CLASSIFICATION OF EQUIVARIANT VECTOR BUNDLES
OVER REAL PROJECTIVE PLANE

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Abstract. We classify equivariant topological complex vector bundles over real projective plane under a compact Lie group (not necessarily effective) action. It is shown that nonequivariant Chern classes and isotropy representations at (at most) three points are sufficient to classify equivariant vector bundles over real projective plane except one case. To do it, we relate the problem to classification on two-sphere through the covering map because equivariant vector bundles over two-sphere have been already classified.

1. Introduction

In the previous paper [Ki], we have classified equivariant topological complex vector bundles over $S^2$. In this paper, we classify equivariant topological complex vector bundles over real projective plane. This explains for one of two exceptional cases of [Ki].

We will consider only projective linear actions on real projective plane. We explain for this. Real projective plane $\mathbb{RP}^2$ is defined as $\mathbb{R}^3 \setminus (0,0,0)/\sim$ where the equivalence relation $\sim$ is given by

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$$

for nonzero real number $\lambda$, and let

$$o : S^2 \to \mathbb{RP}^2$$

be the usual covering map. The projective linear group $\text{PGL}(3, \mathbb{R})$ is defined as the quotient group $\text{GL}(3, \mathbb{R})/\mathbb{Z}(\text{GL}(3, \mathbb{R}))$ where $\mathbb{Z}(\text{GL}(3, \mathbb{R}))$ is the center of $\text{GL}(3, \mathbb{R})$, and let

$$o_{\text{GL}(3, \mathbb{R})} : \text{GL}(3, \mathbb{R}) \to \text{PGL}(3, \mathbb{R})$$

be the quotient map. Let $\text{PGL}(3, \mathbb{R})$ act usually on $\mathbb{RP}^2$. For a compact Lie group $G$, a topological $G$-action on $\mathbb{RP}^2$ is called projective linear if the action is given by a continuous homomorphism $\rho : G \to \text{PGL}(3, \mathbb{R})$. Since $\text{PO}(3, \mathbb{R}) = o_{\text{GL}(3, \mathbb{R})}(O(3))$ is a maximal compact subgroup of $\text{PGL}(3, \mathbb{R})$, we henceforward may assume

$$\rho(G) \subset \text{PO}(3, \mathbb{R}).$$

In Section [2] we show that a topological action on $\mathbb{RP}^2$ by a compact Lie group is conjugate to a projective linear action.

To state main results, we need introduce some terminologies and notations. Let a compact Lie group $G$ act projectively linearly (not necessarily effectively) on $\mathbb{RP}^2$ through a homomorphism $\rho : G \to \text{PGL}(3, \mathbb{R})$ to satisfy (1.1). Let $\text{Vect}_G(\mathbb{RP}^2)$ be the set of isomorphism classes of topological complex $G$-vector bundles over $\mathbb{RP}^2$. For a bundle $E$ in $\text{Vect}_G(\mathbb{RP}^2)$ and a point $x$ in $\mathbb{RP}^2$, denote by $E_x$ the isotropy $G_x$-representation on the fiber at $x$. Put $H = \ker \rho$, i.e. the kernel of the $G$-action on $\mathbb{RP}^2$. 

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\( \mathbb{RP}^2 \). Let \( \text{Irr}(H) \) be the set of characters of irreducible complex \( H \)-representations which has a \( G \)-action defined as
\[
(g \cdot \chi)(h) = \chi(g^{-1}hg)
\]
for \( \chi \in \text{Irr}(H) \), \( g \in G \), \( h \in H \). For \( \chi \in \text{Irr}(H) \), an \( H \)-representation is called \( \chi \)-isotypical if its character is a multiple of \( \chi \). We slightly generalize this concept. For \( \chi \in \text{Irr}(H) \) and a compact Lie group \( K \) satisfying \( H < K < G \) and \( K \cdot \chi = \chi \), a \( K \)-representation \( W \) is called \( \chi \)-isotypical if \( \text{res}_H^K W \) is \( \chi \)-isotypical, and we denote by \( \text{Vect}_K(\mathbb{RP}^2, \chi) \) the set
\[
\{ [E] \in \text{Vect}_K(\mathbb{RP}^2) \mid E_x \text{ is } \chi \text{-isotypical for each } x \in S^2 \}\]
where \( \mathbb{RP}^2 \) delivers the restricted \( K \)-action. As in [K], our classification is reduced to \( \text{Vect}_{G_\chi}(\mathbb{RP}^2, \chi) \) for each \( \chi \in \text{Irr}(H) \). Details are found in [CKMS] Section 2.

The classification of \( \text{Vect}_{G_\chi}(\mathbb{RP}^2, \chi) \) is highly dependent on the \( G_\chi \)-action on the base space \( \mathbb{RP}^2 \). So, we would list all possible actions on \( \mathbb{RP}^2 \) up to conjugacy in terms of covering action, and then assign an equivariant simplicial (or CW) complex structure on \( \mathbb{RP}^2 \) to each action when \( \rho(G_\chi) \) is finite. For these, we first introduce some polyhedra in \( \mathbb{R}^3 \). Let \( P_m \) for \( m \geq 3 \) be the regular \( m \)-gon on \( xy \)-plane in \( \mathbb{R}^3 \) whose center is the origin and one of whose vertices is \((1, 0, 0)\). And,

1. \( |\mathcal{K}_m| \) is defined as the boundary of the convex hull of \( P_m \), \( S = (0, 0, -1) \), \( N = (0, 0, 1) \),
2. \( |\mathcal{K}_T| \) is defined as the regular tetrahedron which is the boundary of the convex hull of four points \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\), and which is inscribed to \( |\mathcal{K}_4| \),
3. \( |\mathcal{K}_I| \) is defined as a regular icosahedron which has the origin as the center.

With these, denote natural simplicial complex structures on \( |\mathcal{K}_m| \), \( |\mathcal{K}_T| \), \( |\mathcal{K}_I| \) by \( \mathcal{K}_m \), \( \mathcal{K}_T \), \( \mathcal{K}_I \), respectively. It is well-known that each closed subgroup of \( \text{SO}(3) \) is conjugate to one of the following subgroups [R] Theorem 11:

1. \( \mathcal{K}_m \) generated by the rotation \( a_n \) through the angle \( 2\pi/n \) around z-axis,
2. \( \mathcal{K}_T \) generated by \( a_n \) and the rotation \( b \) through the angle \( \pi \) around x-axis,
3. \( \mathcal{K}_I \) generated by \( a_n \) and \( b \) through the angle \( \pi \) around x-axis,
4. \( \text{SO}(2) \) which is the set of rotations around z-axis,
5. \( \text{SO}(3) \) itself.

