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Introduction.

This paper concludes the project started in [K6]. The goal of the project is to find a non-commutative generalization of the classical construction of the $p$-typical Witt vectors ring $W(A)$ associated to a commutative associative ring $A$. As we have explained in the introduction to [K6], such a generalization should also produce a generalization of the de Rham-Witt complex of Deligne and Illusie [I], and a generalization of the Witt vectors group $W(A)$ constructed for any associative ring $A$ by Hesselholt [H2], [H3].

The common theme unifying [I] and [H2] is Hochschild homology. The relation between de Rham-Witt complex and Hochschild homology is implicit, and it goes through the identification of differential forms and Hochschild homology classes discovered by Hochschild, Kostant and Rosenberg [HKR].

To take account of the de Rham differential, one has to use cyclic homology ([L], [FT]). The relation between Hesselholt’s construction and Hochschild homology is much more direct, and it goes back to the theory of Topological Cyclic Homology and cyclotomic trace of Bökstedt, Hsiang and Madsen ([BHM], see also an exposition in [HM]). As a part of their theory, Bökstedt, Hsiand and Madsen construct a spectrum $TR(A; p)$ for any ring spectrum $A$, and in particular, for any associative ring $A$. This spectrum is manifestly related to Hochschild homology — in fact, it is constructed starting from Bökstedt’s Topological Hochschild Homology spectrum $THH(A)$ of [B]. What Hesselholt did in [H2] was the following: he took an arbitrary associative ring $A$, and gave a purely algebraic construction of the homotopy group $\pi_0 TR(A; p)$.

It is known that if the ring $A$ is in fact an algebra over a field $k$ of characteristic $p$ — which happens to be the most interesting case for the Witt vector construction — then $TR(A; p)$ is an Eilenberg-Mac Lane spectrum. Thus effectively, $TR(A; p)$ is a chain complex $TR_*(A; p)$, and its homotopy groups $\pi_* TR(A; p)$ are the homology groups of the complex $TR_*(A; p)$. In the situation of the Hochschild-Kostant-Rosenberg Theorem — that is, if $A$ is commutative and smooth — it has been further proved by Hesselholt [H1] that these homology groups are naturally identified with the terms of the de Rham-Witt complex of $X = \text{Spec } A$. This established the link between [I] and [BHM] and became the subject of a lot of further research.

For a general associative unital $k$-algebra $A$, one still has the complex $TR_*(A; p)$, but there is no ready algebraically defined complex one could compare it with. A natural goal is, then, to try to construct such a complex.

This is not what we do in this paper, or at least, not quite. What
we do is construct a functorial “Hochschild-Witt complex” $WCH_q(A)$ for any associative unital algebra over a perfect field $k$ of characteristic $p$ that looks very much like $TR_q(A; p)$ and carries all the additional structures that $TR_q(A; p)$ is known to carry. However, our construction is purely homological, and at present, we do not know how to compare it to equivariant stable homotopy theory used in [BHM]. Lacking a comparison theorem between $WHH_q(A)$ and $TR_q(A; p)$, we at least prove that in degree 0, the homology group $WHH_0(A)$ of the complex $WCH_q(A)$ is indeed Hesselholt’s Witt vectors group, and in the Hochschild-Kostant-Rosenberg situation, our construction recovers the de Rham-Witt complex of Deligne and Illusie.

As explained in the introduction to [K6], our main method is to observe that Hochschild homology is in fact a theory with two variables, an algebra $A$ and an $A$-bimodule $M$. Thus it is natural to expect to have a two-variable “Hochschild-Witt homology” functors $WHH_q(A, M)$. Moreover, if one can manage to equip with functors with an additional structure of a “trace theory” in the sense of [K2], then one only needs to construct $WHH_q(A, M)$ for $A = k$, the base field. This is exactly what has been done in [K6]. After that, the groups $WHH_q(A, M)$ can be simply produced by the general machine of [K2], and taking $M = A$, the diagonal bimodule, one recovers the homology groups $WHH_q(A) = WHH_q(A, A)$ of the Hochschild-Witt complex $WCH_q(A)$.

However, the general machine gives very little information on the groups $WHH_q(A)$, nor allows one to prove any comparison theorems. From this point of view, the Hochschild-Witt complex $WCH_q(A)$ deserves a separate detailed study. This is exactly what the present paper aims to provide.

Let us give a brief overview of the paper. We start with preliminaries, and there is a lot of them — to the point of taking up whole two sections, Section 1 and Section 2. The reason for this is that our approach to Hochschild and cyclic homology is based on the fundamental principle discovered by Connes [C]: a large part of the formalism of cyclic homology has nothing to do with algebras and should be developed for arbitrary functors from the cyclic category $Λ$ to vector spaces or abelian groups. Both Section 1 and Section 2 work in this generality, thus deal essentially with linear algebra.

The material in Section 1 is completely standard. In fact, the only statements for which we do not have ready references are Corollary 1.10 and Lemma 1.4, and even these are too simple to be new. Everything else could be replaced with a reference to any of the standard sources such as [L] or [FT]. However, the results are there not quite in the form we need, the assembly is different, and different notation would require a lot of
translation. In the end, we have decided that it would be cheaper and faster to develop the theory from scratch. The side effect of this is that as far as cyclic homology is concerned, the paper is more-or-less self-contained.

Section 2 still deals with linear algebra but the facts we need are less standard; they are mostly concerned with cyclic covers $\Lambda_l$ of the cyclic category $\Lambda$ that go back to [FT, Appendix]. We do not have ready references for these results, and some of them (e.g. Proposition 2.9) do look new.

Section 3 is still essentially concerned with preliminaries. In Subsection 3.1 we recall the construction of the cyclic homology of an associative algebra and some basic comparison theorems. In Subsection 3.2 we recall a non-commutative generalization of the Cartier isomorphism developed in [K3]. Subsection 3.3 recapitulates the machinery of trace functors of [K2], and Subsection 3.4 recalls the basic properties of the polynomial Witt vectors functors constructed in [K6].

After all these preliminaries, Section 4 finally introduces the main object of our study, the Hochschild-Witt complex $WCH_q(A)$ and the Hochschild-Witt homology groups $WHH_q(A)$ of an associative unital algebra $A$ over a perfect field $k$ of positive characteristic $p$. We do this in Subsection 4.1 and we also construct some additional structures on these groups — namely, the Verschiebung and Frobenius maps $V$, $F$, and the Teichmüller representative map $T$. We prove all sorts of identities these maps satisfy. Then in Subsection 4.2 we illustrate the general theory by computing explicitly the Hochschild-Witt homology groups of a tensor algebra $T^*(M)$ of a $k$-vector space $M$.

The last two sections contain our comparison theorems. Section 5 proves that in degree 0, we recover Witt vectors group constructed by Hesselholt. In Section 6 we prove that in the Hochschild-Kostant-Rosenberg situation, our Hochschild-Witt homology groups are naturally identified with the terms of the de Rham-Witt complex of $[I]$.

Let us mention that even in the commutative smooth case, where we effectively just give a new construction of the de Rham-Witt complex, we believe that our approach is instructive. It is significantly less ad hoc than $[I]$, and it explains some mysteries of the theory. One such is the structure of the associated graded quotient of the de Rham-Witt complex with respect to the natural filtration. While in principle, it was completely described by Illusie, we believe that our general non-commutative expression given in Corollary 4.4 is much simpler, and it is not possible to really understand what is going on without using the cyclic homology approach.

Two comparison results are conspicuously missing from the present paper. One is comparison with $TR(A;p)$ mentioned earlier; as we said, we
expect it to be true, but we currently do not have enough technology to prove it. Another is a very natural lifting result that is very important in \[I\] — namely, if our \(k\)-algebra \(A\) lifts to a flat algebra \(\tilde{A}\) over the Witt vectors ring \(W(k)\), then the periodic cyclic version of our de Rham-Witt homology should coincide with the periodic cyclic homology of \(\tilde{A}\). Unfortunately, we could not prove it, and at present it is not even clear if it is true. If it is true, there is further ambiguity as to what the precise statement should be — instead of the usual periodic cyclic homology, one might need to use the co-periodic cyclic homology introduced recently in \([K4]\). All in all, this certainly deserves further research.

Yet another very intriguing object for comparison is the recent very general construction of Witt vectors and de Rham-Witt complex given by Cuntz and Deninger \([CD]\). Among other things, they managed to define a Witt vectors group for an arbitrary associative ring \(A\) in such a way that it is also an associative ring. This goes against the main trend of our paper since Hochschild homology \(HH_0(A)\) is only a ring for a commutative \(A\), so that the relation between our construction and \([CD]\) cannot be straightforward. At this point, we do not know what the relation might be.

**Acknowledgements.** This paper as well as \([K6]\) took a long time to appear; in fact, the results were announced more than five years ago. I apologize for the delay, and I am very grateful to many people with whom I had the opportunity to discuss the subject. As with \([K6]\), I am especially grateful to L. Hesselholt whose generous input was absolutely crucial for most of this work, and to V. Vologodsky, to whom the main technical idea of \([K6]\) is largely due.

## 1 Preliminaries I.

### 1.1 Generalities.** We follow the same notations and conventions as \([K6]\). In particular, for any category \(\mathcal{C}\), we denote by \(\mathcal{C}^o\) the opposite category, and for any two objects \(c, c' \in \mathcal{C}\), we denote by \(\mathcal{C}(c, c')\) the set of morphisms from \(c\) to \(c'\). We denote by \(\mathsf{pt}\) the point category, and for any group \(G\), we denote by \(\mathsf{pt}_G\) the groupoid with one object with automorphism group \(G\). For any integer \(l \geq 1\), we simplify notation by letting \(\mathsf{pt}_l = \mathsf{pt}_{\mathbb{Z}/l\mathbb{Z}}\).

For any small category \(I\) and any category \(\mathcal{C}\), we will denote by \(\text{Fun}(I, \mathcal{C})\) the category of functors from \(I\) to \(\mathcal{C}\). For any ring \(A\), we denote by \(A\)-mod the category of left \(A\)-modules, and for any small category \(I\), we will simplify notation by letting \(\text{Fun}(I, A) = \text{Fun}(I, A\text{-mod})\). This is an abelian category;
we denote by $\mathcal{D}(I, A)$ its derived category. For any functor $\gamma : I \to I'$ between small categories, we denote by $\gamma^* : \text{Fun}(I', A) \to \text{Fun}(I, A)$ the pullback functor, and we denote by $\gamma_!, \gamma_* : \text{Fun}(I, A) \to \text{Fun}(I', A)$ its left and right adjoint (the left and right Kan extensions along $\gamma$). The functor $\gamma^*$ is exact, hence descends to derived categories, and the derived functors $L^* \gamma_!, R^* \gamma_* : \mathcal{D}(I, A) \to \mathcal{D}(I', A)$ are left and right adjoint to $\gamma^* : \mathcal{D}(I', A) \to \mathcal{D}(I, A)$. The homology and the cohomology of a small category $I$ with coefficients in a functor $E \in \text{Fun}(I, A)$ are given by

$$H_q(I, E) = L^* \tau_! E, \quad H^q(I, E) = R^* \tau_* E,$$

where $\tau : I \to \text{pt}$ is the tautological projection. If the ring $A$ is commutative, then the category $\text{Fun}(I, A)$ is a symmetric unital tensor category with respect to the pointwise tensor product. Its cohomology $H^*(I, A)$ with coefficients in the constant functor $A$ is then an algebra, and for any $E \in \text{Fun}(I, A)$, both $H_*(I, E)$ and $H^*(I, E)$ are modules over $H^*(I, A)$.

We recall that if $I = \Delta^o$ is the opposite of the category $\Delta$ of finite non-empty totally ordered sets, then functors $E \in \text{Fun}(\Delta^o, A)$ are simplicial $A$-modules, and homology groups $HH_*(\Delta^o, E)$ can be computed by the standard chain complex $C_*(E)$ of the simplicial $A$-module $A$. The term $C_i(E)$ of the complex $C_*(E)$ is the value of $E$ at the set $\{0, \ldots, i\}$ with the standard total order, and the differential is the alternating sum of the face maps.

We will assume known the notions of a fibration, a cofibration and a bifibration of small categories originally introduced in [G], together with the correspondence between fibrations and pseudofunctors known as the Grothendieck construction. A fibration, cofibration or bifibration is discrete if its fibers are discrete categories (that is, the only morphisms are identity maps). We also assume known the following useful base change lemma: if we are given a cartesian square

$$\begin{array}{ccc}
I'_1 & \xrightarrow{f_1} & I_1 \\
\pi' \downarrow & & \downarrow \pi \\
I' & \xrightarrow{f} & I \\
\end{array}$$

of small categories, and $\pi$ is a cofibration, then $\pi'$ is a cofibration, and the base change map $L^* \pi'_1 \circ f_1^* \to f^* \circ L^* \pi_1$ is an isomorphism. Dually, if $\pi$ is a fibration, then $\pi'$ is a fibration, and $f^* \circ R^* \pi_* \cong R^* \pi'_* \circ f_1^*$. For a proof, see e.g. [K1, Lemma 1.7]. Another result contained in [K1, Lemma 1.7] is the following projection formula: for any cofibration $\gamma : I' \to I$ and
\( E \in \text{Fun}(I, A) \), we have a natural isomorphism
\[
L^* \pi_! \pi^* E \cong E \otimes_{\mathbb{Z}} L^* \pi_! \mathbb{Z}.
\]

One specific example of a base change situation is a bifibration \( \pi : I' \to I \) whose fiber is equivalent to a groupoid \( \text{pt}_G \). In this case, for any \( E \in \text{Fun}(I', A) \) and any object \( i' \in I' \) with image \( i = \pi(i') \in I \), the \( A \)-module \( E(i') \) carries a natural action of the group \( G \), and we have base change isomorphisms
\[
\pi_! E(i) \cong E(i')^G, \quad \pi_* E(i) \cong E(i')^G,
\]
where in the right-hand side, we have coinvariants and invariants with respect to \( G \). If the group \( G \) is finite, then for any \( A[G] \)-module \( V \), we have a natural trace map
\[
\text{tr}_G = \sum_{g \in G} g : V_G \to V^G
\]
whose cokernel is the Tate cohomology group \( \hat{H}^0(G, V) \). Taken together, these maps then define a natural trace map
\[
\text{tr}_\pi : \pi_! E \to \pi_* E.
\]
This map is functorial in \( E \) and compatible with the base change. In some cases, it is an isomorphism — for example, this happens if for any object \( i' \in I' \), the \( A[G] \)-module \( E(i') \) is of the form \( M[G] \) for some \( A \)-module \( G \) (in such a situation, we will say that \( E \) is \( \pi \)-free). In the general case, we denote by
\[
\pi_* : \text{Fun}(I', A) \to \text{Fun}(I, A)
\]
the functor sending \( E \) to the cokernel of the map \( (1.3) \). For any \( i' \in I' \) with \( i = \pi(i') \), we have a natural identification \( \pi_* (E)(i) \cong \hat{H}^0(G, E(i')) \).

We also note that for any \( E \in \text{Fun}(I', A) \), we can consider the composition
\[
E \xrightarrow{l} \pi^* \pi_! E \xrightarrow{\pi^*(\text{tr}_\pi)} \pi^* \pi_* E \xrightarrow{r} E,
\]
where \( l \) and \( r \) are the adjunction maps, and this defines a functorial map
\[
\text{tr}_\pi^! : E \to E.
\]
Moreover, the pullback functor \( \pi^* : \text{Fun}(I, A) \to \text{Fun}(I', A) \) is fully faithful. Therefore there exists a unique functorial map
\[
\pi^* (e_\pi) \circ E \to \pi^* E
\]
such \( \pi^*(e_\pi) \) coincides with the composition
\[
\pi^* \pi_* E \xrightarrow{r} E \xrightarrow{l} \pi^* \pi_! E.
\]
1.2 Combinatorics. There are many equivalent definitions of Connes’ cyclic category $\Lambda$ originally introduced in [C]; we will use the one that embeds it into the category $\text{Cat}$ of small categories. For alternative approaches, see e.g. [L, Chapter 6], [FT, Appendix].

Denote by $[1]_\Lambda$ the category with one object whose endomorphisms are non-negative integers $a \in \mathbb{Z}_+ \subset \mathbb{Z}$, with composition given by sum. The embedding $\mathbb{Z}_+ \subset \mathbb{Z}$ induces a functor $[1]_\Lambda \to pt\mathbb{Z}$. By the Grothendieck construction, sets with a $\mathbb{Z}$-action correspond to small categories discretely bifibered over $pt\mathbb{Z}$, so that by pullback, every $\mathbb{Z}$-set $S$ induces a discrete bifibration $[1]^S_\Lambda \to [1]_\Lambda$. For any integer $n \geq 1$, we denote $[n]_\Lambda = [1]_{\mathbb{Z}/n\mathbb{Z}}$ the category corresponding to the $\mathbb{Z}$-set $\mathbb{Z}/n\mathbb{Z}$. Explicitly, objects in $[n]_\Lambda$ correspond to residues $y \in \mathbb{Z}/n\mathbb{Z}$, and morphisms from $y$ to $y'$ are non-negative integers $l$ such that $y + l = y' \pmod{n}$. We denote the set of objects of $[n]_\Lambda$ by $Y([n])$. Equivalently, $[n]_\Lambda$ is the path category of the wheel quiver with $n$ vertices and $n$ edges, and $Y([n])$ is the set of vertices.

Any object $y \in Y([n])$ of the category $[n]_\Lambda$ has a natural endomorphism $\tau_y : y \to y$ given by $l = n$, and for any functor $f : [n]_\Lambda \to [m]_\Lambda$, we have

$$f(\tau_y) = \tau_{f(y)}^{\text{deg } f},$$

where $\text{deg } f$ is a non-negative integer independent of $y$, called the degree of the functor $f$. The degree is obviously multiplicative, $\text{deg}(f_1 \circ f_2) = \text{deg}(f_1) \text{deg}(f_2)$.

**Definition 1.1.** A functor $f : [n]_\Lambda \to [m]_\Lambda$ is vertical if it is a discrete bifibration, horizontal if $\text{deg}(f) = 1$, and non-degenerate if $\text{deg}(f) \neq 0$.

**Definition 1.2.** The cyclotomic category $\mathbb{C}_\mathbb{Z}$ is the small category whose objects $[n]$ are numbered by positive integers $n \geq 1$, and whose maps from $[n]$ to $[m]$ are given by non-degenerate functors $f : [n]_\Lambda \to [m]_\Lambda$.

Classes of horizontal resp. vertical maps are obviously closed under compositions and contain all isomorphisms. Denote by $\mathbb{C}_\mathbb{Z}_h, \mathbb{C}_\mathbb{Z}_v \subset \mathbb{C}_\mathbb{Z}$ the subcategories with the same objects as $\mathbb{C}_\mathbb{Z}$ and horizontal resp. vertical maps between them. Then $\mathbb{C}_\mathbb{Z}_v$ is naturally identified with the category of finite $\mathbb{Z}$-orbits — that is, finite sets equipped with a transitive $\mathbb{Z}$-action. On the other hand, $\mathbb{C}_\mathbb{Z}_h$ is the category $\Lambda$.

**Definition 1.3.** The cyclic category $\Lambda$ is the subcategory $\Lambda = \mathbb{C}_\mathbb{Z}_h \subset \mathbb{C}_\mathbb{Z}$.

For any $[n] \in \mathbb{C}_\mathbb{Z}$, the automorphism group $\text{Aut}([n])$ of the object $[n]$ is the cyclic group $\mathbb{Z}/n\mathbb{Z}$ (and all automorphisms are both horizontal and
vertical). Every functor $f : [n]_\Lambda \to [m]_\Lambda$ of degree 1 has left and right-adjoint functors $f^\sharp, f^\sharp : [m]_\Lambda \to [n]_\Lambda$. Explicitly, they are given by

\begin{align}
  f^\sharp(y) &= \min\{y' \in \mathbb{Z} | f(y') \geq \bar{y}\} \mod m, \\
  f^\sharp(y) &= \max\{y' \in \mathbb{Z} | f(y') \leq \bar{y}\} \mod m,
\end{align}

where we choose a representative $\bar{y} \in \mathbb{Z}$ of the residue class $y$, and observe that the result does not depend on this choice. Thus in particular, the category $\Lambda$ is equivalent to its opposite category $\Lambda^o$, $\Lambda \cong \Lambda^o$.

Denote by $\Lambda_\ast$ the category of vertical maps $v : [n] \to [m]$ in $\Lambda R$, with maps from $v : [n] \to [m]$ to $v' : [n'] \to [m']$ given by commutative squares

$$
\begin{array}{c}
[n] \xrightarrow{f} [n'] \\
\downarrow v \downarrow \downarrow v' \\
[m] \xrightarrow{g} [m']
\end{array}
$$

with horizontal $f, g$. Then $\Lambda_\ast$ splits as a disjoint union

\begin{equation}
\Lambda_\ast = \coprod_{l \geq 1} \Lambda_l
\end{equation}

according to the degree $l = \deg(v)$ of the map $v$. The component $\Lambda_1$ is the cyclic category $\Lambda$. Moreover, sending $v : [n'] \to [n]$ to $v' : [n'] \to [m']$ gives two functors

\begin{equation}
i_\ast, \pi_\ast : \Lambda_\ast \to \Lambda
\end{equation}

and their individual components $i_l, \pi_l, l \geq 1$. The functor $\pi_l$ is a bifibration with fiber $pt_l$. To simplify notation, we denote an object $v : [n'] \to [n]$ of $\Lambda_l$ simply by $[n]$; then automatically $n' = nl$, and we have $i_l([n]) = [nl]$, $\pi_l([n]) = [n]$. For any object $[n] \in \Lambda_l$ represented by a vertical arrow $v : [nl] \to [n]$, the arrow $v$ gives a natural map

\begin{equation}
\eta_l : Y(i_l([n])) \to Y(\pi_l([n])).
\end{equation}

The duality $\Lambda \cong \Lambda^o$ extends to dualities $\Lambda_l \cong \Lambda_l^o$, $l \geq 1$.

