Inequalities for Variation Operator

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February 2, 2022

Abstract

Let $f$ be a measurable function defined on $\mathbb{R}$. For each $n \in \mathbb{Z}$ define the operator $A_n$ by

$$A_n f(x) = \frac{1}{2^n} \int_x^{x+2^n} f(y) \, dy.$$

Consider the variation operator

$$\mathcal{V} f(x) = \left( \sum_{n=-\infty}^{\infty} |A_n f(x) - A_{n-1} f(x)|^s \right)^{1/s}$$

for $2 \leq s < \infty$.

It has been proved in [1] that $\mathcal{V}$ is of strong type $(p, p)$ for $1 < p < \infty$ and is of weak type $(1, 1)$, it maps $L^\infty$ to BMO. We first provide a completely different proofs for these known results and in addition we prove that $\mathcal{V}$ maps $H^1$ to $L^1$. Furthermore, we prove that it satisfies vector-valued weighted strong type and weak type inequalities. As a special case it follows that $\mathcal{V}$ satisfies weighted strong type and weak type inequalities.

Mathematics Subject Classification: 26D07, 26D15, 42B20.

Key Words: Variation Operator, $A_p$ Weight, Weak Type $(1, 1)$, Strong Type $(p, p)$, $H^1$ Space, BMO Space.

Introduction:

Let $f$ be a measurable function defined on $\mathbb{R}$. For each $n \in \mathbb{Z}$ define the operator $A_n$ by

$$A_n f(x) = \frac{1}{2^n} \int_x^{x+2^n} f(y) \, dy.$$
It is a well known problem to study the different kinds of convergence of the sequence \( \{A_n f\}_n \) when \( f \in L^p(\mathbb{R}) \) for some \( 1 \leq p < \infty \).

Consider the variation operator

\[ Vf(x) = \left( \sum_{n=-\infty}^{\infty} |A_n f(x) - A_{n-1} f(x)|^s \right)^{1/s} \]

for \( 2 \leq s < \infty \).

Analyzing the boundlessness of the variation operator \( Vf \) is a method of measuring the speed of convergence of the sequence \( \{A_n f\} \).

If a positive function \( w \in L^1_{\text{loc}}(\mathbb{R}) \) satisfies the following condition we say that \( w \) is an \( A_p \) weight for some \( 1 < p < \infty \):

\[ \sup_I \left( \frac{1}{|I|} \int_I w \, dx \right) \left( \frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty, \]

where the supremum is taken over all intervals \( I \) in \( \mathbb{R} \).

\( w \) is an \( A_\infty \) weight if there exists \( \delta > 0 \) and \( \epsilon > 0 \) such that for any measurable \( E \subset Q \),

\[ |E| < \delta \cdot |Q| \implies w(E) < (1 - \epsilon) \cdot w(Q). \]

Here

\[ w(E) = \int_E w \]

It is well known and easy to see that \( w \in A_p \implies w \in A_\infty \) if \( 1 < p < \infty \).

We say that \( w \in A_1 \) if there is a positive constant \( C \) such that

\[ \frac{1}{|Q|} \int_Q w(x) \, dx \leq C w(x) \]

for a.e. \( x \in Q \).

We say that an operator \( T : L^p(X) \to L^p(X) \) maps \( L^p(w) \) to itself if there is a positive constant \( C \) such that

\[ \int_{\mathbb{R}} |Tf(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}} |f(x)|^p w(x) \, dx \]
for all \( f \in L^p \).
We say that \( T \) is of strong type \((p,p)\) if there exists a positive constant \( C \) such that
\[
\|Tf\|_p \leq C\|f\|_p
\]
for all \( f \in L^p(X) \). \( T \) is of weak type \((1,1)\) (or satisfies a weak type \((1,1)\) inequality) if there exists a positive constant \( C \) such that
\[
\mu\{x : |Tf(x)| > \lambda\} \leq \frac{C}{\lambda}\|f\|_1
\]
for all \( f \in L^1(X) \).

