A hodological law causes the evolution of the universe to tend to follow particular types of path. I give simple illustrations in toy models and discuss how Kolmogorov complexity characterises the extent to which hodological laws explain, rather than merely describe, data.

INTRODUCTION

If the probabilities we calculate in quantum theory are probabilities of some well-defined, objective, observer-independent features of nature, then a complete formulation of quantum theory has to include a sample space on which these probabilities are defined. The elements of that sample space form configurations of beables, in Bell’s terminology. Whatever form they take, if they form part of physics as we understand it they presumably have a mathematical structure. It then makes sense to consider generalisations of quantum theory in which the probabilities depend on that structure as well as the Born rule.

This motivates looking at alternatives \cite{1,2} to cosmological theories inspired by the standard understanding of quantum theory. Whatever the fine-grained form of the beables, generalised probability laws associated with them could also affect the probabilities of coarse-grained large-scale features of the universe. The very large scale seems perhaps the most promising regime in which to look for empirical evidence of such deviations from quantum theory, since the strongest evidence for quantum theory comes from small scale phenomena, the relationship between quantum theory and gravity is not known, and there are other outstanding cosmological puzzles that suggest other lacunae in our understanding.

Theories that are based on quantum theory but guide the universe along paths other than those implied by standard unitary quantum dynamics are not yet part of standard mainstream discourse. They impose constraints, in a statistical sense. However, these theories are qualitatively different from quantum theory applied to constrained systems \cite{3}. They are far more general than theories with independent initial and final boundary conditions. They are also far more general than dynamical collapse models, although dynamical collapse models can be seen as examples and can motivate others. Indeed, the generality they allow may raise a concern that considering such theories takes us out of the domain of science: that they can describe data but cannot explain them in any standard scientific sense.

I explain below why this concern is misplaced, using simple models that show why these “hodological” theories are qualitatively different, illustrate their generality and explain the extent to which they could nonetheless be scientifically useful.

HODOLOGY IN THE EHRENFEST URN MODEL

The Ehrenfest urn model \cite{4} nicely illustrates the effect of laws describing a statistical evolution from an initial state. It can be generalized to illustrate laws with independent initial and final boundary conditions \cite{5}. As we discuss below, it can also be generalized to illustrate hodological laws.

The standard version of the Ehrenfest urn model begins with $N$ labelled balls distributed between two urns ($A$ and $B$) in some initial configuration (for example, all in urn $A$, or balls 1 to $\left\lfloor \frac{N}{2} \right\rfloor$ in $A$ and the rest in $B$). The model’s state changes in discrete time steps, at each of which one label is chosen randomly, and the corresponding ball switches urn. It is easy to see (analytically or numerically) that low entropy distributions typically evolve quickly towards and then fluctuate around equipartition, spending nearly all the time close to equipartition and returning to low entropy states very infrequently.

We consider the model with some number $T$ of time steps that is fixed in advance. One might think of this as a toy model of a universe with a fixed cosmological lifetime between its initial and final state. We take the numbers of balls $N_A$ and $N_B = N - N_A$ in urns $A, B$ to be the macro-physical variables of interest, and the locations of each labelled ball to be micro-physical variables. Macrophysically, the possible evolutions from the initial state $N_A = N_A^0$
are thus given by sequences

\[ \overline{N_A} = (N_A^0, N_A^1, \ldots, N_A^T), \]

(1)

where \( N_A^i = N_A^{i-1} \pm 1 \). The sequence \( \overline{N_A} \) has probability

\[
\text{Prob}(\overline{N_A}) = \prod_{i=1}^{T} \left( \delta(N_A^i - N_A^{i-1}, 1) \frac{N - N_A^{i-1}}{N} + \delta(N_A^i - N_A^{i-1}, -1) \frac{N_A^{i-1}}{N} \right). \]

(2)

We are interested in hodological generalisations that alter, and are defined in terms of, the macrophysics. To define such a model, we modify Eqn. (2), reweighting the probabilities by non-negative factors \( w(\overline{N_A}) \) that depend on the form of the path \( \overline{N_A} \) through configuration space. Thus

\[
\text{Prob}_{\text{mod}}(\overline{N_A}) = C w(\overline{N_A}) \text{Prob}(\overline{N_A}),
\]

(3)

where \( C \) is a normalisation constant chosen so that

\[
\sum_{\overline{N_A}} \text{Prob}_{\text{mod}}(\overline{N_A}) = 1.
\]

(4)

**Simple examples**

**Example 1 (fixed macrophysical path points):** Let \( N = 10, T = 20, N_A^0 = 5 \), and take

\[
w(\overline{N_A}) = \delta(N_A^1, 5) \delta(N_A^{20}, 5).
\]

(5)

This weighting ensures that the realised evolution path has an equipartition as its initial and final states and also at the midpoint of its evolution. A sample evolution is shown in Fig. 1.

