HYPERBOLIC POTENTIALS FOR CONTINUOUS NON-UNIFORMLY EXPANDING MAPS

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Abstract. In this work, we give a class of examples of hyperbolic potentials (including the null one) for continuous non-uniformly expanding maps. It implies the existence and uniqueness of equilibrium state (in particular, of maximal entropy measure). Among the maps considered is the important class known as Viana maps.

1. Introduction

The theory of equilibrium states on dynamical systems was firstly developed by Sinai, Ruelle and Bowen in the sixties and seventies. It was based on applications of techniques of Statistical Mechanics to smooth dynamics. Given a continuous map \( f : M \to M \) on a compact metric space \( M \) and a continuous potential \( \phi : M \to \mathbb{R} \), an equilibrium state is an invariant measure that satisfies a variational principle, that is, a measure \( \mu \) such that

\[
h_\mu(f) + \int \phi d\mu = \sup_{\eta \in \mathcal{M}_f(M)} \left\{ h_\eta(f) + \int \phi d\eta \right\},
\]

where \( \mathcal{M}_f(M) \) is the set of \( f \)-invariant probabilities on \( M \) and \( h_\eta(f) \) is the so-called metric entropy of \( \eta \).

In the context of uniform hyperbolicity, which includes uniformly expanding maps, equilibrium states do exist and are unique if the potential is Hölder continuous and the map is transitive. In addition, the theory for finite shifts was developed and used to achieve the results for smooth dynamics.

Beyond uniform hyperbolicity, the theory is still far from complete. It was studied by several authors, including Bruin, Keller, Demers, Li, Rivera-Letelier, Iommi and Todd \([14, 13, 16, 22, 23, 24]\) for interval maps; Denker and Urbanski \([17]\) for rational maps; Leplaideur, Oliveira and Rios \([25]\) for partially hyperbolic horseshoes; Buzzi, Sarig and Yuri \([15, 42]\), for countable Markov shifts and for piecewise expanding maps in one and higher dimensions. For local diffeomorphisms with some kind of non-uniform expansion, there are results due to Oliveira \([26]\); Arbieto, Matheus and Oliveira \([10]\); Varandas and Viana \([39]\). All of whom proved the existence and uniqueness of equilibrium states for potentials with low oscillation. Also, for this type of maps, Ramos and Viana \([32]\) proved it for potentials so-called hyperbolic, which includes the previous ones. The hyperbolicity of the potential is characterized by the fact that the pressure emanates from the hyperbolic region. In all these studies the maps does not have the presence of critical sets and recently, Alves, Oliveira and Santana proved the existence of at most finitely many equilibrium states.

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states for hyperbolic potentials, possible with the presence of a critical set (see \[6\]). More recently, Santana completed this by showing uniqueness in \[33\].

In this work, we give examples of a class of potentials whose Birkhoff sums are uniformly bounded. In particular, we show that the null potentials $\varphi \equiv 0$ is hyperbolic and obtain existence and uniqueness of maximal entropy measure.

2. Preliminaries and Main Result

2.1. Non-uniformly Expanding Maps. We recall the definition of a $(\sigma, \delta)$-hyperbolic time for $x \in M$.

Let $M$ be a connected compact metric space, $f : M \to M$ a continuous map and $\mu$ a reference Borel measure on $M$. Fix $\sigma \in (0, 1), \delta > 0$ and $x \in M$. We say that $n \in \mathbb{N}$ is a $(\sigma, \delta)$-hyperbolic time for $x$ if

- there exists a neighbourhood $V_n(x)$ of $x$ such that $f^n$ sends $V_n(x)$ homeomorphically onto the ball $B_\delta(f^n(x))$;
- $d(f^i(y), f^i(z)) \leq \sigma^{n-i}d(f^n(y), f^n(z)), \forall\ y, z \in V_n(x), \forall\ 0 \leq i \leq n - 1$.

The sets $V_n(x)$ are called hyperbolic pre-balls and their images $f^n(V_n(x)) = B_\delta(f^n(x))$, hyperbolic balls.

We observe that if $n$ is a $(\sigma, \delta)$-hyperbolic time for $x$, then $n$ is a $(\sigma, \delta')$-hyperbolic time for $x$, for every $0 < \delta' < \delta$.

We say that $x \in M$ has positive frequency of hyperbolic times if

$$\limsup_{n \to \infty} \frac{1}{n} \#\{0 \leq j \leq n - 1 | j \text{ is a hyperbolic time for } x\} > 0,$$

and define the expanding set

$$H = \{x \in M | \text{the frequency of hyperbolic times of } x \text{ is positive}\}.$$

We say that a Borel probability measure $\mu$ on $M$ is expanding if $\mu(H) = 1$.

