Relaxed Polar Codes

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Abstract

Polar codes are the latest breakthrough in coding theory, as they are the first family of codes with explicit construction that provably achieve the symmetric capacity of binary-input discrete memoryless channels. Arikan’s polar encoder and successive cancellation decoder have complexities of $l \log l$, for code length $l$. Although, the complexity bound of $l \log l$ is asymptotically favorable, we report in this work methods to further reduce the encoding and decoding complexities of polar coding. The crux is to relax the polarization of certain bit-channels without performance degradation. We consider schemes for relaxing the polarization of both very good and very bad bit-channels, in the process of channel polarization. Relaxed polar codes are proved to preserve the capacity achieving property of polar codes. Analytical bounds on the asymptotic and finite-length complexity reduction attainable by relaxed polarization are derived. For binary erasure channels, we show that the computation complexity can be reduced by a factor of 6, while preserving the rate and error performance. For AWGN channels with medium code lengths, we show that relaxed polar codes can have better bit error probabilities than conventional polar codes, while have reduced encoding and decoding complexities.

I. INTRODUCTION

Polar codes, introduced by Arikan [1], [2], are the most recent breakthrough in coding theory. Polar codes are the first and, currently, the only family of codes with explicit construction (no ensemble to pick from) to asymptotically achieve the capacity of binary input symmetric discrete memoryless channels as the block length goes to infinity. Recent research have considered applying polar codes and the polarization phenomenon in various communications and signal
processing problems, such as wiretap channels [3], data compression [4], [5], multiple access channels [6]–[8], and multi-channels [9].

Polar codes can be encoded and decoded with relatively low complexity. Both the encoding complexity and the successive cancellation (SC) decoding complexity of polar codes are $O(l \log l)$, for code length $l$ [1]. The decoding latency and memory requirements of polar decoders can be reduced to $O(l)$ [10]–[12]. Hardware architectures for polar decoders, with $O(l)$ memory and processing elements, were implemented [10]. A semi-parallel architecture for SC decoding has been recently proposed [11], where efficiency is achieved without a significant throughput penalty by sharing processing resources and taking advantage of the regular structure of polar codes. Latency and algorithmic reductions for polar decoding of Arikan’s polar codes were observed by simplifying the decoder to decode all bits in a rate-one constituent code simultaneously [13], [14]. The encoding and decoding latencies of polar codes can also be reduced to $O(l)$, through multi-dimensional polar transformations [12].

In this paper, we propose methods to reduce both the encoding and decoding computational complexities of polar codes, by means of relaxing the channel polarization. The resultant codes are called relaxed polar codes. Hence, hardware implementations for the encoders and decoders of relaxed polar codes can require smaller area and less power consumptions than conventional polar codes. Efficient methods for the implementation of the SC decoder, as in [10]–[12], can also be applied to to further improve the efficiency of decoding relaxed polar codes.

In practical scenarios, codes have finite block lengths and are designed with a specific information block lengths and rate in order to satisfy a certain error rate. Due to the nature of channel polarization, the error probability of certain bit channels decrease (or increase) exponentially at each polarization step. Hence, encoding and decoding complexity may be reduced by relaxing the polarization of certain channels if their polarization degrees hit suitable thresholds, while satisfying the code rate and error rate requirements. For Arikan’s polar code with length $l$, each bit-channel is polarized $\log l$ times. However, for the proposed relaxed polar codes, some bit
channels will be fully polarized $\log l$ times, and the polarization of the remaining bit-channels will be relaxed, where their polarization is aborted if they become sufficiently good or sufficiently bad with less than $\log l$ polarization steps. Relaxed polarization results in fewer polarizing operations, and hence a reduction in complexity. It is found that with careful construction of relaxed polar codes, there is no error performance degradation. In fact, it is observed that relaxed polar codes can have a lower bit-error rate than conventional polar codes with the same rate.

The rest of this paper is organized as follows. In Section II, we give an overview of channel polarization theory and construction of conventional polar codes, which we call *fully polarized* codes. In Section III, the notion of relaxed channel polarization is introduced and the channel polarization theory is established. The asymptotic bounds on the complexity reduction using relaxed polar codes are discussed in Section IV. Then, upper bounds and lower bounds on the complexity reduction at finite block lengths are derived in Section V. These bounds are evaluated and compared with the actual complexity reductions at certain code parameters, in Section V-D. Constructions of relaxed polar codes for general channels are discussed in Section VI. At the end, we conclude the paper in Section VII.

II. ARIKAN’S FULLY POLARIZED CODES

For any binary-input discrete memoryless channel (B-DMC) $W: \mathcal{X} \to \mathcal{Y}$, let $W(y|x)$ denote the probability of receiving $y \in \mathcal{Y}$ given that $x \in \mathcal{X} = \{0, 1\}$ was sent, for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. For an B-DMC $W$, the *Bhattacharyya parameter* of $W$ is

$$Z(W) \overset{\text{def}}{=} \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)}. \quad (1)$$

The symmetric capacity of a B-DMC $W$ can be written as

$$I(W) \triangleq \sum_{y \in \mathcal{Y}} \frac{1}{2} \sum_{x \in \mathcal{X}} W(y|x) \log \frac{W(y|x)}{\frac{1}{2}W(y|0) + \frac{1}{2}W(y|1)}. \quad (2)$$

For a binary memoryless symmetric (BMS) channel with uniform input, the error probability
of \( W \) can be characterized as

\[
E(W) = \frac{1}{2} \sum_{y \in \mathcal{Y}} \min\{W(y|0), W(y|1)\}.
\]  

(3)

It is easy to show that the Bhattacharyya parameter \( Z(W) \) is always between 0 and 1. Bhattacharyya parameter can be regarded as a measure of the reliability of \( W \). Channels with \( Z(W) \) close to zero are almost noiseless, while channels with \( Z(W) \) close to one are almost pure-noise channels. More precisely, it can be proved that the probability of error of an BMS channel is upper-bounded by its Bhattacharyya parameter

\[
0 \leq 2E(W) \leq Z(W) \leq 1.
\]  

(4)

The construction of polar codes is based on a phenomenon called *channel polarization* discovered by Arikan [1]. Let

\[
P = \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
\]  

(5)

Consider the \( 2 \times 2 \) polarizing transformation \( P \) which takes two independent copies of \( W \) and performs the mapping \( (W, W) \rightarrow (W^-, W^+) \), where \( W : \{0, 1\} \rightarrow \mathcal{Y}, W^- : \{0, 1\} \rightarrow \mathcal{Y}^2 \), and \( W^+ : \{0, 1\} \rightarrow \{0, 1\} \times \mathcal{Y}^2 \), then polarization is defined with the channel transformation

\[
W^-(y_1, y_2|x_1) = \frac{1}{2} \sum_{x_2 \in \{0, 1\}} W(y_1|x_1 \oplus x_2)W(y_2|x_2),
\]  

(6)

\[
W^+(y_1, y_2, x_1|x_2) = \frac{1}{2} W(y_1|x_1 \oplus x_2)W(y_2|x_2),
\]  

(7)

where \( W^- \) and \( W^+ \) are degraded and upgraded channels respectively. Hence, the following is
true for the bit-channel rates [1],

\[ I(W^+) + I(W^-) = 2I(W) \]  \hspace{1cm} (8)

\[ I(W^-) \leq I(W) \leq I(W^+) \]  \hspace{1cm} (9)

The mapping of \((W, W) \rightarrow (W^-, W^+)\) is called one level of polarization. The same mapping is applied to \(W^-\) and \(W^+\) to get \(W^-, W^-, W^-, W^+\), which is the second level of polarization of \(W\). The same process can be continued in order to polarize \(W\) for any arbitrary number of levels. The polarization process can be also described using a binary tree, where the root of the tree is associated with the channel \(W\). Each node in the binary tree is associated with some bit-channel \(W'\) and has two children, where the left child corresponds to \(W'^-\) and the right child corresponds to \(W'^+\).

