‘Effective two dimensionality’ cases bring a new hope to the Kaluza–Klein(like) theories

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Abstract. One step towards realistic Kaluza–Klein(like) theories and a loophole through Witten’s ‘no-go theorem’ is presented for cases that we call effective two dimensionality cases: in $d = 2$, the equations of motion following from the action with the linear curvature leave spin connections and zweibeins undetermined. We present the case of a spinor in $d = (1 + 5)$ compactified on a formally infinite disc with the zweibein that makes a disc curved on an almost $S^2$ and with the spin connection field that allows on such a sphere only one massless normalizable spinor state of a particular charge, which couples the spinor chirally to the corresponding Kaluza–Klein gauge field. We assume no external gauge fields. The masslessness of a spinor is achieved by the choice of a spin connection field (which breaks the left–right symmetry), the zweibein and the normalizability condition for spinor states, which guarantee a discrete spectrum forming the complete basis. We discuss the meaning of the hole, which manifests the non-compactness of the space.

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1. Introduction

The idea of Kaluza and Klein (see [1]) of obtaining the electromagnetism—and, under the
influence of their idea, nowadays also the weak and colour fields ([2–8]; [13] and references
therein)—from purely gravitational degrees of freedom connected with having extra dimensions
is very elegant. More than 25 years ago the Kaluza–Klein(like) theories were studied intensively
by many authors ([8, 9, 13] and the works presented in [14]). Although the breaking of the
symmetry of the starting Lagrange density into low-energy effective ones (that is, to the charges
and correspondingly to the gauge fields assumed by the standard model of the electroweak and
colour interactions) seems very promising, the idea of Kaluza and Klein was almost killed by
the ‘no-go theorem’ of Witten [15] telling that these kinds of Kaluza–Klein(like) theories with
gravitational fields alone (that is, with vielbeins and spin connections) have severe difficulties
in obtaining massless fermions chirally coupled to Kaluza–Klein-type gauge fields in $d = 1 + 3$,
as required by the standard model. There were attempts to escape from the ‘no-go theorem’ in
compact extra spaces by having torsion [5] or by having an orbifold structure [11] or by putting
extra gauge fields by hand in addition to gravity in higher dimensions [12], which is no longer
the pure Kaluza–Klein(like) theory and loses accordingly the elegance.

Since there is the assumption that the space is compact in the ‘no-go theorem’ of Witten,
there have also been attempts to achieve masslessness by appropriate choices of vielbeins in
non-compact spaces [13, 23]; for comments on one such work [13] see the footnote4.

There have been several attempts to point out the importance of non-compact extra
dimensions, for example [16]; many of them are surveyed in [17]. These attempts do not really
try to keep the Kaluza–Klein approach in the original elegant version; rather they embed strings,

4 The author of [13] proposes, for example, the ‘squashed’ $S^2$ sphere, recognizing that with the zweibein of $S^2$
(he calls in this case $S^2$ a compact space) there are no massless spinor states, while with at least a little ‘stronger’
zweibein than that of $S^2$ (like with $f = (1 + (\frac{\rho}{\rho_0})^{2k})$, with $0 < k \leq 2$, $k = 0$ reproduces $S^2$) there are two massless
states. Although the author wrote differently, these two massless states belong to the left- and the right-handed
state with respect to $d = (1 + 3)$ and are therefore not mass protected, and would correspondingly lead to massive
fermion states.
membranes and p-branes into higher dimensional spaces. The most popular models of this kind are probably Randall–Sundrum models [18].

In this paper, we are interested in extra dimensions in the Kaluza–Klein sense: that is, a possibility that the gravity (and only gravity) in extra dimensions manifests as the standard model gauge fields in \((1 + 3)\), coupled to the corresponding charges. In [24], we achieved masslessness of spinors in the pure Kaluza–Klein(like) theory (for the case of \(M^{1+5}\) manifold broken into \(M^{1+3} \times \) an infinite disc) with the appropriate choice of a boundary limiting the extra dimensions on a finite surface on a disc.

In this paper, we take the whole two-dimensional plane, and roll it up into an almost \(S^2\) with one point—the south pole—excluded. It is our choice of a zweibein that forces the two extra dimensions into an almost \(S^2\). Thus, although it has a finite volume (namely the surface of \(S^2\)), the space is non-compact. We require spinor states to be in the fifth and sixth dimensions normalizable\(^5\), proving that the normalizable solutions form a complete set. It is our choice of a particular spin connection field, with the strengths within an interval, which allows only one normalizable massless state of a particular handedness (with respect to \((1 + 3)\)), breaking the parity symmetry. The normalization of states on the disc (curled into almost \(S^2\)) at least into plane waves makes states normalized to at least plane waves also in \(d = (1 + 3)\).

We do not discuss the origin of zweibein and spin connection fields, hoping that there are some spinor currents, which are responsible for the choice we present in equations (2) and (3).

The finite volume of an infinite disc, an appropriate choice of the spin connection field with the strength \(F\) allowed to be within the whole interval \(0 < 2F \leq 1\) and the normalizability requirement make the mass spectrum of our Hermitian Hamiltonian in a non-compact space discrete, with only one massless state of particular charge chirally coupled to the Kaluza–Klein gauge field. It is the sign of \(F\) which makes a choice of the handedness of a massless state, breaking the parity symmetry. The usually expected problem with extra non-compact dimensions having a continuous spectrum is not present in our model.

For a particular choice of the strength of the spin connection field, we find the states and the spectrum (the masses) analytically. This mass spectrum of states forms the complete set on our almost \(S^2\). For the remaining values of the strength, for all of which only one massless solution of a particular handedness in \((1 + 3)\) exists, it is not difficult to find the recursive formulae for normalizable solutions and the masses. Accordingly in this two-dimensional non-compact space, with the spin connections and vielbeins which are both a part of the gravitational gauge fields and with no presence of an (additional) external field, the ‘no-go theorem’ of Witten is not valid.

We also characterize the ‘singularity’ which the spinor solutions ‘feel’ on our infinite disc with the zweibein of an \(S^2\) sphere, when treating the disc as the almost \(S^2\) sphere, i.e. the \(S^2\) sphere with the hole on the southern pole, so that we have the almost \(M^{(1+3)} \times S^2\) case that it is almost a compact space.

Let us add: as it is not difficult to recognize, the two-dimensional spaces are very special [19, 20]. Namely, in dimensions higher than two, when we have no fermions present and only the curvature in the first power in the Lagrange density, the spin connections are normally determined from the vielbein fields, and the torsion is zero. In the two-dimensional spaces, the vielbeins do not determine the spin connection fields. In the present paper, we pay attention \(^5\) In [13], mentioned and discussed in footnote 4, this idea of a finite volume of a non-compact space, as well as the normalizability of states, is already stressed.
to cases, which we call *effective two-dimensionality*, when the spin connections are not fully determined by the vielbeins.

In the types of models proposed, there is a chance of having chirally mass-protected fermions in a theory in which the chirally protecting effective four-dimensional gauge fields are true Kaluza–Klein(like) fields, the degrees of which inherit from the higher-dimensional gravitational ones. We are thus hoping for a revival of true Kaluza–Klein(like) models as candidates for phenomenologically viable models!

One of us (NSMB) has long been trying to develop the approach unifying spins and charges and predicting families [21, 25] so that spinors that carry in \(d \geq 4\) nothing but two kinds of spin (no charges) would manifest in \(d = 1 + 3\) all the properties assumed by the standard model and accordingly share with the Kaluza–Klein(like) theories the problem of masslessness of the fermions before the electroweak-like types of break. The ideas of the approach are briefly presented in the footnote. Our technique in [25], used in this paper, is presented in appendix A.

Let us point out that in odd-dimensional spaces and in even-dimensional spaces divisible by four there is no mass protection in the Kaluza–Klein(like) theories [13, 24]. The spaces, therefore, for which we can have a hope that the Kaluza–Klein(like) theories lead to chirally protected fermions and accordingly to the effective theory of the standard model of the electroweak and colour interactions, have \(2(2n + 1)\) dimensions. And breaking symmetries in such spaces, if starting with one Weyl spinor, and accordingly with the mass protected case, should again lead to mass protected cases in accordance with the standard model.

2. The action, equations of motion, solutions, proofs and comments

In this section, we prove that in \(M^{1+3} \times \) an infinite disc with the particular zweibein and spin connection on the disc there exists only one massless normalizable (on the disc) fermion state of only one handedness and of a particular charge. It is accordingly mass protected. We also present proofs that the Hamiltonian is Hermitian and the spectra of normalizable states correspondingly discrete. For a particular strength of the spin connection field, we present the spectrum and states. We discuss the properties of solutions for the strengths allowed by the normalizability requirement.

6 The approach unifying spin and charges and predicting families [21] proposes in \(d = 1 + (d - 1)\) a simple starting action for spinors with two kinds of spin generators (\(\gamma\) matrices): the Dirac one, which takes care of the spin and the charges, and the second one, anti-commuting with the Dirac one, which generates families. For an explanation of the appearance of the two kinds of spin generators, see [21, 25] and references cited therein. A spinor couples in \(d = 1 + 13\) to the vielbeins and (through two kinds of the spin generators) to the spin connection fields. Appropriate breaks of the starting symmetry lead to the left-handed quarks and leptons in \(d = 1 + 3\), which carry the weak charge, while the right-handed do not carry weak charges. The approach is offering answers to questions about the origin of families of quarks and leptons, about the explicit values of their masses and mixing matrices (predicting the fourth family to be possibly seen at the LHC or at somewhat higher energies) as well as about the masses of the scalar and the weak gauge fields, about the dark matter candidates and about breaking the discrete symmetries. There are many possibilities in the approach for breaking the starting symmetries to those of the standard model. These problems were studied in some crude approximations in [21] and are under consideration [22].
Let us first repeat the four assumptions, stressed already in the introduction.

1. We assume $2(2n + 1)$-dimensional space, in our case $n = 1$, with only gravity, described by the action

$$ S = \alpha \int d^d x \, ER. \quad (1) $$

The Riemann scalar $\mathcal{R} = \mathcal{R}_{abcd} \eta^{ac} \eta^{bd}$ is determined by the Riemann tensor $\mathcal{R}_{abcd} = f^a_{[a} f^b_{b]} (\omega_{cd\alpha} - \omega_{cd\alpha} \omega_{d\beta})$, with vielbeins $^8 f^a_a$ and the spin connections $\omega_{ab\alpha}$ (the gauge fields of $S_{ab} = \frac{1}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a)$). $[a \ b]$ means that the anti-symmetrization must be performed over the two indices $a$ and $b$, and $E$ is the determinant of the inverse zweibein $e^a_\sigma$, $e^a_{\sigma} f^a_{\tau} = \delta^\tau_\sigma$ (equation (2)).

