VOLUME-PRESERVING FLOW BY POWERS OF THE $m$TH MEAN CURVATURE IN THE HYPERBOLIC SPACE

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Abstract. This paper concerns closed hypersurfaces of dimension $n \geq 2$ in the hyperbolic space $\mathbb{H}^{n+1}$ of constant sectional curvature $\kappa$ evolving in direction of its normal vector, where the speed is given by a power $\beta (\geq 1/m)$ of the $m$th mean curvature plus a volume preserving term, including the case of powers of the mean curvature and of the Gauß curvature. The main result is that if the initial hypersurface satisfies that the ratio of the biggest and smallest principal curvature is close enough to 1 everywhere, depending only on $n$, $m$, $\beta$ and $\kappa$, then under the flow this is maintained, there exists a unique, smooth solution of the flow for all times, and the evolving hypersurfaces exponentially converge to a geodesic sphere of $\mathbb{H}^{n+1}$, enclosing the same volume as the initial hypersurface.

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1. Introduction

Let $M^n$ be a smooth, compact oriented manifold of dimension $n \geq 2$ without boundary, $(N^{n+1}, \bar{g})$ be an $(n+1)$-dimensional completed Riemannian manifold, and $X_0 : M^n \to N^{n+1}$ a smooth immersion. Consider a one-parameter family of smooth immersions: $X_t : M^n \to N^{n+1}$.
\( N^{n+1} \) evolving according to
\[
\frac{d}{dt} X(p, t) = \{ \tilde{F}(t) - F(\mathcal{W}(p, t)) \} \nu(p, t), \quad p \in M^n,
\]
(1.1)
where \( \nu(p, t) \) is the outer unit normal to \( M_t = X_t(M^n) \) at the point \( X(p, t) \) in the tangent space \( TN^{n+1} \), \( \mathcal{W}_\lambda(p, t) = -\mathcal{W}_\nu(p, t) \) is the matrix of Weingarten map on the tangent space \( TM^n \) induced by \( X_t \), \( \lambda \) is the map from \( T^n M^n \otimes TM^n \) to \( R^n \) which gives the eigenvalues of the map \( \mathcal{W} \), \( F \) is a smooth symmetric function, and \( \tilde{F}(t) \) is the average of \( F \) on \( M_t \):
\[
\tilde{F}(t) = \frac{\int_{M_t} F(\lambda(\mathcal{W})) \, d\mu_t}{\int_{M_t} \, d\mu_t},
\]
(1.2)
where \( d\mu_t \) denotes the surface area element of \( M_t \). As is clear for the presence of the global term \( \tilde{F}(t) \) in equation (1.1), the flow keeps the volume of the domain \( \Omega_t \) enclosed by \( M_t \) constant.

This paper considers the flow (1.1) with the speed \( F(\lambda) \) given by a power of an \( m \)th mean curvature, namely
\[
F(\lambda_1, \ldots, \lambda_n) = H_m^\beta,
\]
(1.3)
where \( (\lambda_1, \ldots, \lambda_n) \) are the principal curvatures of the evolving hypersurfaces \( M_t \), and for any \( m = 1, \ldots, n \), the \( m \)th mean curvature \( H_m \) is the average of the \( m \)th elementary symmetric functions \( E_m \), namely
\[
H_m = \left( \begin{array}{c} n \end{array} \right)^{-1} E_m = \frac{m!(n-m)!}{n!} \sum_{1 \leq i_1 < \cdots < i_m \leq n} \lambda_{i_1} \cdots \lambda_{i_m}.
\]
(1.4)
Obviously \( H_1 = H/n \) and \( H_n = K \), where \( H \) and \( K \) denote the mean curvature and the Gauß-Kronecker curvature respectively.

For the flow (1.1) without the volume constraint term \( \tilde{F}(t) \), in the case when \( N^{n+1} \) is the Euclidean space \( R^{n+1} \), there are many papers which consider the evolution of convex hypersurfaces, of particular interest here is in the analysis of the flow (1.1) where the speed \( F(\lambda) \) is homogeneous of degree one in the principal curvatures, beginning with a classical result of Huisken [29] who proved that any closed convex hypersurface under mean curvature flow shrinks to a round point in finite time (Huisken’s theorem may be considered as an extension of the theorem of Gage and Hamilton [25] to dimensions bigger than one), and including the similar results on the \( n \)th root of Gauß-Kronecker curvature [20], the square root of scalar curvature [21], and a large family of other speeds [3, 9]. The first such result with degree of homogeneity greater than one was due to Chow [20] who considered flow by powers of the Gauß-Kronecker curvature. He proved that the evolving hypersurfaces by \( K^\beta \) with \( \beta \geq 1/n \) become spherical as they shrink to a point provided the initial hypersurface \( M_0 \) is sufficiently pinched, we also mention that Tso [49] showed the same result for the Gauß-Kronecker curvature flow and Andrews [6] proved that the limit of the solutions under \( K^\beta \)-flow with \( \beta \in \left( \frac{1}{n+2}, \frac{1}{n} \right) \) evolve purely by homothetic contraction to a point in finite time. Later such results were proved by Schulze [46] for the flow by powers of the mean curvature, by Alessandroni and Sinestrari [1] for the flow by powers of the scalar curvature, by Andrews [11] for the flow of convex hypersurfaces with pinched principal curvatures by high powers of curvature, and for such flows in the special case of surfaces in three-dimensional spaces [5, 10, 13, 46], where the lower dimension allows a more complete understanding of the
equation for the evolution of the second fundamental form. However, when $N^{n+1}$ is a more general Riemannian manifold, there are few results on the behavior of these flows: Huisken [30] extended the result of [29] in $\mathbb{R}^{n+1}$ to compact hypersurfaces in general Riemannian manifolds with suitable bounds on curvature. Andrews [4] has considered a flow which takes any compact hypersurface with principal curvatures greater than $\sqrt{c}$ with $c > 0$ in a Riemannian background space with sectional curvatures at least $-c$, and converges to a round point in finite time.

The volume-preserving versions of these flows are the flows (1.1)-(1.2) with an extra term $\bar{F}(t)$ which balances the contraction. In the case of volume-preserving mean curvature flow, Huisken [31] showed that convex hypersurfaces remain convex for all time and converge exponentially fast to round spheres (the corresponding result for curves in the plane is due to Gage [24]), while Andrews [7] extended this result to the smooth anisotropic mean curvature flow, and McCoy showed similar results for the surface area preserving mean curvature flow [38] and the mixed volume preserving mean curvature flows [39]. The volume-preserving flow has been used to study constant mean curvature surfaces between parallel planes [12, 13] and to find canonical foliations near infinity in asymptotically flat spaces arising in general relativity [32] (Rigger [32] showed analogous results in the asymptotically hyperbolic setting). If the initial hypersurface is sufficiently close to a fixed Euclidean sphere (possibly non-convex), Escher and Simonett [23] proved that the flow converges exponentially fast to a round sphere, a similar result for average mean convex hypersurfaces with initially small traceless second fundamental form is due to Li [34]. For a general ambient manifold, Alikakos and Freire [2] proved long time existence and convergence to a constant mean curvature surface under the hypotheses that the initial hypersurface is close to a small geodesic sphere and that it satisfies some non-degenerate conditions. While, Cabezas-Rivas and Miquel exported the Euclidean results of [12, 13] to revolution hypersurfaces in a rotationally symmetric space [18], and showed the same results as Huisken [31] for a hyperbolic background space [17] by assuming the initial hypersurface is horospherically convex (the definition will be given later).

On the other hand, there are few results on speeds different from the mean curvature: McCoy [40] proved the convergence to a sphere for a large class of function $F$ homogeneous of degree one (including the case $F = H^\beta_m$ with $m\beta = 1$), Makowski showed that the mixed volume preserving curvature flow for a function $F$ homogeneous of degree one, starting with a compact and strictly horospheres-convex hypersurface in the hyperbolic space exponentially converges to a geodesic sphere [30], and the volume preserving curvature flow in Lorentzian manifolds for $F$ as a function with homogeneous of degree one exponential converges to a hypersurface of constant $F$-curvature [37] (moreover, stability properties and foliations of such a hypersurface was also examined). In 2010 Cabezas-Rivas and Sinestrari [19] studied the deformation of convex hypersurfaces in $\mathbb{R}^{n+1}$ by a speed of the form (1.4) for some power $\beta \geq 1/m$. In this way $F$ is a homogeneous function of the curvatures with a degree $m\beta \geq 1$. In particular, they proved the following

**Theorem 1.1.** For $m \in \{1, \ldots, n\}$, $m\beta \geq 1$ there exists a positive constant $C = C(n, m, \beta) < 1/n^n$ such that the following holds: If the initial hypersurface of $\mathbb{R}^{n+1}$ is pinched in the sense that

\begin{equation}
K(p) > CH^\sigma(p) > 0 \quad \text{for all} \quad p \in M^n,
\end{equation}
then the flow (1.1)-(1.3) with $F$ given by (1.4), has a unique and smooth solution for all times, inequality (1.5) remains everywhere on the evolving hypersurfaces $M_t$ for all $t > 0$ and the $M_t$’s converge, exponentially in the $C^\infty$-topology, to a round sphere of the same volume as $M_0$.

However, the results of [19] do not close relate to the ambient space, we face the challenges of extending the above results to hypersurface to more general ambient spaces. But not every Riemannian manifold is well suited to deal with the situation analogous to the setting in Euclidean spaces. We want to consider the case that the ambient space is a simply connected Riemannian manifold of constant sectional curvature $\kappa (< 0)$ whose flow behaves quite differently compared to the Euclidean space to a certain extent.

Set $a = \sqrt{|\kappa|}$. $N^{n+1}_\kappa$ is isometric to the hyperbolic space $\mathbb{H}^{n+1}_\kappa$ of radius $1/a$:

$$\mathbb{H}^{n+1}_\kappa := \{p \in L^{n+2} : \langle p, p \rangle = -\frac{1}{a^2}\}.$$ Here $(L^{n+2}, \langle \cdot, \cdot \rangle)$ denotes the $(n+2)$-dimensional Lorentz-Minkowski space. To consider the flow (1.1)-(1.3) in $N^{n+1}_\kappa$ is then equivalent to consider the flow (1.1)-(1.3) in $\mathbb{H}^{n+1}_\kappa$. Now, it is necessary to provided some definitions as in [14, 17] as following.

**Definition 1.2.** A horosphere $\mathcal{H}$ of $\mathbb{H}^{n+1}_\kappa$ is the limit of a geodesic sphere of $\mathbb{H}^{n+1}_\kappa$ as its center goes to the infinity along a fixed geodesic ray.

**Definition 1.3.** An horoball $\mathcal{H}$ is the convex domain whose boundary is a horosphere.

**Definition 1.4.** A hypersurface $M$ of $\mathbb{H}^{n+1}_\kappa$ is said to be convex by horospheres ($h$-convex for short) if it bounds a domain $\Omega$ satisfying that for every $p \in M = \partial \Omega$, there is a horosphere $\mathcal{H}$ of $\mathbb{H}^{n+1}_\kappa$ through $p$ such that $\Omega$ is contained in $\mathcal{H}$ of $\mathbb{H}^{n+1}_\kappa$ bounded by $\mathcal{H}$.

**Remark 1.5.** In fact, Currier in [22] showed that $h$-convex immersions of smooth compact hypersurfaces are embedded spheres, and Borisenko and Miquel in [14] showed that horosphere $\mathcal{H}$ of $\mathbb{H}^{n+1}_\kappa$ is weakly (strictly) $h$-convex if and only if all its principal curvatures are (strictly) bounded from below by $a$ at each point.

Most of the literature mentioned above requires a pinching condition on the initial hypersurface, so that parabolical maximum principles, an important tool in the investigation of evolution equations, can be used to deduce that they can converge and become spherical in shape as the final time is approached under these flows. It is well-known that in hyperbolic spaces the negative curvature of the background space produces terms which introduces the maximum principles either fail or become more subtle for our flow (1.1)-(1.3). So for our purposes a challenge in the hyperbolic ambient setting is how to find a suitable pinching condition on the initial hypersurfaces. However in the hyperbolic space there is an intuitive example, as pointed out by Cabezas-Rivas and Miquel in [17]: a geodesic sphere, moving outward in the radial direction with the speed $H^\beta_m$, its normal curvature decreases, and it becomes nearer and nearer to that of horosphere, but it never gets $h$-convex. This fact leads us to hope for the result by choosing a suitable convex hypersurface which is sufficiently positively curved to overcome the obstructions from the negative curvature imposed by the ambient spaces like that of space of Cabezas-Rivas and Sinestrari [19]. More precisely, denote the turbulent second fundamental form by $\tilde{h}_{ij} := h_{ij} - a g_{ij}$, then the turbulent mean curvature $\tilde{H} = H - na$ and the turbulent Gauß curvature $\tilde{K} = \det \{\tilde{h}^i_j\}$. In this paper the
turbulent geometric quantities are distinguished by a tilde. Compared with the pinching condition $K(p) > CH^p(p) > 0$ on the initial hypersurface in Theorem 1.1 which is analogous to the initial pinching condition in Chow [20] and Schulze [46], it is natural to impose a pinching condition $\tilde{K} > C^*\hat{H}^n > 0$ on the initial hypersurfaces of $\mathbb{H}_\kappa^{n+1}$, where $C^*$ is a suitable positive constant. It is shown in Section 4 that the condition $\tilde{K} > C^*\hat{H}^n > 0$ on a closed hypersurface implies in particular the $h$-convexity of the hypersurface. The aim of this paper is to achieve such extension of the above Theorem 1.1 of Cabezas-Rivas and Sinestrari [19] in Hyperbolic case. Precisely, we prove the following

Theorem 1.6 (main theorem). For $m \in \{1, \ldots, n\}$, $m\beta \geq 1$ there exists a positive constant $C^* = C^*(a,n,m,\beta) < 1/n^\kappa$ such that the following holds: If the initial hypersurface of $\mathbb{H}_\kappa^{n+1}$ is pinched in the sense that

$$K(p) > C^*\hat{H}^n(p) > 0 \quad \text{for all} \quad p \in M^n,$$

then the flow (1.1)-(1.3) with $F$ given by (1.4), has a unique and smooth solution for all times, inequality (1.6) remains everywhere on the evolving hypersurfaces $M_t$ for all $t > 0$ and the $M_t$'s converge, exponentially in the $C^\infty$-topology, to a geodesic sphere of $\mathbb{H}_\kappa^{n+1}$ enclosing the same volume as $M_0$.

Our analysis follows the framework of [19], we make modifications to consider our problem for the background space. The rest of the paper is organized as follows: Section 2 first gives some useful preliminary results employed in the remainder of the paper. Section 3 contains details of the short time existence of the flow (1.1)-(1.3) and the induced evolution equations of some important geometric quantities and the corresponding turbulent quantities, this requires only minor modifications of Euclidean case due to the background curvature. In Section 4 applying the maximum principle to the evolution equation of the turbulent quantity $\tilde{K}/\hat{H}^n$ gives that if the initial hypersurface is pinched good enough then this is preserved for $t > 0$ as long as the flow (1.1)-(1.3) exists. This is a fundamental step in our procedure as in most of the literature quoted above. Furthermore, Section 5 proves the uniform boundedness of the speed $F$ by following a method which was firstly used by Tso [49]. Using more sophisticated results for fully nonlinear elliptic and parabolic partial differential equations, Section 6 obtains uniform bounds on all derivatives of the curvature and proves long time existence of the flow (1.1)-(1.3). Finally Section 7, following the idea in [39], obtains the lower bound for $\hat{H}$, which we infer from a Harnack inequality due to Makowski [36], the estimates obtained so far will then allow us to prove that these evolving hypersurfaces converge to a geodesic sphere of $\mathbb{H}_\kappa^{n+1}$ smoothly and exponentially.

