, 0, \delta, 0, 0] = D_1 \hat{Q}_T[0, 0, \delta, 0, 0]$ which is an isomorphism as we have already discussed in Theorem 3.3. The implicit function theorem gives the the existence and uniqueness as well as the estimate

$$\|w\|_{C^{l,1,\gamma}} \leq C(T) \left[ \|w_0\|_{C^{4,\gamma}} + \|\lambda\|_{C^{1,\gamma}} + \|\tilde{\beta}\|_{C^{1,\gamma}} + \|\tilde{g} - \delta\|_{G^{l,\gamma}} \right].$$

(4.6)
We put \( u := w + \psi_T[w, \tilde{g}] \). After potentially shrinking the neighbourhoods from where \( \lambda, \tilde{\beta}, w_0 \) and \( \tilde{g} \) are taken we can use the second remark after Lemma 3.2 to get
\[
\|u\|_{C_T^4, \gamma} \leq 2\|w\|_{C_T^4, \gamma} + C(T)\|\tilde{g} - \delta\|_{C_T^4}. \tag{4.7}
\]
\[
\leq C(T)(\|w_0\|_{C_T^4, \gamma} + \|\tilde{g} - \delta\|_{C_T^4} + |\lambda| + \|\tilde{\beta}\|_{C_T^4}). \tag{4.8}
\]

We now prove the estimates claimed in the Theorem. \( w \) satisfies the equation \( \dot{Q}_T[w, w_0, \tilde{g}, f] = (0, 0) \) with \( f \) as in (4.4). Thus Theorem 3.3 can be used to obtain better estimates. To apply this theorem there are two terms that we need to estimate:

\[
\left\| \tilde{P}_K(I^4[u, \tilde{g}]\tilde{g}(J_0[\lambda, u, \tilde{\beta}], \tilde{\nu}[u, \tilde{g}]) \right\|_{C_T^{0,0}, \gamma} \quad \text{and} \quad \left\| I^4[u, \tilde{g}]\tilde{g}(J_0[\lambda, u, \tilde{\beta}], \tilde{\nu}[u, \tilde{g}]) \right\|_{C_T^{0,0}, \gamma}. \tag{4.9}
\]

First term
First, we may take the term \( I^4[u, \tilde{g}] \) out of the projection operator, as it only depends on time. Note that \( I^4 \) is of class \( C^{l-8} \subset C^1 \) for \( l \geq 9 \). Using Estimate (4.8) for \( u \) to see that \( u \) is small we get
\[
\|I^4[u, \tilde{g}]\|_{C_T^{0,0}, \gamma} \leq C(T) \left( \|u\|_{C_T^{4, \gamma}} + \|\tilde{g} - \delta\|_{C_T^4} \right). \tag{4.10}
\]

Next we use the Equation (4.2) and can write \( J_0 = e_1 + R_i \) where we know from Equation (4.3) that for small enough \( \tilde{\beta} \) we get \( \|R_i\|_{C_T^{0,0}, \gamma} \leq C(T)|\lambda|. \) Clearly
\[
\tilde{g}(J_0[\lambda, u, \tilde{\beta}], \tilde{\nu}[u, \tilde{g}]) = \tilde{g}(e_1, \tilde{\nu}[u, \tilde{g}]) + \tilde{g}(R_i[\lambda, \xi_p, u], \tilde{\nu}[u, \tilde{g}]). \tag{4.11}
\]

We deal with both terms individually:

1. For \( u = 0 \) and \( \tilde{g} = \delta \) we have \( \tilde{g}(e_1, \tilde{\nu}[u, \tilde{g}]) = -\omega^4 \). As the map \( \tilde{\nu} \) is of class \( C^{l-6} \subset C^1 \) we get
\[
\|\tilde{g}(e_1, \tilde{\nu}[u, \tilde{g}]) + \omega^4\|_{C_T^{0,0}, \gamma} \leq C(T)(\|u\|_{C_T^{4, \gamma}} + \|\tilde{g} - \delta\|_{C_T^4}). \tag{4.12}
\]

2. For the second term we use the estimate we have for \( R_i \) and learn that
\[
\|\tilde{g}(R_i[\lambda, \xi_p, u], \tilde{\nu}[u, \tilde{g}])\|_{C_T^{0,0}, \gamma} \leq C(T)|\lambda|\|\tilde{\nu}\|_{C_T^{0,0}, \gamma}\|\tilde{g}\|_{C_T^4}. \tag{4.13}
\]

The last two factors are not small but bounded.

Combining Estimates (4.9), (4.10), (4.11) and (4.12) and using \( P_{K[0,\delta]}(\omega^4) = 0 \) we learn
\[
\left\| \tilde{P}_K[I^4[u, \tilde{g}]\tilde{g}(J_0[\lambda, u, \tilde{\beta}], \tilde{\nu}[u, \tilde{g}]) \right\|_{C_T^{0,0}, \gamma} \leq C(T) \left( \|u\|_{C_T^{4, \gamma}}^2 + \|\tilde{g} - \delta\|_{C_T^4}^2 + \lambda^2 \right) \leq C(T) \left( \|u\|_{C_T^{4, \gamma}}^2 + \|\tilde{g} - \delta\|_{C_T^4}^2 + \lambda^2 \right). \tag{4.13}
\]
Second term
To estimate $I'$ we can reuse Equation \((4.9)\). Next we note that $\|\tilde{v}[u,\tilde{g}]\|_{C^0,\alpha,\gamma}$ and $\|\tilde{g}\|_{C_T}$ are both bounded. Finally, we can bound $J'$ by using the formula \((4.2)\) and Estimate \((4.3)\). So

\[
\left\| \left[ I'[u, \tilde{g}] \tilde{g}(J[J[\lambda, u, \tilde{\beta}], \tilde{v}[u, \tilde{g}]) \right] \right\|_{C_T}^2 \leq C(T)(\|u\|_{C^{1,\gamma}}^2 + \|\tilde{g} - \delta\|_{C_T}^2 + \lambda^2)
\]

\[
\leq C(T)(\|u\|_{C^{1,\gamma}}^2 + \|\tilde{g} - \delta\|_{C_T}^2 + \lambda^2). 
\]

Combining both estimates
We apply Theorem 3.3 and insert the estimates \((4.13)\) and \((4.14)\) to get

\[
\|w\|_{C^{1,\gamma}} \leq C(T) \left[ \|w_0\|_{C^{1,\gamma}} + \|\tilde{g} - \delta\|_{C_T} + \lambda^2 \right],
\]

which is the first estimate claimed in the Theorem. For the second part we use the Estimate for $\|w(T)\|_{C^{1,\gamma}}$ from Theorem 3.3 and insert Estimates \((4.13)\) and \((4.14)\) to obtain

\[
\|w(T)\|_{C^{1,\gamma}} \leq Ce^{-\delta T} \|w_0\|_{C^{1,\gamma}} + C(T) \left[ \|\tilde{g} - \delta\|_{C_T} + \lambda^2 \right]
\]

After potentially shrinking the neighbourhood to which $w_0$ belongs we may assume that $\|w_0\|_{C^{1,\gamma}} C(T) \leq e^{-\delta T}$ and have therefore established the decay estimate for $w(T)$. \qed

In Theorem 4.1 we deal with an arbitrary metric. In the problem we are investigating we are, however, interested into the particular choice of metric that is $\tilde{g}_{p,\tilde{\beta}}$ with $\tilde{\beta}_p$ being defined as above. As we will now investigate $\tilde{g}$ as a function depending on $\lambda$ and $\tilde{\beta}$ we will write $\tilde{g}_p[\lambda, \tilde{\beta}] := \tilde{g}_{p,\tilde{\beta}}$.

**Corollary 4.2 (Flow with prescribed barycenter curve).**
Let $p \in S$, $T > 1$, $r \geq 2$ and suppose that $\Omega$ is of class $C^{13+2r}$. There exists $\theta(T) > 0$, $\theta'(T) > 0$, $\rho(T) > 0$, $\lambda_0(T) > 0$ and a $C^r$-map $u_p : X_0(\theta(T)) \times [-\lambda_0(T), \lambda_0(T)] \times C^{1,\gamma}_{T,T} (\rho(T)) \to C^{13+2r}_{T,T} (\theta(T))$ such that $u_p[w_0, \lambda, \tilde{\beta}]$ is the unique solution to \((4.1)\) with prescribed barycenter curve $\xi = \tilde{\xi}_{p,\tilde{\beta}}$, metric $\tilde{g} = \tilde{g}_p[\lambda, \tilde{\beta}]$ and initial value $u_0 := w_0 + \psi_0[w_0, \tilde{v}_0]$. $u$ satisfies the estimates

\[
\|u_p\|_{C^{1,\gamma}} \leq C(T) \left[ \|u_0\|_{C^{1,\gamma}} + |\lambda| \right] \quad \text{and} \quad \|u_p(T)\|_{C^{1,\gamma}} \leq Ce^{-\delta T} \|u_0\|_{C^{1,\gamma}} + C(T)|\lambda|.
\]
Proof. The explicit formula for $\tilde{g}^\xi_{p,\lambda}$ in Equation (2.2) as well as the regularity of $\varphi$ combined with the regularity of the map $\beta \mapsto \xi_{p,\beta}$ discussed above readily imply that

$$
\tilde{g}_p : (-1, 1) \times C^{1,\tilde{T}}([0, T], \mathbb{R}^2) \to G^{\delta + r}_T, \ (\lambda, \beta) \mapsto \tilde{g}_p[\lambda, \beta]
$$

is well defined on a small neighbourhood of $(0, 0)$ and of class $C^r$. As long as $\|\beta\|_{C^{1,\tilde{T}}(0)} \leq 1$ it is easy to derive the estimate

$$
\|\tilde{g}_p[\lambda, \beta]\|_{C^{\delta + r}_T} \leq C(S, T) |\lambda|.
$$

(4.16)

Put $l = 8 + r$. Due to (4.16) we may choose $\lambda_0(T)$ and $\rho(T)$ small and then use $\tilde{g}_p[\lambda, \beta]$ in Theorem 4.1. This allows us to define

$$
\bar{w}_p[w_0, \lambda, \beta] := w[w_0, \tilde{g}_p[\lambda, \beta], \lambda, \beta]
$$

with $w$ as in Theorem 4.1. $\bar{w}$ is well defined on a neighbourhood of $(0, 0, 0)$ and of class $C^r$ as $\tilde{g}_p[\cdot, \cdot] \in C^r$ and $w$ is of class $C^{l-\delta} = C^r$. On a small neighbourhood of $(0, 0, 0)$ we now define a map $u_p : X_0 \times (-1, 1) \times C^{1,\tilde{T}}([0, T], \mathbb{R}^2) \to C^{4,1,\gamma}([0, T] \times \mathbb{R}^2)$ by

$$
u_p[w_0, \lambda, \beta] := \bar{w}_p[w_0, \lambda, \beta] + \psi_T[\bar{w}_p[w_0, \lambda, \beta], \tilde{g}_p[\lambda, \beta]].
$$

$u_p \in C^r$ as $\psi_T \in C^{l-\delta} = C^r$ by Lemma 3.2. Using Lemma 3.2 and Theorem 4.1 we see that $u_p$ has the claimed properties. \qed

5 Proof of Theorem 1

We consider the space

$$
C^{1,\tilde{T}}_0([0, T], \mathbb{R}^2) := \{ \beta \in C^{1,\tilde{T}}([0, T], \mathbb{R}^2) \mid \beta(0) = 0 \}.
$$

Given $p \in S$ and small $\lambda > 0$, $w_0 \in X_0$ and $\beta \in C^{1,\tilde{T}}_0([0, T], \mathbb{R}^2)$ we get a function $u_p[w_0, \lambda, \beta]$. The pair $(u_p[w_0, \lambda, \beta], \xi_{p, \beta})$ is a solution to (2.26) with initial values $\xi(0) = p$ and $u_p(0) = w_0 + \psi_0[w_0, \tilde{g}^{p, \lambda}]$ if and only if

$$
(b_1(\xi_{p, \beta}(t)), \xi_{p, \beta}(t)) = -\lambda I[u_p[w_0, \lambda, \beta], \tilde{g}_p[\lambda, \beta]].
$$

(5.1)

To find such $\beta$ we again employ the implicit function theorem. For that we must first define a suitable operator. On a small neighbourhood for $(0, 0, 0)$ we put

$$
\mathcal{T}_p : C^{1,\tilde{T}}_0([0, T], \mathbb{R}^2) \times X_0 \times (-1, 1) \to (C^{0,\tilde{T}}([0, T], \mathbb{R}))^2
$$

$$
(\beta, w_0, \lambda) \mapsto \left( (b_1 \circ \xi_{p, \beta}, \dot{\xi}_{p, \beta}) + \lambda I[u_p[w_0, \lambda, \beta], \tilde{g}_p[\lambda, \beta]] \right)_{i=1,2}.
$$

Clearly (5.1) is equivalent to finding zeros of $\mathcal{T}_p$. 

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Lemma 5.1 (Short time existence of the barycenter curve).
Let $p \in S$, $r \geq 2$ and suppose $\Omega \in C^{13+2r}$. For all $T > 1$ there exist $\theta(T) > 0$, $\rho(T) > 0$, $\lambda(T) > 0$ and a $C^r$-map $\tilde{\beta}_p : [\lambda_0(T), \lambda_0(T)] \times X_0(\theta(T)) \to C^{1,\frac{7}{2}}(\rho(T))$ such that within the respective neighbourhoods $\mathcal{T}_p[\tilde{\beta}, w_0, \lambda] \iff \tilde{\beta} = \tilde{\beta}_p[\lambda, w_0]$. $\tilde{\beta}_p$ satisfies the estimate $\|\tilde{\beta}_p\|_{C^{1,\frac{7}{2}}} \leq C(T)(|\lambda| + \|w_0\|_{C^{4,\gamma}})$.

Proof. Putting $l = 8 + r$ and $n = 13 + 2r$ the operator $\mathcal{T}_p$ is of class $C^r$. To see this we note that $I^t$ and $u_p$ are of class $C^r$. We have already pointed out that $\beta \mapsto \xi_{p,\beta}$ is of class $C^{n-4} \subset C^{l-8}$. Finally, we have the following two remarks:

1. $C^{1,\frac{7}{2}}([0, T], \mathbb{R}^3) \ni \xi \mapsto \dot{\xi} \in C^0,\frac{7}{2}([0, T], \mathbb{R}^3)$ is $C^\infty$ as it is a bounded linear map.
2. As the vector field $b_i$ are $C^{n-1}$ we have that $C^{1,\frac{7}{2}}([0, T], \mathbb{R}^3) \ni \xi \mapsto b_i \circ \xi \in C^0,\frac{7}{2}([0, T], \mathbb{R}^3)$ is of class $C^{n-3} \subset C^r$.

Next we observe $\mathcal{T}_p\left[\tilde{\beta}, 0, 0\right] = \left(\langle b_i \circ \xi_{p,\tilde{\beta}}, \dot{\xi}_{p,\tilde{\beta}} \rangle \right)_{i=1,2}$. In particular $\mathcal{T}_p[0, 0, 0] = 0$. Thus $(0, 0, 0)$ is a zero of $\mathcal{T}_p$. It remains to study the Frechet differential

$$D_1 \mathcal{T}_p[0, 0, 0] \tilde{\beta} = \left. \frac{d}{de} \right|_{e=0} \mathcal{T}_p[e \tilde{\beta}, 0, 0] = \left. \frac{d}{de} \right|_{e=0} \left(\langle b_i \circ \xi_{p,e\tilde{\beta}}, \dot{\xi}_{p,e\tilde{\beta}} \rangle \right)_{i=1,2}$$

Clearly $\xi_{p,0} \equiv p$. Hence $\dot{\xi}_{p,0} \equiv 0$ and

$$D_1 \mathcal{T}_p[0, 0, 0] \tilde{\beta} = \left. \frac{d}{de} \right|_{e=0} \xi_{p,e\tilde{\beta}} = \left. \frac{d}{de} \right|_{e=0} \xi_{p,0} \left(\langle b_i(p), \frac{\partial}{\partial e} \xi_{p,0} \rangle \right)_{i=1,2} = \left. \frac{d}{de} \right|_{e=0} \left(\epsilon \tilde{\beta}^j(t) b_j(p) + \epsilon D_2 \varphi[p, \epsilon \tilde{\beta}(t)] \tilde{\beta}(t) N^S(p) \right)_{i=1,2} = \tilde{\beta}.$$ 

The last step used the fact that $(b_1(p), b_2(p))$ is an orthonormal basis of $\mathcal{T}_p S$. By the definition of the space $C^{1,\frac{7}{2}}([0, T], \mathbb{R}^2)$ we see that $D_1 \mathcal{T}_p[0, 0, 0]$ is an isomorphism into $C^0,\frac{7}{2}([0, T], \mathbb{R}^2)$ with bounded inverse

$$(D_1 \mathcal{T}_p[0, 0, 0])^{-1} \tilde{v} = \left( t \mapsto \int_0^t \tilde{v}(s) ds \right).$$

The lemma follows from the implicit function theorem.

