Characterisation of symmetries of unlabelled triangulations and its applications∗

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Abstract

We give a full characterisation of the symmetries of unlabelled triangulations and derive a constructive decomposition of unlabelled triangulations depending on their symmetries. As an application of these results we can deduce a complete enumerative description of unlabelled cubic planar graphs.

1 Introduction

One of the most studied problems in enumerative combinatorics has been the enumeration of graphs embedded or embeddable on a surface, in particular planar graphs and triangulations. Enumeration of labelled planar graphs, maps, and triangulations [2, 22, 25, 31, 32, 33], properties of random labelled planar graphs like connectedness [22, 25], degree distribution and maximum degree [10, 13, 20, 21, 24, 27], containment of subgraphs [11, 18, 22, 25, 28], and random sampling [6, 19] have been studied intensively. In contrast to this abundance of results, many structural and enumerative problems concerning unlabelled (i.e. non-isomorphic) graphs on a surface are still open. In particular, the fundamental problem of determining the asymptotic number of unlabelled planar graphs remains unsolved. The best known partial results are enumerations of subfamilies of unlabelled planar graphs such as outerplanar graphs [4] or series parallel graphs [12].

In his seminal work [32], Tutte conjectured that almost all planar maps (i.e. graphs embedded on a sphere) are asymmetric—a conjecture that was later proved by Richmond and Wormald [29]. While this tells us that almost all planar maps have no non-trivial automorphisms, the opposite is true for planar graphs: McDiarmid, Steger, and Welsh [25] showed that almost all planar graphs have exponentially many automorphisms. Thus, it is impossible to derive the asymptotic number of unlabelled planar graphs from that of labelled planar graphs.

One of the fundamental tools for the enumeration of graphs and maps is constructive decomposition. The most prominent example is Tutte’s decomposition [32]: 2-connected graphs can be characterised by three disjoint subclasses

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of graphs, each of which is decomposed into smaller building blocks, with 3-connected graphs as one of the base cases, and vice versa, the building blocks construct all possible 2-connected graphs. Constructive decompositions can be interpreted as functional operations of generating functions that encode the enumerative information of the class of graphs or maps that is being decomposed. Following these lines, Chapuy et al. [9] used the decomposition from [32] to derive a grammar that allows to transfer enumeration results for the 3-connected graphs in a given graph class \( \mathcal{G} \) to the whole class \( \mathcal{G} \). As 3-connected planar graphs have a unique embedding \textit{up to orientation} by Whitney’s Theorem [35], the problem of enumerating labelled planar graphs is reduced to the enumeration of labelled 3-connected planar maps.

For unlabelled 3-connected graphs, however, the two embeddings provided by Whitney’s Theorem are not necessarily distinct; whether we have one or two distinct embeddings will depend on the symmetries of the graph. A better understanding of the symmetries of 3-connected planar graphs is therefore the key for the enumeration of unlabelled planar graphs.

In this paper, we derive a complete description of the automorphisms of unlabelled planar triangulations, planar maps in which every face boundary is a triangle (in other words, maximal planar maps). We also develop a constructive decomposition depending on their symmetries. While triangulations are one of the most fundamental classes of planar maps and thus their symmetries are interesting in their own right, the results of this paper can be extended even further: the duals of triangulations are precisely the 3-connected cubic planar maps and thus the constructive decomposition developed in this paper together with the information about the symmetries of the maps in question can be used to obtain an enumeration of unlabelled 3-connected cubic planar graphs. Using the grammar of [9], this provides a complete description of all unlabelled cubic planar graphs [23]. We believe that the insight gained in this work can be applied to study the symmetry and component structure of unlabelled planar graphs, in particular that of 3-connected unlabelled planar graphs, by carefully characterising planar graphs with different types of symmetries.

The constructive decomposition of triangulations will consist of two parts: the characterisation of the building blocks and the construction of how the building blocks will be merged in order to construct the triangulations. The building blocks will depend on the type of symmetries a triangulation \( T \) has: reflective symmetries, rotative symmetries, or both. There will be three classes of basic building blocks, called \textit{girdles}, \textit{fyke nets}, and \textit{spindles}, each one corresponding to one of the three cases for the symmetries of \( T \). In each case, \( T \) will contain a unique subgraph \( G \) from the respective class of base cases. Vice versa, we will show that \( T \) can be constructed from \( G \) by inserting planar maps from some additional classes of maps into some of the faces of \( G \). The construction of inserting maps into faces is similar to the process used to obtain \textit{stack triangulations}, objects that proved to have various applications in geometry [1, 3, 7, 16].

Part of our work is inspired by Tutte [33], who derived decompositions of triangulations with reflective symmetries and of triangulations with rotative symmetries (with the additional property that the order of the automorphism is prime). For our purposes, we need to consider all possible symmetries of triangulations and develop constructive decompositions for all three cases (reflective, rotative, or both types of symmetries). Our decomposition for the case of reflective symmetries will be very close to Tutte’s decomposition. Tutte’s de-
composition for rotative symmetries, however, is not unique (not even when the order is prime) and thus not a constructive decomposition. Our constructive decomposition for rotative symmetries will only bear slight resemblance to Tutte’s decomposition. The case of both types of symmetries has not been considered before.

This paper is organised as follows. After stating the necessary notation and basic facts in Section 2, we prove the aforementioned characterisation of symmetries as reflective or rotative in Section 3. In Sections 4 to 6 we then derive the constructive decomposition of triangulations separately for triangulations with reflective symmetries, with rotative symmetries, and with both types of symmetries. We will then show in Section 7 how to construct the basic building blocks and discuss the results obtained and the further work in Section 8.

2 Preliminaries

All graphs and maps considered in this paper are unlabelled (i.e. are isomorphism classes of labelled graphs) and simple (i.e. no two edges have the same two end vertices). Call a triangulation trivial if it has at most four vertices, so its underlying graph is a triangle or the complete graph $K_4$ on four vertices. In view of the results of this paper, these trivial triangulations represent degenerate cases of the structures considered. In order to keep the results simple, we will thus consider only non-trivial triangulations. Note that in a non-trivial triangulation, no two faces have the same set of vertices. For the rest of this paper, all triangulations are considered to be non-trivial.

A face of a planar map $G$ on a sphere $S$ is a connected component of $S \setminus G$. We refer to the vertices, edges, and faces of $G$ as its cells of dimension 0, 1, and 2, respectively. Two cells of different dimension are called incident if one is contained in the (topological) boundary of the other. Two cells of the same dimension are adjacent if there is a third cell incident with both.

An isomorphism between planar maps $G, H$ is a bijective map $\varphi: G \to H$ that maps each cell to a cell of the same dimension and preserves incidencies. If $G = H$, then we call $\varphi$ an automorphism. Note that for every isomorphism $\varphi$ of planar maps, we can find a homeomorphism of the sphere that maps every point in a cell $c$ to a point in the cell $\varphi(c)$. We can therefore view isomorphisms of planar maps as special homeomorphisms of the sphere. The automorphisms of a given triangulation $T$ form a group which is denoted by $\text{Aut}(T)$. A cell $c$ is invariant under a given automorphism $\varphi$ if $\varphi(c) = c$. We also say that $\varphi$ fixes $c$. A set $A$ of cells is invariant if $\varphi(A) = A$, note that each element of $A$ does not have to be invariant. The automorphisms under which a given cell $c$ is invariant form a group; we denote it by $\text{Aut}(c, T)$.

In enumerative combinatorics, triangulations are often considered with a given rooting; in other words, a certain cell—sometimes even several cells—are required to be invariant under all automorphisms that are considered. In this paper, all triangulations will have a single cell $c_0$ as a root; we will thus consider only automorphisms in $\text{Aut}(c_0, T)$.

The most restrictive kind of rooting is the strong rooting consisting of a vertex, edge, and face that are mutually incident. Isomorphisms between planar maps $G$ and $H$ with a strong rooting are always supposed to map roots of $G$ to roots of $H$. We will later see (Lemma 3.1) that a triangulation with a strong
rooting has only the identity as an automorphism.

An explicit formula for the number of triangulations with a strong rooting has been obtained by Tutte [31]. More generally, Brown [8] derived a formula for the number of near-triangulations. A planar map $N$ with a strong rooting consisting of a face $f_N$, an edge $e_N$, and a vertex $v_N$ is called a near-triangulation if $f_N$ is bounded by a cycle of any length $\geq 3$ while all other faces are bounded by triangles (see Figure 1). The root face $f_N$ is called the outer face of $N$, all vertices and edges on its boundary—in particular the root vertex $v_N$ and the root edge $e_N$—are called outer vertices or outer edges of $N$, respectively. All other vertices, edges, and faces of $N$ are its inner vertices, inner edges, or inner faces, respectively. The number of near-triangulations with a strong rooting with $m+3$ outer vertices and $n$ inner vertices is

$$A(n,m) = \frac{2(2m+3)!}{(m+2)!m!n!(3n+2m+3)!}.$$ 

The number of triangulations with $n+3$ vertices (i.e. $n$ inner vertices) and a strong rooting is obviously given by $A(n,0)$.

![Figure 1: A near-triangulation $N$ with root face $f_N$, root edge $e_N$, and root vertex $v_N$. The outer vertices of $N$ are $v_N, u_1, \ldots, u_6$; the outer edges are $e_N = v_Nu_1, u_1u_2, \ldots, u_5u_6, u_6v_N$.](image)

If a graph $G$ contains a cycle $C$ and an edge $e$ that does not belong to $C$ but connects two vertices of $C$, then we call $e$ a chord of $C$. An inner edge of a near-triangulation $N$ is a chord of $N$ if it is a chord of the cycle bounding the outer face (e.g. the edge $u_1u_4$ in Figure 1 is a chord).

Our goal is to provide a constructive decomposition of triangulations. The reverse direction of this decomposition will rely on the operation of inserting near-triangulations into faces of a given planar map (see Figure 2).

To make this operation precise, let $N$ be a near-triangulation with $m+3$ outer vertices and let $G$ be a planar map; denote by $S_N$ and $S_G$ the spheres on which $N$ and $G$ are embedded, respectively. Suppose that $f$ is a face of $G$ that is bounded by a cycle of length $m+3$; let $e$ be an edge on the boundary of $f$ and let $v$ be one of the end vertices of $e$. We obtain a new planar map $H$ as follows: Deleting the outer face of $N$ from the sphere $S_N$ results in a space $D_N$ homeomorphic to the unit disc; similarly, deleting $f$ from the sphere $S_G$ results in a space $D_G$ homeomorphic to the unit disc. Note that by construction the
boundary $C_N$ of $D_N$ (respectively the boundary $C_G$ of $D_G$) is the boundary of the outer face of $N$ (respectively that of $f$) and thus the point set of a cycle of length $m + 3$. Let $\sigma : C_N \rightarrow C_G$ be a homeomorphism that

- maps vertices to vertices;
- maps the root vertex $v_N$ of $N$ to $v$; and
- maps the (point set of the) root edge $e_N$ of $N$ to (the point set of) $e$.

