ON DIFFERENTIABILITY OF SOBOLEV FUNCTIONS WITH RESPECT TO THE SOBOLEV NORM

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Abstract. We study connections between the $W^{1,p}$-differentiability and the $L^p$-differentiability of Sobolev functions. We prove that, $W^{1,p}$-differentiability implies the $L^p$-differentiability, but the opposite implication is not valid. The notion of approximate differentiability is discussed as well. In addition, we consider the $W^{1,p}$-differentiability of Sobolev functions $\text{cap}_p$-almost everywhere.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. In the classical work [3] it was proved that functions $f : \Omega \to \mathbb{R}$ of the Sobolev space $W^{1,p}_p(\Omega)$, $p > n$, are differentiable almost everywhere in $\Omega$ with respect to the uniform norm: there exists a linear mapping $L : \mathbb{R}^n \to \mathbb{R}$ such that
\[
\lim_{z \to x} \frac{|f(z) - f(x) - L(z - x)|}{|z - x|} = 0
\]
for almost all $x \in \Omega$, see also works [5, 15]. In the case $p = n$ the differentiability of monotone functions of the Sobolev space $W^{1,n}_n(\Omega)$ was obtained in [18]. This result was extended to the case of spaces $W^{1,p}_p(\Omega)$, $n - 1 < p < \infty$, in [19].

The differentiability with respect to the $L_p$-norm was first investigated in [4, 5]. The book [16] is devoted, in particular, to a systematic study of the $L^p$-differentiability, the detailed bibliography can be found in [16]. In addition, in [5] the conception of the $L_p$-differentiability was considered and the following theorem was proved: Let $1 \leq p < \infty$ and $f \in W^{1,p}_p(\mathbb{R}^n)$, then $f$ is $L_p$-differentiable at almost every $x \in \mathbb{R}^n$ with respect to Lebesgue measure. In the work [1], the notion of $L_1$-differentiability for functions of bounded variation was discussed.

In the frameworks of Sobolev space theory, in [17, 19], the differentiability of Sobolev functions with respect to the Sobolev norms was considered. In the work [17] it was proved that for a function $f \in W^{1,p}_p(\Omega)$, the formal differential $Df(x)$, $x \in \Omega$, defined by the weak gradient $\nabla f(x)$, is the differential with respect to convergence in $W^{1,p}_p(\Omega)$ for almost every $x \in \Omega$ with respect to Lebesgue measure.

The first part of the present article is devoted to connections between the $L_p$-differentiability and the $W^{1,p}_p$-differentiability of Sobolev functions. We prove that, $W^{1,p}_p$-differentiability implies the $L_p$-differentiability, but the opposite implication is not valid. The notion of approximate differentiability is discussed as well.

The $L_p$-differentiability of Sobolev functions $\text{cap}_p$-almost everywhere was considered in [2]. The second part of the present article is devoted to the $W^{1,p}_p$-differentiability of Sobolev functions $\text{cap}_p$-almost everywhere, refining the results.

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2.1. Sobolev spaces and capacity. We prove that if \( f \in W^1_p(\Omega), 1 \leq p < \infty \), and there exists a set \( \mathcal{N} \subset \Omega \) with 
\[ \text{cap}_p(\mathcal{N}) = 0, \]
such that every \( x \in \Omega \setminus \mathcal{N} \) is an \( L_p \)-point of the weak gradient of \( f \),
then \( f \) is \( W^1_p \)-differentiable \( \text{cap}_p \)-almost everywhere (up to a set of \( p \)-capacity zero) in \( \Omega \).

As a consequence of the assertion above, we obtain a generalization of the theorem that states Sobolev functions in \( W^2_p \) are \( L_p \)-differentiable \( \text{cap}_p \)-almost everywhere, as referenced in Theorem 3.4.2 of [20]. More precisely, we have the following assertion: If \( f \in W^1_p(\Omega), 1 \leq p < \infty \), and there exists a set \( \mathcal{N} \subset \Omega \) with 
\[ \text{cap}_p(\mathcal{N}) = 0, \]
such that every \( x \in \Omega \setminus \mathcal{N} \) is an \( L_p \)-point of the weak gradient of \( f \),
then, \( f \) is \( L_p \)-differentiable \( \text{cap}_p \)-almost everywhere in \( \Omega \).

Remark that any function of the Sobolev space of the second order \( W^2_p(\Omega) \) satisfies the condition of the above assertion, but the opposite is not true.

2. Sobolev spaces and the differentiability in different topologies

2.1. Sobolev spaces and capacity. Let \( \Omega \) be an open subset of \( \mathbb{R}^n \). The Sobolev space \( W^m_p(\Omega) \), \( m \in \mathbb{N}, 1 \leq p < \infty \), is defined as the normed space of functions \( f \in L_p(\Omega) \) such that the partial derivatives of order less than or equal to \( m \) exist in the weak sense and belong to \( L_p(\Omega) \). The space is equipped with the norm

\[
\|f\|_{W^m_p(\Omega)} = \left( \sum_{|\alpha| \leq m} \left( \int_{\Omega} |D^\alpha f(x)|^p \, dx \right)^\frac{1}{p} \right)^{\frac{1}{p}} < \infty,
\]

\( D^\alpha f \) is the weak derivative of order \( \alpha \) of the function \( f \), where \( \alpha = (\alpha_1, ..., \alpha_n) \) multiindex, \( \alpha_i \in \mathbb{N} \cup \{0\}, 1 \leq i \leq n \).

Sobolev spaces are Banach spaces of equivalence classes \cite{14}. To clarify the notion of equivalence classes of Sobolev functions we use the nonlinear \( p \)-capacity associated with Sobolev spaces \cite{9, 11, 14}. Suppose \( \Omega \) is an open set in \( \mathbb{R}^n \) and \( K \subset \Omega \) is a compact set. The \( p \)-capacity of \( K \) with respect to \( \Omega \) is defined by

\[
\text{cap}_p(K; \Omega) := \inf \int_{\Omega} |\nabla u(x)|^p \, dx,
\]

where the infimum is taken over all functions \( u \in C^\infty_c(\Omega) \), \( u \geq 1 \) on \( K \), which are called admissible functions for the compact set \( K \subset \Omega \). If \( U \subset \Omega \) is an open set, we define

\[
\text{cap}_p(U; \Omega) := \text{sup} \, \text{cap}_p(K; \Omega), \; K \subset U, \; K \text{ is compact}.
\]

In the case of an arbitrary set \( E \subset \Omega \) we define

\[
\text{cap}_p(E; \Omega) := \inf \text{cap}_p(U; \Omega), \; E \subset U \subset \Omega, \; U \text{ is open}.
\]

In case of \( \Omega = \mathbb{R}^n \) we use the notation \( \text{cap}_p(E) = \text{cap}_p(E; \mathbb{R}^n) \). It is well-known that if \( \text{cap}_p(E) = 0 \), then \( |E| = 0 \) for every set \( E \subset \mathbb{R}^n \) \cite{7, 12}, where \( |E| \) denotes the \( n \)-dimensional Lebesgue measure of the set \( E \).

Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( f \in L_{1, \text{loc}}(\Omega) \). The precise representative of \( f \) is defined by

\[
f^* : \Omega \to \mathbb{R}, \quad f^*(x) := \begin{cases} \lim_{r \to 0^+} \int_{B(x,r)} f(y) \, dy, & \text{if the limit exists and belongs to } \mathbb{R}; \\ 0, & \text{otherwise.} \end{cases}
\]
The symbol \( f \) in the definition above stands for the average of the function \( f \):
\[
\int_{B(x, r)} f(y) dy = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy,
\]
where \( B(x, r) \) stands for the open ball around \( x \) with radius \( r \).

Recall that since almost every point in \( \Omega \) is a Lebesgue point with respect to Lebesgue measure for functions \( f \in L_{1, \text{loc}}(\Omega) \), then \( f(x) = f^*(x) \) for almost every point \( x \in \Omega \) with respect to Lebesgue measure. Note also, that if \( f, g \in L_{1, \text{loc}}(\Omega) \) and \( f = g \) almost everywhere in \( \Omega \), then \( f^*(x) = g^*(x) \) for every \( x \in \Omega \). If \( f \) is a continuous function, then \( f(x) = f^*(x) \) for every point \( x \in \Omega \). If \( f \in W^1_p(\Omega) \), then \( \nabla f = \nabla f^* \) almost everywhere in \( \Omega \).

The notion of \( p \)-capacity allows us to refine the concept of Sobolev functions. Let \( f \in W^1_p(\Omega) \). Then, the precise representative \( f^* \) defined by (2.2) is defined quasi-everywhere, i.e., up to a set of \( p \)-capacity zero [10] [14]. If \( f \in W^1_p(\Omega) \), \( f^* \) is called the unique quasicontinuous representation or the canonical representation of the function \( f \).

