Observational constraints of a power spectrum from super-inflation in Loop Quantum Cosmology

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In loop quantum cosmology there may be a super-inflation phase in the very early universe, in which a single scalar field with a negative power-law potential \( V = -M^4 (\phi/M)^{2\beta} \) plays important roles. Since the effective horizon \( \sqrt{SD}/H \) controls the behavior of quantum fluctuation instead of the usual Hubble horizon, we assume the following inflation scenario: the super-inflation starts when the quantum state of the scalar field emerges into the classical regime, and ends when the effective horizon becomes the Hubble horizon, and the effective horizon scale never gets shorter than the Planck length. From consistency with the WMAP 5-year data, we place a constraint on the parameters of the potential (\( \beta \) and \( M \)) and the energy density at the end of the super-inflation, depending on the volume correction parameter \( n \).

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I. INTRODUCTION

In the end of the previous century, loop quantum gravity (LQG) \[1, 2, 3\] inspired loop quantum cosmology (LQC), which is an application of loop quantization to the homogeneous universe models \[4, 5, 6, 7, 8, 9, 10\]. This is featured with singularity avoidance and super-inflation, where the Hubble parameter increases with time. The semiclassical effects of LQG can be incorporated in the forms of the volume correction \[9, 10\] and the energy density correction coming from the holonomy effects \[11, 12, 13, 14, 15, 16\] into the Hamiltonian.

The observation of anisotropy in the cosmic microwave background (CMB) will be the most powerful tool available at present and in the near future to probe the inflationary phase of the universe. It is an interesting possibility that the loop quantum effects in the very early phase of the universe might be imprinted in the CMB anisotropy. In this context, Tsujikawa et al. \[17\] showed that the dynamics during super-inflation due to the volume correction can drive an inflaton field up its potential hill, thus setting the initial conditions for the standard slow-roll inflation and suggested that this transition from the super-inflation to the standard inflation might be responsible for the observed loss of power at the largest angles in the CMB power spectrum but without any explicit calculation of quantum fluctuation in the super-inflationary phase. On the other hand, Zhang and Ling \[18\] considered the slow-roll super-inflation phase due to the energy density correction and calculated quantum fluctuation of the inflaton but without the volume correction effect. They found that signature of loop quantum effects is too weak to detect in the CMB power spectrum with reasonable sensitivity.

If there is a non slow-roll super-inflationary phase due to the loop quantum effects, we can infer that quantum fluctuation generated in that phase might leave imprints in the primordial density perturbation because the statistical properties, say non Gaussianity, of quantum fluctuation generated in the super-inflation would be sufficiently different from that generated in the standard slow-roll inflation. However, for such a scenario to be viable, the predicted power spectrum of the density perturbation must be sufficiently scale-invariant as observed now in the CMB power spectrum and large scale structure. Mulryne and Nunes \[19\] investigated this issue with a scalar field with a power-law potential \( V \propto \phi^\beta \), for which the volume correction is incorporated into the Hamiltonian. They showed that for a non slow-roll solution with constant ratio between the kinetic and potential energies of the scalar field, which is called a scaling solution, if we take the limit \( \beta \to \infty \), the density perturbation generated in the super-inflationary phase is scale-invariant. Copeland et al. \[20\] showed that the scaling solution corresponds to a stable fixed point in terms of dynamical systems theory and that the potential must be negative.

However, taking the limit \( \beta \to \infty \) is not physically acceptable and in reality it is important to determine the allowed region of the parameter(s) for the scenario to be consistent with the presently available observational data, in particular the CMB power spectrum. In this paper, we focus on the following situation according to Refs \[19, 20\]: i) the dynamics is affected from the volume correction both in the matter and the gravitational Hamiltonians \[21\],

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ii) the scalar field has a negative potential and iii) its evolution follows a scaling solution. We adopt the following inflation scenario: the super-inflation starts when the scalar field emerges into the classical regime from the quantum regime, and ends when the semiclassical corrections from loop quantum effects become insignificant. We require that the calculated power spectrum is consistent with the WMAP 5-year data \[22\] and obtain the allowed region for the parameters of the scenario. It is interesting that we can put an upper bound as well as a lower bound on $\beta$ and very large values of $\beta$ are disfavored because of the observed significant deviation from the scale-invariant power spectrum.

This paper is organized as follows. In section II we review Refs \[19, 20\]. In section III we place a constraint on the parameters, and Section IV is a conclusion. In Appendix A we review the calculation of the power spectrum of quantum fluctuation given by Ref \[20\] with volume correction in the gravitational Hamiltonian. In this paper we use the units in which $c = \hbar = 1$.

II. THE QUANTUM FLUCTUATION IN LQC

We here review the derivation of the power spectrum and the scaling solution according to Refs \[19, 20\], but with the other volume correction to the Hamiltonian \[21\].

A. The loop quantized Hamiltonian

We first consider the homogeneous and isotropic universe described by the FRW metric

$$ds^2 = -dt^2 + a(t)^2 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

where $a$ is the scale factor. In LQC the Hamiltonian for gravitation and a single scalar field is given by $H = H_{\text{grav}} + H_{\text{matter}}$, where \[21, 22\] :

$$H_{\text{grav}} = -\frac{3}{8\pi G} aS(q)\dot{a},$$

$$H_{\text{matter}} = \frac{D(q)}{2a}\dot{p}_\phi^2 + a^3 V(\phi).$$

$S(q)$ and $D(q)$ are respectively the volume correction factors in the gravitational and the scalar field Hamiltonians, and the dot denotes the derivative with respect to $t$. We first assume that the scalar field depends only on time, i.e., $\phi = \phi(t)$. $p_\phi$ is a conjugate momentum of $\phi$, which is defined as

$$p_\phi = -\frac{a^3 \dot{\phi}}{D(q)}.$$ 

and $q$ is defined as \[14, 15, 16\] :

$$q \equiv \left( \frac{a}{a_*} \right)^3,$$

where

$$a_* = \left( \frac{2j}{K} \right)^{\frac{1}{2}} \sqrt{\frac{4\pi \gamma}{3} l_{\text{Pl}}}, \quad K = \frac{2\sqrt{2}}{3\sqrt{3 \sqrt{3}}} \quad l_{\text{Pl}}^2 = G, \quad (2.6)$$

