URYSOHN LEMMAS IN TOPOLOGICAL VECTOR SPACES

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Abstract
Two variations of classical Urysohn lemma for subsets of topological vector spaces are obtained in this article. The continuous functions constructed in these lemmas are of quasi-convex type.

Introduction
The classical technique for the proof of Urysohn’s lemma (See: [1]) is applied to derive metrization theorem (See: [2]) for topological vector spaces. It is possible to find the continuous functions of quasi-convex type, when the same technique is applied on certain subsets of a topological vector spaces. The Urysohn lemma is derived on normal spaces or on locally compact spaces. So, two concepts of convex normal and convex regular subsets of a topological vector spaces are introduced; their properties are studied; and two Urysohn lemmas are derived in this article.

Main Results

Definition 1 A subset $A$ of a topological vector space is said to be locally convex if every point has a local base in $A$ consisting of convex subsets of $A$.

Definition 2 Let $A$ be a locally convex subset of a topological vector space $X$. A set $A$ is said to be convex regular, if for a given point $x$ and a given open
convex set $B (\text{open in } A)$ such that $x \in B$, there is an open convex subset $C$ of $A$ such that $x \in C \subseteq \text{cl } C \subseteq B$.

**Definition 3** Let $A$ be a locally convex subset of a topological vector space $X$. A set $A$ is said to be convex normal, if for a given non-empty closed set $A_0$ and a given open convex set $B$ such that $A_0 \subseteq B$, there is an open convex subset $C$ of $A$ such that $A_0 \subseteq C \subseteq \text{cl } C \subseteq B$.

**Proposition 4** Every compact convex set in a locally convex topological vector space is convex normal.

Proof: Let $A_0$ be a closed convex subset of a compact convex set $A$ in a locally convex topological vector space $X$. Let $B$ be an open convex set in $A$ containing $A_0$. To each $x \in A_0$, find an open convex neighbourhood $U_x$ of 0 in $X$ such that $(x + U_x) \cap A \subseteq (x + \text{cl } (U_x)) \cap A \subseteq B$. Since $A_0$ is compact, find a finite subset $\{x_1, x_2, \ldots, x_n\}$ of $A_0$ such that $A_0 \subseteq \bigcup_{i=1}^{n} (x_i + U_{x_i}) \cap A$.

The convex hull $C$ of right hand side of (1) is open in $A$ and it is contained in $B$. Note that, any element in $C$ is of the form $\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n$ with $y_i \in x_i + U_{x_i}, 0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^{n} \lambda_i = 1$. For each $y \in \text{cl } C$, there is net $(\lambda_1^{\delta} y_1^{\delta} + \lambda_2^{\delta} y_2^{\delta} + \cdots + \lambda_n^{\delta} y_n^{\delta})_{\delta \in D}$ in $C$ which converges to $y$. We can find a sub net $(\lambda_1^{\alpha} y_1^{\alpha} + \lambda_2^{\alpha} y_2^{\alpha} + \cdots + \lambda_n^{\alpha} y_n^{\alpha})_{\alpha \in G}$ of this net such that $\lambda_1^{\alpha} \rightarrow \lambda_1, \ y_1^{\alpha} \rightarrow y_1, \ldots, \lambda_n^{\alpha} \rightarrow \lambda_n, y_n^{\alpha} \rightarrow y_n$(say). Since $0 \leq \lambda \leq 1$, $\sum_{i=1}^{n} \lambda_i = 1$, $y_i \in x_i + \text{cl } (U_{x_i})$ and the convex hull of $\bigcup_{i=1}^{n} (x_i + \text{cl } (U_{x_i})) \cap A$ is contained in $B$, we conclude that $y = \sum_{i=1}^{n} \lambda_i y_i \in B$. Thus $\text{cl } C \subseteq B$. This proves the result.

**Definition 5** A function $f$ from a convex subset $A$ of real vector space into the real line is said to be

(i) **convex** if $f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$, for every $x, y \in A$, for every $\lambda \in [0, 1]$.

(ii) **quasi convex** if $\{x \in A : f(x) < r\}$ (or equivalently $\{x \in A : f(x) \leq r\}$) is a convex set, for every real number $r$. 

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Definition 6 Let $A$ be a convex subset of a topological vector space $X$. The set $A$ is said to be ‘(quasi) convex completely regular’ if for a given point $x \in A$ and a given open convex set $B \subseteq A$ such that $x \in B$, there is continuous (quasi)convex function $f : A \to [0, 1]$ by $f(x) = 0$ and $f(A \setminus B) = \{1\} = f(B^c)$.

