General Connectivity Distribution Functions for Growing Networks with Preferential Attachment of Fractional Power

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We study the general connectivity distribution functions for growing networks with preferential attachment (PA) of fractional power, \( \Pi \propto k^\alpha \), using Simon’s method. We first show that the heart of the previously known methods of the rate equations for the connectivity distribution functions is nothing but Simon’s method for word problem. Secondly, we show that for the case of fractional \( \alpha \) the \( Z \)-transformation of the rate equation provides a fractional differential equation of new type, which coincides with that for PA with linear power, when \( \alpha = 1 \). We show that to solve such a fractional differential equation we need define a transdifferential function \( \Upsilon(a, b, c, z) \) that we call \textit{upsilon function}. Most of all the previously known results are obtained consistently in the frame work of a unified theory.

I. INTRODUCTION

Random network models have been a prototype model for a complex network system for a long time\[1\]. However, at nearly the end of 1990’s the scale-free networks were discovered from studying the growth of the internet geometry and topology. After the discovery, scientists have found that many real systems such as internet topology, human sexual relationship, scientific collaboration, economical network, and so on show the scale-free property that the distribution function \( P(k) \propto k^{-\beta} \) is given by \( P(k) \propto k^{-\beta} \) where \( 2 < \gamma < \infty \). To describe the scaling law in the networks there have appeared many studies, using various statistical and mathematical methods \[2, 3, 4, 5, 6, 7, 8\]. The most characteristic property of the scale-free network is the scaling property that the distribution function \( P(k) \) of numbers of links with degree \( k \) is given by \( P(k) \propto k^{-\gamma} \) where \( 2 < \gamma < \infty \). To describe the scaling law in the networks there have appeared many studies, using various statistical and mathematical methods \[9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27\].

In those studies Barabasi and Albert\[9\] first introduced a growing network model (called the BA-model). They introduced the concept of growing networks with preferential attachment (PA) that is proportional to link degree \( k \) and the system exhibits a scaling property of \( P(k) \propto k^{-3} \). In terms of time development of the connectivity distribution function, Krapivsky, Redner and Leyvraz\[11\] and Dorogovtsev and Mendes\[12\] have generalized the PA to include the more general PA that is proportional to \( k^\alpha \) and to that is proportional to attractiveness \( A \) in addition to linear PA such as \( A + k \), respectively. Liu, Lai and Ye\[19\] have given a most direct generalization of the BA-model to include the \( k^\alpha \)-PA in the spirit of Krapivsky, Redner and Leyvraz\[11\].

On the other hand, to show the scale-free nature and to obtain the scaling exponent of the distribution function, Bornholdt and Ebel\[16\] have presented another method based on the famous Simon’s method\[28\]. They obtained the exponent \( \gamma = 1 + 1/(1 - \beta) \), where \( \beta (0 < \beta < 1) \) is the probability that a newly attaching node is exerted into the system, while \( 1 - \beta \) is the probability that a new link is attached between the existing nodes. Therefore, if \( \beta = 1 \) then the system must coincide with the one studied by Krapivsky, Redner and Leyvraz\[11\]. However, it does not do so, because in this limit the former provides \( \gamma = \infty \), while the latter provides \( \gamma = 3 \). Hence, there is something wrong in their study.

Their flaw lies in the two assumptions that were borrowed from those of Simon for the words problem in which the probability that the adding word is a new word is given \( \beta \) while the probability that the adding word is the word that already exists \( \times \) times is given \( 1 - \beta \)\[28\]. In the word problem, there is only a single PA of word to the system and therefore, there is no difference in the sense of attachment of the new word and existing words since both are words. On the other hand, when we apply the method to a network model, we have to be careful because in the network model, there are both attachments of nodes and links. We should not confuse with such attachments so that a node is exerted into the system at each time step and at the same time links are provided accordingly. We cannot separate both procedures using a single parameter \( \beta \) so that the attachment of a node is given by \( \beta \) and the attachment of links by \( 1 - \beta \). This is unacceptable. However, the mathematical ideas that they adopted is very interesting in its own right\[29\].

In this paper, we would like to study the growing network models with PA with fractional power, using Simon’s method\[28\] in order to unify most of the previously known methods and results studied in the literature \[9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27\] and to give new results. We would like to shed a new light on the problem.

The organization of the paper is the following. In Sec.II, we review the growing network models with various types of PA. In Sec.III, we introduce the rate equations for the connectivity distribution functions for the entire network and a particular node, respectively. In Sec.IV, we solve the rate equation for the connectivity distribution function for the entire network. In Sec.V, we solve the rate equation for the connectivity distribution function for \( i \)-th node. In Sec.VI, a conclusion is made.
II. THE EVOLVING NETWORK MODELS

Random network models of Erdős and Rényí[1] have been standard models for complex network systems, until evolving network models have been found by Barabási and Albert[2]. In the former, the size of the network $N$ is fixed and the links are distributed with probability $p$ such that each link has the mean connectivity $\langle k \rangle = pN$.

In the latter, the network grows from a set of $m_0$ seed nodes, putting a newly added node from which $m$ new links are distributed to the existing nodes. The way of adding a link depends on the PA of

$$\Pi_i = \frac{k_i}{\sum_{j=1}^{N-1} k_j}. \quad (1)$$

This yields an evolution equation for the link number $k_i$ of the $i$th node:

$$\frac{dk_i}{dt} = m\Pi_i = \frac{mk_i}{\sum_{j=1}^{N-1} k_j} = \frac{k_i}{2t}, \quad (2)$$

where we have a trivial relation $\sum_{j=1}^{N-1} k_j = 2mt - m$.

