Abstract. We consider the popular matching problem in a roommates instance $G = (V, E)$ with strict preference lists. While popular matchings always exist in a bipartite instance, they need not exist in a roommates instance. The complexity of the popular matching problem in a roommates instance has been an open problem for several years and we prove its NP-hardness here. A sub-class of max-size popular matchings called dominant matchings has been well-studied in bipartite graphs. We show that the dominant matching problem in $G = (V, E)$ is also NP-hard and this is the case even when $G$ admits a stable matching.

1 Introduction

We consider a matching problem in a graph $G = (V, E)$ (need not be complete) where each vertex $u \in V$ ranks its neighbors in a strict order of preference. Such a graph $G$ is usually referred to as a roommates instance. A matching $M$ is stable if there is no blocking edge with respect to $M$, i.e., there is no pair $(a, b)$ such that both $a$ and $b$ prefer each other to their respective assignments in $M$.

Stable matchings always exist when $G$ is bipartite [6], however there are simple roommates instances that do not admit any stable matching. The problem of deciding whether a stable matching exists or not in $G$ is the stable roommates problem. There are several polynomial time algorithms [12,19,21] to solve the stable roommates problem. Here we consider a notion called popularity that is more relaxed than stability.

1.1 Popular Matchings

The notion of popularity was introduced by Gärdenfors [7] in 1975. We say a vertex $u$ prefers matching $M$ to matching $M'$ if either (i) $u$ is matched in $M$ and unmatched in $M'$ or (ii) $u$ is matched in both $M, M'$ and $u$ prefers $M(u)$ to $M'(u)$. For any two matchings $M$ and $M'$, let $\phi(M, M')$ be the number of vertices that prefer $M$ to $M'$.

Definition 1. A matching $M$ is popular if $\phi(M, M') \geq \phi(M', M)$ for every matching $M'$ in $G$, i.e., $\Delta(M, M') \geq 0$ where $\Delta(M, M') = \phi(M, M') - \phi(M', M)$.

Thus there is no matching $M'$ that would defeat a popular matching $M$ in an election between $M$ and $M'$, where each vertex casts a vote for the matching that it prefers. Since there is no matching where more vertices are better-off than in a popular matching, a popular matching can be regarded as a “globally stable matching”.

It is easy to show that every stable matching is popular [7]. Since popularity is a relaxation of stability, popular matchings may exist in roommates instances that admit no stable matchings (see the instance on 4 vertices $d_0, d_1, d_2, d_3$ on the left of Fig. 1). Here we are interested in the complexity of the popular roommates problem, i.e., the problem of deciding if $G = (V, E)$ admits a popular matching or not. This has been an open problem for almost a decade [2] and we show the following result here [7].

Theorem 1. Given a roommates instance $G = (V, E)$ with strict preference lists, the problem of deciding if $G$ admits a popular matching or not is NP-hard.

Very recently, this hardness result also appeared in [8] on the arxiv; our results were obtained independently and our proofs are different.
Popular matchings always exist in a bipartite instance, since stable matchings always exist here. Popular matchings have been well-studied in bipartite graphs, in particular, a subclass of max-size popular matchings called dominant matchings is well-understood \[5,9,14\].

**Definition 2.** A popular matching \(M\) is dominant in \(G\) if \(M\) is more popular than any larger matching in \(G\), i.e., \(\Delta(M, M') > 0\) for any matching \(M'\) such that \(|M'| > |M|\).

Dominant matchings always exist in a bipartite instance and such a matching can be computed in linear time \[14\]. We consider the dominant matching problem in a roommates instance and show the following result.

**Theorem 2.** Given a roommates instance \(G = (V,E)\) with strict preference lists, the problem of deciding if \(G\) admits a dominant matching or not is \(NP\)-hard. Moreover, this hardness holds even when \(G\) admits a stable matching.

### 1.2 Background and Related work

The first polynomial time algorithm for the stable roommates problem was by Irving \[12\] in 1985. A characterization of roommates instances that admit stable matchings was given in \[20\] and new polynomial time algorithms for the stable roommates problem were given in \[19,21\]. As mentioned earlier, Gärdenfors \[7\] introduced the notion of popularity in the stable matching problem in bipartite instances.

Algorithmic questions for popular matchings were initially studied in the one-sided preference lists model: here only one side of the bipartite instance has preferences over its neighbors. Popular matchings need not always exist in this model and there is an efficient algorithm \[1\] to determine if a given instance admits one. Popular **mixed** matchings always exist here \[13\] and such a mixed matching can be computed in polynomial time via linear programming.

In the stable matching problem in bipartite instances (the two-sided preference lists model), popular matchings always exist and a max-size popular matching can be computed efficiently \[9,14\]. These algorithms always compute dominant matchings — it was shown in \[5\] that dominant matchings are essentially stable matchings in a larger bipartite graph. When ties are allowed in preference lists, the problem of deciding if a popular matching problem exists or not is \(NP\)-hard \[24\].

It was shown in \[11\] that the problem of computing a max-weight popular matching in a roommates instance with edge weights is \(NP\)-hard. This was strengthened in \[16\] to show that the problem of computing a max-size popular matching in a roommates instance is \(NP\)-hard. An efficient algorithm was also given in \[16\] to compute a strongly dominant matching in a roommates instance. Strongly dominant matchings are a subclass of dominant matchings; interestingly, in bipartite instances, dominant and strongly dominant are equivalent notions \[14\].

It was shown in \[10\] that every roommates instance \(G = (V,E)\) admits a matching whose unpopularity factor is \(O(\log |V|)\) and it is \(NP\)-hard to compute a least unpopularity matching in \(G\). The complexity of the popular roommates problem was stated as an open problem in several papers/books \[23,10,11,17\].

**Techniques.** We use properties of popular matchings in bipartite instances here — in particular, we use the LP framework of popular matchings that was initiated in \[13\]. Every popular matching \(M\) in a bipartite instance \(H\) is a max-weight perfect matching in a related graph \(\tilde{H}\) and an optimal solution to the dual LP (dual to the max-weight perfect matching LP) is a witness to the popularity of \(M\). It is known that a matching in \(H\) is popular if and only if it has a witness \(\alpha \in \{0, \pm 1\}^n\), where \(n\) is the number of vertices.

A stable matching in our roommates instance \(G\) will correspond to a matching in \(H\) with 0 as a witness and a strongly dominant matching in \(G\) will correspond to a matching in \(H\) with a witness \(\alpha\) such that \(\alpha_u \in \{\pm 1\}\) for all matched vertices \(u\). We show a reduction from 1-in-3 SAT to the popular roommates problem via the problem of deciding if a desired popular matching exists in the bipartite instance \(H\); such a matching is constrained to have a certain witness in \(\{0, \pm 1\}^n\) which will prove its hardness.
Organization of the paper. Section 2 contains an overview of the LP framework of popular matchings in bipartite instances. Section 3 outlines the reduction from 1-in-3 SAT to the popular roommates problem and Section 4 has more details of our hardness reduction. Section 5 shows NP-hardness for dominant matchings.

2 Preliminaries

This section is an overview of the LP framework of popular matchings in bipartite graphs from [13] along with some results from [15,16]. Let $H = (A \cup B, E_H)$ be a bipartite instance with strict preference lists and let $H$ be the graph $H$ augmented with self-loops, i.e., it is assumed that every vertex is its own last choice. Let $M$ be any matching in $H$. Corresponding to $M$, there is a perfect matching $\hat{M}$ in $\hat{H}$ defined as follows: $M = M \cup \{(u,u) : u \text{ is left unmatched in } M\}$.

