THE LOOP PROBLEM FOR REES MATRIX SEMIGROUPS

Mark Kambites
School of Mathematics, University of Manchester
Manchester M60 1QD, England
Mark.Kambites@manchester.ac.uk

Abstract. We study the relationship between the loop problem of a semigroup, and that of a Rees matrix construction (with or without zero) over the semigroup. This allows us to characterize exactly those completely zero-simple semigroups for which the loop problem is context-free. We also establish some results concerning loop problems for subsemigroups and Rees quotients.

1. Introduction

A key facet of both combinatorial group theory and combinatorial semigroup theory is the study of the word problem, that is, the problem of deciding algorithmically whether two words in the generators of a group or semigroup represent the same element [6]. In a group, deciding whether two words represent the same element is algorithmically equivalent to deciding whether one word represents the identity. Hence, the word problem can be viewed from an alternative perspective, as the formal language of all words over the generators which represent the identity. Perhaps surprisingly, there are number of strong results correlating structural properties of groups with language-theoretic properties of their word problems (see, for example, [9, 10, 11, 17, 24, 29]).

While many major group-theoretic results about the word problem as a decision problem have been successfully generalised to semigroups [3, 4, 25], and in some cases beyond [2, 18, 19, 20], it is not so clear how one might apply the language-theoretic approach in a more general setting. In [14], we introduced and began the study of a new way of associating to each finitely generated monoid or semigroup a formal language, called the loop problem. In the case of a group the loop problem is essentially the same as the word problem, and for more general semigroups, it still encodes the structure of the semigroup in a similar way.

The Rees matrix construction, originally used by Rees [28] to give a combinatorial description of completely simple and completely zero-simple semigroups in terms of their maximal subgroups, has become one of the most pervasive ideas in semigroup theory. The groups in the construction can be replaced with more general semigroups [11, 17, 23, 26] or even partial algebras such as small categories and semigroupoids [15, 16, 21], with the resulting construction used either to describe an interesting class of semigroups, or to embed semigroups into other semigroups with nicer properties.
In [14] we started to explore the relationship between the loop problem and a special case of the Rees matrix construction, by demonstrating a connection between the loop problems of completely simple semigroups and the word problems of their maximal subgroups. The main aim of the present paper is to develop this study more systematically, by considering Rees matrix constructions in more generality. In the process, we also establish some results regarding the loop problems of subsemigroups and Rees quotients.

In addition to this introduction, this paper comprises five sections. In Section 2 we briefly recall the definition of the loop problem and the Rees matrix construction, and some basic results concerning the main problem of a semigroup and that of certain of its subsemigroups and Rees quotients. Sections 3 and 4 prove our main theorems about the loop problem and Rees matrix semigroups. Finally, in Section 6 we consider the implications of our main results for the class of completely zero-simple semigroups.

2. Preliminaries

In this section we briefly recall the definition of the loop problem of a monoid or semigroup, and of the Rees matrix construction. We assume the reader to be familiar with basic definitions and foundational results from semigroup theory and from the theory of (both finite and infinite) automata and formal languages. A comprehensive introduction to the loop problem, including a brief recap of the required automata-theoretic prerequisites, can be found in [14]. More detailed introductions to automata and formal languages can be found in [12, 22, 27], while the standard introductions to semigroup theory are [5, 13].

A choice of (monoid) generators for a monoid \( M \) is a surjective morphism \( \sigma : X^* \to M \) from a free monoid onto \( M \). Similarly, a choice of (semigroup) generators for a semigroup \( S \) is a surjective morphism \( \sigma : X^+ \to S \) from a free semigroup onto \( S \). Such a choice of generators is called finite if \( X \) is finite, and a semigroup or monoid which admits a finite choice of generators is called finitely generated.

Recall that the (right) Cayley graph \( \Gamma_\sigma(M) \) of a monoid \( M \) with respect to a choice of generators \( \sigma : X^* \to M \) is a directed graph, possibly with multiple edges and loops, with edges labelled by elements of \( X \). Its vertices are the elements of \( M \), and it has an edge from \( a \in M \) to \( b \in M \) labelled \( x \in X \) exactly if \( a(x) = b \) in the monoid \( M \).

Now let \( \overline{X} = \{ \overline{x} \mid x \in X \} \) be a set of formal inverses for the generators in \( X \), and let \( \bar{X} = X \cup \overline{X} \). We extend the map \( x \to \overline{x} \) to an involution on \( \bar{X} \) by defining \( \overline{x} = x \) for all \( x \in X \), and \( \overline{x_1 \ldots x_n} = \overline{x_n} \ldots \overline{x_1} \) for all \( x_1, \ldots, x_n \in \bar{X} \). The (right) loop automaton \( \bar{\Gamma}_\sigma(M) \) of \( M \) with respect to \( X \) is obtained from the Cayley graph \( \Gamma_\sigma(M) \) by adding for each edge labelled \( x \) an inverse edge, in the opposite direction, labelled \( \overline{x} \). Notice that in the loop automaton, \( w \in \bar{X} \) labels a path from \( p \) to \( q \) if and only if \( \overline{w} \) labels a path from \( q \) to \( p \). We view the loop automaton as a (typically infinite) automaton over \( \bar{X} \), with start state and terminal state the identity of \( M \). The (right) loop problem of \( M \) with respect to \( \sigma \) is the language \( L_\sigma(M) \subseteq \bar{X}^* \) of words recognised by the loop automaton \( \bar{\Gamma}_\sigma(M) \).
Words in $X^*$ and $\overline{X}^*$ are called positive and negative words respectively; words in $\overline{X}^*$ which are neither positive nor negative are called mixed. Similarly, an edge or path in $\tilde{\Gamma}_\sigma(M)$ is called positive [negative, mixed] if it has a positive [respectively negative, mixed] label.

