Jean-Mathieu Teissier, Wolf-Christian Müller

Inverse transfer of magnetic helicity in direct numerical simulations of compressible isothermal turbulence: scaling laws

Open Access via institutional repository of Technische Universität Berlin

Document type
Journal article  |  Accepted version
(i.e. final author-created version that incorporates referee comments and is the version accepted for publication; also known as: Author's Accepted Manuscript (AAM), Final Draft, Postprint)

This version is available at
https://doi.org/10.14279/depositonce-12425

Citation details
Teissier, J.-M., & Müller, W.-C. (2021). Inverse transfer of magnetic helicity in direct numerical simulations of compressible isothermal turbulence: scaling laws. Journal of Fluid Mechanics, 915.
https://doi.org/10.1017/jfm.2021.32.

Terms of use
This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International license: https://creativecommons.org/licenses/by-nc-nd/4.0/
Inverse transfer of magnetic helicity in direct numerical simulations of compressible isothermal turbulence: scaling laws

Jean-Mathieu Teissier\textsuperscript{1} and Wolf-Christian Müller\textsuperscript{1,2}

\textsuperscript{1} Technische Universität Berlin, ER 3-2, Hardenbergstr. 36a, D-10623 Berlin, Germany
\textsuperscript{2} Max-Planck/Princeton Center for Plasma Physics

May 2021

Abstract

The inverse transfer of magnetic helicity is investigated through direct numerical simulations of large-scale-mechanically-driven turbulent flows in the isothermal ideal magnetohydrodynamics (MHD) framework. The mechanical forcing is either purely solenoidal or purely compressive and the turbulent statistically stationary states considered exhibit root mean square (RMS) Mach numbers $0.1 \lesssim M \lesssim 11$. A continuous small-scale electromotive forcing injects magnetic helical fluctuations, which lead to the build-up of ever larger magnetic structures. Spectral scaling exponents are observed which, for low Mach numbers, are consistent with previous research done in the incompressible case. Higher compressibility leads to smaller absolute values of the magnetic helicity scaling exponents. The deviations from the incompressible case are comparatively small for solenoidally-driven turbulence, even at high Mach numbers, as compared to those for compressively-driven turbulence, where strong deviations are already visible at relatively mild RMS Mach numbers $M \gtrsim 3$. Compressible effects can thus play an important role in the inverse transfer of magnetic helicity, especially when the turbulence drivers are rather compressive. Theoretical results observed in the incompressible case can, however, be transferred to supersonic turbulence by an appropriate change of variables, using the Alfvén velocity in place of the magnetic field.

1 Introduction

Magnetic helicity $H^M = \langle a \cdot b \rangle$, defined as the helicity of the magnetic vector potential $a$ (so that $b = \nabla \times a$ is the magnetic field), with $\langle \cdot \rangle$ denoting the volume average, is a quantity expressing topological aspects of the magnetic field lines, such as their degree of linkage, twist, writhe and knottedness [45]. In a closed volume with a vanishing magnetic field component normal to the volume boundaries, or a periodic domain without mean magnetic field, magnetic helicity is gauge invariant [12, 9]. In space, because of the very low resistivity, magnetic field lines are effectively “frozen-in” the ionised gas (plasma) present in various degrees of ionisation throughout the universe [3]. As a consequence, rotational motion is expected to naturally generate helical magnetic fields. Since magnetic helicity is an ideal invariant of the magnetohydrodynamic (MHD) equations [22, 57], it can be an important constraint on the time evolution of magnetic fields, which plays a crucial role in many astrophysical systems of interest. For example, magnetic helicity dynamics are involved in solar flares and coronal mass ejections [38], which transfer magnetic helicity from the sun to the interplanetary medium [34]. The sun also emits magnetic helicity through the solar wind [11], which is manifested at the largest scales through the Parker spiral [10]. Magnetic helicity conservation is also very important in dynamo processes [55, 13, 14]. In laboratory plasmas, it is of relevance for plasma confinement in reversed-field-pinch fusion experiments [23].

Magnetic helicity is subject to an inverse transfer in spectral space, as has been suggested in [29] and subsequently confirmed by various numerical experiments [49, 50, 42, 6, 18, 13, 2, 40, 46, 36, 37]. This key property renders magnetic helicity a potentially important quantity in the generation and sustainment of large-scale magnetic fields in the universe. In direct numerical simulations of astrophysical plasma the achievable Reynolds numbers are far below the ones expected in nature. A possibility to tackle this issue is to use large-eddy simulations (LES) with underlying subgrid-scale models, which are often constructed considering only the direct energy cascade. Understanding better the magnetic helicity dynamics is hence of great importance for the development of more realistic LES subgrid models [43].

Up to the present day, the inverse transfer of magnetic helicity in single-fluid MHD has been studied assuming either an incompressible plasma [2, 40, 46], or in the subsonic/transonic case [6, 18, 13]. Detailed studies on the properties of the transfer of energy and helicity, in particular its spectral locality have also been carried out by [5, 2, 1, 36]. In astrophysical systems however, flows are often highly supersonic. For example, the order of magnitude of the root mean square (RMS) turbulent Mach number ranges from 0.1 to about 10 in the interstellar medium ([21], section 4.2). The present work makes hence one step towards a more realistic
setting by including compressible effects, in the framework of supersonic isothermal ideal MHD. The inverse transfer of magnetic helicity is investigated through direct numerical simulations of large-scale-mechanically-driven compressible plasma under continuous injection of small scale random helical magnetic fluctuations. The large scale mechanical forcing is either purely solenoidal or purely compressive and the considered RMS Mach numbers of the initial hydrodynamic turbulent statistically stationary states range from 0.1 to 11 for solenoidal driving and from 1 to 8 for compressive driving.

A similar numerical setup in the incompressible case, where magnetic helical fluctuations are injected at small scales in the absence of a large-scale mechanical forcing has shown that several quantities, including the kinetic and magnetic energies and helicities, exhibit approximate power-law scaling in Fourier space in the inverse transfer region, i.e. at spatial scales larger than those of the magnetic driving [40, 46]. Furthermore, a dynamic balance leading to a quasi-equipartition between magnetic and kinetic energies on the one hand and current and kinetic helicities on the other hand has been observed in several numerical experiments in the incompressible case [40, 44, 30, 46]. The purpose of the present work is to investigate similarities and differences in the spectral scaling laws and this dynamic balance for turbulent flows exhibiting significant compressible effects. The scaling properties are expected to change, since in compressible MHD, the magnetic field time evolution is governed by:

$$\partial_t b = -(v \cdot \nabla) b + (b \cdot \nabla)v - b(\nabla \cdot v).$$  \hspace{1cm} (1)$$

For a high level of compression, the magnetic field and hence magnetic helicity dynamics are expected to be affected through two aspects. First, through the compression term $-b(\nabla \cdot v)$, which is not present in incompressible MHD. Second, even though the advective $-(v \cdot \nabla)b$ and the stretching $(b \cdot \nabla)v$ terms are also present in the incompressible case, the velocity field’s compressive part may alter them. An analysis of the influence of the compressive velocity field on these three terms is the object of a forthcoming work.

The numerical method used as well as the simulation setup are described in section 2. Section 3 summarises some findings of previous research, which are put in relation with this work’s results in section 4. Finally, section 5 assesses the robustness of the results while concluding remarks are given in section 6.

2 Numerical experiments

2.1 MHD numerical solver

The isothermal ideal (single fluid) MHD equations, in the presence of both a mechanical and an electromotive forcing, can be written as:

$$\partial_t \rho = -\nabla \cdot (\rho v),$$  \hspace{1cm} (2)

$$\partial_t (\rho v) = -\nabla \cdot \left(\rho vv^T + (\rho c_s^2 + \frac{1}{2}|b|^2)I - bb^T\right) + \rho f_V,$$  \hspace{1cm} (3)

$$\partial_t b = \nabla \times (v \times b) + f_M,$$  \hspace{1cm} (4)

$$\nabla \cdot b = 0,$$  \hspace{1cm} (5)

with $\rho$ the mass density, $v$ the fluid velocity, $c_s$ the constant isothermal sound speed so that $p = \rho c_s^2$ is the (thermal) pressure and $b$ the magnetic field so that $E = -v \times b$ is the electric field. The 3 $\times$ 3 identity matrix is denoted by $I$. The mechanical driving occurs through an acceleration field $(f_V)$ and the magnetic helical fluctuations are injected through an electromotive forcing $(f_M)$. These two terms are described in section 2.2.

