Geometric realization of Dynkin quiver type quantum affine Schur-Weyl duality

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Abstract

For a Dynkin quiver $Q$ of type $ADE$ and a sum $\beta$ of simple roots, we construct a bimodule over the quantum loop algebra and the quiver Hecke algebra of the corresponding type via equivariant $K$-theory, imitating Ginzburg-Reshetikhin-Vasserot’s geometric realization of the quantum affine Schur-Weyl duality. Our construction is based on Hernandez-Leclerc’s isomorphism between a certain graded quiver variety and the space of representations of the quiver $Q$ of dimension vector $\beta$. We identify the functor induced from our bimodule with Kang-Kashiwara-Kim’s generalized quantum affine Schur-Weyl duality functor. As a by-product, we verify a conjecture by Kang-Kashiwara-Kim on the simpleness of some poles of normalized $R$-matrices for any quiver $Q$ of type $ADE$.

1 Introduction

For a fixed pair $(n, d)$ of positive integers, we have the following two fundamental objects: the complex simple Lie algebra $\mathfrak{sl}_{n+1}$ of type $A_n$ and the symmetric group $S_d$ of degree $d$. The natural $(\mathfrak{sl}_{n+1}, S_d)$-bimodule structure on the tensor power $\bigotimes^{n+1} \mathbb{C}$ produces a close relationship between their representation theories. This phenomenon is known as the classical Schur-Weyl duality and has many interesting variants.

The quantum affine Schur-Weyl duality is a variant involving their quantum affinizations: the quantum loop algebra $U_q(\mathfrak{sl}_{n+1})$ of $\mathfrak{sl}_{n+1}$ and the affine Hecke algebra $H^\mathbb{A}_d(q)$ of $GL_d$. Both algebras are defined over $k := \mathbb{Q}(q)$. Here we equip the tensor power $\mathbb{V}^\otimes d$ of the natural representation $\mathbb{V} := k^{n+1}[z^\pm 1]$ of $U_q(\mathfrak{sl}_{n+1})$ with a commuting right action of $H^\mathbb{A}_d(q^2)$ using the $R$-matrices. Chari-Pressley [1] proved that the induced functor

$$H^\mathbb{A}_d(q^2)\text{-mod} \to U_q(\mathfrak{sl}_{n+1})\text{-mod}; \quad M \mapsto \mathbb{V}^\otimes d \otimes H^\mathbb{A}_d(q^2) M$$

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gives an equivalence between suitable subcategories of finite-dimensional modules.

The quantum affine Schur-Weyl duality has a beautiful geometric realization due to Ginzburg-Reshetikhin-Vasserot [6]. Here we recall their construction briefly. Let \( \mu_d : \mathcal{F}_d \to \mathcal{N}_d \) be the Springer resolution of the nilpotent cone \( \mathcal{N}_d \) of \( \mathfrak{gl}_d(\mathbb{C}) \), where \( \mathcal{F}_d \) is the cotangent bundle of the full flag variety of \( GL_d(\mathbb{C}) \). The morphism \( \mu_d \) is equivariant with respect to a natural action of the group \( \mathcal{G}_d := GL_d(\mathbb{C}) \times \mathbb{C}^\times \), where \( \mathbb{C}^\times \) acts as the scalar multiplication on the cone \( \mathcal{N}_d \). Due to Ginzburg and Kazhdan-Lusztig, the affine Hecke algebra \( H_\beta^d(q^2) \) is isomorphic to the convolution algebra \( K^{\mathcal{G}_d}(\mathcal{Z}_d) \otimes_A k \) of the equivariant \( K \)-group of the Steinberg variety \( \mathcal{Z}_d := \mathcal{F}_d \times_{\mathcal{N}_d} \mathcal{F}_d \), where \( A = R(\mathbb{C}^\times) = \mathbb{Z}[v^{\pm 1}] \) is the representation ring of \( \mathbb{C}^\times \) and \( - \otimes_A k \) means the specialization \( v \mapsto q \). On the other hand, we consider another Steinberg type variety \( \mathcal{M}_d := \mathcal{M}_d \times_{\mathcal{N}_d} \mathcal{F}_d \) and identified its equivariant \( K \)-group with the bimodule \( \mathcal{V} \otimes \mathcal{Y} \). More precisely, they established an isomorphism \( \mathcal{V} \otimes \mathcal{Y} \cong K^{\mathcal{G}_d}(\mathcal{M}_d \times_{\mathcal{N}_d} \mathcal{F}_d) \otimes_A k \) making the following diagram commute:

\[
\begin{array}{ccc}
U_q(L\mathfrak{sl}_{n+1}) & \longrightarrow & \text{End} (\mathcal{V} \otimes \mathcal{Y}) \\
\phi \downarrow & & \downarrow \cong \\
K^{\mathcal{G}_d}(\mathcal{Z}_d) \otimes_A k & \longleftarrow & \text{End} (K^{\mathcal{G}_d}(\mathcal{M}_d \times_{\mathcal{N}_d} \mathcal{F}_d) \otimes_A k) \longleftarrow K^{\mathcal{G}_d}(\mathcal{Z}_d) \otimes_A k,
\end{array}
\]

where horizontal arrows denote the bimodule structures.

Recently, in a series of papers [9, 10, 11, 12], Kang, Kashiwara, Kim and Oh established some interesting generalized versions of the quantum affine Schur-Weyl duality. One of them (treated in [10] by Kang-Kashiwara-Kim) is associated with a pair \((Q, \beta)\) of a Dynkin quiver \(Q\) of type \(\text{ADE}\) and a sum \(\beta = \sum \alpha_i d_i \alpha_i\) of simple roots, which plays a similar role as the pair \((n, d)\) in the previous paragraphs. One player is the quantum loop algebra \(U_q(\mathfrak{g})\) of the complex simple Lie algebra \(\mathfrak{g}\) whose Dynkin diagram is the underlying graph of \(Q\). The other is the quiver Hecke (KLR) algebra \(H_Q(\beta)\) associated with \((Q, \beta)\), or actually its completion \(\hat{H}_Q(\beta)\) along the grading. The quiver Hecke algebra \(H_Q(\beta)\) is regarded as a generalization of the affine Hecke algebra \(H_\beta^d(q)\) from the viewpoint of the categorification of the quantum group. Inspired by the work of Hernandez-Leclerc [8], Kang-Kashiwara-Kim [10] constructed on a left \(U_q(\mathfrak{g})\)-module \(\mathcal{V} \otimes \mathfrak{g}\) which is a direct sum of some tensor products of affinized fundamental modules a commuting right action of the algebra \(\hat{H}_Q(\beta)\) by using the normalized \(R\)-matrices. However, to make the \(\hat{H}_Q(\beta)\)-action well-defined, we need to assume the simpleness of some poles of the normalized \(R\)-matrices. This assumption was verified for type \(\text{AD}\) in [10] by an explicit computation of the denominators of the normalized \(R\)-matrices. On the other hand, for type
E, this remains a conjecture. Under this well-definedness assumption, Kang-Kashiwara-Kim [10] also proved that the induced functor

$$\hat{H}_Q(\beta) \text{-mod}_{\text{id}} \to U_q(Lg) \text{-mod}_{\text{id}}; \quad M \mapsto \hat{V}^{\otimes \beta} \otimes_{\hat{H}_Q(\beta)} M$$

is exact, factors through the $\beta$-block $C_{Q,\beta}$ of a monoidal full subcategory $C_Q$ of $U_q(Lg) \text{-mod}_{\text{id}}$ introduced by Hernandez-Leclerc [8] and gives a bijection between the simple isomorphism classes. More recently, the author [5] proved that it actually gives an equivalence $\hat{H}_Q(\beta) \text{-mod}_{\text{id}} \simeq C_{Q,\beta}$ by using the notion of affine highest weight category. Note that here we forget the gradings by working with the completion $\hat{H}_Q(\beta)$.

In the present paper, we give a geometric realization of the bimodule $\hat{V}^{\otimes \beta}$ imitating Ginzburg-Reshetikhin-Vasserot’s realization. In our case, the nilpotent cone $N_d$ is replaced by the space $E_\beta$ of representations of the quiver $Q$ over $\mathbb{C}$ of dimension vector $\beta$. The group $G_\beta := \prod_i GL_{d_i}(\mathbb{C})$ naturally acts on $E_\beta$. Instead of the Springer resolution $F_d \to N_d$, we consider a proper morphism $F_\beta \to E_\beta$ from a “quiver flag variety” $F_\beta$ introduced by Lusztig in order to construct the canonical basis of the quantum group. Varagnolo-Vasserot [21] proved that the quiver Hecke algebra $H_Q(\beta)$ is isomorphic to the convolution algebra of the equivariant Borel-Moore homology $H^{G_\beta}_{q}((Z_\beta,k)$, where $Z_\beta := F_\beta \times_{E_\beta} F_\beta$. After completion, it is isomorphic to the completed equivariant $K$-group $\hat{K}^{G_\beta}(Z_\beta)_k$. On the $U_q(Lg)$-side, we consider a canonical $G_\beta$-equivariant proper morphism $M^\bullet_\beta \to M^\bullet_{0,\beta}$ between certain graded quiver varieties. By Nakajima [17], we have an algebra homomorphism $\Phi_\beta : U_q(Lg) \to \hat{K}^{G_\beta}(Z^\bullet_\beta)_k$, where $Z^\bullet_\beta := M^\bullet_\beta \times_{M^\bullet_{0,\beta}} M^\bullet$. The key of our construction is a $G_\beta$-equivariant isomorphism $M^\bullet_{0,\beta} \cong E_\beta$ due to Hernandez-Leclerc [8], which was originally established in order to give a geometric interpretation to their isomorphism between the Grothendieck ring $K(C_Q)$ and the coordinate ring of the maximal unipotent subgroup (see Remark 3.13). This allows us to form the intermediary fiber product $M^\bullet_\beta \times_{E_\beta} F_\beta$.

**Theorem 1.1** (=Theorem 4.4 + 4.6, see also Remark 4.8). There is an isomorphism

$$\hat{V}^{\otimes \beta} \cong \hat{K}^{G_\beta}(M^\bullet_\beta \times_{E_\beta} F_\beta)_k$$

such that the following diagram commutes (up to a twist):

$$\begin{array}{ccc}
U_q(Lg) & \xrightarrow{\phi_\beta} & \text{End}(\hat{V}^{\otimes \beta}) \\
\downarrow & \simeq & \downarrow \simeq \\
\hat{K}^{G_\beta}(Z^\bullet_\beta)_k & \xrightarrow{\Phi_\beta} & \text{End}(\hat{K}^{G_\beta}(M^\bullet_\beta \times_{E_\beta} F_\beta)_k) \\
\end{array}$$

where the horizontal arrows denote the bimodule structures.

Actually, our geometric construction of the $\hat{H}_Q(\beta)$-action is independent of that of [10], which shares the same characterization of the actions. Therefore, their comparison yields a uniform proof of:
Theorem 1.2 (Corollary 4.7). Kang-Kashiwara-Kim’s conjecture [10, Conjecture 4.3.2] on the simpleness of some specific poles of normalized $R$-matrices for tensor products of fundamental modules is true for any quiver $Q$ of type ADE.

Besides, a discussion involving geometric extension algebras yields another proof of the equivalence $\hat{H}_Q(\beta)\text{-mod}_d \simeq C_{Q,\beta}$ given by the bimodule $\hat{V}^\otimes \beta$ without using affine highest weight categories (Theorem 4.9). We would also remark that we do not use any results of [9], [10] for our proofs.

The present paper is organized as follows. In Section 2, we recall the definition of graded quiver varieties $M^\bullet_\beta$ and $M^\bullet_0,\beta$, and Hernandez-Leclerc’s isomorphism $M^\bullet_0,\beta \simeq E_\beta$. In Section 3, we study the convolution algebra $\hat{K}_G^\beta(Z^\bullet_\beta)_k$ (resp. $\hat{K}_G^\beta(Z^\bullet_\beta)_k$) and recall its relation to the quiver Hecke algebra $H_Q(\beta)$ (resp. the quantum loop algebra $U_q(L_{\mathfrak{g}})$). In the final section 4, we study the structure of the bimodule $\hat{K}_G^\beta(M^\bullet_\beta \times E_\beta \mathcal{F}_\beta)_k$.

While the author was writing this paper, there appeared a preprint by Oh-Scrimshaw [19] in arXiv which also proves Theorem 1.2 by a different approach. They compute the denominators of the normalized $R$-matrices for type $E$ explicitly with a computer.

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Convention. An algebra $A$ is associative and unital. We denote by $A^{\text{op}}$ (resp. $A^\times$) the opposite algebra (resp. the set of invertible elements) of $A$ and by $A\text{-mod}$ the category of left $A$-modules. Working over a base field $\mathbb{F}$, the symbol $\otimes$ (resp. $\text{Hom}$) stands for $\otimes_{\mathbb{F}}$ (resp. $\text{Hom}_{\mathbb{F}}$) if there is no other clarification. If $A$ is an $\mathbb{F}$-algebra, we denote by $A\text{-mod}_{\text{id}}$ the category of finite-dimensional left $A$-modules.

