CMB statistical anisotropy, multipole vectors
and the influence of the dipole

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A simple algorithm which gives the multipole vectors in terms of the roots of a polynomial is given. We find that the reported alignment of the low \( l \) multipole vectors can be summarised as an anti-alignment of these with the dipole direction. This anti-alignment is not only present in \( l = 2 \) and \( l = 3 \) but also for \( l = 5 \) and higher. This alignment is likely due to non-linearity in the data processing. Our results are based on the three year WMAP data, we also list corresponding results for the first year data.

I. INTRODUCTION

It is by now a truism that with the availability of high resolution observations of the microwave sky cosmology has grown from a somewhat speculative to an empirical science. Especially the WMAP full sky maps have tightened the error bars on the \( \Lambda \)CDM cosmological standard model.

More recently, it has been observed that there are deviations from the assumption of an isotropic probability distribution of the fluctuations in the cosmic microwave background (CMB) (for a review see [3]). The source of this anisotropy is up to now mysterious be it genuinely cosmological, foreground, a statistical coincidence or a systematic error in the experiment or the data processing.

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In this paper, we contribute to this discussion, although we are also not able to completely resolve it. To this end, there are two steps: First, we make contact with the theory of Bloch coherent states. This leads us to a much simpler algorithm to determine the multipole vectors: We show that they are simply given as the roots of the function

\[ P(z) = \sum_{m=-l}^{l} \sqrt{\frac{2l}{l+m}} a_{lm} z^m, \]

where the \( a_{lm} \) are the coefficients of the temperature map expanded in spherical harmonics.

Secondly, we point out that a more natural interpretation of the alignment of cross products of multipole vectors is to find the multipoles themselves in a preferred plane on the sky. To a very good approximation, this plane is orthogonal to the direction of the dipole originating from the Doppler shift of the CMB due to the sun’s motion relative to it.

As, the dipole is not cosmological, it is likely that the alignment itself is not cosmological either, but due to systematics. We go on to speculate that non-linearities in the data analysis (due to feedback from using the dipole to calibrate the detectors) are a possible cause of such an alignment with the dipole and give evidence for this.

During the late stages of this investigation, the WMAP collaboration has published the data for the first three years of observation. We have updated our computations and use the new data in the main body of the paper. For comparison, in Appendix A, we list the multipole vectors for the first year data, both for the ILC and the Tegmark et al. data sets. Although, the WMAP team does not find evidence for non-gaussianity in the low \( l \) modes, the multipole vectors and thus the conclusions of this paper do not change much which is also anticipated in [6].

The structure of this article is as follows: The next section is mathematical in character and derives our method to compute the multipoles by introducing the Bloch coherent states. It is followed by a section where we describe our findings using this method. The next section presents the Wehrl entropy as a measure of randomness of a distribution of multipole vectors. Before a concluding section we offer some hints on the possible origin of the alignment. In an appendix we give the result of computations based on the first year WMAP data only for comparison with the existing literature.
II. COMPUTING MULTIPOLAR VECTORS

The most direct way to represent the cosmic microwave data is as the temperature as a function of the direction in the sky. Mathematically, it can be thought of as a real function \( T : S^2 \to \mathbb{R} \). Practically, the sky is tessellated into a number of Healpix pixels and the LAMBDA archive provides the temperature data for each of these pixels as obtained from WMAP observations (let us ignore for the moment the complication due to contamination from foreground sources which are not cosmological).

For statistical analysis, this pixelised data however is not convenient and one Fourier transforms (using the anafast program) the temperature data using spherical harmonics

\[
T(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi).
\]

We use a normalisation such that

\[
\int_{S^2} d\Omega Y_{lm}^* Y_{l'm'} = \delta_{ll'} \delta_{mm'}.
\]

The spherical harmonics form a basis of the vector space of complex functions on the sphere and when stressing this aspect we denote \( Y_{lm} \) also as \( |l, m\rangle \). Note that the subspaces spanned by the \( 2l + 1 \) vectors \( |l, m\rangle \) for fixed \( l \) form the irreducible representation of spin \( l \) of \( SO(3) \), the relevant symmetry group for the problem.

The temperature is of course a real function. This is reflected in the fact that

\[
a_{l,-m} = (-1)^m a_{lm}^*.
\]

This will be important later on.