The quotient space $(D_N \cup D_G) / \sigma$ obtained from the union $D_N \cup D_G$ by identifying every point $x \in C_N$ with $\sigma(x)$ is a sphere on which a graph $H$ is embedded. We say that $H$ is obtained from $G$ by inserting $N$ into $f$ at $v$ and $e$. If $G$ is rooted and $f$ is not its root face, then we consider $H$ to have the rooting it inherits from $G$.

If $T$ is a triangulation and $G$ is a 2-connected subgraph of $T$, then $T$ can always be obtained from $G$ by inserting near-triangulations into several of its faces: suppose that for each face $f$ of $G$, we choose an edge $e_f$ on its boundary and one of its end vertices $v_f$. Then the near-triangulation $N_f$ that is inserted into $f$ at $v_f$ and $e_f$ in order to obtain $T$ is uniquely defined. We say that $N_f$ is the near-triangulation induced by $(T, f)$ at $v_f$ and $e_f$.

### 3 Symmetries of triangulations

Throughout this paper, let $T$ be a triangulation and choose a cell $c_0$ as the root of $T$. As mentioned before, we will consider automorphisms in $\text{Aut}(c_0, T)$, i.e. automorphisms of $T$ that fix the root $c_0$. 
For every cell $c$ of $T$ of a given dimension $d$ the numbers of incident cells of dimensions $d+1 \pmod{3}$ and $d+2 \pmod{3}$ are the same. We call this number the degree of $c$ and denote it by $d(c)$. Clearly, for a vertex this notion of degree equals the graph theoretical definition; every edge has degree 2; every face of $T$ has degree 3. The distance of two cells $c, c'$ is the smallest number $\ell$ for which there is a sequence of $\ell + 1$ cells starting at $c$ and ending at $c'$ such that every two consecutive cells in the sequence are incident. Note that every two cells have a distance.

Given a cell $c$ of $T$, the set of cells incident with $c$ has a cyclic order $(c_1, c_2, \ldots, c_{2d(c)})$ in which two cells are consecutive if and only if they are incident in the triangulation (see Figure 3). This order is unique up to orientation. Two cells $c_\alpha, c_\beta$ with $\alpha, \beta \in \{1, 2, \ldots, 2d(c)\}$ are said to lie opposite at $c$ if $|\alpha - \beta| = d(c)$. We observe that if $c$ is a face, then its boundary is a triangle and every vertex $v$ of this triangle is opposite at $c$ to the edge of the triangle that is not incident with $v$. If $c$ is an edge, then its two incident faces lie opposite at $c$ and so do its end vertices. If $c$ is a vertex, the situation depends on the parity of $d(c)$: for even $d(c)$, every incident edge lies opposite to another incident edge while every face lies opposite to a face. For odd $d(c)$, every edge lies opposite to a face.

![Figure 3: A cyclic order $(c_1, c_2, \ldots, c_{2d(c)})$ of the cells incident with a cell $c$.](image)

We first observe some basic properties of automorphisms in $\text{Aut}(T)$ and $\text{Aut}(c_0, T)$.

**Lemma 3.1.** If three mutually incident cells are invariant under an automorphism $\varphi \in \text{Aut}(T)$, then $\varphi$ is the identity.

**Proof.** Let $c$ be one of the cells from the statement and let $(c_1, c_2, \ldots, c_{2d(c)})$ be the cyclic order of its incident cells. The other two cells from the statement are part of this order since they are incident with $c$. Since they are incident with
each other, they have consecutive positions in the order, \( c_1 \) and \( c_2 \), say. Recall that the cyclic order is unique up to orientation; therefore, since \( c_1 \) and \( c_2 \) are invariant by assumption, all cells incident with \( c \) are invariant.

For every cell \( c_i \) incident with \( c \), the same holds: the cells \( c \) and \( c_{i+1} \) are invariant and consecutive in the cyclic order of the incident cells of \( c_i \). Thus, all cells incident with \( c_i \) are invariant. By induction over the distance to \( c \), we obtain that all cells are invariant and therefore \( \varphi \) is the identity.

Lemma 3.1 in particular holds for automorphisms that fix \( c_0 \): if an automorphism \( \varphi \in \text{Aut}(c_0, T) \) fixes two cells that are incident with each other and with \( c_0 \), then \( \varphi \) is the identity. This immediately yields the following.

**Corollary 3.2.** An automorphism in \( \text{Aut}(c_0, T) \) is uniquely determined by its action on the cells incident with \( c_0 \).

Since an automorphism \( \varphi \in \text{Aut}(c_0, T) \) can only map cells of a given dimension to cells of the same dimension and since the cyclic order of the cells incident with \( c_0 \) is unique up to orientation, we obtain the following.

**Corollary 3.3.** For every cell \( c \) of \( T \), \( \text{Aut}(c, T) \) is isomorphic to a subgroup of the dihedral group \( D_{d(c)} \).

By definition, every automorphism \( \varphi \in \text{Aut}(c_0, T) \) fixes \( c_0 \). But, is \( c_0 \) the only invariant cell under \( \varphi \)? It is not hard to prove that it is not:

**Lemma 3.4.** For every \( \varphi \in \text{Aut}(c_0, T) \), there is at least one cell \( c \neq c_0 \) that is invariant under \( \varphi \).

This can be proved by pure combinatorial means (see e.g. [33]), but there is also a simple topological proof, which we provide below.

**Proof.** We can find a cycle in the underlying graph of \( T \) whose set of vertices and edges is invariant (see Figure 4). Indeed, if \( c_0 \) is a face, then its boundary is such a cycle. If \( c_0 \) is an edge, then the two faces incident with \( c_0 \) form an invariant set and hence the union of their boundaries, excluding the edge \( c_0 \), is the desired cycle. Finally, if \( c_0 \) is a vertex, then the vertices adjacent to it form the desired cycle together with all edges that lie opposite to \( c_0 \) at some face incident with \( c_0 \).

![Figure 4: Finding an invariant cycle.](image-url)

The point set of this cycle divides the sphere into two discs, on both of which \( \varphi \) induces a homeomorphism. By the Brouwer fixed-point theorem, both homeomorphisms have a fixed-point and hence the cells containing the fixed-points are invariant under \( \varphi \). Since one of the two discs does not contain \( c_0 \), we have found the desired cell \( c \).

\[\Box\]
Corollary 3.3 provides a nice way of characterising automorphisms \( \varphi \in \text{Aut}(c,T) \) for any given cell \( c \): if \( \varphi \) is not the identity, then either

(i) \( \varphi \) changes the orientation of the cyclic order of the cells incident with \( c \), in which case we call \( \varphi \) \textit{reflective at} \( c \); or

(ii) \( \varphi \) does not change the orientation of the cyclic order, in which case we call \( \varphi \) \textit{rotative at} \( c \).

Note that the distinction between reflective and rotative automorphisms is always up to the cell \( c \) currently considered—if an automorphism \( \varphi \) of \( T \) fixes two different cells \( c, c' \), then it is an element of \( \text{Aut}(c,T) \) as well as of \( \text{Aut}(c',T) \) and the decision whether \( \varphi \) is reflective or rotative at either vertex is performed separately in each automorphism group. We will later see (Corollary 4.3) that an automorphism cannot be reflective at one vertex and rotative at another vertex, but we cannot use this implication yet.

The properties of the automorphism group \( D_{d(c_0)} \) of a regular \( d(c_0) \)-gon immediately implies the following characterisation of reflective and rotative automorphisms.

**Lemma 3.5.** Suppose that \( \varphi \in \text{Aut}(c_0,T) \) is not the identity. Then the following holds.

(i) \( \varphi \) is reflective at \( c_0 \) if and only if it fixes precisely two cells incident with \( c_0 \); these cells lie opposite at \( c_0 \).

(ii) \( \varphi \) is rotative at \( c_0 \) if and only if it fixes no cell incident with \( c_0 \).

We will distinguish whether \( \text{Aut}(c_0,T) \) contains reflective automorphisms, rotative automorphisms, or both. Instead of reflective and rotative automorphisms, we will sometimes shortly speak of \textit{reflections} and \textit{rotations}.

Corollary 3.3 allows us to characterise \( \text{Aut}(c_0,T) \) by the types of automorphisms it contains.

**Theorem 3.6.** For every subgroup \( H \) of \( \text{Aut}(c_0,T) \) that contains at least one non-trivial automorphism, the following holds.

(i) If \( H \) contains a reflection but no rotation, then it is isomorphic to the 2-element group \( \mathbb{Z}_2 \).

(ii) If \( H \) contains \( k \geq 1 \) rotations but no reflection, then it is isomorphic to the cyclic group \( \mathbb{Z}_{k+1} \) where \( k+1 \) is a divisor of \( d(c_0) \).

(iii) If \( H \) contains both reflections and rotations, then it is isomorphic to a dihedral group \( D_n \) where \( n \geq 2 \) is a divisor of \( d(c_0) \).

**Proof.** Claim (i) follows since every reflection has order 2 and there is only one reflection in \( H \) since the composition of two distinct reflections would yield a rotation. Claims (ii) and (iii) follow directly from Corollary 3.3. \( \square \)
4 Reflective symmetries

In this section, suppose that $\text{Aut}(c_0, T)$ contains a reflection $\varphi$.

Our first lemma is a structural result that was first obtained by Tutte [33].

We include (a modified version of) its proof for the sake of completeness.

**Lemma 4.1.** There is a cyclic sequence $(c_0, \ldots, c_\ell)$ of pairwise distinct cells such that for each cell $c$ in the sequence the following holds.

(i) $c$ is invariant under $\varphi$;

(ii) the predecessor and the successor of $c$ in the sequence are incident with $c$ and lie opposite at $c$; and

(iii) no other cell in the sequence is incident with $c$.

**Proof.** Let $I$ be the set of cells that are invariant under $\varphi$. Define an auxiliary graph $F$ with vertex set $I$ by joining two elements of $I$ by an edge whenever they are incident. Note that $\varphi$, although chosen as an element of $\text{Aut}(c_0, T)$, is also an element of $\text{Aut}(c,T)$ for every $c \in I$. Since $\varphi$ is not the identity, Lemma 3.5 implies that every vertex in $F$ has degree 0 or 2 and thus, every component is a cycle or an isolated vertex. Since $\varphi$ is reflective at $c_0$ by assumption, $c_0$ has degree 2 in $F$ and is thus contained in a cycle $C$ of $F$. The vertices of $C$—arranged in the order they appear on $C$—form the desired cyclic sequence: all cells are invariant under $\varphi$ by construction; Lemma 3.5 implies that the predecessor and the successor of a cell $c$ in the sequence lie opposite at $c$ and no other cell in the sequence is incident with $c$.

