Phases of 4D Scalar-tensor black holes coupled to Born-Infeld non-linear electrodynamics

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Abstract

Recent results show that when non-linear electrodynamics is considered the no-scalar-hair theorems in the scalar-tensor theories (STT) of gravity, which are valid for the cases of neutral black holes and charged black holes in the Maxwell electrodynamics, can be circumvented \cite{1,2}. What is even more, in the present work, we find new non-unique, numerical solutions describing charged black holes coupled to non-linear electrodynamics in a special class of scalar-tensor theories. One of the phases has a trivial scalar field and coincides with the corresponding solution in General Relativity. The other phases that we find are characterized by the value of the scalar field charge. The causal structure and some aspects of the stability of the solutions have also been studied. For the scalar-tensor theories considered, the black holes have a single, non-degenerate horizon, i.e., their causal structure resembles that of the Schwarzschild black hole. The thermodynamic analysis of the stability of the solutions indicates that a phase transition may occur.

1 Introduction

Among the most natural generalizations of General Relativity (GR) are the scalar-tensor theories of gravity in which a single or multiple fundamental scalar fields are

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present as a possible remnant of a fundamental unified theory like string theory or higher dimensional gravity theories [3]. Different modifications of scalar-tensor theories are attracting much interest also in cosmology and astrophysics.

Possible deviations from GR, especially the existence of new effects, due to the presence of the scalar field would be of considerable interest. According to the no-scalar-hair conjecture, the black-hole solutions in the STT coincide with the solutions from GR. No-scalar-hair theorems treating the cases of static, spherically symmetric, asymptotically flat, electrically neutral black holes and charged black holes in the Maxwell electrodynamics have been proved for a large class of scalar-tensor theories [4, 5, 6]. The scalar field in these cases is constant, and thus trivial, if one demands that the essential singularity at the center of symmetry is hidden in a regular event horizon.

There are no no-scalar-hair theorems, however, in the non-linear electrodynamics (NLED). It was first introduced by Born and Infeld in 1934 to obtain finite energy density model for the electron [7]. For more information on recent interests in non-linear lagrangians of electrodynamics please see [8], also [9]–[30] and references cited therein for gravitational aspects of NLED.

In the case of NLED, the energy-momentum tensor of the electromagnetic field has a non-vanishing trace which is non-trivially coupled to the scalar field. Hence, the electro-magnetic field acts as a source of the scalar field and allows the existence of asymptotically flat, hairy black holes. Such solutions have been found recently in different non-linear electrodynamics [1, 2]. These solutions are hairy in a sense that the scalar field is not trivial. That hair, however, is secondary since the solutions are determined uniquely by the values of their magnetic charge and mass, and the value of the scalar field at infinity. For other solutions describing asymptotically flat black holes with scalar hair (in which, however, the potential of the scalar field is not positively semi-definite) we refer the reader to [31].

Another interesting effect that occurs when NLED is considered is the presence of multiple black-hole phases in a certain class of STT. Presence of non-unique solutions in STT has been found by Damour and Esposito-Farèse in their works on neutron stars in the STT. One of the most interesting effects that occur in these solutions is the spontaneous scalarization, a scalar field analogue of the spontaneous magnetization of ferromagnets [32, 33, 34]. In the present work we consider the same STT and find multiple-phase black-hole solutions numerically. Unlike the situation in [1, 2], they are not determined in a unique way by the values of their magnetic charge and mass and the value of the scalar field at infinity. The phase diagram of the present solutions is reminiscent of the phase transition between caged black holes and black strings in higher dimensions [35, 36]. The preliminary analysis gives us a reason to suppose that in the system we study a phase transition might occur. The analysis of the thermodynamics of the solutions, however, is not sufficient to make a conclusion about the stability of the solutions and about the existence of phase transitions. The full examination of the problem requires a perturbative analysis.
2 Basic equations and qualitative investigation

The theory we consider is presented [1, 2] both in the Jordan and in the Einstein frames. In the Einstein frame the action of the theory we consider is

$$S = \frac{1}{16\pi G_5} \int d^4x \sqrt{-g} \left[ \mathcal{R} - 2g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - 4V(\varphi) + 4\mathcal{A}^4(\varphi)L(X,Y) \right].$$