Let \( i : \text{SO}(3) \rightarrow \text{PO}(3) \subset \text{PGL}(3, \mathbb{R}) \) be the monomorphism defined as
\[
\begin{align*}
o_{\text{GL}(3, \mathbb{R})} \circ i_{\text{GL}(3, \mathbb{R})} & : \text{SO}(3) \rightarrow \text{GL}(3, \mathbb{R}) \\
& g \mapsto i^{-1}(\rho(g))
\end{align*}
\]
where \( i_{\text{GL}(3, \mathbb{R})} : \text{O}(3) \rightarrow \text{GL}(3, \mathbb{R}) \) is the usual inclusion. Put
\[
\hat{\rho} : G_\chi \rightarrow \text{SO}(3), \quad g \mapsto i^{-1}(\rho(g))
\]
for \( g \in G_\chi \) which is defined by \([11]\). And, define the homomorphism
\[
\hat{\rho} : G_\chi \times Z \rightarrow \text{O}(3), \quad (g, g_0) \mapsto \hat{\rho}(g) \cdot g_0
\]
for \( g \in G_\chi \), \( j \in \mathbb{Z}_2 \) where \( Z \) is the centralizer \( \{id, -id\} \) of \( \text{O}(3) \) and we denote \( -id \in Z \) by \( g_0 \) to avoid symbolic confusion. In Section 2 it is shown that each compact subgroup of \( \text{PGL}(3, \mathbb{R}) \) is conjugate to \( i(\hat{\rho}) \) for an \( \hat{\rho} \)-entry of Table [11]. Denote \( i(\hat{\rho}) \) by \( \hat{\rho} \), and put \( \hat{R} = \hat{\rho} \) where the notation \( \times \) means the internal direct product of two subgroups in \( \text{O}(3) \). Henceforward, it is assumed that \( \rho(G_\chi) = \hat{R} \) for some \( \hat{R} \). Then, images of \( \rho \), \( \hat{\rho} \), \( \hat{\rho} \) are \( R, \hat{R}, \hat{R} \), respectively. Note that the case of \( \hat{R} = \mathbb{Z}_2 \) is conjugate to \( \hat{R} = \text{D}_1 \). So, we exclude \( \text{D}_1 \) in Table [11]. To each
finite \( \hat{R} \), we assign a simplicial complex \( \hat{\cal K}_R \) of Table 1.1 where \( \hat{\cal K}_O \) is defined in the below. The reason for our choice of \( \hat{\cal K}_R \) is explained in Section \( \S \). Let \( O(3) \) and their subgroups act usually on \( \mathbb{R}^3 \). Then, the underlying space \(|\hat{\cal K}_R|\) of \( \hat{\cal K}_R \) is invariant under the \( R \)-action on \( \mathbb{R}^3 \) so that \(|\hat{\cal K}_R|\) inherits the \( R \)-action from which \( \hat{\cal K}_R \) also carries the induced \( \hat{\cal K}_R \)-action. And, \( \hat{\cal K}_R \) and \(|\hat{\cal K}_R|\) deliver the \((G_\chi \times Z)\)-action through \( \hat{\cal p} \). These actions induce \( R \) - and \( G_\chi \)-actions on \(|\hat{\cal K}_R|/Z\). Then, \((G_\chi \times Z)\)- and \( G_\chi \)- actions on \(|\hat{\cal K}_R|\) and \(|\hat{\cal K}_R|/Z\) are equal to \((G_\chi \times Z)\)- and \( G_\chi \)- actions on \( S^2 \) and \( \mathbb{RP}^2 \) when we regard \(|\hat{\cal K}_R|\) and \(|\hat{\cal K}_R|/Z\) as \( S^2 \) and \( \mathbb{RP}^2 \), respectively. And, we regard \( o \) as the orbit map from \(|\hat{\cal K}_R|\) to \(|\hat{\cal K}_R|/Z\). We can give a natural equivariant simplicial (or CW) complex structure on \(|\hat{\cal K}_R|/Z\) so that \( o \) is an equivariant cellular map. Sometimes, we consider the restricted \( G_\chi \)-action on \( S^2 \) and \(|\hat{\cal K}_R|\).

| \( R \) | \( \hat{\cal K}_R \) | \( \hat{D}_R \) | \( \hat{d}^{-1} \) |
|-----|-----|-----|-----|
| \( Z_n, \text{odd } n \) | \( \hat{\cal K}_{2n} \) | \( \{ e^n \} \) | \( S \) |
| \( Z_n, \text{even } n \) | \( \hat{\cal K}_n \) | \( \{ e^n \} \) | \( S \) |
| \( D_n, \text{odd } n, n > 1 \) | \( \hat{\cal K}_{2n} \) | \( \{ v^n, b(e^0) \} \) | \( S \) |
| \( D_n, \text{even } n \) | \( \hat{\cal K}_n \) | \( \{ v^n, b(e^0) \} \) | \( S \) |
| \( T \) | \( \hat{\cal K}_O \) | \( \{ e^n \} \) | \( b(f^{-1}) \) |
| \( O \) | \( \hat{\cal K}_O \) | \( \{ v^n, b(e^0) \} \) | \( b(f^{-1}) \) |
| \( I \) | \( \hat{\cal K}_I \) | \( \{ v^n, b(e^0) \} \) | \( b(f^{-1}) \) |
| \( \text{SO}(2) \) | \( \{ v^0 \} \) | \( S \) |
| \( \text{O}(2) \) | \( \{ v^0 \} \) | \( S \) |
| \( \text{SO}(3) \) | \( \{ S \} \) | \( S \) |

Table 1.1. \( \hat{\cal K}_R, \hat{D}_R, \hat{d}^{-1} \) for each \( \hat{\cal R} \)

![Diagram](image)

**Figure 1.1.** \( \cal K_4 \) and \( \cal K_O \)

In dealing with equivariant vector bundles over \( \mathbb{RP}^2 \), we need to consider isotropy representations at (at most) three points. We will express those points as the image of some points in \( S^2 \) under \( o \). To specify those points, we introduce some more notations. When \( m \geq 3 \), denote by \( v^i \) the vertex exp \( \left( \frac{2\pi i v^i}{m} \right) \) of \( \cal K_m \), and by \( e^i \) the edge of \( \cal K_m \) connecting \( v^i \) and \( v^{i+1} \) for \( i \in \mathbb{Z}_m \). These notations are illustrated in Figure 1.1(a). When we use the notation \( \mathbb{Z}_m \) to denote an index set, it is just
the group $\mathbb{Z}/m\mathbb{Z}$ of integers modulo $m$. In [K1], $\mathcal{K}_m$ and its $v^i$’s, $e^i$’s for $m = 2$ are also defined. We would define similar notations for $\mathcal{K}_T$ and $\mathcal{K}_I$. For $\mathcal{K}_T$ and $\mathcal{K}_I$, pick two adjacent faces in each case, and call them $f^{-1}$ and $f^0$. And, label vertices of $f^{-1}$ as $e^i$ for $i \in \mathbb{Z}_3$ to satisfy

1. $v^0, v^1, v^2$ are arranged in the clockwise way around $f^{-1}$,
2. $v^0, v^1$ are contained in $f^{-1} \cap f^0$.

