CAUSAL FIELD EQUATIONS AND REAL EIGENVALUES FROM A NON-LOCAL LAGRANGIAN

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Abstract. Recently, we proposed a non-local relativistic formulation of MOND (Modified Newtonian Dynamics) [1]. The equations of motion were not derived, rather they were inferred from the result one would obtain by using the Schwinger-Keldysh formalism. The formalism simultaneously ensures causality of the field equations and the reality of in-out operator amplitudes. This point was avoided in [1] as its discussion was too far afield. Here we first demonstrate the features non-local actions generally possess: namely acausal equations of motion and non-real in-out operator amplitudes; and secondly how the Schwinger-Keldysh formalism works to provide the characteristics we usually desire from effective theories.

1 Introduction

Previous relativistic extensions of MOND centered upon scalar-tensor theories [2] thereby introducing additional degrees of freedom. We instead focused on a purely metric formulation and thus considered only gravitational degrees of freedom. A purely metric versus a scalar-tensor approach possesses a very different interpretation of the metric. For the class of theories presented in [2] a distinction was made between a “gravitational” and “physical” metric (see [3] for a complete treatment). The former is responsible for gravitational dynamics whereas the latter determines particle geodesics. Of course, a purely metric approach makes no distinction between “gravitational” and “physical” metric. They are identical and thus the strong equivalence principle is in play.

MOND was proposed by Milgrom in 1983 to offer an alternative to the dark matter description of the rotation curve phenomenon – by altering gravity at low acceleration scales, one could reproduce the asymptotically constant velocities of satellites outside the central galactic bulge [2]. Although a non-relativistic action principle was constructed which possessed the usual symmetries we seek in mechanical systems, no obvious relativistic extension presented itself. The main motivation for undertaking this challenge of formulating a relativistic version of MOND was to answer the gravitational lensing problem. Alone, General Relativity is unable to account for the amount of observed galactic lensing without invoking dark matter [4]. If MOND is to be considered an alternative to dark matter, it must have an impact on lensing. We began the project with every intention and expectation of arriving at a phenomenologically viable theory. Toward the end, however, we came to the conclusion that no purely metric formulation of MOND would be able to produce sufficient lensing (see [5] for the assumptions and analysis which led to this no-go statement). Regardless of how one views this result, there is still the question: can a relativistic, phenomenologically viable theory of MOND be formulated? The scalar-tensor approaches have proven awkward when lensing constraints are taken into account [6]. Although the class of models we considered demonstrated this phenomenological “disaster”, the formalism involved for deriving the field equations is quite instructive and will be relied upon in calculating the correction to Newton’s law of gravity in a locally de Sitter background due to quantum effects [7].

One interested in simplicity, viz. by pure degrees of freedom would certainly want to consider a purely metric extension of MOND, if for no other reason than theoretical completeness. However, as is usually the case in physics, one often gains simplicity in one facet of a theory only to lose it in another. In our case, the “sacrifice” we had to make was locality. The reason rests upon requirements placed on the Newtonian potential in the MOND limit. For MOND to faithfully reproduce the rotation curve data (the very raison d’ˆetre for MOND) it must be that in the non-relativistic regime the Newtonian potential satisfies,

\[ \nabla \cdot \left[ \mu \left( \frac{||\nabla \phi_N||}{a_0} \right) \nabla \phi_N \right] = 4\pi G \rho_m. \] (1)
Here \( \mu(x) \) is to be considered an interpolating function constructed to possess the proper MOND limiting behavior, namely \( \mu(x) \rightarrow x \forall x \ll 1 \). This corresponds to accelerations \( a \sim a_0 \sim 10^{-10} \text{ m/s}^2 \).