Corollary 5.2 (Shorttime existence of the flow).
Let $T > 1$, $p \in S$, $r \geq 2$ and suppose that $\Omega \in C^{13+2r}$. There exist, $\theta(T)$, $\theta'(T)$, $\rho(T) > 0$, and $\lambda_0(T) > 0$ such that for each admissible initial value $(u_0, p) \in C^{4,\gamma}(S^+ \times S) \times S$ satisfying $\|u_0\|_{C^{4,\gamma}} \leq \theta(T)$ and $|\lambda| \leq \lambda_0(T)$ there exists exactly one pair $(u, \tilde{\beta}) \in C^{4,\gamma}([0, T] \times S^+ \times S)$
implies the existence of \( \theta'(T) \) and \( \| \tilde{\beta} \|_{C^1_T([0,T],\mathbb{R}^2)} \leq \rho(T) \).

The graph function \( u \) satisfies the estimates

\[
\|u\|_{C^{4,\gamma}} \leq C(T)(\|u_0\|_{C^4} + |\lambda|) \quad \text{and} \quad \|u(T)\|_{C^{4,\gamma}} \leq C_0 e^{-\delta T} \|u_0\|_{C^4} + C_1(T)\lambda,
\]

The vector field \( \tilde{\beta} \) satisfies the estimate \( \|\tilde{\beta}\|_{C^1_T([0,T],\mathbb{R}^2)} \leq C(T)(|\lambda| + \|u_0\|_{C^{4,\gamma}}) \).

**Proof.** Let \((u_0, p)\) be admissible. We may decompose \( u_0 = w_0 + \bar{w}_0 \). Lemma 3.1 implies \( \|u_0\|_{C^{4,\gamma}} + \|\bar{w}_0\|_{C^{4,\gamma}} \leq C \|u_0\|_{C^{4,\gamma}} \). Hence, for small enough \( \|u_0\|_{C^{4,\gamma}} \) we can ensure \( \|w_0\|_{C^{4,\gamma}} \) and \( \|\bar{w}_0\|_{C^{4,\gamma}} \) to be small enough to apply Lemma 3.2 which implies \( \bar{w}_0 = \psi_0[w_0, \tilde{\beta}^\nu] \).

Choosing \( \theta(T) \) small enough we can ensure \( \|w_0\|_{C^{4,\gamma}} \) to be small enough to apply Lemma 5.1 and derive the existence of \( \bar{\beta} \) satisfying the estimate claimed in the Corollary.

For small enough \( \lambda \) and \( \|u_0\|_{C^{4,\gamma}} \) we may apply Corollary 4.2 to derive the existence of \( u \) as well as the claimed estimates.

Using the Estimates in Corollary 5.2, it is now easy to prove the long-time existence of the flow.

**Corollary 5.3** (Long-time existence of the flow).

There exist \( \theta_0 > 0 \) and \( \lambda_0 > 0 \) such that for all \( \lambda \in [0, \lambda_0] \), \( p \in S \) and admissible initial values \( \|u_0\|_{C^{4,\gamma}} \leq \theta_0 \) there exist:

1. A curve \( \xi \in C^1([0,\infty), S) \) satisfying \( \xi(0) = p \),
2. two \( C^1([0,\infty)) \) curves \( b_1, b_2 : [0,\infty) \to TS \) such that \((b_1(t), b_2(t))\) is an orthonormal basis of \( T_{\xi(t)}S \) for all \( t \geq 0 \),
3. a graph function \( u \in C^{4,\gamma}([0,\infty) \times S^2_T) \) satisfying \( u(0) = u_0 \)

such that system (2.26) is satisfied by \((u, \xi)\) on \([0,\infty)\). The function \( u \) satisfies the decay estimate

\[
\|u\|_{C^{4,\gamma}([\tau,\infty) \times S^2_T)} \leq C(S^2_T, \gamma) \left( e^{-\alpha T} \|u_0\|_{C^{4,\gamma} + \lambda} \right)
\]

for constants \( C > 0 \) and \( \alpha > 0 \) independent of all choices involved and the barycenter curve \( \xi \) satisfies \( \|\xi\|_{C^0([0,\infty),\mathbb{R}^3)} \leq C(\lambda + \|u_0\|_{C^{4,\gamma}}) \).

**Proof.** Keeping the notation from Corollary 5.2, choose a time \( T > 1 \) so large that \( C_0 e^{-\delta T} < \frac{1}{2} \), put \( \theta_0 := \theta(T) \), fix \( \lambda_0(T) > 0 \) such that \( \lambda_0(T) C_1(T) < \frac{1}{2} \theta_0 \) and put \( \lambda_0 := \min(\lambda_0(T), \lambda_0(T)) \).

Let \( p_1 := p \) and choose a frame \((\theta_1^{(1)}, \theta_2^{(1)})\) around \( p_1 \). Corollary 5.2 implies the existence of a curve \( \xi_1 \in C^1([0,T], S) \) with \( \xi_1(0) = p_1 \) and a function \( u_1 \in C^{4,\gamma}([0,T] \times S^2_T) \) solving system (2.26) on the time interval \([0,T]\) and satisfying the estimates

\[
\|u_1(T)\|_{C^{4,\gamma}} \leq \frac{1}{2} \|u_0\|_{C^{4,\gamma}} + C_1(T)\lambda < \theta_0 \quad \text{and} \quad \|\xi_1\|_{C^2_T} \leq C(T)(\lambda + \|u_0\|_{C^{4,\gamma}}).
\]
Let \( p_2 := \xi_1(T) \) and choose a frame \( (b_1^{(2)}, b_2^{(2)}) \) around \( p_2 \) that agrees with \( (b_1^{(1)}, b_2^{(1)}) \) on a small neighbourhood of \( p_2 \). We apply 5.2 with the same choice for \( \lambda \), the point \( p_2 \) and the initial value \( u_1(T) \) to obtain a curve \( \xi_2 \in C^{1,\frac{T}{2}}([0, T], S) \) and a function \( u_2 \in C^{4,1,\gamma}([0, T] \times S^2_+) \) solving system (2.26) on the time interval \([0, T]\) and satisfying the estimates

\[
\|u_2(T)\|_{C^{4,\gamma}} \leq \frac{1}{2}\|u_1(T)\|_{C^{4,\gamma}} + C_1(T)\lambda \quad \text{and} \quad \|\dot{\xi}_2\|_{C^{\frac{T}{2}}_r} \leq C(T)(\lambda + \|u_1(T)\|_{C^{4,\gamma}}).
\]

Inserting the estimate for \( \|u_1(T)\|_{C^{4,\gamma}} \) and using the definition of \( \lambda_0 \) yields

\[
\|u_2(T)\|_{C^{4,\gamma}} \leq \frac{1}{4}\|u_0\|_{C^{4,\gamma}} + \left(1 + \frac{1}{2}\right)C_1(T)\lambda < \frac{1}{4}\theta_0 + \frac{3}{2}C_1(T)\lambda < \theta_0,
\]

\[
\|\dot{\xi}_2\|_{C^{\frac{T}{2}}_r} \leq C(T)\left(\lambda + \frac{1}{2}\|u_0\|_{C^{4,\gamma}} + C_1(T)\lambda\right) = \frac{C(T)}{2}\|u_0\|_{C^{4,\gamma}} + C(T)(1 + C(T))\lambda.
\]

Inductively we now get the existence of points \( p_n \in S \) curves \( \xi_n \in C^{1,\frac{T}{2}}([0, T], S) \) and functions \( u_n \in C^{4,1,\gamma}([0, T] \times S^2_+) \) solving system (2.26) on the time interval \([0, T]\) with initial values \( \xi_n(0) = p_n = \xi_{n-1}(T) \) and \( u_n(0) = u_{n-1}(T) \) that satisfy the estimate

\[
\|u_n(T)\|_{C^{4,\gamma}} \leq \frac{1}{2n}\|u_0\|_{C^{4,\gamma}} + C_1(T)\lambda \sum_{k=0}^{n-1} \frac{1}{2^k} < \theta_0. \tag{5.2}
\]

In the last step we have used the choice \( C_1(T)\lambda < \frac{1}{2}\theta_0 \). Applying Corollary 5.2 as well as Estimate (5.2) gives

\[
\|u_n\|_{C^{4,1,\gamma}} \leq C(T)\|u_n(0)\|_{C^{4,\gamma}} + \lambda
\]

\[
= C(T)\|u_{n-1}(T)\|_{C^{4,\gamma}} + \lambda
\]

\[
\leq C(T)\left(\frac{1}{2n-1}\|u_0\|_{C^{4,\gamma}} + C_1(T)\lambda \sum_{j=0}^{n-2} \frac{1}{2^j}\right) \tag{5.3}
\]

\[
\leq C(T)\|u_0\|_{C^{4,\gamma}} + \lambda.
\]

Finally, we get the following estimate for the barycenter curves:

\[
\|\dot{\xi}_n\|_{C^{\frac{T}{2}}_r} \leq C(T)(\lambda + \|u_{n-1}(T)\|_{C^{4,\gamma}})
\]

\[
\leq C(T)\left(\lambda + \frac{1}{2n-1}\|u_0\|_{C^{4,\gamma}} + C_1(T)\lambda \sum_{j=0}^{n-2} \frac{1}{2^j}\right)
\]

\[
\leq C(T)(\lambda + \|u_0\|_{C^{4,\gamma}})
\]

We can now ‘glue’ the solution on \([0, \infty)\) together by defining functions \( \xi : [0, \infty) \to S \) and
\[ u : [0, \infty) \times S^2_+ \to \mathbb{R} \]

\[ \xi(t) := \begin{cases} 
\xi_1(t) & \text{if } t \in [0, T), \\
\xi_2(t) & \text{if } t \in [T, 2T), \\
\xi_3(t) & \text{if } t \in [2T, 3T), 
\end{cases} \quad \text{and} \quad u(t, \omega) := \begin{cases} 
u_1(t, \omega) & \text{if } t \in [0, T), \\
u_2(t, \omega) & \text{if } t \in [T, 2T), \\
u_3(t, \omega) & \text{if } t \in [2T, 3T), 
\end{cases} \quad (5.4) \]

The fact that \( u \in C^{4,1}([0, \infty) \times S^2_+) \) and \( \xi \in C^1([0, \infty), S) \) is a direct consequence from choosing the \((n+1)-th\) frame to agree with the \(n-th\) frame in a small neighbourhood of \( p_{n+1} \). The fact that \( u \in C^{4,1,\gamma}([0, \infty) \times S^2_+) \) and \( \xi \in C^{1,\gamma}([0, \infty), S) \) is easily derived by distinguishing the cases \(|t_1 - t_2| \geq 1\) and \(|t_1 - t_2| \leq 1\). This gives

\[ \|\xi\|_{C^{0,\frac{3}{2}}([0, \infty), \mathbb{R}^3)} \leq C(T)(\|u_0\|_{C^{4,\gamma}} + \lambda) \quad \text{and} \quad \|u\|_{C^{4,1,\gamma}([0, \infty) \times S^2_+)} \leq C(T)(\|u_0\|_{C^{4,\gamma}} + \lambda). \]

Finally, we prove the decay estimate claimed in the theorem. Let \( m \geq n \in \mathbb{N} \) and use Estimate (5.3) to get

\[ \|u_m\|_{C^{4,1,\gamma}} \leq C(T) \left( \frac{2}{2^m} \|u_0\|_{C^{4,\gamma}} + \lambda \right) \leq C(T) \left( \frac{2}{2^n} \|u_0\|_{C^{4,\gamma}} + \lambda \right). \]

As this is true for all \( m \geq n \) the definition of \( u \) in Equation (5.4) implies

\[ \|u\|_{C^{4,1,\gamma}([nT, \infty) \times S^2_+)} \leq C 2^{-n} \|u_0\|_{C^{4,\gamma}} + C \lambda. \]

Now let \( \tau \geq 0 \) and choose the unique \( n \in \mathbb{N} \) such that \( nT \leq \tau < nT + T \). Then \( 2^{-n} \leq 2^{1-T^{-1}} \). Let \( \alpha := \frac{\ln 2}{T} \in (0, 1) \). Recall that we have chosen a fixed \( T > 1 \) dependent only of \( S^2_+ \) and the Hölder exponent \( \gamma \in (0, 1) \). Thus \( \alpha = \alpha(S^2_+, \gamma) \) and we derive

\[ \|u\|_{C^{4,1,\gamma}([\tau, \infty) \times S^2_+)} \leq \|u\|_{C^{4,1,\gamma}([nT, \infty) \times S^2_+)} \leq C e^{-\alpha \tau} \|u_0\|_{C^{4,\gamma}} + C \lambda. \]

\[ \square \]

Corollary 5.4 (Subconvergence).

Let \( \Omega \in C^{17}, 0 \leq \beta < \gamma < 1, \lambda_0, \theta_0, \theta_1 \) be as in Theorem 1.1, \( \lambda \leq \lambda_0, \phi_0 \in S^{4,\gamma}(\lambda, \theta_0) \) and denote the solution to the area preserving Willmore flow with initial value \( \phi_0 \) by \( \phi(t) \). Then any sequence \( t_n \uparrow \infty \) contains a subsequence \( t_{n_k} \) such that \( \phi(t_{n_k}) \to \phi_{\infty} \) in \( C^{4,\beta}(S^2_+) \) where \( \phi_{\infty} \in S^{4,\beta}(\lambda, \theta_1) \) is a critical point of the elliptic problem (2.9).

Proof. By Corollary 5.3 \( \phi(t) \) is \( C^{4,\gamma}\)-bounded uniformly over time. Hence any sequence \( t_n \uparrow \infty \)

must contain a subsequence \( t_{n_k} \) so that \( \phi(t_{n_k}) \to \phi_{\infty} \) in \( C^{4,\beta} \) for some \( \phi_{\infty} \in C^{4,\beta}(S^2_+) \).

Now let \( t_n \uparrow \infty \) such that \( \phi(t_n) \to \phi_{\infty} \) in \( C^{4,\beta} \). We prove \( P_H^t(W(\phi_{\infty})) = 0 \) by contradiction. Else we may assume that for \( \|\phi - \phi_{\infty}\|_{C^{4,\beta}} \leq 2\delta_0 \) we have \( \rho_0 \leq \|P_H^t(W(\phi))\|_{C^{\gamma,\beta}} \leq 2\rho_0 \) for some \( \delta_0 > 0 \) and \( \rho_0 > 0 \). Clearly \( \phi(t) \) cannot be \( 2\delta_0 \)-close to \( \phi_{\infty} \) for all large times as this would imply

\[ W(\phi(t)) = W(\phi(t_0)) + \int_{t_0}^t |P_H^s(W(\phi(s))|^2 ds \leq W(\phi(t_0)) - \rho_0^2(t - t_0) \to -\infty \]
for some fixed $t_0$ and $t \to \infty$. As $\phi(t_n) \to \phi_\infty$ we may however assume that $\|\phi(t_n) - \phi_*\|_{C^4, \beta} \leq \delta_0$ and choose sequences $\sigma_n$ and $\tau_n$ satisfying $\sigma_n < \tau_n < \sigma_{n+1}$ such that $\|\phi(\sigma_n) - \phi_*\|_{C^4, \beta} = \delta_0$, $\|\phi(\tau_n) - \phi_*\|_{C^4, \beta} = 2\delta_0$ and $\|\phi(t) - \phi_*\|_{C^4, \beta} \leq 2\delta_0$ for all $t \in [\sigma_n, \tau_n]$. Then

$$\|\phi(\tau_n) - \phi(\sigma_n)\|_{C^0, \beta} \leq \int_{\sigma_n}^{\tau_n} \|\phi'(s)\|_{C^0, \beta} ds \leq \int_{\sigma_n}^{\tau_n} \|P_HW(\phi(s))\|_{C_{0, \beta}} ds \leq 2\rho_0(\tau_n - \sigma_n).$$

Note that by Theorem 1.1 we may bound $\|\phi(t)\|_{C^4, \gamma} \leq K$ independent of $t$. For any $\mu > 0$ we may use Ehrlings Lemma to get

$$\delta_0 \leq \|\phi(\tau_n) - \phi(\sigma_n)\|_{C^4, \beta} \leq \mu \|\phi(\tau_n) - \phi(\sigma_n)\|_{C^4, \gamma} + C(\mu) \|\phi(\tau_n) - \phi(\sigma_n)\|_{C^0, \beta}

\leq 2K\mu + C(\mu)2\rho_0(\tau_n - \sigma_n).$$

Choosing $\mu = \frac{\delta_0}{2(1+K)}$ gives $\tau_n - \sigma_n \geq \kappa_0(\delta_0, K) > 0$. Since $W(\phi(t))$ decreases we deduce

$$W(\phi(\sigma_{n+1})) \leq W(\phi(\sigma_n)) - \int_{\sigma_n}^{\tau_n} |P_HW(\phi(s))|^2 \leq W(\phi(t_n)) - \rho_0^2 \kappa_0.$$

Iterating gives $W(\phi(\tau_n)) \to -\infty$ for $n \to \infty$ which is a contradiction. \hfill $\square$

6 Proof of Theorem 2

6.1 Proof of Theorem 2i)

The key-observation to prove Theorem 1.2i) is the following Lemma:

Lemma 6.1. Let $p_0 \in S$, put $q_0 = \frac{\partial \tilde{g}_{p_0, \lambda}}{\partial \lambda}\big|_{\lambda=0}$ and let $u_0 \in C^{4, \gamma}(S^2_+)$ be even.