2. Notation and preliminary results

From now on, use the same notation as in [18, 29, 45] for local coordinates $\{x^i\}$, $1 \leq i \leq n$, near $p \in M^n$ and $\{y^\alpha\}$, $0 \leq \alpha, \beta \leq n$, near $F(p) \in \mathbb{H}_\kappa^{n+1}$. Denote by a bar all quantities on $\mathbb{H}_\kappa^{n+1}$, for example by $\bar{g} = \{\bar{g}_{\alpha\beta}\}$ the metric, by $\bar{g}^{-1} = \{\bar{g}^{\alpha\beta}\}$ the inverse of the metric, by $\nabla$ the covariant derivative, by $\Delta$ the rough Laplacian, and by $\bar{R} = \{\bar{R}_{\alpha\beta\gamma\delta}\}$ the Riemann curvature tensor. Components are sometimes taken with respect to the tangent vector fields $\partial_\alpha (= \frac{\partial}{\partial x^\alpha})$ associated with a local coordinate $\{y^\alpha\}$ and sometimes with respect to a moving orthonormal frame $e_\alpha$, where $\bar{g}(e_\alpha, e_\beta) = \delta_{\alpha\beta}$. The corresponding geometric quantities on $M^n$ will be denoted by $g$ the induced metric, by $g^{-1}$ the inverse of $g$, $\nabla, \Delta, R, \bar{R}, \partial_\alpha$ and $e_\alpha$ the
covariant derivative, the rough Laplacian, the curvature tensor, the natural frame fields and
a moving orthonormal frame field, respectively. Then further important quantities are the
second fundamental form $A(p) = \{h_{ij}\}$ and the Weingarten map $\mathcal{W} = \{g^{ik}h_{kj}\} = \{h^i_j\}$ as a
symmetric operator and a self-adjoint operator respectively. The eigenvalues $\lambda_1(p) \leq \cdots \leq \lambda_n(p)$ of $\mathcal{W}$ are called the principal curvatures of $X(M^n)$ at $X(p)$. The mean curvature is
given by
\[ H := \text{tr}_g \mathcal{W} = h^i_i = \sum_{i=1}^n \lambda_i, \]
the square of the norm of the second fundamental form by
\[ |A|^2 := \text{tr}_g (\mathcal{W}^t \mathcal{W}) = h^i_j h^j_i = \sum_{i=1}^n \lambda_i^2, \]
and the Gauß-Kronecker curvature by
\[ K := \text{det}(\mathcal{W}) = \text{det}\{h^i_j\} = \prod_{i=1}^n \lambda_i. \]
More generally, the $m$th elementary symmetric functions $E_m$ are given by
\[ E_m(\lambda) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} \lambda_{i_1} \cdots \lambda_{i_m} = \frac{1}{m!} \sum_{i_1, \ldots, i_m} \lambda_{i_1} \cdots \lambda_{i_m}, \quad \text{for } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n, \]
and the $m$th mean curvature $H_m$ are given by (1.4). Since $H_m$ is homogeneous of degree $m$, the speed $F$ is homogeneous of degree $m \beta$ in the curvatures $\lambda_i$. Denote the vector $(\lambda_1, \ldots, \lambda_n)$ of $\mathbb{R}^n$ by $\lambda$ and the positive cone by $\Gamma_+ \subset \mathbb{R}^n$, i.e.
\[ \Gamma_+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i > 0, \forall \ i\}. \]
It is clear that $H$, $K$, $H_m$, $F$ may be viewed as functions of $\lambda$, or as functions of $A$, or as functions of $\mathcal{W}$, or also functions of space and time on $M_t$. Since the differentiability properties of these functions are the same in our setting, we do not distinguish between
these notions and write always these functions the same letters in all cases. We use the notation
\[ \dot{F}^i := \frac{\partial F}{\partial \lambda_i}, \quad \dot{F}^{ij} := \frac{\partial F}{\partial h^i_j}, \quad \text{and } \ddot{F}^j := \frac{\partial F}{\partial h^j_i}. \]
If $B$, $C$ are matrices, we write
\[ \dot{F}B = \dot{F}(B) := \dot{F}^j_i B^i_j \quad \text{and } \ddot{F}(B, C) := \frac{\partial^2 F}{\partial h^i_j \partial h^k_l} B^i_j C^k_l. \]
Hess$\mathcal{W}$ will denote the second tensorial derivative as a 2-covariant tensor. If it is contracted
by the standard metric $g$ we have the standard Laplace-Beltrami operator: for any tensor
$T$, we write
\[ g^{-1}\text{Hess}_{\mathcal{W}}(T) = g^{ij} \nabla_i \nabla_j T = \Delta T. \]
More generally, given a $(2,0)$-tensor $w$, we denote
\[ \Delta_w T := w^{ij} \nabla_i \nabla_j T. \]
We also denote
\[ |B|^2_w := w^{ij} B_i B_j. \]
Finally, if $F \in C^2(\Gamma_+)$ is concave, then $F$ is also concave as a curvature function depending on $\{h_j^j\}$.

We note some important properties of $H_m$ (see [19] for a simple derivation).

**Lemma 2.1.** Let $1 \leq m \leq n$ be fixed, the following hold

i) The $m$th roots $H_m^{1/m}$ are concave in $\Gamma_+$.

ii) For $\forall i$, $\frac{\partial H_m}{\partial h_i}(\lambda) > 0$, where $\lambda \in \Gamma_+$.

iii) $H_m^{1/m} \leq \frac{H}{n}$; equivalently, $F \leq \left(\frac{H}{n}\right)^{-\frac{m\beta}{1-m\beta}}$.

iv) $\text{tr}(\dot{F}) \geq m\beta F^{1-\frac{1}{m\beta}}$.

The following algebraic property proved by Schulze in ([46], Lemma 2.5) will be needed in the last section.

**Lemma 2.2.** For any $\varepsilon > 0$ assume that $\lambda_i \geq \varepsilon H > 0$, $i = 1, \ldots, n$, at some point of an $n$-dimensional hypersurface. Then at the same point there exists a $\delta = \delta(\varepsilon, n) > 0$ such that

$$\det\left(\frac{\mathbf{A}}{H_n^2} - H_n^2\right) \geq \delta \left(\frac{1}{n^n} - \frac{K}{H^n}\right).$$

Consider the functions as in [17]:

$$s_\kappa(x) = \frac{\sinh(\sqrt{|\kappa|}x)}{\sqrt{|\kappa|}} = \frac{\sinh(ax)}{a}, \quad c_\kappa(r) = s'_\kappa(x),$$

$$t_\kappa(x) = \frac{s_\kappa(x)}{c_\kappa(x)}, \quad \cot_\kappa(x) = \frac{1}{t_\kappa(x)}.$$ 

Denote $r_p$ the function “distance to $p$” in $\mathbb{H}_n^{n+1}$ and use the notation $\partial_{r_p} = \nabla r_p$. And denote the component of $\partial_{r_p}$ by $\partial_{r_p}^\top$ tangent to $M_t$, which satisfies $\partial_{r_p} = \nabla(r_p|_{M_t})$. Define the inner radius $\rho_-$ by

$$\rho_-(t) = \sup\{r : B_r(q) \text{ is enclosed by } M_t \text{ for some } q \in \mathbb{H}_n^{n+1}\}$$

where $B_r(q)$ is the geodesic ball of radius $r$ with centered at $q$. The following well-known result for $h$-convex hypersurfaces in $\mathbb{H}_n^{n+1}$ will be applied in later sections.

**Lemma 2.3.** Let $\Omega$ be a compact $h$-convex domain, $o$ the center of an inball of $\Omega$, $\rho_-$ its inner radius. Furthermore let $\tau := t_\kappa\left(\frac{a\rho_-}{H_n}\right)$, then

i) the maximal distance $\text{max}d(o, \partial \Omega)$ between $o$ and the points in $\partial \Omega$ satisfies the inequality

$$\text{max}d(o, \partial \Omega) \leq \rho_- + a \ln\left(1 + \sqrt{\tau}\right) < \rho_- + a \ln 2.$$ 

ii) For any interior point $p$ of $\Omega$, $(p, \partial r) \geq a\tan(\text{dist}(p, \partial \Omega))$, where dist denotes the distance in the ambient space $\mathbb{H}_n^{n+1}$.

**Proof.** See ([14], Theorem 3.1) for the proof. \qed
Our analysis relies on many a priori estimates on the Hölder norms of the solutions to elliptic and parabolic partial differential equations in the Euclidean space and regularity issues, which leads to existence for all time and the convergence. We know that, in the case of a function depending on space and time, there is a suitable definition of Hölder norm which is adapted to the purposes of parabolic equations (see e.g. [35]). In addition to the standard Schauder estimates for linear equations, we use in the paper some more recent results which are collected here. The estimates below hold for suitable classes of weak solutions; for the sake of simplicity, we state them in the case of a smooth classical solution, which is enough for our purposes. These are used, for example, by Cabezas-Rivas and Sinestrari [19].

Given \( r > 0 \), we denote by \( B_r \) the ball of radius \( r > 0 \) in \( \mathbb{R}^n \) centered at the origin. First we recall a well known result of Krylov and Safonov, which applies to linear parabolic equations.

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The above result follows from Theorem 3 in [15] (see also Theorem 8.1 in [16]). It generalizes, by a perturbation method, a priori estimates for the solutions of linear elliptic second-order equations to the viscosity solutions, due to Evans and Krylov, about equations with concave dependence on the hessian. In contrast with Evans-Krylov result (see e.g. inequality (17.42) in [26]), Theorem 2.4 gives an estimate in terms of the $C^0$-norm of $f$ rather than the $C^2$-norm, and this is essential for our purposes.

3. Short time existence and evolution equations

This section first consider short time existence for the initial value problem (1.1)-(1.3).

**Theorem 3.1.** Let $X_0 : M^n \to \mathbb{H}^{n+1}$ be a smooth closed hypersurface with mean curvature strictly bounded from below by $m$ everywhere. Then there exists a unique smooth solution $X_t$ of problem (1.1)-(1.3), defined on some time interval $[0,T)$, with $T > 0$.

**Proof.** We can argue exactly as in [19, Theorem 3.1], although the assumptions on the initial hypersurface and the ambient background space in that paper are different, the proof applies to our case as well. \hfill $\square$

Proceeding now exactly as in [29] we derive some evolution equations on $M_t$ from the basic equation (1.1)-(1.3).

**Proposition 3.2.** For the ambient space $N^{n+1} = \mathbb{H}^{n+1}$, on any solution $M_t$ of (1.1)–(1.3) the following hold:

\begin{align}
(3.1) \quad & \partial_t g = 2(\bar{F} - F)A, \\
(3.2) \quad & \partial_t g^{-1} = -2(\bar{F} - F)g^{-1}\mathcal{W}, \\
(3.3) \quad & \partial_t \nu = X_s(\nabla F), \\
(3.4) \quad & \partial_t (d\mu_t) = (\bar{F} - F)Hd\mu_t, \\
(3.5) \quad & \partial_t A = \Delta_{\bar{F}} A + \bar{F}\langle \nabla \mathcal{W}, \nabla \mathcal{W} \rangle + \left[\text{tr}_{\bar{F}}(A\mathcal{W}) + a^2\text{tr}(\bar{F})\right]A \\
& + \left[\bar{F} - (m\beta + 1)F\right]A\mathcal{W} + a^2\left[\bar{F} - (m\beta + 1)F\right]g, \\
(3.6) \quad & \partial_t \mathcal{W} = \Delta_{\bar{F}} \mathcal{W} + \bar{F}\langle \nabla \mathcal{W}, \nabla \mathcal{W} \rangle + \left[\text{tr}_{\bar{F}}(A\mathcal{W}) + a^2\text{tr}(\bar{F})\right]\mathcal{W} \\
& - \left[\bar{F} + (m\beta - 1)F\right]\mathcal{W}^2 + a^2\left[\bar{F} - (m\beta + 1)F\right]\text{Id}.
\end{align}

**Proof.** The first fourth evolution equations under (1.1)–(1.3) follow from straightforward computation as in §3 of [29], and valid in an arbitrary Riemannian manifold.

The evolution of $A$ can be calculated from the definition of $A$:

\[
\partial_t h_{ij} = -\frac{\partial}{\partial t} \langle \nabla_{X_s(\partial_i)} X_s(\partial_j), \nu \rangle.
\]

\[
= -\langle \nabla_{X_s(\partial_i)} \nabla_{X_s(\partial_j)} X_s(\partial_j), \nu \rangle - \langle \nabla_{X_s(\partial_i)} X_s(\partial_j), \nu \rangle - \langle \nabla_{X_s(\partial_i)} X_s(\partial_j), \nu \rangle.
\]

\[
= -\langle \nabla_{X_s(\partial_i)} \nabla_{X_s(\partial_j)} (\bar{F} - F)\nu, \nu \rangle + (F - \bar{F})R_{\partial_i\partial_j} - \nabla_{\nabla_{\partial_i} \partial_j} F.
\]
where $\nu$ is arranged to be $e_0$. Note that the definition of $\dot{F}$ and $\ddot{F}$ allow us to write Hess$\nabla F$ as follows

$$\text{Hess}\nabla F(\partial_i,\partial_j) = \nabla_i \nabla_j F = \nabla_i (\dot{F}_k \nabla_j h^k_l)$$

$$= \dot{F}_k \nabla_i \nabla_j h^k_l + \dot{F}_k \nabla_i h^m_i \nabla_j h^k_l$$

$$= \dot{F}^{kl} \nabla_i \nabla_j h_{kl} + \ddot{F}_k \nabla_i h^m_i \nabla_j h^k_l.$$

Recall a form of Simons’ identity [17] (a simple derivation can be found in [44]), which is a consequence of the Gauß and Codazzi equations

$$\nabla_i \nabla_j h_{kl} = \nabla_k \nabla_i h_{lj} + h_{ij} h_{kp} h^p_l - h_{ip} h^p_i h_{kj} + h_{il} h_{kp} h^p_j - h_{jp} h^p_j h_{kl}$$

$$+ R_{ikjp} h^p_l + R_{iklp} h^p_j - R_{pkj} h^p_i + R_{pjli} h^p_k + R_{ijkl} h_{ij} - R_{0i0j} h_{kl}$$

$$+ \nabla_i R_{0lkj} + \nabla_k R_{0lj}.$$

Therefore

$$(3.7) \quad \partial_l h_{ij} = \dot{F}^{kl} \nabla_k \nabla_i h_{lj} + \dot{F}_k \nabla_i h^m_i \nabla_j h^k_l$$

$$+ \ddot{F}_k \{ h_{ij} h_{kp} h^p_l - h_{ip} h^p_i h_{kj} + h_{il} h_{kp} h^p_j - h_{jp} h^p_j h_{kl}$$

$$+ R_{ikjp} h^p_l + R_{iklp} h^p_j - R_{pkj} h^p_i + R_{pjli} h^p_k + R_{ijkl} h_{ij} - R_{0i0j} h_{kl}$$

$$+ \nabla_i R_{0lkj} + \nabla_k R_{0lj} \} + (\ddot{F} - F) h_{ik} h^k_j - (\ddot{F} - F) R_{0i0j}.$$

Also note that in our case where the background space is a hyperbolic space, the ambient space is locally symmetric ($\nabla R = 0$) and the Riemann curvature tensor takes the form

$$(3.8) \quad \bar{R}_{\alpha \beta \gamma \delta} = -a^2 (g_{\alpha \gamma} g_{\beta \delta} - g_{\alpha \delta} g_{\beta \gamma}).$$

Since $F$ is a homogeneous function of the Weingarten map $\mathcal{W}$ of degree $m\beta$, then

$$(3.9) \quad \dot{F} \mathcal{W} = m\beta F.$$

Then, the relations (3.8) with $\nabla R = 0$ and (3.9) apply to (3.7) to give:

$$\partial_l h_{ij} = \dot{F}^{kl} \nabla_k \nabla_i h_{lj} + \dot{F}_k \nabla_i h^m_i \nabla_j h^k_l + \ddot{F}_k h^k_l h_{ij}$$

$$+ \ddot{F}_k h^k_l h_{ij} + [F - (m\beta + 1) F] h_{ik} h^k_j + a^2 [F - (m\beta + 1) F] g_{ij}.$$

Hence in compact notation we have (3.8).

