Fundamental normal surfaces and the enumeration of Hilbert bases

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Abstract

Normal surfaces are a key tool in computational knot theory and 3-manifold topology, and have featured in significant computational breakthroughs in recent years. Despite this, there has been little practical progress on algorithms that use fundamental normal surfaces, which are described in terms of a Hilbert basis for a pointed rational cone on a high-dimensional integer lattice. In this paper we develop and implement several algorithms to enumerate fundamental normal surfaces, by merging domain-specific techniques from normal surface theory with classical Hilbert basis algorithms. The most successful of these combines a maximal admissible face decomposition with the primal Hilbert basis algorithm of Bruns, Ichim and Koch, and in many cases can solve 168-dimensional enumeration problems (based on 24-tetrahedron knot complements) in a matter of hours. As an application, we use this new algorithm to compute 164 previously unknown crosscap numbers in the KnotInfo database of knot invariants.

Keywords Normal surfaces, fundamental surfaces, algorithms, 3-manifolds, knots, crosscap number

1 Introduction

Normal surfaces are a vital tool for algorithms and complexity in 3-manifold topology and knot theory. For algorithms, they allow us to reduce difficult topological searches to discrete problems on integer points in rational cones [23, 38]. For complexity, they allow us to produce polynomial-sized certificates that prove that unknot recognition and 3-sphere recognition are in NP [23, 43].

Normal surfaces were introduced by Kneser [36], and later developed for algorithmic use by Haken [22]. The key ideas are as follows. We present the input to our topological problem as a 3-manifold triangulation $T$ (so, if the input is a knot, we triangulate the knot complement [23]), and we frame our problem as a search for some embedded surface in $T$ with a particular property (for instance, for the unknot recognition problem we search for a spanning disc in the knot complement). We then show that we can restrict this search to normal surfaces in $T$, which are properly embedded surfaces that intersect the tetrahedra of $T$ in a well-behaved fashion.

Normal surfaces have the useful property that we can encode them arithmetically. Specifically, if $T$ contains $n$ tetrahedra, we can encode normal surfaces as integer points in the normal surface solution cone $C$, which is a pointed rational cone in $\mathbb{R}^{7n}$. For many topological problems, we can construct a finite list $L$ of points in this cone so that, if there is some surface in
with the desired property, then there is one described by a point in our list \( L \). A typical algorithm would then build the cone \( C \), construct the finite list \( L \), rebuild the corresponding surfaces, and test each for the desired property.

A key point of difference between topological algorithms is how we construct this finite list \( L \):

(i) For some algorithms, \( L \) contains the smallest integer point on each extremal ray of the cone \( C \) (sometimes their doubles are also required). The corresponding surfaces are called vertex normal surfaces.

Topological algorithms of this type include unknot recognition \([22, 23]\), 3-sphere recognition \([31, 42]\), computing knot genus in homology 3-spheres \([47]\), connected sum decomposition \([31]\), and Hakeness testing \([30, 34]\).

(ii) For other algorithms, \( L \) is the Hilbert basis for the cone \( C \); that is, all integer points in \( C \) that cannot be expressed as a non-trivial sum of other integer points in \( C \). The corresponding surfaces are called fundamental normal surfaces.

Algorithms of this type include JSJ decomposition \([38]\), computing knot genus in arbitrary 3-manifolds \([1, 44]\), computing the crosscap number of a knot \([12]\), and determining which Dehn fillings of a knot complement are Haken \([33]\).

(iii) For some algorithms, the list \( L \) is much larger again: typically one must first enumerate all fundamental normal surfaces, and then (possibly at great expense) extend this list further in some way. Algorithms of this type include finding an essential planar surface in a link manifold \([32]\), or computing the Heegaard genus of a 3-manifold \([37]\).

Not all integer points in \( C \) correspond to normal surfaces: there are additional quadrilateral constraints that such points must satisfy. As a result, normal surfaces are only obtained from a small subset of the extremal rays or the Hilbert basis of \( C \). This is a powerful means for optimisation that underpins much of this paper.

Case (i) above is a “best case scenario” for normal surface theory. Vertex normal surfaces can be enumerated using highly optimised polytope vertex enumeration techniques, explicitly tailored for the setting of normal surface theory \([7, 13]\). There are practical software implementations available \([11, 19]\), which in turn have enabled new theoretical and experimental advances \([10, 14, 18]\).

Case (ii) is more problematic. Because of the high dimensionality of the problem and the potential for super-exponentially many solutions \([23]\), even state-of-the-art Hilbert basis algorithms \([2, 3]\) are impractically slow for all but the simplest inputs. Although some special cases have been analysed by hand \([24, 28, 29]\), there are no enumeration algorithms or software implementations that exploit the rich structure of normal surface theory, and at present the enumeration of fundamental normal surfaces is still considered impractically difficult.

It is these issues that we address in this paper. We develop new algorithms to enumerate fundamental normal surfaces by merging the structure and constraints of normal surface theory with classical Hilbert basis algorithms. We also develop and compare software implementations of these algorithms, and show that the best of these is practical for “real” non-trivial inputs by using it to compute 164 previously unknown crosscap numbers in the KnotInfo database \([15]\).

\[^1\] The recent paper \([12]\) also computes new crosscap numbers via normal surfaces, but uses integer programming techniques instead. The methods here and in \([12]\) are complementary: each can solve problems that the other cannot.
Case (iii) above is even more difficult, and many such problems remain severely intractable. However, we note that they also benefit from our new algorithms, since the first stage is typically to enumerate all fundamental normal surfaces before extending them to a much larger list \( \mathcal{L} \).

Hilbert basis enumeration is difficult: it generalises polytope vertex enumeration, which is itself NP-hard \([20, 35]\). Nevertheless, several families of algorithms exist, as well as efficient implementations such as the state-of-the-art software package \textit{Normaliz} \([2, 3]\). It is widely noted that no one algorithm for Hilbert basis enumeration is superior in all settings \([2, 17, 40]\), and for this reason we work with three well-known algorithms from the literature:

- We begin with the \textit{primal algorithm} described by Bruns, Ichim and Koch \([2, 3]\) and implemented in \textit{Normaliz}. In essence, this triangulates the cone \( \mathcal{C} \) into simplicial subcones, constructs parallelotopes at the tips of these subcones, locates all integer points within these parallelotopes, and then reduces this list to a final basis.

  In Section 3 we describe a primal algorithm for enumerating fundamental normal surfaces. The key idea is to enumerate all \textit{vertex} normal surfaces, and use these to build up the small region of \( \mathcal{C} \) that satisfies the quadrilateral constraints. We then run each “maximal admissible face” of this region through the primal algorithm of \textit{Normaliz}, and combine the results.

- We follow with the \textit{dual algorithm} of Bruns and Ichim \([2]\), based on work by Pottier \([41]\) and also implemented in \textit{Normaliz}. If we express \( \mathcal{C} \) in the form \( \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \geq 0, A \mathbf{x} = 0 \} \), this algorithm inductively builds Hilbert bases for cones defined by the first \( i \) rows of the matrix \( A \). This generalises the double description method for enumerating extremal rays \([21, 39]\).

