Pfaffian control of some polynomials involving the
\( j \)–function and Weierstrass elliptic functions

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1 Introduction

In this paper, we obtain some new bounds on the number of zeros of polynomials in \( z \) and \( j(z) \), and polynomials in \( z \) and \( \wp(z) \), where \( \wp \) is a Weierstrass \( \wp \)–function associated to a lattice of the form \( (1, i\tau) \), where \( \tau \) is real. By the argument principle, the zeros of a holomorphic function in a closed, compact, simply connected region of the complex plane are controlled by the winding number of the function on its boundary – if the function is tame on the boundary, then we will obtain control over the number of zeros within the region. For \( j \) and \( \wp \), we use Pfaffian definitions of their inverses on suitable contours, and results of Khovanskii [7] to bound the zeros.

For polynomials in \( z \) and \( j(z) \), we obtain the following,

**Theorem 1.** Let \( P(X, Y) \) be a complex polynomial of degree at most \( d \) in either variable. Then \( P(z, j(z)) \) has at most \( 2^{68}d^{10} \) zeros in the standard fundamental domain.

This we believe is a new result, giving a bound on the whole (non–compact) fundamental domain. It may be compared to Binyamini’s result of a similar nature for Noetherian functions on relatively compact domains [6]. An application of this result here would yield a bound which depends on the size of a domain in which the zeros lie, and the size of \( |j| \) on it. Note also that \( j \) is \( o \)–minimal, which gives an ineffective finiteness result. The zero–bound depends polynomially on degree, and in that sense is not too far from the truth, as an obvious lower bound is \( \gg d^2 \).

It is known that the inverse of \( j(ix) \) is real and Pfaffian on the imaginary axis, a fact which may be deduced from its expression in terms of the Gaussian hypergeometric functions, which are themselves Pfaffian on an interval, see for example [6] and [2], who make use of this in the expression of elliptic functions. We make use of a result from Ramanujan’s theory of elliptic functions to alternative bases to obtain a direct expression for the inverse in terms of Gaussian hypergeometric functions. Near the cusps it approximates to the \( q^{-1} \) term in its \( q \)-expansion. Though \( j \) is a Noetherian function, it is unknown whether its real and imaginary parts are Pfaffian in the whole fundamental domain, which would directly furnish a zero–bound, so considering a contour containing the fundamental domain avoids this.

For the Weierstrass \( \wp \) functions, we restrict our attention to those with square lattice, as in this case \( \wp \) is real on the boundary of any fundamental domain, which improves the estimates under consideration, and obtain the following,

**Theorem 2.** Suppose that \( \wp(z) \) is the Weierstrass \( \wp \) function associated to the lattice \( (1, i\tau) \), where \( \tau > 0 \) is real, and let \( P(X, Y) \) be a complex polynomial of degree \( d \) in either variable. Then \( P(z, \wp(z)) \) has at most \( 8d^2 + 14d + 5 \) zeros in each fundamental domain.

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This may be compared with the non–uniform bound \( c(\phi) d^2 \) (see for example [8]), which holds for all \( \phi \). A similar result is that of Jones and Schmidt [6] – they give a Pfaffian definition of \( \phi \) on the whole fundamental domain, which yields a uniform bound of \( cd^{10} \), where \( c \) is an absolute constant. For non–real invariants, a similar result by our method of inferior quality could be obtained, as the need to take real and imaginary parts complicates the Pfaffian definitions involved.

2 Preliminaries
We first have two lemmas bounding the arc integral related to the argument principle, which we will apply to segments of a closed contour, the first to estimate where there is a dominant term, and the second in order to apply bounds on zeros of real functions.

**Lemma 1.** Suppose that \( |f(z)| > C|g(z)| \) for some \( C > 1 \) on the contour \( \Gamma \), and \( f(z), g(z) \) are non–zero on \( \Gamma \). Then

\[
\left| \int_{\Gamma} \frac{f'(z)g(z) + g'(z)f(z)}{f(z)(f(z) + g(z))} \ dz \right| \leq \left| \int_{\Gamma} \frac{f'(z)g(z)}{f(z)} \ dz \right| + C \frac{1}{|\Gamma|} \sup_{z \in \Gamma} \left\{ \frac{|f'(z)||g(z)|}{|f(z)|^2} + \frac{|g'(z)|}{|f(z)|} \right\},
\]

where \(|\Gamma|\) is the length of \( \Gamma \).

