Mott Insulators Without Symmetry Breaking

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(Dated: May 3, 2018)

We present theoretical models, in one and two space dimensions, that exhibit Mott insulating ground states at fractional occupations without any symmetry breaking. The Hamiltonians of these models are non-local in configuration space, but local in phase space.

![Diagram of hopping process](image)

**FIG. 1:** The effect of $H_{1D}$ is to hop a pair of particles while conserving their center of mass position. The hopping range is set by $\kappa^{-1}$. In this figure the nearest neighbor pair hopping is illustrated.

The lesson in constructing models of this kind.

Without further delay let us define these models. First we focus on a one-dimensional (1D) model. Consider a finite lattice with $M = 3N - 2$ sites where $N$ is an odd integer. We label each site by an integer $n$ with the central site $n = 0$. On this lattice we put $N$ spinless fermions. The occupation number is given by $\nu = N/(3N - 2)$ which approaches $1/3$ in the $N \to \infty$ limit. It turns out that this choice of the number of sites ensures that the ground state is non-degenerate for all $N$. The Hamiltonian that describes the interaction between these $N$ fermions is given by

$$H_{1D} = g\kappa^3 \sum_p b_p^+ b_p.$$  \hspace{1cm} (1)

In the above $g$ is an energy parameter, and is positive. The parameter $\kappa$, whose meaning shall be discussed later, is dimensionless ($0 < \kappa < 1$). Since $H_{1D}$ is a sum of positive-definite operators, its eigenvalues are clearly non-negative.

In $H_{1D}$ the index $p$ runs through both integers and half integers, and

$$b_p = \sum_q (qe^{-\kappa^2 q^2})c_{p+q}.$$ \hspace{1cm} (2)

where $q$ takes integer/half-integer value depending on whether $p$ is integer/half-integer. In Eq. (2) $c_{p+q}$ annihilates a fermion at lattice site $p \pm q$. $H_{1D}$ describes the hopping of pairs of fermions from sites $p - q$ and $p + q$ to $p - k$ and $p + k$ (Fig.1). In such a hopping process the center-of-mass position ($p$) of the fermion pair remains unchanged. From Eq. (2) it is apparent that $\kappa^{-1}$ controls the hopping range.

For occupation $\nu = N/(3N - 2)$ and any $\kappa$, the model
defined by Eq. (1) possesses a ground state given by

$$|\Psi_{1D} > = \sum_{\{n\}} \chi(n_1, ..., n_N) |c_{n_1}^+, ..., c_{n_N}^+ |0>, \quad (3)$$

where

$$\chi(\{n\}) \propto \prod_j e^{-\kappa^2 n_j^2/2} \int_{-\infty}^{\infty} dx_j \int_{0}^{2\pi} dy_j e^{-x_j^2/\kappa^2}$$

$$\times e^{n_j(x_j-i\eta_j)} \prod_{j>k} \sinh^3 \left[ (x_j-x_k) + i(y_j-y_k) \right]. \quad (4)$$

This state is annihilated by all the $b_p$'s, i.e.,

$$b_p |\Psi_{1D} > = 0. \quad (5)$$

As a result $H_{1D} |\Psi_{1D} > = 0$ implying $|\Psi_{1D} >$ is a ground state of $H_{1D}$. Eq. (5) can be shown to be true by rewriting it in terms of the wavefunction $\chi(\{n\})$ and using the following identity[9]

$$\int_{-\infty}^{\infty} dx_1 dx_2 \int_{0}^{2\pi} dy_1 dy_2 \Phi_p(z_1, z_2) \prod_{j>k} \sinh^3 \left( \frac{z_j-z_k}{2} \right)$$

$$\times e^{-\kappa^2 \sum k z_k^2} = 0, \text{ for all } p. \quad (6)$$

The $\Phi_p$ in Eq. (6) is given by

$$\Phi_p(z_1, z_2) = \left[ e^{-\kappa^2 p^2} e^{\pi(z_1-z_2)-i\pi(x_1+x_2)^2} \right] \times \left[ \sum_q q e^{-2\kappa^2 q^2} e^{\pi(q(z_1-z_2)-i\pi(x_1-x_2)^2)} \right]. \quad (7)$$

In the latter part of the paper we shall show that for sufficiently small positive values of $\kappa$, $|\Psi_{1D} >$ is the unique ground state of $H_{1D}$, it separated by an energy gap from the lowest excited states, and it does not break translational symmetry. Meanwhile, to get a feeling for $|\Psi_{1D} >$, let us look at a representation of it for $3$ particles, as shown in Fig.2. The result is a coherent superposition of many different configurations. The relative weight and phase of these configurations ensures that they are annihilated by $b_p$. Any breaking of the weight-phase relation causes excited states.