Let us recall the notion of \( L_p \)-points [17]. Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( 1 \leq p < \infty \) and \( f \in L_p, \text{loc}(\Omega) \). Then a point \( x \in \Omega \) is called an \( L_p \)-point of \( f \) if the limit
\[
f^*(x) := \lim_{r \to 0+} \int_{B(x, r)} f(z) dz\text{ exists},
\]
\( f^*(x) \in \mathbb{R} \) and
\[
\lim_{r \to 0+} \int_{B(x, r)} |f(z) - f^*(x)|^p dz = 0.
\]

Remark that by the Lebesgue differentiation theorem we get \( f^* \in L_{p, \text{loc}}(\Omega) \), whenever \( f \in L_{p, \text{loc}}(\Omega) \) for every \( 1 \leq p < \infty \).

### 2.2. The differentiability in different topologies.

Let \( \Omega \subset \mathbb{R}^n \) be an open set, and let \( f : \Omega \to \mathbb{R} \) be a function belonging to \( L_{p, \text{loc}}(\Omega) \) for \( 1 \leq p < \infty \). The function \( f \) is called \( L_p \)-differentiable at \( x \in \Omega \) (see, for example [10]) if there exists a linear mapping \( L : \mathbb{R}^n \to \mathbb{R} \) such that
\[
\lim_{r \to 0+} \int_{B(x, r)} \frac{|f(z) - f^*(x) - L(z - x)|^p}{r^p} dz = 0. \tag{2.3}
\]
This linear mapping, uniquely defined by (2.3), is called the \( L_p \)-differential of the function \( f \) at the point \( x \), denoted by \( D_p f(x) \).

Now we define the notion of approximate differentiability in accordance with [8]. Let \( \Omega \subset \mathbb{R}^n \) be an open set and let \( f : \Omega \to \mathbb{R} \) be a measurable function. We say that \( f \) is approximately differentiable at the point \( x \in \Omega \) if there exist a number \( z \in \mathbb{R} \) and a linear mapping \( L : \mathbb{R}^n \to \mathbb{R} \) such that for every \( \varepsilon > 0 \) the set
\[
A_{\varepsilon} = \{ y \in \Omega \setminus \{ x \} : D_z(y) > \varepsilon \}, \text{ where } D_z(y) := \frac{|f(y) - z - L(y - x)|}{|y - x|},
\]
has density zero at the point \( x \) with respect to the Lebesgue measure.

If \( f \) is approximately differentiable at \( x \), then \( z \) and \( L \) are uniquely determined. The point \( z \) is called the approximate limit of \( f \) at \( x \) and \( L \) is called the approximate differential of \( f \) at \( x \) and is denoted as \( D_{ap} f(x) \).

The notion of \( W^1_p \)-differentiability was introduced in [17]. Let \( 1 \leq p < \infty \), \( \Omega \subset \mathbb{R}^n \) be an open set, \( f \in W^1_{p, \text{loc}}(\Omega) \) and \( x \in \Omega \) an \( L_p \)-point of \( f \). We say that \( f \) is \( W^1_p \)-differentiable at \( x \) if there exists a linear mapping \( L : \mathbb{R}^n \to \mathbb{R} \) such that for
for every open and bounded set \( U \subset \mathbb{R}^n \)
\[
\lim_{h \to 0} \| f_{x,h} - L \|_{W^1_p(U)} = 0, \tag{2.5}
\]
where \( f_{x,h} \) is defined by
\[
f_{x,h}(z) := \frac{f^*(x + h z) - f^*(x)}{h}, \quad h \in \mathbb{R} \setminus \{0\}, z \in \Omega - x.
\]

We call \( L \) the formal differential of \( f \) at \( x \) and denote in by \( L = Df(x) \).

Remark that for each \( x \in \Omega \), the family of functions \( \{ f_{x,h} \}_{h \in \mathbb{R} \setminus \{0\}} \) is well-defined on any non-empty bounded set of \( \mathbb{R}^n \) for every \( h \) such that the value \(|h| > 0\) is sufficiently small: Since \( \Omega \) is open and \( x \in \Omega \), there exists \( r > 0 \) such that \( B(x, r) \subset \Omega \). If \( B \subset \mathbb{R}^n \) is an arbitrary non-empty bounded set, such that \( B \neq \{0\} \), then for every \( h \) such that \(|h| < r/R \), where \( R := \sup_{z \in B}|z| \), we get \( x + hB \subset B(x, r) \). Thus, the function \( f_{x,h} \) is defined on \( B \) for every \( 0 < |h| < r/R \).

3. Comparison for the differentiability in different topologies

In this section we prove that, \( W^1_p \)-differentiability in \( L_p \)-points implies the \( L_p \)-differentiability in \( L_p \)-points, but the opposite implication is not valid.

The first assertion concerns connections between \( L_p \)-differentiability and approximate differentiability.

**Theorem 3.1.** Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( 1 \leq p < \infty \) and \( f \in L_{p,\text{loc}}(\Omega) \). Suppose that \( x \in \Omega \) is an \( L_p \)-point of \( f \). Then:

1. If \( f \) is \( L_p \)-differentiable at \( x \), then it is approximately differentiable at \( x \).
2. If \( f \) is approximately differentiable at \( x \), and there exists an open set \( \Omega_0 \subset \Omega \) containing \( x \) such that the function \( y \mapsto D_x(y) \), as defined in (2.4), is bounded within \( \Omega_0 \), then \( f \) is \( L_p \)-differentiable at \( x \).

**Proof.**

(1) Let \( x \in \Omega \) be an \( L_p \)-point of \( f \) and assume that \( f \) is \( L_p \)-differentiable at \( x \). Let us define for every \( \varepsilon > 0 \)
\[
A_\varepsilon = \{ y \in \Omega \setminus \{ x \} : D_x(y) > \varepsilon \}, \quad D_x(y) := \frac{|f(y) - z - L(y - x)|}{|y - x|},
\]
where \( L \) is the \( L_p \)-differential of \( f \) at \( x \) and \( z := f^*(x) \). We prove that \( A_\varepsilon \) has density zero at \( x \) for every \( \varepsilon > 0 \). Assuming the contrary, we suppose that there exists \( \varepsilon > 0 \) such that the upper density of the set \( A_\varepsilon \) at the point \( x \) is positive, which means that
\[
\limsup_{r \to 0^+} \frac{|A_\varepsilon \cap B(x, r)|}{|B(x, r)|} > 0.
\]
Therefore, there exists a positive number \( \alpha > 0 \) and a sequence \( r_i \to 0^+ \) as \( i \to \infty \) such that
\[
\frac{|A_\varepsilon \cap B(x, r_i)|}{|B(x, r_i)|} > \alpha, \quad \forall i \in \mathbb{N}. \tag{3.1}
\]

Note that for any \( 0 < \sigma < 1 \)
\[
|A_\varepsilon \cap (B(x, r_i) \setminus B(x, \sigma r_i))| = |A_\varepsilon \cap B(x, r_i)| - |A_\varepsilon \cap B(x, \sigma r_i)|. \tag{3.2}
\]

Therefore, using (3.1) and (3.2), we get
\[
\frac{|A_\varepsilon \cap (B(x, r_i) \setminus B(x, \sigma r_i))|}{|B(x, r_i)|} > \alpha - \frac{|A_\varepsilon \cap B(x, \sigma r_i)|}{|B(x, r_i)|}, \quad \forall i \in \mathbb{N}.
\]
Since
\[ |A_x \cap B(x, \sigma r_i)| \leq \sigma^n, \quad \forall i \in \mathbb{N}, \]
we can take the number \( \sigma \) such that \( \sigma^n < \frac{\alpha}{2} \). Then
\[ \frac{|A_x \cap (B(x, r_i) \setminus B(x, \sigma r_i))|}{|B(x, r_i)|} > \frac{\alpha}{2}, \quad \forall i \in \mathbb{N}. \]
Therefore, by the Chebyshev inequality (see, for example, [7]) we get for every \( i \in N \)
\[ \frac{\alpha}{2} < \frac{|A_x \cap (B(x, r_i) \setminus B(x, \sigma r_i))|}{|B(x, r_i)|} \leq \frac{1}{(\varepsilon \sigma)^p} \int_{B(x, r_i)} \frac{|f(y) - f^*(x) - L(y - x)|^p}{|y - x|^p} dy. \]

The last inequality contradicts the assumption that \( x \) is a point of \( L_p \)-differentiability of \( f \). It proves that the set \( A_x \) has density zero at \( x \).

(2) Let \( x \in \Omega \) be an \( L_p \)-point of a function \( f \). Assume that \( f \) is approximately differentiable at \( x \). Then, there exist a number \( z \in \mathbb{R} \) and a linear mapping \( L : \mathbb{R}^n \to \mathbb{R} \) such that for every \( \varepsilon > 0 \) the set
\[ A_x = \{ y \in \Omega \setminus \{ x \} : D_x(y) > \varepsilon \}, \quad \text{where } D_x(y) := \frac{|f(y) - z - L(y - x)|}{|y - x|}, \]
has density zero at the point \( x \) with respect to the Lebesgue measure.

