Efficient quantum algorithm for solving problems with a special structure

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Abstract

Solving a problem on a quantum computer can be transformed to finding the ground state of a problem Hamiltonian that encodes the solution to the problem. We found a method to protect unknown quantum state in an intermediate computation step through quantum entanglement, thus circumvent the restriction of the no-cloning theorem which forbids making copies of an unknown quantum state perfectly. Based on this method, we propose an efficient quantum algorithm for solving problems with a special structure: between an initial and the problem Hamiltonians there exist a sequence of finite number of intermediate Hamiltonians, such that both the overlaps between ground states of any two adjacent Hamiltonians, and the energy gap between the ground and the first excited states of each Hamiltonian are not exponentially small. In comparison, for a specific type of problems where the quantum adiabatic algorithm fails, our algorithm remains to be efficient.

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I. INTRODUCTION

Quantum computing can speedup information processing by harnessing some unique features of quantum mechanics to develop efficient quantum algorithms. Solving a problem on a quantum computer can be transformed to finding the ground state of a problem Hamiltonian that encodes the solution to the problem. The phase estimation algorithm (PEA) [1, 2] projects the ground state of a system out of an initial guess state with success probability proportional to the overlap between them. For a large and complicated system, however, it is very difficult to find a good approximation to its ground state. The quantum adiabatic algorithm (QAA) [3, 4] starts from an initial Hamiltonian $H_0$ whose ground state can be readily prepared, evolve adiabatically to the problem Hamiltonian $H_P$ whose ground state encodes the solution. The QAA has the advantage that, since the evolution Hamiltonian $H(s) = (1 - s)H_0 + sH_P$ includes a single parameter that varies continuously, one simply varies a single parameter to perform computation. The evolution process of the QAA is restricted to a linear interpolation between an initial and the problem Hamiltonians. In some cases, there exists more efficient evolution path that cannot be described by a single parameter.

An analogy of the evolution from an initial Hamiltonian to the problem Hamiltonian is the reaction path in chemical reaction. In theoretical chemistry, a reaction path connects the reactants and the products on the potential energy surface of the chemicals. Sometimes the reaction is difficult to happen because the transition energy along the reaction path is too high. Then instead of going directly from the reactants to the products, one can choose a different path that reaches the products through some intermediate products in a number of steps, and each step has a lower transition energy. Our algorithm is analogous to this method.

A quantum approximate optimization algorithm (QAOA) was developed in ref. [5]. In the QAOA, a quantum state of interest is parameterized using a small set of parameters and prepared on a quantum computer. An objective function whose optimum is sought is efficiently evaluated through quantum measurements. The parameters of the quantum state are then variationally adjusted and optimized on a classical computer according to the outcomes of the quantum measurements. By repeating these procedures iteratively, the algorithm produces approximate solution to some optimization problems. Especially, the
The structure of a problem is considered as the key for whether it can be solved efficiently or not on a quantum computer \([6]\), e.g. the factoring problem \([7]\). In this work, by using a full quantum method, we present an efficient quantum algorithm for solving problems with a special structure. Here the special structure refers to the property that between an initial Hamiltonian \(H_0\) and the problem Hamiltonian \(H_P\), there exist a sequence of finite number of intermediate Hamiltonians \(\{H_l, l = 1, 2, \cdots, m\}\), through which \(H_0\) reaches \(H_P\) as \(H_0 \rightarrow H_1 \rightarrow \cdots \rightarrow H_m = H_P\) such that \(i)\) the overlaps between the ground states of any two adjacent Hamiltonians are not exponentially small; \(ii)\) the energy gap between the ground and the first excited states of each Hamiltonian is not exponentially small. To solve these problems, we start from the ground state \(|\varphi_0\rangle\) of \(H_0\), evolve it through ground states of the intermediate Hamiltonians sequentially, finally reach the ground state \(|\varphi_0^{(m)}\rangle\) of \(H_P\) in \(m\) steps. Here \(m\) characterizes the depth of the circuit for implementing the algorithm. In each step, the ground state of an intermediate Hamiltonian is obtained deterministically in finite time by using a mechanism of simulated quantum resonant transition \([8, 9]\), and is heralded through quantum measurements on a probe qubit. The state is protected through quantum entanglement and can be used repeatedly, thus circumvent the restriction of the no-cloning theorem which forbids making copies of an unknown quantum state perfectly. The procedure of the algorithm is adjusted according the outcomes of the quantum measurements on the probe qubit. The ground state of the problem Hamiltonian is efficiently induced step by step as the intermediate Hamiltonians reach the problem Hamiltonian.

A major difference between the QAA and our algorithm for finding the ground state of
a problem Hamiltonian is illustrated in Fig. 1. While the evolution process of the QAA is restricted to a linear interpolation between an initial and the problem Hamiltonians, our algorithm is more flexible in designing an evolution path. For problems with a special structure that satisfy the above conditions, we can design an efficient evolution path by taking into consideration of the structure of the problem even when the QAA fails. And such a evolution path cannot be described by varying a single parameter.

II. THE ALGORITHM

In our algorithm, for a problem with the special structure, instead of using the global problem Hamiltonian directly, we decompose the problem based on its structure to construct the intermediate Hamiltonians. The solution state to the problem Hamiltonian is induced step by step. In each step, the ground state and the corresponding eigenvalue of an intermediate Hamiltonian are obtained using an optimized algorithm of ref. [9]. We describe the algorithm by using one of its steps as an example.

In the $l$-th step of the algorithm, given the Hamiltonians $H_l$, $H_{l-1}$ and its ground state $|\psi_0^{(l-1)}\rangle$ and the corresponding eigenvalue $E_0^{(l-1)}$ that have been obtained from the previous step, the goal is to obtain the ground state $|\psi_0^{(l)}\rangle$ and the corresponding eigenvalue $E_0^{(l)}$ of $H_l$. The algorithm requires $(n + 1)$ qubits with one probe qubit and an $n$-qubit register $R$ representing the problem of dimension $N = 2^n$. The algorithm Hamiltonian of the $l$-th step is

$$H^{(l)} = -\frac{1}{2}\omega \sigma_z \otimes I_N + H_R^{(l)} + c \sigma_x \otimes I_N,$$

where

$$H_R^{(l)} = \alpha |1\rangle\langle 1| \otimes H_{l-1} + |0\rangle\langle 0| \otimes H_l, \ l = 1, 2, \cdots, m,$$

and $I_N$ is the $N$-dimensional identity operator, $\sigma_x, \sigma_z$ are the Pauli matrices. The first term in Eq. (1) is the Hamiltonian of the probe qubit, the second term contains the Hamiltonian of the register $R$, and the third term describes the interaction between the probe qubit and the register $R$, $\alpha$ is a parameter for setting the energy level of a reference state, and $c \ll 1$ is the coupling strength. The procedures are as follows:

1) Prepare the probe qubit in its excited state $|1\rangle$ and the register $R$ in state $|\psi_0^{(l-1)}\rangle$. The state $|1\rangle|\psi_0^{(l-1)}\rangle$ is an eigenstate of $H_R^{(l)}$ with eigenvalue $\alpha E_0^{(l-1)}$. We estimate the range of the eigenvalue $E_0^{(l)}$ to get the estimated transition frequency range $[\omega_{\text{min}}, \omega_{\text{max}}]$ between states
A resonant transition does not happen. This property allows one to obtain the state \( |\varphi_0^{(l-1)}\rangle \) deterministically by repeating procedures \( \text{ii} \) for a number of times of the procedures needs to be repeated scales as \( \frac{1}{\epsilon} \). The frequency of the probe qubit near the resonant point to obtain accurate value of \( \varphi_0^{(l-1)} \) scales linearly with the number of steps of the algorithm, provided \( \alpha \) is small. The algorithm is efficient even if the overlap \( \langle \varphi_0 | \varphi_0^{(m)} \rangle = 1/2^m \) is exponentially small in some cases, e.g. \( \langle \varphi_0 | \varphi_0^{(m)} \rangle = 1/2^m \) while \( d_0^{(l)} \) in each step is 1/2, the algorithm can be run in time proportional to \( \sum_{i=1}^{m} \frac{1}{\sin^2 (ct/2)} \) and \( t \sim \pi/c \). While if one starts from \( H_0 \) and state \( |\varphi_0\rangle \), evolve directly to the ground state of \( H_P \), the run time scales as \( t \sim 2^{m-1} \pi/c \).

\( \text{ii} \). Set the probe qubit in a frequency from the frequency set, and implement the time evolution operator \( U(t) = \exp \left( -iH^{(l)}t \right) \).

\( \text{iii} \). Read out the state of the probe qubit.

We repeat procedures \( \text{ii}-\text{iii} \) for a number of times. If the probe qubit stays in its excited state \( |1\rangle \), then the register \( R \) remains in state \( |\varphi_0^{(l-1)}\rangle \). Then we run procedures \( \text{ii}-\text{iii} \) by setting the probe in another frequency. If the probe qubit decays to its ground state \( |0\rangle \), it indicates that energy exchange occurs between the probe qubit and the register \( R \), and there is resonant transition from state \( |1\rangle |\varphi_0^{(l-1)}\rangle \) to \( |0\rangle |\varphi_0^{(l)}\rangle \). The ground state eigenvalue \( E_0^{(l)} \) and the corresponding eigenvector \( |\varphi_0^{(l)}\rangle \) of \( H_l \) can be obtained by locating the resonant transition frequency that satisfies the condition \( E_0^{(l)} - \alpha E_0^{(l-1)} = \omega \). Then with \( H_l \), \( |\varphi_0^{(l)}\rangle \) and \( E_0^{(l)} \), we run the algorithm for the next step, finally obtain the ground state of the problem Hamiltonian.

In order to obtain \( E_0^{(l)} \) accurately, we modify the Hamiltonian in Eq. (2) by switching the adjacent Hamiltonians: \( H_R^{(l)} = \alpha |1\rangle \langle 1| \otimes H_l + |0\rangle \langle 0| \otimes H_{l-1} \), and run the procedures above to drive the system back to state \( |\varphi_0^{(l-1)}\rangle \). We repeat this procedure while varying the frequency of the probe qubit near the resonant point to obtain accurate value of \( E_0^{(l)} \). The number of times of the procedures needs to be repeated scales as \( O(1/\epsilon^2) \) where \( \epsilon \) denotes the accuracy of \( E_0^{(l)} \).

In the above procedures, through entanglement between the probe qubit and the register \( R \), the state \( |\varphi_0^{(l-1)}\rangle \) is protected if one fails to obtain the ground state \( |\varphi_0^{(l)}\rangle \) of \( H_l \) when a resonant transition does not happen. This property allows one to obtain the state \( |\varphi_0^{(l)}\rangle \) deterministically by repeating procedures \( \text{ii} \) and \( \text{iii} \) for a number of times without preparing the input state \( |\varphi_0^{(l-1)}\rangle \). The number of times the procedures are repeated is proportional to \( 1/p_0^{(l)} \), where \( p_0^{(l)} = \sin^2 (cd_0^{(l)} t) \) is decay probability of the probe qubit of the \( l \)-th step and \( d_0^{(l)} = |\langle \varphi_0^{(l-1)} | \varphi_0^{(l)} \rangle| \). Then the run time of the algorithm is proportional to \( \sum_{l=1}^{m} 1/p_0^{(l)} \), which scales linearly with the number of steps of the algorithm, provided \( p_0^{(l)} \) are not exponentially small. The algorithm is efficient even if the overlap \( \langle \varphi_0 | \varphi_0^{(m)} \rangle \) is exponentially small in some cases, e.g. \( \langle \varphi_0 | \varphi_0^{(m)} \rangle = 1/2^m \) while \( d_0^{(l)} \) in each step is 1/2, the algorithm can be run in time proportional to \( \sum_{l=1}^{m} 1/\sin^2 (ct/2) \) and \( t \sim \pi/c \). While if one starts from \( H_0 \) and state \( |\varphi_0\rangle \), evolve directly to the ground state of \( H_P \), the run time scales as \( t \sim 2^{m-1} \pi/c \).
We now analyze the efficiency of the algorithm. In the $l$-th step, the probability of the initial state being evolved to state $|0\rangle|\varphi_0^{(l)}\rangle$ reaches maximum at time $t = \pi/(2c d_0^{(l)})$. In the first order approximation of dynamics, the initial state leaks to excited states $|0\rangle|\varphi_j^{(l)}\rangle$ ($j = 1, \ldots, N - 1$) with probability $p_{ij}$ and to states $|1\rangle|\varphi_j^{(l-1)}\rangle$ with probability $p_{0j}$. By setting $t = \pi/(2c d_0^{(l)})$ and assuming $E_1^{(l)} - E_0^{(l)} \gg c d_0^{(l)}$ and $\alpha (E_1^{(l-1)} - E_0^{(l-1)}) \gg c d_0^{(l)}$, we have

$$
\sum_{j=1} \left( p_{1j} + p_{0j} \right) \leq a_t^2 c^2,
$$

where

$$
a_t^2 = \frac{4}{\pi^2} \left( \frac{1 - (d_0^{(l)})^2}{(E_0^{(l)} - E_0^{(l)})^2 - 1)^2} + \frac{1 - (d_0^{(l)})^2}{(\alpha (E_1^{(l-1)} - E_0^{(l-1)}) - 1)^2} \right).$$