$V(\varphi)$ is the potential of the scalar field, $L(X)$ is the Lagrangian of the electromagnetic field, $\mathcal{A}^4(\varphi)$ is the function which determines the coupling between the electromagnetic field and the scalar field. We also have that

$$X = \frac{\mathcal{A}^{-4}(\varphi)}{4} F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}, \quad Y = \frac{\mathcal{A}^{-4}(\varphi)}{4} F_{\mu\nu} (\ast F)^{\mu\nu}$$

where “$\ast$” stands for the Hodge dual with respect to the Einstein frame metric $g_{\mu\nu}$. The type of the STT is determined by the explicit choice of the functions $V(\varphi)$ and $\mathcal{A}^4(\varphi)$. The field equations obtained from action (1) take the following form

$$\mathcal{R}_{\mu\nu} = 2\partial_\mu \varphi \partial_\nu \varphi + 2V(\varphi) g_{\mu\nu} - 2\partial_X L(X,Y) \left( F_{\mu\beta} F_{\nu}^{\beta} - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right)$$

$$- 2\mathcal{A}^4(\varphi) [L(X,Y) - Y \partial_Y L(X,Y)] g_{\mu\nu},$$

$$\nabla_\mu \left( \partial_X L(X,Y) F^{\mu\nu} + \partial_Y L(X,Y) (\ast F)^{\mu\nu} \right) = 0,$$

$$\nabla_\mu \nabla^\mu \varphi = \frac{dV(\varphi)}{d\varphi} - 4\alpha(\varphi) \mathcal{A}^4(\varphi) [L(X,Y) - X \partial_X L(X,Y) - Y \partial_Y L(X,Y)],$$

where $\alpha(\varphi) = \frac{d\ln \mathcal{A}(\varphi)}{d\varphi}$.

In what follows the truncated Born-Infeld electrodynamics described by the Lagrangian

$$L_{BI}(X) = 2b \left( 1 - \sqrt{1 + \frac{X}{b}} \right)$$

will be considered. And $V(\varphi)$ will be equal to zero.

The anzats for metric of a static, spherically symmetric space-time can be taken in the following form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f(r)e^{-2\delta(r)} dt^2 + \frac{dr^2}{f(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right).$$

Since the Born-Infeld NLED is invariant under electric-magnetic duality rotations we will study only the magnetically charged black holes for which the electromagnetic field is given by

$$F = P \sin \theta d\theta \wedge d\phi$$

and the magnetic charge is denoted by $P$.

\footnote{Here we consider the pure magnetic case for which $Y = 0$.}
The field equations reduce to the following coupled system of ordinary differential equations:

\[
\frac{d\delta}{dr} = -r \left( \frac{d\varphi}{dr} \right)^2, \tag{6}
\]

\[
\frac{dm}{dr} = r^2 \left[ \frac{1}{2} f \left( \frac{d\varphi}{dr} \right)^2 - A^4(\varphi)L(X) \right], \tag{7}
\]

\[
\frac{d}{dr} \left( r^2 f \frac{d\varphi}{dr} \right) = r^2 \left\{ -4\alpha(\varphi)A^4(\varphi) \left[ L(X) - X \partial_X L(X) \right] - rf \left( \frac{d\varphi}{dr} \right)^3 \right\}, \tag{8}
\]

where

\[
f = 1 - \frac{2m}{r}
\]

and \(X\) reduces to:

\[
X = \frac{A^{-4}(\varphi)}{2} \frac{P^2}{r^4}. \tag{9}
\]

We will be searching for solutions which have a regular horizon on which the scalar field \(\varphi\) does not diverge. The regularity of the transition between the Einstein and the Jordan conformal frames requires the following restrictions on the coupling function \(0 < A(\varphi) < \infty\) for \(r \geq r_H\), where \(r_H\) is the radius of the horizon, and we will consider STT for which it is satisfied. The diversity and the properties of the solutions depend strongly on the choice of the functions \(A(\varphi)\) (respectively on \(\alpha(\varphi)\)). Even when the functions \(A(\varphi)\) are chosen to satisfy the experimental constraints (see, for example, [33]) we are left with infinitely many possibilities. That is why some restrictions on the functions \(A(\varphi)\) representing the STT should be imposed. In the present work we will consider only theories for which \(\varphi \alpha(\varphi)\) is non-negative or non-positive for all values of \(\varphi\). As we prove below, theories for which \(\varphi \alpha(\varphi) \geq 0\) for all values of \(\varphi\) do not admit (asymptotically flat) black hole solutions with non-trivial scalar field. In theories with \(\varphi \alpha(\varphi) \leq 0\) black holes (if they exist) have a single non-degenerate event horizon when \(\alpha(\varphi) = 0\) admits only the solution \(\varphi = 0\).

