Arc-connectedness for the space of smooth $\mathbb{Z}^d$-actions on 1-dimensional manifolds

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Abstract

We show that the space of $\mathbb{Z}^d$ actions by $C^2$ orientation-preserving diffeomorphisms of a compact 1-manifold is $C^{1+ac}$ arcwise connected.

Centralizers of diffeomorphisms can be viewed as infinitesimal symmetries of a given dynamical system. Starting with the seminal work of Nancy Kopell [10], they have become a central object of study in dynamics. Perhaps the most relevant recent work on this is [1], which solves a longstanding question of Stephen Smale about centralizers of generic diffeomorphisms and, also, is a wonderful “window” to enter into this huge subject. However, despite the effort of many people, several natural problems remain unsolved. Here we deal with a longstanding question raised in the seventies by Harold Rosenberg [15], that partly inspired the thesis of Jean-Christophe Yoccoz [22]: Is the space of (orientation-preserving) commuting circle diffeomorphisms locally arcwise connected? Quite surprisingly, this has revealed as a very difficult question, and only a few (and somewhat recent) results in this direction are known. These may be summarized as follows (in all what follows, all maps in consideration are assumed to preserve the orientation):

1. The space of $\mathbb{Z}^d$ actions by homeomorphisms of either the interval or the circle is arcwise connected (although this seems to be known to the specialists, no written proof exists in the litterature; see §1.2 for a short argument);

2. The space of $\mathbb{Z}^d$ actions by $C^\infty$ diffeomorphisms of the interval is connected [2];

3. The space of $\mathbb{Z}^d$ actions by $C^1$ diffeomorphisms of either the circle or the interval is arcwise connected [16];

4. Any two $\mathbb{Z}^d$ actions by $C^{1+ac}$ circle diffeomorphisms may be connected by a path of $\mathbb{Z}^d$ actions provided one of the generators acts with an irrational rotation number [14]. (Here, $C^{1+ac}$ stands for $C^1$ maps with absolutely continuous derivative.)

It is worth stressing that results in this domain are very sensitive to different degrees of differentiability. The main result of this paper is a general arc-connectedness result in regularity $C^{1+ac}$, which is the first of this type in regularity higher than $C^1$.

Main Theorem. The space of $\mathbb{Z}^d$ actions by $C^2$ diffeomorphisms of a compact 1-manifold is $C^{1+ac}$ arcwise connected.
If $V$ denotes either the circle of the closed interval, such actions arise as holonomy representations of codimension-1 foliations of $\mathbb{T}^d \times V$ transverse to the second factor (and tangent to the boundary if nonempty), where $\mathbb{T}^d$ denotes the $d$-dimensional torus. The above statement can thus be translated in terms of foliations as follows:

**Corollary.** Any two codimension-1 foliations of class $C^2$ of $\mathbb{T}^d \times V$ transverse to the second factor (and tangent to the boundary if nonempty) can be connected by a path of $C^{1+ac}$ foliations.

### 1 Discussion and plan of the proof

#### 1.1 On the technique of proof

Our Main Theorem concerns both the circle and the interval. However, part of the circle case has been already settled. Indeed, path connexion between any action for which a generator has irrational rotation number and the corresponding action by rotations follows from item 4. above, which essentially corresponds to [14, Theorem B] (though some slight modifications in the proof are necessary). Moreover, the case where all generators have a rational rotation number can be reduced to that of the interval by passing to a finite index subgroup (yet this reduction requires several extra arguments). Details will be provided later on.

Thus, our main contribution concerns the case of the interval. In this situation, many fixed points in the interior may arise for the action. However, the restriction to each connected component of the complement of the set of fixed points is an action with no global fixed point. It is hence natural to first deal with (non necessarily faithful) actions of this type, and later check that certain paths of deformations supported on disjoint intervals fit nicely provided a good control for them can be ensured.

In case of absence of global fixed points in the interior, a key role is played by the Mather invariant of group elements. Remind that this captures the failure of a diffeomorphism to arise from a $C^1$ vector field of the closed interval. (A review of this appears in §2, with proofs of new results in the Appendix.) The discussion then splits into two different subcases.

- **The Mather invariants of group elements are trivial.**

  If the endpoints are parabolic fixed points, then we can deform the given action into the trivial one through conjugates. This uses the ideas developed in [6] based on the notion of asymptotic distortion introduced in [14]. Indeed, triviality of the Mather invariant is equivalent to the vanishing of the asymptotic distortion, which is the key (and necessary) ingredient for implementing the conjugacy argument inside the group of $C^{1+ac}$ diffeomorphisms.

  If one or both of the endpoints is hyperbolic, then it is natural to use the classical Sternberg linearization theorem (in Yoccoz' improved version) to transform the original action to one arising from a vector field for which the flow consists of diffeomorphisms that are affine close to these endpoints. The idea is then to locally deform the vector field to one which yields parabolic diffeomorphisms, so that we can apply the previous argument. This actually works, but requires a more subtle strategy because of the lack of a good
control on the norms of the conjugators whenever the maps become less and less hyperbolic. More precisely, we use conjugacies from outside of the group of $C^1$ diffeomorphisms which nevertheless preserve the smooth structure of the original maps (this idea comes from [16]).

It is worth stressing that this use of Sternberg’s theorem is the only issue where we need to assume that the original action is by $C^2$ diffeomorphisms (Sternberg’s result is no longer valid for $C^{1+ac}$ diffeomorphisms; see the Appendix II). All the other arguments are valid in $C^{1+ac}$ regularity. In any case, the proof requires extending to $C^{1+by}$ diffeomorphisms the classical Szekeres construction of generating vector fields (that is, vector fields whose time-1 map is the underlying diffeomorphism), as well as some results from [6] to this context. This is carried out in the Appendix I.

**The Mather invariant of a group element is nontrivial.**

This assumption necessarily implies that the image group is isomorphic to $Z$. Denoting by $f$ its generator, one is tempted to just deform $f$ by a simple linear homotopy (of its graph), and simultaneously deform the whole action in a coherent way (given that every group element is nothing but a power of $f$). However, the major difficulty comes from that $f$ may have a very large $C^{1+by}$ norm even in the case where the norms of the generators are small. (Here and in what follows, by $C^{1+by}$ norm we mean the total variation of the logarithm of the derivative, which for a $C^{1+ac}$ diffeomorphism $f$ will be referred to as the $C^{1+ac}$ norm and corresponds to the $L^1$ norm of its affine derivative $D^2 f / D f$.) This phenomenon is at the core of the classical examples of Sergeraert [19] (see [4, 5] for recent developments on this), and represents a major obstacle to deform a given action in a controlled way. To overcome this difficulty, the key idea consists in using the equivariance properties of the asymptotic distortion and conjugacies in order to first conjugate the original action into another one for which we can ensure that the norm of the corresponding (conjugate) diffeomorphism $f$ is small, and later proceed to the deformation by homotopy. Informally speaking, we first need to put the action in ‘good coordinates” so that the homotopy deformation behaves tamely.

We develop the arguments for each subcase above in the separate sections §3 and §4. The proof of the Main Theorem is then concluded in §5 where we carefully put all the pieces together. It is worth mentioning that this is not at all straighforward; in particular, several of the aforementioned estimates (as those arising from the case of a nontrivial Mather invariant) will be crucial at this step.

### 1.2 The key idea: averaging actions

The strategy of proof above may seems somewhat cryptic since described in technical terms. Nevertheless, we would like to stress the key idea, which consists (whenever possible) in conjugating the original action by a classical averaging procedure so that it becomes closer and closer to an action by isometries. In the present $C^{1+ac}$ setting, this is achieved by using the affine derivative. For actions by $C^1$ diffeomorphisms, the same idea was implemented in [16] via the logarithmic derivative $\log D f$. For completeness of this work, below we give an elementary result in the continuous framework for which the proof uses the same strategy. (Compare [9, Proposition (2.2), Chapitre VII].)
Proposition 1.1. The space of $\mathbb{Z}^d$ actions by homeomorphisms of either the interval or the circle is arcwise connected.

Proof. By identifying the endpoints, the case of the interval can be deduced from that of the circle, so let us only consider this one. Let $f_1, \ldots, f_d$ be the generators of the group, and let $F_i$ be a lift of $f_i$ to the real line. Denote $B(n) := \{F_1^{n_1}F_2^{n_2} \cdots F_d^{n_d} : 0 \leq n_i < n\}$, and consider the homeomorphism $\varphi_n$ defined as

$$\varphi_n(x) := \frac{1}{|B(n)|} \sum_{F \in B(n)} F(x).$$

For each $F_i$ we have

$$\varphi_n(F_i(x)) = \frac{1}{n^d} \sum_{0 \leq n_j < n \atop j \neq i} F_1^{n_1} \cdots F_{i-1}^{n_{i-1}} F_i^{n_i-1} F_{i+1}^{n_{i+1}} \cdots F_d^{n_d}(x),$$

and since the maps $F_j$ commute, this equals

$$\varphi_n(x) + \frac{1}{n^d} \left[ \sum_{0 \leq n_j < n \atop j \neq i} F_i^n (F_1^{n_1} \cdots F_{i-1}^{n_{i-1}} F_{i+1}^{n_{i+1}} \cdots F_d^{n_d}(x)) - F_1^{n_1} \cdots F_{i-1}^{n_{i-1}} F_{i+1}^{n_{i+1}} \cdots F_d^{n_d}(x) \right].$$

Recall that $(F_i^n(y) - y)/n$ uniformly converges to the translation number $\rho(F_i)$. Since there are $n^{d-1}$ terms of type $F_i^n(y) - y$ in the right-side expression above, we deduce the (uniform) convergence

$$\varphi_n(F_i(x)) - \varphi_n(x) \longrightarrow \rho(F_i) \quad \text{as} \quad n \to \infty.$$  

Changing $x$ by $\varphi_n^{-1}(x)$, this yields

$$\varphi_n(F_i(\varphi_n^{-1}(x))) \longrightarrow x + \rho(F_i) \quad \text{as} \quad n \to \infty.$$  

One readily checks that $\varphi_n$ commutes with the integer translations, hence induces a circle homeomorphism, that we still denote by $\varphi_n$. The convergence above translates into that $\varphi_n f_i \varphi_n^{-1}$ uniformly converges to the rotation by $\rho(F_i)$ mod. $\mathbb{Z}$, which is nothing but the rotation number of $f_i$. We have thus produced a sequence of conjugate actions that uniformly converges to an action by rotations. One can then construct a continuous path of such conjugates just by linear interpolation; more precisely, one considers circle homeomorphisms of the form $(1 - s)\varphi_n + s \varphi_{n+1}$, with $s \in [0, 1]$. Finally, having produced continuous paths of conjugate actions ending at actions by rotations, one can connect any two $\mathbb{Z}^d$ actions just by moving the angles of these rotation actions.  

Remark 1.2. The proof above actually shows more: the natural inclusion of $\text{SO}(2, \mathbb{R})^d$ in the space of $\mathbb{Z}^d$-actions on the circle (endowed with the compact-open topology) is a homotopy equivalence. (Compare [Z, Propositions 4.2].) We do not know whether this result extends to higher regularity. 

\footnote{The case of the interval can be also ruled out using the classical Alexander trick, but this doesn’t work for the case of the circle.}
Remark 1.3. One can produce a different proof of Proposition 1.1 by conjugating as in [16] via quasi-invariant probability measures that evolve towards the Lebesgue measure. Both arguments apply more generally to nilpotent group actions. The structural results from [18] can also be adapted to produce still another proof.

