DOUBLE CROSSED BIPRODUCTS AND RELATED STRUCTURES

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ABSTRACT. Let $H$ be a bialgebra. Let $\sigma : H \otimes H \to A$ be a linear map, where $A$ is a left $H$-comodule coalgebra, and an algebra with a left $H$-weak action $\lhd$. Let $\tau : H \otimes H \to B$ be a linear map, where $B$ is a right $H$-comodule coalgebra, and an algebra with a right $H$-weak action $\rhd$. In this paper, we improve the necessary conditions for the two-sided crossed product algebra $A \#^\tau H \# B$ and the two-sided smash coproduct coalgebra $A \times H \times B$ to form a bialgebra (called double crossed biproduct) such that the condition $b_{[1]} \lhd a_0 \otimes b_{[0]} \rhd a_{-1} = a \otimes b$ in Majid’s double biproduct (or double-bosonization) is one of the necessary conditions. On the other hand, we provide a more general two-sided crossed product algebra structure via Brzeziński’s crossed product and give some applications.

CONTENTS

1. Introduction and preliminaries
2. Brzeziński’s two-sided crossed product
   2.1. Right Brzeziński’s crossed product and its special cases
   2.2. Brzeziński’s two-sided crossed product algebra and its special cases
   2.3. Brzeziński’s two-sided crossed product bialgebra
3. An extended version of Majid’s double biproduct
4. Conclusion
References

1. INTRODUCTION AND PRELIMINARIES

Radford’s biproduct ([21]) is one of the celebrated objects in the theory of Hopf algebras, which also can provide examples for Rota-Baxter bialgebras introduced by Ma and Liu in ([13]). This plays a fundamental role in the classification of finite-dimensional pointed Hopf algebras ([1]). Majid realized a categorical interpretation of Radford’s biproduct ([15,16]): $A \star H$ is a Radford’s biproduct if and only if $A$ is a bialgebra in the Yetter-Drinfel’d category $YD^H_H$.

Let $H$ be a bialgebra, $A$ a bialgebra in $H^H_YD$, and $B$ a bialgebra in $YD^H_H$. In [17,18], Majid gave a sufficient condition

$$b_{[1]} \lhd a_0 \otimes b_{[0]} \rhd a_{-1} = a \otimes b$$

(1)

for a two-sided smash product algebra $A \#^\tau H \# B$ and a two-sided smash coproduct coalgebra $A \times H \times B$ to form a bialgebra, named the Majid’s double biproduct (or double-bosonization) and denoted...
by \(A \star H \star B\). Majid’s double biproduct provided a powerful tool to construct quantum groups (see \([7,18]\)), some related studies can be found in the literature \([7,8,11,12,14,20,23,25,26]\).

As an application of our results in this paper, we find that the condition \((\square)\) is also necessary in \(A \star H \star B\). This is a motivation of writing this paper.

In \([24]\), as a generalization of Radford’s results in \([21]\), Wang, Jiao and Zhao presented the necessary and sufficient conditions for the (left) crossed product algebra (introduced in \([3]\) and extended the smash product algebra) and the (left) smash coproduct coalgebra to be a bialgebra (denoted by \(A \sigma^\# H\)) and also gave a mapping description. Similarly, in order to extend Majid’s double biproduct, Ma and Liu in \([14]\) gave the necessary and sufficient conditions for the two-sided crossed product algebra \(A \#^\sigma H \# B\) and the two-sided smash coproduct coalgebra \(A\times H\times B\) to form a bialgebra (denoted by \(A \star^\sigma H \star B\)).

As can be seen from the notations of these structures, if we substitute \(B\) in \(A \star H \star B\) (or \(A \star^\sigma H \star B\)) for the ground field \(K\), then \(A \star H\) (or \(A \star^\sigma H\)) is obtained. Comparing the necessary conditions in \(A \star H \star B\) and \(A \star^\sigma H \star B\), we find that the latter is a little more complicated, especially the condition \((C13)\) in \([14]\) Theorem 2.1 and that the former just contains the necessary and sufficient conditions for the smash product algebra \(A \# H\) and the smash coproduct coalgebra \(A \times H\) to form a bialgebra \(A \star H\). It is worth mentioning that these conditions in the former case are just right for the use of lifting method in the classification of finite-dimensional pointed Hopf algebras. For these reasons, we will improve the necessary conditions in \([14]\) Theorem 2.1 such that the necessary and sufficient conditions for the crossed product algebra \(A \#^\sigma H\) and the smash coproduct coalgebra \(A \times H\) to form a bialgebra \(A \star^\sigma H\) can be included and also \((C13)\) above can be decomposed into several simpler ones (of course containing \((\square)\)). This is another motivation of writing this paper.

In \([3]\), Brzeziński gave the necessary and sufficient conditions for a very extensive crossed product to be an algebra (called Brzeziński’s crossed product) which includes the crossed product \((\square)\) and the twisted tensor product \((\square)\). We will provide a more general two-sided crossed product algebra structure via Brzeziński’s crossed product and list some special cases.

Throughout this paper, we follow the definitions and terminologies in \([23]\), with all algebraic systems supposed to be over the field \(K\). Now, let \(C\) be a coalgebra. We use the simple Sweedler’s notation for the comultiplication: \(\Delta(c) = c_1 \otimes c_2\) for any \(c \in C\). Denote the category of left \(H\)-comodules by \(\text{HMod}\). For \((M,\rho) \in \text{HMod}\), write: \(\rho(x) = x_{-1} \otimes x_0 \in H \otimes M\), for all \(x \in M\). Denote the category of right \(H\)-comodules by \(\text{Mod}^H\). For \((M,\psi) \in \text{Mod}^H\), write: \(\psi(x) = x_{[0]} \otimes x_{[1]} \in M \otimes H\), for all \(x \in M\). We denote the left-left Yetter-Drinfel’d category by \(\text{YD}^H\) and the right-right Yetter-Drinfel’d category by \(\text{YD}_R^H\). Given a \(K\)-space \(M\), we write \(id_M\) for the identity map on \(M\).

First of all, we rewrite the Brzeziński’s crossed product as follows.

**Proposition 1.1.** (Left Brzeziński’s crossed product) \((\square)\): Let \(A\) be an algebra, \(H\) a vector space and \(1_H \in H\). Let \(G : H \otimes H \rightarrow A \otimes H\) (write \(G(x \otimes x') = x^G \otimes x'_G\) for all \(x,x' \in H\)) and \(R : H \otimes A \rightarrow A \otimes H\) (write \(R(x \otimes a) = a_R \otimes x_R\) for all \(x \in H\) and \(a \in A\)) be linear maps such that

\[(\text{LB1})\ a_R \otimes 1_{HR} = a \otimes 1_{H},\ 1_{AR} \otimes x_R = 1_A \otimes x\]

holds. Then \(A \#^G_H = A \otimes H\) as vector space) is an algebra with unit \(1_A \otimes 1_H\) and a product

\[(a \otimes x)(a' \otimes x') = aa'_R x'_R \otimes x_G\]

(2)
if and only if the following conditions hold:

\[ (LB2) \quad (aa')_R \otimes x_R = a_R a'_R \otimes x_R; \]
\[ (LB3) \quad x^G \otimes 1_{HG} = 1_H^G \otimes x_G = 1_A \otimes x; \]
\[ (LB4) \quad x^G \otimes x^G_R \otimes x''_G = x^G x^G_R \otimes x''_G; \]
\[ (LB5) \quad a_R x^G_R \otimes x''_{RG} = x^G a_R \otimes x''_{GR} \]

for all \( a, a' \in A, \ x, x', x'' \in H \), where \( G = g, R = r \).

**Proof.** The sufficiency can be gotten by [3]. So we only prove the necessity. By \( 1_A \otimes 1_H \) is unit and Eqs.\((LB1)\) and \((\frac{3}{3})\), we can get Eq.\((LB3)\). By the associativity and Eq.\((\frac{3}{3})\), we have

\[ a a' x^G_R a''_R \otimes x''_R = a(a' a''_R x^G_R) x^G x^G_R \otimes x''_G. \tag{3} \]

Set \( x' = x'' = 1_H \) in Eq.\((\frac{3}{3})\), one can obtain Eq.\((LB2)\). Let \( a' = a'' = 1_A \) in Eq.\((\frac{3}{3})\), then Eq.\((LB4)\) holds. Likewise, setting \( a' = 1_A, x'' = 1_H \) in Eq.\((\frac{3}{3})\), we can get Eq.\((LB5)\). These finish the proof.

\[ \square \]

**Remark 1.2.** When Eqs.\((LB1)\) and \((LB3)\) are interchanged, Proposition [3] still holds.

In what follows, we list some important special cases of left Brzeziński’s crossed product.

**Example 1.3.** (Left twisted crossed product) Let \( A \) be an algebra, \( H \) a bialgebra, \( \sigma : H \otimes H \longrightarrow A \) and \( R : H \otimes A \longrightarrow A \otimes H \) two linear maps such that Eq.\((LB1)\) holds. Then \( A \#^\sigma_R H \) (= \( A \otimes H \) as a vector space) is an associative algebra with unit \( 1_A \otimes 1_H \) and multiplication

\[ (a \otimes x)(a' \otimes x') = a a' \sigma(x_R, x'_1) \otimes x_{R2}x'_2 \]

for all \( a, a' \in A \) and \( x, x' \in H \) if and only if Eq.\((LB2)\) and the following conditions hold:

\[ (LT C3) \quad \sigma(x, 1_H) = \sigma(1_H, x) = \varepsilon(x) 1_A; \]
\[ (LT C4) \quad \sigma(x'_1, x'_2) \sigma(x_R, x'_{R1}) \otimes x_{R2}x'_1x'_2 = \sigma(x_1, x'_1)\sigma(x_2, x'_2) \otimes x_3x'_3; \]
\[ (LT C5) \quad a_R \sigma(x_R, x'_1) \otimes x_{R2}x'_2 = \sigma(x_1, x'_1)a_R \otimes (x_2x'_2)_R, \]

where \( a, a' \in A, \ x, x', x'' \in H \) and \( r = R \).