Then for every \( r > 0 \) such that \( B(x, r) \subset \Omega_0 \) we get
\[ (3.3) \quad \int_{B(x, r)} \frac{|f(y) - z - L(y - x)|^p}{|y - x|^p} dy \leq \int_{B(x, r)} \frac{|f(y) - z - L(y - x)|^p}{|y - x|^p} dy \]
\[ = \frac{1}{|B(x, r)|} \int_{B(x, r) \cap A_x} (D_x(y))^p dy + \frac{1}{|B(x, r)|} \int_{B(x, r) \setminus A_x} (D_x(y))^p dy \]
\[ \leq M^p |A_x \cap B(x, r)| + \varepsilon^p, \]
where the number \( M \) is a bound on \( D_x \) on the set \( \Omega_0 \). Since \( x \) is a point of approximate differentiability and \( \varepsilon > 0 \) is arbitrary, we obtain that \( x \) is a point of \( L_p \)-differentiability of \( f \). Note that by (2.2) and by (3.3), we get
\[ z = \lim_{r \to 0^+} \int_{B(x, r)} f(y) dy = f^*(x). \]
Due to the uniqueness of \( L_p \)-differential, we get that \( L \) is the \( L_p \)-differential of \( f \) at \( x \).

Recall the notion of the standard mollifier, see, for example, [10]. Let
\[ \eta : \mathbb{R}^n \to \mathbb{R}, \quad \eta(x) := \begin{cases} c_0 \exp \left( \frac{|x|^2}{\varepsilon^2} - 1 \right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}, \]
where the constant \( c_0 \) is chosen for having \( \|\eta\|_{L_1(\mathbb{R}^n)} = 1 \). For every \( \varepsilon > 0 \) we define the function
\[ \eta_\varepsilon : \mathbb{R}^n \to \mathbb{R}, \quad \eta_\varepsilon(x) := \frac{1}{\varepsilon^n} \eta \left( \frac{x}{\varepsilon} \right). \]
The family of functions \( \eta_\varepsilon \) is called the standard mollifier.

Let \( \Omega \subset \mathbb{R}^n \) be an open set. We denote \( \Omega_\varepsilon := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \varepsilon \} \). It is known (see, for example, [20]) that for a function \( f \in L_{1,\text{loc}}(\Omega) \) the convolution

\[
(f_\varepsilon(x) := f \ast \eta_\varepsilon(x) = \int_{\Omega} f(y) \eta_\varepsilon(x-y) dy,
\]

is a smooth function in \( \Omega_\varepsilon \) and \( f_\varepsilon \) converges to \( f \) almost everywhere in \( \Omega \) as \( \varepsilon \to 0^+ \); if \( f \in W^1_{p,\text{loc}}(\Omega) \), \( 1 \leq p < \infty \), then \( f_\varepsilon \) converges to \( f \) as \( \varepsilon \to 0^+ \) in the topology of \( W^1_{p,\text{loc}}(\Omega) \), which means that

\[
\lim_{\varepsilon \to 0^+} \| f - f_\varepsilon \|_{W^1_p(U)} = 0 \quad \text{for every open set} \ U \subset \subset \Omega,
\]

and \( \nabla f_\varepsilon(x) = (\nabla f \ast \eta_\varepsilon)(x), x \in \Omega_\varepsilon \).

Recall also (see, for example, [20]) that if \( f \in L^p(\Omega), 1 \leq p < \infty \), and \( U \subset \Omega \) is an open set such that \( \text{dist}(U, \mathbb{R}^n \setminus \Omega) > 0 \), then for every \( \varepsilon > 0 \) such that \( U \subset \Omega_\varepsilon \)

\[
\| f \ast \eta_\varepsilon \|_{L^p(U)} \leq \| f \|_{L^p(\Omega)}.
\]

Let us formulate the following connection between the convolution and \( L^p \)-points. We give the proof for the convenience of the readers.

**Proposition 3.2.** Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( 1 \leq p < \infty \) and \( f \in L^p_{p,\text{loc}}(\Omega) \). For every \( L^p \)-point \( w \in \Omega \) of \( f \) we have \( \lim_{\varepsilon \to 0^+} f_\varepsilon(w) = f^*(w) \).

**Proof.** By Jensen’s inequality

\[
|f_\varepsilon(w) - f^*(w)|^p = \left| \int_{B(w,\varepsilon)} (f(z) - f^*(w)) \eta_\varepsilon(w-z) dz \right|^p \\
\leq \left( \frac{1}{\varepsilon^n} \int_{B(w,\varepsilon)} |f(z) - f^*(w)| \eta \left( \frac{w-z}{\varepsilon} \right) dz \right)^p \\
\leq \|\eta\|^p_{L^\infty(\mathbb{R}^n)} \omega_n^p \left( \int_{B(w,\varepsilon)} |f(z) - f^*(w)| dz \right)^p \\
\leq \|\eta\|^p_{L^\infty(\mathbb{R}^n)} \omega_n^p \int_{B(w,\varepsilon)} |f(z) - f^*(w)|^p dz,
\]

where \( \omega_n = \|B(0,1)\| \) is the volume of the unit ball \( B(0,1) \subset \mathbb{R}^n \).

In the next assertion we prove that the points of the \( W^1_p \)-differentiability of \( f \) are \( L^p \)-points of its weak gradient \( \nabla f \).

**Theorem 3.3.** Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( 1 \leq p < \infty \) and \( f \in W^1_{p,\text{loc}}(\Omega) \). Suppose \( x \in \Omega \) an \( L^p \)-point of \( f \). Then, \( f \) is \( W^1_p \)-differentiable at \( x \) if and only if \( x \) is an \( L^p \)-point of the weak derivative \( \nabla f \). In this case \( Df(x) = (\nabla f)^*(x) \).

**Proof.** Let \( x \) be an \( L^p \)-point of the weak gradient \( \nabla f \). Therefore, for every open and bounded set \( U \subset \mathbb{R}^n \) it follows that

\[
\lim_{s \to 0} \frac{1}{s^n} \int_{x+sU} |\nabla f(z) - (\nabla f)^*(x)|^p dz = 0.
\]

During the proof we set \( v := (\nabla f)^*(x) \). Let \( U \subset \mathbb{R}^n \) be any non-empty open and bounded set. By the formula (2.0), we get for the convolution \( f_\varepsilon \) that

\[
(f_\varepsilon)_{x,t}(z) = \frac{f_\varepsilon(x + tz) - f_\varepsilon(x)}{t}, \quad t \in \mathbb{R} \setminus \{0\}, z \in \frac{\Omega - x}{t}.
\]
Note that since $f_\varepsilon$ is continuous, then $(f_\varepsilon)^* = f_\varepsilon$.

Then by Jensen’s inequality, Fubini’s theorem and the change of variables formula we get for $t$ with small enough $|t| > 0$:

\begin{align}
\int_U |(f_\varepsilon)_{x,t}(z) - v(z)|^p dz &= \int_U \left| \frac{1}{t} \int_0^t \frac{dx}{dz} f_\varepsilon(x + stz) ds - v(z) \right|^p dz \\
&= \int_U \left| \frac{1}{t} \int_0^1 \nabla f_\varepsilon(x + stz) \cdot tz ds - v(z) \right|^p dz \\
&\leq \sup_{z \in U} |z|^p \int_0^1 \int_U |\nabla f_\varepsilon(x + stz) - v(z)|^p ds dz \\
&= \sup_{z \in U} |z|^p \int_0^1 \int_U |\nabla f_\varepsilon(y) - v(z)|^p dy ds.
\end{align}

Since $f_\varepsilon$ converges to $f$ almost everywhere, $f = f^*$ almost everywhere and $x$ is an $L_p$-point of $f$, then by Proposition 3.2 for almost every $z \in U$

\begin{align}
\lim_{\varepsilon \to 0^+} (f_\varepsilon)_{x,t}(z) &= \lim_{\varepsilon \to 0^+} \frac{f_\varepsilon(x + tz) - f_\varepsilon(x)}{t} = f_{x,t}(z).
\end{align}

By Fatou’s lemma

\begin{align}
\int_U |f_{x,t}(z) - v(z)|^p dz &= \int_U \lim_{\varepsilon \to 0^+} |(f_\varepsilon)_{x,t}(z) - v(z)|^p dz \\
&\leq \liminf_{\varepsilon \to 0^+} \int_U |(f_\varepsilon)_{x,t}(z) - v(z)|^p dz.
\end{align}

Let us denote for every $t$ with small enough $|t| > 0$

\[ F_\varepsilon(s) := \frac{1}{(st)^n} \int_{x+stU} |\nabla f_\varepsilon(z) - v|^p dz, \quad s \in (0,1). \]

