MARKOV-DYCK SHIFTS, NEUTRAL PERIODIC POINTS AND TOPOLOGICAL CONJUGACY

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Abstract. We study the neutral periodic points of Markov-Dyck shifts of finite strongly connected directed graphs. Under certain hypothesis on the structure of the graphs $G$ we show, that the topological conjugacy of their Markov-Dyck shifts implies the isomorphism of the graphs.

1. Introduction. Let $\Sigma$ be a finite alphabet, and let $S$ be the left shift on $\Sigma^\mathbb{Z}$,

$$S((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}, \quad (x_i)_{i \in \mathbb{Z}} \in \Sigma^\mathbb{Z}. $$

The closed shift-invariant subsystems of the shifts $S$ are called subshifts. For an introduction to the theory of subshifts see [6] and [15]. A finite word in the symbols of $\Sigma$ is called admissible for the subshift $X \subset \Sigma^\mathbb{Z}$ if it appears somewhere in a point of $X$. A subshift $X \subset \Sigma^\mathbb{Z}$ is uniquely determined by its language $L(X)$ of admissible words.

In this paper we study the topological conjugacy of subshifts that are constructed from finite directed graphs. We denote a finite directed graph $G$ with vertex set $V$ and edge set $E$ by $G(V,E)$. The source vertex of an edge $e \in E$ or of a directed path in the graph we denote by $s$ and its target vertex by $t$ (or by $s_G$ and $t_G$, in the case that we have to distinguish between graphs.) We consider strongly connected finite directed graphs $G = G(V,E)$. It is assumed that $G$ is not a cycle. We recall the construction of the Markov-Dyck shift $MD(G)$ of $G$ (see [16]). Let $\mathcal{E}^- = \{e^- : e \in E\}$ be a copy of $\mathcal{E}$. Reverse the directions of the edges in $\mathcal{E}^-$ to obtain the edge set $\mathcal{E}^+ = \{e^+ : e \in \mathcal{E}\}$ of the reversed graph of $G(V,\mathcal{E}^-)$. In this way one has defined a graph $\tilde{G}(V,\mathcal{E}^- \cup \mathcal{E}^+)$, where

$$s_{\tilde{G}}(e^-) = s_G(e), \quad t_{\tilde{G}}(e^-) = t_G(e),$$

$$s_{\tilde{G}}(e^+) = t_G(e), \quad t_{\tilde{G}}(e^+) = s_G(e), \quad e \in \mathcal{E}.$$
We denote the vertex set of $\mathcal{V}$, $V \in \mathcal{V}$, the set $\mathcal{E}^{-} \cup \{1_V : V \in \mathcal{V}\} \cup \mathcal{E}^{+}$ is the generating set of the graph inverse semigroup $S(G)$ of $G$ (see [14, Section 10.7]), where, besides $1_{\mathcal{V}} = 1_V, V \in \mathcal{V}$, the relations are

$$1_U1_W = 0, \quad U, W \in \mathcal{V}, U \neq W,$$

$$f^{-}g^+ = \begin{cases} 1_{sG(f)}, & \text{if } f = g, \\ 0, & \text{if } f \neq g, \quad f, g \in \mathcal{E}, \end{cases}$$

$$1_{sG(f)}f^- = f^-1_{tG(f)}, \quad 1_{tG(f)}f^+ = f^+1_{sG(f)}, \quad f \in \mathcal{E}.$$ 

The subsemigroup of the semigroup $S(G)$, that is generated by $\mathcal{E}^{-}(\mathcal{E}^{+})$ we denote by $S^{-}(G)(S^{+}(G))$, and we refer to the elements of $\mathcal{E}^{-}(\mathcal{E}^{+})$ as the generators of $S^{-}(G)(S^{+}(G))$.

The alphabet of $\text{MD}(G)$ is $\mathcal{E}^{-} \cup \mathcal{E}^{+}$, and a word $(e_k)_{1 \leq k \leq K}$ is admissible for $\text{MD}(G)$ precisely if

$$\prod_{1 \leq k \leq K} e_k \neq 0.$$ 

The directed graphs with a single vertex and $N > 1$ loops yield the Dyck inverse monoids (the “polycycliques” of [18]) $D_N$ and the Dyck shifts $D_N$ [7].

Given a subshift $X \subset \Sigma^\mathbb{Z}$ we set

$$x_{[i,j]} = (x_k)_{i \leq k \leq j},$$

and

$$X_{[i,j]} = \{x_{[i,j]} : x \in X\}, \quad i, j \in \mathbb{Z}, i \leq j, \quad x \in X,$$

and we use similar notation in the case that indices range in semi-infinite intervals.

Set

$$\Gamma^+_X(a) = \{x^+ \in X_{[j,\infty)} : ax^+ \in X_{[i,\infty)}\}, \quad a \in X_{[i,j]}, \quad i, j \in \mathbb{Z}, i \leq j.$$ 

The notation $\Gamma^{-}$ has the symmetric meaning. Also set

$$\omega^+_X(a) = \bigcap_{x^+ \in \Gamma^{-}(a)} \{x^+ \in \Gamma^+_X(a) : x^+ax^+ \in X\}, \quad a \in X_{[i,j]}, \quad i, j \in \mathbb{Z}, i \leq j.$$ 

The notation $\omega^{-}$ has the symmetric meaning. Also set

$$A_n(X) = \bigcap_{i \in \mathbb{Z}} \{x \in X : x_{[i,\infty)} \in \omega^+_X(x_{[i-n,i]}n)\} \cap \{x \in X : x_{[i,\infty)} \in \omega^-_X(x_{[i,i+n]}i)\},$$

and

$$A(X) = \bigcup_{n \in \mathbb{N}} A_n(X).$$

The periodic points in $A(X)$ are called the neutral periodic points of $X$. In Section 2 we clarify the structure of the set of neutral periodic points of a Markov-Dyck shift. This includes a characterization of the neutral periodic points of the shift.

For a finite directed graph $G(V, E)$ we denote by $\mathcal{F}_G$ the set of edges that are single incoming edges of their target vertices and we denote by $R_G$ the set of $V \in \mathcal{V}_G$ that have more than one incoming edge. The set $R_G$ is the set of roots of a set of (possibly degenerate) directed rooted trees.\(^1\) We denote the vertex set of the directed tree with root $R \in R_G$ by $V_R$, and its edge set by $F_R$. One has

$$\mathcal{V} = \bigcup_{R \in R_G} V_R, \quad \mathcal{F}_G = \bigcup_{R \in R_G} F_R.$$

