Graph Theoretic Method for Determining non- Hurwitz Equivalence in the Braid Group and Symmetric group

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ABSTRACT. Motivated by the problem of Hurwitz equivalence of $\Delta^2$ factorization in the braid group, we address the problem of Hurwitz equivalence in the symmetric group, obtained by projecting the $\Delta^2$ factorizations into $S_n$. We get $1_S$ factorizations with transposition factors. Looking at the transpositions as the edges in a graph, we show that two factorizations are Hurwitz equivalent if and only if their graphs have the same weighted connected components. The main result of this paper will help us to compute the BMT invariant presented in [1] or [2]. The graph structure gives a weaker but very easy to compute invariant to distinguish between diffeomorphic surfaces which are not deformation of each other.

1 Definitions

Definition 1.1. Braid Group $B_n$

$B_n$ is the group generated by $\sigma_1, \ldots, \sigma_{n-1}$ with the following relations:

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i - j| > 1
\]

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.
\]

Definition 1.2. Half Twist

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Let $H \in B_n$, we say that $H$ is a half twist if $H = P\sigma_iP^{-1}$ for some $1 \leq i \leq n - 2$ and $P \in B_n$.

**Definition 1.3.** Hurwitz move on $G^m (R_k, R_k^{-1})$

Let $G$ be a group, $\overrightarrow{t} = (t_1, ..., t_m) \in G^m$. We say that $\overrightarrow{s} = (s_1, ..., s_m) \in G^m$ is obtained from $\overrightarrow{t}$ by the Hurwitz move $R_k$ (or $\overrightarrow{t}$ is obtained from $\overrightarrow{s}$ by the Hurwitz move $R_k^{-1}$) if

$$s_i = t_i \quad \text{for } i \neq k, k + 1,$$

$$s_k = t_k t_{k+1} t_k^{-1}, \quad s_{k+1} = t_k.$$

**Definition 1.4.** Hurwitz move on factorization

Let $G$ be a group and $t \in G$. Let $t = t_1 \cdot \cdot \cdot t_m = s_1 \cdot \cdot \cdot s_m$ be two factorized expressions of $t$. We say that $s_1 \cdot \cdot \cdot s_m$ is obtained from $t_1 \cdot \cdot \cdot t_m$ by the Hurwitz move $R_k$ if $(s_1, ..., s_m)$ is obtained from $(t_1, ..., t_m)$ by the Hurwitz move $R_k$.

**Definition 1.5.** Hurwitz equivalence of factorization

The factorizations $s_1 \cdot \cdot \cdot s_m, t_1 \cdot \cdot \cdot t_m$ are Hurwitz equivalent if they are obtained from each other by a finite sequence of Hurwitz moves. The notation is $t_1 \cdot \cdot \cdot t_m \overset{HE}{\sim} s_1 \cdot \cdot \cdot s_m$.

## 2 Projecting to $S_n$

Let $\phi : B_n \to S_n$ be the natural homomorphism to $S_n$, given by $\phi(b) \to \pi_b$ where $\pi_b$ is the permutation given by the strings of $b$. In terms of definition [1], $\phi : B_n \to S_n$ is defined by $\phi(\sigma_i) = (i, i + 1) \ 1 \leq i \leq n - 1$.

**Proposition 2.1.** Let $b_1 \cdot \cdot \cdot b_m$, $r_1 \cdot \cdot \cdot r_m$ be two factorizations in $B_n$ s.t. $b_1 \cdot \cdot \cdot b_m \overset{HE}{\sim} r_1 \cdot \cdot \cdot r_m$, then, $\phi(b_1) \cdot \cdot \cdot \phi(b_m) \overset{HE}{\sim} \phi(r_1) \cdot \cdot \cdot \phi(r_m)$.
Proof: It is sufficient to show for the case where the factorization \( r_1 \cdots r_m \) is obtained from \( b_1 \cdots b_m \) by a single Hurwitz move, \( R_i \), and therefore, \( r_i = b_i b_{i+1}^{-1} \) and \( r_{i+1} = b_i \) and \( r_k = b_k \) if \( k \neq i, i + 1 \).