Given a measure $\mu$ on $M$, its Jacobian is a function $J_\mu f : M \to [0, +\infty)$ such that

$$\mu(f(A)) = \int_A J_\mu f d\mu$$

for every $A$ domain of injectivity, that is, a measurable set such that $f(A)$ is measurable and $f_A : A \to f(A)$ is a bijection. We say that the measure has bounded distortion if there exists $\rho > 0$ such that

$$\left| \log \frac{J_\mu f^n(y)}{J_\mu f^n(z)} \right| \leq \rho d(f^n(y), f^n(z)),$$

for every $y, z \in V_n(x)$, $\mu$-almost everywhere $x \in M$, for every hyperbolic time $n$ of $x$. A map with an expanding measure with bounded distortion associated is called non-uniformly expanding.
2.2. Relative Pressure. We recall the definition of relative pressure for non-compact sets by dynamical balls.

Let $M$ be a compact metric space. Consider $f : M \to M$ and $\phi : M \to \mathbb{R}$. Given $\delta > 0$, $n \in \mathbb{N}$ and $x \in M$, we define the dynamical ball $B_\delta(x, n)$ as

$$B_\delta(x, n) := \{ y \in M | d(f^i(x), f^i(y)) < \delta, \; \text{for} \; 0 \leq i \leq n \}.$$

Consider for each $N \in \mathbb{N}$, the set

$$F^\delta_N = \{ B_\delta(x, n) | x \in M, n \geq N \}.$$

Given $\Lambda \subset M$, denote by $F^\delta_N(\Lambda)$ the finite or countable families of elements in $F^\delta_N$ that cover $\Lambda$. Define for $n \in \mathbb{N}$ the Birkhoff sum

$$S_n \phi(x) = \phi(x) + \phi(f(x)) + \cdots + \phi(f^{n-1}(x))$$

and

$$R_{n, \delta} \phi(x) = \sup_{y \in B_\delta(x, n)} S_n \phi(y).$$

Given a $f$-invariant set $\Lambda \subset M$, not necessarily compact, define for each $\gamma > 0$

$$m_f(\phi, \Lambda, \delta, N, \gamma) = \inf_{U \in F^\delta_N(\Lambda)} \left\{ \sum_{B_\delta(y, n) \in U} e^{-\gamma n + R_{n, \delta} \phi(y)} \right\}.$$ 

Define also

$$m_f(\phi, \Lambda, \delta, \gamma) = \lim_{N \to +\infty} m_f(\phi, \Lambda, \delta, N, \gamma).$$

and

$$P_\Lambda(\phi, \delta) = \inf \{ \gamma > 0 | m_f(\phi, \Lambda, \delta, \gamma) = 0 \}.$$ 

Finally, define the relative pressure of $\phi$ on $\Lambda$ as

$$P_\Lambda(\phi) = \lim_{\delta \to 0} P_\Lambda(\phi, \delta).$$

The topological pressure of $\phi$ is, by definition, $P(\phi) = P_M(\phi)$ and satisfies

$$P_f(\phi) = \sup \{ P_f(\phi, \Lambda), P_f(\phi, \Lambda^c) \}$$

where $\Lambda^c$ denotes the complement of $\Lambda$ on $M$.

2.3. Hyperbolic potentials. We say that a continuous function $\phi : M \to \mathbb{R}$ is a hyperbolic potential if the topological pressure $P_f(\phi)$ is located on $H$, i.e.

$$P_f(\phi, H^c) < P_f(\phi).$$

In [24], H. Li and J. Rivera-Letelier consider other type of hyperbolic potentials for one-dimensional dynamics that is weaker than ours. In their context, $\phi$ is a hyperbolic potential if

$$\sup_{\mu \in M_f(M)} \int \phi d\mu < P_f(\phi).$$
2.4. Viana Maps. We recall the definition of the open class of maps with critical sets in dimension 2, introduced by M. Viana in [40]. We skip the technical points. It can be generalized for any dimension (See [1]).

Let $a_0 \in (1, 2)$ be such that the critical point $x = 0$ is pre-periodic for the quadratic map $Q(x) = a_0 - x^2$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $b : S^1 \to \mathbb{R}$ a Morse function, for instance $b(\theta) = \sin(2\pi \theta)$. For fixed small $\alpha > 0$, consider the map

$$f_0 : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$$

$$(\theta, x) \mapsto (g(\theta), q(\theta, x))$$

where $g$ is the uniformly expanding map of the circle defined by $g(\theta) = d\theta (\text{mod } \mathbb{Z})$ for some $d \geq 16$, and $q(\theta, x) = a(\theta) - x^2$ with $a(\theta) = a_0 + \alpha b(\theta)$. It is easy to check that for $\alpha > 0$ small enough there is an interval $I \subset (-2, 2)$ for which $f_0(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Thus, any map $f$ sufficiently close to $f_0$ in the $C^0$ topology has $S^1 \times I$ as a forward invariant region. We consider from here on these maps $f$ close to $f_0$ restricted to $S^1 \times I$. Taking into account the expression of $f_0$ it is not difficult to check that for $f_0$ (and any map $f$ close to $f_0$ in the $C^2$ topology) the critical set is non-degenerate.