Using the following equation

\[
\frac{d}{dr} \left( e^{-\delta} r^2 f \frac{d\varphi}{dr} \right) = 4r^2 e^{-\delta} \alpha(\varphi)A^4(\varphi) \left[ X \partial_X L(X) - L(X) \right], \tag{10}
\]

which is another form of equation (8) and the fact that for the Born-Infeld Lagrangian (3) holds

\[
X \partial_X L(X) - L(X) > 0 \tag{11}
\]

we can draw some conclusions about the general properties of the solutions.

Let us multiply equation (10) by \(\varphi\) and then integrate it in the interval \(r \in [r_H, \infty)\)
where we denote the radius of the outer horizon (the event horizon) with \( r_H \)

\[
\int_{r_H}^{\infty} \varphi \frac{d}{dr} \left( e^{-\delta r^2} f \frac{d\varphi}{dr} \right) dr \\
= 4 \int_{r_H}^{\infty} r^2 e^{-\delta} \varphi \alpha(\varphi) A^4(\varphi) \left[ X \partial_X L(X) - L(X) \right] dr,
\]

and after integrating by parts we get

\[
\lim_{r \to \infty} \left( \varphi e^{-\delta} r^2 f \frac{d\varphi}{dr} \right) - \left. \left( \varphi e^{-\delta} r^2 f \frac{d\varphi}{dr} \right) \right|_{r=r_H} - \int_{r_H}^{\infty} e^{-\delta} r^2 f \left( \frac{d\varphi}{dr} \right)^2 dr \\
= \lim_{r \to \infty} D \varphi - \int_{r_H}^{\infty} e^{-\delta} r^2 f \left( \frac{d\varphi}{dr} \right)^2 dr \\
= 4 \int_{r_H}^{\infty} r^2 e^{-\delta} \varphi \alpha(\varphi) A^4(\varphi) \left[ X \partial_X L(X) - L(X) \right] dr.
\]

In the second line we take advantage of the fact that we are looking for asymptotically flat, black hole solutions so \( f(r_H) = 0 \), \( \lim_{r \to \infty} f(r) = 1 \), \( \lim_{r \to \infty} \delta(r) = 0 \) and for the asymptotic value of the scalar field we impose \( \lim_{r \to \infty} \varphi(r) = \varphi_{\infty} \), where \( \varphi_{\infty} = 0 \). The constant \( D \) denotes the scalar charge which is defined as

\[
D = -\lim_{r \to \infty} r^2 \frac{d\varphi}{dr}.
\]

Since \( \varphi \) is vanishing at infinity we finally get

\[
- \int_{r_H}^{\infty} e^{-\delta} r^2 f \left( \frac{d\varphi}{dr} \right)^2 dr = \\
= 4 \int_{r_H}^{\infty} r^2 e^{-\delta} \varphi \alpha(\varphi) A^4(\varphi) \left[ X \partial_X L(X) - L(X) \right] dr.
\]

The total sign of the left-hand side of (15) is negative for black hole solutions with nontrivial \( \varphi \). The sign of the integral on the right-hand side depends on the sign of \( \varphi \alpha(\varphi) \). If \( \varphi \alpha(\varphi) \geq 0 \) a contradiction is reached so the assumption for the existence of asymptotically flat black holes with non-trivial scalar field in this theory is wrong. Asymptotically flat black-hole solutions with nontrivial scalar field exist in scalar-tensor theories for which \( \varphi \alpha(\varphi) \leq 0 \). Through a similar examination we prove that these black holes have a single, non-degenerate horizon. Let us admit that more than one horizon exists. Then we multiply equation (10) by \( \varphi \) again and integrate it by parts
in the interval \( r \in [r_-, r_{+}] \) where we denote the first inner horizon and the outermost, non-degenerate horizon with \( r_- \) and \( r_+ \), respectively

\[
\left( \varphi e^{-\delta r^2 \frac{d\varphi}{dr}} \right)_{r=r_+} - \left( \varphi e^{-\delta r^2 \frac{d\varphi}{dr}} \right)_{r=r_-} - \int_{r_-}^{r_+} \left[ e^{-\delta r^2 f \left( \frac{d\varphi}{dr} \right)^2} \right] \, dr
\]

\[
= 4 \int_{r_-}^{r_+} r^2 e^{-\delta} \varphi \alpha(\varphi) A^4(\varphi) [X \partial_X L(X) - L(X)] \, dr < 0. \tag{16}
\]

Having in mind that \( f(r_-) = 0 = f(r_+) \) and that \( f < 0 \) in the interval \( r \in [r_-, r_{+}] \) we reach a contradiction, which means that the admission is incorrect.

