SHAPE DERIVATIVE OF THE FIRST EIGENVALUE OF THE 1-LAPLACIAN

NICOLAS SAINTIER

Abstract. We compute the shape derivative of the functional $\Omega \to \lambda_{1,\Omega}$, where $\lambda_{1,\Omega}$ denotes the first eigenvalue of the 1-Laplacian on $\Omega$. As an application, we find that the ball is critical among the volume-preserving deformations.

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$. The 1-Laplacian on $\Omega$ is the formal operator

$$\Delta_1 u = -\text{div} \left( \frac{\nabla u}{|\nabla u|} \right)$$

we get by a formal derivation of $F(u) = \int_{\Omega} |\nabla u| \, dx$, or by letting $p \to 1$ in the definition of the $p$-Laplacian, $p > 1$. By analogy with the definition of the first eigenvalue of the $p$-Laplacian on $\Omega$, we define the first eigenvalue $\lambda_{1,\Omega}$ of the 1-Laplacian on $\Omega$ by the minimization problem

$$\lambda_{1,\Omega} = \inf \left\{ u \in \dot{H}^1_1(\Omega) \right\} \int_{\Omega} |\nabla u| \, dx ,$$

where $\dot{H}^1_1(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in the Sobolev space $H^1(\Omega)$ of functions in $L^1(\Omega)$ with one derivative in $L^1$.

The purpose of this paper is the study of the dependence of $\lambda_{1,\Omega}$ under regular perturbations by diffeomorphisms of $\Omega$, i.e., we want to compute the first variation, the so-called shape derivative, of the functional $\Omega \to \lambda_{1,\Omega}$. General results about the stability of $\lambda_{1,\Omega}$ under perturbations of $\Omega$ have been obtained in [17]. In particular the authors of [17] found the shape derivative of $\Omega \to \lambda_{1,\Omega}$ in the case of regular perturbations by diffeomorphisms close to homotheties. We want to extend this result to the case of a general perturbation by diffeomorphisms.

Let us recall some known facts about $\lambda_{1,\Omega}$ (see e.g. [17, 20]). A natural space to study $\lambda_{1,\Omega}$ is the space $BV(\Omega)$ of functions of bounded variations (see, for instance, [2, 11, 13, 26]). By standard properties of $BV(\Omega)$, we can also define $\lambda_{1,\Omega}$ by

$$\lambda_{1,\Omega} = \inf \left\{ u \in BV(\Omega) \right\} \int_{\Omega} |\nabla u| + \int_{\partial \Omega} |u| \, dH^{n-1} , \quad (1)$$

Date: June 6th, 2007.

Key words and phrases. 1-Laplacian, eigenvalue, shape derivative.

2000 Mathematics Subject Classification: 49Q10 (35P30, 49Q20).
where $|\nabla u|$ is the total variation of the measure $\nabla u$, and $H^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. We refer to [8] for a detailed proof of this assertion. Note here that if $u \in BV(\Omega)$ and $\overline{u}$ is the extension of $u$ by 0 in $\mathbb{R}^n \setminus \overline{\Omega}$, then $\overline{u} \in BV(\mathbb{R}^n)$ and
\[
\int_{\mathbb{R}^n} |\nabla \overline{u}| = \int_{\Omega} |\nabla u| + \int_{\partial \Omega} |u| dH^{n-1}.
\] (2)

By lower semicontinuity of the total variation and compactness of the embedding $BV(\Omega) \hookrightarrow L^1(\Omega)$, it easily follows from (2) that the infimum in (1) is attained by some nonnegative $u \in BV(\Omega)$. Then $u$ is a solution of the equation $\Delta u = \lambda_{1, \Omega}$ in the sense that there exists $\sigma \in L^\infty(\Omega, \mathbb{R}^n)$, $|\sigma|_\infty \leq 1$, such that
\[
\begin{cases}
-\text{div } \sigma = \lambda_{1, \Omega}, \\
\sigma \nabla u = |\nabla u| \text{ in } \Omega, \text{ and} \\
(\sigma \vec{n})u = -u \text{ on } \partial \Omega,
\end{cases}
\] (3)

where $\vec{n}$ is the unit outer normal to $\partial \Omega$, and $\sigma \nabla u$ is the distribution defined by integrating by parts $\int_{\Omega} (\sigma \nabla u) v \, dx$ when $v \in C^\infty_0(\Omega)$ and div $\sigma$ makes sense (see e.g. [8, 3]). We then say that $u$ is an eigenfunction for $\lambda_{1, \Omega}$.

