Fuzzy inequational logic

Vilem Vychodil*

Dept. Computer Science, Palacky University, Olomouc

Abstract

We present a logic for reasoning about graded inequalities which generalizes the ordinary inequational logic used in universal algebra. The logic deals with atomic predicate formulas of the form of inequalities between terms and formalizes their semantic entailment and provability in graded setting which allows to draw partially true conclusions from partially true assumptions. We follow the Pavelka approach and define general degrees of semantic entailment and provability using complete residuated lattices as structures of truth degrees. We prove the logic is Pavelka-style complete. Furthermore, we present a logic for reasoning about graded if-then rules which is obtained as particular case of the general result.

1 Introduction

In this paper, we introduce a general logic for approximate reasoning about atomic predicate formulas which take form of inequalities between terms. Such formulas, called inequalities, are essential in the classic theory of varieties of ordered algebras [10] since the varieties are exactly the classes of ordered algebras which are definable by sets of inequalities. We extend the classic logic for reasoning about inequalities by considering degrees to which one considers the inequalities valid. We would like to stress that our approach is truth-functional and the degrees we use are interpreted as the degrees of truth and they should not be confused or interchanged with degrees that appear in other formalisms and uncertainty theories (like the degrees of belief). We assume that the degrees

*e-mail: vychodil@binghamton.edu, phone: +420 585 634 705, fax: +420 585 411 643
come from general structures of truth degrees. In particular, we use complete residuated lattices [4, 19, 22].

In the proposed logic, we introduce two types of entailment: First, a semantic entailment which is based on evaluating inequalities in particular fuzzy structures called algebras with fuzzy orders [33]. Using algebras with fuzzy orders as models, we are able to introduce degrees to which inequalities semantically follow from collections of partially valid inequalities. Second, we introduce a graded notion of provability (syntactic entailment) which allows us to infer partially valid conclusions from collections of partially valid inequalities. The notion of graded provability is defined using a specific deductive system which consists of axioms and three deduction rules. We prove that our logic is complete in that the degrees of semantic entailment coincide with the degrees of provability. This type of graded completeness is called Pavelka-style completeness [25] after J. Pavelka who, inspired by the influential paper by J. A. Goguen [22], presented the general concept in [29, 30, 31] and studied Pavelka-style complete propositional logics. A thorough and general treatment of logics with this style of completeness is presented in [21].

We consider the completeness result to be the main result of this paper. In addition to that, we present an application of the result showing a complete axiomatization of a logic for reasoning about graded if-then rules called attribute implications. Such rules, sometimes used under different names, are formulas which play important roles in several disciplines concerned with data analysis and management such as the formal concept analysis [20] and relational databases [28]. We show in the paper that the rules can be treated as particular inequalities and that using our general result we may obtain a complete logic for approximate reasoning with such inequalities. By making this observation, we contribute to the area of reasoning with graded if-then rules and present an alternative to the approaches in [9, 32] which may further be explored.

Previous results which are related to our paper include the fuzzy equational logic [3] which introduced Pavelka-style logic for reasoning about graded equalities and fuzzy Horn logic dealing with implications between graded equalities [7]. Our logic can be seen as a generalization of the fuzzy equational logic. Indeed, from the syntactic point of view, it is a logic which results from the fuzzy equational logic by omitting the deduction rule of symmetry. From the semantic
point of view, the present logic uses more general models—algebras with fuzzy orders [33] instead of algebras with fuzzy equalities [6]. A survey of results on fuzzy equational logic can be found in [5].

This paper is organized as follows. In Section 2, we present preliminaries from residuated structures of truth degrees and algebras with fuzzy orders. In Section 3, we introduce our logic and present the central notions of semantic and syntactic entailments. In Section 4, we show that our logic is syntactico-semantically complete in Pavelka style. In Section 5, we present an application of the general completeness result provided in Section 4 by showing a general logic of attribute implications with a complete Pavelka-style axiomatization.

2 Preliminaries

In this section, we present basic notions of complete residuated lattices which appear in the fuzzy inequational logic as the structures of truth degrees. Moreover, we present algebras with fuzzy orders which are used as the basic semantic structures in the fuzzy inequational logic.

2.1 Complete Residuated Lattices

A complete (integral commutative) residuated lattice [4, 19] is an algebra \( L = \langle L, \land, \lor, \otimes, \rightarrow, 0, 1 \rangle \) where \( \langle L, \land, \lor, 0, 1 \rangle \) is a complete lattice, \( \langle L, \otimes, 1 \rangle \) is a commutative monoid, and \( \otimes \) and \( \rightarrow \) satisfy the adjointness property: \( a \otimes b \leq c \) iff \( a \leq b \rightarrow c \) (\( a, b, c \in L \)). Examples of complete residuated lattices include structures on the real unit interval given by left-continuous t-norms [15, 25, 26] as well as finite structures of degrees.

Given \( L \) and \( M \neq \emptyset \), an \( L \)-set \( A \) in \( M \) (or a fuzzy set in \( M \) using degrees in \( L \)) is a map \( A: M \rightarrow L \). For \( a \in M \), the degree \( A(a) \in L \) is interpreted as the degree to which \( a \) belongs to \( A \). Analogously, a binary \( L \)-relation \( R \) on \( M \) is a map \( R: M \times M \rightarrow L \). For \( a, b \in M \), the degree \( R(a, b) \in L \) is interpreted as the degree to which \( a \) and \( b \) are \( R \)-related. Thus, a binary \( L \)-relation on \( M \) may be seen as an \( L \)-set in \( M \times M \). If a symbol like \( \preceq \) denotes a binary \( L \)-relation, we use the usual infix notation and write \( a \preceq b \) instead of \( \preceq(a, b) \).

For \( L \)-sets \( A_1 \) and \( A_2 \) in \( M \), we put \( A_1 \subseteq A_2 \) whenever \( A_1(a) \leq A_2(a) \) for all \( a \in M \) and say that \( A_1 \) is (fully) contained in \( A_2 \). Operations with
**L**-sets are defined componentwise using operations in **L**. For instance, if \( A_1 \) and \( A_2 \) are **L**-sets in \( M \), then \( A_1 \cap A_2 \) and \( A_1 \cup A_2 \) denote **L**-sets in \( M \) such that 
\[
(A_1 \cap A_2)(a) = A_1(a) \land A_2(a)
\]
and 
\[
(A_1 \cup A_2)(a) = A_1(a) \lor A_2(a)
\]
for each \( a \in M \), respectively. Note that \( \cap \) and \( \cup \) may be used for arbitrary arguments. That is, for \( A = \{ A_i; i \in I \} \) where all \( A_i \) \( (i \in I) \) are **L**-sets in \( M \), we consider an **L**-set \( \bigcap A \) in \( M \) which may also be denoted by \( \bigcap_{i \in I} A_i \) so that 
\[
(\bigcap A)(a) = (\bigcap_{i \in I} A_i)(a) = \bigwedge_{i \in I} A_i(a)
\]
for each \( a \in M \). Analogously for \( \bigcup \) and \( \bigvee \).

### 2.2 Algebras with Fuzzy Order

The inequalities we consider as formulas are interpreted in structures called algebras with fuzzy order. These structures represent graded generalizations of the classic ordered algebras. In this section, we recall algebras with fuzzy order and present their basic properties which are needed to establish the completeness theorem. Details on algebraic properties of the structures can be found in [33].

Recall that a type of algebras is given by a set \( F \) of function symbols \( f \in F \) together with their arities. We assume that the arity of each \( f \in F \) is finite. An algebra (of type \( F \), see [12]) is a structure \( M = \langle M, F^M \rangle \) where \( M \) is a non-empty universe set and \( F^M \) is a set of functions interpreting the function symbols in \( F \). That is, for each \( n \)-ary \( f \in F \) there is \( f^M : M^n \rightarrow M \).

Let \( L \) be a complete residuated lattice. An algebra with fuzzy order [33, Definition 1] (of type \( F \)) considering \( L \) as the structure of degrees (shortly, an algebra with **L**-order) is a structure \( M = \langle M, \preceq^M, F^M \rangle \) such that \( \langle M, F^M \rangle \) is an algebra (of type \( F \)) and \( \preceq^M \) is a binary **L**-relation on \( M \) satisfying the following conditions:

\[
\begin{align*}
    a &\preceq^M b \land b \preceq^M a \iff a = b, \quad (1) \\
    a &\preceq^M b \land b \preceq^M c \leq a \preceq^M c, \quad (2) \\
    a_1 &\preceq^M b_1 \land \cdots \land a_n \preceq^M b_n \leq f^M(a_1, \ldots, a_n) \preceq^M f^M(b_1, \ldots, b_n), \quad (3)
\end{align*}
\]

for all \( a, b, c, a_1, b_1, \ldots, a_n, b_n \in M \) and any \( n \)-ary \( f \in F \).

**Remark 1.** (a) Algebras with **L**-order are generalizations of the ordinary ordered algebras in the following sense: If **L** is the two-element Boolean algebra, then [11]
yields that ≼^M is a reflexive and antisymmetric binary relation on M. Moreover, (2) yields that ≼^M is transitive and (3) is the compatibility condition, saying that a function in M is compatible with ≼^M. Thus, setting L to the two-element Boolean algebra, algebras with L-orders become the ordinary ordered algebras.