**Proof.** Considering the difference, we have

\[
\left| \int_{\Gamma} \left( \frac{f'(z)}{f(z)} - \frac{f'(z) + g'(z)}{f(z) + g(z)} \right) \ dz \right| = \left| \int_{\Gamma} \frac{f'(z)g(z) + g'(z)f(z)}{f(z)(f(z) + g(z))} \ dz \right|
\leq \int_{\Gamma} \left| \frac{f'(z)g(z)}{f(z)} - \frac{|g'(z)f(z)|}{|f(z)|} \right| \ dz
\leq \frac{|\Gamma|}{1 - \frac{1}{C}} \sup_{z \in \Gamma} \left\{ \frac{|f'(z)||g(z)|}{|f(z)|^2} + \frac{|g'(z)|}{|f(z)|} \right\}.
\]

\[\square\]

**Lemma 2.**

\[
\frac{1}{2\pi} \left| \int_{\Gamma} \frac{f'(z)dz}{f(z)} \right| \leq \#\{\text{Im}(f(z)) = 0 | z \in \Gamma\}/2 + 1
\]

and

\[
\frac{1}{2\pi} \left| \int_{\Gamma} \frac{f'(z)dz}{f(z)} \right| \leq \#\{\text{Re}(f(z)) = 0 | z \in \Gamma\}/2 + 1.
\]

The following definition is less general that of Pfaffian functions in [7], but is easier to work with (and restricted to one dimension).

**Definition 1.** Let \( f_1, \ldots, f_r \) be a sequence of analytic functions on the interval \((a, b)\). Then \( f_1, \ldots, f_r \) is a Pfaffian chain of degree \( \alpha \) if, for \( 1 \leq i \leq r \) and \( x \in (a, b) \),

\[
\frac{df_i(x)}{dx} = P(x, f_1(x), \ldots, f_i(x)),
\]

and the maximum total degree of each \( P_i \) is \( \alpha \). A Pfaffian function of order \( r \) and degree \((\alpha, \beta)\) is a function \( P(x, f_1(x), \ldots, f_r(x)) \) where \( P \) is a polynomial of total degree at most \( \beta \), and \( f_1(x), \ldots, f_r(x) \) are members of a Pfaffian chain of order \( r \) and degree \( \alpha \).
We make use of the following bound on the number of zeros of a Pfaffian function,

**Theorem 3** ([7]). Let \( f \) be a Pfaffian function of order \( r \) and degree \((\alpha, \beta)\) on the open interval \((a, b)\). Then the number of zeros of \( f \) in \((a, b)\) is at most 
\[
2^{r(r-1)/2} \beta^{r(\alpha + \beta)}.
\]

3 Polynomials in \( z \) and \( j(z) \)

Here we make use of an expression for the inverse of \( j \) in terms of the Gaussian hypergeometric function, which is given, for \(|z| < 1\), by
\[
_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n,
\]
where \((a)_n = \prod_{k=0}^{n-1} (a + k)\) is the rising factorial, and \(c\) not a non–positive integer. Gauss determined 15 relations between those \( _2F_1 \) whose parameters differ by integers. We make use of the following two relations, where we have suppressed the parameters of the function, and use the notation \( F(c \pm 1) = _2F_1(a, b, c \pm 1; z) \),

**Theorem 4** (Gauss, [5], Art. 11, Eq. 16; DLMF [1], Eq. 15.5.21).
\[
z \frac{dF}{dz} = (c-1)F(c-1) - F = z \frac{(c-a)(c-b)F(c+) + c(a+b-c)F}{c(1-z)}.
\]

Now these allow us to construct a Pfaffian chain which \( j^{-1} \) will be defined over on the imaginary axis.

**Lemma 3.** The sequence of functions
\[
\frac{1}{x}, \quad \frac{1}{1-x}, \quad \frac{F}{F(c+)}, \quad F(c+), \quad F, \quad \frac{1}{F}
\]
is a Pfaffian chain of degree 3.

