GENERALIZATIONS OF WAVE EQUATIONS TO MULTIDIMENSIONAL VARIATIONAL PROBLEMS AND THEIR PHYSICAL “SOLUTION”

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Abstract. This is a survey paper based on previous results of the author. In the paper, we define and discuss the generalizations of linear partial differential equations to multidimensional variational problems. Theory of such generalized equations is a new area of mathematics. We consider two examples of such equations: first, the generalized Schrödinger equation which is a natural candidate for the mathematical equation of quantum field theory, and second, the quantum Plato problem which is a natural candidate for a simplest mathematical equation of string theory and, more generally, theory of D-branes. We propose a way to give a mathematical sense to these equations. We also consider a more “physical” approach to the generalized Schrödinger equation which leads to mathematical renormalization beyond the scope of perturbation theory and finally to usual perturbative quantum field theory.

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The purpose of this paper is to introduce the reader to a new area of mathematical physics, called theory of generalizations of wave equations to multidimensional variational problems. This theory arose in connection with numerous attempts of scientists to give a rigorous mathematical sense to constructions of quantum field theory, string theory and, more generally, theory of $D$-branes. Physical intuition behind these areas of physics already gave rise to many remarkable mathematical results, and so providing a mathematical background to these disciplines can be very useful for mathematics as well as for physics.

The formal derivation of the generalized wave equations (see §1 below) is a direct generalization of formal derivation of equations of quantum mechanics and wave optics (the Schrödinger equation and the wave equation) from the equations of classical mechanics and geometric optics (the Hamilton–Jacobi equation and the eikonal equation). The natural examples of generalized wave equations, called the generalized Schrödinger equation and the quantum Plato problem, are given by interesting formulas including determinants.

However, giving a mathematical sense to the generalized wave equations meets problems. It is unclear what are solutions of these equations. The situation here is similar to the situation at the birth of the theory of partial differential equations and the related famous discussion of scientists in the XVIII century on what is the mathematical sense of the notion of function. The generalized wave equations are linear differential equations on infinite dimensional function spaces. But the infinite dimensional differential operators in these equations, as a rule, cannot be multiplied. In order to give a mathematical sense to these equations, one can try to use regularization of the corresponding differential operators. The regularized functional differential equations have a “naive” sense as equations for unknown functionals. However, in such a naive approach, removing regularization leads to divergencies. A way to overcome this difficulty essentially worked out by physicists is to subtract the divergent part of a solution, and consider the remaining finite part; this is heuristically equivalent to adding an infinite summand to the differential operator (see §2 below). Thus, in fact, physicists refuse to consider the initial equation, adding to it infinite “counterterms”. This construction was usually justified by the argument that in the framework of perturbation theory with respect to the interaction constant, the differential operator for zero interaction and its corrections in perturbation theory have no physical sense, and hence may be not considered. However, in our approach to the equations, there is no
perturbation theory at all, and one considers directly the theory with interaction. Therefore this physical argument loses sense, although subtracting the divergent part can be given a mathematically rigorous formulation (see §2).

The construction of subtracting the divergent part of a solution reminds the well known construction due to Hadamard, who constructed fundamental solutions of hyperbolic equations using regularization of integration domain and subtracting the divergent part from the regularized integral. It is well known that this construction of Hadamard (together with other constructions) lead to the birth of theory of distributions.

Thus, one can conclude that generalized wave equations should be considered not as equations for unknown functionals but as equations for other objects — appropriately defined distributions on infinite dimensional function spaces.

The paper consists of three Sections. §1 is devoted to formal derivation of generalized wave equations. In §2 we naively derive the well known functional differential Schrödinger equation from the generalized Schrödinger equation defined in §1, and construct the renormalized evolution operator of the Schrödinger functional differential equation. This construction leads to usual perturbative quantum field theory. Finally, in §3 we give an abstract definition of solutions of generalized wave equations, and discuss the arising problem of constructing appropriate space of distributions on infinite dimensional function space.

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1. Formal derivation of generalized wave equations

This construction first appeared in the papers [1,2,3] and in the book [4] which can serve as a general introduction to the ideas of the present paper. The generalized wave equations appeared in these references as purely formal ones, without definition of their solutions.