\(^1\)The graph with one vertex and no edges is generally considered to be a tree. We refer to this graph as the degenerate directed tree.
The condition, that \( \text{card}(\mathcal{R}_G) = 1 \), is equivalent to the condition, that the graph \( G(\mathcal{V}, \mathcal{F}_G) \) is a (possibly degenerate) directed tree. We denote the graph that is obtained by contracting the non-degenerate trees among the trees \( G(\mathcal{V}_R, \mathcal{F}_R), R \in \mathcal{R}_G \), to their roots \( R \) by \( \hat{G}(\mathcal{R}_G, \mathcal{E} \setminus \mathcal{F}_G) \). In the graph \( \hat{G} \) the source vertex of an edge \( e \in \mathcal{E} \setminus \mathcal{F}_G \) is the root of the tree that has \( s_G(e) \) as a leave, and its target vertex is \( t_G(e) \). In [8] a Property \((A)\) of subshifts, an invariant of topological conjugacy, was introduced, and to a subshift \( X \) with Property \((A)\) a semigroup \( \mathcal{S}(X) \) was invariantly associated. In [5] it was shown that the Markov-Dyck shift \( M_d(G(\mathcal{V}, \mathcal{E})) \) of a graph \( G(\mathcal{V}, \mathcal{E}) \) has Property \((A)\), and that

\[
\mathcal{S}(M_d(G(\mathcal{V}, \mathcal{E}))) = \mathcal{S}(\hat{G}(\mathcal{R}_G, \mathcal{E} \setminus \mathcal{F}_G)).
\]

In Section 2 we show that a topological conjugacy of Markov-Dyck shifts of graphs \( G(\mathcal{V}, \mathcal{E}) \) induces an isomorphism of the graphs \( \hat{G}(\mathcal{R}_G, \mathcal{E} \setminus \mathcal{F}_G) \), that also preserves certain data pertaining to the configuration of the neutral periodic points of the Markov-Dyck shift. For Markov-Motzkin shifts (see [13, Section 4.1]) analogous results hold.

In Section 3 we consider finite directed graphs \( G(\mathcal{V}, \mathcal{E}) \), such that \( \text{card}(\mathcal{R}_G) = 1 \). In this case, following the terminology, that was introduced in [3], we say that a periodic point \( p \) of \( M_d(G) \) and its orbit have negative multiplier \( e \in \mathcal{E} \setminus \mathcal{F}_G \), if there exists an \( i \in \mathbb{Z} \) and an \( M \in \mathbb{N} \), such that

\[
\lambda(p_{i,i+\pi(p)}) = (\hat{\varepsilon}^{-1})^M.
\]

The mapping that assigns to a multiplier \( e \in \mathcal{E} \setminus \mathcal{F}_G \) the set of periodic points of \( M_d(G) \) with negative multiplier \( e \) is an invariant of topological conjugacy [4, Proposition 4.2]. We set

\[
\mathcal{M}(M_d(G)) = \mathcal{E} \setminus \mathcal{F}_G, \quad \nu(M_d(G)) = \text{card}(\mathcal{M}(M_d(G))),
\]

We denote for a multiplier \( e \in \mathcal{E} \setminus \mathcal{F}_G \), by \( I_k^{(e)}(M_d(G)) \) the number of periodic points of \( M_d(G) \) with negative multiplier \( e \) and period \( k \), and we set

\[
\Lambda^{(e)}(M_d(G)) = \min\{k \in \mathbb{N} : I_k^{(e)}(M_d(G)) > 0\}.
\]

The set \( \{\Lambda^{(e)}(M_d(G)) : e \in \mathcal{E} \setminus \mathcal{F}_G\} \) is an invariant of topological conjugacy. We denote by \( I_{2k}^{(0)}(M_d(G)) \) the number of neutral periodic points of period \( 2k \) of \( M_d(G) \), \( k \in \mathbb{N} \). We also denote by \( \Xi_{2k}^{(e)}(M_d(G)) \) the number of orbits of length \( 2k \), \( k \in \mathbb{N} \), with negative multiplier \( e \in \mathcal{M}(M_d(G)) \).

In Section 3 we consider three families, \( \mathbf{F}_1, \mathbf{F}_II \) and \( \mathbf{F}_{III} \) of finite directed graphs \( G(\mathcal{V}, \mathcal{E}) \), such that \( \text{card}(\mathcal{R}_G) = 1 \). For the graphs in each of these families we introduce canonical models. Each canonical model is specified by a set of parameters, that we call the “data” of the model. We then establish for the graphs \( G \) in each of these families, that the topological conjugacy class of the Markov-Dyck shift of \( G \) determines the isomorphism class of the graph \( G \). The proof consists in showing, that the invariants \( \nu(M_d(G)) \), and \( \Lambda^{(e)}, e \in \mathcal{M}(M_d(G)) \), together with \( I_{2k}^{(0)}(M_d(G)), I_k^{(e)}(M_d(G)), \Xi_{2k}^{(e)}, k \in \mathbb{N}, e \in \mathcal{M}(M_d(G)) \) contain sufficient information to determine the “data” of the canonical model of the graph \( G \). We also characterize the Markov-Dyck shifts of the graphs in each of these families within the Markov-Dyck shifts.

The family \( \mathbf{F}_I \) contains the finite strongly connected directed graphs \( G(\mathcal{V}, \mathcal{E}) \), such that \( \text{card}(\mathcal{R}_G) = 1 \), and such that all vertices, except the root of the tree \( G(\mathcal{V}, \mathcal{F}_G) \), have out-degree one.
The family \( \mathbf{F}_{III} \) contains the finite strongly connected directed graphs \( G(\mathcal{V}, \mathcal{E}) \) such that \( \text{card}(\mathcal{R}_G) = 1 \), and such that all leaves of the tree \( G(\mathcal{V}, \mathcal{F}_G) \) are at level one.

The family \( \mathbf{F}_{III} \) contains the finite strongly connected directed graphs \( G(\mathcal{V}, \mathcal{E}) \), such that the graph \( G(\mathcal{V}, \mathcal{F}_G) \) is a tree, that has the shape of a "V", and that is such that the two leaves of the tree \( G(\mathcal{V}, \mathcal{F}_G) \) have the same out-degree, and all interior vertices of the tree \( G(\mathcal{V}, \mathcal{F}_G) \) have out-degree one.

It had been known, that for finite directed graphs, in which every vertex has at least two incoming edges, topological conjugacy of their Markov-Dyck shifts implies the isomorphism of the graphs (see [4, Section 4] and [9, Corollary 3.2]). Actually, for finite directed graphs, in which every vertex has at least two incoming edges, the flow equivalence of their Markov-Dyck shifts implies the isomorphism of the graphs. For Dyck shifts this follows from [17], and for the general case see [1] and [10].

2. Neutral periodic points of Markov-Dyck shifts. We denote the period of a periodic point of a subshift by \( \pi \). Given a (non-degenerate) rooted tree \( \mathcal{T}(\mathcal{V}, \mathcal{F}) \), we denote for \( V \in \mathcal{V} \) by \( b(V) \) the path from the root to \( V \).

