Utility and Privacy of Data Sources: Can Shannon Help Conceal and Reveal Information?

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Abstract—The problem of private information “leakage” (inadvertently or by malicious design) from the myriad large centralized searchable data repositories drives the need for an analytical framework that quantifies unequivocally how safe private data can be (privacy) while still providing useful benefit (utility) to multiple legitimate information consumers. Rate distortion theory is shown to be a natural choice to develop such a framework which includes the following: modeling of data sources, developing application independent utility and privacy metrics, quantifying utility-privacy tradeoffs irrespective of the type of data sources or the methods of providing privacy, developing a side-information model for dealing with questions of external knowledge, and studying a successive disclosure problem for multiple query data sources.

I. INTRODUCTION

Information technology and electronic communications have been rapidly applied to almost every sphere of human activity, including commerce, medicine and social networking. The concomitant emergence of myriad large centralized searchable data repositories has made “leakage” of private information such as medical data, credit card information, or social security numbers via data correlation (inadvertently or by malicious design) an important and urgent societal problem. Unlike the well-studied secrecy problem (e.g., [1]–[3]) in which the protocols or primitives make a sharp distinction between secret and non-secret data, in the privacy problem, disclosing data provides informational utility while enabling possible loss of privacy at the same time. In fact, in the course of a legitimate transaction, a user can learn some public information, which is allowed and needs to be supported, and at the same time also learn/infer private information, which needs to be prevented. Thus every user is (potentially) also an adversary. This drives the need for a unified analytical framework that can tell us unequivocally and precisely how safe private data can be (privacy) while still providing useful benefit (utility) to multiple legitimate information consumers.

It has been noted that utility and privacy are competing goals: perfect privacy can be achieved by publishing nothing at all, but this has no utility; perfect utility can be obtained by publishing the data exactly as received, but this offers no privacy [4]. Utility of a data source is potentially (but not necessarily) degraded when it is restricted or modified to uphold privacy requirements. The central problem of this paper is a precise quantification of the tradeoff between the privacy needs of the respondents (individuals represented by the data) and the utility of the sanitized (published) data for any data source.

Though the problem of privacy and information leakage has been studied for several decades by multiple research communities (e.g., [4]–[8] and the references therein), the proposed solutions have been both heuristic and application-specific. The recent groundbreaking theory of differential privacy [9], [10] from the theoretical computer science community is the first universal model that applies to any statistical database irrespective of application or content. However, the crucial challenges of an analytic characterization of both a utility metric and the privacy-utility tradeoff remains unaddressed. We seek to address these challenges using tools and techniques from information theory.

Rate distortion theory is a natural choice to study the utility-privacy tradeoff; utility can be quantified via fidelity which in turn is related to distortion and privacy can be quantified via equivocation. Our key insight is captured in the following theorem we present in this paper: for a data source with private and public data and desired utility level, maximum privacy for the private data is achieved by minimizing the information disclosure rate sufficient to satisfy the desired utility for the public data. To the best of our knowledge this is the first observation that tightly relates utility and privacy.

In a sparsely referenced paper [11] from three decades ago, Yamamoto developed the tradeoff between rate, distortion, and equivocation for a specific and simple source model. In this paper, we show via the above summarized theorem that Yamamoto’s formalism can be translated into the language of data disclosure. Furthermore, we develop a framework which allows us to model data sources, specifically databases, develop application independent utility and privacy metrics, quantify the fundamental bounds on the utility-privacy tradeoffs, and develop a side-information model for dealing with questions of external knowledge, and study the utility-privacy tradeoffs for multiple query data sources as a successive disclosure problem. The final problem arises in the following context: real-world data sources are in general interactive, that is, they allow users multiple interactions (queries). However, modeling this analytically is particularly challenging. Our framework can handle the non-interactive (single query) case for any given utility and privacy requirements. In this paper, we
study the interactive case as a successive disclosure problem modeled along the information-theoretic successive refinement problem.

II. THE DATABASE PRIVACY PROBLEM

A. Problem Definition

While the problem of quantifying the utility/privacy problem applies to all types of data sources, we start our study with databases because they are highly structured and historically better studied than other types of sources. A database is a table (matrix) whose rows represent the individual entries and whose columns represent the attributes of each entry [5]. For example, the attributes of each entry in a healthcare database typically include name, address, social security number (SSN), gender, and a collection of medical information and each entry contains the information pertaining to an individual. Messages from a user to a database are called queries and, in general, result in some numeric or non-numeric information from the database termed the response.