We can also express $\lambda_{1, \Omega}$ in a more geometric way as an isoperimetric type problem. We recall that a set $C \subset \mathbb{R}^n$ is said of finite perimeter if its characteristic function $\chi_C$ belongs to $BV(\mathbb{R}^n)$. We then define the perimeter $|\partial C|$ of $C$ as $\int_{\mathbb{R}^n} |\nabla \chi_C|$. Using the coarea formula, we can rewrite (1) as
\[
\lambda_{1, \Omega} = \inf_{C \subset \Omega, \chi_C \in BV(\mathbb{R}^n)} \frac{|\partial C|}{|C|}.
\] (4)

We refer e.g. to [18] for a proof of this assertion. This infimum is attained by some set of finite perimeter, e.g. by a level-set of an extremal for (1), called an eigenset or also a Cheeger’s set. Note that minimizers for $\lambda_{1, \Omega}$ touch the boundary $\partial \Omega$ since, if not, we may blow it up by a factor larger than one, which would decrease $\lambda_{1, \Omega}$. Uniqueness and nonuniqueness results of eigensets are in [12, 25]. Concerning regularity, possible references are [1] [10] [15] [18] [25].

To study the variations of $\lambda_{1, \Omega}$ with respect to smooth variations of $\Omega$, we consider a smooth vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ and, for small $t \in \mathbb{R}$, the diffeomorphisms
\[
T_t(x) = x + tV(x),
\] (5)

and eventually the perturbed domains $\Omega_t = T_t(\Omega)$. We want to compute the derivative at $t = 0$ of the map $t \mapsto \lambda_{1, \Omega_t}$.

Shape analysis is the subject of an intense research activity. We refer for example to [10] for an introduction to this field. The shape derivative of the first eigenvalue $\lambda_{p, \Omega}$ of the $p$-Laplacian, $p > 1$, has been computed in [13, 21]:
\[
\frac{d}{dt} \lambda_{p, \Omega_t}_{|t=0} = -(p-1) \int_{\partial \Omega} |\frac{\partial u_p}{\partial \nu}|^p (V, \nu) \, dH^{n-1},
\]

where $u_p$ is the unique positive normalized eigenfunctions for $\lambda_{p, \Omega}$ and $\nu$ is the unit normal vector to $\partial \Omega$. What could be the shape derivative of the first eigenvalue of the 1-Laplacian is thus not obvious from this formula.

Our result is the following:
Theorem. Let $\Omega$ be a smooth bounded open subset of $\mathbb{R}^n$, $V : \mathbb{R}^n \to \mathbb{R}^n$ a smooth vector-field and $\Omega_t = T_t(\Omega)$, where the $T_t$’s are the diffeomorphisms defined by (3). Then

$$\lambda_{1,\Omega_t} \to \lambda_{1,\Omega}$$

as $t \to 0$. Moreover, if we assume that there exists a unique nonnegative eigenfunction $u \in BV(\Omega)$ for $\lambda_{1,\Omega}$ such that $\int_{\Omega} |u| \, dx = 1$, then $u = |A|^{-1} \chi_A$ for some eigenset $A \subset \overline{\Omega}$, and the map $t \to \lambda_{1,\Omega_t}$ is differentiable at $t = 0$ with

$$\frac{d}{dt} \lambda_{1,\Omega_t}|_{t=0} = \int_{\partial A} (\text{div} V - (\nu, DV \nu) + \lambda_{1,\Omega}(V, \nu)) \frac{dH^{n-1}}{|A|},$$

where $\nu$ is the Radon-Nykodym derivative of $\nabla \chi_A$ with respect to $|\nabla \chi_A|$ (i.e. $\nabla \chi_A = \nu |\nabla \chi_A|$ as measures), and $\partial^* A$ denotes the reduced boundary of $A$, i.e. the part of the boundary of $A$ at which can be defined a notion of unit normal vector in a measure theoretic sense (see e.g. [2, 11, 14, 26]).