For $i \in \mathbb{Z}_3$, denote by $e^i$ be the edge connecting $v^i$ and $v^{i+1}$, and by $f^i$ the face which is adjacent to $f^{-1}$ and contains the edge $e^i$. We distinguish the superscripts −1 and 2 only for $f^i$, i.e. $f^{-1} \neq f^2$ in contrast to $v^{-1} = v^2$, $e^{-1} = e^2$. Here, we define one more simplicial complex denoted by $\mathcal{K}_G$ which is the same simplicial complex with $\mathcal{K}_I$ but has the same convention of notations $v^i, e^i, f^{-1}, f^i$ with $\mathcal{K}_T$, $\mathcal{K}_I$. Also, put $|\mathcal{K}_G| = |\mathcal{K}_I|$. These notations are illustrated in Figure 1.1.(b). With these notations, we explain for $\hat{D}_R$-entry of Table 1.1. To each finite $\hat{R}$, we assign a path $\hat{D}_R$ (called the (closed) one-dimensional fundamental domain) in $|\mathcal{K}_R|$ which is listed in the third column of Table 1.1 where $b(\sigma)$ is the barycenter of $\sigma$ for any simplex $\sigma$ and $[x, y]$ is the shortest path in the underlying space $|\mathcal{K}|$ for any simplicial complex $K$ and two points $x, y \in |\mathcal{K}|$. And, let $d^0$ and $d^1$ be boundary points of $\hat{D}_R$ such that $d^0$ is nearer to $v^0$ than $d^1$. For each finite $\hat{R}$, we define one more point $\hat{d}^{-1} \in |\mathcal{K}_R|$ which is listed in the fourth column in Table 1.1. If $\hat{R}$ is one-dimensional, then denote by $\hat{D}_R$ the one point set $\{v^0 = (1,0,0)\}$, and let $\hat{d}^{-1}$, $\hat{d}^0$, $\hat{d}^1$ be equal to $S, v^0, v^0$, respectively. Similarly, if $\hat{R}$ is three-dimensional, then denote by $\hat{D}_R$ the one point set $\{S\}$, and let $\hat{d}^{-1}$, $\hat{d}^0$, $\hat{d}^1$ be equal to $S$. So far, we have defined $d^{-1}$, $d^0$, $d^1$ for each $\hat{R}$. Put $d^i = o(d^i)$ for $i \in \mathbb{Z}$ where $I = \{0, 1\}$ and $I^+ = \{-1, 0, 1\}$. Then, $d^{-1}$, $d^0$, $d^1$ are wanted points of $R\mathbb{P}^2$ according to $\hat{R}$, and we will consider the restriction $E\{d^{-1}, d^0, d^1\}$ for each $E$ in Vect$_{\mathbb{C}}(\mathbb{P}^2, \chi)$. We define the semigroup which will be shown to be equal to the set of all the restrictions. If $\hat{R}$ is finite, let $C(d^i)$ be the shortest path in $|\mathcal{K}_R|$ connecting $\hat{d}^{-1}$ with $\hat{d}^i$ for $i \in \mathbb{Z}$. Otherwise, let $C(d^i)$ be the shortest path in $S^2$ connecting $\hat{d}^{-1}$ with $\hat{d}^i$ for $i \in \mathbb{Z}$. And, denote images

$$
o(\hat{D}_R), \quad o(C(d^0)), \quad o(C(d^1))$$

by

$$D_R, \quad C(d^0), \quad C(d^1),$$

respectively.

**Definition 1.1.** For $\chi \in \text{Irr}(H)$, assume that $\rho(G_\chi) = R$ for some $\hat{R}$ of Table 1.1. Let $A_{G_\chi}(\mathbb{P}^2, \chi)$ be the semigroup of triples $(W_{d^{-1}}, W_{d^0}, W_{d^1})$ in $\text{Rep} \left( (G_\chi)_{d^{-1}} \right)$ × $\text{Rep} \left( (G_\chi)_{d^0} \right)$ × $\text{Rep} \left( (G_\chi)_{d^1} \right)$ satisfying

i) $W_{d^{-1}}$ is $\chi$-isotypical,

ii) $\text{res}_{(G_\chi)_{C(d)}} \quad W_{d^{-1}} \cong \text{res}_{(G_\chi)_{C(d)}} \quad W_{d^i}$ for $i \in I$,

iii) $\text{res}_{(G_\chi)_{D_R}} \quad W_{d^0} \cong \text{res}_{(G_\chi)_{D_R}} \quad W_{d^i}$,

iv) $W_{d^i} \cong gW_{d^i}$ if there exists $g \in G_\chi$ such that $gd^0 = d^1$

where $(G_\chi)_T$ is the subgroup of $G_\chi$ fixing all points of a subset $T \subset \mathbb{P}^2$. And, let $\rho_{\text{vec}} : \text{Vect}_{\mathbb{C}}(\mathbb{P}^2, \chi) \rightarrow A_{G_\chi}(\mathbb{P}^2, \chi)$ be the map defined as $E \mapsto (E_{d^{-1}}, E_{d^0}, E_{d^1})$.

**Remark 1.2.** It might seem that $A_{G_\chi}(\mathbb{P}^2, \chi)$ is defined in a different way with $A_{G_\chi}(S^2, \chi)$ of [K1]. But, these two definitions are defined in the exactly same way by [K1] Lemma 3.10].
Well-definedness of $p'_\text{vect}$ is proved in Proposition 2.1. For notational simplicity, denote a triple $(W, d, e)$ by $(W_E)_{E \in \mathcal{I}}$. Now, we can state main results. Let $c_1 : \text{Vect}_{G}(\mathbb{RP}^2, \chi) \to H^2(\mathbb{RP}^2)$ be the map defined as $[E] \mapsto c_1(E)$.

**Theorem A.** Assume that $p(G, \chi)$ is equal to $R$ for some $R$ of Table 2.7 except $\mathbb{Z}_n$ with odd $n$. Then, $p'_\text{vect}$ is an isomorphism.

**Theorem B.** Assume that $p(G, \chi) = R$ for some $R = \mathbb{Z}_n$ with odd $n$. Then, the preimage $p^{-1}_\text{vect}(W)$ has exactly two elements for each $W$ in $A_{G, \chi}(\mathbb{RP}^2, \chi)$. And, $[E \oplus F] \neq [E \oplus G]$ for any bundles $E, F$ in $\text{Vect}_{G, \chi}(\mathbb{RP}^2, \chi)$ such that $p'_\text{vect}([E_1]) = p'_\text{vect}([F_1])$ and $[E_1] \neq [F_1]$. Also, the following hold:

1. if $\chi(\text{id})$ is even, then Chern classes of two elements of $p^{-1}_\text{vect}(W)$ are the same,

2. if $\chi(\text{id})$ is odd, then Chern classes of two elements of $p^{-1}_\text{vect}(W)$ are different.