We continue to consider a strongly connected finite directed graph \( G = G(\mathcal{V}, \mathcal{E}) \), such that \( \text{card}(\mathcal{E} \setminus \mathcal{F}_G) > 1 \). An edge \( e \in \mathcal{E} \setminus \mathcal{F}_G \), determines a generator of \( \mathcal{S}^- (\hat{G}) \mathcal{S}^+ (\hat{G}) \), that we denote by \( \hat{e}^- (\hat{e}^+) \), and we set

\[
\lambda(e^-) = \hat{e}^-, \quad \lambda(e^+) = \hat{e}^+ , \quad e \in \mathcal{E} \setminus \mathcal{F}_G.
\]

A root \( R \in \mathcal{R}_G \) determines an idempotent of \( \mathcal{S} (\hat{G}) \), that we denote by \( \hat{1}_R \), and we set

\[
\lambda(f^-) = \lambda(f^+) = \hat{1}_R, \quad f \in \mathcal{F}_R, R \in \mathcal{R}_G.
\]

We set

\[
(e^-)^{-1} = e^+, \quad (e^+)^{-1} = e^-, \quad e \in \mathcal{E},
\]

and, more generally, for a path \( b = (b_i)_{1 \leq i \leq 1} \in \mathcal{L}(\text{MD}(G)) \) we use the notation

\[
b^{-1} = ((b_i)^{-1})_{i \geq 1}
\]

for the reversed path of \( b \).

The set of neutral periodic points of \( \text{MD}(G) \) we denote by \( \mathcal{P}^{(0)} (\text{MD}(G)) \). For a vertex \( V \in \mathcal{V} \), we denote by \( \mathcal{P}^{(0)} (V) \) the set of periodic points \( p \) of \( \text{MD}(G) \), such that there is an \( m \in \mathbb{Z} \), such that

\[
V = s_G (p_{|m,m+\pi(p)|}),
\]

and such that

\[
1_V = \prod_{m \leq j < m+\pi(p)} p_j.
\]

Theorem 2.1.

\[
\mathcal{P}^{(0)} (\text{MD}(G)) = \bigcup_{V \in \mathcal{V}} \mathcal{P}^{(0)} (V).
\]

Proof. For the proof let there be given a neutral periodic point \( p \) of \( \text{MD}(G) \). With \( \hat{K}_+ \in \mathbb{Z}_+, \hat{K}_- \in \mathbb{Z}_+ \), and

\[
\hat{e}^+_{\hat{K}_+} \in \hat{E}^+, \quad \hat{K}_+ \geq \hat{K}(+) > 0,
\]

\[
\hat{e}^-_{\hat{K}_-} \in \hat{E}^-, \quad 0 < \hat{K}(-) \leq \hat{K}_-,
\]




and with an $R \in \mathcal{R}_G$, write
\[
\prod_{0 \leq j < \pi(p)} \lambda(p_j) = \left( \prod_{\hat{R}_+ \geq \hat{k}(+) > 0} \hat{c}_\hat{k}(+) \right) \hat{1}_R \left( \prod_{0 < \hat{k}(-) \leq \hat{R}_-} \hat{c}_\hat{k}(-) \right).
\]

Note that
\[
s_G(\hat{c}_{\hat{R}_+}) = t_G(\hat{c}_{\hat{R}_-}).
\]

Assume that
\[
\left( \prod_{0 < \hat{k}(-) \leq \hat{R}_-} \hat{c}_\hat{k}(-) \right) \left( \prod_{\hat{R}_+ \geq \hat{k}(+) > 0} \hat{c}_\hat{k}(+) \right) \in \mathcal{S}^- (\hat{G}) \setminus \{ \hat{1}_R \}. \tag{2.1}
\]

Choose an $\hat{m} \in [0, \pi(p))$, such that
\[
\prod_{0 \leq j < \hat{m}} \lambda(p_j) = \prod_{\hat{R}_+ \geq \hat{k}(+) > 0} \hat{c}_\hat{k}(+).
\]

Then also
\[
\prod_{\hat{m} \leq j < \pi(p)} \lambda(p_j) = \prod_{0 < \hat{k}(-) \leq \hat{R}_-} \hat{c}_\hat{k}(-),
\]

and therefore by (2.1)
\[
\prod_{\hat{m} \leq j < \hat{m} + \pi(p)} \lambda(p_j) = \left( \prod_{0 < \hat{k}(-) \leq \hat{R}_-} \hat{c}_\hat{k}(-) \right) \left( \prod_{\hat{R}_+ \geq \hat{k}(+) > 0} \hat{c}_\hat{k}(+) \right) \in \mathcal{S}^- (\hat{G}).
\]

It follows that the word $p_{[\hat{m}, \hat{m} + \pi(p)]}$ has a suffix $e^- b$, that is uniquely determined by the condition, that $e^- \in \mathcal{E}^- \setminus \mathcal{F}_G$ and that $b$ is the empty word, or $b = (b_i)_{1 \leq i \leq I}$ is a word in $\mathcal{L}(MD(G))$ such that
\[
\prod_{1 \leq i \leq I} \lambda(b_i) = \mathbf{1}_{[e]}.
\]

Let $g \neq e$ be an incoming edge of $t_G(e^-)$. For $K \in \mathbb{N}$ the words
\[
e^{- bp_{[\hat{m}, \hat{m} + K \pi(p)]}}
\]

and
\[
p_{[\hat{m}, \hat{m} + K \pi(p)]} p_{[\hat{m}, \hat{m} + K \pi(p)]}^{-1} b^{-1} g^+
\]

are admissible for $MD(G)$. However the word
\[
e^{- bp_{[\hat{m}, \hat{m} + K \pi(p)]} p_{[\hat{m}, \hat{m} + K \pi(p)]}^{-1} b^{-1} g^+}
\]

is not admissible for $MD(G)$. This contradicts the neutrality of $p$. Under the assumption that
\[
\left( \prod_{0 < \hat{k}(-) \leq \hat{R}_-} \hat{c}_\hat{k}(-) \right) \left( \prod_{\hat{R}_+ \geq \hat{k}(+) > 0} \hat{c}_\hat{k}(+) \right) \in \mathcal{S}^+ (\hat{G}) \setminus \{ \hat{1}_R \},
\]

one has the symmetric argument. We have shown that
\[
\lambda(p_{[\hat{m}, \hat{m} + \pi(p)]}) = \hat{1}_R. \tag{2.2}
\]