Now consider the weak-field expansion of General Relativity about a Minkowski background,

\[
S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R \rightarrow \frac{1}{16\pi G} \int d^4x \left\{ h_{\mu\nu}^{\mu\nu} - h_{\mu}^{\mu} + \mathcal{O}(h^2) \right\}.
\]

(2)

It does not take long before one realizes that there are no local, invariant curvature operators one could add to the Einstein-Hilbert action of General Relativity to recover the appropriate MOND behavior of (1). Since the non-relativistic MOND force law involves \( \| \nabla \phi_N \|^2 \), the weak-field expansion would have to start at cubic order in the action. However, any local curvature invariants added would start at least at quadratic order in the weak-field expansion (e.g. \( R^2, R_{\mu\nu}R^{\mu\nu}, \) etc.), and thus there is no hope in ridding ourselves of even the linear piece.

Abandoning locality is certainly not an uncommon phenomena anymore in theoretical physics. Indeed, effective theories have become more and more commonplace in regimes where ignorance of fundamental principles dominates. Gravity’s effective action, of course, falls in this category; and although we are unable to even say what its full effective action is, nothing prevents us from guessing its form. Ockam’s razor our guide, we chose the simplest class of guesses which would be capable of satisfying the non-relativistic constraint of equation (1),

\[
\mathcal{L} = \frac{1}{16\pi G} \left[ R + a_0^2 \mathcal{F} \left( a_0^{-2} g^{\mu\nu} \phi_{,\mu\nu} \right) \right],
\]

(3)

where the small potential is defined to be \( \varphi[g] \equiv \frac{1}{2} R \) (\( \Box \) is the covariant d’Alembertian). The function \( \mathcal{F}(x^2) \) is chosen as was \( \mu(x) \) from (1) to have the correct limiting behavior in the different acceleration regimes.

In general, one cannot expect (3) to have causal field equations. It is the penalty for varying a temporally non-local action. A more serious concern, however, arises when one wishes to quantize such a theory. Non-local actions are indicative of effective actions, and this is the interpretation we will adopt for this proceeding, as it was when we considered the relativistic extension of MOND in [1] and [5]. Effective actions are related to in-out vacuum amplitudes and therefore even if the operator we are considering is Hermitian, we are in no way guaranteed that its matrix elements are real. We will show using a simple scalar field example how the Schwinger-Keldysh effective action restores not only causality to the field equations, but ensures that in the context of quantum mechanics we deal with real amplitudes.

### 2 A simple scalar field example

Consider a real, massive scalar field in four dimensions with the action,

\[
S_m[\phi] = \int d^4x \mathcal{L}_m(\phi, \partial_\mu \phi).
\]

(4)

We have all learned that the corresponding in-out effective action is,

\[
\Gamma[\phi] = \frac{1}{2} \int d^4y \phi(y)[\Box - m^2] \phi(y) - \frac{1}{2} \int d^4y \int d^4z \phi(y) \Pi^2(y;z) \phi(z) + \mathcal{O}(\hbar^2),
\]

(5)

where \( \Pi^2 \) is the scalar self-energy operator responsible for quantum corrections to the scalar’s mass. Functionally varying this action with respect to \( \phi \) (and ignoring surface terms) gives the operator equation of motion to order \( \hbar \),

\[
[\Box - m^2] \phi(x) - \int d^4y \Pi^2(x;y) \phi(y) = 0.
\]

(6)

Observe the non-local term of equation (6). Since there is nothing preventing \( \Pi^2 \) from being non-zero for spacelike separations of its constituent spacetime points, this term is manifestly acausal. An analogous situation arises, of course, in electrodynamics: the force a charged particle experiences from
a static charge distribution is instantaneous, and therefore acausal. The problem is resolved by using the retarded Green’s function to the Lorentz invariant wave equation.