The numerical solver is a shock-capturing fourth-order finite volume method, described in detail in [53, 54]. The main reconstruction method used is a fourth-order Central Weighted Essentially Non-Oscillatory (CWENO) procedure [35], with a passage through point values in order to maintain the fourth-order accuracy [41, 16]. The interfacial fluxes are computed using the Rusanov approximation [51], also known as “Local Lax-Friedrichs”.

In order to prevent the appearance of negative densities in high Mach number flows, a local reduction of the reconstruction order in the vicinity of strong discontinuities and shocks is used (an approach often referred to as “flattening” or “fallback approach” [19, 8, 47]). This has the effect of adding some numerical dissipation locally, which smoothens the solution and prevents as well oscillations of high-order polynomials on regular grids. The magnetic field solenoidality is maintained up to machine precision by application of the constrained-transport approach [24], making use of a multidimensional version of the Rusanov flux to compute the line-integrated electric field [7]. The time-integration is implemented as a fourth-order ten-stages Strong Stability Preserving Runge-Kutta (SSPRK) method ([31], pseudocode 3). The discrete timestep $\Delta t$ is constrained by the Courant-Friedrichs-Lewy criterion with a Courant number of $C_{CFL} = 1.5$. 


2.2 Simulation and diagnostics procedure

The power spectrum $P(f)$ of a field $f$ shown in this work is defined by a summation over each wavenumber shell $K \in \mathbb{N}$. A shell consists of all the wavevectors $k$ such that $K \leq |k|/\lambda < K + 1$ with $\lambda = 2\pi/L$ the smallest wavenumber in the system, giving the spectrum:

$$P(f)_K = \frac{1}{2} \sum_{K \leq |k|/\lambda < K + 1} |\hat{f}_k|^2,$$

(6)

with $\hat{f}_k$ the Fourier coefficient at wavevector $k$. The compressive part of a power spectrum, $P^{\text{comp}}(f)$, considers only the contributions parallel to the wavevector:

$$P^{\text{comp}}(f)_K = \frac{1}{2} \sum_{K \leq |k|/\lambda < K + 1} |\hat{f}_k \cdot k|^2/|k|^2.$$

(7)

The solenoidal part of a power spectrum is accordingly built through the difference $P^{\text{sol}}(f) = P(f) - P^{\text{comp}}(f)$.

The considered turbulent systems exhibit both a direct cascade of kinetic energy from large to small scales and an inverse transfer of magnetic helicity, from small to large scales. The terminology “cascade” is not used here for the magnetic helicity inverse transfer, since this process displays pronounced spectrally non-local features [2, 36, 53]. The statistically stationary turbulent state is generated and sustained through a continuous use here for the magnetic helicity inverse transfer, since this process displays pronounced spectrally non-local features. The statistically stationary turbulent state is generated and sustained through a continuous use here for the magnetic helicity inverse transfer, since this process displays pronounced spectrally non-local features. The statistically stationary turbulent state is generated and sustained through a continuous use here for the magnetic helicity inverse transfer, since this process displays pronounced spectrally non-local features.

The mechanical forcing is applied after the ten stages of the SSPRK integration method, through:

$$dF^\text{OU}_k(t) = -F^\text{OU}_k(t) \frac{dt}{t_{\text{auto}}} + F_0 \left( \frac{2\sigma |k|^2}{t_{\text{auto}}} \right)^{1/2} \frac{\mathcal{P}_\zeta}{\mathcal{W}} \cdot dW(t),$$

(8)

with $t_{\text{auto}} = L/(2c_sM^*)$ the forcing auto-correlation time, where $M^*$ is an a priori estimate of the expected RMS Mach number during the statistically stationary state, $F_0$ an amplitude, $\sigma$ a spectral profile, $d\mathcal{W}(t) = dt\mathcal{N}(0, dt)$ a Wiener process with $\mathcal{N}(0, dt)$ a 3D Gaussian distribution with zero mean and standard deviation $dt$ representing a three-dimensional continuous random walk. The projection operator $\mathcal{P}_\zeta$ allows to control the forcing’s compressivity: for $\zeta = 0$, only components along $k$ are kept so that the forcing is purely compressive, whereas for $\zeta = 1$, the forcing is projected on the plane orthogonal to $k$ in Fourier space and the resulting field is purely solenoidal. The spectral profile is taken as $\sigma(k) = 1$ for the wavenumber shells $1 \leq K \leq 2$ and 0 otherwise so that only the largest scales are forced.

The mechanical forcing is applied after the ten stages of the SSPRK integration method, through:

$$(pV) \leftrightarrow (pV) + \rho \Delta t f_V,$$

(9)

with $f_V = A_V f_V$, where $f_V$ is the $F^\text{OU}$ field in configuration space, multiplied by an amplitude $A_V$. This amplitude is introduced in order to guarantee a constant energy injection rate $\epsilon_{\text{inj}} = \Delta \mathcal{E}^{K}/\Delta t$ and is determined by the largest root of the second-order equation (cf. [39]):

$$\Delta \mathcal{E}^{K} = \frac{1}{2} A^2 \sum_{i,j,k} \rho_{i,j,k} f_{V,i,j,k}^2 + A_V \Delta t \sum_{i,j,k} \rho_{i,j,k} f_{V,i,j,k},$$

(10)

where $\rho_{i,j,k}$ designates the value of field $\rho$ in the numerical cell indexed by $(i,j,k)$. As a consequence of this normalisation, the choice of the constant amplitude $F_0$ in relation (8) is arbitrary. The injected energy cascades to ever smaller scales where it is finally dissipated due to numerical non-ideal effects, which serve as a simple model for physical viscosity and resistivity. Thus, a statistically stationary state is reached with a roughly constant RMS Mach number $M = \sqrt{(\langle |V|^2 \rangle / c_s^2)} \approx M^*$ on a time scale of the order of the turbulent turnover-time $t_T = L/(2c_s M)$ [26]. Hence the forcing auto-correlation time $t_{\text{auto}}$ is taken close to the turbulent large-eddy turnover time, the natural time of propagation of information from the largest to the smallest scales in the turbulent flow. The energy injection rate determines the RMS Mach number, $M$, of the statistically stationary state. The weak fluctuating mean velocity field which appears as a result of the driving is removed at each iteration. Only the strict conservation of momentum (and energy) are guaranteed up to machine precision by the numerical scheme – this does not prevent the secular generation of a finite mean velocity component by the statistical process that represents the turbulence driver. The removal of a mean velocity is important in the presence of magnetic fields which break the Galilean invariance of the system. At a particular instant in time during the turbulent hydrodynamic statistically stationary state, a small scale electromotive helical driving is switched on while maintaining the large-scale mechanical driving. The electromotive forcing injects
The mechanical forcing and the electromotive driving occur at different spatial scales in order to observe the effects of compressibility. Indeed, compressibility effects enter the magnetic helicity dynamics via a term \( \epsilon_{\text{inj}} \) in the shell around which the electromotive driving takes place. Similarly to the mechanical forcing, the resulting field in configuration space, \( f_M \), is applied after the ten stages of the time integration procedure. It is renormalised in a way analogous to \( \tilde{\zeta} t \) in order to guarantee a constant magnetic energy injection rate \( \epsilon_{\text{inj}}^M \) for different forcing specifications.

Regarding the spectral diagnostics, the helicity spectra are defined as the co-spectrum of a field \( f_H \) and its curl \( g = \nabla \times f \):

\[
Q(f,g)_K = \sum_{K \leq |k|/\kappa < K+1} \hat{f}_k \cdot \hat{g}_k \tag{13}
\]

with the asterisk (*) designating the complex conjugate, so that \( \mathcal{P}(f) = \frac{1}{2} Q(f,f) \). The magnetic vector potential is computed through the Coulomb gauge \( \nabla \cdot a = 0 \) so that \( \hat{a}_k = ik \times \hat{b}_k/k^2 \) in Fourier space. Expressed in the helical basis, the magnetic helicity spectrum is \( H^M = H^M - H^M \) with \( H^M \) the projection of the magnetic field on the positive (respectively, negative) helical eigenvector [15]. The Fourier spectra and co-spectra are furthermore normalised in order to refer more easily to physical quantities. The power spectrum of the velocity (the specific kinetic energy spectrum) and the kinetic helicity spectrum (co-spectrum of velocity and vorticity) are normalised by the isothermal sound speed squared \( c_s^2 \) so that \( \mathcal{P}(v)_K = (1/2) \sum_{K \leq |k|/\kappa < K+1} |\hat{v}_k/c_s|^2 \). Similarly, quantities involving the density, such as \( w = \sqrt{\rho} v \) have an additional normalisation by an appropriate power of the mean density \( \rho_0 \). For example, \( \mathcal{P}(w) \) is normalised by \( \rho_0 c_s^2 \). Since the mean density is \( \rho_0 = 1 \), this additional \( \rho_0 \) factor has no direct effect but is still mentioned since the magnetic field \( b \) has the same dimension as \( w \). For this reason, both the magnetic field power spectrum and the magnetic helicity spectrum are normalised by \( \rho_0 c_s^2 \). These normalisation factors are implicitly assumed in the relations (6) and (7) and not written explicitly in order to reduce the amount of notation.

