On the Solution of Topological Landau-Ginzburg Models with $c = 3$

Z. Maassarani

Physics Department
University of Southern California
Los Angeles, CA 90089-0484

Abstract

The solution is given for the $c = 3$ topological matter model whose underlying conformal theory has Landau-Ginzburg model $W = -\frac{1}{4}(x^4 + y^4) + \alpha x^2 y^2$. While consistency conditions are used to solve it, this model is probably at the limit of such techniques. By using the flatness of the metric of the space of coupling constants I rederive the differential equation that relates the parameter $\alpha$ to the flat coordinate $t$. This simpler method is also applied to the $x^3 + y^6$-model.
1 Solution of the $x^4 + y^4$-model

I first use the methods described in [1, 2] to compute the perturbed topological correlation functions and their prepotential for the superconformal field theory that can be described by the following Landau-Ginzburg potential:

$$W = -\frac{1}{4}(x^4 + y^4) + \alpha x^2 y^2$$

(1)

where the parameter $\alpha$ is complex. When $\alpha$ vanishes the model reduces to the tensor product of two minimal models with two commuting $U(1)$ currents. Define $\varphi_i$, $i = 0, ..., 8$ to be the chiral primary fields such that for $\alpha$ vanishing we have: $\varphi_0 = 1$, $\varphi_1 = x$, $\varphi_2 = y$, $\varphi_3 = x^2$, $\varphi_4 = xy$, $\varphi_5 = y^2$, $\varphi_6 = x^2 y$, $\varphi_7 = xy^2$, $\varphi_8 = x^2 y^2$. Let $t_i$ be the coupling constants corresponding to the chiral primary fields $\varphi_i$, such that $t_i = 0$ for $i = 0$ to 8 corresponds to the unperturbed model with $\alpha = 0$.

The perturbed three-point functions are defined by:

$$C_{ijk} = \langle \varphi_i \varphi_j \varphi_k \exp[\sum t_i \int d^2 z \bar{G} \tilde{G} \varphi_i] \rangle,$$

(2)

from which one can construct a prepotential $F$ such that

$$C_{ijk} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k}.$$

(3)

From the expansion of the exponential in (2) and the use of the symmetries of the unperturbed potential one determines the non-vanishing correlators. Integrating (3) one readily gets the general form of the prepotential $F$:

$$F(t) = \frac{1}{24} t_0^2 + t(t_1 t_7 + t_2 t_6 + t_3 t_5) + \frac{1}{2} t_1 t_2 f_0 + \frac{1}{2} (t_1 t_3 + t_2 t_5) f_1 + \frac{1}{2} (t_1 t_5 + t_2 t_3) f_2 + \frac{1}{4} (t_1^2 t_7 + t_2^2 t_6) f_3$$

$$+ \frac{1}{4} (t_1^2 t_6 + t_2^2 t_5) f_4 + t_1 t_2 t_7 f_5 + \frac{1}{2} (t_1 t_2 t_7 + t_2 t_3 t_6) f_6$$

$$+ t_3 t_5 (t_1 t_7 + t_2 t_6) f_7 + t_4 (t_1 t_3 t_6 + t_2 t_5 t_7) f_8 + \frac{1}{2} t_1^2 (t_1 t_7 + t_2 t_6) f_9$$

$$+ \frac{1}{4} (t_1 t_2 t_7 + t_2 t_3 t_6) f_{10} + t_4 (t_1 t_5 t_6 + t_2 t_3 t_7) f_{11} + \frac{1}{6} t_3 t_5 (t_2^2 + t_3^2) f_{12}$$

$$+ \frac{1}{24} (t_3^2 + t_5^2) f_{13} + \frac{1}{4} t_3^2 (t_3^2 + t_5^2) f_{14} + \frac{1}{2} t_3^2 t_5^2 f_{15}$$

$$+ \frac{1}{24} t_3^2 t_5 f_{17} + \frac{1}{24} t_1^2 f_{19} + \text{(higher order terms)}$$