(b) Note that both (2) and (3) involve ⊗, i.e., the conditions of transitivity and compatibility of ≼^M with the functions in M are formulated in terms of the multiplication ⊗ in L. Condition (1) ensures that the symmetric interior of ≼^M is a compatible fuzzy equality relation, see [33, Theorem 3].

(c) For readers familiar with fuzzy order relations: ≼^M is an L-order in sense of [3 Section 4.3.1], i.e., it is ∧-antisymmetric with respect to a fuzzy equality relation which in our case coincides with the symmetric interior of ≼^M. There are other definitions of fuzzy orders which we do not consider in this paper, e.g., fuzzy orders which are ⊗-antisymmetric with respect to a given similarity relation, cf. [11]. A modestly interesting open problem is whether the subsequent results can be established for such alternative fuzzy orders.

In our considerations on algebras with fuzzy orders, we utilize homomorphisms and factor algebras with fuzzy orders [33]. The notions are introduced as follows. Let M and N be algebras with L-orders (of the same type F). A map h: M → N which satisfies equality

\[ h(f^M(a_1, \ldots, a_n)) = f^N(h(a_1), \ldots, h(a_n)) \] (4)

for any n-ary f ∈ F and all a_1, \ldots, a_n ∈ M; and

\[ a ≼^M b \leq h(a) ≼^N h(b) \] (5)

for all a, b ∈ M is called a homomorphism [33, Section 5] and is denoted by h: M → N. Therefore, homomorphisms are maps which are compatible with the functional parts of M and N and the L-orders of M and N. If h: M → N is surjective, then N is called a (homomorphic) image of M.

Consider an algebra M with L-order. A binary L-relation ξ on M is called an L-preorder compatible with M [33, Section 5] whenever it satisfies

\[ ≼^M \subseteq ξ, \] (6)

\[ ξ(a, b) \otimes ξ(b, c) \leq ξ(a, c), \] (7)

\[ ξ(a_1, b_1) \otimes \cdots \otimes ξ(a_n, b_n) \leq ξ(f^M(a_1, \ldots, a_n), f^M(b_1, \ldots, b_n)), \] (8)
for all \( a, b, c, a_1, b_1, \ldots, a_n, b_n \in M \) and any \( n \)-ary \( f \in F \). Given \( M \) and an \( L \)-preorder \( \xi \) compatible with \( M \), we put

- \( M/\xi = \{ [a]_\xi; a \in M \} \) where \( [a]_\xi = \{ b \in M; \xi(a, b) = \xi(b, a) = 1 \} \);
- \( f^{M/\xi}([a_1]_\xi, \ldots, [a_n]_\xi) = [f^M(a_1, \ldots, a_n)]_\xi \);
- \( [a]_\xi \preceq^{M/\xi} [b]_\xi = \xi(a, b) \);

and call \( M/\xi = (M/\xi, \preceq^{M/\xi}, F^{M/\xi}) \) the factor algebra with \( L \)-order \cite{33} Section 5] of \( M \) modulo \( \xi \). One can show that factor algebras with \( L \)-orders are well defined algebras with \( L \)-orders, see \cite{33} Lemma 4] for details.

The notions of homomorphic images and factor algebras preserve the desirable properties of their classic counterparts \cite{12}. Namely, isomorphic copies of factor algebras can be seen as representations of homomorphic images. Indeed, for an \( L \)-preorder \( \xi \) which is compatible with \( M \), we may introduce a surjective map \( h_\xi: M \to M/\xi \) by putting

\[
h_\xi(a) = [a]_\xi.
\]

The map is called a natural homomorphism \cite{33} Section 5] induced by \( \xi \). Conversely, for a surjective homomorphism \( h: M \to N \), we may introduce a binary \( L \)-relation \( \xi_h \) on \( M \) by putting

\[
\xi_h(a, b) = h(a) \preceq^N h(b),
\]

which is a compatible \( L \)-preorder on \( M \) and \( M/\xi_h \) is isomorphic to \( N \) in terms of the isomorphism of general \( L \)-structures, see \cite{4,33} for details.

### 3 Syntax and Semantics of Fuzzy Inequational Logic

This section introduces the basic notions of fuzzy inequational logic which is developed in Pavelka style. In Subsection 3.1 we introduce formulas, their interpretation in algebras with fuzzy orders, and present some observations which are consequences of the general Pavelka framework. In Subsection 3.2 we introduce a deductive system.
3.1 Formulas, Models, and Semantic Entailment

We consider formulas as syntactic expressions written in a particular language. Namely, a language is defined by a type \( F \) of algebras (i.e., by the collection of function symbols with their arities, cf. Subsection 2.2) and a set \( X \) of object variables. The object variables play the same role as in predicate logics. At this point, we make no assumption on \( X \). Furthermore, the language contains the symbol \( \preceq \) which is the only relation symbol in the language and auxiliary symbols like parentheses and commas.

We consider the usual notion of a term: Given \( F \) and \( X \), each variable \( x \in X \) is a term and if \( t_1, \ldots, t_n \) are terms and \( f \in F \) is an \( n \)-ary function symbol, then \( f(t_1, \ldots, t_n) \) is a term. The set of all terms is then denoted \( T_F(X) \) or simply \( T(X) \) if \( F \) is clear from the context.

A formula (in the language given by \( F \) and \( X \)) is any expression

\[
t \preceq t'
\]

where \( t, t' \in T_F(X) \) and it is called an inequality.

Thus, the notion of inequality is the same as in the case of the classic ordered algebras. For convenience, we may identify formulas with pairs of terms in \( T_F(X) \) and thus the Cartesian product \( T_F(X) \times T_F(X) \) represents the set of all formulas in question. Indeed, each \( t \preceq t' \) may be understood as

\[
(t, t') \in T_F(X) \times T_F(X)
\]

and vice versa. Note that considering formulas as pairs of terms like \( t \preceq t' \) is consistent with the abstract Pavelka approach where formulas are supposed to be abstract objects coming from a predefined set of all formulas which is in our case \( T_F(X) \times T_F(X) \). Therefore, we put

\[
\text{Fml} = T_F(X) \times T_F(X)
\]

and call \( \text{Fml} \) the set of all formulas.

Remark 2. (a) Let us note that in order to be able to consider any formulas, \( T_F(X) \) must be non-empty. Note that \( T_F(X) \neq \emptyset \) whenever \( X \) is non-empty or \( F \) contains nullary function symbols, i.e., symbols for object constants.

(b) Analogously as for the classic algebras, for any complete residuated lattice \( L \), we may consider a term algebra with \( L \)-order \( X \) Example 2]. Namely,
if $T_F(X) \neq \emptyset$, we denote by $T_F(X)$ the algebra $\langle T_F(X), \preceq^{T_F(X)}, F^{T_F(X)} \rangle$ with $L$-order where $\preceq^M$ is the identity, i.e.,

$$t \preceq^{T_F(X)} t' = \begin{cases} 1, & \text{if } t = t', \\ 0, & \text{otherwise,} \end{cases}$$

for all $t, t' \in T_F(X)$. Furthermore, each $f^{T_F(X)}$ is defined by

$$f^{T_F(X)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n).$$

Thus, $T_F(X)$ results from an ordinary term algebra by adding $\preceq^{T_F(X)}$. $L$-relations on $T_F(X)$. We call $T_F(X)$ the (absolutely free) term algebra with $L$-order over variables in $X$.

We now introduce the abstract semantics for our formulas. Recall that in the abstract Pavelka setting [25] an $L$-semantics for $Fml$ is a set $S$ of $L$-sets in $Fml$. Thus, each $E \in S$ is a map $E : Fml \to L$ which defines for each $\varphi \in Fml$ a degree $E(\varphi) \in L$ called the degree to which $\varphi$ is true in $E$. In case of our logic, we introduce $S$ by evaluating inequalities in algebras with fuzzy orders. The details are summarized below.

Let $F$, $X$, and $L$ be fixed. For an algebra $M$ with $L$-order of type $F$, any map $v : X \to M$ is called an $M$-valuation of variables in $X$, i.e., the result $v(x)$ is the value of $x$ in $M$ under $v$. As usual, for each term $t \in T_F(X)$, we define the value $\|t\|_{M,v}$ of $t$ in $M$ under $v$ as follows:

$$\|t\|_{M,v} = \begin{cases} v(x), & \text{if } t \text{ is } x \in X, \\ f^M(\|t_1\|_{M,v}, \ldots, \|t_n\|_{M,v}), & \text{if } t \text{ is } f(t_1, \ldots, t_n). \end{cases} \quad (14)$$