We prove that

\[ \sup_{s \in (0,1)} \sup_{\varepsilon \in (0,\infty)} F_\varepsilon(s) < \infty \]

for application of the dominated convergence theorem to the right-hand side of (3.7) after taking the limit as $\varepsilon \to 0^+$. 
Let $U_0 \subset \mathbb{R}^n$ be an open bounded set such that $\overline{U} \subset U_0$. By (3.5) we get for small enough $\varepsilon > 0$

\begin{equation}
(3.10) \quad \frac{1}{(st)^n} \int_{x+stU} |\nabla f_\varepsilon(z) - v|^p \, dz
\end{equation}

\[
\leq 2^{p-1} \frac{1}{(st)^n} \int_{x+stU} |\nabla f_\varepsilon(z)|^p \, dz + 2^{p-1} |v|^p |U|
\]

\[
= 2^{p-1} \frac{1}{(st)^n} \|\nabla f \ast \eta_\varepsilon\|_{L^p(U_\varepsilon + x + stU)}^p + 2^{p-1} |v|^p |U|
\]

\[
\leq 2^{p-1} \frac{1}{(st)^n} \|\nabla f\|_{L^p(U_\varepsilon + x + stU_\varepsilon)}^p + 2^{p-1} |v|^p |U|
\]

\[
= 2^{p-1} \frac{1}{(st)^n} \int_{x+stU_\varepsilon} |\nabla f(z)|^p \, dz + 2^{p-1} |v|^p |U|
\]

\[
\leq 2^{2p-2} \frac{1}{(st)^n} \int_{x+stU_\varepsilon} |\nabla f(z) - v|^p \, dz + (2^{2p-2} + 2^{p-1}) |v|^p |U_\varepsilon|.
\]

The function

\[ s \mapsto \frac{1}{(st)^n} \int_{x+stU_\varepsilon} |\nabla f(z) - v|^p \, dz \]

is bounded on $(0,1)$ because, by (3.6), there exists $\delta > 0$ such that

\[ \left| \frac{1}{\rho^n} \int_{x+\rho U_\varepsilon} |\nabla f(z) - v|^p \, dz \right| < 1, \quad \forall \rho \in (-\delta, \delta). \]

Hence, for every $-\delta < t < \delta$ and $s \in (0,1)$ we obtain

\begin{equation}
(3.11) \quad \left| \frac{1}{(st)^n} \int_{x+stU_\varepsilon} |\nabla f(z) - v|^p \, dz \right| \leq 1.
\end{equation}

By (3.10), (3.11), the dominated convergence theorem, and the convergence of $f_\varepsilon$ to $f$ in the topology of $W^{1}_{p,\text{loc}}(\Omega)$, we obtain

\begin{equation}
(3.12) \quad \lim_{\varepsilon \to 0^+} \frac{1}{(st)^n} \int_{x+stU_\varepsilon} |\nabla f_\varepsilon(y) - v|^p \, dyds
\end{equation}

\[
= \int_{0}^{1} \frac{1}{(st)^n} \lim_{\varepsilon \to 0^+} \frac{1}{(st)^n} \int_{x+stU_\varepsilon} |\nabla f_\varepsilon(y) - v|^p \, dyds
\]

\[
= \int_{0}^{1} \frac{1}{(st)^n} \int_{x+stU_\varepsilon} |\nabla f(y) - v|^p \, dyds.
\]

Thus, taking the lower limit as $\varepsilon \to 0^+$ in (3.11) and using (3.10) and (3.12), we get

\begin{equation}
(3.13) \quad \int_{U} |f_{x,t}(z) - v(z)|^p \, dz \leq \sup_{z \in U} |z|^p \int_{0}^{1} \frac{1}{(st)^n} \int_{x+stU_\varepsilon} |\nabla f(y) - v|^p \, dyds.
\end{equation}

Therefore, by the dominated convergence theorem, (3.10) and (3.12), we obtain

\begin{equation}
(3.14) \quad \lim_{t \to 0} \int_{U} |f_{x,t}(z) - v(z)|^p \, dz = 0.
\end{equation}

Next, notice that for $t$ with small enough $|t| > 0$ and almost all $z \in U$

\begin{equation}
(3.15) \quad \nabla [f_{x,t} - v](z) = \nabla f(x + tz) - v.
\end{equation}
Hence, by equation (3.15) and the change of variables formula we obtain
\[
\int_U |\nabla [f_{x,t} - v](z)|^p \, dz = \int_U |\nabla f(x + tz) - v|^p \, dz = \frac{1}{t^n} \int_{x+tU} |\nabla f(y) - v|^p \, dy.
\]
Therefore, we get by (3.15)
\[
(3.16) \quad \lim_{t \to 0} \int_U |\nabla [f_{x,t} - v](z)|^p \, dz = \lim_{t \to 0} \frac{1}{t^n} \int_{x+tU} |\nabla f(y) - v|^p \, dy = 0.
\]
By (3.14) and (3.15) we get that \( f \) is \( W^1_p \)-differentiable at \( x \), and \( Df(x) = v \).

Next, suppose that a function \( f \) is \( W^1_p \)-differentiable at \( x \). Then, for every open and bounded set \( U \subset \mathbb{R}^n \) we get
\[
(3.17) \quad 0 = \lim_{t \to 0} \int_U |\nabla [f_{x,t} - Df(x)](z)|^p \, dz = \lim_{t \to 0} \frac{1}{t^n} \int_{x+tU} |\nabla f(y) - Df(x)|^p \, dy.
\]
Multiplying both sides of (3.17) by \( 1/|B(0,1)| \) and choosing \( U = B(0,1) \), we obtain
\[
(3.18) \quad \lim_{t \to 0^+} \int_{B(x,t)} |\nabla f(y) - Df(x)|^p \, dy = 0.
\]
Thus, by (3.18) and (2.2), we get
\[
(\nabla f)^*(x) = \lim_{t \to 0^+} \int_{B(x,t)} \nabla f(y) \, dy = Df(x).
\]
Thus, \( x \) is an \( L_p \)-point of \( \nabla f \) and \( (\nabla f)^*(x) = Df(x) \). \( \square \)

In the following theorem we prove that, at \( L_p \)-points, \( W^1_p \)-differentiability implies \( L_p \)-differentiability.

**Theorem 3.4.** Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( 1 \leq p < \infty \) and \( f \in W^1_{p,loc}(\Omega) \). Let \( x \in \Omega \) be an \( L_p \)-point of \( f \). If \( f \) is \( W^1_p \)-differentiable at \( x \), then it is \( L_p \)-differentiable at \( x \) and \( D_p f(x) = Df(x) \). In particular, \( f \) is approximately differentiable at \( x \).

**Proof.** Let \( x \in \Omega \) be an \( L_p \)-point of \( f \). Assume \( f \) is \( W^1_p \)-differentiable at \( x \). It follows for every small enough \( r > 0 \)
\[
(3.19) \quad \frac{1}{r^n} \int_{B(x,r)} |f(y) - f^*(x) - Df(x)(y-x)|^p \, dy
\]
\[
= \frac{1}{r^n} \int_{B(0,1)} |f(x + rz) - f^*(x) - Df(x)(rz)|^p \, dz
\]
\[
= \frac{1}{r^n} \int_{B(0,1)} \left| \frac{f(x + rz) - f^*(x)}{r} - Df(x)(z) \right|^p \, dz.
\]
Since \( f \) is \( W^1_p \)-differentiable at \( x \), then we get by (3.19)
\[
(3.20) \quad \lim_{r \to 0^+} \frac{1}{r^n} \int_{B(x,r)} |f(y) - f^*(x) - Df(x)(y-x)|^p \, dy = 0,
\]
which means that \( f \) is \( L_p \)-differentiable at \( x \) and, by uniqueness of \( L_p \)-differential, \( D_p f(x) = Df(x) \). By Theorem 3.1 we get that \( f \) is approximately differentiable at \( x \). \( \square \)

As a consequence we have the following result on \( L_p \)-differentiability for Sobolev functions [5].
Corollary 3.5. Let \( 1 \leq p < \infty \), \( \Omega \subset \mathbb{R}^n \) be an open set, \( f \in W^{1,p}_{\text{loc}}(\Omega) \). Then, \( f \) is \( L_p \)-differentiable almost everywhere in \( \Omega \).

**Proof.** Since \( f \in W^{1,p}_{\text{loc}}(\Omega) \), we get \( \nabla f \in L^p_{\text{loc}}(\Omega, \mathbb{R}^n) \). By the Lebesgue differentiation theorem, almost every point in \( \Omega \) is an \( L_p \)-point of \( \nabla f \). By Theorem 3.3 at each such point, \( f \) is \( W^{1,p} \)-differentiable. In addition, by Theorem 3.4 it is also \( L_p \)-differentiable at such points. \( \square \)

The opposite implication of Theorem 3.4 is not true in general. This means that if \( x \) is a point of \( L_p \)-differentiability, it is not necessarily a point of \( W^{1,p}_1 \)-differentiability. Let us provide a counterexample. In the following assertion, we give a function that is differentiable (in the usual sense) at a point \( x \), but the point \( x \) is not an \( L_p \) point of its derivative. Therefore, at such a point, \( f \) is \( L_p \)-differentiable, and by Theorem 3.3 it is not \( W^{1,p}_1 \)-differentiable at such a point.