2. The space $M^{1+2}$ has the symmetry of $M^{1+3}$ × an infinite disc with the zweibein on the disc

$$ e^\sigma_\alpha = f^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f^\sigma_\alpha = f \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2) $$

with

$$ f = 1 + \left( \frac{\rho}{\sqrt{\rho^2 + \rho^2}} \right)^2, $$

$$ x^{(5)} = \rho \cos \phi, \quad x^{(6)} = \rho \sin \phi, \quad E = f^{-2}. $$

The last relation follows from $dx^2 = e^\sigma_{\sigma} e^\sigma_{\alpha} dx^\alpha dx^\sigma = f^{-2} (d\rho^2 + \rho^2 d\phi^2)$. We use indices $s, t = 5, 6$ to describe the flat index in the space of an infinite plane, and $\sigma, \tau = 5, 6$ to describe the Einstein index. $\phi$ determines the angle of rotations around the axis perpendicular to the disc.

3. The spin connection field is chosen to be

$$ f^\sigma_\alpha \omega_{\sigma\alpha\tau} = i F f e^\tau_\alpha x^\sigma, \quad 0 < 2F < 1. \quad (3) $$

4. We require normalizability of states $\psi$ on the disc

$$ \int_0^{2\pi} d\phi \int_0^\infty E \rho \, d\rho \psi^\dagger \psi < \infty, \quad (4) $$

as is usual in quantum mechanics, allowing at most the plane waves normalized to the delta function: $f^{-\infty} dx^{(5)} f^{\infty} dx^{(6)} E e^{i\vec{\phi}(\vec{x} - \vec{x}')}) = \delta^2(\vec{x} - \vec{x}')$. Correspondingly, wavefunctions in $d = (1 + 3) + 2$ are normalized, as is usual in quantum mechanics: states with well-defined four momentum in $d = (1 + 3)$ are normalized to plane waves.

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7 We have proven in [24] that only in even-dimensional spaces of $d = 2$ modulo four dimensions (i.e. in $d = 2(2n + 1), n = 0, 1, 2, \ldots$) do spinors (they are allowed to be in families) of one handedness and with no conserved charges gain no Majorana mass.

8 $f^a_a$ are inverted vielbeins to $e^\sigma_\alpha$ with the properties $e^\sigma_\alpha f^\sigma_\alpha = \delta^\eta_\beta$, $e^\sigma_\alpha f^\sigma_\alpha = \delta^\beta_\alpha$. Latin indices $a, b, c, \ldots, n, m, n, \ldots, s, t, \ldots$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, \ldots, \mu, \nu, \ldots, \sigma, \tau, \ldots$ denote an Einstein index (a curved index). Letters from the beginning of both alphabets indicate a general index $(a, b, c, \ldots$ and $\alpha, \beta, \gamma, \ldots$), those from the middle of both alphabets indicate the observed dimensions $0, 1, 2, 3$ $(m, n, \ldots$ and $\mu, \nu, \ldots$), and indices from the ends of the alphabets indicate the compactified dimensions $(s, t, \ldots$ and $\sigma, \tau, \ldots$). We assume the signature $\eta^{ab} = \text{diag}[1, -1, -1, \ldots, -1]$. 

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Let us make now several statements, proofs of these statements and comments, which will help to clarify the meaning of the assumptions.

**Statement 1.** In the absence of the fermion fields in $d = 2$, any zweibein and any spin connection fulfil the equations of motion.

**Proof.** The action of equation (1) leads to the equations of motion \[ (d - 2)(-i)\omega^c \partial_{\beta} \left( E f^e_{[a} f^\beta_{b]} \right), \] which clearly demonstrates that any spin connection $\omega^c = \omega^c_{ba} f^a_{c}$ (which can in $d = 2$ have only two different indices) satisfies this equation.

**Comment 1.** For $d = 2$ the variation of the action (1) with respect to vielbeins leads to the equation

\[-e_s R + 4 f^{\tau} \omega_{\sigma \tau} = 0\] which is zero for any $R (-2R + 2R = 0)$.

**Statement 2.** The volume of this non-compact space (which looks almost like a $S^2$ sphere) is finite.

**Proof.** The volume is $\int_0^{\infty} f^{-2} \rho \, d\rho = \pi (2\rho_0)^2$.

**Comments 2.** (i) Finite volume helps to ensure the existence of normalizable spinor states on this disc. (ii) The symmetry of this disc, which is the symmetry of $U(1)$ group, determines the charge of spinors in $d = (1 + 3)$.

**Statement 3.** The choice that $M^{1+5}$ breaks into $M^{1+3} \times$ an infinite disc with no gravity in $M^{1+3}$ and with the zweibein of equation (2) and the spin connection of equation (3) on an infinite disc transforms the Lagrange density for a Weyl spinor $L_W^{1/2} = \frac{1}{2}[(\psi^{\dagger} E \gamma^0 \gamma^a p_0 \psi) + (\psi^{\dagger} E \gamma^0 \gamma^a p_0 \psi)]$ into

\[ L_W = \psi^{\dagger} \left\{ E \gamma^0 \gamma^n p_n + E f \gamma^0 \gamma^d \delta^e_{\sigma} \left( p_0 + \frac{1}{2E f} \{p_\sigma, E f\} \right) \right\} \psi, \quad n = 0, 1, 2, 3, \]

\[ p_{0\sigma} = p_\sigma - \frac{1}{2} S^{e\sigma} \omega_{e\sigma}, \]

with $E = \det(e^a_\sigma) = f^{-2}$, $f$ is from equation (3) and with $\omega_{e\sigma}$ from equation (3)\(^9\).

**Proof.** Equation (6) follows from the starting Lagrangian for a Weyl spinor interacting only with the vielbeins and spin connections straightforwardly.

**Comment 3.** The Lagrange density of equation (6) ensures that the Hamiltonian is Hermitian.

**Statement 4.** Normalizability condition for spinors on an infinite disc curled into an almost $S^2$ and with the spin connection of particular choice makes a choice of a spectrum that forms a complete set.

\(^9\) One finds that $\omega_{cda} = \Re e \omega_{cda}$, if $c$, $d$, $a$ are all different, while $\omega_{cda} = \Im m \omega_{cda}$ otherwise.
Proof. The Lagrange density of equation (6) leads to equations of motion (equations (11), (12) and (15))

\[
i f \left\{ e^{i 2 S^{56}} \left[ \frac{\partial}{\partial \rho} + \frac{i 2 S^{56}}{\rho} \left( \frac{\partial}{\partial \phi} - \frac{1}{2 f} \frac{\partial f}{\partial \rho} (1 - 2 F S^{56}) \right) \right] \psi^{(6)} + \gamma^0 \gamma^5 m \psi^{(6)} = 0, \tag{7}
\]

which look for \( F = 1/2 \) like Legendre equations (equation (22)). It is the sign of \( F \) which makes a choice of the handedness of a massless state and breaks accordingly the parity symmetry. One can prove that the only normalizable eigenstates in the interval \( 0 \leq \rho \leq \infty \) are those with integer parameters \( l \) and \( n \), \( (m \rho_0)^2 = l(l + 1) \), in equations (23). These states are Legendre polynomials and form the complete set. Solutions for a non-integer \( n \) are singular at \( \rho = 0 \), while solutions with a non-integer \( l \) are singular at \( \rho = \infty \); both singularities make the corresponding eigenstates unnormalizable. □

Comments 4. (i) In section 2.1, the solutions of equation (7) are discussed for any choice of \( F \) in the interval \( 0 < 2 F \leq 1 \). All the normalizable solutions can, for any \( F \) in this interval, be expressed as a normalizable superposition of a complete set of Legendre polynomials and have the discrete spectrum. (ii) In the limit when \( \rho_0 \rightarrow \infty \), \( f \) (in equation (15), the next section) goes to one and the two equations, equation (15), define the recurrence relations between the Bessel functions of an integer order (\( A_n(\rho m) = J_n(\rho m) \) and \( B_{n+1}(\rho m) = J_{n+1}(\rho m) \)) for any mass \( m \). Making the limit \( \rho_0 \rightarrow \infty \) in equation (22) in the next section, with the discrete mass term \( (m \rho_0)^2 = l(l + 1) \), one again reproduces the Bessel equation, if putting \( l = m \rho_0 \). (Bessel functions can be squared normalized only within a finite radius, determined by zeros.) With \( \rho_0 \) going to infinity the distance between \( m \) values solving this constraint goes to zero, so that in this limit the system of allowed \( m \) values approaches the continuum (all \( m \) values). This is satisfactory because this limit corresponds to our already non-compact space approaches, the usual flat two-dimensional space (with which one would have a truly fully \( 5 + 1 \)-dimensional world in which of course the spectrum seen as a (3 + 1)-dimensional one should be continuous). (iii) For any finite \( \rho_0 \) the plane wave in the fifth and the sixth dimension can be expressed in terms of the Legendre polynomials. To a plane wave, in general, many Legendre polynomials contribute, each corresponding to a different mass. There is the solution for \( 2 F = 1 \) which is independent of \( x^\sigma \), \( \sigma \in \{5\} \), (6). It corresponds to a massless solution. This solution can be called the plane wave with momentum zero. In the limit \( \rho_0 \rightarrow \infty \), the definition for the plane waves in flat space follows. (iv) Accordingly the massless and massive states are in \( d = (1 + 3) + 2 \) normalized in the usual quantum mechanical way. If no gauge fields \( A_\mu \) (equation (52)) and no gravitational fields are present in \( d = (1 + 3) \), localizing (massive) fermions in a finite part of space, the states (being the eigenstates of the four momentum in \( d = (1 + 3) \)) can be normalized to a delta function or to a wave packet. The same would be true also for functions that are wave packets on the almost \( S^2 \) sphere, presented later in footnote 11.

Statement 5. The zweibein (equation (2)) and the spin connection (equation (3)) with the parameter \( F \) within the interval \( 0 < 2 F \leq 1 \) allow only one massless spinor of a particular charge.

Proof. It is proven in the next subsection, in the last paragraph before equation (11), that it is the term \( \psi^+ E f \gamma^0 \gamma^5 \delta_s^\rho (p_\sigma + \frac{1}{2 F} \{ p_\sigma, E f \}) \psi \) in the Lagrange density (equation (6)), which manifests as the mass term \( m \) in equation (7). There is a term in equation (7),
namely \(-i f e^{i\phi S^56} \frac{1}{2f} \frac{\partial}{\partial \phi} (1 - 2F \psi S^56) \psi^{(6)}\), which clearly distinguishes between the two possible values of the spin operator \(S^56\) in \(d = 5, 6\), when this term applies on the state \(\psi^{(6)}\), distinguishing correspondingly also between the two possible types of handedness of the state \(\psi^{(6)}\) in \(d = (1 + 3)\). It is shown in the next subsection that a normalizable massless state \((m = 0\) in equation (7))) must fulfill the condition \((0 \leq (1 - 2F \psi S^56) < 1) \psi^{(6)}\). The sign of \(F\) chooses the handedness of a massless normalizable spinor state.