For every edge in the sequence from Lemma 4.1, its predecessor and its successor are either its two end vertices or its two incident faces. Every face $f$ in the sequence is preceded and followed by a vertex and its opposite edge on the boundary of $f$.

The invariant cells from Lemma 4.1 play a central role in the constructive decomposition of $T$ in the case of a reflective symmetry: we will shortly see that these cells are the only cells invariant under $\varphi$ and thus, they provide a way to define a unique subgraph of $T$ that will be the basic building block in our constructive decomposition.

**Definition 4.1 (Girdle).** Let $G$ be the planar map obtained by taking the union of all vertices and edges that either lie in the sequence from Lemma 4.1 or on the boundary of a face in this sequence. We call this subgraph of $T$ the girdle with respect to $\varphi$. Its cells from the cyclic sequence are called central cells of $G$, the other ones (which are only part of $G$ because they lie on the boundary of a face from the sequence) are called outer cells of $G$. By construction, every face in the sequence is also a face of $G$ (and hence a central cell); the other faces of $G$ are called its sides. For every face in the sequence, precisely one of the edges on its boundary is a central cell and so is the other face incident with this edge. The union of such two faces and their boundaries is called a diamond. Note that every girdle has at least two central vertices; let $j(G)$ be the smallest index for which $c_{j(G)}$ is a vertex.

**Lemma 4.2.** The girdle $G$ has the following properties.
Figure 5: The sequence of cells from Lemma 4.1. The vertices in this picture, together with all black and all dashed edges, form the girdle of $T$ (see Definition 4.1). The central cells of the girdle are the black vertices, the black edges, and the gray faces. The outer cells are the gray vertices and the dashed edges. The girdle has three diamonds.

(i) $G$ has exactly two sides $f_1, f_2$.

(ii) Let $v_1 = v_2 := c_{j(G)}$. If $c_{j(G)+1}$ is an edge, we let $e_1 = e_2 := c_{j(G)+1}$; otherwise $c_{j(G)+1}$ is a face and we let $e_i$ be the unique edge on the boundary of $f_i$ that is incident with $c_{j(G)+1}$. Then $(T, f_i)$ induces a near-triangulation $N_i$ at $v_1$ and $e_i$.

(iii) $\varphi$ is an isomorphism between $N_1$ and $N_2$.

(iv) The central cells of $G$ are precisely the cells that are invariant under $\varphi$.

Proof. By Lemma 4.1, two central cells of $G$ are incident if and only if they are consecutive in the cyclic sequence. We claim that every outer cell is contained in a unique diamond, which implies that the subspace of the sphere consisting of $G$ and the faces in its diamonds is contractible to a Jordan curve, which in turn implies [1] by the Jordan curve theorem. Indeed, every outer cell of $G$ is a vertex or an edge that is contained in a diamond. If two diamonds share an outer edge, they also share an outer vertex $v$. Now $\varphi$ maps $v$ to the other outer vertex of each of the two diamonds, hence they also share their second outer vertex. But then the central edges contained in the two diamonds are distinct and have the same end vertices, contradicting the fact that $T$ has no double edges.

We have thus proved [1]. Since each side is bounded by a cycle, [3] follows immediately. Let $c$ be a cell incident with $c_0$ that is not a central cell of $G$, then $c$ is contained in one of the sides of $G$ or lies on the boundary of precisely one side. The reflection $\varphi$ maps $c$ to a cell that is contained in (or lies on the boundary of) the other side of $G$, which yields [3]. Finally, [4] follows directly from [3]. □
Figure 6: The girdle $G$ of a triangulation $T$ with $j(G) = 3$. The near-triangulations $N_1$ and $N_2$ that $T$ induces on the sides of $G$ at $c_3$ and $c_4$ are isomorphic.

Note that Lemma 4.2(iv) implies that no automorphism is reflective at one vertex and rotative at another vertex:

**Corollary 4.3.** If $\varphi \in \text{Aut}(c_0, T)$ is reflective at $c_0$, then it is reflective at every cell $c$ that is invariant under $\varphi$.

By Lemma 4.2, we have a constructive decomposition of $T$ into its girdle $G$ and two isomorphic near-triangulations $N_1, N_2$. What other properties do $G, N_1,$ and $N_2$ have to satisfy? Clearly, each side of $G$ is bounded by a cycle whose length matches the number of outer vertices of $N_1$ and $N_2$. We call this number the length of the girdle. The following lemma gives a complete characterisation of the near-triangulations that can occur.

**Lemma 4.4.** Let $G$ be a graph that occurs as the girdle of some triangulation and let $N$ be a near-triangulation. There exists a triangulation $T$ with a reflective automorphism $\varphi$, $G$ as its girdle with respect to $\varphi$, and $N$ as the near-triangulation from Lemma 4.2 if and only if

(i) the number of outer vertices of $N$ is the same as the length of $G$ and

(ii) every chord of $N$ has at least one end vertex that is an outer vertex of $G$.

**Proof.** First assume that the triangulation $T$ exists. Property (i) is immediate; in order to prove (ii) let $e = uv$ be a chord of $N$. If $u$ and $v$ are central vertices of $G$, then Lemma 4.2(iii) would imply that $\varphi$ maps $e$ to an edge $e'$ with the same end vertices. Since $e$ is not contained in $G$, Lemma 4.2(iv) shows that $e' \neq e$, contradicting the fact that there are no double edges.

Now assume that $N$ and $G$ satisfy (i) and (ii). Let $f_1, f_2$ be the sides of $G$ and let vertices $v_1, v_2$ and edges $e_1, e_2$ on the boundaries of $f_1$ and $f_2$, respectively, be defined as in Lemma 4.2. By (i) we can insert $N$ into each side $f_i$ of $G$ at $v_i$ and $e_i$. The result of this operation does not have any double edges by (ii) since all its faces are triangular, it is the desired triangulation $T$. \qed
More details about the construction of graphs that can serve as girdles and about the construction of triangulations with reflective symmetry from their girdle and the near-triangulations characterised by Lemma 4.4 will be given in Section 7.1.

5 Rotative symmetries

In this section, suppose that Aut($c_0, T$) contains a rotative automorphism $\varphi$. Then the subgroup $H$ of Aut($c_0, T$) generated by $\varphi$ contains no reflections and hence is isomorphic to a cyclic group by Theorem 3.6. We fix the group $H$ for the rest of this section; let $m$ be its order. For every cell $c$ incident with $c_0$, the cells $c, \varphi(c), \ldots, \varphi^{m-1}(c)$ are distinct since $\varphi, \ldots, \varphi^{m-1}$ are rotations and thus have no invariant cells incident with $c_0$. Without loss of generality, we can choose $\varphi$ in such a way that $c, \varphi(c), \ldots, \varphi^{m-1}(c)$ are arranged around $c_0$ in that order (in clockwise direction, say) for every cell $c$ incident with $c_0$ (see Figure 7).

![Figure 7: The images of a cell $c$ incident with $c_0$ under a rotation $\varphi$ of order 4.](image)

Lemma 3.4 tells us that $c_0$ is not the only invariant cell, so let $c_1$ be such a cell of shortest distance from $c_0$ and consider a shortest path $P$ in $T$ from $c_0$ (or a vertex incident with it—if $c_0$ is an edge or a face) to $c_1$ (or a vertex incident with it).

Lemma 5.1. The paths $P, \varphi(P), \ldots, \varphi^{m-1}(P)$ do not share any internal vertices. If $c_0$ is an edge or a face, all paths have distinct first vertices. The same is true for $c_1$ and the last vertices of the paths.

The special case of Lemma 5.1 in which $m$ is prime has been proved by Tutte [33].

Proof. First note that the paths $P, \varphi(P), \ldots, \varphi^{m-1}(P)$ are distinct, because $\varphi^i(P) = \varphi^j(P)$ for $i \neq j$ would imply that $\varphi^i(c) = \varphi^j(c)$ for some cell $c$ incident with $c_0$, which we already saw to be impossible. The same argument shows that two paths can only share an end vertex if it is $c_0$ or $c_1$.

Suppose two paths $\varphi^i(P), \varphi^j(P)$ with $i \neq j$ share an internal vertex. Its distance from the first vertex has to be the same in both paths, since otherwise the union of the two paths would contain a path from $c_0$ (or a vertex incident
with it) to $c_1$ (or a vertex incident with it) shorter than $P$, a contradiction to the choice of $P$. Choose $i, j$ such that the distance of their first intersection $v$ from their first vertices is as small as possible. The union of the segments of the two paths from the first vertices to $v$ together with $c_0$ separates the sphere into two discs, one of which contains $c_1$ and all its incident cells. Any path $\varphi^k(P)$ starting in the other disc thus has to meet $\varphi^i(P)$ or $\varphi^j(P)$ at the latest in $v$. The minimal choice of $i, j$ implies that every such path goes through $v$.

Therefore, there is a $k$ such that $\varphi^k(P)$ and $\varphi^{k+1}(P)$ meet in $v$. This means that $v$ is invariant under $\varphi$, contradicting the choice of $c_1$ as an invariant cell of minimal distance from $c_0$. This proves the lemma.

Lemma 5.1 implies that, just like the girdle divides the triangulation into two parts in the case of a reflective automorphism, the paths $P, \varphi(P), \ldots, \varphi^{m-1}(P)$ together with $c_0$ and $c_1$ divide the triangulation into $m$ parts. The union of these paths and cells might thus serve as a building block in our constructive decomposition.

**Definition 5.1.** Let $S$ be the union of $c_0$, $c_1$, their boundaries, and paths $P, \varphi(P), \ldots, \varphi^{m-1}(P)$ satisfying the statement of Lemma 5.1 (see Figure 8). We call $S$ a spindle of $T$ with respect to the group $H \subseteq \text{Aut}(c_0, T)$. The cells $c_0$ and $c_1$ are called the north pole and the south pole of the spindle, respectively. A face of a spindle which is neither $c_0$ nor $c_1$ is called a segment of the spindle.

![Figure 8: A triangulation and a spindle (bold) with respect to the group $H = \{\text{id}, \varphi, \varphi^2, \varphi^3\}$ of automorphisms, in which the invariant cells $c_0, c_1$ are both vertices.](image)

Similarly to reflections, we immediately get the following result:

**Lemma 5.2.** A spindle $S$ has the following properties:

(i) $S$ has exactly $m$ segments $f_1, \ldots, f_m$ with each $f_i$ being bounded by a cycle containing $\varphi^{i-1}(P)$ and $\varphi^i(P)$;

(ii) the intersection of $T$ with $f_i$ and its boundary is a near-triangulation $N_i$;
(iii) for every \( i \), \( \varphi \) is an isomorphism from \( N_i \) to \( N_{i+1} \).