2 Hernandez-Leclerc’s isomorphism

2.1 Notation

Throughout this paper, we fix a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ of type ADE and a quiver $Q = (I, \Omega)$ whose underlying graph is the Dynkin diagram of $\mathfrak{g}$, where $I = \{1, 2, \ldots, n\}$ (resp. $\Omega$) is the set of vertices (resp. arrows). For an arrow $h \in \Omega$, let $h', h'' \in I$ denote its origin and goal respectively. We write $i \sim j$ (resp. $i \rightarrow j$) if there is an arrow $h \in \Omega$ such that $\{i, j\} = \{h', h''\}$ (resp. $(i, j) = (h', h'')$). Then the Cartan matrix $(a_{ij})_{i,j \in I}$ of $\mathfrak{g}$ is given by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } i \sim j; \\ 0 & \text{otherwise.} \end{cases}$$

4
Let $P^V = \bigoplus_{i \in I} \mathbb{Z}h_i$ be the coroot lattice of $g$. The fundamental weights $\{\varpi_i\}_{i \in I}$ form a basis of the weight lattice $P = \text{Hom}_\mathbb{Z}(P^V, \mathbb{Z})$ which is dual to $\{h_i\}_{i \in I}$. Let $\alpha_i = \sum_{j \in I} a_{ij} \varpi_j$ be the $i$-th simple root and $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \subset P$ be the root lattice. We put $P^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i$ and $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. The Weyl group is the finite group $W$ of linear transformations on $P$ generated by the set $\{r_i\}_{i \in I}$ of simple reflections, which are given by $r_i(\lambda) := \lambda - \lambda(h_i) \alpha_i$ for $\lambda \in P$. The set $R^+$ of positive roots is defined by $R^+ = \{W \{\alpha_i\}_{i \in I} \cap Q^+\}$.

### 2.2 Representations of Dynkin quiver

For an element $\beta \in Q^+$, we fix an $I$-graded $\mathbb{C}$-vector space $D = \bigoplus_{i \in I} D_i$ such that $\dim D := \sum_{i \in I}(\dim D_i)\alpha_i = \beta$. Let us consider the space

$$E_\beta := \bigoplus_{h \in \Omega} \text{Hom}(D_{h'}, D_{h''})$$

of representations of the quiver $Q$ of dimension vector $\beta$. On the space $E_\beta$, the group $G_\beta := \prod_{i \in I} GL(D_i)$ acts by conjugation. The set $G_\beta \backslash E_\beta$ of $G_\beta$-orbits is naturally in bijection with the set of isomorphism classes of representations of the quiver $Q$ of dimension vector $\beta$. By Gabriel’s theorem, for each $\alpha \in R^+$ there exists an indecomposable representation $M_\alpha$ such that $\dim M_\alpha = \alpha$ uniquely up to isomorphism. The correspondence $\alpha \mapsto M_\alpha$ gives a bijection between the set $R^+$ of positive roots and the set of isomorphism classes of indecomposable objects of the category $\text{Rep} Q$ of finite-dimensional representations of $Q$. Hence, the set

$$\text{KP}(\beta) := \left\{ (m_\alpha) \in (\mathbb{Z}_{\geq 0})^{R^+} \mid \sum_{\alpha \in R^+} m_\alpha \alpha = \beta \right\}$$

of Kostant partitions of $\beta$ labels the set of $G_\beta$-orbits: $G_\beta \backslash E_\beta = \{\Omega_m\}_{m \in \text{KP}(\beta)}$, where for each $m = (m_\alpha) \in \text{KP}(\beta)$, the $G_\beta$-orbit $\Omega_m$ corresponds to the isomorphism class of the representation $\bigoplus_{\alpha \in R^+} M_\alpha^{\sum m_\alpha \alpha}$. We have the natural $G_\beta$-orbit stratification

$$E_\beta = \bigsqcup_{m \in \text{KP}(\beta)} \Omega_m. \quad (2.1)$$

### 2.3 Repetition quiver

We fix a height function $\xi : I \to \mathbb{Z}; i \mapsto \xi_i$ of the quiver $Q$ i.e. it satisfies $\xi_i = \xi_j + 1$ if $i \to j$. Such a function $\xi$ is determined up to adding a constant. Choose a total ordering $I = \{i_1, i_2, \ldots, i_n\}$ such that $\xi_{i_1} \geq \xi_{i_2} \geq \cdots \geq \xi_{i_n}$ and define the corresponding Coxeter element $c := r_{i_1} r_{i_2} \cdots r_{i_n} \in W$.

The repetition quiver $\widehat{Q} = (\widehat{I}, \widehat{\Omega})$ is an infinite quiver defined by

$$\widehat{I} := \{(i, p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\},$$

$$\widehat{\Omega} := \{(i, p) \to (j, p + 1) \mid (i, p), (j, p + 1) \in \widehat{I}, \ i \sim j\}.$$

It is well-known (cf. [7]) that there exists an isomorphism $\phi$ from the Auslander-Reiten quiver of the derived category $D^b(\text{Rep} Q)$ to the repetition quiver $\widehat{Q}$,
which depends on the choice of $\xi$ and is described as follows. Since each indecomposable object of $D^b(\text{Rep} \ Q)$ is isomorphic to a unique stalk complex $M_{a}[k]$ for some $(a, k) \in \mathbb{R}^+ \times \mathbb{Z}$, we have a bijection between the sets of vertices

$$\mathbb{R}^+ \times \mathbb{Z} \ni (a, k) \mapsto \phi(M_{a}[k]) \in \hat{I},$$

which we denote by the same symbol $\phi$. This bijection $\phi : \mathbb{R}^+ \times \mathbb{Z} \to \hat{I}$ is determined inductively as follows:

- For each $i \in I$, we put $\gamma_i := \sum_j \alpha_j$ where $j$ runs all the vertices $j \in I$ such that there is a path in $Q$ from $j$ to $i$. Then $M_{\gamma_i}$ is an injective hull of the 1-dimensional representation $M_{\alpha_i}$. We define $\phi(\gamma_i, 0) := (i, \xi_i);$

- Inductively, if $\phi(\alpha, k) = (i, p)$ for $(\alpha, k) \in \mathbb{R}^+ \times \{0\}$, then we define as:

$$
\begin{align*}
\phi(c^{\pm 1}(\alpha), k) &:= (i, p \mp 2) \quad \text{if } c^{\pm 1}(\alpha) \in \mathbb{R}^+, \\
\phi(-c^{\mp 1}(\alpha), k \mp 1) &:= (i, p \mp 2) \quad \text{if } c^{\pm 1}(\alpha) \in -\mathbb{R}^+.
\end{align*}
$$

In the followings, we only consider the restriction of the bijection $\phi$ on $\mathbb{R}^+ = \mathbb{R}^+ \times \{0\}$, which we denote by the same symbol, i.e. we define $\phi(\alpha) := \phi(\alpha, 0)$ for $\alpha \in \mathbb{R}^+$.

### 2.4 Graded quiver varieties

In this subsection, we recall the definition of the graded quiver varieties. A basic reference is [17].

For elements $\nu = \sum_{i \in I} n_i \alpha_i \in \mathbb{Q}^+$ and $\lambda = \sum_{i \in I} l_i \varphi_i \in \mathbb{P}^+$, we fix $I$-graded $\mathbb{C}$-vector spaces $V = \bigoplus_{i \in I} V_i, W = \bigoplus_{i \in I} W_i$ such that $\dim V_i = n_i, \dim W_i = l_i$ for each $i \in I$. We form the following space of linear maps:

$$
\mathbf{M}(\nu, \lambda) := \left( \bigoplus_{i \in I} \text{Hom}(V_j, V_i) \right) \oplus \left( \bigoplus_{i \in I} \text{Hom}(W_i, V_i) \right) \oplus \left( \bigoplus_{i \in I} \text{Hom}(V_i, W_i) \right).
$$

On the $\mathbb{C}$-vector space $\mathbf{M}(\nu, \lambda)$, the groups $G(\nu) := \prod_{i \in I} GL(V_i)$, $G(\lambda) := \prod_{i \in I} GL(W_i)$ act by conjugation and the 1-dimensional torus $\mathbb{C}^\times$ acts by the scalar multiplication. We write an element of $\mathbf{M}(\nu, \lambda)$ as a triple $(B, a, b)$ of linear maps $B = \bigoplus B_{ij}, a = \bigoplus a_i$ and $b = \bigoplus b_i$. Let $\mu = \bigoplus_{i \in I} \mu_i : \mathbf{M}(\nu, \lambda) \to \bigoplus_{i \in I} \mathfrak{g}(V_i)$ be the map given by

$$
\mu_i(B, a, b) = a_i b_i + \sum_{j \neq i} \varepsilon(i, j) B_{ij} B_{ji},
$$

where $\varepsilon(i, j) := 1$ (resp. $-1$) if $j \to i$ (resp. $i \to j$). A point $(B, a, b) \in \mu^{-1}(0)$ is said to be stable if there exists no non-zero $I$-graded subspace $V' \subset V$ such that $B(V') \subset V'$ and $V' \subset \text{Ker } b$. Let $\mu^{-1}(0)^{st}$ be the set of stable points, on which $G(\nu)$ acts freely. Then we consider a set-theoretic quotient

$$
\mathfrak{M}(\nu, \lambda) := \mu^{-1}(0)^{st} / G(\nu).
$$
It is known that this quotient has a structure of a non-singular quasi-projective variety which is isomorphic to a quotient in the geometric invariant theory. We also consider the affine algebro-geometric quotient
\[ M_0(\nu, \lambda) := \mu^{-1}(0)/\mathcal{G}(\nu) = \text{Spec } \mathbb{C}[\mu^{-1}(0)]^G(\nu), \]

together with a canonical projective morphism \( M(\nu, \lambda) \to M_0(\nu, \lambda) \). These quotients \( M(\nu, \lambda), M_0(\nu, \lambda) \) naturally inherit the actions of the group \( \mathcal{G}(\lambda) := G(\lambda) \times \mathbb{C}^\times \), which makes the canonical projective morphism into a \( \mathcal{G}(\lambda) \)-equivariant morphism.

For \( \nu, \nu' \in \mathbb{Q}^+ \) such that \( \nu' - \nu \in \mathbb{Q}^+ \), there is a natural closed embedding \( M_0(\nu, \lambda) \hookrightarrow M_0(\nu', \lambda) \). With respect to these embeddings, the family \( \{ M_0(\nu, \lambda) \}_\nu \subseteq \mathbb{Q}^+ \) forms an inductive system, which stabilizes at some \( \nu \in \mathbb{Q}^+ \). We consider the union (inductive limit) and obtain the following combined \( \mathcal{G}(\lambda) \)-equivariant morphism:
\[ \pi : M(\lambda) := \bigcup_\nu M(\nu, \lambda) \to M_0(\lambda) := \bigcup_\nu M_0(\nu, \lambda). \]

We denote the fiber \( \pi^{-1}(0) \) of the origin \( 0 \in M_0(\lambda) \) by \( \mathcal{L}(\lambda) = \bigcup_{\nu \in \mathbb{Q}^+} \mathcal{L}(\nu, \lambda) \). Note that \( M(0, \lambda) = \mathcal{L}(0, \lambda) \) consists of a single point.

Next we consider a free abelian monoid \( \mathcal{P}^+ = \mathbb{Z}_{\geq 0} \mathcal{I} \) with the free generating set \( \mathcal{I} \). Define a homomorphism \( \text{cl} : \mathcal{P}^+ \to \mathbb{P}^+ \) by \( \text{cl}(i, p) = \nu_i \). For an element \( \lambda = \sum_i l_i, p_i \in \mathcal{P}^+ \) with \( \text{cl}(\lambda) = \lambda \), we fix a decomposition \( W_i = \bigoplus_p W_{i,p} \) such that \( \dim W_{i,p} = l_i, p \) for each \( (i, p) \in \mathcal{I} \). Define a group homomorphism \( f_i : \mathbb{C}^\times \to \prod_p GL(W_{i,p}) \subset GL(W_i) \) by \( f_i(t)|_{W_{i,p}} := t^{l_i, p} \cdot \text{id}_{W_{i,p}} \) for \( t \in \mathbb{C}^\times \).

We put \( T(\lambda) := (\prod_{i \in I} f_i \times \text{id}):(\mathbb{C}^\times)^I \subset \mathcal{G}(\lambda) \) and consider the subvarieties of \( T(\lambda) \)-fixed points:
\[ \pi^* := \pi^T(\lambda) : M^*(\lambda) := M(\lambda)/T(\lambda) \to M^*_0(\lambda) := M_0(\lambda)/T(\lambda). \]

We refer these varieties as the graded quiver varieties. We put \( \mathcal{L}^*(\lambda) := \mathcal{L}(\lambda)/T(\lambda) = (\pi^*)^{-1}(0) \).

The centralizer of \( T(\lambda) \) inside \( \mathcal{G}(\lambda) \) is
\[ \mathcal{G}(\lambda) \cong G(\lambda) \times \mathbb{C}^\times := \prod_{(i, p) \in \mathcal{I}} GL(W_{i,p}) \times \mathbb{C}^\times \subset \mathcal{G}(\lambda), \]

which naturally acts on the varieties \( M^*(\lambda), M^*_0(\lambda), \mathcal{L}^*(\lambda) \). The morphism \( \pi^* \) is \( \mathcal{G}(\lambda) \)-equivariant.

### 2.5 Hernandez-Leclerc’s isomorphism

Let \( \mathcal{P}_0^+ \subseteq \mathcal{P}^+ \) be the submonoid generated by the subset \( \phi(R^+) \subseteq \mathcal{I} \). For an element \( \beta := \sum_{i \in I} d_i, a_i \in \mathbb{Q}^+ \), we define \( \lambda_\beta := \sum_{i \in I} d_i \phi(a_i) \in \mathcal{P}_0^+ \). In this case, we write \( \pi^*_\beta : M^*_0(\lambda_\beta) \to M^*_0(\lambda_\beta) \) instead of \( \pi^* : M^*_0(\lambda_\beta) \to M^*_0(\lambda_\beta) \) for simplicity. For each \( i \in I \), we identify the vector space \( D_i \) in Subsection 2.2.
with the vector space $W_{\phi(\alpha_i)}$ in Subsection 2.4. This induces the identification $G_\beta = G(\lambda_\beta)$. We write $G_\beta$, $T\beta$ instead of $G(\lambda_\beta)$, $T(\lambda_\beta)$ respectively. By the inclusion $G_\beta = G_\beta \times \{1\} \subset G_\beta \times \mathbb{C}^\times = G_\beta$, the group $G_\beta$ is regarded as a subgroup of the group $G_\beta$. Then the multiplication map $G_\beta \times T\beta \rightarrow G_\beta$ gives an isomorphism of algebraic groups

\[ G_\beta \times T\beta \cong G_\beta. \]  

We equip an action of the group $G_\beta$ on the space $E_\beta$ via the projection $G_\beta \cong G_\beta \times T\beta \rightarrow G_\beta$.

