A $p$-LAPLACIAN SUPERCritical NEUMANN PROBLEM

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Abstract. For $p > 2$, we consider the quasilinear equation $-\Delta_p u + |u|^{p-2} u = g(u)$ in the unit ball $B$ of $\mathbb{R}^N$, with homogeneous Neumann boundary conditions. The assumptions on $g$ are very mild and allow the nonlinearity to be possibly supercritical in the sense of Sobolev embeddings. We prove the existence of a nonconstant, positive, radially nondecreasing solution via variational methods. In the case $g(u) = |u|^{q-2} u$, we detect the asymptotic behavior of these solutions as $q \to \infty$.

1. Introduction. For $p > 2$, we consider the following Neumann problem

$$
\begin{cases}
-\Delta_p u + u^{p-1} = g(u) & \text{in } B, \\
u > 0 & \text{in } B, \\
\partial_\nu u = 0 & \text{on } \partial B.
\end{cases}
$$

Here $B$ is the unit ball of $\mathbb{R}^N$, $N \geq 1$, and $\nu$ is the outer unit normal of $\partial B$. We aim to investigate the existence of nonconstant solutions of (1) under very mild assumptions on the nonlinearity $g$, allowing in particular for Sobolev-supercritical growth.

Quasilinear equations with Neumann boundary conditions and subcritical nonlinearities in the sense of Sobolev embeddings have been studied in several papers, among which we refer to [1, 2, 6, 14, 18, 19, 25, 27, 31] and the references therein.

When $g$ has supercritical growth, a major difficulty in analyzing the existence of solutions of (1) is that, due to the absence of Sobolev embeddings, the energy functional associated to the equation is not well defined in $W^{1,p}(B)$, and so, a priori, it is not possible to apply variational methods. Nonetheless, the problem (1) with the prototype nonlinearity $g(u) = u^{q-1}$ admits the constant solution $u \equiv 1$ for every $q \in (1, \infty)$. This marks a difference with respect to the analogous problem under homogeneous Dirichlet boundary conditions, in which the Pohožaev identity is an insurmountable obstruction to the existence of non-zero solutions when $q \geq p^*$ (see [26, Section 2, pp. 685-686]). Thus it is a natural question to ask whether (1) also admits nonconstant solutions.

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Theorem 1.1. Let $g$ belong to $L^1(0,1)$. Multiplicity results have been obtained in [11, 10]. The strategy used in [29, 28] to obtain existence is that of establishing a priori estimates in some special classes of solutions of (1). This, in turn, allows to provide a variational characterization of the problem in the Sobolev space. On the other hand, in [9, 21], existence is proved by a perturbative method. In [11, 10] the authors apply both a priori estimates and perturbative methods to have multiplicity results. Topological methods have been used in [13] for a related problem.

The case $p \neq 2$ with a supercritical nonlinearity has been treated by S. Secchi in [28], where the right-hand side of the equation in (1) is of the type $a(x)g(u)$, with $a(x)$ nonconstant. Our paper aims to extend the results in [28] to the case $a$ constant and $p > 2$. We remark that our method differs from the one in [28]: whereas S. Secchi adapts to the case $p \neq 2$ the techniques introduced in [29], we take inspiration from the techniques developed in [8].

We also mention that the existence and multiplicity of solutions to supercritical $p$-Laplacian problems under homogeneous Dirichlet boundary conditions have been studied in several papers, see for instance [3, 15, 22] and the references therein.

In order to state our main result, let us introduce our assumptions on $g$. We assume that $g : [0,\infty) \rightarrow \mathbb{R}$ is of class $C^1([0,\infty))$ and satisfies the following hypotheses
\begin{align*}
&\text{(g1)} \lim_{s \to 0^+} \frac{g(s)}{s^p} \in [0,1); \\
&\text{(g2)} \liminf_{s \to \infty} \frac{g(s)}{s^p} > 1; \\
&\text{(g3)} \text{there exists a constant } u_0 > 0 \text{ such that } g(u_0) = u_0^{p-1} \text{ and } g'(u_0) > (p-1)u_0^{p-2}. 
\end{align*}

We remark that by the regularity of $g$ and by (g1) and (g2) we immediately have the existence of an intersection point $u_0 > 0$ between $g$ and the power function $s^{p-1}$, with $g'(u_0) \geq \frac{(p-1)v_0^{p-2}}{v_0^{p-1}}$. Hence, condition (g3) is only needed to prevent the situation in which all intersection points $u_0$ such that $g(s) < s^{p-1}$ for $s \in [u_0 - \epsilon, u_0]$ and $g(s) > s^{p-1}$ for $s \in (u_0, u_0 + \epsilon]$ are double zeros of the function $g(s) - s^{p-1}$, namely they verify also $g'(u_0) = (p-1)u_0^{p-2}$.

**Theorem 1.1.** Let $p > 2$ and $g \in C^1([0,\infty))$ satisfy assumptions (g1)-(g3). Then there exists a positive, nonconstant, radial, nondecreasing solution of (1).

In addition, if $u_{0,1}, \ldots, u_{0,n}$ are $n$ different positive constants satisfying (g3), then (1) admits $n$ different positive, nonconstant, radial, nondecreasing solutions.

Our starting point to prove Theorem 1.1 is to work in the cone of nonnegative, radial, nondecreasing functions
\begin{equation}
\mathcal{C} := \{u \in W^{1,p}_{\text{rad}}(B) : u \geq 0, u(r) \leq u(s) \text{ for all } 0 < r < s \leq 1\},
\end{equation}
introduced by Serra and Tilli in [29], where with abuse of notation we write $u(|x|) := u(x)$. The main advantage of working in this set is the fact that all solutions of (1) belonging to $\mathcal{C}$ are a priori bounded in $W^{1,p}(B)$ and in $L^\infty(B)$. Two strategies are available in literature. The first one, see [29, 28], consists in defining the energy functional $I : \mathcal{C} \rightarrow \mathbb{R}$ associated to the equation and to find a critical point $u$ of $I$, that is to say
\[ I'(u)[\varphi] = 0 \quad \text{for all } \varphi \in \mathcal{C}. \]
This does not imply that $u$ is a weak solution of the problem. Under additional hypotheses on the nonlinearity $g$, the authors prove that it actually is.
In order to weaken the assumptions on the nonlinearity \( g \), we follow a different strategy, see [5]. Thanks to the a priori estimates on the solutions of \([\text{I}]\) belonging to \( \mathcal{C} \), we are allowed to truncate the nonlinearity \( g \). Thus, we deal with a new problem involving a Sobolev-subcritical nonlinearity, with the property that all solutions of the new problem belonging to \( \mathcal{C} \) solve also the original problem \([\text{I}]\). In this way, the energy functional \( I \) associated to the truncated problem is well defined in the whole of \( W^{1,p}(B) \). To get a solution of \([\text{I}]\), we prove that a mountain pass type theorem holds inside the cone \( \mathcal{C} \). The main difficulty here is the construction of a descending flow that preserves \( \mathcal{C} \).

Once the mountain pass solution is found, we need to prove that it is nonconstant. We further restrict our cone, working in a subset of \( \mathcal{C} \) in which the only constant solution of \([\text{I}]\) is the positive constant \( u_0 \) defined in \((g_3)\). In this set, we build an admissible curve on which the energy is lower than the energy of the constant \( u_0 \), which gives immediately that the mountain pass solution is not identically equal to \( u_0 \). We remark that this part of the proof heavily relies on the fact that \( I \) is of class \( C^2 \), thus it cannot be generalized to the case \( 1 < p < 2 \).

In the case in which there is more than one constant \( u_0 \) satisfying condition \((g_3)\), we work in a restricted cone in order to localize the mountain pass solution. This allows us to prove the multiplicity result stated in Theorem 1.1.

We remark that in the setting \( p = 2 \) our hypotheses are slightly more general than the ones in [8]. More precisely, in [8] it is required, for \( p = 2 \),

\[
g(s) \text{ nondecreasing, } \lim_{s \to 0^+} \frac{g(s)}{s} = 0.
\]

Hence our proof also provides the following generalization of [8, Theorem 1.3].

**Theorem 1.2.** Let \( p = 2 \). Let \( g \in C^1([0, \infty)) \) satisfy \((g_1)\), \((g_2)\) and

\((g_3')\) there exists a constant \( u_0 > 0 \) such that \( g(u_0) = u_0 \) and \( g'(u_0) > \lambda_2^{\text{rad}} \),

where \( \lambda_2^{\text{rad}} \) is the second radial eigenvalue of \(-\Delta + 1\) in \( B \) with Neumann boundary conditions. Then there exists an increasing radial solution of \([\text{I}]\).

In addition, if \( u_{0,1}, \ldots, u_{0,n} \) are \( n \) different positive constants satisfying \((g_3')\), then \([\text{I}]\) admits \( n \) different increasing radial solutions.

For \( p = 2 \) and \( g(u) = u^{q-1} \), the result in [11] provides multiple solutions which oscillate around the constant solution \( u = 0 = u_0 \). Similar oscillating solutions can be found via a bifurcation technique, as in [12]. There is a branch bifurcating in correspondence to any power \( q - 1 = \lambda_i^{\text{rad}} \) with \( i \geq 2 \), where \( \lambda_i^{\text{rad}} \) is the \( i \)-th eigenvalue of \(-\Delta + 1\) under homogeneous Neumann boundary conditions in the unit ball. We note in passing that this relation between \( q \) and \( \lambda_i^{\text{rad}} \) seems to be in the same spirit as condition \((g_3')\). It would be interesting to understand whether the bifurcation occurs also when \( p > 2 \).

Theorem 1.1 ensures in particular the existence of a nonconstant, nondecreasing, radial solution of \([\text{I}]\) in the case \( g(u) = u^{q-1} \), for every \( q > p \). Denoting by \( u_q \) such solution, we detect its asymptotic behavior as \( q \to \infty \), in the spirit of [20] (see also [21], [11], and [10]).

**Theorem 1.3.** Let \( p > 2 \) and \( g(u) = u^{q-1} \), with \( q > p \). Denote by \( u_q \) the corresponding positive, nonconstant, radially nondecreasing solution found in Theorem 1.1.

Then, as \( q \to \infty \),

\[
u_q \to G \text{ in } W^{1,p}(B) \cap C^0(\overline{B})
\]
for any $\nu \in (0, 1)$, where $G$ is the unique positive solution of

\[
\begin{cases}
-\Delta_p G + |G|^{p-2}G = 0 & \text{in } B, \\
G = 1 & \text{on } \partial B,
\end{cases}
\]  

(4)

see Lemma 5.7 below for details.

In the proof of this theorem, we use the fact that the solutions $u_q$ are nondecreasing and that the nonlinearity $g$ is a pure power to get an estimate on the $C^1$-norm of $u_q$, which is uniform in $q$. This ensures the existence of a limit profile function $G$ which is nonnegative and radially nondecreasing. We note that it is delicate to prove that $G$ solves the equation in (4) near the boundary $\partial B$. Heuristically, this comes from the fact that $u_q(1) > 1$ for all $q$, and so $\lim_{q \to \infty} u_q(1)^{q-1}$ may be an indeterminate form. In order to prove that $G$ solves actually (4) in the whole ball $B$, we show that the mountain pass levels $c_q$’s tend to a value $c_{\infty}$ which is a critical level for the energy associated to (4). This latter result requires in turn the preliminary proof of the fact that any mountain pass level $c_q$ coincides with the minimum of the energy functional on a Nehari-type set already introduced in [29] (that is to say, the Nehari manifold intersected with the cone $C$). We remark here that the Neumann boundary condition is not preserved in the limit, being $\partial_p G > 0$ on $\partial B$, by Hopf’s Lemma (see for instance [17, Theorem 3.3]). Hence, the convergence $C^{0,\alpha}(B)$ in (3) is optimal.

The paper is organized as follows. In Section 2, we prove a priori estimates for nonnegative, radially nondecreasing solutions of (1). In Section 3 we show the existence of a nonnegative, radially nondecreasing solution of (1) via a mountain pass type argument. Furthermore, in Section 4 we conclude the proof of Theorem 1.1 by proving the nonconstancy of the solution found in Section 3 and the multiplicity result. A sketch of the proof of Theorem 1.2 is also given in the same section. The asymptotic behavior as $q \to \infty$ of the mountain pass solution of (1) in the pure power case is then studied in Section 5. Finally in Appendix A we collect some partial results valid in the case $1 < p < 2$.

2. A priori bounds for nondecreasing radial solutions.

Lemma 2.1. For every $g \in C^1([0, \infty))$ satisfying $(g_1)$–$(g_2)$ there exist $f \in C^1([0, \infty))$ nonnegative and nondecreasing, and a constant $m \geq 1$ for which the following properties hold

$(f_1)$ $\lim_{s \to 0^+} \frac{f(s)}{s^p} \in [m - 1, m]$;

$(f_2)$ $\liminf_{s \to \infty} \frac{f(s)}{s^p} > m$.

Furthermore, if $g$ verifies also $(g_3)$, $f$ verifies

$(f_3)$ $\exists$ a constant $u_0 > 0$ such that $f(u_0) = mu_0^{p-2}$ and $f'(u_0) > m(p-1)u_0^{p-2}$.

Proof. Since $g \in C^1([0, \infty))$ satisfies $(g_1)$ and $(g_2)$, there exists $C \geq 0$ such that $g'(s) \geq -C(p-1)s^{p-2}$ for all $s \in [0, \infty)$.

Hence, if we define $f : [0, \infty) \to \mathbb{R}$ by

\[
f(s) := g(s) + C s^{p-1},
\]

$f \in C^1([0, \infty))$, $f(0) = 0$, $f' \geq 0$, and so $f \geq 0$. Furthermore, by $(g_1)$, $f$ satisfies

\[
\lim_{s \to 0^+} \frac{f(s)}{s^p} \in [C, 1 + C].
\]
Properties \((f_1)-(f_3)\) then follow immediately by \((g_1)-(g_3)\), with \(m := 1 + C\).

As a consequence of the previous lemma, from now on in the paper we consider the equivalent problem

\[
\begin{align*}
-\Delta_p u + mu^{p-1} &= f(u) & \text{in } B, \\
\partial_r u &= 0 & \text{on } \partial B,
\end{align*}
\]

where \(f \in C^1([0, \infty))\) is nonnegative, nondecreasing, and satisfies \((f_1)-(f_3)\). We endow the space \(W^{1,p}(B)\) with the equivalent norm \(\| \cdot \| : W^{1,p}(B) \to \mathbb{R}^+ \) defined by

\[
\|u\| := \left(\|\nabla u\|^p_{L_p(B)} + m\|u\|^p_{L_p(B)}\right)^{1/p}.
\]

We look for solutions to \((5)\) in \(W^{1,p}_{rad}(B)\), that is to say the space of radial functions in \(W^{1,p}(B)\). Since \(p > 1\), we can assume that \(W^{1,p}_{rad}(B)\)-functions are continuous in \((0, 1]\) and define the cone of nonnegative radially nondecreasing functions as in \([2]\). We note that, if \(u \in C\), we can set \(u(0) := \lim_{r \to 0^+} u(r)\) by monotonicity, and consider \(u \in C(B)\). Moreover, being nondecreasing, every \(u \in C\) is differentiable a.e. and \(u'(r) \geq 0\) where it is defined. It is easy to prove that \(C\) is a closed convex cone in \(W^{1,p}(B)\), that is to say, the following properties hold for all \(u, v \in C\) and \(\lambda \geq 0\):

(i) \(\lambda u \in C\);
(ii) \(u + v \in C\);
(iii) if also \(-u \in C\), then \(u \equiv 0\);
(iv) \(C\) is closed for the topology of \(W^{1,p}\).

The cone \(C\) was first introduced in \([29]\) in the case \(p = 2\). It is a useful set when working with Sobolev-supercritical problems because of the following a priori estimates.