**Proposition 3.6.** Let

\[
 f : (-1, 1) \to \mathbb{R}, \quad f(x) = \begin{cases} 
 x^2 \sin \left( \frac{1}{x} \right) & x \in (-1, 1) \setminus \{0\} \\
 0 & x = 0 
\end{cases}.
\]

Then, the function \( f \) is \( L_1 \)-differentiable at 0, but 0 is not a \( W^{1,1}_1 \)-differentiability point of \( f \).

**Proof.** The function \( f \) is differentiable at every \( x \in (-1, 1) \) and

\[
 (3.21) \quad f'(x) = \begin{cases} 
 2x \sin \left( \frac{1}{x} \right) - \cos \left( \frac{1}{x} \right) & x \in (-1, 1) \setminus \{0\} \\
 0 & x = 0 
\end{cases}.
\]

Since \( f \) is continuous at 0, we have that 0 is an \( L_1 \)-point of \( f \). Additionally, as \( f' \) is bounded in \((-1,1)\), \( f \) is Lipschitz continuous on \((-1, 1)\). Therefore, \( f \in W^1((-1,1)) \). The function \( f \) is differentiable at 0, making it \( L_1 \)-differentiable at 0. However, 0 is not an \( L_1 \)-point of \( f' \), as we shall prove below. Thus, by Theorem 3.3 0 is not a \( W^{1,1}_1 \)-differentiability point of \( f \).

Let us prove that 0 is not an \( L_1 \)-point of \( f' \): Note that by the Fundamental Theorem of Calculus, we get

\[
 (f')^*(0) = \lim_{r \to 0^+} \frac{1}{2r} \int_{-r}^{r} f'(y) dy = \lim_{r \to 0^+} \frac{1}{2r} (f(r) - f(-r)) \\
 = \lim_{r \to 0^+} \frac{1}{2r} \left( 2r^2 \sin \left( \frac{1}{r} \right) \right) = 0.
\]
It follows that

\[(3.22)\quad \limsup_{r \to 0^+} \frac{1}{2r} \int_{-r}^{r} |f'(y) - (f')^*(0)| \, dy \]

\[= \limsup_{r \to 0^+} \frac{1}{2r} \int_{-r}^{r} |f'(y)| \, dy = \limsup_{r \to 0^+} \frac{1}{2r} \int_{-r}^{r} 2y \sin \left( \frac{1}{y} \right) - \cos \left( \frac{1}{y} \right) \, dy \]

\[\geq \limsup_{r \to 0^+} \left( \frac{1}{2r} \int_{-r}^{r} \left| \frac{1}{y} \right| \, dy + \frac{1}{2r} \int_{-r}^{r} -2y \sin \left( \frac{1}{y} \right) \, dy \right) \]

\[= \limsup_{r \to 0^+} \frac{1}{2r} \int_{-r}^{r} \left| \frac{1}{y} \right| \, dy + \lim_{r \to 0^+} \frac{1}{2r} \int_{-r}^{r} -2y \sin \left( \frac{1}{y} \right) \, dy, \]

whenever the last limit exists. Notice that

\[\frac{1}{2r} \int_{-r}^{r} 2y \sin \left( \frac{1}{y} \right) \, dy \leq \frac{1}{r} \int_{-r}^{r} |y| \, dy = \frac{2}{r} \int_{0}^{r} y \, dy = r, \]

so

\[(3.23)\quad \lim_{r \to 0^+} \frac{1}{2r} \int_{-r}^{r} 2y \sin \left( \frac{1}{y} \right) \, dy = 0.\]

Let us show that

\[\limsup_{r \to 0^+} \frac{1}{2r} \int_{-r}^{r} \left| \frac{1}{y} \right| \, dy > 0.\]

For every \( r > 0 \), since the function \( \cos \) is an even function, we have by change of variables formula

\[(3.24)\quad \frac{1}{2r} \int_{-r}^{r} \left| \cos \left( \frac{1}{y} \right) \right| \, dy = \frac{1}{r} \int_{0}^{r} \left| \cos \left( \frac{1}{y} \right) \right| \, dy.\]

Denote \( r_k := \frac{1}{2\pi k} \). Note that \( \left| \cos \left( \frac{1}{y} \right) \right| \geq \frac{\sqrt{2}}{2} \) for every \( y \in \left[ \frac{1}{2\pi k + \frac{1}{4}}, \frac{1}{2\pi k} \right] \) and for every \( k \in \mathbb{N} \), and the intervals \( \left[ \frac{1}{2\pi k + \frac{1}{4}}, \frac{1}{2\pi k} \right] \), \( k \in \mathbb{N} \), are pairwise disjoint. It follows that

---

\( ^1 \)Recall that if \( \{a_n\}_{n \in \mathbb{N}} \) is a converging sequence of real numbers and \( \{b_n\}_{n \in \mathbb{N}} \) is an arbitrary sequence of real numbers, then \( \limsup_{n \to \infty} (a_n + b_n) = \limsup_{n \to \infty} a_n + \lim_{n \to \infty} b_n \).
\[(3.25) \quad \frac{1}{r_k} \int_0^{r_k} \left| \cos \left( \frac{1}{y} \right) \right| \ dy \geq 2\pi k \sum_{j=k}^{\infty} \int_{\frac{j+1}{2\pi j + \frac{1}{4}}}^{\frac{j+1}{2\pi j + \frac{3}{4}}} \left| \cos \left( \frac{1}{y} \right) \right| \ dy \]
\[\geq \sqrt{2\pi k} \sum_{j=k}^{\infty} \left( \frac{1}{2\pi j} - \frac{1}{2\pi j + \frac{1}{4}} \right) = \frac{\sqrt{2}}{2} k \sum_{j=k}^{\infty} \left( \frac{1}{j} - \frac{1}{j + \frac{1}{4}} \right) \]
\[= \frac{\sqrt{2}}{2} k \sum_{j=k}^{\infty} \left( \frac{1}{8j} \right) \geq \frac{\sqrt{2}}{2} k \sum_{j=k}^{\infty} \left( \frac{1}{8} \right) = \frac{\sqrt{2}}{32} k \sum_{j=k}^{\infty} \frac{1}{j^2}.
\]

Let us prove here a technical lemma:

**Lemma 3.7.** For every \(k \in \mathbb{N}\) it follows that

\[(3.26) \quad \frac{3}{4k} \leq \sum_{j=k}^{\infty} \frac{1}{j^2} \leq \frac{2}{k}.
\]

**Proof.** Since \(\frac{1}{j^2} \leq \frac{1}{j^2 - \frac{1}{4}}\), and \(\frac{3}{4} \leq \frac{j^2 - \frac{1}{4}}{j^2} \leq 1, j \in \mathbb{N}\), then

\[(3.27) \quad \frac{3}{4} \sum_{j=k}^{\infty} \frac{1}{j^2 - \frac{1}{4}} \leq \sum_{j=k}^{\infty} \frac{1}{j^2} \leq \sum_{j=k}^{\infty} \frac{1}{j^2 - \frac{1}{4}}.
\]

It follows that

\[(3.28) \quad \sum_{j=k}^{\infty} \frac{1}{j^2 - \frac{1}{4}} = \sum_{j=k}^{\infty} \left( \frac{1}{j - \frac{1}{2}} - \frac{1}{j + \frac{1}{2}} \right)
\[= \sum_{j=k}^{\infty} \left( \frac{1}{j - \frac{1}{2}} - \frac{1}{j + \frac{1}{2}} - \frac{1}{(j+1) - \frac{1}{2}} \right) = \frac{1}{k} - \frac{1}{2}.
\]

In the last equality we used telescoping property of sums. Since \(\frac{1}{k} \leq \frac{1}{k - \frac{1}{4}} \leq \frac{2}{k}\), we get \((3.26)\) by combining \((3.27), (3.28)\). \(\square\)

Hence, we conclude by Lemma 3.7 and \((3.22), (3.23), (3.24), (3.25)\)

\[\limsup_{r \to 0^+} \frac{1}{2r} \int_{-r}^{r} |f'(y) - (f')^*(0)| \ dy \geq \frac{\sqrt{2}}{32} \frac{3}{4} > 0.
\]

Therefore, 0 is not an \(L_1\)-point of \(f'\). Thus, by Theorem 3.3 the point 0 is not a \(W^1_1\)-differentiability point of \(f\). \(\square\)

**Remark 3.8.** Notice that the last example demonstrates that differentiability at the point \(x \in \Omega\) (in the usual sense) does not necessarily imply \(W^1_p\)-differentiability at this point \(x \in \Omega\). However, continuous differentiability does imply \(W^1_p\)-differentiability.
4. Sobolev functions with refined weak gradients

In this section, we introduce the space $\text{RW}^1_p(\Omega)$ of Sobolev functions in $W^1_p(\Omega)$ with refined weak gradients, meaning that the weak gradients are $\text{cap}_p$-refined, where $\text{cap}_p$ is the $p$-capacity. We show that the space $\text{RW}^1_p(\Omega)$ lies strictly between the spaces $W^1_p(\Omega)$ and $W^2_p(\Omega)$:

$$W^2_p(\Omega) \subseteq \text{RW}^1_p(\Omega) \subseteq W^1_p(\Omega).$$

This leads to a capacity-based version of Reshetnyak’s theorem \[17\], which asserts that Sobolev functions are $W^1_p$-differentiable almost everywhere with respect to Lebesgue measure. We prove that Sobolev functions with refined gradients are $W^1_p$-differentiable almost everywhere.