Recently, Liu, Lai, and Ye[19] have generalized the above PA to the following:

$$\Pi_i = \frac{k_i^{\alpha}}{\sum_{j=1}^{N-1} k_j^{\alpha}}, \quad (3)$$

and the evolution equation to the following:

$$\frac{dk_i}{dt} = \bar{m}\Pi_i = \frac{\bar{m}k_i^{\alpha}}{\sum_{j=1}^{N-1} k_j^{\alpha}} = \frac{\bar{m}k_i^{\alpha}}{\mu_\alpha t}, \quad (4)$$

where $0 < \alpha < 1$, $\bar{m}$ is the average number of attached links and we have used the relation

$$\sum_{j=1}^{N} k_j^{\alpha} \equiv \mu_\alpha t, \quad (5)$$

where $1 < \mu_\alpha < 2\bar{m}$ for $0 < \alpha < 1$ and $\mu_1 = \langle k \rangle = 2\bar{m}$ for $\alpha = 1$. The solution of the above evolution equation has been given by them.

This formalism is basically the same as that of Krapivsky, Redner and Leyvraz[11] as discussed by Liu, Lai, and Ye[19] in their appendix, since the former solved the time evolution of Eq.(4) while the latter solved the rate equation of the connectivity distribution function which will be discussed in the next section. Furthermore, it is also almost equivalent to the formalism of Dorogovtsev, Mendes and Samukhin[12] and Bianconi and Barabási[17].

To be more specific to this point, the PA for the model of Dorogovtsev, Mendes and Samukhin[12] is defined as follows:

$$\Pi_i = \frac{\sum_{j=1}^{N-1} (A + k_j)}{A + k_i}. \quad (6)$$

where $A$ is called the initial attractiveness of node $i$, which describes the ability of the $i$th node to receive links from other nodes, i.e., the probability for young nodes to get new links. On the other hand, the PA for the model of Bianconi and Barabási[17] is given by

$$\Pi_i = \frac{\eta_i k_i}{\sum_{j=1}^{N-1} \eta_j k_j}, \quad (7)$$

where $\eta_i$ is called the fitness of node $i$, which describes the discrimination of different types of nodes. Although in the original Bianconi-Barabási model[17], the fitness is distributed at random with a distribution of $\rho(\eta)$, we restrict ourselves to the case of the fixed fitness for each node. In this sense, we would like to call such fixed value of fitness the preference throughout this paper. Thus, we are able to recognize that the concepts of attractiveness and preference are essentially equivalent to each other, since one can expand the preference as $\eta_i = 1 + A/k_i$.

In this way, we are led to define the most general form of the PA for a finite system of $N$ nodes:

$$\Pi_i = \frac{\eta_i k_i^\alpha}{\sum_{j=1}^{N-1} \eta_j k_j^\alpha}, \quad (8)$$

where $\alpha$ can be any real number of $-\infty < \alpha < \infty$ in general.

III. THE RATE EQUATION

A. Definition of the Rate Equation

Let us first define that at each instant of time a node is added to the system such that the total number of nodes is given by $N(t) = t$. Let us denote by $N_k(i, t)$ the distribution of the connectivity (= degree) $k$ at the node $i$ and define $N_k(i, t) = \delta_{k,m}$, which means that the node $i$ starts with $k_i(0) = m$ when it is born[30].

Following both Krapivsky, Redner and Leyvraz[11] and Dorogovtsev, Mendes and Samukhin[12], we can define the rate (or master) equation for the time evolution of the connectivity distribution function as follows. Let us define the probability that the node $i$ receives exactly $l$ new links of the $m$ injected as

$$p_{k,l} = \left( \frac{m}{l} \right) (\Pi_k)^l (1 - \Pi_k)^{m-l}, \quad (9)$$

where $\Pi_k$ is the PA given in the previous section, regarding $k_i$ as $k$. Then the rate equation is given by

$$N_k(i, t + 1) = \sum_{l=0}^{m} p_{k,l} N_{k-l}(i, t). \quad (10)$$

We would like to note here that Eq.(9) describes that at each time step, each node can accept $l$ new links from a same single node, in general. However, in the most
of models the number of such receiving links is 0 or 1. Therefore, at this moment the above setting seems somewhat strange but the selection of the receiving link number can be realized from expansion of the PA [see Eq.(13)].

B. The Rate Equation for the Entire Network

Let us define the connectivity distribution of the entire network by

\[ N_k(t) = \frac{1}{t} \sum_{i=1}^{t} N_k(i, t). \]  

(11)

Summing up Eq.(10) over \( i \) from 1 to \( t \), we obtain

\[ (t+1)N_k(t+1) - tN_k(t+1, t+1) = \sum_{i=0}^{m} t \beta \delta_{k,i} N_{k-1}(t). \]  

(12)

Now we expand the right hand side with respect to PA \( \Pi_k \) \[ rhs = \] \[ \text{Eq.}(10) \]. Here, we would like to note that the coefficient of \( \text{Eq.}(15) \) will be discussed in the next subsection[see Eq.(30)]. Here, we would like to note that the

\[ \text{Eq.}(30) \]. Here, we would like to note that the coefficient of the distribution function \( N_k(t) \) will be discussed in the next subsection[see Eq.(30)]. Here, we would like to note that the coefficient of the distribution function \( N_k(t) \)

\[ N_k(t+1) - N_k(t) = \frac{m}{\mu_{\alpha}} \left[-\eta_{k} k^{\alpha} N_k(t) + \eta_{k-1}(k-1)^{\alpha} N_{k-1}(t)\right] + \delta_{k,m}. \]  

(17)

This is the most general form for the rate equation for our purpose here. In these equations, \( N_k(t) \) are defined for \( k \geq m \).

Now, supposing that the time is as sufficiently large as \( t \gg 1 \) such that \( t + 1 \approx t \), the left hand side becomes \( t[N_k(t+1) - N_k(t)] \). Then dividing both sides by \( t \), we obtain

\[ N_k(t+1) - N_k(t) = \frac{m}{\mu_{\alpha} t} \left[-\eta_{k} k^{\alpha} N_k(t) + \eta_{k-1}(k-1)^{\alpha} N_{k-1}(t)\right] + \delta_{k,m}. \]  

(18)

This is exactly the same form of the evolution equation that was studied by Simon [10] \[ \text{2} \]. apart from the coefficient of the last term in the right hand side; in the Simon model, the last term is \( \beta \delta_{k,1} \), where \( \beta \) is the probability that the newly added word is 1 and the probability that it is one of already existing words.

