The generalized Dehn twist along a figure eight

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Abstract

For any unoriented loop on a compact connected oriented surface with one boundary component, the generalized Dehn twist along the loop is defined as an automorphism of the completed group ring of the fundamental group of the surface. If the loop is simple, this is the usual right handed Dehn twist, in particular realized as a mapping class of the surface. We investigate the case when the loop has a single transverse double point, and show that in this case the generalized Dehn twist is not realized as a mapping class.

1 Introduction

The right handed Dehn twist $t_C$ along a simple closed curve $C$ on an oriented surface is a diffeomorphism of the surface, which is illustrated in Figure 1. By definition, a Dehn twist is local; the support of a Dehn twist lies in a regular neighborhood of the curve.

Figure 1: the right handed Dehn twist along $C$

The purpose of this paper is to give a generalization of Dehn twists for not necessarily simple loops and begin the study of it. We will consider on a compact connected oriented surface $\Sigma$ with one boundary component. In this case the mapping class group $\mathcal{M}_{g,1}$ of the surface faithfully acts on the fundamental group $\pi$ of the surface. The results of this paper are summarized as follows:

1. For each unoriented loop $\gamma \subset \Sigma$, the generalized Dehn twist along $\gamma$, denoted by $t_\gamma$, is defined as an automorphism of the completed group ring $\hat{\mathbb{Q}}\pi$ (Definition 3.2.1).

2. When $\gamma$ is simple, $t_\gamma$ is the usual right handed Dehn twist along $\gamma$.

3. We consider whether $t_\gamma$ is a mapping class, i.e., $t_\gamma$ lies in the image of the natural injection $\mathcal{M}_{g,1} \hookrightarrow \text{Aut}(\hat{\mathbb{Q}}\pi)$. We show that if $t_\gamma$ is a mapping class, then it must be local; there is a diffeomorphism representing $t_\gamma$, whose support lies in a regular neighborhood of $\gamma$ (Theorem 3.3.2).

4. Using the criterion above, we show that when $\gamma$ has a single transverse double point and is not homotopic to a power of a simple closed curve (we shall call such a loop a figure eight), then $t_\gamma$ is not a mapping class (Theorem 5.1.1).
Our generalization of Dehn twists is based on the results in [5]. In that paper, the action of Dehn twists on the completed group ring of the fundamental group of the surface is given in terms of an invariant of unoriented loops on the surface $\Sigma$. This invariant is regarded as a derivation on the completed group ring $\hat{\mathbb{Q}}\pi$. When the loop is simple, this invariant turns out to be “the logarithms of Dehn twists”. Even if the loop is not simple, the exponential of this invariant still have its meaning as an automorphism of $\hat{\mathbb{Q}}\pi$. This is our generalization.

The proof of the main theorem presented here is rather ad hoc, since we depend on the classification of the possible configurations of a figure eight on $\Sigma$ (Proposition 4.3.1), and explicit computations of tensors for each configuration. But still, we meet an interesting phenomenon. Namely, if we assume the generalized Dehn twist along a figure eight $\gamma$ is a mapping class, then it should be written as $t_\gamma = t_{C_1}^2 t_{C_2}^2 t_{C_3}^{-1}$, where $C_i$ are the suitable numbered boundary components of a closed regular neighborhood of $\gamma$ (Proposition 5.2.6). The main theorem is proved by looking at this equation in higher degree.

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2 Preliminaries

In this section we prepare what we need to define generalized Dehn twists. We first recall materials from [5], such as symplectic expansion, total Johnson map, Kontsevich’s “associative”, and the Goldman Lie algebra. After that we study symplectic derivations on the completed tensor algebra and algebra automorphisms of the completed tensor algebra preserving the symplectic form. We
end this section by showing that the mapping class group acts on Kontsevich’s “associative” through a symplectic expansion.

All the loops, the homotopies, and the isotopies that we consider are piecewise differentiable. As usual, we often ignore the distinction between a path and its homotopy class.

2.1 Symplectic expansion and total Johnson map

Let Σ be a compact connected oriented $C^\infty$-surface of genus $g > 0$ with one boundary component. Taking a basepoint $\ast$ on the boundary $\partial \Sigma$ we denote by $\pi := \pi_1(\Sigma, \ast)$ the fundamental group of $\Sigma$, which is free of rank $2g$. Let $\zeta \in \pi$ be a based loop parallel to $\partial \Sigma$ and going by counterclockwise manner. If we take symplectic generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \in \pi$ as shown in Figure 2, then $\zeta = \prod_{i=1}^{g} [\alpha_i, \beta_i]$. Here, for based loops $x$ and $y$, their product $xy$ means that $x$ is traversed first, and $[\alpha_i, \beta_i] = \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$.

![Figure 2: symplectic generators for $g = 2$](image)

Let $H := H_1(\Sigma; \mathbb{Q})$ be the first homology group of $\Sigma$, and $\hat{T} := \prod_{m=0}^{\infty} H \otimes^m$ the completed tensor algebra generated by $H$. The algebra $\hat{T}$ has a decreasing filtration given by $\hat{T}_p := \prod_{m \geq p} H \otimes^m$, $p \geq 1$, and is a complete Hopf algebra with respect to the coproduct given by $\Delta(X) = X \otimes 1 + 1 \otimes X$, $X \in H$. We denote by $A_i, B_i \in H$ the homology class represented by $\alpha_i, \beta_i$, respectively. Also, for $x \in \pi$ we denote by $[x] \in H$ the corresponding homology class. The two tensor $\omega = \sum_{i=1}^{g} A_i B_i - B_i A_i \in H \otimes^2$ is independent of the choice of symplectic generators, and called the symplectic form. Here and throughout this paper we often omit $\otimes$ to express tensors.

**Definition 2.1.1** (Kawazumi [3]). A map $\theta: \pi \rightarrow \hat{T}$ is called a Magnus expansion of $\pi$ if

1. for any $x \in \pi$, $\theta(x) \equiv 1 + [x]$ (mod $\hat{T}_2$),

2. for any $x, y \in \pi$, $\theta(xy) = \theta(x)\theta(y)$.

Let $\hat{\mathbb{Q}}\pi$ be the completed group ring of $\pi$. Namely, $\hat{\mathbb{Q}}\pi := \varprojlim_{m \geq p} \mathbb{Q}\pi/(I\pi)^m$, where $I\pi$ is the augmentation ideal of the group ring $\mathbb{Q}\pi$. It has a decreasing filtration given by $\varprojlim_{m \geq p} (I\pi)^p/(I\pi)^m$, $p \geq 1$. Any Magnus expansion $\theta$ induces an isomorphism $\theta: \hat{\mathbb{Q}}\pi \cong \hat{T}$ of complete augmented algebras. Kawazumi [3] introduced this notion to study the automorphism group of a free group or the mapping class group. Taking the fact that $\pi$ is the fundamental group of a surface into consideration, we consider the following notion.

**Definition 2.1.2** (Massuyeau [9]). A Magnus expansion $\theta: \pi \rightarrow \hat{T}$ of $\pi$ is called a symplectic expansion if

1. for any $x \in \pi$, $\theta(x)$ is group-like, i.e., $\Delta(\theta(x)) = \theta(x) \hat{\otimes} \theta(x)$,

2. $\theta(\zeta) = \exp(\omega)$.


Here \( \exp(\omega) = \sum_{m=0}^{\infty}(1/m!)\omega^m \). Symplectic expansions do exist \( \cite{9} \), and they are infinitely many \( \cite{9} \) Proposition 2.8.1. For several constructions, see \( \cite{4} \ [7] \ [9] \). Any symplectic expansion \( \theta \) induces an isomorphism \( \theta: \widehat{\mathbb{Q}} \overset{\cong}{\to} \widehat{\hat{T}} \) of complete Hopf algebras. Moreover, the restriction of \( \theta \) to a cyclic subgroup generated by \( \zeta \) gives an isomorphism \( \widehat{\mathbb{Q}}(\zeta) \overset{\cong}{\to} \mathbb{Q}[\zeta] \). Actually, for our purpose considering Magnus expansions satisfying the second condition of Definition \( 2.1.2 \) is sufficient, due to Proposition \( 2.2.4 \). But still this notion would play a significant role in study of the mapping class groups, as we see in the work of Massuyeau \( \cite{9} \) where he fully used the group-like condition of a symplectic expansion.

We denote by \( \mathcal{M}_{g,1} \) the mapping class group of \( \Sigma \) relative to the boundary, namely the group of orientation preserving diffeomorphisms of \( \Sigma \) which fix the boundary \( \partial \Sigma \) pointwise, modulo isotopies fixing \( \partial \Sigma \) pointwise. By the theorem of Dehn-Nielsen, we can identify

\[
\mathcal{M}_{g,1} = \{ \varphi \in \text{Aut}(\pi); \varphi(\zeta) = \zeta \},
\]

by looking at the natural action of \( \mathcal{M}_{g,1} \) on the fundamental group \( \pi \).

If we fix a Magnus expansion \( \theta \), the associated is the notion of the total Johnson map. We denote by \( \text{Aut}(\hat{T}) \) the group of filter-preserving algebra automorphisms of \( \hat{T} \). Let \( \varphi \in \mathcal{M}_{g,1} \). Then \( \varphi \) induces an isomorphism \( \varphi: \widehat{\mathbb{Q}} \overset{\cong}{\to} \widehat{\mathbb{Q}} \), hence a uniquely determined automorphism \( T^\theta(\varphi) \in \text{Aut}(\hat{T}) \) satisfying \( T^\theta(\varphi) \circ \theta = \theta \circ \varphi \).

**Definition 2.1.3** (Kawazumi \( \cite{3} \)). The automorphism \( T^\theta(\varphi) \in \text{Aut}(\hat{T}) \) is called the total Johnson map of \( \varphi \) associated to \( \theta \).