The goal of privacy protection is to ensure that, to the extent possible, the user’s knowledge is not increased beyond strict predefined limits by interacting with the database. The goal of utility provision is, generally, to maximize the amount of information that the user can receive. Depending on the relationship between attributes, and the distribution of the actual data, a response may contain information that can be inferred beyond what is explicitly included in the response. The privacy policy defines the information that should not be revealed explicitly or by inference to the user and depends on the context and the application. For example, in a database on health statistics, attributes such as name and SSN may be considered private data, whereas in a state motor vehicles database only the SSN is considered private. The challenge for privacy protection is to design databases such that any response does not reveal information contravening the privacy policy.

B. Current Approaches and Metrics

The approaches considered in the literature have centered on perturbation (also called sanitization) which encompasses a general class of database modification techniques that ensure that a user only interacts with a modified database that is derived from the original (e.g.: [4], [6]–[8]). Most of the current perturbation-based approaches are heuristic and application-specific and often focus on additive noise approaches.

Perturbation techniques depend on whether the database is considered interactive (i.e. whether the user can issue more queries after seeing earlier responses) or non-interactive [9]. In the non-interactive model, the database is published after a sanitization process in which personal identifiers are eliminated and the data is perturbed using one of many possible input perturbation approaches; alternately in the interactive model, the database adds noise to the response based on a data model.

In order to quantify the privacy and utility afforded by a data source, metrics are critical. The concept of k-anonymity proposed by Sweeney [7] captures the intuitive notion of privacy that every individual entry should be indistinguishable from (k − 1) other entries for some large value of k. More recently, researchers in the data mining community have proposed to quantify the privacy loss resulting from data disclosure as the mutual information between attribute values in the original and perturbed data sets, both modeled as random variables [8]. Finally, motivated by cryptographic models, the concept of differential privacy from theoretical computer science [9], [10] has created a universal model for privacy which measures the risk of loss of privacy to an individual whose data is in a statistical database. However, this work as well the others described above do not propose a companion universal utility metric that can be guaranteed along with privacy.

C. Privacy vs. Utility

While the privacy problem has been studied by multiple communities using multiple approaches, the companion utility problem has not been studied as analytically and exhaustively except in the context of specific applications. Indeed, most discussions of privacy assume an implicit utility that is left unstated or unmeasured. Utility of a data source is, by necessity, a relative concept and is measured from the point of view of the user: utility is maximal when the user gets full information flow and reduces when the flow of certain information is reduced either by restriction or the addition of noise. The general concept of utility as a measure of the approximation to an underlying (but undisclosed) quantity is a fertile area of research (e.g.: [12], [13]). However, these measures have not been customized for the context of privacy enhancement. Heuristic measures of utility in the context of privacy have been proposed (e.g.: [4]) but they do not yield a general notion of utility. In our proposed work, we will use a working definition of utility as the measure of the distance or divergence (using suitably chosen metrics such as Euclidean or Kullback-Leibler divergence) between the original and sanitized databases.

III. AN INFORMATION-THEORETIC APPROACH

A. Model for Databases

Circumventing the semantic issue: In general, utility and privacy metrics tend to be application specific. Focusing our efforts on developing an analytical model, we propose to capture a canonical database model and representative abstract metrics. Such a model will circumvent the classic privacy issues related to the semantics of the data by assuming that there exist forward and reverse maps of the data set to the proposed abstract format (for e.g., a string of bits or a sequence of real values). Such mappings are often implicitly assumed in the privacy literature [4], [8], [9]: our motivation for making it explicit is to separate the semantic issues from the abstraction and apply Shannon-theoretic techniques.

Model: Our proposed model focuses on large databases with K attributes per entry. Let \( X_k \in \mathcal{X}_k \) be a random variable
A database \( d \) with \( n \) rows is a sequence of \( n \) independent observations of \( X \) with the distribution
\[
p_{X}(x) = p_{X_1, X_2, \ldots, X_K}(x_1, x_2, \ldots, x_K)
\]
which is assumed to be known to the designers of the database. Our assumption of row independence in (1) is justified because correlation in databases is typically across attributes and not across entries. We write \( X^n = (X^n_1, X^n_2, \ldots, X^n_K) \) to denote the \( n \) independent observations of \( X \). This database model is universal in the sense that most practical databases can be mapped to this model.