**Theorem 2.1** (Hernandez-Leclerc [8] Theorem 9.11). There exists a $G_\beta$-equivariant isomorphism of varieties

\[ \mathfrak{M}_{\cdot,0,\beta} \cong E_\beta. \]

Henceforth, we identify the graded quiver variety $\mathfrak{M}_{\cdot,\beta}$ with the space $E_\beta$ under the isomorphism in Theorem 2.1.

We recall some properties of fibers of the $G_\beta$-equivariant morphism $\pi_\beta : \mathfrak{M}_{\cdot,\beta} \rightarrow E_\beta$. By the injective map

\[ \text{KP}(\beta) \ni (m_\alpha) \mapsto \sum_{\alpha} m_\alpha \phi(\alpha) \in \mathcal{P}^+, \]

we regard $\text{KP}(\beta)$ as a subset of $\mathcal{P}^+$.

Then we have a disjoint union decomposition

\[ \mathcal{P}^+ = \bigcup_{\beta \in \mathbb{Q}^+} \text{KP}(\beta). \]

**Proposition 2.2** (cf. [5] Section 3). Let $\mathbf{m} \in \text{KP}(\beta)$ and pick a point $x_\mathbf{m} \in \mathcal{O}_\mathbf{m}$.

1. We have an isomorphism $\pi_\beta^{-1}(x_\mathbf{m}) \cong \mathfrak{L}^*(\mathbf{m})$.
2. The maximal reductive quotient of the stabilizer $\text{Stab}_{G_\beta}(x_\mathbf{m}) \subset G_\beta$ of the point $x_\mathbf{m}$ is isomorphic to $G(\mathbf{m})$.
3. The isomorphism in (1) induces the following commutative diagram:

\[ \begin{array}{ccc}
\text{Aut}(\pi_\beta^{-1}(x_\mathbf{m})) & \cong & \text{Aut}(\mathfrak{L}^*(\mathbf{m})) \\
\uparrow & & \uparrow \\
\text{Stab}_{G_\beta}(x_\mathbf{m}) & \longrightarrow & G(\mathbf{m}) \\
\end{array} \]

where the vertical arrows are the action maps and the lower horizontal arrow is the canonical quotient map in (2).

### 3 Convolution and geometric extension algebras

Let $k$ be a field of characteristic zero. Later in Subsection 3.4, we specialize $k = \mathbb{Q}(q)$. 
3.1 Preliminary on equivariant geometry

For the materials in this subsection, we refer [3] and [4].

Let \( G \) be a complex linear algebraic group. A \( G \)-variety \( X \) is a quasi-projective complex algebraic variety equipped with an algebraic action of the group \( G \). We set \( pt := \text{Spec } \mathbb{C} \) with the trivial \( G \)-action. The equivariant \( K \)-group \( K^G(X) \) is defined to be the Grothendieck group of the abelian category of \( G \)-equivariant coherent sheaves on \( X \) which is a module over the representation ring \( R(G) = K^G(pt) \). We put

\[
K^G(X)_k := K^G(X) \otimes_{\mathbb{Z}} k, \quad R(G)_k := R(G) \otimes_{\mathbb{Z}} k.
\]

Let \( I \subset R(G)_k \) be the augmentation ideal, i.e. the ideal generated by virtual representations of dimension 0. We define the \( I \)-adic completions by

\[
\hat{K}^G(X)_k := \lim_k K^G(X)_k/I^k K^G(X)_k, \quad \hat{R}(G)_k := \lim_k R(G)_k/I^k.
\]

The completed \( K \)-group \( \hat{K}^G(X)_k \) is a module over the algebra \( \hat{R}(G)_k \).

On the other hand, the \( G \)-equivariant Borel-Moore homology with \( k \)-coefficients

\[
H^*_G(X,k) = \bigoplus_{k \in \mathbb{Z}} H^k_G(X,k),
\]

is a module over the \( G \)-equivariant cohomology ring \( H^*_G(pt,k) \) of \( pt \) (with the cup product). Let us define the completion of a \( \mathbb{Z} \)-graded \( k \)-vector space \( V = \bigoplus_{k \in \mathbb{Z}} V_k \) by \( V^\wedge := \prod_{k \in \mathbb{Z}} V_k \). The completion \( H^*_G(pt,k)^\wedge \) naturally becomes a \( k \)-algebra and the completion \( H^*_G(X,k)^\wedge \) becomes a module over \( H^*_G(pt,k)^\wedge \).

Assume that our \( G \)-variety \( X \) is a \( G \)-stable closed subvariety of a non-singular ambient \( G \)-variety \( M \). Then we have the \( G \)-equivariant local Chern character map

\[
(ch^G)_X^M : \hat{K}^G(X)_k \to H^*_G(X,k)^\wedge.
\]

relative to \( M \). We simply write \( ch^G \) instead of \( (ch^G)^M_X \) if the pair \((M,X)\) is obvious from the context. When \( X = M = pt \), the corresponding Chern character map induces an isomorphism of \( k \)-algebras

\[
\hat{R}(G)_k = \hat{K}^G(pt)_k \cong H^*_G(pt,k)^\wedge = H^*_G(pt,k)^\wedge.
\]

We identify \( H^*_G(pt,k)^\wedge \) with \( \hat{R}(G)_k \) via this isomorphism. Then \( (ch^G)_X^M \) is regarded as an \( \hat{R}(G)_k \)-homomorphism.

For a \( G \)-equivariant vector bundle \( E \) on a non-singular \( M \), let \( Td^G(E) \in H^{2*}_G(M,k) \) be the \( G \)-equivariant Todd class. This is an invertible element with respect to the cup product. For the tangent bundle \( T_M \) of \( M \), we put \( Td^G_M := Td^G(T_M) \).

**Theorem 3.1** (Equivariant Riemann-Roch [4]). For \( i = 1, 2 \), let \( X_i \) be a \( G \)-variety which is a \( G \)-stable closed subvariety of a non-singular ambient \( G \)-variety
$M_i$. Assume that a $G$-equivariant morphism $\tilde{f} : M_1 \to M_2$ restricts to a proper morphism $f : X_1 \to X_2$. Then we have

$$f_* \left( \text{Td}^G_{M_1} \cdot (ch^G)^{M_1}_{X_1}(\zeta) \right) = \text{Td}^G_{M_2} \cdot (ch^G)^{M_2}_{X_2}(f_*\zeta), \quad \zeta \in \hat{K}^G(X_1)_k.$$  

The following proposition is standard.

**Proposition 3.2.** Let $M$ be a non-singular $G$-variety. Let $Y \subset X \subset M$ be $G$-stable closed subvarieties, and $i : Y \hookrightarrow X, j : X \setminus Y \hookrightarrow X$ be inclusions. Then we have the following commutative diagram:

$$
\begin{CD}
\hat{K}^G(Y)_k @>i^*>> \hat{K}^G(X)_k @>j^*>> \hat{K}^G(X \setminus Y)_k \\
@VV{(ch^G)^M_Y}V @VV{(ch^G)^M_X}V @VV{(ch^G)^M_{X \setminus Y}}V \\
H^*_c(Y, k)^\wedge @>i^*>> H^*_c(X, k)^\wedge @>j^*>> H^*_c(X \setminus Y, k)^\wedge.
\end{CD}
$$

Next we consider the convolution products. Let $M_i$ be non-singular $G$-varieties for $i = 1, 2, 3$. We denote by $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ the projection to the $(i, j)$-factors for $(i, j) = (1, 2), (2, 3), (1, 3)$. Let $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ be $G$-stable closed subvarieties such that the morphism

$$p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to Z_{13} := p_{13}(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}))$$

is proper. Then we define the convolution product $*: K^G(Z_{12}) \otimes_{R(G)} K^G(Z_{23}) \to K^G(Z_{13})$ relative to $M_1 \times M_2 \times M_3$ by

$$\zeta \ast \eta := p_{13*}(p_{12}^*\zeta \otimes_{M_1 \times M_2 \times M_3} p_{23}^*\eta), \quad \zeta \in K^G(Z_{12}), \eta \in K^G(Z_{23}).$$

This naturally induces the convolution product on the completed $G$-equivariant $K$-groups $\hat{K}^G(Z_{12})_k \otimes_{R(G)_k} \hat{K}^G(Z_{23})_k \to \hat{K}^G(Z_{13})_k$. Similarly, we have the convolution product on the $G$-equivariant Borel-Moore homologies $H^*_c(Z_{12}, k) \otimes H^*_c(pt, k)$

$$H^*_c(Z_{23}, k) \to H^*_c(Z_{13}, k)$$

relative to $M_1 \times M_2 \times M_3$ and its completed version $H^*_c(Z_{12}, k)^\wedge \otimes_{R(G)_k} H^*_c(Z_{23}, k)^\wedge \to H^*_c(Z_{13}, k)^\wedge$.

Under the situation in the previous paragraph, for each $(i, j) = (1, 2), (2, 3), (1, 3)$, we also define the $G$-equivariant Riemann-Roch homomorphism $\text{RR}^G : \hat{K}^G(Z_{ij})_k \to H^*_c(Z_{ij}, k)^\wedge$ relative to $M_i \times M_j$ by

$$\text{RR}^G(\zeta) := (p_i^*\text{Td}^G_M) \cdot (ch^G)^{M_i \times M_j}_{Z_{ij}}(\zeta), \quad \zeta \in \hat{K}^G(Z_{ij})_k,$$

where $p_i : M_i \times M_j \to M_i$ is the projection. By a completely similar discussion as in [3, 5.11.1], we can prove the following.

**Proposition 3.3.** The $G$-equivariant Riemann-Roch homomorphisms are compatible with the convolution product, i.e. we have

$$\text{RR}^G(\zeta \ast \eta) = \text{RR}^G(\zeta) \ast \text{RR}^G(\eta), \quad \zeta \in \hat{K}^G(Z_{12})_k, \eta \in \hat{K}^G(Z_{23})_k.$$
3.2 Quiver Hecke algebra

Fix an element $\beta = \sum_{i \in I} d_i \alpha_i \in \mathbb{Q}^+$ and put $d := \sum_{i \in I} d_i$. Let

$$I^\beta := \{i = (i_1, \ldots, i_d) \in I^d \mid \alpha_{i_1} + \cdots + \alpha_{i_d} = \beta\}.$$ 

The symmetric group $\mathfrak{S}_d$ of degree $d$ acts on the set $I^\beta$ from the right by $(i_1, \ldots, i_d) \cdot w := (i_{w(1)}, \ldots, i_{w(d)})$. Let $s_k \in \mathfrak{S}_d$ denote the transposition of $k$ and $k + 1$ for $1 \leq k < d$.

**Definition 3.4** (Khovanov-Lauda [16], Rouquier [20]). The quiver Hecke algebra $H_Q(\beta)$ is defined to be a $\mathbb{k}$-algebra with the generating set \{e(i) \mid i \in I^\beta \} \cup \{x_1, \ldots, x_d \} \cup \{\tau_1, \ldots, \tau_{d-1}\}$, satisfying the following relations:

$$e(i)e(i') = \delta_{i,i'}e(i), \quad \sum_{i \in I^\beta} e(i) = 1, \quad x_kx_l = x_lx_k, \quad x_ke(i) = e(i)x_k,$$

$$\tau_k e(i) = e(i \cdot s_k) \tau_k, \quad \tau_k \tau_l = \tau_l \tau_k \quad \text{if} \quad |k - l| > 1,$$

$$\tau_k^2 e(i) = \begin{cases} (x_k - x_{k+1})e(i), & \text{if} \quad i_k \leftarrow i_{k+1}, \\ (x_{k+1} - x_k)e(i), & \text{if} \quad i_k \rightarrow i_{k+1}, \\ e(i), & \text{if} \quad d_{i_k,i_{k+1}} = 0, \\ 0, & \text{if} \quad i_k = i_{k+1}, \end{cases}$$

$$(\tau_k x_l - x_{k+l}) \tau_k e(i) = \begin{cases} -e(i), & \text{if} \quad l = k, i_k = i_{k+1}, \\ e(i), & \text{if} \quad l = k + 1, i_k = i_{k+1}, \\ 0, & \text{otherwise}. \end{cases}$$

$$(\tau_{k+1} \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(i) = \begin{cases} e(i), & \text{if} \quad i_k = i_{k+2}, i_k \leftarrow i_{k+1}, \\ -e(i), & \text{if} \quad i_k = i_{k+2}, i_k \rightarrow i_{k+1}, \\ 0, & \text{otherwise}. \end{cases}$$

The quiver Hecke algebra $H_Q(\beta)$ is equipped with a $\mathbb{Z}$-grading given by

$$\deg e(i) = 0, \quad \deg x_k = 2, \quad \deg \tau_k e(i) = -a_{i_k,i_{k+1}}.$$ 

Since the grading is bounded from below (see [16, Theorem 2.5]), the completion $\tilde{H}_Q(\beta) := H_Q(\beta)^\wedge$ inherits a natural structure of $\mathbb{k}$-algebra.

We recall the faithful polynomial right representation of $H_Q(\beta)$ from [16, Section 2.3]. We set

$$P_\beta := \bigoplus_{i \in I^\beta} \mathbb{k}[x_1, \ldots, x_d] 1_i$$

with a commutative $\mathbb{k}[x_1, \ldots, x_d]$-algebra structure $1_i \cdot 1_{i'} = \delta_{ii'} 1_i$. We define $f^w(x_1, \ldots, x_d) := f(x_{w(1)}, \ldots, x_{w(d)})$ for $f \in \mathbb{k}[x_1, \ldots, x_d]$ and $w \in \mathfrak{S}_d$. 

