When Shannon and Khinchin meet Shore and Johnson: equivalence of information theory and statistical inference axiomatics

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We propose a unified framework for both Shannon–Khinchin and Shore–Johnson axiomatic systems. We do it by rephrasing Shannon–Khinchine axioms in terms of generalized arithmetics of Kolmogorov and Nagumo. We prove that the two axiomatic schemes yield identical classes of entropic functionals — Uffink class of entropies. This allows to re-establish the entropic parallelism between information theory and statistical inference that has seemed to be “broken” by the use of non-Shannonian entropies.

PACS numbers: 05.20.-y, 02.50.Tt, 89.70.Cf

I. INTRODUCTION

Entropy is undoubtedly one of the most important concepts in physics, information theory and statistics \cite{1}. The notion of entropy was originally developed by Clausius, Boltzmann, Gibbs, Carathéodory and others in the context of statistical thermodynamics. There it supplemented a new state function that was naturally extensive (due to its connection with the heat 1-form) and in any adiabatically isolated system it represented a non-decreasing function of its state variables (on account of the Clausius theorem). Roughly a half-century after these developments, the entropy paradigm was further conceptualized in information theory by Shannon \cite{2}. In this later context the ensuing entropy (Shannon’s entropy or measure of information) quantitatively represented the minimal number of binary (yes/no) questions which brings us from our present state of knowledge about a given system to the one of certainty. The higher is the measure of information (more questions to be asked) the higher is the ignorance about the system and thus more information will be uncovered after an actual measurement. A proper axiomatization of Shannon’s entropy is encapsulated in the so-called Shannon–Khinchin (SK) axioms \cite{3}. Only one decade after Shannon’s seminal works, Jaynes \cite{4, 5} promoted Shannon’s information measure to the level of inference functional that was able to extract least biased probability distributions from measured data. This procedure is also known as Maximum entropy principle (MEP). Since MEP is, in its essence, a statistical inference method, it needs a proper mathematical qualification to place Jaynes’ heuristic arguments in a sound mathematical framework. The corresponding mathematical qualification was provided by Shore and Johnson (SJ) in the form of axioms that ensure that the MEP estimation procedure is consistent with desired properties of inference methods \cite{6, 7}. At this point one should emphasize that in the statistical inference theory (SIT) entropy functionals serve only as convenient technical vehicles for unbiased assignment of distributions that are compatible with given constraints. In fact, one might say that it is the MEP distribution that is the primary object in SIT while the entropy itself is merely secondary (not having any operational role in the scheme). This is very different from the information theory or thermodynamics where entropies are primary objects with firm operational meanings (given, e.g., in terms of coding theorems or calorimetric measurements). In the original paper \cite{6, 7} Shore and Johnson concluded that their axioms yield only one “measure of bias”, namely Shannon entropy. It might, however, seem a bit puzzling why “measure of bias” should have anything to do with additivity (i.e., one of the defining properties of Shannon’s entropy). In the end, any monotonic function of such a measure should provide the same MEP distribution but might (and as a rule it does) yield non-additive entropy. So, it is perhaps not so surprising that with the advent of generalized entropies \cite{8–16}, the past two decades have seen a renewed interest both in the SJ axiomatics and the associated classes of admissible entropies \cite{17–22}. In particular, in Ref. \cite{21}, it has been shown that the SJ axiomatization of the inference rule does account for substantially wider class of entropic functionals than just SE — the so-called Uffink’s class \cite{22}, which includes Shannon’s entropy as a special case.

The main aim of this paper is to answer the following question: what generalization of the SK axioms would provide the Uffink class of entropic functional. This would not only allow to re-establish the “broken” entropic parallelism between information theory and statistical inference but it should also cast a new light on the Uffink class of entropies and its practical utility.

Let us first recall the original set of SK axioms \cite{3}:

**SK1 Continuity:** Entropy is a continuous function w.r.t.
to probability distribution.

**SK2 Maximality**: Entropy is maximal for uniform distribution.

**SK3 Expandability**: Adding an event with probability zero does not change the entropy.

**SK4S Shannon additivity**: $\mathcal{H}(A \cup B) = \mathcal{H}(A|B) + \mathcal{H}(B) = \mathcal{H}(B|A) + \mathcal{H}(A)$, where $\mathcal{H}(A|B) = \sum_i p_i^A \mathcal{H}(B|A = a_i)$.