Let us assume that $\Omega \subset \mathbb{R}^n$ is strictly convex. We then know from [4, 25] that there exists a unique Cheeger set $A \subset \overline{\Omega}$. Since $A$ has minimum perimeter among all the subsets of $\Omega$ of finite perimeter and of volume $|A|$, it follows from [25] that $\partial A$ is $C^{1,1}$. Hence the unit exterior normal vector to $\partial A$ is defined by (5) almost every point of $\partial A$. Note that this vector coincides with $-\nu H^{n-1}$-a.e. The mean curvature $H_{\partial A}$ of $\partial A$ is defined by $H_{\partial A} = -\text{div}_{\partial A} \nu$, where $\text{div}_{\partial A}$ denotes the tangential derivative on $\partial A$. We can now write that

$$\text{div} V - (\nu, DV \nu) = \text{div}_{\partial A} V = \text{div}_g V_{\partial A} - H_{\partial A}(V, \nu),$$

where $V_{\partial A}$ denotes the tangential part of $V$, and $\text{div}_g$ the divergence operator of the manifold $(\partial A, g)$, $g$ being the metric on $\partial A$ induced by the Euclidean metric (see e.g. [16]). We can thus rewrite (6) as

$$\frac{d}{dt} \lambda_{1,\Omega_t}|_{t=0} = \int_{\partial A} (\text{div}_g V_{\partial A} - H_{\partial A}(V, \nu) + \lambda_{1,\Omega}(V, \nu)) \frac{dH^{n-1}}{|A|}$$

and thus

$$\frac{d}{dt} \lambda_{1,\Omega_t}|_{t=0} = \int_{\partial A} (\lambda_{1,\Omega} - H_{\partial A})(V, \nu) \frac{dH^{n-1}}{|A|}. \tag{7}$$

When $A = \overline{\Omega}$, a situation that happens when $\Omega \subset \mathbb{R}^n$ is smooth convex and its curvature is less that $|\partial \Omega|/(n(n-1)|\Omega|)$ (see [19] when $n = 2$, and [3] for an arbitrary $n$), formula (7) writes as

$$\frac{d}{dt} \lambda_{1,\Omega_t}|_{t=0} = \int_{\partial \Omega} (\lambda_{1,\Omega} - H_{\partial \Omega})(V, \nu) \frac{dH^{n-1}}{|\Omega|},$$

where $\nu$ is the inner unit normal to $\partial \Omega$. In the particular case when $\Omega$ is a ball, $A = \overline{\Omega}$ and $H_{\partial \Omega}$ is constant, so that if we consider a deformation that preserves the volume, i.e. a vector-field $V$ such that $\text{div} V = 0$, we get

$$\frac{d}{dt} \lambda_{1,\Omega_t}|_{t=0} = 0.$$

Hence a ball is critical for such deformations.

The following section is devoted to the proof of the theorem.
Proof of the theorem

To simplify the notations, we let $\lambda = \lambda_{1, \Omega}$ and $\lambda_t = \lambda_{1, \Omega_t}$.