**Proof.** Let \( G(x \otimes x') = \sigma(x_1, x'_1) \otimes x_2x'_2 \) in \( A \#^G_R H \).

\[ \square \]

**Example 1.4.** (Left crossed product) (\([\frac{3}{3}]\)): Let \( H \) be a Hopf algebra with a left weak action \( \triangleright \) on the algebra \( A \). Let \( \sigma : H \otimes H \longrightarrow A \) be a \( K \)-linear map. Let \( A \#^\varepsilon_H \) be the (usually nonassociative) algebra whose underlying space is \( A \otimes H \) and whose multiplication is given by

\[ (a \otimes x)(a' \otimes x') = a(x_1 \triangleright a') \sigma(x_2, x'_1) \otimes x_3x'_2 \]

for all \( a, a' \in A \) and \( x, x' \in H \). This \( A \#^\varepsilon_H \) is an associative algebra with unit \( 1_A \otimes 1_H \) if and only if \( \sigma \) satisfies Eq.\((\text{LC3})\) and the following conditions:

\[ (LC2) \quad (x_1 \triangleright \sigma(x'_1, x''_1)) \sigma(x_2, x'_2x''_2) = \sigma(x_1, x'_1)\sigma(x_2, x'_2, x''); \]
\[ (LC3) \quad (x_1 \triangleright (x'_1 \triangleright a)) \sigma(x_2, x'_2) = \sigma(x_1, x'_1)(x_2x'_2 \triangleright a) \]

for all \( a \in A \) and \( x, x', x'' \in H \).

**Proof.** Let \( R(x \otimes a) = x_1 \triangleright a \otimes x_2 \) in \( A \#^\varepsilon_R H \).

\[ \square \]
Example 1.5. (Left twisted product) (\cite{1}): Let $H$ be a bialgebra and $A$ an algebra. Let $\sigma : H \otimes H \rightarrow A$ be a $K$-linear map. Let $A^{#r}H (= A \otimes H$ as a vector space) is an associative algebra with unit $1_A \otimes 1_H$ and multiplication

$$(a \otimes x)(a' \otimes x') = aa'\sigma(x_1, x'_1) \otimes x_2 x'_2$$

for all $a, a' \in A$ and $x, x' \in H$ if and only if $\sigma$ satisfies Eq.(LTC3) and the following conditions:

\begin{align*}
(L2) \quad & \sigma(x_1', x_1')\sigma(x, x_2 x'_2) = \sigma(x_1, x_1')\sigma(x_2 x'_2, x'), \\
(L3) \quad & a\sigma(x, x') = \sigma(x, x)a
\end{align*}

for all $a \in A$ and $x, x', x'' \in H$.

Proof. Let $x \triangleright a = e(x)a$ in $A^{#r}H$. \hfill \Box

Example 1.6. (Left unified product) (\cite{1}): Let $A, H$ be two bialgebras and $\triangleright : H \otimes A \rightarrow H$, $\triangleright : H \otimes A \rightarrow A$, $\sigma : H \otimes H \rightarrow A$ linear maps such that

\begin{align*}
(\text{LU1}) \quad & 1_H \triangleright a_1 \otimes 1_H \triangleright a_2 = a_1 \otimes 1_H; \quad x_1 \triangleright 1_A \otimes x_2 \triangleright 1_A = 1_A \otimes x \\
(\text{LU2}) \quad & x_1 \triangleright (aa')_1 \otimes x_2 \triangleright (aa')_2 = (x_1 \triangleright a_1)((x_2 \triangleright a_2)_1 \triangleright (x_2 \triangleright a_2)_2 \triangleright a'_2); \\
(\text{LU3}) \quad & \sigma(x_1, 1_H) \otimes x_2 = \sigma(1_H, x_1) \otimes x_2 = 1_A \otimes x; \\
(\text{LU4}) \quad & (x_1 \triangleright \sigma(x_1', x_1''))_1 \sigma((x_2 \triangleright \sigma(x_1', x_1''))_1, (x_2 x_2')_1) \otimes (x_2 \triangleright (x_1', x_1''))_2 = \sigma(x_1, x_1') \sigma((x_2 x_2')_1, x_1'') \otimes (x_2 x_2')_2 x_2''; \\
(\text{LU5}) \quad & (x_1 \triangleright (x_1' \triangleright a_1)_1) \sigma((x_2 \triangleright (x_1' \triangleright a_1)_1)_1, (x_2 \triangleright (x_1' \triangleright a_1)_1)_2) \otimes (x_2 \triangleright (x_1' \triangleright a_1)_2) = (x_2 x_2')_1 \triangleright a_1 \otimes ((x_2 x_2')_2 \triangleright a_2),
\end{align*}

where $a, a' \in A$, $x, x', x'' \in H$.

Proof. Let $R(x \otimes a) = x_1 \triangleright a_1 \otimes x_2 \triangleright a_2$ in $A^{#r}H$. \hfill \Box

Example 1.7. (Left twisted tensor product) (\cite{1}): Let $H, A$ be two algebras and $R : H \otimes A \rightarrow A \otimes H$ a linear map. Then $A^{#r}H (= A \otimes H$ as a vector space) is an associative algebra with unit $1_A \otimes 1_H$ and multiplication

$$(a \otimes x)(a' \otimes x') = aa' \otimes x R x'$$

for all $a, a' \in A$ and $x, x' \in H$ if and only if Eq.(LB1), (LB2) and the following condition hold:

\begin{align*}
(LT3) \quad & a_R \otimes (x x')_R = a_{R_r} \otimes x_r x'R,
\end{align*}

where $a, a' \in A$, $x, x' \in H$ and $r = R$.

Proof. Let $G$ be trivial, i.e., $G(x \otimes x') = 1_A \otimes xx'$ in $A^{#r}H$. \hfill \Box

Remark 1.8. (1) The well known Drinfeld double is also a special case of left Brzeziński’s crossed product.

(2) From above we know that the left Brzeziński’s crossed product is a very general product structure.
**Definition 1.9. (Majid’s double biproduct)** We recall, from [17, 18], the construction of the so-called double biproduct. Let $H$ be a bialgebra, $A$ a bialgebra in $\mathcal{YD}_H^H$, and $B$ a bialgebra in $\mathcal{YD}_B^B$. Adopt the following notation for the structure maps: the counits are $\varepsilon_A$ and $\varepsilon_B$, the comultiplications are $\Delta_A(a) = a_1 \otimes a_2$ and $\Delta_B(b) = b_1 \otimes b_2$, and the actions and coactions are

- $H \otimes A \rightarrow A, x \otimes a \mapsto x \cdot a, A \rightarrow H \otimes A, a \mapsto a_{-1} \otimes a_0,$
- $B \otimes H \rightarrow B, b \otimes x \mapsto b \cdot x, B \rightarrow B \otimes H, b \mapsto b_{[0]} \otimes b_{[1]}$

for all $x \in H, a \in A, b \in B$. The vector space $A \otimes H \otimes B$ becomes an algebra (called the two-sided smash product, $A\#H\#B$) with unit $1_A \otimes 1_H \otimes 1_B$ and multiplication

$$
(a \otimes x \otimes b)(a' \otimes x' \otimes b') = (a_1 \triangleright a') \otimes x_2 \cdot (b \triangleright x_2')b',
$$

and a coalgebra (called the two-sided smash coproduct, $A \times H \times B$) with counit $\varepsilon(a \otimes x \otimes b) = \varepsilon_A(a)\varepsilon_H(x)\varepsilon_B(b)$ and comultiplication

$$
\Delta(a \otimes x \otimes b) = a_1 \otimes a_{2-1} x_1 \otimes b_{[0]} \otimes a_{20} \otimes x_2 b_{[1]} \otimes b_2.
$$

Moreover, assume that Eq. (1) holds. Then $H \otimes B$ with two-sided smash product algebra and two-sided smash product coalgebra is a bialgebra, called the Majid’s double biproduct, denoted by $A \star H \star B$.

**Remark 1.10.** (1) Replacing $A$ (or $B$) with $K$, the double biproduct is exactly the right (or left) variant of Radford’s biproduct.

(2) The double biproduct here is actually the case of Majid’s in [17, Theorem A.1] with a trivial pairing.

2. **Brzeziński’s two-sided crossed product**

In this section, we will first introduce the notion of Brzeziński’s two-sided crossed product generalizing the two-sided smash product. First we need to list the right version of some crossed products.

2.1. **Right Brzeziński’s crossed product and its special cases.**

**Proposition 2.1. (Right Brzeziński’s crossed product) ( [19, Theorem 2.1])** Let $H$ be a vector space with a distinguished element $1_H \in H$ and $B$ an algebra. Let $F : H \otimes H \rightarrow H \otimes B$ (write $F(x \otimes x') = x_F \otimes x'^F$ for all $x, x' \in H$) and $T : B \otimes H \rightarrow H \otimes B$ (write $T(b \otimes x) = x_T \otimes b_T$ for all $x \in H$ and $b \in B$) be two linear maps such that

$$
(RB1) \quad 1_{HT} \otimes b_T = 1_H \otimes b, \quad x_T \otimes 1_{BT} = x \otimes 1_B
$$

holds. Then $H^F \# B (= H \otimes B$ as a vector space) is an associative algebra with unit $1_H \otimes 1_B$ and multiplication

$$
(x \otimes b)(x' \otimes b') = x_F \otimes x'^F_T b_T b'
$$

for all $b, b' \in B$ and $x, x' \in H$ if and only if the following conditions hold:

- $(RB2) \quad x_T \otimes (bb')_T = x_T \otimes b_b';$
- $(RB3) \quad 1_{HF} \otimes x^F = x_F \otimes 1^{HF}_B = x \otimes 1_B;$
- $(RB4) \quad x_F^f \otimes x'^{fi}_T x'^F_T = x_f \otimes x'^{fi}_F x'^{fi}$,
Proof. Let $b, b' \in B$, $x, x', x'' \in H$ and $f = F, t = T$.

Example 2.2. (1) (Right twisted crossed product) Let $B$ be an algebra, $H$ a bialgebra, $\tau : H \otimes H \rightarrow B$ and $T : B \otimes H \rightarrow H \otimes B$ two linear maps such that Eq. (RB1) holds. Then $H^\tau T \# B (= H \otimes B$ as a vector space) is an associative algebra with unit $1_H \otimes 1_B$ and multiplication
\[(x \otimes b)(x' \otimes b') = x_1 x'_{T1} \otimes \tau(x_2, x'_{T2}) b_T b'\]
for all $a, a' \in A$ and $x, x' \in H$ if and only if Eq. (RB2) and the following conditions hold:
\[
(RTC2) \quad \tau(x, 1_H) = \tau(1_H, x) = \varepsilon(x) 1_B;
\]
\[
(RTC3) \quad x_1 x'_1 x''_1 \otimes \tau(x_2, x'_2, x''_2) \tau(x_3, x'_3) = x_1 x'_1 x''_1 \otimes \tau(x_2, x'_2 x''_2) \tau(x'_3, x''_3);
\]
\[
(RTC4) \quad x_{T1} x'_1 \otimes \tau(x_{T2}, x'_2) b_T = (x_1 x'_1) T \otimes b_T \tau(x_2, x'_2),
\]
where $b, b' \in B$, $x, x', x'' \in H$ and $t = T$.

