On the Canonical $c$-Function in 4-d Field Theories Possessing Supergravity Duals

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Abstract

We study monotonicity and other properties of the canonical $c$-function (defined through the correlator of the energy-momentum tensor) in some holographic duals of 4-d quantum field theories. The canonical $c$-function and its derivatives are related to the 5d Green’s function of the dual supergravity theory. While positivity of the canonical $c$-function is obvious, we have not found a general proof of its monotonicity, even though $c$ is monotonic in the few explicit examples we examine in this paper.
Superconformal algebras in 4 dimensions possess two central charges, \( c \) and \( a \), defined as coefficients in the trace anomaly in the background metric \( g_{\mu \nu} \):

\[
\langle T^\mu_\mu \rangle_{\text{ren}} = \frac{c}{16\pi^2} (W_{\mu \nu \rho \sigma})^2 - \frac{a}{16\pi^2} (\tilde{R}_{\mu \nu \rho \sigma})^2,
\]

where \( W_{\mu \nu \rho \sigma} \) is the Weyl tensor and \( \tilde{R}_{\mu \nu \rho \sigma} \) is the dual of the curvature tensor. Outside the critical points, (1) contains additional terms proportional to the beta-function, with \( c \) and \( a \) promoted to the central functions \( c(x) \), \( a(x) \), since they acquire dependence on the running coupling \( g(x) \). One particularly interesting subclass of these theories, characterized by \( c = a \), has properties closely resembling those of the two-dimensional CFTs. All theories admitting a dual description in terms of supergravity fall in this class. The converse has also been conjectured to be true. In this case one can use 5d asymptotically AdS backgrounds to study RG flows between UV and IR conformal fixed points induced by relevant deformations of the 4d SCFT. Just as in two dimensions, the flows are irreversible, having \( c_{\text{UV}} - c_{\text{IR}} \geq 0 \). On the gravity side of the AdS/CFT, there exists a natural candidate for the Zamolodchikov-type c-function monotonically decreasing along the flow as a function of a scale - courtesy of Einstein’s equations of motion. This holographic c-function, \( c_H \), coincides with the canonical c-function defined by the equation:

\[
\langle T_{\mu \nu}(x)T_{\rho \sigma}(0) \rangle = -\frac{1}{48\pi^4} \Pi^{(2)}_{\mu \rho \nu \sigma} \left[ \frac{c(x)}{x^4} \right] + \pi_{\mu \nu} \pi_{\rho \sigma} \left[ \frac{f(x)}{x^4} \right],
\]

where \( \pi_{\mu \nu} = \partial_\mu \partial_\nu - \eta_{\mu \nu} \partial^2 \), and \( \Pi^{(2)}_{\mu \rho \nu \sigma} = 2\pi_{\mu \nu} \pi_{\rho \sigma} - 3(\pi_{\mu \rho} \pi_{\nu \sigma} + \pi_{\mu \sigma} \pi_{\nu \rho}) \), at the critical UV and IR points, where both are equal to the central charges \( c_{\text{UV}} \) and \( c_{\text{IR}} \), respectively. Outside the critical points, the c-function is not unique; in particular, the holographic c-function \( c_H \) does not (and need not to) coincide with the canonical one. There is also no a priori reason for the canonical c-function to be monotonic, although its values at the UV and IR critical points still satisfy inequality \( c_{\text{UV}} \geq c_{\text{IR}} \). In fact, it is known that outside the \( c = a \) corridor the canonical c-function is not monotonic. However, since theories that possess supergravity duals are distinguished by having \( c = a \) (to the leading order in \( N \)), and the supergravity description of RG flows naturally exhibits monotonicity, one may think that the canonical c-function somehow also inherits the monotonicity from the supergravity data. This is indeed so in the models studied so far, but we still do not know whether or not this is true in general.

