We consider both the cases in which the number of solitons called tence of nonlinear modes that interact elastically and are nonlinear wave equations. Integrability implies the existence of nonlinear modes is well described by dispersive integrable physical systems is well described by dispersive integrable random solitons have been numerically investigated in [8–10]. Statistical properties of solutions of large set of generalized hydrodynamic has been recently established in [11]. Incoherent/random nonlinear superpositions of solitons are small dispersion limit or semiclassical limits [1–3]. Incoherence of nonlinear modes that interact elastically and are called solitons. The inverse scattering, also called non-soliton shielding of the focusing nonlinear Schrödinger equation

\[ i \psi_t + \frac{1}{2} \psi_{xx} + |\psi|^2 \psi = 0, \quad (1) \]

We consider both the cases in which the N-soliton spectra is chosen in a deterministic and random way.

Let us recall the one-soliton solution, given by

\[ \psi(x, t) = 2b \sech[2b(x + 2at - x_0)] e^{-2i[ax + (a^2 - b^2)t + \frac{b}{8a} x^2]}, \quad (2) \]

where \( x_0 \) is the initial peak position of the soliton, \( \phi_0 \) is the initial phase, 2b is the modulus of the wave maximal amplitude and \(-2a\) is the soliton velocity. The general N soliton solution can be obtained from the Zakharov-Shabat [20] linear spectral problem, reformulated as a Riemann-Hilbert Problem (RHP) for a 2 \times 2 matrix \( Y_N(z; x, t) \) with the following data [21]: the discrete spectrum \( S := \{z_0; \ldots; z_{N-1}; \bar{z}_0; \ldots; \bar{z}_{N-1}\} \), \( z_j \in \mathbb{C}^+ \) the upper half space, and its norming constants \( \{c_0, \ldots, c_{N-1}\} \) with \( c_j \in \mathbb{C} \). Here and below \( \bar{z} \) stands for the complex conjugate of z.

The matrix \( Y_N(z; x, t) \) is analytic for \( z \in \mathbb{C} \setminus S \) and has simple poles in S with the residue condition

\[ \text{Res}_{z=z_j} Y_N(z) = \lim_{z \to z_j} Y_N(z) \begin{pmatrix} 0 & 0 \\ c_j e^{2\theta(z; x, t)} & 0 \end{pmatrix} \]

\[ \text{Res}_{z=\bar{z}_j} Y_N(z) = \lim_{z \to \bar{z}_j} Y_N(z) \begin{pmatrix} 0 & -\bar{c}_j e^{-2\theta(z; x, t)} \\ 0 & 0 \end{pmatrix} \quad (3) \]

where \( \theta(z; x, t) = i(z^2 t + xz) \) and I is the identity matrix. The equations [3] uniquely determine \( Y_N(z; x, t) \) as a rational matrix function of z in the form

\[ Y_N(z; x, t) = \text{I} + \sum_{j=0}^{N-1} \frac{f_j(x, t)}{z - z_j}, \quad (4) \]

where the coefficients \( f_j(x, t) \) and \( g_j(x, t) \) are determined from a linear system by imposing the residue conditions.
and the phase \( \phi \) main

\[
\psi_N(x, t) = 2i \lim_{z \to \infty} z(Y^N(z; x, t))_{12}
\]

which gives the N-soliton solution in the form \( \psi_N(x, t) = -2i \sum_{j=0}^{N-1} g_j(x, t) \). In the case of one soliton solution, we have that the point spectrum \( \lambda_0 = a + ib \) determines the speed and amplitude of the soliton and the coefficient \( c_0 \) determines the position \( x_0 = \frac{\ln|\lambda_0|}{2b} \) of the soliton peak and the phase \( \phi_0 = \frac{\pi}{2} + \arg(c_0) \) of the soliton.

The FNLS equation can have a soliton of order \( N \) when the matrix function \( Y^N(z) \) has a pole of order \( N \). Such a solution can be viewed as a N-soliton solution where the simple poles coalesce to a pole of order \( N \). The limit as \( N \to \infty \) of such solutions has been studied in [22, 23] where it has been shown that its near field structure is described by the Painlevé III equation. An analogous asymptotical study has been performed for breathers in [24].

