Regular Kiselev black hole with a de Sitter core

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Abstract

We present some new aspects of Kiselev black hole and then study the null and timelike thin shell collapse in this spacetime. For the latter, we show that Kiselev black hole can be matched to de Sitter core with a thin timelike dust shell to produce a non-singular black hole space-time. It is argued that for timelike hypersurface, the equation of state parameter must be non-negative. Using Barrabes-Israel junction conditions, the equation of motion of the shell is obtained. The stability of stationary solutions of the shell is discussed and some appropriate ranges for the parameters of shell and Kiselev geometry are found for which a stable stationary black hole is constructed.

1 Introduction

Existence of black holes (BHs), as one of the predictions of general relativity, have drawn many attentions in theoretical physics [1]. Observing the first image of a BH in the nearby radio galaxy, M87, by the event horizon telescope collaboration [2] for the first time, makes the subject even more interesting. Although it is almost clear to empirical physicists that BHs exist, the interesting problem of their inner structure is not well known. This is partly because of the masking effect of the event horizon. It is believed that the non-eternal BHs can be formed as a consequence of gravitational collapse of a star. According to the Hawking-Penrose singularity theorem [3], in general relativity, the gravitational collapse of reasonable matter leads to geodesically incomplete (i.e. singular) space-time such that this singularity is not naked, i.e. remains hidden behind the event horizon. This is the
result of weak cosmic censorship conjecture. Therefore, imposing some exotic conditions on matter or concerning other extended theories of gravity, the BH singularity may be avoided. This yields a regular BH solution with event horizon but without singularity.

Inspired by Sakharov’s work, who proposed the idea of replacing the Schwarzschild singularity with de Sitter vacuum [4], Bardeen introduced the first ever regular BH [5]. Bardeen solution describes a static spherically symmetric space-time where for small (large) enough radial coordinate, approaches de Sitter (Schwarzschild) space-time. Coupling Einstein equations to a new nonlinear electrodynamics, Ayon-Beato and Garcia [6] generated Bardeen BH from a nonlinear magnetic monopole [7]. Also, they proposed [6] a nonsingular exact BH solution where its corresponding source is a nonlinear electrodynamics satisfying the weak energy condition and in the weak field limit becomes the Maxwell field. Later on, Bronnikov [8] demonstrated that general relativity coupled to some nonlinear electrodynamics where the Lagrangian is a well-defined function of the Maxwell lagrangian, leads to a regular metric if and only if the electric charge is zero. This means that regular solutions can exist with a non-zero magnetic charge. An interesting minimal model of BHs of this type is Hayward BH [10]. Other similar proposals of regular BHs are found in [11].

In the above mentioned regular BHs models, the metric is defined throughout the spacetime without any junction conditions. However there are regular BHs constructed by joining two regions of spacetime, the inner is described by a regular metric and the outer is a known BH solution. These are matched to each other by a smooth junction, boundary surface [12–15], or through a surface layer, thin shell [16–20] which is of interest here. Using Barrabes-Israel junction conditions [21], two distinct space-times can be attached to each other with a timelike, spacelike or null hypersurface. Assuming some universal upper limit for the curvature, Frolov and collaborators [16] proposed a non-singular BH model by matching the Schwarzschild metric to a de Sitter one with a thin spacelike shell. They assumed that as the curvature reaches its upper value, the matter turns into a de Sitter phase and this transition is made through a spacelike thin shell. The stability of their solution is discussed by Balbinot and Poisson [17]. Fitting of de Sitter space into a Schwarzschild BH with a spacelike surface layer of constant curvature is done in [18]. The intrinsic structure of the layer is obtained and

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1. This is in the case that the lagrangian has a correct weak field limit and tends to a finite limit as Maxwell lagrangian goes to infinity.
2. For a classification of different types of regular, asymptotically flat, static and spherically symmetric BHs see [9].
it is shown that the fitting can not be occurred through a boundary surface. As an important result, Poisson and Israel [22] demonstrated that, due to the violation of junction conditions, the Schwarzschild spacetime cannot be matched directly to the de Sitter one with a null hypersurface and a thin shell is required. The matching is done later by Barrabes and Israel [21] and then discussed by many authors, see [23] for a detailed analysis.

Another example is provided by matching a Reissner-Nordström BH to a regular de Sitter core [19] by a dust timelike thin shell where its radius is a function of its proper time such that at a specific radius, the transition between two space-times occurs. Then the stability of solutions is examined and it is shown that solutions with negative shell mass cannot be stable. Taking the massless limit of the shell, the result is the same as obtained before in [23] with a boundary surface. Recently, this work is extended by considering a material layer with pressure in [20].

