REGULARITY RESULTS FOR SOLUTIONS TO AUTONOMOUS OBSTACLE PROBLEMS WITH GENERAL GROWTH

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Abstract. Let us consider the autonomous obstacle problem
\[ \min_v \int_{\Omega} F(Dv(x)) \, dx \]
on a specific class of admissible functions, where we suppose the Lagrangian satisfies proper hypotheses of convexity and superlinearity at infinity. Our aim is to characterize the solution, which exists and it is unique, thanks to a primal-dual formulation of the problem. The proof is based on classical arguments of Convex Analysis and on Calculus of Variations’ techniques.

1. Introduction

In this manuscript we consider autonomous variational obstacle problems of the form
\[ \min \left\{ \int_{\Omega} F(Dv(x)) \, dx : v \in \mathbb{K}_\psi(\Omega) \right\} , \]
where \( \Omega \) is an open and bounded subset of \( \mathbb{R}^n \) and \( \mathbb{K}_\psi(\Omega) \) is the class of admissible functions, defined as
\[ \mathbb{K}_\psi(\Omega) = \left\{ v \in W^{1,1}_{u_0}(\Omega) : v \geq \psi \text{ a.e. on } \Omega, F(Dv) \in L^1(\Omega) \right\} , \quad (K) \]
where \( u_0 \in W^{1,1}(\Omega) \) is a boundary datum such that \( F(Du_0) \in L^1(\Omega) \) and where \( \psi \in W^{1,1}(\Omega) \) is a function called obstacle such that \( F(D\psi) \in L^1(\Omega) \). The main focus we have in this paper is how minimizers of the above-defined constrained problem could be characterized, exploiting a primal-dual formulation of the optimization problem. We suppose the Lagrangian satisfies a super-linear growth condition at infinity, although it is not subjected to any growth condition from above. We also suppose that the integrand inherits the same convexity that we impose on the lower-bound function, i.e. we suppose that the integrand is strongly convex.

There are many works about regularity theory in variational problems and elliptic systems with non-standard growth, but the papers which paved the way were the famous [13] and [14] by Marcellini; since they were published, a lot of new ideas have been applied to this research branch and many results have been proved in several directions (see for example [15] and [16] by Marcellini, or [17] by Mingione for a general exposition and further references).
However, regarding the obstacle problems there are still some issues which have not been studied in an exhaustive way yet. One of these issues deals with the relation between minima and extremals: it is common knowledge that, for both the constrained and unconstrained problems, the regularity of the solutions often comes from the fact that the minimizers are extremals too, i.e. they solve a corresponding variational equality or inequality. Note though that there are examples of variational problems whose minimizers do not satisfy the Euler-Lagrange equation in the weak sense, as proved by Ball and Mizel in [2]. While in the case of standard growth conditions the situation is well established (see for instance the book [9] by Dacorogna), in the case of non-standard growth conditions, the relation between extremals and minima is an issue that requires a careful investigation.

In 2014 Carozza, Kristensen and Passarelli di Napoli investigated exactly this topic in [7] in the case of convex integral functionals, with the aim of showing that their minimizers are characterized to be the energy solutions to the Euler-Lagrange systems for the functionals under non-standard growth conditions. The main tool they use is a particular regularization procedure: the integrand \( F \) is approximated by a sequence of strictly convex and uniformly elliptic integrands \( F_k \) which satisfy standard \( p \)-growth conditions and whose minimizers \( u_k \) strongly converge to the minimizer \( u \) in \( W^{1,p} \). With that said, according to the standard duality theory for convex problems, every such minimizer \( u_k \) is associated to a row-wise solenoidal matrix field denoted by \( \sigma_k \). Finally, for the pairing \( (Du_k,\sigma_k) \), suitable pointwise estimates that are preserved while passing to the limit are proved. Such estimates then provide conditions which allow the Euler-Lagrange system to hold for an \( F \)-minimizer. In a subsequent paper, i.e. [8], the same achievement has been carried on under more general growth assumptions, covering a wide class of functionals, from those with almost-linear growth to the ones with exponential growth and beyond, by the use of Ekeland variational principle and Young measures, to obtain the necessary estimates for the pairing \( (Du_k,\sigma_k) \) to be able to pass to the limit.

In [7], [8], by Carrozza, Kristensen and Passarelli di Napoli, and [11], by Eleuteri and Passarelli di Napoli, the concept of convex duality is exploited. While this seems to be very natural, its use is not so common in the context of convex variational integrals with non-standard growth conditions. We should note though that, before these papers, also in [4] and [5] the authors (respectively Bonfanti and Cellina for the first one and Bonfanti, Cellina and Mazzola for the second) make use of it and, in particular, in [5] is also addressed the question of energy-extremality of minimizers. They work in the context of more general variational integrals \( v \mapsto \int_\Omega F(x,v,Dv)\,dx \) in the multi-dimensional scalar case with \( n \geq 2, \, N = 1 \) under convexity and regularity hypotheses on the gradient. Talking about more general functionals, we remark that the results obtained in [7] can be generalized to
minimizers of the general autonomous convex variational integral \( \int_{\Omega} F(v, Dv) \, dx \), just under
the hypothesis that the integrand \( F = F(\eta, \xi) \) is jointly convex. Similar remarks, together
with the precise statements and sketches of the proofs, were given in [6] by the same authors.
Eventually, in the paper [11] Eleuteri and Passarelli di Napoli address the analogue issue of
[7] in the case of constrained minimizers with a very general obstacle quasi-continuous up
to a subset of zero capacity. Let us mention that relying on techniques of convex analysis,
Scheven and Schmidt in [19] and [20] investigated the Dirichlet minimization problem for
the total variation and the area functional with one-sided obstacle. The main point is that they
were able to identify certain dual maximization problems for bounded divergence-measure
fields and to establish duality formulas and point-wise relations between (generalized) \( BV \)
minimizers and dual maximizers. Their results are very general and apply to very general
obstacles, such as \( BV \) obstacles and the obstacle considered in [11]; the proofs of their re-
sults crucially depend on a new version of the Anzellotti-type pairing (see [1]) which involves
general divergence measure fields and specific representatives of \( BV \) functions, by employing
several fine results on capacities and one-sided approximation. This framework is proved
to be the right one in order to extend the results in [7] to very general problem obstacles,
as long as, by means of the Anzellotti-type pairing, they are able to express the natural
counterpart of the variational inequality in this very general setting, which will reduce to
the usual one once they have the right summability for the functions involved.
We were inspired by them in order to try to extend the results of [8] in the constrained
optimization problem, but we make use of different hypotheses on the obstacle and on the
Lagrangian, in particular a superlinear growth condition at infinity and convexity guaranteed
by the hypotheses on the function that bounds the Lagrangian from below. As in [7], [8] and
[11], we use the concept of convex duality in various steps of the proof. The most challenging
knots in the proof under our hypotheses are the passage to the limit, where we have to pay
attention to the presence of the obstacle, and the proof of the variational inequality.
Our paper is organized as follows: in Section 2 we state the hypotheses and the main re-
sult of the paper, while Section 3 holds the dual formulation of the obstacle problem and
some preliminary results. The proof of the main result is then contained in Section 4 which
is divided in five small parts that are the crucial milestones of the full proof of our Main
Theorem.