For any two integers $m, l \geq 1$, a vertical map $v : [nml] \to [n]$ of degree $ml$ factorizes as

$$
[nml] \xrightarrow{v'} [nm] \xrightarrow{v''} [n],
$$
with $v', v''$ vertical of degrees $l, m$, and such a factorization is unique up to a unique isomorphism. Sending $v$ to the pair $(v', v'')$ then gives a commutative square

\[
\begin{array}{ccc}
\Lambda_{ml} & \xrightarrow{i_m} & \Lambda_l \\
\pi_l & \downarrow & \pi_l \\
\Lambda_m & \xrightarrow{i_m} & \Lambda,
\end{array}
\]  

(1.11)

and this square is obviously Cartesian. By abuse of notation, we also denote by $\pi_l, i_m$ the left vertical and the top horizontal functor in (1.11) whenever there is no danger of confusion.

For any $n, m, l \geq 1$, the functor $i_l$ induces a natural map

\[
\Lambda_l([n], [m]) \rightarrow \Lambda(i_l([n]), i_l([m])) = \Lambda([ln], [lm]),
\]

(1.12)

and this map is obviously injective. We will need the following stronger statement.

**Lemma 1.4.** Consider two objects $[n], [m] \in \Lambda$, and let $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} = \text{Aut}([n]) \times \text{Aut}([m])$ act on $\Lambda([n], [m])$ by

\[
(g_1 \times g_2) \cdot f = g_2 \circ f \circ g_1^{-1}.
\]

Then the stabilizer $\text{Stab}(f) \subset \text{Aut}([n]) \times \text{Aut}([m])$ of any morphism $f \in \Lambda([n], [m])$ is a cyclic group $\mathbb{Z}/l\mathbb{Z}$ whose order divides both $n$ and $m$, its embedding into $\text{Aut}([n]) \times \text{Aut}([m])$ is the product of the standard embeddings $\mathbb{Z}/l\mathbb{Z} \subset \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/l\mathbb{Z} \subset \mathbb{Z}/m\mathbb{Z}$, and the subset of fixed points

\[
\Lambda([n], [m])^{\text{Stab}(f)} \subset \Lambda([n], [m])
\]

coincides with the set $\Lambda_l([n/l], [m/l])$ embedded by the map (1.12).

**Proof.** It is obvious from the definition that both individual actions of $\text{Aut}([n])$ and $\text{Aut}([m])$ on $\Lambda([n], [m])$ are free. Therefore when we project $\text{Stab}(f) \subset \text{Aut}([n]) \times \text{Aut}([m])$ to $\text{Aut}([n])$ resp. $\text{Aut}([m])$, the resulting maps $\text{Stab}(f) \rightarrow \text{Aut}([n]), \text{Stab}(f) \rightarrow \text{Aut}([m])$ are injective. This yields the first claim. Moreover, $f$ fits into a commutative diagram

\[
\begin{array}{ccc}
[n]_\Lambda & \xrightarrow{f} & [m]_\Lambda \\
\downarrow & & \downarrow \\
[n]_{\Lambda/\text{Stab}(f)} & \xrightarrow{7} & [m]_{\Lambda/\text{Stab}(f)},
\end{array}
\]  

(1.13)
and if we denote by \( \sigma \in \text{Stab}(f) \) a generator of the cyclic group \( \text{Stab}(f) \cong \mathbb{Z}/l\mathbb{Z} \), then we have \( \sigma \circ f = f \circ \sigma^{\deg(f)} \). Since both vertical maps in (1.13) have degree \( l \), we have \( \deg(f) = \deg(f) = 1 \), and this gives the second claim. The third claim immediately follows from the definition of the functor \( i_l \). □

For every \([n] \in \Lambda\), a horizontal functor \( h : [1]_\Lambda \to [n]_\Lambda \) is uniquely determined by the image of the unique object \( o \in [1]_\Lambda \), so that we have a natural identification \( Y([n]) = \Lambda([1],[n]) \). Moreover, for any \( y \in Y([n]) \), composing the corresponding functor \( h : [1]_\Lambda \to [n]_\Lambda \) with the unique functor \( \text{pt} \to [1]_\Lambda \) gives an embedding \( i_y : \text{pt} \to [n]_\Lambda \) of the point category \( \text{pt} \) onto the object \( y \). For every map \( f : [m] \to [n] \), we can define the preimage \([f^{-1}(y)]\) by the fibered product

\[
\begin{array}{ccc}
[f^{-1}(y)] & \longrightarrow & [n]_\Lambda \\
\downarrow & & \downarrow f \\
\text{pt} & \xrightarrow{i_y} & [1]_\Lambda.
\end{array}
\]

Then \([f^{-1}(y)]\) is the category corresponding to a finite totally ordered set – its objects are elements \( y' \in f^{-1}(y) \subset Y([m]) \), and the category structure on \([f^{-1}(y)]\) induced from \([n]_\Lambda \) defines a canonical total order on the set \( f^{-1}(y) \). In particular, take \( n = 1 \). Then the category \( \Delta \) of finite non-empty totally ordered sets is a full subcategory in \( \text{Cat} \), and sending \( f : [n] \to [1] \) to \([f^{-1}(o)]\) gives a functor

\[\Lambda/[1] \to \Delta.\]

This functor is an equivalence of categories. Choosing an inverse equivalence and composing it with the forgetful functor, we obtain a natural embedding

\[j : \Delta \to \Lambda.\]

More generally, for any horizontal map \( h : [n] \to [1] \) and any vertical map \( v : [m] \to [1] \) in \( \Lambda R \), we have

\[[n]_\Lambda \times [1]_\Lambda \cong [nm]_\Lambda,\]

and sending \( (h,v) \) to their fibered product gives a functor

\[\widetilde{j} : \Delta \times \Lambda R_v \to \Lambda R.\]

Restricting \( \widetilde{j} \) to \( \Delta \times [l] \) for some \( l \geq 1 \), we obtain natural embeddings

\[(1.15) \quad \widetilde{j}_l : \Delta \times \text{pt}_l \to \Lambda R, \quad j_l : \Delta \to \Lambda_l,\]

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where we identify $\mathbb{Z}/l\mathbb{Z} \cong \text{Aut}([l])$. By duality, (1.14) and (1.15) induce embeddings
\[ j^0 : \Delta^0 \to \Lambda, \quad j_l^0 : \Delta^0 \to \Lambda_l, \quad l \geq 1. \]
The embeddings $j_l$ are compatible with the embeddings $i_l$ of (1.9) in the following sense: for any $l \geq 1$, we have a commutative square
\[
\begin{array}{ccc}
\Delta & \xrightarrow{j_l} & \Lambda_l \\
\downarrow \text{\scriptsize{\textit{7}}}_l & & \downarrow \text{\scriptsize{i}}_l \\
\Delta & \xrightarrow{j} & \Lambda,
\end{array}
\]
where the functor $\text{\scriptsize{7}}_l : \Delta \to \Delta$ is the functor sending a totally ordered set $[n]$ with $n$ elements to the product $[l] \times [n] \cong [nl]$ with lexicographical order. We have a similar commutative square for the categories $\Delta^0$ and the functors $j^0_l$. We note that if we denote by $s : [1] \to [l]$ the map that sends the unique element in $[1]$ to the initial element in $[l]$, then we have a functorial map
\[
s_l = s^0 \times \text{id} : \text{\scriptsize{7}}_l \to \text{id}.
\]
The choice of $s$ breaks cyclic symmetry, so that an analogous map for the functor $i_l$ does not exist.

### 1.3 Homology

Next, we recall briefly the main homological properties of the cyclic categories $\Lambda_l$.

For any ring $A$ and $E \in \text{Fun}(\Lambda, A)$, the functors $j^0 : \Delta^0 \to \Lambda$, $j : \Delta \to \Lambda$ give canonical objects $j^0_\ast j^0 \ast E, j_\ast j^\ast E \in \text{Fun}(\Lambda, A)$. Since $j$ is equivalent to a discrete fibration and $j^0$ is equivalent to a discrete cofibration, the Kan extensions functors $j^0_\ast$ and $j_\ast$ are exact by base change, and the projection formula (1.1) gives canonical identifications
\[
(1.18) \quad j^0_\ast j^0 \ast E \cong E \otimes j^0_\ast \mathbb{Z}, \quad j_\ast j^\ast E \cong E \otimes j_\ast \mathbb{Z},
\]
where $\mathbb{Z}$ is the constant functor with values $\mathbb{Z}$. Explicitly, one has
\[
(1.19) \quad j^0_\ast \mathbb{Z}([n]) \cong \mathbb{Z}[\Lambda([1], [n])), \quad j_\ast \mathbb{Z} \cong \mathbb{Z}[\Lambda([n], [1])]
\]
for any $[n] \in \Lambda$, so that any map $f : [1] \to [n]$ gives a element $[f] \in j^0_\ast \mathbb{Z}([n])$, and any map $f : [n] \to [1]$ gives an element $[f] \in j_\ast \mathbb{Z}([n])$. Define a map $b : j_\ast \mathbb{Z}([n]) \to j^0_\ast \mathbb{Z}([n])$ by
\[
(1.20) \quad b([f]) = [f^\sharp] - [f^\sharp]
\]
for any $f : [n] \to [1]$, where $f^\sharp, f^\sharp$ are as in (1.7).
Lemma 1.5. The maps $b$ of (1.20) are functorial in $[n]$, and we have an exact sequence

$$
0 \longrightarrow \mathbb{Z} \longrightarrow j_!\mathbb{Z} \xrightarrow{b} j_!\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0
$$

of objects in $\text{Fun}(\Lambda, \mathbb{Z})$, where $\mathbb{Z}$ stands for the constant functor.

Proof. Identify $\Lambda([1], [n]) = Y([n])$ with the set $\mathbb{Z}/n\mathbb{Z}$ of objects in $[n]_\Lambda$, and identify $\Lambda([1], [n]) \cong \Lambda([n], [1])$ by sending $f : [n] \to [1]$ to $f_\sharp$. Then $b$ becomes a map $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ sending $[y]$ to $[y] - [y + 1]$ for any $y \in \mathbb{Z}/n\mathbb{Z}$. Therefore $\text{Ker } b = \text{Coker } b = \mathbb{Z}$ irrespective of $[n]$, and this gives (1.21) once we establish functoriality. To do this, consider a map $g : [n] \to [m]$ in $\Lambda$. Then under our identifications $j_!\mathbb{Z}([n]) \cong j_*\mathbb{Z}([n]) \cong \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$, $j_!\mathbb{Z}([m]) \cong j_*\mathbb{Z}([m]) \cong \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$, the maps $g*, g_*$ : $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$ corresponding to $j_!$ resp. $j_*$ are given by

$$
g_*([y]) = [g(y)], \quad g_*([y]) = \sum_{y' \in g^{-1}_\sharp(y)} [a'],
$$

and we note that by (1.7), $g^{-1}_\sharp(y)$ is the (possibly empty) set of residues $g(y), g(y) + 1, \ldots, g(y + 1) - 1$. Therefore we have

$$
b(g_*([y])) = \sum_{y' \in g^{-1}_\sharp(y)} ([y'] - [y' - 1]) = [g(y)] - [g(y + 1)],
$$

and the right-hand side is exactly $g_!(b([y]))$. \hfill \Box

Remark 1.6. Lemma 1.5 has the following geometric meaning. Any $[n] \in \Lambda$ can be interpreted as a cellular decomposition of the circle $S^1$ with 0-cells numbered by elements $y \in Y([n])$. Morphisms in $\Lambda$ give homotopy classes of cellular maps, and duality $\Lambda \cong \Lambda^o$ corresponds to taking the dual cellular decomposition. Then at each $[n] \in \Lambda$, the middle two terms in (1.21) give the cellular chain complex computing the homology $H_*(S^1, \mathbb{Z})$.

Combining (1.18) and (1.21), for any $E \in \text{Fun}(\Lambda, A)$, we obtain a functorial complex

$$
j_*j^*E \longrightarrow j_!j^{o*}E
$$

of length 2, with homology objects in degrees 0 and 1 identified with $E$. \hfill 13
Definition 1.7. For any $E \in \text{Fun}(\Lambda, A)$, the complex $\mathbb{K}_*(E)$ in $\text{Fun}(\Lambda, A)$ is the complex \[(1.22)\], with $\mathbb{K}_0(E) = j_1^0 j^0_* E$ and $\mathbb{K}_1(E) = j_* j^*_1 E$.

For any $l > 1$, the commutative square \[(1.16)\] and the similar square for the category $\Delta^o$ induce base change isomorphisms
\[(1.23)\]
\[
\begin{align*}
  j_l^! \circ i^*_l & \cong i^*_l \circ j_l^!, & i^*_l \circ j_* & \cong j_* \circ i^*_l .
\end{align*}
\]
Thus we can apply $i_l^*$ to the exact sequence \[(1.21)\] and obtain a four-term exact sequence
\[
0 \longrightarrow \mathbb{Z} \longrightarrow j_{l^*} \mathbb{Z} \longrightarrow j_{l!}^* j_l^* E \longrightarrow \mathbb{Z} \longrightarrow 0
\]
and a canonical complex
\[
\begin{align*}
  j_{l^*} j_l^* E \cong E \otimes j_{l^*} \mathbb{Z} \longrightarrow j_{l!}^* j_l^* E \cong E \otimes j_{l!}^* \mathbb{Z}
\end{align*}
\]
for any $E \in \text{Fun}(\Lambda_l, A)$, with homology objects in degrees 0 and 1 identified with $E$. By abuse of notation, we will denote this complex by $\mathbb{K}_*(E)$, same as in Definition 1.7. We then have a canonical identification
\[
i_l^* \mathbb{K}_*(E) \cong \mathbb{K}_*(i_l^* E).
\]
For any object $[n] \in \Lambda$ and any representation $M$ of the cyclic group $\text{Aut}([n]) = \mathbb{Z}/nl\mathbb{Z}$ with generator $\sigma \in \mathbb{Z}/nl\mathbb{Z}$, we will denote by $\mathbb{K}_*(M)$ the complex
\[
(1.24)\]
\[
\begin{align*}
  M \xrightarrow{\text{id} - \sigma} M
\end{align*}
\]
placed in homological degrees 0 and 1. Then for any $E \in \text{Fun}(\Lambda_l, A)$, we have
\[
(1.25) \quad \mathbb{K}_*(E)([n])_\sigma \cong \mathbb{K}_*(E([n])).
\]

Definition 1.8. For any ring $A$, integer $l \geq 1$, and object $E \in \text{Fun}(\Lambda_l, A)$, the Hochschild resp. cyclic homology groups of the object $E$ are given by
\[
\begin{align*}
  HH_*(E) = H_*(\Delta^o, j_l^* E) = H_*(\Lambda_l, j_{l!}^* j_l^* E), & \quad HC_*(E) = H_*(\Lambda_l, E).
\end{align*}
\]
The $Hochschild complex$ $CH_*(E)$ is the standard chain complex $C_*(j_l^* E)$ of the simplicial $A$-module $j_l^* E$. 
Lemma 1.9. For any integer \( l \geq 1 \), any ring \( A \), and any \( E \in \text{Fun}(\Lambda_l, A) \), we have \( H_*(\Lambda_l, j_! j^* E) = 0 \), so that the natural map

\[
HH_*(E) \cong H_*(\Lambda_l, j_! j^* \omega E) \to HC_*(K_*(E))
\]

is an isomorphism. Moreover, for any \( E \in \text{Fun}(\Lambda, A) \), the adjunction maps

\[
HC_*(i^* E) \to HC_*(E), \quad HH_*(i^* E) \to HH_*(E)
\]

are also isomorphisms.

Proof. The first claim is [K1, Lemma 1.10], and the second claim is [K1, Lemma 1.14].

We note that the Hochschild homology isomorphism of (1.27) holds for any simplicial \( A \)-module \( E \), and it can be realized explicitly: map (1.17) induces a map

\[
s_l : i_! E \to E,
\]

and this map becomes a quasiisomorphism after we pass to the standard complexes. In fact, \( E \) does not even need to be an \( A \)-module — the map (1.28) exists equally well for simplicial objects of any nature, for example for simplicial sets (it is known as the edgewise subdivision map and goes back to [S]). Realizing explicitly the cyclic homology isomorphism of (1.27) is more difficult (see e.g. [K4, Subsection 4.1]).

Since the embedding \( j_l : \Delta \to \Lambda_l \) of (1.15) extends to an embedding \( \widetilde{j}_l : \Delta^\circ \times \text{pt} \to \Lambda_l \), for any \( E \in \text{Fun}(\Lambda_l, A) \), the pullback \( j_!^* E \) carries a canonical action of the cyclic group \( \mathbb{Z}/l\mathbb{Z} \). Then by adjunction, \( j_! \mathbb{Z} \) also carries a \( \mathbb{Z}/l\mathbb{Z} \)-action. Analogously, we have a natural \( \mathbb{Z}/l\mathbb{Z} \)-action on the functors \( j_!^* \) and on \( j_!^* \mathbb{Z} \). This is compatible with the differential and turns \( K_*(E) \cong E \otimes K_*(\mathbb{Z}) \) into a complex of \( \mathbb{Z}/l\mathbb{Z} \)-modules in \( \text{Fun}(\Lambda_l, A) \).

Corollary 1.10. For any \( E \in \text{Fun}(\Lambda_l, A) \), the natural \( \mathbb{Z}/l\mathbb{Z} \)-action on the Hochschild homology groups \( HH_*(E) \) induced by the \( \mathbb{Z}/l\mathbb{Z} \)-action on \( j_!^* E \) is trivial.

Proof. Let \( B_0 = \mathbb{Z}[\mathbb{Z}/l\mathbb{Z}] \) be the group algebra of the cyclic group \( \mathbb{Z}/l\mathbb{Z} \), with \( [\sigma] \in B_0 \) being the class of the generator \( \sigma \in \mathbb{Z}/l\mathbb{Z} \), and let \( B_* = B_0(\varepsilon) \) be the free commutative DG algebra over \( B_0 \) generated by one generator \( \varepsilon \) of homological degree 1, with \( \varepsilon^2 = 0 \) and \( d \varepsilon = 1 - [\sigma] \). Then in terms of the identification (1.26), the \( \mathbb{Z}/l\mathbb{Z} \)-action on \( HH_*(E) \) is induced by the
$B_0$-module structure on the complex $\mathbb{K}_*(E) \cong E \otimes \mathbb{K}_*(Z)$, and it suffices to prove that $\mathbb{K}_*(Z)$ is actually a DG module over the whole $B_\ast$. To do this, we have to extend the action map $B_0 \otimes \mathbb{K}_0(Z) \to \mathbb{K}_0(Z)$ to a map of complexes

\[(1.29) \quad B_\ast \otimes \mathbb{K}_0(Z) \to \mathbb{K}_*(Z)\]

in such a way that the resulting $B_\ast$-action is associative and compatible with the differentials. However, by adjunction, we have an identification

$$\text{Hom}(V \otimes \mathbb{K}_0(Z), F) = \text{Hom}(V \otimes j_0^0 Z, F) \cong \text{Hom}(V \otimes Z, j_1^0 F) \cong \text{Hom}(V, F([1]))$$

for any abelian group $V$ and any $F \in \text{Fun}(\Lambda_l, Z)$, so that it suffices to construct the extended map \[(1.29)\] and check that it defines a $B_\ast$-module structure on $\mathbb{K}_*(Z)([1])$. This is easy, since in fact $\mathbb{K}_*(Z)([1]) \cong B_\ast$. \hfill \Box

Now for any ring $A$, integer $l \geq 1$, and $E \in \text{Fun}(\Lambda_l, A)$, consider the adjunction maps

\[(1.30) \quad \kappa_0 : \mathbb{K}_0(E) = j_0^0 j_0^* E \to E, \quad \kappa_1 : E \to j_1^0 j_1^* E = \mathbb{K}_1(E),\]

and let

\[(1.31) \quad B = \kappa_1 \circ \kappa_0 : \mathbb{K}_0(E) \to E \to \mathbb{K}_1(E)\]

be their composition. We can then form a natural periodic resolution

\[(1.32) \quad B \to \mathbb{K}_1(E) \to \mathbb{K}_0(E) \to B \to \mathbb{K}_1(E) \to \mathbb{K}_0(E) \to \cdots\]

of the object $E$. If we denote by $u$ the endomorphism of \[(1.32)\] obtained by shifting to the left by 2 terms, then $u$ gives a natural class $u \in \text{Ext}^2(E, E)$ (equivalently, this is the class represented by Yoneda by the complex $\mathbb{K}_*(E)$). Taking the cone of $u$ and using the isomorphism \[(1.26)\], we obtain the Connes long exact sequence

\[(1.33) \quad HH_\ast(E) \to HC_\ast(E) \xrightarrow{u} HC_{\ast-2}(E) \to \cdots\]

On the other hand, taking the odd-numbered terms of the stupid filtration on the complex \[(1.32)\], we obtain a natural spectral sequence

\[(1.34) \quad HH_\ast(E)[u^{-1}] \Rightarrow HC_\ast(\Lambda_l, E),\]

where the left-hand side is shorthand for “formal polynomials in one variable $u^{-1}$ of homological degree 2 with coefficients in $HH_\ast(E)$”. This is known as
the Hochschild-to-cyclic, or Hodge-to-de Rham spectral sequence. The first non-trivial differential

\[(1.35) \quad B : HH_\ast(E) \to HH_{\ast+1}(E)\]

is known as the Connes-Tsygan differential, or sometimes Rinehart differential. In terms of the identification (1.26), this differential is induced by the map

\[(1.36) \quad B : \mathbb{K}_\ast(E) \to \mathbb{K}_\ast(E)[-1]\]

which is in turn induced by the natural map \(B\) of (1.31) (this is why we use the same notation for all three).

2 Preliminaries II.

2.1 Trace maps. Let us now explore the homological properties of the projections \(\pi_\ast\) of (1.9). Fix an integer \(l \geq 1\), and consider the projection \(\pi_l : \Lambda_l \to \Lambda\). This is a bifibration with fiber \(p_l\), so that we have functorial maps (1.3) and (1.6). We simplify notation by writing

\[(2.1) \quad e_l = e_{\pi_l} : \pi_l E \to \pi_{l!} E, \quad tr_l = tr_{\pi_l} : \pi_l E \to \pi_{l!} E.\]

If \(E \in \Fun(\Lambda_l, A)\) is \(\pi_l\)-free, then the map \(tr_l\) of (2.1) is an isomorphism, and \(E\) is acyclic both for the left-exact functor \(\pi_{l!}\) and for the right-exact functor \(\pi_{l*}\) (that is, \(L^i\pi_{l!} E = \pi_{l!} E, R^i\pi_{l*} E = \pi_{l*} E\)). In particular, by (1.19) and (1.23), this applies to functors of the form \(j_{l!}(E), E \in \Fun(\Delta^o, A)\) and \(j_{l*}(E), E \in \Fun(\Delta, A)\), so that we have a natural isomorphism

\[(2.2) \quad tr_l : \pi_{l!} \mathbb{K}_\ast(E) \cong \pi_{l*} \mathbb{K}_\ast(E)\]

for any \(E \in \Fun(\Lambda_p, E)\). Moreover, since \(\pi_{l!} \mathbb{K}_\ast(E) \cong L^i \pi_{l!} \mathbb{K}_\ast(E)\), we have natural identifications

\[(2.3) \quad HC_\ast(\mathbb{K}_\ast(E)) \cong HC_\ast(\pi_{l!} \mathbb{K}_\ast(E)) \cong HC_\ast(\pi_{l*} \mathbb{K}_\ast(E)).\]

Under the isomorphism (1.26), all these groups are further identified with \(HH_\ast(E)\). Moreover, to simplify notation, we introduce the following.