Note that the usual algebraic view of (14) is that the values of terms in $M$ under $v$ are values of homomorphisms from $T_F(X)$ to $M$. Indeed, as in the classic setting, an $M$-valuation $v : X \to M$ admits a unique homomorphic extension $v^\sharp : T_F(X) \to M$ for which

$$v^\sharp(t) = \|t\|_{M,v}. \quad (15)$$

for all $t \in T_F(X)$. Now, for any inequality $t \preceq t'$, we may introduce the degree to which $t \preceq t'$ is true in $M$ under $v$ by

$$\|t \preceq t'\|_{M,v} = \|t\|_{M,v} \preceq^M \|t'\|_{M,v}. \quad (16)$$
Observe that utilizing (10) and (15), we rewrite (16) as
\[ \| t \preceq t' \|_{M, v} = v^\#(t) = M v^\#(t', t'), \] (17)
where \( \xi_v^\# \) is the compatible \( L \)-preorder on \( T_F(X) \) induced by the homomorphic extension \( v^\# \) of \( v \). By considering the infimum of all degrees (16) ranging over all possible \( M \)-valuations, we define
\[ \| t \preceq t' \|_M = \bigwedge_{v: X \rightarrow M} \| t \preceq t' \|_{M, v} \] (18)
which is called the degree to which \( t \preceq t' \) is true in \( M \) (under all \( M \)-valuations). Since we assume that \( L \) is a complete lattice, (18) is always defined. Utilizing (17), we may rewrite (18) as
\[ \| t \preceq t' \|_M = \bigwedge_{v: X \rightarrow M} \xi_v^\#(t, t') = \left( \bigcap_{v: X \rightarrow M} \xi_v^\# \right)(t, t'). \] (19)
Thus, taking into account the fact that the set of all compatible \( L \)-preorders on any algebra with \( L \)-order is closed under arbitrary intersections [33], we may consider a compatible \( L \)-preorder \( \xi_M \) which is defined as the intersection of \( \xi_v^\# \) for all possible \( M \)-valuations. That is,
\[ \xi_M = \bigcap_{v: X \rightarrow M} \xi_v^\#. \] (20)
Under this notation, we have
\[ \| t \preceq t' \|_M = \xi_M(t, t'). \] (21)
Therefore, \( \xi_M \) can be seen as an algebraic representation of the degrees to which formulas are true in a given algebra \( M \) with \( L \)-order. Using this concept, we introduce the abstract semantics for our logic in Pavelka style as follows:
\[ S = \{ \xi_M: M \text{ is algebra with } L \text{-order of type } F \}. \] (22)

Now, having defined the formulas and their \( L \)-semantics, the abstract Pavelka framework gives us the notions of models and semantic entailment: Let \( \Sigma : Fml \rightarrow L \), i.e., \( \Sigma \) is an \( L \)-set in \( Fml \) and let \( \xi_M \in S \). Under this notation, \( \xi_M \) is called an \( S \)-model of \( \Sigma \) (shortly, a model) whenever \( \Sigma \subseteq \xi_M \). The set of all models of \( \Sigma \) is denoted by \( \text{Mod}(\Sigma) \). That is, using (21), we have
\[ \text{Mod}(\Sigma) = \{ \xi_M: \Sigma(t, t') \preceq \| t \preceq t' \|_M \text{ for all } t, t' \in T_F(X) \}. \] (23)
Notice that Mod(Σ) is indeed a set (not a proper class) which is a subset of S. Moreover, the degree to which \( t \preceq t' \) semantically follows by Σ is defined by

\[
\|t \preceq t'\|_{\Sigma} = (\bigcap \text{Mod}(\Sigma))(t, t')
\]

which by (21) and (23) can be rewritten as

\[
\|t \preceq t'\|_{\Sigma} = \bigwedge \{\|t \preceq t'\|_{\mathcal{M}}; \xi_{\mathcal{M}} \text{ is a model of } \Sigma\},
\]

i.e., \( \|t \preceq t'\|_{\Sigma} \) is the infimum of degrees to which \( t \preceq t' \) is true in all models of Σ which is the usual way of defining degrees of semantic entailment in truth-functional logics using (subclasses of) residuated lattices as the structures of truth degrees.

Remark 3. Note that the mainstream approach in fuzzy logics in the narrow sense [15, 23, 25] considers theories, i.e., the collections of formulas from which we draw consequences, as ordinary sets of formulas, cf. [13, 14] covering recent results. In the Pavelka approach, we consider L-sets of formulas prescribing degrees to which formulas are satisfied in models, i.e., not just degrees 0 and 1 as in the mainstream approach. In our case, for each \( t, t' \in T(X) \), an L-set Σ : Fml → L prescribes a degree Σ(t, t') which can be interpreted as a lower bound of a degree to which \( t \preceq t' \) shall be satisfied in a model. Clearly, the standard understanding of theories as sets of formulas can be viewed as a particular case of the concept of theories as L-sets of formulas since Σ(t, t') = 1 prescribes that \( t \preceq t' \) shall be satisfied (fully) in a model of Σ and Σ(t, t') = 0 means that in a model of Σ the inequality \( t \preceq t' \) need not be satisfied at all.

On the other hand, one can achieve the same goal by considering theories as sets of formulas and introducing formulas of the form \( \overline{a} \Rightarrow t \preceq t' \), where \( \overline{a} \) is (a constant for) a truth degree \( a \in L \) (interpreted by the truth degree itself), and \( \Rightarrow \) is (the symbol for) implication which is interpreted by \( \rightarrow \) in L. This approach is used by Hájek in his Rational Pavelka Logic [24] which extends the Lukasiewicz logic by constants for rational truth degrees in the unit interval and bookkeeping axioms, see also [16]. In our paper, we keep the original Pavelka approach.
3.2 Proofs and Provability Degrees

We characterize the degrees of semantic entailment of inequalities introduced in [24] by suitably defined degrees of provability. In this subsection, we introduce a deductive system for our logic and the next section shows its completeness in Pavelka style. We use a notation which is close to that in [25, Section 9.2].

Let us recall that deduction rules in Pavelka style can be seen as inference rules of the form

\[ \langle \varphi_1, a_1 \rangle, \ldots, \langle \varphi_n, a_n \rangle \to \langle \psi, b \rangle, \]  

(26)

where \( \varphi_1, \ldots, \varphi_n, \psi \) are formulas and \( a_1, \ldots, a_n, b \) are degrees in \( L \). The rule (26) reads: “from \( \varphi_1 \) valid to degree \( a_1 \) and \( \ldots \) and \( \varphi_n \) valid to degree \( a_n \), infer \( \psi \) valid to degree \( b \)”. Hence, unlike the ordinary deduction rules which only have the syntactic component which in our case says that \( \psi \) is derived from \( \varphi_1, \ldots, \varphi_n \), the rule (26) has an additional semantic component which computes the degree \( b \) based on the degrees \( a_1, \ldots, a_n \).

Formally, an \( n \)-ary deduction rule is a pair \( R = \langle R_1, R_2 \rangle \) where \( R_1 \), called the syntactic part of \( R \), is a partial map from \( \text{Fml}^n \) to \( \text{Fml} \) and \( R_2 \), called the semantic part of \( R \) is a map \( R_2 : L^n \to L \). A rule \( R = \langle R_1, R_2 \rangle \) such that \( R_1(\varphi_1, \ldots, \varphi_n) = \psi \) and \( R_2(a_1, \ldots, a_n) = b \) is usually depicted as in (26). The semantic part \( R_2 \) of an \( n \)-ary deduction rule \( R = \langle R_1, R_2 \rangle \) preserves non-empty suprema if

\[ R_2(\ldots, \bigvee_{i \in I} a_i, \ldots) = \bigvee_{i \in I} R_2(\ldots, a_i, \ldots) \]  

(27)

for each \( I \neq \emptyset \) and \( a_i \in L \ (i \in I) \). A deductive system for \( \text{Fml} \) and \( L \) is a pair \( \langle A, \mathcal{R} \rangle \), where

(i) \( A : \text{Fml} \to L \) is an \( L \)-set of axioms, and

(ii) \( \mathcal{R} \) is a set of deduction rules, each preserving non-empty suprema.

In our logic, we use a concrete deductive system \( \langle A, \mathcal{R} \rangle \) where the \( L \)-set \( A \) of axioms is defined by

\[ A(t, t') = \begin{cases} 1, & \text{if } t = t', \\ 0, & \text{otherwise} \end{cases} \]  

(28)
and \( \mathcal{R} \) consists of the following deduction rules:

\[
\begin{align*}
\text{Tra:} & \quad \frac{(t \preceq t', a), (t' \preceq t'', b)}{(t \preceq t'', a \otimes b)}, \quad (29) \\
\text{Com:} & \quad \frac{(t_1 \preceq t'_1, a_1), \ldots, (t_n \preceq t'_n, a_n)}{(f(t_1, \ldots, t_n) \preceq f(t'_1, \ldots, t'_n), a_1 \otimes \cdots \otimes a_n)}, \quad (30) \\
\text{Inv:} & \quad \frac{(t \preceq t', a)}{(h(t) \preceq h(t'), a)}, \quad (31)
\end{align*}
\]

where \( t, t', t'', t_1, t'_1, \ldots, t_n, t'_n \in T_F(X) \), \( f \) is an \( n \)-ary function symbol in \( F \), \( h \) is a homomorphism \( h : T_F(X) \to T_F(X) \), and \( a, b, a_1, \ldots, a_n \in L \). The rules are called the rules of transitivity, compatibility, and invariance, respectively.

**Remark 4.** (a) Note that the rules of compatibility and invariance in (30) and (31) represent in fact multiple rules. Indeed, for each function symbol \( f \in F \), (30) defines a separate deduction rule with the same number of input formulas as the arity of \( f \). In the second case, for each \( h \), (31) defines a separate deduction rule in sense of Pavelka. Note that all the rules have natural meaning. For instance, (29) reads: “from \( t \preceq t' \) valid to degree \( a \) and \( t' \preceq t'' \) valid to degree \( b \), infer \( t \preceq t'' \) valid (at least) to degree \( a \otimes b \).” The compatibility rule can be interpreted analogously. The rule of invariance represents a particular substitution rule when from \( t \preceq t' \) valid to degree \( a \) one infers inequality \( h(t) \preceq h(t') \) valid at least to degree \( a \). Observe that \( h(t) \) represents the result of a simultaneous substitution of each variable \( x \) in term \( t \) by term \( h(x) \).