A direct corollary of Lemma 5.2 is that \( c_0 \) and \( c_1 \) are the only invariant cells under \( \varphi \), even under each \( \varphi^i \) with \( 1 \leq i \leq m - 1 \). We might thus refer to them as the north and south pole of \( T \), not just of the spindle.

By Lemma 5.2 we can obtain all triangulations with rotative symmetry by first constructing all possible spindles and then inserting the same near-triangulation in each segment. However, unlike the girdle, a spindle is not unique since there might be different choices for the path \( P \). Figure 9 shows two different spindles of the same triangulation. Since the near-triangulation inserted in the segments in the first case is not isomorphic to the one used in the second case, we do not have a 1-1 correspondence between triangulations with rotative symmetries and triangulations obtained by taking a spindle and inserting the same near-triangulation in each segment.

![Figure 9: Two spindles (bold) of the same triangulation.](image)

In order to obtain a 1-1 correspondence, we thus have to refine the definition of a spindle. To this end, we will first define a substructure of a triangulation that will be part of our refined spindles.

**Definition 5.2.** A graph is called a cactus if it is connected and every two cycles in it have at most one vertex in common. It is well known that cacti are outerplanar, i.e. there is an embedding on the sphere for which all vertices lie on the boundary of a common face, the outer face. Every block of a cactus—a subgraph that cannot be disconnected by deleting a single vertex—is a cycle or an edge. If a cactus \( G \) has a root vertex, this induces a natural order on its set of blocks, similar to a tree order: Consider the block graph of \( G \)—the graph whose vertices are the vertices separating \( G \) and the blocks of \( G \) and in which a block is adjacent to all separating vertices it contains (see Figure 10). This block graph is always a tree and if we choose its root to be

- the root of \( G \) if it is a separating vertex or otherwise
- the block of \( G \) containing the root,

then this induces a tree order on the block graph and hence in particular an order on the set of blocks of \( G \). In this order the blocks that contain the root are the minimal elements.

Let \( k \geq 2 \) and let \( G \) be an outerplanar subgraph of \( T \) for which the north pole \( c_0 \) lies in its outer face. We call \( G \) a plane symmetric cactus of order \( m = |H| \) if it is a cactus and invariant under all elements of the group \( H \subseteq \text{Aut}(c_0, T) \).
Clearly, the outer face of $G$ is invariant under these automorphisms and by Lemma 3.4, $G$ has another invariant cell. If this cell is a cell of $T$, then it is invariant under rotations and hence the south pole $c_1$ of $T$, in which case we call $G$ antarctic (see Figure 11). If it is not a cell of $T$, then it is a face of $G$ whose boundary is a cycle and hence contains an invariant cell of $T$ by the Brouwer fixed-point theorem. Again, this cell is $c_1$. Either way, we obtain that $G$ has a unique invariant cell $c$ which is not its outer face. The subgraph of $G$ consisting of $c$ (if $c$ is a vertex or an edge) and all vertices and edges on the boundary of $c$ is called the centre of $G$. The maximal connected subgraphs of $G$ that share precisely one vertex with the centre are called branches of $G$ (see Figure 12); the vertex of a branch $B$ that lies in the centre of $G$ is called the base of $B$. Note that if $G$ is antarctic, then it has precisely 1, 2, or 3 branches, depending on whether $c_1$ is a vertex, an edge, or a face (see Figure 11).

![Figure 10: A cactus and its block graph.](image)

Figure 10: A cactus and its block graph.

If $G$ is not antarctic and in addition the boundary of the south pole $c_1$ of $T$ meets the boundary of the centre of $G$, then the south pole has to be a face or an edge and by symmetry all vertices on its boundary lie in the centre of $G$. In this case, we call $G$ pseudo-antarctic (see Figure 13).

![Figure 11: The three types of antarctic plane symmetric cacti.](image)

Figure 11: The three types of antarctic plane symmetric cacti. Note that if $c_1$ is a vertex, then the centre of the antarctic cactus $C$ is $c_1$; thus, the only branch of $C$ is $C$ itself.

Note that the above definition allows the case that the branches of a plane symmetric cactus are just the vertices of its centre, in particular every invariant cycle is a plane symmetric cactus. Furthermore, a plane symmetric cactus of order $k$ is also a plane symmetric cactus of order $\ell$ for every divisor $\ell \geq 2$ of $k$.

Plane symmetric cacti appear in a natural way when we move from the north pole towards the south pole of the triangulation.
Figure 12: A plane symmetric cactus of order 3 with centre $C$ and branches $B_1, \ldots, B_6$.

Figure 13: The two possibilities for a pseudo-antarctic plane symmetric cactus.
Lemma 5.3. Let $C$ be a cycle in $T$ that is invariant under $\varphi$ (and thus a plane symmetric cactus of order $m$). Suppose that $C$ is neither antarctic nor pseudo-antarctic and let $f$ be the face of $C$ that contains the south pole. Denote by $F$ the set of all faces of $T$ that are contained in $f$ and whose boundaries meet $C$. Let $F$ be the subgraph of $T$ consisting of all vertices and edges that lie on the boundary of a face $f' \in F$ but do not lie in $C$ or have an incident vertex in $C$. Then $F$ has a component that is a plane symmetric cactus of order $m$.

Proof. By construction, $F$ is outerplanar and all its edges lie on the boundary of its outer face. Thus, no two of its cycles can meet in more than one vertex, showing that all components of $F$ are cacti. The south pole $c_1$ is not contained in the outer face of $F$ by construction, therefore there is a component $F_1$ of $F$ such that either

- $c_1$ is contained in $F_1$ or
- $c_1$ is contained in a face of $F_1$ that is not its outer face.

In either case, $F_1$ is invariant under $\varphi$ (and hence under all elements of $H$) and thus a plane symmetric cactus of order $m$.

Repeated application of Lemma 5.3 gives rise to a finite sequence $F_0, \ldots, F_k$ of plane symmetric cacti in $T$ as follows. We start by letting $F_0$ be the invariant cycle “closest” to $c_0$ like in Figure 4 if $c_0$ is a face, let $F_0$ be its boundary. If $c_0$ is an edge, let $F_0$ consist of all vertices and edges, apart from $c_0$ itself, that lie on the boundary of a face incident with $c_0$. Finally, if $c_0$ is a vertex, let $F_0$ consist of all vertices adjacent to $c_0$ and all edges that lie opposite to $c_0$ at some face incident with $c_0$. Note that in either case, $F_0$ is a cycle whose length is a multiple of $m$.

If $F_0$ is antarctic or pseudo-antarctic, the sequence ends with $k = 0$; otherwise, by applying Lemma 5.3 with $C = F_0$, we obtain a plane symmetric cactus $F_1$ of order $m$. If $F_1$ is antarctic or pseudo-antarctic, we stop; otherwise, we apply Lemma 5.3 with $C$ being the centre of $F_1$ to obtain another plane symmetric cactus $F_2$ of order $m$. We continue this way until we obtain an antarctic or pseudo-antarctic plane symmetric cactus $F_k$. We call the graphs $F_0, \ldots, F_k$ the levels of $T$ (see Figure 14) and denote their centres by $C_0, \ldots, C_k$.

The idea behind our refined version of a spindle will be as follows: for a constructive decomposition, we shall need a unique substructure of $T$; something that the spindle was not able to provide, since the path $P$ was chosen arbitrarily. Instead of connecting the north pole and the south pole by paths, we will base our construction on the levels of $T$ and connect them by edges. Those edges have to be chosen in a unique way, which we will guarantee by always picking the ‘leftmost’ edge from a given vertex to the next level—a construction that will be made precise shortly. Moreover, it will not always be enough to have $m$ edges from each level to the next. Indeed, if the north pole $c_0$ is a vertex, then its degree might be a multiple of $m$ and there is no criterion which of the $d(c_0)$ edges we should choose. We thus have to start with all these edges.

The starting point of our construction will be vertices $u_0, \ldots, u_{am-1}$ on $F_0 = C_0$ (precise construction follows in Construction 5.3). We would then like to choose an edge from each $u_j$ to the level $F_1$. However, not every vertex $u_j$ necessarily has a neighbour in $F_1$. We will thus walk along the cycle $C_0$ in clockwise direction from each $u_j$ until we find a vertex $v_j$ that has a neighbour
Figure 14: Two triangulations and their levels.

(i) A triangulation with three levels $F_0, F_1, F_2$ (bold), each of which is a plane symmetric cactus of order 3. The last level $F_2$ is antarctic.

(ii) A triangulation with two levels $F_0, F_1$ (bold), both plane symmetric cacti of order 2. The last level $F_1$ is pseudo-antarctic.
in $F_1$. In order to decide which edge from $v_j$ to $F_1$ we will pick, let $e$ be one of the two edges of $C_0$ at $v_j$ and let $e_j = v_jw_j$ be the first edge in clockwise direction around $v_j$, starting at $e$, with $w_j \in F_1$. Note that this definition does not depend on which edge of $C_0$ we choose as $e$. We call $e_j$ the leftmost edge from $v_j$ to $F_1$ and $w_j$ the leftmost neighbour of $v_j$ in $F_1$. We then continue the construction in $F_1$ by first going to the base of the branch that contains $w_j$, then walk along the cycle $C_1$ until we find a vertex that has a neighbour in $F_2$ and so on. We will now make this construction precise.

**Construction 5.3** (liaison edges, sources, targets). We begin our construction by choosing vertices $u_0, \ldots, u_{am-1}$ on $F_0 = C_0$ as follows (see also Figure [15]): if $c_0$ is a vertex, let $a := d(c_0)/m$ and let $u_0, \ldots, u_{am-1}$ be all vertices of $F_0$, where the enumeration is in clockwise direction around the north pole. If $c_0$ is an edge, let $a := 1$ and let $u_0$ and $u_1$ be the vertices of $F_0$ that are not end vertices of $c_0$. Finally, if $c_0$ is a face, let $a := 1$ and let $u_0, u_1, u_2$ be the vertices on its boundary in clockwise direction. Note that by the choice of $u_0, \ldots, u_{am-1}$, we have $\psi(u_j) = u_{j+a} \ (mod\ am)$ for every $j$. With a slight abuse of notation, we will omit the modulo term in the index and simply write $u_i$ instead of $u_i \ (mod\ am)$. We will use this notation also for all other cyclic sequences of vertices throughout this section.

![Figure 15: The vertices $u_0, \ldots, u_{am-1}$ for the north pole $c_0$ being (i) a vertex, (ii) an edge, (iii) a face. Note that in Case (i), we can either have $m = 4$, $a = 1$ or $m = a = 2$.](image)

For each $j = 0, \ldots, am - 1$, we define the vertices $u_j^0 := u_j, u_j^1, \ldots, u_j^k$, $v_j^0, \ldots, v_j^{k-1}$, and $w_j^1, \ldots, w_j^k$ as follows (see Figure [16]): recursively for $0 \leq i \leq k - 1$

(i) let $v_j^i$ be the first vertex starting from $u_j^i$ along the cycle $C_i$ in clockwise direction around the north pole that has an edge to $F_{i+1}$;

(ii) let $v_j^i w_j^{i+1}$ be the leftmost neighbour of $v_j^i$ in $F_{i+1}$; and

(iii) let $u_j^{i+1}$ be the base of the branch of $F_{i+1}$ that contains $w_j^{i+1}$.