1.1. Generalized Hamilton–Jacobi equation. Consider the action functional of the form

\[ J = \int_D F(x^0, \ldots, x^n, \varphi^1, \ldots, \varphi^m, \varphi_x^1, \ldots, \varphi_x^m) dx^0 \ldots dx^n, \]

where \( x = (x^0, \ldots, x^n) \) is a space-time point, \( \varphi^i = \varphi^i(x) \) are real smooth field functions, \( \varphi_x^i = \partial \varphi^i / \partial x^j \), and integration goes over an \((n + 1)\)-dimensional smooth surface \( D \) (the graph of the functions \( \varphi(x) = (\varphi^i(x)) \)) with the boundary \( \partial D = C \) in the space \( \mathbb{R}^{m+n+1} \).
We shall need the well known formula for variation of action on an extremal surface with moving boundary, which is used, for instance, in derivation of the Noether theorem. Let us recall this formula. Let

\[ x = x(s), \quad \varphi = \varphi(s), \quad s = (s^1, \ldots, s^n) \]

be a local parameterization of the boundary surface \( C \). Assume that for each \( n \)-dimensional surface \( C \) sufficiently close to certain fixed surface, there exists a unique \((n + 1)\)-dimensional surface \( D \) with the boundary \( \partial D = C \) which is an extremal of the variational principle (1), i.e., the graph of a solution of the Euler–Lagrange equations. Denote by \( S = S(C) \) the value of integral (1) over the surface \( D \). Then one has the following formula for variation of the functional \( S(C) \):

\[ \delta S = \int_C \left( \sum \pi_i(s) \delta \varphi^i(s) - \sum H_j(s) \delta x^j(s) \right) ds, \]

where

\[ \pi_i = \sum_l F_{\varphi^i x^l} (-1)^l \frac{\partial (x^0, \ldots, \hat{x}^l, \ldots, x^n)}{\partial (s^1, \ldots, s^n)}, \]

\[ H_j = \sum_{l \neq j} \left( \sum_i F_{\varphi^i x^j \varphi^i x^l} \right) (-1)^l \frac{\partial (x^0, \ldots, \hat{x}^l, \ldots, x^n)}{\partial (s^1, \ldots, s^n)} + \left( \sum_i F_{\varphi^i x^j \varphi^i x^j} - F \right) (-1)^j \frac{\partial (x^0, \ldots, \hat{x}^j, \ldots, x^n)}{\partial (s^1, \ldots, s^n)}. \]

Here \( \frac{\partial (x^1, \ldots, x^n)}{\partial (s^1, \ldots, s^n)} \) is the Jacobian, the hat over a variable means that the variable is omitted, and \( F_{\varphi^i x^j} \) is the partial derivative of the function \( F \) with respect to its argument \( \varphi^i x^j \).

Let us derive the equations satisfied by the functional \( S(C) = S(\varphi^i(s), x^j(s)) \).

To this end, note that by formula (4), the quantities

\[ \pi_i(s) = \frac{\delta S}{\delta \varphi^i(s)}, \quad H_j(s) = - \frac{\delta S}{\delta x^j(s)} \]

depend not only on the functions \( \varphi(s), x(s) \), but also on the derivatives \( \varphi^i x^j \) characterizing the tangent plane to the surface \( D \) at the point \( (x(s), \varphi(s)) \). These \( mn(n + 1) \) derivatives are related by \( mn \) equations

\[ \sum_j \varphi^i_{x^j} x^j_k = \varphi^i_{x^k}, \quad i = 1, \ldots, m, \quad k = 1, \ldots, n. \]
Therefore, $m + n + 1$ quantities (5) depend on $m(n + 1) - mn = m$
free parameters. Hence they are related by $n + 1$ equations. $n$ of these
equations are easy to find:

(7) \[ \sum_i \pi_i \varphi_{sk} - \sum_j H_j x_{sk}^j = 0, \quad k = 1, \ldots, n. \]

These equations express the fact that the value of the functional $S(C)$
does not depend on a parameterization of the surface $C$.