We note that the conditional entropy $\mathcal{H}(B|A)$ can be calculated in two ways: (i) from the entropy of the joint distribution of $A \cup B$ and marginal distribution of $A$, or (ii) from the marginal distribution $A$ and entropy of the conditional random variable $B|A = a_i$. This duality is crucial for the internal consistency the SK axiomatic scheme [3]. Aforesaid set of SK axioms has the unique solution — Shannon’s entropy [42]

$$\mathcal{H}(p) = -\sum_i p_i \log p_i \quad (1)$$

With the advent of generalized entropies [8–16] there arose two natural questions. First, is it possible to conceptualize such entropies in terms of information-theoretic axioms (à la SK axioms)? And second, can generalized entropies be used as consistent inference functionals with sound mathematical underpinning (à la SJ axioms)? As for the first question, it is well known that one can “judiciously” generalize the additivity axiom SK4S to produce various generalized entropies. Typical examples are provided by Rényi and Tsallis–Havrda–Charvát (THC) entropies. For instance, for the Rényi entropy, one keeps axioms SK1-3 and substitute SK4S with [8]

**SK4R Rényi additivity**: $\mathcal{R}_q(A \cup B) = \mathcal{R}_q(A|B) + \mathcal{R}_q(B) = \mathcal{R}_q(B|A) + \mathcal{R}_q(A)$, where $\mathcal{R}_q(A|B) = f^{-1}(\sum_i p_i^A f(\mathcal{R}_q(B|A = a_i)))$.

Here, $\rho^A(q) = (p^A)^q / \sum_j (p_j^A)^q$ is the escort (or zooming) distribution [30] and $f$ is an arbitrary invertible and positive function on $[0, \infty)$. Corresponding axiomatics is stringent enough to fix uniquely $f(x)$ to be either $f(x) = e^{(1-q)x}$ (for $q \neq 1$) or $f(x) = x$ (for $q = 1$), and it yields the Rényi entropy

$$\mathcal{R}_q(P) = \log \frac{\sum_i p_i^A}{1 - q}, \quad (2)$$

as the unique solution.

Similarly, for the case of non-additive THC entropy [9, 10] one can augment axioms SK1-3 with [23, 24]

**SK4T Tsallis additivity**: $S_q(A \cup B) = S_q(B|A) + S_q(A) + (1 - q)S_q(B|A)S_q(A)$ where $S_q(B|A) = \sum_i p_i^A \mathcal{S}_q(B|A = a_i)$, where $\mathcal{S}_q(A|B) = (\sum_i p_i^A)^q / \sum_j (p_j^A)^q$.

where $\rho^A(q)$ is again the escort distribution. The unique solution of this axiomatic system gives the THC entropy

$$S_q(P) = \frac{\sum_i p_i^q - 1}{1 - q}. \quad (3)$$

In parallel with this there has been several successful attempts to classify entropic functionals according to various desirable information-theoretic properties. Here we should mention, e.g., the class of strongly pseudo-additive entropies (SPA) based on generalization of Rényi entropy axioms for non-additive entropies [22]. Z-entropies based on group properties of the entropic functionals [27] or classification according to the asymptotic scaling leading to $(c,d)$-entropies [12] and ensuing generalizations [28].

As for the second question, there has been notable progress in recent years in classifying entropic functionals conformant with SJ axioms [21, 22, 24]. Our aim here is to employ generic arithmetical principles to generalize, in a logically sound way, the SK axiomatic scheme. To this end we will use the framework of Kolmogorov–Nagumo (KN) arithmetics [30, 31], KN quasi-arithmetic means [32–34] and escort distributions [36]. Ensuing class of admissible entropies will be compared with the class of entropies solving SJ axioms — Uffink’s class. We will see that both classes not only coincide, and hence bolster the entropic parallelism between information theory and statistical inference, but there also is a close parallelism between the two axiomatic schemes.

The rest of the paper is organized as follows: In Section II, we briefly summarize the concept of generalized arithmetics and outline the key role that Kolmogorov–Nagumo functions play in such a context. In Section III, we introduce the class of Shannon–Khinchin axioms based on the Kolmogorov–Nagumo generalized arithmetics and derive the generic class of entropic functionals satisfying these axioms. In Section IV, we show the equivalence of the aforementioned class and the Uffink’s entropic class. This will, in turn, cast new light on the relationship between SK and SJ axiomatic schemes, and re-establish the entropic parallelism between information theory and statistical inference. The last section is devoted to some further observations, remarks and conclusions.