Comment 5. (i) Having the rotational symmetry around the axis perpendicular to the plane of the fifth and the sixth dimension, it is meaningful to require that \(\psi^{(6)}\) is the eigenfunction of the total angular momentum operator \((M^{56} = x^5 p^6 - x^6 p^5 + S^56)\) in the fifth and the sixth dimension \(M^{56} = (-i \frac{\partial}{\partial \phi} + S^56); M^{56} \psi^{(6)} = (n + \frac{1}{2}) \psi^{(6)}\) (equations (13), (14) and (12)). (ii) The only massless state, which fulfills the normalization condition (see equation (18)) for a positive \(F\), is the state with the property \(2S^56 \psi^{(6)} = \psi^{(6)}\). Its charge (spin on the disc) is for \(0 < 2F \leq 1\) equal to \(\frac{1}{2}\) as is shown in section 4. (iii) All the other states are massive. (iv) The current in the radial direction is for all these cases equal to zero for any \(F\).

Detailed derivations of equations of motion and solutions are presented in section 2.1.

Let us summarize this section. We have a Weyl spinor in \(d = (1 + 5)\)-dimensional space. This space breaks into \(M^{1+5} \times \) an infinite disc with the zweibein that formally looks almost—up to a hole in the southern pole—like an \(S^2\) sphere, while a chosen spin connection allows on such an infinite disc only one normalizable massless state. The Hamiltonian is Hermitian, the mass spectrum of normalizable states is correspondingly discrete and the probability for a fermion to escape out of the disc is zero\(^{10}\).

Allowing the whole interval of the strength of the spin connection fields \((0 < 2F \leq 1)\) the spin connection field is not fine-tuned. For a particular choice of the constant of the spin connection field, i.e. for \(2F = 1\), the normalizable solutions are expressible with the Legendre polynomials and the massive states manifest a spectrum \(m \rho_0 = l(l + 1)\), with \(l = 0, 1, 2, \ldots\) and \(-l \leq n \leq 1\). Note that \(n + 1/2\) is the charge of the spectrum.

A free choice of a zweibein and a spin connection field in the action of equation (1) is possible only in \(d = 2\) dimensional spaces (the presence of fermions might make this possible also for \(d > 2\)).

Let us point out that the ‘two dimensionality’ can be simulated in any dimension larger than two if vielbeins and spin connections are completely flat in all but two dimensions (this point is also discussed in [13]).

2.1. Solutions of the equations of motion for spinors

We look for the solutions of the equations of motion (6) for a spinor in \((1 + 5)\)-dimensional space, which breaks into \(M^{(1+3)} \times \) an infinite disc curved into a non-compact ‘almost’ \(S^2\) sphere as a superposition of all four \((2^6/2 - 1)\) states of a single Weyl representation. (See [25] for technical details of how to write a Weyl representation in terms of the Clifford algebra objects after making a choice of the Cartan subalgebra, for which we take: \(S^{03}, S^{12}, S^{56}\).

A short presentation of our technique can also be found in the appendix.)

\(^{10}\) It is expected that the zweibein curving the infinite disc into an (almost \(S^2\)) sphere and the spin connection, which breaks the parity symmetry and takes part in determining equations of motion, appear dynamically, causing ‘phase transition’. Accordingly dynamical fields, by causing the phase transition, could restore the symmetry of \(M^{1+5}\).

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In our technique one spinor representation—the four states, which all are eigenstates of the chosen Cartan subalgebra with the eigenvalues \( \frac{k}{2} \), correspondingly—are the following four products of projectors \([k]\) and nilpotents \([\bar{k}]\):

\[
\begin{align*}
\varphi_1^1 &= (+)(+)(+\psi_0, \\
\varphi_2^1 &= (+)[-1][-\psi_0, \\
\varphi_1^2 &= [-][-i](+\psi_0, \\
\varphi_2^2 &= [-][-i](+\psi_0,
\end{align*}
\]

where \(\psi_0\) is a vacuum state for the spinor state. If we write the operators of handedness in \(d = (1 + 5)\) as \(\Gamma^{(1+5)} = \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^5 \gamma^6\) (\(= 2^3 i S^{03} S^{12} S^{56}\)), in \(d = (1 + 3)\) as \(\Gamma^{(1+3)} = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3\) (\(= 2^3 i S^{03} S^{12}\)) and in the two-dimensional space as \(\Gamma^{(2)} = i \gamma^5 \gamma^6\) (\(= 2 S^{56}\)), we find that all four states are left handed with respect to \(\Gamma^{(1+5)}\), with the eigenvalue \(-1\), the first two states are right handed and the second two states are left handed with respect to \(\Gamma^{(2)}\), with eigenvalues \(1\) and \(-1\), respectively, while the first two are left handed and the second two right handed with respect to \(\Gamma^{(1+3)}\) with eigenvalues \(-1\) and \(1\), respectively. Taking into account equation (8), we may write the most general wavefunction \(\psi^{(6)}\) obeying equation (7) in \(d = (1 + 5)\) as

\[
\psi^{(6)} = A (\psi_{(+)}^{(4)}) + B [-\psi_{(-)}^{(4)}],
\]

where \(A\) and \(B\) depend on \(x^\sigma\), while \(\psi_{(+)}^{(4)}\) and \(\psi_{(-)}^{(4)}\) determine the spin and the coordinate dependent parts of the wavefunction \(\psi^{(6)}\) in \(d = (1 + 3)\)

\[
\begin{align*}
\psi_{(+)}^{(4)} &= \alpha_+ [+1] (+) + \beta_+ [-1] [-], \\
\psi_{(-)}^{(4)} &= \alpha_- [-1] (+) + \beta_- [+1] [-].
\end{align*}
\]

Using \(\psi^{(6)}\) in equation (7) and separating dynamics in \((1 + 3)\) and on the infinite disc, the following relations follow, from which we recognize the mass term \(m: \frac{\alpha_+}{\bar{\alpha}_+}(p^0 - p^3)\) and \(\frac{\beta_+}{\bar{\beta}_+}(p^1 - i p^2) = m, \frac{\beta_-}{\bar{\beta}_-}(p^0 + p^3) - \frac{\alpha_-}{\bar{\alpha}_-}(p^1 + i p^2) = m, \frac{\alpha_+}{\bar{\alpha}_+}(p^0 + p^3) + \frac{\beta_+}{\bar{\beta}_+}(p^1 - i p^2) = m, \frac{\alpha_-}{\bar{\alpha}_-}(p^0 - p^3) + \frac{\beta_-}{\bar{\beta}_-}(p^1 - i p^2) = m\). One notes that for massless solutions \((m = 0)\), \(\psi_{(+)}^{(4)}\) and \(\psi_{(-)}^{(4)}\) decouple. Taking the above derivation into account, equation (7) transforms into

\[
f \{ f (p_0 + i 2 S^{56} p_6) + \frac{1}{2E} \{ p_5 + i 2 S^{56} p_6, E f \} \} \psi^{(6)} + \gamma^0 \gamma^5 m \psi^{(6)} = 0.
\]

For \(x^{(5)}\) and \(x^{(6)}\) from equation (3) and for the zweibein from equations (2) and (3) and the spin connection from equation (3), one obtains

\[
i f \left\{ e^{i\phi^2 S^{56}} \left[ \frac{\partial}{\partial \rho} + \frac{i 2 S^{56}}{\rho} \left( \frac{\partial}{\partial \phi} \right) - \frac{1}{2F} \frac{\partial f}{\partial \rho} (1 - 2F) S^{56} \right] \right\} \psi^{(6)} + \gamma^0 \gamma^5 m \psi^{(6)} = 0.
\]

Having the rotational symmetry around the axis perpendicular to the plane of the fifth and the sixth dimension, we require that \(\psi^{(6)}\) is the eigenfunction of the total angular momentum operator \(M^{56} = x^5 p^6 - x^6 p^5 + S^{56} = -\frac{3}{\alpha_0} + S^{56}\):

\[
M^{56} \psi^{(6)} = (n + \frac{1}{2}) \psi^{(6)}.
\]
Accordingly we write
\[ \psi^{(6)} = \mathcal{N} \left( A_n \psi^{(4)}_{(+)} + B_{n+1} e^{i\phi} \psi^{(4)}_{(-)} \right) e^{in\phi}. \] (14)

After taking into account that \( S^{56} (+) = \frac{1}{2} [+] \), while \( S^{56} [-] = -\frac{1}{2} [-] \) we end up with the equations of motion for \( A_n \) and \( B_{n+1} \) as follows:
\[ -i f \left\{ \left( \frac{\partial}{\partial \rho} + \frac{n+1}{\rho} \right) - \frac{1}{2 f} \frac{\partial f}{\partial \rho} (1 + 2 F) \right\} B_{n+1} + m A_n = 0, \] (15)
\[ -i f \left\{ \left( \frac{\partial}{\partial \rho} - \frac{n}{\rho} \right) - \frac{1}{2 f} \frac{\partial f}{\partial \rho} (1 - 2 F) \right\} A_n + m B_{n+1} = 0. \]

Let us treat first the massless case \( (m = 0) \). Taking into account that \( f^{-1} = \frac{\partial}{\partial \rho} \ln f \frac{\tau}{\rho} \) and that \( E = f^{-2} \), it follows that
\[ \frac{\partial \ln (B_n \rho^n f^{-F - \frac{1}{2}})}{\partial \rho} = 0, \] (16)
\[ \frac{\partial \ln (A_n \rho^{-n} f^{-F + \frac{1}{2}})}{\partial \rho} = 0. \]

We get correspondingly the solutions
\[ B_n e^{in\phi} = B_0 e^{in\phi} \rho^{-n} f^{F + \frac{1}{2}}, \]
\[ A_n e^{in\phi} = A_0 e^{in\phi} \rho^n f^{-F - \frac{1}{2}}. \] (17)

Requiring that only normalizable (square integrable) solutions are acceptable
\[ 2\pi \int_0^\infty E \rho \, d\rho A_n^* A_n < \infty, \] (18)
\[ 2\pi \int_0^\infty E \rho \, d\rho B_n^* B_n < \infty, \]

it follows that
\[ \text{for } A_n : -1 < n < 2F, \]
\[ \text{for } B_n : 2F < n < 1, \quad n \text{ is an integer.} \] (19)

One immediately sees that for \( F = 0 \) there is no solution for the zweibein from equation (3).