Finally recalling that $\mathcal{W} = g^{-1} A$ we have

$$\partial_l \mathcal{W} = g^{-1} \partial_l A + \partial_l g^{-1} A$$

$$= \Delta_F \mathcal{W} + \dot{F} (\nabla \mathcal{W}, \nabla \mathcal{W}) + [\text{tr}_F(A \mathcal{W}) + a^2 \text{tr}(\dot{F})] \mathcal{W}$$

$$+ [F - (m\beta + 1) F] \mathcal{W}^2 + a^2 [F - (m\beta + 1) F] Id - 2(\ddot{F} - F) \mathcal{W}^2$$

$$= \Delta_F \mathcal{W} + \dot{F} (\nabla \mathcal{W}, \nabla \mathcal{W}) + [\text{tr}_F(A \mathcal{W}) + a^2 \text{tr}(\dot{F})] \mathcal{W}$$

$$- [F + (m\beta - 1) F] \mathcal{W}^2 + a^2 [F - (m\beta + 1) F] Id,$$

which is (3.6).
In the next theorem, we derive the evolution of any homogeneous function of the Weingarten map $\mathcal{W}$ defined on an evolving hypersurface $M_t$ of $\mathbb{H}_n^{n+1}$ under the flow (1.1)-(1.3).

**Theorem 3.3.** If $G$ is a homogeneous function of the Weingarten map $\mathcal{W}$ of degree $\alpha$, then the evolution equation of $G$ under the flow (1.1)-(1.3) in $\mathbb{H}_n^{n+1}$ is the following

$$
\partial_t G = \Delta_F G - \tilde{G}\mathcal{F}(\nabla \mathcal{W}, \nabla \mathcal{W}) + \tilde{G}\mathcal{F}(\nabla \mathcal{W}, \nabla \mathcal{W}) + \alpha \left[ \text{tr}_F(A \mathcal{W}) + a^2 \text{tr}(\mathcal{F}) \right] G \\
- \left[ F + (m\beta - 1)F \right] \tilde{G}\mathcal{W}^2 + a^2 \left[ F - (m\beta + 1)F \right] \text{tr}(\tilde{G}).
$$

**Proof.** The definition of $\tilde{G}$ and $\tilde{G}$ allow us to write $\text{Hess}_{\mathcal{W}}G$ as follows

$$
\text{Hess}_{\mathcal{W}}G = \tilde{G} \text{ Hess}_{\mathcal{W}} \mathcal{W} + \tilde{G}(\nabla \mathcal{W}, \nabla \mathcal{W}),
$$

which gives

$$
\Delta_F G = \tilde{G} g^{-1} \text{ Hess}_{\mathcal{W}} G = \tilde{G} \Delta_F \mathcal{W} + \tilde{G}\mathcal{F}(\nabla \mathcal{W}, \nabla \mathcal{W}).
$$

Therefore, by (3.3)

$$
\partial_t G = \tilde{G} \partial_t \mathcal{W}
$$

$$
= \tilde{G} \Delta_F \mathcal{W} + \tilde{G}\mathcal{F}(\nabla \mathcal{W}, \nabla \mathcal{W}) + \left[ \text{tr}_F(A \mathcal{W}) + a^2 \text{tr}(\mathcal{F}) \right] \tilde{G} \mathcal{W}
- \left[ F + (m\beta - 1)F \right] \tilde{G} \mathcal{W}^2 + a^2 \left[ F - (m\beta + 1)F \right] \text{tr}(\tilde{G}),
$$

where Euler’s theorem $\tilde{G} \mathcal{W} = \alpha \mathcal{W}$ is used in the last line. \hfill \square

An immediate application of the theorem above is to obtain the evolving equations for $H$, and $F$.

**Proposition 3.4.** For the ambient space $N^{n+1} = \mathbb{H}_n^{n+1}$, on any solution $M_t$ of (1.1)-(1.3) the following hold:

$$
\partial_t H = \Delta_F H + \text{tr} \left[ F(\nabla \mathcal{W}, \nabla \mathcal{W}) \right] - \left( F + (m\beta - 1)F \right) |A|^2 \\
+ \left[ \text{tr}_F(A \mathcal{W}) + a^2 \text{tr}(\mathcal{F}) \right] H + na^2 \left[ F - (m\beta + 1)F \right],
$$

(3.10)

$$
\partial_t F = \Delta_F F + (F - \tilde{F}) \left[ \text{tr}_F(A \mathcal{W}) - a^2 \text{tr}(\mathcal{F}) \right].
$$

(3.11)

For the proof of the main theorem, as mentioned in the introduction, it is convenient for us to define some suitable perturbations of the second fundamental form. Define the turbulent second fundamental form

$$
\tilde{h}_{ij} = h_{ij} - ag_{ij}.
$$

Denote $\tilde{A}$ (resp. $\mathcal{W}$) the matrix whose entries are $\tilde{h}_{ij}$ (resp. $\tilde{h}_{ij}$). Then $\tilde{\lambda}_i$ given by

$$
\tilde{\lambda}_i = \lambda_i - a, \quad i \in 1, \ldots, n,
$$

Denote $\tilde{A}$ (resp. $\mathcal{W}$) the matrix whose entries are $\tilde{h}_{ij}$ (resp. $\tilde{h}_{ij}$). Then $\tilde{\lambda}_i$ given by

$$
\tilde{\lambda}_i = \lambda_i - a, \quad i \in 1, \ldots, n,
$$
are the eigenvalues of $\tilde{\mathcal{W}}$. Denote the elementary symmetric functions of the $\tilde{\lambda}_i$ by $\tilde{E}_r, 1 \leq r \leq n$. From the definition it follows that

$$\tilde{H} = \text{tr}_g \tilde{\mathcal{W}} = \tilde{E}_1 = \sum_{i=1}^{n} \tilde{\lambda}_i = H - na,$$

$$|\tilde{A}|^2 = \text{tr}_g (\tilde{\mathcal{W}}^+ \tilde{\mathcal{W}}) = \sum_{i=1}^{n} \tilde{\lambda}_i^2 = |A|^2 + na^2 - 2Ha,$$

$$\tilde{K} = \det \tilde{\mathcal{W}} = \det \{\tilde{h}_{ij}\} = \prod_{i=1}^{n} \tilde{\lambda}_i.$$

It is easy to check that

$$\nabla_k \tilde{h}_{ij} = \nabla_k h_{ij},$$

and therefore the Codazzi equation hold for $\nabla_k \tilde{h}_{ij}$.

The following theorem is easily obtained from (3.1), (3.2), (3.5) and (3.6) by the definitions of $\tilde{A}$ and $\tilde{\mathcal{W}}$.

**Theorem 3.5.** For the ambient space $N^{n+1} = \mathbb{H}^{n+1}_k$, on any solution $M_t$ of (1.1)-(1.3) the following hold

$$\nabla_t \tilde{A} = \Delta_{\tilde{F}} \tilde{A} + \tilde{F} (\nabla \tilde{\mathcal{W}}, \nabla \tilde{\mathcal{W}}) + [\tilde{F} - (m\beta + 1)F]A \tilde{\mathcal{W}}$$

$$+ a [(m\beta + 1)F - F] A + \text{tr}_F( A \tilde{\mathcal{W}}) A, \quad (3.12)$$

$$\nabla_t \tilde{\mathcal{W}} = \Delta_{\tilde{F}} \tilde{\mathcal{W}} + \tilde{F} (\nabla \tilde{\mathcal{W}}, \nabla \tilde{\mathcal{W}}) + [(1 - m\beta)F - F] \tilde{\mathcal{W}} \tilde{\mathcal{W}}$$

$$+ a [(m\beta + 1)F - F] \tilde{\mathcal{W}} + \text{tr}_F( \tilde{A} \tilde{\mathcal{W}}) \tilde{\mathcal{W}}. \quad (3.13)$$

**Proof.** By (3.1) and (3.5)

$$\nabla_t \tilde{A} = \partial_t A - a \partial_t g$$

$$= \Delta_{\tilde{F}} \tilde{A} + \tilde{F} (\nabla \tilde{\mathcal{W}}, \nabla \tilde{\mathcal{W}}) + [\text{tr}_F( A \tilde{\mathcal{W}}) + a^2 \text{tr}(\tilde{F})] A + [\tilde{F} - (m\beta + 1)F] A \tilde{\mathcal{W}}$$

$$+ a^2 [\tilde{F} - (m\beta + 1)F] g - 2a(\tilde{F} - F) A$$

$$= \Delta_{\tilde{F}} \tilde{A} + \tilde{F} (\nabla \tilde{\mathcal{W}}, \nabla \tilde{\mathcal{W}}) + [\text{tr}_F( \tilde{A} \tilde{\mathcal{W}}) + 2am\beta F] A + [\tilde{F} - (m\beta + 1)F] A \tilde{\mathcal{W}}$$

$$+ a [\tilde{F} - (m\beta + 1)F] A + a^2 [\tilde{F} - (m\beta + 1)F] g - 2a(\tilde{F} - F) A$$

$$= \Delta_{\tilde{F}} \tilde{A} + \tilde{F} (\nabla \tilde{\mathcal{W}}, \nabla \tilde{\mathcal{W}}) + [\tilde{F} - (m\beta + 1)F] A \tilde{\mathcal{W}}$$

$$+ a [(m\beta + 1)F - \tilde{F}] A + \text{tr}_F( \tilde{A} \tilde{\mathcal{W}}) A,$$

where the third line follows by the relation

$$\text{tr}_F( A \tilde{\mathcal{W}}) = \text{tr}_F( \tilde{A} \tilde{\mathcal{W}}) + 2am\beta F - a^2 \text{tr}(\tilde{F}).$$

Then (3.12) and (3.2) together imply (3.13). $\Box$

The evolution equation (3.13) of $\tilde{\mathcal{W}}$ applies to give the evolution of any homogeneous function of the $\tilde{\mathcal{W}}$ defined on an evolving hypersurface $M_t$ of $\mathbb{H}^{n+1}_k$ under the flow (1.1)-(1.3).
Theorem 3.6. If $P$ is a homogeneous function of the turbulent Weingarten map $\tilde{\nabla}$ of degree $\gamma$, then the evolution equation of $P$ under the flow of (1.1) in $\mathbb{H}^{n+1}_\kappa$ is the following:

$$\partial_t P = \Delta_F P - \tilde{\nabla}(\nabla \tilde{\nabla}, \nabla \tilde{\nabla}) = \tilde{\nabla}(\nabla \tilde{\nabla}, \nabla \tilde{\nabla}) + \left[ (1 - m\beta) F - \tilde{F} \right] \tilde{\nabla}^2 P + 2a( F - \tilde{F} ) P + \text{tr}_F(\tilde{A} \tilde{\nabla}^2) \tilde{P} \tilde{\nabla}. $$

An immediate application of the theorem above is to obtain the evolving equations for $\tilde{H}$ and $\tilde{K}$.

Proposition 3.7. For the ambient space $N^{n+1} = \mathbb{H}^{n+1}_\kappa$, on any solution $M_t$ of (1.1) the following hold:

$$\partial_t \tilde{H} = \Delta_F \tilde{H} + \text{tr}_F(\tilde{F}(\nabla \tilde{\nabla}, \nabla \tilde{\nabla})) - (\tilde{F} + (m\beta - 1)F)|A|^2 + 2a(F - \tilde{F}) \tilde{H} + \text{tr}_F(\tilde{A} \tilde{\nabla} \tilde{\nabla}) \tilde{H},$$

$$\partial_t \tilde{H}^n = \Delta_F \tilde{H}^n - n(n - 1)\tilde{H}^{n-2}|\tilde{H}|^2 + n\tilde{H}^{n-1}\text{tr}_F(\tilde{F}(\nabla \tilde{\nabla}, \nabla \tilde{\nabla})) + n((1 - m\beta)F - \tilde{F})\tilde{H}^{n-1}|A|^2 + 2an(F - \tilde{F})\tilde{H}^n + n \text{tr}_F(\tilde{A} \tilde{\nabla} \tilde{\nabla}) \tilde{H}^n + an^2 \text{tr}_F(\tilde{A} \tilde{\nabla} \tilde{\nabla}) \tilde{H}^{n-1},$$

$$\partial_t \tilde{K} = \Delta_F \tilde{K} - \tilde{F} \tilde{K}(\nabla \tilde{\nabla}, \nabla \tilde{\nabla}) + \tilde{F}(\nabla \tilde{\nabla}, \nabla \tilde{\nabla}) + [(1 - m\beta)F - \tilde{F}] \tilde{K} \tilde{H}^2 + 2an(F - \tilde{F}) \tilde{K} + \text{tr}_F(\tilde{A} \tilde{\nabla}) \tilde{K} \tilde{\nabla}. $$

Furthermore, (3.16) can be rewritten as

Lemma 3.8. For the ambient space $N^{n+1} = \mathbb{H}^{n+1}_\kappa$, on any solution $M_t$ of (1.1) the following holds

$$\partial_t \tilde{K} = \Delta_F \tilde{K} - \left( \frac{n-1}{n} \right) \left| \nabla \tilde{K} \right|^2_F + \tilde{K} \cdot \frac{\tilde{H} \nabla \tilde{\nabla} - \tilde{\nabla} \nabla \tilde{H}}{F_\tilde{b}} + \frac{\tilde{H}^{2n}}{n \tilde{K}} \left| \nabla(\tilde{K} \tilde{H}^{-n}) \right|^2_F + \tilde{K} \cdot \tilde{\nabla}_b \left( \tilde{F}(\nabla \tilde{\nabla}, \nabla \tilde{\nabla}) \right) + [(1 - m\beta)F - \tilde{F}] \tilde{K} \tilde{H} + 2an(F - \tilde{F}) \tilde{K} + n \text{tr}_F(\tilde{A} \tilde{\nabla}) \tilde{K} + a \text{tr}_F(\tilde{A} \tilde{\nabla}) \tilde{K} \text{tr}_b(\tilde{b}),$$

where $b := \tilde{\nabla}^{-1}$.

Proof. Note that

$$\dot{\tilde{K}} = \tilde{K} \dot{b},$$

this implies

$$\dot{\tilde{K}} \tilde{\nabla}^2 = \tilde{K} \dot{\tilde{H}},$$

and

$$\dot{\tilde{K}} \tilde{F}(\nabla \tilde{\nabla}, \nabla \tilde{\nabla}) = \tilde{K} \dot{b} \tilde{F}(\nabla \tilde{\nabla}, \nabla \tilde{\nabla}) = \tilde{K} \text{tr}_b \left( \tilde{F}(\nabla \tilde{\nabla}, \nabla \tilde{\nabla}) \right).$$
A direct calculation as for example in Lemma 2.2 of [20] gives

\[ -\hat{F} \hat{K} (\nabla \hat{\nu}, \nabla \hat{\nu}) = -\frac{|\nabla \hat{K}|^2}{\hat{K}} - \hat{K} \text{tr}_b (\nabla \hat{\nu} \nabla \hat{\nu}) \]

and

\[ -\hat{K} \text{tr}_b (\nabla \hat{b} \nabla \hat{\nu}) = \frac{\hat{K}}{H^2} |\hat{H} \nabla \hat{\nu} - \hat{\nu} \nabla \hat{H}|^2_{\hat{F}, b} + \frac{|\nabla \hat{K}|^2}{\hat{K}} \left( \frac{\hat{H}^{2n}}{n\hat{K}} - \frac{\hat{H}^2}{n\hat{K}} \right) \nabla (\hat{K} \hat{H}^{-1})^2_{\hat{F}}. \]

Therefore, identities (3.19), (3.20), (3.21) and (3.22) together apply to (3.16) to give (3.17).

\[ \square \]

4. Preserving pinching

To control the pinching of the principal curvature along the flow (1.1)-(1.3) of the Euclidean space, Schulze, in [46], following an idea of Tso [49], explored a test function (3.21) . A direct calculation as for example in Lemma 2.14 SHUNZI GUO, GUANGHAN LI, AND CHUANXI WU

\[ (3.21) \]

\[ (3.22) \]

Therefore, identities (3.19), (3.20), (3.21) and (3.22) together apply to (3.16) to give (3.17).
Furthermore, the first derivative and second derivative term in (4.2) can be computed as follows, the equality
\[ \nabla \left( \frac{\tilde{K}}{H^n} \right) = \frac{\nabla \tilde{K}}{H^n} - \frac{\tilde{K}}{H^{2n}} \nabla \tilde{H}^n \]
implies
\[ \Delta_F \left( \frac{\tilde{K}}{H^n} \right) = \frac{\Delta_F \tilde{K}}{H^n} - 2 \frac{\left< \nabla \tilde{H}^n, \nabla \tilde{K} \right>_F}{H^{2n}} + 2 \frac{\tilde{K}}{H^{3n}} \left| \nabla \tilde{H}^n \right|_F^2 - \frac{\tilde{K}}{H^{2n}} \Delta_F \tilde{H}^n, \]
(4.3)
\[ \left< \nabla \left( \frac{\tilde{K}}{H^n} \right), \nabla \tilde{H}^n \right>_F = \frac{\left< \nabla \tilde{H}^n, \nabla \tilde{K} \right>_F}{H^n} - 2 \frac{\tilde{K}}{H^{2n}} \left| \nabla \tilde{H}^n \right|_F^2, \]
(4.4)
and
\[ \left< \nabla \left( \frac{\tilde{K}}{H^n} \right), \nabla \tilde{K} \right>_F = \frac{\left| \nabla \tilde{K} \right|_F^2}{H^n} - \frac{\tilde{K}}{H^{2n}} \left< \nabla \tilde{H}^n, \nabla \tilde{K} \right>_F. \]
(4.5)
From (4.3), (4.4) and (4.5), it follows
\[ \frac{\Delta_F \tilde{K}}{H^n} = \frac{\tilde{K}}{H^{2n}} \Delta_F \tilde{H}^n - \frac{(n-1)}{n} \left| \nabla \tilde{K} \right|_F^2 \]
(4.6)
\[ = \Delta_F \left( \frac{\tilde{K}}{H^n} \right) + \frac{(n+1)}{n} \left< \nabla \left( \frac{\tilde{K}}{H^n} \right), \nabla \tilde{H}^n \right>_F - \frac{(n-1)}{nK} \left< \nabla \left( \frac{\tilde{K}}{H^n} \right), \nabla \tilde{K} \right>_F - n(n-1) \frac{\tilde{K}}{H^{n+2}} \left| \nabla \tilde{H}^n \right|_F^2. \]
Thus, the equation (4.6) applies to (4.2) to give (4.1). \( \square \)

In order to apply the maximum principle to (4.1) and show that \( \min_{p \in M} \tilde{Q}(p, t) \) is non-decreasing in time some preliminary inequalities are needed in the sequel. The following elementary property is a consequence of ([19], Lemma 4.2) (see also [20] and [46]).

