ERGODIC INFINITE GROUP EXTENSIONS OF
GEODESIC FLOWS ON TRANSLATION SURFACES

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(Communicated by Giovanni Forni)

ABSTRACT. We show that generic infinite group extensions of geodesic flows on square tiled translation surfaces are ergodic in almost every direction, subject to certain natural constraints. K. Frączek and C. Ulcigrai have shown that certain concrete staircases, covers of square-tiled surfaces, are not ergodic in almost every direction. In contrast we show the almost sure ergodicity of other concrete staircases.

1. INTRODUCTION

In 1986, S. Kerckhoff, H. Masur and J. Smillie showed that the geodesic flow on a compact translation surface is uniquely ergodic in almost every direction [18]. On the other hand, the study of the dynamical properties on periodic infinite translation surfaces is in its infancy. There is a natural necessary and sufficient condition for the recurrence of the geodesic flow in almost every direction for $\mathbb{Z}$- or $\mathbb{R}$-covers [13], generic nonperiodic translation surfaces are almost surely recurrent [30], and the almost sure diffusion rate in the periodic full occupancy wind-tree model, a certain important example of infinite translation surface, has also recently been understood [8]. We turn to the question of ergodicity. In [12], a natural example the "simple staircase" was shown to be ergodic in a.e. direction. In fact, the authors noticed that the ergodicity reduces to a classically studied question of ergodicity of cylinder flows (group-valued skew products) over circle rotations initiated by Conze and studied by many authors; see [5] for a good survey. The question of ergodicity for other periodic translation surfaces reduces to the question of ergodicity of cylinder flows over interval-exchange transformations.

Recently, K. Frączek and C. Ulcigrai have shown that certain periodic infinite translation surfaces are not ergodic in almost every direction [10]. More specifically, if the compact square-tiled translation surface $M$ is in the stratum $\mathcal{H}(2)$,
and the infinite translation surface $\tilde{M}$ is an unramified $\mathbb{Z}$-cover of $M$, then the geodesic flow on $\tilde{M}$ is not ergodic in a.e. direction. They showed a similar result for some $\mathbb{Z}^2$ extensions such as the periodic full occupancy wind-tree model.

In this article, we study the case when the surface $\tilde{M}$ is a ramified cover of $M$. The flow on $\tilde{M}$ is a $G$-valued skew product over the geodesic flow on $M$ for some Abelian locally compact group $G$. Our result is a sufficient condition for both recurrence and almost sure ergodicity to hold. This condition is based on ideas developed by M. Boshernitzan [2] and K. Schmidt [28].

While our theorem is somewhat technical to state, it is applicable to many concrete examples. Suppose $M$ is a square tiled surface with a single-cylinder direction (for example the torus). Then any reasonable randomized method of producing a $\mathbb{Z}^d$ (or more generally an Abelian locally compact group $G$) infinite surface $\tilde{M}$ that is a skew product over $M$ is generically ergodic in almost every direction. We also produce concrete examples of staircases ergodic in almost every direction such as the staircase in Figure 1.

![Figure 1](image)

**Figure 1.** The periodic staircase with alternating steps of length 2 and 3 is ergodic in almost every direction. Opposite sides are identified.

J. Chaika and P. Hubert have recently announced the following related result [4]: for a given $f : [0, 1) \to \mathbb{R}$ with integral 0, which is a linear combination of characteristic functions of intervals, for a set of full measure of interval-exchange transformations $T$ the skew product $T_f(x, i) = (Tx, i + f(x))$ into the closed subgroup of $\mathbb{R}$ generated by the values of $f(x)$ is ergodic. There are two significant differences in the results. While their results do not restrict to square-tiled surfaces, particular examples are not accounted for, and for any generic, fixed skew product, ergodicity is only guaranteed in some direction, whereas in this paper, once a square-tiled surface and generic skew product over it are fixed, ergodicity is shown in almost every direction.
2. Translation surfaces

In this section, we will give background material on translation surfaces as well as some definitions for later use, see [16, 21, 32] for details. A translation surface is a 2-dimensional manifold $M$, and a finite set of points $D = \{d_1, d_2, \ldots, d_m\}$ and an open cover of $M \sim D$ by sets $(U_{\alpha})$, together with charts $\phi_\alpha : U_\alpha \to \mathbb{R}^2$ such that for all $\alpha, \beta$, with $U_\alpha \cap U_\beta \neq \emptyset$, $\phi_\alpha \circ \phi_\beta^{-1}$ is a translation on its domain of definition, and at each singular point $d_i \in D$ the surface has a $2\pi n$ cone singularity for some integer $n \geq 1$.

The surface $M$ has a flat metric obtained by pulling back the Euclidean metric on the plane via the coordinate charts. This metric is defined away from the set $D$. In this metric, geodesics that do not go through singularities project via the charts to straight lines in the plane in a fixed direction, and such geodesics are either periodic or simple (do not intersect themselves). If the geodesic in direction $\theta$ starting at $x \in M$ avoids $D$, then we define the geodesic flow $\phi^\theta_t(x)$ to the point obtained after moving in the direction $\theta$ for a time $t$ starting at $x$. Since $\theta$ is usually fixed, we will often suppress the $\theta$ dependence of the flow and simply write $\phi_t(x)$. If the geodesic hits $D$, then the geodesic flow $\phi_t(x)$ is defined up to this time. The geodesic flow preserves the surface area measure $\mu$.

The group $SL_2(\mathbb{R})$ acts on the space of translation surfaces by acting on the charts of $M$. The action preserves the orders of the zeros, and therefore permutes the singularities $D$. The Veech group $\Gamma$ of $M$ is the image of the stabilizer of $M$, $\text{stab}(M)$, under the action of $SL_2(\mathbb{R})$ in $PSL_2(\mathbb{R})$. If $SL_2(\mathbb{R})/\text{stab}(M)$ is of finite volume (i.e., $\text{stab}(M)$ is a lattice), we say that $M$ is a Veech surface.