Now let $S$ be a semigroup, and let $S^1$ be the monoid obtained from $S$ by adjoining an identity 1 (even if $S$ already has an identity). Then a choice of semigroup generators $\sigma : X^+ \to S$ extends uniquely to a choice of monoid generators $\sigma^1 : X^* \to S^1$. We define the loop automaton $\tilde{\Gamma}_\sigma(S)$ and the loop problem $L_\sigma(S)$ of $S$ with respect to $\sigma$ to be respectively the loop automaton $\tilde{\Gamma}_{\sigma^1}(S^1)$ and the loop problem $L_{\sigma^1}(S^1)$.

We now turn our attention to the Rees matrix construction. Let $S$ be a semigroup, $I$ and $J$ sets, 0 a symbol not in $S$ and $P$ an $J \times I$ matrix with entries drawn from $S \cup \{0\}$. The Rees matrix semigroup with zero $M^0(S; I, J; P)$ is the semigroup with set of elements $(I \times S \times J) \cup \{0\}$ and multiplication given by

$$(i_1, g_1, j_1)(i_2, g_2, j_2) = \begin{cases} (i_1, g_1P_{j_1i_2}g_2, j_2) & \text{if } P_{j_1i_2} \in S \\ 0 & \text{if } P_{j_1i_2} = 0. \end{cases}$$

If $P$ contains no 0 entries, $I \times S \times J$ forms a subsemigroup of $M^0(S; I, J; P)$, called a Rees matrix semigroup (without zero) and denoted $M(S; I, J; P)$.

3. Subsemigroups and Ideals

In this section we prove some basic results relating the loop problem of a semigroup to the loop problems of certain of its subsemigroups and quotients.

Let $S$ be a semigroup with a subsemigroup $T$ such that $STS \subseteq T$; such a subsemigroup $T$ is called an ideal of $S$. The Rees quotient of $S$ by $T$ is the semigroup with set of elements $S \setminus T \cup \{0\}$ and multiplication given by

$$st = \begin{cases} 0 & \text{if } s = 0 \text{ or } t = 0; \\ \text{the } S\text{-product } st & \text{otherwise}. \end{cases}$$

Some authors use the symbol $T$ to denote the zero element 0. The map

$$S \to S/T, \ s \mapsto \begin{cases} s & \text{if } s \notin T \\ 0 & \text{if } s \in T. \end{cases}$$

is clearly a morphism from $S$ onto $S/T$. If $\sigma : X^+ \to S$ is a choice of generators for $S$, then composition with this map induces a choice of generators for $S/T$, which we denote by $\sigma/T : X^+ \to S/T$. The map $S \to S/T$ also induces a label-preserving graph morphism from $\tilde{\Gamma}_\sigma(S)$ onto $\tilde{\Gamma}_{\sigma/T}(S/T)$.

Recall that a subset of a semigroup $S$ is called rational if $S$ is the image in $S$ of a regular language under some morphism from a finitely generated free monoid to $S$, or equivalently, under every morphism from a finitely generated free monoid to $S$. A rational subset which is also an ideal is called a rational ideal.
Proposition 3.1. Let $\mathcal{F}$ be a family of languages closed under product, union, Kleene closure and division by finite [respectively, regular] languages. Suppose $\sigma : X^* \to S$ is a choice of generators for a semigroup $S$, and that $L_\sigma(S)$ belongs to $\mathcal{F}$. If $T$ is a finite [respectively, rational] ideal of $S$ then $L_{\sigma/T}(S/T)$ belongs to $\mathcal{F}$.

Proof. Clearly, a path from 1 to 1 in $\hat{\Gamma}_{\sigma/T}(S/T)$ consists of either

(i) a path from 1 to 1 which does not visit 0, and hence which is the image of a path from 1 to 1 in $\hat{\Gamma}_\sigma(S)$ with the same label; or

(ii) the concatenation of

- a path from 1 to 0, which lifts to a path with the same label from 1 to an element of $T$ in $\hat{\Gamma}_\sigma(S)$,
- zero or more paths from 0 to 0, each of which lifts to a path from an element of $T$ to an element of $T$; and
- a path from 0 to 1, which lifts to a path with the same label from an element of $T$ to 1 in $\hat{\Gamma}_\sigma(S)$.

It follows that

$$L_{\sigma/T}(S/T) = L_\sigma(S) \cup L_{1T}L_{TT}L_{T1}$$

where $L_{1T}$, $L_{TT}$ and $L_{T1}$ denote the sets of words which label paths in $\hat{\Gamma}_\sigma(S)$ from 1 to an element of $T$, between two elements of $T$, and from an element of $T$ to 1, respectively.