In addition to acausality, there is a more crucial concern: the operator spectra. Typically, one varies an effective action intending to work with operator expectation values, and in particular vacuum expectation values if working in the Heisenberg picture. The effective action formalism, however, requires the matrix elements of an operator to be in-out amplitudes, where the in and out states exist at asymptotically early and late times, respectively, when any interactions have been turned off. The in-out effective action is related to the in-out vacuum amplitude via the generating functional,

\[ W[J] = -i \ln \langle \Omega_{\text{out}} | \Omega_{\text{in}} \rangle_J, \]  

where, \( \delta W[J] \) and \( J \) is a source current. Usually, the in-out amplitude is expressed in terms of a path integral (up to multiplicative constants and measure factors which for the purposes of this proceeding are irrelevant),

\[ \langle \Omega_{\text{out}} | \Omega_{\text{in}} \rangle \propto \int D\phi \exp \left\{ i \int d^4x \sqrt{-g} \mathcal{L}_m[\phi] \right\}, \]

where the in and out vacua are defined on spacelike hypersurfaces \( \Sigma_i \) and \( \Sigma_f \), respectively. When the in state is identical to the out state the matrix elements really are just expectation values. However, in general there is nothing to prevent the in and out states to differ (even if there are no source currents). The out vacuum is not necessarily free of particles and therefore even if we demand the in vacuum to be such, it does not follow that at asymptotically late times we recover our initial condition. Consequently, we no longer expect real eigenvalues for our operator spectrum.

### 3 The Schwinger-Keldysh Formalism

What we would like is to work with an in-in vacuum amplitude – i.e. a vacuum expectation value. Then, so long as our operators are Hermitian, we need not worry about complex eigenvalues. Further, we wish to have some mechanism in place for which acausal pieces vanish in the expectation value. The Schwinger-Keldysh formalism is expressly constructed to satisfy these requests. Here we give a brief overview. Those interested in a more thorough treatment, however, are referred to [8].

It works by introducing two fields, distinguished by a plus or minus label. One then evolves forward from \( \Sigma_i \) for which only the plus field \( \phi_+ \) is non-zero to an arbitrary but spacelike surface \( \Sigma \), and antithetically backward from \( \Sigma \) to \( \Sigma_i \) for which only \( \phi_- \) is non-zero. The two fields are required to satisfy the boundary condition \( \phi_+|_{\Sigma} = \phi_-|_{\Sigma} \). This has the desired effect of transforming an in-out amplitude into an expectation value. The Schwinger-Keldysh generating functional is defined by inserting a complete set of states in the presence of now two source currents \( J_\pm \),

\[ e^{iW[J_+,J_-]} = \sum_\alpha \langle \Omega_{\text{in}} | \Omega_{\text{out}}^{\alpha} \rangle_{J_-} \langle \Omega_{\text{out}}^{\alpha} | \Omega_{\text{in}} \rangle_{J_+} \propto \int D[\phi_+] D[\phi_-] e^{i(S_m[\phi_+]-S_m[\phi_-]+J_+\phi_+-J_-\phi_-)}. \]

In general, one constructs time-ordered operator expectation values by taking variations of the generating functional and setting \( J_+ = J_- = J \) (it is to be understood that times along the backward evolution are later than those along the forward evolution),

\[ \left. W[J_+,J_-] \right|_{J_+=J_-=0} = \langle \Omega_{\text{in}} | T^\dagger \{ \phi(y_1) \ldots \phi(y_m) \} T\{ \phi(x_1) \ldots \phi(x_n) \} | \Omega_{\text{in}} \rangle, \]

where \( T \) and \( T^\dagger \) are the time and anti-time ordering symbols, respectively. The propagator, for instance, can be expressed as a matrix with elements,

\[ i\Delta_{++}(x,y) = i\langle \Omega_{\text{in}} | T\{ \phi(x)\phi(y) \} | \Omega_{\text{in}} \rangle, \]

\[ i\Delta_{+-}(x,y) = i\langle \Omega_{\text{in}} | \phi(y) \phi(x) | \Omega_{\text{in}} \rangle, \]

\[ i\Delta_{-+}(x,y) = i\langle \Omega_{\text{in}} | \phi(x) \phi(y) | \Omega_{\text{in}} \rangle, \]

