Finding the \(\Theta\)-Guarded Region

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Abstract. We are given a finite set of \(n\) points (guards) \(G\) in the plane \(\mathbb{R}^2\) and an angle \(0 \leq \Theta \leq 2\pi\). A \(\Theta\)-cone is a cone with apex angle \(\Theta\). We call a \(\Theta\)-cone empty (with respect to \(G\)) if it does not contain any point of \(G\). A point \(p \in \mathbb{R}^2\) is called \(\Theta\)-guarded if every \(\Theta\)-cone with its apex located at \(p\) is non-empty. Furthermore, the set of all \(\Theta\)-guarded points is called the \(\Theta\)-guarded region, or the \(\Theta\)-region for short.

We present several results on this topic. The main contribution of our work is to describe the \(\Theta\)-region with \(O(n\Theta)\) circular arcs, and we give an algorithm to compute it. We prove a tight \(O(n)\) worst-case bound on the complexity of the \(\Theta\)-region for \(\Theta \geq \frac{\pi}{2}\). In case \(\Theta\) is bounded from below by a positive constant, we prove an almost linear bound \(O(n^{1+\varepsilon})\) for any \(\varepsilon > 0\) on the complexity. Moreover, we show that there is a sequence of inputs such that the asymptotic bound on the complexity of the \(\Theta\)-region is \(\Omega(n^2)\). In addition we point out gaps in the proofs of a recent publication that claims an \(O(n)\) bound on the complexity for any constant angle \(\Theta\).

Key words: \(\Theta\)-guarded region, unoriented \(\Theta\)-maxima, convex hull generalization, good \(\Theta\)-illumination, \(\alpha\)-embracing contour.

1 Introduction

Illumination and guarding problems have been a popular topic of study in mathematics and computer science for several decades. One instance in this class of problems is the classical one posed by Victor Klee [14]: How many guards are necessary, and how many are sufficient to patrol the paintings and works of art in an art gallery with \(n\) walls? While this particular problem has been solved shortly after by Chvatal [7] proving a tight \(\left\lceil \frac{n}{3} \right\rceil\) bound, many other variants in this problem class have appeared in the literature, see e.g. [18] for a general survey on the topic.

In this paper we consider a guarding problem with a fixed number of guards which have fixed positions in the plane. We concentrate on the mathematical description, the complexity, and the computation of the guarded area.

The model is as follows: We are given a finite set of points (guards) \(G\) in the plane \(\mathbb{R}^2\). A \(\Theta\)-cone is a cone with apex angle \(\Theta\). We call a \(\Theta\)-cone empty (with respect to \(G\)), if it does not contain any point of \(G\) in its interior. A point \(p \in \mathbb{R}^2\) is called \(\Theta\)-guarded (with respect to \(G\)), if every \(\Theta\)-cone with apex located at \(p\)
is non-empty. The set of all Θ-guarded points is called the \( \Theta \)-guardered region, or the Θ-region for short. We consider Θ-cones as open sets, hence the Θ-region is an open set, too. The rationale behind this model is that a point is well-guarded only if it is guarded from all sides.

1.1 Previous Work

For a given set \( G \) of \( n \) points in the plane, Avis et al. [4] were the first to introduce the notion of unoriented \( \Theta \)-maxima. They say that some point \( g \in G \) is a \( \Theta \)-maxima if there exists an empty \( \Theta \)-cone with apex at \( g \). Hence a point \( g \) is \( \Theta \)-maxima if it is not \( \Theta \)-guarded with respect to \( G \). They present an \( O(\frac{n}{\Theta} \log n) \) algorithm for computing the unoriented \( \Theta \)-maximum of the set \( G \), or to put it in other words, an algorithm to query each point in \( G \) if it is \( \Theta \)-guarded or not. A slight variation of their algorithm can actually query any finite point-set \( P \) in \( O(\frac{n+|P|}{\Theta} \log(n+|P|)) \) time as we show in Lemma 7. They further show that the unoriented \( \frac{\pi}{2} \)-maxima can be computed in \( O(n) \) expected time.

Abellanas et al. [1] extent the guarding model (there it is called good \( \Theta \)-illumination) by a range \( r \), i.e., a guard \( g \in G \) can only guard points inside the circle of radius \( r \) that is centered at \( g \). Beneath other results they show how to check if a query point \( p \) is \( \Theta \)-guarded in \( O(n) \) time and output the necessary range and guards as witnesses.

Over years several generalizations of the standard convex hull of a point set have been proposed, like the \( \alpha \)-hull [10], the \( k \)-th iterated hull [6], and the related concept of the \( k \)-hull (\( k \)-depth contour) [8].

Our contribution

After some general observations, we describe the structure of the boundary of the \( \Theta \)-region for different values of \( \Theta \) in Section 2. There we also give an easy and efficient \( O(n \log n) \) time algorithm to compute the boundary in case \( \Theta \geq \pi \). In the main part of the paper we concentrate on the case \( \Theta < \pi \), since for these angles the problem becomes much more involved and the boundary of the \( \Theta \)-region more complex to understand. In Section 3 we show that the boundary of the \( \Theta \)-region is contained in an arrangement of circular arcs. In Section 4 we bound this set of arcs by \( O(\frac{n}{\Theta}) \). Note that in our work \( \Theta \) and \( n \) are independent parameters. In particular, asymptotic bounds are stated in \( n \) and \( \frac{1}{\Theta} \). For \( \Theta \geq \frac{\pi}{2} \) we prove that the complexity of the \( \Theta \)-region is \( O(n) \). If \( \Theta > \delta \) for a positive constant \( \delta > 0 \), we show that the complexity is \( O(n^{1+\varepsilon}) \), for any \( \varepsilon > 0 \). In Section 5 we give a generic example for a \( \Theta \)-region with complexity \( \Omega(n^2) \) where the angle \( \Theta \) is of order \( \frac{1}{n} \). In Section 6 we give an algorithm to compute the \( \Theta \)-region in \( O(n^{\frac{1+\xi}{\Theta}} + \mu \log n) \) time, for any \( \xi > 0 \), where \( \mu \) denotes the complexity of the arrangement of the \( O(\frac{n}{2}) \) arcs. Our algorithm is based on the Partitioning Theorem [13] and on the computation of an arrangement of circular arcs.
Remark: Besides our work there is an independent and very recent publication of Abellanas et al. [2]. There the complexity of the \(\Theta\)-region (called the \(\alpha\)-embracing contour) is claimed to be linear for all constant \(\Theta\), and an algorithm that runs in \(O(n^2 \log n)\) time and \(O(n^2)\) space is proposed. For constant \(\Theta < \frac{\pi}{2}\) our \(O(n^{1+\varepsilon})\) bound on the complexity of the \(\Theta\)-region is weaker than the claimed \(O(n)\) bound. However, we believe that the main results in [2] are not correct, and we will point out two technical mistakes in the Appendix.