The vertices $u_j^0, \ldots, u_j^k$, $v_j^0, \ldots, v_j^{k-1}$, and $w_j^1, \ldots, w_j^k$ are uniquely defined by [i]-[iii]. We have $\psi(u_j^i) = u_{j+a}^i$, $\psi(v_j^i) = v_{j+a}^i$, and $\psi(w_j^i) = w_{j+a}^i$ for all $i, j$ by the symmetry of $T$ and the fact that $\psi(u_j^0) = u_{j+a}^0$. Note that in [i] we
encounter $v^i_j$ before we reach $u^i_{j+a}$: indeed, if the subpath of $C_i$ from $u^i_j$ to $u^i_{j+a}$ contains no vertex that has a neighbour in $F_{i+1}$, then by the fact that $\varphi(u^i_j) = u^i_{j+a}$, no vertex of $C_i$ has a neighbour in $F_{i+1}$, a contradiction to the definition of $F_{i+1}$.

The edges $v^i_j w^{i+1}_j$ are called liaison edges. For every liaison edge, we call $v^i_j$ its source and $w^{i+1}_j$ its target. Note that sources, targets, and bases do not have to be distinct. Clearly, two targets that lie in the same branch will always result in the same base, but also two bases will result in the same source if there is no eligible choice for a source between them on the cycle, and two sources may result in the same target if their leftmost edges lead to the same vertex. It is important to note that the sources $v^i_j, v^i_j + a, \ldots, v^i_j + a(m-1)$ are always distinct since they form an orbit under $\varphi$ by the symmetry of the construction. The same holds for targets and bases up to the $(k-1)$-st level.

With the levels $F_0, \ldots, F_k$ and the liaison edges $v^i_j w^{i+1}_j$, we are now able to define our refined spindles, called fyke nets.

**Definition 5.4 (Fyke net).** Let $\bar{F}$ be the union of

- the levels $F_0, \ldots, F_k$ of $T$,
- all liaison edges $v^i_j w^{i+1}_j$,
- the north pole $c_0$ of $T$,
- all edges from $c_0$ to $F_0$ (if $c_0$ is a vertex), and
- the south pole $c_1$ and its boundary (if the last level $F_k$ is pseudo-antarctic).

The fyke net of $T$ with respect to the group $H \subseteq \text{Aut}(c_0, T)$ is the maximal 2-connected subgraph $F$ of $\bar{F}$ that contains both poles $c_0, c_1$.

The intersection of the fyke net with the level $F_i$ is its $i$th layer and denoted by $H_i$. Note that every layer is a plane symmetric cactus of order $m$ by the
symmetry of the construction. The fyke net has up to five different types of faces:

(i) faces at the north pole $c_0$: either $c_0$ itself (if it is a face) or all faces of $T$ that are incident with $c_0$;

(ii) faces that are bounded by cycles in a branch of a layer; we call such faces leaves;

(iii) faces bounded by two consecutive liaison edges and two subpaths of the two layers connecting their sources and their targets; we call such faces segments;

(iv) if the last layer $H_k$ is pseudo-antarctic, $m$ faces that are bounded by a subpath of the centre of $H_k$ and the south pole $c_1$ (if it is an edge) or one of its incident edges (if it is a face); we call such faces pseudo-antarctic;

(v) the south pole $c_1$ (if it is a face).

The following properties of the fyke net are easy to show, using the 2-connectedness of the fyke net and the structure of the levels of $T$.

**Proposition 5.4.** Let $F$ be the fyke net of a triangulation $T$. Then every segment of $F$ is bounded by a cycle. An edge $e$ of $T$ lies in $F$ if and only if

(i) $e$ is a liaison edge,

(ii) $e$ lies in the centre of a level $F_i$, or

(iii) $e$ and a target $w^j_i$ are contained in the same branch $B$ of a level $F_i$ and $e$ lies on a path from $w^j_i$ to $u^j_i$ in $B$. Equivalently, the block $B(e)$ of $B$ containing $e$ and the smallest block (in the tree order on the block graph induced by choosing the base $u^j_i$ of $B$ as its root) $B(w^j_i)$ containing $w^j_i$ satisfy $B(w^j_i) \geq B(e)$ (in said tree order).
Note that the orbits of $\varphi$ partition the sets of leaves and segments into sets of size $m$. In particular, the near-triangulations that have to be inserted into the leaves and segments in order to re-obtain (i.e. construct) $T$ are isomorphic if the corresponding faces of $F$ are in the same orbit.

Unlike spindles, the fyke net is unique and thus, we can obtain all triangulations with rotative symmetry by first choosing a fyke net and then the near-triangulations that are to be pasted into the leaves and segments.

As in the case of reflective symmetries, we have to be more specific on which near-triangulations we are allowed to paste into leaves and segments.

**Lemma 5.5.** Let $f$ be a segment of the fyke net of $T$. Then there exist unique indices $i,j$ satisfying the following properties (see Figure 18).

(i) The boundary $C$ of $f$ consists of two liaison edges $v^i_j w^{i+1}_j$, $v^{i+1}_j w^{i+1}_{j+1}$ and paths $P_i, P_{i+1}$, where $P_i$ is a path in the centre of the layer $H_i$ from $v^i_j$ to $v^{i+1}_j$ and $P_{i+1}$ is a path in the layer $H_{i+1}$ from $w^{i+1}_j$ to $w^{i+1}_{j+1}$.

(ii) The path $P_i$ has at least one edge and runs along the centre of $H_i$ in clockwise direction around the north pole.

(iii) The base $u^{i+1}_{j+1}$ is a vertex on $P_i \setminus \{v^i_j\}$.

We write $f^i_j = f$ and denote by $N^i_j$ the near-triangulation that $(T, f^i_j)$ induces at the vertex $v^i_j$ and the edge $v^i_j w^{i+1}_j$. The near-triangulation $N^i_j$ has the following properties.

(iv) The edge $v^{i+1}_j w^{i+1}_{j+1}$ is part of the boundary of a face of $N^i_j$ whose third vertex $x$ lies in the subpath of $P_i$ from $v^i_j$ to the predecessor of $u^{i+1}_{j+1}$.

(v) Every edge of $P_{i+1}$ is part of the boundary of a face of $N^i_j$ whose third vertex is in $P_i$.

(vi) No two vertices in $P_{i+1}$ are connected by a chord in $N^i_j$.

(vii) If $m = 2$ and $\varphi(v^i_j) = v^{i+1}_{j+1}$, then there is no edge in $N^i_j$ from $v^i_j$ to $v^{i+1}_{j+1}$.

![Figure 18: The structure of the near-triangulation in a segment of the fyke net.](image-url)
Proof. Property \((i)\) is part of the definition of a segment and \((ii)\) is immediate by the definition of the sources and targets. Property \((iii)\) is clear by the way the source \(v_{i+1}^j\) has been chosen.

Property \((iv)\) follows from the existence of a face having the edge \(w_{j+1}^iw_{j+1}^i+1\) on its boundary and the fact that its third vertex \(x\) cannot be

- a vertex in \(P_{i+1}\), since this would contradict the choice of \(w_{j+1}^i\) as the leftmost neighbour of \(v_{j+1}^i\);
- an internal vertex of \(N_j^i\), since then \(x\) would have been in the \((i+1)\)-st level of \(T\), again contradicting the choice of \(w_{j+1}^i\);
- a vertex on the subpath of \(P_i\) from \(v_{j+1}^i\) to \(v_{j+1}^j\), since by the choice of \(v_{j+1}^i\) no vertex on this path has a neighbour in the \((i+1)\)-st level of \(T\).

In order to prove \((v)\) let \(e\) be an edge of \(P_{i+1}\). It is part of the boundary of a unique face of \(N_1^i\) and by the definition of \(F_{i+1}\) it is also part of the boundary of a face of \(T\) whose third vertex is in \(F_i\). We will show that this latter face is also a face of \(N_1^j\), thus showing \((v)\).

We will prove this for the edges in \(P_{i+1}\) one by one, starting from the edge at \(w_{j+1}^i\). Let \(x_1\) be the last neighbour of \(w_{j+1}^i\) on \(P_i\) (starting from \(v_j^i\)). The edge \(x_1w_{j+1}^i\) divides \(N_j^i\) into two parts, let \(N_1^i\) be the part which contains all of \(P_{i+1}\). The edge \(x_1w_{j+1}^i\) is part of the boundary of a unique face of \(N_1^i\), denote the third vertex of this face by \(y_1\) (see Figure 19). If \(y_1\) is the neighbour of \(w_{j+1}^i\) on \(P_{i+1}\), then we have found the desired face. Otherwise, it cannot be a vertex of \(P_{i+1}\) since the edge \(w_{j+1}^iy_1\) is in \(F_{i+1}\) and would thus also have been in \(H_{i+1}\). Since \(x_1\) was the last neighbour of \(w_{j+1}^i\) on \(P_i\), \(y_1\) has to be an internal vertex of \(N_1^j\). Now repeat the construction with \(y_1\) instead of \(w_{j+1}^i\) to obtain a vertex \(x_2\) on \(P_i\) (possibly \(x_2 = x_1\)), a near-triangulation \(N_2 \subseteq N_1\) and a vertex \(y_2\). As before, \(y_2\) cannot lie on \(P_i\) by the definition of \(x_2\) and not in \(P_{i+1} \setminus \{w_{j+1}^i\}\) by the definition of \(H_{i+1}\). It also cannot be \(w_{j+1}^i\), since then the edge \(x_2w_{j+1}^i\) would either contradict the choice of \(x_1\) as the last neighbour of \(w_{j+1}^i\) on \(P_i\) (if \(x_1 \neq x_2\)) or it would yield a double edge (if \(x_1 = x_2\)), also a contradiction.

We can thus continue the construction and will always obtain internal vertices of \(N_1^j\) for \(y_1, y_2, \ldots\). Since these vertices are distinct and \(N_1^j\) is finite, this is a contradiction, implying that \(y_1\) must have been the neighbour of \(w_{j+1}^i\) on \(P_{i+1}\).

The same construction for every later edge of \(P_{i+1}\) proves \((v)\). Property \((vi)\) follows immediately from \((v)\).

Finally, note that \((vii)\) is immediate since otherwise there would be a double edge in \(T\) between \(v_j^i\) and \(v_{j+1}^i\). \(\square\)

The near-triangulations pasted into leaves or pseudo-antarctic faces, however, do not have any restrictions. Indeed, chords do neither contradict the construction of the layers by Lemma 5.3 nor can they result in double edges.