Equation (19) tells us that the strength \( F \) of the spin connection field \( \omega_{56\sigma} \) can make a choice between the two massless solutions \( A_n \) and \( B_n \): for
\[ 0 < 2F \leq 1 \] (20)
the only massless solution is the left-handed spinor with respect to \( (1 + 3) \):
\[ \psi^{(6)m=0}_{1/2} = \mathcal{N}_0 f^{-F + \frac{1}{2}} 56 \psi^{(4)}_{(+)} \] (21)
It is the eigenfunction of \( M^{56} \) with the eigenvalue \( 1/2 \). No right-handed massless solution is allowed. For the particular choice \( 2F = 1 \) the spin connection field \( -S^{56} \omega_{56\sigma} \) compensates for the term \( \frac{1}{2E^2} \{ p_\sigma, E f \}_- \) and the left-handed spinor with respect to \( d = (1 + 3) \) becomes a constant with respect to \( \rho \) and \( \phi \).

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For $2F = 1$, it is easy to find also all the massive solutions of equation (15). Introducing $u = \frac{\rho}{2\rho_0}$ and assuming that $2F = 1$, one finds from equation (15)

$$B_{n+1} = \frac{i}{2\rho_0 m} (1 + u^2) \left( \frac{d}{du} - \frac{n}{u} \right) A_n^m,$$

$$\left\{ \left( 1 + u^2 \right)^2 \left( \frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} - \frac{n^2}{u^2} \right) + (\rho_0 m)^2 \right\} A_n^m = 0. \quad (22)$$

If one expresses $(\frac{\rho}{2\rho_0})^2 = \frac{1-x}{1+x}$, with $-1 \leq x \leq 1$ for $0 \leq \rho \leq \infty$, it follows that $f = \frac{2}{1+x}$, $\frac{dx}{du} = -\frac{4u}{(1+u^2)^2}$ and $\frac{d^2}{du^2} = (1-x^2)$. Then equation (22) transforms into the equations of motion for the associated Legendre polynomials $A_n^{(\rho_0 m)^2=\ell(l+1)} = P_n^l$ if we assume that $(\rho_0 m)^2 = l(l+1)$:

$$\left( (1-x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} - \frac{n^2}{1-x^2} + l(l+1) \right) A_n^{(\rho_0 m)^2=\ell(l+1)} = 0,$$

$$B_{n+1}^{(\rho_0 m)^2=\ell(l+1)} = \frac{-i}{\rho_0 m} \sqrt{1-x^2} \left( \frac{d}{dx} + \frac{n}{1-x^2} \right) A_n^{(\rho_0 m)^2=\ell(l+1)}. \quad (23)$$

From the above equations we see that for $m = 0$, i.e. for the massless case, the only solution with $n = 0$ exists, which is $A_0^{(\rho_0 m)^2=0}$, a constant (in agreement with our discussions above).

It is not difficult to prove that there is no normalizable solution of equation (23) for an arbitrary $m \rho_0$, which is not of the kind $(m \rho_0)^2 = l(l+1)$, with $l$ being an integer, and also not for a non-integer $n$. The solutions of equation (23) are, namely, not square integrable on the interval $-1 \leq x \leq 1$ when $l \neq \text{an integer}$ and $\nu \neq \text{an integer}$. $P_n^l(x \rightarrow -1+0)$ are unbounded, going to $\infty$, while they are bounded at $(x \rightarrow 1-0)$. One also finds that $P_n^l(x \rightarrow \infty)$ if $(x \rightarrow 1-0)$, unless $\mu = \pm m$, with $m$ being an integer. (See [26], section 5.18, pp 255–8.)

Accordingly the massive solutions with the masses equal to $m = l(l+1)/\rho_0$ (we use the units in which $c = 1 = \hbar$) and the eigenvalues of $M^{26}$ (equation (13))—which is the charge as we see in section 4—equal to $(\frac{1}{2} + n)$, with $-l \leq n \leq l$, $l = 1, 2, \ldots$, are

$$\psi_n^{(6)}(\rho_0 m)^2=\ell(l+1) = \frac{1}{\sqrt{\ell(l+1)}} \left( \frac{1}{\sqrt{4(l^2+1)}} \left[ \frac{1}{\sqrt{\ell(l+1)}} \right] \psi_n^{(4)(l+1)} \right)$$

$$\times e^{i\phi} A_n^{(\rho_0 m)^2=\ell(l+1)} \quad (24)$$

with $A_n^{(\rho_0 m)^2=\ell(l+1)}(x)$, which are the associated Legendre polynomials $P_n^l(x)$, where $x = \frac{1-u^2}{1+u^2}$ and $u = \frac{\rho}{2\rho_0}$. It is not difficult to see that the solutions of equation (15) for

Rewriting the mass operator $\hat{m} = y^0 y^l f^l (\rho_0 - S^{\rho_0} \omega_{56} + \frac{1}{\sqrt{2}} [\rho_0, E] \gamma_5)$ as a function of $\hat{\vartheta}$ and $\phi$:

$$\rho_0 \hat{m} = i y^0 \left[ (+) \frac{1}{\sqrt{4(l^2+1)}} \left[ \frac{1}{\sqrt{\ell(l+1)}} \right] \psi_n^{(4)(l+1)} \right]$$

one can easily show that when applying $\rho_0 \hat{m}$ and $M^{26}$ on $\psi_n^{(6)(l+1)}(x)$, for $l = 1, 2, \ldots$, one obtains from equation (24) $\rho_0 \hat{m} \psi_n^{(6)(l+1)} = (l+1) \psi_n^{(6)(l+1)}$, $M^{26} \psi_n^{(6)(l+1)} = (n+1/2) \psi_n^{(6)(l+1)}$, $l = 1, 2, \ldots$. A wave packet, which is the eigenfunction of $M^{26}$ with the eigenvalue 1/2, for example, can be written as $\psi_n^{(6)} = \sum_{k=0,\infty} C_k \frac{1}{\sqrt{k+1}} (+) \psi_n^{(4)(l+1)}(1 - \delta_{k,0}^2) \frac{1}{\sqrt{2(k+1)}} \left[ \frac{1}{\sqrt{\ell(k+1)}} \right] \psi_n^{(4)(l+1)}$. The expectation value of the mass operator $\hat{m}$ on such a wave packet is

\[ \sum_{k=0,\infty} C_k^2 \frac{1}{\sqrt{k+1}} \sqrt{k+1}/\rho_0. \]
almost zweibein from equation (25) and the spin connections from equation (3) are orthogonal.

One can show as well that the eigenstates, with the discrete eigenvalues \((\rho_0 m)^2 = l(l+1)\), are orthogonal

\[
\int d^2x E \psi^{(6)(\rho_0 m)}_{n+1/2} \psi^{(6)(\rho_0 m)^*}_{n+1/2} = \delta_{n,n'} \delta_{l,l'} \delta_{m,m'} \approx \int d^2x e^{-i(n'-n)\phi} (B^+_{n+1/2} B^0_{n+1} + A^+_{n+1} A^0_{n+1})
\]

for all pairs of \((l, n), (l', n')\); the spectrum is obviously discrete as it should be for the Hermitian Hamiltonian with the boundary conditions determined by normalizability of states.

To obtain solutions for all \(F\) in the interval \(0 < F < \frac{1}{2}\), besides the massless one \(\psi^{(6)m=0}_{1/2}\), is a more difficult work. Yet one can expect that on the space of normalizable functions the Hamiltonian will stay Hermitian and since an infinitesimal change of the constant \(F\) from \(F = \frac{1}{2}\) to a smaller \(F\) cannot spoil the discreteness of the Hamiltonian eigenvalues, the spectrum would stay discrete. One can see that the current in the radial direction is zero for any \(F\). We studied these solutions and found the discrete spectrum; a paper is in preparation.

Let us recognize that \(e^{i\phi} P^l_n\) are spherical harmonics \(Y^l_n\). Expressing \(\rho\) with \(\vartheta\), \(\frac{\rho}{2\rho_0} = \sqrt{\frac{1 - \cos \vartheta}{1 + \cos \vartheta}}\), we rewrite the equations of motion (equations (15)) as follows:

\[
\begin{align*}
\left( \frac{\partial}{\partial \vartheta} + \frac{n + 1 - (F + 1/2)(1 - \cos \vartheta)}{\sin \vartheta} \right) B_{n+1} + i \rho_0 m A_n &= 0, \\
\left( \frac{\partial}{\partial \vartheta} + \frac{-n + (F - 1/2)(1 - \cos \vartheta)}{\sin \vartheta} \right) A_n + i \rho_0 m B_{n+1} &= 0.
\end{align*}
\]

(25)

3. Singularities on an almost \(S^2\) sphere

In this section, we comment on singularities ‘felt’ by a spinor if a non-compact disc with the zweibein from equation (2) and the spin connections from equation (3) is understood as the \(S^2\) sphere with a hole on the southern pole.

Intuitively it is not difficult to see that we are in trouble if we want the chiral fermion field of equation (21), i.e. \(\psi^{(6)m=0}_{1/2} = N_0 f^{F+1/2} 56^6 (\pm) \psi^{(4)}_{(+)}\), on a two-dimensional space to be an eigenstate of some rotational operator \(M^{56}\) if the two-dimensional space has to have the topology of \(S^2\), while the spin of the fermion contributes to \(M^{56}\) in the ‘usual way’

\[
M^{56} = S^{56} + K^{56},
\]

(26)

where \(K^{56}\) is the Killing vector, like in equation (13) \((K^{56} = x^5 p^6 - x^6 p^5)\). Near the starting point (the origin, the northern pole of \(S^2\)) on the topologically \(S^2\) sphere, the Killing operator functions as the orbital angular momentum \((L^{56} = x^5 p^{(6)} - x^6 p^{(5)})\) and has to be added to the spin part \(S^{56}\), just as it is in the flat two-dimensional space. Going away from the starting point the action of \(M^{56}\) may be more complicated than just a simple sum in equation (26). Because of the \(S^2\) topology, there has to be namely yet another point at which the orbital Killing generator eigenvalue goes to zero, since there has to be a point, the south pole, which is left invariant under the orbital Killing transportation as it is at the starting point, at the north pole.

It is also easy to see that on the two-dimensional \(S^2\), the orientation of the Killing transportation in the infinitesimal neighbourhood of this second stable point, the south pole, is in the opposite direction with respect to the orientation of the Killing transportation around the north pole.
If we want to have on $S^2$ only a spinor of one handedness, let us say the spinor $\psi^{(6)\pi/2=0}$ of equation (21), then we should count at the south pole the orbital symmetry generator with the opposite sign relative to $S^{56}$ as we do at the starting point (see equations (45) and (43)). In order to be able to have on the two-dimensional $S^2$ surface a spinor of only one handedness, we have to let the phase rotation generated by the $S^{56}$ part of $M^{56}$ relative to the Killing part at the south pole be of the opposite sign with respect to the north pole. Namely, when we consider smaller and smaller circles around the south pole, the phase of the single handedness spinor state must be rotated under $M^{56}$ so that when extrapolating to the south pole the phase rotation corresponds to the spin, which is inverted relative to the orientation of the two-dimensional space of the $S^2$ surface.

