CONTROLLABILITY FOR A CLASS OF SEMILINEAR FRACTIONAL EVOLUTION SYSTEMS VIA RESOLVENT OPERATORS

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Abstract. This paper deals with the exact controllability for a class of fractional evolution systems in a Banach space. First, we introduce a new concept of exact controllability and give notion of the mild solutions of the considered evolutional systems via resolvent operators. Second, by utilizing the semigroup theory, the fixed point strategy and Kuratowski’s measure of noncompactness, the exact controllability of the evolutional systems is investigated without Lipschitz continuity and growth conditions imposed on nonlinear functions. The results are established under the hypothesis that the resolvent operator is differentiable and analytic, respectively, instead of supposing that the semigroup is compact. An example is provided to illustrate the proposed results.

1. Introduction. This paper aims to establish the sufficient conditions for exact controllability of the following fractional evolution systems:

\[
\begin{align*}
&C D_0^\alpha x(t) = Ax(t) + f(t,x(t)) + Bu(t), & t \in I = [0,a], \\
&x(0) = x_0 \in X,
\end{align*}
\]

where \( C D_0^\alpha \) is the Caputo derivative of order \( 0 < \alpha < 1 \), \( A: D(A) \subset X \to X \) is the infinitesimal generator of a \( C_0 \)-semigroup of bounded linear operators \( \{T(t): t \geq 0\} \) defined on a Banach space \( X \). The control function \( u(\cdot) \) is given in \( L^2(I;U) \) with \( U \) is a Banach space. \( B \) is a bounded linear operator from \( U \) into \( X \), and \( f: I \times X \to X \) is a given continuous function.

Fractional calculus and fractional differential equations have been investigated extensively due mainly to their demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering such as physics, economics, control theory, aerodynamics and electromagnetics, etc. Actually, fractional models can present a more vivid and accurate description over things than integral ones, and fractional derivatives can provide an excellent tool for the description of memory and hereditary properties of various materials and processes. For details, see [2, 4], and the references therein.

Recently, Hernández et al. [11] pointed out that the concept of mild solutions used in some papers such as [7, 13] was not suitable for fractional evolution systems at

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all. Concerning this, we hold that when we investigate the evolutional systems in
infinite-dimensional spaces, the introduction of the concept of mild solutions is the
most important step. Generally speaking, there are two main types of definition of
mild solutions. The first one was constructed in terms of a probability density func-
tion given by El-Borai [10] and was then developed by Zhou et al. [30]. Considering
the convergent domain of the probability density function, we know that this way is
only applicable to the case $0 < \alpha < 1$, where $\alpha$ stands for the fractional derivative
of the considered fractional evolution system. The second one was defined by means
of an $\alpha$-resolvent family provided by Araya et al. [1]. Noticing that the solution of
a fractional evolution system is in fact a Volterra integral equation, we point out
that the concept of the resolvent operator plays an important role in it.

On the other hand, control theory is an interdisciplinary branch of engineering
and mathematics that deals with influence behavior of dynamical systems. Con-
trollability plays an important role in the analysis and design of control systems.
Many practical problems of control theory such as stabilizability and pole assign-
ment may be solved under the controllable hypothesis. Controllability problems
for different kinds of dynamical systems and evolutional systems have been inves-
tigated by many authors. For detailed description of some recent results, we refer
the reader to papers [15, 16, 17, 18, 19, 20, 21, 22, 26] and the references therein.

We point out that there are two basic concepts of controllability. The first one
is exact controllability (complete controllability) and the other one is approximate
controllability. In the case of infinite-dimensional spaces, exact controllability en-
ables to steer the system to arbitrary final state while approximate controllability
means that the system can be steered to an arbitrary small neighborhood of fi-
nal state. The major way of dealing with exact controllability is to transform the
controllability problem into a fixed point problem with hypothesis that the invert-
ability of a controllability operator is satisfied, see [14]. With the help of fixed point
theorem and semigroup theory under the assumption that the linear part of the
associated nonlinear system is approximately controllable, the approximate con-
trollability is also well investigated, see [25]. It is worth mentioning that in case of
finite-dimensional space the concepts of exact and approximate controllability coin-
cide. For example, in [29], J. Wang et al. studied the following fractional differential
systems:

$$\begin{align*}
&\left\{ C^{\alpha}D_{t}^{q}x(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad t \in J = [0, b], \\
x(0) = x_{0} \in X.
\right.
\end{align*}$$

where $C^{\alpha}D_{t}^{q}$ is the Caputo derivative of order $0 < q \leq 1$, $A$ is the infinitesimal
generator of a strongly continuous semigroup, $B$ is a bounded linear operator and $f$
is given $X$-valued functions. The authors obtained the exact controllability under
a compact condition and the assumption that $f$ was Lipschitz continuous.

In [9], by means of Banach contraction mapping principle and the Lipschitz
continuous assumptions imposed on functions $f, g, h, \Phi$ and $I_{i}$, A. Debbouche et
al. established a sufficient condition for the exact controllability of the following
impulsive systems:

$$\begin{align*}
&\left\{ \frac{d^{\alpha}u(t)}{dt^{\alpha}} + A(t, u(t))u(t) + (B\mu)(t) + \Phi(t, f(t, u(\beta(t)))), \int_{0}^{t} g(t, s, u(\gamma(s)))ds, \\
u(0) + h(u) = u_{0}, \\
\Delta u(t_{i}) = I_{i}(u(t_{i})).
\right.
\end{align*}$$
where \(-A(t,.)\) is a closed linear operator, \(B\) is a bounded linear operator, control function \(\mu\) belongs to the spaces \(L^2([0,a];U)\) with \(U\) is a Banach space, and \(f, g, h, \Phi, I_i\) are given functions.

Using the Banach contraction mapping principle together with the Lipschitz continuity of the functions \(f\) and \(g\), V. Vijayakumar et al. in [28] obtained the exact controllability of the following systems:

\[
\begin{cases}
D^\alpha_t (x(t) + f(t, x_t)) = Ax(t) + \int_0^t G(t - s)x(s)ds + (Bu)(t) + g(t, x_t), \quad t \in I, \\
x_0 = \varphi \in B, \quad x'(0) = x_1,
\end{cases}
\]

where \(D^\alpha_t\) is the Caputo derivative of order \(1 < \alpha < 2\), \(A, G(t)\) are closed linear operators, and \(f, g\) are appropriate functions.

In [27], the authors considered a rather general linear evolution equation of fractional type, namely a diffusion type problem in which the diffusion operator is the \(s\)th power of a positive definite operator. In this problem, the fractional parameter \(s\) serves as the control parameter. Using Tikhonov’s compactness theorem, the authors overcame the difficulty that with changing \(s\) also the domain of \(L^S\) changed.

Concerning the above results, we note that when we convert the controllability into a fixed point problem, nonlinear function \(f\) and nonlocal item are usually assumed to be compact or Lipschitz continuous. But these assumptions are not satisfied in many practical applications. Else, Hernández and O'Regan point out that some papers on controllability of abstract control systems contain a similar technical error when the compactness of semigroup and other assumptions are satisfied. Motivated by all above mentioned works, in the present paper, we will introduce a new and weaker concept of exact controllability which can be regarded as an extension of the existing notion. In addition, we introduce a suitable concept for mild solutions of the the fractional evolution system (1.1) via resolvent operators which are well developed for Volterra integral equations. It should also be stressed that for equations with unbounded operators in infinite-dimensional spaces the concept of the resolvent is more appropriate since it is a direct generalization of \(C_0\)-semigroups. Meanwhile, to get the exact controllability of system (1.1) in the infinite-dimensional spaces, the differentiable and analytic conditions will be put on the resolvent operators respectively instead of supposing that the semigroup \(\{T(t) : t \geq 0\}\) is compact. Moreover, by using semigroup theory, the Sadovskii’s fixed point theorem and the Kuratowski’s measure of noncompactness, we investigate the exact controllability of fractional evolution system (1.1) without the Lipschitz continuity and other growth conditions imposed on the nonlinear function \(f\). Actually, the nonlinear function \(f\) is only supposed to be continuous. So our results can be regarded as a development of previous conclusions obtained in [9, 14, 29].

The rest of present paper is organized as follows. Section 2 gives some necessary preliminaries and lemmas. In Section 3, we establish the sufficient conditions for the exact controllability of fractional evolution system (1.1). Finally, an illustrative example is presented to support the new results.

2. Preliminaries and some lemmas. Let \((X, \| \cdot \|)\) be a real Banach space and \(A : D(A) \subset X \rightarrow X\) is the infinitesimal generator of a \(C_0\)-semigroup of bounded linear operators \(\{T(t) : t \geq 0\}\) defined on \(X\). In the sequel we denote by \(L(X,Y)\) the space of bounded linear operators from \(X\) into \(Y\) endowed with the operator norm denoted by \(\| \cdot \|_{L(X,Y)}\) where \(Y\) is another Banach space, and we write simply \(L(X)\) and \(\| \cdot \|_{L(X)}\) when \(X = Y\). Let \(D\) be the domain of the operator \(A\) equipped
with the graph norm \( \|x\|_D = \|x\| + \|Ax\| \). Obviously, \( D \) is a Banach space, and it is continuously and densely embedded into \( X \). We denote by \( C(I; X) \) the space of \( X \)-valued continuous functions on \( I \) with the sup-norm \( \| \cdot \|_{C(I; X)} \). \( C^\alpha(I; X) \), \( \alpha \in (0, 1) \), represents the space formed by all the \( X \)-valued \( \alpha \)-Hölder continuous functions from \( I \) into \( X \) with the norm \( \|x\|_{C^\alpha(I; X)} = \|x\|_{C(I; X)} + \|x\|_{C^\alpha(I; X)} \), where

\[
\|x\|_{C^\alpha(I; X)} = \sup_{t,s \in I, t \neq s} \frac{\|x(t) - x(s)\|}{(t-s)^\alpha}.
\]

In this paper, we assume that the integral equation

\[
x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t A(x(s)) \frac{ds}{(t-s)^{1-\alpha}}, \quad t \geq 0,
\]

has an associated resolvent operator \( \{S(t)\}_{t \geq 0} \) on \( X \).

Let us recall the following known definitions and lemmas.

**Definition 2.1** ([23]). The Caputo fractional derivative of order \( \alpha > 0 \) of a continuous function \( u : (0, \infty) \to \mathbb{R} \) is given by

\[
C_{D_0^+}^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t u^{(n)}(s) \frac{ds}{(t-s)^{n-\alpha+1}},
\]

where \( n = [\alpha] + 1 \), \( [\alpha] \) denotes the integer part of the real number \( \alpha \), and provided the right side integral is pointwise defined on \( [0, \infty) \).

**Remark 1.** (i) Obviously, the Caputo derivative of a constant is equal to zero.
(ii) Especially, when \( 0 < \alpha < 1 \), we have

\[
C_{D_0^+}^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^{1-\alpha}} ds.
\]

(iii) The solutions of Caputo-type equations are in general much more abundant than in the classical case. For instance, the Caputo derivative would represent a memory effect if we consider a function depending on time, pointing out that the state of a system at a given time depends on past events. In addition, the solutions of Caputo-type equations can approximate any given smooth function arbitrarily. For details, see [3].

**Definition 2.2** ([24]). A family \( \{S(t)\}_{t \geq 0} \) of bounded linear operators on \( X \) is called a resolvent operator for (2.1) [or solution operator for (2.1)] if the following conditions are satisfied:

(S1) \( S(t) \) is strongly continuous on \( \mathbb{R}_+ \) and \( S(0) = I \);
(S2) \( S(t) \) commutes with \( A \), which means that \( S(t)D \subset D \) and \( AS(t)x = S(t)Ax \) for all \( x \in D \) and every \( t \geq 0 \);
(S3) the resolvent equation holds

\[
S(t)x = x + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{AS(s)}{(t-s)^{1-\alpha}} x ds.
\]

The inequality in the following lemma will be useful in the proof of the main results.