The group homomorphism \( T^\theta: \mathcal{M}_{g,1} \to \text{Aut}(\hat{T}) \) is injective, since the natural map \( \pi \to \widehat{\mathbb{Q}} \) is injective by the classical fact \( \bigcap_{m=1}^{\infty}(I\pi)^m = 0 \).

### 2.2 Kontsevich’s “associative”

We define a linear map \( N: \hat{T} \to \hat{T}_1 \) by

\[
N(X_1 \cdots X_m) = \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1},
\]

where \( m \geq 1 \), \( X_i \in H \), and \( N(1) = 0 \). The following lemma will be used frequently.

**Lemma 2.2.1** (\( \cite{5} \) Lemma 2.6.2 (1)(2)).

1. For \( u, v \in \hat{T} \), \( N(uv) = N(vu) \).

2. For \( u, v, w \in \hat{T} \), \( N([u, v]w) = N(uv) \).

Here \([u, v] = uv - vu\).

Let us recall Kontsevich’s “associative” \( \cite{6} \). By definition, a derivation on \( \hat{T} \) is a linear map \( D: \hat{T} \to \hat{T} \) satisfying the Leibniz rule:

\[
D(u_1 u_2) = D(u_1)u_2 + u_1 D(u_2),
\]

for \( u_1, u_2 \in \hat{T} \). Since \( \hat{T} \) is freely generated by \( H \) as a complete algebra, any derivation on \( \hat{T} \) is uniquely determined by its values on \( H \), and the space of derivations of \( \hat{T} \) is identified with \( \text{Hom}(H, \hat{T}) \). By the Poincaré duality, \( \hat{T}_1 \cong H \otimes \hat{T} \overset{\cong}{\to} \text{Hom}(H, \hat{T}) \),

\[
X \otimes u \mapsto (Y \mapsto (Y \cdot X)u).
\]

Here \(( \cdot )\) is the intersection pairing on \( H = H_1(\Sigma; \mathbb{Q}) \).
Let \( a_g^- = \text{Der}_\omega(\hat{T}) \) be the space of derivations on \( \hat{T} \) killing the symplectic form \( \omega \). An element of \( a_g^- \) is called a \textit{symplectic derivation}. In view of (2.2.1) any derivation \( D \) is written as

\[
D = \sum_{i=1}^{g} B_i \otimes D(A_i) - A_i \otimes D(B_i) \in H \otimes \hat{T}.
\]

Since \( -D(\omega) = \sum_{i=1}^{g} [B_i, D(A_i)] - [A_i, D(B_i)] \), we have \( a_g^- = \text{Ker}([\ , \ ] : H \otimes \hat{T} \to \hat{T}) \). It is easy to see \( \text{Ker}([\ , \ ] ) = N(\hat{T}_1) \) (see [5] Lemma 2.6.2 (4)). Hence we can write

\[
a_g^- = \text{Ker}([\ , \ ] : H \otimes \hat{T} \to \hat{T}) = N(\hat{T}_1).
\]

The Lie subalgebra \( a_g := N(\hat{T}_2) \) is nothing but (the completion of) what Kontsevich [6] calls \( a_g \).

By a straightforward computation, we have

**Lemma 2.2.2.** Let \( m, n \geq 1 \) and \( X_1, \ldots, X_m, Y_1, \ldots, Y_n \in H \). Then we have

\[
[N(X_1 \cdots X_m), N(Y_1 \cdots Y_n)] = N(N(X_1 \cdots X_m)(Y_1 \cdots Y_n)).
\]

Here the left hand side means the Lie bracket of the two derivations in \( a_g^- = N(\hat{T}_1) \) and in the right hand side \( N(X_1 \cdots X_m)(Y_1 \cdots Y_n) \) means the action of \( N(X_1 \cdots X_m) \in a_g^- \) on the tensor \( Y_1 \cdots Y_n \in \hat{T} \) as a derivation.

**Corollary 2.2.3.** Let \( u \in N(\hat{T}_1) \) and \( v \in \hat{T}_1 \). Then

\[
[u, N(v)] = N(u(v)).
\]

Here the left hand side means the Lie bracket of \( u, N(v) \in a_g^- = N(\hat{T}_1) \) and in the right hand side \( u(v) \) means the action of \( u \in a_g^- \) on the tensor \( v \in \hat{T} \) as a derivation.

**Proof.** If \( u \) and \( v \) are homogeneous, this is Lemma 2.2.2. The general case follows from the bi-linearity. \( \square \)

Let \( \text{IA}(\hat{T}) \) be the subgroup of \( \text{Aut}(\hat{T}) \) consisting of automorphisms which act on \( \hat{T}_1/\hat{T}_2 \cong H \) as the identity. This group is identified with \( \text{Hom}(H, \hat{T}_2) = H \otimes \hat{T}_2 \), by the logarithms:

\[
\text{IA}(\hat{T}) \cong \text{Hom}(H, \hat{T}_2) = H \otimes \hat{T}_2, \ U \mapsto (\log U)|_H.
\]

Let \( \text{IA}_\omega(\hat{T}) \) be the subgroup of \( \text{IA}(\hat{T}) \) consisting of automorphisms preserving \( \omega \). By the same argument in [5] Proposition 2.8.1, we see the bijection (2.2.2) gives a bijection

\[
\text{IA}_\omega(\hat{T}) \cong \text{Ker}([\ , \ ] : H \otimes \hat{T}_2 \to \hat{T}) = N(\hat{T}_3).
\]

The following proposition was communicated to the author by Nariya Kawazumi.

**Proposition 2.2.4.** Assume \( U \in \text{Aut}(\hat{T}) \) satisfies \( U(\omega) = \omega \), and let \( v \in \hat{T}_1 \). Then

\[
U(Nv)U^{-1} = N(Uv).
\]

Here \( U(Nv)U^{-1} \) means the conjugate of the derivation \( Nv \) by the algebra automorphism \( U \).
Proof. Let \( |U| \) be the element of \( \text{Aut}(\hat{T}) \) induced from the action of \( U \) on \( \hat{T}_1/\hat{T}_2 \cong H \). Then \( |U| \) preserves the intersection form \( \omega \), and \( U = U' \circ |U| \), where \( U' \in \text{IA}_\omega(\hat{T}) \). Therefore, it suffices to prove the formula for \( U = |U| \) and \( U \in \text{IA}_\omega(\hat{T}) \).

Suppose \( U = |U| \). We may assume \( v \) is homogeneous and \( v = X_1 \cdots X_m \), where \( m \geq 1 \) and \( X_i \in H \). Then for \( Y \in H \),

\[
U(Nv)U^{-1}(Y) = U(N(X_1 \cdots X_m)(U^{-1}Y)) = U(U(\sum_i((U^{-1}Y \cdot X_i)X_i+1 \cdots X_mX_1\cdots X_{i-1})) = \sum_i(Y \cdot UX_i)U(X_i+1 \cdots X_mX_1\cdots X_{i-1}) = N(U(X_1 \cdots X_m))(Y) = N(Uv)(Y),
\]

hence \( U(Nv)U^{-1} = N(Uv) \). Suppose \( U \in \text{IA}_\omega(\hat{T}) \). By Corollary \( 2.2.3 \), \( \text{ad}(\log U)(Nv) = [\log U, Nv] = N(\log U(v)) \), hence \( \text{ad}(\log U)^k(Nv) = N((\log U)^kv) \), \( k \geq 0 \). Then

\[
U(Nv)U^{-1} = e^{\log U} \circ Nv \circ e^{-\log U} = \sum_{k=0}^\infty \frac{1}{k!}\text{ad}(\log U)^k(Nv) = \sum_{k=0}^\infty \frac{1}{k!}N((\log U)^kv) = N(e^{\log U}v) = N(Uv).
\]

This completes the proof.

\[\square\]

2.3 The Goldman Lie algebra and Kontsevich’s “associative”

Let \( \hat{\pi} \) be the set of conjugacy classes of \( \pi \). In other words, \( \hat{\pi} \) is the set of free homotopy classes of oriented loops on \( \Sigma \). The Goldman Lie algebra \( 2 \) of \( \Sigma \) is the vector space \( \mathbb{Q}\hat{\pi} \) spanned by \( \pi \), equipped with the bracket defined as follows. Let \( \alpha, \beta \) be immersed loops on \( \Sigma \) such that their intersections consist of transverse double points. For each \( p \in \alpha \cap \beta \), let \( [\alpha_p\beta_p] \) be the free homotopy class of the loop first going the oriented loop \( \alpha \) based at \( p \), then going \( \beta \) based at \( p \). Also let \( \varepsilon(p; \alpha, \beta) \in \{\pm 1\} \) be the local intersection number of \( \alpha \) and \( \beta \) at \( p \). Then the bracket of \( \alpha \) and \( \beta \) is given by

\[
[\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta)[\alpha_p\beta_p] \in \mathbb{Q}\hat{\pi}.
\]

Remark that if the loops \( \alpha \) and \( \beta \) are freely homotopic to disjoint curves, then \( [\alpha, \beta] = 0 \).

We recall a main result of \( 5 \), which relates the Goldman Lie algebra of \( \Sigma \) and Kontsevich’s “associative”. Actually, we can give a slightly generalized form by virtue of Proposition \( 2.2.4 \).

**Theorem 2.3.1** \( 5 \) Theorem 1.2.1. Let \( \theta \) be a Magnus expansion of \( \pi \) satisfying \( \theta(\zeta) = \exp(\omega) \). Then the map

\[
-N\theta: \mathbb{Q}\hat{\pi} \to N(\hat{T}_1) = a_\gamma, \pi \ni x \mapsto -N\theta(x) \in N(\hat{T}_1)
\]

is a well-defined Lie algebra homomorphism. The kernel is the subspace \( \mathbb{Q}1 \) spanned by the constant loop 1, and the image is dense in \( N(\hat{T}_1) = a_\gamma \) with respect to the \( \hat{T}_1 \)-adic topology.