**Definition 2.1.** For any \(E \in \Fun(\Lambda, A)\) and any \(l \geq 1\), the complex \(\mathbb{K}_\ast^l(E)\) in \(\Fun(\Lambda, A)\) is given by

\[
\mathbb{K}_\ast^l(E) = \pi_{l!} \mathbb{K}_\ast(i_l^* E) \cong \pi_{l*} \mathbb{K}_\ast(i_l^* E).
\]
Then by definition, for any \( E \in \text{Fun}(\Lambda, A) \) and any \( l \geq 1 \), the complex \( \mathbb{K}_l(E) \) fits into an exact sequence
\[
(2.4) \quad 0 \longrightarrow \pi_{l*}E \longrightarrow \mathbb{K}_1^l(E) \longrightarrow \mathbb{K}_0^l(E) \longrightarrow \pi_l E \longrightarrow 0,
\]
and \((2.3)\) together with the isomorphism of Lemma \((1.9)\) provides a canonical isomorphism
\[
(2.5) \quad HH_*(E) \cong HC_*(\mathbb{K}_*^l(E)).
\]

**Lemma 2.2.** For any \( l \geq 1 \) and any \( E \in \text{Fun}(\Lambda, A) \), we have natural isomorphism of complexes
\[
\mathbb{K}_*(E) \cong \pi_l \mathbb{K}_* (\pi_l^* E) \cong \pi_{l*} \mathbb{K}_* (\pi_l^* E).
\]

**Proof.** By the projection formula \((1.1)\), we have \( \pi_l \mathbb{K}_* (\pi_l^* E) \cong E \otimes \pi_l \pi_l^* Z \) and \( \pi_{l*} \mathbb{K}_* (\pi_l^* E) \cong E \otimes \pi_{l*} \pi_l^* Z \), so that it suffices to consider the case \( E = Z \). In this case, the claim immediately follows from \((1.19)\) and \((1.23)\). \(\Box\)

In particular, for any \( E \in \text{Fun}(\Lambda_l, A) \), the adjunction maps \( E \to \pi_l^* \pi_l E, \pi_l^* \pi_{l*} E \to E \) induce natural maps
\[
\nu_l : \pi_l \mathbb{K}_* (E) \to \pi_l \mathbb{K}_* (\pi_l^* \pi_l E) \cong \mathbb{K}_* (\pi_l^* E),
\]
\[
\varphi_l : \mathbb{K}_* (\pi_l^* E) \to \pi_{l*} \mathbb{K}_* (E).
\]

**Lemma 2.3.** For any \( l \geq 1 \), ring \( A \), and \( E \in \text{Fun}(\Lambda_l, A) \), the map \( \nu_l \) is an isomorphism on homology in degree 0 and equal to the map \( e_l \) of \((2.1)\) in degree 1, and \( \varphi_l \) is an isomorphism on homology in degree 1, and equal to \( e_l \) in degree 0.

**Proof.** The canonical maps \( \kappa_0 \) of \((1.30)\) induce isomorphisms of the 0-th homology objects of the complexes \( \mathbb{K}_* (E) \) resp. \( \mathbb{K}_* (\pi_l E) \) with \( E \) resp. \( \pi_l E \). Since the functor \( \pi_l \) is right-exact, \( \pi_l (\kappa_0) \) then gives an isomorphism between 0-th homology of \( \pi_l \mathbb{K}_* (E) \) and \( \pi_l E \), and since by definition, we have \( \pi_l (\kappa_0) = \kappa_0 \circ \nu_l \), \( \nu_l \) induces an isomorphism on homology in degree 0. Analogously, \( \varphi_l \) induces an isomorphism on homology in degree 1. Then by \((2.2)\), the complexes \( \pi_l \mathbb{K}_* (E) \) and \( \pi_{l*} \mathbb{K}_* (E) \) are one and the same, so that to prove that \( \nu_l \) induces the map \( e_l \) on homology in degree 1, we have to prove that the diagram
\[
\begin{array}{ccc}
\pi_{l*}E & \xrightarrow{\pi_{l*}(\kappa_1)} & \pi_{l*} \mathbb{K}_1 (E) \\
\downarrow e_l & & \downarrow \nu_l \\
\pi_l E & \xrightarrow{\kappa_1} & \mathbb{K}_1 (\pi_l E)
\end{array}
\]
is commutative. But $\pi_! K_1(E) = \pi_! j_! j^*_l E \cong j_* j^*_l E$ and $K_1(\pi_! E) = j_* j^* \pi_! E$, so that by adjunction, it is equivalent to proving that the diagram

\[
\begin{array}{ccc}
j^* \pi_! E & \longrightarrow & j^*_l E \\
\downarrow & & \downarrow \\
j^* \pi_! E & \longrightarrow & j^* \pi_! E
\end{array}
\]

is commutative – in other words, that the composition $j^* \pi_! E \rightarrow j^*_l E \rightarrow j^* \pi_! E$ is the map $j^*(e_l)$. After evaluating at an object $[n] \in \Delta$, this composition reads as

\[
E([n])^\sigma \longrightarrow E([n]) \longrightarrow E([n])_\sigma,
\]
and it is equal to $e_l$ by definition. The argument for $\varphi_l$ is dual. \qed

We will also need a result on compatibility between the Connes-Tsygan differential (1.35) and the trace map (2.1). Consider an object $E$ in the category $\text{Fun}(\Lambda_l, A)$, and denote by $b : K_*(\pi_! E) \rightarrow K_*(\pi_! E)[-1]$ the composition map

(2.7) \[
K_* (\pi_! E) \xrightarrow{\kappa_0} \pi_! E \xrightarrow{\text{tr}_l} \pi_! \pi_! E \xrightarrow{\kappa_1} K_* (\pi_! E)[-1],
\]

where $\kappa_0$, $\kappa_1$ are the adjunction maps (1.30).

**Lemma 2.4.** Under the identifications (2.3), the composition map

\[
\pi_! K_*(E) \xrightarrow{\nu_l} K_*(\pi_! E) \xrightarrow{b} K_*(\pi_! E)[-1] \xrightarrow{\varphi_l} \pi_! \pi_! K_*(E)[-1]
\]

induces the Connes-Tsygan differential $B$ of (1.35).

**Proof.** The Connes-Tsygan differential is by definition induced by the composition

\[
\pi_! K_*(E) \xrightarrow{\pi_! (B)} \pi_! K_*(E)[-1] \xrightarrow{\text{tr}_l} \pi_! \pi_! K_*(E)[-1],
\]

where $B$ is as in (1.36), and $\text{tr}_l$ on the right is the canonical isomorphism (2.2). By (1.31), this composition is in turn given by the composition

\[
\pi_! K_0(E) \xrightarrow{\pi_! (\kappa_0)} \pi_! E \xrightarrow{\pi_! (\kappa_1)} \pi_! K_1(E) \xrightarrow{\text{tr}_l} \pi_! \pi_! K_1(E).
\]

Since the map $\text{tr}_l$ is functorial, this is equal to the composition

\[
\pi_! K_0(E) \xrightarrow{\pi_! (\kappa_0)} \pi_! E \xrightarrow{\text{tr}_l} \pi_! \pi_! E \xrightarrow{\pi_! (\kappa_1)} \pi_! \pi_! K_1(E),
\]

and by definition, $\pi_! (\kappa_0) = \kappa_0 \circ \nu_l$ and $\pi_! (\kappa_1) = \varphi_l \circ \kappa_1$. \qed
2.2 Extra structures. If the ring \( A \) is commutative, then the categories \( \text{Fun}(\Lambda, A) \), \( \text{Fun}(\Delta^\circ, A) \) acquire natural pointwise tensor products, and the derived categories \( D(\Lambda, A) \), \( D(\Delta^\circ, A) \) in turn acquire the derived tensor product \( \otimes^L \). It is well-known that the Hochschild homology functor is multiplicative: for any two objects \( E, E' \in D(\Lambda, A) \), we have a natural Künneth quasiisomorphism

\[
CH_q(E) \otimes^L CH_q(E') \cong CH_q(E \otimes E'),
\]

where we denote by \( CH_q(\cdot) \) the object in the derived category \( D(A\text{-mod}) \) corresponding to the Hochschild homology \( HH_q(\cdot) \) (to make things more canonical, one can agree to take as \( CH_q(E) \) the standard complex of the simplicial \( A \)-module \( j^*E \)). The Connes-Tsygan differential \( B \) of (1.35) is compatible with the Künneth quasiisomorphism (2.8) – that is, we have

\[
B \otimes^L = B \otimes id + id \otimes B'.
\]

For a characteristic-independent proof of this, see e.g. [K3, Lemma 1.3].

We will also need to consider one additional structure on cyclic objects essentially introduced in [K6]. Fix a prime \( p \), and consider the functors \( i_p, \pi_p : \Lambda_p \to \Lambda \) and the map \( \text{tr}_p \) of (2.1). Similarly, denote \( \text{tr}_p^\dagger = \text{tr}^\dagger_p \), where \( \text{tr}^\dagger \) is the map (1.5)

**Definition 2.5.** An \( FV \)-structure on an object \( E \in \text{Fun}(\Lambda, A) \) is given by two maps \( V : i^*_pE \to \pi^*_pE \), \( F : \pi^*_pE \to i^*_pE \) such that

\[
F \circ V = \text{tr}_p^\dagger : i^*_pE \to i^*_pE.
\]

**Remark 2.6.** Formally, Definition 2.5 makes sense for any \( p \geq 1 \); however, the notion is meaningful only when \( p \) is a prime.

For any object \( E \in \text{Fun}(\Lambda, A) \) equipped with an \( FV \)-structure \( \langle V, F \rangle \), the maps \( V \) and \( F \) induce by adjunction natural maps

\[
\nabla : \pi_p i^*_p E \to E, \quad \Phi : E \to \pi_p i^*_p E
\]

whose composition is equal to \( \text{tr}_p \). We can consider the compositions

\[
\pi_p \mathbb{K}_*(i^*_p E) \xrightarrow{\nu_p} \mathbb{K}_*(\pi_p i^*_p E) \xrightarrow{\mathbb{K}_*(\nabla)} \mathbb{K}_*(E),
\]

\[
\mathbb{K}_*(E) \xrightarrow{\mathbb{K}_*(F)} \mathbb{K}_*(\pi_p^* E) \xrightarrow{\varphi_p} \pi_p^* \mathbb{K}_*(i^*_p E),
\]

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where \( \nu_p, \varphi_p \) are the maps (2.6), and by Lemma 1.9 together with the isomorphism (2.5), these induce natural maps

\[
V, F : \text{HH}(E) \to \text{HH}(E).
\]

**Lemma 2.7.** For any \( E \in \text{Fun}(\Lambda, A) \) equipped with an \( FV \)-structure, we have

\[
FV = \text{id} : \text{HH}(E) \to \text{HH}(E)
\]

and

\[
FBV = B : \text{HH}(E) \to \text{HH}_{-1}(E),
\]

where \( V \) and \( F \) are the maps (2.13), and \( B \) is the differential (1.35).

**Proof.** The first identity immediately follows from (2.10) and Corollary 1.10. To prove the second equality, note that by Lemma 2.4 and (2.12), it suffices to prove that the canonical map \( b : \mathbb{K}_0(\pi p i^*_p E) \to \mathbb{K}_1(\pi_* i^*_p E) \) given by (2.7) coincides with the composition

\[
\begin{array}{ccc}
\mathbb{K}_0(\pi p i^*_p E) & \xrightarrow{\kappa_0} & \mathbb{K}_0(E) \\
\mathbb{K}_0(V^\dagger) & \downarrow & \mathbb{K}_0(E) \\
\mathbb{K}_0(E) & \xrightarrow{\kappa_0} & \mathbb{K}_1(E)
\end{array}
\]

\[
\begin{array}{ccc}
E & \xrightarrow{\kappa_1} & \mathbb{K}_1(E) \\
F^\dagger & \downarrow & \mathbb{K}_1(F^\dagger) \\
\mathbb{K}_1(\pi_* i^*_p E) & \xrightarrow{\kappa_1} & \mathbb{K}_1(\pi_* i^*_p E)
\end{array}
\]

are commutative. This immediately follows from the functoriality of the adjunction maps \( \kappa_0, \kappa_1 \). \( \square \)

We note that for any \( E \in \text{Fun}(\Lambda, A) \) equipped with an \( FV \)-structure \( \langle V, F \rangle \), we can iterate the maps \( V \) and \( F \). Namely, for any integer \( m \geq 1 \), we can inductively apply base change to the Cartesian square (1.11) and obtain natural maps

\[
V^m : i^*_{p^m} E \to \pi_*^{p^m} E, \quad F^m : \pi_*^{p^m} E \to i^*_{p^m} E.
\]

By adjunction, these induce maps

\[
\overline{V}^m : \pi_*^{p^m} i^*_{p^m} E \to E, \quad \overline{F}^m : E \to \pi_*^{p^m} i^*_{p^m} E,
\]

where \( p, \varphi \) are the maps (2.6), and by Lemma 1.9 together with the isomorphism (2.5), these induce natural maps

\[
(2.13) \quad V, F : \text{HH}(E) \to \text{HH}(E).
\]
a generalization of (2.11), and taking the compositions (2.12), we obtain natural maps
\[ V^m, F^m : HH_*(E) \to HH_*(E). \]
By base change, these are indeed iterates of the maps (2.13), so that our notation is consistent.

2.3 Yoneda sets. Recall now that sending an object \([n] \in \Lambda\) to the set of objects of the category \([n] \Lambda\) defines a functor \(Y : \Lambda \to \text{Sets}\), and this functor is corepresented by \([1] \in \Lambda\). One can also consider the functors corepresented by different objects \([n] \in \Lambda\); we now prove some results on their structure that we will need in Subsection 4.2.

For any \([n] \in \Lambda\), with automorphism group \(\text{Aut}([n]) = \mathbb{Z}/n\mathbb{Z}\), denote by \(e^n : \text{pt}_n \to \Lambda\) the embedding onto \([n]\). Consider the functor
\[ (2.16) \quad e^n! : A[\mathbb{Z}/n\mathbb{Z}]-\text{mod} \cong \text{Fun}(\text{pt}_n, A) \to \text{Fun}(\Lambda, A). \]
To describe this functor explicitly, let \(\Lambda^n\) be the category of maps \(f : [n] \to [m]\) in \(\Lambda\), with maps from \(f' : [n] \to [m']\) to \(f : [n] \to [m]\) given by commutative diagrams
\[
\begin{array}{ccc}
[n] & \xrightarrow{f'} & [m'] \\
\sim & & \\
[n] & \xrightarrow{f} & [m].
\end{array}
\]
Then sending \(f : [n] \to [m]\) to \([n]\) resp. \([m]\) gives functors \(x : \Lambda^n \to \text{pt}_n\), \(y_n : \Lambda^n \to \Lambda\), the embedding \(e^n\) as \(e^n = y_n \circ \overline{e}^n\) with \(\overline{e}^n\) left-adjoint to \(x\), and we have
\[ e^n! \cong y_n! \circ \overline{e}^n! \cong y_n! \circ x^*. \]
The functor \(y^n : \Lambda^n \to \Lambda\) is a discrete cofibration. Explicitly, define a functor \(\tilde{Y}^n : \Lambda \to \text{Sets}\) by
\[ \tilde{Y}^n([m]) = \Lambda([n],[m]), \quad [m] \in \Lambda, \]
and let \(Y^n = \tilde{Y}^n/\text{Aut}([n])\). Then the \(\text{Aut}([n])\)-action on \(\tilde{Y}^n\) is free, and \(Y^n\) corresponds by the Grothendieck construction to the discrete cofibration \(y^n\). If \(n = 1\), then \(\tilde{Y}^1 = Y^1\) coincides with the functor \(Y : \Lambda \to \text{Sets}\), the category \(\Lambda^1\) is equivalent to \(\Delta^o\), and \(y^1 : \Lambda^1 \to \Lambda\) is then identified with \(j^o : \Delta^o \to \Lambda\).
More generally, take some integer \( l \geq 1 \), denote by \( e^n_\ell : \text{pt}_{nl} \to \Lambda_l \) the embedding onto \([n] \in \Lambda_l\), and consider the functor
\[
e^n_\ell : A[\mathbb{Z}/nl\mathbb{Z}]\text{-mod} \cong \text{Fun}(\text{pt}_{nl}, A) \to \text{Fun}(\Lambda_l, A)
\]
that reduces to (2.16) when \( l = 1 \). Then as in the case \( l = 1 \), we can define the functor \( Y^n_\ell : \Lambda_l \to \text{Sets} \) by
\[
Y^n_\ell([m]) = \Lambda_l([n], [m])/\text{Aut}([n]), \quad [m] \in \Lambda_l,
\]
consider the corresponding discrete cofibration \( y^n_\ell : \Lambda^n_l \to \Lambda_l \), and note that \( e^n_\ell \cong y^n_\ell \circ e^n_\ell \) for some embedding \( e^n_\ell : \text{pt}_{nl} \to \Lambda^n_l \) that admits a right-adjoint projection \( x : \Lambda^n_l \to \text{pt}_{nl} \). We then again have an isomorphism
\[
e^n_\ell \cong y^n_\ell \circ x^*.
\]

**Lemma 2.8.** For any \([n] \in \Lambda_l\), the functor (2.11) is exact. Moreover, for any \( A[\mathbb{Z}/nl\mathbb{Z}]\)-module \( M \), the Hochschild complex \( CH_\ast(e^n_\ell M) \) is naturally quasiisomorphic to the complex \( \mathbb{K}_\ast(M) \) of (1.24), and the Connes-Tsygan differential \( B \) is induced by the trace map \( \text{tr}_{\mathbb{Z}/nl\mathbb{Z}} \).

**Proof.** Since \( y^n_\ell : \Lambda^n_l \to \Lambda_l \) is a discrete cofibration, the first claim immediately follows from (2.18). Moreover, we tautologically have
\[
HC_\ast(e^n_\ell M) = H_\ast(\Lambda_l, e^n_\ell E) \cong H_\ast(\mathbb{Z}/nl\mathbb{Z}, M),
\]
and by (1.23), if we compute \( H_\ast(\mathbb{Z}/nl\mathbb{Z}, M) \) by the standard 2-periodic resolution, then the isomorphism (2.19) is compatible with the periodicity. Then the second claim immediately follows from (1.33). \( \square \)

Now note that for any \( n, l \geq 1 \), we have \( i_\ell \circ e^n_\ell \cong e^{nl} \), so that by adjunction, we have a functorial map
\[
e^n_\ell \to i_\ell^! \circ i_\ell^! \circ e^{nl} \cong i_\ell^! \circ e^{nl}.
\]
Then for any \( A[\mathbb{Z}/nl\mathbb{Z}]\)-module \( M \), the induced map
\[
HH_\ast(e^n_\ell M) \to HH_\ast(i_\ell^! e^{nl} M)
\]
is an isomorphism by Lemma 1.9. What we will need is the following refinement of this fact.

**Proposition 2.9.** For any \( n, l \geq 1 \) and \( M \in A[\mathbb{Z}/nl\mathbb{Z}]\)-mod, the map
\[
HH_\ast(\pi_{nl} e^n_\ell M) \to HH_\ast(\pi_{nl} i_\ell^* e^{nl} M)
\]
induced by the map (2.20) is an isomorphism.
In order to prove this, we need to analyse the structure of the functors $Y^n_l$ in some detail. For any integer $m \geq 1$, denote

\begin{equation}
(2.21) \quad Y^n_{l,m} = Y^n_l \circ i_m : \Lambda_{ml} \to \text{Sets},
\end{equation}

and let $y^n_{l,m} : \Lambda^n_{l,m} \to \Lambda_{ml}$ be the corresponding discrete cofibration. To simplify notation, set $Y^n_{l,m} = Y^n_{l,1}$, $\Lambda_{n,m} = \Lambda_{n,1}$, $y^n_{l,m} = y^n_{l,1}$. Then for any $n, m, l \geq 1$, the embeddings (1.12) taken together define a canonical embedding

\begin{equation}
(2.22) \quad \varphi^n_{l,m} : Y^n_{l,m} \subset Y^n_{nl,ml}
\end{equation}

and a functor $\varphi^n_{l,m} : \Lambda^n_{l,m} \to \Lambda^{nl,ml}$ such that $y^n_{nl,ml} \circ \varphi^n_{l,m} \simeq y^n_{l,m}$. We have a natural projection $x : \Lambda^n_{l,m} \to \text{pt}_{ml}$ such that

\[ i^* \circ e^n_l \simeq y^n_{l,m} \circ x^*, \]

where $i_m : \Lambda_l \to \Lambda_{ml}$ is the functor (1.11), and in terms of this identification, the map (2.20) is induced by the embedding $\varphi^n_{l,1} : \Lambda^n_l \to \Lambda^{nl,l}$.

**Proof of Proposition 2.9.** Denote $E = i^* e^n_l M$, and for any $r \geq 1$ dividing $l$, denote $E_r = i^* e^n_{r,m} M$, where $m = l/r$. Note that the embeddings (2.22) induce injective maps $\nu_r : E_r \to E$. Filter $E$ by setting

\begin{equation}
(2.23) \quad F_i E = \sum_{r|l,l/r \leq i} \nu_r(E_r) \subset E.
\end{equation}

By definition, we have $F_i E = E$, and $F_1 E = E = e^n_l M$, with $\nu_l : E_l \to E$ being the map (2.20). We have $\text{gr}_i F E = 0$ unless $i$ divides $l$, and for any $m|l, r = l/m$, we have $\text{gr}_r^n E \cong \text{gr}_r^n E_r$, where $E_r \subset E$ is equipped with the induced filtration. For any $r|l$, the map $\text{HH}_*(E_l) \to \text{HH}_*(E_r)$ induced by the embedding $E_l \subset E_r$ is an isomorphism by Lemma 1.9, and by induction on $r$, this implies that

\begin{equation}
(2.24) \quad \text{HH}_*(\text{gr}_r^n E) = 0
\end{equation}

for any $m|l, m > 1$.