In an isotropic universe, the \( a_{lm} \) for \( m \geq 0 \) would be independent random variables of zero mean and a variance which only depends on \( l \) but not on \( m \):

\[
\langle a_{lm} \rangle = 0, \quad \langle |a_{lm} a_{l'm'}| \rangle = c_l \delta_{l,l'} \delta_{mm'}.
\]

If the \( a_{lm} \) are drawn from a Gaussian distribution, the \( c_l \) which are estimated as

\[
c_l = \frac{1}{2l + 1} \sum_{m=-l}^{l} |a_{lm}|^2
\]

contain all information about the CMB. In fact, the WMAP measurement of the \( c_l \) has given spectacular credibility to the \( \Lambda CDM \) model, at least for \( l > 10 \). Judging from [11, 12, 13],
the original data analysis pipeline to convert the time series of detector data to pixel data was optimised to obtain the maximally likely values of the $c_l$ rather than the $a_{lm}$.

Isotropy of the CMB (more specifically of the random distribution from which the $a_{lm}$ are drawn) however is an assumption that needs to be checked. As the $c_l$ are invariant under $SO(3)$ rotations, the information about isotropy is independent or “orthogonal” to the $c_l$ in the $a_{lm}$. There is a series of papers including [1, 2, 3, 4] (for a review see [5]) noting that in fact the assumption of isotropy does not hold, at least for small $l$.

For the analysis of isotropy, however, the decomposition of the temperature in terms of spherical harmonics is not optimal as these are defined with respect to a choice of $z$-axis. Thus one uses a representation in terms of “multipoles” (going back to Gauss) that does not require a choice of reference direction but transforms covariantly under rotations.

Let us fix an $l$ and denote by $T_l$ the spin $l$ part of $T$:

$$T_l(\theta, \phi) = \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi).$$

Instead of using spherical coordinates $(\theta, \phi)$ we can consider the celestial sphere as the unit sphere $S^2 = \{\vec{x} \in \mathbb{R}^3: ||\vec{x}|| = 1\}$ in 3-space. Now, $T_l$ can be written as

$$T_l(\vec{x}) = d_l \mathcal{P}_l \prod_{i=1}^{l} (\vec{v}_{li} \cdot \vec{x}).$$

Here $d_l$ is proportional to $\sqrt{c_l}$ (and we will ignore it from now on), the $\vec{v}_{li}$ are unit (“multipole”) vectors and $\mathcal{P}_l = \sum_{m=-l}^{l} |l, m\rangle \langle l, m|$ is the projector to the spin $l$ representation. The latter is needed, as a single $\vec{v}_{li} \cdot \vec{x}$ has spin one and the $l$-fold product is a tensor product of representations which has contributions in spins less than $l$ as well. Concretely, $\mathcal{P}_l$ subtracts terms like $\vec{x}^2 (\vec{v}_{l1} \cdot \vec{v}_{l2}) (\vec{v}_{l3} \cdot \vec{x}) \cdots (\vec{v}_{ll} \cdot \vec{x})$ to make the tensor $\vec{v}_{l1} \otimes \cdots \otimes \vec{v}_{ll}$ totally trace free. Obviously, when computing scalar products with functions of spin $l$, we can leave out the projector $\mathcal{P}_l$.

The $2l + 1$ real components of the $a_{lm}$ contain the same information as $d_l$ and the $l$ unit vectors $\vec{v}_{li}$. Strictly speaking, the $\vec{v}_{li}$ are only defined up to a sign and thus strictly live in $\mathbb{RP}^2$. We shall fix this ambiguity by assuming the multipole vectors to point in directions in the northern hemisphere.

Note that as opposed to the representation of $T_l$ in terms of the $a_{lm}$, the representation in terms of multipole vectors is inherently real and no extra reality condition is needed. In fact, the antipodal ambiguity can be seen as a consequence of this reality.
The existing literature gives several ways to compute the $\vec{v}_l^i$ from the $a_{lm}$ [2, 14, 15]. Here, we will present a different one which seems to be the most straightforward. We claim that the $\vec{v}_l^i$ are just the roots of the polynomial $\sum_{m=-l}^{l} \sqrt{(2l + 1)} a_{lm} z^m$.

Strictly speaking, this is of course not a polynomial, but it is a polynomial divided by $z^l$. Thus, besides $\infty$, it has the same zero points as a polynomial and we will be sloppy in our language and call it a polynomial. A root $z_i \in \mathbb{C} \cup \{\infty\}$ corresponds to a point on $S^2$ via the stereographic projection $z = e^{i\phi} \cot(\theta/2)$.

