Group selection and shrinkage:
Structured sparsity for semiparametric models∗

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Abstract

Sparse regression and classification estimators that respect group structures have application to an assortment of statistical and machine learning problems, from multitask learning to sparse additive modeling to hierarchical selection. This work introduces structured sparse estimators that combine group subset selection with shrinkage. To accommodate sophisticated structures, our estimators allow for arbitrary overlap between groups. We develop an optimization framework for fitting the nonconvex regularization surface and present finite-sample error bounds for estimation of the regression function. As an application requiring structure, we study sparse semiparametric modeling, a procedure that allows the effect of each predictor to be zero, linear, or nonlinear. For this task, the new estimators improve across several metrics on synthetic data compared to alternatives. Finally, we demonstrate their efficacy in modeling supermarket foot traffic and economic recessions using many predictors. These demonstrations suggest sparse semiparametric models, fit using the new estimators, are an excellent compromise between fully linear and fully nonparametric alternatives. All of our algorithms are made available in the scalable implementation \texttt{grpsel}.

1 Introduction

Sparsity over group structures arises in connection with a myriad of statistical and machine learning problems, e.g., multitask learning (Obozinski, Taskar, and Jordan 2006), sparse additive modeling (Ravikumar et al. 2009), and hierarchical selection (Lim and Hastie 2015). Even sparse linear modeling can involve structured sparsity, such as when a categorical predictor is represented as a sequence of dummy variables. In certain domains, groups may emerge naturally, e.g., disaggregates of the same macroeconomic series or genes of the same
biological path. The prevalence of such problems motivates principled estimation procedures capable of encoding structure into the fitted models they produce.

Given response $y = (y_1, \ldots, y_n)^\top \in \mathbb{R}^n$, predictors $X = (x_1, \ldots, x_n)^\top \in \mathbb{R}^{n \times p}$, and nonoverlapping groups $G_1, \ldots, G_g \subseteq \{1, \ldots, p\}$, group lasso (Yuan and Lin 2006; Meier, van de Geer, and Bühlmann 2008) solves

$$
\min_{\beta \in \mathbb{R}^p} \sum_i \ell(x_i^\top \beta, y_i) + \sum_k \lambda_k \|\beta_k\|
$$

where $\ell : \mathbb{R}^2 \to \mathbb{R}_+$ is a loss function (e.g., square loss for regression or logistic loss for classification), $\lambda_1, \ldots, \lambda_g$ are nonnegative tuning parameters, and $\beta_k \in \mathbb{R}^{p_k}$ are the coefficients $\beta$ indexed by $G_k$.\footnote{Here and throughout, the intercept term is omitted to facilitate exposition.} Group lasso couples coefficients via their $l_2$-norm so that all predictors in a group are selected together.

Just as lasso (Tibshirani 1996) is the continuous relaxation of the combinatorially-hard problem of best subset selection (“best subset”), so is group lasso the relaxation of the combinatorial problem of group subset selection (“group subset”):

$$
\min_{\beta \in \mathbb{R}^p} \sum_i \ell(x_i^\top \beta, y_i) + \sum_k \lambda_k 1(\|\beta_k\| \neq 0).
$$

Unlike group lasso, which promotes group sparsity implicitly by nondifferentiability of the $l_2$-norm at the null vector, group subset explicitly penalizes the number of nonzero groups. Consequently, one might interpret group lasso as a compromise made in the interest of computation. However, group lasso has a trick up its sleeve that group subset does not: shrinkage. Shrinkage estimators such as lasso are more robust than best subset to noise (Breiman 1996; Hastie, Tibshirani, and Tibshirani 2020). This consideration motivates one to shrink the group subset estimator:

$$
\min_{\beta \in \mathbb{R}^p, \bar{\nu} \in V} \sum_i \ell(x_i^\top \beta, y_i) + \sum_k \lambda_0 1(\|\beta_k\| \neq 0) + \sum_k \lambda_{1k} \|\bar{\nu}_k\|.
$$

(1.1)

In contrast to group lasso and group subset, (1.1) directly controls both group sparsity and shrinkage via separate penalties—selection via group subset and shrinkage via group lasso. The combination of best subset and lasso in the unstructured setting results in good predictive models with low false positive selection rates (Mazumder, Radchenko, and Dedieu 2022).

Unfortunately, the estimators (1.1), including group lasso and group subset as special cases, do not accommodate overlap among groups. Specifically, if two groups overlap, one cannot be selected independently of the other. To encode sophisticated structures, such as hierarchies or graphs, groups must often overlap. To address this issue, one can introduce group-specific vectors $\bar{\nu}_k \in \mathbb{R}^p$ ($k = 1, \ldots, g$) that are zero everywhere except at the positions indexed by $G_k$. Letting $V$ be the set of all tuples $\bar{\nu} := (\bar{\nu}_1, \ldots, \bar{\nu}_g)$ with elements satisfying this property, group subset with shrinkage becomes

$$
\min_{\beta \in \mathbb{R}^p, \bar{\nu} \in V} \sum_i \ell(x_i^\top \beta, y_i) + \sum_k \lambda_0 1(\|\bar{\nu}_k\| \neq 0) + \sum_k \lambda_{1k} \|\bar{\nu}_k\|.
$$

(1.2)
The vectors $\hat{\nu}_1, \ldots, \hat{\nu}_k$ are a decomposition of $\beta$ into a sum of latent coefficients that facilitate selection of overlapping groups. For instance, if three predictors, $x_1$, $x_2$, and $x_3$, are spread across two groups, $G_1 = \{1, 2\}$ and $G_2 = \{2, 3\}$, then $\beta_1 = \hat{\nu}_{1,1}$, $\beta_2 = \hat{\nu}_{2,1} + \hat{\nu}_{2,2}$, and $\beta_3 = \hat{\nu}_{3,2}$. Since $\beta_2$ has a separate latent coefficient for each group, $G_1$ or $G_2$ can be selected independently of the other. This latent coefficient approach originated for group lasso with Jacob, Obozinski, and Vert (2009) and Obozinski, Jacob, and Vert (2011). When all groups are disjoint, (1.2) reduces exactly to (1.1).

This paper develops computational methods and statistical theory for group subset with and without shrinkage. Via the formulation (1.2), our work accommodates the general overlapping groups setting. On the computational side, we develop algorithms that scale to compute quality (approximate) solutions of the combinatorial optimization problem. Our framework comprises coordinate descent and local search and applies to general smooth convex loss functions (i.e., regression and classification), building on recent advances for best subset (Hazimeh and Mazumder 2020; Dedieu, Hazimeh, and Mazumder 2021). In contrast to existing computational methods for group subset (Guo, Berman, and Gao 2014; Bertsimas and King 2016), which rely on branch-and-bound or commercial mixed-integer optimizers, our methods scale to instances with millions of predictors or groups. We implement our framework in the publicly available R package `grpsel`. On the statistical side, we establish new error bounds for group subset with and without shrinkage. The bounds apply in the overlapping setting and allow for model misspecification. The analysis sheds light on the advantages of structured sparsity and the benefits of shrinkage.

The new estimators have application to a broad range of statistical problems. We focus on sparse semiparametric modeling, a procedure wherein $y$ is modeled via a sum of functions $\sum_j f_j(x_j)$, and $f_j$ can be zero, linear, or nonlinear. Chouldechova and Hastie (2015) and Lou et al. (2016) estimate these flexible models using group lasso with overlapping groups and regression splines. After conducting synthetic experiments on the efficacy of our estimators in fitting these models, we carry out two empirical studies. The first study involves modeling supermarket foot traffic using sales volumes on different products. Only a fraction of supermarket products are traded in volume, necessitating sparsity. The second study involves modeling recessionary periods in the economy using macroeconomic series. The macroeconomic literature contains many examples of sparse linear modeling (De Mol, Giannone, and Reichlin 2008; Li and Chen 2014), yet theory does not dictate linearity. Together these studies suggest semiparametric models are an excellent compromise between fully linear and fully nonparametric alternatives.

Independently and concurrently to this work, Hazimeh, Mazumder, and Radchenko (2022) study computation and theory for group subset with nonoverlapping groups. Their algorithms likewise build on Hazimeh and Mazumder (2020) but apply only to square loss regression. Also related is Zhang et al. (2022) who propose a computational “splicing” technique for group subset that appears promising, though they do not consider overlapping groups or shrinkage.

### 1.1 Organization

The paper is structured as follows. Section 2 presents computational methods. Section 3 provides statistical theory. Section 4 describes simulation experiments. Section 5 reports
data analyses. Section 6 closes the paper. All proofs are relegated to the appendices.

2 Computation

This section introduces our optimization framework and its key components: coordinate descent and local search. The framework applies to any smooth loss function \( \ell(z, y) \) convex in \( z \). The discussion below addresses the specific cases of square loss \( \ell(z, y) = \frac{(y - z)^2}{2} \), which is suitable for regression, and logistic loss \( \ell(z, y) = -y \log(z) - (1 - y) \log(1 - z) \), which is suitable for classification. Both loss functions are implemented in \texttt{grpseq}. Throughout this section, the predictor matrix \( X \) is assumed to have columns with mean zero and unit \( l_2 \)-norm.

2.1 Problem reformulation

From a computational perspective, it helps to reformulate the group subset problem (1.2) as an unconstrained minimization problem involving only the latent coefficients \( \bar{\nu} \). For this task, we denote by \( \nu_k \in \mathbb{R}^{p_k} \) the restriction of \( \bar{\nu}_k \) to the coordinates indexed by group \( k \). No information is lost in this restriction since all elements not indexed by \( G_k \) are zero. We also introduce the vector \( \nu := (\nu_1^\top, \ldots, \nu_g^\top)^\top \in \mathbb{R}^{\sum_k p_k} \) formed by vertically concatenating the vectors \( \nu_1, \ldots, \nu_g \). Consider now the unconstrained minimization problem

\[
\min_{\nu \in \mathbb{R}^{\sum_k p_k}} F(\nu) := L(\nu) + R(\nu).
\]

Here, the function \( L(\nu) \) is the loss term:

\[
L(\nu) := \sum_i \ell \left( \sum_k x_{ik}^\top \nu_k, y_i \right),
\]

where \( x_{ik} \) is the \( i \)th row of the matrix \( X_k \), with \( X_k \) the restriction of \( X \) to the columns indexed by group \( k \). The function \( R(\nu) \) is the regularizer term:

\[
R(\nu) := \sum_k (\lambda_{0k} 1(\|\nu_k\| \neq 0) + \lambda_{1k} \|\nu_k\|).
\]

Observe that the loss \( L(\nu) \) is exactly equivalent to that in (1.2) since

\[
\sum_i \ell \left( \sum_k x_{ik}^\top \nu_k, y_i \right) = \sum_i \ell \left( x_i^\top \sum_k \nu_k, y_i \right) = \sum_i \ell \left( x_i^\top \beta, y_i \right).
\]

The regularizer \( R(\nu) \) is likewise equivalent because \( \|\nu_k\| = \|\bar{\nu}_k\| \). As the equalities immediately above suggest, it is straightforward to recover \( \beta \) from \( \nu \).

