COUNTEREXAMPLES RELATED TO THE KOBAYASHI PSEUDODISTANCE

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Abstract. We present some unexpected examples related to the Kobayashi pseudodistance: For an unramified covering, the vanishing of the Kobayashi pseudodistance on the base does not imply the vanishing on the total space. The vanishing of the Kobayashi pseudodistance does not imply the vanishing of the Kobayashi Royden pseudo metric. Given a locally holomorphic trivial fiber bundle, the Kobayashi pseudodistance may vanish identically on the total space even if the fiber is hyperbolic.

1. Introduction

1.1. Kobayashi Pseudo Distance. For every complex space there exists an intrinsically defined pseudo-distance, the “Kobayashi pseudo distance”. See [1]. It plays an important rôle in complex geometry.

From the definition of the Kobayashi pseudodistance via disc chains one may derive the following “covering formula” (see [1], Theorem. 3.2.8)

Proposition 1.1. Let \( \pi : X \to Y \) be an unramified covering of complex manifolds, \( p, q \in Y, \, \bar{p} \in \pi^{-1}(p) \). Then

\[
d_Y(p, q) = \inf_{\bar{q} \in \pi^{-1}(q)} d_X(\bar{p}, \bar{q})
\]

With the aid of this formula one may deduce: Given an unramified covering \( X \to Y \), the total space \( X \) is hyperbolic (in the sense of Kobayashi) if and only if the base space \( Y \) is hyperbolic.

This raises the question whether the same is also true for total degeneration of the Kobayashi pseudo distance, i.e., given an unramified covering \( \pi : X \to Y \), is the condition \( d_X \equiv 0 \) equivalent to \( d_Y \equiv 0 \)?

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A complex space is (Kobayashi) hyperbolic if its Kobayashi pseudo distance is a distance.
The above stated covering formula (1.1) immediately yields the implication $d_X \equiv 0 \implies d_Y \equiv 0$

But for the opposite direction we construct counter examples: There are infinite unramified coverings $\pi : X \to Y$ with $d_Y \equiv 0$, but $d_X \not\equiv 0$. (Theorem 3.1, Theorem 4.1).

However, for finite coverings, there is a positive answer (Proposition 7.1).

1.2. The Infinitesimal Kobayashi-Royden pseudometric. Let $X$ be a complex manifold, $x \in X$, $v \in T_xX$.

Then the Kobayashi Royden pseudometric $F_X$ (see [3]) is defined as

$$F_X(v) = \inf\{\lambda \in \mathbb{R}^+: \exists f : \Delta \to X \text{ holo. } f(0) = x, \lambda f'(0) = v\}$$

Royden proved that $F_X$ is semi-continuous and therefore locally $L^1$ and demonstrated that $d_X(x,y)$ can be realized as

$$d_X(x,y) = \inf_{\gamma} \int_0^1 F_X(\gamma'(t))dt$$

with the infimum taken over all integration paths $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. (See [3], [1]).

From this it follows that $d_X$ vanishes if $F_X$ vanishes.

We give an example of a complex manifold with vanishing $d_X$ but non-vanishing $F_X$ (Theorem 5.1).

1.3. Fiber bundles. We show that there is a locally trivial holomorphic fiber bundle such that the fiber is Kobayashi hyperbolic, but the total space has vanishing Kobayashi pseudo distance (Theorem 6.1).

2. Preparation

Lemma 2.1. Let $H$ be a closed subgroup of $G = SL_2(\mathbb{R})$.

If $0 < \dim H < 3$, then $H$ is solvable.

