Gravitational decoherence by the apparatus in the quantum-gravity-induced entanglement of masses

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Abstract
One of the outstanding questions in modern physics is how to test whether gravity is classical or quantum in a laboratory. Recently there has been a proposal to test the quantum nature of gravity by creating quantum superpositions of two nearby neutral masses, close enough that the quantum nature of gravity can entangle the two quantum systems, but still sufficiently far away that all other known Standard Model interactions remain negligible. However, preparing superposition states of a neutral mass (the light system) requires the vicinity of laboratory apparatus (the heavy system). We will suppose that such a heavy system can be modelled as another quantum system; since gravity is universal, the lighter system can get entangled with the heavier system, providing an inherent source of gravitational decoherence. In this paper, we will consider a toy model composed of two light and two heavy quantum oscillators prepared in the motional ground state, forming pairs of probe-detector systems, and study under what conditions the entanglement between two light systems evades the decoherence induced by the heavy systems. We conclude by estimating the decoherence in the proposed experiment for testing the quantum nature of gravity.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The theory of General relativity (GR) is one of the most well-tested theories of physics, successfully passing several fundamental tests [1], with its latest success being the observation of gravitational waves [2]. However, at short-distance scales and early times, where quantum effects start playing an important role, GR breaks down [3], and a quantum theory of gravity is needed. There are several candidate quantum gravity (QG) theories, such as string theory [4] and loop QG [5], but despite theoretical progress, the connection with experiments has remained elusive [6].

Although the quantization of gravity is an often-used tool in theoretical physics, forming the backbone of candidate QG theories, thus far, there is no definitive experimental evidence in support of the quantum nature of gravity. The reason is simple—the weakness of the gravitational force makes direct detection of gravitons a formidable challenge, a situation which will likely persist in the foreseeable future [7]. On the other hand, indirect tests of the quantum nature of gravity (with the first discussions dating back to Feynman [8]) have in recent years become a real prospect with the advances in precision sensing and metrology, opening the possibility of probing genuine quantum features of gravity with tabletop experiments.

1.1. The QG-induced entanglement of masses (QGEM) protocol

In 2017 a simple experiment for a definitive test of the quantum nature of gravity was proposed in [9], along with its relevant background and feasibility studies (for a related work see [10]). The idea exploits the QGEM to discern between all classical models of gravity from the quantum one (when talking about a theory of QG, we assume an effective quantum field theory where a massless spin-2 graviton acts as a force carrier for the gravitational force, and which behaves well at low energies [11]). Two nearby masses, each delicately prepared in a spatial superposition, are placed close enough that their mutual gravitational interaction can generate entanglement, but still far enough that all other interactions are strongly suppressed. The generated entanglement can be detected by measuring quantum correlations between the two masses, a genuinely quantum effect with no classical analogue, and, if detected, would provide the first definite evidence for the quantization of the gravitational field.

The argument for the entanglement-based test of the quantization of gravity can be summarized as follows. To generate matter–matter entanglement one requires a quantum interaction coupling the two systems; the quantum matter–matter gravitational interaction (which in the non-relativistic regime is the operator-valued Newtonian potential) corresponds to the shift of the energy of the gravitational field, hence requiring the gravitational field itself to be a quantum operator, ruling out the possibility of a (real-valued) classical gravitational field [12]. Formally, entanglement between two quantum states cannot be increased with local operations and classical communications (LOCCs) [13], as would be the case with a classical gravitational field, and hence, if gravitationally induced entanglement is detected, the gravitational interaction must be ostensibly quantum in nature. This argument has been discussed in detail within
the context of perturbative QG [12, 14], the path-integral approach [15], and the Arnowitt–Desse–Meissner formalism [16].

1.2. Testing the spin of the mediator

To discern the spin character of the graviton it is however not sufficient to consider the non-relativistic matter–matter interactions, one needs to devise an experiment where gravity couples relativistic fields. One promising possibility is to probe the quantum light-bending interaction between a heavy mass and photons in a cavity where the degree of the generated entanglement can be used to distinguish between spin-2 and spin-0 mediators of the gravitational field [17]. Another option is to consider matter-matter interactions beyond the static limit where the post-Newtonian corrections encode the spin character [12]. In this paper, we will quantify the generated entanglement up to the second post-Newtonian contribution.

1.3. Experimental challenges

To realise such an experiment one has to overcome many challenges, such as the preparation of the initial state [18–20], the isolation of the system [21–23] and the reduction of noise [24]. The shielding of the system from spurious interactions will never be completely perfect, and the matter systems will lose their coherence due to interaction with the environment. Methods for battling decoherence have been proposed previously [25–27], and many sources of decoherence have been discussed, such as in [28–30].

1.4. Gravitational decoherence induced by experimental apparatus

There is however one source of inherent decoherence which has thus far not been analysed in detail. To witness the generated entanglement we require the presence of nearby experimental apparatus; while electromagnetic couplings between a neutral mass (the light system) and the lab equipment (the heavy system) can be suppressed with appropriate shielding, their mutual gravitational interaction is unavoidable, and scales unfavourably with the mass of the laboratory apparatus. The heavy laboratory equipment, which can be modelled quantum mechanically, can entangle with the two neutral masses, thus providing an unavoidable source of gravitational decoherence.

When we talk about the ‘apparatus’ or ‘laboratory equipment’ we refer to anything close to the experiment that can be quantum, such as the current carrying wires in the Stern–Gerlach setup [18–20, 31]. We call any such source the ‘heavy mass’. In this paper we consider two heavy systems A and B with mass $m_A = m_B = M$. This paper aims to analyse this gravity-induced decoherence in presence of the heavy masses in a model-independent fashion and to quantify the attenuation of the entanglement between the two light quantum masses.

In this paper we will study decoherence with an entanglement measure, the concurrence, which quantifies how much the laboratory equipment and the test masses are entangled. An often-used approach to analyse decoherence is also to trace out the ‘environment’ system and find the remaining entanglement between the test masses. We briefly discuss this latter approach in section 4, but when we talk about ‘the decoherence’ we refer to the entanglement between the apparatus and the test masses.

First, we will introduce the toy model in section 2. The setup consists of four matter systems: two heavy quantum harmonic oscillators (representing the laboratory apparatus) and two light quantum harmonic oscillators (representing the two test masses), the details of the quantum
systems are introduced in section 2.1. Section 2.2 discusses how the quantum systems are coupled to QG. In section 3 we analyse the decoherence, for which we use the entanglement measure 'concurrence' (section 3.1). To find the concurrence we compute the coupling of the matter systems by the quantized gravitational field within perturbative QG (section 3.2). Then we discuss the induced decoherence on the two light systems in the static limit (section 3.3) as well as for the higher-order momentum corrections by considering the light systems up to the second post-Newtonian contribution (section 3.4). We find the allowed parameter space where the entanglement between the light systems dominates the decoherence (section 4) and we conclude with a discussion of the results (section 5).

2. Modelling quantum matter coupled to the gravitational interaction

Let us consider four massive systems, denoted by \( a, b, A, B \) with light masses \( m_a, m_b \) and heavy masses \( m_A, m_B \), respectively. We wish to understand the entanglement of \( m_a, m_b \) via the quantum nature of gravity, while \( m_A, m_B \) would be responsible for gravitationally decohering the light masses. These massive systems are placed in harmonic traps located at \( \pm \frac{d}{2} \) for systems \( a, b \) and located at \( \pm \frac{D}{2} \) for systems \( A, B \), see figure 1. We will assume \( D > d \).

2.1. Toy model with 2 light and 2 heavy mechanical oscillators

Taking the harmonic oscillators to be well-localized, we obtain:

\[
\hat{x}_a = -\frac{d}{2} + \delta \hat{x}_a, \quad \hat{x}_b = \frac{d}{2} + \delta \hat{x}_b, \quad \hat{x}_A = -\frac{D}{2} + \delta \hat{x}_A, \quad \hat{x}_B = \frac{D}{2} + \delta \hat{x}_B, \tag{1}
\]

with \( \hat{x}_i \) and \( \delta \hat{x}_i \) the position operators and small equilibrium displacement for system \( i = a, b, A, B \). We will further assume that all the masses are neutral to minimize the electromagnetic interactions. Although there will be dipolar interactions between all these systems; the Casimir-induced dipole–dipole interactions between the two systems \( a, A \) and \( b, B \) can be minimised by placing a conducting plate, while the Casimir interaction between a light and a heavy system can be minimised by giving some hierarchy between \( D \) and \( d \). The Hamiltonian for the matter systems is given by:

\[
\hat{H}_m = \sum_{i=a,b,A,B} \frac{\hat{p}_i^2}{2m_i} + \frac{m_i}{2} \omega_i^2 \delta \hat{x}_i^2, \tag{2}
\]

with \( \hat{p}_i \) and \( \omega_i \) the conjugate momenta and the trap’s harmonic frequency for system \( i \), respectively.