(\text{note the we use the convention in which $\theta$ runs from 0 to $\pi$ unlike the latitude which runs from $-90^\circ$ to $90^\circ$). The fact that the multipole vectors are only defined up to a sign is reflected in the invariance of $P(z)$ under the antipodal map $z \mapsto -1/z^*$, each multipole vector corresponds to a pair of antipodal roots of $P(z)$. So there are $l$ multipole vectors as claimed. If $P(z)$ has fewer roots, the remaining multipole vectors point to the north or south poles of the sphere. This can be seen after realising that rescaling $P(z)$ by a constant just corresponds to a rescaling of $d_l$ and does not affect the roots.

To prove this relation between the multipole vectors and the roots of $P(z)$ we make contact with Bloch coherent states on the sphere.

In analogy with Glauber coherent states, which are eigenstates of the annihilation operator of the harmonic oscillator and thus have minimum uncertainty in $x$ and $p$ operators, Bloch coherent states are defined on the 2-sphere. They can be written as a rotation $R \in SO(3)$ applied to the highest weight state $|l, l\rangle$. $R$ is conveniently parametrised in terms of Euler angles $\alpha$, $\beta$, and $\gamma$ as $R(\alpha, \beta, \gamma) = \exp(i\alpha L_z) \exp(i\beta L_y) \exp(i\gamma L_z)$.

As $|l, l\rangle$ is an eigenstate of $L_z$, the $\gamma$-rotation acts only by a phase. As long as we are not interested in overall scalings, we can ignore this part and actually take $R$ as a function of $\alpha$ and $\beta$ only and thus living in the coset $SO(3)/SO(2) = S^2$ which is again a sphere. One observes the coordinates $\alpha$ and $\beta$ are just the spherical coordinates $\phi$ and $\theta$ respectively.

\footnote{This prescription was given before by C. Dennis [7, 8] as we learned after publishing a first version. There, the usefulness for CMB data analysis was also pointed out.}
Now, we want to compute scalar products of Bloch coherent states with functions on the sphere defined in terms of multipole vectors. Let us start with the case \( l = 1 \). We are interested in

\[
f(\alpha, \beta) = \langle 1, 1 | R^{-1}(\alpha, \beta) | \vec{v}_1 \rangle,
\]

where we denoted the function \( \vec{v}_1 \cdot \vec{x} \) as \( | \vec{v}_1 \rangle \). Obviously, if \( \vec{v}_1 \) is in the z-direction, the function is just a spherical harmonic \( | \vec{e}_z \rangle = \sqrt{4\pi/3} | 1, 0 \rangle \). As \( | 1, 0 \rangle \) is orthogonal to \( | 1, 1 \rangle \), the scalar product \( f(\alpha, \beta) \) vanishes if the rotation is the identity or minus the identity. Similarly, for a general vector \( \vec{v}_1 \), the scalar product vanishes if \( R \) rotates the z-axis in the direction of \( \vec{v}_1 \). This is the case, if \( (\alpha, \beta) \) as spherical coordinates and \( \vec{v}_1 \) denote the same point on the sphere.

For higher \( l \), we make use of the fact that \( | l, l \rangle \) can be written as a tensor power of the spin one highest weight: \( | l, l \rangle = | 1, 1 \rangle^{\otimes l} \). Here, as we are taking tensor products of highest weight states only, there are no contributions at lower spins. Thus, we can express the scalar product as the product for the different tensor factors

\[
\langle l, l | R^{-1}(\alpha, \beta) | l, m \rangle \propto \frac{1}{\sin(\beta)} \cdot \frac{1}{\sqrt{l}} \cdot \sqrt{\frac{2l}{l + m}} a_{lm} z^m,
\]

with \( z = \exp(i\alpha) \cot(\beta/2) \). Applying this to the scalar product of the Bloch coherent state with the CMB temperature function we arrive at

\[
\langle l, l | R^{-1}(\alpha, \beta) | T \rangle \propto \frac{1}{\sin(\beta)} \cdot \sum_{m=-l}^{l} \sqrt{\frac{2l}{l + m}} a_{lm} z^m.
\]

Combine this with the conclusion that this has to vanish if \( z \) as a point on the sphere (via the stereographic projection) coincides or is antipodal to one of the multipole vectors to identify the multipole vectors with the roots of the above expression. This concludes our proof of our method to find the multipole vectors as roots of a polynomial.