**Corollary 1.3.** Assume that $p(G, \chi) = R$ for some $R = \mathbb{Z}_n$ with odd $n$. Then, we have

$$\text{Vect}_{G, \chi}(\mathbb{RP}^2, \chi) \cong \text{Vect}_R(\mathbb{RP}^2)$$

as semigroups, and $\text{Vect}_R(\mathbb{RP}^2)$ is generated by line bundles. Also, $A_R(\mathbb{RP}^2, \text{id})$ is generated by all the elements with one-dimensional entries where we simply denote by $\text{id}$ the trivial character of the trivial group. The number of such elements in $A_R(\mathbb{RP}^2, \text{id})$ is equal to $n$.

This paper is organized as follows. In Section 2, we list all closed subgroups of $\text{PGL}(3, \mathbb{R})$ up to conjugacy, and show that a topological action on $\mathbb{RP}^2$ by a compact Lie group is conjugate to a projective linear action. Also, we show that $p'_\text{vect}$ is well-defined. In Section 3, we prove Theorem A through covering action. In Section 4, we prove Theorem B and Corollary 1.3 through equivariant clutching construction.

## 2. Compact subgroups of $\text{PGL}(3, \mathbb{R})$

In this section, we list all compact subgroups of $\text{PGL}(3, \mathbb{R})$ up to conjugacy, and show that a topological action on $\mathbb{RP}^2$ by a compact Lie group is conjugate to a projective linear action. Also, we show that $p'_\text{vect}$ is well-defined.

First, we define covering action of the $G_\chi$-action on $\mathbb{RP}^2$. We call $G_\chi \times Z$ and its action on $S^2$ through $\hat{\rho}$ the covering group of $G_\chi$ and the covering action of the $G_\chi$-action on $\mathbb{RP}^2$, respectively. Here, we note that two kernels of the $(G_\chi \times Z)$-action on $S^2$ and the $G_\chi$-action on $\mathbb{RP}^2$ are equal because $\iota$ is injective. By using covering action, we list all closed subgroups of $\text{PGL}(3, \mathbb{R})$ up to conjugacy.

**Proposition 2.1.** Each compact subgroup of $\text{PGL}(3, \mathbb{R})$ is conjugate to one of $R$ for some $R$-entry in Table 2.7

**Proof.** Let $K$ be an arbitrary compact subgroup of $\text{PGL}(3, \mathbb{R})$, and let $\rho : K \to \text{PGL}(3, \mathbb{R})$ be the inclusion. We may assume that $K \subset \text{PO}(3, \mathbb{R})$ by (1.1). Since the codomain of $\rho$ is equal to $SO(3)$, the image of $\hat{\rho}$ is conjugate to one of $\mathbb{Z}_n$, $\text{D}_n$, $\text{T}$, $\text{O}$, $\text{I}$, $\text{SO}(2)$, $\text{O}(2)$, $\text{SO}(3)$. This shows that the image of $\rho$ is conjugate to one of $R$ for some $R$-entry in Table 2.7 because $\iota$ is injective. So, we obtain a proof.

Now, we prove that a compact Lie group action on $\mathbb{RP}^2$ is conjugate to a projective linear action.

**Proposition 2.2.** If a compact Lie group $G$ act topologically on $\mathbb{RP}^2$, then it is conjugate to a projective linear action.
Proof. By [B Theorem I.9.3], the $G$-action on $\mathbb{RP}^2$ has a covering $G'$-action on $S^2$ where $G'$ satisfies a short exact sequence

$$(\text{id}) \to \mathbb{Z}_2 \to G' \xrightarrow{pr} G \to (\text{id})$$

for some homomorphism $pr$ and the $G'$-action on $S^2$ satisfies

$$o(g' \cdot \hat{x}) = pr(g') \cdot o(\hat{x})$$

for $g' \in G'$ and $\hat{x} \in S^2$. It is well-known that a topological action on $S^2$ by a compact Lie group is conjugate to a linear action [Ko Theorem 1.2], [CK]. By using this, we may assume that the $G'$-action is linear, and we obtain a proof. □

To prove well-definedness of $p'_\text{vect}$, we state a basic lemma.

Lemma 2.3. Let $G$ be a compact Lie group acting topologically on a topological space $X$. And, let $E$ be an equivariant vector bundle over $X$. Then, $\hat{E}_{x} \cong E_{x}$ for each $g \in G$ and $x \in X$. Also,

$$\text{res}^{G_{x}}_{G_{x} \cap G_{x}'} E_{x} \cong \text{res}^{G_{x}'}_{G_{x} \cap G_{x}'} E_{x}$$

for any two points $x, x'$ in the same component of the fixed set $X_{G_{x} \cap G_{x}'}$.

Proposition 2.4. $p'_\text{vect}$ is well-defined.

Proof. To show well-definedness, we should show that $(E_{x})_{\hat{g}} \in I_{+}$ is contained in $A_{G_{x}}(\mathbb{RP}^2, \chi)$ for any $E$ in $\text{Vect}_{G_{x}}(\mathbb{RP}^2, \chi)$. By definition of $\text{Vect}_{G_{x}}(\mathbb{RP}^2, \chi)$, it is easy that $E_{x}$ is $\chi$-isotypical so that $(E_{x})_{\hat{g}}$ satisfies Definition [11]. By the first statement of Lemma 2.3, $(E_{x})_{\hat{g}} \in I_{+}$ satisfies Definition [11][iv]. The remaining are Definition [11][ii], iii). These are satisfied by the second statement of Lemma 2.3. □

3. Relation between equivariant vector bundles over $\mathbb{RP}^2$ and $S^2$.

In this section, we investigate the relation between semigroups $\text{Vect}_{G_{x}}(\mathbb{RP}^2, \chi)$ and $\text{Vect}_{G_{x} \times Z}(S^2, \chi)$, and the relation between $A_{G_{x}}(\mathbb{RP}^2, \chi)$ and $A_{G_{x} \times Z}(S^2, \chi)$. By these, we can prove Theorem [A].

The first relation is easy. Pullback through the covering map gives the semigroup homomorphism

$$(3.1) \quad O : \text{Vect}_{G_{x}}(\mathbb{RP}^2) \longrightarrow \text{Vect}_{G_{x} \times Z}(S^2), \quad E \longmapsto o^{*} E.$$ 

This is an isomorphism because the $Z$-action on $S^2$ is free and $\mathbb{RP}^2 \cong S^2/Z$, see [S p. 132]. Since equivariant vector bundles over $S^2$ have been classified in [KI], we can classify equivariant vector bundles over $\mathbb{RP}^2$ by using this isomorphism. This is the reason why $\tilde{K}_{R}$ of Table [1.1] is defined as the simplicial complex $\tilde{K}_{R}$ of [K3 Table 1.1]. In [K3 Section 3], it is shown that the $(G_{x} \times Z)$-action on $\mathbb{R}^3$ through $\rho$ preserves $\tilde{K}_{R}$ and its underlying space $|\tilde{K}_{R}|$. Also, it is shown that the $(G_{x} \times Z)$-orbit of $\hat{D}_{R}$ covers the underlying space $|\tilde{K}_{R}^{(1)}|$ of the 1-skeleton $\tilde{K}_{R}^{(1)}$ and that $\tilde{D}_{R}$ is a minimal path satisfying such a property.