Proof. Let $F(x \otimes x') = x_1 x'_1 \otimes \tau(x_2, x'_2)$ in Proposition 2.2.

(2) (Right unified product) Let $B, H$ be two bialgebras and $\triangleright : B \otimes H \rightarrow H$, $\blacktriangledown : B \otimes H \rightarrow B$, $\tau : H \otimes H \rightarrow B$ linear maps such that
\[
(RU1) \quad b_1 \triangleright 1_H \otimes b_2 \blacktriangledown 1_H = 1_H \otimes b; \quad 1_B \triangleright x_1 \otimes 1_B \blacktriangledown x_2 = x \otimes 1_B
\]
holds. Then $H^\tau \# B (= H \otimes B$ as a vector space) is an associative algebra with unit $1_H \otimes 1_B$ and multiplication
\[
(x \otimes b)(x' \otimes b') = x_1 (b_1 \triangleright x'_1)_1 \otimes \tau(x_2, (b_1 \triangleright x'_1)_2)(b_2 \blacktriangledown x'_2) b'
\]
for all $x, x' \in H$ and $b, b' \in B$ if and only if the following conditions hold:
\[
(RU2) \quad b_1 b'_1 \triangleright x_1 \otimes b_2 b'_2 \blacktriangledown x_2 = b_1 \triangleright (b'_1 \triangleright x_1)_1 \otimes (b_2 \blacktriangledown x_1)_2)(b_2' \blacktriangledown x_2);
\]
\[
(RU3) \quad x_1 \otimes \tau(1_H, x_2) = x_1 \otimes \tau(x_2, 1_H) = x \otimes 1_B;
\]
\[
(RU4) \quad x_1 x'_1 (\tau(x_3, x'_3)_1 \triangleright x'_1)_1 \otimes \tau(x_2, x'_2, (\tau(x_3, x'_3)_1 \triangleright x'_1)_2)(\tau(x_3, x'_3)_2 \triangleright x'_2)
\]
\[
= x_1 x'_1 x''_1 \otimes \tau(x_2, x'_2 x''_2) \tau(x'_3, x''_3);
\]
\[
(RU5) \quad (b_1 \triangleright x_1)_1 ((b_2 \blacktriangledown x_2_1) \triangleright x'_1)_1 \otimes \tau((b_1 \triangleright x_1)_2, ((b_2 \blacktriangledown x_2_1) \triangleright x'_1)_2)
\]
\[
(x((b_2 \blacktriangledown x_2_2) \triangleright x'_2) = b_1 \triangleright x_1 x'_1 \otimes (b_2 \blacktriangledown x_2 x'_2) \tau(x_3, x'_3),
\]
where $b, b' \in B$, $x, x', x'' \in H$.

Proof. Let $T(b \otimes x) = b_1 \triangleright x_1 \otimes b_2 \blacktriangledown x_2$ in $H^\tau \# B$.

(3) (Right $F$-twist product) Let $H$ be a vector space with a distinguished element $1_H \in H$ and $B$ an algebra. Let $F : H \otimes H \rightarrow A \otimes H$ (write $F(x \otimes x') = x_F \otimes x'^F$ for all $x, x' \in H$) be a linear maps. Then $H^F \# B (= H \otimes B$ as a vector space) is an associative algebra with unit $1_H \otimes 1_B$ and multiplication
\[
(x \otimes b)(x' \otimes b') = x_F \otimes x'^F b b'
\]
for all $b, b' \in B$ and $x, x' \in H$ if and only if Eq. (RB3) and the following conditions hold:
\[
(RF2) \quad x_{Ff} \otimes x'^{ff} x'^F = x_f \otimes x'^F x'^F;
\]
\[
(RF3) \quad x_F \otimes x'^F b = x_F \otimes b x'^F,
\]
where \( b, b' \in B, x, x', x'' \in H \) and \( f = F \).

Proof. Let \( T \) be trivial in Proposition 2.1. \( \square \)

Here we remark that if we set \( F(x, x') = x_1 x'_1 \otimes \tau(x_2, x'_2) \) in \( H^f \# B \), then we get the right twist product \( H^f \# B \) and the conditions (RB3), (RF2), (RF3) become the conditions (RTC2) and

\[
\begin{align*}
(R2) & \quad \tau(x_1 x'_1, x'') \tau(x_2, x'_2) = \tau(x, x'_1 x'') \tau(x'_2, x'_2); \\
(R3) & \quad \tau(x, x') b = b \tau(x, x'),
\end{align*}
\]

which is the right version of \( (LT C 3), (L2), (L3) \).

(4) (Right twisted tensor product) Let \( H, B \) be two algebras and \( T : B \otimes H \to H \otimes B \) (write \( T(b \otimes x) = x_T \otimes b_T \) for all \( x \in H \) and \( b \in B \)) a linear map. Then \( H \# B \) (\( = H \otimes B \) as a vector space) is an associative algebra with unit \( 1_H \otimes 1_B \) and multiplication

\[
(x \otimes b)(x' \otimes b') = xx'_T \otimes b_T b'
\]

for all \( b, b' \in B \) and \( x, x' \in H \) if and only if Eqs. (RB1), (RB2) and the following condition hold:

\[
(R3) \quad x_T x'_1 \otimes b_{T1} = (x'')_T \otimes b_T,
\]

where \( b, b' \in B, x, x' \in H \) and \( t = T \).

Proof. Let \( F \) be trivial, i.e., \( F(x \otimes x') = xx' \otimes 1_B \) in Proposition 2.1. \( \square \)

(5) (Right crossed product) (\( \square \)) Let \( H \) be a Hopf algebra with a right weak action \( \triangleright \) on the algebra \( B \). Let \( \tau : H \otimes H \to B \) be a \( K \)-linear map. Let \( H^f \# B \) be the (usually nonassociative) algebra whose underlying space is \( H \otimes B \) and whose multiplication is given by

\[
(x \otimes b)(x' \otimes b') = x_1 x'_1 \otimes \tau(x_2, x'_2)(a \triangleright x'_3) b'
\]

for all \( b, b' \in B \) and \( x, x' \in H \). This \( H^f \# B \) is an associative algebra with unit \( 1_H \otimes 1_B \) if and only if \( \tau \) satisfies Eq. (RTC2) and the following conditions:

\[
\begin{align*}
(RC2) & \quad \tau(x_1 x'_1, x'') \tau(x_2, x'_2) \triangleleft x''_2 = \tau(x, x'_1 x'') \tau(x'_2, x''_2); \\
(RC3) & \quad \tau(x'_1, x''_1)((b \triangleright x'_2) \triangleleft x'') = (b \triangleright x'_1 x''_1) \tau(x'_2, x''_1)
\end{align*}
\]

for all \( b \in B \) and \( x, x', x'' \in H \).

Proof. Let \( T(b \otimes x) = x_1 \otimes b \triangleright x_2 \) in \( H^f \# B \). \( \square \)

2.2. Brzeziński’s two-sided crossed product algebra and its special cases. Next we give the main result in this section.

**Theorem 2.3.** Let \( H \) be a coalgebra with an element \( 1_H \) such that \( \Delta(1_H) = 1_H \otimes 1_H \) and \( A, B \) two algebras. Let \( G : H \otimes H \to A \otimes H \), \( R : H \otimes A \to A \otimes H \), \( T : B \otimes H \to H \otimes B \) and \( \tau : H \otimes H \to B \) be linear maps, such that Eqs. (LB1)-(LB3), (RB1), (RB2), (RTC2) hold. Then \( A^G \#_R H^T \# B \) (= \( A \otimes H \otimes B \) as vector space) is an associative algebra with unit \( 1_A \otimes 1_H \otimes 1_B \) and multiplication

\[
(a \otimes x \otimes b)(a' \otimes x' \otimes b') = aa'_R x_{R1}^G \otimes x_{T2}^G \otimes \tau(x_{R2}, x_{T2}) b_T b'
\]

for all \( a, a' \in A, x, x' \in H \) and \( b, b' \in B \) if and only if the following conditions hold:

\[
(BT1) \quad x_{G1}^G x_{T1}^G \otimes \tau(x_{G2}, x_{T2}'') \tau(x_2, x'_2)_T
\]
where $a \in A$, $x, x', x'' \in H$, $b \in B$ and $R = r, G = g, T = t$. In this case, we call the algebra $A\#^g_H H \#^g_B$ Brzeziński’s two-sided crossed product algebra.

Proof. ($\iff$) We check the associativity as follows: for all $a, a', a'' \in A$, $x, x', x'' \in H$ and $b, b', b'' \in B$,

\[(a \otimes x \otimes b)(a' \otimes x' \otimes b')(a'' \otimes x'' \otimes b'') = (aa' a'' a')_R x_{R1}^g \otimes x_{1g1}^G \otimes \tau(x_{R2}, x_{1gT2}) b_T b_T b_T b''),
\]

and

\[(a \otimes x \otimes b)((a' \otimes x' \otimes b') (a'' \otimes x'' \otimes b'')) = (a (a' a'') a')_R x_{R1}^g \otimes x_{1gT1G} \otimes \tau(x_{R2}, x_{1gT2}) b_T b_T b_T b''),
\]

so

\[(a \otimes x \otimes b)(a' \otimes x' \otimes b')(a'' \otimes x'' \otimes b'') = (a \otimes x \otimes b)((a' \otimes x' \otimes b')(a'' \otimes x'' \otimes b'')).
\]

We check the unit as follows: for all $a \in A$, $x \in H$ and $b \in B$,

\[(a \otimes x \otimes b)(1_A \otimes 1_H \otimes 1_B) = a_1 a_2 b_T = a_1 a_2 b_T.
\]


Proof. \( a_1 H^G \otimes x_{1G} \otimes \tau(1_H, x_2)b \)\)
\( a \otimes x_1 \otimes \tau(1_H, x_2)b \)
\( a \otimes x \otimes b. \)

\((\implies\) It is straightforward. \(\square\)

Remark 2.4. Taking either \( A = K \) or \( B = K \), we obtain the right Brzeziński crossed product and left twisted crossed product, respectively.

