Quantization of Nambu Brackets from Operator Formalism in Classical Mechanics

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This paper proposes a novel approach to quantizing Nambu brackets in classical mechanics using operator formalism. The approach employs the “Planck derivative” to represent Nambu brackets, from which we derive a commutation relation for their quantization. Notably, this commutation relation aligns with that emerging from the T-duality of closed strings in a twisted torus with a B-field, thereby hinting at a potential connection with Double Field Theory.

1 Introduction

Clarifying the relationship between classical and quantum theory is an activity in which the understanding of the macroscopic world from the perspective of the microscopic world and the understanding of the microscopic world from the perspective of the macroscopic world are mutually reconstructed. It is an actual problem for the foundation of modern physics, a straightforward example of which is the quantization of Poisson brackets. The quantization of more complex structures, such as Nambu brackets, is still controversial and continues to be an exciting and challenging problem that ties together more profound questions about perspectives on the world in theoretical physics.

The Nambu bracket, introduced by Yoichiro Nambu in 1973[1], is an extension of the Poisson bracket to higher dimensions and later applied in various fields, including integrable systems, fluid dynamics, nonequilibrium thermodynamics, and M-theory, and is particularly important in describing membranes[2, 3, 4, 5].

We can point out three main features inherent to the Nambu bracket:

- It is based on multiple Hamiltonians representing conserved quantities.
- It follows the Leibniz rule of the time derivative.
- It satisfies a generalized Jacobi identity called the fundamental identity (F.I.).
Quantization of Nambu brackets, however, is difficult, and most previous attempts have violated at least one of its inherent properties of Nambu brackets [6, 7, 8, 9, 10, 11, 12], see [13] for a review.

Operator formalism in classical mechanics describes classical mechanics in Hilbert spaces studied by Koopman[14] and von Neumann[15] in 1931, a formalism that is now much neglected in physics research. Recently, an attempt has been made to describe quantum mechanics in such a formalism[16, 17]. Such attempts have advanced our understanding of the quantization process by viewing quantization not as a conversion from c-numbers to q-numbers but rather as a two-step conversion process to operator formalism and introducing quantum corrections.

This research explores a new method of quantizing Nambu brackets using operator formalism in classical mechanics.

Central to our method is introducing of the Planck derivative (difference) $D$, which represents the difference between the quantum and classical operators when divided by the Planck constant. From this representation of Nambu brackets by $D$, we derive the commutation relation that must be satisfied by the quantization of Nambu brackets. Coincidentally, the same commutation relation appears in the discussion of noncommutativity obtained by the T dual of a closed string in a twisted torus with a B field, suggesting a connection with Double Field Theory.

The paper is organized as follows: Section 2 gives an overview of Nambu brackets and their role in M-theory. Section 3 introduces the classical operator formalism and explains its importance in the context of our research. Section 4 introduces the Planck derivative to link the operator formalism of classical mechanics with quantum mechanics. Section 5 describes our proposed approach to quantizing of Nambu brackets using classical operator formalism and discusses the resulting commutation relation. Section 6 applies the quantization of Nambu brackets to several examples. Finally, in Section 7, we combine our conclusions with an analysis of the study’s results and point out possible directions for future research.

Appendix A also provides details of the commutation relation.

This research is not only a contribution to the quantization of the M2-brane, a pivotal component of M-theory[18, 19, 20, 21], but also has the potential to stimulate further consideration of the nature of quantum theory from a physical and fundamental point of view: namely, what quantization is.

## 2 Nambu bracket and M-theory

In Hamiltonian mechanics, the state space is spanned by two variables, $x$ and $p$. Nambu mechanics is an extension of this state space to $n$ variables, denoted as $x^i$. In this context, we consider the case of $n = 3$.

The time evolution of mechanical variables is described using multiple “Hamiltonian” as

$$\dot{x}^i = \{ H_1, H_2, x^i \}, \quad \text{(1)}$$

where $\{ A, B, C \}$ denotes the Nambu bracket, defined as:

$$\{ A, B, C \} \equiv \epsilon^{ijk} \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial x^j} \frac{\partial C}{\partial x^k}. \quad \text{(2)}$$
Nambu mechanics possesses three main features. First, the relation
\[ \{A_{\sigma(1)}, A_{\sigma(2)}, A_{\sigma(3)}\} = (-1)^{\sigma} \{A_1, A_2, A_3\}, \] (3)
is based on multiple Hamiltonians that represent conserved quantities of the system. These Hamiltonians are employed to describe the system’s dynamics in terms of its state variables.

Second,
\[ \{A, B, C, D\} = A\{B, C, D\} + \{A, C, D\}B, \] (4)
which signifies that Nambu mechanics adheres to the Leibniz rule for time derivatives. This rule determines how the system’s variables change over time. This rule ensures that the time evolution of the system is consistent with the underlying physical laws.