Since $T$ is finite [rational], there exists a finite [respectively, regular] language $R \subseteq X^+$ with $R\sigma = T$. Let $R = \{r \mid r \in R\}$; then $R$ is also finite [respectively, regular]. Notice that, for any word $r \in X^+$, we have that $r$ labels a path from 1 to an element of $T$ exactly if $r \in R$, and hence that $\overline{r}$ labels a path from an element of $T$ to 1 if and only $\overline{r} \in R$. It follows that

$$L_{1T} = \{w \in \hat{X}^* \mid w\overline{r} \in L_\sigma(S) \text{ for some } r \in R\} = L_\sigma(S)\overline{R}^{-1}.$$ 

Similarly, we have and

$$L_{T1} = \{w \in \hat{X}^* \mid rw \in L_\sigma(S) \text{ for some } r \in R\} = R^{-1}L_\sigma(S)$$

and

$$L_{TT} = \{w \in \hat{X}^* \mid r_1wr_2 \in L_\sigma(S) \text{ for some } r_1, r_2 \in R\} = R^{-1}L_\sigma(S)\overline{R}^{-1}.$$ 

The claim now follows immediately from the presumed closure properties of $\mathcal{F}$. $\square$

Next, we consider loop problems of subsemigroups. We say that a subsemigroup $T$ of $S$ is pseudo-right-unitary if for every element $a \in S$ there exists an element $b \in T$ such that for each element $x \in T$ with $ax \in T$, we have $ax = bx$. Intuitively, $T$ is pseudo-right-unitary if the partial action of $S$ on $T$ by left multiplication is no more complex than the action of $T$ on itself by left multiplication. Note in particular that if $T$ is right unitary then $x \in T$ and $ax \in T$ together imply that $a \in T$, so that right unitary subsemigroups (and in particular unitary subsemigroups) are pseudo-right-unitary.

In fact, a weaker condition will suffice for our purpose. A subsemigroup is weakly pseudo-right-unitary if for every element $a \in S$ and pair of elements $x, y \in T$ such that $ax \in T$, there exists $b \in T$ with $ax = bx$ and $ay = by$. 
Proposition 3.2. Let $T$ be a finitely generated weakly pseudo-right-unitary subsemigroup of a semigroup $S$. Let $\sigma : X^+ \to T$ and $\tau : Y^+ \to S$ be choices of generators such that $X \subseteq Y$ and $\sigma$ is the restriction of $\tau$ to $X^*$. Then $L_\sigma(T) = L_\tau(S) \cap \hat{X}^*$.

Proof. The proof is similar to that of [14] Theorem 5.4. Clearly, $\hat{\Gamma}_\sigma(T)$ is embedded into $\hat{\Gamma}_\tau(S)$, so that any word accepted by the former is also accepted by the latter, and $L_\sigma(T) \subseteq L_\tau(S) \cap \hat{X}^*$.

Conversely, suppose $w$ is a word in $L_\tau(S) \cap \hat{X}^*$. Then $\hat{\Gamma}_\tau(S) = \hat{\Gamma}_{\tau_1}(S^1)$ has a loop at 1 labelled $w$. Write $w = u_0 \overline{v_1} u_1 \overline{v_2} \ldots u_{n-1} \overline{v_n}$ with each $u_i, v_i \in X^*$. By the Zig Zag Lemma [11] Lemma 4.1], there exist $p_0, \ldots, p_n \in S^1$ such that $p_0 = p_n = 1$ and $p_i(u_i \tau) = p_{i+1}(v_{i+1} \tau)$ for $0 \leq i < n$.

We claim that the elements $p_0, \ldots, p_n$ can all be chosen to lie in $T \cup \{1\}$. Indeed, suppose we are given $p_0, \ldots, p_n$ satisfying the above equations and not all lying in $T \cup \{1\}$, and let $k$ be minimal such that $p_k \notin T \cup \{1\}$. Clearly, $1 \leq k \leq n - 1$. Now we have $u_{k-1} \tau \in T$ and $p_{k-1} \in T \cup \{1\}$, so that $p_{k-1}(u_{k-1} \tau) = p_k(v_k \tau)$ lies in $T$. Since $v_k \tau$ and $u_k \tau$ also lie in $T$ and $T$ is weakly pseudo-right-unitary, there exists an element $q_k \in T$ with $q_k(v_k \tau) = p_k(v_k \tau) = p_{k-1}(u_{k-1} \tau)$ and $q_k(u_k \tau) = p_k(u_k \tau) = p_{k+1}(v_{k+1} \tau)$. Replacing $p_k$ with $q_k$, we obtain a sequence $p_0, \ldots, p_n$ with strictly fewer elements outside $T \cup \{1\}$, and continuing in this way we eventually obtain a sequence with the desired properties. Another application of the Zig Zag Lemma shows that $\hat{\Gamma}_{\sigma_1}(T^1)$ has a loop at 1 labelled $w$, so that $w \in L_\sigma(T)$, as required.