A triangulation with rotative symmetry can thus be constructed by first choosing a fyke net, then choosing, for every isomorphism class of leaves or pseudo-antarctic faces, any near-triangulation to be pasted into each of these leaves, and finally, for every isomorphism class of segments, choosing a near-triangulation with properties \((iv)\) and \((v)\) (and \((vii)\) in the case of \(m = 2\)) above.
Figure 19: The construction proving Lemma 5.5(v). Note that the vertices $x_1, x_2, \ldots$ are not necessarily distinct. The vertices $y_1, y_2, \ldots$, however, are mutually distinct.

(recall that (vi) follows immediately). More details about this construction will be given in Section 7.2.

6 Reflective and rotative symmetries

In this section, we assume that Aut($c_0, T$) has a subgroup $H$ that contains both reflective and rotative automorphisms. By Theorem 3.6, $H$ is isomorphic to $D_n$ where $n \geq 2$ is a divisor of $d(c_0)$, i.e., there are $n$ reflections and $n - 1$ rotations (and the identity).

Since the rotations and the identity form a cyclic group, the results of Section 5 can be applied. In particular, there is a unique cell $c_1 \neq c_0$ that is invariant under all rotations. Again, we call $c_0$ the north pole and $c_1$ the south pole of $T$. For each reflection $\varphi$, there is a girdle $G_\varphi$ by the results of Section 4.

Clearly, no two girdles are the same by Corollary 3.2 and every girdle contains the north pole $c_0$ by definition. Thus, there are $2n$ cells incident with $c_0$ that are invariant under some reflection; denote them by $a_0, \ldots, a_{2n-1}$, enumerated in the same order they lie around $c_0$ (in clockwise direction, say). Then for every reflection, there is an $i \in \{0, \ldots, n-1\}$ such that the invariant cells incident with $c_0$ are $a_i$ and $a_{n+i}$; denote this automorphism by $\varphi_i$ and its girdle by $G_i$.

Lemma 6.1. The girdles $G_0, \ldots, G_{n-1}$ have the following properties.

(i) North and south pole are central cells of every girdle.

(ii) The two poles are the only cells that are central cells of more than one girdle.

Proof. The north pole is a central cell of every girdle by definition (Definition 4.1). Let $G_i, G_j$ be two distinct girdles. We first show that there is another cell that is central in both of them and then prove that this cell is the south pole. This will prove both (i) and (ii).
Since for each of $\varphi_i, \varphi_j$, the invariant cells incident with $c_0$ lie opposite, the central cells of $G_i$ incident with the north pole lie in different sides (or on the boundaries of different sides) of $G_j$. Since the central cells of a girdle separate its sides, $G_i$ and $G_j$ meet in at least one central cell apart from the north pole. Let $c$ be such a cell.

Consider the automorphism $\varphi_i \circ \varphi_j$. Since $c$ is invariant both under $\varphi_i$ and under $\varphi_j$, it is also invariant under $\varphi_i \circ \varphi_j$. But the composition of two distinct reflections is always a rotation and thus, the only cells invariant under $\varphi_i \circ \varphi_j$ are the north and south pole, implying that $c$ is the south pole.

Since the cells $a_0, \ldots, a_{2n-1}$ form a cyclic sequence around $c_0$, we will also consider their indices modulo $2n$, similarly to the previous section. For simplicity, we will again write $a_i$ instead of $a_i \text{ (mod } 2n)$. The same kind of notation will be used for the girdles $G_0, \ldots, G_{n-1}$ (modulo $n$ instead of modulo $2n$).

The rotations can be enumerated as $\rho_1, \ldots, \rho_{n-1}$ so that every $\rho_i$ satisfies

\[ \rho_i(a_j) = a_{j+2i} \]

for all $j = 0, \ldots, 2n - 1$. With this notation, we have $\rho_i^1 = \rho_i$ for all $i = 1, \ldots, n - 1$ (and $\rho_i^n = \text{id}$).

Corollary 3.2 and the automorphism $\rho_1$ show that $G_0$ is isomorphic to $G_2$, $G_1$ is isomorphic to $G_3$, and so on. If $n$ is odd, this implies that all girdles are isomorphic; if $n$ is even, all $G_i$ with even $i$ are isomorphic as well as the ones with odd $i$. Moreover, in the latter case every girdle is mapped to itself by the rotation $\rho_{\frac{n}{2}}$. In that case we call the girdles symmetric and $\rho_{\frac{n}{2}}$ a symmetry of each girdle. We thus have proved the following.

**Lemma 6.2.** For every $i = 1, \ldots, n - 1$, the following holds.

(i) For every $j = 0, \ldots, n - 1$, the rotation $\rho_i$ induces an isomorphism between the girdles $G_j$ and $G_{j+2i}$.

(ii) If $n$ is odd, all girdles are isomorphic.

(iii) If $n$ is even, $\rho_{\frac{n}{2}}$ is a symmetry of each girdle and every two girdles $G_i, G_j$ with $i - j$ even are isomorphic.

Recall that Lemma 6.1 tells us that any two girdles cross precisely twice: once at each of the poles. However, while a central cell of a girdle cannot be a central cell of another girdle (unless it is one of the poles), it might well be an outer cell of another girdle.

Since every girdle $G_i$ has both poles as central cells, they divide it into two parts in a natural way: if $(x_j)_{j \in \mathbb{Z}}$ is the cyclic sequence from Lemma 4.1 with $x_0 = c_0$ (note that by Lemmas 4.1 and 4.2 iv) this sequence is unique up to orientation), then $x_k = c_1$ for some $k$ and we can consider the sequences $x_0, x_1, \ldots, x_k$ and $x_k, x_{k+1}, \ldots, x_{m-1}, x_0$. One of the sequences contains $a_i$, so we denote the union of its elements and their boundaries by $M_i$. The other sequence contains $a_{n+i}$, we denote the union of its elements and their boundaries by $M_{n+i}$. We call $M_i$ and $M_{n+i}$ meridians, the cells from the respective sequence of $x_j$'s are the central cells of $M_i$ and $M_{n+i}$, respectively. The other cells are outer cells, as before. Note that a central cell of $G_i$ that lies on the boundary of one of the poles will be contained in both $M_i$ and $M_{n+i}$. However, it will
only be a central cell in one of them. Clearly, \( G_i = M_i \cup M_{i+1} \) and thus \( \bigcup_{i=0}^{n-1} G_i = \bigcup_{i=0}^{2n-1} M_i \).

Like the girdles, the meridians form a cyclic sequence; for simplicity, we will write \( M_i \) instead of \( M_i \mod 2n \).

**Definition 6.1 (Skeleton).** The union \( S := \bigcup_{i=0}^{2n-1} M_i \) is the skeleton of \( T \).

For every \( i = 0, \ldots, 2n-1 \), we say that the meridians \( M_i \) and \( M_{i+1} \) are adjacent. Every face of \( S \) that is not a central cell of at least one of the meridians (equivalently: of one of the girdles) is a segment of \( S \).

Note that the skeleton of \( T \) is unique since all the girdles are.

**Lemma 6.3.** The skeleton \( S \) has the following properties.

(i) Every reflection \( \varphi_i, 0 \leq i \leq n-1 \), induces an isomorphism between \( M_{i-j} \) and \( M_{i+j} \) for every \( j = 1, \ldots, n-1 \).

(ii) Every rotation \( \rho_i, 1 \leq i \leq n-1 \), induces an isomorphism between \( M_j \) and \( M_{j+2i} \) for every \( j = 0, \ldots, 2n-1 \).

(iii) There is an isomorphism in \( H \) that maps \( M_i \) to \( M_j \) if and only if \( i-j \) is even.

(iv) For every central cell \( c \) of a meridian \( M_i, 0 \leq i \leq 2n-1 \), exactly one of the following holds.

(C1) \( c \) is a pole;

(C2) \( c \) lies on the boundary of a pole;

(C3) \( c \) is not contained in any other meridian;

(C4) \( c \) is an outer cell of both meridians adjacent to \( M_i \) and not contained in any other meridian.

(v) Every segment of \( S \) is bounded by a cycle that is contained in the union of two adjacent meridians.

(vi) There is a non-negative integer \( s \) such that for every pair \( (M_i, M_{i+1}) \) of adjacent meridians there are precisely \( s \) such segments.

**Proof.** Claims (i) and (ii) follow from Corollary 3.2 and the way \( \varphi_i \) and \( \rho_i \) act on \( a_1, \ldots, a_{2n} \). Claim (iii) is an immediate corollary of (i) or (ii).

To prove (iv), let \( c \) be a central cell of \( M_i \). Note first that only one of the cases (C1)–(C4) can hold. Now assume that (C1)–(C3) do not hold, i.e., \( c \) is neither a pole nor lies on the boundary of a pole and there is at least one meridian \( M_j \) with \( j \neq i \) that contains \( c \). By Lemma 6.1(C4) \( c \) is an outer cell of every such meridian \( M_j \).

The central cells of \( M_{i-1} \) and \( M_{i+1} \) separate the sphere into two parts, one of which contains the central cells of \( M_i \) (apart from the poles) while the other contains the central cells (apart from the poles) of all other meridians. This implies that \( c \) is an outer cell of at least one of \( M_{i-1}, M_{i+1} \) and not contained in any other meridian; it remains to show that \( c \) is an outer cell of both \( M_{i-1} \) and \( M_{i+1} \). By (ii) \( \varphi_i \) (or \( \varphi_{i-n} \) if \( i > n \)) induces an isomorphism between \( M_{i-1} \) and \( M_{i+1} \) and since \( c \) is invariant under \( \varphi_i \), it is an outer cell of both meridians adjacent to \( M_i \). This proves (iv).
For (v), note first that every segment of $S$ is bounded by a cycle since the graph $S$ is 2-connected. To prove the other half of the statement, choose $2n$ arcs (injective topological paths) from one pole to the other, one in the union of the central cells of each meridian. By (iv), these arcs only meet in the two poles and thus divide the sphere into $2n$ discs, each having a boundary that is contained in the union of two of the arcs, with the corresponding meridians being adjacent. Since every segment of $S$ is contained in such a disc and no other meridian contains a point in this disc, (v) follows.

Finally, (vi) follows by applying rotations and reflections to the segments with their boundaries in $M_1 \cup M_2$.