In this paper, we employ Barrabes-Israel junction conditions to construct a new regular BH space-time. The outer metric is given by Kiselev BH [24] and the core is de Sitter space-time. Also the thin shell is chosen to be a dust timelike hypersurface. The outline of this paper is as follows: In section 2 we describe step by step how one can derive a generalized Kiselev metric. This comes from the fact that we have not restricted ourselves to the linear equation of state for the matter. In section 3, the gravitational collapse of a shell of radiation is studied in Kiselev background. Section 4 is devoted to the main problem of the paper, the gravitational collapse of a timelike shell in Kiselev metric where the interior space-time is de Sitter. After a brief review on the Barrabes-Israel junction conditions in 4.1, section 4.2 is devoted to derive the equation of motion of the thin shell. Section 4.3 deals with shell stability and in section 5 we review highlights of the paper.

Throughout this paper, the signature of the metric tensor is assumed to be $(-,+,+,+)$. Greek indices ($\alpha$, $\beta$, ...) are used to label the four dimensional spacetime described by the metric components $g_{\mu\nu}$ and Latin indices (a, b, ...) are reserved for objects live on the hypersurface $\Sigma$ defined by the three dimensional induced metric $h_{ab}$. The symbol $; \Sigma$ and $\mid_{\Sigma}$ are used to indicate the covariant derivatives in four and three dimension respectively. A dot denotes the derivative with respect to the proper time. For any tensorial quantity like $A$ defined on both sides of $\Sigma$, the notation $[A] \equiv A^{\Sigma} \mid_{\Sigma} - A_{\Sigma} \mid_{\Sigma}$ assigns the jump of the $A$ across $\Sigma$. Einstein field equations coupling constant is given by $\kappa \equiv 8\pi G/c^4$ where $G$ and $c$ are Newton constant and speed of light respectively.
2 Kiselev BH

Kiselev metric firstly proposed in [24] to describe a static spherically space-time in the presence of an anisotropic fluid. It is a well-known fact that in the case of a spherically symmetric space-time of the form

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2 \]  

Einstein equations become linear in \( f \) and a simple calculation gives the components of energy-momentum tensor as

\[ T_{t}^{t} = T_{r}^{r} = -\frac{1}{\kappa r^2} (f + rf' - 1) \]  

\[ T_{\theta}^{\theta} = T_{\phi}^{\phi} = -\frac{1}{2\kappa r} (2f' + rf'') \]

The first equalities in (2) and (3), do not hold for a perfect fluid except for the case of cosmological constant. To satisfy these equations, kiselev’s idea is to construct an energy-momentum tensor via the following steps

- Write a general spherically symmetric energy-momentum tensor in Cartesian coordinate system

\[ T_{\mu}^{\nu} = \begin{pmatrix} A(r) & 0 \\ 0 & C(r) r_{i}r_{j} + B(r)\delta_{i}^{j} \end{pmatrix} \]  

- Take average of it over angles

\[ \langle T_{\mu}^{\nu} \rangle = \begin{pmatrix} A(r) & 0 \\ 0 & \left( \frac{1}{3}r^2C(r) + B(r) \right) \delta_{i}^{j} \end{pmatrix} \]

to obtain the energy-momentum tensor of a perfect fluid. Therefore

\[ A(r) = -\rho(r) \]  

\[ \frac{1}{3}r^2C(r) + B(r) = p(r). \]

\[^{3}\text{It is mentioned in [24] that this BH is surrounded by some quintessential matter in the form of a perfect fluid with a linear equation of state, } p(r) = \omega \rho(r). \text{ The functions } \rho(r) \text{ and } p(r) \text{ are the energy density and pressure of matter such that their ratio, } \omega, \text{ is an arbitrary constant. However a direct substitution of Kiselev metric into the Einstein equations gives a specific anisotropic fluid as the source term [25][26].}\]

\[^{4}\text{Except for the case of a cosmological constant where the mentioned fluid is isotropic.}\]
• Write (4) in spherical coordinates by a coordinate transformation. The result will satisfy first equalities of (2) and (3) if

\[ C(r)r^2 + B(r) = -\rho(r). \quad (8) \]

• Read the unknown functions \( A(r), B(r) \) and \( C(r) \) from (6)-(8) and then find the energy-momentum tensor in spherical coordinates. This yields

\[ T_{\mu\nu} = \text{diag} \left[ -\rho(r), -\rho(r), \frac{1}{2}(\rho(r) + 3p(r)), \frac{1}{2}(\rho(r) + 3p(r)) \right]. \quad (9) \]

• Substitute \( T_{\mu\nu} \) in Einstein equations, (2) and (3), and find the functions \( f(r), \rho(r) \) and \( p(r) \). To do this, either one of these functions or a relation between two of them is needed. Specifying the equation of state, \( p = p(\rho) \) is an example of the latter case.

Following the above mentioned steps, for a linear equation of state \( p(r) = \omega\rho(r) \), one finally arrives at the following expressions for the metric coefficient of Kiselev metric and the corresponding radial and transverse pressures

\[ f = 1 - \frac{2m}{r} - \frac{c}{r^{3\omega+1}} \]

\[ \rho = -p_r = \frac{3\omega}{\kappa r^{3(\omega+1)}}, \quad p_t = T_\theta^\theta = T_\phi^\phi = -\frac{3\omega(1 + 3\omega)}{2\kappa r^{3(\omega+1)}}. \quad (11) \]

where \( 2m \) (i.e. the Schwarzschild radius), and \( c \) are integration constants.

