RECURSIVE MARKOV PROCESS

BY SHOHEI HIDAKA

Japan Advanced Institute of Science and Technology

A Markov process, which is constructed recursively, arises in stochastic games with Markov strategies. In this study, we defined a special class of random processes called the recursive Markov process, which has infinitely many states but can be expressed in a closed form. We derive the characteristic equation which the marginal stationary distribution of an arbitrary recursive Markov process needs to satisfy.

1. Introduction. For $N \in \mathbb{N}$, we consider the evolution of a random variable $X_t \in \mathcal{N} = \{1, 2, \ldots, N\}$ over a discrete time step $t \in \mathbb{Z}$. We call a random process the $k^{th}$ order Markov, if the transition probability of a state at any time step is determined only by its $k$ past states:

$$(1.1) \quad P(X_t \mid X_{t-1}, X_{t-2}, \ldots, X_{t-k}) = P(X_t \mid X_{t-1}, X_{t-2}, \ldots).$$

Denote the series of $k$ states by

$$X_{t-k+1}^t := (X_t, X_{t-1}, \ldots, X_{t-k+1}) \in \mathcal{N}^k.$$

Applying (1.1) $k$ times, we obtain the transition probability $P(X_{t-k+1}^t \mid X_{t-k}^{t-1})$ for the first order Markov process over the series of states $X_{t-k+1}^t$.

Denote the set of $N$ vectors with real entries by $\mathbb{R}^N$ and the set of $N \times M$ matrices with real entries by $\mathbb{R}^{N \times M}$. Denote by $\mathcal{S}^N$ the $N - 1$ dimensional simplex on which an arbitrary probability vector $\theta \in \mathbb{R}^N$ satisfies $1^T_N \theta = 1$ and each of its element satisfies $(\theta)_i \geq 0$. Denote a simplex matrix by $Q \in \mathcal{S}^{N \times M}$ where each of the $M$ column vectors of the matrix $Q \in \mathbb{R}^{N \times M}$ is a simplex vector in $\mathcal{S}^N$. For a transition probability $P(X_{t-k+1}^t \mid X_{t-k}^{t-1})$, there is a corresponding transition matrix $Q \in \mathcal{S}^{N^k \times N^k}$. This is stationary if the probability distribution over $X_{t-k+1}^{t-1}$ satisfies $P(X_{t-k+1}^t) = P(X_{t-k+1}^{t-1})$ for any $t$. The stationary probability vector $\theta = (P(1), P(2), \ldots, P(N^k))^T \in \mathcal{S}^{N^k}$ of a Markov process with a transition matrix $Q$ is a root of the equation

$$(1.2) \quad \theta = Q \theta.$$
In this paper, we consider a class of infinite order Markov processes $X_0^\infty \in \mathcal{N}^\infty$ which can be constructed recursively. We call this class the recursive Markov process. By constructing a $k^{th}$ order transition matrix $Q^{(k)} \in \mathbb{S}^{N^k \times N^k}$ from $Q^{(k-1)} \in \mathbb{S}^{N^{k-1} \times N^{k-1}}$, a recursive Markov process is defined in the limit $\lim_{k \to \infty} Q^{(k)}$. We will give this construction formally in Section 3. This definition of a recursive Markov process is motivated by studying stochastic games [9] with players stochastically reasoning according to past experience. Recently, this class of stochastic games has been studied intensively in game theoretic studies from both theoretical and behavioral points of view [1, 6, 8, 3, 2, 7]. We perform an analysis based on a recursive Markov process for a stochastic game in Section 4.

As the main result of this paper we prove that, for an arbitrary recursive Markov process, the marginal stationary distribution $\omega \in \mathbb{S}^N$ holds an analogous form to (1.2):\[
\omega = Q(\omega)\omega,
\]
where the transition matrix $Q(\omega) \in \mathbb{S}^{N \times N}$ is a function of $\omega$.