(b) All the rules (29)–(31) preserve non-empty suprema since as a consequence of the adjointness property of \( L \), \( \otimes \) is distributive with respect to general suprema \( \bigvee \) in \( L \), see [5, Theorem 1.22].

Using our deduction system, we introduce provability degrees. Recall that in the abstract Pavelka approach, we define proofs consisting of formulas annotated by degrees in \( L \) as follows. Let \( \langle A, \mathcal{R} \rangle \) be a deductive system for \( \text{Fml} \) and \( L \) and let \( \varphi \in \text{Fml} \) and \( a \in L \). A proof (annotated by degrees in \( L \)) of \( \langle \varphi, a \rangle \) by \( \Sigma \) using \( \langle A, \mathcal{R} \rangle \) is a sequence

\[
\langle \varphi_1, a_1 \rangle, \ldots, \langle \varphi_n, a_n \rangle
\]

such that \( \varphi_n \) is \( \varphi \), \( a_n = a \), and for each \( i = 1, \ldots, n \), we have
(i) $a_i = \Sigma(\varphi_i)$, or

(ii) $a_i = A(\varphi_i)$, or

(iii) there are $\langle \varphi_{j_1}, a_{j_1} \rangle, \ldots, \langle \varphi_{j_k}, a_{j_k} \rangle$ such that $j_1, \ldots, j_k < i$ and there is

$\langle R_1, R_2 \rangle \in R$ such that $\varphi_i = R_1(\varphi_{j_1}, \ldots, \varphi_{j_k})$ and $a_i = R_2(a_{j_1}, \ldots, a_{j_k})$.

If there is a proof of $\langle \varphi, a \rangle$ by $\Sigma$ using $\langle A, R \rangle$, we write $\Sigma \vdash \langle A, R \rangle \langle \varphi, a \rangle$ and call $\langle \varphi, a \rangle$ provable by $\Sigma$ using $\langle A, R \rangle$.

Finally, the degree of provability of $\varphi$ by $\Sigma$ using $\langle A, R \rangle$, which is denoted by $|\varphi|^{(A, R)}_{\Sigma}$, is defined as follows:

$$|\varphi|^{(A, R)}_{\Sigma} = \bigvee \{ a \in L; \Sigma \vdash \langle A, R \rangle \langle \varphi, a \rangle \}. \quad (32)$$

That is, $|\varphi|^{(A, R)}_{\Sigma}$ is the supremum of all degrees to which $\varphi$ is provable by $\Sigma$. If we use our deductive system which consists of $A$ defined by (28) and (29)–(31) as the deduction rules, we omit the superscript $\langle A, R \rangle$ and write just $|\varphi|_{\Sigma}$ and $\Sigma \vdash \langle \varphi, a \rangle$.

Remark 5. The rules of compatibility and invariance may be substituted by alternative deduction rules which generalize the classic rules of replacement and substitution often considered in universal algebra and inequational logic and which also appear in [3, 5]. Namely, the rule of replacement is

$$\text{Rep: } \langle t \leq t', a \rangle \quad \langle s \leq s', a \rangle,$$

where $s$ is a term containing $t$ as a subterm and $s'$ results by replacing one occurrence of $t$ by $t'$. A moment’s reflection shows that if (30) derives $f(t_1, \ldots, t_n) \leq f(t'_1, \ldots, t_n)$ valid to degree $a_1 \otimes \cdots \otimes a_n$ then the same result can be achieved by $n$ applications of the replacement rule which derives $f(t_1, \ldots, t_n) \leq f(t'_1, t_2, \ldots, t_n)$ to degree $a_1$, $f(t'_1, t_2, \ldots, t_n) \leq f(t'_1, t'_2, t_3, \ldots, t_n)$ to degree $a_2$, and $\cdots$ and, $f(t'_1, \ldots, t'_{n-1}, t_n) \leq f(t'_1, \ldots, t'_n)$ valid to degree $a_n$ followed by $n$ applications of (29). Conversely, by induction over the rank of $s$ one can show that utilizing the axioms (28), one can produce the result of the replacement rule by applying (30).

The rule of substitution is

$$\text{Sub: } \langle t \leq t', a \rangle \quad \langle t(x/s) \leq t'(x/s), a \rangle,$$
where $t(x/s)$ and $t'(x/s)$ denote terms which result by $t$ and $t'$ by substituting the term $s$ for each occurrence of the variable $x$ in $t$ and $t'$, respectively. Clearly, the rule of substitution is a particular case \[31\]. On the other hand, if $X$ is denumerable, one may obtain the general result of \[31\] by a series of applications of the rule of substitution. Let us note that in order to correctly implement the simultaneous substitution of \[31\], one has to first substitute all variables in $t$ and $t'$ by variables which do not appear in either of $t$, $t'$, and $s$ and thus the assumption on $X$ being (at least) denumerable is essential.

Let us note here that the degrees of semantic entailment and the provability degrees introduced in this section generalize the classic concepts of semantic entailment and provability in the following sense: If $\Sigma$ is a crisp $L$-set, i.e., if $\Sigma(t, t') \in \{0, 1\}$ for all $t, t' \in T_F(X)$, then $\Sigma$ may be seen as an ordinary subset of $Fml$. In addition, $\|t \not< t'\|_\Sigma \in \{0, 1\}$ and $\|t \not< t'\|_\Sigma = 1$ iff $t \not< t'$ follows by $\Sigma$ in the usual sense (i.e., iff $t \not< t'$ is true in each ordered algebra which is a model of $\Sigma$, see the proof of \[33\] Theorem 12 for details). Analogously, $|t \not< t'|_\Sigma \in \{0, 1\}$ and $|t \not< t'|_\Sigma = 1$ iff $t \not< t'$ is provable by $\Sigma$ in the usual sense (i.e., iff $t \not< t'$ is provable by $\Sigma$ using the inference system of the classic inequational logic). This situation occurs in particular if $L$ is the two-element Boolean algebra. Therefore, the graded concepts of semantic and syntactic entailment in the Pavelka approach is what makes our logic non-trivial.

4 Completeness of Fuzzy Inequational Logic

In this section, we show that our logic is Pavelka-style complete over any $L$. It means that the degrees of semantic entailment agree with the degrees of provability. Thus, for each $\Sigma$ and $t \not< t'$, we establish $|t \not< t'|_\Sigma = \|t \not< t'\|_\Sigma$. Note that the $\leq$-part of the claim (Pavelka-style soundness) is more or less evident. We establish the equality by proving that the semantic and syntactic closures associated to any $L$-set of formulas coincide. Some properties of the closures and their relationship to the degrees of semantic entailment and provability follow directly from properties of the abstract Pavelka framework.

We say that $\Sigma: Fml \to L$ is semantically closed whenever

$$\|t \not< t'\|_\Sigma \leq \Sigma(t, t')$$

(33)
for all $t, t' \in T_F(X)$. Since the converse inequality always holds, $\Sigma$ is semantically closed iff $\| t \preceq t' \|_{\Sigma} = \Sigma(t, t')$ for all $t, t' \in T_F(X)$. Note that using (24), $\Sigma$ is semantically closed iff $\Sigma = \bigcap \text{Mod}(\Sigma)$.

As an immediate consequence, we get that the set of all semantically closed $L$-sets of formulas forms a closure system. In order to see that, observe that if $\bigcap \text{Mod}(\Sigma_i) \subseteq \Sigma_i$ ($i \in I$) then $\bigcap_{i \in I} \bigcap \text{Mod}(\Sigma_i) \subseteq \bigcap_{i \in I} \Sigma_i$. Furthermore, $\bigcap_{i \in I} \Sigma_i \subseteq \Sigma_i$ yields $\bigcap \text{Mod}(\bigcap_{i \in I} \Sigma_i) \subseteq \bigcap \text{Mod}(\Sigma_i)$ for all $i \in I$ and thus

$$\bigcap \text{Mod}(\bigcap_{i \in I} \Sigma_i) \subseteq \bigcap_{i \in I} \bigcap \text{Mod}(\Sigma_i) \subseteq \bigcap_{i \in I} \Sigma_i,$$

showing that $\bigcap_{i \in I} \Sigma_i$ is semantically closed. We may therefore consider the semantic closure $\Sigma^b$ of $\Sigma$, i.e., $\Sigma^b$ is the least semantically closed set of formulas containing $\Sigma$:

$$\Sigma^b = \bigcap \{ \Sigma' ; \Sigma \subseteq \Sigma' \text{ and } \bigcap \text{Mod}(\Sigma') \subseteq \Sigma' \}.$$  \hfill (34)