C. The Rate Equation for the Connectivity Distribution Function

We follow the same argument for derivation of the rate equation for the connectivity distribution function, \( N_k(i, t) \). Let us expand the right hand side of Eq.(10) with respect to \( i \). We obtain

\[ N_k(i, t + 1) = p_{k,0} N_k(i, t) + p_{k,1} N_{k-1}(i, t) + \cdots. \]  

(19)

Expanding the \( p_{k,0} \) and \( p_{k,1} \) with respect to \( \Pi_k \) in the same way, we obtain

\[ N_k(i, t + 1) = \left[1 - \frac{m \eta_{k} k^{\alpha}}{\mu_{\alpha} t}\right] N_k(i, t) \]

\[ + \frac{m \eta_{k-1}(k-1)^{\alpha}}{\mu_{\alpha} t} N_{k-1}(i, t) + \cdots. \]  

(20)

This is the most general generalization for the rate equation of the connectivity distribution functions. It includes all the models studied in the previous literature

[10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22] [23] [24] [25] [26] [27].

IV. THE SOLUTION OF THE RATE EQUATIONS FOR THE ENTIRE NETWORK

A. Simon’s method

Let us now consider the steady state solution of the rate equation for the entire network of Eq.(17). Let us
Therefore, we have

Thus, from Eq.(25) we can derive

This means that the $N_k(t)$ grows like $N_k(t) \propto t$. Next, let us define

Substituting the relation of Eq.(21) into the left hand side of Eq.(18) and the relation of Eq.(23) into the right hand side of Eq.(18), we obtain

Hence, we can define the steady state solution $n_k^*$ that is independent of $t$ such as

For $k > m$, we can derive the following for the steady state solution $n_k^*$:

For $k = m$, since $N_k(t) = 0$ for $0 < k < m$, Eq.(18) yields

Therefore, we have

Thus, from Eq.(25) we can derive $n_k^*$ for the steady state:

where $k > m$ and in the last step we have used the gamma function $\Gamma(k) = (k - 1)!$ and Eq.(27).

We can also rewrite the above in the same form that was introduced by Krapivsky, Redner and Leyvraz \cite{11} as follows:

Substituting Eq.(29) into Eq.(15), the definition for $\mu_\alpha$, we have the self-consistency condition for $\mu_\alpha$:

Then by expanding the right hand side with respect to $\mu_\alpha$ and subtracting some terms in both sides, we obtain the simplified relation:

This equation corresponds to Eq.(6) of Krapivsky, Redner and Leyvraz \cite{11} and if we assume $\eta_k = 1$, it coincides with Eq.(A.3) of Liu, Lai and Ye \cite{13} where the relation between $\alpha$ and $\mu_\alpha$ is demonstrated.

If we use the general notation $A_k$ for $m\eta_kk^\alpha$, then we may rewrite Eq.(29) as

which is equivalent to the Eq.(8) in Krapivsky, Redner and Leyvraz \cite{11} if we take $m = 1$. From the definition of the PA, it is obvious that $A_k$ is nothing but the functional of PA. Here we obtain

where $k > m$ and in the last step we have used the gamma function $\Gamma(k) = (k - 1)!$ and Eq.(27).
Multiplying where we have used the definition of the fractional 
produces \( \eta \), if

\[ P(k) \equiv n_k = \frac{2m(m+1)}{k^*(k+1)(k+2)} \propto k^{-3}. \]  

(34)

If \( \alpha = 1 \) and \( \eta_k \neq 1 \), which is the case of Krapivsky, Redner and Leyvraz\(^{11}\) and produces \( A_\infty = mn_\infty \), then

\[ P(k) \equiv n_k \propto k^{-\gamma}, \]

(35)

where \( \gamma = 1 + \mu_1/A_\infty = 1 + \mu_1/mn_\infty \) and \( \mu_1 \) can be calculated by using Eq.(30).

Thus, the heart of Simon’s method is equivalent to that of Krapivsky, Redner and Leyvraz\(^{11}\). It provides a different result from that of Bornholdt and Ebel\(^{10}\). This is due to the fact that both methods adopt different assumptions to make the network systems, although both use Simon’s method.

B. The Method of Dorogovtsev, Mendes and Samukhin

Let us next consider how to solve the Eq.(18), once again. This time we follow and generalize the method of Dorogovtsev, Mendes and Samukhin\(^{12}\). Recently, a similar approach has appeared\(^{26}\).

Going back to Eq.(17) and considering the steady state solution we have

\[ N_k(t) = \frac{m}{\mu_\alpha}[-\eta_k k^\alpha N_k(t) + \eta_{k-1}(k-1)^\alpha N_{k-1}(t)] + t\delta_{k,m}. \]

(36)

Substituting \( N_k(t) = n_k t \) into the above, then we obtain

\[ \left[ \frac{\mu_\alpha}{m} + \eta k^\alpha \right] n_k - \eta_{k-1}(k-1)^\alpha n_{k-1} = \frac{\mu_\alpha}{m}\delta_{k,m}. \]

(37)

For the sake of simplicity for our purpose here, we take \( \eta_k = 1 \). Then Eq.(37) becomes

\[ \left[ \frac{\mu_\alpha}{m} + k^\alpha \right] n_k - (k-1)^\alpha n_{k-1} = \frac{\mu_\alpha}{m}\delta_{k,m}. \]

(38)

Now, let us define a generating function:

\[ \Phi_m(z) \equiv \sum_{k=m}^{\infty} n_k z^k. \]

(39)