11
Theorem 3.5 ([16] Proposition 2.3). The following formulas give a faithful right $H_Q(\beta)$-module structure on the $k$-vector space $P_\beta$:

\[
a \cdot e(i) = a_1, \\
a \cdot x_k = ax_k, \\
(f 1_i) \cdot \tau_k = \begin{cases} 
    f^k - f & \text{if } i_k = i_{k+1}, \\
    x_k - x_{k+1} & \text{if } i_k \leftarrow i_{k+1}, \\
    (x_{k+1} - x_k)f^k1_i \cdot s_k & \text{otherwise}, 
\end{cases}
\]

where $a \in P_\beta$ and $f1_i \in k[x_1, \ldots, x_d]_1$.

Replacing the polynomial ring $k[x_1, \ldots, x_d]$ with the ring $k\llbracket x_1, \ldots, x_d \rrbracket$ of formal power series, we get the completion of the representation $P_\beta$:

\[
\tilde{P}_\beta := \bigoplus_{i \in I^\beta} k[x_1, \ldots, x_d]_1 = P_\beta \otimes_{H_Q(\beta)} \hat{H}_Q(\beta).
\] (3.1)

### 3.3 Varagnolo-Vasserot’s realization

Fix an $I$-graded $\mathbb{C}$-vector space $D = \bigoplus_{i \in I} D_i$ with $\dim D = \beta$, i.e. $\dim D_i = d_i$ as in Subsection 2.2. We consider the following two non-singular $G_\beta$-varieties:

\[
B_\beta = \{ F^\bullet = (D = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^d = 0) \mid F^k \text{ is an } I\text{-graded subspace of } D \}, \\
F_\beta = \{ (F^\bullet, x) \in B_\beta \times E_\beta \mid x(F^k) \subset F^k \text{ for any } 1 \leq k \leq d \}.
\]

The $G_\beta$-action on $F_\beta$ is defined so that the projections $\text{pr}_1 : F_\beta \to B_\beta$ and $\mu_\beta := \text{pr}_2 : F_\beta \to E_\beta$ are $G_\beta$-equivariant. They decompose into connected components as

\[
B_\beta = \bigsqcup_{i \in I^\beta} B_i, \quad F_\beta = \bigsqcup_{i \in I^\beta} F_i,
\]

where we put

\[
B_i := \{ F^\bullet \in B_\beta \mid \dim F^{k-1} = \dim F^k + \alpha_{ik}, \forall k \}, \quad F_i := (\text{pr}_1)^{-1}(B_i)
\]

for $i = (i_1, \ldots, i_d) \in I^\beta$.

We fix a basis $\{v_k\}_{1 \leq k \leq d}$ of the vector space $D$ so that the set $\{v_{i,j}\}_{1 \leq j \leq d_i}$ forms a basis of the vector space $D_i$ for each $i \in I$, where we put $v_{i,j} := v_{d_i+\ldots+d_{i-1}+j}$. Let $H_i \subset GL(D_i)$ be the maximal torus fixing the lines $\{Cv_{i,j}\}_{1 \leq j \leq d_i}$ for each $i \in I$. We set $H_\beta := \prod_{i \in I} H_i \subset G_\beta$.

Let $F^\bullet_0 \in B_\beta$ be the flag defined by $F^k_0 := \bigoplus_{l \geq k} \mathbb{C}v_l$, which belongs to the component $B_0$, with $i_0 := (1^{d_1}, 2^{d_2}, \ldots, n^{d_n}) \in I^\beta$. For each $i \in I^\beta$, we fix an element $w_i \in \mathcal{S}_d$ such that $i = i_0 \cdot w_i$. The set $\{w_i\}_{i \in I^\beta}$ forms a complete system of cost representatives for the quotient $\mathcal{G}_\beta \backslash \mathcal{S}_d$, where $\mathcal{G}_\beta := \text{Stab}_{\mathcal{S}_d}(i_0) = \mathcal{G}_{d_1} \times \cdots \times \mathcal{G}_{d_n}$. For each $w \in \mathcal{S}_d$, we define the flag $F^\bullet_w$ by $F^k_w := \bigoplus_{l \geq k} \mathbb{C}v_{w(l)}$, which belongs to the component $B_{i_0 \cdot w}$. Let $F^\bullet_i := F^\bullet_{w_i} \in B_i$ for $i \in I^\beta$. Then we
have $B_i \cong G_\beta/B_1$ with $B_1 := \text{Stab}_{G_\beta}(F_1^\star) \subset G_\beta$ being the Borel subgroup fixing the flag $F_1^\star$, which contains the maximal torus $H_\beta$. Then we have

$$H^*_G(B_i, k) \cong H^*_G(pt, k) \cong H^*_{H_\beta}(pt, k) \cong k[x_1, \ldots, x_d]_1,$$  \hspace{1cm} (3.2)

where the last isomorphism sends the 1st $H_\beta$-equivariant Chern class of the line $\mathbb{C}v_{u_1(k)}$ to the element $x_k.1_1$. Thus we get an isomorphism

$$H^*_G(B_\beta, k) = \bigoplus_{i \in I^\beta} H^*_G(B_i, k) \cong \bigoplus_{i \in I^\beta} k[x_1, \ldots, x_d]_1 = P_\beta. \hspace{1cm} (3.3)$$

We consider the Steinberg type variety $Z_\beta := F_\beta \times_{E_\beta} F_\beta$ associated with the morphism $\mu_\beta : F_\beta \to E_\beta$. Its $G_\beta$-equivariant Borel-Moore homology group $H^*_G(Z_\beta, k)$ becomes a $k$-algebra with respect to the convolution product relative to $F_\beta \times F_\beta \times F_\beta$. We identify the variety $B_\beta$ with the fiber product \{0\} \times_{E_\beta} F_\beta. Then the convolution product relative to \{0\} \times F_\beta \times F_\beta makes the space $H^*_G(B_\beta, k)$ into a right $H^*_G(Z_\beta, k)$-module.

Let $\mu_1$ denote the restriction of the proper morphism $\mu_\beta : F_\beta \to E_\beta$ to the component $F_i$ for $i \in I^\beta$. We put

$$L_\beta := \bigoplus_{i \in I^\beta} (\mu_1)_* k[\dim F_i],$$

where $k[\dim F_i]$ is the trivial local system (i.e. the constant $k$-sheaf of rank 1) on $F_i$ homologically shifted by $\dim F_i$. By the decomposition theorem, we have

$$L_\beta \cong \bigoplus_{m \in KP(\beta)} L_m \otimes_k IC_m = \bigoplus_{m \in KP(\beta)} \bigoplus_{k \in \mathbb{Z}} L_{m,k} \otimes_k IC_m[k],$$

where $IC_m$ denotes the intersection cohomology complex associated with the trivial local system on the orbit $O_m$ and $L_m = \bigoplus_{k \in \mathbb{Z}} L_{m,k}[k]$ is a self-dual finite-dimensional graded $k$-vector space for each $m \in KP(\beta)$. The vector space $L_m$ is known to be non-zero for all $m \in KP(\beta)$ (see [15, Corollary 2.8]). We consider the Yoneda algebra

$$\text{Ext}^*_G(L_\beta, L_\beta) = \bigoplus_{k \in \mathbb{Z}} \text{Ext}^*_G(L_\beta, L_\beta)$$

in the derived category of $G_\beta$-equivariant constructible complexes on $E_\beta$. This is a $\mathbb{Z}$-graded $k$-algebra whose grading is bounded from below.

By a standard argument (see [3, Section 8.6]), we have an isomorphism of $k$-algebras

$$\text{Ext}^*_G(L_\beta, L_\beta) \cong H^*_G(Z_\beta, k). \hspace{1cm} (3.4)$$

Note that this is not compatible with the $\mathbb{Z}$-grading.

Let $L_i(k)$ be the $G_\beta$-equivariant line bundle on $F_i$ whose fiber at the point $(F^\star, x) \in F_i$ is $F^{k-1}/F^k$ for $i \in I^\beta$ and $1 \leq k \leq d$. 

13
Theorem 3.6 (Varagnolo-Vasserot [21]). There is a unique isomorphism of \( \mathbb{Z} \)-graded \( k \)-algebras

\[
H_Q(\beta) \xrightarrow{\cong} \text{Ext}_{G_\beta}^* (\mathcal{L}_\beta, \mathcal{L}_\beta)
\]

which satisfies the following properties:

1. The composition \( H_Q(\beta) \xrightarrow{\cong} H_{G_\beta}^* (Z_\beta, k) \) of the isomorphisms (3.5) and (3.4) sends the element \( e(1) \) (resp. \( x_k e(1) \)) to the push-forward of the fundamental class \([\mathcal{F}_1]\) (resp. the 1st \( G_\beta \)-equivariant Chern class of the line bundle \( \mathcal{L}_1(k) \)) with respect to the diagonal embedding \( \mathcal{F}_1 \hookrightarrow \mathcal{F}_1 \times_{E_\beta} \mathcal{F}_1 \);

2. We have the following commutative diagram:

\[
\begin{array}{ccc}
H_Q(\beta) & \xrightarrow{\cong} & H_{G_\beta}^* (Z_\beta, k) \\
\downarrow & & \downarrow \\
\text{End} (P_\beta)^{\text{op}} & \xrightarrow{\cong} & \text{End} \left( H_{G_\beta}^* (B_\beta, k) \right)^{\text{op}},
\end{array}
\]

where the lower horizontal arrow denotes the isomorphism induced from (3.3) and the vertical arrows denote the right module structures.

Remark 3.7. Because our convention of the flag variety \( B_\beta \) differs from Varagnolo-Vasserot’s [21], we need a modification. Actually, our isomorphism (3.5) is obtained by twisting the original isomorphism \( H_Q(\beta) \cong \text{Ext}_{G_\beta}^* (\mathcal{L}_\beta, \mathcal{L}_\beta) \) in [21] by a \( k \)-algebra involution on \( H_Q(\beta) \) given by

\[
e(1) \mapsto e(1^{\text{op}}), \quad x_k \mapsto x_{d-k+1}, \quad \tau_k e(1) \mapsto \begin{cases} -\tau_{d-k} e(1^{\text{op}}) & \text{if } i_k = i_{k+1}; \\ \tau_{d-k} e(1^{\text{op}}) & \text{if } i_k \neq i_{k+1}, \end{cases}
\]

where \( 1^{\text{op}} := (i_d, \ldots, i_2, i_1) \) for \( i = (i_1, i_2, \ldots, i_d) \in I^\beta \).

Similarly to the case of the \( G_\beta \)-equivariant Borel-Moore homologies, the \( K \)-group \( K^{G_\beta} (Z_\beta)_k \) becomes an \( R(G_\beta)_k \)-algebra and the \( K \)-group \( K^{G_\beta} (B_\beta)_k \) becomes a right \( K^{G_\beta} (Z_\beta)_k \)-module with respect to the convolution products.

For each \( i \in I^\beta \), we have

\[
K^{G_\beta} (B_i)_k \cong K^{B_i} (\text{pt})_k \cong K^{H_\beta} (\text{pt})_k = R(H_\beta)_k \cong k[y^\pm_1, \ldots, y^\pm_d]_{1_1}
\]

where the last isomorphism sends the class \([Cv_{w_1(k)}]\) of the 1-dimensional \( H_\beta \)-module \( \mathbb{C}v_{w_1(k)} \) to the element \( y_k 1_1 \). The \( G_\beta \)-equivariant Chern character map \((\text{ch}^{G_\beta})_{B_i}\) gives an isomorphism of \( k \)-algebras

\[
\mathcal{K}^{G_\beta} (B_i)_k \cong k[y_1 - 1, \ldots, y_d - 1]_{1_1} \xrightarrow{\cong} \mathbb{K}[x_1, \ldots, x_d]_{1_1} \cong H_{G_\beta} (B_i, k),
\]

where the middle arrow sends the element \( y_k 1_1 \) to the exponential \( e^{x_1} 1_1 \) for \( 1 \leq k \leq d \). Applying the equivariant Riemann-Roch theorem (=Theorem 3.1) to the inclusion \( B_i \hookrightarrow \mathcal{F}_i \), we have

\[
(\text{ch}^{G_\beta})_{\mathcal{F}_i} : C_1 \cdot (\text{ch}^{G_\beta})_{B_i}, \quad C_1 := (\text{Td}_{\mathcal{F}_i}^{G_\beta})^{-1} \text{Td}_{B_i}^{G_\beta} \cdot 1_1 \in \mathbb{K}[x_1, \ldots, x_d]_{1_1}
\]
and hence the map $\left(\text{ch}\ ^G_\beta\right)_{B_\beta}^F : \mathcal{R}(G_\beta)_k \to H^*_G(G_\beta, k)$ is an isomorphism of $\mathcal{R}(G_\beta)_k$-modules. Summing up over $i \in I^\beta$, we obtain an isomorphism of $\mathcal{R}(G_\beta)_k$-modules

$$\left(\text{ch}\ ^G_\beta\right)_{B_\beta}^F : \mathcal{R}(G_\beta)_k \cong H^*_G(G_\beta, k). \quad (3.7)$$

**Proposition 3.8.** The Riemann-Roch homomorphism gives an isomorphism of $\mathcal{R}(G_\beta)_k$-algebras:

$$\text{RR}^G_\beta : \mathcal{R}(Z_\beta)_k \cong H^*_G(Z_\beta, k),$$

which makes the following diagram commute:

$$\begin{array}{ccc}
\mathcal{R}(Z_\beta)_k & \cong & H^*_G(Z_\beta, k) \\
\downarrow & & \downarrow \\
\text{End}\left(\mathcal{R}(B_\beta)_k\right)^{\text{op}} & \cong & \text{End}\left(H^*_G(B_\beta, k)\right)^{\text{op}},
\end{array} \quad (3.8)$$

where the lower horizontal arrow denotes the isomorphism induced from (3.7) and the vertical arrows denote the right module structures.

