The Sunada construction and the simple length spectrum

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Abstract

We show that certain families of iso-length spectral hyperbolic surfaces obtained via the Sunada construction are not generally simple iso-length spectral.

1 Introduction

Let $M$ be a compact Riemannian manifold. The length spectrum $L(M)$ of $M$ is the set of all lengths of closed geodesics on $M$ counted with multiplicities. Two manifolds $M_1$ and $M_2$ are said to be iso-length spectral if $L(M_1) = L(M_2)$.

In [10], Sunada provided a method to construct iso-length spectral manifolds that are frequently not isometric (see also [4, Ch.11-13]). This requires a notion from group theory.

Let $G$ be a finite group. Two subgroups $H$ and $K$ of $G$ are said to be almost conjugate if, for any $g \in G$,

$$|H \cap (g)| = |K \cap (g)|,$$

where $(g)$ denotes the conjugacy class of $g$ in $G$.

**Theorem (Sunada).** Let $M_0$ be a closed Riemannian manifold, $G$ a finite group, and $H$ and $K$ almost conjugate subgroups of $G$. If there is a surjective homomorphism from $\pi_1(M_0)$ onto $G$, then the finite covering spaces $M_H$ and $M_K$ of $M_0$ corresponding to the subgroups $H$ and $K$, respectively, are iso-length spectral.
When $H$ and $K$ are not conjugate in $G$, the manifolds $M_H$ and $M_K$ can often be shown to be nonisometric. For example, when $M_0$ is a surface, a generic hyperbolic metric on $M_0$ will produce nonisometric $M_H$ and $M_K$; see [4, Ch.12.7].

For surfaces, the simple closed geodesics often carry more topological information. Accordingly, the simple length spectrum $L^s(M)$ of $M$ is defined to be the set of all lengths of simple closed geodesics on $M$ counted with multiplicities; see [8]. Two manifolds $M_1$ and $M_2$ are said to be simple iso-length spectral if $L^s(M_1) = L^s(M_2)$.

**Question 1.** Are there nonisometric simple iso-length spectral hyperbolic surfaces?

In [8], McShane and Parlier give example of pairs of 4-holed spheres with geodesic boundary which have the same interior simple length spectrum (one ignores the boundary lengths). They do in fact have different boundary lengths, and so they have different simple length spectrum.

One can ask if Sunada’s construction provides a positive resolution to Question 1.

**Question 2.** Does Sunada’s construction, for a given homomorphism $\rho: \pi_1(M_0) \to G$, generically give simple iso-length spectral surfaces?

To answer Question 2, we choose one of the examples of almost conjugate subgroups Sunada provided in his paper [10].

**Example.** $G = (\mathbb{Z}/8\mathbb{Z})^\times \ltimes \mathbb{Z}/8\mathbb{Z}$ with usual action of $(\mathbb{Z}/8\mathbb{Z})^\times$ on $\mathbb{Z}/8\mathbb{Z}$.

$H = \{(1,0),(3,0),(5,0),(7,0)\}$ and $K = \{(1,0),(3,4),(5,4),(7,0)\}$ are almost conjugate but not conjugate.

Our main theorem is the following.

**Theorem 1.1.** Let $M_0$ be a closed oriented surface of genus 2, $G$, $H$, and $K$ the groups provided in the example above.

There is a surjective homomorphism $\rho: \pi_1(M_0) \to G$ such that, for almost every $[m] \in T(M_0)$, the corresponding iso-length spectral surfaces $M_H$ and $M_K$ are not simple iso-length spectral.

In fact, we prove a little bit more. We define the length set and the simple length set of a manifold $M$ to be the set of all lengths of closed geodesics on $M$ without multiplicities and the set of all lengths of simple closed geodesics on $M$ without multiplicities, respectively. Then from the proof of Theorem 1.1 we have the following corollary.
Corollary 1.1. The surfaces $M_H$ and $M_K$ in Theorem 1.1 have the same length set but they do not have the same simple length set.

This corollary shows that the construction of length equivalent manifolds in [6] does not necessarily give simple length equivalent manifolds.

