A polynomial quantum query lower bound for the set equality problem

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Abstract. The set equality problem is to tell whether two sets $A$ and $B$ are equal or disjoint under the promise that one of these is the case. This problem is related to the Graph Isomorphism problem. It was an open problem to find any $\omega(1)$ query lower bound when sets $A$ and $B$ are given by quantum oracles. We will show that any error-bounded quantum query algorithm that solves the set equality problem must evaluate oracles $\Omega\left(\sqrt{n}\ln n\right)$ times, where $n = |A| = |B|$.

1 Introduction, motivation and results

The amazing integer factoring algorithm of Shor [14] and search algorithm of Grover [7] show that to find quantum lower bounds is more than just a formality. The most popular model of quantum algorithms is the query (oracle) model. Thus, also quantum lower bounds are proved in the query model. There are developed methods that offer tight or nearly tight lower bound for some problems, however for some other problems not. Recently Aaronson [1] found a new method how to get tight quantum query lower bounds for some important problems, for example, the collision problem. This was an open problem since 1997. Aaronson’s method uses symmetrization over the input and therefore can be hard to apply to the problems with asymmetric input. The set equality problem is an example of such problem and it remained unsolved.

In this paper we will find a quantum lower bound for the set equality problem by reduction. We will reduce the collision problem to the set equality problem, therefore getting quantum query lower bound for the set equality problem.

Let assure ourselves that the set equality problem is related with Graph Isomorphism problem. We are given two graphs $G_1, G_2$ and we want to establish whether there exists permutations $p_1, p_2$ over vertices of graphs such that permuted graphs $p_1(G_1), p_2(G_2)$ are equivalent (graphs $G_1, G_2$ are isomorphs). Let $P_i$ denote the set of all graphs gotten by some permutation over graph $G_i$’s vertices ($i \in \{0, 1\}$). It is easy to see that if graphs are isomorphs then $P_1 = P_2$, but if not, then $P_1 \cap P_2 = \emptyset$. Therefore, if one can distinguish between those

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cases, then he can solve the Graph Isomorphism problem. Since there are $n!$ permutations for a graph with $n$ vertices, the sizes of $P_1, P_2$ can be superpolynomial over the number of vertices of graphs $G_1, G_2$.

Let $[n]$ denote the set $\{1, 2, ..., n\}$.

**Definition 1** Let $a : [n] \to [N]$ and $b : [n] \to [N]$ be the functions. Let $A$ be the set of all $a$’s images $A := a([n]) := \{a(1), a(2), ..., a(n)\}$ and $B := b([n]) := \{b(1), b(2), ..., b(n)\}$. There is the promise that either $A = B$ or $A \cap B = \emptyset$.

Let the **general set equality** problem denote the problem to distinguish these two cases, if functions $a$ and $b$ are given by quantum oracles.

By use of Ambainis’ method it is simple to prove $\Omega(\sqrt{n})$ lower bound for the general set equality problem. However, this approach works only if every image can have very many preimages. Graph theorists think that the Graph Isomorphism problem, when graphs are promised not to be equal with themselves by any nonidentical permutation, still is very complex task. This limitation lead us to the set equality problem where $a$ and $b$ are one-to-one functions.

**Definition 2** Let one-to-one set equality problem denote the general set equality problem under promise that $a(i) \neq a(j)$ and $b(i) \neq b(j)$ for all $i \neq j$.

Finding $\omega(1)$ quantum query lower bound for the set equality problem was posed an open problem by Shi. Despite $\omega(1)$ lower bound for the one-to-one set equality problem remaining unsolved task, $\Omega(\frac{\log^{1.93} n}{\log \log n})$ quantum query lower bound was showed for a problem between these two problems when $|a^{-1}(x)| = O(\log n)$ and $|b^{-1}(x)| = O(\log n)$ for all images $x \in [N]$.

In this paper we will show the polynomial quantum query lower bound for the most challenging task: the one-to-one set equality problem.

**Theorem 3** Any error-bounded quantum query algorithm $A$ that solves the one-to-one set equality problem must evaluate functions $\Omega(\frac{\sqrt{n}}{\log \log n})$ times.