**Lemma 2.2.** For every \(1 \leq q < \infty\) there exists \(C(N, q)\) such that

\[
\|u\|_{L^\infty(B)} \leq C(N, q)\|u\|_{W^{1,q}(B)} \quad \text{for all } u \in C.
\]

**Proof.** Since \(u \in C\) is nonnegative and nondecreasing, we get

\[
\|u\|_{L^\infty(B)} = \|u\|_{L^\infty(B \setminus B_{1/2})},
\]
and by the radial symmetry of \(u \in C\),

\[
\|u\|_{L^\infty(B \setminus B_{1/2})} \leq C\|u\|_{W^{1,1}(B \setminus B_{1/2})} \leq C\|u\|_{W^{1,1}(B)}
\]

for some \(C > 0\) depending only on the dimension \(N\). Moreover, being \(B\) bounded, for every \(q \in [1, \infty)\) there exists a constant \(C > 0\) depending only on \(N\) and \(q\), such that

\[
\|u\|_{W^{1,1}(B)} \leq C\|u\|_{W^{1,q}(B)} \quad \text{for all } u \in W^{1,1}(B).
\]

By combining \([4], [7]\) and \([8]\), for every \(q \in [1, \infty)\) we can find a constant \(C(N, q) > 0\) for which the statement holds.

**Lemma 2.3.** For all \(q \in [1, \infty)\), the cone \(C\) endowed with the \(W^{1,p}\)-norm is compactly embedded in \(L^q(B)\).
Proof. If \( N < p \) the conclusion follows at once by the Rellich-Kondrachov theorem.
In the complementary case, we take into account the fact that \( C \)-functions are bounded. More precisely, if we have \( (u_n) \subset C \) bounded in the \( W^{1,p}_\text{loc} \)-norm, there exists \( u \in C \) such that up to a subsequence \( u_n \rightharpoonup u \) in \( W^{1,p}(B) \) and so \( u_n \to u \) in \( L^1(B) \). Therefore, by Lemma 2.2 we get that for every \( q < \infty \),
\[
\int_B |u_n - u|^q dx \leq \|u_n - u\|_{L^\infty(B)}^{q-1}\|u_n - u\|_{L^1(B)}
\leq C(N, p)^{q-1}\|u_n - u\|_{W^{1,p}(B)}^{q-1}\|u_n - u\|_{L^1(B)} \to 0,
\]
that is \( u_n \to u \) in \( L^q(B) \).

Fix \( \delta, M > 0 \) such that
\[
f(s) \geq (m + \delta)s^{p-1} \quad \text{for all } s \geq M. \tag{9}\]
The existence of \( \delta, M > 0 \) follows by \((f_2)\) in Lemma 2.1. We introduce the following set of functions
\[
\mathfrak{G} := \{ \varphi \in C([0, \infty)) : \varphi \text{ nonnegative, } \varphi(s) \geq (m + \delta)s^{p-1} \text{ for all } s \geq M \}. \tag{10}\]
We remark that \( \mathfrak{G} \) depends on \( f \) only through \( \delta \) and \( M \). In the remaining of this section, we shall derive some a priori estimates which are uniform in \( \mathfrak{G} \) and hence depend only on \( \delta \) and \( M \) and not on the specific nonlinearity \( f \) belonging to \( \mathfrak{G} \).

**Lemma 2.4.** There exists a constant \( K_{p-1} > 0 \) such that
\[
\|u\|_{L^{p-1}(B)} \leq K_{p-1}
\]
for every solution \( u \) of
\[
\begin{cases}
-\Delta_p u + mu^{p-1} = \varphi(u) & \text{in } B, \\
u > 0 & \text{in } B, \\
\partial_\nu u = 0 & \text{on } \partial B,
\end{cases} \tag{11}
\]
and for every \( \varphi \in \mathfrak{G} \).

**Proof.** By integrating the equation in \( (11) \) and using the fact that \( \varphi \in \mathfrak{G} \), we have
\[
m \int_B u^{p-1} dx = \int_{\{u < M\}} \varphi(u) dx + \int_{\{u \geq M\}} \varphi(u) dx \geq (m + \delta) \int_{\{u \geq M\}} u^{p-1} dx.
\]
Thus,
\[
m M^{p-1} |B| > m \int_{\{u < M\}} u^{p-1} dx \geq \delta \int_{\{u \geq M\}} u^{p-1} dx,
\]
where \( |B| \) is the volume of the unitary ball of \( \mathbb{R}^N \), and so
\[
\int_B u^{p-1} dx = \int_{\{u < M\}} u^{p-1} dx + \int_{\{u \geq M\}} u^{p-1} dx < \left(1 + \frac{m}{\delta}\right) M^{p-1} |B| =: K_{p-1}^{-1}, \tag{12}
\]
which yields the estimate.

**Lemma 2.5.** There exists a constant \( K_\infty > 0 \) such that
\[
\|u\|_{L^\infty(B)} \leq K_\infty \quad \text{and} \quad \|u\| \leq \left(K_\infty |B| \max_{s \in [0, K_\infty]} \varphi(s)\right)^{1/p}
\]
for every solution \( u \in C \) of \( (11) \) and every \( \varphi \in \mathfrak{G} \).
Proof. Let \( u \in C \) be a solution of (11). We recall that the \( p \)-Laplacian of a radial function is given by
\[
\Delta_p u = \frac{1}{r^{N-1}} \left( r^{N-1} |u'(r)|^{p-2} u'(r) \right)' = |u'(r)|^{p-2} \left( (p-1)u''(r) + \frac{1}{r} (N-1) u'(r) \right).
\]
Hence, since \( u' \geq 0 \) a.e., we can write
\[
\begin{cases}
  \left( r^{N-1} u'(r)^{p-1} \right)' = r^{N-1} (mu^{p-1} - \varphi(u)) & \text{in } (0, 1), \\
  u'(0) = u'(1) = 0.
\end{cases}
\]
Then, by integrating the equation over the interval \((0, r)\) and using the fact that \( \varphi \) is nonnegative, we have
\[
r^{N-1} u'(r)^{p-1} = \int_0^r (mu(t)^{p-1} - \varphi(u(t))) t^{N-1} dt \
\leq m \int_0^r u(t)^{p-1} t^{N-1} dt = m \frac{|\partial B|}{|B|} \int_B u(x)^{p-1} dx,
\]
where \( |\partial B| \) is the \((N-1)\)-dimensional measure of the unitary sphere in \( \mathbb{R}^N \). Together with (12), this gives
\[
\|u\|_{W^{1,p-1}(B)} \leq (1 + m)^{1/(p-1)} K_{p-1}.
\]
The first estimate then follows by Lemma 2.2 (by taking \( q = p - 1 \)), with \( K_{\infty} := (1 + m)^{1/(p-1)} K_{p-1} C(N, p-1) \). Finally, for the last estimate, we multiply the equation of (11) by \( u \), we integrate over \( B \), and we obtain
\[
\|u\|_p = \int_B \varphi(u) u dx \leq K_{\infty} |B| \max_{s \in [0, K_{\infty}]} \varphi(s),
\]
which concludes the proof.

3. Existence of a mountain pass radial solution. In this section we prove the existence of a radial solution of (5) via the Mountain Pass Theorem. Since the nonlinearity \( f \) is possibly supercritical in the sense of Sobolev spaces, we need to truncate and to replace it by a subcritical function which coincides with \( f \) in \([0, K_{\infty}]\), \( K_{\infty} \) being defined in Lemma 2.5. Then, we take advantage of the a priori estimates proved in the previous section to guarantee that the mountain pass solution found with the truncated function is indeed a solution of the original problem (5).

We define the critical Sobolev exponent
\[
p^* := \begin{cases} 
  \frac{Np}{N-p} & \text{if } p < N, \\
  +\infty & \text{otherwise}.
\end{cases}
\]

Lemma 3.1. For every \( \ell \in (p, p^*) \), there exists \( \tilde{f} \in \mathcal{F} \cap C^1([0, \infty)) \) nondecreasing, satisfying (f1)-(f3),
\[
\lim_{s \to \infty} \frac{\tilde{f}(s)}{s^{\ell-1}} = 1, \tag{13}
\]
and with the property that if \( u \in C \) solves
\[
\begin{cases}
  -\Delta_p u + mu^{p-1} = \tilde{f}(u) & \text{in } B, \\
  u > 0 & \text{in } B, \\
  \partial_n u = 0 & \text{on } \partial B,
\end{cases} \tag{14}
\]
then \( u \) solves (5).
Proof. Let \( \delta > 0 \) and \( M > 0 \) be the constants defined in (9), and fix \( s_0 > \max\{K_\infty, M\} \), with \( K_\infty \) given in Lemma 2.5. By (9), two possible cases arise.

**Case** \( f(s_0) = (m + \delta)s_0^{\ell-1} \). By (9), \( f \) is tangent at \( s_0 \) to the curve \((m + \delta)s^{\ell-1}\), hence \( f'(s_0) = (m + \delta)(p-1)s_0^{p-2} \) and we can define the function \( \tilde{f} : [0, \infty) \to [0, \infty) \) as
\[
\tilde{f}(s) := \begin{cases} 
  f(s) & \text{if } s \in [0, s_0], \\
  f(s_0) + (m + \delta)(s^{p-1} - s_0^{p-1}) + (s - s_0)^{\ell-1} & \text{otherwise}.
\end{cases}
\]

**Case** \( f(s_0) > (m + \delta)s_0^{\ell-1} \). First, we modify \( f \) in a right neighborhood of \( s_0 \), that is to say, we consider a \( C^1 \) nondecreasing function \( f_{\mod} : [0, s_0 + \varepsilon] \to [0, \infty) \) in such a way that \( f_{\mod}(s) = f(s) \) in \([0, s_0]\), \( f_{\mod}(s) \geq (m + \delta)s^{p-1} \) in \([s_0, s_0 + \varepsilon]\), and \( f'_{\mod}(s_0 + \varepsilon) = (m + \delta)(p-1)(s_0 + \varepsilon)^{p-2} \). Then, we define \( \tilde{f}(s) \) as in the previous case, with \( f \) replaced by \( f_{\mod} \) and \( s_0 \) by \( s_0 + \varepsilon \).

In both cases, it is easy to check that \( \tilde{f} \) is of class \( C^1([0, \infty)) \), nonnegative, nondecreasing, satisfies (f1), (f2), and (13). Since the constant \( u_0 \) given in (f3) is a solution of (5) in \( C \), we know by Lemma 2.5 that \( u_0 \leq K_\infty < s_0 \). Hence \( \tilde{f} \) verifies also (f3).

Finally, let \( u \in C \) solve (14), we want to show that \( u \) solves (5). To this aim, we notice that \( \tilde{f} \) belongs to \( \mathcal{S} \) by construction. By Lemma 2.5 \( \|u\|_{L^\infty(B)} < K_\infty \). Being \( s_0 > K_\infty \), we have \( \tilde{f}(u) = f(u) \), hence \( u \) solves (5). \( \square \)

As a consequence of the proof of the previous lemma, there exists \( C > 0 \) for which
\[
\tilde{f}(s) \leq C(1 + s^{\ell-1}) \quad \text{for all } s \geq 0.
\]

From now on in the paper, we set \( \tilde{f} = 0 \) in \((-\infty, 0)\). We define the energy functional \( I : W^{1,p}(B) \to \mathbb{R} \) associated to the problem (14) by
\[
I(u) := \int_B \left( \frac{\nabla u|^p + m|u|^p}{p} - \hat{F}(u) \right) \, dx,
\]
(16)
where \( \hat{F}(u) := \int_0^u \tilde{f}(s) \, ds \). Because of (13) and the Sobolev embedding, the functional \( I \) is well defined and of class \( C^2 \), being \( p > 2 \).

We define the operator \( T : (W^{1,p}(B))^p \to W^{1,p}(B) \) as
\[
T(w) = v, \quad \text{where } v \text{ solves } \begin{cases} 
 -\Delta_p v + m|v|^{p-2} v = w & \text{in } B, \\
 \partial_{\nu} v = 0 & \text{on } \partial B.
\end{cases}
\]
(17)
We observe that the definition is well posed because, for all \( w \in (W^{1,p}(B))^p \), the problem \((P_w)\) admits a unique weak solution \( v \in W^{1,p}(B) \). To prove the existence one can apply the direct method of Calculus of Variations, while uniqueness is a consequence of the strict convexity of the map \( u \mapsto \|u\|^p \). Furthermore, by [7, Lemma 2.1] we know that
\[
T \in C((W^{1,p}(B))^p; W^{1,p}(B)).
\]
(18)
We introduce also the operator
\[
\hat{T} : W^{1,p}(B) \to W^{1,p}(B) \quad \text{defined by } \quad \hat{T}(u) = T(\tilde{f}(u)),
\]
(19)
with \( T \) given in (17). Being \( \ell < p^* \), \( u \in W^{1,p}(B) \) implies \( u \in L^{1'}(B) \). Hence, by (15), \( \tilde{f}(u) \in L^{1'}(B) \subset (W^{1,p}(B))^p \), where \( \ell' \) is the conjugate exponent of \( \ell \), and \( \hat{T} \) is well defined.
**Proposition 3.2.** The operator $\tilde{T}$ is compact, i.e. it maps bounded subsets of $W^{1,p}(B)$ into precompact subsets of $W^{1,p}(B)$. Furthermore, there exist two positive constants $a, b$ such that for all $u \in W^{1,p}(B)$ the following properties hold

$$I'(u)[u - \tilde{T}(u)] \geq a\|u - \tilde{T}(u)\|^p,$$

$$\|I'(u)\|_* \leq b\|u - \tilde{T}(u)\|\|u\| + \|\tilde{T}(u)\|_{p^{-2}},$$

where $\|\cdot\|_*$ denotes the norm of the dual space of $W^{1,p}(B)$.

**Proof.** Let $(u_n)$ be a bounded sequence in the reflexive Banach space $W^{1,p}(B)$. Up to a subsequence $u_n \rightharpoonup u$ in $W^{1,p}(B)$ and $u_n \to u$ in $L^\ell(B)$, being $W^{1,p}(B)$ compactly embedded in $L^\ell(B)$.

Now, we claim that $\tilde{f}(u_n) \to \tilde{f}(u)$ in $(W^{1,p}(B))'$. Once the claim is proved, the first part of the statement follows by using the continuity of $\tilde{T}$. We pick any subsequence, still denoted by $(u_n)$, and we know that, up to another subsequence, $u_n \to u$ a.e. in $B$ and that there exists $h \in L^\ell(B)$ such that $|u_n| \leq h$ a.e. in $B$ for all $n$. By the continuity of $\tilde{f}$ we get that $|\tilde{f}(u_n) - \tilde{f}(u)|^\ell \to 0$ a.e. in $B$ and that $|\tilde{f}(u_n) - \tilde{f}(u)|^\ell \leq 2^\ell - 1(\tilde{f}(u_n)^\ell + \tilde{f}(u)\ell) \leq C(1 + h^\ell) \in L^\ell(B)$. Hence, the Dominated Convergence Theorem guarantees that $\tilde{f}(u_n) \to \tilde{f}(u)$ in $L^\ell(B)$. By the arbitrariness of the subsequence picked, we have that the same convergence result holds for the whole sequence $(u_n)$. The claim follows at once from the embedding $L^\ell(B) \hookrightarrow (W^{1,p}(B))'$.

Finally, inequalities (20) follow by (15) as in the proof of [4] Lemmas 3.7, 3.8. □

**Remark 1.** We observe here that (20) implies that $\{u : \tilde{T}(u) = u\}$ coincides with the set of critical points of $I$.

**Lemma 3.3 (Palais-Smale condition).** The functional $I$ satisfies the Palais-Smale condition, i.e. every sequence $(u_n) \subset W^{1,p}(B)$ such that

$$(I(u_n)) \text{ is bounded and } I'(u_n) \to 0 \text{ in } (W^{1,p}(B))'$$

admits a convergent subsequence.