The remaining $(n + 1)$-th equation depends on the form of the func-
tion $F$. Denote it by

(8) \[ \mathcal{H} \left( x^j(s), \varphi^i(s), x_{sk}^j(s), \varphi_{sk}^j(s), \pi_i(s), -H_j(s) \right) = 0. \]

From equalities (5) we obtain the following system of equations on the
functional $S(C)$, called the generalized Hamilton–Jacobi equation (or
the generalized eikonal equation):

(9) \[ \sum_i \frac{\delta S}{\delta \varphi^i(s)} \varphi_{sk}^i + \sum_j \frac{\delta S}{\delta x^j(s)} x_{sk}^j = 0, \quad k = 1, \ldots, n, \]

\[ \mathcal{H} \left( x^j(s), \varphi^i(s), x_{sk}^j(s), \varphi_{sk}^j(s), \frac{\delta S}{\delta \varphi^i(s)}, \frac{\delta S}{\delta x^j(s)} \right) = 0. \]

Examples.

1) (Scalar field with self-action) [3,4] Let $m = 1$ and

(10) \[ F(x^j, \varphi, \varphi_{x^j}) = \frac{1}{2} \left( \varphi_{x^j}^2 - \sum_{j \neq 0} \varphi_{x^j}^2 \right) - V(x, \varphi), \]

\[ = \frac{1}{2} \varphi_{x^j} \varphi_{x^j} - V(x, \varphi), \]

where we have introduced Greek indices $\mu$ instead of $j$, and raising and
lowering indices goes using the Lorentz metric

\[ dx^2 = (dx^0)^2 - \sum_{j \neq 0} (dx^j)^2. \]

Then the generalized Hamilton–Jacobi equation has the following form:

(11) \[ x_{sk}^\mu \frac{\delta S}{\delta x^\mu(s)} + \varphi_{sk} \frac{\delta S}{\delta \varphi(s)} = 0, \quad k = 1, \ldots, n, \]

\[ D_{\mu} \frac{\delta S}{\delta x^\mu(s)} + \frac{1}{2} \left( \frac{\delta S}{\delta \varphi(s)} \right)^2 + D_{\mu} D_{\mu} \left( \frac{1}{2} d \varphi(s)^2 + V(x(s), \varphi(s)) \right) = 0, \]
where
\[ D^\mu = (-1)^\mu \partial(x^0, \ldots, \hat{x}^\mu, \ldots, x^n), \]

and \( d\varphi(s)^2 \) is the scalar square of the differential \( d\varphi(s) \) of the function \( \varphi(s) \) on the surface.

2) (The Plato problem) [4] Assume that one considers \((n + 1)\)-dimensional surfaces \( D \) with the boundary \( C : x = x(s) \) in the Euclidean space \( \mathbb{R}^N \) with coordinates \((x^1, \ldots, x^N)\), and the role of integral (1) is played by the area of the surface \( D \). Then the generalized eikonal equation reads

\[
\sum_i x^i_j \frac{\delta S}{\delta x_j(s)} = 0, \quad 1 \leq k \leq n, \tag{12}
\]

\[
\sum_j \left( \frac{\delta S}{\delta x_j(s)} \right)^2 = \sum_{j_1 < \ldots < j_n} \left( \frac{\partial(x^{j_1}, \ldots, x^{j_n})}{\partial(s^1, \ldots, s^n)} \right)^2.
\]

In particular, in the case \( n = 1, N = 3 \), the equation reads

\[
x_s \frac{\delta S}{\delta x(s)} + y_s \frac{\delta S}{\delta y(s)} + z_s \frac{\delta S}{\delta z(s)} = 0, \tag{13}
\]

\[
\left( \frac{\delta S}{\delta x(s)} \right)^2 + \left( \frac{\delta S}{\delta y(s)} \right)^2 + \left( \frac{\delta S}{\delta z(s)} \right)^2 = x_s^2 + y_s^2 + z_s^2.
\]

The generalized Hamilton–Jacobi equation was first systematically studied, for two-dimensional variational problems, in the book [5]. Regarding the general theory of integration of this equation and, in particular, regarding the characteristics equations which are the generally covariant generalized Hamilton canonical equations equivalent to the Euler–Lagrange equations, see [2, 4].

Let us note that in the case of one-dimensional variational problems, integration of the Hamilton–Jacobi equation is the most powerful method of integration of the canonical Hamilton equations. It would be interesting to check whether one can integrate the Euler–Lagrange equations for some multidimensional variational problems (nonlinear partial differential equations, for example, the Einstein or Yang–Mills equations) using integration of the corresponding generalized Hamilton–Jacobi equations.