**Proof.** We first note that \( \frac{1}{x} \) and \( \frac{1}{1-x} \) are Pfaffian of order 1 and degree \((2, 1)\). By Gauss’ contiguous relations, we have
\[
x \frac{dF(c+)}{dx} = c(F - F(c+))
\]
\[
\frac{dF}{dx} = \frac{(c-a)(c-b)F(c+) + c(a+b-c)F}{c(1-x)}.
\]

Consider
\[
\frac{d}{dx} \frac{F}{F(c+)} = \frac{(c-a)(c-b)F(c+) + c(a+b-c)F}{c(1-x)F(c+)} - \frac{c(F - F(c+))F}{xF(c+)^2}
\]
\[
= \frac{(c-a)(c-b)}{c(1-x)} + \left( \frac{a+b-c}{c(1-x)} - \frac{c-1}{x} \right) \frac{F}{F(c+)} - \frac{c}{x} \left( \frac{F}{F(c+)} \right)^2.
\]
Lemma 4. Let $x$ be the non-zero on this contour. Let $0 = (2, 1)$ be a Pfaffian function of order 4 and degree $3 + 2$. Further, $1/F$ is a Pfaffian function of order 6 and degree $3 + 1$. □

We make use of the following consequence, which is clear considering the above argument for $2F_1 \left( \frac{1}{6}, \frac{5}{6}, 1, \frac{1}{2} + \frac{i}{2} \right)$ and $2F_1 \left( \frac{1}{5}, \frac{5}{5}, 1, \frac{1}{2} + \frac{i}{5} \right)$.

**Lemma 4.** $\frac{2F_1(\frac{1}{5}, \frac{5}{5}, 1, \frac{1}{2} + \frac{i}{5})}{2F_1(\frac{1}{5}, \frac{5}{5}, 1, \frac{1}{2} + \frac{i}{5})}$ is a Pfaffian function of order 9 and degree $2 + 1$ on $(0, 1)$.

Now we express the inverse of $j$ on the imaginary axis in terms of the hypergeometric functions by way of the following theorem, an inversion formula from the theory of elliptic functions to alternative bases – in this case the sextic theory.

**Theorem 5 (Theorem 4.10, $r = 1$).** Let $q$ be a real number in the interval $0 < q < 1$. Then

$$q = \exp \left( -2\pi \frac{2F_1 \left( \frac{1}{5}, \frac{5}{5}, 1, 1 - x \right)}{2F_1 \left( \frac{1}{5}, \frac{5}{5}, 1, x \right)} \right),$$

where, letting $P, Q, R$ be Ramanujan’s Eisenstein series,

$$x(1 - x) = \frac{Q(q)^3 - R(q)^2}{4Q(q)^3}.$$

Letting $q = e^{2\pi i \tau}$, and $\tau$ be purely imaginary, we take

$$\alpha = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{1728}{j(\tau)}},$$

which satisfies

$$\alpha(1 - \alpha) = \frac{1728}{4j(\tau)} = \frac{Q(q(\tau))^3 - R(q(\tau))^2}{4Q(q(\tau))^3},$$

so that by Theorem 5

$$\tau = \frac{2F_1 \left( \frac{1}{5}, \frac{5}{5}, 1, \frac{1}{2} + \frac{i}{2} \sqrt{1 - \frac{1728}{j(\tau)}} \right)}{2F_1 \left( \frac{1}{5}, \frac{5}{5}, 1, \frac{1}{2} - \frac{i}{2} \sqrt{1 - \frac{1728}{j(\tau)}} \right)}.$$

So for $x \geq 1728$, letting $J(x) = j(ix)$,

$$J^{-1}(x) = \frac{2F_1 \left( \frac{1}{5}, \frac{5}{5}, 1, \frac{1}{2} + \frac{i}{2} \sqrt{1 - \frac{1728}{x}} \right)}{2F_1 \left( \frac{1}{5}, \frac{5}{5}, 1, \frac{1}{2} - \frac{i}{2} \sqrt{1 - \frac{1728}{x}} \right)}.$$

**Proof of Theorem 1.** We apply the argument principle to a truncated fundamental domain together with its copies or half–copies under $SL_2(\mathbb{Z})$ indicated by Figure 1. First consider a contour $\Gamma$ within the one indicated, containing all zeros within the original contour, such that $P(z, j(z))$ is non-zero on this contour. Let $0 < \epsilon < \min_{z \in \Gamma}|P(z, j(z))|$. Then by Rouche’s theorem, the
number of zeros of \(P(z, j(z)) + \epsilon e^{i\theta}\) within \(\Gamma\) is equal to that of \(P(z, j(z))\), for any \(\theta\). Choose \(\theta\) so that \(P(z, j(z)) := P(z, j(z)) + \epsilon e^{i\theta}\) is non–zero on the original contour – this is possible by discreteness of zeros of analytic functions.