1.2 Remark on Plotted Pictures

The computer generated pictures are based on the value of the continuous function \(f : \mathbb{R}^2 \setminus G \to (0, 2\pi]\) where \(f(p) = \max \{\Theta : \exists \text{ empty } \Theta\text{-cone with apex } p\}\). The left picture of Figure 1 and the two rightmost pictures in Figure 6 are generated by plotting a grid point shaded, iff \(f\) has a value below the threshold \(\Theta\). In the right picture of Figure 1 we have mapped different intervals of function values in \([0, \pi]\) to different gray scale values to visualize isolines (along the boundary of the gray scale value) of \(f\) in this example. Although these pictures visualize only function values at grid points, one can rely on the pictures, since we deal with cones of a certain angle and not with arbitrarily thin stripes that could somehow pass between grid points.

2 The Shape of the \(\Theta\)-Guarded Region

We start with some observations. A point \(p \in \mathbb{R}^2\) does not belong to the \(\Theta\)-region, if there is an empty \(\Theta\)-cone with apex \(p\). Hence, no point inside an empty cone can belong to the region, and hence, the region can not contain holes. A point \(p\) lies on the boundary of a \(\Theta\)-region, if the closure\(^3\) of each \(\Theta\)-cone with apex \(p\) is non-empty, and there is at least one empty (open) \(\Theta\)-cone with apex at \(p\). The example in Figure 1 shows that the \(\Theta\)-region is not necessarily connected for \(0 < \Theta < \pi\). The shape of the \(\Theta\)-region is invariant under translation, rotation, and scaling of \(G\).

The shapes of all \(\Theta\)-regions can be grouped according to \(\Theta\). The boundary of the \(\pi\)-region is just the convex hull \(CH(G)\), because the intersection of all half-planes containing \(G\) (convex hull) is the same as removing every half-plane from \(\mathbb{R}^2\) that does not contain any point of \(G\) (\(\pi\)-region). However, for \(0 < \Theta < \pi\), empty (convex) \(\Theta\)-cones can enter the convex hull through the edges, while for \(\pi < \Theta < 2\pi\) the apexes of empty (concave) \(\Theta\)-cones do not even have to touch the convex hull (see Figure 2). Therefore, the \(\Theta\)-region is connected, if \(\Theta \geq \pi\). Trivially, the \(2\pi\)-region is the plane \(\mathbb{R}^2\) and the \(0\)-region is the empty set.

Before we discuss the \(\Theta\)-region for \(0 < \Theta < \pi\) in Sections 3–6, we discuss the simpler case \(\pi < \Theta < 2\pi\) below. Throughout the paper we use the property about inscribed angles: Given a circular arc \(C_{l,r}\) from \(l\) to \(r\), then \(\angle lpr = \angle lqr\) holds for all \(p, q \in C_{l,r}\). We write \(C_{l,r}^\alpha\) if the inscribed angle is \(\alpha\). The arc end points are always given in counterclockwise order.

\(^3\) Exceptionally we consider closed cones here.
Fig. 1. An example with |G| = 50. The Θ-region is not necessarily connected for 0 < Θ < π (left). The isolines of function \( f \) show how components of the Θ-region disconnect for decreasing Θ in this example (right).

Fig. 2. For Θ < π (resp. Θ > π) the region lies inside (outside) the convex hull \( CH(G) \) and the bounding arcs are bend inside (outside) the region.

2.1 Finding the Θ-Guarded Region for Θ > π

As already discussed (see Figure 2), for Θ > π every point in the convex hull interior of \( G \) is Θ-guarded. Intuitively, the boundary of the Θ-region is drawn by the apex of an empty Θ-cone which is rotated around the convex hull \( CH(G) \) such that its rays are always tangent to \( CH(G) \). The following algorithm computes the boundary of the Θ-region.

We first compute the clockwise sequence of guards \( G' = \{g_1, \ldots, g_k\} \) defining the convex hull (see for example [16]). Formalizing the intuition given above, we construct an algorithm that outputs circular arcs defining the boundary of the Θ-region as follows. We identify all pairs \( (g_i, g_j) \in G' \times G' \) with \( g_i \neq g_j \), for which there exists an empty Θ-cone that is tangent to \( g_i \) and \( g_j \), and has its apex outside the convex hull. We say that the apex of the Θ-cone can \"see\" the polygonal chain of \( CH(G) \) from \( g_i \) to \( g_j \). Such a pair \( (g_i, g_j) \) will always have the property, that the lines supporting the convex hull edges \( (g_j, g_{\text{succ}(j)}) \) and \( (g_{\text{pred}(j)}, g_i) \) have an angle of intersection not greater than Θ, and that the lines supporting \( (g_i, g_{\text{succ}(i)}) \) and \( (g_{\text{pred}(i)}, g_j) \) have an angle of intersection
greater than $\Theta$. The sequence of all these pairs $(g_i, g_j)$ and the corresponding circular arcs $C_{g_j, g_i}^{2\pi - \Theta}$, that are defined by $g_i, g_j$, and the apex of the $\Theta$-cone that is tangent to $g_i$ and $g_j$, can be computed by a cyclic scan over the sequence $G'$. The arc end points of the $\Theta$-region boundary can be computed as the intersection points of each circular arc $C_{g_j, g_i}^{2\pi - \Theta}$ with the supporting lines through $(g_j, g_{\text{succ}(j)})$ and $(g_{\text{pred}(i)}, g_i)$. Consequently the $\Theta$-region has the same complexity than the convex hull. The running time of the algorithm is dominated by the convex hull construction in $O(n \log n)$ time. We summarize the above in the following Lemma.

Lemma 1. The boundary of the $\Theta$-region for $\Theta > \pi$ can be computed in time $O(n \log n)$ and its complexity is $|CH(G)|$.

3 The Boundary of the $\Theta$-Region

From now on we assume that the angle is $0 < \Theta < \pi$. Here we give a mathematical description of the $\Theta$-region. First we come back to the inscribed angles and explain its meaning for our setting. Let $e = (l, r) \in G \times G$ be any pair of guards. Then the set of points where we can place the apex of an empty $\Theta$-cone passing through the line segment $(l, r)$ in the same direction is bounded by the circular arc, incident to $l$ and $r$ having inscribed angles $\Theta$, and its chord $lr$. We denote this closed circular segment with $D_{l,r}^\Theta$ (or $D_e$ for short) and its bounding circular arc, as above, with $C_{l,r}^\Theta$ (or $C_e$ for short). Because of the orientation, the circular segment is described uniquely.