Using the change of variable formula for functions of bounded variations [14], we can rewrite $\lambda_t$ as

$$\lambda_t = \inf_{v \in BV(\Omega)} \frac{\int_{\bar{\Omega}} |D(x, t) \cdot \nu_v| C(x, t) \nabla \bar{v}|\nabla \bar{u}|}{\int_{\bar{\Omega}} C(x, t)|v| \, dx},$$

(8)

where $D(x, t) = (DT_t(x))^{-1}$, $C(x, t) = |\det (DT_t(x))|$, and $\nu_v$ is the Radon-Nikodym derivative of $\nabla v$ with respect to $|\nabla v|$. Recall that $\bar{v}$ denotes the extension of $v$ to $\mathbb{R}^n$ by 0 - see (2). As $|\nu_v| = 1$ $|\nabla v|$ - a.e,

$$\lambda_t \leq \inf_{v \in BV(\Omega)} \frac{\int_{\bar{\Omega}} |D(x, t) \cdot C(x, t) \nabla \bar{v}|}{\int_{\bar{\Omega}} C(x, t)|v| \, dx}. $$

(9)

Since $|D(x, t)|, C(x, t) \to 1$ as $t \to 0$ uniformly in $x \in \bar{\Omega}$, we deduce from (9) that

$$\limsup_{t \to 0} \lambda_t \leq \lambda.$$  

(10)

We let $u_t \in BV(\Omega_t)$ be a nonnegative eigenfunction for $\lambda_t$ normalized by $\int_{\Omega_t} u_t \, dx = 1$, and $v_t = u_t \circ T_t \in BV(\Omega)$. Then $(\bar{v}_t)$ is bounded in $BV(\mathbb{R}^n)$. Indeed if we denote by $\tilde{u}_t$ the Radon-Nikodym derivative of $\nabla u_t$ with respect to $|\nabla u_t|$, we have

$$\int_{\mathbb{R}^n} \nabla \bar{v}_t = \int_{\Omega_t} |(DT_t^{-1})^{-1} \tilde{u}_t| |\det DT_t^{-1}| \nabla \bar{u}_t| \leq (1 + o(1)) \int_{\Omega_t} \nabla \bar{u}_t|$$

(11)

and

$$\int_{\Omega} v_t \, dx = \int_{\Omega_t} u_t |\det DT_t^{-1}| \, dx = 1 + o(1).$$

(12)

We can thus assume that the $v_t$'s converge to some nonnegative $v \in L^1(D)$ in $L^1(D)$ and a.e. in $D$, where $D$ is a smooth bounded open subset of $\mathbb{R}^n$ containing both $\Omega$ and the $\Omega_t$'s. We denote by $u$ the restriction of $v$ to $\Omega$. Then $u \in BV(\Omega), u \geq 0$, and in view of (12),

$$\int_{\Omega} u \, dx = 1.$$

The lower semi-continuity of the total variation and (11) then give

$$\lambda \leq \int_{\Omega} |\nabla \bar{u}| \leq \int_{\mathbb{R}^n} |\nabla v| \leq \liminf_{t \to 0} \int_{\mathbb{R}^n} |\nabla \bar{v}_t| \leq \liminf_{t \to 0} \lambda_t.$$

We then deduce with (10) that

$$\lambda_t \to \lambda = \int_{\Omega} |\nabla \bar{u}|,$$

as $t \to 0$, and then that

$$\int_{\Omega} |\nabla \bar{v}_t| \to \lambda = \int_{\Omega} |\nabla \bar{u}|,$$

(13)

as $t \to 0$. This proves the first part of the theorem.
Let us note for a future use that, as in [17], it follow from (13) that \( \| \nabla \bar{v}_t \| \to \| \nabla \bar{u} \| \)
weakly in the sense that
\[
\int_{\mathbb{R}^n} \phi |\nabla \bar{v}_t| \to \int_{\mathbb{R}^n} \phi |\nabla \bar{u}| \tag{14}
\]
for any \( \phi \in C(\mathbb{R}^n) \) with compact support.