2.2 Coordinate descent

Coordinate descent algorithms are optimization routines that minimize along successive coordinate hyperplanes. The coordinate descent scheme developed here iteratively fixes all
but one group of coordinates (a coordinate group) and minimizes in the directions of these coordinates.

The objective function $F(\nu)$ is a sum of smooth convex and discontinuous nonconvex functions and is hence discontinuous nonconvex. The minimization problem with respect to group $k$ is

$$\min_{\xi \in \mathbb{R}^{p_k}} F(\nu_1, \ldots, \nu_{k-1}, \xi, \nu_{k+1}, \ldots, \nu_g).$$

(2.1)

The complexity of this coordinate-wise minimization depends on the type of loss function and the properties of the group matrix $X_k$. In the case of square loss, the minimization involves a least-squares fit in $p_k$ coordinates, taking $O(p_k^2 n)$ operations. To bypass these involved computations, a partial minimization scheme is adopted whereby each coordinate group is updated using a single gradient descent step taken with respect to that group. This scheme results from a standard technique of replacing the objective function with a surrogate function that is an upper bound. To this end, we require Lemma 1.

**Lemma 1.** Let $L : \mathbb{R}^{\sum_k p_k} \rightarrow \mathbb{R}$ be a continuously differentiable function. Suppose there exists a $c_k > 0$ such that the gradient of $L$ with respect to the $k$th coordinate group satisfies the Lipschitz property

$$\| \nabla_k L(\nu) - \nabla_k L(\tilde{\nu}) \| \leq c_k \| \nu_k - \tilde{\nu}_k \|,$$

for all $\nu \in \mathbb{R}^{\sum_k p_k}$ and $\tilde{\nu} \in \mathbb{R}^{\sum_k p_k}$ that differ only in group $k$. Then it holds

$$L(\nu) \leq \bar{L}_{\bar{c}_k}(\nu; \tilde{\nu}) := L(\tilde{\nu}) + \nabla_k L(\tilde{\nu})^\top (\nu_k - \tilde{\nu}_k) + \frac{\bar{c}_k}{2} \| \nu_k - \tilde{\nu}_k \|^2,$$

(2.2)

for any $\bar{c}_k \geq c_k$.

Lemma 1 is the block descent lemma of Beck and Tetruashvili (2013), which holds under a Lipschitz condition on the group-wise gradients of $L(\nu)$. This condition is satisfied for square loss with $c_k = \gamma_k^2$ and for logistic loss with $c_k = \gamma_k^2/4$, where $\gamma_k$ is the maximal eigenvalue of $X_k^\top X_k$. Using the result of Lemma 1, an upper bound of $F(\nu)$, treated as a function in group $k$, is given by

$$\bar{F}_{\bar{c}_k}(\nu; \tilde{\nu}) := \bar{L}_{\bar{c}_k}(\nu; \tilde{\nu}) + R(\nu).$$

(2.3)

Thus, in place of the minimization (2.1), we use the minimization

$$\min_{\xi \in \mathbb{R}^{p_k}} \bar{F}_{\bar{c}_k}(\nu_1, \ldots, \nu_{k-1}, \xi, \nu_{k+1}, \ldots, \nu_g; \tilde{\nu}).$$

(2.4)

This new problem admits a simple analytical solution, given by Proposition 1.

**Proposition 1.** Define the thresholding function

$$T_c(\xi; \lambda_0, \lambda_1) := \begin{cases} 
(1 - \frac{\lambda_1}{c \| \xi \|}) \xi & \text{if } (1 - \frac{\lambda_1}{c \| \xi \|}) \| \xi \| \geq \sqrt{\frac{2 \lambda_0}{c}} \\
0 & \text{otherwise},
\end{cases}$$

(2.5)

where $(x)_+$ is shorthand for $\max(x, 0)$. Then the coordinate-wise minimization problem (2.4) is solved by

$$\nu^*_k = T_{\bar{c}_k} \left( \tilde{\nu}_k - \frac{1}{\bar{c}_k} \nabla_k L(\tilde{\nu}); \lambda_{0k}, \lambda_{1k} \right).$$
Proposition 1 states that a minimizer is given by appropriately thresholding a gradient descent update to coordinate group $k$. For both square and logistic loss, the gradient $\nabla_k L(\tilde{\nu})$ can be expressed as
\[
\nabla_k L(\tilde{\nu}) = -X_k^T r,
\]
where $r = y - \sum_k X_k \tilde{\nu}_k$ for square loss and $r = y - (1 + \exp(-\sum_k X_k \tilde{\nu}_k))^{-1}$ for logistic loss. Hence, a solution to (2.4) can be computed in as few as $O(p_k n)$ operations.

Algorithm 1 now presents the coordinate descent scheme. Several algorithmic optimizations and heuristics can improve performance; these are discussed in Appendix A.5.

While Algorithm 1 may appear as an otherwise standard coordinate descent algorithm, the presence of the group subset penalty complicates the analysis of its convergence properties. No standard convergence results directly apply; e.g., Tseng (2001) that applies to group lasso cannot be invoked immediately here. Hence, we work towards establishing some properties tailored to Algorithm 1. Two results are presented. Lemma 2 establishes convergence of the sequence of objective values. Theorem 1 establishes convergence of the sequence of iterates to a stationary point of $F(\nu)$ that satisfies a certain coordinate-wise property. A point $\nu^*$ with active coordinate groups $A^*$ is said to be a stationary point of $F(\nu)$ if $\nabla_{A^*} L(\nu^*) = 0$.

Lemma 2. Let $\bar{c}_k \geq c_k$ for all $k = 1, \ldots, g$. Then the sequence of objective values $\{F(\nu^{(m)})\}_{m \in \mathbb{N}}$ produced by Algorithm 1 is decreasing, convergent, and satisfies the inequality
\[
F(\nu^{(m)}) - F(\nu^{(m+1)}) \geq \sum_k \frac{\bar{c}_k - c_k}{2} \|\nu_k^{(m+1)} - \nu_k^{(m)}\|^2.
\]

Lemma 2 is derived from the result of Lemma 1.

Theorem 1. Let $\bar{c}_k > c_k$ for all $k = 1, \ldots, g$. Then the sequence of iterates $\{\nu^{(m)}\}_{m \in \mathbb{N}}$ produced by Algorithm 1 converge to a solution $\nu^*$ that is a stationary point of $F(\nu)$ satisfying the fixed point equations
\[
\nu^*_k = T_{\bar{c}_k} \left( \nu^*_k - \frac{1}{\bar{c}_k} \nabla_k L(\nu^*); \lambda_{0k}, \lambda_{1k} \right), \quad k = 1, \ldots, g.
\] (2.6)

To prove Theorem 1, Lemma 2 is used to show that the active set stabilizes during coordinate descent. This property allows the group subset penalty to be treated as a fixed
quantity after some finite number of iterations, and in turn, opens up the results of Tseng (2001). The fixed point equations in Theorem 1 have the interpretation that the limit point of the iterates of Algorithm 1 cannot be improved by partially minimizing in the directions of any coordinate group. This notion is stronger than stationarity alone because all points satisfying (2.6) are stationary, but not all stationary points satisfy (2.6).

2.3 Local search

Local search methods have a long history in combinatorial optimization. Here we present a local search method tailored specifically to the group subset problem. The proposed method generalizes an algorithm that first appeared in Beck and Eldar (2013) for solving instances of unstructured sparse optimization. Hazimeh and Mazumder (2020) and Dedieu, Hazimeh, and Mazumder (2021) adapt it to best subset with promising results. The core idea is simple: given an incumbent solution, search a neighborhood local to that solution for a minimizer with lower objective value by discretely optimizing over a small set of coordinate groups. This scheme turns out to be useful when the predictors are strongly correlated, a situation in which coordinate descent alone may produce a poor solution.

Define the group sparsity pattern of the vector $\nu$ to be the set of nonzero group indices:

$$gs(\nu) := \{ k \in \{1, \ldots, g\} : \|\nu_k\| \neq 0\},$$

and define the constraints sets

$$C_1^s(\nu) := \left\{ e \in \{0, 1\}^{\sum_k p_k} : gs(e) \subseteq gs(\nu), \sum_k 1(\|e_k\| \neq 0) \leq s \right\},$$

and

$$C_2^s(\nu) := \left\{ e \in \{0, 1\}^{\sum_k p_k} : gs(e) \nsubseteq gs(\nu), \sum_k 1(\|e_k\| \neq 0) \leq s \right\}. $$

Now, consider the optimization problem

$$\min_{\xi \in \mathbb{R}^{\sum_k p_k}, e^1 \in C_1^s(\nu), e^2 \in C_2^s(\nu)} F(\nu - e^1\nu + e^2\xi). \quad (2.7)$$

Given a fixed vector $\nu$, a solution to (2.7) is a minimizer among all ways of replacing a subset of active coordinate groups in $\nu$ with a new subset. The complexity of the problem is dictated by $s$, which controls the size of these subsets. When $s = g$, the full combinatorial problem whose solution is a global minimizer of the group subset problem is recovered. For $s \ll g$, a reduced combinatorial problem is obtained whose solution space is usually orders of magnitude smaller than that of the full problem.