**Definition 2.3** ([24]). A resolvent operator \( S(t) \) for (2.1) is called differentiable, if \( S(\cdot)x \in W_{loc}^{1,1}(\mathbb{R}_+; X) \) for all \( x \in D \) and there is \( \varphi \in L_{1,loc}^1(\mathbb{R}_+) \) such that

\[
\|\dot{S}(t)x\| \leq \varphi(t)\|x\|_D \quad a.e. \text{ on } \mathbb{R}_+, \forall x \in D.
\]
Definition 2.4 ([24]). A resolvent operator $S(t)$ for (2.1) is called analytic, if the function $S(\cdot) : [a, b] \to \mathcal{L}(X)$ admits analytic extension to a sector $\sum(0, \theta_0) = \{ \lambda \in \mathbb{C} : |\arg(\lambda)| < \theta_0 \}$ for some $0 < \theta_0 \leq \pi/2$.

Remark 2. Analytic resolvent operator has been introduced by Da Prato and Iannelli, see [8] for more details.

Consider the following nonlinear Volterra integral equation:

$$
\dot{x}(t) = A(t)x(t) + f(t, x(t)), \quad t \in (0, T),
$$

where $h \in L^1(I; X)$.

According to [24], we introduce the concept of mild solution of (2.2) as follows.

Definition 2.5 ([24]). A function $x \in C(I; X)$ is called a mild solution of the nonlinear Volterra integral equation (2.2) if $\int_0^t x(s) \, ds \in D$ for all $t \in I$ and

$$
x(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t A(t-s)x(s) \, ds, \quad t \in I,
$$

holds on $I$.

The next result compiles different properties related to the mild solution of (2.2).

Lemma 2.6 ([24]). (i) Suppose (2.2) admits a differentiable resolvent operator $S(t)$ and $h \in C(t; D)$. Then the function $x(t)$ defined by

$$
x(t) = \int_0^t S(t-s)h(s) \, ds + h(t), \quad t \in I,
$$

is a mild solution of (2.2).

(ii) Suppose (2.2) admits a analytic resolvent operator $S(t)$ and $h \in C^\alpha(t; X)$. Then the function $x(t)$ defined by

$$
x(t) = S(t)(h(t) - h(0)) + \int_0^t S(t-s)[h(s) - h(t)] \, ds + S(t)h(0), \quad t \in I,
$$

is a mild solution of (2.2).

To establish our next theorems, we need the following lemmas which are the useful tools to study the equicontinuity of the considered operators.

Lemma 2.7 ([12]). If $w \in C(I; D)$ and $W : I \to X$ is the function defined by $W(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{w(s)}{(t-s)^{1-\alpha}} \, ds$, then $W \in C^\alpha(I; D)$ and

$$
\|W\|_{C^\alpha(I; D)} \leq \frac{2}{\alpha \Gamma(\alpha)} \|w\|_{C(I; D)}.
$$

Lemma 2.8 ([12]). If $w \in C(I; X)$ and $W : I \to X$ is the function defined by $W(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{w(s)}{(t-s)^{1-\alpha}} \, ds$, then $W \in C^\alpha(I; X)$ and

$$
\|W\|_{C^\alpha(I; X)} \leq \frac{2}{\alpha \Gamma(\alpha)} \|w\|_{C(I; X)}.
$$

To end this section, we recall some properties of Kuratowski’s measures of noncompactness and several lemmas which will be used in the next section. The presentation here can be found in, for example, [6].
Lemma 2.9. Let us denote by $X$ a Banach space.
(i) Let $S, T$ be bounded sets of $X$ and $\lambda \in \mathbb{R}$. Then
\begin{itemize}
  \item[(i)] $\alpha(S) = 0$ if and only if $S$ is relatively compact;
  \item[(ii)] $S \subset T$ implies $\alpha(S) \leq \alpha(T)$;
  \item[(iii)] $\alpha(\lambda S) = |\lambda| \alpha(S)$, where $\lambda S = \{x = \lambda z : z \in S\}$;
  \item[(iv)] $\alpha(S + T) \leq \alpha(S) + \alpha(T)$, where $S + T = \{x = y + z : y \in S, z \in T\}$;
\end{itemize}
(2) Let $D \subset C(I; X)$ be bounded. Then $D(t)$ is bounded in $X$ and $\alpha(D(t)) \leq \alpha(D)$.
(3) Let $D \subset C(I; X)$ be bounded and equicontinuous. Then $\alpha(D(t))$ is continuous on $I$, and
$\alpha(D) = \max_{t \in I} \alpha(D(t))$.
(4) Let $D = \{u_n\} \subset C(I; X)$ be a bounded and countable set. Then $\alpha(D(t))$ is integrable on $I$, and
$\alpha\left(\left\{\int_t u_n(t)dt : n \in \mathbb{N}\right\}\right) \leq 2 \int_I \alpha(D(t))dt$.

The following two fixed-point theorems and the theorem of Ascoli-Arzelà play a key role in our proof of exact controllability.

Lemma 2.10 (Sadovskii). Let $\mathcal{P}$ be a condensing operator on a Banach space $X$, which is continuous and takes bounded sets into bounded sets, and $\alpha(\mathcal{P}(D)) < \alpha(D)$ for every bounded set $D$ of $X$ with $\alpha(D) > 0$. If $\mathcal{P}(S) \subset S$ for a convex, closed and bounded set $S$ of $X$, then $\mathcal{P}$ has a fixed point in $S$.

Lemma 2.11 (Mönch). Let $D$ be a closed and convex subset of a Banach space $E$ and $x_0 \in D$. Assume that the continuous operator $A : D \to D$ has the following property: $C \subset D$ countable, $C \subset \overline{\text{co}}(\{x_0\} \cup A(C)) \to C$ is relatively compact. Then $A$ has a fixed point in $D$.

Lemma 2.12 (Ascoli-Arzelà). $H \subset C[J, E]$ is relatively compact if and only if $H$ is equicontinuous and for any $t \in J$, $H(t)$ is a relatively compact set in $E$.

3. Exact controllability results. In this section, we establish sufficient conditions for the exact controllability of the fractional evolution system (1.1). As an application, an example is given to illustrate our main results. In the sequel we assume the resolvent operator $\{S(t)\}_{t \geq 0}$ is differentiable and denote by $\varphi_A$ the function introduced in Definition 2.3.

We first introduce the mild solution of the fractional evolution system (1.1). Obviously, if $x(\cdot)$ is a solution of (1.1), $t \in I$, then one has
\begin{equation}
  x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{A x(s)}{(t-s)^{1-\alpha}} ds + x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, x(s))}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{B u(s)}{(t-s)^{1-\alpha}} ds.
\end{equation}

By Definition 2.5 and the representation (3.1), we give the following concept of mild solution for system (1.1).

Definition 3.1. For each $u \in L^2(I; U)$, a mild solution of the fractional evolution system (1.1) on $J$ we mean a function $x \in C(J; X)$ which satisfies $\int_0^t \frac{x(s)}{(t-s)^{1-\alpha}} ds \in \mathcal{D}$ for all $t \in J$ and
\begin{align*}
  x(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, x(s))}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{B u(s)}{(t-s)^{1-\alpha}} ds,
\end{align*}
where $t \in J = [0, \tau]$, $\tau \in (0, a]$.

Definition 3.2 (Exact controllability). The fractional evolution system (1.1) is said to be exactly controllable on the interval $I = [0, a]$ if for every $x_0, x_1 \in X$, there exists a control function $u \in L^2(I; U)$ and a constant $\tau \in (0, a]$ such that a mild solution $x$ of system (1.1) on $J = [0, \tau]$ satisfies $x(\tau) = x_1$. 
Remark 3. Compared to the existing concept in [9, 14, 29] and so on in which \( \tau \) is equal to \( a \), our definition in which \( \tau \in (0, a] \) is weaker and can be regarded as an extension of the present notion of exact controllability.

For the sake of convenience, we list the main hypotheses and some notations to be used in the paper as follows:

\( (H1) \) \( f \in C(I \times X; D) \) and takes bounded sets into bounded sets, \( x_0 \in D \).

\( (H2) \) The linear operator \( B : L^2(I; U) \rightarrow L^1(I; D) \) is bounded, \( W(t) \) defined by

\[
W(t)u = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{Bu(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^t \dot{S}(t-s) \left( \int_0^s \frac{Bu(y)}{(s-y)^{1-\alpha}} dy \right) ds, \quad t \in I,
\]

has an induced inverse operator \( W^{-1}(t) \) which takes values in \( L^2(I; U)/\ker W(t) \) for every \( t \in I \) and there exist two positive constants \( M_1, M_2 > 0 \) such that \( \|B\|_{L(U, D)} \leq M_1 \) and \( \sup_{t \in I} \|W^{-1}(t)\|_{L(X; L^2(I; U)/\ker W(t))} \leq M_2 \).

\( (H3) \) For any bounded subset \( D \subset B_R = \{x \in X : \|x\| \leq R \} \) \( (R > 0) \), there exists a positive constant \( L \) such that \( \alpha(f(t, D)) \leq L \alpha(D), \ t \in I \).

Let us take

\[
F_x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(s, x(s)) \frac{(t-s)^{1-\alpha}}{\Gamma(\alpha)} ds, \quad B_u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t Bu(s) \frac{(t-s)^{1-\alpha}}{\Gamma(\alpha)} ds.
\]

Theorem 3.3. Suppose that the hypotheses \( (H1), (H2) \) and \( (H3) \) are satisfied. Then the fractional evolution system (1.1) is exactly controllable on \( I \).

Proof. Denote \( C = \sup\{\|f(t, x(t))\|_D : \|x(t)\| \leq R_0, \ t \in I\} \), where

\[
R_0 = (\|\varphi_A\|_{L^1(I)} + 1)(2\|x_0\|_D + 1).
\]

Set

\[
\tau = \min \left\{ a, \left( \frac{\alpha(\|x_0\|_D + 1)}{C + M_1 M_3} \right)^\frac{1}{\alpha}, \left( \frac{\Gamma^2(\alpha + 1)}{(4L + 1)(2\|\varphi_A\|_{L^1(I)} + 1)^2(2M_1 M_2 a^\alpha + 1)} \right)^\frac{1}{\alpha} \right\},
\]

where \( M_3 = M_2 \left[ \left( \|x_0\|_D + \frac{C a^\alpha}{\alpha} \right)^2 + \|x_1\| \right] \).

Using hypothesis \( (H2) \), for an arbitrary function \( x(\cdot) \in C(I; X) \), we define the control

\[
u_x(t) = W^{-1}(\tau) \left( x_1 - x_0 - F_x(\tau) - \int_0^\tau \dot{S}(\tau-s)(x_0 + F_x(s))ds \right)(t), \quad t \in I.
\]

Put \( J = [0, \tau] \). By considering Lemma 2.6 (i), we shall show that, the operator \( T : C(J; X) \rightarrow C(J; X) \) defined by

\[
(Tx)(t) = x_0 + F_x(t) + \int_0^t \dot{S}(t-s)(x_0 + F_x(s))ds + B_{u_x}(t) + \int_0^t \dot{S}(t-s)B_{u_x}(s)ds,
\]

has a fixed point. This fixed point is then a mild solution of the system (1.1) on \( J \).

Obviously, \( (Tx)(\tau) = x_1 \) which implies that \( u_x \) steers the system (1.1) from \( x_0 \) to \( x_1 \) in finite time \( \tau \). This means that system (1.1) is exactly controllable on \( J \).