To repeat this reasoning, with $K_+ \in \mathbb{Z}_+$, $K_- \in \mathbb{Z}_+$, and
\[
e_{k(+) \in \mathcal{E}^+}, \quad K_+ \geq k(+) > 0,
\]
\[
e_{k(-) \in \mathcal{E}^-}, \quad 0 < k(-) \leq K_-,
\]
and with a $V \in \mathcal{V}$, write
\[
\prod_{\hat{m} \leq j < \hat{m}+\pi(p)} p_j = \left( \prod_{K_+ \geq k(+) > 0} e_{k(+)}^+ \right) 1_V \left( \prod_{0 < k(-) \leq K_-} e_{k(-)}^- \right).
\]
Note that
\[
s_G(e_{K(+)}^+) = t_G(e_{K(-)}^-).
\]
Assume that
\[
\left( \prod_{0 < k(-) \leq K_-} e_{k(-)}^- \right) \left( \prod_{K_+ \geq k(+) > 0} e_{k(+)}^+ \right) \in \mathcal{S}^-(G) \setminus \{1_V\}. \tag{2.3}
\]
Choose an $m \in [0, \pi(p))$, such that
\[
\prod_{\hat{m} \leq j < \hat{m}+m} p_j = \prod_{K_+ \geq k(+) > 0} e_{k(+)}^+.
\]
Then also
\[
\prod_{\hat{m}+m \leq j < \hat{m}+\pi(p)} p_j = \prod_{0 < k(-) \leq K_-} e_{k(-)}^-,
\]
and therefore by (2.3)
\[
\prod_{m \leq j < m+\pi(p)} p_j = \left( \prod_{0 < k(-) \leq K_-} e_{k(-)}^- \right) \left( \prod_{K_+ \geq k(+) > 0} e_{k(+)}^+ \right) \in \mathcal{S}^-(G).
\]
It follows from this and from (2.2), that there is an $R \in \mathcal{R}_G$, such that there is a directed path $(f_l)_{1 \leq l \leq L}, L \in \mathbb{N}$, in the tree $G(V_R, F_R)$, such that
\[
\prod_{m \leq j < m+\pi(p)} p_j = \prod_{1 \leq l \leq L} f_l^-.
\]
This contradicts the periodicity of $p$. Under the assumption that
\[
\left( \prod_{0 < k(-) \leq K_-} e_{k(-)}^- \right) \left( \prod_{K_+ \geq k(+) > 0} e_{k(+)}^+ \right) \in \mathcal{S}^+(G) \setminus \{1_V\},
\]
one has the symmetric argument. This confirms that
\[
1_V = \prod_{m \leq j < m+\pi(p)} p_j,
\]
and completes the proof. \qed

The set of neutral periodic points of a subshift $X \subset \Sigma^\mathbb{Z}$ carries a pre-order relation $\preceq(X)$ (see [8]). For neutral periodic points $q$ and $r$ of $X$ one has $q \preceq(X) r$, if there exists a point in $A(X)$, that is left asymptotic to the orbit of $q$ and right asymptotic to the orbit of $r$. The equivalence relation that is derived from $\preceq(X)$ we denote by $\approx(X)$.

We set
\[
P^{(0)}_R = \bigcup_{V \in \mathcal{V}_R} P^{(0)}(V), \quad R \in \mathcal{R}_G.
\]
The proof of the following lemma is similar to the proof of Theorem 3.2 of [5].

**Lemma 2.2.** The $\approx(X)$-equivalence class of $p \in P^{(0)}_R, R \in \mathcal{R}_G$, coincides with $P^{(0)}_R$. 

Proof. The proof comes in two parts. For the first part let \( R \in \mathcal{R}_G \), let \( U, W \in \mathcal{V}_R \), and \( q, r \in P^{(0)}_R \), and let \( j, k \in \mathbb{Z} \), be such that
\[
U = s_G(p_{[j,j+\pi(q)]}), \quad W = s_G(r_{[k,k+\pi(r)]}),
\]
and such that
\[
1_U = \prod_{j\leq i < j+\pi(p)} q_i, \quad 1_W = \prod_{k\leq i < k+\pi(r)} r_i.
\]
Let a point \( x \in Md(G) \) be given by
\[
x_{(-\infty,0]} = q_{(-\infty,j)}b^{-1}(U), \quad x_{(0,\infty)} = b(W)r_{(k,\infty)}.
\]
One has
\[
x \in A_{\max\{\pi(q),\pi(r)\}}(Md(G)).
\]
This follows since the edges in the paths \( b(U) \) and \( b(W) \) are by construction the only incoming edges of their target vertices. We have proved that \( q \approx (X)r \).

For the second part let \( R, R' \in \mathcal{R}_G \),
\[
R \neq R', \quad (2.4)
\]
let
\[
V \in \mathcal{V}_R, p \in P^{(0)}_R, \quad V' \in \mathcal{V}_{R'}, p' \in P^{(0)}_{R'},
\]
and let \( j, j' \in \mathbb{Z} \) be such that
\[
V = s_G(p_{[j,j+\pi(p)]}), \quad V' = s_G(p'_{[j',j'+\pi(p')]}),
\]
and
\[
1_V = \prod_{j\leq i < j+\pi(p)} p_i, \quad 1_{V'} = \prod_{j'\leq i < j'+\pi(p')} p'_i.
\]
We prove that \( p \) and \( p' \) are \( \lesssim (Md(G)) \)-incomparable. Assume that
\[
p \lesssim (Md(G)) p', \quad (2.5)
\]
and let \( J \in \mathbb{N} \), and
\[
x \in A_{f}(Md(G)),
\]
and \( m, m' \in \mathbb{Z}, m < m' \), be such that
\[
x_{(\infty, m)} = p_{(\infty,j)}, \quad x_{(m',\infty)} = p'_{(m',\infty)}.
\]
With \( K, K' \in \mathbb{Z}_+ \), and
\[
e_k \in \mathcal{E} \setminus \mathcal{F}_G, \quad K \geq k > 0,
\]
\[
e'_{k'}, \in \mathcal{E} \setminus \mathcal{F}_G, \quad 0 < k' \leq K',
\]
and with \( Q \in \mathcal{R}_G \),
\[
t_G(e_K) = Q = t_G(e_{K'}),
\]
write
\[
\prod_{m \leq i < m'} \lambda(x_i) = (\prod_{K \geq k > 0} \mathring{e}^+_k) \mathring{1}_Q (\prod_{0 < k' \leq K'} \mathring{e}^+_k).
\]
Assumption \((2.4)\) implies that \((K, K') \neq (0,0)\). Assume \( K > 0 \). Then we have an \( M \in [m, m') \), such that \( x_M = e^+_K \), and
\[
\prod_{m \leq i < M} \lambda(x_i) = \mathring{1}_Q.
\]
Let $g \neq e_K$, be an incoming edge of $Q$. The word
\[ g^{-b(V)^{-1}}p_{(m-J\pi(p),m)}x_{(m,M)} \]
is admissible for $MD(G)$. However the word
\[ g^{-b(V)^{-1}}p_{(m-J\pi(p),m)}x_{(m,M)}e_K^+ \]
is not. This contradicts (2.5). Under the assumption that $K' > 0$, one has the symmetric argument. We have shown that $p \not\preceq (MD(G))p'$. \hfill $\Box$

Denoting by $\Pi_n(Y)$ the number of points of period $n$ of a shift-invariant set $Y \subset \Sigma^\mathbb{Z}$, the zeta function of $Y$ is given by
\[ \zeta_Y(z) = e^{\sum_{n \in \mathbb{N}} \Pi_n(Y)z^n}. \]

For the finite strongly connected directed graph $G(V,E)$, we vertex weigh the graph $\widehat{G}(R_G,E \setminus F_G)$ by assigning to its vertices $R \in R_G$ the zeta function of $P(0)_R$.

**Corollary 2.3.** For finite strongly connected directed graphs $G(V,E)$ the topological conjugacy of the Markov-Dyck shifts $MD(G)$ implies the isomorphism of the vertex weighted graphs $\widehat{G}(R_G,E \setminus F_G)$ with weights $(\zeta_P(0)_R)_{R \in R_G}$.