(see Appendix A for details). If the overlap $d_0^{(l)}$ and energy gap $E_1^{(l)} - E_0^{(l)}$ are not exponentially small, i.e. bounded by a polynomial function of the problem size, then $a_t$ is finite and the error in the $l$-th step is bounded by $a_t^2 c^2$. Considering errors accumulated in all steps, the success probability of the algorithm

$$
P_{\text{succ}} \geq \prod_{j=1}^m \left[ 1 - a_t^2 c^2 \right] \geq \left[ 1 - (a_{\text{max}} c)^2 \right]^m,$$

where $a_{\text{max}}$ is the maximum value of $a_t$. The coefficient $c$ can be set such that $a_{\text{max}} c < 1/\sqrt{m}$, then $P_{\text{succ}} > 1/e$ in the asymptotic limit of $m$.

We consider the cost when there is deviation $\epsilon$ of the probe frequency from the resonant transition frequency. By ignoring the off-resonant transition, the transition probability from the state $|1\rangle|\varphi_0^{(l-1)}\rangle$ to $|0\rangle|\varphi_0^{(l)}\rangle$ is $\sin^2 \left( \frac{\Omega_0 t}{2} \right) \frac{(2c d_0^{(l)})^2}{(2c d_0^{(l)})^2 + \epsilon^2}$ where $\Omega_0 = \sqrt{\left( 2c d_0^{(l)} \right)^2 + \epsilon^2}$. The transition probability is finite if $\epsilon < 2c d_0^{(l)}$. The number of times that this procedure needs to be repeated scales as $O(1/\epsilon^2)$, the cost is finite as long as $d_0^{(l)}$ and $c$ are not exponentially small.

The run time of our algorithm is proportional to the number of steps of the algorithm, and the run time in each step is proportional to $\pi/(2c d_0^{(l)})$, therefore the total run time of the algorithm scales as $O(\sum_{l=1}^m \frac{\pi}{2 (E_1^{(l)} - E_0^{(l)}) d_0^{(l)} c^2})$. 

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III. SEARCH PROBLEM WITH A SPECIAL DATA STRUCTURE

We apply the algorithm for solving a type of problems with a special structure. The general search problem is to find a marked item in an unsorted database of \( N \) items using an oracle that recognizes the marked item. The oracle is defined in terms of a problem Hamiltonian \( H_P = -|q⟩⟨q| \), where \( |q⟩ \) is the marked state associated with the marked item. The initial Hamiltonian is defined as \( H_0 = -|ψ₀⟩⟨ψ₀| \), where \( |ψ₀⟩ = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j⟩ \). If a problem has a special structure that allows us to construct a sequence of intermediate Hamiltonians

\[
H_j = \frac{N_j}{N} H_0 + \left( 1 - \frac{N_j}{N} \right) H_{P_j}, \quad j = 1, 2, \ldots, m - 1
\]

where

\[
H_{P_j} = - \sum_{q_j \in M_j} |q_j⟩⟨q_j|,
\]

and \( M_1 \supset \cdots \supset M_{m-1} \), with size \( N_1, \ldots, N_{m-1} \), respectively, and \( N_j/N_j-1 \) is not exponentially small, then our algorithm can solve this type of problems efficiently.

Define \( |q_j^⊥⟩ = \frac{1}{\sqrt{N - N_j}} \sum_{j \notin M_j} |j⟩ \), in basis \( \{|q_j⟩\}_{q_j \in M_j}, \{q_j^⊥⟩\} \) the intermediate Hamiltonian \( H_j \) in Eq. (4) can be written as

\[
H_j = \begin{pmatrix}
\frac{s-1}{N} & \cdots & \frac{s-1}{N} & (s-1)\sqrt{N-N_j} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{s-1}{N} & \cdots & \frac{s-1}{N} & (s-1)\sqrt{N-N_j} \\
(s-1)\sqrt{N-N_j} & \cdots & (s-1)\sqrt{N-N_j} & s^2
\end{pmatrix} - sI_j,
\]

where \( s = 1 - \frac{N_j}{N} \) and \( I_j \) is the identity matrix of dimension \( N_j + 1 \). Let \( |e⟩ = (1, \cdots, 1)^T \) and \( |0⟩ = (0, \cdots, 0)^T \) be \( N_j \times 1 \) vectors, respectively. The intermediate Hamiltonian \( H_j \) has \( N_j - 1 \) degenerate eigenstates \( |e_j'⟩ = (|e_j⟩, 0)^T \) with eigenvalue 0, where \( ⟨e_j|e⟩ = 0 \), and two eigenstates with eigenvalues \( E_{±}^{(j)} = \frac{-1 \pm \Delta E_j}{2} \), where \( \Delta E_j = \sqrt{\left(1 - \frac{2N_j}{N}\right)^2 + 4\frac{N_j^2}{N^2} \left(1 - \frac{N_j}{N}\right)} \) is the energy gap and has a minimum of \( \sqrt{1/3} \sqrt{3} \approx 0.638 \) as \( N_j/N = 1/3 \). The ground state of \( H_j \) is \( |V_{j}⟩ = x_1^{(j)}(|e⟩, 0)^T + x_2^{(j)}(|0⟩, 1)^T \). In the \( j \)-th step of the algorithm, by setting \( ω = 1 \) and \( α = \left(E_2^{(j)} - 1\right)/E_1^{(j)} \), the condition for resonant transition between states \( |1⟩|V_{j-1}⟩ \) and \( |0⟩|V_{j}⟩ \) is satisfied. The overlap between ground states of two adjacent intermediate Hamiltonians is

\[
d_0^{(j)} = ⟨V_{j}⟩ = N_j x_1^{(j)*} x_1^{(j)} + \sqrt{\frac{N-N_j}{N-N_j}} x_2^{(j)*} x_2^{(j)}.
\]
where $N_j \left| x_1^{(j)} \right|^2 + \left| x_2^{(j)} \right|^2 = 1$. The components $x_1^{(j)}$ and $x_2^{(j)}$ are continuous functions of $N_j/N$. If the ratio $N_j/N_{j-1}$ are finite, where $N_0 = N$, then $d_0^{(j)}$ are finite, the conditions of our algorithm are satisfied. The overlap between the ground state of the intermediate Hamiltonian $H_j$ and that of the problem Hamiltonian $H_P$ is $\left| x_1^{(j)} \right|$. It increases monotonically as $j$ increases and $H_j$ reaching the problem Hamiltonian.