Let \( \Omega = \{x \in C(J; X) : \|x(t)\| \leq R_0, \ t \in J\} \), then \( \Omega \) is obviously a closed ball in \( C(J; X) \). In the following, we divide the proof into several steps.

**Step 1.** We shall show \( T(\Omega) \subseteq \Omega \).
From hypothesis (H2), we obtain

\[
\|u_x(t)\| \leq \|W^{-1}(\tau)\|_{L^2(U\cap D)} \|x\|_{D} + \int_0^\tau \frac{\|f(s, x(s))\|_{D}}{(\tau - s)^{1-\alpha}} ds + \int_0^\tau \|x(t)\|_{D} + \int_0^s \frac{\|f(y, x(y))\|_{D}}{(s - y)^{1-\alpha}} dy ds \]
\[
\leq M_2 \left( \|x\|_{D} + \frac{Ca_\alpha}{\alpha} + \|\varphi_A\|_{L^1(I)} \left( \|x\|_{D} + \frac{Ca_\alpha}{\alpha} \right) + M_1 M_T \frac{\tau^\alpha}{\alpha} \right)
\]
\[
\leq \|x\|_{D} \left( \|\varphi_A\|_{L^1(I)} + 1 \right) + C_T \left( \|\varphi_A\|_{L^1(I)} + 1 \right) \left( \|x\|_{D} + \frac{Ca_\alpha}{\alpha} \right) + M_1 M_T \frac{\tau^\alpha}{\alpha}
\]

which implies \(T x \in \Omega\). Thus we deduce that \(T(\Omega) \subseteq \Omega\).

**Step 2.** We demonstrate that the operator \(T : \Omega \to \Omega\) is equicontinuous. For any \(x \in \Omega\), \(0 < t < t + h \leq \tau\), we get

\[
(T x)(t + h) - (T x)(t) = F_x(t + h) - F_x(t)
\]
\[
= \int_0^{t+h} \dot{S}(t + h - s) F_x(s) ds - \int_0^t \dot{S}(t - s) F_x(s) ds
\]
\[
+ \int_0^{t+h} \dot{S}(t + h - s) x_0 ds - \int_0^t \dot{S}(t - s) x_0 ds
\]
\[
+ B_{u_x}(t + h) - B_{u_x}(t)
\]
\[
+ \int_0^{t+h} \dot{S}(t + h - s) B_{u_x}(s) ds - \int_0^t \dot{S}(t - s) B_{u_x}(s) ds.
\]

Denote

\[
I_1 = F_x(t + h) - F_x(t),
\]
\[
I_2 = \int_0^{t+h} \dot{S}(t + h - s) F_x(s) ds - \int_0^t \dot{S}(t - s) F_x(s) ds,
\]
\[
I_3 = \int_0^{t+h} \dot{S}(t + h - s) x_0 ds - \int_0^t \dot{S}(t - s) x_0 ds,
\]
\[
I_4 = B_{u_x}(t + h) - B_{u_x}(t),
\]
\[ I_5 = \int_0^{t+h} \dot{S}(t + h - s)B_{u_x}(s)ds - \int_0^t \dot{S}(t - s)B_{u_x}(s)ds. \]

It is obviously that
\[
\| (Tx)(t + h) - (Tx)(t) \| \leq \| I_1 \| + \| I_2 \| + \| I_3 \| + \| I_4 \| + \| I_5 \|. 
\]

Therefore, we only need to check \( \| I_i \| \to 0 \) independently of \( x \in \Omega \) as \( h \to 0 \), \( i = 1, 2, 3, 4, 5 \).

As a matter of fact,
\[
\| I_1 \| \leq \int_0^t \left[ \frac{1}{(t - s)^{1-\alpha}} - \frac{1}{(t + h - s)^{1-\alpha}} \right] \| f(s, x(s)) \| ds \\
+ \int_{t+h}^{t} \frac{1}{(t + h - s)^{1-\alpha}} \| f(s, x(s)) \| ds \\
\leq C \frac{(t + h)\alpha - t^\alpha + h^\alpha}{\alpha} + C \frac{h^\alpha}{\alpha} \to 0, \text{ as } h \to 0.
\]

From Lemma 2.7 and
\[
I_2 = \int_0^{t+h} \dot{S}(t + h - s)F_x(s)ds - \int_0^t \dot{S}(t - s)F_x(s)ds \\
= \int_0^h \dot{S}(t + h - s)F_x(s)ds + \int_0^{t+h} \dot{S}(t + h - s)F_x(s)ds \\
- \int_0^t \dot{S}(t - s)F_x(s)ds \\
= \int_0^h \dot{S}(t + h - s)F_x(s)ds + \int_0^t \dot{S}(s)F_x(t + h - s)ds - \int_0^t \dot{S}(s)F_x(t - s)ds,
\]

it follows that
\[
\| I_2 \| \leq \int_0^h \| \dot{S}(t + h - s)F_x(s) \| ds + \int_0^t \| \dot{S}(s)(F_x(t - s + h) - F_x(t - s)) \| ds \\
\leq \int_0^h \varphi_A(t + h - s)\| F_x(s) \| ds + \int_0^t \varphi_A(s)\| F_x \|_{C^0([J, T])}h^\alpha ds \\
\leq C \frac{\tau^\alpha}{\alpha \Gamma(\alpha)} \int_0^h \varphi_A(t + h - s)ds + C \frac{2Ch^\alpha}{\alpha \Gamma(\alpha)} \int_0^t \varphi_A(s)ds \\
\leq \frac{C}{\Gamma(\alpha + 1)} \left( \tau^\alpha \int_0^h \varphi_A(t + h - s)ds + 2h^\alpha \| \varphi_A \|_{L^1([J])} \right) \to 0, \text{ as } h \to 0.
\]

Since
\[
I_3 = \int_0^{t+h} \dot{S}(t + h - s)x_0 ds - \int_0^t \dot{S}(t - s)x_0 ds \\
= \int_0^h \dot{S}(t + h - s)x_0 ds + \int_0^{t+h} \dot{S}(t + h - s)x_0 ds - \int_0^t \dot{S}(t - s)x_0 ds \\
= \int_0^h \dot{S}(t + h - s)x_0 ds,
\]
on one can obtain
\[
\| I_3 \| \leq \| x_0 \|_D \int_0^h \varphi_A(t + h - s)ds \to 0, \text{ as } h \to 0.
\]
As already done in the estimation of $\|I_1\|$, we have
\[
\|I_4\| \leq \int_0^t \left[ \frac{1}{(t-s)^{1-\alpha}} - \frac{1}{(t+h-s)^{1-\alpha}} \right] \|B_{ux}(s)\| ds
+ \int_t^{t+h} \|B_{ux}(s)\| ds
\leq M_1M_3 \frac{(t+h)^{\alpha} - t^\alpha + h^\alpha}{\alpha} + M_1M_3 \frac{h^\alpha}{\alpha} \to 0, \text{ as } h \to 0.
\]

Note that
\[
I_5 = \int_0^{t+h} \hat{S}(t+h-s)B_{ux}(s)ds - \int_0^t \hat{S}(t-s)B_{ux}(s)ds
= \int_0^t \hat{S}(t+h-s)B_{ux}(s)ds + \int_t^{t+h} \hat{S}(t+h-s)B_{ux}(s)ds
- \int_0^t \hat{S}(t-s)F_{ux}(s)ds
= \int_0^t \hat{S}(t+h-s)B_{ux}(s)ds + \int_0^t \hat{\Phi}(s)B_{ux}(t+h-s)ds
- \int_0^t \hat{S}(s)B_{ux}(s)ds,
\]

As already done in the discussion of $\|I_2\|$, we get
\[
\|I_5\| \leq \int_0^h \|\hat{S}(t+h-s)B_{ux}(s)\| ds
+ \int_0^t \|\hat{S}(s)(B_{ux}(t+s+h) - B_{ux}(t+s))\| ds
\leq M_1 M_3 \frac{\alpha h^\alpha}{\Gamma(\alpha)} \int_0^h \|\varphi_A(t+h-s)\|_{\mathcal{D}} ds + \frac{2M_1 M_3 h^\alpha}{\alpha \Gamma(\alpha)} \int_0^t \|\varphi_A(s)\|_{L^1(J)} ds
\leq \frac{M_1 M_3}{\Gamma(\alpha+1)} \left( r^\alpha \int_0^h \|\varphi_A(t+h-s)\| ds + 2h^\alpha \|\varphi_A\|_{L^1(J)} \right) \to 0, \text{ as } h \to 0.
\]

Consequently, $\|((\mathcal{T}x)(t+h)) - (\mathcal{T}x)(t)\| \to 0$ independently of $x \in \Omega$ as $h \to 0$, which indicates that the operator $\mathcal{T} : \Omega \to \Omega$ is equicontinuous.

**Step 3.** We prove that $\mathcal{T}$ is continuous on $\Omega$.

For this purpose, we assume that $y_n \in \Omega$ with $y_n \to y$ in $\Omega$. Obviously, for each $t \in J$,
\[
(\mathcal{T}y_n)(t) - (\mathcal{T}y)(t) = J_1 + J_2 + J_3 + J_4,
\]
where
\[
J_1 = \int_0^t (F_{y_n}(t) - F_y(t)),
J_2 = \int_0^t (\hat{S}(t-s)(F_{y_n}(s) - F_y(s)))ds,
J_3 = B_{uy_n}(t) - B_{uy}(t),
J_4 = \int_0^t (\hat{S}(t-s)(B_{uy_n}(s) - B_{uy}(s)))ds.
\]

It is easy to see that
\[
\|J_1\| \leq \int_0^t (t-s)^{\alpha-1} \|f(s, y_n(s)) - f(s, y(s))\| ds, \quad t \in J,
\]

and
\[
\|J_2\| \leq \int_0^t \|\hat{S}(t-s)(F_{y_n}(s) - F_y(s))\| ds, \quad t \in J.
\]

Finally,\[
\|J_3\| \leq \int_0^t \|B_{uy_n}(s) - B_{uy}(s)\| ds, \quad t \in J.
\]

It remains to estimate $\|J_4\|$. Since $\mathcal{T}$ is measurable, we have
\[
\|J_4\| \leq \int_0^t \|\hat{S}(t-s)(B_{uy_n}(s) - B_{uy}(s))\| ds.
\]

By using Hölder’s inequality, we get
\[
\|J_4\| \leq \left( \int_0^t \|\hat{S}(t-s)\|^{\frac{\alpha}{\alpha+1}} ds \right)^{\frac{\alpha+1}{\alpha}} \left( \int_0^t \|B_{uy_n}(s) - B_{uy}(s)\|^{\alpha+1} ds \right)^{\frac{1}{\alpha+1}}.
\]

By using the continuity of $\mathcal{T}$ on $\Omega$, we have
\[
\|J_1\| \to 0, \quad \|J_2\| \to 0, \quad \|J_3\| \to 0, \quad \|J_4\| \to 0, \quad \text{ as } h \to 0.
\]
and
\[ \|J_2\| \leq \int_0^t \varphi_A(t-s) \left( \int_0^s (s-z)^{\alpha-1} \|f(z, y_n(z)) - f(z, y(z))\| dz \right) ds, \quad t \in J. \]

Note that
\[ (\cdot - s)^{\alpha-1} \|f(s, y_n(s)) - f(s, y(s))\| \leq 2C(\cdot - s)^{\alpha-1} \in L^1(J, R^+), \]
and
\[ \|\varphi_A(\cdot - s) \left( \int_0^s (s-z)^{\alpha-1} \|f(z, y_n(z)) - f(z, y(z))\| dz \right) \| \leq \frac{2C\gamma^\alpha}{\alpha} \|\varphi_A(\cdot - s). \]