1. For small $\epsilon, \mu > 0$ the following functions are even: $B_0[\epsilon u_0, \delta + \mu q_0], W[\epsilon u_0, \delta + \mu q_0], H[\epsilon u_0, \delta + \mu q_0]$.

2. For small $\epsilon, \mu > 0$ and $i = 1, 2$ the function $\nabla C^i[\epsilon u_0, \delta + \mu q_0]$ is odd.

3. $D_2 C[0, \delta]q_0 = 0$ and $D_2 \psi_0[0, \delta]q_0$ (from Lemma 3.2) is even.

If $T > 1$ and $\alpha \in C^1(\hat{\mathbb{R}}^3)$ put $q := \frac{\partial \tilde{g}_{p_0, \lambda}}{\partial \alpha}\big|_{\lambda=0}$. Then $D_2 \psi_T[0, \delta]q$ is even.

Proof. Let $h_{ij}$ denote the second fundamental form of $S$ in the chart $f[p_0, \cdot]$. Recalling Equation (2.2) we get

$$q_0 = \frac{\partial \tilde{g}_{p_0, \lambda}}{\partial \lambda}\big|_{\lambda=0} = \begin{bmatrix} 0 & 0 & h_{1a}x^a \\ 0 & 0 & h_{2a}x^a \\ h_{1a}x^a & h_{2a}x^a & 0 \end{bmatrix}$$

and put $T := \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$.

It is easy to check that $(T^*q_0)(x) = q_0(x)$. For any $u_0 \in C^{4, \gamma}(S^2_+)$ and $\tilde{g}_0 \in C^1(\hat{\mathbb{R}}^3)$ we have $O[u_0, T^*\tilde{g}_0] = O[u_0 \circ T, \tilde{g}_0] \circ T$ for $O \in \{ B_0, W \}$, $A[u_0, T^*\tilde{g}_0] = A[u_0 \circ T, \tilde{g}_0]$ and $C[u_0, T^*\tilde{g}_0] = \cdots$. 

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$-C[u_0 \circ T, \xi_0]$. See e.g. Lemma 10 in [2]. These relations readily imply the first two statements. Exploiting the parity of the barycenter we get

$$D_2 C[0, \delta]q_0 = \frac{d}{d\mu} \bigg|_{\mu=0} C[0, \delta + \mu q_0] = -\frac{d}{d\mu} \bigg|_{\mu=0} C[0 \circ T, \delta + \mu q_0] = -D_2 C[0, \delta]q_0.$$  

Hence the first part of the third statement is established. For the second part let $\tilde{w}_0 := D_2 \psi_0[0, \delta]q_0$. Then by construction of $\psi_0$ in Lemma 3.2 and the first part of this Lemma

$$\left( \frac{\partial \tilde{w}_0}{\partial \eta}, \frac{\partial}{\partial \eta} (\Delta + 2) \tilde{w}_0 \right) = -D_2 B_0[0, \delta] \frac{\partial \tilde{q}_0^{\rho, \lambda}}{\partial \lambda} \bigg|_{\lambda=0} = \frac{d}{d\mu} \bigg|_{\mu=0} B_0[0, \delta + \mu q_0] = \text{even}.$$  

By definition of the space $Y_0$ we deduce that $\tilde{w}_0$ is the unique solution to a problem of the form

$$\Delta^2 \tilde{w}_0 \equiv \text{const.}, \quad \partial_\eta \tilde{w}_0, \partial_\eta \Delta \tilde{w}_0 = \text{even} \quad \text{and} \quad \int_{S^2} \tilde{w}_0 d\mu_{g_\Sigma} = 0$$

which implies that $\tilde{w}_0$ is also even. A similar argument proves the claimed parity statement of $D_2 \psi_T[0, \delta]q$.

Let $p \in S$, $u_0 \in C^{4, \gamma}(S^2_+) \text{ and } \lambda > 0 \text{ be as in Corollary 5.3. Let } u \in C^{4,1, \gamma}([0, \infty) \times S^2_+) \text{ and } \xi \in C^{1, \gamma}([0, \infty), S) \text{ denote the unique solutions discussed in Theorem 5.3. The time-dependent metric } g^{\xi, \lambda} \text{ is now defined for all } t \in [0, \infty). \text{ By the decay estimate provided in Theorem 5.3 we deduce that there exists a time } T_0(\lambda) > 0 \text{ such that } \|u\|_{C^{4,1, \gamma}([T_0, \infty) \times S^2_+)} \leq M_0 \lambda. \text{ Thus, without loss of generality, we may assume}

$$\|u\|_{C^{4,1, \gamma}([0, \infty) \times S^2_+)} \leq M_0 \lambda. \tag{6.1}$$

for the proof of Theorem 1.2i). Given an arbitrary $T > 1$ the graph function $u$ satisfies the Equation

$$\begin{align*}
Q_T[u, u_0, g^{\xi, \lambda}, f] &= 0, \\
B_T[u, g^{\xi, \lambda}] &= 0, \\
u(0) &= u_0,
\end{align*} \tag{6.2}$$

where $Q_T$ was defined in Equation (3.6) and $f$ is chosen as

$$f = P_T[u, \tilde{g}] P_{K, u, \tilde{g}}^+ \left[ \tilde{g} \left( \lambda \left( D_2 F^{\lambda} b_1(\xi), \tilde{F}_\lambda^\dagger \right) D_1 F^{\lambda} b_1(\xi), \tilde{F}_\lambda^\dagger \right) \right]$$

as we described in Equation (4.4). We recall Estimates (4.13) and (4.14):

$$\|P_{K, [0, \delta]}^+ f\|_{C^{0,0, \gamma}_T} + \|f\|^2_{C^{0,0, \gamma}_T} \leq C(T) \left( \|u\|^2_{C^{4,1, \gamma}_T} + \|\tilde{g} - \delta\|^2_{C^{4,1, \gamma}_T} + \lambda^2 \right) \leq C(T, M_0) \lambda^2 \tag{6.3}$$

In the last step we have used (6.1). The idea of the proceeding analysis is the following: We expand the nonlinear equation (6.2) to first order in $u$ and $\lambda$. Making use of Lemma 6.1 and
Estimate (6.1) we find
\[
\begin{align*}
\dot{u} + \frac{1}{2} \Delta (\Delta + 2) u &= \text{even} \cdot \lambda + \mathcal{O}(\lambda^2), \\
\frac{\partial u}{\partial \eta} - \frac{\partial}{\partial \eta} (\Delta + 2) u &= \text{even} \cdot \lambda + \mathcal{O}(\lambda^2), \\
u(0) &= u_0,
\end{align*}
\]
where ‘even’ = \( \mathcal{O}(\lambda^0) \). This allows the derivation of the improved decay estimate
\[
\| u^{-}(T) \|_{C^{4,\gamma}} \leq C e^{-6T} \| u_0^{-} \|_{C^{4,\gamma}} + C(T) \lambda^2. \tag{6.4}
\]
Once this is established we may follow the \textit{gluing together} strategy demonstrated in the proof of Corollary 5.3 to deduce
\[
\| u^{-} \|_{C^{4,1,\gamma}(\mathbb{R} \times \mathbb{S}^2_+)} \leq C \left( e^{-\alpha \tau} \| u^{-}(0) \|_{C^{4,\gamma}} + \lambda^2 \right)
\]
which implies Theorem 1.2i).

Finally, recall the direct decompositions discussed in Lemma 3.1. Defining the spaces
\[
X_0^\parallel := \text{span}_\mathbb{R} \{ 1, \omega^1, \omega^2 \} \quad \text{and} \quad X_0^\perp := X_0 \cap \left( X_0^\parallel \right)^\perp.
\]
we may refine the decompositions from Lemma 3.1 by writing
\[
C^{4,\gamma}(\mathbb{S}^2_+) = X_0^\parallel \oplus X_0^\perp \oplus Y_0 \quad \text{and} \quad C^{4,1,\gamma}([0, T] \times \mathbb{S}^2_+) = X_T^\parallel \oplus X_T^\perp \oplus Y_T
\]
where for \( * = \parallel, \perp \) we introduced \( X_T^* := \{ u \in X_T \mid u(t) \in X_0 \forall t \in [0, T] \} \). We denote the projections from \( C^{4,\gamma}(\mathbb{S}^2_+) \) / \( C^{4,1,\gamma}([0, T] \times \mathbb{S}^2_+) \) onto the direct summands by \( \pi_{X_T^\parallel}, \pi_{X_T^\perp}, \pi_{X_T^\parallel} \) and \( \pi_{Y_T} \) respectively. It is readily checked that all these projections preserve parity and hence e.g. \( \pi_{X_T^\parallel} u^{-} = (\pi_{X_T^\parallel} u)^{-} \).

\textbf{Lemma 6.2.} Let \( \Omega \in \mathcal{C}^{13+2r} \) with \( r \geq 2 \) and \( T > 1 \). There exists \( \lambda_0(T) > 0 \) such that for \( \lambda \in [0, \lambda_0] \) and a solution \( (u, \xi) \) from 5.3 satisfying (6.1) we have
\[
\begin{align*}
\| u^{-} \|_{C_T^{4,\gamma}} &\leq C(T, M_0) \left( \| u_0^{-} \|_{C^{4,\gamma}} + \lambda^2 \right) \\
\| u^{-}(T) \|_{C_T^{4,\gamma}} &\leq C(M_0) e^{-6T} \| u_0^{-} \|_{C^{4,\gamma}} + C(T, M_0) \lambda^2.
\end{align*}
\]
\textbf{Proof.} Put \( t := 8 + r \). The operator \( Q_T \) from (3.6) is of class \( \mathcal{C}^{4-8} = \mathcal{C}^r \) and the map \( \lambda \to \tilde{g}^{\xi, \lambda} \in G_T^{8+r} \) is of class \( \mathcal{C}^r \). Putting \( q := \frac{\partial \tilde{g}^{\xi, \lambda}}{\partial \lambda} \big|_{\lambda=0}, q_0 := q(0) \) and expanding Equation (6.2) around \( (u, u_0, \lambda, f) = (0, 0, 0, 0) \) then gives
\[
\begin{align*}
\dot{u} + \frac{1}{2} \Delta (\Delta + 2) u &= \sum_{\mu=0}^{2} \frac{D_2 C^\mu[0, \delta]}{\| \nabla C^\mu[0, \delta] \|_{L^2(\mathbb{S}^2_+)}^2} \nabla C^\mu[0, \delta] \\
&\quad + P_k^{\parallel}[0, \delta] \left[ D_2 W[0, \delta] q - f \right] + R_1, \\
\left( \frac{\partial u}{\partial \eta} - \frac{\partial}{\partial \eta} (\Delta + 2) u \right) &= - D_2 B_T[0, \delta] q + R_2, \\
u(0) &= u_0,
\end{align*}
\]
where \( \pi_{X_T^\parallel} u^{-} = (\pi_{X_T^\parallel} u)^{-} \).
Using $Q_T \in C^2$ as well as Estimates (6.1) and (6.3) we get

$$\|R\|_{C_T^{0,0,\gamma}} \leq C(T) \left[ \|u\|^2_{C_T^{4,1,\gamma}} + \lambda^2 + \|f\|^2_{C_T^{2,0,\gamma}} \right] \leq C(T)\lambda^2.$$ 

We apply the projections $\pi_{X_T^\perp}$ and $\pi_{X_T}$ to Equation (6.6). Using Lemma 6.1 we may infer that $v^- := \pi_{X_T^\perp}(u^-)$ and $\tilde{v}^- := \pi_{X_T}(u^-)$ satisfy the following equations:

$$\begin{align*}
\dot{v}^- + \frac{1}{2}\Delta(\Delta + 2)v^- &= -P_{K[0,\delta]}[f]^- + \pi_{X_T^\perp}(R_1), \\
\frac{\partial v^-}{\partial \eta} &= 0, \\
v^- &= \pi_{X_T^\perp}(u_0).
\end{align*}$$

and

$$\begin{align*}
\dot{\tilde{v}}^- + \frac{1}{2}\Delta(\Delta + 2)\tilde{v}^- &= \pi_{X_T}(R^-), \\
\frac{\partial \tilde{v}^-}{\partial \eta} &= 0, \\
\tilde{v}^- &= \pi_{X_T}(u_0).
\end{align*}$$

Applying the improved Schauder estimates from Appendix C.2 to the first equation and regular Schauder estimates (see e.g. [7, 17]) to the second one immediately gives

$$\|v^-\|_{C_T^{4,1,\gamma}} \leq C(T) \left[ \|u_0\|_{C_T^{4,\gamma}} + \lambda^2 \right], \quad \|v^-\|_{C_T^{4,\gamma}} \leq C e^{-6T} \|u_0\|_{C_T^{4,\gamma}} + C(T)\lambda^2 \quad (6.7)$$

as well as

$$\|\tilde{v}^-\|_{C_T^{4,1,\gamma}} \leq C(T) \|\pi_{X_0}\|_{C_T^{4,\gamma}} + \lambda^2 \|u_0\|_{C_T^{4,\gamma}} \quad (6.8)$$

By definition $\tilde{v}_0^- \in \text{span}_\mathbb{R} \{\omega^1, \omega^2\}$. Using $C^i \in C^2$, $C[u_0, \tilde{g}^{p,\lambda}] = 0$, $\nabla C^i[0,\delta] = -\frac{3}{\pi} \omega^i$ and Lemma 6.1 we get

$$\|\tilde{v}_0^-\|_{C_T^{4,\gamma}} \leq C \max_{i=1,2} \left| \int_{S^2_+} \omega^i u_0 \right| \leq C \left| C[u_0, \tilde{g}^{p,\lambda}] - D_2 C[0,\delta]q_0 \right| + C(\|u_0\|_{C_T^{4,\gamma}}^2 + \lambda^2) \leq C\lambda^2 \quad (6.9)$$

Let $w^- := v^- + \tilde{v}^- = \pi_{X_T}(u^-)$ and $\tilde{w}^- := \pi_{Y_T}(u^-)$. Combining Estimates (6.7), (6.8) and (6.9) we derive

$$\|w^-\|_{C_T^{4,1,\gamma}} \leq C(T) \|u_0\|_{C_T^{4,\gamma}} + \lambda^2 \quad \text{and} \quad \|w^-\|_{C_T^{4,\gamma}} \leq C e^{-6T} \|u_0\|_{C_T^{4,\gamma}} + C(T)\lambda^2. \quad (6.10)$$

Finally, we use $\tilde{w}^- = \psi_T[w, \tilde{g}^{p,\lambda}]$, $D_1 \psi_T[0,\delta] = 0$, the parity of $D_2 \psi_T[0,\delta]q$ derived in Lemma 6.1 and Estimates (6.1) and (6.10) to get

$$\|\tilde{w}^-\|_{C_T^{4,1,\gamma}} \leq C(T)(\|w\|_{C_T^{4,1,\gamma}}^2 + \lambda^2) \leq C(T)\lambda^2 \quad (6.11)$$

which, when combined with Estimate (6.10), implies the Lemma as $u^- = w^- + \tilde{w}^-$. \hfill \Box

As noted above, Lemma 6.2 implies Theorem 1.2i).
6.2 Proof of Theorem 2ii)

Recall that for a map \( v : S^2_+ \to \mathbb{R} \) we denote the even part by \( v^+ \) and the odd part by \( v^- \). We follow an idea from Mattuschka [14]. For \( p \in S, \lambda > 0 \) and \( M > 0 \) put

\[
\mathcal{A}_{p,\lambda}^M := \left\{ u_0 \in C^{4,\gamma}(S^2_+) \mid (u_0, p) \text{ is admissible, } \|u_0\|_{C^{4,\gamma}} \leq \lambda M \text{ and } \|u_0\|_{C^{4,\gamma}} \leq \lambda M^2 \right\}.
\]

Note that for \( u \in \mathcal{A}_{p,\lambda}^M \) the immersion \( \phi_{p,u}^\lambda \) (recall Subsection 2.1) satisfies \( \phi_{p,u}^\lambda \in S^-(\lambda, M) \).