**Proof.** Let $(u_n) \subset W^{1,p}(B)$ be a (PS)-sequence for $I$ as in the statement. By (15) and L'Hôpital’s rule, we get

$$\lim_{s \to +\infty} \frac{\tilde{f}(s)}{s^\ell} = \lim_{s \to +\infty} \frac{\tilde{f}(s)}{\ell s^{\ell - 1}} = \frac{1}{\ell}.$$

Thus,

$$\lim_{s \to +\infty} \frac{\tilde{f}(s)s}{F(s)} = \lim_{s \to +\infty} \frac{\tilde{f}(s)}{\ell s^{\ell - 1} F(s)} = \ell$$

and so, there exist $\mu \in (p, \ell]$ and $R_0 > 0$ such that $\tilde{f}(s)s \geq \mu F(s)$ for all $s \geq R_0$.

Now, we estimate

$$I(u_n) - \frac{1}{\mu} I'(u_n)[u_n] \geq \left(\frac{1}{p} - \frac{1}{\mu}\right) \|u_n\|^p + \int_{\{u_n \leq R_0\}} \left(\frac{1}{\mu} \tilde{f}(u_n) u_n - \tilde{F}(u_n)\right) dx$$

and, being $(u_n)$ a (PS)-sequence,

$$I(u_n) - \frac{1}{\mu} I'(u_n)[u_n] \leq |I(u_n)| + \frac{1}{\mu} \|I'(u_n)\|_* \|u_n\| \leq C(1 + \|u_n\|),$$
for some \( C > 0 \). Since we know that \( \int_{\{u_n \leq R_n\}} \left( \frac{1}{p} \tilde{f}(u_n) u_n - \tilde{F}(u_n) \right) dx \) is uniformly bounded in \( n \), we get

\[
\left( \frac{1}{p} - \frac{1}{\mu} \right) \|u_n\|^p \leq C(1 + \|u_n\|).
\]

Therefore, \((u_n)\) is bounded in \( W^{1,p}(B) \) and there exists \( u \in W^{1,p}(B) \) such that \( u_n \rightharpoonup u \) in \( W^{1,p}(B) \). Hence, Proposition \ref{prop:weak_convergence} guarantees that, up to a subsequence, \( T(u_n) \to \tilde{T}(u) \) in \( W^{1,p}(B) \). This implies, by the triangle inequality, that

\[
\limsup_{n \to \infty} \|u_n - \tilde{T}(u)\| \leq \limsup_{n \to \infty} \|u_n - \tilde{T}(u_n)\|.
\]

On the other hand, by the first inequality of \eqref{eq:uniform_boundedness}, we obtain

\[
\|u_n - \tilde{T}(u_n)\|^p \leq \frac{C}{a} \|I'(u_n)\|_s \to 0.
\]

Together with \eqref{eq:weak_convergence}, we conclude that \( u_n \to \tilde{T}(u) = u \) in \( W^{1,p}(B) \). \( \square \)

We define

\[
\begin{align*}
\text{if } v(t) &= m t^{p-1}, \quad &t &\in (0, u_0), \\
\text{if } v(t) &= m t^{p-1}, \quad &t &\in (u_0, +\infty).
\end{align*}
\]

By Lemma \ref{lem:existence}, \( \tilde{f} \) satisfies \((f_3)\), so that \( u_0 \) is an isolated zero of \( \tilde{f}(t) - m t^{p-1} \), hence

\[
\text{if } u_+ = +\infty \text{ is possible. Next, we define the set}
\]

\[
C_* := \{ u \in C : u_- \leq u \leq u_+ \text{ in } B \}.
\]

Clearly, \( C_* \) is closed and convex.

**Lemma 3.4.** The operator \( \tilde{T} \) defined in \eqref{eq:tilde_T} satisfies \( \tilde{T}(C_*) \subseteq C_* \).

**Proof.** We first note that \( u \in C_* \) implies \( \tilde{f}(u) \in C \), by the properties of \( \tilde{f} \). Now, let \( u \in C_* \) and \( v := \tilde{T}(u) \). By standard regularity theory (see e.g. \[23\] Theorem 2]), \( v \in C^{1,\alpha}(\overline{B}) \) for some \( \alpha \in (0, 1) \). Therefore, by \[15\] Theorem 1.1], we know that \( v \geq 0 \) in \( B \). Furthermore, due to uniqueness, \( v \) is radial. Now we prove that \( v \) is nondecreasing. It is enough to show that for every \( r \in (0, 1) \) one of the following cases occurs:

\[
\begin{align*}
(a) \quad v(s) \leq v(r) \text{ for all } s \in (0, r), \\
(b) \quad v(s) \geq v(r) \text{ for all } s \in (r, 1).
\end{align*}
\]

Indeed, if \( v(t) > v(r) \) for some \( t < r \), by the continuity of \( v \), there exists \( s \in (t, r) \) for which \( v(t) > v(s) > v(r) \) which violates both \((a)\) and \((b)\). Now, we fix \( r \in (0, 1) \). If \( \tilde{f}(u(r)) \leq mv(r)^{p-1} \), we consider the test function

\[
\varphi(x) := \begin{cases} 
(v(|x|) - v(r))^+ & \text{if } |x| \leq r, \\
0 & \text{otherwise} 
\end{cases}
\]

and we have

\[
\begin{align*}
\int_{B_r} (|\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \varphi + mv^{p-1} \varphi) dx &= \int_{B_r} \tilde{f}(u) \varphi dx \leq \tilde{f}(u(r)) \int_{B_r} \varphi dx \\
&\leq mv(r)^{p-1} \int_{B_r} \varphi dx.
\end{align*}
\]
Hence,
\[ \int_{B_r} |\nabla \varphi|^p \, dx + m \int_{B_r} (v(|x|)^{p-1} - v(r)^{p-1})(v(|x|) - v(r))^+ \, dx \leq 0, \]
that is \( \varphi \equiv 0 \), i.e., the case (a) occurs. Analogously, if \( \tilde{f}(u(r)) > mv(r)^{p-1} \), we consider the test function
\[ \varphi(x) := \begin{cases} 
0 & \text{if } |x| \leq r, \\
(v(|x|) - v(r))^+ & \text{otherwise}
\end{cases} \]
and we prove that (b) holds. Therefore, we have proved that \( v \) is nondecreasing.

It remains to show that \( u^- \leq v \leq u^+ \). By the fact that \( \tilde{f}(u^-) = mv_{\pi}^{p-1} \) and that \( f \) is nondecreasing we get
\[ -\Delta_p (v - u^-) + m(v^{p-1} - u_{\pi}^{p-1}) = \tilde{f}(u) - \tilde{f}(u^-) \geq 0. \]
Hence, if we multiply the equation above by \( (v - u^-)^- \) and integrate it over \( B \), we obtain
\[ -\|\nabla (v - u^-)^-\|_p^p - m \int_B (u_{\pi}^{p-1} - v^{p-1})(v - u^-)^- \, dx \geq 0, \]
that is \((v - u^-)^- \equiv 0 \) in \( B \). Similarly, if \( u_+ < +\infty \) we prove that \( v \leq u_+ \) in \( B \).

**Lemma 3.5 (Locally Lipschitz vector field).** Let \( W := W^{1,p}(B) \setminus \{ u : \tilde{T}(u) = u \} \). There exists a locally Lipschitz continuous operator \( K : W \to W^{1,p}(B) \) satisfying the following properties:
(i) \( K(C_\pi \cap W) \subset C_\pi \);
(ii) \( \frac{1}{2}\|u - K(u)\| \leq \|u - \tilde{T}(u)\| \leq 2\|u - K(u)\| \) for all \( u \in W \);
(iii) for all \( u \in W \)
\[ \Gamma(u)[u - K(u)] \geq \frac{a}{2}\|u - \tilde{T}(u)\|_p, \]
where \( a > 0 \) is the constant given in Proposition 3.2.

**Proof.** We follow the arguments in the proofs of [4] Lemma 4.1 and [3] Lemma 2.1.
We define the continuous functions \( \delta_1, \delta_2 : W \to \mathbb{R} \) as
\[ \delta_1(u) := \frac{1}{2}\|u - \tilde{T}(u)\| \quad \text{and} \quad \delta_2(u) := \frac{a\|u - \tilde{T}(u)\|^{p-1}}{2b\|u\| + \|\tilde{T}(u)\|^{p-2}}, \]
where \( a, b \) are the constants introduced in Proposition 3.2. First we claim that for every \( u \in W \), we can find a radius \( \varrho(u) > 0 \) such that for every \( v, \, w \in N(u) := \{ \phi \in W^{1,p}(B) : \|\phi - u\| < \varrho(u) \} \) it results that
\[ \|\tilde{T}(v) - \tilde{T}(w)\| < \min\{\delta_1(v), \delta_2(v), \delta_1(w), \delta_2(w)\}. \]
We argue by contradiction. We suppose first that the inequality in 25 involving \( \delta_1(v) \) is not satisfied, that is to say for every \( n \in \mathbb{N} \) we can find \( v_n, w_n \in N_n(u) := \{ \phi \in W^{1,p}(B) : \|\phi - u\| < \frac{1}{n} \} \) for which
\[ \|\tilde{T}(v_n) - \tilde{T}(w_n)\| \geq \delta_1(v_n) = \frac{1}{2}\|v_n - \tilde{T}(v_n)\|. \]
Since \( v_n, w_n \in N_n(u) \) for every \( n \), and by the continuity of \( \tilde{T} \), we get
\[ \lim_{n \to \infty} \|v_n - u\| = \lim_{n \to \infty} \|w_n - u\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|\tilde{T}(v_n) - \tilde{T}(w_n)\| = 0. \]
Hence, passing to the limit in (26), we obtain by the second inequality in (20)

\[ 0 \geq \frac{1}{2} \| u - \bar{T}(u) \| \geq \frac{\| I'(u) \|}{2b(\| u \| + \| \bar{T}(u) \|)^{p-2}} > 0, \]

where we have used Remark 1 and the fact that \( u \in W \) implies that \( u \not\equiv 0 \). This is absurd, and so (26) cannot hold. Proceeding analogously, one reaches a contradiction in each of the remaining three cases in (25), and the claim is proved.

Now, let \( \mathcal{U} \) be a locally finite open refinement of \( \{ N(u) : u \in W \} \) and let \( \{ \pi_U : U \in \mathcal{U} \} \) be the standard partition of unity subordinated to \( \mathcal{U} \), i.e.

\[ \pi_U(u) := \frac{\alpha_U(u)}{\sum_{V \in \mathcal{U}} \alpha_V(u)}, \quad \alpha_U(u) := \text{dist}(u, W \setminus U). \]

Clearly, \( \sum_{U \in \mathcal{U}} \pi_U(u) = 1 \) for any \( u \in W \), \( \pi_U \) is Lipschitz continuous, it satisfies \( \text{supp}(\pi_U) \subseteq U \) and \( 0 \leq \pi_U \leq 1 \) for any \( U \in \mathcal{U} \). Furthermore, since \( \mathcal{U} \) is a refinement of \( \{ N(u) : u \in W \} \), given any \( U \in \mathcal{U}, \) (25) holds in particular for any \( u, w \in U \).

For every \( U \in \mathcal{U} \) we choose an element \( a_U \in U \) with the property that, if \( U \cap C_* \neq \emptyset \), then \( a_U \in U \cap C_* \). We define \( K : W \to W^{1,p}(B) \) as

\[ K(u) := \sum_{U \in \mathcal{U}} \pi_U(u) \bar{T}(a_U). \]

Therefore, \( K \) is locally Lipschitz continuous due to the Lipschitz continuity of any \( \pi_U \) and to the local finiteness of the refinement \( \mathcal{U} \).

Moreover, (i) holds thanks to the facts that \( \bar{T} \) preserves the cone \( C_* \) (see Lemma 3.4), that \( K \) is a convex combination of points \( \bar{T}(a_U) \), and that \( C_* \) is convex.

In order to prove (ii), by using the properties of the functions \( \pi_U \) and (25), we estimate

\[ \| K(u) - \bar{T}(u) \| \leq \sum_{U \in \mathcal{U}} \pi_U(u) \| \bar{T}(a_U) - \bar{T}(u) \| < \delta_1(u) = \frac{1}{2} \| u - \bar{T}(u) \|. \tag{27} \]

This gives immediately

\[
\begin{align*}
\| u - K(u) \| &\leq \| K(u) - \bar{T}(u) \| + \| u - \bar{T}(u) \| < \frac{3}{2} \| u - \bar{T}(u) \|, \\
\| u - \bar{T}(u) \| &\leq \| u - K(u) \| + \| K(u) - \bar{T}(u) \| \leq \| u - K(u) \| + \frac{1}{2} \| u - \bar{T}(u) \|,
\end{align*}
\]

which imply the two inequalities of (ii).

By using the definition of \( \delta_2 \) and (20), we can finally prove (iii). Indeed, for every \( u \in W \)

\[
I'(u)\| u - K(u) \| \geq I'(u)\| u - \bar{T}(u) \| - \| I'(u) \|_p \| K(u) - \bar{T}(u) \| \\
> a\| u - \bar{T}(u) \|^p - b\| u - \bar{T}(u) \| (\| u \| + \| \bar{T}(u) \|)^{p-2} \delta_2(u) \\
= \frac{a}{2}\| u - \bar{T}(u) \|^p,
\]

and the proof is concluded. \( \square \)

Without loss of generality, we will take from now on

\[ \ell \in \left( p, \min \left\{ \frac{(p-1)^2 + p - 2}{p - 2}, p^* \right\} \right), \tag{28} \]

where \( \ell \) is the subcritical growth of \( \ell \) defined in Lemma 3.1. This further condition is needed in the following two lemmas.
Lemma 3.6. For all \( c \in \mathbb{R} \) there exists \( C = C(c) > 0 \) for which
\[
\|u\| + \|\tilde{T}(u)\| \leq C \left( 1 + \|u - \tilde{T}(u)\|^{\beta} \right), \quad \beta := \frac{\ell - 1}{(p - 1)^2 - (p - 2)(\ell - 1)} \tag{29}
\]
holds for every \( u \in W^{1,p}(B) \) with \( I(u) \leq c \).

Proof. Reasoning as in the first part of the proof of Lemma 3.3 we easily get, by the fact that \( I(u) \leq c \), that
\[
\|u\|^p \leq C(1 + \|I'(u)\|_* \|u\|),
\]
for some positive constant \( C \). Then, by the second inequality of (20) and by Young’s inequality with exponents \((p',p)\),
\[
\|u\|^p \leq C(1 + \|u - \tilde{T}(u)\|^{p'}(\|u\| + \|\tilde{T}(u)\|^{p'-2}(\|u\|)) \leq C(1 + \|u - \tilde{T}(u)\|^{p'}(\|u\| + \|\tilde{T}(u)\|^{p'-2} + \varepsilon \|u\|^{p'}),
\]
where \( C > 0 \) may change from line to line, and \( \varepsilon > 0 \) is sufficiently small. Hence,
\[
\|u\| \leq C(1 + \|u - \tilde{T}(u)\|^{p'/p}(\|u\| + \|\tilde{T}(u)\|^{1-p'/p}). \tag{30}
\]
Young’s inequality with exponents \((p/p', (p-1)/(p-2))\) then gives
\[
\|u\| \leq C[1 + \|u - \tilde{T}(u)\|^{p'/p}(\|u\| + \|\tilde{T}(u)\|)] \text{ for some } C > 0, \varepsilon > 0 \text{ small}. \tag{31}
\]
Now, consider the equation satisfied by \( v := \tilde{T}(u) \)
\[
-\Delta_p v + m|v|^{p-2}v = \tilde{f}(u) \quad \text{in } B.
\]
By testing it with \( v \) and using (15), Hölder’s inequality, and the Sobolev embedding, we get
\[
\|v\|^p = \int_B \tilde{f}(u)v \, dx \leq \int_B C(1 + u^{\ell-1})v \, dx \leq C \left( \int_B (1 + u^{\ell}) \, dx \right)^{1/\ell'} \left( \int_B v^{\ell} \, dx \right)^{1/\ell}
\leq C\|v\|_{L^{\ell'}(B)} \left( |B|^{1/\ell'} + \|u\|_{L^{\ell'}(B)}^{\ell/\ell'} \right) \leq C\|v\| \left( 1 + \|u\|^{\ell/\ell'} \right),
\]
that is \( \|\tilde{T}(u)\| \leq C(1 + \|u\|^{(\ell-1)/(p-1)}) \). By (30), we obtain
\[
\|\tilde{T}(u)\| \leq C \left( 1 + \|u - \tilde{T}(u)\|^{\ell-1/(p-1)}(\|u\| + \|\tilde{T}(u)\|)^{(\ell-1)(p-2)} \right). \tag{32}
\]
By applying Young’s inequality with exponents \((\ell-1)/(p-1), (p-2)/\ell(p-1)\), we have
\[
\|\tilde{T}(u)\| \leq C \left[ 1 + \|u - \tilde{T}(u)\|^{(\ell-1)/(p-1)}(\|u\| + \|\tilde{T}(u)\|)^{(\ell-1)(p-2)} + \varepsilon \|u\| + \|\tilde{T}(u)\| \right]
\]
for some \( C > 0, \varepsilon > 0 \) small. Together with (31), this implies the thesis.