In this letter we study properties of the canonical c-function defined by the Eq.(2) by relating it and its derivatives to the 5d Green’s function of the Klein-Gordon equation in the asymptotically AdS background. We shall use coordinate system in which 5d metric has the form

\[
ds^2 = dy^2 + e^{2\phi(y)} dx^\mu dx_\mu.
\]
In the supergravity description of the dual 4d RG flows the metric (3) is typically characterized by the following properties:

i) It is asymptotically AdS, i.e. \( \phi(y) = y/R_{UV} + O(e^{-2y/R_{UV}}) \) for \( y \to \infty \), where \( R_{UV} \) is the “radius” of the AdS space at \( y \to \infty \). We shall set \( R_{UV} = 1 \) henceforth.

ii) It has an IR singularity at some finite value of \( y \) which we can set to zero without loss of generality.

iii) If \( \phi(y) \) is the function describing the background of the supergravity dual to a certain gauge theory with 4d coordinate \( x \), then \( \phi(y + \lambda) - \lambda \) is the background corresponding to the rescaled 4d coordinate \( e^{-\lambda}x \) (with the IR singularity now located at \( y = -\lambda \), \( \phi(y) \to -\infty \) for \( y \to -\lambda \)). From points i) and iv) we also have:

iv) The derivative of \( \phi \) with respect to \( y \) is a monotonic function, \( \phi''_{yy} \leq 0 \), \( \phi''_{yy} = 0 \) at the critical points.

v) \( d\phi(y)/dy \geq 1 \).

vi) \( d\phi(y)/d\lambda = d\phi(y)/dy - 1 \).

The transverse traceless part of the energy-momentum tensor,

\[
T_{\mu\nu} = T_{\mu\nu} + \frac{1}{3} \square \pi_{\mu\nu} T, \tag{4}
\]

where \( T = T_{\mu}^{\mu} \), obeys

\[
\langle T_{\mu\nu}(x)T_{\rho\sigma}(0) \rangle = -\frac{1}{48\pi^4} \Pi^{(2)}_{\mu\nu\rho\sigma} \left( \frac{c(x)}{x^4} \right). \tag{5}
\]

With the Euclidean 4-momentum \( k \) oriented along \( z \)-axis we find

\[
c(|x|) = |x|^4 \int e^{-ikx} \frac{F(k^2)}{k^4} d^4k = 4\pi^2|x|^3 \int_0^\infty \frac{J_1(k|x|)}{k^2} F(k^2) dk, \tag{6}
\]

where \( F(k^2) = \langle T_{xy}(k)T_{xy}(0) \rangle \) is the momentum space two-point function of a minimally coupled massless scalar in an asymptotically AdS background. The function \( F(k^2) \) can be computed following the standard procedure \[8\], if the solution of the free wave equation in the asymptotically AdS background is known. For pure \( AdS_5 \) we have \( F(k^2) = -N^2k^4\log k^2/64\pi^2 \), and therefore \( G(x) = 3N^2/\pi^4x^8 \) and \( c(x) = N^2/4 \) (in the normalization of \[8\]). By dimensional analysis, the canonical central function \( c(x) \) depends on \( \lambda \) through the combination \( xe^{-\lambda} \). Since \( \dot{c}(x, \lambda) \equiv x\partial_x c(x, \lambda) = -dc(x, \lambda)/d\lambda \), the desired monotonicity property of \( c(x) \), \( \dot{c}(x, \lambda) \leq 0 \), is translated into the monotonicity of \( c(x, \lambda) \) with respect to \( \lambda \), \( \partial_\lambda c(x, \lambda) \geq 0 \). We now try to establish this property by looking at the 5d Green’s function \( G(x, y|x', y'|) \). Due to the 4d translation invariance,
\(G(x, y|x', y')\) depends on \(|x - x'|\); we put \(x' = 0\) and use \(x\) instead of \(|x|\) to simplify notations. The function \(G(x, y|0, y')\) obeys

\[-\partial_y e^{4\phi(y)} \partial_y G - \Box e^{2\phi(y)} G = \delta^{(4)}(x) \delta(y - y'),\]

from which we deduce

\[e^{4\phi(y)} \partial_y G(x, y|0, y') \mid_{\text{ren}} = -\Box \int_0^y dy_1 e^{2\phi(y_1)} \int_{y_1}^\infty dy_2 e^{-4\phi(y_2)} \int_{-\lambda}^{y_2} dy_3 e^{2\phi(y_3)} G(x, y_3|0, y') ,\]