**Proof.** By Proposition 3.3, the map $\text{RR}^G_\beta : \mathcal{R}(Z_\beta)_k \to H^*_G(Z_\beta, k)$ is an algebra homomorphism and the diagram (3.8) commutes. To prove that the map $\text{RR}^G_\beta : \mathcal{R}(Z_\beta)_k \to H^*_G(Z_\beta, k)$ is an isomorphism, it suffices to check that the equivariant Chern character map $\left(\text{ch}\ ^G_\beta\right)_{Z_\beta}^F : \mathcal{R}(Z_\beta)_k \to H^*_G(Z_\beta, k)$ gives an isomorphism of $\mathcal{R}(Z_\beta)_k$-modules since $\text{RR}^G_\beta$ is obtained from $\left(\text{ch}\ ^G_\beta\right)_{Z_\beta}^F$ by multiplying the $G_\beta$-equivariant Todd class $\rho^*_1\text{Td}^G_\beta$, which is an invertible element. Because we have the connected component decomposition

$$Z_\beta = \bigcup_{i \in I^\beta} Z_{i, i'}, \quad Z_{i, i'} := F_i \times E_{i, i'},$$

we focus on a connected component

$$Z_{i, i'} = \{(F^*, F^{'*}, x) \in B_i \times B_i \times E_{i, i'} \mid x(F^k) \subset F^k, x(F^{nk}) \subset F^{nk}, \forall k\}.$$

For each $w \in \mathcal{S}_\beta w_{1, i'}$, we define a locally closed $G_\beta$-subvariety $Z_{i, i'}^w = G_\beta \times B_i \{(F^*, F^{'*}, x) \in Z_{i, i'} \mid \forall w \in B_i F^w_x \in B_i F^{'*} \}$ which is a $G_\beta$-equivariant affine bundle over $B_i$. They give a $G_\beta$-stable stratification $Z_{i, i'} := \bigcup_{w \in \mathcal{S}_\beta w_{1, i'}} Z_{i, i'}^w$. Fix a total ordering $\mathcal{S}_\beta w_{1, i'} = \{w_1, w_2, \ldots, w_m\}$ such that we have $w_k w^{-1} < w_l w^{-1}$ in the Bruhat ordering only if $k < l$. We simply write $Z_{i, i'}^k := Z_{i, i'}^w$ and set $Z_{i, i'}^{\leq} := \bigcup_{j \leq k} Z_{i, i'}^j$. Then for each $k$, the variety $Z_{i, i'}^{\leq k}$ is closed in $Z_{i, i'}$ and its complement is $Z_{i, i'}^{> k}$. Since $Z_{i, i'}^{> k}$ is a $G_\beta$-equivariant affine bundle over $B_i$, its homology of odd degree vanishes:
\[ H_{\text{odd}}(Z_{i,l}^k, k) = 0. \] Therefore an inductive argument with respect to \( k \) yields \[ H_{\text{odd}}(Z_{i,l}^{\leq k}, k) = 0. \] Using the cellular fibration lemma \([3, 5.5.1]\) for equivariant \( K \)-groups and Proposition 3.2, we obtain the following commutative diagram with exact rows for each \( k \):

\[
\begin{array}{cccccc}
0 & \rightarrow & \hat{K}^{G_{\beta}}(Z_{i,l}^{\leq k-1})_k & \rightarrow & \hat{K}^{G_{\beta}}(Z_{i,l}^k)_k & \rightarrow & 0 \\
\text{ch}^{G_{\beta}} & & \downarrow & & \text{ch}^{G_{\beta}} & & \downarrow \\
0 & \rightarrow & H^*_{\ast}(Z_{i,l}^{\leq k-1}, k)^\wedge & \rightarrow & H^*_{\ast}(Z_{i,l}^k, k)^\wedge & \rightarrow & 0.
\end{array}
\]

Note that the map \( \text{ch}^{G_{\beta}} : \hat{K}^{G_{\beta}}(Z_{i,l}^k)_k \rightarrow H^*_{\ast}(Z_{i,l}^k, k)^\wedge \) is an isomorphism for any \( k \) since again the variety \( Z_{i,l}^k \) is an affine bundle over \( B_l \). Hence, by induction on \( k \), we conclude that \( \text{ch}^{G_{\beta}} : \hat{K}^{G_{\beta}}(Z_{i,l}^{\leq k})_k \rightarrow H^*_{\ast}(Z_{i,l}^{\leq k}, k)^\wedge \) is an isomorphism for all \( k \).

Note that the isomorphism (3.4) induces an isomorphism between the completions:

\[ \text{Ext}^*_{G_{\beta}}(L_{\beta}, L_{\beta})^\wedge \cong H^*_{\ast}(Z_{\beta}, k)^\wedge. \]

As a summary of this subsection, we have the following.

**Corollary 3.9.** We have the following isomorphisms of \( k \)-algebras:

\[ \hat{H}_Q(\beta) \cong \text{Ext}^*_{G_{\beta}}(L_{\beta}, L_{\beta})^\wedge \cong H^*_{\ast}(Z_{\beta}, k)^\wedge \cong \hat{K}^{G_{\beta}}(Z_{\beta}, k). \]

### 3.4 Nakajima’s homomorphism and the category \( \mathcal{C}_{Q, \beta} \)

Henceforth, we specialize \( k \) to be the field \( \mathbb{Q}(q) \) of rational functions in an indeterminate \( q \). In this subsection, we consider the quantum loop algebra \( U_q \equiv U_q(Lg) \) defined over \( k \). The quantum loop algebra \( U_q(Lg) \) is isomorphic to the level zero quotient of the quantum affine algebra \( U_q(\hat{g}) \) without the degree operator. We do not recall the definitions here. See e.g. \([5, 10, 17]\) for the precise definitions of \( U_q(Lg) \) or \( U_q(\hat{g}) \).

Recall the quiver varieties with proper \( G(\lambda) \)-equivariant morphism \( \pi : \mathfrak{M}(\lambda) \rightarrow \mathfrak{M}_0(\lambda) \) for each \( \lambda \in P^+ \) (see Subsection 2.4). We consider the Steinberg type variety \( Z(\lambda) := \mathfrak{M}(\lambda) \times_{\mathfrak{M}_0(\lambda)} \mathfrak{M}(\lambda) \). Then its \( G(\lambda) \)-equivariant \( K \)-group \( K^{G(\lambda)}(Z(\lambda)) \) becomes an \( R(G(\lambda)) \)-algebra with respect to the convolution product relative to \( \mathfrak{M}(\lambda) \times \mathfrak{M}(\lambda) \) and \( \mathfrak{M}(\lambda) \). We identify the fiber \( \mathcal{L}(\lambda) = \pi^{-1}(0) \) with the fiber product \( \mathfrak{M}(\lambda) \times_{\mathfrak{M}_0(\lambda)} \{0\} \). Then the convolution product relative to \( \mathfrak{M}(\lambda) \times \mathfrak{M}(\lambda) \times \{0\} \) makes the \( K \)-group \( K^{G(\lambda)}(\mathcal{L}(\lambda)) \) into a left \( K^{G(\lambda)}(Z(\lambda)) \)-module.

Recall that \( G(\lambda) = G(\lambda) \times \mathbb{C}^\times \). We set \( A := R(\mathbb{C}^\times) \) and identify \( A = \mathbb{Z}[u, u^{-1}] \) in the standard way. Specializing \( v \) to \( q \), we regard \( k \) as an \( A \)-algebra.

**Theorem 3.10** (Nakajima \([17]\) Theorem 9.4.1). There exists a \( k \)-algebra homomorphism

\[ \Phi_A : U_q(Lg) \rightarrow K^{G(\lambda)}(Z(\lambda)) \otimes_A k \]
such that the pull-back
\[ \mathcal{W}(\lambda) := \Phi^*_A \left( K^{G(\lambda)}(\mathcal{O}(\lambda)) \otimes_A \mathbb{k} \right) \]
is a cyclic $U_q(L\mathfrak{g})$-module generated by an extremal weight vector $w_{\lambda} := [\mathcal{O}_{\Sigma(0,\lambda)}] \in K^{G(\lambda)}(\mathcal{O}(0,\lambda)) \otimes_A \mathbb{k}$ of weight $\lambda$. Moreover the module $\mathcal{W}(\lambda)$ is free of finite rank over $\text{End}_{U_q}(\mathcal{W}(\lambda)) \cong R(\mathcal{G}(\lambda)) \otimes_A \mathbb{k}$.

**Remark 3.11.** The module $\mathcal{W}(\lambda)$ is known to be isomorphic to the global Weyl module defined by Chari-Pressley [2] and also to the level 0 extremal weight module defined by Kashiwara [13]. In particular, if $\lambda = \varpi_i$ for some $i \in I$, the module $\mathcal{W}(\varpi_i)$ is isomorphic to the affinization of the fundamental module $W(\varpi_i)$ (see [14]).

Take an element $\lambda \in \mathcal{P}^+$ with $\text{cl}(\lambda) = \lambda$ and recall the 1-dimensional subtorus $T(\lambda) \subset G(\lambda) \subset G(\lambda)$. We identify $R(T(\lambda)) = A$ via the isomorphism $\prod_{i \in I} f_i \times \text{id} : \mathbb{C}^\times \cong T(\lambda)$. Let $m_\lambda$ be the kernel of the restriction $R(G(\lambda)) \otimes_A \mathbb{k} \to R(T(\lambda)) \otimes_A \mathbb{k} = \mathbb{k}$. The corresponding specialization $\mathcal{W}(\lambda)/m_\lambda \mathcal{W}(\lambda)$ (known as the local Weyl module defined in [2]) has a unique simple quotient $L(\lambda)$ in $U_q$-modfd.

**Definition 3.12** (Hernandez-Leclerc [8]). We define the category $C_Q$ (resp. $C_{Q,\beta}$ for each $\beta \in Q^+$) to be the minimal Serre full subcategory of the category $U_q$-modfd of finite-dimensional $U_q(L\mathfrak{g})$-modules containing the simple objects $\{L(\lambda) \mid \lambda \in \mathcal{P}_0^+\}$ (resp. $\{L(m) \mid m \in KP(\beta)\}$), where $\mathcal{P}_0^+ = \bigcup_{\beta \in Q^+} KP(\beta) \subset \mathcal{P}^+$ is as in Subsection 2.5.

**Remark 3.13.** Let $G$ be a linear algebraic group whose Lie algebra is $\mathfrak{g}$ and $N$ be the maximal unipotent subgroup of $G$ corresponding to the positive roots. Hernandez-Leclerc [8] proved that the category $\mathcal{C}_Q$ is a monoidal subcategory and there is an isomorphism from the complexified Grothendieck ring $K(C_Q)_C$ to the coordinate ring $\mathbb{C}[N]$, which sends the classes of simple objects to the elements of the dual canonical basis bijectively. Actually, Hernandez-Leclerc established an isomorphism between their quantizations. We have a block decomposition $C_Q = \bigoplus_{\beta \in Q^+} C_{Q,\beta}$ satisfying $C_{Q,\beta} \otimes C_{Q,\beta'} \subset C_{Q,\beta + \beta'}$ (see [5, Section 2.6]). This decomposition corresponds to the weight decomposition $\mathbb{C}[N] = \bigoplus_{\beta \in Q^+} \mathbb{C}[N]_\beta$. The isomorphism $\mathfrak{M}^*_\beta \cong E_\beta$ in Theorem 2.1 was originally established in order to give a geometric interpretation to the isomorphism $K(C_{Q,\beta})_C = \mathbb{C}[N]_\beta$.

Now we fix an element $\beta \in Q^+$. In Subsection 2.5, we defined the graded quiver variety $\mathfrak{M}^*_\beta$ with a canonical $G_\beta$-equivariant proper morphism $\pi_\beta : \mathfrak{M}^*_\beta \to E_\beta$, which is obtained from $\pi : \mathfrak{M}(\lambda) \to \mathfrak{M}_0(\lambda)$ with $\lambda = \text{cl}(\lambda_\beta)$ by taking the fixed locus with respect to the action of the 1-dimensional torus $T_\beta \subset G_\beta \subset G(\lambda)$. We form the Steinberg type variety $Z^*_\beta := \mathfrak{M}^*_\beta \times_{E_\beta} \mathfrak{M}^*_\beta = Z(\lambda)^{T_\beta}$. Let $\tau_\beta$ be the kernel of the restriction $R(G_\beta) \otimes_A \mathbb{k} \to R(T_\beta) \otimes_A \mathbb{k} = \mathbb{k}$. Note that the decomposition (2.2) $G_\beta \cong G_\beta \times T_\beta$ yields an isomorphism
\[ K^{G_\beta}(X) \otimes_A \mathbb{k} \cong K^{G_\beta}(X)_{\mathbb{k}} \]
for any $\mathbb{G}_\beta$-variety $X$ with a trivial $T_\beta$-action. In particular, we have an isomorphism $R(\mathbb{G}_\beta) \otimes_A k \cong R(G_\beta)_k$ of $k$-algebras, via which the maximal ideal $\mathfrak{r}_\beta \subset R(\mathbb{G}_\beta) \otimes_A k$ corresponds to the augmentation ideal $I \subset R(G_\beta)_k$. Therefore we have an isomorphism

$$[K^{G_\beta}(X) \otimes_A k]_{\mathfrak{r}_\beta} \cong \hat{K}^{G_\beta}(X)_k,$$

where $[-]_{\mathfrak{r}_\beta}$ denotes the $\mathfrak{r}_\beta$-adic completion. We define the $k$-algebra homomorphism $\Phi_\beta : U_q(Lg) \to \hat{K}^{G_\beta}(Z^\bullet_\beta)_k$ as the following composition:

$$U_q(Lg) \xrightarrow{\Phi_\beta} K^{G(\lambda)}(Z(\lambda)) \otimes_A k \xrightarrow{\text{(restriction to $G_\beta \subset G(\lambda)$)}} K^{G_\beta}(Z(\lambda)) \otimes_A k \xrightarrow{(\mathfrak{r}_\beta \text{-adic completion)}} K^{G_\beta}(Z(\lambda)) \otimes_A k \xrightarrow{(\text{localization theorem})} K^{G_\beta}(Z^\bullet_\beta) \otimes_A k \xrightarrow{(\text{isomorphism } (3.9))} \hat{K}^{G_\beta}(Z^\bullet_\beta)_{\mathfrak{r}_\beta} \cong \hat{K}^{G_\beta}(Z^\bullet_\beta)_k.$$

**Theorem 3.14** ([5] Theorem 4.9). The pull-back along the homomorphism $\Phi_\beta : U_q(Lg) \to \hat{K}^{G_\beta}(Z^\bullet_\beta)_k$ induces an equivalence

$$\Phi_\beta^* : \hat{K}^{G_\beta}(Z^\bullet_\beta)_k \rightleftarrows \mathcal{C}_{Q_\beta} \text{-mod}_{\text{id}}$$

between the category $\hat{K}^{G_\beta}(Z^\bullet_\beta)_k$-modules and the category $\mathcal{C}_{Q_\beta} \subset U_q$-modules.

