The space–time line element for static ellipsoidal objects

Ranchhaigiri Brahma1 · A. K. Sen1

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Abstract
In this paper, we solved the Einstein’s field equation and obtained a line element for static, ellipsoidal objects characterized by the linear eccentricity ($\eta$) instead of quadrupole parameter ($q$). This line element recovers the Schwarzschild line element when $\eta$ is zero. In addition to that it also reduces to the Schwarzschild line element, if we neglect terms of the order of $r^{-2}$ or higher which are present within the expressions for metric elements for large distances. Furthermore, as the ellipsoidal character of the derived line element is maintained by the linear eccentricity ($\eta$), which is an easily measurable parameter, this line element could be more suitable for various analytical as well as observational studies.

Keywords Einstein’s field equation · Space–time geometry · Line element · Ellipsoidal objects

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1 Department of Physics, Assam University, Silchar, Assam 788011, India
1 Introduction

The space–time line element plays an important role in general theory of relativity (GR). Such line element for the static and spherical objects was found by Schwarzschild [1] in 1916 right after the GR has been formulated. On the other hand, Weyl [2] found another form of the space–time line element for the objects which possess axial symmetry and was extended by Lewis [3], Papapetrou [4] and others. Later that form of the line element was known as Weyl–Lewis–Papapetrou form [5, 6].

The astronomical objects which possess only axial symmetry (but not central symmetry) have non-zero values of quadrupole moment and linear eccentricity. Therefore in that case, it is necessary to take into account the quadrupole moment and linear eccentricity while solving the Einstein’s field equations. For the first time in 1959 Erez and Rosen [7] solved the Einstein’s field equation for the non-spherical objects with quadrupole parameter \( q \) by using the Weyl metric. Later this solution was investigated by Doroshkevich et al. [8], Winicour et al. [9], Young and Coulter [10], Quevedo and Parkes [11] among others. In addition to that, a few other line elements for the non-spherical objects with quadrupole parameter \( q \) were also obtained by different authors through different approaches and those have been discussed in [6, 12]. One of the such solutions known as the \( q \)-metric\(^1\) [13] was obtained by performing Zipoy–Voorhees (ZV) transformation on the Schwarzschild metric [14, 15]. The \( q \)-metric is the most compact form of the line element of this kind in terms of its mathematical structure. However, though an explicit expression for the parameter \( q \) for this metric in principle can be obtained by comparing with Newtonian quadrupole moment, but it has limited applications, which is reported in [16, p. 7554]. The other quadrupolar line elements have complicated mathematical structure which make them disadvantageous in various analytical studies. Additionally the accurate measurement of the quadrupole moment \( J_2 \) and hence the quadrupole parameter \( q \) (e.g. in the case of Erez-Rosen metric \( q = 2J_2 (R_0/M_0)^2 \) where \( R_0 \) and \( M_0 \) are the average radius and the mass of the gravitating object respectively) are relatively difficult [11, 17, 18] than that of the linear eccentricity \( \eta \).

In view of the above situation, it is necessary to study the space–time geometry due to non-spherical objects with quadrupole moment using alternative approach. The objective of the current article is to derive the space–time line element outside that kind of the static ellipsoidal gravitating object without incorporating the quadrupole parameter \( q \). Such notion of finding a space–time line element for static and ellipsoidal objects was observed in [19]. In the current study we sincerely carried out the calculations to obtain the line element where we used the ellipsoidal coordinates which are suitable in present context. Consequently the ellipsoidal symmetry of the space–time is maintained by the parameter called linear eccentricity \( \eta \) instead of quadrupole parameter \( q \) used in different static, axially symmetric solutions. This line element may describe the space–time geometry produced by a static object with non-zero linear eccentricity or quadrupole moment. Such form of the line element may be simpler and easier to use in various analytical studies and astronomical measurements. From the

\(^1\) In literature \( q \)-metric is also known as the Zipoy–Voorhees metric, \( \delta \)-metric and \( \gamma \)-metric.
geometrical shapes (and sizes) of various objects as measured astronomically, it will be possible to calculate the parameter $\eta$. Whereas, the calculation of the parameter $q$ in such cases requires a more complicated theoretical approach. For example, we have various disk shaped galaxies and rotationally flattened astronomical objects (which include Sun as well), where it will be easier to calculate $\eta$ rather than $q$.