Example 2.5. (1) (Two-sided twisted crossed product) Let \( H \) be a bialgebra and \( A, B \) two algebras. Let \( \sigma : H \otimes H \rightarrow A, \tau : H \otimes H \rightarrow B, R : H \otimes A \rightarrow A \otimes H \) and \( T : B \otimes H \rightarrow H \otimes B \) be linear maps such that Eqs.\((LR1), (LR2), (RB1), (RB2), (LTC3), (RTC3)\) hold. Then \( A^\# H^R \# B \) (= \( A \otimes H \otimes B \) as vector space) is an associative algebra with unit \( 1_A \otimes 1_H \otimes 1_B \) and multiplication
\[
(a \otimes x \otimes b)(a' \otimes x' \otimes b') = aa'_R \sigma(x_{1R}, x'_{1R}) \otimes x_{2R} x'_{2R} \otimes \tau(x_{3R}, x'_{3R})b_T b'
\]
for all \( a, a' \in A \), \( x, x' \in H \) and \( b, b' \in B \) if and only if the conditions \((BT4)\) and
\begin{align*}
(TC1) & \quad \sigma(x_1, x_1') a_R \otimes (x_2 x_2')_R \otimes \tau(x_3, x'_3) = a_R \sigma(x_1, x_1') \otimes x_2 x_2' \otimes \tau(x_3, x'_3) \\
(TC2) & \quad \sigma(x_1, x_1') \sigma(x_2 x_2', x'_{1T}) \otimes x_3 x'_3 \otimes \tau(x_4, x'_4, x'_3) \tau(x_5, x'_5) = \sigma(x_1, x'_{1T}) \sigma(x_2 x_2', x'_{1T}) \otimes x_3 x'_3 \otimes \tau(x_4, x'_4, x'_3) \tau(x_5, x'_5) \\
(TC3) & \quad \sigma(x_{1T}, x'_{1T}) x_{2T} x'_{2T} \otimes \tau(x_{3T}, x'_{3T}) b_T b' = \sigma(x_{1T}, x'_{1T}) \otimes (x_2 x_2')_T \otimes \tau(x_3, x'_3) b_T b'
\end{align*}
are satisfied for all \( a \in A, x, x' \in H, b, b' \in B \) and \( R = r, T = t \).

Proof. Let \( G(x \otimes x') = \sigma(x_1, x'_{1T}) \otimes x_2 x_2' \) in Theorem 2.3. \(\square\)

(2) (Two-sided twisted product) Let \( H \) be a bialgebra and \( A, B \) two algebras. Let \( \sigma : H \otimes H \rightarrow A, \tau : H \otimes H \rightarrow B \) be linear maps and \( A^\# H \) left twisted product, \( H^\# B \) right twisted product. Then \( A^\# H^R \# B \) (= \( A \otimes H \otimes B \) as vector space) is an associative algebra with unit \( 1_A \otimes 1_H \otimes 1_B \) and multiplication
\[
(a \otimes x \otimes b)(a' \otimes x' \otimes b') = aa'_R \sigma(x_{1R}, x'_{1R}) \otimes x_2 x'_2 \otimes \tau(x_3, x'_3) b b'
\]
for all \( a, a' \in A \), \( x, x' \in H \) and \( b, b' \in B \).

Proof. Let \( R \) and \( T \) be trivial in \( A^\# H^R \# B \). And the rest is straightforward. \(\square\)

(3) (Two-sided twisted tensor product) Let \( H, A, B \) be three algebras. Let \( R : H \otimes A \rightarrow A \otimes H, T : B \otimes H \rightarrow H \otimes B \) be linear maps and \( A^\# H \) left twisted product, \( H^R \# B \) right twisted tensor product. Then \( A^\# R H^T \# B \) (= \( A \otimes H \otimes B \) as vector space) is an associative algebra with unit \( 1_A \otimes 1_H \otimes 1_B \) and multiplication
\[
(a \otimes x \otimes b)(a' \otimes x' \otimes b') = aa'_R \otimes x_R x'_R \otimes b_T b'
\]
for all \( a, a' \in A \), \( x, x' \in H \) and \( b, b' \in B \) if and only if Eq.\((BT4)\) holds.

Proof. We only prove the necessity of \((BT4)\) as follows. And the rest is straightforward by letting \( \sigma \) and \( \tau \) be trivial in \( A^\# R H^T \# B \). By the associativity in \( A^\# R H^T \# B \), we can get
\[
aa''_R a'_R \otimes (x_R x'_R), x''_R \otimes (b_T b'), b''_R = a(a' a''_R), x R x'_R, b_T b', b''_R.
\]
Let $a = a' = 1_A$, $b'' = b'' = 1_B$ and $x' = x'' = 1_H$ in the above equation. Then we obtain (BT4).

(4) (Two-sided crossed product) ([4]) Let $H$ be a bialgebra with a left weak action $\triangleright$ on the algebra $A$ and a right weak action $\triangleleft$ on the algebra $B$. Let $\sigma : H \otimes H \to A$ and $\tau : H \otimes H \to B$ be linear maps and $A\#^r H \# B$ left crossed product, $H \#^l B$ right crossed product. Then $A\#^r H \# B$ is a crossed product in $H$ as vector space, which is an associative algebra with unit $1_A \otimes 1_H \otimes 1_B$ and multiplication

$$(a \otimes x \otimes b)(a' \otimes x' \otimes b') = a(x_1 \triangleright a')\sigma(x_2, x'_1) \otimes x_3x'_2 \otimes \tau(x_3, x'_2)(b \triangleleft x'_2)b'$$

for all $a, a' \in A$, $x, x' \in H$ and $b, b' \in B$.

Proof. The result can be proved by letting $R(a \otimes x) = x_1 \triangleright a \otimes x_2$ and $T(b \otimes x) = x_1 \otimes b \triangleleft x_2$ in Example 2.5 (1). □

Remark 2.6. (1) When $\sigma$ and $\tau$ are trivial in $A\#^r H \# B$, we can obtain the two-sided smash product $A\# H \# B$ in [7, 8].

(2) Two-sided twisted tensor product $A\#^r H \# B$ in Example 2.5 (3) is exactly the iterated twisted tensor product in [9, Theorem 2.1] with the map $B \otimes A \longrightarrow A \otimes B$ trivial. In this case, Eq.(BT4) is exactly [9, Eq.(2.1)]. We note that Eq.(2.1) in [9, Theorem 2.1] is only a sufficient condition for $A\#^r H \# B$ to be an associative algebra. Here Eq.(BT4) is also necessary.

(3) Two-sided twisted tensor product $A\#^r H \# B$ in Example 2.5 (3) can be seen as the iterated crossed product in [9, Theorem 2.3] with the structure maps $Q : B \otimes A \longrightarrow A \otimes H \otimes B$, $b \otimes a \mapsto a \otimes 1_H \otimes b$, $\nu : A \otimes A \longrightarrow A \otimes H$, $a \otimes a' \mapsto aa' \otimes 1_H$ and $\sigma : B \otimes B \longrightarrow H \otimes B$, $b \otimes b' \mapsto 1_H \otimes bb'$. In this case, Eq.(BT4) is exactly [9, Eq.(2.9)] and [9, Eqs.(2.10) and (2.11)] hold automatically. Eq.(2.9) in [9, Theorem 2.3] is just one sufficient condition for $A\#^r H \# B$ to be an associative algebra.

2.3. Brzeziński’s two-sided crossed product bialgebra.

**Theorem 2.7.** Let $A$, $B$ be bialgebras and $H$ a coalgebra with an element $1_H$ such that $\Delta(1_H) = 1_H \otimes 1_H$. Let $G : H \otimes H \longrightarrow A \otimes H$, $R : H \otimes A \longrightarrow A \otimes H$, $T : B \otimes H \longrightarrow H \otimes B$ and $\tau : H \otimes H \longrightarrow B$ be linear maps such that $\varepsilon_B(\tau(x, x')) = \varepsilon_H(x)\varepsilon_H(x')$. Then $A\#^G H \# B$ equipped with the two-sided tensor product coalgebra $A \otimes H \otimes B$ becomes a bialgebra if and only if the following conditions hold:

\begin{enumerate}
  \item[(G1)] $\varepsilon_A(x^G)\varepsilon_H(x') = \varepsilon_H(x)\varepsilon_H(x')$;
  \item[(G2)] $\varepsilon_A(a_R)\varepsilon_H(x_R) = \varepsilon_A(a)\varepsilon_H(x)$; $\varepsilon_B(b_T)\varepsilon_H(x_T) = \varepsilon_B(b)\varepsilon_H(x)$;
  \item[(G3)] $a_{R_1} \otimes x_{R_1} \otimes a_{R_2} \otimes x_{R_2} = a_{1R} \otimes x_{1R} \otimes a_{2R} \otimes x_{2R}$;
  \item[(G4)] $x_{T_1} \otimes b_{T_1} \otimes x_{T_2} \otimes b_{T_2} = x_{1T} \otimes b_{1T} \otimes x_{2T} \otimes b_{2T}$;
  \item[(G5)] $x_{1G_1} \otimes x'_{1G_2} \otimes \tau(x_2, x'_2) \otimes x_{1G_2} \otimes x'_{1G_2} \otimes \tau(x_3, x'_3) = x_{1G_1} \otimes x'_{1G_2} \otimes \tau(x_2, x'_2) \otimes x_{3G_2} \otimes x'_{3G_2} \otimes \tau(x_4, x'_4)$,
\end{enumerate}

where $a \in A$, $x, x' \in H$ and $b \in B$. In this case, we call this bialgebra Brzeziński’s two-sided crossed product bialgebra and denoted by $A \#^G H \# B$. 