**Proof.** If \( \theta \) is symplectic, this is \( 5 \) Theorem 1.2.1. We just remark that in the proof of \( 5 \) Theorem 1.2.1, we use (co)homology theory of Hopf algebras, hence we need \( \theta \) to be group-like.

Fix a symplectic expansion \( \theta \) and let \( \theta' \) be a Magnus expansion satisfying \( \theta'(\zeta) = \exp(\omega) \). Then there exists \( U \in \text{IA}_\omega(\hat{T}) \) such that \( \theta' = U \circ \theta \) (see \( 5 \) §2.8). The map \( a_\gamma \to a_\gamma, D \to U \circ D \circ U^{-1} \) is a Lie algebra automorphism, and for any \( x \in \pi \) we have

\[
U \circ (-N\theta(x)) \circ U^{-1} = -N(U\theta(x)) = -N\theta'(x),
\]

by Proposition \( 2.2.4 \). This completes the proof.

\[\square\]
There is an action of the mapping class group $\mathcal{M}_{g,1}$ on the Goldman Lie algebra $\mathfrak{q}\hat{\pi}$. The action is induced from the action on $\pi$.

**Theorem 2.3.2.** Let $\theta$ be a Magnus expansion of $\pi$ satisfying $\theta(\zeta) = \exp(\omega)$. Then the mapping class group $\mathcal{M}_{g,1}$ acts on $\mathfrak{a}_g^-$ by $f \cdot D = T^\theta(f) \circ D \circ T^\theta(f)^{-1}$, where $f \in \mathcal{M}_{g,1}$, $D \in \mathfrak{a}_g^-$. Moreover, the Lie algebra homomorphism $-N\theta: \mathfrak{q}\hat{\pi} \to \mathfrak{a}_g^-$ in Theorem 2.3.1 is $\mathcal{M}_{g,1}$-equivariant.

**Proof.** It suffices to prove that $-N\theta: \mathfrak{q}\hat{\pi} \to \mathfrak{a}_g^-$ is $\mathcal{M}_{g,1}$-equivariant. Let $x \in \pi$ and $f \in \mathcal{M}_{g,1}$. Since $\theta(\zeta) = e^\omega$, the total Johnson map $T^\theta(f)$ satisfies $T^\theta(f) e^\omega = T^\theta(f) \theta(x) = \theta(f(x)) = \theta(\zeta) = e^\omega$, hence $T^\theta(f) e = \omega$. By Proposition 2.2.4, we have $-N\theta(f(x)) = -N(T^\theta(f) \theta(x)) = T^\theta(f) \circ (-N\theta(x)) \circ T^\theta(f)^{-1}$.

This completes the proof. □

3 A generalization of Dehn twists

In this section we first recall another main result of [5], which describes the total Johnson map of the Dehn twist along a simple closed curve. Motivated by this result, we introduce a generalization of Dehn twists for not necessarily simple loops on $\Sigma$, as automorphisms of the completed group ring of $\pi$.

3.1 The logarithms of Dehn twists

For a Magnus expansion $\theta$ of $\pi$, we denote $\ell^\theta := \log \theta$. This is a map from $\pi$ to $\mathfrak{t}T_1$. Note that the logarithm is defined on the set $1 + \mathfrak{t}T_1$.

**Definition 3.1.1.** For $x \in \pi$, set

$$L^\theta(x) := \frac{1}{2} N(\ell^\theta(x) \ell^\theta(x)) \in \hat{T}_2.$$  

In fact, $L^\theta$ descends to an invariant of unoriented loops on $\Sigma$.

**Lemma 3.1.2 ([5] Lemma 2.6.4).** For $x, y \in \pi$, we have

1. $L^\theta(x^{-1}) = L^\theta(x)$,
2. $L^\theta(yxy^{-1}) = L^\theta(x)$.

Hence for any unoriented loop $\gamma \subset \Sigma$, we can define $L^\theta(\gamma)$ as $L^\theta(x)$, where $x \in \pi$ is freely homotopic to $\gamma$. As in § 2.2, we regard $L^\theta(x) \in \hat{T}_2$ as a derivation on $\hat{T}$.

For a simple closed curve $C$ on $\Sigma$, we denote by $t_C \in \mathcal{M}_{g,1}$ the right handed Dehn twist along $C$. The following was proved in [5]. In fact what is given here is a slightly generalized form.

**Theorem 3.1.3 ([5], Theorem 1.1.1).** Let $\theta$ be a Magnus expansion of $\pi$ satisfying $\theta(\zeta) = \exp(\omega)$, and $C$ a simple closed curve on $\Sigma$. Then the total Johnson map $T^\theta(t_C)$ is described as

$$T^\theta(t_C) = e^{-L^\theta(C)}.$$  

Here, the right hand side is the algebra automorphism of $\hat{T}$ defined by the exponential of the derivation $-L^\theta(C)$.  

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Proof. If \( \theta \) is symplectic, this is \[3\] Theorem 1.1.1. Fix a symplectic expansion \( \theta \) and let \( \theta' \) be a Magnus expansion satisfying \( \theta'(\zeta) = \exp(\omega) \). As in the proof of Theorem \[2, 3\] there exists \( U \in \text{IA}_\omega(\hat{T}) \) such that \( \theta' = U \circ \theta \). Note that this condition implies \( \ell^{\theta'} = U \circ \ell^\theta \). Then for any \( \varphi \in \mathcal{M}_{g,1} \), we have \( T^{\theta'}(\varphi) = U \circ T^\theta(\varphi) \circ U^{-1} \), since \( T^{\theta'}(\varphi) \circ \theta' = \theta' \circ \varphi = U \circ \theta \circ \varphi = U \circ T^\theta(\varphi) \circ \theta = U \circ T^\theta(\varphi) \circ U^{-1} \circ \theta' \) and the linear span of the image \( \theta'(\pi) \) is dense in \( \hat{T} \) (see \[3\]). On the other hand, for \( x \in \pi \) we have

\[
L^{\theta'}(x) = \frac{1}{2} N(\ell^{\theta'}(x) \ell^{\theta'}(x)) = \frac{1}{2} N(U \ell^{\theta}(x) U \ell^{\theta}(x)) = \frac{1}{2} N(U \ell^{\theta}(x) \ell^{\theta}(x))
\]

by Proposition \[2.2.1\] hence \( e^{-L^{\theta'}(C)} = U \circ e^{-L^{\theta}(C)} \circ U^{-1} \). Now the formula for \( \theta' \) follows from the formula for \( \theta \).

Remark 3.1.4. Fix a Magnus expansion \( \theta \) of \( \pi \) satisfying \( \theta(\zeta) = \exp(\omega) \). Let \( f \in \mathcal{M}_{g,1} \) and \( C \) a simple closed curve on \( \Sigma \). By an argument similar to the proof of \( L^{\theta'}(x) = U \circ L^\theta(x) \circ U^{-1} \) above, we have \( L^{\theta}(f(C)) = T^\theta(f) \circ L^\theta(C) \circ T^\theta(f)^{-1} \). Therefore we might expect a possibility of another proof of \[3\] Theorem 1.1.1. For example, let us restrict our attentions to non-separating simple closed curves. Any two non-separating simple closed curves are in the same orbit of the action of \( \mathcal{M}_{g,1} \) on the set of unoriented loops (up to homotopy) on \( \Sigma \). If one could prove the formula \( T^\theta(t_C) = e^{-L^\theta(C)} \) for a particular choice of a Magnus expansion \( \theta \) satisfying \( \theta(\zeta) = \exp(\omega) \), say one of the symplectic expansions in \[3, 4, 7, 9\], and a particular choice of a simple closed curve \( C \), then the formula for any such \( \theta \) and any \( C \) non-separating follows.

3.2 Generalized Dehn twists

Now we introduce generalized Dehn twists. Let \( \theta \) be a Magnus expansion of \( \pi \) satisfying \( \theta(\zeta) = \exp(\omega) \). We denote by \( \text{Aut}(\hat{\pi}) \) the group of filter-preserving algebra automorphisms of \( \hat{\pi} \), which is isomorphic to \( \text{Aut}(\hat{T}) \) as a group through \( \theta \).

Let \( \gamma \) be an unoriented loop on \( \Sigma \). Then the exponential \( e^{-L^\theta(\gamma)} \) of the derivation \( -L^\theta(\gamma) \) is a filter-preserving algebra automorphism of \( \hat{T} \). Thus the map

\[
t_\gamma := \theta^{-1} \circ e^{-L^\theta(\gamma)} \circ \theta
\]

lies in \( \text{Aut}(\hat{\pi}) \). As we see in the proof of Theorem \[3.1.3\] if \( \theta' \) is another Magnus expansion satisfying \( \theta'(\zeta) = \exp(\omega) \) then \( L^{\theta'}(\gamma) = U \circ L^\theta(\gamma) \circ U^{-1} \), where \( U \in \text{Aut}_\omega(\hat{T}) \) satisfies \( \theta' = U \circ \theta \). This shows that \( t_\gamma \) is actually independent of the choice of \( \theta \).

Definition 3.2.1. We call \( t_\gamma \in \text{Aut}(\hat{\pi}) \) the generalized Dehn twist along \( \gamma \).

We remark that the generalized Dehn twists have the following natural property. Let \( f \in \mathcal{M}_{g,1} \). Then we have \( L^\theta(f(\gamma)) = T^\theta(f) \circ L^\theta(\gamma) \circ T^\theta(f)^{-1} \) (see Remark 3.1.4), thus

\[
t_{f(\gamma)} = f \circ t_\gamma \circ f^{-1}.
\]

Since we have a natural injective homomorphism

\[
\mathcal{M}_{g,1} \hookrightarrow \text{Aut}(\pi) \hookrightarrow \text{Aut}(\hat{\pi})
\]

we can ask whether \( t_\gamma \) gives an element of \( \mathcal{M}_{g,1} \).