Interestingly, one can find out that although the source fluid of Kiselev BH is anisotropic, i.e. \( p_r \neq p_t \), the average pressure \( \bar{p} \) satisfies a linear equation of state \[ \bar{p} = p_r + 2p_t = \omega \rho. \quad (12) \]

The weak energy condition implies \( \rho \geq 0 \), therefore from (11) the multiplication of \( \omega \) and \( c \) must be negative. This also leads to a negative radial pressure in contrast to the transverse pressures. In the next sections, we

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5In [26], the authors have used the equation of state of modified Chaplygin gas and found the analytical expressions for energy density, pressure and metric coefficient.

6Although this metric is time independent, it is usually called that Kiselev metric is sourced by quintessential dark energy if \(-1 \leq \omega < -\frac{1}{3}\). This only means that one deals with scales more smaller than the cosmological scale.

7For a detailed discussion on energy conditions of energy momentum tensor supporting the Kiselev BH see [27].
shall see that this condition plays a central role in studying a collapsing shell in Kiselev space-time.

In summary, the mentioned steps allow one to find a family of solutions that become unique for a specific equation of state of matter. Combining (2), (3) and (9), one gets

$$f = 1 - \frac{a}{r} - \frac{\kappa}{r} \int_a^r dr \, r^2 \rho(r), \quad \frac{r \, d\rho}{3 \, dr} = \rho + p$$

(13)

where the horizon is assumed to be $r = a$. Above equations are linear so their solutions corresponding to the different energy density, can be superposed [28]. For the case of a linear equation of state, this means that for a sum of different sources with different values of state parameters, the corresponding coefficient of metric would be

$$f = 1 - \frac{2m}{r} - \sum_n \frac{c_n}{r^{3\omega_n+1}}$$

(14)

Here, it is instructive to write Kiselev metric (10) in other coordinate systems. Introducing the tortoise radial coordinate $r^* = \int f(r)^{-1} dr$ and a new coordinate $\tau(t, r) = t + \psi(r)$, the different forms of Kiselev metric are summarized in table 1.

| $\psi(r)$ | Name | Line Element |
|-----------|------|--------------|
| $er^*$    | Eddington-Finklestein | $ds^2 = -(1 - \frac{2m}{r} - \frac{c}{r^{3\omega+1}}) \, d\tau^2 + 2\epsilon d\tau dr + r^2 d\Omega^2$ |
| $\int \frac{1+\epsilon f(r)}{f(r)} dr$ | Kerr-Schild | $g_{\alpha\beta} = \eta_{\alpha\beta} + \left(\frac{2m}{r} + \frac{c}{r^{3\omega+1}}\right) \times \frac{\partial_{\alpha}(\tau + \epsilon r)}{\partial_{\beta}(\tau + \epsilon r)}$ |
| $\int \frac{\sqrt{1+\epsilon f(r)}}{f(r)} dr$ | Painleve-Gullstrand | $ds^2 = -d\tau^2 + \epsilon \left( dr + \sqrt{\frac{2m}{r} \frac{c}{r^{3\omega+1}} \, d\tau} \right)^2 + r^2 d\Omega^2$ |

Table 1: Different choices of $\psi(r)$ lead to different coordinates. Ingoing (outgoing) geodesics correspond to $\epsilon = 1$ ($\epsilon = -1$).
At the end of this section, we present a qualitative description of the number and places of the event horizons for Kiselev space-time. It is clear from (10) that one can not determine the Kiselev horizons for arbitrary values of $\omega$ analytically therefore many authors treat this solution by selecting some particular values for $\omega$. For the case of $\omega = -\frac{2}{3}$, a detailed analysis of null geodesics is done in [29] and the structure of horizon is discussed in [30] and [31]. Moreover it is shown that this choice of $\omega$ gives a Nariai type BH [32].

Introducing some dimensionless variables $u \equiv \frac{r}{2m}$ and $\tilde{c} \equiv \frac{c}{(2m)^{\omega+1}}$, therefore the sign of $\tilde{c}$ is the same as $c$, $u$ is positive and the metric component can be written as

$$f = 1 - \frac{1}{u} - \frac{\tilde{c}}{u^{3\omega+1}}. \quad (15)$$

The horizon is now at $u = u_0 = \text{const}$ where $\tilde{c} = u_0^{3\omega}(u_0 - 1)$. Combining this with the positiveness of the energy density condition mentioned above, the multiplication of $\omega$ and $\tilde{c}$ must be negative. This gives

$$\omega(1 - u_0) > 0. \quad (16)$$

Equation (16) reveals that for $\omega > 0$, the Kiselev BH horizon(s) is (are) larger than $2m$ and vice versa. Moreover, note that the extremum of $f$ is at $\tilde{u} = [-1 + 3\omega]\tilde{c}^{1/3\omega}$. Solving this for $\tilde{c}$ and substituting it into the condition $\tilde{c}\omega < 0$, one finds that

