Viscous singular shock profiles for a system of conservation laws modeling two-phase flow

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Abstract

This paper is concerned with singular shocks for a system of conservation laws via the Dafermos regularization \( u_t + f(u)_x = \epsilon u_{xx} \). For a system modeling incompressible two-phase fluid flow, the existence of viscous profiles is proved using Geometric Singular Perturbation Theory. The weak convergence and the growth rate of the viscous solution are also derived; the weak limit is the sum of a piecewise constant function and a \( \delta \)-measure supported on a shock line, and the maximum value of the viscous solution is of order \( \exp(1/\epsilon) \).

Keywords: Conservation laws; Singular shocks; Viscous profiles; Dafermos regularization; Geometric Singular Perturbation Theory.

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1. Introduction

Keyfitz et al \cite{22,23} considered the system of conservation laws

\[
\begin{align*}
\beta_t + (vB_1(\beta))_x &= 0 \\
v_t + (v^2B_2(\beta))_x &= 0
\end{align*}
\]

(1)

where \( t \geq 0, x \in \mathbb{R}, v \in \mathbb{R}, \beta \in [\rho_1, \rho_2] \) with \( 0 < \rho_2 < \rho_1 \) and

\[
B_1(\beta) = \frac{(\beta - \rho_1)(\beta - \rho_2)}{\beta}, \quad B_2(\beta) = \frac{\beta^2 - \rho_1\rho_2}{2\beta^2}.
\]

(2)

For Riemann problems with data in feasible regions, they constructed uniquely defined admissible solutions. It can be readily shown that this system is not

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everywhere hyperbolic, and hence standard methods does not apply (see e.g. [6, 37]). To resolve this problem, along with rarefaction waves and regular shocks, the concept of singular shocks was adopted. A singular shock solution, roughly speaking, is a distribution which contains delta measures and is the weak limit of a sequence of approximate viscous solutions. For details of the definition, we refer to [21, 35].

The existence of singular shocks for (1) was proved in [23]. Their proof was based on a construction of approximate solution of the regularized system via Dafermos regularization

\[
\begin{align*}
\beta_t + (v B_1(\beta))_x &= \epsilon t \beta_{xx} \\
v_t + (v^2 B_2(\beta))_x &= \epsilon t v_{xx}
\end{align*}
\]

for certain Riemann data

\[
(\beta, v)(x, 0) = \begin{cases} (\beta_L, v_L), & x < 0 \\ (\beta_R, v_R), & x > 0. \end{cases}
\]

It was shown that the approximated solutions approach a distribution containing a delta function.

A family of exact solutions of (3ε)-(4), rather than approximate solutions, is called a viscous profile. In this paper, besides proving existence of viscous profile, we also give descriptions of their limiting behavior, including weak convergence and growth rates. The main tool in our study is Geometric Singular Perturbation Theory (GSPT), which will be introduced in Section 4. The use of this tool on singular shocks was first introduced in the pioneering work of Schecter [32].

The system (1) is equivalent to a two-fluid model for incompressible two-phase flow [8, p.248] of the form

\[
\begin{align*}
\partial_t (\alpha_i) + \partial_x (\alpha_i u_i) &= 0 \\
\partial_t (\alpha_i \rho_i u_i) + \partial_x (\alpha_i \rho_i u_i^2) + \alpha_i \partial_x p_i &= F_i, & i = 1, 2,
\end{align*}
\]

where the drag terms \( F_i \) are neglected, the densities \( \rho_i \) are assumed to be constant. The six unknowns are reduced to four by the relation \( \alpha_1 + \alpha_2 = 1 \) and the assumption

\[
p_1 = p_2.
\]

We follow [22] to replace the volume fractions \( \alpha_1 \) and \( \alpha_2 \) by a density-weighted volume element \( \beta = \rho_2 \alpha_1 + \rho_1 \alpha_2 \) and the momentum equations by a single
equation for the momentum difference \( v = \rho_1 u_1 - \rho_2 u_2 - (\rho_1 - \rho_2) K \), where \( K = \alpha_1 u_1 + \alpha_2 u_2 \) is taken to be zero. That is,

\[
\alpha_1 = \frac{\rho_1 - \beta}{\rho_1 - \rho_2}, \quad \alpha_2 = \frac{\beta - \rho_2}{\rho_1 - \rho_2}, \quad u_1 = \frac{\beta - \rho_2}{\rho_1 - \rho_2} v, \quad u_2 = \frac{\beta - \rho_1}{\rho_1 - \rho_2} v. \tag{7}
\]

After a rescaling of the space variable, we assume that \( \rho_1 - \rho_2 = 1 \). Plugging (7) into (5) yields (1) under the assumption (6) when \( \alpha_1 \alpha_2 \neq 0 \).

The system (1) is a simple example of continuous model for two-phase flow, but it shares with other continuous models the property of changing type – that is, it is not hyperbolic for some (in this case, most) states. The purpose of this study is to shed light on the mathematical properties of the change-of-type system that appear in continuous models of two-phase flow. The original studies [22, 23] showed the existence of self-similar solutions with reasonable properties. Specifically, the singular shocks that appear can be considered to be propagating phase boundaries. In this paper, we focus on viscous profiles of singular shocks and unveil some of their limiting behavior.

In Section 2, we state our main result. The validity of the assumptions of the theorem is discussed in Section 3 with some proofs for the sufficient conditions postponed to Appendix A. In Section 4, we review and enhance some tools in GSPT, including Fenichel’s Theorems and the Exchange Lemma. Section 5 is devoted to describing the limiting subsystems corresponding to multiple time scales. The proof of the main theorem is completed in Section 6, and numerical simulations are shown in Section 7.

### 2. Main Result

In standard notation for conservation laws, we write (1) as

\[
u_t + f(u)_x = 0, \tag{8}\]

where \( u = (\beta, v) \), and write Riemann data (4) in the form

\[
u(x, 0) = u_L + (u_R - u_L)H(x), \tag{9}\]

where \( H(x) \) is the step function taking value 0 if \( x < 0 \); 1 if \( x > 0 \).

We study the systems that approximate (8) via the Dafermos regularization:

\[
u_t + f(u)_x = \epsilon u_{xx} \tag{10}\]
for small $\epsilon > 0$. Using the self-similar variable $\xi = x/t$, the system is converted to
\[
-\xi \frac{d}{d\xi} u + \frac{d}{d\xi} \left( f(u) \right) = \epsilon \frac{d^2}{d\xi^2} u,
\] (11c)
and the initial condition (9) becomes
\[
u(-\infty) = u_L, \; \nu(+\infty) = u_R.
\] (12)
The system (11c) is equivalent to
\[
\epsilon u_\xi = f(u) - \xi u - w \\
w_\xi = -u
\] (13c)
or, up to a rescaling of time,
\[
\dot{u} = f(u) - \xi u - w \\
\dot{w} = -\epsilon u \\
\dot{\xi} = \epsilon.
\] (14c)
The time variable in (14c) is implicitly defined by the equation of $\dot{\xi}$. When $\epsilon = 0$, (14c) is reduced to
\[
\dot{u} = f(u) - \xi u - w \\
\dot{w} = 0, \; \dot{\xi} = 0.
\] (15)
Returning to the $(\beta, v)$ notation, the system (14c) is written as
\[
\dot{\beta} = B_1(\beta)v - \xi \beta - w_1 \\
\dot{v} = B_2(\beta)v^2 - \xi v - w_2 \\
\dot{w}_1 = -\epsilon \beta \\
\dot{w}_2 = -\epsilon v \\
\dot{\xi} = \epsilon,
\] (16c)
and (15) becomes
\[
\dot{\beta} = B_1(\beta)v - \xi \beta - w_1 \\
\dot{v} = B_2(\beta)v^2 - \xi v - w_2 \\
\dot{w}_1 = 0, \; \dot{w}_2 = 0, \; \dot{\xi} = 0.
\] (17)
The linearization at any equilibrium \((\beta, v, w_1, w_2, \xi)\) for \((17)\) has eigenvalues \(\lambda_\pm(\beta, v) - \xi\), where

\[
\lambda_\pm(u) = 2vB_2(\beta) \pm v\sqrt{B_1(\beta)B'_2(\beta)}.
\]  

(18)

Note that \(\text{Re}(\lambda_\pm(u)) = 2vB_2(\beta)\) since \(B_1(\beta)B'_2(\beta) \leq 0\) when \(\rho_2 \leq \beta \leq \rho_1\). Moreover, the system is nonhyperbolic everywhere in the physical region except on the lines \(\{\beta = \rho_1\}\), \(\{\beta = \rho_2\}\), and \(\{v = 0\}\).

An over-compressive shock region is a region where the condition (H1) defined below holds. It was shown in [23] that any data in an over-compressive shock region admits a singular shock solution, and the shock speed \(s\) is defined by \((19)\) below. Our main theorem confirms Dafermos profiles in a subset of this region.

**Main Theorem.** Consider the Riemann problem \((1), (4)\). Let \(u_L = (\beta_L, v_L)\) and \(u_R = (\beta_R, v_R)\) be two points in \([\rho_1, \rho_2] \times (0, \infty)\) with \(\beta_R \neq \beta_L\), and let

\[
s = \frac{v_LB_1(\beta_L) - v_RB_1(\beta_R)}{\beta_L - \beta_R},
\]  

(19)

\[
w_L = f(u_L) - su_L, \quad w_R = f(u_R) - su_R
\]  

(20)

\[
e_0 = w_{2L} - w_{2R}
\]  

(21)

where we denote \(w_L = (w_{1L}, w_{2L})\) and \(w_R = (w_{1R}, w_{2R})\). Assume

(H1) \(\text{Re}(\lambda_\pm(u_R)) < s < \text{Re}(\lambda_\pm(u_L))\), where \(\lambda_\pm(u)\) are defined in \((18)\).

(H2) \(e_0 > 0\).

(H3) For the system \((17)\), there exists a trajectory joining \((\beta_L, v_L, w_L, s)\) and \((\rho_1, +\infty, w_L, s)\), and a trajectory joining \((\beta_R, v_R, w_R, s)\) and \((\rho_2, +\infty, w_R, s)\).

Then there is a singular shock with Dafermos profile for the Riemann data \((u_L, u_R)\). That is, for each small \(\epsilon > 0\), there is a solution \(\tilde{u}^\epsilon(\xi)\) of \((11a) - (12)\), and \(\tilde{u}_\epsilon(\xi)\) becomes unbounded as \(\epsilon \to 0\). Indeed,

\[
\max_\xi \left( \epsilon \log \tilde{u}_\epsilon(\xi) \right) = \left( \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right) \left( \frac{w_{2L} - w_{2R}}{\epsilon} \right) + o(1) \quad \text{as} \ \epsilon \to 0,
\]  

(22)

where we write \(\tilde{u}_\epsilon = (\tilde{\beta}^\epsilon, \tilde{v}^\epsilon)\). Moreover, if we set \(u_\epsilon(x, t) = \tilde{u}_\epsilon(x/t)\), then \(u_\epsilon(x, t)\) is a solution of \((10c)\) and

\[
\beta_\epsilon \to \beta_L + (\beta_R - \beta_L)H(x - st)
\]  

(23a)

\[
v_\epsilon \to v_L + (v_R - v_L)H(x - st) + \frac{e_0}{\sqrt{1 + s^2}}t\delta_{x=st}
\]  

(23b)

in the sense of distributions as \(\epsilon \to 0\).
The trajectories in (H3) are illustrated in Fig 1.

Remark 1. A similar result holds if \(v_L < 0\) and \(v_R < 0\). In that case, the assumption \(e_0 > 0\) in (H2) is replaced by \(e_0 < 0\), and \(+\infty\) in (H3) is replaced by \(-\infty\).

The notation \(t\delta_{\{x=st\}}\) in (23b) denotes, following [38, 2], the functional on \(C^\infty_c(\mathbb{R} \times \mathbb{R}^+)\) defined by

\[
\langle t\delta_{\{x=st\}}, \varphi \rangle = \int_0^\infty t\varphi(st,t)\sqrt{1 + s^2} \, dt.
\] (24)

The weight \(\sqrt{1 + s^2}\) in the integral is to normalize the functional so that it is independent of parametrization of the line \(\{x=st\}\).