Proof. (\(\iff\)) First by Eqs.(G1) and (G2), we can show that \(\varepsilon_{A\otimes B}\) is an algebra map. Next, we check that \(\Delta_{A\otimes B}\) is an algebra map. In fact, for all \(a, a' \in A, x, x' \in H\) and \(b, b' \in B\), we have

\[
\Delta_{A\otimes B}(a \otimes x \otimes b)(a' \otimes x' \otimes b') = \Delta_{A\otimes B}(aa'_{R}x_{L}^{G} \otimes x'_{T1G1} \otimes \tau(x_{R2}, x'_{T2})b_{T}b')
\]

\[
= a_{1}a'_{R}x_{R1}^{G} \otimes x'_{T1G1} \otimes \tau(x_{R2}, x'_{T2})b_{T1}b'_{1} \otimes a_{2}a'_{R}x_{R2}^{G} \otimes x'_{T1G2} \otimes \tau(x_{R2}, x'_{T2})b_{T2}b'_{2}
\]

\[
(\text{G5}) = a_{1}a'_{R}x_{R1}^{G} \otimes x'_{T1G1} \otimes \tau(x_{R12}, x'_{T12})b_{T1}b'_{1} \otimes a_{2}a'_{R}x_{R1}^{G} \otimes x'_{T21G} \otimes \tau(x_{R22}, x'_{T22})b_{T2}b'_{2}
\]

\[
(\text{G3}) = a_{1}a'_{R}x_{R1}^{G} \otimes x'_{T1G1} \otimes \tau(x_{R12}, x'_{T12})b_{T1}b'_{1} \otimes a_{2}a'_{R}x_{R1}^{G} \otimes x'_{T21G} \otimes \tau(x_{R22}, x'_{T22})b_{T2}b'_{2}
\]

\[
(\text{G4}) = a_{1}a'_{R}x_{R1}^{G} \otimes x'_{T1G1} \otimes \tau(x_{R12}, x'_{T12})b_{T1}b'_{1} \otimes a_{2}a'_{R}x_{R1}^{G} \otimes x'_{T21G} \otimes \tau(x_{R22}, x'_{T22})b_{T2}b'_{2}
\]

\[
= \Delta_{A\otimes B}(a \otimes x \otimes b)\Delta_{A\otimes B}(a' \otimes x' \otimes b')
\]

and

\[
\Delta_{A\otimes B}(1_{A} \otimes 1_{H} \otimes 1_{B}) = 1_{A1} \otimes 1_{H1} \otimes 1_{B1} \otimes 1_{A2} \otimes 1_{H2} \otimes 1_{B2}
\]

\[
= 1_{A} \otimes 1_{H} \otimes 1_{B} \otimes 1_{A} \otimes 1_{H} \otimes 1_{B}.
\]

(\(\implies\)) Since \(\varepsilon_{A\otimes B}\) is an algebra map, then we have

\[
(\text{BA2}) \quad \varepsilon_{A}(aa'_{R}x_{R1}^{G}e_{H}(x'_{T1G1})e_{B}(\tau(x_{R2}, x'_{T2})b_{T}b')) = \varepsilon_{A}(a)e_{H}(x)e_{B}(b)e_{A}(a')e_{H}(x')e_{B}(b').
\]

Let \(a = a' = 1_{A}, b = b' = 1_{B}\) in Eq.(BA2), we obtain (G1). Similarly, (G2) holds.

By \(\Delta((a \otimes x \otimes b)(a' \otimes x' \otimes b')) = \Delta(a \otimes x \otimes b)\Delta(a' \otimes x' \otimes b')\), we have

\[
(\text{BA3}) \quad (aa'_{R}x_{R1}^{G})_{1} \otimes x'_{T1G1} \otimes (\tau(x_{R2}, x'_{T2})b_{T}b')_{1} \otimes (aa'_{R}x_{R1}^{G})_{2} \otimes x'_{T1G2} \otimes (\tau(x_{R2}, x'_{T2})b_{T}b')_{2}
\]

\[
= a_{1}a'_{R}x_{R1}^{G} \otimes x'_{T1G1} \otimes \tau(x_{R12}, x'_{T12})b_{T1}b'_{1} \otimes a_{2}a'_{R}x_{R1}^{G} \otimes \tau(x_{R22}, x'_{T22})b_{T2}b'_{2}.
\]

Let \(a = 1_{A}, b = b' = 1_{B}, x' = 1_{H}\) in Eq.(BA3), we have

\[
(a'_{R}x_{R1}^{G})_{1} \otimes 1_{HT1G1} \otimes (\tau(x_{R2}, 1_{HT2})b_{1})_{1} \otimes (a'_{R}x_{R1}^{G})_{2} \otimes 1_{HT1G2}
\]

\[
\otimes (\tau(x_{R2}, 1_{HT2})b_{1})_{2} = a'_{R}x_{R1}^{G} \otimes 1_{HT1G} \otimes \tau(x_{R12}, 1_{HT2})b_{1}
\]

\[
\otimes a'_{R}x_{R1}^{G} \otimes 1_{HT1G} \otimes \tau(x_{R22}, 1_{HT2})b_{1}.
\]

Then we can obtain (G3) by \((LB1), (LB3)\) and \((RB1), (RTC2)\). Lieewise, one has (G4) and (G5).

\(\square\)

**Definition 2.8.** Let \(A, B\) be two algebras, and \(H\) be a coalgebra with a distinguished element \(1_{H} \in H\). Let \(G : H \otimes H \longrightarrow A \otimes H, \tau : H \otimes H \longrightarrow B\) and \(S : H \longrightarrow H\) be linear maps. Then \(S\) is called a \((G, \tau)-\text{antipode of } H\) if for all \(x \in H\), the following conditions hold:

\[
(\text{I1}) \quad S(x_{1})_{1}^{G} \otimes x_{2}G \otimes \tau(S(x_{1})_{2}, x_{3}) = 1_{A} \otimes 1_{H} \otimes 1_{B}\varepsilon_{H}(x);
\]

\[
(\text{I2}) \quad x_{1}^{G} \otimes S(x_{3})_{1}G \otimes \tau(x_{2}, S(x_{3})_{2}) = 1_{A} \otimes 1_{H} \otimes 1_{B}\varepsilon_{H}(x).
\]

In this case, we call \(H\) a \((G, \tau)\)-Hopf algebra.

**Remark 2.9.** (1) Let \(G\) and \(\tau\) be trivial, i.e., \(x^{G} \otimes y_{G} = 1_{A} \otimes xy\) and \(\tau(x, y) = 1_{B}\varepsilon(x)v(y)\) hold for all \(x, y \in H\). Then (I1) and (I2) can ensure that \(S\) is the antipode of \(H\). Therefore in this case, the Hopf algebra \(H\) is a \((G, \tau)\)-Hopf algebra.

(2) Let \(\sigma : H \otimes H \longrightarrow A\) be linear map. When \(G(x, x') = \sigma(x_{1}, x'_{1}) \otimes x_{2}x'_{2}\) in Definition 2.8, we call \(H\) a \((\sigma, \tau)-\text{Hopf algebra}\).
Proposition 2.10. Let $A, B$ be two Hopf algebras with antipode $S_A, S_B$, $H$ a coalgebra with an element $1_H$ such that $\Delta(1_H) = 1_H \otimes 1_H$ and $S_H : H \rightarrow H$ a linear map. Suppose that $A \#^G_H B$ is a Brzeziński's two-sided crossed product bialgebra. Then $A \#^G_H B$ is a Hopf algebra with antipode $\overline{S}$ defined by

$$\overline{S}(a \otimes x \otimes b) = (1_A \otimes 1_H \otimes S_B(b))(1_A \otimes S_H(x) \otimes 1_B)(S_A(a) \otimes 1_H \otimes 1_B)$$

if and only if $H$ is a $(G, \tau)$-Hopf algebra.

Proof. $(\Longleftarrow)$ For all $a \in A, b \in B, x \in H$, and $R = r, T = t$, we have

$$\overline{S} \ast (id_A \ast_{r}^G H \ast_B)(a \otimes x \otimes b)$$

and

$$\overline{S} = a_1 S_A(a_2) R, x_{1r1}^G \otimes S_H(x_2) T R r_1 G \otimes \tau(x_{1r2}, S_H(x_2) T R r_2) b_1 S_B(b_2) T$$

Then $\overline{S}$ is the antipode of $A \#^G_H B$.

$(\Longrightarrow)$ Obvious. \hfill \qed

Next we only list two special cases. Other cases can be given similarly.

Corollary 2.11. Let $H, A, B$ be bialgebras. Let $\sigma : H \otimes H \rightarrow A \otimes H$, $R : H \otimes A \rightarrow A \otimes H$, $T : B \otimes H \rightarrow H \otimes B$ and $\tau : H \otimes H \rightarrow B$ be linear maps such that $\epsilon_B(\tau(x, x')) = \epsilon_H(x)\epsilon_H(x')$. Then $A \#^G_H B$ equipped with the two-sided tensor product coalgebra $A \otimes H \otimes B$ becomes a bialgebra if and only if Eqs. (G2)-(G4) and the following conditions hold:

$$(J1) \quad \epsilon_A(\sigma(x, x')) = \epsilon_H(x)\epsilon_H(x');$$

$$(J2) \quad \sigma(x, x')_1 \otimes \sigma(x, x')_2 = \sigma(x_1, x'_1) \otimes \sigma(x_2, x'_2);$$

$$(J3) \quad \tau(x, x')_1 \otimes \tau(x, x')_2 = \tau(x_1, x'_1) \otimes \tau(x_2, x'_2);$$

$$(J4) \quad \tau(x_1, x'_1) \otimes x_2 x'_2 = \tau(x_2, x'_2) \otimes x_1 x'_1;$$

$$(J5) \quad \sigma(x_1, x'_1) \otimes x_2 x'_2 = \sigma(x_2, x'_2) \otimes x_1 x'_1.$$
(J6) \( \tau(x_1, x'_1) \otimes \sigma(x_2, x'_2) = \tau(x_2, x'_2) \otimes \sigma(x_1, x'_1) \),

where \( x, x' \in H \). In this case, we call this bialgebra two-sided twisted crossed product bialgebra and denoted by \( A \ast \ast_{\mathcal{R}} H^*_T \ast \ast B \). Furthermore, \( A \ast \ast_{\mathcal{R}} H^*_T \ast \ast B \) is a Hopf algebra with antipode Eq.(7) if and only if \( H \) is a \((\sigma, \tau)\)-Hopf algebra.

Proof. Note here Eqs.(J2)-(J6) are equivalent to Eq.(G5) for \( G(x, x') = \sigma(x_1, x'_1) \otimes x_2 x'_2 \). The rest is obvious. \( \Box \)

Corollary 2.12. Let \( H \) be a bialgebra. Let \( \sigma : H \otimes H \rightarrow A \) be a linear map, where \( A \) is a bialgebra with a left \( H \)-weak action such that \( \varepsilon_A(\sigma(x, x')) = \varepsilon_H(x) \varepsilon_H(x') \). Let \( \tau : H \otimes H \rightarrow B \) be a linear map, where \( B \) is a bialgebra with a right \( H \)-weak action such that \( \varepsilon_B(1_B) = 1 \), \( \varepsilon_B(\tau(x, x')) = \varepsilon_H(x) \varepsilon_H(x') \). The two-sided crossed product \( A \#^\sigma H \#^\tau B \) equipped with the two-sided tensor product coalgebra \( A \otimes H \otimes B \) becomes a bialgebra if and only if Eqs.(J2)-(J6), (D4), (F3), (F4) and the following condition holds:

\[ (P1) \quad x \otimes (x_1 \triangleright a) = x_1 \otimes (x_2 \triangleright a), \quad (b \triangleleft x_2) \otimes x_1 = (b \triangleleft x_1) \otimes x_2, \]

where \( x \in H, a \in A \). In this case, we call this bialgebra two-sided crossed product bialgebra and denoted by \( A \ast \ast_{\mathcal{R}} H^\tau \ast B \). Furthermore, \( A \ast \ast_{\mathcal{R}} H^\tau \ast B \) is a Hopf algebra with antipode Eq.(7) if and only if \( H \) is a \((\sigma, \tau)\)-Hopf algebra.