where the symbol \(\mid_{\text{ren}}\) means that terms proportional to \(\delta^{(4)}(x)\) are discarded. Since

\[c(x, \lambda) = \kappa x^4 \lim_{y \to \infty} \lim_{y' \to \infty} e^{4\phi(y')} \partial_y' \int_{-\lambda}^y dy_1 \int_{y_1}^\infty dy_2 \int_{-\lambda}^{y_2} dy_3 F(x, y_1, y_2, y_3, y', \lambda) ,\]

\[x \partial_x c(x, \lambda) = -\frac{d}{d\lambda} c(x, \lambda) = \kappa x^4 \lim_{y \to \infty} \lim_{y' \to \infty} e^{4\phi(y')} \partial_y' \int_{-\lambda}^y dy_1 \int_{y_1}^\infty dy_2 \int_{-\lambda}^{y_2} dy_3 F' ,\]

where

\[F(x, y_1, y_2, y_3, y', \lambda) = e^{2\phi(y_1) - 4\phi(y_2) + 2\phi(y_3)} G(x, y_3|0, y') .\]

The above expression can also be written as

\[x \partial_x c(x, \lambda) = \kappa x^4 \lim_{y \to \infty} \lim_{y' \to \infty} e^{4\phi(y')} \partial_y' \int_{-\lambda}^y dy_1 e^{2\phi(y_1)} \int_{y_1}^\infty dy_2 e^{-4\phi(y_2)} \int_{-\lambda}^{y_2} dy_3 e^{2\phi(y_3)}\]

\[\left[ \frac{dG(x, y_3|0, y')}{d\lambda} + 2G(x, y_3|0, y')(\phi'(y_1) - 2\phi'(y_2) + \phi'(y_3)) \right] ,\]

where \(\phi'(y)\) denotes the derivative of \(\phi\) with respect to \(\lambda\). We would like to see that \(dc(x, \lambda)/d\lambda \geq 0\). Since for \(y' \to \infty\) \(G(x, y_3|0, y') \sim e^{-4y'}\), the sign of \(dc(x, \lambda)/d\lambda\) is the same as the sign of \(dG(x, y_3|0, y')/d\lambda + 2G(x, y_3|0, y')(\phi'(y_1) - 2\phi'(y_2) + \phi'(y_3))\). We have also \(y_1 \leq y_2, y_3 \leq y_2\) and therefore \(\phi'(y_1) - 2\phi'(y_2) + \phi'(y_3) \geq 0\) because of properties \(v)\) and \(vi)\) of \(\phi(y)\) itemized above. Since \(G(x, y_3|0, y') > 0\), the second term in the square bracket in Eq.\((13)\) is nonnegative.

We shall investigate now the monotonicity of \(G(x, y|0, y')\) with respect to the parameter \(\lambda\). The derivative \(\partial_\lambda G(x, y|0, y')\) obeys

\[\left( -\partial_y e^{4\phi(y)} \partial_y - e^{2\phi(y)} \right) \partial_\lambda G(x, y|0, y') = R(x, y, y') ,\]
where
\[ R(x, y; y') = 4\partial_y \phi'(y) e^{4\phi} \partial_y G + 2\phi'(y) \partial_y (e^{4\phi} \partial_y G) - 2\phi'(y) \delta^{(4)}(x) \delta(y - y') . \]  

(15)

Then
\[ \partial_\lambda G(x, y|0, y') = \int_\lambda^\infty d\tilde{y} \int \tilde{x} G(x, y|\tilde{x}, \tilde{y}) R(\tilde{x}, \tilde{y}, y') . \]  

(16)

To prove monotonicity of \( G(x, y|0, y') \) with respect to \( \lambda \) we have to show that \( R(x, y, y') \mid_{\text{ren}} \) given by
\[ R(x, y, y') \mid_{\text{ren}} = 4\partial_y \phi'(y) e^{4\phi} \partial_y G + 2\phi'(y) \partial_y (e^{4\phi} \partial_y G) \]  

(17)

(the term with delta-functions is irrelevant since upon integration it produces \(-2\phi'(y') G(x, y|0, y') \rightarrow 0 \) for \( y' \rightarrow \infty \) and \( x > 0 \)) is nonnegative. Even though this is certainly true for large values of \( y \), it appears to be false in general (see the counterexample below).