By performing \( R_i \) on \( \phi(b_1) \cdots \phi(b_i) \cdot \phi(b_{i+1}) \cdots \phi(b_m) \) we get

\[ \phi(b_1) \cdots \phi(b_i) \phi(b_{i+1}) \phi(b_i)^{-1} \cdot \phi(b_i) \cdots \phi(b_m) \]

which is equal to,

\[ \phi(b_1) \cdots \phi(b_i b_{i+1} b_i^{-1}) \cdot \phi(b_i) \cdots \phi(b_m) \]

which is the same as \( \phi(r_1) \cdots \phi(r_m) \).

From Propositions 2.1, we are interested in the properties of the Hurwitz equivalence relation on factorizations in \( S_n \). In \( B_n \) we are interested in \( \Delta^2 \) factorizations where all factors are powers of half-twists.

\( \Delta^2 = (\sigma_1 \cdots \sigma_{n-1})^n \) is a full \( 2\pi \) twist of all the strings (in \( B_n \)) and therefore, \( \phi(\Delta^2) = 1_{S_n} \). If \( H \) is a half twist by definition 1.2, we get that \( \phi(H) = (i, j) \) \( 1 \geq i, j \geq n \). As a result, when projecting to \( S_n \), we are interested in the properties of \( 1_{S_n} \) factorizations with transpositions or \( 1_{S_n} \) (when the power of the half twist is even) as factors.

### 3 Hurwitz Equivalence Properties in \( S_n \)

**Definition 3.1.** Let \( \Gamma_1 \cdots \Gamma_m \), \( \Gamma_i = (a_i, b_i) \) \( 1 \leq a_i, b_i \leq n \) be a factorization. We define the graph of the factorization \( G_F = (V_F, E_F) \) where \( V_F = \{1, \ldots, n\} \) are the vertices and \( E_F = \{(i, j) \mid \exists k \text{ s.t. } \Gamma_k = (i, j)\} \) are the edges of the factorization graph.

**Definition 3.2.** We define the weight of an edge \( (i, j) \in E_F \) as the number of elements \( \Gamma_k \) s.t. \( \Gamma_k = (i, j) \). The weight of \( (i, j) \) in the factorization \( F \) will be noted as \( W_F((i, j)) \).

For a given graph \( G_F \) we denote the graphs of its connected components as \( G^1_F, \ldots, G^d_F \) where \( d \) is the number of connected components in the graph. For each connected component, let \( G^i_F = (V^i_F, E^i_F) \), where \( V^i_F \) are the vertices of \( G^i_F \) and \( E^i_F \) are the edges.
Definition 3.3. We define the weight of the connected component $G^i_F$ to be:

$$W(G^i_F) = \sum_{e_r \in E^i_F} W(e_r)$$

Theorem 3.4. Let $F_1, F_2$ be two $1_{S_n}$ factorizations with the same number of factors. Then $F_1 \simhe F_2$ if and only if $G_{F_1}$ and $G_{F_2}$ have the same number of connected components $G^1_{F_1}, \ldots, G^d_{F_1}$ and $G^1_{F_2}, \ldots, G^d_{F_2}$ respectively, and there exists a permutation $\pi$ s.t. $V^i_{F_1} = V^\pi(i)_{F_2}$ and $W(G^i_{F_1}) = W(G^\pi(i)_{F_2})$ for each $i \leq d$.

In other words, two factorizations are Hurwitz equivalent if and only if the connected components of the factorizations graphs contain the same nodes and have the same weights.

Example 3.5.

The $1_{S_n}$ factorizations,

$$F_1 = (2,6) \cdot (1,4) \cdot (1,5) \cdot (3,6) \cdot (4,5) \cdot (1,5) \cdot (2,3) \cdot (3,6)$$

$$F_2 = (2,6) \cdot (1,5) \cdot (3,6) \cdot (3,6) \cdot (2,6) \cdot (1,5) \cdot (1,4) \cdot (1,4)$$

have connected components with the same nodes and and weights as shown in Figure 3, and by Theorem 3.4 they are Hurwitz equivalent.

