On the Lebesgue measure of the Feigenbaum Julia set

Artem Dudko

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The Feigenbaum quadratic polynomial

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Filled Julia set $\mathcal{K}(f) = \{ z \in \mathbb{C} : \{ f^n(z) \}_{z \in \mathbb{N}} \text{ is bounded} \}$.

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Figure: The airplane map $p(z) = z^2 + c$, $c \approx -1.755$. 
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Figure: The Julia set of $f_{\text{Feig}}$
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Figure: The Julia set of \( f_{\text{Feig}} \)

**Theorem (D.-Sutherland)**

The Julia set of \( f_{\text{Feig}} \) has Hausdorff dimension less than two (and hence its Lebesgue measure is zero).
Renormalization

A quadratic-like map is a ramified covering \( f : U \to V \) of degree 2, where \( U \subseteq V \) are topological disks in \( \mathbb{C} \).
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A quadratic-like map \( f \) is called renormalizable with period \( n \) if there exist domains \( U' \subset U \) for which \( f^n : U' \to V' = f^n(U') \) is a quadratic-like map. The map \( f^n|_{U'} \) is called a pre-renormalization of \( f \); the map \( R_n f := \Lambda \circ f^n|_{U'} \circ \Lambda^{-1} \), where \( \Lambda \) is an appropriate rescaling of \( U' \), is the renormalization of \( f \).
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The map $f_{Feig}$ is infinitely renormalizable. The sequence of germs $R^k(f_{Feig})$ converges geometrically fast to $F$ (Lanford, Sullivan, McMullen).
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**Definition**

A *Feigenbaum map* is an infinitely renormalizable quadratic-like map with bounded combinatorics and a priori bounds.
Denote by $f_n$ the $n$-th prerenormalization of $f$, by $\mathcal{J}_n$ the Julia set of $f_n$ and by $\mathcal{O}(f)$ the critical orbit of $f$. 
Nice domains

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Avila and Lyubich constructed domains \( U^n \subset V^n \) (called nice domains) for which

- \( f_n(U^n) = V^n \);
- \( U^n \supset \mathcal{J}_n \cap \mathcal{O}(f) \);
- \( V^{n+1} \subset U^n \);
- \( f^k(\partial V^n) \cap V^n = \emptyset \) for all \( n, k \);
- \( A^n = V^n \setminus U^n \) is “far” from \( \mathcal{O}(f) \);
- \( \text{area}(A^n) \asymp \text{area}(U^n) \asymp \text{diam}(U^n)^2 \asymp \text{diam}(V^n)^2 \).
Escaping and returning set

For each $n \in \mathbb{N}$, let $X_n$ be the set of points in $U^0$ that land in $V^n$ under some iterate of $f$, and let $Y_n$ be the set of points in $A^n$ that never return to $V^n$ under iterates of $f$. Introduce the quantities

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\eta_n = \frac{\text{area}(X_n)}{\text{area}(U^0)}, \quad \xi_n = \frac{\text{area}(Y_n)}{\text{area}(A^n)}.
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Theorem (Avila-Lyubich)

Let \( f \) be a periodic point of renormalization (\( \mathcal{R}^p f = f \) for some \( p \)). Then exactly one of the following is true:

1. \( \eta_n \) converges to 0 exponentially fast, \( \inf \xi_n > 0 \), and \( \dim H(J f) < 2 \) (Lean case);
2. \( \eta_n \approx \xi_n \approx \frac{1}{n} \) and \( \dim H(J f) = 2 \) with \( \text{area}(J f) = 0 \) (Balanced case);
3. \( \inf \eta_n > 0 \), \( \xi_n \) converges to 0 exponentially fast, and \( \text{area}(J f) > 0 \) (Black Hole case).
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An approach to prove $\dim_H(\mathcal{J}_{\text{Feig}}) < 2$.