The limiting case $s = 1$ admits an efficient computational scheme, given in Algorithm 2, referred to hereafter as local search. Algorithm 2 comprises two low-complexity loops: an outer loop over the active set and an inner loop over the inactive set. Within the inner loop, an active coordinate group is removed, and the objective is minimized in the directions of an inactive coordinate group. This minimization problem can be solved by iterating the thresholding operator (2.5).
Algorithm 2: Local search

\begin{algorithm}
\textbf{input} : $\nu \in \mathbb{R}^\sum p_k$
$A \leftarrow \text{gs}(\nu)$
\For{$k \in A$}{
\For{$j \notin A$}{
$\nu^{(j)} \leftarrow \nu$
$\nu_k^{(j)} \leftarrow 0$
$\nu_j^{(j)} \leftarrow \arg \min_{\xi \in \mathbb{R}^{pk}} F(\nu_1^{(j)}, \ldots, \nu_{j-1}^{(j)}, \xi, \nu_{j+1}^{(j)}, \ldots, \nu_g^{(j)})$
}
$j^* \leftarrow \arg \min_{j \notin A} F(\nu^{(j)})$
\If{$F(\nu^{(j^*)}) < F(\nu)$}{
$\nu \leftarrow \nu^{(j^*)}$
\break
}
}
\Return{$\nu$}
\end{algorithm}

Since Algorithm 2 involves optimizing in one coordinate group only, it need not produce a stationary point even if initialized at one. Algorithm 3 thus combines local search with coordinate descent. The combined algorithm first produces a candidate solution using coordinate descent and then follows up with local search. This scheme is iterated until the solution cannot be improved. Compared with coordinate descent alone, coordinate descent with local search can yield significantly lower objective values in high-correlation scenarios. Algorithm 3 is guaranteed to converge because Algorithm 2 never increases the objective value (by construction), Algorithm 1 is convergent, and the objective function is bounded below.

2.4 Regularization sequence

To ensure larger groups are not unfairly penalized more strongly than smaller groups, the parameters $\lambda_{0k}$ and $\lambda_{1k}$ are configured to reflect the group size $p_k$. Suitable default choices
are \( \lambda_{0k} = p_k \lambda_0 \) and \( \lambda_{1k} = \sqrt{p_k} \lambda_1 \), where \( \lambda_0 \) and \( \lambda_1 \) are nonnegative. For fixed \( \lambda_1 \), we take a sequence \( \{ \lambda_0^{(t)} \}_{t=1}^T \) such that \( \lambda_0^{(0)} \) yields \( \hat{\nu} = 0 \), and sequentially warm start the algorithms. That is, the solution for \( \lambda_0^{(t+1)} \) is obtained by using the solution from \( \lambda_0^{(t)} \) as an initialization point. The sequence \( \{ \lambda_0^{(t)} \}_{t=1}^T \) is computed in such a way that the active set of groups corresponding to \( \lambda_0^{(t+1)} \) is always different to that corresponding to \( \lambda_0^{(t)} \). Proposition 2 presents the details of this method, extending an idea of Hazimeh and Mazumder (2020) for best subset.

**Proposition 2.** Suppose that \( \hat{\nu}^{(t)} \) is the result of running Algorithm 1 with \( \lambda_0 = \lambda_0^{(t)} \). Let \( A^{(t)} \) be the active set of groups. Then running Algorithm 1 initialized to \( \hat{\nu}^{(t)} \) and using \( \lambda_0 = \lambda_0^{(t+1)} \) where

\[
\lambda_0^{(t+1)} = \alpha \cdot \max_{k \notin A^{(t)}} \left( \frac{\| \nabla L(\hat{\nu}^{(t)}) \| - \lambda_{1k}}{2 p_k \bar{c}_k} \right)^2
\]

produces a solution \( \hat{\nu}^{(t+1)} \) such that \( \hat{\nu}^{(t+1)} \neq \hat{\nu}^{(t)} \) for any \( \alpha \in [0, 1) \).

### 3 Error bounds

This section presents a finite-sample analysis of the proposed estimators. In particular, we state probabilistic upper bounds for the error of estimating the underlying regression function. These bounds accommodate overlapping groups and model misspecification. The role of structure and shrinkage is discussed, and comparisons are made with known bounds for other estimators.

#### 3.1 Setup

The data is assumed to be generated according to the regression model

\[
y_i = f^0(x_i) + \varepsilon_i, \quad i = 1, \ldots, n,
\]

where \( f^0 : \mathbb{R}^p \to \mathbb{R} \) is a regression function, \( x_i \in \mathbb{R}^p \) are fixed predictors, and \( \varepsilon_i \sim \mathcal{N}(0, \sigma^2) \) is iid stochastic noise. This flexible specification encompasses the semiparametric model \( f^0(x) = \sum_j f_j(x_j) \) (with \( f_j \) zero, linear, or nonlinear) that is the focus of our empirical studies, and the linear model \( f^0(x) = x^\top \beta^0 \). Let \( f^0 := (f^0(x_1), \ldots, f^0(x_n))^\top \) be the vector of function evaluations at the sample points. The goal of this section is to place probabilistic upper bounds on \( \| f^0 - \hat{f} \|^2 / n \), the estimation error of \( \hat{f} := X\beta \).

The objects of our analysis are the group subset estimators (1.2). We allow the predictor groups \( G_1, \ldots, G_q \) to overlap. To facilitate comparisons against existing results, we constrain the number of nonzero groups rather than penalize them. To this end, let \( \mathcal{V}(s) \) be the set of all \( \hat{\nu} \) such that at most \( s \) groups are nonzero:

\[
\mathcal{V}(s) := \left\{ \hat{\nu} \in \mathcal{V} : \sum_k 1(\| \hat{\nu}_k \| \neq 0) \leq s \right\}.
\]

\(^2\)For all values of the group subset penalty parameter \( \lambda_0 \), there exists a constraint parameter \( s \) which yields an identical solution.
We consider the regular group subset estimator:

\[
\min_{\beta \in \mathbb{R}^p, \bar{\nu} \in \mathcal{V}(s)} \frac{1}{n} \|y - X\beta\|^2,
\]

and the shrinkage estimator:

\[
\min_{\beta \in \mathbb{R}^p, \bar{\nu} \in \mathcal{V}(s)} \frac{1}{n} \|y - X\beta\|^2 + 2 \sum_k \lambda_k \|\bar{\nu}_k\|.
\]

The results derived below apply to global minimizers of these nonconvex problems. The algorithms of the preceding section cannot guarantee such minimizers in general. If global optimality is of foremost concern, the output of our algorithms can be used to initialize a mixed-integer optimizer which can guarantee a global solution at additional computational expense.

### 3.2 Bound for group subset selection

We begin with Theorem 2, which characterizes an upper bound for group subset with no shrinkage. The notation \(p_{\max} := \max_k p_k\) represents the maximal group size. As is customary, we absorb numerical constants into the term \(C > 0\).

**Theorem 2.** Let \(\delta \in (0, 1]\) and \(\alpha \in (0, 1)\). Then, for some numerical constant \(C > 0\), the group subset estimator (3.1) satisfies

\[
\frac{1}{n} \|f^0 - \hat{f}\|^2 \leq \min_{\beta \in \mathbb{R}^p, \bar{\nu} \in \mathcal{V}(s)} \frac{1 + \alpha}{(1 - \alpha)n} \|f^0 - X\beta\|^2 + \frac{C\sigma^2}{\alpha(1 - \alpha)n} \left[ sp_{\max} + s \log \left( \frac{g}{s} \right) + \log(\delta^{-1}) \right]
\]

with probability at least \(1 - \delta\).

The first term on the right-hand side of (3.3) is the error incurred by the oracle in approximating \(f^0\) as \(X\beta\). In general, this error is unavoidable in finite-dimensional settings. The three terms inside the brackets have the following interpretations. The first term is the cost of estimating \(\beta\); with \(s\) active groups, there are at most \(s \times p_{\max}\) parameters to estimate. The second term is the price of selection; it follows from an upper bound on the total number of group subsets. The third term controls the trade-off between the tightness of the bound and the probability it is satisfied. Finally, the scalar \(\alpha\) appears in the bound due to the proof technique (as in, e.g., Rigollet 2015). When \(f^0 = X\beta^0\), \(\alpha\) need not appear.

Hazimeh, Mazumder, and Radchenko (2022) obtain a similar bound for \(f^0 = X\beta^0\) in the case of equisized nonoverlapping groups. In the special case that all groups are singletons, (3.3) matches the well-known bound for best subset (Raskutti, Wainwright, and Yu 2011).

Theorem 2 confirms that group subset is preferable to best subset in structured settings. Consider the following example. Suppose we have \(g\) groups each of size \(p_0\) so that the total
number of predictors is $p = g \times p_0$. It follows for group sparsity level $s$ that the ungrouped selection problem involves choosing $s \times p_0$ predictors. Accordingly, the ungrouped bound scales as $sp_0 + s p_0 \log(p/(sp_0)) = sp_0 + sp_0 \log(g/s)$. On the other hand, the grouped bound scales as $sp_0 + s \log(g/s)$, i.e., it improves by a factor $p_0$ of the logarithm term.

### 3.3 Bounds for group subset selection with shrinkage

We now establish bounds for group subset with shrinkage. The results are analogous to those established in Mazumder, Radchenko, and Dedieu (2022) for best subset with shrinkage. Two results are given, a bound where the error decays as $1/\sqrt{n}$, and another where the error decays as $1/n$. Adopting standard terminology (e.g., Hastie, Tibshirani, and Wainwright 2015), the former bound is referred to as a “slow rate” and the latter bound as a “fast rate.”

The slow rate is presented in Theorem 3.

**Theorem 3.** Let $\delta \in (0, 1]$. Let $\gamma_k$ be the maximal eigenvalue of the matrix $X_k^\top X_k/n$ and

$$
\lambda_k \geq \sqrt{\frac{\gamma_k \sigma}{n}} \sqrt{p_k + 2 \sqrt{p_k \log(p_k)} + 2 \log(g) + 2 \log(\delta^{-1})}, \quad k = 1, \ldots, g.
$$

Then the group subset estimator (3.2) satisfies

$$
\frac{1}{n} \|f^0 - \hat{f}\|^2 \leq \min_{\beta \in \mathbb{R}^p, \nu \in V(s)} \frac{1}{n} \|f^0 - X\beta\|^2 + 4 \sum_k \lambda_k \|\nu_k\|
$$

with probability at least $1 - \delta$.