The basis is chosen such that the matter systems are uncoupled, which will simplify our computations. In general, as an initial system we can choose a Hamiltonian where there is a coupling between systems \( a \) (\( b \)) and \( A \) (\( B \)) given by:

\[
\hat{H}_m = \sum_{i=1,2,3,4} \frac{\hat{p}_i^2}{2m_i} + \frac{k_0}{2} (\delta \hat{x}_1^2 + \delta \hat{x}_2^2) + \frac{k_1}{2} (\delta \hat{x}_1 - \delta \hat{x}_2)^2 + \frac{k_2}{2} (\delta \hat{x}_3^2 + \delta \hat{x}_4^2) + \frac{k_3}{2} (\delta \hat{x}_3 - \delta \hat{x}_4)^2. \tag{3}
\]

There exists a unitary transformation such that the Hamiltonian becomes decoupled. After the transformation, the matter Hamiltonian can be written as

\[
\hat{H}_m = \hat{H}_a + \hat{H}_b + \hat{H}_A + \hat{H}_B. \tag{4}
\]
Figure 1. A graphical representation of the setup that visualizes the introduced parameters $D, d$. With $a$ and $b$ denoting the light systems and $A$ and $B$ denoting the heavy systems. We assume the hierarchy $D > d > 0$. Furthermore, $d$ is taken large enough such that the gravitational coupling dominates other couplings between the masses (for example from the Casimir–Polder and dipole–dipole interactions).

with $\hat{H}_i = \frac{\hat{p}_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 \hat{x}_i^2$, and with $\omega_a^2 = k_{0(2)}/m_a$, $\omega_A^2 = [k_{0(2)} + 2k_{1(3)}]/m_A$. The change of basis is given as:

$$
\hat{x}_a = \frac{1}{\sqrt{2}} \left( \hat{x}_1 + \hat{x}_2 \right), \quad \hat{x}_b = \frac{1}{\sqrt{2}} \left( \hat{x}_3 + \hat{x}_4 \right)
$$

Thus we can write the Hamiltonian as in (2).

The mode operators for the harmonic oscillator systems are given by:

$$
\delta \hat{x}_j = \sqrt{\frac{\hbar}{2m_j\omega_j}} \left( \hat{\phi} + \hat{\phi}^\dagger \right), \quad \hat{p}_j = i \sqrt{\frac{\hbar m_j\omega_j}{2}} \left( \hat{\phi} - \hat{\phi}^\dagger \right),
$$

with $\hat{\phi} = \hat{a}, \hat{b}, \hat{A}, \hat{B}$, and the operators satisfying the usual commutation relations$^3$. Thus the Hamiltonian can be written as:

$$
\hat{H}_m = \hbar \omega_a \hat{a}^\dagger \hat{a} + \hbar \omega_b \hat{b}^\dagger \hat{b} + \hbar \omega_A \hat{A}^\dagger \hat{A} + \hbar \omega_B \hat{B}^\dagger \hat{B}.
$$

2.2. Coupling quantum matter to quantized linearized gravity

We now introduce a gravitational field and study the interaction Hamiltonian $\hat{H}_{\text{int}}$ between the gravitational and matter fields.

We work in linearized gravity where the metric is given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $\eta_{\mu\nu}$ the flat Minkowski background with signature $(-,+,+,+)$ and with $h_{\mu\nu}$ a perturbation which is small in magnitude around the Minkowski background. The metric fluctuations are then promoted to quantum operators:

$^3$ These commutation relations are:

$$
[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = [\hat{A}, \hat{A}^\dagger] = [\hat{B}, \hat{B}^\dagger] = 0,
$$

$$
[\hat{a}, \hat{b}] = [\hat{A}, \hat{B}] = [\hat{b}^\dagger, \hat{A}^\dagger] = [\hat{b}^\dagger, \hat{B}^\dagger] = 0,
$$

$$
[\hat{a}^\dagger, \hat{b}^\dagger] = [\hat{A}^\dagger, \hat{B}] = [\hat{A}^\dagger, \hat{B}^\dagger] = 1.
$$
These two modes can be treated independently. The operators $\hat{P}^{\mu\nu}$ and $\hat{P}^{\mu\nu}_{\ast}$ denote the graviton annihilation and creation operators, respectively, and satisfy the following commutation relations [32]:

$$
\left[\hat{P}^{\mu\nu}_{\ast}(\vec{k}), \hat{P}^{\mu\nu}(\vec{k}')\right] = (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})\delta(\vec{k} - \vec{k}'),
$$

In the weak field regime, we can decompose the metric fluctuation operator into two modes: the spin-2 mode $\gamma_{\mu\nu}$ and the spin-0 mode $\gamma = \eta_{\mu\nu}\gamma^{\mu\nu}$ [32]⁴. Such that: $\hat{h}_{\mu\nu} = \hat{\gamma}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\hat{\gamma}$. Consequently the spin-2 and spin-0 decomposed parts of the graviton can be promoted to operators as well, and they are given in terms of the graviton creation- and annihilation operators [32]:

$$
\hat{\gamma}_{\mu\nu} = A \int d^3k \sqrt{\frac{\hbar}{2\omega_k(2\pi)^3}} \left( \hat{P}^{\mu\nu}_{\ast}(\vec{k}) e^{-i\vec{k}\cdot\vec{r}} + \text{H.c.} \right),
$$

$$
\hat{\gamma} = 2A \int d^3k \sqrt{\frac{\hbar}{2\omega_k(2\pi)^3}} \left( \hat{P}^{\mu\nu}(\vec{k}) e^{-i\vec{k}\cdot\vec{r}} + \text{H.c.} \right),
$$

which satisfy the commutation relations in (8)⁵. The gravity Hamiltonian can then be written in terms of graviton creation and annihilation operators [32]. Now that both the matter systems and graviton system have been introduced, we continue by studying their interaction and in the next section the consequential entanglement generation. The interaction term is given by the graviton coupling to the stress–energy tensor $T_{\mu\nu}$ (which specifies the matter system contents):

$$
\hat{H}_{\text{int}} = -\frac{1}{2} \int d^3r \hat{h}_{\mu\nu}(\vec{r}) \hat{T}_{\mu\nu}(\vec{r}).
$$

We consider the two harmonically trapped particles $a, b$ to be moving along the $x$-axis, and the two heavy systems $A, B$ to be static. Systems $A, B$ are taken to be static because we consider these systems to be very massive systems such that their motion remains negligible when perturbed by the two light systems. The four systems thus generate the following currents:

$$
\hat{T}_{00}(\vec{r}) \equiv \sum_{n=a,b,A,B} m_n c^2 \delta(\vec{r} - \vec{r}_n), \quad \hat{T}_{ij}(\vec{r}) \equiv \sum_{n=a,b} \frac{\hat{p}_n \hat{p}_{nj}}{E/c^2} \delta(\vec{r} - \vec{r}_n),
$$

with the position of the matter systems $\vec{r}_n = (\hat{x}_n, 0, 0)$, with the momentum $\hat{p}_n = (-E/c, \vec{p})$, energy $E = \sqrt{\vec{p}^2 c^2 + m^2 c^4}$, and with $i, j = 1, 2, 3$.

Since we specified the movement of the oscillators $a, b$ to be along the $x$-axis, the only non-zero $\hat{T}_{\mu\nu}$-components are $\hat{T}_{01}, \hat{T}_{10}$ and $\hat{T}_{11}$. Therefore, the only relevant $\hat{h}_{\mu\nu}$ components in the coupling are $\hat{h}_{00} = \hat{\gamma}_{00} + \frac{1}{2}\hat{\gamma}, \hat{h}_{10} = \hat{\gamma}_{01}$ and $\hat{h}_{11} = \hat{\gamma}_{11} - \frac{1}{2}\hat{\gamma}$. Writing the interaction

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⁴ These two modes can be treated independently. $\gamma_{\mu\nu}$ is sometimes called the trace-reversed metric since $\hat{h} = -\gamma$.

⁵ Following (8) and the definition $\gamma = \eta_{\mu\nu}\gamma^{\mu\nu}$, the additional commutation relation is: $[\hat{P}(\vec{k}), \hat{P}^{\dagger}(\vec{k}')] = -\delta(\vec{k} - \vec{k}')$. 

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Hamiltonian in terms of the decomposed metric perturbation, while exploiting the symmetries $\tilde{T}_{01} = \tilde{T}_{10}$ and $\tilde{\gamma}_{01} = \tilde{\gamma}_{10}$, gives:

$$\tilde{H}_{\text{int}} = \int d^3 r \left( \frac{1}{2} \left[ \tilde{\gamma}_{00} (\vec{r}) + \frac{1}{2} \tilde{\gamma} (\vec{r}) \right] \tilde{T}_{00} (\vec{r}) + \frac{1}{2} \left[ \tilde{\gamma}_{11} (\vec{r}) - \frac{1}{2} \tilde{\gamma} (\vec{r}) \right] \tilde{T}_{11} (\vec{r}) + \tilde{\gamma}_{10} (\vec{r}) \tilde{T}_{10} (\vec{r}) \right).$$

(13)

As explained in [12], the energy shift in the graviton vacuum due to the above interaction can only induce entanglement when the gravitational field is quantized, with $h_{\mu\nu}$ or equivalently $\gamma_{\mu\nu}$ and $\gamma$. This can be formalized using the LOCC principle, which states that a LOCC channel (such as a classical real-valued gravitational field) cannot increase the entanglement between the two systems. Only quantum communication can increase entanglement between the systems [12]. The graviton here acts as a quantum communicator between the two systems and, therefore, is able to induce a coupling that entangles previously unentangled oscillators. This entanglement and decoherence are studied in the next sections.