A joint distribution in (1) models the fact that the attributes in general are correlated and can reveal information about one another. In addition to the revealed information, a user of a database can have access to correlated side information from other information sources. We model the side-information as an \( n \)-length sequence \( Z^n \) which is correlated with the database entries via a joint distribution \( p_{XZ}(x,z) \).

**Public and private variables:** We consider a general model in which some attributes need to be kept private while the source can reveal a function of some or all of the attributes. We write \( K_r \) and \( K_h \) to denote sets of private (subscript \( h \) for hidden) and public (subscript \( r \) for revealed) attributes, respectively, such that \( K_r \cup K_h = K \equiv \{1, 2, \ldots, K\} \). We further denote the corresponding collections of public and private attributes by \( X_r = \{X_k\}_{k \in K_r} \) and \( X_h = \{X_k\}_{k \in K_h} \), respectively. Our notation allows for an attribute to be both public and private; this is to account for the fact that a database may need to reveal a function of an attribute while keeping the attribute itself private. In general, a database can choose to keep public (or private) one or more attributes \((K > 1)\). Irrespective of the number of private attributes, a non-zero utility results only when the database reveals an appropriate function of some or all of its attributes.

**Special cases:** For \( K = 1 \), the lone attribute of each entry (row) is both public and private, and thus, we have \( X \equiv X_r \equiv X_h \). Such a model is appropriate for data mining [8] and census [4], [6] data sets in which utility generally is achieved by revealing a function of every entry of the database while simultaneously ensuring that no entry is completely revealed. For \( K = 2 \) and \( K_h \cup K_r = K \) and \( K_h \cap K_r = \emptyset \), we obtain the Yamamoto model in [11].

**B. Metrics: The Privacy and Utility Principle**

Even though utility and privacy measures tend to be specific to the application, there is a fundamental principle that unifies all these measures in the abstract domain. The aim of a privacy-preserving database is to provide some measure of utility to the user while at the same time guaranteeing a measure of privacy for the entries in the database.

A user perceives the utility of a perturbed database to be high as long as the response is similar to the response of the original database; thus, the utility is highest of an original (unperturbed) database and goes to zero when the perturbed database is completely unrelated to the original database. Accordingly, our utility metric is an appropriately chosen average ‘distance’ function between the original and the perturbed databases. Privacy, on the other hand, is maximized when the perturbed response is completely independent of the data. Our privacy metric measures the difficulty of extracting any private information from the response, i.e., the amount of uncertainty or equivocation about the private attributes given the response.

**C. A Privacy-Utility Tradeoff Model**

We now propose a privacy-utility model for databases. *Our primary contribution is demonstrating the equivalence between the database privacy problem and a source coding problem with additional privacy constraints.* For our abstract universal database model, sanitization is thus a problem of mapping a set of database entries to a different set subject to specific utility and privacy requirements. Our notation below relies on this abstraction.

Recall that a database \( d \) with \( n \) rows is an instantiation of \( X^n \). Thus, we will henceforth refer to a real database \( d \) as an *input sequence* and to the corresponding sanitized database \((SDB) d'\) as an *output sequence*. When the user has access to side information, the reconstructed sequence at the user will in general be different from the SDB sequence.

Our coding scheme consists of an encoder \( F_E \) which is a mapping from the set of all input sequences (i.e., all databases \( d \) picked from an underlying distribution) to a set of indices \( W \equiv \{1, 2, \ldots, M\} \) and an associated table of output sequences (each of which is a \( d' \)) with a one-to-one mapping to the set of indices given by
\[
F_E : (X^n_1 \times X^n_2 \times \ldots \times X^n_K)_{k \in K_{enc}} \rightarrow W \equiv \{SDB_k\}_{k=1}^M
\]
where \( K_r \subseteq K_{enc} \subseteq K \) and \( M = 2^{nR} \) is the number of output (sanitized) sequences created from the set of all input sequences. The encoding rate \( R \) is the number of bits per entry (without loss of generality, we assume \( n \) entries in \( d \) and \( d' \)) of the sanitized database. The encoding \( F_E \) in (2) includes both public and private attributes in order to model the general case in which the sanitization depends on a subset of all attributes.