In Section 2, we introduce our notation, and formulate a Markov process in a linear algebraic form. Introducing three basic operators, we give an extension of the transition matrix, called a shift matrix. Our analysis on the shift matrix illustrates the general properties which any stationary distribution satisfies. In Section 3, we define the recursive Markov process, and show the main result (1.3). In Section 4, we give an application of a recursive Markov process for a stochastic game.

2. Markov process

2.1. Transition matrix. For each $i$, let us write $X_i \in \mathcal{N}$ and $X = (X_1, X_2, \ldots, X_k) \in \mathcal{N}^k$. For the $k^{th}$ order Markov process (1.1), we assign each state in $\mathcal{N}^k$ an integer in the set $\mathcal{C}_{N,k} := \{1, 2, \ldots, N^k\}$ by the indexing map $h_{N,k} : \mathcal{N}^k \mapsto \mathbb{Z}$ defined by
\[
h_{N,k}(X) := 1 + \sum_{j=1}^k (X_j - 1)N^j.
\]
This encoding of the states is done without loss of generality and is used throughout this paper.

For $i \in \mathcal{C}_{N,k}$, denote the set of integers by
\[
\mathcal{H}_{N,k}(i) := \left\{ h_{N,k}(X_1^k) : i = h_{N,k}(X_0^{k-1}) \right\}.
\]
This set consists of the \( N \) indices of those states in the \( k \)-th order Markov process which can be reached from the state \( i \). First we define the transition matrix \( Q^{(k)} \in \mathbb{S}^{N^k \times N^k} \) for the \( k \)-th order Markov process with respect to this encoding as follows.

**Definition 2.1 (Transition matrix).** The transition matrix \( Q^{(k)} \in \mathbb{S}^{N^k \times N^k} \) for the \( k \)-th order Markov process is defined by

\[
(Q^{(k)})_{i,j} := P \left( h_{N,k}^{-1}(i) \mid h_{N,k}^{-1}(j) \right),
\]

where \((Q)_{i,j}\) is the \((i,j)\) element of the matrix \( Q \). Observe that, unless \( i \in \mathcal{H}_{N,k}(j) \), \((Q^{(k)})_{i,j} = 0\).

In order to analyze the properties of the transition matrix \( Q^{(k)} \), let us introduce the vectors and matrices as follows. Let us denote the zero vector by \( 0_N = (0, 0, \ldots, 0)^T \in \mathbb{R}^N \), the identity matrix by \( E_N \in \mathbb{S}^{N \times N} \) and the unit vector by

\[
e_{N,i} := \begin{pmatrix} 0, \ldots, 0, \overbrace{1}^i, 0, \ldots, 0 \end{pmatrix}^T \in \mathbb{S}^N,
\]

and let \( E_{N,i} := e_{N,i} e_{N,i}^T \). We define a special permutation matrix called the *commutation matrix* \([5]\) by:

\[
C_{n,m} := \sum_{i=1}^m e_{m,i}^T \otimes E_n \otimes e_{m,i}.
\]

where \( \otimes \) denotes the Kronecker product.

For the state \( i \in \mathcal{C}_{N,k} \) of a \( k \)-th order Markov process with \( Q^{(k)} \in \mathbb{S}^{N^k \times N^k} \) and its corresponding indices \( m_1 < \ldots < m_N \in \mathcal{H}_{N,k}(i) \), we define

\[
q^{(k)}_i := (P(m_1 \mid i), \ldots, P(m_N \mid i))^T \in \mathbb{S}^N,
\]

and for \( i \in \mathcal{C}_{N,k-1} \), we define the simplex matrix

\[
Q^{(k)}_i := \left( q^{(k)}_{N(i-1)+1}, \ldots, q^{(k)}_{N(i-1)+N} \right) \in \mathbb{S}^{N \times N}.
\]

Using the above notation, Hidaka and his colleagues \([4]\) showed an arbitrary \( k \)-th order transition matrix \( Q^{(k)} \) can be decomposed as:

\[
Q^{(k)} = C_{N,N^{k-1}} \sum_{i \in \mathcal{C}_{N,k-1}} E_{N^{k-1},i} \otimes Q^{(k)}_i.
\]
2.2. Linear operators. In this section, we introduce three linear operators in their matrix forms to show the basic properties of the \( k \)th order transition matrix serving as the foundation of our main result.