3. Gravitationally induced entanglement and decoherence

We assume that initially the quantum matter systems are in the ground state (denoted by $|0\rangle_i$, with $i$ specifying the system $i = a, b, A, B$):

$$|\psi_i\rangle = |0\rangle_a |0\rangle_b |0\rangle_A |0\rangle_B .$$

(14)

Since gravity will couple all the systems, it will induce interaction between the heavy and light oscillators, $\hat{H}_hl$ (which is presented in (36) and (46)). As a result of this interaction, the final state will evolve to:

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{n_a,n_b,n_A,n_B} C_{n_a,n_b,n_A,n_B} |n_a\rangle |n_b\rangle |n_A\rangle |n_B\rangle .$$

(15)

The number states are denoted by $|n\rangle$, and the normalisation is given by $N = \sum_{n_a,n_b,n_A,n_B} |C_{n_a,n_b,n_A,n_B}|^2$.

In first-order perturbation theory the coefficients for the final wavefunction are given by:

$$C_{n_a,n_b,n_A,n_B} = \lambda \frac{\langle n_a | | n_b | | n_A | | n_B | | H_{hl} | | 0 | | 0 | | 0 \rangle}{\sum_{i = a, b, A, B} \left( E_{h_i} - E_{n_i} \right)} ,$$

(16)

for the perturbed states, and $C_{0000} = 1$ for the unperturbed state. The interaction is scaled by a bookkeeping parameter $\lambda$. In the above equation $E_{0i}$ is the ground-state energy and $E_{n_i}$ denotes the $n$th excited state energy, for system $i = a, b, A, B$.

At this point, it is important to take note that $H_{hl}$ is a quantum operator. If it were classical, so not operator-valued, then $C_{n_a,n_b,n_A,n_B} = 0$ for any perturbed state, due to the orthogonality of the states. Thus the final wavefunction would be $|\psi\rangle = |0\rangle |0\rangle |0\rangle |0\rangle$, the initial wavefunction. No entanglement can be generated in an initially unentangled system from a classical interaction. Since we are working in the framework of a perturbative quantum field theory of gravity we expect an entanglement, which we will quantify by an entanglement measure named concurrence.

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6 Here we have left out the subscripts on the kets to ease the notation. In the remainder of the paper, the order of the states is always $a, b, A, B$. 

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7
3.1. Basic formulae for quantifying entanglement using concurrence

We use concurrence as a measure of entanglement between bipartite systems. The larger the concurrence, the more strongly entangled the subsystems are, where a maximally entangled state gives the value $\sqrt{2}$, and an unentangled state gives the value 0 [33]. In the context of the QGEM experiment different measures for the entanglement have been used, such as the entanglement entropy (or von Neumann entropy) [25, 26], which is related to the concurrence via a simple relation [33], or the negativity [34, 35]. By using concurrence we obtain simple expressions that can be connected to the concurrence in the absence of the heavy oscillators, which was computed in [12]. It should be noted that concurrence becomes problematic in the general case, and if we were to for example consider next order results in section 4, another measure of entanglement would be needed.

We are interested in the decoherence of systems $a$, $b$ due to their coupling to the more massive systems $A$, $B$. In sections 3.3 and 3.4 we will analyse the coupling of the heavy and light systems by finding the concurrence between the heavy and light subsystems. Since entanglement and decoherence are two sides of the same coin, by studying the concurrence for the light-heavy bipartition, we gain information about the effects of the apparatus (the heavy oscillators) on the coherence of the QGEM experiment (the two light particles). The state of the full system consisting of both the heavy and the light subsystems is a pure (unentangled) state. Therefore we use the pure state definition of concurrence. For a system of arbitrary dimensions that is described by a pure state $\rho$, the concurrence along a bipartition can be computed using: [36, 37]

$$C_{hl} \equiv \sqrt{2 - 2\text{Tr}(\rho^2)}, \quad (17)$$

where $\rho_l = \text{Tr}_h(\rho)$ is the partial density matrix representing the light subsystem $l$, which is found by tracing out the heavy subsystem $h$ in the full density matrix $\rho = |\psi_f\rangle\langle\psi_f|$. In section 4 we will find the concurrence within the light system (so the entanglement between the two light harmonic oscillators) in the presence of the heavy system. We first trace out the heavy system, resulting in a mixed state $\rho_l$ that describes the light system in the presence of the heavy system. Then we can find the entanglement between the two light systems using the mixed state concurrence definition. As shown in section 3.3, the light systems are in a superposition of the ground state and the first excited state, forming a two-level system (ignoring the suppressed higher-order corrections). Treating this system as consisting of effective qubits, the partial density matrix $\rho_l$ is of rank 4. The concurrence of a mixed system that is represented by a rank-4 density matrix $\rho_l$ is given by: [33, 38]

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \quad (18)$$

where the $\lambda_i$’s are the ordered eigenvalues (highest to lowest) of the matrix $\sqrt{\hat{\rho}}\rho\hat{\rho}$ with $\hat{\rho} = (Y \otimes Y)\rho^* (Y \otimes Y)$, where $\rho^*$ is the complex conjugate of $\rho$, and $Y$ is the Pauli matrix. In section 4 we trace out the heavy systems $A, B$ and use (18) to compute the concurrence in the bipartition $a - b$.

First, we compute $\rho_l$ by tracing out the heavy system. The partial density matrix for the light system is found to be:

$$\rho_l = \frac{1}{N^2} \sum_{n_a, n_a, N_a, N_a} C_{n_a, n_a, n_a, N_a} C_{N_a, N_a, n_a} |n_a, n_a\rangle\langle n_a, n_a| \quad (19)$$
using the notation $|n_a n_b⟩ = |n_a⟩|n_b⟩$. Inserting this expression into (17), the heavy-light concurrence, denoted $C_{hl}$, can be expressed in terms of the coefficients $C$ defined in (16):

$$C_{hl} = \left[ 2 - \frac{2}{N_z^2} \sum_{n_a, n_A, n_B, n_C} C_{n_a n_B n_C} C_{n_A n_B n_C} C_{n_A n_B n_C} C_{n_A n_B n_C} \right]^{1/2}. \quad (20)$$

Finding all the relevant expressions of the coefficients in (16) would result in the quantification of decoherence/entanglement at first order in the perturbation theory. For this, we need to find the interaction Hamiltonian between the heavy and light system, $\hat{H}_{hl}$, which is generated by the exchange of the virtual graviton.

### 3.2. Derivation of the quantum matter–matter gravitational interaction

The interaction between gravity and matter is given in (13), from which we can compute the shift in energy to the graviton vacuum at second order in perturbation theory$^7$:

$$\Delta \hat{H}_{2} \equiv \int d^3k \frac{\langle 0|\hat{H}_{\text{int}}|\vec{k}\rangle\langle \vec{k}|\hat{H}_{\text{int}}|0⟩}{E_0 - E_\vec{k}}, \quad (21)$$

with $E_0$ the energy of the vacuum state and $E_\vec{k} = E_0 + \hbar \omega_\vec{k}$ the energy of the one-particle state $|\vec{k}\rangle$ representing the intermediate graviton, which is created from the vacuum with the graviton creation operators. The collection of normalized projectors $|\vec{k}\rangle\langle \vec{k}|$ is given by:

$$|\vec{k}\rangle\langle \vec{k}| = \frac{1}{2} P_{00}^\dagger (\vec{k}) |0⟩⟨0| P_{00} (\vec{k}) + \frac{1}{2} P_{01}^\dagger (\vec{k}) |0⟩⟨0| P_{01} (\vec{k})$$

$$- P_{10}^\dagger (\vec{k}) |0⟩⟨0| P_{10} (\vec{k}) - P_{11}^\dagger (\vec{k}) |0⟩⟨0| P_{11} (\vec{k}). \quad (22)$$

For each projector summed in the above expression we can evaluate $⟨0|\hat{H}_{\text{int}}|\vec{k}⟩$, with the interaction given in (13):

$$⟨0|\hat{H}_{\text{int}} P_{00} (\vec{k}) |0⟩ = A \sqrt{\frac{\hbar}{2 \omega_\vec{k}}} \hat{T}_{00} (\vec{k}), \quad (23)$$

$$⟨0|\hat{H}_{\text{int}} P_{11} (\vec{k}) |0⟩ = A \sqrt{\frac{\hbar}{2 \omega_\vec{k}}} \hat{T}_{11} (\vec{k}), \quad (24)$$

$$⟨0|\hat{H}_{\text{int}} P_{01} (\vec{k}) |0⟩ = A \sqrt{\frac{\hbar}{2 \omega_\vec{k}}} \hat{T}_{01} (\vec{k}), \quad (25)$$

$$⟨0|\hat{H}_{\text{int}} P_{10} (\vec{k}) |0⟩ = A \sqrt{\frac{\hbar}{2 \omega_\vec{k}}} \left[ \hat{T}_{00} (\vec{k}) - \hat{T}_{11} (\vec{k}) \right]. \quad (26)$$