Finally,
\[ \{|A, B, C, D, E\} = \{|A, D, E\}, B, C\} + \{|A, B, \{C, D, E\}\}, C\} + \{|A, B, \{C, D, E\}\}, \] (5)
denoted as the fundamental identity (F.I.), signifies that Nambu mechanics satisfies a generalized Jacobi identity. This identity guarantees the consistency and well-defined nature of the Nambu bracket operation, which is crucial for describing the behavior of complex physical systems.

Membranes serve as fundamental objects in the description of M-theory. The Nambu bracket is closely connected to membrane theory in its Lagrangian formalism, as elaborated below.

In the mechanics of a point particle, the action can be expressed as follows:
\[ S = \int_{C_1} p dx - H dt = \int_{C_2} dp \wedge dx - dH \wedge dt \]
\[ = \int_{C_2} \left( \epsilon_{ij}\epsilon^{ab}\partial_a x^i \partial_b x^j - \frac{\partial H_1}{\partial x^i} \partial_\sigma x^i \right) d\sigma dt \]
\[ = \int_{C_3} \left( \epsilon_{ij}\{x^i, x^j\} - \{H_1, t\} \right) d\sigma dt, \] (6)
where \( C_1 = \partial C_2, \) \( x^1 = x, \) \( x^2 = p, \) and the subscripts \( i \) are \( i = x, p \) and \( a = t, \sigma. \) The symbol \( \epsilon_{ij} \) and \( \epsilon^{ab} \) represent the Levi-Civita symbol.

The dynamics of a point particle can be represented in two dimensions, where the additional parameter can be interpreted as a parameter of change arising from virtual work. Consequently, in this context, the dynamics of a point particle possess an inherent description akin to closed string theory, characterized by two parameters.

In Nambu dynamics, the action is expressed as:
\[ S = \int_{C_3} \left( \epsilon_{ijk}\epsilon^{abc}\partial_a x^i \partial_b x^j \partial_c x^k - \frac{\partial H_1}{\partial x^i} \frac{\partial H_2}{\partial x^j} \partial_x x^i x_j \partial_x x^k \right) d^2\sigma dt. \] (7)

The original action originally included two string-like parameters, but it can be rewritten as a three-parameter membrane-like action by introducing an additional virtual displacement parameter.

2In particular, we can determine the form of the Hamiltonian so that it satisfies the following,
From the above, it can be observed that Nambu dynamics is associated with a \((D - 1)\)-dimensional membrane where \(D\) represents the number of dynamical variables.

### 3 Operator formalism in classical mechanics

The operator formalism of classical mechanics was initially developed in the 1930s by Koopman\[14\] and von Neumann\[15\]. However, it has undergone revisions in recent years and offers the advantage of dealing with hybrid systems that combine classical and quantum theory\[16, 17, 22, 29, 30, 31\]. A key characteristic of the operator formalism of classical mechanics is the differentiation between quantum and classical operators.

In quantum mechanics, the relationship between position and momentum in Poisson brackets

\[
\{x^i, x^j\} = \epsilon^{ij}
\]

is quantized by replacing it with the corresponding equation

\[
[i\hbar \hat{x}^{(q)i}, \hat{x}^{(q)j}] = \hbar \epsilon^{ij}.
\]

Here, the variable is denoted with a superscript \((q)\) to emphasize that it represents a quantum physical quantity.

The operator formalism of classical mechanics is characterized by replacing position and momentum with operators while preserving their commutativity

\[
[i\hbar \hat{x}^{(c)i}, \hat{x}^{(c)j}] = 0,
\]

where \(e\) is the Lagrange multiplier and if \(\epsilon^{ij}\) depends on \((t, \sigma^1, \sigma^2)\), the action is

\[
S_s = \int_{C_2} \left( \epsilon_{ij} \{x^i, x^j\} - e \epsilon_{ij} \epsilon^{ij} - e \right) d\sigma dt.
\]

We integrate out \(\epsilon_{ij}\) and obtain

\[
S_s = \int_{C_2} \left( \frac{1}{e} \{x^i, x^j\} \{x_i, x_j\} - e \right) d\sigma dt.
\]

In Nambu dynamics, we rewrite as

\[
S = \left( \int_{C_3} \epsilon_{ijk} \{x^i, x^j, x^k\} - \{H_1, H_2, t\} \right) d^2\sigma dt.
\]

If we set the form of the Hamiltonian so that it satisfies the following,

\[
\{H_1, H_2, t\} \equiv e \left( \epsilon_{ijk} \epsilon^{ijk} + 1 \right)
\]

where \(e\) is the Lagrange multiplier and if \(\epsilon_{ijk}\) depends on \((t, \sigma^1, \sigma^2)\), the action is

\[
S_m = \int_{C_3} \left( \epsilon_{ijk} \{x^i, x^j, x^k\} - e \epsilon_{ijk} \epsilon^{ijk} - e \right) d^3\sigma.
\]