1.2. **Generalized wave equations.** Generalized wave equations are obtained from the generalized Hamilton–Jacobi equation (9) by formal
substitution

\[
\frac{\delta S}{\delta x^j(s)} \rightarrow -i\hbar \frac{\delta}{\delta x^j(s)},
\]
\[
\frac{\delta S}{\delta \varphi^i(s)} \rightarrow -i\hbar \frac{\delta}{\delta \varphi^i(s)}.
\]

We obtain the following system of linear functional differential equations:

\[
\sum_i \varphi^i_{,k} \frac{\delta \Psi}{\delta \varphi^i(s)} + \sum_j x^j_{,k} \frac{\delta \Psi}{\delta x^j(s)} = 0, \quad k = 1, \ldots, n,
\]
\[
H \left( x^j(s), \varphi^i(s), x^j_{,k}(s), \varphi^i_{,k}(s), -i\hbar \frac{\delta}{\delta \varphi^i(s)}, -i\hbar \frac{\delta}{\delta x^j(s)} \right) \Psi = 0.
\]

The first \( n \) of these equations express independence of the “functional” \( \Psi(C) \) on a parameterization of the surface \( C : x = x(s), \varphi = \varphi(s) \).

**Examples.**

1) In the case of scalar field with self-action (see Subsect. 1.1), we have the following generalized wave equation which we call the *generalized Schrödinger equation*:

\[
x^\mu_{,k} \frac{\delta \Psi}{\delta x^\mu(s)} + \varphi^i_{,k} \frac{\delta \Psi}{\delta \varphi^i(s)} = 0, \quad k = 1, \ldots, n,
\]
\[
-\hbar \frac{\delta^2 \Psi}{2 \delta \varphi^i(s)^2} + D^\mu D_\mu \left( \frac{1}{2} d \varphi^i(s)^2 + V(x(s), \varphi(s)) \right) \Psi.
\]

2) In the case of the Plato problem, we obtain the following generalized wave equation which we call the *quantum Plato problem*:

\[
\sum_j x^j_{,k} \frac{\delta \Psi}{\delta x^j(s)} = 0, \quad 1 \leq k \leq n,
\]
\[
\sum_j -\hbar \frac{\delta^2 \Psi}{\delta x^j(s)^2} = \sum_{j_1 < \ldots < j_n} \left( \frac{\partial (x^{j_1}, \ldots, x^{j_n})}{\partial (s^1, \ldots, s^n)} \right)^2 \Psi.
\]

In particular, in the case \( n = 1, N = 3 \), the equation reads

\[
x_s \frac{\delta \Psi}{\delta x(s)} + y_s \frac{\delta \Psi}{\delta y(s)} + z_s \frac{\delta \Psi}{\delta z(s)} = 0,
\]
\[
-\hbar^2 \left( \frac{\delta^2 \Psi}{\delta x(s)^2} + \frac{\delta^2 \Psi}{\delta y(s)^2} + \frac{\delta^2 \Psi}{\delta z(s)^2} \right) = (x_s^2 + y_s^2 + z_s^2) \Psi.
\]
2. The physical “solution”

The material of this Section is revised mostly from the paper [6], obtained by development of beautiful Connes–Kreimer’s paper [7].

2.1. Functional differential Schrödinger equation. Let us consider the question on what are solutions of generalized wave equations, for example, of the generalized Schrödinger equation (16). Let us first suppose that solutions are usual complex valued functions \( \Psi(x^j(s), \varphi_i(s)) \). Then we can restrict the solutions to the family of surfaces of the form

\[
    x^0 = t, \quad x^j = s^j \text{ for } j = 1, \ldots, n.
\]

On these surfaces, the system (16) turns into a single equation for a functional \( \Psi(t, \varphi(x)) \), \( x = (x^1, \ldots, x^n) \), called the functional differential Schrödinger equation:

\[
    i\hbar \frac{\partial \Psi}{\partial t} = \widehat{H}(t) \Psi,
\]

where

\[
    \widehat{H}(t) = H(t, \hat{\varphi}, \hat{\pi}), \quad \hat{\varphi}(x) = \varphi(x), \quad \hat{\pi}(x) = -i\hbar \frac{\delta}{\delta \varphi(x)}
\]

is the functional differential operator corresponding to the classical Hamiltonian

\[
    H(t, \varphi, \pi) = \int \left( \frac{1}{2} \pi(x)^2 + \frac{1}{2} \sum_{j=1}^{n} \varphi_{xj}(x)^2 + V(t, x, \varphi(x)) \right) dx.
\]

