Convergence of a mixed finite element–finite volume scheme for the isentropic Navier-Stokes system via dissipative measure-valued solutions

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July 31, 2018

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Abstract

We study convergence of a mixed finite element–finite volume numerical scheme for the isentropic Navier-Stokes system under the full range of the adiabatic exponent. We establish suitable stability and consistency estimates and show that the Young measure generated by numerical solutions represents a dissipative measure-valued solutions of the limit system. In particular, using the recently established weak–strong uniqueness principle in the class of dissipative measure-valued solutions we show that the numerical solutions converge strongly to a strong solutions of the limit system as long as the latter exists.

Keywords: Compressible Navier–Stokes system, finite volume scheme, finite element scheme, stability, convergence, measure-valued solution

*The research of E.F. leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

†The research of M.L.-M. has been supported by the German Science Foundation under the grants LU 1470/2-3 and the Collaborative Research Centers TRR 146 and TRR 165.
1 Introduction

Time evolution of the density $\rho = \rho(t,x)$ and the velocity $\mathbf{u} = \mathbf{u}(t,x)$ of a compressible barotropic viscous fluid can be described by the Navier–Stokes system

\begin{align}
\partial_t \rho + \text{div}_x (\rho \mathbf{u}) &= 0, \\
\partial_t (\rho \mathbf{u}) + \text{div}_x (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p &= \text{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \\
\mathbb{S}(\nabla_x \mathbf{u}) &= \mu \left( \nabla_x \mathbf{u} + \nabla^T_x \mathbf{u} - \frac{2}{3} \text{div}_x \mathbf{u} \mathbb{I} \right) + \eta \text{div}_x \mathbf{u} \mathbb{I}.
\end{align}
We assume the fluid is confined to a bounded physical domain \( \Omega \subset \mathbb{R}^3 \), where the velocity satisfies the no-slip boundary conditions

\[ u|_{\partial \Omega} = 0. \quad (1.4) \]

For the sake of simplicity, we ignore the effect of external forces in the momentum equation \((1.2)\).

In the literature there is a large variety of efficient numerical methods developed for the compressible Euler and Navier-Stokes equations. The most classical of them are the finite volume methods, see, e.g., \cite{8, 25, 29}, the methods based on a suitable combination of the finite volume and finite element methods \cite{2, 9, 10, 16, 17}, or the discontinuous Galerkin schemes, e.g. \cite{11, 12} and the references therein. Although these methods are frequently used for many physical or engineering applications, there are only partial theoretical results available concerning their analysis for the compressible Euler or Navier-Stokes systems. We refer to the works of Tadmor et al. \cite{13, 28, 30} for entropy stability in the context of hyperbolic balance laws and to the works of Gallouët et al. \cite{16, 17} for the stability analysis of the mixed finite volume–finite element methods based on the Crouzeix–Raviart elements for compressible viscous flows. In \cite{20} Jovanović and Rohde obtained the error estimate for entropy dissipative finite volume methods applied to nonlinear hyperbolic balance laws under (a rather restrictive) assumption of the global existence of a bounded, smooth exact solution.

Our goal in this paper is to study convergence of solutions to the numerical scheme proposed originally by Karlsen and Karper \cite{21, 22, 23, 24} to solve problem \((1.1–1.4)\) in polygonal (numerical) domains, and later modified in \cite{4} to accommodate approximations of smooth physical domains. The scheme is implicit and of mixed type, where the convective terms are approximated via upwind operators, while the viscous stress is handled by means of the Crouzeix–Raviart finite element method. As shown by Karper \cite{24} and in \cite{4}, the scheme provides a family of numerical solutions containing a sequence that converges to a weak solution of the Navier-Stokes system as the discretization parameters tend to zero. Recently, Gallouët et al. \cite{18} established rigorous error estimates on condition that the limit problem admits a smooth solution. Numerical experiments illustrating theoretical predictions have been performed in \cite{6}.

We consider the problem under physically realistic assumptions, where theoretical results are still in short supply. In particular, our results cover completely the isentropic pressure–density state equation

\[ p(\rho) = a\rho^\gamma, \quad 1 < \gamma < 2. \quad (1.5) \]

Note that the assumption \( \gamma < 2 \) is not restrictive in this context as the largest physically relevant exponent is \( \gamma = \frac{5}{3} \). Let us remark that the available theoretical results concerning global-in-time existence of weak solutions cover only the case \( \gamma > \frac{5}{2} \), \cite{7}, see also the recent result by Plotnikov and Weigant \cite{27} for the borderline case in the 2D setting. Similarly, the error estimates obtained by Gallouët et al. \cite{18} provide convergence under the same conditions yielding explicit convergence rates for \( \gamma > \frac{3}{2} \) and mere boundedness of the numerical solutions in the limit case \( \gamma = \frac{5}{2} \).

Our goal is to establish convergence of the numerical solutions in the full range of the adiabatic exponent \( \gamma \) specified in \((1.3)\). The main idea is to use the concept of dissipative measure-valued solution to problem \((1.1–1.4)\) introduced recently in \cite{3, 19}. These are, roughly speaking, measure-
valued solutions satisfying, in addition, an energy inequality in which the dissipation defect measure dominates the concentration remainder in the equations. Although very general, a dissipative measure-valued solution coincides with the strong solution of the same initial-value problem as long as the latter exists, see [3]. Our approach is based on the following steps:

- We recall the numerical energy balance identified in Karper’s original paper.
- We use the energy estimates to show stability of the numerical method.
- A consistency formulation of the problem is derived involving numerical solutions and error terms vanishing with the time step \( \Delta t \) and the spatial discretization parameter \( h \) approaching zero.
- We show that the family of numerical solutions generates a dissipative measure-valued solution of the problem. Such a result is, of course, of independent interest. As claimed recently by Fjordholm et al. [14], [15] the dissipative measure-valued solutions yield, at least in the context of hyperbolic conservation laws, a more appropriate solution concept than the weak entropy solutions.
- Finally, using the weak–strong uniqueness principle established in [3], we infer that the numerical solutions converge (a.a.) pointwise to the smooth solution of the limit problem as long as the latter exists.

The paper is organized as follows. The numerical scheme is introduced in Section 2. In Section 3, we recall the numerical counterpart of the energy balance and derive stability estimates. In Section 4, we introduce a consistency formulation of the problem and estimate the numerical errors. Finally, we show that the numerical scheme generates a dissipative measure-valued solution to the compressible Navier–Stokes system and state our main convergence results in Section 5.

## 2 Numerical scheme

To begin, we introduce the notation necessary to formulate our numerical method.

### 2.1 Spatial domain, mesh

We suppose that \( \Omega \subset \mathbb{R}^3 \) is a bounded domain. We consider a polyhedral approximation \( \Omega_h \), where \( \Omega_h \) is a polygonal domain,

\[
\overline{\Omega}_h = \bigcup_{E^j \in E_h} E^j, \quad \text{int}[E^i] \cap \text{int}[E^j] = \emptyset \text{ for } i \neq j,
\]

where each \( E^j \in E_h \) is a closed tetrahedron that can be obtained via the affine transformation

\[
E^j = h A_{E^j} \tilde{E} + a_{E^j}, \quad A_{E^j} \in \mathbb{R}^{3 \times 3}, \quad a_{E^j} \in \mathbb{R}^3,
\]
where $\tilde{E}$ is the reference element
\[
\tilde{E} = \text{co} \{ [0, 0, 0], [1, 0, 0], [0, 1, 0], [0, 0, 1] \},
\]
and where all eigenvalues of the matrix $A_{E_j}$ are bounded above and below away from zero uniformly for $h \to 0$. The family $E_h$ of all tetrahedra covering $\Omega_h$ is called mesh, the positive number $h$ is the parameter of spatial discretization. We write
\[
a \lesssim b \iff a \leq cb, \quad c > 0 \text{ independent of } h,
\]
\[
a \gtrsim b \iff a \geq cb, \quad c > 0 \text{ independent of } h,
\]
\[
a = b \iff a \lesssim b \text{ and } a \gtrsim b.
\]
Furthermore, we suppose that:

- a non-empty intersection of two elements $E^j, E^i$ is their common face, edge, or vertex;
- for all compact sets $K_i \subset \Omega, K_e \subset \mathbb{R}^3 \setminus \overline{\Omega}$ there is $h_0 > 0$ such that $K_i \subset \Omega_h, K_e \subset \mathbb{R}^3 \setminus \overline{\Omega}_h$ for all $0 < h < h_0$.

The symbol $\Gamma_h$ denotes the set of all faces in the mesh. We distinguish exterior and interior faces:
\[
\Gamma_h = \Gamma_{h, \text{int}} \cup \Gamma_{h, \text{ext}}, \quad \Gamma_{h, \text{ext}} = \left\{ \Gamma \in \Gamma_h \mid \Gamma \subset \partial \Omega \right\}, \quad \Gamma_{h, \text{int}} = \Gamma_h \setminus \Gamma_{h, \text{ext}}.
\]