We now define an edge weight function $w_M$ in $\hat{H}$. For any vertex $u$ and neighbors $v, v'$ in $\hat{H}$, let $\operatorname{vote}_u(v, v')$ be 1 if $u$ prefers $v$ to $v'$, it is -1 if $u$ prefers $v'$ to $v$, else it is 0 (i.e., $v = v'$).

The matching $M$ is defined as follows:

$$w_M(u,v) = \operatorname{vote}_u(v, \hat{M}(u)) + \operatorname{vote}_u(u, \hat{M}(v)) \quad \text{for } (u,v) \in E_H.$$  

Thus $w_M(u,v) \in \{0, \pm 1\}$. We need to define $w_M$ on self-loops as well: for any $u \in V$, let $w_M(u,u) = 0$ if $u$ is unmatched in $M$, else let $w_M(u,u) = -1$. Thus $w_M(u,u)$ is $u$’s vote for itself versus $\hat{M}(u)$.

It is easy to see that for any matching $N$ in $H$, $\Delta(N,M) = w_M(\hat{N})$. Thus $M$ is popular if and only if every perfect matching in $\hat{H}$ has weight at most 0. Let $n = |A \cup B|$.

Theorem 3 ([13]). Let $M$ be any matching in $H = (A \cup B, E_H)$. The matching $M$ is popular if and only if there exists a vector $\alpha \in \mathbb{R}^n$ such that $\sum_{u \in A \cup B} \alpha_u = 0$ and

$$\alpha_a + \alpha_b \geq w_M(a,b) \quad \forall (a,b) \in E_H$$

$$\alpha_u \geq w_M(u,u) \quad \forall u \in A \cup B.$$  

The vector $\alpha$ will be an optimal solution to the LP that is dual to the max-weight perfect matching LP in $\hat{H}$ (with edge weight function $w_M$). For any popular matching $M$, a vector $\alpha$ as given in Theorem 3 will be called a witness to $M$. The following lemma will be useful to us.

**Lemma 1 ([15]).** Any popular matching in $H = (A \cup B, E_H)$ has a witness in $\{0, \pm 1\}^n$.

Call any $e \in E_H$ a popular edge if there is some popular matching in $H$ that contains $e$. Let $M$ be a popular matching in $H$ and let $\alpha \in \{0, \pm 1\}^n$ be a witness of $M$.

**Lemma 2 ([16]).** If $(a,b)$ is a popular edge in $H$, then $\alpha_a + \alpha_b = w_M(a,b)$. If $u$ is an unstable vertex in $H$ then $\alpha_u = 0$ if $u$ is left unmatched in $M$, else $\alpha_u = -1$.

The popular subgraph $F_H$ is a useful subgraph of $H$ defined in [16].

**Definition 3.** The popular subgraph $F_H = (A \cup B, E_F)$ is the subgraph of $H = (A \cup B, E_H)$ whose edge set $E_F$ is the set of popular edges in $E_H$.

The graph $F_H$ need not be connected. Let $C_1, \ldots, C_h$ be the various components in $F_H$.

**Lemma 3 ([16]).** For any connected component $C_i$ in $F_H$, either $\alpha_u = 0$ for all vertices $u \in C_i$ or $\alpha_u \in \{\pm 1\}$ for all vertices $u \in C_i$. Moreover, if $C_i$ contains one or more unstable vertices, either all these unstable vertices are matched in $M$ or none of them is matched in $M$.

The following definition marks the state of each connected component $C_i$ in $F_H$ as “stable” or “dominant” in $\alpha$ — this classification will be useful to us in our hardness reduction.

**Definition 4.** A connected component $C_i$ in $F_H$ is in stable state in $\alpha$ if $\alpha_u = 0$ for all vertices $u \in C_i$. Similarly, $C_i$ in $F_H$ is in dominant state in $\alpha$ if $\alpha_u \in \{\pm 1\}$ for all vertices $u \in C_i$.  

3
3 Hardness of the popular roommates problem

Our reduction will be from 1-in-3 SAT. Recall that 1-in-3 SAT is the set of 3CNF formulas with no negated variables such that there is a satisfying assignment that makes exactly one variable true in each clause. Given an input formula $\phi$, to determine if $\phi$ is 1-in-3 satisfiable or not is NP-hard [18].

We will now build a roommates instance $G = (V, E)$. The vertex set $V$ will consist of vertices in 4 levels: levels 0, 1, 2, and 3 along with 5 other vertices $d_0, d_1, d_2, d_3, z$. The vertices $d_0, d_1, d_2, d_3$ form a gadget $D$ (on the left of Fig. 1) described below. The vertex $d_0$ will be the last choice neighbor of vertex $v$ for every $v \in V \setminus \{z\}$.

Vertices in $V \setminus \{d_0, d_1, d_2, d_3, z\}$ are partitioned into gadgets that appear in some level $i$, for $i \in \{0, 1, 2, 3\}$. Every edge $(u, v)$ in $G$ where $u, v$ are in $V \setminus \{d_0, z\}$ is either inside a gadget or between 2 gadgets in consecutive levels. We describe these gadgets below.

Level 1 vertices. Every gadget in level 1 is a variable gadget. Corresponding to each variable $X_i$, we will have the gadget on the right of Fig. 1. The preference lists of the 4 vertices in the gadget corresponding to $X_i$ are as follows:

- $x_i$: $y_i \succ y_i' \succ z \succ \cdots$
- $x_i'$: $y_i \succ y_i' \succ \cdots$
- $y_i$: $x_i \succ x_i' \succ z \succ \cdots$
- $y_i'$: $x_i \succ x_i' \succ \cdots$

The vertices in the gadget corresponding to $X_i$ are also adjacent to vertices in the “clause gadgets” corresponding to $X_j$: these neighbors belong to the “···” part of the preference lists. Note that the order among the vertices in the “···” part in the above preference lists does not matter. Also, $d_0$ is the last choice of each of the above vertices.

![Fig. 1](image)

Each of $d_1, d_2, d_3$ is a top choice neighbor for another vertex here and $d_0$ is the last choice of $d_1, d_2, d_3$. On the right is the gadget corresponding to variable $X_i$: vertex preferences are indicated on edges.

The gadget $D$. There will be 4 vertices $d_0, d_1, d_2, d_3$ that form the gadget $D$ (see the left of Fig. 1). The preferences of vertices in $D$ are given below.

- $d_1$: $d_2 \succ d_3 \succ d_0$
- $d_2$: $d_3 \succ d_1 \succ d_0$
- $d_3$: $d_1' \succ d_2 \succ d_0$
- $d_0$: $d_1 \succ d_2 \succ d_3 \succ \cdots$

The vertex $d_0$ will be adjacent to all vertices in $G$ other than $z$. The order of other neighbors in $d_0$’s preference list does not matter. Let $c = X_i \lor X_j \lor X_k$ be a clause in $\phi$. We will describe the gadgets that correspond to $c$.