$^7$ The first-order term corresponding to the emission/absorption of a graviton is given by $⟨0|\hat{H}_{\text{int}}|0⟩$. This contribution vanishes since $\hat{H}_{\text{int}}$ depends linearly on the graviton creation and annihilation operators, and $\hat{P}(0) = \hat{P}^\dagger (0) = 0$. In the second-order term (corresponding to the exchange of a virtual graviton) $⟨0|\hat{H}_{\text{int}}|\vec{k}⟩$ is quadratically dependent on the creation and annihilation operators. The operator commutation rules show that this contribution is non-vanishing.
The momentum-space stress–energy tensor components are given by the Fourier transform of the components in position space, which from (12) are found to be:

\[
\hat{T}_{00}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \left[ m_A c^2 e^{-i\vec{k} \cdot \vec{x}_a} + m_B c^2 e^{-i\vec{k} \cdot \vec{x}_b} + E_a e^{-i\vec{k} \cdot \vec{r}_a} + E_b e^{-i\vec{k} \cdot \vec{r}_b} \right],
\]

(28)

\[
\hat{T}_{01}(\vec{k}) = -\frac{c}{(2\pi)^{3/2}} \left[ p_u e^{-i\vec{k} \cdot \vec{r}_a} + \hat{p}_u e^{-i\vec{k} \cdot \vec{r}_a} \right],
\]

(29)

\[
\hat{T}_{11}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \left[\frac{\hat{p}_a^2 c^2}{E_a} e^{-i\vec{k} \cdot \vec{r}_a} + \frac{\hat{p}_b^2 c^2}{E_b} e^{-i\vec{k} \cdot \vec{r}_a} \right].
\]

(30)

Filling in (28)–(30) and (23)–(26) into (21) gives an expression for the graviton energy shift from the vacuum, \(\Delta H_g\). This expression can be simplified by performing the integral over \(\vec{k}\) (in (21))\(^9\). Furthermore we restrict the movement to the \(x\)-axis, meaning that \(p_{1,y} = p_{1,z} = 0\), \(\hat{p}_{1,x} \equiv \hat{p}_1\), and \(\hat{r}_1 = (\hat{x}_1, 0, 0)\) for \(i = a, b, A, B\), to find the expression:

\[
\begin{align*}
\Delta H_g &= -\mathcal{A}^2 \frac{1}{16\pi c^2} \left[ \frac{m_A E_c c^2 + m_A \hat{p}_a^2}{|\chi_A - \chi_a|} + \frac{m_A E_b c^2 + m_A \hat{p}_b^2}{|\chi_A - \chi_b|} + \frac{m_A m_B c^2 + m_A \hat{p}_c^2}{|\chi_A - \chi_b|} \right] \\
&\quad + \frac{E_a E_b - 4\hat{p}_a \hat{p}_b c^2}{|\chi_a - \chi_b|} + \frac{\hat{p}_a^2 c^2}{|\chi_a - \chi_b|} + \frac{\hat{p}_b^2 c^2}{|\chi_a - \chi_b|} + \frac{\hat{p}_c^2 c^2}{|\chi_a - \chi_b|} \\
&\quad + \frac{m_B E_c c^2 + m_B \hat{p}_c^2}{|\chi_B - \chi_b|} + \frac{m_B E_b c^2 + m_B \hat{p}_b^2}{|\chi_B - \chi_b|}. \\
\end{align*}
\]

(32)

Taking \(m_2 = m_b = m\) and \(m_1 = m_B = M\), and expanding (32) in powers of \(1/c^2\) gives the non-relativistic couplings among the 4 oscillators up to order \(1/c^4\), and in first-order in \(G\), the full expression is presented in (33):

\[
\begin{align*}
\hat{\Delta}H_g &= -G \left[ \frac{mM}{|\chi_a - \chi_A|} + \frac{mM}{|\chi_b - \chi_A|} + \frac{mM}{|\chi_a - \chi_B|} + \frac{mM}{|\chi_b - \chi_B|} + \frac{M^2}{|\chi_a - \chi_B|} \right] \\
&\quad - \frac{G M^3}{2m} \left( \frac{\hat{p}_a^2}{|\chi_a - \chi_A|} + \frac{\hat{p}_a^2}{|\chi_b - \chi_A|} + \frac{\hat{p}_b^2}{|\chi_a - \chi_B|} + \frac{\hat{p}_b^2}{|\chi_b - \chi_B|} \right) + \frac{3M^3}{2|\chi_a - \chi_b|} \\
&\quad - G \left[ \frac{5M}{8m^2} \left( \frac{\hat{p}_a^4}{|\chi_a - \chi_A|} + \frac{\hat{p}_a^4}{|\chi_b - \chi_A|} + \frac{\hat{p}_b^4}{|\chi_a - \chi_B|} + \frac{\hat{p}_b^4}{|\chi_b - \chi_B|} \right) + \frac{5\hat{p}_a^4 - 18\hat{p}_a^2 \hat{p}_b^2 + 5\hat{p}_b^4}{8m^2|\chi_a - \chi_b|} \right] \\
&\quad + \left( C \frac{1}{c^6} \right). \\
\end{align*}
\]

(33)

\(^8\) The momentum-space stress–energy tensor components are given by the Fourier transform of the components in position space:

\[
\hat{T}_{\mu\nu}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^4r e^{-i\vec{k} \cdot \vec{r}} \hat{T}_{\mu\nu}(\vec{r}).
\]

(27)

\(^9\) This integration is simply

\[
\int \frac{d^4k}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{r}} = \frac{1}{4\pi^2} \delta(\vec{r}).
\]

(31)

and the expression was rewritten such that \(\hat{r} = \hat{x}_1 - \hat{x}_f\).
We can reach the classical point particle limit by substituting \( \vec{r} \equiv \vec{x}_i - \vec{x}_j \) with the number-valued distances discussed in section 2, the potential becomes:

\[
\Delta H_g = -G \left[ \frac{m^2}{d} + \frac{M^2}{D} - \frac{8mM}{D^2 - D^2} \right] \left[ \frac{3p_a^2 - 8p_ap_b + 3p_b^2}{2d} - \frac{6DM(p_a^2 + p_b^2)}{(d^2 - D^2)m} \right] 
- \frac{G}{c^4} \left[ \frac{5p_a^2 - 18p_a^2p_b^2 + 5p_b^2}{8dm^2} + \frac{20DM(p_a^2 + p_b^2)}{8(d^2 - D^2)m^3} \right] + \left( O \left( \frac{1}{c^6} \right) \right). \tag{34}
\]

If the heavy systems are not taken into account, i.e. \( M = 0 \), (34) reduces to the same expression found in [12] for the interaction between two harmonic oscillators. Furthermore, in the centre-of-mass frame, i.e. \( p \equiv p_a = -p_b \), (34) gives a potential that matches known results for the non-relativistic potential between classical point particles [39–41].

### 3.3. Quantifying decoherence—leading order effect

In this section, we give the expression for the decoherence due to the gravitational interaction between the heavy and light systems. We find the decoherence using an entanglement measure, the concurrence, given in (20), which quantifies the information of the light system shared with the heavy system. We start by finding the first-order interaction terms between the heavy and light systems. Since we are considering a bipartite heavy–light system, only the interaction between heavy and light is taken into account to find the decoherence. Any heavy–heavy or light–light interaction can be viewed as ‘self-interaction’ since it only causes entanglement within the subsystem.

However, the light–light entanglement is important to analyse for the decoherence due to the presence of the heavy system. Taking into account only the light–light couplings in (33) and following the same procedure as described in this section, we find the concurrence between the two light oscillators at the lowest order to be:

\[
C_{ll} = \frac{Gm}{d\omega_l^2} + \frac{2Gm}{c^2d}, \tag{35}
\]

where we have taken the first-order coupling, which consists of a static contribution (from the position operator coupling) and a non-static contribution (from momentum operator coupling), with the momentum contribution being suppressed by \( 1/c^2 \).