**II. GENERALIZED ARITHMETICS AND KOLMOGOROV AND NAGUMO FUNCTIONS**

Let us now introduce the concept of generalized arithmetics. From abstract algebra it is known that arithmetic operations can be defined in various ways, even if one assumes commutativity and associativity of addition and multiplication, and distributivity of multiplication with respect to addition [30, 31]. In consequence, whenever one encounters “plus” or “times” one has certain flexibility in interpreting these operations. A change of realization of arithmetic, without altering the remaining
structures of equations involved, plays an analogous role as a symmetry transformation in physics.

Let us considering a bijection \( f: M \rightarrow N \subset \mathbb{R} \), where \( M \) and \( N \) are some sets. The map \( f \) allows us to define addition, subtraction, multiplication, and division in \( M \), as follows

\[
\begin{align*}
x \oplus y &= f(f^{-1}(x) + f^{-1}(y)), \\
x \odot y &= f(f^{-1}(x) - f^{-1}(y)), \\
x \oslash y &= f(f^{-1}(x)f^{-1}(y)), \\
x \odot y &= f(f^{-1}(x)/f^{-1}(y)).
\end{align*}
\]

One can readily verify the following standard properties:

(1) associativity \((x \oplus y) \oplus z = x \oplus (y \oplus z)\), \((x \odot y) \odot z = x \odot (y \odot z)\), \((x \odot y) \odot z = (x \odot z) \odot (y \odot z)\).

For a future convenience we will explicitly affiliate with the arithmetic operations \(\oplus, \odot, \oslash\) and \(\odot\) the symbol of the function \(f\), so for instance, we will write \(\oplus_f\) instead of \(\oplus\), etc.

This generalized arithmetical structure motivated Kolmogorov and Nagumo \cite{32} to formulate the most general class of means, so-called quasi-linear means, that are fully compatible with the usual Kolmogorov postulates of probability theory \cite{32}.

The aforementioned generalized arithmetics can be extended to real multivariate functions in a rather natural way. For instance, for a function of two variables \(G(x, y)\) it is defined as

\[
G_f(x, y) \equiv f(G(f^{-1}(x), f^{-1}(y))).
\]

Let us state in this connection a couple of important consequences that can be easily verified: \(i)\) when \(z = x \oplus_f y,\) then \(g(z) = g(x) \odot_f g(y),\) \(ii)\) \(x \odot_f y = x \odot_f \log y.\)

Here, by \(f \cdot g\) we implicitly mean the composition of two functions. Particularly important for our purposes will be the so-called \(q\)-deformed algebra where

\[
f(x) \equiv f_q(x) = \log_q x = \frac{x^{1-q} - 1}{(1-q)}.\]

Ensuring operation \(\otimes_{f_2}\) is traditionally denoted as \(q\)-addition and the notation \(\otimes_q\) is often used instead. \(iii)\) For the generalized product \(\otimes_f\) the function \(f\) is not determined uniquely. In fact, there exists a two-parametrical class of functions \(f_{a, b}\) so that \(f(x) \mapsto f_{a, b}(x) = f(ax^b)\), which yield the same product. Indeed,

\[
x \otimes_{f_{a, b}} y = f \left( a \left[ f^{-1}(x)/a \right]^{1/b} (f^{-1}(y)/a) \right)^{1/b} = x \otimes_f y.
\]

This result will be particularly important in Section III.

### III. KOLMOGOROV–NAGUMO GENERALIZATION OF SHANNON–KHINCHIN AXIOMS

Let us assume that we have a discrete set of events \(A = \{a_i\}_{i=1}^M\) and \(B = \{b_j\}_{j=1}^N\) with ensuing probabilities

\[
P_A = \{p_i\}_{i=1}^n \text{ and } P_B = \{q_j\}_{j=1}^m.\]

Corresponding joint and conditional probabilities are \(P_{A\cup B} = \{r_{ij}\}_{i,j=1}^{n,m}\) and \(P_{A\mid B} = \{r_{ij}/q_j\}_{i,j=1}^{n,m}\), respectively. So for instance, \(P_{A\mid B} = \{r_{ij}/q_j\}_{i,j=1}^{n,m}\), respectively. For this notation we can generalize the Shannon–Khinchin (SK) entropic axioms in terms of the Kolmogorov–Nagumo arithmetics in the following way:

**SK1** Continuity: Entropy is a continuous function w.r.t. to probability distribution.

**SK2** Maximaliy: Entropy is maximal for uniform distribution.

**SK3** Expandability: Adding an event with probability zero does not change the entropy.

**SK4** Composability: Entropy of a joined system \(A \cup B\) can be expressed as \(S(A \cup B) = S(A\mid B) \otimes_f S(B)\), where \(S(A\mid B)\) is conditional entropy satisfying consistency requirements I), II) (see below).