Outline of the paper. Section 2 contains the relevant background. In Section 3 we give the proof of the main theorem. The sketch of the proof is as follow. We begin by defining a surjective homomorphism $\rho : \pi_1(M_0) \to G$ and a closed curve $\alpha$ in $M_0$. By Sunada’s construction, the covering spaces $\pi_H : M_H \to M_0$ and $\pi_K : M_K \to M_0$ corresponding to the subgroups $H$ and $K$ are iso-length spectral. We then show that, for almost every $[m] \in \mathcal{T}(M_0)$, the induced metrics on $M_H$ and $M_K$ have the following property. In each of these two covering spaces $M_H$ and $M_K$, there are exactly four closed geodesics having the same length as $\alpha$, namely the two degree-one components of $\pi_H^{-1}(\alpha)$ (and $\pi_K^{-1}(\alpha)$) and their images under the lifts of the hyperelliptic involution $\tau : M_0 \to M_0$. We also show that these four closed geodesics on $M_H$ are nonsimple while the other four closed geodesics on $M_K$ are simple. Therefore $M_H$ and $M_K$ are not simple iso-length spectral.

We remark on one subtlety of the proof. According to [9], there are curves $\gamma, \gamma'$ on $M_0$ such that for every hyperbolic metric $m$ on $M_0$, $\text{length}_m(\gamma) = \text{length}_m(\gamma')$. Although these are nonsimple on $M_0$, they become simple in a finite sheeted cover, so must be accounted for in our proof.

2 Background

Let $M$ be a closed oriented surface of genus $g \geq 2$. We denote the Teichmüller space of $M$ by

$$\mathcal{T}(M) = \{ [m] \mid m \text{ is a hyperbolic metric on } M \},$$

where $[m]$ represents the equivalence class via the equivalence relation $m \sim m'$ if there exists an isometry $f : (M, m) \to (M, m')$ such that $f \simeq \text{id}_M$, see e.g. [4].

Given $[m] \in \mathcal{T}(M)$, the holonomy homomorphism

$$\rho_m : \pi_1(M) \to \text{PSL}_2(\mathbb{R})$$
is well defined up to conjugation in $\text{PSL}_2(\mathbb{R})$. This determines an embedding

$$\mathcal{T}(M) \to \text{Hom}(\pi_1(M), \text{PSL}_2(\mathbb{R}))/\text{conjugation}$$

by $[m] \mapsto [\rho_m]$.

Let $\gamma$ be an essential closed curve on $M$. The length function of $\gamma$

$$\text{length}_\gamma(\cdot) : \mathcal{T}(M) \to \mathbb{R}_+$$

is defined as the length of the $m$-geodesic homotopic to $\gamma$. Using the holonomy homomorphism, one can compute

$$\text{length}_{[m]}(\gamma) = 2 \cosh^{-1}\left(\frac{|\text{tr}(\rho_m(\gamma))|}{2}\right).$$

The embedding (1) makes $\mathcal{T}(M)$ into a real analytic manifold. By (2), the length functions are analytic (see e.g. [3] or [1]). Since $\mathcal{T}(M)$ is connected, we then have the following theorem; see [8].

**Theorem 2.1.** Let $c \in \mathbb{R}$, $\alpha$ and $\beta$ be closed curves on $M$. The function

$$f = c \cdot \text{length}_\gamma(\beta) - \text{length}_\gamma(\alpha) : \mathcal{T}(M) \to \mathbb{R}$$

is real analytic, in particular, $f \neq 0$ almost everywhere or $f = 0$ everywhere.

Let $\gamma$ and $\gamma'$ be closed curves on $M$. The geometric intersection number of $\gamma$ and $\gamma'$ is defined by

$$i(\gamma, \gamma') = \min_{\overline{\gamma}, \overline{\gamma}'} |(\overline{\gamma} \times \overline{\gamma}')^{-1}(\Delta)|,$$

where $\overline{\gamma}$ and $\overline{\gamma}'$ are in the homotopy classes $[\gamma]$ and $[\gamma']$, respectively, $\overline{\gamma} \times \overline{\gamma}' : S^1 \times S^1 \to M \times M$, and $\Delta \subset M \times M$ is diagonal.

The next theorem provides a tool for dealing with the phenomenon arising from [3].