### 2.3 Simulation runs

The forcing parameters used for the simulated runs and the resulting time-averaged RMS Mach number in the hydrodynamic statistically stationary state \( M \) are displayed in Table 1. The number in the runs’ labels stands for their approximate Mach number and the letter (‘s’ or ‘c’) for the forcing used (purely solenoidal with \( \zeta = 1 \) or purely compressive with \( \zeta = 0 \)). The main runs are performed at resolution 512^3 and confirmation at higher resolution.

| Label | \( K_{\text{inj}} \) | \( \zeta \) | \( t_{\text{auto}} \) | \( M \) | \( \epsilon_{\text{inj}}^M \) |
|-------|-----------------|-------|-----------------|-------|-----------------|
| M01s4 | \( 7 \times 10^{-4} \) | 1     | 50              | 0.116 | \( 28 \times 10^{-4} \) |
| M1s2  | \( 7 \times 10^{-4} \) | 1     | 5               | 1.09  | \( 14 \times 10^{-4} \) |
| M5s   | 0.11            | 1     | 1               | 5.06  | 0.11            |
| M7s   | 0.31            | 1     | 5/7             | 7.03  | 0.31            |
| M11s  | 1.21            | 1     | 5/12            | 11.1  | 1.21            |

Table 1: Parameters of the simulation: kinetic energy injection rate \( \epsilon_{\text{inj}}^K \), spectral weight \( \zeta \) governing the forcing’s compressivity, forcing auto-correlation time \( t_{\text{auto}} \), which result in a time-averaged root mean square Mach number \( M \). During the hydrodynamically statistically stationary state, magnetic helicity injection is switched on with a constant magnetic energy injection rate \( \epsilon_{\text{inj}}^M \).
has been done for selected cases, see section 5. In order to investigate subsonic, transonic and supersonic turbulence at various compressibility, \( M \) varies in the range \( \approx 0.1 - 11 \) for the solenoidally-driven runs and \( \approx 1 - 8 \) for the compressively-driven ones. The magnetic-to-kinetic energy injection ratio \( \epsilon_{\text{inj}}^M/\epsilon_{\text{inj}}^K \) is taken as unity for all runs, apart for the M01s4 and M1s2 runs where it is 4 and 2 respectively (as written in the run’s label) in order to obtain a faster convergence of the scaling laws in the inverse transfer region. A parameter study in order to assess the influence of the magnetic-to-kinetic energy injection ratio has been performed in [53], section 7.1 and suggests that the approximate scaling exponents observed in the spectral region of inverse transfer converge to the same values as long as \( \epsilon_{\text{inj}}^M \) and \( \epsilon_{\text{inj}}^K \) are of the same order of magnitude. The rate of convergence increases with a greater magnetic energy injection rate \( \epsilon_{\text{inj}}^M \).

Figure 1 shows mass density slices for the M1s2, M11s, M1c and M8c runs during the hydrodynamic statistically stationary state. Since the mass density is governed by \( \partial_t \rho = -\nabla \cdot (\rho v) \), a compressive forcing with a high \( \nabla \cdot v \) component induces strong mass density variations, visible through regions of very low density for the M8c run and pronounced shock fronts, already present at a RMS Mach number around unity (M1c run). Probability distribution functions (PDF) of the mass density statistics during the hydrodynamic statistically stationary state are shown in figure 2 and display a considerably larger density spread for compressively-driven runs at relatively lower RMS Mach numbers as compared to the solenoidally-driven ones: for example, the spread of the M3c run is comparable to the M11s one. These curves are obtained by averaging over at least forty snapshots equally spaced in time over roughly \( 4T \). A sampling rate higher than the turbulent turnover time is chosen because of the high variability of the density PDFs for the supersonic compressively-driven runs. For compressively-driven turbulence, the snapshot-to-snapshot variations of the density PDFs are significantly greater, as illustrated in figures 2.(c-d). For this reason, the influence of the particular snapshot taken as initial condition for magnetic helicity injection is assessed by taking two different instants in the M5c hydrodynamic
Figure 2: (a-b) Mass density PDF of all transonic and supersonic runs, time-averaged over at least forty $512^3$ snapshots equally spaced in a time interval of about $4T$. (c-d) Snapshot-to-snapshot variations of the mass density PDFs for the M11s and M5c hydrodynamic statistically stationary state. The thick line corresponds to the time average and the thinner lines to selected snapshots during the statistically stationary state. Among these snapshots, the red dashed and blue dotted curves for the M5c run correspond to different initial conditions for the magnetic helicity injection: the M5cA run’s initial condition corresponds to the red dashed curve, whereas the M5cB run’s initial condition corresponds to the blue dotted one.

statistically stationary state. These runs are hereafter labelled respectively “M5cA” and “M5cB” and make use of the initial hydrodynamic state corresponding to the red dashed (respectively blue dotted) curve in figure 2.(d). For all the other runs, only one snapshot in the hydrodynamic statistically stationary state is taken as initial condition for the magnetic helicity injection.

Even though higher-order numerics allow to see comparatively many details as compared to lower-order schemes, the present simulations also exhibit well-known numerical inaccuracies such as, for example, phase and amplitude errors (numerical dissipation, [52]), spurious generation of smallest-scale fluctuations (energy excess towards the grid-scale, [25, 20, 52, 26, 53]) as well as statistical errors due to the finite extent in discretized space and time. Section 5 addresses this issue by testing the robustness of the obtained results.

3 Previous research

An experimental setup similar to the one presented here has led to the observation of several spectral scaling laws for incompressible MHD turbulence [40, 46]. In these works, pseudo-spectral direct numerical simulations with $1024^3$ collocation points have been considered, with a magnetic helicity injection at the wavenumber shells $203 \leq K \leq 209$. Among other quantities, the kinetic helicity $H^V = Q(v, \nabla \times v)$, magnetic helicity $H^M = Q(a, b)$, kinetic energy (which is the specific kinetic energy $E^V = P(v)$ in the incompressible case with a uniform $\rho = 1$) and magnetic energy $E^M = P(b)$ exhibit scaling laws $\sim K^m$ with an exponent $m \approx -0.4, -3.3, -1.2$ and $-2.1$ respectively. For the magnetic helicity, the only ideal invariant among the above-mentioned quantities, a dimensional argument “à la Kolmogorov” leads to a spectral exponent $m = -2$, which is clearly at variance from the observed $-3.3$. This exponent can be derived assuming a constant flux of magnetic helicity and the
Figure 3: (a-b) Time evolution of the kinetic, potential, magnetic and total energies for two representative M5s and M5cA runs. (c-d) Time evolution of the magnetic field-velocity alignment (normalised cross-helicity).

locality of transfers so that $[a]/[T] = \frac{L^2}{T}$ with $[\cdot]$ meaning the dimension, $T$ and $L$ time and length dimensions respectively and using $[\cdot] \sim [v] = L/T$ in the incompressible case [49].

Furthermore, based on the Eddy Damped Quasi-Normal Markovian closure model ([32, 48] extended to helical MHD in [49]), a dynamical equilibrium between shearing and twisting effects leads to the following relation [40, 46]:

$$\left( \frac{E^V}{E^M} \right)^\gamma \propto \frac{H^V}{H^M}, \quad (14)$$

with $H^J = Q(b, \nabla \times b) = \left( \frac{2\pi K}{2} \right)^2 H^M$ the current helicity and $\gamma$ a constant exponent, discussed below. This balance has also been observed (with $\gamma = 1$) in the direct transfer region of decaying MHD turbulence with a large scale helical magnetic field [44, 30] and has been interpreted as “partial Alfvénization of the flow” [30]. It is henceforth referred to as “Alfvénic balance”.

While the above-mentioned power-law exponents for the kinetic and magnetic energies and helicities are consistent with the Alfvénic balance for $\gamma = 1$ [40], a later work mentions that relation (14) is better verified with an exponent $\gamma = 2$ [46]. This could be due to the fact that the spectral domains where the scaling laws are followed are slightly different for the different quantities.

The present work looks for scaling laws and aims at extending relation (14) to the compressible case.