$j$ is an SU(2) parameter which is associated with the link of the spin network state in LQG (we assume $j$ is sufficiently large), and $\gamma$ is the Barbero-Immirzi parameter which is here assumed $\gamma = \ln(2)/\left(\pi \sqrt{3}\right)$ by the black hole entropy argument in LQG \[1\], but see also Ref \[24\]. Note that $a_\ast$ is the characteristic scale factor in LQC: when the scale factor is smaller than $a_\ast$, the LQC effects are remarkable. In the semiclassical region ($l_{\text{Pl}} \ll a \ll a_\ast$), $S(q)$ and $D(q)$ take the following forms \[25\] :

$$S(q) \sim \frac{3}{2} q, \quad D(q) \sim \left( \frac{9}{2l + 3} \right)^{\frac{3}{2}} q^{\frac{3(3-l)}{2}}.$$  

(2.7)
while, in the classical region \((a \gg a_* )\), these are
\[
S(q) \sim 1, \quad D(q) \sim 1,
\]
so that the classical theory is recovered. It should be noted that the characteristic scale factor \(a_*\) is in fact a problematic object in LQC as indicated in Refs \([26, 27]\). We will caution this in section IV. We can conveniently parametrize the correction factors in the semiclassical region as follows:
\[
S(q) = S_* a^r, \quad D(q) = D_* a^n,
\]
where
\[
S_* = \frac{3}{2} a_*^{-r}, \quad D_* = \left( \frac{9(9 - n)}{81 - 5n} \right)^{\frac{n - 9}{r}} a_*^{-n},
\]
\[
n = \frac{9(3 - l)}{1 - l}, \quad r = 3.
\]

We assume \(0 < l < 1\) and hence \(9 < n < \infty\) to remove the divergence of the inverse volume factor. Notice that for Eq. (2.11) to be physical, we impose the constraint \(81/5 < n < \infty\) on the parameter \(n\).

The scalar field in general depends on time and position, i.e. \(\phi = \phi(t, x)\). So in this situation we need to consider the following scalar field Hamiltonian in LQC:
\[
\mathcal{H}_{\text{matter}} = \frac{D^2(q)}{2a^3} \rho^2 + aS(q) \delta_{\phi}^2 \partial_\phi \partial_\phi \phi + a^3 V(\phi),
\]
where \(a\) and \(b\) run over all spatial induces or 1 to 3. We will consider quantum fluctuation of the scalar field for this Hamiltonian. From the gravitational and the scalar field Hamiltonians (2.2) and (2.3), we can obtain the following modified Friedmann equation and the scalar field equation:
\[
H^2 = \frac{8\pi G}{3} S(q) \rho, \quad \dot{\phi} + 3H \left( 1 - \frac{1}{3} \frac{d \ln D}{d \ln a} \right) \phi + DV_{,\phi} = 0,
\]
where \(\dot{\phi}\) denotes the derivative with respect to \(\phi\) and \(H \equiv \dot{a}/a\) is the Hubble parameter, and the energy density \(\rho\) is given by
\[
\rho = \frac{\dot{\phi}^2}{2D(q)} + V(\phi).
\]

Using the conformal time \(\tau\) where \(dt = a d\tau\), Eq. (2.16) is rewritten as
\[
\phi'' + \left( 2 - \frac{d \ln D}{d \ln a} \right) \frac{a'}{a} \phi' + a^2 DV_{,\phi} = 0,
\]
where the prime denotes the derivative with respect to \(\tau\). In the classical region, since \(D(q) \simeq S(q) \simeq 1\), these equations reduce to the classical ones, on the other hand, in the semiclassical region, the second term on the left-hand side of Eq. (2.18) acts an anti-friction term and enables the scalar field to climb the potential.

More remarkably, the time derivative of the Hubble parameter
\[
\dot{H} = -\frac{4\pi GS \dot{\phi}^2}{D} \left[ 1 - \left( \frac{1}{6} \frac{d \ln D}{d \ln a} + \frac{1}{6} \frac{d \ln S}{d \ln a} \right) \right] + \frac{4\pi GS}{3} \frac{d \ln S}{d \ln a} V
\]
is positive in the semiclassical region. The accelerated expansion with this feature is called super-inflation.

Since we have \(c = \hbar = 1\), the length, the time and the mass are all of the same dimension, say \(L\). The dimensions of the scale factor \(a\) and the Newtonian constant \(G\) are then \(L\) and \(L^2\). From this argument, Eq. (2.15) shows that the scalar field \(\phi\) and its potential \(V\) are of dimension \(L^{-1}\) and \(L^{-4}\), respectively.
B. The scaling solution

We review the scaling solution and its stability according to Refs. [19, 20]. We write the dynamics of the homogeneous system in terms of the following three variables:

\[ x = \frac{\dot{\phi}}{\sqrt{2D\rho}}, \quad y = \sqrt{\frac{|V|}{\rho}}, \quad \lambda = -\sqrt{\frac{3D}{16\pi GS}} \frac{V_{,\phi}}{V}, \]

(2.20)

where we use Eq. (2.17) with the negative potential, and we also need a constraint

\[ x^2 - y^2 = 1 \]

(2.21)

from the Friedmann equation (2.15). From Eqs. (2.15), (2.16) and (2.20), we can derive the following set of ordinary differential equations:

\[ x, N = 3x\bar{\alpha} - 3x^3\bar{\alpha} + \sqrt{\frac{3}{2}} \lambda y^2, \]

(2.22)

\[ y, N = -3\bar{\alpha}yx^2 - \sqrt{\frac{3}{2}} \lambda xy, \]

(2.23)

\[ \lambda, N = \lambda \left( n - r \right) + \sqrt{6}x\lambda^2 (1 - \Gamma), \]

(2.24)

where \( N \equiv \ln a \) and

\[ \bar{\alpha} = \frac{n}{6} - 1, \quad \Gamma = \frac{VV_{,\phi}}{V_{,\phi}^2}. \]

(2.25)

Because of the constraint (2.21), \( x \) and \( y \) are not independent. Hence we consider only \( x \) and \( \lambda \). This system has several fixed points but here we concentrate on stable ones only. If

\[ \Gamma < \frac{n + r - 12}{2(n - 6)}, \]

(2.26)

a couple of stable fixed points \((x, \lambda)\) are given by

\[ \left( 1, -\frac{n - r}{2\sqrt{6(1 - \Gamma)}} \right), \quad \left( -1, \frac{n - r}{2\sqrt{6(1 - \Gamma)}} \right). \]

(2.27)

These points correspond to the kinetic-term dominant solutions. If

\[ \frac{n + r - 12}{2(n - 6)} < \Gamma < \frac{12 + n - 3r}{2(6 - r)}, \]

(2.28)

a couple of stable fixed points are given by

\[ \left( -\sqrt{\frac{n - r}{12\bar{\alpha}(1 - \Gamma)}}, \sqrt{\frac{\bar{\alpha}(n - r)}{2(1 - \Gamma)}} \right), \quad \left( \sqrt{\frac{n - r}{12\bar{\alpha}(1 - \Gamma)}}, -\sqrt{\frac{\bar{\alpha}(n - r)}{2(1 - \Gamma)}} \right). \]

