Axisymmetric non-abelian BPS monopoles from $G_2$ metrics

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Abstract

Exact $SU(2) \times U(1)$ self-gravitating BPS global monopoles in four dimensions are constructed by dimensional reduction of eight dimensional metrics with $G_2$ holonomy asymptotic to cones over $S^3 \times S^3$. The solutions carry two topological charges in an interesting way. They are generically axially but not spherically symmetric. This last fact is related to the isometries and asymptotic topology of the $G_2$ metrics. It is further shown that some $G_2$ metrics known numerically reduce to supersymmetric cosmic strings.
1 Introduction

Self-gravitating non-abelian solitons [1] for a review) are natural generalisations of flat space solitons. From the original Bartnik-McKinnon solutions [2] onwards, interesting behaviour has been found in static Einstein-Yang-Mills systems. For example, non-abelian black holes can violate many no-hair and uniqueness theorems [1]. However, not many exact self-gravitating non-abelian static solutions are known. One such solution [3, 4], of $\mathcal{N} = 4$ gauged supergravity, has been used recently to construct a supergravity dual to large $N$, $\mathcal{N} = 1$ super Yang-Mills [5].

Recent work on manifolds with $G_2$ holonomy [6, 7, 8, 9, 10, 11, 12, 13, 14] provides an easy way to obtain supersymmetric solitons by dimensional reduction. Manifolds of $G_2$ holonomy are seven dimensional manifolds that admit a parallel spinor. They are therefore important in supersymmetric compactifications of eleven dimensional supergravity or M-theory. Here we will trivially extend the Riemannian $G_2$ manifolds to eight dimensions by adding a time direction and then dimensionally reduce to obtain supersymmetric monopole solutions of a four dimensional theory.

Four families of noncompact $G_2$ manifolds asymptotic to cones over $S^3 \times S^3$ are known, denoted $\mathbb{B}_7, \mathbb{C}_7, \overline{\mathbb{C}}_7, \mathbb{D}_7$ in [13]. The metrics all have an isometry group containing $SU(2) \times SU(2) \times U(1)$. We will dimensionally reduce on $SU(2) \times U(1)$ contained in the $SU(2) \times SU(2)$. This will result in static four dimensional manifolds with an $SU(2)/U(1) = S^2$ factor and $U(1)$ isometry, corresponding to axially symmetric monopole or cosmic string solutions. There will also be $SU(2) \times U(1)$ gauge fields and scalars. The Bogomol'nyi equations have effectively already been solved in constructing the $G_2$ metric and we will see that the four dimensional solutions are automatically BPS. This method allows the construction of exact supersymmetric solutions that would be hard to guess directly in four dimensions. The correspondence between special holonomy metrics in higher dimensions and BPS solutions has been used recently in the other direction in [15] in which $SU(2)$ instantons on $S^4$ were shown to give rise to Spin(7) metrics on the chiral spinor bundle of $S^4$.

The dimensionally reduced Lagrangian density in four dimensions will be [16]

$$
\mathcal{L} = -R - \frac{1}{4} e^{\phi} F_{\mu\nu}^{ab} F_{\mu\nu}^{ab} + \frac{1}{4} g^{\mu\nu} \phi_{\mu} \phi_{\nu} \Phi^{ab} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} e^{-\phi} \left[ -4 \Phi^{ii} + \epsilon_{ijk} \epsilon_{lmn} \Phi_{ij} \Phi^{jm} \Phi^{kn} \right],
$$

(1)
where we have written $e^{2\phi} \equiv \det \Phi$ to emphasise the dilatonic behaviour of the determinant. The Higgs fields $\Phi_{ab}$ are in the second symmetric power of the adjoint representation of the gauge group $SU(2) \times U(1)$, transforming as (21) below. This is somewhat uncommon in the context of monopoles but arises naturally in dimensional reduction [16]. The Lagrangian has standard Einstein-Yang-Mills-Higgs(-dilaton) terms with a potential that is unbounded below. Here and throughout, indices $a, b, \ldots$ run from 1 to 4 and indices $i, j, \ldots$ run from 1 to 3. The metric convention is $(-, +, +, +)$.

The terms in the Lagrangian (1) come from dimensional reduction of the Einstein-Hilbert action in eight dimensions. The full supersymmetric theory will have fermionic fields and also more bosonic fields coming from an eight dimensional supergravity (e.g. [17]). These other fields are set to zero in the solutions discussed here.

Section 2 reviews the relevant results on $G_2$ metrics. Section 3 is the dimensional reduction from eight to four dimensions. Section 4 discusses the metric, gauge fields and scalars. It is seen that the solutions correspond to global monopoles as they are asymptotically conical. They are generically not spherically symmetric. This result is discussed in the context of previous work on non-spherically symmetric black holes [15, 19, 20, 21] and is related to recent work on the asymptotic topology of $G_2$ metrics [6, 11, 14]. It is shown that although defining magnetic charges by integration of field strengths over spheres at infinity is problematic, nontrivial topological charges may be associated with the scalar fields. Section 5 considers some aspects of four dimensional solutions corresponding to more general $G_2$ metrics than considered in the previous sections. Solutions corresponding to cosmic strings are found. Section 6 is the conclusion.

2 $G_2$ metrics asymptotic to cones over $S^3 \times S^3$

Three families of complete nonsingular seven dimensional metrics with $G_2$ holonomy are known, based on generalisations of [6, 7]. They are asymptotic to cones over $\mathbb{C}P^3$, $SU(3)/T^2$ and $S^3 \times S^3$, and hence noncompact. These topological spaces belong to a restricted set of possibilities for cohomogeneity one $G_2$ metrics [22]. The last of these cases has isometries appropriate for dimensional reduction to a self-gravitating non-abelian monopole in four dimensions.
A general ansatz for the $G_2$ metric with nine radial functions was introduced in [8].

$$ds_7^2 = dr^2 + a_i(r)^2(\tilde{\sigma}_i + g_i(r)\sigma_i)^2 + b_i(r)^2\sigma_i^2,$$

where $\sigma_i$ and $\tilde{\sigma}_i$ are the left invariant one forms on each copy of $SU(2) = S^3$. That is,

$$\sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi \quad \tilde{\sigma}_1 = \cos \tilde{\psi} d\tilde{\theta} + \sin \tilde{\psi} \sin \tilde{\theta} d\tilde{\phi},$$
$$\sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi \quad \tilde{\sigma}_2 = -\sin \tilde{\psi} d\tilde{\theta} + \cos \tilde{\psi} \sin \tilde{\theta} d\tilde{\phi},$$
$$\sigma_3 = d\psi + \cos \theta d\phi \quad \tilde{\sigma}_3 = \tilde{d}\psi + \cos \tilde{\theta} d\tilde{\phi}.$$