4 Results

4.1 Structure formation

As magnetic energy is injected and transferred to larger scales, the total energy of the system tends to grow secularly (figure 3.(a-b)). In compressible isothermal MHD, the total energy $E^T$ is the sum of the kinetic
\[ \mathcal{E}^K = \frac{1}{2} \rho |\mathbf{v}|^2, \] magnetic \[ \mathcal{E}^M = \frac{1}{2} |\mathbf{b}|^2 \] and potential energy \[ \mathcal{E}^\rho = \rho \frac{\rho_e}{\rho_0} \ln \rho/\rho_0, \] stored as density fluctuations. Since no large-scale energy removal occurs, the asymptotic state of the system would be a condensate of magnetic helicity at the largest scales. This would strongly affect the locality of transfers and the spectral exponents \[ [2, 36]. \] For this reason, in order to avoid the strong modification of the inverse transfer range by finite-size effects, the spectra are taken at an instant when an energetically dominant largest-scale magnetic structure has not yet emerged. The restricted spectral bandwidth of the numerical experiments thus requires caution with regard to the assumed stationarity of the established asymptotic scaling interval at \( 20 \leq K \leq 30 \) or \( 15 \leq K \leq 25 \).

Any large-scale damping of magnetic energy or a magnetic large-scale condensate of finite energy would have a finite spectral width of influence reducing the observable scaling region from the large-scale side as does the finite spectral support of the magnetic driving from the small-scale side. The chosen setup, although certainly not ideal, represents a pragmatic compromise between unquestionable statistical stationarity and a perfectly developed scaling region. Nevertheless, spectral scaling laws consistent with results from previous research in incompressible MHD and natural extensions of the same to compressible MHD have been found. Furthermore, the main results are confirmed through higher resolution runs (section 5) and the main measurements of spectral exponents are presented as functions in time displaying acceptable saturation to constant levels (cf. figures 7, 8, 9 and 11).

For all the runs, the magnetic field-velocity alignment,

\[ \rho_C = \frac{(\mathbf{v} \cdot \mathbf{b})}{\sqrt{|(\mathbf{v} \cdot \mathbf{v})|(\mathbf{b} \cdot \mathbf{b})}} \] (15)

grows in time but remains very low (under 0.035, see figure 3,(c-d)), so that the average alignment of magnetic and velocity field is small and the dynamics of the system are dominated by the direct cascades of kinetic and magnetic energies and the inverse transfer of magnetic helicity.

Figure 4.(a) shows the total magnetic helicity in the system as a function of time for the M1s2, M1l3 and M8c runs, taken as representatives. The helicity is normalised by an estimate of its injection rate: \( \epsilon_{\text{inj}}^M = 2e_{\text{inj}}^M/(K_HT) \) with \( K_H = 50 \) the shell around which the electromotive forcing takes place. Therefore, if all the injected magnetic helicity would be conserved, the normalised magnetic helicity in the system should be 1 at \( t/T = 1 \). Indeed, at early times, the total magnetic helicity follows this ideal evolution (indicated through the orange dotted line). However, due to numerical non-idealities, magnetic helicity is dissipated already at the forcing scale, and even partially transferred to smaller scales, where it is dissipated more rapidly, giving an effective rate of increase smaller than unity. The rate of increase is around 40 to 60% for these runs, since at \( t/T = 1 \) about half of the injected magnetic helicity is present in the system. These values are consistent with the ones obtained by considering the magnetic helicity flux \( \Pi^M(K) = \sum_{K_0=1}^{K} T^M(K_0) \) with \( T^M = 2Q(b, v \times b)/\epsilon_{\text{inj}}^M \) the magnetic helicity transfer function normalised by its injection rate (figures 4.(b-d)). For these curves, an approximate plateau is visible at larger scales, showing that about 50 till 70 percent of the injected magnetic helicity experience the inverse transfer, the rest being transferred to smaller scales or dissipated at the forcing scale. This plateau hints at a roughly constant flux of magnetic helicity to larger scales, even though a visible deviation exists that is probably due to nonlocal transport \([2, 36, 53]\) and the finite size of the system. Not all the magnetic helicity transferred to larger scales is conserved, giving hence a lower rate of increase, since dissipation still occurs, especially when magnetic helicity has not yet reached the largest scales. This is why at later times the curves are not linear but convex, as can be seen through the tangent of the M1s2 curve, which lies under it. Indeed, as magnetic helicity is transferred to ever larger scales, dissipative effects are smaller so that magnetic helicity is better conserved.

This inverse transfer of magnetic helicity in supersonic flows is illustrated in figure 5, where the time evolution of the magnetic helicity spectra \( H^M \) is shown for the least and most compressible cases considered. Figure 6 shows the time evolution of the magnetic helicity integral scale, defined as:

\[ I_{H^M} = L \int_K K^{-1} H^M_K dK \]

(16)

Magnetic helicity is not a positive-definite quantity. Even though division by zero is in theory possible, since only magnetic helicity of one sign is injected, positive magnetic helicity dominates the system at all scales so that division by zero does not occur and considering this integral scale remains meaningful. The spectrum of the positive helical magnetic field \( H^+_M \) is about an order of magnitude larger than its negative part \( H^-_M \) at all scales. The time is measured in units of the turbulent turnover time \( T = L/(2c_L\mathcal{M}) \) of the hydrodynamic statistically stationary state of the flow, so that \( \mathcal{M} \) designates the RMS Mach number before the injection of magnetic helicity. The inverse transfer occurs faster with increasing compressibility. While for the solenoidally-driven runs, more than 1.5\( T \) are required to reach the integral scale \( I_{H^M} = \frac{L}{2}, \) less than about 0.6\( T \) are needed for the compressively-driven runs with \( \mathcal{M} \gtrsim 5 \). For lower densities indeed, the Lorentz force has a greater impact on the velocity field’s temporal evolution, since \( \partial_t \mathbf{v} = \ldots + \frac{1}{\rho} (\nabla \times \mathbf{b}) \times \mathbf{b}, \) which backreacts on the
Figure 4: (a) Evolution of the total magnetic helicity in the system, normalised by its estimated injection rate for some representative runs. The orange dotted line has a slope of 1, indicating the ideal evolution. The black dotted line is the tangent at the instant $t = 0.4t_f$ of the M1s2 run, showing that these curves are convex at later times. (b-d) Magnetic helicity fluxes as a function of time for these three runs, 5 curves regularly spaced in the interval $[t_i, t_f]$ with $t_i$ the instant at which the electromotive driving is switched on and $t_f$ an instant at which $H_M \approx L/6$ – see relation (16). The darker the curve, the further time has advanced. The dotted line indicates zero on the ordinate.
Figure 5: Time evolution of the magnetic helicity spectra for the least and most compressible runs. The darker the curve, the later it is. The ten curves are equally-spaced in time and correspond to the time-intervals $t \in [0.335т, 0.35т], [0.22т, 0.24т], [0.24т, 0.26т]$ and $[0.16т, 0.18т]$ for the M01s4, M11s, M1c and M8c runs, respectively.

Figure 6: Magnetic helicity integral scale time evolution.
magnetic field. At high compressibility, especially for compressively-driven runs, large regions of low density are present (figure 1), which are associated with higher Alfvén velocities $v_A = b/\sqrt{\rho}$ and hence faster dynamical timescales of magnetic field reconfiguration $\propto l_{rec}/v_A$, with $l_{rec}$ characterising the field-parallel extent of the magnetic reconnection region. Magnetic reconnection processes are required to achieve the topological changes of magnetic structures in the course of the inverse transport of magnetic helicity. Those changes are communicated along the affected field-lines by Alfvén-waves propagating at speed $v_A$. This explains why for the M5cB run, whose initial condition presents a peak of the density PDF at lower densities (figure 2,(d)), the growth of the magnetic helicity integral scale is faster than for the M5cA run. This is also manifest in the time evolution of the magnetic helicity spectra: for the M5c run, in contrast to the M01s4, M1s2, M5s, M7s, M11s runs respectively, expressed in units of their respective large-eddy turbulent turnover time. For the compressively-driven M1c, M3c, M5cA, M5cB, M8c runs, the respective instants are $t/\tau_T = 1.71, 0.95, 0.66, 0.42, 0.62$. This scale of $L/6$ is chosen in order to have a sufficiently large spectral distance between the magnetically forced and the largest scales, since a strong modification of the scaling range by the electromotive forcing at small scales and through large-scale condensation of magnetic helicity at the largest scales at late times due to the limited box size are expected. The magnetic-to-kinetic energy ratio $\mathcal{P}(b)/\mathcal{P}(\sqrt{\rho}v)$ is of the order $10^{-2}$ at the largest scales $K < 3$ due to the dominance of the kinetic forcing and the deliberate avoidance of a large-scale magnetic condensate. The spectral power-law exponent is determined by a logarithmic linear least squares fit (LSF) to the spectrum in a relevant domain, which depends on the run and the considered quantity. This domain is shown through vertical dashed lines in the figures.