The estimate (22) confirms the asymptotic behavior conjectured in [22]. In [23], some approximate solutions for the Dafermos regularization were constructed, but they were not exact solutions to (11\(\epsilon\)). The results in the main theorem can also be compared to [32] and [24], where Dafermos profiles were constructed for a system motivated by gas dynamics. Those authors obtained families of unbounded solutions to (11\(\epsilon\)), but they did not give descriptions of asymptotic behaviors of solutions.

The assumption (H3) says that there exist solutions of (17) of the form

\[
\gamma_1 = \left(\beta_1(\zeta), v_1(\zeta), w_{1L}, w_{2L}, s\right), \quad \gamma_2 = \left(\beta_2(\zeta), v_2(\zeta), w_{1R}, w_{2R}, s\right)
\] (25)
satisfying
\[
\lim_{\zeta \to -\infty} (\beta_1(\zeta), v_1(\zeta)) = (\beta_L, v_L), \quad \lim_{\zeta \to 0^-} (\beta_1(\zeta), v_1(\zeta)) = (\rho_1, \infty)
\] (26)

and
\[
\lim_{\zeta \to 0^+} (\beta_2(\zeta), v_2(\zeta)) = (\rho_2, \infty), \quad \lim_{\zeta \to \infty} (\beta_2(\zeta), v_2(\zeta)) = (\beta_R, v_R).
\] (27)

Note that the limits in (26) and (27) as \( \zeta \to 0^\pm \) corresponds to the fact that the trajectories go to infinity at finite time. Since the system \((17)\) is autonomous, the blow up time is arbitrary due to invariance of time translation. We pick the blow up time to be \( \zeta = 0 \) for convenience.

Note that \( \gamma_1 \) and \( \gamma_2 \), if they exist, are unique. The follows from a local analysis for \((17)\) at \((\beta, v, w, \xi) = (\rho_1, \infty, w_L, s)\) and \((\rho_2, \infty, w_R, s)\). See Appendix A for details.

A sample set of data for which \((H1)-(H3)\) holds is, following [22],
\[
\rho_1 = 2, \quad \rho_2 = 1, \quad u_L = (1.9, 1.0), \quad u_R = (1.1, 1.1/1.9).
\] (28)
This will be verified in the next section.

3. Sufficient Conditions for \((H1)-(H3)\)

The regions at which \((H1)\) holds, or the over-compressive shock regions, can be described by the following

**Proposition 1.** In the Riemann problem \((1), (9)\), let \( u_L = (\beta_L, v_L) \) and \( u_R = (\beta_R, v_R) \) be two points in \([\rho_1, \rho_2] \times (0, \infty)\). Then \((H1)\) holds if and only if \( u_R \) lies in the interior of a cusped triangular region bounded by the curves
\[
v = v_L \left( \frac{B_1(\beta_L) - 2B_2(\beta_L)(\beta_L - \beta)}{B_1(\beta)} \right), \quad \rho_2 < \beta < \beta_L,
\] (29)

and
\[
v = v_L \left( \frac{B_1(\beta_L)}{B_1(\beta) + 2B_2(\beta)(\beta_L - \beta)} \right), \quad \rho_2 < \beta < \beta_L.
\] (30)

On the boundary segment \((29)\), \( s = \text{Re}(\lambda_\pm(u_L)) \), and on \((30)\), \( s = \text{Re}(\lambda_\pm(u_R)) \).

The curves defined by \((29)\) and \((30)\) and the region where over-compressive shock solution exist are illustrated in Fig 2.
Proof. This follows from a direct calculation. See [22, Corollary 3.1].

The following proposition asserts that (H2) is implied by (H1).

**Proposition 2.** In the Riemann problem (1), (9), if the Riemann data lie in an over-compressive shock region in \([\rho_1, \rho_2] \times (0, \infty)\), then (H2) holds.

**Proof.** See [22, Section 3.1].

The assumption (H3) is a condition on dynamics of 2-dimensional systems. Analyzing phase portraits we have the following

**Proposition 3.** Given Riemann data in an over-compressive shock region in \([\rho_1, \rho_2] \times (0, \infty)\), if \(\beta_R < \sqrt{\rho_1 \rho_2} < \beta_L\), \(w_{1L} < 0\), \(w_{2R} < 0 < w_{2L}\), and \(|s|\) is sufficiently small, then (H3) holds.

**Proof.** See Appendix A.

Proposition 2 says that (H2) holds whenever (H1) holds, so the Main Theorem requires only (H1) and (H3). The author believes that (H3) is also a consequence of (H1). This needs further work to be verified.

For the sample set of data (28), we have

\[(w_{1L}, w_{2L}) = (-.05, .22), \ (w_{1R}, w_{2R}) = (-.05, -.11), \ s = 0. \] (31)

Since \(w_{2R} < w_{2L}\), (H2) holds. From Proposition 1 and 3 (H1) and (H3) also hold. Hence the main theorem applies. Note that the conditions (H1)-(H3) persist under perturbation of the Riemann data \((u_L, u_R)\), so those assumptions still hold for any data close to (28).
4. Geometric Singular Perturbation Theory

Our main goal in this paper is to solve the boundary value problem (11)–(12) corresponding to (1). Note that (11) is a singularly perturbed equation since the perturbation $\epsilon \frac{d^2}{d\xi^2} u$ has a higher order derivative than the other terms in the equation. To deal with singularly perturbed equations, we will apply Geometric Singular Perturbation Theory (GSPT). The idea of GSPT is to first study a set of subsystems which forms a decomposition of a system, and then to use the information for the subsystems to conclude results for the original system. Prototypical examples include relaxation oscillations for forced Van der Pol Equations [9, 25, 26] and FitzHugh-Nagumo Equations [17, 22, 30]. Surveys on this topic can be found in [14, 19, 20, 31].

In Section 4.1 and 4.2, we recall some fundamental theorems in GSPT, including Fenichel’s Theorems and an existence theorem for solutions to Silnikov problems. In Section 4.3 we state a version of the Exchange Lemma and give a new proof for it.

4.1. Fenichel’s Theory for Fast-Slow Systems

Note that (14) is a fast-slow system, which means that the system is of the form
\[
\dot{x} = f(x, y, \epsilon) \\
\dot{y} = \epsilon g(x, y, \epsilon)
\]
where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^l$, and $\epsilon$ is a parameter. In order to deal with fast-slow systems, Fenichel’s Theory was developed in [10, 11, 12]. Some expositions for that theory can be found in [14, 41].

An important feature of a fast-slow system is that the system can be decomposed into two subsystems: the limiting fast system and the limiting slow system. The limiting fast system is obtained by taking $\epsilon = 0$ in (32); that is,
\[
\dot{x} = f(x, y, 0) \\
\dot{y} = 0.
\]

On the other hand, note that the system (32) can be converted to, after a rescaling of time,
\[
\epsilon \dot{x}' = f(x, y, \epsilon) \\
\dot{y} = g(x, y, \epsilon).
\]
Taking $\epsilon = 0$ in (34), we obtain the limiting slow system

$$
\begin{align*}
0 &= f(x, y, 0) \\
y' &= g(x, y, 0).
\end{align*}
$$

(35)

Note that the limiting slow system (35) describes dynamics on the set of critical points of the limiting fast system (33), and we need to piece together the information of the limiting fast system and the limiting slow system in the vicinity of the set of critical points. To piece this information together, normal hyperbolicity defined below is a crucial condition.

**Definition 1.** A critical manifold $S_0$ for (33) is an $l$-dimensional manifold consisting of critical points of (33). A critical manifold is normally hyperbolic if $D_x f(x, y, 0)|_{S_0}$ is hyperbolic. That is, at any point $(x_0, y_0) \in S_0$, all eigenvalues of $D_x f(x, y, 0)|_{(x_0, y_0)}$ have nonzero real part.

Now we turn to normal hyperbolicity for general systems

$$
\dot{z} = F(z),
$$

(36)

where $z \in \mathbb{R}^N, N \geq 1$. Recall that a manifold $S \subset \mathbb{R}^N$ is locally invariant if for any point $p \in S \setminus \partial S$, there exist $t_1 < 0 < t_2$ such that $p \cdot (t_1, t_2) \in S$, where $\cdot$ denotes the flow for (36). A locally invariant $C^r$ manifold $S \subset \mathbb{R}^N, r \geq 1$, is normally hyperbolic for the system (36) if the growth rate of vectors transverse to the manifold dominates the growth rate of vectors tangent to the manifold. (Note that this definition is consistent with Definition 1.) In this case, from the standard theory for normally hyperbolic manifolds (see e.g. [5, 13, 40]) it is assured that stable and unstable manifolds $W^s(S)$ and $W^u(S)$ are defined.

For a locally invariant manifold $\Lambda \subset \mathbb{R}^N$ for (36) which is not necessarily normally hyperbolic, a center manifold is a normally hyperbolic locally invariant manifold, with the smallest possible dimension, containing $\Lambda$. In classical cases, $\Lambda = \{p_0\}$ is an isolated non-hyperbolic equilibrium, and a center manifold for $p_0$ has dimension equal to the number of generalized eigenvalues of $DF(p_0)$ with zero real part. For instance, the planar system $(\dot{x}, \dot{y}) = (x^3, y)$ has a non-hyperbolic isolated equilibrium $p_0 = (0, 0)$, and the $x$-axis is a center manifold for $p_0$. For the theory on the center manifolds of general invariant sets $\Lambda$, we refer to [31, 4].

Fenichel’s Theory is a center manifold theory for fast-slow systems. For a normally hyperbolic critical manifold $S_0$ for (33), the stable and unstable
manifolds $W^s(S_0)$ and $W^u(S_0)$ can be defined in the natural way. We denote them by $W^s_0(S_0)$ and $W^u_0(S_0)$ to indicate their invariance under (32) with $\epsilon = 0$. Fenichel’s Theory assures that the hyperbolic structure of $S_0$ persists under perturbation (32). Below we state three fundamental theorems of Fenichel’s Theory following Jones [14].

**Theorem 4** (Fenichel’s Theorem 1). Consider the system (32), where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^l$, and $f, g$ are $C^r$ for some $r \geq 2$. Let $S_0$ be a compact normally hyperbolic manifold for (33). Then for any small $\epsilon \geq 0$ there exist locally invariant $C^r$ manifolds, denoted by $S_\epsilon$, $W^s_\epsilon(S_\epsilon)$ and $W^u_\epsilon(S_\epsilon)$, which are $C^1$ $O(\epsilon)$-close to $S_0$, $W^s_0(S_0)$ and $W^u_0(S_0)$, respectively. Moreover, for any continuous families of compact sets $I_\epsilon \subset W^u_\epsilon(S_\epsilon)$, $J_\epsilon \subset W^s_\epsilon(S_\epsilon)$, $\epsilon \in [0, \epsilon_0]$, there exist positive constants $C$ and $\nu$ such that

\[
\text{dist}(z \cdot t, S_\epsilon) \leq Ce^{\nu t} \quad \forall \ z \in I_\epsilon, \ t \leq 0 \tag{37a}
\]

\[
\text{dist}(z \cdot t, S_\epsilon) \leq Ce^{-\nu t} \quad \forall \ z \in J_\epsilon, \ t \geq 0 \tag{37b}
\]

where $\cdot$ denotes the flow for (32).

**Proof.** See [14, Theorem 3].

**Remark 2.** If $S_0$ is locally invariant under (32) for each $\epsilon$, then the proof of [14, Theorem 3] allows $S_\epsilon$ to be $S_0$.

Note that $W^u_\epsilon(S_\epsilon)$ and $W^s_\epsilon(S_\epsilon)$ can be interpreted as a decomposition in a neighborhood of $S_0$ in $(x, y)$-space. The following theorem asserts that this induces a change of coordinates $(a, b, c)$ such that $W^u_\epsilon(S_\epsilon)$ and $W^s_\epsilon(S_\epsilon)$ correspond to $(a, c)$-space and $(b, c)$-space, respectively.