  In Section 4 we describe a dual algorithm for enumerating fundamental normal surfaces, by filtering out intermediate points that break the quadrilateral constraints. This in turn generalises the filtering technique of Letscher for enumerating vertex normal surfaces \([7]\).

- We finish with the \textit{Contejean-Devie} algorithm \([17]\), which “grows” potential basis vectors by incrementing their coordinates one at a time. Although this is inefficient for high-dimensional problems, it has an extremely small memory footprint and so we include it in our list.

  In Section 5 we present a variant of the Contejean-Devie algorithm for fundamental normal surfaces, using a filtering technique similar to that used in the dual algorithm above.

Through experimental testing, we find in Section 6 that both the primal and dual algorithms for fundamental normal surfaces run significantly faster than generic state-of-the-art Hilbert basis algorithms. Of these, the primal algorithm is found to have the most consistent performance.

Regarding bounds on running times: For the simpler problem of enumerating vertex normal surfaces, the best theoretical bounds are still severely exaggerated when compared with real behaviour \([13]\), and we expect the same here. We do not derive explicit bounds in this short paper, but we do note that the best known bound on the \textit{output size} is \( \exp(O(n^2)) \) \([23]\), and so any bounds on running time should be at least this large.

All implementations are written using the freely-available software package \textit{Regina} \([4, 11]\), and can be obtained from \textit{Regina}’s online source repository or the forthcoming version 4.91. The primal algorithm also incorporates portions of \textit{Normaliz}, as described above.
2 Preliminaries

Here we give a very brief overview of triangulations and normal surfaces. In keeping with the focus of this paper, we concentrate on the algebraic formulation at the expense of geometric insight; for further information the reader is referred to the excellent summary in [23].

Throughout this paper, the input for a topological problem is a 3-manifold triangulation $T$, built from $n$ tetrahedra by affinely identifying (or “gluing together”) some or all of the $4n$ tetrahedron faces in pairs so that the resulting topological space is a 3-manifold (possibly with boundary).

Such triangulations are more general than simplicial complexes: as a result of the face gluings, different edges of the same tetrahedron might be identified together, and likewise with vertices. Two faces of the same tetrahedron may even be glued together. This general definition allows us to express rich topological structures using few tetrahedra, which is important for computation.

A normal surface in the 3-manifold triangulation $T$ is a properly embedded surface in $T$ that meets each tetrahedron in a (possibly empty) disjoint union of normal discs, each of which is a curvilinear triangle or quadrilateral. Each triangle separates one vertex of the tetrahedron from the others, and each quadrilateral separates two vertices from the others, as illustrated in Figure 1(a). Normal surfaces may be disconnected or empty.

There are seven types of disc in each tetrahedron, defined by which edges of the tetrahedron a normal disc meets. These include four triangle types and three quadrilateral types, all illustrated in Figure 1(b). The vector representation of a normal surface $S$ is a vector $\mathbf{v}(S) \in \mathbb{Z}^{7n}$, where the $7n$ elements of $\mathbf{v}(S)$ count the number of discs of each type in each tetrahedron. Given $\mathbf{v}(S)$, it is simple to reconstruct the surface $S$ (up to an isotopy that preserves the simplices of $T$).

For each normal surface $S$, $\mathbf{v}(S)$ satisfies the matching equations. These are $3f$ linear homogeneous equations derived from $T$, where $f$ is the number of non-boundary faces of $T$. We express these equations as $A\mathbf{v}(S) = 0$, where $A$ is a $3f \times 7n$ matching matrix. In essence, these equations ensure that triangles and quadrilaterals in adjacent tetrahedra can be joined together. The matching matrix $A$ is sparse with small integer entries. The normal surface solution cone $\mathcal{C}$ is the rational cone $\mathcal{C} = \{ \mathbf{x} \in \mathbb{R}^{7n} \mid A\mathbf{x} = 0, \ \mathbf{x} \geq 0 \}$, which is a pointed cone with vertex at the origin.

No normal surface $S$ can have two different types of quadrilateral in the same tetrahedron, since these would necessarily intersect (contradicting the requirement that $S$ be properly embedded). We say that a vector $\mathbf{x} \in \mathbb{R}^{7n}$ satisfies the quadrilateral constraints if, for each tetrahedron $\Delta$ of $T$, at most one of the three coordinates of $\mathbf{x}$ that counts quadrilaterals in $\Delta$ is non-zero. Running through all $n$ tetrahedra, this gives us $n$ distinct constraints of the form “at most one of $x_1, x_2, x_3$ is non-zero”. The quadrilateral constraints are non-linear, and their solution set is non-convex.
A point \( x \in \mathbb{R}^7 \) is called \emph{admissible} if it lies in the cone \( C \) and satisfies the quadrilateral constraints. By a theorem of Haken \cite{22}, an integer vector \( x \in \mathbb{Z}^7 \) represents a normal surface if and only if \( x \) is admissible. The \emph{admissible region} of \( C \) is the set of all admissible points in \( C \).

Let \( P \subset \mathbb{R}^d \) be any pointed rational cone with vertex at the origin, and let \( P \cap \mathbb{Z}^d \) denote all integer points in \( P \). Then the \emph{Hilbert basis} of \( P \cap \mathbb{Z}^d \), denoted \( \text{Hilb}(P \cap \mathbb{Z}^d) \) or just \( \text{Hilb}(P) \) for convenience, is a minimal set of integer points that generates all of \( P \cap \mathbb{Z}^d \) under addition. Equivalently, \( \text{Hilb}(P) \) is the set of all \( x \in P \cap \mathbb{Z}^d \) for which, if \( x = y + z \) with \( y, z \in P \cap \mathbb{Z}^d \), then either \( y = 0 \) or \( z = 0 \). Hilbert bases are finite and unique \cite{25, 48}, and have many applications \cite{10}. They can be defined more generally, but for this paper the setting given here is sufficient.

A \emph{fundamental normal surface} is a normal surface \( S \) for which, if \( \nu(S) = \nu(T) + \nu(U) \) for normal surfaces \( T \) and \( U \), then either \( \nu(T) = 0 \) or \( \nu(U) = 0 \). The fundamental normal surfaces are represented by the admissible points in \( \text{Hilb}(C) \), i.e., those points of \( \text{Hilb}(C) \) that satisfy the quadrilateral constraints.

A \emph{vertex normal surface} is a normal surface \( S \) for which \( \nu(S) \) lies on an extremal ray of \( C \) and the elements of \( \nu(S) \) have no common factor.\footnote{There are several definitions of \emph{vertex normal surface} in the literature; others allow common factors \cite{5, 31}, or use the double cover if \( S \) is one-sided in \( T \) \cite{14, 46}. Our definition here follows that of Jaco and Oertel \cite{30}.} It is clear that every vertex normal surface is also a fundamental normal surface, but in general the converse is not true.