By Lemma 6.3(vi), we can denote the segments of $S$ whose boundaries are contained in the union of $M_i$ and $M_{i+1}$ by $f_{i1}, \ldots, f_{is}$. Note that the cycle from Lemma 6.3(v) bounding $f_{ij}$ is the union of a subpath of $M_i$ and a subpath of $M_{i+1}$. These paths meet in their end vertices; denote by $v_{ij}$ their end vertex closer to the north pole and by $w_{ij}$ the one closer to the south pole. Without loss of generality, we assume that the enumeration of $f_{i1}, \ldots, f_{is}$ is chosen so that $v_{ij}$ is closer to the north pole than $v_{ij}'$ whenever $j < j'$. Finally, let $e_{ij}$ be the edge on the boundary of $f_{ij}$ that is incident with $v_{ij}$ and

(i) contained in $M_i$ if $i$ is even, or
(ii) contained in $M_{i+1}$ if $i$ is odd.

With this notation and Lemma 6.3(i) and (ii), we obtain the following.

**Lemma 6.4.** Let $j \in \{1, \ldots, s\}$.

(i) The pair $(T, f_{ij})$ induces a near-triangulation $N_{ij}$ at $v_{ij}$ and $e_{ij}$ for every $i$.

(ii) The near-triangulations $N_j^0, \ldots, N_j^{2n-1}$ are isomorphic.

For a complete description of all possible skeletons, we need to characterise their structure at the poles and at other points where two adjacent meridians meet.

**Lemma 6.5.** Let $S$ be a skeleton and $c$ be one of the poles of $T$. Then the structure of $S$ at $c$ is the following.

(i) If $c$ is a vertex, then either

(a) no two meridians meet in a cell incident with $c$ or
(b) there is a number $k \geq 1$ such that every two adjacent meridians meet in their first $k$ edges starting from $c$.

(ii) If $c = uv$ is an edge, then two non-adjacent meridians, say $M_0$ and $M_2$, have $u$ respectively $v$ as a central cell and the other two have its incident faces as central cells. No two meridians meet in an edge $e \neq c$ incident with $u$ or $v$.

(iii) If $c$ is a face $f$ with vertices $u, v, w$ on its boundary, then three mutually non-adjacent meridians, say $M_0, M_2, M_4$, have $u$, $v$, respectively $w$ as a central cell and the other three have its incident edges as central cells. Either
(a) no two meridians meet in a cell incident with exactly one of $u, v, w$

or

(b) there is a number $k \geq 1$ such that every two adjacent meridians meet in their first $k$ edges starting from $c$.

Proof. Statements (i) and (iii) follow from Lemma 6.3(i) and (ii). The first claim in (ii) is immediate, since each of the four meridians contains a different cell incident with $c$ as a central cell. For $i \in \{1, 3\}$, denote by $f_i$ the face incident with $c$ that is a central cell of $M_i$ (see Figure 20(ii)). Since $u, v$ are incident with $f_1$ and $f_3$, there are unique vertices $w_1, w_3$ different from $u$ and $v$ that are incident with $f_1$ and $f_3$, respectively. Note that $w_1$ and $w_3$ are distinct, since otherwise $f_1$ and $f_3$ would have the same set of incident vertices, which is not possible as our triangulations are simple and non-trivial. Suppose that $M_0$ meets $M_1$ at an edge $e \neq c$ incident with $u$; this has to be the edge $uw_1$. By applying $\varphi_0$, we see that $M_0$ meets $M_3$ in the edge $uw_3$. Thus, $w_1$ and $w_3$ are connected by an edge $e_0$ that is central in $M_0$. Applying $\varphi_2$ shows that $M_2$ also has a central edge $e_2$ that connects $w_1$ and $w_3$. Since our triangulations are simple, the edges $e_0$ and $e_2$ are identical and $T$ is a $K_4$. Since we assume all triangulations to be non-trivial, this is a contradiction. We have thus shown (ii).

Lemma 6.5 describes the structure of the skeleton at the poles. The following lemma deals with the intersections of adjacent meridians between two segments.

Lemma 6.6. Let $j \in \{1, \ldots, s-1\}$ be fixed. Then there is a number $k_j \geq 0$, such that for every $i$, the intersection of $M_i$ and $M_{i+1}$ has a component that is a path of length $k_j$ from $v^j_i$ to $v^j_{i+1}$.

Proof. This follows immediately from Lemma 6.3(i).

All possible skeletons can thus be constructed by first choosing the numbers $n \geq 2$ and $s$ and the dimensions of the poles. Note that a pole can only be an edge if $n = 2$ and it can only be a face if $n = 3$. Then choose the structure at the poles according to Lemma 6.5 and between the segments according to Lemma 6.6.

All triangulations with both reflective and rotative symmetry can be obtained by first taking a skeleton and then inserting the same near-triangulation in each type of segment according to Lemma 6.4. Similarly to the case of reflective symmetries, the near-triangulation inserted into a segment is only allowed to have chords that do not produce double edges by reflecting. In this case, this means that for every chord of the near-triangulation and each meridian bounding the corresponding segment, not both end vertices of the chord are contained in the meridian and central in it. More details about this construction will be given in Section 7.3.

7 Constructions

In this section we formalise the constructive decompositions developed in the previous sections and show how to construct the basic graphs arising in them: girdles, fyke nets, and skeletons.
Figure 20: The possible structures of a skeleton at a pole as stated in Lemma 6.5, with \( k = 1 \) in the cases (i)(b) and (iii)(b).
Figure 21: The structure of a skeleton between two segments as described in Lemma 6.6. In Case (i), we have $k_j = 0$, in Case (ii) $k_j = 2$.

7.1 Reflective symmetries

The construction of all possible girdles is rather easy. Once the length $\ell$ of the girdle and the number $d$ of diamonds are fixed, all that is left is to consider all arrangements of $d$ diamonds on a girdle of length $\ell$. Note that $d \leq \frac{\ell}{2}$ is necessary; in the case of $c_0$ being a face, we furthermore have $d \geq 1$.

Let a girdle $G$ be given. The near-triangulations that can be inserted into the sides of $G$ in order to give rise to a triangulation with reflective symmetry have to satisfy the conditions of Lemma 4.4. In particular, the distribution of chords is restricted.

Definition 7.1. Let $N$ be a near-triangulation and let $D$ be a subset of its set of outer vertices. We call $N$ chordless outside $D$ if every chord of $N$ has at least one end vertex in $D$.

More generally, let a cycle $C$ with a root vertex $v_C$ and a root edge $e_C$ incident with $v_C$ be given and let $D_C$ be a set of vertices in $C$. Suppose that the length of $C$ is the same as the number of outer vertices of $N$ and let $\alpha$ be the unique isomorphism from $C$ to the boundary $C_N$ of the outer face of $N$ that maps $v_C$ to the root vertex $v_N$ of $N$ and $e_C$ to the root edge $e_N$ of $N$. We call $N$ chordless outside $D_C$ if it is chordless outside $\alpha(D_C)$.

Recall that $j(G)$ is the smallest index for which $c_{j(G)}$ is a vertex and let $v_1, v_2, e_1, e_2$ be given as in Lemma 4.2(ii). Denote by $C_G$ the cycle in $G$ bounding $f_1$ and let $D_G$ be the set of outer vertices of $G$ in $C$. With this notation, Lemmas 4.2 and 4.4 give rise to the following.

Theorem 7.1. The triangulations $T$ with a reflective symmetry in $\text{Aut}(c_0, T)$ are precisely the ones that can be constructed by choosing

- a girdle $G$ that contains $c_0$ as a central cell and
- a near-triangulation $N$ that is chordless outside $D_G$
and inserting a copy of $N$ into each side $f_1$ (respectively $f_2$) of $G$ at $v_1$ and $e_1$ (respectively at $v_2$ and $e_2$).

**Remark 7.2.** Since the girdle $G_\varphi$ of a triangulation $T$ with respect to a given reflection $\varphi$ is only unique up to orientation, some triangulations have two ways of constructing them by inserting near-triangulations into the sides of a girdle. More precisely, the construction in Theorem 7.1 is a 1-1 correspondence if and only if there is another reflection $\psi \neq \varphi$ in $\text{Aut}(G_0, T)$ that fixes $G_\varphi$. By Theorem 3.6, this is equivalent to $\text{Aut}(G_0, T)$ being isomorphic to $D_n$ with $n$ even. For all other triangulations, the construction is a 1-2 correspondence, i.e. all triangulations $T$ with $\text{Aut}(T, c_0) \simeq \mathbb{Z}_2$ can be constructed in precisely two different ways.

### 7.2 Rotative symmetries

As opposed to girdles, the construction of a graph $F$ that can serve as a fyke net requires several steps. Suppose that the desired order $m$ of the automorphism group is already given, as well as the dimensions of the poles and the number $k + 1$ of layers $H_0, \ldots, H_k$. We construct $F$ in the following steps.

- Choose the layer $H_0$ to be a cycle depending on the dimension of the north pole $c_0$ like in Figure 3. If $c_0$ is a vertex, let $am$ be the length of $H_0$, otherwise we put $a = 1$.

- For $i = 1, \ldots, k - 1$, let $C_i$ be a cycle whose length is a multiple of $m$. These cycles will serve as the centres of the layers. The choice of $C_k$ depends on the dimension of the south pole $C_1$: if $c_1$ is a vertex, then $C_k$ only consists of $c_1$, and the layer $H_k$ will be antarctic. If $c_1$ is an edge, $C_k$ can either be a cycle of even length, in which case $H_k$ will be pseudo-antarctic, or the edge $c_1$ itself, in which case $H_k$ will be antarctic. Finally, if $c_1$ is a face, then $C_k$ has to be a cycle whose length is divisible by 3. In that case, $H_k$ will be antarctic if $H_k$ is a triangle and pseudo-antarctic otherwise.

- Choose the bases $v^0_0, \ldots, v^0_{am-1}$ in $H_0$ in a clockwise order like in Definition 5.4 if $c_0$ is an edge, then choose two opposite vertices as $v^0_0, v^0_1$ (the other two will then be the end vertices of $c_0$), otherwise choose all vertices of $H_0$.

- Choose the sources $v^0_0, \ldots, v^0_{am-1}$ as follows: For $j = 0, \ldots, a - 1$, choose $v^0_j$ to be a vertex on the path starting at $v^0_0$ and running along $H_0$ in clockwise direction around the north pole to the predecessor of $v^0_{j+a}$ so that $v^0_1, \ldots, v^0_a$ appear in clockwise order on $H_0$, but are not necessarily distinct. The remaining sources $v^0_1, \ldots, v^0_{am-1}$ are obtained by recursively applying the rotative symmetry $\varphi$. The set of sources in $H_0$ is denoted by $S_0$.

- Recursively for $i = 1, \ldots, k$, choose the bases $u^i_0, \ldots, u^i_{am-1}$ and the sources $v^i_0, \ldots, v^i_{am-1}$ on $C_i$ as follows: if $C_i$ has length $am$, pick a subpath consisting of $a_i$ vertices and choose $u^i_0, \ldots, u^i_{a_i-1}$ from this subpath so that they appear in clockwise order on $C_i$. Again, the bases do not have to be distinct. Furthermore, if the sources $v^i_{i-1}$ and $v^{i+1}_{i+1}$ were identical, the corresponding bases $u^i_i$ and $u^{i+1}_{i+1}$ should also be identical. The other
bases follow again by applying symmetry. After choosing the bases, we can pick the sources like in the previous step (but note that we do not need any sources for \( i = k \)). We denote the sets of bases and sources in \( C_i \) by \( B_i \) and \( S_i \), respectively.