Now, we only have to prove the non-existence of extremal black holes (black holes with degenerate event horizon). For the scalar-tensor theories we consider \( \alpha(\varphi) \) turns to zero only when \( \varphi = 0 \). Let us admit that an extremal black hole with non-trivial scalar field exists. In this case, the left-hand side of (10) is equal to zero. The right-hand side is equal to zero only when \( \alpha(\varphi_H) = 0 \), where \( \varphi_H \) is the value of the scalar field on the horizon, which means that \( \varphi_H = 0 \). We also require that \( \varphi_\infty = 0 \), where \( \varphi_\infty \) is the value of the scalar field at the spacial infinity. In this situation, the only possibility to have a solution with non-trivial scalar field is the scalar field to have at least one extremum. So let us integrate equation (10) in the interval \( r \in [r_H, r_e] \), where \( r_e \) is the point of the leftmost (the one which is nearest to the event horizon) extremum of \( \varphi \) which is on the right of the event horizon

\[
0 = \left( e^{-\delta r^2 f \frac{d\varphi}{dr}} \right)_{r=r_H} - \left( e^{-\delta r^2 f \frac{d\varphi}{dr}} \right)_{r=r_e}
\]

\[
= 4 \int_{r_e}^{r_H} r^2 e^{-\delta} \alpha(\varphi) A^4(\varphi) [X \partial_X L(X) - L(X)] \, dr. \tag{17}
\]

Since \( \varphi \neq 0 \) in the interval \( (r_H, r_e) \), the sign of \( \alpha(\varphi) \) also does not change in this interval. Hence, the integral on the right-hand side of (17) is non-zero and has a fixed sign which depends on the sign of \( \alpha(\varphi) \). The contradiction we reach means that no extremal solutions with non-trivial scalar field can exist.

To sum up, we can say that if a black hole exists it will have a single horizon, i.e., its causal structure will be Schwarzschild-like\(^2\). In both conformal frames, inside the event horizon a space-like singularity is hidden.

Finishing this section it is worth noting that the differential equations system \( \mathcal{E} \mathcal{S} \) is invariant under the rigid rescaling \( r \to \lambda r \), \( m \to \lambda m \), \( P \to \lambda P \) and \( b \to \lambda^{-2} b \) where \( 0 < \lambda < \infty \). Therefore, given a solution to \( \mathcal{E} \mathcal{S} \) with one set of physical parameters \( (r_h, M, P, b, D, T) \), the rigid rescaling produces new solutions with parameters \( (\lambda r_h, \lambda M, \lambda P, \lambda^{-2} b, \lambda D, \lambda^{-1} T) \). Here \( T \) denotes the temperature of the horizon.

\(^2\)Another class of scalar-tensor theories which admit black holes of the Schwarzschild type are those with negative function \( \beta(\varphi) = \frac{d\alpha(\varphi)}{d\varphi} \) for all values of \( \varphi \) and \( \alpha(\varphi_\infty)=0 \). This can be shown by a method similar to that presented above. One can also show that scalar-tensor theories with \( \beta(\varphi) > 0 \) for all values of \( \varphi \) do not admit asymptotically flat black holes with nontrivial scalar field.
3 Numerical results

The nonlinear system (6)-(8) is inextricably coupled and the event horizon $r_H$ is a priori unknown boundary. In order to be solved numerically, it is recast as an equivalent first order system of ordinary differential equations. Following the physical assumptions of the matter under consideration the asymptotic boundary conditions are set, i.e.,

$$\lim_{r \to \infty} m(r) = M \quad (M \text{ is the mass of the black hole in the Einstein frame}),$$

$$\lim_{r \to \infty} \delta(r) = \lim_{r \to \infty} \varphi(r) = 0.$$  

At the horizon both the relationship $f(r_H) = 0$ and the regularization condition

$$\left. \left( \frac{df}{dr} \cdot \frac{d\varphi}{dr} \right) \right|_{r=r_H} = \left\{ 4\alpha(\varphi)A^4(\varphi)[X\partial_X L(X) - L(X)] \right\}|_{r=r_H}$$

cconcerning the spectral quantity $r_H$ must be held. For the treating the above posed boundary-value problem (BVP) the Continuous Analog of Newton Method (see, for example [37],[38],[20]) is used. After an appropriate linearization the original BVP is rendered to solving a vector two-point BVP. On a discrete level sparse (almost diagonal) linear algebraic systems with regard to increments of sought functions $\delta(r), m(r),$ and $\varphi(r)$ have to be inverted.

