LEBESGUE-RIESZ NORM ESTIMATES
FOR FRACTIONAL LAPLACE TRANSFORM

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Abstract.
We obtain in this short article the bilateral non-asymptotic estimations for the norm in Lebesgue-Riesz and bilateral Grand Lebesgue spaces of the so-called fractional Laplace integral transform.
We give also examples to show the sharpness of these inequalities.

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1. Introduction. Notations. Statement of problem.

Let $f : (0, \infty) \to \mathbb{R}$ be measurable function and $\kappa, r$ be constant real numbers. The following linear operator (transform) $L_{\kappa,r}[f](s)$ is named in the article [23] as a "Fractional Laplace Transform (FLT)"

\[ L_{\kappa,r}[f](s) \overset{\text{def}}{=} \int_{0}^{\infty} (1 + st/\kappa)^{-\kappa-r} f(t) \, dt, \quad s \geq 0. \quad (1.1) \]

The classical Laplace transform $L[f](s)$ may be obtained formally as a limit

\[ L[f](s) = \lim_{\kappa \to \infty} L_{\kappa,r}[f](s) = \int_{0}^{\infty} e^{-st} f(t) \, dt. \quad (1.2) \]

The applications of these transform in the statistical mechanics, differential equations, geophysics etc. are described in the articles [20], [21], [22], [23].
Many properties of FLT: inversion, convolution identity, transform of derivatives etc. are investigated in the article [23].

We intent to obtain in this report the $L_p \to L_q$ operator norm estimates for the fractional Laplace operator of the form

\[ | L_{\kappa,r}[f](\cdot) |_q \leq K_{\kappa,r}(p, q) \, |f|_p, \quad p, q \geq 1. \quad (1.3) \]
Hereafter

$$|f|_p := \left[ \int_0^\infty |f(t)|^p \, dt \right]^{1/p};$$

the weight case will be considered further.

We agree to take as a capacity for the number $K_{\kappa,r}(p,q)$ its minimal value:

$$K_{\kappa,r}(p,q) \overset{\text{def}}{=} \sup_{f:|f|_p \in (0,\infty)} \left[ \frac{|L_{\kappa,r}[f](\cdot)|}{|f|_p} \right]. \quad (1.4)$$

The estimates of a form (1.3) for "pure" Laplace transform is described in [16], [18], [12], [14].

We use symbols $C(X,Y)$, $C(p,q;\psi)$, etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like $C_1(X,Y)$ and $C_2(X,Y)$.

The relation $g(\cdot) \sim h(\cdot)$, $p \in (A,B)$, where $g = g(p)$, $h = h(p)$, $g, h : (A,B) \to \mathbb{R}_+$, denotes as usually

$$0 < \inf_{p \in (A,B)} h(p)/g(p) \leq \sup_{p \in (A,B)} h(p)/g(p) < \infty.$$ 

The symbol $\sim$ will denote usual equivalence in the limit sense.

We will denote as ordinary the indicator function

$$I(x \in A) = 1, x \in A, \quad I(x \in A) = 0, x \notin A;$$

here $A$ is a measurable set.

All the passing to the limit in this article may be grounded by means of Lebesgue dominated convergence theorem.

2. Main result: upper estimations for fractional Laplace operator

Some notations. Let $p = \text{const} \geq 1$; we denote as ordinary by $p'$ its conjugate number: $p' = p/(p-1)$, where $1' := +\infty$.

We impose the following restriction on the values $(p,q)$ in the inequality (1.3):

$$q = p' = p/(p-1) \iff \frac{1}{p} + \frac{1}{q} = 1. \quad (2.1)$$

Denote also for the values $(\kappa,r)$ : $\kappa + r > 1/2$

$$v(-1/2, \kappa + r) = \int_0^{\infty} \frac{x^{-1/2}}{(1 + x)\kappa+r} \, dx = \sqrt{\pi} \cdot \frac{\Gamma(\kappa + r - 1/2)}{\Gamma(\kappa + r)}, \quad (2.2)$$

$$w(\kappa,r) = \sqrt{\kappa} \cdot v(-1/2, \kappa + r) = \sqrt{\pi} \cdot \frac{\Gamma(\kappa + r - 1/2)}{\Gamma(\kappa + r)},$$

$$z_{\kappa,r}(p) = w(\kappa,r)^{2/p'} = w(\kappa,r)^{2/q}. \quad (2.3)$$

**Theorem 2.1.** Suppose in addition
\[ p \in [1, 2], \ \kappa + r > 1/2. \]  
(2.4)

Then we deduce the following upper estimate:

\[ K_{\kappa,r}(p,q) \leq z_{\kappa,r}(p), \]  
(2.5a)

or, in detail:

\[
|L_{\kappa,r}[f]|_q = |L_{\kappa,r}[f]|_{p'} \leq \left[\sqrt{\pi \kappa} \cdot \frac{\Gamma(\kappa + r - 1/2)}{\Gamma(\kappa + r)}\right]^{2/p'} \cdot |f|_p.
\]  
(2.5b)