## 3 The primal algorithm

The primal algorithm of Bruns, Ichim and Koch comes in several forms \cite{2, 3}; here we describe the variant most relevant to us. Let \( P \subset \mathbb{R}^d \) be a pointed rational cone with vertex at the origin. The Bruns-Ichim-Koch algorithm takes as input the extremal rays of \( P \), and computes the Hilbert basis \( \text{Hilb}(P) \). The following is an overview of the procedure; full details can be found in \cite{2}.

1. Use a variant of Fourier-Motzkin elimination to compute the support hyperplanes for \( P \) and triangulate \( P \) into simplicial subcones (cones whose extremal rays are linearly independent).

2. For each simplicial subcone \( \mathcal{S} \), build the semi-open parallelotope spanned by the smallest integer vector on each extremal ray of \( \mathcal{S} \), as shown in Figure 2(a). Specifically, if the extremal rays are defined by integer vectors \( v_1, \ldots, v_k \), we build the parallelotope \( \{ \sum \lambda_i v_i \mid 0 \leq \lambda_i < 1 \} \).

![Building a semi-open parallelotope](figure2a.png)

(a) Building a semi-open parallelotope

![All integer points in the parallelotope](figure2b.png)

(b) All integer points in the parallelotope

Figure 2: Building parallelotopes in the primal algorithm
3. Identify all integer points within each semi-open parallelootope, as illustrated in Figure 2(b), and combine these into a single list, which generates all of $\mathcal{P} \cap \mathbb{Z}^d$. Reduce both the intermediate and final lists by removing redundant generators (i.e., vectors that can be expressed as non-trivial sums of others in the lists). The final reduced list is the Hilbert basis $\text{Hilb}(\mathcal{P})$.

In our case, the pointed rational cone is the normal surface solution cone $\mathcal{C} \subseteq \mathbb{R}^{7n}$. Running the Bruns-Ichim-Koch algorithm directly over $\mathcal{C}$ can be slow, since $\mathcal{C}$ is high-dimensional and the triangulation can be extremely large. Moreover, it is wasteful: our aim is to enumerate fundamental normal surfaces, and so we do not need all of $\text{Hilb}(\mathcal{C})$, but just its admissible points.

Our solution, instead of triangulating all of $\mathcal{C}$, is to just triangulate the admissible region of $\mathcal{C}$. We call a face $F \subseteq \mathcal{C}$ admissible if every point $x \in F$ is admissible. In particular, we note that the vertex normal surfaces correspond precisely to the admissible extremal rays of $\mathcal{C}$. A maximal admissible face of $\mathcal{C}$ is an admissible face that is not a strict subface of another admissible face.

One can show that the admissible region of $\mathcal{C}$ is precisely the union of all maximal admissible faces [9, 30]. Note that maximal admissible faces may have different dimensions, and if we truncate the origin (which is common to all faces) then the admissible region may even be disconnected.

Our strategy now is to enumerate all maximal admissible faces of $\mathcal{C}$, and use the Bruns-Ichim-Koch algorithm to triangulate and compute the Hilbert basis for each. This has the advantage that maximal admissible faces are often significantly simpler than the full cone $\mathcal{C}$: they have lower dimension [9], and are often generated by very few admissible extremal rays [6].

The following result underpins our strategy:

**Lemma 1.** A vector $x \in \mathbb{R}^{7n}$ represents a fundamental normal surface if and only if there is some maximal admissible face $F \subseteq \mathcal{C}$ for which $x \in \text{Hilb}(F)$.

**Proof.** Let $x = v(S)$ where $S$ is a fundamental normal surface. Since $x$ is an admissible point of $\mathcal{C}$, it must belong to some maximal admissible face $F \subseteq \mathcal{C}$. If $x \notin \text{Hilb}(F)$ then there are non-zero integer vectors $y, z \in F$ for which $x = y + z$. Since $F$ is an admissible face of $\mathcal{C}$ it follows that $y = v(T)$ and $z = v(U)$ for some normal surfaces $T$ and $U$, contradicting the fundamentality of $S$.

Conversely, suppose that $x \in \text{Hilb}(F)$ for some maximal admissible face $F \subseteq \mathcal{C}$. Then $x$ is an admissible integer vector of $\mathcal{C}$, and so $x = v(S)$ for some normal surface $S$. If $S$ is not fundamental then $v(S) = v(T) + v(U)$ for normal surfaces $T$ and $U$ with $v(T), v(U) \neq 0$. Since they represent normal surfaces, both $v(T), v(U) \in \mathcal{C}$, and so any face of the cone $\mathcal{C}$ that contains $v(S)$ must contain both summands $v(T)$ and $v(U)$ as well. In particular we have $v(T), v(U) \in F$, contradicting the assumption that $x = v(T) + v(U) \in \text{Hilb}(F)$. \hfill \Box

In our algorithms, we represent faces $F \subseteq \mathcal{C}$ using zero sets [21]. For each face $F \subseteq \mathcal{C}$, the zero set of $F$ is a subset of $\{1, \ldots, 7n\}$ indicating which coordinate positions are zero throughout $F$. We denote this zero set by $Z(F)$, and define it formally as $Z(F) = \{k \mid x_k = 0 \text{ for all } x \in F\}$.

Because $\mathcal{C}$ is of the form $\{x \in \mathbb{R}^{7n} \mid Ax = 0, \ x \geq 0\}$, zero sets effectively encode the support hyperplanes for each face: it is then clear that for any two faces $F, G \subseteq \mathcal{C}$ we have $Z(F) = Z(G)$ if and only if $F = G$, and $Z(F) \subseteq Z(G)$ if and only if $G \subseteq F$. Zero sets can be stored and manipulated efficiently in code using bitmasks of length $7n$. 


The following algorithm makes use of admissible zero sets, which correspond to admissible faces of $\mathcal{C}$. We call a set $z \subseteq \{1, \ldots, 7n\}$ admissible if the corresponding zero/non-zero coordinate pattern satisfies the quadrilateral constraints in $\mathbb{R}^{7n}$; that is, for each tetrahedron $\Delta$ of our triangulation, at most one of the three coordinate positions that counts quadrilaterals in $\Delta$ does not appear in $z$.

**Algorithm 2** (Maximal admissible face decomposition). Given the set $\mathcal{V}$ of all admissible extremal rays of $\mathcal{C}$, the following algorithm enumerates all maximal admissible faces of $\mathcal{C}$.

1. Build zero sets for all rays $R \in \mathcal{V}$ and store these in the set $\mathcal{S}_1$. Initialise an empty output set $\mathcal{M}$, which will eventually contain the zero sets for all maximal admissible faces of $\mathcal{C}$.

2. Inductively build sets $\mathcal{S}_2, \mathcal{S}_3, \ldots$ as follows. To build the set $\mathcal{S}_k$:

   (a) For each zero set $z \in \mathcal{S}_{k-1}$, find all $\Gamma \in \mathcal{S}_1$ for which $z \not\subseteq \Gamma$. For each such $\Gamma$, insert $z \cap \Gamma$ into $\mathcal{S}_k$. If no such $\Gamma$ exists, insert $z$ into the output set $\mathcal{M}$.