We also get a slight generalization to the theorem about $L_p$-differentiability $\text{cap}_p$-almost everywhere for Sobolev functions within $W^2_p$, refer to Theorem 3.4.2 in \[20\]. We establish that this result holds for a broader class of functions, specifically those in $\text{RW}^1_p$.

We extend the notion of $W^1_p$-differentiability and introduce a notion of $W^k_p$-differentiability, $k \in \mathbb{N}$. We represent the space $\text{RW}^k_p$, where $k \in \mathbb{N}$, and prove that functions in $\text{RW}^k_p$ are $W^k_p$-differentiable $\text{cap}_p$-almost everywhere.

4.1. The space $\text{RW}^1_p$. Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $1 \leq p < \infty$. We write $f \in \text{RW}^1_p(\Omega)$ if $f \in W^1_p(\Omega)$ and the weak gradient $\nabla f$ is $\text{cap}_p$-refined, meaning that for

$$\lim_{r \to 0^+} \int_{B(x,r)} |\nabla f(z) - (\nabla f)^*(x)|^p dz = 0 \quad \text{for} \quad \text{cap}_p - \text{almost every} \quad x \in \Omega.$$  \((4.1)\)

Recall the following fine property of Sobolev functions \[7\,12\]:

**Theorem 4.1.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p < \infty$. If $f \in W^1_p(\Omega)$, then there exists a Borel set $N \subset \Omega$ such that

$$\text{cap}_p(N) = 0 \quad \text{and} \quad \lim_{r \to 0^+} \int_{B(x,r)} |f(z) - f^*(x)|^p dz = 0 \quad \forall x \in \Omega \setminus N. \quad (4.2)$$

**Remark 4.2.** Notice that functions of the space $W^2_p(\Omega)$ have $\text{cap}_p$-refined weak gradients. Indeed, let $f \in W^2_p(\Omega)$, then $\nabla f \in W^1_p(\Omega, \mathbb{R}^n)$, hence by Theorem 4.1, it follows that $\text{cap}_p - \text{almost every} \ x \in \Omega$ is an $L_p$-point of $\nabla f$, thus $f \in \text{RW}^1_p(\Omega)$.

**Example 4.3.** We provide simple examples that demonstrate that the inclusions $W^2_p(\Omega) \subset \text{RW}^1_p(\Omega)$ and $\text{RW}^1_p(\Omega) \subset W^1_p(\Omega)$ can also be strict.

1. We give an example for function $f \in \text{RW}^1_p(\Omega) \setminus W^2_p(\Omega)$. We choose $\Omega = B(0,1) \subset \mathbb{R}^n$, $n > 1$, $p = 1$ and let us look at the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by the rule $f(x) = |x|$. Since $f$ is a Lipschitz function, then $f \in W^1_1(B(0,1))$. The weak gradient of $f$ is given by $\nabla f(x) = \frac{x}{|x|}$, which is not in $W^1_p(B(0,1), \mathbb{R}^n)$. Therefore, $f \not\in W^2_p(B(0,1))$.

Since every point $x \neq 0$ is a continuous point of $\nabla f$, then it is a Lebesgue point, so

$$(\nabla f)^*(x) = \lim_{r \to 0^+} \int_{B(x,r)} \nabla f(z) dz = \frac{x}{|x|}, \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$
Therefore
\[
\lim_{r \to 0^+} \int_{B(x,r)} \frac{z}{|z|} - \frac{x}{|x|} \, dz = 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{cap}_1(\{0\}) = 0.
\]

Thus, \( f \in RW^1_p(B(0,1)) \). We use the assumption \( n > 1 \) to get \( \text{cap}_1(\{0\}) = 0 \) from \( \mathcal{H}^{n-1}(\{0\}) = 0 \) using inequality \( \text{cap}_p(E) \leq C(n,p)\mathcal{H}^{n-p}(E) \), where \( E \subset \mathbb{R}^n \), \( C(n,p) \) is a constant dependent on \( n, p \) only.

(2) To construct a function \( f \in W^1_p(\Omega) \setminus RW^1_p(\Omega) \) we choose \( \Omega = B(0,1) \subset \mathbb{R} \), \( p > 1 \) and the same function as above \( f : \mathbb{R} \to \mathbb{R}, f(x) = |x| \). As above
\[
\lim_{r \to 0^+} \int_{B(x,r)} \frac{z}{|z|} - \frac{x}{|x|} \, dz = 0, \quad \forall x \in \mathbb{R} \setminus \{0\},
\]
and
\[
(\nabla f)^*(0) = \lim_{r \to 0^+} \int_{B(0,r)} \frac{z}{|z|} \, dz = 0, \quad \lim_{r \to 0^+} \int_{B(0,r)} \left| \frac{z}{|z|} - 0 \right| \, dz = 1 \neq 0.
\]

Since \( p > 1 \) we have \( \text{cap}_p(\{0\}) > 0 \), because the (outer) measure \( \text{cap}_p \) is an atomic measure in the case where the parameter \( p \) is strictly bigger than the dimension \( n \) (for proof see for example [12]). Thus \( f \notin RW^1_p(B(0,1)) \).

In fact, \( f \in RW^1_p(\Omega) \) for \( p > n \) if and only if \( f \in W^1_p(\Omega) \) and every point \( x \in \Omega \) is an \( L_\infty \)-point of \( \nabla f \).

By using standard methods one can get:

**Proposition 4.4.** Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( 1 \leq p < \infty \). The set \( RW^1_p(\Omega) \) is a vector subspace of \( W^1_p(\Omega) \). Moreover, the space \( RW^1_p(\Omega) \cap L_\infty(\Omega) \) is an algebra with respect to the pointwise product.

4.2. Fine differentiability of functions in \( RW^1_p \). Now we proceed to prove the capacitary version of Reshetnyak's theorem [17].

**Theorem 4.5.** Let \( 1 \leq p < \infty, \Omega \subset \mathbb{R}^n \) be an open set and let \( f \in RW^1_p(\Omega) \).

Then \( f \) is \( W^1_p \)-differentiable \( \text{cap}_p \)-almost everywhere in \( \Omega \). In particular, \( f \) is \( L_\infty \)-differentiable \( \text{cap}_p \)-almost everywhere in \( \Omega \).

**Proof.** Since \( f \in RW^1_p(\Omega) \), then there exists a set \( E \subset \Omega \) such that \( \text{cap}_p(E) = 0 \) and for every \( x \in \Omega \setminus E \)
\[
(4.3) \quad \lim_{r \to 0^+} \int_{B(x,r)} |f(y) - f^*(x)|^p \, dy = 0 \quad \text{and} \quad \lim_{r \to 0^+} \int_{B(x,r)} |\nabla f(y) - (\nabla f)^*(x)|^p \, dy = 0.
\]

By Theorem 4.3 we get that \( f \) is \( W^1_p \)-differentiable at every point \( x \in \Omega \setminus E \). \( \square \)

By Remark 4.2 and Theorem 4.5 we get the following corollary:

**Corollary 4.6.** Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( 1 \leq p < \infty \) and \( f \in W^2_p(\Omega) \). Then, \( f \) is \( W^1_p \)-differentiable \( \text{cap}_p \)-almost everywhere in \( \Omega \). In particular, \( f \) is \( L_\infty \)-differentiable \( \text{cap}_p \)-almost everywhere in \( \Omega \).
4.3. The space $RW^k_p$. We say that $\alpha \in \mathbb{R}^n$ is a multi-index if $\alpha = (\alpha_1, \ldots, \alpha_n)$, where for every $1 \leq i \leq n$, $\alpha_i \in \mathbb{N} \cup \{0\}$. Recall the operations $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$ and for $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

**Definition 4.7.** Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $1 \leq p < \infty$ and $k \in \mathbb{N}$. We define the space $RW^k_p(\Omega)$ as a set of functions $f \in W^k_p(\Omega)$ which have cap$_p$-refined weak derivatives of order $k$: for every multi-index $\alpha$ such that $|\alpha| = k$

$$\lim_{r \to 0^+} \int_{B(x, r)} |D^\alpha f(z) - (D^\alpha f)(x)|^p dz = 0 \quad \text{for} \quad \text{cap}_p - \text{almost every} \quad x \in \Omega.$$ 

**Remark 4.8.** The space $RW^k_p(\Omega)$ is a vector subspaces of $W^k_p(\Omega)$.

**Remark 4.9.** Note that for a function $f \in RW^k_p(\Omega)$, we get by Theorem 4.1 that almost every point with respect to cap$_p$ is an $L_p-$point of $D^\alpha f$ for every multi-index $|\alpha| \leq k$.