In passing, it should be observed that the two illustrative axiomatic schemes aforementioned before imply that one should require from that the entropic functionals should obey two natural properties:

I) For independent variables \(A, B\), the joint entropy \(S(A \cup B)\) should be composable from entropies \(S(A)\) and \(S(B)\), i.e., \(S(A \cup B) = F(S(A), S(B))\)

II) Conditional entropy should be decomposable into entropies of conditional distributions, i.e.,

\[
S(B\mid A) = G \left( P_{A}, \{S(B\mid A = a_i)\}_{i=1}^m \right).
\]

Here \(F\) and \(G\) are functional to be determined shortly.

Let us also note that the conditional entropy \(S(A\mid B)\) automatically fulfills several important properties:

a) **Entropic Bayes’ rule**: \(S(A\mid B) = S(B\mid A) \otimes_f S(B)\).

b) “2nd law of thermodynamics”: \(S(A\mid B) \leq S(A)\).

Moreover, we can define the mutual information as

\[
I(A, B) = S(A \cup B) \otimes_f (S(B) \otimes_f S(A)).
\]

The composition requirement I) is equivalent to \(I(A, B) = f(1)\) for independent events. We might note that the requirement I) is equivalent to strict composability axiom introduced in Ref. \cite{27}.

Let us now prove the following theorem:

**Theorem 1.** The most general class of entropic functionals \(S\) satisfying the aforesaid axioms SK1-4 can be expressed as

\[
S'_q(P) = f \left( \exp_q \sum_i p_i \log_q \left( \frac{1}{p_i} \right) \right),
\]

where \(f(x)\) is a generic monotonic, continuous function defined on \(x \in [0, \infty)\).
Proof: First, we show that the functional has to be symmetric in all components of \( P \). Since the relabeling of the events should not change the information \( \{A_i\} \) corresponding to \( \{p_i\} \), we get that \( S \) must be symmetric. Second, the entropy of the uniform distribution \( S(n) \equiv S(1/n, \ldots, 1/n) \) can be obtained from composability axiom. To this end we denote the random variable with uniform distribution as \( U_{nm} = U_n \cup U_m \). We abbreviate \( S(U_n) \) as \( S(n) \). Then [see Eq. \((12)\)]

\[
S(nm) = S(n) \otimes f S(m) \Rightarrow S(n) = f(n).
\]

Third, let us take two random events \( A \) and \( B \) with distributions \( p_i \) and \( q_i = 1/m \). Let us also introduce the so-called Daróczy mapping \([23, 24]\), i.e., \( S \rightarrow f^{-1} S \). After this mapping we get multiplicative entropy. From the definition of \( S(A|B) \) we we obtain that

\[
m f^{-1} S(p_1/m, \ldots, p_n/m) = f^{-1} S(p_1, \ldots, p_n),
\]

since the conditional entropy is for each event just the usual unconditional one. Therefore, entropy must be a first order homogeneous, symmetric function. According to \([37]\) the solution of homogeneous equation \((11)\) can be (under mildly restrictive assumptions) expressed as \( f^{-1} S \equiv F \) where

\[
F(x_1, \ldots, x_n) = b \prod_{i=1}^{n} x_i^{a_i} \quad \text{where} \quad \sum_i a_i = 1.
\]

Here \( a \) and \( b \) are constants to be specified later. However, this solution is not symmetric in its variables. This can be achieved by symmetrization of Eq. \((12)\) that can be then rewritten in the following form

\[
F(p_1, \ldots, p_n) = b \prod_{i=1}^{n} \left( \sum_{k_i} p_{i}^{a_{i,k}} \right).
\]

At this point we apply the Daróczy mapping in the following form

\[
f^{-1} S(p_1, \ldots, p_n) = b \prod_{i=1}^{n} \left( \sum_{k_i} p_{i}^{q_{i,k}} \right)^{c/(1-a_i)},
\]

which still keeps the entropy to be a homogeneous function of the first order. Note that this representation is also mentioned in \([38]\).