Throughout the article, we will assume that $M$ is a compact translation surface; by [18], the geodesic flow is (uniquely) ergodic in almost every $\theta$. We will form skew products over these flows in the following manner: let $G$ be a locally compact Abelian group with translation-invariant metric $\|\cdot\|$, whose identity element we denote $0$, with Haar measure $\nu$, and for a finite collection of geodesic path segments $\{\gamma_i\}$ (disjoint, without loss of generality), whose union we denote $\gamma$, let $f : \gamma \to G$ be constant on each $\gamma_i$ and 0 elsewhere. Given a point $x \in M$ and a direction $\theta$ that is not parallel to any $\gamma_i$, denote

$$S_t(x) = \sum_{0 \leq s < t \atop f(\phi_s(x)) \neq 0} f(\phi_s(x)),$$

$$S_t^{-1}(g) = \{x \in M : S_t(x) = g\}, \quad B_\epsilon(g) = \{y \in G : \|y - g\| < \epsilon\}.$$

Then the transformation $\tilde{\phi}_t$, from $\tilde{M} = M \times G$ to itself, is given by

$$\tilde{\phi}_t(x, g) = (\phi_t(x), g + S_t(x)),$$

and it is this continuous time flow that we study; $\tilde{\phi}_t$ preserves the measure $\tilde{\mu} = \mu \times \nu$. An essential value of a general skew product $\{\tilde{M}, \tilde{\mu}, \tilde{\phi}_t\}$ is some $g \in G$ such that for every $\epsilon > 0$ and $A \subset M$ with $\mu(A) > 0$, there is some $t > 0$ such that

$$\mu\left(A \cap \phi_t(A) \cap S_t^{-1}(B_\epsilon(g))\right) > 0.$$
We denote by $E(\tilde{\varphi}_t)$ the union of all essential values. Then if $E(\tilde{\varphi}_t)$ is nonempty, it is a closed subgroup of $G$, and $\{M, \varphi_t\}$ is ergodic if and only if $E(\tilde{\varphi}_t) = G$ and $\{M, \varphi_t\}$ is ergodic (see [1] for example). If $0 \in E(\tilde{\varphi}_t)$, then the system is said to be recurrent.

**Definition 2.1.** We will call a sequence of sets $C_n$ a quasi-rigidity sequence of sets in the direction $\theta$ if there is a sequence $t_n \to \infty$ and a fixed $\epsilon > 0$ such that

- $\mu(C_n) \geq \epsilon$,
- $\sup_{x \in C_n} d(x, \phi_{t_n} x) \to 0$, and
- $\forall t_0 \in \mathbb{R}^+, \mu(C_n \triangle \varphi_{-t_0} C_n) \to 0$.

That is, each set $C_n$ is of nontrivial measure, the map $\phi_{t_n}$ acts almost like the identity on these sets, and they are nearly invariant under the flow $\varphi_{t_n}$ for fixed $t_0$.

**Lemma 2.2.** Suppose that $\{M, \varphi_t\}$ is ergodic and that for some $g \in G$ we have a quasi-rigidity sequence of sets $C_n$ such that

$$\lim_{n \to \infty} \sup_{x \in C_n} \| S_{t_0} \varphi_{t_n} x - g \| = 0.$$ 

Then $g \in E(\tilde{\varphi}_t)$.

**Proof.** This is merely a restatement for continuous flows of [6, Corollary 2.8], which itself is a reworking of [9, §4]. In particular, by [25], for all but countably many $t \in \mathbb{R}$ the $\mathbb{Z}$-action generated by $\varphi_t$ is ergodic; fix such a $t_0 \in \mathbb{R}^+$. By assumption, for this time $t_0$ we have

$$\lim_{n \to \infty} \mu(C_n \triangle \varphi_{-t_0} C_n) = 0.$$ 

In [6, Corollary 2.8], it is assumed that $S_{t_0} \varphi_{t_n} x = g_n$ is constant on the set $C_n$, with $g_n \to g$; this assumption may be replaced with

$$\lim_{n \to \infty} \sup_{x \in C_n} \| S_{t_0} \varphi_{t_n} x - g \| = 0$$

with no modification to the proof (see also [6, Lemma 7] and the prior discussion in that citation). If the $t_n$ were all integer multiples of $t_0$, then we could directly apply that result, but we make no such restriction. Referring back to the results of [9, §4], however, the transformations $\varphi_{t_0}$ and $\varphi_{t_n}$ play different roles; the former is ergodic, and the latter acts nearly as identity on the set $C_n$. So long as the two transformations commute, and both preserve the measure $\mu$, the underlying techniques in no way require any particular relation between the two times. An essential value for the $\mathbb{Z}$-action $\varphi_{t_0}$ immediately extends to an essential value for the $\mathbb{R}$-action $\{\varphi_t\}$.

A similar remark appears as [17, Theorem 11]; the notion of restricting a search for essential values to certain subsets whose measures are bounded away from zero is a common step in the study of periodic group extensions of finite-measure systems.

**Definition 2.3.** If the geodesic flow $\varphi_t$ in a particular direction $\theta$ decomposes the space $M$ into a finite union of sets on which $\varphi_t$ is periodic (see Figure 2),
the direction $\theta$ is said to \textit{admit a representation by periodic cylinders}. Given a sequence $\{\theta_i\}$ of distinct directions that admit a representation by periodic cylinders, denote the cylinders by $A_{i,j}$, the width of $A_{i,j}$ by $\alpha_{i,j}$, and the height of each cylinder (i.e., the period of the geodesic flow in this cylinder) by $h_{i,j}$.

Consider a direction $\theta$. For $i = 1, 2, 3, \ldots$, the quantity

$$E_i = \max \left\{ \frac{h_{i,j} \tan |\theta - \theta_i|}{\alpha_{i,j}} : j = 1, 2, \ldots, n_i \right\}$$

will be called the \textit{error of the approximation $\theta_i$}. If $\theta_i \to \theta$ and $E_i \in O(1)$, we say that $\{\theta_i\}$ is a \textit{sequence of periodic approximations} to the flow in the direction $\theta$. Note that if $\theta_i \to \theta$ and $\theta$ does not admit a representation by periodic cylinders (and only countably many directions, those of \textit{saddle connections}, i.e., orbit segments that begin and end at singular points, may have such a representation), then, since for any $T > 0$, there are only a finite number of saddle connections of length at most $T$, we necessarily have

$$\lim_{i \to \infty} \min_j h_{i,j} = \infty.$$ 

If, furthermore, $E_i \to 0$, we call the sequence $\{\theta_i\}$ a \textit{good sequence of periodic approximations}. If there is some $\delta > 0$ such that for all $i$ we have $E_i \in (\delta, 1/2)$, we will call the sequence a \textit{fairly good sequence of periodic approximations}. See again Figure 2. If $\inf \{ \mu(A_{i,j}) = h_{i,j} \cdot \alpha_{i,j} : i = 1, 2, \ldots, j = 1, 2, \ldots, n_i \} > 0$, we will say that this sequence \textit{has cylinders of comparable measures}.