**Definition 2.2.** For \( 0 \leq m \leq k \), we define the \( k \)th order marginalization matrix
\[
M^{(k)}_m := E_{N^m} \otimes 1_N^T \otimes E_{N^{k-m}} \in \mathbb{S}^{N^k \times N^{k+1}}.
\]

For \( 0 \leq m \leq k \) and the tuple of vectors \( Q^{(k)}_N := (q^{(k)}_1, \ldots, q^{(k)}_{N^k}) \), we define the \( k \)th order branching matrix
\[
B^{(k)}_m \left( Q^{(k)}_N \right) := \sum_{i \in C_{N,m}, j \in C_{N,k-m}} E_{N^m,i} \otimes q^{(k)}_{N^{k-m}(i-1)+j} \otimes E_{N^{k-m},j} \in \mathbb{S}^{N^{k+1} \times N^k}.
\]

For \( 0 \leq m \leq k \), we define the cycling matrix
\[
C^{(k)}_m := C_{N^m,N^{k-m}} = \sum_{i \in C_{N,k-m}} e_{N^{k-m},i}^T \otimes E_{N^m} \otimes e_{N^{k-m},i}.
\]

Let each element of the vector \( \theta^{(k)} \in \mathbb{S}^{N^k} \) be an arbitrary stochastic vector consisting of the probability \( P(X_1, X_2, \ldots, X_k) \) for \( h_{N,k} (X_1, X_2, \ldots, X_k) \in C_{N,k} \). Let \( Q^{(k)}_N := (q^{(k)}_1, \ldots, q^{(k)}_{N^k}) \) be the tuple of simplex vectors, such that the vector \( \left(\begin{array}{c} (q^{(k)}_1)^T \\ \vdots \\ (q^{(k)}_{N^k})^T \end{array}\right)^T \) consists of the conditional probability \( P(Y | X_1, X_2, \ldots, X_k) \). Then, the three types of matrices introduced above correspond to the operators on the stochastic vector \( \theta^{(k)} \) as follows.

1. Marginalization \( M^{(k-1)}_m \theta^{(k)} \in \mathbb{S}^{N^{k-1}}: P(X_1, \ldots, X_m, X_{m+2}, \ldots, X_k) \).
2. Branching \( B^{(k)}_m \left( Q^{(k)}_N \right) \theta^{(k)} \in \mathbb{S}^{N^{k+1}}: P(X_1, \ldots, X_m, Y, X_{m+1}, \ldots, X_k) \)
3. Cycling \( C^{(k)}_m \theta^{(k)} \in \mathbb{S}^{N^k}: P(X_{m+1}, X_{m+2}, \ldots, X_k, X_1, \ldots, X_m) \)

Figure 1 illustrates the branching and marginalization matrices for \( N = 2 \) and \( k = 1, 2, 3 \).

The reader can confirm the properties of these operators above by finding the following identities. For an arbitrary tuple of simplex vectors \( Q^{(k)}_N \) and \( 0 \leq m, n \leq k \), we have the identity
\[
M^{(k)}_m = C^{(k)}_{m-n} M^{(k)}_n C^{(k+1)}_{n-m},
\]
\[
M^{(k)}_m M^{(k+1)}_m = \begin{cases}
M^{(k)}_m M^{(k+1)}_m & \text{for } n \geq m \\
M^{(k)}_m M^{(k+1)}_m & \text{otherwise},
\end{cases}
\]
RECURSIVE MARKOV PROCESS

Fig 1. Branching and marginalization matrix for \( N = 2 \) and \( k = 1, 2, 3 \).

and

\[
B_m^{(k)} \left( Q_N^{(k)} \right) = C_{m-n}^{(k+1)} D_n^{(k)} \left( Q_N^{(k)} \right) C_{n-m}^{(k)}.
\]

For an arbitrary integer \( m \),

\[
C_{mod(k)}^{(nm)} = \left( C_m^{(k)} \right)^m.
\]

Now it is easy to understand that the \( k \)th order transition matrix \( Q^{(k)} \) (2.1), which is the "shift" operator \( P \left( X_1^k \right) \to P \left( X_2^{k+1} \right) \), can be written with the three matrices as follows.