Proof. Let $H^0$ denote the connected component of $H$ containing the neutral element. This is a connected Lie subgroup. Since connected Lie subgroups of a given Lie group correspond to the Lie subalgebras of the corresponding Lie algebra, $H^0$ is conjugate to one of the following:

$$B = \left\{ \begin{pmatrix} \lambda & c \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{R}^*, c \in \mathbb{R} \right\}, \quad U = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} : c \in \mathbb{R} \right\},$$

$$K = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right\} \text{ or } T^+ = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{R}^+ \right\}$$
Observe that conjugation by any element of $H$ stabilizes $H^0$. Therefore $H$ is contained in

$$N_G(H^0) = \{g \in G : gH^0g^{-1} = H^0\}$$

A case-by-case check verifies that $N_G(H^0)$ is always solvable, if $H^0$ is a non-trivial (i.e. $H \neq G$) connected Lie subgroup of $SL_2(\mathbb{R})$: $H^0$ is conjugate to one of the above listed subgroups $B$, $U$ and $T^+$, and

$$N_G(B) = B, \quad N_G(U) = B,$$

$$N_G(T^+) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{R}^* \right\} \cup \left\{ \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} : \lambda \in \mathbb{R}^* \right\}$$

and all these groups are easily seen to be solvable. □

**Proposition 2.2.** Let $F_2$ be the free group with two generators. Then:

- There is an embedding $j_1$ of $F_2$ into $SL_2(\mathbb{R})$ as a discrete subgroup.
- There is an embedding $j_2$ of $F_2$ into $SL_2(\mathbb{R})$ as a dense subgroup.

**Proof.** Due to [5], [4] $SL_2(\mathbb{R})$ contains both discrete and non-discrete free subgroups with 2 generators. We claim that a non-discrete free subgroup $\Gamma$ of $SL_2(\mathbb{R})$ must be dense. Indeed, let $H$ denote the closure of $\Gamma$ in $SL_2(\mathbb{R})$. The closure of a subgroup is again a subgroup. Since $H$ contains a free subgroup, it cannot be solvable. Due to lemma [2.1] we may conclude that dim $H \in \{0, 3\}$. Since the free subgroup was assumed to be non-discrete, we have dim $H \neq 0$. Hence dim($H$) = 3 and consequently $H = SL_2(\mathbb{R})$, i.e., the free subgroup $\Gamma$ is dense in $SL_2(\mathbb{R})$. □

2.1. Modular group.

**Remark.** A well-known explicit example of a discrete free subgroup with 2 generators of $PSL_2(\mathbb{R})$ is the modular group

$$\Gamma(2) = \{A \in PSL_2(\mathbb{Z}) : A \equiv I \mod 2\}.$$

In fact, the corresponding modular curve $C = H^+ / \Gamma(2)$ may be compactified by adding one point for every element of

$$\mathbb{P}_1(\mathbb{Q}) / \Gamma(2) \simeq \mathbb{P}_1(\mathbb{F}_2).$$

Then $C$ equals $\mathbb{P}_1$ with three points removed ($\#\mathbb{P}_1(\mathbb{F}_2) = 3$). Since $\Gamma(2)$ is torsion-free, we obtain that

$$\Gamma(2) \cong \pi_1(C) \cong \pi_1(\mathbb{P}_1 \setminus \{0, 1, \infty\})$$

is a free group with two generators.
Since $\Gamma(2)$ is free, the natural embedding of $\Gamma(2)$ into $PSL_2(\mathbb{R})$ lifts to an embedding into $SL_2(\mathbb{R})$ by choosing an arbitrary lift for its two generators.

3. Main construction

Theorem 3.1. Let $X = H^+ \times SL_2(\mathbb{C})$ with $H^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$. There exists a properly discontinuous and free action $\mu$ of the free group $F_2$ with 2 generators on $X$ such that $d_Y \equiv 0$ for $Y = X/\mu$.

Remark. Note that $d_X \not\equiv 0$, since $H^+$ is hyperbolic. In fact, it is well-known that

$$d_{H^+}(z, w) = \text{arcosh} \left( 1 + \frac{|z - w|^2}{2\Im(z)\Im(w)} \right)$$

Hence this gives an example of an unramified covering $X \to Y$ with $d_Y \equiv 0$, but $d_X \not\equiv 0$.