A user with a view of the SDB (i.e., an index \( w \in W \) for every \( d \)) and with access to side information \( Z^n \), whose entries \( Z_i, i = 1, 2, \ldots, n \), take values in the alphabet \( Z \), reconstructs the database \( d' \) via the mapping
\[
F_D : W \times Z^n \rightarrow \{\hat{X}^n_{r,m}\}_{m=1}^M \in \left( \prod_{k \in K_r} X^n_k \right)
\]
where \( \hat{X}^n_{r,m} = F_D (F_E (X^n)) \).

A database may need to satisfy multiple utility constraints for different (disjoint) subsets of attributes, and thus, we consider a general framework with \( L \geq 1 \) utility functions that need to be satisfied. Relying on the distance based utility principle, we model the \( l^{th} \) utility, \( l = 1, 2, \ldots, L \), via the requirement that the average distortion \( \Delta_l \) of a function \( f_l \) of
the revealed variables is upper bounded, for some $\epsilon > 0$, as

$$u_l : \Delta_l \equiv \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} g \left( f_l (X_{r,i}) ; f_l \left( \hat{X}_{r,i} \right) \right) \right] \leq D_l + \epsilon, \quad l = 1, 2, \ldots, L,$$

where $g(\cdot, \cdot)$ denotes a distortion function, $\mathbb{E}$ is the expectation over the joint distribution of $(X_r, \hat{X}_r)$, and the subscript $i$ in $X_{r,i}$ and $\hat{X}_{r,i}$ denotes the $i^{th}$ entry of $X_r$ and $\hat{X}_r$, respectively. Examples of distance-based distortion functions include the Euclidean distance for Gaussian distributed database entries, the Hamming distance for binary input and output sequences, and the Kullback-Leibler (K-L) ‘distance’ comparing the input and output distributions.

Having argued that a quantifiable uncertainty captures the privacy of a database, we model the uncertainty or equivocation about the private variables using the entropy function as

$$p : \Delta_p \equiv \frac{1}{n} H(X^n_k | W, Z^n) \geq E - \epsilon,$$

i.e., we require the average number of uncertain bits per dimension to be lower bounded by $E$. The case in which side information is not available at the user is obtained by simply setting $Z^n = 0$ in (3) and (5). While our general problem allows separate constraints on the privacy and utility, we show later that for specific canonical databases (census and data mining) a constraint on only one of them (utility or privacy) suffices (see Corollary 6 in Section III-E).

The utility and privacy metrics in (4) and (5), respectively, capture two aspects of our universal model: a) both represent averages by computing the metrics across all database instantiations $d$, and b) the metrics bound the average distortion and privacy per entry. Thus, as the likelihood of the non-typical sequences decreases exponentially with increasing $n$ (very large databases), these guarantees apply nearly uniformly to all (typical) entries. Our general model also encompasses the fact that the exact mapping from the distortion and equivocation domains to the utility and privacy domains, respectively, can depend on the application domain. We write $D \equiv (D_1, D_2, \ldots, D_L)$ and $\Delta \equiv (\Delta_1, \Delta_2, \ldots, \Delta_L)$. Based on our notation thus far, we define the utility-privacy tradeoff region as follows.

**Definition 1:** The utility-privacy tradeoff region $T$ is the set of all feasible utility-privacy tuples $(D, E)$ for which there exists a coding scheme $(F_E, F_D)$ given by (2) and (3), respectively, with parameters $(n, M, \Delta, \Delta_p)$ satisfying the constraints in (4) and (5).

**D. Equivalence of Utility-Privacy and Rate-Distortion-Equivocation**

We now present an argument for the equivalence of the above utility-privacy tradeoff analysis with a rate-distortion-equivocation analysis of the same source. For the database source model described here, a classic lossy source coding problem is defined as follows.

**Definition 2:** The set of tuples $(R, D)$ is said to be feasible (achievable) if there exists a coding scheme given by (2) and (3) with parameters $(n, M, \Delta)$ satisfying the constraints in (4) and a rate constraint

$$M \leq 2^{n(R + \epsilon)}.$$  

When an additional privacy constraint in (5) is included, the source coding problem becomes one of determining the achievable rate-distortion-equivocation region defined as follows.

