Conjectured Exact Locations of Dynamical Transition Points for the $\pm J$ Ising Spin Glass Model

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The conjectured exact locations of the dynamical transition points for the $\pm J$ Ising spin glass model are theoretically shown based on a percolation theory and conjectures. The dynamical transition is a transition for the time evolution of the distance between two spin configurations. The distance is called the damage or the Hamming distance. The conjectured exact locations of the dynamical transition points are obtained by using the values of the threshold fractions of the random bond percolation problem. The present results are obtained as locations of points on the Nishimori line which is a special line on the phase diagram. We obtain $T_D = 2/\ln(z/z-2)$ and $p_D = z/(z-1)$ for the Bethe lattice, $T_D \to \infty$ and $p_D \to 1/2$ for the infinite-range model, $T_D = 2/\ln 3$ and $p_D = 3/4$ for the square lattice, $T_D \sim 3.9347$ and $p_D \sim 0.62441$ for the 4-dimensional hypercubic lattice, and $T_D = 2/\ln[1 + 2\sin(\pi/18)/(1 - 2\sin(\pi/18))]$ and $p_D = [1 + 2\sin(\pi/18)]/2$ for the triangular lattice, when $J/k_B = 1$, where $z$ is the coordination number, $J$ is the strength of the exchange interaction between spins, $k_B$ is the Boltzmann constant, $T_D$ is the temperature at the dynamical transition point, and $p_D$ is the probability, that the interaction is ferromagnetic, at the dynamical transition point.

§1. Introduction

To establish reliable analytical theories of spin glasses has been one of the most challenging problems in statistical physics for years. Our main interest in this article does not lie directly in the issue of the properties of the phases in spin glasses. We instead will concentrate ourselves on the precise determination of the structure of phase diagram for dynamical properties of spin glasses. This problem is of practical importance for numerical studies, since exact locations of transition points greatly facilitate reliable estimates of physical properties around the transition points.

The dynamical transition mentioned in this article is a transition for the time evolution of the distance between two spin configurations. The distance is called the damage or the Hamming distance. In the distance measurement, the time evolution of two spin configurations is studied under the same thermal noise, where, as for the initial condition, the two configurations are generally set to have a relationship with each other.

The $\pm J$ model is known as one of the Ising spin glass models. A line on the phase diagram for the $\pm J$ model is called the Nishimori line. Interestingly, the internal energy, the upper bound of the specific heat and so forth are exactly calculated on the line. In addition, the internal energy does not depend on any lattice shape and instead depends on the number of nearest-neighbor pairs in the whole system. The location of the multicritical point is conjectured on the Nishimori line of the square lattice, and the conjectured location is in good agreement with the results of other numerical estimates. The present results are also obtained
as locations of points on the Nishimori line.

We use two conjectures in this article. One of the conjectures has already been proposed and discussed in Refs. 8), 9), and another will be proposed in this article.

In order to obtain the conjectured exact locations of the dynamical transition points, the values of the threshold fractions of the random bond percolation problem are used. If a threshold fraction of the random bond percolation problem is calculated, one is able to calculate the conjectured location of the dynamical transition point in the $\pm J$ model by using the present theory. Generally, calculation of the threshold fractions of the random bond percolation problem is easier than that of the dynamical transition points in the $\pm J$ model. Therefore, the present theory can be promising in this respect.

This article is organized as follows. In §2, the $\pm J$ model is explained. In §3, our conjectures are described, and conjectured exact equations are shown. The present results by using the obtained equations are given in §4. In §5, the concluding remarks of this article are described.

§2. Model

The Hamiltonian for the $\pm J$ model, $\mathcal{H}$, is given by

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{i,j} S_i S_j,$$

(2.1)

where $\langle i, j \rangle$ denotes nearest-neighbor pairs, $S_i$ is a state of the spin at the site $i$, and $S_i = \pm 1$. $J_{i,j}$ is the strength of the exchange interaction between the spins at the sites $i$ and $j$. The value of $J_{i,j}$ is given with a distribution $P(J_{i,j})$. The distribution $P(J_{i,j})$ is given by

$$P(J_{i,j}) = p \delta_{J_{i,j}, J} + (1 - p) \delta_{J_{i,j}, -J},$$

(2.2)

where $J > 0$, and $\delta$ is the Kronecker delta. $p$ is the probability that the interaction is ferromagnetic, and $1 - p$ is the probability that the interaction is antiferromagnetic.