2. Statement of the Main Result

Let us consider the problem

\[
\min_{v \in K_v(\Omega)} \int_{\Omega} F(Dv(x)) \, dx,
\]  

(2.1)
where \( \Omega \subset \mathbb{R}^n \) is an open and bounded set, \( F : \mathbb{R}^n \to \mathbb{R} \) is a \( C^1 \) function and \( u_0 \in W^{1,1}(\Omega) \) is a boundary datum such that \( F(Du_0) \in L^1(\Omega) \). Moreover, the function \( \psi \in W^{1,1}(\Omega) \) is called obstacle and it is such that \( F(D\psi) \in L^1(\Omega) \). The class of the admissible functions \( K_\psi(\Omega) \) is defined as in the introduction, but we remark it here:

\[
K_\psi(\Omega) := \{ v \in W^{1,1}(\Omega) : v \geq \psi \) a.e. on \( \Omega, \ F(Dv) \in L^1(\Omega) \}.
\]

We suppose that there exists a \( C^1 \) and strictly convex function \( \phi : \mathbb{R}^n \to [0, +\infty) \) such that

\[
\phi(\xi) := \theta(|\xi|) \quad \forall \xi \in \mathbb{R}^n
\]

for a function \( \theta : [0, +\infty) \to [0, +\infty) \) superlinear at infinity. Moreover, we suppose that it holds true that

\[
\langle \phi'(\xi) - \phi'(\eta), \xi - \eta \rangle \geq \nu (1 + |\xi| + |\eta|)^{-1} |\xi - \eta|^2 \quad \forall \xi, \eta \in \mathbb{R}^n
\]

for some \( \nu > 0 \). We also suppose that

\[
F - \phi \quad \text{is a convex function.}
\]

As a consequence of hypothesis (H2), there exists \( c \in \mathbb{R} \) such that

\[
c \leq F(\xi) - \phi(\xi)
\]

for all \( \xi \in \mathbb{R}^n \) such that \(|\xi|\) is large enough.

Indeed, since \( F - \phi \) is a convex function, then it holds true that

\[
F(\xi) - \phi(\xi) \geq F(0) - \phi(0) + \langle (F - \phi)'(0), \xi \rangle.
\] (2.2)

If we define \( \nu := (F - \phi)'(0) \), then we have that there exists \( r > 0 \) such that

\[
|\langle \nu, \xi \rangle| \leq |\nu| \cdot |\xi| \leq 2 \phi(\xi)
\]

for every \( \xi \in \mathbb{R}^n \) such that \(|\xi| \geq r\), since \( \phi \) is superlinear at infinity. This implies that

\[
\langle \nu, \xi \rangle \geq -2 \phi(\xi)
\]

for every \( \xi \in \mathbb{R}^n \) such that \(|\xi| \geq r\) and, if we use it in (2.2), we obtain that

\[
F(\xi) - \phi(\xi) \geq F(0) - \phi(0) - 2 \phi(\xi) = c - 2 \phi(\xi),
\]

still for every \( \xi \in \mathbb{R}^n \) such that \(|\xi| \geq r\). From the last inequality, we obtain that

\[
F(\xi) - \phi(\xi) \geq c \quad \forall \xi \in \mathbb{R}^n : |\xi| \geq r_2 > r
\]

thanks to (H2) and the superlinearity at infinity of \( \phi \).

Clearly, the hypothesis (H2*) implies that \( F \) is superlinear at infinity and, moreover, the
hypotheses (H1) and (H2) imply that $F$ also satisfies the (H1)-inequality. Indeed, since $F - \phi$ is convex and $C^1$, then

$$
\langle (F - \phi)'(\xi) - (F - \phi)'(\eta), \xi - \eta \rangle \geq 0,
$$
guaranteeing us that

$$
\langle F'(\xi) - F'(\eta), \xi - \eta \rangle \geq \langle \phi'(\xi) - \phi'(\eta), \xi - \eta \rangle \geq \nu (1 + |\xi| + |\eta|)^{-1} |\xi - \eta|^2.
$$

(2.3)

In the end, it is worth noticing that the hypothesis (H2) implies the strong convexity of $F$.

We now define the space

$$
S^\infty(\Omega) := \{ \sigma \in L^\infty(\Omega) : \text{div} \sigma \leq 0 \text{ in distributional sense} \}
$$

and, fixed $U \in W^{1,1}(\Omega)$, we define the measure $[\sigma, U]_{w_0}(\Omega)$ on $\Omega$ as

$$
[\sigma, U]_{w_0}(\Omega) := \int_{\Omega} (U - u_0) d(-\text{div} \sigma) + \int_{\Omega} \langle \sigma, Du_0 \rangle \ dx.
$$

(2.5)

It is worth noticing that, fixed $U \in W^{1,1}(\Omega)$, then $[\sigma, U]_{w_0}(\Omega)$ is equal to $\langle \sigma, DU \rangle \in L^1(\Omega)$ since

$$
\int_{\Omega} \varphi (-\text{div} \sigma) \ dx = \int_{\Omega} \langle \sigma, D\varphi \rangle \ dx \quad \forall \varphi \in W^{1,1}_0(\Omega).
$$

(2.6)

The functional

$$
\mathcal{F}(v) := \int_{\Omega} F(Dv(x)) \ dx
$$

under the assumptions (H1) and (H2) is a proper, convex and lower semicontinuous functional on $W^{1,1}(\Omega)$ and thus, given $u_0 \in W^{1,1}(\Omega)$ such that $F(Du_0) \in L^1(\Omega)$, the existence and uniqueness of the minimizer $u$ in the convex space $K_\psi(\Omega)$ are granted. A characterization of this minimizer is stated in the following Theorem, which is our main Theorem.