Next, we would show that $A_{G_{x}}(\mathbb{RP}^2, \chi)$ is isomorphic to $A_{G_{x} \times Z}(S^2, \chi)$. For this, we state an easy lemma on isotropy subgroups of $G_{x}$ and $G_{x} \times Z$. For any $\hat{x} \in S^2$ and $x = o(\hat{x})$, let

$$p_{1} : G_{x} \times Z \to G_{x} \quad \text{and} \quad p_{1, \hat{z}} : (G_{x} \times Z)_{\hat{z}} \to (G_{x})_{\hat{z}}$$

be the projection to the first components. And, let $p_{1, \hat{z}} : R((G_{x})_{\hat{z}}) \to R((G_{x} \times Z)_{\hat{z}})$ be the isomorphism sending a $(G_{x})_{\hat{z}}$-representation $W$ to $W$ itself regarded as a $(G_{x} \times Z)_{\hat{z}}$-representation through $p_{1, \hat{z}}$. 

\[\text{proof}\]
Lemma 3.1. For any points \( \hat{x}, \hat{y} \) in \( S^2 \) and their images \( x = o(\hat{x}), y = o(\hat{y}) \) in \( \mathbb{RP}^2 \), the following hold

1. \( p_1((G_\chi \times Z)_{\hat{z}}) = (G_\chi)_x \) and \( p_1|_{(G_\chi \times Z)_{\hat{z}}} \) is injective,
2. \( p_1((G_\chi \times Z)_{C(\hat{d})}) = (G_\chi)_{C(d')} \) and \( p_1|_{(G_\chi \times Z)_{C(\hat{d})}} \) is injective for \( i \in I \),
3. \( p_1((G_\chi \times Z)_{D_R}) = (G_\chi)_{D_R} \) and \( p_1|_{(G_\chi \times Z)_{D_R}} \) is injective,
4. \( p^*_1,\hat{x}(E_x) \cong (\alpha E)_{\hat{x}} \) as \((G_\chi \times Z)_{\hat{z}}\)-representations for each \( E \) in the set \( \text{Vect}_{G_\chi}(\mathbb{RP}^2, \chi) \).

Proof. (1) It is easy that \( p_1((G_\chi \times Z)_{\hat{z}}) \) is contained in \( (G_\chi)_x \). For each \( g \in (G_\chi)_x \), one of \((g, \text{id})\) and \((g, g_0)\) fixes \( \hat{x} \) because \( g \cdot \hat{x} = \pm \hat{x} \). So, \( p_1((G_\chi \times Z)_{\hat{z}}) = (G_\chi)_x \) is obtained.

Now, we prove injectivity of \( p_1|_{(G_\chi \times Z)_{\hat{z}}} \). Pick arbitrary two elements \((g, g_0')\) and \((g', g_0'')\) of \((G_\chi \times Z)_{\hat{z}}\) for \( j, j' \in \mathbb{Z}_2 \). If \( p_1((g, g_0')) = p_1((g', g_0'')) \), then \( g = g' \). This shows injectivity.

(2) and (3) are obtained by applying (1) to each point \( \hat{x} \) in \( C(d') \) and \( D_R \).

(4) is easy by definition of pullback. \( \square \)

Proposition 3.2. Let \( P_1 : A_{G_\chi}(\mathbb{RP}^2, \chi) \to A_{G_\chi \times Z}(S^2, \chi) \) be the map defined as

\[
(W_{d'})_{i \in I^+} \mapsto (p^*_1, d)(W_{d'})_{i \in I^+}.
\]

Then, \( P_1 \) is bijective.

Proof. For each \( W = (W_{d-1}, W_{d0}, W_{d1}) \) in \( A_{G_\chi}(\mathbb{RP}^2, \chi) \), Lemma 3.1 says that the element \( P_1(W) \) satisfies Definition 1(iii)~(iiii). Put \( W_{di} = p^*_1(d)(W_{di}) \) for \( i \in I \). It is easily shown that there exists \( g \) in \( G_\chi \times Z \) such that \( gd^0 = d^1 \) if and only if there exists \( g' \) in \( G_\chi \) such that \( gd^0 = d^1 \) where \( g, g' \) satisfy \( p_1(g) = g' \). Then, we can also show that if \( g' W_{d0} \cong W_{d1} \) holds, then \( g' W_{d0} \cong W_{d1} \). So, \( P_1(W) \) satisfies Definition 1(iv), and \( P_1(W) \) is contained in \( A_{G \times Z}(S^2, \chi) \). Similarly, we can also show that \( P_1 \) has an inverse. \( \square \)

Now, we can prove Theorem A

Proof of Theorem A Observe that \( p'_\text{vect} = P^{-1}_1 \circ \text{vect} \circ O \) where the map \( \text{vect} \) on \( \text{Vect}_{G \times Z}(S^2, \chi) \) is defined in [K]. Note that \( O, \text{vect}, P_1 \) are all isomorphisms by 3.1, [K] Theorem C, Proposition 3.2 respectively. Therefore, \( p'_\text{vect} \) is also an isomorphism.

(3.2)

\[
\begin{array}{c}
\text{Vect}_{G_\chi}(\mathbb{RP}^2, \chi) \\
\downarrow \text{vect} \\
A_{G_\chi}(\mathbb{RP}^2, \chi) \\
\downarrow P_1 \\
A_{G_\chi \times Z}(S^2, \chi)
\end{array}
\]

4. PROOF OF THEOREM B

In this section, we deal with the case of \( R = \mathbb{Z}_n \) with odd \( n \). In this case, we can check that

\[
R_{d-1} \cong \mathbb{Z}_n \quad \text{and} \quad R_{d0} = R_{d1} = (\text{id}).
\]
Pick an arbitrary $W = (W_i)_{i \in I}$ in $A_{G}, (\mathbb{RP}^2, \chi)$. By [K1] Theorem B, the map $p_{\text{vect}}$ of diagram (3.2) is surjective and $p_{\text{vect}}^1(\hat{P}_1(W))$ consists of two elements. Also since $O$ and $P_1$ are isomorphic by (3.1) and Proposition 3.2, the diagram (3.2) shows that $p_{\text{vect}}^1$ is surjective and $p_{\text{vect}}^1(W)$ consists of two elements. To prove Theorem B we should calculate Chern classes of these two. Pick an element $g_1$ of $G_{\chi}$ satisfying $\hat{\rho}(g_1) = a_n$. To calculate Chern classes, we would describe bundles in $p_{\text{vect}}^1(W)$ as equivariant clutting construction. For this, we first express $\mathbb{RP}^2 \cong |\hat{K}_R|/Z$ as the quotient of more simple equivariant simplicial complex. To use results of [K1], we do similar things for $p_{\text{vect}}^1(\hat{P}_1(W))$ and $S^2 \cong |\hat{K}_R|$.