**Theorem 2.2.** Let $\gamma$, $\gamma'$ be closed curves on $M$ and $k \in \mathbb{R}$.

If $\text{length}_{m}(\gamma) = k \cdot \text{length}_{m}(\gamma')$, for all $[m] \in \mathcal{T}(M)$, then $i(\gamma, \alpha) = k \cdot i(\gamma', \alpha)$, for all simple closed curves $\alpha$ on $M$.
Proof. For \( k = 1 \), a proof can be found in [7], for example. The same idea works here, and we sketch it.

Given a simple closed curve \( \alpha \), there exists a sequence \( \{ \[m_n]\} \subset \mathcal{T}(M) \) such that

\[
\frac{1}{n} \cdot \text{length}_{[m_n]}(\eta) \to i(\eta, \alpha),
\]

for all closed curves \( \eta \) on \( M \).

Now suppose \( \text{length}_{[m]}(\gamma) = k \cdot \text{length}_{[m]}(\gamma') \) for all \( [m] \in \mathcal{T}(M) \). Then

\[
\frac{1}{n} \cdot \text{length}_{[m_n]}(\gamma) \to i(\gamma, \alpha)
\]

and

\[
\frac{k}{n} \cdot \text{length}_{[m_n]}(\gamma') \to k \cdot i(\gamma', \alpha).
\]

So \( k \cdot i(\gamma', \alpha) = i(\gamma, \alpha) \). \( \square \)

The following theorem is shown in [7].

**Theorem 2.3.** Given \( \gamma \) and \( \gamma' \) closed curves on \( M \), if

\[
\text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\gamma'),
\]

for all \( [m] \in \mathcal{T}(M) \), then \( [\gamma] = \pm [\gamma'] \) in \( H_1(M) \).

### 3 Proof of the main theorem

Let \( M_0 \) be a closed oriented surface of genus 2. We write the fundamental group of \( M_0 \) as \( \pi_1(M_0) = \langle a, b, c, d | [a, b][c, d] = 1 \rangle \), see Figure 1.

![Figure 1: \( M_0 \) with the generators of \( \pi_1(M_0) \).](image)

Let \( G \), \( H \) and \( K \) be groups given in the example in Section [1]. We define a surjective homomorphism \( \rho : \pi_1(M_0) \to G \) by

\[
\rho(a) = (3, 0), \quad \rho(b) = (5, 0), \quad \rho(c) = (1, 0), \quad \text{and} \quad \rho(d) = (1, 1).
\]
Let $\pi : M \to M_0$, $\pi_H : M_H \to M_0$ and $\pi_K : M_K \to M_0$ be the covering spaces of $M_0$ corresponding to $\ker(\rho)$, $\rho^{-1}(H)$ and $\rho^{-1}(K)$, respectively.

To help visualizing the covering space $M$, first we construct the covering space $\pi : M_N \to M_0$ corresponding to the subgroup $N = \mathbb{Z}/8\mathbb{Z}$ of $G$, as shown in Figure 2. Then we construct $M$ from the surjective homomorphism $\sigma : \pi_1(M_N) \to N$, the restriction of $\rho$ to $\pi_1(M_N) < \pi_1(M_0)$, see Figure 3. Observe that the generator of $\mathbb{Z}/8\mathbb{Z} \cong N < G$ translates each piece in Figure 3 to the right, and sends the last piece to the first piece.

![Figure 2: The covering space $M_N$.](image)

![Figure 3: The covering space $M$.](image)
Lemma 3.1. Let $\alpha = abd[c^{-1}]d^{-1}$ be a closed curve on $M_0$. Then $\pi^{-1}_H(\alpha) = \beta_1^H \cup \cdots \cup \beta_5^H$, $\pi^{-1}_K(\alpha) = \beta_1^K \cup \cdots \cup \beta_5^K$ where $\pi_H|_{\beta_1^H}$, $\pi_K|_{\beta_1^K}$ are degree one, for $i = 1, 2$, and degree two, for $i = 3, 4, 5$. Furthermore $\beta_1^H$, $\beta_2^H$ are nonsimple and $\beta_1^K$, $\beta_2^K$ are simple.

Figure 4: The closed curve $\alpha$ on $M_0$.