**Theorem 2.1.** Let $F$ be a function in $C^1(\mathbb{R}^n)$, satisfying (H1) and (H2) with a function $\phi$ defined as before, and let $u_0 \in W^{1,1}(\Omega)$ be such that $F(Du_0), F(tDu_0) \in L^1(\Omega)$ for some $t > 1$. Then, the minimizer $u \in W^{1,1}_{w_0}(\Omega)$ of the minimum problem (2.1) is characterized by

$$
F^*(F'(Du)) \in L^1(\Omega), \quad \langle F'(Du), Du \rangle \in L^1(\Omega)
$$

(2.7)

and

$$
\text{div} F'(Du) \leq 0
$$

(2.8)

in distributional sense. Moreover it holds the following equality

$$
\int_{\Omega} F(Du) \ dx = \left[ F'(Du), \psi \right]_{w_0}(\Omega) - \int_{\Omega} F^*(F'(Du)) \ dx.
$$

(2.9)
3. Preliminaries

Given a continuous function $F : \mathbb{R}^n \rightarrow \mathbb{R}$, its polar function is the function $F^* : \mathbb{R}^n \rightarrow \mathbb{R}$ defined as

$$F^*(z) := \sup_{\xi \in \mathbb{R}^n} [\langle z, \xi \rangle - F(\xi)] \quad \forall z \in \mathbb{R}^n. \quad (3.1)$$

We know that $F^*$ is always a convex and lower semicontinuous function and we can also define the bipolar function of $F$, $F^{**} : \mathbb{R}^n \rightarrow \mathbb{R}$, as follows:

$$F^{**}(\xi) := \sup_{z \in \mathbb{R}^n} [\langle z, \xi \rangle - F^*(z)] \quad \forall \xi \in \mathbb{R}^n. \quad (3.2)$$

It is possible to prove that $F^{**} = F$ if and only if $F$ is convex and lower semicontinuous for every $\xi \in \mathbb{R}^n$ (Fenchel-Moreau Theorem). Thanks to Fenchel’s inequality applied on the polar function $F^*$, it also holds that

$$\langle \xi, \eta \rangle \leq F^*(\eta) + F^{**}(\xi), \quad (3.3)$$

for each $\eta, \xi \in \mathbb{R}^n$. We also remark that the equality holds true when $\eta \in \partial F^{**}(\xi)$, where $\partial F^{**}(\xi)$ is the subgradient of $F^{**}$ in $\xi$. In particular, if $F \in C^1(\mathbb{R}^n)$ and it is convex, then in (3.3) the equality holds true also for $\eta = F'(\xi)$, i.e.

$$\langle \xi, F'(\xi) \rangle = F^*(F'(\xi)) + F^{**}(\xi). \quad (3.4)$$

Another important result concerning the polar of a function is given by the following Lemma, which can be found in [8] and whose proof can be found in [10], even if the Lemma is not explicitly stated.

**Lemma 3.1.** If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, then its polar $F^* : \mathbb{R}^n \rightarrow \mathbb{R}$ is superlinear at infinity.

We now state the following Lemma, whose proof can be founded in [20], that will be necessary for the proof of Theorem 3.3 stated later and for the proof of the Main Theorem.

**Lemma 3.2.** Let $\varphi \in W^{1,p}(\Omega)$ and let $u_0 \in W^{1,p}(\Omega)$, with $p \geq 1$. Suppose that there exists $g \in W^{1,p}_{u_0}(\Omega)$ such that

$$g(x) \geq \varphi(x) \quad \text{a.e. in } \Omega.$$

Then, there exists a non-increasing sequence of functions $(\varphi_k)_k \subset W^{1,p}_{u_0}(\Omega)$ such that

$$\varphi_k \rightarrow \varphi \quad \text{a.e. in } \Omega.$$

The next Theorem will be used in the following in order to generate a sequence of approximating problems which will lead us to the proof of our main Theorem.
Theorem 3.3. Given \( G \in \mathcal{C}^1(\mathbb{R}^n) \) strictly convex, Lipschitz continuous and such that it satisfies the variational inequality
\[
\int_{\Omega} \langle G'(Du), D\varphi - Du \rangle \, dx \geq 0 \quad \forall \varphi \in \mathbb{K}_\psi(\Omega)
\] (3.5)
where \( u \in \mathbb{K}_\psi(\Omega) \), then
\[
\min_{v \in \mathbb{K}_\psi(\Omega)} \int_{\Omega} G(Dv) \, dx = \max_{\sigma \in S^{\infty}(-\Omega)} \left( [\sigma, \psi]_{u_0}(\Omega) - \int_{\Omega} G^*(\sigma) \, dx \right).
\] (3.6)
Moreover, if \( u \in \mathbb{K}_\psi(\Omega) \) is a solution of (2.1) then
\[
\int_{\Omega} G(Du) \, dx \geq \langle G'(Du), \psi \rangle_{u_0}(\Omega) - \int_{\Omega} G^*(G'(Du)) \, dx.
\] (3.7)

Proof. We consider \( \sigma \in S^{\infty}(-\Omega) \) and \( v \in \mathbb{K}_\psi(\Omega) \). Since \(-\text{div} \, \sigma\) is a non-negative Radon measure and \( v \geq \psi \) a.e. in \( \Omega \), it holds that
\[
\int_{\Omega} (v - \psi) \, d(-\text{div} \, \sigma) \geq 0.
\]
By the definition (2.5) and by the previous inequality we get that
\[
[\sigma, \psi]_{u_0}(\Omega) = \int_{\Omega} (\psi - u_0) \, d(-\text{div} \, \sigma) + \int_{\Omega} \langle \sigma, Du_0 \rangle \, dx
\]
\[
= \int_{\Omega} (\psi - v + v - u_0) \, d(-\text{div} \, \sigma) + \int_{\Omega} \langle \sigma, Du_0 \rangle \, dx
\]
\[
\leq \int_{\Omega} (v - u_0) \, d(-\text{div} \, \sigma) + \int_{\Omega} \langle \sigma, Du_0 \rangle \, dx.
\]
Since \( v, u_0 \in W^{1,1}_{u_0}(\Omega) \), then \( v = u_0 \) on \( \partial \Omega \) and using (2.6) on the first integral of the last expression, we get that
\[
[\sigma, \psi]_{u_0}(\Omega) = \int_{\Omega} \langle \sigma, Dv - Du_0 \rangle \, dx + \int_{\Omega} \langle \sigma, Du_0 \rangle \, dx
\]
\[
= \int_{\Omega} \langle \sigma, Dv \rangle \, dx
\]
\[
\leq \int_{\Omega} G(Dv) \, dx + \int_{\Omega} G^*(\sigma) \, dx
\]
by means of (3.3) and the fact that \( G = G^{**} \). The previous inequality implies that
\[
\int_{\Omega} G(Dv) \, dx \geq [\sigma, \psi]_{u_0}(\Omega) - \int_{\Omega} G^*(\sigma) \, dx
\]
so, passing to the minimum on \( v \in \mathbb{K}_\psi(\Omega) \) in the left-hand side and to the maximum on \( \sigma \in S^{\infty}(-\Omega) \) in the right-hand side, we get
\[
\min_{v \in \mathbb{K}_\psi(\Omega)} \int_{\Omega} G(Dv) \, dx \geq \max_{\sigma \in S^{\infty}(-\Omega)} \left( [\sigma, \psi]_{u_0}(\Omega) - \int_{\Omega} G^*(\sigma) \, dx \right).
\] (3.8)
We have now to prove the reverse inequality. Let us choose \( u \in \mathbb{K}_\psi(\Omega) \) as the unique solution of

\[
\min_{v \in \mathbb{K}_\psi(\Omega)} \int_\Omega G(Dv) \, dx
\]
and let us consider (3.4) with \( F = G \) and \( \xi = Du \) and integrating we obtain that

\[
\int_\Omega G(Du) \, dx = \int_\Omega \langle G'(Du), Du \rangle \, dx - \int_\Omega G^*(G'(Du)) \, dx
\]
\[
= \int_\Omega \langle G'(Du), Du - Du_0 \rangle \, dx + \int_\Omega \langle G'(Du), Du_0 \rangle \, dx - \int_\Omega G^*(G'(Du)) \, dx.
\]