The next proposition is a counterpart of Proposition 3.8.

**Proposition 3.15.** The Riemann-Roch homomorphism gives an isomorphism of $\hat{R}(G_\beta)_k$-algebras:

$$RR^{G_\beta} : \hat{K}^{G_\beta}(Z^\bullet_\beta)_k \cong H^{G_\beta}_*(Z^\bullet_\beta)_k.$$

**Proof.** As in the proof of Proposition 3.8, it suffices to prove that the equivariant Chern character map $(\text{ch}^{G_\beta})_{Z^\bullet_\beta}^{	ext{gr}^\bullet} : \hat{K}^{G_\beta}(Z^\bullet_\beta)_k \to H^{G_\beta}_*(Z^\bullet_\beta)_k$ is an isomorphism.

Note that the $G_\beta$-orbit stratification (2.1) yields a stratification of $Z^\bullet_\beta$:

$$Z^\bullet_\beta = \bigsqcup_{m \in \text{KP}(\beta)} Z^\bullet_\beta|_{O_m}, \quad Z^\bullet_\beta|_{O_m} \cong G_\beta \times_{\text{Stab}_{G_\beta}(x_m)} \left( \pi_{-1}(x_m) \times \pi_{-1}^{-1}(x_m) \right).$$

Fix a total ordering $\text{KP}(\beta) = \{m_1, m_2, \ldots, m_s\}$ such that we have $O_k \subset O_l$ only if $k < l$. Set $Z^k_\beta := Z^\bullet_\beta|_{O_{m_k}}$ and $Z^{k-1}_\beta := \bigsqcup_{l \leq k} Z^l_\beta$. Then the variety $Z^{k-1}_\beta$ is a closed subvariety of $Z^k_\beta$ whose complement is $Z^k_\beta$. By Proposition 2.2 and the reduction, we have

$$K^{G_\beta}(Z^k_\beta) \cong K^{G_\beta(m_k)}(\mathfrak{L}^\bullet(m_k) \times \mathfrak{L}^\bullet(m_k)),$$

$$H^{G_\beta}_*(Z^k_\beta, k) \cong H^{G_\beta}_*(\mathfrak{L}^\bullet(m_k) \times \mathfrak{L}^\bullet(m_k), k).$$
for each \( k \). Then, using [17, Theorem 7.4.1], we can prove that the equivariant Chern character map gives an isomorphism \( \text{ch}^G_{\beta} : \hat{K}^G_{\beta}(Z^k_{\beta})_k \xrightarrow{\sim} H^*_G(Z^k_{\beta},k)^{\wedge} \) for each \( k \). Moreover, we obtain the following commutative diagram with exact rows for each \( k \):

\[
\begin{array}{cccccc}
0 & \rightarrow & \hat{K}^G_{\beta}(Z^\leq_{\beta}^{k-1})_k & \rightarrow & \hat{K}^G_{\beta}(Z^k_{\beta})_k & \rightarrow & 0 \\
\downarrow \text{ch}^G_{\beta} & & \downarrow \text{ch}^G_{\beta} & & \downarrow & & \downarrow \text{ch}^G_{\beta} \\
0 & \rightarrow & H^*_G(Z^\leq_{\beta}^{k-1},k)^{\wedge} & \rightarrow & H^*_G(Z^k_{\beta},k)^{\wedge} & \rightarrow & 0.
\end{array}
\]

By induction on \( k \), the equivariant Chern character map gives an isomorphism
\[
\text{ch}^G_{\beta} : \hat{K}^G_{\beta}(Z^\leq_{\beta}^{k})_k \xrightarrow{\sim} H^*_G(Z^\leq_{\beta}^{k},k)^{\wedge}
\]
for all \( k \).

We consider the proper push-forward
\[
\mathcal{L}^\bullet_{\beta} := (\pi_{\beta})_! \mathbb{k}
\]
of the trivial local system \( \mathbb{k} \) on \( \mathcal{M}^\bullet_{\beta} \). By the decomposition theorem, we have
\[
\mathcal{L}^\bullet_{\beta} \cong \bigoplus_{m \in \text{KP}(\beta)} L^m_{m,k} \otimes_k \mathcal{T} \mathcal{C}_m = \bigoplus_{m \in \text{KP}(\beta)} \bigoplus_{k \in \mathbb{Z}} L^m_{m,k} \otimes_k \mathcal{T} \mathcal{C}_m[k],
\]
where \( L^m_{m,k} = \bigoplus_k L^m_{m,k} \) is a finite-dimensional graded \( \mathbb{k} \)-vector space, which is known to be non-zero for each \( m \) (see [17, Theorem 14.3.2]). Similarly to the previous subsection, we have a standard isomorphism of \( \mathbb{k} \)-algebras
\[
\text{Ext}^*_G(\mathcal{L}^\bullet_{\beta}, \mathcal{L}^\bullet_{\beta}) \cong H^*_G(Z^\bullet_{\beta}, \mathbb{k}), \tag{3.10}
\]
which also induces an isomorphism between completions.

**Corollary 3.16.** We have the following isomorphisms of \( \mathbb{k} \)-algebras:
\[
\text{Ext}^*_G(\mathcal{L}^\bullet_{\beta}, \mathcal{L}^\bullet_{\beta})^{\wedge} \cong H^*_G(Z^\bullet_{\beta}, \mathbb{k})^{\wedge} \cong \hat{K}^G_{\beta}(Z^\bullet_{\beta})_k.
\]

### 4 Dynkin quiver type quantum affine Schur-Weyl duality

#### 4.1 Geometric construction of a bimodule and a Morita equivalence

We keep the notation in the previous sections. In particular, \( k = \mathbb{Q}(q) \). We fix an element \( \beta = \sum_{i \in I} d_i \alpha_i \in \mathbb{Q}^+ \) and put \( \lambda := \text{cl}(\lambda_{\beta}) \in P^+ \). From the two \( G_{\beta} \)-equivariant proper morphisms \( \pi_{\beta} : \mathcal{M}^\bullet_{\beta} \rightarrow E_{\beta} \) and \( \mu_{\beta} : \mathcal{F}_\beta \rightarrow E_{\beta} \), we form the fiber product \( \mathcal{M}^\bullet_{\beta} \times_{E_{\beta}} \mathcal{F}_\beta \). The convolution products make its completed
$G_\beta$-equivariant $K$-group $\hat{R}^{G_\beta}(\mathfrak{M}_\beta^i \times_{E_\beta} \mathcal{F}_\beta)_k$ into a $(\hat{R}^{G_\beta}(Z_\beta^i)_k, \hat{R}^{G_\beta}(Z_\beta)_k)$-bimodule. More precisely, the convolution products give $k$-algebra homomorphisms

$$\hat{R}^{G_\beta}(Z_\beta^i)_k \to \text{End} \left( \hat{R}^{G_\beta}(\mathfrak{M}_\beta^i \times_{E_\beta} \mathcal{F}_\beta)_k \right) \leftarrow \hat{R}^{G_\beta}(Z_\beta)_k^\text{op},$$

whose images commute with each other. In the rest of this subsection, we prove that this bimodule induces a Morita equivalence.

For a moment, we focus on a component $\mathfrak{M}_\beta^i \times_{E_\beta} \mathcal{F}_i$ for a fixed $i \in I^\beta$. Using the isomorphism $B_1 \cong G_\beta/B_1$ with $B_1 = \text{Stab}_{G_\beta}(F_i^\bullet)$, we have

$$\mathfrak{M}_\beta^i \times_{E_\beta} \mathcal{F}_i \cong \mathfrak{M}_\beta^i \times_{E_\beta} \left( G_\beta \times B_1 \> \text{pr}_1^{-1}(F_i^\bullet) \right) \cong G_\beta \times B_1 \left( \mathfrak{M}_\beta^i \times_{E_\beta} \> \text{pr}_1^{-1}(F_i^\bullet) \right), \quad (4.1)$$

where $\text{pr}_1$ is the projection $\mathcal{F}_i \ni (F^\bullet, x) \mapsto F^\bullet \in B_1$. We define a 1-parameter subgroup $\rho_t : C^\times \to H_\beta$ by $\rho_t(v_{\mu}(k)) := t^k v_{\mu}(k)$ for $t \in C^\times$. Note that this depends on the choice of $w_1 \in \mathfrak{S}_d$ fixed in Subsection 3.3. We observe that

$$\text{pr}_1^{-1}(F_i^\bullet) \cong \left\{ x \in E_\beta \mid x(F_i^k) \subset F_i^k, \forall k \right\} = \left\{ x \in E_\beta \mid \lim_{t \to 0} \rho_1(t) x = 0 \right\}.$$

Therefore we get

$$\mathfrak{M}_\beta^i \times_{E_\beta} \text{pr}_1^{-1}(F_i^\bullet) \cong \left\{ x \in \mathfrak{M}_\beta^i \mid \lim_{t \to 0} \rho_1(t) x = 0 \right\}.$$

Since the morphism $\pi_\beta : \mathfrak{M}_\beta^i \to E_\beta$ is the $T_\beta$-fixed part of $\pi : \mathfrak{M}(\lambda) \to \mathfrak{M}_0(\lambda)$, it is natural to consider the following subvariety of $\mathfrak{M}(\lambda)$:

$$\tilde{\mathfrak{M}}(\lambda; w_1) := \left\{ x \in \mathfrak{M}(\lambda) \mid \lim_{t \to 0} \rho_1(t) x = 0 \in \mathfrak{M}_0(\lambda) \right\},$$

which turns out to be the tensor product variety introduced by Nakajima [18]. Since the subgroups $T_\beta$ and $\rho_1(C^\times)$ commute with each other, we have

$$\mathfrak{M}_\beta^i \times_{E_\beta} \text{pr}_1^{-1}(F_i^\bullet) \cong \tilde{\mathfrak{M}}(\lambda; w_1)^{T_\beta}. \quad (4.2)$$

Using (4.1), (4.2) and the reduction, we obtain

$$K^{G_\beta}(\mathfrak{M}_\beta^i \times_{E_\beta} \mathcal{F}_i) \cong K^{H_\beta}(\tilde{\mathfrak{M}}(\lambda; w_1)^{T_\beta}), \quad (4.3)$$

$$H_*^{G_\beta}(\mathfrak{M}_\beta^i \times_{E_\beta} \mathcal{F}_i, k) \cong H_*^{H_\beta}(\tilde{\mathfrak{M}}(\lambda; w_1)^{T_\beta}, k). \quad (4.4)$$

**Proposition 4.1.** The $G_\beta$-equivariant Chern character map gives an isomorphism:

$$\text{ch}^{G_\beta} : \hat{R}^{G_\beta}(\mathfrak{M}_\beta^i \times_{E_\beta} \mathcal{F}_i)_k \cong \text{End} \left( \hat{R}^{G_\beta}(\mathfrak{M}_\beta^i \times_{E_\beta} \mathcal{F}_i)_k \right) \leftarrow \hat{R}^{G_\beta}(\mathfrak{M}_\beta^i \times_{E_\beta} \mathcal{F}_i, k)^\text{op}.$$
The \( G_\beta \)-equivariant Borel-Moore homology \( H^*_{\alpha}({\mathcal{M}}^*_{\alpha} \times E_\beta, F_\beta, k) \) becomes a \((H^*_{\alpha}(Z^*_{\alpha}, k), H^*_{\alpha}(Z^*_{\beta}, k))\)-bimodule by the convolution products, similarly to the case of \( K \)-groups. On the other hand, the Ext-group \( \text{Ext}_{G_\beta}^*(L^*_\beta, L^*_\beta) \) becomes a \((\text{Ext}_{G_\beta}^*(L^*_\beta, L^*_\beta), \text{Ext}_{G_\beta}^*(L^*_\beta, L^*_\beta))\)-bimodule by the Yoneda products. This bimodule \( \text{Ext}_{G_\beta}^*(L^*_\beta, L^*_\beta) \) gives a Morita equivalence between \( \text{Ext}_{G_\beta}^*(L^*_\beta, L^*_\beta) \) and \( \text{Ext}_{G_\beta}^*(L^*_\beta, L^*_\beta) \) because \( IC_m \) appears as a non-zero direct summand of both \( L^*_\beta \) and \( L^*_\beta \) for each \( m \in K^*(\beta) \). Moreover, we have a standard isomorphism

\[
H^*_{\alpha}(M^*_\beta \times E_\beta, F_\beta, k) \cong \text{Ext}_{G_\beta}^*(L^*_\beta, L^*_\beta) \tag{4.5}
\]

**Theorem 4.2.** We have the following commutative diagram:

\[
\begin{array}{cccc}
\hat{K}^G_{\beta}(Z^*_\beta)_k & \longrightarrow & \text{End} \left( \hat{K}^G_{\beta}(M^*_\beta \times E_\beta, F_\beta) \right) & \longleftarrow & \hat{K}^G_{\beta}(Z^*_\beta)_{\text{op}}^k \\
\text{RR}^G_{\beta} & \cong & \text{RR}^G_{\beta} & \cong & \text{RR}^G_{\beta} \\
H^*_{\alpha}(Z^*_\beta, k)^\wedge & \longrightarrow & \text{End} \left( H^*_{\alpha}(M^*_\beta \times E_\beta, F_\beta, k) \right)^\wedge & \longleftarrow & H^*_{\alpha}(Z^*_\beta, k)^{\text{op}} \tag{4.5} \\
(\text{4.10}) & \cong & (\text{4.5}) & \cong & (\text{3.4}) \\
\text{Ext}_{G_\beta}^*(L^*_\beta, L^*_\beta)^\wedge & \longrightarrow & \text{End} \left( \text{Ext}_{G_\beta}^*(L^*_\beta, L^*_\beta) \right)^\wedge & \longleftarrow & \text{Ext}_{G_\beta}^*(L^*_\beta, L^*_\beta) \tag{4.5} \text{op},
\end{array}
\]

where each row denotes the bimodule structure defined above. In particular, the bimodule \( \hat{K}^G_{\beta}(M^*_\beta \times E_\beta, F_\beta)_k \) gives a Morita equivalence between two convolution algebras \( \hat{K}^G_{\beta}(Z^*_\beta)_k \) and \( \hat{K}^G_{\beta}(Z^*_\beta)_k \).