$$\frac{\omega}{3\omega + 1} > 0. \quad (17)$$

Thus function $f$ has no extreme within $-1/3 < \omega < 0$ whereas for other values of $\omega$, it has exactly one extremum. Putting these all together, we can divide the parameter space, $\tilde{c}$ and $\omega$, into different regions depending on the number of horizons and the positivity of energy density. First, consider the case that $-1/3 < \omega < 0$. For this interval, $\lim_{u \to 0^+} f(u) = -\infty$ and $\lim_{u \to \infty} f(u) = 1$. Thereby, Kiselev BH is not naked and has exactly one horizon. For the case of $\omega < -1/3$ or $\omega > 0$, the BH has at most two horizons. The extremal case occurs once we have $f(u_0) = 0$ and this means that $\tilde{c}$ has the following value [32]

$$\tilde{c}_{\text{ext}} = -\frac{1}{3\omega + 1} \left( \frac{3\omega}{3\omega + 1} \right)^{3\omega}. \quad (18)$$
If \( \omega < 0 \) and \( \tilde{c} > \tilde{c}_{ext} \), then \( f(u) > 0 \) and the BH is naked. A similar argument can be applied when \( \omega > 0 \) and \( \tilde{c} < \tilde{c}_{ext} \). Figure 1 presents a summary of the results in the parameter space.

![Figure 1: The properties of Kiselev BH in parameter space (\( \tilde{c}, \omega \)).](image)

In the next sections, we are interested in studying a collapsing spherical thin shell, both null and timelike, in Kiselev space-time.

### 3 Null thin shell collapse

Here we consider the simplest model of gravitational collapse which is a collapsing thin shell of null matter. It is convenient to use the ingoing Eddington-Finkelstein coordinates which are adopted to the ingoing null geodesics. We assume that the geometry is flat inside the shell and its...
exterior space-time is described by Kiselev metric. Therefore

\[ ds^2 = -f(r)dv^2 + 2dvdr + r^2d\Omega^2, \quad f(r) = \begin{cases} \frac{2m}{r} - \frac{c}{r^{3\omega+1}} & v \geq v_0 \\ 1 & v < v_0 \end{cases} \] (19)

where \( v = t + \int f(r)^{-1}dr \). Suppose that the shell moves along the null trajectory \( v = v_0 \) in both space-times, inside and outside the shell. Therefore

\[ f(v, r) = 1 - \left( \frac{2m}{r} - \frac{c}{r^{3\omega+1}} \right) \Theta(v - v_0) \] (20)

in which \( \Theta \) is the step function. This is a particular case of Vaidya generalization of Kiselev metric [33] defined by \( m(v) = m\Theta(v - v_0) \) and \( c(v) = c\Theta(v - v_0) \). Moreover, the above mentioned metric for \( \omega \neq -1 \) is a special case of a large family of dynamical BH introduced in [34]. Substituting (19) into Einstein equations gives the following non-vanishing components for energy-momentum tensor

\[ T_{v v} = \frac{1}{\kappa} \left( \frac{2m}{r^2} + \frac{c}{r^{3\omega+2}} \right) \delta(v - v_0) - \frac{3c\omega}{\kappa r^{3(\omega+1)}} \left[ 1 - \left( \frac{2m}{r} - \frac{c}{r^{3\omega+1}} \right) \Theta(v - v_0) \right] \] (21)

\[ T_{r r} = \frac{3c\omega}{\kappa r^{3(\omega+1)}} \Theta(v - v_0) \] (22)

\[ T_{\theta \theta} = -\frac{3c\omega(1 + 3\omega)}{2\kappa r^{3\omega+1}} \Theta(v - v_0), \quad T_{\phi \phi} = \sin \theta T_{\theta \theta} \] (23)

By introducing two future-pointing null vectors \( v_\mu = (1, 0, 0, 0) \) and \( w_\mu = (g_{vv}/2, -1, 0, 0) \) [35], one can write the above energy-momentum tensor as

\[ T_{\mu \nu} = \left( \frac{2m}{\kappa r^2} + \frac{c}{\kappa r^{3\omega+2}} \right) \delta(v - v_0) v_\mu v_\nu + \left( \rho + p_t \right) (v_\mu w_\nu + v_\nu w_\mu) \Theta(v - v_0) \] (24)

\[ \left( \rho + p_t \right) (v_\mu w_\nu + v_\nu w_\mu) + p_t g_{\mu \nu} \] (25)

It is evident from this relation that the energy flows only along the null direction \( w_\mu \) since \( T_{\mu \nu} v^\mu v^\nu = 0 \). As expected, in the static case, the above energy momentum tensor reduces to

\[ T_{\mu \nu} = \left( \rho + p_t \right) (v_\mu w_\nu + v_\nu w_\mu) + p_t g_{\mu \nu}. \] (25)

This is the source of Kiselev space-time and as mentioned before, it has not the form of a perfect fluid energy-momentum tensor.