Proposition 7 Every convex subset of a locally convex space is convex completely regular, when it contains more than one point.

Proof: Let $A$ be a locally convex subset of a locally convex space. Let $x \in A$ and $B$ be an open convex subset of $A$ such that $x \in B$. Without loss of generality, let us assume that there is a continuous semi norm $p$ on $X$ such that $\inf \{p(x - y) : y \in A \setminus B\} \geq 1$. Define a map $f : A \to [0, 1]$ by $f(y) = \min\{p(x - y), 1\}$. Then $f$ is continuous convex mapping such that $f(x) = 0$ and $f(y) = 1$, for every $y \in A \setminus B$.

First Urysohn lemma 8 Let $A$ be a convex normal subset of a topological vector space $X$. Let $A_0$ be a non-empty closed convex subset of $A$ and $B$ be an open convex subset of $A$ containing $A_0$. Then there is a continuous quasi convex function $f : A \to [0, 1]$ such that $f(A_0) = \{0\}$ and $f(A \setminus B) = \{1\}$.

Proof: Let $P$ be the set of all rational numbers in the interval $[0, 1]$. Define $U_1 = B$. Find an open convex subset $U_0$ in $A$ such that $A \subseteq U_0 \subseteq \text{cl}(U_0) \subseteq U_1 \subseteq B$. Find $(U_r)_{r \in P}$ of open convex sets such that ‘$p < q \Rightarrow \text{cl}(U_p) \subseteq U_q$’.

Define $f : A \to [0, 1]$ by $f(x) = \begin{cases} \inf \{p \in P : x \in U_p\}, & \text{if } x \in U_1 \\ 1, & \text{if } x \notin U_1 \end{cases}$

By the proof of the classical Urysohn lemma (See: [1, Theorem 33.1]), $f$ is a continuous function. Since $\{x \in A : f(x) \leq \epsilon\}$ is convex, for every $\epsilon \geq 0$, $f$ is a quasi convex function.

Lemma 9 Let $A$ be a locally convex locally compact subset subset of a locally convex space $(X, (p_i)_{i \in I})$. Suppose further that $A$ is complete as a uniform space with the uniformity induced by $(p_i)_{i \in I}$. Let $A_0$ be a nonempty compact convex subset of $A$ and $B$ be an open (in $A$) convex subset of $A$ such that $A_0 \subseteq B$. Then there is an open convex subset $C$ of $A$ such that $A_0 \subseteq C \subseteq \text{cl}C \subseteq B$ and $\text{cl}C$ is compact.

Proof: To each $x \in A_0$, find an open convex neighbourhood $U_x$ of $0$ in $X$ such that $(x + U_x) \cap A \subseteq (x + \text{cl}(U_x)) \cap A \subseteq B$, and $(x + \text{cl}(U_x)) \cap A$ is compact.
Find a finite subset $\{x_1, x_2 \cdots x_n\}$ of $A_0$ such that

$$A_0 \subseteq \bigcup_{i=1}^{n} (x_i + U_{x_i}) \cap A \quad (2)$$

The convex hull $C$ of right hand side of (2) is open in $A$ and it is contained in $B$. By theorem in [2] and by completeness of $A$, the convex hull of $\bigcup_{i=1}^{n} (x_i + cl(U_{x_i})) \cap A$ has a compact closure. Thus $clC$ is contained in a compact subset of $A$. So, as in the proof of proposition 4, we see that $clC \subseteq B$. This completes the proof.

**Second Urysohn lemma 10** Let $A$ be a locally convex locally compact subset of a locally convex space. Suppose further that $A$ is complete. Let $A_0$ be a compact convex subset of $A$, and $B$ be an open convex subset of $A$ such that $A_0 \subseteq B$. Then, there is a continuous function $f : A \rightarrow [0, 1]$ with compact support such that $f(A_0) = \{1\}$, $f(A \setminus B) = \{0\}$, and $1 - f$ is a quasi convex function.

Proof: Find convex open sets $U_0$ and $U_1$ such that $cl(U_0)$ and $cl(U_1)$ are compact and $A \subseteq U_0 \subseteq cl(U_0) \subseteq U_1 \subseteq cl(U_1) \subseteq B$. As in the proof of first Urysohn lemma, we can find a function $g : A \rightarrow [0, 1]$ such that $g(A_0) = \{0\}, \quad g(A \setminus U_1) = \{1\}, \quad g$ is a quasi convex and $g$ is continuous. The required function $f$ is $1 - g$.

**References**

[1] James R. Munkres, Topology, 2nd edition, Prentice Hall of India, New Delhi, 2000

[2] W. Rudin, *Functional Analysis*, second ed., International series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991.