These together with the continuity of \( f \) and Lebesgue’s domination convergence theorem indicate
\[ \|J_1\| \to 0, \quad \|J_2\| \to 0, \quad \text{as} \ n \to +\infty. \]

On the other hand, for each \( t \in J \), we can easily get
\[
\begin{align*}
\|J_3\| & \leq \frac{M_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u_{y_n}(s) - u_y(s)\| ds \\
& \leq \frac{M_1 M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \|F_{y_n}(\tau) - F_y(\tau)\| + \int_0^\tau \varphi_A(\tau - s) \|F_{y_n}(s) - F_y(s)\| ds \right) ds,
\end{align*}
\]
\[
\begin{align*}
\|J_4\| & \leq \int_0^t \varphi_A(t-s) \|B_{u_{y_n}}(s) - B_u(s)\| ds \\
& \leq \frac{M_1}{\Gamma(\alpha)} \int_0^t \varphi_A(t-s) \left( \int_0^s (s-z)^{\alpha-1} \|u_{y_n}(z) - u_y(z)\| dz \right) ds \\
& \leq \frac{M_1 M_2}{\Gamma(\alpha)} \int_0^t \varphi_A(t-s) \left( \int_0^s (s-z)^{\alpha-1} \|F_{y_n}(z) - F_y(z)\| dz \right) ds \\
& \quad + \frac{M_1 M_2}{\Gamma(\alpha)} \int_0^t \varphi_A(t-s) \left( \int_0^s \varphi_A(\tau - z) \|F_{y_n}(z) - F_y(z)\| dz \right) ds.
\end{align*}
\]

Since
\[
\begin{align*}
(\cdot - s)^{\alpha-1} \left( \int_0^s (s-z)^{\alpha-1} \|F_{y_n}(z) - F_y(z)\| dz \right) & \leq \frac{2C\gamma^\alpha}{\alpha} \|\varphi_A(\cdot - s) \| L^1(J) \in L^1(J, R^+), \\
\varphi_A(\cdot - s) \left( \int_0^s (s-z)^{\alpha-1} \|F_{y_n}(z) - F_y(z)\| dz \right) & \leq \frac{2C\gamma^\alpha}{\alpha^2} \varphi_A(\cdot - s) \in L^1(J, R^+), \\
\varphi_A(\cdot - s) \left[ \int_0^s (s-z)^{\alpha-1} \left( \int_0^\tau \varphi_A(\tau - z) \|F_{y_n}(z) - F_y(z)\| dz \right) dz \right] & \leq \|\varphi_A\| L^1(J) \frac{2C\gamma^\alpha}{\alpha^2} \varphi_A(\cdot - s) \in L^1(J, R^+),
\end{align*}
\]
and
\[
\begin{align*}
(\cdot - s)^{\alpha-1} \|F_{y_n}(\tau) - F_y(\tau)\| & \leq \frac{2C\gamma^\alpha}{\alpha} (\cdot - s)^{\alpha-1} \in L^1(J, R^+), \\
(\cdot - s)^{\alpha-1} \left( \int_0^\tau \varphi_A(\tau - z) \|F_{y_n}(z) - F_y(z)\| dz \right) & \leq \|\varphi_A\| L^1(J) \frac{2C\gamma^\alpha}{\alpha} (\cdot - s)^{\alpha-1},
\end{align*}
\]
it follows from the similar discussion of \(|\|J_1\|\) and \(|\|J_2\|\) that  
\[ |\|J_3\| | \rightarrow 0, |\|J_4\| | \rightarrow 0, \text{ as } n \rightarrow +\infty. \]
Therefore, for each \(t \in J\), we have  
\[ ((T y_n)(t) - (T y)(t)) \rightarrow 0, \text{ as } n \rightarrow +\infty. \]  
(3.4)
Thus, from Lemma 2.12, it is not difficult to see that \(\{T y_n\}\) is relatively compact in \(C(J; X)\).
We now prove  
\[ |\|T y_n - T y| |_{C(J; X)} \rightarrow 0, \text{ as } n \rightarrow +\infty. \]  
(3.5)
As a matter of fact, if (3.5) is not true, then there exists a positive constant \(\varepsilon_0\) and a sequence \(\{y_{n_i}\} \subset \{y_n\}\) such that  
\[ |\|T y_{n_i} - T y| |_{C(J; X)} \geq \varepsilon_0 \quad (i = 1, 2, 3, \ldots). \]  
(3.6)
Since \(\{T y_n\}\) is relatively compact, there is a subsequence of \(\{T y_{n_i}\}\) which converges in \(C(J; X)\) to some \(v \in C(J; X)\). No loss of generality, we assume that \(\{T y_{n_i}\}\) itself converges to \(v\):  
\[ |\|T y_{n_i} - v| |_{C(J; X)} \rightarrow 0, \text{ as } i \rightarrow +\infty. \]  
(3.7)
In view of (3.4) and (3.7), we have \(v = T y\), and so, (3.7) contradicts (3.6). Then, (3.5) holds, and the continuity of \(T\) on \(\Omega\) is proved.
Denote \(B = \overline{\text{co}}T(\Omega)\) (i.e. the convex closure of \(T(\Omega)\)). It is easy to check \(T(B) \subseteq B\) and \(B\) is equicontinuous.

**Step 4.** We shall prove that \(T : B \rightarrow B\) is a condensing operator (i.e. \(\alpha(T D) < \alpha(D)\) for any bounded \(D \subset B\) which is not relatively compact).
Introduce the decomposition  
\[ (T x)(t) = (T_1 x)(t) + (T_2 x)(t) + (T_3 x)(t) + (T_4 x)(t), \quad x \in B, \]
where  
\[ (T_1 x)(t) = x_0 + \int_0^t \dot{S}(t - s) x_0 ds + F_\varepsilon (t), \]
\[ (T_2 x)(t) = \int_0^t \dot{S}(t - s) F_\varepsilon (s) ds, \]
\[ (T_3 x)(t) = B_{u_\varepsilon} (t), \]
\[ (T_4 x)(t) = \int_0^t \dot{S}(t - s) B_{u_\varepsilon} (s) ds. \]
From [5], for an arbitrary small positive constant \(\varepsilon\) and any bounded set \(D \subset B\), there exits a countable sequence \(D_0 = \{z_n\}_{n=1}^\infty \subset D\), such that  
\[ \alpha(T(D)) \leq 2\alpha(T(D_0)) + \varepsilon. \]  
(3.8)
Clearly,  
\[ \alpha((T D_0)(t)) \leq \alpha((T_1 D_0)(t)) + \alpha((T_2 D_0)(t)) + \alpha((T_3 D_0)(t)) + \alpha((T_4 D_0)(t)). \]  
(3.9)
In view of Lemma 2.9 (4) and hypothesis (H3), we obtain that
\[ \begin{align*}
\alpha ((T_1 D_0)(t)) &= \alpha \left( \left\{ x_0 + \int_0^t \dot{S}(t-s)x_0 ds + F_{zn}(t) \right\} \right) \\
&\leq \frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\alpha (\{f(s, z_0(s))\}) \, ds \\
&\leq \frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}L\alpha (D_0(s)) \, ds \\
&\leq \frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \cdot \alpha (D) \\
&\leq \frac{2}{2L\tau^\alpha} \cdot \alpha (D),
\end{align*} \]

and
\[ \begin{align*}
\alpha ((T_2 D_0)(t)) &= \alpha \left( \left\{ \int_0^t \dot{S}(t-s)F_{zn}(s) ds \right\} \right) \\
&\leq \frac{2}{\Gamma(\alpha)} \int_0^t \alpha \left( \left\{ \dot{S}(t-s) \left( \int_0^s \int_{(s-y)^{1-\alpha}} f(y, z_0(y)) \, dy \right) \right\} \right) \, ds \\
&\leq \frac{2}{\Gamma(\alpha)} \int_0^t \dot{\varphi}_A(t-s) \alpha \left( \left\{ \int_0^s \int_{(s-y)^{1-\alpha}} f(y, z_0(y)) \, dy \right\} \right) \, ds \\
&\leq \frac{4}{\Gamma(\alpha)} \int_0^t \dot{\varphi}_A(t-s) \left( \int_0^s (s-y)^{\alpha-1} \alpha (\{f(y, z_0(y))\}) \, dy \right) \, ds \\
&\leq \frac{4}{\Gamma(\alpha)} \int_0^t \dot{\varphi}_A(t-s) \left( \int_0^s (s-y)^{\alpha-1} L\alpha (D_0(y)) \, dy \right) \, ds \\
&\leq \frac{4L}{\alpha \Gamma(\alpha)} \int_0^t \dot{\varphi}_A(t-s) ds \cdot \alpha (D) \\
&\leq \frac{4L\|\dot{\varphi}_A\|_{L^1}(t) \tau^\alpha}{\alpha \Gamma(\alpha)} \cdot \alpha (D).
\end{align*} \]

For brevity, we take
\[ F_{zn} = x_1 - x_0 - F_{zn}(\tau) - \int_0^\tau \dot{S}(\tau-s) (x_0 + F_{zn}(s)) \, ds. \]

Note that
\[ \begin{align*}
\alpha (\{F_{zn}\}) &\leq \alpha (\{F_{zn}(\tau)\}) + \alpha \left( \left\{ \int_0^\tau \dot{S}(\tau-s)F_{zn}(s) ds \right\} \right) \\
&\leq \frac{2}{\Gamma(\alpha)} \int_0^\tau (\tau-s)^{\alpha-1}\alpha (\{f(s, z_0(s))\}) \, ds \\
&\quad + \frac{4}{\Gamma(\alpha)} \int_0^\tau \dot{\varphi}_A(\tau-s) \left( \int_0^s (s-y)^{\alpha-1} \alpha (\{f(y, z_0(y))\}) \, dy \right) \, ds \\
&\leq \frac{2}{\Gamma(\alpha)} \int_0^\tau (\tau-s)^{\alpha-1}L\alpha (D_0(s)) \, ds \\
&\quad + \frac{4}{\Gamma(\alpha)} \int_0^\tau \dot{\varphi}_A(\tau-s) \left( \int_0^s (s-y)^{\alpha-1} L\alpha (D_0(y)) \, dy \right) \, ds \\
&\leq \frac{2L}{\alpha \Gamma(\alpha)} \int_0^\tau (\tau-s)^{\alpha-1} ds \cdot \alpha (D) + \frac{4L}{\alpha \Gamma(\alpha)} \int_0^\tau \dot{\varphi}_A(\tau-s) ds \cdot \alpha (D), \\
&\leq \frac{2L}{\alpha \Gamma(\alpha)} \alpha (D) + \frac{4L\|\dot{\varphi}_A\|_{L^1}(t) \tau^\alpha}{\alpha \Gamma(\alpha)} \cdot \alpha (D) \\
&= \frac{2L}{\Gamma(\alpha+1)} \cdot \alpha (D).
\end{align*} \]
This together with Lemma 2.9 (4), hypotheses (H2) and (H3) implies

\[ \alpha \left( (T_4 D_0)(t) \right) = \alpha \left( \left\{ B_{u_{\alpha}^n}(t) \right\} \right) \leq \frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \alpha \left( \{ B_{u_{\alpha}^n}(s) \} \right) ds \]

\[ \leq \frac{2M_1M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \alpha \left( \{ F_{z_{\alpha}} \} \right) ds \]

\[ \leq \frac{2M_1M_2\alpha^\alpha}{\alpha \Gamma(\alpha)} \cdot \frac{2L \| \varphi_A \| _{L^1(I)} + 1}{\alpha} \cdot \alpha(D) \]

\[ = \frac{4LM_1M_2\alpha^\alpha(2\| \varphi_A \| _{L^1(I)} + 1)}{\Gamma^2(\alpha + 1)} \cdot \alpha(D), \tag{3.12} \]

and

\[ \alpha \left( (T_4 D_0)(t) \right) = \alpha \left( \left\{ \int_0^t \tilde{S}(t-s)B_{u_{\alpha}^n}(s) ds \right\} \right) \leq \frac{2}{\Gamma(\alpha)} \int_0^t \varphi_A(t-s) \alpha \left( \left\{ \int_0^t B_{u_{\alpha}^n}(y) (s-y)^{1-\alpha} dy \right\} \right) ds \]

\[ \leq \frac{4}{\Gamma(\alpha)} \int_0^t \varphi_A(t-s) \left( \int_0^t (s-y)^{\alpha-1} \alpha \left( \{ B_{u_{\alpha}^n}(s) \} \right) \right) dy ds \]

\[ \leq \frac{4M_1M_2}{\Gamma(\alpha)} \int_0^t \varphi_A(t-s) \left( \int_0^t (s-y)^{\alpha-1} \alpha \left( \{ F_{z_{\alpha}} \} \right) \right) dy ds \]

\[ \leq \frac{4M_1M_2\alpha^\alpha}{\alpha \Gamma(\alpha)} \cdot \frac{2L \| \varphi_A \| _{L^1(I)} + 1}{\alpha} \int_0^t \varphi_A(t-s) ds \cdot \alpha(D) \]

\[ = \frac{8LM_1M_2\alpha^\alpha\| \varphi_A \| _{L^1(I)} (2\| \varphi_A \| _{L^1(I)} + 1)}{\Gamma^2(\alpha + 1)} \cdot \alpha(D). \tag{3.13} \]