Therefore, embedding the $S^2$ sphere in a three-dimensional Euclidean space, it is not surprising that if we want a spinor of one handedness and succeed in implementing it at the north pole in an outward normal direction, we can hardly implement it at the south pole. We might hope for the compensation by the orbital part of $M^{56}$, except at the poles. This means that we could have a state of a handed spinor if the wavefunction goes to zero at at least one of the poles, say the southern pole (see equations (19) and (21)).

### 3.1. Formal introduction of a singular point

We might formally introduce at the south pole a special singularity, so that we require the wavefunction instead to behave at the south pole in the usual differentiable way, to be differentiable only after being multiplied (corrected) by a phase factor: instead of $\psi$ we require that $e^{i\phi}\psi$ is our wave differentiable function in the neighbourhood of the singular point at the south pole, the phase factor $e^{i\phi}$ itself behaving singularly. By making this modified requirement of the differentiability, we effectively change the orbital angular momentum of the wavefunction by 1 unit of $\hbar$ before we require the wavefunction to be smooth or differentiable. Thereby we have made the requirement that the actual wavefunction should have a rather unphysical extra bit of negative angular momentum around the south pole. We must admit that it looks rather strange from the physical point of view, unless we recognize that this smoothness condition is to simulate the non-compactness of the $S^2$ space, which only after adding a singular point becomes an $S^2$ at all.

When changing the differentiability of the wavefunction in the neighbourhood of the singular point with the requirement that the wavefunction must be multiplied by a phase, we recognize that such a phase multiplication of the wavefunction appears when transforming the coordinate system from the northern to the southern pole, as we can see in equation (39) below. This phase transformation of the wavefunction requires the appearance of the spin connection field, as can be seen in equation (35): the gauge transformation of any spin connection field (when transforming the coordinate system) appears even if the spin connection field is zero and manifests in the second term of this equation.

### 3.2. Gauge transformations from the northern to the southern pole

To demonstrate further what the hole does in the non-compact space of an almost $S^2$ sphere, let us transform the coordinate system from the northern to the southern pole of the sphere $S^2$ as the $S^2$ would be a sphere made out of an infinite plane with the zweibein of a sphere and look at how the equations of motion and the wavefunctions transform correspondingly and how they
\[ \rho^{NP} \cdot \rho^{SP} = (2\rho_0)^2 \]

\[ \sqrt{(2\rho_0)^2 + (\rho^{NP})^2} = \sqrt{(2\rho_0)^2 + (\rho^{SP})^2} \]

**Figure 1.** Transforming coordinates from the north to the south pole on \( S^2 \).

demonstrate the non-compactness of our space. From figure 1, we read

\[ x^{NP(5)} = \left( \frac{2\rho_0}{\rho^{SP}} \right)^2 x^{SP(5)}, \quad x^{NP(6)} = -\left( \frac{2\rho_0}{\rho^{SP}} \right)^2 x^{SP(6)}, \quad (27) \]

and

\[ \rho^{SP} \rho^{NP} = (2\rho_0)^2, \quad E^{NP} d^2 x^{NP} = E^{SP} d^2 x^{SP}, \quad (28) \]

where \( x^{NP\sigma}, \sigma = 5, 6 \), stay for the up to now used \( x^\sigma, \sigma = 5, 6 \), while \( x^{SP\sigma}, \sigma = 5, 6 \), stay for coordinates when we put our coordinate system at the southern pole and \( \rho_0 \) is the radius of \( S^2 \) as before. We have \( E^{SP} = (1 + (\rho^{SP} / 2\rho_0)^2)^{-2} \) and \( E^{NP} = (1 + (\rho^{NP} / 2\rho_0)^2)^{-2} = \left( \frac{2\rho_0}{\rho^{SP}} \right)^4 E^{SP} \). We can also write \( x^{NP\sigma} = \left( \frac{2\rho_0}{\rho^{SP}} \right)^2 \left( -1 + \sigma \right) x^{SP\sigma} \).

We ought to transform the Lagrange density (equation (6)) expressed with respect to the coordinates at the northern pole

\[ L^W = \psi^{NP\dagger} E^{NP} \gamma^0 \gamma^4 \left( f^s_{NP\sigma} P_{0\sigma} + \frac{1}{2 E^{NP}} \{ P_{\sigma}, E^{NP} f^s_{NP\sigma} \} \right) \psi^{NP}, \quad (29) \]

to the corresponding Lagrange density \( L^{SP} \) expressed with respect to the coordinates at the southern pole by assuming

\[ \psi^{NP} = S \psi^{SP}. \quad (30) \]

We use the anti-symmetric tensor \( \varepsilon^{(5)(6)} = 1 = -\varepsilon^{(5)} \). We recognize that

\[ f^s_{NP\sigma} = f^s_{SP\sigma} \frac{\partial x^{NP\sigma}}{\partial x^{SP\sigma}}, O_{-1}^{-1}, \]

\[ f^s_{SP\sigma} = f^s_{SP} \delta^\sigma_0, \quad f^{SP} = \left( 1 + \left( \frac{\rho^{SP}}{2\rho_0} \right)^2 \right). \quad (31) \]
The matrix $O$ takes care that the zweibein expressed with respect to the coordinate system at the southern pole is diagonal: $f^s_{SP} = f^{SP} \delta^s_\sigma$

$$O = \begin{pmatrix} -\cos(2\phi + \pi) & -\sin(2\phi + \pi) \\ \sin(2\phi + \pi) & -\cos(2\phi + \pi) \end{pmatrix}. \quad (32)$$

Requiring that

$$S^{-1} \gamma^0 \gamma^t S O^{-1t} = \gamma^0 \gamma^t,'$$

from which it follows that $S^{-1} \gamma^0 \gamma^t S O^{-1t} = S^{0t'}$, and recognizing that $p^s_{\sigma} = \frac{\partial x^{s\sigma'}}{\partial x^{s\sigma'}} p^s_{\sigma'}$, with $p^s_{\sigma} = \frac{\partial x^{s\sigma'}}{\partial x^{s\sigma'}}$, we find that $\gamma^s f^s_{NP} p^s_{0\sigma}(\gamma^s f^s_{NP} (p^s_{\sigma} - \frac{1}{2} S^{0t'} \omega^{s0}_{s0\sigma}))$ transforms into $\gamma^s f^s_{NP} p^s_{0\sigma}$.

$$\gamma^s f^s_{NP} p^s_{0\sigma} = \gamma^s f^s_{NP} \left\{ \frac{p^s_{SP}}{2} - \frac{1}{2} S^{0t'} \epsilon x^t_{\sigma'} \left( \frac{F x^t_{\sigma}}{f^{SP} (f^{SP} - 1) \rho^2} + 2i \frac{\epsilon x^t_{\sigma}}{(2 \rho^2)^2 (f^{SP} - 1)} \right) \right\}. \quad (34)$$

In the above equation, we took into account that $\omega^{s0}_{s0\sigma}$ transforms into $O^{-1t'} \omega^{s0}_{s0\sigma} \left( \frac{\partial x^{s\sigma'}}{\partial x^{s\sigma'}} \right) (\omega^{s0}_{s0\sigma} + O^{-1t'} (\frac{\partial x^{s\sigma'}}{\partial x^{s\sigma'}}) (\frac{\partial x^{s\sigma'}}{\partial x^{s\sigma'}}))$, from which it follows that $\omega^{s0}_{s0\sigma}$ transforms into

$$O^{-1t'} \omega^{s0}_{s0\sigma} \left( \frac{\partial x^{s\sigma'}}{\partial x^{s\sigma'}} \right) \left( \frac{\partial x^{s\sigma'}}{\partial x^{s\sigma'}} \right). \quad (35)$$

Similarly we transform the term $\gamma^s \left\{ \frac{1}{2} E^s f^s_{NP} \right\}_{\gamma^t}^{\gamma^t}$ into

$$\gamma^s \left\{ \frac{1}{2} E^s f^s_{NP} \right\}_{\gamma^t}^{\gamma^t} \left\{ \frac{1}{2} f^s_{NP} \left\{ p^s_{0\sigma}, \ln \left( \frac{\rho^2}{2 \rho^2} \right) \right\}_{\gamma^t}. \quad (36)$$

The action $\int d^2 x^{NP} L^{NP}_{\psi}$, with the density from equation (6), transforms, when the coordinate system is put at the southern pole, as follows:

$$\int d^2 x^{NP} L^{NP}_{\psi} = \int d^2 x^{SP} \psi^{SP} E^S \gamma^t \gamma^0 \gamma^t \left( f^s_{NP} \epsilon x^t_{\sigma} \left( \frac{F x^t_{\sigma}}{f^{SP} (f^{SP} - 1) \rho^2} + 2i \frac{\epsilon x^t_{\sigma}}{(2 \rho^2)^2 (f^{SP} - 1)} \right) \right) S \psi^{SP}, \quad (37)$$

which leads to the Lagrange density

$L^{SP}_{\psi} = \psi^{SP} E^S \gamma^t \gamma^0 \gamma^t \left( f^s_{NP} \epsilon x^t_{\sigma} \left( \frac{F x^t_{\sigma}}{f^{SP} (f^{SP} - 1) \rho^2} + 2i \frac{\epsilon x^t_{\sigma}}{(2 \rho^2)^2 (f^{SP} - 1)} \right) \right) \psi^{SP}$. \quad (38)

The requirement that $S^{-1} \gamma^0 \gamma^t S O^{-1t} = \gamma^0 \gamma^t'$ is fulfilled by the operator $S = e^{-i 56 \omega_{56}}$, and $\omega_{56} = 2\phi + \pi$, so that in the space of the two vectors $\{+\} \psi^{(4)}_{(+)}$, $\{-\} \psi^{(4)}_{(-)}$

$$S = \begin{pmatrix} e^{i (\phi^{NP} + (\pi/2))} & 0 \\ 0 & e^{-i (\phi^{NP} + (\pi/2))} \end{pmatrix}. \quad (39)$$
with $\phi^{\text{NP}} = -\phi^{\text{SP}}$, while we have
\[ \gamma^0 \gamma^5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^0 \gamma^6 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \] (40)