For some problems, the degeneracy of eigenstates is known, e.g., the factoring problem, and the conditions of our algorithm are satisfied, then they can be solved efficiently. The general search problem has a structure that the number of the marked states is very small and that of the unmarked states is exponentially large. It cannot be divided further, the two conditions of our algorithm cannot be satisfied simultaneously, therefore cannot be solved efficiently. Our algorithm has the same efficiency as that of the Grover’s algorithm \[10\] and the QAA \[11, 12\] (see Appendix B & C).

In the following, we present a problem where the QAA fails while our algorithm remains to be efficient.

We consider the following problem: let $W[0, \cdots, N-1]$ be an unsorted table of $N$ items, each is associated with an integer in the range $[0, \cdots, n-1]$, where $N = 2^n$. There are $N/2 + 1$ items correspond to the integer $n-1$, $N/2^2$ items correspond to $n-2$, $\cdots$, $2$ items correspond to $1$, and one item corresponds to $0$. The problem is to find the index $y$ such that $W[y]$ is the minimum. This problem can be mapped to a general search problem with a unique solution by using an oracle that recognizes whether an integer that an item associated with is zero or not. But the general search problem cannot be mapped to this problem using one oracle. Here by using $n-1$ different oracles, our algorithm solves this problem efficiently.

The problem Hamiltonian $H_P$ of this problem is constructed as

$$H_P|k\rangle = h|k\rangle, \quad k = 0, 1, \cdots, N - 1,$$

let $h_k$ be the integer associated with state $|k\rangle$, then

$$h = \begin{cases} 
-1, & \text{if } h_k < 1 \\
0, & \text{if } h_k \geq 1
\end{cases}.$$ 

This can be achieved by using an oracle that recognizes whether an integer $h_k$ is larger or less than 1. It is a comparison logic circuitry and can be implemented efficiently on a quantum
The problem Hamiltonian can also be written as $H_P = -|y\rangle\langle y|$, where $|y\rangle$ is the state associated with 0. To solve this problem, we prepare a division set \{\(v_1 = n-1, v_2 = n-2, \ldots, v_{n-1} = 1\)\}, and construct $H_{P_j}$ by using \(n-1\) different oracles as

$$H_{P_j}|k\rangle = h_{k_j}|k\rangle, \quad j = 1, \ldots, n-1,$$

where

$$h_{k_j} = \begin{cases} -1, & \text{if } h_k < v_j \\ 0, & \text{if } h_k \geq v_j \end{cases}.$$

The Hamiltonians $H_{P_j}$ can also be written as $H_{P_j} = -\sum_{q_j \in M_j} |q_j\rangle\langle q_j|$, and $M_1 \supset \cdots \supset M_{n-1}$. The size of $M_1, \ldots, M_{n-1}$ are $N_1 = N/2 + 1$, $N_2 = N/2^2$, $\ldots$, $N_{n-1} = 1$, respectively. The set $M_j$ contains all the basis states associated with integers that are less than $v_j$. The ratio $N_j/N_{j-1} \approx 1/2$, therefore by constructing intermediate Hamiltonians as shown in Eq. (4), the overlap between the ground states of any two adjacent Hamiltonians are finite ($\approx 1/\sqrt{2}$). And the energy gap between the ground and the first excited states of each Hamiltonian is finite. Therefore this problem can be solved efficiently using our algorithm.

The above problem can be solved by the QAA. We construct the problem Hamiltonian as $H'_P|k\rangle = h_k|k\rangle$, and evaluate the upper bound of the minimum energy gap of the adiabatic evolution Hamiltonian $(1-s)H_0 + sH'_P$. For $H'_P$, the contribution to the gap from the lower energy levels is larger than that of the higher energy levels, therefore we can obtain the upper bound of the gap by assuming all excited states are degenerate at the highest energy level with eigenvalue $n-1$. The problem Hamiltonian in this case can be written as $(n-1)(I - |y\rangle\langle y|)$. The minimum energy gap of the adiabatic evolution Hamiltonian is $\Delta E_{\text{min}} = \frac{2(n-1)}{N} \sqrt{\frac{(N-1)N}{n^2N-4(n-1)}}$ at $s = \frac{nN-2(n-1)}{n^2N-4(n-1)}$. It equals $2(n-1)/n\sqrt{N}$ at the asymptotic limit of $N$, and decreases exponentially with the problem size, thus cannot be solved efficiently using the QAA. For this problem, as long as the energy level scales polynomially with the problem size, the QAA cannot solve it efficiently. The problem in ref. [5] where the QAA fails and the QAOA succeeds has similar structure and can be solved efficiently using our algorithm (see Appendix D).

We have performed numerical simulation for three cases: (a) general search problem with a unique solution, the eigenvalue of the ground state is zero and that of the excited states is one; (b) the problem above; and (c) search problem with a unique solution where the eigenvalue of the ground state is zero and that of excited states is $(n-1)$. We simulate
these cases for $n = 10, 12, 16$ qubits, respectively. For $n = 10$, the minimum energy gaps are 0.03125, 0.05089, 0.05623, at $s = 1/2$, 0.1166, 0.09986, respectively. For $n = 12$, the minimum energy gaps are 0.01563, 0.02715, 0.02864, at $s = 1/2$, 0.0940, 0.0833, respectively. For $n = 16$, the minimum energy gaps are 0.003906, 0.007188, 0.007324, at $s = 1/2$, 0.0676, 0.0625, respectively. The minimum energy gap in case (b) is between those of (a) and (c) as expected.

IV. DISCUSSION

Quantum computation has its limitations, e.g. the no-cloning theorem forbids to copy an unknown quantum state perfectly. Therefore intermediate calculation results stored in an unknown quantum state cannot be used by making copies as in classical computation. Here, we found a way to protect and use unknown quantum state in an intermediate step to circumvent the problem of making copies of an unknown quantum state. The calculation result of an intermediate step is preserved in an entangled state, and is heralded through quantum measurements on the probe qubit without destroying the state. In each step, the procedures of the algorithm can be repeated until the desired results are obtained. Therefore the run time of the algorithm is proportional to the number of steps. Unlike projective methods, one does not have to run the algorithm from beginning when fails to obtain desired results in an intermediate step.