(2.29)

These points correspond to the scaling solutions where the ratio of the kinetic term to the potential term is kept constant. For simplicity, we only consider a constant \( \Gamma \) and then we can determine the potential form by Eq. (2.25). For \( \Gamma \neq 1 \), the potential is given by

\[ V = -V_0|\phi|^{\beta}, \]

(2.30)

where \( \beta \) and \( V_0 \) are constants. For \( \Gamma = 1 \), we have an exponential potential. Notice that since the kinetic-term dominated solution (2.27) and the scaling solution (2.29) can not be defined for \( \Gamma = 1 \), we can only use the power-law potential (2.30) and have \( 1 - \Gamma = 1/\beta \). Moreover, since the scaling solutions can be responsible for the fluctuation of the present CMB radiation as we will see later, we hereafter adopt the scaling solutions.
To calculate quantum fluctuation we rewrite Eq. (2.20) as
\[ x = \sqrt{\frac{4\pi G S}{D}} \phi, \Phi. \] (2.31)

The two scaling solutions expressed by fixed points (2.29) can be analyzed in the same manner, we take the first one
\[ x_0 = -\sqrt{\frac{(n-r)\beta}{2(n-6)}}. \] (2.32)

where we have used Eqs. (2.25) and (2.30). Substituting \( x = x_0 \) into Eq. (2.31), and integrating it with respect to \( a \), we obtain \( \phi \) as
\[ \phi = \frac{2x_0}{n-r} \sqrt{\frac{3D}{4\pi GS}}. \] (2.33)

Differentiating the above with respect to \( \tau \), we obtain
\[ \phi' = x_0 \sqrt{\frac{3D}{4\pi GS}} a'. \] (2.34)

On the other hand, we rewrite Eq. (2.20) as
\[ \phi' = x_0 a \sqrt{2D\rho}. \] (2.35)

Here we can write \( \rho \) in terms of \( x_0 \) using Eqs. (2.20) and (2.21). For consistency between Eqs. (2.34) and (2.35), we can obtain the differential equation of the scale factor. Integrating it with respect to \( \tau \), the scale factor is obtained as
\[ a = A(-\tau)^p, \] (2.36)
where
\[ A = \left[ -\frac{1}{p} \sqrt{\frac{8\pi f \tilde{S}_s}{3(x_0^2 - 1)}} \right] \left[ \frac{2x_0}{n-r} \sqrt{\frac{3\tilde{D}_s}{4\pi \tilde{S}_s}} \right]^\beta \gamma \left( \frac{2j}{K} \right) \sqrt{\frac{4\pi \gamma}{3}} l_{\text{Pl}}, \] (2.37)
\[ p = -\frac{4}{2(r+2) + (n-3)\beta}, \] (2.38)
\[ \tilde{S}_s = \frac{3}{2}, \] (2.39)
\[ \tilde{D}_s = \left( \frac{9(n-9)}{5n-81} \right)^{-\beta/2}. \] (2.40)

Since \( \phi \) and \( V_0 \) have dimension of \( L^{-1} \) and \( L^{\beta-4} \), respectively, we put this \( V_0 \) as
\[ V_0 = \frac{f}{l_{\text{Pl}}^{4-\beta}} = M^{4-\beta}, \] (2.41)
where \( f \) is a dimensionless constant and \( M \) gives the mass scale of the scalar field \( \phi \). We have used this form to get Eq. (2.37).

C. The effective horizon

In the usual inflation scenario, quantum fluctuation is frozen when the fluctuation scale \( a/k \) gets longer than the Hubble horizon scale \( 1/H \). In LQC we will see below that the behavior of quantum fluctuation may be controlled by the effective horizon \( \sqrt{SD}/H \) instead of the Hubble horizon \( 1/H \).
To get insight into the physical properties of the effective horizon we consider a massless scalar field. The field equations for the massless scalar field are given by putting $V = 0$ into the equations in section II A. The equation of motion is then given by

$$\ddot{\phi} + (3 - n) H \dot{\phi} = 0.$$  \hspace{1cm} (2.42)

The above can be easily integrated to give

$$\dot{\phi} = C a^{n-3},$$ \hspace{1cm} (2.43)

where $C$ is an integral constant. Substituting Eq. (2.43) into the Friedmann equation (2.15) with the massless scalar field, and integrating it with respect to $t$, we can obtain the following scale factor:

$$a = \left\{ \frac{(r + n) - 6}{2} \right\} C \sqrt{\frac{4\pi G S_*}{3 D_s}} (-t)^{-\frac{1}{n+1}} + C_1,$$ \hspace{1cm} (2.44)

where $C_1$ is an integral constant, and it should be noted that the scale factor increases with time for $-\infty < t < 0$.

We here consider the following perturbation for the scalar field:

$$\phi = \phi(t) + \delta \phi(t, x).$$ \hspace{1cm} (2.45)

Using Eq. (2.14), we can obtain the equation for the perturbation of the massless scalar field as

$$\delta \ddot{\phi} + (3 - n) H \delta \dot{\phi} - \frac{DS}{a^2} \nabla^2 \delta \phi = 0.$$ \hspace{1cm} (2.46)

Here, using the Fourier transformation

$$\delta \phi = \sum_k \delta \phi_k \exp(ikx),$$ \hspace{1cm} (2.47)

and substituting Eq. (2.46) into Eq. (2.46), we obtain

$$\delta \ddot{\phi}_k + (3 - n) H \delta \dot{\phi}_k + \frac{DS k^2}{a^2} \delta \phi_k = 0.$$ \hspace{1cm} (2.48)

It should be noted that since we observe the density perturbation as a functional of the Fourier mode $\delta \phi_k$, from Eq. (2.17) the density perturbation of the massless scalar field is given by

$$\frac{\delta \rho}{\rho} \simeq \frac{\delta \phi_k}{\phi}.$$ \hspace{1cm} (2.49)

First, we consider the short-wave-length limit. Assuming $\delta \phi_k = \exp (i\omega t)$ and inserting this into Eq. (2.48), then it becomes

$$- \omega^2 + i(3 - n) H \omega + \frac{DS k^2}{a^2} = 0.$$ \hspace{1cm} (2.50)

We compare three terms on the left-hand side of Eq. (2.50). If

$$\omega H \ll \frac{DS k^2}{a^2}$$ \hspace{1cm} (2.51)

is satisfied, the second term is much smaller than the third term. In that case, using Eqs. (2.50) and (2.51), we can take $\omega$ as follows

$$\omega = \frac{\sqrt{DSk}}{a},$$ \hspace{1cm} (2.52)

and substituting this into Eq. (2.51), we obtain

$$1 \ll \frac{\sqrt{DSk}}{aH}.$$ \hspace{1cm} (2.53)
Using Eqs. (2.43) and (2.44), and \( \delta \phi_k = \exp (i\omega t) \), we can obtain the density perturbation in the short-wave-length limit for the massless scalar field as follows:

\[
\frac{\delta \phi_k}{\phi} = \frac{i\omega}{C} \frac{\delta \phi_k}{a^{n-3}} \propto e^{i\omega t}(-t)^2,
\]

(2.54)

where we put \( r = 3 \). Notice that the density perturbation decreases with time in the limit of Eq. (2.53).