Lemma 3.7. Let \( c \in \mathbb{R} \) be such that \( I'(u) \neq 0 \) for all \( u \in \mathcal{C}_* \) with \( I(u) = c \). Then, there exist two positive constants \( \bar{\varepsilon} \) and \( \bar{\delta} \) such that the following inequalities hold
(i) \( \|I'(u)\|_* \geq \bar{\delta} \) for all \( u \in \mathcal{C}_* \) with \( |I(u) - c| \leq 2\bar{\varepsilon} \);
(ii) \( \|u - K(u)\| \geq \bar{\delta} \) for all \( u \in \mathcal{C}_* \) with \( |I(u) - c| \leq 2\bar{\varepsilon} \).

Proof. (i) The proof follows by Lemma 3.3. Indeed, suppose by contradiction that (i) does not hold, then we can find a sequence \( (u_n) \subset \mathcal{C}_* \) such that \( \|I'(u_n)\|_* < \frac{1}{n} \) and \( c - \frac{1}{n} \leq I(u_n) \leq c + \frac{1}{n} \) for all \( n \). Hence, \( (u_n) \) is a Palais-Smale sequence, and since \( I \) satisfies the Palais-Smale condition at level \( c \), up to a subsequence, \( u_n \to u \) in \( W^{1,p}(B) \). Since \( (u_n) \subset \mathcal{C}_* \) and \( \mathcal{C}_* \) is closed, \( u \in \mathcal{C}_* \). The fact that \( I \) is of class
$C^1$ then gives $I(u_n) \to c = I(u)$ and $I'(u_n) \to 0 = I'(u)$, which contradicts the hypothesis.

(ii) Let

$$I^{c+2\varepsilon}_{c-2\varepsilon} := \{ u \in C_* : |I(u) - c| \leq 2\varepsilon \}.$$  

By the part (i), $I^{c+2\varepsilon}_{c-2\varepsilon} \subset W$, where $W$ is defined in Lemma 3.5. Hence, for all $u \in I^{c+2\varepsilon}_{c-2\varepsilon}$, $\|u - K(u)\| \geq \frac{1}{2} \|u - \tilde{T}(u)\|$ by Lemma 3.5(ii). By the second inequality of (20) and by (i), we have for all $u \in I^{c+2\varepsilon}_{c-2\varepsilon}$

$$\|u - \tilde{T}(u)\| \geq \frac{\delta}{b(\|u\| + \|\tilde{T}(u)\|)^{p-2}}.$$  

This implies by (29), that

$$\|u - \tilde{T}(u)\| \geq \frac{\delta}{bc\|u - \tilde{T}(u)\|^{\beta} + \|u - \tilde{T}(u)\|^{p-2}}.$$  

which in turn gives $\|u - \tilde{T}(u)\|^\| \geq M$ for some positive $M$ and for all $u \in I^{c+2\varepsilon}_{c-2\varepsilon}$. Indeed, if by contradiction we had $\inf \|u - \tilde{T}(u)\| = 0$ over all $u \in I^{c+2\varepsilon}_{c-2\varepsilon}$, we could find a sequence $(u_n) \subset I^{c+2\varepsilon}_{c-2\varepsilon}$ such that $\|u_n - \tilde{T}(u_n)\| \to 0$, and so by passing to the limit as $n \to \infty$ in

$$\|u_n - \tilde{T}(u_n)\| \geq \frac{\delta}{bc\|u_n - \tilde{T}(u_n)\|^{\beta} + \|u_n - \tilde{T}(u_n)\|^{p-2}}.$$  

we would have the contradiction $0 \geq \delta/(bc\|u\|^{\beta} + \|u\|^{p-2})$, being $\beta > 0$ thanks to the choice of $\ell$ in (28). Therefore, for all $u \in I^{c+2\varepsilon}_{c-2\varepsilon}$, $\|u - K(u)\| \geq \frac{M}{2} \geq \min\{\delta, \frac{M}{2}\}$, still denoted by $\delta$, and the proof is concluded.

**Lemma 3.8 (Descending flow argument).** Let $c \in \mathbb{R}$ be such that $I'(u) \neq 0$ for all $u \in C_*$, with $I(u) = c$. Then, there exists a function $\eta : C_* \to C_*$ satisfying the following properties:

(i) $\eta$ is continuous with respect to the topology of $W^{1,p}(B)$;

(ii) $I(\eta(u)) \leq I(u)$ for all $u \in C_*$;

(iii) $I(\eta(u)) \leq c - \varepsilon$ for all $u \in C_*$ such that $|I(u) - c| < \varepsilon$;

(iv) $\eta(u) = u$ for all $u \in C_*$ such that $|I(u) - c| > 2\varepsilon$,

where $\varepsilon$ is the positive constant corresponding to $c$ given in Lemma 3.7.

**Proof.** Let $\chi_1 : \mathbb{R} \to [0, 1]$ and $\chi_2 : W^{1,p}(B) \to [0, 1]$ be two smooth cut-off functions such that

$$\chi_1(t) = \begin{cases} 1 & \text{if } |t - c| < \varepsilon, \\ 0 & \text{if } |t - c| > 2\varepsilon, \end{cases} \quad \chi_2(u) = \begin{cases} 1 & \text{if } \|u - K(u)\| \geq \bar{\delta}, \\ 0 & \text{if } \|u - K(u)\| \leq \frac{1}{2}, \end{cases}$$  

where $\bar{\delta}$ and $\varepsilon$ are given in Lemma 3.7. Recalling the definition of $K$ in Lemma 3.5, let $\Phi : W^{1,p}(B) \to W^{1,p}(B)$ be the map defined by

$$\Phi(u) := \begin{cases} \chi_1(I(u))\chi_2(u) \frac{u - K(u)}{\|u - K(u)\|} & \text{if } |I(u) - c| \leq 2\varepsilon, \\ 0 & \text{otherwise}. \end{cases}$$
Note that the definition of $\Phi$ is well posed by Lemma 3.7. For all $u \in C_*$, we consider the Cauchy problem

$$
\begin{align*}
\frac{d}{dt} \eta(t, u(x)) &= -\Phi(\eta(t, u(x))) & (t, x) \in (0, \infty) \times B, \\
\partial_x \eta(t, u(x)) &= 0 & (t, x) \in (0, \infty) \times \partial B, \\
\eta(0, u(x)) &= u(x) & x \in B.
\end{align*}
$$

(32)

Being $K$ locally Lipschitz continuous by Lemma 3.5, for all $u \in C_*$ there exists a unique solution $\eta(\cdot, u) \in C^1([0, \infty); W^{1,p}(B))$.

We shall prove that for all $t > 0$, $\eta(t, C_*) \subset C_*$. Fix $T > 0$. For every $u \in C_*$ and $n \in \mathbb{N}$ with $n \geq T/\delta$, let

$$
\begin{align*}
\tilde{\eta}_n(0, u) &:= u, \\
\tilde{\eta}_n(t_{i+1}, u) &:= \tilde{\eta}_n(t_i, u) - \frac{T}{n} \Phi(\bar{\eta}_n(t_i, u)) \\
\end{align*}
$$

for all $i = 0, \ldots, n - 1$,

$$
t_i := i \cdot \frac{T}{n} \quad \text{for all } i = 0, \ldots, n.
$$

Let us prove that for all $i = 0, \ldots, n - 1$, $\tilde{\eta}_n(t_{i+1}, u) \in C_*$. If $|I(u) - c| > 2\bar{\varepsilon}$, then $\tilde{\eta}_n(t_{i+1}, u) = u \in C_*$ for every $i = 0, \ldots, n - 1$. Otherwise, let

$$
\lambda := \frac{T}{n} \cdot \frac{\chi_1(I(\bar{\eta}_n(t_i, u))) \chi_2(\bar{\eta}_n(t_i, u))}{\|\bar{\eta}_n(t_i, u) - K(\bar{\eta}_n(t_i, u))\|}.
$$

Clearly, $\lambda \leq 1$ by Lemma 3.7(ii), being $n \geq T/\delta$. Therefore, it results for every $i = 0, \ldots, n - 1$

$$
\tilde{\eta}_n(t_{i+1}, u) = (1 - \lambda) \tilde{\eta}_n(t_i, u) + \lambda K(\bar{\eta}_n(t_i, u)) \in C_ *
$$

by induction on $i$, by Lemma 3.5(i), and by the convexity of $C_*$. For every $i = 0, \ldots, n - 1$, we can now define the line segment

$$
\eta_n^{(i)}(t, u) := \left(1 - \frac{t}{T} + i\right) \tilde{\eta}_n(t_i, u) + \left(\frac{t}{T} - i\right) \tilde{\eta}_n(t_{i+1}, u)
$$

for all $t \in [t_i, t_{i+1}]$. We denote by $\eta_n := \bigcup_{i=0}^{n-1} \eta_n^{(i)}$ the whole Euler polygonal defined in $[0, T]$. Being $C_*$ convex, we get immediately that for all $t \in [0, T]$, $\eta_n(t, u) \in C_*$. We claim that $\eta_n(\cdot, u)$ converges to the solution $\eta(\cdot, u)$ of the Cauchy problem (32) in $W^{1,p}(B)$. Indeed, for all $i = 0, \ldots, n - 1$, we integrate by parts the equation of (32) in the interval $[t_i, t_{i+1}]$ and we obtain

$$
\eta(t_{i+1}, u) = \eta(t_i, u) - \frac{T}{n} \Phi(\eta(t_i, u)) + \int_{t_i}^{t_{i+1}} (s - t_i) \frac{d}{ds} \Phi(\eta(s, u)) ds.
$$

On the other hand, we define the error

$$
\varepsilon_i := \|\eta(t_i, u) - \eta_n(t_i, u)\| \quad \text{for every } i = 0, \ldots, n.
$$

Hence, for every $i = 0, \ldots, n - 1$, we get

$$
\varepsilon_{i+1} \leq \varepsilon_i + \frac{T}{n} \|\Phi(\eta(t_i, u)) - \Phi(\eta_n(t_i, u))\| + \left\|\int_{t_i}^{t_{i+1}} (t_{i+1} - s) \frac{d}{ds} \Phi(\eta(s, u)) ds\right\|. 
$$

(33)

Now, since $\Phi$ is locally Lipschitz and $\eta([0, T]) \subset W^{1,p}(B)$ is compact,

$$
\|\Phi(\eta(t_i, u)) - \Phi(\eta_n(t_i, u))\| \leq \varepsilon_i L_{\Phi}
$$

(34)
for some $L_{\Phi} = L_{\Phi}(\eta([0, T])) > 0$. Furthermore,
\[
\left\| \int_{t_i}^{t_{i+1}} (t_{i+1} - s) \frac{d}{ds} \Phi(\eta(s, u))ds \right\| \leq \int_{t_i}^{t_{i+1}} (t_{i+1} - s) \left\| \frac{d}{ds} \Phi(\eta(s, u)) \right\| ds
\]
\[
\leq \frac{T}{n} \int_0^T \left\| \Phi'(\eta(s, u)) \right\| \| \Phi(\eta(s, u)) \| ds
\]
\[
\leq \frac{T^2}{n} \sup_{s \in [0, T]} \| \Phi'(\eta(s, u)) \|_* = \frac{T^2}{n} L_{\Phi}.
\]
Thus, combining the last inequality with (34) and (33), we have
\[
\varepsilon_{i+1} \leq \varepsilon_i + \frac{T}{n} \varepsilon_i L_{\Phi} + \frac{T^2}{n} L_{\Phi} \quad \text{for all } i = 0, \ldots, n - 1.
\]
This implies that
\[
\varepsilon_{i+1} \leq T^2 L_{\Phi} \sum_{j=0}^{i} \left( 1 + \frac{T}{n} L_{\Phi} \right)^j = T \left[ \left( 1 + \frac{T}{n} L_{\Phi} \right)^{i+1} - 1 \right] \to 0 \quad \text{as } n \to \infty,
\]
where we have used the fact that $\varepsilon_0 = 0$. By the triangle inequality and the continuity of $\eta(t, u)$ and $\eta_n(t, u)$, this yields the claim.

Hence, for all $t \in [0, T]$, $\eta(t, u) \in C_s$ by the closedness of $C_s$.

For all $u \in C_s$ and $t > 0$ we can write
\[
I(\eta(t, u)) - I(u) = \int_0^t \frac{d}{ds} I(\eta(s, u))ds
\]
\[
= - \int_0^t \frac{\chi_1(I(\eta(s, u)))\chi_2(\eta(s, u))}{\| \eta(s, u) - K(\eta(s, u)) \|} I'(\eta(s, u))[\eta(s, u) - K(\eta(s, u))]ds
\]
\[
\leq - \frac{a}{2} \int_0^t \| \eta(s, u) - T(\eta(s, u)) \|^{p-2} \chi_1(I(\eta(s, u)))\chi_2(\eta(s, u))ds \leq 0,
\]
where we have used the inequality in Lemma 3.5(iii).

Now, let $u \in C_s$ be such that $|I(u) - c| < \bar{\varepsilon}$ and let $t \geq 2^{p+2}\bar{\varepsilon}/(a\delta^{p-1})$. Then, two cases arise: either there exists $s \in [0, t]$ for which $I(\eta(s, u)) \leq c - \bar{\varepsilon}$ and so, by the previous calculation we get immediately that $I(\eta(t, u)) \leq c - \bar{\varepsilon}$, or for all $s \in [0, t]$, $I(\eta(s, u)) > c - \bar{\varepsilon}$. In this second case,
\[
c - \bar{\varepsilon} < I(\eta(s, u)) \leq I(u) < c + \bar{\varepsilon}.
\]
In particular, by Lemma 3.7(i), $\eta(s, u) \in W$, by the definitions of $\chi_1$ and $\chi_2$, and by Lemma 3.7(ii), it results that for all $s \in [0, t]$
\[
\chi_1(I(\eta(s, u))) = 1, \quad \| \eta(s, u) - K(\eta(s, u)) \| \geq \delta, \quad \text{and} \quad \chi_2(\eta(s, u)) = 1.
\]
Hence, by (34) and Lemma 3.5(ii) and (iii), we obtain
\[
I(\eta(t, u)) \leq I(u) - \int_0^t \frac{a}{2^{p+1}} \delta^{p-1} ds \leq c + \bar{\varepsilon} - \frac{a}{2^{p+1}} \delta^{p-1} t \leq c - \bar{\varepsilon}.
\]
Finally, if we define with abuse of notation
\[
\eta(u) := \eta \left( \frac{2^{p+2}\bar{\varepsilon}}{a\delta^{p-1}}, u \right),
\]
it is immediate to verify that $\eta$ satisfies (i)-(iv). \qed

**Lemma 3.9 (Mountain pass geometry).** Let $\tau > 0$ be such that $\tau < \min \{ u_0 - u_-, u_+ - u_0 \}$. Then there exists $\alpha > 0$ such that
(i) \( I(u) \geq I(u_-) + \alpha \) for every \( u \in \mathcal{C}_* \) with \( \|u - u_-\|_{L^\infty(B)} = \tau \); (ii) if \( u_+ < \infty \), then \( I(u) \geq I(u_+) + \alpha \) for every \( u \in \mathcal{C}_* \) with \( \|u - u_+\|_{L^\infty(B)} = \tau \).