(3)

The ranges for the coordinates are $0 \leq \theta, \tilde{\theta} \leq \pi$, $0 \leq \phi, \tilde{\phi} < 2\pi$ and $0 \leq \psi, \tilde{\psi} \leq 4\pi$. The metric (2) generically has isometry group $SU(2) \times SU(2)$ corresponding to left multiplication of $SU(2)$ on each of the $S^3$s. The condition for $G_2$ holonomy becomes a set of nine first order equations for $a_i, b_i, g_i$. The general solution to these coupled nonlinear equations is not known. This is one reason to consider consistent truncations of the nine function ansatz by setting various of the radial functions to be equal. Another reason is that an extra $U(1)$ isometry is needed to give the M-theory solution a Kaluza-Klein interpretation in terms of type IIA string theory.

Various truncated solutions involving six radial functions, providing the desired $U(1)$ isometry, have been studied and a unified description has recently been given [12, 13, 14]. Four families of solutions are known numerically, denoted $\mathbb{B}_7, \mathbb{C}_7, \tilde{\mathbb{C}}_7, \mathbb{D}_7$. A generic member of any family is asymptotically locally conical (ALC) as opposed to asymptotically conical (AC), meaning that there is an $S^1$ that stabilises as $r \to \infty$. This will be the orbit of the $U(1)$ isometry. All four families of metrics have a limiting case in which they become AC. The families have different behaviours at the origin. Thus $\mathbb{B}_7$ and $\mathbb{D}_7$ have an $S^3$ bolt whilst $\mathbb{C}_7$ and $\tilde{\mathbb{C}}_7$ have a $T^{1,1} = (S^3 \times S^3)/S^1$ bolt. A bolt is a subspace that remains of finite size in a degenerate orbit at the origin, the principal orbits here are $S^3 \times S^3$. For the cases with an $S^3$ bolt, the AC limit corresponds to the original $G_2$ metric of [3, 4] and for this metric the $U(1)$ isometry is enhanced to a third $SU(2)$, as will be discussed below. From an M-theory perspective, the most interesting result is a unified treatment of the deformed - corresponding to $\mathbb{B}_7$ - and resolved - corresponding to $\mathbb{D}_7$ - conifolds in type IIA string theory [12, 13, 14].

We will concentrate on the few cases in which a closed form solution is known. Other cases will be considered in §5. This begins with an ansatz for the metric with six radial
functions studied in [8, 10],

\[ ds^2 = dr^2 + a_i(r)^2(\tilde{\sigma}_i - \sigma_i)^2 + b_i(r)^2(\tilde{\sigma}_i + \sigma_i)^2. \] (4)

In [9] it was shown that for the case of a collapsing \( S^3 \) at the origin, i.e. the \( B_7 \) family in the notation of the previous paragraph, the only regular solutions of the six-function equations are also solutions of a reduced set of four-function equations, with the metric ansatz now written as

\[ ds^2 = \frac{dr^2}{c(r)^2} + a(r)^2 \left[ (\tilde{\sigma}_1 - \sigma_1)^2 + (\tilde{\sigma}_2 - \sigma_2)^2 \right] + b(r)^2 \left[ (\tilde{\sigma}_1 + \sigma_1)^2 + (\tilde{\sigma}_2 + \sigma_2)^2 \right] + c(r)^2(\tilde{\sigma}_3 + \sigma_3)^2 + d(r)^2(\tilde{\sigma}_3 - \sigma_3)^2. \] (5)

This metric has an \( \tilde{SU}(2) \times SU(2) \times U(1) \times \mathbb{Z}_2 \) symmetry, corresponding to the symmetries of an M-theory lift of \( N \) D6-branes wrapping the \( S^3 \) of the deformed conifold geometry.

An exact solution of the four-function equations was found to be [10]

\[ a(r) = \frac{\sqrt{(r - r_0)(r + 3r_0)}}{\sqrt{8r_0}}, \quad c(r) = \frac{\sqrt{2r_0(r^2 - 9r_0^2)}}{\sqrt{3(r^2 - r_0^2)}}, \]

\[ b(r) = \frac{\sqrt{(r + r_0)(r - 3r_0)}}{\sqrt{8r_0}}, \quad d(r) = \frac{r}{\sqrt{6r_0}}, \] (6)

where \( r_0 \) is a scale parameter, present for any Ricci-flat metric. The range of the radial coordinate here is \( 3r_0 \leq r < \infty \), so it will generally be more convenient to work with the shifted coordinate \( \tilde{r} = r - 3r_0 \). The corresponding metric is ALC because \( c(r) \) remains finite at infinity. It was shown [10, 8] that this solution extends to a two parameter family of solutions, \( B_7 \), that are not known explicitly. A limiting case of this family is the AC solution which is known exactly\(^1\)

\[ a(r) = d(r) = \sqrt{r}, \]

\[ b(r) = c(r) = \frac{r}{3} \sqrt{1 - \left( \frac{r_0}{r} \right)^{3/2}}, \] (7)

where again \( r_0 \) is a scale parameter and \( r_0 \leq r < \infty \), so it will be convenient to use the shifted coordinate \( \tilde{r} = r - r_0 \). The singular conifold is obtained in the limit \( r_0 \to 0 \). The AC metric has an enhanced isometry group \( SU(2)^3 \times \mathbb{Z}_2 \). We will see below that the fact that the third term in the isometry group for the generic ALC metric is \( U(1) \) and not \( SU(2) \) will mean that the global monopole in four dimensions is axially but not spherically

\(^1\)The AC metric is more commonly expressed in terms of the radial variable \( \rho = \sqrt{r} \).
symmetric. Finally, note that in the AC case, the $\mathbb{Z}_2$ symmetry can be interpreted as a spontaneously broken “triality” symmetry that is important in the physics of M-theory on the $G_2$ manifold [23].