Let us examine how an eigenstate of $M^{ab}$ from equation (13), expressed with respect to the coordinate at the northern pole
\[ \psi_n^{\text{NP}(6)} = \left( \alpha_n (\rho^{\text{NP}})^{56} \psi^{(4)}_\alpha + i \beta_n (\rho^{\text{NP}})^{56} \psi^{(4)}_\beta \right) e^{i\phi^{\text{NP}}} e^{i\gamma^{\text{NP}}}, \] (41)

with the property
\[ M^{\text{NP}6} \psi_n^{\text{NP}(6)} = (n + \frac{1}{2}) \psi_n^{\text{NP}(6)}, \] (42)

where $M^{\text{NP}6} = (S^{66} - i \frac{\partial}{\partial \phi^{\text{SP}}})$, look like when we put the coordinate system at the southern pole. When putting the coordinate system at the southern pole, not only $\phi^{\text{NP}}$ transforms into $-\phi^{\text{SP}}$, but also $\gamma^6$ goes into $-\gamma^6$, accordingly
\[ 56^6 (\alpha) \text{ goes into } 56^{-6} (\alpha), \] (43)
\[ 56^{-6} (\beta) \text{ goes into } 56^6 (\beta), \]

therefore $S^{66}_5 (-) = -\frac{1}{2} 56^6$ and $S^{66}_5 (+) = \frac{1}{2} 56^6$. Taking into account equations (43), (39) and (32), we obtain
\[ \psi_n^{\text{SP}(6)}(x^{\text{NP}}) = S \psi_n^{\text{NP}(6)}(x^{\text{NP}}, x^{\text{SP}}) \]
\[ = \left( i \alpha_n \left( \frac{2\rho_0}{\rho^{\text{SP}}} \right)^2 \right) e^{-i\phi^{\text{SP}}} 56^6 \psi^{(4)}_\alpha + \beta_n \left( \frac{2\rho_0}{\rho^{\text{SP}}} \right)^2 56^6 \psi^{(4)}_\beta e^{-i\gamma^{\text{SP}}}, \]
\[ = \left( i \alpha_n \left( \frac{2\rho_0}{\rho^{\text{SP}}} \right)^2 \right) e^{-i\phi^{\text{SP}}} \psi^{(4)}_\alpha + \beta_n \left( \frac{2\rho_0}{\rho^{\text{SP}}} \right)^2 \psi^{(4)}_\beta e^{-i\gamma^{\text{SP}}}. \]
(44)

When evaluating $M^{\text{SP}6} = (S^{66} + i \frac{\partial}{\partial \phi^{\text{SP}}})$ on $S \psi_n^{\text{NP}(6)}(x^{\text{NP}}, x^{\text{SP}})$, it follows that
\[ \left( S^{66} + i \frac{\partial}{\partial \phi^{\text{SP}}} \right) S \psi_n^{\text{NP}(6)}(x^{\text{NP}}, x^{\text{SP}}) = \left( n + \frac{1}{2} \right) S \psi_n^{\text{NP}(6)}(x^{\text{NP}}, x^{\text{SP}}). \]
(45)

Accordingly the massless state $\psi_n^{\text{NP}(6)m=0} = N_0^{\text{NP}} f^{\text{NP}(-F+(1/2))} 56^6 \psi^{(4)}_\alpha$ from equation (21) looks, when transforming the coordinate system from the northern to the southern pole, like
\[ \psi_n^{\text{SP}(6)m=0} = N_0^{\text{SP}} \left( f^{\text{SP}(-F+(1/2))} 56^6 \psi^{(4)}_\alpha \right) e^{-i\phi^{\text{SP}}}. \]
(46)

Taking into account that $x^{\text{SP}(5)} + i2S^{66}x^{\text{SP}(6)} = \rho^{\text{SP}} e^{-i2S^{66}\phi^{\text{SP}}}$ and
\[ \frac{\partial}{\partial x^{\text{SP}(5)}} + i2S^{66} \frac{\partial}{\partial x^{\text{SP}(6)}} = e^{-i2S^{66}\phi^{\text{SP}}} \left( \frac{\partial}{\partial \rho^{\text{SP}}} - i2S^{66} \frac{1}{\rho^{\text{SP}}} \frac{\partial}{\partial \phi^{\text{SP}}} \right), \]
we can write the equations of motion as
\[ i f e^{-i\phi^{\text{SP}}2S^{66}} \left\{ \left( \frac{\partial}{\partial \rho^{\text{SP}}} - i2S^{66} \frac{\partial}{\partial \phi^{\text{SP}}} \right) + S^{66} \frac{4F}{f^{\text{SP}}} - 2 \times 2S^{66} \right\} \psi^{(6)} + \gamma^5 m \psi^{(6)} = 0. \]
(47)
For $\psi_{n+1}^{SP(6)} = (A_{- (n+1)} e^{-i \theta_{SP}} - \psi_{(+)}^{(4)} + B_{-n} \psi_{(-)}^{(4)}) e^{-i \theta_{SP}}$, we find the equations for $A_{- (n+1)}$ and $B_{-n}$:

$$\begin{align*}
-i f \left\{ \left( \frac{\partial}{\partial \rho_{SP}} + \frac{n - 1}{\rho_{SP}} + \frac{1}{\rho_{SP}} \left( \frac{2F + 1}{f_{SP}} - 1 \right) \right) B_{-n} + m A_{- (n+1)} = 0, \\
-i f \left\{ \left( \frac{\partial}{\partial \rho_{SP}} + \frac{n + 1}{\rho_{SP}} + \frac{1}{\rho_{SP}} \left( \frac{-2F + 1}{f_{SP}} - 1 \right) \right) A_{- (n+1)} + m B_{-n} = 0. 
\end{align*}$$

(48)

Using $f_{SP}^{\rho} \frac{\partial}{\partial \rho_{SP}} = \frac{1}{\rho_{0}} \frac{\partial}{\partial \rho_{SP}}$ and $f_{SP}^{\theta} = \frac{1}{\rho_{0}} \frac{1}{\sin \theta_{SP}}$, equation (48) transforms into

$$\begin{align*}
&\left( \frac{\partial}{\partial \rho_{SP}} + \frac{n - 1 + (F + \frac{1}{2})(1 + \cos \theta_{SP})}{\sin \theta_{SP}} \right) B_{-n} + i \rho_{0} m A_{- (n+1)} = 0, \\
&\left( \frac{\partial}{\partial \rho_{SP}} + \frac{n + (F - \frac{1}{2})(1 + \cos \theta_{SP})}{\sin \theta_{SP}} \right) A_{- (n+1)} + i \rho_{0} m B_{-n} = 0.
\end{align*}$$

(49)

Again we find for $2F = 1$

$$\begin{align*}
&\left\{ \frac{1}{\sin \theta_{SP}} \frac{\partial}{\partial \theta_{SP}} \left( \sin \theta_{SP} \frac{\partial}{\partial \theta_{SP}} \right) + \left[ (\rho_{0} m)^{2} - \frac{n^{2}}{\sin^{2} \theta_{SP}} \right] \right\} A_{- (n+1)} = 0, \\
&B_{-n} = i \frac{1}{(\rho_{0} m)^{2}} \left( \frac{\partial}{\partial \theta_{SP}} + \frac{n}{\sin \theta_{SP}} \right) A_{- (n+1)}. 
\end{align*}$$

(50)

Let us conclude this section by recognizing that we have at the south pole allowed a certain special singularity which is of the following type: around a point in the two-dimensional space—the singular point—we let the phase of the wavefunction rotate so that it turns around $2\pi$ as one goes around $2\pi$ in the direction to the singular point, i.e. as $\phi$ goes around. This would for a properly smooth function only be allowed provided that the magnitude of the wavefunction decreases linearly with the distance to the singular point. Of course, from the point of view of the structure of the singularity we can make a gauge transformation and replace the just mentioned phase rotation of the wavefunction by a singular (essentially constant) value of the spin connection on the circles around the singular point.

4. Spinors and the gauge fields in $d = (1 + 3)$

To study how spinors couple to the Kaluza–Klein gauge fields in the case of $M^{(1+5)}$, ‘broken’ into $M^{(1+3)} \times S^{2}$ with the radius of $S^{2}$ equal to $\rho_{0}$ and with the spin connection field $\omega_{\mu\nu} = \text{image}[\text{vector}]/\rho_{0}$, we first look for (background) gauge gravitational fields, which preserve the rotational symmetry around the axis through the northern and the southern poles. Requiring that the symmetry determined by the Killing vectors of equation (B.1) (following [24]) with $f_{s}^{\sigma} = f^{s}_{\sigma}$, $f_{s}^{\mu} = 0$, $e_{s}^{\sigma} = f^{-1}_{s} \delta_{s}^{\sigma}$, $e_{s}^{m} = 0$, be preserved, we find for the background vielbein field

$$\begin{align*}
e_{\mu}^{a} &= \left( \begin{array}{c} \hat{e}_{\mu}^{m} \\
\hat{e}_{\mu}^{s} \end{array} \right), \quad f_{\mu}^{a} = \left( \begin{array}{c} \hat{f}_{\mu}^{m} \\
\hat{f}_{\mu}^{s} \end{array} \right) 
\end{align*}$$

(51)
with
\[ f^\sigma_m = K^{(56)\sigma} B^{(5)}_{\mu} f^\mu_m = \varepsilon^\sigma_\tau x^\tau A_\mu \delta^\mu_m, \]
\[ e^\mu_{\epsilon\sigma} = -\varepsilon^\sigma_\tau x^\tau A_\mu \epsilon_{\epsilon\sigma}, \]
\[ s = 5, 6; \sigma = 5, 6. \]
Requiring that correspondingly the only nonzero torsion fields are those from equation (B.2), we find for the spin connection fields
\[ \omega_{\mu 1\mu} = \varepsilon_\sigma A_\mu, \quad \omega_{\mu m\mu} = \frac{1}{2} f^{-1} \varepsilon_{\epsilon\sigma} x^\sigma \delta^\mu_m F_{\mu\nu}, \]
\[ F_{\mu\nu} = A_{[\nu, \mu]}. \]
The U(1) gauge field \( A_\mu \) depends only on \( x^\mu \). All the other components of the spin connection fields, except (by the Killing symmetry preserved \( \omega_{\mu\sigma} \)) from equation (6), are zero, since for simplicity we allow no gravity in \((1 + 3)\)-dimensional space. The corresponding nonzero torsion fields \( T^\mu_\nu \) are presented in equation (B.2) and in the expressions following this equation, all the other components are zero.