2 On the asymptotic distortion and Mather invariant

2.1 Asymptotic distortion and conjugacies

Given a diffeomorphism $f$ of a compact (connected) 1-manifold $V$ (i.e. the circle or the closed interval), we let $\text{var}(\log D f)$ be the total variation of the logarithm of its derivative. (We use the notations $C^{1+bv}$ and $\text{Diff}^{1+bv}$ to refer to maps for which this is a finite number.) The asymptotic distortion of $f$ is defined as

$$\text{dist}_\infty(f) := \lim_{n \to \infty} \frac{\text{var}(\log D f^n)}{n}.$$ 

Notice that this limit exists because of the subadditivity of $\text{var}(\log D (\cdot))$. Moreover, for each integer $n \geq 1$, one has $\text{var}(\log D f^n) \leq n \text{var}(\log D f)$, and therefore

$$\text{dist}_\infty(f) \leq \text{var}(\log D f).$$

(1)

Also notice that $\text{var} \log D (\cdot)$ is invariant under passing to the inverse; as a consequence,

$$\text{dist}_\infty(f) = \text{dist}_\infty(f^{-1}).$$

(2)

Unlike $\text{var}(\log D (\cdot))$, the quantity $\text{dist}_\infty(\cdot)$ is invariant under conjugacy. Actually, it arises as an infimum along conjugates: for every $C^{1+bv}$ diffeomorphism, one has

$$\text{dist}_\infty(f) = \inf_{h \in \text{Diff}^{1+bv}(V)} \text{var}(\log D (h f h^{-1})).$$

(3)

This appears as Proposition 1.2 in [6] for the case of the interval, yet the very same proof applies to the case of the circle.

Because of the equality above, asymptotic distortion is crucial in regard to the problem of approximating either the identity (in the case of the interval) or a rotation (in the case of the circle) by conjugates in the $C^{1+bv}$ topology. This is reflected by the next result, which appears as Theorem B in [14] for the case of the circle, but whose proof works verbatim for the case of the interval (the relevant hypothesis is the vanishing of the asymptotic distortion of maps); see Proposition 2.2 below for more details.

Proposition 2.1. Let $f_1, \ldots, f_d$ be commuting $C^{1+bv}$ diffeomorphisms of a compact 1-manifold. If the asymptotic distortion of each $f_i$ vanishes, then there exists a continuous path (for the $C^{1+bv}$ topology) of simultaneous conjugates $h_t f_i h_t^{-1}$ that starts at the given $f_i$ and finishes at isometries. Besides, along this path, each function $t \mapsto \text{var}(\log D (h_t f_i h_t^{-1}))$ is bounded from above by $\text{var}(\log D f_i)$. Finally, if $f_1, \ldots, f_d$ are of class $C^{1+ac}$, then the path is continuous for the $C^{1+ac}$ topology.
The hypothesis of vanishing asymptotic distortion arises in two relevant cases. On the one hand, it is shown in [14] that it holds for every $C^{1+ac}$ circle diffeomorphism of irrational rotation number. (Warning: this result is false for $C^{1+bv}$ diffeomorphisms; see Remark 2 therein.) On the other hand, it also holds if $f$ is a $C^{1+bv}$ diffeomorphism of the interval with no interior fixed point that has a $C^{1+bv}$ centralizer non-isomorphic to $\mathbb{Z}$ and for which the endpoints are parabolic. This follows from the relation between $\text{dist}_\infty$ and the Mather invariant, as explained later on.

We next give a more general version of Proposition 2.1 whose proof follows the very same lines but still applies in case of nonvanishing asymptotic distortion.

**Proposition 2.2.** Given any family of commuting $C^{1+bv}$ (resp. $C^{1+ac}$) diffeomorphisms $f_1, \ldots, f_d$ of a compact 1-manifold and $\varepsilon > 0$, there exists a $C^{1+bv}$-continuous (resp. $C^{1+ac}$-continuous) path of simultaneous conjugates $h_t f_i h_{t}^{-1}$ that starts at the given $f_i$ and finishes at (commuting) diffeomorphisms $\bar{f}_i$ such that $\text{var}(\log D\bar{f}_i) \leq \text{dist}_\infty(f_i) + \varepsilon$. Besides, along this path, each function $t \mapsto \text{var}(\log D(h_t f_i h_t^{-1}))$ is bounded from above by $\text{var}(\log D f_i)$.

**Proof.** For each $n \geq 1$, let $g_n$ be defined by letting $g_n(0) = 0$ and

$$Dg_n(x) := \frac{\prod_{0 \leq n_j < n} D(f_1^{n_1} \cdots f_d^{n_d})(x)}{\prod_{0 \leq n_j < n} D(f_1^{n_1} \cdots f_d^{n_d})(y)} dy.$$  

This defines a diffeomorphism, since the prescribed value for $Dg_n$ is everywhere positive and the total integral of this function equals 1. Using commutativity and the chain rule

$$D(g_n \circ f_i \circ g_n^{-1})(g_n(x)) = \frac{Dg_n(f_i(x))}{Dg_n(x)} \cdot Df_i(x),$$

we compute:

$$D(g_n \circ f_i \circ g_n^{-1})(g_n(x)) = \frac{\prod_{0 \leq n_j < n} D(f_1^{n_1} \cdots f_d^{n_d})(f_i(x))}{\prod_{0 \leq n_j < n} D(f_1^{n_1} \cdots f_d^{n_d})(x)} \cdot Df_i(x)$$

$$= \frac{\prod_{0 \leq n_j < n} D(f_1^{n_1} \cdots f_d^{n_d})(f_i(x)) \cdot Df_i(x)}{\prod_{0 \leq n_j < n} D(f_1^{n_1} \cdots f_d^{n_d})(x)}$$

$$= \frac{\prod_{0 \leq n_j < n} D(f_1^{n_1} \cdots f_i^{1+n_i} \cdots f_d^{n_d})(x)}{\prod_{0 \leq n_j < n} D(f_1^{n_1} \cdots f_i^{n_i} \cdots f_d^{n_d})(x)}$$

$$= \frac{\prod_{0 \leq n_j < n; j \neq i} D(f_1^{n_1} \cdots f_i^{n_i-1} f_{i+1}^{n_{i+1}} \cdots f_d^{n_d})(x)}{\prod_{0 \leq n_j < n; j \neq i} D(f_1^{n_1} \cdots f_i^{n_i-1} f_{i+1}^{n_{i+1}} \cdots f_d^{n_d})(x)}.$$
Thus,
\[
D(g_n \circ f_i \circ g_n^{-1})(g_n(x)) = \left[ \prod_{0 \leq n_j < n; j \neq i} D(f^n_i)(f^{n_1}_1 \cdots f^{n_{i-1}}_{i-1} f^{n_{i+1}}_{i+1} \cdots f^n_d)(x) \right]^{\frac{1}{n^d}},
\]
and therefore
\[
\log \left( D(g_n \circ f_i \circ g_n^{-1})(g_n(x)) \right) = \frac{1}{n^d} \sum_{0 \leq n_j < n; j \neq i} \log(D(f^n_i)(f^{n_1}_1 \cdots f^{n_{i-1}}_{i-1} f^{n_{i+1}}_{i+1} \cdots f^n_d)(x)).
\]
Since $\text{var}(\log D(\cdot))$ is invariant under change of coordinates, a triangle inequality yields
\[
\text{var}(\log D(g_n \circ f_i \circ g_n^{-1})) \leq \frac{1}{n^d} \sum_{0 \leq n_j < n; j \neq i} \text{var}(\log D f^n_i).
\]
Finally, by an elementary counting argument,
\[
\text{var}(\log D(g_n \circ f_i \circ g_n^{-1})) \leq \frac{n^{d-1}}{n^d} \text{var}(\log D f^n_i) = \frac{\text{var}(\log D f^n_i)}{n}.
\]
Now, by definition, the right-side expression above converges to $\text{dist}_\infty(f_i)$. Therefore, we may fix an integer $N_i$ so that it becomes smaller than or equal to $\text{dist}_\infty(f_i) + \varepsilon$. Letting $N := \max N_i$, we obtain a sequence of conjugate actions with the desired properties ending at the conjugate action by $g_N$. To obtain a continuous path, it suffices to interpolate between (the derivatives) of $g_n$ and $g_{n+1}$: for $s \in [0, 1]$, define $g_s$ by letting $g_s(0) = 0$ and
\[
Dg_s(x) = C_s Dg_n(x)^{1-s} Dg_{n+1}(x)^s
\]
for a well-chosen constant $C_s$ so that $\int_0^1 Dg_s(x) \, dx = 1$. Indeed, this does not increase the $C^{1+ac}$ norm beyond those of $g_n, f_i g_n^{-1}$ and $g_{n+1}, f_i g_{n+1}^{-1}$. The details are left to the reader. □

2.2 A detour on drift of cocycles in Banach spaces

Most of the analysis done in \cite{14} leading to Proposition 2.1 works for cocycles with respect to isometric actions on Banach spaces (see Lemma 2 therein). In the same way as Proposition 2.2 extends Proposition 2.1 to the case of nonvanishing asymptotic distortion, Lemma 2 from \cite{14} can be extended to cocycles with nonzero drift, as we explain below.

Let $U$ be a linear isometric action of a group $\Gamma$ on a Banach space $\mathbb{B}$. A cocycle for $U$ is a map $c : \Gamma \to \mathbb{B}$ that, for all $g_1, g_2$ in $\Gamma$, satisfies the relation
\[
c(g_1 g_2) = c(g_2) + U(g_2)(c(g_1)).
\]
For each $f \in \Gamma$, we define the drift of $c$ at $f$ as
\[
\text{drift}_c(f) := \lim_{n \to \infty} \frac{\|c(f^n)\|_\mathbb{B}}{n}.
\]
The limit above exists because the sequence \( n \mapsto \|c(f^n)\|_\mathbb{B} \) is subadditive; indeed,
\[
\|c(f^{m+n})\| = \|c(f^n) + U(f^n)(c(f^m))\| \leq \|c(f^n)\| + \|U(f^n)(c(f^m))\| = \|c(f^n)\| + \|c(f^m)\|.
\]
Moreover, it equals the coboundary defect of \( c(f) \) (see Lemma 2.3 below):
\[
drift_c(f) = \inf_{\psi \in \mathbb{B}} \|c(f) - (\psi - U(f)(\psi))\|_\mathbb{B}.
\] (5)

Notice that if \( \Gamma \) is an Abelian group of \( C^{1+ \alpha} \) diffeomorphisms of a 1-manifold \( V \), then \( U: (f, \varphi) \mapsto (\varphi \circ f) \cdot Df \) is an isometric action on \( L^1(V) \), and \( c(f) := \frac{Df}{D} \) is a cocycle for this action. The drift of this cocycle at \( f \) is nothing but the asymptotic distortion of \( f \). In this view, equality (5) above should be compared to (3).

The next lemma should be compared to [3], and is suitable for applications in wide contexts. As the reader will notice, the proof is an adaptation of that of Proposition 2.2 to this broader context.

**Lemma 2.3.** Let \( U \) be a linear isometric action of a finitely-generated Abelian group \( \Gamma \) on a Banach space \( \mathbb{B} \), and let \( c: \Gamma \to \mathbb{B} \) be a cocycle. Then there exists a sequence of vectors \( \psi_n \in \mathbb{B} \) such that, for all \( f \in \Gamma \), the coboundary defect
\[\|c(f) - (\psi_n - U(f)(\psi_n))\|_\mathbb{B}\]
converges to \( \text{drift}_c(f) \) as \( n \) goes to infinity.

**Proof.** We number the elements of \( \Gamma \) as \( f_1, f_2, \ldots \). Let us denote
\[B(n) := \{f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} : 0 \leq m_i < n\}\]
For each \( n \geq 1 \), define
\[\psi_n := \frac{1}{|B(n)|} \sum_{g \in B(n)} c(g).\] (6)
Each \( f \in \Gamma \) equals \( f_i \) for a certain index \( i \). Then, for each \( n \geq i \),
\[
U(f)(\psi_n) = \frac{1}{|B(n)|} \sum_{g \in B(n)} U(f)(c(g)) = \frac{1}{|B(n)|} \sum_{g \in B(n)} [c(gf) - c(f)] = -c(f) + \frac{1}{|B(n)|} \sum_{g \in B(n)} c(gf) = -c(f) + \frac{1}{|B(n)|} \sum_{g \in B(n)} c(fg).
\]
Therefore,
\[
\|c(f) - (\psi_n - U(f)(\psi_n))\|_\mathbb{B} \leq \frac{1}{|B(n)|} \sum_{g \in B(n)} [c(gf) - c(g)]\|_\mathbb{B},
\]
and the last expression equals
\[
\frac{1}{|B(n)|} \sum_{0 \leq m_j < n, j \neq i} \left\| c(f_1^{m_1} f_2^{m_2} \cdots f_{i-1}^{m_{i-1}} f_{i+1}^{m_{i+1}} \cdots f_n^{m_n}) - c(f_1^{m_1} f_2^{m_2} \cdots f_{i-1}^{m_{i-1}} f_{i+1}^{m_{i+1}} \cdots f_n^{m_n}) \right\|_\mathbb{B}.
\]
By the cocycle relation, this reduces to

\[
\frac{1}{|B(n)|} \left| \sum_{0 \leq m_j < n, j \neq i} U(f_1^{m_1} \cdots f_{i-1}^{m_{i-1}} f_i^{m_i+1} \cdots f_n^{m_n})(c(f^n)) \right|, 
\]

which, by the triangular inequality, is smaller than or equal to

\[
\frac{1}{|B(n)|} \sum_{0 \leq m_j < n, j \neq i} \|U(f_1^{m_1} \cdots f_{i-1}^{m_{i-1}} f_i^{m_i+1} \cdots f_n^{m_n})(c(f^n))\|_B = \frac{1}{|B(n)|} \sum_{0 \leq m_j < n, j \neq i} \|c(f^n)\|_B. 
\]

A simple counting argument yields the following upper bound for the last expression:

\[
\frac{1}{n^n} n^{n-1} \|c(f^n)\|_B = \frac{\|c(f^n)\|_B}{n}. 
\]

By definition, this converges to drift\(_c(f)\) as \(n \to \infty\). Therefore, the lim sup of

\[
\|c(f) - (\psi_n - U(f)(\psi_n))\|_B
\]

is at most drift\(_c(f)\).