Proof. It is straightforward by setting the left and right coactions are trivial in Theorem 3.2. \( \Box \)

Example 2.13. Let \( H = K C_2 = K[1, a] \) and \( A = B = K C_4 = K[1, x, x^2, x^3] \) be group Hopf algebras \((\mathbb{K})\), where \( C_i \) is cyclic group with order \( i, i = 2, 4 \). Define the linear maps \( \triangleright : H \otimes A \rightarrow A, \triangleright : B \otimes H \rightarrow B, \sigma : H \otimes H \rightarrow A \) and \( \tau : H \otimes H \rightarrow B \) by

\[
1_H \triangleright 1_A = 1_A, \quad 1_H \triangleright x = x, \quad 1_H \triangleright x^2 = x^2, \quad 1_H \triangleright x^3 = x^3, \\
a \triangleright 1_A = 1_A, \quad a \triangleright x = x^3, \quad a \triangleright x^2 = x^2, \quad a \triangleright x^3 = x;
\]

\[
1_B \triangleleft 1_H = 1_B, \quad x \triangleleft 1_H = x, \quad x^2 \triangleleft 1_H = x^2, \quad x^3 \triangleleft 1_H = x^3, \\
1_B \triangleleft a = 1_B, \quad x \triangleleft a = x^3, \quad x^2 \triangleleft a = x^2, \quad x^3 \triangleleft a = x;
\]

and

\[
\sigma(1_H, 1_H) = 1_A, \quad \sigma(1_H, a) = 1_A, \quad \sigma(a, 1_H) = 1_A, \quad \sigma(a, a) = x; \\
\tau(1_H, 1_H) = 1_A, \quad \tau(1_H, a) = 1_A, \quad \tau(a, 1_H) = 1_A, \quad \tau(a, a) = x,
\]

then by \([\mathbb{K}]\), we know that \( \triangleright \) is a weak action of \( H \) on \( A \) and \( \triangleleft \) is a weak action of \( H \) on \( B \). Thus we can get two-sided crossed product \( A \#^\sigma H \#^\tau B \). Also by Corollary 2.12, two-sided crossed product \( A \#^\sigma H \#^\tau B \) equipped with the two-sided tensor product coalgebra \( A \otimes H \otimes B \) becomes a bialgebra, since \( H \) and \( A, B \) are cocommutative. But unfortunately \( \tilde{S} \) defined by Eq.(7) is not an antipode of \( A \ast \ast_{\mathcal{R}} H^\tau \ast B \), because Eq.(11) doesn’t hold for \( a \in H \).

3. An extended version of Majid’s double biproduct

Firstly, we recall the construction of double crossed biproduct in \([\mathbb{K}]\).
Lemma 3.1. ([74]) Let $H$ be a bialgebra. Let $\sigma : H \otimes H \to A$ be a linear map, where $A$ is a left $H$-comodule coalgebra such that $\epsilon_A(1_A) = 1$, $\epsilon_A(\sigma(x, x')) = \epsilon_H(x)\epsilon_H(x')$, and an algebra with a left $H$-weak action. Let $\tau : H \otimes H \to B$ be a linear map, where $B$ is a right $H$-comodule coalgebra such that $\epsilon_B(1_B) = 1$, $\epsilon_B(\tau(x, x')) = \epsilon_H(x)\epsilon_H(x')$, and an algebra with a right $H$-weak action. The two-sided crossed product $A \#^\sigma H \#^\tau B$ equipped with the two-sided smash coproduct $A \times H \times B$ becomes a bialgebra if and only if the following conditions hold:

(C1) $\epsilon_A, \epsilon_B$ are algebra maps;

(C2) $\Delta_A(1_A) = 1_A \otimes 1_A$, $\Delta_B(1_B) = 1_B \otimes 1_B$;

(C3) $1_{A-1} \otimes 1_{A0} = 1_H \otimes 1_A$, $1_{B[0]} \otimes 1_{B[1]} = 1_B \otimes 1_H$;

(C4) $\epsilon_A(x \triangleright a) = \epsilon_A(a)\epsilon_H(x)$, $\epsilon_B(b \triangleleft x) = \epsilon_B(b)\epsilon_H(x)$;

(C5) $a_1 \otimes a_{2-1} \otimes a_{20} = a_1\sigma(a_{2-1}, x_1) \otimes a_{2-12} x_2 \otimes a_{20}$;

(C6) $b_{1[0]} \otimes x b_{1[1]} \otimes b_2 = b_{1[0]} \otimes x b_{1[1]} \otimes \tau(x_2, b_{1[2]}) b_2$;

(C7) $(aa')_1 \otimes (aa')_2 = a_1 a_2, a_{2-1} \otimes a_{20} = a_1 a_{2-1 \otimes} a_{2-12} \otimes a_{2-1} a_{20}$;

(C8) $(bb')_1 \otimes (bb')_2 = b_{1[0]} b_{1[1]} \otimes b_{1[2]} b_{1[3]}$;

(C9) $\sigma(x_1, x'_1) \otimes \sigma(x_1, x'_1) \otimes \sigma(x_1, x'_1) \otimes x_2 x'_2 \otimes \sigma(x_1, x'_1)$;

(C10) $\tau(x_2, x'_2) \otimes x_1 x'_1 \otimes \tau(x_2, x'_2) \otimes \tau(x_2, x'_2) = \tau(x_1, x'_1) \otimes x_2 x'_2 \otimes \tau(x_1, x'_1)$;

(C11) $(a \triangleright a)_1 \otimes (a \triangleright a)_2 = (a_1 a_2) \otimes x_2 \otimes (a_1 a_2)$;

In this case, we call this bialgebra the double crossed biproduct denoted by $A \#^\sigma H \#^\tau B$.

Next we give an improved version of the above result.

Theorem 3.2. Under the assumption of Lemma 3.1. The two-sided crossed product $A \#^\sigma H \#^\tau B$ equipped with the two-sided smash coproduct $A \times H \times B$ becomes a bialgebra if and only if

(C1) – (C6) and the following conditions hold:

(D7) $(aa')_1 \otimes (aa')_2 = a_1 a_2, a_{2-1} \otimes a_{2-12} \otimes a_{20}$;

(D8) $(aa')_1 \otimes (aa')_2 = a_1 a_2, a_{2-1} \otimes a_{2-12} \otimes a_{20}$;

(D9) $(bb')_1 \otimes (bb')_2 = b_{1[0]} b_{1[1]} \otimes b_{1[2]} b_{1[3]}$;

(D10) $(bb')_1 \otimes (bb')_2 = b_{1[0]} b_{1[1]} \otimes b_{1[2]} b_{1[3]}$;

(D11) $\sigma(x_1, x'_1) \otimes \sigma(x_1, x'_1) = x_1 x'_1 \otimes \sigma(x_1, x'_1)$;

(D12) $\sigma(x_1, x'_1) \otimes x_1 x'_1 \otimes \sigma(x_1, x'_1)$;

(D13) $\tau(x_2, x'_2) \otimes x_1 x'_1 \otimes x_2 x'_2$;

(D14) $\tau(x, x'_1) \otimes x_1 x'_1 \otimes x_2 x'_2$;

In this case, we call this bialgebra the double crossed biproduct denoted by $A \#^\sigma H \#^\tau B$. 


(D15) \((x \triangleright a_1) \otimes (x \triangleright a_2) = (x_1 \triangleright a_1)\sigma(x_2, a_{2-1}) \otimes x_3 \triangleright a_{20};\)

(D16) \((x_1 \triangleright a)_{-1} x_2 \otimes (x_1 \triangleright a)_0 = x_1 a_{-1} \otimes x_2 \triangleright a_0;\)

(D17) \((b \triangleleft x_2)_0 \otimes (x_1 \triangleleft x_2)_1 = b_{[0]} \triangleleft x_1 \otimes b_{[1]} x_2;\)

(D18) \((b \triangleleft x_1) \otimes (b \triangleleft x_2) = b_{[0]} \triangleleft x_1 \otimes \tau(b_{[1]} x_1, x_2)(b_2 \triangleleft x_3);\)

(D19) \(a_{-1} x \otimes 1_B \otimes a_0 = a_{-1} a_{1} \otimes \tau(a_{-12}, x_2) \otimes a_0;\)

(D20) \(b_{[0]} \otimes 1_A \otimes x b_{[1]} = b_{[0]} \otimes \sigma(x_1, b_{[1]} x) \otimes x_2 b_{[1]};\)

(D21) \(b_{[1]} \triangleright a_0 \otimes b_{[0]} \triangleright a_{-1} = a \otimes b;\)

(D22) \(\tau(x_3, x'_1)(b \triangleleft x'_2)b' \otimes a(x_1 \triangleright a')\sigma(x_2, x'_1)
\quad = \tau(a_{-1} x_1, a'_{-1} x'_1)(b_{[0]} \triangleleft x'_2) b'_{[0]} \otimes a_0 (x_2 \triangleright a_0') \sigma(x_3 b_{[1]} x'_3 b'_{[1]})).\)

**Proof.** By Lemma 5.1, we only need to prove that the conditions (C7)-(C13) are equivalent to (D7)-(D22).

(a) Step 1. Firstly, apply \(id \otimes \varepsilon \otimes id\) to Eq.(C7), we can get Eq.(D7); apply \(\varepsilon \otimes id \otimes id\) to Eq.(C7), then Eq.(D8) holds. On the other hand, by Eqs.(D7) and (D8), we have

\[
(aa')_1 \otimes (aa')_{2-1} \otimes (aa')_{20} \overset{(D7)}{=} a_1 (a_{2-1} \triangleright a'_1) \otimes (a_{2-12} a'_{2-1}) \otimes (a_{20} a'_{20})_0
\]

\[
\overset{(D8)}{=} a_1 (a_{2-1} \triangleright a'_1) \otimes (a_{2-12} a'_{2-1}) \otimes (a_{20} a'_{20})_0
\]

\[
= a_1 (a_{2-1} \triangleright a'_1) \sigma(a_{2-12}, a'_{2-1}) \otimes (a_{20} a'_{20})_0
\]

i.e., Eq.(C7) holds. Thus Eq.(C7) \(\iff\) Eqs.(D7) and (D8).