The channel polarization process can be also represented using the Kronecker powers of \(P\) defined as follows. \(P^\otimes 1 = P\) and for any \(i > 1\),

\[ P^\otimes i = \begin{bmatrix} P^\otimes(i-1) & 0 \\ 0 & P^\otimes(i-1) \end{bmatrix}, \]

where \(P^\otimes i\) is a \(2^i \times 2^i\) matrix. Let \(n = \log_2 l\). Then the \(l \times l\) polarization transform matrix is defined as \(P_l \triangleq R_lP^\otimes n\), where \(R_l\) is the bit-reversal permutation matrix [1]. Let \(u'_1\) denote the vector \((u_1, u_2, \ldots, u_l)\) of \(l\) independent and uniform binary random variables. Let \(x'_1 = u'_1P_l\) be transmitted through \(l\) independent copies of a binary-input discrete memoryless channel (B-DMC) \(W\) to form channel output \(y'_1\). Let \(W^l: \mathcal{X}^l \rightarrow \mathcal{Y}^l\) denote the channel that results from \(l\) independent copies of \(W\) in the polar transformation i.e. \(W^l(y'_1|x'_1) \triangleq \prod_{i=1}^l W(y_i|x_i)\) The combined channel \(W_l\) is defined with transition probabilities given by

\[ W_l(y'_1|u'_1) \triangleq W^l(y'_1|u'_1P_l) = W^l(y'_1|u'_1R_lP^\otimes n). \]  \hspace{1cm} (10)
This is the channel that the random vector \( u_1^l \) observes through the polar transformation.

Assuming uniform channel input and a genie-aided successive cancellation decoder, the bit-channel \( W^{(i)}_l \) is defined with the following transition probability:

\[
W^{(i)}_l \left( y^i_1, u_1^{i-1} | u_i \right) \triangleq \frac{1}{2^{l-1}} \sum_{u_{i+1}^{l-1} \in \{0,1\}^{l-1}} W_i \left( y^i_1 | u^i_1 \right).
\] (11)

Notice that \( W^{(i)}_l \) gives the transition probabilities of \( u_i \) assuming all the preceding bits \( u_1^{i-1} \) are already decoded and are available, together with the \( l \) observations at the channel output \( y^i_1 \).

This is actually the channel that \( u_i \) observes and is also referred to as \( i \)-th bit-channel. It can be observed that \( W^{(i)}_l \) corresponds to the \( i \)-node in the \( n \)-th level of polarization of \( W \). The following recursive formulas hold for Bhattacharyya parameters of individual bit-channels in the polar transformation [1]

\[
Z(W^{(2l-1)}_l) \leq 2Z(W^{(i)}_l) - Z(W^{(i)}_l)^2
\] (12)

\[
Z(W^{(2l)}_l) = Z(W^{(i)}_l)^2
\] (13)

with equality iff \( W \) is a binary erasure channel.

The channel polarization theorem states that as the code length \( l \) goes to infinity, the bit-channels become polarized, meaning that they either become noise-free or very noisy. Define the set of good bit-channels according to the channel \( W \) and a positive constant \( \beta < 1/2 \) as

\[
G_l(W, \beta) \triangleq \left\{ i \in [l] : Z(W^{(i)}_l) < 2^{-i\beta/l} \right\},
\] (14)

where \([l] \triangleq \{1, 2, \ldots, l\}\), then the main channel polarization theorem follows [1], [2]:

**Theorem 1:** For any BMS channel \( W \), with capacity \( C(W) \), and any constant \( \beta < 1/2 \),

\[
\lim_{l \to \infty} \frac{|G_l(W, \beta)|}{l} = C(W).
\]
Theorem 1 readily leads to a construction of capacity-achieving polar codes. The crux of polar codes is to carry the information bits on the upgraded noise-free channels and freeze the degraded noisy channels to a predetermined value, e.g. zero. The following theorem shows the error exponent under successive cancellation decoding [1]:

**Theorem 2:** Let $W$ be a BMS channel and let $k = |G_l(W, \beta)|$ be the cardinality of the information bits, which are encoded using a polar code of length $l$, and transmitted over $W^i$, then the probability of decoder error under successive cancellation decoding satisfies $P_e \leq \sum_{i \in G_l(W, \beta)} Z(W_l(i)) \leq 2^{-l\beta}$.

Similar to the set of good bit-channels, the set of bad bit-channels is defined according to the channel $W$ and a positive constant $\beta < \frac{1}{2}$ as

$$\mathcal{B}_l(W, \beta) \overset{\text{def}}{=} \left\{ i \in [l] : Z(W_l(i)) > 1 - 2^{-l\beta} \right\}, \quad (15)$$

The following corollary can be derived by specializing the Theorem 3 of [15]:

**Corollary 3:** Let $W$ be an arbitrary BSM channel. Then, for any positive constant $\beta < \frac{1}{2}$,

$$\lim_{n \to \infty} \frac{|\mathcal{B}_l(W, \beta)|}{l} = 1 - C(W). \quad (16)$$

**III. RELAXED POLARIZATION THEORY**

In this section, we define relaxed polarization. We prove that, similar to conventional polar codes, relaxed polar codes can asymptotically achieve the capacity of a binary memoryless symmetric channel. We also prove that the bit-channel error probability of relaxed polar codes is at most that of conventional polar codes without rate-loss.

Let us observe the definition of good channels in Theorem 1. Let us also observe that the Bhattacharyya parameters (BP) approach 0 or 1 exponentially with the block length $l$. Let $\widetilde{W}$ denote the bit-channels of the relaxed polar code. Consider two independent copies of a parent bit-channel $i$ at polarization level $t$ to be polarized into two bit-channel children at level $t + 1$, July, 2014 DRAFT
corresponding to a code of length $2^{t+1}$, via the following channel transformation

$$
\left(\tilde{W}_{2^t}^{(i)}, \tilde{W}_{2^t}^{(j)}\right) \rightarrow \left(\tilde{W}_{2^{t+1}}^{(2i-1)}, \tilde{W}_{2^{t+1}}^{(2j)}\right).
$$

(17)

Consider the case, when the polarized channel $\tilde{W}_{2^t}^{(i)}$ at level $t < n$, where $n = \log l$, is sufficiently good, such that it satisfies the definition of a good channel at the target length $2^l$, i.e. $Z(\tilde{W}_{2^t}^{(i)}) < 2^{-l^\beta}/l$. Then, the idea of relaxed polarization is to stop further polarization of this good channel, and the corresponding node in the polarization tree is called a relaxed node, such that the channels of all the descendents of a relaxed node are the same as that of the relaxed parent node and will also be relaxed. Then, the bit-channel transformation at the relaxed node is given by

$$
\tilde{W}_{2^{t+1}}^{(2i-1)}(y_{2^t}, u_{2i-1}^{2i-2}|u_{2i-1}) = \tilde{W}_{2^t}^{(i)}(y_{2^t}, u_{2i-2}^{2i-2}|u_{2i-1})
$$

$$
\tilde{W}_{2^{t+1}}^{(2i)}(y_{2^t}, u_{2i-1}^{2i-1}|u_{2i}) = \tilde{W}_{2^t}^{(i)}(y_{2^t}, u_{2i}^{2i}|u_{2i}).
$$

Relaxing the further polarization of sufficiently good channels is called good-channel relaxed polarization. For the good-channel relaxed polar code, define the set of good bit-channels according to the channel $W$ and a positive constant $\beta < 1/2$ as

$$
\tilde{G}_l(W, \beta) \overset{\text{def}}{=} \{ i \in [l] : Z(\tilde{W}_i^{(i)}) < 2^{-l^\beta}/l \}.
$$

(18)

Next, we show that relaxed polar codes, similar to fully polarized codes, asymptotically achieve the capacity of BMS channels.