For our numerical solutions we have considered the STT studied by Damour and Esposito-Farèse in their works on neutron stars in the STT of gravity. In this particular theory, the coupling function has the following form

$$A(\varphi) = e^{\frac{1}{2}\beta\varphi^2}, \quad (18)$$

where $\beta$ is a negative constant. Observational data from binary-pulsar and solar-system experiments restricts the admissible values of $\beta$. When $\alpha(\varphi = 0) = 0$, as it is in our case, the coupling constant should be $\beta > -5$ (see [33] for more details). We have studied the solutions for values of the parameters which are in agreement with the current observations but also for such values that are out the admissible interval since the later have qualitatively different behavior from the former which makes them interesting for the theory.

For this coupling function the field equations possess the discrete symmetry $\varphi \rightarrow -\varphi$. Let us also note that every general relativistic solution is a solution to the scalar tensor theory under consideration with $\varphi = 0$.

A thorough study of the phase space would be difficult due to the large number of parameters $(\beta, b, P, M)$ in the problem. So in order to illustrate the general behavior of the obtained solutions we give several representative figures considering several values of $\beta, b$ and $P$ and varying the mass $M$ of the black hole. For the cases presented here $b = 0.01; 0.2$. 

7
3.1 General description of the phase space

Even if the values of all four parameters $\beta, b, P, M$ and the boundary conditions are fixed the solutions of (6-8) are not uniquely determined, i.e. the solutions are not unique. An additional parameter should be used for labeling of the different solutions. One natural choice would be to label the different solution by the value of the scalar charge $\mathcal{D}$, which here unlike the cases in [1, 2] is independent.

The global structure of the phase space changes with the variation of $\beta$. For $\beta > \beta_{\text{crit}}$, where $\beta_{\text{crit}} \approx -14.9$ when $b = 0.01$ and $P = 1.0$, the $M - \mathcal{D}$ phase diagram consists of three branches, while for $\beta < \beta_{\text{crit}}$ the number of branches is five. The change of the qualitative structure of that phase diagram with the variation of $\beta$ is given in Figure 1. The different cases have been studied in more details in the succeeding subsections.

![Figure 1](image.png)

Figure 1: The $M - \mathcal{D}$ relation for several different values of $\beta$, $b = 0.01$ and $P = 1.0$. For $\beta > \beta_{\text{crit}}$ the Middle branch disappears.

3.2 Cases with three solutions, $\beta = -4.0$

In the left panel of Figure 2 the dependence $\mathcal{D}(M)$ of the scalar charge on the mass for $\beta = -4.0$ is presented. There are two special points to be considered, namely $A$ and $E$. For $P = 1.0$ the points $A$ and $E$ lie at masses $M_A \approx 0.794$ and $M_E \approx 0.46$. For values of the black hole mass in the interval $M \in (M_E, M_A)$ three solutions co-exist. We call them Outer and Trivial. The equations posses a discrete symmetry $\varphi \rightarrow -\varphi$ as a result of which the Outer solution has a mirror image with respect to the abscissa, whose scalar field charge $\mathcal{D}$ has an opposite sign. The symmetric solutions we will denote as $\text{Outer}_\pm$ where the indices $+$ and $-$ refer to the sign of the scalar field charge of the solutions. The radius and the temperature of the event horizon, are the same for both solutions in the couple so when we comment on them will usually omit the $+$ and $-$ indices.

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3 An alternative choice would be the value of the scalar field on the horizon.
Figure 2: The $M - D$ and the $M - r_H$ relations, for $\beta = -4.0$, $b = 0.01$ and $P = 1.0$.

Figure 3: The $M - T^{-1}$ relation for the same values of the magnetic charge as in Figure 2.
The *Trivial* brunch has scalar field $\varphi \equiv 0$ and represents the Einstein-Born-Infeld solution in GR. For masses lower than $M_E$ only the solutions which we call *Outer* exist and for $M > M_A$ only the *Trivial* solution remains.