We denote by $S^0$ the semigroup obtained from $S$ by adjoining a zero. Since $S$ is a right unitary subsemigroup of $S^0$, Proposition 3.2 has the following immediate corollary.

Corollary 3.3. Let $\sigma : X^+ \to S$ be a choice of generators for a semigroup $S$. Let $Y = X \cup \{z\}$ and let $\tau : Y^+ \to S^0$ be the (unique) extension of $\sigma$ to a choice of generators for $S^0$. Then $L_\sigma(S) = L_\tau(S^0) \cap \hat{X}^*$.

Conversely, we can also obtain the loop problem of $S^0$ from that of $S$, but to do so we need some more operations.

Theorem 3.4. Let $\mathcal{F}$ be a family of languages closed under union, product, Kleene closure and rational transductions, and let $S$ be a semigroup. Then the loop problem for $S$ belongs to $\mathcal{F}$ if and only if the loop problem for $S^0$ belongs to $\mathcal{F}$.

Proof. Let $\sigma : X^+ \to S$ be a choice of generators for $S$, let $Y = X \cup \{z\}$ and let $\tau : Y^+ \to S^0$ be the unique extension of $\sigma$ to a choice of generators for $S^0$. Since intersection with a regular language can be performed by a rational transduction, one implication is immediate from Corollary 3.3.

Conversely, the loop automaton $\hat{\Gamma}_{\sigma}(S^0)$ consists of the loop automaton $\hat{\Gamma}_{\sigma}(S)$ with an extra vertex 0 adjoined. In addition to the edges in $\hat{\Gamma}_{\sigma}(S)$, it has an edge from every vertex to 0 labelled $z$, a corresponding edge from 0 to every vertex labelled $\tau$, and an edge from 0 to 0 labelled $x$ for each $x \in \hat{X}$. 


We define languages
\[ L_{10} = \{ uz \mid u \text{ is a prefix of a word in } L_\sigma(S) \}, \]
\[ L_{00} = \{ zuz \mid u \text{ is a factor of a word in } L_\sigma(S) \}, \]
and
\[ L_{01} = \{ zu \mid u \text{ is a suffix of a word in } L_\sigma(S) \}. \]

It is a routine exercise to verify that \( L_{10}, L_{00} \) and \( L_{01} \) can be obtained using rational transductions from \( L_\sigma(S) \), and hence lie in \( \mathcal{F} \). One can also check that
1. \( L_{10} \) is the set of all words labelling paths in \( \hat{\Gamma}_\tau(S^0) \) from 1 to 0 which do not visit 0 except at the end;
2. \( L_{00} \) is the set of all words labelling paths in \( \hat{\Gamma}_\tau(S^0) \) from 0 to 0 which do not visit 0 except at the beginning and end; and
3. \( L_{01} \) is the set of all words labelling paths in \( \hat{\Gamma}_\tau(S^0) \) from 0 to 1 which do not visit 0 except at the beginning.

Now if \( \pi \) is a path from 1 to 1 in \( \hat{\Gamma}_\tau(S^0) \), then \( \pi \) either does or does not visit the vertex 0. If \( \pi \) does not visit zero then it is entirely contained in \( \hat{\Gamma}_\sigma(S) \) and its label lies in \( L_\sigma(S) \). If \( \pi \) does visit 0 then it must decompose into a path from 1 to 0 which does not visit 0 except at the end, followed by zero or more paths from 0 to 0 which do not visit 0 except at the end, followed by a path from 0 to 1 which does not visit 0. It follows that
\[ L_\tau(S^0) = L_\sigma(S) \cup L_{10} L_{00}^* L_{01}. \]

Thus, \( L_\tau(S^0) \) can be obtained from languages in \( \mathcal{F} \) using the operations of union, product and Kleene closure, and so itself lies in \( \mathcal{F} \). \( \square \)

4. From Semigroups to Rees Matrix Semigroup

In this section, we show how to describe the loop problem of a finitely generated Rees matrix semigroup in terms of the loop problem of the underlying semigroup. We first do this for Rees matrix constructions without zero; we shall subsequently apply one of the results of Section 3 extend this result to the case of Rees matrix constructions with zero.

**Theorem 4.1.** Let \( S \) be a semigroup and \( M = M(S; I, J; P) \) a finitely generated Rees matrix semigroup without zero. Then the loop problem of \( M \) is the Kleene closure of a rational transduction of the loop problem of \( S \).

**Proof.** By [1, Main Theorem], we may assume that \( I \) and \( J \) are finite and \( S \) is finitely generated. Let \( \sigma : X^+ \to S \) be a finite choice of generators for \( S \), and \( \tau : Y^+ \to M \) a finite choice of generators for \( M \). We shall show that \( L_\tau(M) \) is a rational transduction of \( L_\sigma(S) = L_\sigma(S^1) \); the argument is a refinement of that used to prove [14, Theorem 5.5].

For each \( y \in Y \), suppose \( y\tau = (i_y, g_y, j_y) \) and let \( w_y \in X^+ \) be a word representing \( g_y \in S \). For each \( i \in I \) and \( j \in J \), let \( w_{ji} \in X^+ \) be a word representing \( P_{ji} \in S \).

