On the Replica Fourier Transform

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Abstract

The Replica Fourier Transform introduced previously is related to the standard definition of Fourier transforms over a group. Its use is illustrated by block-diagonalizing the eigenvalue equation of a four-replica Parisi matrix.

1 The replica group

Given a Parisi scenario with \( R \) replica symmetry breaking steps (Parisi, 1980), each replica \( \alpha = 1, 2, \ldots, n \) can be univocally written as a series of \( R + 1 \) numbers

\[
\alpha \equiv (a_0, a_1, \ldots, a_R) \quad \text{with} \quad a_j = 0, 1, \ldots, p_j/p_{j+1} - 1
\]

where the \( p_j \)'s are the sizes of the Parisi sub-boxes (by definition \( p_0 \equiv n \) and \( p_{R+1} \equiv 1 \)). If we impose periodic conditions, the set of these sequences has the structure of an additive group and precisely it can be represented as the following direct product

\[
\mathbb{F}_{p_0/p_1} \times \mathbb{F}_{p_1/p_2} \times \cdots \times \mathbb{F}_{p_R/p_{R+1}}
\]

where \( \mathbb{F}_p \) is the additive group of the first \( p \) integers modulo \( p \).

On this group, one can define a (co-)distance (overlap in spin glass jargon) by saying that two replicas, \( \alpha \equiv (a_0, \ldots, a_R) \) and \( \beta \equiv (b_0, \ldots, b_R) \), have overlap \( \alpha \cap \beta = t \) if \( (\alpha - \beta) \equiv (0, 0, \ldots, 0, a_t - b_t, \ldots) \) with \( a_t \neq b_t \) and the first \( t \) entries equal to zero. It is easy to check that, given three replicas, their mutual distances satisfy the well-known ultrametric properties.

On the other hand, it will turn out to be useful to define another kind of distance (co-overlap): given two replicas, \( \mu \equiv (m_0, \ldots, m_R) \) and \( \nu \equiv (n_0, \ldots, n_R) \), we say that they have co-overlap \( \mu^* \nu = k \) if \( (\mu-\nu) \equiv (\ldots, m_{k-1} - n_{k-1}, 0, \ldots, 0) \) with \( m_{k-1} \neq n_{k-1} \) and the last \( R+1-k \) entries equal to zero. The co-overlap also satisfies ultrametric properties, with the difference that the base of the ultrametric isosceles triangle is smaller than the other sides.

Finally, since we deal with a direct product of groups, the Haar measure is simply the product of the Haar measures of each \( \mathbb{F}_{p_i/p_{i+1}} \)

\[
\int d\alpha \overset{\text{def}}{=} \sum_{a_0} \sum_{a_1} \cdots \sum_{a_R}
\]

and for later convenience we choose to keep it normalised to \( p_0 = n \).

\(^1\)Without loss of generality we identify the replica \( n \) as the sequence \( n \equiv (0, 0, \ldots, 0) \).
2 Character and Fourier Transform

From (2), it follows immediately that the character \( \chi(\alpha) \) of the replica group is simply the product

\[
\chi(\alpha) = \prod_{j=0}^{R} \exp \left\{ 2\pi i \frac{a_j}{p_j/p_{j+1}} \right\}
\]

for each \( \alpha \equiv (a_0, a_1, \ldots, a_r) \in A \). The Fourier Transform of a generic function \( f(\alpha) \) at momentum \( \mu \in A \) reads

\[
\hat{f}(\mu) = \int d\alpha \chi(\alpha \mu) f(\alpha).
\]

(5)

In many interesting cases where one needs the Fourier Transform of a function depending only on the overlap, viz. \( f(\alpha) \equiv f_{\alpha \cap n} \), the integration in (5) can be worked out shell by shell

\[
\hat{f}(\mu) = \sum_{t=0}^{R+1} \int_{S_t} d\alpha \chi(\alpha \mu) f_{\alpha \cap n=0}
\]

(6)

where each integration is restricted to the subgroup \( S_{t} \equiv \{ \alpha : \alpha \cap n = t \} \).

In order to compute the Fourier Transform of a shell, let us consider the set \( B_{t} \equiv \{ \alpha : \alpha \cap n \geq t \} \), i.e. the complementary of a ball with radius \( t \).