**Proof.** Suppose by contradiction that there exists a sequence \((w_n)_n \subset \mathcal{C}_*\) such that
\[
\|w_n\|_{L^\infty(B)} = w_n(1) = \tau > 0 \quad \text{for all } n\tag{36}
\]
and \( \limsup_{n \to \infty} [I(u_- + w_n) - I(u_-)] \leq 0 \). Since
\[
\frac{1}{p} \int_B (|\nabla w_n|^p + m_+ w_n - u_+ w_n) dx = \int_B \int_0^1 (u_- + tw_n)^{p-1} w_n dt dx,
\]
we get
\[
I(u_- + w_n) - I(u_-) = \frac{1}{p} \int_B \int_0^1 (m_+ w_n - u_+ w_n) |\nabla w_n|^p + m_+ w_n - u_+ w_n w_n dt dx.
\]
Therefore, since by \((f_3)\) and the definition of \( u_- \)
\[
ms^{p-1} - \tilde{f}(s) > 0 \quad \text{for } s \in (u_-, u_0), \tag{37}
\]
we conclude that \( \|\nabla w_n\|_{L^p(B)} \to 0 \) and that \( |\nabla w_n| \to 0 \) a.e. in \( B \) up to a subsequence. Together with \((36)\), this ensures that \((w_n)\) is bounded in \( W^{1,p}(B) \) and so, up to a subsequence, it is weakly convergent to some \( w \in W^{1,p}(B) \). In particular,
\[
\lim_{n \to \infty} \int_B |\nabla (w_n - w)|^{p-2} \nabla (w_n - w) \cdot \nabla w dx = 0.
\]
By the Dominated Convergence Theorem, we now get that \( \nabla w = 0 \) a.e. in \( B \) and so the sequence \((w_n)\) converges to the constant solution \( w \equiv \tau \) in the \( W^{1,p}\)-norm. Again by the Dominated Convergence Theorem we can conclude that
\[
0 = \lim_{n \to \infty} \int_B \int_0^1 (m_+ w_n - u_+ w_n)^{p-1} - \tilde{f}(u_- + tw_n) w_n dt dx
\]
\[
= \int_B \int_0^1 (m_+ t\tau - \tilde{f}(u_- + t\tau)\tau) dt dx,
\]
which contradicts \((37)\). Hence there exists \( \alpha_1 > 0 \) such that (i) holds.

In a similar way, now using the fact that \( ms^{p-1} - \tilde{f}(s) < 0 \) for \( s \in (u_0, u_+) \), we find \( \alpha_2 > 0 \) such that (ii) holds if \( u_+ < \infty \). The claim then follows with \( \alpha := \min\{\alpha_1, \alpha_2\} \).

Let
\[
U_- := \left\{ u \in \mathcal{C}_* : I(u) < I(u_-) + \frac{\alpha}{2}, \|u - u_-\|_{L^\infty(B)} < \tau \right\},
\]
\[
U_+ := \begin{cases} \left\{ u \in \mathcal{C}_* : I(u) < I(u_+) + \frac{\alpha}{2}, \|u - u_+\|_{L^\infty(B)} < \tau \right\}, & \text{if } u_+ < \infty, \\ \left\{ u \in \mathcal{C}_* : I(u) < I(u_-), \|u - u_-\|_{L^\infty(B)} > \tau \right\}, & \text{if } u_+ = \infty. \end{cases} \tag{38}
\]
where \( \tau \) and \( \alpha \) are given by Lemma \( 3.9 \)
\[
\Gamma := \{ \gamma \in C([0,1];C_{s}) : \gamma(0) \in U_{-}, \gamma(1) \in U_{+} \},
\]
and
\[
c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)). \tag{39}
\]

**Proposition 3.10** (Mountain Pass Theorem). The value \( c \) defined in \( 39 \) is finite and there exists a critical point \( u \in C_{s} \setminus \{u_{-}, u_{+}\} \) of \( I \) with \( I(u) = c \). In particular, \( u \) is a weak solution of \( \mathbf{1} \).

**Proof.** We first observe that, by Lemma \( 3.1 \) any critical point of \( I \) solves weakly \( 5 \) which is equivalent to \( \mathbf{1} \).

**Case** \( u_{+} < \infty \). Pick any \( \gamma \in \Gamma \). We note that \( \tau < \min\{u_{0} - u_{-}, u_{+} - u_{0}\} \) implies \( u_{+} - u_{-} > 2\tau \). Hence, by the definition of \( \Gamma \), \( U_{-} \), and \( U_{+} \), and by the reverse triangle inequality we get
\[
\|\gamma(1) - u_{-}\|_{L^{\infty}(B)} \geq \|\gamma(1) - u_{+}\|_{L^{\infty}(B)} - (u_{+} - u_{-}) > \tau.
\]
Analogously, \( \|\gamma(0) - u_{+}\|_{L^{\infty}(B)} > \tau \). Now, since \( \gamma \) is continuous with respect to the \( W^{1,p} \)-norm, by Lemma \( 2.2 \) it is continuous also with respect to the \( L^{\infty} \)-norm. So, there exist \( t_{-}, t_{+} \in (0,1) \) such that \( \|\gamma(t_{-}) - u_{-}\|_{L^{\infty}(B)} = \tau \) and \( \|\gamma(t_{+}) - u_{+}\|_{L^{\infty}(B)} = \tau \). Hence, by Lemma \( 3.9 \) \( I(\gamma(t_{-})) \geq I(u_{-}) + \alpha \) and \( I(\gamma(t_{+})) \geq I(u_{+}) + \alpha \), which imply immediately that
\[
c \geq \max\{I(u_{-}), I(u_{+})\} + \alpha > \max\{I(u_{-}), I(u_{+})\}. \tag{40}
\]
On the other hand, \( \Gamma \) is not empty, since it contains at least the path \( t \in [0,1] \mapsto (1 - t)u_{-} + tu_{+} \), hence \( c < +\infty \). Therefore, \( c \) is a finite number.

Now, assume by contradiction that there does not exist a critical point \( u \in C_{s} \) for which \( I(u) = c \). Then, there exists a deformation \( \eta : C_{s} \to C_{s} \) satisfying (i)-(iv) of Lemma \( 3.8 \) with \( \varepsilon = \varepsilon(c) > 0 \) given by Lemma \( 3.7 \). Without loss of generality, we assume that \( 4\varepsilon < \alpha \). By the definition \( 39 \) of \( c \), there exists a curve \( \gamma \in \Gamma \) such that
\[
\max_{t \in [0,1]} I(\gamma(t)) < c + \varepsilon \tag{41}
\]
and we define the curve \( \widetilde{\gamma} : [0,1] \to C_{s} \), by \( \widetilde{\gamma}(t) := \eta(\gamma(t)) \). We check that also \( \widetilde{\gamma} \in \Gamma \).
Indeed, clearly \( \widetilde{\gamma} \in C([0,1];C_{s}) \). Moreover, \( \widetilde{\gamma}(0) \in U_{-} \) because, by the definition of \( U_{-} \) and by \( 40 \),
\[
I(\gamma(0)) \leq I(u_{-}) + \frac{\alpha}{2} \leq c - \alpha + \frac{\alpha}{2} < c - 2\varepsilon
\]
and so, by Lemma \( 3.8 \) (iv), \( \widetilde{\gamma}(0) = \gamma(0) \in U_{-} \). Analogously, \( \widetilde{\gamma}(1) = \gamma(1) \in U_{+} \). Furthermore, by \( 41 \) and by Lemma \( 3.8 \) (iii), we get \( I(\eta(\gamma(t))) \leq c - \varepsilon \) for all \( t \in [0,1] \). Hence
\[
\max_{t \in [0,1]} I(\widetilde{\gamma}(t)) \leq c - \varepsilon,
\]
which yields a contradiction with \( 39 \). Finally, \( 40 \) ensures that the critical point \( u \in C_{s} \) at level \( c \) cannot be \( u_{-} \) or \( u_{+} \).

**Case** \( u_{+} = \infty \). As in the previous case, for any \( \gamma \in \Gamma \) there exists \( t_{-} \in (0,1) \) for which \( I(\gamma(t_{-})) \geq I(u_{-}) + \alpha \), and so we have
\[
c \geq I(u_{-}) + \alpha > I(u_{-}). \tag{43}
\]
Furthermore, let $t \cdot 1$ denote the constant function of constant value $t$. For any $t > M$ we get by the fact that $\tilde{f} \in \tilde{\mathcal{G}}$

$$I(t \cdot 1) = |B| \left( \frac{t^p}{p} - \int_0^t \tilde{f}(s) ds \right)$$

$$\leq |B| \left( \frac{t^p}{p} - \int_0^M \tilde{f}(s) ds - (m + \delta) \int_M^t s^{p-1} ds \right)$$

$$\leq \frac{|B|}{p} \left( t^p - pM \min \{s \in [0,M] \} \tilde{f}(s) - (m + \delta)(t^p - M^p) \right)$$

$$= C - \frac{|B|(m + \delta - 1)}{p} t^p \to -\infty \quad \text{as} \quad t \to \infty,$$

being $m + \delta - 1 > 0$. Hence, we can find a sufficiently large constant $k > 0$ such that the curve

$$\gamma : t \in [0,1] \mapsto u_- + kt \in \mathcal{C},$$

is such that $\gamma(0) = u_- \in U_-$ and $\gamma(1) = u_- + k \in U_+$. Therefore, $\gamma \in \Gamma$ and consequently $c < +\infty$. Now, suppose by contradiction that there does not exist any critical point $u \in \mathcal{C}$ of $I$, such that $I(u) = c$. Then, by Lemma 3.7-(i), $\|I'(u)\|_* \leq \delta$ for any $u \in \mathcal{C}$ such that $|I(u) - c| \leq 2\varepsilon$. Without loss of generality, we can take $4\varepsilon < \alpha$. In correspondence of $\varepsilon$, consider the deformation $\eta$ built in Lemma 3.8 and a curve $\gamma \in \Gamma$ such that $\max_{t \in [0,1]} I(\gamma(t)) < c + \varepsilon$. Now, let $\gamma(t) := \eta(\gamma(t))$. We claim that $\gamma \in \Gamma$. Indeed, $\gamma(0) = \gamma(1) \in U_-$, since (42) holds by (43). Analogously, being $\gamma \in \Gamma$, $\gamma(1) \in U_+$ and so

$$I(\gamma(1)) < I(u_-) \leq c - 4\varepsilon.$$ 

This yields, by Lemma 3.8-(iv), that $\eta(\gamma(1)) = \gamma(1)$ and so $\gamma \in \Gamma$. Now, since $I(\gamma(t)) < c + \varepsilon$, by Lemma 3.8-(iii), $I(\gamma(t)) \leq c - \varepsilon$ holds for all $t \in [0,1]$. This contradicts the definition of $c$. Hence, there exists a critical point $u \in \mathcal{C}$ of $I$ at level $c$, which is not equal to $u_-$ by (43).

4. The mountain pass solution is nonconstant. We are now ready to prove that the mountain pass solution $u \in \mathcal{C} \setminus \{u_-, u_+\}$ found in the previous section is nonconstant. To this aim, we observe that since in $\mathcal{C}$ the only constant solutions are $u_-$, $u_+$, and $u_0$, it remains to prove that $u \neq u_0$.

Lemma 4.1. Let $v \in W^{1,p}(B) \setminus \{0\}$ be such that

$$\int_B vdx = 0,$$  

and let

$$\psi : \mathbb{R}^2 \to \mathbb{R}, \quad \psi(s,t) := I'(t(u_0 + sv))[u_0 + sv].$$

There exist $\varepsilon_1, \varepsilon_2 > 0$ and a $C^1$-function $h : (-\varepsilon_1, \varepsilon_1) \to (1 - \varepsilon_2, 1 + \varepsilon_2)$ such that for $(s,t) \in V := (-\varepsilon_1, \varepsilon_1) \times (1 - \varepsilon_2, 1 + \varepsilon_2)$ we have

$$\psi(s,t) = 0 \quad \text{if and only if} \quad t = h(s).$$

Moreover,  

(i) $h(0) = 1$, $h'(0) = 0$;  

(ii) $I(h(s)(u_0 + sv)) < I(u_0)$ for $s \in (-\varepsilon_1, \varepsilon_1), s \neq 0$;  

(iii) $\frac{\partial}{\partial s}\psi(s,t) < 0$ for $(s,t) \in V$. 

Proof. Since $I$ is a $C^2$-functional, $\psi$ is of class $C^1$ with $\psi(0,1) = 0$. By $(f_3)$ we get
\[
\frac{\partial}{\partial t} \bigg|_{(0,1)} \psi(s,t) = I''(u_0)[u_0, u_0] = [m(p-1)u_0^{p-2} - \tilde{f}'(u_0)] \int_B u_0^2 \, dx < 0 \tag{47}
\]
and by $(45)$
\[
\frac{\partial}{\partial s} \bigg|_{(0,1)} \psi(s,t) = I'(u_0)[v] + I''(u_0)[v, v] = [m(p-1)u_0^{p-2} - \tilde{f}'(u_0)]u_0 \int_B v \, dx = 0.
\]
Thus the existence of $\varepsilon_1, \varepsilon_2$ and $h$, as well as property (i), follow from the Implicit Function Theorem. To prove (ii), we write $h(s) = 1 + o(s)$, for $s \in (-\varepsilon_1, \varepsilon_1)$, $s \neq 0$, so that
\[
h(s)(u_0 + sv) - u_0 = sv + o(s)
\]
and therefore, by Taylor expansion and $(f_3)$,
\[
I(h(s)(u_0 + sv)) - I(u_0) = \frac{1}{2} I''(u_0)[sv + o(s), sv + o(s)] + o(s^2)
\]
\[
= \frac{s^2}{2} I''(u_0)[v, v] + o(s^2)
\]
\[
= \frac{s^2}{2} \int_B [m(p-1)u_0^{p-2} - \tilde{f}'(u_0)]v^2 \, dx + o(s^2) < 0.
\]
Then, property (ii) holds after making $\varepsilon_1, \varepsilon_2$ smaller if necessary, and property (iii) is a consequence of $(47)$ and of the regularity of $\psi$. \qed

Remark 2. Let $\mathcal{N}_*$ be the following Nehari-type set (see also $(63)$ ahead)
\[
\mathcal{N}_* := \{ u \in C_* \setminus \{0\} : I'(u)[u] = 0 \}. \tag{48}
\]
The previous lemma shows that $u_0$ is not a local minimum of the functional $I$ restricted to $\mathcal{N}_*$, that is to say, for every $\varepsilon > 0$ there exists $u_{\varepsilon} \in \{ u \in W^{1,p}(B) : ||u - u_0|| < \varepsilon \} \cap \mathcal{N}_*$ such that $I(u_{\varepsilon}) < I(u_0)$. Indeed, if $v \in W^{1,p}(B) \setminus \{0\}$ is radial, nondecreasing and satisfies $(45)$, then for every $s \in (-\varepsilon_1, \varepsilon_1)$, $s \neq 0$, it holds
\[
h(s)(u_0 + sv) \in \mathcal{N}_* \quad \text{and} \quad I(h(s)(u_0 + sv)) < I(u_0).
\]
Furthermore, since $h(s) \in C^1((-\varepsilon_1, \varepsilon_1))$ and $h(0) = 1$,
\[
\lim_{s \to 0} \| h(s)(u_0 + sv) - u_0 \| = 0,
\]
so that for every $\varepsilon > 0$ there exists $s_{\varepsilon} \in (-\varepsilon_1, \varepsilon_1)$ such that $\| h(s_{\varepsilon})(u_0 + s_{\varepsilon}v) - u_0 \| < \varepsilon$. The statement then follows with $u_{\varepsilon} = h(s_{\varepsilon})(u_0 + s_{\varepsilon}v)$.