The main properties of $f$ in a $C^3$ neighbourhood of $f$ that we will use here are summarized below (See [1], [9], [31]):

(1) $f$ is **non-uniformly expanding**, that is, there exist $\lambda > 0$ and a Lebesgue full measure set $H \subset S^1 \times I$ such that for all point $p = (\theta, x) \in H$, the following holds

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| Df(f^i(p))^{-1} \| < -\lambda.$$

(2) Its orbits have **slow approximation to the critical set**, that is, for every $\epsilon > 0$ the exists $\delta > 0$ such that for every point $p = (\theta, x) \in H \subset S^1 \times I$, the following holds

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} - \log \text{dist}_\delta(p, C) < \epsilon.$$

where

$$\text{dist}_\delta(p, C) = \begin{cases} 
\text{dist}(p, C), & \text{if } \text{dist}(p, C) < \delta \\
1 & \text{if } \text{dist}(p, C) \geq \delta
\end{cases}$$

(3) $f$ is topologically mixing;

(4) $f$ is strongly topologically transitive;

(5) it has a unique ergodic absolutely continuous invariant (thus SRB) measure;

(6) the density of the SRB measure varies continuously in the $L^1$ norm with $f$.

**Remark 1.** We observe that this definition of non-uniformly expansion is included in ours by neighbourhoods. Details can be found in [6] or [33]. Other obvious references are [1], [31] and the original work by M. Viana [40].
2.5. Main Result.

**Theorem 1.** Let \( f : M \to M \) be a continuous non-uniformly expanding map and \( H \) its expanding set and \( \varphi : M \to \mathbb{R} \) a continuous potential with Birkhoff sums uniformly bounded. If \( H \) and \( H^c \) are both dense on \( M \) and the topological entropy \( h(f) \) is positive, then the potential \( \varphi \) is hyperbolic. In particular, if \( \varphi \) is Hölder there exists a unique equilibrium state.

**Corollary 1.** Let \( f : M \to M \) be a continuous non-uniformly expanding map and \( H \) its expanding set. If \( H \) and \( H^c \) are both dense on \( M \) and \( h(f) > 0 \), the null potential is hyperbolic. In particular, there exists a unique measure of maximal entropy for \( f \).

**Corollary 2.** Let \( f : S^1 \times I \to S^1 \times I \) be a Viana map. There exists a unique measure of maximal entropy for \( f \).

3. Proof of the Main Result

We begin by proving the next Theorem, which is the base to prove our main result.

**Theorem 2.** Let \( f : M \to M \) be a continuous non-uniformly expanding map and \( H \) its expanding set. If \( H \) and \( H^c \) are both dense on \( M \) and \( h(f) > 0 \), the null potential is hyperbolic. In particular, there exists a unique measure of maximal entropy for \( f \).

**Proof.** We will show that the null potential is hyperbolic. In order to do that we will use the power of hyperbolic times. We divide the proof in several lemmas.

**Lemma 1.** If \( n \in \mathbb{N} \) is a \((\sigma, \delta)\)-hyperbolic time for \( x \in M \), then \( B_\delta(x,n) = V_n(x) \).

**Proof.** Firstly, given \( y \in V_n(x) \), by definition we have

\[
d(f^i(y), f^i(x)) \leq \sigma^{n-i}d(f^n(y), f^n(x)) < \delta, \forall 0 \leq i \leq n - 1.
\]

and it means that \( y \in B_\delta(x,n) \). So, \( V_n(x) \subset B_\delta(x,n) \).

We observe that if \( y \in B_\delta(x,n) \) then \( f^n(y) \in B_\delta(f^n(x)) \). If \( y \in V_n(x)^c \) there exists \( 0 \leq i \leq n - 1 \) such that \( d(f^i(y), f^i(x)) > \sigma^{n-i}d(f^n(y), f^n(x)) \). Once \( f^n \) sends \( V_n(x) \) homeomorphically to \( B_\delta(f^n(x)) \), it means that

\[
f^n(y) \notin B_\delta(f^n(x)) \implies y \in B_\delta(x,n)^c \implies V_n(x)^c \subset B_\delta(x,n)^c.
\]

Since \( V_n(x) \subset B_\delta(x,n) \) and \( V_n(x)^c \subset B_\delta(x,n)^c \), we have \( V_n(x) = B_\delta(x,n) \). \( \square \)

As as consequence of the previous lemma, if \( n \) is a \((\sigma, \delta)\)-hyperbolic time for \( x \), then \( B_\delta(x,n) \) is always an open set. Since \( H^c \) is dense on \( M \) and for every dynamical ball \( B_\delta(x,n) \) where \( n \) is a \((\sigma, \delta)\)-hyperbolic time for \( x \) we can always find \( y(x) \in B_\delta(x,n) \cap H^c \). So, we have the following lemma.