The construction of the $\Theta$-region is motivated by the idea of locally removing sets $T_i$ of unguarded points from the convex hull $CH(G)$ such that the remaining part matches the $\Theta$-region (see Figure 2, middle), i.e. we aim for

$$\Theta\text{-region} = CH(G) \setminus \bigcup_{i \in I} T_i$$

for specific sets $T_i$. Next we give the construction for the sets $T_i$. Consider any empty $\Theta$-cone $c$ that has at least a guard on each ray (see Figure 3, left). First we turn the cone clockwise while pushing the cone towards the point set, such that it always stays empty but touches a guard on each boundary (see Figure 3,
We end this motion when the apex of the cone reaches the position of a guard, say \( l_0 \). Afterwards we start again with cone \( c \), i.e. in the original position, and rotate the cone in a similar way counterclockwise until the apex reaches the position of another guard, say \( r_0 \). We extend our notions. With \( L_i \) (resp. \( R_i \)) we denote the set of guards that are incident to the left (resp. right) ray of a cone during the construction (the white points in Figure 3, right). We call the closure of the union of all cones, which are used during the construction, the *tunnel* \( T_i \) \textit{with respect to} \( L_i \) and \( R_i \), or tunnel for short (shaded region in Figure 3, right).

Note that the index set \( I \) in Formula (1) enumerates over all tunnels.

Note that Formula (1) describes the \( \Theta \)-region of \( G \), because each empty \( \Theta \)-cone that intersects \( \text{CH}(G) \) lies in at least one tunnel \( T_i \); Let \( c \) be such a cone. We can identify a tunnel by moving \( c \) in the direction of its medial axis until one of its rays is tangent to a guard. Then we let the empty cone slide along that point without rotation until the second ray is also tangent to a guard. According to our construction there is a tunnel that contains this cone and hence the cone \( c \) in its original position.

No point in \( T_i \) is \( \Theta \)-guarded, but only its boundary can contribute to the boundary of the \( \Theta \)-region. First we consider its straight-line boundaries. Since \( \Theta < \pi \), each point of a straight-line boundary can be crossed infinitesimally by an empty \( \Theta \)-cone. That means, there are open neighborhoods of unguarded points around each point of a straight-line boundary, and hence they can not contribute to the \( \Theta \)-boundary. Points beyond the straight-line boundaries belong to different tunnels and will be processed independent from \( T_i \).

Therefore we only have to consider the curved boundary of \( T_i \). Observe that during the construction, the apex of the rotating cone is drawing a sequence of circular arcs between \( l_0 \) and \( r_0 \) which we will formalize next. We define the set

\[
U_i := \bigcap_{(l,r) \in L_i \times R_i} D_{l,r}^\Theta
\]

as the intersection of all circular segments for guard pairs in \( L_i \times R_i \). In the following Lemma we state that we can derive the curved boundary of \( T_i \) from these circular segments. Let \( h_i \) be the closed half plane which is bounded by the line through \( l_0 \) and \( r_0 \) and contains the sequence of arcs.

**Lemma 2.** Let \( T_i, U_i, \) and \( h_i \) be as defined above. Then \( T_i \cap h_i = U_i \).

**Proof.** (Superset.) Let \( p \in U_i \). Assume there is no empty cone with apex \( p \) through tunnel \( T_i \). This means that there is at least a pair \((l,r) \in L_i \times R_i\) with the property that \( \angle lpr < \Theta \). Hence \( p \notin D_{l,r}^\Theta \), which is a contradiction. (Subset.) Let \( p \) be the apex of an empty \( \Theta \)-cone through tunnel \( T_i \). This means that \( \angle lpr \geq \Theta \) for all \((l,r) \in L_i \times R_i\), and hence \( p \) lies in all corresponding circular segments \( D_{l,r}^\Theta \). \( \square \)
It follows that the $\Theta$-region boundary is contained in the curved boundary of the union of the sets $U_i$, i.e.

$$
\partial \Theta \text{-region} \subseteq \partial \bigcup_{i \in I} U_i = \partial \bigcup_{i \in I} \left( \bigcap_{(l,r) \in L_i \times R_i} D_{l,r}^{\Theta} \right),
$$

(3)

where $i$ enumerates over all tunnels. We observe that the intersections of $|L_i| \cdot |R_i|$ circular segments $D_{l,r}$ in Formulae (2) and (3) are too pessimistic. During the construction of a tunnel we collect all guard pairs $(l,r) \subset L_i \times R_i$, that are incident to the rotating cone simultaneously, in $E_i$. Since the touching point of $L_i$ (resp. $R_i$) can only change in one direction to its neighbor in the sequence of $L_i$ (resp. $R_i$), that leads to a set $E_i$ of size $|L_i| + |R_i| - 1$. Therefore we may reduce the intersection of the circular segments in Formula (2) to

$$
U_i := \bigcap_{(l,r) \in E_i} D_{l,r} \cap h_i
$$

for which Lemma 2 is still valid. With $C$ we denote the set of all circular arcs that appear in the boundary of a set $U_i$.

4 Upper Bounds on the Worst-Case Complexity

Now we discuss the worst-case complexity of the $\Theta$-region and state the asymptotic bounds in the number $n$ of guards and the reciprocal value of the angle, i.e. $\frac{1}{\Theta}$. During the analysis of the complexity, we distinguish cases according to the value of $\Theta$. We already know that the 0-region is the empty set. Since $G$ is a discrete set the $\Theta$-region is also the empty set for values close to 0.

**Lemma 3.** The $\Theta$-region for $\Theta \leq \frac{\pi}{n}$ is the empty set.

*Proof.* Consider the $n$ rays emanating from a point $p \in \mathbb{R}^2 \setminus G$ through the guards in $G$. Then the rays form at least one empty cone with angle of at least $\frac{\pi}{n}$ which contains an empty $\Theta$-cone. Hence $p$ is unguarded. We can argue similarly for the guards $p \in G$. \qed

According to the right term in Formula (3) the complexity of the $\Theta$-region is hidden in an arrangement of circular arcs. Since there are at most $O(n^2)$ different circular arcs, two for each guard pair, the complexity of the $\Theta$-region is trivially $O(n^4)$. Now we show that the set $C$ of circular arcs is of $O(\frac{n^2}{\Theta})$ size. Hence the complexity of the $\Theta$-region is $O(\frac{n^2}{\Theta^2})$.

**Theorem 1.** The set $C$ of circular arcs, which defines the boundary of the $\Theta$-region, is of $O(\frac{n^2}{\Theta})$ size.