In this letter we consider the case when the norming constants \( \{c_j\}_{j=0}^{N-1} \) scale as \( 1/N \) as the number \( N \) of simple poles (i.e. the number of solitons) tends to infinity. On the physical side, scaling the norming constants to be small means that the individual solitons are centered at positions that are logarithmically large in \( N \), so that in the finite part of the \( (x, t) \) plane only the tails of the solitons add up. The resulting gas of solitons is a condensate in the terminology of [6].

Differently from [18], [19] where the infinite set of solitons is obtained by letting the soliton spectra accumulate on one or more simply connected bounded domains \( D \) of the complex upper plane \( \mathbb{C}^+ \) and their complex conjugate \( \overline{D} \). We let the number of solitons goes to infinity in such a way that their point spectrum \( z_j (\overline{z}_j) \) fills uniformly the domain \( D \). The corresponding norming constants \( c_j \) are interpolated by a smooth function \( \beta(z, \overline{z}) \), namely

\[
c_j = \frac{A}{\pi N} \beta(z_j, \overline{z}_j),
\]

where \( A \) is the area of the domain \( D \) and \( N \) is the total number of solitons.

The remarkable emerging feature is that as \( N \to \infty \), for certain types of domains and densities, we have a “soliton shielding”, namely, the gas behaves as a finite number of solitons. This happens for example if the distribution function is \( \beta(z, \overline{z}) = \pi^{-1} r(z) \) with \( r(z) \) an analytic function in \( D \), and the domain is described by \( \mathcal{D} := \{z \in \mathbb{C} \text{ s.t. } |(z - d_0)^n - d_1| < \rho \} \), \( n \in \mathbb{N} \), with \( d_0 \in \mathbb{C}^+ \), \( |d_1| \) and \( \rho > 0 \) sufficiently small so that \( \mathcal{D} \subset \mathbb{C}^+ \). Then the deterministic soliton gas is equivalent to a n-soliton solution. In the case \( n = 1 \), the domain \( \mathcal{D} \) is a disk centered at \( \lambda_0 = d_0 + d_1 \) and the infinite number of solitons superimpose nonlinearly in their tails to produce a single soliton solution with point spectrum \( \lambda_0 \) and norming constant equal to \( \rho^2 r(\lambda_0) \). We are going to see that this behaviour persists also when the N soliton spectrum is a random variable distributed according to the Ginibre ensemble [25] or the uniform distribution on the disk.

When the domain \( D \) is an ellipse we show that such deterministic soliton gas is a step-like periodic elliptic wave at \( x = -\infty \) and rapidly decreasing at \( x = +\infty \) as in [18].

**Deterministic soliton gas.** In order to obtain the limit as the N-soliton solution as \( N \to \infty \), we impose that the norming constants \( c_j \) scale as \( 1/N \). Then we use a transformation that removes the singularities of \( Y^N \). Indeed let \( \gamma_+ \) be a closed anticlockwise oriented contour that encircles all the poles in the upper half space and \( D_{\gamma_+} \) the finite domain with boundary \( \gamma_+ \) and similarly we define \( \gamma_- = -\overline{\gamma_+} \) and \( D_{\gamma_-} \) encircles all the poles in the lower half space.

One ends up with the RHP for the matrix function \( Y^N(z; x, t) \) analytic in \( \mathbb{C} \setminus \{\gamma_+ \cup \gamma_-\} \), subject to the conditions

\[
\begin{align*}
\tilde{Y}^N(z; x, t) &= \tilde{Y}^{-N}(z; x, t)\tilde{J}_N(z; x, t), & z \in \gamma_+ \cup \gamma_- \\
\tilde{Y}^N(z; x, t) &= I + O\left(\frac{1}{z}\right), & z \to \infty,
\end{align*}
\]

where the subscripted \( Y_\pm \) denote the left/right boundary values along the oriented contour and

\[
\tilde{J}_N(z; x, t) = \begin{cases} 
1 & \text{if } z \in \mathbb{C}_+ \\
\sum_{j=0}^{N-1} e^{2i \pi j} z_j e^{2i \pi j} & \text{if } z \in \mathbb{C}_- \\
1 & \text{if } z \in \mathbb{C}_+ \\
0 & \text{if } z \in \mathbb{C}_-.
\end{cases}
\]