   (b) Reduce $\mathcal{S}_k$ to its maximal elements by set inclusion. That is, remove all $z \in \mathcal{S}_k$ for which there exists some strict superset $z' \in \mathcal{S}_k$.

   (c) If $\mathcal{S}_k$ is empty, terminate the algorithm and return the output set $\mathcal{M}$.

The key observation is that each set $\mathcal{S}_i$, once constructed, contains precisely the zero sets for all admissible faces of dimension $i$. A full proof of correctness and termination will be given shortly.

First, however, we make a further observation regarding zero sets. Recall from standard polytope theory that, for any two faces $F$ and $G$ of a polyhedron $\mathcal{P}$, the join $F \vee G$ is the unique smallest-dimensional face of $\mathcal{P}$ that contains both $F$ and $G$, and that for any face $H \subseteq \mathcal{P}$ with $F, G \subseteq H$ we have $F \vee G \subseteq H$.

**Lemma 3.** For any two faces $F, G$ of the normal surface solution cone $\mathcal{C}$, we have $Z(F \vee G) = Z(F) \cap Z(G)$.

**Proof.** Since $F \vee G \supseteq F, G$, any coordinate position that takes a non-zero value in either $F$ or $G$ also takes a non-zero value somewhere in $F \vee G$. Therefore $Z(F \vee G) \subseteq Z(F) \cap Z(G)$.

Let $H$ be the face of $\mathcal{C}$ obtained by intersecting $\mathcal{C}$ with the supporting hyperplanes $\{x_i = 0\}$ for all $i \in Z(F) \cap Z(G)$. It is clear that $Z(F) \cap Z(G) \subseteq Z(H)$. Moreover, $F, G \subseteq H$ since every point in $F$ or $G$ lies on all of the supporting hyperplanes listed above. Therefore $F \vee G \subseteq H$, and so $Z(F \vee G) \subseteq Z(H) \subseteq Z(F \vee G)$.

We now have $Z(F \vee G) \subseteq Z(F) \cap Z(G) \subseteq Z(F \vee G)$, and so $Z(F \vee G) = Z(F) \cap Z(G)$. □

We can now give a full proof of correctness and termination for Algorithm 2.

**Proof of correctness and termination.** We first prove by induction that, once constructed, each set $\mathcal{S}_i$ contains precisely the zero sets for all admissible faces of $\mathcal{C}$ of dimension $i$.

This is clearly true for $i = 1$, since the 1-dimensional faces of $\mathcal{C}$ are precisely the extremal rays, and we use all admissible extremal rays of $\mathcal{C}$ to fill $\mathcal{S}_1$ in step 1 of the algorithm.

Consider now some $i > 1$. We assume our inductive hypothesis for both $\mathcal{S}_{i-1}$ and $\mathcal{S}_1$, and prove it for $\mathcal{S}_i$ in three stages:
(i) Every \( y \in S_i \) is the zero set \( Z(F) \) for some admissible face \( F \subseteq \mathcal{C} \) of dimension \( \geq i \).

Suppose that \( y \) is constructed in step 2(a) of the algorithm as \( y = z \cap v \), where \( z \in S_{i-1} \) and \( v \in S_1 \). By the inductive hypothesis we have \( z = Z(G) \) and \( v = z(R) \) for some admissible faces \( G, R \subseteq \mathcal{C} \) with \( \dim(G) = i - 1 \) and \( \dim(R) = 1 \).

By Lemma \( \mathbb{K} \) we have \( y = Z(G \lor R) \). Because step 2(a) of the algorithm only constructs admissible zero sets \( y = z \cap v \), it follows that \( G \lor R \) must be an admissible face. Because step 2(a) only considers pairs for which \( z \not\subseteq v \), it follows that \( R \not\subseteq G \) and therefore \( G \lor R \) has strictly higher dimension than \( G \); that is, \( \dim(G \lor R) \geq i \).

(ii) For every admissible \( i \)-dimensional face \( F \subseteq \mathcal{C} \), we have \( Z(F) \in S_i \).

Let \( G \subseteq F \) be any \((i-1)\)-dimensional subface of \( F \), and let \( R \subseteq F \) be any extremal ray of \( F \) not contained in \( G \). Since \( F \supseteq G, R \) we have \( F \supseteq G \lor R \), and since \( i - 1 = \dim(G) < \dim(G \lor R) \leq \dim(F) = i \) it follows that \( \dim(F) = \dim(G \lor R) \), and hence \( F = G \lor R \).

By Lemma \( \mathbb{K} \) we then have \( Z(F) = Z(G) \lor Z(R) \).

Since \( F \) is admissible, so are \( G \) and \( R \), and it follows from the inductive hypothesis that \( Z(G) \in S_{i-1} \) and \( Z(R) \in S_1 \). Because \( R \not\subseteq G \) we have \( Z(G) \not\subseteq Z(R) \), and we already know from \( F \) that \( Z(F) = Z(G) \lor Z(R) \) is admissible. Therefore the set \( Z(F) = Z(G) \lor Z(R) \) is inserted into \( S_i \) in step 2(a) of the algorithm.

Furthermore, \( Z(F) \) is not removed from \( S_i \) in step 2(b) of the algorithm. This is because, by stage (i) above, any strict superset \( z' \supset Z(F) \) that appears in \( S_i \) must correspond to a strict subface \( F' \subset F \) of dimension \( \geq i \). This is impossible, since \( F \) itself has dimension \( i \).

(iii) For every admissible \( j \)-dimensional face \( F \subseteq \mathcal{C} \) with \( j > i \), we have \( Z(F) \notin S_i \).

For any such \( F \), let \( G \subseteq F \) be any \( i \)-dimensional subface of \( F \). Because \( F \) is admissible, \( G \) is also, and by stage (ii) above it follows that \( Z(G) \in S_i \). Since \( G \) is a strict subface of \( F \), \( Z(G) \) is a strict superset of \( Z(F) \), and so even if \( Z(F) \) is constructed in step 2(a) of the algorithm, it will be subsequently removed in step 2(b).

Together these three stages establish our inductive claim.

From here it is simple to see that Algorithm \( \mathbb{2} \) terminates: since the cone \( \mathcal{C} \) has finitely many faces, there are only finitely many non-empty sets \( S_k \). Note also that Algorithm \( \mathbb{2} \) does not terminate until all non-empty sets \( S_k \) have been constructed, since if \( S_k \) is empty then there are no admissible \( k \)-faces, which means there can be no admissible \( i \)-faces for any \( i \geq k \).

To finish, we show that the output of Algorithm \( \mathbb{2} \) is correct. Let \( F \subseteq \mathcal{C} \) be any maximal admissible face of dimension \( i \); by our induction above we have \( Z(F) \subseteq S_i \). Suppose there is some \( v \in S_1 \) for which \( Z(F) \not\subseteq v \) and for which \( Z(F) \cap v \) is admissible. Following the argument in stage (i) above, \( Z(F) \cap v \) must be the zero set for some admissible face of dimension \( \geq i + 1 \). Moreover, this must be a strict superface of \( F \), which contradicts the maximality of \( F \). Therefore there can be no such \( v \in S_1 \), and so we include \( Z(F) \) in the output set \( \mathcal{M} \) as required.