In the case of no overlap, Theorem 3 demonstrates that the shrinkage estimator satisfies the same slow rate as group lasso (Lounici et al. 2011, Theorem 3.1). An identical expression to Lounici et al. (2011) for $\lambda_k$ can be stated here using a more intricate chi-squared tail bound in the proof. In the case of overlap, the same slow rate can be obtained for group lasso from Percival (2012, Lemma 4).

The following assumption is required to establish the fast rate.

**Assumption 1.** Let $s < \min(n/p_{max}, g)/2$. Then there exists a $\phi(2s) > 0$ such that

$$
\min_{\theta \in \mathbb{R}^p, \nu \in V(2s)} \frac{\|X\theta\|}{\sqrt{n} \sum_k \|\nu_k\|} \geq \phi(2s).
$$

Assumption 1 is satisfied provided no collection of $2s$ groups have linearly dependent columns in $X$. This condition is a weaker version of the restricted eigenvalue condition used in Lounici et al. (2011) and Percival (2012) for group lasso, which (loosely speaking) places additional restrictions on the correlations of the columns in $X$.

The fast rate is presented in Theorem 4. The notation $\lambda_{max} := \max_k \lambda_k$ represents the maximal shrinkage parameter.
Theorem 4. Let Assumption 1 hold. Let $\delta \in (0,1]$ and $\alpha \in (0,1)$. Let $\lambda_1,\ldots,\lambda_g \geq 0$. Then, for some numerical constant $C > 0$, the group subset estimator (3.2) satisfies
\[
\frac{1}{n} \| f^0 - \hat{f} \|^2 \leq \min_{\beta \in \mathbb{R}^p, \nu \in V(s)} \frac{1 + \alpha}{(1 - \alpha)n} \| f^0 - X\beta \|^2 + \frac{C\sigma^2}{\alpha(1 - \alpha)n} \left[ sp_{\max} + s \log \left( \frac{g}{s} \right) + \log(\delta^{-1}) \right] + \frac{C\lambda_{\max}^2}{\alpha(1 - \alpha)\phi(2s)^2}
\]
with probability at least $1 - \delta$.

Theorem 4 establishes that the shrinkage estimator achieves the bound of the regular estimator up to an additional term that depends on $\lambda_{\max}$ and $\phi(2s)$. By setting the shrinkage parameters to zero, the dependence on these terms vanishes, and the bounds are identical.

Theorems 3 and 4 together show that group subset with shrinkage does no worse than group lasso or group subset. This property is helpful because group lasso tends to outperform when the noise is high or the sample size is small, while group subset tends to outperform in the opposite situation. This empirical observation is consistent with the above bounds since the slow rate (3.4) depends on $\sigma/\sqrt{n}$ while the fast rate (3.5) depends on $\sigma^2/n$. Hence, (3.4) is typically the tighter of the two bounds for large $\sigma$ or small $n$.

4 Simulations

This section investigates the statistical and computational performance of the proposed estimators for sparse semiparametric modeling on synthetic data. They are compared against group lasso, group SCAD, and group MCP, the latter two estimators being group versions of the smoothly clipped absolute deviation penalty (Fan and Li 2001) and the minimax concave penalty (Zhang 2010). The group subset estimators are fit using grpsel, our R implementation of the algorithms presented in Section 2. Group lasso, group SCAD, and group MCP are fit using the popular R package grpreg (Breheny and Huang 2015).

4.1 Tuning parameters and implementation

The range of tuning parameters for each estimator is:

- Group subset: a grid of $\lambda_0$ chosen adaptively using the method of Proposition 2, where the first $\lambda_0$ sets all coefficients to zero;
- Group subset+lasso: a grid of $\lambda_1$ containing logarithmically spaced points between $\lambda_1^{\max}$ and $\lambda_1^{\min} = 10^{-4}\lambda_1^{\max}$, where $\lambda_1^{\max}$ is the smallest value that sets all coefficients to zero, and for each value of $\lambda_1$, a grid of $\lambda_0$ chosen as above;
- Group lasso: a grid of $\lambda$ containing logarithmically spaced points between $\lambda^{\max}$ and $\lambda^{\min} = 10^{-4}\lambda^{\max}$, where $\lambda^{\max}$ is the smallest value that sets all coefficients to zero;
- Group SCAD: the same grid of $\lambda$ as above, and for each value of $\lambda$, a grid of the nonconvexity parameter $\gamma$ containing logarithmically spaced points between $\gamma^{\max} = 100$ and $\gamma^{\min} = 2 + 10^{-4}$; and
• Group MCP: the same grid of \( \lambda \) as above, and for each value of \( \lambda \), a grid of the nonconvexity parameter \( \gamma \) containing logarithmically spaced points between \( \gamma^{\text{max}} = 100 \) and \( \gamma^{\text{min}} = 1 + 10^{-4} \).

Grids of 100 points are used for the primary tuning parameters \((\lambda_0, \lambda)\) and grids of 30 points for the secondary tuning parameters \((\lambda_1, \gamma)\).

Unlike \texttt{grpsel}, \texttt{grpreg} does not have native support for overlapping groups. To this end, we use the approach proposed in Jacob, Obozinski, and Vert (2009) of expanding the predictor matrix \( X \in \mathbb{R}^{n \times p} \) by replicating a predictor each time it appears in a new group to get \( \tilde{X} \in \mathbb{R}^{n \times \sum_k p_k} = (X_1, \ldots, X_g) \). \texttt{grpreg} is then run on the expanded predictor matrix \( \tilde{X} \) with nonoverlapping groups. The downside of this approach compared with that of \texttt{grpsel} is the additional memory required to store \( \tilde{X} \), which can be much wider than \( X \).

### 4.2 Sparse semiparametric modeling

Recall that in a sparse semiparametric model, the response \( y \) is modeled via a sum of univariate functions \( \sum_j f_j(x_j) \), where \( f_j \) can be zero, linear, or nonlinear. We follow the approach of Chouldechova and Hastie (2015) in using overlapping groups and regression splines to fit this model. Briefly, for every \( x_j \), an orthogonal spline is computed and two groups are formed: a linear group containing the first term of the spline (assumed equal to \( x_j \)) and a nonlinear group containing all terms of the spline. Due to the linear and nonlinear groups overlapping, the fit \( \hat{f}_j \) is nonlinear whenever the nonlinear group is selected regardless of whether the linear group is also selected. The group penalty parameters are scaled to control the trade-off between fitting \( f_j \) as linear or nonlinear. For group subset penalty parameter \( \lambda \), we set \( \lambda_k = \lambda \) for \( k \) a linear group and \( \lambda_k = 2\lambda \) for \( k \) a nonlinear group. For group lasso penalty parameter \( \lambda \), we set \( \lambda_k = \sqrt{2}\lambda \) for nonlinear groups to achieve an equivalent penalization.

### 4.3 Simulation design

We study regression and classification. For regression, the response is generated according to

\[
y_i = f^0(x_i) + \varepsilon_i, \quad i = 1, \ldots, n,
\]

while, for classification, it is generated as

\[
y_i = \begin{cases} 
0 & \text{if } f^0(x_i) + \varepsilon_i < 0 \\
1 & \text{if } f^0(x_i) + \varepsilon_i \geq 0, \quad i = 1, \ldots, n.
\end{cases}
\]

In both cases, \( f^0(x_i) = \sum_j f^0_j(x_{ij}) \) and \( \varepsilon_i \sim \mathcal{N}(0, \sigma^2) \). The covariates \( x_i \) are treated fixed and constructed by (1) drawing samples iid from \( \mathcal{N}(0, \Sigma) \), (2) applying the standard normal distribution function to produce uniformly distributed variables that conform to \( \Sigma \) (see Falk 1999), and (3) min-max scaling to the interval \([-1, 1]\). The correlation matrix \( \Sigma \) is defined elementwise as \( \Sigma_{i,j} = \rho^{|i-j|} \). The number of covariates is 2,500. For regression, 50 of the these covariates are selected at random to be nonzero: 40 relate to the response linearly as \( f(x) = x \) and 10 relate nonlinearly as \( f(x) = \cos(\pi x), f(x) = \sin(\pi x), \) or \( f(x) = \exp(10x) \).

For classification, support recovery is more difficult, so for that task 10 covariates are linear
and 5 are nonlinear. The functions evaluations are scaled to mean zero and variance one so that all functions are on the same footing. Each of the 2,500 covariates are expanded using a cubic thin-plate spline containing three knots at equispaced quantiles. Four terms are in each spline so that the number of predictors \( p = 10,000 \). The number of groups \( g = 5,000 \), consisting of 2,500 linear groups and 2,500 nonlinear groups. The sample size \( n = 1,000 \) is fixed and the noise parameter \( \sigma \) is varied to alter the signal-to-noise ratio (SNR).

### 4.4 Statistical performance

For regression, we measure out-of-sample test loss by the mean square loss on a testing set and report it relative to that of the best-performing estimator:

\[
\text{Relative test loss} := \frac{\text{Mean square loss} - \text{Mean square loss}^*}{\text{Mean square loss}^*},
\]

where \( \ast \) indicates the minimal mean square (test) loss among the five estimators considered. The best possible value of this metric is zero. For classification, we report the same metric but measured in terms mean logistic loss. As a measure of sparsity, we report the number of fitted functions that are nonzero:

\[
\text{Sparsity} := \sum_j \hat{f}_j \neq 0.
\]

Finally, as a measure of support recovery, we report the micro F1-score for the classification of linear and nonlinear functions:

\[
\text{Support recovery} := \frac{2 \cdot \text{True positives}}{2 \cdot \text{True positives} + \text{False positives} + \text{False negatives}}.
\]

The best possible value of this metric is one and the null value is zero. These metrics are all evaluated using tuning parameters that minimize mean square loss or mean logistic loss over a separate validation set of size \( n \).

The metrics under consideration are aggregated over 30 simulations. The regression results are reported in Figure 1 and the classification results in Figure 2. The vertical bars are averages and the error bars are standard errors.