\textbf{Lemma 2.4.} Suppose that $\theta$ admits a sequence of good periodic approximations $\{\theta_i\}$ with cylinders of comparable measures, and let $\{i_k\}$ be an increasing sequence.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{Decomposition of $\{M, \varphi_t\}$ into periodic cylinders in direction $\theta_i$ of widths $\alpha_{i,j}$ and heights $h_{i,j}$; the top and bottom of each rectangle are identified, and $\varphi_t$ flows vertically at unit speed. The error compared to the flow in direction $\theta$ relative to the width of the cylinders is bounded for periodic approximations, bounded and bounded away from zero for fairly good periodic approximations, and converging to zero for good periodic approximations.}
\end{figure}
of positive integers. For some fixed $\delta > 0$, consider a sequence of subcylinders $C_k$; connected subsets of the cylinder $A_{i_k,j}$ (for some $j$) that are invariant under the flow in direction $\theta_{i_k,j}$, and whose measure is at least $\delta \mu(A_{i_k,j})$. Assume further that the boundary of each $C_k$ is separated from the boundary of $A_{i_k,j}$ by a distance of at least $\delta \alpha_{i_k,j}$. Then the {$C_k$} form a quasi-rigidity sequence of sets.

**Proof.** That $\mu(C_k)$ is bounded away from zero follows from the condition that the cylinders have comparable measures. As the sequence $\theta_{i_k,j}$ was taken to be a sequence of good periodic approximations, the relative error converges to zero. It now follows from the fact that each $C_k$ is bounded away from the boundary of $A_{i_k,j}$ that for each $x \in C_k$, the orbit through time $h_{i_k,j}$ of $x$ remains in $A_{i_k,j}$, from which it follows that

$$d(\varphi_{h_{i_k,j}}(x), x) \leq E_{i_k,j} \alpha_{i_k,j} \xrightarrow{k \to \infty} 0.$$ 

Finally, we have

$$\mu(\varphi_{t_0} C_k \Delta C_k) \leq 2 t_0 E_{i_k,j},$$

(the difference is exactly those $x$ so close to the boundary of $C_k$ that they flow out of the subcylinder in time $t_0$), which converges to zero. 

**Definition 2.5.** If the geodesic flow in direction $\theta$ admits a representation by a single periodic cylinder, then $\theta$ will be called a single-cylinder direction. If all nonsingular orbits in the direction $\theta$ are periodic with the same period, then the direction is said to be periodic.

Single-cylinder directions are of great convenience in studying translation surfaces. It was shown in [14] that if $M$ is square-tiled with a prime number of tiles and in $\mathcal{H}(2)$ (genus two with a single singularity), then $M$ has a single-cylinder direction. In [23, Corollary A.2], McMullen extended this result to all square-tiled surfaces in $\mathcal{H}(2)$. Kontsevich and Zorich showed that there is a dense set of square tiled surfaces in any stratum with single cylinder directions [19, Lemma 18]. Finally Lanneau and Nguyen have identified certain substrata of $\mathcal{H}(4)$ and $\mathcal{H}(6)$ that have single cylinder directions [20].

The following is a standard argument in the study of translation surfaces, but hardly elementary:

**Lemma 2.6.** Suppose that $M$ is a Veech surface with Veech group $\Gamma$, and that $\{ M, \varphi_t \}$ has at least one direction $\theta'$ that admits a representation by $N$ periodic cylinders. Then for almost every direction $\theta$, there is a sequence {$\theta_i$} of directions, chosen from the directions $\Gamma(\theta')$, which is a good sequence of periodic approximations, with $N$ cylinders of comparable measures, to the flow in the direction $\theta$. Furthermore, for almost every direction $\theta$, there is a sequence {$\theta_i$}, taken from the directions $\Gamma(\theta')$, of fairly good periodic approximations with $N$ cylinders of comparable measures.

**Proof.** Any element $\gamma \in \Gamma$ transforms the flow in direction $\theta'$ to a flow in another direction that necessarily also admits a representation by periodic cylinders. The number of cylinders and their measures are not changed by the action of $\gamma \in \Gamma$. 


SL(2, R), so we must show that for almost every θ, a sequence may be chosen out of this orbit that is a good or fairly good sequence of periodic approximations to the flow in the direction of θ.

Since M is a Veech surface, the Veech group Γ is necessarily Fuchsian of the first kind. The fact that almost every direction has a sequence of good approximations coming from the orbit under Γ of the single-cylinder direction follows directly from a theorem of Patterson [24, §9] (see [29] for a more general case). In the language of that article, an application of Patterson’s Theorem states: for every ε > 0, there exists a set of full measure Θ(ε) ⊂ S¹ such that for any θ ∈ Θ(ε), there exists a sequence of N cylinder directions with slopes pₙ/qₙ such that |θ − pₙ/qₙ| ≤ ε/qₙ². The assumptions of this statement then can be checked as in [15], and thus the conclusions for good approximations follow from Patterson’s Theorem.

The appropriate lower bound for fairly good approximations follows from the ergodicity techniques used to prove similar statements in [17, §2.2]: the orbit of a periodic direction under Γ will intersect arbitrary sets of positive measure infinitely often in SL₂(R)/Γ. This quotient is of finite volume and has finitely many cusps by assumption, so a nontrivial annular neighborhood around a generic slope will also contain infinitely many points from the orbit Γ(θ').

The use of a fixed annulus as opposed to a sequence of shrinking neighborhoods in the proof of Lemma 2.6 is what distinguishes fairly good approximations from good approximations. The conclusion of Lemma 2.6 can in fact be strengthened to state that the set of exceptional directions (those that cannot be approximated to the desired accuracy from the orbit under Γ of any given single-cylinder direction) is not just of measure zero, but indeed of Hausdorff dimension less than one. See again [17] for similar statements; we state only that almost every direction can be so approximated in the interest of simplicity and generic appeal.

3. Generic Constructions

**Definition 3.1.** Let θ, the direction of the geodesic flow, be fixed such that {M, ϕₜ} admits a sequence of periodic approximations {θᵢ}. Let ˜D denote an arbitrary finite subset of M. Then a point x ∈ (˜D ∼ D) is said to avoid D ∪ ˜D in direction θ if for some fixed ε > 0 there is a sequence {εᵢ} so that each of the following holds:

- for each i, x belongs to the interior of some Aᵢ,j,
- the εᵢ-neighborhood around the hᵢ,j sec|θᵢ − θ| orbit of x in the direction of θ remains in Aᵢ,j (restricted to those j for which x ∈ Aᵢ,j) and does not intersect D ∪ (˜D ∼ x), and
- infᵢ→∞ εᵢhᵢ,j ≥ ε.