![FIG. 1: Single run of \( N = 10 \) balls, constrained to \( N_A = N_B = 5 \) at \( t = 0, 10 \) and 20.](image)

**Example 2 (weighting towards a given macrophysical path):** If, again with \( N = 10, T = 20, N_A^0 = 5 \), we take

\[
w(\overline{N_A}) = \exp\left( -\frac{1}{6} \sum_{t=1}^{20} (N_A(t) - (5 - \frac{t}{4}))^2 \right),
\]

(6)

then the realised evolution path is likely to be relatively close to the line \( N_A(t) = (5 - \frac{t}{4}) \). Sample evolutions are shown in Fig. 2.
FIG. 2: 5 runs of $N = 10$ balls, initial state $N_A = N_B = 5$ at $t = 0$, drawn from an ensemble with evolution probabilities modified by the weight factor (6).

**Testing hodological laws**

Suppose now, for the sake of discussion, that we observe a new physical system whose properties are opaque to us, except for one discrete physical parameter that appears to evolve as though following some type of Ehrenfest urn model. That is, there is one observable discrete parameter, $N_A$, which appears always to lie in the range $0 \leq N_A \leq 10$. We observe its value only at discrete time steps $t$, after each of which it increases or decreases by 1. Suppose we cannot measure anything else about the system (perhaps it is effectively a black box, or a very distant cosmological object that regularly emits a discrete signal). Suppose also that, while we cannot directly observe the system’s internal structure, extrapolating our knowledge of other better understood systems, and examining the evolution statistics of $N_A$, lead us to the strong hypothesis that it is characterised by some Ehrenfest model, with labelled subsystems playing roles analogous to those of the balls and urns. Suppose also that we have no information or good hypothesis about any interaction with other systems. And suppose that the system goes through repeated runs of 20 time steps, apparently resetting (say after a gap of 10 time steps, so that individual runs are identifiable) after each, with each run starting with $N_A = 5$.

After a while, we will conclude that, so long as we learn nothing more about the system, the only immediately scientifically fruitful theories we can make about it are defined by generalised Ehrenfest urn models of the form (3). We can evaluate these by Bayesian reasoning. Informally, this would run roughly as follows. First, if our physical theories (the new system aside) take their current form, defined by initial states and standard evolution laws, then before we examine the data we would assign a high prior weight to the standard Ehrenfest probability law (2), i.e. to $Cw(N_A) = 1$ for all paths $N_A$. We might assign a lower prior weight to the hypothesis that any modification of the form (3) gives a better theory, and we would almost certainly assign low prior weights to specific modified laws like (5) and (6). But since the system is novel and mysterious, we should and probably would be undogmatic: every specific law $L$ would be assigned a non-zero prior weight $\text{Prob}_{\text{prior}}(L)$.

Suppose that on the first run we observed an evolution of the form of Fig. 1. According to the standard Ehrenfest probability law (2), the probability of equipartition of 10 balls at $t = 10$, given initial equipartition, is $\frac{964533}{1953125} \approx \frac{1}{2}$. The probability of equipartition at both $t = 10$ and $t = 20$, given initial equipartition, is thus $\approx \frac{1}{4}$.

Bayesian hypothesis testing, given data $D$, assigns the posterior probability weight

$$\text{Prob}_{\text{post}}(L) = \frac{\text{Prob}(D | L)\text{Prob}_{\text{prior}}(L)}{\sum_i \text{Prob}(D | L_i)\text{Prob}_{\text{prior}}(L_i)},$$

where the sum is over the set (which we assume countable) of all laws considered.

After the resulting Bayesian reweighting, our posterior weights for some of our modified laws would thus be smaller or zero, and our weight for (5) would (for sensible values of $\text{Prob}_{\text{prior}}(L_i)$) be somewhat larger. If our prior confidence in the law defined by Eqn. (2) was high, our posterior confidence would still be high. However, if we saw $M$ runs, all of which produced evolutions with equipartition at $t = 10$ and $t = 20$, the numerator in our posterior weight for (2) will be scaled by $(\frac{1}{4})^M$, while the corresponding expression for Eqn. (5) remains unchanged. If the evolutions appear to be otherwise random, then our posterior weights for Eqn. (5) increase with $M$, tending to 1 for large $M$. In other words, we would eventually become very confident that the system is in fact governed by Eqn. (5).
Suppose instead that we saw an evolution of the type illustrated by Fig. 2. According to the standard Ehrenfest urn model, the probability of an evolution as close as these to the line \( N_A(t) = (5 - \frac{4}{t}) \) is roughly 1 in 50000. Even after a single run, unless our prior weight for any law other than (2) was significantly less than \( 2 \times 10^{-5} \), we would significantly lose confidence in (2) and begin considering alternative laws seriously. After a small number of runs, we would likely arrive at something like Eqn. (6) as our best fit to the data.