**Lemma 4.2.** For any \( \varepsilon \in (0, 1/n) \) and any \( \tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) \in \mathbb{R}^n \) with \( \tilde{\lambda}_i > 0 \) for all \( i = 1, \ldots, n \), there exists a constant \( C = C(\varepsilon, n) \in (0, 1/n^n) \) satisfies
\[ \tilde{Q}(\tilde{\lambda}) > C \]
such that
\[ \tilde{\lambda}_1 > \varepsilon \tilde{H}(\tilde{\lambda}). \]

The following estimate which is a stronger version of Lemma 2.3 (ii) in [29] can be viewed as a generalisation by Cabezas-Rivas and Miquel [19].

**Lemma 4.3.** If \( \tilde{H} > 0 \) and the inequality \( \tilde{\mathbf{W}} > \varepsilon \tilde{H} \text{Id} \) is valid with some \( \varepsilon > 0 \) at a point on a hypersurface immersed in \( \mathbb{H}^{n+1}_\kappa \), then \( \varepsilon \leq 1/n \) and
\[ \left| H \nabla \tilde{\mathbf{W}} - \tilde{\mathbf{W}} \nabla H \right|^2 \geq \frac{n-1}{2} \varepsilon^2 \tilde{H}^2 \left| \nabla \tilde{\mathbf{W}} \right|^2. \]
Proof. The proof of the Lemma can be argued exactly as in ([19], Lemma 4.1), only define $\tilde{\mathcal{W}} := \mathcal{W} - a \text{Id}$ at a point on a hypersurface immersed in $\mathbb{H}_{n+1}^n$. \qed

Also as in [19], the preceding two lemmas allow us to prove the pinching estimate for our flow, which is one of the key steps in the proof of our main result.

**Theorem 4.4.** There exists a constant $C^* = C^*(a, n, m, \beta) \in (0, 1/n^n)$ with the following property: if $X : M^n \times [0, T) \to \mathbb{H}_{n+1}^n$, with $t \in [0, T)$, is a smooth solution of (1.1) in (1.3), with $F$ given by (1.4) for some $\beta \geq 1/m$, such that

- the initial immersion $X_0$ satisfies (1.6) with the constant $C^*$,
- the solution $M_t = X(M^n, t)$ satisfies $\tilde{H} > 0$ for all times $t \in [0, T)$,

then the minimum of $\tilde{K}/\tilde{H}^n$ on $M_t$ is nondecreasing in time.

Proof. The assumption $\tilde{H} > 0$ on evolving hypersurface ensures that the quotient $\tilde{Q}$ is well-defined for $t \in [0, T)$. For the proof of the Theorem, it is suffices to prove that the minimum of $\tilde{Q}$ (denote by $\tilde{Q}$) is nondecreasing in time. First, by (1.6), $\lambda_1 > 0$ on $M_t$ for $t = 0$, then this implies that $\lambda_1 > 0$ on $M_t$ for $t \in [0, T)$ by a contradiction argument. In fact, suppose to the contrary that there exists a first time $t_0 > 0$ at which $\lambda_1 = 0$ as some point, then $\tilde{Q}(t_0) = 0$. On the other hand, by $\lambda_1 > 0$ on $M_t$ for $t \in [0, t_0)$, $\tilde{H} > 0$ for all times $t \in [0, t_0)$. Thus applying the theorem on $[0, t_0)$ implies that $\tilde{Q}(t)$ is nondecreasing in $[0, t_0)$. So it cannot decrease from $C^*$ to zero as $t$ goes to $t_0$, which gives a contradiction. Now applying the maximum principle to equation (1.1) for $\tilde{Q}$ gives

$$
\partial_t \tilde{Q} \geq \frac{\tilde{Q}}{H_2^2} \left[ \tilde{H} \nabla \tilde{\mathcal{W}} - \tilde{\mathcal{W}} \nabla \tilde{H} \right]_{\tilde{F}, b}^2 + \tilde{Q} \text{tr}_b - \tilde{Q} \text{Id} \left( \tilde{F}(\nabla \tilde{\mathcal{W}}, \nabla \tilde{\mathcal{W}}) \right) \\
+ \left[ (m\beta - 1) F + \tilde{F} \right] \frac{\tilde{Q}}{H} \left( n |\tilde{A}|^2 - \tilde{H}^2 \right) + a \tilde{Q} \text{tr}_F (\tilde{A} \tilde{\mathcal{W}}) \left( \text{tr}(\tilde{b}) - \frac{n^2}{H} \right)
$$

(4.7)

The various terms appearing here can be estimated as follows, as in [19, Theorem 4.3]. The $h$-convexity of $M_t$ implies that the third term of RHS in inequality (4.7) can be dropped with the strictly $h$-convexity on $M_t$. The last term can also be dropped by the arithmetic-harmonic mean inequality,

$$
\sum_{i=1}^n \tilde{b}_i - \frac{n^2}{H} \geq 0
$$

on $M_t$. It remains to estimate the first two terms of RHS in inequality (4.7), now proceeding exactly as in [19, 20] and [46], choose orthonormal frame which diagonalises $\tilde{\mathcal{W}}$ so that

$$
\left[ \tilde{H} \nabla \tilde{\mathcal{W}} - \tilde{\mathcal{W}} \nabla \tilde{H} \right]_{\tilde{F}, b}^2 = \sum_{i, m, n} \tilde{F}^i \frac{1}{\lambda_m \lambda_n} \left( \tilde{H} \nabla_i \tilde{h}_m^n - \tilde{h}_m^n \nabla_i \tilde{H} \right)^2 \geq \frac{1}{H^2} \sum_{i, m, n} \tilde{F}^i \left( \tilde{H} \nabla_i \tilde{h}_m^n - \tilde{h}_m^n \nabla_i \tilde{H} \right)^2
$$

(4.8)
where $\tilde{\lambda}_m \leq \tilde{H}$ was used in the last inequality by strictly $h$-convexity of $M_h$, i.e., $\tilde{\lambda}_m > 0$ for any $m$. Now the property that each $\tilde{F}^i$ is positive in the interior of the positive cone can be used. More precisely, for any $\varepsilon \in (0, 1/n]$

$$\Xi_\varepsilon := \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \min_{1 \leq i \leq n} \tilde{\lambda}_i \geq \varepsilon (\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_n) > 0 \},$$

$$W_1(\varepsilon) = \min \{ \tilde{F}^i(\lambda) : 1 \leq i \leq n, \lambda \in \Xi_\varepsilon, |\lambda| = 1 \}.$$ 

By homogeneity of $\tilde{F}^i$ with degree $m\beta - 1$ and Lemma 4.3 together imply that the inequality (4.8) can be estimated as the top of p.453 of [19], the following inequality holds:

$$\tilde{F}^i(\lambda) \geq W_1(\varepsilon)|\lambda|^{m\beta - 1}, \quad \lambda \in \Xi_\varepsilon,$$

where $W_1(\varepsilon)$ is an increasing positive function of $\varepsilon$. This estimation, $h$-convexity of a hypersurface and Lemma 4.3 together imply that the inequality (4.8) can be estimated as follows:

$$\left| \tilde{H} \nabla \tilde{\mathbf{w}} - \tilde{\mathbf{w}} \nabla \tilde{H} \right|^2_{\tilde{F}, \tilde{\mathbf{h}}} \geq \frac{n-1}{2} W_1(\varepsilon) \varepsilon^2 |\tilde{\mathbf{w}}|^{m\beta - 1} \left| \nabla \tilde{\mathbf{w}} \right|^2,$$

for some $\varepsilon \in (0, 1/n]$. 

The term $|\tilde{F}(\nabla \tilde{\mathbf{w}}, \nabla \tilde{\mathbf{w}})|$ is smooth as long as $\tilde{\lambda}_i > 0$ for any $i$, homogeneous of degree $m\beta - 2$ in $\lambda_i$ and quadratic in $\nabla \tilde{\mathbf{w}}$. The following estimation from above the term $|\tilde{F}(\nabla \tilde{\mathbf{w}}, \nabla \tilde{\mathbf{w}})|$ can be derived as in [19] inequality (4.7): For any $\varepsilon \in (0, 1/n]$, there exists a constant $W_2(\varepsilon)$ such that, at any point where $\tilde{\mathbf{w}} \geq \varepsilon \tilde{H} \text{Id}$,

$$|\tilde{F}(\nabla \tilde{\mathbf{w}}, \nabla \tilde{\mathbf{w}})| \leq W_2(\varepsilon) |\tilde{\mathbf{w}}|^{m\beta - 2} \left| \nabla \tilde{\mathbf{w}} \right|^2,$$

where $W_2(\varepsilon)$ is decreasing in $\varepsilon$.

A next step is to show that $|\tilde{b} - \frac{n}{\tilde{H}} \text{Id}|$ is small if the principal curvatures are pinched enough. It is clear that

$$|\tilde{b} - \frac{n}{\tilde{H}} \text{Id}| \leq \sqrt{n} \max \left\{ \left( \frac{1}{\tilde{\lambda}_1} - \frac{n}{\tilde{H}} \right), \left( \frac{n}{\tilde{H}} - \frac{1}{\tilde{\lambda}_n} \right) \right\}.$$

Since for some $\varepsilon \in (0, 1/n]$

$$\tilde{\lambda}_1 \geq \varepsilon \tilde{H},$$

then

$$\frac{1}{\tilde{\lambda}_1} - \frac{n}{\tilde{H}} \leq \frac{1 - \varepsilon n}{\varepsilon \tilde{H}}.$$ 

On other hand, (4.11) gives

$$\tilde{\lambda}_n \leq (1 - (n - 1)\varepsilon) \tilde{H}$$

which implies that

$$\frac{n}{\tilde{H}} - \frac{1}{\tilde{\lambda}_n} \leq \frac{(n - 1)(1 - n\varepsilon)}{\tilde{H} (1 - (n - 1)\varepsilon)}.$$ 

This combines with estimate (4.12) to give

$$|\tilde{b} - \frac{n}{\tilde{H}} \text{Id}| \leq \frac{C(\varepsilon)}{\tilde{H}},$$
where

$$\mathcal{N}(\varepsilon) = \begin{cases} \frac{\sqrt{n}(1 - \varepsilon n)}{\varepsilon}, & 0 < \varepsilon \leq \frac{1}{2(n-1)}; \\ \frac{\sqrt{n}(n-1)(1-n\varepsilon)}{(1-(n-1)\varepsilon)}, & \frac{1}{2(n-1)} < \varepsilon < \frac{1}{n}. \end{cases}$$

Thus, the inequalities \( \tilde{H} < H, \left| H \right|^2 \leq n \| \psi \|^2 \), estimations (4.8), (4.9), (4.10) and (4.15) together give:

$$\begin{align*}
\frac{1}{H^2} \left| \tilde{H} \nabla \tilde{\psi} - \tilde{\psi} \nabla \tilde{H} \right|^2_{F,h} - \left| \tilde{\partial}_{n} - \frac{n}{H} \text{Id} \right| \left| \tilde{\partial}(\nabla \tilde{\psi}, \nabla \tilde{\psi}) \right| \\
\geq \frac{1}{H} \| \psi \|^2 m^{\beta - 2} \left| \nabla \tilde{\psi} \right|^2 \left( \frac{n-1}{2\sqrt{n}} W_1(\varepsilon)^2 - W_2(\varepsilon) \right) \mathcal{N}(\varepsilon)
\end{align*}$$

To achieve our purpose by application of the maximum principle, it is necessary that \( \mathcal{N}'(\varepsilon) := \left( \frac{n-1}{2\sqrt{n}} W_1(\varepsilon)^2 - W_2(\varepsilon) \right) \) is non-negative on \( M_t \). In fact, \( \mathcal{N}(\varepsilon) \) is a strictly decreasing function of \( \varepsilon \); in addition, \( \mathcal{N}(\varepsilon) \) is arbitrarily large as \( \varepsilon \) goes to zero and tends to zero as \( \varepsilon \) goes to \( 1/n \) by its definition, \( W_1(\varepsilon) \) is increasing and \( W_2(\varepsilon) \) is decreasing. Therefore, \( \mathcal{N}'(\varepsilon) \) is a strictly increasing function of \( \varepsilon \), it is negative as \( \varepsilon \) goes to zero and positive as \( \varepsilon \) goes to \( 1/n \). So there exists a unique value \( \varepsilon_0 \in (0, 1/n) \) such that

$$\mathcal{N}'(\varepsilon_0) = 0.$$  

By Lemma 4.2 there exists a constant \( C^* \in (0, 1/n^2) \) satisfies \( \tilde{Q}(\tilde{\lambda}) > C^* \) such that \( \tilde{\lambda}_i > \varepsilon \tilde{H}(\tilde{\lambda}) \) with a \( \varepsilon_0 \in (0, 1/n^2) \) given by (4.17). Thus, if \( \tilde{Q} > C^* \geq 0 \) everywhere on the initial hypersurface, applying the maximum principle for \( \tilde{Q} \) implies that \( \partial_t \tilde{Q} \geq 0 \), i.e., \( \tilde{Q} \) is non-decreasing in time. This guarantees that \( \tilde{Q} > C^* \) is preserved under the flow (1.1)-(1.3) in \( \mathbb{H}^{n+1}_\kappa \).