### 2.2 Function spaces

Our scheme utilizes spaces of piecewise smooth functions, for which we define the traces
\[
v^{\text{out}} = \lim_{\delta \to 0} v(x + \delta n_\Gamma), \quad v^{\text{in}} = \lim_{\delta \to 0} v(x - \delta n_\Gamma), \quad x \in \Gamma, \quad \Gamma \in \Gamma_{h, \text{int}},
\]
where $n_\Gamma$ denotes the outer normal vector to the face $\Gamma \subset \partial E$. Analogously, we define $v^{\text{in}}$ for $\Gamma \subset \Gamma_{h, \text{ext}}$. We simply write $v$ for $v^{\text{in}}$ if no confusion arises. We also define
\[
[v] = v^{\text{out}} - v^{\text{in}}, \quad \langle v \rangle_\Gamma = \frac{v^{\text{out}} + v^{\text{in}}}{2}, \quad \langle v \rangle_\Gamma = \frac{1}{|\Gamma|} \int_{\Gamma} v \, dS_x.
\]
Next, we introduce the space of piecewise constant functions
\[
Q_h(\Omega_h) = \left\{ v \in L^1(\Omega_h) \mid v|_E = \text{const} \in R \text{ for any } E \in E_h \right\},
\]
with the associated projection
\[
\Pi^Q_h : L^1(\Omega_h) \to Q_h(\Omega_h), \quad \Pi^Q_h[v] = \langle v \rangle_E = \frac{1}{|E|} \int_E v \, dx, \quad E \in E_h.
\]
We shall occasionally write

$$\Pi^Q_h[v] = \langle v \rangle.$$  

Finally, we introduce the Crouzeix–Raviart finite element spaces

$$V_h(\Omega_h) = \left\{ v \in L^2(\Omega_h) \mid v|_E = \text{affine function } E \in E_h, \int_{\Gamma} v^{\text{in}} \, dS_E = \int_{\Gamma} v^{\text{out}} \, dS_E \text{ for } \Gamma \in \Gamma_{h,\text{int}} \right\},$$

$$V_{0,h}(\Omega_h) = \left\{ v \in V_h(\Omega_h) \mid \int_{\Gamma} v^{\text{in}} \, dS_E = 0 \text{ for } \Gamma \in \Gamma_{h,\text{ext}} \right\},$$

along with the associated projection

$$\Pi^V_h : W^{1,1}(\Omega_h) \to V_h(\Omega_h), \quad \int_{\Gamma} \Pi^V_h[v] \, dS_E = \int_{\Gamma} v \, dS_E \text{ for any } \Gamma \in \Gamma_h.$$  

We denote by $\nabla_h v, \text{div}_h v$ the piecewise constant functions resulting from the action of the corresponding differential operator on $v$ on each fixed element in $E_h$:

$$\nabla_h v \in Q_h(\Omega_h; \mathbb{R}^3), \quad \nabla_h v = \nabla x v \text{ for } E \in E_h, \quad \text{div}_h v \in Q_h(\Omega_h), \quad \text{div}_h v = \text{div}_x v \text{ for } E \in E_h.$$  

### 2.3 Discrete time derivative, dissipative upwind

For a given time step $\Delta t > 0$ and the (already known) value of the numerical solution $v_h^{k-1}$ at a given time level $t_{k-1} = (k-1)\Delta t$, we introduce the discrete time derivative

$$D_t v_h = \frac{v_h^k - v_h^{k-1}}{\Delta t}$$

to compute the numerical approximation $v_h^k$ at the level $t_k = t_{k-1} + \Delta t$.

To approximate the convective terms, we use the dissipative upwind operators introduced in [4] (see also [3]), specifically,

$$\text{Up}[r_h, u_h] = \left\{ r_h \right\} \left\langle u_h \cdot n \right\rangle_{\Gamma} - \frac{1}{2} \max\left\{ a; \mid \left\langle u_h \cdot n \right\rangle_{\Gamma} \right\} \left[ r_h \right]$$

$$= r_h^{\text{out}} \left\langle u_h \cdot n \right\rangle_{\Gamma}^{\text{out}} + r_h^{\text{in}} \left\langle u_h \cdot n \right\rangle_{\Gamma}^{\text{in}} + \frac{h^a}{2} \left[ r_h \right] \chi \left( \frac{\left\langle u_h \cdot n \right\rangle_{\Gamma}}{h^a} \right),$$

where

$$\chi(z) = \begin{cases} 
0 & \text{for } z < -1, \\
1 - z & \text{if } -1 \leq z \leq 0, \\
z + 1 & \text{if } 0 < z \leq 1, \\
0 & \text{for } z > 1. 
\end{cases}$$
2.4 Numerical scheme

Given the initial data
\[ \rho_h^0 \in Q_h(\Omega_h), \quad u_h^0 \in V_{0,h}(\Omega_h; R^3), \]  
(2.2)
and the numerical solution
\[ \rho_h^{k-1} \in Q_h(\Omega_h), \quad u_h^{k-1} \in V_{0,h}(\Omega_h; R^3), \quad k \geq 1, \]
the value \([\rho_h^k, u_h^k] \in Q_h(\Omega_h) \times V_{0,h}(\Omega_h; R^3)\) is obtained as a solution of the following system of equations:
\[
\int_{\Omega_h} D_t \rho_h^k \phi \ dx - \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} U p(\rho_h^k, u_h^k) [[\phi]] \ dS_x = 0 \quad (2.3)
\]
for any \(\phi \in Q_h(\Omega_h);\)
\[
\int_{\Omega_h} D_t (\rho_h^k \langle u_h^k \rangle) \cdot \phi \ dx - \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} U p(\rho_h^k, u_h^k) \cdot [[\langle \phi \rangle]] \ dS_x - \int_{\Omega_h} p(\rho_h^k) \text{div}_h \phi \ dx
\]
\[+ \mu \int_{\Omega_h} \nabla_h u_h^k : \nabla_h \phi \ dx + \left( \frac{\mu}{3} + \eta \right) \int_{\Omega_h} \text{div}_h u_h^k \text{div}_h \phi \ dx = 0 \quad (2.4)
\]
for any \(\phi \in V_{0,h}(\Omega_h; R^3).\) The specific form of the viscous stress in (2.4) reflects the fact that the viscosity coefficients are constant.

It was shown in [24] (see also [5, Part II]) that system (2.3), (2.4) is solvable for any choice of the initial data (2.2). In addition, \(\rho_h^0 > 0\) whenever \(\rho_h^0 > 0.\) In general, the solution \([\rho_h^k, u_h^k]\) may not be uniquely determined by \([\rho_h^{k-1}, u_h^{k-1}]\) unless the time step \(\Delta t\) is conveniently adjusted by a CFL type condition. We make more comments on this option in Remark 4.3 below.

As shown in [4] (see also [5, Part II]), the family of numerical solutions converges, up to a suitable subsequence, to a weak solution of the Navier-Stokes system (1.1–1.4) as \(h \to 0\) if

- the time step is adjusted so that \(\Delta t \approx h;\)
- the viscosity coefficients satisfy \(\mu > 0, \eta \geq 0,\)
- the pressure satisfies \(p(\rho) = a\rho^\gamma + b\rho, \ a, b > 0, \ \gamma > 3.\)

If the limit solution of the Navier–Stokes system is smooth, then qualitative error estimates can be derived on condition that \(p\) satisfies (1.3) with \(\gamma \geq 3/2,\) see Gallouët et al. [18]. Unfortunately, many real world applications correspond to smaller adiabatic exponents, the most popular among them is the air with \(\gamma = 7/5.\) It is therefore of great interest to discuss convergence of the scheme in the physically relevant range \(1 < \gamma < 2.\)
3 Stability - energy estimates

It is crucial for our analysis that the numerical scheme \[22, 24\] admits a certain form of total energy balance. For the pressure potential

\[ P(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma, \quad P''(\varrho) = \frac{p'(\varrho)}{\varrho} = a\gamma \varrho^{\gamma - 2}, \]

the total energy balance reads

\[
\int_{\Omega_h} D_t \left[ \frac{1}{2} \varrho_h^k \left| \langle \mathbf{u}_h^k \rangle \right|^2 + P(\varrho_h^k) \right] \, dx + \int_{\Omega_h} \left[ \mu |\nabla_h \mathbf{u}_h^k|^2 + (\mu/3 + \eta) |\text{div}_h \mathbf{u}_h^k|^2 \right] \, dx \\
= -\frac{1}{2} \int_{\Omega_h} P''(\varrho_h^k) \left( \varrho_h^k - \varrho_h^{k-1} \right)^2 \, dx - \int_{\Omega_h} \frac{\Delta t}{2} \varrho_h^{k-1} \left| \frac{\langle \mathbf{u}_h^k \rangle - \langle \mathbf{u}_h^{k-1} \rangle}{\Delta t} \right|^2 \, dx \\
- \frac{h^\alpha}{2} \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} \left[ \left[ \varrho_h^k \right] \right] \left[ \left[ P'(\varrho_h^k) \right] \right] \chi \left( \frac{\langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_{\Gamma}}{h^\alpha} \right) \, dS_x \\
- \frac{1}{2} \sum_{\Gamma \in \Gamma} \int_{\Gamma} P''(\varrho_h^k) \left[ \left[ \varrho_h^k \right] \right]^2 \left| \langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_{\Gamma} \right| \, dS_x \\
- \frac{h^\alpha}{2} \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} \left\{ \varrho_h^k \cdot \left[ \left[ \langle \mathbf{u}_h^k \rangle \right] \right]^2 \chi \left( \frac{\langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_{\Gamma}}{h^\alpha} \right) \right\} \, dS_x \\
- \frac{1}{2} \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} \left( \left[ \langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_{\Gamma} \right]^+, \left[ \langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_{\Gamma} \right]^-, \langle \varrho_h^k \rangle^\text{in}, \langle \varrho_h^k \rangle^\text{out} \right) \left[ \left[ \langle \mathbf{u}_h^k \rangle \right] \right]^2 \, dS_x,
\]

with

\[ s_h^k \in \text{co}\{\varrho_h^k, \varrho_h^{k-1}\}, \quad z_h^k \in \text{co}\{(\varrho_h^k)^\text{in}, (\varrho_h^k)^\text{out}\}, \]

see \[5\] Chapter 7, Section 7.5.4]. As the numerical densities are positive, all terms on the right-hand side of (3.1) representing numerical dissipation are non-positive. For completeness, we remark that the scheme conserves the total mass, specifically,

\[ \int_{\Omega_h} \varrho_h^k \, dx = \int_{\Omega_h} \varrho_h^0 \, dx, \quad k = 1, 2, \ldots \] (3.2)

3.1 Dissipative terms and the pressure growth

It is easy to check that

\[ P''(z)(\varrho_1 - \varrho_2)^2 \geq a\gamma(\varrho_1^{\gamma/2} - \varrho_2^{\gamma/2})^2 \]

whenever \( z \in \text{co}\{\varrho_1, \varrho_2\}, \varrho_1, \varrho_2 > 0, 1 < \gamma < 2. \) (3.3)