Multiplying \( z^k \) to Eq.(38) and summing up over \( k \) from \( m \) to \( \infty \), we obtain

\[ \left[ \frac{\mu_\alpha}{m} + (1-z) \left( z \frac{d}{dz} \right)^\alpha \right] \Phi_m(z) = \frac{\mu_\alpha}{m} z^m, \]

(40)

where we have used the definition of the fractional derivative\(^ {31} \, 32 \, 33\):

\[ \left( z \frac{d}{dz} \right)^\alpha \Phi_m(z) \equiv \sum_{k=m}^{\infty} k^\alpha n_k z^k. \]

(41)

When \( \alpha = 1, \mu_1 = 2m \) and therefore, Eq.(40) turns out to be

\[ \left[ 2 + (1-z) z \frac{d}{dz} \right] \Phi_m(z) = 2z^m. \]

(42)

This is essentially equivalent to Eq.(7) in the paper of Dorogovtsev, Mendes and Samukhin\(^ {12}\).

Let us show this. Define as \( \Phi(z) = \sum_{k=0}^{\infty} n_k z^k = n_m + n_{m+1} z + \cdots \), which means that \( k \) starts not from \( m \) but from \( 0 \). Therefore, we can rewrite \( \Phi_m(z) \) as \( \Phi_m(z) \equiv z^m \Phi(z) \). From this, we find

\[ \frac{d}{dz} \Phi_m(z) = \frac{d}{dz} z^m \Phi(z) = z^m \left( \frac{d}{dz} \frac{m}{z} \right) \Phi(z). \]

(43)

Substituting Eq.(43) into Eq.(42), we obtain the following equation for \( \Phi(z) \):

\[ z(1-z) \frac{d}{dz} \Phi(z) + (m+2-mz) \Phi(z) = 2. \]

(44)

By differentiation for both sides of Eq.(44) with respect to \( z \), we obtain

\[ z(1-z) \frac{d^2}{dz^2} \Phi(z) + (m+3 -(m+2)z) \frac{d}{dz} \Phi(z) - m \Phi(z) = 0. \]

(45)

The solution of the above equation can be obtained by the Gauss’ hypergeometric function:

\[ _2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} x^n, \]

(46)

which is the solution of the following ordinary differential equation:

\[ x(1-x) \frac{d^2}{dx^2} y(x) + [c - (a+b+1)] \frac{dy(x)}{dx} - aby(x) = 0. \]

(47)

Using the boundary condition that \( \Phi(0) = \text{const} \) and \( \Phi'(0) = 0 \) for Eq.(10), we get \( \Phi(0) = 2/(m+2) \). Using this and with replacement of \( a = 1, b = m \), and \( c = m+3 \), \( \Phi(z) \) can be obtained as

\[ \Phi(z) = \frac{2}{m+2} _2F_1(1, m, m+3; z) \]

\[ = \sum_{k=0}^{\infty} \frac{2m(m+1)\Gamma(k+m)}{\Gamma(k+m+3)} x^k. \]

(48)

Therefore, comparing it with the definition of \( \Phi(z) = \sum_{k=m}^{\infty} n_{k+m} z^k \), we find

\[ P(k) \equiv n_{k+m} = \frac{2m(m+1)\Gamma(k+m)}{\Gamma(k+m+3)} \]

\[ = \frac{2m + m + 1}{(k+m)(k+m+1)(k+m+2)}. \]

(49)
Since in this expression \( k \) starts from \( k = 0 \), if we change it to start from \( k = m \) (i.e., we change the variable of \( k \) from \( k + m \) to \( k \)), then we obtain

\[
P(k) \equiv n_k = \frac{2m(m+1)}{k(k+1)(k+2)} \propto k^{-3}. \tag{50}
\]

Hence, our result above is exactly equivalent to Eq.(34) and coincides with that of Dorogovtsev, Mendes and Samukhin\[12\]. We can repeat the similar procedure for the integer \( \alpha \).

Thus, we can understand that both methods of Krapivsky, Redner and Leyvraz\[11\] and Dorogovtsev, Mendes and Samukhin\[12\] give the identical result when \( \alpha = 1 \) and, therefore, they are essentially equivalent to each other. However, this method is not directly applicable to the case of fractional \( \alpha \). In this case we need a more sophisticated method such as the fractional calculus\[31, 32, 33\]. This is a really challenging problem for physicists. Some hints are discussed in Appendix A.

**V. THE SOLUTION OF THE RATE EQUATION FOR THE DISTRIBUTION FUNCTION**

In this section we are going to solve the rate equation of Eq.(20), unifying both methods of Krapivsky, Redner and Leyvraz\[11\] and Dorogovtsev, Mendes and Samukhin\[12\].

**A. The Discrete Rate Equation**

Let us first apply the method of Krapivsky, Redner and Leyvraz\[11\]. Now, we want to solve the discrete rate equation:

\[
N_k(i, t + 1) = N_k(i, t) - \frac{m \eta_k k^{\alpha}}{\mu \alpha t} N_k(i, t) + \frac{m \eta_{k-1} (k - 1)^{\alpha}}{\mu \alpha t} N_{k-1}(i, t). \tag{51}
\]

Let us define the average connectivity \( \overline{k}(i, t) \) by

\[
\overline{k}(i, t) \equiv \sum_{k=m}^{\infty} k N_k(i, t). \tag{52}
\]

Multiplying Eq.(20) by \( k \) and summing up over \( k \), and using this definition we obtain

\[
\overline{k}(i, t + 1) = \overline{k}(i, t) + \frac{m}{\mu \alpha t} \eta_k k^{\alpha}(i, t), \tag{53}
\]

where \( \eta_k k^{\alpha}(i, t) \equiv \sum_{k=m}^{\infty} \eta_k k^{\alpha} N_k(i, t) \). If we set \( \eta_k = 1 \), then it becomes

\[
\overline{k}(i, t + 1) = \overline{k}(i, t) + \frac{m}{\mu \alpha t} \overline{k}(i, t), \tag{54}
\]

where \( \overline{k}(i, t) \equiv \sum_{k=m}^{\infty} k^{\alpha} N_k(i, t) \). Hence, if we can further impose \( \alpha = 1 \), then we obtain

\[
\overline{k}(i, t + 1) = \overline{k}(i, t) + \frac{1}{2t} \overline{k}(i, t), \tag{55}
\]

where we have used \( \mu_1 = 2m \).