- Having fixed the bases and sources for every \( i \), we can now extend the \( C_i \) to layers \( H_i \). To that end, we need to add a suitable plane cactus at every base in \( C_i \). For every \( i \), denote by \( \pi_i : S_{i-1} \to B_i \) the function that maps \( v \) to \( u \) if there is a \( j \) with \( v = v^i_j \) and \( u = w^i_j \). For every base \( u \) in \( C_i \), we now attach a plane cactus at \( u \) that has at most as many maximal blocks in its natural order as \( u \) has preimages under \( \pi_i \). Again, it is sufficient to choose the cacti for the first \( a \) bases, the others are isomorphic by symmetry.

- Finally, we choose the targets and the liaison edges. For every \( u \in B_i \) and every \( v^i_j \in (\pi_i)^{-1}(u) \), we choose a vertex in the cactus at \( u \) to be the target \( w^i_j \) for the liaison edge \( v^i_j w^i_j \) according to the following rules:
  
  - The targets are arranged in clockwise order for increasing index \( j \) and
  
  - every maximal block has at least one vertex that does not belong to any other block and is chosen as a target.

The near-triangulations contained in a segment and its boundary are characterised by Lemma 5.5

**Definition 7.2.** Let \( C \) be a cycle that consists of a path \( P_0 \) from \( v_1 \) to \( v_2 \), a path \( P_u \) from \( w_1 \) to \( w_2 \), and two edges \( v_1 w_1, v_2 w_2 \). Denote the length of \( C \) by \( \ell \) and let \( u \) be a vertex on \( P_u \). We choose \( v_1 \) as the root vertex of \( C \) and \( v_1 w_1 \) as the root edge. If \( N \) is a near-triangulation with \( \ell \) outer edges, root vertex \( v_N \), and root edge \( e_N \), let us denote by \( \alpha \) the isomorphism from \( C \) to the boundary of the outer face of \( N \) that respects the rooting. We call \( N \) 2-layered with respect to \( v_1 \), \( v_2 \), \( w_1 \), \( w_2 \), and \( u \) if it satisfies Conditions (vi)–(iv) of Lemma 5.5 with \( v^j_j := \alpha(v_1), v^j_{j+1} := \alpha(v_2), w^j_j := \alpha(w_1), w^j_{j+1} := \alpha(w_2) \), and \( w^j_{j+1} := \alpha(u) \).

If \( T \) is a triangulation with rotative symmetry and \( f \) is a segment of its fyke net \( F \), then by Lemma 5.5 we have \( f = f^i_j \) for some \( i \) and \( j \) and \( N^j_j \) is 2-layered with respect to \( v^j_j, v^j_{j+1}, w^j_j, w^j_{j+1} \), and \( w^j_{j+1} \). Here the boundary of \( f^i_j \) is rooted at \( v^i_j \) and \( v^i_j w^i_{j+1} \). For every leaf \( f \) of the fyke net we choose the root vertex \( v_f \) of its boundary to be its vertex closest to the base of the branch it is contained in. As root edge \( e_f \) we choose the left of the two edges at \( v_f \) on this boundary. Finally, if \( F \) has pseudo-antarctic faces, we choose for each such face \( f \) the root edge \( e_f \) to be either the south pole \( c_1 \) (if it is an edge) or the unique edge incident to both \( f \) and \( c_1 \) (if \( c_1 \) is a face). As the root vertex \( v_f \) we choose the end vertex of \( e_f \) that lies in clockwise direction from \( e_f \) around \( f \).

**Theorem 7.3.** The triangulations \( T \) with rotative symmetries in \( \text{Aut}(c_0, T) \) are precisely the ones that can be constructed by choosing

- a fyke net \( F \),

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• for every isomorphism class \([f]\) (under rotation) of leaves of \(F\) a strongly rooted near-triangulation whose boundary is isomorphic to the boundaries of those leaves with respect to the rooting,

• for every isomorphism class \([f]\) of pseudo-antarctic faces of \(F\) a strongly rooted near-triangulation whose boundary is isomorphic to the boundaries of those faces with respect to the rooting, and

• for every isomorphism class \(\{f^i_j, f^i_{j+1}, \ldots, f^i_{j+(m-1)h}\}\) of segments of \(F\) a 2-layered near-triangulation with respect to \(v^i_j, v^i_{j+1}, w^i_{j+1}, w^i_{j+1}\), and \(w^i_{j+1}\)

and inserting a copy of each near-triangulation into the corresponding faces of \(F\) at their root vertices and edges. This construction is a 1-1 correspondence.

7.3 Reflective and rotative symmetries

The graphs that can serve as a skeleton of a triangulation can be constructed as follows. Suppose that the number \(n\) of reflections is given. Then we can choose

• the number \(s\) of isomorphism classes of segments of the skeleton,

• the structure of the skeleton at the poles according to Lemma 6.5,

• the numbers \(k_1, \ldots, k_{s-1}\) from Lemma 6.6 and

• the distances of \(v^i_j\) and \(w^i_j\) on \(M_1\) and on \(M_2\) for every \(j = 1, \ldots, s\) as well as the number and distribution of diamonds on these meridians between this two vertices. For arbitrary \(i\), the structure of \(M_i\) at the boundaries of the segments is identical to that of \(M_1\) or \(M_2\), depending of the parity of \(i\), by Lemma 6.3(iii).

The near-triangulations that can be inserted into a segment are similar to those that can be inserted into a side of a girdle: if such a near-triangulation had a chord both of whose end vertices are central cells of the same meridian, then applying the reflection that corresponds to that meridian shows that there is a double edge, a contradiction. In other words, a chord is only allowed if its end vertices are not in the same meridian or if they are in the same meridian, but at least one of them is an outer vertex of that meridian.

Definition 7.3. Let \(N\) be a near-triangulation with root vertex \(v_N\) and root edge \(e_N\). Suppose that a vertex \(w_N \neq v_N\) is fixed, then the boundary of \(N\) is the union of two paths from \(v_N\) to \(w_N\); denote the path that contains \(e_N\) by \(R\) and the other path by \(L\). We call \(L\) and \(R\) the sides of the boundary. If vertex sets \(D_L\) and \(D_R\) on \(L\) and \(R\) are given, we call \(N\) 2-sided chordless outside \(D_L\) and \(D_R\) if every chord of \(N\) whose end vertices both lie on \(L\) or both lie on \(R\) has least one end vertex in \(D_L\) or in \(D_R\), respectively.

More generally, let a cycle \(C\) with a root vertex \(v_C\) and a root edge \(e_C\) incident with \(v_C\) be given. If \(w_C \neq v_C\) is given, let us define subpaths \(L_C\) and \(R_C\) as before. Suppose that \(D_{L_C}\) and \(D_{R_C}\) are sets of vertices on \(L_C\) and \(R_C\), respectively. If there is an isomorphism \(\alpha\) from \(C\) to the boundary of \(N\) that respects the rooting and maps \(w_C\) to \(w_N\), then we call \(N\) 2-sided chordless outside \(\alpha(D_{L_C})\) and \(\alpha(D_{R_C})\).
Consider a segment $f^0_j$ of the skeleton, let $C_j$ be its boundary. The two sides of $C_j$ are its intersections $R_j$ with $M_0$ and $L_j$ with $M_1$, the set $D_{R_j}$ (respectively $D_{L_j}$) is the set of outer vertices of $M_0$ on $R_j$ (respectively of $M_1$ on $L_j$). The near-triangulations that can be inserted into $f^0_j$ are precisely those that are 2-sided chordless outside $D_{L_j}$ and $D_{R_j}$. We thus have the following characterisation of triangulations with both reflective and rotative symmetries.

**Theorem 7.4.** The triangulations $T$ for which $\text{Aut}(c_0, T)$ has a subgroup $H$ isomorphic to $D_n$ are precisely those that can be constructed by choosing

- a skeleton $S_H$ and
- for every $j = 1, \ldots, s$, a near-triangulation $N_j$ that is 2-sided chordless outside $D_{L_j}$ and $D_{R_j}$,

and inserting a copy of $N_j$ into $f^0_j$ at $v^i_j$ and $e^i_j$ for every $j = 1, \ldots, s$ and $i = 0, \ldots, 2n - 1$.

**Remark 7.5.** The construction from Theorem 7.4 is a 1-2 correspondence. Indeed, the skeleton $S$ with respect to $\text{Aut}(c_0, T)$ is unique up to the enumeration of its meridians. By Lemma 6.3(iii), the $|\text{Aut}(c_0, T)|$ many choices for the enumeration result in two different decompositions. Since the choice of $S_H$ only depends on which meridian of $S$ is chosen as $M_0$ for $S_H$, we have shown that all triangulations $T$ with $\text{Aut}(T, c_0) \supseteq H \cong D_n$ can be constructed in precisely two different ways.

8 Discussion and outlook

The constructive decomposition presented in this paper is the key to enumerate triangulations with specific symmetries (and to sample them uniformly at random, based on a recursive method [17] or on Boltzmann sampler [5, 15]). For this end, it will be necessary to translate the decomposition into functional equations for the cycle index sums [30, 34] that enumerate these triangulations and the basic structures arising in their decomposition. This will be done in another paper [23].

In Section 7 we showed how to construct the basic structures of the decomposition: girdles, fyke nets, and skeletons. These constructions will be enough to determine their cycle index sums, but for a complete set of functional equations we still need to provide a construction for the different types of near-triangulations that are to be inserted into the faces of the basic structures. In [23], we will present such a construction, thus completing the constructive decomposition as well as its interpretation as functional equations.

Our final aim, however, is not to enumerate triangulations, but cubic planar graphs. This can be achieved along the following lines. From the enumeration of triangulations, we can obtain an enumeration of their duals: cubic planar maps. More precisely, since the triangulations considered in this paper are simple, their duals are precisely the 3-connected cubic planar maps. From 3-connected cubic planar maps, we can go to 3-connected cubic planar graphs, since by Whitney’s Theorem, every such graph has a unique embedding up to orientation. However, the correspondence between maps and graphs implied by Whitney’s Theorem is not a 2-1 correspondence. Indeed, if a 3-connected cubic
planar graph has a reflective symmetry, then it has only one embedding. Since we distinguished triangulations with reflective symmetries and triangulations without reflective symmetries, we will be able to obtain relations between graphs and maps separately for each of the two cases, thus resulting in an enumeration of all 3-connected cubic planar graphs. Using the grammar developed in [9], we will then obtain an enumeration of all cubic planar graphs.

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