Now note that the filtration (2.23) is induced by the filtration $F_*$ on $Y^{nl,l}$ given by the unions of the images of the embeddings (2.22), and after evaluating at any object $[s] \in \Lambda_l$, the latter filtration splits: we have

\begin{equation}
(2.25) \quad Y^{nl,l}([s]) = \coprod_{m \leq i, m|l} Y^{nm,m}([s]),
\end{equation}

where $Y^{nm,m}([s])$ is the object of $\text{gr}_r^n E_r$ corresponding to $[s]$. Thus, by induction on $r$, we have $\text{HH}_*(Y^{nl,l}) = 0$.
where we denote $Y^{nm,m}_{r}(s) = Y^{nm,m}_{r-1}(s) \setminus (F_{m-1} Y^{nl,l}(s) \cap Y^{nm,m}_{r}(s))$.

Moreover, by Lemma 1.4, the splitting (2.25) is invariant under the action of $\mathbb{Z}/l\mathbb{Z} \subset \text{Aut}([s])$. Therefore the filtration $F_\ast E([s])$ admits a $\mathbb{Z}/l\mathbb{Z}$-equivariant splitting. By base change, this implies that the map $\pi_{l!} F_\ast E \to \pi_{l!} E$ is injective for any $i$, and $\pi_{l!} E$ acquires a filtration $F_\ast$ such that

$$gr_{l}^{i} \pi_{l!} E \cong \pi_{l!} gr_{l}^{i} E, \quad 1 \leq i \leq l.$$  

Then to prove the Proposition, it suffices to prove that $HH_{\ast}(\pi_{l!} gr_{l}^{i} E) = 0$ for any $i > 1$. Since this is tautologically true if $gr_{l}^{i} E = 0$, we may further assume that $i = m$ divides $l$.

We now observe that by Lemma 1.4, we have a commutative diagram with Cartesian squares

$$\begin{array}{cccccc}
\text{pt}_{nl} & \xleftarrow{x} & \Lambda_{nm,m}^{r} & \xrightarrow{y_{nm,m}^{r}} & \Lambda_{l} \\
\pi_{r} & \downarrow & \downarrow & \downarrow \pi_{r} \\
\text{pt}_{nl} & \xleftarrow{x} & \Lambda_{nm,m}^{l} & \xrightarrow{y_{nm,m}^{l}} & \Lambda_{m},
\end{array}$$

where the rightmost vertical arrow is the functor (1.11), the leftmost arrow denotes by abuse of notation the tautological projection $\pi_{r} : \text{pt}_{nl} \to \text{pt}_{nl}$, and the arrow in the middle is a bifibration with fiber $\text{pt}_{r}$. Therefore if we denote $E' = e_{m!}^{n} \pi_{rs}^{*} M$, then by base change, we have

$$\pi_{l!} E_{r} = \pi_{l!} y_{r}^{nm,m} x^{*} M = y_{l}^{nm,m} x^{*} \pi_{rs}^{*} M = E',$$

so that

$$\pi_{l!} gr_{l}^{i} E \cong \pi_{l!} \pi_{r}^{*} gr_{F}^{m} E = \pi_{m!} \pi_{r}^{*} gr_{F}^{m} E_{r} \cong \pi_{m!} gr_{F}^{m} \pi_{r}^{*} E_{r} \cong \pi_{m!} gr_{F}^{m} E'.$$

where we define the filtration $F_{\ast}$ on $E'$ in the same way as on $E$. Thus by induction, we may further assume that $l = m$.

It remains to notice that by Lemma 1.4 for any $[s] \in \Lambda_{m}$, the action of $\mathbb{Z}/l\mathbb{Z} \subset \text{Aut}([s])$ on $Y_{r}^{nl,l}([s])$ is free. Therefore the object $gr_{F}^{l} E \in \text{Fun}(\Lambda_{l}, A)$ is $\pi_{l!}$-free, and

$$HH_{\ast}(\pi_{l!} gr_{l}^{l} E) \cong HH_{\ast}(L^{\ast} \pi_{l!} gr_{l}^{l} E) \cong H_{\ast}(\Delta^{\circ} \times \text{pt}_{l!} gr_{l}^{l} E).$$

This vanishes by (2.24). \hfill \Box
3 Constructions.

3.1 Algebras. Fix a commutative ring $k$. To any associative unital algebra $A$ over $k$ one associates a canonical object $A^\natural \in \text{Fun}(\Lambda, k)$ as follows:

- on objects, $A^\natural([n]) = A^{\otimes_k Y([n])} = A^{\otimes_k n}$, with copies of $A$ numbered by elements $y \in Y([n])$,
- for any map $f : [n'] \to [n]$, the map
  $$A^\natural(f) : A^{\otimes_k n'} = \bigotimes_{y \in Y([n])} A^{\otimes_k f^{-1}(y)} \to A^{\otimes_n}$$
  is given by
  $$A^\natural(f) = \bigotimes_{y \in Y([n])} m_{f^{-1}(y)},$$
  where $m_{f^{-1}} : A^{\otimes_k f^{-1}(y)} \to A$ is the map which multiplies the entries in the canonical order.

Note that (3.1) makes sense in an arbitrary symmetric monoidal category. In particular, for any associative monoid $G$, we can define the natural functor $G^\natural : \Lambda \to \text{Sets}$ by setting $G^\natural([n]) = G^n$, and with the structure maps given by (3.1). We will say that a $k$-algebra $A$ is monomial if $A = k[G]$ for such a monoid $G$. In this case, we have

$$A^\natural \cong k[G^\natural].$$

Now take a general associative $k$-algebra $A$, and assume that it is flat as a $k$-module. Then by definition, the Hochschild homology $HH_\ast(A)$ and the cyclic homology $HC_\ast(A)$ are given by

$$HH_\ast(A) = HH_\ast(A^\natural), \quad HC_\ast(A) = HC_\ast(A^\natural).$$

The spectral sequence (1.34) reads as

$$\text{HH}_\ast(A)[u] \Rightarrow HC_\ast(A).$$

Note that by Lemma 1.9, one can equally well replace $\Lambda$ with $\Lambda_l$, $l \geq 1$, and $A_\natural$ with $i_l^\ast A_\natural$; the resulting spectral sequence is the same.

For any two flat $k$-algebras $A$, $B$, we have a natural isomorphism

$$\text{(3.4)} \quad (A \otimes_k B)^\natural \cong A^\natural \otimes B^\natural,$$
and the Künneth quasiisomorphism provides natural multiplication maps
\begin{equation}
HH_i(A) \otimes_k HH_j(B) \to HH_{i+j}(A \otimes_k B).
\end{equation}

By \(\text{(2.9)}\), the Connes-Tsygan differential \(B\) is a derivation with respect to these multiplication maps.

In general, both \(HH_i(A)\) and \(HC_i(A)\) are just \(k\)-modules. However, if \(A\) is commutative, then the product map \(m: A \otimes_k A \to A\) is a map of algebras. Then \(\text{(3.4)}\) together with the map \(m^\natural: (A \otimes_k A)^\natural \to A^\natural\) turns \(A^\natural\) into an associative commutative algebra object in \(\text{Fun}(\Lambda, k)\), and the multiplication maps \(\text{(3.5)}\) turn \(HH_i(A)\) into an an associative commutative \(k\)-algebra. The Connes-Tsygan differential \(B\) is then a derivation with respect to this algebra structure.

The main comparison theorem for Hochschild homology in the commutative case is the classic result of Hochschild, Kostant, and Rosenberg.

**Theorem 3.1 (\[HKR\]).** Assume that the algebra \(A\) is commutative and finitely generated over \(k\), and that \(X = \text{Spec} A\) is smooth over \(k\). Then we have natural isomorphisms
\begin{equation}
HH_i(A) \cong \Omega^i_A = H^0(X, \Omega^i_X)
\end{equation}
for any \(i \geq 0\), and these isomorphisms are compatible with multiplication. \(\square\)

We want to emphasize that the Hochschild-Kostant-Rosenberg Theorem requires no assumptions on \(\text{char } k\). As for cyclic homology, the main result is the following (for a characteristic-independent proof, see e.g. \[K3\] Theorem 2.2, or an essentially equivalent \[L\] Theorem 3.4.11).

**Theorem 3.2.** In the assumptions and under the identifications of Theorem \[3.1\] the Connes-Tsygan differential
\[B: HH_i(A) \to HH_{i+1}(A),\quad i \geq 0\]
becomes de Rham differential \(d: \Omega^i_A \to \Omega^{i+1}_A\). \(\square\)

### 3.2 Cartier isomorphism.
Now assume that \(k\) is a perfect field of positive characteristic \(\text{char } k = p\). Then in the assumptions of Theorem \[3.1\] the classic Cartier isomorphism gives an expression for the de Rham cohomology groups of the variety \(X = \text{Spec} A\). Namely, denote by
\[B\Omega^*_A \subset Z\Omega^*_A \subset \Omega^*_A\]
the image resp. kernel of the de Rham differential. Then one has a functorial isomorphism

\( C : \Omega^*_{A}/B\Omega^*_A \cong \Omega^*_{A(1)}, \)

where \( A^{(1)} \) is the twist of \( A \) by the Frobenius endomorphism \( k \to k \) (that is, \( A \) with the \( k \)-vector space structure given by \( \lambda \cdot a = \lambda^p a, \lambda \in k, a \in A \)).

We will need a non-commutative generalization of this given in [K3].

For any \( k \)-vector space \( E \), denote

\( C^{(1)}((E) = (E^{\otimes p})_{\sigma}, \quad C^{(1)}((E) = (E^{\otimes p})^{\sigma}, \)

where \( \sigma \) is the permutation of order \( p \), and let \( E^{(1)} \) be the Frobenius twist of \( E \). Then one has a natural map

\( C^{(1)}(E) \to C^{(1)}(E), \quad e \mapsto e^{\otimes p}, \)

and one observes that the map is additive and \( k \)-linear. If \( E \) is finite-dimensional, then one can dualize the construction and obtain a functorial \( k \)-linear map

\( R : C^{(1)}(E) \to E^{(1)} \).

Taking filtered colimits, one extends the map \( R \) to arbitrary vector spaces.

The cokernel \( \Phi(E) \) of the map \( (3.8) \) coincides with the kernel of the map \( (3.9) \), and the composition

\[ C^{(1)}(E) \longrightarrow \Phi(E) \longrightarrow C^{(1)}(E) \]

of the natural projection and the natural embedding is the trace map \( \text{tr}_{\mathbb{Z}/p\mathbb{Z}} \) for the group \( \mathbb{Z}/p\mathbb{Z} \) generated by \( \sigma \).

If one now takes a \( k \)-algebra \( A \) and considers the object \( A^\natural \in \text{Fun}(\Lambda, k) \), then the maps \( C, R \) taken together define functorial maps

\( C : A^{(1)^{\natural}} \to \pi_p i^* p_* A^{\natural}, \quad R : \pi p_* i^* p_* A^{\natural} \to A^{(1)^{\natural}}. \)

The cokernel \( \Phi A^\natural \) of the map \( C \) coincides with the kernel of the map \( R \), and the composition map

\[ \pi_p i^* p_* A^{\natural} \longrightarrow \Phi A^\natural \longrightarrow \pi_p i^* p_* A^{\natural} \]

is the trace map \( \text{tr}_p \text{ pf } (2.1) \). One then considers the complex \( \mathbb{K}_p(A^\natural) \) of Definition \( (2.1) \) and the augmentation maps

\[ \pi_p i^* p_* A^{\natural}[1] \longrightarrow \mathbb{K}_p(A^\natural) \longrightarrow \pi_p i^* p_* A^{\natural} \]

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induced by the exact sequence (2.4), and one defines subcomplexes $B\mathbb{K}_p^q(A^2)$ and $Z\mathbb{K}_p^q(A^2)$ in $\mathbb{K}_p^q(A^3)$ by

(3.11) \quad B\mathbb{K}_p^q(A^2) = \widetilde{\kappa}_1(\text{Ker } R), \quad Z\mathbb{K}_p^q(A^2) = \widetilde{\kappa}_0^{-1}(\text{Im } C).

Note that since $\widetilde{\kappa}_0 \circ \widetilde{\kappa}_1 = 0$, we have $B\mathbb{K}_p^q(A^2) \subset Z\mathbb{K}_p^q(A^2) \subset \mathbb{K}_p^q(A^2)$. In fact, $B\mathbb{K}_p^q(A^2) \cong \Phi A^2[-1]$, and we have a short exact sequence

$$0 \longrightarrow Z\mathbb{K}_p^q(A^2) \longrightarrow \mathbb{K}_p^q(A^2) \longrightarrow \Phi A^2 \longrightarrow 0.$$  

The quotient $Z\mathbb{K}_p^q(A^2)/B\mathbb{K}_p^q(A^2)$ is a complex of length 2, and its homology objects in degrees 0 and 1 are identified with $A^{(1)}z$ by the maps (3.10). In this, it is similar to the complex $\mathbb{K}_s(A^{(1)}z)$.

If the algebra $A = k[G]$ is monomial, then it is easy to identify the two complexes. Namely, the diagonal maps $G^n \to G^np$, $n \geq 1$ commute with the structure maps (3.1) and define a map of functors $\pi_p^*G^n \to i_p^*G^n$. By (3.2), it induces maps $\pi_p^*A^{(1)}z \to i_p^*A^z$, $\mathbb{K}_s(\pi_p^*A^{(1)}z) \to \mathbb{K}_s(i_p^*A^z)$, and by adjunction, we obtain a map from $\mathbb{K}_s(A^{(1)}z)$ to $\mathbb{K}_p^q(A^2)$. By [K3] Lemma 5.2, it lands in $Z\mathbb{K}_p^q(A^2) \subset \mathbb{K}_p^q(A^3)$, and it is easy to check that the induced map

(3.12) \quad \mathbb{K}_s(A^{(1)}z) \to Z\mathbb{K}_p^q(A^2)/B\mathbb{K}_p^q(A^2)

is a quasiisomorphism. For a general algebra $A$, there no obvious map of complexes like this. Nevertheless, we have the following.

**Theorem 3.3.** Assume given a perfect field $k$ of characteristic $p = \text{char } k > 0$ and an associative unital $k$-algebra $A$.

(i) Denote by $\nu : \mathcal{D}(A, k) \to \mathcal{D}(A, W_2(k))$ the tautological functor that takes a $k$-vector spaces and treats it as a module over the second Witt vectors ring $W_2(k)$. Then there exists a canonical isomorphism

(3.13) \quad \nu(\mathbb{K}_s(A^{(1)}z)) \cong \nu(Z\mathbb{K}_p^q(A^2)/B\mathbb{K}_p^q(A^2))

in the derived category $\mathcal{D}(A, W_2(k))$. If $A = k[G]$ is monomial, then this isomorphism coincides with (3.12).

(ii) Assume that $A$ is commutative and satisfies the assumptions of Theorem 3.1. Then the isomorphisms (3.6) and (2.5) induce isomorphisms

$$HC_i(B\mathbb{K}_p^q(A^2)) \cong B\Omega^i_A, \quad HC_i(Z\mathbb{K}_p^q(A^2)) \cong Z\Omega^i_A,$$

and with these identifications, the map induced by the map (3.13) is inverse to the Cartier isomorphism (3.7).
Start of the proof. If the characteristic \( p \) is odd, then the isomorphism (3.13) is \([K3\text{ Lemma 4.4}]\), it coincides with (3.12) by \([K3\text{ Lemma 5.2}]\), and there is no need to apply the functor \( \nu \). For \( p = 2 \), one has to apply \( \nu \) and modify the argument as in \([K5\text{ Section 4}]\). This gives (3.13) but does not prove its compatibility with (3.12). Moreover, we will need a more explicit form of (3.13), and for this, we need some notions to be introduced later. Thus we postpone the end of the proof of Theorem 3.3 (i) to the end of Subsection 4.1.

As for (ii), it is \([K3\text{ Proposition 5.1}]\). While formally, that paper requires \( p \) to be odd throughout, this is not really used in Section 5 that contains Proposition 5.1 — all one needs is \([K3\text{ Lemma 5.2}]\) that reduces to the compatibility between (3.13) and (3.12).

□

Finally, in Section 6, we will need one simple result that morally belongs to \([K3]\) but is not contained explicitly in that paper; we prove it here.

Lemma 3.4. For any associative unital \( k \)-algebra \( A \) and integer \( n \geq 1 \), the object \( i^*_p \Phi A^2 \) is \( \pi_{p^n} \)-free.

Proof. By definition, it suffices to prove that for any \( k \)-vector space \( E \), the \( \mathbb{Z}/p^n\mathbb{Z} \)-action on \( \Phi(E^\otimes p^n) \) induced by the \( \mathbb{Z}/p^{n+1}\mathbb{Z} \)-action on \( E^\otimes p^{n+1} \) is free. Choose a basis in \( E \), so that \( E = k[S] \) for some set \( S \), and decompose

\[
E^\otimes p^{n+1} = k[S^p] = E^\otimes p^n \oplus E',
\]

where \( E^\otimes p^n = k[S^p] \) is embedded by the diagonal embedding \( S^p \subset S^{p^{n-1}} \), and \( E' = k[S'] \) is spanned by the complement \( S' = S^{p^{n+1}} \setminus S^p \). Then we have

\[
\Phi(E^\otimes p^n) \cong E_{\sigma p^n},
\]

where \( \sigma \) is the generator of the group \( \mathbb{Z}/p^{n+1}\mathbb{Z} \), and this group acts on \( S' \) without fixed points. □

3.3 Trace functors. It is useful to categorify the construction of the cyclic object \( A^2 \) in the following way. Assume given a unital monoidal category \( \mathcal{C} \), and define a category \( \mathcal{C}^2 \) as follows.

- Objects are pairs \( \langle [n], \{c_y\} \rangle \) of an object \( [n] \in \Lambda \) and a collection of objects \( c_y \in \mathcal{C} \) numbered by elements \( y \in Y([n]) \).

- Morphisms from \( \langle [n'], \{c'_y\} \rangle \) to \( \langle [n], \{c_y\} \rangle \) are given by a collection of a morphism \( f : [n'] \to [n] \) and morphisms

\[
f_y : \bigotimes_{y' \in f^{-1}(y)} c'_{y'} \to c_y, \quad y \in Y([n]),
\]
where the product is taken in the canonical order on \( f^{-1}(y) \). The forgetful functor \( \rho : C^2 \to \Lambda \) is a cofibration. Its fiber \( C^2_{\lfloor n \rfloor} \) over \( [n] \in \Lambda \) is naturally identified with the product \( C^Y([n]) \) of copies of \( C \) numbered by elements \( y \in Y([n]) \), and the transition functors are induced by the tensor product in \( C \) via an obvious generalization of (3.1). A morphism \( \langle f, \{ f_y \} \rangle \) in \( C^2 \) is cocartesian with respect to \( \rho \) if and only if all its components \( f_y \) are invertible maps.

**Definition 3.5.** A *trace functor* from \( C \) to some category \( E \) is a functor \( F^\natural : C^\natural \to E \) that sends all maps cocartesian with respect to \( \rho : C^\natural \to \Lambda \) to invertible maps in \( E \). Definition 3.5 is a version of [K2, Definition 2.1], with the equivalence proved in [K2, Lemma 2.5]. More precisely, every trace functor \( F^\natural : C^\natural \to E \) induces a functor \( F : C \to E \) by restriction to \( C = C^\natural_{[1]} \subset C^\natural \). Then [K2, Lemma 2.5] shows that extending an arbitrary \( F : C \to E \) to a trace functor is equivalent to giving functorial isomorphisms

\[
\tau_{M,N} : F(M \otimes N) \cong F(N \otimes M), \quad M, N \in C
\]

that satisfy some compatibility constraints listed in [K2, Definition 2.1].

Any associative unital algebra object \( A \) in a unital monoidal category \( C \) canonically defines a section

\[
\alpha : \Lambda \to C^\natural
\]

of the projection \( \rho \). On objects, \( \alpha \) sends \( [n] \in \Lambda \) to \( \langle [n], \{ A \} \rangle \) (that is, we take \( n \) copies of \( A \)). On morphisms, \( \alpha \) is essentially given by (3.1) (for a more invariant categorical definition, see [K2, Subsection 1.4]). Given a trace functor \( F^\natural : C^\natural \to E \) to some category \( E \), we can compose it with \( \alpha \) and obtain a canonical functor

\[
FA^\natural = F^\natural \circ \alpha : \Lambda \to E.
\]

For any \( [n] \in \Lambda \), we can choose a morphism \( f : [n] \to [1] \), and its cocartesian lifting proves an isomorphism

\[
FA^\natural([n]) \cong F(A^\otimes n).
\]

Note, however, that the left-hand side carries a canonical action of \( \mathbb{Z}/n\mathbb{Z} = \text{Aut}([n]) \), and the right-hand side does not: the symmetry is broken by the
choice of the morphism $f$. To recover the $\mathbb{Z}/n\mathbb{Z}$-action on $F(A^\otimes n)$, one has to use the maps (3.14) (specifically, $\tau_{A,A^\otimes n-1}$ generates the group).

If the monoidal category $\mathcal{C}$ is symmetric (for example, $\mathcal{C}$ is the category of flat modules over a commutative ring $k$), then the tautological functor $I : \mathcal{C} \to \mathcal{C}$ carries natural isomorphisms (3.14) provided by the symmetry map; these satisfy the necessary constraints and promote $I$ to a trace functor $I^\natural : \mathcal{C}^\natural \to \mathcal{C}$. Then $A^\natural = IA^\natural$ is exactly the canonical cyclic object of Subsection 3.1.