Recall Taylor formula with remainder of integral form for functions $f$ of the class $C^k$: If $\Omega \subset \mathbb{R}^n$ is an open set and $f \in C^k(\Omega)$, then for every $x \in \Omega$ there exists $r > 0$ such that $B(x, r) \subset \Omega$ and for every $y \in B(x, r)$ the following formula holds:

$$(4.4) \quad f(y) = \sum_{|\alpha| \leq k-1} \frac{D^\alpha f(x)}{\alpha!} (y-x)^\alpha + \sum_{|\alpha| = k} \frac{k}{\alpha!} (y-x)^\alpha \int_0^1 (1-t)^{k-1} D^\alpha f(x+t(y-x)) dt.$$ 

Writing $y = x + hz$ for $|h| < r, z \in B(0, 1)$, we get

$$(4.5) \quad f(x + hz) = \sum_{|\alpha| \leq k-1} \frac{D^\alpha f(x)}{\alpha!} (hz)^\alpha + h k \sum_{|\alpha| = k} \frac{k}{\alpha!} z^\alpha \int_0^1 (1-t)^{k-1} D^\alpha f(x + thz) dt.$$ 

The Taylor polynomial of order $k$ of $f$ around the point $x$ is given by

$$\mathcal{P}^k_{f,x} : \mathbb{R}^n \to \mathbb{R}, \quad \mathcal{P}^k_{f,x}(y) := \sum_{|\alpha| \leq k} \frac{D^\alpha f(x)}{\alpha!} (y-x)^\alpha,$$

and substituting $y = x + hz$ we get

$$(4.6) \quad \mathcal{P}^k_{f,x}(x + hz) = \sum_{|\alpha| \leq k} \frac{D^\alpha f(x)}{\alpha!} (hz)^\alpha.$$ 

The remainder of order $k$ of $f$ around $x$ is given by

$$(4.7) \quad \mathcal{R}^k_{f,x} : \Omega \to \mathbb{R}, \quad \mathcal{R}^k_{f,x}(y) := f(y) - \mathcal{P}^k_{f,x}(y).$$ 

We get by (4.5), (4.6) and (4.7)

$$(4.8) \quad \mathcal{R}^k_{f,x}(x + hz) = h^k \sum_{|\alpha| = k} \frac{k}{\alpha!} z^\alpha \int_0^1 (1-t)^{k-1} D^\alpha f(x + thz) dt - \sum_{|\alpha| = k} \frac{D^\alpha f(x)}{\alpha!} (hz)^\alpha$$

$$= h^k \sum_{|\alpha| = k} \frac{k}{\alpha!} z^\alpha \int_0^1 (1-t)^{k-1} D^\alpha f(x + thz) dt - \sum_{|\alpha| = k} \frac{D^\alpha f(x)}{\alpha!} (hz)^\alpha \left( k \int_0^1 (1-t)^{k-1} dt \right)$$

$$= k h^k \sum_{|\alpha| = k} \frac{z^\alpha}{\alpha!} \int_0^1 (1-t)^{k-1} (D^\alpha f(x + thz) - D^\alpha f(x)) dt, \quad |h| < r, z \in B(0, 1).$$
Now we give definitions of the Taylor polynomial and the remainder for Sobolev functions $f \in W^k_p(\Omega)$ in terms of the precise representative:

**Definition 4.10.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $k \in \mathbb{N}$. Let $f \in W^k_p(\Omega)$, and let $x \in \Omega$ be an $L_1$-point of all the weak derivatives of $f$ up to order $k$. We define the Taylor polynomial of order $k$ of the function $f$ at the point $x$ to be the following function:

$$P^k_{f,x} : \mathbb{R}^n \to \mathbb{R}, \quad P^k_{f,x}(z) := \sum_{|\alpha| \leq k} \frac{(D^\alpha f)^*(x)}{\alpha!} (z - x)^\alpha.$$  

We define the remainder of order $k$ of the function $f$ at the point $x$ to be the following function:

$$R^k_{f,x} : \Omega \to \mathbb{R}, \quad R^k_{f,x}(z) := f^*(z) - P^k_{f,x}(z).$$

We define the remainder family by

$$(4.9) \quad \{ R^k_{f,x,h} \}_{h \in \mathbb{R} \setminus \{0\}}, \quad R^k_{f,x,h}(z) := R^k_{f,x}(x + hz), \quad \forall z \in \frac{\Omega - x}{h}.$$  

**Remark 4.11.** The function $z \mapsto R^k_{f,x,h}(z)$ is defined on $\frac{\Omega - x}{h}$ and, in particular, the family of functions $\{ R^k_{f,x,h} \}_{h \in \mathbb{R} \setminus \{0\}}$ is defined on any bounded set $B \subset \mathbb{R}^n$ for every small enough $|h|$.

**Definition 4.12.** Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \leq p < \infty$, $k \in \mathbb{N}$ and $f \in W^k_p(\Omega)$. Let $x \in \Omega$ be an $L_p$-point of all the weak derivatives, $D^\alpha f$, for every multi-index $|\alpha| \leq k$. We say that $f$ is $W^k_p$-differentiable at $x$ if for every open and bounded set $V \subset \mathbb{R}^n$ we get

$$(4.10) \quad \lim_{h \to 0} \left\| \frac{1}{h^k} R^k_{f,x,h} \right\|_{W^k_p(V)} = 0,$$

where $R^k_{f,x,h}$ is the remainder family defined in (4.9). More explicitly,

$$(4.11) \quad \lim_{h \to 0} \left\| \frac{1}{h^k} \left[ f(x + h(\cdot)) - \sum_{|\alpha| \leq k} \frac{(D^\alpha f)^*(x)}{\alpha!} (h(\cdot))^\alpha \right] \right\|_{W^k_p(V)} = 0,$$

where in $(\cdot)$ we put the norm variable.

**Remark 4.13.** Recall that the Sobolev norm $\| f \|_{W^k_p(U)}$ is equivalent to the norm $\| f \|_{L_p(U)} + \sum_{|\alpha| = k} \| D^\alpha f \|_{L_p(U)}$ for every open and bounded set $U \subset \mathbb{R}^n$ with Lipschitz boundary. This equivalence means that there exist constants $c, C$ such that for every $f \in W^k_p(U)$

$$c \| f \|_{W^k_p(U)} \leq \| f \|_{L_p(U)} + \sum_{|\alpha| = k} \| D^\alpha f \|_{L_p(U)} \leq C \| f \|_{W^k_p(U)}.$$  

In particular, this equivalence holds for open balls. A proof of this equivalence can be found in [9].

**Lemma 4.14.** Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \leq p < \infty$, $k \in \mathbb{N}$ and $f \in W^k_p(\Omega)$. Suppose $x \in \Omega$ is a point such that for every multi-index $|\alpha| = k$

$$(4.12) \quad \lim_{r \to 0^+} \int_{B(x,r)} |D^\alpha f(y) - (D^\alpha f)^*(x)|^p dy = 0,$$
and for every multi-index $|\alpha| \leq k - 1$

(4.13) \[
\lim_{\epsilon \to 0^+} D^\alpha f_\epsilon(x) = (D^\alpha f)^+(x), \quad f_\epsilon = f * \eta_\epsilon.
\]

Then, $f$ is $W^k_p$-differentiable at $x$.

**Remark 4.15.** Note that, by Proposition 3.2, we can assume in Lemma 4.14 that $x$ is an $L_p$-point of the weak derivatives $D^\alpha f$ for every $|\alpha| \leq k$ to obtain equations (4.12) and (4.13).