*Proof.* Instead of counting the arcs directly we count their end points. Let $p$ be an arc end point of a tunnel as shown in Figure 4, left. In this position a ray of
the rotating cone is incident to two guards at once. We assume without loss of
generality that two guards lie on the left ray. We focus on the guard \( l_k \) that is
closer to the apex and count how often a guard can be in this situation. Clearly
the number is bounded by \( n - 1 \) because there are no more other guards. On the
other hand we observe that the empty \( \Theta \)-cones in this situation can not intersect
each other beyond the second guard on the left ray (see Figure 4, middle). Hence
there can be at most \( \floor{\frac{2\pi}{\Theta}} \) different such cones. With the same argumentation
for the right ray we bound the number of arc end points per guard by \( 2\floor{\frac{2\pi}{\Theta}} \) and
hence the total number of arc end points by \( O(n\Theta) \).
\( \square \)

From the last Theorem we can derive, that if the angle \( \Theta > \delta \) is bounded by a
constant \( \delta > 0 \), then the number of arcs in \( \mathcal{C} \) is \( O(n) \) and the complexity of the
\( \Theta \)-region is \( O(n^2) \). With an auxiliary construction we can even further improve
this result.

**Theorem 2.** If the angle \( \Theta > \delta \) is bounded by a constant \( \delta > 0 \), then the
complexity of the \( \Theta \)-region is \( O(n^{1+\varepsilon}) \), for any \( \varepsilon > 0 \).

**Proof.** We make use of the following construction. Let \( a \in \mathcal{C} \) be an arc in the
boundary of tunnel \( T_i \), and let \( u \) and \( v \) be the end points of \( a \). The line segment
\((u,v)\) and \( a \) are the boundary of a circular segment, say \( d_a \). Now we glue a
triangle \( t_a \) at the edge \((u,v)\) of \( d_a \), which has an angle of \( \min\{\Theta, \frac{\pi}{4}\} \) at \( u \) and
\( v \), and denote this new object with \( F_a := d_a \cup t_a \) (see Figure 4, right). We state
that the triangle \( t_a \) is a subset of \( T_i \): Assume there is a guard \( g \in t_a \). Then
the angles \( \angle gwv \) and \( \angle gvw \) are smaller than \( \Theta \). Hence two empty \( \Theta \)-cones with
apexes \( u \) and \( v \) would belong to different tunnels what is a contradiction to
\( a \subset \partial T_i \). Furthermore, because of the angle at \( u \) and \( v \) the set of empty \( \Theta \)-cones
with apexes at points in \( a \) have to cover \( t_a \), what completes the proof of the
statement.

We repeat the above construction for each arc \( a \in \mathcal{C} \) and collect the new
objects \( F_a \) in the set \( \mathcal{F} \). We repeat the definition of \( \alpha \)-fatness from Efrat et
al. [11]: An object \( F \) is \( \alpha \)-fat for some fixed \( \alpha > 1 \), if there exist two concentric
disks \( D \subseteq F \subseteq D' \) such that the ratio \( \frac{\rho}{\rho'} \) between the radii of \( D' \) and \( D \) is at most
\( \alpha \). We state that there is an \( \alpha > 1 \) such that the objects \( F_a \in \mathcal{F} \) are \( \alpha \)-fat: The

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{An end point \( p \) of a circular arc in the boundary of a tunnel (left). A situation
that is described in the proof of Theorem 1 (middle). The auxiliary construction that
is described in the proof of Theorem 2 (right).}
\end{figure}
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The worst-case scenario occurs when the arc $a$ is almost a straight-line. Hence we concentrate on the proof that the triangle $t_a$ is $\alpha$-fat (see again Figure 4, right). Remember that the angle at $u$ is $\min\{\Theta, \frac{\pi}{4}\}$. Then the ratio between the radii of the circumcircle and the inscribed circle is

$$\frac{\rho'}{\rho} = \frac{1}{\sin\left(\frac{1}{2} \cdot \min\{\Theta, \frac{\pi}{4}\}\right)} \leq \frac{1}{\sin\left(\delta_{\frac{\pi}{2}}\right)} =: \alpha,$$

which is a constant. Without loss of generality we assume $\delta \leq \frac{\pi}{4}$.

The main Theorem in Efrat et al. [11] states, that the combinatorial complexity of the union of a collection $F$ of $\alpha$-fat objects, whose boundary intersect pairwise in at most $s$ points, is $O(|F|^{1+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on $\varepsilon$, $\alpha$, and $s$.

It is already shown that $\alpha$ is a constant and that the objects in $F$ are $\alpha$-fat. The boundary of each convex object $F_a \in F$ has always three edges: two line segments and a circular arc. Therefore the boundary of each pair of objects in $F$ intersect in at most $s = 10$ points. As we said above $|C| \in O(n)$, and hence $|F| \in O(n)$, because $\Theta$ is bounded from below by a constant $\delta$. Therefore the construction fulfills all preconditions to apply the Theorem of Efrat et al. which completes the proof. \qed

Now we show that the complexity of the $\Theta$-guarded region is linear for angles $\Theta$ at least $\frac{\pi}{2}$.

**Theorem 3.** The complexity of the $\Theta$-region is $O(n)$ for $\frac{\pi}{2} \leq \Theta < \pi$.

**Proof.** Let $J$ be a set of $m$ Jordan curves, i.e. simply-closed curves. Kedem et al. [12] proved that if any two curves in $J$ intersect in at most two points then the complexity of their union is $O(m)$.

For each set $U_i$ we define a Jordan curve $J_i$. Let $J_i$ be the curved boundary of $U_i$ from $l_0$ to $r_0$ connected with an auxiliary half circle $C_{\left(\frac{\pi}{2}r_0,l_0\right)}$, i.e.

$$J_i := \partial \left( \bigcap_{(l,r) \in E_i} D_{l,r}^{\Theta} \cap h_i \right) \cup C_{\left(\frac{\pi}{2}r_0,l_0\right)}.$$

Note that the auxiliary half-circle lies in $T_i$ because $\Theta$ is obtuse. Note further that $J_i$ is the boundary of a convex region and that $J_i$ lies inside the circle that supports $C_{\left(\frac{\pi}{2}r_0,l_0\right)}$. We repeat this construction for all tunnels $T_i$, with $i \in I$, and collect all curves $J_i$ in $J$. We state that any two curves in $J$ intersect at most twice. Now assume that there are two curves $J_i, J_j \in J$ which intersect in more than two points. We distinguish the following cases as they are shown in Figure 5. Let $A_i$ (resp. $A_j$) denote the sequence of circular arcs of $U_i$ (resp. $U_j$).