Conversely, consider any output \( z \in \mathcal{M} \). We must have \( z \in S_i \) for some \( i \), and so \( z = Z(G) \) for some admissible \( i \)-dimensional face \( G \subseteq \mathcal{C} \). If \( G \) is not maximal, there must be some admissible \((i + 1)\)-dimensional superface \( F \supseteq G \), and some admissible extremal ray \( R \subseteq F \) not contained in \( G \). Following the argument in stage (ii) above, for the pair \( Z(G) \in S_i \) and \( Z(R) \in S_1 \), we have \( Z(G) \not\subseteq Z(R) \) and \( Z(G) \lor Z(R) \) is admissible, which means we would not insert \( z \) into the output set \( \mathcal{M} \) in step 2(a) of the algorithm. Therefore \( z = Z(G) \) for some maximal admissible face \( G \subseteq \mathcal{C} \). \( \square \)
Note that each zero set in step 1 can be built by examining any non-zero representative of
the corresponding ray \( R \). In step 2(a) a set might be constructed from many different \((z, v)\)
pairs, and so it is important when inserting \( z \cap v \) into \( S_k \) to avoid duplicates. Each \( S_k \) is only
required for the following inductive step (building \( S_{k+1} \)), and can then be discarded.

We can now present a full algorithm for enumerating fundamental normal surfaces.

**Algorithm 4** (Primal algorithm for fundamental normal surfaces). To enumerate all funda-
mental normal surfaces in a given triangulation:

1. Enumerate all vertex normal surfaces—that is, all admissible extremal rays of \( \mathcal{C} \)—using
the algorithms in [5] and [7], which are optimised specifically for normal surface theory.
Let \( \mathcal{V} \) be the resulting set of admissible extremal rays.

2. Using \( \mathcal{V} \) as input, enumerate all maximal admissible faces of \( \mathcal{C} \) using Algorithm 2.

3. For each maximal admissible face \( F \subseteq \mathcal{C} \), identify which rays in \( \mathcal{V} \) are subfaces of
\( F \), and pass these rays as input to the Bruns-Ichim-Koch algorithm which will then enumerate
\( \text{Hilb}(F) \).

4. To finish, take the union \( \bigcup \text{Hilb}(F) \) over all maximal admissible faces \( F \). This is pre-
cisely the set of all vector representations of fundamental normal surfaces in the given
triangulation.

In step 3, we can use zero sets to quickly identify which rays are subfaces of \( F \): a ray \( R \in \mathcal{V} \)
is a subface of \( F \) if and only if \( Z(R) \supseteq Z(F) \).

From Lemma 1, it is clear that this algorithm is correct. The Bruns-Ichim-Koch algorithm
does indeed enumerate \( \text{Hilb}(F) \) in step 3, because each extremal ray of \( F \) is admissible, and so
the rays in \( \mathcal{V} \) that are subfaces of \( F \) are precisely the extremal rays of \( F \).

### 4 The dual algorithm

We turn now to the dual algorithm of Bruns and Ichim [2], which is based on earlier work of
Pottier [41]. Again this algorithm arrives in several forms, and we describe the variant most
relevant to us. Consider the cone \( \mathcal{P} = \{x \in \mathbb{R}^d \mid Ax = 0, x \geq 0\} \), where \( A \) is some rational
\( m \times d \) matrix. The dual algorithm takes as input the matrix \( A \), and computes the Hilbert basis
\( \text{Hilb}(\mathcal{P}) \).

It constructs a series of cones \( \mathcal{P}^{(0)}, \ldots, \mathcal{P}^{(m)} \), beginning with the non-negative orthant
\( \mathcal{P}^{(0)} = \mathbb{R}^d_{\geq 0} \), and finishing with the target cone \( \mathcal{P}^{(m)} = \mathcal{P} \). In general, \( \mathcal{P}^{(k)} \) is defined as for
\( \mathcal{P} \) above but using only the first \( k \) rows of the matrix \( A \). At each stage, the dual algorithm
uses the previous basis \( \text{Hilb}(\mathcal{P}^{(k-1)}) \) to inductively compute the next basis \( \text{Hilb}(\mathcal{P}^{(k)}) \). In this
sense, the dual algorithm generalises the double description method for enumerating extremal rays
[21, 39].

The key inductive step of the dual algorithm is the following [2].

**Algorithm 5** (Bruns-Ichim-Pottier inductive step). Let \( \mathcal{P} \subset \mathbb{R}^d \) be a pointed rational cone
with vertex at the origin, let \( h \in \mathbb{R}^d \) be a rational vector, and define the cones \( \mathcal{P}_+ = \{x \in
\mathcal{P} \mid h \cdot x \geq 0\}, \mathcal{P}_- = \{x \in \mathcal{P} \mid h \cdot x \leq 0\}, \) and \( \mathcal{P}_0 = \{x \in \mathcal{P} \mid h \cdot x = 0\} \). Then, given the input
basis \( B = \text{Hilb}(\mathcal{P}) \), the following algorithm computes the Hilbert bases \( \text{Hilb}(\mathcal{P}_+), \text{Hilb}(\mathcal{P}_-) \)
and \( \text{Hilb}(\mathcal{P}_0) \).
1. Initialise candidate bases $B_+ = \{ x \in B \mid h \cdot x \geq 0 \}$ and $B_- = \{ x \in B \mid h \cdot x \leq 0 \}$.

2. Expand these candidate bases: for all pairs $x \in B_+$ and $y \in B_-$ with $h \cdot x > 0 > h \cdot y$, insert the sum $x + y$ into $B_+$ and/or $B_-$ according to whether $h \cdot (x + y) \geq 0$ and/or $h \cdot (x + y) \leq 0$.

3. Reduce the candidate bases: remove all $b \in B_+$ for which there exists some different $b' \in B_+$ with $b - b' \in P_+$. Reduce $B_-$ in a similar fashion.

4. If $B_+$ and $B_-$ are both unchanged after the most recent round of expansion and reduction (steps 2 and 3), terminate with $\text{Hilb}(P_+) = B_+$, $\text{Hilb}(P_-) = B_-$, and $\text{Hilb}(P_0) = B_+ \cap B_-$. Otherwise return to step 2 for a new round of expansion and reduction.

In essence, step 2 expands the set of integer points that are generated under addition by each individual set $B_+$ and $B_-$, and step 3 removes redundant vectors from each generating set. See [2] for full proofs of correctness and termination, both of which are non-trivial results.

There are many ways to optimise this algorithm. In practice, one can split $B_+$ and $B_-$ into three sets according to whether $h \cdot x > 0$, $h \cdot x < 0$ or $h \cdot x = 0$, to avoid duplication along the common hyperplane $h \cdot x = 0$. The expansion and reduction steps can be partially merged, so that we first test each new sum $x + y$ for reduction before inserting it into $B_+$ and/or $B_-$. Several other optimisations are possible: see [2] Remark 16, and the “darwinistic reduction” in [2] Section 3.