**Proposition 2.1 (Transition matrix as the shift operator).** Denote an arbitrary transition matrix by \( Q^{(k)} \in \mathcal{S}_N^{k \times N^k} \), and the corresponding tuple of vectors \( Q_N^{(k)} := \left( q_N^{(k)}(i-1)+j \right)_{i \in \mathcal{C}_{N,k-1}, j \in \mathcal{C}_{N,1}} \). Then we have

\[
Q^{(k)} = M_k^{(k)} C_1^{(k+1)} B_k^{(k)} \left( Q_N^{(k)} \right) = C_1^{(k)} M_{k-1}^{(k)} B_k^{(k)} \left( Q_N^{(k)} \right).
\]

**Proof.**

\[
M_{k-1}^{(k)} B_k^{(k)} \left( Q_N^{(k)} \right) = \sum_{i \in \mathcal{C}_{N,k-1}, j \in \mathcal{C}_{N,1}} E_{N^{k-1}, j} \otimes e_N^T \otimes q_N^{(k)}(i-1)+j
\]

\[
= \sum_{i \in \mathcal{C}_{N,k-1}} E_{N^{k-1}, i} \otimes Q_i^{(k)} = C_{N, N^{k-1}}^{-1} Q_i^{(k)}
\]
2.3. *k*-shift matrix. Let us introduce a *k*-shift operator, which is an extension of Proposition 2.1 as follows.

**Definition 2.3.** For a series of tuples \( Q^{(1)}, \ldots, Q^{(k)} \), define a *k*-shifting transition matrix:

\[
S \left( Q^{(1)}, \ldots, Q^{(k)} \right) := M_1^{(1)} \ldots M_k^{(k)} C_1^{(k+1)} B_k^{(k)} \left( Q^{(k)} \right) \ldots B_1^{(1)} \left( Q^{(1)} \right).
\]

We can easily see that the transition matrix is identical to the 1-shift matrix, \( Q^{(k)} = S \left( Q^{(k)} \right) \), by this definition. The *k*-shift matrix is written in the recursive form as follows.

**Lemma 2.1** (Recursive property of the marginal matrix). Given tuples of vectors \( Q_N^{(m)} := (q_1^{(m)}, \ldots, q_N^{(m)}) \in \mathbb{S}^{N \times N^m} \) for \( m = 0, 1, \ldots, k \), we can write the corresponding *k*-shifting transition matrix in a recursive form:

\[
S \left( Q_N^{(m)}, \ldots, Q_N^{(k)} \right) = C_1^{(m)} \sum_{i=1}^{N^{m-1}} E_{N^{m-1},i} \otimes \overline{Q}_i^{(m)}
\]

where for \( 1 \leq m \leq k \) and \( 1 \leq i \leq N^{m-1} \)

\[
\overline{Q}_i^{(m)} := \left( \overline{Q}_N^{(m+1)} \right)_{N(i-1)+1} q_N^{(m)} \left( \overline{Q}_N^{(m+1)} \right)_{N(i-1)+2} q_N^{(m)} \cdots \left( \overline{Q}_N^{(m+1)} \right)_{N(i-1)+N} q_N^{(m)},
\]

and \( \overline{Q}_i^{(k+1)} := I_N \) for \( 1 \leq i \leq N^k \).