Next, we consider the long-wave-length limit and assume that the third term is much smaller than the other terms on the left-hand side of Eq. (2.48). Then we obtain the following relation:

\[
(3 - n)H \delta \dot{\phi} \gg \frac{DSk^2}{a^2} \delta \phi_k.
\]

(2.55)

In this case, Eq. (2.48) implies \( \delta \dot{\phi}_k / \delta \phi_k \sim H \), and hence we can rewrite Eq. (2.55) as follows:

\[
1 \gg \sqrt{DS/H}.
\]

(2.56)

where we have neglected the constant \( (3 - n) \). In the above limit, we can solve Eq. (2.48) and then obtain \( \delta \dot{\phi}_k \) as follows:

\[
\delta \dot{\phi}_k = C_2 a^{n-3},
\]

(2.57)

where \( C_2 \) is an integral constant. Using this \( \delta \dot{\phi}_k \) and Eq. (2.43), we can obtain the following density perturbation in the long-wave-length limit for the massless scalar field:

\[
\frac{\delta \phi_k}{\phi} \propto \frac{C_2}{C} = \text{const.}
\]

(2.58)

The density perturbation is constant in the region of Eq. (2.56).

The above argument means that the behavior of the density perturbation strongly depends on whether its length scale is larger or smaller than \( \sqrt{DS/H} \). Thus \( \sqrt{DS/H} \) acts an effectively horizon for the super-inflation in LQC. From now on, we call \( \sqrt{DS/H} \) the effective horizon scale, and the scales satisfying Eqs. (2.53) and (2.56) are said to be sub-horizon and super-horizon, respectively.

D. The evolution of the fluctuation scale and the effective horizon scale

In the super-inflation, the scale factor is given by Eq. (2.36) on the scaling solutions. Using this, the fluctuation scale \( a/k \), the effective horizon scale \( \sqrt{DS/H} \) and the Hubble horizon scale \( 1/H \) behave as

\[
\frac{a}{k} \propto (-\tau)^p, \quad \frac{\sqrt{DS}}{H} \propto (-\tau)^{\frac{n+r+2}{n+r}+1}, \quad \frac{1}{H} \propto (-\tau)^{p+1},
\]

(2.59)

respectively. We can see that the fluctuation scale gets longer than the Hubble horizon scale as time proceeds. For \( p > -2/(n + r) \) the fluctuation scale becomes longer than the effective horizon scale, while for \( p < -2/(n + r) \) the fluctuation scale gets shorter than the effective horizon scale. In our scenario we assume that the scale of fluctuation gets longer than the effective horizon scale to become classical through some decoherence processes. Then we impose the following condition on \( p \):

\[
p > \frac{-2}{n + r}.
\]

(2.60)

We call this condition the super-horizon condition.

E. The power spectrum in LQC

We calculate the power spectrum in the same way in Ref. [19]. However, note that we also incorporate the correction factor \( S(q) \) into the matter Hamiltonian as seen in Eq. (2.14). We put the scalar field perturbation given by Eq. (2.45) into Eq. (2.14) as

\[
H_{\phi + \delta \phi} = \frac{1}{2} D(q) p_\phi^2 + \frac{1}{2} aS(q) \delta^{ab} \partial_a (\phi + \delta \phi) \partial_b (\phi + \delta \phi) + a^3 V(\phi + \delta \phi),
\]

(2.61)
where \( p_{\phi+\delta\phi} = a^3(\dot{\phi} + \delta\dot{\phi})/D \). Based on the above Hamiltonian, we can calculate the power spectrum \( P_{\delta\phi} \) of the density perturbation induced by quantum fluctuation of the scalar field. In particular, the spectral index \( n_s \) for the density perturbation or the tilt \( b \), which is defined \(^{28}\) by

\[
b \equiv n_s - 1 \equiv \frac{dP_{\delta\phi}}{dk},
\]

is a crucial observable to see the consistency between the prediction of the theory and the observation. The result is given by

\[
b = 3 - \frac{2\sqrt{9 - (6 - 4n - 3r)2p - (12 + 4n - 2nr - r^2 - 2n^2)p^2}}{2 + (n + r)p},
\]

where we can see the scale-invariant power spectrum is achieved by \( p = 0 \), corresponding to \( \beta \to \infty \) with \( r = 3 \) in Eq. (2.38). The detailed calculation is described in Appendix A.

### III. THE CONSISTENCY WITH THE WMAP 5-YEAR DATA

#### A. The potential of the scalar field

According to the WMAP 5-year data, the CMB power spectral index is within the range \(^{22}\)

\[
n_s = 0.963^{+0.014}_{-0.015}.
\]

Hence we use the following range of the tilt:

\[
b = -0.037^{+0.014}_{-0.015}.
\]

To constrain the power index \( \beta \) of the potential from the observational data we use the relation between \( \beta \) and the observed tilt \( b \). Then, with \( r = 3 \), we solve Eq. (2.63) for \( p \) and obtain

\[
p = \frac{2\sqrt{\zeta - \xi}}{\chi},
\]

where

\[
\zeta = (b^2 - 6b + 1)n^2 + (2b^2 - 12b - 42)n + 33b^2 - 198b + 441, \quad \xi = (2b^2 - 12b + 2)n + 6b^2 - 36b + 42, \quad \chi = (b^2 - 6b + 1)n^2 + (6b^2 - 36b + 46)n + 9b^2 - 54b + 93.
\]

We can see that only this root satisfies the super-horizon condition \(^{26\text{a}}\) and hence we have discarded another one. Besides, it should be noted that the effective horizon scale decreases with time.