3 From the $G_2$ metric to the global monopole

First, trivially extend the $G_2$ metric of (5) to an eight dimensional Lorentzian manifold

$$ds^2_{1,7} = -dt^2 + ds^2_7$$

This will be a solution to the eight dimensional Einstein vacuum equations. We want to dimensionally reduce on an internal group manifold $G = SU(2) \times U(1)$ to get non-abelian gauge fields for $G$, scalars transforming in an adjoint representation of $G$, and a four dimensional Lorentzian metric. Part of the isometry group of the metric (6) is $\tilde{SU}(2) \times SU(2)$, by action of $SU(2)$ on the two sets of left invariant one forms (3). We will take the $\tilde{SU}(2)$ to be part of the internal group manifold. Then observe [24] that there are three commuting $U(1)$ Killing vectors of the metric (6), which can be taken to be $\frac{\partial}{\partial \phi}$, $\frac{\partial}{\partial \tilde{\phi}}$, and $\frac{\partial}{\partial \psi} + \frac{\partial}{\partial \tilde{\psi}}$. The last of these cannot be simultaneously reduced with $\tilde{SU}(2)$ whilst the first two are related by the $\mathbb{Z}_2$ symmetry of the metric. So we further reduce on the $U(1)$ generated by $\frac{\partial}{\partial \psi}$. This leaves a remaining $SU(2)/U(1) = S^2$, so we expect there to be an $S^2$ factor in the reduced metric, at least topologically. Further, we make the identification $\psi \sim \psi + 2\pi$, to half the range of $\psi$. This is a symmetry of the metric (6) without fixed points - the bolt $S^3$ is of finite size at the origin - on the manifold and so will not introduce orbifold singularities.

The eight dimensional metric had an unbroken supersymmetry, due to the special holonomy, and the reduction process commutes with the supersymmetry transformations. We can see this as follows. The isometry group acts on the parallel spinor to give another parallel spinor. By considering the supersymmetry transformations, we see that the transformed parallel spinor will be the same as the original parallel spinor if and only if the isometry transformation commutes with supersymmetry. But it is a fact that $G_2$ metrics admit only one parallel spinor [23]. Therefore supersymmetry commutes with the isometry group and therefore the dimensional reduction does not ruin supersymmetry. In other words the lower dimensional solution will have unbroken supersymmetries, in fact one supercharge, and will be a BPS soliton. It should thus be stable. The Bogomol'nyi

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equations for gravitating systems are the first order consistency equations for the existence of parallel spinors on the background \[1\]. To explicitly find the corresponding generalised BPS bound it is likely that the arguments of \[26, 27\] will need to be modified because the spacetime in the present case is not asymptotically flat.

To dimensionally reduce, we want to rewrite the metric \((5)\) in the usual form for Scherk-Schwarz reductions \[16, 28\]

\[
ds_{1,7}^2 = (\det \Phi)^{-1/2} ds_{1,3}^2 + \Phi_{ab}[\lambda^a + A^a][\lambda^b + A^b],
\]

where the left invariant forms for the internal \(SU(2) \times U(1)\) are \(\lambda^i = \tilde{\sigma}_i\) and \(\lambda^4 = d\phi\). Note that \(\Phi, A, ds_{1,3}^2\) are functions of \((t, r, \theta, \psi)\) whilst dependence on the internal coordinates \((\phi, \tilde{\theta}, \tilde{\phi}, \tilde{\psi})\) is restricted to the \(\lambda^a\).

By comparing \((5)\) with \((8)\) one can read off the scalar fields

\[
\Phi_{ab} = \begin{pmatrix}
  a^2 + b^2 & 0 & 0 & (b^2 - a^2) \sin \psi \sin \theta \\
  0 & a^2 + b^2 & 0 & (b^2 - a^2) \cos \psi \sin \theta \\
  0 & 0 & c^2 + d^2 & (c^2 - d^2) \cos \theta \\
  (b^2 - a^2) \sin \psi \sin \theta & (b^2 - a^2) \cos \psi \sin \theta & (c^2 - d^2) \cos \theta & (a^2 + b^2) \sin^2 \theta + (c^2 + d^2) \cos^2 \theta
\end{pmatrix}_{ab},
\]

with determinant

\[
det \Phi = 4(a^2 + b^2)^2 c^2 d^2 \left[(\alpha(\bar{r}) - 1) \sin^2 \theta + 1\right],
\]

and the gauge fields are

\[
A^1 = -\frac{4(b^2 - a^2)(a^2 + b^2)c^2 d^2}{\det \Phi} \left[\sin \psi \sin \theta \cos \theta d\psi - \left[(\alpha(\bar{r}) - 1) \sin^2 \theta + 1\right] \cos \psi d\theta\right],
\]

\[
A^2 = -\frac{4(b^2 - a^2)(a^2 + b^2)c^2 d^2}{\det \Phi} \left[\cos \psi \sin \theta \cos \theta d\psi + \left[(\alpha(\bar{r}) - 1) \sin^2 \theta + 1\right] \sin \psi d\theta\right],
\]

\[
A^3 = \frac{4(a^2 + b^2)(c^2 - d^2)a^2 b^2}{\det \Phi} \sin^2 \theta d\psi,
\]

\[
A^4 = \frac{4(a^2 + b^2)^2 c^2 d^2}{\det \Phi} \cos \theta d\psi,
\]

where

\[
\alpha(\bar{r}) = \frac{a^2 b^2 (c^2 + d^2)}{c^2 d^2 (a^2 + b^2)}.
\]

And the four dimensional metric - note that in \((3)\) we have already rescaled to be in the Einstein frame - is

\[
ds_{1,3}^2 = \det \Phi(\bar{r}, \theta)^{1/2} \left(-dt^2 + \frac{d\bar{r}^2}{c^2} + R^2(\bar{r})d\Omega^2_{1,3}\right),
\]

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where

$$R^2(\bar{r}) = \frac{4a^2b^2}{(a^2 + b^2)},$$  \hspace{1cm} (15)$$

and \(d\Omega^2_\bar{r}\) is a metric on the sphere at radius \(\bar{r}\),

$$d\Omega^2_\bar{r} = d\theta^2 + \frac{\sin^2 \theta}{(\alpha(\bar{r}) - 1) \sin^2 \theta + 1} d\psi^2. \hspace{1cm} (16)$$

It is already clear that the special AC metric \([3]\) will result in a simpler solution because in this case \(\alpha(\bar{r}) = 1\) and much of the angular dependence will vanish. For this family of solutions, \(\alpha(\bar{r}) = 1\) is the condition for spherical symmetry.

4 Discussion of metric, scalars and gauge fields

4.1 Special case: the metric

Consider first the AC case \([3]\) in which \(a(\bar{r}) = d(\bar{r})\) and \(b(\bar{r}) = c(\bar{r})\). The metric has the expected \(S^2\) factor, with coordinates \(0 \leq \theta \leq \pi\) and \(0 \leq \psi < 2\pi\). Because \(\alpha(\bar{r}) = 1\) in this special case, the metric on the sphere is the usual round metric. Furthermore, \(\det \Phi = 4(a^2 + b^2)a^2b^2\), with no \((\theta, \psi)\) dependence, and therefore the metric is spherically symmetric.