Theorems asserts that inequality \( \tilde{Q} > C^* \) holds for all \( t \in [0, T) \), furthermore, the definition of \( C^* \) and Lemma 4.2 together shows that

$$\begin{align*}
\tilde{\lambda}_i &\geq \varepsilon_0 \tilde{H} \text{ on } M^n \times [0, T) \quad \text{for each } i, \\
\lambda_i &\geq \varepsilon_0 \tilde{H} \text{ on } M^n \times [0, T) \quad \text{for each } i.
\end{align*}$$

5. Upper bound on \( F \)

In this section uniform bounds from above on the speed for the flow and for the curvature of the hypersurface are derived, depending only on the initial data. The bounds on curvatures together with the estimates in the next section will imply the long time existence of the flow by well-known arguments. In order to achieve this, the method is to study the evolution under the flow (1.1)-(1.3) of the function

$$Z_t = \frac{F}{\Phi - \varepsilon}.$$ 

Here \( \Phi = s_\kappa(r_p) (\nu, \partial_{r_p}) \), which could be seen as “support function” of \( M^n \) in \( \mathbb{H}^{n+1}_\kappa \), and \( \varepsilon \) is a constant to be chosen later. The method used to obtain these bounds is very robust.
and applies to the Gauß curvature flow in [19], the flow with a general class of speeds in [3], the volume-preserving anisotropic mean curvature flow [7], the mixed volume preserving mean curvature flows in [39], the mixed volume preserving curvature flow in [40], the volume preserving mean curvature flow in the hyperbolic space in [17] and the volume preserving flow by powers of the mth mean curvature in [19].

Given a function \( f : \mathbb{R} \to \mathbb{R} \), \( f(r_p) \) will mean \( f \circ r_p \). An extension of [17] Lemma 3 will be needed later.

**Lemma 5.1.** In \( \mathbb{H}^{n+1}_\kappa \),

\[
(5.2) \quad (\nabla_X \partial_{r_p}, Y) = \nabla^2 r_p(X, Y) = \begin{cases} 0 & \text{if } X = \partial_{r_p}, \\ \cos_\kappa(r_p) \langle X, Y \rangle & \text{if } \langle X, \partial_{r_p} \rangle = 0 \end{cases}
\]

\[
(5.3) \quad \Delta_F r_p = \text{tr}(F) \cos_\kappa(r_p).
\]

Moreover, if \( f : \mathbb{R} \to \mathbb{R} \) is a \( C^2 \) function,

\[
(5.4) \quad \bar{\Delta}_F (f(r_p)) = f''(r_p) |\partial_{r_p}|_F^2 + f'(r_p) \Delta_F r_p.
\]

And, for the restriction of \( r_p \) to a hypersurface \( M \) of \( \mathbb{H}^{n+1}_\kappa \), one has

\[
(5.5) \quad \Delta_F r_p = -\text{tr}(\bar{F}) (\nu, \partial_{r_p}) + \cos_\kappa(r_p) \left( \text{tr}(\bar{F}) - |\partial_{r_p}|_F^2 \right).
\]

\[
(5.6) \quad \Delta_F f(r_p) = f''(r_p) |\partial_{r_p}|_F^2 + f'(r_p) \Delta_F r_p
\]

\[
= (f''(r_p) - f'(r_p) \cos_\kappa(r_p)) |\partial_{r_p}|_F^2
\]

\[
+ f'(r_p) \left( \text{tr}(\bar{F}) \cos_\kappa(r_p) - \text{tr}(\bar{M}) (\nu, \partial_{r_p}) \right).
\]

**Proof.** First (5.2) and (5.3) follow from [31, page 46] (see also [27]), and (5.4) follows form a direct calculation. On the other hand, the Gauß and Codazzi equations give the following

\[
\text{Hess}_{\nabla} r_p(X, Y) = \nabla^2 r_p(X, Y) = \langle \nabla_X \partial_{r_p}, Y \rangle
\]

\[
= \langle \nabla_X \nabla r_p, Y \rangle + A(X, Y) \langle \partial_{r_p}, \nu \rangle
\]

\[
= \nabla^2 r_p(X, Y) + A(X, Y) \langle \partial_{r_p}, \nu \rangle
\]

\[
= \text{Hess}_{\nabla} r_p(X, Y) + A(X, Y) \langle \partial_{r_p}, \nu \rangle.
\]

This combines with (5.2) and (5.3) to give (5.5). (5.8) gives (5.6) by a direct calculation. \( \square \)

**Corollary 5.2.** For \( t \in [0, T) \) and any constant \( \epsilon \), on any solution \( M_t \) of \( \text{(1.1)-(1.3)} \) in \( \mathbb{H}^{n+1}_\kappa \), the following holds

\[
(5.7) \quad \partial_t Z = \Delta_F Z + \frac{2 \langle \nabla Z, \nabla \Phi \rangle}{\Phi - \epsilon} \Phi - \epsilon \left( \text{tr}(A \bar{\Phi}) - a^2 \text{tr}(\bar{F}) \right) - c_\kappa(r) \frac{Z}{\Phi - \epsilon} \bar{F}
\]

\[
- \epsilon \frac{Z}{\Phi - \epsilon} \text{tr}(A \bar{\Phi}) - a^2 \text{tr}(\bar{F}) Z + (1 + m\beta) c_\kappa(r) Z^2.
\]

**Proof.** Using (1.1) and (5.2) a direct calculation gives

\[
(5.8) \quad \nabla_t (\cos_\kappa(r_p) \partial_{r_p}) = c_\kappa(r_p) (\bar{F} - F) \nu,
\]

which implies that

\[
(5.9) \quad \partial_t \Phi = \cos_\kappa(r_p) (\partial_{r_p}, \nabla F) + c_\kappa(r_p) (\bar{F} - F)
\]
by combining (3.3). On the other hand, a direct calculation gives
\begin{equation}
\Delta_F \Phi = \langle \nu, \partial_{\nu} \rangle \Delta_F s_n(r_p) + 2 \langle \nabla s_n(r_p), \nabla \langle \nu, \partial_{\nu} \rangle \rangle + s_n(r_p) \Delta_F \langle \nu, \partial_{\nu} \rangle.
\end{equation}

Taking \( f = s_n \) and using (5.6) give
\begin{equation}
\Delta_F (s_n(r_p)) = -\frac{1}{s_n(r_p)} \langle \partial_{\nu} \rangle^2 + c_n(r_p) \text{tr}_F(\mathcal{W}) \langle \nu, \partial_{\nu} \rangle + tr(\hat{F}) \frac{c_n(r_p)}{s_n(r_p)}.
\end{equation}

We choose a frame \( \{ e_i \} \) at \( p \) which is normal to \( \nu \) and tangent to \( M_t \). With respect to this frame field, let \( \{ e^i \} \) be the field of dual frames. Direct computations having into account (5.2) give
\begin{equation}
\langle \nabla s_n(r_p), \nabla \langle \partial_{\nu} \rangle \rangle = \frac{s_n(r_p)}{c_n(r_p)} \langle \partial_{\nu}, \nu \rangle \langle \partial_{\nu}^T \rangle^2 + c_n(r_p) \hat{F}^i_j \ A(\dot{\partial}_{\nu}^T, \partial_{\nu}^T, e^i) e_i.
\end{equation}

Since
\begin{equation}
\Delta_F \langle \nu, \partial_{\nu} \rangle = \langle \nu, \Delta_F \partial_{\nu} \rangle + \langle \Delta_F \nu, \partial_{\nu} \rangle + 2 \langle \nabla \nu, \nabla \partial_{\nu} \rangle,
\end{equation}

\begin{equation}
\langle \nu, \nabla_i \nabla_j \partial_{\nu} \rangle = \frac{1}{s_n^2(r_p)} \langle \partial_{\nu}, e_i \rangle \langle \partial_{\nu}, e_j \rangle \langle \partial_{\nu}, \nu \rangle - c_n(r_p) h_{ij}
\end{equation}

\begin{equation}
- c_n^2(r_p) g_{ij} \langle \nu, \partial_{\nu} \rangle + 2 c_n^2(r_p) \langle \partial_{\nu}, e_i \rangle \langle \partial_{\nu}, e_j \rangle \langle \partial_{\nu}, \nu \rangle + c_n(r_p) h_{ij} \langle \nu, \partial_{\nu} \rangle^2,
\end{equation}

\begin{equation}
\langle \nabla_j \nu, \nabla_i \partial_{\nu} \rangle = c_n(r_p) h_{ij} - c_n(r_p) h(\partial_{\nu}^T, \partial_{\nu}^T, e_i) e_i,
\end{equation}

\begin{equation}
\langle \nabla_i \nabla_j \nu, \partial_{\nu} \rangle = \langle \partial_{\nu}, e_k \rangle \nabla_k(h_{ij}) - \langle \nu, \partial_{\nu} \rangle h_{kj}^k h_{ij},
\end{equation}

combination of (5.13), (5.14), (5.15) and (5.10) together implies
\begin{equation}
\Delta_F \langle \partial_{\nu}, \nu \rangle = \frac{1}{s_n^2(r_p)} \langle \partial_{\nu}, \nu \rangle \langle \partial_{\nu}^T \rangle^2 + c_n(r_p) \text{tr}_F(\mathcal{W})
\end{equation}

\begin{equation}
- \text{tr}(\hat{F}) \ c_n^2(r_p) \langle \partial_{\nu}, \nu \rangle + 2 c_n^2(r_p) \langle \nu, \partial_{\nu} \rangle \langle \partial_{\nu}^T \rangle^2 + c_n(r_p) \langle \nu, \partial_{\nu} \rangle^2 \text{tr}_F(\mathcal{W}) - 2 c_n(r_p) \hat{F}^i_j A(\dot{\partial}_{\nu}^T, \partial_{\nu}^T, e^i) e_i
\end{equation}

\begin{equation}
+ \langle \partial_{\nu}^T, \nabla F \rangle - \langle \partial_{\nu}, \nu \rangle \text{tr}_F(A\mathcal{W}).
\end{equation}

From (5.10), (5.11), (5.12) and (5.17), it follows
\begin{equation}
\Delta_F \Phi = c_n(r_p) \text{tr}_F(\mathcal{W}) + s_n(r_p) \langle \partial_{\nu}, \nabla F \rangle - \Phi \text{tr}_F(A\mathcal{W}).
\end{equation}

Combining this with (5.9) yields
\begin{equation}
\partial_t \Phi = \Delta_F \Phi + \Phi \text{tr}_F(A\mathcal{W}) + c_n(r_p) (\dot{F} - F - \text{tr}_F(\mathcal{W})).
\end{equation}
From (5.18), (5.11) and (5.1), it follows
\begin{equation}
\partial_t Z = \frac{1}{\Phi - \epsilon} \left( \Delta_F F + (F - \tilde{F}) \left[ \text{tr}_F(A\mathscr{W}) - a^2 \text{tr}(\tilde{F}) \right] \right)
- \frac{F}{(\Phi - \epsilon)^2} \left( \Delta_F \Phi + \Phi \text{ tr}_F(A\mathscr{W}) + c_\kappa(r_p) \left( \tilde{F} - F - \text{tr}_F(\mathscr{W}) \right) \right).
\end{equation}
Another computation leads to
\begin{equation}
\Delta_F Z = \frac{\Delta F}{\Phi - \epsilon} - \frac{F \Delta_F \Phi}{(\Phi - \epsilon)^2} - 2 \frac{1}{\Phi - \epsilon} \langle \nabla Z, \nabla \Phi \rangle_F.
\end{equation}
Replacing (5.20) into (5.19), a few more computations having into account \( \text{tr}_F(\mathscr{W}) = m \beta F \) by Euler’s theorem gives the desired evolution equation (5.7) of \( Z \).

In order to get a uniform upper bound on \( Z \), previously we have to give the bounds on \( r_p \) and \( (\partial_{p^i}, \nu) \). The following estimate on \( r_p \) for the preserving volume mean curvature flow in [17] is also valid in our case with the help of Lemma 2.3 i).

**Lemma 5.3.** Let \( \psi \) be the inverse of the function \( s \mapsto \text{vol}(S^n) \int_0^s s(t) d\ell \) and \( \xi \) the inverse function of \( s \mapsto s + a \ln \left( \frac{1 + \sqrt{\tan(\frac{\pi}{2})}}{1 + \tan(\frac{\pi}{2})} \right)^2 \). If \( V_0 = \text{vol}(\Omega_0) \) and \( \rho_-(t) \) is the inner radius of \( \Omega_t \), then
\begin{equation}
\xi(\psi(V_0)) \leq \rho_-(t) \leq \psi(V_0),
\end{equation}
for every \( t \in [0, T) \).

An immediate consequence of the lemma above and Lemma 2.3 i) is
**Corollary 5.4.** For every \( t \in [0, T) \), if \( p, q \in \Omega_t \), then
\begin{equation}
\text{dist}(p, q) < 2(\psi(V_0) + a \ln 2).
\end{equation}

Now, if \( p_{t_0} \in \Omega_t \) for an arbitrary fixed \( t_0 \in [0, T) \), then using (5.22) gives an upper bound \( r_{p_{t_0}}(x) \leq 2(\psi(V_0) + a \ln 2) \) for every \( x \in M_t \). Thus, for an upper bound on \( F \), it is necessary to show that a geodesic ball with fixed center remains inside the evolving \( \Omega_t \) for a short time.

**Lemma 5.5.** If \( B(p_{t_0}, \rho_{t_0}) \subset \Omega_{t_0} \) for some \( t_0 \in [0, T) \), where \( \rho_{t_0} = \rho_-(t_0) \) is the inner radius of \( M_{t_0} \), then there exists some constant \( \tau = \tau(a, n, m, \beta, V_0) > 0 \) such that \( B(p_{t_0}, \rho_{t_0}/2) \subset \Omega_t \) for every \( t \in [t_0, \min\{t_0 + \tau, T\}) \).

**Proof.** Proceeding similarly as in [17] Lemma 8], our procedure is to compare the deformation of \( M_t \) by the equation (1.1)-(1.3) with a geodesic sphere shrinking under the \( H_\beta \)-flow.

For convenience, let \( r_B(t) \) be the radius at time \( t \) of a geodesic sphere \( \partial B(p_{t_0}, r_B(t)) \) centered at \( p_{t_0} \), evolving under \( H_\beta \)-flow and with the initial condition \( r_B(t_0) = \rho_{t_0} \). The radius of the evolving geodesic sphere \( \partial B(p_{t_0}, r_B(t)) \) satisfies
\begin{equation}
\frac{d r_B(t)}{dt} = - \text{co}_\kappa^\beta(r_B(t)).
\end{equation}
with the initial condition \( r_B(t_0) = \rho_{t_0} \), this ODE has solution
\[
\int_{\rho_{t_0}}^{r} \tan^m_\kappa(s) ds = -(t - t_0).
\]

Denote \( \mathcal{F}(r) := \int_{\rho_{t_0}}^{r} \tan^m_\kappa(s) ds \). Since \( \mathcal{F}(r) \) is an increasing function in \( r \), then for \( t \geq t_0 \), \( r_B(t) \geq \rho_{t_0}/2 \) if and only if
\[
t \leq t_0 + \int_{\rho_{t_0}}^{\rho_{t_0}/2} \tan^m_\kappa(s) ds.
\]

On the other hand, let \( \mathcal{G}(s) = \int_{s/2}^{s} \tan^m_\kappa(u) du \), since \( s \mapsto \tan_\kappa(s) \) is increasing, then
\[
\frac{d\mathcal{G}(s)}{ds} > 0
\]
which shows that \( \mathcal{G}(s) \) is increasing function in \( s \). Now using (5.21), this gives that if
\[
t - t_0 \leq \int_{\xi(\psi(\kappa))}^{\xi(\psi(\kappa))/2} \tan^m_\kappa(s) ds := \tau,
\]
then
\[
r_B(t) \geq \rho_{t_0}/2.
\]

For any \( x \in M \), let \( r(x, t) = r_{p_{t_0}}(X_t(x)) \), from (1.1), it follows
\[
\frac{dr}{dt} = (\bar{F}(t) - F) \left( \nu_t, \partial_{r_{p_{t_0}}} \right).
\]

If \( \varphi : \mathbb{R} \to \mathbb{R} \) is a \( C^2 \) function, set \( f(x, t) = \varphi(r(x, t)) - \varphi(r_B(t)) \), from (5.23) and (5.27), it follows
\[
\partial_t f = \varphi'(r_{p_{t_0}}) (\bar{F}(t) - F) \left( \nu_t, \partial_{r_{p_{t_0}}} \right) + \varphi'(r_B) \cos^m_\kappa(r_B).
\]