**Proof** is at the same as one for the Laplace transform, see, e.g. [18], p. 341-350. Namely, let us consider the following linear operator:

\[
L_0 V_p[f](s) = \int_0^{\infty} t^2 s^{-2} h(s/t) f(t) \, dt,
\]

\[
h(x) = h_{\kappa,r}(x) = (1 + x/\kappa)^{-\kappa - r}.
\]

It is proved in fact in [18], p. 346

\[
|L_{\kappa,r}[f]|_q = |L_0 V_p[f]|_q \leq \left[\int_0^{\infty} t^{-1/2} h(t) \, dt\right]^{2/p'} \cdot |f|_p = z_{\kappa,r}(p) \cdot |f|_p,
\]

Q.E.D.

**Remark 2.1.** Note that we derive as \( \kappa \to \infty \) from the inequality (2.5b) the \( L_p - L_q \) estimation for "pure" Laplace transform.

3. **Main result: inverse estimations for fractional Laplace operator**

**A. Necessity of our conditions.**

The necessity of restrictions (2.4) is proved in [18], p. 220-222; we need to prove the relation (2.1).

**Theorem 3.1.** Suppose for some values \((p,q), \ p, q > 0\) there exists a finite constant (function) \( K = K_{\kappa,r}(p,q) \) such that the inequality (1.3) is satisfied for arbitrary function \( f \) from the Schwartz class \( C_0^\infty(R_+) \). Then the relation (2.1) there holds.

**Proof.** We will apply the well known scaling method; see, e.g. [15], p. 285; [19], [12], [13], [14]. Indeed, let \( f \neq 0 \) be any function from the set \( C_0^\infty(R_+) \), then

\[
|y|_q \leq K_{\kappa,r}(p,q) |f|_p, \ y := L_{\kappa,r}[f].
\]  
(3.1)

Let \( \lambda \) be arbitrary positive number. Introduce the so-called dilation operator, more exactly, the family of operators \( T_\lambda : \)

\[
T_\lambda[f](x) = f(\lambda \cdot x).
\]
Obviously, $T_\lambda[f] \in C^\infty_0(R_+)$. We derive substituting into (3.1) the function $T_\lambda[f]$ instead the initial function $f$:

$$|y_\lambda|_q \leq K_{\kappa,r}(p,q) \mid T_\lambda[f] \mid_p, \; y := L_{\kappa,r}[T_\lambda f].$$

(3.2)

But

$$|T_\lambda f|_p = \lambda^{-1/p} |f|_p, \quad |y_\lambda|_q = \lambda^{-1+1/q} |L_{\kappa,r}[f]|_q \leq \lambda^{-1+1/q} K|f|_p,$$

Therefore

$$\lambda^{-1+1/q} K|f|_p \geq \lambda^{-1/p} |f|_p.$$  

(3.3)

Since the number $\lambda$ is arbitrary positive, we deduce from (3.3)

$$-1 + 1/q = -1/p, \iff 1/p + 1/q = 1.$$

**Remark 3.1.** The proposition of theorem 3.1 remains true, with at the same proof, for the more general integral transforms of the form

$$y(s) = \int_R h(s \cdot t) f(t) \, dt.$$  

**B. Lower estimates.**

Note that the unique critical point of the values $p$ is the value $p = 2 - 0$. We can restrict ourselves the values $p$ by the set $1 \leq p < 2$.

Let us choose as a capacity of trial function

$$f_0(x) := x^{-1/2} I_{(0,1)}(x),$$

then

$$|f_0|^p_p = \int_0^1 x^{-p/2} dx = \frac{2}{2 - p}; \quad |f_0|_p = \left[ \frac{2}{2 - p} \right]^{1/p} \asymp \left[ \frac{2}{2 - p} \right]^{1/2};$$

$$g_0(s) := L_{\kappa,r}[f](s) = \int_0^1 (1 + sx/\kappa)^{-\kappa-r} \frac{dx}{\sqrt{x}} = s^{-1/2} \int_0^s (1 + z/\kappa)^{-\kappa-r} \frac{dz}{\sqrt{z}}$$

and we have as $s \to \infty$:

$$g_0(s) \sim s^{-1/2} \int_0^\infty (1 + z/\kappa)^{-\kappa-r} \frac{dz}{\sqrt{z}} = \sqrt{\kappa/s} \psi(-1/2, \kappa + r) = \sqrt{\frac{\kappa}{s}} \cdot \frac{\Gamma(\kappa + r - 1/2)}{\Gamma(\kappa + r)}.$$