The dynamical transition mentioned in this article is characterized by a distance between two spin configurations $\{S_i\}$ and $\{\tilde{S}_i\}$. It has been pointed out that, for the cubic lattice, there are three phases, i.e., a high-temperature phase, an intermediate phase and a low-temperature phase. In the high-temperature phase, the two configurations become identical quickly, so that the distance between them vanishes. In the intermediate phase and the low-temperature phase, the distance between the two configurations remains a finite in the long-time limit if the system size is large enough. In the intermediate phase, the distance between the two configurations does not depend on the initial conditions of the two configurations. In the low-temperature phase, the distance between the two configurations depends on the initial conditions of the two configurations. We concentrate ourselves on the dynamical transition between the high-temperature phase and the intermediate phase, and especially concentrate ourselves on the location of the dynamical transition point.

We explain the measurement of the distance. The distance $D(t)$ is given
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\[ D(t) = \frac{1}{2N}[\langle \sum_{i} |S_i(t) - \tilde{S}_i(t)| \rangle_{\text{MC}}]_R, \]  

where \( t \) is the time, \( N \) is the number of sites, \( \langle \rangle_{\text{MC}} \) denotes the sample average by the Monte Carlo method, and \([\cdot]_R\) denotes the random configuration average for the exchange interactions. As for the initial condition of \( \{ S_i \} \) and \( \{ \tilde{S}_i \} \), \( \{ S_i(0) \} \) is set to random, and \( \{ \tilde{S}_i(0) \} = -\{ S_i(0) \} \) is set, for example. As for the Monte Carlo method, a heat-bath method is used, and is as follows. By using the uniform pseudo-random number \( r(t) \) (0 \( \leq \) \( r(t) < 1 \)), update of \( \{ S_i(t) \} \) and \( \{ \tilde{S}_i(t) \} \) is given by

\[ S_i(t + \Delta t) = \text{sign}\left\{ \frac{1}{1 + \exp[-2\beta \sum_j J_{i,j} S_j(t)]} - r(t) \right\}, \]

\[ \tilde{S}_i(t + \Delta t) = \text{sign}\left\{ \frac{1}{1 + \exp[-2\beta \sum_j J_{i,j} \tilde{S}_j(t)]} - r(t) \right\}, \]

where \( \Delta t \) is a time step per spin, the summations of the right-hand sides of Eqs. (2.4) and \( (2.5) \) are over the nearest-neighbor sites of the site \( i \), \( \beta = 1/k_B T \), \( T \) is the temperature, and \( k_B \) is the Boltzmann constant. Note that, in order to measure \( D(t) \), the two spin configurations \( \{ S_i(t) \} \) and \( \{ \tilde{S}_i(t) \} \) are updated with the same pseudo-random number sequence \( \{ r(t) \} \).

We apply a percolation theory. We use the Fortuin-Kasteleyn (FK) cluster. The FK clusters consist of the FK bonds which are probabilistically put between spins. The number of the FK bonds is rigorously related to the internal energy. We define the probability for putting the FK bond as \( P_{FK} \). \( P_{FK} \) is given by

\[ P_{FK} = 1 - e^{-\beta J_{i,j} S_i S_j - \beta |J_{i,j}|}. \]

For calculating \( \langle P_{FK} \rangle_T \), a gauge transformation is used, where \( \langle \rangle_T \) denotes the thermal average. The gauge transformation is performed by

\[ J_{i,j} \rightarrow \sigma_i \sigma_j J_{i,j}, \quad S_i \rightarrow S_i \sigma_i, \]

where \( \sigma_i = \pm 1 \). The gauge transformation has no effect on thermodynamic quantities. By performing the gauge transformation, the \( \mathcal{H} \) part becomes \( \mathcal{H} \rightarrow \mathcal{H} \) and the \( P_{FK} \) part becomes \( P_{FK} \rightarrow P_{FK} \). By using Eq. (2.2), the distribution \( P(J_{i,j}) \) is rewritten as

\[ P(J_{i,j}) = \frac{e^{\beta P J_{i,j}}}{2 \cosh(\beta P J)}, \quad J_{i,j} = \pm J, \]

where \( \beta P \) is given by

\[ \beta P = \frac{1}{2J} \ln \frac{p}{1 - p}. \]