**Theorem 5** (Fenichel’s Theorem 2). Suppose the assumptions in Theorem 4 hold. Then under a $C^r$ $\epsilon$-dependent coordinate change $(x, y) \mapsto (a, b, c)$, the system (32) can be brought to the form

\[
\begin{align*}
\dot{a} &= A^u(a, b, c, \epsilon)a \\
\dot{b} &= A^s(a, b, c, \epsilon)b \\
\dot{c} &= \epsilon(h(c) + E(a, b, c, \epsilon))
\end{align*}
\]

in a neighborhood of $S_\epsilon$, where the coefficients are $C^{r-2}$ functions satisfying

\[
\inf_{\lambda \in \text{Spec}A^u(a,b,c,0)} \text{Re} \lambda > \nu, \quad \sup_{\lambda \in \text{Spec}A^s(a,b,c,0)} \text{Re} \lambda < -\nu \tag{39}
\]
for some $\nu > 0$, and

$$E = 0 \quad \text{on} \quad \{a = 0\} \cup \{b = 0\}. \quad (40)$$

**Proof.** See [14, Section 3.5] or [18, Proposition 1].

The family of trajectories for (35) forms a foliation of $S_0$. The following theorem says that this induces a foliation of $W^u_\epsilon(S_\epsilon)$ and $W^s_\epsilon(S_\epsilon)$.

**Theorem 6** (Fenichel’s Theorem 3). Suppose the assumptions in Theorem 4 hold. Let $\Lambda_0$ be a submanifold in $S_0$ which is locally invariant under (35). Then there exist locally invariant manifolds $\Lambda_\epsilon$, $W^u_\epsilon(\Lambda_\epsilon)$, and $W^s_\epsilon(\Lambda_\epsilon)$ for (32) which are $C^{r-2}$ $O(\epsilon)$-close to $\Lambda_0$, $W^u_0(\Lambda_0)$, and $W^s_0(\Lambda_0)$, respectively. Moreover, for any continuous families of compact sets $I_\epsilon \subset W^u_\epsilon(\Lambda_\epsilon)$, $J_\epsilon \subset W^s_\epsilon(\Lambda_\epsilon)$, $\epsilon \in [0, \epsilon_0]$, there exist positive constants $C$ and $\nu$ such that (37) holds with $S_\epsilon$ replaced by $\Lambda_\epsilon$. Suppose in addition that $S_0$ is invariant under (32) for each $\epsilon$. Then $\Lambda_\epsilon$ can be chosen to be $\Lambda_0$.

**Proof.** Using Fenichel’s coordinates $(a,b,c)$ in Theorem 5 for the splitting of $S_0$, we can take $W^u_\epsilon(\Lambda_\epsilon)$ and $W^s_\epsilon(\Lambda_\epsilon)$ to be the pre-images of the sets $\{(a,b,c) : a = 0, c \in \Lambda_0\}$ and $\{(a,b,c) : b = 0, c \in \Lambda_0\}$, respectively, in $(x,y)$-space. From (39) we obtain (37) with $S_\epsilon$ replaced by $\Lambda_\epsilon$. Suppose $S_0$ is invariant under (32) for each $\epsilon$, then from the remark after Theorem 4 we can take $S_\epsilon = S_0$ and hence $\Lambda_\epsilon = \Lambda_0$.

The system (38) is called a **Fenichel normal form** for (32), and the variables $(a,b,c)$ are called **Fenichel coordinates**.

### 4.2. Silnikov Boundary Value Problem

We have seen in Section 4.1 that fast-slow systems (32) can locally be converted into normal forms (38), where $A^u$ and $A^s$ satisfy the gap condition (39), and $E$ is a small term satisfying (40). If we append the system with the equation $\dot{\epsilon} = 0$ and then replace $c$ by $\tilde{c} = (c, \epsilon)$, we obtain a system of the form

$$\begin{align*}
\dot{a} &= A^u(a,b,\tilde{c})a \\
\dot{b} &= A^s(a,b,\tilde{c})b \\
\dot{\tilde{c}} &= \tilde{h}(\tilde{c}) + \tilde{E}(a,b,\tilde{c}),
\end{align*}$$

(41)
Fig 3: Trajectories in the rectangle \( \{ 0 \leq a \leq a^1, 0 \leq b \leq b^0 \} \) can be parametrized in \( T \geq 0 \) by \( a(T) = a^1, b(0) = b^0 \).

for which (39) and (40) are satisfied with \( E \) replaced by \( \tilde{E} \). Therefore, in this subsection we consider systems of the form

\[
\begin{align*}
\dot{a} &= A^u(a,b,c)a \\
\dot{b} &= A^s(a,b,c)b \\
\dot{c} &= h(c) + E(a,b,c)
\end{align*}
\] (42)

satisfying (39) and (40).

A Silnikov problem is the system (42) along with boundary data of the form

\[
(b, c)(0) = (b^0, c^0), \quad a(T) = a^1,
\] (43)

where \( T \geq 0 \). This boundary value problem was posed in [36] to study homoclinic bifurcation. A heuristic reason for the existence of solutions of a Silnikov problem is illustrated in Fig 3. Consider the simple case \( \dot{a} = a, \dot{b} = -b \) and \( \dot{c} = 0 \). There are infinitely many trajectories contained in the box \( \{ 0 \leq a \leq a^1, 0 \leq b \leq b^0 \} \). We may parametrize the set of trajectories in \( T \geq 0 \) by \( b(0) = b^0 \) and \( a(T) = a^1 \). On the \( a \)-axis and \( b \)-axis, the trajectories tend to the origin in backward and forward time, respectively. This suggests that trajectories near the axes can stay for an arbitrarily long time in the box, which implies that for any large \( T \) there exists a trajectory satisfying \( b(0) = b^0 \) and \( a(T) = a^1 \). When \( T \) grows to infinity, the trajectories approach the axes. In the general case \( \dot{a} = A^u a \) and \( \dot{b} = A^s b \) in arbitrary dimension, both \( a \)- and \( b \)-spaces consist of solutions tending to the origin in forward or backward time, so we have the same conclusion.
The critical manifold for (42) is \( \{ a = 0, b = 0 \} \), on which the system is governed by the limiting slow system

\[
\dot{c} = h(c). 
\] (44)

For a solution \((a(t), b(t), c(t))\) of (42)-(43), the condition (39) suggests that \(a(t)\) and \(b(t)\) decay to 0 in backward time and forward time, respectively. In this case, the condition (40) suggests that \(c(t)\) is approximately the solution of (44). The above description is confirmed by a theorem from Schecter [33]:

**Theorem 7** (Generalized Deng's Lemma [33]). Consider the system (42) satisfying (39) and (40) with \(C^r\) coefficients, \(r \geq 1\), defined on the closure of a bounded open set \(B_{k,\Delta} \times B_{m,\Delta} \times V \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l\), where \(B_{k,\Delta} = \{ a \in \mathbb{R}^k : |a| < \Delta \}, \Delta > 0\), and \(V\) is a bounded open set in \(\mathbb{R}^l\).

Let \(K_0\) and \(K_1\) be compact subsets of \(V\) such that \(K_0 \subset \text{Int}(K_1)\). For each \(c^0 \in K_0\) let \(J_{c^0}\) be the maximal interval such that \(\phi(t, c^0) \in \text{Int}(K_1)\) for all \(t \in J_{c^0}\), where \(\phi(t, c^0)\) is the solution of (44) with initial value \(c^0\). Let \(\nu > 0\) be the number in (39). Suppose there exists \(\beta > 0\) such that \(\tilde{\nu} := \nu - \beta > 0\) and

\[
|\phi(t, c^0)| \leq Me^{\beta |t|} \quad \forall t \in J_{c^0}.
\]

Then there is a number \(\delta_0 > 0\) such that if \(|a^1| < \delta_0, |b^0| < \delta_0, c^0 \in V_0,\) and \(T > 0\) is in \(J_{c^0}\), then the Silnikov boundary value problem (42) and (43) has a solution \((a(t), b(t), c(t))\) on the interval \(0 \leq t \leq T\). Moreover, there is a number \(K > 0\) such that for all \((t, T, a^1, b^0, c^0)\) as above and for all multi-indices \(i\) with \(|i| \leq r\),

\[
|D_i a(t, T, a^1, b^0, c^0)| \leq Ke^{-\tilde{\nu}(T-t)} \\
|D_i b(t, T, a^1, b^0, c^0)| \leq Ke^{-\tilde{\nu}t} \\
|D_i c(t, T, a^1, b^0, c^0) - D_i \phi(t, c^0)| \leq Ke^{-\tilde{\nu}T}.
\] (45)

**Sketch of Proof.** Here we sketch the proof in [33]. Write (42) as

\[
\dot{a} = \tilde{A}^a(t, c^0) a + f(t, c^0, a, b, z) \\
\dot{b} = \tilde{A}^s(t, c^0) b + g(t, c^0, a, b, z) \\
\dot{z} = \tilde{A}^c(t, c^0) z + \theta(t, c^0, z) + \tilde{E}(t, c^0, a, b, z),
\]

where

\[
\tilde{A}^i(t, c^0) = A^i(0, 0, \phi(t, c^0)) \quad i = u, s \\
\tilde{A}^c(t, c^0) = Dh|_{\phi(t, c^0)}/
\]

14
and

$$\tilde{E}(t, c^0, a, b, z) = E(a, b, \phi(t, c^0) + z).$$

Let $\Phi_i(t, s, c^0)$ be the solution operator for $\tilde{A}_i(t, c^0)$, $i = u, s, c$. Then $(a(t), b(t), c(t))$ is a solution of Silnikov problem (42) and (43) if and only if $c(t) = \phi(t, c^0) + z(t)$ and $\eta(t) = (a(t), b(t), z(t))$ satisfies

$$a(t) = \Phi^u(t, T, c^0)a^1 - \int_t^T \Phi^u(t, s, c^0)f(s, c^0, \eta(s)) \, ds$$

$$b(t) = \Phi^s(t, 0, c^0)b^0 + \int_0^t \Phi^s(t, s, c^0)g(s, c^0, \eta(s)) \, ds$$

$$z(t) = \int_0^t \Phi^c(t, s, c^0)(\theta(s, c^0, z(s)) + \tilde{E}(s, c^0, \eta(s))) \, ds.$$  \hspace{1cm} (46)

Define an linear operator $L$ by the right-hand side of (46) for functions $\eta(t) = (a(t), b(t), z(t))$. It can be shown that the restriction of $L$ on a neighborhood of 0 in the space of functions $\eta(t) = (a(t), b(t), z(t))$ equipped with the norm

$$\|\eta\| = \sum_{|i| \leq r} \left[ \sup_{0 \leq t \leq T} (e^{\tilde{\nu}(T-t)}|D_i a(t)| + e^{\tilde{\nu}t}|D_1 b(t)| + e^{\tilde{\nu}T}|D_1 z(t)|) \right]$$

is a contraction mapping. Hence the existence of solution of (42) and (43) follows from the standard Banach fixed point theorem.

**Remark 3.** Theorem 7, which was asserted by Schecter [33] in a more delicate form, is a generalization of the Strong $\lambda$-Lemma in Deng [7], and $C^r$-Inclination Theorem in Brunovsky [1]. In Deng’s work, the boundary data lie near an equilibrium that may nonhyperbolic. In Brunovsky’s work, the boundary data lie near a solution of a rectifiable slow flow on a normally hyperbolic invariant manifold. Schecter’s work allows considering more general flows on normally hyperbolic invariant manifolds.

4.3. The Exchange Lemma

Note that (38) can be considered as a special case of (42), and recall that (38) is the normal form of fast-slow systems (32). This enables us to use Theorem 7 to analyze Silnikov problems for fast-slow systems (32).

The Silnikov problem for (38) corresponds to the boundary data

$$a(\tau/\epsilon) = a^1, \quad (b, c)(0) = (b^0, c^0),$$

(47)
with given $(a^1, b_0, c^0) \in \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l$ and $\tau > 0$. It can be interpreted as finding trajectories for (38ε) connecting the sets $\{b = b_0, c = c^0\}$ and $\{a = a^1\}$, with prescribed time interval $0 \leq t \leq \tau/\epsilon$; see Fig 4. Note that the set $\{b = b_0, c = c^0\}$ is of dimension $k$. The Exchange Lemma is a tool tracking the $(k+1)$-manifold $I^*_\epsilon$ that evolves from a $k$-manifold $I_\epsilon$ which is transverse to the center-stable manifold $\{a = 0\}$. The theory of Exchange Lemma was first developed in [17, 15, 16] to study singularly perturbed systems near a normally hyperbolic, locally invariant manifold. Some generalizations of the Exchange Lemma for a broader class of systems were given by W. Liu [28] and Schecter [34].