Since known physical laws are based on probabilistic or deterministic evolution from initial conditions, we might think a system apparently described by a modified Ehrenfest urn model such as (5) or (6) must very likely have some additional internal mechanism and associated variables hidden from us, so that the complete system is described by a more conventional law. We might then continue to search for ways of observing the hidden variables and obtaining a better and more detailed model. Still, unless and until we succeeded, the relevant modified Ehrenfest urn model would be our best description. And we might not succeed: there need not necessarily be any internal mechanism that gives any deeper explanation.

Formally, these calculations can be underpinned by the theory of Solomonoff induction and the principle of minimum description length (MDL) for hypothesis identification [6]. Roughly speaking, according to the MDL principle, the best hypothesis to fit the data is the one that minimizes the sum of the length of the program required to frame the hypothesis and the length of the string required to characterize the data given the hypothesis. The latter is approximately the Shannon entropy \( S(H) \) of a program mapping \( \approx S(H) \) bit strings to paths that, according to hypothesis \( H \), are typical. If \( H_0 \) is given by (2), \( H_1 \) by (5) and \( H_2 \) by (6), then for a single run

\[
S(H_0) - S(H_1) \approx 2, \quad S(H_0) - S(H_2) \approx 16.
\]  

For \( M \) runs, \( L(H_i) \) is fixed, while

\[
S(H_0) - S(H_1) \approx 2M, \quad S(H_0) - S(H_2) \approx 16M.
\]

Hence, if \( H_1 \) or \( H_2 \) fit the data, their description length becomes less than that of \( H_0 \) for large \( M \), and they become preferred to \( H_0 \); if no more refined hypothesis fits the data then they become the MDL hypothesis. The same is true of any hypothesis \( H \) such that \( S(H_0) - S(H) > 0 \).

**DISCUSSION**

The Ehrenfest urn model illustrates how a model with probabilistic microdynamics can be modified by macrodynamical laws that guide macroscopic variables towards particular paths. Such laws themselves may be either deterministic (as in example (5)) or probabilistic (as in example (6)). It also illustrates how standard scientific inference can identify such laws, if they offer a compressed description of the observed data.

Exactly the same points apply when we consider a microdynamics given by any version of quantum theory that makes probabilistic predictions about the microdynamics underpinning the physics of a macroscopic system, including in principle the evolution of the universe. As we have seen, one can model the evolution of a physical system via an Ehrenfest urn model without committing to identifying specific subsystems as balls and urns, or even committing to the belief that such subsystems necessarily exist. Similarly, one can search for modified macrodynamical laws in nature while remaining agnostic about precisely which events, beables, or histories define the fundamental sample space for quantum theory. The task is more complicated, because there are many more possible relevant variables and types of law. Nonetheless, the methodology of Solomonoff induction applies. A systematic search for modified macrodynamical laws that might fit observation better than standard quantum theory should be a major goal of cosmological science, since in a strong sense it would necessarily advance our knowledge. Null results, excluding all laws of a given type up to a given degree of complexity, would solidify and parametrise our confidence in the standard paradigm. New laws would qualitatively change our understanding of nature.

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References

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[1] Adrian Kent. Beyond boundary conditions: General cosmological theories. In COSMO-97, pages 562–564. World Scientific, 1998 (arxiv:0905.0632).

[2] Adrian Kent. Beable-guided quantum theories: Generalizing quantum probability laws. Physical Review A, 87(2):022105, 2013.

[3] Paul Adrien Maurice Dirac. Lectures on quantum mechanics, volume 2. Courier Corporation, 2001.

[4] Paul Ehrenfest and Ehrenfest Tatjana. Über zwei bekannte einwände gegen das boltzmannsche h-theorem. Physikalische Zeitschrift, 8:311–314, 1907.

[5] Murray Gell-Mann and James B Hartle. Time symmetry and asymmetry in quantum mechanics and quantum cosmology. Physical origins of time asymmetry, 1:311–345, 1994.

[6] Ming Li, Paul Vitányi, et al. An introduction to Kolmogorov complexity and its applications, volume 3. Springer, 2008.

[7] Examples of relevant versions of quantum theory include theories with some form of Copenhagen collapse rule, quantum theory supplemented by mass-energy beables determined mathematically by (fictitious) asymptotically late time measurements, a consistent history version of quantum theory defined via some appropriate set selection rule, an one-world version of quantum theory defined by some appropriate selection rule for Everettian branches, or some version of de Broglie-Bohm theory.