Level 0 vertices. There will be three level 0 gadgets, each on 4 vertices, corresponding to clause $c$. See Fig. 2 We describe below the preference lists of the 4 vertices $a_1', b_1', a_2', b_2'$ that belong to the leftmost gadget. For the sake of readability, we have dropped the superscript $c$ from these vertices.
Fig. 2. Corresponding to clause \( c = X_1 \lor X_2 \lor X_3 \) we have the above 3 gadgets in level 0. The vertex \( a_i^c \)'s second choice is \( y_j' \) and \( b_i^c \)'s is \( x_i^k \). Similarly, \( a_3^c \)'s is \( y_k' \) and \( b_3^c \)'s is \( x_1' \). Also \( a_5^c \)'s is \( y_t' \) and \( b_5^c \)'s is \( x_2' \).

\[
\begin{align*}
a_1: & \quad b_1 \succ y_j' \succ b_2 \succ z \\
a_2: & \quad b_2 \succ b_1 \\
b_1: & \quad a_2 \succ x_1' \succ a_1 \succ z \\
b_2: & \quad a_1 \succ a_2
\end{align*}
\]

Though \( d_0 \) is not explicitly listed in the above preference lists, recall that \( d_0 \) is the last choice of each of these vertices. Neighbors that are outside this gadget are underlined. The preferences of vertices in the other 2 gadgets in level 0 corresponding to \( c \) (\( a_t^c, b_t^c \) for \( t = 3, 4 \) and \( a_t^c, b_t^c \) for \( t = 5, 6 \)) are analogous. We will now describe the three level 2 gadgets corresponding to clause \( c \). See Fig. 3.

Fig. 3. We have the above 3 gadgets in level 2 corresponding to \( c = X_1 \lor X_2 \lor X_3 \). The vertex \( p_2^c \)'s second choice is \( y_j \) and \( q_2^c \)'s is \( x_k \). Similarly, \( p_6^c \)'s is \( y_4 \) and \( q_6^c \)'s is \( x_1 \). Similarly \( p_8^c \)'s is \( y_t \) and \( q_8^c \)'s is \( x_j \).

Level 2 vertices. There will be three level 2 gadgets, each on 6 vertices, corresponding to clause \( c \). The preference lists of the vertices \( p_i^c, q_i^c \) for \( 0 \leq t \leq 2 \) are described below. Also, \( d_0 \) is the last choice of each of these vertices. For the sake of readability, we have again dropped the superscript \( c \) from these vertices.

\[
\begin{align*}
p_0: & \quad q_0 \succ q_2 \\
p_1: & \quad q_1 \succ q_2 \succ z \\
p_2: & \quad q_0 \succ y_j \succ q_1 \succ q_2 \\
q_0: & \quad p_0 \succ p_2 \succ z \succ s_0 \\
q_1: & \quad p_1 \succ p_2 \\
q_2: & \quad p_1 \succ x_k \succ p_0 \succ p_2
\end{align*}
\]

Let us note the preference lists of \( p_2 \) and \( q_2 \): they are each other’s fourth choices. The vertex \( p_2 \) regards \( q_0 \) as its top choice, \( y_j \) as its second choice, and \( q_1 \) as its third choice. The vertex \( q_2 \) regards \( p_1 \) as its top choice, \( x_k \) as its second choice, and \( p_0 \) as its third choice.

The preferences of vertices in the other 2 gadgets in level 2 corresponding to \( c \) (\( p_t^c, q_t^c \) for \( 3 \leq t \leq 5 \) and \( p_t^c, q_t^c \) for \( 6 \leq t \leq 8 \)) are analogous to the above preference lists. The vertex \( s_0 \) that appears in \( q_0 \)'s preference list is a vertex from the level 3 gadget corresponding to clause \( c \). Note that \( s_0 \) also appears in \( q_3 \)'s preference list and the vertex \( t_0 \) appears in the preference lists of \( p_1 \) and \( q_7 \).

Level 3 vertices. Gadgets in level 3 are again clause gadgets. There is exactly one level 3 gadget on 8 vertices \( s_i^c, t_i^c \), for \( 0 \leq i \leq 3 \), corresponding to clause \( c \). As before, \( d_0 \) is the last choice of each of these vertices.
\[ s_0 : t_1 \succ q_0 \succ t_2 \succ q_3 \succ t_3 \quad t_0 : s_3 \succ p_2 \succ s_2 \succ p_4 \succ s_1 \]
\[ s_1 : t_1 \succ t_0 \quad t_1 : s_1 \succ s_0 \]
\[ s_2 : t_2 \succ t_0 \quad t_2 : s_2 \succ s_0 \]
\[ s_3 : t_3 \succ t_0 \quad t_3 : s_3 \succ s_0 \]

The preference lists of the 8 vertices in the level 3 gadget corresponding to clause $c$ are described above. For the sake of readability, we have again dropped the superscript $c$ from these vertices.

It is important to note the preference lists of $s_0$ and $t_0$ here. Among neighbors in this gadget, $s_0$’s order is $t_1 \succ t_2 \succ t_3$ while $t_0$’s order is $s_3 \succ s_2 \succ s_1$. Also, $s_0$’s order is interleaved with $q_0 \succ q_3$ (these are vertices from level 2 gadgets) and $t_0$’s order is interleaved with $p_7 \succ p_4$.

There is one more vertex in $G$. This is the vertex $z$, the neighbors of $z$ are $\cup_i \{x_i, y_i\}$ \cup_{\phi} \{a_{2i-1}^c, b_{2i-1}^c : i = 1, 2, 3\} \cup \{p_{3j+1}^c, q_{3j}^c : j = 0, 1, 2\}$. The preference order of these neighbors in $z$’s preference list is as follows:

\[ z : x_1 \succ y_1 \succ \cdots \succ x_n \succ y_n \succ a_1^c \succ b_1^c \succ \cdots \]

Here $n_0$ is the number of variables in $\phi$. Note that $z$ prefers any neighbor in a level 1 gadget to other neighbors. Thus the vertex set $V$ is $\{z\} \cup \{d_0, d_1, d_2, d_3\} \cup_{i=0}^3 \{\text{level } i \text{ vertices}\}$. We will partition the set $\cup_{i=0}^3 \{\text{level } i \text{ vertices}\}$ into $X \cup Y$ where

\[ X = \cup_i \{x_i, x_i'\} \cup_{\phi} \{a_0^c, a_1^c, \ldots, a_6^c, p_6^c, \ldots, p_8^c, s_0^c, \ldots, s_5^c\} \]
\[ Y = \cup_i \{y_i, y_i'\} \cup_{\phi} \{b_0^c, b_1^c, \ldots, b_6^c, q_6^c, \ldots, q_8^c, t_0^c, \ldots, t_5^c\} \]

Lemma 4. For any popular matching $M$ in $G$, the following properties hold:

1. either $\{(d_0, d_1), (d_2, d_3)\} \subset M$ or $\{(d_0, d_2), (d_1, d_3)\} \subset M$.
2. $M$ matches all vertices in $X \cup Y$.

Proof. Since each of $d_1, d_2, d_3$ is a top choice neighbor for some vertex in $G$, a popular matching in $G$ cannot leave any of these 3 vertices unmatched. Since these 3 vertices have no neighbors outside themselves other than $d_0$, a popular matching has to match $d_0$ to one of $d_1, d_2, d_3$. Thus $d_0, d_1, d_2, d_3$ are matched among each other in $M$.

The only possibilities for $M$ when restricted to $d_0, d_1, d_2, d_3$ are the pair of edges $(d_0, d_1), (d_2, d_3)$ or $(d_0, d_2), (d_1, d_3)$. The third possibility $(d_0, d_3), (d_1, d_2)$ is “less popular than” $(d_0, d_1), (d_2, d_3)$ as $d_0, d_2$, and $d_3$ prefer the latter to the former. This proves part (1) of the lemma.