Definition 3.2.2. Let \( \gamma \) be an unoriented loop on \( \Sigma \). We say \( t_\gamma \in \text{Aut}(\hat{\pi}) \) is a mapping class if \( t_\gamma \) lies in the image of \( \mathcal{M}_{g,1} \).
For example, for any simple closed curve $C$, the generalized Dehn twist $t_C$ is a mapping class, and it is the usual right handed Dehn twist along $C$ by Theorem 3.1.3. Moreover, for any power of $C$, the generalized Dehn twist along it is a mapping class. For, we have $L^\theta(C^m) = m^2 L^\theta(C)$ hence $t_{C^m} = (t_C)^{m^2}$, where $m \in \mathbb{Z}$.

### 3.3 The support of a generalized Dehn twist

We shall give a criterion for the realizability of $t_\gamma$ as a mapping class. We use the following, which would be well-known to experts. Similar statements can be found in several literatures, but we do not find a suitable reference.

**Lemma 3.3.1.** Let $S$ be a compact connected oriented surface, and $\varphi$ an orientation preserving diffeomorphism of $S$ fixing the boundary $\partial S$ pointwise. If $\varphi$ preserves every homotopy class of oriented loops on $S$, then $\varphi$ is isotopic relative to the boundary $\partial S$ to a product of boundary-parallel Dehn twists. Here a boundary-parallel Dehn twist is meant a Dehn twist along a simple closed curve which is parallel to one of the components of $\partial S$.

**Proof.** Let $g$ be the genus of $S$ and $r$ the number of the boundary components of $S$. In case $S$ is the 2-sphere or a disk or an annulus, the assertion is trivial. Thus we may assume if $g = 0$, $r \geq 3$. Then we can choose a collection $\{C_i\}_{i}$ of $2g + r$ (if $g = 0$, then $r - 1$) simple closed curves on $S$ satisfying the three properties from [1] Proposition 2.8, see Figure 3. By [1] Lemma 2.9, we can deform $\varphi$ by an isotopy into a diffeomorphism fixing the union $\bigcup_i C_i$ pointwise. The complement $S \setminus \bigcup_i C_i$ is a disjoint union of three disks and $r$ annuli. The restriction of $\varphi$ to each disk component is isotopic relative to the boundary to the identity, and the restriction to each annulus component is isotopic relative to the boundary to a power of the Dehn twist along a simple closed curve parallel to the boundary of the annulus. Thus we can deform $\varphi$ by an isotopy into a product of boundary-parallel Dehn twists. This completes the proof. □

![Figure 3: $\{C_i\}_{i}$ for $g = 2, r = 3$](image)

**Theorem 3.3.2.** Let $\gamma$ be an unoriented loop on $\Sigma$ and suppose the generalized Dehn twist $t_\gamma$ is a mapping class. Then $t_\gamma \in \mathcal{M}_{g,1}$ is represented by a diffeomorphism whose support lies in a regular neighborhood of $\gamma$.

**Proof.** We claim that if $\delta$ is an oriented loop on $\Sigma$ disjoint from $\gamma$, then $t_\gamma(\delta) = \delta$. Let $x \in \pi$ be a representative of $\delta$. By Theorem 2.3.2

\[
-N\theta(t_\gamma(x)) = t_\gamma \cdot (-N\theta(x)) = T^\theta(t_\gamma) \circ (-N\theta(x)) \circ T^\theta(t_\gamma)^{-1} = e^{-L^\theta(\gamma)} \circ (-N\theta(x)) \circ e^{L^\theta(\gamma)} = \sum_{m=0}^{\infty} \frac{1}{m!} \text{ad}(-L^\theta(\gamma))^m (-N\theta(x)).
\]
But $-L^\theta(\gamma) = (1/2)N((\log \theta(\gamma))^2) = \sum_{n=1}^{\infty} a_n N\theta((\gamma - 1)^n)$, where $\gamma \in \pi$ is a representative of $\gamma$ and $\sum_{n=1}^{\infty} a_n (z - 1)^n$ is the Taylor expansion of $(1/2)(\log z)^2$ at $z = 1$. Since $\gamma$ and $\delta$ are disjoint, the Goldman bracket $[\gamma^\alpha, \delta]$ is 0 for $n \geq 0$. By Theorem 2.3.1 we have $[-\theta(\gamma)^n], -\theta(\gamma)] = 0$ for $n \geq 0$. Thus we obtain $[-L^\theta(\gamma), -\theta(\gamma)] = 0$ and $-\theta(t^m(\gamma)) = -\theta(\gamma^m)$. By Theorem 2.3.1 we have $t^m(\gamma(x)) = t^m(\gamma(\delta)) - \delta \in \mathbb{Q}$. Since the action of $M_{\mathbb{Q},1}$ on $\hat{\mathbb{Q}}$ preserves the augmentation $\hat{\mathbb{Q}} \to \mathbb{Q}$, $\hat{\mathbb{Q}} \ni x \to 1$, we conclude $t^m_\gamma(\delta) = \delta$. The claim is proved.

Let $\Sigma$ be a closed regular neighborhood of $\gamma$. By the claim, each oriented component of the boundary $\partial \Sigma$ is preserved by $t^m_\gamma$. By an isotopy, we may assume $t^m_\gamma$ is represented by a diffeomorphism $\varphi$ fixing $\partial \Sigma$ pointwise. Also, if $\delta$ is an oriented loop on $\Sigma \setminus \text{Int}(N)$ then $t^m_\gamma(\delta)$ is ambient isotopic to $\delta$. Since $\partial \Sigma$ is preserved by $t^m_\gamma$, this ambient isotopy can be chosen to have its support in $\Sigma \setminus \text{Int}(N)$. By Lemma 3.3.1 applying a suitable isotopy to $\varphi$ we can write $\varphi = \varphi' \circ t^m_\zeta$, $m \in \mathbb{Z}$, where $\varphi'$ is a diffeomorphism whose support lies in $N$, and $t^m_\zeta$ is the Dehn twist along a simple closed curve parallel to $\partial \Sigma$. It should be remarked that $\partial(\Sigma \setminus \text{Int}(N)) = \partial(\Sigma \setminus \partial N)$. Let $\Sigma'$ be the connected component of $\Sigma \setminus \text{Int}(N)$ containing $\partial \Sigma$. If $\Sigma'$ is an annulus, then $t^m_\zeta$ is isotopic to a diffeomorphism whose support lies in $\Sigma$, so is $\varphi$.

Finally, we show that if $\Sigma'$ is not an annulus, then $m$ should be 0. We can choose a based loop $y$ on $\Sigma'$, freely homotopic to a simple closed curve, so that $[y] \in H = H_1(\Sigma; \mathbb{Q})$ is zero. If $\Sigma'$ is not an annulus, then $y$ is a boundary component $D \subset \partial \Sigma'$ which is non-separating on $\Sigma$, because $\Sigma'$ is not an annulus and $N(\gamma)$ is connected. We can take $y$ as a simple based loop on $\Sigma'$ representing $D$. Since $[y]$ is primitive, we can take symplectic basis $A_1, B_1, \ldots, A_g, B_g \in H$ such that $[y] = A_1$. Now we have $t^m_\gamma(y) = \varphi' \circ t^m_\zeta(y) = t^m_\zeta(y)$. Choosing a symplectic expansion $\theta$ and applying $\theta$ to this equation, we have

$$e^{-L^\theta(\gamma)}(\theta(y)) = T^\theta(t^m_\gamma(y)) = \theta(t^m_\zeta(y)) = T^\theta(t^m_\zeta(y)) = e^{-mL^\theta(\zeta)}\theta(y).$$

Taking the logarithm, we get $L^\theta(\gamma)\theta(y) = mL^\theta(\zeta)\theta(y)$. By [5] Corollary 6.5.3, we have $L^\theta(\zeta)\theta(y) = 0$. On the other hand, modulo $T_1$, we have

$$L^\theta(\zeta)\theta(y) = \frac{1}{2} N(\omega \omega)(A_1) = \omega A_1 - A_1 \omega \neq 0,$$

thus $m = 0$. This completes the proof.

## 4 Loops with a single transverse double point

In this section we consider a class of loops on a surface, which we could say the simplest next to simple closed curves. We will call such a loop a figure eight. First we consider the case when the surface is a pair of pants. After that we classify the possible configurations of a figure eight on $\Sigma$.

### 4.1 Figure eight on a surface

**Definition 4.1.1.** Let $S$ be an oriented surface and $\gamma$ an unoriented immersed loop on $S$. We say $\gamma$ is a figure eight on $S$ if its self-intersection consists of a single transverse double point and $\gamma$ is not homotopic to a simple closed curve or a square of a simple closed curve.

A simple closed curve or a square of a simple closed curve can be deformed into an immersed loop with a single transverse double point. But in view of the remark after Definition 3.2.1, we exclude them from the above definition.
4.2 Figure eight on a pair of pants

Let $P$ be a pair of pants. Fixing a base point $* \in \text{Int}(P)$, let $\delta_1, \delta_2, \delta_3$ be simple based loops on $P$ such that each $\delta_i$ is freely homotopic to one of the oriented boundary component of $P$, and $\delta_1 \delta_2 \delta_3 = 1 \in \pi_1(P, *)$. See Figure 4. We denote by $A_i$ the boundary component of $P$ which is freely homotopic to $\delta_i$. Let $\gamma_1, \gamma_2,$ and $\gamma_3$ be immersed loops on $P$ which are homotopic to $\delta_2 \delta_3^{-1}$, $\delta_3 \delta_1^{-1}$, and $\delta_1 \delta_2^{-1}$, respectively. The underlying unoriented loop of $\gamma_i$ is a figure eight.