We integrate out \(\epsilon_{ijk}\) and obtain

\[
S_m = \int_{C_3} \left( \frac{1}{e} \{x^i, x^j, x^k\} \{x_i, x_j, x_k\} - e \right) d^3\sigma,
\]

which is the relativistic membrane action.
and introducing their corresponding canonical counterparts, denoted as $\hat{\xi}_i$:

$$[\hat{x}^{(c)}_i, \hat{\xi}_j] = i\delta^i_j, \quad [\hat{\xi}_i, \hat{\xi}_j] = 0.$$  \hspace{1cm} (18)

Additionally, the Liouvillian operator, denoted as $\hat{L}$, is introduced in correspondence with the Hamiltonian:

$$\hat{L} = e^{i\frac{\partial H(\hat{x})}{\partial \hat{x}_i}} \hat{x}_i. \hspace{1cm} (19)$$

Since position and momentum are commute and can be simultaneously diagonalized by eigenstates, the wave function can be expressed as a function of position and momentum:

$$|\psi\rangle = \int dx dp |x,p\rangle \langle x,p|\psi\rangle. \hspace{1cm} (20)$$

The Schrödinger equation can be written as:

$$i\frac{\partial}{\partial t} |\psi\rangle = \hat{L} |\psi\rangle. \hspace{1cm} (21)$$

As a specific example, for a free particle\cite{22}, the Liouvillian operator $\hat{L}$ can be represented as

$$\hat{L} = \frac{\hat{p}}{m} \hat{\xi}_x. \hspace{1cm} (22)$$

The corresponding Schrödinger equation becomes:

$$i\frac{\partial}{\partial t} |\psi\rangle = \int d\xi_x dp' \frac{p'}{m} |\xi_x, p'\rangle \langle \xi_x, p' |\psi\rangle. \hspace{1cm} (23)$$

By applying $\langle \xi_x, p |$ from left-hand side, we obtain,

$$i\frac{\partial}{\partial t} \langle \xi_x, p |\psi\rangle = \frac{p}{m} \xi_x \langle \xi_x, p |\psi\rangle. \hspace{1cm} (24)$$

By introducing a proportionality factor $A$, the solution can be expressed as:

$$\langle \xi_x, p |\psi\rangle = Ae^{i\frac{p}{m} \xi_x t}. \hspace{1cm} (25)$$

Performing the Fourier transform with respect to $\xi_x$, we obtain:

$$\langle x, p |\psi\rangle = A\delta(x - \frac{p}{m}). \hspace{1cm} (26)$$

This solution reproduces the linear trajectory of a free particle in classical mechanics.

4 Planck derivative $D$

In the previous section, we discussed the operator formalism of classical mechanics, where we observed the commutativity of position and momentum operators. However, the Poisson bracket-like structures that are essential in canonical quantization are not explicitly present in this framework. To establish a connection between the operator formalism of classical mechanics and quantum mechanics,
it is important to consider structures analogous to Poisson brackets within the classical operator formalism.

To establish the relationship between the operator formalism of classical mechanics and quantum mechanics, we introduce the difference between classical and quantum operators denoted as $D$

\[
\hat{x}(q)_i = \hat{x}(c)_i + \hbar D(\hat{x}(c)_i),
\]  

(27)

We refer to $D$ as the Planck derivative,

\[
D = \frac{\hat{x}(q)_i - \hat{x}(c)_i}{\hbar}.
\]  

(28)

Next, we can express the Poisson bracket within the operator formalism of classical mechanics using $D$ as follows:

\[
[\hat{x}(c)_i, -2iD(\hat{x}(c)_j)] = \epsilon_{ij}.
\]  

(29)

Note that the right-hand side of the equation represents the usual antisymmetric epsilon symbol, indicating the Poisson bracket relationship between the position operator $\hat{x}(c)_i$ and the Planck derivative $-2iD(\hat{x}(c)_j)$.

The results of the previous section indicate that $D$ can be expressed as:

\[
D(\hat{x}(c)_j) = -\frac{1}{2}\epsilon_{jk}\hat{\xi}_k.
\]  

(30)

Therefore, the Schrödinger equation of quantum mechanics can be alternatively expressed using classical operators\[16\]:

\[
i\hbar \frac{\partial}{\partial t} |\psi(q)\rangle = H \left( \hat{x}(c)_i - \frac{1}{2}\hbar \epsilon^{ij}\hat{\xi}_j \right) |\psi(q)\rangle.
\]  

(31)

In this equation, $H$ represents the Hamiltonian, and $|\psi(q)\rangle$ denotes the quantum state. The expression inside the parentheses indicates the modified classical operators incorporating the Planck derivative term.

Note that there exists a well-known approach to the representation of operators in phase space called the Bopp operator\[23, 24, 25, 26\]. Also, the Schrödinger equation on such a phase space has already been examined and discussed in previous work\[27, 28\].