\[ i\Delta_{--}(x,y) = i\langle \Omega_{\text{in}} | T^\dagger \{ \phi(x)\phi(y) \} | \Omega_{\text{in}} \rangle. \]
Working to order $\hbar$, the Schwinger-Keldysh effective action (or the in-in effective action) is,

$$
\Gamma[\bar{\phi}_+; \bar{\phi}_-] \equiv W[J_+, J_-] - J_+ \bar{\phi}_+ - J_- \bar{\phi}_-,
$$

$$
= S_m[\bar{\phi}_+] - S_m[\bar{\phi}_-] - \frac{1}{2} \int d^4y \int d^4z \bar{\phi}_+(y) \Pi^2_{++}(y; z) \bar{\phi}_+(z)
$$

$$
- \frac{1}{2} \int d^4y \int d^4z \bar{\phi}_+(y) \Pi^2_{+-}(y; z) \bar{\phi}_-(z) - \frac{1}{2} \int d^4y \int d^4z \bar{\phi}_-(y) \Pi^2_{-+}(y; z) \bar{\phi}_+(z)
$$

$$
+ \frac{1}{2} \int d^4y \int d^4z \bar{\phi}_-(y) \Pi^2_{--}(y; z) \bar{\phi}_-(z),
$$

(16)

where now $\bar{\phi}_\pm \equiv \pm \frac{\delta W[J_+, J_-]}{\delta J_{\pm}}$. The self-energy operators satisfy,

$$
\Pi^2_{++}(x; y) = \Pi^2_{++}(y; x), \quad \Pi^2_{--}(x; y) = \Pi^2_{--}(y; x),
$$

(17)

$$
\Pi^2_{+-}(x; y) = \Pi^2_{-+}(y; x).
$$

(18)

In addition to the $\pm$ labels, the operators are related by simple rules which alter the $i\epsilon$ terms in propagators and change the sign of some vertices. One obtains the field equations by varying (16),

$$
\left. \frac{\delta \Gamma[\bar{\phi}_+; \bar{\phi}_-]}{\delta \bar{\phi}_+} \right|_{\bar{\phi}_+ = \bar{\phi}_- = \phi} = \left[ \Box - m^2 \right] \phi(x) - \int d^4y \left[ \Pi^2_{++}(x; y) + \Pi^2_{--}(x; y) \right] \phi(y).
$$

(19)

It is not readily obvious that equation (19) is causal and pure real. In (19), these properties are realized via the relations,

$$
\Pi^2_{+-}(x; y) = -\Pi^2_{++}(x; y) \quad \forall \ x, y \text{ spacelike separated},
$$

(20)

$$
\Pi^2_{-+}(x; y) = \Pi^2_{++}(x; y) \quad \forall \ x, y \text{ timelike separated}.
$$

(21)

One can easily verify that (20)-(21) are satisfied by the propagator equations (12)-(15). For an example of this procedure and a more detailed treatment of the technical aspects which arise during calculations, the reader is referred to [9].

4 Conclusions

Deriving MOND from purely gravitational principles forces one to consider non-local actions. The result is that causality of the field equations and reality of the operator spectra are no longer ensured. Using a real scalar field example, we have shown how the Schwinger-Keldysh formalism overcomes these two obstacles. The result is causal and real field equations.

In our example, the Schwinger-Keldysh self-energy operators satisfied certain relationships with each other dependant upon the separation of their constituent spacetime points. If spacelike, they were exactly minus each other thus ensuring causality. If timelike, they were complex conjugates and therefore the sum real. These properties ultimately derive from the vertex signs and the regularization adopted but not shown in (12)-(15).

The analysis here is easily adapted to higher spin theories, and it is the intent to use this formalism to calculate the quantum corrections to $G_N$ in a locally de Sitter background [7]. Although we did not concern ourselves with quantizing the relativistic MOND theory, it is important to not only have an understanding of what the quantum implications of adding operators to classical actions have but also to have a formalism that generates for us physically meaningful quantities.

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