We now prove the differentiability of the map \( t \to \lambda_t \) and the formula (6). We use \( u \) as a test-function in (8) to estimate \( \lambda_t \), so that
\[
\lambda_t - \lambda \leq \frac{\int_{\Omega} |D(x, t)\nu| C(x, t)|\nabla \bar{u}|}{\int_{\Omega} C(x, t)u \, dx} - \lambda, \tag{15}
\]
where \( \nu \) is the Radon-Nikodym derivative of \( \nabla u \) with respect to \( |\nabla u| \). Since \( |\nu| = 1 \) \( |\nabla u| \)-a.e., we can assume that \( |\nu| = 1 \) everywhere. Direct computations give that
\[
C(x, t) = \det(DT_t(x)) = 1 + t \text{div} V(x) + o(t), \tag{16}
\]
and
\[
|D(x, t)\nu| = 1 - (\nu; DV(x)\nu) + o(t), \tag{17}
\]
where the \( o(t) \) is uniform in \( x \). Thus (15) becomes
\[
\lambda_t - \lambda \leq \frac{\lambda + t \int_{\Omega} \text{div} V (\nu - (\nu; DV\nu)) |\nabla \bar{u}| + o(t)}{1 + t \int_{\Omega} \text{div} V u \, dx + o(t)} - \lambda
= t \left( \int_{\Omega} \text{div} V (|\nabla \bar{u}| - \lambda u) \, dx - \int_{\Omega} (\nu, DV\nu)|\nabla \bar{u}| \right) + o(t).
\]
Hence
\[
\limsup_{t \to 0^+} \frac{\lambda_t - \lambda}{t} \leq \int_{\Omega} \{(\text{div} V - (\nu, DV\nu))|\nabla \bar{u}| - \lambda u \text{div} V \, dx \}, \tag{18}
\]
and
\[
\liminf_{t \to 0^-} \frac{\lambda_t - \lambda}{t} \geq \int_{\Omega} \{(\text{div} V - (\nu, DV\nu))|\nabla \bar{u}| - \lambda u \text{div} V \, dx \}. \tag{19}
\]