Eq.(54) can be regarded as a discrete version of the continuous time equation such as Eq.(2) with the PA \( \Pi_i \) of fractional exponent \( \alpha \):

\[
\frac{dk_i}{dt} = m \Pi_i = \frac{mk^\alpha_i}{\sum_{j=1}^{N-1} k_j} = \frac{mk^\alpha_i}{\mu \alpha t}. \tag{56}
\]

In general, since \( \overline{k}(i, t) \) is not equal to \( \overline{k}(i, t)^\alpha \), the discretization of Eq.(56) is not equivalent to Eq.(54). However, for \( \alpha = 1 \), Eq. (56) becomes identical to Eq.(2) and therefore, the discretization of Eq.(2) becomes equivalent to Eq.(55). Eq.(55) can be solved directly as follows: From Eq.(55) we find

\[
\overline{k}(i, t + 1) = \left( 1 + \frac{1}{2t} \right) \overline{k}(i, t) = \left( 1 + \frac{1}{2t} \right) \left( 1 + \frac{1}{2(t - t_i)} \right) \overline{k}(i, t_i). \tag{57}
\]

Taking logarithm of both sides, we obtain

\[
\ln \overline{k}(i, t) = \sum_{j=t_i}^{t-1} \ln \left( 1 + \frac{1}{2j} \right) + \ln \overline{k}(i, t_i) \approx \frac{1}{2} \sum_{j=t_i}^{t-1} \frac{1}{j} + \ln \overline{k}(i, t_i) \approx \frac{1}{2} \ln \left( \frac{t}{t_i} \right) + \ln \overline{k}(i, t_i), \tag{58}
\]

where we have assumed that \( t \) is sufficiently large such that \( t \gg t_i \). Hence, we approximately obtain

\[
\overline{k}(i, t) = m \left( \frac{t}{t_i} \right)^{1/2}, \tag{59}
\]

where we have used \( \overline{k}(i, t_i) = m \). This exactly coincides with the result of time development of the seminal work of Barabási and Albert’s continuous model\[9\].

To go further to the case of fractional \( \alpha \), the discrete model is not so convenient since Eq.(53) is not simple enough for physicists. Some hints are discussed in Appendix A.

**B. The Fractional Differential Rate Equation**

Let us now take a continuous time limit, by which we obtain the following continuous differential equation:

\[
\frac{\mu \alpha}{m} \frac{\partial}{\partial t} N_k(i, t) = -\eta_k k^{\alpha} N_k(i, t) + \eta_{k-1} (k-1)^{\alpha} N_{k-1}(i, t), \tag{60}
\]

However, for \( \alpha = 1 \), Eq. (56) becomes identical to Eq.(2) and therefore, the discretization of Eq.(2) becomes equivalent to Eq.(55). Eq.(55) can be solved directly as follows: From Eq.(55) we find

\[
\overline{k}(i, t + 1) = \left( 1 + \frac{1}{2t} \right) \overline{k}(i, t) = \left( 1 + \frac{1}{2t} \right) \left( 1 + \frac{1}{2(t - t_i)} \right) \overline{k}(i, t_i). \tag{57}
\]

Taking logarithm of both sides, we obtain

\[
\ln \overline{k}(i, t) = \sum_{j=t_i}^{t-1} \ln \left( 1 + \frac{1}{2j} \right) + \ln \overline{k}(i, t_i) \approx \frac{1}{2} \sum_{j=t_i}^{t-1} \frac{1}{j} + \ln \overline{k}(i, t_i) \approx \frac{1}{2} \ln \left( \frac{t}{t_i} \right) + \ln \overline{k}(i, t_i), \tag{58}
\]

where we have assumed that \( t \) is sufficiently large such that \( t \gg t_i \). Hence, we approximately obtain

\[
\overline{k}(i, t) = m \left( \frac{t}{t_i} \right)^{1/2}, \tag{59}
\]

where we have used \( \overline{k}(i, t_i) = m \). This exactly coincides with the result of time development of the seminal work of Barabási and Albert’s continuous model\[9\].

To go further to the case of fractional \( \alpha \), the discrete model is not so convenient since Eq.(53) is not simple equation of \( \overline{k}(i, t) \). Thus, for this purpose, we have to use the continuous rate equation.
where \( N_k(i, t) \) is the connectivity distribution function of node \( i \) at time \( t \). For our purpose here, we will set \( \eta_k = \eta = \) constant, for the sake of simplicity.

Let us define the Z-transformation of the connectivity distribution function \( N_k(i, t) \) by

\[
\Phi_m(i, t; z) \equiv \sum_{k=m}^{\infty} N_k(i, t) z^k \equiv z^m \Phi(i, t; z),
\]

(61)

where \( \Phi(i, t; z) \equiv \sum_{k=0}^{\infty} N_{k+m}(i, t) z^k \). Multiplying both sides of Eq.(60) by \( z^k \) and summing up over \( k \) and using the fractional derivative that was introduced in the previous section, we obtain the following fractional differential equation:

\[
\frac{1}{\beta_\alpha} t \frac{\partial}{\partial t} \Phi_m(i, t; z) + (1 - z) \left( z \frac{\partial}{\partial z} \right)^\alpha \Phi_m(i, t; z) = 0,
\]

(62)

where \( \beta_\alpha \equiv \frac{m}{\mu_\alpha} \). This corresponds to Eq.(40) in Sec.IV and also it corresponds to Eq.(14) in Krapivsky, Redner and Leyvraz[11].