Conversely, if for \(f \in \Gamma\) and \(\psi \in B\) we let \(C := \|c(f) - (\psi - U(f)(\psi))\|\), then, for each \(i \geq 1\), we have

\[
C = \|U(f_i)(c(f)) - (U(f^i)(\psi) - U(f^{i+1})(\psi))\|. 
\]

The triangular inequality and the cocycle relation (together with \(c(id) = 0\)) then yield

\[
nC = \sum_{i=0}^{n-1} \|U(f_i)(c(f)) - (U(f^i)(\psi) - U(f^{i+1})(\psi))\| 
\geq \left\| \sum_{i=0}^{n-1} U(f_i)(c(f)) - (U(f^i)(\psi) - U(f^{i+1})(\psi)) \right\| 
= \left\| \sum_{i=0}^{n-1} [c(f^{i+1}) - c(f^i)] - (\psi - U(f^n)(\psi)) \right\| 
\geq \|c(f^n)\| - \|\psi\| - \|U(f^n)(\psi)\|. 
\]

Therefore,

\[
C \geq \frac{\|c(f^n)\|}{n} - \frac{\|\psi\|}{n}. 
\]

Passing to the limit in \(n\) this yields \(C \geq \text{drift}_c(f)\). □
2.3 Mather invariant and the fundamental inequality

For every $C^2$ diffeomorphism $f$ of $[0, 1)$ with no fixed point in the interior, George Szekeres has built in [21] a $C^1$ generating vector field, that is, a vector field for which $f$ is the time-1 map of its flow. In the Appendix of this work, we carry out a non straightforward extension of this classical construction to $C^{1+bv}$ diffeomorphisms and study its continuity properties.

**Warning.** In order to reduce the amount of notations, in what follows we will often confuse a vector field $X$ on an interval $I$ of $\mathbb{R}$ with the function $dx(X)$, where $x$ denotes the coordinate on $\mathbb{R}$. With this abuse, given a diffeomorphism $h$, the pushforward $h^*X$ will become the function $(Dh \times X) \circ h^{-1}$, and the pull-back $h^*X$ the function $X \circ Dh$.

With the extension of Szekeres’ vector fields at hand, we can proceed to extend the definition of the Mather invariant to interval diffeomorphisms of regularity lower than $C^2$ along the classical lines. Namely, we let $\text{Diff}^{1+\Delta}_{1+bv}(\mathbb{R})$ (resp. $\text{Diff}^{1+\Delta}_{1+ac}(\mathbb{R})$) be the set of $C^2$ (resp. $C^{1+bv}$, $C^{1+ac}$) diffeomorphisms of the interval with no fixed point in the interior. (The letter $\Delta$ stands for the latter condition.) For $f \in \text{Diff}^{1+\Delta}_{1+bv}(\mathbb{R})$, let $X$ and $Y$ be the left and right vector fields of $f$, respectively. (The former arises by seeing $f$ as a diffeomorphism of $[0, 1)$, and the latter as a diffeomorphism of $(0, 1]$. Together with them comes a Mather diffeomorphism $M_f := P_Y \circ P_X^{-1}$, where $P_X$ (resp. $P_Y$) is the $C^{1+\Delta}$ diffeomorphism from $[0, 1)$ to $\mathbb{R}$ induced by $X$ (resp. $Y$) sending some fundamental interval $[a, f(a)]$ of $f$ to $[0, 1]$. In concrete terms,

$$P_X = P : x \in (0, 1) \mapsto \int_a^x \frac{du}{X(u)},$$

and a similar formula stands for $P_Y$ (with perhaps a different choice for the point $a$). Since $f$ is the time-1 map of the flows of both $X$ and $Y$, the map $M_f$ commutes with the integer translations, and it is hence the lift of a circle diffeomorphism. The Mather invariant of $f$ is the class of this circle diffeomorphism (also denoted $M_f$) modulo composition with rotations on the left and right. (These naturally come from the choice of the point $a$ in [8]; see [6] §2 for further details.)

Although the Mather invariant is not a genuine circle diffeomorphism (but an equivalence class of them), the total variation of the logarithm of its derivative is well defined. The next result that relates this with the asymptotic distortion was obtained in [6] for $C^2$ diffeomorphisms. The proof of this extended version is given in the Appendix.

**Theorem 2.4.** For every $f \in \text{Diff}^{1+\Delta}_{1+bv}(\mathbb{R})$, one has

$$|\text{var}(\log DM_f) - \text{dist}_\infty(f)| \leq |\log Df(0)| + |\log Df(1)|.$$

The following corollary of the previous theorem will be very useful for us.

**Corollary 2.5.** For every $f \in \text{Diff}^{1+\Delta}_{1+bv}(\mathbb{R})$ with trivial Mather invariant, one has

$$\text{dist}_\infty(f) = |\log Df(0)| + |\log Df(1)|.$$
Proof. Triviality of the Mather invariant of $f$ is equivalent to $\text{var}(\log DM_f) = 0$. By Theorem 2.4 this implies
\[
\text{dist}_\infty(f) \leq |\log Df(0)| + |\log Df(1)|.
\]
To show the reverse inequality, just notice that, for every $n \geq 1$,
\[
\text{var}(\log Df^n) \geq |\log Df^n(1) - \log Df^n(0)| = n[|\log Df(1)| + |\log Df(0)|].
\]
Dividing by $n$ both sides of this inequality and letting $n \to \infty$ yields the desired estimate. □

Mather invariant remains unchanged under $C^1$ conjugacy. Besides, together with the conjugacy classes of the germs at the endpoints, it totally describes $C^1$ conjugacy classes of diffeomorphisms. In case it is trivial, that is, when it coincides with (the class of) a rotation, the $C^1$ centralizer of the diffeomorphism is isomorphic to $\mathbb{R}$ (and coincides with the flow of the generating vector field). Otherwise, it reduces to $\mathbb{Z}$, and its generator is a root of the diffeomorphism. All these results were established by John Mather, and are carefully developed in Chapter V of Yoccoz’ thesis [22]. (Proofs for $C^2$ diffeomorphisms therein extend with no changes to $C^{1+bv}$ maps once the existence of generating vector fields is established.)

3 The case of a trivial Mather invariant

We consider a nonnecessarily faithful (yet nontrivial) action of $\mathbb{Z}^d$ by $C^{1+bv}$ diffeomorphisms on the interval $[0,1]$ with no global fixed point in the interior. By Kopell’s lemma [17], every element acting nontrivially admits no fixed point in the interior, hence has a well-defined Mather invariant.

Throughout this section, we assume that an element acting non trivially has a trivial Mather invariant. By Mather’s theory, if this happens, then it holds for every nontrivial diffeomorphism in the image group. We will refer to this setting just as a $\mathbb{Z}^d$ action with trivial Mather invariant.

Assume first that the endpoints are parabolic fixed points for all elements. (It is easy to see that this holds provided a nontrivial element has parabolic endpoints; see [6, Proposition 8.1] for a short argument.) In this framework, by Corollary 2.5 the asymptotic distortion of every element vanishes. Therefore, Proposition 2.1 implies the following.

Lemma 3.1. Consider a $\mathbb{Z}^d$ action by $C^{1+bv}$ (resp. $C^{1+ac}$) diffeomorphisms of $[0,1]$ with no global fixed point in the interior and trivial Mather invariant. If all elements are parabolic at the endpoints, then there exists a $C^{1+bv}$-continuous (resp. $C^{1+ac}$-continuous) path of simultaneous conjugates $h_t f_i h_t^{-1}$ starting at the original action and finishing at the trivial one along which the $C^{1+bv}$ norms of the generators do not increase.

To deal with hyperbolic fixed points, we first remind some elementary facts about germs of hyperbolic, 1-dimensional linear diffeomorphisms. We state them as a remark for future reference.
Remark 3.2. As it is very well known (see for instance [8]), the centralizer of a non trivial linear germ of diffeomorphism of the real line fixing the origin coincides with the group of germs of linear maps fixing the origin. Indeed, if \( g \) commutes with \( x \mapsto \lambda x \) then, for every \( n \in \mathbb{Z}, \)

\[
g(x) = \frac{g(\lambda^n x)}{\lambda^n} = \left( \frac{g(\lambda^n x) - g(0)}{\lambda^n x - 0} \right) x.
\]

Letting \( n \to \infty \) or \( n \to -\infty \) according to whether \( \lambda < 1 \) or \( \lambda > 1 \), respectively, the right-side expression converges to \( Dg(0) x \), which shows that \( g \) is linear.

Now, given \( \alpha > 0 \), we let \( h^\alpha \) be the germ (at the origin) of the map \( x \mapsto x^\alpha \). Notice that this is not a \( C^1 \) diffeomorphism for \( \alpha \neq 1 \), but the only failure of differentiability arises at the origin (away from it, the map is actually a \( C^\infty \) diffeomorphism). The crucial point that we will exploit is that \( h^\alpha \) conjugates the linear map \( x \mapsto \lambda x \) to \( x \mapsto \lambda^\alpha x \), which is still a linear map but of a different multiplier.

Let us again consider a \( \mathbb{Z}^d \) action on \([0, 1]\), but this time we assume that an endpoint (say, the origin) is hyperbolic for a certain (equivalently, every nontrivial) group element \( f \). If the action is by \( C^2 \) diffeomorphisms, then we may use a classical theorem of Shlomo Sternberg [20] in its sharp version (due to Yoccoz [22]): there exists a germ of \( C^2 \) diffeomorphism \( \hat{g} \) such that \( \hat{g} f \hat{g}^{-1} \) is linear about the origin. By Remark 3.2, conjugacy by \( \hat{g} \) transforms the centralizer of \( f \) inside the group of germs (which contains the image group of \( \mathbb{Z}^d \)) into the group of linear transformations. Now, a conjugacy by \( h^\alpha \) transforms this affine action into another action along which the multipliers of group elements at the origin change, and become closer to 1 as \( \alpha \) goes to zero. Finally, a conjugacy by \( \hat{g}^{-1} \) transforms this new affine action into a new \( \mathbb{Z}^d \) action by \( C^2 \) diffeomorphisms.

We denote \( \hat{g}^\alpha := (\hat{g}^{-1} h^\alpha \hat{g}) \), and we state the relevant features of this procedure: Conjugacy by \( \hat{g}^\alpha \) transforms the original \( \mathbb{Z}^d \) action into another smooth \( \mathbb{Z}^d \) action for which the multipliers of group elements at the origin become closer to 1. Moreover, the map that sends \( \alpha \) to the \( \mathbb{Z}^d \) action conjugated by \( \hat{g}^\alpha \) is a continuous deformation (starting at \( \alpha = 1 \)) of the original action. Furthermore, the Mather invariant of the new action remains trivial.