We call the matrix \( \tilde{J}_N(z; x, t) \) the *jump matrix*. The solution \( \tilde{Y}^N(z; x, t) \) is obtained from \( Y^N(z; x, t) \) by the relation \( \tilde{Y}^N(z; x, t) = Y(z; x, t) \tilde{J}_N(z; x, t) \) for \( z \in \mathbb{C} \setminus \{D_{\gamma_+} \cup D_{\gamma_-}\} \) in [6] we have shown that its near field structure is described by the Painlevé III equation. An analogous asymptotical study has been performed for breathers in [24].
\[ \frac{\text{d}w}{\text{d}t} = \psi(z, t) \] Consequently the RH-problem \[ \tilde{Y}_+^\infty(z, x, t) = \tilde{Y}_-^\infty(z, x, t) \tilde{J}_-^\infty(z, x, t), \quad \tilde{J}_-^\infty(z, x, t) = \left( \frac{1}{\pi} \int_{\partial D} \frac{e^{-2\theta(w, x, t)} \beta(w, x, t) d^2w}{\pi(z - w)} \chi_{\gamma_-} \right) d^2w \]
\[ \tilde{Y}_-^\infty(z, x, t) = \mathcal{I} + \mathcal{O} \left( \frac{1}{z} \right), \quad z \to \infty, \quad (9) \]
with \( \beta^*(w, \bar{w}) = \overline{\beta(w, \bar{w})} \). The limiting FNLS solution is given by
\[ \psi_\infty(x, t) = 2i \lim_{z \to \infty} z(\tilde{Y}_-^\infty(z, x, t))_{12}. \quad (10) \]
For a general bounded domain \( D \) and smooth function \( \beta(z, \bar{z}) \), the class of solutions of FNSL obtained from \( [9] \) and \( [10] \) is unexplored. In the case \( \beta(z, \bar{z}) = n \pi^{-1} r(z) \), with \( r(z) \) analytic in \( D \), we can apply Green theorem for \( z \notin D \) and obtain
\[ \int_{\partial D} e^{2\theta(w, x, t)} \beta(w, x, t) d^2w = \int_{\partial D} \frac{r(w) e^{2\theta(w, x, t)} d\omega}{z - w} \frac{d\omega}{2\pi i}, \quad (11) \]
and similarly for the integral over \( \overline{D} \).

For sufficiently smooth simply connected domains \( D \) the boundary \( \partial \overline{D} \) can be described by the so-called Schwarz function \( S(z) \) \[ 26 \] of the domain \( D \) through the equation
\[ \tau = S(z). \]
The Schwarz function admits analytic extension to a maximal domain \( \overline{D}^0 \subset D \). For example, for quadrature domains, \( \overline{D}^0 \) is just \( D \) minus a finite collection of points \[ 26 \]. The simplest such quadrature domain is the disk, which is one of our examples below. For other classes of domains we have that \( \partial D \setminus \overline{D}^0 \) may consist of a \textit{mother-body}, i.e., a collection of smooth arcs \[ 27 \]. An example of this is the ellipse, which will be our second example.

**Shielding of soliton gas for quadrature domains.**

We start by considering the class of domains
\[ D := \left\{ z \in \mathbb{C} \text{ s.t. } \left| (z - d_0)m - d_1 \right| < \rho \right\}, \quad m \in \mathbb{N}, \quad (12) \]
with \( d_0 \in \mathbb{C}^+ \) and \( |d_1|, \rho > 0 \) sufficiently small so that \( D \in \mathbb{C}^+ \). When \( m = 1 \) such domain coincides with the disk \( D_\rho(\lambda_0) \) of radius \( \rho > 0 \) centred at \( \lambda_0 = d_0 + d_1 \). When \( m > 1 \) the domain \( D \) has a \( m \)-fold symmetry about \( d_0 \) and is simply connected if \( |d_1| \leq \rho \), and otherwise it has \( m \) connected components \[ 28 \]. The boundary of \( D \) is described by
\[ \tau = S(z), \quad S(z) = \overline{d_0} + \left( d_1 + \frac{\rho^2}{(z - d_0)m - d_1} \right) \frac{z}{\rho}. \quad (13) \]