![Diagram](image)

**Figure 4.1.** $\pi: \hat{K}_{2n} \to K_{2n}$ when $n = 3$

In Table 1.1 $\hat{K}_R = K_{2n}$. Denote by $K^S_{2n}$ the lower simplicial subcomplex of $K_{2n}$ such that

$$|K^S_{2n}| = |K_{2n}| \cap \left\{ (x, y, z) \in \mathbb{R}^3 \mid z \leq 0 \right\},$$

and by $K^N_{2n}$ the upper part. Denote by $B$ the set $\{S, N\}$, and by $\hat{K}_{2n}$ the disjoint union $\bigsqcup_{q \in B} K^q_{2n}$. Denote $K^q_{2n} \subset \hat{K}_{2n}$ by $K^q_{2n}$. The $(G_{\chi} \times Z)$-action and its restricted $G_{\chi}$-action on $\hat{K}_{2n}$ induce $(G_{\chi} \times Z)$- and $G_{\chi}$-actions on $K_{2n}$ and $|K_{2n}|$ where the $G_{\chi}$-action on $\hat{K}_{2n}$ preserves $K^q_{2n}$. Then, $K_{2n}$ can be regarded as a quotient of $\hat{K}_{2n}$. This is expressed by the simplicial map $\pi: \hat{K}_{2n} \to K_{2n}$ such that $\pi|K^q_{2n}$ is the identity to $K^q_{2n}$ for $q \in B$. Denote by $|\pi|: |K_{2n}| \to |K_{2n}|$ the underlying space map of $\pi$. Easily, $\pi$ and $|\pi|$ are $(G_{\chi} \times Z)$-maps. Here, we introduce the following notations:

$$\hat{P}^q_{2n} = |K^q_{2n}| \cap |\pi|^{-1}(P_{2n}),$$

$$\hat{P}_{2n} = B^S_{2n} \cup B^N_{2n} = \hat{P}^S_{2n} \cup \hat{P}^N_{2n},$$

$$\hat{e}^i_q = \pi^{-1}((e^i)),\quad \hat{e}^i_q = \pi^{-1}((e^i))$$

for $q \in B$ and $i \in I$. Also, let $c: \hat{P}_{2n} \to \hat{P}_{2n}$ be the map satisfying $c(\hat{x}) \neq \hat{x}$ and $|\pi|(\hat{x}) = |\pi|(c(\hat{x}))$ for each $\hat{x} \in \hat{P}_{2n}$. It is easy that $c$ is $(G_{\chi} \times Z)$-equivariant. In
this time, we do the similar thing for $|\mathcal{K}_{2n}|/Z$.

\[
\begin{array}{c}
|\mathcal{K}_{2n}| \xrightarrow{o} |\mathcal{K}_{R}|
\\
\pi
\\
|\mathcal{K}_{2n}| \xrightarrow{o} |\mathcal{K}_{2n}|/Z
\end{array}
\]

Put $\mathcal{K}_{R} = \mathcal{K}_{2n}^{S}$. Let $|\pi'| : |\mathcal{K}_{R}| \to |\mathcal{K}_{2n}|/Z$ be the map defined by $\bar{x} \mapsto (o \circ |\pi'|)(\bar{x})$. Then, $|\pi'|$ is a cellular $G_{\chi}$-map, and we consider $|\mathcal{K}_{2n}|/Z$ as the quotient of $|\mathcal{K}_{R}|$ through $|\pi'|$. Also, let $\bar{o} : |\mathcal{K}_{2n}| \to |\mathcal{K}_{R}|$ be the map defined by

\[
\bar{o}(|\bar{x}|) = \begin{cases}
\bar{x} & \text{for } \bar{x} \in |\mathcal{K}_{S}^{2n}|, \\
g_{0}\bar{x} & \text{for } \bar{x} \in |\mathcal{K}_{N}^{2n}|.
\end{cases}
\]

So defined maps $|\pi'|$ and $\bar{o}$ satisfy the commutative diagram (4.1).

Now, we describe equivariant vector bundles over $|\mathcal{K}_{2n}|/Z$ as an equivariant clutching construction of an equivariant vector bundle over $|\mathcal{K}_{R}|$, and then do the similar thing for equivariant vector bundle over $|\mathcal{K}_{2n}|$. Put $W = (W_{d-1}, W_{d_0}, W_{d_1})$.

Let $F$ be the equivariantly trivial $G_{\chi}$-bundle $|\mathcal{K}_{R}| \times W_{d-1}$ over $|\mathcal{K}_{R}|$. Here, note that $|\pi'|/|S| = d-1$, and that both $(G_{\chi})_{S}$ and $(G_{\chi})_{d-1}$ are equal to $G_{\chi}$. Then, we can show that

\[
F \cong |\pi'|^{*}E' \quad \text{for each } E' \in p_{\text{vect}}^{-1}(W)
\]

because $\left(|\pi'|^{*}E\right)_{S} \cong E_{d-1}^{S} \cong W_{d-1}$ and $|\mathcal{K}_{R}|$ is equivariant homotopically equivalent to the one point set $\{S\} \subset |\mathcal{K}_{R}|$. Let $\Phi : P^{S}_{d_0} \to \text{Iso}(W_{d-1})$ be an arbitrary continuous map which is called a preclutching map with respect to $F$ where $\text{Iso}$ means the set of nonequivariant isomorphisms. We try to glue $F$ along $P^{S}_{d_0}$

\[
P^{S}_{d_0} \times W_{d-1} \to P^{S}_{2n} \times W_{d-1}, \quad (\bar{x}, u) \mapsto \left(g_{0}\Phi(\bar{x}), \Phi(\bar{x})u\right)
\]

for $\bar{x} \in P^{S}_{2n}$ and $u \in W_{d-1}$ via a preclutching map $\Phi$. This gluing gives an equivariant vector bundle in $p_{\text{vect}}^{-1}(W)$ if and only if the following two conditions hold:

\[
\begin{align*}
E1'. & \quad \Phi(g_{0}\Phi(\bar{x})) = \Phi(\bar{x})^{-1} \quad \text{for each } \bar{x} \in P^{S}_{d_0}, \\
E2'. & \quad \Phi(g\bar{x}) = g\Phi(\bar{x})g^{-1} \quad \text{for each } g \in G_{\chi} \text{ and } \bar{x} \in P^{S}_{d_0}.
\end{align*}
\]