This representation significantly simplifies the evaluation of multipole vectors as it can be handed to any polynomial root finder such as \texttt{mathmatica}. A list of multipole vectors can be downloaded from \url{http://mathphys.iu-bremen.de/cmb}
The realisation of multipole vectors as roots of a polynomial also allows for a simple error analysis: We can just add a small amount of noise to the $a_{lm}$ and compute how much the multipole vectors move on the sky. We found that, on average, the position of the multipole vectors changed by $(0.13 + 0.04 \cdot l) \text{deg}/\mu\text{K}$ upon adding random noise of amplitude $1\mu\text{K}$.

Another advantage of our way of computing the multipole vectors as roots of $P(z)$ is that it has a direct generalization to complex functions on the sphere that the inherently real representation as products of terms of the form $\vec{v} \cdot \vec{x}$ has not. For a generic complex function, the $a_{lm}$ no longer obey $a_{l,-m} = (-1)^{m}a_{lm}^{*}$. Thus, the roots of $P(z)$ do no longer come in pairs of antipodal points. Rather the $2l$ roots now are multipole spinors and no longer just live on the northern hemisphere. This is clear from the perspective of [16], where $P(z)$ was constructed from elementary spinors of spin $1/2$ rather than the multipole vectors $\vec{v} \cdot \vec{x}$ of spin one.

This has a direct application to temperature data in conjunction with polarisation: We can treat the polarization as the complex phase of the temperature and work out the multipole spinors for this combined function. Unfortunately, the current status of the noise in the polarisation data for low $l$ does not warrant such an analysis, yet, and we postpone it to future investigation.

### III. ANISOTROPY IN WMAP MULTIPOLe VECTORS

We apply our algorithm to the internal linear combination (ILC) map of the combined three year WMAP data [6]. As explained there, this foreground cleaned map is trustworthy for low $l$. The existing literature mainly discusses alignment in the quadratic Doppler shift corrected map of Tegmark et al. [4] which was based on the first year data. Therefore, in appendix A we also list the multipole vectors for computations based on that data as well as on the first year ILC map.

In table I and II we list the multipole vectors for the three year run WMAP data. Figure I shows a graphical representation of the multipole vectors up to $l = 5$ on the northern hemisphere in galactic coordinates (remember that the vectors are only defined up to reversal). The colours encode the different $l$: Red, blue, black, green, yellow stand for $l = 1, 2, 3, 4, 5$ respectively. Note that by choosing the sine of latitude as a coordinate we have an area preserving map. This was done to have a proper impression of probabilities of points falling
in a given area on the celestial sphere.

**TABLE I**: Multipole vectors for three year ILC map. The last column lists the angle to the dipole measured such that it is always below 90°.

| l  | latitude | longitude | angle to dipole |
|----|----------|-----------|----------------|
| 1  | -96.15   | 48.25     | 0              |
| 2  | 2.86     | 13.55     | **85.79**      |
|    | 124.60   | 7.93      | 66.63          |
| 3  | 93.83    | 39.51     | **88.21**      |
|    | 23.89    | 8.26      | **77.13**      |
|    | -46.28   | 11.66     | 55.17          |
| 4  | -25.56   | 28.25     | 56.76          |
|    | -111.24  | 1.40      | 48.63          |
|    | -158.68  | 66.83     | 36.22          |
|    | -141.18  | 34.75     | 35.72          |
| 5  | 176.69   | 2.10      | **86.54**      |
|    | 39.92    | 36.79     | **86.40**      |
|    | 99.76    | 35.50     | **84.94**      |
|    | -71.77   | 32.60     | 24.09          |
|    | -131.13  | 53.63     | 22.44          |
| 6  | 76.41    | 36.95     | **85.46**      |
|    | 32.70    | 52.10     | 70.60          |
|    | -28.79   | 16.25     | 62.95          |
|    | -137.11  | 8.86      | 52.28          |
|    | -72.33   | 16.99     | 36.82          |
|    | -119.85  | 56.76     | 16.62          |
| 7  | 24.71    | 8.40      | **76.77**      |
|    | 92.99    | 20.92     | 69.65          |
|    | 175.65   | 26.30     | 69.55          |
|    | 5.45     | 41.08     | 67.09          |
|    | -11.98   | 33.74     | 61.93          |
|    | -73.11   | 20.44     | 33.41          |
|    | -136.07  | 66.75     | 27.49          |
| 8  | 117.20   | 40.10     | **86.84**      |
|    | 12.15    | 4.10      | **81.07**      |
|    | 26.74    | 40.34     | **78.04**      |
|    | -27.13   | 26.76     | 56.72          |
|    | -65.83   | 3.99      | 51.30          |
|    | 73.59    | 85.79     | 45.90          |
|    | -95.43   | 19.89     | 28.36          |
|    | -139.63  | 53.41     | 27.50          |
| 9  | 73.09    | 40.06     | **88.82**      |
|    | 9.56     | 11.45     | **88.36**      |
|    | 124.20   | 43.40     | **81.73**      |
|    | 149.11   | 32.43     | **80.51**      |
|    | -9.58    | 16.84     | **75.27**      |
As noted in the existing literature, there is no obvious clustering of the multipole vectors at any single point in the sky. There is, however, a planarity and a more indirect correlation: One can form vector products of multipole vectors and those are close for \( l = 2 \) and \( l = 3 \) (see [2, 4], for a discussion of dependence on various choices, see [18], for the specifics of the map making procedure [19] and for a review [5]).