The rest of the section will be devoted to the proof of Theorem 3.4. Starting with the first direction of the theorem.

Proof of the first direction: In the proof of the first direction we prove that if two factorizations are Hurwitz equivalent the factorizations have the same graph components with the same weights. Therefore, it is sufficient to show that when operating a single Hurwitz move, the vertices and weights of the graph’s connected components will remain the same.

Let $F_1 = \Gamma_1 \cdots \Gamma_m$ and $F_2 = \Gamma_1 \cdots \Gamma_i \Gamma_{i+1} \Gamma_i^{-1} \cdot \Gamma_i \cdots \Gamma_m$ the factorization obtained from $F_1$ by performing Hurwitz move $R_i$.

Let $\Gamma_j = (a_j, b_j), \ 1 \leq a_j, b_j \leq n, \ j \leq m$, then, in the cases where:

$$\{a_i, b_i\} \bigcap \{a_{i+1}, b_{i+1}\} = \phi$$
Figure 1: The graphs of the factorizations $F_1$ and $F_2$

\[
\{a_i, b_i\} \cap \{a_{i+1}, b_{i+1}\} = \{a_i, b_i\}
\]

We get that,

\[
\Gamma_i \Gamma_{i+1} \Gamma_i^{-1} = \Gamma_{i+1}
\]

and the two factorizations have the same factors in a different order, so the factorizations graphs $G_{F_1}$ and $G_{F_2}$ are the same.

We are left with the case where $\Gamma_i = (a_i, b)$ and $\Gamma_{i+1} = (a_{i+1}, b)$. $\Gamma_i \Gamma_{i+1} \Gamma_i^{-1} = (a_i, a_{i+1})$ replaces $\Gamma_{i+1}$. We will show that the theorem still holds for this case.

**Lemma 3.6.**

1. If $v_1, v_2 \in V_{F_1}^r$ then $v_1, v_2 \in V_{F_2}^{r_1}$ for some $r_1$.

2. If $v_1 \in V_{F_1}^{r_2}$ and $v_2 \in V_{F_1}^{r_2}$, $r_1 \neq r_2$ then $v_1 \in V_{F_2}^{t_1}$ and $v_2 \in V_{F_2}^{t_2}$, for some $t_1, t_2$ s.t. $t_1 \neq t_2$.

3. If $V_{F_1}^{r_1} = V_{F_2}^{r_1}$ then $W(G_{F_1}^{r_1}) = W(G_{F_2}^{r_1})$.

**Proof 1:** Since $v_1, v_2 \in V_{F_1}^r$ there is a sequence of edges connecting them in $G_{F_1}^r$, say, $\{e_s\}_{s=1}^p$.

Since $G_{F_1}$ is a connected component, $\{e_s\}_{s=1}^p \subset E_{F_1}^r$. So if $\Gamma_i \Gamma_{i+1} \Gamma_i^{-1}$ all $\{e_s\}_{s=1}^p$ remain as elements in the factorization $F_2$, since only $\Gamma_{i+1}$ is replaced by $\Gamma_{i+1} \Gamma_i \Gamma_i^{-1}$.
In the case where $\Gamma_{i+1} \in E_{F_1}$, every $s'$ s.t. $e_{s'} = \Gamma_{i+1}$ will be replaced in the sequence by $\Gamma_i \Gamma_{i+1} \Gamma_i^{-1} = (a_i, a_{i+1})$ and $\Gamma_i = (a_i, b)$ which are elements in $F_2$ and connect $a_{i+1}$ with $b$.

**Proof 2:** From (1), we conclude that the number of vertices in a connected component can only increase. Therefore, if $v_1 \in V_{F_1}^{r_1}$ and $v_2 \in V_{F_1}^{r_2}$, $r_1 \neq r_2$ and they are in the same connected component $V_{F_2}^r$ in $G_{F_2}$, then $V_{F_1}^{r_1}, V_{F_1}^{r_2} \subset V_{F_2}^r$. Therefore, there exists an edge $(v_1', v_2') \in E_{F_2}$ s.t. $v_1', v_2'$ belongs to a different connected components in $G_{F_1}$. But the only edge that was added is $(a_{i+1}, a_i)$ and $a_{i+1}, a_i$ are in the same connected component in $G_{F_1}$.