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8 Setting the arbitrary functions and parameters of [34] as: \( M(v) = m\Theta(v - v_0) \), \( k = -(1 + 3\omega)/2 \) and \( C(v) = -3\omega c \Theta(v - v_0)/8\pi \), the metric [20] is resulted.
4 Timelike thin shell collapse

Here, we want to consider the gravitational collapse of a timelike spherical thin shell in Kiselev space-time. In contrast to the null case, there is no a single coordinate covering both regions, inside and outside the shell and one should introduce two different coordinates. This means that one has to apply the Barrabes-Israel formalism \cite{21} to join two space-times separated by the shell and determine the surface energy-momentum of it. Below, first we review Barrabes-Israel junction conditions briefly and then we join outer Kiselev and inner de Sitter space-times assuming the shell is made of pressureless matter.

4.1 Junction Conditions

Let \( \Sigma \) be a timelike hypersurface that partitions space-time \( \mathcal{V} \) into two parts \( \mathcal{V}^\pm \). In region \( \mathcal{V}^\pm \), the metric and coordinate charts are \( g^\pm_{\alpha\beta} \) and \( x^\pm_\alpha \) respectively. The unit normal vector to \( \Sigma \) is \( n^\alpha \) pointing from \( \mathcal{V}^- \) to \( \mathcal{V}^+ \) and defined such that

\[
 n^\alpha n_\alpha = 1, \quad n_\alpha e^\alpha(a) = 0
\]

where \( e^\alpha(a) \) are three basis vectors on \( \Sigma \) and have zero jump across \( \Sigma \), i.e. \([e^\alpha(a)] = 0\). The first junction condition dictates the continuity of the metric across \( \Sigma \): \( [g_{\alpha\beta}] = 0 \). Defining the induced metric on \( \Sigma \) as \( h_{ab} = g_{\alpha\beta} e^\alpha(a) e^\beta(b) \), this condition can be written as \( [h_{ab}] = 0 \). The second junction condition relates the energy-momentum tensor of \( \Sigma \) to the discontinuity of extrinsic curvature, \( K_{ab} \),

\[
 [K_{ab}] = \kappa \left( S_{ab} - \frac{1}{2} h_{ab} S \right)
\]

where \( S_{ab} \) is the energy-momentum of the surface layer \( \Sigma \) defined as \( T_\Sigma^{\alpha\beta} = \delta(\tau) S_{ab} e^\alpha(a) e^\beta(b) \) and the traces of \( K_{ab} \) and \( S_{ab} \) are indicated by \( K \) and \( S \) respectively.

Now, let us find the equation of motion of the shell. To do so, it is straightforward to verify that the energy momentum conservation equation on the hypersurface reduces to

\[
 S^a_{b(a)} + \left[ T_{\alpha\beta} e^\alpha_b n^\beta \right] = 0.
\]

Here we restrict ourselves to the case that the shell is composed of a pressureless perfect fluid. We will show that such surface energy momentum tensor is required to have a smooth transition across the layer. So, assume

\[
 S_{ab} = \sigma u_a u_b
\]
where $\sigma$ is the surface energy density of the shell and $u^a$ is the three-velocity of it. Inserting (29) into (28) leads to

$$
(\sigma u^a)_{|a} = \left[ T_{\alpha\beta} u^\alpha n^\beta \right].
$$

(30)

The equation of motion of the shell can be found by calculating its acceleration

$$
a^\alpha \equiv u^\alpha_{;\beta} u^\beta = a^\alpha e^\alpha_{(\alpha)} + u^a u^b K_{ab} n^\alpha.
$$

(31)

Projecting it along the layer gives an internal motion of the shell while its normal component $n_{\alpha} a^\alpha = u^a u^b K_{ab}$ describes the motion of the shell. It is also evident that the jump of the normal acceleration, $n_{\alpha} a^\alpha$, across $\Sigma$ is related to the jump of extrinsic curvature. Therefore, making use of (27), we are able to find the shell equation of motion as follows

$$
[n_{\alpha} a^\alpha] = \frac{\kappa}{2} \sigma.
$$

(32)

In the next section we utilize (30) and (32) to investigate a collapsing timelike shell in Kiselev space-time.

### 4.2 The motion of a collapsing timelike shell

Here, we study the collapsing of a timelike thin shell immersed in Kiselev spacetime. To do this, we consider that the space-time inside the shell is described by de Sitter geometry. In this way, we can show that the Kiselev BH can be matched to the de Sitter core by a timelike dust shell and therefore, in principle, an infinite number of stationary non-singular BH can be constructed. Each of which is labeled by parameter $\omega$. We will return to this point in the next section.