Similarly, we can obtain that Eq.(C8) \(\iff\) Eqs.(D9) and (D10); Eq.(C9) \(\iff\) Eqs.(D11) and (D12); Eq.(C10) \(\iff\) Eqs.(D13) and (D14); Eq.(C11) \(\iff\) Eqs.(D15) and (D16); Eq.(C12) \(\iff\) Eqs.(D17) and (D18).

(b) Step 2. Secondly, let \(x = 1, a' = 1, b = b' = 1\) in Eq.(C13), we have

\[
a_{-1} x'_1 \otimes 1_B \otimes a_0 \otimes x'_2 = a_{-1} x'_1 \otimes \tau(a_{-12}, x'_2) \otimes a_0 \otimes x'_3
\]

(7)

Using \(id \otimes id \otimes id \otimes \varepsilon\) to Eq.(7), one can get Eq.(D19). Applying \(\varepsilon \otimes id \otimes id\) to Eq.(19), we have

\[
a \otimes 1_B \varepsilon(x) = a_0 \otimes \tau(a_{-1}, x).
\]

(8)

Set \(a = a' = 1, b = 1, x' = 1\) in Eq.(C13), we have

\[
x_1 \otimes b'_{[0]} \otimes 1_A \otimes x_2 b'_{[1]} = x_1 \otimes b'_{[0]} \otimes \sigma(x_2, b'_{[1]}) \otimes x_3 b'_{[2]}
\]

(9)

Applying \(\varepsilon \otimes id \otimes id \otimes id\) to Eq.(9), one can get Eq.(D20). Applying \(id \otimes id \otimes \varepsilon\) to Eq.(D20), we have

\[
b \otimes 1_A \varepsilon(x) = b_{[0]} \otimes \sigma(x, b_{[1]}).
\]

(10)

Setting \(a = 1, b' = 1, x = x' = 1\) in Eq.(C13), we have

\[
a'_{-1} \otimes b_{[0]} \otimes a'_0 \otimes b_{[1]} = a'_{-11} \otimes (b_{[0]} \triangleright a'_{-12}) \otimes (b_{[1]} \triangleright a'_0) \otimes b_{[2]}.
\]

(11)

Applying \(\varepsilon \otimes id \otimes id \otimes \varepsilon\) to Eq.(11), we have Eq.(D21).

Applying \(\varepsilon \otimes id \otimes id \otimes \varepsilon\) to Eq.(C13), we have

\[
\tau(x_3, x'_2)(b \triangleleft x'_3)b' \otimes a(x_1 \triangleright a')\sigma(x_2, x'_1)
\]

(12)
While

\[
\tau(a_{-1}x_1, a'_{-1}x'_1)(b[0] \prec a'_{-12}x'_2)b'[0] \otimes a_0(x_2b[1]_1) \succ a'_0)\sigma(x_3b[1]_2, x'_3b'[1]_1).
\]

RHS of Eq.(12)

\[\begin{align*}
(D^{19})(D^{20}) & \equiv \tau(a_{-1}x_1, a'_{-11}x'_1)(b[0] \prec a'_{-12}x'_2)\tau(a'_{-13}, x'_3)b'[0] \\
& \otimes a_0\sigma(x_2, b[1]_1)(x_3b[1]_2 \succ a'_0)\sigma(x_4b[1]_3, x'_4b'[1]_1)
\end{align*}\]

\[\begin{align*}
(LC^3)(RC^3) & \equiv \tau(a_{-1}x_1, a'_{-11}x'_1)\tau(a'_{-12}, x'_2)((b[0] \prec a'_{-13}) \prec x'_3)b'[0] \\
& \otimes a_0(x_2 \succ (b[1]_1 \succ a'_0))\sigma(x_3, b[1]_2)\sigma(x_4b[1]_3, x'_4b'[1]_1)
\end{align*}\]

\[\begin{align*}
& = \tau(a_{-1}x_1, a'_{-11}x'_1)\tau(a'_{-12}, x'_2)((b[0][0] \prec a'_{0-1}) \prec x'_3)b'[0] \\
& \otimes a_0(x_2 \succ (b[0][1] \succ a'_0))\sigma(x_3, b[1]_1)\sigma(x_4b[1]_2, x'_4b'[1]_1)
\end{align*}\]

\[\begin{align*}
(D^{21}) & \equiv \tau(a_{-1}x_1, a'_{-11}x'_1)(b[0] \prec x'_3)b'[0] \otimes a_0(x_2 \succ a'_0)\sigma(x_3b[1]_1, x'_3b'[1]_1)
\end{align*}\]

RHS of Eq.(D22)

and LHS of Eq.(12) is exactly LHS of Eq.(D22), so we obtain Eq.(D22).

On the other hand, we have

RHS of Eq.(C13)

\[\begin{align*}
(D^{20}) & \equiv a_{-11}x_1a'_{-11}x'_1 \otimes \tau(a_{-12}x_2, a'_{-12}x'_2)(b[0] \prec a'_{-13}x'_3)b'[0] \\
& \otimes a_0\sigma(x_3, b[1]_1)(x_4b[1]_2 \succ a'_0)\sigma(x_5b[1]_3, x'_5b'[1]_1) \otimes x_6b[1]_4x'_6b'[1]_2
\end{align*}\]

\[\begin{align*}
(D^{19}) & \equiv \tau(a_{-11}x_1a'_{-11}x'_1) \otimes \tau(a_{-12}x_2, a'_{-12}x'_2)(b[0] \prec a'_{-13}x'_3)\tau(a'_{-14}, x'_4)b'[0] \\
& \otimes a_0\sigma(x_3, b[1]_1)(x_4b[1]_2 \succ a'_0)\sigma(x_5b[1]_3, x'_5b'[1]_1) \otimes x_6b[1]_4x'_6b'[1]_2
\end{align*}\]

\[\begin{align*}
(LC^3)(RC^3) & \equiv \tau(a_{-11}x_1a'_{-11}x'_1) \otimes \tau(a_{-12}x_2, a'_{-12}x'_2)\tau(a'_{-13}, x'_3)((b[0][0] \prec a'_{0-1}) \prec x'_3)b'[0] \\
& \otimes a_0(x_3 \succ (b[0][1] \succ a'_0))\sigma(x_4, b[1]_1)\sigma(x_5b[1]_2, x'_5b'[1]_1) \otimes x_6b[1]_3x'_6b'[1]_2
\end{align*}\]

\[\begin{align*}
(D^{21}) & \equiv \tau(a_{-11}x_1a'_{-11}x'_1) \otimes \tau(a_{-12}x_2, a'_{-12}x'_2)\tau(a'_{-13}, x'_3)(b[0][0] \prec x'_4)b'[0] \\
& \otimes a_0(x_3 \succ a'_0)\sigma(x_4, b[1]_1)\sigma(x_5b[1]_2, x'_5b'[1]_1) \otimes x_6b[1]_3x'_6b'[1]_2
\end{align*}\]

\[\begin{align*}
(D^{22}) & \equiv \tau(a_{-11}x_1a'_{-11}x'_1) \otimes \tau(a_{-12}x_2, a'_{-12}x'_2)(b[0] \prec x'_3)b'[0] \\
& \otimes a_0(x_3 \succ a'_0)\sigma(x_4b[1]_1, x'_4b'[1]_1) \otimes x_5b[1]_2x'_5b'[1]_2
\end{align*}\]

\[\begin{align*}
(D^{19})(D^{20}) & \equiv \tau(a_{-11}x_1a'_{-11}x'_1) \otimes \tau(a_{-12}x_2, a'_{-12}x'_2)(b[0][0] \prec x'_3)b'[0] \\
& \otimes a_0(x_3 \succ a'_0)\sigma(x_4b[1]_1, x'_4b'[1]_1) \otimes x_5b[1]_2x'_5b'[1]_2
\end{align*}\]

\[\begin{align*}
(D^{22}) & \equiv \tau(a_{-11}x_1a'_{-11}x'_1) \otimes \tau(a_{-12}x_2, a'_{-12}x'_2)(b[0][0] \prec x'_3)b'[0] \\
& \otimes a_0(x_3 \succ a'_0)\sigma(x_4b[0][1], x'_4b'[0][1]) \otimes x_5b[1]_2x'_5b'[1]_2
\end{align*}\]

\[\begin{align*}
(D^{22}) & \equiv \text{LHS of Eq.(C13).}
\end{align*}\]
DOUBLE CROSSED BIPRODUCTS AND RELATED STRUCTURES

Thus Eq.(C13) ⇔ Eqs.(D19)-(D22). These finish the proof. \square

Remark 3.3. (1) Eq.(D21) is exactly Eq.[1] in the Majid’s double biproduct.

(2) The first half parts in Eqs.(C1)-(C4) and Eqs.(D5), (D7), (D8), (D11), (D12), (D15), (D16) are the conditions A1)-A9) and twisted comodule cocycle in [24, Theorem 2.5].

(3) The second half parts in Eqs.(C1)-(C4) and Eqs.(D6), (D9), (D10), (D13), (D14), (D17), (D18) are the conditions (G1)-(G9) in [14, Remark 2.3] and (C6) in [14, Theorem 2.1]. These are the necessary and sufficient conditions for the right crossed product $H \# B$ equipped with the right smash coproduct $H \times B$ becomes a bialgebra.

Proposition 3.4. Let $H$ be a bialgebra. Let $\sigma : H \otimes H \to A$ be a linear map, where $A$ is a left $H$-comodule coalgebra such that $\varepsilon_A(1_A) = 1$, and an algebra with a left $H$-weak action. Let $B$ be a right $H$-module algebra and also a comodule coalgebra such that $\varepsilon_B(1_B) = 1$. The one-sided crossed product $A \#^\sigma H \# B$ equipped with the two-sided smash coproduct $A \times H \times B$ becomes a bialgebra if and only if Eqs.(C1)-(C5), (D7), (D8), (D10)-(D12), (D15)-(D17), (D20), (D21) and the following conditions hold:

\[
\begin{align*}
(E1) \quad (bb')_1 \otimes (bb')_2 &= b_1b'_{[0]} \otimes (b_2 \triangleleft b'_{[1]})b'_2; \\
(E2) \quad (b \triangleleft x)_1 \otimes (b \triangleleft x)_2 &= (b_1 \triangleleft x_1) \otimes (b_2 \triangleleft x_2); \\
(E3) \quad (b_{[0]} \triangleleft x'_2)b'_{[1]} \otimes (x_1 \triangleright a)\sigma(x_2, x'_1) &= (b_{[0]} \triangleleft x'_1)b'_{[0]} \otimes (x_1 \triangleright a)\sigma(x_2b_{[1]}, x'_2b'_{[1]}).
\end{align*}
\]

Proof. Let $\tau$ be trivial, i.e., $\tau(x, y) = \varepsilon(x)\varepsilon(y)1_B$ in Theorem 3.2. \square

Remark 3.5. (1) By Eq.(D21), Eq.[1] in the Majid’s double biproduct is not only sufficient but also necessary.