Consider any vertex $v \in X \cup Y$. If $v$ is left unmatched in $M$ then we either have an alternating path $p_1 = (v, d_0)-(d_0, d_1)-(d_1, d_3)$ or an alternating path $p_2 = (v, d_0)-(d_0, d_2)-(d_2, d_1)$ with respect to $M$: in each of these alternating paths, the starting vertex $v$ is unmatched in $M$, the middle edge belongs to $M$, and the third edge is a blocking edge with respect to $M$.

Suppose $p_1$ is an alternating path with respect to $M$. Consider $M \oplus p_1$ versus $M$: the vertices $v, d_1, d_3$ prefer $M \oplus p_1$ to $M$ while $d_0$ and $d_2$ prefer $M$ to $M \oplus p_1$; the other vertices are indifferent between $M$ and $M \oplus p_1$. Thus $M \oplus p_1$ is more popular than $M$, a contradiction to $M$’s popularity. Similarly, if $p_2$ is an alternating path with respect to $M$ then $M \oplus p_2$ is more popular than $M$. Hence every vertex in $X \cup Y$ has to be matched in $M$. This proves part (2). \qed

Since the total number of vertices in $G$ is odd, at least 1 vertex has to be left unmatched in any matching in $G$. Lemma 4 implies that the vertex $z$ will be left unmatched in $M$.

Let $G_0$ be the subgraph of $G$ induced on $X \cup Y \cup \{z\}$. The matching $M$ restricted to $G_0$ has to be popular on $G_0$, otherwise it would contradict the popularity of $M$ in $G$. We will now show the following converse of Lemma 4.

Lemma 5. If $G_0$ admits a popular matching that matches all vertices in $X \cup Y$ then $G$ admits a popular matching.
Proof. Let \( M_0 \) be a popular matching in \( G_0 \) that matches all vertices in \( X \cup Y \). We claim \( M = M_0 \cup \{(d_0, d_1), (d_2, d_3)\} \) is a popular matching in \( G \).

Let \( G'_0 \) be the subgraph obtained by removing all negative edges to \( M_0 \) from \( G_0 \). Since \( M_0 \) is popular in \( G_0 \), it satisfies the following three necessary and sufficient conditions for popularity (from [9]) in \( G'_0 \).

1. There is no alternating cycle that contains a blocking edge.
2. There is no alternating path with \( z \) as an endpoint that contains a blocking edge.
3. There is no alternating path that contains two blocking edges.

We need to show that \( M \) obeys the above 3 conditions in the subgraph \( G' \) obtained by deleting negative edges to \( M \) from \( G \). The graph \( G' \) is the graph \( G'_0 \) along with some edges within the gadget \( D \). There is no edge in \( G' \) between \( D \) and any vertex in \( G_0 \) since every edge in \( G \) between \( D \) and a vertex in \( G_0 \) is negative to \( M \). This is because for any such edge \((d_0, v)\), the vertex \( d_0 \) prefers \( d_1 \) (its partner in \( M \)) to \( v \) and similarly, \( v \) prefers each of its neighbors in \( G_0 \) to \( d_0 \). Since \( v \in X \cup Y \), note that \( M_0 \) matches \( v \) to one of its neighbors in \( G_0 \).

It is easy to check that the edge set \( \{(d_0, d_1), (d_2, d_3)\} \) satisfies the above 3 conditions in the subgraph of \( D \) obtained by pruning negative edges to \( M_0 \). We know that \( M_0 \) satisfies the above 3 conditions in \( G'_0 \). Thus \( M \) satisfies the above 3 conditions in \( G' \). Hence \( M \) is popular in \( G \). \( \Box \)

We will show the following theorem in Section 4.

Theorem 4. \( G_0 \) admits a popular matching that matches all vertices in \( X \cup Y \) if and only if \( \phi \) is 1-in-3 satisfiable.

Since \( G \) admits a popular matching if and only if the instance \( G_0 \) admits a popular matching that matches all vertices in \( X \cup Y \), Theorem 4 implies the NP-hardness of the popular matching problem in a roommates instance \( G = (V, E) \). Thus we can conclude Theorem 4 stated in Section 1.

4 Proof of Theorem 4

Our goal now is to use the LP framework for bipartite matchings from Section 2. However the graph \( G_0 \) is non-bipartite. This is due to the presence of the vertex \( z \). So let us convert the graph \( G_0 \) on vertex set \( X \cup Y \cup \{z\} \) into a bipartite instance \( H \) by splitting the vertex \( z \) into 2 vertices \( z, z' \). That is, every occurrence of \( z \) in the preference lists of vertices in \( Y \) will be replaced by \( z' \).

Thus \( H = (X' \cup Y', E_H) \) where \( X' = X \cup \{z'\} \) and \( Y' = Y \cup \{z\} \). The edge set \( E_H \) of \( H \) is the same as the edge set of \( G_0 \), except that each edge \((z, v)\) where \( v \in Y \) gets replaced by the edge \((z', v)\) in \( H \).

The graph \( H \) is a bipartite graph with \( X \cup \{z'\} \) on the left and \( Y \cup \{z\} \) on the right. The preference list of \( z \) (similarly, \( z' \)) is the original preference list of \( z \) restricted to neighbors in \( X \) (resp., \( Y \)). The vertices of \( H \setminus \{z, z'\} \) are level \( i \) vertices in \( G \), for \( i = 0, \ldots, 3 \). Let \( F_H \) be the popular subgraph of \( H \).

Lemma 6. Let \( C \) be any level \( i \) gadget in \( H \), where \( i \in \{0, 1, 2, 3\} \). All the vertices in \( C \) belong to the same connected component in \( F_H \).

Proof. Consider a level 0 gadget in \( H \), say on \( a_0^z, b_0^z, a_0^z, b_0^z \). The “men-optimal” (or \( X' \)-optimal) stable matching in \( H \) contains the edges \((a_0^z, b_0^z)\) and \((a_0^z, b_0^z)\) while the “women-optimal” (or \( Y' \)-optimal) stable matching contains the edges \((a_0^z, b_0^z)\) and \((a_0^z, b_0^z)\). Thus there are popular edges among these 4 vertices and so these 4 vertices belong to the same connected component in \( F_H \).

Consider a level 1 gadget in \( H \), say on \( x_i, y_i, x_i', y_i' \). A stable matching in \( H \) contains \((x_i, y_i)\) and \((x_i', y_i')\) while a dominant matching in \( H \) contains \((x_i, y_i)\) and \((x_i', y_i)\). Thus there are popular edges among these 4 vertices and so these 4 vertices belong to the same connected component in \( F_H \).
Consider a level 2 gadget in $H$, say on $p_i^\ell, q_i^\ell$ for $i = 0, 1, 2$. There is a dominant matching in $H$ that contains the edges $(p_0^\ell, q_2^\ell)$ and $(p_2^\ell, q_0^\ell)$. There is also another dominant matching in $H$ that contains the edges $(p_1^\ell, q_2^\ell)$ and $(p_2^\ell, q_1^\ell)$. Thus there are popular edges among these 6 vertices and so these 6 vertices belong to the same connected component in $F_H$.

Consider a level 3 gadget in $H$, say on $s_i^\ell, t_i^\ell$ for $i = 0, \ldots , 3$. There is a dominant matching in $H$ that contains $(s_0^\ell, t_1^\ell)$, and $(s_1^\ell, t_0^\ell)$. There is another dominant matching in $H$ that contains $(s_0^\ell, t_2^\ell)$ and $(s_2^\ell, t_0^\ell)$. There is yet another dominant matching in $H$ that contains $(s_0^\ell, t_3^\ell)$ and $(s_3^\ell, t_0^\ell)$. Thus there are popular edges among these 8 vertices and so these 8 vertices belong to the same connected component in $F_H$.