Another generalization, given by Tin [39], is the $(k+\sigma)$-Exchange Lemma, $1 \leq \sigma \leq l$, which tracks the $(k+\sigma)$-manifold $I^*_\epsilon$ evolving from a $(k+\sigma-1)$-manifold $I_\epsilon = \{b = b_0, c^0 \in \Lambda\}$, where $\Lambda$ is a $(\sigma-1)$-manifold. A major difference between the $(k+\sigma)$-Exchange Lemma and the general Exchange Lemma in [34] is that the estimates (45) for the derivatives in slow variables were not considered in [34].

We analyze Silnikov problems for fast-slow systems in normal form (38ε) in the lemma below. Then, in Theorem 9, we will return to (32ε) to present a version of the $(k+\sigma)$-Exchange Lemma.

**Lemma 8.** Consider a system of the form (38ε) satisfying (39) and (40) defined on the closure of a bounced open set $B_{k,\Delta} \times B_{m,\Delta} \times V \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l$, where the coefficients are $C^r$ for some integer $r \geq 0$. Let $\Lambda \subset V$ be a $(\sigma-1)$-
dimensional manifold, $1 \leq \sigma \leq l$ and $\tau_0 > 0$. Suppose
\[
c \circ [0, \tau_0] \subset V \quad \forall c \in \Lambda,
\]
where $\circ$ denotes the flow for the limiting slow system (35). Let $J \subset (0, \tau_0)$ be a closed interval and $\mathcal{A} \subset B_k, \Delta \setminus \{0\}$ be a compact set. Then for each small $\epsilon > 0$ and $(a^1, c^0, \tau) \in \mathcal{A} \times \Lambda \times J$, the boundary value problem (38ε)-47 has a unique solution for $t \in [0, \tau/\epsilon]$. Moreover, if we denoted the solution by $(a, b, c)(t; \tau, a^1, b^0, c^0, \epsilon)$ and set
\[
p_\epsilon = (a, b, c)(0; \tau, a^1, b^0, c^0, \epsilon), \quad q_\epsilon = (a, b, c)(\tau/\epsilon; \tau, a^1, b^0, c^0, \epsilon),
\]
then
\[
\left\| p_\epsilon - (0, b^0, c^0) \right\|_{C^r(\mathcal{A} \times \Lambda \times J)} + \left\| q_\epsilon - (a^1, 0, c^0 \circ \tau) \right\|_{C^r(\mathcal{A} \times \Lambda \times J)} \leq C e^{-\tilde{\nu}/\epsilon} \tag{49}
\]
for some positive constants $\tilde{C}$ and $\tilde{\nu}$.

See Fig 4 for an illustration of (49).

**Sketch of Proof.** The existence of solutions follows directly from Theorem 7, so it remains to prove (49). Write $p_\epsilon = (a^{in}_\epsilon, b^0_\epsilon, c^0_\epsilon)$ and $q_\epsilon = (a^1_\epsilon, b^0_\epsilon, c^0_\epsilon)$, then (49) is equivalent to
\[
\left\| (a^{in}_\epsilon, b_\epsilon, (c_\epsilon - c^0 \circ \tau)) \right\|_{C^r(\mathcal{A} \times \Lambda \times J)} \leq \tilde{C} e^{-\tilde{\nu}/\epsilon}. \tag{50}
\]
The estimate of the derivatives in $(a^1, c^0) \in \mathcal{A} \times \Lambda$ in (50) follows directly from (45). To prove the estimate of the derivatives in $\tau \in J$, note that from (46) we have
\[
a^{in}_\epsilon = \Phi^u(0, \tau/\epsilon, c^0)a^1 - \int_0^{\tau/\epsilon} \Phi^u(0, s, c^0)f(s, c^0, \eta(s)) \, ds
\]
\[
b_\epsilon = \Phi^s(\tau/\epsilon, 0, c^0)b^0 + \int_0^{\tau/\epsilon} \Phi^s(\tau/\epsilon, s, c^0)g(s, c^0, \eta(s)) \, ds \tag{51}
\]
\[
c_\epsilon = c^0 \circ \tau + \int_0^{\tau/\epsilon} \Phi^c(\tau/\epsilon, s, c^0)(\theta(s, c^0, z(s)) + \tilde{E}(s, c^0, \eta(s))) \, ds.
\]
As in the proof of Theorem 7, it can be shown that the derivatives of the integrands in (51) are exponentially small, so we obtain (50).
The following theorem is a modification of the \((k + \sigma)\)-Exchange Lemma. A main difference in this version is that we assert the existence of the trajectories under consideration as a result of the lemma, while in the original version those trajectories were assumed to exist. The proof of the original theorem \cite{39} is based on tracking tangent spaces to an invariant manifold using linearized differential equations in terms of differential forms, while the approach we present below relies on estimates for solution operators, following closely to the proof of the general Exchange Lemma in \cite{33}.

**Theorem 9.** Consider a system of the form \((32)\) where \((x, y) \in \mathbb{R}^n \times \mathbb{R}^l\), and \(f\) and \(g\) are \(C^r\) functions for some \(r \geq 2\). Let \(S_0\) be a normally hyperbolic critical manifold for \((33)\), and suppose \(D_x f|_{S_0}\) has a splitting of \(k\) unstable eigenvalues and \(m\) stable eigenvalues, \(k + m = n\). Let \(\bar{q}_0 \in W^u(S_0) \setminus S_0\), \(\bar{p}_0 \in W^s(S_0) \setminus S_0\), \(\tau_0 > 0\), and assume

\[
\pi^s(\bar{p}_0) \circ [0, \tau_0] \subset S_0 \quad \text{and} \quad \pi^u(\bar{q}_0) = \pi^s(\bar{p}_0) \circ \tau_0, \tag{52}
\]

where \(\circ\) denotes the flow for the limiting slow system \((35)\), and \(\pi^{s,u}\) are the projections into \(S_0\) along stable/unstable fibers with respect to the limiting fast system \((33)\). Let \(\{\mathcal{I}_\epsilon\}_{\epsilon \in [0, \epsilon_0]}\) be a \(C^r\) family of \((k + \sigma - 1)\)-dimensional manifolds, \(1 \leq \sigma \leq l\), and suppose

\((T1)\) \(\mathcal{I}_0\) is transverse to \(W^s(S_0)\) at \(p_0\), and \(\Lambda := \pi^s(\mathcal{I}_0 \cap W^s(S_0))\) is of dimension \((\sigma - 1)\).
(T2) the slow flow \(35\) is not tangent to \(\Lambda\) at \(\pi^*(\bar{p}_0)\).

(T3) The trajectory \(\pi^*(p_0) \circ [0, \tau_0]\) is rectifiable and not self-intersecting.

Let
\[ I^*_\epsilon = \mathcal{I}_\epsilon \cdot [0, \infty), \tag{53} \]
where \(\cdot\) denotes the flow for \(32\). Choose a compact interval \(J \subset (0, \infty)\) containing \(\bar{\tau}_0\) satisfying \(\Lambda \circ J \subset S_0\), and set \(\tilde{\Lambda} = \Lambda \circ J\). Then there exists a neighborhood \(V_0\) of \(\bar{q}_0\) such that
\[ I^*_\epsilon \cap V_0 \text{ is } C^{r-2} \text{-close to } W^u_0(\tilde{\Lambda}) \cap V_0. \tag{54} \]
Moreover, given any sequence \(\bar{q}_\epsilon \in I^*_\epsilon \cap V_0\) such that \(\bar{q}_\epsilon \to \bar{q}_0\), there exists a sequence \((\bar{p}_\epsilon, \bar{\tau}_\epsilon) \in \mathcal{I}_\epsilon \times J\) which converges to \((\bar{p}_0, \bar{\tau}_0)\) and satisfies that, setting \(T_\epsilon = \bar{\tau}_\epsilon / \epsilon\),
\[ \bar{q}_\epsilon = \bar{p}_\epsilon \cdot T_\epsilon \quad \forall \, \epsilon > 0, \tag{55} \]
and
\[ T_\epsilon = (\tau_0 + o(1)) \epsilon^{-1}. \tag{56} \]

See Fig. 5 for an illustration of (54).

**Proof.** Since \(S_0\) is normally hyperbolic for \(33\), from [7, Lemma 2.2], after a \(C^{r-2}\) change of of coordinates, we can convert \(32\) to \(38\). Moreover, from (T1) we may assume
\[ \mathcal{I}_\epsilon = B_{k, \Delta} \times \{b^0\} \times \Lambda \tag{57} \]
for some constant \(b^0 \in B_{m, \Delta} \setminus \{0\}\).

Since \(\bar{q}_0 \in W^u_0(S_0) \setminus S_0\), we have \(a(\bar{q}_0) \neq 0\) and \(b(\bar{q}_0) = 0\), where \(a(\bar{q}_0)\) and \(b(\bar{q}_0)\) denote the \(a\)- and \(b\)-coordinates of \(\bar{q}_0\). Set
\[ \mathcal{A} = \{a \in \mathbb{R}^k : |a - a(\bar{q}_0)| < \Delta_1\} \tag{58} \]
for some positive number \(\Delta_1 < \frac{1}{2} \min\{\Delta, |a(\bar{q}_0)|\}\), so that \(\mathcal{A} \subset B_{k, \Delta} \setminus \{0\}\). Let \(p_\epsilon\) and \(q_\epsilon\) be the functions of \((a^1, e^0, \tau) \in \mathcal{A} \times \Lambda \times J\) defined by (48). From (57) we see that \((p_\epsilon, \tau)\) parametrizes \(\mathcal{I}_\epsilon \times J\) in a neighborhood of \((\bar{p}_0, \tau_0)\). Hence \(q_\epsilon\) parametrizes \(\mathcal{I}_\epsilon^*\) in neighborhoods of \(\bar{q}_0\). The estimate (49) holds with \(r\) replaced by \(r - 2\). In particular,
\[ \|q_\epsilon - (a^1, 0, e^0 \circ \tau)\|_{C^{r-2}(\mathcal{A} \times \Lambda \times J)} \leq C e^{-\rho/\epsilon}. \]
Note that
\[ W^u(\Lambda) = \{(a, b, c) : b = 0, c \in \Lambda \} \]
\[ = \{(a, b, c^0 \circ \tau) : b = 0, c^0 \in \Lambda, \tau \in J \}, \] (59)
so we obtain (54).

Next we consider the sequence \( \bar{q}_\epsilon \in I_\epsilon \) given in the statement. Choose \((a_1^{\epsilon}, c_0^{\epsilon}, \tau_\epsilon) \in \mathcal{A} \times \Lambda \times J \) such that \( \bar{q}_\epsilon = q_\epsilon(a_1^{\epsilon}, c_0^{\epsilon}, \tau_\epsilon) \), and set \( \bar{p}_\epsilon \equiv p_\epsilon(a_1^{\epsilon}, c_0^{\epsilon}, \tau_\epsilon) \).

Then by definition \( \bar{q}_\epsilon = \bar{p}_\epsilon \cdot (\bar{\tau}_\epsilon/\epsilon) \). From (T2) and (T3), \( \tilde{\Lambda} \) is a \( \sigma \)-dimensional manifold, and for any \( c^1 \in \tilde{\Lambda} \), there exists unique \((c_0, \tau_0) \in \Lambda \times J \) such that \( c^1 = c^0 \circ \tau_0 \). Hence (55) uniquely determines \( \bar{p}_0 \in I_0 \cap W^u(\Lambda) \) and \( \bar{\tau}_0 \in J \).