We solve Eq. (2.68) for \( \beta \) and obtain

\[
\beta = \frac{2(2 + 5p)}{(n - 3)p}.
\]

Substituting Eq. (3.3) into Eq. (3.7) and using Eqs. (3.2)–(3.6), we plot the allowed region for \( \beta \) against \( n \) in FIG. 1. If we can specify the value for \( n \), then we can place a rather stringent constraint on \( \beta \). For example, if we put \( n = 100 \), then \( \beta = 25.2 \). However, even if we do not have any information about \( n \), \( \beta \) is weakly constrained to \( 15 \lesssim \beta \lesssim 80 \).

In any case, \( \beta \) is bounded from above in the present scenario and an infinitely large value is not allowed from the WMAP 5-year data. This result is not so sensitive even if we slightly expand the range of \( b \) around the observed one \(^{32}\). Actually, we can see that the dependence of \( \beta \) on \( b \) is not monotonic. For the observed range \(^{32}\) of \( b \), \( \beta \) decreases as \( b \) increases. However, as \( b \) increases further to 0, \( \beta \) turns to increase to infinity. This behavior is common at least for all \( n \) in the range shown in FIG. 1.
FIG. 1: The allowed region of $\beta$ against $n$. The solid line corresponds to the best-fit value, while the dashed lines denote the boundary of measurement error ranges.

B. The energy density at the end of the super-inflation

In LQG, there is the smallest area element $\Delta = 2\sqrt{3}\pi\gamma l_P^2$ [1]. Since the Hubble parameter increases with time, the Hubble horizon scale $1/H$ decreases and the energy density increases. Thus, if the Hubble horizon characterizes causality and the nature of quantum fluctuation, we need to show that the Hubble horizon scale is never smaller than the square root of the smallest area element to validate the present analysis. However, in the super-inflation scenario, causality and the nature of quantum fluctuation may be controlled by the effective horizon $\sqrt{SD/H}$ and it decreases as time proceeds. Therefore, we need to find the condition that the effective horizon scale is never smaller than the square root of the smallest area element for the super-inflation in LQC.

In calculating the energy density at the end of the super-inflation, we assume the following super-inflation scenario: the super-inflation starts when the quantum state of the scalar field emerges into the classical regime [17, 29] (we call this time $\tau_{\text{start}}$), and this ends when the effective horizon becomes the Hubble horizon $\sqrt{SD/H} = 1/H$ (we call this time $\tau_{\text{end}}$) and before the effective horizon scale gets shorter than the square root of the smallest area element (we call this time $\tau_{Pl}$).

We calculate the energy density from Eq. (2.15) as

$$\rho = \frac{3}{8\pi GS} \frac{(-p)^2}{A^2} (-\tau)^{-2(p+1)},$$

(3.8)

where we have used Eqs. (2.7), (2.10) and (2.36) with $r = 3$, and

$$a_* = \tilde{a}_* l_P, \quad A = \tilde{A}_s f^2 a_*, \quad \dot{a} = \frac{-p}{\tau},$$

(3.9)

where $\tilde{a}_*$ and $\tilde{A}_s$ are calculated as

$$\tilde{a}_* = \left(\frac{2j}{K}\right)^{\frac{3}{4}} \sqrt{\frac{4\pi\gamma}{3}},$$

(3.10)

$$\tilde{A}_s = \left[ \frac{1}{p} \sqrt{\frac{8\pi S_*}{3(x_0 - 1)}} \right] \left[ \frac{2x_0}{n - r} \sqrt{\frac{3D_*}{4\pi S_*}} \right]^\frac{2}{p} \left[ \frac{(2j/3)^{\frac{3}{4}}}{\left(\frac{2j}{K}\right)^{\frac{3}{4}} \sqrt{\frac{4\pi\gamma}{3}}} \right]^p.$$

(3.11)

From Eqs. (3.2)–(3.6), we can show $p+1 > 0$ and see the energy density given by Eq. (3.8) is monotonically increasing with time. Incidentally if we take $p \to 0$ in Eq. (3.8), $\rho$ approaches to a zero at the all time.
The energy density at the end of the super-inflation is

\[
\rho_{\text{end}} = \frac{3}{8\pi G} \frac{(-p)^2}{a_*^2} S_*^{-\frac{1}{4}} D_*^{-\frac{3}{4}} \left( \dot{A}_* \right)^{\frac{2}{3}} \left( \ddot{S}_* \dot{D}_* \right) - \frac{2(\nu+1)}{\nu+3} f,
\]

(3.12)

where we have used Eq. (3.8) and the ending time

\[
(-\tau_{\text{end}}) = \left( \dot{A}_* f \right)^{\frac{2}{3}} \left( \ddot{S}_* \dot{D}_* \right)^{-\frac{1}{4(\nu+3)}},
\]

(3.13)

where we have used Eqs. (3.9) and (2.36) so that the effective horizon is equal to the Hubble horizon at \( \tau = \tau_{\text{end}} \).

1. the energy density at the end of the super-inflation

Since we have the temperature fluctuation amplitude as \( \delta p_k / \rho \sim \delta \phi / \dot{\phi} \sim 10^{-5} \) in the WMAP 5-year data, using the amplitude and the spectral index, we will obtain the information about the energy density at the end of the super-inflation. For this purpose, we calculate \( \delta \phi_k \) and \( \dot{\phi} \) in terms of \( f \). From Eqs. (2.34), (2.35), and (2.36), \( \dot{\phi} \) is given by

\[
\dot{\phi} = \frac{(-p)x_0}{(-\tau)a} \sqrt{3D} \frac{4\pi G}{\nu}
\]

(3.14)

and \( \delta \phi_k \) can be calculated from the power spectrum as Eq. (A.10)

\[
< 0 |(\delta \phi_k)^2|0 \rangle = \frac{k^3}{2 \pi^2} = \mathcal{P}_{\delta \phi}(k).
\]

(3.15)

So far the dimension of the scale factor is \( L \) and the wave number \( k \) is dimensionless. Here, since we would like to make the dimensionless scale factor and the wave number of dimension \( L^{-1} \), we define

\[
a_0 = \frac{a}{a_*}, \quad \hat{k} = \frac{k}{a_*},
\]

(3.16)

where the characteristic scale factor \( a_* \) has dimension \( L \). Using Eqs. (3.15), (3.16) and (A.19), we calculate \( \delta \phi_k \) as

\[
\delta \phi_k = \sqrt{\frac{\Gamma(|\nu|)}{4\pi^2}} \left[ \frac{p}{2 + (n+3)p} \right]^{\frac{1-2|\nu|}{2}} \frac{(-p)^{\nu-q}}{S^{3/4} D^{1/4} \tilde{a}^2 a_*} \left( \sqrt{SDa_0 \hat{k}} \right)^{\frac{3-2|\nu|}{2}} (-\tau)^{-\frac{1+2|\nu|}{2}},
\]