Indeed it is enough to assume \( 0 < \varrho_1 \leq z \leq \varrho_2; \) whence

\[ P''(z)(\varrho_1 - \varrho_2)^2 \geq a\gamma \varrho_2^{\gamma - 2}(\varrho_1 - \varrho_2)^2, \]
and (3.3) reduces to showing
\[ \hat{\varphi}^{\gamma/2-1} (\varphi_2 - \varphi_1) \geq (\hat{\varphi}^{\gamma/2} - \hat{\varphi}_1^{\gamma/2}) \text{ or, equivalently, } \varphi_1 \hat{\varphi}^{\gamma/2-1} \leq \hat{\varphi}_1^{\gamma/2}, \]
where the last inequality follows immediately as \( \varphi_1 \leq \varphi_2, 1 < \gamma < 2. \)

Consequently, the terms on the right-hand side of (3.1) representing the numerical dissipation and containing \( P'' \) satisfy
\[
\frac{1}{2} \int_{\Omega_h} P''(s^k_h) \frac{(\varphi^k_h - \varphi^{k-1}_h)^2}{\Delta t} \, dx \geq \frac{a_2}{2} \int_{\Omega_h} \frac{(\varphi^k_h)^{\gamma/2} - (\varphi^{k-1}_h)^{\gamma/2})^2}{\Delta t} \, dx,
\]
\[
\frac{h^\alpha}{2} \sum_{\Gamma_{h,int}} \int_{\Gamma} \left[ \left[ (\partial \varphi^k_h) \right] \right] \left[ \left[ (\partial^\prime \varphi^k_h) \right] \right] \chi \left( \frac{\langle \mathbf{u}^k_h \cdot \mathbf{n} \rangle_{\Gamma}}{h^\alpha} \right) \, dS_x \geq \frac{a_2 h^\alpha}{2} \sum_{\Gamma_{h,int}} \int_{\Gamma} \left[ \left[ (\varphi^k_h)^{\gamma/2} \right] \right] \left[ \left[ (\varphi^{k-1}_h)^{\gamma/2} \right] \right] \chi \left( \frac{\langle \mathbf{u}^k_h \cdot \mathbf{n} \rangle_{\Gamma}}{h^\alpha} \right) \, dS_x,
\]
\[
\frac{1}{2} \sum_{\Gamma_{h}} \int_{\Gamma} P''(\varphi^k_h) \left[ (\varphi^k_h)^2 \right] \left( \langle \mathbf{u}^k_h \cdot \mathbf{n} \rangle_{\Gamma} \right) \, dS_x \geq \frac{a_2}{2} \sum_{\Gamma_{h}} \int_{\Gamma} \left[ \left[ (\varphi^k_h)^{\gamma/2} \right] \right] \left[ \left[ \langle \mathbf{u}^k_h \cdot \mathbf{n} \rangle_{\Gamma} \right] \right] \, dS_x.
\]
\]

In particular, the energy balance (3.1) gives rise to
\[
\int_{\Omega_h} D_t \left[ \mathbf{u}^k_h \right] \left[ \mathbf{u}^k_h \right]^2 + P(\varphi^k_h) \, dx + \int_{\Omega_h} \left[ \mu |\nabla_h \mathbf{u}^k_h|^2 + (\mu/3 + \eta) |\operatorname{div} \mathbf{u}^k_h|^2 \right] \, dx
+ a \int_{\Omega_h} \frac{(\varphi^k_h)^{\gamma/2} - (\varphi^{k-1}_h)^{\gamma/2})^2}{\Delta t} \, dx + \Delta t \int_{\Omega_h} \varphi^{k-1}_h \left| \frac{\langle \mathbf{u}^k_h \rangle - \langle \mathbf{u}^{k-1}_h \rangle}{\Delta t} \right|^2 \, dx
+ a \sum_{\Gamma_{h}} \int_{\Gamma} \left[ \left[ (\varphi^k_h)^{\gamma/2} \right] \right] \max \left\{ h^\alpha; |\langle \mathbf{u}^k_h \cdot \mathbf{n} \rangle_{\Gamma} | \right\} \, dS_x
+ a h^\alpha \sum_{\Gamma_{h,int}} \int_{\Gamma} \left[ \langle \mathbf{u}^k_h \rangle \right] \left[ \left[ (\varphi^k_h)^{\gamma/2} \right] \right] \chi \left( \frac{\langle \mathbf{u}^k_h \cdot \mathbf{n} \rangle_{\Gamma}}{h^\alpha} \right) \, dS_x
+ \sum_{\Gamma_{h,int}} \int_{\Gamma} \left( (\varphi^k_h)^{\text{in}} [\langle \mathbf{u}^k_h \cdot \mathbf{n} \rangle_{\Gamma}]^+ - (\varphi^k_h)^{\text{out}} [\langle \mathbf{u}^k_h \cdot \mathbf{n} \rangle_{\Gamma}]^- \right) \left[ \left[ (\mathbf{u}^k_h) \right] \right] \, dS_x \lesssim 0.
\]

4 Consistency

Our goal is to derive a consistency formulation for the discrete solutions satisfying (2.3), (2.4). To this end, it is convenient to deal with quantities defined on \( R \times \Omega_h \). Accordingly, we introduce
\[
\varphi_h(t, \cdot) = \varphi^0_h \text{ for } t < \Delta t, \quad \varphi_h(t, \cdot) = \varphi^k_h \text{ for } t \in [k\Delta t, (k+1)\Delta t), \quad k = 1, 2, \ldots,
\]
\[
\mathbf{u}_h(t, \cdot) = \mathbf{u}^0_h \text{ for } t < \Delta t, \quad \mathbf{u}_h(t, \cdot) = \mathbf{u}^k_h \text{ for } t \in [k\Delta t, (k+1)\Delta t), \quad k = 1, 2, \ldots,
\]
and
\[ D_t v_h = \frac{v(t, \cdot) - v(t - \Delta t, \cdot)}{\Delta t}, \quad t > 0. \] (4.3)

For the sake of simplicity, we keep the time step \( \Delta t \) constant, however, a similar ansatz obviously works also for \( \Delta t = \Delta t_k \) adjusted at each level of iteration.

A suitable consistency formulation of equation (2.3) reads
\[ -\int_{\Omega_h} \varphi(0, \cdot) \, dx = \int_0^T \int_{\Omega_h} [\varrho_h \partial_t \varphi + \varrho_h \mathbf{u}_h \cdot \nabla \varphi] \, dx \, dt + \mathcal{O}(h^\beta), \quad \beta > 0, \] \[ (4.4) \]
for any test function \( \varphi \in C_∞((0, \infty) \times \Omega_h) \), where \( \beta \) denotes a generic positive exponent, and, accordingly, the remainder term \( \mathcal{O}(h^\beta) \), that may depend also on the test function \( \varphi \), tends to zero as \( h \to 0 \). Similarly, we want to rewrite (2.4) in the form
\[ -\int_{\Omega_h} \mathbf{u}_h \cdot \varphi(0, \cdot) \, dx = \int_0^T \int_{\Omega_h} \left[ \varrho_h \langle \mathbf{u}_h \rangle \partial_t \varphi + \varrho_h \langle \mathbf{u}_h \rangle \otimes \mathbf{u}_h : \nabla \varphi + p(\varrho_h) \text{div} \mathbf{u}_h \right] \, dx \, dt \
- \int_0^T \int_{\Omega_h} \left[ \mu \nabla \mathbf{u}_h : \nabla \varphi + (\mu/3 + \eta) \text{div} \mathbf{u}_h \cdot \text{div} \varphi \right] \, dx \, dt + \mathcal{O}(h^\beta) \] \[ (4.5) \]
for any \( \varphi \in C_∞((0, \infty) \times \Omega_h; \mathbb{R}^3) \).

### 4.1 Preliminaries, some useful estimates

We collect certain well-known estimates used in the subsequent analysis. We refer to [5, Part II, Chapters 8,9] for the proofs.

#### 4.1.1 Discrete negative and trace estimates for piecewise smooth functions

The following inverse inequality
\[ \|v\|_{L^p(\Omega_h)} \lesssim h^{3\left(\frac{1}{p} - \frac{1}{q}\right)}\|v\|_{L^q(\Omega_h)}, \quad 1 \leq q \leq p \leq \infty, \] \[ (4.6) \]
holds for any \( v \in Q_h(\Omega_h) \).

The trace estimates read
\[ \|v\|_{L^p(\Gamma)} \lesssim h^{1/p}\|v\|_{L^p(E)} \quad \text{whenever} \quad \Gamma \subset \partial E, \quad 1 \leq p \leq \infty \] \[ (4.7) \]
for any \( v \in Q_h(\Omega_h) \).

Finally, we report a discrete version of Poincaré’s inequality
\[ \|v - \langle v \rangle\|_{L^2(E)} \equiv \|v - \Pi_h^Q[v]\|_{L^2(E)} \lesssim h\|\nabla_h v\|_{L^2(E)} \quad \text{for any} \quad v \in V_h(\Omega_h). \] \[ (4.8) \]
4.1.2 Sobolev estimates for broken norms

We have

\[ \|v\|_{L^6(\Omega_h)}^2 \lesssim \sum_{\Gamma_{h,\text{int}}} \int_{\Gamma} \frac{[[v]]^2}{h} \, dS_x + \|v\|_{L^2(\Omega_h)}^2 \]  

(4.9)

for any \( v \in Q_h(\Omega_h) \). In particular, we may combine the negative estimates (4.6) with (4.9) to obtain

\[ \|\varrho_h\|_{L^\infty(\Omega_h)} = \left( \left\| \varrho_h^{\gamma/2} \right\|_{L^\infty(\Omega_h)} \right)^{2/\gamma} \lesssim h^{-1/\gamma} \left( \left\| \varrho_h^{\gamma/2} \right\|_{L^6(\Omega_h)} \right)^{1/\gamma} \]

\[ \lesssim h^{-1/\gamma} \left( \sum_{\Gamma_{h,\text{int}}} \int_{\Gamma} \frac{[\varrho^{\gamma/2}]^2}{h} \, dS_x \right)^{1/\gamma} + h^{-1/\gamma} \left( \left\| \varrho_h^{\gamma/2} \right\|_{L^2(\Omega_h)} \right)^{1/\gamma} \]  

(4.10)

Next, we have the discrete variant of Sobolev’s inequality

\[ \|v\|_{L^6(\Omega_h)}^2 \lesssim \sum_{E \in E_h} \left\| \nabla_h v \right\|_{L^2(E;\mathbb{R}^3)}^2 \equiv \left\| \nabla_h v \right\|_{L^2(\Omega_h;\mathbb{R}^3)}^2 \]  

(4.11)

for any \( v \in V_{0,h}(\Omega_h) \).