Indeed, if the property \( dG(x, y|0, y')/d\lambda \geq 0 \) would hold in general, it would imply \( dG(x)/d\lambda \geq 0 \), where \( G(x) \) is the 4d two-point function (the zeroth-order term in the expansion of \( G(x, y|0, y') \) in powers of \( e^{-4y}, e^{-4y'} \). The behavior of \( G(x) \) with respect to \( \lambda \) is related to the behavior of the scale-invariant function \( x^8 G(x) \) with respect to \( x \), since \( dG/d\lambda = -x \partial_x (x^8 G(x))/x^8 \). We shall demonstrate that in one of the three known explicit supergravity realizations of the dual RG flows, the function \( x^8 G(x) \) is not monotonic in \( x \), and therefore \( dG(x, y|0, y')/d\lambda \geq 0 \) can not be true in general. Intriguingly, in spite of this negative result, \( c(x) \) turns out to be monotonic in all explicit examples described below.

Flow to \( \mathcal{N} = 1 \) SYM

In our first example, we calculate the 5d Green’s function and the central function \( c(x, \lambda) \) for the flow to the \( \mathcal{N} = 1 \) SYM studied in [9]. The metric is of the form \( (3) \), where \( \phi(y) \) is given by
\[ \phi(y) = \frac{1}{2} \left( y + \log [2 \sinh(y + \lambda)] - \lambda \right) . \]  

(18)

The function \( \phi(y) \) has properties \( i) \) - \( vi) \) listed in the beginning of the paper. The solutions of the Klein-Gordon equation in the background given by (18) are
\[ \phi_1(y) = \frac{(1 - A^2)\pi A}{\sin \pi A} {}_2F_1 \left( A, -A; 2; 1 - e^{-2y-2\lambda} \right) , \]  

(19)

\[ 2 \text{ If } G \text{ is represented by } G = G_0 e^{-4y} + G_1 e^{-6y} + \ldots \text{, where } G_0 \text{ is positive by unitarity, then } e^{4\phi(y)} \partial_y G \rightarrow -4G_0 < 0 \text{ for large } y, \text{ whereas the term } 2\phi'(y) \partial_y (e^{4\phi} \partial_y G) \text{ is suppressed by the additional power of } e^{-2y} \text{ (note that } \phi'(y) \text{ and } \partial_y \phi'(y) \text{ have the same order of magnitude } \sim e^{-2y} \). \text{ Since } \partial_y \phi'(y) \leq 0, \text{ } R(x, y, y') \mid_{\text{ren}} = -16e^{-4y} \partial_y \phi(y) G_0(x) [1 + O(e^{-2y}, e^{-2y'})] \geq 0 \text{ and therefore } dG/d\lambda \geq 0. \]
Figure 1: Function $c(x)/N^2$ for the flow to $\mathcal{N} = 1$ SYM (solid line), $\mathcal{N} = 4$ Coulomb branch, discrete spectrum, $n = 4$ case (dotted line), $\mathcal{N} = 4$ Coulomb branch, continuous spectrum with a mass gap, $n = 2$ case (dashed line).

$$\phi_2(y) = -e^{-4y}F_1\left(2 + A, 2 - A; 3; e^{-2y-2\lambda}\right),$$

where $A = ike^\lambda/2$. The Green's function obeying Dirichlet condition at the boundary ($y = \infty$) is constructed in a standard way, and in the region $-\lambda \leq y \leq y'$ with $y' \to \infty$ is represented by a series

$$G(k^2, y, y') = e^{-4y'} \frac{(1 - A^2)\pi A}{\sin \pi A} F_1\left(1 - A; 1 - e^{-2y - 2\lambda}\right) + O(e^{-6y'}).$$

Then the 4d two-point function, $\mathcal{F}(k^2)$, can be obtained as

$$\mathcal{F}(k^2) = \lim_{y \to \infty} \lim_{y' \to \infty} e^{4(\phi(y') + \phi(y))} \partial_y \partial_{y'} G(k^2, y, y').$$