Consider first the regression results. In line with our theory, group subset exhibits excellent performance when the signal is strong but fares poorly when the signal is weak. Group lasso behaves contrarily, performing capably at low SNRs but poorly at high SNRs. The transition between the two estimators in terms of relative test loss occurs at \( \text{SNR} \approx 2 \) in the moderate-correlation scenario \( (\rho = 0.5) \) and later at \( \text{SNR} \approx 3 \) in the high-correlation scenario \( (\rho = 0.9) \). Group subset+lasso achieves the best of both worlds. It improves the performance of group subset when the signal is weak and, by tapering off shrinkage, eventually behaves like group subset when the signal is strong. Group SCAD and group MCP also try to unwind shrinkage via their nonconvexity parameters. Even so, there remains a gap between these estimators and group subset+lasso. The latter converges earlier to the right sparsity level and the correct model. The gap is most stark in the high-correlation scenario, where their support recovery is roughly half that of the group subset estimators at \( \text{SNR} = 10 \).
Turning our attention to the classification results, we again see that group lasso maintains an edge in prediction over group subset when the signal is weak and vice-versa when the signal is strong. As before, the gap between group lasso and group subset at low SNRs is closed once shrinkage is introduced. The group subset estimators also continue to have a clear upper hand in support recovery. Closer inspection reveals this upper hand is because they make fewer false positive selections than the competing estimators, similar to the regression setting. There is, however, one interesting difference to the regression setting: group subset+lasso has an advantage over group subset when the SNR is high. In this regime, it has noticeably lower relative test loss and marginally better support recovery. This result corresponds to a well-known phenomenon where the maximum likelihood estimator diverges as the classes become separable (see Hastie, Tibshirani, and Wainwright 2015). Shrinkage has the desirable side-effect of preventing this divergence.
Figure 2: Comparisons of estimators for sparse semiparametric classification. Metrics are aggregated over 30 synthetic datasets generated with $n = 1,000$, $p = 10,000$, and $g = 5,000$. Vertical bars represent averages and error bars denote (one) standard errors. Dashed lines indicate the true number of nonzero functions.

4.5 Computational performance

We now compare the computational performance of grpsel against grpreg for regression. Our estimators—group subset and group subset + lasso—and those of grpreg—group lasso, group SCAD, and group MCP—each solve different optimization problems, so it does not make sense to ask whether one implementation is faster than another for the same problem. Rather, the purpose of these comparisons is to provide indications of run time and computational complexity for alternative approaches to structured sparsity. Both grpsel and grpreg are set with a convergence tolerance of $10^{-4}$. All run times and iteration counts are measured with reference to the coordinate descent algorithms of each package over a grid of the primary tuning parameter. Where there is a secondary tuning parameter, the figures reported are averaged over the secondary parameter, e.g., the total time taken to evaluate a $100 \times 30$ grid of parameters divided by 30. The simulation design is as before, but we now
fix the SNR and vary the number of covariates to measure scalability.

The results as aggregated over 30 synthetic datasets are reported in Figure 3. The vertical bars are averages and the error bars are standard errors. For $p = 10,000$, \texttt{grpsel}

![Figure 3: Comparisons of packages and estimators. Metrics are aggregated over 30 synthetic datasets generated with SNR = 1, $\rho = 0.5$, and $n = 1,000$. Vertical bars represent averages and error bars denote (one) standard errors.](image)

...can compute an entire solution path in one to two seconds. Group subset+lasso achieves marginally lower run times than group subset, requiring slightly fewer iterations to converge thanks to the additional regularization from shrinkage. For $p = 25,000$ the run times from \texttt{grpsel} are still less than five seconds to fit a path, while for $p = 100,000$ the times are around 10 seconds. \texttt{grpreg} is also impressive in these scenarios, though relative to \texttt{grpsel} it is slower. Group lasso converges in the fewest iterations, followed by group subset+lasso. Group SCAD and group MCP always take several thousand iterations to converge.

## 5 Data analyses

This section studies two contemporary problems: modeling foot traffic in major supermarkets and modeling recessions in the business cycle. Both problems are characterized by the availability of many candidate predictors and the possibility for misspecification of linear models. These characteristics motivate consideration of sparse semiparametric models.
5.1 Supermarket foot traffic

The first dataset contains anonymized data on foot traffic and sales volumes for a major Chinese supermarket (see Wang 2009). The task is to model foot traffic using the sales volumes of different products. To facilitate managerial decision-making, the fitted model should identify a subset of products that well-predict foot traffic (i.e., it should be sparse).

The sample contains \( n = 464 \) days. We randomly hold out 10\% of the data as a testing set and use the remaining data as a training set. Sales volumes are available for 6,398 products. To fit sparse semiparametric models, the approach described in Section 4.2 is applied. A four-term thin-plate spline is used for each product, resulting in \( p = 25,592 \) predictors and \( g = 12,796 \) groups (6,398 linear groups plus 6,398 nonlinear groups). As a measure of predictive accuracy, we report mean square loss on the testing set. We also report the number of fitted functions that are nonzero. As benchmarks, we include random forest and lasso, which respectively produce dense nonparametric models and sparse linear models. In addition, the (unconditional) mean is evaluated as a predictive method to assess the value added by the predictors. Ten-fold cross-validation is used to choose tuning parameters.

The metrics under consideration are aggregated over 30 training-testing set splits and reported in Table 1. Averages are reported with standard errors in parentheses.

| Method              | Mean square loss | Total   | Linear  | Nonlinear |
|---------------------|-----------------|--------|---------|-----------|
| Group subset + lasso| 0.650 (0.026)   | 178.6  | 115.7   | 62.9      |
| Group lasso         | 0.651 (0.025)   | 267.0  | 132.9   | 134.1     |
| Group MCP           | 0.658 (0.026)   | 215.1  | 120.9   | 94.1      |
| Lasso               | 0.712 (0.028)   | 165.2  | 165.2   | -         |
| Random forest       | 1.330 (0.057)   | -      | -       | -         |
| Mean                | 9.856 (0.367)   | -      | -       | -         |

**Table 1:** Comparisons of methods for modeling supermarket foot traffic. Metrics are aggregated over 30 splits of the data into training and testing sets. Averages are reported next to (one) standard errors in parentheses. Group subset + lasso used to fit sparse semiparametric models leads to the lowest mean square loss, though within statistical precision of group lasso and group MCP. Nonetheless, group subset + lasso has the edge of selecting fewer products than either of these estimators, which is advantageous for decision-making. Lasso yields models that are slightly more sparse but at the same time substantially less predictive, signifying linearity might be too restrictive. Random forest is markedly worse than any sparse method. It appears only a small fraction of products explain foot traffic, around 2–4%.

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4Available at [https://personal.psu.edu/ril4/DataScience](https://personal.psu.edu/ril4/DataScience).
5.2 Economic recessions

The second dataset contains monthly data on macroeconomic series for the United States (see McCracken and Ng 2016). The dataset is augmented with the National Bureau of Economic Research recession indicator, a dummy variable that indicates whether a given month is a period of recession or expansion. The task is to model the recession indicator using the macroeconomic series. Such models are useful for scenario analysis and for assessing economic conditions in the absence of low-frequency variables such as quarterly gross domestic product growth.

The sample contains \( n = 746 \) months, ending in October 2021. It includes the COVID-19 recession. We again randomly hold out 10\% of the data as a testing set and use the remaining data as a training set. Because there are relatively few recessionary periods, a stratified split is applied so that the proportion of recessions in the testing and training sets are equal. The dataset has 127 macroeconomic series. We add to this dataset six lags of each series, leading to a total set of 889 series. Applying a four-term thin-plate spline to each series yields \( p = 3556 \) predictors and \( g = 1778 \) groups (889 linear groups plus 889 nonlinear groups). To evaluate predictive accuracy, we report mean logistic loss on the testing set. The remaining metrics and methods are as before.

The results as aggregated over 30 training-testing set splits are reported in Table 2. The

| Method            | Mean logistic loss | Sparsity       |
|-------------------|--------------------|----------------|
|                   |                    | Total | Linear | Nonlinear |
| Group subset+lasso| 0.968 (0.103)      | 44.0  | 16.2  | 27.8      |
| Group lasso       | 1.063 (0.102)      | 57.5  | 30.7  | 26.8      |
| Group MCP         | 1.141 (0.092)      | 31.5  | 18.7  | 12.8      |
| Lasso             | 1.093 (0.107)      | 56.8  | 56.8  | -         |
| Random forest     | 1.294 (0.039)      | -     | -     | -         |
| Mean              | 3.703 (0.000)      | -     | -     | -         |

| Method            | Mean logistic loss | Sparsity       |
|-------------------|--------------------|----------------|
|                   |                    | Total | Linear | Nonlinear |
| Group subset+lasso| 0.968 (0.103)      | 44.0  | 16.2  | 27.8      |
| Group lasso       | 1.063 (0.102)      | 57.5  | 30.7  | 26.8      |
| Group MCP         | 1.141 (0.092)      | 31.5  | 18.7  | 12.8      |
| Lasso             | 1.093 (0.107)      | 56.8  | 56.8  | -         |
| Random forest     | 1.294 (0.039)      | -     | -     | -         |
| Mean              | 3.703 (0.000)      | -     | -     | -         |

**Table 2:** Comparisons of methods for modeling economic recessions. Metrics are aggregated over 30 splits of the data into training and testing sets. Averages are reported next to (one) standard errors in parentheses.

sparse semiparametric models from group subset+lasso predict recessionary periods well. They perform better than those from group lasso and, at the same time, depend on fewer macroeconomic series. Group MCP is sparser still but also less predictive. Lasso also yields models worse than those from group lasso and group subset+lasso, highlighting the value in allowing for nonlinearity. As with the supermarket dataset, the dense models from random forest underperform relative to the sparse models from other methods. All methods improve on the mean, illustrating the predictive content of monthly series.

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5 The January 2022 vintage is used, available at [https://research.stlouisfed.org/econ/mccracken](https://research.stlouisfed.org/econ/mccracken). See Appendix C.1 for preprocessing steps.

6 Available at [https://fred.stlouisfed.org/series/USREC](https://fred.stlouisfed.org/series/USREC).
6 Concluding remarks

Despite a broad array of applications, structured sparsity via group subset selection is not well-studied, especially in high dimensions where it is computationally taxing. This paper represents an effort to close the gap. Our optimization framework consists of low complexity algorithms that come with convergence results. A theoretical analysis of the proposed estimators illuminates some of their finite-sample properties. The estimators behave favorably in simulation, exhibiting excellent support recovery when fitting sparse semiparametric models. In real-world modeling tasks, they improve on popular benchmarks.