**Proof.** Observe the recurrent relationship between

\[
S \left( Q_N^{(k)} \right) = Q^{(k)} = \sum_{i=1}^{N^{k-1}} e_{N^{k-1},i}^T \otimes Q_i^{(k)} \otimes e_{N^{k-1},i}
\]

and

\[
S \left( Q_N^{(k-1)}, Q_N^{(k)} \right) = C_1^{(k-1)} M_{k-1}^{(k-1)} \left( C_1^{(k)} \right)^{-1} Q_N^{(k)} B_{k-1}^{(k-1)} \left( Q_N^{(k)} \right)
= \sum_{i=1}^{N^{k-2}} e_{N^{k-2},i}^T \otimes \overline{Q}_i^{(k-1)} \otimes e_{N^{k-2},i}
\]

where \( \overline{Q}_i^{(k-1)} = \sum_{j=1}^{N} e_{N,j}^T \otimes Q_i^{(k-1)} q_N^{(k-1)} \). For \( 1 \leq m \leq k \), find the recursive relationship between \( S \left( Q_N^{(m-1)}, \ldots, Q_N^{(k)} \right) \) and \( S \left( Q_N^{(m)}, \ldots, Q_N^{(k)} \right) \) by inductively writing \( \overline{Q}_i^{(m-1)} := \sum_{j=1}^{N} e_{N,j}^T \otimes Q_i^{(m-1)} q_N^{(m-1)} \). \( \square \)
2.4. Marginal stationary distribution. As we often need the marginal distribution rather than the full stationary distribution, it is crucial to describe the property of the marginal stationary distribution. With the $k$-shift matrix defined in the previous section, we can now analyze a general property of an arbitrary marginal stationary distribution. Before stating the marginal stationary distribution, let us note that an arbitrary stationary vector is uniquely expressed with branching matrices as follows.

**Proposition 2.2.** For every $\theta \in \mathbb{S}^N$, there is a unique tuple of vectors $\Theta_N^{(m)} := \left(\theta_{1}^{(m)}, \ldots, \theta_{N}^{(m)}\right)$ for $m = 0, 1, \ldots, k - 1$, which holds

$$\theta = B_{k-1}^{(k-1)} \left(\Theta_N^{(k-1)}\right) B_{k-2}^{(k-2)} \left(\Theta_N^{(k-2)}\right) \ldots B_0^{(0)} \left(\Theta_N^{(0)}\right).$$

**Proof.** Given $\Theta(k) \in \mathbb{S}^N$, for $i \in \mathcal{C}_{N,k-2}$ and $j \in \mathcal{C}_{N,1}$ define for $i \in \mathcal{C}_{N,k-1}$

$$\left(\theta(k)_{(i-1)j}\right) := \sum_{m=1}^{N} \left(\theta(k)_{i+m}\right),$$

and

$$\theta_{N(i-1)+j}^{(k)} := \left(\theta_{N(i-1)+j+1}^{(k)}, \ldots, \theta_{N(i-1)+j+N}^{(k)}\right)^T / \left(\theta(k)_{N(i-1)+j}\right) \in \mathbb{S}^N.$$

Apply this definition recursively for $k, k - 1, \ldots, 0$. \hfill \Box

Now we are ready to state the lemma as follows.

**Lemma 2.2.** According to Proposition 2.2, with $\Theta_N^{(m)} := \left(\theta_{1}^{(m)}, \ldots, \theta_{N}^{(m)}\right)$ for $m = 0, 1, \ldots, k$, write $\theta = B^{(k)} \left(\Theta_N^{(k)}\right) \ldots B^{(0)} \left(\Theta_N^{(0)}\right) \in \mathbb{S}^N$. For a transition matrix $Q(k) \in \mathbb{S}^{N \times N}$, which holds $\theta = Q(k) \theta$, we have

$$\theta(0) = \Theta(1) \theta(0) = \Theta(2) \theta(0) = \ldots = \Theta(k) \theta(0) = Q(k) \theta(0),$$

where $\Theta(m) := \mathcal{S} \left(\Theta(1), \ldots, \Theta(m)\right)$ and $Q(k) := \mathcal{S} \left(\Theta(1), \ldots, \Theta(k), Q(k)\right)$.