Since \( T(D_0) \subset B \) is bounded and equicontinuous, one gets from the Lemma 2.9 (3) that

\[ \alpha \left( T(D_0) \right) = \max_{t \in J} \alpha \left( (T D_0)(t) \right). \tag{3.14} \]

Then, from (3.8)-(3.14), it follows that

\[ \alpha \left( (TD) \right) \leq 2 \max_{t \in J} \left[ \alpha \left( (T_1 D_0)(t) \right) + \alpha \left( (T_2 D_0)(t) \right) + \alpha \left( (T_3 D_0)(t) \right) + \alpha \left( (T_4 D_0)(t) \right) \right] + \varepsilon \]

\[ \leq \frac{4L\tau^\alpha}{\Gamma^2(\alpha + 1)} \cdot \alpha(D) + \frac{8L \| \varphi_A \| _{L^1(I)} \tau^\alpha}{\alpha \Gamma(\alpha)} \cdot \alpha(D) \]

\[ + \frac{8LM_1M_2\alpha^\alpha(2\| \varphi_A \| _{L^1(I)} + 1)}{\Gamma^2(\alpha + 1)} \cdot \alpha(D) \]

\[ + \frac{16LM_1M_2\alpha^\alpha\| \varphi_A \| _{L^1(I)} (2\| \varphi_A \| _{L^1(I)} + 1)}{\Gamma^2(\alpha + 1)} \cdot \alpha(D) + \varepsilon \]

\[ \leq \frac{4L \| \varphi_A \| _{L^1(I)} + 1}{\Gamma^2(\alpha + 1)} \cdot \alpha(D) \| \varphi_A \| _{L^1(I)} + 1 \leq \frac{2M_1M_2\alpha^\alpha + 1}{\alpha \Gamma(\alpha)} + \varepsilon. \]

By the arbitrariness of \( \varepsilon \) and (3.2), we have

\[ \alpha \left( (TD) \right) \leq \frac{4L \| \varphi_A \| _{L^1(I)} + 1}{\Gamma^2(\alpha + 1)} \cdot \alpha(D) \leq \frac{2M_1M_2\alpha^\alpha + 1}{\alpha \Gamma(\alpha)} \cdot \alpha(D) < \alpha(D), \]

which implies \( T : B \to B \) is a condensing operator.

Therefore, due to Lemma 2.10, \( T \) has at least one fixed point \( x \) on \( B \). It is easy to see that \( x \) is a mild solution of the fractional evolution system (1.1) on \( J \) satisfying
$x(\tau) = x_1$ and thus system (1.1) is exactly controllable on $I$. This completes the proof.

**Remark 4.** In Theorem 3.3, $f$ is supposed to be continuous instead of compact or Lipschitz continuous, and $f$ has no other growth conditions. We put the differentiable condition on resolvent operator instead of assuming the semigroup $\{T(t) : t \geq 0\}$ is compact. The concept of exact controllability here is weaker than the existing notion. In addition, if $f$ is compact or Lipschitz continuous, then condition (H3) is satisfied. Therefore, our results extend some previous conclusions such as in [9, 14, 29].

In the following, we shall suppress the assumptions that $f$ and $x_0$ are $\mathcal{D}$-valued and suppose that the resolvent operator $\{S(t)\}_{t \geq 0}$ is analytic. To this end, we need to replace the hypothesis (H1) and (H2) with the following hypotheses:

(H1) $f \in C(I \times X; X)$ and takes bounded sets into bounded sets, $x_0 \in X$.

(H2) The linear operator $B : L^2(I; U) \rightarrow L^1(I; X)$ is bounded, $W(t)$ defined by

$$W(t)u = S(t)Bu(t) + \int_0^t \dot{S}(t-s)(Bu(s) - Bu(t)) \, ds,$$

has an induced inverse operator $W^{-1}(t)$ which takes values in $L^2(I; U)/\ker W(t)$ for every $t \in I$ and there exist two positive constants $M_1, M_2 > 0$ such that $\|B\|_{L(C(U,X))} \leq M_1$ and $\sup_{t \in I} \|W^{-1}(t)\|_{L(C(X; L^2(I; U))/\ker W(t))} \leq M_2$.

According to [24, Chapter 1] and [24, Chapter 2], there exist positive numbers $M_0, M_1$ and $M_2$ such that $\|S(t)\|_{L(X)} \leq M_0$ for all $t \in I$ and $\|\dot{S}(t)\|_{L(X)} \leq M_1 t^{-1}$, $\|\dot{S}(t)\|_{L(X)} \leq M_2 t^{-2}$ for all $t \in (0, a]$.

**Theorem 3.4.** Suppose that the hypotheses (H1)', (H2)' and (H3) are satisfied. Then the fractional evolution system (1.1) is exactly controllable on $I$.

**Proof.** We let $C = \sup \{\|f(t, x(t))\| : \|x(t)\| \leq R_0, \ t \in I\}$, where $R_0 = M_0(2\|x_0\| + 1)$. Set

$$\tau = \min\{a, \left(\frac{\alpha^2 M_0(\|x_0\| + 1)}{(M_0 \alpha + 2M_1)(C + M_1 M_3)}\right)^{\frac{1}{2}}, \left(\frac{4L(M_0 \alpha + 6M_1)\Gamma^2(\alpha + 1) + 2M_1 M_2 \alpha (M_0 \Gamma(\alpha + 1) + 12M_1 \alpha \Gamma)}{\alpha^2 C + 2M_1 C} \right)^{\frac{1}{2}}\},$$

where $M_3 = M_2 \left(\|x_1\| + M_0 \|x_0\| + \frac{(M_0 \alpha + 2M_1 \alpha \Gamma)}{\alpha^2 C + 2M_1 C}\right)$.

For an arbitrary function $x(\cdot) \in C(I; X)$, $t \in I$, by hypothesis (H2)', we can define the following control

$$u_x(t) = W^{-1}(\tau) \left(x_1 - S(\tau)x_0 - S(\tau)F_x(\tau) - \int_0^\tau \dot{S}(\tau-s)(F_x(s) - F_x(\tau)) \, ds\right)(t).$$

Put $J = [0, \tau]$. By Lemma 2.6 (ii), we shall show that the operator $T : C(J; X) \rightarrow C(J; X)$ defined by

$$(Tx)(t) = S(t)x_0 + S(t)F_x(t) + S(t)Bu_x(t) + \int_0^t \dot{S}(t-s)(F_x(s) - F_x(t)) \, ds + \int_0^t \dot{S}(t-s)(Bu_x(s) - Bu_x(t)) \, ds$$

(3.15)
has a fixed point.

Denote \( \Omega = \{ x \in C(J; X) : \| x(t) \| \leq R_0, \ t \in J \} \). Next, we also divide the proof into 4 steps.

**Step 1.** \( T(\Omega) \subseteq \Omega \).

For any \( x \in \Omega, s, t \in J \) with \( s \leq t \), it is easy to see that
\[
\| F_x(s) - F_x(t) \| \leq \int_0^s \left( \frac{1}{(s - y)^{1-\alpha}} - \frac{1}{(s + (t - s) - y)^{1-\alpha}} \right) \| f(y, x(y)) \| dy
\]
\[
+ \int_s^{s + (t - s)} \frac{1}{C(t - s)^{\alpha}} \| f(y, x(y)) \| dy
\]
\[
\leq \frac{C(t - s)^{\alpha}}{\alpha} + \frac{2}{\alpha} = \frac{2C(t - s)^{\alpha}}{\alpha},
\]
and
\[
\| Bux(s) - Bux(t) \| \leq \frac{2M1M3}{\alpha}(t - s)^{\alpha}.
\]

From the hypothesis (H2)' and (3.16), it follows that
\[
\| u_x(t) \| \leq M2 \left( \| x_1 \| + M0\| x_0 \| + M0 \frac{Ca^{\alpha}}{\alpha} + \int_0^t \frac{M1}{\alpha} \| f_x(\tau) - F_x(s) \| d\tau \right)
\]
\[
\leq M2 \left( \| x_1 \| + M0\| x_0 \| + M0 \frac{Ca^{\alpha}}{\alpha} + \frac{2M1C}{\alpha} \int_0^t \frac{1}{\alpha} (t - s)^{\alpha} d\tau \right)
\]
\[
\leq M2 \left( \| x_1 \| + M0\| x_0 \| + \frac{(M0Ca + 2M1C)\alpha}{\alpha^2} \right) = M3, \ t \in I.
\]

Then, for any \( x \in \Omega \) and \( t \in J \), by (H2)', (3.15) and (3.17), we infer that
\[
\|(Tx)(t)\| \leq M0\| x_0 \| + \frac{M0C\tau^{\alpha}}{\alpha} + \frac{M0M1M3\tau^{\alpha}}{\alpha^2}
\]
\[
+ \int_0^t \frac{M1}{\alpha} \frac{2C(t - s)^{\alpha}}{\alpha^2} ds + \int_0^t \frac{M1}{\alpha} \frac{2M1M3(t - s)^{\alpha}}{\alpha^2} ds
\]
\[
\leq M0\| x_0 \| + \frac{M0C\tau^{\alpha}}{\alpha} + \frac{M0M1M3\tau^{\alpha}}{\alpha^2}
\]
\[
+ \frac{2M1C\tau^{\alpha}}{\alpha^2} + \frac{2M1M1M3\tau^{\alpha}}{\alpha^2}
\]
\[
\leq M0\| x_0 \| + \frac{M0\alpha}{\alpha^2} + \frac{M0M1M3\alpha}{\alpha^2} + \frac{2M1C\tau^{\alpha}}{\alpha^2} + \frac{2M1M1M3\tau^{\alpha}}{\alpha^2}
\]
\[
\leq M0(2\| x_0 \| + 1) = R_0,
\]
which indicates \( Tx \in \Omega \). Hence, the conclusion \( T(\Omega) \subseteq \Omega \) follows.