**Theorem 4:** For any BMS channel $W$, with capacity $C(W)$, and any constant $\beta < 1/2$,

$$
\lim_{l \to \infty} \frac{|\tilde{G}_l(W, \beta)|}{l} = C(W).
$$

**Proof:** Consider a relaxed channel at level $t < n$, where $n = \log l$. Then $Z(\tilde{W}_{2^t}^{(i)}) < 2^{-l^\beta}/l$. 

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Then the BPs of all its $2^{n-t}$ descendents at level $n$ are equal to $Z(\tilde{W}_{2t}^{(i)})$ and are in $\tilde{G}_t(\tilde{W}, \beta)$. In case of full polarization, if a channel belongs to $G(W, \beta)$, then it must have polarized to a good channel at level $n$ or earlier. If it polarized at level $n$, then by definition it also belongs to $\tilde{G}_t(\tilde{W}, \beta)$. Otherwise, its parent has polarized to a good channel at level $t < n$. With relaxed polarization, this channel and all its $2^{n-t} - 1$ siblings will also be in $\tilde{G}_t(\tilde{W}, \beta)$. Therefore, $G(W, \beta) \subset \tilde{G}_t(\tilde{W}, \beta)$ and hence $|\tilde{G}_t(\tilde{W}, \beta)| \geq |G(W, \beta)|$. Then the proof follows from Theorem 1.

The upper bound $2^{-l^\beta}$ on the probability of error as in Theorem 2 is still valid for the relaxed polar code constructed with respect to $\tilde{G}_t(\tilde{W}, \beta)$. Hence, Theorem 4 shows that it is possible to construct good-channel relaxed polar codes, which are still capacity achieving.

The remaining question is to actually compare the bit-error rate of relaxed polar code with that of Arikan’s polar code. Consider the special case when $\tilde{G}_t(\tilde{W}, \beta) = G_t(W, \beta)$. Consider the channel $W_{t/2}^{(i)}$, then we have the following inequalities (the proof is provided in the Appendix.)

$$E \left( W_{t/2}^{(2i-1)} \right) = 2E \left( W_{t/2}^{(i)} \right) - 2E \left( W_{t/2}^{(i)} \right)^2$$

$$E \left( W_{t/2}^{(2i)} \right) \geq 2E \left( W_{t/2}^{(i)} \right)^2$$

Hence, the next theorem follows:

**Theorem 5:** Consider a good-channel relaxed polar code with good-channel set $\tilde{G}_t(\tilde{W}, \beta)$, and let $G_t(W, \beta)$ be the good channel set of the fully polarized polar code. If $\tilde{G}_t(\tilde{W}, \beta) = G_t(W, \beta)$, then the bit error probability of the relaxed polar code is at most equal to that of the fully polarized code, i.e.

$$\sum_{i \in \tilde{G}_t(\tilde{W}, \beta)} E(\tilde{W}_t^{(i)}) \leq \sum_{i \in G_t(W, \beta)} E(W_t^{(i)})$$

**Proof:** It suffices to consider the last level of channel polarization e.g. channel $W_{t/2}^{(i)}$ and its children, $W_{t/2}^{(2i-1)}$ and $W_{t/2}^{(2i)}$, assuming that $W_{t/2}^{(i)}$ is a relaxed node. Hence, both indices $2i - 1$
and 2i are contained in \( \tilde{G}_l(W, \beta) \) and \( G_l(W, \beta) \). For the relaxed code, it follows that sum error probability of these two channels is

\[
E\left(\tilde{W}_t^{(2i-1)}\right) + E\left(\tilde{W}_t^{(2i)}\right) = 2E\left(\tilde{W}_{t/2}^{(i)}\right) = 2E\left(W_{t/2}^{(i)}\right).
\]

Together with summing (19) and (20), it follows that

\[
2E\left(\tilde{W}_{t/2}^{(i)}\right) \leq E\left(W_{t}^{(2i-1)}\right) + E\left(W_{t}^{(2i)}\right).
\]

Hence, the sum error probability of both channels in the relaxed code is at most their sum in the fully polarized code. Since the remaining good bit-channels are the same in both codes, the proof follows.

Remark 1: The concept of good-channel relaxed polarization, discussed so far, can be extended to bad-channel relaxed polarization as follows. Consider the bit-channels in the polarization tree, at a level \( t < n \), that are very bad. A bit-channel can be considered very bad if its Bhattacharyya parameter is very close to 1, or if none of its descendents will fall into the set of good bit-channels at the last level of polarization \( n \). Hence, the polarization of very bad bit-channels can be stopped without affecting the final set of good bit-channels. Thus, with careful bad-channel relaxed polarization, more complexity reductions can be possible, without degrading the code rate and error performance.

Remark 2: The obvious advantage of relaxed polarization is savings in both encoding and decoding complexities, since there will be no channel transformations done at relaxed nodes while encoding, and there will be no need to calculate new likelihood ratios (LRs) at relaxed nodes while decoding. Hence, relaxed polarization will result in a reduction in the encoding and decoding computational complexities of polar codes. Another advantage of relaxed polarization is the reduction in space (area and memory) complexity in practical implementations. That is because decoding of relaxed polar codes will result in smaller LRs (or log-likelihood ratios (LLRs) in case the computation is done in the log domain [10]) than that for fully polarized polar codes, and hence smaller bit-width will be required for LR calculation and storage. Relaxed polarization has the effect of reducing the number of processing nodes required at lower levels of the polarization tree, and hence one can expect even more efficient implementations (or less throughput penalty) with semi-parallel hardware architectures [11]. Also, by appropriate
permutation of the bit channels, one expects to be able to eliminate the wiring for the wider butterflies in FFT-like SC decoders for relaxed polar codes. The reduction in bit-width and number of processing nodes required for relaxed polar code decoders has the compound effect of reduction in power consumption.

The reduction in encoding and decoding complexity will be addressed in the following section.

IV. ASYMPTOTIC ANALYSIS OF COMPLEXITY REDUCTION

In this section, we establish bounds on the asymptotic complexity reduction (as the code’s block length goes to infinity) in polar code encoders and decoders, made possible by relaxed polarization.

First, we elaborate the notion of complexity reduction. For Arıkan’s polar codes, the total number of channel polarization operations required using Arıkan’s butterfly polarization structure, is $A(n) = nl$, where $l = 2^n$ is the length of the code. As a result, the encoding procedure consists of $nl$ binary XOR operations and decoding procedure consists of $nl$ LLR combinations. Therefore, each skipped polarization operation is equivalent to one unit of complexity reduction in both encoding and decoding, where the unit corresponds to a binary XOR when encoding and an LLR combining operation when decoding. The complexity reduction $R(n)$ is defined to be the ratio of the number of polarization operations that are skipped due to relaxed polarization to the total number of polarization operations ($A(n)$) required for full polarization. The complexity reduction (CR) can be directly translated into encoding and decoding complexity ratios of $(1 - R(n))^{-1}$.

For the asymptotic analysis throughout this section, a family of capacity-achieving polar codes is assumed which is constructed with respect to a fixed parameter $\beta < \frac{1}{2}$ and the set of good bit-channels $\mathcal{G}_l(W, \beta)$, for any block length $l = 2^n$.