**Proof.** There is a canonical projection $\chi$ of the set of points in $MD(G)$, that are left-asymptotic to a point in $P(0)(MD(G))$, and also right-asymptotic to point in $P(0)(MD(G))$ onto the associated semigroup $S(MD(G)) = S(\widehat{G}(R_G,E \setminus F_G))$.

(see [9, Section 3] and [8, Section 3]). A topological conjugacy induces an isomorphism of the associated semigroups and it acts accordingly on the inverse images under $\chi$ of each of the elements in the set
\[ \{1_R : R \in R_G\} \cup \{e^- : e \in E \setminus F_G\} \subset S(\widehat{G}(R_G,E \setminus F_G)), \]

(see [11, Section 2]). By Lemma 2.2
\[ \chi^{-1}(1_R) = P(0)_R, \quad R \in R_G, \]
and
\[ \bigcup_{R \in R_G} \chi^{-1}(1_R) = P(0)(MD(G)). \quad \Box \]

To a vertex $V \in V$ we associate the circular code $C_V$ that contains the words $(c_i)_{1 \leq i \leq I} \in L(MD(G))$ such that
\[ s_{\tilde{G}}(c) = t_{\tilde{G}}(c) = V, \]
and
\[ \prod_{1 \leq i \leq I} c_i = 1_V, \]
\[ \prod_{1 \leq i \leq J} c_i \neq 1_V, \quad 1 < J < I. \]

The generating function of $C_V$ we denote by $\varphi_V$.

**Corollary 2.4.** For $G(V,E)$,
\[ \zeta_{P(0)(MD(G))} = \prod_{V \in V} \frac{1}{1 - \varphi_V}. \]
Proof. The corollary follows from Theorem 2.1 (e.g. see [19, Section 5] or [12, Section 2]).

3. Families of finite directed graphs. In this section we consider the case of strongly connected finite directed graphs \( G = G(\mathcal{V}, \mathcal{E}) \), such that \( \text{card}(\mathcal{R}_G) = 1 \). These graphs are precisely the graphs, that have a Dyck inverse monoid as the associated semigroup of their Markov-Dyck shifts. Complete invariants for the isomorphism for these directed graphs are known for the case that all source vertices \( s(e), e \in \mathcal{M}(\mathrm{Md}(G)) \), have the same out-degree (see [2]). We denote the root of \( \mathcal{F}_G \) by \( V_0 \), and the out-degree of \( V_0 \) by \( D(V_0) \). We set
\[
\mathcal{M}_\ell(\mathrm{Md}(G)) = \{ e \in \mathcal{M}(\mathrm{Md}(G)) : \Lambda(e) = \ell \}, \quad \ell \in \mathbb{N}.
\]
By the use of the notation \( \Lambda(\mathrm{Md}(G)) \) we indicate that all lengths \( \Lambda(e)(\mathrm{Md}(G)), e \in \mathcal{E} \setminus \mathcal{F}_G \), are equal and that \( \Lambda(\mathrm{Md}(G)) \) is their common value.

3.1. A family of finite directed graphs I. Set
\[
\Pi_I = \{ (S_\ell)_{\ell \in \mathbb{N}} \in \mathbb{Z}_+^\mathbb{N} : 1 < \sum_{\ell \in \mathbb{N}} S_\ell < \infty \}.
\]
The data \( (S_\ell)_{\ell \in \mathbb{N}} \in \Pi_I \) determine canonical models \( G((S_\ell)_{\ell \in \mathbb{N}}) \) of the graphs in \( \mathcal{F}_I \).

We define \( G((S_\ell)_{\ell \in \mathbb{N}}) \) as the graph with vertices \( V_0 \) and
\[
V_{t,s,t}, \quad 1 \leq t < \ell, 1 \leq s \leq S_\ell, \quad \ell > 1,
\]
and edges
\[
f_{t,s,t}, \quad 1 \leq t < \ell, 1 \leq s \leq S_\ell, \quad \ell > 1,
\]
and
\[
e_{t,s}, \quad 1 \leq s \leq S_\ell, \quad \ell \in \mathbb{N}.
\]
The source and target mappings are given by
\[
s(f_{t,s,1}) = V_0,
\]
\[
s(f_{t,s,t}) = V_{t,s,t-1}, \quad 1 < t \leq \ell,
\]
\[
t(f_{t,s,t}) = V_{t,s,t}, \quad 1 \leq t < \ell, \quad 0 < s \leq S_\ell, \quad \ell > 1,
\]
and
\[
s(e_{1,s}) = V_0 = t(e_{1,s}), \quad 0 < s \leq S_1,
\]
\[
s(e_{t,s}) = V_{t,s,t-1}, t(e_{t,s}) = V_0, \quad 0 < s \leq S_\ell, \quad \ell > 1.
\]

One has
\[
\mathcal{F}_{G((S_\ell)_{\ell \in \mathbb{N}})} = \{ f_{t,s,t} : 1 \leq t < \ell, 1 \leq s \leq S_\ell, \quad \ell > 1 \}.
\]
The Dyck shifts \( D_N, N > 1 \), belong here with the data \( S_1 = N, S_\ell = 0, \ell > 1 \). Also the Fibonacci-Dyck shift belongs here with the data \( S_1 = 1, S_2 = 1, S_\ell = 0, \ell > 2 \).

Theorem 3.1. For a finite directed graph \( G = G(\mathcal{V}, \mathcal{E}) \) there exist data
\[
(S_\ell)_{\ell \in \mathbb{N}} \in \Pi_I,
\]
such that there is a topological conjugacy
\[
\mathrm{Md}(G) \simeq \mathrm{Md}(G((S_\ell)_{\ell \in \mathbb{N}})), \tag{3.1.1}
\]
if and only if the associated semigroup of \( \mathrm{Md}(G) \) is a Dyck inverse monoid, and
\[
\frac{1}{2} \nu_f(f(0)) (\mathrm{Md}(G)) = \nu(\mathrm{Md}(G)) + \sum_{\ell > 1} (\ell - 1) \text{card}(\mathcal{M}_\ell(\mathrm{Md}(G)));
\]
and in this case (3.I.1) holds for
\[ S_\ell = \text{card}(\mathcal{M}_\ell(MD(G))), \quad \ell \in \mathbb{N}. \] (3.I.2)

**Proof.** The statement holds if \( G \) is a single vertex graph. Consider a graph \( \tilde{G} = G(\tilde{V}, \tilde{E}) \), such that the graph \( G(\tilde{V}, F_{\tilde{G}}) \) is a non-degenerate tree. Denote by \( J_{\tilde{G}}(\ell) \) the number of leafs of \( F_{\tilde{G}} \) that have level \( \ell - 1 \). One has that
\[ \text{card}(F_{\tilde{G}}) \leq \sum_{\ell > 1} (\ell - 1)J_{\tilde{G}}(\ell), \] (3.I.3)
and
\[ J_{\tilde{G}}(\ell) \leq \text{card}(\mathcal{M}_\ell(MD(\tilde{G}))), \quad \ell > 1, \] (3.I.4)
Therefore
\[ \text{card}(F_{\tilde{G}}) \leq \sum_{\ell > 1} (\ell - 1)\text{card}(\mathcal{M}_\ell(MD(\tilde{G}))). \] (3.I.5)
Equality holds simultaneously in (3.I.3) and (3.I.4) if and only if it holds in (3.I.5) if and only if \( G \) belongs to \( \mathcal{F}_I \). Equation (3.I.2) implies equality in (3.I.5) and the theorem follows.