It remains to prove the opposite inequalities. We use \( \bar{v}_t \) as a test-function to estimate \( \lambda \), so that
\[
\lambda_t - \lambda = \int_{\Omega_t} |\nabla \bar{v}_t| - \lambda \geq \int_{\Omega} |D(x, t)\nu_t| C(x, t) |\nabla \bar{u}_t| - \int_{\Omega} \frac{|\nabla \bar{v}_t|}{\nu_t \, dx},
\]
where \( \nu_t \) denotes the Radon-Nykodym derivative of \( \nabla \bar{v}_t \) with respect to \( |\nabla \bar{v}_t| \). As previously, we can assume that \( |\nu_t| = 1 \) everywhere. In view of (16) and (17), we obtain
\[
\lambda_t - \lambda \geq \int_{\Omega} |\nabla \bar{v}_t| + t \int_{\Omega} \text{div} V (\nu_t, DV\nu_t) |\nabla \bar{v}_t| - \int_{\Omega} \frac{|\nabla \bar{v}_t|}{\nu_t \, dx} + o(t). \tag{20}
\]
Since $\text{div} \, V \in C(\Omega)$, we get thanks to (14) that
\[
\int_{\Omega} \text{div} \, V |\nabla \bar{v}_t| \rightarrow \int_{\Omega} \text{div} \, V |\nabla \bar{u}|.
\]
Independently,
\[
\int_{\Omega} v_t \, dx = \int_{\Omega} u_t |\det D T^{-1}_t| \, dx
\]
with
\[
\det D T^{-1}_t = 1 - t \text{div} \, V + o(t),
\]
so that
\[
\int_{\Omega} v_t \, dx = 1 - t \int_{\Omega} u_t \text{div} \, V \, dx + o(t) = 1 - t \int_{\Omega} u \text{div} \, V \, dx + o(t).
\]
Thus
\[
\frac{\int_{\Omega} |\nabla \bar{v}_t|}{\int_{\Omega} v_t \, dx} = \frac{\int_{\Omega} |\nabla \bar{v}_t| + t \int_{\Omega} |\nabla \bar{v}_t| \int_{\Omega} u \text{div} \, V \, dx + o(t)}{\int_{\Omega} |\nabla \bar{v}_t| + t \lambda \int_{\Omega} u \text{div} \, V \, dx + o(t)},
\]
where the last equality follows from (13). Hence (20) becomes
\[
\lambda_t - \lambda \geq t \int_{\Omega} \{(\text{div} \, V - (\nu, D V \nu)) |\nabla \bar{u}| - \lambda u \text{div} \, V \, dx\} + o(t).
\]
Eventually, in view of (13) and the weak convergence of $\nabla \bar{v}_t$ to $\nabla \bar{u}$, which follows from the $L^1$ convergence of the $\bar{v}_t$ to $\bar{u}$, we can apply Reshetnyak theorem [2, 23, 22] to get that
\[
\int_{\Omega} g(x, \nu_t(x)) |\nabla \bar{v}_t| \rightarrow \int_{\Omega} g(x, \nu(x)) |\nabla \bar{u}|
\]
for any continuous function $g : \bar{\Omega} \times S \rightarrow \mathbb{R}$, where $S$ denotes the unit sphere of $\mathbb{R}^n$. In particular,
\[
\int_{\Omega} (\nu_t, D V \nu_t) |\nabla \bar{v}_t| = \int_{\Omega} (\nu, D V \nu) |\nabla \bar{u}| + o(1).
\]
Hence
\[
\lambda_t - \lambda \geq t \int_{\Omega} \{(\text{div} \, V - (\nu, D V \nu)) |\nabla \bar{u}| - \lambda u \text{div} \, V \, dx\} + o(t),
\]
and thus
\[
\limsup_{t \to 0^+} \frac{\lambda_t - \lambda}{t} \geq \int_{\Omega} \{(\text{div} \, V - (\nu, D V \nu)) |\nabla \bar{u}| - \lambda u \text{div} \, V \, dx\},
\]
and
\[
\liminf_{t \to 0^-} \frac{\lambda_t - \lambda}{t} \leq \int_{\Omega} \{(\text{div} \, V - (\nu, D V \nu)) |\nabla \bar{u}| - \lambda u \text{div} \, V \, dx\}.
\]
Since by assumption $u$ is the unique normalized eigenfunctions for $\lambda$, we deduce from these two inequalities and (18), (19) that the map $t \to \lambda_t$ is differentiable at $t = 0$ with
\[
\frac{d}{dt} \lambda_t|_{t=0} = \int_{\Omega} \{(\text{div} \, V - (\nu, D V \nu)) |\nabla \bar{u}| - \lambda u \text{div} \, V \, dx\}.
Eventually, as there always exists an extremal for the problem \( \{ \text{1} \} \), there exists a set of finite perimeter \( A \subset \bar{\Omega} \) such that \( u = |A|^{-1} \chi_A \). It then follows from geometric measure theory that \( |\nabla \bar{u}| = |A|^{-1} H^{n-1}_{\partial^* A} \) (see e.g. \( \{ 2 \} \)). The previous formula becomes

\[
\frac{d}{dt} \lambda_{t=0} = \int_{\partial^* A} (\text{div} V - (\nu, D V \nu)) \frac{dH^{n-1}}{|A|} - \lambda \int_A \text{div} V \frac{dx}{|A|},
\]

from which we deduce \( \{ 6 \} \) using Green’s formula for sets of finite perimeter. This ends the proof of the theorem.

**Acknowledgments.** The author acknowledges the support of the grants FONCYT PICT 03-13719 (Argentina).

**References**

[1] F.J. Almgren, Existence and regularity almost everywhere of solutions to elliptic variational problems, *Mem. Am. Math. Soc.*, 165, vol.4, 1976.

[2] L. Ambrosio, N. Fusco, D. Pallara, Functions of bounded variations and free discontinuity problems, *Oxford Mathematical Monographs*, The Clarendon Press, Oxford University Press, New-York, 2000.

[3] F. Alter, V. Caselles, A. Chambolle, A characterization of convex calibrable sets in \( \mathbb{R}^n \), *Math. Ann.*, 332 (2), 2005, 329-366.

[4] V. Caselles, A. Chambolle, M. Novaga, Uniqueness of the Cheeger set of a convex body, *preprint*.