**Proof.** Define for $1 \leq m \leq k$

$$\overline{\Theta}_m^{(k)} := M_m^0 \times M_1^{(n+1)} \ldots M_{m-n+1}^{(m-1)} \times M_{m-n}^{(m)} \ldots M_{k-1}^{(k-1)},$$

and denote $\overline{\Theta}_m^{(k)}$ for $m = 0, 1, \ldots, m-1$. Then we have

$$\overline{\Theta}_m^{(k)} C_1^{(k)} = \overline{\Theta}_m^{(k)} \theta(0) = \overline{\Theta}_m^{(k)} \theta.$$
As \( \theta = Q\theta \), we have \( \Theta^{(m-1)}\theta^{(0)} = \Theta^{(m)}\theta^{(0)} \) for \( 1 < m \leq k \), as
\[
M^{(k)}_m \theta = \Theta^{(m-1)}\theta^{(0)} = M^{(k)}_{m-1}M^{(k)}_k B^{(k)} \left( Q^{(k)}_N \right) \theta = \Theta^{(m)}\theta^{(0)}.
\]

For \( m = 1 \), we have \( \theta^{(0)} = \overline{Q}^{(k)}\theta^{(0)} \).

Lemma 2.2 implies the necessary condition that the marginal stationary vector \( \theta^{(0)} \) holds. As each of these conditions includes a part of the full stationary distribution in the term \( \Theta^{(m)} \), we still need to know the full stationary distribution to calculate its marginal distribution. This requirement, however, can be relaxed when we have a converging series of stationary vectors \( \theta^{(k)} \to M_k^{(k)}\theta^{(k+1)} \) as \( k \to \infty \). This is formally stated by the following theorem which replaces the stationary distribution with its corresponding transition matrices for the condition of the marginal vector.

**Theorem 2.1.** Suppose that, for each integer \( k > 0 \), we have a \( k \)th order transition matrix \( Q^{(k)} \in \mathbb{S}^{N_k \times N_k} \) and its corresponding tuples of vectors \( Q^{(k)}_N = \left( q^{(k)}_1, \ldots, q^{(k)}_{N_k} \right) \). Denote the stationary vectors by \( \theta^{(k)} \in \mathbb{S}^{N_k} \), which holds \( \theta^{(k)} = Q^{(k)}\theta^{(k)} \), and denote the marginal stationary vector by \( \omega^{(m)} := M^{(m)}_m \ldots M^{(k)}_k \theta^{(k)} \in \mathbb{S}^{N_m} \). Then we have
\[
\lim_{k \to \infty} \omega^{(k)}_m - S \left( Q^{(m)}_N, \ldots, Q^{(k)}_N \right) \omega^{(k)}_m = 0_{N^m},
\]
if we have the convergence
\[
\lim_{k \to \infty} \theta^{(k)} - M^{(k)}_k \theta^{(k+1)} = 0_{N^k}.
\]

**Proof.** By Proposition 2.2, for \( m \geq 1 \) we can uniquely write the \( m \)th order stationary vector
\[
\theta^{(m)} = B^{(m-1)}_1 \left( \Theta^{(m)}_1 \right) \ldots B^{(0)}_0 \left( \Theta^{(m)}_0 \right) \in \mathbb{S}^{N_k}
\]
where \( \Theta^{(m)}_n := \left( \theta^{(m,n)}_1, \ldots, \theta^{(m,n)}_{N^n} \right) \) for \( n = 0, 1, \ldots, m \). With the tuple \( Q^{(k)}_N \) corresponding with \( Q^{(k)} \), define
\[
\overline{\Theta}^{(k)}_{n,m} := S \left( \Theta^{(k)}_n, \ldots, \Theta^{(k)}_m \right), \overline{Q}^{(k)}_n := S \left( \Theta_1^{(k)}, \ldots, \Theta^{(k)}_k, Q^{(k)}_N \right).
\]