We can find the lowest order coupling between the light and heavy matter systems by substituting the expression (1) for the position operators in terms of their displacements into the Hamiltonian in (33). Then, by Taylor expanding the small displacements \( \delta \hat{x}_i \), the lowest order interaction terms are given by:

\[
\hat{H}_{hl} = 16GmM \left[ \frac{\delta \hat{x}_a \delta \hat{x}_A + \delta \hat{x}_b \delta \hat{x}_B}{(D - d)^3} + \frac{\delta \hat{x}_a \delta \hat{x}_b + \delta \hat{x}_a \delta \hat{x}_B}{(D + d)^3} \right]. \tag{36}
\]

Note that in the above expression, there is no coupling between the momentum and the position operators, even though the light system is taken to be non-static. This is because the lowest order coupling is between one heavy position/momentum operator and one light position/momentum operator. The coupling with momentum operators at this order appear as \(-4G\hat{p}_a\hat{p}_b/dc^2\), it only gives a coupling between the two light particles instead of the light and heavy subsystems.
We will now use the mode operators in (5) to write $\hat{H}_{hl}$ in terms of the mode operators $j, j^\dagger$ with $j = a, b, A, B$. The resulting Hamiltonian is:

$$\hat{H}_{hl}^{\text{op}} = \frac{8G \hbar \sqrt{Mm}}{\sqrt{\omega \omega_l}} \left[ \frac{a^\dagger A^\dagger + b^\dagger B^\dagger}{(D - d)^3} + \frac{a^\dagger B^\dagger + A^\dagger a^\dagger}{(D + d)^3} \right],$$

where all irrelevant terms (the terms that annihilate the vacuum) have been left out for simplicity. Filling the Hamiltonian in (37) into (16), we find the non-zero coefficients are: (any terms of the form $|00n_A n_B\rangle$ and $|n_a n_b 00\rangle$ are omitted because they arise from the self-interaction within the light and heavy subsystems, respectively, and are therefore not relevant to our analysis)

$$C_{1010} = C_{0101} = -\frac{g - \omega_h + \omega_l}{\omega_h + \omega_l}, \quad C_{0110} = C_{1001} = -\frac{g + \omega_h + \omega_l}{\omega_h + \omega_l},$$

where we assumed $\omega_a = \omega_b = \omega_l$ and $\omega_A = \omega_B = \omega_h$ for simplicity, and we set $\lambda = 1$. With the couplings given by:

$$g_{\pm} = \frac{8G}{(D \pm d)^3} \sqrt{Mm} \sqrt{\omega h \omega_l},$$

The final state of equation (15) (up to the first-order in the perturbation theory) is thus given by:

$$|\psi_f\rangle = \frac{1}{\sqrt{N}} \left( |0000\rangle - \frac{g - \omega_h + \omega_l}{\omega_h + \omega_l} |1010\rangle - \frac{g + \omega_h + \omega_l}{\omega_h + \omega_l} |0110\rangle - \frac{g}{\omega_h + \omega_l} |1001\rangle - \frac{g}{\omega_h + \omega_l} |0101\rangle + \ldots, \right),$$

with the normalization $N = 1 + 2(g_+^2 + g_-^2)/(\omega_h + \omega_l)^2$, and using the notation $|n_a n_b n_A n_B\rangle = |n_a n_b n_A n_B\rangle$. The ‘.’’ in (40) indicate the states that do not contribute to the entanglement [42]. The final state is an entangled state between the ground states and first excited states of the light and heavy subsystems. Due to the pair-wise interactions taken here, in each of the perturbed states, one heavy and one light system are in the first excited states. Using equation (17) the concurrence is found to be:

$$C_{hl}^{(1)} = \frac{1 + 2 \left( \frac{g_{\pm}^2}{\omega^2} \right) + 8 \frac{g_{\pm}^2}{\omega^2}}{1 + 2 \frac{g_{\pm}^2}{\omega^2}},$$

where $\omega \equiv \omega_h + \omega_l$ for simplicity, and the superscript (1) denotes that we have taken the lowest order contributions to the entanglement (i.e. linear equations of motion). In the limit where $g_{\pm}/\omega \ll 1$ the concurrence becomes:

$$C_{hl}^{(1)} \approx 2\sqrt{2} \frac{g_{\pm}^2}{\omega^2}.$$
We now consider two special cases representing different experimental setups: $D \gg d$ and $D = 2d$. Taking the limit $D \gg d$, we can Taylor expand the couplings $g_{\pm} \approx \frac{8G}{D^2} \frac{Mm}{\sqrt{\omega_l \omega_i}} \left(1 \mp \frac{d}{D} + O\left(\frac{d^2}{D^2}\right)\right)$. The expression for the concurrence simplifies to:  

$$C_{hl}^{(1)} (D \gg d) \approx \frac{32G}{(\omega_h + \omega_i)D^3} \sqrt{\frac{Mm}{\omega_l \omega_i}}.$$  

(43)

The degree of entanglement grows with the masses of the light and heavy system ($m, M$, respectively), but it grows inversely with the harmonic trap frequencies and inversely (inverse cubic) with the distance between the light and heavy system.

We now explore another possible configuration of the four oscillators where the spacing between any adjacent oscillators will be $d$, by setting $D = 2d$. In this case, the concurrence in (42) simplifies to:

$$C_{hl}^{(1)} (D = 2d) \approx \frac{32\sqrt{365}G}{27(\omega_h + \omega_i)} d^3 \sqrt{\frac{Mm}{\omega_l \omega_i}}.$$  

(44)

We note that the dependence on the masses, frequencies and distance between the oscillators is identical to the behaviour of the concurrence in (43).

Instead of limiting $D$ and choosing a specific setup to simplify the results, we could also note that the coupling between two neighbouring oscillators, $g_\pm$, will dominate over the coupling between two maximally separated oscillators, $g_\mp$, (see the denominator of the first factor in (39)). We can thus simplify the expression for the concurrence by considering only the coupling $g = \sqrt{g_+^2 + g_-^2} \approx g_-$ (and $g_-$ as in (39)), giving:

$$C_{hl}^{(1)} (g) \approx \frac{16\sqrt{2}G}{(D - d)^3 (\omega_h + \omega_i)} d^3 \sqrt{\frac{2mM}{\omega_l \omega_i}}.$$  

(45)

These three limits work in different domains of $d/D$. In figure 2 we compare the different approximations as a function of $D$. The range of $D$ shown is from $d$ (which is taken to be of the order $10^{-3} m$, following [12]) to $10^{-3} m$, the lines continue to be a constant for a larger $D$. As one would expect, the concurrence $C_{D=2d}$ is the worst approximation (except when $D = 2d$). The concurrence $C_{g}$ performs the best across the whole range. Although the concurrence $C_{D=0}$ starts performing well around $D \sim 10^{-3} m$ as well. We have explored and analysed these limits in order to be able to perform an analytical analysis in section 4.

The concurrence quantifies the entanglement due to the coupling between the light and heavy systems. Since entanglement and decoherence are two sides of the same coin, the concurrence between the subsystems provides a handle on the decoherence behaviour of the test masses due to the presence of the apparatus. If there is no interaction between the heavy and light subsystems ($g_\pm = 0$), then there is no gravitational decoherence from the experimental apparatuses. However, in any experiment the gravitational decoherence due to the experimental apparatus is unavoidable. Minimizing the mass $M$, and maximizing the trap frequency $\omega$, as well as the distance $D$, minimizes the decoherence from the apparatuses.

---

10 It turns out that when keeping these second-order terms, the approximation $g_{\pm} / \omega \ll 1$ simplifies the concurrence to:

$$C_{hl}^{(1)} (D \gg d) \approx 2\sqrt{\frac{g}{\omega}} \sqrt{2 + \frac{4D^2}{2D^2}} \approx 4\sqrt{\frac{g}{\omega}} \left(1 + \frac{21}{2} \frac{d^2}{D^2} \right).$$

Neglecting $O(d/D)^2$ terms we recover (43).
Figure 2. Difference between the concurrence in (42) and different approximations of this concurrence given in (43)–(45), as a function of the distance $D$. The figure shows the validity of each approximation as a function of $D$ (when $\Delta C = 0$ the approximation matches the expression (42)). The approximation denoted $C^{(1)}(h)$ is applicable over the whole range of $D$, while $C^{(1)}(D \gg d)$ works increasingly better as $D$ becomes large compared to $d$, and $C^{(1)}(D = 2d)$ holds exactly at $D = 2d$ and becomes a worse approximation away from this value, as expected. For $d = 10^{-4}$ m, $m = 10^{-14}$ kg, $\omega_l = 10^8$ Hz, $\omega_h = 10^8$ Hz, $M = 10^{-8}$ kg.