Figure 4: a pair of pants

![Diagram of a pair of pants](image)

Lemma 4.2.1. Let $\gamma$ be a figure eight on $P$, and $N(\gamma) \subset P$ a closed regular neighborhood of $\gamma$. Then the complement $P \setminus \text{Int}(N(\gamma))$ is a disjoint union of three annuli. Each annulus contains one boundary component of $P$ and one boundary component of $N(\gamma)$. In particular, the inclusion $N(\gamma) \subset P$ is a strong deformation retract.

Proof. The regular neighborhood $N(\gamma)$ is diffeomorphic to a pair of pants. Cutting at the double point, we can divide the loop $\gamma$ into two simple closed curves. Let $C_1$ and $C_2$ be the boundary components of $N(\gamma)$ which are homotopic to these simple closed curves, and $C_3$ the remaining boundary component of $N(\gamma)$. The complement $P \setminus \text{Int}(N(\gamma))$ has six boundary components, $\{A_i, C_i\}$. The Euler characteristic of $P \setminus \text{Int}(N(\gamma))$ is equal to $\chi(P) - \chi(N(\gamma)) = 0$, and each connected component has genus 0.

We denote by $\Sigma_{0,r}$ the compact connected oriented surface of genus 0 with $r$ boundary components. Computing the Euler characteristic, we see that the possible topological types of $P \setminus \text{Int}(N(\gamma))$ are $\Sigma_{0,4} \amalg \Sigma_{0,1} \amalg \Sigma_{0,1} \amalg \Sigma_{0,2} \amalg \Sigma_{0,2} \amalg \Sigma_{0,2}$, or $\Sigma_{0,2} \amalg \Sigma_{0,2} \amalg \Sigma_{0,2} \amalg \Sigma_{0,2}$. For, if $P \setminus \text{Int}(N(\gamma)) \cong \Sigma_{0,r_1} \amalg \cdots \amalg \Sigma_{0,r_k}$, we must have $r_i \leq 6$ and $\sum r_i(2 - r_i) = 0$. Suppose $P \setminus \text{Int}(N(\gamma)) \cong \Sigma_{0,4} \amalg \Sigma_{0,1} \amalg \Sigma_{0,1}$. Since $A_i$ are not homologous to 0 as a homology class of $P$, $A_i$ does not bound a disk. Also, if $C_1$ or $C_2$ bounds a disk, then $\gamma$ is homotopic to a simple closed curve. Therefore neither $C_1$ nor $C_2$ bounds a disk. But there are two disk components in $P \setminus \text{Int}(N(\gamma))$, a contradiction. Suppose $P \setminus \text{Int}(N(\gamma)) \cong \Sigma_{0,3} \amalg \Sigma_{0,2} \amalg \Sigma_{0,1}$. From what we have just seen, the boundary of $\Sigma_{0,1}$ must be $C_3$. Since $A_i$ are not homologous to each other, any two of them do not appear in the boundary of $\Sigma_{0,2}$. Therefore, by a suitable renumbering, we may assume the boundary of $\Sigma_{0,2}$ is $C_1 \amalg C_2$ or $A_1 \amalg C_1$. If $\delta \Sigma_{0,2} \cong C_1 \amalg C_2$, then we can construct a simple closed curve in $\Sigma_{0,2} \cup N(\gamma) \subset P$ meeting $C_1$ in a transverse double point.

This contradicts the fact that the genus of $P$ is 0. If $\delta \Sigma_{0,2} \cong A_1 \amalg C_1$, $\gamma$ is contained in the annulus $(N(\gamma) \cup \Sigma_{0,1}) \cup (\Sigma_{0,2})$, which contains $A_1$ as a strong deformation retract. This implies that $\gamma$ is homotopic to a power of the simple closed curve $A_1$, a contradiction. Therefore, we have $P \setminus \text{Int}(N(\gamma)) \cong \Sigma_{0,2} \amalg \Sigma_{0,2} \amalg \Sigma_{0,2}$.

The remaining part of the lemma follows from the fact $A_i$ are not homologous to each other.

\[ \square \]

Proposition 4.2.2. Any figure eight on $P$ is isotopic relative to the boundary $\partial P$ to one of $\gamma_1$, $\gamma_2$, or $\gamma_3$. 

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Proof. Let \( \gamma \) be a figure eight on \( P \) and \( p \) the unique self intersection of \( \gamma \). By an isotopy, we may assume \( p = * \). Let \( \gamma : [0,1] \to P \) be a parametrization of \( \gamma \) satisfying \( \gamma(0) = \gamma(1/2) = \gamma(1) = * \). Let \( f_1 := \gamma|_{[0,1/2]} \) and \( f_2 := \gamma|_{[1/2,1]} \), and consider the map \( f := f_1 \lor f_2 : S^1 \lor S^1 \to P \). Since \( \gamma \) is not homotopic to a simple closed curve, the image of \( f_i \) are essential simple closed curves on \( P \), thus parallel to one of the boundaries of \( P \). Also, by Lemma 4.2.1, the induced map \( f_* : \pi_1(S^1 \lor S^1) \to \pi_1(P) \) is an isomorphism. By a suitable renumbering and taking the inverse of the parametrization of \( f_1 \) or \( f_2 \), we may assume \( (f_1, f_2) \) are free homotopic to one of \( (\delta_1, \delta_2) \), \( (\delta_2, \delta_3) \), or \( (\delta_3, \delta_1) \), respectively.

Suppose \( f_1 \) and \( f_2 \) are free homotopic to \( \delta_1 \) and \( \delta_2 \), respectively. Note that \( \pi_1(P, *) \) is a free group of rank two generated by \( \delta_1 \) and \( \delta_2 \). If we regard \( f_1 \) and \( f_2 \) as elements of \( \pi_1(P, *) \), the endomorphism defined by \( \delta_1 \mapsto f_1 \) and \( \delta_2 \mapsto f_2 \) is an isomorphism which act trivially on the abelianization of \( \pi_1(P, *) \). But it is classically known that such an isomorphism is in fact an inner automorphism, see [8] §3.5, Corollary N4. Thus there exists \( x \in \pi_1(P, *) \) such that \( f_1 = x^{-1}\delta_1 x \), \( f_2 = x^{-1}\delta_2 x \).

Representing \( x \) as a loop based at \( * \), we regard \( x \) as an isotopy of \( * \). Let \( \{\Psi_t\}_{t \in [0,1]} \) be an ambient isotopy of \( P \) relative to \( \partial P \) extending \( x \). Then \( \Psi_1(f_1) = \delta_1 \) and \( \Psi_1(f_2) = \delta_2 \). This implies \( \Psi_1(\gamma) \) is isotopic to \( \gamma_3 \), so is \( \gamma = \Psi_0(\gamma) \).

By the same way, if \( (f_1, f_2) \) are free homotopic to \( (\delta_2, \delta_3) \) or \( (\delta_3, \delta_1) \), we conclude \( \gamma \) is isotopic to \( \gamma_1 \) or \( \gamma_2 \), respectively. This completes the proof.

4.3 Figure eight on the surface \( \Sigma \)

Lemma 4.3.1. Let \( Q \) be a closed subset of \( \Sigma \) which lies in \( \text{Int}(\Sigma) \) and is diffeomorphic to a pair of pants. Then the pair \( (\Sigma, Q) \) is diffeomorphic to one of the pairs in Figure 5.

Figure 5: the pair \( (\Sigma, Q) \)

- **case 1**

- **case 2** \( (1 \leq h \leq g) \)

- **case 3** \( (1 \leq h \leq g) \)

- **case 4** \( (k_1, k_2, h \geq 0, k_1 + k_2 + h = g) \)
Proof. The assertion is obtained by the classification theorem of surfaces. Consider the complement $\Sigma' := \Sigma \setminus \text{Int}(Q)$. This is a compact oriented surface with four boundary components. The number of connected components is 1, 2, or 3. Let $\Sigma_{h,r}$ be a compact connected oriented surface of genus $h$ with $r$ boundary components. By computing the Euler characteristic of each component, we can determine the topological types of $\Sigma'$. If $|\pi_0(\Sigma')| = 1$, then $\Sigma' \cong \Sigma_{g,4}$. This is the case 1. If $|\pi_0(\Sigma')| = 2$, $\Sigma' \cong \Sigma_{h-1,2} \amalg \Sigma_{g-h,2}$ or $\Sigma' \cong \Sigma_{h-1} \amalg \Sigma_{g-h,3}$ where $1 \leq h \leq g$ and $\Sigma_{g-h,2}$ or $\Sigma_{g-h,3}$ is the connected component containing $\partial \Sigma$. This is the case 2 or 3, respectively. If $|\pi_0(\Sigma')| = 3$, then $\Sigma' \cong \Sigma_{k_1,1} \amalg \Sigma_{k_2,1} \amalg \Sigma_{h,2}$, where $k_1, k_2, h \geq 0$, $k_1 + k_2 + h = g$, and $\Sigma_{h,2}$ is the connected component containing $\partial \Sigma$. This is the case 4. 

Figure 6: the pair $(\Sigma, \gamma)$

case I

case II-a ($1 \leq h \leq g$)  

case II-b ($1 \leq h \leq g$)

case III-a ($2 \leq h \leq g$)  

case III-b ($2 \leq h \leq g$)

case IV-a ($k_1, k_2 > 0$)

case IV-b ($k_1, k_2 > 0$)
Proposition 4.3.2. Let \( \gamma \) be a figure eight on \( \Sigma \). Then the pair \((\Sigma, \gamma)\) is diffeomorphic to one of the pairs in Figure 6.