On the other hand, from (5.26), it follows
\[
\Delta f = \Delta(\varphi(r_{p_{t_0}}))
= (\varphi''(r_{p_{t_0}}) - \varphi'(r_{p_{t_0}}) \cos_\kappa(r_{p_{t_0}})) \left| \partial^\perp_{\nu_t} \right|^2
+ \varphi'(r_{p_{t_0}}) (n \cos_\kappa(r_{p_{t_0}}) - H(\nu, \partial_{r_{p_{t_0}}}))
\]
Therefore, (5.28) can be rewritten as
\[
\partial_t f = \frac{F}{H} \Delta f + \varphi'(r_{p_{t_0}}) \left( \nu_t, \partial_{r_{p_{t_0}}} \right) \bar{F}(t) + \varphi'(r_B) \cos^m_\kappa(r_B)
- \frac{n}{H} \varphi'(r_{p_{t_0}}) \cos_\kappa(r_{p_{t_0}}) + \frac{F}{H} \left[ \varphi'(r_{p_{t_0}}) \cos_\kappa(r_{p_{t_0}}) - \varphi''(r_{p_{t_0}}) \right] \left| \partial^\perp_{\nu_t} \right|^2.
\]
Taking \( \varphi'(u) = \tan_\kappa(u) \) in (5.29) gives
\[
\partial_t f = \frac{F}{H} \Delta f + \tan_\kappa(r_{p_{t_0}}) \left( \nu_t, \partial_{r_{p_{t_0}}} \right) \bar{F}(t) + \cos^{m-1}_\kappa(r_B)
- \frac{n}{H} \frac{F}{H} \left( 1 - \frac{1}{\cos^2_\kappa(r_{p_{t_0}})} \right) \left| \partial^\perp_{\nu_t} \right|^2.
\]
Now, set $t_1 = \inf\{t > t_0 : p_{t_0} \notin \Omega_t\}$. Because $\Omega_t$ is $h$-convex, Lemma 2.3 ii) implies $\langle \nu_t, \partial_{p_{t_0}} \rangle \geq 0$ for any $t \in [t_0, t_1]$. Thus, (5.30) combines with Lemma 2.1 iii) and the initial condition to give
\[
(5.31) \quad \left\{ \begin{array}{l}
\partial_t f \geq \frac{F}{\Pi} \Delta f + \co_k m^{\beta-1} (r_B) - (\frac{H}{\eta})^{m^{\beta-1}}, \\
f(x, t_0) = \varphi(r(x, t_0)) - \varphi(p_{t_0}) \geq 0.
\end{array} \right.
\]

Next, set $r(t) := \min_{x \in M} r(x, t)$ for any $t \in [t_0, t_1]$ and $\Theta(t) := \{x \in M \mid r(x, t) = r(t)\}$.

Applying the minimum of $r$ to (5.32) gives
\[
(5.33) \quad \left\{ \begin{array}{l}
\partial_t f_{\min} \geq \co_k m^{\beta-1} (r_B) - (\frac{H_{\max}}{\eta})^{m^{\beta-1}}, \\
f_{\min}(t_0) \geq 0.
\end{array} \right.
\]

Note that any point where the minimum of $f$ is attained is the point where the minimum of $r$ is attained for any $t \in [t_0, t_1]$, and at the point the hypersurface is tangent to an inball of radius $r(t)$, which implies that $H_{\max} = nco_k(r)$ on any point of $\Theta(t)$. Thus, using a standard comparison principle concludes that
\[
(5.34) \quad f(x, t) \geq 0
\]
for any $t \in [t_0, t_1]$ as long as $f(x, t)$ is well defined for $t \in [0, T)$, and it follows from (5.24) that $r_B(t)$ is positive for $t \in [t_0, t_0 + \int_0^1 \co_k m^{\beta} (s) ds [r_0, t_0 + \tau)]$. Then $f(x, t) \geq 0$ for any $t \in [t_0, \min\{t_0 + \tau, T, t_1\}]$.

To complete the proof, assume that $t_1 < \min\{t_0 + \tau, T\}$. By (5.33),
\[
r(x, t_1 - \zeta) \geq r_B(t_1 - \zeta) \text{ for all } \zeta \in (0, t_1 - \tau).
\]
Hence by (5.26),
\[
r(x, t_1) = \lim_{\zeta \rightarrow 0^+} r(x, t_1 - \zeta) \geq r_B(t_1) \geq \rho_{t_0}/2,
\]
which is a contradiction with $r(x, t_1) = r_{p_{t_0}}(t_1) = 0$ by definition of $t_1$. Therefore, $t_1 \geq \min\{t_0 + \tau, T\}$, which, together with (5.33) and (5.26), implies
\[
r(t) \geq \rho_{t_0}/2 \text{ on } [t_0, \min\{t_0 + \tau, T\}],
\]
which completes the proof. \(\square\)

The above lemma assists us by allowing us to consider a uniform bound on the speed of the flow.

**Theorem 5.6.** For $t \in [0, T)$,
\[
(5.34) \quad F(\cdot, t) < C_1 = C_1(n, m, \beta, a, M_0),
\]
moreover,
\[
(5.35) \quad H_m(\cdot, t) < C_2 := C_1^{1/\beta}.
\]

**Proof.** For any fixed $t_0 \in [0, T)$, let $\rho_{t_0}$ and $p_{t_0}$ be as in Lemma 5.5. Then by Corollary 5.4 and Lemma 5.3 on the hypersurface $M_t$ for every $t \in [t_0, \min\{t_0 + \tau, T\})$
\[
D_1 := \frac{\xi(\psi(V_0))}{2} \leq r_{p_{t_0}} \leq \xi(\psi(V_0)) =: D_2.
\]
Moreover, having into account Lemma 2.3 ii),
\[ \Phi = s_n(r_{pi_0}) \left\{ \nu, \partial r_{pi_0} \right\} \geq \alpha s_n(D_1) t a_{n}(D_1). \]
Then, taking the constant \( \epsilon = \alpha s_n(D_1) t a_{n}(D_1)/2 \) leads to
\[ (5.36) \quad \Phi - \epsilon \geq \epsilon > 0, \]
which ensures \( Z_t = \frac{F}{\Phi - \epsilon} \) is well-defined on the same time interval.

Let us go back to the equation (5.7), since strict \( h \)-convexity holds for each \( M_t, F, \bar{F} \) and \( \text{tr}_F(A\mathcal{W}) - a^2 \text{tr}(F) \) are all positive, which together with (5.36), the two terms containing \( \bar{F} \) and the term \( a^2 \text{tr}(F)Z \) can be neglected. Furthermore, note that \( F \) is homogeneous of degree \( m_{\beta} \), Euler’s theorem and (4.19) together give the following
\[ \text{tr}_F(A\mathcal{W}) = \bar{F}^i \lambda_i^2 \geq \epsilon_0 H \bar{F}^i \lambda_i = \epsilon_0 m_{\beta} H F. \]
Now from the above remark,
\[ (5.37) \quad \partial_t Z \leq \Delta_F Z + \frac{2 \left\{ \nabla Z, \nabla \Phi \right\} F}{\Phi - \epsilon} - \epsilon \epsilon_0 m_{\beta} H Z^2 + (1 + m_{\beta}) C(D_2) Z^2. \]
On the other hand, from (5.36) and Lemma 2.1 iii), it follows
\[ Z \leq \frac{F}{\epsilon} \leq \frac{1}{\epsilon} \left( \frac{H}{n} \right)^{m_{\beta}}. \]
Applying this to (5.37) gives
\[ \partial_t Z \leq \Delta_F Z + \frac{2 \left\{ \nabla Z, \nabla \Phi \right\} F}{\Phi - \epsilon} + \left( (1 + m_{\beta}) C(D_2) - \epsilon^{1+ \frac{1}{c_{n_{\epsilon}}} n m_{\beta} \epsilon_0 Z^{\frac{1}{m_{\beta}}}} \right) Z^2. \]
Assume that in \((\bar{x}, \bar{t}), \bar{t} \in [t_0, \min\{t_0 + \tau, T\}]\), \( Z \) attains a big maximum \( C \gg 0 \) for the first time. Then
\[ Z(\bar{x}, \bar{t}) \geq C(\Phi - \epsilon)(\bar{x}, \bar{t}) \geq \epsilon C, \]
which gives a contradiction if
\[ C > \max_{x \in M^n} \left\{ Z(x, t_0), \frac{1}{\epsilon} \left( \frac{C_n(D_2)(m_{\beta} + 1)}{n_0 \epsilon m_{\beta}} \right)^{m_{\beta}} \right\}. \]
Thus,
\[ Z(x, t) \leq \max_{x \in M^n} \left\{ Z(x, t_0), \frac{1}{\epsilon} \left( \frac{C_n(D_2)(m_{\beta} + 1)}{n_0 \epsilon m_{\beta}} \right)^{m_{\beta}} \right\}, \]
on \([t_0, \min\{t_0 + \tau, T\}].\)

From the definition of \( Z(x, t) \) and the upper bound \( D_2 \) of \( \rho_t \), it follows
\[ F(x, t) \leq (s_n(D_2) - \epsilon) \max_{x \in M^n} \left\{ Z(x, t_0), \frac{1}{\epsilon} \left( \frac{C_n(D_2)(m_{\beta} + 1)}{n_0 \epsilon m_{\beta}} \right)^{m_{\beta}} \right\}, \]
on \([t_0, \min\{t_0 + \tau, T\}].\) Since \( t_0 \) is arbitrary, and \( \tau \) does not depend to \( t_0 \), this implies
\[ F(x, t) = (s_n(D_2) - \epsilon) \max_{x \in M^n} \left\{ Z(x, t_0), \frac{1}{\epsilon} \left( \frac{C_n(D_2)(m_{\beta} + 1)}{n_0 \epsilon m_{\beta}} \right)^{m_{\beta}} \right\} =: C_1(n, m, \beta, a, M_0) \]
on $[0, T)$, which is (5.34), and so (5.35) by the definition of $F$.

Inserting the estimate (5.34) into (1.2) immediately gives the following

**Corollary 5.7.** For $t \in [0, T)$,

$$\bar{F}(t) < C_1.$$  

Hence the speed of the evolving hypersurfaces is bounded.

**Corollary 5.8.** For $t \in [0, T)$,

$$\left| \frac{\partial}{\partial t} X(p,t) \right| < C_3 := 2C_1.$$  

The curvature of $M_t$ also remains bounded.

**Corollary 5.9.** For $t \in [0, T)$,

$$\left| \mathcal{W} \right| < H \leq C_4.$$  

**Proof.** The homogeneity of $F$, (4.19) and the inequality Lemma 2.1 iv) imply that

$$m\beta F = \dot{F} \lambda_i \geq \varepsilon_0 H \text{tr}(\dot{F}) \geq \varepsilon_0 H m\beta F^{1 - \frac{1}{m\beta}}.$$  

Thus, by (5.34)

$$H \leq \frac{1}{\varepsilon_0} F^{\frac{1}{m\beta}} \leq \frac{1}{\varepsilon_0} C_1^{\frac{1}{m\beta}} =: C_4,$$

and so with the $h$-convexity of $M_t$

$$\left| \mathcal{W} \right| < C_4.$$  

6. **Long time existence**

In this section, it will be shown that the solution of the initial value problem (1.1)-(1.3) with the pinching condition (1.6) exists for all positive times. As usual, the first step is to obtain suitable bounds on the solution on any finite time interval $[0, T)$, which guarantees the problem (1.1)-(1.3) has a unique solution on the time interval such that the solution converges to a smooth hypersurface $M_T$ as $t \to T$. Thus, it is necessary to show that the solution remains uniformly convex on the finite time interval which ensures the parabolicity assumption of (1.1)-(1.3).

First it is to show the preserving $h$-convexity of the evolving hypersurface $M_t$. Recall that Theorem 4.4 and Lemma 4.2 together imply the strictly $h$-convexity of $M_t$. However, comparing with the initial assumptions of Theorem 1.6 there is a priori assumption $\bar{H} > 0$ in Theorem 4.4. As Cabezas-Rivas and Sinestrari mentioned in [19], note that for small times such an assumption holds due to the smoothness of the flow for small times and the initial pinching condition (1.6), but it is possible that at some positive time both $\min \bar{K}$ and $\min \bar{H}$ tend to zero such that $\bar{K}/\bar{H}^n$ remains bounded. Thus, to exclude such a possibility, following [19], it is necessary to complement Theorem 4.4 by establishing a positive lower bound on $\bar{H}$ for the finite time.
Lemma 6.1. Under the hypotheses of Theorem 4.4, there exist \( C_5, C_6 > 0 \) depending on \( n, m, \beta, a, M_0 \) such that

\[
\min_{M_t} \ddot{H} \geq C_5 e^{-C_6 t} \quad \forall t \in [0, T).
\]

Remark 6.2. Here the lower bound on \( \ddot{H} \) is enough for our purposes, and we will give later a stronger lower bound on \( \ddot{H} \) in Lemma 7.4.

Proof. Since under the hypotheses of Theorem 4.4, the evolving hypersurfaces \( M_t \) remains \( h \)-convex for every \( t \in [0, T) \). Here it is shown that \( \ddot{H} \) satisfies the lower bound (6.1), which in particular implies that \( \min \ddot{H} \) cannot go to zero as \( t \to T \). Let us go back to the evolution equation (3.14) of \( \ddot{H} \),

\[
\partial_t \ddot{H} = \Delta \ddot{H} + \text{tr} [F(\nabla \ddot{H}, \nabla \ddot{H})] - (\ddot{F} + (m\beta - 1)F) |\dot{A}|^2 + 2a(F - \ddot{F}) \ddot{H} + \text{tr} (\dot{A} \ddot{W}) \ddot{H}.
\]

The various terms appearing here are easily estimated. First, convexity of \( F \) implies

\[
\text{tr} [F(\nabla \ddot{H}, \nabla \ddot{H})] = \text{tr} [\ddot{F}(\nabla \ddot{H}, \nabla \ddot{H})] \geq 0.
\]

The next term can be estimated using the \( h \)-convexity of \( M_t \), (5.34) and (5.38),

\[
-(\ddot{F} + (m\beta - 1)F) |\dot{A}|^2 \geq -(\ddot{F} + m\beta F) \dot{H}^2 \geq -(1 + m\beta) C_1 \dot{H}^2.
\]

A similar calculation applies to the third term,

\[2a(F - \ddot{F}) \ddot{H} \geq -2aC_1 \ddot{H}.
\]

The last term here is positive by the \( h \)-convexity of \( M_t \). Consequently,

\[
\partial_t \ddot{H} \geq \Delta \ddot{H} - (1 + m\beta) C_1 \dot{H}^2 - 2aC_1 \ddot{H} \geq \Delta \ddot{H} - \max \left\{ (1 + m\beta) C_1, 2aC_1 \right\} (\dot{H}^2 + \ddot{H}).
\]

Now, if \( \ddot{H} \geq 1 \), then nothing is needed to prove. Thus, if we assume \( \ddot{H} < 1 \) and set \( C_6 = 2 \max \left\{ (1 + m\beta) C_1, 2aC_1 \right\} \), then

\[
\partial_t \ddot{H} \geq \Delta \ddot{H} - C_6 \ddot{H}.
\]

The parabolic maximum principle now gives

\[
\min_{M_t} \ddot{H} \geq C_5 e^{-C_6 t}, \quad \text{where} \ C_5 \text{ is given by} \ \min_{M_0} \ddot{H}. \]

\[\square\]

Corollary 6.3. Let \( X : M^n \times [0, T_{\text{max}}) \to \mathbb{R}^{n+1} \) be the solution of (1.1) - (1.3) with an initial value which satisfies the pinching condition (1.6). Then, the hypersurfaces \( M_t \) are strictly \( h \)-convex on any finite time interval; that is, for any \( t \in [0, T) \), with \( T < +\infty \) and \( T \leq T_{\text{max}} \), we have

\[
\inf_{M^s \times [0, T)} \dot{\lambda}_i > 0, \quad \forall i = 1, \ldots, n.
\]

Therefore, Theorem 4.4 is valid also without the hypothesis that \( \ddot{H} > 0 \) for \( t \in (0, T) \). The same holds for the other results that have been obtained until here under the same assumptions of Theorem 4.4.