**Proof.** The commutativity of the upper half (resp. lower half) of the diagram follows from Proposition 3.3 (resp. an equivariant version of [3, Theorem 8.6.7]).

\[\square\]

### 4.2 The left action of \( U_q(Lg) \)

In this subsection, we fix \( i = (i_1, \ldots, i_d) \in I^\beta \) and investigate the \( U_q(Lg) \)-module structure of the pull-back \( \tilde{\Phi}_{\beta}^*(\hat{K}^G_{\beta}(M^*_\beta \times E_\beta, F_\beta)) \).

We use the following notation. For each \( i \in I \), we define \( \lambda_i := c_i(\phi(\alpha_i)) = \varpi_j \) and \( a_i := q^p \) if \( \phi(\alpha_i) = (j, p) \in \tilde{I} \). Recall from Theorem 3.10 that we have

\[
\text{End}_{_{\tilde{\Phi}_{\beta}^*(\hat{K}^G_{\beta}(M^*_\beta \times E_\beta, F_\beta))}}(\omega(\lambda_i)) \cong R(\mathcal{G}(\lambda_i)) \otimes_A k = R(G(\lambda_i))_k \cong k[z_i^{\pm 1}], \tag{4.6}
\]

where \( z_i \) denotes the class of the 1-dimensional representation of \( G(\lambda_i) = \mathbb{C}^\times \) of weight 1.

We recall some properties of the tensor product variety \( \mathcal{F}(\lambda; w_1) \). Let

\[ \mathbb{H}_\beta := H_\beta \times \mathbb{C}^\times \subset G_\beta \times \mathbb{C}^\times = G_\beta \subset \mathcal{G}(\lambda) \]

be a maximal torus. By construction, the subvariety \( \mathcal{F}(\lambda; w_1) \subset \mathcal{M}(\lambda) \) is stable under the action of \( \mathbb{H}_\beta \). The convolution product makes the \( \mathbb{H}_\beta \)-equivariant
$K$-group $K^{H^\beta}(\tilde{\mathcal{G}}(\lambda; w_1))$ into a left $K^{H^\beta}(Z(\lambda))$-module. Via the composition of the homomorphisms

$$U_q(L\mathfrak{g}) \xrightarrow{\Phi} K^{G(\lambda)}(Z(\lambda)) \otimes_\mathbb{A} \mathbb{A} \mathbb{k} \rightarrow K^{H^\beta}(Z(\lambda)) \otimes_\mathbb{A} \mathbb{k},$$

where the latter one is the restriction to $H^\beta \subset G(\lambda)$, we regard the $H^\beta$-equivariant $K$-group $K^{H^\beta}(\tilde{\mathcal{G}}(\lambda; w_1)) \otimes_\mathbb{A} \mathbb{k}$ as a $U_q(L\mathfrak{g})$-module.

**Theorem 4.3** (Nakajima [18]). There is a $U_q(L\mathfrak{g})$-module isomorphism

$$K^{H^\beta}(\tilde{\mathcal{G}}(\lambda; w_1)) \otimes_\mathbb{A} \mathbb{k} \cong V^\otimes_1 := \mathbb{W}(\lambda_i) \otimes \cdots \otimes \mathbb{W}(\lambda_{i_d}),$$

where the action of $R(H^\beta) \otimes_\mathbb{A} \mathbb{k}$ on the LHS is translated into the action on the RHS via the isomorphism

$$R(H^\beta) \otimes_\mathbb{A} \mathbb{k} \cong \mathcal{O}_1 := k[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \subset \text{End}_{U_q}(V^\otimes_1); \quad (4.7)$$

where we set $X_k := z_{\chi_k}$ using the notation in (4.6).

The decomposition (2.2) $G^\beta \cong G^\beta \times T^\beta$ induces the decomposition $H^\beta \cong H^\beta \times T^\beta$ of the maximal torus $H^\beta$. Similarly to the case of $G^\beta$-equivariant $K$-groups in Subsection 3.4, this decomposition yields a natural isomorphism

$$K^{H^\beta}(X) \otimes_\mathbb{A} \mathbb{k} \cong K^{H^\beta}(X)_k$$

for any $H^\beta$-variety $X$ with a trivial $T^\beta$-action. When $X = \text{pt}$, we have the following commutative diagram:

$$
\begin{array}{ccc}
R(H^\beta) \otimes_\mathbb{A} \mathbb{k} & \xrightarrow{\cong} & R(H^\beta) \mathbb{k} \\
(4.7) \downarrow \cong & & \downarrow \cong \\
\mathcal{O}_1 = k[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] & \xrightarrow{\cong} & k[y_1^{\pm 1}, \ldots, y_d^{\pm 1}]_{1},
\end{array}
$$

where the bottom horizontal arrow sends the element $a^{-1}_{i_k}X_k$ to $y_k1_i$ for $1 \leq k \leq d$. Under this isomorphism, the maximal ideal $\mathfrak{r}_\beta \subset R(H^\beta) \otimes_\mathbb{A} \mathbb{k}$ defined as the kernel of the restriction $R(H^\beta) \otimes_\mathbb{A} \mathbb{k} \rightarrow R(T^\beta) \otimes_\mathbb{A} \mathbb{k} = \mathbb{k}$ corresponds to the augmentation ideal of $R(H^\beta)$. Therefore we have a natural isomorphism

$$[K^{H^\beta}(X) \otimes_\mathbb{A} \mathbb{k}]^\wedge \mathfrak{r}_\beta \cong \hat{K}^{H^\beta}(X)_k, \quad (4.9)$$

where $[-]^\wedge$ denotes the $\mathfrak{r}_\beta$-adic completion. In particular, completing the diagram (4.8), we get

$$
\begin{array}{ccc}
[R(H^\beta) \otimes_\mathbb{A} \mathbb{k}]^\wedge & \xrightarrow{\cong} & \hat{R}(H^\beta) \mathbb{k} \\
\cong & & \cong \\
\hat{\mathcal{O}}_1 := k[X_1 - a_{i_1}, \ldots, X_d - a_{i_d}] & \xrightarrow{\cong} & k[y_1 - 1, \ldots, y_d - 1]_{1}.
\end{array}
$$
Theorem 4.4. We have the following isomorphism of $U_q(Lg)$-modules:

$$\hat{\Phi}_\beta^* \left( \widehat{K}^{G_\beta}(M_\beta^r \times_{E_\beta} F_1)_k \right) \cong \hat{\mathcal{V}}^\otimes i := \mathcal{V}^\otimes i \otimes \mathcal{O}_1 \hat{O}_1.$$ 

Proof. Actually, there is the following isomorphism:

$$\hat{\mathcal{V}}^\otimes i \cong \left[ K^{H_\beta}(\widetilde{Z}(\lambda; w_1)) \otimes_k A_k \right]_{\tau_\beta} \hat{\mathcal{K}}^{G_\beta}(Z_\beta^*),$$

(4.3)

$$(\text{isomorphism } (4.9))$$

We need to show that this is a $U_q(Lg)$-homomorphism. By construction, the following diagram of $k$-algebras commutes:

$$\begin{array}{ccc}
K^{G(\lambda)}(Z(\lambda)) \otimes_k A_k & \xrightarrow{\cong} & K^{G_\beta}(Z_\beta^*) \otimes_k A_k \\
\downarrow & & \downarrow \\
K^{H_\beta}(Z(\lambda)) \otimes_k A_k & \xrightarrow{\cong} & K^{H_\beta}(Z_\beta^*),
\end{array}$$

(4.3)

where the vertical arrows denote the restrictions to the maximal tori. Moreover, by using an $H_\beta$-equivariant version of [17, Proposition 8.2.3], we can see that the following diagram also commutes:

$$\begin{array}{ccc}
K^{G_\beta}(Z_\beta^*) \otimes_k K^{G_\beta}(M_\beta^r \times_{E_\beta} F_1)_k & \xrightarrow{\ast} & K^{G_\beta}(M_\beta^r \times_{E_\beta} F_1)_k \\
\downarrow & & \downarrow \\
K^{H_\beta}(Z_\beta^*) \otimes_k K^{H_\beta}(\widetilde{Z}(\lambda; w_1)_{T_\beta})_k & \xrightarrow{\ast} & K^{H_\beta}(\widetilde{Z}(\lambda; w_1)_{T_\beta})_k,
\end{array}$$

(4.3)

where the horizontal arrows denote the convolution products. From these commutative diagrams, combined with the definition of $\hat{\Phi}_\beta$ and Theorem 4.3, we obtain the conclusion. \qed

4.3 The right action of $\widehat{H}_Q(\beta)$

Summarizing the discussion so far, we have obtained a $(U_q(Lg), \widehat{H}_Q(\beta))$-bimodule structure on the left $U_q(Lg)$-module

$$\hat{\mathcal{V}}^\otimes i := \bigoplus_{i \in I^\beta} \hat{\mathcal{V}}^\otimes i.$$
such that the following diagram commutes:

\[
\begin{array}{ccc}
U_q(L\mathfrak{g}) & \xrightarrow{\Phi_\beta} & \text{End}(\tilde{V}^{\otimes \beta}) \\
\downarrow & & \downarrow \cong \\
\tilde{R}^{G_\beta}(Z_{\beta})_k & \xrightarrow{\cong} & \text{End}\left(\tilde{R}^{G_\beta}(\mathfrak{m}^\bullet_\beta \times_{E_\beta} F_\beta)_k\right) \cong \tilde{R}^{\beta}(Z_{\beta})_k^{\text{op}}.
\end{array}
\]

In this subsection, we describe the right action \( \psi : \hat{H}_Q(\beta) \to \text{End}_{U_q}(\tilde{V}^{\otimes \beta})^{\text{op}} \) of the quiver Hecke algebra \( \hat{H}_Q(\beta) \) on the space \( \tilde{V}^{\otimes \beta} \).

For each \( i = (i_1, \ldots, i_d) \in I^\beta \), we set

\[ v_i := (w_{\lambda_i} \otimes \cdots \otimes w_{\lambda_{i_d}}) \otimes 1 \in \tilde{V}^{\otimes i} = (\mathbb{W}(\lambda_{i_1}) \otimes \cdots \otimes \mathbb{W}(\lambda_{i_d})) \otimes O_1 \hat{O}_i. \]

**Proposition 4.5.** The highest weight space \( \bigoplus_{i \in I^\beta} \hat{O}_i v_i \subset \tilde{V}^{\otimes \beta} \) of weight \( \lambda \) is stable under the right action of \( \hat{H}_Q(\beta) \). Moreover it is isomorphic to the completed polynomial representation \( \hat{P}_\beta \) defined in (3.1).

**Proof.** Note that the connected component of the graded quiver variety \( \mathfrak{m}^\bullet_\beta = \mathfrak{m}(\lambda)^{T_\beta} \) corresponding to the highest weight space is \( \mathfrak{m}(0, \lambda)^{T_\beta} = \text{pt} \) and hence \( \mathfrak{m}(0, \lambda)^{T_\beta} \times_{E_\beta} F_\beta = \mathcal{B}_\beta \). Therefore we have

\[ \bigoplus_{i \in I^\beta} \hat{O}_i v_i \cong \tilde{R}^{G_\beta}(\mathfrak{m}(0, \lambda)^{T_\beta} \times_{E_\beta} F_\beta)_k \cong \tilde{R}^{G_\beta}(\mathcal{B}_\beta)_k \cong \hat{P}_\beta \]

as \( \hat{H}_Q(\beta) \)-module, where the last isomorphism comes from (3.3) and (3.7). \( \square \)

Henceforth, we normalize the isomorphism \( \tilde{R}^{G_\beta}(\mathfrak{m}^\bullet_\beta \times_{E_\beta} F_\beta)_k \cong \tilde{V}^{\otimes i} \) of \( U_q(L\mathfrak{g}) \)-modules in Theorem 4.4 by multiplying the element of \( \hat{O}_i \) corresponding to the ratio \( C^{-1}_i \) of Todd classes defined in (3.6) for each \( i \in I^\beta \) so that the isomorphism

\[ \bigoplus_{i \in I^\beta} \hat{O}_i v_i = \bigoplus_{i \in I^\beta} \mathbb{k}[X_1 - a_{i_1}, \ldots, X_d - a_{i_d}] v_i \cong \hat{P}_\beta = \bigoplus_{i \in I^\beta} \mathbb{k}[x_1, \ldots, x_d] 1_i \]

in Proposition 4.5 above sends the element \( v_i \) to \( 1_i \).