Let \( (u, \xi) \) denote a solution as it is described in 5.3 and denote the basis vectors along \( \xi \) by \( b_i : [0, \infty) \to TS \). By assumption we have \( u(t) \in \mathcal{A}_{\xi(t),\lambda}^M \) for all \( t \geq 0 \). The strategy to prove Theorem 1.2ii) is to define a family of maps

\[
\hat{u}_t : [0, \lambda] \to C^{4,\gamma}(S^2_+) \quad \text{such that } \hat{u}_t[\lambda] = u(t), \hat{u}_t[0] = 0 \text{ and } \hat{u}_t[s] \in \mathcal{A}_{\xi(t),s}^{C(M)}.
\]

Once this is established we may use the evolution equation \((6.10)\) to get the Equation

\[
\langle \dot{\xi}(t), b_i(t) \rangle = -\lambda \Gamma[u(t), \tilde{g}(\xi(t),\lambda)] = -\lambda \Gamma[\hat{u}_t[\lambda], \tilde{g}(\xi(t),\lambda)],
\]

expand the right hand side in \( \lambda \) and obtain the Theorem. The first step towards the construction of a suitable map is the following lemma.

**Lemma 6.3.** Let \( l \geq 8 \). There exist \( \theta_0, \theta_1, \eta_0 > 0 \) and \( C^{l-7} \) maps \( w_0^l : X_0^\perp(\theta_0) \times G_0^l(\delta; \sigma_0) \to X_0^\perp(\theta_1) \) and \( \tilde{w}_0 : X_0^\perp(\theta_0) \times G_0^l(\delta; \sigma_0) \to Y_0(\theta_2) \) such that for all \( (w_0^l, \tilde{w}_0, \tilde{g}_0) \in X_0^\perp(\theta_0) \times X_0^\perp(\theta_1) \times Y_0(\theta_2) \times G_0^l(\delta; \sigma_0) \) we have

\[
B_0[w_0^l + \tilde{w}_0, \tilde{g}_0] = 0
\]

\[
C[w_0^l + \tilde{w}_0, \tilde{g}_0] = 0 \iff w_0^l = w_0^l[w_0^l, \tilde{g}_0] \text{ and } \tilde{w}_0 = \tilde{w}_0[w_0^l, \tilde{g}_0].
\]

\[
A[w_0^l + \tilde{w}_0, \tilde{g}_0] = 2\pi
\]

The maps satisfy \( D_1 w_0^l[0, \delta] = 0 \) and \( D_1 \tilde{w}_0[0, \delta] = 0 \).

**Proof.** From Lemma 3.2 we immediately find that \( \tilde{w}_0 = w_0^l + w_0^l + \psi_0[w_0^l + \tilde{w}_0, \tilde{g}_0] \) must be true to satisfy \( B_0[w_0^l + \tilde{w}_0, \tilde{g}_0] = 0 \). On a neighbourhood of \( (0, 0, \delta) \) we now define a map

\[
F : X_0^\perp \times X_0^\parallel \times G_0^l(\delta; \sigma_0) \to \mathbb{R}^3 \text{ by }
\]

\[
(w_0^l, w_0^l, \tilde{g}_0) \mapsto \left( A \left[ w_0^l + w_0^l + \psi_0[w_0^l + w_0^l, \tilde{g}_0], \tilde{g}_0 \right] - 2\pi, C \left[ w_0^l + w_0^l + \psi_0[w_0^l + w_0^l, \tilde{g}_0], \tilde{g}_0 \right] \right).
\]

As \( \psi_0, A \) and \( C \) are of class \( C^{l-7} \) we have \( F \in C^{l-7} \). Clearly \( F[0, 0, \delta] = 0 \). Additionally

\[
D_2 F[0, 0, \delta] \phi = \left( 2 \int_{S_+^l} \varphi d\mu_{S^2}, -\frac{3}{2\pi} \int_{S_+^l} \varphi \omega^1 d\mu_{S^2}, -\frac{3}{2\pi} \int_{S_+^l} \varphi \omega^2 d\mu_{S^2} \right).
\]
The implicit function theorem implies the existence of the neighbourhoods and the map $w_0$ as well as its regularity. To establish $D_1w_0[0,\delta] = 0$ we use $D_1\psi_0[0,\delta] = 0$ and argue as in the proof of Lemma 3.2. $D_1\tilde{w}_0[0,\delta] = 0$ then follows from

$$
\tilde{w}_0[w_0,\delta] = \psi_0[w_0 + w_0[w_0,\delta]],
$$

(6.12)

We need the following properties of the maps $w_0$ and $\tilde{w}_0$ that we just constructed.

**Lemma 6.4.** Let $\Omega \in C^{13+2r}$ so that $\lambda \mapsto \tilde{g}^{p,\lambda} \in C^{8+r}$ is a map of class $C^r$. Let $M > 0$: There exists $\lambda_0(M) > 0$ such that for $\lambda \in [0,\lambda_0(M)]$:

1. The following are even functions:

$$
D_2w_0^\|0,\delta] \frac{\partial \tilde{g}^{p,\lambda}}{\partial \lambda} \bigg|_{\lambda=0} \quad \text{and} \quad D_2\tilde{w}_0[0,\delta] \frac{\partial \tilde{g}^{p,\lambda}}{\partial \lambda} \bigg|_{\lambda=0}
$$

2. For $(w_0^\|,p) \in X_0 \times S$ satisfying $\|w_0^\| \|_{C^4,\gamma} \leq M\lambda$ put $u_0 := w_0^\| + w \| [w_0^\|,\tilde{g}^{p,\lambda}] + \tilde{w}[w_0^\|,\tilde{g}^{p,\lambda}]$. Then $\|u_0 - (w_0^\|)^-\| \leq C(M)\lambda^2$.

3. For admissible $(u_0,p) \in C^4,\gamma(S^2_+) \times S$ satisfying $\|u_0\|_{C^4,\gamma} \leq M\lambda$ and $w_0^\| := \pi X_0^\| (u_0)$ we have $\|u_0^+ - (w_0^\|)^-\| \leq C(M)\lambda^2$.

**Proof.** We exploit $D_2C^4[0,\delta] \frac{\partial \tilde{g}^{p,\lambda}}{\partial \lambda} \bigg|_{\lambda=0} = 0$ from Lemma 6.1. Using $D_2\tilde{w}_0[0,\delta] = D_2\psi_0[0,\delta]$ from Equation (6.12) and the parity of $D_2\psi_0[0,\delta] \frac{\partial \tilde{g}^{p,\lambda}}{\partial \lambda} \bigg|_{\lambda=0}$ from Lemma 6.1 we deduce the second claimed parity and can compute

$$
0 = \frac{d}{d\lambda} \bigg|_{\lambda=0} C^4[w_0^\|,\tilde{g}^{p,\lambda}] + \tilde{w}_0[0,\tilde{g}^{p,\lambda}], \tilde{g}^{p,\lambda}]
$$

$$
= -\frac{3}{2\pi} \int_{S^2_+} \omega^2 D_2w_0^\|[0,\delta] \frac{\partial \tilde{g}^{p,\lambda}}{\partial \lambda} \bigg|_{\lambda=0} \ d\mu_{S^2} - \frac{3}{2\pi} \int_{S^2_+} \omega^2 D_2\tilde{w}_0[0,\delta] \frac{\partial \tilde{g}^{p,\lambda}}{\partial \lambda} \bigg|_{\lambda=0} \ d\mu_{S^2}
$$

(6.13)

As $D_2w_0^\|0,\delta] \frac{\partial \tilde{g}^{p,\lambda}}{\partial \lambda} \bigg|_{\lambda=0} \in X_0^\|$ = span $\{1,\omega^1,\omega^2\}$ Equation (6.13) implies the claimed parity. For the second part we use Lemma 6.3 and part 1 of this Lemma to estimate

$$
\|u_0^+ - (w_0^\|)^-\|_{C^4,\gamma} \leq \|w_0^\| [w_0^\|,\tilde{g}^{p,\lambda}]^\| \|_{C^4,\gamma} + \|\tilde{w}_0[w_0^\|,\tilde{g}^{p,\lambda}]^-\|_{C^4,\gamma}
$$

$$
\leq C \left( \|w_0^\|_{C^4,\gamma}^2 + \lambda^2 \right)
$$

$$
\leq C(M)\lambda^2.
$$

The final statement follows from the second by noting that admissibility implies $u_0 = w_0^\| + w_0^\| [w_0^\|,\tilde{g}^{p,\lambda}] + \tilde{w}_0[w_0^\|,\tilde{g}^{p,\lambda}]$ and that the continuity of the projection $u_0 \mapsto w_0^\|$ gives the necessary estimate.

\[\square\]
Lemma 6.5. Let \( \Omega \in C^{13+2r} \) with \( r \geq 2 \) and \( M > 0 \). There exists \( \lambda_0(M) > 0 \) such that for all \( p \in S \), \( \lambda \in [0, \lambda_0(M)] \) and \( u_0 \in \mathcal{A}_{p,\lambda}^M \) there exists a map \( \hat{u}_0 : [0, \lambda] \to C^{4,\gamma}(\mathcal{S}_4^2) \) of class \( C^r \) that has the following properties:

1. \( \hat{u}_0(0) = 0 \) and \( \hat{u}(\lambda) = u \). Additionally \( \hat{u}_0(s) \in \mathcal{A}_{p,s}^{C(M)} \) for all \( s \in [0, \lambda] \).

2. \( \hat{u}_0(0) \) is an even function.

3. \( \|\hat{u}_0\|_{C^r([0,\lambda], C^{4,\gamma}(\mathcal{S}_4^2))} \leq C(M, \Omega, r) \).

Proof. Using the direct decomposition (6.5) we may write \( u_0 = w_0^+ + w_0^- + \tilde{w}_0 \) for unique \( w_0^+ \in X_0^+ \), \( w_0^- \in X_0^- \) and \( \tilde{w}_0 \in Y_0 \). As \( \|u_0\|_{C^{4,\gamma}} \leq M\lambda \) we may conclude \( \|w_0^+_0\|_{C^{4,\gamma}} + \|w_0^-\|_{C^{4,\gamma}} + \|\tilde{w}_0\|_{C^{4,\gamma}} \leq C(M)\lambda \) by using continuity of the projections. Since \( u_0 \) is admissible we may conclude that for small enough \( \lambda_0(M) \) and \( \lambda \leq \lambda_0(M) \) we have \( w_0^0 = w_0^0[w_0^+, \tilde{g}^{p,\lambda}] \) and \( \tilde{w}_0 = \tilde{w}_0[w_0^+, \tilde{g}^{p,\lambda}] \). Additionally, Lemma 6.4 implies \( \|w_0^\perp\|_{C^{4,\gamma}} \leq C(M)\lambda^2 \). Combining these estimates we derive that the \( C^\infty \)-curve

\[
\hat{w}_0^+ : [0, \lambda] \to C^{4,\gamma}(\mathcal{S}_4^2), \quad \hat{w}_0(s) := s \lambda \left( w_0^+ \right) + s^2 \lambda^2 \left( w_0^- \right)
\]

has all derivatives with respect to \( s \) bounded by a constant \( C(M) \) independent of \( \lambda \). Also note that \( \|\hat{w}_0^+(s)\|_{C^{4,\gamma}} \leq \|w_0^+\|_{C^{4,\gamma}} \leq C(M)\lambda \). Hence, for \( \lambda \) small enough, we may define the curve

\[
\hat{u}_0 : [0, \lambda] \to C^{4,\gamma}(\mathcal{S}_4^2), \quad \hat{u}_0(s) := \hat{w}_0^+(s) + w_0^\perp[\hat{w}_0^+(s), \tilde{g}^{p,\lambda}] + \tilde{w}_0[\hat{w}_0^+(s), \tilde{g}^{p,\lambda}].
\]

Note that for \( \Omega \in C^{13+2r} \) the map \( s \mapsto \tilde{g}^{p,\lambda} \in G^S_0 \) is of class \( C^r \) and that \( w_0^\perp \) and \( \tilde{w}_0 \) are both maps of class \( C^r \). Hence \( \hat{u}_0 \in C^r([0, \lambda], C^{4,\gamma}(\mathcal{S}_4^2)) \) and

\[
\|\hat{u}_0\|_{C^r([0, \lambda], C^{4,\gamma}(\mathcal{S}_4^2))} \leq C(M, \Omega, r)
\]

(6.14)

independent of \( \lambda \) as this is the case for the curve \( s \mapsto \hat{w}_0^+(s) \). The facts \( \hat{u}_0(0) = 0 \) and \( \hat{u}_0(\lambda) = u_0 \) are trivial. Also note that by construction \( \hat{u}_0(s) \in \mathcal{A}_{s}^{C(M)} \). Indeed this follows from the last statement of Lemma 6.4. Finally, \( \hat{u}_0'(0) \) is even due to Lemma 6.4 as

\[
\hat{u}_0'(0) = \frac{1}{\lambda} \left( w_0^+ \right) + D_2 w_0^\perp[0, \delta] \frac{\partial \tilde{g}^{p,\lambda}}{\partial s} \bigg|_{s=0} + D_2 \tilde{w}_0[0, \delta] \frac{\partial \tilde{g}^{p,\lambda}}{\partial s} \bigg|_{s=0}.
\]

\( \square \)

Using the map \( \hat{u}_0 \) constructed in Lemma 6.5 we may now prove the second part of Theorem 1.2.

Theorem 6.6. Let \( \Omega \in C^{21} \) and \( M > 0 \). There exists \( \lambda_0(M) > 0 \) such that any solution \((u, \xi)\) to (2.26) with \( u(t) \in \mathcal{A}_{\xi(t),\lambda}^M \) for all times \( t \geq 0 \) satisfies

\[
\sup_{t \geq 0} \left| \dot{\xi}(t) - \frac{3}{2} \lambda^3 \nabla H^S(\xi(t)) \right| \leq C(M)\lambda^4.
\]

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Proof. By assumption $\Omega \in C^{13+2r}$ with $r := 4$. For $t \geq 0$ let $u_t(\omega) := u(t, \omega)$. Then, for any $t \geq 0$, we may define the map $\hat{u}_t : [0, \lambda] \to C^{4, \gamma}(S^2_+)$ as in Lemma 6.5 and get $\hat{u}_t(s) \in A^{C(M)}_{\xi(s), \lambda}$ for all $t \geq 0$. Now pick $t \geq 0$ and use Equation (2.26) to write

$$\langle \dot{\xi}(t), b_i(t) \rangle = \langle -\lambda I^i[u(t), \tilde{g}^{(t), \lambda}], -s I^i[\hat{u}_t(s), \tilde{g}^{(t), s}] \rangle\bigg|_{s=\lambda}. $$

In the last step we have used $\hat{u}_t(\lambda) = u_t = u(t)$, $\hat{u}_t(0) = 0$ and $I^i[0, \delta] = 0$. Note:

1. $I^i : C^{4, \gamma}(S^2_+) \times G_0^{8+r} \to \mathbb{R}$ is of class $C^r$ on a small neighbourhood of $(0, \delta)$ as stated in Section 4.

2. $\hat{u}_t : [0, \lambda] \to C^{4, \gamma}(S^2_+)$ is of class $C^r$ by Lemma 6.5.

3. $s \mapsto \tilde{g}^{p, s} \in G_0^{8+r}$ is of class $C^r$ as we discussed in the proof of Corollary 4.2.

Hence $y_{t,i} : [0, \lambda] \to \mathbb{R}$, $y_{t,i}(s) := -s I^i[\hat{u}_t(s), \tilde{g}^{(t), s}]$ is of class $C^r \subset C^4$ and thus

$$y_{t,i}(\lambda) = y_{t,i}(0) + y_{t,i}''(0)\lambda + \frac{1}{2}y_{t,i}'''(0)\lambda^2 + \frac{1}{6}y_{t,i}''''(0)\lambda^3 + \frac{\lambda^4}{6} \int_0^1 y_{t,i}^{(4)}(\lambda \rho)(1 - \rho)^3 d\rho. $$

The last term can be bounded by $C(M)\lambda^4$ using Lemma 6.5. A direct computation in Appendix A shows $y_{t,i}(0) = y_{t,i}''(0) = y_{t,i}'''(0) = 0$ and $y_{t,i}''''(0) = 9\partial_i H^S(\xi(t))$ with $\partial_i$ taken in the chart $f[\xi(t), \cdot]$. Thus for all $t \geq 0$

$$\left|\langle \dot{\xi}(t), b_i(\xi(t)) \rangle - \frac{3}{2} \lambda^3 \partial_i H^S(\xi(t)) \right| \leq C(M)\lambda^4. $$

This is Theorem 1.2ii). Just recall that we rescaled time by a factor of $\lambda^4$ right before Equation (2.24). 