The last point deserves some attention. A different view of the previous procedure is that we have changed the Szekeres vector field \( \mathcal{X} = \mathcal{Y} \) of \( f \) to another smooth vector field that coincides with \( \alpha \mathcal{X} \) in a neighborhood of the origin. The new action restricted to this neighborhood consists just in integrating this new vector field \( \alpha \mathcal{X} \) to the same times of integration than the original action in regard to \( \mathcal{X} \). Since we always keep a smooth vector field defined on the whole interval \([0, 1]\), the Mather invariant remains trivial.

If there is also hyperbolicity at the right endpoint, we simultaneously perform a similar deformation about it (otherwise, we keep untouched a neighborhood of this point). We let \( g^\alpha \) be the resulting conjugating map that involves eventual deformation at both endpoints, and we summarize all of this in the lemma below.

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2The Sternberg-Yoccoz theorem still holds for hyperbolic germs of \( C^{1+\tau} \) diffeomorphisms (see for instance [11]). The use of this more general version allows extending our Main Theorem from \( C^2 \) to \( C^{1+\tau} \) commuting diffeomorphisms (with absolutely continuous derivative), with the exact same proof. In particular, this applies whenever the affine derivatives lie not only in \( L^1 \) but also in \( L^p \) for some \( p > 1 \).
Lemma 3.3. Conjugacy by $g^\alpha$ yields a continuous deformation of the original action (in the parameter $\alpha < 1$) along which Mather invariant remains always trivial. Besides, after conjugacy by $g^\alpha$, multipliers (at the endpoints) change from a value $Df(\cdot)$ to $Df(\cdot)^\alpha$.

We are now ready to implement the deformation in the general case of trivial Mather invariant. We stress that the conjugating maps $h_t$ we will obtain below do not belong to $\text{Diff}^{1+ac}([0,1])$ in case of hyperbolic fixed points, yet they conjugate the original action into another smooth one.

Proposition 3.4. For every $\mathbb{Z}^d$ action by $C^2$ diffeomorphisms of $[0,1]$ with no global fixed point in the interior and trivial Mather invariant, there exists a $C^{1+ac}$-continuous path of simultaneous conjugates $h_t f_i h_t^{-1}$ starting at the original action and finishing at the trivial one along which the $C^{1+bv}$ norms of the generators do not increase more than twice the $C^{1+bv}$ norms of the original action.

Proof. The case of parabolic endpoints is settled by Lemma 3.1. For the non-parabolic case, the heuristic idea is as follows: we first conjugate the action by the map $g^\alpha$ above, where $\alpha < 1$ is close-enough to 1 so that the $C^{1+bv}$ norms of the generators remain controlled. We next perform the conjugacy procedure of Proposition 2.2 until we make the total variation of the logarithm of the derivatives of the generators smaller than twice the corresponding asymptotic distortion, namely

$$2 \alpha \cdot [| \log Df_i(0) | + | \log Df_i(1) |].$$

Notice that this is a genuine reduction only if $\alpha < 1/2$; however, this is not a major problem, since the factor 2 can be easily replaced by any factor strictly larger than 1.

Observe that the previous deformation occurs along a $C^{1+ac}$-continuous path. The idea now is to repeat this argument many times so that, in the limit, we obtain the desired path by concatenation. Nevertheless, it is not clear that this process will actually converge (it may keep trapped before reaching the end). To overcome this problem we will slightly change our viewpoint using an idea from [16]. Roughly speaking, instead of directly producing the conjugating maps, we concatenate between their affine derivatives via affine paths: this allows keeping a good control on the corresponding $L^1$ norms just by convexity (triangle inequality).

For concreteness, fix any $\alpha < 1/2$ and, for each $n \geq 1$, consider the diffeomorphism $g_n$ defined by a formula similar to (4) but replacing $f_i$ by $\tilde{f}_i := g^\alpha f_i(g^\alpha)^{-1}$. For $N$ large enough, the value of each

$$\text{var}(\log D(g_N \tilde{f}_i g_N^{-1}) = \text{var}(\log D(g_N g^\alpha f_i(g_N g^\alpha)^{-1}))$$

becomes smaller than or equal to

$$2 \text{dist}_\infty(\tilde{f}_i) = 2\alpha [| \log Df_i(0) | + | \log Df_i(1) |].$$
Set $G_1 := g_N g^\alpha$, and consider its affine derivative $\frac{D^2 G_1}{DG_1}$. Although this is not an $L^1$ function, the $L^1$ norm of each

$$\frac{D^2 G_1}{DG_1} \circ f_i \cdot D f_i + \frac{D^2 f_i}{Df_i} - \frac{D^2 G_1}{DG_1}$$

is finite and, actually, smaller than or equal to $2\alpha \left[ |\log D f_i(0)| + |\log D f_i(1)| \right]$. Notice that this $L^1$ norm is nothing but

$$\text{var}(\log D(G_1 f_i G_1^{-1})).$$

Repeat this procedure starting with the $\mathbb{Z}^d$ action with generators $G_1 f_i G_1^{-1}$. One thus obtains a map $G_2$ for which

$$\frac{D^2 G_2}{DG_2} \circ (G_1 f_i G_1^{-1}) \cdot D(G_1 f_i G_1^{-1}) + \frac{D^2 (G_1 f_i G_1^{-1})}{DG_1} - \frac{D^2 G_2}{DG_2}$$

is smaller than or equal to

$$2\alpha \left[ |\log D(G_1 f_i G_1^{-1})(0)| + |\log D(G_1 f_i G_1^{-1})(1)| \right] = 4\alpha^2 \left[ |\log D f_i(0)| + |\log D f_i(1)| \right].$$

Again, this $L^1$ norm is nothing but

$$\text{var}(\log D((G_2 G_1)_i (G_2 G_1)_i^{-1})).$$

Proceeding this way, we get a sequence of conjugating maps $H_n := G_n \cdots G_2 G_1$ for which

$$\text{var}(\log D(H_n f_i H_n^{-1})) \leq (2\alpha)^n \left[ |\log D f_i(0)| + |\log D f_i(1)| \right].$$

Now, for $t \in [1 - 1/n, 1 - 1/(n + 1)]$, let $h_t$ be defined by $h_t(0) = 0$ and

$$Dh_t = C_t \left( DH_n \right)^{s_t} \left( DH_{n+1} \right)^{1 - s_t},$$

where $s_t := (1 + nt - n)(n + 1)$ is the affine function in $t$ with value 0 at $1 - 1/n$ and 1 at $1 - 1/(n + 1)$, and $C_t$ is the unique constant for which $\int_0^1 Dh_t(x) \, dx = 1$. Then,

$$D \log Dh_t = s_t \log DH_n + (1 - s_t) \log DH_{n+1}.$$ 

By the cocycle identity of the affine derivative $D^2/D = D(\log D)$, this implies that the $L^1$ norm of

$$\frac{D^2 h_t}{Dh_t} \circ f_i \cdot D f_i + \frac{D^2 f_i}{Df_i} - \frac{D^2 h_t}{Dh_t}$$

is smaller than or equal to the sum of the $L^1$ norms of

$$s_t \left[ \frac{D^2 H_n}{DH_n} \circ f_i \cdot D f_i + \frac{D^2 f_i}{Df_i} - \frac{D^2 H_n}{DH_n} \right] \quad \text{and} \quad (1 - s_t) \left[ \frac{D^2 H_{n+1}}{DH_{n+1}} \circ f_i \cdot D f_i + \frac{D^2 f_i}{Df_i} - \frac{D^2 H_{n+1}}{DH_{n+1}} \right].$$

This is bounded from above by

$$s_t (2\alpha)^n \left[ |\log D f_i(0)| + |\log D f_i(1)| \right] + (1 - s_t) (2\alpha)^{n+1} \left[ |\log D f_i(0)| + |\log D f_i(1)| \right],$$

hence by

$$(2\alpha)^n \left[ |\log D f_i(0)| + |\log D f_i(1)| \right].$$

Summarizing, for $t \in [1 - 1/n, 1 - 1/(n + 1)],$

$$\text{var}(\log D(h_t f_i h_t^{-1})) \leq (2\alpha)^n \left[ |\log D f_i(0)| + |\log D f_i(1)| \right],$$

and this estimate allows closing the proof. \(\square\)
4 The case of a nontrivial Mather invariant

Again, we consider a non necessarily faithful \( \mathbb{Z}^d \) action by \( C^{1+ac} \) diffeomorphisms of \([0, 1]\) with no global fixed point in the interior, but we now assume that the Mather invariant is not trivial. By Mather’s theory, this implies that the image group is isomorphic to \( \mathbb{Z} \). We let \( f \) be the generator of the image group. As we already mentioned, we would like to deform \( f \) and, simultaneously, the whole action. However, the \( C^{1+bv} \) norm of \( f \) may be very large, and having no control for it would lead to losing any control for the deformation.

To solve this problem, we apply Proposition 2.2 to the original action in which we include \( f \) as a generator. Notice that since the Mather invariant of nontrivial elements is nontrivial, their asymptotic distortion is positive. Proposition 2.2 restated as follows will imply that, at the end, the corresponding (conjugate) \( f \) attains a small \( C^{1+bv} \) norm.

**Lemma 4.1.** Let \( f_1, \ldots, f_\ell \) be \( C^{1+bv} \) commuting diffeomorphisms of \([0, 1]\) (not necessarily generating a group isomorphic to \( \mathbb{Z}^\ell \)) so that the Mather invariant of the action is not trivial. Then there exists a path \((h_t)_{t \in [0,1]}\) of \( C^{1+bv} \) diffeomorphisms of \([0, 1]\) starting at the identity and such that, for each \( 1 \leq i \leq \ell \), the conjugates \( h_t \circ f_i \circ h_t^{-1} \) form a continuous path of \( C^{1+bv} \) diffeomorphisms, each of which has \( C^{1+bv} \) norm smaller than or equal to that of \( f_i \); and such that

\[
\operatorname{var}(\log D(h_1 \circ f_i \circ h_1^{-1})) \leq 2 \dist_{\infty}(f_i).
\]

If all the \( f_i \) are \( C^{1+ac} \), then this deformation occurs along \( C^{1+ac} \) diffeomorphisms, and is continuous for the \( C^{1+ac} \) norm.

We are now in position to proceed to the whole deformation.

**Proposition 4.2.** Assume that a \( \mathbb{Z}^d \) action by \( C^{1+bv} \) (resp. \( C^{1+ac} \)) diffeomorphisms of \([0, 1]\) with no global fixed point at the interior does not have a trivial Mather invariant. Then there is a continuous path of actions of \( \mathbb{Z}^d \) starting at the given one and ending at the trivial action which is continuous with respect to the \( C^{1+bv} \) (resp. \( C^{1+ac} \)) topology. Besides, along this path, the \( C^{1+bv} \) (resp. \( C^{1+ac} \)) norm of the generators remains bounded from above by twice their \( C^{1+bv} \) (resp. \( C^{1+ac} \)) norm for the original action.

**Proof.** We first apply the previous lemma for \( \ell := d + 1 \) letting \( f_{d+1} := f \), where \( f \) is the generator of the image group. The outcome is a path of conjugate actions by \( C^{1+bv} \) (resp. \( C^{1+ac} \)) diffeomorphisms \( h_t \) along which the \( C^{1+ac} \) norms do not increase and, at the end, finishes with an action for which the conjugate \( F := h_1 f h_1^{-1} \) of \( f \) satisfies

\[
\operatorname{var}(\log D F) \leq 2 \dist_{\infty}(f).
\]

Now, for each \( 1 \leq i \leq d \), there exists an integer \( m_i \) such that \( f_i = f^{m_i} \). Using the homogeniity of \( \dist_{\infty} \), for those \( i \) for which \( m_i \neq 0 \), we obtain

\[
\operatorname{var}(\log D F) \leq 2 \dist_{\infty}(f) = \frac{2 \dist_{\infty}(f^{m_i})}{|m_i|} = \frac{2 \dist_{\infty}(f_i)}{|m_i|}.
\]
Let $F_t$ be the homotopy of $F$ to the identity that is linear on $\log D(\cdot)$. More precisely, we let $F_t$ be so that $F_t(0) = 0$ and
\[
DF_t(x) := \frac{e^{t+(1-t)} \log DF(x)}{\int_0^1 e^{t+(1-t)} \log DF(y) \, dy}.
\]
We concatenate the path of conjugates of the given action by $h_t$ with the path of actions that associate to the $i$th generator of $\mathbb{Z}^d$ the map $F_t^{m_i}$. Since, for a certain constant $c$,
\[
\log DF_t = t + (1 - t) \log DF + c,
\]
we have $\text{var}(\log DF_t) = (1 - t) \text{var}(\log DF)$. Therefore, by (7),
\[
\text{var}(\log DF_t^{m_i}) \leq |m_i| \text{var}(\log DF_t) = (1 - t) |m_i| \text{var}(\log DF) \leq 2 (1 - t) \text{dist}_\infty(f_i),
\]
and this yields the desired path. □

5 End of the proof

We now proceed to the proof of our Main Theorem in the general case.