To determine the current, which couples the spinor to the Kaluza–Klein gauge fields \( A_\mu \), we analyse (as in (24)) the spinor action (equation (6))
\[ S = \int d^d x \psi^{(6)} \gamma^\alpha P_{0\alpha} \psi^{(6)} \]
\[ = \int d^d x \bar{\psi}^{(6)} \gamma^\tau p_\tau \psi^{(6)} + \int d^d x \bar{\psi}^{(6)} \gamma^m \delta^\mu_m \bar{p}_\mu \psi^{(6)} \]
\[ + \int d^d x \bar{\psi}^{(6)} \gamma^m \delta^\mu_m (\varepsilon^\sigma_\tau x^\tau \bar{p}_\sigma + S^{56}) \psi^{(6)} \]
\[ + \text{terms } \propto x^\sigma \text{ or } \propto x^5 x^6. \]
Here \( \psi^{(6)} \) is a spinor state in \( d = (1 + 5) \) after the break of \( M^{1+5} \) into \( M^{1+3} \times S^2 \). Note that \( E \) is for \( f^{a}_{\mu} \) from equation (51) equal to \( f^{-2} \). The term in the second row in equation (54) is the mass term (equal to zero for the massless spinor), the term in the third row is the kinetic term, which together with the term in the fourth row defines the covariant derivative \( p_{0\mu} \) in \( d = (1 + 3) \). The terms in the last row contribute nothing when integration over the disc (curved into a sphere \( S^2 \)) is performed, since they are all proportional to \( x^\sigma \) or to \( \varepsilon_{\epsilon\sigma} x^\tau x^\tau \)
\[-\gamma^m \frac{1}{2} S^{5m} \omega_{5\mu} = -\gamma^m \frac{1}{2} f^{-1} F_{mn} \varepsilon_{\epsilon\sigma} x^\alpha \]
and
\[-\gamma^m f^{\rho\sigma}_{a m} \frac{1}{2} S^{5\tau} \omega_{5\tau} = \gamma^m A_m x^5 x^5 S^{56} \varepsilon_{\epsilon\sigma} \frac{4iF(f-1)}{f^2}. \]
We end up with the current in \((1 + 3)\)
\[ J^\mu = \int E d^2 x \bar{\psi}^{(6)} \gamma^\sigma S^\mu M^{56} \psi^{(6)}. \]
The charge in \((1 + 3)\) is proportional to the total angular momentum \( M^{56} = L^{56} + S^{56} \) around the axis from the southern to the northern pole of \( S^2 \), but since for the choice of \( 2F = 1 \) (and for any \( 0 < 2F \leq 1 \)) in equation (19) only a left-handed massless spinor exists, with the angular momentum zero, the charge of a massless spinor in \((1 + 3)\) is equal to \( 1/2 \).

The Riemann scalar is for the vielbein of equation (51) equal to \( R = -\frac{1}{4} \rho^2 f^{-2} F_{mn} F_{mn} \).

If we integrate the Riemann scalar over the fifth and the sixth dimensions, we obtain
\[ \frac{8\pi}{3} (p_0)^4 F_{mn} F_{mn}. \]

5. Conclusions

In this paper, we prove that one can escape from the ‘no-go theorem’ of Witten [15], i.e. one can guarantee the masslessness of spinors and their chiral coupling to the Kaluza–Klein(like)
The normalizable massive states have masses equal to the theories of the elegant version, with only the gravity, and will help to revive them.

The normalizable massless state which is correspondingly mass protected and which couples to with one point missing and with a particular spin connection field) exists allowing only one additional interaction (or a dynamical restoration of the symmetry

\[ M^{(n+3)} \times M^{d-4} \geq 2 \] occurs in a way that vielbeins and spin connections are completely flat in all but two dimensions, while the two-dimensional space, although of finite volume, is non-compact with a particular spin connection contributing to the properties of spinors. In our particular case, it is the zweibein (the zweibein of the \( S^2 \) sphere with a hole at the southern pole) on an infinite disc which guarantees that the non-compact space has finite volume and enables, together with the spin connection field on this disc \( \omega_{st} = i F \frac{3 \epsilon_{srt}}{F_0} \), the \( \omega_{st} \) field breaks the parity symmetry and the sign of \( F \) makes a choice of the handedness of the massless state), that only one normalizable spinor state (of particular handedness) is massless, carrying the Kaluza–Klein charge of \( \frac{1}{2} \) and coupling chirally to the corresponding Kaluza–Klein gauge field. We do not discuss what generates this particular spin connection and zweibein, assuming that some fermion currents could be responsible for this choice. Let us add that requiring normalizability of states in extra dimensions guarantees that states are normalizable in the whole \( d = (1 + (d - 1)) \) space.

Since the spin connection strength is determined only within an interval \((0 < 2F \leq 1)\), what we proposed is not a fine-tuning. Taking (in the absence of fermions) the action for the gravitational gauge fields with the linear curvature for \( d = 2 \) (when any zweibein and any spin connection fulfills the corresponding equations of motion), we are allowed to make any choice of a zweibein and spin connection. (This choice leads to nonzero torsion.)

There is the discrete spectrum of normalizable eigenstates of the Hermitian Hamiltonian on the infinite disc for the chosen zweibein and spin connection of any strength \( F \) in the interval \((0 < 2F \leq 1)\), as we proved in section 2.

The normalizable eigenstates, which are chosen to be at the same time the eigenstates of the total angular momentum on the disc \( M^{56} = x^5 p^6 - x^6 p^5 + S^{56} \), with the eigenvalues \((n + 1/2)\), carry the Kaluza–Klein charge \((n + 1/2)\). The only massless state carries the charge \((\frac{1}{2})\). For the choice of \( 2F = 1 \), the normalizable massless state is independent of the coordinates on the disc. The normalizable massive states have masses equal to \( k(k + 1)/\rho_0 \), \( k = 1, 2, 3, \ldots \), with \(-k \leq n \leq k\). The spectrum is obviously discrete and stays discrete for all \( F \) in the interval \( 0 < 2F \leq 1 \) and for any finite \( \rho_0 \). The current is for all the solutions and also for all \( F \) equal to zero. As long as the Hamiltonian is Hermitian on a disc, fermions cannot leave the disc, unless an additional interaction (or a dynamical restoration of the symmetry \( M^{(n+5)} \), i.e. the phase transition) would force them to go out of the disc, which is not the case for our toy model.

Understanding the infinite disc as an \( S^2 \) sphere with the southern pole missing, a singularity of the type should be recognized: around a point in the two-dimensional space of \( S^2 \)—the singular point—we let the phase of the wavefunction rotate so that it turns around \( 2\pi \) as one goes around \( 2\pi \) in the direction to the singular point. But from the point of view of the structure of the singularity we can make a gauge transformation and replace the just mentioned phase rotation of the wavefunction by a singular (essentially constant) value of the spin connection on the circles around the singular point.

The possibility that after the break a two-dimensional manifold (with the zweibein of \( S^2 \), with one point missing and with a particular spin connection field) exists allowing only one normalizable massless state which is correspondingly mass protected and which couples to the Kaluza–Klein charge, opens, to our understanding, a new hope for the Kaluza–Klein(like) theories of the elegant version, with only the gravity, and will help to revive them.
Let us finally comment on renormalizability of the Kaluza–Klein-like theories, i.e. on theories with gravity only, just as in our case. The quantization of gravity is, as is well known, still an unsolved problem. There are several attempts [27] in the literature to quantize gravity in the weak field limit. In our case gravity is in its weak field limit, since the break of symmetries (not only in our toy model but also in the spin-charge-family theory of one of us [21]) is expected to occur far below the Planck scale, i.e. below $10^{16}$ GeV. Two of the authors of this paper [28] have started to develop the model in which the complex action model [29] could function as a cut off in loop diagrams.

Appendix A. A short presentation of the technique [25]

We make in this appendix a short review of the technique, initiated and developed by one of the authors (NSMB) when proposing an approach [21] in which all the internal degrees of freedom of either spinors or vectors can be described in the space of $d$-anticommuting (Grassmann) coordinates if the dimension of ordinary space is also $d$ and further developed in the present shape by the two authors of this paper (NSMB and HBN) [25] to construct a spinor basis as products of nilpotents and projections formed from the objects $\gamma^a$ for which we only need to know that they obey the Clifford algebra. Nilpotents and projections are odd and even objects of $\gamma^a$s, respectively, and are chosen to be eigenstates of a Cartan subalgebra of the Lorentz group.

The technique can be used to construct a spinor basis for any dimension $d$ and any signature in an easy and transparent way. Equipped with the graphic presentation of basic states, the technique offers an elegant way of seeing all the quantum numbers of states with respect to the Lorentz group, as well as transformation properties of the states under the Clifford algebra objects.

We assume the objects $\gamma^a$, which fulfil the Clifford algebra

$$\{\gamma^a, \gamma^b\} = I 2\eta^{ab}, \text{ for } a, b \in \{0, 1, 2, 3, 5, \ldots, d\},$$

(A.1)

for any $d$, even or odd. $I$ is the unit element in the Clifford algebra, while $\{\gamma^a, \gamma^b\}_\pm = \gamma^a \gamma^b \pm \gamma^b \gamma^a$.

We assume the ‘Hermiticity’ property for $\gamma^a$s

$$\gamma^{a\dagger} = \eta^{aa} \gamma^a,$$

(A.2)

in order that $\gamma^a$ are compatible with (A.1) and formally unitary, i.e. $\gamma^{a\dagger} \gamma^a = I$.

We also define the Clifford algebra objects

$$S^{ab} = \frac{i}{4} [\gamma^a, \gamma^b] := \frac{i}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a),$$

(A.3)

which close the Lie algebra of the Lorentz group $\{S^{ab}, S^{cd}\}_\pm = i(\eta^{ad} S^{bc} + \eta^{be} S^{ad} - \eta^{ac} S^{bd} - \eta^{bd} S^{ac})$. One finds from equation (A.2) that $(S^{ab})^\dagger = \eta^{aa} \eta^{bb} S^{ab}$.

Recognizing from equation (A.3) and the Lorentz algebra relation that two Clifford algebra objects $S^{ab}, S^{cd}$ with all indices different commute, we select the Cartan subalgebra of the algebra of the Lorentz group

$$S^{03}, S^{12}, S^{56}, \ldots, S^{d-1}, \text{ if } d = 2n \geq 4,$$

$$S^{03}, S^{12}, \ldots, S^{d-2}, \text{ if } d = (2n + 1) > 4.$$

(A.4)
The choice for the Cartan subalgebra in $d < 4$ is straightforward. It is useful to define one of the Casimirs of the Lorentz group—the handedness $\Gamma (\{\Gamma^a, S^{ab}\}_- = 0)$ in any $d$

$$
\Gamma := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}} \gamma^a), \quad \text{if } d = 2n,
$$

$$
\Gamma := (i)^{(d-1)/2} \prod_a (\sqrt{\eta^{aa}} \gamma^a), \quad \text{if } d = 2n + 1.
$$

(A.5)

We understand the product of $\gamma^a$s in the ascending order with respect to the index $a$: $\gamma^0 \gamma^1 \cdots \gamma^d$. It follows from equation (A.2) for any choice of the signature $\eta^{aa}$ that $\Gamma^\dagger = \Gamma$, $\Gamma^2 = I$. We also find that for $d$ even the handedness anti-commutes with the Clifford algebra objects $\gamma^a (\{\gamma^a, \Gamma\}_+ = 0)$, while for $d$ odd it commutes with $\gamma^a (\{\gamma^a, \Gamma\}_- = 0)$.