Proof. Using Proposition 2.2 we choose embeddings $j_1 : F_2 \to SL_2(\mathbb{R})$ and $j_2 : F_2 \to SL_2(\mathbb{R}) \subset SL_2(\mathbb{C})$ such that $j_1$ has dense image in $SL_2(\mathbb{R})$ and $j_2(F_2)$ is discrete. The action $\mu$ of $F_2$ on $X$ is now defined by

$$\mu(\alpha) : (z, A) \mapsto (\phi(j_1(\alpha))(z), j_2(\alpha) \cdot A)$$

where $\phi$ is the usual action by Möbius transformations, i.e.,

$$\phi \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) (z) = \frac{az + b}{cz + d},$$

and $j_2(\alpha) \cdot A$ denotes the product with respect to the group law on $SL_2(\mathbb{C})$. Discreteness of $j_2(F_2)$ implies that this action $\mu$ is free and properly discontinuous. Thus we obtain an unramified covering $\pi : X \to X/\mu \overset{\text{def}}{=} Y$.

Let $(x, A), (y, B) \in H^+ \times SL_2(\mathbb{C}) = X$ be arbitrary. Since $j_1(F_2)$ is dense and since $SL_2(\mathbb{R})$ acts transitively on $H^+$, there is a sequence $\alpha_n \in F_2$ with

$$\lim_{n \to \infty} (\phi(j_1(\alpha_n))(y)) = x$$

Now

$$d_X ((x, A), (y, B)) = d_{H^+}(x, y),$$

because $SL_2(\mathbb{C})$ is a complex Lie group and therefore has vanishing Kobayashi pseudodistance ([1], Ex. 3.1.22). By the covering formula
we have
\[ d_Y(\pi(x, A), \pi(y, B)) = \inf_{\alpha \in F_2} d_X((x, A), \mu(\alpha)(y, B)) = \inf_{\alpha \in F_2} d_{H^+}(x, \phi(j_1(\alpha))(y)) \]

But
\[ \inf_{\alpha \in F_2} d_{H^+}(x, \phi(j_1(\alpha))(y)) = 0 \]
because
\[ \lim_{n \to \infty} (\phi(j_1(\alpha_n))(y)) = x. \]
Therefore
\[ d_Y(\pi(x, A), \pi(y, B)) = 0 \]
for every \((x, A)\) and \((y, B)\) in \(X\). Thus \(d_Y \equiv 0\). \(\square\)

4. Another example, based on the construction of Margulis

Semisimple Lie groups like \(SL_2(\mathbb{C})\) always carry non-trivial topology. Thus one might ask, whether there is an example of an unramified cover as in Theorem 3.1 above with a contractible total space.

This is indeed the case. We recall that Margulis constructed a properly discontinuous free action of the free group with two generators \(F_2\) on \(\mathbb{R}^3\) by affin-linear transformations (2).

If in the preceding construction we replace \(SL_2(\mathbb{C})\) by \(\mathbb{C}^3\) and the action of \(F_2\) via \(j_2\) and right multiplication by the complexified action of Margulis, we obtain an unramified covering \(X' \to Y'\) with \(X' = H^+ \times \mathbb{C}^3\) and \(d_{Y'} \equiv 0\). In this way we establish the below result.

**Theorem 4.1.** There exists a complex manifold \(Y'\) with identically vanishing Kobayashi pseudo distance \(d_{Y'}\), whose universal covering is biholomorphic to
\[ H^+ \times \mathbb{C}^3 = \{ z \in \mathbb{C}^4 : \Im(z_1) > 0 \}. \]

5. Infinitesimal Pseudodistance

Let \(\pi : X \to Y\) be an unramified covering, \(x \in X, y = \pi(x)\). Every holomorphic map \(f\) from the unit disc \(\Delta\) to \(X\) mapping 0 to \(x\) yields a map \(g\) from the unit disc to \(Y\) by concatenation with \(\pi\) (i.e., \(g = \pi \circ f\)). Conversely, since \(\Delta\) is simply-connected, every holomorphic map from \(Y\) lifts to a map to \(X\).
Therefore, the definition of the Kobayashi Royden pseudodistance implies that $\pi^*F_Y = F_X$, i.e.,

$$F_Y ((D\pi)(v)) = F_X(v) \quad \forall v \in T_x X$$

Thus, given an unramified covering $\pi : X \to Y$, we have $F_X \equiv 0$ if and only if $F_Y \equiv 0$.

In the preceding section we constructed unramified coverings $\pi : X \to Y$ where the Kobayashi pseudo distance of $Y$ is vanishing identically, but $d_X \not\equiv 0$.