In Case 1, the sequence of circular arcs $A_i$ and $A_j$ intersect in point $p$ as is shown in Pictures (1a) or (1b). That means that for each tunnel $T_i$ and $T_j$ there exists an empty $\Theta$-cone with apex in $p$. Therefore the angle $\angle apb$ has to be at least $2\Theta$ which is at least $\pi$. That is a geometrical contradiction.
In Case 2, we consider a point \( p \) that lies on the sequence of arcs \( A_j \) outside \( J_i \) as shown in Picture (2). The angle \( \angle cpd \) is at least \( \Theta \). By construction the angle \( \angle bpa \) is larger than \( \angle cpd \) and hence \( \angle apb > \Theta \). It is a contradiction that \( p \) does not lie inside \( J_i \).

In Case 3, we consider an empty \( \Theta \)-cone with apex \( c \) through tunnel \( T_j \). Assume this cone passes between \( a \) and \( b \). Then the angle \( \angle bca \) is at least \( \Theta \) and hence \( c \) has to lie inside \( J_i \). This is a contradiction. In case that the empty \( \Theta \)-cone \( c \) does not pass between \( a \) and \( b \), but \( b \) and \( c \), or \( a \) and \( d \), similar geometric contradictions can be shown.

Other cases are excluded since no guard can lie inside \( J_i \) or \( J_j \). This completes the proof. \( \square \)

5 Lower Bound on the Worst-Case Complexity

In the following we show that there is a sequence of inputs such that the asymptotic bound on the complexity of their \( \Theta \)-guarded region is \( \Omega(n^2) \). For this purpose we give a generic construction for point sets \( G_i \) with \( n_i \) guards and angles \( \Theta_i \) for all \( i \in \mathbb{N} \), such that the complexity of the \( \Theta_i \)-region of the point set \( G_i \) is lower bounded by \( c \cdot n_i^2 \) for some constant \( c \) and \( \lim_{i \to \infty} n_i = \infty \). In fact \( n_i \) is a linear function in \( i \), and \( \Theta_i \) is of order \( \frac{1}{i} \). Therefore the complexity bound can also be interpreted as \( \Omega\left(\frac{n^2}{i}\right) \).

First we motivate the construction for a given \( i \in \mathbb{N} \). To achieve the desired complexity, we construct the point set \( G_i \) in such a way that the \( \Theta_i \)-region is fragmented into \( c \cdot n_i^2 \) connected components, each of constant complexity. Figure 6 illustrates the idea of the construction. The area, where the \( \Theta_i \)-region is highly fragmented is at the center of the convex hull. The decomposition is forced by long, thin tunnels that enter this area ‘axis parallel’ from above, below, left, and right; more precisely the medial axis of the cones that enter these tunnels deepest are parallel to the principal axes. In the first step of the construction we determine the tunnels that force the fragmentation and implicitly determine the area (bounded by the box in Figure 6) that contains these connected component. Unfortunately the same guards, that define these tunnels, define an even larger number of unwanted tunnels which can enter this area as well. Therefore we
have to place additional guards in the second step with the intention to prevent unwanted tunnels from entering this area, because they could erase some of the connected components, hence reducing the total complexity (see Figure 6, middle and right). We show how to place a linear number of additional guards as obstacles in the plane to keep out a squared number of tunnels from this area. We note that because of the construction in the second step the convex hull can be huge compared to the box in which we count the connected components. For simplicity reasons we disregard the shape of the $\Theta_i$-region outside this box. The construction details are given below.

5.1 First step: To determine the wanted tunnels.

We denote the square of edge length $2i$ that is centered at the origin and is oriented parallel to the principal axes with $B_i$. In this step guards are placed on the boundary of the boxes $B_{4i}$ and $B_{2i}$. The area, in which we will count the connected components, is $B_i$ (see Figure 7, left). The entire construction is symmetric to the origin as well as to the principal axes. For this reason we only give the construction for the upper half of box $B_{4i}$; the constructions for the lower, left, and right half of this box are done analogously.

Now we introduce the guard patterns A and B (see Figure 7, right), that define two ways to place guards inside a cell of width 1 and height $4i$, which we will use later on to stamp the upper half of the box with. First we define $\Theta_i$ as the angle$^4$ between the rays emanating from $(\frac{1}{2}, 0)$ through the upper corners $(0, 4i)$ and $(1, 4i)$. To get guard pattern A we place four guards on the boundary of this cone: two with $y = 2i$ and two with $y = 4i$. These four guards define a wanted tunnel which is thin in the sense that the boundary of the tunnel stays in the box of width 1 for values $0 \leq y \leq i$; remember that we only care for

\[^4\text{Note that } \Theta \text{ and } i \text{ are not independent because } \Theta = \arctan(\frac{1}{8i}) \leq \frac{1}{8i}.\]
the interior of $B_i$. For technical reasons we add guards at $(0,2i)$ and $(1,2i)$ to avoid unwanted tunnels between neighboring guard patterns A. In case we do not need a tunnel inside the cell we use pattern B: three guards that are placed equidistant on the top edge of the pattern make it impossible for any cone to enter this box from above deeper than $y = 2i$. Next we subdivide the upper half of the box $B_{2i}$ in $8i$ cells of width 1. The medial quarter is stamped with pattern A, the remaining cells are stamped with pattern B (see Figure 8).

This way we can guarantee $2i$ wanted tunnels from above which intersect $B_i$ and touch the $x$-axis. After repeating this construction for the lower, left, and right half of $B_{2i}$, tunnels from above and below touch at the $x$-axis as well as tunnels from the left and right touch at the $y$-axis. This follows immediately from the symmetric construction. Removing these tunnels from the box $B_i$, yield to a fragmentation into $(2i + 1)^2$ connected components.

Finally we remark that we can save up to 2 guards per stamped pattern if guards overlap with the neighboring pattern. Hence the number of guards, that are placed in the entire first step, sum up to $80i + 4$. 

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**Fig. 7.** Placement of the guards in the first step for $i = 2$ (left). Guard patterns A and B (right).

**Fig. 8.** Subdivision of the upper half of $B_{4i}$ in $8i$ cells (left). The guard pattern for the upper half (right).
5.2 Second step: To exclude the unwanted tunnels.

Again we start with the construction for the upper half of $B_{4i}$. Figure 9 shows the situation of the $2i$ neighboring guard patterns $A$. We denote the guard pairs on the line $y = 4i$ with $P_1, \ldots, P_{2i}$ and the guard pairs on the line $y = 2i$ with $Q_1, \ldots, Q_{2i}$. An empty cone that enters $B_i$ from above therefore has to pass through $P_k$ and $Q_\ell$ for some $k, \ell \in \{1, \ldots, 2i\}$. If $k = \ell$, the tunnel is wanted. So the task is to hinder all cones through tunnels with $k \neq \ell$ to intersect the box $B_i$. We introduce a new notion to reformulate the problem.