In the coordinate space (21) becomes $G(x, y|0, y') = e^{-4y'} G(x, y) + O(e^{-6y'})$, where

$$G(x, y) = \sum_{n=0}^\infty G_n(x, y),$$

$$G_n(x, y) = -\frac{N^2e^{-7\lambda}}{\pi^4x} \left(\frac{e^{-2y}}{e^{2\lambda}}\right)^n \frac{2e^{-4y}}{n!(n + 2)!} \sum_{t=n+2}^\infty \frac{4}{t^4} \left((n + 1)^2 - t^2\right) K_1(2txe^{-\lambda}).$$

The first term, $G_0(x, y) = e^{-4y} G(x)$, gives the 4d two-point function:

$$G(x) = \frac{N^2e^{-7\lambda}}{\pi^4x} \sum_{n=2}^\infty n^4(n^2 - 1)K_1(2nx e^{-\lambda}) = \frac{N^2e^{-7\lambda}}{\pi^4x} \int_0^\infty \frac{f(xe^{-7\lambda}, t) \cosh t}{(e^{2x} - e^{-\lambda} \cosh t - 1)^2} dt,$$
where \( f(u,t) = 48e^{4u\cosh t}(1 + e^{6u\cosh t}) + 312e^{6u\cosh t}(1 + e^{2u\cosh t}) > 0 \). The 5d Green’s function is monotonic with respect to the parameter \( \lambda \) (see Fig.(2)) which according to the discussion in the preceding section implies monotonicity of the canonical central function. The explicit expression for \( c(x) \) was obtained in [10]:

\[
c(x) = N^2 x^3 e^{-3\lambda} \sum_{n=2}^{\infty} (n^2 - 1) K_1(2nx e^{-\lambda}) = N^2 x^3 e^{-3\lambda} \int_0^\infty \frac{3e^{2x \exp -\lambda \cosh t} - 1}{(e^{2x \exp -\lambda \cosh t} - 1)^3} \cosh t \, dt. \tag{26}
\]

Using Mellin summation technique\( \cite{12} \) we obtain the following representation for \( c(x) \)

\[
c(x) = N^2 x^3 e^{-3\lambda} \sum_{n=2}^{\infty} (n^2 - 1) K_1(2nx e^{-\lambda}) = N^2 x^3 e^{-3\lambda} \sum_{n=0}^{\infty} \frac{x^{2n} e^{-2n\lambda}}{n!(n+1)!} \left[ \zeta'(-2n-3) - \zeta'(-2n-1) \right] \tag{27}
\]

\[
- N^2 x^4 e^{-4\lambda} \sum_{n=0}^{\infty} \frac{x^{2n} e^{-2n\lambda}}{n!(n+1)!} \left[ \frac{B_{2n+2}}{n+1} - \frac{B_{2n+4}}{n+2} \right] \psi(n+1) + \frac{1}{2(n+1)} - \log(x e^{-\lambda}) \]

where \( \zeta'(s) \) is the derivative of the Riemann zeta-function, \( B_{2n} \) are Bernoulli numbers. The central function is positive definite, monotonic (Fig. (1)), and has the correct UV and IR limits: for small \( x \) \( c(x) \to c_{UV} = N^2/4 \) and for \( x \to \infty \) \( c(x) \sim x^{5/2} e^{-4x} \to c_{IR} = 0 \). It is also non-analytic in \( x \) as the expansion (27) shows.