The lemma below shows that no edge between a level $\ell$ vertex and a level $(\ell + 1)$ vertex is used in any popular matching in $H$, for $\ell \in \{0, 1, 2\}$.

**Lemma 7.** There is no popular edge in $H$ between a level $\ell$ vertex and a level $(\ell + 1)$ vertex for $\ell \in \{0, 1, 2\}$.

**Proof.** Let $c = X_i \cup X_j \cup X_k$ be a clause in $\phi$. We will first show that no edge between a level 0 vertex and a level 1 vertex can be popular. Consider any such edge in $H$, say $(a_1^c, y_j^c)$. In order to show this edge cannot be present in a popular matching, we will show a popular matching $S$ along with a witness $\alpha$ such that $\alpha_{a_1^c} + \alpha_{y_j^c} > wt_S(a_1^c, y_j^c)$. Then it will immediately follow from the slackness of this edge that $(a_1^c, y_j^c)$ does not belong to any popular matching (by Lemma 2).

Let $S$ be the men-optimal stable matching. The vector $\alpha = 0$ is a witness to $S$. The edges $(a_1^c, b_1^c)$ and $(x_j^c, y_j^c)$ belong to $S$, so we have $wt(a_1^c, y_j^c) = -2$ while $\alpha_{a_1^c} = \alpha_{y_j^c} = 0$. Thus $(a_1^c, y_j^c)$ is not a popular edge. We can similarly show that $(x_k^c, b_1^c)$ is not a popular edge by considering the women-optimal stable matching $S'$.

We will now show that no edge between a level 1 vertex and a level 2 vertex is popular. Consider any such edge in $H$, say $(p_2^c, y_j^c)$. Consider the dominant matching $N$ that contains the edges $(p_0^c, q_2^c)$ and $(p_2^c, q_0^c)$. All dominant matchings in $H$ contain the edges $(x_j^c, y_j^c)$ and $(x_j^c, y_j^c)$.

Any witness $\beta$ to $N$ sets $\beta_{p_2^c} = \beta_{q_2^c} = -1$ and $\beta_{x_j} = \beta_{y_j} = 1$. This is because $(x_j, y_j)$ and $(p_0^c, q_0^c)$ are blocking edges to $N$, so $\beta_{x_j} = \beta_{y_j} = 1$ and similarly, $\beta_{p_0^c} = \beta_{q_0^c} = 1$ (this makes $\beta_{p_2^c} = \beta_{q_2^c} = -1$). Consider the edge $(p_2^c, y_j)$. We have $wt_N(p_2^c, y_j) = -2$ while $\beta_{p_2^c} + \beta_{y_j} = 0$. Thus this edge is slack and so it cannot be a popular edge. We can similarly show that the edge $(x_k, q_2^c)$ is not popular by considering the dominant matching $N'$ that includes the edges $(p_1^c, q_2^c)$ and $(p_2^c, q_1^c)$.

We will now show that no edge between a level 2 vertex and a level 3 vertex is popular. Consider any such edge in $H$, say $(s_0^c, q_0^c)$. Consider the dominant matching $T$ that includes the edges $(s_0^c, t_1^c)$ and $(s_1^c, t_0^c)$. Here $q_0^c$ is matched either to $p_0^c$ or to $p_2^c$. In both cases, we have $wt_T(s_0^c, q_0^c) = -2$ while $\gamma_{s_0^c} = -1$ and $\gamma_{q_0^c} = 1$, where $\gamma$ is a witness to the matching $T$. Hence $(s_0^c, q_0^c)$ is not a popular edge. It can similarly be shown for any edge $e$ between a level 2 vertex and a level 3 vertex in $H$ that $e$ is not a popular edge.

\[\Box\]

### 4.1 Desired popular matchings in $H$

It is simple to see that $M$ is a popular matching in $G_0$ that matches all vertices in $X \cup Y$ and leaves $z$ unmatched if and only if $M$ is a popular matching in $H$ that matches all vertices in $X \cup Y$ and leaves $z$ and $z'$ unmatched. We will call such a matching $M$ in $H$ a “desired popular matching” here. Let $M$ be such a matching and let $\alpha \in \{0, \pm1\}^n$ be a witness of $M$, where $n = |X' \cup Y'|$.

The following two observations will be important for us. Recall Definition 4 from Section 2

1. All level 3 gadgets have to be in dominant state in $\alpha$.
2. All level 0 gadgets have to be in stable state in $\alpha$.

The vertices $s_0$ and $t_0$, for all clauses $c$, are left unmatched in any stable matching in $H$. Since $M$ has to match the unstable vertices $s_0^c$ and $t_0^c$ for all clauses $c$, $\alpha_{s_0^c} = \alpha_{t_0^c} = -1$ for all $c$ (by Lemma 2). Thus the first observation follows from Lemma 3. We prove the second observation below.
**Claim.** Any level 0 gadget has to be in stable state in $\alpha$.

**Proof.** Consider any level 0 gadget, say on vertices $a_0, b_0, a'_0, b'_0$. Since $M$ is a popular matching, we have $\alpha_{a_0} + \alpha_{b_0} = \text{wt}_M(a_0, z)$ and $\alpha_{a'_0} + \alpha_{b'_0} = \text{wt}_M(z', b'_0)$. Since $z$ and $z'$ are unmatched in $M$, it follows from Lemma 2 that $\alpha_{z} = \alpha_{z'} = 0$. We also have $\text{wt}_M(a_0, z) = 0$ since $z$ prefers $a_0$ to being unmatched while $a'_0$ likes any of its neighbors in $Y$ (one of them is its partner in $M$) to $z$. Similarly, $\text{wt}_M(z', b'_0) = 0$. Thus $\alpha_{a_0} \geq 0$ and similarly, $\alpha_{b_0} \geq 0$.

The edge $(a'_0, b'_0)$ is a popular edge. Thus $\alpha_{a'_0} + \alpha_{b'_0} = \text{wt}_M(a'_0, b'_0)$ (by Lemma 2). Observe that $\text{wt}_M(a'_0, b'_0) = 0$ since either $(a'_0, b'_0) \in M$ or $(a'_0, b'_0), (a'_1, b'_0)$ are in $M$. Thus $\alpha_{a'_0} + \alpha_{b'_0} = 0$. Since $\alpha_{a_0}$ and $\alpha_{b_0}$ are non-negative, it follows that $\alpha_{a_0} = \alpha_{b_0} = 0$. Thus this gadget is in stable state in $\alpha$. \hfill $\square$

The following lemmas are easy to show and are crucial to our NP-hardness proof. Let $c = X_i \lor X_j \lor X_k$ be any clause in $\phi$. In our proofs below, we are omitting the superscript $c$ from vertex names for the sake of readability. Recall that $\alpha \in \{0, \pm 1\}^n$ is a witness of our desired popular matching $M$.

**Lemma 8.** For every clause $c$ in $\phi$, at least two of the three level 2 gadgets corresponding to $c$ have to be in dominant state in $\alpha$.

**Proof.** Let $c$ be any clause in $\phi$. We know from observation 1 that the level 3 gadget corresponding to $c$ is in dominant state in $\alpha$. So $\alpha_{s_0} = \alpha_{t_0} = -1$. Also, one of the following three cases holds:

1. $(s_0, t_1)$ and $(s_1, t_0)$ are in $M$, (2) $(s_0, t_2)$ and $(s_2, t_0)$ are in $M$, (3) $(s_0, t_3)$ and $(s_3, t_0)$ are in $M$.