By performing the gauge transformation, the distribution \( P(J_{i,j}) \) part becomes

\[ \prod_{(i,j)} P(J_{i,j}) = \frac{e^{\beta P \sum_{(i,j)} J_{i,j}}}{[2 \cosh(\beta P J)]^{N_B}}, \]
$\sum \sigma_i e^{\beta P \sum_{\langle i,j \rangle} \sigma_i \sigma_j}$

where $N_B$ is the number of nearest-neighbor pairs in the whole system. When the value of $\beta P$ is consistent with the value of the inverse temperature $\beta$, the line for $\beta = \beta P$ on the phase diagram is called the Nishimori line. By using the gauge transformation, $\langle P_{FK} \rangle_T$ on the Nishimori line is obtained as

$$\langle P_{FK} \rangle_T = \tanh(\beta P J),$$

where $\beta P = 1/k_B T_P$, $T_P$ is the temperature on the Nishimori line. The value of $\langle P_{FK} \rangle_T$ on the Nishimori line does not depend on any lattice shape.

§3. Conjectures

The present theory uses two conjectures. If the two conjectures are correct, the present theory gives the exact values. Here, we describe the two conjectures and show conjectured exact equations derived by using the two conjectures.

(1) We conjecture

$$T_{FK} = T_D$$

for arbitrary lattices, where $T_{FK}$ is the percolation transition temperature of the FK cluster, and $T_D$ is the dynamical transition temperature for the distance between two spin configurations. Two-spin correlations of nearest neighbor pairs are correctly estimated by using the FK bonds, while, in the case of the $\pm J$ model, two-spin correlations of no nearest neighbor pairs are not correctly estimated from simple connection of the FK bonds, however, the transition point for the time evolution of the distance between two spin configurations may be consistent with the percolation transition point of the clusters for the net two-spin correlations by the FK bonds. This conjecture for $T_{FK} = T_D$ has already been proposed and discussed, and the confirmations of this conjecture have been numerically performed. The previous studies indicate that $T_{FK}$ may exactly agree with $T_D$.

(2) We conjecture

$$\langle P_{FK} \rangle_T = P_C$$

at the percolation transition point of the FK cluster on the Nishimori line of arbitrary lattices, where $P_{FK}$ is the probability for putting the FK bond between spins, and $P_C$ is the threshold fraction of the random bond percolation problem. In the random bond percolation problem, bonds for generating clusters are randomly put on the edges of the lattice, and one of the clusters is percolated at the threshold fraction $P_C$. From Eq. (2.11), the value of $\langle P_{FK} \rangle_T$ on the Nishimori line does not depend on any lattice shape. From the fact, Eq. (3.2) is conjectured. We propose this conjecture in this article as a conjecture which may be exact.

By using Eqs. (2.9), (2.11), (3.1) and (3.2), we obtain

$$T_D = \frac{2J}{k_B \ln(1 + P_C/1 - P_C)},$$

$$p_D = \frac{1}{2}(1 + P_C).$$

(3.3)

(3.4)
Eqs. (3.3) and (3.4) are conjectured exact equations. By using Eqs. (3.3), (3.4) and the value of the threshold fraction $P_C$ of the random bond percolation problem, the values of the dynamical transition temperature $T_D$ and the dynamical transition probability $p_D$ are calculated as the location of a point on the Nishimori line. The obtained values are conjectured exact values. Note that the dynamical transition probability $p_D$ is the probability that the interaction is ferromagnetic at the dynamical transition point.

Campbell and Bernardi have derived an equation for the energy $E$ in the $\pm J$ model and the threshold fraction $P_C$ of the random bond percolation problem, i.e., $E = J N_B \{ 1 - 2 P_C / [1 - \exp(-2\beta J)] \}$ on the assumption of a random active-bond spatial distribution. By applying the energy and the temperature on the Nishimori line to this equation, the same equations (Eqs. (3.3) and (3.4)) are obtained where the energy on the Nishimori line is $-N_B J \tanh(\beta P J)$.