Finally, we recall the projection estimates for the Crouzeix–Raviart spaces

\[ \|\Pi_h^V[v] - v\|_{L^q(\Omega_h)} + h \|\nabla_h \Pi_h^V[v] - \nabla x v\|_{L^q(\Omega_h;\mathbb{R}^3)} \lesssim h^j \|\nabla^j v\|_{L^q(\Omega_h;\mathbb{R}^3)} \]  

(4.12)

4.1.3 Upwind consistency formula

We report the universal formula

\[ \int_{\Omega_h} r \mathbf{u} \cdot \nabla x \phi \, dx = \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} U_{p}[r, \mathbf{u}] [[F]] \, dS_x \]

\[ + \frac{h^\alpha}{2} \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} [r] [[F]] \chi \left( \frac{\langle \mathbf{u} \cdot \mathbf{n} \rangle_{\Gamma}}{h^\alpha} \right) \, dS_x \]

\[ + \sum_{E \in E_h} \sum_{\Gamma_{E} \subset \partial E} \int_{\Gamma_{E}} (F - \phi) [r] [\langle \mathbf{u} \cdot \mathbf{n} \rangle_{\Gamma}]^- \, dS_x \]

\[ + \sum_{E \in E_h} \sum_{\Gamma_{E} \subset \partial E} \int_{\Gamma_{E}} \phi r (\mathbf{u} \cdot \mathbf{n} - \langle \mathbf{u} \cdot \mathbf{n} \rangle_{\Gamma}) \, dS_x + \int_{\Omega_h} r (F - \phi) \text{div}_h \mathbf{u} \, dx \]  

(4.13)

for any \( r, F \in Q_h(\Omega_h), \mathbf{u} \in V_{0,h}(\Omega_h;\mathbb{R}^3), \phi \in C^1(\Omega_h) \), see [5] Chapter 9, Lemma 7].
4.2 Consistency formulation of the continuity method

Our goal is to derive the consistency formulation (4.4) of the discrete equation of continuity (2.3).

4.2.1 Time derivative

We consider test functions of the form \( \psi(t)\phi(x) \) to obtain

\[
\int_0^T \int_{\Omega_h} D_t(\varrho_h) \langle \psi \phi \rangle \, dx \, dt = \int_0^T \int_{\Omega_h} D_t(\varrho_h) \phi \, dx \, dt
\]

\[
= - \int_0^T \int_{\Omega_h} \frac{\psi(t + \Delta t) - \psi(t)}{\Delta t} \varrho_h \phi \, dx \, dt - \frac{1}{\Delta t} \int_{\Omega_h} \varrho_h \psi(t + \Delta t) \phi \, dx \, dt
\]

whenever the function \( \psi \in C^\infty_c[0,T] \) and \( \Delta t \) is small enough so that the interval \([T - \Delta t, \infty)\) is not included in the support of \( \psi \). By means of the mean-value theorem we get that

\[
\int_0^T \int_{\Omega_h} D_t(\varrho_h) \langle \psi \phi \rangle \, dx \, dt = - \int_0^T \int_{\Omega_h} \partial_t \psi \varrho_h \phi \, dx \, dt - \int_{\Omega_h} \varrho_h \psi(0) \phi \, dx + O(\varrho)
\]

(4.14)

for any \( \phi \in C(\Omega_h) \), \( \psi \in C^\infty_c[0,T] \). Note that the \( O(\varrho) \) term depends on the second derivative of \( \psi \).

4.2.2 Convective term - upwind

Relation (4.13) evaluated for \( r = \varrho_h, u = u_h^k, F = \langle \phi \rangle, \phi \in C^1(\Omega_h) \) gives rise to

\[
\int_{\Omega_h} \varrho_h^k u_h^k \cdot \nabla_x \phi \, dx = \sum_{\Gamma \in \Gamma_h, \text{int}} \int_{\Gamma} \text{Up}[\varrho_h^k, u_h^k][[\langle \phi \rangle]] \, dS_x
\]

\[
= \frac{h^\alpha}{2} \sum_{\Gamma \in \Gamma_h, \text{int}} \int_{\Gamma} [\varrho_h^k][[[\langle \phi \rangle]]] \chi \left( \frac{\langle u_h^k \cdot n \rangle}{h^\alpha} \right) \, dS_x
\]

\[
+ \sum_{E \in E_h} \sum_{\Gamma_E \subseteq \partial E} \int_{\Gamma_E} (\langle \phi \rangle - \phi) [[\varrho_h^k]] [[\langle u_h^k \cdot n \rangle]] \, dS_x
\]

\[
+ \sum_{E \in E_h} \sum_{\Gamma_E \subseteq \partial E} \int_{\Gamma_E} \varrho_h^k (u_h^k \cdot n - \langle u_h^k \cdot n \rangle) \, dS_x + \int_{\Omega_h} \varrho_h (\langle \phi \rangle - \phi) \text{div}_h u_h^k \, dx.
\]

(4.15)

Using an elementary inequality

\[
|\varrho_1 - \varrho_2| \leq \left| (\varrho_1)^{\gamma/2} - (\varrho_2)^{\gamma/2} \right| (\varrho_1)^{1-\gamma/2} + (\varrho_2)^{1-\gamma/2}, \ 1 \leq \gamma \leq 2
\]

(4.16)
we get

\[
\frac{h^\alpha}{2} \left| \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} [[\mathcal{g}_h^k]] [[\langle \phi \rangle]] \chi \left( \frac{\langle u_h^k \cdot n \rangle_{\Gamma}}{h^\alpha} \right) \ dS_x \right| \lesssim h^{1 + \alpha} \| \phi \|_{C^1(\overline{\Omega}_h)} \left| \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} [[\mathcal{g}_h^k]] \ dS_x \right|
\]

\[
\lesssim h^{1 + \alpha} \| \phi \|_{C^1(\overline{\Omega}_h)} \left( \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} \left( [\mathcal{g}_h^k]^{(\gamma)/2} \right)^2 \ dS_x + \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} \{ [\mathcal{g}_h^k]^{1 - (\gamma)/2} \}^2 \ dS_x \right),
\]

where, by virtue of (3.5),

\[
h^{1 + \alpha} \| \phi \|_{C^1(\overline{\Omega}_h)} \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} \left( [\mathcal{g}_h^k]^{(\gamma)/2} \right)^2 \ dS_x \leq c(\phi) h g_k, \quad \Delta t \sum_k g_k < \infty,
\]

and, in accordance with (3.2) and the trace estimates (4.7),

\[
h^{1 + \alpha} \| \phi \|_{C^1(\overline{\Omega}_h)} \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} \left\{ [\mathcal{g}_h^k]^{1 - (\gamma)/2} \right\}^2 \ dS_x \lesssim h^{\alpha} c(\phi) \sum_{\Gamma \in \Omega_h} \int_{\Gamma} [\mathcal{g}_h^k]^{2 - \gamma} \ dx \lesssim h^\alpha.
\]

We may infer that

\[
\frac{h^\alpha}{2} \left| \sum_{\Gamma \in \Gamma_{h,\text{int}}} \int_{\Gamma} [[\mathcal{g}_h]] [[\langle \phi \rangle]] \chi \left( \frac{\langle u_h \cdot n \rangle_{\Gamma}}{h^\alpha} \right) \ dS_x \right|_{L^1(0,T)} = O(h^\beta), \; \beta > 0 \text{ whenever } \alpha > 0. \tag{4.17}
\]

Next, using (3.5) again, we deduce

\[
\left| \sum_{E \in \mathcal{E}_h} \sum_{\Gamma \in \partial \mathcal{E}_h} \int_{\Gamma} \left( \langle \phi \rangle - \phi \right) [[\mathcal{g}_h^k]] [\langle u_h^k \cdot n \rangle_{\Gamma}]^- \ dS_x \right|
\]

\[
\lesssim h \| \phi \|_{C^1(\overline{\Omega}_h)} \sum_{E \in \mathcal{E}_h} \sum_{\Gamma \in \partial \mathcal{E}_h} \int_{\Gamma} \left( [\mathcal{g}_h^k]^{\gamma/2} \right)^2 \ dS_x \left( \sum_{E \in \mathcal{E}_h} \int_{\Gamma} [\mathcal{g}_h^k]^{2 - \gamma} \ | \langle u_h^k \cdot n \rangle_{\Gamma} \ | \ dS_x \right)^{1/2}
\]

\[
\lesssim h^{1/2} \left( \sum_{E \in \mathcal{E}_h} \sum_{\Gamma \in \partial \mathcal{E}_h} \int_{\Gamma} \left( [\mathcal{g}_h^k]^{\gamma/2} \right)^2 \ dS_x \right)^{1/2} \left( \sum_{E \in \mathcal{E}_h} \int_{\Gamma} [\mathcal{g}_h^k]^{2 - \gamma} \ | \langle u_h^k \rangle \ | \ dx \right)^{1/2}
\]