We define a finite state transducer from \( X^* \) to \( Y^* \) with
1. vertex set \( (I \times J) \cup \{ A, Z \} \) where \( A \) and \( Z \) are new symbols;
2. initial state \( A \);
3. terminal state \( Z \);
4. for each generator \( y \in Y \), an edge from \( A \) to \( (i_y, j_y) \) labelled \( (w_y, y) \);
• for each generator \( y \in Y \), an edge from \((i_y, j_y)\) to \(Z\) labelled \((w_y, y)\);
• for each generator \( y \in Y \), each \( k \in J \) and each \( i \in I \), an edge from \((i, k)\) to \((i, j_y)\) labelled \((w_{ki_y}w_y, y)\); and
• for each generator \( y \in Y \), each \( j \in J \) and each \( i \in I \), an edge from \((i, j_y)\) to \((i, j)\) labelled \((w_yw_{ji_y}, \overline{y})\).

We say that a path starting at 1 in \(\hat{\Gamma}_\sigma(M)\) is non-returning if it does not visit the vertex 1 except at the start and (possibly) the end. Now let \( g \in S \), \( i \in I \), \( j \in J \) and \( v \in \hat{Y}^+ \) and suppose \( n \) is a positive integer. We claim that the following conditions are equivalent:

(i) the loop automaton \(\hat{\Gamma}_\sigma(M)\) has a non-returning path of length \( n \) from 1 to \((i, g, j)\) labelled \(v\);

(ii) the transducer has a path of length \( n \) from \(A\) to \((i, j)\) labelled \((u, v)\)

for some \( u \in X^+ \) such that the loop automaton \(\hat{\Gamma}_\sigma(S)\) has a non-returning path of length \( n \) from 1 to \( g \) labelled \( u \).

We prove this claim by induction on \( n \). The case \( n = 1 \) is immediate from the definition of the transducer, so suppose \( n > 1 \) and that the claim holds for smaller \( n \).

Suppose first that (i) holds, and let \( \pi \) be the path given by the hypothesis. Let \( e \) be the last edge the path \( \pi \), and let \( \pi' \) be the path \( \pi \) with the last edge removed, so that \( \pi = \pi'e \). Let \( \ell \) be the label of \( \pi' \). Since \( n > 1 \) and \( \pi \) is non-returning, the path \( \pi' \) must end at a vertex of the form \((i', g', k)\).

It follows easily from the definition of the multiplication in a Rees matrix semigroup that the vertices in the loop automaton corresponding to elements with first coordinate \( i \) are connected to the rest of the automaton only via the vertex 1. Hence, since the path \( \pi \) is non-returning, we must have \( i = i' \), so that \( \pi \) actually ends at \((i, g', k)\). Now \( \pi \) is a path of length \( n - 1 \), so by the inductive hypothesis, the transducer has a path of length \( n - 1 \) from \( A \) to \((i, k)\) labelled \((u', v')\) for some word \( u' \in X^+ \) such that \(\hat{\Gamma}_\sigma(S)\) has a path from 1 to \( g' \) labelled \( u' \).

We treat separately the case where \( e \) is a positive edge, and that where \( e \) is a negative edge. Suppose first that \( e \) is a positive edge, with label \( y \in Y \). Then clearly \( j = j_y \) and by definition the transducer has an edge from \((i, k)\) to \((i, j)\) with label \((w_{ki_y}w_y, y)\). Hence, the transducer has a path of length \( n \) from 1 to \((i, j)\) with label

\[
(u', v')(w_{ki_y}w_y, y) = (u'w_{ki_y}w_y, v'y) = (u'w_{ki_y}w_y, v).
\]

Let \( u = u'w_{ki_y}w_y \). Now from the definition of the loop automaton, we must have

\[
(i, g, j) = (i, g', k)(y\tau) = (i, g', k)(i_y, g_y, j_y) = (i, g'w_{ki_y}, g_y, j_y).
\]

Equating second coordinates, we see that \( g = g'w_{ki_y}w_y \).

Now since \( w_{ki_y}w_y \in X^+ \) represents \( P_{ki_y}g_y \), it follows that \(\hat{\Gamma}_\sigma(S)\) has a path from \( g' \) to \( g \) labelled \( w_{ki_y}w_y \). Combining with the path given by the inductive hypothesis, we see that \(\hat{\Gamma}_\sigma(S)\) has a path from 1 to \( g \) labelled \( u'w_{ki_y}w_y = u \), which shows that (ii) holds in the case that \( e \) is a positive edge.