We say that the kernel \( K \) satisfies \( D_r \) condition for \( 1 \leq r < \infty \), and write \( K \in D_r \), if there exists a sequence \( \{c_l\}_{l=1}^\infty \) of positive numbers \( \sum_l c_l < \infty \) and any \( l \geq 2 \) and \( x > 0 \),
\[
\left( \int_{S_l(x)} \|K(x-y) - K(-y)\|_r^r \, dy \right)^{1/r} \leq c_l |S_l(x)|^{-1/r'},
\]
where \( S_l(x) = (2^lx, 2^{l+1}x) \).

When \( K \in D_1 \) we have the Hörmander condition:
\[
\int_{\{y > 4x\}} \|K(x-y) - K(-y)\|_\infty \, dy \leq C
\]
where \( C \) is a positive constant which does not depend on \( y \in \mathbb{R} \). Let \( A \) and \( B \) be Banach spaces. A linear operator \( T \) mapping \( A \)-valued functions into \( B \)-valued functions is called a singular integral operator of convolution type if the following conditions are satisfied:

(i) \( T \) is a bounded operator from \( L^q_A(\mathbb{R}) \) to \( L^q_B(\mathbb{R}) \) for some \( q \), \( 1 \leq q \leq \infty \).

(ii) There exists a kernel \( K \in D_1 \) such that
\[
Tf(x) = \int K(x-y) \cdot f(y) \, dy
\]
for every \( f \in L^q_A(\mathbb{R}) \) with compact support and for a.e. \( x \notin \text{supp}(f) \).
Given a locally integrable function $f$ we define the sequence-valued operator $T$ as follows:

$$Tf(x) = \left\{ A_n f(x) - A_{n-1} f(x) \right\}_n$$

$$= \left\{ \int_{\mathbb{R}} \left( \frac{1}{2^n} \chi_{(-2^n,0)}(x-y) - \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x-y) \right) f(y) \, dy \right\}_n$$

$$= \int_{\mathbb{R}} K(x-y) \cdot f(y) \, dy,$$

where $K$ is the sequence-valued function

$$K(x) = \left\{ \frac{1}{2^n} \chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x) \right\}_n.$$

It is clear that

$$\|Tf(x)\|_{l^s} = Vf(x).$$

**Lemma 1.** The kernel operator

$$K(x) = \left\{ \frac{1}{2^n} \chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x) \right\}_n$$

satisfies $D_r$ condition for $r \geq 1$.

**Proof.** Let $x_0 \in \mathbb{R}$ and $i \in \mathbb{Z}$ be given, consider $x$ and $y$ in $\mathbb{R}$ such that $x_0 < x \leq x_0 + 2^i$ and $x_0 + 2^i < y \leq x_0 + 2^{i+1}$ with $j > i$. Let $\phi_n(y) = \chi_{(-2^n,0)}(y)$. Then $\phi_n(x - y) - \phi_n(x_0 - y) = 0$ unless $n = j$ in which case

$$\phi_j(x - y) - \phi_j(x_0 - y) = \chi_{(x_0 + 2^j, x_0 + 2^{j+1})}(y).$$

To see this first it is clear that $\phi_n(x - y) = \chi_{(x,x+2^n)}(y)$. Now if $n < i$ then

$$x + 2^n < x - x_0 + x_0 + 2^i \leq x_0 + 2^i < y.$$

Thus $\phi_n(x - y) = 0$. Clearly, the same holds for $\phi_n(x_0 - y)$. If $i \leq n < j$ then

$$x + 2^n \leq x_0 + 2^i + 2^n \leq x_0 + 2 \cdot 2^n \leq x_0 + 2^j,$$

and $\phi_n(x - y) = \phi_n(x_0 - y) = 0$. If $n > j$ then

$$x + 2^n > x_0 + 2^n \geq x_0 + 2^{j+1} \geq y.$$
and since $y > x > x_0$ we have
\[ \phi_n(x - y) - \phi_n(x_0 - y) = 1 - 1 = 0. \]

Finally, if $n = j$ then
\[ \phi_j(x_0 - y) = \chi_{(x_0,x_0+2^j)}(y) = 0, \]
while
\[ \phi_j(x - y) = \chi_{(x,x+2^j)}(y) = 1 \]
whenever
\[ x_0 + 2^j \leq y \leq x + 2^j. \]