We will also need a version of the isomorphism (3.4) for the objects $FA^\natural$. To obtain it, recall that a pseudotensor structure on a functor $F : \mathcal{C} \to \mathcal{E}$ between unital monoidal categories $\mathcal{C}$, $\mathcal{E}$ with unit objects $1_\mathcal{C}$, $1_\mathcal{E}$ and product functors $m_{\mathcal{C}} : \mathcal{C}^2 \to \mathcal{C}$, $m_{\mathcal{E}} : \mathcal{E}^2 \to \mathcal{E}$ is given by functorial maps

$$\varepsilon : 1_\mathcal{E} \to F(1_\mathcal{C}), \mu_{M,N} : F(M \otimes N) \to F(M) \otimes F(N), \quad M, N \in \mathcal{C},$$

such that $\mu_{1,M} = \varepsilon \otimes \text{id}$, $\mu_{M,1} = \text{id} \otimes \varepsilon$ (unitality), and $(\text{id} \otimes \mu_{N,L}) \circ \mu_{M,N \otimes L} = (\mu_{M,N} \otimes \text{id}) \circ \mu_{M \otimes N,L}$ for any $M, N, L \in \mathcal{C}$ (associativity). If $\mathcal{C}$ and $\mathcal{E}$ are symmetric monoidal categories, then a pseudotensor structure is symmetric if $\mu$ commutes with commutativity isomorphisms. Now assume given a unital monoidal category $\mathcal{E}$ and a symmetric unital monoidal category $\mathcal{C}$. Then $\mathcal{C}^2 = \mathcal{C} \times \mathcal{C}$ is also a unital monoidal category, and since $\mathcal{C}$ is symmetric, the tensor product functor $m_{\mathcal{C}} : \mathcal{C}^2 \to \mathcal{C}$ is a tensor functor. Therefore it extends to a functor

$$m^\natural : \mathcal{C}^{2\natural} \cong \mathcal{C}^\natural \times_{\Lambda} \mathcal{C}^\natural \to \mathcal{C}$$

that commutes with projections $\rho$. Moreover, this relative tensor product $m^\natural$ is symmetric in the obvious sense, and the unit object $1_{\mathcal{C}} \in \mathcal{C}$ is an algebra object, so that we obtain a section $1^\natural : \Lambda \to \mathcal{C}^\natural$ of the projection $\rho$. This section is obviously cocartesian.

**Definition 3.6.** A pseudotensor structure on a trace functor $F^\natural : \mathcal{C}^\natural \to \mathcal{E}$ is given by functorial maps

$$\varepsilon : 1_{\mathcal{E}} \to F^\natural(1^\natural), \quad \mu : m_{\mathcal{E}} \circ F^{2\natural} \to F^\natural \circ m^\natural$$

satisfying the unitality and associativity conditions. If $\mathcal{E}$ is symmetric, then a pseudotensor structure is symmetric if $\mu$ commutes with the commutativity isomorphisms.

For any pseudotensor structure on a trace functor $F^\natural$, the functor $F : \mathcal{C} \to \mathcal{E}$ induced by $F^\natural$ inherits a pseudotensor structure in the usual sense.
Conversely, a pseudotensor structure on $F$ extends to $F^\natural$ if it is compatible with the maps (3.14).

Now for any two associative unital algebra objects $A$, $B$ in a symmetric monoidal category $C$, and for any trace functor $F^\natural : C^\natural \to \mathcal{E}$ equipped with a pseudotensor structure in the sense of Definition 3.6, we obtain a canonical map

$$\mu : FA^\natural \otimes FB^\natural \to F(A \otimes B)^\natural,$$

a version of the isomorphism (3.4). If the pseudotensor structure on $F^\natural$ is tensor — that is, if the maps (3.19) are isomorphisms — then (3.20) is also an isomorphism. This is the case, for instance, when $F^\natural$ is the tautological functor $I^\natural$. If $\mathcal{E}$ and the pseudotensor structure on $F^\natural$ are symmetric, then the map (3.20) commutes with the symmetry maps. If, moreover, $A$ is commutative, then $FA^\natural$ becomes a commutative associative unital algebra object in the category of functors from $\Lambda$ to $\mathcal{E}$.

### 3.4 Polynomial Witt vectors.

Assume now given a perfect field $k$ of positive characteristic $p$. For any integer $m \geq 1$, let $W_m(k)$ be the ring of $m$-truncated $p$-typical Witt vectors of $k$, and let $W(k)$ be their inverse limit. Then [K6, Section 2] constructs a series of polynomial functors

$$W_m : k\text{-mod} \to W_m(k)\text{-mod}, \quad m \geq 1$$

and surjective restriction maps $R : W_{m+1} \to W_m$, $m \geq 1$, called polynomial Witt vectors functors. It is further proved in [K6, Section 3] that one has injective co-restriction maps $C : W_m \to W_{m+1}$, $m \geq 1$ such that $C \circ R = p \text{id}$, and non-additive but functorial Teichmüller maps

$$T : E \to W_m(E), \quad E \in k\text{-mod}$$

such that $R^{m-1} \circ T = \text{id}$. Moreover, $W_m$ together with the maps $R$, $C$ and $T$ canonically extend to trace functors $W_m^\natural : k\text{-mod}^\natural \to W_m(k)\text{-mod}$.

To describe the construction, recall from [K6, Subsection 4.1] that for any unital monoidal category $C$ and integer $l \geq 1$, the functors $i_l$, $\pi_l$ of (1.9) give rise to a natural commutative diagram

$$C^\natural \xleftarrow{i_l^\natural} C_l^\natural \xrightarrow{\pi_l^\natural} C^\natural$$

such that $\rho$ is a natural transformation $\rho : i_l \Rightarrow \pi_l$. If $\Lambda$ is a unital algebra object in $C$, then the functors $i_l$, $\pi_l$ of (1.9) give rise to a natural commutative diagram

$$\Lambda \xleftarrow{i_l} \Lambda_l \xrightarrow{\pi_l} \Lambda^\natural.$$
where the square on the right-hand side is Cartesian, and the functor \( i^C_l \) sends \( \langle [n], c_\gamma \rangle \) to \( \langle i_l([n]), c_\gamma \rangle \), with the collection \( c_\gamma \) given by \( c_\gamma y = c_\eta_i(y) \), where \( \eta_i \) is the map (1.10).

We now fix an integer \( m \geq 1 \), and let \( C_m \) be the category of flat finitely generated \( W_m(k) \)-modules, with the natural quotient functor \( q: C_m \to C, E \mapsto E/p \) to the category \( C = C_1 \) of finite-dimensional \( k \)-vector spaces. Moreover, let \( l = p^m \), and simplify notation by writing \( i^{(m)} = i^{C_m}_{p^m}, \pi^{(m)} = \pi^{C_m}_{p^m} \), with \( \pi^{(m)} \) being the corresponding functor (1.4). Then \( C_m \) is a symmetric unital tensor category, and the tautological embedding \( C_m \subset W_m(k) \)-mod defines a canonical trace functor \( I^\natural \in \text{Fun}(C_m, W_m(k)) \). By base change, the functor

\[
Q^\natural_m = \pi^{(m)}_* i^{(m)*} F^\natural \in \text{Fun}(C_m^*, W_m(k))
\]

is also a trace functor. Here is, then, the main result of [K6].

**Theorem 3.7 ([K6, Proposition 4.3 (i)])**. There exists a unique trace functor \( W^\natural_m \in \text{Fun}(C, W_m(k)) \) such that \( Q^\natural_m \cong q^* W^\natural_m \). \( \square \)

This completely defines \( W^\natural_m \) on finite-dimensional \( k \)-vector spaces; one then extends it to all vector spaces by requiring that the extension commutes with filtered colimits. In addition, [K6, Proposition 4.3 (ii)] constructs the restriction maps

\[
R: W^\natural_{m+1} \to W^\natural_m
\]

and the Teichmüller maps \( T: W^\natural_1 \to W^\natural_m \), and [K6, Lemma 3.1] provides co-restriction maps \( C: W^\natural_m \to W^\natural_{m+1} \). Passing to the limit with respect to the restriction maps, one obtains a trace functor \( W^\natural \) from \( k \)-vector spaces to modules over the Witt vectors ring \( W(k) \).

By definition, the functors \( W^\natural_m \) carry a decreasing standard filtration \( F^i W^\natural_m \) and an increasing co-standard filtration \( F_i W^\natural_m \) given by

\[
F^i W^\natural_m = \text{Ker} R^{m-i}, F_i W^\natural_m = \text{Im} C^{m-i} \subset W^\natural_m,
\]

where \( R \) and \( C \) are the restriction and the co-restriction maps. In the limit \( W^\natural \), only the standard filtration survives, and we have \( W^\natural_m = W^\natural/F^m W^\natural \).
It has been also proved in [K6, Subsection 3.1] that in fact the values of the limit functor $W^\natural$ are torsion-free, and for any $k$-vector space $E$, we have

$$W(E)/p \cong \lim_{R \arr R} C^i(E),$$

where $C^i(E) \subset E^{\otimes p^i}$ is the subspace of $\mathbb{Z}/p^i\mathbb{Z}$-invariant vectors.

If $m = 1$, then $W^\natural_1$ is simply the tautological trace functor $I^\natural$. In particular, it carries a pseudotensor structure. This has been extended to all $m \geq 1$ in [K6, Proposition 4.7]. Namely, it has been proved that the functors $W^\natural_m$, $m \geq 1$ carry natural pseudotensor structures in the sense of Definition 3.6, and the restriction maps $R : W^\natural_{m+1} \arr W^\natural_m$ together with the Teichmüller maps (3.21) are compatible with the pseudotensor structures. Passing to the limit, one obtains a pseudotensor structure on the trace functor $W^\natural$.

Moreover, consider the diagram (3.22) for $l = p$, and simplify notation by writing $i = i_p$, $\pi = \pi_p$. Then we have the following.

**Proposition 3.8.** For any $m \geq 1$, there exists functorial additive maps

$$V : W^\natural_m \circ i \arr W^\natural_{m+1} \circ \pi, \quad F : W^\natural_{m+1} \circ \pi \arr W^\natural_m \circ i$$

such that $RV = VR$, $FR = RF$, $VF = p\text{id}$, and $FV$ coincides with the trace map $\text{tr}_1^\natural$ of (1.5).

We can again pass to the limit with respect to the maps $R$ and obtain maps $V$, $F$ for the limit trace functor $W^\natural$. They define an $FV$-structure on $W^\natural$ in the sense of [K6, Definition 4.5], and by [K6, Proposition 4.7], this structure is compatible with the pseudotensor structure in the sense of [K6, Definition 4.6].

### 4 Definitions and properties.

**4.1 General case.** As in Subsection 3.4, fix a perfect base field $k$ of positive characteristic $p$. Consider the trace functors $W^\natural_m$, $m \geq 1$ provided by Theorem 3.7, the restriction maps $R$ of (3.23), the Teichmüller maps $T$ of (3.21), and the co-restriction maps $C$ such that $RC = CR = p\text{id}$. Then for any associative unital $k$-algebra $A$ and integer $m \geq 1$, we have a natural object

$$W_mA^\natural \in \text{Fun}(A, W_m(k))$$
of (3.16), and we have natural maps
\[ R : W_{m+1}A^2 \to W_mA^2, \quad C : W_mA^3 \to W_{m+1}A^3 \]
and the Teichmüller map
\[ A^2 \to W_mA^2 \]
of simplicial sets. We also have the limit trace functor \( W \) and the limit object
\[ W^A^2 \cong \lim_{R} W_mA^2 \]
in the category Fun(Λ, W(k)). For any \( m \), \( W_mA^2 \) carries the standard and costandard filtrations \( F^\ast W_mA^2 \), \( F^\ast W_mA^2 \) induced by the corresponding filtrations on the trace functor \( W^A_m \). The standard filtration survives in the limit, and we have \( W_mA^2 = WA^2/F^mWA^2 \) for any \( m \geq 1 \). Moreover, \( WA^2 \) is torsion-free, and we have
\[ WA^2/p = \lim_{R} \pi^{p^\ast}\pi^{p^\ast}A^2, \]
where \( R \) is the map (3.10). If \( m = 1 \), then \( W_1^A = I^2 \) is the tautological trace functor, so that \( W_1A^2 \cong A^2 \).

**Definition 4.1.** The **Hochschild-Witt homology groups** of the algebra \( A \) are given by
\[ W_mHH_\ast(A) = HH_\ast(W_mA^2), \quad m \geq 1, \quad WHH_\ast(A) = HH_\ast(WA^2), \]
where \( W_mA^2, WA^2 \) are the natural cyclic objects (4.1), (4.4), and \( HH_\ast(-) \) are the Hochschild homology groups of Definition 1.8. The **Hochschild-Witt complex** \( WCH_\ast(A) \) and its truncated versions \( W_mCH_\ast(A) \) are given by
\[ W_m(CH_\ast(A) = CH_\ast(W_mA^2), \quad m \geq 1, \quad WCH_\ast(A) = CH_\ast(WA^2), \]
where \( CH_\ast(-) \) is the Hochschild complex of Definition 1.8.

Since the trace functors \( W_m^A \), \( m \geq 1 \) are equipped with pseudotensor structures, (3.20) gives a natural map
\[ W_mA^2 \otimes W_mB^2 \to W_m(A \otimes B)^2, \quad m \geq 1 \]
for any two associative unital $k$-algebras $A$, $B$, and these maps are compatible with the restriction maps $R$. They are also compatible with the Teichmüller maps (4.3), in the sense that by [K6 (3.16)], we have a commutative diagram

\[
\begin{array}{ccc}
A^\sharp \times B^\sharp & \longrightarrow & (A \otimes B)^\sharp \\
\downarrow T \times T & & \downarrow T \\
W_m A^\sharp \times W_m B^\sharp & \longrightarrow & W_m (A \otimes B)^\sharp
\end{array}
\]

of cyclic sets. In the limit, we obtain a map

\[WA^\sharp \otimes WB^\sharp \to W(A \otimes B)^\sharp.\]

Neither this map nor the maps (4.5) are isomorphisms (unless $m = 1$). However, coupled with the Künneth map (2.8), they do provide product maps

\[
\begin{align*}
W_m HH_*(A) \otimes W_m HH_*(A) & \to W_m HH_*(A \otimes B), \quad m \geq 1, \\
W_m HH_*(A) \otimes HH_*(B) & \to HH_*(A \otimes B)
\end{align*}
\]

on the level of Hochschild-Witt homology; these maps are associative and unital, and compatible with the restriction maps. For a commutative algebra $A$, $HH_*(A)$ and $W_m HH_*(A)$, $m \geq 1$ thus become unital graded-commutative algebra, and the Connes-Tsygan differential $B$ is its derivation.

On the other hand, note that by definition, the functor $\alpha : \Lambda \to k\text{-mod}^\sharp$ of (3.15) corresponding to the algebra $A$ induces a functor $\alpha_p : \Lambda_p \to k\text{-mod}_p^\sharp$ such that

\[\alpha \circ \pi_p \cong i_p^k \circ \alpha_p, \quad \alpha \circ i_p \cong i_p^k \circ \alpha_p.\]

Then Proposition 3.8 provides natural functorial maps

\[
\begin{align*}
V : i_p^* W_m A^\sharp & \to \pi_p^* W_{m+1} A^\sharp, \\
F : \pi_p^* W_{m+1} A^\sharp & \to i_p^* W_m A^\sharp
\end{align*}
\]

for any $m \geq 1$, and in the limit, we obtain a natural $FV$-structure on the cyclic abelian group $WA^\sharp$. On the level of homology, the maps (4.8) induce natural group maps

\[
\begin{align*}
V : W_m HH_*(A) & \to W_{m+1} HH_*(A), \\
F : W_{m+1} HH_*(A) & \to W_m HH_*(A)
\end{align*}
\]

for any $m \geq 1$, and the corresponding maps in the limit. By definition, we have $VR = RV$, $FR = RF$, $VC = CV$, $FC = CF$, where $R$ and $C$ are the maps (1.2).
Lemma 4.2. For any integer \( m \geq 1 \) and two unital associative \( k \)-algebras \( A, B \), we have

\[
F(a \cdot b) = F(a) \cdot F(b), \quad a \in W_{m+1}HH_q(A), b \in W_{m+1}HH_q(B),
\]
\[
V(a \cdot F(b)) = V(a) \cdot b, \quad a \in W_mHH_q(A), b \in W_{m+1}HH_q(B),
\]
\[
V(F(b) \cdot a) = a \cdot V(b), \quad a \in W_{m+1}HH_q(A), b \in W_mHH_q(B),
\]
where the product is taken with respect to the maps (4.9). If \( A \) is commutative, and we take \( B = A \), then all three equalities hold with respect to the product in the algebras \( W_mHH_q(A) \).

Proof. Immediately follows from [K6, Definition 4.6]. □

Moreover, we can iterate the maps (4.8) as in (2.14). In particular, for any integers \( m, n \geq 1 \), we obtain natural maps of cyclic abelian groups

\[
\nabla^n: \pi_{p^m}i_{p^n}^* W_mA^\natural \rightarrow W_{m+n}A^\natural, \quad \nabla^n: W_{m+n}A^\natural \rightarrow \pi_{p^n}i_{p^n}^* W_mA^\natural,
\]
a version of the maps (2.15).

Proposition 4.3. For any integers \( m, n \geq 1 \) and any associative unital \( k \)-algebra \( A \), the maps (2.14) and (4.2) fit into short exact sequences of cyclic abelian groups

\[
0 \longrightarrow \pi_{p^m}i_{p^n}^* W_mA^\natural \overset{\nabla^m}{\longrightarrow} W_{m+n}A^\natural \overset{R^m}{\longrightarrow} W_nA^\natural \longrightarrow 0,
\]
\[
0 \longrightarrow \pi_{p^m}i_{p^n}^* W_mA^\natural \overset{\nabla^m}{\longrightarrow} W_{m+n}A^\natural \overset{R^m}{\longrightarrow} \pi_{p^n}i_{p^n}^* W_mA^\natural \longrightarrow 0.
\]

Moreover, for any \( m \geq 1 \), we have commutative diagrams

\[
\begin{array}{ccc}
\pi_{p^m}i_{p^m}^* A^\natural & \longrightarrow & \pi_{p^m}i_{p^m}^* A^\natural \\
\nabla^{m-1} & \downarrow & \nabla^{m} \\
W_mA^\natural & \longrightarrow & W_{m+1}A^\natural
\end{array}
\]
\[
\begin{array}{ccc}
\pi_{p^m}i_{p^m}^* A^\natural & \longrightarrow & \pi_{p^m}i_{p^m}^* A^\natural \\
R & \downarrow & R \\
W_mA^\natural & \longrightarrow & W_{m+1}A^\natural
\end{array}
\]

where \( C \) and \( R \) in the top row are obtained by applying \( \pi_{p^m-1}i_{p^m-1}^* \) resp. \( \pi_{p^m-1}i_{p^m-1}^* \) to the maps \( C \) resp. \( R \) of (3.10).

Proof. It suffices to prove everything after evaluation at an arbitrary object \([n] \in \Lambda\). Choosing a map \( f: [n] \rightarrow [1] \) provides an identification \( W_mA^\natural([n]) \cong W_m(A^\otimes n) \), and by [K6, Proposition 4.7 (iii)], our sequences then become the exact sequences of [K6, Lemma 3.7] for \( E = A^\otimes m \). The commutative diagrams are those of [K6, Lemma 3.4]. □
Corollary 4.4. The associated graded quotients $\text{gr}_F^i$ and $\text{gr}_F^F$ of the objects $WA^i$, $W_mA^i$ with respect to the standard and co-standard filtrations are given by

$$
\text{gr}_F^i W_mA^i \cong \text{gr}_F^F W_RA^i \cong \pi_{p^i} \iota_{p^i} A^i, \quad \text{gr}_F^F W_mA^i \cong \pi_{p^i} \iota_{p^i} A^i.
$$

Proof. Clear. □

Lemma 4.5. For any associative unital $k$-algebra $A$ and integer $m \geq 1$, we have

$$
FV = VF = p \text{id} : W_mHH_*(A) \to W_mHH_*(A),
$$

and

$$
FBV = B : W_mHH_*(A) \to W_mHH_{*+1}(A),
$$

where $F$ and $V$ are the maps (4.9), and $B$ is the Connes-Tsygan differential. The same equalities hold in the limit groups $\text{WH}_*(A)$.

Proof. Everything except for the equality $VF = p \text{id}$ is Lemma 2.4. The equality $VF = p \text{id}$ follows from [K6, Lemma 3.11] by the same argument as in the proof of Proposition 4.3. □

End of the proof of Theorem 3.3 (i). For any complex $E_*$ in an abelian category $E$, and any integers $n \leq m$, denote by $\tau_{[n,m]}E_*$ the truncation of $E_*$ with respect to the canonical filtration — that is, let $\overline{E}_* = \tau_{[n,m]}E_*$ be the subquotient of $E_*$ given by

$$
\overline{E}_n = \text{Ker } d_n, \quad \overline{E}_m = \text{Coker } d_{m+1}, \quad \overline{E}_i = E_i, \quad n < i < m,
$$

and $\overline{E}_i = 0$ otherwise, where $d_i : E_i \to E_{i-1}$ is the differential. Recall that the functors $\tau_{[n,m]}$ descend to the derived category $\mathcal{D}(E)$.

For any ring $O$ and any object $E \in \text{Fun}(\Lambda p, O)$, extend the resolution (1.32) to the right to obtain a 2-periodic complex $P_*(E)$ in $\text{Fun}(\Lambda p, O)$ with terms

$$
K_{2i}(E) = \mathbb{K}_0(E), \quad K_{2i+1}(E) = \mathbb{K}_1(E), \quad i \in \mathbb{Z},
$$

and denote by $\pi_{p^i}E \in \mathcal{D}(\Lambda p, O)$ the object represented by the complex $\pi_{p^i}P_*(E)$. Then [K5, (3.10)] provides a functorial map

$$
\mathbb{K}_* (\mathbb{Z})[1] \otimes (\tau_{[-1,0]} \pi_{p^i}E)[1] \to \tau_{[0,1]} \pi_{p^i}E.
$$

Moreover, for any object $E' \in \mathcal{D}(\Lambda p, O)$ equipped with a map $b : E' \to (\tau_{[-1,0]} \pi_{p^i}E)[1]$, we can compose $b$ with the map (4.11) and obtain a map

$$
\mathbb{K}_* (\mathbb{Z}) \otimes E' \to \tau_{[-1,0]} \pi_{p^i}E.
$$
If the composition map

\[(4.13) \quad E' \xrightarrow{b} (\tau_{[-1,0]} \pi_{p^b} E)[1] \xrightarrow{} (\tau_{[-1,-1]} \pi_{p^b} E)[1]\]

is an isomorphism, then (4.12) is an isomorphism in \(\mathcal{D}(\Lambda, O)\).