7 Prove of Theorem 3

We need the following uniqueness result of Alessandroni and Kuwert from [2], Theorem 3:

**Theorem 7.1.** Let $\Omega \in C^{12}$ and $q \in S$ be a nondegenerate critical point of $H^S$. There exists $\lambda_0 > 0$, a neighbourhood $U$ of $q$ and a $C^1$ curve $\gamma : [0, \lambda_0] \to U$ satisfying $\gamma(0) = q$ such that for each $\lambda \in [0, \lambda_0)$ there exists a unique solution $\phi_\lambda \in M_{\lambda}^{4, \gamma}(S)$ of (2.9) with area $2\pi \lambda^2$ and barycenter $C[\phi_\lambda] \in U$. The barycenter satisfies $C[\phi_\lambda] = \gamma(\lambda)$.

To deduce the convergence of the flow, we first prove that under the assumptions of Theorem 1.3 any barycenter curve must stabilise in a small neighbourhood of a critical point of $H^S$. For that purpose we introduce $U(a, r) := B_r(a) \cap S$ for $a \in S$ and $r > 0$. 

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Lemma 7.2. Let \( \Omega \in C^{23} \) and assume that \( H^S \) only has finitely many critical points \( p_1, \ldots, p_m \) all of which are nondegenerate. Then for all \( \epsilon_0 > 0 \) small enough, there exists \( \lambda_0(\epsilon_0) > 0 \) such that the following is true:

Let \( \lambda \leq \lambda_0, \phi_0 \in S(\lambda, \theta_0) \), denote the solution to the area preserving Willmore flow with initial value \( \phi_0 \) by \( \phi(t) \) and its barycenter curve by \( \xi(t) \). There exists a time \( T = T(\phi_0, \lambda, \epsilon_0) \) and \( 1 \leq i \leq m \) such that \( \xi(t) := C[\phi(t)] \in U(p_i, \epsilon_0 \lambda^\frac{1}{2}) \) for all \( t \geq T \).

Proof. For \( h > 0 \) let \( \mathcal{N}(\epsilon_0, h) := \bigcup_{i=1}^{m} U(p_i, \epsilon_0 h) \) and \( \mathcal{N}^c(\epsilon_0, h) := S \setminus \mathcal{N}(\epsilon_0, h) \). As \( H^S \) only has finitely many critical points that are all nondegenerate we get \( \inf_{\mathcal{N}^c(\epsilon_0, h)} |\nabla H^S| \geq \kappa_0 \epsilon_0 h \) for some \( \kappa_0 > 0, h \leq 1 \) and small enough values of \( \epsilon_0 \). Putting \( a(t) := \xi(\lambda t) \) we may use Theorem 1.2(ii) for large enough times to estimate

\[
\frac{d}{dt} H^S(a(t)) = \nabla H^S(a(t)) \left( \frac{3}{2} \nabla H^S(a(t)) + R(t) \right) \geq \frac{3}{2} |\nabla H^S(a(t))|^2 - C \lambda. \tag{7.15}
\]

Let \( t_n \uparrow \infty \). Then there exists a subsequence \( (t_{n_k}) \) so that \( a(t_{n_k}) \in \mathcal{N}(\epsilon_0, \lambda^\frac{1}{2}) \). Indeed if this was not true there would be a time \( T_0 > 0 \) with the property that \( a(t) \in \mathcal{N}^c(\epsilon_0, \lambda^\frac{1}{2}) \) for all times \( t \geq T_0 \). Estimate (7.15) and the lower bound for \( \nabla H^S \) on \( \mathcal{N}^c(\epsilon_0, \lambda^\frac{1}{2}) \) give for small \( \lambda(\epsilon_0) \) and \( t \to \infty \)

\[
H(a(t)) - H(a(T_0)) = \int_{T_0}^{t} \frac{d}{ds} H^S(a(s))ds \geq \int_{T_0}^{t} \frac{3}{2} \kappa_0^2 \epsilon_0^2 \lambda^\frac{3}{2} - C \lambda ds \to \infty
\]

which is a contradiction.

Now let \( p \) and \( q \) be (not necessarily distinct) critical points of \( H^S \) and let us examine what happens when \( a \) travels from \( a(t_1) \in U(p, \epsilon_0 \lambda^\frac{1}{2}) \) to \( a(t_2) \in U(q, \epsilon_0 \lambda^\frac{1}{2}) \) when it exists \( \mathcal{N}(\epsilon_0, \lambda^\frac{1}{2}) \) during its journey. We may choose times \( \tilde{t}_1 < \tilde{t}_2 \) such that \( a(\tilde{t}_1) \in \partial U(p, \epsilon \lambda^\frac{1}{2}) \to a(\tilde{t}_2) \in \partial U(q, \epsilon \lambda^\frac{1}{2}) \) as well as \( a(s) \in \mathcal{N}^c(\epsilon_0, \lambda^\frac{1}{2}) \) for all times \( s \in [\tilde{t}_1, \tilde{t}_2] \). As there are only finitely many critical points we may bound \( d(a(\tilde{t}_1), a(\tilde{t}_2)) \geq C(S) > 0 \) for \( \epsilon_0 \) and \( \lambda(\epsilon_0) \) small. Using Theorem 1.2 to bound \( \dot{a} \) we conclude

\[
0 < C(S) \leq d(a(\tilde{t}_1), a(\tilde{t}_2)) \leq \int_{\tilde{t}_1}^{\tilde{t}_2} |\dot{a}(s)| ds \leq \tilde{C}(S, \phi_0)[\tilde{t}_2 - \tilde{t}_1]. \tag{7.16}
\]

It is easy to see that \( |H(a(t_1)) - H(p)| \leq C(S) \epsilon_0^2 \lambda^\frac{3}{2} \) and that for small enough \( \lambda \)

\[
\int_{t_1}^{t_2} \nabla H(a(s))|\dot{a}(s)| ds \geq \int_{t_1}^{t_2} \frac{3}{2} |\nabla H(a(s))|^2 - C \lambda ds \geq \int_{t_1}^{t_2} \frac{3}{2} \kappa_0^2 \epsilon_0^2 \lambda^\frac{3}{2} - C \lambda ds \geq 0 \tag{7.17}
\]

by Theorem 1.2(ii). Similar statements are true for \( t_2 \) and \( \tilde{t}_2 \). Using the Expansion for \( \dot{a} \) from
Theorem 1.2ii) we compute
\[ H(q) - H(p) \geq H(a(t_2)) - H(a(t_1)) - C(S)\epsilon_0^2 \lambda^{\frac{3}{2}} \]
\[ \geq \frac{\int_{t_1}^{t_2} \nabla H(a(s)) \dot{u}(s) ds}{\epsilon_0^2 \lambda^{\frac{3}{2}}} - C(S)\epsilon_0^2 \lambda^{\frac{3}{2}} \]
\[ \geq C(\epsilon_0, S)\lambda^{\frac{3}{2}}|t_2 - t_1| - C(S)\epsilon_0^2 \lambda^{\frac{3}{2}} \]

In the second to last step we used the gradient estimate for \( \nabla H^S \) on \( \mathcal{N}^c(\epsilon_0, \lambda^{\frac{3}{2}}) \). Using (7.16) we get \( H(q) > H(p) \) for small enough \( \lambda \) and as there are only finitely many critical points \( a(t) \) must stabilise in \( U(p, \epsilon_0 \lambda^{\frac{3}{2}}) \) for some critical point \( p \).

We now prove Theorem 1.3

**Proof.** Let \( \phi(t) \) be a flow line. Lemma 7.2 implies that \( \xi(t) := C[\phi(t)] \) must stabilize in a neighbourhood \( U(p, \epsilon_0 \lambda^{\frac{3}{2}}) \) of some critical point \( p \) of \( H^S \). Using Lemma 5.4 choose \( t_n \to \infty \) such that \( \phi(t_n) \) converges to a critical point \( \phi_\infty \) of the elliptic problem (2.9). Now let \( \tau_n \to \infty \) be any other sequence and assume that \( \phi(\tau_n) \not\to \phi_\infty \). Lemma 5.4 implies that there exists a subsequence \( \tau_{n_k} \) and a solution \( \phi'_{\infty} \not\equiv \phi_\infty \) of the elliptic problem 2.9 that lies in \( C^{2,\beta}(\Sigma_+^4) \) such that \( \phi(\tau_{n_k}) \to \phi'_{\infty} \) in \( C^{2,\beta} \). As \( \phi'_{\infty} \) and \( \phi_\infty \) both have their barycenter in \( U(p, \epsilon_0 \lambda^{\frac{3}{2}}) \) this contradicts Theorem 7.1 for small enough \( \lambda \). The fact, that the limit \( \phi_\infty \) is one of the solutions constructed in [2] also follows from the uniqueness result in Theorem 7.1.

### A Expansion

For the duration of this section we will write \( u(s) := \hat{u}_t(s), p := \xi(t), \tilde{g}(s) := \tilde{g}^{p,s} \) and \( d\mu[u, \tilde{g}] := d\mu_{\hat{u}_t \tilde{g}} \). We consider the map
\[ z(s) := I^i[u(s), \tilde{g}(s)] = \int_{\mathbb{S}^2} W[u(s), \tilde{g}(s)] P_H^i[u(s), \tilde{g}(s)] \nabla C^i[u(s), \tilde{g}(s)] d\mu[u(s), \tilde{g}(s)] \]
and prove that \( z(0) = z'(0) = 0 \) and \( z''(0) = -3\partial_t H^S(p) \). This implies the identities for \( \gamma_{i,t}(s) = -sz(s) \).

First we note that as \( W[0, \delta] = 0, H[0, \delta] = 2 \) and \( \nabla C^i[0, \delta] = -\frac{3}{2\pi} \omega^i \) we may conclude \( z(0) = 0 \) and
\[ z'(0) = -\frac{3}{2\pi} \int_{\mathbb{S}^2} \left( D_1 W[0, \delta] \frac{\partial u}{\partial s} \bigg|_{s=0} + D_2 W[0, \delta] \frac{\partial \tilde{g}}{\partial s} \bigg|_{s=0} \right) \omega^i d\mu_{\hat{g}, \tilde{g}}. \]

As \( u'(0) \) is even \( D_1 W[0, \delta] u'(0) = -\Delta(D + 2) u'(0) \) is also even and the first integral vanishes. For the second integral we recall Lemma 6.1 to learn that \( D_2 W[0, \delta] \tilde{g}'(0) \) is an even function.
and deduce that the second integral also vanishes. It remains to compute $z''(0)$. Again using $W[0, \delta] = 0$ we get

$$z''(0) = \frac{d^2}{ds^2} \left|_{s=0} \int_{S^2_+} \left( W[su'(0), \delta + s\tilde{g}'(0)] \left( P_{\tilde{H}} \nabla C^{i} \right) [su'(0), \delta + s\tilde{g}'(0)] + \int_{S^2_+} \left( D_1 W[0, \delta]u''(0) + D_2 W[0, \delta]\tilde{g}''(0) \right) \left( P_{\tilde{H}} \nabla C^{i} \right) [0, \delta]d\mu_{S^2}. \right. \right.$$ (A.1)

We claim that the first line vanishes. First note that by Lemma 6.1 the integrand in the first line is an odd function. We now need to check that the derivative of the measure at $s = 0$ is even\(^1\).

For this we use the following two formulas from [2] (see Lemma 1\(^2\) and the proof of Lemma 7)

$$D_1 d\mu[0, \delta]\varphi = 2\varphi d\mu_{S^2},$$
$$D_2 d\mu[0, \delta]q = \frac{1}{2} tr_{S^2} q.$$

Inserting $\tilde{g}'(0)$ from the proof of Lemma 6.1 we see that

$$D_1 d\mu[0, \delta]u'(0) = 2u'(0)$$
$$D_2 d\mu[0, \delta]\tilde{g}'(0) = tr_{S^2} \tilde{g}'(0) - g'(0)(\omega, \omega) = -2h_{ab}(p)\omega^a\omega^b$$

are both even. Thus the first line in Equation (A.1) vanishes and we must only compute the last line. By definition

$$W[u, \tilde{g}] = \frac{1}{2} \left( \Delta_2 H + |h^0|^2 H + Ric^2(\tilde{\nu}, \tilde{\nu})H \right).$$

As $\tilde{g}(s)$ is the pullback of a flat metric we may drop the last term. Next we note that $h^0[0, \delta] = 0$ and hence $D_1|h^0|^2[0, \delta] = 0$ and similarly $D_2|h^0|^2[0, \delta] = 0$. Using $H[0, \delta] = 2$ we get

$$(D_1 \Delta[0, \delta]q) H[0, \delta] = (D_2 \Delta[0, \delta]q) H[0, \delta] = 0$$

for arbitrary $\varphi, q$. Combining these considerations we deduce

$$z''(0) = \frac{1}{2} \int_{S^2_+} \Delta \left( D_1 H[0, \delta]u''(0) + D_2 H[0, \delta]\tilde{g}''(0) \right) \frac{3}{2\pi} \omega^i d\mu_{S^2}. \right.$$ (A.2)

Using [2] Lemma 7 and the standard result for the variation of $H$, we get

$$D_2 H[0, \delta] \varphi = -(\Delta + |h^0|^2) \varphi = -(\Delta + 2) \varphi,$$ (A.3)
$$D_2 H[0, \delta] q = -\frac{1}{2} tr_{S^2} \nabla \cdot q + tr_{S^2} \nabla \cdot q(\tilde{\nu}, \cdot) + q(\tilde{\nu}, \cdot) - tr_{S^2} q.$$ (A.4)

\(^1\)As $W[0, \delta] = 0$ one of the $s$-derivatives must act on the Willmore operator. Hence it suffices to check the first derivative of the measure.

\(^2\)Note that our variation is along the outer normal.
We apply (A.4) to \( q = \tilde{g}''(0) \). Denoting the second fundamental form of \( S \) in the chart \( f[p, \cdot] \) by \( h_{ij} \) we can use Formula (2.2) to get (**-entries are to be inferred from symmetry)

\[
\tilde{g}''(0) = \begin{bmatrix}
2h_{1a}h_{1b}x^a x^b & 2h_{1a}h_{2b}x^a x^b & \partial_1 h_{ab} x^a x^b \\
* & 2h_{2a}h_{2b}x^a x^b & \partial_2 h_{ab} x^a x^b \\
* & * & 0
\end{bmatrix}.
\]

Inserting into (A.4) and repeatedly using \( \text{tr}_{S^2} A = \text{tr}_{S^3} A - A(\omega, \omega) \) gives

\[
\int_{S^2_+} \Delta D_2 H[0, \delta] \tilde{g}''(0) \omega^j d\mu_{S^2} = \int_{S^2_+} \Delta \left( 6\partial_a h_{kj} \omega^k \omega^j \omega^3 - 2\partial_j h_{ja} \omega^a \omega^3 \right) \omega^j d\mu_{S^2} = -3\pi \partial_i H(p).
\]

(A.5)

Next we must consider

\[
\int_{S^2_+} \Delta D_1 H[0, \delta] u''(0) d\mu_{S^2} = -\int_{S^2_+} \Delta (\Delta + 2) u''(0) \omega^j d\mu_{S^2} = \int_{S^2_+} \frac{\partial \Delta u''(0)}{\partial \eta} \omega^j d\mu_{S^2}.
\]

(A.6)

To evaluate this integral we must exploit the boundary condition \( B_0[u(s), \tilde{g}(s)] = 0 \) and differentiate twice.