whence, using (4.10) to control the last term, we conclude

\[
\left| \sum_{E \in \mathcal{E}_h} \sum_{\Gamma \in \partial \mathcal{E}_h} \int_{\Gamma} \left( \langle \phi \rangle - \phi \right) [[\mathcal{g}_h]] [\langle u_h \cdot n \rangle_{\Gamma}]^- \ dS_x \right|_{L^2(0,T)} = O(h^\beta). \tag{4.18}
\]
Furthermore,
\[ \sum_{E \in \mathcal{E}_h} \sum_{E \in \mathcal{E}_h} \int_{\Gamma_E} \phi \partial_h^k \left( \mathbf{u}_h^k \cdot \mathbf{n} - \langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_{\Gamma} \right) \, dS_x = \sum_{E \in \mathcal{E}_h} \sum_{E \in \mathcal{E}_h} \int_{\Gamma_E} (\phi - \langle \phi \rangle_{\Gamma}) \partial_h^k \left( \mathbf{u}_h^k \cdot \mathbf{n} - \langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_{\Gamma} \right) \, dS_x, \]
where, by virtue of Poincaré’s inequality and the trace estimates (4.7),
\[ \left| \sum_{E \in \mathcal{E}_h} \sum_{E \in \mathcal{E}_h} \int_{\Gamma_E} (\phi - \langle \phi \rangle_{\Gamma}) \partial_h^k \left( \mathbf{u}_h^k \cdot \mathbf{n} - \langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_{\Gamma} \right) \, dS_x \right| \]
\[ \lesssim h \| \nabla \phi \|_{L^{\infty}(\Omega_h)} \sum_{E \in \mathcal{E}_h} \sum_{E \in \mathcal{E}_h} \int_{\Gamma_E} \partial_h^k \left| \mathbf{u}_h^k \cdot \mathbf{n} - \langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_{\Gamma} \right| \, dS_x \lesssim \sum_{E \in \mathcal{E}_h} \int_{\Gamma_E} \partial_h^k \left| \mathbf{u}_h^k \cdot \mathbf{n} - \langle \mathbf{u}_h^k \cdot \mathbf{n} \rangle_{\Gamma} \right| \, dS_x \]
\[ \lesssim h \sum_{E \in \mathcal{E}_h} \| \nabla \mathbf{u}_h^k \|_{L^2(E)} \| \partial_h^k \|_{L^2(E)} \lesssim h \| \nabla \mathbf{u}_h^k \|_{L^2(\Omega_h)} \| \partial_h^k \|_{L^2(\Omega_h)} \lesssim h \| \nabla \mathbf{u}_h^k \|_{L^2(\Omega_h)} \| \partial_h^k \|_{L^\infty(\Omega_h)}. \]

Going back to (4.10) we observe that the right-hand side is controlled as soon as
\[ 1 - \frac{2 + \alpha}{2\gamma} > 0 \text{ meaning } \alpha < 2(\gamma - 1). \] (4.19)

Finally, it is easy to check that the last integral in (4.15) can be handled in the same way. Thus we conclude that the consistency formulation (4.4) holds for any test function \( \varphi \in C^\infty_c([0, \infty) \times \Omega_h) \) as long as \( \alpha > 0, \gamma > 1 \) are interrelated through (4.19).

### 4.3 Consistency formulation of the momentum method

Our goal is to take \( \Pi_h^V [\varphi], \varphi \in C^\infty_c(\Omega_h; \mathbb{R}^3) \) as a test function in the momentum scheme (2.4). To begin, observe that
\[ \int_{\Omega_h} \nabla_h \mathbf{u}_h : \nabla_h \Pi_h^V [\varphi] \, dx = \int_{\Omega_h} \nabla_h \mathbf{u}_h : \nabla \varphi \, dx, \int_{\Omega_h} \text{div}_h \mathbf{u}_h \text{div}_h \Pi_h^V [\varphi] \, dx = \int_{\Omega_h} \text{div}_h \mathbf{u}_h \text{div}_x \varphi \, dx \]
\[ \int_{\Omega_h} p(\rho_h) \text{div}_h \Pi_h^V [\varphi] \, dx = \int_{\Omega_h} p(\rho_h) \text{div}_x \varphi \, dx, \]
see [5], Chapter 9, Lemma 8.

#### 4.3.1 Time derivative

We compute
\[ \int_{\Omega_h} D_t (\partial_h^k \langle \mathbf{u}_h^k \rangle) : \varphi \, dx = \int_{\Omega_h} D_t (\partial_h^k \langle \mathbf{u}_h^k \rangle) \cdot \Pi_h^V [\varphi] \, dx \]
\[ + \int_{\Omega_h} \partial_h^{k-1} (\mathbf{u}_h^k - \langle \mathbf{u}_h^{k-1} \rangle) \cdot \frac{\partial_h^{k-1} \rho_h - \rho_h^{k-1}}{\Delta t} \cdot (\varphi - \Pi_h^V [\varphi]) \, dx \]
\[ + \int_{\Omega_h} \frac{\partial_h^{k-1} \rho_h - \rho_h^{k-1}}{\Delta t} \cdot (\partial_h^k \langle \mathbf{u}_h^k \rangle) \cdot (\varphi - \Pi_h^V [\varphi]) \, dx, \] (4.20)
where

\[
\left| \int_{\Omega_h} \frac{\theta_h^{k-1} \langle u_h^k \rangle - \langle u_h^{k-1} \rangle}{\Delta t} \cdot (\phi - \Pi^V_h [\phi]) \, dx \right| \lesssim h^2 \| \phi \|_{C^2(\Omega_h)} \left( \int_{\Omega_h} \frac{\theta_h^{k-1} \langle u_h^k \rangle - \langle u_h^{k-1} \rangle}{\Delta t} \, dx \right) \lesssim h^2 \left( \int_{\Omega_h} \frac{\theta_h^{k-1} \langle u_h^k \rangle - \langle u_h^{k-1} \rangle}{\Delta t} \, dx \right)^{1/2} \lesssim h^2 (\Delta t)^{-1/2} \left( \int_{\Omega_h} \frac{\theta_h^{k-1} \langle u_h^k \rangle - \langle u_h^{k-1} \rangle}{\Delta t} \, dx \right)^{1/2},
\]

where the most right integral is controlled in \( L^2(0,T) \) by the numerical dissipation in (3.5).

As for the remaining integral, we may use inequality (4.16) to obtain

\[
\left| \int_{\Omega_h} \frac{\theta_h^k - \theta_h^{k-1}}{\Delta t} \langle u_h^k \rangle \cdot (\phi - \Pi^V_h [\phi]) \, dx \right| \lesssim h^2 \left( \int_{\Omega_h} \left( \frac{\theta_h^k - \theta_h^{k-1}}{\Delta t} \right)^2 \, dx \right)^{1/2} \lesssim h^2 (\Delta t)^{-1/2} \left( \int_{\Omega_h} \left( \frac{\theta_h^k - \theta_h^{k-1}}{\Delta t} \right)^2 \, dx \right)^{1/2}.
\]

Finally, we may repeat the same argument as in Section 4.2.1 to conclude that

\[
\int_0^T \int_{\Omega_h} \psi D_t(\theta_h \langle u_h \rangle) \Pi^V_h [\phi] \, dx \, dt = - \int_0^T \int_{\Omega_h} \theta_h \langle u_h \rangle \cdot \phi \partial_t \psi \, dx \, dt - \int_{\Omega_h} \psi(0) \theta_h^0 \langle u_h^0 \rangle \cdot \phi \, dx + O(h^3)
\]

provided \( \psi \in C^\infty_c[0,T), \phi \in C^\infty_c(\Omega_h; R^3) \).

### 4.3.2 Convective term - upwind

Applying formula (4.13) we obtain
\[
\int_{\Omega_h} \varphi_h \cdot \nabla \phi \, dx - \sum_{\Gamma \in \Gamma_h, \text{int}} \int_{\Gamma} U_p[\varphi_h(\langle u_h \rangle) \cdot \langle u_h \rangle] \cdot [\langle \Pi^h \phi \rangle] \, dx
\]
\[
= \frac{h^\alpha}{2} \sum_{\Gamma \in \Gamma_h, \text{int}} \int_{\Gamma} \left[ [\varphi_h(\langle u_h \rangle)] \cdot [\langle \Pi^h \phi \rangle] \right] \chi \left( \frac{\langle u_h \cdot n \rangle_{\Gamma}}{h^\alpha} \right) \, dS_x
\]
\[
+ \sum_{E \in E_h} \sum_{\Gamma \in \partial E} \int_{\Gamma} \varphi_h \cdot \langle u_h \rangle \left( \langle u_h \cdot n \rangle_{\Gamma} - \langle u_h \cdot n \rangle_{\Gamma} \right) \, dS_x
\]
\[
+ \int_{\Omega_h} \varphi_h \cdot \langle \langle \Pi^h \phi \rangle \rangle \, dx
\]

(4.22)

We proceed in several steps.

**Step 1**

Applying (4.12) we get

\[
\frac{h^\alpha}{2} \sum_{\Gamma \in \Gamma_h, \text{int}} \int_{\Gamma} \left[ [\varphi_h(\langle u_h \rangle)] \cdot [\langle \Pi^h \phi \rangle] \right] \chi \left( \frac{\langle u_h \cdot n \rangle_{\Gamma}}{h^\alpha} \right) \, dS_x
\]

\[
\lesssim h^{1+\alpha} \sum_{\Gamma \in \Gamma_h, \text{int}} \int_{\Gamma} \left[ [\varphi_h(\langle u_h \rangle)] \right] \chi \left( \frac{\langle u_h \cdot n \rangle_{\Gamma}}{h^\alpha} \right) \, dS_x,
\]

where

\[
[\varphi_h(\langle u_h \rangle)] = (\varphi_h)^{\text{out}} [\langle u_h \rangle] + \langle u_h \rangle [\varphi_h].
\]

Consequently

\[
\frac{h^\alpha}{2} \sum_{\Gamma \in \Gamma_h, \text{int}} \int_{\Gamma} \left[ [\varphi_h(\langle u_h \rangle)] \cdot [\langle \Pi^h \phi \rangle] \right] \chi \left( \frac{\langle u_h \cdot n \rangle_{\Gamma}}{h^\alpha} \right) \, dS_x
\]

\[
\lesssim h^{1+\alpha} \left( \sum_{\Gamma \in \Gamma_h, \text{int}} \int_{\Gamma} \left[ \varphi_h(\langle u_h \rangle) \right]^2 \chi \left( \frac{\langle u_h \cdot n \rangle_{\Gamma}}{h^\alpha} \right) \, dS_x \right)^{1/2} \left( \sum_{\Gamma \in \Gamma_h, \text{int}} \int_{\Gamma} \varphi_h^2 \, dS_x \right)^{1/2}
\]

\[
+ h^{1+\alpha} \sum_{\Gamma \in \Gamma_h, \text{int}} \int_{\Gamma} \langle u_h \rangle [\varphi_h] \, dS_x,
\]

16
where the first integral on the right-hand side is controlled by the numerical dissipation in (3.5) and the trace estimates.