C. The Solution of the Rate Equation with Linear Preferential Attachment

When \( \alpha \) is fractional we meet the same difficulty that has been discussed in the end of the previous section. Therefore, let us restrict ourselves to consider the simplest case of linear PA with \( \alpha = 1 \). In this case, \( \mu_1 = 2m \) such that \( \beta_1 = \frac{1}{2\eta} \) and therefore, Eq.(62) becomes

\[
\frac{1}{\beta_1} t \frac{\partial}{\partial t} \Phi_m(i, t; z) + (1 - z) z \frac{\partial}{\partial z} \Phi_m(i, t; z) = 0.
\]

(63)

If we take \( \eta = 1 \) this corresponds to Eq.(42) and if \( \eta > 1 \) then it corresponds to Eq.(14) in Dorogovtsev, Mendes and Samukhin[12].

Let us solve Eq.(63). Using the standard method for solving linear partial differential equation, it yields the following differential equations for the characteristic curve:

\[
\frac{dz}{z(1 - z)} = \beta_1 \frac{dt}{t} = \frac{d\Phi_m}{0},
\]

(64)

where 0 in the denominator in the last equation means that \( d\Phi_m = 0 \) such that \( \Phi_m(i, i; z) = \Phi_0 = \) constant. Since by definition \( N_k(i, i) = \delta_{k,m} \), we have the initial condition

\[
\Phi_m(i, i; z) = \Phi_0 \equiv z_0^m = 1,
\]

(65)

which then gives

\[
\frac{z_i}{1 - z_i} = c_2,
\]

(67)

Substituting Eq.(67) into Eq.(65), we obtain the relation:

\[
c_1 = \left( \frac{c_2}{1 + c_2} \right)^m,
\]

(68)

which is the characteristic curve that we seek for Eq.(64). Since the solution moves as a point along this curve from the initial time \( t = t_i \), the solution of Eq.(63) is given by

\[
\Phi_m(i, t; z) = \left( \frac{c_2(t_i/t)^{\beta_1}}{1 - z(1 - (t_i/t)^{\beta_1})} \right)^m.
\]

(69)

Now, solving Eq.(66) for \( c_2 \) and substituting it into Eq.(69), we obtain

\[
\Phi_m(i, t; z) = \left[ \frac{z (t_i/t)^{\beta_1}}{1 - z (1 - (t_i/t)^{\beta_1})} \right]^m.
\]

(70)

Adjusting with our definition of Eq.(61), we finally obtain the solution of Eq.(62) as

\[
\Phi(i, t; z) = \left[ \frac{(t_i/t)^{\beta_1}}{1 - z (1 - (t_i/t)^{\beta_1})} \right]^m.
\]

(71)

Now expanding the denominator of Eq.(71) using the formula:

\[
\frac{1}{(1 - z)^\gamma} = \sum_{k=0}^{\infty} \frac{\Gamma(k + \gamma)}{\Gamma(\gamma) k!} z^k,
\]

(72)

we obtain

\[
\Phi(i, t; z) = \sum_{k=0}^{\infty} \frac{\Gamma(k + m)}{\Gamma(m) k!} \left( \frac{t_i}{t} \right)^{m\beta_1} \left[ 1 - \left( \frac{t_i}{t} \right)^{\beta_1} \right]^k z^k.
\]

(73)

Comparing this with Eq.(61), we end up with the following distribution function:

\[
N_{k+m}(i, t) = \frac{\Gamma(k + m)}{\Gamma(m) k!} \left( \frac{t_i}{t} \right)^{m\beta_1} \left[ 1 - \left( \frac{t_i}{t} \right)^{\beta_1} \right]^k,
\]

(74)

where \( \sum_{k=0}^{\infty} N_{k+m}(i, t) = 1 \) is obviously satisfied. The Eq.(74) corresponds to Eq.(16) in Krapivsky, Redner and Leyvraz[11] and Eq.(15) in Dorogovtsev, Mendes and Samukhin[12].

D. Time Development of the Average Connectivity

Applying Eq.(74) together with the following formula:

\[
\frac{\gamma z}{(1 - z)^\gamma} = \sum_{k=0}^{\infty} \Gamma(k + \gamma) \frac{1}{\Gamma(\gamma) k!} z^k,
\]

(75)
the time development of the average connectivity of node $i$ is calculated as

$$\overline{k}(i, t) = \sum_{k=0}^{\infty} k N_{k+m}(i, t) = m \left( \frac{t}{t_i} \right)^{\beta_1},$$ (76)

which corresponds to Eq.(59) for the case of the discrete rate equation. If there is no effect of the preference so that $\eta = 1$, then $\beta_1 = 1/2$. In this case, Eq.(76) recovers the same time development of the seminal work of Barabási and Albert[9] once again. And also it corresponds to Eq.(16) in Dorogovtsev, Mendes and Samukhin[12], respectively.

Notify Eq.(35) for the scale-free exponent of $\gamma_1$ for $\alpha = 1$, where $\gamma_1 = 1 + \frac{m}{m_{\infty}} = 1 + \frac{\eta}{\eta_{\infty}}$. If we set $\eta = \eta_{\infty}$ then from the definition of $\beta_1 = \eta/2$ we find the relationship between $\gamma_1$ and $\beta_1$:

$$\beta_1 = \frac{1}{\gamma_1 - 1}.$$ (77)

This recovers Eq.(17) in Dorogovtsev, Mendes and Samukhin[12]. As pointed out by them, this relation is exact for any finite preference of $\eta$ as long as $\alpha = 1$. However, we do not know yet whether or not this relation holds true even for the fractional $\alpha$ such as $\beta_\alpha = 1/(\gamma_\alpha - 1)$.