Next we present a two dimensional (2D) model with similar properties. The lattice consists of $M'$ chains, with each chain made up of $M = 3N - 2$ sites. In each chain we place $N$ spinless fermions. As the result, the 2D occupation number is also given by $\nu = N/(3N-2)$. In the limit $N, M' \to \infty$ the occupation number is also 1/3. The Hamiltonian of the 2D model is given by

$$H_{2D} = \sum_{n,p} \left[ g_1 \kappa^3 b_{n,p}^+ b_{n,p} + g_2 \kappa^3 (b_{n,p}^+ b_{n,p})^2 + b_{n+1,p}^+ b_{n,p} \right]. \quad (8)$$

where $\nu > 1$ is the previous $|\Psi_{1D} >$ defined for the n-th chain, is a zero-energy eigen state of $H_{2D}$. This is because $b_p$ annihilates $|\Psi_{1D} >$. Let us imagine obtaining $H_{2D}$ by turning on $g_2$ from zero while holding $g_1$ fixed. At non-zero $g_2$ the excited states of adjacent chains are mixed. However, so long as $g_2$ is small compared to the excitation gap at $g_2 = 0$, this mixing will not produce negative energy eigen states.[10] Under that condition $|\Psi_{2D} >$ continues to be the exact ground state of $H_{2D}$, and it is separated by an energy gap from the excited states. Since $|\Psi_{1D} >$ does not break translation symmetry, so does'nt $|\Psi_{2D} >$.

The remaining task is to collaborate the statements we have made about $|\Psi_{1D} >$. To accomplish this we will make use of a mapping from the 1D Hamiltonian to a well-studied problem in two dimensions, namely, the spinless fermions in a strong magnetic field.

It is well known that when the lowest Landau level (LLL) is 1/3 filled, and if the interaction between fermions falls off sufficiently fast as a function of the interparticle distance, the ground state is an incompressible quantum fluid. It is also well-known that when a particle...
is confined to the lowest Landau level the two components of its coordinate no longer commute. This implies that the dynamics of the fermions is effectively that of a 1D system. This intrinsic one-dimensionality can be made explicit through a mapping of the LLL problem to one dimension. What has not been appreciated is that although the time reversal symmetry is explicitly broken in the 2D problem, the corresponding 1D Hamiltonian is time-reversal invariant. Moreover, after the mapping to 1D, the ground state is an example of a featureless Mott insulator at 1/3 occupation number.

To start, let us consider a spin-polarized two dimensional electron gas where the electrons interact through a two-body potential

\[ H_{\text{int}} = V_0 \int d^2 r d^2 r' V(\mathbf{r} - \mathbf{r}') \psi^+(\mathbf{r}) \psi^+(\mathbf{r}') \psi^-(\mathbf{r}') \psi^-(\mathbf{r}) \]

\[ V(\mathbf{r}) = \nabla^2 \delta(\mathbf{r}). \]  

(11)

Although this interaction potential is a singular function of the coordinates, it has non-singular matrix element between states in the lowest Landau level. Laughlin’s \( \nu = 1/3 \) wavefunction [11] is known to be the exact ground state of this potential [12]. The gap to the lowest excited state is proportional to the parameter \( V_0/l_B^2 \), where \( l_B \) is the magnetic length.