(4)

where $f_n \equiv f_n(t)$. There are fifty four unknown functions to be determined in the complete expression for $F$! The $f_n$’s explicitly appearing in equation (4) correspond to all the three and four-point functions of the relevant perturbations. The remaining $f_n$’s, which determine the higher order perturbed functions, can all be easily found in terms of the three and four-point functions. The functions $f_n$ all have the following form: $f_n(t) = \sum_{m=0}^{\infty} a_m t^{2m+\pi}$ where $\pi \equiv n \ mod \ 2$ and $t = t_8$. These functions are determined by solving the highly redundant set of equations obtained from requiring that the $C_{ijk}$ be the structure constants of an associative algebra

$$C_{ijp} g^{pq} C_{klq} = C_{ikp} g^{pq} C_{jql}$$

(5)
where \( g^{pq} = g_{pq} = C_{pq0} = \delta_{p+q,8} \) is the \( t_i \)-independent metric on the space of the chiral primary fields \[4\]. The equations yielded by the conditions (5) are quadratic in the \( f_n \) and the \( f'_n \). The redundant equations serve as consistency checks. Once the \( f_n \) corresponding to three and four-point functions are determined, the remaining functions \( f_n \) can be determined by solving elementary sets of linear equations that can be obtained from equations (5).

While solving these equations it turns out useful to introduce the ratio

\[
\alpha(t) \equiv \frac{f_1(t)f_2(t)}{(f_0(t))^2}. \tag{6}
\]

This ratio involves the only non-vanishing three-point functions. This apparent redefinition of \( \alpha \) has a reason: the \( \alpha \) of equation (6) will turn out to be equal to that of equation (1).

After making some eliminations in a subset of equations obtained from equations (5), one finds:

\[
\begin{align*}
f_5 - f_7 &= \frac{\alpha \alpha'}{1 - 4\alpha^2}, \\
f'_7 + f''_7 &= (f_5 - f_7)^2, \quad \alpha^2(f'_5 + f''_5) = (f_5 - f_7)^2.
\end{align*}
\]

These equations imply that \( \alpha \) satisfies a Schwarzian differential equation:

\[
\{\alpha; t\} = -(\alpha')^2 \frac{8\alpha^2 + 6}{(1 - 4\alpha^2)^2} \tag{7}
\]

where

\[
\{\alpha; t\} \equiv \frac{\alpha''}{\alpha'} - \frac{3}{2}(\frac{\alpha''}{\alpha'})^2. \tag{8}
\]

Taking \( \alpha \) instead of \( t \) as the variable and using the properties of the Schwarzian derivative one can rewrite (7) as

\[
\{t; \alpha\} = \frac{8\alpha^2 + 6}{(1 - 4\alpha^2)^2}. \tag{9}
\]

The general solution of equation (9) can be written as the ratio of two independent solutions of

\[
[(1 - 4\alpha^2)\frac{d^2}{d\alpha^2} - 8\alpha \frac{d}{d\alpha} - 1] y = 0. \tag{10}
\]

Equation (10) can be cast into a standard hypergeometric form by making the change of variable \( z = \alpha + \frac{1}{2} \):

\[
[z(z - 1)\frac{d^2}{dz^2} + (2z - 1) \frac{d}{dz} + \frac{1}{4}] y = 0. \tag{11}
\]

The solutions of this equation can be expressed using the hypergeometric function \( F(1/2, 1/2; 1; \alpha + 1/2) \). Equations (10) and (11) can also be derived by the methods given in \[4, 5\].

A remark about the coupling constant \( t \) is in order. The vanishing of the potential describes the target space of the SCFT. More precisely define \( \tilde{W} \equiv W + z^2 \); then \( \tilde{W} = 0 \)
describes a $Z_4$ orbifoldized torus in a weighted projective space whose weights are (1,1,2) for
the coordinates $(x, y, z)$. Take the modular parameter of the torus to be $\tau$. Since $\tau$ is also a
flat coordinate it must satisfy equation (9) and can therefore be written as $\tau = \frac{a y_1 + b y_2}{c y_1 + d y_2}$
where $y_{1,2}$ are two independent solutions of equation (10) such that $t \equiv \frac{y_1}{y_2}$. One therefore has:

$$\tau = \frac{at + b}{ct + d}.$$  \hspace{1cm} (12)

To determine $(a, b, c, d)$ I utilize two facts. There is an obvious inversion symmetry $t \to -t$
which corresponds to $\tau \to -\frac{1}{\tau}$. Moreover for $\alpha = 0$ (corresponding to $t = 0$) the torus has a
further $Z_4$ symmetry and is in fact rectangular; thus $t = 0$ corresponds to $\tau = i$. Equation
(12) is then reduced to equation (13)

$$\tau = -i \mu t + \frac{1}{\mu t - 1}$$ \hspace{1cm} (13)

where $\mu$ is an undetermined scale parameter.