Suppose now that \( e \) is a negative edge, with label \( \overline{y} \) for some \( y \in Y \). In this case \( k = j_y \) and the transducer has an edge from \((i, k)\) to \((i, j)\) with
label \((w_y \overline{w_{ji_y}} y)\). Hence, the transducer has a path of length \(n\) from \(A\) to \((i, j)\) with label 
\[(u', v')(w_y \overline{w_{ji_y}} y) = (u' w_y \overline{w_{ji_y}}, v'y) = (u' \overline{w_y \overline{w_{ji_y}}}, v).\]

Let \(u = u' \overline{w_y \overline{w_{ji_y}}}\). Now from the definition of the loop automaton, we must have 
\[(i, g', k) = (i, g', j)(yτ) = (i, g', j)(iy, gy, jy) = (i, g'P_{ji_y}gy, jy).\]

Again equating second coordinates, we see this time that \(gP_{ji_y}gy = g'.\) Since \(w_{ji_y}w_y\) represents \(P_{ji_y}gy\), it follows that \(\hat{L}_\sigma(S)\) has a path from \(g\) to \(g'\) labelled \(w_{ji_y}w_y\), and hence a path from \(g'\) to \(g\) with label 
\[w_{ji_y}w_y = w_y \overline{w_{ji_y}}.\]

Combining with the path given by the inductive hypothesis, we obtain a path from \(1\) to \(g\) labelled \(u = u' \overline{w_y \overline{w_{ji_y}}}\), so that (ii) again holds.

Conversely, suppose (ii) holds, and let \(π\) be a path of length \(n\) in the transducer from \(A\) to \((i, j)\) labelled \((u, v)\) for some \(u \in X^+\) such that \(\hat{Γ}_\sigma(S)\) has a path from \(1\) to \(g\) with label \(u\). Let \(e\) be the last edge of \(π\) and let \(π'\) be the path \(π\) with the final edge removed. Then \(π'\) is a path of length \(n − 1\) from \(A\) to some vertex \((i', k)\) with label of the form \((u', v')\). Moreover, it follows easily from the definition of the transducer that \(i = i'\), so that \(π'\) ends at \((i, k)\). Let \(g' \in G\) be the element represented by \(u'\). Then by the inductive hypothesis, there exists a path of length \(n - 1\) in the loop automaton from \(1\) to \((i, g', k)\) with label \(v'\). Now \(e\) is an edge from \((i, k)\) to \((i, j)\). From the definition of the edges in the transducer, we see that there exists \(y \in Y\) such that either \(jy = j\) and \(e\) has label \((w_{ki_y}w_y, y)\), or else \(jy = k\) and \(e\) has label \((w_y \overline{w_{ji_y}} y)\). As before, we treat these two cases separately.

In the former case, observe that we have \(u = u' w_{ki_y}w_y\) from which we deduce that \(g = g' P_{ki_y}gy\). But now 
\[(i, g', k)(yτ) = (i, g' P_{ji_y}gy, jy) = (i, g, j)\]
so we see that the loop automaton has an edge from \((i, g', k)\) to \((i, g, j)\) labelled \(y\). Combining this with the path whose existence we deduced using the inductive hypothesis, we conclude that the loop automaton has a path from \(1\) to \((i, g, j)\) labelled \(v = v'y\), so that (i) holds as required.

Next we consider the case in which \(jy = k\) and \(e\) has label of the form \((w_y \overline{w_{ji_y}} y)\). Here we have \(u = u' \overline{w_y \overline{w_{ji_y}}}\), so \(\hat{Γ}_σ(S)\) has a path from \(g'\) to \(g\) with label \((w_y \overline{w_{ji_y}} y)\), and hence a path from \(g\) to \(g'\) with label \(w_{ji_y}w_y\). It follows that 
\[(i, g, k)(yτ) = (i, gP_{ji_y}gy, jy) = (i, g', j)\]
so we see that the \(\hat{Γ}_σ(M)\) has an edge from \((i, g, j)\) to \((i, g', k)\) labelled \(y\), and hence an inverse edge from \((i, g', k)\) to \((i, g, j)\) labelled \(y\). Combining this with the path whose existence we deduced using the inductive hypothesis, we conclude that the loop automaton has a path from \(1\) to \((i, g, j)\) labelled \(v = v'y\), so that (i) holds as required. This completes the proof that (i) and (ii) are equivalent.

Now let \(K\) be the set of all non-empty words labelling non-returning loops at \(1\) in \(\hat{Γ}_σ(M)\). We claim that \(K\) is exactly the image of \(L_σ(S)\) under the
transduction defined by our transducer. Since $L_{\tau}(M)$ is clearly the Kleene closure of $K$, this will suffice to complete the proof.

Suppose first that $v \in K$. Then by definition the loop automaton $\hat{\Gamma}_{\tau}(M)$ has a non-returning loop at 1 labelled $v$. Note that all edges in $\Gamma_{\tau}(M)$ which end at 1 run from vertices corresponding to generators $y$ and have label $\overline{v}$, so we may assume that the last edge of the path runs from $y\tau$ to 1, and has label $\overline{y}$. Let $\pi$ be the path without this last edge, so that $\pi$ runs from 1 to $y\tau$, and let $v' \in \hat{Y}^+$ be the label of this path, so that $v = v'\overline{y}$. Then by the equivalence above, the transducer has a path from $A$ to $(i_y, j_y)$ labelled $(u, v')$ for some $u \in \hat{X}^+$ such that $u$ labels a path in $\hat{\Gamma}_\sigma(S)$ from 1 to $g_y$. But directly from the definition, the transducer also has an edge from $(i_y, j_y)$ to $Z$ labelled $(\overline{wu}_y, \overline{v})$. Hence, we deduce that $(u\overline{wu}_y, v)$ is accepted by the transducer, where $u\overline{wu}_y$ labels a path from 1 to 1 in $\hat{\Gamma}_\sigma(S)$. Thus, $v$ lies in the image under the transduction of $L_\sigma(S)$.