**Definition 3:** The rate-distortion-equivocation region $R$ is the set of all tuples $(R, D, E)$ for which there exists a coding scheme given by (2) and (3) with parameters $(n, M, \Delta, \Delta_p)$ satisfying the constraints in (4), (5), and (6). The set of all feasible distortion-equivocation tuples $(D, E)$ is denoted by $R_{D-E}$, the equivocation-distortion function in the $D-E$ plane is denoted by $\Gamma(D)$, and the equivocation-distortion function which quantifies the rate as a function of both $D$ and $E$ is denoted by $\Gamma(D, E)$.

Thus, a rate-distortion-equivocation code is by definition a (lossy) source code satisfying a set of distortion constraints that achieves a specific privacy level for every choice of the distortion tuple. In the following theorem, we present a basic result capturing the precise relationship between $T$ and $R$. To the best of our knowledge, this is the first analytical result that quantifies a tight relationship between utility and privacy. We briefly sketch the proof here; details can be found in [14].

**Theorem 4:** For a database with a set of utility and privacy metrics, the tightest utility-privacy tradeoff region $T$ is the distortion-equivocation region $R_{D-E}$.

**Proof:** The crux of our argument is the fact that for any feasible utility level $D$, choosing the minimum rate $R(D)$, ensures that the least amount of information is revealed about the source via the reconstructed variables. This in turn ensures that the maximum privacy of the private attributes is achieved for that utility since, in general, the public and private variables are correlated. For the same set of utility constraints, since such a rate requirement is not a part of the utility-privacy model, the resulting privacy achieved is at most as large as that in $R_{D-E}$ (see Fig. 1(a)).

Implicit in the above argument is the fact that a utility-privacy achieving code does not perform any better than a rate-distortion-equivocation code in terms of achieving a lower rate (given by $\log_2 M/n$) for the same distortion and privacy constraints. This is because if such a code exists then we can always find an equivalent source coding problem for which the code would violate Shannon’s source coding theorem [15]. An immediate consequence of this is that a distortion-constrained source code suffices to preserve a desired level of privacy; in other words, the utility constraints require revealing data which in turn comes at a certain privacy cost that must be borne and vice-versa. We capture this observation in Fig. 1(b) where we contrast existing privacy-exclusive and utility-exclusive regimes (extreme points of the utility-privacy tradeoff curve) with our more general approach of determining the set of feasible utility-privacy tradeoff points.

From an information-theoretic perspective, the power of Theorem 4 is that it allows us to study the larger problem
of database utility-privacy tradeoffs in terms of a relatively familiar problem of source coding with privacy constraints. As noted previously, this problem has been studied for a specific source model by Yamamoto and here we expand his elegant analysis to arbitrary database models including those with side information at the user.

E. Capturing the Effects of Side-Information

It has been illustrated that when a user has access to an external data source (which is not part of the database under consideration) the level of privacy that can be guaranteed changes [7], [9]. We cast this problem in information-theoretic terms as a side information problem.

In an extended version of this work [14], we have developed the tightest utility-privacy tradeoff region for the three cases of a) no side information ($L = 1$ case studied in [11]), b) side information only at the user, and c) side information at both the source (database) and the user. We present a result for the case with side information at the user only and for simplicity, we assume a single utility function, i.e., $L = 1$. The proof mimics that of source coding with side information in [16] and therefore, involves the use of an auxiliary random variable $U$. The proof also includes bounds on the equivocation along the lines of those in [11, Appendix 1]. The following theorem defines the bounds on the region $\mathcal{R}$ in Definition 4 via the functions $\Gamma(D)$ and $R(D, E)$ where $\Gamma(D)$ bounds the maximal achievable privacy and $R(D, E)$ is the minimal information rate (see Fig. 1(a)) for very large databases ($n \to \infty$). The proof is omitted due to space and can be found in [14].

**Theorem 5:** For a database with side information available only at the user, the functions $\Gamma(D)$ and $R(D, E)$ and the regions $\mathcal{R}_{D-E}$ and $\mathcal{R}$ are given by

$$\Gamma(D) = \sup_{p(x, x_s, x_h) : p(x|x, x_s, z) \in P(D)} H(X_h|UZ) \quad (7)$$

$$R(D, E) = \inf_{p(x, x_s, x_h) : p(u|x, x_s) \in P(D, E)} I(X_h X_r U) - I(Z; U) \quad (8)$$