A sequence of neighborhoods Cᵢ for which x satisfies this definition for the same ε > 0 will be called a sequence of orbit-neighborhoods for x. See Figure 4 for an illustration (the sets Cᵢ,1 and Cᵢ,2 will be defined later). If every point in
\[ \tilde{D} \text{ avoids } D \cup \tilde{D}, \text{ and if } D \cap \tilde{D} = \emptyset, \text{ then we will say that } \tilde{D} \text{ is a self-avoiding set in direction } \theta. \]

Note that we merely need some sequence of periodic approximations for each \( \theta \); in situations where a canonical method for finding approximations exists, a subsequence of this canonical set suffices.

**Proposition 3.2.** Suppose \( \theta \) has a good sequence of periodic approximations and fix \( P \in \mathbb{N} \). Then almost every element \( \tilde{D} \in M^P \) (with respect to the product measure) is a self-avoiding set in the direction \( \theta \).

**Proof.** It suffices to show that the proposition is true for \( P = 1 \); larger sets may then be inductively constructed one point at a time by presuming \( x_1 \) through \( x_{P-1} \) are in \( D \) for the purposes of finding \( x_P \). Let \( \theta \) be fixed such that \( \{ \theta_i \} \) is a good sequence of periodic approximations. For a fixed \( \epsilon > 0 \), let \( B_i = B_i(\epsilon) \) be the complement of the \( \epsilon' \)-neighborhood of the boundaries of the cylinders, where

\[ \epsilon' = \frac{\epsilon \min_j \{ \alpha_{i,j} \}}{2}, \]

so that \( \mu(B_i) \geq (1 - \epsilon) \). See Figure 3. Now note that if \( x \) belongs to infinitely many \( B_i(\epsilon) \), it is self-avoiding singleton: for sufficiently large \( i \) we have the error \( E_i < \epsilon \). As

\[ \mu \left( \limsup_{i \to \infty} B_i \right) \geq \limsup_{i \to \infty} \mu(B_i), \]

for this choice of \( \epsilon \) the set of self-avoiding singletons in direction \( \theta \) is of measure at least \((1 - \epsilon)\), and \( \epsilon \) was arbitrary. \( \square \)

**Corollary 3.3.** Suppose that almost every \( \theta \) admits a sequence of good approximations. Then almost every subset \( \tilde{D} \in M^P \) is self-avoiding in almost every direction.

**Figure 3.** The \( \epsilon \)-proportion removed in constructing \( B_i(\epsilon) \).
Proof: We certainly have almost surely that $\tilde{D} \cap D = \emptyset$. For a full-measure set of $\theta$, a full-measure set of $\tilde{D}$ is self-avoiding in direction $\theta$. Apply the Fubini Theorem to reverse the order of quantifiers. 

Let $S = \{s_1, \ldots, s_n\}$ be a finite subset of $M$ (possibly including points in $D$), $L_i > 0$ and $\tau_i \in S^1$ ($i = 1, \ldots, n$) be, for the moment, arbitrary. Let $\gamma_r$ be a geodesic of length $L_r > 0$ in direction $\tau_r$ originating at $s_r$. Note that we write “a” geodesic since, if $s_r \in D$, there may be several possible such geodesics, and if a geodesic originating at $s_r$ arrives at a point of $D$, there are multiple possible continuations. Although generically this will not happen, a priori we allow all such geodesic segments. Let $a_{r,1}, \ldots, a_{r,N(r)}$ be a finite set of points in $\gamma_r$ ($N(r)$ is arbitrary but finite); assume that $a_{r,N(r)} = \varphi_{L_r}(s_r)$ (the terminus of $\gamma_r$) and set $a_{r,0} = s_r$. Denote by $A$ the (finite) collection of all $a_{r,i}$ together with any $d \in D$ such that $d \in \gamma_i$ for some $i$.

**Corollary 3.4.** For fixed $n$, for almost every construction of $\gamma_r$ and $A$ (with initial points $s_i$ chosen according to a probability measure on $M$ absolutely continuous with respect to $\mu$, lengths $l_i$ chosen according to some probability measure absolutely continuous with respect to Lebesgue on $\mathbb{R}^+$ and directions $\tau_i$ by a measure absolutely continuous with respect to Lebesgue measure on $S^1$, and $A$ then chosen according to a measure on $\gamma$ which is absolutely continuous with respect to Lebesgue measure), the set $A$ is a self-avoiding set.

Proof. We apply Proposition 3.2, noting that almost surely no $\gamma_i$ crosses a singularity $d \in D$. 

**Definition 3.5.** Let $G$ be an Abelian locally compact group with translation-invariant metric endowed with Haar measure $\nu$, and we will use the segments $\gamma_i$ to define our skew product $\{\tilde{M}, \tilde{\varphi}_i\}$ as in §2. Let $f: M \rightarrow G$ be piecewise constant on each $\gamma_i$ with discontinuities a subset of $A$; denote the value of $f$ between $a_{i,j-1}$ and $a_{i,j}$ by $f_{i,j}$. Finally, denote

$$
\sigma_{i,j} = \begin{cases} 
 f_{i,1} & (j = 0) \\
 f_{i,j+1} - f_{i,j} & (j = 1, 2, \ldots, N(i) - 1) \\
 -f_{i,N(i)} & (j = N(i)).
\end{cases}
$$

The function $f$ is defined to be identity off the segments $\gamma_i$; the values $\sigma_{i,j}$ are the ‘jumps’ seen in $f$ while flowing along the path $\gamma_i$.

Suppose that $\theta'$ is a single cylinder direction, and let $q_{i,j}$ denote the projected length of the geodesic segment connecting $a_{i,j-1}$ to $a_{i,j}$ in the direction transverse to $\theta'$. Then we say that *the flow in the single cylinder direction $\theta'$ sees an average value of the identity in $G$ if*

$$
\sum_{j=1}^{N(i)} \sum_{i=1}^{n} f_{i,j} q_{i,j} = 0.
$$

A function $f$ defined in this way that sees an average value of the identity we call *nontrivial on the collection of cuts $\gamma_i$.*
**Definition 3.6.** Let \( \theta \) be fixed. A point \( x \in (\tilde{D} \sim D) \) that avoids \((\tilde{D} \cup D)\) with orbit neighborhoods \( C_k \) of heights \( h_k \to \infty \) is said to be essential in direction \( \theta \) if there is some \( P \) such that along an infinite subsequence of orbit neighborhoods we have

\[
\sup_{k=1,2,...} \sup_{y \in C_k} |S_{h_k}(y)| \leq P.
\]

Showing that points are essential is, in general, a demanding requirement. Single-cylinder directions, however, provide a ready approach.