**Proof 3:** From (1) and (2), we see that the connected components remain the same. The weights of the connected components remain the same since all edges are the same except for $(a_i, b)$ that was replaced by $(a_i, a_{i+1})$. But $a_i, a_{i+1}, b$ are all in the same connected component. Therefore the weight of all connected components remain the same.

The Lemma proves that when performing a Hurwitz move on two transpositions the nodes of the connected components remain the same and so are the weights of the connected components. If one or both of the factors are $1_{S_n}$, the Hurwitz move does not change the factors only the order and therefore, the theorem still holds.

This concludes the proof of the first direction. \hfill \qed

**Proof of the second direction:** To complete Theorem 3.4 we need to show that if two factorizations have the same connected components with the same weights they are Hurwitz equivalent. To prove that we will show that each factorization is Hurwitz equivalent to a standard canonical factorization which depends only on the nodes of the factorization’s connected components and their weights.

**Lemma 3.7.** Let $(a, b) \cdot (c, d)$ be a factorization is $S_n$ then,

1. By performing Hurwitz move $R_0$ we get:
\( (a, b) \cdot (c, d) \overset{\text{HE}}{\sim} \begin{cases} (c, d) \cdot (a, b), & \text{if } \{a, b\} \cap \{c, d\} = \emptyset \\ (c, d) \cdot (a, b), & \text{if } \{a, b\} \cap \{c, d\} = \{a, b\} \\ (a, d) \cdot (a, b), & \text{if } b = c \text{ and } a \neq d \end{cases} \)

2. By performing Hurwitz move \( R_0^{-1} \) we get:

\( (a, b) \cdot (c, d) \overset{\text{HE}}{\sim} \begin{cases} (c, d) \cdot (a, b), & \text{if } \{a, b\} \cap \{c, d\} = \emptyset \\ (c, d) \cdot (a, b), & \text{if } \{a, b\} \cap \{c, d\} = \{a, b\} \\ (c, d) \cdot (a, d), & \text{if } b = c \text{ and } a \neq d \end{cases} \)

3. \( (a, b) \cdot 1_{S_n} \overset{\text{HE}}{\sim} 1_{S_n} \cdot (a, b) \)

**Proof:** Trivial.

From Lemma 3.7 (3) the \( 1_{S_n} \) factors commutes with all other factors. Each factorization is Hurwitz equivalent to a factorization where all \( 1_{S_n} \) factors are on the left of the factorization and the two factorizations have the same transpositions as factors. Since the theorem requires that the weights of the connected components is equal and that both factorizations have the same number of factors, the number of \( 1_{S_n} \) factor is equal.

Therefore, to prove Theorem 3.4 we can ignore the \( 1_{S_n} \) factors, and find standard canonical form to the transposition factors only.

Let \( F_1 = \Gamma_1 \cdots \Gamma_m \) be a factorization of \( 1_{S_n} \) where all factors are of transpositions.

**Lemma 3.8.** If \( \Gamma_j \notin E_{F_1}^{t_1} \) and \( \Gamma_j+1 \notin E_{F_1}^{t_2} \), \( t_1 \neq t_2 \) then \( \Gamma_1 \cdots \Gamma_j \Gamma_{j+1} \cdots \Gamma_m \overset{\text{HE}}{\sim} \Gamma_1 \cdots \Gamma_{j+1} \Gamma_j \cdots \Gamma_m \).

**Proof:** Since \( \Gamma_j, \Gamma_{j+1} \) belong to a different connected component, they do not connect the same vertex and therefore, \( \Gamma_j \Gamma_{j+1} \Gamma_j^{-1} = \Gamma_{j+1} \) (See Lemma 3.4). Therefore, by operating Hurwitz move \( R_t \) they commute.

As a result, for each connected component, all elements of the component commutes with all elements of other components. Therefore, factorization is Hurwitz equivalent to a factorization with the same factors ordered according
to the component they belong too. For example, order the connected components by the lowest vertex they contain, then gather all factors of the first component to the left, and after them the factors of the second component and so on.