More specifically, one computes normalised vector products like \( \vec{w}_2 = (\vec{v}_{21} \times \vec{v}_{22})/\|\vec{v}_{21} \times \vec{v}_{22}\| \) and similarly for pairs of \( l = 3 \) multipole vectors. These happen to be confined to a
FIG. 2: First year multipole vectors from Tegmark et al. map

relatively small region of the sky and this apparent anisotropy of the random distribution has been termed “axis of evil” in [20]. In [3], the cosine of the angle between \( \vec{w}_2 \) and the three pairs of \( l = 3 \) vectors have been computed and found to be 0.9531, 0.8719, and 0.8377 and from these relatively large numbers (note that \( \arccos(0.8377) = 33.1^\circ \)) it was concluded that the quadrupole and the octopole are aligned.

The obvious question is about the origin of this alignment. It could come from a systematic error in the measurement or post-processing of the data, it could be a statistical fluke or it could be of genuine cosmological importance. In this note, we want to argue for the first possibility.

In order to settle this question, it is worth understanding if there is another astronomical or cosmological datum in the region in which the \( \vec{w} \)'s are pointing. [3] suggests the pole of the ecliptic plane, however Bielewicz et al. [18] conclude “While the nominal significance of these results are confirmed in this paper, we also found that it is not at all unusual to observe such a strong alignment with one of the three major axes (ecliptic, galactic or super-galactic), given the peculiar internal arrangements of the quadrupole and octopole. This, it is not the ecliptic correlation per se that is anomalous, but rather the quadrupole-octopole alignment.”.

Furthermore, computing the vector products appears to be quite a derived quantity from the original data. Thus the question about the geometric significance of the alignment arises. Geometrically, \( \vec{w}_2 \) is the vector that is orthogonal to the plane spanned by \( \vec{v}_{21} \) and \( \vec{v}_{22} \). Thus, an alignment of different vector products means that all the vectors that cross-multiplied to
these $\bar{w}$'s lay in a plane on the sky that is orthogonal to the point on which the $\bar{w}$'s cluster. So it is rather this plane than the direction of the vector products that appears to be of significance.

Therefore, it is possible to subsume the different alignments into the statement that there appears to be a special plane in the sky in which a lot of multipole vectors happen to be.

What to our mind has not been stressed enough in the previous literature is that this plane happens to be the plane orthogonal to the dipole: In Table III we thus present as well the angle between the multipole vectors and the dipole. There appears to be an unnaturally large number of multipoles which are nearly orthogonal to the dipole (highlighted in boldface): Both quadrupole vectors as well as two of the three octopole vectors and three of the five $l = 5$ multipole happen to be nearly at right angles! (The third octopole is off by quite a margin even given the alignment claimed in [3]. However one should realise that the scalar products of the $\bar{w}$'s involving this multipole correspond to angles of $29^\circ$ and $33^\circ$.) For $l = 4$, there is no strong indication of multipole vectors being close to that plane. For a graphic presentation see Figure IV where the line indicates the plane orthogonal to the dipole. One can see that not only multipoles for $l = 2$ and $3$ fall on this line, but also three of the five vectors at $l = 5$, a result so far not reported in the literature\(^2\).

As far as relevance is concerned, we just mention that the fraction of a sphere that is at an angle larger than $\delta$ from a given vector (or its antipodal such that $0 \leq \delta \leq 90^\circ$) is given by $\cos(\delta)$. Take for example $l = 5$, where the probability to find three vectors at least $\theta = 85^\circ$ from the dipole direction is 0.006.