Since \( G \) satisfies the variational inequality (3.5), then it holds true that

\[
\int_\Omega \langle G'(Du), Dw \rangle \, dx \geq 0 \quad \forall \; w \in \mathcal{C}_0^\infty(\Omega) : w \geq 0.
\]

Thanks to that, if we set \( \sigma = G'(Du) \), then we have that \( \sigma \in S_\infty^-(\Omega) \), thus we have

\[
\int_\Omega G(Du) \, dx = \int_\Omega \langle \sigma, Du - Du_0 \rangle \, dx + \int_\Omega \langle \sigma, Du_0 \rangle \, dx - \int_\Omega G^*(\sigma) \, dx
\]
\[
= \int_\Omega (u - u_0) d(-\text{div} \sigma) + \int_\Omega \langle \sigma, Du_0 \rangle \, dx - \int_\Omega G^*(\sigma) \, dx
\]
\[
= \int_\Omega (\psi - \psi + u - u_0) d(-\text{div} \sigma) + \int_\Omega \langle \sigma, Du_0 \rangle \, dx - \int_\Omega G^*(\sigma) \, dx
\]
\[
= \int_\Omega (\psi - u_0) d(-\text{div} \sigma) + \int_\Omega \langle \sigma, Du_0 \rangle \, dx
\]
\[
- \int_\Omega (\psi - u) d(-\text{div} \sigma) - \int_\Omega G^*(\sigma) \, dx
\]
\[
= [\sigma, \psi]_{u_0}(\Omega) - \int_\Omega \langle \sigma, D\psi - Du \rangle \, dx - \int_\Omega G^*(\sigma) \, dx
\]
\[
\leq [\sigma, \psi]_{u_0}(\Omega) - \int_\Omega G^*(\sigma) \, dx,
\]

where we used once again (2.6) and the fact that it holds

\[
\int_\Omega \langle \sigma, D\psi - Du \rangle \, dx \geq 0. \tag{3.9}
\]

Indeed, since \( u \in W^{1,1}_{w_0}(\Omega) \) and \( u \geq \psi \), then we can use the Lemma 3.2 and consider the sequence \( (\psi_j)_j \subset W^{1,1}_{w_0}(\Omega) \) such that

\[
\psi_j \longrightarrow \psi \quad \text{a.e. on } \Omega.
\]

Also, we can notice that \( (\psi_j)_j \subset \mathbb{K}_\psi(\Omega) \), thanks to the fact that \( G \) is a Lipschitz continuous function, so

\[
\int_\Omega \langle \sigma, D\psi_j - Du \rangle \, dx \geq 0 \quad \forall \; j \in \mathbb{N}.
\]
Now, thanks to the Monotone Convergence Theorem, we deduce the validity of (3.9).

Finally, we have that
\[
\min_{v \in K_\psi(\Omega)} \int_\Omega G(Dv) \, dx = \int_\Omega G(Du) \, dx
\]
\[
\leq [\sigma, \psi]_{u_0}(\Omega) - \int_\Omega G^*(\sigma) \, dx
\]
\[
\leq \max_{\sigma \in S^\infty(\Omega)} \left( [\sigma, \psi]_{u_0}(\Omega) - \int_\Omega G^*(\sigma) \, dx \right).
\]

Combining previous estimate with (3.8), we establish (3.6) and, recalling that \( \sigma = G'(Du) \), the equality at (3.7).

\[\square\]

4. PROOF OF THEOREM 1.1

We define \( G := F - \frac{1}{2} \phi \). \( G \) is clearly a strongly convex function since there exists \( \phi_1 := \frac{1}{2} \phi \) such that \( G - \phi_1 = F - \phi \) is convex. It is easy to prove that \( G \) possesses the properties of \( F \), i.e. \( G \) is superlinear at infinity and there exists \( \nu_1 > 0 \) such that
\[
\langle G'(\xi) - G'(\eta), \xi - \eta \rangle \geq \langle \phi'(\xi) - \phi'(\eta), \xi - \eta \rangle \geq \nu_1 (1 + |\xi| + |\eta|)^{-1} |\xi - \eta|^2
\]
for all \( \xi, \eta \in \mathbb{R}^n \). It also holds true that there exists \( r_1 > 0 \) such that
\[
G(\xi) \geq \frac{1}{2} \phi(\xi) + c \quad \forall \xi \in \mathbb{R}^n : |\xi| \geq r_1.
\]

Since \( G \) is superlinear at infinity, its polar
\[
G^*(z) := \sup_{\mathbb{R}^n} (\langle z, \xi \rangle - G(\xi))
\]
is a real-valued and convex function and, since \( G \) is convex and \( C^1 \), then \( G^* \) is strictly convex and superlinear at infinity thanks to Lemma 3.1. Fixed \( k \in \mathbb{N} \) and \( \xi \in \mathbb{R}^n \), we can define
\[
G_k^{**}(\xi) := \sup_{|z| \leq k} (\langle z, \xi \rangle - G^*(z)).
\]

We can observe that \( G_k^{**} \) is a real-valued, convex and Lipschitz function and, since \( G^{**} \) is lower semicontinuous, then
\[
G_k^{**} \xrightarrow{k \to \infty} G^{**} = G \quad \text{pointwise}.
\]