$$\mathcal{R}_{D-E} = \{(D, E) : D \geq 0, 0 \leq E \leq \Gamma(D)\} \quad (9)$$

$$\mathcal{R} = \{(R, D, E) : D \geq 0, 0 \leq E \leq \Gamma(D), R \geq R(D, E)\} \quad (10)$$

where $P(D, E)$ is the set of all $p(x, x_s, z) : p(u|x, x_s) \in P(D, E)$ such that $E[d(X_r, g(U, Z))] \leq D$ and $H(X_h|UZ) \geq E$ while $P(D)$ is defined as

$$P(D) = \bigcup_{H(X_h|X_r U) \leq E} P(D, E) \quad (11)$$

While Theorem 5 applies to a variety of database models, it is extremely useful in quantifying the utility-privacy tradeoff for the following special cases of interest.

i) **The single database problem** (i.e., no side information): SDB is revealed. Here, we have $Z = 0$ and $U = X_r$, i.e., the reconstructed vectors seen by the user are the same as the SDB vectors.

ii) **Completely hidden private variables:** Privacy is completely a function of the statistical relationship between public, private, and side information data. The expression for $R(D, E)$ in (8) assumes the most general model of encoding both the private and the public variables. When the private variables can only be deduced from the revealed variables, i.e., $X_h - X_r - U$ is a Markov chain, the expression for $R(D, E)$ in (8) will simplify to the Wyner-Ziv source coding formulation [16], thus clearly demonstrating that the privacy of the hidden variables is a function of both the correlation between the hidden and revealed variables and the distortion constraint.

iii) **Census and data mining problems without side information:** Information rate completely determines privacy achievable. For $Z = 0$, setting $X_r = X_h \equiv X$ (such that
With this substitution, from Theorem 5, we have the following corollary. This fundamental result is captured in the preceding corollary.

**Corollary 6:** For the special case of \( K = 1 \), i.e., \( X_r = X_h = X \), the utility-privacy problem is completely defined by a utility constraint since the maximum achievable equivocation is directly obtainable from the minimal information transfer rate.

### F. A Successive Disclosure Problem

As mentioned earlier, databases can be broadly categorized as non-interactive and interactive depending on whether the data is sanitized once before publishing or repeatedly in response to each query, respectively. For census and similar statistical databases a one-shot sanitization is typical whereas for more interactive databases multiple queries can lead to multiple sanitizations.

**Single-query model:** The model and analysis proposed in Sections III-A-III-D capture the non-interactive database model and the resulting utility-privacy tradeoff region. For this one-shot model, sanitization is determined by the choice of the utility and privacy metrics defined *a priori*. In contrast to existing approaches that are dominantly focused on additive noise perturbations satisfying a large set of queries [17], [18], our one-shot approach is independent of queries and is designed to satisfy specific utility and privacy constraints. Such a model is relevant for databases such as those with medical and clinical data that may find repeated uses in the future but with queries that cannot be predicted ahead of time or which require query-independent strict sanitization prior to interaction to ensure regulatory compliance (e.g., US HIPAA privacy policies [19]).

**Multiple-query model:** For a large majority of data repositories, utility is a function of their usage and as such the problem of addressing the utility-privacy tradeoffs in a multiple query model is imperative. A side-effect of allowing multiple queries is that a user can refine her query to learn more information at each step, which in turn can lead to privacy breaches. Our aim is to determine if a certain level of overall utility can be guaranteed while preserving a desired overall privacy threshold. In the absence of disclosure controls, a database will typically respond to each query independently of the previous queries. We seek to develop a model in which the database is cognizant of current and past queries in responding to future queries. To this end, we assume the existence of a data collector that provides an interface for the user to submit queries and collate the responses over multiple queries, a common assumption in the multi-query literature [9], [10], [18]. For this model, under the assumption that the user wishes to obtain a refined view of the source, we propose to determine whether a source can be successively disclosed, i.e., whether a set of overall utility and privacy constraints can be satisfied via multiple disclosures with increasing refinement at each stage and without any information loss relative to an equivalent single-shot model with the same overall utility and privacy constraints.