Conversely, suppose $u \in \hat{X}^+$ lies in the loop problem $L_\sigma(S)$, and that the transducer accepts $(u, v)$. Then the transducer has a path $\pi$ from $A$ to $Z$ labelled $(u, v)$. Again, we proceed by letting $\pi'$ be the path obtained from $\pi$ by deleting the last edge. Then there must exist a generator $y$ such that $\pi'$ ends at $(i_y, j_y)$. Moreover, $\pi'$ must be labelled $(u', v')$ where $u = u'\overline{w}_y$ and $v = v'\overline{y}$. Now since $u \in L_\sigma(S)$ and $w_y$ represents $g_y$, we deduce that $u'$ labels a path from 1 to $g_y$ in $L_\sigma(S)$. By the equivalence above, it follows that the loop automaton $\hat{\Gamma}_{\tau}(S)$ has a non-returning path from 1 to $(i_y, g_y, j_y)$ labelled $v'$. Now it certainly also has an edge from $(i_y, g_y, j_y)$ to 1 labelled $\overline{v}$, so we deduce that $v = v'\overline{y} \in K$, which completes the proof.

We now use some results from Section 3 to establish a result corresponding to Theorem 4.1 for the case of Rees matrix constructions with zero.

**Theorem 4.2.** Let $\mathcal{F}$ be a family of languages closed under rational transduction, union, product and Kleene closure. Let $S$ be a semigroup and let $M = M^0(S; I, J; P)$ be a finitely generated Rees matrix semigroup with zero over $S$. If the loop problem for $S$ belongs to $\mathcal{F}$ then the loop problem for $M$ belongs to $\mathcal{F}$.

**Proof.** Let $S^0$ denote the semigroup $S$ with an additional zero element 0 adjoined, and let $M'$ be the Rees matrix semigroup (without zero) $M' = M(S^0; I, J; P)$. By Theorem 3.4, the loop problem for $S^0$ belongs to $\mathcal{F}$, and so by Theorem 4.1 the loop problem for $M'$ also belongs to $\mathcal{F}$.

Now let
\[ T = I \times \{0\} \times J \subseteq M'. \]
Since $M$ is finitely generated, we deduce by [1 Main Theorem] that $I$ and $J$ are finite, and hence that $T$ is finite. It is readily verified that $T$ is an ideal of $M'$ and that $M$ is isomorphic to the Rees quotient $M'/T$.

Now since the operations of union and division by finite languages can easily be realised as rational transductions, we may assume that $\mathcal{F}$ is closed under these operations as well as product and Kleene closure. Thus, Proposition 3.1 tells us that the loop problem for $M'/T$, and hence that for $M$, belongs to $\mathcal{F}$ as required. □
5. From Rees Matrix Semigroups to Subsemigroups

Our objective in this section is to describe the loop problem for a semigroup $S$ in terms of the loop problem of a Rees matrix semigroup $M = M(S; I, J; P)$ or $M = M^0(S; I, J; P)$. In the most general case this is not possible, since $M$ may not contain sufficient information about $S$. For example, if the entries in the sandwich matrix $P$ are all drawn from an ideal of $S$, then $M$ will clearly retain no information about the structure of $S$ outside this ideal, beyond its cardinality. Thus, for our results in this section it will be necessary to impose some restrictions on the elements of the sandwich matrix, and the ideals they generate.

**Proposition 5.1.** Let $M = M(S; I, J; P)$ or $M = M^0(S; I, J; P)$ be a finitely generated Rees matrix semigroup. Choose $i \in I$ and $j \in J$ with $P_{ji} \neq 0$ and let

$$T = \{(i, s, j) \mid s \in S\} \subseteq M.$$  

If for every $j' \in J$ we have either $P_{j'i} = 0$ or $SP_{j'i} \subseteq SP_{ji}$ then $T$ is a pseudo-right-unitary subsemigroup of $M$.

**Proof.** Clearly, $T$ is a subsemigroup of $M$. Now let $a \in M$. To show that $T$ is pseudo-right-unitary, we must find an element $b \in T$ such that for every element $x \in T$ with $ax \in T$, we have $ax = bx$. If $M$ is a Rees matrix semigroup with zero and $a = 0$ then $ax$ can never lie in $T$, so any element $b \in T$ will fulfill the condition vacuously. So assume $a \neq 0$, say $a = (i_a, s_a, j_a)$.

Now by assumption, either $P_{j_a'i} = 0$, or $SP_{j_a'i} \subseteq SP_{ji}$. In the former case we have $ax = 0$ for all $x \in T$, so that choosing any $b \in T$ again suffices vacuously to fulfill the condition.