Let us start with the case of the interval. Let $f_i$ be commuting $C^2$ diffeomorphisms of $[0, 1]$, and denote by $\mathcal{I}$ the family of connected components of the complement of the set of their common fixed points. Since the $C^{1+\text{ac}}$ norm is invariant under affine rescaling, we may apply Propositions 3.4 and 4.1 to the action restricted to each $\bar{I}$, for $I \in \mathcal{I}$. We thus obtain $C^{1+\text{ac}}$-continuous paths of conjugate actions on each $\bar{I}$ ending at the trivial action along which the $C^{1+\text{ac}}$ norms of the generators are always bounded from above by $2 \text{var}(\log Df_i|_I)$. Putting all these deformations together, we obtain a path of conjugate actions $(F_t)_i$ ending at the trivial action. Notice that in presence of interior hyperbolic fixed points, conjugacies on the left and right keep the action smooth provided we chose along the path the same parameter $\alpha$ for the conjugator $x \mapsto x^\alpha$ (in case of a trivial Mather invariant; see §3) and/or the parameter for the linear homotopy (in case of a nontrivial Mather invariant; see §4). This can be easily achieved.

Continuity of this path is straightforward to prove. Indeed, given $\varepsilon > 0$, we can choose a finite subfamily $\mathcal{J}$ of $\mathcal{I}$ such that, for each $i$,
\[
\sum_{I \notin \mathcal{J}} \text{var}(\log Df_i; I) < \frac{\varepsilon}{4}.
\]
Let $N$ denote the cardinal of $\mathcal{J}$. Given a time $t_0 \in [0, 1]$, we can choose $\delta > 0$ so that, for all $|t - t_0| < \delta$, all $I \in \mathcal{J}$ and all $i$, the restriction to $I$ of $(F_t)_i$ is $\varepsilon/2N$-close to that of $(F_{t_0})_i$ in the $C^{1+\text{ac}}$ topology. Since along the deformation the $C^{1+\text{ac}}$ norm of the $i$th generator restricted to each $I$ remains bounded from above by $2 \text{var}(\log Df_i; I)$, this implies that $(F_t)_i$ and $(F_{t_0})_i$ are at a distance at most
\[
2 \sum_{I \notin \mathcal{J}} \text{var}(\log Df_i; I) + \frac{\varepsilon}{2N} \cdot |\mathcal{J}| < \varepsilon,
\]
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thus showing continuity at $t_0$.

Suppose now that $f_1, \ldots, f_d$ are commuting $C^{1+ac}$ circle diffeomorphisms. If one of them has irrational rotation number, then the existence of a $C^{1+ac}$-continuous path of conjugates $h_t f_i h_t^{-1}$ ending at an action by rotations follows from [14, Theorem B], which may be seen as an application of Proposition 2.1 (Just notice that, although this result is stated for faithful actions in [14], it does not use faithfulness along the proof.) The key point here is knowing that the asymptotic distortion vanishes, and for this it is crucial to assume $C^{1+ac}$ regularity rather than $C^{1+bv}$ (see [14, Remark 2] on this).

The case where all the $f_i$ have a rational rotation number is much less trivial than expected. In this situation, the rotation number function $\rho$ yields a group homomorphism into $\mathbb{T}^1$ with finite image (see [17] for this and other structure results used below). One is then tempted to apply the arguments of the interval case to the action of $\ker(\rho)$, which is the finite-index subgroup formed by the elements having fixed points. Nevertheless, this requires several adjustments.

Let $f_*$ be such that $\rho(f_*)$ generates the image group $\rho(\Gamma)$, say $\rho(f_*) = 1/n$. We assume that $n \geq 2$, as $n = 1$ is essentially the same as the case of the interval and can be settled in a similar way: one should just take care in preserving the same multiplier at the endpoints along the deformation in case this comes from an hyperbolic periodic point in the circle, and this is ensured by the method we have employed.

Every group element uniquely writes as a product $f_i^i \bar{f}$, with $0 \leq i < n$ and $\rho(\bar{f}) = 0$. Besides, if $p_0$ denotes a point that is fixed by all elements in $\ker(\rho)$, then letting $p_i := f_i^i(p_0)$, all the intervals $[p_i, p_{i+1}]$ are fixed by these elements.

Assume for a while that the group $\ker(\rho)$ admits no global fixed point in the interior of $[p_0, p_1]$. By Kopell’s lemma, this is the case of every element therein acting nontrivially. Then there are two cases to consider.

The action of $\ker(\rho)$ on $[p_0, p_1]$ has a trivial Mather invariant. If the endpoints are parabolic, then we are in the case of vanishing asymptotic distortion, and Proposition 2.1 yields a deformation into an action by a finite-order rotation (that factors throughout $\mathbb{Z}/n\mathbb{Z}$). If $p_0$ is hyperbolic for a certain element in $\ker(\rho)$, then this is also the case for every nontrivial element in $\ker(\rho)$, and (because of the commutativity with $f_*$) this also holds at every point $p_i$. We then conjugate by maps of type $g^a$ as before at each of these points in an equivariant way. Notice that this can be done because, by the Sternberg-Yoccoz’ theorem, there is a (unique) linear coordinate around each of these points, and $f_*$ must conjugate the one at $p_i$ into that at $p_{i+1}$ by commutativity. In this way, we can build a $C^{1+ac}$-continuous path of conjugate actions (with conjugating maps that are no longer of class $C^{1+ac}$) along which asymptotic distortion becomes smaller and smaller, and the concatenation trick of the corresponding affine derivatives allows closing the proof as in §3.

The action of $\ker(\rho)$ on $[p_0, p_1]$ does not have a trivial Mather invariant. In this case, the restriction of $\ker(\rho)$ to $[p_0, p_1]$ is either trivial, in which case the group is finite, hence conjugate to a group of rotations (this conjugacy can be obviously achieved along a path), or generated by a single map. In the last case, we would like to apply the argument of §4. Nevertheless, we cannot proceed so easily since we need to preserve the equivariance under
the action of $f_*$. To do this, we let $f$ be the generator of the action of $\ker (\rho)$ on $[p_0, p_1]$. Notice that $f^n|_{[p_0, p_1]} = f^k$ for a certain integer $k \neq 0$. We start by conjugating the circle by affine maps so that, in the end, all the intervals $[p_i, p_{i+1}]$ have the same length. Then, we conjugate $f_*$ so that it becomes a rotation from $[p_i, p_{i+1}]$ onto $[p_{i+1}, p_{i+2}]$ for $0 \leq i < n - 1$ (this can be easily achieved by means of a cohomological equation on $\log(Df)$). Notice that all this procedure can be done through conjugacy by a $C^{1+ac}$-continuous path of diffeomorphisms $h_t$. If we denote $\hat{f}_*$ (resp. $\hat{f}$) the conjugate by $h_1$ of $f_*$ (resp. $f$, where it is defined), then $\hat{f}_*|_{[p_{n-1}, p_n]}$ becomes a composition of a rotation and $f^k$.

Now we apply Proposition 2.2 in order to conjugate the action of $\ker (\rho)$ on $[p_0, p_1]$ along a path so that the $C^{1+bv}$ norm of $\hat{f}$ becomes very close to $\text{dist}_{\infty}(\hat{f})$. We perform the very same deformation on the remaining intervals, just by conjugating by the corresponding rotations. Next, as in §4 we deform the conjugate version of $\hat{f}$ along its graph until reaching the identity, and we extend this deformation to the rest of the intervals via conjugacy by the corresponding rotations. Finally, we extend this deformation to $\hat{f}_*$ so that it coincides with a rotation except for the last interval, where it is forced to coincide with the composition of a rotation with the $k^{th}$ power of the corresponding deformed version of $\hat{f}$. We leave to the reader to check that this deformation is well behaved (with a good control on the $C^{1+ac}$ norm) and ends with an action by a single rotation of order $n$.

Now we no longer assume that $\ker (\rho)$ acts with no global fixed point in the interior of $[p_0, p_1]$. In this case, we have a countable family $\mathcal{I}$ of closed intervals $I$ in $[p_0, p_1]$ with disjoint interior on each of which $\ker (\rho)$ acts with no global fixed point inside. It is not hard to see that we may apply the arguments above to the restriction of the action on the union of intervals $I \cup f_*(I) \ldots \cup f^{n-1}_*(I)$. (Indeed, most of our results work verbatim for non-connected compact 1-manifolds...) Pasting together the corresponding deformations along different $I \in \mathcal{I}$ yields the desired $C^{1+ac}$-continuous path. The only delicate issue is, again, that of the multipliers at the hyperbolic periodic points, but this is still ensured by the uniqueness of linear coordinates around them.

In all cases, we have connected the original action with an action by isometries. Since any two actions by rotations can be connected just by moving the angles, this allows connecting any two $\mathbb{Z}^d$-actions, thus closing the proof.

6 Appendix I: vector fields for $C^{1+bv}$ diffeomorphisms of the interval

Here we proceed to give proofs of the results announced in §2.3 and later proceed to some further developments.

6.1 The construction of the vector field

Remind that $\text{Diff}_+^{1+bv, \Delta}([0, 1])$ (resp. $\text{Diff}_+^{1+ac, \Delta}([0, 1])$) denotes the space of $C^{1+bv}$ (resp. $C^{1+ac}$) diffeomorphisms of $[0, 1]$ with no fixed point in the interior. For simplicity, whenever
it is defined, we will denote the affine derivative $D \log D f$ simply by $L f$. (For a $C^{1+ac}$
diffeomorphism, this is an $L^1$ function.)

**Proposition 6.1.** Given $f \in \text{Diff}^{1+\text{bv}, \Delta}_1([0, 1])$ such that $f(x) > x$ for every $x \in (0, 1)$, let

$\Delta(x) := f(x) - x$, and let

$c_0(f) := \begin{cases} \frac{\log D f(0)}{D f(0) - 1} & \text{if } D f(0) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$

For each $n \geq 0$, let $X_n := c_0(f)(f^n)_*(\Delta) = c_0(f) \frac{\Delta f^n}{D f^n}$. Then:

1. the sequence of vector fields $X_n$ uniformly converges on every compact subset of $[0, 1]$;
2. its limit $X$ is uniquely integrable and complete, and $f$ is the time-1 map of the corresponding flow;
3. the (well-defined) flow $(f^t)$ of $X$ coincides with the $C^1$ centralizer of $f$;
4. for every $c > 0$, the function $\log X_n$ converges to $\log X$ in the BV topology on the fundamental interval $[c, f(c)]$, and

$$|\text{var}(\log X; [c, f(c)]) - \log D f(0)| \leq \text{var}(\log D f; [0, c]).$$

Furthermore, if $f \in \text{Diff}^{1+\text{ac}, \Delta}_1([0, 1])$, then

5. the function $\log X$ is absolutely continuous on every fundamental interval $[c, f(c)]$, and

$$\left\| D \log X - \frac{\log D f(0)}{X} \right\|_{L^1([c, f(c)])} \leq \|L f\|_{L^1([0, c])}.$$  

**Proof.** We proceed step by step.

1. For every $k \in \mathbb{N}$, we have $\log \frac{X_{k+1}}{X_k} = \theta \circ f^{-k}$ with, for every $x \in (0, 1)$,

$$\theta(x) := \log \left( \frac{x - f^{-1}(x)}{D f^{-1}(x) (f(x) - x)} \right) = \log \left( \int_0^1 D f^{-1} (x + s(f(x) - x)) \, ds \right) - \log(D f^{-1}(x))$$

(the last equality follows from Taylor’s integral formula). In particular, $\theta$ extends to $[0, 1)$ as a continuous map, with $\theta(0) = 0$. By the mean value theorem, for every $x \in [0, 1)$,

$$\int_0^1 D f^{-1} (x + s(f(x) - x)) \, ds = D f^{-1}(y_x) \quad \text{for some } y_x \in [x, f(x)].$$

Therefore, given $c \in [0, 1)$, for every $x \in [0, c]$ and every $0 \leq i \leq j$, we have

$$\sum_{k=i}^{j-1} |\theta \circ f^{-k}(x)| = \sum_{k=i}^{j-1} \left| \log D f^{-1}(y_{f^{-k}(x)}) - \log D f^{-1}(f^{-k}(x)) \right|$$

$$\leq \text{var}(\log D f^{-1}; [0, f^{-i+1}(c)]) = \text{var}(\log D f; [0, f^{-i}(c)]) \xrightarrow[i \to +\infty]{} 0.$$
As a consequence, $\sum_k \theta \circ f^{-k}$ converges absolutely and uniformly on $[0, c]$. Denote by $\Sigma$ this sum (which is thus continuous on $[0, 1]$). Then $(X_n)$ converges uniformly on every compact subset of $[0, 1]$ towards $X := c_0(f) \Delta e^\Sigma$. In particular, $X$ vanishes only at 0, and is strictly positive at all points of $(0, 1)$.

