A Generalization of Brown’s Construction for the Degree/Diameter Problem

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Abstract

The degree/diameter problem is the problem of finding the largest possible number of vertices \( n_{\Delta,D} \) in a graph of given degree \( \Delta \) and diameter \( D \). We consider the problem for the case of diameter \( D = 2 \). William G Brown gave a lower bound of the order of \((\Delta, 2)\)-graph. In this paper, we give a generalization of his construction and improve the lower bounds for the case of \( \Delta = 306 \) and \( \Delta = 307 \). One is \((306, 2)\)-graph with 88723 vertices, the other is \((307, 2)\)-graph with 88724 vertices.

1 Introduction

The degree/diameter problem is the problem of finding the largest possible number \( n_{\Delta,D} \) of vertices in a graph of given degree \( \Delta \) and diameter \( D \). Let \( G \) be a graph with degree \( \Delta \) (\( \Delta > 2 \)) and diameter \( D \), then we have

\[
|G| \leq n_{\Delta,D} \leq 1 + \Delta \sum_{k=1}^{D-1} (\Delta - 1)^k
\]

where \( |G| \) is the number of vertices of \( G \). The right hand side of the above inequation is called Moore bound. On the other hand, a lower bound of \( n_{\Delta,D} \) for small degree and small diameter are available at http://combinatoricswiki.org. Especially for case of \( D = 2 \) and large degree, there exists the general construction that gives a lower bound of \( n_{\Delta,2} \), which is called Brown’s construction [3]. Let \( F_q \) be the finite field, where \( q \) is a power of a prime. Brown’s construction gives the graph \( B(F_q) \) whose vertices are lines in \( F_q^2 \) and two lines are adjacent if and only if they are orthogonal. It follows that

\[
|B(F_q)| = q^2 + q + 1, \quad \Delta(B(F_q)) = q + 1, \quad D(B(F_q)) = 2.
\]

The degree of each vertex of \( B(F_q) \) is \( q + 1 \) or \( q \). Among \( q^2 + q + 1 \) vertices, \( q + 1 \) vertices are of degree \( q \) and \( q^2 \) vertices are of degree \( q + 1 \). If \( q \) is a power of 2, there exists \((q + 1, 2)\)-graph with \( q^2 + q + 2 \) vertices [2].

In this paper, we generalize Brown’s construction by replacing a field with a commutative ring, and search new records of the degree/diameter problem.
2 Generalized Brown’s Construction

We give some definitions for generalized Brown’s construction. A graph \( G = (V, E) \) consists of a set \( V \) of vertices and a set \( E \subset \{(v, w) \in V^2 | v \neq w \} \) of edges. If \((v, w)\) is in \( E \), it is said that \( v \) and \( w \) are adjacent, which is denoted by \( v \sim w \). The order \(|G|\) of the graph is the number of vertices. The neighbours \( N(v) \) of the vertex \( v \) is a set of vertices which are adjacent to \( v \). The degree \( \delta(v) \) of the vertex \( v \) is the number of neighbours \(|N(v)|\). The degree \( \Delta(G) \) of the graph \( G \) is the maximum degree of vertices, namely \( \Delta(G) = \max \{\delta(v) | v \in V \} \).

The distance \( d(v, w) \) of vertices is the shortest path length between \( v \) and \( w \). The diameter \( D(G) \) of the graph is the maximum distance of all pairs of vertices.

Let \( R \) be a commutative ring with unity. \( R^3 \) denotes the set of invertible elements of \( R \). \( R^3 \) is naturally seen as \( R \)-module. The addition and \( R \)-action are defined by coordinate-wise. The inner product \( \cdot : R^3 \times R^3 \Rightarrow R \) is defined as follows:

\[
(v_1, v_2, v_3) \cdot (w_1, w_2, w_3) = v_1w_1 + v_2w_2 + v_3w_3.
\]

\( v \) and \( w \) are orthogonal if and only if the inner product vanishes, namely \( v \cdot w = 0 \). The cross product \( \times : R^3 \times R^3 \Rightarrow R \) is defined as follows:

\[
(v_1, v_2, v_3) \times (w_1, w_2, w_3) = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1).
\]

Definition 1. Let \( R \) be a commutative ring with unity. The vertex set \( V \) of the graph \( B(R) \) is

\[
V = (R^3 \setminus \{v | \exists r \in R, r \cdot v = 0 \}) / \sim
\]

where \( v \sim w \) if and only if there exists \( k \in R^* \) such that \( k \cdot v = w \). The two vertices \([v] \) and \([w] \) are adjacent if and only if \( v \cdot w = 0 \).

The definition of adjacency above is well-defined because the orthogonality does not depend on the selection of representatives. We call the above construction of a graph from a ring a generalized Brown’s construction. It is clear that the new construction coincides with Brown’s one when the ring \( R \) is a field.

Lemma 1. Let \( E \) be a Euclidean domain and \( u \) be a prime element in \( E \). If \( E/(u^k) \) is a finite ring, then the degree of each vertex of \( B(E/(u^k)) \) is \( \Delta \) or \( \Delta - 1 \), where \((u^k)\) is the principal ideal generated by \( u^k \).