**The \( n \)-soliton solution.** This solution is obtained from \( [11] \) by choosing \( m = n \) in \( [13] \). We then substitute \( w = \mathcal{S}(w) \) in the contour integral \( [11] \) and use the residue theorem at the \( n \) poles given by the solution \( \{ \lambda_0, \ldots, \lambda_{n-1} \} \) of the equation \( (z - d_0)^n = d_1 \). Then
\[ \int_{\partial D} \frac{e^{2\theta(\mathcal{S}(w), x, t)} d^2w}{\mathcal{S}(w) - z - w} \frac{d\omega}{2\pi i} = \int_{\partial D} \frac{S(w) e^{2\theta(\mathcal{S}(w), x, t)} d\omega}{\mathcal{S}(w) - z - w} \frac{d\omega}{2\pi i}, \quad (14) \]
which gives, up to a sign the entry 21 of the jump matrix \[ 9 \]. Namely the solution \( \psi_\infty(x, t) \) in \( [10] \) coincides with the \( n \) soliton solution \( \psi_\infty(x, t) \) in \( [9] \) with spectrum \( \{ \lambda_0, \ldots, \lambda_{n-1} \} \) and corresponding norming constants \( c_j = \rho^2 r(\lambda_j)/\prod_{k \neq j} (\lambda_j - \lambda_k) \) for \( j = 0, \ldots, n - 1 \).

**One soliton solution.** In particular, in the case \( n = m = 1 \) and \( D = D_\rho(\lambda_0) \) the disk centred at \( \lambda_0 = d_0 + d_1 \) of radius \( \rho \), we obtain exactly the RH-problem \( [7] \) for \( N = 1 \) and \( c_0 = \rho^2 r(\lambda_0) \). Namely we recover the one soliton solution \( [2] \) of the FNLS \( [1] \) equation with \( \lambda_0 = d_0 + d_1 = a + ib \), with peak position \( x_0 \) and phase shift \( \phi_0 \) given respectively by
\[ x_0 := \log\left( |\rho^2 r(\lambda_0)| \right) - \log(2b), \quad \phi_0 := \arg(r(\lambda_0)) - \frac{\pi}{2}. \quad (14) \]
We observe that the radius \( \rho \) of the disk and the value of the function \( r(\lambda) \) at \( \lambda_0 \) contribute to the phase shift of the soliton but not to its amplitude or velocity, which are uniquely determined by the center of the disk \( \lambda_0 \).

**Soliton solution of order \( n \).** By considering \( m = 1 \), namely, the disk \( D_\rho(\lambda_0) \) and \( \beta(z, \bar{z}) = n(z - \lambda_0)^{n-1} r(z) \) for \( n > 1 \) one obtains the soliton solution of order \( n \). This degenerate solution and the limit \( n \to \infty \) has been extensively analyzed in \[ 23 \].

**Remark.** In Figure \[ 1 \] we plot the resulting “effective” soliton using an approximation of the uniform measure on the unit disk by means of \( N \) Fejé points, namely the set of \( N \) points described by the vector \( w = (w_0, \ldots, w_N) \) that minimizes the energy
\[ E(w) = -2 \sum_{0 \leq j < k \leq N - 1} \log |w_j - w_k| + N \sum_{j=0}^{N-1} |w_j|^2 \quad (15) \]
(suitably translated/rescaled) over all possible configurations. Then the uniform measure on the disk \( D_\rho(\lambda_0) \) is obtained by the rescaling \( z_j = \rho w_j - \lambda_0 \). The train of solitons on the left (albeit slowly) will move towards \( -\infty \) as \( \Omega(\log N) \).