VI. CONCLUSIONS

In conclusion, we have studied an analytical method of how to obtain the connectivity distribution functions for the various growing networks with PA of fractional power. First, in order to unify the growing network models in a general point of view, we have discussed the various types of PA [See Eq.(8)]. Second, we have presented a general version of the rate equations for the growing networks with PA of fractional power [See Eq.(16)]. Here we have presented the rate equations for the connectivity distribution functions for the entire network and for an arbitrary node, respectively. Third, we have presented the way of solving the rate equations for connectivity distribution functions for the entire network from the point of view of Simon’s method[28]. We have shown that our method unifies both methods of Krapivsky, Redner and Leyvraz[11] and Dorogovtsev, Mendes and Samukhin[12] into the same mathematical framework. We have introduced an idea of fractional calculus that the rate equation for the connectivity distribution function with PA of fractional power becomes the fractional differential equation. We have presented a scheme to solve the problem in the appendix, where we have introduced a new type of transidental function that we call the $\upsilon$ function, $\Upsilon(a, b, c; z)[See A.13]$. Fourth, we have presented the way of solving the rate equations for connectivity distribution functions for the $i$th node, using the generalize method.

Thus, we have unified some of the previously known methods and results of connectivity distribution functions for such growing networks into a single analytical theory using fractional calculus in the spirit of the seminal Simon’s method of word problem[28]. In this context, we have emphasized that in spirit, Simon’s method is identical not only to the method of Bornholdt and Ebel[16] but also to both methods of Krapivsky, Redner and Leyvraz[11] and Dorogovtsev, Mendes and Samukhin[12]. However, the former provides a different scale-free exponent from the latter, since the defining conditions for the growing networks are distinct to each other. We have been able to establish a new method of fractional calculus to the problem of growing network models with PA of fractional power of $\alpha$. This type of the fractional differential equation [See Eq.(40)] seems to be not yet known in this field of growing network models, which is different from the one that is known as the fractional diffusion equations etc in the fractional calculus[31, 32, 33]. Therefore, we believe that our approach may shed a new light on solving such a novel type of fractional differential equations in its own right.

Acknowledgments

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APPENDIX A: FRACTIONAL CALCULUS

For the noninteger case of $\alpha$, we need a different way of solving the rate equation. We have to invent a new method. This case seems to require the method of fractional calculus[31, 32, 33].

Let us see this point. Go back to Eq.(40), once again.

$$\left[ \frac{\mu_\alpha}{m} + (1 - z) \left( z \frac{d}{dz} \right)^{\alpha} \right] \Phi_m(z) = \frac{\mu_\alpha}{m} z^{m}. \quad (40)$$

This type of differential equation has been called the fractional differential equation when the degree of the derivative is fractional, which is called fractional derivative. Recently, there have appeared many applications of the concept to various areas of physics[31, 32, 33].

From this context, our problem is one of such application to physics as well. How to solve the equation such as Eq.(40) with fractional or irrational $\alpha$ would be a mathematical challenge to physicists.

Using the properties of fractional differentials that $(\frac{d}{dz})^{\alpha} = (\frac{d}{dz})(\frac{d}{dz})^{\alpha-1}$, let us rewrite Eq.(40) as

$$\left[ \frac{\mu_\alpha}{m} + z(1 - z) \left( z \frac{d}{dz} \right)^{\alpha-1} \right] \Phi_m(z) = \frac{\mu_\alpha}{m} z^{m}, \quad (A.1)$$

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which corresponds to Eq.(42). Using the relation of
Eq.(43), Eq.(A.1) becomes
\[
\left[ \frac{\mu_\alpha}{m} + z(1 - z) \left( \frac{d}{dz} + \frac{m}{z} \right) \left( z \frac{d}{dz} + m \right)^{\alpha - 1} \right] \Phi(z) = \frac{\mu_\alpha}{m},
\]
which corresponds to Eq.(44). Operating both sides of
Eq.(A.2) by \( \frac{dz}{z} \), it turns out to be the following fractional differential equation:
\[
\frac{\mu_\alpha}{m} z \frac{d}{dz} \Phi(z) + (1 - z) \left( z \frac{d}{dz} + m \right)^{\alpha + 1} \Phi(z)
+ [-m + (m - 1)z] \left( z \frac{d}{dz} + m \right)^\alpha \Phi(z) = 0,
\]
which corresponds to Eq.(45). Comparing this with
Eq.(47), we now easily see that if \( \alpha = 1 \), then Eq.(A.3)
turns out to be Eq.(48). Thus, the problem is mapped to
how to solve the fractional differential equation such as
Eq.(40) or Eq.(A.2) when \( \alpha \) is fractional.

Let us denote \( as = e^u \). Then, Eq.(40) and Eq.(A.2)
can be written as
\[
\left[ \frac{\mu_\alpha}{m} + (1 - e^u) \left( \frac{d}{du} + m \right)^\alpha \right] \Phi_m(u) = \frac{\mu_\alpha}{m} e^{mu},
\]
and
\[
\left[ \frac{\mu_\alpha}{m} + (1 - e^u) \left( \frac{d}{du} + m \right)^\alpha \right] \Phi(u) = \frac{\mu_\alpha}{m},
\]
respectively. Similarly, Eq.(A.3) is transformed into the following:
\[
\frac{\mu_\alpha}{m} \frac{d}{du} \Phi(u) + (1 - e^u) \left( \frac{d}{du} + m \right)^{\alpha + 1} \Phi(u)
+ [-m + (m - 1)e^u] \left( \frac{d}{du} + m \right)^\alpha \Phi(u) = 0,
\]
Let us now define the Laplace transformation:
\[
\mathcal{L}[\Phi(u)] \equiv \Psi(s) = \int_0^{\infty} \Phi(u) e^{-su} du.
\]
The definition of fractional derivative requires the following operation:
\[
\mathcal{L} \left[ \left( \frac{d}{du} + m \right)^\alpha \Phi(u) \right] = (s + m)^\alpha \Psi(s).
\]
The Laplace transforms of other terms are given as follows:
\[
\mathcal{L} \left[ e^u \Phi(u) \right] = \Psi(s - 1),
\]
\[
\mathcal{L} \left[ e^u \left( \frac{d}{du} + m \right)^\alpha \Phi(u) \right] = (s+m-1)^\alpha \Psi(s-1).
\]
The proof of these relations is straightforward. Using
these, the Laplace transform of Eq.(A.6) becomes
\[
\frac{\mu_\alpha}{m} s \Psi(s) + (s + m)^{\alpha + 1} \Psi(s) - (s + m - 1)^{\alpha + 1} \Psi(s - 1)
- m(s + m)^\alpha \Psi(s) + (m - 1)(s + m - 1)^\alpha \Psi(s - 1) = 0,
\]
which yields a recursion relation:
\[
\Psi(s) = \frac{(s + m - 1)^\alpha}{m} \Psi(s - 1) + \frac{m}{m} (s + m)^\alpha \Psi(s - 1).
\]