**Step 2.** The operator \( T : \Omega \to \Omega \) is equicontinuous.

For any \( x \in \Omega, 0 < t < t + h \leq \tau \), one has
\[
(Tx)(t + h) - (Tx)(t)
\]
\[
= (S(t + h) - S(t)x_0 + S(t + h)F_x(t + h) - S(t)F_x(t) + S(t + h)B_{ux}(t + h) - S(t)B_{ux}(t)
\]
\[
+ \int_0^{t + h} \hat{S}(t + h - s)(F_x(s) - F_x(t + h))ds - \int_0^t \hat{S}(t - s)(F_x(s) - F_x(t))ds
\]
\[
+ \int_0^{t + h} \hat{S}(t + h - s)(B_{ux}(s) - B_{ux}(t + h))ds
\]
\[
- \int_0^t \hat{S}(t - s)(B_{ux}(s) - B_{ux}(t))ds.
\]

Denote
\[
I_1 = (S(t + h) - S(t))x_0.
\]
\[I_2 = S(t+h)F_x(t+h) - S(t)F_x(t),\]
\[I_3 = S(t+h)B_{u_x}(t+h) - S(t)B_{u_x}(t),\]
\[I_4 = \int_0^{t+h} \dot{S}(t+h-s)(F_x(s) - F_x(t+h))ds - \int_0^t \dot{S}(t-s)(F_x(s) - F_x(t))ds,\]
\[I_5 = \int_0^{t+h} \dot{S}(t+h-s)(B_{u_x}(s) - B_{u_x}(t+h))ds - \int_0^t \dot{S}(t-s)(B_{u_x}(s) - B_{u_x}(t))ds.\]

Clearly,
\[\| (T_x)(t+h) - (T_x)(t) \| \leq \| I_1 \| + \| I_2 \| + \| I_3 \| + \| I_4 \| + \| I_5 \|.\]

Next, we shall check \( \| I_i \| \to 0 \) independently of \( x \in \Omega \) as \( h \to 0 \), \( i = 1, 2, 3, 4, 5 \).

Since \( S(\cdot)x_0 \) is continuous, one has \( \| I_1 \| \to 0 \), as \( h \to 0 \). Using Lemma 2.8, we have
\[\| I_2 \| \leq \| S(t+h)\|_{\mathcal{L}(X)}\| F_x(t+h) - F_x(t) \| + \| (S(t+h) - S(t))F_x(t) \| \leq M_0\| F_x \|_{C^0}h^\alpha + \int_t^{t+h} \| \dot{S}(s)F_x(t) \| ds\]
\[\leq M_0\| F_x \|_{C^0}h^\alpha + M_1\| F_x \|_{C^0} \int_t^{t+h} \frac{s^\alpha}{s} ds\]
\[\leq M_0 \frac{2C}{\alpha \Gamma(\alpha)}h^\alpha + M_1 \frac{2C}{\alpha \Gamma(\alpha)}h^\alpha \to 0, \text{ as } h \to 0.\]

As already done in the discussion of \( \| I_i \| \) in Theorem 3.3, one can obtain
\[\| I_2 \| \leq \| S(t+h)\|_{\mathcal{L}(X)}\| B_{u_x}(t+h) - B_{u_x}(t) \| + \| (S(t+h) - S(t))B_{u_x}(t) \| \leq M_0 \int_0^t \left[ \frac{1}{(t-s)^{1-\alpha}} - \frac{1}{(t+h-s)^{1-\alpha}} \right] \| B_{u_x}(s) \| ds\]
\[+ M_0 \int_t^{t+h} \| B_{u_x}(s) \| (t+h-s)^{1-\alpha} ds + \| (S(t+h) - S(t))\|_{\mathcal{L}(X)} \frac{M_1\bar{M}_3^\alpha}{\alpha}\]
\[\leq M_0M_1\bar{M}_3 \frac{(t+h)^{1-\alpha} - t^{1-\alpha} + h^\alpha}{\alpha} + M_0M_1\bar{M}_3 \frac{h^\alpha}{\alpha} + \| (S(t+h) - S(t))\|_{\mathcal{L}(X)} \frac{M_1\bar{M}_3^\alpha}{\alpha}\]
\[\to 0, \text{ as } h \to 0.\]

Note that
\[I_4 = \int_0^{t+h} \dot{S}(t+h-s)(F_x(s) - F_x(t+h))ds - \int_0^t \dot{S}(t-s)(F_x(s) - F_x(t))ds\]
\[= \int_0^{t+h} \dot{S}(t+s-h)(F_x(s) - F_x(t+h))ds\]
\[+ \int_t^{t+h} \dot{S}(t+s-h)(F_x(s) - F_x(t+h))ds - \int_0^t \dot{S}(t-s)(F_x(s) - F_x(t))ds\]
\[= \int_0^{t+h} \dot{S}(t+s-h)(F_x(t) - F_x(t+h))ds\]
\[+ \int_t^{t+h} \dot{S}(t+s-h)(F_x(s) - F_x(t+h))ds - \int_0^t \dot{S}(t+s-h)(F_x(t) - F_x(t+h))ds\]
\[= \int_0^{t+h} (\dot{S}(t+s) - \dot{S}(s))(F_x(t+s) - F_x(t+h))ds\]
\[+ \int_0^{t+h} \dot{S}(t+s-h)(F_x(s) - F_x(t))ds\]
\[+ \int_t^{t+h} \dot{S}(t+s-h)(F_x(s) - F_x(t+h))ds.]
From Lemma 2.8, it follows that

\[
\|I_4\| \leq \int_0^t \|\hat{S}(h + s) - \hat{S}(s)\|_{L(X)} \|F_x(t - s) - F_x(t)\| ds \\
+ \left\| \int_0^t \hat{S}(h + s)(F_x(t) - F_x(t + h)) ds \right\| \\
+ \int_0^{t+h} \|\hat{S}(t + h - s)\|_{L(X)} \|F_x(s) - F_x(t + h)\| ds \\
\leq \int_0^t \int_s^{s+h} \|\hat{S}(\xi)\|_{L(X)} \|F_x\|_{C^0} s^\alpha d\xi ds \\
+ \|\hat{S}(h) - S(h)\|_{L(X)} \|F_x(t) - F_x(t + h)\| \\
+ \frac{M_1 \|F_x\|_{C^0}}{h} \int_t^{t+h} \frac{d\xi ds}{(t + h - s)^{1-\alpha}} \\
\leq \frac{2}{\alpha(1 - \alpha)} (\|F_x\|_{C^0} M_2 + 2M_0 \|F_x\|_{C^0} h^\alpha + M_1 \|F_x\|_{C^0}) \frac{h^\alpha}{\alpha} \\
\leq \frac{2}{\alpha(1 - \alpha)} \frac{M_1 M_3 M_2}{\Gamma(\alpha)} + 2M_0 \frac{2}{\alpha(1 - \alpha)} \frac{M_1}{\Gamma(\alpha)} h^\alpha + M_1 \frac{2}{\alpha(1 - \alpha)} \frac{h^\alpha}{\alpha} \rightarrow 0, \text{ as } h \rightarrow 0.
\]

Just by the same way, as \(h \rightarrow 0\), we also have

\[
\|I_5\| \leq \frac{2h^\alpha}{\alpha(1 - \alpha)} \|B_{u^y}(\cdot)\|_{C^0} M_2 + 2M_0 \|B_{u^y}(\cdot)\|_{C^0} h^\alpha + M_1 \|B_{u^y}(\cdot)\|_{C^0} \frac{h^\alpha}{\alpha} \\
\leq \frac{2h^\alpha}{\alpha(1 - \alpha)} \frac{M_1 M_3 M_2}{\Gamma(\alpha)} + 2M_0 \frac{2M_1 M_3 h^\alpha}{\alpha(1 - \alpha)} + M_1 \frac{2M_1 M_3 h^\alpha}{\alpha(1 - \alpha)} \alpha \rightarrow 0.
\]

Then, we assert that \(\|(T x)(t+h) - (T x)(t)\| \rightarrow 0\), as \(h \rightarrow 0\), for all \(x \in \Omega\). Therefore, the operator \(T : \Omega \rightarrow \Omega\) is equicontinuous.

**Step 3.** The operator \(T\) is continuous on \(\Omega\).

Assume that \(y_n \in \Omega\) with \(y_n \rightarrow y\) in \(\Omega\). Obviously, for each \(t \in J\),

\[
\|(T y_n)(t) - (T y)(t)\| \\
\leq M_0 \|F_{y_n}(t) - F_y(t)\| + M_0 \|B_{u_{y_n}}(t) - B_{u_y}(t)\| \\
+ \int_0^t \|\hat{S}(t - s)\| [(F_{y_n}(s) - F_{y_n}(t)) - (F_y(s) - F_y(t))]|ds \\
+ \int_0^t \|\hat{S}(t - s)\| [(B_{u_{y_n}}(s) - B_{u_{y_n}}(t)) - (B_{u_y}(s) - B_{u_y}(t))]|ds.
\]

Note that

\[
\|F_{y_n}(t) - F_y(t)\| \leq \int_0^t (t - s)^{\alpha-1} \|f(s, y_n(s)) - f(s, y(s))\| ds, \quad t \in J,
\]

\[
\|B_{u_{y_n}}(t) - B_{u_y}(t)\| \leq \frac{M_1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \|u_{y_n}(s) - u_y(s)\| ds, \quad t \in J,
\]

\[
\leq M_2 (M_0 \|F_{y_n}(\cdot) - F_y(\cdot)\| \\
+ \int_0^t \|\hat{S}(t - s)\| [(F_{y_n}(s) - F_{y_n}(t)) - (F_y(s) - F_y(t))]|ds),
\]

and

\[
(\cdot - s)^{\alpha-1} \|f(s, y_n(s)) - f(s, y(s))\| \leq 2C(\cdot - s)^{\alpha-1} \in L^1(J, R^+),
\]

\[
(\cdot - s)^{\alpha-1} \|u_{y_n}(s) - u_y(s)\| \leq 2M_3 (\cdot - s)^{\alpha-1} \in L^1(J, R^+),
\]

Therefore,

\[
\|(T y_n)(t) - (T y)(t)\| \leq M_0 + M_1 - M_2 (M_0 + M_1 - M_3) \|u_y(\cdot)\|_{L^1(J, R^+)} + M_1 - M_3 (M_0 + M_1 - M_2) \|u_y(\cdot)\|_{L^1(J, R^+)} \\
+ M_2 (M_0 + M_1 - M_3) \|F_y(\cdot) - F_{y_n}(\cdot)\|_{L^1(J, R^+)} + M_1 - M_2 (M_0 + M_1 - M_3) \|F_y(\cdot) - F_{y_n}(\cdot)\|_{L^1(J, R^+)},
\]

as \(h \rightarrow 0\), we have

\[
\|(T y_n)(t) - (T y)(t)\| \rightarrow 0, \quad t \in J.
\]

Hence, \((T y_n)(t) \rightarrow (T y)(t)\) in \(L^1(J, R^+)\), which implies that \(T : \Lambda \rightarrow \Lambda\) is continuous for \(\alpha \in (0, 1)\). This completes the proof.
theorem, we have that

\[
\| \hat{S}(\tau - s)[(F_{y_n}(s) - F_{y_n}(\cdot)) - (F_{y}(s) - F_{y}(\cdot))] \|
\leq \frac{\| \hat{S}(\tau - s) \|}{M_1} \frac{4C(\cdot - s)^{\alpha}}{\alpha}
\leq \frac{4M_1C}{\alpha} \cdot s^{-\alpha - 1} \in L^1(J, R^+),
\]

\[
\| \hat{S}(\cdot - s)[(F_{y_n}(s) - F_{y_n}(\cdot)) - (F_{y}(s) - F_{y}(\cdot))] \|
\leq \frac{\| \hat{S}(\cdot - s) \|}{M_1} \frac{4C(\cdot - s)^{\alpha}}{\alpha}
\leq \frac{4M_1C}{\alpha} \cdot s^{-\alpha - 1} \in L^1(J, R^+),
\]

\[
\| \hat{S}(\cdot - s)[(B_{u_n}(s) - B_{u_n}(\cdot)) - (B_{u}(s) - B_{u}(\cdot))] \|
\leq \frac{\| \hat{S}(\cdot - s) \|}{M_1} \frac{4M_1S(\cdot - s)^{\alpha}}{\alpha}
\leq \frac{4M_1M_3}{\alpha} \cdot s^{-\alpha - 1} \in L^1(J, R^+).
\]

For each \( t \in J \), by hypothesis (H1)' and Lebesgue’s domination convergence theorem, we have that \( \| (T_{ny})(t) - (Ty)(t) \| \to 0 \), as \( n \to +\infty \). Similar to the proof of Step 3 in Theorem 3.3, we can, by Lemma 2.12, demonstrate \( \| T_{ny} - Ty \|_{C(J, X)} \to 0 \), as \( n \to +\infty \), i.e., \( T \) is continuous on \( \Omega \).