(3.17)

where

\[
q = -\left( \frac{2 + 3|\nu| - n(1 - |\nu|)}{2} p - 1 + 2|\nu| \right).
\]

(3.18)

From Eqs. (2.49), (3.14), and (3.17), we can obtain the fluctuation of the energy density as

\[
\frac{\delta p_k}{\rho} = \frac{(-q)}{a_0 x_0 (SD)^{3/4}} \sqrt{\frac{\Gamma(|\nu|)}{3\pi}} \left[ \frac{p}{2 + (n+3)p} \right]^{\frac{1-2|\nu|}{2}} \left( \sqrt{SDa_0 \hat{k}} \right)^{\frac{3-2|\nu|}{2}} (-\tau)^{-\frac{1+2|\nu|}{2}},
\]

(3.19)

and we assume that the fluctuation observed by WMAP is created at the end time of the super-inflation (\( \tau = \tau_{\text{end}} \)),

\[
\left( \frac{\delta p_k}{\rho} \right)_{\text{obs}} = \frac{(-q)}{a_* x_0} \left( \ddot{S}_* \dot{D}_* \right)^{\frac{1-2|\nu|}{2}} \sqrt{\frac{\Gamma(|\nu|)}{3\pi}} \left[ \frac{p}{2 + (n+3)p} \right]^{\frac{1-2|\nu|}{2}} \left( \frac{a_0 \hat{k}}{-p} \right)^{\frac{3-2|\nu|}{2}} \left( \dot{A}_* \right)^{\frac{1-2|\nu|}{2}} \left( \ddot{S}_* \dot{D}_* \right)^{-\frac{1+2|\nu|}{2}} \frac{1}{a_*^{3/4}} f^{-\frac{1-2|\nu|}{2}},
\]

(3.20)

where we have used Eq. (3.13) at this time, and \( (\delta p_k / \rho)_{\text{obs}} \) is the observational data of the energy density fluctuation. Here, we rewrite Eq. (3.20) as follows:

\[
f = W \dot{A}_*^{\frac{2}{\nu+3}} \left( \ddot{S}_* \dot{D}_* \right)^{-\frac{2}{\nu+3}},
\]

(3.21)
where

\[ W = \left[ \left( \frac{\delta \rho_k}{\rho} \right)^{-1} \frac{(-q)}{a_* x_0} \left( \tilde{S}_* \tilde{D}_* \right)^{\frac{1}{n+3}} \sqrt{\frac{G \Gamma([\nu])}{3 \pi}} \sqrt{\frac{\sqrt{\frac{a_*}{n+3}} \tilde{D}_*^{\frac{1}{n+3}}}{\frac{3}{n+3}}} \right]^{\frac{1}{n+3}} \]  

Substituting Eq. (3.12) into Eq. (3.21), we obtain the energy density at the end of the super-inflation as follows:

\[ \rho_{\text{end}} = \frac{3(-p)^2}{8\pi l_p^4 a_*^4} W \tilde{S}_*^{-\frac{2}{n+3}} \tilde{D}_*^{\frac{2}{n+3}} \left( \tilde{S}_* \tilde{D}_* \right)^{\frac{2}{n+3}}. \]  

Because the effective horizon scale must not be shorter than the square root of the smallest area element in the present assumption, we need the following condition of the super-inflation in LQC:

\[ \frac{\sqrt{DS}}{H} > \sqrt{\Delta}, \]  

where

\[ \Delta = 2 \sqrt{3} \pi l_p^2. \]  

At the end of the super-inflation, we substitute Eq. (3.13) into Eq. (3.24), and rewrite this equation as

\[ f < \frac{a_*^2}{\Delta(-p)^2} \left( \tilde{A}_* \right)^{-\frac{2}{n+3}} \left( \tilde{S} \tilde{D} \right)^{\frac{2(p+1)}{n+3}+\frac{1}{p}}. \]  

By substituting Eq. (3.26) into Eq. (3.12), we obtain the upper limit of the energy density at the end of the super-inflation,

\[ \rho_{\text{end}} < \frac{3}{8\pi G \Delta} \left( \tilde{S}^{-\frac{2}{n+3}} \tilde{D}_*^{\frac{2}{n+3}} \right), \]  

and we call this the effective horizon condition.

In FIG. 2, we plot the allowed region of \( \rho_{\text{end}}/\rho_{\text{Pl}} \) with \( \rho_{\text{Pl}} = l_p^{-4} \) and \( (\delta \rho_k/\rho)_{\text{obs}} = 10^{-5} \) for the different values of \( n \). At least for the region shown in this figure, we can see that the effective horizon condition (3.27) is well satisfied. Moreover, the energy scale \( \rho_{\text{end}} \) is far beyond that for the nucleosynthesis constraint. In this region the energy density is much smaller than the Planck energy density. We can see that if we know the value of \( n \), \( \rho_{\text{end}} \) is strongly constrained from the observational data. For example, if we put \( n = 100 \), then \( \rho_{\text{end}}/\rho_{\text{Pl}} = 10^{-22.6} \).

2. The period of the super-inflation

Since the effective horizon decreases with time, the super-inflation must end before the effective horizon scale gets equal to the square root of the smallest area element to justify the present calculation. Thus we have the following condition:

\[ \tau_{\text{start}} < \tau_{\text{end}} < \tau_{\text{Pl}}. \]  

Substituting Eq. (3.13) into Eq. (3.28), and using Eq. (2.11), we can obtain the following condition for the mass scale of the scalar field \( M \):

\[ \left[ (-\tau_{\text{start}}) \tilde{A}_*^{\frac{1}{n+3}} \left( \tilde{S}_* \tilde{D}_* \right)^{\frac{1}{n+3}} \right]^{\frac{1}{n+3}} > \left( \frac{M}{m_{\text{Pl}}} \right) > \left[ (-\tau_{\text{Pl}}) \tilde{A}_*^{\frac{1}{n+3}} \left( \tilde{S}_* \tilde{D}_* \right)^{\frac{1}{n+3}} \right]^{\frac{1}{n+3}}, \]  

where we assume \( M > 0 \) and \( 15 \lesssim \beta \lesssim 80 \). Here, substituting Eq. (3.21) into Eq. (2.11), we can take the scalar field mass scale with WMAP data.