Now we define
\[
\overline{G}_k^{**}(\xi) := \max \left\{ G_k^{**}(\xi), \frac{1}{2} \theta(|\xi|) + c \right\}.
\]
Again, \( \overline{G}_k^{**} \xrightarrow{k \to \infty} G^{**} = G \) pointwise, but for each \( k \in \mathbb{N} \) there also must exists \( r_k > 0 \) such that \( r_k \xrightarrow{k \to \infty} +\infty \) and such that
\[
\overline{G}_k^{**}(\xi) = \frac{1}{2} \theta(|\xi|) + c.
\]
when $|\xi| \geq r_k$. Now we define
\[
H_k(\xi) := \begin{cases} 
\mathcal{G}_{r_k}^*(\xi) & \text{if } |\xi| < r_k, \\
\frac{\theta(r_k)}{2r_k} |\xi| + c & \text{if } |\xi| \geq r_k.
\end{cases}
\]
Again, it is possible to prove that $H_k$ is a convex and $m_k$--Lipschitz function, with
\[m_k := \frac{\theta(r_k)}{r_k}\]
for all $k \in \mathbb{N}$. Now we regularize $H_k$ by means of the convolution kernels
\[\Phi_\varepsilon(\xi) := \varepsilon^{-n} \Phi\left(\frac{\xi}{\varepsilon}\right),\]
where
\[\Phi(\xi) := \begin{cases} 
\exp\left(\frac{1}{1 - |\xi|^2}\right) & \text{if } |\xi| < 1 \\
0 & \text{if } |\xi| \geq 1
\end{cases}\]
and where $c$ is chosen such that $\int_{\mathbb{R}^n} \Phi(\xi) \, d\xi = 1$. In particular, we consider the function $\Phi_\varepsilon * H_k(\xi)$ and we remark that this is a convex, $C^\infty$ and $m_k$--Lipschitz function for which it holds
\[H_k(\xi) \leq \Phi_\varepsilon * H_k(\xi) \leq H_k(\xi) + \varepsilon m_k \tag{4.1}\]
for each $k \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$. Now we define
\[\delta_k := \frac{1}{k^2 m_k},\]
\[\mu_k := \frac{1}{k - 1}\]
and
\[F_k(\xi) := \Phi_{\delta_k} * H_k(\xi) - \mu_k + \frac{1}{2} \phi(\xi) \tag{4.2}\]
and we notice that it holds true that $F_k(\xi) \leq F_{k+1}(\xi)$ for all $k \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$ (see [7]). It is important to notice that, for every $k \in \mathbb{N}$, $F_k$ is a strongly convex function since
\[F_k - \frac{1}{2} \phi(\xi) = \Phi_{\delta_k} * H_k(\xi) - \mu_k \text{ is convex.}\]
Since $F_k$ are $C^\infty$, $m_k$--Lipschitz continuous and convex functions, then they satisfy the variational inequality
\[\int_{\Omega} \langle F_k'(Du_k), D\eta - Du_k \rangle \, dx \geq 0 \quad \forall \eta \in \mathcal{K}_\psi(\Omega) \quad \forall k \in \mathbb{N}, \tag{4.3}\]
where $u_k$ is the solution to the minimum problem (2.1) for $F_k$. Indeed $\mathcal{K}_\psi(\Omega)$ is a convex space and thus we have that, for any $\eta \in \mathcal{K}_\psi(\Omega)$ and $k \in \mathbb{N}$, the function
\[v_k := \varepsilon \eta + (1 - \varepsilon) u_k = u_k + \varepsilon (\eta - u_k)\]
is an element of $K_\psi(\Omega)$ and it yields

$$\frac{1}{\varepsilon} \int_\Omega [F_k(Du_k + \varepsilon Dv_k) - F_k(Du_k)] \, dx \geq 0,$$

so

$$\frac{1}{\varepsilon} \int_\Omega [F_k(Du_k + \varepsilon Dv_k) - F_k(u_k)] \, dx = \int_\Omega \int_0^1 \langle F'_k(Du_k + s \varepsilon Dv_k), Dv_k \rangle \, ds \, dx. \quad (4.4)$$

But $Dv_k = D\eta - Du_k$, leading to

$$\frac{1}{\varepsilon} \int_\Omega [F_k(Du_k + \varepsilon Dv_k) - F_k(Du_k)] \, dx = \int_\Omega \int_0^1 \langle F'_k(Du_k + s \varepsilon D(\eta - u_k)), D(\eta - u_k) \rangle \, ds \, dx.$$

Since $F_k$ is Lipschitz then $F'_k$ is bounded and thus, from the integrability of $D\eta - Du_k$, we can use the Dominated Convergence Theorem, which leads us to

$$\int_\Omega \int_0^1 \langle F'_k(Du_k), D(\eta - u_k) \rangle \, ds \, dx = \int_\Omega \langle F'_k(Du_k), \eta - Du_k \rangle \, dx \geq 0.$$

4.1. Approximation of the problem. Now we shall construct a sequence of obstacle problems whose dual problem is given by Theorem 3.3. Let $(F_k)_k$ be the sequence of functions obtained in (4.2). We can observe that $F_k \not> F$ pointwise and that each $F_k$ is $C^\infty$ and strictly convex, moreover they satisfy hypothesis (H2*).

Let us fix $k \in \mathbb{N}$ and let $u_k \in K_\psi(\Omega)$ be the solution to the obstacle problem

$$\min_{w \in K_\psi(\Omega)} \int_\Omega F_k(Dw) \, dx \quad (4.5)$$

and let

$$\sigma_k := F'_k(Du_k) \in S^{-\infty}_\infty(\Omega) \quad (4.6)$$

be the solution of the dual problem given by (3.6), i.e. $\sigma_k$ is such that

$$\max_{\sigma \in S_\infty(\Omega)} \left( [\sigma, \psi]_{u_k(\Omega)} - \int_\Omega F^*_k(\sigma) \, dx \right) = [\sigma_k, \psi]_{u_k(\Omega)} - \int_\Omega F^*_k(\sigma_k) \, dx$$

where $F^*_k$ is the polar function of $F_k$. By (4.3) and the fact that the $F_k$ are strictly convex, we are able to apply Theorem 3.3 to each $F_k$. Therefore, from (3.7) with $G = F_k$, $u = u_k$ and remembering the definition of $\sigma_k$ in (4.6), we have that the following equality holds true for all $k \in \mathbb{N}$

$$\int_\Omega F_k(Du_k) \, dx = [\sigma_k, \psi]_{u_k(\Omega)} - \int_\Omega F^*_k(\sigma_k) \, dx.$$
Now we can notice that, since $F_k(\xi) \nearrow F(\xi)$, it is not difficult to check that $F_k^*(\xi) \searrow F^*(\xi)$ as $k \to \infty$, pointwise in $\zeta$. Furthermore, since $F_k$ satisfies (4.3), we also have that $\sigma_k$ solves the following variational inequality

$$\int_\Omega \langle \sigma_k, D\eta \rangle \, dx \geq 0 \quad \forall \eta \in C_0^\infty(\Omega) : \eta \geq 0, \quad \forall k \in \mathbb{N}. \tag{4.7}$$