In the QAOA, the quantum state of interest is prepared on a quantum computer and optimized on a classical computer according to the outcomes of the quantum measurements. In our algorithm, the desired quantum state in each step is obtained through quantum simulation of resonant transitions. We construct the intermediate Hamiltonians in advance by decomposing the problem based on its structure through a number of different oracles, the ground state of the problem Hamiltonian is induced step by step. The evolution procedure is adjusted according to the outcomes of the quantum measurements on the probe qubit, either repeat the present step or run the next step of the algorithm. Such an “evolve-measure-evolve” method and the use of different oracles play important roles in the speedup of the algorithm for solving problems with the special structure, even when the QAA fails. In the QAOA, an integer $p$ is used to characterize the depth of the algorithm, and the quality of the approximation improves as $p$ is increased. In our algorithm for solving the
search problem with special data structure, the parameters that characterize the structure of the problem \(N_j\) are encoded in the intermediate Hamiltonians. This guarantees that the overlaps between the ground states of the intermediate Hamiltonians and that of the problem Hamiltonian increase monotonically as the intermediate Hamiltonians reach the problem Hamiltonian.

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**Appendix A: Error analysis**

In the \(l\)-th step of the algorithm, we are given Hamiltonians \(H_l, H_{l-1}\) and its ground state \(|\varphi^{(l-1)}_0\rangle\) and the ground state eigenvalue \(E^{(l-1)}_0\) that have been obtained from the previous step, the goal is to prepare the ground state \(|\varphi^{(l)}_0\rangle\) and obtain its corresponding eigenvalue \(E^{(l)}_0\) of the Hamiltonian \(H_l\). This can be achieved via a mechanism of simulated quantum resonant transition. The algorithm requires \((n+1)\) qubits with one probe qubit and an \(n\)-qubit register \(R\) representing a Hamiltonian of dimension \(N = 2^n\). The Hamiltonian for the \(l\)-th step of the algorithm is constructed as

\[
H^{(l)} = -\frac{1}{2}\omega\sigma_z \otimes I_N + H_R^{(l)} + c\sigma_x \otimes I_N, \tag{A1}
\]

where

\[
H_R^{(l)} = \alpha|1\rangle\langle 1| \otimes H_{l-1} + |0\rangle\langle 0| \otimes H_l, \quad l = 1, 2, \cdots, m, \tag{A2}
\]

and \(I_N\) is the \(N\)-dimensional identity operator, \(\sigma_{x, z}\) are the Pauli matrices. The first term in Eq. (A1) is the Hamiltonian of the probe qubit, the second term represents the Hamiltonian of the register \(R\), and the third term describes the interaction between them, \(\alpha\) is an adjustable parameter and \(c \ll 1\) is the coupling strength.

In basis \(\{|1\rangle|\varphi^{(l-1)}_0\rangle, |0\rangle|\varphi^{(l-1)}_0\rangle, |1\rangle|\varphi^{(l-1)}_j\rangle, |0\rangle|\varphi^{(l)}_j\rangle\},\ j = 1, 2, \cdots, N - 1,\) where \(|\varphi^{(l-1)}_j\rangle\) and \(|\varphi^{(l)}_j\rangle\) are eigenstates of the Hamiltonians \(H_{l-1}\) and \(H_l\), respectively, and \(H_{l-1}|\varphi^{(l-1)}_j\rangle =
$E^{(l-1)}_j|\varphi_j^{(l-1)}\rangle$, $H_l|\varphi_j^{(l)}\rangle = E_j^{(l)}|\varphi_j^{(l)}\rangle$, the algorithm Hamiltonian $H^{(l)}$ of the $l$-th step can be written as

$$H^{(l)} = \left(\begin{array}{cccccccc}
\frac{\omega}{2} + \alpha E_0^{(l-1)} & cd_{00} & 0 & \cdots & 0 & cd_{10} & \cdots & cd_{N-1,0} \\
\omega + E_0^{(l)} & cd_{01} & \cdots & cd_{0,N-1} & 0 & \cdots & 0 \\
0 & 0 & \omega + \alpha E_1^{(l-1)} & \cdots & 0 & cd_{11} & \cdots & cd_{N-1,1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & cd_{0,N-1} & 0 & \cdots & 0 & 0 & \cdots & cd_{N-1,N-1} \\
0 & cd_{10} & 0 & \cdots & 0 & cd_{1,N-1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
cd_{N-1,0} & 0 & cd_{N-1,1} & \cdots & cd_{N-1,N-1} & 0 & \cdots & \frac{\omega}{2} + E_1^{(l)} \\
\end{array}\right)$$

(A3)

Here we have denoted $d_{jk} = \langle \varphi_j^{(l)} | \varphi_k^{(l-1)} \rangle$ and $d_{00} = d_{00} > 0$. Suppose the resonance condition $E_0^{(l)} - \alpha E_0^{(l-1)} = \omega$, i.e., $\omega + \alpha E_0^{(l-1)} = \omega + E_0^{(l)} = \epsilon_0$ and let the total Hamiltonian of the system be written as $H^{(l)} = \epsilon_0 + H_0 + cV$ where

$$H_0 = \left(\begin{array}{cccccccc}
0 & cd_0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & cd_0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \alpha \delta_{1,0}^{(l-1)} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \alpha \delta_{N-1,0}^{(l-1)} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{array}\right), \quad (A4)$$

with energy differences being $\delta_{j0}^{(l-1)} = E_1^{(l-1)} - E_0^{(l-1)}$ and $\delta_{j0}^{(l)} = E_j^{(l)} - E_0^{(l)}$ and

$$V = \left(\begin{array}{cccccccc}
0 & 0 & 0 & \cdots & 0 & d_{10} & \cdots & d_{N-1,0} \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & d_{01} & 0 & \cdots & 0 & d_{11} & \cdots & d_{N-1,1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
d_{10} & 0 & d_{11} & \cdots & d_{1,N-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
d_{N-1,0} & 0 & d_{N-1,1} & \cdots & d_{N-1,N-1} & 0 & \cdots & 0 \\
\end{array}\right). \quad (A5)$$
Let us treat $c$ as small and according the first order approximation for evolution we have

\[
e^{-it(H_0+cV)} = e^{-iH_0 t} \left( 1 - ic \int_0^1 ds \ e^{isH_0 V} e^{-isH_0} \right) + o(c^2). \tag{A6}
\]