2. Since by definition $X_{n+1} = f_* X_n$, in the limit we obtain $X = f_* X$, or equivalently $X = f^* X$, that is, $X = X_{[0, f]}$. This implies that the derivative of $x \mapsto \int_x^{f(x)} \frac{du}{x(u)}$ is identically 0 on $(0, 1)$, so this map is constant, say equal to some $\tau \in \mathbb{R}^*$. The proof of the equality $\tau = 1$ in the proof of the “usual” Szekeres theorem (i.e. when $f$ is assumed $C^2$) reproduced in [17, Proposition 4.1.14] works without any change in the present setting.

Let us now show that $X |_{(0, 1)}$ is complete and uniquely integrable. Fix $a \in (0, 1)$, and let

$$P_X = P : x \in (0, 1) \mapsto \int_a^x \frac{du}{X(u)}.$$ (8)

The map $P$ is of class $C^1$, with positive derivative, and $P(f^n(a)) = n$ for every $n \in \mathbb{Z}$. Therefore, $P$ defines a $C^1$ diffeomorphism between $(0, 1)$ and $\mathbb{R}$. Its inverse $\psi := P^{-1}$ hence sends $\mathbb{R}$ into $(0, 1)$.

We claim that, for every $(t_0, x_0) \in \mathbb{R} \times (0, 1)$, the Cauchy problem

$$\begin{cases} \dot{x} = X(x) \\ x(t_0) = x_0 \end{cases}$$

has a unique maximal solution, defined on all of $\mathbb{R}$ by $\gamma(t) := P^{-1}(t - t_0 + P(x_0))$. (In particular, the only solution of $\dot{x} = X(x)$ passing through 0 is constant equal to 0.) Indeed, given $(t_0, x_0) \in \mathbb{R} \times (0, 1)$, one immediately checks that the map $\gamma$ above is a (maximal) solution to the above Cauchy problem. To prove uniqueness, first notice that one cannot invoke the Cauchy-Lipschitz theorem, since $X$ is not necessarily locally Lipschitz. However, the one-dimensional setting provides a more elementary argument, as shown below.

Assume $\tilde{\gamma} : I \to [0, 1)$ is a maximal solution of the same Cauchy problem, and let $I_\star \subset I$ be the maximal interval containing $t_0$ on which $\tilde{\gamma}$ does not vanish. Then for every $t \in I_\star$, we have $\dot{\tilde{\gamma}}(t) = X(\tilde{\gamma}(t))$, with $X(\tilde{\gamma}(t)) \neq 0$, hence $\frac{\dot{\tilde{\gamma}}(t)}{X(\tilde{\gamma}(t))} = 1$. By integration and change of variables, we obtain

$$\int_{\tilde{\gamma}(t_0)}^{\tilde{\gamma}(t)} \frac{du}{X(u)} = t - t_0,$$

that is, $P(\tilde{\gamma}(t)) - P(x_0) = t - t_0$, and so $\tilde{\gamma}(t) = \gamma(t)$. It hence remains to justify that $I_\star = \mathbb{R}$. Assume by contradiction that one of the extrema of $I_\star$, say its infimum to fix ideas, is finite. Then the restriction of $\tilde{\gamma}$ to $I_\star$ is not maximal, since it can be extended until $-\infty$ by $\gamma$. Therefore, the infimum $\alpha$ of $I_\star$ is not that of $I$. This implies that $\lim_{t \to \alpha} \tilde{\gamma}(t) = \lim_{t \to \alpha} \gamma(t) = 0$. However, this is in contradiction to $\lim_{t \to \alpha} \gamma(t) = \gamma(\alpha) \in (0, 1)$. A similar argument shows that sup $I_\star = +\infty$, thus closing the proof of the uniqueness.

Summarizing, the flow $(t, x) \mapsto \phi(t, x)$ of $X$ is well-defined on $\mathbb{R} \times [0, 1)$ and given by $\phi(t, x) = P^{-1}(t + P(x))$ for $x \neq 0$ and $\phi(t, 0) = 0$ for every $t$. In particular, the equality

$$\int_x^{f(x)} \frac{du}{X(u)} = 1,$$
which is equivalent to $P(f(x)) - P(x) = 1$, means that $f$ is the time-1 map of $\mathcal{X}$.

3. For every $t \in \mathbb{R}$, the time-$t$ map $f^t = \phi(t, \cdot)$ of the flow of $\mathcal{X}$ commutes with $f$. Therefore, in order to derive 3. from Kopell’s Lemma [10], it is enough to prove that $f^t$ is a $C^1$ diffeomorphism of $[0, 1)$ for each $t$. The formula for $\phi(t, x)$ given in the proof of 2. shows that $f^t$ is $C^1$ and $Df^t = \frac{\lambda^t}{\lambda} f$ on $(0, 1)$. We are thus reduced to proving that $Df^t(x)$, or rather $\log Df^t(x)$, has a limit when $x$ goes to 0, namely $t \log Df(0)$. To do this, it is enough to restrict to $t \in [0, 1]$. For every $x > 0$,

$$
\log Df^t(x) = \log \left( \frac{\mathcal{X}(f^t(x))}{\mathcal{X}(x)} \right) = \log \left( \frac{\Delta(f^t(x))}{\Delta(x)} \right) + \Sigma(f^t(x)) - \Sigma(x).
$$

Since $\Sigma$ is continuous at 0 and vanishes at this point, what we need to prove is that

$$
\lim_{x \to 0} \log \left( \frac{\Delta(f^t(x))}{\Delta(x)} \right) = t \log Df(0).
$$

If $Df(0) = 1$ then, for some $u \in [x, f^t(x)]$,

$$
\left| \frac{\Delta(f^t(x))}{\Delta(x)} - 1 \right| = \left| \frac{\Delta'(u) \times f^t(x) - x}{\Delta(x)} \right| \leq \max_{y \in [x, f^t(x)]} |\Delta'(y)| \xrightarrow{x \to 0} |\Delta'(0)| = 0.
$$

If $\lambda := Df(0) > 1$, then $\Delta'(0) = \lambda - 1 > 0$, so $\frac{\Delta(y)}{y} \xrightarrow{y \to 0} \lambda - 1 \neq 0$. This implies $\frac{\Delta(f^t(x))}{\Delta(x)} \sim_{x \to 0} \frac{f^t(x)}{x}$. Thus, we need to prove that $\log \left( \frac{f^t(x)}{x} \right) \xrightarrow{x \to 0} t \log \lambda$. To do this, first observe that

$$
\mathcal{X}(x) = \frac{\log \lambda}{\lambda - 1} \Delta(x) e^{\Sigma(x)} \sim_{x \to 0} (\log \lambda) x.
$$

Thus, given $\varepsilon > 0$, we may let $\delta > 0$ be such that

$$
\frac{1 - \varepsilon}{(\log \lambda)u} < \frac{1}{\mathcal{X}(u)} < \frac{1 + \varepsilon}{(\log \lambda)u}
$$

for every $u \in (0, \delta]$. Assuming that $f^t(x)$ (and thus $x$) is in this interval, we obtain

$$
\int_x^{f^t(x)} \frac{1 - \varepsilon}{(\log \lambda)u} du \leq \int_x^{f^t(x)} \frac{du}{\mathcal{X}(u)} \leq \int_x^{f^t(x)} \frac{1 + \varepsilon}{(\log \lambda)u} du,
$$

hence

$$
\frac{1 - \varepsilon}{\log \lambda} \log \left( \frac{f^t(x)}{x} \right) \leq t \leq \frac{1 + \varepsilon}{\log \lambda} \log \left( \frac{f^t(x)}{x} \right),
$$

and thus

$$
\frac{t \log \lambda}{1 + \varepsilon} \leq \log \left( \frac{f^t(x)}{x} \right) \leq \frac{t \log \lambda}{1 - \varepsilon}.
$$

Letting $\varepsilon \to 0$, this gives the desired limit for $\log \left( \frac{f^t(x)}{x} \right)$. 

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4. Since \( \mathcal{X} \) is bounded away from zero on \([c, f(c)]\), by item 1, we have that \( \log \mathcal{X}_k \) converges uniformly towards \( \log \mathcal{X} \) on this segment. Now for every \( k \geq j \in \mathbb{N} \),

\[
\var\left( \log \mathcal{X}_k - \log \mathcal{X}_j; [c, f(c)] \right) \leq \var\left( \log(\Delta \circ f^{-k}) - \log(\Delta \circ f^{-j}); [c, f(c)] \right) + \var\left( \log Df^{-k} - \log Df^{-j}; [c, f(c)] \right).
\] (9)

Concerning the last term,

\[
\var(\log Df^{-k} - \log Df^{-j}; [c, f(c)]) = \var\left( \sum_{i=j}^{k-1} \log(Df^{-1} \circ f^{-i}); [c, f(c)] \right)
\leq \sum_{i=j}^{k-1} \var(\log(Df^{-1} \circ f^{-i}); [c, f(c)])
= \sum_{i=j}^{k-1} \var(\log(Df^{-1}; [f^{-i}(c), f^{-i+1}(c)]))
\leq \var(\log(Df^{-1}; [f^{-k+1}(c), f^{-j+1}(c)]))
= \var(\log(Df; [f^{-k}(c), f^{-j}(c)])) \xrightarrow{j \to +\infty} 0.
\]

Concerning the previous term in (9),

\[
\var(\log(\Delta \circ f^{-k}) - \log(\Delta \circ f^{-j}); [c, f(c)]) = \| D \log(\Delta \circ f^{-k}) - D \log(\Delta \circ f^{-j}) \|_{L^1([c, f(c)])}
= c_0(f) \| \frac{D\Delta f^{-k}}{\mathcal{X}_k} - \frac{D\Delta f^{-j}}{\mathcal{X}_j} \|_{L^1([c, f(c)])} \xrightarrow{j \to +\infty} 0,
\]

since \( \frac{D\Delta f^{-k}}{\mathcal{X}_k} \) converges uniformly towards \( \frac{Df^{-1}}{\mathcal{X}_k} \) on \([c, f(c)]\). By completeness of \( BV([c, f(c)]) \), we get that \( \log \mathcal{X} \) belongs to this space. Observe furthermore that

\[
\var(\log(\Delta \circ f^{-k}); [c, f(c)]) = c_0(f) \int_c^{f(c)} \left| \frac{D\Delta f^{-k}}{\mathcal{X}_k} \right| \xrightarrow{k \to +\infty} c_0(f) \int_c^{f(c)} \frac{D\Delta f(0)}{\mathcal{X}} = \log Df(0),
\]

and that (because of the previous estimate with \( j = 0 \))

\[
\var(\log Df^{-k}; [c, f(c)]) \leq \var(\log(Df; [f^{-k}(c), c])).
\]

Therefore, letting \( k \) go to infinity in

\[
\var(\log \mathcal{X}_k; [c, f(c)]) \leq \var(\log(\Delta \circ f^{-k}); [c, f(c)]) + \var(\log Df^{-k}; [c, f(c)])
\]

and

\[
\var(\log(\Delta \circ f^{-k}); [c, f(c)]) \leq \var(\log \mathcal{X}_k; [c, f(c)]) + \var(\log Df^{-k}; [c, f(c)])
\]

(which both follow from the definition of \( \mathcal{X}_k \)), we get

\[
\var(\log \mathcal{X}; [c, f(c)]) \leq \log Df(0) + \var(\log Df; [0, c])
\]

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and
\[\log Df(0) \leq \text{var}(\log \mathcal{X}; [c, f(c)]) + \text{var}(\log Df; [0, c]),\]
which yield the desired estimate.