Therefore, we have as $p \to 2 - 0$ or equally as $q \to 2 + 0$:

$$|L_{\kappa,r}[f_0]|_q = |L_{\kappa,r}[f_0]|_{p'} \sim \left[ \sqrt{\frac{\kappa}{\pi}} \cdot \frac{\Gamma(\kappa + r - 1/2)}{\Gamma(\kappa + r)} \right] \cdot |f_0|_p,$$

which coincides with upper estimate (2.5b), of course when $p \to 2 - 0$. 


We now give a more accurate lower bound holds for all the values $p$, $p \in (1, 2)$. Closer look closely to the function $g_0(s)$. Consider only the case $s \geq 1$, suppose in addition

$$\kappa > 0, \quad \kappa + r > 1$$

and introduce a new variable

$$Y = Y(\kappa, r) = \frac{\kappa}{\kappa + r - 1} \left[ 1 - \left( \frac{\kappa}{\kappa + 1} \right)^{\kappa + r - 1} \right];$$

(3.5)

$$\int_0^s (1 + z/\kappa)^{-\kappa - r} \frac{dz}{\sqrt{z}} \geq \int_0^1 \frac{dz}{\sqrt{z} \left( 1 + z/\kappa \right)^{\kappa + r}} \geq \int_0^1 \frac{dz}{(1 + z/\kappa)^{\kappa + r}} = Y(\kappa, r);$$

hence

$$|g_0|_q \geq Y^q \cdot \int_1^\infty |g_0(s)|^q \, ds \geq Y^q(\kappa, r) \cdot \frac{2}{q - 2};$$

$$|g_0|_q : |f_0|_p \geq (p - 1)^{1-1/p} \cdot (1 - p/2)^{2/p - 1} \cdot Y(\kappa, r).$$

Thus, we proved in fact that under additional conditions (3.5)

$$K_{\kappa, r}(p, q) \geq (p - 1)^{1-1/p} \cdot (1 - p/2)^{2/p - 1} \cdot Y(\kappa, r) = (p - 1)^{1-1/p} \cdot (1 - p/2)^{2/p - 1} \cdot \frac{\kappa}{\kappa + r - 1} \left[ 1 - \left( \frac{\kappa}{\kappa + 1} \right)^{\kappa + r - 1} \right].$$

(3.7)

Obviously, this result remains true even in the extremal cases $p = 1$ and $p = 2$.

4. Weight estimates for FLT

Let us consider in this section the action of the fractional Laplace transform operator $L_{\kappa, r}$ on the weight function $f_\mu(t) = t^{\mu - 1} f(t)$, where $f(\cdot) \in L_p(R_+)$:

$$\Psi[f](s) = \Psi_{\mu, \kappa, r}[f](s) := \int_0^\infty t^{\mu - 1} \left( 1 + ts/\kappa \right)^{-\kappa - r} f(t) \, dt = L_{\kappa, r}[f_\mu](s).$$

(4.1)

Notations and restrictions: $1/\sigma := 1 + \mu - \frac{2}{p}$, $p \leq Q$,

$$\frac{1}{Q} = \mu - \frac{1}{p}, \quad p' = p/(p - 1),$$

(4.2)

$$0 < \mu < 1, \quad \sigma < p' \Rightarrow 1/\mu < p \leq 2/\mu, \quad (\kappa + r) \cdot \sigma + \frac{\sigma}{p'} > 1,$$

(4.3)

$$\Theta(p) = \Theta_{\mu, \kappa, r}(p) := \left[ \kappa^{1-\sigma/p} \, B(1 - \sigma/p', \ (\kappa + r) \cdot \sigma + \sigma/p' - 1) \right]^{1/\sigma},$$

(4.4)

where $B(\alpha, \beta)$ denotes the ordinary Beta function.

Notice that the relation (4.2) is necessary for the inequality of a form
\[ \Psi_{\mu,\kappa,r}[f] |Q \leq K_{\mu,\kappa,r}^\nu(p,Q) \cdot |f|_p, \]

where we accept as before

\[ K_{\mu,\kappa,r}^\nu(p,Q) = \sup_{0 < |f|_p < \infty} \left[ \frac{|\Psi_{\mu,\kappa,r}[f]|_Q}{|f|_p} \right]. \]

This proposition may be proved by means of scaling method.