**Corollary 3.2.** For finite directed graphs \( G = G(\mathcal{V}, \mathcal{E}) \) such that the associated semigroup of \( MD(G) \) is a Dyck inverse monoid, and such that
\[ \frac{1}{2} f^{(0)}(MD(G)) = \nu(MD(G)) + \sum_{\ell > 1} (\ell - 1)\text{card}(\mathcal{M}_\ell(MD(G))) \]
the topological conjugacy of their Markov-Dyck shifts implies the isomorphism of the graphs.

**Proof.** In (3.I.2) the data \( (S_\ell)_{\ell \in \mathbb{N}} \) are expressed in terms of invariants of topological conjugacy.

### 3.2. A family of finite directed graphs II.

We set
\[ \Pi = \{(R,(Q_M)_{M \in \mathbb{N}}) \in \mathbb{Z}_+ \times \mathbb{N}^\mathbb{N} : 1 < R + \sum_{M \in \mathbb{N}} Q_M < \infty\}. \]
The data \( (R,(Q_M)_{M \in \mathbb{N}}) \in \Pi \) determine canonical models \( G((R,(Q_M)_{M \in \mathbb{N}})) \) of the graphs in \( \mathcal{F}_II \). We define \( G((R,(Q_M)_{M \in \mathbb{N}})) \) as the directed graph with a vertex \( V(0) \) and with vertices
\[ V_{M,q}(1), \quad 1 \leq q \leq Q_M, \quad M \in \mathbb{N}, \]
and with edges
\[ f_{M,q}, \quad 1 \leq q \leq Q_M, \quad M \in \mathbb{N}, \]
and
\[ e_r, \quad 1 \leq r \leq R, \]
and
\[ e_{M,q,m}, \quad 1 \leq m \leq M, \quad 1 \leq q \leq Q_M, \quad M \in \mathbb{N}. \]
The source and target mappings are given by
\[ s(f_{M,q}) = V(0), \quad t(f_{M,q}) = V_{M,q}(1), \quad 1 \leq q \leq Q_M, \quad M \in \mathbb{N}, \]
and
\[ s(e_r) = t(e_r) = V(0), \quad 1 \leq r \leq R, \]
and
\[ s(e_{M,q,m}) = V_{M,q}, \quad t(e_{M,q,m}) = V(0), \quad 1 \leq m \leq M, 1 \leq q \leq Q_M, \quad M \in \mathbb{N}. \]
One has
\[ \mathcal{F}_{G(R,(Q_M)_{M \in \mathbb{N}})} = \{ f_{M,q}, 1 \leq q \leq Q_M, M \in \mathbb{N} \}. \]
Note the non-empty intersection of \( \mathbf{F}_{II} \) with \( \mathbf{F}_{I} \).

![Figure 2. G(1,(1,0,3,0,\ldots))](image)

**Theorem 3.3.** For a finite directed graph \( G = G(V,E) \) there exist data
\[(R,(Q_M)_{M \in \mathbb{N}}) \in \Pi, \]
such that there is a topological conjugacy
\[ MD(G) \simeq MD(G((R,(Q_M)_{M \in \mathbb{N}}))), \tag{3.II.1} \]
if and only if the associated semigroup of \( MD(G) \) is a Dyck inverse monoid, and
\[ \Lambda^{(e)} \leq 2, \quad e \in E \setminus \mathcal{F}_G, \tag{3.II.2} \]
and in this case (3.1) holds for
\[ R = \text{card}\{ e \in E \setminus \mathcal{F}_G : \Lambda^{(e)} = 1 \}, \tag{3.II.3} \]
\[ Q_M = \frac{1}{M} \text{card}\{ e \in E \setminus \mathcal{F}_G : \Xi_4^{(e)} = M + I_{2}^{(0)}(MD(G)) - \nu(MD(G)) + \text{card}(\mathcal{M}_1(G)) \}, \quad M \in \mathbb{N}. \]
The source and target mappings are given by
\[
G(e) = \{ e' \in E \setminus F_G : s(e') = s(e) \} + D(V_0).
\]
This follows from the observation, that in this case every orbit of length 4 with negative multiplier \( e \in E \setminus F_G \) either the word \( f^- f^+ \), \( f \in F_G \), \( t(f) = s(e) \), or a word \( \bar{e}^- \bar{e}^+ \), \( \bar{e} \in M_1(G) \), or a word \( \bar{e}^- \bar{e}^+ \), \( \bar{e} \in M_2(G) \), \( s(\bar{e}) = s(e) \). It is
\[
D(V_0) = I^{(0)}(MD(G)) - \nu(MD(G)) + \text{card}(M_1(G)),
\]
and the equations (3.II.3) follow. \( \square \)

**Corollary 3.4.** For finite directed graphs \( G = G(V, E) \) such that the associated semigroup of \( MD(G) \) is a Dyck inverse monoid, and such that
\[
\Lambda(e) \leq 2, \quad e \in E \setminus F_G,
\]
the topological conjugacy of their Markov-Dyck shifts implies the isomorphism of the graphs.

**Proof.** In (3.II.3) the data \( (R, (Q_M)_{M \in N}) \) are expressed in terms of invariants of topological conjugacy. \( \square \)

### 3.3. A family of finite directed graphs III

We consider the family \( F_{III} \) of finite directed graphs
\[
G[\ell, M] = G(V[\ell, M], E[\ell, M]), \quad \ell, M \in N, M \in N,
\]
such that \( G(V[\ell, M], F_G[\ell, M]) \) is a non-degenerate tree. We describe the graphs \( G[\ell, M], \ell, M \in N \) as follows: For \( \ell, M \in N \) the graph \( G[\ell, M] \) has a vertex \( V(0) \), vertices
\[
V_0(0), V_1(0), \quad 1 \leq l < \ell,
\]
and edges
\[
f_0(0), f_1(0), \quad 1 \leq l < \ell,
\]
and
\[
e_0(m), e_1(m), \quad 1 \leq m \leq M.
\]
The source and target mappings are given by
\[
s(f_0(1)) = s(f_1(1)) = V(0),
\]
\[
s(f_0(l)) = V_0(l-1), \quad s(f_1(l)) = V_1(l-1), \quad t(f_0(l)) = V_0(l), \quad t(f_1(l)) = V_1(l),
\]
\[
1 \leq l < \ell,
\]
and
\[
s(e_0(m)) = V_0(\ell-1), \quad s(e_1(m)) = V_1(\ell-1), \quad t(e_0(m)) = t(e_1(m)) = V(0),
\]
\[
1 \leq m \leq M.
\]
The tree \( G(V[\ell, M], F_G[\ell, M]) \) has the root \( V_0 \) and one has
\[
F_G[\ell, M] = \{ f_0(l), f_1(l), 1 \leq l < \ell \}, \quad \ell, M \in N, M \in N.
\]
Note the non-empty intersection of \( F_{III} \) with \( F_I \) and \( F_{II} \).
Lemma 3.5. For $G = G[\ell, M]$ one has that
\[
\frac{1}{2} F_2(0)(\nu(G)) = \nu(M_D(G)) + 2\Lambda(M_D(G)) - 2.
\]