Please compare this with Eq.(25) in Sec.IV. Since
we take \( \eta_k = 1 \) for our purpose here, it is clear that
Eq.(A.12) in the language of the (continuous) Laplace
transformation corresponds to Eq.(25) in the language
of the (discrete) Z-transformation, where the continuous
variable \( s + m \) in Eq.(A.12) corresponds to the discrete
variable \( k \) in Eq.(25).

Looking at Eq.(A.12), if we are able to Laplace trans-
form \( \Psi\) into the original function \( \Phi(z) \), then the
problem can be solved. However, it is not so simple since
Eq.(A.12) is a recursion relation for \( \Psi(s) \) and the
variable \( s \) is a continuous one. Therefore, we cannot simply
represent \( \Psi(s) \) in terms of \( s \) such as \( \eta_k \) in the case
of Eq.(25). Rather, it should be regarded as a fun-
cational relation. For example, it is supposed to be like a func-
tional relation such as \( \Gamma(z + 1) = z \Gamma(z) \), which defines
the Gamma-function \( \Gamma(z) \), where if \( z = n \), integer, then
\( \Gamma(n + 1) = n! \).

From this understanding, we would like to regard
the relation (A.12) as a defining relation for a new (unknown)
type of function. Let us define the following function
\( \Upsilon_\alpha(a, b; c; s) \), upsilon function of \( s \), such that
\[
\Upsilon_\alpha(a, b; c; s) = \frac{b(s + c - 1)^\alpha}{a + b(s + c)^\alpha} \Upsilon_\alpha(a, b; c; s - 1,
\]
where \( a, b, c \) are real numbers. From this, as suggested
by Aomoto[34], we find that the upsilon function can be
represented in terms of an infinite product:
\[
\frac{1}{\Upsilon_\alpha(a, b; c; s)} \equiv \frac{1}{b(s + c)^\alpha} \prod_{n=1}^{\infty} \left( \frac{1}{1 + \frac{a}{b(s + c + n)^\alpha}} \right).
\]
The convergence of the upsilon function can be domi-
nated by the function of an infinite product in the
denominator. If we denote it by
\[
f_\alpha(s; x) \equiv \prod_{n=1}^{\infty} \left( 1 + \frac{x}{(s + n)^\alpha} \right),
\]
then from Eq.(A.14) the upsilon function can be obtained as
\[
\Upsilon_\alpha(a, b; c; s) = b(s + c)^\alpha f_\alpha(s + c; a/b).
\]
The analytical properties and convergence of this func-
tion \( f_\alpha(s; x) \) may be obtained by the standard argument
of entire functions[33].
At this moment, we do not know much about the mathematical properties of the function $\Upsilon_\alpha(a, b, c; s)$ \cite{34}. However, if we define this function, then we can formally represent the Z-transformed function $\Phi(z)$ in terms of the zeta function as follows. Adjusting Eq. (A.12) with our definition of (A.13), our $\Psi(s)$ can be written in terms of $\Upsilon_\alpha(a, b, c; s)$ as

$$\Psi(s) \equiv \Upsilon_\alpha \left( \frac{\mu_a}{m}, 1, m; s \right). \quad (A.17)$$

Now, we invert $\Psi(s)$ by the inverse Laplace transformation:

$$\Phi(u) = \frac{1}{2\pi i} \int_{-\infty+ci}^{\infty+ci} \Psi(s) e^{su} ds$$

$$= \frac{1}{2\pi i} \int_{-\infty+ci}^{\infty+ci} \Upsilon_\alpha \left( \frac{\mu_a}{m}, 1, m; s \right) e^{su} ds, \quad (A.18)$$

where $c$ is a constant. Replacing as $z = e^u$, we finally obtain the following formula for $\Phi(z)$:

$$\Phi(z) = \frac{1}{2\pi i} \int_{-\infty+ci}^{\infty+ci} \Upsilon_\alpha \left( \frac{\mu_a}{m}, 1, m; s \right) z^s ds. \quad (A.19)$$

This is supposed to be the solution of the fractional differential equation of Eq. (A.3).

Next, if we can expand $\Phi(z)$ in terms of $z$ such as

$$\Phi(z) = \sum_{k=0}^{\infty} \Upsilon_\alpha \left( \frac{\mu_a}{m}, 1, m; k \right) z^k,$$

where $\Upsilon_\alpha \left( \frac{\mu_a}{m}, 1, m; k \right)$ denote the expansion coefficients, then comparing the definition of the Z-transformation:

$$\Phi(z) = \sum_{k=0}^{\infty} n_{k+m+1} z^k,$$

we obtain

$$n_{k+m+1} = \Upsilon_\alpha \left( \frac{\mu_a}{m}, 1, m; k \right). \quad (A.21)$$

Hence, our problem is solved.

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in their model each node starts with \( k = 1 \). On the other hand, if we want to adjust with the results of Dorogovtsev, Mendes and Samukhin\[12\], then we adopt \( \delta_{k,0} \), since in their model each node starts with \( k = 0 \). Thus, the value of \( m \) in the \( \delta \)-function depends on the model that we adopt.

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