\[ \left( \frac{M}{m_{\text{Pl}}} \right) = \left[ W \tilde{A}_*^{\frac{1}{n+3}} \left( \tilde{S}_* \tilde{D}_* \right)^{\frac{1}{n+3}} \right]^{\frac{1}{n+3}}. \]
Since the super-inflation starts when the quantum state of the scalar field emerges into the classical regime, we can require the uncertainty principle condition of the scalar field

$$|\phi \cdot p_\phi| > 1$$  \hspace{1cm} (3.31)

at $\tau = \tau_{\text{start}}$. Substituting Eqs. (2.33) and (2.24) into Eq. (3.31), and using Eq. (2.36), we have

$$\frac{3x_0^2a^2}{2(n-3)\pi GS}(-p)>1.$$  \hspace{1cm} (3.32)

By substituting Eq. (3.21) into Eqs. (3.32) and (3.24), we can determine $(-\tau_{\text{start}})$ and $(-\tau_{\text{Pl}})$ as

$$( -\tau_{\text{start}} ) > \left[ \frac{3x_0^2a^2}{2(n-3)\pi S_*} \right]^{\frac{1}{p+1}} W^\frac{p}{n(p+1)} \left( S_*\tilde{D}_* \right)^{\frac{1}{n+3(p+1)}},$$  \hspace{1cm} (3.33)

$$( -\tau_{\text{Pl}} ) = \left[ \frac{\Delta}{S_*\tilde{D}_*} \right]^{\frac{2}{(n+5)p+2}} W^{-\frac{2}{(n+5)p+2}} \left( S_*\tilde{D}_* \right)^{\frac{2}{n+5(p+1)}}.$$  \hspace{1cm} (3.34)

It turns out that the uncertainty principle condition does not essentially constrain the parameters of the scenario. Then, we substitute Eq. (3.34) into Eq. (3.29), which gives a lower bound of the scalar field mass scale $M$. In FIG. 3, we plot the allowed region from the observational constraint (3.30) with $b = -0.037$ and $\delta \rho_k/\rho_{\text{obs}} = 10^{-5}$ for the different values of $n$. At last in the region shown in FIG. 3, we can easily show that the allowed region is far beyond the lower bound in Eq. (3.29) and the scalar filed mass scale has the Planck mass order. If we can know the volume correction parameter $n$, then we can constrain the scalar field mass scale $M$ rather stringently. For example, if we put $n = 100$, then $M/m_{\text{Pl}} = 0.150$.

**IV. CONCLUSION**

We have considered a single scalar field with the negative power-law potential $V = -M^4(\phi/M)^\beta$ in LQC, and determined the allowed region of the potential power index $\beta$ and the energy density at the end of the super-inflation $\rho_{\text{end}}$, and the scalar field mass scale $M$ by using the consistency with the WMAP 5-year data.
FIG. 3: The allowed region of the scalar field mass scale $M$ constrained from the observational data for the density perturbation. The solid line denotes the best-fit value, while the dashed lines denote the boundary of the allowed region. There is almost no dependence on the value of SU(2) parameter $j$ for $j = 100, 10^{10}$.

First we have reviewed Ref [20]. Using dynamical systems theory we have found scaling solutions which are stable fixed points and satisfy the super-horizon condition. Second, we have determined the super-horizon condition (2.60) by the behavior of the density perturbation of a massless scalar field, and then considered the effective horizon scale instead of the Hubble horizon scale. Third, we have assumed the super-inflation scenario: the super-inflation starts when the quantum state of the scalar field emerges into the classical regime and ends when the effective horizon gets equal to the Hubble horizon before the effective horizon scale gets shorter than the square root of the smallest area element. Finally, by using the above inflation scenario and the consistency with the WMAP 5-year data, we have reached the following conclusion. If we can specify the volume correction parameter $n$, we can constrain the potential parameters $\beta$ and $M$, and the energy density at the end of the super-inflation $\rho_{\text{end}}$ rather stringently. Even if we only know the volume correction parameter $n$ in the range as $81/5 < n < \infty$, we can constrain $\beta$, $M$ and $\rho_{\text{end}}$ as follows: $\beta$ exists in the region as $15 \lesssim \beta \lesssim 80$, $M$ is the order of the Planck mass, and $\rho_{\text{end}}$ is smaller than the Planck energy density, respectively. Besides, for example, if we put $n = 100$, we can constrain $\beta$, $M$ and $\rho_{\text{end}}$ as follows: $\beta = 25.2$, $\rho_{\text{end}}/\rho_{\text{Pl}} = 10^{-22.6}$, and $M/m_{\text{Pl}} = 0.150$. The reason why we have obtained the upper bound on the power index $\beta$ in contrast to the previous works [19, 20] is that we have considered the observed spectral index for the CMB power spectrum in the WMAP 5-year data, which significantly favors a red power spectrum.

In this paper we have only considered the scalar field perturbation and directly related it to the density perturbation, and have assumed the negative potential only in the super-inflation. Since the semiclassical LQG effects will become insignificant as the universe expands, the super inflation lasts only for a finite interval of time. After that the motion of the scalar field will be irrelevant and the potential will not be described by the negative potential, so the super-inflation might turn to the standard chaotic inflation when it ends. We would simply assume that the scalar field perturbation amplitude may be of the same order as the temperature perturbation in the observed CMB anisotropy. However, the observed CMB anisotropy is actually the temperature perturbation on the last scattering surface, so we need the consistent formulation of matter and curvature perturbations in LQC with the appropriate treatment of gauge freedom and the detailed analysis of their evolution in different scales. To formulate the consistent perturbation formulation, Bojowald et al. [30, 31] recently indicated that anomaly cancellation should occur in the effective theory and this strongly restricts the possible effective theory. From these point of view, our analysis here considered is a toy model which does not take into account backreaction and anomaly cancellation. Therefore, the present assumption that the density perturbation of the scalar field in this simplified framework is directly comparable with the observed CMB power spectrum at least in order of magnitude is to be under careful investigation. To get more robust constraint on the super-inflation scenario, we will need to use other independent observations: large scale structure, non-Gaussian, gravitational waves (cf. [32]) and so on. These problems will be our next work. As mentioned in section II, there is an open problem in introducing the characteristic scale factor $a_*$ into the flat FRW universe, in which the overall constant factor in the scale factor is just a gauge freedom. Although this problem deserves careful attention and more...
work is required to clarify the validity of the dynamical equations, we have chosen here to proceed by demonstrating a method of obtaining the observational constraint in the present scenario, which can easily be employed once progress is made on this currently uncertain section of the theory. See Ref [26, 27] for a recent interesting attempt to resolve this important issue.