4.2. Passage to the limit. Now we want to prove that $u_k \to u$ strongly in $W^{1,1}(\Omega)$, where $u$ is the solution of the obstacle problem (2.1). Firstly, we observe that, by the definition of $F_k$ in (4.2), the definition of $H_k$ and by (4.1), we have that

$$F_k(\xi) = \Phi_{\delta_k} * H_k(\xi) - \mu_k + \frac{1}{2} \phi(\xi) \geq H_k(\xi) - \mu_k + \frac{1}{2} \phi(\xi) \geq \frac{1}{2} \theta(|\xi|) + c - \mu_k + \frac{1}{2} \phi(\xi)$$

guaranteeing us, remembering the definition of $\phi$, that

$$F_k(\xi) \geq \theta(|\xi|) + c - \mu_k \geq \theta(|\xi|) - (c + 1) = \theta(|\xi|) - C, \tag{4.8}$$

with the choice of $\mu_k$ done before. Thanks to that, we can observe that

$$\int_\Omega \theta(|Du_k|) \, dx \leq \int_\Omega [F_k(Du_k) + C] \, dx \leq \int_\Omega [F_k(Du_0) + C] \, dx \leq \int_\Omega [F(Du_0) + C] \, dx = C |\Omega| + \int_\Omega F(Du_0) \, dx < +\infty$$

where we used the minimality of $u_k$ and the fact that $F_k \nearrow F$, in addition to the hypotheses of boundedness of $\Omega$ and that $F(Du_0) \in L^1(\Omega)$. This grant us that

$$\int_\Omega F_k(Du_k) \, dx \leq \int_\Omega F(Du_0) \, dx < +\infty \tag{4.9}$$

This also tells us that $(Du_k)_k$ is equi-integrable by De La Vallée-Poussin Theorem so, by Dunford-Pettis Theorem, there exists a non-relabeled subsequence $(u_k)_k$ that converges weakly to some $v$ in $W^{1,1}(\Omega)$. We have that $v \in K_\psi(\Omega)$ because the $u_k$ are in $K_\psi(\Omega)$ and it is a convex closed set, therefore weakly closed. Now, fixed $k_0 \in \mathbb{N}$, by the lower semicontinuity of $F_{k_0}$, we have that

$$\liminf_{k \to +\infty} \int_\Omega F_{k_0}(Du_k) \, dx \geq \int_\Omega F_{k_0}(Dv) \, dx$$
and the monotonicity of the sequence $F_k$ yields

$$\int_{\Omega} F_{k_0} (Du_k) \, dx \leq \int_{\Omega} F_k (Du_k) \, dx$$

for every $k > k_0$. Therefore

$$\int_{\Omega} F_{k_0} (Dv) \, dx \leq \liminf_{k \to +\infty} \int_{\Omega} F_k (Du_k) \, dx$$

and since $F_k \nearrow F$, taking the limit as $k_0 \to +\infty$, we deduce that, by the Monotone Convergence Theorem and the minimality of $u$,

$$\int_{\Omega} F(Du) \, dx \leq \int_{\Omega} F(Dv) \, dx \leq \liminf_{k \to +\infty} \int_{\Omega} F_k (Du_k) \, dx. \quad (4.10)$$

On the other hand, by the minimality of $u_k$ we have that

$$\int_{\Omega} F_k (Du_k) \, dx \leq \int_{\Omega} F_k (Du) \, dx.$$

Using once more the Monotone Convergence Theorem, we have that

$$\limsup_{k \to +\infty} \int_{\Omega} F_k (Du_k) \, dx \leq \limsup_{k \to +\infty} \int_{\Omega} F_k (Du) \, dx = \int_{\Omega} F(Du) \, dx. \quad (4.11)$$

If we compare (4.10) and (4.11), we can easily notice that

$$\int_{\Omega} F_k (Du_k) \, dx \xrightarrow{k \to +\infty} \int_{\Omega} F(Du) \, dx = \int_{\Omega} F(Dv) \, dx, \quad (4.12)$$

but the strict convexity of $F$ implies the uniqueness of the solution and, therefore, that $u = v$.

To deduce the convergence is actually strong, we use the uniform convexity of the $F_k$, i.e. we use that $F_k - \phi$ is convex for all $k > 1$, where $\phi$ is the function appearing in assumption (H1). Thanks to H"older’s inequality and the equi-integrability of $(Du_k)_k$ we showed before, we obtain that

$$\left[ \int_{\Omega} |Du - Du_k| \, dx \right]^2 \leq \left[ \int_{\Omega} |Du - Du_k| (1 + |Du| + |Du_k|)^{\frac{1}{2}} (1 + |Du| + |Du_k|)^{\frac{1}{2}} \, dx \right]^2 \leq \int_{\Omega} |Du - Du_k|^2 (1 + |Du| + |Du_k|)^{-1} \, dx \int_{\Omega} (1 + |Du| + |Du_k|) \, dx \leq \nu \int_{\Omega} |Du - Du_k|^2 (1 + |Du| + |Du_k|)^{-1} \, dx$$
for a \( \nu > 0 \). Now thanks to (2.3) and the variational inequality (4.3), since \( F_k \) is \( C^1 \) and convex and \( Du_k \to Du \) weakly in \( L^1(\Omega) \), we have that
\[
\left( \int_\Omega |D u - Du_k| \, dx \right)^2 \leq \nu \int_\Omega |D u - Du_k|^2 (1 + |D u| + |Du_k|)^{-1} \, dx
\]
\[
\leq \int_\Omega \langle F'_k(Du_k) - F'_k(Du), D u - Du_k \rangle \, dx
\]
\[
\leq \int_\Omega |F_k(Du) - F_k(Du_k)| \, dx
\]
\[
\leq \int_\Omega |F_k(Du) - F_k(Du_k)| \, dx \xrightarrow{k \to +\infty} 0
\]
It follows that \( Du_k \to Du \) strongly in \( L^1(\Omega) \). Since \( u_k - u \in W^{1,1}_0(\Omega) \), we have shown that the non-relabeled subsequence \( (u_k)_k \) converges strongly to \( u \) in \( W^{1,1}(\Omega) \) but, by the uniqueness of the limit, we conclude by a standard argument that the full sequence \( (u_k)_k \) converges strongly in \( W^{1,1}(\Omega) \) to \( u \).