It will be convenient to generalize the problem a little, which will not change the essential points of the solution: consider equation (20, 21) with an arbitrary local classical Hamiltonian:

\[
    H(t, \varphi, \pi) = \int \mathcal{F}(t, x, \varphi(x), \pi(x), \ldots) dx,
\]

where the Hamiltonian density \( \mathcal{F} \) is an arbitrary polynomial of \( \varphi(x) \), \( \pi(x) \), and finite number of their derivatives at the point \( x \), smoothly depending on \( t \) and \( x \). Here firstly the following question arises: in what order should one put the non-commuting operators \( \hat{\varphi}(x), \hat{\pi}(x) \) in the quantum Hamiltonian (21). Let us adopt the convention (which is inessential for the forthcoming constructions) that all the operators \( \hat{\pi}(x) \) stand to the right of the operators \( \hat{\varphi}(x) \).
2.2. The evolution operator and its graph decomposition. Let us decompose the Hamiltonian \( H(t) \) into its homogeneous components:

\[
H(\varphi, \pi) = \sum_{n=0}^{N} H_n(\varphi, \pi), \quad H_n(\varphi, \pi) = \sum_{k+l=n} H_{k,l}(\varphi, \pi),
\]

(24)

\[
H_{k,l}(\varphi, \pi) = \frac{1}{k!l!} \int a_{k,l}(x_1, \ldots, x_k; x'_1, \ldots, x'_l) \times 
\varphi(x_1) \ldots \varphi(x_k) \pi(x'_1) \ldots \pi(x'_l) \, dx_1 \ldots dx_k \, dx'_1 \ldots dx'_l.
\]

Here \( a_{k,l} \) are certain distributions of the sets of variables \( x_1, \ldots, x_k; x'_1, \ldots, x'_l \), symmetric in the variables \( x_i \) and in \( x'_j \), called the coefficient functions of the Hamiltonian. If all these functions are smooth functions rapidly decreasing at infinity, i.e. elements of the Schwartz space, then the Hamiltonian is called *regular*. In contrast, the Hamiltonian (23) is called local, it is irregular.

Let us introduce the following notation: \( V \) is the Schwartz space of functions \((\varphi, \pi)\); \( V' \) is the dual space of distributions; \( SV \) is the space of regular Hamiltonians (the topological symmetric algebra of the space \( V \)); \( SV' \) is the space of all polynomial Hamiltonians (polynomial functionals on the space \( V \), i.e., the topological symmetric algebra of the space \( V' \)).

Let us first assume that our Hamiltonians are regular.

Then one has the following usual product formula for symbols of functional differential operators:

\[
H^{(1)} * H^{(2)}(\varphi, \pi) = \exp i\hbar \int \frac{\delta}{\delta \pi_1(x)} \frac{\delta}{\delta \varphi_2(x)} \, dx \times 
H^{(1)}(\varphi_1, \pi_1) H^{(2)}(\varphi_2, \pi_2) |_{\varphi_1=\varphi_2=\pi_1=\pi_2=\pi}.
\]

(25)

The evolution operator \( U = U(T_0, T_1) \) of equation (20) from the time moment \( t = T_0 \) till the time moment \( t = T_1 \), is given by the well known chronological exponent:

\[
U = T \exp \int_{T_0}^{T_1} \frac{1}{i\hbar} \hat{H}(t) \, dt
\]

(26)

\[
U \overset{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{k!(i\hbar)^k} \int_{T_0 \leq t_1, \ldots, t_k \leq T_1} T \hat{H}(t_1) \cdots \hat{H}(t_k) \, dt_1 \cdots dt_k.
\]