**Theorem 6:** Let $C(W)$ be the capacity of the channel $W$. Then, for any $\epsilon > 0$, small enough $\delta > 0$, and large enough $l$, the complexity reduction ratio using the relaxed polar code, constructed
with $\tilde{G}_l(W, \beta)$, is at least
\[(C - \epsilon)(1 - (2 + \delta)\beta)\]

**Proof:** Pick a fixed $\delta$ such that $0 < \delta < 1/\beta - 2$. Consider the polarization level $\lceil (2 + \delta)\beta n \rceil$. Let $l' = 2^{\lceil (2+\delta)\beta n \rceil}$ be the total number of nodes at this level. Then for large enough $n$, the nodes at this level with index belonging to the set $G_{l'}(W, 1/(2 + \delta))$ will be relaxed. Notice that the fraction of these nodes, i.e. $|G_{l'}(W, 1/(2 + \delta))|/l'$, approaches the capacity $C$ by Theorem 1. The fraction of bit-channel polarizations between the level $\lceil (2 + \delta)\beta n \rceil$ and the last level $n$ is $(1 - (2 + \delta)\beta)$ of the total $nl'$, among which a fraction of $C - \epsilon$ of them are relaxed, for large enough $n$. Therefore, the complexity reduction will be at least $(C - \epsilon)(1 - (2 + \delta)\beta)$.

In the next theorem, a bound on the asymptotic complexity reduction using the bad-channel relaxed polarization is provided. The following scenario is considered for bad-channel relaxed polarization: if none of the descendants of a certain node will belong to $G_l(W, \beta)$, then the polarization at that node, and consequently all of its descendants, will be relaxed.

**Theorem 7:** For any $\epsilon, \delta > 0$ and large enough $l = 2^n$, the complexity reduction ratio using the bad-channel relaxed polarization is at least
\[(1 - C - \epsilon)(1 - \frac{(2 + \delta)\log n}{n})\]

**Proof:** Consider the polarization level $t = \lceil (2 + \delta)\log n \rceil$. Then by Corollary 3 for large enough $n$, the fraction of nodes with Bhattacharyya parameter at least $1 - 2^{-t/(2+\delta)} \geq 1 - 2^{-n}$ is at least $1 - C - \epsilon$. Consider such a node $V$ with Bhattacharyya parameter $Z \geq 1 - 2^{-n}$. Then the best descendant of $V$ at the last level of polarization has Bhattacharyya parameter
\[Z^{2^n-t} > Z^{2^n} > (1 - 2^{-n})^{2^n} > \frac{1}{\epsilon},\]
which implies that it can not be a good-bit-channel. Therefore, the total fraction of complexity.
reduction is at least
\[(1 - C - \epsilon) \frac{n - t}{n},\]
and the theorem follows.

Observe that, by neglecting $\epsilon$ and $\delta$ in the bounds given in Theorem 6 and Theorem 7 and by assuming large enough $n$, the complexity reduction ratio from good-channel and bad-channel relaxed polarization is $C(1 - 2\beta)$ and $1 - C$, respectively, which are both positive constant factors. By combining both good and bad channel relaxation, the ratio of saved operations approaches $1 - 2\beta C$. Hence, relaxed polarization can provide a non-vanishing scalar reduction in complexity, even as the code length grows infinitely.

V. Finite Length Analysis of Complexity Reduction

In this section, we derive bounds on the complexity reduction resulted from good-channel and bad-channel relaxed polarization at finite block lengths.

A. Relaxed polar code constructions using Bhattacharyya parameters

In general, finite block length polar codes are constructed by fixing either a target frame error probability (FER) $E$ or target code rate $R$. We consider construction of polar codes with code length $l = 2^n$, at a target FER of $E$. To simplify notation, let $Z_{i,t} \triangleq Z(\hat{W}^{(i)}_{2t})$. At finite block lengths, we need to specify certain thresholds for Bhattacharyya parameters in order to establish criteria for good-channel and bad-channel relaxed polarization. As a result, the following scenarios are considered for relaxed polarization:

1) Good-Channel Relaxed Polarization (GC-RP):

Node $i$ at polarization level $t$ is not further polarized if $Z_{i,t} < T_g$

2) Bad-Channel Relaxed Polarization (BC-RP):

Node $i$ at polarization level $t$ is not further polarized if $Z_{i,t} > T_b$
3) All-Channel Relaxed Polarization (AC-RP):

Node $i$ at polarization level $t$ is not further polarized if $Z_{i,t} < T_g$ or $Z_{i,t} > T_b$ where $T_g$ and $T_b$ are thresholds that can be considered as parameters of the construction.

Remark 3: If $T_g = 2E/l$, then GC-RP constructed codes satisfy the target FER $E$. Let $\Gamma$ be the set of good bit-channel channels which are used to transmit the information bits. Then, this can be observed by (4) and the fact that $|\Gamma| \leq l$, which gives the following

$$\text{FER} \leq \sum_{i \in \Gamma} Z_{i,n}/2 \leq |\Gamma| E/l \leq E. \quad (21)$$

In the proposed bad-channel relaxed polarization (BC-RP), the bad channels are not further polarized if they become sufficiently bad, where the bad-channel relaxation threshold can be set to be $T_b = 1 - T_g$. To guarantee no rate loss from BC-RP, it should only be performed if $n - t \leq \lceil \log_2 \log_2 T_g \log_2 T_b \rceil$. The proposed all-channel relaxed polarization (AC-RP) relaxes the polarization of a bit-channel if it becomes either sufficiently good or bad. Since the bad bit-channels do not contribute to the FER, the target FER is still maintained with AC-RP.

In Fig. 1, the achieved complexity reduction ratio for a binary erasure channel with erasure probability $p$, BEC($p$), is shown. It is observed that up to 85% CR is achievable, i.e. fully polarized (FP) code requires 6.6-fold the complexity of RP code. AC-RP will result in more complexity reduction than GC-RP as the channel becomes worse. The rate-loss is calculated as

$$R_{Loss} = R_{FP} - R_{RP}, \quad (22)$$

where $R_{FP}$ and $R_{RP}$ are the rates of the codes which are constructed by full and relaxed polarization, respectively. The rate at a target FER $E$ is calculated by aggregating the maximum number of bit-channels such that their accumulated BPs does not exceed $2E$. In this simulation result shown in Fig. 1 the rate loss is always less than $10^{-4}$. Another important observation, from Fig. 1 is the symmetry of the CR curve around $p = 0.5$. This is explained by the following
Theorem 8, which is a direct result of Lemma 9 and the description of GC-RP and BC-RP.

**Theorem 8:** The complexity reduction with bad-channel relaxed polarization of BEC($p$) with threshold $1 - T$ is the same as that of good-channel relaxed polarization of BEC($1 - p$) with threshold $T$.

**Lemma 9:** Let the nodes in the polarization tree be labeled by their BPs. Then, the polarization tree for BEC($p$) and the polarization tree for BEC($1 - p$) are isomorphic, where a node $V$ with BP $Z$ in the first tree is isomorphic to a node with BP $1 - Z$ in the second tree.

**Proof:** We show that the one-to-one mapping is nothing but mirroring i.e. the $i$-th node at the polarization level $t$ will be mapped to the node indexed by $2^t - i$ at the same level. It is sufficient to show this for one polarization level and then the rest follows from induction. Let
Fig. 2. Upper bound (UB) on Complexity Reduction.

\[ W_1 = \text{BEC}(p) \text{ and } W_2 = \text{BEC}(1 - p). \text{ Then} \]

\[ Z(W_2^+) = (1 - p)^2 = 1 - (2p - p^2) = 1 - Z(W_1^-) \]

And

\[ Z(W_2^-) = 2(1 - p) - (1 - p)^2 = 1 - p^2 = 1 - Z(W_1^+) \]

Therefore, by induction on the polarization level, it is shown that each polarized node \( V \) in the polarization tree of \( W_1 \) can be mapped to the polarization tree of \( W_2 \) by reversing the sequence of +’s and −’s during its polarization. Furthermore, the BP \( Z \) of \( V \) will be mapped to BP \( 1 - Z \) at the image of \( V \).