3.4. Quantifying decoherence—contribution from higher-order couplings

The first-order coupling between the heavy and light systems is only between the position operators (it is a quadratic coupling in the Hamiltonian, i.e, linear equations of motion). We now look at the post-Newtonian corrections which contains also momentum operators, focusing on cubic couplings in the Hamiltonian (quadratic couplings in the equations of motion). Inserting the position operators in (1) into the Hamiltonian given in (33), we obtain the cubic couplings in (46):

$$
\hat{H}_{hl}^{(2)} = 48GmM \left[ \frac{\delta \hat{x}_A (\delta \hat{x}_A)^2 - (\delta \hat{x}_A)^2 \delta \hat{x}_A}{(D - d)^4} + \frac{\delta \hat{x}_A (\delta \hat{x}_B)^2 - (\delta \hat{x}_A)^2 \delta \hat{x}_B}{(d + D)^4} \right]
+ \delta \hat{x}_A (\delta \hat{x}_B)^2 - (\delta \hat{x}_A)^2 \delta \hat{x}_B
\right]
+ 6GM \left[ \frac{(\hat{p}_A)^2 \delta \hat{x}_A + (\hat{p}_B)^2 \delta \hat{x}_B}{(d + D)^2} - \frac{(\hat{p}_A)^2 \delta \hat{x}_A + (\hat{p}_B)^2 \delta \hat{x}_B}{(D - d)^2} \right].
$$

(46)

Here we consider only the next order coupling between the light and heavy matter systems (the heavy–heavy and light–light couplings can be seen as self-interactions for the light–heavy bipartition used to calculate $C_{hl}$). This expression contains the couplings between three operators: two light momentum/position operators and one heavy position operator, or two heavy position operators and one light position operator. The relevant non-zero coefficients for the
In this regime the concurrence simplifies to:
\[ C_{0102} = C_{1020} = \frac{\theta_1^-}{2\omega_h + \omega_l}, \quad C_{0120} = C_{1020} = \frac{\theta_1^+}{2\omega_h + \omega_l}, \quad C_{0201} = C_{2010} = \frac{\theta_2^- - \theta_2^-}{\omega_h + 2\omega_l}, \quad C_{0210} = C_{2010} = \frac{\theta_2^+ - \theta_2^+}{\omega_h + 2\omega_l}, \]
(47)
with the six different couplings defined by:
\[ g_1^\pm = \frac{12\sqrt{2G}}{\omega_h (D \pm d)} \sqrt{\frac{m}{\omega_l}}, \quad g_2^\pm = \frac{12\sqrt{2G}}{\omega_l (D \pm d)} \sqrt{\frac{M}{\omega_h}}, \quad g_3^\pm = \frac{3G\omega_l}{\sqrt{2}c^2 (D \pm d)} \sqrt{\frac{M}{\omega_h}}. \]
(49)
The ‘−’-labelled couplings arise due to interactions between neighbouring heavy and light oscillators, while the ‘+’-labelled couplings arise due to maximally separated heavy and light oscillators. Moreover, we underline the fact that the \( g_3 \) couplings represent the interaction of two momentum operators with a position operator, while the \( g_1 \) and \( g_2 \) couplings are attributable to the product of three position operators.

Recalling that \( C_{0000} = 1 \), the perturbed wavefunction up to first order from (15) is given by:
\[
|\psi_1\rangle = \frac{1}{\sqrt{N}} \left[ (0000) + \frac{\theta_1^-}{2\omega_h + \omega_l} (|0102\rangle + |1020\rangle) - \frac{\theta_1^+}{2\omega_h + \omega_l} (|0120\rangle + |1002\rangle) \\
+ \frac{\theta_2^- - \theta_2^-}{\omega_h + 2\omega_l} (|0201\rangle + |2010\rangle) + \frac{\theta_2^+ - \theta_2^+}{\omega_h + 2\omega_l} (|0210\rangle + |2001\rangle) \right] + \ldots,
\]
(50)
where the normalization constant is now given by \( N = 1 + 2\left(\frac{(\theta_1^-)^2 + (\theta_1^+)^2}{(2\omega_h + \omega_l)^2} + (\theta_2^- - \theta_2^-)^2 + (\theta_2^+ - \theta_2^+)^2\right) \). The ‘\ldots’ in (50) indicates the additional terms which do not contribute to entanglement [42]. The concurrence is calculated using its definition in (20) and presented in (51):
\[
C_{hl}^{(2)} = \left\{ 2 - \frac{2}{\sqrt{\pi}} \left[ \left[ \frac{2}{(2\omega_h + \omega_l)^4} \right] \right. \right.
+ \left. \left. \left[ \frac{2}{(2\omega_h + \omega_l)^4} + \frac{8(\theta_1^-)^2 + (\theta_1^+)^2}{(2\omega_h + \omega_l)^4} \right] \right] \right\}^{1/2}.
\]
(51)
The expression simplifies based on the assumption that the characteristic couplings over the associated frequency is significantly smaller than one, i.e.
\[ \frac{\theta_1^-}{2\omega_h + \omega_l} \ll 1, \quad \frac{\theta_1^+}{\omega_h + 2\omega_l} \ll 1. \]
(52)
In this regime the concurrence simplifies to:
\[ C_{hl}^{(2)} \approx 2 \sqrt{\frac{(\theta_1^-)^2 + (\theta_1^+)^2}{(2\omega_h + \omega_l)^2} + \frac{(\theta_2^- - \theta_2^-)^2 + (\theta_2^+ - \theta_2^+)^2}{(\omega_h + 2\omega_l)^2}}. \]
(53)
Figure 3. Concurrence as a function of the separation $D$. For $m \sim 10^{-14} \text{kg}$, $d \sim 10^{-4} \text{m}$ and $\omega \sim 10^8 \text{Hz}$. For $M = 10^{-8} \text{kg}$, and for different values of $\omega_h = 10^7, 10^8, 10^9 \text{Hz}$. The solid lines represent the concurrence due to the first-order couplings in (45). The dash-dotted represent the concurrence due to the next order couplings in (54).

Again, the concurrence quantifies the decoherence of the light oscillators due to the heavy oscillators. From (51) we see that the concurrence decreases as the couplings $g_i$ are set to zero, with the concurrence being zero when there is no more coupling between the system and the environment, meaning that there is no loss of coherence in the light subsystem.

In order to get a better idea of the parameter dependence we explore the approximation where the couplings $g_{1,2,3}$ dominate the respective $g_{1,2,3}^{+}$ couplings. In addition, we use the fact that the coupling $g_3$ is suppressed by a factor $1/c^2$ (for typical values of the distances and trap frequencies), leaving us with the couplings $g_{1,2}$. The concurrence then simplifies to:

$$C_{hl}^{(2)}(g) \approx \frac{24G\sqrt{2\hbar}}{(D-d)^4\omega_h\omega_l} \sqrt{\frac{m\omega_l}{(\omega_l + 2\omega_h)^2} + \frac{M\omega_h}{(2\omega_l + \omega_h)^2}}.$$  

We see that the second-order coupling contribution is suppressed by $\sqrt{\hbar}$, and has an inverse quartic dependence on the distance.

In figure 3 we plot the different order contributions to the concurrence given in (45) and (54) for different $\omega_h$ and as a function of $D$. The light oscillator system is taken to be as in [9, 12]. The heavy frequencies are taken over a range $10^7 - 10^9 \text{Hz}$, which are experimentally viable [43]. The heavy mass is taken to be $10^{-8} \text{kg}$, such that $M > m$. We see that the first-order concurrence dominates the next order concurrence with about ten orders of magnitude. As $D$ increases the concurrence goes to zero and both order concurrences become zero eventually. This plot shows clearly that the next order coupling contributions to the decoherence are negligible. The entanglements in figure 3 are quantities of our toy model with harmonic oscillators. Nevertheless, current technologies are not able to measure this amount of entanglement, although entanglement based tomography methods may be used in the future to detect small entanglements [23, 44].

In this section, we have calculated the decoherence due to next order momentum and position couplings of the system and environment. We saw that the dominant contribution comes
from the coupling of the position operators, not the position-momentum operator coupling. In (35) we saw that the momentum-contributions (at first order) also do not increase the light–light concurrence, $C_{ll}$ much, they are suppressed by a factor $1/c^2$. The contribution of the momentum terms in the decoherence scales as $\sqrt{\hbar/c^2}$, which is approximately an order of $1/c^2$ smaller.

Additionally, we saw that these next order couplings entangle states where one of the light oscillators is in the first excited state and one of the heavy oscillators is in the second excited state. This contribution is however dominated by the first-order position couplings, which give rise to entanglement with the first excited states.

### 4. Restrictions on the experimental parameters

In the above sections we found the concurrence between the heavy oscillators and the light oscillators, which we will now compare to the concurrence between the two light test masses. We want the entanglement between the heavy and the light system ($C_{hl}$) to be smaller than the entanglement between the light systems ($C_{ll}$). If we wish to measure the entanglement between the light systems (representing the light quantum masses), then we will want the light systems to share more information between them than with the heavy system (representing the environment). By requiring $C_{ll} > C_{hl}$, we aim to restrict the parameter space of the heavy system such that the entanglement between the light systems dominates over the decoherence.

As we have seen that the momentum terms in $C_{ll}$ and the second-order couplings giving $C_{hl}^{(2)}$ are heavily suppressed, we simply compare the $C_{ll}$ and $C_{hl}^{(1)}$ in the static case. So we require the first term in (35) to be larger than (45) (which uses the approximation that one of the coupling terms can be neglected, which was shown to be the best approximation across the range of $D$ considered). The resulting inequality is:

$$D > \left( \frac{16 \sqrt{2M} \omega_l^2}{\sqrt{M} \omega_l \omega_h (\omega_l + \omega_h)} \right)^{1/3} d + d.$$  

(55)

This inequality is plotted in figure 4, where the light oscillator system parameters are chosen as found in previous works, $m \sim 10^{-14}$ kg, $d \sim 10^{-4}$ m and $\omega_l \sim 10^8$ Hz [12]. In this figure, the area above the curve is the parameter space such that the light-light entanglement dominates the decoherence. The range of $M$ is chosen such that $M \gg m$. We see that as $\omega_h$ increases, the allowed parameter space increases. Furthermore, a heavier apparatus mass requires a higher separation $D$ for the internal entanglement to dominate, as one would expect.