As \(\bar{r} \to 0\), the metric is, up to an overall constant and with \(\rho = 4\bar{r}^{3/4}/3\),

$$ds^2_{1,3} \sim -\frac{1}{2} \left(\frac{3\rho}{4}\right)^{2/3} dt^2 + d\rho^2 + \frac{9}{16} \rho^2(d\theta^2 + \sin^2 \theta d\psi^2). \hspace{1cm} (17)$$

There is a conical singularity at the origin, hidden as a point of infinite redshift. An infinite redshift at the origin is seen in other contexts such as \([29]\) pg. 683, which is the metric describing a self-similar star cluster. Interestingly, the star cluster metric is asymptotically conical, which also turns out to be the case here because as \(\bar{r} \to \infty\), the metric is, letting \(\rho = 2\bar{r}^{3/2}/3\) and up to an overall constant,

$$ds^2_{1,3} \sim -\frac{1}{3} \left(\frac{3\rho}{2}\right)^{4/3} dt^2 + d\rho^2 + \frac{3}{4} \rho^2(d\theta^2 + \sin^2 \theta d\psi^2). \hspace{1cm} (18)$$

This metric is asymptotically conical (in the three dimensional sense now) with a deficit solid angle. Asymptotically conical metrics are characteristic of global monopoles \([30]\). In the manifestly asymptotically conical coordinate system one must have \(T_{00} \sim \frac{1}{\rho^2}\), where \(\rho\) is the radial coordinate in \([18]\). Thus the monopole has an infinite positive energy.
The singular conifold in seven dimensions is obtained as the limit \( r_0 \to 0 \). Near the origin in this limit, the metric in four dimensions will just be \( (18) \). So the four dimensional metric will also have a conical singularity at the origin in this limit.

### 4.2 Generic case: the metric

Consider now the generic ALC case, concentrating on the closed form four-function solution \( (3) \). Again, we have the expected \( S^2 \) factor. However, the metric on the sphere is not the round metric. Thus the solution is not spherically symmetric. We have a well defined metric on \( S^2 \), because as \( \theta \to 0 \) then \( d\Omega_2^2 \) goes as \( (1 + O(\theta^2))d\theta^2 + (\theta^2 + O(\theta^3))d\psi^2 \) for all \( \tilde{r} \). In \( (14) \) note that \( \alpha(\tilde{r}) \to 1 \) as \( \tilde{r} \to 0 \) and \( \alpha(\tilde{r}) \to \infty \) as \( \tilde{r} \to \infty \), this implies that the \( S^2 \) is round near the origin and becomes increasingly stretched and cylindrical in the \( z \) direction as we move out radially.

As \( \tilde{r} \to 0 \), the metric takes the following form up to a constant with \( \rho = 4\tilde{r}^{3/4}/3 \)

\[
ds^2 \sim -\frac{1}{2} \left(\frac{3\rho}{4}\right)^{2/3} dt^2 + d\rho^2 + \frac{9}{16} \rho^2 (d\theta^2 + \sin^2 \theta d\psi^2).
\] (19)

The metric is the same as the special case \( (17) \), a fact we will see directly in §5 below. Again there is a conical singularity and point of infinite redshift at the origin. As \( \tilde{r} \to \infty \), the metric is up to constant and letting \( \rho = \tilde{r}^3/3 \)

\[
ds^2 \sim \sin \theta \left(- (3\rho)^{4/3} dt^2 + \frac{3}{2r_0} \left[ d\rho^2 + \frac{9\rho^2}{6} d\Omega_2^2 \right]\right).
\] (20)

This metric is not quite asymptotically conical as \( d\Omega_2^2 \) has a dependence on \( \rho \) \( (13) \). The \( S^2 \) is increasingly cylindrical as we move out. This fact and the overall \( \sin \theta \) term means that there is not a straightforward way of defining the energy of the solution. It is in the same family, \( \mathbb{B}_7 \), as the special solution we found before, and could be called by analogy a global monopole.

The most interesting feature of the metric is the lack of spherical symmetry. As mentioned in §2 above, this can be understood from the symmetries of the eight dimensional metric. We reduced on a \( U(1) \subset SU(2) \times U(1) \), where the embedding is entirely into the first term of the direct product. The isometry group of the reduced metric will be the normaliser of this \( U(1) \) subgroup modulo the \( U(1) \) itself. This is because the remaining symmetry must commute with the symmetry we are reducing, otherwise it would not have a well-defined action on the reduced manifold. We quotient out the \( U(1) \), which
trivially normalises itself, because it has been quotiented out of the metric in the reduction. It is immediately seen that the reduced normaliser is just the second term of the direct product, i.e. $U(1)$. This is the axial symmetry. In the special case above we had started with $SU(2) \times SU(2)$ in the original metric, the reduced normaliser was then $SU(2) \simeq SO(3)$ and we obtained spherical symmetry.

Lack of spherical symmetry has been discussed before in the context of hairy black holes with Yang-Mills-Higgs matter. It was shown in [18] that a magnetically charged Reissner-Nordström black hole embedded in a theory with additional massive charged vector fields is unstable under perturbations in these vector fields if the horizon radius is less than the radius of the magnetic monopole core. Physically this is the fact that production of charged vector particles is energetically favourable in a sufficiently strong magnetic field because these particles carry magnetic moments that can be aligned to partially shield the magnetic field [19].

In [20], perturbative static solutions away from the Reissner-Nordström solution were found with nonzero massive vector fields. As the Reissner-Nordström metric is spherically symmetric with magnetic charge $n$, these vector fields can be expanded in monopole vector harmonics. The total angular momentum of these harmonics cannot be zero unless $n = 1$, due to a contribution to the angular momentum of magnitude $eg = n$ directed along the line from the monopole to the charged particle. This implies that if $n > 1$ then the solution cannot be spherically symmetric. Recently [21], exact solutions for these axially symmetric black holes have been found numerically and it was shown that they should be classically stable. In the present situation, the massive vector fields are due to spontaneous symmetry breaking as we will see below. We have an exact analytic solution for an axially symmetric metric with Yang-Mills and Higgs fields.

Another result closely related to the present situation is that spherically symmetric global monopoles of scalar fields have been shown to be at best marginally stable under certain axially symmetric perturbations [31, 32].