Having found $\alpha(t)$, one determines:

$$f_0 = C(\frac{\alpha'}{1-4\alpha^2})^{1/2}, f_1 = \frac{1}{2} f_0((1 + 2\alpha)^{1/2} - (1 - 2\alpha)^{1/2})$$
$$f_2 = \frac{1}{2} f_0((1 + 2\alpha)^{1/2} + (1 - 2\alpha)^{1/2})$$
$$f_3 = 2f_7 = -\left(\frac{\alpha''}{\alpha'} + \frac{6\alpha\alpha'}{1-4\alpha^2}\right), f_4 = \frac{1}{C^2} f_0^2$$
$$f_5 = f_9 = -\frac{1}{2}\left(\frac{\alpha''}{\alpha'} + 4\frac{\alpha\alpha'}{1-4\alpha^2}\right)$$
$$f_6 = \frac{1}{C^2} f_2^2, f_8 = \frac{1}{C^2} f_0 f_2, f_{10} = \frac{1}{C^2} f_1^2$$
$$f_{11} = \frac{1}{C^2} f_0 f_1, f_{12} = 0, f_{13} = \frac{2}{C^2} f_1 f_2, f_{14} = f_4$$
$$f_{15} = f_3, f_{17} = f_5, f_{19} = -\frac{3}{2}\left(\frac{\alpha''}{\alpha'} + \frac{16\alpha\alpha'}{3 - 1 - 4\alpha^2}\right)$$  \hspace{1cm} (14)

where $f_n \equiv f_n(t)$ and $C$ is an integration constant. The relations $f_3 = 2f_7, f_5 = f_9, ..$ and
particularly $f_{12} \equiv 0$ suggests some further symmetry of this model that was not employed
in making the ansatz (4) for $\mathcal{F}$.

To calculate the effective Landau-Ginzburg potential in the flat coordinates $t_i$, one makes
the most general ansatz for $W$ that is consistent with the symmetries of the model:

$$W = -\frac{1}{3}(x^4 + y^4) + \gamma_0(t)x^2y^2 + \gamma_1(t_6x^2y + t_7xy^2) + \gamma_2(t_3x^2 + t_5y^2)$$
$$+ \gamma_3(t_5x^2 + t_3y^2) + \frac{1}{2}\gamma_4(t_6^2x^2 + t_7^2y^2) + \frac{1}{2}\gamma_5(t_7^2x^2 + t_6^2y^2)$$
$$+ t_4\gamma_6xy + t_6t_7\gamma_7xy + \gamma_8(t_1x + t_2y) + \gamma_9(t_3t_7x + t_5t_6y)$$
$$+ \gamma_{10}(t_5t_7x + t_3t_6y) + t_4\gamma_{11}(t_6x + t_7y) + \frac{1}{2}t_6t_7\gamma_{12}(t_6x + t_7y)$$

3
linear in the coupling constants $\mu$ and then the concentrate upon the behaviour of the marginal parameter. I first study the given these $\gamma$'s, solving for the remaining unknown $\gamma_i$'s is straightforward. If one merely uses the consistency conditions on the Landau-Ginzburg to solve for the $\gamma_i$'s, one would obtain a second possible solution for $\gamma_2$ and $\gamma_3$: the $\gamma_2 \leftrightarrow \gamma_3$ solution. However the choice $\lim_{t_i \to 0} \varphi_3 = x^2$ (and similarly for $\varphi_5$) made earlier, forces $\gamma_2$ to be even and $\gamma_3$ to be odd in $\alpha$. 

Using the consistency conditions to solve for the $f_i$'s and the $\gamma_i$'s is extremely tedious in general. Another procedure for obtaining the same information can be inferred from the work of [2, 3, 4]. We now give a brief exposition of this method and show that specific information such as the equation for $\alpha$ can be extracted.