The rest of the paper will be organized as follows. In the section 2 will be notations and previous results that we will use. In the section 3 we will preview the main idea of the proof of Theorem 3. The section 4 will start the proof, the section 5 will prepare for continuing proof and the section 6 will finish the proof.

# 2 Preliminaries

## 2.1 Quantum query algorithms

The most popular model of the quantum computing is a query (or oracle, or black box) model where the input is given by the oracle. For more details, see a survey by Ambainis or a textbook by Gruska. In this paper we are able to skip them because our proof will be based on reduction to solved problems.

In this paper we consider only the worst case complexity for error-bounded quantum algorithms. Thus, without loss of generality, we can assume that any
quantum algorithm makes the same number of queries for any input. If we say that algorithm has two input functions \(a\) and \(b\) then for technical reasons somewhere it can be comprehend with one input function denoted as \(a, b\).

One of the most amazing quantum algorithms is Grover’s search algorithm \([7]\). It shows how a given \(x_1 \in \{0, 1\}, x_2 \in \{0, 1\}, \ldots, x_n \in \{0, 1\}\) to find the \(i\) such that \(x_i = 1\) with \(O(\sqrt{n})\) queries under promise that there exists at most one such \(i\).

This algorithm can be considerably generalized to so called amplitude amplification \([6]\). Using amplitude amplification one can make good quantum algorithms for many problems till the quadratic speed-up over classical algorithms.

By straightforward use of amplitude amplification we get quantum algorithm for the general set equality problem making \(O(\sqrt{n})\) queries and quantum algorithm for the one-to-one set equality problem making \(O(n^{1/3})\) queries. Therefore our lower bound probably is not tight.

### 2.2 Quantum query lower bounds

There are two main approaches to get good quantum query lower bounds. The first is Ambainis’ \([2]\) quantum adversary method, other is lower bound by polynomials introduced by Beals et al. \([5]\) and substantially generalized by Aaronson \([4]\), Shi \([13]\) and others. Although explicitly we will use only Ambainis’ method, the lower bound we will get by the reduction to the problem, solved by polynomials’ method.

The basic idea of the adversary method is, if we can construct a relation \(R \subseteq X \times X\), where \(X\) and \(Y\) consist of 0-instances and 1-instances and there is a lot of ways how to get from an instance in \(X\) to an instance in \(Y\) that is in the relation and back by flipping various variables, then query complexity must be high.

**Theorem 4** \([2]\) Let \(f(x_1, \ldots, x_n)\), be a function of \(n\) variables with values from some finite set and \(X, Y\) be two sets of inputs such that \(f(x) \neq f(y)\) if \(x \in X\) and \(y \in Y\). Let \(R \subseteq X \times Y\) be such that

- For every \(x \in X\), there exist at least \(m\) different \(y \in Y\) such that \((x, y) \in R\).
- For every \(y \in Y\), there exist at least \(m'\) different \(x \in X\) such that \((x, y) \in R\).
- For every \(x \in X\) and \(i \in \{1, \ldots, n\}\), there are at most \(l\) different \(y \in Y\) such that \((x, y) \in R\) and \(x_i \neq y_i\).
- For every \(y \in Y\) and \(i \in \{1, \ldots, n\}\), there are at most \(l'\) different \(x \in X\) such that \((x, y) \in R\) and \(x_i \neq y_i\).

Then, any quantum algorithm computing \(f\) uses \(\Omega(\sqrt{mn'})\) queries.

Actually, original Ambainis’ formulation was about \(\{0, 1\}\)-valued variables but we can use any finite set as it is implied by the next, more general theorem in Ambainis’ paper \([2]\).
2.3 The collision problem

Finding $\omega(1)$ quantum lower bound for the collision problem was an open problem since 1997. In 2001 Scott Aaronson [11] solved it by showing polynomial lower bound. Later his result was improved by Yaoyun Shi [13]. Recently, Shi’s result was extended by Samuel Kutin [10] and by Andris Ambainis [4] in another directions.

Below is an exact formulation of the collision problem due to Shi [13].

**Definition 5** Let $n > 0$ and $r \geq 2$ be integers with $r|n$, and let a function $f$ of domain size $n$ be given as an oracle with the promise that it is either one-to-one or $r$-to-one. Let **r-to-one collision problem** denote the problem to distinguishing these two cases.