**Lemma 2.** Given \( 0 < \delta < \delta_1 \) and a covering \( \mathcal{U} \) of \( H \), such that each \( B_\delta(x,n) \in \mathcal{U} \) with \( x \in H \) and \( n \geq 1 \) is a \((\sigma, \delta)\)-hyperbolic time for \( x \), let \( y(x) \in B_{\delta_1}(x,n) \cap H^c \). We have \( B_\delta(x,n) \subset B_{\delta_1}(y(x),n) \), \( \forall \delta < \delta_1 \) and the collection \( \mathcal{V}(\mathcal{U}) \) of dynamical balls \( B_{\delta_1}(y(x),n) \) is a covering of \( H^c \) with the same cardinality as \( \mathcal{U} \).
Given Lemma 3. 

Proof. Let \( z \in B_\delta(x, n) = V_n(x) \). We know that 
\[
d(f^i(z), f^i(x)) \leq \sigma^{n-i} d(f^n(z), f^n(x)) < \delta, \quad \forall i \leq n.
\]
Also 
\[
d(f^i(y), f^i(x)) \leq \sigma^{n-i} d(f^n(y), f^n(x)) < \delta, \quad \forall i \leq n.
\]
It implies that 
\[
d(f^i(z), f^i(y)) < 2\delta, \quad \forall i \leq n \implies z \in B_{2\delta}(y, n).
\]
So, \( B_\delta(x, n) \subset B_{2\delta}(y, n) \) and \( \overline{B_\delta(x, n)} \subset B_{2\delta}(x, n) \subset B_{4\delta}(y, n) \), as we claimed.

We know that \( H \) and \( H^c \) are both dense on \( M \). Given a covering \( \mathcal{U} \in \mathcal{F}_N^\delta(H) \), we have 
\[
H \subset \bigcup_{B_\delta(x, n) \in \mathcal{U}} B_\delta(x, n) \implies M = \bigcup_{B_\delta(x, n) \in \mathcal{U}} \overline{B_\delta(x, n)} \subset \bigcup_{B_{4\delta}(y(x), n) \in \mathcal{V}(\mathcal{U})} B_{4\delta}(y(x), n).
\]
and 
\[
H^c \subset \bigcup_{B_{4\delta}(y(x), n) \in \mathcal{V}(\mathcal{U})} B_{4\delta}(y(x), n) \implies \mathcal{V}(\mathcal{U}) := \{B_{4\delta}(y(x), n)\} \in \mathcal{F}_N^\delta(H^c).
\]

Since \( M \) is compact, given \( \overline{X} \subset M \), there exists a countable set \( X_0 \subset X \) such that \( \overline{X}_0 \subset M \). So, since \( H \) and \( H^c \) are both dense on \( M \), there exist countable sets \( X_0 \subset H \) and \( Y_0 \subset H^c \) both dense on \( M \). Let \( X_0 = \{x_1, \ldots, x_k, \ldots\} \) and \( Y_0 = \{y_1, \ldots, y_k, \ldots\} \).

Lemma 3. Given \( \theta > 0 \) and a covering \( \mathcal{U} = \{B_\delta(x, n)\} \in \mathcal{F}_N^\delta(H) \), there exists a covering \( \mathcal{U}' = \{B_{2\delta}(x_i, n_i) \mid x_i \in X_0\} \in \mathcal{F}_N^\delta(H) \) such that \( n_i \) is a \((\sigma, \delta)\)-hyperbolic time for \( x_i \) and 
\[
\sum_{B_{2\delta}(x_i, n_i) \in \mathcal{U}'} e^{-\frac{\tau n_i}{\theta}} \leq \sum_{B_\delta(x, n) \in \mathcal{U}} e^{-\gamma n}.
\]