We now bound the winding number of \(P(\epsilon, j(z))\) on this boundary. The contour is chosen to be dominated by the \(q^{-1}\) term of \(j\) near the cusps. Let \(j(it) = J(t)\). We will refer to elements of \(SL_2(\mathbb{Z})\) by \(g\), with entries

\[
\begin{pmatrix}
\tilde{a} & \tilde{b} \\
\tilde{c} & \tilde{d}
\end{pmatrix}
\]

acting on \(z\) by

\[
g(z) = \frac{\tilde{a}z + \tilde{b}}{\tilde{c}z + \tilde{d}}.
\]

The contour is composed of lines or curves near the cusps, and images of the imaginary axis under the action of some \(g\).

**Copies of the imaginary axis \(i\mathbb{R}_{\geq 1}\):** Here we have \(\text{Im}(P(g(it), j(g(it)))) = \text{Im}\left(P\left(\frac{\tilde{a}it + \tilde{b}}{\tilde{c}it + \tilde{d}}, j(it)\right)\right) = \text{Im}\left(P\left(\frac{\tilde{a}it + \tilde{b}}{\tilde{c}it + \tilde{d}}, j(it)\right)\right)\) = \(\text{Im}\left(P\left(\frac{\tilde{a}it + \tilde{b}}{\tilde{c}it + \tilde{d}}, j(it)\right)\right)\), for some real \(Q\), as \(j(it)\) is real. \(Q\) is of degree \(\leq 2d\), and has the same number of zeros as \(\text{Im}(P(g(it), j(g(it))))\).

As \(J(t)\) is \(1–1\) from \((1, \infty)\) to \((1728, \infty)\), letting \(x = J(t)\), the number of zeros of \(Q(t, J(t))\) is equal to that of \(Q(J^{-1}(x), x)\) on \(x > 1728\), and as \(\sqrt{1 - \frac{1728}{x}}\) is \(1–1\) on \(x > 1728\), letting
\[
y = \sqrt{1 - \frac{1728}{x}}, \quad \text{the number of zeros of } Q(J^{-1}(x), x) \text{ is equal to that of}
\]
\[
Q \left( J^{-1} \left( \frac{1728}{1 - y^2} \right), \frac{1728}{1 - y^2} \right) = \left( \frac{1}{1 - y^2} \right)^{2d} R \left( J^{-1} \left( \frac{1728}{1 - y^2} \right), y \right)
\]
on (0, 1), where \( R \) is of degree \( \leq 4d \). Now using the expression for \( J^{-1} \) in terms of the hypergeometric series \( _2F_1 \), we have that the number of zeros of \( \text{Im}(P(g(it), j(g(it)))) \) is bounded by that of
\[
R \left( \frac{2F_1 \left( \frac{\frac{1}{2}, \frac{3}{2}, 1, \frac{1}{2} + 
\frac{y}{2}} \right)}{2F_1 \left( \frac{\frac{1}{2}, \frac{3}{2}, 1, -\frac{y}{2}} \right)}, y \right)
\]
for \( y \in (0, 1) \). Now by Lemma \[2\] this is a Pfaffian function of order 9 and degree \( (3, 4d) \). By Theorem \[3\] the number of zeros is then bounded by \( 2^{6d}d^{10} \), so by Lemma \[2\] on this copy of the imaginary axis, we have
\[
\left| \frac{1}{2\pi} \int_{\gamma} \frac{d}{dz} (P(z, j(z))) \frac{1}{P(z, j(z))} dz \right| \leq 2^{6d}d^{10}.
\]

**Copies of \( \text{Im}(z) = Y \):** For these sections of the boundary, we have
\[
P(g(x + iY), j(g(x + iY))) = \left( \frac{1}{e(x + iY) + d} \right)^d Q(x + iY, j(x + iY)),
\]
so that
\[
\int_{\gamma} \frac{d}{dz} \left( P(g(z), j(z)) \right) dz = \int_{\gamma} \frac{d}{dz} \left( Q(z, j(z)) \right) dz - \frac{dc}{e z + d} dz.
\]
For sufficiently large \( Y, \frac{dc}{ez + d} < 0.01 \), so we consider the remaining term in \( Q \).