We organized the present paper as: In Sect. 2 we constructed a geometrical form of the space–time line element for the static ellipsoidal objects. Then using this form of the line element we calculated the Ricci tensors. Then in Sect. 3 we solved the Einstein’s field equations. Here firstly we obtained the time–time metric coefficient of the line element denoted by $A$. The resulting expression of $A$ helps to establish a relation between the unknown metric coefficients. Consequently we solved for the other metric coefficients using this relation and at the end we found all the unknown metric coefficients. In Sect. 4 we discussed the results under various conditions. At last in Sect. 5 we concluded and summarized our current study.

2 Construction of the line element

It is physically relevant that the space–time geometry produced by a static and ellipsoidally symmetric object is also ellipsoidally symmetric which is a kind of axially symmetric space–time. This is similar to that a centrally symmetric object produces a centrally symmetric space–time geometry [20, p. 282]. Therefore in order to derive the space–time line element due to ellipsoidal gravitating objects we chose the ellipsoidal coordinate system $(u, \theta, \phi)$ which is suitable in current context. The space–time line element in such a system of coordinates can be found by starting from the Minkowski space–time in Cartesian coordinates as below:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

(1)

Then transforming the coordinates $(x, y, z) \rightarrow (u, \theta, \phi)$ by the relations [21, pp. 39–40]:

$$x = \sqrt{u^2 + \eta^2 \sin \theta \cos \phi}; \quad y = \sqrt{u^2 + \eta^2 \sin \theta \sin \phi}; \quad z = u \cos \theta$$

(2)

we obtained the Minkowski space–time in ellipsoidal coordinates as [22, p. 15]:

$$ds^2 = c^2 dt^2 - \left[ \frac{u^2 + \eta^2 \cos^2 \theta}{u^2 + \eta^2} \right] du^2 - \left( u^2 + \eta^2 \cos^2 \theta \right) d\theta^2 - \left( u^2 + \eta^2 \right) \sin^2 \theta d\phi^2$$

(3)

Here the parameter $\eta = \sqrt{v^2 - u^2}$ is a constant known as the linear eccentricity where $v$ and $u$ represents the semi-major and the semi-minor axis of the ellipsoid respectively; and $\theta, \phi$ are the polar and the azimuthal angle coordinates respectively.
Therefore the line element of the curved space–time outside the ellipsoidal object characterized by the linear eccentricity ($\eta$) with mass $M$ can be written in the form as below:

\[
ds^2 = A^2c^2dt^2 - B^2\left[\frac{u^2 + \eta^2\cos^2\theta}{u^2 + \eta^2}\right]du^2 - D_1^2\left(u^2 + \eta^2\right)\sin^2\theta d\phi^2 - D_2^2\left(u^2 + \eta^2\right)\sin^2\theta d\phi^2
\]

(4)

where $A, B, D_1$ and $D_2$ are unknown metric coefficients and are functions of $(u, \theta)$ only which indicates the axially symmetric character of the metric and possesses the boundary condition that $A, B, D_1, D_2 \rightarrow 1$ when $M \rightarrow 0$ (i.e. if there is no mass).

It is also to be mentioned that in ellipsoidal coordinate system denoted by $(u, \theta, \phi)$, the surface at $u = \text{constant}$ represents an ellipsoid with semi-minor axis $u$ [21, p. 41]. Therefore in order to maintain the ellipsoidally symmetric character of the line element the metric coefficients $D_1$ and $D_2$ in Eq. (4) should be equal as described in [23, 24]. So, now we write as:

\[
D_1 = D_2 = D
\]

(5)

Then the Eq. (4) with $D_1 = D_2 = D$ represents a space–time line element due to static and ellipsoidal objects [23, 24] where now we defined $\eta$ as the linear eccentricity of the gravitating object with mass $M$. This definition of $\eta$ indicates that at $\eta = 0$ the Eq. (4) should recover the Schwarzschild line element.