**Definition 1.** Let $t$ be a tunnel through $P_k$ and $Q_\ell$. We say that an empty cone enters tunnel $t$ the deepest if the $y$-value of its apex is minimal among all empty cones in $t$.

Note that the deepest cone in a tunnel is unique and is tangent to at least one guard on each ray. We define the slope of a cone as the slope of its medial axis. Because of the regular structure of the cells with guard pattern $A$ we can make the following Observation. Informally it states that the slope of a deepest cone through $P_j$ and $Q_{j+h}$ is independent of $j$ and implicitly given by $h$.

**Observation 1** Let $h \in \{0, \ldots, 2i-1\}$. Let $c_j$ be the deepest cone through $P_j$ and $Q_{j+h}$ and let $d_j$ be the deepest cone through $P_{j+h}$ and $Q_j$ for all $j \in \{1, \ldots, 2i-1\}$. Then all cones $c_j$ have the same slope, and all cones $d_j$ have the same slope.

W.l.o.g. we concentrate on the cones $c_j$, but we can argue in the same way for the cones $d_j$. We derive from Observation 1 that for fixed $h$ the intersection of all deepest cones $c_j$ is again a cone with the same slope (see Figure 10, left). Assume we place a guard at a random position inside this intersection. Then none of the cones $c_j$ is empty anymore. That means that there are new deepest cones with different slopes, since they have to be tangent to the new guard (see Figure 10, middle). Now we pull the new guard in the direction of the medial axis of the former deepest cones towards infinity. Then we observe, that while we move this point, the deepest cones are rotated and this way are pulled away from the $x$-axis. Since the limes of the rotation, compared to the slope of the original deepest cones $c_j$, has absolute value $\Theta_{2\pi}$, we can force a rotation by
Fig. 10. The deepest cones $c_j$ used in Observation 1. Here the vertical arrow marks the direction of their medial axis (left). The deepest cones that are tangent to the new vertex (middle). The deepest cones, in the case that the new guard is pulled far out in the direction of the former medial axis (right).

Fig. 11. Construction of the second step for all parts: above, below, left, and right (left). Construction used in the proof of Lemma 4 (right).

an angle that is arbitrarily close to half of the apex angle, i.e. $\frac{\Theta_i}{2} - \varepsilon$ for any $\varepsilon > 0$ (see Figure 10, right). Note the generality of the above discussion for all $h = \{0, \ldots, 2i - 1\}$. We will definitely not place a guard in the union of the deepest cones for $h = 0$, since these tunnels force the decomposition of the $\Theta_i$-region in the center. In spite of this we have used exactly this case in the drawings in Figure 10, since it depicts the worst-case scenario we will consider later in the proof.

Now we are able to complete the construction. First we compute the slope of the medial axes of the deepest cones $c_j$ as well as $d_j$ of Observation 1 for all $h = \{1, \ldots, 2i - 1\}$. For each of these $4i - 2$ slopes we place a guard at the intersection point of the ray emanating from the origin having this slope and the boundary of a new box $B_x$ (see Figure 11, left). Box $B_x$ has necessarily to be large enough to guarantee, that each intersection point lies

1. outside the union of the wanted cones and
2. inside the intersection of the deepest cones of the given slope.
The existence of box \( B_x \) with these properties follows from the discussion above. But this is not sufficient. It remains to prove the following lemma.

**Lemma 4.** Box \( B_x \) can be chosen large enough such that no empty \( \Theta_i \)-cone, but the wanted cones, can intersect \( B_i \).

**Proof.** It is sufficient to prove the claim for the deepest cones with the minimum apex \( y \)-value amongst all deepest cones according to Observation 1. These are the cones for \( h = 0 \). (Please note that we do not block tunnels for \( h = 0 \) in practice; we just prove that we could even hinder cones through these tunnels from entering \( B_i \).)

Remember that we can place the guard which blocks the deepest cones such that the deepest cones are rotated by an angle arbitrarily close to \( \frac{\Theta_i}{2} \). W.l.o.g. we assume that the cone is rotated clockwise. Consider the empty cone \( c \) of maximum angle with apex at \( a = (\frac{i}{4}, i) \) inside a cell with guard pattern \( A \) (shaded region in Figure 11, right). This cone touches the boundary of \( B_i \) and its left ray is vertical as it is the case for maximal rotated deepest cones. If we can show that the angle of \( c \) is smaller than \( \Theta_i \) it follows that \( c \) can not enter \( B_i \). We crop \( c \) at the line \( y = 4i \) to make it a rectangular triangle. Now we take the \( \Theta_i \)-cone from pattern \( A \), Figure 7, and move its apex to \( a \). We divide the triangle along the right boundary of the \( \Theta_i \)-cone through point \( (\frac{5}{8}, 4i) \). Consequently the left sub-triangle has angle \( \Theta_i \) at point \( a \). Since the opposite leg of the entire triangle is 2-times the opposite leg of the left sub-triangle, the total angle of \( c \) at \( a \) is less than 2-times \( \Theta_i \). \( \square \)

After repeating this construction for the lower, left, and right half we have placed \( 16i - 8 \) additional guards. Together with the guards from the first step they define the set \( G_i \) with \( n_i = 96i - 4 \) guards in total. The generic example presented in this section proves the following Theorem.

**Theorem 4.** There is a sequence of inputs \((\Theta_i, n_i, G_i)_{i \in \mathbb{N}} \) with \( \lim_{i \to \infty} \Theta_i = 0 \) such that the asymptotic bound on the complexity of their \( \Theta \)-region is \( \Omega(n^2) \) where \( n \) is the number of guards.

6 Algorithm

Here we discuss a way to compute the boundary of the \( \Theta \)-region. Note that we gave bounds on the worst-case complexity of the \( \Theta \)-region above. Clearly, for any \( n \) and any \( \Theta \) there are sets \( G \) for which the \( \Theta \)-region is empty or extremely simple. Despite of this our algorithm will consider the \( O(n^2) \) arcs in \( C \) and hence can not be output-sensitive. We allow a simplification in the presentation of the algorithm: We will consider a set \( C' \) of arcs which are longer on one side, i.e. \( |C| = |C'| \) and \( \bigcup C \subset \bigcup C' \).

First we compute the convex hull \( CH(G) \) and add for each hull edge \((u, v)\) the circular arc \( C_{u,v}^{\Theta} \) to the set \( C' \). For each guard \( g \), that is not a vertex of \( CH(G) \), we compute all empty cones of maximal angle with apex at \( g \) together with two guards (witnesses) \( g_{\min} \) and \( g_{\max} \) per empty cone, which lie on its rays. (See the
light shaded cone in Figure 12.) This can be done by using the algorithm of Avis et al. [4] in \(O((\frac{n}{r}) \log n)\) time and \(O(n)\) space.