### 2 The flat metric method

The basic idea in this section is to consider a general parametrization of the Landau-Ginzburg potential, compute the metric, and impose flatness. The result is a simpler method of obtaining the dependence of $W$ and $\mathcal{F}$ in terms of flat coordinates. For simplicity I shall concentrate upon the behaviour of the marginal parameter. I first study the $x^4 + y^4$-model and then the $x^3 + y^6$-model.

The simplest form for a Landau-Ginzburg potential is one whose perturbation terms are linear in the coupling constants $\mu_i$,

$$W = \frac{1}{3}(x^4 + y^4) + \mu_8 x^2 y^2 + \frac{1}{2} \mu_7 x y^2 + \frac{1}{2} \mu_6 x^2 y + \frac{1}{2} \mu_5 y^2 + \mu_4 x y + \frac{1}{2} \mu_3 x^2 + \mu_2 y + \mu_1 x + \mu_0 1$$

$$+ \frac{1}{6} \gamma_1 (t_7^2 x + t_6^2 y) + \gamma_1 (t_1 t_7 + t_2 t_6) + t_3 t_5 \gamma_15 + \frac{1}{2} t_4^2 \gamma_16$$

$$+ \frac{1}{2} \gamma_1 (t_3 t_7^2 + t_5 t_6^2) + \frac{1}{2} \gamma_18 (t_3 t_7^2 + t_5 t_6^2) + \frac{1}{2} \gamma_{10} (t_3 t_6^2 + t_5 t_7^2)$$

$$+ t_4 t_6 t_7 t_20 + \frac{1}{2} t_6^2 t_5^2 \gamma_{21} + \frac{1}{24} \gamma_{22} (t_4^4 + t_5^4) + t_0$$

where $\gamma_i \equiv \gamma_i(t)$ are, as yet, arbitrary functions. The chiral primary fields are given by

$$\varphi_i(x, y) = \frac{\partial W}{\partial t_i}$$

(16)

The structure constants $C^{ij}_k$ are extracted from

$$\varphi_i \varphi_j = C^{ij}_k \varphi_k \mod \nabla W.$$ 

(17)

One obtains in this way a redundant set of equations from which one can solve for the $\gamma_i$'s in terms of $\gamma_0(t)$. However it is simpler to use the values of the $C_{ijk}$ given by equations (3), (4) and (14) to determine the $\gamma_i$'s in terms of $\alpha$. In this way one obtains additional consistency checks. As mentionned earlier, one finds $\gamma_0 \equiv \alpha$. I give below the $\gamma_i$ for the terms in $W$ linear in the coupling constants $t_1$ to $t_7$:

$$\gamma_1 = C^{i=1/2} (\alpha')^{1/2} (1 - 4\alpha^2)^{1/4}, \quad \gamma_2 = \frac{1}{C} f_2, \quad \gamma_3 = -\frac{1}{C} f_1$$

$$\gamma_6 = (\alpha')^{1/2}, \quad \gamma_8 = C^{1/2} \frac{\alpha'}{1 - 4\alpha^2}.$$ 

(18)

Given these $\gamma_i$'s, solving for the remaining unknown $\gamma_i$'s is straightforward. If one merely uses the consistency conditions on the Landau-Ginzburg to solve for the $\gamma_i$'s, one would obtain a second possible solution for $\gamma_2$ and $\gamma_3$: the $\gamma_2 \leftrightarrow \gamma_3$ solution. However the choice $\lim_{t_i \to 0} \varphi_3 = x^2$ (and similarly for $\varphi_5$) made earlier, forces $\gamma_2$ to be even and $\gamma_3$ to be odd in $\alpha$. 