The semantic closure $\Sigma^b$ of $\Sigma$ determines the degrees of semantic entailment. Indeed, $\Sigma \subseteq \Sigma^b$ yields $\bigcap \text{Mod}(\Sigma) \subseteq \bigcap \text{Mod}(\Sigma^b) \subseteq \Sigma^b$. Moreover, $\bigcap \text{Mod}(\Sigma)$ is semantically closed owing to the fact that $\text{Mod}(\Sigma) \subseteq \text{Mod}(\bigcap \text{Mod}(\Sigma))$ which is easy to see since $\xi_M \in \text{Mod}(\Sigma)$ implies $\bigcap \text{Mod}(\Sigma) \subseteq \xi_M$ and so $\xi_M \in \text{Mod}(\bigcap \text{Mod}(\Sigma))$. Altogether, we get that

$$\Sigma^b = \bigcap \text{Mod}(\Sigma)$$  \hfill (35)

which using (24) means that

$$\Sigma^b(t, t') = \| t \preceq t' \|_{\Sigma}$$  \hfill (36)

for all $t, t' \in T_F(X)$. Since $\Sigma^b = (\Sigma^b)^b$, we further derive

$$\| t \preceq t' \|_{\Sigma} = \Sigma^b(t, t') = \| t \preceq t' \|_{\Sigma^b}$$  \hfill (37)

Recall that $\Sigma : Fml \rightarrow L$ is called syntactically closed under $\langle A, R \rangle$ if $A \subseteq \Sigma$ and for any $n$-ary deduction rule $\langle R_1, R_2 \rangle$ in $R$ and arbitrary formulas $\varphi_1, \ldots, \varphi_n$, we have

$$R_2(\Sigma(\varphi_1), \ldots, \Sigma(\varphi_n)) \leq \Sigma(R_1(\varphi_1, \ldots, \varphi_n))$$  \hfill (38)

provided that $R_1(\varphi_1, \ldots, \varphi_n)$ is defined. In case of the deduction system of our logic which consists of $A$ defined by (28) and deduction rules (29)–(31), the
previous condition of $\Sigma$ being syntactically closed translates into

\[
\begin{align*}
\Sigma(t,t) &= 1, & (39) \\
\Sigma(t,t') \otimes \Sigma(t',t'') &\leq \Sigma(t,t''), & (40) \\
\Sigma(t_1,t'_1) \otimes \cdots \otimes \Sigma(t_n,t'_n) &\leq \Sigma(f(t_1,\ldots,t_n), f(t'_1,\ldots,t'_n)), & (41) \\
\Sigma(t,t') &\leq \Sigma(h(t),h(t')), & (42)
\end{align*}
\]

which all must be satisfied for all $t,t',t'',t_1,t'_1,t_2,t'_2,\ldots,t_n,t'_n \in T_F(X)$, any $n$-ary function symbol $f \in F$, and any homomorphism $h : T_F(X) \rightarrow T_F(X)$.

It can be shown that the set of all syntactically closed $L$-sets of formulas forms a closure system, see [29] and [25, Lemma 9.2.5]. The syntactic closure $\Sigma^+$ of $\Sigma$ is thus introduced by

\[
\Sigma^+ = \bigcap \{\Sigma' ; \Sigma \subseteq \Sigma' \text{ and } \Sigma' \text{ is syntactically closed} \}.
\]

As a consequence of the fact that the syntactic parts of deduction rules in deductive systems preserve non-empty suprema, it follows that

\[
\Sigma^+(t,t') = |t \preceq t'|_\Sigma
\]

for all $t,t' \in T_F(X)$, see [29] and [25, Theorem 9.2.8].

We now turn our attention to the completeness of our logic. By the previous observations on the relationship between the syntactic/semantic entailments and syntactic/semantic closures of $L$-sets of formulas, in order to prove that our logic is Pavelka-style complete, it suffices to show the equality of syntactic and semantic closures for any $\Sigma$. The proof is elaborated by the following two lemmas.

**Lemma 1.** For any $\Sigma : Fml \rightarrow L$, we have $\Sigma^+ \subseteq \Sigma^\#$.

**Proof.** It suffices to check that $\Sigma^\#$ contains $\Sigma$ and is syntactically closed because $\Sigma^+$ is the least syntactically closed $L$-set in $Fml$ containing $\Sigma$.

Obviously, $\Sigma \subseteq \Sigma^\#$ and thus it suffices to check that $\Sigma^\#$ satisfies all (39)–(42). Trivially, $\Sigma^\#$ satisfies (39) because $\xi_M(t,t) = 1$ for any $\xi_M \in \text{Mod}(\Sigma)$. In order to see that (40) is satisfied, take $t,t',t'' \in T_F(X)$ and observe that (7),
[20], and [35] together with the fact that $a \otimes \bigwedge_{i \in I} b_i \leq \bigwedge_{i \in I}(a \otimes b_i)$ yield

$$\Sigma^ \omega(t, t') \otimes \Sigma^ \omega(t', t'') = \left(\bigcap \text{Mod}(\Sigma)\right)(t, t') \otimes \left(\bigcap \text{Mod}(\Sigma)\right)(t', t'')$$

$$\leq \bigwedge_{\xi \in \text{Mod}(\Sigma)} \left(\xi_{\mathcal{M}}(t, t') \otimes \xi_{\mathcal{M}}(t', t'')\right)$$

$$\leq \bigwedge_{\xi \in \text{Mod}(\Sigma)} \bigwedge_{v: \mathcal{X} \rightarrow \mathcal{M}} \left(\xi_{v^t}(t, t') \otimes \xi_{v^t}(t', t'')\right)$$

$$\leq \bigwedge_{\xi \in \text{Mod}(\Sigma)} \bigwedge_{v: \mathcal{X} \rightarrow \mathcal{M}} \xi_{v^t}(t, t'')$$

$$= \Sigma^ \omega(t, t').$$

Analogously, one may check (41) utilizing (8). Finally, (42) is satisfied because for every homomorphism $h: \mathcal{T}_F(X) \rightarrow \mathcal{T}_F(X)$ and $\mathcal{M}$-valuation $v: \mathcal{X} \rightarrow \mathcal{M}$ one can take an $\mathcal{M}$-valuation $w: \mathcal{X} \rightarrow \mathcal{M}$ satisfying $w(x) = v^t(h(x))$ for all $x \in \mathcal{X}$. For $w$, by induction over the rank of terms, we get that $w^t(t) = v^t(h(t))$ for all $t \in \mathcal{T}_F(X)$. Therefore,

$$\Sigma^ \omega(t, t') = \bigwedge_{\xi \in \text{Mod}(\Sigma)} \bigwedge_{v: \mathcal{X} \rightarrow \mathcal{M}} \xi_{w^t}(t, t')$$

$$\leq \bigwedge_{\xi \in \text{Mod}(\Sigma)} \bigwedge_{v: \mathcal{X} \rightarrow \mathcal{M}} \xi_{v^t}(h(t), h(t'))$$

$$= \Sigma^ \omega(h(t), h(t')).$$

Therefore, $\Sigma^ \omega$ is syntactically closed.

Note that using [36], [44], and Lemma 1 we get that our logic is sound:

$$|t \not\asymp t'_{\Sigma} = \Sigma^+ (t, t') \leq \Sigma^ \omega (t, t') = \|t \not\asymp t'\|_{\Sigma}. \quad (45)$$

The next lemma proves the converse inequality.

**Lemma 2.** For any $\Sigma: Fml \rightarrow L$, we have $\Sigma^ \omega \subseteq \Sigma^+$.  

**Proof.** It suffices to check that $\Sigma^+$ contains $\Sigma$ and is semantically closed because $\Sigma^+$ is the least semantically closed $L$-set in $Fml$ containing $\Sigma$.

Observe that since $\Sigma^+$ satisfies (39)–(41), it is a compatible $L$-preorder on $\mathcal{T}_F(X)$ and by definition it contains $\Sigma$. Therefore, we may consider the factor algebra $\mathcal{T}_F(X)/\Sigma^+$ with $L$-order. For the factor algebra we now prove that $\Sigma^+ = \xi_{\mathcal{T}_F(X)/\Sigma^+}$ by checking both inclusions.

Take a $\mathcal{T}_F(X)/\Sigma^+$-valuation $v: \mathcal{X} \rightarrow \mathcal{T}_F(X)/\Sigma^+$ such that $v(x) = [x]_{\Sigma^+}$. For its homomorphic extension $v^t: \mathcal{T}_F(X) \rightarrow \mathcal{T}_F(X)/\Sigma^+$, we have $v^t(t) = [t]_{\Sigma^+}$ for all $t \in \mathcal{T}_F(X)$. As a consequence

$$\xi_{\mathcal{T}_F(X)/\Sigma^+}(t, t') \leq \xi_{v^t}(t, t') = [t]_{\Sigma^+} \leq [t']_{\Sigma^+} = \Sigma^+ (t, t'),$$

where $[t]_{\Sigma^+}$ denotes the $\Sigma^+$-equivalence class of $t$.
which proves that $\xi_{T_F(X)/\Sigma^r} \subseteq \Sigma^r$. Conversely, take $v : X \to T_F(X)/\Sigma^r$ and let $h : X \to T_F(X)$ be a map such that $h(x) \in v(x)$ for all $x \in X$. For the homomorphic extension $h^v$ of $h$, we get $v^v(t) = [h^v(t)]_{\Sigma^r}$ for all $t \in T_F(X)$. As a consequence of (42), it follows that

$$\Sigma^r(t, t') \leq \Sigma^r(h^v(t), h^v(t')) = [h^v(t)]_{\Sigma^r} \leq [h^v(t')]_{\Sigma^r} = \xi_{v^v}(t, t'),$$

showing $\Sigma^r \subseteq \xi_{v^v}$. Since $v$ is arbitrary, we get $\Sigma^r \subseteq \xi_{T_F(X)/\Sigma^r}$.