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APPENDIX A: THE POWER SPECTRUM IN LQC

We calculate the power spectrum of quantum fluctuation based on the Hamiltonian (2.61). We can obtain the equation of motion for the perturbation as

$$\delta \phi'' = \left[ -2 \frac{a'}{a} + \frac{D'}{D} \right] \delta \phi' + D \left[ S \nabla^2 - a^2 \frac{d^2 V}{d \phi^2} \right] \delta \phi. \quad (A1)$$

Now, we use the following variables in Eq. (A1):

$$u = \frac{a}{\sqrt{D}} \delta \phi, \quad (A2)$$

$$m_{\text{eff}}^2 = - \left( \frac{a}{\sqrt{D}} \right)'' \frac{\sqrt{D}}{a} + a^2 D \frac{\partial^2 V}{\partial \phi^2}, \quad (A3)$$

and then we obtain

$$u'' + (SD \nabla^2 + m_{\text{eff}}^2) u = 0. \quad (A4)$$

Besides, we obtain a plane wave expansion of $u$ as

$$u = \int \frac{d^3 k}{(2\pi)^{3/2}} \left[ w_k(\tau) \hat{a}_k e^{ikx} + w_k^*(\tau) \hat{a}_k^\dagger e^{-ikx} \right], \quad (A5)$$

where $\hat{a}_k^\dagger$ and $\hat{a}_k$ are creation and annihilation operators, which satisfy the usual commutation relations

$$[\hat{a}_k, \hat{a}_k^\dagger] = \delta(k - k'), \quad [\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0. \quad (A6)$$

Substituting Eq. (A5) into Eq. (A4), we have

$$w_k'' + (SDk^2 + m_{\text{eff}}^2) w_k = 0. \quad (A7)$$

The canonical quantization for $w_k$ and its conjugate momentum requires

$$w_k \frac{dw_k}{d\tau} - w_k^* \frac{dw_k^*}{d\tau} = -i. \quad (A8)$$

We use the Fourier transformation of $\hat{u}_k(\tau)$ as

$$\hat{u}_k(\tau) = \int d^3 x e^{-ikx} u, \quad (A9)$$

and then the power spectrum $\mathcal{P}_u$ of quantum fluctuation

$$\langle 0 | \hat{u}_k^\dagger \hat{u}_k | 0 \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_u(k) \quad (A10)$$

is given by

$$\mathcal{P}_u(k) = \frac{k^3}{2\pi^2} |w_k|^2, \quad (A11)$$

where we have defined the vacuum as $\hat{a}_k | 0 \rangle = 0$.

We need to solve Eq. (A7) to calculate the right-hand side of Eq. (A11). We use the following variables:

$$\nu = -\frac{\sqrt{1 - 4m_{\text{eff}}^2 D^2}}{2 + (n + r)p}, \quad (A12)$$

$$\psi = \alpha k (\tau)^{(2 + (n + r)p)/2} = \left| \frac{2p}{2 + (n + r)p} \right| \frac{\sqrt{SDk}}{aH}, \quad (A13)$$
where we have used $\alpha = (2\sqrt{S_D}D_\lambda A^{\lambda+1})/(2 + (n + r)p)$. Using these variables, the left-hand side of Eq. (A7) is rewritten as

$$w_k'' + (SDk^2 + m_{eff}^2)w_k = \frac{d^2w_k}{d\psi^2} + \left[\frac{1}{\psi} + \frac{1}{\sqrt{SDk}}\right]\frac{dw_k}{d\psi} + \left[1 + \frac{1}{4\tau^2SDk^2} - \frac{\nu^2}{\psi^2}\right]w_k,$$

and then the solution of Eq. (A7) is given by

$$w_k(-\tau) = \sqrt{\frac{\pi}{2(2 + (n + r)p)}}\left\{d_1\sqrt{-\tau}H^{(1)}_{\nu}(\psi) + d_2\sqrt{-\tau}H^{(2)}_{\nu}(\psi)\right\},$$

(A15)

where $d_1$ and $d_2$ are constants and have the relation $|d_1|^2 - |d_2|^2 = 1$ to satisfy Eq. (A9). $H^{(1)}_{\nu}(\psi)$ and $H^{(2)}_{\nu}(\psi)$ are the Hankel functions which are given by $H^{(1)}_{\nu}(\psi) = J_{\nu}(\psi) + iY_{\nu}(\psi)$ and $H^{(2)}_{\nu}(\psi) = J_{\nu}(\psi) - iY_{\nu}(\psi)$, respectively, where $J_{\nu}(\psi)$ is the first Bessel function and $Y_{\nu}(\psi)$ is the second Bessel function. We chose the mode function so that the vacuum state becomes the Bunch-Davis like one in the short-wave-length limit $\psi \ll 1$ or $\sqrt{HD}/H \gg a/k$, (if we take $n = r = 0$ at the classical region, then the solution corresponds to the Bunch-Davis vacuum [33]). Then we get

$$w_k = \frac{(-\tau)^{-(n+r)p/2}}{\sqrt{2 + (n + r)p/ak}}e^{iak(-\tau)(2+(n+r)p)/2}.$$

(A16)

In this limit, Eq. (A15) becomes as follows,

$$w_k(-\tau) = \frac{(-\tau)^{-(n+r)p/2}}{\sqrt{2 + (n + r)p/ak}}\left\{d_1\exp[iak(-\tau)(2+(n+r)p)/2] + d_2\exp[-iak(-\tau)(2+(n+r)p)/2]\right\}.$$

(A17)

In the long-wave-length limit $\psi \ll 1$ or $\sqrt{HD}/H \ll a/k$, the first and second Bessel functions are

$$J_{\nu} \rightarrow \frac{1}{\Gamma(\nu + 1)} \left(\frac{\psi}{2}\right)^{|\nu|},$$

$$Y_{\nu} \rightarrow -\frac{\Gamma(|\nu|)}{\pi} \left(\frac{\psi}{2}\right)^{-|\nu|}.$$

(A18)

Hence $H^{(1)}_{\nu}(\psi) \sim iY_{\nu}(\psi)$, the power spectrum becomes

$$P_{\delta\phi}(k) = \frac{\Gamma(|\nu|)}{4\pi^2} \frac{p}{2 + (n + r)p} \left|\frac{H^2}{S^{3/2}D^{1/2}}aH\right|^{3-2|\nu|} k^{3-2|\nu|-1-|\nu|(n+r)p+2},$$

(A19)

where we have used $P_{\delta\phi}(k) = P_{\delta\phi}(k)D/a^2$. We define the tilt $b$ as the exponent of $k$ in the above equation [23]. To calculate this in terms of $p$, we substitute Eq. (A3) into Eq. (A12) using Eqs. (2.29), (2.30) and (2.30). Then the tilt becomes the following:

$$b = 3 - \frac{2\sqrt{9 - (6 - 4n - 3r)p2 - (12 + 4n - 2nr - r^2 - 2n^2)p^2}}{2 + (n + r)p}.$$

(A20)

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