The results derived from figure 4 can be considered the results for the ‘static case’, where the light oscillator system is considered to have no momentum. We can also consider the case in which it does have momentum contributions, still at first order in the couplings. This results in the inequality:

$$D > \left( \frac{1}{d \omega_l^2} + \frac{16 \sqrt{2M}}{\pi \omega_l \omega_h (\omega_l + \omega_h)} \right)^{1/3} d,$$

(56)

which is similar to the one in (55). The second term in the denominator of the first fraction is the contribution from the momentum coupling in the light system. If this term is taken to be zero (so that it reduces to the static case), then we recover the (55). For the parameter space of the light system considered here ($d \sim 10^{-4}$ m, $\omega_l \sim 10^8$ Hz), the momentum contribution is of the order $10^{-12}$, and is thus negligibly small compared to the first term (which is of the order $10^{-4}$). In this range of experimental parameters, the contribution to the entanglement from the
momentum coupling within the light system is so heavily suppressed that it does not change the parameter space much.

The analysis we have done so far has compared the entanglement between the heavy and light systems with the entanglement between the two light systems in the absence of the heavy system. Comparing these two concurrences has provided a way to put restrictions on the parameter space. However, we should also have a look at the entanglement of the two light systems in the presence of the heavy systems. By tracing out the heavy systems we can take the effects of the heavy system into account and then compute the concurrence within the light system.

We consider again the pure density matrix of the full system and we want to find the concurrence within the light subsystem, represented by the density matrix $\rho_l$ in (19). Since $\rho_l$ represents a mixed state, we cannot use the pure state definition of the concurrence in (20).

Instead, we use the definition for the concurrence for mixed states which was presented in equation (18):

$$C = \max (0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),$$

where the $\lambda_i$’s are the ordered eigenvalues (highest to lowest) of the matrix $\sqrt{\rho_l \rho_l^*}$ with $\rho_l = (Y \otimes Y) \rho_l^* (Y \otimes Y)$, where $\rho_l^*$ is the complex conjugate of $\rho_l$, and $Y$ is the Pauli matrix we need to use the mixed state definition. (Previously we made the bipartition light–heavy, where the total density matrix is pure. Therefore we were allowed to use the pure state definition of concurrence along this bipartition. Here we have traced out the heavy system, and have a density matrix $\rho_l$ that is mixed. So for the bipartition light $a$–light $b$). Note that this is different from taking a bipartition where system 1 contains light system $a$ and system 2 contains the heavy systems and light system $b$, which has a total pure state, and which regards the heavy systems as a part of the quantum system as opposed to the environment.

We take the coefficients presented in (38) to find the density matrix $\rho_l$ at first order in the couplings. The light systems are in a superposition of the ground state and first excited state, this two-level system results in a rank-4 light density matrix, allowing the application of (57).
Figure 5. The concurrence between the two light oscillators in the presence of two heavy systems (resulting in decoherence), as a function of the heavy system’s mass, $M$. For light systems with parameters $m \sim 10^{-14}$ kg, $d \sim 10^{-4}$ m and $\omega_l \sim 10^8$ Hz. The heavy systems have $D = 10^{-3}$ m and $\omega_h = 10^8 - 10^9$ Hz.

In figure 5 we plot the concurrence between the two light systems with the effect of the heavy systems taken into account, as a function of the heavy mass $M$. As expected, we see that as the heavy mass increases, the coupling between the heavy and light systems increases and thus the entanglement between the two light systems decreases due to decoherence. As the heavy mass goes to zero, we recover the value for a static light–light system given by $C_{ll}$ in (35). For the distance $D$ to be of the order of millimetres, the system fully decoheres if $M > 10^{-8}$ kg for $\omega_h = 10^9$ Hz and for $M > 10^{-6}$ kg for $\omega_h = 10^9$ Hz. This is in line with the parameter space plotted in figure 4 (at the order of millimetres, the lines have the mass values mentioned above). Additionally from this plot, we could require that the decoherence reduces the entanglement to maximally 80% of the original value, which would require the heavy mass to be approximately of order $10^{-9}$ kg or smaller for $\omega_h = 10^8$ Hz. Knowing the experimental parameters of the heavy system can provide us with information about the expected coherence of the light system.

The magnitude of $C_{ll}$ is a quantity of our toy model with harmonic oscillators. With current technologies we cannot measure this amount of entanglement, but efforts are being made to set up a detailed scheme based on Stern–Gerlach interferometry using a coupling to a spin NV-centre in diamond in order to witness gravitationally induced entangled in the lab (see for example [17, 45–47]).

5. Discussion

The main results of this paper are the concurrence formulae for various configurations of the toy model shown in figure 1. Such a model captures the key features of the QGEM scheme, and the concurrence formulae shows the expected dependencies on the experimental parameters. The concurrence arising from the lowest-order coupling was found to be dominant over the higher-order contributions; a large separation between the test masses and the apparatuses, high trap frequencies, and low masses of the apparatuses will reduce the gravitationally-induced decoherence. The main concurrence formulae, which we will now discuss, are listed in table 1.
Table 1. In the table, this notation is explained in more detail for all different concurrence expressions that were considered in the text. The notation of the concurrence $C_{ij}(l)$, where $b$ indicates the bi-partition (the second column), $o$ denotes the order of the non-relativistic (NR) expansion of $\Delta H_g$ in (33) (the third column), and $l$ indicates the limit that is taken to simplify the expression (the fourth column). The couplings labelled ‘$-$’ (‘$+$’) arise due to interactions between neighbouring (maximally separated) heavy and light oscillators. The coupling $g_{-} << \omega$ are given in (39), and the couplings $g_{1/2}^{l} >> 2\omega_{l} + \omega_{l}$ and $g_{1}^{l} >> \omega_{l} + 2\omega_{l}$ are given in (49) ($g_{1,2}$ come from the coupling between position operators, while $g_{1}$ comes from the coupling between two momentum and one position operator). The distances $d,D$ are illustrated in figure 1. *The limit $g << \omega$ is also taken, except for $C_{(h)b}$.

| Bi-partition                          | Order in NR expansion $\Delta H_g$ | Limit$^*$ | Equation |
|---------------------------------------|------------------------------------|-----------|----------|
| $C_{ll}$                              | light | light (heavy absent)            | zeroth & $1/c^2$ | None     | (35)     |
| $C_{(lb)}^{(1)}$                      | light | light (heavy present)          | zeroth     | None     | (57)     |
| $C_{hl}^{(1)}$                        | heavy | light                           | zeroth     | None     | (42)     |
| $C_{hl}^{(1)}(D \gg d)$              | heavy | light                           | zeroth     | $D \gg d$ | (43)     |
| $C_{hl}^{(1)}(D = 2d)$               | heavy | light                           | zeroth     | $D = 2d$ | (43)     |
| $C_{hl}^{(1)}(g)$                    | heavy | light                           | zeroth     | $g_+ \gg g_-$ | (45) |
| $C_{hl}^{(2)}$                       | heavy | light                           | $1/c^2$    | None     | (53)     |
| $C_{hl}^{(2)}(g)$                    | heavy | light                           | $1/c^2$    | $g^{+}_{1,2} \gg g^{+}_{3,3}, g^{+}_{1,2}$ | (54) |

We explored the limits $D \gg d$ and $D = 2d$ corresponding to different setups (in (43) and (43), respectively), resulting in the same dependence on the experimental parameters, but resulting in a bigger decoherence for the $D = 2d$ setup due to the smaller $D$. We also approximated the concurrence by assuming that the nearest neighbour coupling dominates (in (45)), which turned out to be the best approximation, and we used this to restrict the parameter space for the apparatus.

We explored the first-order momentum contributions to the decoherence (in (51)), which appeared in the next order couplings and are therefore suppressed by a factor $\sqrt{k}$ compared to the momentum contributions to the light-light entanglement, which entered at the lowest order couplings. We found that relative to the static contributions to the entanglement, the non-static contributions are negligible.

By requiring the decoherence to be smaller than the light-light concurrence, we found that the separation $D$ will be of the order of centimetres for the masses up to $M \sim 100$ kg if the trap frequency is larger than $10^5$ Hz. A smaller trap frequency for the same range of masses requires a larger separation. Of course, a larger separation $D$, a smaller mass $M$, and a higher frequency $\omega_b$, decrease the decoherence. This is illustrated in figure 5 in which we plotted the light–light entanglement under the influence of interactions with the environment (i.e. the heavy system).

By modelling the apparatuses as harmonic oscillators, we can make an approximate prediction about the allowed separation between the detectors and the test masses that does not completely destroy the coherence of the test particles. For example, the typical spacing of ion traps is of the order of millimetres, which is smaller than the scale found here, and the decoherence is smaller than the light–light entanglement only for masses $M$ up to $10^{-6}$ kg, for the considered frequencies (as seen from figure 4).
Setting one of the heavy masses to be zero, $M_B = 0$, we can also use our method to find the decoherence due to a single massive oscillator. At no point in the calculations have we assumed that $M > m$, therefore the resulting decoherence rates hold for any mass $M$. However, in the range where $M < m$, we expect the light–light entanglement to be dominant since the gravitational coupling scales with the masses, assuming that the distances are such that $D > d$. In other words, these light sources of decoherence might become relevant at very short distances. Similarly, we have not explored masses of $M \sim m$, where the coupling between the heavy and light systems is of the same strength. These sources are expected to become relevant at $D \sim d$.