Since

$$d_X(x, y) = \inf_{\gamma} \int_0^1 F_X(\gamma'(t))dt$$

where the infimum is taken over all paths $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$ (see [3]), the non-vanishing of $d_X$ implies that $F_X$ is not vanishing. The non-vanishing of $F_X$ in turn implies that $F_Y$ is not vanishing.

Thus every unramified covering $\pi : X \to Y$ with $d_Y \equiv 0$ and $d_X \not\equiv 0$ yields an example of a complex manifold $Y$ with $d_Y \equiv 0$ but $F_Y \not\equiv 0$.

In combination with Theorem 3.1 or Theorem 4.1 this yields:

**Theorem 5.1.** There exists a complex manifold $Y$ with identically vanishing Kobayashi pseudo distance $d_Y$ whose infinitesimal Kobayashi-Royden pseudo metric is not vanishing ($F_Y \not\equiv 0$).

### 6. Locally trivial fiber bundles

**Theorem 6.1.** There exists a locally trivial holomorphic fiber bundle $\rho : Y \to B$ with fiber isomorphic to the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ such that $d_Y \equiv 0$.

**Proof.** The starting point is the construction of Theorem 3.1. Recall that $j_2$ denoted an embedding of the free group $F_2$ into $SL_2(\mathbb{C})$ as a discrete subgroup. Hence the quotient $B = j_2(F_2)\backslash SL_2(\mathbb{C})$ is a complex manifold. The natural projection of $X = \mathbb{H}^+ \times SL_2(\mathbb{C})$ onto its second factor, $SL_2(\mathbb{C})$, induces a projection from $X/\mu = Y$ onto $B$. By construction this is a locally trivial holomorphic fiber bundle with $\mathbb{H}^+ \simeq \Delta$ as fiber. This yields the assertion of the proposition, since $d_Y \equiv 0$ by Theorem 3.1. $\square$

### 7. Finite coverings

**Proposition 7.1.** Let $\pi : X \to Y$ be a finite unramified covering

Then the Kobayashi pseudodistance $d_X$ vanishes if and only if $d_Y$ vanishes.
Proof. Assume $d_Y \equiv 0$.
Since the covering is finite, the infimum in $[1,1]$ is actually a minimum. It follows that for every $\tilde{p} \in X$ and $q \in Y$ there exists an element $\tilde{q} \in \pi^{-1}(q)$ with $d_X(\tilde{p}, \tilde{q}) = 0$. As a consequence, for every $\tilde{q} \in Y$ we have $\pi(E_{\tilde{p}}) = Y$ if we define
$$E_{\tilde{p}} \overset{\text{def}}{=} \{ x \in X : d_X(x, \tilde{p}) = 0 \}.$$These sets $E_{\tilde{q}}$ are the equivalence classes for the natural equivalence relation defined by $x \sim y \iff d_X(x, y) = 0$. Hence for $x, y \in X$ either $E_x = E_y$ or $E_x \cap E_y$ is empty. It follows that the set $E_x$ define a partition of $X$ into finitely many disjoint closed subsets. But $X$ is connected, therefore there exists no non-trivial finite partition of $X$ into disjoint closed subsets. Thus $E_x = X$ for any $x$, which implies $d_X \equiv 0$. □

References

[1] Shoshichi Kobayashi, Hyperbolic complex spaces, volume 318 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] (Springer-Verlag, Berlin, 1998).
[2] G. A. Margulis, ‘Free completely discontinuous groups of affine transformations’, Dokl. Akad. Nauk SSSR (4) 272 (1983), 785–788.
[3] H. L. Royden, ‘Remarks on the Kobayashi metric’, in: Several complex variables, II (Proc. Internat. Conf., Univ. Maryland, College Park, Md., 1970) (1971) pp. 125–137. Lecture Notes in Math., Vol. 185.
[4] Jörg Winkelmann, ‘How frequent are discrete cyclic subgroups of semisimple Lie groups?’, Doc. Math. 6 (2001), 31–37.
[5] , ‘Generic subgroups of Lie groups’, Topology (1) 41 (2002), 163–181.

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