To show \( p_\epsilon \to \bar{p}_0 \) and \( \tau_\epsilon \to \bar{\tau}_0 \), since \((p_\epsilon, \tau_\epsilon)\) lies in the compact set \( \Lambda \times J \), it suffices to show that every convergent subsequence of \( \{(c_\epsilon, \tau_\epsilon)\} \) converges to \((c^0, \tau_0)\). Note that from the equation for \( \hat{c}_\epsilon \) in (51), we have
\[ \hat{c}_\epsilon = c_\epsilon^0 \circ \tau_\epsilon + o(1). \] (60)
Since \( q_\epsilon \to \bar{q}_0 \equiv (\bar{a}^1, 0, \bar{c}^1) \), given any convergent subsequence \((c_{\epsilon_j}, \tau_{\epsilon_j})\) of \((c_\epsilon, \tau_\epsilon)\), say \((c_{\epsilon_j}, \tau_{\epsilon_j}) \to (c^0, \tau_0)\), from (60) we obtain \( \bar{c}^1 = \bar{c}^0 \circ \tau_0 \). From (52) we have \( c^1 = \bar{c}^0 \circ \tau_0 \). Hence \( (\bar{c}^0, \tau_0) = (c^0, \tau_0) \). This completes the proof.

5. Singular Configuration

The fast-slow system (14\( \epsilon \)) has multiple natural time scales, which leads to multiple limiting subsystems. In this section, we will find trajectories in those subsystems so that the union of the trajectories joins the end states \( u_L \) and \( u_R \). Those trajectories are called singular trajectories, and their union is called a singular configuration. In Section 6 we will show that there are solutions of (14\( \epsilon \)) close to the singular configuration.

5.1. End States \( U_L \) and \( U_R \)

The system (17) has a normally hyperbolic critical manifold
\[ \mathcal{S}_0 = \{(u, w, \xi) : f(u) - \xi u - w = 0, \xi \neq \text{Re}(\lambda_{\pm}(u))\}, \] (61)
where \( \lambda_{\pm}(u) \) are the eigenvalues of \( Df(u) \), defined in (18). The limiting slow system for (16\( \epsilon \)) is
\[ 0 = f(u) - \xi u - w \]
\[ w' = -u \]
\[ \xi' = 1. \] (62)
From (H1) we have $s < \text{Re}(\lambda\pm(u_L))$, so $(u_L, w_L, s) \in S_0$. Choose $\delta > 0$ so that $s + 2\delta < \text{Re}(\lambda\pm(u_L))$, and set
\[
U_L = (u_L, w_L, s) \circ (-\infty, \delta] \ \ (62)
\]
\[
= \{ (u, w, \xi) : u = u_L, w = w_L - \alpha_1 u_L, \xi = s + \alpha_1, \alpha_1 \in (-\infty, \delta] \},
\]
where $\circ$ denotes the flow for (62). It is clear that $U_L \subset S_0$ is normally hyperbolic with respect to (17), and is locally invariant with respect to (14). Note that each point in $U_L$ is a hyperbolic equilibrium for the 2-dimensional system (15), and the unstable manifold $W_u^0(U_L)$ is naturally defined.

Proposition 10. Assume (H1). Let $U_L$ be defined by (63). Fix any $k \geq 1$. There exists a family of invariant manifolds $W^u_\epsilon(U_L)$ which are $C^k O(\epsilon)$-close to $W^u_0(U_L)$ such that for any continuous family $\{ \mathcal{I}_\epsilon \}_{\epsilon \in [0, \epsilon_0]}$ of compact sets $\mathcal{I}_\epsilon \subset W^u_\epsilon(U_L)$,
\[
\text{dist}(p \circ t, U_L) \leq Ce^{\mu t} \ \ \forall \ p \in \mathcal{I}_\epsilon, \ t \leq 0, \ \epsilon \in [0, \epsilon_0].
\] (64)
for some positive constants $C$ and $\mu$.

Proof. This follows from Theorem 9 by taking $U_L$ to be $U_0$. Although $U_L$ is not compact, it is uniformly normally hyperbolic since $\xi - \text{Re}(\lambda\pm(u_L)) < -\delta$ on $U_L$, and the proof of Theorem 6 in [14, Theorem 4] is still valid. \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ [21]

Remark 4. Proposition 10 was also asserted in [32, 29, 24].

From (H1) we also have, by decreasing $\delta$ if necessary, $s - 2\delta > \text{Re}(\lambda\pm(u_R))$, and hence a similar result holds for the set $U_R$ defined by
\[
U_R = (u_R, w_R, s) \circ [-\delta, \infty) \ \ (65)
\]
\[
= \{ (u, w, \xi) : u = u_R, w = w_R - \alpha_2 u_R, \xi = s + \alpha_2, \alpha_2 \in [-\delta, \infty) \}.
\]

Proposition 11. Assume (H1). Let $U_R$ be defined by (65). Fix any $k \geq 1$. There exists a family of invariant manifolds $W^s_\epsilon(U_R)$ which are $C^k O(\epsilon)$-close to $W^s_0(U_R)$ such that for any continuous family $\{ \mathcal{J}_\epsilon \}_{\epsilon \in [0, \epsilon_0]}$ of compact sets $\mathcal{J}_\epsilon \subset W^s_\epsilon(U_R)$,
\[
\text{dist}(p \circ t, U_R) \leq Ce^{-\mu t} \ \ \forall \ p \in \mathcal{J}_\epsilon, \ t \geq 0, \ \epsilon \in [0, \epsilon_0].
\] (66)
for some positive constants $C$ and $\mu$. \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ [21]
5.2. Intermediate States $\mathcal{P}_L$ and $\mathcal{P}_R$

Consider the system (16). In order to study the dynamics at $\{v = +\infty\}$, we set $r = 1/v$ and $\kappa = \epsilon \log(1/r)$. Then (16) is converted, after multiplying the equations by $r$, to

\[
\begin{align*}
\dot{\beta} &= B_1(\beta) - \xi \beta r - w_1 r \\
\dot{r} &= -r B_2(\beta) + \xi r^2 + w_2 r^3 \\
\dot{w}_1 &= -\epsilon \beta r \\
\dot{w}_2 &= -\epsilon \\
\dot{\xi} &= \epsilon r \\
\dot{\kappa} &= \epsilon (B_2(\beta) + \xi r + w_2 r^2).
\end{align*}
\] (67)

Note that the time variable in (67) is different from that of (16). We use the same dot symbol to denote derivatives, but there should be no ambiguity since the different time scales can be distinguished by comparing the term $\dot{\xi}$.

The limiting fast system for (67) is

\[
\begin{align*}
\dot{\beta} &= B_1(\beta) - \xi \beta r - w_1 r \\
\dot{r} &= -r B_2(\beta) + \xi r^2 + w_2 r^3 \\
\dot{w}_1 &= 0, \quad \dot{w}_2 = 0, \quad \dot{\xi} = 0, \quad \dot{\kappa} = 0.
\end{align*}
\] (68)

The obvious equilibria for (68), besides $(\beta_L, r_L, w_{1L}, w_{2L}, s)$ and $(\beta_R, r_R, w_{1L}, w_{2L}, s)$, where $r_L = 1/v_L$ and $r_R = 1/v_R$, are

\[
\begin{align*}
\mathcal{P}_L &= \{ (\beta, r, w_1, w_2, \xi, \kappa) : \beta = \rho_1, r = 0 \}, \\
\mathcal{P}_R &= \{ (\beta, r, w_1, w_2, \xi, \kappa) : \beta = \rho_2, r = 0 \}.
\end{align*}
\] (69, 70)

The limiting slow system on $\mathcal{P}_L$ is

\[
w_1' = 0, \quad w_2' = -1, \quad \xi' = 0, \quad \kappa' = B_2(\rho_1),
\] (71)

and on $\mathcal{P}_R$ is

\[
w_1' = 0, \quad w_2' = -1, \quad \xi' = 0, \quad \kappa' = B_2(\rho_2)
\] (72)

The Fenichel coordinates near $\mathcal{P}_L$ can be described as follows.

**Proposition 12.** Let $W^{u,s}(\mathcal{P}_L)$ be the $C^k$ unstable/stable manifolds of $\mathcal{P}_L$ for (67), $k \geq 1$. Then there exists a $C^k$ function $\hat{\beta} = \hat{\beta}(\beta, r, w_1, w_2, \xi, \kappa, \epsilon)$ such that

\[
\hat{\beta} = \beta \quad \text{when } r = 0
\] (73)
and \((\hat{\beta}, r, w_1, w_2, \xi, \kappa)\) is a change of coordinates near \(P_L\) satisfying
\[
W^s_\epsilon(P_L) = \{(\hat{\beta}, r, w_1, w_2, \xi, \kappa) : \hat{\beta} = \rho_1\}
\]  
(74)
\[
W^u_\epsilon(P_L) = \{(\hat{\beta}, r, w_1, w_2, \xi, \kappa) : r = 0\}.
\]  
(75)

Moreover, the projection \(\pi^s_{\epsilon, P_L}\) into \(P_L\) along stable fibers with respective to
\[
\pi^s_{\epsilon, P_L}((\rho_1, r, w_1, w_2, \xi, \kappa)) = (\rho_1, 0, w_1, w_2, \xi, \kappa)
\]  
(76)

in \((\hat{\beta}, r, w_1, w_2, \xi, \kappa)\)-coordinates.

Proof. The linearization of (68) at \(P_L\) corresponds to the matrix
\[
\begin{pmatrix}
B'(\rho_1) & -\xi \rho_1 \\
0 & -B_2(\rho_1)
\end{pmatrix} = \begin{pmatrix}
1 - \frac{\rho_2}{\rho_1} & -\frac{\xi \rho_1}{\rho_1} \\
0 & -\frac{1}{2} (1 - \frac{\rho_2}{\rho_1})
\end{pmatrix}
\]  
(77)

which has one positive and one negative eigenvalue. Note that \(P_L\) is invariant under (67) for each \(\epsilon\). From Theorem 4 and the remark following it, \(W^s_\epsilon(P_L)\) and \(W^u_\epsilon(P_L)\) are well defined and both have dimension 1, and we may take \(W^u_\epsilon(P_L) = \{r = 0\}\). Note that \(\{\beta = \rho_1\}\) is transverse to \(W^s_\epsilon(P_L)\), so we can choose Fenichel coordinates \((a, b, c)\) corresponding to this splitting with \(b = r\) and
\[
a = \beta - \rho_1 + \phi(w_1, w_2, \xi, \kappa, \epsilon, r)
\]
for some \(C^k\) function \(\phi\). Let \(\hat{\beta} = a + \rho_1\). Then the desired result follows.

An analogous result holds for \(P_R\). We omit it here.

5.3. Transversal Intersections

Fix small \(r^0 > 0\) such that \(\gamma_1\) intersects \(\{r = r^0\}\) at a unique point. Denote this point by \(p^\text{in}_0\). That is,
\[
p^\text{in}_0 = \gamma_1 \cap \{r = r^0\}.
\]  
(78)

We set
\[
\mathcal{I}_\epsilon = W^u_\epsilon(U_L) \cap \{r = r^0\} \cap V_1,
\]  
(79)

where \(V_1\) is an open neighborhood of \(p^\text{in}_0\) such that \(\mathcal{I}_\epsilon\) can be parametrized as
\[
\mathcal{I}_\epsilon = \{(\hat{\beta}, r, w_1, w_2, \xi, \kappa) : r = r^0, \kappa = \epsilon \log(1/r^0), \\
w_1, w_2, (w_1L, w_2L, s) + \alpha_1(-\beta_L, -v_L, 1) + \epsilon \theta(a, \alpha_1, \epsilon), \\
|a| < \Delta_1, |\alpha_1| < \Delta_1\},
\]  
(80)
where the coordinates \((\hat{\beta}, r, w_1, w_2, \xi, \kappa)\) are defined in Proposition 12. From (74) we see that \(I_0\) and \(W^s_0(\mathcal{P}_L)\) intersect transversally at \(p^\text{in}_0\), and if we set

\[
\Lambda_L = \pi^s_{0,\mathcal{P}_L}(I_0 \cap W^s_0(\mathcal{P}_L)),
\]

where \(\pi^s_{0,\mathcal{P}_L}\) is the projection into \(\mathcal{P}_L\) along stable fibers with respect to (68), then

\[
\Lambda_L = \{(\beta, r, w_1, w_2, \xi, \kappa) : \beta = \rho_1, r = 0, \kappa = 0,
(w_1, w_2, \xi) = (w_{1L}, w_{2L}, s) + \alpha_1(-\beta_L, -v_L, 1),
|\alpha_1| < \Delta_1\}.
\]

Similarly, by shrinking \(r^0\) if necessary, \(\gamma_2\) intersects \(\{r = r^0\}\) at a unique point

\[
p^\text{out}_0 = \gamma_2 \cap \{r = r^0\}.
\]

Set

\[
\mathcal{J}_e = W^s_\epsilon(\mathcal{U}_R) \cap \{r = r^0\} \cap V_2,
\]

where \(V_2\) is an open neighborhood of \(p^\text{out}_0\) such that \(\mathcal{J}_e\) has a parametrization analogous to (80). Then \(\mathcal{J}_0\) is transverse to \(W^u_\epsilon(\mathcal{P}_R)\) at \(p^\text{out}_0\), and we set

\[
\Lambda_R = \pi^u_{0,\mathcal{P}_R}(\mathcal{J}_0 \cap W^u_0(\mathcal{P}_R)),
\]

where \(\pi^u_{0,\mathcal{P}_R}\) is the projection into \(\mathcal{P}_R\) along unstable fibers with respect to (68).