Figure 5: The covering space $M$ and a component $\gamma_1$ of $\pi^{-1}(\alpha)$.

Proof. First we look at a component $\gamma_1$ of $\pi^{-1}(\alpha)$ in $M$, see Figure 5. Observe that the preimage of $\alpha$ is sixteen simple closed curves on $M$ denotes $X = \{\gamma_1, \ldots, \gamma_{16}\}$. $G$ acts on $X$ and this action is equivalent to the action of $G$ on the cosets of $L = \text{Stab}_G(\gamma_1) = \{(1,0), (7,0)\}$. More precisely, the bijection

$$G//L \to X$$
given by

\[ gL \mapsto g \cdot \gamma_1 \]

is equivariant with respect to the actions of \( G \). We assume \( \{\gamma_1, \ldots, \gamma_{16}\} \) are numbered so that

\[
\begin{align*}
\gamma_1 & \to L, \quad \gamma_2 \to (1, 1)L, \quad \gamma_3 \to (1, 2)L, \quad \gamma_4 \to (1, 3)L, \\
\gamma_5 & \to (1, 4)L, \quad \gamma_6 \to (1, 5)L, \quad \gamma_7 \to (1, 6)L, \quad \gamma_8 \to (1, 7)L, \\
\gamma_9 & \to (3, 0)L, \quad \gamma_{10} \to (3, 3)L, \quad \gamma_{11} \to (3, 6)L, \quad \gamma_{12} \to (3, 1)L, \\
\gamma_{13} & \to (3, 4)L, \quad \gamma_{14} \to (3, 7)L, \quad \gamma_{15} \to (3, 2)L, \quad \gamma_{16} \to (3, 5)L.
\end{align*}
\]

We use the above representations to compute \( H \) and \( K \) orbits under the actions of \( H \) and \( K \) on \( X \). Then the \( H \) orbits partition \( \{\gamma_1, \ldots, \gamma_{16}\} \) as

\[
\{\gamma_1, \gamma_9\}, \{\gamma_5, \gamma_{13}\}, \{\gamma_2, \gamma_8, \gamma_{10}, \gamma_{16}\}, \{\gamma_3, \gamma_7, \gamma_{11}, \gamma_{15}\}, \{\gamma_4, \gamma_6, \gamma_{12}, \gamma_{14}\}
\]

and the \( K \) orbits partition \( \{\gamma_1, \ldots, \gamma_{16}\} \) as

\[
\{\gamma_1, \gamma_{13}\}, \{\gamma_5, \gamma_9\}, \{\gamma_2, \gamma_8, \gamma_{10}, \gamma_{14}\}, \{\gamma_3, \gamma_7, \gamma_{11}, \gamma_{15}\}, \{\gamma_4, \gamma_6, \gamma_{10}, \gamma_{16}\}.
\]

All closed curves in each \( H \) orbit lie above exactly one closed curve on \( M_H \) and all closed curves in each \( K \) orbit lie above exactly one closed curve on \( M_K \). So we can write \( \pi_H^{-1}(\alpha) = \beta_1^H \cup \cdots \cup \beta_5^H \) and \( \pi_K^{-1}(\alpha) = \beta_1^K \cup \cdots \cup \beta_5^K \).

We may associate \( \beta_1^H, \beta_2^H, \beta_1^K \) and \( \beta_2^K \) with the orbits \( \{\gamma_1, \gamma_9\}, \{\gamma_5, \gamma_{13}\}, \{\gamma_1, \gamma_{13}\} \) and \( \{\gamma_5, \gamma_9\} \), respectively.

Next we observe that \( \pi_H |_{\beta_i^H}, \pi_K |_{\beta_i^K} \) are degree one, for \( i = 1, 2 \), and degree two, for \( i = 3, 4, 5 \).