Given \( \omega^{(k)}_n = \overline{Q}^{(k)}_n \omega^{(k)}_n \) by Lemma 2.2, the convergence \( \lim_{k \to \infty} \theta^{(k)} - M^{(k)}_k \theta^{(k+1)} = 0_{N^k} \) implies
\[
\lim_{k \to \infty} \overline{Q}^{(k)}_n - S \left( \Theta^{(k-1)}_n, \ldots, \Theta^{(k-1)}_{k-1}, Q^{(k-1)}_N, Q^{(k)}_N \right) = 0_{N,N}.
\]

Applying this recursively, we obtain the theorem. \( \square \)
3. **Recursive Markov process**. Theorem 2.1 implies that the marginal stationary vector $\omega^{(k)}_m$ is obtained by the $k$-shift matrix $S \left( Q^{(m)}_N, \ldots, Q^{(k)}_N \right)$ in the limit $k \to \infty$. This theorem motivates us to consider a special class of Markov processes with a converging series of transition matrices in a certain form as follows.

**Definition 3.1 (Recursive Markov process).** We call a $k^{th}$ order Markov process with the transition matrix $Q^{(k)}_i$ recursive, if each element of the block matrix $Q^{(m)}_i$ is a function of the elements of $q^{(m-1)}_i \in S^N$ for $1 < m \leq k$, $i \in C_{N,m}$.

This definition of the recursive Markov process is motivated by the fact that we can analyze the convergence of such a series of transition matrices in a closed form. The following corollary states that this class is characterized by a closed-form equation of the marginal stationary distribution.

**Corollary 3.1.** Suppose there is a map $f : S^N \mapsto S^{N \times N}$, with which an infinite order recursive Markov process satisfies $Q^{(m+1)}_i = f \left( q^{(m)}_i \right)$ for $m = 1, 2, \ldots$ and $i \in C_{N,m}$. Denote the fixed point $\omega \in S^N$ for the linear transformation $f(\omega)$, which satisfies $\omega = f(\omega) \omega$. Then, the marginal stationary vector of the $k^{th}$ order stationary vector $\theta^{(k)}$ in the limit $k \to \infty$ corresponds with $\omega$ as follows:

$$\omega = \lim_{k \to \infty} M^{(1)}_1 \ldots M^{(k-1)}_{k-1} \theta^{(k)} \in S^N,$$

if the limit shift matrix $\bar{Q} := \lim_{k \to \infty} S \left( Q^{(1)}_N, \ldots, Q^{(k)}_N \right)$ of this recursive Markov process is irreducible.

**Proof.** Denote $\bar{Q} = (q_1, \ldots, q_N) \in S^{N \times N}$. According to Theorem 2.1, the marginal stationary distribution $\omega$ holds $\omega = \bar{Q} \omega$, and the recursive Markov process holds $q_i = \bar{Q} q_i = f(q_i) q_i$ for $i \in C_{N,1}$. As $\bar{Q}$ is irreducible, $\omega = q_i$ and $\omega = f(\omega) \omega$. □

4. **Application**. As a minimal example which motivates the recursive Markov process, we provide an analysis of a two-armed bandit problem in this section. Consider a gambler with two options, betting on either arm, denoted as 0 or 1. By betting one dollar on either arm at each step, he may win one dollar or lose his dollar. Suppose that two arms 0 and 1 have constant winning rates $p_0$ and $p_1$, respectively, which the gambler does not know beforehand. This gambler has a sufficiently large number of dollars,
and wishes to find out the arm with the certain best winning rate in the long term.

As an example, let us consider the betting strategy with the quantitative confidence level \((q_0, q_1) \in \mathbb{R}^2, q_0, q_1 > 0\). At every step, the gambler bets on arm 0 with probability \(q_0/(q_0 + q_1)\). Given the confidence level \((q_0, q_1)\), the gambler updates his confidence level to \((q_0\Delta, q_1)\) with a multiplier \(\Delta > 1\) if he wins with arm 0; otherwise, set \((q_0/\Delta, q_1)\). This is similar for \(q_1\), when he chooses arm 1.