A preclutching map satisfying these two conditions is called an equivariant clutching map with respect to $F$. By (4.2), two elements of $p_{\text{vect}}^{-1}(W)$ can be constructed in this way. We do the similar thing for equivariant vector bundle over $|\mathcal{K}_{2n}|$. Let $\bar{F}$ be the $(G_{\chi} \times Z)$-bundle $\bar{o} \circ F$ over $|\mathcal{K}_{2n}|$. Easily, $\bar{F}$ is equal to the equivariantly trivial bundle $|\mathcal{K}_{2n}| \times p_{1}^{1}W_{d-1}$. The $(G_{\chi} \times Z)$-action on it is expressed by

\[
g \cdot (\bar{x}, u) = (g\bar{x}, gu),
\]

\[
g_{0} \cdot (\bar{x}, u) = (g_{0}\bar{x}, u)
\]

for $g \in G_{\chi}$, $\bar{x} \in |\mathcal{K}_{2n}|$, $u \in p_{1}^{1}W_{d-1}$. And, we can show that

\[
\bar{F} \cong |\pi'|^{*}E \quad \text{for each } E \in p_{\text{vect}}^{-1}(P_{1}(W))
\]

because $|\mathcal{K}_{2n}|$ is equivariant homotopically equivalent to the set $B \subset |\mathcal{K}_{2n}|$. Let $\bar{\Phi} : P_{2n} \to \text{Iso}(p_{1}^{1}W_{d-1})$ be an arbitrary continuous map. We call it a preclutching map with respect to $\bar{F}$. We try to glue $\bar{F}$ along $P_{2n}$

\[
P_{2n} \times p_{1}^{1}W_{d-1} \to P_{2n} \times p_{1}^{1}W_{d-1}, \quad (\bar{x}, u) \mapsto (c(\bar{x}), \bar{\Phi}(\bar{x})u)
\]

for $\bar{x} \in P_{2n}$ and $u \in p_{1}^{1}W_{d-1}$ via a preclutching map $\bar{\Phi}$. This gluing gives an equivariant vector bundle in $p_{\text{vect}}^{-1}(P_{1}(W))$ if and only if the following two conditions hold:

\[
E1. \quad \bar{\Phi}(c(\bar{x})) = \bar{\Phi}(\bar{x})^{-1} \quad \text{for each } \bar{x} \in P_{2n},
\]

\[
E2. \quad \bar{\Phi}(g\bar{x}) = g\bar{\Phi}(\bar{x})g^{-1} \quad \text{for each } g \in G_{\chi} \text{ and } \bar{x} \in P_{2n}.
\]
E2. \( \Phi(\bar{g}\bar{x}) = \bar{g}\Phi(\bar{x})\bar{g}^{-1} \) for each \( \bar{g} \in G_x \times Z \) and \( \bar{x} \in \bar{P}_{2n} \).

A preclutching map satisfying these two conditions is called an **equivariant clutching map** with respect to \( \bar{F} \). Note that equivariant clutching maps \( \Phi, \bar{\Phi} \) with respect to \( \bar{F}, F \) are actually \( \text{Iso}_H(p^*_1W_{d-1}) \), \( \text{Iso}_H(W_{d-1}) \)-valued by equivariance, respectively. Here, \( \text{Iso}_H(\cdot) \) is the subgroup of \( H \)-equivariant elements in \( \text{Iso}(\cdot) \). Also since 

\[
\text{res}_{G \times Z}^{G_x \times Z} p^*_1W_{d-1} = \text{res}_{G}^{G} W_{d-1},
\]

we have \( \text{Iso}_H(p^*_1W_{d-1}) = \text{Iso}_H(W_{d-1}) \). Details on equivariant clutching construction can be found in [Ki, Section 4].

Let \( \Omega_{\bar{F}} \) and \( \Omega_F \) be sets of equivariant clutching maps with respect to \( \bar{F} \) and \( F \), respectively. To use results of [Ki], we need relate \( \Omega_F \) to \( \Omega_{\bar{F}} \). Consider the map 

\[
q_{\Omega} : \Omega_{\bar{F}} \rightarrow \Omega_F, \quad \bar{\Phi} \mapsto \Phi(\bar{x}) = g_0\bar{\Phi}(\bar{x})
\]

for \( \bar{x} \in \bar{P}_{2n}^S \). Then, we have the following:

**Lemma 4.1.** \( q_{\Omega} \) is a well-defined one-to-one correspondence.

**Proof.** First, we show that \( q_{\Omega}(\Phi) \) for each \( \Phi \) is actually contained in \( \Omega_F \). Put \( \Phi = q_{\Omega}(\bar{\Phi}) \). For this, we show that \( \Phi \) satisfies Condition E1', E2'. Condition E1' is proved as follows:

\[
\Phi(g_0c(\bar{x})) = g_0\bar{\Phi}(g_0c(\bar{x})) = \bar{\Phi}(c(\bar{x}))g_0^{-1} = \bar{\Phi}(\bar{x})^{-1}g_0^{-1} = (g_0^{-1}\Phi(\bar{x}))^{-1}g_0^{-1} = \Phi(\bar{x})^{-1}
\]

for \( \bar{x} \in \bar{P}_{2n}^S \). And, Condition E2', is proved as follows:

\[
\Phi(g\bar{x}) = g_0\Phi(g\bar{x}) = g_0g\bar{\Phi}(g\bar{x})g^{-1} = g_0g\Phi(\bar{x})g^{-1} = g\Phi(\bar{x})g^{-1}
\]

for each \( g \in G_x \) and \( \bar{x} \in \bar{P}_{2n}^S \). So, \( q_{\Omega}(\Phi) \) is contained in \( \Omega_F \).

Second, we construct the inverse of \( q_{\Omega} \) to show bijectivity of it. Consider the map 

\[
q_{\Omega}^{-1} : \Omega_F \rightarrow \Omega_{\bar{F}}, \quad \Phi \mapsto \bar{\Phi}(\bar{x}) = \begin{cases} 
\Phi(g\bar{x}) & \text{for } \bar{x} \in \bar{P}_{2n}^S, \\
g_0^{-1}\Phi(\bar{x})c(\bar{x}) & \text{for } \bar{x} \in \bar{P}_{2n}^S.
\end{cases}
\]

We show that \( \bar{\Phi} = q_{\Omega}^{-1}(\Phi) \) for each \( \Phi \) is actually contained in \( \Omega_{\bar{F}} \). For this, we show that \( \Phi \) satisfies Condition E1', E2. Condition E1' is proved by definition of \( \Phi \). We
prove Condition E2. only when \( \bar{\phi} \in P_{2n}^{c} \) and \( \bar{g} = gg_{0} \) for some \( g \in G_{\chi} \) as follows:

\[
\Phi(\bar{g} \bar{x}) = \Phi(gg_{0} \bar{x}) = \left(g_{0}^{-1} \Phi(c(gg_{0} \bar{x}))\right)^{-1} = \Phi(c(gg_{0} \bar{x}))^{-1}g_{0} = \Phi(g \bar{x})g_{0} = g \Phi(\bar{x})g^{-1}g_{0} = gg_{0}^{-1} \Phi(\bar{x})g^{-1}g_{0} = gg_{0} \Phi(\bar{x})g_{0}^{-1}g^{-1} = \bar{g} \Phi(\bar{x})g^{-1}
\]

where we use equivariance of \( c \). Proof of Condition E2. for other \( \bar{x} \)'s and \( \bar{g} \)'s is proved similarly. Since \( q_{0} \) and \( q_{0}^{-1} \) are inverses of each other, we obtain a proof. \( \square \)