One can construct a number $C > 0$ (depending on the geometry of $A^n$ and the critical orbit $O(f)$) such that if $\eta_n/\xi_n < C$ for some $n$ then $\dim_H(\mathcal{J}_{\text{Feig}}) < 2$. But ...
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need a different approach.
The structure of $F$.

The Cvitanović-Feigenbaum equation:

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\begin{align*}
F(z) &= -\frac{1}{\lambda} F^2(\lambda z), \\
F(0) &= 1, \\
F(z) &= H(z^2),
\end{align*}
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with $H^{-1}(z)$ univalent in $\mathbb{C} \setminus ((-\infty, -\frac{1}{\lambda}] \cup [\frac{1}{\lambda^2}, \infty))$, where $\frac{1}{\lambda} = 2.5029\ldots$ is one of the Feigenbaum constants.
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Proposition (H. Epstein)

*The map $F$ has a maximal analytic extension to $\hat{F} : \hat{W} \to \mathbb{C}$, where $\hat{W} \supset \mathbb{R}$ is open, simply connected and dense in $\mathbb{C}$.*
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Theorem (H. Epstein, X. Buff)

All critical points of $\hat{F}$ are simple. The critical values of $\hat{F}$ are contained in real axis. Moreover, $\hat{F}$ is a ramified covering.
Partition of $\hat{\mathcal{W}}$
Central tiles

Denote by $P_I$, $P_{II}$, $P_{III}$ and $P_{IV}$ the connected components of $\hat{F}^{-1}(\mathbb{H}_\pm)$ containing 0 on the boundary. Set

$$W = \text{Int}(\overline{P_I \cup P_{II} \cup P_{III} \cup P_{IV}}).$$
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Denote by \( F \) the quadratic like restriction of \( \hat{F} \)

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Denote by $F$ the quadratic like restriction of $\hat{F}$

$$W \rightarrow \mathbb{C} \setminus ((-\infty, -\frac{1}{\lambda}] \cup [\frac{1}{\lambda^2}, \infty)).$$

For $n \in \mathbb{N}$ and any set $A$ let $A^{(n)} = \lambda^n A$ and denote by $F_n = F^{2^n}|_{W^{(n)}}$ the $n$-th pre-renormalization of $F$. 

The (new) returning sets

\[ \tilde{X}_n = \{ z \in W^{(1)} : F^k(z) \in W^{(n)} \text{ for some } k \}, \quad \tilde{\eta}_n = \frac{\text{area}(\tilde{X}_n)}{\text{area}(W^{(1)})}. \]
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Using Avila-Lyubich trichotomy we obtain:

**Proposition**

\[ \dim_H(\mathcal{J}_F) < 2 \text{ if and only if } \tilde{\eta}_n \to 0 \text{ exponentially fast}. \]
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Idea to prove \( \tilde{\eta}_n \to 0 \): construct recursive estimates of the form

\[ \tilde{\eta}_{n+m} \leq C_{n,m} \tilde{\eta}_n \tilde{\eta}_m. \]
Copies of central tiles

We will call any connected component of $P_j^{(m)}$, where $k, m \in \mathbb{Z}_+$, $J$ is a quadrant, a *copy of $P_j^{(m)}$ under $F^k$*. 
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![Graph showing copies of central tiles]
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A copy $Q$ of $P_{j}^{(m)}$ under $F^{k}$ is primitive if $F^{i}(Q) \cap W^{(m)} = \emptyset$ for all $0 \leq i < k$. 
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A copy $Q$ of $P_j^{(m)}$ under $F^k$ is separated if there exists $0 \leq i < k$ with $F^i(Q) \subset W^{(m)}$ and $F^i(Q) \cap J_F^{(n-1)} = \emptyset$ for maximal such $i$. 
Comparing the returning sets