**Proof.** Using (4.8) for the smooth function $f_\epsilon$ we get:

\[
\frac{1}{h^k} R^k_{f_\epsilon,x,h}(z) = k \sum_{|\alpha|=k} \frac{z^\alpha}{\alpha!} \int_0^1 (1 - t)^{k-1} (D^\alpha f_\epsilon(x + th) - D^\alpha f_\epsilon(x)) \, dt.
\]

Therefore,

(4.14) \[
\left| \frac{1}{h^k} R^k_{f_\epsilon,x,h}(z) \right|^p = \left| k \sum_{|\alpha|=k} \frac{z^\alpha}{\alpha!} \int_0^1 (1 - t)^{k-1} (D^\alpha f_\epsilon(x + th) - D^\alpha f_\epsilon(x)) \, dt \right|^p
\]

\[
\leq k^p |z|^p C(k,p) \sum_{|\alpha|=k} \left( \frac{1}{\alpha!} \right)^p \int_0^1 (1 - t)^{(k-1)p} \left| D^\alpha f_\epsilon(x + th) - D^\alpha f_\epsilon(x) \right|^p \, dt
\]

\[
\leq k^p |z|^p C(k,p) \sum_{|\alpha|=k} \left( \frac{1}{\alpha!} \right)^p \int_0^1 |D^\alpha f_\epsilon(x + th) - D^\alpha f_\epsilon(x)|^p \, dt,
\]

where $C(k,p)$ is a constant dependent on $k,p$ only.

Let $U \subset \mathbb{R}^n$ be an open ball. Then, by Fubini’s theorem, the change of variables formula and inequality (4.14) we get

(4.15) \[
\int_U \left| \frac{1}{h^k} R^k_{f_\epsilon,x,h}(z) \right|^p \, dz
\]

\[
\leq k^p C(k,p) \sup_{w \in U} |w|^p \sum_{|\alpha|=k} \left( \frac{1}{\alpha!} \right)^p \int_0^1 \left( \int_U |D^\alpha f_\epsilon(x + th) - D^\alpha f_\epsilon(x)|^p \, dz \right) \, dt
\]

\[
= k^p C(k,p) \sup_{w \in U} |w|^p \sum_{|\alpha|=k} \left( \frac{1}{\alpha!} \right)^p \int_0^1 \left( \frac{1}{(th)^n} \int_{x + thU} |D^\alpha f_\epsilon(y) - D^\alpha f_\epsilon(x)|^p \, dy \right) \, dt.
\]

Note that for almost every $z \in U$ we get

(4.16) \[
\lim_{\epsilon \to 0^+} R^k_{f_\epsilon,x,h}(z) = \lim_{\epsilon \to 0^+} (f_\epsilon(x + h) - P^k_{f_\epsilon,x}(x + h))
\]

\[
= f^*(x + h) - P^k_{f,x}(x + h) = R^k_{f,x,h}(z).
\]

Indeed, since $f_\epsilon$ converges to $f$ almost everywhere in $\Omega$ and $f = f^*$ almost everywhere, then $\lim_{\epsilon \to 0^+} f_\epsilon(x + h) = f^*(x + h)$ for almost every $z \in U$; by the
Thus, by equation (4.23) and the change of variables formula we get
\begin{equation}
\lim_{\varepsilon \to 0^+} D^\alpha f_\varepsilon(x) = (D^\alpha f)^*(x).
\end{equation}

Thus, taking into account the assumption (4.13), we obtain for every \( z \in \mathbb{R}^n \)
\begin{equation}
\lim_{\varepsilon \to 0^+} \mathcal{P}^k_{f_\varepsilon, x}(x + hz) = \lim_{\varepsilon \to 0^+} \frac{D^\alpha f_\varepsilon(x)}{\alpha!} (hz)^\alpha
= \sum_{|\alpha| \leq k} \frac{(D^\alpha f)^*(x)}{\alpha!} (hz)^\alpha = \mathcal{P}^k_{f, x}(x + hz).
\end{equation}

Thus, by (4.16) and Fatou's lemma
\begin{equation}
\int_{U} \left| \frac{1}{h^n} R^k_{f, x, h}(z) \right|^p dz \leq \liminf_{\varepsilon \to 0^+} \int_{U} \left| \frac{1}{h^n} R^k_{f_\varepsilon, x, h}(z) \right|^p dz.
\end{equation}

For every multi-index \( \alpha \) such that \( |\alpha| = k \) we get by the dominated convergence theorem, the convergence of \( f_\varepsilon \) to \( f \) in the topology of \( W^k_{p, \text{loc}}(\Omega) \) and (4.17)
\begin{equation}
\lim_{\varepsilon \to 0^+} \int_0^1 \left( \frac{1}{(th)^n} \int_{x + thU} |D^\alpha f_\varepsilon(y) - D^\alpha f_\varepsilon(x)|^p dy \right) dt
= \int_0^1 \left( \frac{1}{(th)^n} \int_{x + thU} |D^\alpha f(y) - (D^\alpha f)^*(x)|^p dy \right) dt.
\end{equation}

Therefore, by taking the lower limit as \( \varepsilon \to 0^+ \) in the inequality (4.16) and using (4.19), (4.20), we obtain
\begin{equation}
\int_{U} \left| \frac{1}{h^n} R^k_{f, x, h}(z) \right|^p dz \leq k^p C(k, p) \sup_{w \in U} |w|^p \sum_{|\alpha| = k} \left( \frac{1}{\alpha!} \right)^p \int_0^1 \left( \frac{1}{(th)^n} \int_{x + thU} |D^\alpha f(y) - (D^\alpha f)^*(x)|^p dy \right) dt.
\end{equation}

By dominated convergence theorem, the assumption (4.12), and (4.21), we obtain
\begin{equation}
\lim_{h \to 0} \int_{U} \left| \frac{1}{h^n} R^k_{f, x, h}(z) \right|^p dz = 0.
\end{equation}

Next, let \( \alpha \) be a multi-index such that \( |\alpha| = k \). Then, for almost every \( z \in U \)
\begin{equation}
D^\alpha \left( \frac{1}{h^n} R^k_{f, x, h}(z) \right) (z) = D^\alpha f(x + hz) - (D^\alpha f)^*(x).
\end{equation}

Thus, by equation (4.23) and the change of variables formula we get
\begin{equation}
\int_{U} \left| D^\alpha \left( \frac{1}{h^n} R^k_{f, x, h}(z) \right) \right|^p dz = \int_{U} |D^\alpha f(x + hz) - (D^\alpha f)^*(x)|^p dz
= \frac{1}{h^n} \int_{x + hU} |D^\alpha f(y) - (D^\alpha f)^*(x)|^p dy.
\end{equation}

\footnote{Which means that \( \lim_{\varepsilon \to 0^+} \| f - f_\varepsilon \|_{W^k_p(U)} = 0 \) for every open set \( U \subset \subset \Omega \).}
Taking the limit as $h \to 0$ on both sides of the equation (4.24) and using assumption (4.12) we get
\begin{equation}
\lim_{h \to 0} \int_U \left| D^\alpha \left( \frac{1}{h^k} R^k_{f,x,h} \right)(z) \right|^p \, dz = 0.
\end{equation}

Now, let $V \subset \mathbb{R}^n$ be any open and bounded set. Let $U$ be an open ball such that $V \subset U$. Using Remark 4.13, there exists a constant $C$ such that
\begin{equation}
\int_U \left| D^\alpha \left( \frac{1}{h^k} R^k_{f,x,h} \right)(z) \right| \, dz \leq C \left( \int_U \left| \frac{1}{h^k} R^k_{f,x,h} \right|_{L^p(U)} + \sum_{|\alpha| = k} \left| D^\alpha \left( \frac{1}{h^k} R^k_{f,x,h} \right) \right|_{L^p(U)} \right).
\end{equation}

Taking the limit as $h \to 0$ in inequality (4.26) and using (4.22), (4.25), we obtain
\begin{equation}
\lim_{h \to 0} \left| \frac{1}{h^k} R^k_{f,x,h} \right|_{W^k_p(U)} = 0.
\end{equation}

The following theorem is capacitory version of Reshetnyk's theorem [17]:

**Theorem 4.16.** Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \leq p < \infty$, $k \in \mathbb{N}$ and $f \in RW^k_p(\Omega)$. Then, $f$ is $W^k_p-$differentiable at $\text{cap}_p-$almost every $x \in \Omega$.

**Proof.** By the assumption that $f \in RW^k_p(\Omega)$ and Remark 4.9 there exists $E \subset \Omega$ such that $\text{cap}_p(E) = 0$ and for every $x \in \Omega \setminus E$ and multi-index $|\alpha| \leq k$ we get
\begin{equation}
\lim_{r \to 0^+} \int_{B(x,r)} \left| D^\alpha f(y) - (D^\alpha f)^*(x) \right|^p \, dy = 0.
\end{equation}

By Proposition 3.2 and the fact that $D^\alpha f_\varepsilon = (D^\alpha f) * \eta_\varepsilon = (D^\alpha f)_\varepsilon$ we also know that for every $x \in \Omega \setminus E$ and every multi-index $|\alpha| \leq k$
\begin{equation}
\lim_{\varepsilon \to 0^+} D^\alpha f_\varepsilon(x) = (D^\alpha f)^*(x).
\end{equation}

By Lemma 4.14 each $x \in \Omega \setminus E$ is a point of $W^k_p-$differentiability of $f$. □

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