### 4.1 Dual operator of $\hat{x}(q)_i$

In this section, we will delve into the concept of the dual operator associated with $\hat{x}(q)_i$. The dual operator, denoted as $\hat{\tilde{x}}(q)_i$, is obtained by interchanging the classical and the Planck derivative operator $D(\hat{x}(c)_i)$:

\[
\hat{\tilde{x}}(q)_i = \hat{x}(c)_i - hD(\hat{x}(c)_i),
\]  

(32)

3We can also check that the commutation relation $x^i$ satisfies

\[
[\hat{x}(q)_i, \hat{\tilde{x}}(q)_j] = \frac{1}{2} \{\hat{x}(c)_i, -2iD(\hat{x}(c)_j)\} + \frac{1}{2} \{[\hat{x}(c)_i, -2iD(\hat{x}(c)_j)], \hat{x}(c)_j\}
\]

\[
= \frac{1}{2}ihe^{ij} + \frac{1}{2}ihe^{ij} = ihe^{ij}.
\]
\[ D(\hat{x}^{(c)i}) = \frac{\hat{x}^{(c)i} - \hat{x}^{(q)i}}{\hbar}. \]  

(33)

Notably, these operators independently satisfy the Heisenberg algebra, reflecting the fundamental commutation relations of quantum mechanics:

\[
[\hat{x}^{(q)i}, \hat{x}^{(q)j}] = -i\hbar \delta^{ij}
\]

(34)

but are commute with \( \hat{x}^{(q)i} \):

\[
[\hat{x}^{(q)i}, \hat{x}^{(q)j}] = 0.
\]

(35)

See Appendix A for detailed calculations.

Consequently, the classical operator \( \hat{x}^{(c)i} \) can be regard as the center of mass between the quantum operator \( \hat{x}^{(q)i} \) and the dual operator \( \hat{x}^{(q)i} \):

\[
\hat{x}^{(c)i} = \frac{\hat{x}^{(q)i} + \hat{x}^{(q)i}}{2}.
\]

(36)

Furthermore, the Planck derivative \( \hbar D(\hat{x}^{(c)i}) \) represents their relative coordinate:

\[
\hbar D(\hat{x}^{(c)i}) = \frac{\hat{x}^{(q)i} - \hat{x}^{(q)i}}{2}.
\]

(37)

In summary, in classical mechanics, states are represented by simultaneous eigenstates of position and momentum,

\[
|x^{(c)}, p^{(c)}\rangle
\]

(38)

or

\[
|x^{(c)}, \xi_p\rangle, \text{ or } |\xi_x, p^{(c)}\rangle \text{ or } |\xi_x, \xi_p\rangle.
\]

(39)

On the other hand, in the equivalent quantum mechanics, states are described by simultaneous eigenstates of position \( x \) and its dual coordinate \( \tilde{x} \):

\[
|x^{(q)}, \tilde{x}^{(q)}\rangle
\]

(40)

or

\[
|p^{(q)}, \tilde{p}^{(q)}\rangle, \text{ or } |x^{(q)}, \tilde{p}^{(q)}\rangle \text{ or } |p^{(q)}, \tilde{x}^{(q)}\rangle.
\]

(41)

As a result, the wavefunction in quantum mechanics includes both coordinates and dual coordinates:

\[
|\hat{x}^{(c)i} - \hbar D(\hat{x}^{(c)i}), \hat{x}^{(c)j} - \hbar D(\hat{x}^{(c)j})\rangle = -\frac{1}{2}i[\hat{x}^{(c)i}, -2\hbar D(\hat{x}^{(c)j})] + \frac{1}{2}i[-2\hbar D(\hat{x}^{(c)i}), \hat{x}^{(c)j}] \\
= -\frac{1}{2}i\hbar \epsilon^{ij} + \frac{1}{2}i\hbar \epsilon^{ij} = -i\hbar \epsilon^{ij}
\]

(5)

\[
|\hat{x}^{(c)i} - \hbar D(\hat{x}^{(c)i}), \hat{x}^{(c)j} + \hbar D(\hat{x}^{(c)j})\rangle = \frac{1}{2}i[\hat{x}^{(c)i}, 2\hbar D(\hat{x}^{(c)j})] + \frac{1}{2}i[-2\hbar D(\hat{x}^{(c)i}), \hat{x}^{(c)j}] \\
= \frac{1}{2}i\hbar \epsilon^{ij} + \frac{1}{2}i\hbar \epsilon^{ij} = 0
\]
\[ \psi(q)(x(q), \bar{x}(q)). \] (42)

By recombining these coordinates, similar to the Wigner function, we obtain:

\[ \psi(q) \left( \frac{x(q) + \bar{x}(q)}{2}, x(q) - \frac{\bar{x}(q)}{2} \right) = \psi(q)(x^{(c)}, 2hD(x)) = \psi(q)(x^{(c)}, h\xi), \] (43)

which can be viewed as the wavefunction of classical mechanics. Moreover, by performing a Fourier transform on the second term, we obtain:

\[ \psi(q)(x^{(c)}, p^{(c)}), \] (44)

representing the wavefunction of classical mechanics.