Our implementation grpsel is available on the R repository CRAN.

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Appendix A  Computation

A.1  Proof of Proposition 1

Proof. The subscript $k$ is dropped from $c_k$, $\lambda_{0k}$, and $\lambda_{1k}$ to simplify the notation. Since the objective is treated as a function in the $k$th group of coordinates $\nu_k$ only, we have

$$
\bar{F}_c(\nu; \tilde{\nu}) = \nabla_k L(\tilde{\nu})^T (\nu_k - \tilde{\nu}_k) + \frac{c}{2} \|\nu_k - \tilde{\nu}_k\|^2 + \lambda_0 1(\|\nu_k\| \neq 0) + \lambda_1 \|\nu_k\|
$$

$$
= \frac{c}{2} \|\nu_k - \tilde{\nu}_k\|^2 + \lambda_0 1(\|\nu_k\| \neq 0) + \lambda_1 \|\nu_k\|
$$

where $\tilde{\nu}_k = \tilde{\nu}_k - 1/c \nabla_k L(\tilde{\nu})$. When $\lambda_1 = 0$, it is not hard to see a minimizer of $\bar{F}_c(\nu; \tilde{\nu})$ is

$$
\nu_k^* = \begin{cases} 
\tilde{\nu}_k & \text{if } \|\tilde{\nu}_k\| \geq \sqrt{\frac{2\lambda_0}{c}} \\
0 & \text{otherwise}.
\end{cases}  \tag{A.1}
$$

When $\lambda_0 = 0$ and $\lambda_1 > 0$, the minimizer is

$$
\nu_k^* = \begin{cases} 
\left(1 - \frac{\lambda_1}{c\|\tilde{\nu}_k\|}\right) \tilde{\nu}_k + \left(1 - \frac{\lambda_1}{c\|\tilde{\nu}_k\|}\right)_+ \|\tilde{\nu}_k\| & \|\tilde{\nu}_k\| \geq 0 \\
0 & \text{otherwise}.
\end{cases} \tag{A.2}
$$

This expression follows from the proximal operator for the $l_2$-norm (Beck 2017). Combining (A.1) with (A.2) leads to the result of the proposition. \qed
A.2 Proof of Lemma 2

Proof. Denote by $\nu^*$ the result of applying the thresholding function (2.5) to $\tilde{\nu}$. Starting from the inequality (2.2) with $\nu = \nu^*$, we add $R(\nu^*)$ to both sides to obtain

$$F(\nu^*) \leq L(\tilde{\nu}) + \nabla_k L(\tilde{\nu})^\top (\nu^*_k - \tilde{\nu}_k) + \frac{c_k}{2} \|\nu^*_k - \tilde{\nu}_k\|^2 + R(\nu^*),$$

where the left-hand side follows from definition (2.3). Adding $\bar{c}_k/2\|\nu^*_k - \tilde{\nu}_k\|^2$ to both sides and rearranging terms leads to

$$F(\nu^*) \leq L(\tilde{\nu}) + \nabla_k L(\tilde{\nu})^\top (\nu^*_k - \tilde{\nu}_k) + \frac{\bar{c}_k}{2} \|\nu^*_k - \tilde{\nu}_k\|^2 + R(\nu^*) + \frac{c_k - \bar{c}_k}{2} \|\nu^*_k - \tilde{\nu}_k\|^2$$

$$= \bar{F}_{\bar{c}_k}(\nu^*; \tilde{\nu}) + \frac{\bar{c}_k - c_k}{2} \|\nu^*_k - \tilde{\nu}_k\|^2.$$ 

Using $\bar{F}_{\bar{c}_k}(\nu^*; \tilde{\nu}) \leq \bar{F}_{\bar{c}_k}(\nu^*; \nu^*) = F(\nu^*)$, we reorganize terms to get

$$F(\tilde{\nu}) - F(\nu^*) \geq \frac{\bar{c}_k - c_k}{2} \|\nu^*_k - \tilde{\nu}_k\|^2. \tag{A.3}$$

Now, define the vector

$$\eta_k^{(m)} := \begin{cases} (\nu_1^{(m+1)}), \ldots, \nu_k^{(m+1)}), \nu_k^{(m)}, \nu_{k+1}, \ldots, \nu_g^{(m)} \end{cases} \text{ if } k > 0$$

$$\nu(m) \text{ otherwise.}$$

Take $\nu = \eta_k{(m)}$ and $\nu^* = \eta_k^{(m)}$ and sum both sides of the inequality (A.3) over $1 \leq k \leq g$ to get

$$\sum_k [F(\eta_k^{(m-1)}) - F(\eta_k^{(m)})] = F(\eta_0^{(m)}) - F(\eta_g^{(m)}) \geq \sum_k \frac{\bar{c}_k - c_k}{2} \|\nu_k^{(m+1)} - \nu_k^{(m)}\|^2.$$ 

By definition $\eta_0^{(m)} = \nu^{(m)}$ and $\eta_g^{(m)} = \nu^{(m+1)}$, establishing $\{F(\nu^{(m)})\}_{m \in \mathbb{N}}$ is decreasing. Since $F(\nu)$ is bounded below, $\{F(\nu^{(m)})\}_{m \in \mathbb{N}}$ must converge. \hfill \Box

A.3 Proof of Theorem 1

The proof relies on the following lemma that the active set must stabilize in finitely many iterations. The lemma is established by contradiction along the lines of Dedieu, Hazimeh, and Mazumder (2021, Theorem 1).

Lemma 3. Let $\bar{c}_k > c_k$ for all $k = 1, \ldots, g$. Then the sequence of iterates $\{\nu^{(m)}\}_{m \in \mathbb{N}}$ stabilizes to a fixed support within a finite number of iterations.

Proof. Suppose the support does not stabilize in finitely many iterations. Choose an $m$ such that $\text{gs}(\nu^{(m+1)}) \neq \text{gs}(\nu^{(m)})$. Then at least one group was added or removed from the support, i.e., there is a $k$ such that either (1) $\nu_k^{(m)} = 0$ and $\nu_k^{(m+1)} \neq 0$ or (2) $\nu_k^{(m)} \neq 0$ and $\nu_k^{(m+1)} = 0$. Consider case (1). It follows from Lemma 2

$$F(\nu^{(m)}) - F(\nu^{(m+1)}) \geq \frac{\bar{c}_k - c_k}{2} \|\nu_k^{(m+1)}\|^2,$$
and, because \( \nu_k^{(m+1)} \) is the output of the thresholding function (2.5), it holds \( \nu_k^{(m+1)} \geq \sqrt{2\lambda_{0k}/\bar{c}_k} \). These inequalities together imply

\[
F'(\nu^{(m)}) - F'(\nu^{(m+1)}) \geq (\bar{c}_k - c_k) \lambda_{0k}/\bar{c}_k.
\]

Similar working yields the same inequality for case (2). For \( \bar{c}_k > c_k \), the quantity on the right-hand side is strictly positive. Hence, a change to the support yields a strict decrease in the objective value. However, if the support changes infinitely many times, this contradicts that \( F(\nu) \) is bounded below. Thus, the support must stabilize in finitely many iterations.

We are now ready to prove Theorem 1.

**Proof.** From Lemma 3, there exists a finite \( m^* \) such that the subsequence of iterates \( \{\nu^{(m)}\}_{m \geq m^*} \) share the same active set, say \( A \). Hence, for all \( m \geq m^* \) and \( k \in A \) we have

\[
\bar{F}_{\bar{c}_k}(\nu; \nu^{(m)}) \propto \frac{\bar{c}_k}{2} \left\| \nu_k - \left( \nu_k^{(m)} - \frac{1}{\bar{c}_k} \nabla_k L(\nu^{(m)}) \right) \right\|^2 + \lambda_{1k} \|\nu_k\|
\]

i.e., the group subset penalty can be treated as fixed. Denote by \( \nabla_k^2 \bar{F}_{\bar{c}_k}(\nu; \nu^{(m)}) \) the second directional derivative of \( \bar{F}_{\bar{c}_k}(\nu; \nu^{(m)}) \) along the vector \( \nu \in \mathbb{R}^{\sum_k p_k} \). The infimum of the minimal eigenvalue of \( \nabla_k^2 \bar{F}_{\bar{c}_k}(\nu; \nu^{(m)}) \) over all \( \nu_k \) and \( \nu \) is \( \bar{c}_k \). Since \( \bar{c}_k > 0 \), \( \bar{F}_{\bar{c}_k}(\nu; \nu^{(m)}) \) is strictly convex. Furthermore, Lemma 2 implies the level set \( \{\nu \in \mathbb{R}^{\sum_k p_k} : F(\nu) \leq F(\nu^{(0)})\} \) is bounded when the initialization \( \nu^{(0)} \in \mathbb{R}^{\sum_k p_k} \), and hence \( \{\nu^{(m)}\}_{m \geq m^*} \) is bounded. These conditions are sufficient to invoke Tseng (2001, Theorem 5.1) and establish \( \{\nu^{(m)}\}_{m \geq m^*} \) converges to a stationary point \( \nu^* \) of \( \bar{F}_{\bar{c}_k}(\nu; \nu^*) \). We conclude by the equality \( \bar{F}_{\bar{c}_k}(\nu^*; \nu^*) = F(\nu^*) \) that \( \nu^* \) is also a stationary point of \( F(\nu) \).