As expected from the presence of $\nabla \cdot \mathbf{v}$ terms in the nonlinear evolution equations of magnetic helicity and magnetic energy (cf. equation (1)), the respective spectra display systematic changes when augmenting the Mach number of the flow. Those are more pronounced for compressively-driven turbulence than for solenoidally-driven flows. The low Mach number solenoidally-driven M01s4 and M1s2 runs exhibit an approximate magnetic helicity power-law scaling $H^M \sim K^\nu$ with $\nu \approx -3.3$, gained from a LSF in the region $20 \leq K \leq 30$ (figure 7). This exponent is consistent with the observations in the incompressible case [40, 46]. For higher RMS Mach numbers, increasing deviations are observed, namely $\nu \approx -3.2, -3.1$ and $-3.0$ for the M5s, M7s and M11s runs respectively. Although significant, these deviations are clearly smaller than those obtained in the compressively-forced runs. There, the spectra are clearly flatter at lower Mach numbers, see figure 7. For the M1c, M3c, M5cA and M8c runs, the exponents computed through a LSF in the region $15 \leq K \leq 25$ are $\nu \approx -3.2, -2.6, -2.2$ and $-2.3$ respectively. For supersonic compressively-driven flows, the scaling exponent is very sensitive to the initial conditions: for example, the M5cB run, for which the magnetic helicity injection starts at a different instant during the same hydrodynamic statistically stationary state as the M5cA run, presents a significantly flatter spectrum with $\nu \approx -1.5$. This is linked with a peak of the initial condition’s mass density PDF at lower densities (figure 2.(d)), so that faster timescales are expected for the M5cB run. This illustrates that the single realisations considered here could still be in a transient state. One would either need a significantly larger scale separation between the electromotive driving scale and the largest scales (which implies a very large resolution of the computational domain), or perform ensemble averaging over many realisations in order to obtain a less approximate dependency of the magnetic helicity exponent as a function of $M$ when using a compressive forcing.

These studies would imply significantly higher computational costs, which are beyond the resource available to the present investigation. Nevertheless, independently of possible transients, a clear tendency towards flatter exponents with increasing compressibility is observable, which can be largely compensated by an appropriate choice of variables, even with very different initial conditions, as shown below.

The spectra of the positive helical magnetic field $H^M_+ = H^M - H^M_-$ are close to those of the total magnetic helicity $H^M = H^M_+ - H^M_-$, which is expected since $H^M_+$ is about an order of magnitude greater than $H^M_-$ at all scales. After a time of the order of $\tau_T$, the exponent measured are close to $-3.2, -3.2, -3.1, -3.0$ and $-2.9$ for the M01s4, M1s2, M5s, M7s and M11s runs respectively and close to $-3.0, -2.6, -2.1, -1.5$ and $-2.3$ for the M1c, M3c, M5cA, M5cB and M8c runs respectively. The negative helical magnetic field $H^M_-$ scales as $-2.7, -2.5, -2.1, -2.2, -2.0$ (M01s4, M1s2, M5s, M7s, M11s runs) and $-2.4, -2.0, -1.7, -1.1, -1.7$ (M1c, M3c, M5cA, M5cB, M8c runs).

The same tendency is observed for the magnetic energy spectra: the scaling exponents are close to the $-2.1$ observed in the incompressible case [40, 46] for the low Mach number M01s4, M1s2 and M1c runs, flatter with
Figure 7: Top: Time evolution of the magnetic helicity power-law exponents. Bottom: magnetic helicity spectra at an instant when $I_M \approx \frac{t}{t_T}$ (shown through circles in the top plots), compensated by $K^{3.3}$, since $H^M \sim K^{-3.3}$ in the incompressible case. The exponents measured in the top plots are gained through a LSF in the region delimited with vertical dashed lines in the bottom plots (here $20 \leq K \leq 30$ and $15 \leq K \leq 25$ for the solenoidally- and compressively-driven runs respectively).
Figure 8: Same as figure 7, but for the Alfvénic helicity $H^A$ which exhibits a $K^{-1.3}$ scaling in the incompressible case.
increasing compressibility for the solenoidally-driven runs and significantly flatter for the compressively-driven runs already at lower initial RMS Mach numbers (curves not shown here).

In the incompressible case, the Alfvén velocity $v_A = b/\sqrt{\rho}$ (with $\rho$ a constant) and the magnetic field are in essence the same quantity. However, in the compressible case, low density regions are associated with higher Alfvén velocities, so that the dynamical timescales are shorter there. For this reason, it is tempting to consider the power spectra of the Alfvén velocity $E_A = P(v_A)$ as well as the co-spectrum of the Alfvén velocity and its curl $H_A = Q(v_A, \nabla \times v_A)$, which is the helicity of the Alfvén velocity, a quantity hereafter called “Alfvénic helicity”. In the incompressible case, the Alfvénic helicity is $b \cdot j/\rho$ and corresponds thus to the current helicity $H_J = \left( \frac{2\pi L}{K} K \right)^2 H_M$, which exhibits an approximate scaling law $H_J \sim K^{-1.3}$ [40, 46].

The Alfvénic helicity and Alfvén velocity power spectra display only a weak dependence on the compressible character of the forcing and the flow. While the magnetic energy and helicity spectra become flatter with increasing compressibility, for the Alfvénic helicity the incompressible scaling exponent $-1.3$ is approximately observed for all simulations with $M \gtrsim 3$ as well, see figure 8). This is in particular the case for the M5cA and M5cB runs, which present very different $H_M$ scaling exponents. The scaling exponents of $E_A$, which approach asymptotically $-1.8$ for the most compressible runs (see figure 9), are also closer to the $-2.1$ scaling for the magnetic energy in the incompressible case [40, 46] and present a weaker dependence on the flow’s compressibility as compared to $E_M$.

This universality of the exponents over a wide range of compressibility suggests a systematic scale-dependent correlation between the density and the magnetic fields, which is the object of further investigations. The M1c run may present different dynamics since its velocity compressive ratio $P_{comp}(v)/P(v)$ remains large in the inverse transfer region (see figure 10). This is reflected by the lack of low density regions in this simulation.

4.3 Scaling relations: kinetic quantities

In the incompressible case, the kinetic energy and helicity exhibit approximate scaling laws with exponents close to $-1.2$ and $-0.4$ respectively [40, 46]. In the compressible case, several alternative definitions for the incompressible kinetic energy have been considered in the literature, e.g. $E^V = P(u)$, i.e. the specific kinetic energy spectrum, $E^K = P(w)$, with $w = \sqrt{\rho} v$, which is dimensionally a kinetic energy spectrum, $E^U = P(u)$, with $u = \rho^{1/3} v$, a quantity for which the Kolmogorov spectrum with exponent $-\frac{5}{3}$ is recovered in compressible
Figure 10: (a-b) Time-averaged velocity compressive ratio during the hydrodynamic statistically stationary state, exhibiting values close to 1/3 for the solenoidally-driven supersonic runs and 1/2 for the supersonic compressively-driven ones in a reasonable wavenumber range, consistently with [27]. (c-d) Snapshots of the compressive ratio at an instant when $\frac{I_{HM}}{L} \approx L/6$: all supersonic runs exhibit a compressive ratio close to 0.2, independently of the mechanical forcing type. The M1c run continues to exhibit a large compressive ratio.
Figure 11: Same as figure 7, but for the solenoidal part of the specific kinetic energy spectrum $E_{V,\text{sol}}$ which exhibits a $K^{-1.2}$ scaling in the incompressible case.
Figure 12: Top: Kinetic helicity spectra $H^V$ at an instant when $I_{HM} = \frac{L}{19}$ compensated by $K^{0.4}$. Bottom: Time evolution of $H^V$, compensated by $K^{0.4}$, for a low Mach number run without large-scale mechanical driving, with $\mu_{M} = 7 \times 10^{-4}$ (for which $M \approx 0.2$). The curves are equally spaced in time from an instant when $I_{HM} \approx L/19$ till $I_{HM} \approx L/3$. The darker the curve, the larger $I_{HM}$.

From the above-mentioned quantities, the specific kinetic energy spectrum $E^V = P(v)$ and its solenoidal part $E^{V,\text{sol}}$ (shown in figure 11) present the most universal behaviour, with an exponent close to $-1.2$ for all the runs. The deviations from the $-1.2$ exponent become weaker at later times for the M01s4 and M1s2 runs and the stronger deviations for the M1c run are due to the high velocity spectrum compressive ratio. The other above-mentioned spectra, as well as that of the density, are displayed in appendix A.