In the latter case, where $SP_{j_a'i} \subseteq SP_{ji}$, we can choose an element $s_b \in S$ such that $s_aP_{j_a'i} = s_bP_{ji}$. Let $b = (i, s_b, j)$. Now suppose $x = (i, s_x, j) \in T$ is such that

$$ax = (i_a, s_aP_{j_a'i}s_x, j) \in T.$$  

Then from the definition of $T$, we have $i_a = i$. Now by the choice of $s_b$, we have

$$bx = (i, s_bP_{ji}s_x, j) = (i, s_aP_{j_a'i}s_x, j) = ax$$  

as required to complete the proof. \hfill \Box

Combining Propositions 3.2 and 5.1 immediately yields the following.

**Theorem 5.2.** Let $M = M(S; I, J; P)$ or $M = M^0(S; I, J; P)$ be a finitely generated Rees matrix semigroup. Choose $i \in I$ and $j \in J$ with $P_{ji} \neq 0$ and such that for every $j' \in J$ we have $P_{j'i} = 0$ or $SP_{j'i} \subseteq SP_{ji}$ and let

$$T = \{(i, s, j) \mid s \in S\} \subseteq M.$$  

Let $\sigma : X^+ \to T$ and $\tau : Y^+ \to M$ be choices of generators such that $X \subseteq Y$ and $\sigma$ is the restriction of $\tau$ to $X^+$. Then $L_\sigma(T) = L_\tau(M) \cap \hat{X}^+$.

Suppose now that $S$ is a monoid with identity 1. Recall that an element $g \in S$ is called a unit if there exists an element $g^{-1} \in S$ with $gg^{-1} = g^{-1}g = 1$; the collection of units in $S$ forms a subgroup, called the group of units of $S$. As a corollary of Theorem 5.2 we obtain the following.
The Loop Problem for Rees Matrix Semigroups

Theorem 5.3. Let $M = M(S; I, J; P)$ or $M = M^0(S; I, J; P)$ be a finitely generated Rees matrix semigroup over a monoid $S$. If the sandwich matrix $P$ contains a unit then there exist choices of generators for $S$ and $M$ such that the loop problem for $S$ is the intersection of the loop problem for $M$ with a regular language.

Proof. Choose $i \in I$ and $j \in J$ such that $P_{ji}$ is a unit, and let $T = \{(i, s, j) \mid s \in S\}$. Define $\rho : S \to T$ by $s^\rho = (i, sP^{-1}ji, j)$. It is a routine exercise to verify that $\rho$ is an isomorphism. By [1, Main Theorem], $S$ is finitely generated, and so $T$ is finitely generated. Let $\sigma : X^+ \to S$ be a choice of semigroup generators for $S$; it follows easily that $\sigma \rho : X^+ \to T$ is a choice of semigroup generators for $T$, and that $L_\sigma(S) = L_{\sigma \rho}(T)$. Choose now a finite choice of generators $\tau : Y^+ \to M$ such that $X \subseteq Y$ and $\sigma \rho$ is the restriction of $\tau$ to $X^+$.

Since $P_{ji}$ is a unit, we have $SP_{ji} = S$, so that $SP_{ji'} \subseteq SP_{ji}$ for every $j' \in J$ with $P_{ji'} \neq 0$. Now from Theorem 5.2 we deduce that $L_\sigma(S) = L_{\sigma \rho}(T) = L_\tau(M) \cap \tilde{X}^+$ as required. □

6. Completely Zero-simple Semigroups

Theorems 4.2 and 5.3 are of particular interest in one special case. Recall that a finitely generated semigroup with zero is called completely zero-simple if it has finitely many idempotents and no non-zero ideals. A theorem of Rees [28] says that every completely zero-simple semigroup is isomorphic to a Rees matrix semigroup with zero over any of its non-zero maximal subgroups, with a sandwich matrix in which every row and every column contains a non-zero entry. Hence, we obtain the following description of the loop problem for completely zero-simple semigroups.

Theorem 6.1. Let $F$ be a family of languages closed under union, rational transduction, product and Kleene closure. Let $S$ be a finitely generated completely zero-simple semigroup. Then the following are equivalent:

- $S$ has loop problem in $F$;
- any non-zero maximal subgroup of $S$ has loop problem in $F$;
- every non-zero maximal subgroup of $S$ has loop problem in $F$.

(In fact it is possible to eliminate from the hypothesis of Theorem 6.1 the requirements that $F$ be closed under union and product. This can be proved directly using a modification of the proof of [14, Theorem 5.4]; the technique makes use of the group structure of the base semigroup, and so does not seem to lead to a corresponding strengthening of Theorem 4.2. Since the proof of the stronger result is rather lengthy, and since very few interesting language classes are closed under Kleene closure and transductions but not under product and union, we content ourselves here with the slightly weaker statement which can be derived easily from Theorem 4.2.)

In [14] we posed the question of which semigroups have context-free loop problem. A theorem of Muller and Schupp [24], combined with a subsequent
result of Dunwoody [8], says that a finitely generated group has context-free word problem exactly if it is virtually free, that is, has a free subgroup of finite index. In [14] we applied this result to give a complete characterization of completely simple semigroups with context-free loop problem. We are now in a position to state a corresponding result for completely zero-simple semigroups.

**Corollary 6.2.** A finitely generated completely zero-simple semigroup has context-free loop problem if and only if its maximal subgroups are virtually free.

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