To summarise, for the family of metrics considered here we have that $AC \leftrightarrow$ spherical symmetry and $ALC \leftrightarrow$ axial symmetry. A result on $G_2$ metrics reviewed in §2 was that such metrics asymptotic to cones over $S^3 \times S^3$ were generically $ALC$. A result

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2We will see in §5 below that this correspondence only holds for the metrics with an $S^3$ bolt, $B_7$ and $D_7$. The metrics with a $T^{1,1}$ bolt do not have enhanced isometry in the $AC$ case.
reviewed in the preceding paragraphs is that spherically symmetric (global) monopoles with charged massive vector fields may not be stable under certain axial perturbations. This suggests that one might be able to understand the preference for ALC metrics by lifting results about instabilities of spherically symmetric monopole configurations to the higher dimensional metrics.

4.3 Generic case: the scalars and gauge fields

Does the solution carry topological charges? This will depend firstly upon how much gauge symmetry is broken by the scalar fields. The scalars are in the second symmetric power of the adjoint representation of the gauge group so they transform as

$$\delta_\xi \Phi_{ab} = f_{acd} \xi_c \Phi_{db} + f_{bcd} \xi_c \Phi_{ad},$$  \hspace{1cm} (21)

where $f_{abc}$ are the structure constants. In our case $G = SU(2) \times U(1)$ so $f_{ijk} = \epsilon_{ijk}$ and all other components are zero. Symmetries are preserved if $\delta_\xi \Phi_{ab} = 0$, which then allows construction of a gauge invariant field strength $\xi^a F^a$.

The situation is subtle, because although scalar fields transforming in adjoint representations are usually interpreted as Higgs fields associated with symmetry breaking, here they couple to the gauge fields and metric in such a way that $\det \Phi$ behaves rather like a dilaton coupling $e^{2\phi}$. So one needs to disentangle ‘dilaton’ and ‘Higgs’ type behaviour of these fields. The scalar potential, in the Einstein frame Lagrangian \((16)\), is

$$U(\Phi) = \frac{1}{4} \left( \frac{1}{(\det \Phi)^{1/2}} \right) \left[ -4\Phi_{ij} + \epsilon_{ijk} \epsilon_{lmn} \Phi_{il} \Phi_{jm} \Phi_{kn} \right],$$  \hspace{1cm} (22)

which is unbounded below (the internal space was not Ricci-flat) and without critical points and so it is at first unclear how to define the vacua. This is characteristic of dilaton potentials. Correspondingly, the scalar fields \((10)\) diverge at infinity. However, if we normalise the scalar fields to make them all finite at infinity by dividing by $\bar{r}^2$ then we can consider them to belong to a vacuum moduli space corresponding to minimising the ‘Higgs’ part of the potential.

Inserting \((10)\) as $\bar{r} \to \infty$ divided by $\frac{1}{\bar{r}^2}$ into \((21)\) one finds that the $SU(2)$ symmetry is broken at infinity to a $U(1)$, with symmetry generator $\xi = (0, 0, \epsilon, 0)$. This depends crucially on the fact that in this normalisation $\Phi_{14} \sim \Phi_{24} \to 0$ as $r \to \infty$ because $a^2 - b^2 \to \infty$ slower than $a^2, b^2$ or $d^2$. Further, the remaining $U(1)$, generated by $\xi' = (0, 0, 0, \epsilon)$, is unbroken throughout all of space. Thus there are potentialy two magnetic charges.
A naive attempt to define the magnetic charge fails. As $\bar{r} \to \infty$ then $F^4 \sim \frac{1}{\bar{r}^4}$, and similarly the gauge invariant field strength of the $U(1) \subset SU(2)$ goes as $F \equiv F^a \xi^a \sim \frac{1}{\bar{r}^4}$. Therefore we cannot form a magnetic charge by integrating over the sphere at spatial infinity. This is reminiscent of the situation for gravitational sphalerons [2, 34] which are unstable. However, the probable stability of the present solution means that either there is a topological charge with the usual relationship between the magnetic integral at infinity and the charge not holding or the situation is similar to the Chamseddine-Volkov solution, which doesn’t have any Higgs fields and hence no topological charge. A good indication that the former possibility is the case here is that the usual asymptotic relationship relating the gauge and scalar fields for monopoles, $D_\mu \xi^a = 0$, does not hold for this solution.

Consider first the $U(1)$ not coming from the $SU(2)$. Because the symmetry is unbroken throughout space, we can consider the field near the origin $\bar{r} \to 0$ and find a potential, in cartesian coordinates,

$$A^4 \sim \frac{z}{\sqrt{x^2 + y^2 + z^2}} \frac{(x dy - y dx)}{x^2 + y^2}. \quad (23)$$

This is clearly the potential corresponding to a Dirac magnetic monopole with charge $m = 2$. And this charge is topological in nature and so is the charge of this $U(1)$.

For the other $U(1)$ we need to look at the homotopy class of the map from $S^2_\infty$ to the moduli space of Higgs vacua which is $G/H$, where $H$ is the unbroken subgroup of the gauge group $G$. Here this is $SU(2)/U(1) = S^2$. This can be topologically nontrivial because for $G$ simply connected $\pi_2(G/H) \cong \pi_1(H) \cong \mathbb{Z}$. We can find the degree of the map $S^2 \to S^2$ in two ways. For the first method, decompose the matrix of scalar fields into various $SU(2)$ representations

$$\frac{1}{r^2} \Phi = \begin{pmatrix} \mathbf{A} & \mathbf{v} \\ \mathbf{v}^t & s \end{pmatrix}, \quad (24)$$

where $\mathbf{A}$ is a 3 by 3 matrix, $\mathbf{v}$ is a 3 component vector and $s$ is a scalar. The matrix can be visualised as an ellipsoid, defined by its three eigenvalues and the vector gives an oriented 3

The Chamseddine-Volkov solution also does not, in fact, have a gauge invariant magnetic charge defined by an integral at infinity. Its stability will presumably follow along the lines of [33].
direction. So from (10)

\[
A = \begin{pmatrix}
    k_1 & 0 & 0 \\
    0 & k_1 & 0 \\
    0 & 0 & k_2
\end{pmatrix},
\]

(25)

and

\[
v = \left(-\frac{1}{\bar{r}}k_3 \sin \theta \cos \psi, -\frac{1}{\bar{r}}k_3 \sin \theta \sin \psi, -k_2 \cos \theta\right).
\]

(26)

In these expressions, \(k_1, k_2, k_3\) are positive constants. We see that the ellipsoid \(A\) breaks the \(SU(2) \simeq SO(3)\) symmetry to rotations about one axis, i.e. \(U(1) \cong SO(2)\). Then the vector \(v\) will completely break the symmetry, unless it is pointing along the axis of symmetry of the ellipsoid. And this is precisely what happens as \(\bar{r} \to \infty\) and the first two components vanish! This gives a geometric understanding of the partial symmetry breaking at infinity found above. The vacua are thus defined by an aligned ellipsoid and vector. And hence just by the direction of the vector (the normalisation is unimportant).