**Lemma A.1.** The following identities hold:

\[
D_1 B[0, \delta] \varphi = \left( \frac{\partial \varphi}{\partial \eta} - \frac{\partial}{\partial \eta} (\Delta + 2) \varphi \right)
\]

\[
D_2 B[0, \delta] \tilde{g}''(0) = \left( \partial_a h_{bc} \omega^a \omega^b \omega^c, 10\partial_a h_{bc} \omega^a \omega^b \omega^c - 2\partial_ah_{ab} \omega^b \right)
\]

Denote arbitrary even functions from \( S^2_+ \) to \( \mathbb{R} \) by \( E \). Once Lemma A.1 is proven we may combine it with Lemma 6.1 to get

\[
0 = D_1 B_0[0, \delta] u''(0) + D_2 B_0[0, \delta] \tilde{g}''(0) + \frac{d^2}{ds^2} B_0[su''(0), \delta + s\tilde{g}''(0)]
\]

\[
= \left( \frac{u''(0)}{\eta} + \partial_a h_{bc} \omega^a \omega^b \omega^c + E, -\frac{\partial}{\partial \eta} (\Delta + 2) u''(0) + 10\partial_a h_{bc} \omega^a \omega^b \omega^c - 2\partial_ah_{ab} \omega^b + E \right).
\]

(A.7)

Combining both components of Equation (A.7) gives

\[
\frac{\partial \Delta u''(0)}{\partial \eta} = 12\partial_a h_{bc} \omega^a \omega^b \omega^c - 2\partial_ah_{ab} \omega^b + E.
\]

Inserting into Equation (A.6) and combining with Equation (A.5) yields

\[
\frac{1}{2} \int_{S^2_+} \Delta \left( D_1 H[0, \delta] u''(0) + D_2 H[0, \delta] \tilde{g}''(0) \right) d\mu_{S^2} = 2\pi \partial_i H^S(p).
\]

Multiplying with \(-\frac{3}{2\pi}\) and recalling Equation (A.2) shows \( z''(0) = -3\partial_i H^S(p) \). It remains to prove Lemma A.1.
Proof. Let \( \omega_0 \in \partial S_3^+ \) and \( \phi : U \subset \mathbb{R}^2 \rightarrow \phi(U) \subset S_3^+ \) be a parameterization near \( \omega_0 \) so that \( g_{ij}(\omega) := \langle \partial_i \phi, \partial_j \phi \rangle \) satisfies \( g_{ij}(\omega_0) = \delta_{ij} \) and \( \partial_0 g_{ij}(\omega_0) = 0 \). Put \( f_\epsilon(\omega) := (1 + \epsilon \varphi(\omega)) \omega \), \( g(\mu) := \delta + \mu q \) with \( q := \tilde{g}'(0) \) and let \( \tilde{\nu}(\epsilon, \mu) \) denote the inner normal of \( f_\epsilon \) with respect to \( g(\mu) \) (so \( \tilde{\nu}(0, 0) = -\omega \)). Then at \( \omega_0 \)

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon, \mu = 0} \tilde{\nu} = \partial_\epsilon \varphi \partial_\epsilon \phi \quad \text{and} \quad \frac{d}{d\mu} \bigg|_{\epsilon, \mu = 0} \tilde{\nu} = q(\phi, \partial_\epsilon \phi) \partial_\epsilon \phi + \frac{1}{2} q(\phi, \phi) \phi. \tag{A.8}
\]

The second formula is proven in [2] (see Lemma 7). To get the first one first note that \( \langle \tilde{\nu}(\epsilon, 0), \tilde{\nu}(\epsilon, 0) \rangle = 1 \) implies that \( \partial_\epsilon \tilde{\nu}(0, 0) \) must be tangential. Hence

\[
\frac{\partial \tilde{\nu}}{\partial \epsilon} \bigg|_{\epsilon, \mu = 0} = \langle \partial_\epsilon \phi, \frac{\partial \tilde{\nu}}{\partial \epsilon} \bigg|_{\epsilon, \mu = 0} \rangle \partial_\epsilon \phi = -\langle \tilde{\nu}(0, 0), \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon, \mu = 0} \rangle \partial_\epsilon \phi = \partial_\epsilon \varphi \partial_\epsilon \phi,
\]

where we used that \( \tilde{\nu}(0, 0) = -\phi \). For \( \tilde{g} = \delta \) we have \( \tilde{\nu}_{\mathbb{R}^2} = e_3 \) and \( \tilde{h}_{\mathbb{R}^2} = 0 \). Using Equations (A.3) and (A.8) we get

\[
D_1 B_0[0, \delta] \varphi = \left( \langle \frac{\partial \tilde{\nu}}{\partial \epsilon} \bigg|_{\epsilon, \mu = 0}, e_3 \rangle, \frac{\partial}{\partial \eta} D_1 H[0, \delta] \varphi \right) = \left( \langle \nabla \omega^3, \nabla \varphi \rangle, -\frac{\partial (\Delta + 2) \varphi}{\partial \eta} \right)
\]

at \( \omega_0 \). As \( \nabla \omega^3 = e_3 = \eta \) the first identity follows.

To establish the second formula we use Lemma 3 from [2], \( H[0, \delta] = 2, \tilde{\nu}(0, 0) = -\phi \) and note \( \tilde{h}_{ij}^2 \equiv 0 \) for \( \mu = 0 \) to get

\[
D_2 B[0, \delta] g'(0) = \left( \frac{d}{d\mu} \bigg|_{\mu = 0} \frac{\langle \tilde{\nu}(0, \mu), e_3 \rangle}{\sqrt{g(\mu)^{33}}}, \frac{\partial}{\partial \eta} D_2 H[0, \delta] g'(0) + 2 \frac{d\tilde{h}_{\mathbb{R}^2}}{d\mu} \bigg|_{\mu = 0} (\phi, \phi) \right). \tag{A.9}
\]

\( \tilde{\nu}(0, 0) = -\omega \perp e_3 \) and \( g'(0) = q \). Using Equation (A.8) then gives

\[
\frac{d}{d\mu} \bigg|_{\mu = 0} \frac{\langle \tilde{\nu}(0, \mu), e_3 \rangle}{\sqrt{g(\mu)^{33}}} = q(\phi, \partial_\epsilon \phi) \partial_\epsilon \phi^3 = q_{\mu \nu} \omega^\mu \langle \nabla \omega^\nu, \nabla \omega^3 \rangle = q_{\mu 3} \omega^\mu \tag{A.10}
\]

at \( \omega_0 \). As \( q(t\omega) = t^2 q(\omega) \) we learn \( \nabla \tilde{\nu} = -2q \). Using Equation (A.4) then gives \( D_2 H[0, \delta] q = 3q(\tilde{\nu}, \tilde{\nu}) + \text{tr}_{\mathbb{R}^3} \nabla q(\tilde{\nu}, \cdot) \). Inserting \( \tilde{\nu} = -\omega \) we may write \( \text{tr}_{\mathbb{R}^3} \nabla q(x, \cdot) = \partial_\mu q_{\mu \nu} x^\nu \) and \( q(\tilde{\nu}, \tilde{\nu}) = q_{\mu 3} x^\mu x^3 \). As we take \( \partial_\eta \) we only need terms that contain precisely one \( x^3 \). Hence at \( \omega_0 \)

\[
D_2 \left( \frac{\partial H}{\partial \eta} \right) [\delta] q = \frac{\partial}{\partial \eta} \left( 6\partial_a h_{ij} \omega^i \omega^j \omega^3 - 2\partial_j h_{ja} \omega^a \omega^3 \right) = 6\partial_a h_{ij} \omega^i \omega^j \omega^a - 2\partial_j h_{ja} \omega^a. \tag{A.11}
\]

Finally, we must linearize \( \tilde{h}_{ij}^2 = g_{\alpha 3} \Gamma_{ij}^\alpha \). As \( \Gamma_{ij}^\alpha \) vanishes for \( \mu = 0 \) we must only take the derivatives of the Christoffel symbols into account. Using \( \partial_3 q = 0 \) an easy computation then shows

\[
D_2 \left( \tilde{h}_{\mathbb{R}^2}(\tilde{\nu}, \tilde{\nu}) \right) q = 2\omega^i \omega^j D \Gamma_{ij}^3 [\delta] q = (2\partial_3 q + \partial_3 q_{ij}) \omega^i \omega^j = 4\partial_a h_{ij} \omega^i \omega^j \omega^a. \tag{A.12}
\]

Equations (A.9), (A.10), (A.11) and (A.12) imply the second formula in the Lemma.
B Solution of the elliptic problem

Alessandroni and Kuwert [2] study the elliptic problem (2.9) for \( \phi \in S(\lambda, \theta) \) by making the ansatz

\[
\phi^{p,\lambda} = F^\lambda[p, f_u].
\]

For given \( p \in S \) they first derive a solution to the elliptic problem with prescribed barycenter \( C[\phi] = p \). This is achieved by applying the diffeomorphism \( F^\lambda[p, \cdot] \) and studying the Equation in \( \mathbb{R}^3 \) with the background metric \( \tilde{g} = \tilde{g}^{p,\lambda} \):

\[
\begin{align*}
P^i_{K[u, \tilde{g}]} W[u, \tilde{g}] &= 0, \\
B_0[u, \tilde{g}] &= 0, \\
C[u, \tilde{g}] &= 0, \\
A[u, \tilde{g}] &= 2\pi.
\end{align*}
\]

It is shown\(^3\) that the unique solution \( u(\lambda) \in C^{4,\gamma}(S^2_+) \) depends regularly on \( \lambda \). In fact, it is shown that for \( \Omega \in C^m \) and \( l = m - 1 \geq 6 \) we have \( \tilde{g}^{p,\lambda} \in G^l_0(\delta; \sigma_0) \) for \( \lambda \leq \lambda_0(\Omega, \sigma_0) \) and that \( u \in C^{l-4}([0, \lambda_1), C^{4,\gamma}(S^2_+)) \) for some small \( \lambda_1(\Omega) \). Putting \( \varphi := u'(0) \) and using Lemma 6.1 we see that

\[
\begin{align*}
&\Delta(\Delta + 2) \varphi = \text{even}, \\
&\frac{\partial \varphi}{\partial \eta} - 2\pi \int_{S^2_+} \varphi\omega^i &= 0, \\
&2\int_{S^2_+} \varphi = - \frac{d}{d\lambda}|_{\lambda=0} A[0, \tilde{g}^{p,\lambda}].
\end{align*}
\]

The uniqueness of this linear problem implies that \( \varphi \) is even and hence \( \|u^-\|_{C^{4,\gamma}} \leq C\lambda^2 \). Thus \( \phi^{p,\lambda} \in S^-(\lambda, K) \) for suitable \( K \). Finally, in Section 3 of their paper a suitable choice for \( p \) is derived that makes \( \phi^{p,\lambda} \) a solution of (2.9).

C Parabolic Schauder Theory and regularity of meta maps

The following definitions are taken from [7]. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( T > 0 \) and \( \Omega_T := [0, T] \times \Omega \). For \( (l, \alpha) \in \mathbb{N}_0 \times \mathbb{N}_0^\alpha \) we abbreviate \( D^{l,\alpha} u(t, x) := \partial_t^l \partial_x^\alpha u(t, x) \). For \( m \in \mathbb{N}_0 \) let

\[
C^m(\bar{\Omega}_T) := \{ u : \bar{\Omega}_T \to \mathbb{R} \mid D^{l,\alpha} u \text{ is defined in } \text{int} \Omega_T \text{ for } 4l + |\alpha| \leq m \text{ and continuous on } \bar{\Omega}_T \}.
\]

\[
\|u\|_{C^m(\bar{\Omega}_T)} := \sum_{4l+|\alpha| \leq m} \|D^{l,\alpha} u\|_{C^0(\bar{\Omega}_T)}.
\]

\(^3\)See Lemma 6 and Proposition 1 in their paper. Also note the varying definitions of \( B \).
For \( u \in C^m(\overline{\Omega_T}) \) and \( \gamma \in (0, 1) \) we define the *temporal* and *spatial*-Hölder seminorms

\[
[u]^*_\gamma := \sup_{4t+|\alpha|=m} \sup_{(x,t) \neq (y,t)} \frac{|D^{l,\alpha} u(x, t) - D^{l,\alpha} u(y, t)|}{|x-y|^\gamma},
\]

\[
[u]_\gamma := \sup_{0<\gamma-4l-|\alpha| \leq 4} \sup_{(x,t) \neq (x,s)} \frac{|D^{l,\alpha} u(x, t) - D^{l,\alpha} u(x, s)|}{|s-t|^\gamma},
\]

and put \( \|u\|_{C^{m,\gamma}(\overline{\Omega_T})} = \|u\|_{C^m(\overline{\Omega_T})} + \|u\|_{C^\gamma} \). Then the parabolic Hölder space \( C^{m,\gamma}(\overline{\Omega_T}) \) is defined as

\[
C^{m,\gamma}(\overline{\Omega_T}) := \{ u \in C^m(\overline{\Omega_T}) \mid \|u\|_{C^{m,\gamma}(\overline{\Omega_T})} < \infty \}.
\]

We also refer to this space as \( C^{m,\gamma}(\overline{\Omega_T}) \) where \([\cdot]\) denotes the floor function. The definition of boundary spaces such as \( C^{3,0,\gamma}([0, T] \times \partial \Omega) \) is analogue and also covered in [7]. Lifting the definitions onto a manifold \( M \) works as usual and is e.g. covered in [7] for the case when \( M = \partial U \) for a sufficiently regular domain \( U \).

**Improved Schauder estimates**

All Schauder theory except for the decay estimate used e.g. in Theorem 3.3 are standard and may be derived by following Simon’s scaling argument [17]. To prove the decay consider a function \( \varphi \in X_T \) satisfying \( \varphi(t, \cdot) \in X^+_0 \) (recall Equation (6.5)) for all \( t \in [0, T] \) and

\[
\begin{aligned}
\varphi + \frac{1}{2} \Delta (\Delta + 2) \varphi &= 0 \quad \text{on } [0, T] \times S^2_+ \\
\varphi(0, \cdot) &= \psi_0 \quad \text{on } \{0\} \times S^2_+.
\end{aligned}
\] (C.1)

Following Lemma 4 in [2] we see that for all \( u_0 \in X^+_0 \) we have

\[
\int_{S^2_+} u_0 \Delta (\Delta + 2) u_0 d\mu_{S^2} \geq \lambda_2 (\lambda_2 - 2) \int_{S^2_+} u_0^2 \geq 24 \int_{S^2_+} u_0^2 d\mu_{S^2}
\]

where \( \lambda_2 = 6 \) is the second non-zero eigenvalue of \(-\Delta\). Let \( \lambda' := 24 \), pick \( \mu \in (0, \frac{\lambda_2}{2}) \) and define \( \varphi_\mu(t, \omega) := e^{\mu t} \varphi(t, \omega) \). Then \( \varphi_\mu \) has zero boundary conditions along \( \partial S^2_+ \) and solves

\[
\begin{aligned}
\varphi_\mu + \frac{1}{2} \Delta (\Delta + 2) \varphi_\mu - \mu \varphi_\mu &= 0 \quad \text{on } [0, T] \times S^2_+ \\
\varphi_\mu(0, \cdot) &= \psi_0 \quad \text{on } \{0\} \times S^2_+.
\end{aligned}
\]

Standard Schauder theory then gives

\[
\|\varphi_\mu\|_{C^{2,1,\gamma}_T} \leq C(S^2_+, \gamma, \mu) T^N \left( \|\psi_0\|_{C^{2,\gamma}} + \sup_{t \in [0, T]} \|\varphi_\mu(t)\|_{L^2(S^2_+)} \right) \quad \text{ (C.2)}
\]

for some \( N = N(S^2_+, \gamma) \in \mathbb{N} \). The scaling of the Schauder constant with \( T \) follows by scaling arguments. Indeed, an estimate of the form (C.2) is usually derived and a suitable \( L^2 \)-estimate
inserted to derive the usual Schauder estimate. Using the fact that \( \varphi(t, \cdot) \in X_0^1 \) for all \( t \in [0, T] \) and remembering \( (C.1) \) we compute

\[
\frac{d}{dt} \frac{1}{2} \int_{S_+^2} \varphi^2 \, d\mu_{S^2} = -\frac{1}{2} \int_{S_+^2} \varphi_{\mu} (\Delta + 2) \varphi \, d\mu_{S^2} + \mu \int_{S_+^2} \varphi^2 \, d\mu_{S^2}
\]

\[
\leq \left( -\frac{\lambda'}{2} + \mu \right) \int_{S_+^2} \varphi^2 \, d\mu_{S^2} \leq 0.
\]

This implies \( \sup_{t \in [0,T]} \| \varphi_{\mu}(t, \cdot) \|^2_{L^2(S_+^2)} = \| \psi_0 \|^2_{L^2(S_+^2)} \) and therefore

\[
e^{-\lambda T} \| \varphi(T, \cdot) \|_{C^{4, \gamma}} = \| \varphi_{\mu}(T, \cdot) \|_{C^{4, \gamma}} \leq \| \varphi_{\mu} \|_{C^{4,1, \gamma}} \leq C(S_+^2, \gamma) T^N \| \psi_0 \|_{C^{4, \gamma}}.
\]

Choose \( \mu := \frac{1}{3} \lambda' \) and note \( e^{-\lambda' T} S_+^2 \gamma \leq C(S_+^2, \gamma) \). Thus

\[
\| \varphi(T) \|_{C^{4, \gamma}} \leq C(S_+^2, \gamma) e^{-\lambda' T} \| \psi_0 \|_{C^{4, \gamma}} = C(S_+^2, \gamma) e^{-6T} \| \psi_0 \|_{C^{4, \gamma}}.
\]