We now finish the proof as follows. Using the inclusion $\Sigma^r \subseteq \xi_{T_F(X)/\Sigma^r}$, we get $\xi_{T_F(X)/\Sigma^r} \in \text{Mod}(\Sigma^r)$ and thus $\bigcap \text{Mod}(\Sigma^r) \subseteq \xi_{T_F(X)/\Sigma^r} \subseteq \Sigma^r$ on account of $\xi_{T_F(X)/\Sigma^r} \subseteq \Sigma^r$. This proves that $\Sigma^r$ is semantically closed. 

To sum up, we have established the following completeness theorem:

**Theorem 3** (completeness). For any $\Sigma : \text{Fml} \to L$ and $t, t' \in T_F(X)$, we have

$$|t \preceq t'|_\Sigma = \|t \preceq t'\|_\Sigma.$$  

(46)

**Proof.** Consequence of (36), (44), Lemma 1 and Lemma 2.

We conclude the section by remarks on the completeness.

**Remark 6.** (a) Our inequational logic can be seen as a particular fragment of a first-order fuzzy logic which only uses atomic formulas and a single relation symbol—the symbol for inequality. For this particular fragment, we have established Pavelka-style completeness over arbitrary $L$. This is in contrast to the full first-order logic (with all connectives in the language including the implication) where Pavelka-style completeness depends on the continuity of the truth functions of logical connectives, cf. 25, 29, 30, 31.

(b) As a consequence of Theorem 3, we get that $\Sigma^n$ (which is equal to $\Sigma^r$) is a compatible $L$-preorder on $T_F(X)$ which in addition satisfies (42), i.e., it is a **fully invariant** compatible $L$-preorder on $T_F(X)$ and the factor algebra $T_F(X)/\Sigma^n$ with $L$-order fully describes the degrees of entailment by $\Sigma$ because

$$|t \preceq t'|_\Sigma = \|t \preceq t'\|_\Sigma = \|t \preceq t'\|_{T_F(X)/\Sigma^n}.$$  

This generalizes the well-known property of syntactically/semantically closed sets of inequalities in case of the classic inequational logic.

(c) Also note that the notion of provability degree is not finitary in the usual sense: $|t \preceq t'|_\Sigma = a$ does not guarantee that $\Sigma \vdash (t \preceq t', a)$. The arguments are the same as in the case of the fuzzy equational logic 3, cf. 3 Example 3.32.
5 Application: Abstract Logic of Graded Attributes

We now show how the general result in the previous section can be used to obtain complete axiomatizations of logics dealing with particular problem domains. For illustration, we show a general logic for reasoning with graded if-then rules which generalize the ordinary attribute implications which appear in formal concept analysis\[20\] of relational object-attribute data. In this section, we first recall the notions related to attribute implications and their entailment and then we present their generalization which exploits the results from Section\[4\].

Consider a finite set $Y$ of symbols called attributes. An attribute implication over $Y$ is an expression $A \Rightarrow B$\(47\) such that $A, B \subseteq Y$. The intended meaning of $A \Rightarrow B$ is to express a dependency “if an object has all the attributes in $A$, then it has all the attributes in $B$” and if $A = \{p_1, \ldots, p_m\}$ and $B = \{q_1, \ldots, q_n\}$, the attribute implication\(47\) is written as $\{p_1, \ldots, p_m\} \Rightarrow \{q_1, \ldots, q_n\}$.\(48\)

For $A, B, M \subseteq Y$, we call $A \Rightarrow B$ satisfied by $M$ (or true in $M$) whenever $A \subseteq M$ implies $B \subseteq M$ (i.e., $A \not\subseteq M$ or $B \subseteq M$) and denote the fact by $M \models A \Rightarrow B$. Note that if $M$ is considered as a set of attributes of an object, then $M \models A \Rightarrow B$ means that “If the object has all the attributes in $A$, then it has all the attributes in $B$” which corresponds with the intended meaning outlined above.

Remark 7. Let us note that formulas like\(48\) appear in other disciplines and are extensively used for knowledge representation and reasoning about data dependencies. For instance, they are known under the name functional dependencies in relational databases\[28\] and can be seen as particular definite clauses used in logic programming\[27\]. Interestingly, even if the database semantics of the rules differs from the one introduced above, it yields the same notion of semantic entailment\[18\] and thus a common axiomatization. Rules like\(48\) are also used in data mining as association rules\[1\,35\], with their validity in data being defined using constraints such as confidence and support.
Semantic entailment of attribute implications is introduced as follows. A set \( M \subseteq Y \) is called a *model* of a set \( \Sigma \) of attribute implications whenever \( M \models A \Rightarrow B \) for all \( A \Rightarrow B \in \Sigma \). Furthermore, \( A \Rightarrow B \) is *semantically entailed* by \( \Sigma \), written \( \Sigma \models A \Rightarrow B \), if \( M \models A \Rightarrow B \) for each model \( M \) of \( \Sigma \).

The semantic entailment of attribute implications has an axiomatization which is based on the following deduction rules:

\[
\begin{align*}
\text{Ax:} & \quad A \cup B \Rightarrow A, \\
\text{Tra:} & \quad A \Rightarrow B, B \Rightarrow C \quad \Rightarrow A \Rightarrow C, \\
\text{Aug:} & \quad A \Rightarrow B, A \cup C \Rightarrow B \cup C, 
\end{align*}
\]

(49)

where \( \cup \) denotes the set-theoretic union and \( A, B, C \subseteq Y \). Note that Ax is a nullary rule, i.e., each \( A \cup B \Rightarrow A \) is an axiom. Using the deduction rules, we define the usual notion of provability of attribute implications from sets of attribute implications: for \( \Sigma \) and \( A \Rightarrow B \), we put \( \Sigma \vdash A \Rightarrow B \) whenever there is a sequence (a proof) \( \varphi_1, \ldots, \varphi_n \) such that \( \varphi_n \) is \( A \Rightarrow B \) and each \( \varphi_i \) in the sequence is in \( \Sigma \) or results by preceding formulas in the sequence using Ax, Tra, or Aug. The usual completeness theorem is established: \( \Sigma \models A \Rightarrow B \) iff \( \Sigma \vdash A \Rightarrow B \).

The axiomatization based on Ax, Tra, and Aug was discovered by Armstrong [2]. There are other equivalent systems of deduction rules which are even simpler. For instance, Tra (transitivity), and Aug (augmentation) can be equivalently replaced by the rule of cut (also known as pseudo-transitivity [28]):

\[
\begin{align*}
\text{Cut:} & \quad A \Rightarrow B, B \cup C \Rightarrow D \quad \Rightarrow A \cup C \Rightarrow D
\end{align*}
\]

(50)

for all \( A, B, C, D \subseteq Y \).

In this section, we propose a general form of formulas like (48) with general semantics and a complete Pavelka-style axiomatization. In particular, we focus on a generalization where *attributes are graded*. That is, instead of considering the presence/absence of attributes as in the classic setting, we allow attributes to be present to degrees and we allow graded entailment of rules from \( L \)-sets of other rules, following Pavelka’s approach. The presented extension is motivated by the fact that in many situations, a data analyst may want to express validity of rules to degrees and may want to be able to make an approximate inference based on partially true rules.

Remark 8. There are approaches which generalize attribute implications in a
graded setting. Most notably, the early approach by Polandt \[32\] which introduces attribute implications as formulas in the formal concept analysis of graded object-attribute data and the more general approach by Belohlavek and Vychodil \[9\] which parameterizes the semantics of the rules by linguistic hedges \[8, 17, 34\]. The approaches are different from the generalization presented below. Namely, \[9\] uses rules which may be seen as implications between graded L-sets of attributes, i.e., the (constants for) truth degrees appear explicitly in the antecedents and consequents of the rules. In contrast, the generalization in this section does not use (constants for) truth degrees in formulas but, on the other hand, it offers a more general interpretation of the rules, e.g., \( \Rightarrow \) may have other interpretations than the residuum in L.

We start by considering formulas of our general logic of attribute implications. Although it is widely used, the set-theoretic treatment of attribute implications like \((48)\) is somewhat limiting. For instance, it implies that the (interpretation of) conjunction which is tacitly used in the definition of \( M |= A \Rightarrow B \) is idempotent. Of course, this is true in the classic setting but it may not be desirable in a graded generalization. Therefore, we view \((48)\) as a (propositional) formula of the form

\[
(p_1 \& \cdots \& p_m \& \top) \Rightarrow (q_1 \& \cdots \& q_n \& \top),
\]

where \( \Rightarrow \) is a symbol for material implication, \( \& \) is a symbol for conjunction, and \( \top \) is the truth constant denoting 1 (the truth value “true”). Observe that \( \top \) is needed to correctly handle the case of \( m = 0 \) or \( n = 0 \). Thus, in the narrow sense, an attribute implication can be seen as a (propositional) formula in the form of an implication between conjunctions of attributes in \( Y \) (which are considered as propositional variables). Since the classic \( \& \) is commutative, associative, and idempotent, the order of variables, additional parentheses, or duplicities of variables may be neglected.

Formula \((51)\) is true under a given evaluation \( e \) of propositional variables in sense of the classical propositional logic, if the value of the antecedent (under the evaluation \( e \)) is \textit{less than or equal to} the value of the consequent (under the evaluation \( e \)). Therefore, \((51)\) being true may be expressed via the \textit{ordering of truth degrees}. The main idea of our approach is to utilize general L-orders to evaluate such formulas instead of the standard order of the truth values 0 and 1.