As before, running the dual algorithm of Bruns, Ichim and Pottier directly over the normal surface solution cone $C \subset \mathbb{R}^{7n}$ can be extremely slow. Like the double description method for vertex enumeration, the key problem is combinatorial explosion: even if the final Hilbert basis $\text{Hilb}(P)$ is small, the intermediate bases $\text{Hilb}(P^{(k)})$ can be extremely large. The high dimension of the normal surface solution cone $C$ exacerbates this problem.

Our solution, as with the primal algorithm, is to restrict our attention to just the admissible region of $C$. This time we do not decompose the admissible region into maximal admissible faces; instead we embed the quadrilateral constraints directly into the algorithm itself. In this way, a single run through the dual algorithm can compute all admissible Hilbert basis elements for $C$.

The idea is simple, and mirrors Letscher’s filter for the double description method [7]: when expanding candidate bases, ignore sums $x + y$ that do not satisfy the quadrilateral constraints.

Algorithm 6 (Dual algorithm for fundamental normal surfaces). To enumerate all fundamental normal surfaces in a triangulation with the $3f \times 7n$ matching matrix $A$:

1. Reorder the rows of $A$ using a good heuristic (as discussed further below).

2. Initialise the candidate basis $B^{(0)}$ with all unit vectors in $\mathbb{R}^{7n}$.

3. Inductively construct candidate bases $B^{(1)}, \ldots, B^{(3f)}$ as follows. For each $i = 1, \ldots, 3f$, run a modified Algorithm 5 with input basis $B^{(i-1)}$, using the vector $h \in \mathbb{R}^{7n}$ defined by the $i$th row of $A$. The specific modifications to Algorithm 5 are:

   • In step 2, only consider pairs $x, y$ for which $x + y$ satisfies the quadrilateral constraints.

   • In step 3, instead of testing for $b - b' \in P_+$, test for $b - b' \geq 0$ and $h \cdot (b - b') \geq 0$. Instead of testing for $b - b' \in P_-$, test for $b - b' \geq 0$ and $h \cdot (b - b') \leq 0$. 


When this modified Algorithm 5 terminates with output \( B_+ \) and \( B_- \), set \( B^{(i)} = B_+ \cap B_- \).

4. The final set \( B^{(3f)} \) is then the set of all vector representations of fundamental normal surfaces.

Although we use the Bruns-Ichim-Pottier inductive process in step 3 of our algorithm, we do not identify any cone \( \mathcal{P} \subset \mathbb{R}^{7n} \) for which the input set \( B^{(i-1)} \) is the Hilbert basis. This is why we modify step 3 of Algorithm 5, we must remove any reference to the unknown cone \( \mathcal{P} \).

The order in which we process the rows of \( A \) can have a significant effect on performance, by affecting the severity of the combinatorial explosion [2, 21]. In step 1 of Algorithm 6 we reorder the rows using position vectors [7], which essentially favours rows that begin with long strings of zeroes. Position vectors help optimise our quadrilateral constraint filter, and have been found to perform well as a sorting heuristic for the related double description method; see [7] for details.

We can prove correctness and termination for Algorithm 6 as follows.

**Proof of correctness and termination.** Let \( A^{(k)} \) denote the \( k \times 7n \) submatrix of \( A \) obtained by taking the first \( k \) rows (after the reordering in step 1 of the algorithm), and let \( \mathcal{P}^{(k)} = \{ x \in \mathbb{R}^{7n} | A^{(k)}x = 0, x \geq 0 \} \). We prove by induction that each candidate basis \( B^{(k)} \) constructed in Algorithm 6 consists of all elements of \( \text{Hilb}(\mathcal{P}^{(k)}) \) that satisfy the quadrilateral constraints.

The initial case is simple: \( \mathcal{P}^{(0)} \) is just the non-negative orthant \( \mathbb{R}_{\geq 0}^7 \), whose Hilbert basis consists of the unit vectors in \( \mathbb{R}^7 \), all of which satisfy the quadrilateral constraints. This matches the initialisation of \( B^{(0)} \) in step 2 of the algorithm.

Assume now that \( B^{(i-1)} \) contains all elements of \( \text{Hilb}(\mathcal{P}^{(i-1)}) \) that satisfy the quadrilateral constraints, and consider the construction of \( B^{(i)} \) in step 3 of Algorithm 5. This involves running a modified Algorithm 5 (the Bruns-Ichim-Pottier inductive step), with input basis \( B^{(i-1)} \) and with the vector \( h \in \mathbb{R}^7 \) defined by the \( i \)th row of the (sorted) matrix \( A \).

Let \( \mathcal{P}_+ = \{ x \in \mathcal{P}^{(i-1)} | h \cdot x \geq 0 \} \), \( \mathcal{P}_- = \{ x \in \mathcal{P}^{(i-1)} | h \cdot x \leq 0 \} \), and \( \mathcal{P}_0 = \mathcal{P}_+ \cap \mathcal{P}_- = \{ x \in \mathcal{P}^{(i-1)} | h \cdot x = 0 \} \). We claim that the modified Algorithm 5 terminates with output sets \( B_+ \) and \( B_- \) containing those elements of \( \text{Hilb}(\mathcal{P}_+) \) and \( \text{Hilb}(\mathcal{P}_-) \) respectively that satisfy the quadrilateral constraints. If this is true, then setting \( B^{(i)} = B_+ \cap B_- \) will fill \( B^{(i)} \) with those elements of \( \text{Hilb}(\mathcal{P}_0) = \text{Hilb}(\mathcal{P}^{(i)}) \) that satisfy the quadrilateral constraints, as required. This will complete our induction, and the final output \( B^{(3f)} \) will then consist of all admissible elements of \( \text{Hilb}(\mathcal{P}^{(3f)}) = \text{Hilb}(\mathcal{C}) \); that is, all vector representations of fundamental normal surfaces.

It remains to prove our claim above, i.e., that the modified Algorithm 5 terminates with output sets \( B_+ \) and \( B_- \) containing those elements of \( \text{Hilb}(\mathcal{P}_+) \) and \( \text{Hilb}(\mathcal{P}_-) \) respectively that satisfy the quadrilateral constraints.

Consider running the modified Algorithm 5 and the original Algorithm 4 side-by-side, where in the original algorithm we use the full basis \( \text{Hilb}(\mathcal{P}^{(i-1)}) \) as input. Let \( B_+^m \) and \( B_-^m \) denote the working sets \( B_+ \) and \( B_- \) in our modified algorithm, and let \( B_+^o \) and \( B_-^o \) denote the working sets \( B_+ \) and \( B_- \) in the original algorithm. From Bruns and Ichim [2], we know that the original algorithm terminates with \( B_+^o = \text{Hilb}(\mathcal{P}_+) \) and \( B_-^o = \text{Hilb}(\mathcal{P}_-) \).