Proof. The conclusion follows the argument as in [19] Corollary 6.2, only with obvious change of \( \lambda_i \) by \( \dot{\lambda}_i = \lambda_i - a \). \[\square\]
Preserving $h$-convexity of the evolving hypersurface leads to the following lower bound on the term $F$.

**Corollary 6.4.** $F \geq a^{m\beta}$ for all $t \in [0,T)$, with $T < +\infty$ and $T \leq T_{\text{max}}$.

**Proof.** In view of Lemma 2.1 iv), (5.34) and (5.40),

$$0 \leq \left[ \text{tr}_F (A'W') - a^2 \text{tr}(\dot{F}) \right] = \sum_{i=1}^{n} F_i \left( \lambda_i^2 - a^2 \right)$$

$$\leq (C_4 + a) \sum_{i=1}^{n} (F_i \lambda_i - F_i a) \leq (C_4 + a) \left( m\beta F - m\beta F^{1-\frac{1}{m\beta}} a \right)$$

$$\leq m\beta C_{\frac{1}{1-\frac{1}{m\beta}}} (C_4 + a) \left( \frac{F}{m\beta} - a \right),$$

which implies

$$F - a^{m\beta} \geq 0. \quad \Box$$

From Corollary 6.4 we have the following lower bound on $\bar{F}(t)$ by its definition.

**Corollary 6.5.** $\bar{F}(t) \geq a^{m\beta}$ for all $t \in [0,T)$, with $T < +\infty$ and $T \leq T_{\text{max}}$.

Since our flow is different from the volume preserving mean curvature flow, we cannot follow the induction argument of Hamilton as in [17, 28, 29, 30, 31, 38, 39], etc, to obtain uniform estimates on all orders of curvature derivatives and hence smoothness and convergence of the $M_t$ for the flow (1.1)-(1.3). Instead we use a more PDE theoretic approach, following an argumentation similar to the one in [19].

Before proceeding further, we adopt a local graph representation for a $h$-convex hypersurface as in [17]. For each fixed $t_0$, let $p_{t_0}$ be a center of an inball of $\Omega_{t_0}$, and $S^n$ the unit sphere in $T_{p_{t_0}} \mathbb{H}^{n+1}_\kappa$. For each $t$, since $M_t$ is $h$-convex, there exists a function $r : S^n \to \mathbb{R}^+$ such that $M_t$ can be written as a map: $S^n \to \mathbb{H}^{n+1}_\kappa$, again denoted by $X_t$, satisfying

$$X_t(x) = \exp_{p_{t_0}} r(t, u(t,x)) u(t,x),$$

where $u(t,x) = \frac{\exp^{-1}_t X_t(x)}{r_{p_{t_0}}(X_t(x))}$ and $r(t, u(t,x)) = r_{p_{t_0}}(X_t(x))$. At least, from Lemma 5.3 there exists some constant $\tau = \tau(a, n, m, \beta, V_0) > 0$ such that for $t \in [t_0, \min\{t_0 + \tau, T\})$ (near $t_0$), $p_{t_0} \in \Omega_t$, and so the map $u_t : M^n \to S^n \subset T_{p_{t_0}} \mathbb{H}^{n+1}_\kappa$ defined by $u_t(x) = u(t,x)$ is a diffeomorphism. On the other hand, the map

$$\dot{X}_t(x) = \exp_{p_{t_0}} r(t, u(t_0,x)) u(t_0,x)$$

is another parametrization of $M_t$. Incorporating a tangential diffeomorphism $\chi_t = u_{p_{t_0}}^{-1} \circ u_t : M^n \to M^n$ into the flow (1.1)-(1.3) to ensure that this parametrization is preserved; that is, if $X_t$ is a solution of (1.1)-(1.3), $X_t$ satisfies the equation

$$\left( \partial_t \dot{X}_t, v_t \right) = \dot{F}_t - F_t.$$
\(X_t\) can be considered as a map from \(S^n\) into \(\mathbb{R}^{n+1}\) by using the diffeomorphism \(u_t^{-1}\), i.e.,
\[
(6.5) \quad \tilde{X}_t(u) = \exp_{p_0} r(t, u) \ u \ \text{for every } u \in S^n,
\]
where \(r(u) = r_{p_0}(\tilde{X}_t(u))\) is a function on \(S^n\). For any local orthonormal frame \(\{e_i\}\) of \(S^n\), let \(D\) be the Levi-Civita connection on \(S^n\), a basis \(\{\tilde{e}_i\}\) of the tangent space to \(M_t\) is given by
\[
(6.6) \quad \tilde{e}_i = \tilde{X}_{t*}e_i = D_i(r)\partial_{\tau p_0} + s_\kappa(r)\tau_s e_i, \quad 1 \leq i \leq n,
\]
where \(\tau_s\) denotes the parallel transport along the geodesic starting from \(p_0\) in the direction of \(u\), and until \(\exp_{p_0}(r(u))\). As in \([17]\), by using Lemma 2.3 and (5.21) we deduce that
\[
|\tilde{X}_{t*}e_i| < \frac{s_\kappa(\psi(V_0) + a \ln 2)}{a \tan(\psi(V_0))}.
\]
Furthermore, (6.6) implies
\[
|e_i(r)| \leq |\tilde{X}_{t*}e_i|.
\]
Therefore, both the first derivatives of \(\tilde{X}_t\) and \(r\) are bounded independently of \(t\). The outward unit normal vector of \(M_t\) can be expressed as
\[
(6.7) \quad \nu = \frac{1}{|\xi|}(s_\kappa(r)\partial_r - \sum_{i=1}^n D_i r e_i)
\]
with
\[
|\xi| = \sqrt{s_\kappa^2(r) + |D r|^2}.
\]
After a standard computation, the second fundamental form of \(M_t\) can be expressed as
\[
(6.8) \quad h_{ij} = -\frac{1}{|\xi|} \left( s_\kappa(r)D_j D_i r - s_\kappa^2(r)c_\kappa(r)\sigma_{ij} - 2c_\kappa(r)D_i r D_j r \right),
\]
and the metric \(g_{ij}\) is
\[
(6.9) \quad g_{ij} = D_i r D_j r + s_\kappa^2(r)\sigma_{ij},
\]
where \(\sigma_{ij}\) is the canonical metric of \(S^n\). From this, the inverse metric can be expressed as
\[
(6.10) \quad g^{ij} = \frac{1}{s_\kappa^2(r)} \left( \sigma^{ij} - \frac{1}{|\xi|^2} D^i r D^j r \right),
\]
where \((\sigma^{ij})=(\sigma_{ij})^{-1}\) and \(D^i r = \sigma^{ij} D_j r\). Then equations (6.8) and (6.10) imply that
\[
(6.11) \quad h_j^i = -\frac{1}{|\xi|s_\kappa(r)} \left[ \frac{1}{s_\kappa(r)} \left( D_j D^i r - D_j r D^i r D^j r - c_\kappa(r)(\delta^i_j + \frac{D^i j D^j r}{|\xi|^2}) \right) \right]
\]
and
\[
(6.12) \quad H = -\frac{1}{|\xi|s_\kappa(r)} \left( \Delta \sigma r - \frac{1}{|\xi|^2} \nabla_\sigma^2 r(D r, D r) \right) + \frac{c_\kappa(r)}{|\xi|} \left( n + \frac{|D r|^2}{|\xi|^2} \right).
\]
Using (6.9) and (6.10) the Christoffel symbols have the expression:
\[
(6.13) \quad \Gamma^k_{ij} = \frac{1}{s_\kappa^2(r)} \left[ D_i D_j r r + s_\kappa(r)c_\kappa(r)(D_i r \sigma_{ij} + D_j r \sigma_{il} - D_l r \sigma_{ij}) \right] \left( \sigma^{kl} - \frac{1}{|\xi|^2} D^k r D^l r \right).
\]
Lemma 6.6. Let \( \phi := H_{m}^{1/m} \) and 
\[ \Gamma = \{ \lambda = (\lambda_1, \ldots, \lambda_n) : M_0 \leq H(\lambda) \leq M_1, \quad \min_{1 \leq i \leq n} \lambda_i \geq \epsilon H(\lambda) \}, \]
which is a compact symmetric subset of the positive cone \( \Gamma_+ \). There exists constants \( m_2 > m_1 > 0 \) depending only on \( n, M_0 \) such that for every \( t \in [0, T) \) and \( x \in M_t \), the following inequality

\[ m_1 \leq \frac{\partial \phi}{\partial \lambda_i} (\lambda) \leq m_2, \quad i = 1, \ldots, n, \quad \lambda \in \Gamma, \]
holds as long as the hypersurfaces \( M_t \) are strictly \( h \)-convex.

\[ \text{Proof.} \quad \text{Since} \quad \frac{\partial \phi}{\partial \lambda_i} (\lambda) > 0 \quad \text{for any} \quad \lambda \in \Gamma, \quad \text{and} \quad \Gamma \quad \text{is compact, there exist} \quad m_2 > m_1 > 0 \quad \text{such that} \]

\[ m_1 \leq \frac{\partial \phi}{\partial \lambda_i} (\lambda) \leq m_2, \quad i = 1, \ldots, n, \quad \lambda \in \Gamma. \]

□

Lemma 6.7. Let \( M \subset \mathbb{H}^{n+1}_\kappa \) be an embedded hypersurface satisfying at every point \( D_3 < H < D_4, \lambda_1 \geq \epsilon H \) for given positive constants \( D_3, D_4, \epsilon \). Given any \( p \in M \), let \( r \) be a local graph representation of \( M \) over a unit ball \( S^n \subset T_p M \). Then \( r \) satisfies

\[ ||r||_{C^{2,\alpha}(S^n)} \leq C_7 (1 + ||F||_{C^\alpha(S^n)}) \]

for some \( C_7 > 0 \) and \( 0 < \alpha < 1 \) depending only on \( n, D_3, D_4, \epsilon \) and the parameters \( \beta, m \) in the definition of \( F \).

\[ \text{Proof.} \quad \text{In exactly the same way as} \quad [19, \text{Lemma 6.3}], \quad \text{let us set} \quad \phi := H_{m}^{1/m}. \quad \text{Recalling Lemma} \quad \text{(2.1)}, \quad \phi \quad \text{is concave in} \quad \Gamma_+. \quad \text{Then in this case, the Bellman’s extension} \quad \tilde{\phi} \quad \text{of} \quad \phi \quad \text{takes the form} \]

\[ \tilde{\phi}(\lambda) := \inf_{\lambda \in \Gamma} \left[ \phi(\lambda) + D\phi(\lambda) (\bar{\lambda} - \lambda) \right] \]

for any \( \bar{\lambda} \in \Gamma_+ \). Notice that \( \phi \) is homogeneous of degree one, the extension simplifies to

\[ \tilde{\phi}(\lambda) = \inf_{\lambda \in \Gamma} D\phi(\lambda) \bar{\lambda}. \]

The Bellman extension preserves concavity, by definition, and homogeneity, since it is the infimum of linear functions. Importantly, \( \tilde{\phi} \) coincides with \( \phi \) on \( \Gamma \) by homogeneity of \( \phi \). Furthermore, using the definition \( \tilde{\phi} \) and Lemma 6.6, \( \tilde{\phi} \) is uniformly elliptic, that

\[ m_1 |\bar{\eta}| \leq \tilde{\phi}(\hat{\lambda} + \bar{\eta}) - \tilde{\phi}(\hat{\lambda}) \leq \sqrt{m_2} |\bar{\eta}|, \quad \text{for all} \quad \hat{\lambda}, \bar{\eta} \in \mathbb{R}^n, \quad \bar{\eta} \geq 0. \]

Now the hypotheses imply that the principal curvatures of \( M \) at every point are contained between two fixed positive constants. So \( M \) can be written as a local graph representation \( r \) over a unit ball \( S^n \subset T_p M \) at a given point \( p \in M \) with \( ||r||_{C^2} \) bounded in terms of \( D_4 \). Let us consider the function \( \tilde{\phi}(\hat{\lambda}(u)) \), where \( \hat{\lambda}(u) \) are the principal curvatures of \( M \) at the point \( (u, r(u)) \). Since \( \hat{\lambda}(u) \) are the eigenvalues of a matrix depending on \( D_{r}, D^2_{r} \) in view of (6.11), \( \tilde{\phi}(\hat{\lambda}(u)) \) can be expressed as \( \tilde{\Phi}(D_{r}(u), D^2_{r}(u)) \) for a suitable function \( \tilde{\Phi} = \tilde{\Phi}(u, A) \), with \( (u, A) \in S^n \times S \), \( S \) being the set of symmetric \( n \times n \) matrices. The dependence of \( \tilde{\Phi} \) on \( A \) is related to the dependence of \( \Phi \) on \( \hat{\lambda} \). In fact, it is well known that the concavity of \( \Phi \) with respect to \( \hat{\lambda} \) implies the concavity of \( \tilde{\Phi} \) with respect to \( D^2_{r} \), and that ellipticity on \( \tilde{\phi} \) implies the ellipticity condition (2.3) for \( \Phi \). In addition, \( \Phi \) is homogeneous of degree one with respect
to $D^2 r$. Furthermore, set $G(u, A) := \tilde{\Phi}(Dr(u), A)$ and $f(u) = \tilde{\Phi}(Dr(u), D^2 r(u))$. The above argument implies that the elliptic equation for $r$

$$G(D^2 r(u), u) = f(u), \quad u \in S^n, \ r \in C^2(S^n)$$

satisfies the conditions of Theorem 2.5. This theorem gives that there exists $\alpha \in (0, 1)$ such that

$$||r||_{C^{2,\alpha}(S^n)} \leq C(1 + ||f||_{C^\alpha(S^n)})$$

where $C$ depends on $n, D_3, D_4, \varepsilon, \beta$.

Our assumptions say that $\bar{\lambda}(u)$ belongs for every $u$ to the set $\bar{\Gamma}$ where $\bar{\phi}$ and $\bar{\Phi}$ coincide. Thus, $f$ coincides with $H_m^{1/m} = F^{1/\beta m}$ at $\bar{\lambda}(u)$. Observe that our assumptions; that is the uniform bounds on the curvatures both from below and above imply that $F$ is contained between two positive values depending only on $n, D_3, D_4, \varepsilon, \beta$. Therefore $||F^{1/\beta m}||_{C^\alpha}$ is estimated by $||F||_{C^\alpha}$ times a constant depending only on these quantities. we obtain Lemma 6.7.