Let us now show that in order to fulfill the decomposability axiom \((11)\), only one \( a_i \) must be non-zero. To this end, we explicitly express \( f^{-1} S(A|B) \) as

\[
f^{-1} S(A|B) = \frac{\left( \sum_{k_1, l_1} (r_{k_1,l_1} | q_{l_1})^{a_{1}} \right)^{c/(1-a_1)}}{\left( \sum_{l_1} q_{l_1}^{a_{1}} \right)^{c/(1-a_1)}} \times \cdots \times \frac{\left( \sum_{k_n, l_n} (r_{k_n,l_n} | q_{l_n})^{a_{n}} \right)^{c/(1-a_n)}}{\left( \sum_{l_n} q_{l_n}^{a_{n}} \right)^{c/(1-a_n)}}.
\]

This can be more explicitly rewritten as

\[
f^{-1} S(A|B) = \left( \sum_{l_1} \rho^B_{l_1}(a_1) \sum_{k_1} (r_{k_1,l_1} | q_{l_1})^{a_{1}} \right)^{c/(1-a_1)} \times \cdots \times \left( \sum_{l_n} \rho^B_{l_n}(a_n) \sum_{k_n} (r_{k_n,l_n} | q_{l_n})^{a_{n}} \right)^{c/(1-a_n)},
\]

where \( \rho^B_{l_1}(a) = q_{l_1}^{a} / \sum q_{l_1}^{a} \) is the escort distribution \([36]\).

This expression is an unconditional entropy of the conditional distribution only if one \( a_j \) is non-zero and the rest is zero. With this we get that

\[
f^{-1} S(A|B) = \left( \sum_i \rho^B_{l_1}(a) \sum_k (r_{k_1,l_1} | q_{l_1})^{a} \right)^{1/(1-a)} = \left\{ \sum_i \rho^B_{l_1}(a) [S(A|B = b_i)]^{1-a} \right\}^{1/(1-a)}.
\]

Hence, Eq. \((15)\) boils down to

\[
f^{-1} S(p_1, \ldots, p_n) = b \left( \sum_k p_k^{a} \right)^{c/(1-a)} = \exp_q \left( \sum_k p_k \log_q \frac{1}{p_k} \right).
\]

which by Eq. \((7)\) is equivalent to \((9)\). This concludes the proof.

IV. EQUIVALENCE WITH SHORE-JOHNSON AXIOMS

Let us now turn our attention to MEP and corresponding consistency requirements. The MEP can be formulated in the following way \([4, 8, 9]\):

Theorem 2 (Maximum entropy principle). Given the set of linear constraints \( \sum_i p_i E_i = (E^{(k)}) \), the least biased estimate of the underlying distribution \( P = \{p_i\} \) is obtained from maximization of the entropic functional \( S(P) \) under normalization constraint and set of constraints \( (E^{(k)}) \), i.e., by maximizing the Lagrange functional

\[
S(P) - \alpha \sum_{i=1}^{N} p_i - \nu \sum_{k=1}^{N} \beta^{(k)} \sum_{i=1}^{N} p_i E^{(k)}.
\]

Shore and Johnson formulated the set of consistency requirements that the MEP should satisfy \([8, 9]\):

SJ1 Uniqueness: the result should be unique.

SJ2 Permutation invariance: the permutation of states should not matter.
**SJ3** *Subset independence*: It should not matter whether one treats disjoint subsets of system states in terms of separate conditional distributions or in terms of the full distribution.

**SJ4** *System independence*: It should not matter whether one accounts for independent constraints related to independent systems separately in terms of marginal distributions or in terms of full-system.

**SJ5** *Maximality*: In absence of any prior information, the uniform distribution should be the solution.

Let us now state without proof the theorem that provides the most general class of admissible entropic functionals consistent with aforesaid **SJ axioms**:  

**Theorem 3** (Uffink theorem). *The class of entropic functionals* $S$ *satisfying the axioms * **SJ1-5** *can be expressed as* 

$$ S_q^f(P) = f \left( \left( \sum_i p_i^q \right)^{1/(1-q)} \right), $$

for any $q > 0$ and for any strictly increasing function $f$.