**B. Analysis of complexity reduction for GC-RP**

In this subsection, bounds on the complexity reduction from good-channel relaxed polarization are discussed. Let \( \mathcal{T} = \mathcal{T}_g \). In the next theorem, a simple upper bound is provided, which is also illustrated in Fig. 2.
Theorem 10: The good-channel relaxed polarization complexity reduction is upper bounded by

$$R_g(n) \leq \frac{(n - t_g)}{n},$$

where $t_g = \left\lceil \log_2 \frac{\log_2 T}{\log_2 p} \right\rceil$ and $p$ is the erasure channel parameter.

Proof: The upper bound follows by considering the minimum number of polarization levels required for the best polarized channel to reach the threshold $T$. Notice that $Z_{1,0} = p$. Then, after $t$ polarization levels, the minimum BP among all $Z_{i,t}$ is indeed $Z_{2^t, t} = p^{2^t}$. Hence, $t_g$ polarization levels are required for the BP of at least one bit channel to be less than $T$. The upper bound on saved operations follows by skipping all polarization steps at all remaining $n - t_g$ levels.

Next, we derive lower bounds on the complexity reduction with relaxed polarization for $BEC(p)$. For any polarization level $t$ and some threshold $B$, let $G_{B,t}$ denote the set of bit channels at polarization level $t$ with BP at most $B$ i.e. $G_{B,t} = \{i : Z_{i,t} \leq B\}$.

![Diagram](attachment:fig3.png)

Fig. 3. Lower bound (LB1) on Complexity Reduction.

Theorem 11: Let $t_b = \left\lceil \log_2 \frac{\log_2 B}{\log_2 p} \right\rceil$, for an arbitrary threshold $B$. For a polarization level
\( t_\gamma \geq t_b \), let \( \gamma = \left| G_{B,t_\gamma} \right| / 2^{t_\gamma} \). Then the good-channel relaxed polarization complexity reduction is lower bounded by

\[
R_g(n) \geq \gamma 2^{-t_r}(n - t_r - t_\gamma)/n
\]

where \( t_r = \left\lceil \log_2 \frac{\log_2 T}{\log_2 B} \right\rceil \).

**Proof:** Notice that \( t_\gamma \geq t_b \) guarantees that \( G_{B,t_\gamma} \) is a non-empty set. In polarization level \( t_r + t_\gamma \), any node in \( G_{B,t_\gamma} \) has at least one descendant with BP less than \( T \), i.e. the right-most descendant which has BP at most \( B^{2t_r} \leq T \). Therefore, there are at least \( \gamma 2^{t_\gamma} = \left| G_{B,t_\gamma} \right| \) nodes at polarization level \( t_r + t_\gamma \) that have BP less than \( T \), and will be relaxed. Relaxing each of these nodes is equivalent to skipping \( S = (n - t_r - t_\gamma)2^{n-t_r-t_\gamma} \) polarization steps. Then the total number of polarization steps skipped is \( \gamma 2^{t_\gamma} S \), and the proof follows.

---

**Fig. 4.** Lower bound (LB2) on Complexity Reduction.

**Corollary 12:** Let \( t_g = \left\lceil \log_2 \frac{\log_2 T}{\log_2 p} \right\rceil \). Then the good-channel relaxed polarization complexity reduction is lower bounded by

\[
R_g(n) \geq 2^{-t_g}(n - t_g)/n
\]
Proof: The corollary follows by taking $B = p$, and $t_{\gamma} = t_b = 0$ in Theorem 11.

The following provides a tighter lower bound on the GC-RP complexity reduction, which is also illustrated in Figure 4.

Theorem 13: Let $t_b = \left\lceil \log_2 \frac{\log_2 B - 1}{\log_2 p} \right\rceil + 1$, for an arbitrary threshold $B$. For a polarization level $t \geq t_b$, let $\gamma' = \min_{t \geq t_b} | \{ i \text{ odd} : i \in \mathcal{G}_{B,t} \} | / 2^t$. Then the good-channel relaxed polarization complexity reduction is lower bounded by

$$R_g(n) \geq \frac{\gamma' 2^{-t_r}}{2n} ((n - t_r)(n - t_r - 2t + 1) + t(t - 1))$$

where $t_r = \left\lceil \log_2 \frac{\log_2 T}{\log_2 B} \right\rceil$.

Proof: Consider the right-most node at polarization level $t - 1$ which has BP $p^{2^{t-1}} \leq p^{2^{t_b-1}} \leq B/2$. Therefore, the left child of this node is contained in $\mathcal{G}_{B,t}$ which means that the set of odd-indexed nodes in $\Gamma_{B,t}$ is always non-empty for $t \geq t_b$. The right-most descendant of any of these nodes, after $t_r$ more polarization levels, will have BP less than $T$ and will be relaxed by eliminating the polarizing subtrees emanating from them. The total reduction of polarization steps for each of these relaxed nodes is at least $S(t) = \gamma' 2^{n - t_r} (n - t - t_r)$ polarization steps. The bound follows by summation of $S(t)$ for all $t$ with $t_b \leq t \leq n - t_r$. Notice that since the right-polarized children of those odd-indexed nodes at $t$ are even indexed, they are not counted among the odd-indexed nodes in $\mathcal{G}_{B,t'}$, for any other $t'$.

Notice that in the above lower bound, a necessary condition is that $t + 2t_r > n$ to guarantee that no double counting occurs. In many practical operation scenarios, this condition holds. If the condition does not hold, one can modify the bound by limiting the computed summation to $\sum_{t_0 = t}^{t + t_r} S(t_0)$.

Remark 4: In Theorem 11 for large enough $t_{\gamma}$, the parameter $\gamma$ is independent of $B$ and is approximately equal to $C$, the capacity of the underlying channel $W$. Also, $\gamma'$ in Theorem 13 is approximately $C/2$. For our practical applications, we can always pick a proper value of $B$.
such that these approximations still hold at the desired block length.

C. Analysis of complexity reduction for AC-RP

In this subsection, we analyze the complexity reduction from bad-channel relaxed polarization, as well as all-channel (both good and bad) relaxed polarization. As opposed to the previous subsection, we limit our attention in this subsection to binary erasure channels (BEC), wherein the exact computation of Bhattacharyya parameters is applicable at finite block lengths.

Throughout this subsection, we always assume the channel BEC($p$). For a function $F(p)$, let $F^t(p)$ denote the output from recursive application of the function $F$ $t$ times, with initial input $p$.

**Theorem 14:** The bad-channel relaxed polarization complexity reduction is upper bounded by

$$R_b(n) \leq \frac{(n - t_b)}{n},$$

(24)

where $t_b = \min_t \{F^t(p) > 1 - \mathcal{T}\}$ and $F(p) = 2p - p^2$.

**Proof:** The left child of a node with BP $Z$ is associated with a bit-channel with BP $F(Z)$. Hence, it requires $t_b$ polarization levels for the worst left-polarized bit-channel to have BP greater than $1 - \mathcal{T}$. The rest of the proof follows as for the GC-RP case.

Theorem 14 can also be proved by combining the results of Theorem 8 and Theorem 10. The bounds derived for the good-channel relaxed polarization in the previous subsection, can be turned into bounds for bad-channel relaxed polarization of BEC($p$) by replacing $p$ with $1 - p$ in the bounds, and modifying other parameters accordingly. Hence, to avoid writing similar proofs, we only mention the theorems and skip the proofs.