To connect \(p^\text{in}_0\) and \(p^\text{out}_0\), we have the following

**Proposition 13.** The system \((68)\) has a trajectory

\[
\gamma_0 = \{(\beta, 0, w_{1L}, w_{20}, s, \kappa_0) : \beta \in (\rho_2, \rho_1)\},
\]

which joins the points

\[
\pi^s_{\mathcal{P}_R}(p^\text{out}_0) \bullet \tau_{10} \in \mathcal{P}_L \quad \text{and} \quad \pi^u_{\mathcal{P}_L}(p^\text{in}_0) \bullet (-\tau_{20}) \in \mathcal{P}_R,
\]

where

\[
w_{20} = w_{2L} - \frac{\rho_2}{\rho_1 + \rho_2}(w_{2L} - w_{2R}), \quad \kappa_0 = \frac{\rho_1(\rho_1 - \rho_2)}{2\rho_2(\rho_1 + \rho_2)},
\]

and

\[
\tau_{10} = \frac{\rho_2}{\rho_1 + \rho_2}(w_{2L} - w_{2R}), \quad \tau_{20} = \frac{\rho_1}{\rho_1 + \rho_2}(w_{2L} - w_{2R}).
\]
Moreover, if we set
\[ \Lambda_L = \Lambda_L \bullet [\tau_1-, \tau_1+] \quad \text{and} \quad \Lambda_R = \Lambda_R \bullet (-[\tau_2-, \tau_2+]), \quad (89) \]
where \( \tau_1- < \tau_{10} < \tau_1+ \) and \( \tau_2- < \tau_{20} < \tau_2+ \), then \( W^u_0(\Lambda_L) \) and \( W^s_0(\Lambda_R) \) intersect transversally along \( \gamma_0 \) in the space \( \{r = 0\} \).

**Proof.** Note that the restriction of the system (68) on \( \{r = 0\} \) is simply \( \beta = B_1(\beta) \), so every trajectory of (68) joins \( \{\beta = \rho_1\} \) and \( \{\beta = \rho_2\} \). Also note that
\[
\begin{align*}
\pi^u_{\tau R}(p_0^{\text{out}}) \bullet \tau &= (\rho_1, 0, w_{1L}, w_{2L}, s, 0) + \tau(0, 0, -1, 0, B_2(\rho_1)) \\
\pi^u_{\tau L}(p_0^{\text{out}}) \bullet \tau &= (\rho_2, 0, w_1, w_{2R}, s, 0) + \tau(0, 0, 0, -1, 0, B_2(\rho_2)), \quad \forall \tau \in \mathbb{R},
\end{align*}
\]
in \((\rho, w_1, w_2, \xi, \kappa)\)-coordinates. Hence \( \gamma_0 \) defined in (86) joins \( \pi^u_{\tau R}(p_0^{\text{out}}) \bullet \tau_{10} \) and \( \pi^u_{\tau L}(p_0^{\text{out}}) \bullet (-\tau_{20}) \) if
\[
w_{20} = w_{2L} - \tau_{10} = w_{2R} + \tau_{20}, \quad \kappa_0 = B_2(\rho_1)\tau_{10} = B_2(\rho_2)\tau_{20}, \quad (90)
\]
which gives (87) and (88).

Let \( \hat{\Lambda}_L \) and \( \hat{\Lambda}_R \) be defined in (89). From the parameterizations (75) and (82), we have
\[
W^u_0(\hat{\Lambda}_L) = \{ (\beta, r, w_1, w_2, \xi, \kappa) : r = 0, (w_1, w_2, \xi, \kappa) = (w_{1L}, w_{2L}, s, 0) + \alpha_1(-\beta_L, -v_L, 1, 0) + \tau_1(0, -1, 0, B_2(\rho_1)), \beta \in (\rho_2, \rho_1), |\alpha_1| < \Delta_1, \tau_1 \in [\tau_1-, \tau_1+] \}
\]
and
\[
W^s_0(\hat{\Lambda}_R) = \{ (\beta, r, w_1, w_2, \xi, \kappa) : r = 0, (w_1, w_2, \xi, \kappa) = (w_{1R}, w_{2R}, s, 0) + \alpha_1(-\beta_R, -v_R, 1, 0) - \tau_2(0, -1, 0, B_2(\rho_2)), \beta \in (\rho_2, \rho_1), |\alpha_2| < \Delta_1, \tau_2 \in [\tau_2-, \tau_2+] \}.
\]

Fix any \( q_0 \in \gamma_0 \), we have
\[
T_{q_0}W^u_0(\hat{\Lambda}_L) = \text{Span} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\beta_L & -v_L \\ 0 & -\beta_R & -v_R \\ 0 & 1 & 0 \\ 0 & 0 & B_2(\rho_1) \end{pmatrix} \right\}
\]

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and
\[
T_{q_0}W^s_0(\tilde{\Lambda}_R) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\beta_R \\ 0 \\ -v_R \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ B_2(\rho_2) \end{pmatrix} \right\}.
\]
Hence \( T_{q_0}W^u_0(\tilde{\Lambda}_L) \) and \( T_{q_0}W^s_0(\tilde{\Lambda}_R) \) span the space \( \{ r = 0 \} \). This means \( W^u_0(\tilde{\Lambda}_L) \) and \( W^s_0(\tilde{\Lambda}_R) \) intersect transversally in the space \( \{ r = 0 \} \) at \( q_0 \). \( \square \)

**Remark 5.** The transversality in Proposition 13 can also be proved using the method of Melnikov functions, as shown in [32, Proposition 6.3] and [24, Proposition 4.4]. See also [27].

Let \( \gamma_0 \) be the trajectory defined in Proposition 13. We set
\[
q_0 = \gamma_0 \cap \Gamma,
\]
where
\[
\Gamma = \{(\beta, r, w_1, w_2, \xi, \kappa) : \beta = \frac{\rho_1 + \rho_2}{2}\}.
\]
Then
\[
\pi^u_{PL}(q_0) = \pi^s_{PL}(p^\text{in}_{0L}) \bullet \tau_{10}, \quad \pi^s_{PL}(q_0) = \pi^u_{PL}(p^\text{out}_{0L}) \bullet (-\tau_{20}).
\]
Let
\[
\sigma_1 = \pi^s_{PL}(p^\text{in}_{0L}) \bullet [0, \tau_{10}]
= \{(\rho_2, 0, w_{1L}, w_{2L} - \tau, s, B_2(\rho_1)\tau) : \tau \in [0, \tau_{10}]\}
\]
and
\[
\sigma_2 = \pi^u_{PL}(p^\text{in}_{0L}) \bullet [-\tau_{20}, 0]
= \{(\rho_1, 0, w_{1R}, w_{2R} + \tau, s, -B_2(\rho_2)\tau) : \tau \in [0, \tau_{20}]\}
\]
in \((\beta, r, w_1, w_2, \xi, \kappa)\)-coordinates. Then we obtain the singular configuration
\[
\gamma_1 \cup \sigma_1 \cup \gamma_0 \cup \sigma_2 \cup \gamma_2
\]
connecting \( U_L \) and \( U_R \). See Fig 6.
6. Completing the Proof of the Main Theorem

We split the proof of the main theorem into two parts. In the first subsection we prove the existence of solutions of the boundary value problem \( 11 \), \( 12 \), and show that \( 22 \) holds. In the second subsection we derive the weak limit \( 23 \).

6.1. Existence of Trajectories

**Proposition 14.** Assume (H1)-(H3). Let \( p^{\text{in}}_0, p^{\text{out}}_0, q_0, \mathcal{I}_\epsilon, \mathcal{J}_\epsilon \) and \( \Sigma \) be defined in Section 5.3. Then for each small \( \epsilon > 0 \), there exist \( p^{\text{in}}_\epsilon \in \mathcal{I}_\epsilon, p^{\text{out}}_\epsilon \in \mathcal{J}_\epsilon, q_\epsilon \in \Gamma \) and \( T_{1\epsilon}, T_{2\epsilon} > 0 \) such that

\[
p^{\text{in}}_\epsilon = q_\epsilon \cdot (-T_{1\epsilon}), \quad p^{\text{out}}_\epsilon = q_\epsilon \cdot T_{2\epsilon},
\]

where \( \cdot \) denotes the flow of \( 67 \), satisfying

\[
(p^{\text{in}}_\epsilon, p^{\text{out}}_\epsilon, q_\epsilon) = (p^{\text{in}}_0, p^{\text{out}}_0, q_0) + o(1)
\]

and

\[
T_{1\epsilon} = (\tau_{10} + o(1))\epsilon^{-1}, \quad T_{2\epsilon} = (\tau_{20} + o(1))\epsilon^{-1},
\]

as \( \epsilon \to 0 \), where \( \tau_{10} \) and \( \tau_{20} \) are defined in \( 88 \). Moreover, if we set \( \kappa_\epsilon(\sigma) \) to be the \( \kappa \)-coordinate of \( q_\epsilon \cdot \sigma, \sigma \in [-T_{1\epsilon}, T_{2\epsilon}] \), then

\[
\max_{\sigma \in [-T_{1\epsilon}, T_{2\epsilon}]} \kappa_\epsilon(\sigma) = \kappa_0 + o(1),
\]
where \( \kappa_0 \) is defined in (87).

**Proof.** We will apply the Exchange Lemma (Theorem 9) with \((k, m, l, \sigma) = (1, 1, 3, 1)\). From (80) and (74), we know that the \((k + \sigma)\)-manifold \( I_0 \) is transverse to the \((m + l)\)-manifold \( W_0^s(P_L) \) at \( p_0^0 \), and the image of the projection

\[
\pi_{P_L}^* (I_0 \cap W_0^s(P_L)) = \Lambda_L
\]

is \( \sigma \)-dimensional, so (T1) in the Exchange Lemma holds. The limiting slow system on \( P_L \) is governed by (71), and by the parametrization (82) of \( \Lambda_L \) it follows that (T2) holds. Also it is clear that (T3) holds with \( \tau_0 = \tau_{10} \), where \( \tau_{10} \) is defined in (88). Theorem 9 implies that there exists a neighborhood \( V_0 \) of \( q_0 \) such that

\[
I_\epsilon^* \cap V_0 \text{ is } C^1 O(\epsilon) \text{-close to } W_0^u(\tilde{\Lambda}_L) \cap V_0, \tag{103}
\]

where \( I_\epsilon^* = I_\epsilon \cdot [0, \infty) \). Similarly,

\[
J_\epsilon^* \cap V_0 \text{ is } C^1 O(\epsilon) \text{-close to } W_0^s(\tilde{\Lambda}_R) \cap V_0, \tag{104}
\]

where \( J_\epsilon^* = J_\epsilon \cdot (-\infty, 0] \). From Proposition 13, it follows that the projections of \( I_\epsilon^* \) and \( J_\epsilon^* \) in the 5-dimensional space \( \{ r = 0 \} \) intersect transversally at a unique point in \( \Gamma \) near \( q_0 \). For the relation \( r = \exp(-\kappa/\epsilon) \), we then recover a unique intersection point

\[
q_\epsilon \in I_\epsilon^* \cap J_\epsilon^* \cap \Gamma
\]

in \( (\beta, r, w_1, w_2, \xi, \kappa) \)-space. By construction we have (99) and (100). The estimates (101) follows from (56). Note that

\[
\max_{\sigma_1 \cup \gamma_0 \cup \sigma_2} \kappa = \kappa_0,
\]

where \( \sigma_1, \sigma_2 \) and \( \gamma_0 \) are defined in Section 5.3, so we obtain (102). \( \square \)