To make the technique simple, we introduce the graphic presentation as follows:

$$
\begin{align*}
(ab) & := \frac{1}{2} (\gamma^a + \frac{\eta^{aa}}{ik} \gamma^b), \quad [ab] := \frac{1}{2} \left(1 + \frac{i}{k} \gamma^a \gamma^b\right), \\
\gamma^a & := \frac{1}{2} (1 + \Gamma), \quad \gamma^a & := \frac{1}{2} (1 - \Gamma),
\end{align*}
$$

where $k^2 = \eta^{aa} \eta^{bb}$. One can easily check by taking into account the Clifford algebra relation (equation (A.1)) and the definition of $S^{ab}$ (equation (A.3)) that if one multiplies from the left-hand side by $S^{ab}$ the Clifford algebra objects $(k)ab$ and $[k]ab$, it follows that

$$
S^{ab}(k) = \frac{1}{2} k \frac{ab}{ab}, \quad S^{ab}[k] = \frac{1}{2} k \frac{ab}{ab},
$$

(A.7)

which means that we obtain the same objects back multiplied by the constant $\frac{1}{2}k$. This also means that when $(k)$ and $[k]$ acting from the left-hand side on anything (on a vacuum state $|\psi_0\rangle$, for example) are eigenvectors of $S^{ab}$. We further find

$$
\begin{align*}
\gamma^a(k) & = \eta^{aa} [-k], \quad \gamma^b(k) = -ik [-k], \\
\gamma^a[k] & = (-k), \quad \gamma^b[k] = -ik \eta^{aa} (-k),
\end{align*}
$$

(A.8)

from which it follows

$$
S^{ac}(k)[k] = \frac{1}{2} \eta^{ac} \eta^{bc} [-k][-k], \quad S^{ac}[k][k] = \frac{1}{2} (-k)(-k), \quad S^{ac} (k)[k] = -\frac{1}{2} \eta^{ac} (-k)(-k), \quad S^{ac} [k](k) = \frac{1}{2} \eta^{ac} (-k)[k].
$$

It is useful to deduce the following relations:

$$
ab(ab)(ab) = \eta^{aa} (-k), \quad [ab] = [ab],
$$

(A.9)

and

$$
\begin{align*}
(ab)(ab) & = 0, \quad (ab)(ab) = \eta^{aa} [k], \quad (-k)(ab) = \eta^{aa} [-k], \quad (-k)(ab) = 0, \\
(ab)(ab) & = 0, \quad (ab)(ab) = \eta^{aa} [k], \quad (-k)(ab) = \eta^{aa} [-k], \quad (-k)(ab) = 0,
\end{align*}
$$

(A.10)

$$
\begin{align*}
(ab)(ab) & = 0, \quad (ab)(ab) = \eta^{aa} [k], \quad (-k)(ab) = \eta^{aa} [-k], \quad (-k)(ab) = 0, \\
(ab)(ab) & = 0, \quad (ab)(ab) = \eta^{aa} [k], \quad (-k)(ab) = \eta^{aa} [-k], \quad (-k)(ab) = 0,
\end{align*}
$$

$$
\begin{align*}
(ab)(ab) & = 0, \quad (ab)(ab) = \eta^{aa} [k], \quad (-k)(ab) = \eta^{aa} [-k], \quad (-k)(ab) = 0,
\end{align*}
$$

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We recognize in the first equation of the first row and the first equation of the second row the demonstration of the nilpotent and the projector character of the Clifford algebra objects $^{ab}\gamma^{k}$ and $^{ab}\delta^{k}$, respectively.

Let us point out that whenever the Clifford algebra objects apply from the left-hand side, they always transform $^{ab}\gamma^{k}$ to $\frac{1}{2}k$, never to $\frac{1}{2}k$, and similarly $^{ab}\delta^{k}$ to $\frac{1}{2}k$, never to $\frac{1}{2}k$.

Accordingly we define a vacuum state $|\psi_{0}\rangle$ so that one finds

$$\langle k | k \rangle = 1,$$

$$\langle [k] | [k] \rangle = 1.$$

Taking into account the above equations, it is easy to find a Weyl spinor irreducible representation for $d$-dimensional space, with $d$ even or odd.

For $d$ even we simply make a starting state as a product of $d/2$, let us say, only nilpotents $^{ab}\gamma^{k}$, one for each $S^{ab}$ of the Cartan subalgebra elements (equation (A.4)), applying it to an (unimportant) vacuum state. For $d$ odd the basic states are products of $(d - 1)/2$ nilpotents and a factor $(1 \pm \Gamma)$. Then the generators $S^{ab}$, which do not belong to the Cartan subalgebra, being applied to the starting state from the left, generate all the members of one Weyl spinor.

$$\langle 0d|^{12}35|^{d-1d-2}(k_{0d})(k_{12})(k_{35})\cdots(k_{d-1d-2})\psi_{0}\rangle$$

$$\langle 0d|^{12}35|^{d-1d-2}(-k_{0d})(-k_{12})(k_{35})\cdots(k_{d-1d-2})\psi_{0}\rangle$$

$$\langle 0d|^{12}35|^{d-1d-2}(-k_{0d})(k_{12})(-k_{35})\cdots(k_{d-1d-2})\psi_{0}\rangle$$

$$\vdots$$

$$\langle 0d|^{12}35|^{d-1d-2}(-k_{0d})(k_{12})(k_{35})\cdots(-k_{d-1d-2})\psi_{0}\rangle$$

$$\langle 0d|^{12}35|^{d-1d-2}(k_{0d})(-k_{12})(-k_{35})\cdots(k_{d-1d-2})\psi_{0}\rangle$$

All the states have the handedness $\Gamma$, since $\{\Gamma, S^{ab}\} = 0$. States belonging to one multiplet with respect to the group $SO(q, d - q)$, i.e. to one irreducible representation of spinors (one Weyl spinor), can have any phase. We made a choice of the simplest one, taking all phases equal to one.

The above graphic representation demonstrates that for $d$ even all the states of one irreducible Weyl representation of a definite handedness follow from a starting state, which is, for example, a product of nilpotents $(k_{ab})^{mn}$, by transforming all possible pairs of $(k_{ab})(k_{mn})$ into $[-k_{ab}][k_{mn}]$. There are $S^{am}$, $S^{an}$, $S^{bm}$, $S^{bn}$, which do this. The procedure gives $2^{(d/2 - 1)}$ states. A Clifford algebra object $\gamma^{a}$, being applied from the left-hand side, transforms a Weyl spinor of one handedness into a Weyl spinor of the opposite handedness. Both Weyl spinors form a Dirac spinor.
For $d$ odd a Weyl spinor has, besides a product of $(d-1)/2$ nilpotents or projectors, also either the factor $\frac{\gamma^a}{\gamma^a} := \frac{1}{2}(1 + \Gamma)$ or the factor $\frac{\gamma^a}{\gamma^a} := \frac{1}{2}(1 - \Gamma)$. As in the case of $d$ even, all the states of one irreducible Weyl representation of a definite handedness follow from a starting state, which is, for example, a product of $(1 + \Gamma)$ and $(d-1)/2$ nilpotents $(k_{ab})$, by transforming all possible pairs of $(k_{ab})(k_{mn})$ into $[-k_{ab}][-k_{mn}]$. But $\gamma^a$s, being applied from the left-hand side, do not change the handedness of the Weyl spinor, since $[\Gamma, \gamma^a] = 0$ for $d$ odd. A Dirac and a Weyl spinor are for $d$ odd identical and a ‘family’ has accordingly $2^{(d-1)/2}$ members of basic states of a definite handedness.

We shall speak about left handedness when $\Gamma = -1$ and about right handedness when $\Gamma = 1$ for either $d$ even or odd.

**Appendix B. The Killing vectors and the torsion terms for our model**

The infinitesimal coordinate transformations manifesting the symmetry of $M^{1+3}$ and the $S^2$ are $x^{\mu} = x^{\mu}, x^{\nu} = x^{\nu} + \phi_1 K^{A\nu}$, with $\phi_1$ being the parameter of rotations around the axis which goes through both poles and with the infinitesimal generators of rotations around this axis $M^{(5)(6)} = x^{(5)} p^{(6)} - x^{(6)} p^{(5)} + S^{(5)(6)}$:

$$K^{A\nu} = K^{(5)(6)} = -iM^{(5)(6)} x^\nu = e^\nu_\xi x^\xi, \quad (B.1)$$

with $e^\nu_\xi = -\xi = -\xi^a_\xi, e^{(5)(6)} = 1$. The operators $K^{A}_\sigma = f^{-2} e_{\sigma\tau} x^\tau$ fulfill the Killing relation $K^{A}_\sigma + \Gamma^a_{\sigma\tau} K^{A}_{\tau\sigma} + K^{A}_{\tau\sigma} + \Gamma^a_{\sigma\tau} = 0$ (with $\Gamma^a_{\sigma\tau} = -\frac{1}{2} g^a_{\rho\sigma} (g^a_{\rho\tau} + g^a_{\tau\rho})$).

From $\gamma^a p_{0a} \gamma^a p_{0b} = p_{0a} p_{0b} - i S^{ab} S^{cd} R_{abc} + S^{cd} T^{\beta}_{ab} p_0 b$, we find for the torsion

$$T^{\beta}_{ab} = f^{\alpha}_{[a} (f^{\beta}_{b])}_{\sigma} + \omega_{[\sigma} f^{\beta}_{[a} f^{\gamma}_{b]} \cdot (B.2)$$

From equation (B.2), we read that to the torsion on $S^2$ both the zweibein $f^{\alpha}_{[a}$ and the spin connection $\omega_{\sigma a}$ contribute. While we have on $S^2$ for $R_{\sigma\tau} = f^{-2} \eta_{\sigma\tau} \frac{1}{2^{\frac{3}{2}}} \rho_6$, and correspondingly for the curvature $R = \frac{2}{(\rho 6)}$, we find for the torsion $T^{\sigma}_{is} = T^{\sigma}_{is} f^{s}_{i}$ with $T^{5}_{is} = 0 = T^{6}_{is}, \quad s = 5, 6, \quad T^{5}_{65} = -T^{5}_{65} = -(f_{6} + \frac{\phi F_j (f_{1})}{\rho_3}, x_5)$, $T^{6}_{56} = -T^{6}_{56} = -f_{5} + \frac{\phi F_j (f_{1})}{\rho_3}, x_6$. The torsion $T^{2}_{is} = T^{2}_{is}$ is for our particular choice of the zweibein and spin connection fields from equations (2) and (3) correspondingly equal to $-\frac{2 \rho_2}{(\rho_6)} (1 - (2F)^2)$.

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