Finally, applying the inequality (4.6), trace inequality (4.7) and Sobolev’s inequality (4.11), we obtain

\[
\begin{align*}
&h^{1+\alpha} \sum_{\Gamma \in \Gamma_{i,n}} \int_{\Gamma} \left| \langle \mathbf{u}_h^k \rangle \left[ \left[ \partial_h^k \right] \right] \right| \, dS_x 
\lesssim h^{1+\alpha} \sum_{\Gamma \in \Gamma_{i,n}} \int_{\Gamma} \left| \langle \mathbf{u}_h^k \rangle \right| \left| \left[ \partial_h^k \right] \right| \left| \left[ \left[ \partial_h^k \right] \right] \right| \, dS_x \\
&\lesssim h^{1+\alpha} \sum_{\Gamma \in \Gamma_{i,n}} \left( \int_{\Gamma} \left[ \left[ \partial_h^k \right] \right] \, dS_x \right)^{1/2} \left\| \langle \mathbf{u}_h^k \rangle \right\|_{L^6(\Gamma)} \left\| \left[ \partial_h^k \right] \right\|_{L^3(\Gamma)} \\
&\lesssim h^{1+\alpha} \sum_{E \in E_h} \left( h^{\alpha} \int_{\partial E} \left[ \left[ \partial_h^k \right] \right] \, dS_x \right)^{1/2} \left\| \langle \mathbf{u}_h^k \rangle \right\|_{L^6(E)} \left\| \left[ \partial_h^k \right] \right\|_{L^3(E)} \\
&\lesssim h^{1+\alpha} \left\| \nabla_h \mathbf{u}_h \right\|_{L^2(\Omega_h)} \left\| \left[ \partial_h^k \right] \right\|_{L^3(\Omega_h)},
\end{align*}
\]

where we have used the numerical dissipation in (3.5). Thus, in order to complete the estimates we have to control

\[
\left\| \left[ \partial_h^k \right] \right\|_{L^3(\Omega_h)}
\]
uniformly in \( k \). As \( 1 < \gamma < 2 \), it is enough to consider the critical case \( \gamma = 1 \), for which the inverse inequality (4.6) gives rise to

\[
\left\| \left[ \partial_h^k \right] \right\|_{L^3(\Omega_h)} = \left( \left\| \left[ \partial_h^k \right] \right\|_{L^{3/2}(\Omega_h)} \right)^{1/2} \lesssim h^{-1/2} \left\| \partial_h^k \right\|_{L^1(\Omega_h)}.
\]

**Step 2**

Using (4.23) we deduce

\[
\sum_{E \in E_h} \sum_{\Gamma \in \partial E} \int_{\Gamma_E} \left( \langle \Pi_h^V \mathbf{f} \rangle - \mathbf{f} \right) \cdot \left[ \left[ \partial_h^k \right] \left[ \langle \mathbf{u}_h^k \rangle \right] \left[ \langle \mathbf{n} \rangle \right] \right] \, dS_x
\]

\[
= \sum_{E \in E_h} \sum_{\Gamma \in \partial E} \int_{\Gamma_E} \left( \langle \Pi_h^V \mathbf{f} \rangle - \mathbf{f} \right) \cdot \left( \left[ \partial_h^k \right] \left[ \left[ \langle \mathbf{u}_h^k \rangle \right] \right] \left[ \langle \mathbf{n} \rangle \right] \right) \, dS_x,
\]

where, furthermore,

\[
\left| \sum_{E \in E_h} \sum_{\Gamma \in \partial E} \int_{\Gamma_E} \left( \langle \Pi_h^V \mathbf{f} \rangle - \mathbf{f} \right) \left( \left[ \partial_h^k \right] \left[ \left[ \langle \mathbf{u}_h^k \rangle \right] \right] \left[ \langle \mathbf{n} \rangle \right] \right) \, dS_x \right|
\]

\[
\lesssim h^2 \left\| \mathbf{f} \right\|_{C^2(\Gamma_h; \mathbb{R}^3)} \left( \sum_{E \in E_h} \sum_{\Gamma \in \partial E} \int_{\Gamma_E} \left( \left[ \partial_h^k \right] \left[ \left[ \langle \mathbf{u}_h^k \rangle \right] \right] \left[ \langle \mathbf{n} \rangle \right] \right) \, dS_x \right)^{1/2}
\]

\[
\times \left( \sum_{E \in E_h} \sum_{\Gamma \in \partial E} \int_{\Gamma_E} \left( \left[ \partial_h^k \right] \left[ \langle \mathbf{u}_h^k \rangle \right] \, dS_x \right)^{1/2}
\]

17
where the former integral in the product on the right-hand is controlled by the numerical dissipation in (3.5), while
\[
\sum_{E \in E_h} \sum_{\Gamma \subset \partial E} \int_{\Gamma_E} (\hat{\theta}_h^k)^{\text{out}} \mid \langle \mathbf{u}_h^k \rangle \mid \text{d}S_x \lesssim h^{-1} \| \mathbf{u}_h^k \|_{L^6(\Omega_h; \mathbb{R}^3)} \| \hat{\theta}_h^k \|_{L^{6/5}(\Omega_h)} \lesssim h^{-3/2} \| \mathbf{u}_h^k \|_{L^6(\Omega_h; \mathbb{R}^3)} \| \hat{\theta}_h^k \|_{L^1(\Omega_h)}.
\]

Finally,
\[
\left| \sum_{E \in E_h} \sum_{\Gamma \subset \partial E} \int_{\Gamma_E} \langle \Pi_h^V \phi \rangle - \phi \right| \cdot \langle \mathbf{u}_h^k \rangle \left[ \left[ \hat{\theta}_h^k \right] \left[ \langle \mathbf{u}_h^k \cdot n \rangle \Gamma \right] \right]^{-} \text{d}S_x \lesssim h^2 \sum_{E \in E_h} \sum_{\Gamma \subset \partial E} \| \langle \mathbf{u}_h^k \rangle \|_{L^6(\Gamma)} \| \hat{\theta}_h^k \|_{L^{3/2}(\Gamma)} \lesssim h \| \mathbf{u}_h^k \|_{L^6(\Omega_h)} \| \hat{\theta}_h^k \|_{L^{3/2}(\Omega_h)} \lesssim h^{3-3/\gamma} \| \mathbf{u}_h^k \|_{L^6(\Omega_h)} \| \hat{\theta}_h^k \|_{L^\gamma(\Omega_h)},
\]
where the exponent \(3 - 3/\gamma > 0\) as soon as \(\gamma > 1\).

**Step 3**

We write
\[
\sum_{E \in E_h} \sum_{\Gamma \subset \partial E} \int_{\Gamma_E} \hat{\theta}_h^k \phi \cdot \langle \mathbf{u}_h^k \rangle \left( \mathbf{u}_h^k \cdot n - \langle \mathbf{u}_h^k \cdot n \rangle \Gamma \right) \text{d}S_x = \sum_{E \in E_h} \sum_{\Gamma \subset \partial E} \int_{\Gamma_E} \hat{\theta}_h^k (\phi - \langle \phi \rangle \Gamma) \cdot \langle \mathbf{u}_h^k \rangle \left( \mathbf{u}_h^k \cdot n - \langle \mathbf{u}_h^k \cdot n \rangle \Gamma \right) \text{d}S_x,
\]
where, by virtue of the trace inequality (4.7) and Poincaré’s inequality (4.8),
\[
\left| \sum_{E \in E_h} \sum_{\Gamma \subset \partial E} \int_{\Gamma_E} \hat{\theta}_h^k (\phi - \langle \phi \rangle \Gamma) \cdot \langle \mathbf{u}_h^k \rangle \left( \mathbf{u}_h^k \cdot n - \langle \mathbf{u}_h^k \cdot n \rangle \Gamma \right) \text{d}S_x \right| \lesssim h \left\| \hat{\theta}_h^k \right\|_{L^\infty(\Omega_h)} \sum_{E \in E_h} \sum_{\Gamma \subset \partial E} \int_{\Gamma_E} \sqrt{\hat{\theta}_h^k} \left| \langle \mathbf{u}_h^k \rangle \right| \left| \mathbf{u}_h^k \cdot n - \langle \mathbf{u}_h^k \cdot n \rangle \Gamma \right| \text{d}S_x \lesssim h \left\| \hat{\theta}_h^k \right\|_{L^\infty(\Omega_h)} \left\| \sqrt{\hat{\theta}_h^k} \langle \mathbf{u}_h^k \rangle \| \nabla_h \mathbf{u}_h^k \|_{L^2(\Omega_h; \mathbb{R}^3)} \right\|
\]
where, in view of (4.10)
\[
\left\| \sqrt{\hat{\theta}_h^k} \right\|_{L^\infty(\Omega_h)} \lesssim h^{-\frac{2+\alpha}{2\gamma}}, \text{ with } \frac{2+\alpha}{2\gamma} < 1 \text{ or } 0 < \alpha < 2(\gamma - 1).
\]

**Step 4**

18
Finally,
\[
\int_{\Omega_h} \varphi_h^k \langle u_h^k \rangle \cdot (\langle \Pi_h^{V} [\phi] \rangle - \phi) \, \text{div}_h u_h^k \, dx \leq h^2 \left\| \varphi_h^k \right\|_{L^\infty(\Omega_h)} \left\| \varphi_h^k \right\|_{L^2(\Omega_h)} \left\| \nabla_h u_h^k \right\|_{L^2(\Omega_h; R^3)};
\]
whence the rest of the proof follows exactly as in Step 3.

Summing up the previous observations, we obtain the consistency formulation of the momentum method (4.5).

**Remark 4.1.** As \( \varphi \) has compact support, equation (4.5) is satisfied also on the limit domain \( \Omega \) for all \( h \) small enough.

Thus we have shown the following result.