**Proposition 15.** Assume (H1)-(H3) hold. Let \( q_\epsilon = (\beta_\epsilon, r_\epsilon, w_{1\epsilon}, w_{2\epsilon}, \xi_\epsilon, \kappa_\epsilon) \in \Gamma \) be defined in Proposition 14. Let \( v_\epsilon^0 = \exp(\kappa_\epsilon^0/\epsilon) \) and

\[
(\tilde{\beta}_\epsilon, \tilde{v}_\epsilon, \tilde{w}_{1\epsilon}, \tilde{w}_{2\epsilon})(\xi) = (\beta_\epsilon^0, v_\epsilon^0, w_{1\epsilon}^0, w_{2\epsilon}^0) \cdot (\xi - \xi_\epsilon^0), \tag{105}
\]

or equivalently,

\[
(\tilde{\beta}_\epsilon, \tilde{v}_\epsilon, \tilde{w}_{1\epsilon}, \tilde{w}_{2\epsilon}, \xi) = (\beta_\epsilon^0, v_\epsilon^0, w_{1\epsilon}^0, w_{2\epsilon}^0, \xi_\epsilon^0) \cdot \left( \frac{\xi - \xi_\epsilon^0}{\epsilon} \right). \tag{106}
\]

Then \( (\tilde{\beta}_\epsilon, \tilde{v}_\epsilon) \) is a solution of (11e) and (12), and it satisfies (22).
Proof. Since (11e) and (13e) are equivalent, and \((\tilde{\beta}_\epsilon, \tilde{v}_\epsilon, \tilde{w}_1, \tilde{w}_2)(\xi)\) is a solution of (13e), we know \((\tilde{\beta}_\epsilon, \tilde{v}_\epsilon)\) is a solution of (11e).

Let \(T_1, T_2 \in \mathbb{R}\) be defined in Proposition 14. Then
\[
q_{\epsilon} \bullet (-T_1/\epsilon) \in \mathcal{I}_\epsilon, \quad q_{\epsilon} \bullet (T_2/\epsilon) \in \mathcal{J}_\epsilon.
\]

Since \(\mathcal{I}_\epsilon \subset W^{u}(U_L)\) and \(\mathcal{J}_\epsilon \subset W^{s}(U_R)\), from (64) and (66) we have
\[
\lim_{t \to -\infty} \text{dist}(q_{\epsilon} \bullet t, U_L) = 0, \quad \lim_{t \to \infty} \text{dist}(q_{\epsilon} \bullet t, U_R) = 0,
\]
which implies (12). Since \(\kappa_{\epsilon} = \epsilon \log(v_{\epsilon})\), from (102) we obtain (22). \qed

6.2. Convergence of Trajectories

Based on the results in Proposition 14, we first derive some estimates for the self-similar solution \(\tilde{u}_{\epsilon}(\xi)\).

Proposition 16. Let \(\tilde{u}_{\epsilon} = (\tilde{\beta}_{\epsilon}, \tilde{v}_{\epsilon})\) be the solution of (11e) and (12) in Proposition 15. Let \(p^{in}_{\epsilon}\) and \(p^{out}_{\epsilon}\) be defined in Proposition 14. Then
\[
\left| \xi_{\epsilon}^{\text{in}} - s \right| + \left| \xi_{\epsilon}^{\text{out}} - s \right| = o(1) \tag{107}
\]
\[
\int_{-\infty}^{\xi_{\epsilon}^{\text{in}}} |\tilde{u}(\xi) - u_L| \, d\xi + \int_{\xi_{\epsilon}^{\text{out}}}^{\infty} |\tilde{u}(\xi) - u_R| \, d\xi = o(1) \tag{108}
\]
\[
\int_{\xi_{\epsilon}^{\text{in}}}^{\xi_{\epsilon}^{\text{out}}} \tilde{u}(\xi) \, d\xi = (0, e_0) + o(1) \tag{109}
\]
as \(\epsilon \to 0\), where \(\xi_{\epsilon}^{\text{in}}\) and \(\xi_{\epsilon}^{\text{out}}\) are \(\xi\)-coordinates of \(p^{in}_{\epsilon}\) and \(p^{out}_{\epsilon}\), respectively.

Proof. Note that \(s\) is the \(\xi\)-coordinate of \(p^0_{\epsilon}\), so
\[
\left| \xi_{\epsilon}^{\text{in}} - s \right| \leq |p^{in}_{\epsilon} - p^0_{\epsilon}|,
\]
which tends to zero by (100). Similarly, \(\xi_{\epsilon}^{\text{out}} - s \to 0\). This gives (107).

Since every point in \(U_L\) has \(u\)-coordinate equal to \(u_L\),
\[
|\tilde{u}(\xi) - u_L| \leq \text{dist}((\tilde{u}(\xi), \tilde{v}(\xi), \xi), U_L) = \text{dist}((u^0_{\epsilon}, w^0_{\epsilon}, \xi^0_{\epsilon}) \bullet \xi_{\epsilon}^0 / \epsilon, \xi_{\epsilon}^0 / \epsilon, U_L),
\]
where the last equality follows from (106). Using (64), the last term is \(\leq C \exp\left(\nu \frac{\xi - \xi^0_{\epsilon}}{\epsilon}\right)\). Since \(\xi_{\epsilon}^{\text{in}} < \xi^0_{\epsilon}\), it follows that
\[
\int_{-\infty}^{\xi_{\epsilon}^{\text{in}}} |\tilde{u}(\xi) - u_L| \, d\xi \leq \int_{-\infty}^{\xi_{\epsilon}^{\text{in}}} C \exp\left(\nu \frac{\xi - \xi^0_{\epsilon}}{\epsilon}\right) \, d\xi \leq \int_{-\infty}^{\xi^0_{\epsilon}} C \exp\left(\nu \frac{\xi - \xi^0_{\epsilon}}{\epsilon}\right) \, d\xi = \frac{\epsilon}{\nu} C,
\]
29
A similar inequality holds for $\int_{\xi_{\text{out}}}^{\infty} |\tilde{u}(\xi) - u_R| \, d\xi$, so we obtain (108).

Since $\tilde{\beta}_{\epsilon}(\xi)$ is uniformly bounded in $\epsilon$, its integral between $\xi_{\text{in}}^{\epsilon}$ and $\xi_{\text{out}}^{\epsilon}$ is $o(1)$ by (107), and this proves the first part of (109). From the equation of $\dot{\xi}$ in (67), denoting the time variable by $\zeta$, we can write

$$\xi(0) = \xi_0^{\epsilon}, \quad \frac{d\xi}{d\zeta} = \epsilon \tilde{r}_{\epsilon}(\xi), \quad (110)$$

where $\tilde{r}_{\epsilon}(\xi) = 1/\tilde{v}_{\epsilon}(\xi)$. From (99) we have

$$\xi(-T_{1\epsilon}) = \xi_{\text{in}}^{\epsilon}, \quad \xi(T_{2\epsilon}) = \xi_{\text{out}}^{\epsilon}. \quad (111)$$

From (110) and (111) it follows that

$$\int_{\xi_{\text{in}}^{\epsilon}}^{\xi_{\text{out}}^{\epsilon}} \tilde{v}(\xi) \, d\xi = \int_{\xi_{\text{in}}^{\epsilon}}^{\xi_{\text{out}}^{\epsilon}} \frac{1}{\tilde{r}(\xi)} \, d\xi = \int_{-T_{1\epsilon}}^{T_{2\epsilon}} \epsilon \, d\zeta = \epsilon(T_{1\epsilon} + T_{2\epsilon}),$$

which converges to $w_{2L} - w_{2R} = e_0$ by (101). This proves (109).

From the estimates in Proposition 16 we can derive the weak convergence of $\tilde{u}(\xi)$ as follows.

**Proposition 17.** Let $\tilde{u}_{\epsilon} = (\tilde{\beta}_{\epsilon}, \tilde{v}_{\epsilon})$ be the solution of (11e) and (12) given in Proposition 15. Then

$$\tilde{u}_{\epsilon} \rightharpoonup u_L + (u_R - u_L)H(\xi - s) + (0, e_0)\delta_0(\xi - s) \quad (112)$$

in the sense of distributions as $\epsilon \to 0$.

**Proof.** Let $\psi \in C_\infty_c(\mathbb{R})$ be a smooth function with compact support. From (108) we have

$$\left| \int_{-\infty}^{\xi_{\text{in}}^{\epsilon}} \psi(\xi)(\tilde{u}(\xi) - u_L) \, d\xi \right| \leq \|\psi\|_{L^\infty} \int_{-\infty}^{\xi_{\text{in}}^{\epsilon}} |\tilde{u}(\xi) - u_L| \, d\xi \leq \|\psi\|_{L^\infty} C\epsilon,$$

which implies

$$\int_{-\infty}^{\xi_{\text{in}}^{\epsilon}} \psi(\xi) \tilde{u}(\xi) \, d\xi = \left( \int_{-\infty}^{\xi_{\text{in}}^{\epsilon}} \psi(\xi) \, d\xi \right) u_L + o(1) = \left( \int_{-\infty}^{s} \psi(\xi) \, d\xi \right) u_L + o(1).$$

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A similar inequality holds for \( \int_{\xi_{\text{out}}}^{\infty} \psi \, d\xi \), so

\[
\int_{\mathbb{R} \setminus \left[ \xi_{\text{in}}, \xi_{\text{out}} \right]} \psi(\xi) \tilde{u}(\xi) \, d\xi = \left( \int_{-\infty}^{s} \psi(\xi) \, d\xi \right) u_L + \left( \int_{s}^{\infty} \psi(\xi) \, d\xi \right) u_R + o(1). \tag{113}
\]

On the other hand, from (107) and (109) we have

\[
\int_{\xi_{\text{in}}}^{\xi_{\text{out}}} |(\psi(\xi) - \psi(s)) \tilde{u}_e(\xi)| \, d\xi \leq \left( \max_{\xi \in [\xi_{\text{in}}, \xi_{\text{out}}]} |\psi(\xi) - \psi(s)| \right) \left( \int_{\xi_{\text{in}}}^{\xi_{\text{out}}} \tilde{u}_e(\xi) \, d\xi \right)
\]
\[= o(1) \left( o(0) + o(1) \right) = o(1) \] (114)

which implies

\[
\int_{\xi_{\text{in}}}^{\xi_{\text{out}}} \psi(\xi) \tilde{u}_e(\xi) \, d\xi = \psi(s) \int_{\xi_{\text{in}}}^{\xi_{\text{out}}} \tilde{u}_e(\xi) \, d\xi + o(1) = \psi(s)(0, e_0) + o(1). \tag{115}
\]

Combining (113) and (114) we obtain

\[
\int_{-\infty}^{\infty} \psi(\xi) \tilde{u}_e(\xi) \, d\xi = u_L \int_{-\infty}^{s} \psi(\xi) \, d\xi + u_R \int_{s}^{\infty} \psi(\xi) \, d\xi + (0, e_0)\psi(s) + o(1). \tag{116}
\]

This holds for all \( \psi \), so we have (112).

Converting the results of Proposition 17 from self-similar variables to physical space variables, we obtain the following

**Proposition 18.** Let \( \tilde{u}_e = (\tilde{\beta}_e, \tilde{v}_e) \) be the solution of (11c) and (12) given in Proposition 15. Let \( u_e(x, t) = \tilde{u}_e(x/t) \). Then the weak convergence (23) holds.