Using the consistency conditions to solve for the $f_i$'s and the $\gamma_i$'s is extremely tedious in general. Another procedure for obtaining the same information can be inferred from the work of [2, 3, 4]. We now give a brief exposition of this method and show that specific information such as the equation for $\alpha$ can be extracted.
where \( \mu_8 \equiv \alpha \). The chiral primary fields are now taken to be \( \varphi_i = \frac{\partial W(x,y,\mu)}{\partial \mu_i} \). There is a natural metric \( \tilde{g}_{ij} \) defined on the space of coupling constants \([1, 4]\) obtained by setting \( \tilde{g}_{ij} = C_{ij}^{\text{max}} \) where \( \varphi_{\text{max}} \) is the unique (up to a scaling factor) chiral primary field of the unperturbed theory of highest dimensional charge. The structure constants are extracted from equations (17). However the choice of the field \( \varphi_{\text{max}} \) is ambiguous in the following sense: any linear combination of the chiral primary field of highest charge (\( \beta \varphi_{\text{max}} \) with \( \beta \neq 0 \)) with fields of lower charges can be used, resulting in a conformally related metric, i.e. \( C_{ij}^{\text{max}} = \frac{1}{\beta} C_{ij}^{\text{lin.comb.}} \). However, conformal perturbation theory gives a natural set of flat coordinates, the \( t_i \)'s, in which \( \tilde{g}_{0\text{max}} = 1 \). This means that to get the flat metric for (19) as a function of the flat coordinate \( t \), one should take \( \varphi_{\text{max}} = \left( \frac{dt}{\alpha} \right) x^2 y^2 \). For simplicity in what follows I shall take \( \varphi_{\text{max}} = x^2 y^2 \) and remember to multiply the resulting metric, \( \tilde{g}_{ij} \), by the flattening factor \( \beta \) of \( \left( \frac{dt}{\alpha} \right) \) later.

It is easy to see that the metric elements \( \tilde{g}_{ij} \) are polynomials in the \( \mu_i \)'s, for \( i = 1 \) to 7, with coefficient being rational fractions of \( \mu_8 = \alpha \). Because I am interested in the \( \mu_8 \)-dependence of \( t \), I need only keep the linear and quadratic terms in \( \mu_1 \) to \( \mu_7 \) as I will send these coupling constants to zero at the end of the calculation. The inversion of this metric is also done at the origin (\( \mu_i = 0 \) for \( i = 1 \) to 7).

To obtain the differential equation for \( t(\alpha) \) it is enough to calculate one non-trivially vanishing component of the Riemann tensor of the kind \( R_{s8s}^s \), at the origin. The choice \( 2R_{s8s}^s \) simplifies the calculations by respecting the \( x \leftrightarrow y \) symmetry. The following connections

\[
\Gamma_{s8}^s = 2\Gamma_{s4}^4 = \frac{tt''}{t'}
\]

are the only non-vanishing terms in the \((\Gamma)^2\)-parts of \( R_{s8s}^s \), at the origin. The derivatives of the relevant connections are:

\[
\begin{align*}
\partial_8 \Gamma_{s4}^4 &= \frac{1}{2} \left( \frac{t''}{t'} - \left( \frac{t''}{t'} \right)^2 \right), \quad \partial_4 \Gamma_{s8}^s = \frac{4\alpha^2 + 3}{(1 - 4\alpha^2)^2}.
\end{align*}
\]

The only contributions away from the origin come from a linear \( \mu_4 \)-term in \( g_{48} \) and a quadratic \( \mu_4 \)-term in \( g_{88} \) which both appear in \( \partial_4 \Gamma_{s4}^4 \). The vanishing of \( 2R_{s8s}^s \) yields equation (9). I could have chosen \( R_{18s}^1 \) (with \( g_{17}, g_{18}, g_{78} \) contributing) and obtained the same result.

Consider now the following Landau-Ginzburg potential:

\[
W = -(\frac{1}{3} x^3 + \frac{1}{6} y^6) + \mu_9 x y^4 + \mu_8 x y^3 + \mu_7 x y^2 + \mu_6 y^4 + \mu_5 y^3 + \mu_4 x y + \mu_3 x + \mu_2 y^2 + \mu_1 y + \mu_0 \quad (22)
\]

where \( \mu_9 \equiv \alpha \). The relevant terms in the component \( R_{998}^8 \) are:

\[
\begin{align*}
\Gamma_{98}^8 &= \frac{1}{2} \left( \frac{32}{1 - 16\alpha^3} \alpha^2 + \frac{t''}{t'} \right), \quad \Gamma_{99}^9 = \frac{t''}{t'}
\end{align*}
\]

\[
\begin{align*}
\partial_9 \Gamma_{98}^8 &= \frac{1}{2} \left( \frac{t''}{t'} - \frac{t''}{t'} \right) + 16 \partial_9 \left( \frac{\alpha^2}{1 - 16\alpha^3} \right), \quad \partial_9 \Gamma_{99}^9 = \frac{73\alpha + 432\alpha^4}{(1 - 16\alpha^3)^2}.
\end{align*}
\]
The vanishing of $2R_{998}^8$ yields

$$\{t; \alpha\} = 2\alpha \frac{41 - 80\alpha^3}{(1 - 16\alpha^3)^2}. \quad (24)$$