As we did in the proof of Theorem 1, we find the arcs in \(\mathcal{C}\) via their end points. If we move an empty \(\Theta\)-cone with apex \(g\) and its left ray through \(g_{\text{min}}\) along the line through \(g\) and \(g_{\text{min}}\) until a guard, say \(g_r\), is tangent to the other ray, the new apex marks an end point \(p_r\) of two arcs in the set \(\mathcal{C}\) (see Figure 12). Since we do not know the second end points of the arcs, we add the piece of \(C_{g_{\text{min}},g_r}\) to \(\mathcal{C}'\) that ends in \(g_r\) and \(p_r\), and we add the piece of \(C_{g,g_r}\) to \(\mathcal{C}'\) that ends in \(g\) and \(p_r\). A similar construction for the line through \(g\) and \(g_{\text{max}}\) will add another two arcs to \(\mathcal{C}'\).

Note that by fixing the line through \(gg_{\text{min}}\) we can find guard \(g_r\) naively by simply inspecting all guards in \(G\), and similarly we can find \(g_l\) for the line through \(gg_{\text{max}}\). However, one can compute guards \(g_r\) and \(g_l\) faster with the help of the well-know Partition Theorem that has been extensively used in the context of range searching. We cite the theorem for a planar point set.

**Theorem 5.** (Partition Theorem [13].) Any set \(S\) of \(n\) points in the plane can be partitioned into \(O(r)\) disjoint classes by a simplicial partition, such that every simplex (i.e. triangle) contains between \(\frac{n}{r}\) and \(\frac{2n}{r}\) points and every line crosses at most \(O(r^{\frac{2}{3}})\) simplices (crossing number). Moreover, for any \(\xi > 0\) such a simplicial partition can be constructed in \(O(n^{1+\xi})\) time.

Using this Theorem recursively one can construct a tree which is called a partition tree (e.g. the root of the tree, associated with \(S\), has \(O(r)\) children, each associated with a simplex from the first level, and so on). From now on we assume that \(r\) is a constant. Observe that if \(r\) is a constant, the partition tree is of \(O(n)\) size and it can be constructed in \(O(n^{1+\xi})\) time for any \(\xi > 0\).

**Lemma 5.** For any \(\xi > 0\), there is a data structure of \(O(n \log n)\) size and \(O(n^{1+\xi})\) construction time such that for the given lines through \(gg_{\text{min}}\) and \(gg_{\text{max}}\), corresponding guards \(g_l\) and \(g_r\) can be computed in additional \(O(n^{\frac{2}{3}+\xi})\) time.

**Proof.** Assume we are given a partition tree and suppose that we fix the line through \(gg_{\text{max}}\). Clearly by Theorem 5 we have the bound on the number of triangles that intersect the line which is \(O(\sqrt{n})\). On those triangles we recur, which leads to a total of \(O(\sqrt{n})\) triangles intersected by the line. But still there
might be $O(r)$ triangles lying completely to the left of the line $gg_{\text{max}}$. For those triangles we can precompute a convex hull for the points inside each triangle. This will increase the total space of the partition tree by a $O(\log n)$ factor since every level in the tree now will be of $O(n)$ size. However, this way we avoid recursing on the triangles that lie completely to the left of $gg_{\text{max}}$. Namely, for every triangle that lies to the left of $gg_{\text{max}}$, guard $g_l$ can be found as an extreme point of the precomputed convex hull in the direction perpendicular to the line that forms the $\Theta$-cone with the line through $gg_{\text{max}}$ in $O(\log n)$ total time (see [15], Section 7.9). The case for the line through $gg_{\text{min}}$ is similar. \hfill \square

Therefore we can state the following Lemma.

**Lemma 6.** For any $\xi > 0$, the set $C'$ can be computed in $O(n^{\frac{2}{3} + \xi}/\Theta)$ time and $O(n \log n)$ space.

Next we discuss how to compute the $\Theta$-guarded region from the set of circular arcs $C'$. For each connected component of the $\Theta$-region the algorithm outputs a sequence $p_1, \ldots, p_k$ of points in the plane and circular arcs incident with pairs $p_{i-1}, p_i$ for $i = 2, \ldots, k$ and $p_k, p_1$ as edges of the $\Theta$-region.

We start with computing the arrangement $A(C')$ of set $C'$. Let $\psi$ denote the number of cells in $A(C')$ and let $\mu$ denote the total complexity of the arrangement $A(C')$, which upper bounds the complexity of the $\Theta$-region. Edelsbrunner et al. [9] showed that $\mu$ is at most $O(\sqrt{\psi} \cdot (\frac{1}{\Theta}) \cdot 2^{\alpha(n)})$, where $\alpha(\cdot)$ is the inverse Ackerman function which is an extremely slow-growing function. Moreover, the arrangement $A(C')$ can be constructed in $O((n + \mu) \log n)$ time by the plane-sweep algorithm of Bentley and Ottman [5].

Since arcs in $C'$ are bounding circular segments from the formula (3), cells in the arrangement $A(C')$ will have the property that they are either $\Theta$-guarded or not $\Theta$-guarded. Hence, if some point from the cell is $\Theta$-guarded then the whole cell belongs to the $\Theta$-region and opposite. Let $P$ denote the set of $\psi$ different points such that each point is taken from the interior of $\psi$ different cells in $A(C')$. To detect the cells that belong to the $\Theta$-region, we use the following lemma.

**Lemma 7 (Avis et al. [4]).** Let $G$ be a set of $n$ guards and let $P$ be a set of $\psi$ query points in $\mathbb{R}^2$. The $\Theta$-unguarded points of $P$ can be reported together with their witnesses $g_{\text{min}}$ and $g_{\text{max}}$ in $O(\frac{n+\psi}{\Theta} \log(n + \psi))$ time and $O(n)$ space.