Proof. Take \( \gamma > 0 \) and \( \mathcal{U} \) such that 
\[
\tau := \sum_{B_\delta(x, n) \in \mathcal{U}} e^{-\gamma n} < \infty.
\]
Given \( a > 0 \), we have that 
\[
\sum_{i=1}^{\infty} e^{-ia} = \frac{e^{-a}}{1 - e^{-a}}
\]
We take \( a > N \) and \( n_i \in \mathbb{N} \) a \((\sigma, \delta)\)-hyperbolic time for \( x_i \) such that \( \frac{n_i}{1+\delta} \geq ia \) and also 
\[
\frac{e^{-a}}{1 - e^{-a}} \leq \tau \iff \frac{1}{e^a - 1} \leq \tau \iff \frac{1}{\tau} + 1 \leq e^a \iff \ln \left( \frac{\tau + 1}{\tau} \right) \leq a.
\]
We also can take the sequence \( n_i \) increasing. By considering the collection \( \mathcal{U}' = \{B_{2\delta}(x_i, n_i) \mid x_i \in X_0\} \), we have 
\[
X_0 \subset \bigcup_{i=1}^{\infty} B_\delta(x_i, n_i) \implies M = X_0 = \bigcup_{i=1}^{\infty} \overline{B_\delta(x_i, n_i)} \subset \bigcup_{B_{2\delta}(x, n) \in \mathcal{U}'} B_{2\delta}(x, n).
\]
and it means that \( \mathcal{U}' = \{ B_{2\delta}(x_i, n_i) \mid x_i \in X_0 \} \in \mathcal{F}_N^{2\delta}(H) \) is such that

\[
\sum_{B_{2\delta}(x_i, n_i) \in \mathcal{U}'} e^{-\frac{\gamma}{1 + \theta} m_i} \leq \sum_{i=1}^{\infty} e^{-\alpha - \gamma n} = \frac{e^{-\alpha}}{1 - e^{-\alpha}} \leq \sum_{B_{\delta}(x, n) \in \mathcal{U}} e^{-\gamma n}.
\]

□

**Remark 2.** We have that the collection \( \mathcal{U}' \) is a covering of \( M \) by open subsets and since \( M \) is compact, there exists a finite subcovering \( \mathcal{U}'' \). Then

\[
\sum_{B_{2\delta}(x, n) \in \mathcal{U}''} e^{-\gamma n} \leq \sum_{B_{\delta}(x, n) \in \mathcal{U}''} e^{-\gamma n}.
\]

It means that it is enough to consider finite coverings \( \mathcal{U}'' = \{ B_{2\delta}(x_i, n_i) \} \) of \( H \) where \( n_i \) is a \((\sigma,\delta)\)-hyperbolic time for \( x_i \) because we will consider the infimum of those sums.

**Lemma 4.** We have that \( P_{H^c}(0) = 0 < P_H(0) \).

**Proof.** Let \( \delta < \delta_1 \). We have

\[
m_f \left( 0, H^c, 8\delta, N, \frac{\gamma}{1 + \theta} \right) = \inf_{V \in \mathcal{F}_N^{2\delta}(H)} \left\{ \sum_{B_{8\delta}(y, n) \in V} e^{-\frac{\gamma}{1 + \theta} n} \right\} \leq \inf_{U \in \mathcal{F}_N^{2\delta}(H)} \left\{ \sum_{B_{8\delta}(x, n) \in U} e^{-\frac{\gamma}{1 + \theta} n} \right\} = \inf_{U \in \mathcal{F}_N^{2\delta}(H)} \left\{ \sum_{B_{2\delta}(x, n) \in U} e^{-\gamma n} \right\} = m_f \left( 0, H, \delta, N, \gamma \right).
\]

It implies that

\[
m_f \left( 0, H^c, 8\delta, N, \frac{\gamma}{1 + \theta} \right) \leq m_f \left( 0, H, \delta, N, \gamma \right) \Rightarrow m_f \left( 0, H^c, 8\delta, \frac{\gamma}{1 + \theta} \right) \leq m_f \left( 0, H, \delta, \gamma \right).
\]

So,

\[
P_{H^c}(0, 8\delta) = \inf \{ \rho > 0 \mid m_f \left( 0, H^c, 8\delta, \rho \right) = 0 \} \leq \inf \{ \gamma / (1 + \theta) > 0 \mid m_f \left( 0, H, \delta, \gamma \right) = 0 \} = \inf \{ \gamma > 0 \mid m_f \left( 0, H, \delta, \gamma \right) = 0 \} / (1 + \theta) = P_H(0, \delta) / (1 + \theta).
\]

By taking the limits when \( \delta \to 0 \), we have

\[
P_{H^c}(0) \leq P_H(0) / (1 + \theta) < P_H(0).
\]

Since it holds for every \( \theta > 0 \) we have

\[
P_{H^c}(0) = 0 < P_H(0).
\]

□

Since Lemma 4 shows that the null potential is hyperbolic, we obtain the Theorem 2. It is the base for our main result.