**Theorem 4.1.**

\[ | \Psi_{\mu,\kappa,r}[f] |Q \leq \Theta(p) \cdot |f|_p, \tag{4.5} \]

or equally

\[ K_{\mu,\kappa,r}^\nu(p,Q) \leq \Theta_{\mu,\kappa,r}(p). \tag{4.5a} \]

**Proof** follows immediately after simple calculations from [18], p. 220-222.

Consider the following linear operator

\[ U_{\sigma,\mu}[f](x) := \int_R |t|^{\mu-1} h(x \cdot t) f(t) \, dt, \tag{4.6} \]

where

\[ 1 \leq p \leq Q; \quad \frac{1}{Q} = \mu - \frac{1}{p}, \quad \frac{1}{\sigma} = 1 + \mu - \frac{2}{p}, \quad t, x \in R. \tag{4.6a} \]

Denote

\[ M(p) = \left[ \int_R |t|^{-\sigma(p-1)/p} |h(t)|^\sigma \, dt \right]^{1/\sigma}. \tag{4.7} \]

It is proved in [18], p. 220-222 that

\[ |U_{\sigma,\mu}[f]|_Q \leq M(p) \cdot |f|_p. \tag{4.8} \]

It remains to calculate the integral \( M(p) \) in (4.7), in which we substitute

\[ h(x) = (1 + |x|/\kappa)^{-\kappa-r}, \quad f(x) = f(|x|). \]

We now turn to the conclusion of the lower bound for the value \( K_{\mu,\kappa,r}^\nu(p,Q) \). Note that the unique critical value of the parameter \( p \) is the value \( p_0 = 2/\mu - 0 \) and correspondingly \( Q = 2/\mu + 0 \). We therefore consider the following trial function

\[ f_0(x) = x^{-\mu/2} \cdot I_{(0,1)}(x); \]

then

\[ |f_0|_p = \int_0^1 x^{-\mu/2} \, dx = \frac{2}{2-\mu}, \quad |f_0|_p = \left[ \frac{2}{2-\mu} \right]^{1/p}. \]

Further, let us denote for the values \( s \geq 1 \) only
\[ g_0(s) = \Psi[f_0](s) = \int_0^1 t^{\mu/2-1} (1 + ts/\kappa)^{-\kappa-r} \, dt = \]
\[ s^{-\mu/2} \int_0^s z^{\mu/2-1} (1 + z/\kappa)^{-\kappa-r} \, dz; \]
then
\[ g_0(s) \geq s^{-\mu/2} \int_0^1 (1 + z/\kappa)^{-\kappa-r} \, dz = s^{-\mu/2} Y(\kappa, r); \]
\[ |g_0|_Q \geq Y(\kappa, r) \cdot \left[ \frac{2}{\mu Q - 2} \right]^{1/Q}; \]
\[ |g_0|_Q : |f_0|_p \geq Y(\kappa, r) \cdot (1 - \mu p/2)^{2/p - \mu} \cdot (\mu p - 1)^{\mu - 1/p}. \]
Thus,
\[ K^{(\mu)}(p, Q) \geq Y(\kappa, r) \cdot (1 - \mu p/2)^{2/p - \mu} \cdot (\mu p - 1)^{\mu - 1/p}, \quad 1/\mu \leq p \leq 2/\mu. \quad (4.9) \]

5. Generalization on the Grand Lebesgue Spaces (GLS).

We recall first of all here for reader conventions some definitions and facts from the theory of GLS spaces.

Recently, see [1], [2], [3], [4], [5], [6], [7], [8], [10], [11] etc. appear the so-called Grand Lebesgue Spaces (GLS)
\[ G(\psi) = G = G(\psi; A; B); \quad A; B = \text{const}; \quad A \geq 1, \quad B \leq \infty \]
spaces consisting on all the measurable functions \( f : X \to R \) with finite norms
\[ ||f||_{G(\psi)} \stackrel{\text{def}}{=} \sup_{p \in (A; B)} \left[ \frac{|f|_p}{\psi(p)} \right]. \quad (5.1) \]

Here \( \psi = \psi(p), \quad p \in (A, B) \) is some continuous positive on the open interval \( (A; B) \) function such that
\[ \inf_{p \in (A; B)} \psi(p) > 0. \quad (5.2) \]

We will denote
\[ \text{supp}(\psi) \stackrel{\text{def}}{=} (A; B). \]

The set of all such a functions with support \( \text{supp}(\psi) = (A; B) \) will be denoted by \( \Psi(A; B) \).

This spaces are rearrangement invariant; and are used, for example, in the theory of Probability, theory of Partial Differential Equations, Functional Analysis, theory of Fourier series, Martingales, Mathematical Statistics, theory of Approximation etc.