In order to make things concrete, we will write the metric in both regions as

$$
\begin{align*}
(ds^2)^\pm &= -f_\pm(r)dt^2 + f_\pm^{-1}(r)dr^2 + r^2 d\Omega^2. \\
\end{align*}
$$

(33)

where the functions $f_\pm$ are defined as

$$
\begin{align*}
f_- (r) &= 1 - \frac{\Lambda}{3} r^2, \quad r < R(\tau) \\
f_+ (r) &= 1 - \frac{2m}{r} - \frac{e}{r^{2\omega+1}}, \quad r > R(\tau)
\end{align*}
$$

(34)

$\Lambda$ is the cosmological constant and the shell radius is denoted by $R(\tau)$ parameterized by the proper time, $\tau$, of comoving particle on the shell. The line element on $\Sigma$ is then given by

$$
(ds^2)_\Sigma = -d\tau^2 + R(\tau)^2 d\Omega^2
$$

(35)
The hypersurface $\Sigma$ is assumed to be timelike throughout the space-time, i.e. $n^\alpha n_\alpha = 1$. Thus the shell radius must be smaller than the de Sitter horizon $L = \sqrt{3} \Lambda$. Regarding the region $V^+$, as mentioned before, the positivity of the energy density requires $\omega < 0$. In the case of $\omega > 0$, the function $f_+(r)$ blows up at $r \to 0$ and tends to 1 at enough large values of $r$. This means, either we have a naked BH, which we have excluded from this study, or we have a BH with at least one horizon, see figure 1. Therefore, there is at least an interval of $r$ in which the function $f_+(r)$ is positive. This interval is $0 < r < r_-$ for a non-naked BH and thus the hypersurface $\Sigma$ is timelike in this range where $r_-$ is the innermost horizon radius. For the case that $\omega < 0$, the function $f_+$ tends to minus infinity when $r \to 0$, so the hypersurface $\Sigma$ would be spacelike for $r < r_-$. Therefore, here, we only consider the case that $\omega$ is positive.

According to the first junction condition, the induced metric on both sides of $\Sigma$ must be the same, $[h_{ab}] = 0$. This relation along with equations (33)-(35), gives
\[
\dot{t} = \frac{\beta(R, \dot{R})}{f(R)}
\] (36)
where $\beta(R, \dot{R}) \equiv \sqrt{f(R) + \dot{R}^2}$. It is convenient to choose $e^\alpha_\tau = u^\alpha$, then from (26)
\[
n_\alpha = \left(-\dot{R}, \frac{\beta}{f(R)}, 0, 0\right)
\] (37)
By considering (36) and (37), after some straightforward calculations, the non-zero components of extrinsic curvature are derived as follows
\[
n_\alpha a^\alpha = K^\tau_\tau = \frac{\dot{\beta}}{\dot{R}}, \quad K^\theta_\theta = K^\phi_\phi = -\frac{\beta}{\dot{R}}
\] (38)
Substituting the above relations into the shell equation of motion (32), it can be simplified as
\[
\dot{\beta}_+ - \dot{\beta}_- = \frac{\kappa}{2} \dot{R} \sigma.
\] (39)
Another useful equation is (30). By noting (1), (9) and (10), it can be easily seen that the two terms in the right hand side of (30) are individually
\[
\text{This is equivalent to say that the four-velocity of the shell, } u^\alpha = (i, \dot{R}, 0, 0) \text{ is a normalized timelike vector, } u^\alpha u_\alpha = -1.
\]
Consequently we have
\[(R^2 \sigma)' = 0.\] (40)

Making use of the two latter equations, one can show that
\[\beta_- - \beta_+ = \frac{M}{R} + \text{const.}\] (41)

where \(M \equiv \frac{\kappa}{2} R^2 \sigma\) is the proper shell mass which is constant by virtue of equation (40). Also the constant of (41) is equal to zero. This can be easily verified by substituting (27) and (29) into the second relation of (38).

Now, a question may be raised here. Is it possible to have a stable stationary shell by adjusting the free parameters of shell and geometry? This is the subject of the next section.

4.3 Stable regular BH

In this section, we have found some appropriate ranges for shell radius and its mass and also for three parameters of Kiselev metric \((m, c, \omega)\) for which a stable stationary BH is constructed. To do this, by aid of (34), we insert the definition of \(\beta\) into (41). This reduces (41) in the form of a conservation law
\[\dot{R}^2 + V(R) = -1\] (42)

where
\[V(R) = -\left[\frac{-2m - \frac{c^3}{R^2} + \frac{R^3}{2L}}{2M} - \frac{M}{2R} \right]^2 - \frac{R^2}{L^2}.\] (43)

is the effective potential of shell. For stationary BHs, \(\dot{R} = 0\), and so \(V(R) = -1\) and \(dV(R)/dR = 0\). The stability of this solution will be guaranteed by the constraint that the sign of \(d^2V(R)/dR^2\) should be positive.