   - In case (1), the vertex $t_0$ prefers $p_1$ and $p_2$ to its partner $s_1$ in $M$. Thus $\text{wt}_M(p_4, t_0) = \text{wt}_M(p_7, t_0) = 0$. Since $\alpha_{t_0} = -1$, we need to have $\alpha_{p_4} = \alpha_{p_7} = 1$ so that $\alpha_{p_4} + \alpha_{t_0} \geq \text{wt}_M(p_4, t_0)$ and $\alpha_{p_7} + \alpha_{t_0} \geq \text{wt}_M(p_7, t_0)$. Thus the middle and rightmost level 2 gadgets corresponding to $c$ (see Fig. 3) have to be in dominant state in $\alpha$.
   
   - In case (2), the vertex $t_0$ prefers $p_7$ to its partner $s_2$ in $M$ and the vertex $s_0$ prefers $q_0$ to its partner $t_2$ in $M$. Thus $\alpha_{p_7} = \alpha_{q_0} = 1$ so that $\alpha_{p_7} + \alpha_{t_0} \geq \text{wt}_M(p_7, t_0)$ and $\alpha_{s_0} + \alpha_{q_0} \geq \text{wt}_M(s_0, q_0)$. Thus the leftmost and middle level 2 gadgets corresponding to $c$ (see Fig. 3) have to be in dominant state in $\alpha$.

   - In case (3), the vertex $s_0$ prefers $q_0$ and $q_3$ to its partner $t_3$ in $M$. Thus $\alpha_{s_0} = \alpha_{q_3} = 1$ so that $\alpha_{s_0} + \alpha_{q_0} \geq \text{wt}_M(s_0, q_0)$ and $\alpha_{s_0} + \alpha_{q_3} \geq \text{wt}_M(s_0, q_3)$. Thus the leftmost and middle level 2 gadgets corresponding to $c$ (see Fig. 3) have to be in dominant state in $\alpha$. \hfill $\square$

**Lemma 9.** For any clause $c$ in $\phi$, at least one of the level 1 gadgets corresponding to variables in $c$ is in dominant state in $\alpha$.

**Proof.** We showed in Lemma 8 that at least two of the three level 2 gadgets corresponding to $c$ are in dominant state in $\alpha$. Assume without loss of generality that these are the leftmost gadget and middle gadget (see Fig. 3).

In particular, we know from the proof of Lemma 8 that $\alpha_{q_0} = \alpha_{q_3} = 1$. This also forces $\alpha_{p_1} = \alpha_{p_4} = 1$. This is because $\alpha_{p_1}$ and $\alpha_{p_4}$ have to be non-negative since $p_1$ and $p_4$ are neighbors of the unmatched vertex $z$.

As $q_0$ and $p_1$ are the most preferred neighbors of $p_2$ and $q_2$, we have $\text{wt}_M(p_2, q_0) = \text{wt}_M(p_1, q_2) = 0$. Since $(p_2, q_0)$ and $(p_1, q_2)$ are popular edges, it follows from Lemma 2 that $\alpha_{p_2} = \alpha_{q_2} = -1$. Thus either (i) $(p_2, q_0)$ and $(p_0, q_2)$ are in $M$ or (ii) $(p_2, q_1)$ and $(p_1, q_2)$ are in $M$. This means that either $\text{wt}_M(p_2, y_j) = 0$ or $\text{wt}_M(x_k, q_2) = 0$. That is, either $\alpha_{y_j} = 1$ or $\alpha_{x_k} = 1$.

Similarly, $\text{wt}_M(p_5, q_3) = 0$ and we can conclude that $\alpha_{p_5} = \alpha_{q_3} = -1$. Thus either (i) $(p_5, q_3)$ and $(p_3, q_3)$ are in $M$ or (ii) $(p_5, q_4)$ and $(p_4, q_3)$ are in $M$. This means that either $\text{wt}_M(p_5, y_k) = 0$ or $\text{wt}_M(x_j, q_3) = 0$. That is, either $\alpha_{y_k} = 1$ or $\alpha_{x_j} = 1$.

Thus either (i) the gadgets corresponding to variables $X_i$ and $X_j$ are in dominant state or (ii) the gadget corresponding to $X_k$ is in dominant state in $\alpha$. Thus at least one of the level 1 gadgets corresponding to variables in $c$ is in dominant state in $\alpha$. \hfill $\square$
Lemma 10. For any clause $c$ in $\phi$, at most one of the level 1 gadgets corresponding to variables in $c$ is in dominant state in $\alpha$.

Proof. We know from observation 2 made at the start of this section that all the three level 0 gadgets corresponding to $c$ are in stable state in $\alpha$. So $\alpha_{a_i} = \alpha_{b_i} = 0$ for $1 \leq t \leq 6$. Either (i) $(a_1, b_1)$ and $(a_2, b_2)$ are in $M$ or (ii) $(a_1, b_2)$ and $(a_2, b_1)$ are in $M$. So either $\text{wt}_M(a_1, b_1') = 0$ or $\text{wt}_M(a_2, b_1) = 0$. So either $\alpha_{b_1'} \geq 0$ or $\alpha_{a_2} \geq 0$.

Consider any variable $X_i$. Either $\{(x_r, y_r'), (x_r', y_r')\} \subset M$ or $\{(x_r, y_r), (x_r', y_r')\} \subset M$. It follows from Lemma 2 that $\alpha_{x_r} + \alpha_{y_r'} = \text{wt}_M(x_r, y_r') = 0$ and $\alpha_{x_r'} + \alpha_{y_r} = \text{wt}_M(x_r', y_r) = 0$. Also due to the vertices $z$ and $z'$, we have $\alpha_{x_r} \geq 0$ and $\alpha_{y_r} \geq 0$. Thus $\alpha_{y_r'} \leq 0$ and $\alpha_{x_r'} \leq 0$.

Hence we can conclude that either $\alpha_{y_r'} = 0$ or $\alpha_{x_r'} = 0$. In other words, either the gadget corresponding to $X_j$ or the gadget corresponding to $X_k$ is in stable state. Similarly, by analyzing the level 0 gadget on vertices $a_t, b_t$ for $t = 3, 4$, we can show that either the gadget corresponding to $X_k$ or the gadget corresponding to $X_i$ is in stable state. Also, by analyzing the level 0 gadget on vertices $a_t, b_t$ for $t = 5, 6$, either the gadget corresponding to $X_i$ or the gadget corresponding to $X_j$ is in stable state.

Thus at least two of the three level 1 gadgets corresponding to variables in clause $c$ are in stable state in $\alpha$. Hence at most one of these three gadgets is in dominant state in $\alpha$. □

Lemma 11. If $H$ admits a desired popular matching then $\phi$ has a 1-in-3 satisfying assignment.

Proof. Let $M$ be a desired popular matching in $H$. That is, $M$ matches all in $X \cup Y$ and leaves $z, z'$ unmatched. Let $\alpha \in \{0, \pm 1\}^n$ be a witness of $M$.

We will now define a true/false assignment for the variables in $\phi$. For each variable $X_i$ in $\phi$ do:

- set $X_r$ to true if its level 1 gadget is in dominant state in $\alpha$, i.e., if $\alpha_{x_r} = \alpha_{y_r} = 1$ or equivalently, $(x_r, y_r')$ and $(x_r', y_r)$ are in $M$.
- else set $X_r$ to false, i.e., here $\alpha_{x_r} = \alpha_{y_r} = 0$ or equivalently, $(x_r, y_r)$ and $(x_r', y_r')$ are in $M$.