5. We now assume \(f \in \text{Diff}^{1+ac,\Delta}(0, 1)\). Then, by the definition, \(\log \mathcal{X}_k\) is locally absolutely continuous on \((0, 1)\). Since, for every \(c > 0\), the subspace \(\mathcal{AC}([c, f(c)])\) is closed in \(BV([c, f(c)])\), the function \(\log \mathcal{X}\) is also absolutely continuous on \([c, f(c)]\), and so is \(\mathcal{X}\). One can then improve the estimate of item 4, as follows: Given \(k \geq 0\), the functions \(\mathcal{X}_k\) and \(Df^k\) are almost everywhere differentiable, and the following equalities hold almost everywhere on \([0, 1]\):
\[
D \log \mathcal{X}_k = D \left(\log(\Delta \circ f^{-k}) - \log(Df^{-k})\right) = \frac{D\Delta(f^{-k})}{\Delta(f^{-k})} \cdot Df^{-k}(x) - D \log Df^{-k} = c_0(f) \frac{D\Delta(f^{-k})}{\mathcal{X}_k} - Lf^{-k}.
\]
Therefore,
\[
\left\|D \log \mathcal{X}_k - c_0(f) \frac{D\Delta(f^{-k})}{\mathcal{X}_k}\right\|_{L^1([c, f(c)])} = \left\|Lf^{-k}\right\|_{L^1([c, f(c)])} = \left\| \sum_{i=0}^{k-1} L(f^{-1}) \circ f^{-i} \cdot Df^{-i}\right\|_{L^1([c, f(c)])} \leq \sum_{i=0}^{k-1} \left\|L(f^{-1})\right\|_{L^1([f^{-i+1}(c), f^{-i}(c)])} \leq \left\|Lf\right\|_{[0, c]},
\]
and taking the limit when \(k\) goes to infinity gives the desired estimate. \(\square\)

### 6.2 Mather invariant and the fundamental inequality revisited

Our goal here is to prove Theorem [2,4]. Namely, for every \(f \in \text{Diff}^{1+bv,\Delta}(0, 1)\),
\[
|\text{var}(\log DM_f) - \text{dist}_\infty(f)| \leq |\log Df(0)| + |\log Df(1)|.
\] (10)

As mentioned in [2,3] this corresponds to an extension of [6] Theorem B] to the \(C^{1+bv}\) setting.

Assume that \(f(x) > x\) for \(x \in (0, 1)\) to fix ideas (otherwise, just use [2] and the fact that, by definition, the Mather invariant of \(f^{-1}\) equals that of \(f\) up to a reflexion.) Since \(f\) is the time-1 map of the flow of both \(\mathcal{X}\) and \(\mathcal{Y}\), the maps \(\psi_\mathcal{X} = P_\mathcal{X}^{-1}\) and \(\psi_\mathcal{Y} = P_\mathcal{Y}^{-1}\) satisfy \(\psi \circ T = f \circ \psi\) for \(T := T_1\), the translation by 1. Therefore, for each positive \(m, n\) we have, letting \(k := m + n:\)
\[
M_f = T_{-m} \circ (\psi_\mathcal{Y})^{-1} \circ f^k \circ \psi_\mathcal{X} \circ T_{-n}.
\] (11)
This easily implies that
\[
\| \langle \varphi(t) \rangle - \langle \varphi(t-n) \rangle \| \leq \| \varphi(t) - \varphi(t-n) \|.
\]
This yields,
\[
\left| \varphi(t) - \varphi(t-n) \right| = \frac{\left| \varphi(t) - \varphi(t-n) \right|}{\left| \varphi(t) - \varphi(t-n) \right|} \cdot \left| \varphi(t) - \varphi(t-n) \right|.
\]
By item 4. of Proposition 1, the latter expression is smaller than or equal to
\[
\left| \varphi(t) - \varphi(t-n) \right| = \frac{\left| \varphi(t) - \varphi(t-n) \right|}{\left| \varphi(t) - \varphi(t-n) \right|} \cdot \left| \varphi(t) - \varphi(t-n) \right|.
\]
Letting \( m = n = N \rightarrow \infty \), the last two terms above converge to 0, and Proposition 5.1 of
\[
\left| \varphi(t) - \varphi(t-n) \right| \leq \| \varphi(t) - \varphi(t-n) \|.
\]
Putting everything together, we finally obtain
\[
\left| \varphi(t) - \varphi(t-n) \right| = \frac{\left| \varphi(t) - \varphi(t-n) \right|}{\left| \varphi(t) - \varphi(t-n) \right|} \cdot \left| \varphi(t) - \varphi(t-n) \right|.
\]
From \( M_f = (\psi^{-1}) \circ \psi \) we obtain
\[
\log M_f = (\log \mathcal{X} - \log \mathcal{Y} \circ \psi).n
\]
It readily follows from item 4. in Proposition \ref{prop} (and its analog for \( \mathcal{Y} \)) that \( M_f \) belongs to
\[
\text{Diff}^{1+\bullet}(\mathbb{R}/\mathbb{Z}) \] for \( f \in \text{Diff}^{1+\bullet}(\mathbb{R}/\mathbb{Z}) \), where \( \bullet \) stands for either \( b \) or \( a \); moreover,
\[
\varphi(t) - \varphi(t-n) = \frac{\varphi(t) - \varphi(t-n)}{\varphi(t) - \varphi(t-n)} \cdot \varphi(t) - \varphi(t-n).
\]

\textbf{Remark 6.2.} The proof above has the disadvantage of using a result from \cite{6}. Notice also that the first proof of \cite{10} given in \cite{6} for \( C^2 \) diffeomorphisms applies with some minor adjustments to \( C^{1+ac} \) diffeomorphisms, but fails in \( C^{1+bv} \) regularity. Below we propose a direct argument at least for half of the inequality.

\[
\varphi(t) - \varphi(t-n) = \frac{\varphi(t) - \varphi(t-n)}{\varphi(t) - \varphi(t-n)} \cdot \varphi(t) - \varphi(t-n).
\]
Now, since $M_f$ is invariant under conjugacy, this implies
\[
\var(\log DM_f) \leq |\log Df(0)| + |\log Df(1)| + \inf_h \var(\log Dhf h^{-1}),
\]
where the infimum runs over all $h \in \text{Diff}^{1+\text{bv}}_+(\mathbb{R})$. By (3), this infimum is nothing but the asymptotic distortion of $f$. Thus, the previous inequality becomes
\[
\var(\log DM_f) \leq |\log Df(0)| + |\log Df(1)| + \text{dist}_\infty(f),
\]
which is one of the inequalities involved in (10).

6.3 The case of piecewise smooth homeomorphisms

Equality (11) allows thinking of the Mather invariant as a renormalization of the action of high powers of $f$. For concreteness, assume again that $f(x) > x$ for all $x \in (0,1)$, and suppose that both $m, n$ are positive. Equality (11) then says that, in order to compute $M_f$ on $[0,1]$ (which is identified to $[a,f(a)]$ via $\psi_X$) we may proceed by going to the translated point $t - n$, look for the image under $k$ iterates of $f$ of $\psi_X(t - n)$, coming back to the real line by $\psi_Y^{-1}$, and finally translating by $-m$. This is nothing but looking at the action of $f^k$ from the interval $[f^{-n}(a), f^{-n+1}(a)]$ into $[f^m(a), f^{m+1}(a)]$, both identified to the unit segment, the former via $\psi_X^{-1}$ and the latter via $\psi_Y^{-1}$.

There are two applications of this view. The first is that, if we know a priori the vector fields $\mathcal{X}$ and $\mathcal{Y}$ in neighborhoods of the corresponding endpoints, then (11) explicitly gives the Mather invariant. This is particularly useful in the case where $f$ is of class $C^2$ and the endpoints are hyperbolic fixed points of $f$. Indeed, in this situation, the Sternberg-Yoccoz linearization theorem establishes that the germs of $f$ at these points are $C^2$ linearizable. Therefore, up to a $C^2$ change of coordinates, we may assume that $\mathcal{X}(y) = \lambda y$ (resp. $\mathcal{Y}(z) = \mu (1 - z)$) in a neighborhood of 0 (resp. 1), where $\lambda := \log(Df(0)) > 0$ (resp. $\mu := \log(Df(1)) < 0$). Taking $m, n$ large enough so that the intervals $[f^{-n}(a), f^{-n+1}(a)]$ and $[f^m(a), f^{m+1}(a)]$ lie inside the interior of these domains of linearization, this yields a particularly simple expression for (11).

Another application of this view of the Mather invariant is the extension of its definition to homeomorphisms that are $C^{1+\text{bv}}$ except for finitely many points in the interior and have nonvanishing left and right derivatives (piecewise $C^{1+\text{bv}}$ diffeomorphisms, for short). Of course, one way to proceed in this case is to allow the vector fields $\mathcal{X}$ and $\mathcal{Y}$ to have discontinuities. Indeed, both $\mathcal{X}$ and $\mathcal{Y}$ are well defined in neighborhoods of the corresponding endpoints (because vector fields exist for germs of diffeomorphisms, as easily follows from Proposition 6.1), and starting from there they can be extended to the whole interval in a unique way by using the equivariance relations
\[
\mathcal{X}(f(x)) = \mathcal{X}(x) \cdot Df(x), \quad \mathcal{Y}(f(x)) = \mathcal{Y}(x) \cdot Df(x).
\]

However, equality (11) is in many cases easier to handle. In particular, it leads to the fundamental inequality (10) in this broader context, the proof of which follows along the
same lines of the one given above. Notice that both the variation of the logarithm of the derivative and the asymptotic distortion are well defined for piecewise $C^{1+bv}$ diffeomorphisms (at break points, we keep the value of the right derivative).

A particularly relevant example of the previous discussion is the space $\text{PL}_+^\Delta([0,1])$ of piecewise-affine homeomorphisms of the interval with no fixed point in the interior. In this context, the variation of the Mather invariant described above has been considered by many authors. A nice review of all of this may be found in [13]. In particular, one can find therein a proof of the fact that $M_f$, together with the multipliers $Df(0)$ and $Df(1)$, are a complete invariant of $PL_+^\Delta$ conjugacy in $\text{PL}_+^\Delta([0,1])$. (These are analogous results to those of Mather that hold for $C^2$ diffeomorphisms.)

There are several other special features of piecewise-affine homeomorphisms in this context. One is that the conjugating maps that realize the asymptotic distortion as the infimum of the total variation of the logarithm of the derivative along the conjugacy class may be also taken to be piecewise-affine. This immediately follows from the explicit formula (4) that defines them. In this regard, it would be interesting to further study the case of piecewise-projective homeomorphisms: can the conjugating maps be taken also being piecewise-projective?

Another special feature concerns equality (12), namely

$$\lim_{N \to \infty} \text{var}(\log Df; [f^{-N}(a), f^{N+1}(a)]) = \text{dist}_\infty (f).$$

Indeed, for a large-enough $N$, the left-hand side expression above obviously stabilizes. More generally, let $f$ be piecewise $C^{1+bv}$ so that it is affine on neighborhoods of both $[0, \varepsilon]$ and $[1 - \varepsilon, 1]$ for a certain $\varepsilon > 0$. Let $a$ be a point in the interior of one of these intervals such that $f(x)$ also lies therein, and let $k$ be a positive integer such that either $f^k(a) > 1 - \varepsilon$ or $f^k(a) < \varepsilon$, according to whether $f$ moves interior points to the right or to the left. If we denote by $I$ the interval with endpoints $a, f(a)$, then

$$\text{dist}_\infty (f) = \text{var}(\log Df^k; I).$$

A third special feature concerns the fundamental inequality (10), which in this case becomes an exact equality. More precisely, remind that for $C^2$ (and, more generally, for $C^{1+ac}$) diffeomorphisms, one always has the strict inequality

$$\text{var}(\log Df) < |\log Df(0)| + |\log Df(1)| + \text{dist}_\infty (f)$$

whenever the endpoints are hyperbolic fixed points; see [6, Proposition 4.4]. However, for piecewise-affine homeomorphisms, the left and right-hand-side expressions above become equal.