In conclusion, by incorporating dual coordinates, quantum mechanics becomes entirely equivalent to classical mechanics.

5 Quantization of Nambu brackets

This section presents the main result of the paper.

In the operator formalism of classical mechanics, we introduced the operator \( D \) to preserve the Poisson brackets and establish a relationship between quantum mechanical operators and classical mechanical operators. In a similar manner, we introduce \( D \) to preserve the Nambu bracket in Nambu mechanics:

\[ [\hat{x}(c)^i, \hat{x}(c)^j, -2D(\hat{x}(c)^k)] = \epsilon^{ijk}. \] (45)

To satisfy this relationship, we require the following condition:

\[ [\hat{x}(c)^i, D(\hat{x}(c)^k)] = \frac{1}{2}\epsilon^{ijk} \xi_i. \] (46)

Similar to classical mechanics, we define the quantum operator as:

\[ \hat{x}(q)^i = \hat{x}(c)^i + \hbar D \left( \hat{x}(c)^i \right), \] (47)

\[ D = \frac{\hat{x}(q)^i - \hat{x}(c)^i}{\hbar}. \] (48)

From the given definitions, we can derive the following commutation relations for quantum operators:

\[ [\hat{x}(q)^i, \hat{x}(q)^j] = i\hbar \epsilon^{ijk} \xi_k, \] (49)

\[ [\hat{x}(q)^i, [\hat{x}(q)^j, \hat{x}(q)^k]] = i\hbar \epsilon^{ijk}. \] (50)

Similar to ordinary quantum mechanics, the time evolution of operators in the Heisenberg representation can be described as:

\[ i\hbar \frac{\partial}{\partial t} \hat{O}(q) = \frac{1}{\hbar} [\hat{O}(q), [H_1(\hat{x}(c)^i, H_2(\hat{x}(c)^j)]]], \] (51)
where \( \hat{O}^{(q)} \) represents a quantum operator and \( H_1(\hat{x}^{(c)} + hD(\hat{x}^{(c)i})) \) and \( H_2(\hat{x}^{(c)} + hD(\hat{x}^{(c)i})) \) are the corresponding classical Hamiltonian operators.

In the Schrödinger representation, the time evolution of a quantum state \(|\psi\rangle\) can be expressed as:

\[
i\frac{\partial}{\partial t}|\psi\rangle = \frac{1}{\hbar}[H_1(\hat{x}^{(c)} + hD(\hat{x}^{(c)i}))), H_2(\hat{x}^{(c)} + hD(\hat{x}^{(c)i}))]|\psi\rangle
\]

where \(|\psi\rangle\) represents the quantum state.

These equations describe the time evolution of operators and states within the framework of the operator formalism of classical Nambu mechanics extended to include quantum operators and the Planck derivative.

By substituting the following expression for the Hamiltonians,

\[
H_i((\hat{x}^{(c)} + hD(\hat{x}^{(c)i})) = H_i(\hat{x}^{(c)}) + \frac{\partial H_i(\hat{x}^{(c)})}{\partial \hat{x}_i} hD(\hat{x}^{(c)}) + \ldots,
\]

we obtain:

\[
\frac{1}{\hbar}[H_1, H_2] = \frac{1}{2}i\epsilon_{ijk} \frac{\partial H_1(\hat{x}^{(c)})}{\partial \hat{x}_i} \frac{\partial H_2(\hat{x}^{(c)})}{\partial \hat{x}_j} \hat{\xi}_k - \frac{1}{2}\hbar \Delta_Q,
\]

where \( \Delta_Q \) is defined as:

\[
\Delta_Q \equiv \left[ \frac{\partial H_1(\hat{x}^{(c)})}{\partial \hat{x}_i} D(\hat{x}^{(c)}) - \frac{\partial H_2(\hat{x}^{(c)})}{\partial \hat{x}_j} D(\hat{x}^{(c)}) \right] + \ldots
\]

Here, \( \Delta_Q \) represents a quantum effect arising from the commutation of the Planck derivative operator \( D(\hat{x}^{(c)i}) \) with the derivatives of the classical Hamiltonians. The term \( \frac{1}{2}\hbar \Delta_Q \) captures the quantum corrections to the classical bracket structure and contributes to the overall evolution of the system.

Eventually, the following Schrödinger equation is obtained:

\[
i\frac{\partial}{\partial t}|\psi\rangle = \frac{[H_1, H_2]}{\hbar} |\psi\rangle = \frac{1}{2}i\epsilon_{ijk} \frac{\partial H_1(\hat{x}^{(c)})}{\partial \hat{x}_i} \frac{\partial H_2(\hat{x}^{(c)})}{\partial \hat{x}_j} \hat{\xi}_k |\psi\rangle - \frac{1}{2}\hbar \Delta_Q |\psi\rangle + \ldots
\]

If we consider the position representation, we obtain:

\[
\frac{\partial}{\partial t}\psi(x^{(c)}, t) = \{H_1, H_2, \psi(x^{(c)}, t)\} - \frac{1}{2} \int d^3x^{(c)} (x^{(c)}) |\Delta_Q| \psi(x^{(c)}, t) + \ldots
\]

From this equation, the time evolution governed by Nambu brackets is reproduced in the classical limit.