Let \( B = \overline{\sigma(T(\Omega))} \). Obviously, \( T(B) \subseteq B \) and \( B \) is equicontinuous.

**Step 4.** \( T : B \to B \) is a condensing operator.

Set

\[
(Tx)(t) = (T_1x)(t) + (T_2x)(t) + (T_3x)(t) + (T_4x)(t), \quad x \in B,
\]

where

\[
(T_1x)(t) = S(t)x_0 + S(t)F_x(t),
\]

\[
(T_2x)(t) = \int_0^t \hat{S}(t-s)(F_x(s) - F_x(t))ds,
\]

\[
(T_3x)(t) = (t)B_{u_x}(t),
\]

\[
(T_4x)(t) = \int_0^t \hat{S}(t-s)(B_{u_x}(s) - B_{u_x}(t))ds.
\]

Denoted by \( D_0 = \{ z_n \}_{n=1}^{\infty} \) the countable sequence in bounded set \( D \subseteq B \) satisfying

\[
\alpha(T(D)) \leq 2\alpha(T(D_0)) + \varepsilon, \quad (3.18)
\]

for arbitrary \( \varepsilon > 0 \). Thus,

\[
\alpha ((TD_0)(t)) \leq \alpha ((T_1D_0)(t)) + \alpha ((T_2D_0)(t)) + \alpha ((T_3D_0)(t)) + \alpha ((T_4D_0)(t)) \cdot (3.19)
\]
From Lemma 2.9 (4) and hypothesis (H3), it follows that

\[
\alpha ((T_1 D_0)(t)) = \alpha \left( \left\{ S(t)x_0 + S(t) \frac{1}{\Gamma(\alpha)} \int_0^t f(s, z_n(s)) \, ds \right\} \right)
\leq \frac{2M_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \alpha \left( \{ f(s, z_n(s)) \} \right) \, ds
\leq \frac{2M_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \alpha (D_0(s)) \, ds
\leq \frac{2M_0 L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, ds \cdot \alpha (D)
\leq \frac{2M_0 L \tau^\alpha}{\Gamma(\alpha+1)} \cdot \alpha (D).
\]

Moreover, by the well known inequality

\[
|s^\alpha - t^\alpha| \leq (t-s)^\alpha, \quad \alpha \in (0, 1), \quad 0 < s \leq t,
\]

we have

\[
\alpha ((T_2 D_0)(t)) = \alpha \left( \left\{ \int_0^t S(t-s) \frac{1}{\Gamma(\alpha)} \left( \int_0^s f(y, z_n(y)) \, dy - \int_0^t f(y, z_n(y)) \, dy \right) \, ds \right\} \right)
\leq \frac{2}{\Gamma(\alpha)} \int_0^t \frac{M_1}{t-s} \alpha \left( \left\{ \int_0^s f(y, z_n(y)) \, dy - \int_0^t f(y, z_n(y)) \, dy \right\} \right) \, ds
\leq \frac{2M_1}{\Gamma(\alpha)} \int_0^t \frac{1}{t-s} \alpha \left( \left\{ \int_0^s (s-y)^{\alpha-1} - (t-y)^{\alpha-1} \, f(y, z_n(y)) \, dy \right\} \right) \, ds
\leq \frac{4M_1}{\Gamma(\alpha)} \int_0^t \frac{1}{t-s} \alpha \left( \left\{ \int_0^t (s-y)^{\alpha-1} - (t-y)^{\alpha-1} \, \alpha \{ f(y, z_n(y)) \} \, dy \right\} \right) \, ds
\leq \frac{2M_1 \alpha (D)}{\Gamma(\alpha)} \int_0^t \frac{1}{t-s} \left( \int_0^t (s-y)^{\alpha-1} - (t-y)^{\alpha-1} \, \alpha \{ f(y, z_n(y)) \} \, dy \right) \, ds
\leq \frac{4M_1 \alpha (D)}{\Gamma(\alpha)} \int_0^t \frac{1}{t-s} \left( \int_0^t (s-y)^{\alpha-1} - (t-y)^{\alpha-1} \, \alpha \{ f(y, z_n(y)) \} \, dy \right) \, ds
\leq \frac{4M_1 \alpha (D)}{\Gamma(\alpha)} \int_0^t \frac{1}{t-s} \left( \int_0^t (s-y)^{\alpha-1} - (t-y)^{\alpha-1} \, \alpha \{ f(y, z_n(y)) \} \, dy \right) \, ds
\leq \frac{4M_1 \alpha (D)}{\Gamma(\alpha)} \int_0^t \frac{1}{t-s} \left( \int_0^t (s-y)^{\alpha-1} - (t-y)^{\alpha-1} \, \alpha \{ f(y, z_n(y)) \} \, dy \right) \, ds
\leq \frac{4M_1 \alpha (D)}{\Gamma(\alpha+1)} \int_0^t \frac{1}{t-s} \left( \int_0^t (s-y)^{\alpha-1} + (t-s)^{\alpha} \, \alpha \right) \, ds
\leq \frac{4M_1 \alpha (D)}{\Gamma(\alpha+1)} \int_0^t \frac{1}{t-s} \left( \int_0^t (s-y)^{\alpha-1} + (t-s)^{\alpha} \, \alpha \right) \, ds
\leq \frac{12M_1 \alpha (D)}{\Gamma(\alpha+1)} \int_0^t \frac{1}{t-s} \left( \int_0^t (s-y)^{\alpha-1} \, \alpha \right) \, ds
\leq \frac{12M_1 \alpha (D)}{\Gamma(\alpha+1)} \int_0^t \frac{1}{t-s} \left( \int_0^t (s-y)^{\alpha-1} \, \alpha \right) \, ds
\leq \frac{\alpha (\alpha+1)}{\alpha (\alpha+1)} \cdot \alpha (D).
\]

\[(3.22)\]
By means of (3.20) and (3.22), one has

\[
\alpha \left( (\mathcal{T}_3 D_0)(t) \right) \leq 2 M_0 M_1 M_2 \int_0^t (t-s)^{\alpha-1} \alpha \left( \{ x_1 - S(\tau)x_0 - S(\tau)F_{z_n}(\tau) \} \right) ds \\
+ \frac{2 M_0 M_1 M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \alpha \left( \left\{ \int_0^t \tilde{S}(\tau-s)(F_{z_n}(s) - F_{z_n}(\tau)) ds \right\} \right) ds \\
\leq \frac{2 M_0 M_1 M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \alpha \left( \{ (\mathcal{T}_1 D_0)(\tau) \} + \alpha \{ (\mathcal{T}_2 D_0)(\tau) \} \right) ds \\
\leq \frac{2 M_0 M_1 M_2 \alpha^n L (M_0 \alpha + 6 M_1) \tau^\alpha}{\alpha^{\Gamma(2)(\alpha+1)}} \cdot \alpha(D),
\]

which implies

\[
\alpha \left( \{ B_{z_n}(t) \} \right) \leq \frac{4 M_1 M_2 \alpha^n L (M_0 \alpha + 6 M_1) \tau^\alpha}{\alpha^{\Gamma(2)(\alpha+1)}} \cdot \alpha(D).
\]

Combining Lemma 2.9 (4), hypothesis (H3)' with (3.21) and (3.24), we can obtain

\[
\begin{align*}
\alpha \left( (\mathcal{T}_3 D_0)(t) \right) & = \alpha \left( \left\{ \int_0^t \tilde{S}(t-s) \frac{1}{\Gamma(\alpha)} \left( \int_0^s B_{u_{z_n}}(y) \right) ds \right\} \frac{(s-y)^{1-\alpha}}{y} \right) \\
& \leq \frac{2}{\Gamma(\alpha)} \int_0^t \frac{M_1}{t-s} \alpha \left( \left\{ \int_0^s \frac{B_{u_{z_n}}(y)}{(s-y)^{1-\alpha}} ds \right\} \left( \int_0^s \frac{y}{(t-y)^{1-\alpha}} dy \right) \right) ds \\
& \leq \frac{4 M_1}{\Gamma(\alpha)} \int_0^t \frac{1}{t-s} \alpha \left( \left\{ \int_0^s (s-y)^{1-\alpha} - (t-y)^{1-\alpha} B_{u_{z_n}}(y) dy \right\} \right) ds \\
& \leq \frac{4 M_1}{\Gamma(\alpha)} \int_0^t \frac{1}{t-s} \alpha \left( \left\{ \int_0^s (s-y)^{1-\alpha} - (t-y)^{1-\alpha} \alpha(B_{u_{z_n}}(y)) dy \right\} \right) ds \\
& \leq \frac{4 M_1}{\Gamma(\alpha)} \int_0^t \frac{1}{t-s} \left( \frac{s^\alpha}{\alpha} - \frac{t^\alpha - (t-s)^{\alpha}}{\alpha} + \frac{(t-s)^{\alpha}}{\alpha} \right) ds \\
& \leq \frac{4 M_1}{\Gamma(\alpha)} \int_0^t \frac{1}{t-s} (s^\alpha - t^\alpha + 2(t-s)^\alpha) ds \\
& \leq \frac{4 M_1}{\Gamma(\alpha+1)} \int_0^t \frac{1}{t-s} \cdot 3(t-s)^\alpha ds \\
& \leq \frac{12 M_1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} ds \\
& \leq \frac{48 M_1 M_2 M_1 \alpha^{2n} L (M_0 \alpha + 6 M_1) \tau^\alpha}{\alpha^{\Gamma(3)(\alpha+1)}} \cdot \alpha(D).
\end{align*}
\]

As already done in Theorem 3.3, we have

\[
\alpha \left( \mathcal{T}(D_0) \right) = \max_{t \in J} \alpha \left( (\mathcal{T} D_0)(t) \right).
\]
Then, from (3.18)-(3.20), (3.22), (3.23), (3.25) and (3.26), it follows that
\[
\alpha((\mathcal{T}D)) 
\leq 2 \max_{t \in J} [\alpha((\mathcal{T}_1D_0)(t)) + \alpha((\mathcal{T}_2D_0)(t)) + \alpha((\mathcal{T}_3D_0)(t)) + \alpha((\mathcal{T}_4D_0)(t))] + \varepsilon
\]
\[
\leq 4M_0 L \tau^\alpha \frac{1}{\Gamma(\alpha + 1)} \cdot \alpha(D) + \frac{24M_1L \tau^\alpha}{\alpha \Gamma(\alpha + 1)} \cdot \alpha(D) 
+ \frac{8M_0M_1a^\alpha L(M_0 \alpha + 6 \bar{M}_1) \tau^\alpha}{\alpha \Gamma^2(\alpha + 1)} \cdot \alpha(D)
\]
\[
+ \frac{96M_1M_2 \bar{M}_1a^2 \alpha L(M_0 \alpha + 6 \bar{M}_1) \tau^\alpha}{\alpha \Gamma^3(\alpha + 1)} \cdot \alpha(D) + \varepsilon
\]
\[
\leq \frac{4L(M_0 \alpha + 6 \bar{M}_1)[\Gamma^2(\alpha + 1) + 2M_1M_2 \alpha^\alpha (M_0 \Gamma(\alpha + 1) + 12 \bar{M}_1 a^\alpha)] \tau^\alpha \alpha(D) + \varepsilon}{\alpha \Gamma^3(\alpha + 1)}.
\]
Therefore, by the choice of \( \tau \) and the arbitrariness of \( \varepsilon \), we have
\[
\alpha((\mathcal{T}D)) \leq \frac{4L(M_0 \alpha + 6 \bar{M}_1)[\Gamma^2(\alpha + 1) + 2M_1M_2 \alpha^\alpha (M_0 \Gamma(\alpha + 1) + 12 \bar{M}_1 a^\alpha)] \tau^\alpha \alpha(D) + \varepsilon}{\alpha \Gamma^3(\alpha + 1)} < \alpha(D),
\]
which implies \( \mathcal{T} : B \rightarrow B \) is a condensing operator.