**Elliptic Domain.** We now consider the case in which \( D \) coincide with an elliptic domain \( E \) and with \( \beta(z, \bar{z}) = r(z) \) analytic. For the sake of simplicity, we assume that the focal points \( ia_1 \) and \( ia_2 \) of the ellipse lie on the imaginary
where \( R(z) := \sqrt{(z - i\alpha_1)(z - i\alpha_2)} \), \( y_0 = \frac{\alpha_1 + \alpha_2}{2} \), and \( c = \frac{\sqrt{\alpha_1 \alpha_2}}{2} \). The function \( S(z) \) is analytic in \( \mathcal{E} \) away from the segment \( \mathcal{I} := [i\alpha_1, i\alpha_2] \), with boundary values \( S_{\pm}(z) \). For \( z \not\in \mathcal{E} \cup \mathcal{E} \), the integral along the boundary \( \partial \mathcal{E} \) (\( \partial \mathcal{E} \)) of the ellipse in [11] can be deformed to a line integral on the segment \( \mathcal{I} := [i\alpha_1, i\alpha_2] \), namely
\[
\int_{\partial \mathcal{E}} \frac{r(w) e^{2\theta(w;x,t)}}{z - w} \frac{dw}{2\pi i} = \int_{\mathcal{I}} \frac{r(w) \delta S(w) e^{2\theta(w;x,t)}}{z - w} \frac{dw}{2\pi i},
\]
where \( \delta S(z) = S_+(z) - S_-(z) \). Next we define
\[
\Gamma(z) := \begin{cases} \tilde{Y}_\infty(z), & z \in \mathbb{C} \setminus \{D_{\tau_+} \cup D_{\tau_-} \} \\ \tilde{Y}_\infty(z) J(z), & z \in D_{\tau_+} \cup D_{\tau_-} \end{cases}
\]
where
\[
J(z) = \begin{pmatrix} 1 & \int_{\mathcal{I}} \frac{r(w) \delta S(w) e^{2\theta(w;x,t)}}{z - w} \frac{dw}{2\pi i} \chi_{D_{\tau_+}} \\ \int_{\mathcal{I}} \frac{r(w) \delta S(w) e^{2\theta(w;x,t)}}{z - w} \frac{dw}{2\pi i} \chi_{D_{\tau_-}} & 1 \end{pmatrix}
\]
In this way \( \Gamma(z) \) does not have a jump on \( \gamma_+ \cup \gamma_- \). Since \( J(z) \) has a jump in \( \mathcal{I} \cup \mathcal{I} \) it follows that \( \Gamma(z) \) is analytic in \( \mathbb{C} \setminus \{\mathcal{I} \cup \mathcal{I} \} \) with jump conditions
\[
\Gamma_+(z) = \Gamma_-(z) e^{\theta(z;x,t) \sigma_3} G(z) e^{-\theta(z;x,t) \sigma_3},
\]
where
\[
G(z) = \begin{pmatrix} \frac{1}{1 - i\chi \delta S(z) \tau_3} \\ \chi \delta S(z) \tau_3 \end{pmatrix}.
\]
order $n$. When the domain $\mathcal{D}$ is an ellipse, we showed that the spectral measure concentrates on lines connecting the foci of the ellipse and the soliton gas initial datum is asymptotically step-like oscillatory.

**Acknowledgments.** We are grateful to K. McLaughlin for useful discussions and the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the program “Dispersive hydrodynamics: mathematics, simulations and experiments, with application in nonlinear waves”, EPSRC Grant Number EP/R014604/1. T.G. and G.O. acknowledge the support from H2020 grant No. 778010 IPaDEGAN, the support of INdAM/GNFM and the research project Mathematical Methods in Non Linear Physics (MMNLIP), Gruppo 4-Fisica Teorica of INFN.