In Nambu quantum mechanics, the dual coordinate is introduced similarly:

\[
\hat{x}^{(c)i} = \hat{x}^{(c)i} - hD(\hat{x}^{(c)i}).
\]

The commutation relations for the dual coordinate are given by:

\[
[\hat{x}^{(q)i}, \hat{x}^{(q)j}] = -i\hbar \epsilon_{ijk} \hat{\xi}_k,
\]

\[
[\hat{x}^{(q)i}, [\hat{x}^{(q)j}, \hat{x}^{(q)k}]] = -i\hbar \epsilon_{ijk}.
\]
It should be noted that the dual coordinates also commute with the original coordinates,
\[ [\hat{x}^{(q)i}, \hat{\tilde{x}}^{(q)j}] = 0. \tag{61} \]

It is important to highlight the difference between Nambu quantum mechanics and ordinary quantum mechanics. In Nambu quantum mechanics, the operator \( D \) is not explicitly defined in terms of the wave number operator. To express \( D \) explicitly, the dual operator must be introduced. In this case, it can be written as:
\[ \hbar D(\hat{x}^{(c)i}) = \frac{\hat{x}^{(q)i} - \hat{\tilde{x}}^{(q)i}}{2}. \tag{62} \]

The quantization of Nambu brackets was discussed in a two-step approach: representing Nambu brackets in the operator formalism of classical mechanics and introducing a quantum operator via the previously introduced \( D \).

The resulting commutation relation are:
\[
\begin{align*}
[\hat{x}^{(q)i}, [\hat{x}^{(q)j}, \hat{\tilde{x}}^{(q)k}]] &= i\hbar e^{ijk}, \tag{63} \\
[\hat{\tilde{x}}^{(q)i}, [\hat{x}^{(q)j}, \hat{\tilde{x}}^{(q)k}]] &= -i\hbar e^{ijk}, \tag{64} \\
[\hat{x}^{(q)i}, \hat{x}^{(q)j}] &= i\hbar e^{ijk} \hat{\xi}_k, \tag{65} \\
[\hat{\tilde{x}}^{(q)i}, \hat{\tilde{x}}^{(q)j}] &= -i\hbar e^{ijk} \hat{\xi}_k, \tag{66} \\
[\hat{x}^{(q)i}, \hat{\tilde{x}}^{(q)j}] &= 0. \tag{67}
\end{align*}
\]

where \( \hat{\xi}_k \) represents the wave number operator of \( \hat{x}^{(c)i} \).

We note that a similar commutation relation appears in the context of non-geometric string theory, specifically when considering the T-duality of a closed string in a twisted torus with a B-field. In this case, the commutation relation takes the form:
\[ [\hat{x}^{i}, \hat{x}^{j}] = \frac{\hbar l_s^4}{3\hbar} R^{ijk} \hat{p}_k. \tag{68} \]

where \( l_s \) represents the string length \( l_s = \sqrt{\alpha'} \), and \( R^{ijk} \) is referred as the R-flux[35].

In the theory of closed strings, the solution of the equation of motion of a string is given by
\[ X(\tau, \sigma) = X_L(\tau - \sigma) + X_R(\tau + \sigma) \tag{69} \]
and its dual field is defined as:
\[ \tilde{X}(\tau, \sigma) = X_L(\tau - \sigma) - X_R(\tau + \sigma). \tag{70} \]

T-duality is achieved by interchanging the fields and dual fields:
\[ X(\tau, \sigma) \leftrightarrow \tilde{X}(\tau, \sigma). \tag{71} \]
The zero-mode part of the fields can be expressed as:

$$X_L(\tau - \sigma) = x_L + \alpha' p_L(\tau - \sigma) + \ldots,$$

$$X_R(\tau - \sigma) = x_R + \alpha' p_R(\tau + \sigma) + \ldots.$$  

Therefore, we have:

$$X(\tau, \sigma) = x + \alpha' (p\tau + \bar{p}\sigma) + \ldots,$$

$$\bar{X}(\tau, \sigma) = \bar{x} + \alpha' (\bar{p}\tau + p\sigma) + \ldots,$$

where

$$x = x_L + x_R,$$

$$p = p_L + p_R,$$

$$\bar{x} = x_L - x_R,$$

and

$$\bar{p} = p_L - p_R.$$  

Here, $\bar{p}$ represents the number of windings of the string but corresponds to the momentum of the dual coordinate.