□

With Lemma 4 and the next lemma, we can see that the constant potentials are all hyperbolic.

**Lemma 5.** We have that \( P_{\Lambda}(\phi + c) = P_{\Lambda}(\phi) + c \), for all potential \( \phi \) and constant \( c \in \mathbb{R} \).
Proof. Let \( \psi := \phi + c \). We have \( R_{n,\delta}\psi(x) = R_{n,\delta}\phi(x) + nc \). So,

\[
m_f(\psi, \Lambda, \delta, N, \gamma) = \inf_{U \in \mathcal{F}_N(\Lambda)} \left\{ \sum_{B_\delta(y,n) \in U} e^{-\gamma n + R_{n,\delta}\psi(y)} \right\} = m_f(\phi, \Lambda, \delta, N, \gamma - c).
\]

It implies that \( m_f(\psi, \Lambda, \delta, \gamma) = m_f(\phi, \Lambda, \delta, N, \gamma - c) \). If \( \gamma > 0 \) is such that \( m_f(\psi, \Lambda, \delta, \gamma) = 0 \), then \( m_f(\phi, \Lambda, \delta, N, \gamma - c) = 0 \Rightarrow \gamma - c \geq P_\Lambda(\phi, \delta) \) or \( \gamma \geq P_\Lambda(\phi, \delta) + c \Rightarrow P_\Lambda(\psi, \delta) \geq P_\Lambda(\phi, \delta) + c \).

In the other hand, if \( m_f(\phi, \Lambda, \delta, N, \beta) = 0 \), then \( m_f(\psi, \Lambda, \delta, \beta + c) = 0 \) and \( P_\Lambda(\psi, \delta) \leq \beta + c \Rightarrow P_\Lambda(\psi, \delta) \leq P(\phi, \delta) + c \). It means that \( P_\Lambda(\psi, \delta) = P(\phi, \delta) + c \) or \( P_\Lambda(\psi) = P(\phi) + c \). \( \Box \)

The previous lemma also shows that if \( \phi \) is a hyperbolic potential, so is \( \phi + c, \forall c \in \mathbb{R} \). We can also obtain the following lemma.

**Lemma 6.** Let \( \varphi \) be a potential such that there exist \( \beta \in \mathbb{R} \) with

\[
R_{n,\delta}\varphi(x) \leq \beta, \forall x \in M, \forall n \geq 1.
\]

Then, \( P_\Lambda(\varphi) \leq P_\Lambda(0) \).

**Proof.**

\[
m_f(\varphi, \Lambda, \delta, N, \gamma) = \inf_{U \in \mathcal{F}_N(\Lambda)} \left\{ \sum_{B_\delta(y,n) \in U} e^{-\gamma n + R_{n,\delta}\varphi(y)} \right\} \leq \inf_{U \in \mathcal{F}_N(\Lambda)} \left\{ \sum_{B_\delta(y,n) \in U} e^{-\gamma n + \beta} \right\} \leq e^\beta \inf_{U \in \mathcal{F}_N(\Lambda)} \left\{ \sum_{B_\delta(y,n) \in U} e^{-\gamma n} \right\} = e^\beta m_f(0, \Lambda, \delta, N, \gamma).
\]

It implies that

\[
m_f(\varphi, \Lambda, \delta, \gamma) = \lim_{N \to +\infty} m_f(\varphi, \Lambda, \delta, N, \gamma) \leq e^\beta m_f(0, \Lambda, \delta, \gamma).
\]

So, \( m_f(0, \Lambda, \delta, \eta) = 0 \Rightarrow m_f(\varphi, \Lambda, \delta, \eta + \theta) = 0 \) and

\[
P_\Lambda(\varphi, \delta) = \inf\{ \gamma \mid m_f(\varphi, \Lambda, \delta, \gamma) = 0 \} \leq \inf\{ \eta \mid m_f(0, \Lambda, \delta, \eta) = 0 \} = P_\Lambda(0, \delta).
\]

Finally,

\[
P_\Lambda(\varphi) = \lim_{\delta \to 0} P_\Lambda(\varphi, \delta) \leq \lim_{\delta \to 0} P_\Lambda(0, \delta) = P_\Lambda(0).
\]

and the lemma is proved. \( \Box \)

**Lemma 7.** Let \( \varphi \) be a potential such that there exist \( \alpha, \beta > 0 \) with

\[
\alpha \leq R_{n,\delta}\varphi(x) \leq \beta, \forall x \in M, \forall n \geq 1.
\]

Then, \( P_{H^\varphi}(\varphi) \leq P_{H^\varphi}(0) = 0 < P_H(\varphi) \leq P_H(0) \).
Proof. It is enough to show that \(0 < P_H(\varphi)\) because Lemma 6 gives the other inequalities. We follow along the same lines as in Lemma 4.