Let \( \{G^r_{F_1}\}_{r=1}^s \) be the distinct connected components of \( G_{F_1} \). From Lemma 3.8, \( F_1 \overset{HE}{\sim} f_1 \cdots f_s \) where \( f_r \) is a factorization with elements from \( E^r_{F_1} \). The length of the factorization \( f_r \) is equal to \( W(G^r_{F_1}) \) and \( s \) is the number of connected components. Therefore, to conclude the proof, it is sufficient to show that each \( f_r \) is Hurwitz equivalent to a standard canonical factorization which depends only on the length of \( f_r \) (which can never be changed by Hurwitz moves) and \( V^r_{F_1} \).

We define an order on \( V^r_{F_1} \) vertices, \( V^r_{F_1} = \{v_{t_1}, ..., v_{t_l}\} \). Note that since \( F_1 = 1_{S_n} \) then, \( f_r = 1_{S_n} \) as a product.

**Lemma 3.9.** Let \( f = \Gamma_1 \cdots \Gamma_m \) be a factorization with a single connected component, \( G^r_f \), then \( \forall v_1, v_2 \in V^r_f, \quad f \overset{HE}{\sim} (v_1, v_2) \cdot \gamma_1 \cdots \gamma_{m-1} \).

**Proof:** Proof by induction on the minimal length of the path connecting \( v_1 \) with \( v_2 \). In the case where the minimal length is 1, there exists \( 1 \leq j \leq m \) s.t. \( \Gamma_j = (v_1, v_2) \). Operating \( \{R_{k-1}\}_{k=j-2}^0 \) sequence of Hurwitz moves, we get a factorization \( (v_1, v_2) \cdot \gamma_1 \cdots \gamma_{m-1} \) (See Lemma 3.7). We will assume that the lemma is true for a path with length less than \( n \), we will prove that the factorization where the minimal path between \( v_1 \) and \( v_2 \) is \( n \), is Hurwitz equivalent to a factorization which the path between \( v_1 \) and \( v_2 \) is of length \( n - 1 \):

Let \( (a_1, a_2), (a_2, a_3), ..., (a_n, a_{n+1}) \) be the minimal path between \( v_1 = a_1 \) and \( v_2 = a_{n+1} \). To prove the above we will perform another induction, on the number of factors which are in between \( (a_1, a_2) \) and \( (a_2, a_3) \). We will assume that \( (a_1, a_2) \) is left to \( (a_2, a_3) \):

\[
f = \cdots (a_1, a_2) \cdot (b_1, c_1) \cdot (b_2, c_2) \cdots (b_k, c_k) \cdot (a_2, a_3) \cdots
\]

Let \( k \) be the number of factors between \( (a_1, a_2) \) and \( (a_2, a_3) \).

In the case where \( k = 0 \), \( (a_1, a_2) \cdot (a_2, a_3) \overset{HE}{\sim} (a_1, a_3), (a_1, a_2) \) and we are done
since the new factorization contains the path \((a_1, a_3), (a_3, a_4), ..., (a_n, a_{n+1})\) which is of length \(n - 1\).

In the case where \(k > 0\):

- If \(\{a_1, a_2\} \cap \{b_1, c_1\} = \emptyset\) then, \((a_1, a_2) \cdot (b_1, c_1) \overset{HE}{\sim} (b_1, c_1) \cdot (a_1, a_2)\) (By Lemma 3.9), and now \((a_1, a_2)\) and \((a_2, a_3)\) are separated by \(k - 1\) factors, and so we are done.

- If \(a_1 = b_1\) then, \((a_1, a_2) \cdot (a_1, c_1) \overset{HE}{\sim} (a_2, c_1) \cdot (a_1, a_2)\) (By Lemma 3.7), and now \((a_1, a_2)\) and \((a_2, a_3)\) are separated by \(k - 1\) factors. Note that \((a_1, c_1)\) is not an element in the path (since the path is minimal).