Proof. One has that
\[
\text{card}(F_G[\ell, M]) = 2\Lambda(M_D(G[\ell, M])) - 2, \quad \ell, M \in \mathbb{N}, M \in \mathbb{N}.
\]

For $\ell, L, M \in \mathbb{N}$, $\ell \geq 4$, $L < \ell - 2$, we introduce auxiliary graphs
\[
G_{2,M}[\ell, L] = G(V_{2,M}[\ell, L], E_{2,M}[\ell, L]), \quad G_{M,2}[\ell, L] = G(V_{M,2}[\ell, L], E_{M,2}[\ell, L]).
\]
In both vertex sets $V_{2,M}[\ell, L]$ and $V_{M,2}[\ell, L]$ there are vertices
\[
V(l), \quad 0 \leq l \leq L,
\]
and
\[
V_0(l, m), V_1(l, m), \quad 1 \leq m \leq M, L + 2 \leq l < \ell,
\]
and in both edge sets $E_{2,M}[\ell, L]$ and $E_{M,2}[\ell, L]$ there are edges
\[
f(l), \quad 0 \leq l \leq L,
\]
and
\[
f_0(l, m), f_1(l, m), \quad 1 \leq m \leq M, L + 2 \leq l < \ell,
\]
and
\[
e_0(m), e_1(m), \quad 1 \leq m \leq M.
\]
with source and target vertices partially given by
\[
s(f(l)) = V(l - 1), \quad 1 \leq l \leq L,
\]
\[
s(f_0(l, m)) = V_0(l - 1, m), \quad s(f_1(l, m)) = V_1(l - 1, m), \quad 1 \leq m \leq M, L + 1 < l < \ell,
\]
\[
t(f(l)) = V(l), \quad 1 \leq l \leq L,
\]
\[
t(f_0(l, m)) = V_0(l, m), \quad t(f_1(l, m)) = V_1(l, m), \quad 1 \leq m \leq M, L + 1 < l < \ell,
\]
and
\[
s(e_0(m)) = V_0(\ell - 1, m), s(e_1(m)) = V_0(\ell - 1, m), t(e_0(m)) = t(e_1(m)) = V(0),
\]
\[
1 \leq m \leq M.
\]
In addition, the graph $G_{2,M}[\ell, L]$ has vertices
\[
V_0(L + 1), V_1(L + 1),
\]
and edges
\[
f_0(L + 1), f_1(L + 1),
\]
and the definition of its source and target mappings is completed by setting
\[
s(f_0(L + 1)) = s(f_1(L + 1)) = V(L),
\]
\[
t(f_0(L + 1)) = V_0(L + 1), \quad t(f_1(L + 1)) = V_1(L + 1),
\]
and
\[
s(f_0(L + 2, m)) = V_0(L + 1), \quad s(f_1(L + 2, m)) = V_1(L + 1), \quad 1 \leq m \leq M.
\]
In addition, the graph $G_{M,2}[\ell, L]$ has vertices
\[
V(L + 1, m), \quad 1 \leq m \leq M,
\]
and edges
\[
f(L + 1, m), \quad 1 \leq m \leq M,
\]
and the definition of its source and target mappings is completed by setting
\[ s(f(L + 1, m)) = V(L), \quad t(f(L + 1, m)) = V(L + 1, m), \quad 1 \leq m \leq M, \]
and
\[ s(f_0(L + 2, m) = s(f_1(L + 2, m) = V(L + 1, m), \quad 1 \leq m \leq M. \]

The graphs \( G(V_{2,M}[\ell, L], \mathcal{F}_{G_{2,M}[\ell, L]}) \) and \( G(V_{M,2}[\ell, L], \mathcal{F}_{G_{M}[\ell, L]}) \) are directed trees.

**Figure 3.** \( G_{2,3}(4.1) \)

**Figure 4.** \( G_{3,2}(4.1) \)

The invariants of topological conjugacy \( \nu \) and \( \Lambda \), or, equivalently, \( \nu \) and \( I_2^{(0)} \), together do not separate the graphs \( G[\ell, M] \) from the graphs \( G_{2,M}[\ell, L] \) nor from the graphs \( G_{M,2}[\ell, L] \), but together with the invariant \( I_4^{(0)} \) they do, as the next lemma shows.

**Lemma 3.6.** (a) For \( \ell, M \in \mathbb{N} \), and \( G = G[\ell, M] \) one has
\[
I_4^{(0)}(Md(G)) = 8\Lambda(Md(G)) + \nu(Md(G))^2 + 10\nu(Md(G)) + 4.
\]
(b1) For $\ell, M \in \mathbb{N}, \ell > 3, 2 \leq L < \ell - 4$, and $G = G_{2,M}[\ell, L]$, one has
\[
I_4^{(0)}(MD(G)) = 4\nu(MD(G))\Lambda(MD(G)) + \nu(MD(G))^2 - 2\nu(MD(G)) + 8 + 4(1 - \nu(MD(G)))L.
\]

(b2) For $\ell, M \in \mathbb{N}, \ell > 3, 2 \leq L < \ell - 4$, and $G = G_{M,2}[\ell, L]$ one has
\[
I_4^{(0)}(MD(G)) = 4\nu(MD(G))\Lambda(MD(G)) + \frac{1}{2}\nu(MD(G))^2 + 3\nu(MD(G)) - 4 + 4(1 - \nu(MD(G)))L.
\]

Proof. The number of neutral periodic orbits of length 4 of
\[
MD(G[\ell, M]) = (MD(G_{2,M}[\ell, L]), MD(G_{M,2}[\ell, L])),
\]
that contain the points that carry the infinite concatenation of words of the form $e^{-}e^{+}\tilde{e}^{-}\tilde{e}^{+}, e \neq \tilde{e}$, is equal to
\[
1 + M(M - 1) (1 + M(M - 1)), \frac{1}{2}M(M + 1),
\]
and the number of neutral periodic orbits of length 4 of
\[
MD(G[\ell, M]) = (MD(G_{2,M}[\ell, L]), MD(G_{M,2}[\ell, L])),
\]
that contain the points that carry the infinite concatenation of words of the form $e^{-}e^{-}\tilde{e}^{+}\tilde{e}^{-}, t(e) = s(\tilde{e})$, is equal to
\[
2\ell + 6M - 2 (2M\ell + 2 - 2ML, 2\ell M + M - 1 - L).
\]

\[\square\]

Lemma 3.7. For $\ell > 4$, and $M \in \mathbb{N}$ and for $G(V, E) = G[\ell, M]$ one has that
\[
\Xi_{\ell+2}^{(e)}(MD(G)) = \Lambda(MD(G)) + \frac{1}{2}\nu(MD(G)),
\]
\[
\Xi_{\ell+2}^{(e)}(MD(G)) = (\Lambda(MD(G)) + \frac{1}{2}\nu(MD(G)))^2 + \Lambda(MD(G)) + 2\nu(MD(G)) - 2, \quad e \in E \setminus F_G.
\]

Proof. Let $e \in E \setminus F_G$, and let $O^{(e)}$ be the shortest periodic orbit of $MD(G)$ with negative multiplier $e$.