**Regularity of meta maps**

Let \( U \) be a bounded domain and \( V \) be a bounded convex domain. Put \( A := C^{0,\frac{\gamma}{2}}([0, T], C^l(V, \mathbb{R})) \), \( B := C^{0,\gamma}([0, T] \times U, V) \) and \( C := C^{0,\gamma}([0, T] \times U) \). For \( g \in A \) and \( f \in B \) put \( (g \circ f)(t, x) := g(t, f(t, x)) \) and consider the meta map

\[
T : A \times B \rightarrow C, \ (g, f) \mapsto g \circ f.
\]

It is easy to see that \( T \) is well defined if \( l \geq 1 \). \( T \) is even continuous if \( l \geq 2 \). As an example we show that \( [g \circ f - g \circ \tilde{f}]_\gamma \) is small if \( f \approx \tilde{f} \). Indeed

\[
\Delta := |g(t, f(t, x)) - g(t, \tilde{f}(t, x)) - g(s, f(s, x)) + g(s, \tilde{f}(s, x))|
\]

\[
\leq |g(t, f(t, x)) - g(t, \tilde{f}(t, x)) - g(t, f(s, x)) + g(t, \tilde{f}(x, x))| + |g(t, f(s, x)) - g(t, \tilde{f}(s, x)) - g(s, f(s, x)) + g(s, \tilde{f}(s, x))|
\]

\[
\leq |g(t, \lambda f(t, x) + (1 - \lambda) \tilde{f}(t, x)) - g(t, \lambda f(s, x) + (1 - \lambda) \tilde{f}(s, x))|_{\lambda=0}^{\lambda=1} + |g(t, \lambda f(s, x) + (1 - \lambda) \tilde{f}(s, x)) - g(s, \lambda f(s, x) + (1 - \lambda) \tilde{f}(s, x))|_{\lambda=0}^{\lambda=1}.
\]

Convexity of \( V \) and the chain rule then readily imply

\[
\Delta \leq \left( \| D_2 g \|_{C^0} [f - \tilde{f}]_\gamma + \| D_2 g \|_{C^0} \| f - \tilde{f} \|_{C^\gamma} ([f]_\gamma + [\tilde{f}]_\gamma) + \| D_2 g \|_{C^0} \| f - \tilde{f} \|_{C^\gamma} \right) |t - s|^{2l}.
\]

This shows the continuity of \( T \). Similarly one shows that for \( T \in C^k \) it suffices if \( l \geq 2 + k \) as differentiating \( g \circ f \) with respect to \( f \) produces \( D g \circ f \) which must be continuous in \( f \). Similar arguments allow one to study \( g \circ f \in C^{4,1,\gamma}([0, T] \times U) \) for \( f \in C^{4,1,\gamma}([0, T] \times U, V) \). The corresponding meta map is of class \( C^k \) as long as \( l \geq 6 + k \) as four additional derivatives of \( g \) are required.
D The Riemannian barycenter

Let \( \tilde{g} \in G_0^l(\delta; \epsilon) \) with \( l \geq 2 \). This appendix serves the purpose of constructing the Riemannian barycenter of the immersion \( f_u : S^2_+ \to (\mathbb{R}^3, \tilde{g}) \) for small enough \( u \). The analysis follows the construction in [2] but changes the euclidean projection \( \pi_{\mathbb{R}^2} \) used in [2] to the Riemannian projection \( \pi_{\mathbb{R}^2}[\tilde{g}, \cdot] \). We recall \( D_r := \{ x \in \mathbb{R}^2 \mid |x| < r \} \) and \( Z_r := D_r \times [-r, r] \).

**Lemma D.1.** For \( \epsilon > 0 \) small enough the following is true: For each \( p \in Z_2 \) there exists a unique \( x \in D_2 \), denoted by \( \pi_{\mathbb{R}^2}[\tilde{g}, p] \), such that \( \tilde{g}_x(p - x, \mathbb{R}^2) = 0 \). The map \( \pi_{\mathbb{R}^2} : G_0^l(\delta; \epsilon) \times Z_2 \to D_2, (\tilde{g}, p) \mapsto x \) is of class \( C^4 \).

**Proof.** First we prove uniqueness. Fix a point \( p = (\tilde{p}, p^3) \in Z_2 \) and consider the map

\[
\Phi : D_2(0) \subset \mathbb{R}^2 \to \mathbb{R}^2, \; f(q) := \tilde{p} - (\delta - \tilde{g}_q)(p - q, e_i)e_i.
\]

For small \( \epsilon > 0 \) it is easy to show that \( \Phi : D_2 \to D_2 \) defines a contraction which implies the existence and uniqueness. To check the regularity we consider the \( C^1 \)-map \( \bar{\Phi} : Z_2^2 \times D_2 \times G_\epsilon \to \mathbb{R}^2, \; \bar{\Phi}(p, q, \omega) := \tilde{g}_q(p - q, e_i)e_i \). Given any \( p_0 = (\tilde{p}_0, p^3_0) \in Z_2 \) we note \( \bar{\Phi}(p_0, \tilde{p}_0, \delta) = 0 \) and \( D_2\bar{\Phi}(p_0, \tilde{p}_0, \delta) = -\text{id}_{\mathbb{R}^2} \). The regularity of the local and hence global inverse follows from the implicit function theorem.

We briefly motivate the modified definition of the Riemannian barycenter. For an immersion \( f_u : S^2_+ \to (\mathbb{R}^3, \tilde{g}) \) we wish to define the Riemannian barycenter \( x \in \mathbb{R}^2 \) by the implicit equation

\[
I := \int_{S^2_+} (\exp_{\tilde{g}})^{-1}(f_u(\omega))d\mu_g(\omega) \perp_{\tilde{g}} \mathbb{R}^2.
\]

Note that by definition \( I \in T_x \mathbb{R}^3 \) and hence it is sensible to demand orthogonality with respect to \( \tilde{g}_x \). We may reformulate the condition as \( \tilde{g}_x(x + I - x, \mathbb{R}^2) = 0 \) which implies \( \pi_{\mathbb{R}^2}[\tilde{g}, x + I] = x \). Hence we study zeros of the map

\[
X : U_\epsilon \times G_\epsilon \times D_\epsilon \to \mathbb{R}^2, \; X[u, \tilde{g}, x] := \pi_{\mathbb{R}^2} \left[ \tilde{g}, x + \int_{S^2_+} (\exp_{\tilde{g}})^{-1}(f_u(\omega))d\mu_g(\omega) \right] - x,
\]

which reduces to \( X' \) (we include the prime to distinguish the map from [2] to the one studied here) from [2] if we replace \( \pi_{\mathbb{R}^2}[\tilde{g}, \cdot] \) with \( \pi_{\mathbb{R}^2}[\delta, \cdot] \) as is used there. In particular both maps are identical if \( \tilde{g} = \delta \) is inserted. Repeating the analysis from [2] we get the following Theorem:

**Theorem D.2** (The two dimensional barycenter).

There exist \( \epsilon > 0 \) and \( \rho > 0 \) such that for \( u \in C^1(S^2_+, \mathbb{R}) \) and \( \tilde{g} \in C^2(Z_2, M_3(\mathbb{R})) \) satisfying \( \|u\|_{C^1} < \epsilon \) and \( \|\tilde{g} - \delta\|_{C^2} < \epsilon \) there exists a unique point \( x = C[u, \tilde{g}] \in D_\rho(0) \subset \mathbb{R}^2 \) such that

\[
|X[u, \tilde{g}, C[u, \tilde{g}]]| = 0 \quad \text{and} \quad |X[u, \tilde{g}, C[u, \tilde{g}]]| = 0
\]

\[
|X[u, \tilde{g}, C[u, \tilde{g}]]| \leq C(\|u\|_{C^1(S^2_+)} + \|\tilde{g} - \delta\|_{C^2}).
\]

As map from \( G_0^l \times C^{4,\gamma}(S^2_+) \to \mathbb{R}^2 \) the map \( C \) is of class \( C^{l-1} \).
Proof. As \( X'[,\delta,\cdot] = X[,\delta,\cdot] \) the proof from [2] carries over.

As \( X[,\delta,\cdot] = X'[,\delta,\cdot] \) we conclude the same explicit formula for \( C[u,\delta] \) as is given in [2]:

\[
C[u,\delta] = \pi_{R^2} \left( \int f_u d\mu_{f_u \delta} \right)
\]

Following [2] we now compute the \( L^2 \)-gradient of \( C^i \). The only difference in the analysis that is required is to also take the derivative of the projection into account. Let \( f(\epsilon) \) be a variation of \( f_u \) with \( f(0) = \varphi \tilde{\nu} \) where \( \tilde{\nu} \) is the inner normal of \( f_u \). We set \( g := f_u \tilde{g}, x(\epsilon) := C[f(\epsilon), \tilde{g}], \)

\[
I[\tilde{g}, x, f] := \int_{S^2_i} \left( \exp_{\tilde{g}}^{-1}(f) d\mu_{\tilde{g}} \right)
\]

We investigate the derivative:

\[
0 = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \pi_{R^2} \left[ \tilde{g}, x + \int_{S^2_i} \left( \exp_{\tilde{g}}^{-1}(f(\epsilon)) d\mu_{f(\epsilon) \tilde{g}} \right) \right] = \left. \frac{dx}{d\epsilon} \right|_{\epsilon=0} = D_2 \pi_{R^2} \left[ \tilde{g}, x + I[\tilde{g}, x, f_u] \right] \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left( x + \int_{S^2_i} \left( \exp_{\tilde{g}}^{-1}(f(\epsilon)) d\mu_{f(\epsilon) \tilde{g}} \right) \right) = \left. \frac{dx}{d\epsilon} \right|_{\epsilon=0} \tag{D.1}
\]

Inserting (D.2) into (D.1) and dropping the arguments on \( I \) for spatial reasons gives

\[
\left. \frac{dx}{d\epsilon} \right|_{\epsilon=0} = \left( D_2 \pi_{R^2}[\tilde{g}, x + I] \left( \int_{S^2_i} D_x \left( \left( \exp_{\tilde{g}}^{-1}(f_u(\omega)) d\mu_{\tilde{g}} + \mathbb{I}_{R^2} \right) \right) - \mathbb{I}_{R^2} \right) \right)^{-1}
\]

As the Riemannian barycenter is invariant under reparameterization the \( L^2 \)-gradient of \( C^i \) is normal along \( f_u \). We may therefore multiply the actual \( L^2 \)-gradient with \( \tilde{\nu} \) to obtain a scalar function which we denote by \( \nabla C^i[u, \tilde{g}] \). By definition we then have

\[
\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} C^i[f(\epsilon), \tilde{g}] = \int_{S^2_i} \nabla C^i[u, \tilde{g}] \tilde{g} \left( \frac{\partial f}{\partial \epsilon} \right)_{\epsilon=0} (\tilde{\nu}) d\mu_{\tilde{g}}.
\]
The analysis above gives the explicit formula
\[\sum_{i=1}^{2} \nabla C^i[u, \tilde{g}]e_i = -\frac{1}{\mu_g(S^2_+)} \left( D_2\pi_{\mathbb{R}^2}[\tilde{g}, x + I] \left( \int_{S^2_+} D_x \left( (\exp_{\tilde{g}}^x)^{-1} \right) (f_u(\omega))d\mu_g + id_{\mathbb{R}^2} \right)^{-1} \right. \]
\[\left. - D_2\pi_{\mathbb{R}^2}[\tilde{g}, x + I] \left( \int_{S^2_+} (\exp_{\tilde{g}}^x)^{-1} (f_u)d\mu_g H \right. \right. \]
\[\left. + D \left( (\exp_{\tilde{g}}^x)^{-1} \right) (f_u(\omega)) - (\exp_{\tilde{g}}^x)^{-1} (f_u(\omega))H \right) \].

Specializing to \(\tilde{g} = \delta\) gives
\[\nabla C^i[u, \delta] = -\frac{1}{\mu_g(S^2_+)} \pi_{\mathbb{R}^2} \left( \int_{S^2_+} (\exp_{\delta}^x)^{-1} (f_u)d\mu_g H + D \left( (\exp_{\delta}^x)^{-1} \right) (f_u) - (\exp_{\delta}^x)^{-1} (f_u)H \right)^i \].

Note that \(\int_{S^2_+} (\exp_{\delta}^x)^{-1} (f_u)d\mu_\gamma \perp \delta \mathbb{R}^2\) by definition of \(x = C[u, \delta]\). This eliminates the first term and thus establishes the same formula that is derived in [2]. In particular we recover
\[\nabla C^i[0, \delta] = -\frac{3}{2\pi} \omega^i.\]

A note on regularity

We consider \(\nabla C^i\) as a map \(\nabla C^i : G_T^l \times C_4^{\alpha,\gamma}([0, T] \times \mathbb{S}^2_+) \to C_T^{0,\gamma}([0, T], \mathbb{R}^2)\) defined on a neighbourhood of \((\delta, 0)\). We use the fact that the map \((x, v, \tilde{g}) \mapsto \exp_{\tilde{g}}^x v \in \mathbb{R}^3\) defined on a suitably large neighbourhood of \((0, 0, \delta) \in \mathbb{R}^2 \times \mathbb{R}^3 \times G^0_0\) is of class \(C^{l-1}\). This is proven in Appendix 4 in [2]. Combined with the explicit formula for \(\nabla C^i\) we get that \(\nabla C^i\) is of class \(C^{l-7}\) as long as \(l \geq 7\).

Application

We now define \(C[\phi^\lambda_{p,u}]\) by applying Theorem D.2 to \(f_u\) with the pullback metric \(\tilde{g}^\rho,\lambda\). This gives a point \(x = C[u, \tilde{g}^\rho,\lambda] \in \mathbb{R}^2\) and we wish to define
\[C[\phi] := F^\lambda[p, x].\]

We must now prove that this definition does not depend on \(p\) and the chosen frame.

Proof. Abbreviate \(\tilde{g} := \tilde{g}^\rho,\lambda\). Rewriting the defining equation for \(x\) gives
\[x = C[f_u, \tilde{g}] \iff \int_{S^2_+} (\exp_{\tilde{g}}^x)^{-1} (f_u(\omega))d\mu_{f_u,\tilde{g}} \perp \mathbb{R}^2\] with respect to \(g_x\)
\[\iff \int_{S^2_+} (\exp_{\tilde{g}}^x)^{-1} (f_u(\omega))d\mu_{f_u,\tilde{g}} = k\nu_{\mathbb{R}^2}(x)\text{ for some }k \in \mathbb{R}.\]
We note that $F^\lambda[p, \cdot]$ is an isometry up to the scaling factor $\lambda^2$ in the definition of $\tilde{g}^{p,\lambda}$. Applying the differential $(DF^\lambda[p, x])^{-1}$ to (D.3) we learn

$$
\begin{align*}
x = C[f_u, \tilde{g}] &\iff (DF^\lambda[p, x])^{-1} \left[ \int_{S^2_+} (\exp_\tilde{g}^\delta)^{-1} (f_u(\omega))d\mu_{F^\lambda[p, x]} \right] = \lambda k N^S(F^\lambda[p, x]) \\
&\iff \int_{S^2_+} (\exp_{F[p, x]}^\delta)^{-1} (F^\lambda[p, f_u(\omega)])d\mu_{\phi^\star} = \lambda k N^S(F^\lambda[p, x]) \\
&\iff \int_{S^2_+} (f_u(\omega) - F[p, x])d\mu_{\phi^\star} = \lambda N^S(F^\lambda[p, x]).
\end{align*}
$$

For $\lambda$ small enough we see that $x = C[f_u, \tilde{g}]$ is equivalent to

$$
\pi_S \int_{S^2_+} \phi(\omega)d\mu_{\phi^\star} = F^\lambda[p, x],
$$

where $\pi_S$ is the nearest point projection onto $S$. As the left hand side of equation (D.4) is independent of the choices for $p$ and the frame the right hand side must be too.

An immediate consequence is that for $\phi = F^\lambda[p, f_u]$ we have $C[\phi] = p$ if and only if $C[f_u, \tilde{g}^{p,\lambda}] = 0$. We close by proving that we may always parameterize $\phi$ over its barycenter.

**Theorem D.3.** There exists $\lambda_0 > 0$, $\theta_0 > 0$ such that for $\lambda \leq \lambda_0$ each $\phi \in S'(\lambda, \theta_0)$ may be parameterized over its barycenter $C[\phi]$. That is, there exists a unique graph function $\tilde{u} \in C^{4,\gamma}(S^2_+)$ such that

$$
\phi = F^\lambda[C[\phi], f_{\tilde{u}}].
$$

**Proof.** As $|\varphi[p, \lambda x]| \leq C\lambda^2$ we derive $F^\lambda[p, Z_2] = \text{im } F^\lambda[p, \cdot] \supset B_{\frac{1}{2}\lambda}(p)$ for small enough $\lambda$. Next we have the following claim:

**Claim # 1:** There exist $\epsilon, \delta, \lambda > 0$ such that $F^\lambda[p, x] \in \text{im } F^\lambda[q, \cdot]$ for $p, q \in S$ and $x \in Z_2$ satisfying $d(p, q) < \lambda \delta$, $|x| \leq 1 + \epsilon$.