**Coulomb branch of \( \mathcal{N} = 4 \) SYM**

The metric corresponding to the supergravity description of the \( n \)-dimensional (\( 1 \leq n \leq 5 \)) transverse distribution of the \( D3 \) branes (associated with the Coulomb branch of the \( \mathcal{N} = 4 \) SYM with \( SO(n) \times SO(6 - n) \) local gauge symmetry) was obtained in [11]. For \( n = 2, 4 \) the s-wave Klein - Gordon equation for a minimally coupled massless scalar,

\[
\frac{1}{r} \partial_r \left[ r^5 \left( 1 \pm \frac{\ell^2}{r^2} \right) \partial_r \xi_k(r) \right] - k^2 L^4 \xi_k(r) = 0 , \tag{28}
\]

where \( \pm \) corresponds to \( n = 2, 4 \), respectively, \( L \) is the radius of the asymptotic \( AdS_5 \) and \( \ell \) characterizes the size of the branes distribution, can be solved exactly. We consider the \( n = 2 \) and the \( n = 4 \) cases separately.

\[3\]Mellin summation (rather than the Euler-Maclaurin formula) is convenient in this case since the function of \( n \) multiplying the modified Bessel function in [20] is a polynomial. This is not the case in our third example, where it is the Euler-Maclaurin formula which allows to obtain an explicit expression whilst the Mellin summation is useless.
Figure 2: $\mathcal{N} = 1$ flow: five-dimensional Green’s function $G(x, y = 1)$ versus $\lambda$ for (from top to bottom) $x = 1.41$, $x = 1.55$, $x = 1.7$

Figure 3: $\mathcal{N} = 4$ flow, $n = 2$ (continuous spectrum): four-dimensional Green’s function $G(x)$ versus $\lambda$ for (from top to bottom) $x = 1$, $x = 1.1$, $x = 1.2$

Figure 4: $\mathcal{N} = 4$ flow, $n = 4$ (discrete spectrum): four-dimensional Green’s function $G(x)$ versus $\lambda$ for (from top to bottom) $x = 1$, $x = 1.1$, $x = 1.2$
Two-dimensional distribution of $D3$ branes

The solutions of (28) are

\[ \xi_k^{(1)}(r) = v a_+ F_1(a_+, a_+; 2a_+ + 2; v), \]  
\[ \xi_k^{(2)}(r) = v a_- F_1(a_-, a_-; 2a_- + 2; v), \]

where $a_\pm = -1/2 \pm \sqrt{1 + e^{2\lambda}k^2}$, $e^\lambda = L^2/\ell$, $v = 1/(1 + \ell^2/r^2)$. The first solution is regular at the location of the branes, and is used to construct the two-point function [11]:

\[ \mathcal{F}(k^2) = -\frac{N^2 k^4}{32\pi^2} \Psi \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + e^{2\lambda}k^2} \right). \]

Then the central function is given by

\[ c(x) = -\frac{N^2 x^3}{8} \int_0^\infty k^2 J_1(kx) \Psi \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + e^{2\lambda}k^2} \right) dk. \]

Using the Binet expression [13],

\[ \Psi(z) = \log z - 1/2z - 2 \int_0^\infty (t^2 + z^2)^{-1} (e^{2\pi t} - 1)^{-1} t dt, \]

and the result\(^4\)

\[ \int_0^\infty x^2 \log \left( 1 + \frac{z^2}{x^2} \right) J_1(cx) dx = \frac{4}{c^3} - \frac{2\pi^2}{c} K_2(cz), \]

one can obtain the following spectral representation for $c(x)$:

\[ c(x) = \frac{N^2 x^4 e^{-4\lambda}}{4} \int_1^\infty \rho(\mu^2) \Delta_E(e^{-\lambda}x, \mu^2) d\mu^2, \]

where

\[ \rho(\mu^2) = \pi^2 \left( 1 - \frac{2}{e^{2\pi \sqrt{\mu^2 - 1}} - 1} + \frac{4}{e^{2\pi \sqrt{\mu^2 - 1}} - 1} \right). \]

\(^4\) This integral can be obtained by taking a derivative with respect to $c$ of the expression

\[ \int_0^\infty x \log \left( 1 + \frac{z^2}{x^2} \right) J_0(cx) dx = \frac{2}{c^2} - \frac{2z}{c} K_1(cz), \]

given in 2.12.28.9 of [13].
and $\Delta_E(x, m^2) = \sqrt{m^2 x^2} K_1(\sqrt{m^2 x^2})/4\pi^2 x^2$ is a four-dimensional free Euclidean propagator. For small $x$ we get

$$c(x) = \frac{N^2}{4} - \frac{7N^2 x^2 e^{-2\lambda}}{96} + O\left(x^4, x^4 \log x\right).$$  \hfill (36)