In Ref. [15], the exact values of the percolation threshold of the FK cluster are also conjectured. The conjecture proposed in this article is more general, so that the present study includes finite-dimensional cases that are not mentioned in Ref. [15].

§4. Results

We show the present results by applying the conjectured exact equations obtained in [3], i.e., Eqs. (3.3) and (3.4), to several lattices. The present results are obtained as locations of points on the Nishimori line.

For the Bethe lattice, the threshold fraction $P_C$ of the random bond percolation problem is obtained as $P_C = 1 / (z - 1)$ where $z$ is the coordination number. By using Eqs. (3.3) and (3.4), we obtain

$$T_D = \frac{2 J}{k_B \ln(z/z - 2)}, \quad p_D = \frac{z}{2(z - 1)}.$$  

This is the result for the Bethe lattice. $T_D$ agrees with the ferromagnetic transition temperature for the pure system, $T_C^{[20]}$. When $z = N - 1$ and $J \to J / \sqrt{N}$, this model becomes the infinite-range model. Then, in the thermodynamic limit, we obtain $T_D \to \infty$ and $p_D \to 1/2$ when $J / k_B = 1$. This result for the infinite-range model agrees with the previous results in Refs. [11], [21].

For the square lattice, the threshold fraction of the random bond percolation problem is obtained as $1/2^{[11]}$. By using Eqs. (3.3) and (3.4), we obtain

$$T_D = \frac{2}{\ln 3} \sim 1.820478, \quad p_D = \frac{3}{4} = 0.75,$$

when $J / k_B = 1$. This is the result for the square lattice. When $J / k_B = 1$, the ferromagnetic transition temperature for the pure system, $T_C$, is $2 / \ln(1 + \sqrt{2})$ ($\sim 2.269^{[20]}$). $T_D$ does not agree with $T_C$ in this case. The previous numerical results are $T_D \sim 1.8^{[11]}$ for $p = 1/2$, $T_D \sim 1.7^{[21]}$ for $p = 1/2$, $T_{FK} \sim 1.81$ and $1.82^{[22]}$ for $p = 0.7$, and $T_{FK} \sim 1.8^{[22]}$ for $p = 0.8$. It seems that the obtained result is valid.

Fig. [1] shows a schematic phase diagram for the $\pm J$ model. $p$ is the probability that the interaction is ferromagnetic, and $1 - p$ is the probability that the interaction
Fig. 1. A schematic phase diagram for the ±J model. $p$ is the probability that the interaction is ferromagnetic, and $1 - p$ is the probability that the interaction is antiferromagnetic. $T$ is the temperature. The paramagnetic phase (‘Para’), the ferromagnetic phase (‘Ferro’), the Griffiths phase (‘Griffiths Phase’) and the spin glass phase (‘Spin Glass’) are depicted. The Nishimori line (the dashed line), the Griffiths temperature (the dotted line) and the $T_d$ line (the short dashed line) are also depicted. $T_C$ is the Curie temperature for the ferromagnetic model. The point ‘A’ is the dynamical transition point for the distance between two spin configurations on the Nishimori line. The point ‘B’ is the multicritical point.