(2) The conditions in Proposition 3.4 are equivalent to the ones in [11, Theorem 3.2].

Proof. By Eq.(D22), we have

\[
(b_{[0]} \triangleleft x'_1)b'_{[0]} \otimes a(x_1 \triangleright a')\sigma(x_2b_{[1]}, x'_2b'_{[1]}) = (b \triangleleft x'_2)b' \otimes a(x_1 \triangleright a')\sigma(x_2, x'_1). \tag{13}
\]

Setting $a = 1 = a'$, $b' = 1$ in Eq.(13), we have

\[
(b_{[0]} \triangleleft x'_1) \otimes \sigma(xb_{[1]}, x'_2) = (b \triangleleft x'_2) \otimes \sigma(x, x'_1). \tag{14}
\]

Setting $a = 1 = a'$, $b = 1$ in Eq.(13), we get

\[
b'_{[0]} \otimes \sigma(x, x'b'_{[1]}) = b' \otimes \sigma(x, x'). \tag{15}
\]

On the other hand, we have

\[
(b_{[0]} \triangleleft x'_1)b'_{[0]} \otimes a(x_1 \triangleright a')\sigma(x_2b_{[1]}, x'_2b'_{[1]}) \tag{16}
\]

\[
(b_{[0]} \triangleleft x'_1)b' \otimes a(x_1 \triangleright a')\sigma(x_2b_{[1]}, x'_2) \tag{17}
\]

\[
(b \triangleleft x'_2)b' \otimes a(x_1 \triangleright a')\sigma(x_2, x'_1). \tag{18}
\]

Therefore Eq.(13) ⇔ Eqs.(16) and (17).
(⇒) We only check that Eq.(14) in [10], Theorem 3.2] holds. In fact, one can compute as follows.

LHS of Eq.(14) = (b_{[0]} ⊲ x'_2) b'_{[0]} (x_1 > a') \sigma(x_2, x'_1) \otimes x_2 b_{[1]} x'_2 b_{[1]}

(b_{[0]} [0] ⊲ x'_2 b'_{[0]} (x_1 > a') \sigma(x_2 b_{[0][1]}, x'_2 b'_{[0][1]}) \otimes x_2 b_{[1]} x'_2 b_{[1]}

= (b_{[0]} ⊲ x'_2 b'_{[0]} (x_1 > a') \sigma(x_2 b_{[1][1]}, x'_2 b'_{[1][1]}) \otimes x_2 b_{[1]2} x'_2 b_{[1]2}

= (b_{[0]} ⊲ x'_2 b'_{[0]} (x_1 > a') \sigma(x_2 b_{[1][1]}, x'_2 b'_{[1][1]}) \otimes x_4 b_{[1][2]} x'_2 b_{[1][2]}

= (D21) (b_{[0]} ⊲ a' x'_2) b'_{[0]} [x_1 > (b_{[0][1]} > a'_0)] \sigma(x_2, b_{[1][1]} \sigma(x_3 b_{[1][1]}, x'_2 b'_{[1][1]})

= (b_{[0]} ⊲ a' x'_2) b'_{[0]} [x_1 > (b_{[1][1]} > a'_0)] \sigma(x_2, b_{[1][1]} \sigma(x_3 b_{[1][1]}, x'_2 b'_{[1][1]})

= (LC3) (b_{[0]} ⊲ a' x'_2) b'_{[0]} \sigma(x_1, b_{[1][1]}(x_2 b_{[1][2]} > a'_0) \sigma(x_3 b_{[1][1]}, x'_2 b'_{[1][1]})

= (b_{[0]} ⊲ a' x'_2) b'_{[0]} \sigma(x_1, b_{[1][1]}(x_2 b_{[1][2]} > a'_0) \sigma(x_3 b_{[1][1]}, x'_2 b'_{[1][1]})

= RHS of Eq.(14)

(⇐) Let b = 1, a' = 1, x' = 1 in Eq.(14), then we can get Eq.(20); Applying id ⊗ id ⊗ ε to Eq.(14), one can obtain

(b ⊲ x'_2) b' ⊲ (x_1 > a') \sigma(x_2, x'_1) = (b_{[0]} ⊲ a' x'_2) b'_{[0]} (x_1 b_{[1]} > a'_0) (x_2 b_{[1][1]} x'_2 b_{[1][1]}).

(16)

Let a' = 1, b' = 1 in Eq.(14), we get Eq.(13). Likewise, we can obtain Eqs.(15) and (21).

Up to now, we have checked that Eqs.(20), (21), (E3) ⇒ Eq.(14) in [10], Theorem 3.2]. And the rest is obvious.

\[\square\]

**Corollary 3.6.** Let H be a bialgebra. Let A be a left H-comodule coalgebra such that ε_A(1) = 1, and an algebra with a left H-action ⊲. Let B be a right H-comodule coalgebra such that ε_B(1) = 1, and an algebra with a right H-action ⊳. The two-sided smash product A#H#B equipped with the two-sided smash coproduct A × H × B becomes a bialgebra if and only if Eqs.(C1)-(C4), (D8),(D10),(D16),(D17),(D21) and the following conditions hold:

(F1) (aa')_1 ⊗ (aa')_2 = a_1(a_{21} ⊲ a'_1) ⊗ a_{20} a'_2;

(F2) (bb')_1 ⊗ (bb')_2 = b_1 b'_{[0]} ⊗ (b_2 ⊲ b'_{[1][1]}) b'_2;

(F3) (x > a)_1 ⊗ (x > a)_2 = (x_1 > a_1) ⊗ (x_2 > a_2);

(F4) (b ⊲ x)_1 ⊗ (b ⊲ x)_2 = (b_1 ⊲ x_1) ⊗ (b_2 ⊲ x_2).

**Proof.** Let σ and τ be trivial, i.e., σ(x, y) = ε(x)ε(y)1_A and τ(x, y) = ε(x)ε(y)1_B in Theorem [3.2].

\[\square\]

**Remark 3.7.** (1) By Eq.(D21), Eq.(11) in the Majid’s double biproduct is not only sufficient but also necessary.
(2) The first half parts in Eqs.(C1)-(C4) and Eqs.(F1), (D8),(F3), (D16) and the assumption of left module algebra and left comodule coalgebra are equivalent to A is a bialgebra in $\mathcal{YD}^H$. Similarly, the second half parts in Eqs.(C1)-(C4) and Eqs.(F2), (D10), (D17), (F4) and the assumption of right module algebra and right comodule coalgebra are equivalent to B is a bialgebra in $\mathcal{YD}^H$.

**Corollary 3.8.** ([24]) Let $H$ be a bialgebra. Let $\sigma : H \otimes H \rightarrow A$ be a linear map, where $A$ is a left $H$-comodule coalgebra, and an algebra with a left $H$-weak action. The left crossed product $A \#^\sigma H$ equipped with the left smash coproduct $A \times H$ becomes a bialgebra if and only if the first half parts in Eqs.(C1)-(C4) and Eqs.(D5), (D7), (D8), (D11), (D12), (D15), (D16) hold.

**Proof.** Let $B = K$ in Theorem 3.2. □

**Remark 3.9.** (1) Note that condition (D5) is exactly the condition of the twisted comodule cocycle in [24]. Although this condition is the only prerequisite for the bialgebra $A \star^\sigma H$ in [24], it is just one of the necessary conditions.

(2) The conditions in Corollary 3.8 are simpler than the ones in [10], Corollary 3.3, but obvious they are equivalent.

**Corollary 3.10.** Let $H$ be a bialgebra. Let $\tau : H \otimes H \rightarrow B$ be a linear map, where $B$ is a right $H$-comodule coalgebra, and an algebra with a right $H$-weak action. The right crossed product $H \;^\tau \#^{\tau} B$ equipped with the right smash coproduct $H \times B$ becomes a bialgebra if and only if the second half parts in Eqs.(C1)-(C4) and Eqs.(D6), (D9), (D10), (D13), (D14), (D17), (D18) hold. In this case, we denote this bialgebra by $H \;^\tau \star B$.

**Proof.** Let $A = K$ in Theorem 3.2. □

**Remark 3.11.** The conditions in Corollary 3.10 are simpler than the ones in [14], Proposition 2.2, but obvious they are equivalent.

4. **Conclusion**

We end this paper by three questions:

**Question 4.1.** Majid realized a categorical interpretation of Radford’s biproduct ([15]); $A \;^\star H$ is a Radford’s biproduct if and only if $A$ is a bialgebra in the Yetter-Drinfel’d category $\mathcal{YD}^H$. This ensure that $A \;^\star H$ can play a central role in the classification of finite-dimensional pointed Hopf algebras. By Theorem 3.2 and Remark 3.3, we know that the first half parts in Eqs.(C1)-(C4) and Eqs.(D5), (D7), (D8), (D11), (D12), (D15), (D16) are the necessary and sufficient conditions for $A \;^\star H$ to be bialgebra. These conditions correspond to the conditions for Radford’s biproduct one by one given in [24]. Here we note that Eq.(D16) is compatible condition for Yetter-Drinfeld module. Then whether there exists an appropriate category $\mathcal{YD}^H$ such that the conditions above for $A \;^\star H$ hold if and only if $A$ is a bialgebra in the category $\mathcal{YD}^H$.

**Question 4.2.** In Theorem 3.2, we give the necessary and sufficient conditions for the two-sided crossed product $A \#^\sigma H \;^\tau \# B$ equipped with the two-sided smash coproduct $A \times H \times B$ becomes a bialgebra. In Theorem 3.2, a more general two-sided crossed product algebra $A \#^G H \;^\tau \# B$ is constructed. Replaced $A \#^\sigma H \;^\tau \# B$ by $A \#^G H \;^\tau \# B$, what are the new conditions for the new bialgebra?
Question 4.3. In Theorem 3.2, we only obtain the bialgebra structure $A \star^\sigma H \star B$. It is natural to ask when $A \star^\sigma H \star B$ is a Hopf algebra, i.e., how to construct the antipode for $A \star^\sigma H \star B$. Although we provide the antipode in Proposition 2.10 for $A \star F \star H \star T \star B$, unfortunately, the coalgebra structure of $A \star F \star H \star T \star B$ is just the two-sided tensor product coalgebra. Recently, the authors in [23] give the antipode for a class of Majid’s double biproduct by (co)triangular Hopf quasigroups.

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