Using this expansion, we can calculate the anomalous dimension of the operator inducing the flow in the UV limit,

$$h_{UV} = -\lim_{UV} \frac{\ddot{c} - \dot{c}}{2\dot{c}} = -1. \hfill (37)$$

The coordinate representation of the scale-invariant function $x^8 G(x)$ is given by

$$x^8 G(x) = \frac{3N^2}{\pi^4} \left(1 + xK_1(x)\frac{(x^2 + 8)^2}{64} + x^2 K_0(x)\frac{x^2 + 8}{16}\right)$$

$$+ \frac{x^6}{384} K_0(x) - x [K_0(x) I_1(x) + K_1(x) I_0(x)]$$

$$+ \frac{x^8}{192} \int_1^\infty \frac{\mu^4}{e^{2\pi\sqrt{\mu^2 - 1}} - 1} \frac{\mu K_1(\mu x)}{x} d\mu^2 - \frac{x^8}{384} \int_1^\infty \frac{\mu^4}{e^{\pi\mu^2 - 1}} \frac{\mu K_1(\mu x)}{x} d\mu^2. \hfill (40)$$

Note that $x^8 G(x)$ (or $x^8 e^{-8\lambda} G(x e^{-\lambda})$ versus $\lambda$) in this case is monotonic (Fig.3) which again implies monotonicity of $c(x)$ (Fig.1).

**Four-dimensional distribution of D3 branes**

The solutions of (28) for $n = 4$ are

$$\xi^{(1)}_k(r) = (1 - u)^2 F_1 \left(a + 2, 1 - a; 1; u\right), \hfill (38)$$

$$\xi^{(2)}_k(r) = 2 F_1 \left(a, -1 - a; 1; u\right), \hfill (39)$$

where $a = -1/2 + \sqrt{1 - e^{2\lambda k^2}/2}$, $u = 1 - \ell^2 / r^2$. The second solution (regular at $r = 0$) leads to the two-point function with the discrete spectrum [11]:

$$\mathcal{F}(k^2) = -\frac{N^2 k^4}{64\pi^2} \left[\Psi\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - e^{2\lambda k^2}}\right) + \Psi\left(\frac{1}{2} - \frac{1}{2} \sqrt{1 - e^{2\lambda k^2}}\right)\right]. \hfill (40)$$

We shall see that in this case the Green’s function is not monotonic with respect to the parameter $\lambda$. The coordinate representation of (40) is given by

$$G(x) = \frac{N^2}{2\pi^4 x} e^{-7\lambda} \sum_{n=1}^\infty (2n + 1)[n(n + 1)]^{5/2} K_1\left(2\sqrt{n(n + 1)} xe^{-\lambda}\right). \hfill (41)$$
The explicit expression for $G(x)$ can be obtained from (41) using Euler - Maclaurin formula:

$$G(x) = \frac{N^2 e^{-7\lambda}}{\pi^4 x} F(x, \lambda) + \frac{3\sqrt{2}N^2 e^{-7\lambda}}{\pi^4 x} K_1(2\sqrt{2}xe^{-\lambda})$$
$$- \frac{N^2 e^{-7\lambda}}{2\pi^4 x} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} f^{(2n-1)}(0),$$

(42)

where $f(t) = (2t+3)[(t+1)(t+2)]^{5/2} K_1(2\sqrt{(t+1)(t+2)}xe^{-\lambda})$, and the integral involved is calculated in the Appendix with the result

$$F(x, \lambda) = \int_{\sqrt{2}}^{\infty} t^6 K_1(2tx\lambda)dt$$
$$= \frac{3e^{7\lambda}}{x^7} \left\{ 1 + \left(1 + x^2 e^{-2\lambda}\right)^2 2\sqrt{2}xe^{-\lambda} K_1(2\sqrt{2}xe^{-\lambda}) 
+ 4x^2 e^{-2\lambda} \left(1 + x^2 e^{-2\lambda} + \frac{x^4 e^{-4\lambda}}{3}\right) K_0(2\sqrt{2}xe^{-\lambda})
- 2\sqrt{2}xe^{-\lambda} \left[K_0(2\sqrt{2}xe^{-\lambda}) I_1(2\sqrt{2}xe^{-\lambda}) + I_0(2\sqrt{2}xe^{-\lambda}) K_1(2\sqrt{2}xe^{-\lambda})\right] \right\}.$$