The radii of the black holes are shown in the right panel of Figure (2). An object with zero radius of the event horizon (a naked singularity) is reached for a finite value of $M$ for all three solutions - the $\text{Outer}_\pm$ and the *Trivial*.

The inverse temperature $T^{-1}$ of the solutions is presented in Figure (3).

### 3.3 Cases with five solutions, $\beta = -50.0$

#### 3.3.1 Solutions with Born-Infeld parameter $b=0.01$

![Figure 4](https://via.placeholder.com/150)

*Figure 4:* The value of scalar field charge $D$ as a function of the mass $M$ of the black hole, for two different values of the magnetic charge $P = 1.5$ and $P = 3.0$.

![Figure 5](https://via.placeholder.com/150)

*Figure 5:* The $M - r_H$ relation for the same values of the magnetic charge as in Figure (4). A magnification in the encircled region is presented in Figure (6).

For $\beta = -50.0$ and $b = 0.01$ the $D(M)$ dependence is given in Figure (4). Here, the special points to be considered are four, $A$, $B_\pm$ and $E$. For $P = 1.5$ the points $A$ and $B_\pm$ lie at masses $M_A \approx 2.023$ and $M_{B_\pm} = M_B \approx 2.37$ while in the case of $P = 3.0$ we
Figure 6: A magnification of the encircled region in Figure (5).

Figure 7: The $M - T^{-1}$ relation for the same values of the magnetic charge as in Figure (4).
have $M_A \approx 3.60$ and $M_{B\pm} = M_B \approx 4.53$. As it can be seen, for values of the black hole mass in the interval $M \in (M_A, M_B)$ five solutions co-exist. They can be separated in three groups which we name *Outer*, *Middle* and *Trivial*. The symmetric solutions we will denote as $Outer_{\pm}$ and $Middle_{\pm}$ with the same convention for the + and − indices as in the previous case. Both points $B_+$ and $B_-$ are projected on the same point on the $M - r_H$ and $M - T^{-1}$ diagrams, which we denote simply as $B$.

The *Trivial* has scalar field $\varphi \equiv 0$ and represents the Einstein-Born-Infeld solution in GR. For masses lower than $M_E \approx 1.00$ and $M_E \approx 2.42$ for $P = 1.5$ and $P = 3.0$, respectively, only the solutions which we call *Outer* exist and for $M > M_B$ only the *Trivial* solution remains.

The radii of the black holes are shown in Figures (5) and (6). Again, for three of the solutions - the *Outer* and the *Trivial* a naked singularity is reached for a finite value of $M$. The *Outer* solutions have a larger radius than the *Trivial* for low masses, but as it can be seen in Figure (6), with the increase of the mass, in point $C$, the situation changes and the black hole with zero scalar charge becomes larger. The approximate position of point $C$ is shown also on Figure (4) with a dotted vertical line. The *Middle* solution black holes are smaller than the other three for all values of $M$.

The inverse temperature $T^{-1}$ of the solutions is presented in Figure (7). With the decrease of the mass $M$ the inverse temperature of the *Trivial* solution passes through a local maximum which gets sharper with the increase of the magnetic charge $P$ and leads to numerical calculation difficulties. In the limit of vanishing radii of the black holes their inverse temperature decreases steeply.

### 3.3.2 Solutions with Born-Infeld parameter $b=0.2$

![Figure 8](image)

Figure 8: The $M - \mathcal{D}$ and the $M - r_H$ relations, for $b = 0.2$ and $P = 1.0$.

Here we present an example of solutions for values of the parameters for which the *Trivial* solution reaches an extremal black hole instead of a naked singularity with the decrease of the mass $M$. In Figure (8) the dependence $\mathcal{D}(M)$ of the scalar charge on the mass and the radii of the black holes are presented. The points $A$ and $B_\pm$ lie at masses $M_A \approx 1.13$ and $M_{B\pm} = M_B \approx 1.54$. Again, for values of the black hole mass
Figure 9: A magnification in the encircled region of the $M - r_H$ relation from Figure (8) and the $M - T^{-1}$ relation, for $b = 0.2$ and $P = 1.0$. 

in the interval $M \in (M_A, M_B)$ five solutions co-exist. The Trivial reaches an extremal black hole at $M_E \approx 0.93$ and this can be seen on the $M - r_H$ diagram.