Here, we perform a numerical analysis of (42) and (43) to get more insight regarding a stable regular Kiselev BH. First, without loss of generality, we set \(L = 1\) and normalize other parameters as follows: \(R/L \to R, m/L \to m, M/L \to M\) and \(c/\sqrt{L} \to c\) to get dimensionless parameters. Solving \(V(R) + 1 = 0\) and \(dV(R)/dR = 0\) simultaneously, gives us two set of solutions for \(m\) and \(c\) in terms of \(M, R\) and \(\omega\).

\(^{10}\)This is because every component of energy momentum tensor in both regions of spacetime is proportional to the corresponding coefficient of metric and also the fact that the velocity and acceleration vectors of the shell are orthogonal.
In obtaining the above results, we have assumed that $M > 0$. As it is evident from (43), the effective potential is symmetric under the change of $M \rightarrow -M$. Also according to (44) and (45), under $M \rightarrow -M$, we have $m_1 \leftrightarrow m_2$ and $c_1 \leftrightarrow c_2$. Therefore, the same sets of solutions would have been obtained if we had imposed the condition $M < 0$. It should be noted that, inserting the values of $M$, $R$ and $\omega$ from their acceptable range of $M$, $\omega > 0$ and $R < 1$, into equations (44) and (45), will lead to the stationary solutions as long as $m_1, m_2 > 0$. In addition, the resulted solutions will be stable if they satisfy $d^2V(R)/dR^2|_{m=m_{1,2},c=c_{1,2}} > 0$. Moreover, we assume that the BH is not naked. This condition strongly affects the acceptable range of the shell radius and therefore its mass. Also, as mentioned before, the shell radius must be smaller than the de Sitter and the innermost Kiselev horizon to have a timelike shell. These constraints can be shown diagrammatically. Figures 2(a) and 2(b) illustrate the allowed ranges of $R$, $\omega$ and $M$ obtained from (44) and (45) by taking into account that the BH is not naked and the shell is timelike. These figures indicate that for the first solution, the ranges of validity of $M$ and $R$ are significantly limited but this is not so for the second solution.
Figure 2: Range of validity of $M$, $R$ and $\omega$ for stable stationary solutions obtained from (a) the first solution (44) (b) the second solution (45)

Also, we have plotted the allowed regions in the parameter space $(m, R, \omega)$ and $(c, R, \omega)$ in figure 3 and 4 for solutions (44) and (45).
Figure 3: Range of validity of $m$, $R$ and $\omega$ for stable stationary solutions obtained from (a) solution (44) and (b) solution (45).

Figure 4: Range of validity of $c$, $R$ and $\omega$ for stable stationary solutions obtained from (a) solution (44) and (b) solution (45).
In order to be more specific, let us consider the case of \( \omega = 1/3 \). In this case, equations (44a) and (45a) reduce to
\[
\begin{align*}
m_1 &= m = 2R^3 + M_1 \frac{1 - 2R^2}{\sqrt{1 - R^2}}, \\
m_2 &= m = 2R^3 + M_2 \frac{2R^2 - 1}{\sqrt{1 - R^2}}.
\end{align*}
\]
(46)

In view of equation (46), for the fixed values of \( m \) and \( R \), we get two different values of the shell mass, \( M_1 \) and \( M_2 \) as follows
\[
\begin{align*}
M_1 &= \sqrt{1 - R^2} \frac{(2R^3 - m)}{2R^2 - 1}, \\
M_2 &= -\sqrt{1 - R^2} \frac{(2R^3 - m)}{2R^2 - 1}.
\end{align*}
\]
(47)

Inserting \( M_1 \) and \( M_2 \), into \( c_1 \) and \( c_2 \), equations (44b) and (45b), we interestingly find out that the parameter \( c \) for both sets of solutions are identical and given by
\[
c = \frac{m^2 (R^2 - 1) + m (6R^3 - 8R^5) + (4R^2 - 3) R^4}{1 - 2R^2)^2}.
\]
(48)

Then, making use of (48), we are able to find the horizons of Kiselev BH, (10), as a function of its mass and the shell radius as follows
\[
r_{\pm} = m \pm R \frac{|m - R| \sqrt{4R^2 - 3}}{|1 - 2R^2|}.
\]
(49)

Therefore for having non-naked BH solutions, the shell radius must satisfy \( R \geq \frac{\sqrt{3}}{2} \). Further examinations reveals that the hypersurface \( \Sigma \) remains timelike only if
\[
m \geq R \geq \frac{\sqrt{3}}{2}, \quad m \neq \frac{\sqrt{3}}{2} \quad \text{or} \quad R \geq \frac{\sqrt{3}}{2}, \quad m > 1 \quad (50)
\]

The upper limit of the BH mass can be deduced from positivity of \( d^2V(R)/dR^2 \). In this case, it is straightforward to verify that
\[
\frac{d^2V(R)}{dR^2} = 2 \left( 4 + \frac{1}{R^2 - 1} + \frac{6(R - m)}{m - 2R^3} \right),
\]
(51)

which is again the same for both sets of solutions (44) and (45). We observe that \( d^2V(R)/dR^2 \) is negative for \( 0 < R < 1, \ m \geq 2 \). However, for \( m < 2 \), there always exists an interval within \( \sqrt{3}/2 < R < 1 \) in which \( d^2V(R)/dR^2 \) > 0. Comparing these with the one obtained in (50), we conclude that the valid range of the BH mass is \( \frac{\sqrt{3}}{2} < m < 2 \). Using (47), it is easy to see that the
shell mass lies on the interval \(0 < |M| < \frac{\sqrt{3}}{4}\). The parameter \(c\) is also restricted to the range \(-3 < c < 1 - \sqrt{3}\) by noting (48).