Representation (42) allows us to study the region of small $x$:

$$G(x, \lambda) = \frac{3N^2}{\pi^4 x^8} + \frac{2N^2 e^{-6\lambda}}{315\pi^4 x^2} + O \left(1, \log (xe^{-\lambda})\right).$$

(43)

Note that the scale-invariant function $x^8 G(x)$ is not monotonic (see Fig. 4). Analytically it can be seen from (43), where the second term is positive definite, so that for small $x$ the function $x^8 G(x)$ first increases before decreasing (exponentially) down to zero for $x \to \infty$.

Nevertheless, the canonical $c$-function is monotonic. The $c$-function can be obtained by slightly modifying the procedure described in [10]:

$$c(x) = \frac{N^2 x^3 e^{-3\lambda}}{2} \sum_{n=1}^{\infty} (2n+1) \sqrt{n(n+1)} K_1(2\sqrt{n(n+1)}xe^{-\lambda}).$$

(44)

Applying Euler - Maclaurin summation formula, we get

$$c(x) = \frac{N^2 x^2 e^{-2\lambda}}{2} K_2(2\sqrt{2}xe^{-\lambda}) + \frac{3\sqrt{2}N^2 x^3 e^{-3\lambda}}{4} K_1(2\sqrt{2}xe^{-\lambda})$$
$$- \frac{N^2 x^3 e^{-3\lambda}}{2} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} f^{(2n-1)}(0),$$

(45)
where \( f(t) = (2t + 3)\sqrt{(t + 1)(t + 2)}K_1(2\sqrt{(t + 1)(t + 2)}xe^{-\lambda}) \). The expansion for small \( x \) is

\[
c(x) = \frac{N^2}{4} - \frac{N^2x^2e^{-2\lambda}}{6} + O\left(x^4, x^4\log x\right).
\]

The function \( c \) is positive definite and monotonic (Fig.[1]).

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**Appendix: Calculation of the integral**

\[
I(c, a) = \int_a^\infty x^6K_1(cx)dx
\]

Volume 2 of \[14\] (1.12.1.1, p.47) provides an expression for an indefinite integral, \( \int x^6K_1(x)dx \), in terms of Lommel’s functions which for our purposes can be rewritten as

\[
I(c, a) = \frac{1}{c^7} \left[ \frac{x^7}{6}K_1(x) \frac{1}{4} \right]_a^\infty
\]

where

\[
_1F_2(a; b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n(c)_n} \frac{z^n}{n!}
\]

is one of the hypergeometric functions, \( (a)_n = \Gamma(n + a)/\Gamma(a) \). It is known (see 7.14.2.98 in Volume 3 of \[14\]) that

\[
_1F_2\left(1; 4, 4; z^2\right) = \frac{9}{z^6} \left[ 4I_0(2z) - 4 - 4z^2 - z^4 \right].
\]

Integrating the series (48) using (49) we obtain

\[
_1F_2\left(1; 4, 5; z^2\right) = \frac{36}{z^8} \left[ 4zI_1(2z) - 2z^2 - 2z^4 - \frac{z^6}{3} \right].
\]

Substituting (48) - (50) into (47) we get

\[
\int_a^\infty x^6K_1(cx)dx = \frac{384}{c^7} \left\{ 1 + \left(1 + \frac{a^2c^2}{8}\right)^2 \right\} acK_1(ac) + \frac{a^2c^2}{2} \left( 1 + \frac{a^2c^2}{8} + \frac{a^4c^4}{192} \right) K_0(ac)
\]

\[
- ac [K_0(ac)I_1(ac) + I_0(ac)K_1(ac)].
\]

For \( a = 0 \) (51) reduces to the well-known result, \( I(c, 0) = 384/c^7 \) \[14\].
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