Now we recall the normalized R-matrices. For any pair \( (i_1, i_2) \in I^2 \), we simplify \( z_k := z_{\lambda_i} \) for \( k = 1, 2 \). Then it is known (see e.g. [14, Section 8]) that there is a unique \( (U_q \otimes \mathbb{k}[z_1^{\pm 1}, z_2^{\pm 1}]) \)-homomorphism, called the normalized R-matrix

\[ R_{i_1,i_2}^{\text{norm}} : \mathbb{W}(\lambda_{i_1}) \otimes \mathbb{W}(\lambda_{i_2}) \to \mathbb{k}(z_2/z_1) \otimes_{\mathbb{k}[z_2/z_1]} (\mathbb{W}(\lambda_{i_2}) \otimes \mathbb{W}(\lambda_{i_1})) ; \]

such that \( R_{i_1,i_2}^{\text{norm}}(w_{\lambda_{i_1}} \otimes w_{\lambda_{i_2}}) = w_{\lambda_{i_2}} \otimes w_{\lambda_{i_1}} \). The denominator of the normalized R-matrix \( R_{i_1,i_2}^{\text{norm}} \) is defined as the monic polynomial \( d_{i_1,i_2}(u) \in \mathbb{k}[u] \) of the smallest degree among polynomials satisfying

\[ \text{Im } R_{i_1,i_2}^{\text{norm}} \subset d_{i_1,i_2}(z_2/z_1)^{-1} \otimes (\mathbb{W}(\lambda_{i_2}) \otimes \mathbb{W}(\lambda_{i_1})) \].

24
By [14, Proposition 9.3], we have
\begin{equation}
   d_{i_1, i_2}(1) \neq 0. \tag{4.10}
\end{equation}

Let $K_i$ be the fraction field of the ring $\hat{O}_i$ for each $i \in I^\beta$. It is known that the $U_q \otimes K_i$-module
\begin{equation*}
   \hat{V}_K^{\otimes i} := V^{\otimes i} \otimes_{\hat{O}_i} K_i = \hat{V}^{\otimes i} \otimes_{\hat{O}_i} K_i
\end{equation*}
is irreducible (see e.g. [14, Proposition 9.5]). For each $w \in \mathcal{S}_d$, the $k$-algebra isomorphism
\begin{equation*}
   \varphi_w : \hat{O}_1 \cong \hat{O}_1(w); \quad f(X_1, \ldots, X_d) \mapsto f^w(X_1, \ldots, X_d) := f(X_{w(1)}, \ldots, X_{w(d)})
\end{equation*}
duces an isomorphism $K_i \cong K_{i,w}$ of the fraction fields, which we denote by the same symbol $\varphi_w$. The pull-back $\varphi_w^* \hat{V}_K^{\otimes i,w}$ is an irreducible $U_q \otimes K_i$-module.

For each $i \in I^\beta$ and $1 \leq k < d$, we define the following non-zero $U_q \otimes K_i$-homomorphism
\begin{equation*}
   R_k^i := (1^\otimes (k-1) \otimes R_{i_k,i_{k+1}+1}^{\text{norm}} \otimes 1^\otimes (d-k-1)) \otimes \varphi_{s_k} : \hat{V}_K^{\otimes i} \rightarrow \varphi_{s_k}^* \hat{V}_K^{\otimes i \cdot s_k}.
\end{equation*}
By the irreducibility, this is an isomorphism and we have
\begin{equation}
   \text{Hom}_{U_q \otimes K_i} \left( \hat{V}_K^{\otimes i}, \varphi_{s_k}^* \hat{V}_K^{\otimes i \cdot s_k} \right) = K_i \cdot R_k^i. \tag{4.11}
\end{equation}

Let $\hat{V}_K^{\otimes \beta} := \bigoplus_{i \in I^\beta} \hat{V}_K^{\otimes i}$. We regard $\hat{V}^{\otimes \beta} \subset \hat{V}_K^{\otimes \beta}$ naturally.

**Theorem 4.6.** The right action of the quiver Hecke algebra $\hat{H}_Q(\beta)$ on the space $\hat{V}^{\otimes \beta}$ is given by the following formulas:
\begin{align*}
   v \cdot e(i) &= \delta_{i,i} v \quad \tag{4.12} \\
   v \cdot x_k &= \log(a_{i_k}^{-1} X_k)v \quad \tag{4.13} \\
   v \cdot \tau_k &= \begin{cases} 
   \left( \log(a_{i_k}^{-1} X_k) - \log(a_{i_k+1}^{-1} X_{k+1}) \right)^{-1}(R_k^1(v) - v) & \text{if } i_k = i_{k+1}, \\
   \left( \log(a_{i_k}^{-1} X_{k+1}) - \log(a_{i_k+1}^{-1} X_k) \right) R_k^1(v) & \text{if } i_k \leftarrow i_{k+1}, \\
   R_k^1(v) & \text{otherwise,}
   \end{cases} \quad \tag{4.14}
\end{align*}
\begin{equation*}
   \text{where } v \in \hat{V}^{\otimes i} \text{ with } i = (i_1, \ldots, i_d) \in I^\beta \text{ and } \log(X) := \sum_{m=1}^{\infty} (-1)^{m+1}(X - 1)^m/m.
\end{equation*}

**Proof.** The first formula (4.12) is clear from Theorem 3.6 (1) and the construction.

To prove the second formula (4.13), we assume that the vector $v \in \hat{V}^{\otimes i}$ corresponds to an element $\zeta \in \hat{K}^{G_\beta}(M_{\beta}^* \times E_\beta, F_1)_k$ under the isomorphism in Theorem 4.4. By Theorem 3.6 (1), the right action of $e^x s \in \hat{H}_Q(\beta)$ on $\hat{V}^{\otimes i}$
corresponds to the convolution with the class $\Delta_*[\mathcal{L}_i(k)] \in \tilde{R}^{G_\beta}(\mathcal{F}_i \times E_\beta \mathcal{F}_i)_k$ from the right, where $\mathcal{L}_i(k)$ is the line bundle on $\mathcal{F}_i$ defined in Subsection 3.3 and $\Delta : \mathcal{F}_i \to \mathcal{F}_i \times E_\beta \mathcal{F}_i$ is the diagonal embedding. By [17, Lemma 8.1.1], we have $\zeta \ast (\Delta_*[\mathcal{L}_i(k)]) = \zeta \otimes p_2^*[(\mathcal{L}_i(k))]$, where $p_2 : \mathfrak{M}_i^* \times E_\beta \mathcal{F}_i \to \mathcal{F}_i$ is the second projection. The isomorphism (4.3) translates the operation $- \otimes p_2^*[(\mathcal{L}_i(k))]$ on $\tilde{R}^{G_\beta}(\mathfrak{M}_i^* \times E_\beta \mathcal{F}_i)$ into the multiplication of the element $y_k 1_i \in R(H_\beta)$ on $K^{H_\beta}(\mathcal{S}(\lambda; u_1)^{T_\beta})$. Thus we have $v \ast e_{x_k} = (a_{i_k}^{-1}x_k)v$ (see (4.8)).

Let us verify the third formula (4.14). Let $\psi : \tilde{H}_Q(\beta) \to \text{End}_{U_q}(\tilde{V}^{\otimes \beta})^{op}$ be the structure morphism. First, we consider the case $i_k = i_{k+1}$. From the commutation relation between $e(1)\tau_k$ and $x_1$ in $H_Q(\beta)$, and the formula (4.13) for $\psi(x_1)$ which we have proved in the previous paragraph, we see that

$$(D\psi(e(1)\tau_k) + 1)f = f^{x_k}(D\psi(e(1)\tau_k) + 1)$$

holds in $\text{End}_{U_q}(\tilde{V}^{\otimes i})$ for any $f \in \mathcal{O}_i$, where we put $D := \log(a_{i_k}^{-1}X_k) - \log(a_{i_{k+1}}^{-1}X_{k+1})$. In other words, the operator $D\psi(e(1)\tau_k) + 1$ belongs to $\text{Hom}_{U_q \otimes \mathcal{O}_i}(\tilde{V}^{\otimes i}, \tilde{V}^{\otimes i})$. Therefore it extends to an operator on the localizations. Namely, we can regard $D\psi(e(1)\tau_k) + 1 \in \text{Hom}_{U_q \otimes \mathcal{K}_i}(\tilde{V}^{\otimes i}, \tilde{V}^{\otimes i})$ as an operator on $\tilde{V}^{\otimes i}$.

The case $i_k \neq i_{k+1}$ is easier. In this case, the commutation relation in $H_Q(\beta)$ and the formula (4.13) for $\psi(x_1)$ show that the operator $\psi(e(1)\tau_k)$ already belongs to $\text{End}_{U_q \otimes \mathcal{O}_i}(\tilde{V}^{\otimes i}, \tilde{V}^{\otimes i})$. Therefore it extends to an element in $\text{Hom}_{U_q \otimes \mathcal{K}_i}(\tilde{V}^{\otimes i}, \tilde{V}^{\otimes i})$. Then we proceed just as in the previous paragraph to obtain the desired formula (4.14), taking Proposition 4.5, the formulas in Theorem 3.5 and (4.11) into consideration.

\[\Box\]

**Corollary 4.7** (= [10] Conjecture 4.3.2). For any $i_1, i_2 \in I$, the order of zero of the denominator $d_{i_1, i_2}(u)$ at the point $u = a_{i_2}/a_{i_1}$ is at most one.

**Proof.** Since we know (4.10), we may assume that $i_1 \neq i_2$. We consider a sequence $i = (i_1, i_2) \in I^\beta$ with $\beta = \alpha_{i_1} + \alpha_{i_2}$. When $i_1 \leftrightarrow i_2$, the formula (4.14) tells us that the operator $(\log(a_{i_1}^{-1}z_1) - \log(a_{i_2}^{-1}z_2))R_{\mathfrak{K}}^i$ belongs to $\text{Hom}_{U_q}(\tilde{V}^{\otimes i}, \tilde{V}^{\otimes i-s_1})$, where we put $z_k = z_{1_k}$ for $k = 1, 2$ as before. Notice that

$$\log(a_{i_1}^{-1}z_1) - \log(a_{i_2}^{-1}z_2) \in (z_2/z_1 - a_{i_2}/a_{i_1}) \cdot \mathcal{O}_i^\times.$$ 

Therefore we find that the order of zero of $d_{i_1, i_2}(u)$ at $u = a_{i_2}/a_{i_1}$ is at most one. For the other case $i_k \neq i_{k+1}$, by the formula (4.14), the operator $R_{\mathfrak{K}}^i$ already belongs to $\text{Hom}_{U_q}(\tilde{V}^{\otimes i}, \tilde{V}^{\otimes i-s_1})$. Therefore the order of zero of $d_{i_1, i_2}(u)$ at $u = a_{i_2}/a_{i_1}$ is zero. \[\Box\]
Remark 4.8. For each \( i \in I^\beta \), we define a topological \( k \)-algebra automorphism \( \sigma_i \) of \( \hat{\mathcal{O}}_i \) by setting
\[
\sigma_i(\log(a_{i_k}^{-1}X_k)) := a_{i_k}^{-1}X_k - 1
\]
for all \( k \). This induces a \( U_q(Lg) \)-module automorphism \( \sigma := \bigoplus_{i \in I^\beta} (1 \otimes \sigma_i) \) on the module \( \hat{V} \otimes^\beta \). If we twist our right \( \hat{H}_Q(\beta) \)-action by this automorphism \( \sigma \) (i.e. we replace the structure map \( \psi \) with \( \sigma \psi(-)^{-1} \)), we get a new right \( \hat{H}_Q(\beta) \)-action on \( \hat{V} \otimes^\beta \) given by the following formulas:
\[
v \cdot e(i') = \delta_{i,i'} v \tag{4.15}
v \cdot x_k = (a_{i_k}^{-1}X_k - 1)v \tag{4.16}
v \cdot \tau_k = \begin{cases} (a_{i_k}^{-1}X_k - a_{i_{k+1}}^{-1}X_{k+1})^{-1}(R^i_k(v) - v) & \text{if } i_k = i_{k+1}, \\
(a_{i_k}^{-1}X_{k+1} - a_{i_{k+1}}^{-1}X_k)\hat{R}^i_k(v) & \text{if } i_k \leftarrow i_{k+1}, \\
\hat{R}^i_k(v) & \text{otherwise}, \end{cases} \tag{4.17}
\]
where \( v \in \hat{V} \otimes^i \) with \( i = (i_1, \ldots, i_d) \in I^\beta \). This new action is same as Kang-Kashiwara-Kim’s action in [9, 10].

Theorem 4.9. The formulas (4.12), (4.13) and (4.14) (or the formulas (4.15), (4.16) and (4.17)) define a structure of a \( (U_q(Lg), \hat{H}_Q(\beta)) \)-bimodule on the left \( U_q(Lg) \)-module \( \hat{V} \otimes^\beta \). The functor \( M \mapsto \hat{V} \otimes^\beta \otimes_{\hat{H}_Q(\beta)} M \) gives an equivalence of categories:
\[
\hat{H}_Q(\beta)-\text{mod}_{fd} \xrightarrow{\sim} C_{Q,\beta}.
\]

Proof. This follows from the discussions in this subsection, Theorem 3.14 and Theorem 4.2. \( \square \)

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