**Proposition 3.7** (Koksma-type inequality). Suppose that \( M \) is a translation surface with a single cylinder direction \( \theta' \), with the cylinder height given by \( h \) (and width \( 1/h \)). We suppose that \( f \) is nontrivial on the collection of cuts (Definition 3.5). Then, for every \( x \) not on the boundary of the cylinder we have

\[
\|S_{h}(x)\| \leq 4 \sum_{j=1}^{N(i)} \sum_{i=1}^{n} \|f_{i,j}\|,
\]

where the sum \( S_{h}(x) \) is with respect to the flow in the direction \( \theta' \).

**Proof.** Note that the condition that the average value of the sums in the cylinder is identity (1) is sufficient for recurrence of the flow in the event that \( G = \mathbb{Z} \) or \( \mathbb{R} \) (see [27]), and is clearly necessary in all cases where \( G \) contains elements of infinite order.

Given a particular segment, the points \( a_{i,j-1} \) and \( a_{i,j} \) are in the cylinder (possibly on the boundary), and the segment between them completely crosses the cylinder some number of times. Denote this number by \( P_{i,j} \). There are also possibly two partial crossings corresponding to the segment leaving \( a_{i,j-1} \) and arriving into \( a_{i,j} \) (in \( P_{i,j} = 0 \) then there is only one such segment); denote by \( p_{i,j}' \) and \( p_{i,j}'' \), respectively the lengths of these two segments when measured in the transverse direction to the flow. We have

\[
0 \leq p_{i,j}', p_{i,j}'' < \frac{1}{h}, \quad 0 \leq q_{i,j} - \frac{P_{i,j}}{h} \leq \frac{2}{h},
\]

the latter of which we rewrite as

\[
|P_{i,j} - q_{i,j}h| \leq 2.
\]

As the entire space is shown in this single periodic cylinder, the entire segment is represented by these pieces, so (1) translates to

\[
\sum_{j=1}^{N(i)} \sum_{i=1}^{n} f_{i,j} \left( \frac{P_{i,j}}{h} + p_{i,j}' + p_{i,j}'' \right) = 0. \tag{2}
\]

The periodic orbit of any point \( x \) in the direction \( \theta' \) (ignoring the orbits containing the points \( a_{i,j} \)) crosses the segment from \( a_{i,j-1} \) to \( a_{i,j} \) a total of
\(P_{i,j} + \phi_{i,j}(x)\) times, where \(\phi_{i,j}(x) \in \{0, 1, 2\}\), according to whether it crosses neither ‘partial segment,’ one of them, or both. Then we have

\[
S_h(x) = \sum_{j=1}^{N(i)} \sum_{i=1}^{n} f_{i,j} (P_{i,j} + \phi_{i,j}(x))
\]

\[
\|S_h(x)\| \leq \left\| \sum_{j=1}^{N(i)} \sum_{i=1}^{n} f_{i,j} P_{i,j} \right\| + 2 \sum_{j=1}^{N(i)} \sum_{i=1}^{n} \|f_{i,j}\|
\]

\[
\leq h \left\| \sum_{j=1}^{N(i)} \sum_{i=1}^{n} f_{i,j} q_{i,j} \right\| + 4 \sum_{j=1}^{N(i)} \sum_{i=1}^{n} \|f_{i,j}\|
\]

\[
= 4 \sum_{j=1}^{N(i)} \sum_{i=1}^{n} \|f_{i,j}\|. \quad \square
\]

**Theorem 3.8.** Let \(a = a_{j,i}\) be an essential point with respect to a good sequence of periodic approximations. Then \(\sigma = \sigma_{i,j}\) is an essential value of the skew product \([M, \phi]\) for the flow in direction \(\theta\). In particular, the skew product is recurrent.

**Proof.** Using a good sequence of periodic approximations allows us to consider for almost every \(x\), for sufficiently large \(i\) the flow in direction \(\theta\) and the flow in direction \(\theta_i\) are indistinguishable through length \(h_{i,j}\), at least insofar as the sums \(S_{h_{i,j}}(x)\) are concerned. The point \(a\) has a sequence of orbit-neighborhoods \(C_k\) that remain in the same periodic cylinder \(A_{i,j}\). The only discontinuity of \(f\) in \(C_k\) is the point \(a\) by definition. So we split the orbit-neighborhood into two sets of equal measure \(C_{k,1}\) and \(C_{k,2}\) so that the sum \(S_{h_{i,j}}(x)\) is constant on each. Furthermore, the difference between the two values taken is exactly \(\sigma\). See Figure 4. Without loss of generality, then, assume that for \(x \in C_{k,1}\) and \(y \in C_{k,2}\) we have

\[
S_{h_{i,j}}(y) = S_{h_{i,j}}(x) + \sigma.
\]

As there is a uniform bound \(P\) across all values and \(G\) is locally compact, by passing to a subsequence (which we continue to denote \(\{i_k\}\) we may find some limit

\[
\lim_{k \to \infty} S_{h_{i_k}}(y \in C_{k,1}) = g, \quad \lim_{k \to \infty} S_{h_{i_k}}(y \in C_{k,1}) = g + \sigma.
\]

Both \(C_{k,1}\) and \(C_{k,2}\) are a quasi-rigidity sequence sets by Lemma 2.4, so by Lemma 2.2, both \(g\) and \(g + \sigma\) are in \(E(\hat{\phi}_i)\). As the set of essential values is a group, \(\sigma \in E(\hat{\phi}_i)\). \(\square\)

**Corollary 3.9.** Suppose that \(f\) is nontrivial on a collection of cuts (see Definition 3.5), and in a direction \(\theta\) assume both that all \(\alpha_{i,j}\) are essential points and the collection of all \(\alpha_{i,j}\) is a self-avoiding set. Then every value \(f_{i,j}\) is an essential value for \([M, \phi]\). In particular, if \([M, \phi]\) is ergodic in direction \(\theta\) and the values \(f_{i,j}\) generate a dense subgroup of \(G\), then \([M, \phi]\) is ergodic.