Let $t_g = \left\lfloor \frac{\log_2 \frac{\log_2 \mathcal{T}}{\log_2 p}}{\log_2 1 - p} \right\rfloor$ and $t_b$ as in Theorem 14. In fact, $t_b = \left\lfloor \log_2 \frac{\log_2 \mathcal{T}}{\log_2 1 - p} \right\rfloor$. Observe that if $p > 0.5$, then $t_b < t_g$, if $p < 0.5$, then $t_b > t_g$, and if $p = 0.5$, then $t_b = t_g$. Combining Theorem 14 with upper-bounds on GC-RP of Theorem 10 results in the following upper bound on AC-RP complexity reduction.
Corollary 15: The all-channel relaxed polarization complexity reduction is upper bounded by

\[ R_a(n) \leq (n - t')/n, \]  

(25)

where \( t' = t_g \) for \( p \leq 0.5 \) and \( t' = t_b \) for \( p > 0.5 \).

The next theorem can be also regarded as the counterpart of Theorem 11 for bad-channel relaxed polarization.

Theorem 16: Let \( t_c = \left\lceil \log_2 \frac{\log_2 G}{\log_2 (1 - p)} \right\rceil \), for an arbitrary threshold \( G \). For a polarization level \( t_\beta \geq t_c \), let \( \beta = \left\lceil \frac{\left| \{ i : \text{Z}_{i,t_\beta} \geq 1 - G \} \right|}{2^t} \right\rceil \). Then, the bad-channel relaxed polarization complexity reduction is lower bounded by

\[ R_g(n) \geq \beta 2^{-t_l} (n - t_l - t_\beta) / n \]

where \( t_l = \left\lceil \log_2 \frac{\log_2 T}{\log_2 G} \right\rceil \).

For AC-RP, the lower bounds of Theorem 11 and Theorem 16 can be combined as in the following corollary. It is assumed that the set of relaxed nodes in GC-RP and the set of relaxed nodes in BC-RP do not intersect. This is a valid assumption as long as the good-channel and bad-channel relaxation thresholds, \( T \) and \( 1 - T \), are far apart enough, as characterized in subsection V-A.

Corollary 17: The all-channel relaxed polarization is lower bounded by

\[ R_a(n) \geq \frac{1}{n} \left( \beta 2^{-t_l} (n - t_l - t_\beta) + \gamma 2^{-t_r} (n - t_r - t_\gamma) \right). \]

Theorem 18: Let \( t_c = \left\lceil \log_2 \frac{\log_2 G - 1}{\log_2 (1 - p)} \right\rceil + 1 \), for an arbitrary threshold \( G \). For a polarization level \( t \geq t_c \), let \( \beta' = \min_{t \geq t_c} \left| \{ i \ odd : \text{Z}_{i,t} \geq 1 - G \} \right| / 2^t \). Then the bad-channel relaxed polarization complexity reduction is lower bounded by

\[ R_b(n) \geq \frac{\beta' 2^{-t_l}}{2n} ((n - t_l)(n - t_l - 2t + 1) + t(t - 1)) \]
where \( t_l = \left\lceil \log_2 \left( \log_2 \frac{T}{G} \right) \right\rceil \).

Similar to Theorem 13, the above theorem holds under the condition that \( t + 2t_l > n \). Also, similar to Corollary 19, the following corollary holds by combining Theorem 13 and Theorem 18. To make notations consistent, let \( t'_\gamma \) denote the level \( t \) in Theorem 13 and \( t'_\beta \) denote the level \( t \) in Theorem 18.

**Corollary 19:** The all-channel relaxed polarization is lower bounded by

\[
R_a(n) \geq \frac{\beta' 2^{-t_l}}{2n} \left( (n - t_l)(n - t_l - 2t'_\beta + 1) + t'_\beta^2 - t'_\beta \right) \\
+ \frac{\gamma' 2^{-t_r}}{2n} \left( (n - t_r)(n - t_r - 2t'_\gamma + 1) + t'_\gamma^2 - t'_\gamma \right).
\]

### D. Numerical evaluation of complexity reduction by relaxed polarization on erasure channels

In this subsection, we compute the complexity reduction of different scenarios of relaxed polarization over binary erasure channels and compare them with the bounds provided in this section.

The block length of the constructed relaxed polar code is assumed to be \( l = 2^{20} \) and the FER of \( E = 10^{-5} \) is assumed for the code construction. The erasure probability \( p \) will be varying between 0 and 1. We have observed that the thresholds \( B = 2p - p^2 \) and \( 1 - G = p^2 \) will result in desired values for \( \gamma \) in Theorem 11 and \( \beta \) in Theorem 16. With these values of \( B \) and \( G \) we have \( \gamma = 1 - p \) and \( \beta = p \). We have also observed that \( \gamma' \) in Theorem 13 and \( \beta' \) in Theorem 18 can be well approximated by \( \gamma/2 \) and \( \beta/2 \). The results of Fig. 5 show that actual CR of GC-RP can be characterized using the derived upper and lower bounds, and up to 70% complexity reduction is achievable at a target FER of \( 10^{-5} \).

The performance of AC-RP is analyzed in Fig. 6 at the same target FER of \( 10^{-5} \), where the analytical bounds are compared to the numerical results from actual construction. It is observed that the bounds give a good approximation of the actual complexity reduction. GC-RP is effective with good channel parameters and BC-RP is more effective with bad channel parameters. The
VI. RELAXED POLARIZATION ON GENERAL BMS CHANNELS

In this section, we describe how a code can be constructed and decoded on general binary memoryless channels using relaxed polarization.

A. Construction of relaxed polar codes on general BMS channels

For general binary memoryless channels, the Bhattacharyaa parameters are exponentially hard to compute as block length increases. This is due to the exponential output alphabet size of the polarized bit-channels. Instead of exact calculation of Bhattacharyya parameters, they can be well approximated by bounding the output alphabet size of bit-channels via channel degrading.
and channel upgrading transformations [16]. The channel degrading and upgrading operations provide tight lower bounds and upper bounds on the corresponding Bhattacharyya parameters. In order to construct polar codes for continuous-output BMS channels (e.g. additive white Gaussian noise (AWGN) channels), the channel can be first quantized. Then, the degrading and upgrading operations will be performed for the bit-channels resulting from polarization of the quantized channel [16]. For AWGN channels, the BC-EP can also be reasonably approximated using density evolution and a Gaussian approximation [17]. Alternatively, for short codes, the BC-EP can be numerically evaluated using Monte-Carlo simulations, assuming a genie-aided SC decoder. For generality of description, the error probability of the $j$-th bit-channel at the $t$-th polarization level can be bounded by

$$P_{t,j} \leq P_{t,j} \leq \overline{P}_{t,j}$$

(26)
where $P_{t,j}$ is the probability of error of the upgraded version of $W_2^{(j)}$ and $\overline{P}_{t,j}$ is the probability of error of its degraded version.

The values of $P_{t,j}$ and $\overline{P}_{t,j}$ can be computed using upgraded and degraded versions of polarization tree. In the upgraded polarization tree, after each level of polarization the resulting bit-channels will be upgraded to have a limited output alphabet size. Then in the next level, the upgraded bit-channels will be polarized. As a result, all the bit-channels in the upgraded polarization tree will have a limited output alphabet size. Therefore, $P_{t,j}$ can be easily computed. The same procedure is repeated to get a degraded polarization tree and to compute $\overline{P}_{t,j}$. The construction of fully polarized codes can be done according to either lower bounds or upper bounds on the probability of error of the bit-channels at the last level of polarization. For instance, in case of using upper bounds, bit-channels are sorted according to their error probabilities $\overline{P}_{n,j}$ in ascending order. Accumulate as many good bit-channels in the set $\Gamma$, such that $\sum_{j \in \Gamma} P_{n,j} \leq E$, where $E$ is the target FER. Then, the FP code is defined by $\Gamma$ and has rate $R = |\Gamma|/l$.