---

1. Kamvissis, S.; McLaughlin, K.D.T.-R.; Miller, P.D.; Annals of Mathematics Studies, 154. Princeton University Press, Princeton, NJ, 2003. xii+265 pp.
2. Jenkins, R.; McLaughlin, K. D. T.-R., Comm. Pure Appl. Math. 67 (2014), no. 2, 246-320.
3. Lax, P.D.; Levermore, C.D.; Comm. Pure Appl. Math. 36 (1983), no. 3, 253-290.
4. Zakharov V.E. “Kinetische equation for solitons”. In: Sov. Phys. JETP 33.3 (1971), pp. 538-541.
5. El, G.A.;[https://arxiv.org/pdf/2104.05812.pdf](https://arxiv.org/pdf/2104.05812.pdf)
6. El G.A. and Tovbis A. Phys. Rev. E 101.5 (2020), pp. 052207, 21. issn: 2470-0045.
7. El G.A. and Kamchatov A. M. Phys. Rev. Lett. 95, 204101 (2005).
8. Congy, T.; El, G.A. and Roberti G. Physical Review E 103 (2021), p. 042201.
9. Doyon, B.; Yoshimura T. and Caux, J. S. Phys. Rev. Lett. 120, 045301 (2018).
10. Bonnemain, T.; Doyon, B.; El, G.A.;[https://arxiv.org/pdf/2203.08551.pdf](https://arxiv.org/pdf/2203.08551.pdf)
11. Shurgalina E.G. and Pelinovsky E.N.; Phys. Lett. A 380.24 (2016), pp. 2049–2053.
12. Dutkay D. and Pelinovsky E. Phys. Lett. A. 378.42 (2014), pp. 3102–3110.
13. Gelash, A.; Agafontsev, D.; Zakharov, V.; El, G.A.; Randoux, S. and Suret, P. Phys. Rev. Lett. 123, 234102 (2019).
14. Gelash, A.; Agafontsev, D.; Suret, P. and Randoux, S. Phys. Rev E 104, 044213 (2021).
15. El, G.A.; Krylov, A.L.; Molchanov, S. and Venakides, S.; Physica D: Nonlinear Phenomena, vol. 152-153, pp. 653-664, May 2005.
16. Suret, P.; Tikan, A.; Bonnefoy, F.; Copie, F.; Ducrozé, G.; Gelash, A.; Prabhudesai, G.; Michel, G.; Cazaubiel, A.; Falcon, E.; El, G. A. and Randoux, S. Phys. Rev. Lett 125, 264101 (2020).
17. Redor, I.; Barthélemy, E.; Michallet, H., Onorato, M. and Mordant, N., Phys. Rev. Lett. 122, 214502 (2019).
18. Girotti, M.; Grava, T.; Jenkins, R.; McLaughlin, K. D. T., Comm. Math. Phys. 384 (2021), no. 2, 733-784.
19. Girotti, M.; Grava, T.; Jenkins, R.; McLaughlin, K. D. T.,[https://arxiv.org/pdf/2205.02601.pdf](https://arxiv.org/pdf/2205.02601.pdf).
[20] Zakharov V.E. and Shabat A.B. 1980 Functional Analysis and Its Applications 13(3) 166–74
[21] Faddeev L.D. and Takhtajan L.A. 2007 Hamiltonian Methods in the Theory of Solitons. Springer-Verlag, Berlin, Heidelberg.
[22] Bilman D.; Buckingham R.; Wang, R.J.; Differential Equations 297 (2021), 320 - 369.
[23] Bilman D.; Buckingham, R.J.; Nonlinear Sci. 29 (2019), no. 5, 2185 - 2229.
[24] Bilman, D.; Ling, L.; Miller, P.D. Duke Math. J. 169 (2020), no. 4, 671 - 760.
[25] Ginibre, J. Journal of Mathematical Physics 6 (1965): 440–449.
[26] Gustafsson, B. Acta Appl. Math., 1 (1980), 209 - 240.
[27] Ameur, Y.; Hedenmalm, H. and Makarov, N. Ann. Probab. 43 (2015), no. 3, 1157–1201.
[28] Balogh, F.; Grava, T.; Merzi, D.; Constr. Approx. 46 (2017), no. 1, 109 - 169.
[29] Grava, T.; Minakov, A.; Orsatti, G.; In progress (2023).
[30] Rider, B. and Virág, B. Int. Math. Res. Not. IMRN 2007, no. 2, Art. ID rnm006, 33 pp.
[31] Deift, P.; Kriecherbauer, T.; McLaughlin, K. T.-R.; Venakides, S.; Zhou, X.; Comm. Pure Appl. Math. 52 (1999), no. 11, 1335–1425.