We find the fact that the multipole vectors seem to avoid the direction of the dipole is a strong indication that the alignment is not of cosmological origin but rather is an artefact of the data processing. The dipole (which is roughly a factor of hundred stronger than the higher spin contributions) is thought to be the first order Doppler shift contribution from the monopole (again a factor of thousand stronger) due to the motion of the sun relative to the rest-frame of the CMB. Thus, any genuine correlation of the CMB with the dipole could only hold in a rather Ptolemean cosmological model. In section V we therefore discuss possible systematic origins of such an alignment.

\(^2\) [21] applies wavelet methods to the three year WMAP data and also finds a number of preferred directions in the plane orthogonal to the dipole direction.
IV. WEHRL ENTROPY

So far, we have used Bloch coherent states to prove that the roots of $P(z)$ are in fact the multipole vectors. In [17], they are employed to define a notion of entropy for quantum mechanical states vectors in $SU(2)$ representations. This “Wehrl entropy”[22] is a function on the irreducible representations of spin $l$ is monotonic, strongly subadditive and positive and thus has all properties of an entropy. In our notation, it is defined as

$$S_W(T) = - \int_{SO(3)/SO(2)} d\Omega \left| \langle l, l | R^{-1}(\alpha, \beta) | T \rangle \right|^2 \ln \left( \left| \langle l, l | R^{-1}(\alpha, \beta) | T \rangle \right|^2 \right)$$

where $\Omega = (\alpha, \beta)$ is a point on the sphere. This entropy was introduced as a measure of how “quantum” a state is and it is a conjecture due to E. Lieb[17] that it is minimised if the state is coherent.[16] used the function $P(z)$ to prove this conjecture for low spin. Here we ignore the quantum mechanical interpretation and propose the Wehrl entropy as a measure of “randomness” of an ensemble of multipole vectors. Note, that the coherent states are necessarily complex and thus the CMB temperature function $T$ cannot attain the minimal value assuming the conjecture holds.
We computed the Wehrl entropy for the multipole vectors of the first year data up to \( l = 20 \). To have a comparison we computed as well the Wehrl entropy at each \( l \) for 50 realisations of random vectors on the sphere. From these we computed the maximum. Figure 4 shows the Wehrl entropy for the CMB as measured by WMAP in blue. The maximum obtained from random multipole vectors is indicated by the solid black line. The shaded regions contain 68.3%, 95.4%, and 99.7% of the simulated collection of random vectors.

V. POSSIBLE SOURCES OF ANTI-ALIGNMENT WITH THE DIPOLE

Finding an apparent correlation between the cosmological multipole vectors and the dipole arising from the sun’s movement calls for an understanding of an underlying systematic error. In this section, we want to give some indications on what might be the cause.

Before we come to specifics of the data post-processing, we would like to point out a peculiarity of the multipole representation: The ordinary representation of functions on the sphere in terms of coefficients of spherical harmonics transform in a simple (linear) way under addition of functions. In contrast, the spherical harmonic coefficients of a product of functions is quite involved and requires (as representation theoretically it is a tensor product) Clebsch-Gordan coefficients

\[
Y_{l_1 m_1}(\theta, \phi)Y_{l_2 m_2}(\theta, \phi) = \sum_{LM} CG_{l_1 m_1, l_2 m_2}^{LM} Y_{LM}(\theta, \phi).
\]

The multipole representation, in contrast, is in itself multiplicative. This implies that the multipoles of a sum of two functions transform in a complicated way (as the roots of the sum of two polynomials) and by themselves are not even well defined without specifying the weights \( d_l \). The only thing one can say is that they will not change too much if a small function is added (see also the section on noise dependence of the multipole vectors).

Under products however, the multiple vectors of the product are just the union of the multipole vectors for the two factors (up to some complications arising due to the lower order terms arising from the projector \( P_l \)). Thus taking products of two functions is the natural way for multipoles to spread between sectors of different \( l \).

We thus consider the following scenario likely for the cause of the anti-alignment: Instead of recording the proper temperature distribution in the sky \( T(\theta, \phi) \), there is a small
multiplicative error

\[ \hat{T}(\theta, \phi) = (1 + \epsilon f(\theta, \phi)) T(\theta, \phi), \]

(such an error term was also considered in [23]). This would lead to a mixing between different \( l \) modes of \( f \) and \( T \). The mode that would be the strongest in the mixing would be the dipole as the monopole does not mix and the dipole is two orders of magnitude stronger than the remaining modes.

This would however directly lead to an appearance of the dipole as a multipole vector for the higher \( l \). The anti-alignment would only appear if there would be a correction for the error \( f(\theta, \phi) \) which is slightly off or overcompensating.