We now have
\[
\| K(x - y) - K(x_0 - y) \|_{l^s} = \sum_n \left| \frac{1}{2^n} \phi_n(x - y) - \frac{1}{2^{n-1}} \phi_{n-1}(x - y) \right|^s
- \left( \frac{1}{2^n} \phi_n(x_0 - y) - \frac{1}{2^{n-1}} \phi_{n-1}(x_0 - y) \right)^s
= \sum_n \left| \frac{1}{2^n} \phi_n(x - y) - \frac{1}{2^n} \phi_n(x_0 - y) \right|^s
- \left( \frac{1}{2^{n-1}} \phi_{n-1}(x - y) - \frac{1}{2^{n-1}} \phi_{n-1}(x_0 - y) \right)^s
= 2 \left| \frac{1}{2^j} \phi_j(x - y) - \frac{1}{2^j} \phi_j(x_0 - y) \right|^s
= 2 \left| \frac{1}{2^j} \chi_{(x_0+2^j,x+2^j)}(y) \right|^s.
\]

Thus we get
\[
\| K(x - y) - K(x_0 - y) \|_{l^s} = 2^{1/s} \left| \frac{1}{2^j} \chi_{(x_0+2^j,x+2^j)}(y) \right|.
\]

Given $x$, choose an integer $i$ such that $2^{i-1} \leq x < 2^i$. By using our previous
observation we obtain

\[
\left( \int_{2^l x}^{2^{l+1} x} \| K(x - y) - K(-y) \|_{L^r}^r \, dy \right)^{1/r} 
\leq \left( \int_{2^{l+i-1}}^{2^{l+i}} \| K(x - y) - K(-y) \|_{L^r}^r \right)^{1/r} 
+ \left( \int_{2^{l+i}}^{2^{l+i+1}} \| K(x - y) - K(-y) \|_{L^r}^r \right)^{1/r} 
\leq 2^{2/s} \frac{2^{s/r}}{2^{l+i}} = C 2^{s/r} |S_i(x)|^{-1/r'}
\]

and this completes our proof. \[
\]

It is easy to check that the Fourier transform of the kernel of our vector-valued operator \(T\) is bounded and since we also have \(D_1\) condition we deduce that \(T\) is a singular integral operator of convolution type.

**Lemma 2.** A singular integral operator \(T\) mapping \(A\)-valued functions into \(B\)-valued functions can be extended to an operator defined in all \(L^p_A\), \(1 \leq p < \infty\), and satisfying

(a) \(\| Tf \|_{L^p_B} \leq C_p \| f \|_{L^p_A}, \quad 1 < p < \infty\),

(b) \(\| Tf \|_{W^{1,p}_B} \leq C_1 \| f \|_{L^1_A}\),

(c) \(\| Tf \|_{L^1_B} \leq C_2 \| f \|_{H^1_A}\),

(d) \(\| Tf \|_{\text{BMO}(B)} \leq C_3 \| f \|_{L^\infty(A)}, \quad f \in L^\infty_c(A)\),

where \(C_p, C_1, C_2, C_3 > 0\).

**Proof.** See J. L. Rubio de Francia et al [2].

Our first result is the following:

**Theorem 1.** The variation operator \(\nabla f\) maps \(H^1\) to \(L^1\) for \(2 \leq s < \infty\).
Proof. We have proved that the operator
\[
\begin{align*}
Tf(x) &= \left\{ \int_{\mathbb{R}} \left( \frac{1}{2^n} \chi(-2^n, 0)(x) - \frac{1}{2^{n-1}} \chi(-2^{n-1}, 0)(x) \right) f(y) \, dy \right\}_n \\
&= \int_{\mathbb{R}} K(x-y) \cdot f(y) \, dy,
\end{align*}
\]
is a singular integral operator. Since also \( \|Tf(x)\|_{L^p} = \mathcal{V}f(x) \) applying Lemma 2 (c) to our operator \( T \) shows that there exists a constant \( C > 0 \) such that
\[
\|\mathcal{V}f\|_1 \leq C \|f\|_{H^1}
\]
for all \( f \in H^1 \).