- If \(a_2 = b_1\) then, \((a_1, a_2) \cdot (a_2, c_1) \overset{HE}{\sim} (a_1, c_1) \cdot (a_1, a_2)\) (By Lemma 3.7), and now \((a_1, a_2)\) and \((a_2, a_3)\) are separated by \(k - 1\) factors. Note that if \((a_2, c_1)\) is in the path then \(c_1 = a_3\) and then \(k = 0\).

This concludes the proof of Lemma 3.9.

**Lemma 3.10.**

1. \((a, b) \cdot (a, b) \cdot (a, c) \cdot (a, c) \overset{HE}{\sim} (a, c) \cdot (a, c) \cdot (a, b) \cdot (a, b)\).

2. \((a, b) \cdot (a, b) \cdot (a, c) \cdot (a, c) \overset{HE}{\sim} (a, b) \cdot (a, b) \cdot (b, c) \cdot (b, c)\).

**Proof 1:** \((a, b) \cdot (a, b) \cdot (a, c) \overset{HE}{\sim} (a, c) \cdot (a, b) \cdot (a, b)\) By operating Hurwitz moves \(R_1\) and \(R_0\) and therefore, \((a, b) \cdot (a, b) \cdot (a, c) \cdot (a, c) \overset{HE}{\sim} (a, c) \cdot (a, c) \cdot (a, b) \cdot (a, b)\).

**Proof 2:** By performing the Hurwitz moves \(R_1^{-1}, R_2^{-1}, R_1^{-1}\).

Now we are ready to start forming \(f_r\) into a standard canonical form:

\[v_{t_1}, v_{t_2} \in V_f, \text{ from Lemma 3.9} \]

\[f_r \overset{HE}{\sim} (v_{t_1}, v_{t_2}) \cdot f_r^1\]

where \(f_r^1\) is the factorization with the \(W(G_{f_r}^1) - 1\) other factors.

\[f_r^1 = (v_{t_1}, v_{t_2})\] since \(f_r = 1_{S_n}\) and \(f_r^1 (v_{t_1}, v_{t_2})^{-1} f_r\).

Because \(f_r^1 = (v_{t_1}, v_{t_2})\), \(f_r^1\) contains a path connecting \(v_{t_1}\) with \(v_{t_2}\). Again, using Lemma 3.9 we get,

\[f_r \overset{HE}{\sim} (v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdot f_r^2\text{ and } f_r^2 = 1_{S_n}\]

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By the first direction of Theorem 3.4 \((v_1, v_2) \cdot (v_1, v_2) \cdot f_r^2\) still creates a single connected component. Therefore, there is a path between \(v_t\) and \(v_{t_3}\), which means that \(f_r^2\) contains a path from \(v_t\) to \(v_{t_3}\) or from \(v_t\) to \(v_{t_3}\) (for example in some cases where the path in \(G_r^f\) includes \((v_1, v_2)\)).

In the first case we get \(f_r \overset{HE}{\sim} (v_1, v_2)(v_1, v_2)(v_{t_2}, v_{t_3})(v_{t_2}, v_{t_3}) \cdot f_r^4\) and in the second case we get \(f_r \overset{HE}{\sim} (v_1, v_2)(v_1, v_2)(v_{t_1}, v_{t_3})(v_{t_1}, v_{t_3}) \cdot f_r^4\) which is by Lemma 3.10 Hurwitz equivalent to the first case.

We continue with this process to bring \(f_r\) to the form:
\[(v_1, v_2) \cdot (v_1, v_2) \cdot (v_{t_2}, v_{t_3}) \cdot (v_{t_2}, v_{t_3}) \cdots (v_{t_m}, v_{t_m}) (v_{t_{m-1}}, v_{t_{m-1}}) \cdot f_r^{2m-2}\]
Assume we came to the point where:
\[f_r \overset{HE}{\sim} (v_1, v_2) \cdot (v_1, v_2) \cdots (v_{t_{k-1}}, v_{t_{k-1}}) \cdot (v_{t_{k-1}}, v_{t_{k-1}}) \cdot f_r^{2k-2}\]
where \(k < m\) and \(f_r^{2k-2}\) is a factorization with \(W(G_r^f) - 2k + 2\) factors.