We claim that, for as long as both algorithms are running side-by-side, the working sets \( B_+^m \) and \( B_-^m \) contain precisely those elements of \( B_+^o \) and \( B_-^o \) respectively that satisfy the quadrilateral constraints. We show this again using induction:

- It is true when the algorithms begin, since we assume from our “outer induction” on \( B^{(i)} \) that the input basis \( B^{(i-1)} \) for the modified algorithm contains precisely those elements...
that satisfy the quadrilateral constraints from the input basis Hilb(\(P^{(i-1)}\)) for the original algorithm.

- We now show that, if at some point \(B^+_m\) and \(B^-_m\) contain the elements of \(B^+_o\) and \(B^-_o\) that satisfy the quadrilateral constraints, then this remains true after a single round of expansion (step 2 of Algorithm 5).

It is clear from our modification in step 3 of Algorithm 5 that we do not insert any new sums \(x + y\) into \(B^+_m\) or \(B^-_m\) that break the quadrilateral constraints. Moreover, in the original algorithm, any sum \(x + y\) that does satisfy the quadrilateral constraints must have summands \(x \in B^+_o\) and \(y \in B^-_o\) that satisfy them also (since \(x, y \geq 0\)); therefore \(x \in B^+_m\) and \(y \in B^-_m\), and the sum \(x + y\) is also considered in the modified algorithm.

- Likewise, we can show that if at some point \(B^+_m\) and \(B^-_m\) contain the elements of \(B^+_o\) and \(B^-_o\) that satisfy the quadrilateral constraints, then this remains true after a single round of reduction (step 3 of Algorithm 5).

First, we note that our modification to the reduction step is sound: for any \(b, b' \in B^+_m\), we have \(b - b' \in P^+\) if and only if \(b - b' \geq 0\) and \(h \cdot (b - b') \geq 0\), since we already have \(A^{(i-1)}b = A^{(i-1)}b' = 0\) from our outer induction. Next, we observe that a vector \(b \in B^+_m\) is removed in step 3 of the modified Algorithm 5 if and only if it is removed in the original algorithm: this is because \(b\) satisfies the quadrilateral constraints, and so any \(b'\) for which \(b - b' \geq 0\) must satisfy them also, and therefore \(b' \in B^+_m\) if and only if \(b' \in B^+_o\). Similar arguments apply to \(P^-\) and \(B^-_m\).

By induction, \(B^+_m\) and \(B^-_m\) contain precisely those elements of \(B^+_o\) and \(B^-_o\) that satisfy the quadrilateral constraints for as long as both algorithms are running.

It follows that the modified algorithm terminates no later than the original algorithm, since if \(B^+_o\) and \(B^-_o\) are unchanged under some round of expansion and reduction, then their elements \(B^+_m\) and \(B^-_m\) that satisfy the quadrilateral constraints are unchanged also. The modified algorithm might terminate earlier, but this will not change the final output: if \(B^+_m\) and \(B^-_m\) are unchanged under some earlier round of expansion and reduction, then running any additional rounds will leave them unchanged.

Either way, we see that the final output sets \(B^+_m\) and \(B^-_m\) from the modified Algorithm 5 contain precisely those elements of the final output sets \(B^+_o = \text{Hilb}(P^+)\) and \(B^-_o = \text{Hilb}(P^-)\) that satisfy the quadrilateral constraints, thus establishing our original claim.

\[\square\]

5 The Contejean-Devie algorithm

Our final algorithm is based on the Contejean-Devie method for enumerating Hilbert bases \[17\]. In essence, we “grow” candidate basis elements out from the origin by incrementally adding unit vectors. Like the dual algorithm, the Contejean-Devie method works with a rational cone \(P = \{x \in \mathbb{R}^d \mid Ax = 0, x \geq 0\}\), taking as input the matrix \(A\) and computing the Hilbert basis \(\text{Hilb}(P)\).

**Algorithm 7** (General Contejean-Devie algorithm). *To compute the Hilbert basis \(\text{Hilb}(P)\) as described above given the rational matrix \(A\) as input:*

1. Initialise an empty output set \(B\), and a “working set” \(S\) that contains all unit vectors in \(\mathbb{R}^d\).
2. While \( S \neq \emptyset \), update \( B \) and \( S \) as follows:

- For all \( x \in S \) with \( Ax = 0 \), insert \( x \) into \( B \).
- Create a new working set \( S' \). For all \( x \in S \) with \( Ax \neq 0 \), if there is no \( b \in B \) for which \( x - b \geq 0 \), then insert \( x + e_i \) into \( S' \) for all unit vectors \( e_i \in \mathbb{R}^d \) with \( Ax \cdot A e_i < 0 \).
- Replace \( S \) with the new working set \( S' \).

3. The final set \( B \) is then the Hilbert basis \( \text{Hilb}(\mathcal{P}) \).

The algorithm works breadth-first: the \( k \)th iteration of step 2 builds vectors with \( \sum x_i = k+1 \) by incrementing earlier vectors for which \( \sum x_i = k \). The condition \( Ax \cdot A e_i < 0 \) roughly means that incrementing the \( i \)th coordinate of \( x \) will move us towards the null space of the matrix \( A \).

This breadth-first structure creates some problems. First, there is significant redundancy in how vectors are generated: each new vector \( x' \in S' \) is typically constructed from many different source vectors \( x \in S \). Moreover, the intermediate working sets \( S \) can grow to be extremely large.

Contejean and Devie address these issues by modifying their breadth-first algorithm into a depth-first, stack-based algorithm [17, Section 6]. This stack-based algorithm eliminates redundancy, and at most \( d \) working vectors need to be stored at any one time.

Although the Contejean-Devie method is no longer state-of-the-art, we include it here because it uses different techniques, and because the stack optimisation avoids the severe space requirements of the other algorithms: its memory use is essentially dominated by the output size only.

For our setting of normal surfaces, we again embed the quadrilateral constraints directly into the algorithm. The modification is simple, and the resulting algorithm, instead of “growing” vectors through all of \( \mathbb{R}^7 \), only grows vectors through the much smaller region of \( R \) in which the quadrilateral constraints are met.

We present our modified breadth-first algorithm below. The stack-based variant is too complex to describe in this short paper (see [17] for details), but the essential modification is the same.

**Algorithm 8** (Contejean-Devie algorithm for fundamental normal surfaces). To enumerate all fundamental normal surfaces in a triangulation with matching matrix \( A \), run Algorithm 7 as above with \( A \) as input, but modified so that in step 2 we only consider sums \( x + e_i \) that satisfy the quadrilateral constraints. The output set \( B \) is then the set of all vector representations of fundamental normal surfaces.

Proof of correctness and termination. We use the fact that the general Contejean-Devie algorithm is known to terminate and correctly compute \( \text{Hilb}(\mathcal{P}) \), as described in Algorithm 7. Recall that Algorithm 8 is a simple modification of Algorithm 7: we only consider sums \( x + e_i \) that satisfy the quadrilateral constraints.