21
In our setting, we formalize attribute implications as atomic formulas in a language of algebras with \( \mathbf{L} \)-order: Consider a set \( Y = \{f_1, \ldots, f_n\} \) of attributes. Each attribute \( f_i \) will be considered as a nullary function symbol, i.e., as a symbol of an object constant. In addition to that, we consider a binary function symbol \( \cdot \) (called a composition which may be viewed as a symbol for a fuzzy conjunction) and a nullary function symbol \( \top \) (called an identity). Therefore, 

\[
F = \{\cdot, f_1, \ldots, f_n, \top\}.
\] 

Any inequality written in the language given by \( F \) and \( X = \emptyset \) is called a (general) attribute implication.

**Example 1.** The role of the composition \( \cdot \) is to express antecedents and consequent of attribute implications consisting of more than one attribute. For instance, (48) can be seen as inequality \( p_1 \cdot (p_2 \cdot (\cdots p_m) \cdots) \preceq q_1 \cdot (q_2 \cdot (\cdots q_n) \cdots) \).

In addition, \( \top \) may be seen as the counterpart of the empty antecedents and consequents, e.g., \( p \preceq \top \) and \( \top \preceq q \) may represent \( \{p\} \Rightarrow \emptyset \) and \( \emptyset \Rightarrow \{q\} \).

In each algebra with \( \mathbf{L} \)-order which is considered a reasonable interpretation of the generalized attribute implications, \( \cdot \) and \( \top \) shall satisfy some basic properties. It is reasonable to assume that \( \top \) is neutral with respect to \( \cdot \), \( \top \) is the greatest element, \( \cdot \) is associative (to make parentheses in terms irrelevant) and commutative (to make the order of \( f_1, \ldots, f_n \) in terms irrelevant). We therefore postulate the following laws:

\[
t \cdot \top \preceq t, \quad (53)
\]

\[
t \preceq t \cdot \top, \quad (54)
\]

\[
t \preceq \top, \quad (55)
\]

\[
r \cdot (s \cdot t) \preceq (r \cdot s) \cdot t, \quad (56)
\]

\[
(r \cdot s) \cdot t \preceq r \cdot (s \cdot t), \quad (57)
\]

\[
t \cdot s \preceq s \cdot t, \quad (58)
\]

where \( r, s, t \in T_F(\emptyset) \). Two remarks are in order: First, (55) does not ensure that an algebra \( \mathbf{M} \) with \( \mathbf{L} \)-equality satisfying (55) to degree 1 has \( \top^\mathbf{M} \) as the greatest element. On the other hand, for each \( f_i \), we have \( f_i^\mathbf{M} \preceq^\mathbf{M} \top^\mathbf{M} = 1 \). Second, (58) may be considered superfluous. It is the opinion of the author.
that \( \cdot \) should be commutative but the logic can be developed in a more general setting without (58) in much the same way as it is presented below.

**Definition 4.** An algebra with \( L \)-order of type (52) which satisfies inequalities (53)–(58) for \( r, s, t \in T_F(\emptyset) \) to degree 1 is called an \( L \)-structure for general attribute implications over attributes \( Y = \{ f_1, \ldots, f_n \} \).

**Remark 9.** Since we always consider the generalized attribute implications to be evaluated in \( L \)-structures which are algebras with \( L \)-orders satisfying (53)–(58), we may accept the usual rules of simplifying the inequalities. Namely, we disregard parentheses and the order of symbols in terms, and we may omit \( \top \) if it is a part of a compound term. In addition, we may omit the symbol of composition and write just \( ts \) instead of \( t \cdot s \). Therefore, (48) may be written as

\[
p_1 p_2 \cdots p_m \preceq q_1 q_2 \cdots q_n.
\]

Note that \( \cdot \) is not idempotent and thus \( p \preceq p \) and \( p \preceq pp \) represent different general attribute implications.

**Example 2.** Let us show that particular \( L \)-structure for general attribute implications can be derived directly from \( L \). Indeed, for a complete residuated lattice \( L = \langle L, \land, \lor, \otimes, \rightarrow, 0, 1 \rangle \), we may consider a structure

\[
M = \langle M, \preceq^M, \cdot^M, f_1^M, \ldots, f_n^M, \top^M \rangle,
\]

where \( M = L \), \( f_i^M \in L \) for each \( i = 1, \ldots, n \), \( \top^M = 1 \), and

\[
a \preceq^M b = a \rightarrow b, \quad a \cdot^M b = a \otimes b,
\]

for all \( a, b \in M \). It is easy to check that \( M \) is an algebra with \( L \)-order and it satisfies each inequality (53)–(58) to degree 1. First, \( M \) is indeed an algebra with \( L \)-order: (1) is satisfied because \( a \rightarrow b = b \rightarrow a = 1 \) is true iff \( a \leq b \) and \( b \leq a \) and thus iff \( a = b \); (2) is satisfied because \( (a \rightarrow b) \otimes (b \rightarrow c) \leq a \rightarrow c \) follows by the adjointness property; (3) is satisfied for \( \cdot \) because \( (a \rightarrow b) \otimes (c \rightarrow d) \leq (a \otimes c) \rightarrow (b \otimes d) \) holds in \( L \); the case of (3) and the nullary operations is trivial since \( 1 \leq f_i \rightarrow f_i \) and \( 1 \leq 1 \rightarrow 1 \). In addition, each (53)–(58) is obviously satisfied to degree 1 since \( \langle L, \otimes, 1 \rangle \) is a commutative monoid with 1 being the greatest element in \( L \). Thus, \( M \) represents an \( L \)-structure for general attribute implications where \( \cdot^M \) is not idempotent in general. Note that an \( L \)-structure for general attribute implications with idempotent \( \cdot^M \) may be obtained by putting \( a \cdot^M b = a \land b \) for all \( a, b \in L \) and leaving the rest as in the previous case. Again,
using \((a \rightarrow b) \otimes (c \rightarrow d) \leq (a \land c) \rightarrow (b \land d)\), it follows that the structure is indeed an \(L\)-structure for general attribute implications.

The framework of the inequational logic gives us the notions of semantic entailment and provability of general attribute implications:

**Definition 5.** Let \(\Sigma\) by an \(L\)-set of general attribute implications and let

\[
\Sigma^{AI}(t, t') = \begin{cases} 
1, & \text{if } t \preceq t' \text{ is in the form of some formula in (53)–(58)}; \\
0, & \text{otherwise}.
\end{cases}
\] (59)

The degree \(\|t \preceq t'\|^{AI}_\Sigma\) to which a general attribute implication \(t \preceq t'\) is *semantically entailed* by \(\Sigma\) is defined by

\[
\|t \preceq t'\|^{AI}_\Sigma = \|t \preceq t'\|_{\Sigma, \Sigma^{AI}}
\] (60)

and the degree \(|t \preceq t'|^{AI}_\Sigma\) to which \(t \preceq t'\) is *provable* by \(\Sigma\) is defined by

\[
|t \preceq t'|^{AI}_\Sigma = |t \preceq t'|_{\Sigma, \Sigma^{AI}},
\] (61)

where \(\Sigma \cup \Sigma^{AI}\) denotes the union of \(L\)-sets \(\Sigma\) and \(\Sigma^{AI}\).

Applying Theorem 3 we obtain the following completeness of the logic of general attribute implications.

**Theorem 6.** Let \(\Sigma\) by an \(L\)-set of general attribute implications. Then, for any general attribute implication \(t \preceq t'\), we have

\[
|t \preceq t'|^{AI}_\Sigma = \|t \preceq t'\|^{AI}_\Sigma.
\]

*Proof.* Consequence of (60), (61), and Theorem 3. \(\square\)

Let us conclude this section by remarks on the consequence of Theorem 6 and properties of the proposed logic of general attribute implications.

**Remark 10.** Owing to the general notion of \(L\)-structure for general attribute implications, the fact that \(\|t \preceq t'\|^{AI}_\Sigma \geq a\) should be understood so that \(t \preceq t'\) is true at least to degree \(a\) under any possible interpretation of the composition \(\cdot\) and the ordering \(\preceq\) which makes all formulas true at least to the degrees prescribed by \(\Sigma\). This is in contrast with the other approaches such as [9] where the analogues of the composition and ordering are given directly by the structure of degrees. In our setting, the structure of degrees just puts a constraint on the mutual relationship of \(\cdot\) and \(\preceq\). Namely, since \(M\) is supposed to be an algebra...
with \(L\)-order, \(M\) is compatible with \(\leq_M\), i.e., the condition (3) with \(\cdot\) in place of \(f\). The condition is quite natural and generalizes the monotony property: If \(t\) is less than or equal to \(t'\) (under some evaluation) and \(s\) is less than or equal to \(s'\) (under the same evaluation), then \(ts\) (i.e., the conjunction of \(t\) and \(s\)) is less than or equal to \(t's'\) (i.e., the conjunction of \(t'\) and \(s'\)).