We can view this strategy either as a first order Markov process with infinitely many states or as a recursive Markov process. For this problem, analysis of the marginal stationary distribution of the recursive Markov process is sufficient as we are only interested in whether this gambler may end up betting on the best-wining-rate arm almost every time.

According to Corollary 3.1, this random process, if it has a stationary distribution, is characterized by the equation

\[
\begin{pmatrix}
\frac{p_0q_0}{q_0 + q_1} \\
\frac{p_0q_0}{q_0 + q_1} \\
\frac{p_0q_0}{q_0 + q_1}
\end{pmatrix} =
\begin{pmatrix}
p_0q_0/\Delta & p_0q_0/\Delta & p_0q_0/\Delta \\
p_0q_0/\Delta & p_0q_0/\Delta & p_0q_0/\Delta \\
p_0q_0/\Delta & p_0q_0/\Delta & p_0q_0/\Delta
\end{pmatrix}
\begin{pmatrix}
\frac{p_0q_0}{q_0 + q_1} \\
\frac{p_0q_0}{q_0 + q_1} \\
\frac{p_0q_0}{q_0 + q_1}
\end{pmatrix},
\]

where \(\overline{p}_i := 1 - p_i\) for \(i = 0, 1\). Without loss of generality, we can reduce (4.1) as follows:

\[
\begin{pmatrix}
1 \\
1
\end{pmatrix} =
\begin{pmatrix}
\Delta/(q_0\Delta + q_1) & 1/(q_0 + q_1\Delta) \\
1/(q_0\Delta + q_1) & \Delta/(q_0 + q_1\Delta)
\end{pmatrix}
\begin{pmatrix}
p_0q_0 + \overline{p}_1q_1 \\
p_0q_0 + \overline{p}_1q_1
\end{pmatrix}.
\]

Solving this equation for given \(p_0, p_1, \Delta\), we have the unique solution

\[
\frac{q_0}{q_1} = \frac{\Delta\overline{p}_1 - p_1}{\Delta\overline{p}_0 - p_0},
\]

if the right hand side is positive. This implies that this strategy successfully converges to the desired choice for sufficiently large \(\Delta \to \infty\):

\[
\lim_{\Delta \to \infty} \frac{q_0}{q_1} = \begin{cases} 
\infty & \text{for } p_0 > p_1 \\
1 & \text{for } p_0 = p_1 \\
0 & \text{for } p_0 < p_1
\end{cases}.
\]

This simple case study demonstrates how powerful the analysis of a Markov process can be if it is recursive.
Acknowledgments. This study was supported by JSPS KAKENHI 23300099.

References.
[1] Borgers, T. and Sarin, R. (1997). Learning Through Reinforcement and Replicator Dynamics. *Journal of Economic Theory* **77** 1–14.
[2] Camerer, C. F. (2003). Behavioural studies of strategic thinking in games. *Trends in Cognitive Sciences* **7** 225–231.
[3] Camerer, C. F. and Hua Ho, T. (1999). Experience-weighted Attraction Learning in Normal Form Games. *Econometrica* **67** 827–874.
[4] Hidaka, S., Torii, T. and Masumi, A. (2015). Which types of learning make a simple game complex? *Complex Systems* **24**.
[5] Magnus, J. R. and Neudecker, H. (1988). *Matrix differential calculus*. Cambridge Univ Press, New York.
[6] Press, W. H. and Dyson, F. J. (2012). Iterated Prisoner’s Dilemma contains strategies that dominate any evolutionary opponent. *Proceedings of the National Academy of Sciences* **109** 10409–10413.
[7] Roth, A. E. and Erev, I. (1995). Learning in extensive-form games: Experimental data and simple dynamic models in the intermediate term. *Games and Economic Behavior* **8** 164–212.
[8] Sato, Y. and Crutchfield, J. P. (2003). Coupled replicator equations for the dynamics of learning in multiagent systems. *Physical Review E* **67** 015206.
[9] Shapley, L. S. (1953). Stochastic Games. *Proceedings of the National Academy of Sciences* **39** 1095-1100.