For the simplicity of \( \beta_1^H, \beta_2^H, \beta_1^K \) and \( \beta_2^K \), we look at their associated orbits. We observe that \( \gamma_1 \) intersects \( \gamma_0 = (3, 0) \cdot \gamma_1 \) nontrivially by inspecting Figure 3 for the actions of \( G \) and Figure 5 for the picture of \( \gamma_1 \). Similarly we can compute

\[
\begin{align*}
\gamma_1 \cap \gamma_9 & \neq \emptyset, \quad \gamma_5 \cap \gamma_{13} \neq \emptyset, \\
\gamma_1 \cap \gamma_{13} & = \emptyset, \quad \gamma_5 \cap \gamma_9 = \emptyset.
\end{align*}
\]

Since the \( H \) orbit \( \{\gamma_1, \gamma_9\} \) corresponding to \( \beta_1^H \) contains intersecting curves, \( \beta_1^H \) is nonsimple. Similarly, \( \beta_2^H \) is also nonsimple. Since the \( K \) orbit \( \{\gamma_1, \gamma_{13}\} \) corresponding to \( \beta_1^K \) contains pairwise disjoint curves, \( \beta_1^K \) is simple. Similarly, \( \beta_2^K \) is also simple.

To prove Theorem 1.1 we will show that generically a hyperbolic metric on \( M_0 \) lifted to a hyperbolic metric on \( M_H \) has the property that there are
exactly four closed curves on $M_H$ having the same length as $\beta_1^H$ (and $\beta_2^H$) and these four closed curves are nonsimple. In the previous Lemma, we found two such closed curves, namely $\beta_1^H$ and $\beta_2^H$. Lemma 3.2 provides the other two closed curves and we will use Lemma 3.3 to show that there are exactly four such closed curves. Since $M_K$ has a simple closed curve, $\beta_1^K$, of the same length in its lifted metric, $M_H$ and $M_K$ cannot be simple iso-length spectral.

Let $\tau : M_0 \to M_0$ be the hyperelliptic involution. $\tau$ is isotopic to an isometry for any hyperbolic metric on $M_0$. So for any curve $\lambda$ on $M_0$, $\text{length}_{M_0}(\lambda) = \text{length}_{M_0}(\tau(\lambda))$. For a specific basepoint, the induced map $\tau_* : \pi_1(M_0) \to \pi_1(M_0)$ can be computed to be

$$
\tau_*(a) = a^{-1}, \quad \tau_*(b) = b^{-1}, \quad \tau_*(c) = ac^{-1}dc^{-1}a^{-1}, \quad \tau_*(d) = b^{-1}ad^{-1}ba^{-1}.
$$

We have the following lemma.

**Lemma 3.2.** The hyperelliptic involution $\tau : M_0 \to M_0$ lifts to $\tau_H : M_H \to M_H$ and $\tau_K : M_K \to M_K$. In particular, $\tau_H (\beta_i^H) \subset M_H$ is nonsimple and $\tau_K (\beta_i^K) \subset M_K$ is simple, for $i = 1, 2$.

**Proof.** Let $\psi : G \to G$ be the automorphism of $G$ defined by $\psi(j, k) = (j, -k)$, for any element $(j, k) \in G$. Then we can compute $\psi \circ \rho = \rho \circ \tau_*$ and $H = \psi^{-1}(H)$. So $\rho^{-1}(H) = \rho^{-1}(\psi^{-1}(H)) = \tau_*^{-1}(\rho^{-1}(H))$. Thus

$$
\tau_* (\pi_H)(\pi_1(M_H))) = \tau_* (\rho^{-1}(H)) = \rho^{-1}(H) = (\pi_H)(\pi_1(M_H)).
$$

Hence the lifting criterion implies that we may lift $\tau$ to $\tau_H$. The existence of a lift $\tau_K$ to $M_K$ is proven in the same way. \qed

**Lemma 3.3.** For almost every $[m] \in T(M_0)$, if $\gamma$ is a closed curve, $k \in \mathbb{Q}$ and $k \cdot \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$ then $k = 1$ and $\gamma = \alpha$ or $\tau(\alpha)$.

**Proof.** For any $\gamma$ and any $k$, either $k \cdot \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$ is true for every $[m]$ or $k \cdot \text{length}_{[m]}(\gamma) \neq \text{length}_{[m]}(\alpha)$ for almost every $[m]$, by Theorem 2.1. So it suffices to show that if $k \cdot \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$, for every $[m]$, then $k = 1$ and $\gamma = \alpha$ or $\tau(\alpha)$. 