For convenience we denote $|\phi_0\rangle = |1\rangle|\varphi^{(l-1)}_0\rangle$, $|\phi_1\rangle = |0\rangle|\varphi^{(l)}_0\rangle$, $|\phi_{ij}\rangle = |1\rangle|\varphi^{(l-1)}_j\rangle$, $|\phi_{ij}\rangle = |0\rangle|\varphi^{(l)}_j\rangle$, $j = 1, 2, \ldots, N - 1$. From the initial state $|\phi_0\rangle = |1\rangle|\varphi^{(l-1)}_0\rangle$ it follows

\[
e^{-it(H_0+cV)}|\phi_0\rangle = -i|\phi_1\rangle - ic \int_0^1 ds \ e^{isH_0 V} \left( \cos \frac{s \pi}{2} |\phi_0\rangle - i \sin \frac{s \pi}{2} |\phi_1\rangle \right)
= -i|\phi_1\rangle - ic \int_0^1 ds \ e^{isH_0 V} \sum_{j=1}^{N-1} \left( \cos \frac{s \pi}{2} d_{j0}^* |\phi_{1j}\rangle - i \sin \frac{s \pi}{2} d_{0j} |\phi_{0j}\rangle \right)
= -i|\phi_1\rangle - ic \sum_{j=1}^{N-1} d_{j0}^* \left( \frac{ie^{it\delta^{(l-1)}_{j0}} - 1}{t\delta^{(l-1)}_{j0} + \frac{\pi}{2}} - \frac{ie^{it\delta^{(l-1)}_{j0}} - 1}{t\delta^{(l-1)}_{j0} - \frac{\pi}{2}} \right) |\phi_{1j}\rangle
- ic \sum_{j=1}^{N-1} d_{0j} \left( \frac{ie^{it\delta^{(l-1)}_{j0}} - 1}{t\delta^{(l-1)}_{j0} + \frac{\pi}{2}} - \frac{ie^{it\delta^{(l-1)}_{j0}} - 1}{t\delta^{(l-1)}_{j0} - \frac{\pi}{2}} \right) |\phi_{0j}\rangle. \tag{A7}
\]

We can see that in the first order approximation of dynamics, there are leaks to excited states $|0\rangle|\varphi^{(l)}_j\rangle$ with probability $p_{1j}$ as well as to states $|1\rangle|\varphi^{(l-1)}_j\rangle$ with probability $p_{0j}$, where

\[
p_{1j} = c^2 |d_{j0}|^2 \left( \frac{\pi}{2} \frac{ie^{it\delta^{(l)}_{j0}} - t\delta^{(l)}_{j0}}{((t\delta^{(l)}_{j0})^2 - (\frac{\pi}{2})^2)^2} \right)^2 \leq \frac{4c^2 |d_{j0}|^2}{\pi^2 (\frac{E^{(l-1)}_{j0} - E^{(l)}_{j0}}{cd_0 - 1})^2}, \tag{A8}
\]

and

\[
p_{0j} = c^2 |d_{0j}|^2 \left( \frac{\pi}{2} \frac{t\alpha\delta^{(l-1)}_{j0} ie^{it\delta^{(l)}_{j0}} + \frac{\pi}{2}}{(t\alpha\delta^{(l-1)}_{j0})^2 - (\frac{\pi}{2})^2)^2} \right)^2 \leq \frac{4c^2 |d_{0j}|^2}{\pi^2 (\frac{\alpha(E^{(l-1)}_{j0} - E^{(l)}_{j0})}{cd_0 - 1})^2}. \tag{A9}
\]

As a result

\[
\sum_j p_{1j} + \sum_j p_{0j} \leq \frac{4c^2(1 - d_{0j}^2)}{\pi^2} \left( \frac{1}{(\frac{E^{(l-1)}_{j0} - E^{(l)}_{j0}}{cd_0 - 1})^2} + \frac{1}{(\frac{\alpha(E^{(l-1)}_{j0} - E^{(l)}_{j0})}{cd_0 - 1})^2} \right). \tag{A10}
\]
In the above error control we have assumed $t = \pi/(2cd_0)$ and $E_1^{(l)} - E_0^{(l)} \gg cd_0$ and $\alpha(E_1^{(l-1)} - E_0^{(l-1)}) \gg cd_0$.

If the overlap $d_0$ and the energy gap $E_1^{(l)} - E_0^{(l)}$ are not exponentially small, i.e., bounded by a polynomial function of the problem size, then $a_l$ is a finite number and the error in the $l$-th step is bounded by $a_l^2c^2$. Considering the errors accumulated in all steps of the algorithm, the success probability of the algorithm for preparing ground state of the problem Hamiltonian

$$P_{\text{succ}} \geq \prod_{j=1}^{m} \left[ 1 - a_l^2c^2 \right] \geq \left[ 1 - (a_{\text{max}}c)^2 \right]^m,$$

where $a_{\text{max}}$ is the maximum value of $a_l$ for $l = 1, \ldots, m$. The coupling coefficient $c$ can be set such that $a_{\text{max}}c < 1/\sqrt{m}$, then $P_{\text{succ}} > 1/e$ in the asymptotic limit of $m$.

**Appendix B: Application of the algorithm for the general search problem**

The general search problem is to find a marked item in an unsorted database of $N$ items using an oracle that recognizes the marked item. The oracle is defined in terms of a problem Hamiltonian

$$H_P = -|q\rangle\langle q|,$$

where $|q\rangle$ is the marked state associated with the marked item. The initial Hamiltonian is defined as

$$H_0 = -|\psi_0\rangle\langle \psi_0|,$$

where $|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j\rangle$. Define $|q^+\rangle = \frac{1}{\sqrt{N-1}} \sum_{j\neq q} |j\rangle$, let $H(s) = (1-s)H_0 + sH_P$, there are $N-2$ degenerate eigenstates with eigenvalue 0 of $H(s)$, and they are orthogonal to states $|q\rangle$, $|q^+\rangle$, and $|\psi_0\rangle$. With the initial state being set as $|\psi_0\rangle$, the system evolves in the subspace spanned by $\{|q\rangle, |q^+\rangle\}$. The energy gap between the ground and first excited states of $H(s)$ is

$$\Delta E(s) = \sqrt{(1-2s)^2 + \frac{4}{N}s(1-s)}.$$

and the minimum energy gap is $\Delta E(1/2) = 1/\sqrt{N}$ at $s = 1/2$.

In Fig. 2, we plot the energy gap v.s. the parameter $s$. And Fig. 3 shows the change of the fidelity between the initial state $|\psi_0\rangle$ and the instantaneous ground state of the Hamiltonian $H(s)$ v.s. $s$. From these figures, we can see that as the energy gap reaches its
FIG. 2. Energy gap vs the parameter $s$ for the search problem with a unique solution by setting $N = 10^8$.

minimum at $s = 1/2$, the fidelity between the initial state and the ground state of $H(s)$ drops dramatically. The two conditions of our algorithm: both the overlaps between ground states of any two adjacent Hamiltonians and the energy gap between the ground and the first excited states of each Hamiltonian are not exponentially small, cannot be satisfied simultaneously for the general search problem.