Since $M$ is our desired popular matching, it follows from Lemmas 9 and 10 that for every clause $c$ in $\phi$, exactly one of the three level 1 gadgets corresponding to variables in $c$ is in dominant state in $\alpha$. That is, for each clause $c$ in $\phi$, exactly one of the three variables in $c$ is set to true. □

4.2 The converse

Suppose $\phi$ admits a 1-in-3 satisfying assignment. We will now use this assignment to construct a desired popular matching $M$ in $H$. For each variable $X_i$ in $\phi$ do:

- if $X_i = \text{true}$ then include the edges $(x_r, y_r')$ and $(x_r', y_r)$ in $M$.
- else include the edges $(x_r, y_r)$ and $(x_r', y_r')$ in $M$.

Consider a clause $c = X_i \lor X_j \lor X_k$. We know that exactly one of $X_i, X_j, X_k$ is set to true in our assignment. Assume without loss of generality that $X_j = \text{true}$.

We will include the following edges in $M$ from all the gadgets corresponding to $c$. Corresponding to the level 0 gadgets for $c$ (see Fig. 2), we do:

- Add the edges $(a_1^i, b_1), (a_2^i, b_2), (a_3^i, b_3), (a_4^i, b_4)$ from the leftmost gadget and $(a_5^i, b_5), (a_6^i, b_6)$ from the rightmost gadget to $M$.
- We will select $(a_5^i, b_5), (a_6^i, b_6)$ from the middle gadget. (Note that we could also have selected $(a_3^i, b_3), (a_4^i, b_4)$ from the middle gadget.)

Corresponding to the level 2 gadgets for $c$ (see Fig. 3), we do:

- Add the edges $(p_0^c, q_0^c), (p_0^c, q_0^c), (p_2^c, q_4^c), (p_5^c, q_5^c)$ from the leftmost gadget, $(p_0^c, q_0^c), (p_2^c, q_4^c), (p_5^c, q_5^c)$ from the middle gadget, and $(p_0^c, q_0^c), (p_2^c, q_4^c), (p_5^c, q_5^c)$ from the rightmost gadget to $M$.

Since the leftmost and rightmost level 2 gadgets (see Fig. 3) are dominant, we will include $(s_0^c, t_2^c)$ and $(s_2^c, t_0^c)$ in $M$. Hence
Theorem 5. The matching $M$ described above is a popular matching.

Proof. We will prove $M$’s popularity by describing a witness $\alpha \in \{0, \pm 1\}^n$. That is, $\sum_{u \in X' \cup Y'} \alpha_u = 0$ and every edge will be covered by the sum of $\alpha$-values of its endpoints. i.e., $\alpha_u + \alpha_v \geq wt_M(u, v)$ for all edges $(u, v)$ in $H$. We will also have $\alpha_u \geq wt_M(u, u)$ for all vertices $u$.

Set $\alpha_z = \alpha_{z'} = 0$. Also set $\alpha_u = 0$ for all vertices $u$ in gadgets that are in stable state. That is, there are no blocking edges to $M$ in these gadgets. This includes all level 0 gadgets, and the gadgets in level 1 that correspond to variables set to false, and also the level 2 gadgets in stable state, i.e., such as the gadget with vertices $p_5^c, q_3^c, p_4^c, q_1^c, p_5^c, q_0^c$ (the middle gadget in Fig. 3) since we assumed $X_j = \text{true}$.

For every variable $X_r$ assigned to true: set $\alpha_{x_r} = \alpha_{y_r} = 1$ and $\alpha_{x_r'} = \alpha_{y_r'} = -1$. For every clause, consider the level 2 gadgets corresponding to this clause that are in dominant state: for our clause $c$, these are the leftmost and rightmost gadgets in Fig. 3 (since we assumed $X_j = \text{true}$).

Recall that we included in $M$ the edges $(p_0^c, q_0^c), (p_1^c, q_2^c), (p_2^c, q_1^c)$ from the leftmost gadget. We will set $\alpha_{p_0} = \alpha_{p_1} = \alpha_{p_2} = 1$ and $\alpha_{q_0} = \alpha_{q_1} = \alpha_{q_2} = -1$. We also included in $M$ the edges $(p_0^c, q_0^c), (p_1^c, q_1^c), (p_2^c, q_2^c)$ from the rightmost gadget. We will set $\alpha_{p_0} = \alpha_{p_1} = \alpha_{p_2} = 1$ and $\alpha_{q_0} = \alpha_{q_1} = \alpha_{q_2} = -1$.

In the level 3 gadget corresponding to $c$, we included the edges $(s_0^c, t_2^c), (s_1^c, t_1^c), (s_2^c, t_0^c), (s_3^c, t_3^c)$ in $M$. We will set $\alpha_{t_1} = \alpha_{t_2} = \alpha_{t_3} = 1$ and $\alpha_{s_1} = \alpha_{s_2} = \alpha_{s_3} = -1$.

The claim below shows that $\alpha$ is indeed a valid witness to $M$. Thus $M$ is a popular matching. □

Claim. The vector $\alpha$ defined above is a witness to $M$.

Proof. For any edge $(u, v) \in M$, we have $\alpha_u + \alpha_v = 0$, thus $\sum_{u \in X' \cup Y'} \alpha_u = 0$. For any neighbor $v$ of $z$ or $z'$, we have $\alpha_v \geq 0$. Thus all edges incident to $z$ or $z'$ are covered by the sum of $\alpha$-values of their endpoints. It is also easy to see that for every intra-gadget edge $(u, v)$, we have $\alpha_u + \alpha_v \geq wt_M(u, v)$.

In particular, the endpoints of every blocking edge to $M$ have $\alpha$-value set to 0. When $X_j = \text{true}$, in the gadgets involving clause $c$, $(x_j, y_j), (p_0^c, q_0^c), (p_2^c, q_2^c), (s_0^c, t_2^c)$ are blocking edges to $M$.

So we will now check that the edge covering constraint holds for all edges $(u, v)$ where $u$ and $v$ belong to different levels. Consider edges in $H$ between a level 0 gadget and a level 1 gadget. When $X_j = \text{true}$, the edges $(a_1^c, y_1^c)$ and $(x_j', b_j^c)$ are most interesting as they have one endpoint in a gadget in stable state and another endpoint in a gadget in dominant state.

Observe that both these edges are negative to $M$. This is because $a_1^c$ prefers its partner $b_j^c$ to $y_1^c$ and $y_1^c$ prefers its partner $x_j$ to $a_1^c$. Thus $wt_M(a_1^c, y_1^c) = -2 < \alpha_{a_1} + \alpha_{y_1} = 0 - 1$. Similarly, $b_j^c$ prefers its partner $a_1^c$ to $x_j'$ and $x_j'$ prefers its partner $y_j$ to $b_j^c$. Thus $wt_M(x_j', b_j^c) = -2 < \alpha_{x_j'} + \alpha_{b_j} = -1 + 0$.

We will now consider edges in $H$ between a level 1 gadget and a level 2 gadget. We have $wt_M(p_2^c, y_j) = 0$ since $p_2^c$ prefers $y_j$ to its partner $q_1^c$ while $y_j$ prefers its partner $x_j'$ to $p_2^c$. We have $\alpha_{p_2} + \alpha_{y_j} = -1 + 1 = \alpha_{x_j'} + \alpha_{y_j} = 0$. The edge $(x_j, q_1^c)$ is negative to $M$ and so this is covered by the sum of $\alpha$-values of its endpoints. Similarly, $(p_0^c, y_j)$ is negative to $M$ while $wt_M(x_j, q_0^c) = 0 = 1 - 1 = \alpha_{p_0} + \alpha_{y_j}$. We have $wt_M(p_0^c, y_j) = 0$ and $\alpha_{p_0} = \alpha_{y_j} = 0$. Similarly, $wt_M(x_i, q_0^c) = 0$ and $\alpha_{x_i} = \alpha_{q_0} = 0$. Thus all these edges are covered.