Thus the map \(S^2 \to S^2\) is

\[
(\theta, \psi) \mapsto \lim_{\bar{r} \to \infty} \frac{1}{\sqrt{\frac{k_1^2}{\bar{r}} \sin^2 \theta + k_2^2 \cos^2 \theta}} \left(-\frac{1}{\bar{r}}k_3 \sin \theta \cos \psi, -\frac{1}{\bar{r}}k_3 \sin \theta \sin \psi, -k_2 \cos \theta\right) = (0, 0, \pm 1),
\]

(27)

with the positive value for \(\theta > \frac{\pi}{2}\) and negative for \(\theta < \frac{\pi}{2}\). At \(\theta = \frac{\pi}{2}\) the vector is zero, so the direction is determined by the ellipsoid, which is ambiguous up to a \(\mathbb{Z}_2\) flip in direction. This map has degree 1. This is easiest seen by considering \(\bar{r}\) as defining a homotopy of maps with \(\bar{r}\) going from \(\bar{r} = k_3/k_2\), which is just the identity map from the sphere to the sphere with degree one, to \(\bar{r} \to \infty\) which is the map we are interested in. The degree is homotopy invariant so the map in (27) has degree one.

Alternatively, we can construct the moduli space of vacua explicitly by acting with \(G = SU(2)\) on a particular vacuum element that we know from our solution

\[
\Psi_0(0, 0) = \begin{pmatrix}
    k_1 & 0 & 0 & 0 \\
    0 & k_1 & 0 & 0 \\
    0 & 0 & k_2 & -k_2 \\
    0 & 0 & -k_2 & k_2
\end{pmatrix},
\]

(28)
where $k_1, k_2$ are fixed positive constants, and without loss of generality we are considering the point $\theta = 0$. Varying $\theta$ must keep us within the space of vacua, after a suitable normalisation. Strictly the action is of $SU(2) \times U(1)$ but we have already shown that the $U(1)$ leaves these fields invariant. The most general nontrivial finite transformation is by a group element $g = e^{-\alpha T_1} e^{-\beta T_2}$. Where $\{ T_a \}$ are the generators of the adjoint representation of $SU(2) \times U(1)$. Under this transformation, (28) becomes

$$\Psi_0(\alpha, \beta) = k_1 \begin{pmatrix} \cos^2 \beta + \sin^2 \alpha \sin^2 \beta & \cos \alpha \sin \alpha \sin \beta & - \cos^2 \alpha \cos \beta \sin \beta & 0 \\ \cos \alpha \sin \alpha \sin \beta & \cos^2 \alpha & \cos \alpha \sin \alpha \cos \beta & 0 \\ - \cos^2 \alpha \cos \beta \sin \beta & \cos \alpha \sin \alpha \cos \beta & \sin^2 \beta + \sin^2 \alpha \cos^2 \beta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + k_2 \begin{pmatrix} \cos^2 \alpha \sin^2 \beta & - \cos \alpha \sin \alpha \sin \beta & \cos^2 \alpha \cos \beta \sin \beta & - \cos \alpha \sin \beta \\ - \cos \alpha \sin \alpha \sin \beta & \sin^2 \beta & - \cos \alpha \sin \alpha \cos \beta & \sin \alpha \\ \cos^2 \alpha \cos \beta \sin \beta & - \cos \alpha \sin \alpha \cos \beta & \cos^2 \alpha \cos^2 \beta & - \cos \alpha \cos \beta \\ - \cos \alpha \sin \beta & \sin \alpha & - \cos \alpha \cos \beta & 1 \end{pmatrix}.$$  

This gives us the full moduli space of vacua, $S^2(\alpha, \beta)$, with $0 \leq \alpha \leq \pi, 0 \leq \beta < 2\pi$. Now we need to find the degree of the map $S^2_{\infty} \rightarrow S^2(\alpha, \beta)$ given by $\alpha(\theta, \psi), \beta(\theta, \psi)$ such that

$$\lim_{\bar{r} \rightarrow \infty} \frac{1}{\bar{r}^2} \Phi(\bar{r}, \theta, \psi) = \Psi_0(\alpha(\theta, \psi), \beta(\theta, \psi)).$$ (30)

Note that there is in fact no dependence on $\psi$ in the case under consideration. In this equation, the positive constants $k_i$ will in general be different in $\Phi$ and $\Psi_0$. The solution to (30) is given by

$$\alpha = 0 \text{ if } \theta < \frac{\pi}{2}, \quad \alpha = \pi \text{ if } \theta > \frac{\pi}{2}, \quad \beta = 0.$$ (31)

Which is exactly what we found in (27) above.

So the enhanced gauge symmetry at infinity, in which the $SU(2)$ is not completely broken, does in fact result in a topological charge. This is clearly related to the possibility of defining a gauge invariant magnetic charge, even though this vanishes.

### 4.4 Special case: the scalars and gauge fields

The special spherically symmetric case similarly has two topological charges. Furthermore, one can also define corresponding charges by integrals at spatial infinity.
Consider first the $U(1)$ not in the $SU(2)$. The gauge field is just $A^4 = \cos \theta d\psi$ which is exactly the potential for a Dirac monopole with magnetic moment $m = 2$. Because this is now true as $\bar{r} \to \infty$, the topological charge arises in the usual fashion and coincides with the integral of $F^4 = dA^4 \sim \frac{1}{\bar{r}}$ over the sphere at spatial infinity.

Consider now the $U(1) \subset SU(2)$. With the notation of (24), one has for the scalar fields,

$$A = (a^2 + b^2)I_3,$$

$$v^t = (b^2 - a^2)(\sin \psi \sin \theta, \cos \psi \sin \theta, \cos \theta).$$

(32)

In the geometrical language of the previous subsection, the ellipsoid is in fact a sphere and therefore the spontaneous symmetry breaking is due solely to the vector $v$. The generator of the unbroken $U(1)$ is clearly $\xi = \epsilon(\sin \psi \sin \theta, \cos \psi \sin \theta, \cos \theta, 0)$. Because the ellipsoid does not contribute to the symmetry breaking, the gauge symmetry is everywhere broken to $U(1)$. The corresponding gauge invariant field strength is

$$F \equiv \xi^a F^a = \xi^a (dA - A \wedge A)^a = -3 \frac{4\bar{r}^3 - r_0^3}{(4\bar{r}^{3/2} - r_0^{3/2})^2} d\theta \wedge \sin \theta d\psi.$$

(33)

This can be integrated to give a finite magnetic charge

$$Q_B = \int_{S^2_{\infty}} F = -3\pi.$$ 

(34)

The existence of a corresponding topological charge is seen as the unit winding number of the identity map $S^2 \to S^2$, exactly as for the usual Yang-Mills-Higgs monopole.