For convenience in calculation, we parameterize each \( |e^{i}|, |e_{0}^{i}| \) in \( |K_{2n}|, |\bar{K}_{2n}| \) for \( 0 \leq i \leq 2n - 1 \) by \( t \in [i, i+1] \) linearly to satisfy \( e_{0}^{i} \mapsto i \) and \( e_{0}^{i} \mapsto i \). Vertices \( e_{0}^{i}, e_{0}^{i} \) are parameterized by 0 or 2 according to context. Pick a path \( \sigma : [0, 1] \to \text{Iso}_{H}(W_{d-1}) \) such that \( \sigma(0) = \sigma(1) = \text{id} \) and \( [\sigma] \) is a generator of \( \pi_{1}\left(\text{Iso}_{H}(W_{d-1}), \text{id}\right) \). For a path \( \gamma \) defined on \( [0, 1] \), define a path \( \gamma^{tr} \) on \( [0, 1] \) as \( \gamma^{tr}(t) = \gamma(1-t) \) for \( t \in [0, 1] \), and a path \( \gamma_{-i} \) on \( [i, i+1] \) as \( \gamma_{-i}(t) = \gamma(t-i) \) for \( i \in \mathbb{Z} \). Now, we prove Theorem B and Corollary 4.3.

Proof of Theorem B. We obtain the first and second statement through the isomorphism \( \Psi_{1} \) because the preimage \( p_{\text{vect}}^{-1}\left(P_{1}(W)\right) \) contained in \( \text{Vect}_{G_{\chi} \times S^{2}, \chi} \) has two elements by \([K] \) Theorem B). In \([K] \) Prooof of Theorem B), these two are described as equivariant clutching constructions of \( F \) via equivariant clutching maps \( \Phi, \Phi' \) with respect to \( F \) satisfying the following:

1. \( \Phi|_{P_{2n}^{c}} \equiv \text{id} \),
2. \( \Phi'|_{P_{2n}^{c}}(t) = \left\{ \begin{array}{ll}
\sigma(t) & \text{for } t \in [0, 1], \\
(\sigma^{tr})^{-1}(t) & \text{for } t \in [1, 2].
\end{array} \right. \)

Since \( \bar{\Phi}'|_{P_{2n}^{c}}(t) = \sigma \vee (\sigma^{tr})^{-1}(t) \) for \( t \in [0, 2] \) and \( \Phi' \) is equivariant, we also have

\[
\Phi'|_{P_{2n}^{c}}(t) = g_{1} \left( \sigma \vee (\sigma^{tr})^{-1}(t-2i) \right) g_{1}^{i}
\]

for an integer \( 0 \leq i \leq n-1 \) and \( t \in [2i, 2i+2] \) where \( \rho(g_{1}) = a_{n} \). Put \( \Phi = q_{0}(\Phi) \) and \( \Phi' = q_{0}(\Phi') \). By Lemma 4.1 they are equivariant clutching maps with respect to \( F \). Denote by \( E \) and \( E' \) the equivariant vector bundles determined by \( \Phi \) and \( \Phi' \), respectively. It is easy that the \( E \) is equivariantly trivial. Since \( \Phi' \) is equivariant, it is determined by its values on the half \( [0, n] \) of \( P_{2n}^{c} \). Observe that \( \Phi' \) on \( [1, n] \cup [n+1, 2n] \) does not contribute to \( c_{1}(E') \) because \( n \) is odd and the pair \( g_{1}^{i} \sigma^{i}g_{1}^{i} \) and \( g_{1}^{i} \sigma^{i-1}g_{1}^{i} \) for \( 0 \leq i, i' \leq n-1 \) cancel each other in the fundamental group \( \pi_{1}\left(\text{Iso}_{H}(W_{d-1}), \text{id}\right) \). In other words, \( E' \) is nonequivariantly isomorphic to the nonequivariant vector bundle \( E'' \) determined by the preclutching map \( \Phi'' \) with respect to \( F \) defined by

\[
\Phi''(t) =\begin{cases}
\text{id} & \text{for } t \in [1, n] \cup [n+1, 2n],
\sigma(t) & \text{for } t \in [0, 1],
\sigma(t-n)^{-1} & \text{for } t \in [n, n+1].
\end{cases}
\]
Then, we can show that
\[ c_1(E'') \equiv \chi(id) \mod 2 \]
in \( H^2(\mathbb{R}P^2, \mathbb{Z}) \) by \[\text{Kl}\] Lemma 7.1 because \([\sigma]\) is a generator of the fundamental group \(\pi_1(\text{Iso}_H(W_{d-1}), \text{id})\). Therefore, we obtain a proof. \(\square\)

**Proof of Corollary 1.3.**
By \[\text{Kl}, \text{Theorem D}\], \(\text{Vect}_{G_x \times \mathbb{Z}}(S^2, \chi)\) has a rank \(\chi(id)\)-bundle so that \(\text{Vect}_{G_\chi}(\mathbb{R}P^2, \chi)\) has a rank \(\chi(id)\)-bundle by the isomorphism (3.1). From this, we have
\[ \text{Vect}_{G_x}(\mathbb{R}P^2, \chi) \cong \text{Vect}_{\mathbb{R}}(\mathbb{R}P^2) \]
as semigroups by \[\text{CKMS}\] Lemma 2.2. Also, we obtain
\[ \text{Vect}_{R \times \mathbb{Z}}(S^2) \cong \text{Vect}_{\mathbb{R}}(\mathbb{R}P^2) \]
by the isomorphism (3.1). It is easy that \(\text{Vect}_{\mathbb{R}}(\mathbb{R}P^2)\) is generated by line bundles and that \(A_R(\mathbb{R}P^2, \text{id})\) is generated by all the elements with one-dimensional entries by \[\text{Kl}\] Theorem D. Since \(R_x = \langle \text{id} \rangle\) for each \(x \in o(P_{2n})\), any triple \((W_{d'}, x)\) in \(\text{Rep}(R_{d-1}) \times \text{Rep}(R_{d'}) \times \text{Rep}(R_{d''})\) is contained in \(A_R(\mathbb{R}P^2, \text{id})\) if and only if \(W_{d'}\)'s are same dimensional. So, the number of all the elements in \(A_R(\mathbb{R}P^2, \text{id})\) with one-dimensional entries is equal to \(n\) because \(R_{d-1} \cong \mathbb{Z}_n\) and \(R_{d'} = R_{d''} = \langle \text{id} \rangle\). Therefore, we obtain a proof. \(\square\)

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