To apply our algorithm for solving the search problem, we insert an intermediate Hamiltonian $H_1 = H (s = 1/2)$ between $H_0$ and $H_P$. The problem can be solved in two steps: start with the ground state $|\psi_0\rangle$ of $H_0$ and evolve it to the ground state of $H_1$, then evolve to the ground state of $H_P$. The ground and first excited states of $H_1$ are denoted as $|V_0\rangle$ and $|V_1\rangle$, with corresponding eigenvalues $E_0 = \frac{1}{2} - \frac{1}{2\sqrt{N}}$, $E_1 = \frac{1}{2} + \frac{1}{2\sqrt{N}}$, respectively. By setting $\omega = 1$ and $\alpha = 1 - E_0$, the condition for resonant transition between states $|1\rangle|\psi_0\rangle$ and $|0\rangle|V_0\rangle$ is satisfied. In order to separate states $|0\rangle|V_0\rangle$ and $|0\rangle|V_1\rangle$, the coupling coefficient is set $c < \Delta E (1/2) = 1/\sqrt{N}$. Therefore the evolution time for the initial state $|1\rangle|\psi_0\rangle$ to be
FIG. 3. Fidelity of the initial state $|\psi_0\rangle$ and the ground state of the Hamiltonian $H(s)$ vs the parameter $s$ for the search problem with a unique solution by setting $N = 10^8$.

evolved to state $|0\rangle|V_0\rangle$ is $t \sim 1/ \left( c \sqrt{\frac{N+1}{2N}} \right)$, which scales as $O \left( \sqrt{N} \right)$.

In the second step, by setting $\omega = 1$, in order for the resonant transition between states $|1\rangle|V_0\rangle$ and $|0\rangle|q\rangle$ to occur, the parameter $\alpha$ is set as $\alpha = 4 \sqrt{N}/(\sqrt{N}+1)$. Therefore the total run time of the algorithm for solving the search problem scales as $O \left( \sqrt{N} \right)$, which has the same scaling as that of the Grover’s algorithm [10] and adiabatic evolution algorithm [11, 12].

In the case of multiple marked items in a database, the problem Hamiltonian is defined as $H_P = -\sum_{q\in M} |q\rangle\langle q|$, where $M$ is a set of marked states associated with the marked items of size $N_q$. Let $|q^{\perp}\rangle = \frac{1}{\sqrt{N-N_q}} \sum_{j\notin M} |j\rangle$; there are $N-N_q-1$ degenerate eigenstates of the Hamiltonian $H(s) = (1-s) H_0 + s H_P$ that are orthogonal to states $|q\rangle$, $|q^{\perp}\rangle$ and $|\psi_0\rangle$ with eigenvalue 0. It has two eigenstates, $x_1^\pm (|e\rangle, 0)^T + x_2^\pm (|0\rangle, 1)^T$ with corresponding eigenvalues:

$$\lambda_{\pm} = \frac{-1 \pm \sqrt{(2s-1)^2 - 4 N_q N (s-1)}}{2},$$  

(B4)
respectively. As \( s = \frac{1}{2} \), the minimum energy gap is \( \Delta \lambda = \sqrt{N_q/N} \), the corresponding eigenvectors in basis \( \{(|e\rangle, 0)^T, (|0\rangle, 1)^T\} \) are

\[
|V_-\rangle = \left( \frac{1}{\sqrt{2N_q}} \sqrt{1 + \sqrt{N_q/N}}, \frac{1}{\sqrt{2}} \sqrt{1 - \sqrt{N_q/N}} \right)
\]

and

\[
|V_+\rangle = \left( \frac{1}{\sqrt{2N_q}} \sqrt{1 - \sqrt{N_q/N}}, -\frac{1}{\sqrt{2}} \sqrt{1 + \sqrt{N_q/N}} \right)
\]

, respectively. In the first step of the algorithm, where \( d = \langle V_-|\psi_0\rangle = \frac{1}{\sqrt{2}} \sqrt{\frac{N_q}{N}} + \frac{1}{\sqrt{2}} \sqrt{\frac{N_q}{N}} \), \( \langle V_+|\psi_0\rangle = \sqrt{1 - d^2} \). With \( \omega = 1 \), the parameter \( \alpha \) is set as \( \alpha = 1 - \lambda_- (1/2) = \frac{3}{2} + \frac{1}{2} \sqrt{\frac{N_q}{N}} \) to satisfy the condition for resonant transition. In order to separate states \( |0\rangle|V_-\rangle \) and \( |0\rangle|V_+\rangle \), the coupling coefficient is set \( c < \sqrt{\frac{N_q}{N}} \). Then the evolution time from the initial state \( |1\rangle|\psi_0\rangle \) to state \( |0\rangle|V_-\rangle \) scales as \( O\left(\sqrt{N/N_q}\right) \). In the second step, with the initial state being set as \( |1\rangle|V_-\rangle \), and \( \omega = 1 \), we have to set \( \alpha = -2\lambda_- (1/2) = 4/(1 + \sqrt{N/N_q}) \) to satisfy the condition for resonant transition. The overlap \( d = \langle V_-|q\rangle = \frac{1}{\sqrt{2N_q}} \sqrt{1 + N_q/N} \). Therefore the total run time of the algorithm scales as \( O\left(\sqrt{N/N_q}\right) \), which is the same as that of the Grover’s algorithm \([10]\) and the quantum adiabatic algorithm \([11, 12]\).

**Appendix C: Application of the algorithm for the factoring problem**

Factoring an integer \( Z = x \times y \) on a quantum computer can achieve exponential speedup over its classical counterparts by using Shor’s algorithm \([7]\). An integer \( a < Z \) and co-prime with \( Z \) is used, the order of \( a \) is the smallest integer \( r \) that satisfies \( a^r = 1 (\text{mod } Z) \), and can be found efficiently through quantum Fourier transform. Then the factors of \( Z \) can be calculated as \( \gcd(a^{r/2} \pm 1, Z) \). The cost of the algorithm scales as \( O(L^3) \), where \( L = \lceil \log_2 Z \rceil \).