Lemma 4.2. Fix $0 < t_- < 1 < t_+$ such that
\[
t_- u_0 \in U_-, \quad t_+ u_0 \in U_+ \quad \text{and} \quad u_- < t_- u_0 < u_0 < t_+ u_0 < u_+,
\]
where $U_{\pm}$ are defined in $(38)$. Let $v \in W^{1,p}(B) \setminus \{0\}$ radial, nondecreasing, satisfy $(45)$. For $s \geq 0$ define
\[
\gamma_s : [t_-, t_+] \to W^{1,p}(B) \quad \gamma_s(t) := t(u_0 + sv).
\]
Then there exists $\bar{s} > 0$ such that $\gamma_{\bar{s}}(t_{\pm}) \in U_{\pm}$, $\gamma_{\bar{s}}(t) \in C_*$ for $t_- \leq t \leq t_+$ and
\[
\max_{t_- \leq t \leq t_+} I(\gamma_{\bar{s}}(t)) < I(u_0). \tag{51}
\]
Proof. Case $u_+ < \infty$. First, we notice that such $t_-$ and $t_+$ exist. Indeed, by Lemma 3.9 we know that $I(tu_0) \geq I(u_-) + \alpha$ for $t = (u_- + \tau)/u_0$. Hence, the continuity of $I$ implies that

$$\exists \ t_- \in \left( \frac{u_-}{u_0}, \frac{u_- + \tau}{u_0} \right) \text{ such that } I(t_-u_0) < I(u_-) + \frac{\alpha}{2}.$$  

The existence of $t_+$ can be proved analogously.

We claim that there exists a positive constant $s_0 \leq \varepsilon_1$ (as in Lemma 4.1), such that

$$I(\gamma_s(t)) < I(u_0) \quad \text{for all } (s, t) \in [-s_0, s_0] \times [t_-, t_+].$$  

We first observe that the function $t \mapsto I(u_0)$ has a unique strict maximum point at 1. Indeed,

$$\frac{d}{dt} I(\gamma_0(t)) = I'(tu_0)[u_0] = |B|(m(tu_0)^{p-1} - \tilde{f}(tu_0))u_0$$

and

$$m(tu_0)^{p-1} - \tilde{f}(tu_0) \begin{cases} > 0 & \text{if } t \in [t_-, 1), \\ < 0 & \text{if } t \in (1, t_+]. \end{cases}$$  

Since, being $\tilde{f}(tu_0) > m(p-1)u_0^{-2} = m(u^{p-1})|_{u=u_0}$, the inequalities hold locally near $t = 1$ and then, by (52) and the definition of $t_-$ and $t_+$, they hold in the whole intervals $[t_-, 1)$ and $(1, t_+]$, respectively. As a consequence,

$$I(\gamma_0(t)) < I(u_0) \quad \text{for all } t \in [t_-, 1) \cup (1, t_+].$$

Now, by the continuity in $s$ of the function $I(t(u_0 + sv))$, there exists $s_0 \in (0, \varepsilon_1)$ such that

$$I(\gamma_s(t)) < I(u_0) \quad \text{for all } (s, t) \in [-s_0, s_0] \times [t_-, t_+].$$  

We notice that such $t_-$ and $t_+$ exist. Indeed, by Lemma 3.9 we know that $I(tu_0) \geq I(u_-) + \alpha$ for $t = (u_- + \tau)/u_0$. Hence, the continuity of $I$ implies that

$$\exists \ t_- \in \left( \frac{u_-}{u_0}, \frac{u_- + \tau}{u_0} \right) \text{ such that } I(t_-u_0) < I(u_-) + \frac{\alpha}{2}.$$  

Therefore, for all $s \in (-\varepsilon_1, \varepsilon_1)$, $h(s)$ is the unique maximum point of the map $t \mapsto I(\gamma(s)(t))$, so that

$$I(\gamma_s(t)) \leq I(\gamma_s(h(s))) < I(u_0) \quad \text{for all } (s, t) \in V \setminus \{(0, 1)\}$$

by Lemma 4.1(iii). By (54) and (55), the claim (52) follows.

Furthermore, by (49) and since $v$ is radial and nondecreasing, we may choose $\bar{s} \in (0, s_0)$ so small that

$$\gamma(\bar{s})(t_-) = t_-(u_0 + \bar{s}v) \in U_- \quad \text{and} \quad \gamma(\bar{s})(t_+) = t_+(u_0 + \bar{s}v) \in U_+.$$  

By the convexity of $C_\varepsilon$, for all $t \in [0, 1]$

$$t\gamma(\bar{s})(t_-) + (1-t)\gamma(\bar{s})(t_+) = (u_0 + \bar{s}v)[t_+(t_- - t_+)] = \gamma(\bar{s})(t_+ + t(t_- - t_+)) \in C_\varepsilon,$$

that is $\gamma(\bar{s})(t) \in C_\varepsilon$ for all $t \in [t_-, t_+]$, and we conclude the proof in this case.

Case $u_+ = \infty$. The existence of $t_-$ follows as in the previous case, while the existence of $t_+$ is a consequence of the facts that $tu_0 - u_- > \tau$ for all $t > (u_- + \tau)/u_0$ and $I(tu_0) \to -\infty$ as $t \to +\infty$, see (44). The rest of the proof is analogous to case above, with the only change in the definition of $U_+$.  

• Proof of Theorem 1.1. By Proposition 3.10, there exists a mountain pass solution \(u \in C_* \setminus \{ u_-, u_+ \}\) of (1) such that \(I(u) = c\). Furthermore, \(u > 0\) by [30, Theorem 5]. It only remains to prove that \(u \neq u_0\). To this aim, let \(\gamma_t\) be the curve given in Lemma 4.2 and define \(\bar{C}^*_t\) and \(\bar{C}^{*+}_t\) for each \(t \in [0, 1]\). Clearly, \(\bar{C}^*_t \subseteq \bar{C}^{*+}_t\) and \(c \leq \max_{t \in [0, 1]} I(\bar{C}(t)) \leq 0\) by the previous lemma. Hence, the mountain pass solution \(u\) is different from the constant \(u_0\). Since \(u \in C_*\), and the only constant solutions of (1) in \(C_*\) are \(u_-, u_+,\) and \(u_0\), this implies in particular that \(u\) is nonconstant.

The second part of the statement is proved by reasoning in the same way for each \(u_{0,i}\), with \(i = 1, \ldots, n\). We define \(u^{(i)}_\pm\) and the cone of nonnegative, radial, nondecreasing functions \(C^{(i)}\), corresponding to each \(u_{0,i}\). In this way, for every \(i\), we get a nonconstant positive mountain pass solution \(u^{(i)}_\pm\) in \(C^{(i)}\). Hence, \(u^{(i)}_- \leq u^{(i)}_+\). Assume without loss of generality that \(u_{0,1} < u_{0,2} < \cdots < u_{0,n}\), then \(u^{(1)}_- < u^{(2)}_+ < \cdots < u^{(n)}_+\) and so the \(n\) solutions found are distinct.

• Proof of Theorem 1.2. The proof of Theorem 1.1 works also for the case \(p = 2\), with the only exception of Lemma 4.1. In order to prove this lemma, we need the stronger assumption \((g'_3)\) instead of \((g_3)\), and we can proceed as in [8, Lemma 4.9].

5. Asymptotic behavior in the pure power case. Let \(q > p > 2\). In this section we study the problem (1) for \(g(u) = u^{q-1}\), namely

\[
\begin{aligned}
-\Delta_p u + u^{p-1} &= w^{q-1} \quad \text{in } B, \\
u > 0 &= \text{in } B, \\
\partial_B u &= 0 \quad \text{on } \partial B.
\end{aligned}
\]  

(56)

By Theorem 1.1 there exists a radial nondecreasing solution of (56) for every \(q > p\). We remark that, concerning the notation in Sections 2–4, in this specific case, \(f = g, m = 1, u_0 = 1, u_- = 0, u_+ = \infty\) and \(C_* = C\). In this section we aim to find the asymptotic behavior of this solution of (56) as \(q \to \infty\).

For all \(q \geq p+1\), the functions \(f_q(s) := s^{q-1}\) belong to the same set \(\mathcal{F}\) defined in (10), with \(m = 1\) and \(\delta = M - 1\) for a fixed \(M > 1\), i.e.

\(f_q \in \mathcal{F} = \{ \varphi \in C([0, \infty)) : \varphi \text{ nonnegative, } \varphi(s) \geq Ms^{p-1} \text{ for all } s \geq M \}\), \(q \geq p+1\).

For our analysis we need an additional property (namely (57) below) on the truncated function \(\tilde{f}\) introduced in Lemma 3.1, in order to ensure it, we provide here a more explicit construction of \(\tilde{f}\).

Lemma 5.1. For every \(q \geq p+1\), there exists \(\tilde{f}_q \in \mathcal{F} \cap C^1([0, \infty))\) nondecreasing, satisfying \((g_1)-(g_3)\),

\[
\begin{aligned}
\text{fixed any } s > 0, \text{ the map } t \in (0, \infty) &\mapsto \frac{\tilde{f}_q(ts)}{p^{s-1}} \text{ is increasing}, \\
\exists \ell \in \left( p, \min \left\{ \frac{(p-1)^2 + p - 2}{p-2}, p^{s-1} \right\} \right) \text{ such that } \lim_{s \to \infty} \frac{\tilde{f}_q(s)}{s^{\ell-1}} = d,
\end{aligned}
\]  

(57)

for some \(d > 0\), and with the property that if \(u \in C\) solves

\[
\begin{aligned}
-\Delta_p u + u^{p-1} &= \tilde{f}_q(u) \quad \text{in } B, \\
u > 0 &= \text{in } B, \\
\partial_B u &= 0 \quad \text{on } \partial B,
\end{aligned}
\]  

(59)
then \( u \) solves \((56)\).

**Proof.** By Lemma 2.5, there exists \( K_\infty \) such that \( \|u\|_{L^\infty(B)} \leq K_\infty \) for every \( u \in \mathcal{C} \) solution of \((56)\). Notice that \( K_\infty \geq 1 \), because \( 1 \in \mathcal{C} \) is a solution of \((56)\) for every \( q \).

Fix \( s_0 > \max\{K_\infty, M\} \) and \( \ell \in \left(p, \min\left\{\left(\frac{(p-1)^2+p-2}{p-2}, p^*\right)\right\}\right). \) We define

\[
\hat{f}_q(s) := \begin{cases} 
  s^{q-1} & \text{if } s \in [0, s_0], \\
  s_0^{q-1} \left(\frac{q-1}{\ell-1} \right) s^{\ell-1} - s_0^{q-1} \left(\frac{q-1}{\ell-1} \right) s^{\ell-1} & \text{otherwise.}
\end{cases}
\]

(60)

It is straightforward to verify that \( \hat{f}_q \) is of class \( C^1 \), nonnegative and nondecreasing; it satisfies \((57)\) and \((58)\), with \( d = (q-1)s_0^{q-\ell}/(\ell-1) \), which implies also \((g_2)\). Since \( \hat{f}_q(s) = s^{q-1} \) in \([0, 1]\), \( \hat{f}_q \) also satisfies \((g_1)\) and \((g_3)\).

In order to prove that \( \hat{f}_q \in \hat{\mathcal{G}} \), it remains to show that

\[
\hat{f}_q(s) \geq Ms^{p-1} \quad \text{for all } s \geq M.
\]

(61)

Since \( \hat{f}_q = f_q \) in \([M, s_0] \), it is enough to verify that \((61)\) for all \( s > s_0 \). This is equivalent to show that

\[
\xi(s) := [(q-1)s_0^{q-\ell} s^{\ell-p} - M(\ell-1)]s^{p-1} \geq (q-\ell)s_0^{q-1} \quad \text{for all } s > s_0.
\]

(62)

Now, \( \xi(s_0) \geq (q-\ell)s_0^{q-1} \), being \( s_0 > M \). Moreover, \( \xi(s) \geq 0 \) for all \( s > s_0 \). Therefore, \((62)\) holds and so, by Lemma 2.5, all solutions of \((59)\) solve also \((56)\). \qed

We denote by \( \hat{F}_q \) the primitive of \( \hat{f}_q \) and by \( I_q \) the associated energy functional. We introduce the Nehari-type set

\[
\mathcal{N}_q := \left\{ u \in \mathcal{C} \setminus \{0\} : \int_B (|\nabla u|^p + |u|^p)dx = \int_B \hat{f}_q(u)udx \right\}.
\]

(63)

**Lemma 5.2.** There exists \( \sigma > 0 \) such that

\[
\inf_{q \geq p+1} \inf_{u \in \mathcal{N}_q} \|u\|_{L^\infty(B)} \geq \sigma.
\]

**Proof.** Suppose by contradiction that there exist \((q_n)\), with \( q_n \geq p+1 \) for any \( n \), and \((u_n) \subset \mathcal{N}_{q_n} \), such that \( \|u_n\|_{L^\infty(B)} \to 0 \) as \( n \to \infty \). Then, for \( n \) sufficiently large \( \hat{f}_{q_n}(u_n) = u_n^{q_n-1} \) and, being \( q_n \geq p+1 \), there exists \( \varepsilon > 0 \) such that \( \hat{f}_{q_n}(u_n)u_n = u_n^{q_n-1} \leq (1-\varepsilon)u_n^p \) for every \( n \). Therefore, since \( u_n \in \mathcal{N}_{q_n} \), for \( n \) large we get

\[
0 = \int_B (|\nabla u_n|^p + u_n^p - u_n^{q_n})dx \geq \int_B (|\nabla u_n|^p + u_n^p - (1-\varepsilon)u_n^p)dx
\]

\[
= \int_B (|\nabla u_n|^p + \varepsilon u_n^p)dx \geq 0,
\]

which is impossible, since \( 0 \notin \mathcal{N}_{q_n} \). \qed

Let \( c_q \) be the mountain pass level corresponding to \( q \) as in \((69)\), that is to say

\[
c_q = \inf_{\gamma \in \Gamma_q} \max_{t \in [0,1]} I_q(\gamma(t)),
\]

(64)

where

\[
\Gamma_q := \{ \gamma \in C([0,1];\mathcal{C}) : \gamma(0) \in U_{q,-}, \gamma(1) \in U_{q,+} \},
\]

and

\[
U_{q,-} = \left\{ u \in \mathcal{C} : I_q(u) < \frac{c_q}{2}, \|u\|_{L^\infty(B)} < \tau \right\},
\]

\[
U_{q,+} = \left\{ u \in \mathcal{C} : I_q(u) < 0, \|u\|_{L^\infty(B)} > \tau \right\},
\]

(65)
with $\tau < \min\{\sigma, 1\}$, $\sigma$ given in Lemma 5.2 and $\alpha_q$ is given as in Lemma 3.9 for $g(s) = s^{q-1}$.

We notice that property (57) is crucial for the proof of the following lemma.

**Lemma 5.3.** For every $u \in C \setminus \{0\}$ there exists a unique $h_q(u) > 0$ such that $h_q(u)u \in N_q$. It holds

$$I_q(tu) > 0 \quad \text{for all } t \in (0, h_q(u)].$$

Furthermore, if $(u_n) \subset C \setminus \{0\}$ is such that $u_n \to u \in C \setminus \{0\}$ with respect to the $W^{1,p}$-norm, then $h_q(u_n) \to h_q(u)$. Finally, the map

$$H : u \in C \cap \mathcal{S}^1 \mapsto h_q(u)u \in N_q,$$

where $\mathcal{S}^1 := \{u \in W^{1,p}(B) : \|u\| = 1\}$

is a homeomorphism.