Since \( F_k \nrightarrow F \) pointwise and \( F_k+1 \geq F_k \) for all \( k \in \mathbb{N} \), it follows in particular from Dini’s Lemma that the convergence is locally uniform in \( \xi \), so we can now prove that \( \sigma_k = F'_k(Du_k) \to F'(Du) \) locally uniformly. To that end, we consider \( (F'_k(\xi_k))_k \) where \( \xi_k := Du_k \) and \( \xi := Du \). Because difference-quotients of convex functions are increasing in the increment, we have for all \( \eta \in \mathbb{R}^n \) and \( 0 < |t| \leq 1 \) that
\[
| \langle F'_k(\xi_k) - F'(\xi), \eta \rangle | \leq \left| \frac{F_k(\xi_k + t \eta) - F_k(\xi_k) - \langle F'(\xi), t \eta \rangle}{t} \right|
\]
\[
\leq |F_k(\xi_k + \eta) - F_k(\xi_k) - \langle F'(\xi), \eta \rangle|.
\]
Consequently we get
\[
\limsup_{k \to +\infty} | \langle F'_k(\xi_k) - F'(\xi), \eta \rangle | \leq | F(\xi + \eta) - F(\xi) - \langle F'(\xi), \eta \rangle |,
\]
for all \( \eta \in \mathbb{R}^n \). Hence, for all \( 0 < s \leq 1 \), we obtain that
\[
\limsup_{k \to +\infty} \left| \frac{\langle F'_k(\xi_k) - F'(\xi), s \eta \rangle}{s} \right| \leq \left| \frac{F(\xi + s \eta) - F(\xi) - \langle F'(\xi), s \eta \rangle}{s} \right|,
\]
which is equivalent to
\[
\limsup_{k \to +\infty} | \langle F'_k(\xi_k) - F'(\xi), \eta \rangle | \leq \left| \frac{F(\xi + s \eta) - F(\xi)}{s} - \langle F'(\xi), \eta \rangle \right|.
\]
If we let \( s \) tend to 0 and recall that \( F \) is differentiable in \( \xi \), we conclude that the left-hand side must vanish. This proves the local uniform convergence of derivatives, so it follows that \( \sigma_k = F'_k(Du_k) \to F'(Du) \) locally uniformly and, in particular, in measure on \( \Omega \).
We now consider the equality
\[ \langle \sigma_k, Du_k \rangle = F_k^*(\sigma_k) + F_k(Du_k), \] (4.13)
which has been deduced by (3.4) using the definition of \( \sigma_k \), remembering that \( F_k^{**} = F_k \) and choosing \( \xi = Du_k \). Passing to the limit, we deduce the pointwise extremality relation
\[ \langle \sigma^*, Du \rangle = F^*(\sigma^*) + F(Du), \] (4.14)
with \( \sigma^* := F'(Du) \).

4.3. The validity of (2.7). Since \( u_0 \in \mathbb{K}_\psi(\Omega) \), we can use (4.3) choosing \( \eta = u_0 \), thus getting
\[ \hat{\Omega} \langle \sigma_k, Du_0 - Du_k \rangle \, dx \geq 0 \quad \forall k \in \mathbb{N}. \]
Now, integrating (4.13) over \( \Omega \) and using the previous inequality, we have that, chosen \( t > 1 \),
\[ \int_{\Omega} F_k^*(\sigma_k) \, dx = \int_{\Omega} \langle \sigma_k, Du_k \rangle \, dx - \int_{\Omega} F_k(Du_k) \, dx \]
\[ \leq \int_{\Omega} \langle \sigma_k, Du_0 \rangle \, dx - \int_{\Omega} F_k(Du_k) \, dx \]
\[ = \frac{1}{t} \int_{\Omega} \langle \sigma_k, t Du_0 \rangle \, dx - \int_{\Omega} F_k(Du_k) \, dx \]
\[ \leq \frac{1}{t} \int_{\Omega} F_k^*(\sigma_k) \, dx + \frac{1}{t} \int_{\Omega} F_k(t Du_0) \, dx - \int_{\Omega} F_k(Du_k) \, dx, \]
where we also used (3.3) exploiting the convexity of \( F_k^* \) and the strong convexity of \( F_k \).
Reabsorbing the first term in the right-hand side by the left-hand side and using the fact that \( F_k(\xi) \leq F(\xi) \) for all \( \xi \in \mathbb{R}^n \) we obtain that
\[ \int_{\Omega} F_k^*(\sigma_k) \, dx \leq \frac{1}{t - 1} \int_{\Omega} F_k(t Du_0) \, dx - \frac{t}{t - 1} \int_{\Omega} F_k(Du_k) \, dx \]
\[ \leq \frac{1}{t - 1} \int_{\Omega} F(t Du_0) \, dx - \frac{t}{t - 1} \int_{\Omega} F_k(Du_k) \, dx \]
\[ \leq \left\lvert \frac{1}{t - 1} \int_{\Omega} F(t Du_0) \, dx - \frac{t}{t - 1} \int_{\Omega} F_k(Du_k) \, dx \right\rvert \]
\[ \leq \frac{1}{t - 1} \int_{\Omega} |F(t Du_0)| \, dx + \frac{t}{t - 1} \int_{\Omega} |F_k(Du_k)| \, dx \]
\[ \leq C_1 \int_{\Omega} |F(t Du_0)| \, dx + C_2 \int_{\Omega} F(Du_0) \, dx \] (4.15)
by (4.9). Recalling that \( F_k^* \searrow F^* \) and by the hypothesis that \( F(t Du_0) \in L^1(\Omega) \) with \( t > 1 \), we also obtain that
\[ \int_{\Omega} F^*(\sigma_k) \, dx \leq C_1 \int_{\Omega} |F(t Du_0)| \, dx + C_2 \int_{\Omega} F(Du_0) \, dx < +\infty. \]
Since we already observed that \( \sigma_k \rightarrow F'(Du) \) locally uniformly, by the previous estimate and Fatou’s lemma we have that
\[
\int_{\Omega} F^*(F'(Du)) \, dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} F^*(\sigma_k) \, dx < +\infty.
\]
Thus
\[
F^*(F'(Du)) \in L^1(\Omega).
\]
Whence, by (4.14), we also have
\[
\langle F'(Du), Du \rangle \in L^1(\Omega)
\]
since \( F(Du) \in L^1(\Omega) \) by the definition of minimizer.

4.4. The validity of the variational inequality. Now we want to prove the validity of the variational inequality. Since \( F^* \) is superlinear at infinity and since \( F^*_k \searrow F^* \), then there exists \( \theta : [0, +\infty] \rightarrow [0, +\infty] \) increasing, convex and superlinear at infinity such that
\[
\theta(|\xi|) \leq F^*(\xi) \leq F^*_k(\xi) \quad \forall \xi \in \mathbb{R}^n.
\] (4.16)
Moreover, using (4.15) we have that
\[
\sup_{k \in \mathbb{N}} \int_{\Omega} \theta(|\sigma_k|) \, dx \leq \int_{\Omega} F^*_k(\sigma_k) \, dx \leq C \int_{\Omega} F(Du_0) \, dx < +\infty,
\]
thus we can use De La Vallé-Poussin Theorem in order to obtain the equi-integrability for \( (\sigma_k)_k \). Since \( \sigma_k \) converges in measure to \( \sigma^* \), then we can apply Vitali’s Convergence Theorem which proves that \( \sigma_k \) converges to \( \sigma^* \) in \( L^1(\Omega) \) and this also entails that \( \sigma_k \) converges to \( \sigma^* \) a.e. on \( \Omega \) up to subsequences. Now we can pass to the limit as \( k \rightarrow +\infty \) in (4.7) and, by the weak convergence of \( \sigma_k \) to \( \sigma^* \), this yields
\[
\int_{\Omega} \langle \sigma^*, D\eta \rangle \, dx \geq 0 \quad \forall \eta \in C_0^\infty(\Omega), \quad \eta \geq 0.
\] (4.17)
This implies that \( \text{div} \sigma^* \leq 0 \) in the distributional sense, i.e. (2.8).