The idea, that the dipole cannot be ignored when mode mixing is considered and indeed gives the main contribution was stressed in [24]. However, there the mixing was proposed to be due to gravitational lensing which was shown in [25] to be too weak to explain the alignment.

Another possible source of such multiplicative mode mixing is the filter used to eliminate galactic foreground emissions [11, 13] using the MASTER algorithm. Decomposing the filter into spherical harmonics confirms the expectation that due to rotational invariance, the low \( l \) modes have a large fraction of their power in the \( m = 0 \) modes in galactic coordinates and that virtually there are only modes with even \( l \) due to invariance of the filter under parity reflections. Our understanding of the MASTER algorithm is that due to computational complexity it only gives corrections depending on \( l \) and not on \( m \) and thus only corrects the \( d_l \) or \( c_l \) but does not change the directions of the multipole vectors. Furthermore it was developed for the BOOMERANG experiment which sees only a small portion of the sky and thus [13] describes Monte Carlo testing only for \( l \) of at least medium size and not the small values of \( l \) that show the alignment.

In our analysis of the filter, we were however not able to show conclusively that an application of the galactic filter directly leads to an anti-alignment of multipole vectors and the dipole.

Besides multiplicative corrections, there is another possibility of making the dipole influence the higher \( l \) modes: The sampling function could be slightly non-linear:

\[ \hat{T}(\theta, \phi) = T(\theta, \phi) + \epsilon T^2(\theta, \phi). \]

Again, the main contribution would come from the dipole (due to its strength compared to
the other modes), it would even be of a similar effect if instead of $T^2$ there were a contribution that is schematically $\text{dipole} \cdot T$.

Let us investigate such a contribution. The multipole representation is covariant under rotations. So it is simplest to go to a frame in which the dipole points in the $z$-direction. Let us study the polynomial $P(z)$ in this frame.

Any complex polynomial can be factored into linear factors $(z - a_i)$ in terms of the roots $a_i$. In our case, as noted above, from the reality of $T^* = T$ we have invariance of $P$ under the antipodal map $P(z) = P(-1/z)^\ast$. Thus, the roots always come in pairs $a_i$ and $-1/a_i^\ast$. Such a pair contributes a quadratic factor (conveniently normalised)

$$P_a(z) = \frac{a^\ast}{z}(z - a)(z + 1/a^*) = a^\ast z + (|a|^2 - 1) - a/z$$

and the total polynomial $P(z)$ is the product of such factors.

The dipole itself has its roots at the poles, 0 and $\infty$, thus it comes with the polynomial

$$P_0(z) = 1.$$ 

This polynomial obviously lacks the highest and lowest power of $z$.

Similarly, multipoles on the equator (in this frame, orthogonal to the dipole in a general frame) have $|a| = 1$ and thus the constant term vanishes:

$$P_a(z) = z/a - a/z \quad \text{for } |a| = 1.$$

In general, if we treat the roots of a degree $N$ polynomial as independent random variables, the coefficient of the $z^k$ term is expected to be of size $\sqrt{\binom{N}{k}}$ relative to the other terms (just from counting the number of terms contributing to $z^k$ when expanding the product of linear factors). The general form of $P(z)$ above reflects this, yielding an isotropic distribution of multipole vectors for independent $a_{lm}$. A polynomial with roots at special locations (the poles of the sphere, or the unit circle) will thus deviate from the general form. For example, if all roots lay on the unit circle (all are orthogonal to the dipole in the frame independent language), every other coefficient is missing, as for $|a| = 1$ the polynomial in $z$ is effectively in $z^2$.

Let us now see how a small quadratic contamination influences the location of the multipoles. Consider adding to $P(z)$ at $l$ a small contribution of $T^2 \approx \text{dipole} \cdot T$, the result will be the $P_{l-1}(z)$ for $l - 1$ times $P_0(z) = 1$. As a result, there will be a change in the coefficients
of $z^{±l}$ relative to the intermediate powers of $z$. We found numerically that such a shift is well able to move the multipoles towards the equatorial plane.

What could be the origin of such a non-linear contribution? A possible source of feedback of the dipole into the higher modes comes from the real time calibration of WMAP on the Earth velocity modulation of the dipole as described in section 3 of [12]. This procedure uses a priori knowledge of the dipole from COBE to compare with the raw WMAP data (without correcting for the galactic cut). Given the relative strength of the dipole even a comparably small error in this prior could couple the the dipole into the low $l$ spectrum via non-linear feedback. For example, it could be possible that minimisation of the dipole could also decrease the $a_{lm}$ for $|m| < l$ for $l > 1$ in the frame aligned with the dipole.