Fix $n, m \in \mathbb{N}$. Let $\mathcal{P}$ and $\mathcal{S}$ be the collection of all primitive and separated copies of $P_j^{(m)}$, where $J$ is any quadrant. Modulo zero measure one has:

$$\tilde{X}_n = \bigcup_{Q \in \mathcal{P}} Q.$$

For a copy $Q$ of $P_j$ under $F_k$ set

$$X_Q = F_k^-(\lambda n - 1) \tilde{X}_{m+1} \cap Q.$$
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Modulo zero measure one has

$$\tilde{X}_{n+m} = \bigcup_{Q \in \mathcal{P} \cup \mathcal{S}} X_Q.$$
Proposition

Let $T$ be a primitive or a separated copy of $P_j^{(m)}$ under $F^k$ with $m \geq 2$. Then the inverse branch $\phi : P_j^{(m)} \to T$ of $F^k$ analytically continues to a univalent map on $\text{sign}(P_j^{(m)}) \lambda^m \mathbb{C}_\lambda$ where

$$\mathbb{C}_\lambda = \mathbb{C} \setminus \left( (-\infty, -\frac{1}{\lambda}] \cup \left[ \frac{F(\lambda)}{\lambda^2}, \infty \right) \right).$$
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Koebe distortion

We construct numbers $M(A)$ such that

**Corollary**

Let $A, B$ be two measurable subsets of $P_J$ of positive measure and let $T$ be a primitive or a separated copy of $P_L^{(m)}$ under $F^k$ for some $k \geq 0$ and $m \geq 2$. Then

$$\frac{\text{area}(F^{-k}(B^{(m)}) \cap T)}{\text{area}(F^{-k}(A^{(m)}) \cap T)} \leq M(A)\text{area}(B).$$
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Notice, $\lambda^{n-1} \tilde{X}_{m+1} \subset W^{(n)}$. 
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$$\frac{\text{area}(F^{-k}(B^{(m)}) \cap T)}{\text{area}(F^{-k}(A^{(m)}) \cap T)} \leq M(A)\text{area}(B).$$

Notice, $\lambda^{n-1}\tilde{X}_{m+1} \subset W^{(n)}$. Set

$$\Sigma_{n,m} = P^{(n)}_1 \setminus (\lambda^{n-1}\tilde{X}_{m+1} \cup \bigcup_{Q \in \mathcal{G}} Q),$$

$$M_{n,m} = M(\lambda^{-n}\Sigma_{n,m} \cap P_1).$$
Recursive estimates

Using the identities

\[
\text{area}(\tilde{X}_n) = \sum_{Q \in \mathcal{P}} \text{area}(Q), \\
\text{area}(\tilde{X}_{n+m}) = \sum_{Q \in \mathcal{P} \cup \mathcal{S}} \text{area}(X_Q),
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we show:

**Proposition**

*For every \( n \geq 2 \) and \( m \geq 1 \), one has*

\[
\tilde{\eta}_{n+m} \leq M_{n,m} \text{area}(P_i) \tilde{\eta}_n \tilde{\eta}_{m+1}.
\]
Results of computations

Using rigorous computer estimates we prove:

\[ M_6 = \lim_{m \to \infty} M_{6,m} < 9.4, \quad \tilde{\eta}_6 = \frac{\text{area}(\tilde{X}_6 \cap P_{1}^{(1)})}{\text{area}(P_{1}^{(1)})} < \frac{0.09}{\text{area}(P_{1})}. \]

We obtain \( \tilde{\eta}_6 M_6 \text{area}(P_{1}) < 0.846 < 1 \), so \( \mathcal{J}_F \) has Hausdorff dimension less than 2.
Computing the escaping set

Let $V_2 = (-\infty, -\frac{1}{\lambda}] \cup F^{-3}(V) \cup \left[\frac{1}{\lambda^2}, \infty\right)$. 
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Lemma

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**Figure:** While the preimage labeled $D_{16}$ partially intersects $W^{(3)}$, it lies completely inside $\tilde{W}_3 = W^{(1)}$. 
Thank you!