Same as before, the new factorization creates a connected graph and \(f_r^{2k-2} = 1_s\). Since the graph is connected, \(f_r^{2k-2}\) contains a path from \(v_{t_{k+1}}\) to one of the vertices \(v_s\), \(s \leq k\). So, there is a path from \(v_s\) to \(v_{t_{k+1}}\) which does not include the factors left to \(f_r^{2k-2}\) because they create a connected graph which does not include \(v_{t_{k+1}}\). So, from Lemma 3.9,
\[f_r \overset{HE}{\sim} (v_1, v_2) \cdot (v_1, v_2) \cdots (v_{t_{k-1}}, v_{t_{k-1}}) \cdot (v_{t_{k-1}}, v_{t_{k-1}}) \cdot f_r^{2k-1},\]
and since \(f_r^{2k-1} = (v_{t_{k+1}}, v_s)\) as a product, there is a path from \(v_{t_{k+1}}\) to \(v_s\).

By Lemma 3.9,
\[f_r \overset{HE}{\sim} (v_1, v_2) \cdot (v_1, v_2) \cdots (v_{t_{k-1}}, v_{t_{k-1}}) \cdot (v_{t_{k-1}}, v_{t_{k-1}}) \cdot (v_{t_{k-1}}, v_{t_{k-1}}) \cdot (v_{t_{k-1}}, v_{t_{k-1}}) \cdot f_r^{2k}.\]

Now, only using the factors left to \(f_r^{2k}\) we need to change \((v_{t_{k+1}}, v_s)\cdot (v_{t_{k+1}}, v_s)\) to \((v_t, v_{t_{k+1}})\cdot (v_t, v_{t_{k+1}})\). Since \(s \leq k\) there is a path from \(v_s\) to \(v_t\) in the graph created by the factors on the left, from this fact and using Lemma 3.10 we see that the factorizations:
\[(v_1, v_2) \cdot (v_1, v_2) \cdots (v_{t_{k-1}}, v_{t_{k-1}}) \cdot (v_{t_{k-1}}, v_{t_{k-1}}) \cdot (v_{t_{k-1}}, v_{t_{k-1}}) \cdot (v_{t_{k-1}}, v_{t_{k-1}})\]
and
\[(v_1, v_2) \cdot (v_1, v_2) \cdots (v_{t_{k-1}}, v_{t_{k-1}}) \cdot (v_{t_{k-1}}, v_{t_{k-1}}) \cdot (v_{t_{k-1}}, v_{t_{k-1}}) \cdot (v_{t_{k-1}}, v_{t_{k-1}})\]
are Hurwitz equivalent, since Lemma 3.9 allows us to commute couples of
transpositions, or to change one vertex in the couple if the two couples have a common vertex.

To complete the proof of Theorem 3.4 we need to show that we can also bring the right factors, $f_r^{2m-2}$ to a standard form. This can be done in a similar way to the above procedure:

Take the first factor in $f_r^{2m-2}$, i.e. $f_r^{2m-2} = (v_{t_x}, v_{t_y}) \cdot f_r^{2m-1}$ and again by using Lemma 3.9 we get

$$f_r^{2m-2} \sim (v_{t_x}, v_{t_y}) \cdot (v_{t_x}, v_{t_y}) f_r^{2m}.$$  

Using Lemma 3.10 and the factors on the left, we can change $(v_{t_x}, v_{t_y}) \cdot (v_{t_x}, v_{t_y})$ to $(v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2})$ since the graph of the factors on the left is the same as $f_r$. This concludes the proof of Theorem 3.4 since every factorization $f_r$ is Hurwitz equivalent to:

$$(v_{t_1}, v_{t_2}) \cdot (v_{t_1}, v_{t_2}) \cdots (v_{t_{m-1}}, v_{t_m}) \cdot (v_{t_{m-1}}, v_{t_m}) (v_{t_1}, v_{t_2}) \cdots (v_{t_1}, v_{t_2})$$

Which depends only on the factorization graph and the number of factors in the factorization. \qed
References

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