Once again, consider running the modified Algorithm 8 and the original Algorithm 7 side-by-side. We denote the working sets \( B \) and \( S \) in the modified algorithm by \( B^m \) and \( S^m \), and in the original algorithm by \( B^o \) and \( S^o \). We claim that, for as long as both algorithms are running side-by-side, \( B^m \) and \( S^m \) contain precisely those elements of \( B^o \) and \( S^o \) respectively that satisfy the quadrilateral constraints.

We prove this by a simple induction. The claim is clearly true at the beginning of the algorithms, where \( B^m = B^o = \emptyset \) and both \( S^m \) and \( S^o \) contain all unit vectors in \( \mathbb{R}^7 \). It also
remains true each time we update the sets $B$ and $S$ in step 2 of the algorithm, since we only ever consider non-negative vectors: therefore if $x + e_i$ satisfies the quadrilateral constraints then $x$ must also, and if $x - b \geq 0$ for some $x$ that satisfies the quadrilateral constraints then $b$ must satisfy them also.

The modified algorithm terminates no later than the original, since if $S^o = \emptyset$ then we must have $S^m = \emptyset$, but again the modified algorithm might terminate earlier. However, this does not affect the output, since if $S^m = \emptyset$ at some stage of the modified algorithm then further iterations of step 2 will maintain $S^m = \emptyset$, and will produce no additional output vectors in $B^m$.

Therefore the final output $B^m$ of the modified algorithm contains precisely those elements of the final output $B^o = \text{Hilb}(\mathcal{P})$ that satisfy the quadrilateral constraints. Since $\mathcal{P} = \mathcal{C}$ in our case, this gives the vector representations of all fundamental normal surfaces. □

6 Performance

We now compare the performance of the new primal, dual and Contejean-Devie algorithms for enumerating fundamental normal surfaces (Algorithms 4, 6 and 8) for “typical” input triangulations.

All algorithms are implemented directly in Regina [4,11], except for step 3 of the primal algorithm which uses Normaliz [2,3] to run the Bruns-Ichim-Koch algorithm over each maximal admissible face. The dual algorithm incorporates the relevant optimisations described by Bruns and Ichim [2,3], and the Contejean-Devie algorithm uses the optimised stack-based implementation [17]. All implementations use arbitrary-precision integer arithmetic.

As a benchmark, we also compare these new algorithms against a state-of-the-art implementation of generic Hilbert basis enumeration. For this we run Normaliz directly over the normal surface solution cone to compute the basis $\text{Hilb}(\mathcal{C})$, and then identify which of its members are admissible. This benchmark comparison shows whether our new algorithms are successful in using the context of normal surface theory to their advantage.

![Figure 3: Comparison of running times for different algorithms](image)

Figure 3: Comparison of running times for different algorithms

- Not all optimisations are found to be effective in the normal surface setting, due to the high dimensionality and the quadrilateral constraints. See the full version of this paper for details.
triangulations that represent 13,400 distinct closed 3-manifolds. The data are grouped by the number of tetrahedra $n$, and within each group we plot the median and worst-case times on a log scale. All measurements use single-threaded code running on a 2.93GHz Intel Xeon X5570 CPU.

We only plot the Contejean-Devie and benchmark algorithms for $n \leq 4$ and $n \leq 5$ respectively, since beyond these limits the algorithms did not finish within a week. Two of the 50,817 inputs for the dual algorithm also failed to finish in a week (both with $n = 11$), and these two pathological cases are omitted from the dual algorithm plot.

The results are extremely pleasing: the new primal and dual algorithms run significantly faster than the benchmark algorithm, and can process much larger inputs as a result. The median times for the primal and dual algorithms are comparable, but the worst-case plot highlights an interesting feature: for the dual algorithm, the distribution of running times has a long tail, and the worst cases have running times many orders of magnitude slower than the median. In contrast, the primal algorithm was consistently fast, with all 50,817 cases finishing in under 1 second.

Both the Contejean-Devie and primal algorithms show significantly smaller memory footprints than the dual and benchmark algorithms. See the full version of this paper for details.

The success of the primal algorithm may be because the cone $\mathcal{C}$ typically has only few maximal admissible faces and few admissible extremal rays [6], and these are represented by integer vectors with small coordinates. As a result, the subproblems passed to the Bruns-Ichim-Koch algorithm become relatively simple. Again, we explore these issues further in the full version of this paper.

7 Application to crosscap numbers

To illustrate the practicality of the new primal algorithm, we use it to compute previously-unknown crosscap numbers of knots. The genus of a knot $K$ is an invariant describing the smallest-genus orientable surface that the knot bounds; on the other hand, the crosscap number $C(K)$ is an invariant describing the smallest-genus non-orientable surface that the knot bounds.

Crosscap numbers are difficult to compute: of the 2977 non-trivial prime knots with $\leq 12$ crossings, 2661 crosscap numbers are still unknown; in contrast, the genus is known for all of these knots [15]. Although there are specialised methods for certain classes of knots [26, 27, 45], there is no general algorithm yet for computing crosscap numbers. However, we can come close: by enumerating fundamental normal surfaces, it is possible to either compute the crosscap number of a knot precisely or else reduce it to one of two possible values [12].

Due to space limitations we only give a very brief summary of the computations here. For more information on crosscap numbers and how they can be computed, see [12]. The website http://www.maths.uq.edu.au/~bab/code/ includes data files giving full details of the computations below, including all of the relevant triangulations and their vertex and fundamental normal surfaces.

The crosscap number algorithm [12, Algorithm 4.4] works with triangulated knot complements: these are triangulated 3-manifolds with boundary, obtained by removing a small regular neighbourhood of a knot from the 3-sphere. The algorithm requires the triangulation $\mathcal{T}$ to have specific properties (an “efficient suitable triangulation”), which can be tested by enumerating
vertex normal surfaces. It then enumerates all fundamental normal surfaces in T, runs some simple tests over each (such as whether the knot bounds the surface, and computing the orientable or non-orientable genus), and then either determines C(K) precisely or declares that C(K) ∈ \{t, t + 1\} for some t.

We process our knot complements in order by increasing number of tetrahedra, and stop at the point where the computations take over two days to run. In total, we run the algorithm over 293 knots, whose number of tetrahedra n ranges from 5 to 24 (with mean n ≃ 19.7 and median n = 21). All but 20 of these computations run in under 12 hours, which is very pleasing given that these enumeration problems involve up to 7 × 24 = 168 dimensions.

The final outcomes are also pleasing: we are able to compute 164 previously-unknown crosscap numbers precisely (for the remaining 129 knots we obtain information that is already known). A detailed table of results can be found at [http://www.maths.uq.edu.au/~bab/code/](http://www.maths.uq.edu.au/~bab/code/).

It should be noted that the paper [12] also computes new crosscap numbers, but using integer programming methods instead. Our techniques here require knots whose complements do not have too many tetrahedra, but do not need any prior information about the knots themselves. In contrast, the integer programming techniques of [12] can work with very large knot complements, but they require knots for which good lower bounds on C(K) are already known. In this sense, the two methods complement each other well.

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