**Proof.** Fix $u \in C \setminus \{0\}$ and consider the corresponding map $\phi : t \in [0, \infty) \mapsto I_q(tu)$. Clearly, for all $t > 0$, it results that

$$\phi'(t) = 0 \iff I_q'(tu)[u] = 0 \iff \|u\|_p = \frac{1}{tp^{1-1}} \int_B \tilde{f}_q(tu)u dx.$$  

(67)

Furthermore, $\phi(0) = 0$ and, since $1 < p < \ell < q$,

$$\phi(t) = \frac{t^p}{p} \|u\|_p^p - \frac{t^q}{q} \|u\|_{L^q(B)}^q > 0 \quad \text{for } t > 0 \text{ small,}$$

$$\phi(t) = \frac{t^p}{p} \|u\|_p^p + t^{p-\ell} s_0^{q-1} \|u\|_{L^1(B)} - t^{\ell} s_0^{q-\ell} \|u\|_{L^q(B)} < 0 \quad \text{for } t \text{ large.}$$

(68)

Therefore, by the continuity of $\phi$, there exists $h_q(u) \in (0, \infty)$ such that

$$\phi(h_q(u)) = \max_{t \in [0, \infty)} \phi(t)$$

and consequently, $\phi'(h_q(u)) = 0$. We can prove that the maximum point $h_q(u)$ is the unique non-zero critical point of $\phi$. Indeed, suppose by contradiction that $\phi$ admits another critical point $0 < \tilde{h} \neq h_q(u)$, then by (67)

$$\|u\|_p^p = \frac{1}{h_q(u)} \int_B \tilde{f}_q(hu)u dx = \frac{1}{h_q(u)} \int_B \tilde{f}_q(h_q(u)u)u dx,$$

which contradicts (57), being $\tilde{f}_q(ts)/t^{p-1}$ strictly increasing in $t$, for all $s > 0$. Thus, $h_q(u)$ is unique and (66) holds. Furthermore, $H$ is well defined.

Now, let $(u_n) \subset C \setminus \{0\}$, $u_n \to u \in C \setminus \{0\}$. Suppose by contradiction that the corresponding sequence $(h_q(u_n))$ is unbounded. Then,

$$\|u_n\|_p^p = \frac{1}{h_q(u_n)^{p-1}} \int_B \tilde{f}_q(h_q(u_n)u_n)u_n dx \not\to \infty \quad \text{as } n \to \infty,$$

by (57). This contradicts the fact that $(u_n)$ is convergent. Hence, $(h_q(u_n))$ is bounded and we can find a subsequence, still indexed by $n$, for which $h_q(u_n) \to \tilde{h}$. For the Dominated Convergence Theorem we obtain

$$\|u_n\|_p^p = \frac{1}{h_q(u_n)^{p-1}} \int_B \tilde{f}_q(h_q(u_n)u_n)u_n dx \to \frac{1}{\tilde{h}^{p-1}} \int_B \tilde{f}_q(hu)u dx.$$ 

By the uniqueness of the limit, this yields

$$\frac{1}{\tilde{h}^{p-1}} \int_B \tilde{f}_q(hu)u dx = \|u\|_p^p,$$

that is $\tilde{h} = h_q(u)$ and in particular $H$ is continuous.
Finally, the continuous map \( v \in \mathcal{N}_q \mapsto v/\|v\| \in \mathcal{C} \cap \mathcal{S}^1 \) is the inverse of \( H \), by the uniqueness of \( h_q(u) \) and by the fact that \( h_q(u) = 1 \) if and only if \( u \in \mathcal{N}_q \).

The preceding lemma allows to prove that the mountain pass level in the cone coincides with a Nehari-type level in the cone.

**Lemma 5.4.** The following equalities hold

\[
c_q = \inf_{u \in \mathcal{C}\backslash \{0\}} \sup_{t \geq 0} I_q(tu) = \inf_{u \in \mathcal{N}_q} I_q(u). \tag{69}
\]

**Proof.** We shall split the proof of (69) into three steps.

**Step 1.** We first prove that \( \inf_{u \in \mathcal{C}\backslash \{0\}} \sup_{t \geq 0} I_q(tu) = \inf_{u \in \mathcal{N}_q} I_q(u) \). From Lemma 5.3, we know that

\[
\inf_{u \in \mathcal{C}\backslash \{0\}} \sup_{t \geq 0} I_q(tu) = \inf_{u \in \mathcal{C}\backslash \{0\}} I_q(h_q(u)u) \geq \inf_{u \in \mathcal{N}_q} I_q(u)
\]

being \( h_q(u)u \in \mathcal{N}_q \). On the other hand,

\[
\inf_{u \in \mathcal{C}\backslash \{0\}} \sup_{t \geq 0} I_q(tu) \leq \inf_{u \in \mathcal{C}\backslash \{0\}} I_q(tu) = \inf_{u \in \mathcal{C}\backslash \{0\}} I_q(h_q(u)u) = \inf_{u \in \mathcal{N}_q} I_q(u),
\]

where we have used the fact that \( H \) defines a homeomorphism between \( \mathcal{C} \cap \mathcal{S}^1 \) and \( \mathcal{N}_q \).

**Step 2.** Now we prove that \( c_q \leq \inf_{u \in \mathcal{C}\backslash \{0\}} \sup_{t \geq 0} I_q(tu) \). Indeed, for all \( u \in \mathcal{C}\backslash \{0\} \), by (68), there exists \( \bar{t}_u \) so large that \( I_q(t_u u) < 0 \) and \( \|\bar{t}_u u\|_{L^\infty(B)} > \tau \). Hence, we can consider the curve \( \gamma : t \in [0,1] \mapsto t\bar{t}_u u \in \mathcal{C} \). Clearly \( \gamma \in \Gamma_q \), so that we get

\[
c_q \leq \inf_{u \in \mathcal{C}\backslash \{0\}} \max_{t \in [0,1]} I_q(t\bar{t}_u u) \leq \inf_{u \in \mathcal{C}\backslash \{0\}} \sup_{t \geq 0} I_q(t\bar{t}_u u) = \inf_{u \in \mathcal{C}\backslash \{0\}} \sup_{t \geq 0} I_q(tu).
\]

**Step 3.** Finally we show that \( c_q \geq \inf_{u \in \mathcal{N}_q} I_q(u) \). Let \( \gamma \) be any curve in \( \Gamma_q \), we claim that \( \gamma([0,1]) \cap \mathcal{N}_q \neq \emptyset \). If the claim holds true, we know that for any \( \gamma \in \Gamma_q \) there exists \( t_\gamma \in [0,1] \) such that \( \gamma(t_\gamma) \in \mathcal{N}_q \), and so we can conclude that

\[
c_q \geq \inf_{\gamma \in \Gamma_q} I_q(\gamma(t_\gamma)) \geq \inf_{u \in \mathcal{N}_q} I_q(u).
\]

It remains to prove the claim. Pick any \( \gamma \in \Gamma_q \), then \( \|\gamma(0)\|_{L^\infty(B)} < \tau < \sigma \), with \( \sigma \) given in Lemma 5.2. If \( \gamma(0) \neq 0 \), by Lemma 5.3 we know that there exists a unique \( h_q(\gamma(0)) > 0 \) such that \( h_q(\gamma(0)) \gamma(0) \in \mathcal{N}_q \). Hence, together with Lemma 5.2 we obtain

\[
\sigma h_q(\gamma(0)) > \tau h_q(\gamma(0)) > \|h_q(\gamma(0))\gamma(0)\|_{L^\infty(B)} \geq \sigma,
\]

so that \( h_q(\gamma(0)) > 1 \). If \( \gamma(0) \equiv 0 \), let \( \varepsilon \in (0,\tau) \). By the continuity of \( \gamma \) in the \( L^\infty \)-norm (see Lemma 2.2) and the fact that \( \|\gamma(1)\|_{L^\infty(B)} > \tau \), there exists \( \bar{t} \in (0,1) \) such that \( \|\gamma(\bar{t})\|_{L^\infty(B)} = \varepsilon \). Then, by Lemma 5.3, there is a unique \( h_q(\gamma(\bar{t})) \) for which \( h_q(\gamma(\bar{t})) \gamma(\bar{t}) \in \mathcal{N}_q \). Proceeding as in the case \( \gamma(0) \neq 0 \), by replacing 0 by \( \bar{t} \), we get immediately that \( h_q(\gamma(\bar{t})) > 1 \).

Furthermore, by (56), \( I_q(\gamma(1)) > 0 \) for all \( t \in (0,h_q(\gamma(1))) \). Suppose by contradiction that \( h_q(\gamma(1)) \geq 1 \), then \( I_q(\gamma(1)) > 0 \), but this is absurd, being \( \gamma \in \Gamma_q \).

In conclusion, we have that there exists \( \bar{t} \in [0,1] \) for which \( h_q(\gamma(\bar{t})) > 1 \), and that \( h_q(\gamma(1)) < 1 \). By the continuity of \( h_q \) proved in Lemma 5.3 and the continuity of \( \gamma \), there exists \( t_\gamma \in (\bar{t},1) \) for which \( h_q(\gamma(t_\gamma)) = 1 \), that is \( \gamma(t_\gamma) \in \mathcal{N}_q \).
By Theorem \[1.1\] there exists a nonconstant, nondecreasing, radial solution \(u_q\) of \(\text{(56)}\), which by Proposition \[3.10\] can be characterized as a mountain pass solution, that is to say
\[
e_c = I_q(u_q) \quad \text{and} \quad I_q'(u_q) = 0.
\]
We shall now provide some a priori bounds on \(u_q\), uniform in \(q\).

Lemma 5.5. There exists \(C > 0\) independent of \(q\) such that, for all \(q \geq p + 1\),
\[
\|u_q\|_{C^1(\bar{B})} \leq C.
\]

Proof. By integrating the equation satisfied by \(u_q\), we get
\[
\int_B u_q^{p-1}(1 - u_q^{q-p})dx = 0.
\]
Since \(u_q \neq 1\) is positive and nondecreasing, we deduce that
\[
u_q(0) < 1, \quad u_q(1) > 1 \quad \text{for all} \quad q \geq p + 1.
\]

Consider the equation satisfied by \(u_q\) in radial form. We multiply it by \(u_q' \geq 0\) to obtain
\[
\left(\frac{p-1}{p} (u_q')^p + \frac{u_q^q}{q} - \frac{u_q^p}{p}\right)' = -\frac{N-1}{r} (u_q')^p.
\]
We deduce that the function
\[
L_q(r) := \frac{p-1}{p} (u_q'(r))^p - \frac{u_q(r)^p}{p} + \frac{u_q(r)^q}{q}, \quad r \in [0,1]
\]
is nonincreasing in \(r\), and hence, using \(\text{(71)}\),
\[
L_q(r) \leq L_q(0) = -\frac{u_q(0)^p}{p} + \frac{u_q(0)^q}{q} \leq 0 \quad \text{for all} \quad r \in [0,1].
\]
We note that \(L_q(r) \leq 0\) is equivalent to
\[
(u_q(r), u_q'(r)) \in \Sigma := \left\{ (x,y) \in \mathbb{R}^2 : x \geq 0, 0 \leq y \leq \left[\frac{p}{p-1} \left(\frac{x^p}{p} - \frac{x^q}{q}\right)\right]^{1/p}\right\}.
\]
This implies (see Figure \[1\])
\[
u_q \leq \left(\frac{q}{p}\right)^{\frac{1}{q-p}} \rightarrow 1 \quad \text{as} \quad q \rightarrow \infty,
\]
\[
u_q' \leq \left(\frac{q-p}{q(p-1)}\right)^{\frac{1}{q}} \rightarrow p^{-\frac{1}{q}} \quad \text{as} \quad q \rightarrow \infty.
\]
The previous a priori bounds ensure the existence of a limit profile.

Lemma 5.6. There exists a function \(u_\infty \in \mathcal{C}\) for which
\[
u_q \rightarrow u_\infty \quad \text{in} \quad W^{1,p}(B), \quad u_q \rightarrow u_\infty \quad \text{in} \quad C^{0,\nu}(\bar{B}) \quad \text{as} \quad q \rightarrow \infty,
\]
for any \(\nu \in (0,1)\). Furthermore, \(u_\infty(1) = 1\).

Proof. The existence of \(u_\infty\) and the convergence are consequences of the previous lemma, together with the compactness of the embedding \(C^1 \hookrightarrow C^{0,\nu}\). Since, up to a subsequence, \(u_q \rightarrow u_\infty\) pointwise, we deduce that \(u_\infty \in \mathcal{C}\). From \(\text{(71)}\) we immediately get that \(u_\infty(1) \geq 1\).
It only remains to show that $u_\infty(1) = 1$. To this aim, suppose by contradiction that $u_\infty(1) > 1$. Then there exist $s \in (0, 1)$ and $\delta > 0$ such that

$$u_q(r) \geq 1 + \delta \quad \text{for every } s \leq r \leq 1,$$

and for every $q$ sufficiently large. We integrate the equation satisfied by $u_q$ in the interval $(s, 1)$

$$s^{N-1}(u_q'(s))^{p-1} = \int_s^1 u_q^{p-1}(u_q^{q-p} - 1) r^{N-1} \, dr$$

and we replace (73) to obtain

$$s^{N-1}(u_q'(s))^{p-1} \geq \int_s^1 (1 + \delta)^{p-1} ((1 + \delta)^{q-p} - 1) r^{N-1} \, dr \to +\infty$$

as $q \to \infty$, in contradiction with Lemma 5.5.

**Lemma 5.7.** The quantity

$$c_\infty := \inf \left\{ \frac{\|v\|_p}{p} : v \in \mathcal{C}, \ v = 1 \text{ on } \partial B \right\}$$

is achieved by the unique radial function $G$ satisfying (4).