It should be noted that the commutation relation we discussed breaks the Jacobi identity:

$$[\hat{x}^{(q)j}, [\hat{x}^{(q)j}, \hat{x}^{(q)k}]] + [\hat{x}^{(q)j}, [\hat{x}^{(q)k}, \hat{x}^{(q)i}]] + [\hat{x}^{(q)k}, [\hat{x}^{(q)i}, \hat{x}^{(q)j}]] = 3i\hbar\epsilon^{ijk},$$

and this breaking is solely due to the presence of the Planck constant. This breaking of the Jacobi identity is analogous to the phenomenon of Bianchi breaking by monopoles in gauge theory[32]. In the case of monopoles, it is discussed that the quantization of monopoles should occur under a condition that avoids nonassociativity even if the Jacobi law is not satisfied. Similarly, in our case, the breaking of the Jacobi identity does not imply nonassociativity.

6 Application

In this section, we will explore some applications of the quantization of Nambu brackets.
6.1 Models that can be reduced to quantum mechanics

In this section, we will discuss models that can be reduced to quantum mechanics within the framework of Nambu brackets. Specifically, we consider a scenario where one of the Hamiltonians, denoted as \( H_3 \), leads to the Poisson bracket of ordinary classical mechanics. By choosing \( H_1 \) as the harmonic oscillator, the Hamiltonians can be defined as follows:

\[
\hat{H}_1 = \frac{1}{2m} \left( \hat{x}_2^{(q)} \right)^2 + \frac{k}{2} \left( \hat{x}_1^{(q)} \right)^2 , \\
\hat{H}_2 = \hat{x}_3^{(q)} .
\] (81)

The Heisenberg equations of motion for this system are given by:

\[
\dot{\hat{x}}^{(q)1} = \frac{1}{i\hbar} [\hat{H}_1, [\hat{H}_2, \hat{x}^{(q)1}]] = \hat{x}_1^{(q)} \hat{x}_3^{(q)} - \hat{x}_2^{(q)} \hat{x}_3^{(q)},
\] (82)

\[
\dot{\hat{x}}^{(q)2} = \frac{1}{i\hbar} [\hat{H}_1, [\hat{H}_2, \hat{x}^{(q)2}]] = -[\hat{H}_1, \hat{\xi}_1],
\] (83)

\[
\dot{\hat{x}}^{(q)3} = \frac{1}{i\hbar} [\hat{H}_1, [\hat{H}_2, \hat{x}^{(q)1}]] = 0.
\] (84)

We observe that the equations of motion for \( x^{(q)1} \) and \( x^{(q)2} \) are identical to the equations of motion in quantum harmonic oscillators. The time evolution of \( x^{(q)3} \) is constant, indicating that it remains unchanged over time.

This is the same as the quantum theory of harmonic oscillators. Therefore, in this particular case, the model reduces to the familiar quantum theory of harmonic oscillators. This demonstrates the applicability of the Nambu bracket formalism in capturing known quantum mechanical systems and reproducing their dynamics.

6.2 Lotka-Volterra Systems

In this section, we consider the application of Nambu brackets to describe Lotka-Volterra systems. It is known that the Lotka-Volterra model can be described using Nambu brackets[36]. The corresponding Hamiltonians for this system are given by:

\[
\hat{H}_1 = \hat{x}_1^{(q)} \hat{x}_2^{(q)} \hat{x}_3^{(q)}, \\
\hat{H}_2 = \hat{x}_1^{(q)} + \hat{x}_2^{(q)} + \hat{x}_3^{(q)}.
\] (85)

The Heisenberg equations of motion for this system are:

\[
\dot{x}_1^{(q)} = \frac{1}{i\hbar} [\hat{H}_1, [\hat{H}_2, \hat{x}_1^{(q)}]] = \hat{x}_1^{(q)} \hat{x}_3^{(q)} - \hat{x}_2^{(q)} \hat{x}_3^{(q)},
\] (86)
\[ \dot{x}_2^{(q)} = \frac{1}{i\hbar} [\hat{H}_1, [\hat{H}_2, \dot{x}_2^{(q)}]] = \dot{x}_2^{(q)} x_3^{(q)} - \dot{x}_1^{(q)} x_2^{(q)}, \tag{87} \]

\[ \dot{x}_3^{(q)} = \frac{1}{i\hbar} [\hat{H}_1, [\hat{H}_2, \dot{x}_3^{(q)}]] = \dot{x}_1^{(q)} x_2^{(q)} - \dot{x}_1^{(q)} x_3^{(q)}. \tag{88} \]

It is worth noting that these equations can also be written in the following form:

\[ \dot{x}_1^{(q)} = \dot{x}_3^{(q)} x_1^{(q)} - \dot{x}_1^{(q)} x_2^{(q)} + i\hbar \left( -\xi_2 + \xi_1 \right), \tag{89} \]

\[ \dot{x}_2^{(q)} = \dot{x}_1^{(q)} x_2^{(q)} - \dot{x}_1^{(q)} x_3^{(q)} + i\hbar \left( -\xi_3 + \xi_2 \right), \tag{90} \]

\[ \dot{x}_3^{(q)} = \dot{x}_1^{(q)} x_3^{(q)} - \dot{x}_2^{(q)} x_1^{(q)} + i\hbar \left( -\xi_1 + \xi_3 \right), \tag{91} \]

where we have used the commutation relation.