Next we place the above two dimensional electron gas on a cylinder with circumference \( L \) with the magnetic field perpendicular to the surface. In the Landau gauge the LLL basis orbitals are of the following form

\[ \phi_n(\mathbf{r}) = \frac{1}{\sqrt{\pi^{1/2} L l_B}} e^{i2\pi n y/L} e^{-(x - 2\pi n l_B^2/L^2)/2l_B^2}, \]  

(12)

These orbitals are delocalized around the cylinder (\( y \)) while localized along its axis (\( x \)). The quantization of the \( y \)-momentum in units of \( 2\pi/L \) explains why we have a lattice. If we expand the field operator \( \psi(\mathbf{r}) \), projected to the lowest Landau level, as \( \psi(\mathbf{r}) = \sum_n \phi_n(\mathbf{r}) c_n \), substitute the result into Eq. (11), and rescale the coordinates by \( L/2\pi \), we obtain the Hamiltonian (1) with \( \kappa = 2\pi l_B/L \), and \( g = \frac{4V_0}{(2\pi)^{1/2} l_B^2} \). Thus \( \kappa \) measures the ratio between the magnetic length and the circumference of the cylinder. Since \( \kappa^{-1} \) controls the hopping lengths of the 1D and 2D Hamiltonians, it is clearly desirable to restrict \( \kappa > 0 \). On the other hand, to preserve the liquid property of the Laughlin state we need to keep \( L > l_B \) (or \( \kappa < 2\pi \)). To satisfy both requirements we assume \( 0 < \kappa < 1 \).

In Landau gauge, the \( \nu = 1/3 \) Laughlin wavefunction on a cylinder reads [13, 14]

\[ \psi_{1/3}(x_j, y_j) \propto \prod_{j > k} \sin^3 \left( \frac{x_j - z_k}{L/\pi} \right) e^{-\sum_j x_j^2/2l_B^2}, \]  

(13)

where \( z = x + iy \). The \( \chi(\{n_k\}) \) given in Eq. (4) are the coefficients of \( \psi_{1/3} \) when expanded in terms of products of the lowest Landau level orbitals (Eq. (12)).

At this point we have mapped \( H_{\text{int}} \) to \( H_{1D} \), and the Laughlin state \( |\Psi_{1/3} \rangle > |\Psi_{1D} \rangle > \). Since the above mapping is a unitary transformation, all known properties of \( |\Psi_{1/3} \rangle \) and \( H_{\text{int}} \) are preserved. For example, the fact that \( |\Psi_{1/3} \rangle \) describes a quantum liquid at magnetic filling factor 1/3 translates into the statement that \( |\Psi_{1D} \rangle \) is a quantum liquid at occupation number 1/3. The fact that \( |\Psi_{1/3} \rangle \) is the non-degenerate ground state of \( H_{\text{int}} \) implies that \( |\Psi_{1D} \rangle \) is the non-degenerate ground state of \( H_{1D} \). Finally, the fact that \( H_{\text{int}} \) possesses an energy gap and fractional charge quasiparticles at filling factor 1/3 implies the same for \( H_{1D} \) at lattice occupation 1/3. In this way we have established that the model presented by \( H_{1D} \) possesses a featureless Mott insulating ground state at fractional occupation number!

By using the 1D to 2D mapping discussed previously it is simple to map \( H_{2D} \) to a 3D Hamiltonian \( H'_{\text{int}} \) describing coupled layers:

\[ H'_{\text{int}} = \sum_n H_n \]

\[ H_n = \int d^2 r d^2 r' V(\mathbf{r} - \mathbf{r}') \left[ \psi^+_n(\mathbf{r}) \psi^+_n(\mathbf{r}') \psi_n(\mathbf{r}') \psi_n(\mathbf{r}) + V_2 (\psi^+_{n+1}(\mathbf{r}) \psi^+_{n+1}(\mathbf{r}') \psi_{n+1}(\mathbf{r}') \psi_{n+1}(\mathbf{r}) + h.c.) \right]. \]  

(14)

In the above equation \( V(\mathbf{r}) \) is the same as that given in Eq. (11), and \( V_{1,2} \propto g_{1,2} \) respectively. With \( V_2 = 0 \) the problem becomes that of many independent layers. At \( \nu = 1/3 \) each layer is in the liquid ground state described by \( |\Psi_{1/3} \rangle \). The term proportional to \( V_2 \) hops a pair of electrons between adjacent layers. As discussed below Eq. (10), so long as \( V_2 \) is sufficiently smaller than \( V_1 \), the ground state is the direct product of the 1/3 Laughlin liquid in each layer.