**A.4 Proof of Proposition 2**

**Proof.** Under the conditions of Theorem 1, Algorithm 1 is guaranteed to converge to a stationary point \( \hat{\nu}^{(t)} \) such that for all \( k \not\in A^{(t)} \) it holds

\[
\left( \|\nabla_k L(\hat{\nu}^{(t)})\| - \lambda_{1k} \right)_+ \leq \sqrt{\frac{2\lambda_{0k}^{(t)}}{\bar{c}_k}} = \sqrt{\frac{2\lambda_{0k}^{(t)}}{\bar{c}_k} P_k}.
\]

Then initializing Algorithm 1 with \( \hat{\nu}^{(t)} \) and using \( \lambda_0 = \lambda_0^{(t+1)} \) such that

\[
\lambda_0^{(t+1)} < \max_{k \in A^{(t)}} \left( \left( \frac{\|\nabla L(\hat{\nu}^{(t)})\| - \lambda_{1k}}{2p_k \bar{c}_k} \right)^2 \right)
\]

leads to \( \hat{\nu}^{(t+1)} \neq \hat{\nu}^{(t)} \).
Gradient screening

Rather than cycling through all $g$ groups in each coordinate descent round, it is convenient to restrict the updates to a smaller set of screened groups. The initialization $\nu^{(0)}$ can be used to compute the group-wise gradients $\left\{ \| \nabla_k L(\nu^{(0)}) \| \sqrt{p_k} \right\}_{k \in \mathcal{A}^{(0)}}$ which are already available as a consequence of selecting $\lambda_0$ dynamically (see Proposition 2). The inactive groups whose gradients are among the top 500 largest are classed as “strong,” in addition to the active set of groups. The remaining groups are classed as “weak.” The coordinate descent updates are restricted to the strong groups until convergence is achieved, at which time a further round over the weak groups is performed. If the solution does not change after this further round, convergence is declared. Otherwise, any weak groups that have become active are shifted to the strong set, and the process is repeated.

Gradient screening is also used in local search. Rather than searching through all inactive groups in the inner loop of Algorithm 2, only the inactive groups whose gradients are among the largest 5% are enumerated.

Gradient ordering

The solutions produced by coordinate descent often benefit from greedily ordering the groups. At the beginning of the algorithm, the groups are sorted according to their gradients. Any groups in $\mathcal{A}^{(0)}$ are placed first. The coordinate descent updates then proceed using this new ordering.

Active set updates

The set of active groups typically stabilizes after several rounds of coordinate descent updates. At this time, several additional rounds are required for the nonzero coefficients to converge. Rather than cycling through the full set (or screened set) of groups, the updates are restricted to the active groups only, usually a small subset. Once convergence is achieved on the active set, a further round is performed over the inactive set to confirm overall convergence.

Appendix B Error bounds

B.1 Proof of Theorem 2

The proof requires the following lemma.

**Lemma 4.** Let $\delta \in (0, 1]$. Let $X \in \mathbb{R}^{n \times p}$ be a fixed matrix and $\varepsilon \in \mathbb{R}^n$ be a $\mathcal{N}(0, \sigma^2 I)$ random vector. Define $\theta := \sum_k \nu_k$ and the random event

$$A_{\theta} := \left\{ |\varepsilon^T X \theta| \geq \|X \theta\| C_1 \sigma \sqrt{sp_{\max} + s \log \left( \frac{g}{s} \right) + \log(\delta^{-1})} \right\}.$$

Then, for some numerical constant $C_1 > 0$, the probability of the union $\bigcup_{\theta \in \mathcal{V}(2s)} A_{\theta}$ is at most $\delta$.

**Proof.** For $\mathcal{A} \subseteq \{1, \ldots, g\}$, a set of active groups, denote by $S_{\mathcal{A}} := \bigcup_{k \in \mathcal{A}} G_k$ the set of active predictors. Denote the singular value decomposition of $X_{\mathcal{A}}$ by $U_{\mathcal{A}} D_{\mathcal{A}} V_{\mathcal{A}}^T$, where $X_{\mathcal{A}}$ are
the columns of \( \mathbf{X} \) indexed by \( \mathcal{S}_A \). Define the set of unit vectors \( \mathcal{B}_2^{\mathcal{S}_A} := \{ \mathbf{u} \in \mathbb{R}^r : \| \mathbf{u} \| \leq 1 \} \), and the set of \( s \) group-sparse subsets \( \mathcal{P}(s) := \{ A \subseteq \{ 1, \ldots, g \} : |A| = s \} \). For all \( \mathbf{v} \in \mathcal{V}(2s) \) such that \( \mathbf{v} \not\equiv \mathbf{0} \), it holds

\[
\frac{\| \mathbf{v}^\top \mathbf{X} \mathbf{\theta} \|}{\| \mathbf{X} \mathbf{\theta} \|} = \frac{|\mathbf{v}^\top \mathbf{U}_A \mathbf{D}_A \mathbf{V}_A^\top \mathbf{\theta}_A|}{\| \mathbf{D}_A \mathbf{V}_A^\top \mathbf{\theta}_A \|} \leq \max_{A \in \mathcal{P}(2s)} \sup_{u \in \mathcal{B}_2^{\mathcal{S}_A}} |\mathbf{v}^\top \mathbf{U}_A u|.
\]

For any \( t \in \mathbb{R} \), this inequality implies

\[
P \left( \bigcup_{\mathbf{v} \in \mathcal{V}(2s)} \left\{ |\mathbf{v}^\top \mathbf{X} \mathbf{\theta}| \geq \| \mathbf{X} \mathbf{\theta} \| t \right\} \right) \leq P \left( \max_{A \in \mathcal{P}(2s)} \sup_{u \in \mathcal{B}_2^{\mathcal{S}_A}} |\mathbf{v}^\top \mathbf{U}_A u| \geq t \right).
\]

Applying Boole’s inequality to the right-hand side yields

\[
P \left( \max_{A \in \mathcal{P}(2s)} \sup_{u \in \mathcal{B}_2^{\mathcal{S}_A}} |\mathbf{v}^\top \mathbf{U}_A u| \geq t \right) \leq \sum_{A \in \mathcal{P}(2s)} P \left( \sup_{u \in \mathcal{B}_2^{\mathcal{S}_A}} |\mathbf{v}^\top \mathbf{U}_A u| \geq t \right).
\]

We bound the supremum over \( \mathcal{B}_2^{\mathcal{S}_A} \) using an \( \epsilon \)-net argument. Let \( \mathcal{E}^{\mathcal{S}_A} \) be an \( \epsilon \)-net of \( \mathcal{B}_2^{\mathcal{S}_A} \) with respect to \( \ell_2 \)-norm that satisfies \( |\mathcal{E}^{\mathcal{S}_A}| \leq (3/\epsilon)^{|\mathcal{S}_A|} \). Such an \( \mathcal{E}^{\mathcal{S}_A} \) is guaranteed to exist for \( \epsilon \in (0, 1) \) (Rigollet 2015, Lemma 1.18). Setting \( \epsilon = 1/2 \), it holds for any \( A \in \mathcal{P}(2s) \) and any \( \mathbf{z} \in \mathcal{E}^{\mathcal{S}_A} \)

\[
\sup_{u \in \mathcal{B}_2^{\mathcal{S}_A}} |\mathbf{v}^\top \mathbf{U}_A u| \leq 2 \sup_{\mathbf{z} \in \mathcal{E}^{\mathcal{S}_A}} |\mathbf{v}^\top \mathbf{U}_A \mathbf{z}|.
\]

Applying Boole’s inequality to this bound yields

\[
\sum_{A \in \mathcal{P}(2s)} P \left( 2 \sup_{\mathbf{z} \in \mathcal{E}^{\mathcal{S}_A}} |\mathbf{v}^\top \mathbf{U}_A \mathbf{z}| \geq t \right) \leq \sum_{A \in \mathcal{P}(2s)} \sum_{\mathbf{z} \in \mathcal{E}^{\mathcal{S}_A}} P \left( 2|\mathbf{v}^\top \mathbf{U}_A \mathbf{z}| \geq t \right).
\]

The cardinality of \( \mathcal{P}(2s) \) satisfies \( |\mathcal{P}(2s)| = (2^g - 1) \leq \log((eg)/(2s))^2s \). For any \( A \in \mathcal{P}(2s) \), the cardinality of \( \mathcal{E}^{\mathcal{S}_A} \) satisfies \( |\mathcal{E}^{\mathcal{S}_A}| \leq 6^{|\mathcal{S}_A|} \leq 6^{2s \max} \). Since \( \mathbf{U}_A \) is orthonormal and \( \mathbf{z} \) has unit length, the random variable \( \mathbf{v}^\top \mathbf{U}_A \mathbf{z} \sim \mathcal{N}(0, \sigma^2) \). Using a standard Gaussian tail bound (Rigollet 2015, Lemma 1.4), we have

\[
P \left( 2|\mathbf{v}^\top \mathbf{U}_A \mathbf{z}| \geq t \right) \leq 2 \exp \left( -\frac{t^2}{8\sigma^2} \right).
\]

It follows from the chain of inequalities above

\[
P \left( \bigcup_{\mathbf{v} \in \mathcal{V}(2s)} \left\{ |\mathbf{v}^\top \mathbf{X} \mathbf{\theta}| \geq \| \mathbf{X} \mathbf{\theta} \| t \right\} \right) \leq 2 \exp \left( -\frac{t^2}{8\sigma^2} + 2s \max \log(6) + 2s \log \left( \frac{eg}{2s} \right) \right).
\]

Setting \( t \geq \sqrt{8\sigma^2 \log(2) + 2s \max \log(6) + 2s \log(eg/(2s)) + \log(\delta^{-1})} \) concludes the proof.