Letting \( j(z)^lh(z) \) be the term with the largest power of \( j \) occurring in \( Q(z, j(z)) \), where \( h(z) \) is its coefficient over the polynomials in \( z \), let the degree of \( h \) be \( n \). For sufficiently large \( Y \), \( j(x + iY)(iY)^n \) is the dominant term in \( Q(x + iY, j(x + iY)) \), i.e. \( |j(x + iY)(iY)^n| > 2|Q(x + iY, j(x + iY)) - j(x + iY)(iY)^n| \). In the same way the \( e^{-2\pi i(x+iY)} \) term in the \( q \)-expansion of \( j \) dominates \( j(x + iY) \) when \( Y \) is large, and so as \( Y \to \infty \),
\[
\frac{|Q(x + iY, j(x + iY)) - e^{-2\pi i(x+iY)}(iY)^n|}{|e^{-2\pi i(x+iY)}(iY)^n|} \to 0,
\]
and the same holds for the derivative of the numerator. So taking \( f(z) = (iY)^ne^{-2\pi iz} \) and \( g(z) = Q(z, j(z)) - f(z) \), for sufficiently large \( Y \), we may take \( C = 2 \) in Lemma \[1\] and obtain the bound
\[
\left| \int_{\gamma} \frac{d}{dz} \left( Q(z, j(z)) \right) dz \right| \leq \left| \int_{\gamma} \frac{d}{dz} \left( e^{-2\pi iz}(iY)^n \right) \right| e^{-2\pi iz}(iY)^n dz + 2 \sup_{z \in \gamma} \left\{ \left| f'(z) \right| |g(z)| + \left| g'(z) \right| \right\}
\]
\[
\leq 2\pi l + 0.01
\]
\[
\leq 2\pi d + 0.01.
\]
We also take \( Y \) sufficiently large to ensure all zeros in the fundamental domain (and its copies) are interior to the contour, which is possible by virtue of the dominating term in \( P(z, j(z)) \) at the various cusps (or by the \( o \)-minimality of \( j \) implying there are only finitely many). Finally, the integral over the entire contour is bounded in absolute value by \( 8 \cdot 2^{6d}d^{10} + 10d + 0.2 \leq 2^{68}d^{10} \), as there are 8 copies of the imaginary axis, and 10 copies or half-copies of the line \((-1/2 + iY, 1/2 + iY)\).
4 Polynomials in $z$ and the Weierstrass $\wp$–function

Here we make use of the following theorem of Khovanskii,

**Theorem 6** (§2.3, Theorem 2, [7]). Let $G : R^{n+1} \to R^1$ be a smooth function with nondegenerate level set $M^n$. Let $F : R^{n+1} \to R^n$ be a smooth proper map, and $\hat{F} : M^n \to R^n$ its restriction to $M^n$. Let, further, $\hat{J}$ be any smooth function on $R^n$ that coincides on $M^n$ with the Jacobian $J$ of the map $(F, G) : R^{n+1} \to R^n \times R^1$. Under these conditions the following holds: the maximum number of nondegenerate preimages of any point in the range of of the map $\hat{F} : M^n \to R^n$ is bounded by that of the map $(F, \hat{J}) : R^{n+1} \to R^n \times R^1$.

We apply this to a polynomial in $x$ and $\wp^{-1}(x)$, where $\wp$ is a Weierstrass $\wp$–function, and use the argument principle to bound the zeros of $P(z, \wp(z))$ in its fundamental domain. We consider $\wp$ with lattices of the form $(1, i\tau)$, $\tau > 0$ real. The bound on the number of zeros follows in a similar way to §2.3 Theorem 1 of [7] – we proceed in this manner to give a better bound than simply applying Theorem 6.

First we have the differential equation satisfied by $\wp$,

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

and $\wp(x) := \wp(ix)$ satisfies the equation

$$\wp'(x)^2 = -4\wp(x)^3 + g_2\wp(x) + g_3$$

where $g_2, g_3$ are the Weierstrass invariants of $\wp$, which are real as $\wp$ is associated to the lattice $(1, i\tau)$. For the derivatives of the inverses of $\wp(x)$ and $\wp(x)$, we have the expressions

$$\frac{d}{dx}\wp^{-1}(x) = \frac{1}{\wp'(\wp^{-1}(x))} = \frac{1}{\sqrt{4x^3 - g_2x - g_3}},$$

$$\frac{d}{dx}\wp^{-1}(x) = \frac{1}{\wp'(\wp^{-1}(x))} = \frac{1}{\sqrt{-4x^3 + g_2x + g_3}}$$

where the sign of the square root depends on the branch of the inverse of $\wp$ or $\wp$ which is under consideration. Note that both expressions will be real in the domains under consideration.