All periodic orbits of $MD(G)$ of length $\Lambda(MD(G)) + 2$ with multiplier $e^{-}$ are obtained by inserting a word of the form $g^{-}g^{+}$, where the source vertex of the edge $g^{-}$ is transversed by $O^{(e)}$, into $O^{(e)}$. The number of these words is $\ell + M$.

All periodic orbits of $MD(G)$ of length $\Lambda(MD(G)) + 4$ with multiplier $e^{-}$ are obtained by either inserting two words of the form $g^{-}g^{+}$, where the source vertex of the edge $g^{-}$ is transversed by $O^{(e)}$, into $O^{(e)}$, or by inserting a word of the form $g^{-}\tilde{g}^{-}\tilde{g}^{+}g^{+}, t(g) = s(\tilde{g})$, into $O^{(e)}$, where the source vertex of the edge $g^{-}$ is transversed by $O^{(e)}$, into $O^{(e)}$, and the number of these words is $\ell + 4M - 2$.

\[\square\]

Theorem 3.8. For a finite directed graph $G(V, E)$ there exist $\ell > 4, M \in \mathbb{N}$, such that there is a topological conjugacy
\[
MD(G) \simeq MD(G[\ell, M]), \tag{3.11}
\]
if and only if there is a Dyck inverse monoid associated to $MD(G)$, all $\Lambda(e), e \in E \setminus F_G$, have the same value, and
\[
\frac{1}{2}I_2^{(0)}(MD(G)) = 2\Lambda(MD(G)) + \nu(MD(G)) - 2, \tag{A}
\]
\[
I_4^{(0)}(MD(G)) = 8\Lambda(MD(G)) + \nu(MD(G))^2 + 10\nu(MD(G)) + 4, \tag{B}
\]
\[ \Xi_{\ell+2}(MD(G)) = \Lambda(MD(G)) + \frac{1}{2} \nu(MD(G)), \quad e \in \mathcal{E} \setminus \mathcal{F}_G, \quad (C) \]

\[ \Xi_{\ell+4}(MD(G)) = (\Lambda(MD(G)) + \frac{1}{2} \nu(MD(G)))^2 + \Lambda(MD(G)) + 2 \nu(MD(G)) - 2, \quad e \in \mathcal{E} \setminus \mathcal{F}_G. \quad (D) \]

If conditions (A)(B)(C)(D) are satisfied, then (3.III.1) holds for \( \ell = \Lambda(MD(G)), \quad M = \frac{1}{2} \nu(MD(G)). \quad (3.III.2) \)

**Proof.** Necessity follows from Lemma 3.5, Lemma 3.6. and Lemma 3.7. To prove sufficiency, let \( G = G(V, \mathcal{E}) \) be a graph that satisfies the conditions of the theorem. We denote the root of the tree \( G(V, \mathcal{F}_G) \) by \( V_0 \). The out-degree of a vertex we denote by \( D(V) \). It follows from (A) that \( D(V_0) \leq 2 \).

In the case \( D(V_0) = 2 \), one has by (A) and (C) that the tree \( G(V, \mathcal{F}_G) \) has two leaves, that have the same out-degree, and equations (3.III.2) follow. The task is to exclude the case \( D(V_0) = 1 \).

Assume, that \( D(V_0) = 1 \). Let \( L \) be maximal, such that the tree \( G(V, \mathcal{F}_G) \) has a single vertex \( V_L \) at level \( L \). One has that

\[ L < \Lambda(MD(G)) - 2, \]

since otherwise by (A),

\[ D(V_L) = \Lambda(MD(G)), \]

which is either by (C) incompatible with \( L > 0 \), or it contradicts (B).

By deriving a contradiction to (A)(B)(C)(D) we will exclude each of the following cases (c1 - 5):

\[ D(V_L) > \frac{1}{2} \nu(MD(G)) + 1, \quad (c1) \]

\[ D(V_L) = \frac{1}{2} \nu(MD(G)) + 1, \quad (c2) \]

\[ D(V_L) = \frac{1}{2} \nu(MD(G)), \quad (c3) \]

\[ \frac{1}{2} \nu(MD(G)) > D(V_L) > 2, \quad (c4) \]

\[ D(V_L) = 2. \quad (c5) \]

We consider cycles \( b = (e_k)_{1 \leq k \leq \Lambda(MD(G))} \) in \( G \), such that \( s(e_1) = V_L \). By (C)

\[ \sum_{1 \leq k \leq \Lambda(MD(G))} (D(s(e_k)) - 1) = \frac{1}{2} \nu(MD(G)). \quad (3.III.3) \]

In case (c1) one has a contradiction to (3.III.3) and therefore to (C). Case (c2) is by (C) only possible if

\[ \nu(MD(G)) = D(V_L) = 2, \]

which contradicts (A).

In case (c3) it follows from (3.III.3) for the cycle \( b = (e_k)_{1 \leq k \leq \Lambda(MD(G))} \), that

\[ \sum_{1 < k \leq \Lambda(MD(G))} (D(s(e_k)) - 1) = 1, \]
and by (D) the only other vertex besides \( s(e_1) \), that is traversed by \( b \), that has an out-degree, that exceeds one (and is equal to two), is necessarily \( s(e_2) \). This means that \( G \) is isomorphic to \( G_1 \), and by Lemma 3 (A) and (B) yield a contradiction.

For case (c4) we set
\[
d = D(V_L) - 2,
\]
and we have from (3.III.3) that
\[
\sum_{1 < k \leq \Lambda(MD(G))} (D(s(e_k)) - 1) = \frac{1}{2} \nu(MD(G)) - 1 - d.
\]
One has
\[
(2 + d)(\frac{1}{2} \nu(MD(G)) - d) > \nu(MD(G)), \quad 0 < d < \frac{1}{2} \nu(MD(G)) - 2.
\]
It follows from (3.III.4) that
\[
\nu(MD(G)) \geq D(V_L)(\frac{1}{2} \nu(MD(G)) - d),
\]
which contradicts (3.III.5).

With \( G_1 \) in place of \( G_1 \) and with statement (b2) of Lemma 3.6 in place of statement (b1), one has for case (c5) the same argument as for case (c3).

Corollary 3.9. For directed graphs \( G(V, E) \), such that \( S(MD(G(V, E))) \) is a Dyck inverse monoid, and that satisfy conditions (A)(B)(C)(D), the topological conjugacy of the Markov-Dyck shifts \( MD(G(V, E)) \) implies the isomorphism of the graphs \( G(V, E) \).

Proof. In (3.III.2) the data \([\ell, M]\) are expressed in terms of invariants of topological conjugacy.

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