5 Monopoles and cosmic strings from other $G_2$ metrics

So far we have only considered reductions of the $B_7$ family of $G_2$ manifolds, and only the two cases which have a known closed form solution for the radial functions. However, Taylor series expansions about the origin are known for all the families of metrics discussed in §2 [9, 11, 12, 13, 14] and these can be used to examine the behaviour of the reduced four dimensional metrics near the origin. Further, the asymptotic behaviour of the radial functions can be found from their first order equations and this allows us to study the asymptotics of the four dimensional metrics. All of these seven dimensional metrics have a $U(1)$ isometry in the generic ALC case and therefore will have axial symmetry. If they
did not have the \( U(1) \) isometry, the reduced metric would not even be axially symmetric. We will see now that the cases with an \( S^3 \) bolt (collapsing \( S^3 \)) are global monopoles and the cases with a \( T^{1,1} \) bolt (collapsing \( S^1 \)) are cosmic strings. These are all solutions of the same four dimensional theory (1).

5.1 \( \mathbb{B}_7 \) and \( \mathbb{C}_7 \)

The metrics for the \( \mathbb{B}_7 \) and \( \mathbb{C}_7 \) families with a \( U(1) \) symmetry both have the form

\[
\begin{align*}
\text{ds}_7^2 &= dr^2 + a_1(r)^2 \left[ (\tilde{\sigma}_1 - \sigma_1)^2 + (\tilde{\sigma}_2 - \sigma_2)^2 \right] \\
&+ b_1(r)^2 \left[ (\tilde{\sigma}_1 + \sigma_1)^2 + (\tilde{\sigma}_2 + \sigma_2)^2 \right] + a_3(r)^2(\tilde{\sigma}_3 - \sigma_3)^2 + b_3(r)^2(\tilde{\sigma}_3 + \sigma_3)^2. \quad (35)
\end{align*}
\]

The Taylor expansions of the radial functions about the origin, to the order that we need them, are for \( \mathbb{B}_7 \) (36):

\[
\begin{align*}
a_1(r) &\sim a_3(r) \sim 1 + \frac{1}{16} r^2, \\
b_1(r) &\sim b_3(r) \sim -\frac{1}{4} r,
\end{align*}
\]

where a trivial scale factor has been fixed to one. We see that an \( S^3 \) collapses at the origin. For \( \mathbb{C}_7 \) the Taylor expansions are (37):

\[
\begin{align*}
a_1(r) &\sim 1 - \frac{\mathcal{q}}{8} r + \frac{(16 - 3\mathcal{q}^2)r^2}{128}, \quad a_3(r) \sim -r, \\
b_1(r) &\sim 1 + \frac{\mathcal{q}}{8} r + \frac{(16 - 3\mathcal{q}^2)r^2}{128}, \quad b_3(r) \sim q + \frac{\mathcal{q}^3 r^2}{16},
\end{align*}
\]

where a trivial scale factor has again been fixed to one and \( q \) is a constant. Regularity requires \( |q| \leq q_0 = 0.91 \cdot \cdot \cdot \) with \( |q| = q_0 \) corresponding to the AC solution. We see that now an \( S^1 \) collapses at the origin.

The dimensional reduction is now done as in §3. We are interested here in the behaviour of the spatial metric near the origin, so we will not consider the time component. For the \( \mathbb{B}_7 \) case one obtains up to a constant, after putting \( \rho = 2r^{3/2}/3 \),

\[
\begin{align*}
\text{ds}_3^2 &\sim d\rho^2 + \frac{9\rho^2}{16} (d\theta^2 + \sin^2 \theta d\psi^2). \quad (38)
\end{align*}
\]

This is just the metric at the origin that we found in §4. Note that the \( r \) here is a different radial coordinate to that used above. For \( \mathbb{C}_7 \) one obtains up to a constant

\[
\begin{align*}
\text{ds}_3^2 &\sim \sin \theta (dr^2 + 2d\theta^2 + 4r^2 d\psi^2). \quad (39)
\end{align*}
\]
The metric has become cylindrical, with axial not spherical symmetry at the origin. Further, there is a surplus plane angle suggesting that the solution corresponds to a cosmic string. This can be understood as the fact that in the $G_2$ metric only an $S^1$ collapses at the origin. Note that this solution remains axially not spherically symmetric for all allowed values of $q$. In particular the AC limit does not restore spherical symmetry.

The asymptotic behaviour as $r \to \infty$ of the radial functions can be found from the first order equations they satisfy. From the equations in \[9, 11\] one has for both $B_7$ and $C_7$ that in the ALC case

$$
a_1(r)^2 \sim \frac{r^2}{12}, \quad a_3(r)^2 \sim \frac{r^2}{9},
$$

$$
b_1(r)^2 \sim \frac{r^2}{12}, \quad b_3(r)^2 \sim k^2,
$$

where $k$ is a constant. Doing the dimensional reduction and setting $\rho = r^3/3$ the metric up to a constant is

$$
ds_3^2 \sim \sin \theta (dp^2 + \frac{9\rho^2}{6}d\Omega_6^2).
$$

This is in agreement with what we found before in (20). It is interesting that the monopole and the string have the same asymptotic behaviour in the ALC case. This is possible because the sphere becomes increasingly cylindrical asymptotically, as was commented in the previous section. The asymptotic behaviour for the AC case may be found similarly.