**Proof.** By the Direct Method of the Calculus of Variations,

$$c'_\infty := \inf \left\{ \frac{\|v\|_p}{p} : v \in W^{1,p}(B), \ v = 1 \text{ on } \partial B \right\}$$

is achieved by any function $G$ solving (4). In particular, we can choose $G \geq 0$. By the strict convexity of $\| \cdot \|_p$, such $G$ is unique, see for example [24]. Let us prove that $c_\infty = c'_\infty$. Clearly, $c'_\infty \leq c_\infty$. On the other hand, by the strong maximum principle (see [30, Theorem 5]), $G > 0$ in $B$, and by the radial symmetry $G'(0) = 0$. If we integrate the equation in (4) in its radial form, we get

$$r^{N-1} |G'(r)|^{p-2} G''(r) = \int_0^r t^{N-1} G(t)^{p-1} \, dt > 0.$$
Hence, \( G \in \mathcal{C} \) and so \( c_\infty \leq c'_\infty \). This concludes the proof. \( \square \)

**Lemma 5.8.** It holds \( c_\infty \leq \liminf_{q \to \infty} c_q \).

**Proof.** We take the function \( u_\infty \) introduced in Lemma 5.6 as test function in the definition of \( c_\infty \) and we get for some constant \( C > 0 \) independent of \( q \),

\[
c_\infty \leq \frac{\|u_\infty\|^p}{p} \leq \liminf_{q \to \infty} \frac{\|u_q\|^p}{p} = \liminf_{q \to \infty} \left( I_q(u_q) + \int_B \frac{u_q^q}{q} \, dx \right) = \liminf_{q \to \infty} \left( c_q + \frac{\|u_q\|^p}{q} \right) \leq \liminf_{q \to \infty} \left( c_q + \frac{C}{q} \right) = \liminf_{q \to \infty} c_q,
\]

where we used (56), (70), and Lemma 5.5. \( \square \)

- **Proof of Theorem 1.3.** Let \( G \) be the unique solution of (4). Since \( G \in \mathcal{C} \setminus \{0\} \), by Lemma 5.3 there exists a unique \( h_q(G) > 0 \) such that \( h_q(G) G \in \mathcal{N}_q \). Since \( \tilde{f}_q(s) = s^{q-1} \) for \( s \leq 1 = \|G\|_{L^\infty(B)} \), we have

\[
h_q(G) = \left( \frac{\|G\|^p}{\int_B G^q \, dx} \right)^{\frac{1}{q-p}} \to \frac{1}{\|G\|_{L^\infty(B)}} = 1 \quad \text{as } q \to \infty.
\]

This implies

\[
c_\infty = \frac{\|G\|^p}{p} = \lim_{q \to \infty} \frac{\|h_q(G) G\|^p}{p} = \lim_{q \to \infty} \left( I_q(h_q(G) G) + \frac{h_q(G)^q}{q} \int_B G^q \, dx \right).
\]

Now, since \( h_q(G) G \in \mathcal{N}_q \), we can rewrite the last term as

\[
c_\infty = \lim_{q \to \infty} \left( I_q(h_q(G) G) + \frac{\|h_q(G) G\|^p}{q} \right) = \lim_{q \to \infty} I_q(h_q(G) G),
\]

by (75). Then, since \( h_q(G) G \in \mathcal{N}_q \), Lemma 5.4 implies that

\[
c_q = \inf_{u \in \mathcal{N}_q} I_q(u) \leq I_q(h_q(G) G).
\]

The previous two equations provide \( c_\infty \geq \limsup_{q \to \infty} c_q \). By combining this inequality with Lemma 5.8 we obtain that

\[
c_\infty = \lim_{q \to \infty} c_q.
\]

As a consequence, the inequalities in (74) are indeed equalities, so that

\[
\lim_{q \to \infty} \|u_q\| = \|G\| \quad \text{and} \quad \|u_\infty\| = \|G\|.
\]

Hence, \( u_\infty \) achieves \( c_\infty \) and, by Lemma 5.7, \( u_\infty = G \). Together with the \( W^{1,p} \)-weak convergence and the uniform convexity of \( W^{1,p}(B) \), this implies that \( u_q \to G \) in \( W^{1,p}(B) \). By Lemma 5.6, the convergence is also \( C^{0,\nu}(B) \) for any \( \nu \in (0,1) \). \( \square \)

**Appendix A. Some remarks in the case** \( 1 < p < 2 \). In this appendix we consider the case \( 1 < p < 2 \). We prove that Proposition 3.10 holds under an additional assumption on \( g \), that is to say, a mountain pass solution exists also in this case. Nevertheless, we do not know whether the mountain pass solution is nonconstant. In particular, we prove that Lemma 4.1(ii) does not hold for \( 1 < p < 2 \) and \( g(u) = u^{q-1} \).

We require \( g \) to be of class \( C^1((0,\infty)) \cap C([0,\infty)) \), to satisfy \((g_1)-(g_3)\) and \((g_4)\) \( \inf\{t \in (u_0, \infty) : g(t) = t^{p-1}\} \) < \( \infty \).
We remark that the assumption on the regularity of $g$ is slightly weaker than in case $p \geq 2$. This allows us to cover the nonlinearities which behaves like $s^{q-1}$ ($q > p$) near the origin.

The results in Section 2 hold also in this setting with exactly the same proofs. The only difference is that the function $f$ in Lemma 2.1 is of class $C^1([0, \infty)) \cap C([0, \infty))$ as $g$.

Furthermore, proceeding as in Lemma 3.1 we can build, also in this case, the subcritical nonlinearity $\tilde{f}$.

**Lemma A.1.** For every $\ell \in (p, p^*)$, there exists $\tilde{f} \in F$ nondecreasing, satisfying

$$\tilde{f} = f \quad \text{in} \quad [0, s_0] \quad \text{for some} \quad s_0 > \max\{K_\infty, M\}$$

($K_\infty$ as in Lemma 2.5 and $M$ as in (6)), $(f_1)$-$(f_3)$,

$$\lim_{s \to \infty} \frac{\tilde{f}(s)}{s^{\ell-1}} = 1,$$

and with the property that, if $u \in C$ solves (14), then $u$ solves (5).

The associated energy functional $I$ is defined as in (16) and is of class $C^1$.

In this setting there exists a mountain pass solution of the problem, as stated in the following proposition.

**Proposition A.2 (Mountain Pass Theorem).** Let $1 < p < 2$. Let $g \in C^1((0, \infty)) \cap C([0, \infty))$ satisfy $(g_1)$-$(g_4)$. Then the value $c$ defined in (39) is finite and there exists a critical point $u \in C_*$ of $I$ with $I(u) = c$.

The proof of Proposition A.2 relies on several preliminary results, which hold under the same assumptions.

**Proposition A.3.** The operator $\tilde{T}$ defined in (19) is compact. Furthermore, there exist two positive constants $a$, $b$ such that for all $u \in W^{1,p}(B)$ the following properties hold

$$I'(u)[u - \tilde{T}(u)] \geq a\|u - \tilde{T}(u)\|^2(\|u\| + \|\tilde{T}(u)\|)^{p-2},$$

$$\|I'(u)\|_* \leq b\|u - \tilde{T}(u)\|^{p-1},$$

(78)

**Proof.** The proof is analogous to the one of Proposition 3.2.

**Lemma A.4 (Palais-Smale condition).** $I$ satisfies the Palais-Smale condition, i.e. every sequence $(u_n) \subset W^{1,p}(B)$ such that $(I(u_n))$ is bounded and $I'(u_n) \to 0$ in $(W^{1,p}(B))^*$ admits a convergent subsequence.

**Proof.** Reasoning as in Lemma 3.3 we obtain that any (PS)-sequence $(u_n)$ is weakly converging to some $u$ in $W^{1,p}(B)$ and that (21) holds. Now, by the first inequality of (78) we get

$$\|u_n - \tilde{T}(u_n)\|^2(\|u_n\| + \|\tilde{T}(u_n)\|)^{p-2} \leq \frac{1}{a}\|I'(u_n)\|_*\|u_n - \tilde{T}(u_n)\|.$$

Hence, being $(u_n)$ bounded and $\tilde{T}$ compact (see Proposition A.3), we have

$$\|u_n - \tilde{T}(u_n)\| \leq \frac{1}{a}\|I'(u_n)\|_*\|u_n\| + \|\tilde{T}(u_n)\|^{2-p} \to 0.$$

We conclude that $u_n \to \tilde{T}(u) = u$ in $W^{1,p}(B)$.

**Lemma 3.4** holds for all $1 < p < \infty$, hence also in this case the operator $\tilde{T}$ preserves the cone $C_*$ defined in (24).
Lemma A.5 (Locally Lipschitz vector field). Let $W := W^{1,p}(B) \setminus \{u : \bar{T}(u) = u\}$. There exists a locally Lipschitz continuous operator $K : W \to W^{1,p}(B)$ satisfying the following properties:

(i) $K(C_\ast \cap W) \subset C_\ast$;
(ii) $\frac{1}{2} \|u - K(u)\| \leq \|u - \bar{T}(u)\| \leq 2\|u - K(u)\|$ for all $u \in W$;
(iii) let $\alpha > 0$ be the constant given in Proposition A.3 then

$$I'(u)[u - K(u)] \geq \frac{\alpha}{2} \|u - \bar{T}(u)\|^2 (\|u\| + \|\bar{T}(u)\|)^{p-2} \quad \text{for all } u \in W.$$

Proof. By (78) it is possible to proceed as in Lemma 2.1, with $D^+ := C_\ast$ and $D^- := \emptyset$.

Lemma A.6. Let $c \in \mathbb{R}$ be such that $I'(u) \neq 0$ for all $u \in C_\ast$ with $I(u) = c$. Then there exist two positive constants $\overline{\varepsilon}$ and $\delta$ such that the following inequalities hold:

(i) $\|I'(u)\| \geq \overline{\varepsilon}$ for all $u \in C_\ast$ with $|I(u) - c| \leq 2\overline{\varepsilon}$;
(ii) $\|u - K(u)\| \geq \delta$ for all $u \in C_\ast$ with $|I(u) - c| \leq 2\overline{\varepsilon}$.

Proof. The proof of part (i) is analogous to the one given in Lemma 3.7. We prove now (ii). Let

$$I_{c-\overline{\varepsilon}}^{\overline{\varepsilon}} := \{u \in C_\ast : |I(u) - c| \leq 2\overline{\varepsilon}\}.$$

By the part (i), $I_{c-\overline{\varepsilon}}^{\overline{\varepsilon}} \subset W$, where $W$ is defined in Lemma A.5. Furthermore, for all $u \in I_{c-\overline{\varepsilon}}^{\overline{\varepsilon}}$, $\|u - K(u)\| \geq \frac{1}{2} \|u - \bar{T}(u)\|$ by Lemma A.5(ii). Now, by the second inequality of (78) and by the (i) part of the present lemma, we have for all $u \in I_{c-\overline{\varepsilon}}^{\overline{\varepsilon}}$

$$\|u - \bar{T}(u)\| \geq \left(\frac{\|I'(u)\|}{b}\right)^{\frac{1}{p-1}} \geq \left(\frac{\delta}{b}\right)^{\frac{1}{p-1}}.$$

Hence, $\|u - K(u)\| \geq \min \left\{\frac{\delta}{2}, \left(\frac{\delta}{\overline{\varepsilon}}\right)^{\frac{1}{p-1}}\right\}$, still denoted by $\delta$.

Lemma A.7. Let $c \in \mathbb{R}$. The set

$$\{\|u\| : u \in C_\ast \text{ and } I(u) \leq c\}$$

is bounded by a constant depending only on $c$.

Proof. Let $u \in C_\ast$, then $u \leq u_+$, where $u_+$ is defined in (22). Since the function $\tilde{f}$ introduced in Lemma A.1 belongs to $\mathfrak{F}$, we have

$$u_+ = \inf \{t \in (u_0, \infty) : g(t) = t^{p-1}\} < \infty$$

by (g4). If in addition $I(u) \leq c$, relation (i) provides

$$\frac{\|u\|^p}{p} \leq c + C \int_B (u + u^t)dx \leq c + C|B|(u_++u_+^t).$$

Lemma A.8 (Descending flow argument). Let $c \in \mathbb{R}$ be such that $I'(u) \neq 0$ for all $u \in C_\ast$ with $I(u) = c$. Then there exists a function $\eta : C_\ast \to C_\ast$ satisfying the following properties:

(i) $\eta$ is continuous with respect to the topology of $W^{1,p}(B)$;
(ii) $I(\eta(u)) \leq I(u)$ for all $u \in C_\ast$;
(iii) $I(\eta(u)) \leq c - \overline{\varepsilon}$ for all $u \in C_\ast$ such that $|I(u) - c| < \overline{\varepsilon}$;
(iv) $\eta(u) = u$ for all $u \in C_\ast$ such that $|I(u) - c| > 2\overline{\varepsilon}$,

where $\overline{\varepsilon}$ is the positive constant given by Lemma A.6.
Proof. We define \( \eta(t, u) \) as in the first part of the proof of Lemma A.4. For all \( u \in C_* \) and \( t > 0 \) we can write
\[
I(\eta(t, u)) - I(u) = \int_0^t \frac{ds}{ds} I(\eta(s, u)) ds
\]
\[
= - \int_0^t \chi_1(I(\eta(s, u))) \chi_2(\eta(s, u)) \left( \|\eta(s, u) - K(\eta(s, u))\| \right) ds
\]
\[
\leq - \frac{a}{2} \int_0^t \|\eta(s, u) - T(\eta(s, u))\|^2 \chi_1(I(\eta(s, u))) \chi_2(\eta(s, u)) ds
\]
\[
\leq \frac{a}{2} \int_0^t \|\eta(s, u) - K(\eta(s, u))\|^2 \chi_1(I(\eta(s, u))) \chi_2(\eta(s, u)) ds
\]
\[
\leq \frac{a}{2} \int_0^t \|\eta(s, u) - K(\eta(s, u))\|^2 \chi_1(I(\eta(s, u))) \chi_2(\eta(s, u)) ds
\]
where we have used the inequality in Lemma A.5-(iii).

Now, let \( u \in C_* \) be such that \( I(u) - c \) is not constant. In particular, Proposition A.9 below implies that \( I(\eta(t, u)) \leq c - \varepsilon \), where \( \varepsilon > 0 \) is arbitrary.

Moreover, being \( \eta(s, u) \in C_* \), for every \( s \in [0, t] \), Lemmas A.7 and A.3 provide the existence of a constant \( \tilde{C} \) such that \( \|\eta(s, u)\| \leq \tilde{C} \) for all \( s \in [0, t] \). Hence, by (79) and Lemma A.5-(ii) we obtain
\[
I(\eta(t, u)) \leq I(u) - \frac{a\tilde{C}^2 - p}{a\tilde{C}}
\]
so that \( I(\eta(t, u)) \leq c - \varepsilon \) for
\[
t \geq \frac{16\tilde{C}^2 - p}{a\tilde{C}}.
\]

Finally, if we define with abuse of notation
\[
\eta(u) := \left( \frac{16\tilde{C}^2 - p}{a\tilde{C}}, u \right),
\]
we have proved that \( \eta \) satisfies (ii) and (iii). Properties (i) and (iv) are immediate. Hence, by Lemma A.4, we proved that \( \eta \) preserves the cone can be proved as in Lemma A.4, since \( T(C_*) \subset C_* \) also for \( 1 < p < 2 \).

• Proof of Proposition A.2. The preliminary results shown in this appendix allow us to prove Proposition A.2 by proceeding as in the proof of Proposition A.10.

In the case \( 1 < p < 2 \), we cannot conclude that the mountain pass solution found in Proposition A.2 is nonconstant. In particular, Proposition A.9 below implies that Lemma A.10 does not hold for \( 1 < p < 2 \) and \( g(u) = u^{q-1} \).

Since we are in the pure power case, we refer to the truncated nonlinearity \( \tilde{f} \) defined in (60), namely
\[
\tilde{f}(s) := \begin{cases} 
 s^{q-1} & \text{if } s \in [0, s_0], \\
 s^{q-1} + \frac{q-1}{\ell-1}s^{-\ell}(s^{\ell-1} - s_0^{\ell-1}) & \text{otherwise}.
\end{cases}
\]
for some fixed \( s_0 > \max\{K_\infty, M\} \) and \( \ell \in (p, p^*) \). We introduce the Nehari manifold
\[
\mathcal{N} := \{ u \in W^{1,p}(\Omega) \setminus \{0\} : I'(u)[u] = 0 \}.
\]
Proposition A.9. Let $1 < p < 2$ and $g(u) = u^{q-1}$, $q > p$. For every nonconstant $v \in W^{1,p}(B) \cap L^\infty(B)$ there exists $\varepsilon_1(v) > 0$ such that the following properties hold for any $0 < s < \varepsilon_1(v)$:

(i) there exists a unique $h(s) > 0$ such that $h(s)(1 + sv) \in \mathcal{N}$;
(ii) $I(h(s)(1 + sv)) - I(1) > 0$.

Proof. Let $v \in W^{1,p}(B) \cap L^\infty(B)$ be nonconstant. Since $\tilde{f} \equiv g$ in $[0, s_0]$, from the definition of $\mathcal{N}$ we can compute explicitly

$$h(s) = \left( \frac{\int_B (\sum |\nabla v|^p + |1 + sv|^p)dx}{\int_B |1 + sv|^q dx} \right)^\frac{1}{q-p}$$

for every $0 < s < \varepsilon_1(v) = \frac{s_0 - 1}{\|v\|_{L^\infty(B)}}$.

Hence we see that $h(s)$ is unique and regular in $s$. Therefore the proof of (i) is concluded.

In order to prove (ii), we write the Taylor expansion at the first order of $h$

$$h(s) = 1 + h'(0)s + o(s).$$

By explicit calculations, we arrive at

$$I(h(s)(1 + sv)) - I(1) = \frac{s^p}{p} \int_B |\nabla v|^p dx + o(s).$$

Since $v$ is nonconstant, the statement follows.

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