As for the kinetic helicity, the $-0.4$ incompressible exponent appears to be the asymptotic behaviour for all levels of higher compressibility (figure 12). The negative kinetic helicity values at large scales for some runs are not an established feature of the inverse transfer experiment: their sporadic occurrence is a consequence of the random mechanical forcing. The spectral transient region influenced by the small-scale magnetic driving is wider at lower Mach numbers. This is probably linked with the presence of a large-scale mechanical driving in the present experiments, as compared to those of [46]. In hydrodynamic turbulence, the energy cascade tends to an equipartition of energy in the positive and negative helical parts of the velocity field at small scales, so that the kinetic helicity tends to zero [17]. On the other hand, the injection of magnetic helicity at small scales leads to the generation of kinetic helicity. It is plausible that for runs with pronounced compressible effects and low density regions, the dynamical timescale of the kinetic helicity production is faster than that of the direct energy cascade leading to vanishing kinetic helicity, so that the $-0.4$ exponent observed in the absence of large-scale mechanical driving is recovered, but only for the supersonic flows. As shown in figure 12, bottom, a low Mach number run performed in the absence of a large-scale mechanical driving gives a kinetic helicity scaling compatible with $-0.4$.  

hydrodynamic turbulence with a low $\nabla \cdot v$ component [28, 33, 27, 26], $\frac{1}{2} Q(\rho v, v)$, the co-spectrum of velocity and momentum (or more generally, any co-spectrum $\frac{1}{2} Q(\rho^\alpha v, \rho^{1-\alpha} v)$ with $\alpha \in [0, 1]$ [4]), or $E^{V,\text{sol}} = P^{\text{sol}}(v)$, the solenoidal part of the specific kinetic energy spectrum.
4.4 Alfvénic balance

The observations presented so far suggest that the Alfvénic balance found in the incompressible case (relation (14)) can be extended to the compressible case when considering, instead of the current helicity $H^J$ and the magnetic energy $E^M$, the Alfvénic helicity $H^A$ and the power spectrum of the Alfvén velocity $E^A$ respectively, and by considering the specific kinetic energy $E^V$ or its solenoidal part. Since the energy associated with $H^A$ and the kinetic helicity $H^V$ is the solenoidal part of $E^A$ and of $E^V$, respectively, the following extension of the Alfvénic balance in the compressible case is proposed:

$$\left(\frac{E^{V,sol}}{E^{A,sol}}\right)^\gamma \propto \frac{H^V}{H^A}.$$

(17)

This choice is a straightforward extension of the balance in the incompressible case, where only solenoidal modes exist. It suggests that the shear, not the compressive waves, balance the twisting in compressible turbulence as well (cf. text around relation (14)). To test this relation, the ratio

$$\Lambda(r_H,r_E,\gamma) = \frac{r_E^{\gamma}}{r_H},$$

with $r_H = r_{H^A} = H^V/H^A$ and $r_E = r_{E^{V,sol}} = E^{V,sol}/E^{A,sol}$ is plotted in figure 13 for the different runs and different $\gamma$. This relation is well followed for the least compressible M01s4, M1s2 and M1c runs with $\gamma = 2$, with $\Lambda \approx 1.1 \pm 0.3$ on the domain $20 \leq K \leq 44$, consistently with [46]. For the most compressible cases, the curves for the M3c, M5cA M5cB and M8c runs show a similar behaviour for $\gamma = 1$, with $\Lambda \approx 2.7 \pm 0.5$ on the $13 \leq K \leq 33$ domain. Regarding the supersonic solenoidally-driven runs, they present an intermediate behaviour: for $\gamma = 2$, the M5s curve would be included in the “M01s4-M1s2-M1c bundle” on the domain $12 \leq K \leq 40$, whereas for $\gamma = 1$, the M11s curve would be included in the “M3c-M5c-M8c bundle” on the domain $16 \leq K \leq 31$.

Replacing the solenoidal parts $E^{V,sol}$ and $E^{A,sol}$ by the total energies $E^V$ and $E^A$ in $\Lambda$ also gives a good horizontal for the high Mach number compressively-driven runs. The “M3c-M5c-M8c bundle” verifies then $\Lambda \approx 3 \pm 0.5$ on the same $13 \leq K \leq 33$ domain. With this choice however, the M1c curve does not stay in the “M01s4-M1s2” bundle, which is why the choice of relation (17) is preferred.

Although other variants of relation (17) are partially consistent with the data as well, this choice shows overall the best agreement with the simulations, as well as the least spread between the curves. This also holds at higher numerical resolution, which is not the case for some other possibilities, as shown in section 5.

5 Confirmation at higher resolution

In addition to the already mentioned inaccuracies of the present numerical experiments (see section 2.3), in the framework of the fallback approach, not all cells are reconstructed at higher order. For the most compressible M11s and M8c runs, a little more than 50% of the reconstructions occur indeed at third order or lower. It is hence important to check that the results presented here are reasonably well converged, both with respect to the resolution and the numerical scheme’s order. Therefore, some additional runs are considered: the M1s2LR and M1s2HR runs, which are lower resolution (256$^3$) and higher resolution (1024$^3$) runs respectively and the M1s2LO run which is done at resolution 512$^3$ but using a second-order scheme. Similarly, additional runs for the M8c case are considered, labelled analogously “M8cLR”, “M8cHR” and “M8cLO”. For this supersonic compressively-driven run, the timestep $\Delta t$ becomes very small at higher resolutions, due to very high Alfvén speeds in low density regions, so that only an early instant in time is considered, when $T_{HM} \approx 0.045L$, but where scaling behaviour is already observed.

Finer structures are visible for the M1s2 family in the magnetic helicity density slices with increasing resolution and/or using a higher-order scheme (figure 14), as a result of accuracy gain and significant reduction of numerical dissipation. The level of detail of the 512$^3$ lower-order run is very similar to the one of the 256$^3$ higher-order run, showing the benefits of using higher-order numerics: even though they are computationally more expensive at a given resolution, they allow a comparable accuracy at lower resolution, resulting in a net performance gain.

For the M1s2 family, a scaling exponent consistent with $H^M \sim K^{-3.3}$ is observed as well for the higher resolution run (figure 15.(a)). The lower resolution and lower order runs also seem to converge to this exponent, but more slowly (figure 15.(b)). The Alfvénic balance (relation (17)) is also well verified with $\gamma = 2$ (figure 15.(c)). Similarly, for the M8c family, the scaling exponents for the Alfvénic helicity are consistent with $H^A \sim K^{-1.3}$ for all the runs and the Alfvénic balance (17) looks well converged for $\gamma = 1$ (figures 15.(d, e)). For these supersonic runs, the lower-order run M8cLO seems to deviate less than the lower resolution M8cLR one as compared to the M8cHR run, in contrast with the M1s2LR and M1s2LO runs, which both give significant deviations from the 512$^3$ and 1024$^3$ fourth-order runs. This means that the effect of lowering the order has less impact on numerical dissipation, than changing the spatial resolution in case of shock fronts which should actually
Figure 13: Test of the Alfvénic balance (relation (17)) by plotting $\Lambda(r_H, r_E, \gamma) = \frac{r_E}{r_H}$ (relation (18)) with $r_H = r_{H^V} = H^V/H^A$ and $r_E = r_{E_{A,\text{vol}}} = E_{V,\text{sol}}/E_{A,\text{sol}}$ for different runs and different exponents $\gamma$: (a) For the subsonic and transonic runs, $\gamma = 2$, compatible with the results in the incompressible case. (b) For the supersonic compressively-driven runs, which verify the Alfvénic balance for $\gamma = 1$. (c-d) For the supersonic solenoidally-driven runs, which present an intermediate behaviour with respect to the $\gamma$ exponent.
Figure 14: Magnetic helicity density slices, normalised by the mean magnetic helicity in the system, at an instant when $I_{h_M} \approx \frac{1}{2} L$. 
dominate dissipation for M8c. Hence, at second order, the reconstruction is already at its limit in resolving the shock fronts at the given resolution. Thus the results regarding shocks and dissipation can not be improved by raising the order. This does however not mean that we do not get significant improvements in smooth regions.

Other variants of the Alfvénic balance, shown in figures 15, (f-h) with different choices for $(r_H, r_E, \gamma)$ in relation (18), which also present a horizontal line at resolution $512^3$, present deviations from the horizontal at the higher $1024^3$ resolution as well as a greater resolution dependence. This hints to a better validity of the variant expressed in relation (17).

These observations suggest that even though studies at higher resolutions would be beneficial to capture all the turbulent fine structures, the observed scaling laws are quite robust.