We will now consider edges in $H$ between a level 2 gadget and a level 3 gadget. These edges are $(s_0^c, q_0^c), (s_0^c, q_1^c), (p_2^c, t_0^c), (p_2^c, t_0^c)$. We have $wt_M(s_0^c, q_0^c) = 0$ and $\alpha_{s_0} = -1, \alpha_{q_0} = 1$, so this edge is covered. Similarly, $wt_M(p_2^c, t_0^c) = 0$ and $\alpha_{p_2} = 1, \alpha_{t_0} = -1$. The edges $(s_0^c, q_1^c)$ and $(p_2^c, t_0^c)$ are negative to $M$, so they are also covered. Thus it can be checked that $\alpha$ is a witness for $M$. □

Thus $H$ admits a desired popular matching if and only if $\phi$ has a 1-in-3 satisfying assignment. This completes the proof of Theorem 4.
5 Dominant matchings

Recall that a popular matching \( M \) is dominant if \( M \) is more popular than every larger matching. Observe that every popular matching \( M \) in our roommates instance \( G = (V, E) \) is a max-size matching; this is because \( M \) matches all vertices in \( G \) except the vertex \( z \) (by Lemma 4). Thus every popular matching in \( G \) is dominant and so it follows from Theorem 1 that the dominant matching problem in \( G \) is NP-hard.

Note that the instance \( G \) does not admit a stable matching. This is due to the gadget \( D = \{d_0, d_1, d_2, d_3\} \). However the instance \( G_0 = G \setminus D \) admits stable matchings. It is easy to see that a stable matching in \( G_0 \) matches all vertices in \( X \cup Y \) except the vertices \( s_0^c, t_0^c \) for all clauses \( c \).

Lemma 12. A popular matching \( N \) in \( G_0 \) is dominant if and only the set of vertices matched in \( N \) is \( X \cup Y \).

Proof. Let \( N \) be any popular matching in \( G_0 \). Any popular matching has to match all stable vertices in \( G_0 \) (those matched in any stable matching) \( \mathcal{H} \), thus \( N \) matches all stable vertices in \( G_0 \). Suppose some unstable vertex in \( X \cup Y \) (say, \( s_0^c \)) is left unmatched in \( N \). We claim that \( t_0^c \) also has to be left unmatched in \( N \). Since \( s_0^c \) and \( t_0^c \) have no other neighbors, the edge \( (s_0^c, t_0^c) \in N \) and so there is an augmenting path \( \rho = s_0^c - t_0^c \) with respect to \( N \). Observe that \( N \) is not more popular than \( N \cup \rho \), a larger matching. Thus \( N \) is not a dominant matching in \( G_0 \).

In order to justify that \( t_0^c \) also has to be left unmatched in \( N \), let us view \( N \) as a popular matching in \( H \). We know that \( s_0^c \) and \( t_0^c \) belong to the same connected component in the popular subgraph \( F_H \) (by Lemma 5). So if \( s_0^c \) is left unmatched in \( N \), then \( t_0^c \) is also unmatched in \( N \) (by Lemma 3).

Conversely, suppose \( N \) is a popular matching in \( G_0 \) that matches all vertices in \( X \cup Y \). Then there is no larger matching than \( N \) in \( G_0 \) and thus \( N \) is a dominant matching.

Thus a dominant matching exists in \( G_0 \) if and only if there is a popular matching in \( G_0 \) that matches all vertices in \( X \cup Y \). Hence it follows from Theorem 1 that the dominant matching problem is NP-hard even in roommates instances that admit stable matchings. Thus Theorem 2 stated in Section 1 follows.

References

1. D.J. Abraham, R.W. Irving, T. Kavitha, and K. Mehlhorn. Popular matchings. SIAM Journal on Computing, 37(4): 1030–1045, 2007.
2. P. Biró, R. W. Irving, and D. F. Manlove. Popular Matchings in the Marriage and Roommates Problems. In the 7th International Conference in Algorithms and Complexity (CIAC): 97–108, 2010. (Technical Report TR-2009-306, University of Glasgow, 2009)
3. Á. Cseh. Popular Matchings. In Trends in Computational Social Choice, Edited by Ulle Endriss, COST (European Cooperation in Science and Technology): 105–122, 2017.
4. Á. Cseh, C.-C. Huang, and T. Kavitha. Popular matchings with two-sided preferences and one-sided ties. In the 42nd International Colloquium on Automata, Languages, and Programming (ICALP): Part I, 367–379, 2015.
5. Á. Cseh and T. Kavitha. Popular edges and dominant matchings. In the 18th International Conference on Integer Programming and Combinatorial Optimization (IPCO): 138–151, 2016.
6. D. Gale and L.S. Shapley. College admissions and the stability of marriage. American Mathematical Monthly, 69(1): 9–15, 1962.
7. P. Gårdenfors. Match making: assignments based on bilateral preferences. Behavioural Sciences, 20(3): 166–173, 1975.
8. S. Gupta, P. Misra, S. Saurabh, and M. Zehavi. Popular Matching in Roommates Setting is NP-hard. https://arxiv.org/pdf/1803.09370.pdf
9. C.-C. Huang and T. Kavitha. Popular matchings in the stable marriage problem. Information and Computation, 222: 180–194, 2013.
10. C.-C. Huang and T. Kavitha. Near-Popular Matchings in the Roommates Problem. SIAM Journal on Discrete Mathematics, 27(1): 43–62, 2013.
11. C.-C. Huang and T. Kavitha. Popularity, Self-Duality, and Mixed matchings. In the 28th ACM-SIAM Symposium on Discrete Algorithms (SODA): 2294-2310, 2017.
12. R. W. Irving. *An efficient algorithm for the stable roommates problem*. Journal of Algorithms, 6: 577–595, 1985.

13. T. Kavitha, J. Mestre, and M. Nasre. *Popular mixed matchings*. Theoretical Computer Science, 412(24): 2679–2690, 2011.

14. T. Kavitha. *A size-popularity tradeoff in the stable marriage problem*. SIAM Journal on Computing, 43(1): 52–71, 2014.

15. T. Kavitha. *Popular half-integral matchings*. In the 43rd International Colloquium on Automata, Languages, and Programming (ICALP): 22.1-22.13, 2016.

16. T. Kavitha. *Max-size popular matchings and extensions*. [http://arxiv.org/abs/1802.07440](http://arxiv.org/abs/1802.07440)

17. D. J. Manlove. *Algorithmics of Matching Under Preferences*. World Scientific, 2013.

18. T. J. Schaefer. The complexity of satisfiability problems. In the 10th Annual ACM Symposium on Theory of Computing, 216–226, 1978.

19. A. Subramanian. *A new approach to stable matching problems*. SIAM Journal on Computing, 23(4): 671–700, 1994.

20. J. J. M. Tan. A necessary and sufficient condition for the existence of a complete stable matching. *Journal of Algorithms*, 12: 154–178, 1991.

21. C.-P. Teo and J. Sethuraman. *The geometry of fractional stable matchings and its applications*. Mathematics of Operations Research, 23(4): 874–891, 1998.