Here \( T \hat{H}(t_1) \cdots \hat{H}(t_k) \) denotes the chronological product, i.e. the product of the operators \( \hat{H}(t_i) \) in the decreasing order of time moments \( t_i \).
Let us decompose the chronological exponent (26) over the homogeneous components of the Hamiltonians $H(t)$. It is not difficult to see that one obtains certain sum over graphs. More precisely, consider the graphs $\Gamma$ consisting of several vertices and edges. An edge of the graph can join two vertices (an internal edge), or go from a vertex outside (an external edge). Multiple edges are allowed. To each such graph with $k$ vertices one assigns a summand of the sum (26) by the following rule. Let us assign numbers $1, \ldots, k$ to the vertices of the graph $\Gamma$ (the summand of the sum (26) will not depend on this numeration). Let us assign to the $i$-th vertex the homogeneous component $H_{n_i}(t_i)$ of degree $n_i$ of the Hamiltonian $H(t_i)$, where $n_i$ is the number of edges (internal and external) going to the vertex $i$. To each internal edge joining the vertices $i$ and $j$, let us assign the operator

$$
\int \frac{\delta}{\delta \pi_1(x)} \frac{\delta}{\delta \varphi_2(x)} \, dx,
$$

called the pairing, applied to the two Hamiltonians $H_{n_i}(t_i)$ and $H_{n_j}(t_j)$, the first Hamiltonian being the one with the greater value of time $t$, and the second Hamiltonian being the one with the lower value of $t$. After that, let us multiply all the homogeneous Hamiltonians, integrate over $t_1, \ldots, t_k$, and multiply by the constant responsible for combinatorics of the graph and for the powers of $i\hbar$. We obtain the summand $U(\Gamma)$ in the sum (26), so that

$$
U = \sum_{\Gamma} U(\Gamma).
$$

2.3. Mathematical renormalization. We need to extend these formulas to the case of irregular local Hamiltonian (23). It is not difficult to see that in this case, the summand $U(\Gamma) \in SV'$ makes sense for the graphs $\Gamma$ without loops, or, in other words, for disjoint unions of tree graphs. And for the graphs with loops, divergencies arise.

To overcome this difficulty, let us regularize the Hamiltonian $H(t)$, i.e. replace it by a regular Hamiltonian $H_\varepsilon(t)$, $\varepsilon > 0$, which tends to $H(t)$ as $\varepsilon \to 0$. Then the regularized evolution operator $U_\varepsilon$ is well defined for $\varepsilon > 0$, but diverges as $\varepsilon \to 0$. Let us make the following assumption on the Hamiltonian $H(t)$ and the regularization: assume that for each graph $\Gamma$, the divergent part of the operator $U_\varepsilon(\Gamma)$ is a polynomial in $\varepsilon^{-1}$ and $\log \varepsilon$ without constant term. Thus, $U_\varepsilon(\Gamma)$ is decomposed into an asymptotic series in $\varepsilon$, belonging to the algebra $A = \mathbb{C}[[\varepsilon, \varepsilon^{-1}, \log \varepsilon]]$. (Here the double square brackets mean a formal power series in $\varepsilon, \varepsilon^{-1}$, and $\log \varepsilon$, which is a polynomial in $\varepsilon^{-1}$ and $\log \varepsilon$.) This algebra decomposes into the direct sum of two subalgebras:
\[ \mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-, \text{ where } \mathcal{A}_+ = \mathbb{C} + \varepsilon \mathbb{C}[\varepsilon, \log \varepsilon] \text{ and } \mathcal{A}_- = \varepsilon^{-1} \mathbb{C}[\varepsilon^{-1}, \log \varepsilon] + \log \varepsilon \mathbb{C}[\varepsilon^{-1}, \log \varepsilon]. \] Denote by \( T : \mathcal{A} \to \mathcal{A}_- \) the projection with the kernel \( \mathcal{A}_+ \) (extracting the divergent part of a series).

The procedure of extraction of finite part from the operators \( U_\varepsilon(\Gamma) \), called the renormalization procedure, was developed, in the framework of quantum field theory, by Bogoliubov, Parasiuk (see, for instance, the book [8] and references therein), Hepp [9], and others. This procedure has been related with the Birkhoff decomposition by Connes and Kreimer [7], and given a final mathematical form (in particular, taken out of the scope of perturbation theory) by the author [6]. A “drawback” of this procedure from the point of view of the present paper, is that it does not solve the Schrödinger equation (20). Actually it is not difficult to see that this procedure is heuristically equivalent to adding an “infinite summand” to the Hamiltonian, so that the evolution operator becomes finite. On the other hand, this procedure is equivalent, as pointed out in [6], to renormalization in perturbative quantum field theory, and hence it has a physical sense. If one adopts the optimistic assumption that theory of differential equations of quantum field theory exists, then it must be related with this procedure. The question on giving it a mathematical sense in the framework of theory of differential equations, will be discussed in §3.