**Proposition 6.3.** For every $f \in \text{PL}_+([0,1])$ one has

$$\text{var}(\log Df) = |\log Df(0)| + |\log Df(1)| + \text{dist}_\infty (f).$$

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**Proof.** We assume that $f(x) > x$ for $x \in (0, 1)$, and we use again (11). By differentiation, this becomes

$$DM_f(t) = \frac{\mathcal{Y}(\psi_X(M(t) + m))}{\mathcal{X}(\psi_X(t - n))} \cdot Df^k(\psi_X(t - n))$$

(at discontinuity points of the derivative, this equality holds for left and right derivatives).

Let $p_0 < p_1 < \ldots < p_n$ be a finite family of points of $[a, f(a)]$ that includes all discontinuity points of $Df^k$ therein, as well as $a$ and $f(a)$. Slightly changing $a$ if necessary, we may ensure that neither $a$ nor $f(a)$ are among these discontinuity points. Then

$$\text{var} (\log DM_f) = \sum_{i=1}^{n-1} \text{var} \left( \log \left( \frac{\mathcal{Y}^k}{\mathcal{X}} \right) + \log Df^k; [p_{i-1}, p_i] \right).$$

Notice that $\log(Df^k)$ has no variation on $[p_{i-1}, p_i]$, since its variation may only arise as Dirac jumps at the points $p_i$. Moreover, such a point adds $| \log(Df^k(p_i)) - \log(Df^k(p_{i+1}))|$ to the variation of $\log DM_f$. All these terms add up to $\text{dist}_\infty (f)$, because of (13). Hence,

$$\text{var} (\log DM_f) = \sum_{i=1}^{n-1} \text{var} \left( \log \left( \frac{\mathcal{Y}^k}{\mathcal{X}} \right) \right) + \text{dist}_\infty (f)$$

$$= \text{var} \left( \log \left( \frac{\mathcal{Y}^k}{\mathcal{X}} \right); [a, f(a)] \right) + \text{dist}_\infty (f).$$

Finally, on the interval $[a, f(a)]$, the function $\mathcal{X}$ is strictly increasing (equal to $\lambda x$, with $\lambda = \log Df(0) > 0$), while $\mathcal{Y} \circ f^k$ is strictly decreasing (equal to $\mu(1 - f^k(x))$, with $\mu = \log Df(1) < 0$). This yields

$$\text{var} \left( \log \left( \frac{\mathcal{Y}^k}{\mathcal{X}} \right); [a, f(a)] \right) = \left| \log(e^{\lambda} \lambda a) - \log(\lambda a) + \log(\mu(1 - f^k(a))) - \log(e^{\mu} \mu(1 - f^k(a))) \right|$$

$$= \lambda - \mu.$$

Putting everything together, we obtain the announced equality (14).

The last special feature of $\text{PL}_+^\Delta([0, 1])$ is the well-known fact that the Mather invariant can never be (the class of) a rotation (this is closely related to the fact that the centralizer of an element $f$ in $\text{PL}_+^\Delta([0, 1])$ is a finite extension of the group generated by $f$, and it is infinite cyclic). Indeed, this follows from the fundamental equality (14), which in its turn implies that

$$\text{var} (\log DM_f) \geq 2 \left[ | \log Df(0) | + | \log Df(1) | \right];$$

see [13] for an alternative (less quantitative) argument.

**Example 6.4.** Let $f$ be the piecewise-affine homeomorphisms considered in Example 4.3 from [6]. It can be readily checked that $\text{dist}_\infty (f) = \lambda - \mu$, while $\text{var} (\log DM_f) = 2 (\lambda - \mu)$.
6.4 A remark concerning the Mather homomorphism

Another remarkable object introduced by Mather is a group homomorphism from the group of \( C^{1+bv} \) diffeomorphisms of a 1-manifold into \( \mathbb{R} \). (The motivation was to prove the non-simplicity of such a group; see [12]). Although the original construction concerns diffeomorphisms of the real line with compact support, it also applies to the circle and the interval.

To be more concrete, given a diffeomorphism \( f \in \text{Diff}^{1+bv}([0,1]) \), we denote \( \mu_f \) the (finite) signed measure induced by the Riemann-Stieljes integration with respect to \( \log(Df) \). (The fact that \( \log(Df) \) has bounded variation implies that this integration is well defined.) The measure \( \mu_f \) has a unique decomposition

\[
\mu_f = \mu_f^{ac} + \mu_f^s,
\]

where \( \mu_f^{ac} \) (resp. \( \mu_f^s \)) is absolutely continuous (resp. totally singular) with respect to the Lebesgue measure. We then let

\[
\phi_M(f) := \int_0^1 d\mu_f^{ac}.
\]

It is straightforward to check that \( \phi_M \) defines a continuous group homomorphism from \( \text{Diff}^{1+bv}([0,1]) \) onto \( \mathbb{R} \) (see [12] for further details).

**Proposition 6.5.** If a diffeomorphism \( f \in \text{Diff}^{1+bv,\Delta}([0,1]) \) has parabolic fixed points and its image under the Mather homomorphism \( \phi_M \) is nonzero, then its Mather invariant \( M_f \) is nontrivial.

**Proof.** If \( M_f \) is trivial for \( f \in \text{Diff}^{1+bv,\Delta}([0,1]) \) with parabolic fixed points then, according to the fundamental inequality [14], one has \( \text{dist}_\infty(f) = 0 \). By [6], there exists a sequence of \( C^{1+bv} \) diffeomorphisms \( h_n \) of \([0,1]\) such that \( h_n f h_n^{-1} \) converges to the identity in the \( C^{1+bv} \) topology. By the continuity of \( \phi_M \), this implies that \( \phi_M(h_n f h_n^{-1}) \) converges to zero. However, since \( \phi_M \) is a group homomorphism, for each \( n \) we have \( \phi_M(h_n f h_n^{-1}) = \phi_M(f) \). Therefore, if \( M_f \) is trivial, then \( \phi_M(f) = 0 \).

\( \Box \)

**Question 6.6.** Is it possible to have \( \phi_M(f) = 0 \neq \phi_M(g) \) for two \( C^2 \) diffeomorphisms \( f \) and \( g \) that are \( C^1 \)-conjugate? Compare [6, Theorem D], which establishes the invariance of the asymptotic distortion under \( C^1 \) conjugacy.

7 Appendix II: A remark concerning \( C^1 \) vector fields

Sternberg gave in [20] an example of an hyperbolic germ of \( C^1 \) diffeomorphism that is not \( C^1 \) (even bi-Lipschitz) linearizable, namely,

\[
x \mapsto e^\lambda x \left( 1 - \frac{1}{\log(x)} \right),
\]
where $\lambda < 0$. Inspired on this, and following [17, Exercise 4.1.12], let us consider a $C^1$ vector field on $[0, 1[$ that vanishes only at 0 and 1 and satisfies on a neighborhood of 0 the equality

$$\mathcal{X}(x) := \lambda x \left(1 - \frac{1}{\log(x)}\right) \frac{\partial}{\partial x}.$$  

It is easy to see that $\mathcal{X}$ is hyperbolic at the origin, with linear part $\lambda x \frac{\partial}{\partial x}$. We claim, however, that $\mathcal{X}$ is not $C^1$ linearizable.

To show this, we first claim that if $f$ denotes the time-1 map of the flow of $\mathcal{X}$, then for every $x > 0$ close-enough to the origin, one has

$$f(x) = e^{\lambda x} \left(1 - \frac{1}{\log(f(x))}\right). \quad (15)$$

Indeed, let us fix such a $x > 0$, and let us denote by $h(t)$ the solution of

$$\frac{dh}{dt}(t) = \lambda h(t) \left(1 - \frac{1}{\log(h(t))}\right), \quad \text{with} \quad h(0) = x.$$  

If we put $\varphi(t) := \log(h(t))$, then we have $\varphi' = \lambda \left(1 - \frac{1}{\varphi}\right)$, and so

$$(\varphi^{-1})'(t) = \frac{1}{\lambda(1 - t)} = \frac{t}{\lambda(t - 1)}.$$  

Since $h(0) = x$, we have $\varphi(0) = \log(x)$, hence $\varphi^{-1}(\log(x)) = 0$. Therefore,

$$\varphi^{-1}(t) = -\int_{\log(x)}^{t} \frac{s}{\lambda(1 - s)} \, ds = \frac{t}{\lambda} - \frac{\log(x)}{\lambda} + \frac{1}{\lambda} \log\left(\frac{1 - t}{1 - \log(x)}\right).$$

Since $\varphi^{-1}(\log(h(t))) = t$, this gives

$$t = \frac{\log(h(t))}{\lambda} - \frac{\log(x)}{\lambda} + \frac{1}{\lambda} \log\left(\frac{1 - \log(h(t))}{1 - \log(x)}\right),$$

and so

$$\log(h(t)) = \lambda t + \log(x) - \log\left(\frac{1 - \log(h(t))}{1 - \log(x)}\right).$$

Since $h(1) = f(x)$, we have

$$\log(f(x)) = \lambda + \log(x) - \log\left(\frac{1 - \log(f(x))}{1 - \log(x)}\right),$$

which proves (15).
Let us now suppose by contradiction that \( X \) is \( C^1 \) conjugate to its linear part. If it was, then \( f \) would also be \( C^1 \) conjugate to its linear part. However, as we next show, this is not the case. Indeed, from (15), one easily concludes that
\[
\frac{f^k(x)}{e^{\lambda k}x} = \frac{f(x)}{e^{\lambda x}} \cdot \frac{f^2(x)}{e^{\lambda f(x)}f(x)} \cdots \frac{f^k(x)}{e^{\lambda f^{k-1}(x)}}
\]
\[
= \frac{1 - \log(x)}{1 - \log(f(x))} \cdot \frac{1 - \log(f(x))}{1 - \log(f^2(x))} \cdots \frac{1 - \log(f^k(x))}{1 - \log(f^{k-1}(x))} = \frac{1 - \log(x)}{1 - \log(f^k(x))}.
\]
The right-hand-side expression converges to zero as \( k \) goes to infinity (since \( f^k(x) \) converges to the origin). However, if \( f \) was conjugated to \( x \mapsto e^\lambda x \) by some bi-Lipschitz homeomorphism \( \phi \) with bi-Lipschitz constant \( M \), then from \( f^k(x) = \phi(e^{\lambda k}\phi^{-1}(x)) \) one would obtain a.e. close to the origin: \( Df^k \geq e^{\lambda k}/M^2 \). By integration, this would yield
\[
f^k(x) \geq \frac{e^{\lambda k}x}{M^2},
\]
which contradicts the convergence of \( f^k(x)/e^{\lambda k}x \) to zero.

To close this discussion, we claim that \( f \) can be explicitly conjugated to its linear part (by a non bi-Lipschitz map). Indeed, letting \( \Phi \) be so that \( \Phi(x) := x (1 - \log(x)) \) close to the origin (and extending it in an equivariant way), equality (15) may be read as \( \Phi(f(x)) = e^\lambda \Phi(x) \), which is the announced conjugacy relation. Notice that \( D\Phi(x) = -\log(x) \). Using this relation (or by a direct analysis), a straightforward computation shows that \( f \) is of class \( C^{1+ac} \). It is worth to stress that \( f \) is not of class \( C^{1+\alpha} \) for any \( \alpha > 0 \), because the Sternberg-Yoccoz linearization theorem still holds in this setting [11].

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References

[1] Bonatti, C., Crovisier, S. & Wilkinson, A. The \( C^1 \) generic diffeomorphism has trivial centralizer. *Publications Mathématiques de l’IHÉS* **109** (2009), 185-244.

[2] Bonatti, C. & Eynard-Bontemps, H. Connectedness of the space of smooth actions of \( \mathbb{Z}^n \) on the interval. *Erg. Theory and Dyn. Sysytems* **36** (2016), 2076-3106.

[3] De Cornulier, Y., Tessera, R. & Valette, A. Isometric group actions on Hilbert spaces: growth of cocycles. *Geometric and Functional Analysis (GAFA)* **17** (2007), 770-792.
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