Proof. It is clear that the degrees of vertices represented by \((1, 0, 0), (0, 1, 0), (0, 0, 1) \) are the same. Let \( v = ([a], [b], [c]) \) be a representative of any vertex where \( a, b, c \in E \). If any element of \([a], [b], [c] \) is not invertible in \( E/(u^k) \), there exist natural numbers \( 1 \leq l, m, n < k \) and some elements \( a', b', c' \in E \) such that \( a = u^la', b = u^mb', c = u^nc' \). \( v \) is not a representative of vertices because the equation \([u^{\min(l, m, n)}] \cdot v = ([0], [0], [0]) \) holds. This is a contradiction. Therefore, at least one of \([a], [b], [c] \) is invertible. If \([a] \) is invertible, there exists one-to-one correspondence \( \overline{U} : N([1, 0, 0]) \to N([v]) \) such that for all \([w] \in N([1, 0, 0]) \),
$U(w) = [U^{-1}w]$ where $U$ is an invertible matrix defined as follows

$$U = \begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{pmatrix}$$

If $v \cdot v = 0$, then $\delta([w]) = \delta([(1, 0, 0)]) - 1$. If not so, $\delta([w]) = \delta([(1, 0, 0)])$. In the same way, if $[b]/[c]$ is invertible, then $\delta([w]) = \delta([(0, 1, 0)])/\delta([(0, 0, 1)])$ or $\delta([(1, 0, 0)])/\delta([(0, 0, 1)]) - 1$. Therefore, for all the vertex $[w]$,

$$\delta([w]) = \begin{cases} \delta([(1, 0, 0)]) & (v \cdot v \neq 0) \\ \delta([(1, 0, 0)]) - 1 & \text{(otherwise)} \end{cases}$$

\[\square\]

**Lemma 2.** Let $E$ be a Euclidean domain and $I$ be an ideal of $E$. The diameter of $B(E/I)$ is 2.

**Proof.** For any two distinct vertices represented by $v = ([v_1], [v_2], [v_3])$ and $w = ([w_1], [w_2], [w_3])$, consider the cross product $v \times w$. If $v \times w = 0$, then $[v_i] \cdot w = [v_i] \cdot v$ for $i = 1, 2, 3$. There exists $e \in E$ such that $I = (e)$ because any Euclidean domain is a principal ideal domain. If $gcd(v_1, v_2, v_3, e)$ is not a unity, where $gcd$ is a greatest common divisor, there exists $e' \neq 1$ in $E$ such that $e = de'$. $v$ is not a representative because $[e'] \cdot v = 0$. This is a contradiction. Therefore $d$ is a unity, namely $v_1$ and $v_2$, $v_3$, $e$ are coprime. Then there exist $a, b, c, d \in E$ such that $av_1 + bv_2 + cv_3 + de = 1$ in $E$. Seeing this formula in $E/I$, we get $[a][v_1] + [b][v_2] + [c][v_3] = [1]$. $v = [1] \cdot v = ([a][v_1] + [b][v_2] + [c][v_3])v = ([a][v_1] + [b][v_2] + [c][v_3])w$ means $[v] = [w]$, which is a contradiction to that two vertices are distinct, then $v \times w \neq 0$. If $v \times w$ is a representative of vertex, $[v \times w]$ is adjacent to $[v]$ and $[w]$. If $v \times w = ([k_1], [k_2], [k_3])$ is not a representative of vertex, $v \times w = [gcd(k_1, k_2, k_3)] \cdot u$ and $u$ is a representative of vertex $[u]$ is adjacent to $[v]$ and $[w]$. $\square$

**Theorem.** The following equations hold.

1. $|B(Z_{pk})| = p^{2k} + p^{2k-1} + p^{2k-2}$
2. $\Delta(B(Z_{pk})) = p^k + p^{k-1}$
3. $D(Z_{pk}) = 2$

**Proof.** It is straightforward to show the formula of the order of $B(Z_{pk})$.

$$|B(Z_{pk})| = \frac{|Z_{pk}|^3 - |\{mp | 0 \leq m < p^{k-1}\}|^3}{|Z_{pk}| - |\{mp | 0 \leq m < p^{k-1}\}|}$$

$$= \frac{(p^k)^3 - (p^{k-1})^3}{p^{k} - p^{k-1}} = p^{2k} + p^{2k-1} + p^{2k-2}$$
Using Lemma 1, it is only enough to show that the degree of the vertex represented by \((1,0,0)\) satisfy the formula of the degree of \(B(\mathbb{Z}_p^k)\).

\[
\Delta(B(\mathbb{Z}_p^k)) = \delta([(1,0,0)]) = \frac{|\mathbb{Z}_p^k|^2 - |\{mp|0 \leq m < k\}|^2}{|\mathbb{Z}_p^k| - |\{mp|0 \leq m < k\}|} = \frac{(p^k)^2 - (p^{k-1})^2}{p^k - p^{k-1}} = p^k + p^{k-1}
\]

Using Lemma 2 we get \(D(B(\mathbb{Z}_p^k)) = 2\)

We search new records of the degree/diameter problem among graphs by generalized Brown’s construction of \(\mathbb{Z}_n\) where \(2 \leq n \leq 10000\). Using the above theorem, \(B(\mathbb{Z}_{17}^2)\) has degree 306 and diameter 2 and 88723 vertices. It is a new record of \((306,2)\) of the degree/diameter problem because it cannot be obtained from ordinary Brown’s construction. The power of a prime less than 305 = 306 − 1 is 293\(^1\) and the graph \(B(\mathbb{Z}_{293})\) obtained from ordinary Brown’s construction of 293\(^1\) has 294 = 293 + 1 degree and 86143 = 293\(^2\) + 293 + 1 vertices. The old record of 306 = 294 + 12 is 86156 = 86143 + 12 obtained from \(B(\mathbb{Z}_{293})\) by duplicating vertices. In the same way, the graph obtained from \(B(\mathbb{Z}_{17}^2)\) by duplicating any one vertex, whose order is 88724, is a new record of \((307,2)\) because the power of a prime less than 306 = 307 − 1 is 293\(^1\).

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