**Proposition 1.** Let $\wp$ have lattice $(1, i\tau)$, where $\tau > 0$ is real, and $P(X, Y)$ be a complex polynomial of total degree at most $d$. Then the number of zeros of $\text{Im}(P(z, \wp(z)))$ on the open line $(\beta, \beta + \gamma)$, $\gamma \in \{1, i\tau\}$ between two adjacent poles of $\wp$ is bounded by $4d^2 + 6d + 1$.

**Proof.** By periodicity, $\wp(z) = \wp(z - \beta)$, so we may take a transformation of $z$ to consider the lines $(0, 1)$, and $(0, i\tau)$. As $\wp$ has lattice $(1, i\tau)$, and $\tau$ is real, it is real on these lines, and $\text{Im}(P(z, \wp(z))) = Q(x, \wp(\gamma x))$, $\gamma \in \{1, i\}$ for some real polynomial $Q(X, Y)$ of degree at most $d$. The argument proceeds identically for either line, so we consider $(0, 1)$.

Let $x_0 \in (0, 1)$ be such that $\wp'(x_0) = 0$. Then $\wp(x)$ is $1$–$1$ on the interval $(0, x_0)$, so the number of zeros of $Q(x, \wp(x))$ is equal to that of $Q(\wp^{-1}(x), x)$ on the interval $(\wp(x_0), \infty)$. Considering the system

$$Q(u, x) = 0$$
$$u - \wp^{-1}(x) = 0$$

we have, letting $J$ be the Jacobian of the system $(Q, u - \wp^{-1}(x))$, by Theorem 6 that the number of nondegenerate solutions the system is bounded by an upper bound of the number of
nondegenerate preimages of any point in the range of the system

\[
Q(u, x) \\
J(u, x).
\]

Letting \( Q(X, Y) = \frac{\partial}{\partial X} Q(X, Y) \), and similarly for \( Q_Y(X, Y) \), \( J(x, u) \) is given by

\[
\frac{Q_X(u, x)}{\sqrt{4x^3 - g_2x - g_3}} + Q_Y(u, x).
\]

Taking some point \((a, b)\) in the range of the system \((Q, J)\), we bound the number of nondegenerate preimages. If \( J(x, u) = b \), then

\[
J(u, x) = \frac{Q_X(u, x)}{\sqrt{4x^3 - g_2x - g_3}} + Q_Y(u, x) = b,
\]

and this holds iff

\[
Q_X(u, x) + (Q_Y(u, x) - b)\sqrt{4x^3 - g_2x - g_3} = 0,
\]

and if this holds, then

\[
(Q_X(u, x) + (Q_Y(u, x) - b)\sqrt{4x^3 - g_2x - g_3}) \\
\cdot (Q_X(u, x) - (Q_Y(u, x) - b)\sqrt{4x^3 - g_2x - g_3})
\]

holds, so that the number of preimages of \((a, b)\) is bounded by the number of nondegenerate solutions of

\[
Q(u, x) = a = 0
\]

\[
Q_X(u, x)^2 - (Q_Y(u, x) - b)^2(4x^3 - g_2x - g_3) = 0,
\]

which by Bézout’s theorem is bounded by \(2d^2 + 3d\). The interval \((x_0, 1)\) is similar.

\[\square\]

Proof of Theorem 2. Let \( P(z) := P(z, \varphi(z)) \). We first take a box \( B \) within the fundamental domain such that all zeros of \( P \) in the interior of the fundamental domain lie within \( B \). Next,
define $\epsilon, \theta$ such that $0 < \epsilon < \min_{\partial B} |P|$, and $P_\epsilon(z) := P(z) + \epsilon e^{i\theta} \neq 0$ for $z \in \partial F$. By Rouché’s theorem, $P_\epsilon(z)$ has the same number of zeros in $B$ counting multiplicity as $P(z)$.