[5] I. Chavel, *Riemannian geometry – a modern introduction*, Cambridge Tracts in Mathematics, 108, Cambridge University Press, Cambridge, 1993.

[6] J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian, in *Problems in Analysis, A Symposium in honor of S. Bochner*, Princeton Univ. Press, 1970, 195-199.

[7] F. Demengel, On some nonlinear equation involving the 1-Laplacian and trace map inequalities, *Nonlinear Analysis*, 48, 2002, 1151-1163.

[8] F. Demengel, On some nonlinear partial differential equations involving the 1-Laplacian and critical Sobolev exponent, *ESAIM*, 4, 1999, 667-686.

[9] F. Demengel, F. De Vyust, M. Motron, A numerical approach of the first eigenvalue for the 1-Laplacian on the square and other particular sets, *Preprint*, 2002.

[10] E. De Giorgi, Frontiere orientate di misura minima, Seminario di Matematica della Scuola Normale Superiore di Pisa, Editrice Tecnico Scientifica, Pisa, 1961.

[11] L.C. Evans, R.F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Math., CRC Press, Ann Harbor, 1992.

[12] V. Fridman, B. Kawohl, Isoperimetric estimates for the first eigenvalue of the \( p \)-Laplace operator and the Cheeger constant, *Comment. Math. Univ. Carolinae*, 44, 2003, 659-667.

[13] J. Garcia Melian, J. Sabina De Lis, On the perturbation of eigenvalues for the \( p \)-Laplacian, *C.R. Acad. Sci. Paris*, Série 1, 332, 2001, 893-898.

[14] Giusti, E., *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, Birkhäuser, 1984.

[15] E. Gonzalez, U. Massari, I. Tamanini, On the regularity of boundaries of sets minimizing perimeter with a volume constraint, *Indiana Univ. Math. J.*, 32, 1983, 25-37.

[16] A. Henrot, M. Pierre, *Variation et optimisation de formes - une analyse geométrique*, *Mathmatiques et applications* 48, Berlin, New York, Springer, 2005.

[17] E. Hebey, N. Saintier, Stability and perturbations of the domain for the first eigenvalue of the 1-Laplacian, *Archiv der Mathematik*, 86, (4), 2006, 340-351.

[18] I. R. Ionescu, T. Lachand-Robert, Generalized Cheeger sets related to landslides, *Calc. Var. and PDE’s*, 23 (2005), 227-249.

[19] B. Kawohl, T. Lachand-Robert, Characterization of Cheeger sets for convex subsets of the plane, *Pacific J. Math.*, 225 (1), 2006, 163-118.
8 NICOLAS SAINTIER

[20] B. Kawohl, F. Schuricht, Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem, Comm. Contemp. Math., to appear.

[21] P.D. Lamberti, A differentiability result for the first eigenvalue of the $p$-Laplacian upon domain perturbation, Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday, vol. 1, 2, Kluwer Acad. Publ., Dordrecht, 2003, 741-754.

[22] S. Luckhaus, L. Modica, The Gibbs-Thompson relation within the gradient theory of phase transitions, ARMA, 107 (1), 71-83.

[23] Yu. G. Reshetnyak, Weak convergence of completely additive vector functions on a set, Siberian Math. J., 9, 1968, 1039-1045; translated from Sibirskii Matematicheskii Zhurnal, 9, 1968, 1386-1394.

[24] N. Saintier, Estimates of the best Sobolev constant of the embedding of $BV(\Omega)$ into $L^1(\partial\Omega)$ and related shape optimization problems, submitted.

[25] E. Stredulinsky, W.P. Ziemer, Area minimizing sets subject to a volume constraint in a convex set, J. Geom. Anal., 7, 1997, 653-677.

[26] W.P. Ziemer, Weakly differentiable functions. Sobolev spaces and functions of bounded variations, Graduate Texts in Mathematics 120, Springer-Verlag, 1989.

DEPARTAMENTO DE MATEMÁTICA, FCEyN UBA (1428), BUENOS AIRES, ARGENTINA

E-mail address: nsaintie@dm.uba.ar