Shi [13] showed following quantum lower bound for the $r$-to-one collision problem.

**Theorem 6** [13] Any error-bounded quantum algorithm that solves $r$-to-one collision problem must evaluate the function $\Omega((n/r)^{1/3})$ times.

Kutin [10] and Ambainis [4] extended his result for functions with any range.

2.4 Notations

Let $F^* := F^*(n, N)$ denote the set of all partial functions from $[n]$ to $[N]$. Then any $f^* \in F^*$ can be conveniently represented as a subset of $[n] \times [N]$, i.e., $f^* = \{(i, f^*(i)) : i \in \text{dom}(f^*)\}$.

For a finite set $K \subseteq \mathbb{Z}^+$, let $SG(K)$ denote the group of permutations on $K$. For any integer $k > 0$, $SG(k)$ is a shorthand for $SG([k])$. For each $\sigma \in SG(n)$ and $\tau \in SG(N)$ define $\Gamma^*_{\sigma, \tau} : F^* \rightarrow F^*$ as $\Gamma^*_{\sigma, \tau}(f^*) := \{(\sigma(i), \tau(j)) : (i, j) \in f^*\}, \forall f^* \in F^*$.

3 The idea behind the proof

The rest of the paper is proof of the Theorem [3] In this section we will discuss the main idea behind this proof. The key is to reduce some problem with known quantum query lower bound to the one-to-one set equality problem. Unfortunately, a simple reduction does not work. Therefore we must make a chain of reductions and in the end get the problem, which can be solved by arbitrary methods.

The problem, which we will try to reduce to the one-to-one set equality problem, is the collision problem. All steps of reduction will be probabilistic. One of that steps Midrijanis [11] used to prove quantum query lower bound for modified set equality problem. We will conclude that any quantum query algorithm that solves the one-to-one set equality problem either solves the collision problem or some other problem that will be presented later. For the collision problem we
have a quantum query lower bound and for this other problem we will prove it using Ambainis' adversary method. This implies lower bound for the one-to-one set equality problem.

Unfortunately, since these reductions are probabilistic ones, but Theorem 4 tells about ordinary functions, a lot of technical work must be done to provide the correctness of the last reduction. We will analyze properties of those reductions and show, informally, that they are very similar (in sense of query complexity).

There will be two kinds of reduction from the collision problem to the set equality problem. Let $f$ denote $r$-to-one function which the collision problem has in the input. From $f$ we will randomly get two functions, $a$ and $b$. The both reductions will randomly permutate range and domain of $f$ and divide domain into 2 disjoint halves. The first reduction will takes those halves of domain as domains for functions $a$ and $b$. The second reduction will take only the first half for both functions, just it will make additional permutation over domain for both functions.

Informally, it is clear that both reductions makes "almost" equal pair of functions $a$ and $b$ whenever $r$ is big "enough". We will show that any quantum algorithm that can make distinction between them must make "quite many" queries. On the other hand, every algorithm for the set equality problem that don't make distinction between them can be used to solve the collision problem that is proved to be hard.

4 Framework of the proof

We have some $n$ and $1 < r < n$, such that $2|n$ and $r|n$. From the conditions of the collision problem we have function $f : [n] \rightarrow [N]$ with promise that $f$ is either one-to-one or $r$-to-one. Let us choose random variables $\sigma \in SG[n], \sigma_1 \in SG[n/2], \sigma_2 \in SG[n/2]$ and $\tau \in SG[N]$.

With complementary reduction we will denote the process deriving functions $a$ and $b$ such that $a(i) = \Gamma^\sigma_\tau(f)(i)$ and $b(i) = \Gamma^\sigma_\tau(f)(n/2+i)$ for all $i \leq n/2$.

With equivalent reduction we will denote the process deriving functions $a$ and $b$ such that $a(\sigma_1(i)) = \Gamma^\sigma_\tau(f)(i)$ and $b(\sigma_2(i)) = \Gamma^\sigma_\tau(f)(i)$ for all $i \leq n/2$.

Lemma 7 For any quantum algorithm $A$ that solves the set equality problem with $T$ queries either there exists quantum algorithm that solves $r$-to-one collision and makes $O(T)$ queries or there exists quantum algorithm that makes distinction between complimentary and equivalent reduction and makes $O(T)$ queries.