We now bound the number of zeros of $P_\epsilon$ within the contour $\Gamma$ given by the truncations of the lines on the boundary of $F$ united with interior quarter–circles about the poles of $\wp(z)$, as indicated in Figure 2 – the common radii of these circles is taken so that there are no zeros of $P_\epsilon$ in a disc of this radius about the poles, and such that they do not intersect $B$. The radii will be taken sufficiently small subject to this. As $B$ lies in the interior of $\Gamma$, a bound upon the number of zeros of $P_\epsilon$ within $\Gamma$ bounds the number within $B$, and so bounds that of $P(z, \wp(z))$ within the fundamental domain.

By the argument principle, the number of zeros of $P_\epsilon$ within $\Gamma$ is equal to

$$
\frac{1}{2\pi i} \int_{\Gamma} \frac{P_\epsilon'(z)}{P_\epsilon(z)} \, dz,
$$

which we now estimate. As $\wp(z)$ is real on the lines of $\Gamma$, considering for example the line $z = ix$, for $x$ real,

$$
\text{Re}(P_\epsilon(z)) = \text{Re}(P(ix, \wp(ix))) + \epsilon \cos(\theta) = Q(x, \wp(ix)).
$$

The number of zeros of which is, by Proposition 1, bounded by $4d^2 + 6d + 1$, and holds for any radius of quarter-circles. The other lines are similar, and so our bound for the integral over all of the lines is $8d^2 + 12d + 6$ by Lemma 2.

On the quarter–circles, we may expand $P_\epsilon$ into its Laurent series – given a sufficiently small radius $\delta$, the term of smallest power will dominate. We let $\gamma$ be the path $p + \delta e^{i\xi}$, where $\xi$ ranges over the appropriate interval of length $\pi/2$ in $[0, 2\pi]$, for the particular pole $p$, to be the interior quarter–circle. This path has length $\delta \pi/4$. Writing

$$
P_\epsilon(z) = a_k(z - p)^k + \sum_{n=k+1}^{\infty} a_n(z - p)^n,
$$

we have, as the radius $\delta \to 0$,

$$
\frac{P_\epsilon(z) - a_k(z - p)^k}{a_k(z - p)^k} \to 0,
$$

$$
\frac{\frac{d}{dz}(P_\epsilon(z) - a_k(z - p)^k)}{a_k(z - p)^k} \to \frac{(k + 1)a_{k+1}}{a_k}.
$$

So for sufficiently small $\delta$, we have, letting $f(z) = a_k(z - p)^k$, and $g(z) = P_\epsilon(z) - f(z)$, by Lemma 1 with $C = 2$,

$$
\frac{1}{2\pi} \left| \int_{\gamma} \frac{P_\epsilon'(z)}{P_\epsilon(z)} \, dz \right| \leq \frac{1}{2\pi} \left| \int_{\gamma} \frac{k}{z - p} \, dz \right| + \delta \frac{1}{2} \left( \frac{(k + 1)a_{k+1}}{a_k} + 0.1 \right)
$$

$$
\leq \frac{|k|}{4} + 0.1
$$

$$
\leq d/2 + 0.1,
$$

where the last inequality follows from the fact that $\wp$ has poles of order 2, so $-2d \leq k \leq d$. As there are 4 quarter–circles about the poles of $\wp$, and 4 line segments, the absolute value of the whole integral is bounded by $8d^2 + 14d + 7$. 

\[ \square \]
References

[1] NIST digital library of mathematical functions. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[2] R. Bianconi. Some model theory of hypergeometric and Pfaffian functions. *South Amer. J. Log.*, 2(2):297–318, 2016.

[3] G. Binyamini. Density of algebraic points on Noetherian varieties. *Geom. Funct. Anal.*, 29(1):72–118, 2019.

[4] S. Cooper. Inversion formulas for elliptic functions. *Proc. Lond. Math. Soc. (3)*, 99(2):461–483, 2009.

[5] C. F. Gauss. Circa seriem infinitam 1 + \( \frac{\alpha x}{1.\gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)} x^3 + \cdots \) etc. *Commentationes societatis regiae scientiarum Gottingensis recentiores*, II, 1813.

[6] G. Jones and H. Schmidt. Pfaffian definitions of weierstrass elliptic functions. *Mathematische Annalen*, 2020.

[7] A. G. Khovanski˘ı. *Fewnomials*, volume 88 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1991. Translated from the Russian by Smilka Zdravkovska.

[8] D. Masser. *Auxiliary Polynomials in Number Theory*. Cambridge Tracts in Mathematics. Cambridge University Press, 2016.