Proof. Let \( N(\gamma) \) be a closed regular neighborhood of \( \gamma \). By Lemma 4.3.1, the pair \((\Sigma, N(\gamma))\) is diffeomorphic to one of the cases in Figure 5. If we identify \( N(\gamma) \) with \( P \), then \( \gamma \subset N(\gamma) \cong P \) is isotopic to one of the \( \gamma_i \), by Proposition 4.2.2. Note that the choice of \( \gamma_i \) corresponds to the choice of two of the boundary components of \( P \). As we did in Lemma 4.2.1, let \( C_1 \) and \( C_2 \) be the boundary components of \( N(\gamma) \) which are homotopic to the simple closed curves obtained by dividing \( \gamma \) at its unique self intersection point.

In the case 1, the curves \( C_i \) are all non-separating. Therefore, we can arrange that \( C_1 \) and \( C_2 \) correspond to undotted circles in the case 1. This is the case I. In the case 2, if \( C_1 \) and \( C_2 \) are non-separating, this is the case II-a. The other case is the case II-a. The case 3 is treated similarly, and we have the cases III-a or III-b. Note that in this case \( h = 1 \) is excluded, since \( \gamma \) is not homotopic to a power of a simple closed curve. In the case 4, if none of \( C_1 \) or \( C_2 \) appears on the boundary of \( \Sigma_{h,2} \), this is the case IV-a. The other case is the case IV-b. Again, to ensure \( \gamma \) to be a figure eight, we need \( k_1 \) and \( k_2 \) to be positive. This completes the proof.

We shall call a figure eight \( \gamma \) on \( \Sigma \) non-separating if \( \Sigma \setminus \text{Int}(N(\gamma)) \) is connected, and separating if \( \Sigma \setminus \text{Int}(N(\gamma)) \) is not connected. In Figure 6, the case I is non-separating, and the others are separating.

5 Proof of the main theorem

In this section we show the main theorem of this paper.

5.1 Statement and outline of proof

Theorem 5.1.1. Let \( \gamma \) be a figure eight on \( \Sigma \). Then the generalized Dehn twist \( t_\gamma \) is not a mapping class in the sense of Definition 3.2.2.

Let \( \gamma \subset \Sigma \) be a figure eight and \( p \in \gamma \) the unique self intersection of \( \gamma \). Let \( \gamma: [0,1] \to \Sigma \) be a parametrization of \( \gamma \) such that \( \gamma(0) = \gamma(1/2) = \gamma(1) = p \). Taking a path \( \delta \) from \( * \in \partial \Sigma \) to \( p \), we denote

\[
x := \delta \cdot \gamma|_{[0,1/2]} \cdot \delta^{-1}, \quad y := \delta \cdot (\gamma|_{[1/2,1]})^{-1} \cdot \delta^{-1} \in \pi.
\]

Let \( N(\gamma) \) be a closed regular neighborhood of \( \gamma \). Then \( N(\gamma) \) is diffeomorphic to a pair of pants. We denote by \( C_1, C_2, \) and \( C_3 \) the boundary component of \( N(\gamma) \) freely homotopic to \( x, y, \) and \( xy \), respectively. Note that \( \gamma \) is freely homotopic to \( xy^{-1} \).

For each configuration of a figure eight given in Proposition 4.3.2, \( x \) and \( y \) can be represented in terms of symplectic generators, as Table 1. Here we identify the surfaces in Figure 6 with the surface in Figure 2 by a natural way, and the parametrization of \( \gamma \) is indicated in Figure 6.

|   | \( x \) | \( y \) |
|---|---|---|
| I | \( \alpha_1 \) | \( \alpha_2 \) |
| II-a | \((\prod_{i=1}^{h}[\alpha_i, \beta_i])\beta_h^{-1} \)
| II-b | \((\prod_{i=1}^{h}[\alpha_i, \beta_i])\alpha_h\beta_h^{-1} \)
| III-a | \((\prod_{i=1}^{h}[\alpha_i, \beta_i])\beta_h^{-1} \)
| III-b | \((\prod_{i=1}^{h}[\alpha_i, \beta_i])\alpha_h \)
| IV-a | \(\prod_{i=1}^{k_1}[\alpha_i, \beta_i] \)
| IV-b | \(\prod_{i=1}^{k_1}[\alpha_i, \beta_i] \)
The proof of Theorem 5.1.1 depends on Proposition 4.3.2 and explicit computations of the invariant \( L^\theta \) for \( x, y, xy, \) and \( xy^{-1} \). Suppose \( t_\gamma \) is a mapping class. By Theorem 3.3.2, \( t_\gamma \) is represented by a diffeomorphism whose support lies in \( N(\gamma) \). It is known that the mapping class group of a pair of pants is the free abelian group of rank three generated by the boundary-parallel Dehn twists. See for example, [1] §3.6. It follows that there exist \( m_1, m_2, m_3 \in \mathbb{Z} \) such that

\[ t_\gamma = t_{C_1}^{m_1} t_{C_2}^{m_2} t_{C_3}^{m_3}. \]  

(5.1.1)

Let us choose a Magnus expansion \( \theta \) satisfying \( \theta(\zeta) = \exp(\omega) \). Since the Dehn twists \( t_{C_1}, t_{C_2}, \) and \( t_{C_3} \) commute with each other, so do the derivations \( L^\theta(C_1), L^\theta(C_2), \) and \( L^\theta(C_3) \). Therefore, applying \( T^\theta \) to (5.1.1) and taking the logarithm, we obtain

\[ L^\theta(xy^{-1}) = m_1 L^\theta(x) + m_2 L^\theta(y) + m_3 L^\theta(xy). \]  

(5.1.2)

In fact, for any configuration of a figure eight, we get

\[ t_\gamma = t_{C_1}^{2} t_{C_2}^{2} t_{C_3}^{-1} \]

as an intermediate result, see Proposition 5.2.6. Looking at this equation in higher degree leads us to a contradiction. To do this, we will need the lower terms of the invariant \( L^\theta \). For simplicity we write \( \ell = \ell^\theta, L = L^\theta \) and

\[ \ell(x) = \sum_{m=1}^{\infty} \ell_m(x), \quad L(x) = \sum_{m=2}^{\infty} L_m(x), \]

where \( \ell_m(x), L_m(x) \in H^{\otimes m} \). We have

\[ L_m(x) = \frac{1}{2} \sum_{i=1}^{m-1} N(\ell_i(x)\ell_{m-i}(x)). \]

By the Baker-Campbell-Hausdorff formula, in the completed free Lie algebra generated by variables \( u \) and \( v \), we have

\[
\log(\exp u \exp v) = u + v + \frac{1}{2}[u,v] + \frac{1}{12}[u-v,[u,v]] - \frac{1}{24}[u,[v,[u,v]]] + \text{(higher terms)}. \tag{5.1.3}
\]

For \( x, y \in \pi \), we denote \( X = [x], Y = [y] \in H \). By (5.1.3), we get the lower terms of \( \ell(xy) = \log(\theta(x)\theta(y)) \). For example, we have

\[
\begin{align*}
\ell_1(xy) &= \ell_1(x) + \ell_1(y) \\
\ell_2(xy) &= \ell_2(x) + \ell_2(y) + \frac{1}{2}[X,Y] \\
\ell_3(xy) &= \ell_3(x) + \ell_3(y) + \frac{1}{2}([X,\ell_2(y)] + [\ell_2(x),Y]) + \frac{1}{12}[X - Y, [X,Y]],
\end{align*} \tag{5.1.4}
\]

and if \( [X,Y] = 0 \), we have

\[
\begin{align*}
\ell_2(xy) &= \ell_2(x) + \ell_2(y) \\
\ell_3(xy) &= \ell_3(x) + \ell_3(y) + \frac{1}{2}([X,\ell_2(y)] + [\ell_2(x),Y]) \\
\ell_4(xy) &= \ell_4(x) + \ell_4(y) + \frac{1}{2}([X,\ell_3(y)] + [\ell_2(x),\ell_2(y)] + [\ell_3(x),Y]) + \frac{1}{12}[X - Y, [X,\ell_2(y)] + [\ell_2(x),Y]]. \tag{5.1.5}
\end{align*}
\]

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and
\[\ell_5(xy) = \ell_5(x) + \ell_5(y) + \frac{1}{2}([X, \ell_4(y)] + [\ell_2(x), \ell_3(y)] + [\ell_3(x), \ell_2(y)] + [\ell_4(x), Y]) + \frac{1}{12}[\ell_2(x) - \ell_2(y), [X, \ell_2(y)] + [\ell_2(x), Y]] + \frac{1}{12}[X - Y, [X, \ell_3(y)] + [\ell_2(x), \ell_2(y)] + [\ell_3(x), Y]] - \frac{1}{24}[X, [Y, [\ell_2(x)] + [\ell_2(x), Y]]].\] (5.1.6)

### 5.2 determination of coefficients

The goal of this subsection is to prove Proposition \[\text{(5.2.0)}\]

**Lemma 5.2.1.**

\[L_2(xy) = L_2(x) + L_2(y) + N(XY)\]
\[L_2(xy^{-1}) = L_2(x) + L_2(y) - N(XY)\]

**Proof.** Since \(\ell_1(x) = [x] = X\),
\[L_2(xy) = \frac{1}{2}N(\ell_1(xy)\ell_1(xy)) = \frac{1}{2}N((X + Y)(X + Y)) = \frac{1}{2}(N(XX) + N(YY) + N(XY) + N(YX)) = L_2(x) + L_2(y) + N(XY).\]

Here we use Lemma \[\text{(2.2.1)}\] The other one follows from the first one by replacing \(y\) with \(y^{-1}\). \[\square\]