**Proof:** For small enough $\lambda$ we have $\text{im } F^\lambda[q, \cdot] \supset B_{\frac{1}{2}\lambda}(q)$. Now if $|x| < 1 + \epsilon$ then

$$
|F^\lambda[p, x] - q| \leq |p - q| + \lambda |x| + |\varphi[q, \lambda x]| \leq \lambda \delta + \lambda(1 + \epsilon) + C\lambda^2
$$

which implies the claim for a sufficiently small choice of $\epsilon$, $\delta$ and $\lambda$.

Now suppose that $\phi \in S'(\epsilon, \lambda)$. Then we may write $\phi = F^\lambda[p, f_u]$ for some $p \in S$ and $\|u\|_{C^{4,\gamma}} < \epsilon$. Let $q \in S$ denote the barycenter of $\phi$. By definition $q = C[\phi] = F^\lambda[p, C[f_u, \tilde{g}^{p,\lambda}]]$ and using the Estimate from Theorem D.2 we get $|C[f_u, \tilde{g}^{p,\lambda}]| < \delta(\epsilon, \lambda)$ with $\delta \to 0$ as $\lambda, \epsilon \to 0$. This gives $d(p, q) \leq C\lambda \delta(\epsilon, \lambda)$. As $|f_u(\omega)| < 1 + \epsilon$ we may use Claim # 1 for small enough $\epsilon$ and $\lambda$ to define $\tilde{u}$ by

$$
f_{\tilde{u}}(\omega) = \left( F^\lambda[q, \cdot] \right)^{-1} \left( F^\lambda[p, f_u(\omega)] \right).
$$

Uniqueness is easily established.
E  The constraint space

Let \( \tilde{g} \in C^5(Z_2, M_3(\mathbb{R})) \) be a metric that is close to the euclidean metric \( \| \tilde{g} - \delta \|_{C^5} \leq \epsilon \). Also consider a function \( u \in C^{1,\gamma}(S^2_+) \) that satisfies \( \| u \|_{C^{1,\gamma}} \leq \epsilon \). We consider the \( L^2 \)-gradients of the area functional \( A[u, \tilde{g}] \) and the barycenter components \( C^i[u, \tilde{g}] \) and put

\[
K[u, \tilde{g}] := \text{span}_\mathbb{R} \{ \nabla A[u, \tilde{g}], \nabla C^i[u, \tilde{g}] \}.
\]

In the Appendix of [2] the following formulas are derived for \( \tilde{g} = \delta \) and \( u = 0 \):

\[
\nabla A[0, \delta] = -2 \quad \text{and} \quad \nabla C^i[0, \delta] = -\frac{3}{2\pi} \omega^i.
\]

There it is also shown that the maps \( (\tilde{g}, u) \mapsto \nabla A[u, \tilde{g}], \nabla C^i[u, \tilde{g}] \in C^{0,\gamma}(S^2_+) \) are of class \( C^2 \).

Hence we learn that for \( \epsilon \) small enough the following quantities are well defined:

\[
\psi_0[u, \tilde{g}] := \frac{\nabla A[u, \tilde{g}]}{\| \nabla A[u, \tilde{g}] \|_{L^2(f^*_u g)}} \quad \text{and} \quad \psi_i[u, \tilde{g}] := \frac{\nabla C^i[u, \tilde{g}]}{\| \nabla C^i[u, \tilde{g}] \|_{L^2(f^*_u g)}}.
\]

Clearly \( (\psi_\mu[u, \tilde{g}])^2_{\mu=0} \) constitutes a generating system for \( K[u, \tilde{g}] \). (E.2) implies that for \( u = 0 \) and \( \tilde{g} = \delta \) the functions \( (\psi[0, \delta])^2_{\mu=0} \) even provide an \( L^2(S^2_+) \)-orthonormal basis of \( K[0, \delta] \). Let

\[
A_{\mu\nu}[u, \tilde{g}] := \langle \psi_\mu[u, \tilde{g}], \psi_\nu[u, \tilde{g}] \rangle_{L^2(f^*_u g)}.
\]

As \( A_{\mu\nu}[0, \delta] = \delta_{\mu\nu} \) we learn that for \( \epsilon \) small enough \( (\psi_\mu[u, \tilde{g}])^2_{\mu=0} \) constitutes a basis of \( K[u, \tilde{g}] \) and that the matrix \( A_{\mu\nu}[u, \tilde{g}] \) is invertible. As a consequence we may write the \( L^2(f^*_u g) \)-projection of any \( X \) onto the space \( K[u, \tilde{g}] \) as

\[
P_{K[u, \tilde{g}]}(X) := \sum_{\mu, \nu=0}^2 A_{\mu\nu}[u, \tilde{g}][\psi_\mu[u, \tilde{g}], X]_{L^2(f^*_u g)}\psi_\nu[u, \tilde{g}].
\]

We denote the complementary projection onto the \( L^2(f^*_u g) \)-orthogonal complement \( K[u, \tilde{g}]^\perp \) by \( P_{K^\perp[u, \tilde{g}]} \). As \( u \) and \( \tilde{g} \) are usually clear from context we often drop them in the notation as simply write \( P_K \) and \( P_K^\perp \).

The metric \( \tilde{g}^{p,\lambda} \) satisfies \( \| \tilde{g}^{p,\lambda} \|_{C^{n-1}} \leq C(\Omega)\lambda \) if \( \Omega \) is of class \( C^n \). Thus we learn that for \( \lambda \leq \lambda_0(\Omega, \epsilon) \) we may always achieve \( \| \tilde{g} - \delta \|_{C^5} \leq \epsilon \).

F  Computation of metric derivative

We abbreviate \( f := f_u, \tilde{g}(t) := \tilde{g}^{\xi(t),\lambda} \) and denote the \( L^2(f^* \tilde{g}) \) scalar product by \( \langle \cdot, \cdot \rangle \). We also put \( \phi(t) := F^\lambda[\xi(t), f(t)] \). We have to compute \( D_2 C^i[u, \tilde{g}][\tilde{g}] \) as well as \( D_2 A[u, \tilde{g}][\tilde{g}] \).
Barycenter
We begin by investigating the barycenter. As $C[\lambda, \tilde{f}(t), \tilde{g}(t)] = 0$ we may write

$$D_2 C[\lambda, \tilde{f}(t), \tilde{g}(t)] \tilde{g}(t) = -\langle \nabla C[\lambda, \tilde{f}(t), \tilde{g}(t)], \tilde{g}(\tilde{f}(t), \tilde{v}) \rangle.$$

The geometric object the evolves in time is $\phi$. Its evolution is then decomposed into two parts. $f$ only represents the evolution of the ‘sphere-shape’ while the movement of the barycenter is not contained in $f$. However, to exploit that $C[f(t)] = \xi(t)$ we must take the full evolution of $\phi$ into account. For that purpose we fix $t$ and consider small time displacements $\epsilon$ from $t$. We may then write

$$\phi(t + \epsilon) = F^\lambda \xi(t), h(t, \epsilon)$$

where $h(t, 0) = f(t)$. Unlike $f$ the quantity $h$ encodes the full evolution of $\phi$ as $t$ is a fixed time and the dynamical variable is now $\epsilon$. Expressing $\xi(t + \epsilon) = C[\phi(t + \epsilon)]$ inside the chart centered at $\xi(t)$ as $\xi(t + \epsilon) = f[\xi(t), (\xi^1(t + \epsilon), \xi^2(t + \epsilon))$ and differentiating at $\epsilon = 0$ gives

$$\lambda^{-1} \dot{\xi}^i(t) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \lambda^{-1} \xi^i(t + \epsilon)$$

$$= \frac{d}{d\epsilon} \bigg|_{\epsilon=0} C^i[h(t, \epsilon), \tilde{g}\xi(t), \lambda]$$

$$= \langle \nabla C^i[f(t), \tilde{g}(t)], \tilde{g} \big(\frac{\partial h}{\partial \epsilon} \big|_{\epsilon=0}, \tilde{v}\rangle. \quad (F.2)$$

Here $\tilde{v}$ is the inner normal of $h$. Next we must relate $\partial_h h(t, 0)$ to $\tilde{f}(t)$. For that purpose we use the relation $F^\lambda \xi(t + \epsilon), f(t + \epsilon)] = \phi(t) = F^\lambda \xi(t), h(t, \epsilon)]$ to obtain

$$D_2 F^\lambda \xi(t), h(t, 0) \bigg|_{\epsilon=0} \frac{\partial h}{\partial \epsilon} = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} F^\lambda \xi(t), h(t, \epsilon)]$$

$$= \frac{d}{d\epsilon} \bigg|_{\epsilon=0} F^\lambda \xi(t + \epsilon), f(t + \epsilon)]$$

$$= D_1 F^\lambda \xi(t), f(t) \dot{\xi}(t) + D_2 F^\lambda \xi(t), f(t) \tilde{f}(t). \quad (F.3)$$

Remembering $h(t, 0) = f(t)$, multiplying with $(D_2 F^\lambda \xi(t), f(t))]^{-1}$, recalling $X$ from Equation (2.22) and inserting into equations (F.1) and (F.2) yields

$$D_2 C^i[f(t), \tilde{g}(t)] \tilde{g}(t) \quad (F.1)$$

$$= -\langle \nabla C^i[f(t), \tilde{g}(t)], \tilde{g}(\tilde{f}(t), \tilde{v}) \rangle$$

$$= -\langle \nabla C^i[f(t), \tilde{g}(t)], \tilde{g} \big(\frac{\partial h}{\partial \epsilon} \big|_{\epsilon=0}, \tilde{v}) - X \rangle$$

$$= -\frac{1}{\lambda} \dot{\xi}^i(t) + \langle \nabla C^i[f(t), \tilde{g}(t)], X \rangle. \quad (F.2)$$

Area
We copy the derivation from above. This time we use that $A[f(t), \tilde{g}(t)] = 2\pi$. Taking the derivative then gives

$$D_2 C^i[f(t), \tilde{g}(t)] \tilde{g}(t) = -\langle \nabla C^i[f(t), \tilde{g}(t)], \tilde{g}(\tilde{f}(t), \tilde{v}) \rangle. \quad (F.3)$$
Next we must use that $A[\phi(t)] = 2\pi \lambda^2$ and again use $h(t, \epsilon)$ for small $\epsilon$ to obtain

$$0 = \langle \nabla A[f(t), \tilde{g}(t)], \tilde{g} \left( \frac{\partial h}{\partial \epsilon} \right)_{\epsilon=0}, \tilde{v} \rangle.$$ 

We reuse Equation (F.3) to derive the analogue of Equation (F.4) given by

$$D_2 A[f(t), \tilde{g}(t)] \tilde{g} = \langle \nabla A[f(t), \tilde{g}(t)], X \rangle.$$ 

**(F.6)**

**Conclusion**

We may now substitute Equations (F.4) and (F.6) into the definition of $\tau$ in Equation (2.17) to get

$$\tau[u, \tilde{g}] \overset{(2.17)}{=} - \sum_{\mu, \nu = 0}^{3} A^{\mu \nu}[u, \tilde{g}] \frac{D_2 C^{\mu}[u, \tilde{g}][\tilde{g}]}{\|\nabla C^{\mu}[u, \tilde{g}]\|_{L^2(f_* g)}} \frac{\nabla C^{\nu}[u, \tilde{g}][\tilde{g}]}{\|\nabla C^{\nu}[u, \tilde{g}]\|_{L^2(f_* g)}}$$

**(F.7)**

$$= - \sum_{\mu, \nu = 0}^{3} A^{\mu \nu}[u, \tilde{g}] \frac{\langle \nabla C^{\mu}[u, \tilde{g}], X \rangle}{\|\nabla C^{\mu}[u, \tilde{g}]\|_{L^2(f_* g)}} \frac{\nabla C^{\nu}[u, \tilde{g}][\tilde{g}]}{\|\nabla C^{\nu}[u, \tilde{g}]\|_{L^2(f_* g)}}$$

**(F.8)**

$$+ \sum_{\nu = 0}^{2} \sum_{i = 1}^{2} A^{i \nu}[u, \tilde{g}] \frac{\xi^i}{\lambda \|\nabla C^{i}[u, \tilde{g}]\|_{L^2(f_* g)}} \frac{\nabla C^{\nu}[u, \tilde{g}][\tilde{g}]}{\|\nabla C^{\nu}[u, \tilde{g}]\|_{L^2(f_* g)}}$$

$$= - PK(X) + \sum_{\nu = 0}^{2} A^{i \nu}[u, \tilde{g}] \frac{\xi^i}{\lambda \|\nabla C^{i}[u, \tilde{g}]\|_{L^2(f_* g)}} \frac{\nabla C^{\nu}[u, \tilde{g}][\tilde{g}]}{\|\nabla C^{\nu}[u, \tilde{g}]\|_{L^2(f_* g)}}.$$

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**References**

[1] Shmuel Agmon, Avron Douglis, and Louis Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. i. Communications on pure and applied mathematics, 12(4):623–727, 1959.

[2] Roberta Alessandroni and Ernst Kuwert. Local solutions to a free boundary problem for the Willmore functional. Calculus of Variations and Partial Differential Equations, 55(2):1–29, 2016.

[3] Nicholas D Alikakos and Alexandre Freire. The normalized mean curvature flow for a small bubble in a Riemannian manifold. Journal of Differential Geometry, 64(2):247–303, 2003.

[4] Abbas Bahri and Jean-Michel Coron. The scalar-curvature problem on the standard three-dimensional sphere. Journal of Functional Analysis, 95(1):106–172, 1991.
[5] Giovanni Bellettini and Giorgio Fusco. Some aspects of the dynamic of $V = H - \bar{H}$. *Journal of Differential Equations*, 157(1):206–246, 1999.

[6] Simon Eberle, Barbara Niethammer, and André Schlichting. Gradient flow formulation and longtime behaviour of a constrained Fokker–Planck equation. *Nonlinear Analysis*, 158:142–167, 2017.

[7] Šamuil D. Ejdel’man. Parabolic equations. In *Partial Differential Equations VI*, pages 203–316. Springer, 1994.

[8] Norihisa Ikoma, Andrea Malchiodi, and Andrea Mondino. Area-constrained Willmore surfaces of small area in Riemannian three-manifolds: an approach via Lyapunov-Schmidt reduction. In *Regularity and singularity for partial differential equations with conservation laws*, pages 31–50. Res. Inst. Math. Sci. (RIMS), Kyoto, 2017.

[9] Norihisa Ikoma, Andrea Malchiodi, and Andrea Mondino. Embedded area-constrained Willmore tori of small area in Riemannian three-manifolds: minimization. *Proceedings of the London Mathematical Society*, 115(3):502–544, 2017.

[10] Norihisa Ikoma, Andrea Malchiodi, and Andrea Mondino. Foliation by Area-constrained willmore spheres near a Nondegenerate Critical Point of the Scalar Curvature. *International Mathematics Research Notices*, 2020(19):6539–6568, 2020.

[11] Hermann Karcher. Riemannian center of mass and mollifier smoothing. *Communications on pure and applied mathematics*, 30(5):509–541, 1977.

[12] Tobias Lamm and Jan Metzger. Small surfaces of Willmore type in Riemannian manifolds, 2010.

[13] Tobias Lamm, Jan Metzger, and Felix Schulze. Foliations of asymptotically flat manifolds by surfaces of Willmore type, 2011.

[14] Marco Mattuschka. *The Willmore flow with prescribed area for a small bubble in a Riemannian manifold*. PhD thesis, University of Freiburg, 2018.

[15] Andrea Mondino. Some results about the existence of critical points for the Willmore functional. *Mathematische Zeitschrift*, 266(3):583–622, 2010.

[16] Frank Pacard and Xingwang Xu. Constant mean curvature spheres in Riemannian manifolds. *Manuscripta Mathematica*, 128(3):275–295, 2009.

[17] Leon Simon. Schauder estimates by scaling. *Calculus of Variations and Partial Differential Equations*, 5(5):391–407, 1997.

[18] Gieri Simonett. The Willmore flow near spheres. *Differential and Integral Equations*, 14(8):1005–1014, 2001.
[19] Axel Stahl. Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition. *Calculus of Variations and Partial Differential Equations, 4*(4):385–407, 1996.

[20] Rugang Ye. Foliation by constant mean curvature spheres. *Pacific Journal of Mathematics, 147*(2):381–396, 1991.