**Proposition 4.2.** Let the pressure \( p \) satisfy (1.5), with \( 1 < \gamma < 2 \). Suppose that \( [\varrho_h, u_h] \) is a family of numerical solutions given through (4.1), (4.2), where \( [\varrho_h^k, u_h^k] \) satisfy (2.2–2.4), where
\[
\Delta t \approx h, \quad 0 < \alpha < 2(\gamma - 1).
\]

Then
\[
- \int_{\Omega_h} \varrho_h^0 \varphi(0, \cdot) \, dx = \int_0^T \int_{\Omega_h} [\varrho_h \partial_t \varphi + \varrho_h u_h \cdot \nabla x \varphi] \, dx \, dt + O(h^\beta), \quad \beta > 0,
\]
for any test function \( \varphi \in C_\infty^\infty([0, \infty) \times \bar{\Omega}_h) \),
\[
- \int_{\Omega_h} \varrho_h^0 \langle u_h^0 \rangle \cdot \varphi(0, \cdot) \, dx = \int_0^T \int_{\Omega_h} [\varrho_h \langle u_h \rangle \partial_t \varphi + \varrho_h \langle u_h \rangle \otimes u_h \cdot \nabla x \varphi + P(\varrho_h) \text{div}_h \varphi] \, dx \, dt
- \int_0^T \int_{\Omega_h} [\mu \nabla_h u_h : \nabla x \varphi + (\mu/3 + \eta) \text{div}_h u_h \cdot \text{div}_h \varphi] \, dx \, dt + O(h^\beta), \quad \beta > 0,
\]
for any \( \varphi \in C_\infty^\infty([0, \infty) \times \Omega_h; R^3) \).
Moreover, the solution satisfies the energy inequality
\[
\int_{\Omega_h} \left[ \frac{1}{2} \varrho_h |\langle u_h \rangle|^2 + P(\varrho_h) \right] \, (\tau, \cdot) \, dx + \int_0^T \int_{\Omega_h} \mu |\nabla_h u_h|^2 + (\mu/3 + \eta) |\text{div}_h u_h|^2 \, dx \, dt \leq \int_{\Omega_h} \left[ \frac{1}{2} \varrho_h^0 |\langle u_h^0 \rangle|^2 + P(\varrho_h^0) \right] \, dx
\]
for a.e. \( \tau \in [0, T] \).
Remark 4.3. A close inspection of the previous discussion shows that the same method can be used to handle a variable time step $\Delta t_k$ adjusted for each step of iteration by means of a CFL-type condition, such as $||u_h^{k-1} + c_h^{k-1}||_{L^\infty(\Omega)} \Delta t_k / h \leq \text{CFL}$. Here $\text{CFL} \in (0, 1]$ and $c_h^{k-1} \equiv \sqrt{p'(\rho_h^{k-1})}$ denotes the sound speed. Though this condition is necessary for stability of time-explicit numerical schemes, it still may be appropriate even for implicit schemes for areas of high-speed flows. Note that the only part that must be changed in the proof of Proposition 4.2 is Section 4.3.1, where the time derivative in the momentum method is estimated.

5 Measure-valued solutions

Our ultimate goal is to perform the limit $h \to 0$. For the sake of simplicity, we consider the initial data

$$\rho_0 \in L^\infty(R^3), \quad 0 \geq \rho > 0 \ a.a. \ in \ R^3, \ u_0 \in L^2(R^3).$$

With this ansatz, it is easy to find the approximation $[\rho_h^0, u_h^0]$ such that

$$\rho_h^0 \to \rho_0 \ in \ L^\infty(\Omega), \quad \rho_h^0 > 0, \quad \int_{\Omega_h} \rho_h^0 \phi \ dx \to \int_{\Omega} \rho_0 \phi \ dx \ for \ any \ \phi \in L^\infty(R^3),$$

$$\rho_h^0 \langle u_h^0 \rangle \to \rho_0 u_0 \ in \ L^2(\Omega; R^3), \quad \int_{\Omega_h} \rho_h^0 \langle u_h^0 \rangle \cdot \phi \ dx \to \int_{\Omega} \rho_0 u_0 \cdot \phi \ dx \ for \ any \ \phi \in L^\infty(R^3; R^3),$$

$$\int_{\Omega_h} \left[ \frac{1}{2} \rho_h^0 \langle u_h^0 \rangle^2 + P(\rho_h^0) \right] \ dx \to \int_{\Omega} \left[ \frac{1}{2} \rho_0 |u_0|^2 + P(\rho_0) \right] \ dx \ as \ h \to 0. \quad (5.1)$$

5.1 Weak limit

Extending $\rho_h$ by $\bar{\rho} > 0$ and $u_h$ to be zero outside $\Omega_h$, we may use the energy estimates (4.26) to deduce that, at least for suitable subsequences,

$$\rho_h \to \rho \ \text{weakly-(*) in} \ L^\infty(0, T; L^\gamma(\Omega)), \ \rho \geq 0$$

$$\langle u_h \rangle, \ u_h \to u \ \text{weakly in} \ L^2((0, T) \times \Omega; R^3),$$

where $u \in L^2(0, T; W^{1,2}_0(\Omega)), \ \nabla_h u_h \to \nabla_x u \ \text{weakly in} \ L^2((0, T) \times \Omega; R^{3 \times 3}),$

$$\rho_h \langle u_h \rangle \to \bar{\rho}_h \bar{u}_h \ \text{weakly-(*) in} \ L^\infty(0, T; L^{\frac{3}{2}}(\Omega; R^3)),$$

see [4] or [5, Part II, Section 10.4].

Remark 5.1. Note that, by virtue of Poincaré’s inequality (4.8) and the energy estimates (4.26),

$$\|u_h - \langle u_h \rangle\|_{L^2(0, T; L^2(K; R^3))} \lesssim h \ for \ any \ compact \ K \in \Omega,$$

in particular, the weak limits of $u_h, \langle u_h \rangle$ coincide in $\Omega$.  

20
In addition, the limit functions satisfy the equation of continuity in the form

$$-\int_{\Omega} \varphi(0, \cdot) \, dx = \int_{0}^{T} \int_{\Omega} [\varrho \partial_{t} \varphi + \varrho \mathbf{u} \cdot \nabla_{x} \varphi] \, dx \, dt$$

(5.2)

for any test function $\varphi \in C^\infty_c([0, \infty) \times \overline{\Omega})$. It follows from (5.2) that $\varrho \in C^{\text{weak}}([0, T]; L^\gamma(\Omega))$; whence (5.2) can be rewritten as

$$\left[ \int_{\Omega} \varrho \varphi(\tau, \cdot) \, dx \right]_{t=0}^{t=\tau} = \int_{0}^{\tau} \int_{\Omega} [\varrho \partial_{t} \varphi + \varrho \mathbf{u} \cdot \nabla_{x} \varphi] \, dx \, dt$$

(5.3)

for any $0 \leq \tau \leq T$ and any $\varphi \in C^\infty([0, T] \times \Omega)$.

### 5.2 Young measure generated by numerical solutions

The energy inequality (3.1), along with the consistency (4.4), (4.5) provide a suitable platform for the use of the theory of measure-valued solutions developed in [3]. Consider the family $[\varrho_h, \mathbf{u}_h]$. In accordance with the weak convergence statement derived in the preceding part, this family generates a Young measure - a parameterized measure

$$\nu_{t,x} \in L^\infty((0, T) \times \Omega; \mathcal{P}([0, \infty) \times \mathbb{R}^3))$$

for a.a. $(t, x) \in (0, T) \times \Omega$,

such that

$$\langle \nu_{t,x}, g(\varrho, \mathbf{u}) \rangle = g(\varrho_h, \mathbf{u}_h)(t, x)$$

for a.a. $(t, x) \in (0, T) \times \Omega$,

whenever $g \in C([0, \infty) \times \mathbb{R}^3)$, and

$$g(\varrho_h, \mathbf{u}_h) \rightharpoonup g(\varrho, \mathbf{u})$$

weakly in $L^1((0, T) \times \Omega)$.

Moreover, in view of Remark 5.1, the Young measures generated by $[\varrho_h, \mathbf{u}_h]$ and $[\varrho, \langle \mathbf{u}_h \rangle]$ coincide for a.a. $(t, x) \in (0, T) \times \Omega$.

Accordingly, the equation of continuity (5.3) can be written as

$$\left[ \int_{\Omega} \varrho \varphi(\tau, \cdot) \, dx \right]_{t=0}^{t=\tau} = \int_{0}^{\tau} \int_{\Omega} [\varrho \partial_{t} \varphi + \nu_{t,x} \mathbf{u}_h \cdot \nabla_{x} \varphi] \, dx \, dt$$

(5.4)

In order to apply a similar treatment to the momentum equation (4.25), we have to replace the expression $\varrho_h \langle \mathbf{u}_h \rangle \otimes \mathbf{u}_h$ in the convective term by $\varrho_h \langle \mathbf{u}_h \rangle \otimes (\mathbf{u}_h - \langle \mathbf{u}_h \rangle)$. This is possible as

$$\|\varrho_h \langle \mathbf{u}_h \rangle \otimes \mathbf{u}_h - \varrho_h \langle \mathbf{u}_h \rangle \otimes \langle \mathbf{u}_h \rangle\|_{L^1(\Omega_h; \mathbb{R}^{3 \times 3})} = \|\varrho_h \langle \mathbf{u}_h \rangle \otimes (\mathbf{u}_h - \langle \mathbf{u}_h \rangle)\|_{L^1(\Omega_h; \mathbb{R}^{3 \times 3})} \lesssim h \|\sqrt{\varrho_h} \langle \mathbf{u}_h \rangle\|_{L^2(\Omega_h; \mathbb{R}^3)} \|\nabla_h \mathbf{u}_h\|_{L^2(\Omega_h; \mathbb{R}^{3 \times 3})} \|\sqrt{\varrho_h}\|_{L^\infty(\Omega_h)}$$

where, by virtue of (4.10),

$$h \|\sqrt{\varrho_h}\|_{L^\infty(\Omega_h)} \lesssim h^{1 - \frac{2}{3\gamma}},$$