Proof. This tabular shows the acceptance probability of algorithm $A$ running on $a$ and $b$.

| reduction's type | function's $f$ type | one-to-one | $r$-to-one |
|-----------------|---------------------|------------|------------|
| complimentary   | $p_1^c > 4/5$       | $p_2^c$    |
| equivalent      | $p_1^e < 1/5$       | $p_2^e$    |

There are two possibilities. If $p_2^c \geq 2/5$ or $p_2^e \leq 3/5$ then algorithm $A$ can be used to solve the collision problem. But if $p_2^c < 2/5$ and $p_2^e > 3/5$ then algorithm
$A$ can be used to make distinction between complementary and equivalent reduction.

In the next sections we will prove the following lemma:

**Lemma 8** Any quantum algorithm $A$ that makes distinction between complementary and equivalent reduction makes $\Omega(\sqrt{\frac{1}{\log n}})$ queries.

Choosing $r = n^{2/5} \log^{3/5} n$ Lemma 7 together with Lemma 8 and Theorem 6 will finish the proof of Theorem 3.

5 The lower bound of distinction, preparation

In this section we will start to prove Lemma 8. Informally, both reductions, complementary and equivalent, make "quite similar" pairs of functions. So we have to define what means "similar" and to proof exactly how similar. Also, Theorem 4 deals with ordinary input not distributions over inputs, therefore we have to define what means "similar" and to proof exactly how similar.

In this subsection we will investigate properties of both reductions. We will speak only about pairs of functions $(a, b)$ that can be result of either complementary or equivalent reduction with nonzero probability $p(a, b) > 0$. We will investigate what pairs $(a, b)$ can appear.

For any function $a$, which is in some pairs, let $INV(a) := (a_i | 0 \leq i \leq r) := (a_0, a_1, ..., a_{r-1}, a_r)$ denote the tuple where $a_i$ is the number of image's elements $x \in \mathbb{N}$, such that cardinality of the set of preimages of $x$ is $i$, formally $a_i := \#x(|a^{-1}(x)| = i)$ for $0 < i \leq r$ and $a_0 := \frac{r}{r} - \sum_{i=1}^{r} a_i$, where $\frac{r}{r}$ is just the total count of images.

Let $INV(a, b)$ denote $(INV(a), INV(b))$. $INV(a, b)$ is quite good way to describe the structure of some pair of functions $(a, b)$ because of many reasons. Firstly, one can see, that $INV(a, b) = INV(G^a_\tau(a), G^b_\tau(b))$ for any pair of functions $(a, b)$ and any $\sigma_1 \in SG[n/2], \sigma_2 \in SG[n/2]$ and $\tau \in SG[\mathbb{N}]$.

Also, the probability for any pair $(a, b)$ to appear after reduction $p(a, b)$ depends only on $INV(a, b)$. Moreover, if there exists pairs of functions $(a_1, b_1)$ and $(a_2, b_2)$ such that $INV(a_1, b_1) = INV(a_2, b_2)$ then there exists variables $\sigma_1 \in SG[n/2], \sigma_2 \in SG[n/2]$ and $\tau \in SG[\mathbb{N}]$ such that $(a, b) = (G^a_\tau, G^b_\tau).$

Now we will show, that, for any pair $(a, b)$, $INV(a)$ and $INV(b)$ are closely related. For any $INV(a) = (a_0, a_1, ..., a_{r-1}, a_r)$ let $INV(\tilde{a})$ denote the tuple $(a_{r-i} | 0 \leq i \leq r) := (a_r, a_{r-1}, ..., a_1, a_0)$.

It is evident that for any pair of functions $(a, b)$ that occurs after complementary reduction holds $INV(a) = INV(b)$ but for any pair of functions $(a, b)$ that occurs after equivalent reduction holds $INV(a) = INV(b)$.

We will use these facts to show that complementary and equivalent reductions are quite similar, in other words, any functions $b_1$ and $b_2$ such that $INV(b_1) = INV(b_2)$ differ in many bits only with very small probability. So we will be able to use Ambainis’ Theorem 4 about bit’s block flip to show lower bound.
Let \( a \) be any function that stand in some pair \((a, b)\) with \( INV(a) = (a_0, a_1, \ldots, a_r) \). Let \( DISP(INV(a)) := \max_{0 \leq i \leq r} (|i - r/2| : a_i > 0) \).