Let \(\delta < \delta_1\). We have

\[
\begin{align*}
  m_f \left( 0, H^c, 8\delta, N, \frac{\gamma}{1 + \theta} \right) &= \inf_{\nu \in F_N^0(H^c)} \left\{ \sum_{B_{8\delta}(y,n) \in \nu} e^{-\frac{\gamma}{1 + \theta} n} \right\} \\
  &\leq \inf_{\nu(\mathcal{U}') \in \mathcal{F}_N^0(H)} \left\{ \sum_{B_{8\delta}(y,n) \in \nu(\mathcal{U}')} e^{-\frac{\gamma}{1 + \theta} n} \right\} \\
  &= \inf_{\mathcal{U}' \in \mathcal{F}_N^0(H)} \left\{ \sum_{B_{8\delta}(x,n) \in \mathcal{U}'} e^{-\gamma n} \right\} \\
  &\leq e^\alpha \inf_{\mathcal{U}' \in \mathcal{F}_N^0(H)} \left\{ \sum_{B_{8\delta}(x,n) \in \mathcal{U}'} e^{-\gamma n + R_n, 8\delta \varphi(x)} \right\} = m_f (\varphi, H, \delta, N, \gamma).
\end{align*}
\]

It implies that

\[
\begin{align*}
  m_f \left( 0, H^c, 8\delta, N, \frac{\gamma}{1 + \theta} \right) &\leq m_f (\varphi, H, \delta, N, \gamma) \\
  \iff m_f \left( 0, H^c, 8\delta, \frac{\gamma}{1 + \theta} \right) &\leq m_f (\varphi, H, \delta, \gamma).
\end{align*}
\]

So,

\[
\begin{align*}
P_{H^c}(0, 8\delta) &= \inf \{ \rho > 0 \mid m_f(0, H^c, 8\delta, \rho) = 0 \} \\
  &\leq \inf \{ \gamma/(1 + \theta) > 0 \mid m_f(\varphi, H, \delta, \gamma) = 0 \} \\
  &= \inf \{ \gamma > 0 \mid m_f(\varphi, H, \delta, \gamma) = 0 \} / (1 + \theta) = P_H(\varphi, \delta)/(1 + \theta).
\end{align*}
\]

By taking the limits when \(\delta \to 0\), we have

\[
P_{H^c}(0) \leq P_H(\varphi)/(1 + \theta) < P_H(\varphi).
\]

Since it holds for every \(\theta > 0\) we have

\[
P_{H^c}(0) = 0 < P_H(\varphi).
\]

and the lemma is proved. \(\square\)

In particular, we have

\[
P_{H^c}(\varphi) \leq P_{H^c}(0) \text{ and } P_H(\varphi) \leq P_H(0).
\]

We recall that a hyperbolic potential \(\varphi\) satisfies

\[
P_{H^c}(\varphi) < P_H(\varphi).
\]

It means that the potential \(\varphi\) is hyperbolic:

\[
P_{H^c}(\varphi) \leq P_{H^c}(0) = 0 < P_H(\varphi) \leq P_H(0).
\]

Lemma 8. Let \(\phi : M \to N\) with its Birkhoff sums uniformly bounded, that is, there exists \(r > 0\) such that

\[
|S_n \phi(x)| < r, \forall n \in \mathbb{N}, \forall x \in M.
\]

Then, \(\phi\) is a hyperbolic potential.
Proof. In fact, by following the proof of Lemma 6 we can see that \( P_{H^c}(\phi) \leq P_{H^c}(0) = 0 \). Also, Birkhoff’s Ergodic Theorem implies that \( \int \phi \, d\eta = 0 \) for every \( f \)-invariant probability \( \eta \). It means that \( P(\phi) = \sup_{\eta \in M_f(M)} \left\{ h_\eta(f) + \int \phi \, d\eta \right\} = \sup_{\eta \in M_f(M)} \left\{ h_\eta(f) \right\} = h(f) = P(0) = P_{H}(0) > 0 \).

So, \( \phi \) is a hyperbolic potential. □

Remark 3. The previous lemma proved that the potential \( \phi \) is hyperbolic if we have \( \int \phi \, d\eta = 0 \) for every invariant measure \( \eta \). In particular, if we have the Birkhoff sums uniformly bounded.

Lemma 9. If \( \phi \leq \psi \), then \( P_{\Lambda}(\phi) \leq P_{\Lambda}(\psi) \).