Now take \(O = W_2(k)\) and \(E = \nu(i_p^* A^2)\). Then by definition, we have an isomorphism of complexes

\[\nu(Z \mathbb{K}_p^p(A^2)/B \mathbb{K}_p^p(A^2)) \cong \tau_{[0,1]} \pi_{p^b} P_\ast(E).\]

On the other hand, the shifted truncation \((\tau_{[-1,0]} \pi_{p^b} P_\ast(E))[1]\) is the complex \(\tilde{E}_*\) with terms \(\tilde{E}_1 = \pi_{p^b} E, \tilde{E}_0 = \pi_{p^b} E\), and with the differential given by the trace map \(\text{tr}_p\).

To obtain the isomorphism (3.13), one takes as \(E'\) the complex of length 2 given by \(E'_1 = \tilde{E}_1, E'_0 = W_2 A^2\), and with the differential given by the map \(V\) of (4.8). Then we have a map \(b : E'_* \rightarrow \tilde{E}_*\) equal to \(\text{id}\) in degree 1 and \(F\) in degree 0, and the composition

\[E'_* \xrightarrow{b} \tilde{E}_* \xrightarrow{} \tau_{[0,0]} \pi_{p^b} E \cong \nu(A^{(1)}_E)\]

is a quasiisomorphism by Proposition 4.3. The corresponding map (4.12) is the isomorphism (3.13).

To see that this isomorphism is compatible with (3.12), note that for any set \(S\), sending an element \(s \in S\) to its Teichmüller representative \(T(s)\) gives a natural map \(W_2(k)[S] \rightarrow W_2(k[S])\), and if we compose it with the projection \(F : W_2(k[S]) \rightarrow C^{(1)}(k[S]),\) then by [K6, (3.18)], the resulting map is induced by the diagonal embedding \(S \subset S^p\). If \(A = k[G]\), we can collect these maps for all powers \(G^n, n \geq 1\), and obtain a natural map

\[(4.14) \quad E''_* \rightarrow E'_*\]

where \(E''_*\) is given by \(E''_0 = W_2(k)[G^2], E''_1 = k[G^2]\), with the differential induced by the embedding \(V : k \rightarrow W_2(k)\). The map (4.14) is a quasiisomorphism, and the map (4.12) for the object \(E''_*\) exactly coincides with the map (4.12).

4.2 Free algebras. Now we want to give some idea as to how Hochschild-Witt homology groups look like. In order to do this, we compute them for free algebras over \(k\). So, let \(M\) be a \(k\)-vector space, and let

\[A = T^\ast(M) = \bigoplus_{n \geq 0} M^\otimes n\]
be the free associative unital $k$-algebra generated by $M$. By definition, the algebra $A$ is graded by non-negative integers, and its Hochschild homology $HH_*(A)$ inherits the grading. Recall that for any integer $n \geq 1$, the component $HH_i(A)_n \subset HH_*(A)$ of degree $n$ is given by

$$HH_i(A)_n \cong \begin{cases} (M^\otimes n)_\sigma, & i = 0, \\ (M^\otimes n)_\sigma, & i = 1, \\ 0, & i \geq 2, \end{cases}$$

where $\sigma : M^\otimes n \to M^\otimes n$ is the permutation of order $n$ that generates the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. The component $HH_i(A)_0$ is $k$ for $i = 0$ and 0 otherwise. The Connes-Tsygan differential $B$ vanishes on $HH_0(A)_0$, and for $n \geq 1$, we have

$$B = \text{tr}_{\mathbb{Z}/n\mathbb{Z}} : (M^\otimes n)_\sigma \to (M^\otimes n)_\sigma,$$

where the $\mathbb{Z}/n\mathbb{Z}$-action is generated by $\sigma$.

To see the isomorphism (4.15) explicitly, one can use the functors $e^n_!$ of (2.16). For any $n \geq 1$, the map $M \to A$ induces a $\mathbb{Z}/n\mathbb{Z}$-equivariant map $M^\otimes n \to A_n^\otimes = e^n_! A_n^\otimes$, where $A_n^\otimes \subset A^\otimes$ is the component of degree $n$, and by adjunction, we obtain a natural map

$$e^n_! M^\otimes n \to A_n^\otimes,$$

An easy computation shows that this map is in fact an isomorphism, so that we have

$$HH_*(e^n_! M^\otimes n) \cong HH_*(A)_n.$$ 

Then Lemma 2.8 immediately implies (4.15).

In order to generalize (4.15) to Hochschild-Witt homology, recall from [K0 Subsection 3.4] that a $\mathbb{Z}$-grading on a $k$-vector space $E$ tautologically induces a $\mathbb{Z}$-grading on $W_m(E)$ for any $m \geq 1$, but these gradings are not compatible with the restriction maps $R$. In order to make them compatible, one has to rescale them, so that the natural grading on $W_m(E)$ and on the limit group $W(E)$ is indexed by elements $a \in \mathbb{Z}[1/p]$ in the localization at $p$ of the ring $\mathbb{Z}$. Thus $W_m HH_*(A)$ and $W_m CH_*(A)$ are also graded by $\mathbb{Z}[1/p]$.

Note that the Frobenius and Verschibung maps $F$, $V$ of (4.9) do not preserve the rescaled degree: $F$ has degree $p$, while $V$ has degree $1/p$.

For any $n \in \mathbb{Z}$, $m \geq 1$, we can equip $W_m(M^\otimes n)$ with a $\mathbb{Z}/n\mathbb{Z}$-action by (3.17), and we then have a $\mathbb{Z}/n\mathbb{Z}$-equivariant map $W_m(M^\otimes n) \to e^n_! W_m A_n^\otimes,$
where as before, $W_m A_n^\natural$ stands for the component of degree $n$. By adjunction, we get a natural map

$$e_i^n W_m (M^{\otimes n}) \to W_m A_n^\natural. \quad \text{(4.16)}$$

If $m \geq 2$, this map is no longer an isomorphism. However, we still have the following result.

**Lemma 4.6.** For any $n \geq 1$, the map

$$K_\ast (W_m (M^{\otimes n})) \to W_m CH_\ast (A)_n$$

induced by (4.16) and Lemma 2.8 is a quasiisomorphism.

**Proof.** It suffices to prove that the map (4.16) induces a quasiisomorphism on Hochschild homology of the associated graded quotients $\text{gr}_F$ with respect to the standard filtration. Since the functor $e_i^n$ is exact, we have $\text{gr}_F e_i^n W_m (M^{\otimes n}) \cong e_i^n \text{gr}_F W_m (M^{\otimes n})$, while the quotient $\text{gr}_F W_m CH_\ast (A)$ is described by Corollary 4.4. With these identifications, the claim immediately follows from Proposition 2.9. \qed

Assume now given $a \in \mathbb{Z}[1/p]$ of the form $a = np^{-i}$, $n > 0$ prime to $p$, $i > 0$. We then have the following result.

**Lemma 4.7.** For any $m \geq 1$, the iterated Verschiebung and Frobenius maps

$$V^i : W_m HH_0 (A)_n \to W_{m+i} HH_0 (A)_n,$$

$$F^i : W_{m+i} HH_1 (A)_n \to W_m HH_1 (A)_n \quad \text{(4.17)}$$

are isomorphisms, and $W_m HH_i (A)_n = 0$ for $i \neq 0, 1$.

**Proof.** Denote $q = p^i$. Corollary 4.4 and Proposition 4.3 immediately show that the map $V^i : \pi q \iota_q^\ast W_m A_n^\natural \to W_{m+i} A_n^\natural$ becomes an isomorphism after taking the associated graded quotients with respect to the standard filtration. Therefore it is itself an isomorphism, and together with Lemma 4.6 we obtain a quasiisomorphism

$$W_{m+i} CH_\ast (A)_n \cong CH_\ast (\pi q \iota_q^\ast e_i^n W_m (M^{\otimes n})). \quad \text{(4.18)}$$

For any $\mathbb{Z}/\mathbb{Z}/n\mathbb{Z}$-module $E$, denote $K_n^i (E) = CH_\ast (i_q^\ast e_i^n E)$. Then explicitly, we have

$$i_q^\ast e_i^n E ([m]) \cong E [Y_n^q ([m])]_{\mathbb{Z}/n\mathbb{Z}}, \quad [m] \in \Lambda_q,$$

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where $Y^{n,q} : \Lambda_q \rightarrow \text{Sets}$ is the functor (2.21). By Lemma 1.4, the $\mathbb{Z}/q\mathbb{Z}$-action on $Y^{n,q}([m])$ is free for any $[m] \in \Lambda_q$, so that $i_q^* e^n_i E$ is $\pi_q$-free. Moreover, $Y^{n,q} \circ j_q^n : \Delta^n \rightarrow \text{Sets}$ is a simplicial set with a free $\mathbb{Z}/q\mathbb{Z}$-action and only finitely many non-degenerate simplices. Therefore the Tate cohomology $\hat{H}^*(\mathbb{Z}/q\mathbb{Z}, K^{n,q}_r(E))$ vanishes, and we have natural quasiisomorphisms

$$C_*(\mathbb{Z}/q\mathbb{Z}, K^{n,q}_r(E)) \cong K^{n,q}_r(E)_{\mathbb{Z}/q\mathbb{Z}} \cong H^*(\mathbb{Z}/q\mathbb{Z}, K^{n,q}_r(E)),$$

where $C_*(\mathbb{Z}/q\mathbb{Z}, -)$, $C^*(\mathbb{Z}/q\mathbb{Z}, -)$ are the complexes computing homology and cohomology of the group $\mathbb{Z}/q\mathbb{Z}$. If we take $E = W_m(M^{\otimes n})$, then $K^{n,q}_r(E)$ is quasiisomorphic to $W_mCH_*(A)_a$ by Lemma 4.6 and Lemma 1.9 (4.18) identifies the quasiisomorphic complexes (4.19) with $W_{m+1}CH_*(A)_a$, and the maps $V^i$ resp. $F^i$ of (4.17) are induced by the natural adjunction maps

$$K^{n,q}_r(E) \rightarrow C_*(\mathbb{Z}/q\mathbb{Z}, K^{n,q}_r(E)), \quad C^*(\mathbb{Z}/q\mathbb{Z}, K^{n,q}_r(E)) \rightarrow K^{n,q}_r(E).$$

But the complex $K^{n,q}_r(E)$ only has non-trivial homology groups in degrees $0$ and $1$, and the $\mathbb{Z}/q\mathbb{Z}$-action on these groups is trivial by Corollary 1.10. This proves the claim. \hfill $\square$

We can now combine Lemma 4.6 and Lemma 4.7 to obtain a complete description of the Hochschild-Witt homology groups of the algebra $A = T^*(M)$. For simplicity, we only treat the limit case $\text{WHH}_*(A)$ and leave it to the reader to figure out the truncated versions. We note that any positive element $a \in \mathbb{Z}[1/p]$, $a > 0$ can be uniquely represented as $a = np^i$, where $n > 0$ is prime to $p$, and we denote $i = \|a\|$.

**Theorem 4.8.** The Hochschild-Witt homology $\text{WHH}_i(A)$ vanishes if $i \neq 0, 1$, and $\text{WHH}_i(A)_a = 0$ for negative $a \in \mathbb{Z}[1/p]$. For $a = 0$, we have $\text{WHH}_0(A)_0 = k$ and $\text{WHH}_1(A)_0 = 0$. For any $a > 0$ with $\|a\| \geq 0$, we have natural identifications

$$\text{WHH}_0(A)_a \cong W(M^{\otimes a})_{\mathbb{Z}/a\mathbb{Z}}, \quad \text{WHH}_1(A)_a \cong W(M^{\otimes a})_{\mathbb{Z}/a\mathbb{Z}},$$

where $\mathbb{Z}/a\mathbb{Z}$ acts on $W(M^{\otimes a})$ via the isomorphism (3.17). The Frobenius map $F : \text{WHH}_1(A)_a \rightarrow \text{WHH}_1(A)_{pa}$ is an isomorphism for any $a > 0$. The Verschiebung map $V : \text{WHH}_0(A)_{pa} \rightarrow \text{WHH}_0(A)_a$ is an isomorphism if $\|a\| < 0$. If $\|a\| \geq 0$, so that $a = np^{\|a\|}$ is an integer, the iterated Verschiebung map

$$V^{\|a\|} : \text{WHH}_0(A)_a \rightarrow \text{WHH}_0(A)_n$$

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is injective and identifies $W\text{HH}_0(A)_a$ with the image of the iterated Verschiebung map

\[
V^{\|a\|}: W(M^\otimes a)_{Z/aZ} \to W(M^\otimes n)_{Z/nZ} = W\text{HH}_0(A)_n.
\]

**Proof.** We note that the information contained in Theorem 4.8 together with the identities of Lemma 4.5 allow one to reconstruct immediately the maps $F$ and $V$ in all the other degrees, and also compute the Connes-Tsygan differential $B$. In particular, in the situation when isomorphisms (4.20) hold, $B$ must coincide with the trace map $\text{tr}_{Z/aZ}$. This also immediately follows from Lemma 4.6 and Lemma 2.8, as indeed do the isomorphisms (4.20). The fact that $V$ and $F$ are isomorphisms when $\|a\| < 0$ follows from Lemma 4.7 by taking the limit with respect to $m$. Finally, the fact that the maps (4.21) and (4.22) agree immediately follows from the construction of the quasiisomorphisms of Lemma 4.6, and since $W(E)$ has no $p$-torsion for any $E$, Lemma 4.5 together with [K6, Lemma 3.11] then implies that the Frobenius map $F^{\|a\|}$ coincides with the corresponding Frobenius map $F^{|a|}$. The latter is an isomorphism by [K6, Corollary 3.9]. □

5 **Comparison I.**

We now want to compare the Hochschild-Witt homology $W\text{HH}_0(A)$ of degree 0 of an associative unital algebra $A$ over a perfect field $k$ of characteristic $p = \text{char} k > 0$ to the group $W(A)$ of non-commutative Witt vectors of $A$ constructed in [H2]. We start by reviewing Hesselholt’s construction.

5.1 **Recollection.** Hesselholt starts with an arbitrary associative ring $A$, possibly non-unital, and defines the ghost map $w : A^N \to (A/[A,A])^N$ by

\[
w(a_0 \times a_1 \times \ldots) = w_0 \times w_1 \times \ldots, \quad w_i = \sum_{j=0}^{i} p^j a_j^{p^i-j},
\]

where $A/[A,A] = HH_0(A)$ is the quotient of $A$ by the abelian subgroup spanned by all commutators $aa' - a'a$, $a, a' \in A$. One observes that $w$ is well-defined as a map of sets. One also observes that for any $n \geq 0$, $w_n$ only depends on $a_i$ with $0 \leq i \leq n$, so that we also have truncated ghost maps $w : A^{n+1} \to (A/[A,A])^{n+1}$, where the terms in the product correspond to the integers $0, \ldots, n \in \mathbb{N}$. If we denote by $R : A^{n+1} \to A^n$ the restriction map obtained by projecting to the first $n$ coordinates, then $w \circ R = R \circ w$. 

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With these data, Hesselholt constructs a series of abelian groups $W_n(A)$, $n \geq 1$, and surjective set-theoretic maps $q : A^n \to W_n(A)$ such that

(I) the ghost map $w : A^n \to (A/[A,A])^n$ factors as

$$A^n \xrightarrow{q} W_n(A) \xrightarrow{\overline{w}} (A/[A,A])^n,$$

where $\overline{w}$ is a map of abelian groups,

(II) $W_n(A)$ and $\overline{w}$ are functorial in $A$,

(III) if $A/[A,A]$ has no $p$-torsion, then $\overline{w}$ is injective, and

(IV) we have functorial surjective restriction map $R : W_{n+1}(A) \to W_n(A)$ such that $R \circ q = q \circ R$ and $\overline{w} \circ R = R \circ \overline{w}$.

The group $W(A)$ is then the inverse limit of $W_n(A)$ with respect to $R$.

The construction of the groups $W_n(A)$ given in [H2] was not quite correct, so let us present a corrected construction given in [H3]. The construction is inductive. For $n = 1$, we just take $W_1(A) = A/[A,A]$, $\overline{w} = \text{id}$, and we observe that all the properties are tautologically satisfied. We then fix $n \geq 1$ and assume given groups $W_i(A)$ as above for all $i \leq n$. The construction proceeds in two steps. In the first step, consider the set

$$\widehat{W}_{n+1}(A) = A \times W_n(A)$$

and the map

$$\tilde{q} = \text{id} \times q : A^{n+1} = A \times A^n \to \widehat{W}_{n+1}(A),$$

where the first component in $A \times A^n$ corresponds to the 0-th component in $A^{n+1}$. Denote by $\tilde{R} : \widehat{W}_{n+1}(A) \to A$ the projection onto this first component, so that we have $\tilde{R} \circ \tilde{q} = R^n$. Note that by (5.1) and the condition (I), the ghost map $w : A^{n+1} \to (A/[A,A])^{n+1}$ factors as $w = \tilde{w} \circ \tilde{q}$, with $\tilde{w} : \widehat{W}_{n+1}(A) \to (A/[A,A])^{n+1}$ given by

$$\tilde{w}(a \times b) = w(a \times 0) + (0 \times p\overline{w}(b)), \quad a \in A, b \in W_n(A).$$

**Lemma 5.1.** There exist universal polynomials $c_i(s_0, s_1)$, $i \geq 1$ of degrees $p^i$ in two non-commuting variables $s_0$, $s_1$ such that for any $n \geq 1$ and elements $a_0, a_1 \in A$ in an associative ring $A$, we have

$$\sum_{i=1}^{n} p^i c_i(a_0, a_1)p^{n-i} \equiv 0 \mod [A, A].$$
Proof. It obviously suffices to consider the universal case \( A = T^*(M) \), the tensor algebra generated by the free abelian group \( M = \mathbb{Z}[\{s_0, s_1\}] \). Then \([K6\] Lemma 3.15 (i)] provides elements \( c_i \in M^\otimes p^i \) such that for any \( n \geq 1 \), we have

\[
(s_0 + s_1)^\otimes p^n = s_0^\otimes p^n + s_1^\otimes p^n + \sum_{i=1}^{p^i-1} \sum_{j=0}^{\sigma^j(c_i^\otimes p^{n-i})} \in M^\otimes p^n,
\]

where for any \( l \geq 0 \), \( \sigma : M^\otimes l \to M^\otimes l \) is the cyclic permutation. To deduce the claim, it remains to recall that for any \( m \in M^\otimes l \subset A \), the difference \( m - \sigma(m) \) lies in the subspace \([A, A] \subset A\).

Alternatively, one can use the polynomials \( \delta_i \) of \([H2\]; these work by \([H2, Proposition 1.2.3]\). \( \square \)

**Lemma 5.2.** There exists a unique functorial abelian group structure on \( \tilde{W}_{n+1}(A) \) such that

(i) \( \tilde{R} : \tilde{W}_{n+1}(A) \to A \) is a group map, and

(ii) \( \tilde{w} : \tilde{W}_{n+1}(A) \to (A/[A, A])^{n+1} \) is also a group map.

Moreover, in this group structure, we have

\[
(a \times b) + (a' \times b') = (a + a') \times (b + b' - q(c_1(a, a') \times \cdots \times c_n(a, a'))) \quad (5.4)
\]

for any \( a, a' \in A, b, b' \in W_n(A) \), where \( c_\bullet(-,-) \) are the polynomials provided by Lemma \([5.4]\).

Proof. In principle, existence follows from \([H3\]. Namely, Hesselholt constructs a binary operation on \( A^{n+1} \) given by explicit universal non-commutative polynomials \( s_i(a_0, \ldots, a_n, a'_0, \ldots, a'_n), \) \( 0 \leq i \leq n \), and then shows that it descends to an abelian group structure on \( \tilde{W}_{n+1}A \). He then checks that \( w : A^{n+1} \to (A/[A, A])^{n+1} \) factors as \( w = \tilde{w} \circ \tilde{q} \) for some group map \( \tilde{w} : \tilde{W}_{n+1}(A) \to (A/[A, A])^{n+1} \). Since \( \tilde{q} \) is surjective, the factorization is unique, so that Hesselholt’s \( \tilde{w} \) must coincide with the map \([5.2]\). This gives (ii). Moreover, it is immediately clear from his construction that \( s_i \) does not depend on \( n \) and only depends on \( a_0, \ldots, a_i, a'_0, \ldots, a'_i \), and in particular, \( s_0 = a_0 + a'_0 \). This is (i).

Alternatively, it is easy to see existence directly. Namely, one can simply define the group structure on \( \tilde{W}_{n+1}(A) \) by \([5.4]\). Then (i) is given, and (ii)
immediately follows from (5.1), (5.2) and (5.3). The only thing to check is that we indeed have an abelian group structure — that is, that the map

\[(5.5)\quad A \times A \to W_n(A), \quad a \times a' \mapsto q(c_1(a, a') \times \cdots \times c_n(a, a'))\]

is a symmetric 2-cocycle of the group $A$ with values in $W_n(A)$. For every ring $A$, we can find a ring $A'$ and a surjective map $A' \to A$ such that $A'/[A', A']$ has no $p$-torsion, and since the map (5.5) is functorial, it suffices to check the symmetric cocycle condition after replacing $A$ with $A'$. But then the ghost map $\overline{w} : W_n(A) \to (A/[A, A])^n$ is injective by (III), and by (5.2), $\tilde{w}$ is also injective on $a \times W_n(A) \subset \tilde{W}_{n+1}(A)$ for any $a \in A$. Therefore it suffices to check the symmetric cocycle condition after applying the map $\tilde{w}$, and then it immediately follows from (ii).

Uniqueness follows from (5.4), and to prove (5.4), note that by (i), the group law in $\tilde{W}_{n+1}(A)$ is in any case given by

\[(a \times b) + (a' \times b') = (a + a') \times (b + b' - F(a, a', b, b')),
\]

where $F$ is a certain functorial map

\[F : A \times A \times W_n(A) \times W_n(A) \to W_n(A).\]

We need to prove that $F(a, a', b, b') = q(c_1(a, a') \times \cdots \times c_n(a, a'))$. As before, by functoriality, it suffices to prove this claim for rings $A$ such that $A/[A, A]$ has no $p$-torsion, and in this case, $\tilde{w}$ is injective on $W_n(A) \times a \subset \tilde{W}_{n+1}(A)$ for any $a \in A$. Thus it suffices to prove the claim after applying the map $\tilde{w}$, and then it immediately follows from Lemma 5.1.