**Lemma 5.2.2.** Suppose \([X, Y] = 0\). Then
\[L_4(xy) = L_4(x) + L_4(y) + N(X\ell_3(y) + Y\ell_3(x)) + N(\ell_2(x)\ell_2(y))\]
\[L_4(xy^{-1}) = L_4(x) + L_4(y) - N(X\ell_3(y) + Y\ell_3(x)) - N(\ell_2(x)\ell_2(y))\]

**Proof.** By (5.1.5), we have
\[L_4(xy) = N(\ell_1(xy)\ell_3(xy)) + \frac{1}{2}N(\ell_2(xy)\ell_2(xy)) = N((X + Y)(\ell_3(x) + \ell_3(y)) + \frac{1}{2}([X, \ell_2(y)] + [\ell_2(x), Y])) + \frac{1}{2}N((\ell_2(x) + \ell_2(y))(\ell_2(x) + \ell_2(y))).\]

By Lemma \[\text{(2.2.1)}\] we have \(N(X[\ell_2(y)]) = N([X, X]\ell_2(y)) = 0\), \(N(X[\ell_2(x), Y]) = -N(X[Y, \ell_2(x)]) = -N([X, Y]\ell_2(x)) = 0\) (using \([X, Y] = 0\), etc. Therefore
\[L_4(xy) = N((X + Y)(\ell_3(x) + \ell_3(y)) + \frac{1}{2}N((\ell_2(x) + \ell_2(y))(\ell_2(x) + \ell_2(y))).\]

Expanding the right hand side, we obtain the first formula. The other one follows from the first one by replacing \(y\) with \(y^{-1}\). \[\square\]
To advance our computation we need explicit values of a symplectic expansion. In [9], Massuyeau gave some lower terms of a symplectic expansion. If we denote it by $\theta^0$, then the values of $\ell^0 = \log \theta^0$ on symplectic generators are as follows: modulo $T_4$,

$$
\ell^0(a_i) = A_i + \frac{1}{2}[A_i, B_i] - \frac{1}{12}[B_i, [A_i, B_i]] + \frac{1}{2} \sum_{j<i} [A_i, [A_j, B_j]],
$$

$$
\ell^0(\beta_i) = B_i - \frac{1}{2}[A_i, B_i] + \frac{1}{12}[A_i, [A_i, B_i]] + \frac{1}{4}[B_i, [A_i, B_i]] + \frac{1}{2} \sum_{j<i} [B_i, [A_j, B_j]].
$$

Note that our conventions is different from [9] Definition 2.15.

Lemma 5.2.3. Suppose $h \geq 2$. Then the tensors $u_1 := N(\sum_{i=1}^h [A_i, B_i] [A_i, B_i])$, $u_2 := N(\sum_{i=1}^h [A_i, B_i][A_i, B_i])$, and $u_3 := N([A_i, B_i][A_i, B_i])$ are linearly independent.

Proof. Note that the tensors $X_1, \ldots, X_{m}$, $X_{i} \in \{A_i, B_i\}$, constitute a basis of $H^{\otimes m}$. Writing $u_i$ in terms of this basis, we see that the coefficients of $A_1 B_1 A_1 B_1$ in $u_1$, $u_2$, and $u_3$ are 0, 1, and 0, respectively; the coefficients of $A_1 B_1 A_1 B_1$ in $u_1$, $u_2$, and $u_3$ are 2, 1, and 0, respectively; the coefficients of $A_1 B_1 A_1 B_1$ in $u_1$, $u_2$, and $u_3$ are all 4. This shows that $u_1$, $u_2$, and $u_3$ are linearly independent.

Lemma 5.2.4. Let $k_1, k_2 \geq 1$ and set $\omega_1 := \sum_{i=1}^{k_1} [A_i, B_i]$ and $\omega_2 := \sum_{i=k_1+1}^{k_2} [A_i, B_i]$. Then the tensors $N(\omega_1 \omega_1)$, $N(\omega_1 \omega_2)$, and $N(\omega_1 \omega_2)$ are linearly independent.

Proof. This is proved by the same way as in Lemma 5.2.3.

Lemma 5.2.5. Let $x$ and $y$ be one of the pair other than the case 1 in Table 1. We assume $h \geq 2$ in the cases II-a and II-b. Then for $L = L^0$, the tensors $L_4(x)$, $L_4(y)$, and $N(X \ell_5(y) + Y \ell_3(x)) + N(\ell_2(x) \ell_2(y))$ are linearly independent.

Proof. For simplicity we denote $M := N(X \ell_5(y) + Y \ell_3(x)) + N(\ell_2(x) \ell_2(y))$. By a direct computation using (5.2.1), we get the values of $L_4(x)$, $L_4(y)$, and $M$ for $\theta = \theta^0$ as Table 2.

|        | $L_4(x)$                                          | $L_4(y)$                                          |
|--------|---------------------------------------------------|---------------------------------------------------|
| II-a   | $(1/2)u_1 - (1/2)u_2 + (1/24)u_3$                 | $(1/24)u_3$                                       |
| II-b   | $(1/2)u_1 - (1/2)u_2 + (1/24)u_3$                 | $(1/2)u_1$                                       |
| III-a  | $(1/2)u_1 - (1/2)u_2 + (1/24)u_3$                 | $(1/24)u_3$                                       |
| III-b  | $(1/2)u_1 - (1/2)u_2 + (1/24)u_3$                 | $(1/2)u_1 - u_2 - (1/2)u_3$                       |
| IV-a   | $(1/2)N(\omega_1 \omega_1)$                      | $(1/2)N(\omega_2 \omega_2)$                     |
| IV-b   | $(1/2)N(\omega_1 \omega_2)$                      | $(1/2)N(\omega_1 \omega_1) + (1/2)N(\omega_2 \omega_2) + N(\omega_1 \omega_2)$ |

These computations together with Lemmas 5.2.3 and 5.2.4 give the result.
Now we have an intermediate result to deduce a contradiction.

**Proposition 5.2.6.** Suppose \( t_\gamma \) is a mapping class. Then we have \( t_\gamma = t^{2}_{C_1} t^{2}_{C_2} t^{-1}_{C_3} \).

**Proof.** Let \( \theta^0 \) be the symplectic expansion of \( (5.2.1) \) and \( L = L^{\theta^0} \). If \( \gamma \) is non-separating, \( X = [\alpha_1] \) and \( Y = [\alpha_2] \), hence \( L_2(x) = XX, L_2(y) = YY \), and \( N(XY) \) are linearly independent. By Lemma 5.2.1, the degree two part of \( (5.1.2) \) is equivalent to

\[
L_2(x) + L_2(y) - N(XY) = m_1 L_2(x) + m_2 L_2(y) + m_3(L_2(x) + L_2(y) + N(XY))
\]

Comparing the coefficients, we get \( (m_1, m_2, m_3) = (2, 2, -1) \), i.e., \( t_\gamma = t^{2}_{C_1} t^{2}_{C_2} t^{-1}_{C_3} \). Suppose \( \gamma \) is separating. If the configuration of \( \gamma \) is neither II-a nor II-b with \( h = 1 \), the same argument applied to the degree four part of \( (5.1.2) \), together with Lemmas 5.2.2 and 5.2.3 gives the result.

Suppose \( h = 1 \) and the configuration of \( \gamma \) is II-a or II-b. These cases are rather special, since \( C_1 = C_2 \) for the former case and \( C_1 = C_3 \) for the latter. If the configuration of \( \gamma \) is II-a, \( x = \alpha_h \beta_h \alpha^{-1}_h \) and \( y = \beta^{-1}_h \). Then \( L_2(x) = L_2(y) = B_h B_h \) and \( L_2(xy) = 0, L_2(xy^{-1}) = 4B_h B_h \). Looking at the degree two part of \( (5.1.2) \), we get \( m_1 + m_2 = 4 \). Also we have \( L_4(x) = L_4(y) = (1/24)u_3, L_4(xy) = (1/4)u_3, \) and \( L_4(xy^{-1}) = -(1/12)u_3 \). Looking at the degree four part of \( (5.1.2) \), we get \( m_3 = -1 \). Since \( t_{C_1} = t_{C_2} \), we have \( t_\gamma = t^{2}_{C_1} t^{2}_{C_2} t^{-1}_{C_3} \). If the configuration of \( \gamma \) is II-b, then \( x = \alpha_h \beta_h \alpha^{-1}_h \), \( y = [\beta_h, \alpha_h] \). By a similar computation, we get \( m_1 + m_3 = 1 \) and \( m_2 = 2 \). Since \( t_{C_1} = t_{C_3} \), we have \( t_\gamma = t^{2}_{C_1} t^{2}_{C_2} t^{-1}_{C_3} \). This completes the proof. \( \square \)

### 5.3 deduce a contradiction

By Proposition 5.2.6 if \( t_\gamma \) is a mapping class, we have

\[
L(xy) + L(xy^{-1}) = 2L(x) + 2L(y).
\]  \( (5.3.1) \)

Suppose \( \gamma \) is non-separating. Let \( \theta = \theta^0 \) be the symplectic expansion of \( (5.2.1) \). By a straightforward computation, for \( x = \alpha_1 \) and \( y = \alpha_2 \) we have

\[
L_4(xy) + L_4(xy^{-1}) - 2L_4(x) - 2L_4(y) = -\frac{1}{12} N([A_1, A_2][A_1, A_2]) \neq 0.
\]

This contradicts to \( (5.3.1) \), which completes the proof of Theorem 5.1.1 for non-separating \( \gamma \).

Hereafter we assume \( \gamma \) is separating.