**Proof.** Let \( \varphi \in C^\infty_c(\mathbb{R} \times \mathbb{R}^+) \). From (112) we have

\[
\int_0^\infty \int_{-\infty}^{\infty} \varphi(x, t) u_e(x, t) \, dx \, dt = \int_0^\infty t \int_{-\infty}^{\infty} \varphi(t\xi, t) \tilde{u}_e(\xi) \, d\xi \, dt
\]
\[= \int_0^\infty \left\{ u_L \int_{-\infty}^{s} \varphi(t\xi, t) \, d\xi + (0, e_0)\varphi(st, t) + u_R \int_{s}^{\infty} \varphi(t\xi, t) \, d\xi \right\} \, dt + o(1)
\]
\[= u_L \int_0^\infty \int_{-\infty}^{st} \varphi(x, t) \, dx \, dt + u_R \int_0^\infty \int_{st}^{\infty} \varphi(x, t) \, dx \, dt + (0, e_0) \int_0^\infty t \varphi(st, t) \, dt + o(1).
\]

From the definition (24), we conclude that (23) holds.

Now Propositions 15 and 18 complete the proof of the Main Theorem.
7. Numerical Simulations

Some numerical solutions for (1) using the Lax-Friedrichs scheme are shown in Fig 7. The solutions appear to grow unboundedly as the number of steps increases.

Also some numerical approximations for (10ε) are shown in Fig 8. The algorithm was a shooting method following the descriptions in [22]. Since $w_1$ and $ξ$ are essentially constant near the shock, we project the trajectories in the $(β, r, w_2)$ space. Note that $w_2(ξ)$ does not converge as $ξ → ±∞$ while $x_2 = w_2 + (ξ - s)v$ converges, we replace $w_2$ by $x_2$ (again, following [22]). Note that $x_2$ is a mild modification of $w_2$ near the shock since within the $ε$-neighborhood of $ξ = s$ the difference between $x_2$ and $w_2$ is of order $o(1)$.

As $ε$ decreases, the minimal value of $r$-coordinate on the trajectories in Fig 8 tends to zero. This means the maximum of $v$ tends to infinity. Also observe that the change of the value of $x_2$ concentrates in the vicinity of $β = ρ_1$ and $β = ρ_2$. This is consistent with our proof for the main theorem.

Appendix A. Proof of Proposition 3

If the section we prove the sufficiency of the conditions in Proposition 3 for (H3), which says that there is a trajectory for (17) connecting $(u_L, w_L, s)$ and $\{β = ρ_1, v = +∞\}$, and a trajectory connecting $(u_R, w_R, s)$ and $\{β = ρ_2, v = +∞\}$. We will focus on finding the first trajectory while finding the second one is similar.

We will switch back and forth between $(β, v)$- and $(β, r)$-coordinates, where $r = 1/v$. The system (17) is converted to (68) in $(β, r)$-coordinates.
It suffices to find trajectories connecting $u_L = (\beta_L, r_L)$ and $p_L \equiv (\rho_1, 0)$ for $(68)$ with $(w, \xi) = (w_L, s)$. From (H1) we know that $u_L$ is a source, and we will also see that $p_L$ is a saddle. Our strategy is to construct a negatively invariant region in which every trajectory goes backward to $u_L$, and one of those trajectories goes forward to $p_L$. See Fig A.9.

To construct such a region, we first study the flow on the boundary of the feasible region \( \{ \rho_2 \leq \beta \leq \rho_1 \} \). The equation of $\dot{\beta}$ in $(17)$ with $(w, \xi) = (w_L, s)$ is

$$
\dot{\beta} = -s\rho_1 - w_{1L}
$$
on \{ \beta = \rho_1 \}.

Hence the proposition below implies that the region \( \{ \beta \leq \rho_1 \} \) is negatively invariant. Similarly, for $(17)$ with $(w, \xi) = (w_R, s)$, the region \( \{ \beta \geq \rho_2 \} \) is positively invariant.

**Lemma 19.** If (H1) holds, then

$$
sp_1 + w_{1L} < 0 \quad \text{and} \quad sp_2 + w_{1R} < 0,
$$

(A.1)

where $s$, $w_{1L}$ and $w_{1R}$ are as defined in (19) and (20).

**Proof.** By definition of $s$ and $w_{1L}$ we have

$$
sp_1 + w_{1L} = sp_1 + v_L B_1(\beta_L) - s\beta_L = \frac{v_L B_1(\beta_L) - v_R B_1(\beta_R)}{\beta_L - \beta_R}(\rho_1 - \beta_L) + v_L B_1(\beta_L).
$$

Fig 8: Trajectories for $\epsilon u'' = (Df(u) - s)u'$ as $\epsilon$ decreases from 1 to 0.01. The variable $x_2$ is a modification of $w_2$. 
Fig A.9: Phase portraits for (A.2) in $(\beta, v)$ space and (A.8) in $(\beta, r)$ space. The shaded region $V$ is a backward invariant region in which every backward trajectory tends to $u_L$, and $\gamma_1$ is the unique trajectory in $V$ which tends to $p_L = (\rho_1, 0)$.

From Proposition 1, we know (H1) implies $\beta_R < \beta_L$ and

$$v_R \leq v_L \left( \frac{B_1(\beta_L) - 2B_2(\beta_L)(\beta_L - \beta_R)}{B_1(\beta_R)} \right).$$

Since $B_1(\beta_L) < 0$, it follows that

$$s\rho_1 + w_{1L} \leq \frac{\rho_1 - \beta_L}{\beta_L - \beta_R} v_L \left( 2B_2(\beta_L)(\beta_L - \beta_R) \right) + v_L B_1(\beta_L)$$

$$= \left( \frac{(\rho_1 - \beta_L)(\beta_L^2 - \rho_1 \rho_2)}{\beta_L^2} \right) v_L$$

$$= \frac{\rho_1 - \beta_L}{\beta_L^2} v_L < 0.$$ 

Similarly, using $s\rho_2 + w_{1R} = s\rho_2 + v_R B_1(\beta_R) - s\beta_R$, one obtains $s\rho_2 + w_{1R} < 0$. □

Proposition 3. Suppose (H1) holds. If $\beta_R < \sqrt{\rho_1 \rho_2} < \beta_L$, $w_{1L} < 0$, $w_{2R} < 0$, $w_{2L}$, and $|s|$ is sufficiently small, then (H3) holds.

Proof. We focus on $u_L$ while the proof for $u_R$ is similar. As mentioned at the beginning of this section and Fig A.9, we first construct a negatively invariant region in which every trajectory goes backward to $u_L$, and then we will show
that the region contains an unbounded trajectory which approaches $p_L$ in forward time.

Consider (17) with $(w, \xi) = (w_L, s)$. That is,

$$\dot{\beta} = vB_1(\beta) - s\beta - w_{1L}$$
$$\dot{v} = v^2B_2(\beta) - sv - w_{2L}. \tag{A.2}$$

The null-clines for this system are

$$\dot{\beta} = 0 : v = \vartheta_1(\beta) := \frac{\beta(s\beta + w_{1L})}{(\beta - \rho_1)(\beta - \rho_2)} \tag{A.3}$$
$$\dot{v} = 0 : v = \vartheta_2(\beta) := \frac{s\beta + s\beta^2 + 2w_{2L}(\beta^2 - \rho_1\rho_2)}{\beta^2 - \rho_1\rho_2} \beta. \tag{A.4}$$

First we consider the case $s = 0$. Then the equations of $\vartheta_1$ and $\vartheta_2$ reduce to

$$\vartheta_1(\beta) = \frac{\beta w_{1L}}{(\beta - \rho_1)(\beta - \rho_2)}, \quad \vartheta_2(\beta) = \frac{\sqrt{2w_{2L}}}{\sqrt{\beta^2 - \rho_1\rho_2}} \beta, \tag{A.5}$$

which yield

$$\frac{d\vartheta_1}{d\beta} = \frac{-w_{1L}(\beta^2 - \rho_1\rho_2)}{(\rho_1 - \beta)^2(\beta - \rho_2)^2}, \quad \frac{d\vartheta_2}{d\beta} = \frac{-\rho_2\rho_2\sqrt{2w_{2L}}}{(\beta^2 - \rho_1\rho_2)^{3/2}}. \tag{A.6}$$

From the assumption $w_{1L} < 0$, it follows that $\vartheta_1$ is strictly increasing and $\vartheta_2$ is strictly decreasing on the interval $(\sqrt{\rho_1\rho_2}, \rho_1)$. See Fig. A.9.

Let $\sigma(\tau)$ be the solution to (A.2) with the initial condition $\sigma(0) = (\rho_1, v_L)$. By the monotonicity of $\vartheta_1$ and $\vartheta_2$ and the assumption $\beta_L \in (\sqrt{\rho_1\rho_2}, \rho_1)$, the backward trajectory of $\sigma$ hits the half-line $l_1 = \{(\beta, v) : v \geq v_L\}$ at some time $\tau_0 < 0$. Let $l_2 = \{(\rho_1, v) : v \geq v_L\}$ and $V$ be the region enclosed by the curves

$$l_1 \cup \{\sigma(\tau) : \tau_0 \leq \tau \leq 0\} \cup l_2. \tag{A.7}$$

Then $V$ forms a backward invariant region.

For the case $s \neq 0$, since strictly monotonicity on compact sets persist under perturbations, with $|s|$ sufficiently small we know $\vartheta_1$ is strictly increasing and $\vartheta_2$ is strictly decreasing on the interval $[\sqrt{\rho_1\rho_2} + \delta, \rho_1]$ for any fixed $\delta > 0$. We choose $\delta > 0$ small enough so that the backward trajectory $\sigma$ defined above lies in the region $(\sqrt{\rho_1\rho_2} + \delta, \rho_1) \times \mathbb{R}$. Therefore, when $|s|$ is
small, the backward trajectory passing through \((\rho_1, v_L)\), also denoted by \(\sigma\), hits the line \(l_1\) at some negative time. Hence the region \(V\) enclosed by the curves \((A.7)\) is still backward invariant under \((A.2)\).

We claim that \(u_L\) attracts every point in \(V\) in backward time. Note that
\[
\frac{\partial}{\partial \beta}(vB_1(\beta) - s\beta - w_{1L}) + \frac{\partial}{\partial v}(v^2B_2(\beta) - sv - w_{2L}) = vB_1'(\beta) - s + 2vB_2(\beta) - s = 4vB_2(\beta) - 2s,
\]
which is positive when \(\beta \in (\beta_L, \rho_1)\) and \(s\) is small. In the last equality we used \(B_1'(\beta) = 2B_2(\beta)\). By Bendixson’s negative criterion, the system has no periodic orbit inside \(V\). Since \(V\) is backward invariant and \(u_L\) is the only equilibrium on the closure of \(V\), it follows from the Poincaré-Bendixson Theorem that every trajectory in \(V\) tends to \(u_L\) in backward time.

It remains to show that there is a trajectory in \(V\) tending to \(\{\beta = \rho_1, v = \infty\}\). Let \(r = 1/v\). Then \((A.2)\) is converted to, after multiplying by \(r\),
\[
\dot{\beta} = B_1(\beta) - s\beta r - w_{1L}r, \\
\dot{r} = -rB_2(\beta) + sr^2 + w_{2L}r^3.
\]
(A.8)

At the equilibrium \(p_L = (\rho_1, 0)\), the eigenvalues of the linearized system are \(\lambda_+ = 1 - \frac{\rho_2}{\rho_1}\) and \(\lambda_- = \frac{1}{2}(1 - \frac{\rho_2}{\rho_1})\), and the corresponding eigenvectors are \(y_+ = (1, 0)\) and \(y_- = (\frac{2\rho_1(w_{1L} + sp_1)}{3\rho_1 - \rho_2}, 1)\), so \(p_L\) is a hyperbolic saddle, and hence there exists a trajectory, denoted by \(\gamma_1\), which tends to \(p_L\). The trajectory of \(\gamma_1\) is tangent to the line \(\{p_L + ty_+ : t \in \mathbb{R}\}\) at \(p_L\). Since \(s\rho_1 + w_{1L} < 0\) by Lemma \(19\), we know \(p_L + ty_1, t \geq 0\), lies in the region \(\{\beta < \rho_1, r \geq 0\}\). Therefore, converting \((A.8)\) back to \((A.2)\), the solution converted from \(\gamma_1(\tau)\), also denoted by \(\gamma_1(\tau)\), lies in \(V\). Now we conclude that \(\gamma_1(\tau)\) approaches \(\{\beta = \rho_1, v = \infty\}\) in forward time and approaches \(u_L\) in backward time.

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