We are now ready to prove Theorem 2.
Proof. Take any $\tilde{\nu} \in \mathcal{V}(s)$ and any $\beta \in \mathbb{R}^p$ such that $\beta = \sum_k \tilde{\nu}_k$. Optimality of $\hat{\nu}$ and $\hat{\beta} = \sum_k \hat{\nu}_k$ implies

$$\frac{1}{n} \|y - X\hat{\beta}\|^2 \leq \frac{1}{n} \|y - X\beta\|^2,$$

which, after some algebra, leads to

$$\frac{1}{n} \|\mathbf{f}^0 - X\hat{\beta}\|^2 \leq \frac{1}{n} \|\mathbf{f}^0 - X\beta\|^2 + \frac{2}{n} \|\varepsilon^\top X(\hat{\beta} - \beta)\|.$$

Observe $\hat{\beta} - \beta = \sum_k (\hat{\nu}_k - \tilde{\nu}_k)$, with at most $2s$ components of $(\hat{\nu}_1 - \tilde{\nu}_1, \ldots, \hat{\nu}_s - \tilde{\nu}_s)$ not equal to 0. An application of Lemma 4 thus yields the high-probability upper bound

$$\frac{2}{n} \|\mathbf{f}^0 - X\hat{\beta}\| C_1\sigma \sqrt{sp_{\text{max}} + s \log \left(\frac{g}{s}\right) + \log(\delta^{-1})} \leq \frac{2}{n} \left(\|\mathbf{f}^0 - X\hat{\beta}\| + \|\mathbf{f}^0 - X\beta\|\right) C_1\sigma \sqrt{sp_{\text{max}} + s \log \left(\frac{g}{s}\right) + \log(\delta^{-1})},$$

where the last line follows from Minkowski’s inequality for $l_p$-norms. Using Young’s inequality $(2ab \leq \alpha a^2 + \alpha^{-1}b^2$ for $\alpha > 0$), the first term on the right-hand side is bounded as

$$\frac{2}{n} \|\mathbf{f}^0 - X\hat{\beta}\| C_1\sigma \sqrt{sp_{\text{max}} + s \log \left(\frac{g}{s}\right) + \log(\delta^{-1})} \leq \frac{\alpha}{n} \|f^0 - X\hat{\beta}\|^2 + \frac{C_1^2\sigma^2}{\alpha n} \left[sp_{\text{max}} + s \log \left(\frac{g}{s}\right) + \log(\delta^{-1})\right].$$

A bound for the second term on the right-hand side follows similarly. Putting the results together and rearranging terms, we arrive at

$$\frac{1}{n} \|\mathbf{f}^0 - X\hat{\beta}\|^2 \leq \frac{1 + \alpha}{(1 - \alpha)n} \|\mathbf{f}^0 - X\beta\|^2 + \frac{2C_1^2\sigma^2}{\alpha(1 - \alpha)n} \left[sp_{\text{max}} + s \log \left(\frac{g}{s}\right) + \log(\delta^{-1})\right],$$

holding with probability at least $1 - \delta$ for $\alpha \in (0, 1)$. Taking $C \geq 2C_1^2$ completes the proof. \qed

### B.2 Proof of Theorem 3

The proof requires the following lemma.

**Lemma 5.** Let $\delta \in (0, 1]$. Let $X \in \mathbb{R}^{n \times p}$ be a fixed matrix and $\varepsilon \in \mathbb{R}^n$ be a $\mathcal{N}(0, \sigma^2 I)$ random vector. Let $\gamma_k$ be the maximal eigenvalue of $X_k^\top X_k/n$, where $X_k$ is the submatrix of $X$ corresponding to group $k$. Define the random event

$$A_k = \left\{ \|X_k^\top \varepsilon\| \geq \sqrt{n\gamma_k}\sigma \left[p_k + 2\sqrt{p_k \log(p_k) + p_k \log(\delta^{-1})} + 2\log(p_k) + 2\log(\delta^{-1})\right] \right\}.$$

Then the probability of the union $\cup_k A_k$ is at most $\delta$. 

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Proof. Denote the singular value decomposition of $X_k$ by $U_k D_k V_k^\top$. Using the properties of the operator norm, it holds
\[ \|X_k^\top \varepsilon\| = \|V_k D_k U_k^\top \varepsilon\| \leq \sqrt{n \gamma_k} \|U_k^\top \varepsilon\|. \]
For any $t \in \mathbb{R}$, this inequality implies
\[ P\left( \bigcup_k \left\{ \|X_k^\top \varepsilon\| \geq \sqrt{n \gamma_k t} \right\} \right) \leq P\left( \bigcup_k \left\{ \|U_k^\top \varepsilon\| \geq t \right\} \right). \]
Applying Boole’s inequality to the right-hand side yields
\[ P\left( \bigcup_k \left\{ \|U_k^\top \varepsilon\| \geq t \right\} \right) \leq \sum_k P\left( \|U_k^\top \varepsilon\| \geq t \right). \]
Since $U_k$ is orthonormal, the random variable $\|U_k^\top \varepsilon\|^2/\sigma_k^2 \sim \chi^2(p_k)$. Using a standard chi-squared tail bound (Laurent and Massart 2000, Lemma 1), we have for $t = p_k + \sqrt{2p_k x + 2x}$ and $x > 0$
\[ P\left( \|U_k^\top \varepsilon\| \geq \sigma \sqrt{t} \right) \leq \exp(-x). \]
It follows from the chain of inequalities above
\[ P\left( \bigcup_k \left\{ \|X_k^\top \varepsilon\| \geq \sqrt{n \gamma_k \sigma (p_k + \sqrt{2p_k x + 2x})} \right\} \right) \leq \exp(-x + \log(g)). \]
Setting $x \geq \log(g) + \log(\delta^{-1})$ concludes the proof.

We are now ready to prove Theorem 3.

Proof. For any $\tilde{\nu} \in \mathcal{V}(s)$ and any $\beta \in \mathbb{R}^p$ such that $\beta = \sum_k \hat{\nu}_k$, we have
\[ \frac{1}{n} \|y - X\hat{\beta}\|^2 + 2 \sum_k \lambda_k \|\hat{\nu}_k\| \leq \frac{1}{n} \|y - X\beta\|^2 + 2 \sum_k \lambda_k \|\nu_k\|, \]
which leads to
\[ \frac{1}{n} \|f^0 - X\hat{\beta}\|^2 \leq \frac{1}{n} \|f^0 - X\beta\|^2 + \frac{2}{n} \|\varepsilon^\top X(\hat{\beta} - \beta)\| + 2 \sum_k \lambda_k \left( \|\hat{\nu}_k\| - \|\nu_k\| \right). \tag{B.1} \]
The Cauchy-Schwarz inequality and Minkowski’s inequality are applied in turn to get
\[ \frac{2}{n} \|\varepsilon^\top X(\hat{\beta} - \beta)\| \leq \frac{2}{n} \sum_k \|X_k^\top \varepsilon\| \|\hat{\nu}_k - \nu_k\| \leq \frac{2}{n} \sum_k \|X_k^\top \varepsilon\| (\|\hat{\nu}_k\| + \|\nu_k\|). \]
Applying Lemma 5, and using the assumed lower bound on $\lambda_k$, yields
\[ \frac{2}{n} \sum_k \|X_k^\top \varepsilon\| (\|\hat{\nu}_k\| + \|\nu_k\|) \leq 2 \sum_k \lambda_k (\|\hat{\nu}_k\| + \|\nu_k\|) \]
with high-probability. Plugging this bound into (B.1), we arrive at
\[ \frac{1}{n} \|f^0 - X\hat{\beta}\|^2 \leq \frac{1}{n} \|f^0 - X\beta\|^2 + 4 \sum_k \lambda_k \|\nu_k\|, \]
holding with probability at least $1 - \delta$. \qed
B.3 Proof of Theorem 4

Proof. Begin with inequality (B.1). First, we bound the term $2/n\varepsilon^T X(\hat{\beta} - \beta)$. Lemma 4 gives the high-probability upper bound

$$
\frac{2}{n} |\varepsilon^T X(\hat{\beta} - \beta)| \leq \frac{2}{n} \|X(\hat{\beta} - \beta)\| C_1 \sigma \sqrt{s p_{\text{max}}} + s \log \left( \frac{g}{s} \right) + \log(\delta^{-1})
$$

Using Young’s inequality ($2ab \leq a^2 + 2ab^2$), the first term on the right-hand side is bounded as

$$
\frac{2}{n} \|f^0 - X\hat{\beta}\| C_1 \sigma \sqrt{s p_{\text{max}}} + s \log \left( \frac{g}{s} \right) + \log(\delta^{-1})
$$

The second term on the right-hand side is bounded similarly. Now, we bound the remaining term $2 \sum_k \lambda_k(\|\hat{\nu}_k\| - \|\hat{\nu}_k\|)$ in (B.1). Minkowski’s inequality and Assumption 1 give

$$
2 \sum_k \lambda_k(\|\hat{\nu}_k\| - \|\hat{\nu}_k\|) \leq 2 \lambda_{\text{max}} \sum_k \|\hat{\nu}_k - \hat{\nu}_k\| \leq \frac{2 \lambda_{\text{max}}}{\sqrt{n} \phi(2s)} \|X(\beta - \hat{\beta})\|
$$

Using Minkowski’s inequality again, we have

$$
\frac{2 \lambda_{\text{max}}}{\sqrt{n} \phi(2s)} \|X(\beta - \hat{\beta})\| \leq \frac{2 \lambda_{\text{max}}}{\sqrt{n} \phi(2s)} \left( \|f^0 - X\hat{\beta}\| + \|f^0 - X\beta\| \right).
$$

Two applications of Young’s inequality yields

$$
\frac{2 \lambda_{\text{max}}}{\sqrt{n} \phi(2s)} \left( \|f^0 - X\hat{\beta}\| + \|f^0 - X\beta\| \right) \leq \frac{4 \lambda_{\text{max}}^2}{\alpha \phi(2s)^2} + \frac{\alpha}{2n} \|f^0 - X\hat{\beta}\|^2 + \frac{\alpha}{2n} \|f^0 - X\beta\|^2.
$$

Finally, putting the bounds together and simplifying the resulting expression, we have

$$
\frac{1}{n} \|f^0 - X\hat{\beta}\|^2 \leq \frac{1 + \alpha}{(1 - \alpha)n} \|f^0 - X\beta\|^2 + \frac{4 \lambda_{\text{max}}^2}{\alpha (1 - \alpha) n} \left[ s p_{\text{max}} + s \log \left( \frac{g}{s} \right) + \log(\delta^{-1}) \right] + \frac{4 \lambda_{\text{max}}^2}{\alpha (1 - \alpha) \phi(2s)^2},
$$

holding with probability at least $1 - \delta$ for $\alpha \in (0, 1)$.

\hfill \Box

Appendix C Data analyses

C.1 Macroeconomic data preprocessing

All series are made stationary using standard transformations given in McCracken and Ng (2016). Some series contain missing observations and outliers, which are also treated as missing. These missing values are imputed using the na_kalman function of the \texttt{R} package \texttt{imputeTS}. An observation $x_i$ is treated as an outlier if $|x_i - Q_2|/(Q_3 - Q_1) > 4.5$ where $Q_1$, $Q_2$, and $Q_3$ are the respective quartiles of the data.

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