That is, for a generic method of making ‘cuts’ \(\gamma_i\) on the surface \(M\) and using them to form a \(G\)-fold cover, as long as the values \(f_{i,j}\) include generators for
the group $G$, the extension is ergodic in almost every direction provided we may show the discontinuities are essential points in almost every direction; by Proposition 3.7, for Veech surfaces that admit a single-cylinder direction we have the necessary bound.

**Example 3.10.** Let $M$ be the two-torus $T = [0,1)^2$ with standard identifications (which is flat without any singularities). Almost every direction $\theta$ has a good sequence of single-cylinder periodic directions given by a subsequence of the continued fraction convergents to $\theta$, and the flow $\{ M, \phi_t \}$ is ergodic in any irrational direction.

Let $\{ g_i \}$ generate the group $G$, and for each $i$ construct a pair of parallel geodesics of equal length, at random starting points, in random directions, and of random lengths (with the relevant distributions are all absolutely continuous with respect to Lebesgue). To see that this can be done, fix a direction $\theta$ and a length $\ell$. Let $E(d, \theta, \ell)$ be the other end-point of the segment starting at $d \in D$ in the direction $\theta$ of length $\ell$. Then $\bigcup_{d \in D} E(d, \theta, \ell)$ is of full measure, thus for almost all $d \in D$, the point $E(d, \theta, \ell)$ is in $D$ as well.

Define $f$ to be $\pm g_i$ on these pairs. Then in any direction not parallel to such a segment, the projection of $f$ to the base segment $[0,1) \times \{0\}$ will be of bounded variation ($\text{Var}(f) = 4 \sum |g_i|$) and have integral zero, so by a combination of Corollary 3.3, Proposition 3.7, Theorem 3.8, and Corollary 3.9, for generic such constructions the skew product $\{ \tilde{M}, \tilde{\phi}_t \}$ is ergodic in almost every direction.

We turn to the question of periodic orbits; in addition to the zero integral condition, we suppose that a $\mathbb{Z}$-cover is formed using disjoint cuts with constant value 1 or $-1$ on each slit. Consider a periodic cylinder on $T$ and one of the cuts. The cut entirely crosses the cylinder a certain number of times, and then there are zero (no partial crossings), one or two hanging edges. The same happens for the other cuts. Together the partial crossings partition the cylinder into a

![Figure 4. The two halves of the orbit-neighborhood $C_i = C_{i,1} \cup C_{i,2}$ which do not intersect any discontinuities of $f$ and whose sums differ by exactly $\sigma$ in the proof of Theorem 3.8.](image-url)
certain number of subcylinders with constant ergodic sum; we will call them strips. By the generic assumption, for almost every direction $\theta$, there is a good sequence $\theta_i$ of periodic approximations. If $\theta_i$ is good, then each pair of neighboring strips is separated by the orbit of exactly one end-point of a slit; thus, the jump in essential value is by $\pm 1$ for neighboring strips. Thus, at least one of the strips must have essential value 0, since the projection of the cuts onto any transverse is assumed to have zero integral. The strip with 0 essential value lifts to a periodic orbit in $\tilde{M}$. We have shown that for almost every direction $\theta$, there is a periodic orbit on $\tilde{M}$ with direction arbitrarily close to $\theta$. Now consider only the orbits in $M$ that lift to periodic orbits in $\tilde{M}$, we have shown that they have dense directions in $M$. Furthermore, almost every direction in $M$ is uniquely ergodic. These two ingredients immediately imply that these periodic orbits are in fact dense in $M$, just repeat the proof (without change) of the main theorem of [3] (see [22, Theorem 4.3]). Thus we have shown that periodic orbits are, in fact, dense in $\tilde{M}$.

**Example 3.11.** Let $M$ be square-tiled: a finite cover of the torus such that $M$ may be considered to be tiled by squares that share corners. Then $M$ is a Veech surface; this fact was first shown by Veech using Boshernitzan’s criterion in [31], but modern proofs center around the use of arithmetic subgroups of $\text{SL}_2(\mathbb{R})$ (see e.g., [14, Appendix C], [11, Theorem 5.5]). Assume, further, that $M$ has a single-cylinder direction, so that almost every direction $\theta$ admits a good sequence of periodic single-cylinder directions. By the same argument as in the previous example, periodic orbits are dense for the $\mathbb{Z}$-extension case formed by taking disjoint slits satisfying the zero integral condition with constant value $\pm 1$.

While convenient, the assumption that $M$ admits a single-cylinder direction is not necessary to apply the ideas and results of this section; an example of a class of surfaces with no single-cylinder directions for which certain constructions generically produce ergodic covers is addressed in Example 4.7 and the remark following that example.

We remark that the idea of comparing ergodic sums on two different sides of a discontinuity goes back at least as far as [28] and can be found more recently in [7], and the notion of finding generic points bounded a positive proportional distance from the orbits of singularities can be found in [2]. A similar approach to studying ergodicity of real-valued skew products over rank-one interval-exchange transformations can be found in [4].

**4. Generalized Staircases**

In this section, we shall apply the same principles as in our generic constructions to certain specific skew products of square-tiled surfaces called generalized staircases.

**Definition 4.1.** Let $M$ be a translation surface, and let $\tilde{M}$ be a $\mathbb{Z}^d$-cover of $M$ formed by taking, for each $i = 1, 2, \ldots, d$, a pair of parallel cuts of equal length, with $f = \pm e_i$ on each cut, where $e_i$ is the $i$-th standard basis element for $\mathbb{Z}^d$. 


If all of the cuts intersect at most at their end-points, then \( \tilde{M} \) will be called a **generalized staircase** over \( M \); there is a natural visualization of the flow on the surface with some number of cuts that transfer the flow along a \( \mathbb{Z}^d \)-periodic cover. If, furthermore, \( M \) is square-tiled and each cut is of length one (the length of the sides of the squares) and all end-points are integer points, \( \tilde{M} \) will be called a **natural staircase** over \( M \).

Note that if each cut is nontrivial in homology, then the \( \mathbb{Z}^d \)-cover of \( M \) is **unramified**, and if \( M \in \mathcal{H}(2) \), by [10, Theorem 1.4], the system \( \{\tilde{M}, \tilde{\phi}_t\} \) is not ergodic in almost every direction. However, if some of the cuts have end-points that are **not** in \( D \) (the singularities of the flat metric on \( M \)), then we have constructed a ramified cover over a set of marked points \( D \). It is direct to see that if \( M \) consists of a single square, then only trivial natural staircases are possible. If we represent the torus with two squares, however, then a natural staircase that is a ramified cover becomes possible. This cover of the torus was presented in [12], where it was shown to be equivalent to a previously studied **cylinder transformation**, known to be ergodic in all irrational directions.