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Let $y_1$ be a simple closed curve as shown in Figure 6. The geometric intersection number of $\alpha$ and $y_1$ is $i(\alpha, y_1) = 1$. Since $k \cdot \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha)$, by Theorem 2.2, $k \cdot i(\gamma, y_1) = i(\alpha, y_1) = 1$. Since the geometric intersection numbers are nonnegative integers, $k = 1$. To prove that $\gamma = \alpha$ or $\tau(\alpha)$, we find some necessary conditions for $\gamma$ to have the same length as $\alpha$, for every $[m] \in \mathcal{T}(M_0)$

Let $y_2$ be the simple closed curve shown in Figure 6. Since $i(\gamma, y_2) = i(\alpha, y_2) = 0$ by Theorem 2.2, $\gamma$ and $\alpha$ are contained in $M_0 - y_2$.

We cut $M_0$ along the simple closed curve $y_2$ to get a torus with two holes and change the basis $\{a, b, d\}$ to the basis $\{a, b, x = da^{-1}\}$, see Figure 7. Then $\alpha = abxaba^{-1}b^{-1}x^{-1}$ and $\tau_s(\alpha) = a^{-1}b^{-1}b^{-1}x^{-1}ba^{-1}b^{-1}axb$. Consider the spine as shown in Figure 8, we homotope $\alpha$ and $\gamma$ into spine, as edge loops without backtracking. Then by considering metrics on $M_0$ where length of some of the edges are bounded and others tend to infinity, we see that in order for $\gamma$ to have the same length as $\alpha$ in $M_0$,

\[
\# \{a_1 \text{ edges of } \gamma\} = \# \{a_1 \text{ edges of } \alpha\} = 3, \\
\# \{x_1 \text{ edges of } \gamma\} = \# \{x_1 \text{ edges of } \alpha\} = 3, \\
\# \{b_1 \text{ edges of } \gamma\} + \# \{b_2 \text{ edges of } \gamma\} = \# \{b_1 \text{ edges of } \alpha\} + \# \{b_2 \text{ edges of } \alpha\} = 8.
\]
Since \( \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha) \) and \([\alpha] = [ab] \in H_1(M_0)\), \([\gamma] = \pm [ab] \in H_1(M_0)\), by Theorem 2.3. Thus from the observation of the edge counts above (replacing \( \gamma \) with \( \gamma^{-1} \) if necessary), we have the following conditions:

1. \( \gamma \) consists of exactly two \( a \)'s, one \( a^{-1} \), one \( x \), and one \( x^{-1} \),
2. \( \# \{b^{-1}\text{s in } \gamma\} = \# \{b\text{'s in } \gamma\} - 1 \), and
3. \( \# \{b_1 \text{ edges of } \gamma\} + \# \{b_2 \text{ edges of } \gamma\} = 8 \).

Next we find all closed curves on \( M_0 \) satisfying these three conditions. By the conditions above we know the exact number of \( a \)'s, \( a^{-1} \)'s, \( x \)'s, and \( x^{-1} \) that appear in \( \gamma \). So we only need to determine the possible number of \( b \)'s and \( b^{-1} \). To do this, we note that while the number \( a_1 \)-edge and the number of \( x_1 \)-edge can be computed directly by counting the number of \( \{a, a^{-1}\} \) and \( \{x, x^{-1}\} \), respectively, some combinations of \( x \)'s and \( b \)'s provide cancellations in the sum of \( b_1 \) and \( b_2 \)-edge count. One example is that \( x \) alone contributes 2 to the sum of \( b_1 \) and \( b_2 \)-edge count, \( b \) alone also contributes 2 to the sum of \( b_1 \) and \( b_2 \)-edge count but \( xb \) contributes only 2 to the sum of \( b_1 \) and \( b_2 \)-edge count.

Taking this type of cancellation into consideration, we can produce a list \( A \) of 4320 words in \( \{a_{\pm 1}, b_{\pm 1}, x_{\pm 1}\} \) that contains all curves satisfying the three conditions.