With the variable change $x = \frac{1}{16}\alpha^{-3}$ equation (24) becomes

$$\{t; x\} = \frac{3}{8} \frac{1}{x^2} + \frac{1}{2} \frac{1}{(x - 1)^2} - \frac{31}{72} \frac{1}{x(x - 1)}. \quad (25)$$

The solutions of equation (25) can be expressed as a ratio of two independent solutions of the hypergeometric equation

$$[x(x - 1)\frac{d^2}{dx^2} + (\frac{3}{2}x - \frac{1}{2})\frac{d}{dx} + \frac{5}{144}]y = 0. \quad (26)$$

A possible choice of such independent solutions of equation (26) is:

$$F(5/12, 1/12; 1/2; x), \quad x^\frac{3}{2}F(11/12, 7/12; 3/2; x).$$

3 Conclusion

The method of section 1 for calculating the perturbed correlators is systematic but tedious. It becomes quickly impractical when the number of functions $f_n$ rise sharply. Consider for instance the $x^3 + y^6$-potential. The total number of functions $f_n$ to be considered is well above 200!! Another obstacle arises while obtaining the consistency conditions for the $f_n$; equations (5) generate many redundant equations (the number of functions $\gamma_i$ is smaller but their consistency conditions have a more complicated form). And one often needs to consider equations involving higher order functions to get a closed set of equations for the three and four-point functions. One then solves for $\alpha(t)$, the coefficient of the marginal perturbation. Identifying $\alpha$, for the same model, as a ratio of perturbed three-point functions is difficult: there are ten three-point functions and one has to consider some four-point functions to find the differential equation sought. In the $x^4 + y^4$-model the number of unknowns was substantially reduced by the $x \leftrightarrow y$ symmetry but this is not the case in general.

The flat metric method is thus markedly more direct. While I have applied the technique to determining the function $\alpha(t)$, one can easily determine the other functions $\gamma_i(t)$ in a general perturbation of the potential. Using the flatness of the metric also has one other significant advantage: the technique can be used very selectively, in that it is easy to apply it to solving for one particular unknown in the potential.

Acknowledgement

I would like to thank N. Warner for many enlightening discussions, for reading the manuscript, and for advising me of some of the preliminary results of [5].
Note Added

I have been advised \footnote{9} that there is some unpublished work by N. Noumi in which the flat coordinates for elliptic singularities have also been calculated.

References

[1] R. Dijkgraaf, E. Verlinde, H. Verlinde, ‘Topological Strings in $d < 1$ ’, IAS-preprint IASSNS-HEP-90/71 (Oct. 1990).

[2] E. Verlinde, N.P. Warner, ‘Topological Landau-Ginzburg Matter at $c = 3$’, USC-preprint USC-91/005, IASSNS-HEP-91/16.

[3] A. Strominger, ‘Special Geometry’, Santa-Barbara-preprint UCSBTH-89-61.

[4] C. Vafa, ‘Topological Landau-Ginzburg Models’, HUTP-preprint HUTP-90/A084.

[5] W. Lerche, D.-J. Smit, N.P. Warner, Differential Equations for Periods and Flat Coordinates in two-dimensional Topological Matter Theories, USC-preprint USC/91-022, LBL-31104, CALT-68-1738.

[6] A. Giveon and D.-J. Smit, Mod. Phys. Lett. A 6 (24) (1991) 2211.

[7] B. Blok and A. Varchenko, ‘Topological Conformal Field Theories and the Flat Coordinates’, IAS-preprint IASSNS-HEP-91/5.

[8] K. Saito, Publ. RIMS, Kyoto Univ., 19 (1983) 1231; K. Saito, Publ. RIMS, 26 (1990) 15.

[9] B. Blok, Private Communication.