In proposed good-channel relaxed polar codes, a node will not be further polarized if the upper bound on its bit-channel error probability is lower than a certain threshold $E_g$. For bad-channel relaxed polar codes, a bit-channel will not be further polarized if the lower bound on its error probability exceeds an upper threshold $E_b$. Numerically, it was found for BMS channels that the error performance of the constructed code is closer to that of the upper-bound calculated using the degraded channel. Hence, when polarization is relaxed for a node, the error probability of the children of a non-polarized node is set to the degraded-channel error probability of the parent. As a result, the procedure for designing relaxed polar codes of length $l$ at a target FER $E$ on general BMS channels is specified below. Each node $j$ at level $t$ in the polarization tree is associated with a label $\text{Relaxed}(t, j)$ which is initialized to 0, and will be set to 1 only if this node will not be polarized. The error probability (EP) of each node in the RP tree is initialized to that of the fully polarized tree $\overline{P}_{t,j}^R = \overline{P}_{t,j}$. 

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Algorithm 1 Relaxed Construction for General BMS Channels

1: **Stage 1:** Calculate AC-RP bit-channel EP for target FER $E$ and rate $R$
   2: Set the GC relaxation threshold as $E_g = E/(Rl)$
   3: Set the BC relaxation threshold as $E_b = H^{-1}(1 - H(E_g))$
   4: for $t = 1 : n$ do
      5: for $j = 1 : 2^t$ do
         6: if Relaxed($t-1$, $\lceil j/2 \rceil$) = 1 then
            7: Relaxed($t$, $j$) = 1, $P_{t,j} = P_{t,\lceil j/2 \rceil}$
         8: else if $\{P_{t,j} < E_g\}$ or $\{P_{t,j} > E_b\}$ then
            9: Relaxed($t$, $j$) = 1
         10: end if
      11: end for
   12: end for
13: **Stage 2:** Construct AC-RP code with rate $R$
   14: Sort bit-channel EPs $P_{t,j}^R$ in ascending order
   15: Select $\Gamma_R$ to have the $RL$ bit channels with smallest EP

For the case of AWGN channel, first the channel parameter is calculated to satisfy the condition $C \geq R$, where $R$ is the target rate and $C$ is the capacity of the channel. Then the channel is quantized using the method of [16] to get a channel with discrete output alphabet. Then Algorithm 1, discussed above, will be applied to this channel.

With target FER $E$, the good-channel relaxation threshold is chosen to be $E_g = E/(Rl)$ to satisfy the target FER, i.e. $\sum_{j \in \Gamma_R} P_{n,j} \leq E$. Let $H(E_W)$ be the entropy of the channel $W$ with fidelity $E_W$, such that $H(E_W) = 1 - I(W)$, where $I(W)$ is capacity of channel $W$. Then, the bad-channel relaxation threshold is chosen such that $H(E_b) = 1 - H(E_g)$. For general BMS channels $W$ with error probability $E_W$, the approximation $H(W) \approx h_2(E_W)$ can be used, based on the inequality $H(W) \leq h_2(E_W)$ [11, 18], where $h_2(\epsilon) = -\epsilon \log_2(\epsilon) - (1 - \epsilon) \log_2(1 - \epsilon)$ is the binary entropy function. To guarantee that there is no rate loss from bad-channel relaxation if $\{P_{t,j} > E_b\}$ is satisfied at Step 13, then relaxation may only be done after verifying that the lower bound on the error probability of the best upgraded descendent channel of that node is still higher than $E_g$, which will be satisfied for practical frame errors $E$.

The same procedure above can be used to construct GC-RP codes, by neglecting the bad-
channel relaxation condition \( \{P_{t,j} > E_b\} \) in step 13.

In case of erasure channels with erasure probability \( \epsilon \), the channel parameter (erasure probability) for the target channel capacity is \( \epsilon = 1 - C \). The upper and lower bound on the bit-channel error probabilities coincide, and can be calculated exactly by the BPs, \( P_{t,j} = Z_{t,j}/2 \). To calculate the relaxation thresholds, the error probability is \( E = \epsilon/2 \), and the entropy is \( H(E) = 2E \).

Algorithm 1 constructs the relaxed polar codes for general BMS channels, using bounds on the bit-channel error probability. However, for short block lengths, the bit-channel error probability can be numerically calculated to be \( \tilde{P}_{t,j} \) using Monte-Carlo simulations, assuming a genie-aided successive cancellation decoder. In such a case, Algorithm 1 is modified by letting \( P_{t,j} = \tilde{P}_{t,j} \).

**B. Decoding of relaxed polar codes on general BMS channels**

Decoding of relaxed polar codes can be done by a modified successive cancellation decoder. For a polar code of length \( l \) and BMS channel \( W \), suppose that \( u_1^l \) is the input vector and \( y_1^l \) is the received word.

Consider a relaxed polar code constructed as explained in the previous subsection. At each level \( t = \log l \), for \( 1 \leq i \leq l \), Relaxed(\( t, i \)) = 1 means that \( \tilde{W}_{i}^{(i)} \) is not polarized and Relaxed(\( t, i \)) = 0 means that \( \tilde{W}_{i}^{(i)} \) is fully polarized. In practical application of relaxed polar codes, the decoder will have prior knowledge of the polarization map by Relaxed(\( t, i \)), which requires at most storage of \( 2l \) bits. For communication systems, the polarization map can be specified by the communication standard, similar to the specification of the parity-check matrices of block codes.

For \( i = 1, 2, \ldots, l \), the decoder computes the likelihood ratio (LR) \( L_{i}^{(i)} \) of \( u_i \), given the channel outputs \( y_1^l \) and previously decoded bits \( \hat{u}_1^{i-1} \)

\[
L_{i}^{(i)}(y_1^l, \hat{u}_1^{i-1}) = \frac{\tilde{W}_{i}^{(i)}(y_1^l, \hat{u}_1^{i-1}|u_i = 0)}{\tilde{W}_{i}^{(i)}(y_1^l, \hat{u}_1^{i-1}|u_i = 1)}.
\]

Let \( u_1^{j,o} \) and \( u_1^{j,e} \) denote the sub-vectors with odd and even indices, respectively. For FP polar
codes, Arikan observed that calculation of the LRs at length \( l \) require another \( l \) calculations at length \((l/2)\), where the LRs from the pair \( \left(L^{(2i-1)}_l(y^{i}_{1}, \hat{u}^{2i-2}_{1}) , L^{(2i)}_l(y^{i}_{1}, \hat{u}^{2i-1}_{1}) \right) \) are assembled from the pair \( \left(L^{(i)}_{l/2}(y^{l/2}_{1}, \hat{u}^{2i-2}_{1,e} \oplus \hat{u}^{2i-2}_{1,o} ) , L^{(i)}_{l/2}(y^{l/2+1}_{1}, \hat{u}^{2i-2}_{1,e} ) \right) \), via a straightforward calculation using the bit-channel recursion formulas for \( n \geq 1 \) \[1\].

Hence, if Relaxed\((t,i) = 0\), the likelihood ratio (LR) \( L^{(i)}_t \) can be computed recursively as follows.

\[
L^{(2i-1)}_l(y^{i}_{1}, \hat{u}^{2i-2}_{1}) = 1 + L^{(i)}_{l/2}(y^{l/2}_{1}, \hat{u}^{2i-2}_{1,e} \oplus \hat{u}^{2i-2}_{1,o} ) L^{(i)}_{l/2}(y^{l/2+1}_{1}, \hat{u}^{2i-2}_{1,e} ) \]

\[
L^{(2i)}_l(y^{i}_{1}, \hat{u}^{2i-1}_{1}) = \left[ L^{(i)}_{l/2}(y^{l/2}_{1}, \hat{u}^{2i-2}_{1,e} \oplus \hat{u}^{2i-2}_{1,o} ) \right]^{1-2\hat{u}^{2i-1}_{1}} L^{(i)}_{l/2}(y^{l}_{1,2+1}, \hat{u}^{2i-2}_{1,e} ).
\]

Otherwise, if Relaxed\((t,i) = 1\) the decoding equations are modified as follows:

\[
L^{(2i-1)}_l(y^{i}_{1}, \hat{u}^{2i-2}_{1}) = L^{(i)}_{l/2}(y^{l/2}_{1}, \hat{u}^{2i-2}_{1,o} ) \]

\[
L^{(2i)}_l(y^{i}_{1}, \hat{u}^{2i-1}_{1}) = L^{(i)}_{l/2}(y^{l}_{1,2+1}, \hat{u}^{2i-2}_{1,e} ).
\]

At the last stage when \( l = 1 \), the LRs are simply \( L^{(i)}_1(y_i) = W(y_i|0)/W(y_i|1) \). At the end, hard decisions are made on \( L^{(i)}_l \), except for frozen bit-channels \( W^{(i)}_l \) where \( \hat{u}_i = u_i = 0 \).