**Definition 9** We say that \( a \) is "bad" (and denote \( BAD(a) \)) if \( DISP(INV(a)) > 15\sqrt{r} \ln \frac{n}{r} \).

Informally, \( a \) is bad if there exists image such that after reduction most of its preimages are in domain of either \( a \) or \( b \).

It is easy to see that for any pair \((a, b)\) holds: \( a \) is bad if and only if \( b \) is bad.

This Lemma shows, that the difference between complementary and equivalent reductions is quite small:

**Lemma 10** The sum \( \sum_{(a, b), BAD(a)} p(a, b) \) is less than small constant if \( r \gg \ln \frac{n}{r} \).

**Proof.** Let us see only the case when \((a, b)\) occurs after complementary reduction. Let \( f : n \to N \) be the function before reduction. Let choose some fixed image \( j \) of function \( f \), thus \( j \in f([n]) \). We say that \( j \) is "bad" (denote by \( BAD(j) \)) if \( |a^{-1}(j)| - r/2 > 15\sqrt{r} \ln \frac{n}{r} \). Let \( p_j \) denote the probability that \( j \) is bad. It is easy to see that for all \( j \in f([n]) \) \( p_j \) is equal with some \( p \). It is easy to see that \( BAD(a) \Leftrightarrow BAD(j) \) for some \( j \in f([n]) \). Therefore probability for \( a \) to be bad is less than \( n/r \times p \) where \( n/r \) is the total count of images. Now it remains only to show that \( p \ll r/n \).

There are two cases how \( j \) can be bad, the first is that \( |a^{-1}(j)| \) is too big and the second is that \( |a^{-1}(j)| \) is too small. Obviously, that both of these cases holds with similar probability. Let’s count the probability that \( |a^{-1}(j)| \) is too big. Let enumerate all preimages of \( j \) as \( x_1, x_2, \ldots, x_r \). Let \( \chi' \) denote the random variable that is 1 if \( x_i \) become a member of domain of \( a \) and 0 elsewhere. Let \( \chi' := \chi'_1 + \chi'_2 + \ldots + \chi'_r \). Thus we reach out for \( Pr[\chi' > E(\chi') + 15\sqrt{r} \ln \frac{n}{r}] \), since \( E(\chi') = r/2 \).

Let \( \chi_i \) denote the random variable that is 1 with probability 1/2 and 0 with probability 1/2. Let \( \chi := \chi_1 + \ldots + \chi_r \). It is easy to see that for all \( s \geq r/2 = E(\chi') = E(\chi) \) holds \( Pr[\chi' > s] \leq Pr[\chi > s] \). Now we can apply Chernoff’s inequality

\[
Pr[\chi > (1 + \epsilon)E(\chi)] \leq e^{-\epsilon^2 E(\chi)/3}
\]

if \( 0 \leq \epsilon \leq 1 \). It is easy to see that \( E(\chi) = r/2 \). Let \( \epsilon := 30\sqrt{\ln \frac{n}{r}} \). It is easy to see that \( \epsilon E(\chi) = 15\sqrt{r} \ln \frac{n}{r} \). It remains to evaluate the probability \( e^{-\epsilon^2 E(\chi)/3} \ll r/n \) if \( r \gg \ln \frac{n}{r} \).

\[\square\]

6 Proof’s completing

In this section we will reduce the problem to distinguish between complementary and equivalent reduction (with distribution over input) to problem of the ordinary input.
**Definition 11** Let $n > 0$ and $r \geq 2$ be integers such that $n|2$ and $n|r$. Let $a : n/2 \to N$ and $b : n/2 \to N$ be functions given by an oracle, such that the pair $(a, b)$ can occur after complementary or equivalent reduction. $\text{INV}(a)$ is known and it is promised that $a$ is not bad. Let $\text{ComesFrom}$ problem denote the problem to decide whether the pair $(a, b)$ occurred after complementary or equivalent reduction.