Because of the relation

\[
\sum_{a_j=0}^{p_j/p_{j+1}} \exp \left\{ 2\pi i \frac{a_j}{p_j/p_{j+1}} \right\} = \frac{p_j}{p_{j+1}} \delta_{r,0},
\]

(7)

it is straightforward to verify that the Fourier Transform of \( B_t \) at momentum \( \mu \) \((\mu \cap n = k)\) reads

\[
\int_{B_t} d\alpha \chi(\alpha \mu) = \begin{cases} \sum_{t=0}^{k} \frac{p_j}{p_{j+1}} & \text{if } t \geq k \\ 0 & \text{if } t < k \end{cases}
\]

(8)

Consequently, the Fourier Transform of a shell is derived as the difference of two successive balls, thus giving

\[
\int_{S_t} d\alpha \chi(\alpha \mu) = \int_{B_t} d\alpha \chi(\alpha \mu) - \int_{B_{t+1}} d\alpha \chi(\alpha \mu) = \begin{cases} p_t - p_{t+1} & \text{if } t \geq k \\ -p_{t+1} & \text{if } t = k - 1 \\ 0 & \text{if } t < k - 1 \end{cases}
\]

(9)

Note that, because of the translational invariance of the function \( f \), its Fourier transform depends only on the co-overlap \( \mu \cap n \). By making use of the previous result and taking as null the values out of the range of definition, equation (3) becomes

\[
\hat{f}(\mu) = \hat{f}_{\mu \cap n=0} = \sum_{t=k}^{R+1} p_t (f_t - f_{t-1})
\]

(10)

thus retrieving the Replica Fourier Transform introduced in (De Dominicis et al., 1996. De Dominicis et al., 1997). In the limit \( R \to \infty \), (10) coincides with the transform defined by Ménard and Parisi (1991) for the continuum.

Besides this simple case, the same formalism can be extended to the situations where one needs to (Fourier) transform in the presence of a passive overlap, which is not to be summed over. For example, for a 3-index matrix \( A^{\alpha \beta \gamma} \), parametrised in the usual way

\[
A^{\alpha \beta / \gamma} = A^{\gamma}_t \quad \text{where} \quad r = \alpha \cap \beta, \quad t = \max(\alpha \cap \gamma; \beta \cap \gamma),
\]

(11)
the Fourier Transform with respect to the lower index at $\alpha \cap \beta$ fixed, can be written

$$\hat{A}^r(\mu) = \int d\zeta \chi(\zeta) A^r_{\zeta\cap \gamma}$$

$$\zeta = \begin{cases} \alpha - \gamma & \alpha \cap \gamma \geq \beta \cap \gamma \\ \beta - \gamma & \text{otherwise} \end{cases}$$ (12)

As above one can perform the integration shell by shell, with the slight difference that the occurrence of different ultrametric regions must be taken into account. Indeed, one finds that the region $\zeta \cap n > r$ occurs twice and equation (12) becomes

$$\hat{A}^r(\mu) \equiv \hat{A}^r_{\mu_n = k} = \sum_{t=k}^{R+1} p_t^{(r)} (A^r_t - A^r_{t-1})$$ where

$$p_t^{(r)} = \begin{cases} p_t & t \leq r \\ \frac{p_t}{2p_t} & t > r \end{cases}$$ (13)

In the same way, one can generalise this procedure for RFT in the presence of two passive overlaps e.g. for the longitudinal anomalous configuration of 4-index ultrametric matrices as shown in ref. (De Dominicis et al., 1997). In other words, one can say that the Replica Fourier Transform has no “perception” of the ultrametric tree (the character does not depend on the matrix which it acts upon), but it extracts out the right weights according to the passive overlaps involved. This property is useful when dealing with convolution products with several replicas, where it reduces drastically an otherwise quite tedious procedure. We illustrate it on the Longitudinal-Anomalous eigenvalue equation for a 4-replica Parisi matrix. It writes

$$\frac{1}{2} \int d\rho d\sigma M^{\alpha \beta; \rho \sigma} f^{\rho \sigma; \gamma} = \lambda_A f^{\alpha \beta; \gamma}$$ (14)

The left hand side has contributions from the Replicon-like and Longitudinal Anomalous like configurations. The former is diagonal, the latter is more delicate to treat. It can be written

$$\frac{1}{2} \int d(\rho - \sigma) d\xi M^{\alpha \beta; \rho \sigma} f^{\rho \sigma; \gamma}$$

where $\zeta$ is defined in (12) and $\alpha \cap \beta = r$ is fixed. In (15) one recognizes a convolution which under Fourier Transform over $\zeta$ yields the product of $f$ and the Fourier Transformed of $M$ (called $K$). So we get

$$\sum_{s=0}^{k} \left\{ \delta^{K_{r-t}}_{s} K_{t+r+1}^{r} + \frac{1}{4} K_{k}^{r} \delta^{(k-1)}_{s} \right\} \hat{f}_k = \lambda_A \hat{f}_k.$$ (16)

where the $M$ matrix is now block-diagonalized (Temesvári et al., 1994. De Dominicis et al., 1997.). The diagonal term is the contribution from the Replicon configuration, the weight $\delta^{(k-1)}_s / 2 \equiv (p_{s+1}^{(k-1)} - p_{s+1}) / 2$ coming out from the sum over $(\rho - \sigma)$.

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