Generalized staircases can be parameterized by the base points and lengths of their respective cuts. The results of the previous section may be adapted readily under the additional restriction that the cuts are parallel to sides to ensure that generic generalized staircases are ergodic in almost every direction. However, specific directions can have quite different behavior.

**Example 4.2.** Let \( M = \mathbb{T} \) be the two-torus. Suppose that \( \theta \) is a direction that admits a good sequence of periodic approximations. Then there exist uncountably many choices of \( \beta, r \) such that, with cuts as in Figure 5, the skew product model of the corresponding generalized staircase \( \{\tilde{T}, \tilde{\phi}_t\} \) is **nonregular**. This staircase is the same as the skew product considered by Conze in [5, Theorem 4.2], where exactly this conclusion is shown. Therefore the geodesic flow is **recurrent** (0 is an essential value), but not ergodic (no other integers are essential values).

The situation of natural staircases is, in some sense, more interesting: the severe restrictions imposed by the integer end-points of the cuts present an
immediate obstacle to directly applying the techniques of the previous section without modification. If the staircase is given by an unramified cover, then all end-points of the cuts are singularities, and there is no adapting our techniques (indeed, for \( M \in \mathcal{H}(2) \) the flow will not be ergodic in almost every direction by \([10]\)). So assume that at least one of the cuts has an end-point (with integer coordinates, as we consider a natural staircase) that is not a singularity. In a square-tiled surface, the directions that admit representation by periodic cylinders correspond to rational-slope flows on the plane. The end-points of our cuts are all integer points in the plane, as are all of the singularities \( \tilde{D} \); the end-points of our cuts will always belong to the edge of the periodic cylinders and therefore orbit into singularities in the directions of the periodic approximations, a rather insurmountable impediment to directly apply Definition 3.1.

Thus far, however, the only identifications of the edges of the cylinders we have used is the identification of the top to the bottom, but there are also certain identifications on the sides of the cylinders. It is illustrative to make explicit which directions for square-tiled surfaces form different types of periodic approximation. Let \( M \) be square-tiled on \( k \) squares. The flow in direction \( \theta \) induces a rotation by \( \theta' = 1/\theta \) mod 1 on \( S^1 \) when the bases of all squares are projected to a single circle. Let the continued-fraction expansion of \( \theta' \) be given by \( \theta' = [a_1, a_2, a_3, \ldots] \).

Then the periodic approximations correspond to the sequence of the convergents \( p_n/q_n \) to \( \theta' \). The relative error \( E_n \) of the approximation is no more than \( k q_n \| q_n \theta' \| \) (the standard error of \( \| q_n \theta' \| \) multiplied by the number of squares, as a single cylinder must wind through all of them, and divided by the width \( 1/q_n \)), and it is standard that

\[
\frac{k}{a_{n+1} + 1} \leq k q_n \| q_n \theta' \| \leq \frac{k}{a_{n+1}}.
\]

Therefore, a good sequence of periodic approximations corresponds to a subsequence of partial quotients diverging, while a fairly good sequence of periodic approximations corresponds to a subsequence of partial quotients at least as large as \( 2k + 1 \), but bounded above. Both situations are clearly satisfied for generic \( \theta' \). It is not immediately clear that, if \( M \) admits a single-cylinder direction, both conditions are generically satisfied along the subsequences of \( p_n/q_n \) that correspond to these single-cylinder directions, but this fact holds by the Patterson–Sullivan Theorem used in Lemma 2.6.

**Theorem 4.3.** Suppose that \( M \) is square-tiled and \( f \) defines a natural staircase. Assume that \( d \) is not a singularity of the flat metric, but is an end-point of a cut along which \( f \) takes the value \( e_i \) (and not the end-point of any other cut). Assume, further, that along a subsequence of fairly good periodic approximations, there exist cylinders \( A_{i,j} \) with \( d \) on both sides of \( A_{i,j} \), and for each of these approximations \( \theta_i \), there is some \( P \) (independent of \( i \) ) such that for each \( x \in A_{i,j} \),

\[
\| S_{h_{i,j}}(x) \| \leq P.
\]
where the sum is with respect to the flow in the periodic direction $\theta_i$. Then $e_i$ is an essential value of $\{M, \tilde{\varphi}_t\}$ in direction $\theta$. If $d$ is the end-point of several cuts, then the essential value is given by the sum of the values of $f$ on the cuts terminating at $d$. In both cases, the staircase is recurrent.

![Figure 6](image)

**Figure 6.** A modified orbit-neighborhood used in staircases: $C_{i,1}$ and $C_{i,2}$ are the two colored parallelograms. The region outside the cylinder appears somewhere inside via the appropriate identification and there are no singularities within $C_{i,1}$ or $C_{i,2}$. No cuts have end-points within these neighborhoods other than the specified cuts with values $\pm e_i$ with common end-point $d$. The time $h'$ flow (to traverse the entire vertical distance) acts as horizontal translation by $E_n/q_n \in [\delta/q_n, 1/2q_n]$ on this set. The constant $\delta$ is from the definition of fairly good approximation.

**Proof.** The singularities and ramification points are arranged along the edges of the periodic cylinders $A_{i,j}$. The flow in the direction $\theta_i$ induces the rotation by $p_i/q_i$ on the projection of all horizontal bases of the $k$ squares to a single circle. Note that the vertical distance between successive singularities and ramification points on the boundary of the cylinder (i.e., the geodesic distance in direction $\theta_i$ between them) must be at least $q_i$, and the height of any cylinder is no more than $kq_i$, where $k$ is the number of squares in $M$. Consider the flow of a vertical interval of length $q_i$ both above and below $d$ on both sides of $A_{i,j}$, and arrange the cylinder so that this interval is at the bottom of one side. We flow this interval forward until intersecting the height of $d$ within the cylinder, and
backwards to the height of the base of the cylinder; this notion is well-defined despite the flow exiting the cylinder in one of these directions as there are no singularities near $d$ (see Figure 6). No other discontinuities of $f$ are in these vertical segments, then, and, furthermore, the cut with end-point $d$ traverses the cylinder $A_{i,j}$ from exactly one of the two copies of $d$ on the boundary of $A_{i,j}$ (because $d$ is an end-point of the cut). See Figure 6; the two halves of this orbit neighborhood are $C_{i,1}$ and $C_{i,2}$; because there are no singularities within height $q_i$ of $d$ along the boundary of the cylinder, there is no problem extending these neighborhoods outside the boundary. The distances are maintained as drawn, and any cut besides those with end-point $d$ that might intersect these neighborhoods must pass completely through them; the only end-point of any cut seen in $C_{i,1} \cup C_{i,2}$ is $d$. The difference in the ergodic sums is therefore exactly a sum of $\pm e_i$, the values of $f$ on the cuts terminating at $d$. As the error $E_i$ is not larger than half the width of $A_{i,j}$ (as we have a fairly good sequence of periodic directions), these neighborhoods may flow for time to traverse the entire height of the cylinder without intersecting the sides (except for the small bit at the bottom where there are no singularities, as mentioned prior). The measure of each orbit neighborhood can be measured by the product of the vertical length of the segment between $d$ and the nearest singularity (which is at least $q_i$ as we are square-tiled) and the horizontal drift (the error term $E_i$, which is a positive portion of $1/q_i$, as we have assumed only a fairly good sequence of periodic approximations); the measure of each $C_{i,m}$ is at least $\delta$, where $\delta > 0$ is the fixed constant from the definition of a fairly good sequence of periodic approximations.