In Figure (9) a magnification of the encircled region from Figure (8) and the inverse temperature are shown. The point C is once again indicated with a vertical line. With the approaching of the extremal black hole the inverse temperature $T^{-1}$ rises unboundedly.

4 Thermodynamics

Black holes have long been known to be thermodynamical systems [39]. The First Law (FL) of black hole thermodynamics in the presence of a scalar field has the following form [40]

$$\delta M = T \delta S + \Psi_H \delta P + D \delta \varphi_{\infty},$$

where $T$, $S$ and $P$ are the temperature, the entropy, and the magnetic charge of the black hole, respectively, and $\varphi_{\infty}$ is the value of the scalar field at spacial infinity.

The quantity $\Psi$ conjugate to the magnetic charge is the potential of the magnetic field which is given by the following definition

$$H_{\mu} = \partial_{\mu} \Psi.$$

On the other hand the magnetic field is defined as

$$H_{\mu} = - \star G_{\mu\nu} \xi^\nu,$$  

where

$$G_{\mu\nu} = -2 \frac{\partial (A^4(\varphi)L)}{\partial F_{\mu\nu}},$$

$\xi = \frac{\partial}{\partial t}$ is the Killing vector generating time translations and “$\star$” is the Hodge star operator.
Since in our case the asymptotic value of scalar field is fixed, the term which contains its variation vanishes and the FL reduces to the form it has in GR.

In the situation when the solutions are not unique a natural question is which of them are stable. Certainly, the answer of this question requires linear perturbative analysis of our system of coupled differential equations and solving the corresponding eigenvalue problem. In certain cases, however, some information on the stability can be inferred by using only the equilibrium thermodynamical characteristics of the solutions via the so-called “Poincare” or “turning point” method. For a nice discussion of the method we refer the reader to [41] and references therein. The advantage of this method is its remarkable simplicity. The “turning point” method, however, may hide many uncertainties and should be applied with caution. The method consists in the following. Consider a system with thermodynamical parameters $\mu^i$. The equilibrium states (stable or not) are extrema of an appropriate Massieu function $S$. At equilibrium state we can define the conjugate variables $\beta_i(\mu^j) = \frac{\partial S_{eq}}{\partial \mu^i}$. According to the method the change of stability can only occur at turning points or bifurcations in the equilibrium sequence on the conjugate diagram $\beta_i(\mu^i)$. In the absence of bifurcations the stability character changes only when the equilibrium curve meets a turning point and if one single point of an equilibrium sequence is shown to be fully stable, then all equilibria in the sequence are fully stable up to the first turning point. At the turning point the branch with negative slope is always unstable while the branch with a positive slope is more stable than the one with negative slope.

Concerning the application of the “turning point” method to our case we shall consider scalar-tensor black holes in the micro-canonical ensemble. In this case the corresponding Massieu function is the entropy $S(M)$. We will comment on the case with five branches first. The conjugate variable is $\beta_M = T^{-1}$. For $\beta < \beta_{\text{crit}}$ the conjugate diagram $M - T^{-1}$ is shown on Figures (7) and (9). The uncertainties in our case come from the fact that we do not know the full diagram, i.e., whether there are other branches and bifurcation points different from point A. Assuming however that there are no other bifurcation points and taking into account that there is only one turning point $B$ we may conclude that the Middle branch is unstable while the Outer branch is more stable. The cusp which appears in point $B$ on the $M - r_H$ diagram in Figures (6) and (9) also indicates a change in the stability of the solutions. Since the Outer branch is the unique solution (up to the discrete degeneracy $\text{Outer}_{\pm}$) for sufficiently small masses one might accept that the Outer branch is probably stable there. Then according to the “turning point” method the Outer branch should be fully stable up to the turning point $B$. The Trivial branch is unique solution for sufficiently large masses and as a GR solution is known to be stable there [42, 43]. So, assuming that Trivial branch is stable for large masses, the “turning point” method asserts that the Trivial branch should be stable up to the bifurcation point $A$. Since the entropy of a black hole is proportional to the area of its event horizon, among the three black-hole

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4The turning points are points where two equilibrium branches merge with a vertical tangent. A bifurcation point is point where branching of equilibrium sequences occurs.

5We keep the magnetic charge fixed.