Proof. If \( \phi \leq \psi \), we obtain \( R_{n,\delta} \phi(x) \leq R_{n,\delta} \psi(x) \). So,

\[
m_f(\phi, \Lambda, \delta, N, \gamma) = \inf_{U \in F_N(\Lambda)} \left\{ \sum_{B_{\delta}(y,n) \in U} e^{-\gamma n + R_{n,\delta} \phi(y)} \right\} \\
\leq \inf_{U \in F_N(\Lambda)} \left\{ \sum_{B_{\delta}(y,n) \in U} e^{-\gamma n + R_{n,\delta} \psi(y)} \right\} = m_f(\psi, \Lambda, \delta, N, \gamma) \Rightarrow
\]

\[
m_f(\phi, \Lambda, \delta, \gamma) = \lim_{N \to \infty} m_f(\phi, \Lambda, \delta, N, \gamma) \leq \lim_{N \to \infty} m_f(\psi, \Lambda, \delta, N, \gamma) = m_f(\psi, \Lambda, \delta, \gamma).
\]

It implies that \( m_f(\phi, \Lambda, \delta, \gamma) = 0 \) if \( m_f(\psi, \Lambda, \delta, \gamma) = 0 \), that is,

\[
P_{\Lambda}(\phi, \delta) = \inf \{ \gamma \mid m_f(\phi, \Lambda, \delta, \gamma) = 0 \} \leq \inf \{ \gamma \mid m_f(\psi, \Lambda, \delta, \gamma) = 0 \} = P_{\Lambda}(\psi, \delta).
\]

Finally,

\[
P_{\Lambda}(\phi) = \lim_{\delta \to 0} P_{\Lambda}(\phi, \delta) \leq \lim_{\delta \to 0} P_{\Lambda}(\psi, \delta) = P_{\Lambda}(\psi).
\]

□

With the previous lemma we can obtain the following example.

Example 1. Let \( \varphi : M \to \mathbb{R} \) be a hyperbolic potential and \( \phi : M \to \mathbb{R} \) such that

\[
\max \phi - \min \phi < P_H(\varphi) - P_{H^c}(\varphi).
\]

It implies that

\[
P_{H^c}(\varphi + \phi) \leq P_{H^c}(\varphi + \max \phi) = P_{H^c}(\varphi) + \max \phi < P_H(\varphi) + \min \phi = P_H(\varphi + \min \phi) \leq P_H(\varphi + \phi).
\]

So, \( \varphi + \phi \) is a hyperbolic potential.

If \( |t| \leq 1 \), we also have

\[
\max t \phi - \min t \phi < P_H(\varphi) - P_{H^c}(\varphi).
\]

and \( \varphi + t \phi \) is also a hyperbolic potential.

In particular, since the null potential is hyperbolic, if we have

\[
\max \phi - \min \phi < P_H(0) = P(0) = h(f),
\]

then \( \phi \) is also a hyperbolic potential.
Remark 4. We observe that the set of continuous hyperbolic potentials is open with respect to the $C^0$ topology, as can be see in [1], Proposition 3.1.

Example 2. Now, for Viana maps, we construct a potential with Birkhoff sums uniformly bounded.

Let $B$ be an open set and $V = f^{-1}(B)$ such that $V \cap B = \emptyset$ and $V \cap \mathcal{C} = \emptyset$, where $\mathcal{C}$ is the critical set. Let $\phi : \overline{B} \to \mathbb{R}$ be a $C^\infty$ function such that $\phi_{|\partial B} \equiv 0$ and we define a potential $\varphi : X \to \mathbb{R}$ as

$$
\varphi(x) = \begin{cases} 
\phi(x), & \text{if } x \in B \\
-\phi(f(x)), & \text{if } x \in V \\
0, & \text{if } x \in (V \cup B)^c 
\end{cases}
$$

Claim 1. The Birkhoff sums $S_n \varphi$ are uniformly bounded.

Proof. For $x \in V$, we have that

$$
S_n \varphi(x) = \varphi(x) + \varphi(f(x)) + \cdots + \varphi(f^{n-1}(x)) = \\
-\phi(f(x)) + \phi(f(x)) + 0 + \cdots + \phi(f^{n-1}(x)) \leq \sup \phi,
$$

For $x \in B$, we have

$$
S_n \varphi(x) = \varphi(x) + \varphi(f(x)) + \cdots + \varphi(f^{n-1}(x)) \leq \sup \phi - \inf \phi,
$$

if $f^{n-1}(x) \in V$ and it is equal to $\phi(x)$, otherwise.

For $x \in (V \cup B)^c$, we have at most the same estimate for $S_n \varphi(x)$ because the orbit of $x$ may intersect $V \cup B$.

□

So, the Birkhoff sums are uniformly bounded and Lemma 3 guarantees that $\varphi$ is hyperbolic. Moreover, $\varphi$ is Hölder, which means that we have existence and uniqueness of equilibrium state.

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