Therefore, the sets $C_{i,1}$ and $C_{i,2}$ again form a quasi-rigidity sequence of sets (their near-invariance under any fixed $\varphi_{t_0}$ is clear from $q_i \to \infty$). As in Theorem 3.8, the sum of the values of $f$ on the cuts terminating at $d$ is the difference in sums between infinitely many of the two $C_{i,1}$ and $C_{i,2}$, and these sums are bounded by assumption.

**Corollary 4.4.** Suppose that, for each $i$, at least one end-point one of the cuts where $f$ takes value $e_i$ is not a singularity and is not the end-point of any other cut, and assume that $M$ admits a single-cylinder direction. Then the natural staircase given by $f$ is ergodic in almost every direction. Furthermore, every single-cylinder periodic direction on $M$ is a periodic direction on the staircase; in particular, periodic directions are dense for the staircase.

**Proof.** As $M$ admits a single-cylinder periodic direction, almost every $\theta$ has a fairly good sequence of single-cylinder periodic directions by Lemma 2.6. As there is only one cylinder to consider, the relevant bound on sums used in Theorem 4.3 is satisfied by Proposition 3.7. As we assume that each cut has a nonsingular end-point not shared by any other cut, the group of essential values therefore contains each $e_i$.

Consider a single-cylinder direction on $M$. No end-point of any cut belongs to the interior of the single cylinder, so the ergodic sum over every point is the
same constant, which must be zero, as the projection of the cuts onto any transverse is assumed to have zero integral. Thus, every point that has nonsingular orbit in this direction on $M$ has ergodic sum zero, and every single-cylinder periodic direction on $M$ lifts to a periodic direction on the staircase.

**Example 4.5.** Consider the surface given by Figure 7; the bottom row consisting of two squares and the top row having $n \geq 3$ squares. The first presentation with cuts as marked on the left and its associated staircase is not ergodic in almost every direction, by [10, Theorem 1.4]; the surface is in $\mathcal{H}(2)$ and both cuts are nontrivial in $H_1(M, \mathbb{Z})$. However, if $n$ is additionally odd, then a straightforward check shows that the direction with slope 1/2 is a single cylinder direction. Thus, for the presentation on the right, for which both cuts have an end-point that is not a singularity, Theorem 4.3 implies that the flow on the corresponding staircase is ergodic in almost every direction and periodic directions are dense. Note that the two compact surfaces are the same, and the only difference is in the choice of cuts!

**Figure 7.** Both surfaces are equivalent (given the same number of omitted squares), but the two different realizations as a natural staircase are quite different. Identification is by opposite-side, except where marked with $a, b$.

**Example 4.6.** A straightforward check shows that the direction with slope 1/2 is a single cylinder direction for the surface shown in Figure 8 having $n$ squares in the top row and $n-1$ squares in the bottom row for $n = 3 \mod 4$. Thus, since, as in the previous example, both cuts have an end-point that is not a singularity, Theorem 4.3 implies that the flow on the corresponding staircase is ergodic in almost every direction and periodic directions are dense.

**Figure 8.** Identification is by opposite-side, except where marked with $a, b$. 
The condition that there is some direction $\theta'$ such that $\{M, \varphi_t\}$ decomposes into a single periodic cylinder is convenient but not necessary:

**Example 4.7.** Let $M$ be the square-tiled surface in Figure 9, and assume that the number of tiles in each row is odd and at least five. Then if we construct a $\mathbb{Z}$-staircase using the cuts marked, this staircase is ergodic in almost every direction. This surface, however, admits no single-cylinder directions.

**Proof.** That each row has more than three squares ensures that the two cuts with value $+1$ do not join together to form a loop; each cut of value $+1$, therefore, has both end-points not singularities. The assumption that the number of squares in each row is odd guarantees the existence of an integer involution point: a vertex around which we may rotate the entire figure by $\pi$. Suppose that some flow in slope $q/p > 1$ is a single-cylinder direction. This flow induces a rotation by $p/q$ on the bases of the squares; a periodic cylinder will be of width $1/q$ and may be taken to begin with an interval of length $1/q$ extending horizontally from this involution point. This surface has no single-cylinder directions (see the appendix to the arXiv version of this publication [26, Corollary A.5]).

The surface clearly has a decomposition into two cylinders of equal measure and length: the horizontal flow achieves such a decomposition, and the two cylinders are exchanged by the involution. By setting the involution point to be the origin, the involution itself may be represented as $-\text{Id}$. Of course $-\text{Id}$ is in the center of $\Gamma$, the Veech group. It follows that any image of these two cylinders under any element of $\Gamma$ also consists of two components that are exchanged by the involution.

Note that $f$, the function used to define the staircase, is invariant under this involution. Combining this fact with the fact that the involution exchanges the two cylinders, we see that the ergodic sum in each periodic cylinder is equal to the other. As their sum must be the average of the function, each periodic cylinder sees a total sum of zero. We may then consider the end-point of the cut with value $+1$ and directly apply the same construction in Figure 6 to find an essential value of $\pm1$ for this staircase.

![Figure 9](image-url)
Note that via the techniques of §3, any probabilistic technique of generating covers (not restricted to natural staircases, so the cuts are simple geodesic segments as in §3) on these surfaces will generically produce covers that are ergodic in almost every direction, provided

- the flow in the horizontal direction of every point sees a sum of zero across all cuts traversed through one period;
- the cuts are generated pairwise, invariant under the involution;
- the values taken on the cuts generate the group $G$.

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