Consequently, using Lemma 2.10, \( \mathcal{T} \) has at least one fixed point \( x \) on \( B \) and \( x \) is then a mild solution of the fractional evolution system (1.1) on \( J \) satisfying \( x(\tau) = x_0 \) which implies that \( u_0 \) steers the system (1.1) from \( x_0 \) to \( x_1 \) in finite time \( \tau \). This means that system (1.1) is exactly controllable on \( I \). The conclusion of this theorem follows.

\[\Box\]

**Remark 5.** (i) We suppress the hypotheses that \( f, x_0 \) are \( D \)-valued and assume that \( f, x_0 \) are \( X \)-valued. The analytic condition is put on the resolvent operator instead of the compactness of semigroup \( \{\mathcal{T}(t) : t \geq 0\} \).

(ii) We can also use Mönch fixed point theorem to demonstrate our results. In fact, one can assume that \( C \subset B \) is countable, and \( C \subset \overline{\mathbb{C}}(z) \cup \mathcal{T}C \), where \( z \in B \). It is easy to see that
\[
\alpha(C(t)) \leq \alpha((\mathcal{T}C)(t)).
\]
Equation (3.27), and the estimation way of Kuratowskis measures of noncom- pactness \( \alpha \) in Step 4, we can choose a appropriate constant \( \tau \in (0, \alpha] \) such that
\[
\alpha((\mathcal{T}C)(t)) \leq \frac{1}{2} \alpha(TC), \ t \in [0, \tau],
\]
which indicates that \( \alpha(TC) \leq \frac{1}{2} \alpha(TC) \). So, we obtain \( \alpha(TC) = 0 \). Thus, \( \alpha(C) = 0 \), and this implies that \( C \) is relatively compact. By means of Lemma 2.11, \( \mathcal{T} \) has a fixed point \( x \) on \( B \) and \( x \) is then a mild solution of system (1.1) on \( [0, \tau] \) satisfying \( x(\tau) = x_1 \). Therefore, system (1.1) is exactly controllable on \( I \).

4. An example. Consider the following fractional evolution system:
\[
\begin{cases}
C^\alpha x(t, \xi) = \frac{\partial}{\partial \xi} x(t, \xi) + \frac{2e^{-t} x(t, \xi)}{1 + |x(t, \xi)|} + \rho \zeta(t, \xi), & t \in [0, 1], \ \alpha \in (0, 1),
x(t, 0) = x(t, \pi) = 0,
x(0, \xi) = 0, \quad 0 < \xi < \pi,
\end{cases}
\]
for \( (t, \xi) \in [0, 1] \times [0, \pi] \), where \( \zeta : [0, 1] \times [0, \pi] \rightarrow [0, \pi] \) is continuous, and \( \rho > 0 \).
Let us take $X = C([0, \pi])$ and the operator $A : D \subset X \to X$ given by $Aw = w'$ with domain $D = \{ w \in X : w' \in X, w(0) = w(\pi) = 0 \}$. Then, $A$ is an infinitesimal generator of a semigroup $\{ T(t) : t \geq 0 \}$ on $X$ which is given by $T(t)w(s) = w(t+s)$ for $w \in X$, and $T(t)$ is not compact on $X$. From [24], we know that the equation

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{A(x(s))}{(t-s)^{1-\alpha}} \, ds, \quad t \geq 0,$$

has an associated analytic resolvent operator $\{ S(t) \}_{t \geq 0}$ on $X$ which is given by

$$S(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{\theta}} e^{\lambda t} (\lambda^{\alpha} - A)^{-1} d\lambda, & t > 0, \\ I, & t = 0, \end{cases}$$

where $\theta \in (\frac{\pi}{2}, \pi)$ and $\Gamma_{\theta}$ denotes a contour $\{ Re^{i\theta} \} \cup \{ Re^{-i\theta} \}$.

(3.28) can be regarded as a control system of the form (1.1), where

$$x(t)(\xi) = x(t, \xi), \quad f(t, x(t))(\xi) = \frac{2e^{-i\xi} x(t, \xi)}{1 + |x(t, \xi)|^2}, \quad Bu(t)(\xi) = \rho \xi(t, \xi), 0 < \xi < \pi.$$

For $\xi \in (0, \pi)$, we assume that the linear operator $W(t)$ is given by

$$W(t)u(\xi) = \rho S(t) \int_{0}^{t} \frac{(t-s)^{\alpha-1} \xi(s, \xi)}{\Gamma(\alpha)} ds$$

$$+ \rho \int_{0}^{t} S(t-s) \left( \int_{0}^{s} (t-y)^{\alpha-1} \xi(y, \xi) dy - \int_{0}^{t} (t-s)^{\alpha-1} \xi(s, \xi) ds \right) ds,$$

has an induced inverse operator $W^{-1}(t)$ which takes values in $L^{2}(I; U/kerW(t))$ for every $t \in [0, 1]$ and $\sup_{t \in I} ||W^{-1}(t)||_{L^{2}(I; U)/kerW(t)} \leq M_{2}$. (H1)' holds obviously.

Moreover, it is easy to check that (H3) are satisfied with $L = 2$. Consequently, from Theorem 3.4, it follows that the fractional evolution system (3.28) is exactly controllable on $[0, 1]$.

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**REFERENCES**

[1] D. Araya and C. Lizama, *Almost automorphic mild solutions to fractional differential equations*, *Nonlinear Anal.*, **69** (2008), 3692–3705.

[2] B. Ahmad, J. J. Nieto, A. Alsaedi and M. El-Shahed, *A study of nonlinear Langevin equation involving two fractional orders in different intervals*, *Nonlinear Anal.*, **13** (2012), 599–606.

[3] C. Bucur, *Local density of Caputo-stationary functions in the space of smooth functions*, *ESAIM Control Optim. Calc. Var.*, **23** (2017), 1361–1380.

[4] C. Bucur and E. Valdinoci, *Nonlocal Diffusion and Applications*, Lecture Notes of the Unione Matematica Italiana, Springer, Bologna, 2016.

[5] D. Bothe, *Multivalued perturbations of m-accretive differential inclusions*, *Israel J. Math.*, **108** (1998), 109–138.

[6] J. Banas and K. Goebel, *Measure of Noncompactness in Banach Spaces*, Lect. Notes Pure Appl. Math., Marcel Dekker, New York, 1980.

[7] K. Balachandran and J. Y. Park, *Controllability of fractional integrodifferential systems in Banach spaces*, *Nonlinear Anal.*, **3** (2009), 363–367.

[8] G. Da Prato and M. Iannelli, *Existence and regularity for a class of integrodifferential equations of parabolic type*, *J. Math. Anal. Appl.*, **112** (1985), 36–55.

[9] A. Debbouche and D. Baleanu, *Controllability of fractional evolution nonlocal impulsive quasi-linear delay integro-differential systems*, *Comput. Math. Appl.*, **62** (2011), 1442–1450.
[10] M. M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations, *Chaos Solitons Fract.*, 14 (2002), 433–440.

[11] E. Hernández, D. O’Regan and K. Balachandran, On recent developments in the theory of abstract differential equations with fractional derivatives, *Nonlinear Anal.*, 73 (2010), 3462–3471.

[12] E. Hernández, D. O’Regan and K. Balachandran, Existence results for abstract fractional differential equations with nonlocal conditions via resolvent operators, *Indagationes mathematicae*, 24 (2013), 68–82.

[13] O. K. Jaradat, A. Al-Omari and S. Momani, Existence of the mild solution for fractional semilinear initial value problems, *Nonlinear Anal.*, 69 (2008), 3153–3159.

[14] S. Ji, G. Li and M. Wang, Controllability of impulsive differential systems with nonlocal conditions, *Appl. Math. Comput.*, 217 (2011), 6981–6989.

[15] X. Li and J. Cao, An impulsive delay inequality involving unbounded time-varying delay and applications, *IEEE Trans. Autom. Contr.*, 62 (2017), 3618–3625.

[16] X. Li and S. Song, Impulsive control for existence, uniqueness and global stability of periodic solutions of recurrent neural networks with discrete and continuously distributed delays, *IEEE Trans. Neural Net. Learn. Sys.*, 24 (2013), 868–877.

[17] X. Li and S. Song, Stabilization of delay systems: delay-dependent impulsive control, *IEEE Trans. Autom. Contr.*, 62 (2017), 406–411.

[18] H. Li and Y. Wang, Lyapunov-based stability and construction of Lyapunov functions for Boolean networks, *SIAM J. Control Optim.*, 55 (2017), 3437–3457.

[19] H. Li and Y. Wang, Further results on feedback stabilization control design of Boolean control networks, *Automatica*, 83 (2017), 303–308.

[20] X. Li and J. Wu, Stability of nonlinear differential systems with state-dependent delayed impulses, *Automatica*, 64 (2016), 63–69.

[21] H. Li, L. Xie and Y. Wang, On robust control invariance of Boolean control networks, *Automatica*, 68 (2016), 392–396.

[22] H. Li, L. Xie and Y. Wang, Output regulation of Boolean control networks, *IEEE Trans. Autom. Contr.*, 62 (2017), 2993–2998.

[23] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, Academic Press, New York, 1999.

[24] J. Prüss, *Evolutionary Integral Equations and Applications*, Monographs in Mathematics, Birkhäuser Verlag, Basel, 1993.

[25] R. Sakthivel and Y. Ren, Approximate controllability of fractional differential equations with state-dependent delay, *Results in Mathematics*, 63 (2013), 949–963.

[26] I. Stamova, T. Stamov and X. Li, Global exponential stability of a class of impulsive cellular neural networks with Supremums, *Inter. J. Adapt. Contr. Signal Process.*, 28 (2014), 1227–1239.

[27] J. Sprekels and E. Valdinoci, A new type of identification problems: optimizing the fractional order in a nonlocal evolution equation, *SIAM J. Control Optim.*, 55 (2017), 70–93.

[28] V. Vijayakumar, A. Selvakumar and R. Murugesu, Controllability for a class of fractional neutral integro-differential equations with unbounded delay, *Appl. Math. Comput.*, 232 (2014), 303–312.

[29] J. Wang and Y. Zhou, Complete controllability of fractional evolution systems, *Commun. Nonlinear Sci. Numer. Simulat.*, 17 (2012), 4346–4355.

[30] Y. Zhou and F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, *Nonlinear Anal.*, 11 (2010), 4465–4475.

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