21
where the exponent is positive as soon as (4.24) holds, specifically, \(0 < \alpha < 2(\gamma - 1)\). Moreover, we have
\[
\varrho_h \langle \mathbf{u}_h \rangle \otimes \langle \mathbf{u}_h \rangle + p(\varrho_h) \mathbb{I} \to \{ \varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} \} \text{ weakly-}^* \text{ in } [L^\infty(0, T; \mathcal{M}(\Omega))]^{3 \times 3};
\]
whence letting \(h \to 0\) in (4.25) gives rise to
\[
- \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \, dx = \int_0^T \int_\Omega \left[ \langle \nu_{t,x}; \varrho \mathbf{u} \rangle \cdot \partial_t \varphi + \{ \varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} \} : \nabla_x \varphi \right] \, dx \, dt
- \int_0^T \int_\Omega \left[ \mu \nabla \mathbf{u} : \nabla_x \varphi + (\mu/3 + \eta) \text{div} \mathbf{u} \cdot \text{div} \varphi \right] \, dx \, dt
\]
or, equivalently,
\[
\left[ \int_\Omega \langle \nu_{t,x}; \varrho \mathbf{u} \rangle \cdot \varphi(0, \cdot) \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[ \langle \nu_{t,x}; \varrho \mathbf{u} \rangle \cdot \partial_t \varphi + \{ \varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} \} : \nabla_x \varphi \right] \, dx \, dt
- \int_0^\tau \int_\Omega \left[ \mu \nabla \mathbf{u} : \nabla_x \varphi + (\mu/3 + \eta) \text{div} \mathbf{u} \cdot \text{div} \varphi \right] \, dx \, dt
\]
for any \(0 \leq \tau \leq T\), \(\varphi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^3)\), where we have set
\[
\nu_{0,x} = \delta_{[\varrho_0(x), \mathbf{u}_0(x)]}.
\]

Finally, we introduce the concentration remainder
\[
\mathcal{R} = \{ \varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} \} - \{ \nu_{t,x}; \varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} \} \in [L^\infty(0, T; \mathcal{M}(\Omega))]^{3 \times 3}
\]
and rewrite (5.5) in the form
\[
\left[ \int_\Omega \langle \nu_{t,x}; \varrho \mathbf{u} \rangle \cdot \varphi(0, \cdot) \, dx \right]_{t=0}^{t=\tau}
= \int_0^\tau \int_\Omega \left[ \langle \nu_{t,x}; \varrho \mathbf{u} \rangle \cdot \partial_t \varphi + \{ \nu_{t,x}; \varrho \mathbf{u} \otimes \mathbf{u} \} : \nabla_x \varphi + \langle \nu_{t,x}, p(\varrho) \rangle \text{div} \varphi \right] \, dx \, dt
- \int_0^\tau \int_\Omega \left[ \mu \nabla \mathbf{u} : \nabla_x \varphi + (\mu/3 + \eta) \text{div} \mathbf{u} \cdot \text{div} \varphi \right] \, dx \, dt
+ \int_0^\tau \int_\Omega \mathcal{R} : \nabla_x \varphi \, dx \, dt
\]
for any \(0 \leq \tau \leq T\), \(\varphi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^3)\).

Similarly, the energy inequality (4.20) can be written as
\[
\left[ \int_\Omega \left[ \frac{1}{2} \langle \nu_{t,x}; \varrho |\mathbf{u}|^2 + P(\varrho) \rangle \right] \, dx \right]_{t=0}^{t=\tau} + \int_0^\tau \int_{\Omega_h} \mu |\nabla \mathbf{u}|^2 + (\mu/3 + \eta) |\text{div} \mathbf{u}|^2 \, dx \, dt
+ \mathcal{D}(\tau) \leq 0
\]
(5.7)
for a.e. \( \tau \in [0, T] \), with the dissipation defect \( D \) satisfying
\[
\int_0^\tau \| R \|_{\mathcal{M}(\Omega)} \ dt \lesssim \int_0^\tau D(t) \ dt, \quad D(\tau) \geq \liminf_{h \to \infty} \int_0^\tau |\nabla_h u_h|^2 \ dx \ dt - \int_0^\tau |\nabla_x u|^2 \ dx \ dt,
\]
\[\text{cf. [3, Lemma 2.1].}\]

At this stage, we recall the concept of dissipative measure valued solution introduced in [3]. These are measure–valued solutions of the Navier-Stokes system (1.1–1.4) satisfying the energy inequality (5.7), where the concentration remainder in the momentum equation is dominated by the dissipation defect as stated in (5.8) and the following analogue of Poincaré’s inequality holds:
\[
\lim_{h \to 0} \int_0^\tau \int_{\Omega_h} |u_h - u|^2 \ dx \ dt \leq \liminf_{h \to \infty} \int_0^\tau \int_{\Omega_h} |\nabla_h u_h|^2 \ dx \ dt - \int_0^\tau \int_{\Omega} |\nabla_x u|^2 \ dx \ dt (\leq D(\tau)),
\]
where \( u \) is a weak limit of \( u_h \), or, equivalently, of \( \langle u_h \rangle \). Consequently, relations (5.4), (5.6–5.8) imply that the Young measure \( \{ \nu_{t,x} \}_{t,x \in (0,T) \times \Omega} \) represents a dissipative measure-valued solution of the Navier-Stokes system (1.1–1.4) in the sense of [3] as soon as we check (5.9).

By standard Poincaré’s inequality in \( \Omega_h \) we get, on one hand,
\[
\int_{\Omega_h} |u_h - u|^2 \ dx = \int_{\Omega_h} |u_h - \Pi^V_h [u]|^2 \ dx + \int_{\Omega_h} |\Pi^V_h [u] - u|^2 \ dx \lesssim \int_{\Omega_h} |\nabla_h u_h - \nabla_x u|^2 \ dx + O(h^\beta).
\]
On the other hand,
\[
\liminf_{h \to \infty} \int_0^\tau \int_{\Omega_h} |\nabla_h u_h|^2 \ dx \ dt - \int_0^\tau \int_{\Omega} |\nabla_x u|^2 \ dx \ dt = \liminf_{h \to \infty} \int_0^\tau \int_{\Omega_h} |\nabla_h u_h - \nabla_x u|^2 \ dx \ dt.
\]
Thus it is enough to observe that, by virtue of (4.12),
\[
\nabla_h \Pi^V_h [u] \to \nabla_x u \text{ (strongly) in } L^2(\Omega_h; R^3) \text{ whenever } u \in W^{1,2}_0(\Omega; R^3).
\]

Seeing that validity of (5.6) as well as the bound on the dissipation remainder (5.8) can be extended to the class of test functions \( \varphi \in C^1(\Omega) \), \( \varphi|_{\partial \Omega} = 0 \), we have shown the following result.

**Theorem 5.2.** Let the pressure \( p \) satisfy (1.5), with \( 1 < \gamma < 2 \). Suppose that \( [\varphi_h, u_h] \) is a family of numerical solutions given through (4.1), (4.2), where \( [\varphi^k_h, u^k_h] \) satisfy (2.2–2.4), where
\[
\Delta t \approx h, \quad 0 < \alpha < 2(\gamma - 1),
\]
and the initial data satisfy (5.1).

Then any Young measure \( \{ \nu_{t,x} \}_{t,x \in (0,T) \times \Omega} \) generated by \( \varphi^k_h, u^k_h \) for \( h \to 0 \) represents a dissipative measure-valued solution of the Navier-Stokes system (1.1–1.4) in the sense of [3].
Of course, the conclusion of Theorem 5.2 is rather weak, and, in addition, the Young measure need not be unique. On the other hand, however, we may use the weak-strong uniqueness principle established in [3, Theorem 4.1] to obtain our final convergence result.

**Theorem 5.3.** In addition to the hypotheses of Theorem 5.2, suppose that the Navier-Stokes system (1.1–1.4) endowed with the initial data \([\varrho_0, \mathbf{u}_0]\) admits a regular solution \([\varrho, \mathbf{u}]\) belonging to the class

\[
\varrho, \nabla_x \varrho, \mathbf{u}, \nabla_x \mathbf{u} \in C([0, T] \times \Omega), \quad \partial_t \mathbf{u} \in L^2(0, T; C(\Omega; R^3)), \quad \varrho > 0, \quad \mathbf{u}|_{\partial \Omega} = 0.
\]

Then

\[
\varrho_h \rightarrow \varrho \text{ (strongly) in } L^\gamma((0, T) \times K), \quad \mathbf{u}_h \rightarrow \mathbf{u} \text{ (strongly) in } L^2((0, T) \times K; R^3)
\]

for any compact \(K \subset \Omega\).

Indeed, the weak–strong uniqueness implies that the Young measure generated by the family of numerical solutions coincides at each point \((t, x)\) with the Dirac mass supported by the smooth solution of the problem. In particular, the numerical solutions converge strongly and no oscillations occur. Note that the Navier–Stokes system admits local-in-time strong solutions for arbitrary smooth initial data, see e.g. Cho et al. [1], and even global-in-time smooth solutions for small initial data, see, e.g., Matsumura and Nishida [20], as soon as the physical domain \(\Omega\) is sufficiently smooth.

**6 Conclusions**

We have studied the convergence of numerical solutions obtained by the mixed finite element–finite volume scheme applied to the isentropic Navier-Stokes equations. We have assumed the isentropic pressure–density state equation \(p(\varrho) = a \varrho^\gamma\) with \(\gamma \in (1, 2)\). Remind that this assumption is not restrictive, since the largest physically relevant exponent is \(\gamma = 5/3\). In order to establish the convergence result we have used the concept of dissipative measure-valued solutions. These are the measure-valued solutions, that, in addition, satisfy an energy inequality in which the dissipation defect measure dominates the concentration remainder in the equations. The energy inequality (3.1), along with the consistency (4.4), (4.5) gave us a suitable framework to apply the theory of measure-valued solutions. As shown in Section 5.2, the numerical solutions \([\varrho_h, \mathbf{u}_h]\) generate a Young measure - a parameterized measure \(\{\nu_{t,x}\}_{t,x \in (0, T) \times \Omega}\), that represents a dissipative measure-valued solution of the Navier-Stokes system (1.1–1.4), cf. Theorem 5.2. Finally, using the weak-strong uniqueness principle established in [3, Theorem 4.1] we have obtained the convergence of the numerical solutions to the exact regular solution, as long as the latter exists, cf. Theorem 5.3. The present result is the first convergence result for numerical solutions of three-dimensional compressible isentropic Navier-Stokes equations in the case of full adiabatic exponent \(\gamma \in (1, 2)\).
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