Remark 11. It is interesting to observe how the inference system simplifies in case of \(F\) given by (52) and \(X = \emptyset\). First, \(X = \emptyset\) means that (31) is superfluous because it infers \(\langle t \leq t', a \rangle\) from \(\langle t \leq t', a \rangle\). In addition, in case of \(f_1, \ldots, f_n\) or \(\top\), (30) becomes a nullary rule which infers \(\langle f \leq f, 1 \rangle\) or \(\langle \top \leq \top, 1 \rangle\) from no input formulas. Since both are axioms to degree 1, see (28), it makes sense to consider (30) only for the composition. That is, our deductive system for general attribute implications reduces to

\[
\text{Tra: } \frac{\langle t \leq t', a \rangle, \langle t' \leq t'', b \rangle}{\langle t \leq t'', a \otimes b \rangle}, \quad \text{Com: } \frac{\langle t \leq t', a \rangle, \langle s \leq s', b \rangle}{\langle ts \leq t's', a \otimes b \rangle},
\]

for all \(t, t', t'', s, s' \in T_F(\emptyset)\) and \(a, b \in L\). Observe that by a particular case of Com for \(s = s'\) and \(b = 1\), we get a derived deduction rule

\[
\text{Aug: } \frac{\langle t \leq t', a \rangle}{\langle ts \leq t's, a \rangle},
\]

where \(t, t', s \in T_F(\emptyset)\) and \(a \in L\). Conversely, Tra and Aug yield Com. Indeed, applying Aug twice, we get \(\langle ts \leq t's, a \rangle\) and \(\langle st' \leq s't', b \rangle\) from \(\langle t \leq t', a \rangle\) and \(\langle s \leq s', b \rangle\), respectively. Now, using the axiom of commutativity (58) and Tra, we infer \(\langle t's \leq t's', b \rangle\) and thus \(\langle ts \leq t's', a \otimes b \rangle\) by Tra. This shows that the deductive system can be reduced to Tra and Aug. This is an interesting observation because it means that the two deduction rules in our logic are in fact Pavelka-style extensions of the two main Armstrong deduction rules of transitivity and augmentation, see (49). In addition, our system proves each \(ts \leq t\) to degree 1 which generalizes the nullary Armstrong rule Ax. Indeed, we infer \(\langle st \leq \top, 1 \rangle\) from \(\langle s \leq \top, 1 \rangle\) by Aug and thus \(\langle ts \leq t, 1 \rangle\) is derivable by (53) and (58) using Tra. We can simplify the system even more by considering a single deduction rule which generalizes (50). Namely, we may introduce

\[
\text{Cut: } \frac{\langle t \leq t', a \rangle, \langle t's \leq s', b \rangle}{\langle ts \leq s', a \otimes b \rangle},
\]

for all \(t, t', s \in T_F(\emptyset)\) and \(a \in L\).
for all $t, t', s, s' \in T_F(\emptyset)$ and $a, b \in L$ with the possibility of $s$ being omitted. Clearly, Tra is then a particular case of Cut with $s$ omitted and Aug results by Cut for $s' = t's$ and $b = 1$. Conversely, one can infer $\langle ts \preceq t's, a \rangle$ from $\langle t \preceq t', a \rangle$ by Aug and then apply Tra with $\langle t's \preceq s', b \rangle$ to obtain the result of Cut. Therefore, Tra and Aug can be replaced by Cut. As a result, our logic has a Pavelka-style complete deductive system which results by attaching a non-trivial semantic part to the deduction rules of the ordinary Armstrong system in both the original version and the simplified version using Cut.

Conclusions

We showed a Pavelka-style complete logic for reasoning with graded inequalities using any complete residuated lattices as the structure of truth degrees. The results generalize the previous results on completeness of fuzzy equational logic by considering more general semantics given by algebras with fuzzy orders and omitting the deduction rule of symmetry. In addition, we showed an application of the general completeness result showing a way to generalize the ordinary attribute implications in a graded setting with a general semantics and Pavelka-style complete inference system which generalizes the well-known Armstrong system of inference rules.

Acknowledgment

Supported by grant no. P202/14-11585S of the Czech Science Foundation.

References

[1] Rakesh Agrawal, Tomasz Imieliński, and Arun Swami, Mining association rules between sets of items in large databases, Proceedings of the 1993 ACM SIGMOD International Conference on Management of Data (New York, NY, USA), SIGMOD ’93, ACM, 1993, pp. 207–216.

[2] William Ward Armstrong, Dependency structures of database relationships, Information Processing 74: Proceedings of IFIP Congress (Amsterdam) (J. L. Rosenfeld and H. Freeman, eds.), North Holland, 1974, pp. 580–583.

[3] Radim Belohlávek, Fuzzy equational logic, Archive for Mathematical Logic 41 (2002), no. 1, 83–90.

[4] Radim Belohlávek, Fuzzy Relational Systems: Foundations and Principles, Kluwer Academic Publishers, Norwell, MA, USA, 2002.
[5] Radim Belohlavek and Vilem Vychodil, *Fuzzy Equational Logic*, Studies in Fuzziness and Soft Computing, vol. 186, Springer, 2005.

[6] ______, *Algebras with fuzzy equalities*, Fuzzy Sets and Systems 157 (2006), no. 2, 161–201.

[7] ______, *Fuzzy Horn logic I*, Archive for Mathematical Logic 45 (2006), no. 1, 3–51.

[8] ______, *Formal concept analysis and linguistic hedges*, International Journal of General Systems 41 (2012), no. 5, 503–532.

[9] ______, *Attribute dependencies for data with grades*, CoRR abs/1402.2071 (2014), http://arxiv.org/abs/1402.2071.

[10] Stephen L. Bloom, *Varieties of ordered algebras*, Journal of Computer and System Sciences 13 (1976), no. 2, 200–212.

[11] Ulrich Bodenhofer, Bernard De Baets, and János Fodor, *A compendium of fuzzy weak orders: Representations and constructions*, Fuzzy Sets and Systems 158 (2007), no. 8, 811–829.

[12] Stanley Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York, 1981.

[13] Petr Cintula, Petr Hájek, and Carles Noguera (eds.), *Handbook of Mathematical Fuzzy Logic, Volume 1*, Studies in Logic, Mathematical Logic and Foundations, vol. 37, College Publications, 2011.

[14] Petr Cintula, Petr Hájek, and Carles Noguera (eds.), *Handbook of Mathematical Fuzzy Logic, Volume 2*, Studies in Logic, Mathematical Logic and Foundations, vol. 38, College Publications, 2011.

[15] Francesc Esteva and Lluís Godo, *Monoidal t-norm based logic: Towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems 124 (2001), no. 3, 271–288.

[16] Francesc Esteva, Lluís Godo, and Carles Noguera, *Expanding the propositional logic of a t-norm with truth-constants: Completeness results for rational semantics*, Soft Computing 14 (2010), no. 3, 273–284.

[17] ______, *A logical approach to fuzzy truth hedges*, Information Sciences 232 (2013), 366–385.

[18] Ronald Fagin, *Functional dependencies in a relational database and propositional logic*, IBM Journal of Research and Development 21 (1977), no. 6, 534–544.

[19] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Volume 151*, 1st ed., Elsevier Science, San Diego, USA, 2007.

[20] Bernhard Ganter and Rudolf Wille, *Formal concept analysis: Mathematical foundations*, 1st ed., Springer-Verlag New York, Inc., Secaucus, NJ, USA, 1997.

[21] Giangiacomo Gerla, *Fuzzy Logic. Mathematical Tools for Approximate Reasoning*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.
[22] Joseph A. Goguen, *The logic of inexact concepts*, Synthese 19 (1979), 325–373.

[23] Siegfried Gottwald, *Mathematical fuzzy logics*, Bulletin of Symbolic Logic 14 (2008), no. 2, 210–239.

[24] Petr Hájek, *Fuzzy logic and arithmetical hierarchy*, Fuzzy Sets and Systems 73 (1995), no. 3, 359–363.

[25] ———, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998.

[26] Erich Peter Klement, Radko Mesiar, and Endre Pap, *Triangular Norms*, 1 ed., Springer, 2000.

[27] John W. Lloyd, *Foundations of Logic Programming*, Springer-Verlag New York, Inc., New York, NY, USA, 1984.

[28] David Maier, *Theory of Relational Databases*, Computer Science Pr, Rockville, MD, USA, 1983.

[29] Jan Pavelka, *On fuzzy logic I: Many-valued rules of inference*, Mathematical Logic Quarterly 25 (1979), no. 3–6, 45–52.

[30] ———, *On fuzzy logic II: Enriched residuated lattices and semantics of propositional calculi*, Mathematical Logic Quarterly 25 (1979), no. 7–12, 119–134.

[31] ———, *On fuzzy logic III: Semantical completeness of some many-valued propositional calculi*, Mathematical Logic Quarterly 25 (1979), no. 25–29, 447–464.

[32] Silke Pollandt, *Fuzzy-Begriffe: Formale Begriffsanalyse unscharfer Daten*, Springer, 1997.

[33] Vilem Vychodil, *Variety theorem for algebras with fuzzy order*, CoRR abs/1406.7702 (2014). [http://arxiv.org/abs/1406.7702](http://arxiv.org/abs/1406.7702)

[34] Lotfi A. Zadeh, *A fuzzy-set-theoretic interpretation of linguistic hedges*, Journal of Cybernetics 2 (1972), no. 3, 4–34.

[35] Mohammed J. Zaki, *Mining non-redundant association rules*, Data Mining and Knowledge Discovery 9 (2004), 223–248.