**Remark 5.3.** While we need Lemma 5.2 for comparison purposes, the reader may notice that it can also be used as an alternative for Hesselholt’s construction of the group structure on $\tilde{W}_{n+1}(A)$ and of the map $\tilde{w}$. It is essentially the same argument, but it needs a smaller number of formulas.

In the second step of the construction, Hesselholt considers the free abelian group $\mathbb{Z}[A \times A]$ spanned by the set $A \times A$, and the map

\[(5.6)\quad \tilde{d} : \mathbb{Z}[A \times A] \to \tilde{W}_{n+1}(A), \quad \tilde{d}(a \times a') = (aa' - a'a) \times 0.
\]

One then sets $W_{n+1}(A) = \text{Coker} \tilde{d}$. By (5.2), the composition $\overline{w} \circ \tilde{d}$ vanishes, so that $\overline{w}$ factors through a functorial map

\[\overline{w} : A^{n+1} \to W_n(A).
\]
This gives (I), and (II), (IV) are also obvious from the construction. To finish the inductive step, one has to check (III). There seems to be no direct argument for this statement, so Hesselholt proves it by topological methods (and also proves a comparison theorem between $W_n(A)$ and a certain functorial group $TR^0_n(A;p)$ that comes from algebraic topology).

One additional thing that comes out of Hesselholt’s construction is a functorial map $V: W_n(A) \to W_{n+1}(A)$ induced by the embedding $0 \times W_n(A) \subset A \times W_n(A) = \tilde{W}_{n+1}(A)$. By definition, it fits into a functorial exact sequence

\[ W_n(A) \xrightarrow{V} W_{n+1}(A) \xrightarrow{R^\alpha} A \xrightarrow{} 0. \]

In general, the sequence need not be exact on the left (and an example when it is not is contained in [13]). Hesselholt further observes that the map (5.6) factors through the quotient $A \otimes A$ of the group $\mathbb{Z}[A \times A]$, and that it then induces a functorial map $\partial: HH_1(A) \to W_n(A)$ that extends the exact sequence (5.7) one step to the left.

5.2 Comparison. Assume now given an associative unital algebra $A$ over a perfect field $k$ of characteristic $p = \text{char } k > 0$, and consider the Hochschild-Witt homology groups $W_nHH_0(A)$, $n \geq 1$. By definition, we have

\[ W_nHH_0(A) = H_0(\Delta^o, j^{oo}W_nA^2) = \text{colim}_{\Delta^o} j^{oo}W_nA^2, \]

so that by adjunction, we have a natural augmentation map $\tau: j^{oo}W_nA^2 \to W_nHH_0(A)$, where $W_nHH_0(A)$ is understood as a constant simplicial group. We also have the Teichmüller map $T: A^3 \to W_nA^3$ of [13]. Composing the two maps and evaluating at $[1] \in \Delta^o$, we obtain a natural map of sets

\[ \bar{q} = (\tau \circ T)([1]) : A \to W_nHH_0(A). \]

In addition, we have the restriction maps $R: W_nHH_0(A) \to W_{n-1}HH_0(A)$ and the Verschiebung maps $V: W_{n-1}HH_0(A) \to W_nHH_0(A)$ of (4.9). Combining the maps $V$ with (5.8), we obtain a functorial map of sets

\[ q: A^n \to W_nHH_0(A), \quad q(a_1 \times \cdots \times a_n) = \sum_{i=1}^n V^{i-1}(\bar{q}(a_i)). \]

To avoid confusion with the polynomial Witt vectors $W_n$ of Subsection 3.4, let us from now denote Witt vectors of Subsection 5.1 by $W^H_n(A)$ (where $H$ stands for “Hesselholt”). The comparison theorem that we want to prove is the following one.
Theorem 5.4. There exists functorial isomorphisms

\[ \iota : W_n HH_0(A) \cong W_n^H(A), \quad n \geq 1 \]

such that \( R \circ \iota = \iota \circ R, \) \( V \circ \iota = \iota \circ V, \) and \( \iota \circ q = q. \)

In order to prove this, we need to find Hochschild-Witt counterparts of all the steps in the inductive construction presented in Subsection 5.1. For \( n = 1, \) Theorem 5.4 is clear: we have

\[ W_1^1 HH_0(A) = HH_0(A) = A/\langle A, A \rangle, \]

and this coincides with \( W_1^H(A) \) by definition. Take an integer \( n \geq 2. \) Denote

\[ W_1^n A := \pi_p i^*_p W_{n-1} A \]

and

\[ W_1^n CH_q(A) = CH_q(W_1^n A), \quad W_1^n HH_q(A) = HH_q(W_1^n A). \]

Then by Proposition 4.3, we have a natural short exact sequence

\[
\begin{array}{cccccc}
0 & \to & W_1^n A & \to & W_{n} A & \to & A \to 0
\end{array}
\]

in \( \text{Fun}(\Lambda, W_n(k)) \) that induces an exact sequence of Hochschild homology complexes. By Lemma 1.9, we also have

\[ W_1^n HH_0(A) = \text{colim} \Delta_0 j^{\omega p} \pi_p i^*_p W_{n-1} A \cong \text{colim} \Delta_0 \times \text{pt} p_j \pi_p i^*_p W_{n-1} A \cong \text{colim} \text{pt} p_j \pi_p i^*_p W_{n-1} HH_0(A) = W_{n-1} HH_0(A) \mathbb{Z}/p \mathbb{Z}, \]

and since the \( \mathbb{Z}/p \mathbb{Z} \)-action on \( W_{n-1} HH_0(A) \) is trivial by Corollary 1.10, this group is identified with \( W_{n-1} HH_0(A) \). Let \( \tau : j^{\omega p} W_1^n A \to W_{n-1} HH_0(A) \) be the augmentation map, and let \( W_n A = j^{\omega p} W_n A \mathbb{V}(\text{Ker} \tau). \) Then (5.10) induces a short exact sequence

\[
\begin{array}{cccccc}
0 & \to & W_{n-1} HH_0(A) & \to & W_n A & \to & j^{\omega p} A \to 0
\end{array}
\]

of simplicial \( W_n(k) \)-modules, where \( W_{n-1} HH_0(A) \) is understood as a constant simplicial module. Thus if we denote by

\[ W_n CH_0(A) = C_0(W_n A) \]

the standard complex of the simplicial \( W_n(k) \)-module \( W_n A \), then in degree 0, we have a natural short exact sequence

\[
\begin{array}{cccccc}
0 & \to & W_{n-1} HH_0(A) & \to & W_n CH_0(A) & \to & \mathbb{V}_n CH_0(A) \to 0,
\end{array}
\]

where \( A = CH_0(A) \) is the degree-0 term of the Hochschild complex \( CH_0(A). \) The Teichmüller map \( T \) provides a set-theoretic splitting \( T : j^{\omega p} A \to W_n A \) of the sequence (5.11) and an isomorphism of sets

\[ W_n CH_0(A) \cong A \times W_{n-1} HH_0(A). \]
Lemma 5.5. In terms of the isomorphism \((5.13)\), the group law in the abelian group \(W_n CH_0(A)\) is given by
\[(5.14) \quad (a \times b) + (a' \times b') = (a + a') \times (b + b' - q(c_1(a, a') \cdots c_n(a, a'))),\]
where \(q\) is the map \((5.9)\), and \(c_i\) are the polynomials of Lemma 5.1.

Proof. By the uniqueness clause of Lemma 5.2, it suffices to prove the claim for one particular choice of the polynomials \(c_i\); let us choose those provided by [K6, Lemma 3.15 (i)]. For any \(k\)-vector space \(E\), [K6, Lemma 3.7] provides a short exact sequence
\[0 \longrightarrow W^1_n(E) \longrightarrow W_n(E) \overset{R_{n-1}}{\longrightarrow} E \longrightarrow 0,\]
where \(W^1_n(E) = W_{n-1}(E \otimes \mathbb{Z}/p\mathbb{Z})\), and \(\mathbb{Z}/p\mathbb{Z}\) acts on \(W_{n-1}(E \otimes p)\) via the isomorphism \((3.17)\). The Teichmüller map \(T : E \to W_n(E)\) splits this sequence set-theoretically, so that the group law in \(W_n(E) = E \times W^1_n(E)\) is given by
\[(5.15) \quad (a \times b) + (a' \times b') = (a + a') \times (b + b' - \tilde{c}(a, a'))\]
for any \(a, a' \in E\), \(b, b' \in W^1_n(E)\), where \(\tilde{c} : E \times E \to W^1_n(E)\) is a certain 2-cocycle of the group \(E\) with coefficients in \(W^1_n(E)\). The cocycle \(\tilde{c}\) has been computed in [K6, Lemma 3.15 (ii)] — one has
\[(5.16) \quad \tilde{c}(e, e') = \sum_{m=1}^{n} V^m(T(c_m(e, e'))), \quad e, e' \in E,\]
where \(T : E \otimes p^m \to W_{n-m}(E \otimes p^m)\) is the Teichmüller splitting, the polynomials \(c_m\) are evaluated in the tensor algebra \(T^* E\), so that \(c_m(e, e')\) lies in \(E \otimes p^m\), and \(V^m : W_{n-m}(E \otimes p^m) \to W_n(E)\) is the iteration of the Verschiebung map. Now observe that for any \(m \geq 1\), we have a commutative diagram of simplicial sets
\[
\begin{array}{ccc}
\bar{i}_{p^m}^* j^{\text{op}} A^2 & \xrightarrow{T} & \bar{i}_{p^m}^* j^{\text{op}} W_{n-m} A^2 \\
\downarrow c_{p^m} & & \downarrow c_{p^m} \\
j^{\text{op}} A^2 & \xrightarrow{T} & j^{\text{op}} W_{n-m} A^2
\end{array}
\]
and a commutative diagram of simplicial \(W_n(k)\)-modules
\[
\begin{array}{ccc}
\bar{i}_{p^m}^* j^{\text{op}} W_{n-m} A^2 & \xrightarrow{V^m} & j^{\text{op}} W_n A^2 & \xrightarrow{\tau} & W_n HH_0(A) \\
\downarrow c_{p^m} & & \| & & \|
\end{array}
\]
and
\[
\begin{array}{ccc}
j^{\text{op}} W_{n-m} A^2 & \xrightarrow{\tau} & W_{n-m} HH_0(A) & \xrightarrow{V^m} & W_n HH_0(A),
\end{array}
\]

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where $T$ are the Teichmüller maps, $c_p^m$ are the egdewise subdivision maps \((1.28)\), and $\tau$ are the augmentation maps. It remains to notice that after evaluation at $[1] \in \Lambda$, the map

$$
c_p^m : A^{\otimes p^m} = \tau_p^m \circ A^2([1]) \to A = j_{\otimes} A^2([a])
$$

sends $a_1 \otimes \cdots \otimes a_{p^m}$ to the product $a_1 \cdots a_{p^m} \in A$, and compare (5.14), (5.8) and (5.9) with (5.15) and (5.16). □

Proof of Theorem 5.4. Fix an integer $n \geq 1$, and assume by induction that the isomorphisms $\iota$ are already constructed for $W_i \text{HH}_0(A)$ with $i \leq n$. Then comparing Lemma 5.5 and Lemma 5.2, we see that the isomorphism $\iota : W_n \text{HH}_0(A) \cong W_n^H(A)$ canonically extends to a functorial isomorphism

$$
\tilde{\iota} : \overline{W}_{n+1} \text{CH}_0(A) \cong \widetilde{W}_{n+1}(A)
$$

that automatically commutes with the maps $V$ and $R$. To show that $\tilde{\iota}$ then descends to an isomorphism $\iota$ between $W_{n+1} \text{HH}_0(A)$ and $W_{n+1}^H(A)$, note that by definition, the homology of the complex $\overline{W}_{n+1} \text{CH}_1(A)$ in degree 0 is isomorphic to $W_{n+1} \text{HH}_0(A)$. The differential $d : \overline{W}_{n+1} \text{CH}_1(A) \to \overline{W}_{n+1} \text{CH}_0(A)$ is given by $d = \partial_0 - \partial_1$, where for $i = 0, 1$,

$$
\partial_i : \overline{W}_{n+1} \text{CH}_1(A) = \overline{W}_{n+1} A^2([2]) \to \overline{W}_{n+1} \text{CH}_0(A) = \overline{W}_{n+1} A^2([1]),
$$

are the face maps corresponding to the two maps $[1] \to [2]$ in $\Delta$. Then the Teichmüller map provides an isomorphism

$$
T : A^{\otimes 2} = A^2([2]) \cong \overline{W}_{n+1} A^2([2]),
$$

and since $T$ is a map of simplicial sets, we have $T \circ \partial_i = \partial_i \circ T$, $i = 0, 1$. The differential $d$ is therefore given by

$$
d(a \otimes a') = \partial_0(T(a \otimes a')) - \partial_1(T(a \otimes a')) =\bigg(\partial_0(b(a \otimes a')) - \partial_1(b(a \otimes a')) = T(aa') - (aa'),
$$

for any $a \otimes a' \in A^{\otimes 2} \cong \overline{W}_{n+1} \text{CH}_1(A)$, and in terms of the identification (5.13), this coincides on the nose with the map (5.6). □

Remark 5.6. Hesselholt’s differential $\partial : \text{HH}_1(A) \to W_n^H(A)$ is also visible in our approach — this is simply the connecting differential for the long exact sequence of Hochschild homology corresponding to (5.11) (or (5.10)).

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6 Comparison II.

To finish the paper, we now show that in the setup of the Hochschild-Kostant-Rosenberg Theorem, our Hoshchild-Witt complex reduces to de Rham-Witt complex of Deligne and Illusie [I].

6.1 Recollection. We start by recalling the relevant material from [I]. We fix a finite field \( k \) of characteristic \( \text{char} k = p \), and a commutative associative \( k \)-algebra \( A \). Recall that classically, one associates to \( A \) the commutative associative ring \( W(A) \) of \( p \)-typical Witt vectors of \( A \) (see e.g. [I, Section 0.1] or any of the other standard references). The construction is functorial, and since \( A \) is a \( k \)-algebra, \( W(A) \) is a \( W(k) \)-algebra. As a set, \( W(A) \) is identified with the product \( A^\mathbb{N} \) of a countable number of copies of \( A \), so that its elements are infinite sequences \((x_0, x_1, \ldots)\) of elements of \( A \). The operations are given by certain universal polynomials in \( x_i \). One defines \( n \)-truncated Witt vectors \( W_n(A) \) by forgetting \( x_i, i \geq n \); the operations on \( W(A) \) are compatible with the truncations, so that \( W_n(A) \) are also rings. We have natural additive multiplicative restriction maps \( R : W_n(A) \to W_{n-1}(A) \). Moreover, one defines the Verschiebung map \( V : W_{n-1}(A) \to W_n(A) \) by \( V((x_0, \ldots, x_n)) = (0, x_0, \ldots, x_n) \), and the Frobenius map \( F : W_{n+1}(A) \to W_n(A) \) by \( F((x_0, \ldots, x_n)) = (x_0^p, \ldots, x_{n-1}^p) \). One has \( FV = VF = p, FR = RF, RV = VR \).

From now on, to avoid confusion with our polynomial Witt vectors, we will denote Witt vectors ring \( W_n(A) \) by \( W_n^{cl}(A) \) (where \( cl \) stands for “classical”). Hesselholt’s non-commutative construction of [H2] is a generalization of the classical construction, so that for a commutative associative ring \( A \), one has \( W_n^H(A) \cong W_n^{cl}(A) \) and \( W_n^H(A) \cong W_n^{cl}(A), n \geq 1 \). These isomorphisms are compatible with the restriction maps \( R \) and the Verschiebung maps \( V \).

Now assume that as in Theorem 3.1 \( A \) is finitely generated over \( k \), and \( X = \text{Spec} A \) is smooth over \( k \).
Definition 6.1 ([I, Definition I.1.1]). A $V$-de Rham procomplex $M^*$ on $X = \text{Spec} A$ is a collection $M^*_n$, $n \geq 1$ of commutative associative DG algebras and maps 

$$R : M_{n+1} \longrightarrow M_n, \quad V : M_n \longrightarrow M_{n+1}$$

between them such that $RV = VR$, $R$ is surjective, multiplicative and commutes with the differential, and

(i) the rings $M^0_n$ are identified with the rings $W^{cl}_n(A)$ in such a way that $R$ and $V$ are the restriction and the Verschiebung maps,

(ii) we have 

$$V(xdy) = V(x)dV(y)$$

for any $n, i, j, x \in M^i_n, y \in M^j_n$, and

(iii) we have 

$$V(y)dx = V(x^{p-1}y)dV(x)$$

for any $n, x, y \in A$.

Definition 6.2. An $FV$-de Rham procomplex $M^*$ on $X$ is a collection $M^*_n$, $n \geq 1$ of commutative associative DG algebras and maps 

$$R, F : M_{n+1} \longrightarrow M_n, \quad V : M_n \longrightarrow M_{n+1}$$

between them such that $RV = VR$, $RF = FR$, $FV = VF$, $R$ is multiplicative and commutes with the differential, and

(i) the rings $M^0_n$ are identified with the rings $W^{cl}_n(A)$ in such a way that $R$, $F$ and $V$ are the restriction, the Frobenius and the Verschiebung maps, and

(ii) we have 

$$FV = VF = p, \quad FdV = d,$$

(iii) we have 

$$F(xy) = F(x)F(y), \quad xV(y) = V(F(x)y)$$

for any $n, i, j, x \in M^i_n, y \in M^j_n$. 

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We note that Definition 6.2 (ii) immediately yields \( pF \circ d = dF, \ p \circ dV = Vd. \) We also note that every \( FV \)-de Rham procomplex such that the restriction maps \( R \) are surjective and

\[ M^*_1 = \lim_{\leftarrow} M^*_n \]

has no \( p \)-torsion is automatically a \( V \)-procomplex – for example, to check Definition 6.1 (iii), note that

\[ F(V(y)dx) = F (V(y)F(dx) = ydF(x) = py^{p-1}dx = F(V(y)p^{-1}dx)) = F(V(y)p^{-1}V(dx)), \]

so that both sides become equal after applying \( F \), hence also after multiplying by \( p = FV. \)

All \( V \)-de Rham procomplexes on \( X \) form a category in an obvious way. A major result of \([1]\) that we will need can be summarized as follows.

**Theorem 6.3 ([1]).**

(i) The category of \( V \)-de Rham procomplexes on \( X \) admits an initial object \( W_\Omega^* \).

(ii) This \( V \)-de Rham procomplex \( W_\Omega^* \) has no \( p \)-torsion and uniquely extends to an \( FV \)-de Rham procomplex.

**Proof.** (i) is \([1] \) Théorème I.1.3]. In (ii), lack of \( p \)-torsion is \([1] \) Corollaire I.3.5], and existence is \([1] \) Théorème I.2.17 and \([1] \) Proposition I.2.18]. Uniqueness immediately follows from the lack of \( p \)-torsion, since \( VF = p. \)

**Definition 6.4.** The \( FV \)-de Rham procomplex \( W_\Omega^* \) of Theorem 6.3 (ii) is the de Rham-Witt complex of the scheme \( X = \text{Spec} \ A. \)

**Remark 6.5.** In fact, \([1]\) works for any smooth finite-type scheme \( X/k \), not necessarily an affine one; we will not need this.

### 6.2 Normalization

Next, we need a convenient way to check whether a given \( FV \)-de Rham procomplex is the de Rham-Witt complex or not.

Recall that the de Rham complex \( \Omega_A^* \) has the following universal property: for any commutative DG algebra \( M^* \) over \( k \), any algebra map \( A \to M^0 \) extends uniquely to a DG algebra map \( \Omega_A^* \to M^* \). In particular, for any \( FV \)-de Rham procomplex \( M^*_1 \), the isomorphism \( M^0_1 \cong W_1(A) = A \) induces a map

\[ (6.1) \quad \Omega_A^* \to M^*_1. \]
As in Subsection 3.2, let $B\Omega_A^* \subset \Omega_A^*$ be the subcomplexes of exact resp. closed forms. Recall that we have the Cartier isomorphism $C$ of (3.7); composing $C$ and its inverse with the natural embedding and natural projection, we obtain canonical maps

\[
\tilde{C} : Z\Omega_A^* \to \Omega_A^*, \quad \tilde{C}' : \Omega_A \to \Omega_A^*/B\Omega_A^*
\]
where from now on, we will ignore $k$-vector space structures and Frobenius twists. By induction, denote $B_1\Omega_A^* = B\Omega_A^*$, $Z_1\Omega_A^* = Z\Omega_A^*$ and

\[
B_n\Omega_A^* = \tilde{C}^{-1}(B_{n-1}\Omega_A^*), \quad Z_n\Omega_A^* = \tilde{C}^{-1}(Z_{n-1}\Omega_A^*) \subset Z\Omega_A^*,
\]
for any $n \geq 2$. Assume given an $FV$-de Rham procomplex $M_i^*$ on $X$, and define $\overline{M}_i^*$ by

\[
\overline{M}_n^* = \text{Ker} R \subset M_n^*
\]
for any $n \geq 1$. Note that $\overline{M}_1^* = M_1^*$, and since $V$ commutes with $R$, the natural map $V^n$ induces a map

\[
\overline{V}^n : M_1^* \to \overline{M}_n^*
\]
for any $n \geq 1$.

**Lemma 6.6.** Assume given an $FV$-de Rham procomplex $M_i^*$ such that the natural map (6.1) is an isomorphisms, and for every $i$ and $n' < n$ with some fixed $n$, we have

\[
\overline{M}_n^* = \text{Im} \overline{V}^n + d\text{Im} \overline{V}^n.
\]
Then the restriction maps $R : M_{n'+1}^* \to M_n^*$ are surjective for any $n' < n$.