It is interesting to note that unlike usual Hamiltonians, \( H_{1D} \) and \( H_{2D} \) do not have the single-fermion hopping term. One can in fact add a single particle hopping term

\[ H'_{1D} = -te^{-\kappa^2/4} \sum_n [c^+_n c_n + h.c.] \]  

(15)

to \( H_{1D} \) without changing qualitatively any of the results. To understand this we note that after mapping to 2D Eq. (15) becomes

\[ -t \int d^2 r \cos(2\pi y/L) \psi^+(\mathbf{r}) \psi(\mathbf{r}), \]  

(16)

i.e., a periodic potential. It is clear that so long as \( |t| \) is much smaller than the energy gap of \( H_{\text{int}} \), it will not change the quantum liquid nature of the ground state and will not collapse the excitation gap. Similar single-particle terms can by added to \( H_{2D} \). Again, so long as the strength of the single particle hopping terms is sufficiently small, the qualitative results we discussed above will be preserved.
Models that gives featureless Mott insulating ground state at fractional occupation number have been difficult to find. What is special about the models presented in this paper? To answer this question let us focus on $H_{1D}$. A standard (1D) interaction potential is local in configuration space (q-space), i.e.,

$$
\hat{V} = \prod_j \int dq_j \sum_{i<j} U(q_i - q_j)|q_1, ..., q_N > < q_1, ..., q_N | (17)
$$

When this type of potential dominates the Hamiltonian, it favors particles to assume a fixed q-space configuration. For a repulsive potential such a configuration often take the form of a regular lattice. $H_{1D}$, on the other hand, is non-local in q-space and the particles do not necessarily have to pay a high price in energy to get close. However, there is a hidden locality in $H_{1D}$ which is revealed not in configuration space, but in phase space.

The quantum description of phase space dynamics is most conveniently done in terms of coherent states. For a single particle with coordinate $q$ and momentum $p$, the coherent state $|z>$ satisfies

$$
[\hat{q} + i \frac{\lambda^2}{\hbar} \hat{p}] |z> = z |z>,
$$

where $z = x + i y$ and $\lambda$ is any pre-chosen length scale serving to make $\hat{q}$ and $(\lambda^2/\hbar) \hat{p}$ the same dimension. If $|q>$ denotes the position eigenstate, it is simple to show that

$$
<q|z> \propto e^{iyq/\lambda^2} e^{-(x-q)^2/2\lambda^2}.
$$

and this function defines the transformation between the phase space and configuration space descriptions. It is interesting to note if we identify $\lambda$ with the magnetic length and $q/\lambda^2$ with the y-momentum, the functions given in Eq. (19) become precisely the LLL basis orbitals in Eq. (12). Thus a 2D quantum mechanical problem defined in the LLL is completely equivalent to the coherent state representation of a 1D quantum problem.

Let us go back to the non-local Hamiltonian $H_{1D}$. When expressed in terms of the coherent state basis, it gets a local form

$$
H_{1D} = \sum_{i<j} \int \frac{d^2 z_k}{2\pi \lambda^2} V(z_i - z_j)|z_1, ..., z_N > < z_1, ..., z_N | (20)
$$

The $V(z_i - z_j)$ in the above equation is the potential given in $H_{int}$. This potential prevents particles from getting close together in the phase space. However due to the non-commutativity of the phase space coordinates, the particles are not frozen in any particular configuration. We believe the fact that $H_{1D}$ and $H_{2D}$ are linked to phase-space local potential is the main reason that they have featureless Mott insulating ground states at fractional occupation numbers.

In conclusion, we have constructed 1D and 2D lattice models that exhibit featureless Mott insulating ground state at fractional occupation number. The Hamiltonians of these models have finite range interactions, and respect lattice translation and time reversal symmetries. These Hamiltonians are local in phase space instead of configuration space. We believe phase space local interaction could be a missing key for constructing models of this type of novel insulators.

DHL is supported by DOE grant DE-AC03-76SF00098. JML thanks the Miller Institute for Basic Research in Science for financial support and hospitality during his recent visit. Supports from the Research Council of Norway and the Fulbright foundation are also acknowledged.

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