In summary, we have investigated the gravitational decoherence induced by the experimental apparatus in the QGEM scheme. In particular, we have considered harmonic oscillators coupled via gravity: two light oscillators coupled to two heavy oscillators, the former (latter) two playing the role of the system (experimental apparatus). While such an analysis is a toy model for the QGEM scheme (a proper analysis would require a precise modelling of the geometry and composition of the experimental apparatus, etc), it nonetheless highlights that the apparatuses have to be carefully considered in the experimental design. We quantify the gravitationally induced entanglement between the two light systems as well as between the light and heavy systems using concurrence, and show that the heavy systems are a source of unavoidable gravitational decoherence. Measuring this decoherence with current technology is extremely challenging. The best candidate would be to enhance the concurrence by using an initial state with larger delocalization (such as a squeezed state or large superposition state).

**Data availability statement**

No new data were created or analysed in this study.

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**References**

[1] Will C 2014 The confrontation between general relativity and experiment *Living Rev. Relativ.* **17** 1–117
[2] Abbott B P *et al* (LIGO Scientific and Virgo) 2016 Observation of gravitational waves from a binary black hole merger *Phys. Rev. Lett.* **116** 061102
[3] Hawking S W and Ellis G F R 2011 *The Large Scale Structure of Space-Time* (Cambridge University Press)
[4] Bjerrum-Bohr N E 2004 Quantum gravity, effective fields and string theory (arXiv:hep-th/0410097 [hep-th])
[5] Thiemann T 2007 *Approaches to Fundamental Physics (Lecture Notes in Physics* vol 721) (Springer) pp 185–263
[6] Amelino-Camelia G 2013 Quantum-spacetime phenomenology *Living Rev. Relativ.* 16 1–137
[7] Dyson F 2013 Is a graviton detectable? *Int. J. Mod. Phys.* A 28 1330041
[8] Dewitt C M and Rickles D 2011 The role of gravitation in physics : report from the 1957 Chapel Hill Conference
[9] Bose S, Mazumdar A, Morley G W, Ulbricht H, Toroš M, Paternostro M, Geraci A, Barker P, Kim M S and Milburn G 2017 Spin entanglement witness for quantum gravity *Phys. Rev. Lett.* 119 240401
[10] Marletto C and Vedral V 2017 Gravitationally-induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity *Phys. Rev. Lett.* 119 240402
[11] Donoghue J F 1994 General relativity as an effective field theory: the leading quantum corrections *Phys. Rev.* D 50 3874–88
[12] Bose S, Mazumdar A, Schut M and Toroš M 2022 Mechanism for the quantum natured gravitons to entangle masses *Phys. Rev.* D 105 106028
[13] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Mixed state entanglement and quantum error correction *Phys. Rev.* A 54 3824–51
[14] Marshman R J, Mazumdar A and Bose S 2020 Locality and entanglement in table-top testing of the quantum nature of linearized gravity *Phys. Rev.* A 101 052110
[15] Christodoulou M, Di Biagio A, Aspelmeyer M, Brukner Časlav, Rovelli C and Howl R 2022 Locally mediated entanglement through gravity from first principles (arXiv:2202.03368)
[16] Danielson D L, Satsichandran G and Wald R M 2022 Gravitationally mediated entangled: Newtonian field versus gravitons *Phys. Rev.* D 105 086001
[17] Biswas D, Bose S, Mazumdar A and Toroš M 2022 Gravitational optomechanics: photon-matter entanglement via graviton exchange (arXiv:2209.09273 [gr-qc])
[18] Marshman R J, Mazumdar A, Folman R and Bose S 2022 Constructing nano-object quantum superpositions with a Stern-Gerlach interferometer *Phys. Rev. Res.* 4 023087
[19] Margalit Y et al 2020 Realization of a complete Stern-Gerlach interferometer: towards a test of quantum gravity (arXiv:2011.10928 [quant-ph])
[20] Zhou R, Marshman R J, Bose S and Mazumdar A 2022 Catapulting towards massive and large spatial quantum superposition (arXiv:2206.04088 [quant-ph])
[21] van de Kamp T W, Marshman R J, Bose S and Mazumdar A 2020 Quantum gravity witness via entanglement of masses: Casimir screening *Phys. Rev.* A 102 062807
[22] Chevalier H, Paige A J and Kim M S 2020 Witnessing the nonclassical nature of gravity in the presence of unknown interactions *Phys. Rev.* A 102 022428
[23] Barker P F, Bose S, Marshman R J and Mazumdar A 2022 Entanglement based tomography to probe new macroscopic forces (arXiv:2203.00038 [hep-ph])
[24] Toroš M, Van De Kamp T W, Marshman R J, Kim M S, Mazumdar A and Bose S 2021 Relative acceleration noise mitigation for nanocrystal matter-wave interferometry: applications to entangling masses via quantum gravity *Phys. Rev. Res.* 3 023178
[25] Schut M, Tilly J, Marshman R J, Bose S and Mazumdar A 2022 Improving resilience of quantum-gravity-induced entanglement of masses to decoherence using three superpositions *Phys. Rev.* A 105 032411
[26] Tilly J, Marshman R J, Mazumdar A and Bose S 2021 Qudits for witnessing quantum-gravity-induced entanglement of masses under decoherence *Phys. Rev.* A 104 052416
[27] Pedernales J S, Morley G W and Plenio M B 2020 *Phys. Rev. Lett.* 125 023602
[28] Toroš M, Mazumdar A and Bose S 2020 Loss of coherence of matter-wave interferometer from fluctuating graviton bath (arXiv:2008.08609 [gr-qc])
[29] Rijavec S, Carlsson M, Bassi A, Vedral V and Marletto C 2021 Decoherence effects in nonclassicality tests of gravity *New J. Phys.* 23 043040
[30] Torrieri G 2022 arXiv:2210.08586 [gr-qc]
[31] Zhou R, Marshman R J, Bose S and Mazumdar A 2022 Mass independent scheme for large spatial quantum superpositions (arXiv:2210.05689 [quant-ph])
[32] Gupta S N 1952 Quantization of Einstein’s gravitational field: linear approximation *Proc. Phys. Soc.* A 65 161
[33] Wootters W K 1998 Entanglement of formation of an arbitrary state of two qubits *Phys. Rev. Lett.* 80 2245–8
[34] Matsumura A 2022 Path-entangling evolution and quantum gravitational interaction Phys. Rev. A 105 042425
[35] Miki D, Matsumura A and Yamamoto K 2021 Entanglement and decoherence of massive particles due to gravity Phys. Rev. D 103 026017
[36] Rungta P, Bužek V, Caves C M, Hillery M and Milburn G J 2001 Universal state inversion and concurrence in arbitrary dimensions Phys. Rev. A 64 042315
[37] Bhaskara V S and Panigrahi P K 2017 Generalized concurrence measure for faithful quantification of multiparticle pure state entanglement using Lagrange’s identity and wedge product Quantum Inf. Process. 16 1–15
[38] Hill S and Wootters W K 1997 Entanglement of a pair of quantum bits Phys. Rev. Lett. 78 5022–5
[39] Grignani G, Harmark T, Orselli M and Placidi A 2020 Fixing the non-relativistic expansion of the 1PM potential J. High Energy Phys. JHE12(2020)142
[40] Iwasaki Y 1971 Quantum theory of gravitation vs. classical theory:—fourth-order potential Prog. Theor. Phys. 46 1587–609
[41] Cristofoli A, Bjerrum-Bohr N E J, Damgaard P H and Vanhove P 2019 Post-Minkowskian Hamiltonians in general relativity Phys. Rev. D 100 084040
[42] Balasubramanian V, McDermott M B and Van Raamsdonk M 2012 Momentum-space entanglement and renormalization in quantum field theory Phys. Rev. D 86 045014
[43] Slezak B R, Lewandowski C W, Hsu J-F and D’Urso B 2018 Cooling the motion of a silica microsphere in a magneto-gravitational trap in ultra-high vacuum New J. Phys. 20 063028
[44] Bose S, Mazumdar A, Schut M and Toros M 2023 Entanglement witness for the weak equivalence principle Entropy 25 448
[45] Schut M, Grinin A, Dana A, Bose S, Geraci A and Mazumdar A 2023 (arXiv:2307.07536 [quant-ph])
[46] Schut M, Geraci A, Bose S and Mazumdar A 2023 Micron-size spatial superpositions for the QGEM-protocol via screening and trapping (arXiv:2307.15743 [quant-ph])
[47] Direkci S, Winkler K, Gut C, Hammerer K, Aspelmeyer M and Chen Y 2023 Macroscopic quantum entanglement between an optomechanical cavity and a continuous field in presence of non-Markovian noise (arXiv:2309.12532 [quant-ph])