One can explicitly construct \([m] \in \mathcal{T}(M_0)\), a hyperbolic metric on \( M_0 \) such that

\[
\rho_m(a) = \begin{pmatrix} 5/3 & 3/4 \\ 3/4 & 5/4 \end{pmatrix},
\]

\[
\rho_m(b) = \begin{pmatrix} 4 & 0 \\ 0 & 1/4 \end{pmatrix},
\]
\[ \rho_m(x) = \begin{pmatrix} 5/3 & -16/3 \\ -1/3 & 5/3 \end{pmatrix}. \]

Then the trace of \( \rho_m(\alpha) \) is

\[ \text{tr}(\rho_m(\alpha)) = 109505/2048. \]

By using Mathematica, we have that the elements in \( A \) having the same trace squared as \( \alpha \) are \( \alpha \) and \( \tau(\alpha) \).

Thus if \( \text{length}_{[m]}(\gamma) = \text{length}_{[m]}(\alpha) \), for every \([m] \in T(M_0)\), then \( \gamma = \alpha \) or \( \tau(\alpha) \).

**Proof of Theorem 1.1.** Let \( \rho : \pi_1(M_0) \to G \) be the surjective homomorphism defined in this section.

Let \( \alpha = abd[d, c^{-1}]d^{-1} \) be a closed geodesic on \( M_0 \).

By Lemma 3.1 and Lemma 3.2, for almost every \([m] \in T(M_0)\), there are four nonsimple closed geodesics \( \{\beta_1^H, \beta_2^H, \tau_H(\beta_1^H), \tau_H(\beta_2^H)\} \) on \( M_H \) having length \( l = \text{length}_{[m]}(\beta_1^H) = \text{length}_{[m]}(\alpha) \) and there are four simple closed geodesics \( \{\beta_1^K, \beta_2^K, \tau_K(\beta_1^K), \tau_K(\beta_2^K)\} \) on \( M_K \) having length \( l \).

If \( \gamma^H \) is a closed geodesic on \( M_H \) having length

\[ l = \text{length}_{[m]}(\beta_1^H) = \text{length}_{[m]}(\alpha), \]

then \( \pi_H(\gamma^H) \) is a closed geodesic on \( M_0 \) having length

\[ k \cdot l = k \cdot \text{length}_{[m]}(\beta_1^H) = k \cdot \text{length}_{[m]}(\gamma), \]

for some \( k = 1, 1/2, 1/4, \) or \( 1/8 \), since the degree of \( \pi_H \) and \( \pi_K \) is \( 8 \).

By Lemma 3.3, \( k = 1 \) and \( \pi_H(\gamma^H) = \alpha \) or \( \tau(\alpha) \). Thus \( \gamma^H \) is one of the four nonsimple closed curves above. Hence there are exactly four closed curves on \( M_H \) having length \( l \) and those four closed curves are nonsimple. Similarly, there are exactly four closed curves on \( M_K \) having length \( l \) and those four closed curves are simple.

Therefore \( M_H \) and \( M_K \) are not simple iso-length spectral. \( \square \)

**Proof of Corollary 1.1.** As the proof of Theorem 1.1 shows, for almost every \([m] \in T(M_0)\), there is a simple closed geodesic on \( M_K \) with the same length as \( \alpha \) on \( M_0 \), but no such simple geodesic on \( M_H \). Therefore, \( M_H \) and \( M_K \) are not simple length equivalent. \( \square \)
4 Final discussion

Theorem 1.1 should hold for any surjective homomorphism $\rho : \pi_1(M_0) \to G$ and for any closed surface $M_0$. Indeed, it can be shown that for $G$ as in Theorem 1.1 and any $\rho$, there is a genus 2 or 3 subsurface $\Sigma \subset M_0$ so that the restriction $\rho|_{\pi_1(\Sigma)}$ is surjective. Then, one can list all such surjective homomorphisms and try to construct a curve $\alpha$ in $\Sigma$ playing the role of $\alpha$ in the proof of Theorem 1.1. This does not seem to provide much new information, and even for the cases analyzed by the author, the resulting presentation is significantly more complicated. It would be interesting to find an approach that works for all homomorphisms simultaneously.

Another class of examples that would be interesting to analyze with respect to Question 2 are those given in [2] and [3], as the proof that the surfaces are iso-length spectral is more directly geometric.

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