### 6.1 Reduction

**Lemma 12** If there exists quantum algorithm $A$ that makes distinction between complementary and equivalent reduction with $T$ queries then there exists quantum algorithm $A'$ that solves $\text{ComesFrom}$ problem with $O(T)$ queries.

**Proof.** Firstly, we can ignore all pairs $(a, b)$ that have $\text{BAD}(a)$ because they appear with very small probability (Lemma 10). If we want to improve the probability we can just repeat $A$ several times.

Secondly, without loss of generality, we can assume that the accepting probability of $A$ depends only on $\text{INV}(a, b)$. If not, we can modify algorithm $A$, such that it choose random variables $\sigma_1 \in \text{SG}[n/2], \sigma_2 \in \text{SG}[n/2] \text{ and } \tau \in \text{SG}[N]$ at the beginning and further just deal with pair of functions $(\Gamma_{\sigma_1}^a(a), \Gamma_{\sigma_2}^b(b))$.

Thirdly, since $A$ makes distinction between complementary and equivalent reduction and for any pair of functions $(a, b)$ depends only on $\text{INV}(a, b)$, there exists some $I := \text{INV}(a)$ such that $A$ makes distinction between $(a, b_1)$ such that $	ext{INV}(b_1) = \text{INV}(a) = I$ and $(a, b_2)$ such that $\overline{\text{INV}}(b_2) = \text{INV}(a) = I$ for any function $b_1$ and $b_2$.

It follows, that for this particular $I$ we can solve $\text{ComesFrom}$ problem using algorithm $A$. 

\[\square\]

### 6.2 Lower bound for the $\text{ComesFrom}$ problem

**Lemma 13** Any quantum algorithm $A$ that solves the $\text{ComesFrom}$ problem makes $\Omega\left(\frac{r}{\sqrt{\log n}}\right)$ queries.

**Proof.** Let $I = (a_0, ..., a_r)$ denote the known $\text{INV}(a)$. We will use Theorem 12 to prove lower bound quite similarly to Ambainis’ proof about lower bound for counting. Let $X$ be the set of all $(a, b)$ such that $\text{INV}(b) = \text{INV}(a) = I$ and let $Y$ be the set of all $(a, b)$ such that $\overline{\text{INV}}(b) = \text{INV}(a) = I$.

Let $\Psi := \Psi(I) := \sum_{i : a_i > r/2} a_i - r/2$. Since $a$ is not bad, it implies that $\Psi = O(\sqrt{r \log n})$. $2\Psi$ is just the number of points that must be changed to to switch from $\text{INV}(b)$ to $\overline{\text{INV}}(b)$. Let $\Phi := \prod_{i : a_i < r/2} (2\Psi - (2\Psi))$.

Let $R$ be the set of all $((a, b_1), (a, b_2))$ such that $b_1$ differs from $b_2$ exactly in $2\Psi$ points and $\text{INV}(b_1) = \overline{\text{INV}}(b_2)$. Then, $m = m' = C^{2\Psi}_{n/4 + \Psi} \Phi$ and $l = l' = \frac{r}{\sqrt{\log n}}$. 

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Therefore,

\[
\frac{mn'}{ll'} = \left( \frac{C^{2\Psi - 1}_{n/4 + \Psi - 1} \Phi}{C^{2\Psi}_{n/4 + \Psi} \Phi} \right)^2 = \left( \frac{(n/4 + \Psi)!}{(2\Psi)!} \right)^2 = \left( \frac{n/4 + \Psi - 1}{2\Psi} \right)^2 = \left( \frac{n}{8\Psi} + \frac{1}{2} \right)^2 = \Omega \left( \left( \frac{n}{\Psi} \right)^2 \right) = \Omega \left( \frac{r}{\log n} \right)
\]

Now we can apply Theorem 4 and get that any quantum algorithm makes \( \Omega \left( \sqrt{\frac{r}{\log n}} \right) \) queries.

7 Conclusion

We showed a polynomial quantum query lower bound for the set equality problem. It was done by reduction. Arguments that allowed reduction was very specific to the set equality problem. It would be nice to find some more general approach to find quantum query lower bounds for this and other similar problems. Also, it would be fine to make smaller difference between quantum lower and upper bounds for the set equality problem.

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