From the above description, \( l \) calculations for \((1 + \log l)\) levels are required for decoding conventional FP polar codes. However, the decoding complexity of relaxed polar codes is linearly reduced by the ratio of relaxed nodes in the polarization tree, since no operation is required at relaxed nodes.
C. Numerical results

The complexity reduction achievable is analyzed by actual construction of the relaxed polar codes on AWGN channels in Fig. 7. An AWGN channel with binary-input capacity $C = 0.7$ is used to calculate upper and lower bounds on the bit-channel error probabilities. The CR at different code lengths $2^{10}$, and $2^{14}$ are logged at different target FER $E$. The rate achievable by construction of the FP code at each FER $E$ is also logged. It is observed that at larger target FER $E$, a higher rate is possible, due to possible accumulation of more good-channel bits. The CR also increases with target FER $E$, due to the increase of the relaxation threshold $E_g$, despite the increase in the code rate. Since the number of polarization levels increases with the code block length $l$, the achievable CR from RP increases with $l$. The effect of CR due to bad-channel relaxation becomes more visible, at higher target FER, as $E_b$ also becomes lower.
The error-rate performance by actual successive cancellation decoding of the RP codes is shown in Fig. 8 as a function of SNR = $10 \log_{10}(1/\sigma^2)$. Practical code length of $l = 4096$ is assumed with near half-rate code of $R = 0.59$, respectively. The code is constructed with Algorithm 1, assuming an AWGN channel with binary-input capacity $C = 0.7$, and with $E = 0.1$.

The FP good-channel set is optimized for the FP polar code. For simpler comparison, both the GC-RP and AC-RP codes use the same good bit-channel set found by construction (Stage 4) of the AC-RP code. It is observed that the GC-RP code has similar error-rate performance to the AC-RP code. The performance curves show that both GC-RP and AC-RP have similar performance. RP codes can have a Frame Error Rate (FER) performance degradation compared to FP codes, due to the higher LR magnitudes of FP codes. However, RP codes have a better bit error rate (BER) than FP codes. This is justified by Theorem 5 and verifies the proper design and selection of relaxation thresholds. Another important observation is that the proposed construction of relaxed polar codes is robust enough, such that it performs well over the whole range of simulated SNRs, although the codes are constructed for a certain SNR.

VII. CONCLUSIONS

In this work, a new paradigm for polar codes, called relaxed polar coding, is investigated. In relaxed polar codes, a bit-channel will not be further polarized if it has been already polarized to be sufficiently good or sufficiently bad. Hence, encoding and decoding of relaxed polar (RP) codes have lower computational and time complexity than those of conventional polar codes. RP codes also have lower space complexity than conventional polar codes in fixed point hardware implementations, due to the less number of bits required to store the likelihood ratios. This has the compound effect of decoder implementations with less power consumption. It is proved in this work that, similar to conventional polar codes, RP codes are capacity achieving. It is also proved that with proper design RP codes will have lower information bit error rates than conventional polar codes of the same rate. Constructions of RP codes on erasure channels, and
on general BMS channels are described. Asymptotic and finite-length bounds on the complexity reduction achievable by relaxed polar coding are derived and verified for binary erasure channel against actual constructions. Successive cancellation decoding of RP codes on AWGN channels is described and its performance is investigated. It is verified by numerical simulations on the AWGN channel that the information bit error rates of properly designed RP codes are at least as good as those of conventional polar code with the same rate.

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For any DMC $W$, let $E(W)$ denote the probability of error of $W$ under ML decoder. Let $W : \mathcal{X} \to \mathcal{Y}$ be a BMS channel, where $\mathcal{X}$ is the binary alphabet. Then

$$E(W) = \sum_{y \in \mathcal{Y}} \frac{1}{2} \min \{W(y|0), W(y|1)\}$$

Let the channel $W$ be polarized using the Arıkan’s Butterfly. Let the polarized bit-channels be denoted by $W^+$ and $W^-$. Then it is known that

$$W^-(y_1, y_2|u_1) = \frac{1}{2} \sum_{u_2 \in \mathcal{X}} W(y_1|u_1 \oplus u_2)W(y_2|u_2)$$

(31)

and

$$W^+(y_1, y_2, u_1|u_2) = \frac{1}{2} W(y_1|u_1 \oplus u_2)W(y_2|u_2)$$

(32)

Lemma 20: For any BMS $W$,

$$E(W^-) = 2E(W) - 2E(W)^2$$

(33)

and

$$E(W^+) \geq 2E(W)^2$$

(34)

Proof: Suppose that the size of output alphabet $\mathcal{Y}$ is $M$. Let $\mathcal{Y} = \{y_1, y_2, \ldots, y_M\}$. Then for $i = 1, 2, \ldots, M$, let

$$a_i = \min \{W(y|0), W(y|1)\}$$

and

$$b_i = \max \{W(y|0), W(y|1)\}$$
Then
\[ E(W) = \frac{1}{2} \sum_{i=1}^{M} a_i \] (35)

and
\[ \sum_{i=1}^{M} a_i + \sum_{i=1}^{M} b_i = 2 \] (36)

The size of output alphabet of $W^-$ is $M^2$. For any pair $(y_i, y_j) \in \mathcal{Y}^2$,

\[ \{ W^-(y_i, y_j | 0), W^-(y_i, y_j | 1) \} = \left\{ \frac{1}{2} (a_i a_j + b_i b_j), \frac{1}{2} (a_i b_j + a_j b_i) \right\} \]

Notice that $a_i a_j + b_i b_j \geq a_i b_j + a_j b_i$. Therefore,

\[ E(W^-) = \frac{1}{4} \sum_{i=1}^{M} \sum_{j=1}^{M} (a_i b_j + a_j b_i) = \frac{1}{2} (\sum_{j=1}^{M} a_i) (\sum_{j=1}^{M} b_j) \]
\[ = E(W)(2 - 2E(W)) \]

where the last equality follows by (35) and (36). This proves the first part of the lemma.

The size of the output alphabet of $W^+$ is $2M^2$. For any pair $(y_i, y_j) \in \mathcal{Y}^2$, there are two corresponding elements in the output alphabet of $W^+$. Then

\[ \left\{ \left\{ W^+(y_i, y_j, 0|0), W^+(y_i, y_j, 0|1) \right\}, \left\{ W^+(y_i, y_j, 1|0), W^+(y_i, y_j, 1|1) \right\} = \right\{ \left\{ \frac{1}{2} a_i a_j, \frac{1}{2} b_i b_j \right\}, \left\{ \frac{1}{2} a_i b_j, \frac{1}{2} a_j b_i \right\} \} \]

(37)
Then

\[ 2E(W^+) = \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} a_i a_j + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \min \{a_i b_j, a_j b_i\} \]

\[ \geq \sum_{i=1}^{M} \sum_{j=1}^{M} a_i a_j = (\sum_{i=1}^{M} a_i)^2 = 4E(W)^2 \]

which proves the second part of the lemma.

\[ \blacksquare \]