4.5. The validity of (2.9). We should observe that
\[
\int_{\Omega} \langle \sigma^*, Du \rangle \, dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} \langle \sigma_k, Du_k \rangle \, dx.
\] (4.18)
Indeed, by (4.8), (4.13), (4.16) and the fact that \( \theta(|\xi|) \geq 0 \) and \( \theta(|\xi|) \geq 0 \) for all \( \xi \in \mathbb{R}^n \), we get that
\[
\langle \sigma_k, Du_k \rangle = F^*_k(\sigma_k) + F_k(Du_k) \geq \overline{\theta}(|\sigma_k|) + \theta(|Du_k|) + c - \mu_k \geq -\mu_k
\]
Therefore we can apply Fatou’s Lemma to the sequence of functions \( (\langle \sigma_k, Du_k \rangle + \mu_k)_k \) that, thanks to the definition of \( (\mu_k)_k \), converges a.e. to \( \langle \sigma^*, Du \rangle \) letting us deduce that
\[
\int_{\Omega} \langle \sigma^*, Du \rangle \, dx \leq \liminf_{k \to +\infty} \int_{\Omega} (\langle \sigma_k, Du_k \rangle + \mu_k) \, dx \leq \liminf_{k \to +\infty} \int_{\Omega} \langle \sigma_k, Du_k \rangle \, dx.
\]
as we wanted to prove. Since \( u_0 \in W^{1,1}(\Omega) \) and since \( \psi \in W^{1,1}(\Omega) \), by Lemma 3.2 there exists a non-increasing sequence \( (\psi_j)_j \subset W^{1,1}_{\text{loc}}(\Omega) \) such that \( \psi_j \to \psi \) a.e. in \( \Omega \). Using (4.3) with \( \psi_j \) in place of \( \eta \), since \( (\psi_j)_j \subset \mathbb{K}_\psi(\Omega) \), we get that
\[
\int_{\Omega} \langle \sigma_k, Du_k \rangle \, dx \leq \int_{\Omega} \langle \sigma_k, D\psi_j \rangle \, dx
\]
and the weak convergence of \( \sigma_k \) to \( \sigma^* \) implies that
\[
\liminf_{k \to +\infty} \int_{\Omega} \langle \sigma_k, Du_k \rangle \, dx \leq \liminf_{k \to +\infty} \int_{\Omega} \langle \sigma_k, D\psi_j \rangle \, dx = \int_{\Omega} \langle \sigma^*, D\psi_j \rangle \, dx.
\]
Combining this with (4.14) and (4.18), we obtain that
\[
\int_{\Omega} F(Du) \, dx + \int_{\Omega} F^*(\sigma^*) \, dx = \int_{\Omega} \langle \sigma^*, Du \rangle \, dx \leq \int_{\Omega} \langle \sigma^*, D\psi_j \rangle \, dx = [\sigma^*, \psi_j]_{u_0(\Omega)}(\Omega).
\]
Now, passing to the limit as \( j \to +\infty \), the Monotone Convergence Theorem yields
\[
\int_{\Omega} F(Du) \, dx + \int_{\Omega} F^*(\sigma^*) \, dx \leq [\sigma^*, \psi]_{u_0(\Omega)}(\Omega) \tag{4.19}
\]
On the other hand, we have that
\[
[\sigma^*, \psi]_{u_0(\Omega)} = \int_{\Omega} (\psi - u_0) d(-\text{div } \sigma^*) + \int_{\Omega} \langle \sigma^*, Du_0 \rangle \, dx
\]
\[
= \int_{\Omega} (\psi - u + u - u_0) d(-\text{div } \sigma^*) + \int_{\Omega} \langle \sigma^*, Du_0 \rangle \, dx
\]
\[
\leq \int_{\Omega} (u - u_0) d(-\text{div } \sigma^*) + \int_{\Omega} \langle \sigma^*, Du_0 \rangle \, dx
\]
where we used the fact that
\[
\int_{\Omega} (u - \psi) d(-\text{div } \sigma^*) \geq 0
\]
because \(-\text{div } \sigma^*\) is a non-negative Radon measure and \( u \geq \psi \) a.e. on \( \Omega \). For a standard sequence of mollifiers \((\varphi_\varepsilon)_\varepsilon\), where we choose the mollifiers lower than 1, we define \( u_\varepsilon = u * \varphi_\varepsilon \). By Fatou’s Lemma and then integrating by parts, we get that
\[
[\sigma^*, \psi]_{u_0(\Omega)} \leq \liminf_{\varepsilon \to 0} \int_{\Omega} (u_\varepsilon - u_0) d(-\text{div } \sigma^*) + \int_{\Omega} \langle \sigma^*, Du_0 \rangle \, dx
\]
\[
= \liminf_{\varepsilon \to 0} \int_{\Omega} \langle \sigma^*, Du_\varepsilon \rangle \, dx
\]
\[
\leq \liminf_{\varepsilon \to 0} \int_{\Omega} F(Du_\varepsilon) \, dx + \int_{\Omega} F^*(\sigma^*) \, dx \tag{4.20}
\]
where we used (3.3) with $F^{**} = F$, thanks to the properties we have on $F$. Jensen’s inequality implies also that
\[
\int_{\Omega} F(Du_\varepsilon) \, dx = \int_{\Omega} F(Du \ast \varphi_\varepsilon) \, dx \leq \int_{\Omega} \varphi_\varepsilon \ast F(Du) \, dx
\]
granting us that
\[
\liminf_{\varepsilon \to 0} \int_{\Omega} F(Du_\varepsilon) \, dx \leq \int_{\Omega} F(Du) \, dx.
\]
If we use this inequality in (4.20), we obtain
\[
[\sigma^*, \psi]_{u_0}(\Omega) \leq \int_{\Omega} F(Du) \, dx + \int_{\Omega} F^*(\sigma^*) \, dx,
\]
which combined with (4.19), grants us that it is in fact an equality, i.e.
\[
[\sigma^*, \psi]_{u_0}(\Omega) = \int_{\Omega} F(Du) \, dx + \int_{\Omega} F^*(\sigma^*) \, dx.
\]

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