Notice that the classical Lebesgue - Riesz spaces \( L_p \) are extremal case of Grand Lebesgue Spaces, see [11], [12].

Let a function \( \xi : R_+ \to R \) be such that
\[ \exists(A, B) : \quad 1 \leq A < B \leq \infty \Rightarrow \forall p \in (A, B) \ |f|_p < \infty. \]
Then the function $\psi = \psi_\xi(p)$ may be naturally defined by the following way:

$$\psi_\xi(p) := |\xi|_p, \ p \in (A, B). \ (5.3)$$

Let now the (measurable) function $f : R_+ \to R, f \in G\psi$ for some $\psi(\cdot)$ with support $\text{supp} \psi = (A, B)$ for which

$$(a, b) := (A, B) \cap (1, 2) \neq \emptyset. \ (5.4)$$

We define the function

$$\lambda(q) = \frac{q}{q-1}, \ q \in (b', a')$$

and introduce a new $\psi$ - function, say $\nu = \nu_z = \nu_z(q)$ as follows.

$$\nu_z(q) = z_{\kappa,r}(\lambda(q)) \cdot \psi(\lambda(q)), \ q \in (b', a'). \ (5.5)$$

**Theorem 5.1.** We assert under condition (5.4):

$$\|L_{\kappa,r}[f]\|G\nu \leq 1 \cdot \|f\|G\psi_{a,b}, \ (5.6)$$

where the constant "1" is the best possible.

**Proof. Upper bound.**

Let further in this section $p \in (a, b)$. We can and will suppose without loss of generality $\|f\|G\psi_{a,b} = 1$. Then

$$|f|_p \leq \psi_{a,b}(p), \ p \in (a, b). \ (5.7)$$

We conclude after substituting into the proposition of theorem 2.1

$$|L_{\kappa,r}[f]|_q \leq K_{\kappa,r}(\lambda(q) \cdot \psi_{a,b}(\lambda(q))) = \nu_z(q), \ q \in (b', a'). \ (5.8)$$

The inequality (5.6) follows from (5.8) after substitution $p = \lambda(q)$.

**Proof. Exactness.**

The exactness of the constant "1" in the proposition (5.8) follows immediately from the theorem 2.1 in the article [12].

6. Concluding remarks

A. The majority of obtained results may be obtained for the more general operators of the form

$$T_{K}[f](x) = \int_{0}^{\infty} K(x \cdot y) f(y) \, dy, \ x \geq 0, \ (6.1)$$

or more generally

$$T_{K}^{(\mu-1)}[f](x) = \int_{0}^{\infty} x^{\mu-1} K(x \cdot y) f(y) \, dy. \ (6.1a)$$
For instance, assume \( K(z) \geq 0 \) and define the Mellin’s transform of the kernel \( K(\cdot) \):
\[
\zeta(s) = \int_0^\infty x^{s-1} K(x) \, dx.
\]

Then
\[
T_K[f]^p_p \leq \zeta(1/p) \int_0^\infty (x|f(x)|)^p \frac{dx}{x^2},
\]
\[
\int_0^\infty x^{p-2} |T_K[f](x)|^p \, dx \leq \zeta^p(1 - 1/p) |f|^p_p,
\]
see [16], p. 256; [9], p.146.

Another approach. Let \( p, q = \text{const} \geq 1 \), \( 1/p + 1/q = 1 \). We derive using Hölder’s inequality:
\[
|T_K[f](x)| \leq \left[ \int_0^\infty |K(xy)|^q \, dy \right]^{1/q} \cdot |f|_p = x^{-1/q} \cdot |K|_q \cdot |f|_p,
\]
or equally
\[
\sup_{x > 0} \left[ x^{1/q} |T_K[f](x)| \right] \leq |K|_q \cdot |f|_p, \quad (6.2)
\]
if of course \( K(\cdot) \in L_q \). We do not suppose in (6.2) the non-negativity of the function \( K(\cdot) \).

**B.** Analogously may be investigated the so-called ”multivariate” case:
\[
T_K[f](\vec{x}) = \int_{R^d} K(\vec{x} \circ \vec{y}) f(\vec{y}) \, d\vec{y}, \; x, y \in R^d,
\]
where
\[
\vec{x} \circ \vec{y} = \{x_1y_1, x_2y_2, \ldots, x_dy_d\}.
\]

Needed here \( L_q(R^d) - L_p(R^d) \) estimations for operator \( T_K[\cdot] \) may be found in the book of G.O.Okikiolu [18], p. 222-224, 341-345.

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