**Proof.** Avis et al. [4] presented an algorithm to compute all $\Theta$-unguarded guards of $G$ in $O(\frac{n}{\Theta} \log n)$ time and $O(n)$ space. So far the set of query points and the set of guards are the same. But since their algorithm actually distinguishes between query points and guards, it can be extended immediately: In steps 2 and 3 of procedure $\text{Unoriented Maxima}$ on page 284f. only guards are inserted into the convex hull constructions, while tangents to these convex hulls are only computed through query points. The running time of the algorithm is dominated by sorting the points in $G \cup P$ for $\frac{\pi}{\Theta}$ many directions which takes $O(\frac{n+\psi}{\Theta} \log(n + \psi))$ time. For more details see Section 2 and Section 6 (Appendix) in [4]. \hfill \square
\begin{center}
\begin{tabular}{|c|c|}
\hline
angle \(\Theta\) & worst-case complexity \\
\hline
\(\pi \leq \Theta < 2\pi\) & \(|CH(G)|\) vertices \\
\(\frac{\pi}{2} \leq \Theta < \pi\) & \(O(n)\) \\
\(\delta < \Theta < \frac{\pi}{2}\), for constant \(\delta > 0\) & \(O(n^{1+\varepsilon})\), for any \(\varepsilon > 0\) \\
\hline
\end{tabular}
\end{center}

Table 1. The worst-case complexity of the \(\Theta\)-region in dependency on the angle \(\Theta\).

At the end we collect all \(\Theta\)-guarded cells and output the sequence of nodes and edges on the boundary of the union of them. We conclude with the following Theorem.

**Theorem 6.** For any \(\xi > 0\), the \(\Theta\)-region for \(\Theta < \pi\) can be computed in time \(O(n^{\frac{2}{\Theta}} + \xi/\Theta + \mu \log n)\), where \(\mu\) denotes the complexity of the arrangement \(A(C')\).

7 Conclusion

In this paper we consider a point to be guarded, if it is guarded from 'all' sides by a given finite set of guards \(G\). Our main goals were to analyze the shape and the complexity of the \(\Theta\)-region, i.e. the set of all \(\Theta\)-guarded points, and give a mathematical description of it.

As a result, we showed that the \(\Theta\)-region is defined by a set of at most \(O(n^{\frac{2}{\Theta}})\) many circular arcs. The difficulty in the complexity analysis of the \(\Theta\)-region itself appeared while arguing about the complexity of the union of convex sets \(U_i\) which are bounded by these arcs (cf. Formula 3). In dependency on \(\Theta\) we summarize our results on the worst-case complexity of the \(\Theta\)-region in Table 1. Furthermore, we could give a series of inputs with decreasing angle and increasing number of guards whose asymptotic complexity is \(\Omega(n^2)\). Finally we gave an algorithm to compute the \(\Theta\)-region.

8 Appendix: Technical Mistakes in a Recent Publication

Abellanas et al. [2] claim the complexity of the \(\Theta\)-region to be \(O(n)\) for a fixed value of \(\Theta \in (0, \pi]\) and suggest an algorithm to compute it. Because the paper is not precise in several places, we will point out mistakes that show gaps in the proofs of the main results.

**Mistake in the Construction.** The entire construction, and hence the results, are based on the existence of certain arc chains, one per convex hull edge, which are introduced in Definition 4 on page 367. But Definition 4 is actually a characterization of a point-set. That means Definition 4 contains a *hidden and unproved Lemma*, stating that the defined point-set forms a chain of arcs. In the context of other mistakes we will argue below that these chains do not exist in general for acute \(\Theta < \frac{\pi}{2}\).
We begin with Definition 3 on page 366. (Be aware of the index typos.) It is stated that a guard pair \((s_i, s_j)\) can only be entered by empty cones from one direction, and hence one of the two connecting arcs is excluded a priori. But this is not generally true for \(\Theta < \frac{\pi}{2}\) as the guard pair \((a, c)\) in Figure 13 shows. In the following we assume that, however, both directions can be distinguished and handled somehow.

Next we focus on the construction of the chain as is given in the proofs of Lemma 3 and Lemma 4 on pages 368–370. There vertices are connected via two arc pieces to a predecessor and a successor only, guaranteeing a vertex degree of 2 by construction.

Because of the mentioned mistake in Definition 3, there is now a misunderstanding of the terms predecessor and successor of an arc end point: The predecessor (resp. successor) is the left (right) end point of the connecting arc piece \(\text{viewed from the direction of the empty cone}\) that supports this arc. (By the way, this cannot be decided only by the \(x\)-coordinate of guards, as is assumed for the partitioning of the set \(M_{s_k, s_{k+1}}\) in the left column, above Figure 3, on page 367.)

Now we have a look at our Figure 13. There \((s_k, s_{k+1})\) is a convex hull edge and \(a, b, c\) denote guards. There may be also additional (shaded) guards but we rely on the existence of the three empty (shaded) cones. Then \(D, E,\) and \(F\) denote non-empty arc pieces on the pairwise arc chains (dotted lines) between these guards. During the construction the guards (even if they are removed later on) and arc pieces get their position in the chain that belongs to the convex hull edge. We derive an order on these objects from the empty cones: \(c \prec D \prec a\) and \(a \prec E \prec b\) and \(b \prec F \prec c\). These constraints on the order can only be satisfied by a loop that contains these guards and arc pieces, since every vertex has degree 2. This contradicts the assumption of the existence of an arc chain.

Remark: Because of this general argumentation, we are convinced that the task of finding the \(\Theta\)-region has to be motivated, however, rather by the directions of empty cones than by the convex hull edges. (Note that each instance of the generic example in Section 5 can be embedded in a huge box with just four hull edges.)
Mistake in the Complexity Statement. In Proposition 2 on page 370 the complexity of the $\Theta$-region is claimed to be linear for any fixed value of $\Theta$. Below we point out that this complexity bound is not justified in the most crucial step of the proof, hence leaving a gap in the proof.

The proof begins with a consideration on the boundary of the $\Theta$-region restricted to a convex hull edge $(s_k, s_{k+1})$, i.e. cones can now only enter the hull through $(s_k, s_{k+1})$. It is stated, that the boundary is the chain of arcs that corresponds to this edge and that it has linear size. Although we have proven above that these chains do not exist in general for $\Theta < \frac{\pi}{2}$, and hence we do not know anything about the complexity of a proper replacement for these chains, we assume just for the sake of argumentation that this would be true, however.

In the second half of the proof the total number of vertices in all arc chains is estimated. But what is missing in the counting argument for the complexity of the union of the $\Theta$-regions, that are restricted to a convex hull edge, are the intersections between these restricted $\Theta$-regions. (Remember that we have built our examples with quadratic complexity only on this kind of intersections in Section 5.) No characteristic of the objects is given here, why the complexity of the union should be $O(n)$ for fixed values of $\Theta$; even fatness and convexity seem not to be strong enough for this claim.

The whole proof of this Proposition seems to go much more along the line of proving that the $\Theta$-region can be described by a linear number of arcs, which we have also proven in Theorem 1, than arguing about the complexity of the $\Theta$-region itself. Without further argumentation there is just an upper bound of $O(n^2)$ on the complexity for constant angles since the $\Theta$-region is embedded in the arrangement of $O(n)$ arcs.

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