This problem of successive disclosure has a natural relationship to a problem of *successive refinement* in information theory, which pertains to determining whether successively revealing data from a source with decreasing distortion at each stage can ensure no rate loss relative to a one-shot approach with the same final distortion [20]–[22]. We demonstrate this analogy in Fig. 2 where, at the first stage, the user obtains a specific view (denoted \( X_1 \) of a source \( X \)) of the source which in conjunction with the second stage provides a final refined view \( X_2 \). While the successive refinement problem is to determine whether \( \tilde{X}_2 = R(D_2) \), the successive disclosure problem is that of determining whether \( R_2 = R(D_2, E_2) \) where \( D_2 < D_1 \) and \( E_2 < E_1 \). As with the successive refinement problem, our results can help determine the conditions and relationships between the input and output sequences under which a source can be disclosed successively.

Analogous to successive refinement, we start by studying a *multiple disclosure* problem in which we seek to determine the rates \( R_0 \) and \( R_1 \) at which the database responds with distortion (utility) and privacy levels \( (D_0, E_0) \) and \( (D_1, E_1) \) to two queries, respectively, such that a user using both query responses can reconstruct a response at a distortion-privacy level of \( (D_2, E_2) \). Analogous to the relationship between multiple description and successive refinement, the successive disclosure problem described here is a special case of the multiple disclosure problem for which there is no rate loss, i.e., \( R_1 = R(D_1, E_1) \) and \( R_0 + R_1 = R(D_2, E_2) \).

While a detailed analysis of this problem can be found in an extended version of this work [14], we now present two example privacy problems for which the successive refinement problem presents immediate insights on the effects of refined disclosure. The two problems are privacy preservation in census and data mining databases, and in both cases, we briefly argue that the successive disclosure problem simplifies to the successive refinement problem. Recall that in Corollary 6 we showed that the census and data mining problems are special cases for which the rate-distortion-equivocation region is directly obtainable from the rate-distortion curve because for both problems the public and the private variables are the same as a result of which the maximum achievable equivocation is directly obtainable from the rate-distortion function. The following theorem summarizes our result.

**Theorem 7:** For \( K = 1 \) databases, successive disclosure with distortion-privacy pairs \( (D_1, E_1) \) and \( (D_2, E_2) \) are achievable if and only if there exists a conditional distribution...
**Successive Refinement**

1st Refinement:
Index $W_1 \in \{1, 2, \ldots, 2^{m(k_R-R_k)}\}$

User: $\hat{X}_1(W_1)$, distortion $D_1$

2nd Refinement:
Index $W_2 \in \{1, 2, \ldots, 2^{m(R_k-R_k)}\}$

User: $\hat{X}_2(W_1, W_2)$, distortion $D_2$

**Successive Disclosure**

1st Query:
Index $W_1 \in \{1, 2, \ldots, 2^{m(k_R-R_k)}\}$

User: $\hat{X}_1(W_1)$, distortion $D_1$

Privacy $E_1$

2nd Query:
Index $W_2 \in \{1, 2, \ldots, 2^{m(R_k-R_k)}\}$

User: $\hat{X}_2(W_1, W_2)$, distortion $D_2$

Privacy $E_2$

![Fig. 2. Successive Refinement and Successive Disclosure Problems.](image)

$p(\hat{x}_1, \hat{x}_2|x)$ with

$$\mathbb{E} \left[ g \left( X, \hat{X}_k \right) \right] \leq D_k, \quad k = 1, 2, \quad (12)$$

such that

$$R(D_k, E_k) = I(X; \hat{X}_k), \quad k = 1, 2, \quad (13)$$

and $X - \hat{X}_2 - \hat{X}_1$ form a Markov chain, i.e.,

$$p(\hat{x}_1, \hat{x}_2|x) = p(\hat{x}_2|x)p(\hat{x}_1|\hat{x}_2). \quad (14)$$

Thus, for these two special but fundamentally important problems, we can show that the Markov condition $X - \hat{X}_2 - \hat{X}_1$ (see Fig. 2) required for successive refinement [20, Theorem 2] also hold here and in fact suffices to satisfy the successive disclosure requirement of no additional rate or privacy leakage. More work is needed to address questions such as the practical implications of the above Markov condition [21] and generalizing the solution to arbitrary sources.

**IV. CONCLUDING REMARKS**

We have presented an abstract model for databases with an arbitrary number of public and private variables, developed application-independent privacy and utility metrics, used rate distortion theory to determine the fundamental utility-privacy tradeoff limits, and introduced a successive disclosure problem to study utility-privacy tradeoffs and determine the conditions for no privacy loss for multiple query data sources. Future work includes generalizing the results to distributed data sources and relating current approaches in computer science and our universal approach.

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