**Lemma 5.3.1.** Suppose \( [X, Y] = 0 \). Then

\[
L_6(xy) + L_6(xy^{-1}) - 2L_6(x) - 2L_6(y) = -\frac{1}{12} N([X, \ell_2(y)] + [\ell_2(x), Y])([X, \ell_2(y)] + [\ell_2(x), Y]).
\]  \( (5.3.2) \)

**Proof.** We have

\[
L_6(z) = N(Z\ell_5(z)) + N(\ell_2(z)\ell_4(z)) + \frac{1}{2} N(\ell_3(z)\ell_3(z)).
\]

We denote \( L''_6(z) := N(Z\ell_5(z)), L''_6(z) := N(\ell_2(z)\ell_4(z)), \) and \( L''_6(z) := \frac{1}{2} N(\ell_3(z)\ell_3(z)) \). By
Finally, by (5.1.5),

\[ L'_6(xy) = N((X + Y)\ell_5(xy)) \]
\[ = N((X + Y)(\ell_5(x) + \ell_5(y))) \]
\[ + \frac{1}{2} N((X + Y)(\ell_2(x), \ell_2(y)) + [\ell_3(x), \ell_2(y)] + [\ell_3(y), \ell_2(x)] + [\ell_3(x), \ell_2(y)]) \]
\[ + \frac{1}{12} N((X + Y)[\ell_2(x) - \ell_2(y), [X, \ell_2(y)] + [\ell_3(x), Y]]) \]
\[ + \frac{1}{12} N((X + Y)[X - Y, [X, \ell_2(y)] + [\ell_2(x), \ell_2(y)] + [\ell_3(x), Y]]) \]
\[ - \frac{1}{24} N((X + Y)[X, [X, \ell_2(y)] + [\ell_2(x), Y]]). \]

By Lemma 2.2.1 and \([X, Y] = 0\), the fourth term vanish:

\[ N((X + Y)[X - Y, [X, \ell_3(y)] + [\ell_2(x), \ell_2(y)] + [\ell_3(x), Y]]) \]
\[ = N([X + Y, X - Y][[X, \ell_3(y)] + [\ell_2(x), \ell_2(y)] + [\ell_3(x), Y]]) \]
\[ = 0, \]

so does the fifth term. Also, we have \(N((X + Y)[X, \ell_4(y)]) = N([X + Y, X]\ell_4(y)) = 0\) and \(N((X + Y)[\ell_4(x), Y]) = 0\). Therefore,

\[ L'_6(xy) = N((X + Y)(\ell_5(x) + \ell_5(y))) \]
\[ + \frac{1}{2} N((X + Y)(\ell_2(x), \ell_3(y)] + [\ell_3(x), \ell_2(y)]) \]
\[ + \frac{1}{12} N((X + Y)[\ell_2(x) - \ell_2(y), [X, \ell_2(y)] + [\ell_2(x), Y]]). \]

Next, by (5.1.3),

\[ L'_6(xy) = N(\ell_2(xy)\ell_4(xy)) \]
\[ = N((\ell_2(x) + \ell_2(y))(\ell_4(x) + \ell_4(y))) \]
\[ + \frac{1}{2} N((\ell_2(x) + \ell_2(y))[X, \ell_3(y)] + [\ell_2(x), \ell_2(y)] + [\ell_3(x), Y]) \]
\[ + \frac{1}{12} N((\ell_2(x) + \ell_2(y))[X - Y, [X, \ell_2(y)] + [\ell_2(x), Y]]). \]

By Lemma 2.2.1 we have \(N((\ell_2(x) + \ell_2(y))[\ell_2(x), \ell_2(y)]) = 0\). Therefore,

\[ L'_6(xy) = N((\ell_2(x) + \ell_2(y))(\ell_4(x) + \ell_4(y))) \]
\[ + \frac{1}{2} N((\ell_2(x) + \ell_2(y))[X, \ell_3(y)] + [\ell_3(x), Y]) \]
\[ + \frac{1}{12} N((\ell_2(x) + \ell_2(y))[X - Y, [X, \ell_2(y)] + [\ell_2(x), Y]]). \]

Finally, by (5.1.5),

\[ L''_6(xy) = \frac{1}{2} N(\ell_3(xy)\ell_3(xy)) \]
\[ = \frac{1}{2} N\left( (\ell_3(x) + \ell_3(y) + \frac{1}{2}([X, \ell_2(y)] + [\ell_2(x), Y])) \right)^2 \]
\[ = \frac{1}{2} N((\ell_3(x) + \ell_3(y))(\ell_3(x) + \ell_3(y))) \]
\[ + \frac{1}{2} N((\ell_3(x) + \ell_3(y))[X, \ell_2(y)] + [\ell_2(x), Y])) \]
\[ + \frac{1}{8} N([X, \ell_2(y)] + [\ell_2(x), Y]) \]
Proof. This is proved by the same way as in Lemma 5.2.1 we get

\[ L_6(xy) = N((X + Y)(\ell_5(x) + \ell_5(y)) + N((\ell_2(x) + \ell_2(y))(\ell_4(x) + \ell_4(y))) + \frac{1}{2}N((\ell_3(x) + \ell_3(y))(\ell_3(x) + \ell_3(y))) - \frac{1}{24}N(([X, \ell_2(y)] + [\ell_2(x), Y])([X, \ell_2(y)] + [\ell_2(x), Y])). \]

Summing all the three terms and using Lemma 2.2.1, we get

\[ L_6(xy) = N((X - Y)(\ell_5(x) - \ell_5(y)) + N((\ell_2(x) - \ell_2(y))(\ell_4(x) - \ell_4(y))) + \frac{1}{2}N((\ell_3(x) - \ell_3(y))(\ell_3(x) - \ell_3(y))) - \frac{1}{24}N(([X, \ell_2(y)] + [\ell_2(x), Y])([X, \ell_2(y)] + [\ell_2(x), Y])). \]

Hence

\[ L_6(xy) + L_6(xy^{-1}) = 2N((X - Y)(\ell_5(x) + Y\ell_5(y)) + 2N((\ell_2(x) - \ell_2(y))(\ell_4(x) + \ell_2(y)\ell_4(y))) + 2N((\ell_3(x) + \ell_3(y))(\ell_3(x) + \ell_3(y))) - \frac{1}{12}N(([X, \ell_2(y)] + [\ell_2(x), Y])([X, \ell_2(y)] + [\ell_2(x), Y])). \]

Expanding the right hand side, we get the formula. \( \square \)

Lemma 5.3.2. Let \( h \geq 1 \). Then

\[ N([B_h, \sum_{i=1}^{h} [A_i, B_i]][B_h, \sum_{i=1}^{h} [A_i, B_i]]) \neq 0. \]

If \( h \geq 2 \), then

\[ N([B_h, \sum_{i=1}^{h-1} [A_i, B_i]][B_h, \sum_{i=1}^{h-1} [A_i, B_i]]) \neq 0. \]

Proof. This is proved by the same way as in Lemma 5.2.3 \( \Box \)

Suppose the configuration of \( \gamma \) is one of II-a, II-b, III-a, and III-b. By a straightforward computation, we see the right hand side of (5.3.2) in Lemma 5.3.1 for \( L = L^\theta \) is

\[ -\frac{1}{12}N([B_h, \sum_{i=1}^{h} [A_i, B_i]][B_h, \sum_{i=1}^{h} [A_i, B_i]]) \]

if the configuration of \( \gamma \) is II-a or II-b, and

\[ -\frac{1}{12}N([B_h, \sum_{i=1}^{h-1} [A_i, B_i]][B_h, \sum_{i=1}^{h-1} [A_i, B_i]]) \]

if the configuration of \( \gamma \) is III-a or III-b. By Lemma 5.3.2 this contradicts to (5.3.1).

Finally, we consider the cases IV-a and IV-b.

Lemma 5.3.3. Suppose \( X = Y = 0 \). Then

\[ L_8(xy) + L_8(xy^{-1}) - 2L_8(x) - 2L_8(y) = -\frac{1}{12}N([\ell_2(x), \ell_2(y)][\ell_2(x), \ell_2(y)]). \] (5.3.3)
Proof. This is proved by the same way as in Lemma 5.3.1. We just remark that if $X = Y = 0$, then by (5.1.3)

$$
\ell_2(xy) = \ell_2(x) + \ell_2(y) \\
\ell_3(xy) = \ell_3(x) + \ell_3(y) \\
\ell_4(xy) = \ell_4(x) + \ell_4(y) + \frac{1}{2}[[\ell_2(x), \ell_2(y)]] \\
\ell_5(xy) = \ell_5(x) + \ell_5(y) + \frac{1}{2}[[\ell_2(x), \ell_3(y)] + [\ell_3(x), \ell_2(y)]] \\
\ell_6(xy) = \ell_6(x) + \ell_6(y) + \frac{1}{2}[[\ell_2(x), \ell_4(y)] + [\ell_3(x), \ell_3(y)] + [\ell_4(x), \ell_2(y)]] \\
+ \frac{1}{12}[[\ell_2(x) - \ell_2(y), [\ell_2(x), \ell_2(y)]]].
$$

If the configuration of $\gamma$ is IV-a or IV-b, the right hand side of (5.3.3) for $L = L^a$ is

$$
-\frac{1}{12} N([\sum_{i=1}^{k_1+k_2}[A_i, B_i], \sum_{i=k_1+1}^{k_1+k_2}[A_i, B_i] [\sum_{i=1}^{k_1+k_2}[A_i, B_i], \sum_{i=k_1+1}^{k_1+k_2}[A_i, B_i]]].
$$

By the same way as in Lemma 5.2.3, we see that this tensor of degree eight is not zero. This contradicts to (5.3.1).

Now we have deduced a contradiction for any configuration of a figure eight. This completes the proof of Theorem 5.1.1.

We end this paper by posing a question regarding the characterization of simple closed curves.

**Question 5.3.4.** Let $\gamma$ be an unoriented loop on $\Sigma$ and suppose the generalized Dehn twist $t_\gamma$ is a mapping class. Is $\gamma$ homotopic to a power of a simple closed curve?

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