Due to our lack of direct information on the details of this calibration procedure, we have not attempted to test this hypothesis using for example Monte Carlo data. Still, we consider it a plausible source of non-linearity which mixes the dipole into the sectors for $2 \leq l \leq 5$ which in turn is in principle able to move the multipoles away from the direction of the dipole.

VI. CONCLUSIONS

We have reanalysed the low $l$ multipole vectors of the three year WMAP data. To this end, we presented a simple algorithm to compute these multipole vectors which reduces this problem to finding the roots of a polynomial. We argued for this algorithm by making contact with Bloch coherent states on the sphere.

We pointed out that the alignment found earlier between the $l = 2$ and $l = 3$ multipole vectors is really an anti-alignment with the dipole. This anti-alignment is thus likely not cosmological as the dipole itself is not cosmological but due to the sun’s motion. Furthermore, we found that this anti-alignment extends to three of the five multipole vectors at $l = 5$ and higher.

This anti-alignment with the dipole points towards a systematic error stemming from the data processing. We could not conclusively pinpoint the source but argued that a likely possibility is a non-linear mixing the dipole and higher multipoles due to the running recalibration of the WMAP detectors on the modulation of the dipole.

It would be interesting to test this hypothesis with simulations using the real data pro-
cessing pipeline.

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APPENDIX A: COMPARISON WITH FIRST YEAR DATA

The main text was based on the recently published temperature data of the first three years of WMAP operation. The relevant foreground cleaned map is the “internal linear combination”.

For comparison with the existing literature which is mainly based on the quadratic Doppler shift corrected map of Tegmark et al. [4], cleaned map including the DQ higher order Doppler correction [2]3. Table II shows the resulting multipole vectors while Figure 2 shows them in a plot of the northern hemisphere in galactic coordinates. There is no equivalent to the Tegmark et al. map for the three year data. For the reader to judge the influence

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3 We thank Max Tegmark and Greg Weeks for useful correspondence.
TABLE II: Multipole vectors up to $l = 5$ using the first year Tegmark et al. map

| $l$ | latitude | longitude | angle to dipole |
|-----|----------|-----------|----------------|
| 1   | -96.15   | 48.25     | 0              |
| 2   | 11.26    | 16.64     | **88.69**      |
|     | 118.87   | 25.13     | **79.81**      |
| 3   | 86.94    | 39.30     | **87.59**      |
|     | 22.63    | 9.18      | **78.61**      |
|     | -44.92   | 8.20      | 58.73          |
| 4   | -26.49   | 26.86     | 57.07          |
|     | 74.74    | 5.46      | 54.31          |
|     | -153.02  | 31.93     | 45.30          |
|     | -151.36  | 76.73     | 35.58          |
| 5   | 44.67    | 33.54     | **88.97**      |
|     | 172.84   | 3.07      | **88.39**      |
|     | 98.70    | 38.50     | **87.75**      |
|     | -74.21   | 31.43     | 23.63          |
|     | -122.69  | 57.54     | 18.34          |

of the different schemes for foreground removal, we also list in table III the multipole vectors for the ILC using the first year data only. The corresponding plot is figure 4.

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TABLE III: Multipole vectors for first year ILC map data

| ℓ | latitude | longitude | angle to dipole |
|---|----------|-----------|----------------|
| 2 | 15.55    | 3.21      | **78.23**      |
|   | 120.95   | 19.79     | **75.69**      |
| 3 | 95.27    | 37.04     | **85.89**      |
|   | 21.73    | 9.39      | **79.31**      |
|   | -47.02   | 10.71     | 55.47          |
| 4 | 71.93    | 6.96      | 56.20          |
|   | -28.31   | 30.23     | 53.66          |
|   | -160.38  | 70.63     | 36.88          |
|   | -142.64  | 39.46     | 34.10          |
| 5 | 100.79   | 38.52     | **88.06**      |
|   | 176.05   | 1.20      | **87.64**      |
|   | 40.12    | 37.00     | **86.29**      |
|   | -73.53   | 34.23     | 21.90          |
|   | -128.61  | 54.54     | 21.00          |
| 6 | 84.08    | 34.58     | **82.83**      |
|   | 34.55    | 53.56     | 69.98          |
|   | -23.00   | 17.06     | 66.21          |
|   | -147.13  | 5.36      | 60.85          |
|   | -74.51   | 20.96     | 32.34          |
|   | -120.34  | 55.52     | 16.49          |

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