Let us proceed to exposition of the procedure of mathematical renormalization. Denote by \( \mathcal{H} \) the complex vector space with the basis consisting of all graphs \( \Gamma \). The space \( \mathcal{H} \) has a structure of commutative algebra in which the product is given by disjoint union of graphs: \( \Gamma \Gamma' = \Gamma \cup \Gamma' \). The unity of this algebra is the empty graph \( \Gamma_0 = 1 \). This space also has a coproduct \( \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \) given by the formula

\[ \Delta(\Gamma) = \sum_{\gamma \subset \Gamma} \tilde{\gamma} \otimes \Gamma/\tilde{\gamma}. \]

Here we adopt the following notation. \( \gamma \subset \Gamma \) is a full subgraph of the graph \( \Gamma \), i.e. the vertices of the graph \( \gamma \) form an arbitrary subset of the set of vertices of the graph \( \Gamma \), and the edges of \( \gamma \) are the internal or external edges of the graph \( \Gamma \) which pass through at least one vertex of the subgraph. This subgraph \( \gamma \) can be uniquely decomposed into a disjoint union of tree graphs whose “vertices” are either usual vertices or one-particle irreducible full subgraphs (i.e., connected graphs with more than one vertex which remain connected after removing any edge). The graph \( \tilde{\gamma} \) is defined as the disjoint union of these one-particle irreducible full subgraphs. Further, the graph \( \Gamma/\tilde{\gamma} \) is defined as the quotient graph of the graph \( \Gamma \) by \( \tilde{\gamma} \), i.e. each connected component of the graph \( \tilde{\gamma} \) is contracted in the graph \( \Gamma/\tilde{\gamma} \) to a subgraph with one vertex.
whose edges are identified with the external edges of this connected component of the graph \( \tilde{\gamma} \).

**Proposition.** [6] The bialgebra \( \mathcal{H} \) is a bigraded Hopf algebra.

We shall not recall here the definition and properties of Hopf algebras (see [7] and references therein). The only fact that we need is the Milnor–Moore theorem: a commutative (graded) Hopf algebra is dual to the universal enveloping algebra of certain Lie algebra. The corresponding Lie group consists of the characters of the algebra \( \mathcal{H} \), i.e., of homomorphisms of commutative algebras \( \chi : \mathcal{H} \to \mathbb{C} \). The product of characters \( \chi_1 \) and \( \chi_2 \) in this Lie group is given by the formula

\[
(\chi_1 \star \chi_2)(X) = \langle \chi_1 \otimes \chi_2, \Delta(X) \rangle.
\]

Denote by \( U : \mathcal{H} \to SV' \otimes A \) the linear map given by the formula \( U(\Gamma) = U_\varepsilon(\Gamma) \). It is easy to see that \( U \) is a homomorphism of commutative algebras. Let us now define the counterterm operation \( C(\Gamma) \) and the Bogoliubov–Parasiuk \( R \)-operation \( R(\Gamma) \) by the following recursive formulas. For disjoint unions of tree graphs \( \Gamma \), put \( C(\Gamma) = 0 \), \( R(\Gamma) = U(\Gamma) \). For one-particle irreducible graphs \( \Gamma \) without proper one-particle irreducible subgraphs, put

\[
C(\Gamma) = -T(U(\Gamma))
\]

and

\[
R(\Gamma) = U(\Gamma) + C(\Gamma).
\]

For general graphs \( \Gamma \), put

\[
C(\Gamma) = -T(R(\Gamma)),
\]

\[
R(\Gamma) = R(\Gamma) - T(R(\Gamma)),
\]

where the \( R \)-operation is defined by the recursive formula

\[
R(\Gamma) = U(\Gamma) + \sum_{\tilde{\gamma} \subset \Gamma \neq \Gamma} C(\tilde{\gamma}) U(\Gamma/\tilde{\gamma}),
\]

in the notation (29).