**Theorem 6.8.** Let $M_t$ be a solution of (1.1), (1.3), defined on any finite time interval $[0, T)$, with initial condition satisfying (1.6). Then, for any $0 < t_0 < T, \alpha \in (0, 1)$ and every integer $k \geq 0$, there exist constant $C_8$, depending on $n, m, \beta, a, M_0$ and $C_9 = C_9(n, m, \beta, a, M_0, k)$ such that

$$\|F\|_{C^{0}(M^n \times (t_0, T])} \leq C_8, \quad (6.14)$$

$$\|r\|_{C^{k}(M^n \times (t_0, T])} \leq C_9. \quad (6.15)$$

**Proof.** The cases $k = 0, 1$ of (6.15) follow the fact that $r$ and its first order derivatives are uniformly bounded. Then, by (6.10), (6.11) implies that its second order derivatives are uniformly bounded. For each fixed $t_0 \in [0, T)$, $M_t$ can be locally reparameterized as graphs over the unit sphere $S^n$ with center $p_{t_0}$ in $T_{p_{t_0}} H^{\alpha+1}_n$ as (6.5). Then, from (6.11), (6.7) and (6.3), a short computation yields that the distance function $r$ on $S^n$ satisfies the following parabolic PDE

$$\partial_t r = s_{-1}(r)|\xi|(\bar{F}_t - F_t), \quad (6.16)$$

where the length of the outward normal vector $|\xi|$ is given by the expression (6.7). The function $F = H_m^{\beta}$ in the coordinate system under consideration is a function of $D^2 r$ and $Dr$. The RHS of (6.16) is a fully nonlinear operator, furthermore, observe that (6.16) can be rewritten as

$$\partial_t r = -s_{-1}(r)|\xi|H_m^{\beta} + s_{-1}(r)|\xi|F_t$$

$$= s_{-1}(r)H_{m}^{\beta-1/m}(|\xi|^m H_m)^{1/m} + s_{-1}(r)|\xi|F_t. \quad (6.17)$$

Since $H_{m}^{\beta-1/m}$ is bigger than $a^{m\beta-1}$ in view of Corollary 6.4 and bounded above by $C_{2}^{\beta - \frac{1}{m}}$ coming from (6.3), and $r$ can be bounded independently of $t$, this implies that $s_{-1}(r)H_{m}^{\beta-1/m}$ are also uniformly Hölder continuous functions. Then, from (6.11) this ensures that (6.16) is a linear, strictly parabolic partial differential equation. However, the higher order regularity does not follow by the general theory of Krylov and Safonov [33] because the operator is not a concave function of $D^2 r$. Here, we use instead the arguments in [19] who followed the procedure of [8, 48, 40], which consists of first proving regularity in space at a fixed time and then regularity in time.
The first step is to derive $C^\alpha$-estimate (6.14) of $F$. In the coordinate system under consideration, (6.14) can be written as
\begin{equation}
\partial_t F = a^{ij} D_i D_j F + b^i D_i F + e F, \quad (u, t) \in S^n \times J,
\end{equation}
where $J = [t_0, \min\{t_0 + \tau, T\}]$, and the estimates are given by
\begin{equation}
 a^{ij} = \beta H_m^{\beta - 1} c^{ij}, \quad b^i = \beta H_m^{\beta - 1} c^i \Gamma^i_j.
\end{equation}
and
\begin{equation}
e = \beta (F - \bar{F}) H_m^{\beta - 1} (\text{tr}_c(A \Psi) - a^2 \text{tr}(c)).
\end{equation}
Here
\begin{equation}
c = \{c^{ij}\} = \left\{\frac{\partial H_m}{\partial h_i^k} g^{kj}\right\}.
\end{equation}
Since $a^{ij}, b^i$ and $e F$ are uniformly bounded on the curvatures both from above on any finite time interval in view of (5.40), (6.11), (6.13) and from below by $h$-convexity of $M_t$, the equation (6.18) is uniformly parabolic with uniformly bounded coefficients. Then applying Theorem 2.4 to (6.18) gives that for any $0 < r' < 1$ and $J' = J - t_0$ there exist some constants $D_6 > 0$ and $\alpha \in (0, 1)$ depending on $n, m, \beta, a, M_0$ such that
\begin{equation}
\|F\|_{C^\alpha(B_{r'} \times J')} \leq D_6 \|F\|_{C^0(M^n \times [0, T])} \leq D_6.
\end{equation}
Therefore, covering $M_t$ by finite many graphs on balls of radius $r'$ can give (6.14).

Furthermore, for any fixed time $t \in [t_0, T)$, applying (6.14) to Lemma 6.7 on $M_t$ implies that
\begin{equation}
\|r\|_{C^{2, \alpha}(M_t)} \leq D_7 := D_7(n, m, \beta, a, M_0, k).
\end{equation}
From this estimate on $r(\cdot, t)$ for any fixed $t$, following the procedure of [8, §3.3, §3.4] or [48, Theorem 2.4] to equation (6.16), a $C^{2, \alpha}$ estimate for $r$ with respect to both space and time can be obtained. Once $C^{2, \alpha}$ regularity is established, standard parabolic theory yields uniform $C^k$ estimates (6.15) for any integer $k > 2$, which implies uniform $C^k$ estimates (6.15) for any integer $k \geq 0$ with the fact that $r$, its first order derivatives and its second order derivatives are uniformly bounded.

**Theorem 6.9.** If $M^n$ is a closed $n$-dimensional smooth manifold and $M_0: M^n \to \mathbb{H}^{n+1}_K$ is an immersion pinched in the sense of (1.6), then the solution $M_t$ of (1.1)-(1.3) with initial condition $X_0$, is smooth and everywhere satisfies (1.6) on $[0, \infty)$.

**Proof.** As we have already seen, the preserving pinching condition (1.6) and smoothness throughout the interval of existence follows from Theorem 4.4 and Lemma 6.1.

On the other hand, it is clear from the expression (6.5) for $\tilde{X}_t$ that all the higher order derivatives of $\tilde{X}_t$ are bounded if and only if the corresponding derivatives of $r$ are bounded. Thus, uniform $C^k$ estimates (6.15) of $r$ implies that all the derivatives of $\tilde{X}_t$ are also uniformly bounded. So the smoothness of $\tilde{X}_t$ implies the same smoothness of the reparametrization $X_t$ of $\tilde{X}_t$ given by (6.2).

It remains to show that the interval of existence is infinite. Suppose to the contrary there is a maximal finite time $T$ beyond which the solution cannot be extended and we derive a contradiction. Then the evolution equation (1.1)-(1.3) implies that
\begin{equation}
\|X(p, \sigma) - X(p, \tau)\|_{C^0(X_0)} \leq \int_\tau^\sigma |\bar{F} - F| (p, t) dt
\end{equation}
for $0 \leq \tau \leq \sigma < T$. By (5.34) and (5.38), $X(\cdot, T)$ tends to a unique continuous limit $X(\cdot, T)$ as $t \to T$. In order to conclude that $X(\cdot, T)$ represents a hypersurface $M_T$, next under this assumption and in view of the evolution equation (3.1) the induced metric $g$ remains comparable to a fix smooth metric $\tilde{g}$ on $M^n$:

$$\left| \frac{\partial}{\partial t} \left( \frac{g(\xi, \xi)}{\tilde{g}(\xi, \xi)} \right) \right| = \left| \frac{\partial g(\xi, \xi)}{\partial g(\xi, \xi)} \frac{g(\xi, \xi)}{\tilde{g}(\xi, \xi)} \right| \leq 2|F - F| \|A\| \|\frac{g(\xi, \xi)}{\tilde{g}(\xi, \xi)}\|,$$

for any non-zero vector $\xi \in TM^n$, so that ratio of lengths is controlled above and below by exponential functions of time, and hence since the time interval is bounded, there exists a positive constant $C$ such that

$$\frac{1}{C} \tilde{g} \leq g \leq C \tilde{g}.$$

Then the metrics $g(t)$ for all different times are equivalent, and they converge as $t \to T$ uniformly to a positive definite metric tensor $g(T)$ which is continuous and also equivalent by following Hamilton’s ideas in [28]. Therefore using the smoothness of the hypersurfaces $M_t$ and interpolation,

$$\|X(p, \sigma) - X(p, \tau)\|_{C^k(X_0)} \leq C\|X(p, \sigma) - X(p, \tau)\|^{1/2}_{C^0(X_0)} \|X(p, \sigma) - X(p, \tau)\|^{1/2}_{C^{2k}(X_0)} \leq C|\sigma - \tau|^{1/2},$$

so the sequence $\{X(t)\}$ is a Cauchy sequence in $C^k(X_0)$ for any $k$. Therefore $M_t$ converge to a smooth limit hypersurface $M_T$ which must be a compact embedded hypersurface in $\mathbb{H}^{n+1}_\kappa$. Finally, applying the local existence result, the solution $M_T$ can be extended for a short time beyond $T$, since there is a solution with initial condition $M_T$, contradicting the maximality of $T$. This completes the proof of Theorem 6.9.

7. Exponential convergence to a geodesic sphere

Observe that, to finish the proof of Theorem 1.6 it remains to deal with the issues related to the convergence of the flow (1.1)-(1.3). It should be proved that solutions of equation (1.1)-(1.3) with initial conditions (1.6) converge, exponentially in the $C^\infty$-topology, to a geodesic sphere in $\mathbb{H}^{n+1}_\kappa$ as $t$ approaches infinity.

The first step is to show that, if a smooth limit exists, it has to be a geodesic sphere of $\mathbb{H}^{n+1}_\kappa$. To address the first step, let

$$f = \frac{1}{n^\kappa} - \frac{K}{H^\kappa},$$

and we will show that the principal curvature come close together when time tends to infinity.

Then as remarked in Section 4, $f \geq 0$ with equality holding only at umbilic points, which is the value assumed on a geodesic sphere of $\mathbb{H}^{n+1}_\kappa$. The following Lemma is an immediate consequence of the evolution equation (1.1) of $\dot{Q}$.

Lemma 7.1. The quantity $f$ evolves under (1.11)-(1.13) satisfying

$$(7.1) \quad \partial_t f = \Delta f + \frac{(n + 1)}{n H^n} \left( \nabla f, \nabla \dot{H}^n \right)_F - \frac{(n - 1)}{n K} \left( \nabla f, \nabla \dot{K} \right)_F - \frac{\dot{H}^n}{n K} |\nabla f|^2_F.$$
In view of (3.11), such that there holds

\[- \left( \frac{\bar{Q}}{H^2} \left| \bar{H} \nabla \bar{\psi} - \bar{\psi} \nabla \bar{H} \right|^2_{F,b} + \bar{Q} \text{tr}_{b} - \frac{\bar{Q}}{H^2} \left( \bar{F} (\nabla \bar{\psi}, \nabla \bar{\psi}) \right) \right) \]

\[- \left[ (m \beta - 1) F + \bar{F} \right] \frac{\bar{Q}}{H} \left( n |A|^2 - \bar{H}^2 \right) - a \bar{Q} \text{tr}_{\bar{F}} (\bar{A} \bar{\psi}) \left( \text{tr}(\bar{b}) - \frac{n^2}{H} \right). \]

Corollary 7.2. Under the conditions of Theorem 1.6.

(7.2) \( \partial_t f \leq \Delta_{\hat{F}} f + \frac{(n+1)}{nH^2} \langle \nabla f, \nabla \hat{H} \rangle_{\hat{F}} - \frac{(n-1)}{nK} \langle \nabla f, \nabla \bar{K} \rangle_{\hat{F}} - \frac{\bar{H}^n}{nK} \langle \nabla f \rangle_{\hat{F}}^2 \)

where \( C_{11} = \delta m \beta a^\alpha C^\alpha \)

Proof. Applying the similar argument as in Theorem 4.4, Corollary 6.5, inequality (1.6), and Proposition 7.3. For \( x \in (0, t_0) \), we let \( G((x_0, t_0), (x_0, t)) \) be a nonnegative solution of an equation of the form

(7.3) \( (\partial_t - \Delta_{\hat{F}}) \theta = -\dot{\theta} + a^\alpha \theta + b^s \theta_\zeta = g. \)

Here \( g = g(x, t, \theta(x, t)) \) and we assume that there exists \( \zeta \in \mathbb{R}_+ \) such that \( f \) satisfies the inequality \(-\zeta \theta(x, t) \leq g(x, t, \theta(x, t)) \leq \zeta \theta(x, t) \) for all \( (x, t) \in G(4R) \). We assume the coefficients are measurable and bounded by a constant \( c_0 \in \mathbb{R}_+ \) and there exist \( 0 < \Lambda_1 \leq \Lambda_2 < \infty \) such that \( \Lambda_1 \delta^{ij} \leq (a^{ij}) \leq \Lambda_2 \delta^{ij} \). Then there exists \( c = c(n, \Lambda_1, \Lambda_2, R, \|b\|_{L^\infty}, c_0, \zeta) > 0 \) such that there holds

(7.4) \( \sup_{G((0, -4R^2), \Delta)} \theta \leq c \cdot \inf_{G(R)} \theta. \)

From this the following lemma can be obtained.

Lemma 7.4. There exists a constant \( 0 < C_{12} = C_{12}(n, a, m, \beta, M_0) \) such that for all \( t \in [0, \infty) \)

(7.5) \( \hat{H} \geq C_{12}. \)

Proof. We will follow an idea from [36, Lemma 6.2]. For \( t \in [0, \infty) \) let \( x_t \in M_t \) be a point in contact with an enclosing sphere of radius \( D_1 < \rho < D_2 \). This implies

(7.6) \( \sup_{x \in M_t} F(x) \geq F(x_t) \geq c^0_{\mu_\zeta} (\rho_\zeta) \geq c^0_{\mu_\zeta} (D_1). \)

In view of (3.11), \( \theta := F - a^\alpha \) satisfies the evolution equation

(7.7) \( (\partial_t - \Delta_{\hat{F}})(F - a^\alpha) = (F - \bar{F}) \left[ \text{tr}_{\bar{F}} (A \bar{\Psi}) - a^2 \text{tr}(\bar{F}) \right]. \)
Since \((\partial_t - \Delta F)\) is uniformly parabolic in view of Corollary 6.4 and Lemma 6.6, we can apply Proposition 7.3 together with (7.6) to obtain the lower bound for \(F - a^m\beta\) for all times
\[
F - a^m\beta \geq c \left( co_k m^\beta(D_1) - a^m\beta \right),
\]
Therefore, with help of Lemma 2.1, we have
\[
H \geq n \left[ c \left( co_k m^\beta(D_1) - a^m\beta \right) + a^m\beta \right] \frac{1}{m\beta},
\]
which implies
\[
\hat{H} \geq C_{12},
\]
where
\[
C_{12} = n \left\{ c \left( co_k m^\beta(D_1) - a^m\beta \right) + a^m\beta \right\} \frac{1}{m\beta} - a > 0.
\]

**Proposition 7.5.** Under the conditions of Theorem 1.6, the rate of convergence of \(f\) to 0 as \(t \to \infty\) is exponential.

**Proof.** Applying the similar argument as in Theorem 4.4 and Lemma 7.4 to (7.1) gives
\[
\partial_t f_{\text{max}}(t) \leq - C_{11} C_{12} f_{\text{max}}(t),
\]
which implies that
\[
f_{\text{max}}(t) \leq C_{13} e^{-C_{14} t},
\]
where \(C_{13} = f_{\text{max}}(0)\), \(C_{14} = C_{11} C_{12}\). This proves the Proposition.

As we have seen, since \(M_t\) is strictly \(h\)-convex on \([0, \infty)\), from Lemma 5.5 there exists some constant \(\tau = \tau(a, n, m, \beta, V_0)(> 0)\) such that for each \(t_0 \in [0, \infty)\), on the time interval \([t_0, t_0 + \tau]\), \(p_0 \in \Omega_t\), and \(M_t\) can be represented as graph \((r)\) (6.2). Let us go back to the evolution equation (6.17) of \(r\), we know that it is uniformly parabolic because of \(h\)-convexity of \(M_t\) for any \(t \in [0, \infty)\). So repeating the same arguments as in Section 6 implies that all the derivatives of \(r\) are uniformly bounded. Thus, we are now in conditions to apply Arzel-Ascoli Theorem to conclude the existence of sequences of maps \(r_t\) and \(\hat{X}_t\) solves (6.3) with \(t_i \to \infty\), which \(C^\infty\)-converge to smooth maps \(r_\infty : S^n \to \mathbb{R}_+\) and \(\hat{X}_\infty : S^n \to \mathbb{H}^{n+1}_{\kappa}\) satisfying \(\hat{X}_\infty(u) = \exp_{p_0} r_\infty(u) u\). Therefore, the reparametrization \(X_t\) of \(\hat{X}_t\) given by (6.2) has the same convergence properties. Thus we conclude that \(X_t, C^\infty\)-converges to a map \(X_\infty : S^n \to \mathbb{H}^{n+1}_{\kappa}\), and that two immersions \(X_\infty\) and \(\hat{X}_\infty\) are equivalent in the sense of an isometry of \(\mathbb{H}^{n+1}_{\kappa}\) carrying \(p_\infty\) to \(p_0\). And since the convergence is smooth and all the hypersurfaces \(X_t(S^n)\) satisfy (1.6), we can assure that \(\mathcal{S} = X_\infty(S^n)\) must be a compact embedded hypersurface and satisfies (1.6).

On the other hand, Proposition 7.3 says that all points on \(\mathcal{S}\) are umbilic points. In conclusion, the only possibility is that \(\mathcal{S}\) represents a geodesic sphere in \(\mathbb{H}^{n+1}_{\kappa}\) and, by the volume-preserving properties of the flow, such sphere has to enclose the same volume as the initial condition \(X_0(M)\).

Finally, from Proposition 7.3, we can conclude with the standard arguments as in [40, Theorem 3.5], [19, Theorem 7.3] and [36, Corollary 7.3] that the flow converges exponentially to the geodesic sphere \(\mathcal{S}\) in \(\mathbb{H}^{n+1}_{\kappa}\) in the \(C^\infty\)-topology.
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