The shell radius \(R(M)\) and \(c(M)\) for stable stationary BH for different values of \(m\) are plotted in figure 5(a) and figure 5(b). We see that as \(m\) increases, the maximum values of \(R\) and \(|c|\) are also increase whereas the maximum of \(|M|\) decreases. It is also evident that for fixed values of \(m\), larger \(|M|\) leads to larger \(R\) and smaller \(|c|\).

Now we turn back to the general case in which \(\omega\) is arbitrary. In this case, the coefficient of \(M^2\) in two sets of solutions, (44a) and (45a), are nonzero and an analytical analysis is impossible. Consequently, we confine ourselves to the numerical analysis in this case. To obtain sequences of stable stationary regular BHs with fixed values of \(m\) and \(\omega\), the shell masses are found from equations (44a) and (45a). Once \(M(R)\) is determined, \(c\) can be obtained from (44b) and (45b). According to the figures 2, 3 and 4 for \(\omega < 1/3\), both sets of solutions (44) and (45) give the stable stationary BHs, therefore we expect that the plot of \(R(M)\) and \(c(M)\) be degenerate in this interval of \(\omega\) for appropriate values of \(m\) and \(M\). To see this, \(R(M)\) and \(c(M)\) are plotted in 6(a), 6(c) with \(\omega = 0.1\) and 6(b), 6(d) with \(\omega = 0.2\) for different values of \(m\). It is evident that for \(1.9 \lesssim m \lesssim 5.4\) (\(\omega = 0.1\)) and \(1.2 \lesssim m \lesssim 2.9\) (\(\omega = 0.2\)), two sets of solutions are appeared. For example, for \(\omega = 0.1\), \(M = 0.31\) and \(m = 2.3\), two stable stationary regular solutions are \(R \approx 0.864\), \(c \approx -3.593\) and \(R \approx 0.043\), \(c \approx -2.9\). For each of these, the effective potentials are plotted in figures 7(a) and 7(b) respectively. The relative minimum in these figures satisfy all the requirement mentioned
before for a stable stationary regular BH. Also from figure 6, it turns out that for fixed values of $\omega$ and $m$, by increasing $|M|$, the shell remains stable if $R$ and also $|c|$ increases. Moreover we see that for a specific value of $\omega$, increasing $m$ leads to a larger upper limit for $R$ and $|c|$ and also increasing the maximum of $|M|$.

Figure 6: The plot of $R(M)$ and $c(M)$ for a sequence of stable stationary regular BH with (a), (c) $\omega = 0.1$ and (b), (d) $\omega = 0.2$. 
Figure 7: Effective potential for $\omega = 0.1$, $M = 0.31$ and $m = 2.3$ (a) First solution with $c \approx -3.593012$. (b) Second solution with $c \approx -2.900303$. The dots indicate the location of event horizons.

The functions $R(M)$ and $C(M)$ for the sequence of stable stationary solutions with $\omega = 0.5$ and $\omega = 0.8$ are plotted in figure 8a,c) and 8b,d) respectively. Here, we see that for a fixed $\omega$ and $m$, increasing $M$, leads to increasing $R$ and decreasing $|c|$ for a stable shell. For fixed values of $\omega$, by increasing the value of $m$, the maximum values of $R$ and $|c|$ increase whereas the maximum value of $|M|$ decreases.
Figure 8: The plot of $R(M)$ and $c(M)$ for a sequence of stable stationary regular BH with (a), (c) $\omega = 0.5$ and (b), (d) $\omega = 0.8$.

5 Concluding Remarks

In this paper, we have studied the gravitational collapse of a timelike thin shell in Kiselev BH with a de Sitter core. We have shown that the sign of equation of state parameter, a free parameter of Kiselev metric, must be negative from the requirement that the shell is timelike and the weak energy condition is satisfied. Moreover there are two stationary configurations, for each of which the other two free parameters of Kiselev BH, $m$ and $c$, are found from (44) and (45). However these configurations are not necessarily stable. Satisfying this condition numerically, restricts the range of parameters. The resulted regions are symmetric by changing the sign of shell mass although these solutions of are not physically acceptable. This is in contrast with the case of a charged regular BH constructed in a similar way in [19] and the solutions with negative shell mass are unstable. For the particular
choice of $\omega = 1/3$, our analytical result, for the valid range of BH mass, is approximately same as that obtained by [19], however stable stationary solutions exist even for the negative shell mass.

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