It is shown [6,7] that the counterterm operation \( C(\Gamma) \) and the \( R \)-operation \( R(\Gamma) \) can be extended to homomorphisms of commutative algebras (characters)

\[
C : \tilde{\mathcal{H}} \to SV' \otimes A_-, \quad R : \mathcal{H} \to SV' \otimes A_+;
\]

here \( \tilde{\mathcal{H}} = \mathrm{Ker} \bar{\tau} \) is the augmentation ideal in \( \mathcal{H} \), where \( \bar{\tau}(\Gamma_\emptyset) = 1 \), \( \bar{\tau}(\Gamma) = 0 \) for \( \Gamma \neq \Gamma_\emptyset \). Moreover, these maps satisfy the relation

\[
R(X) = (C \star U)(X),
\]
where \( \star \) is the product (30) of characters.

**Definition.** The element

\[
\tilde{U} = \sum_\Gamma R(\Gamma)|_{\varepsilon = 0}
\]

is called the renormalized evolution operator corresponding to the time interval \([T_0, T_1]\).

Thus, the renormalized evolution operator is in general determined non-uniquely, it depends on a regularization of the Hamiltonian.

### 3. On mathematical solutions

In this Section we discuss and give the definitions of mathematical solutions of generalized wave equations. This material is new.

We have seen in §2 that the procedure of solution of the generalized Schrödinger equation (16) in a space of functionals \( \Psi(x^j(s), \varphi^i(s)) \) leads to divergencies, and requires subtracting the divergent part from the evolution operator. But the procedure of subtracting the divergent part from solutions of differential equations is well known in mathematics since the times of the work of Hadamard [10] (see also a short exposition in [4]). This procedure lead afterwards to creation of theory of distributions. Therefore, it is natural to assume that the “true” mathematical solutions of generalized wave equations are not functionals but appropriately defined distributions on infinite dimensional function space. In this definition, the space-time variables and the field variables should be considered in equal rights.

Let us proceed to definitions. In this Section, similarly to §2, we denote by \( V \) the Schwartz space (or any nuclear space [11]) of functions \((\varphi^i(s), x^j(s), \pi_i(s), -H_j(s))\) (in the case of generalized wave equations (15)), or functions \((x^j(s), -H_j(s))\) (in the case of quantum Plato problem (17)). The variables \( \pi_i(s) \) are called *canonically conjugate* to the variables \( \varphi^i(s) \), and the variables \( H_j(s) \) are called canonically conjugate to \( x^j(s) \). But, for shortness, we shall, as before, call these functions simply by \((\varphi(s), \pi(s))\), and omit discrete indices, as in §2. Let us introduce, as in §2, the space \( SV \) of symbols of regular differential operators and the space \( SV' \) of symbols of all differential operators.
We shall need to replace the algebra (25) of symbols of regular differential operators by the Weyl–Moyal algebra $SV$ with the Moyal product

$$H^{(1)} *_{\text{Moyal}} H^{(2)}(\phi, \pi) = \exp \frac{i\hbar}{2} \int \left( \frac{\delta}{\delta \pi_1(s)} \frac{\delta}{\delta \phi_2(s)} - \frac{\delta}{\delta \pi_2(s)} \frac{\delta}{\delta \phi_1(s)} \right) ds \times H^{(1)}(\phi_1, \pi_1) H^{(2)}(\phi_2, \pi_2)|_{\phi_1 = \phi_2 = \phi, \pi_1 = \pi_2 = \pi}.$$ 

This algebra is isomorphic to the algebra (25), but is seemingly better adapted for passing to irregular local operators (cf. [4]).

**Definition.** Let us call by a general space of distributions a representation of the Weyl–Moyal algebra $(SV, *_{\text{Moyal}})$ in a topological vector space $E$. Let us call by a solution of a system of differential equations ($\hat{D}_\alpha \Psi = 0$), $D_\alpha \in SV'$, a distribution $\Psi \in E$ such that for any regularization $D_{\alpha, \varepsilon} \in SV$, $\varepsilon > 0$ of symbols $D_\alpha$, $D_{\alpha, \varepsilon} \to D_\alpha$ as $\varepsilon \to 0$, we have $\hat{D}_{\alpha, \varepsilon} \Psi \to 0$ in the space $E$ as $\varepsilon \to 0$.

Numerous examples of general spaces of distributions can be obtained from the unitary representations of the infinite dimensional Heisenberg group constructed in the book [11] (they are called in [11] by representations of canonical commutation relations) by passing to the dual space of the space of smooth vectors. Another example of a general space of distributions is constructed in the paper [12]. See also the book [13] and references therein. However, up to now no examples (or proofs of existence) of nonzero solutions of generalized wave equations are known.

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