5.2 $\tilde{C}_7$ and $D_7$

The metrics for the $\tilde{C}_7$ and $D_7$ families with a $U(1)$ symmetry both have the form

$$
\begin{align*}
\frac{ds_i^2}{a(r)^2} &= dr^2 + a(r)^2 \left[ (\tilde{\sigma}_1 + g(r)\sigma_1)^2 + (\tilde{\sigma}_2 + g(r)\sigma_2)^2 \right] + b(r)^2(\sigma_1^2 + \sigma_2^2) + c(r)^2(\tilde{\sigma}_3 + g_3(r)\sigma_3)^2 + f(r)^2\sigma_3^2.
\end{align*}
$$

The Taylor expansions about the origin are, for $D_7$ \[12, 14\]

$$
\begin{align*}
a(r) &\sim \frac{r}{2}, \quad b(r) \sim 1 - \frac{(q^2 - 2)r^2}{16},
\end{align*}

$$
\begin{align*}
c(r) &\sim -\frac{r}{2}, \quad f(r) \sim q + \frac{q^3r^2}{16},
\end{align*}

$$
\begin{align*}
g(r) &\sim -\frac{a(r)f(r)}{2b(r)c(r)}, \quad g_3(r) \sim -1 + 2g(r)^2,
\end{align*}

where a trivial parameter has been fixed to one and $q$ is a free parameter. The AC solution is recovered when $q = 1$. There is an $S^3$ collapsing at the origin.
For $\tilde{C}_7$, the Taylor expansions about the origin are 

\[ a(r) \sim 1 + \frac{(4 - c_0^4)r^2}{16}, \quad b(r) \sim b_0 + \frac{(4 - 3b_0^2c_0^2)r^2}{16b_0}, \]
\[ c(r) \sim c_0 + \frac{(2 + c_0^4)r^2}{4}, \quad f(r) \sim (1 + b_0^2)r, \]
\[ g(r) \sim -\frac{b_0c_0r}{2}, \quad g_3(r) \sim b_0^2 - \frac{(1 + b_0^2)r^2}{c_0}, \]

setting a trivial parameter to one and with $b_0$ and $c_0$ as free parameters. There is an $S^1$ collapsing at the origin.

Now compute the metrics near the origin as previously. There is a subtlety which is that these families of metrics do not have a $\mathbb{Z}_2$ symmetry interchanging $\sigma_i \leftrightarrow \tilde{\sigma}_i$. Therefore we will get different metrics depending on which of the $SU(2)$s we reduce on. To get a sensible metric in four dimensions we should reduce on the copy that does not (partially) collapse at the origin, which corresponds to the $\sigma_i$. The result for $D_7$ up to a constant, after putting $\rho = 2r^{3/2}/3$,

\[ ds^2_{3} \sim d\rho^2 + \frac{9\rho^2}{16}(d\theta^2 + \sin^2 \theta d\psi^2). \] (45)

This is exactly as for $B_7$ in (38). Thus this family also give global monopoles. The result for $\tilde{C}_7$ up to a constant is

\[ ds^2_{3} \sim \sin \theta \left[ dr^2 + d\theta^2 + \left( \frac{1}{b_0^2} + 1 \right)^2 r^2 d\psi^2 \right]. \] (46)

This metric is very similar to that for $C_7$ in (39). It is cylindrical with a surfeit plane angle that depends on $b_0$. There is therefore a natural interpretation as a cosmic string. Also as for $C_7$, spherical symmetry cannot be restored for any value of $(b_0, c_0)$ and in particular it will not be restored in the AC limit.

As in the previous subsection, we can also calculate the asymptotic behaviour of the metric. We will do this for $D_7$ in the ALC case. From the first order equations for the radial functions [13] one has that as $r \to \infty$

\[ a(r)^2 \sim \frac{\sqrt{3} - 1}{2} r^2, \quad b(r)^2 \sim \frac{\sqrt{3} - 1}{2} r^2, \]
\[ c(r)^2 \sim (1 - \sqrt{3})^2 r^2, \quad f(r)^2 \sim k^2, \]
\[ g(r) \sim -\frac{a(r)f(r)}{2b(r)c(r)}, \quad g_3(r) \sim -1 + 2g(r)^2, \] (47)
where $k$ is a constant. Doing the dimensional reduction one obtains the asymptotic metric up to a constant with $\rho = 2r^{7/2}/7$

$$ds^2 \sim \sin \theta (d\rho^2 + \alpha \rho^2 d\tilde{\Omega}_\rho^2),$$

with $\alpha = 4.48 \cdots$ and $d\tilde{\Omega}_\rho^2$ is a metric on the $S^2$ similar to $[G]$, increasingly cylindrical at infinity.

6 Conclusion

We have shown how one can obtain BPS monopoles in four dimensions by dimensional reduction of special holonomy metrics of higher dimension. In particular, we have considered seven dimensional Riemannian metrics of $G_2$ holonomy extended trivially to eight dimensions by adding a time direction.

The $G_2$ metrics we considered all had $SU(2) \times SU(2) \times U(1)$ isometry group in the generic case. Four families of such metrics are known, denoted $\mathbb{B}_7, \mathbb{C}_7, \tilde{\mathbb{C}}_7, \mathbb{D}_7$. We concentrated on the $\mathbb{B}_7$ case for which a couple of closed form solutions are known. We reduced on $SU(2) \times U(1) \subset SU(2) \times SU(2)$ to get static four dimensional metrics with an $S^2$ factor and a $U(1)$ isometry group corresponding to axial symmetry.

The $G_2$ metrics considered are generically asymptotically locally conical (ALC), with the $U(1)$ isometry corresponding to a stabilised $S^1$. There is a special limiting case in each family of metrics in which the metric becomes asymptotically conical (AC) with the $S^1$ blowing up at infinity. For $\mathbb{B}_7$ and $\mathbb{D}_7$ the AC case has an enhancement of isometry to $SU(2)^3$. This is seen in the four dimensional solution as an enhancement from axial to spherical symmetry. It was suggested that the preference for ALC $G_2$ metrics could be understood from instabilities of certain spherically symmetric four dimensional solutions with charged massive gauge fields under axially symmetric perturbations.

The families $\mathbb{B}_7$ and $\mathbb{D}_7$ have an $S^3$ that collapses at the origin and the four dimensional solution is spherically symmetric at the origin, with a conical singularity at the origin hidden as a point of infinite redshift. These are monopoles. The metric becomes increasingly cylindrical asymptotically. For $\mathbb{C}_7$ and $\tilde{\mathbb{C}}_7$, only an $S^1$ collapses at the origin and this results in the four dimensional metric having axial symmetry at the origin. These are cosmic strings.
The $SU(2) \times U(1)$ gauge and scalar fields were also examined. It was shown that in the generic case the $SU(2)$ gauge symmetry is completely broken except at infinity where it is broken to $U(1)$. This allows a topological charge to be associated with the solution by constructing a map $S^2_\infty \to S^2$. This is not the usual map because the Higgs fields are not in the usual adjoint representation. There is a second topological charge from the $U(1)$ gauge symmetry.

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