It is interesting to note that the Lotka-Volterra equations can be recovered from these equations of motion by neglecting the quantum correction terms. The terms involving the wave number operators \( \xi_i \) can be treated as quantum corrections that arise from the quantization of Nambu brackets.

By considering the classical limit where \( \hbar \) approaches zero, the quantum corrections vanish, and we are left with the classical Lotka-Volterra equations. This demonstrates the correspondence between the quantum description using Nambu brackets and the classical description of the Lotka-Volterra system.

In summary, the quantization of Nambu brackets provides a framework to describe Lotka-Volterra systems in a quantum mechanical setting. The equations of motion obtained from Nambu brackets capture the dynamics of the system, including both classical and quantum effects.

## 7 Discussion

In this paper, we present a new approach to the quantization of Nambu brackets using operator formalism in classical mechanics. Our approach introduces a Planck derivative (differential) \( D \) that bridges classical and quantum mechanics. By using \( D \) to represent the Nambu brackets, we derive a commutation relation that satisfies the quantization of Nambu brackets. Interestingly, this commutation relation formally coincides with the one manifested in the discussion of noncommutativity in the context of double field theory.

The significance of this research is that it demonstrates how exploring the correspondence between classical and quantum theory can contribute to the quantization of the M2-brane and inspire further investigations into the nature of quantum theory from a physical and fundamental perspective. Reexamining the classical operator formalism and linking it to quantum mechanics via the Planck derivative, as well as incorporating the two-step process of converting to operator formalism and introducing quantum corrections, offers a novel perspective on quantization.

Our research suggests several avenues for future exploration. One such avenue is investigating the connection between our approach and double field
theory. Examining the potential implications of this relationship could provide valuable insights into the fundamental structure of M-theory and, consequently, the unification of quantum mechanics and gravity.

Another promising direction is extending our approach beyond Nambu brackets to encompass other complex structures. Further refinement of the operator formalism may enhance our understanding of the quantization processes in more intricate systems and facilitate a better comprehension of the relationship between the classical and quantum realms.

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**Appendix A. Confirmation of commutation relation**

In this section, the calculation of the commutation relation by section 4 is detailed. For the sake of simplicity, we use symbols like the following

\[ Q = \hat{x}^{(q)1}, \quad P = \hat{x}^{(q)2}, \quad \tilde{Q} = \hat{x}^{(q)1}, \quad \tilde{P} = \hat{x}^{(q)2}, \]

\[ Q_c = \hat{x}^{(c)1}, \quad P_c = \hat{p}^{(c)2}, \]

\[ \xi_q = \hat{\xi}_1, \quad \xi_p = \hat{\xi}_2. \]

Therefore, \(Q, P, \tilde{Q}, \tilde{P}\) are

\[ Q = Q_c - \frac{1}{2}\xi_p, \]

\[ P = P_c + \frac{1}{2}\xi_q, \]

\[ \tilde{Q} = Q_c + \frac{1}{2}\xi_p, \]

\[ \tilde{P} = P_c - \frac{1}{2}\xi_q. \]

From the above, the commutation relation is calculated as follows

\[ [Q, P] = [Q_c, \frac{1}{2}\xi_q] + [\frac{1}{2}\xi_p, P_c] = i, \]

\[ [\tilde{Q}, \tilde{P}] = [Q_c, -\frac{1}{2}\xi_q] + [\frac{1}{2}\xi_p, P_c] = -i, \]
\[ [Q, \bar{Q}] = [Q_c - \frac{1}{2}\xi_p, Q_c + \frac{1}{2}\xi_p] = 0, \quad (102) \]

\[ [P, \bar{Q}] = [P_c + \frac{1}{2}\xi_q, Q_c + \frac{1}{2}\xi_p] \]
\[ = [P_c + \frac{1}{2}\xi_p, Q_c] + [\frac{1}{2}\xi_q, Q_c] = i\frac{1}{2} - i\frac{1}{2} = 0, \quad (103) \]

\[ [P, \bar{P}] = [P_c + \frac{1}{2}\xi_q, P_c - \frac{1}{2}\xi_q] = 0, \quad (104) \]

\[ [Q, \bar{P}] = [Q_c - \frac{1}{2}\xi_q, P_c - \frac{1}{2}\xi_q] \]
\[ = [Q_c - \frac{1}{2}\xi_q] + [-\frac{1}{2}\xi_p, P_c] = -i\frac{1}{2} + i\frac{1}{2} = 0. \quad (105) \]

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