Our algorithm can also be applied for factoring large integers efficiently. The problem Hamiltonian for factoring an integer \( Z \) is defined as

\[
H_p^i|k\rangle = h_k|k\rangle = a^k (\text{mod } Z)|k\rangle, \ k = 0, 1, \cdots, N - 1,
\]

where \( N = 2^n \geq Z \) contains a few periods of the integer \( a \). \( H_p^i \) is diagonal in computational basis states (CBS) of qubits and its ground state eigenvalue is 1. The CBS with eigenvalue
1 of $H_P$ are in form of $|p \cdot r\rangle$, where $p = 0, 1, \ldots, \lfloor (N - 1)/r \rfloor$. The order $r$ of $a$ can be determined if the ground state of the problem Hamiltonian is obtained.

The eigenvalues of the problem Hamiltonian are integers and distribute uniformly in the period of $a$, since $k$ ranges from 0 to $N - 1$ uniformly. The ratio between the degeneracies of any two energy levels of the problem Hamiltonian is about one. Therefore we construct a division set \( \{v_1 = \lfloor \frac{Z}{2} \rfloor, v_2 = \lfloor \frac{v_1}{2} \rfloor, \cdots, v_L = \lfloor \frac{v_{L-1}}{2} \rfloor \} \), then the Hamiltonians $H_{P_j}$ can be constructed using $L$ different oracles as

$$H_{P_j}|k\rangle = h_{k_j}|k\rangle, \quad j = 1, \cdots, L, \quad \text{(C2)}$$

where

$$h_{k_j} = \begin{cases} -1, & \text{if } h_k \leq v_j \\ 0, & \text{if } h_k > v_j \end{cases} \quad \text{(C3)}$$

We construct a sequence of intermediate Hamiltonians

$$H_j = \frac{N_j}{N} H_0 + \left( 1 - \frac{N_j}{N} \right) H_{P_j}, \quad j = 1, 2, \cdots, L \quad \text{(C4)}$$

where $N_j$ are degeneracies of ground states of the Hamiltonian $H_{P_j}$. We insert the intermediate Hamiltonians between the initial Hamiltonian $H_0 = -|\psi_0\rangle\langle\psi_0|$, where $|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle$ and the problem Hamiltonian $H_P$, which is defined as

$$H_P|k\rangle = E_k|k\rangle, \quad k = 0, 1, \cdots, N - 1 \quad \text{(C5)}$$

where

$$E_k = \begin{cases} -1, & \text{if } h_k \leq 1 \\ 0, & \text{if } h_k > 1 \end{cases} \quad \text{(C6)}$$

As we have discussed in the main text, the energy gap between the ground and first excited states of the algorithm Hamiltonian is finite. The ratio $N_j/N_{j-1}$ for the factoring problem Hamiltonian is about 1/2 in each step of the algorithm, thus the overlap between ground states of two adjacent Hamiltonians are finite. Therefore the runtime of each step of the algorithm is finite, and the total run time of the algorithm is proportional to the number of steps $L$ of the algorithm. The order $r$ of $a$ can be obtained by measuring the ground state of the problem Hamiltonian, thus the factoring problem can be solved efficiently.
Appendix D: Application of the algorithm for a symmetrized cost function

In Ref. [5], the authors show an example where the quantum adiabatic algorithm fails and the quantum approximate optimization algorithm succeeds. Here we apply our algorithm for this problem [15]. The problem has a symmetrized function only of the Hamming weight \( w = z_1 + z_2 + \cdots + z_n \) as a result of all bits being treated symmetrically. Here \( z_j \) are bits with value either 0 or 1. The cost function has the form
\[
h(w) \equiv h(z_1, \ldots, z_n) = q^2 w(n-w)(n-w-1) + \frac{1}{2} w(w-1)(n-w) + \frac{1}{6} w(w-1)(w-2) \quad (D1)
\]
where \( q \) is an integer greater than or equal to 3. The goal is to find the minimum of the cost function.

The number of the computational basis states \( |z_1, \ldots, z_n\rangle \) that have Hamming weight \( w \) is \( N_w = \binom{n}{w} \), therefore as \( w \leq \lfloor n/2 \rfloor \), the ratio \( N_w/N_{w-1} \propto n \) is not exponentially small. And the cost function has the same structure. It satisfies the conditions we discussed in the main text when apply the algorithm for solving a search problem with a special data structure. Therefore can be solved efficiently using our algorithm in the same way.

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[1] D. S. Abrams and S. Lloyd, Quantum algorithm providing exponential speed increase for finding eigenvalues and eigenvectors. Phys. Rev. Lett. 83, 5162 (1999).

[2] A. Y. Kitaev, Quantum measurements and the Abelian stabilizer problem. e-print quant-ph/9511026 (1995).

[3] E. Farhi, J. Goldstone, S. Gutmann, A numerical study of the performance of a quantum adiabatic evolution algorithm for satisfiability. e-print quant-ph/0007071v1 (2000)

[4] E. Farhi, J. Goldstone, S. Gutmann, J. Lapan, A. Lundgren, D. Preda, A quantum adiabatic evolution algorithm applied to random instances of an NP-Complete problem. Science 292, 472 (2001).

[5] E. Farhi, J. Goldstone, S. Gutmann, A quantum approximate optimization algorithm. e-print quant-ph/arxiv: 1411.4028 (2014).

[6] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information (Cambridge Univ. Press, Cambridge, England, 2000).

[7] P. Shor, Algorithms for quantum computation: discrete logarithms and factoring. Proc. 35th
Ann. Symp. on Found. of Comp. Sci., 124-134 (IEEE Comp. Soc. Press, Los Alamitos, CA, 1994).

[8] H. Wang, S. Ashhab and Franco Nori, Quantum algorithm for obtaining the energy spectrum of a physical system, Phys. Rev. A 85, 062304 (2012).

[9] Z. Li et al., Quantum simulation of resonant transitions for solving the eigenproblem of an effective water Hamiltonian, Phys. Rev. Lett. 122, 090504 (2019).

[10] L. K. Grover, Quantum mechanics helps in searching for a needle in a haystack. Phys. Rev. Lett. 79, 325-328 (1997).

[11] T. Albash and D. A. Lidar, Adiabatic quantum computation, Rev. Mod. Phys. 90, 015002 (2018).

[12] J. Roland and N. J. Cerf, Quantum search by local adiabatic evolution. Phys. Rev. A, 65, 042308 (2002).

[13] C. Dürr and P. Høer, A quantum algorithm for finding the minimum. e-print quant-ph/9607014 (1996).

[14] W. P. Baritompa, D. W. Bulger and G. R. Wood, Grover’s quantum algorithm applied to global optimization. SIAM J. Optim. 15, 1170 (2005).

[15] E. Farhi, J. Goldstone, S. Gutmann, Quantum adiabatic evolution algorithms versus simulated annealing. e-print arXiv:quant-ph/0201031 (2002).