A detailed proof can be found in Ref. [21]. Let us now discuss some salient results of the proof. First two axioms lead to the fact that the entropic functional is a symmetric functional in the probability components. Third axiom determines the function in the sum form, i.e., in the form $S(P) = f(\sum_k g(p_k))$, with $g$ being an arbitrary increasing concave function. The fourth axiom gives us the final form of the entropic functional (without specifying the range for $q$'s), and finally the fifth axiom guarantees that $q > 0$. Note that the class obtained from Theorem 1 and epitomized by Eq. 9 is the same as the class given by Eq. [21] from Uffink theorem, since $\sum p_i \log_q(1/p_i) = (\sum p_i^q - 1)/(1-q)$ and $\exp_q(y) = 1 + (1-q)y^{1/(1-q)}$. Therefore, we immediately see that in axiom **SK4** the requirement **II** (decomposability) corresponds to axiom **SJ3**, while requirement **I** (composability) corresponds to axiom **SJ4**. Moreover, the interpretation of $f$ and $q$ is now clear. The function $f$ determines the scaling of the entropy for uniform distribution (as it is independent of $q$), see also [28], while the parameter $q$ determines the correlations in the system through MaxEnt distribution, which can be expressed as (see Ref. [21])

$$ p_i = \frac{1}{Z_q} \exp_q (-\beta \Delta E_i), $$

$$ Z_q = \sum_i \exp_q (-\beta \Delta E_i), $$

$$ \beta = \frac{\sum_i p_i^q \log_q(1/p_i)}{q f'(Z_q) Z_q}, $$

where $\Delta E_i = E_i - \langle E \rangle$. As discussed, e.g., in [33], a monotonic function of an entropic functional gives the same MEP distribution and redefines only the Lagrange multipliers but does not change the actual form of the distribution. This can be interpreted as a sort of *gauge invariance* $S(P) \rightarrow f(S(P))$. Finally, let us mention that $q = 1$ corresponds to uncorrelated MEP distributions for disjoint systems, for which we get a stronger version of system independence axiom [21]:

**SJ4SSI** *Strong system independence*: Whenever two subsystems of a system are disjoint, we can treat the subsystems in terms of independent distributions.

The solution is then

$$ S_q^f(P) = f \left( \exp \left[ \sum_i p_i \log_q (1/p_i) \right] \right), $$

which is equivalent (through Daróczy mapping) to Shannon entropy — as expected. On the other hand, if we require that the entropy must be in the trace form [13, 27], i.e., $S(P) = \sum q(p_i)$, then we get that $f(x) = \log_q(x)$ and we end up with the class of THC entropies

$$ S_{\log_q}(P) = \sum p_i \log_q (1/p_i). $$

**V. CONCLUSIONS**

In this paper, we have formulated a generalized set of axioms arising from information theory (originally formulated by Shannon and Khinchin) and compared it with the statistical-inference axioms of Shore and Johnson. We have shown that the class of entropies fulfilling one set of axioms automatically satisfies also the second set, so that the axioms are equivalent. The class $S_f^q$ of entropic functionals obtained is characterized by the Kolmogorov–Nagumo function $f$ and a positive parameter $q$, where $f$ determines a scaling behavior of entropy for uniform distributions and $q$ quantifies correlations for MEP distributions for disjoint sets. Let us mention that the class of $S_f^q$ can also be found in the literature under the name *strongly pseudo-additive* (SPA) entropies [40] or $Z$-entropies [27]. However, the novelty of this work is to point out that these two axiomatic systems are in fact logically equivalent, as that they produce the same class of entropies.

It might be interesting to investigate to what extent the aforementioned equivalence between the two axiomatic systems can be broken by working with more general constraints (e.g., non-inductive inference constraints such as non-linear constraints or scalings) or by relaxing some of the presented axioms. In fact, it is well-known that many complex systems do not satisfy SK axioms, not even in our generalized sense [13, 28, 41]. By relaxing some of these axioms, one might obtain further maneuvering space allowing for entropies of such complex systems as path-dependent or super-exponential systems.
Acknowledgments.

P.J. and J.K. were supported by the Czech Science Foundation (GAČR), Grant No. 19-16066S. J.K. was also supported by the Austrian Science Fund (FWF) under Project No. I3073.

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[42] Here and throughout we use the base of natural logarithms. Entropy thus defined is then measured in natural units — nats, rather than bits. To convert, note that 1 bit = 0.693 nats.