6 Conclusion

Similarly to the incompressible case, spectral scaling laws are observed for the magnetic helicity inverse transfer in supersonic compressible isothermal MHD turbulence. This has been shown through direct numerical simulations, starting with mechanically driven statistically stationary states of hydrodynamic turbulence, with RMS Mach numbers ranging from about 0.1 to 11 for a purely solenoidal forcing and from about 1 to 8 for a purely compressive driving, to which small scale magnetic helical fluctuations are injected. The higher-order numerics allow results of convincing accuracy already at resolution $512^3$: in spite of well-known numerical inaccuracies (numerical dissipation, energy excess towards the grid-scale or "bottleneck effect"), the main results seem well converged when comparing to $256^3$ and $1024^3$ runs. Of course, more investigations are needed to solidify the findings of this study.

The scaling exponents obtained are generally in good agreement with those found in the incompressible case for subsonic and transonic solenoidally-driven flows. In particular, the magnetic helicity spectrum goes as $K^m$ with $m \approx -3.3$. The spectra become increasingly flatter with higher compressibility. The deviations are however relatively small for the solenoidally-driven runs even at RMS Mach numbers of the order of ten as compared to the purely compressive driving, where the spectra are significantly flatter already at a RMS Mach number around three in the initial hydrodynamic statistically stationary state. Due to the relatively limited scale separation between the electromotive driving and the largest scales, these results should be considered with appropriate caution, since transient behaviours as well as finite-resolution effects could still play a role.

In the interstellar medium, both cases of a purely solenoidal or a purely compressive forcing are very unlikely to happen [27]. The present results suggest that effects of compressibility have a significant impact on the magnetic helicity dynamics in astrophysical systems of interest already at relatively low RMS Mach numbers, especially in situations where the turbulence drivers are rather compressive.

An appropriate change of variable, namely considering the Alfvén velocity $v_A = b/\sqrt{\rho}$ instead of the magnetic field, alleviates the differences between the compressible and the subsonic/transonic solenoidally-driven runs. The co-spectrum of the Alfvén velocity and its curl, named here “Alfvénic helicity” presents for all the runs but the M1c one a similar scaling, consistent with the incompressible case one, going as $K^m$ with $m \approx -1.3$. Even though this result is observed over a small range in spectral space, it suggests an universality in the inverse transfer dynamics, valid over a wide range of compressibility.

Lastly, the “Alfvénic balance” (a quasi-equipartition in terms of magnetic and kinetic energies and helicities) observed in incompressible MHD has been extended using an appropriate change of variables. Further investigations of the balance relation are under way.

In the present numerical experiments, magnetic helicity of one sign dominates the system at all scales. In astrophysical systems of interest however, magnetic helical components of mixed signs are expected to be present. In this context, the scaling laws may differ. Nevertheless, the preliminary study presented here in the simpler case of one helical sign injection provides insights which should help the interpretation of future results in more general cases.

The authors acknowledge the North-German Supercomputing Alliance (HLRN) for providing HPC resources that have contributed to the research results reported in this paper. Computing resources from the Max Planck Computing and Data Facility (MPCDF) are also acknowledged. JMT gratefully acknowledges support by the Berlin International Graduate School in Model and Simulation based Research (BIMoS).

Declaration of Interests. The authors report no conflict of interest.

A Spectra linked with kinetic energy

As mentioned in section 4.3, the (specific) kinetic energy spectrum in the incompressible case where the mass density $\rho$ is constant can correspond to several quantities in the compressible case. Figure 16 shows some possibilities not displayed in the main text, at an instant when $L_t \approx L/6$. The spectra are compensated by $K^{1.2}$ since the scaling observed in the incompressible case is $E^V = P(v) \sim K^m$ with $m \approx -1.2$ [46]. This $E^V$
Figure 15: Confirmation of results through a resolution/numerical order study. The LR and HR labels stand for lower ($256^3$) and higher ($1024^3$) resolution respectively, and the LO for lower-order (second-order) numerics. The (a-c) subfigures above the dashed line are for the M1s2 family, the (d-h) subfigures under the line for the M8c family. (a) Compensated magnetic helicity spectra for the M1s2 family; the dashed vertical lines delimiting $20 \leq K \leq 30$ show the domain where the LSF occurs in subfigure (b), which shows the time evolution of the magnetic helicity scaling exponent. (c) Verification of the Alfvénic balance. (d) Compensated Alfvénic helicity spectra for the M8c family. (e-h) Different Alfvénic balance variants (where $\Lambda(^r_r H, ^r_E \gamma) = r_{E}^{\gamma} / r_{H}$, see relation (18)) for the M8c family, with $r_{H}^{\gamma} = H^{\gamma} / H$, and similarly $r_{E}^{A} = E^{A} / E_{M}$, $r_{E}^{K} = E^{K} / E^{M}$. 

22
scaling is well verified in the supersonic flows studied here as well. Deviations are observed when considering \( P(\rho^{1/3} v) \), \( P(\rho^{1/2} v) \) or the co-spectrum \( \frac{1}{2} Q(\rho v, v) \).

Spectral data of the mass density field are displayed in figure 17. The time evolution of the exponents displays large variations on timescales smaller than \( t_T \), especially for the supersonic compressively-driven runs, which are very probably associated with sound waves and shocks. The spectral exponents are generally between \(-1\) and \(-0.5\) for solenoidally-driven flows and steeper (between \(-1.7\) and \(-1\)) for compressively-driven ones. As the Mach number is decreased the spectral interval directly influenced by the small-scale forcing appears to become wider, reducing the observable scaling region.

References

[1] A. Alexakis and L. Biferale. Cascades and transitions in turbulent flows. *Physics Reports* 767-769, pages 1–101, 2018.

[2] A. Alexakis, P. D. Mininni, and A. Pouquet. On the inverse cascade of magnetic helicity. *The Astrophysical Journal* 640, pages 335–343, 2006.

[3] H. Alfvén. On the existence of electromagnetic-hydrodynamic waves. *Arkiv för Matematik, Astronomi och Fysik* 29B(2), pages 1–7, 1942.

[4] H. Aluie. Scale decomposition in compressible turbulence. *Physica D* 247, pages 54–65, 2013.

[5] H. Aluie. Coarse-grained incompressible magnetohydrodynamics: analyzing the turbulent cascades. *New Journal of Physics* 19, 025008, 2017.

[6] D. Balsara and A. Pouquet. The formation of large-scale structures in supersonic magnetohydrodynamic flows. *Physics of Plasmas Vol. 6 No. 1*, pages 89–99, 1999.

[7] D. S. Balsara. Multidimensional HLL E Riemann solver: Application to Euler and magnetohydrodynamic flows. *Journal of Computational Physics* 229, pages 1970–1993, 2010.

[8] D. S. Balsara. Self-adjusting, positivity preserving high order schemes for hydrodynamics and magnetohydrodynamics. *Journal of Computational Physics* 231, pages 7504–7517, 2012.

[9] M. A. Berger. Magnetic helicity in a periodic domain. *Journal of Geophysical Research 102 No. A2*, pages 2637–2644, 1997.

[10] M. A. Berger. Introduction to magnetic helicity. *Plasma Physics and Controlled Fusion 41*, pages B167–B175, 1999.

[11] J. W. Bieber, P. A. Evenson, and W. H. Matthaeus. Magnetic helicity of the Parker field. *The Astrophysical Journal, 315*, pages 700–705, 1987.

[12] D. Biskamp. *Nonlinear Magnetohydrodynamics*. Cambridge University Press, 1993.

[13] A. Brandenburg. The inverse cascade and nonlinear alpha-effect in simulations of isotropic helical hydro-magnetic turbulence. *The Astrophysical Journal* 550, pages 824–840, 2001.

[14] A. Brandenburg and A. Lazarian. Astrophysical hydromagnetic turbulence. *Space Science Reviews 178*, pages 163–200, 2013.

[15] A. Brandenburg and K. Subramanian. Astrophysical magnetic fields and nonlinear dynamo theory. *Physics Reports* 417, pages 1–209, 2005.

[16] P. Buchmüller and C. Helzel. Improved accuracy of high-order WENO finite volume methods on cartesian grids. *Journal of Scientific Computing 61*, pages 343–368, 2014.

[17] Q. Chen, S. Chen, and G. L. Eyink. The joint cascade of energy and helicity in three-dimensional turbulence. *Physics of Fluids 15*, pages 361–374, 2003.

[18] M. Christensson and M. Hindmarsh. Inverse cascade in decaying three-dimensional magnetohydrodynamic turbulence. *Physical Review E 64*, 056405, 2001.