Remark 1. It is clear that our argument also provides a different proof for the following known facts (see [1]):

(a) \( \|\mathcal{V}f\|_p \leq C_p \|f\|_p, \quad 1 < p < \infty \),

(b) \( \|\mathcal{V}f\|_{WL^1} \leq C_1 \|f\|_1 \),

(c) \( \|\mathcal{V}f\|_{BMO} \leq C_2 \|f\|_{\infty}, \quad f \in L_c^\infty(\mathbb{R}) \),

where \( C_p, C_1, C_2 > 0 \).

Lemma 3. Let \( T \) be a singular integral operator with kernel \( K \in D_r \), where \( 1 < r < \infty \). Then, for all \( 1 < \rho < \infty \), the weighted inequalities
\[
\left\| \left( \sum_j \|Tf_j\|_B^\rho \right)^{1/\rho} \right\|_{L^p(w)} \leq C_{p, \rho}(w) \left\| \left( \sum_j \|f_j\|_A^\rho \right)^{1/\rho} \right\|_{L^p(w)}
\]
hold if \( w \in A_{p/r'} \) and \( r' \leq p < \infty \), or if \( w \in A_p^{r'} \) and \( 1 < p \leq r' \). Likewise, if \( w(x)^{r'} \in A_1 \), then the weak type inequality
\[
w \left( \left\{ x : \left( \sum_j \|Tf_j(x)\|_B^\rho \right)^{1/\rho} > \lambda \right\} \right) \leq C_{\rho}(w) \frac{1}{\lambda} \int \left( \sum_j \|f_j(x)\|_A^\rho \right)^{1/\rho} w(x) \, dx
\]
holds.
Proof. See J. L. Rubio de Francia et al [2].

Our next result is the following:

**Theorem 2.** Let \(2 \leq s < \infty\). Then, for all \(1 < \rho < \infty\), the weighted inequalities

\[
\left\| \left( \sum_j (\mathcal{V}f_j)^\rho \right)^{1/\rho} \right\|_{L^p(w)} \leq C_{p,\rho}(w) \left\| \left( \sum_j |f_j|^\rho \right)^{1/\rho} \right\|_{L^p(w)}
\]

hold if \(w \in A_{p/r'}\) and \(r' \leq p < \infty\), or if \(w \in A_p'\) and \(1 < p \leq r'\). Likewise, if \(w(x)^{r'} \in A_1\), then the weak type inequality

\[
w \left( \left\{ x : \left( \sum_j (\mathcal{V}f_j(x))^\rho \right)^{1/\rho} > \lambda \right\} \right) \leq C_{\rho}(w) \frac{1}{\lambda} \int \left( \sum_j |f_j(x)|^\rho \right)^{1/\rho} w(x) \, dx
\]

holds.

**Proof.** We have proved that the operator

\[
Tf(x) = \left\{ \int_{\mathbb{R}} \left( \frac{1}{2^n} \chi_{(-2^n,0)}(x-y) - \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x-y) \right) f(y) \, dy \right\}_n
\]

is a singular integral operator and its kernel operator \(K\) satisfies \(D_r\) condition for \(1 \leq r < \infty\). Since also \(\|Tf(x)\|_{L^r} = \mathcal{V}f(x)\) applying Lemma 3 to our operator \(T\) gives the result of our theorem. \(\square\)

In particular we have the following corollary:

**Corollary 3.** Let \(2 \leq s < \infty\). Then the weighted inequalities

\[
\|\mathcal{V}f\|_{L^p(w)} \leq C_{p,\rho}(w) \|f\|_{L^p(w)}
\]

hold if \(w \in A_{p/r'}\) and \(r' \leq p < \infty\), or if \(w \in A_p'\) and \(1 < p \leq r'\). Likewise, if \(w(x)^{r'} \in A_1\), then the weak type inequality

\[
w \left( \{ x : \mathcal{V}f(x) > \lambda \} \right) \leq C_{\rho}(w) \frac{1}{\lambda} \int |f(x)| w(x) \, dx
\]

holds.
References

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