6In general, the stability of the trivial solution within the framework of GR does not guarantee its stability within the “larger” scalar-tensor theory.
phases we consider, the one with the biggest radius would have the maximal entropy (see Figures (5), (6), (8) and (9)). So for \( M < M_C \) the Outer solutions would be thermodynamically favorable and for \( M > M_C \) - the Trivial. The Middle solutions are thermodynamically unstable for all values of the mass since their radius is smaller than the radii of the other three solutions. The point \( C \) is a candidate for a point of a first order phase transition. Let us stress again that above analysis based on the “turning point” method is only suggestive and cannot serve for a definitive solution of the stability of the solutions. Reliable analysis of the risen questions will be given elsewhere together with the solution of corresponding eigenvalue problem.

For \( \beta > \beta_{\text{crit}} \) the \( M - T^{-1} \) diagram is given in Figure (3). No turning points can be seen on it. As in the previously discussed case, since Outer± are the only solutions for sufficiently small masses one might accept that they are probably stable there. Then according to the “turning point” method the Outer branch should be fully stable up to the bifurcation point \( A \). Again, assuming that Trivial branch is stable for large enough masses one can expect that to the left it should be stable at least up to the bifurcation point \( A \). From the right panel of Figure (2) we can see that the Outer solutions have larger radius of the event horizon so they would be thermodynamically favorable.

The thermodynamical stability considerations were made in the Einstein frame. In order to transfer the conclusions to the physical, Jordan frame properly, we need to clarify the connection between the thermodynamic properties in the two conformal frames. The temperature of the event horizon is invariant under conformal transformations of the metric that are unity at infinity [44]. The properly defined entropy is also the same in both conformal frames. It has been proved that in the Jordan frame the entropy of the black hole is not simply one fourth of the horizon area [45, 46] as in the Einstein frame and needs to be generalized. The entropy in the Jordan frame is defined as

\[
S_J = \frac{1}{4G_*} \int d^2 x \sqrt{-\tilde{g}} F(\Phi). \tag{23}
\]

Passing to the Einstein frame we get

\[
S_J = \frac{1}{4G_*} \int d^2 x \sqrt{-g} = S_E = S. \tag{24}
\]

In the last two equations \( \tilde{g} \) and \( g \) are the determinants of the induced metrics on the horizon in the Jordan and in the Einstein frame, respectively.

The term in the FL [19] connected with the magnetic charge is also preserved under the conformal transformations.

In order for the FL of thermodynamics to be satisfied in the Jordan frame the mass should be properly chosen since the Arnowitt-Deser-Misner (ADM) masses in both frames are not equivalent. It can be easily shown that in the STT considered the ADM mass in the Jordan frame \( M_J \) is equal ADM mass in the Einstein frame \( M \)

\[
M_J = M. \tag{25}
\]

For the Jordan frame, the proper mass in the FL of thermodynamics is the ADM mass in the Einstein frame \( M \). Similarly, for boson and fermion stars the proper measure for the energy of the system is again the ADM mass in the Einstein frame \( M \). For more details on the subject we would refer the reader to the works [47, 48, 49, 50].
5 Conclusion

In the present work new numerical solutions describing multiple-phase, charged black holes coupled to non-linear electrodynamics in the scalar-tensor theories with massless scalar field were found. Since an electric-magnetic duality is present, here only the purely magnetically charged case was studied. For the Lagrangian of the non-linear electrodynamics the truncated Born-Infeld Lagrangian was chosen and a special STT was considered. As a result of the numerical and analytical investigations, some general properties of the solutions were found. In the class of STT for which $\varphi \alpha(\varphi) \leq 0$ for all values of $\varphi$, black holes with a single horizon exist, i.e., their causal structure resembles that of the Schwarzschild black hole and is simpler than the corresponding solution in GR. In dependance of the black hole mass, different black hole phases, labeled by the scalar charge coexist. For sufficiently small masses there are two phases and only one phase for sufficiently large masses. In the intermediate range of black hole mass, there are three or five coexisting phases.

The thermodynamical stability of the solutions was also discussed but the present data is not sufficient for a final conclusion. The full understanding of the stability of the phases and the dynamics of possible phase transitions requires perturbative analysis.

Presence of multiple phases is also expected in cases of other sources of gravity whose tensor of energy-momentum has a non-vanishing trace, such as: linear and non-linear electrodynamics in higher dimensions (or even in 3D gravity), gauge theories and so on.

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