[19] P. Colella and P. R. Woodward. The piecewise parabolic method (PPM) for gas-dynamical simulations. *Journal of Computational Physics* 54, pages 174–201, 1984.
Figure 16: Different variants corresponding to the (specific) kinetic energy in the incompressible case (where $P(v) \sim K^{-1.2}$), taken at an instant when $I_{H_M} \approx L/6$. 
Figure 17: Power spectrum of the density at the same instant for which $\mathcal{I}_M \approx L/6$ (bottom), and time evolution of its scaling exponent in the region delimited by vertical black dashed lines, $15 \leq K \leq 25$ (top). The subsonic run M01s4 for which the density is everywhere close to the mean density $\rho_0 = 1$ is not shown.
[20] W. Dobler, N. E. L. Haugen, T. A. Yousef, and A. Brandenburg. Bottleneck effect in three-dimensional turbulence simulations. *Physical Review E* 68, 026304, 2003.

[21] B. G. Elmegreen and J. Scalo. Interstellar turbulence I: Observations. *Annual Review of Astronomy and Astrophysics* 42, pages 211–273, 2004.

[22] W. M. Elsässer. Hydromagnetic dynamo theory. *Reviews of Modern Physics* 28, No. 2, pages 135–163, 1956.

[23] D. F. Escande, P. Martin, S. Ortolani, A. Buffa, P. Franz, L. Marrelli, E. Martines, G. Spizzo, S. Cappello, A. Murari, R. Pasqualotto, and P. Zanca. Quasi-single-helicity reversed-field-pinch plasmas. *Physical Review Letters* 85, No. 8, pages 1662–1665, 2000.

[24] C. R. Evans and J. F. Hawley. Simulation of magnetohydrodynamic flows: a constrained transport method. *The Astrophysical Journal* 332, pages 659–677, 1988.

[25] G. Falkovich. Bottleneck phenomenon in developed turbulence. *Physics of Fluids* 6, pages 1411–1414, 1994.

[26] C. Federrath. On the universality of supersonic turbulence. *Monthly Notices of the Royal Astronomical Society* 436, pages 1245–1257, 2013.

[27] C. Federrath, J. Roman-Duval, R. S. Klessen, W. Schmidt, and M.-M. Mac Low. Comparing the statistics of interstellar turbulence in simulations and observations, solenoidal versus compressive turbulence forcing. *Astronomy and Astrophysics* 512, A81, 2010.

[28] R. C. Fleck, Jr. Scaling relations for the turbulent, non-self-gravitating, neutral component of the interstellar medium. *The Astrophysical Journal* 458, pages 739–741, 1996.

[29] U. Frisch, A. Pouquet, J. Léorat, and A. Mazure. Possibility of an inverse cascade of magnetic helicity in magnetohydrodynamic turbulence. *Journal of Fluid Mechanics* 68 Part 4, pages 769–778, 1975.

[30] J. Graham, P. D. Mininni, and A. Pouquet. High Reynolds number magnetohydrodynamic turbulence using a Lagrangian model. *Physical Review E* 84, 016314, 2011.

[31] D. I. Ketcheson. Highly efficient strong stability-preserving Runge-Kutta methods with low-storage implementations. *Society for Industrial and Applied Mathematics Journal on Scientific Computing* 30, No. 4, pages 2113–2136, 2008.

[32] R. H. Kraichnan and S. Nagarajan. Growth of turbulent magnetic fields. *The Physics of Fluids* 10, pages 859–870, 1967.

[33] A. G. Kritsuk, M. L. Norman, P. Padoan, and R. Wagner. The statistics of supersonic isothermal turbulence. *The Astrophysical Journal* 665, pages 416–431, 2007.

[34] A. Kumar and D. M. Rust. Interplanetary magnetic clouds, helicity conservation, and current-core fluxropes. *Journal of Geophysical Research* 101, pages 667–684, 1996.

[35] D. Levy, G. Puppo, and G. Russo. Central WENO schemes for hyperbolic systems of conservation laws. *Mathematical Modelling and Numerical Analysis* 33, No. 3, pages 547–571, 1999.

[36] M. Linkmann and V. Dallas. Large-scale dynamics of magnetic helicity. *Physical Review E* 94, 053209, 2016.

[37] M. Linkmann, G. Sahoo, M. McKay, A. Berera, and L. Biferale. Effects of magnetic and kinetic helicities on the growth of magnetic fields in laminar and turbulent flows by helical Fourier decomposition. *The Astrophysical Journal* 836:26, 2017.

[38] B. C. Low. Magnetohydrodynamic processes in the solar corona: Flares, coronal mass ejections, and magnetic helicity. *Physics of Plasmas* 1, pages 1684–1690, 1994.

[39] M.-M. Mac Low. The energy dissipation rate of supersonic, MHD turbulence in molecular clouds. *The Astrophysical Journal* 524, pages 169–178, 1999.

[40] S. K. Malapaka. A Study of Magnetic Helicity in Decaying and Forced 3D-MHD Turbulence. PhD thesis, Universität Bayreuth, 2009.

[41] P. McCorquodale and P. Colella. A high-order finite-volume method for conservation laws on locally refined grids. *Communications in Applied Mathematics and Computational Science* 6, No. 1, pages 1–25, 2011.
[42] M. Meneguzzi, U. Frisch, and A. Pouquet. Helical and nonhelical turbulent dynamos. *Physical Review Letters* **47**, No. 15, pages 1060–1064, 1981.

[43] M. Miesch, W. Matthaeus, A. Brandenburg, A. Petrosyan, A. Pouquet, F. Cambon, C. ANDJenko, D. Uzdensky, J. Stone, S. Tobias, J. Toomre, and M. Velli. Large-eddy simulations of magnetohydrodynamic turbulence in astrophysics and space physics. *Space Science Reviews* **194**, pages 97–137, 2015.

[44] P. D. Mininni and A. Pouquet. Finite dissipation and intermittency in magnetohydrodynamics. *Physical Review E* **80**, 025401, 2009.

[45] H. K. Moffatt. The degree of knottedness of tangled vortex lines. *Journal of Fluid Mechanics* **35**, pages 117–129, 1969.

[46] W.-C. Müller, S. K. Malapaka, and A. Busse. Inverse cascade of magnetic helicity in magnetohydrodynamic turbulence. *Physical Review E* **85**, 015302, 2012.

[47] J. Núñez-de la Rosa and C.-D. Munz. XTROEM-FV: A new code for computational astrophysics based on very high-order finite volume methods- I. Magnetohydrodynamics. *Monthly Notices of the Royal Astronomical Society* **455**, pages 3458–3479, 2016.

[48] S. A. Orszag. Analytical theories of turbulence. *Journal of Fluid Mechanics* **41**, pages 363–386, 1970.

[49] A. Pouquet, U. Frisch, and J. Léorat. Strong MHD helical turbulence and the nonlinear dynamo effect. *Journal of Fluid Mechanics* **77**, part 2, pages 321–354, 1976.

[50] A. Pouquet and G. S. Patterson. Numerical simulation of helical magnetohydrodynamic turbulence. *Journal of Fluid Mechanics* **85**, part 2, pages 305–323, 1978.

[51] V. V. Rusanov. The calculation of the interaction of non-stationary shock waves with barriers. *Zhurnal Vychislitel’noi Matematiki i Matematicheskoi Fiziki*, 1:2, pages 267–279, 1961. English: USSR Computational Mathematics and Mathematical Physics, 1:2, pp. 304-320, 1962.

[52] W. Schmidt, W. Hillebrandt, and J. C. Niemeyer. Numerical dissipation and the bottleneck effect in simulations of compressible isotropic turbulence. *Computer and Fluids* **35**, pages 353–371, 2006.

[53] J.-M. Teissier. Magnetic helicity inverse transfer in isothermal supersonic magnetohydrodynamic turbulence. PhD thesis, Technische Universität Berlin, 2020.

[54] P. S. Verma, J.-M. Teissier, O. Henze, and W.-C. Müller. Fourth-order accurate finite-volume CWENO scheme for astrophysical MHD problems. *Monthly Notices of the Royal Astronomical Society* **482**, pages 416–437, 2019.

[55] E. T. Vishniac and J. Cho. Magnetic helicity conservation and astrophysical dynamos. *The Astrophysical Journal* **550**, pages 752–760, 2001.

[56] F. Waleffe. The nature of triad interactions in homogeneous turbulence. *Physics of Fluids A: Fluid Dynamics* **4**, pages 350–363, 1992.

[57] L. Woltjer. A theorem on force-free magnetic fields. *Proceedings of the National Academy of Sciences* **44**, No. 6, pages 489–491, 1958.