**Proof.** For every $n$, any element $x \in M_n^0$ can be lifted to an element of $M_n^0 = W_n(M_n^0)$ (for example, to the Teichmüller representative $\overline{\alpha}$). Thus $R : M_n^0 \to M_1^0$ is surjective. Since $\Omega_i^X$ is spanned by forms $x_0dx_1 \wedge \cdots \wedge dx_i$, $R : M_n^i \to M_1^i$ is surjective for every $i$. To deduce that $R : M_n^i \to M_n^i$, is surjective for every $n' < n$, use induction on $n'$ and (6.6). \qed

**Definition 6.7.** An $FV$-de Rham procomplex $M_i^*$ is **normalized** if the following holds.

(i) The natural map (6.1) is an isomorphism.
(ii) We have a commutative diagram

\[
\begin{array}{ccc}
M_2^* & \xrightarrow{F} & M_1^* \\
\downarrow R & & \downarrow \pi \\
M_1^* \cong \Omega_A^* & \xrightarrow{\tilde{C}'} & \Omega_A^*/B\Omega_A^*,
\end{array}
\]

where \(\pi\) is the natural projection map, and \(\tilde{C}'\) is the map \((6.2)\).

(iii) For any \(n \geq 1\), the map

\[
\eta = \nabla^n \oplus (d \circ \nabla^n) : M_1^* \oplus M_1^{*-1} \rightarrow \overline{M}_n
\]

is surjective, and its kernel \(\text{Ker } \eta \subset \Omega_A^* \oplus \Omega_A^{*-1} = M_1^* \oplus M_1^{*-1}\) is an extension of \(Z_n\Omega_A^{*-1} \subset \Omega_A^{*-1}\) by \(B_n\Omega_A^* \subset \Omega_A^*\)—that is, we have a commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & B_n\Omega_A^* & \longrightarrow & \text{Ker } \eta & \longrightarrow & Z_n\Omega_A^{*-1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Omega_A^* & \longrightarrow & \Omega_A^* \oplus \Omega_A^{*-1} & \longrightarrow & \Omega_A^{*-1} & \longrightarrow & 0
\end{array}
\]

with exact rows.

**Proposition 6.8.** An FV-de Rham procomplex is normalized if and only if it is isomorphic to the de Rham-Witt complex \(W_\cdot \Omega_A^*\).

The difficult part of this statement is the “if” part, but fortunately, this has been proved in [1]. To prove the “only if” part, we need the following lemma.

**Lemma 6.9.** If an FV-de Rham procomplex \(M^*_i\) is normalized, then

\[
M^*_i = \lim_{\longrightarrow} M^*_n
\]

has no \(p\)-torsion for any \(i\).

**Proof.** Since we have \(p = FV\), all the groups \(\overline{M}^*_n\) are annihilated by \(p\), so that multiplication by \(p\) induces a map

\[
\overline{p}_n : \overline{M}^*_n \rightarrow \overline{M}^*_n
\]

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It suffices to prove that this map is injective for every \( n \geq 1 \). Moreover, we have \( \overline{p}_n \circ \overline{V}^n = \overline{V}^{n+1} \circ F \), so that \( \overline{p}_n \) sends \( \text{Im} V^n \) into \( \text{Im} V^{n+1} \), and it suffices to prove that the induced maps

\[
\overline{p}_n : \text{Im} V^n \to \text{Im} V^{n+1}, \quad \overline{p}_n : \text{Coker} V^n \to \text{Coker} V^{n+1}
\]

are injective. Using Definition 6.7 (iii) and (iv), we rewrite these maps as maps

\[
\Omega^i_A / B_n \Omega^i_A \to \Omega^i_A / B_{n+1} \Omega^i_A, \quad \Omega^{i-1}_A / Z_{n-1} \Omega^{i-1}_A \to \Omega^{i-1}_A / Z_n \Omega^{i-1}_A,
\]

and since \( pV^n = V^{n+1}F \) and \( pdV^n = V^{n+1}F \), by Definition 6.7 (ii), both maps are induced by the inverse Cartier map \( \overline{C}' \) of (6.2). Hence both are injective. □

**Proof of Proposition 6.8.** For the “if” part, Definition 6.7 (ii) is [I, Proposition I.3.3], and the rest is [I, Corollaire I.3.9]. Conversely, assume given a normalized \( F \)-de Rham procomplex \( M_n \) on \( X \). Then by Definition 6.7 (i), (iii), \( M_n \) satisfies the assumptions of Lemma 6.6 for any \( n \geq 1 \), and by Lemma 6.6 and Lemma 6.9 forgetting \( F \) turns it into a \( V \)-de Rham procomplex in the sense of Definition 6.1. Then by Theorem 6.3, we have a natural map

\[
\varphi : W_\cdot \Omega_A \to M_\cdot
\]

from the de Rham-Witt complex \( W_\cdot \Omega_A \). This map is compatible with the restriction maps, so that it suffices to prove that for any \( n \geq 1 \), \( \varphi \) identifies the kernel of the map

\[
R : W_{n+1} \Omega_A \to W_n \Omega_A
\]

with \( \overline{M}_n \). Since the de Rham-Witt complex is normalized, this immediately follows from Definition 6.7 (iii) and (iv). □

### 6.3 Comparison.

Now keep the assumptions and notation of the last Subsection, and consider Hochschild-Witt homology groups \( W_n HH_\cdot (A) \) of the algebra \( A \). Since \( A \) is commutative, \( W_n HH_\cdot (A) \) is a graded-commutative algebra with respect to the product (4.7), and the Connes-Tsygan differential \( B \) is a derivation for this product. Moreover, we have natural maps \( V \) and \( F \) of (4.9).

**Lemma 6.10.** The groups \( M_n^i = W_n HH_i (A) \) with the product (4.7), the differential \( d = B \), and the maps \( V \), \( F \) of (4.9) form an \( F \)-de Rham procomplex on \( X = \text{Spec} A \) in the sense of Definition 6.2.
Proof. Definition 6.2 (ii) is Lemma 4.5 and (iii) immediately follows from Lemma 4.2. It remains to check (i). We have the functorial isomorphisms $W_nHH_0(A) \cong W_n^H(A)$ provided by Theorem 5.4, and they are compatible with the maps $R$ and $V$. Combining these with the identifications $W_n^H(A) \cong W_n^{cl}(A)$, we obtain isomorphisms $\iota : W_nHH_0(A) \cong W_n^{cl}(A), \ n \geq 1$ that are also compatible with $R$ and $V$. In the limit, we get an identification $\iota : WHH_0(A) \cong W(A)$. Since this group has no $p$-torsion, $FV = VF = p$ implies that $\iota$ is also compatible with $F$, and then the same is true for its restrictions $\iota : W_nHH_0(A) \cong W_n^{cl}(A)$. It remains to show that $\iota$ is multiplicative. But every element $x \in W_nHH_0(A)$ is of the form $V(y) + T(a), \ y \in W_{n-1}HH_0(A), \ a \in A$, where $T$ is the Teichmüller map (4.3). Therefore by induction and (iii), it suffices to prove that $\iota$ is multiplicative on the image of $T$. But $T$ is multiplicative by (4.6), and $\iota(T(a)), \ a \in A$ is the Teichmüller representative $a$ of $a$. □

Lemma 6.11. The $FV$-de Rham procomplex $W_*HH_*(A)$ of Lemma 6.10 satisfies the condition (ii) of Definition 6.7

Proof. By definition, in terms of the identifications of Lemma 1.9 and (2.5), the Frobenius map $F : W_2HH_*(A) \to HH_*(A)$ is induced by the composition map

$$\mathbb{K}_*(W_2A^2) \xrightarrow{\mathbb{K}_*(F)} \mathbb{K}_*(\pi_p \gamma_p^* A^2) \xrightarrow{\varphi_p} \mathbb{K}_*(A^2),$$

where $\varphi_p$ is the map (2.6). By Lemma 2.3, this map coincides with the map $\varepsilon_p$ on homology of degree 0, and by [K3, Lemma 4.1 (iii)], we have $\varepsilon_p = C \circ R$. Therefore $\varphi_p$ takes values in the subcomplex $ZK^p(A^2) \subset K^p(A^2)$. Moreover, we have a tautological commutative diagram

$$\begin{array}{c}
ZK^p(A^2) \xrightarrow{\xi} K^p(A^2) \\
\downarrow \phi \\
ZK^p(A^2)/BK^p(A^2) \xrightarrow{\xi \circ \phi} K^p(A^2)/BK^p(A^2),
\end{array}$$

where $\xi$ is the natural projection, so that to prove the claim, it suffices to prove that we have a commutative diagram

$$\begin{array}{c}
\mathbb{K}_*(W_2A^2) \xrightarrow{\mathbb{K}_*(F)} \mathbb{K}_*(\pi_p \gamma_p^* A^2) \\
\downarrow \mathbb{K}_*(R) \downarrow \xi \circ \phi \\
\mathbb{K}_*(A^2) \xrightarrow{\sim} ZK^p(A^2)/BK^p(A^2)
\end{array}$$

(6.7)
in $D(\Lambda, W_2(k))$, where the map in the bottom is the isomorphism (3.13). As in the proof of Theorem 3.3 (i) given in Subsection 4.1, let $E = i^*_p A^\natural$, and consider the complexes $E'_q, \tilde{E}_q$. Then bottom map in (6.7) is the map (4.12) associated to the map $b : E'_q \to \tilde{E}_q$, and the rightmost map is the map (4.12) associated to the tautological embedding $\pi_p E \cong \tilde{E}_0 \subset \tilde{E}_q$. Thus to prove that the diagram is commutative, it suffices to observe that we have a commutative diagram

$$
\begin{array}{ccc}
W_2 A^\natural & \xrightarrow{\pi_p} & \pi_p E \cong \tilde{E}_0 \\
\downarrow & & \downarrow \\
E'_q & \xrightarrow{b} & \tilde{E}_q,
\end{array}
$$

where the vertical arrows are the natural embedding maps, and the map

$$
W_2 A^\natural \longrightarrow E'_q \longrightarrow A^\natural
$$

obtained by composing the embedding $E'_0 \subset E'_q$ with the quasiisomorphism (4.13) is the restriction map $R$.

To proceed further, we need to adapt (3.11) to describe the subgroups $B_n \Omega^*_A, Z_n \Omega^*_A$ of (6.3). To this effect, consider the iterates

(6.8) $C^n : A^\natural \to \pi_p^n i^*_p A^\natural, \quad R^n : \pi_p^n i^*_p A^\natural \to A^\natural.$

of the maps (3.10), and simplify notation by setting $\mathbb{K}^{(n)}(A^\natural) = \mathbb{K}^{(n)}_p(A^\natural)$ for any $n \geq 1$. Then (2.4) induces natural augmentation maps

$$
\pi_p^n i^*_p A^\natural[1] \xrightarrow{\tilde{\kappa}_1} \mathbb{K}^{(n)}(A^\natural) \xrightarrow{\tilde{\kappa}_0} \pi_p^n i^*_p A^\natural
$$

and we can define subcomplexes $B\mathbb{K}^{(n)}(A^\natural), Z\mathbb{K}^{(n)}(A^\natural) \subset \mathbb{K}^{(n)}(A^\natural)$ by

(6.9) $B\mathbb{K}^{(n)}_* (A^\natural) = \tilde{\kappa}_1(\text{Ker } R^n), \quad Z\mathbb{K}^{(n)}_* (A^\natural) = \tilde{\kappa}_0^{-1}(\text{Im } C^n).$

Since $\tilde{\kappa}_0 \circ \tilde{\kappa}_1 = 0$, we have $B\mathbb{K}^{(n)}_* (A^\natural) \subset Z\mathbb{K}^{(n)}_* (A^\natural)$. Moreover, (2.5) provides a canonical identification

(6.10) $HC_*(\mathbb{K}^{(n)}_*(A^\natural)) \cong HH_*(A),$

and if we let

$$
B_n HH_*(A) = HC_*(BK^{(n)}_*(A^\natural)), \quad Z_n HH_*(A) = HC_*(ZK^{(n)}_*(A^\natural)),$$

then we have natural maps

(6.11) $B_n HH_*(A) \to Z_n HH_*(A) \to HH_*(A).$
Lemma 6.12. For any integer $n \geq 1$, the natural maps (6.11) are injective, and if we identify $HH_*(A) \cong \Omega^*_A$ by Theorem 3.13, then $B_nHH_*(A) \cong B_n\Omega^*_A \subset \Omega^*_A$ and $Z_nHH_*(A) \cong Z_n\Omega^*_A \subset \Omega^*_A$.

Proof. Induction on $n$. For $n = 1$, (6.9) coincides with (3.11), so that we are done by Theorem 3.3 (ii). For any $n \geq 2$, Lemma 3.4 shows that $\pi_{p^n-1}i_{p^n-1}^*\Phi A^\natural \cong \pi_{p^n-1}i_{p^n-1}^*\Phi A^\natural$, and moreover, if we denote this object by $\Phi^{(n)}A^\natural$, then we have natural short exact sequences

$$0 \longrightarrow \pi_{p^n-1}i_{p^n-1}^*A^\natural \overset{C}{\longrightarrow} \pi_{p^n}i_{p^n}^*A^\natural \longrightarrow \Phi^{(n)}A^\natural \longrightarrow 0,$$

$$0 \longrightarrow \Phi^{(n)}A^\natural \longrightarrow \pi_{p^n}i_{p^n}^*A^\natural \longrightarrow \pi_{p^n-1}i_{p^n-1}^*A^\natural \longrightarrow 0.$$

Furthermore, we have $K_\nu^{(n)}(A^\natural) \cong \pi_{p^n-1}i_{p^n-1}^*K^{(1)}_\nu(A^\natural)$ by base change, and if we denote

$$\overline{B}K^{(n)}_\nu(A^\natural) = \overline{\kappa}_1(\text{Ker } R), \quad \overline{Z}K^{(n)}_\nu(A^\natural) = \overline{\kappa}_0^{-1}(\text{Im } C),$$

then we have

$$\overline{B}K^{(n)}_\nu(A^\natural) \cong \Phi^{(n)}A^\natural[-1] \cong L^* \pi_{p^n-1}i_{p^n-1}^*B K^{(1)}_\nu(A^\natural)$$

and

$$\overline{Z}K^{(n)}_\nu(A^\natural) \cong L^* \pi_{p^n-1}i_{p^n-1}^*Z K^{(1)}_\nu(A^\natural).$$

Therefore when we pass to homology, the maps

$$HC_*(\overline{B}K^{(n)}_\nu(A^\natural)) \to HC_*(\overline{Z}K^{(n)}_\nu(A^\natural)) \to HC_*(K^{(n)}_\nu(A^\natural))$$

are injective, and under (6.10), these are exactly the maps (6.11) for $n = 1$. Moreover, possibly after applying the tautological functor $\nu$, the isomorphism (3.13) induces a natural quasiisomorphism

(6.12) $$\overline{Z}K^{(n)}_\nu(A^\natural)/\overline{B}K^{(n)}_\nu(A^\natural) \cong K^{(n-1)}_\nu(A^\natural)$$

that gives the inverse Cartier map under the identification (6.10). It remains to notice that by definition, we have

$$\overline{B}K^{(n)}_\nu(A^\natural) \subset B K^{(n)}_\nu(A^\natural) \subset Z K^{(n)}_\nu(A^\natural) \subset \overline{Z}K^{(n)}_\nu(A^\natural),$$

and the quasiisomorphism (6.12) provides exact triangles

$$\overline{B}K^{(n)}_\nu(A^\natural) \longrightarrow B K^{(n)}_\nu(A^\natural) \longrightarrow B K^{(n-1)}_\nu(A^\natural) \longrightarrow$$

$$\overline{B}K^{(n)}_\nu(A^\natural) \longrightarrow Z K^{(n)}_\nu(A^\natural) \longrightarrow Z K^{(n-1)}_\nu(A^\natural) \longrightarrow$$

Comparing this with (6.3) and applying induction, we get the claim. \[\square\]
Remark 6.13. Note that (6.9) makes sense for any associative unital \( k \)-algebra \( A \), so that the maps (6.11) also exist in full generality. Moreover, for any \( m \leq n \), one can consider the subcomplexes
\[
B_m \mathbb{K}^{(n)}_*(A^2) = \kappa_1(\text{Ker} \, R^m), \quad Z_m \mathbb{K}^{(n)}_*(A^2) = \kappa_0^{-1}(\text{Im} \, C^m),
\]
and then the same argument as in the proof of Lemma 6.12 shows that we have natural identifications
\[
B_m \HH_{*}(A) \cong HC_{*}(B_m \mathbb{K}^{(n)}_*(A^2)), \quad Z_m \HH_{*}(A) \cong HC_{*}(Z_m \mathbb{K}^{(n)}_*(A^2)).
\]
Thus for any associative unital \( k \)-algebra \( A \), we have an infinite sequence of natural maps
\[
(6.13) \quad B_1 \HH_{*}(A) \to \cdots \to B_n \HH_{*}(A) \to \cdots \to Z_n \HH_{*}(A) \to \cdots \to Z_1 \HH_{*}(A) \to \HH_{*}(A),
\]
and for any \( n \), the first \( n \) terms in the sequence are represented explicitly by subcomplexes in \( \mathbb{K}^{(n)}_*(A^2) \). We also have natural long exact sequences
\[
B_n \HH_{*}(A) \to Z_n \HH_{*}(A) \to \HH_{*}(A)
\]
obtained by iterating the quasiisomorphism (3.13). We do not know under what assumptions the maps (6.13) are injective.

Theorem 6.14. Assume given a \( k \)-algebra \( A \) satisfying the assumptions of Theorem 3.1, and let \( W_q \Omega_q^i \) be the de Rham-Witt complex of the scheme \( X = \text{Spec} \, A \), as in Definition 6.4. Then the Hochschild-Kostant-Rosenberg isomorphism (3.6) extends to a series of functorial multiplicative isomorphisms
\[
W_n \HH_{*}(A) \cong W_n \Omega_q^i
\]
that commute with the maps \( V \), \( F \), and send the Connes-Tsygan differential \( B \) to the de Rham-Witt differential \( d \).

Proof. By Proposition 6.8 it suffices to prove that the \( FV \)-de Rham pro-complex \( M' = W_q \HH_{*}(A) \) of Lemma 6.10 is normalized in the sense of Definition 6.7. Definition 6.7 (i) is Theorem 4.2 and Definition 6.7 (ii) is Lemma 5.11 so what we have to check is Definition 6.7 (iii). Use induction on \( n \). The base case \( n = 0 \) is trivial. Assume the statement proved for all \( n' < n \). The first of the exact sequences of Proposition 4.3 induces a long exact sequence
\[
\HH_{*}(\pi_p^* \Theta_{p!}(A^2)) \to W_n \HH_{*}(A) \xrightarrow{R} W_{n-1} \HH_{*}(A)
\]
of homology groups. Note that Definition 6.7 (iii) implies (6.6), so that in particular, $M_n'$ satisfy the assumptions of Lemma 6.6. Then $R$ is surjective, so that the connecting differential in the long exact sequence vanishes, and we have

$$\overline{M}_n' \cong HH_*(\pi_{p^n}i_{p^n}^*A^\natural) \cong HC_*(\mathbb{K}_*(\pi_{p^n}i_{p^n}^*A^\natural)).$$

By definition, the map $\nabla^n$ is then induced by the canonical map

$$(6.14) \quad \nu_{p^n} : \mathbb{K}^{(n)}_*(A^\natural) \rightarrow \mathbb{K}_*(\pi_{p^n}i_{p^n}^*A^\natural)$$

of (2.6), where we have used the identification (6.10). By Lemma 2.3, this map is an isomorphism in homological degree 0, and equal to the canonical map $e_{p^n}$ in homological degree 1. By [K3] Lemma 4.1 (iii), we have $e_{p^n} = C^n \circ R^n$, where $C^n$ and $R^n$ are the maps (6.8). In terms of the complexes (6.9), this means that we have $\text{Ker} \nu_{p^n} \cong B\mathbb{K}^{(n)}_*(A^\natural) \subseteq \mathbb{K}^{(n)}_*(A^\natural)$, and the cokernel $\text{Coker} \nu_{p^n}$ of the map (6.14) fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}\mathbb{K}^{(n)}_*(A^\natural)[1] \longrightarrow \mathbb{K}^{(n)}_*(A^\natural)[1] \xrightarrow{B \circ \nu_{p^n}} \text{Coker} \nu_{p^n} \longrightarrow 0,$$

where $B : \mathbb{K}_*(\pi_{p^n}i_{p^n}^*A^\natural)[1] \rightarrow \mathbb{K}_*(\pi_{p^n}i_{p^n}^*A^\natural)$ is the map (1.36). We conclude that the map

$$(6.15) \quad \eta = \nu_{p^n} \oplus (B \circ \nu_{p^n}) : \mathbb{K}^{(n)}_*(A^\natural) \oplus \mathbb{K}^{(n)}_*(A^\natural)[1] \rightarrow \mathbb{K}_*(\pi_{p^n}i_{p^n}^*A^\natural)$$

is surjective, and its kernel is an extension of $\mathbb{Z}\mathbb{K}^{(n)}_*(A^\natural)[1]$ by $B\mathbb{K}^{(n)}_*(A^\natural)$ in the same sense as in Definition 6.7 (iii). Therefore we have a commutative diagram

$$
\begin{array}{ccc}
B_nHH_*(A) & \longrightarrow & HH_*(\text{Ker} \eta) \\
\downarrow & & \downarrow \\
HH_*(A) & \longrightarrow & HH_*(A) \oplus HH_{-1}(A)
\end{array}
$$

whose rows are exact in the middle term. By Theorem 3.3 (iii), the leftmost and the rightmost vertical maps in this diagram are injective. Since the top row comes from a long exact sequence, and the bottom row is exact on the left and the right, the top row is then also exact on the left and on the right. Moreover, the middle vertical map is also injective, so that the connecting differential

$$HC_*(\mathbb{K}_*(\pi_{p^n}i_{p^n}^*A^\natural)) \cong HC_*(\text{Coker} \eta) \rightarrow HC_{-1}(\text{Ker} \eta)$$

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in the long exact sequence induced by the surjective map (6.15) vanishes. Therefore the map
\[
HH_\ast(A) \oplus HH_{\ast-1}(A) \cong HC_\ast(K_\ast(n)(A^F) \oplus K_\ast(n)(A^F)[1]) \to
HC_\ast(K_\ast(\pi_{p^m n^m} A^F)) \cong M_n
\]
induced by the map (6.15) is surjective, and its kernel \(HC_\ast(Ker \eta)\) is an extension of \(B_n HH_\ast(A)\) by \(Z_n HH_{\ast-1}(A)\). This is Definition 6.7 (iii). □

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Affiliations (in the precise form required for legal reasons):

1. Steklov Mathematics Institute, Algebraic Geometry Section (main affiliation).
2. Laboratory of Algebraic Geometry, National Research University Higher School of Economics.
3. Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang, Korea.

E-mail address: kaledin@mi.ras.ru