NON-SOLVABLE GROUPS GENERATED BY INVOLUTIONS IN WHICH EVERY INVOLUTION IS LEFT 2-ENGEL

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Abstract. The following problem is proposed as Problem 18.57 in [The Kourovka Notebook, No. 18, 2014] by D. V. Lytkina:
Let $G$ be a finite 2-group generated by involutions in which $[x, u, u] = 1$ for every $x \in G$ and every involution $u \in G$. Is the derived length of $G$ bounded?
The question is asked of an upper bound on the solvability length of finite 2-groups generated by involutions in which every involution (not only the generators) is also left 2-Engel. We negatively answer the question.

1. Introduction and Result

The following problem is proposed as Problem 18.57 of [2] by D. V. Lytkina:

Question 1.1. Let $G$ be a finite 2-group generated by involutions in which $[x, u, u] = 1$ for every $x \in G$ and every involution $u \in G$. Is the derived length of $G$ bounded?

Question 1.1 is asked of an upper bound on the solvability length of finite 2-groups generated by involutions in which all involutions of groups (not only the generators) are also left 2-Engel elements. We negatively answer the question. In the proof we need some well-known facts about the groups of exponent 4.

It is known that groups of exponent 4 are locally finite [6] and the free Burnside group $\mathcal{B}$ of exponent 4 with infinite countable rank is not solvable [5]. In [1], it is proved that the solvability of $\mathcal{B}$ is equivalent to the one of the group $H$ defined as follows:

Let $H$ be the freest group generated by elements $\{x_i \mid i \in \mathbb{N}\}$ with respect to the following relations:
1. $x_i^2 = 1$ for all $i \in \mathbb{N}$;
2. The normal closure $\langle x_i \rangle^H$ is abelian for all $i \in \mathbb{N}$.
3. $h^4 = 1$ for all $h \in H$.

Therefore $H$ is a non-solvable group of exponent 4 generated by involutions $x_i$ ($i \in \mathbb{N}$). Note that the relation (2) above is equivalent to say that $x_i$ is a left 2-Engel element of $H$ that is $[x, x_i, x_i] = 1$ for all $x \in H$. We do not know if $H$ has the property requested in Question 1.1, that is, whether every involution $u \in H$ is a left 2-Engel element of $H$. Instead we find a quotient of $H$ which is still non-solvable but it satisfies the latter property. The latter quotient of $H$ will provide a counterexample for Question 1.1. To introduce the quotient we need to recall some definitions and results on right 2-Engel elements.

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For any group $G$, $R_2(G)$ denotes the set of all right 2-Engel elements of $G$, i.e.

$$R_2(G) = \{ a \in G \mid [a, x, x] = 1 \text{ for all } x \in G \}.$$ 

It is known [3] that $R_2(G)$ is a characteristic subgroup of $G$. The subgroup $R_2(G)$ is a 2-Engel group that is $[x, y, y] = 1$ for all $x, y \in R_2(G)$. Thus $R_2(G)$ is nilpotent of class at most 3 [4] and so it is of solvable length at most 2.

**Theorem 1.2.** Let $H$ be the freest group defined above. Then $\overline{H} = H/R_2(H)$ satisfies the following condition:

(4) all involutions $u \in \overline{H}$ are left 2-Engel in $\overline{H}$.

Furthermore, $\overline{H}$ is not solvable so that there is no upper bound on solvability lengths of finite 2-groups $\overline{H}_n = \frac{(x_1, \ldots, x_n)}{R_2((x_1, \ldots, x_n))}$ which satisfy all conditions (1), (2), (3) and (4) above.

2. **Proof of Theorem 1.2**

The following is the key lemma of the paper.

**Lemma 2.1.** Let $G$ be any group of exponent 4 and $b \in G$ is such that $b^2 \in R_2(G)$. Then $[a, b, b] \in R_2(G)$ for all $a \in G$. This means that, in every group $G$ of exponent 4, every involution of the quotient $G/R_2(G)$ has an abelian normal closure.

**Proof.** Let $N$ be the freest group generated by elements $a, b, c$ subject to the following relations:

(1) $x^4 = 1$ for all $x \in N$;
(2) $[b^2, x, x] = 1$ for all $x \in N$.

By [6] it is known that $N$ is finite. Now by nq package [7] one can construct $N$ in GAP [8] by the following commands:

```gap
LoadPackage("nq");
F:=FreeGroup(4);
a:=F.1;b:=F.2;c:=F.3;x:=F.4;
G:=F/[x^4,LeftNormedComm([b^2,x,x])];
N:=NilpotentQuotient(G,[x]);
gen:=GeneratorsOfGroup(G,[x]);
LeftNormedComm([gen[1],gen[2],gen[2],gen[3],gen[3]]);
```

Note that in above gen[1], gen[2] and gen[3] correspond to the free generators $a$, $b$ and $c$, respectively. The output of last command in above (which is id the trivial element of $N$) shows that $[a, b, b, c] = 1$. This completes the proof. □

It may be interesting in its own right that the group $N$ defined in the proof of Lemma 2.1 is nilpotent of class 7 and order $2^{41}$.

**Proof of Theorem 1.2.** It follows from Lemma 2.1 that $\overline{H} = H/R_2(H)$ has the property (4) mentioned in the statement of Theorem 1.2. Since $H$ is not solvable by [1] and [5], it follows that there is no upper bound on the solvable lengths of finite 2-groups $\overline{H}_n$. By construction $\overline{H}_n$ is generated by involutions and by Lemma 2.1 all involutions in $\overline{H}_n$ are left 2-Engel. This completes the proof. □
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