is antiferromagnetic. $T$ is the temperature. The paramagnetic phase (‘Para’), the ferromagnetic phase (‘Ferro’), the Griffiths phase (‘Griffiths Phase’) and the spin glass phase (‘Spin Glass’) are depicted. In this article, we do not mention the existence of a mixed phase between the ferromagnetic phase and the spin glass phase. The Nishimori line (the dashed line), the Griffiths temperature (the dotted line) and the $T_d$ line (the short dashed line) are also depicted. $T_C$ is the Curie temperature for the ferromagnetic model. The value of the Griffiths temperature corresponds to the value of $T_C$. The $T_d$ line represents the dynamical transition temperature for the distance between two spin configurations. It has been pointed out that the value of $T_D$ numerically agrees very well with the value of $T_C$ when $p = 1$. The phase between the Griffiths temperature and the $T_d$ line can also be called the Griffiths phase, but, in this phase, the behavior of the distance is the same as that in the paramagnetic phase. The point ‘A’ is the dynamical transition point for the distance between two spin configurations on the Nishimori line. The point ‘B’ is the multicritical point. In a certain lattice case, the dynamical transition temperature $T_D$ does not agree with $T_C$. On the other hand, in a certain lattice case, the dynamical transition tempera-
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ture $T_D$ agrees with $T_C$. For example, $T_D$ for the square lattice does not agree with $T_C$ for the same lattice as mentioned above. On the other hand, $T_D$ for the Bethe lattice agrees with $T_C$ for the same lattice as mentioned above. For the multicritical point, it seems that the temperature at the multicritical point generally does not agree with $T_D$. For example, when $J/k_B = 1$, the temperature at the multicritical point for the square lattice is roughly equal to 0.957(12) while $T_D$ for the square lattice is $2J/k_B \ln(\sqrt{2} - 1 + 1/\sqrt{2} - 1 - 1)$ from the result in Ref. 23, while $T_D$ for the Bethe lattice is $2J/k_B \ln(z/z - 1 + 1/\sqrt{2} - 1)$ from the present result.

For the simple cubic lattice, the threshold fraction of the random bond percolation problem is numerically estimated as $p_C \sim 0.248813$. By using Eqs. (3.3) and (3.4), we obtain

$$T_D \sim 3.9347, \quad p_D \sim 0.62441,$$

when $J/k_B = 1$. This is the result for the simple cubic lattice. The previous numerical results are $T_D \sim 4.10$ for $p = 1/2$, $T_D \sim 4.03$ for $p = 1/2$, $T_D \sim 3.91$ for $p = 1/2$, and $T_{FK} \sim 3.92$ for $p = 1/2$. It seems that the obtained result is valid.

For the 4-dimensional hypercubic lattice, the threshold fraction of the random bond percolation problem is numerically estimated as $p_C \sim 0.16013$. By using Eqs. (5.3) and (5.4), we obtain

$$T_D \sim 6.191, \quad p_D \sim 0.5801,$$

when $J/k_B = 1$. This is the result for the 4-dimensional hypercubic lattice. The previous numerical results are $T_D \sim 6.11$ for $p = 1/2$, and $T_D \sim 6.03$ for $p = 1/2$. The obtained result does not contradict with the previous results.

For the triangular lattice, the threshold fraction of the random bond percolation problem is obtained as $2\sin(\pi/18)$. By using Eqs. (5.3) and (5.4), we obtain

$$T_D = \frac{2}{\ln\left[1 + 2\sin(\pi/18) \left/ 1 - 2\sin(\pi/18)\right.\right]} \sim 2.759641, \quad p_D = \frac{1 + 2\sin(\pi/18)}{2} \sim 0.6736482,$$

when $J/k_B = 1$. This is the result for the triangular lattice. The previous numerical results are $T_{FK} \sim 2.74$ and $2.73$ for $p = 0.6$, and $T_{FK} \sim 2.75$ and $2.76$ for $p = 0.7$. It seems that the obtained result is valid.

From these results, it seems reasonable to consider that the present theory is roughly good, although more precise numerical confirmations are needed in order to decide the correctness of the present theory.

§5. Concluding Remarks

We theoretically showed the conjectured exact locations of the dynamical transition points for the $\pm J$ Ising spin glass model based on a percolation theory and conjectures.

The present theory may be applied directly to the Gaussian Ising spin glass model and the Potts gauge glass model. Concretely, the solution of $\langle P_{FK}(T) \rangle_R$ on
the Nishimori line for the Gaussian Ising spin glass model is obtained in Ref. \[15\], and the solution of \[\langle P_{FK} \rangle_R\] on the Nishimori line for the Potts gauge glass model is obtained in Ref. \[27\]. By using these solutions instead of the solution for the ±J Ising spin glass model, the conjectured exact equations in the Gaussian Ising spin glass model and the Potts gauge glass model are obtained.

We studied the transition for the time evolution of the distance between two spin configurations on the Nishimori line. This study is different from the study of the aging phenomena on the Nishimori line as in Ref. \[5\].

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