Now we write the metric components in Eq. (4) as follows (by putting $D_1 = D_2 = D$):

\[
e^{2\nu} = A^2,
\]

\[
e^{2\psi} = B^2\left[\frac{u^2 + \eta^2\cos^2\theta}{u^2 + \eta^2}\right],
\]

\[
e^{2\mu_1} = D^2\left(u^2 + \eta^2\cos^2\theta\right)
\]

and

\[
e^{2\mu_2} = D^2\left(u^2 + \eta^2\right)\sin^2\theta
\]

(6)

And then we obtain from Eq. (4) as below:

\[
ds^2 = e^{2\nu}c^2dt^2 - e^{2\psi}du^2 - e^{2\mu_1}d\theta^2 - e^{2\mu_2}d\phi^2
\]

(7)

Also the line element (7) can be written in the form as below:

\[
ds^2 = e^\beta\left[\chi c^2dt^2 - \frac{1}{\chi}d\phi^2\right] - \frac{e^{(\psi + \mu_1)}}{\Delta^{1/2}}\left[du^2 + \Delta d\theta^2\right]
\]

(8)

where

\[
\beta = \nu + \mu_2; \quad \Delta = e^{2(\mu_1 - \psi)} \quad \text{and} \quad \chi = e^{\nu - \mu_2}
\]

(9)
The line element (8) is known as Chandrasekhar’s form [5], [25, p. 277] of an axially symmetric space–time. The metric functions $v$, $\psi$, $\mu_1$ and $\mu_2$ depend only on $u$ and $\theta$ coordinates. These are the coefficients associated with each of the coordinates described by the metric tensors $g_{ij}$ in the line element, which are independent of $t$ and $\phi$ exhibiting static and axial symmetry of the space–time$^2$ (i.e. $\partial g_{ij}/\partial t = 0$ and $\partial g_{ij}/\partial \phi = 0$) [6], [25, p. 66]. In the present context the covariant ($g_{ij}$) is diagonal and hence the elements of its contravariant are simply reciprocal to that of covariant ($g^{ij}$) i.e. $g^{ij} = \frac{1}{g_{ij}}$.

On the other hand, in general theory of relativity the geometry of the space–time is given by the Einstein’s field equation:

$$R_{ij} - \frac{1}{2} R g_{ij} = \kappa T_{ij}$$

(10)

where $R_{ij}$ is called the Ricci tensor, $R$ is the Ricci scalar (the trace of Ricci tensor), $T_{ij}$ is the energy-momentum tensor, and $\kappa = 8\pi G/c^4$ is a constant of proportionality; $G$ is the gravitational constant and $c$ is the speed of light in vacuum. The Ricci tensor is the contraction of two upper and lower middle indices of the Riemann curvature tensor $R^n_{ijk} = \frac{\partial \Gamma^n_{ik}}{\partial x^j} - \frac{\partial \Gamma^n_{ij}}{\partial x^k} + \Gamma^n_{ikp} \Gamma^p_{j} - \Gamma^n_{ip} \Gamma^p_{jk}$, that is

$$R_{ik} = R^n_{ink} = \frac{\partial \Gamma^n_{ik}}{\partial x^k} - \frac{\partial \Gamma^n_{ik}}{\partial x^n} + \Gamma^n_{ikp} \Gamma^p_{in} - \Gamma^n_{ip} \Gamma^p_{nk}$$

(11)

where the symbol $\Gamma^n_{ik}$ is known as the Christoffel symbol of second kind that is written as:

$$\Gamma^n_{ik} = \frac{1}{2} g^{np} \left[ \frac{\partial g_{ip}}{\partial x^k} + \frac{\partial g_{pk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^p} \right]$$

(12)

Now for the line element (7) we obtained the non-vanishing Ricci tensors as follows (where (‘) and (’) denote the derivatives w.r.t. $u$, and $\theta$ respectively):

$$R_{00} = - e^{2(\nu-\psi)} \left[ \nu'' + \nu' \left( \nu' - \psi' + \mu'_1 + \mu'_2 \right) \right]$$

$$- e^{2(\nu-\psi)} \left[ \tilde{\nu} + \tilde{\nu} \left( \tilde{\nu} - \tilde{\mu}_1 + \tilde{\psi} + \tilde{\mu}_2 \right) \right]$$

(13)

$$R_{11} = \nu'' + \mu''_1 + \mu''_2 + (\nu')^2 + (\mu'_1)^2 + (\mu'_2)^2 - \nu' \psi' - \mu'_1 \psi' - \mu'_2 \psi'$$

$$+ e^{2(\psi-\mu_1)} \left[ \tilde{\psi} + \tilde{\psi} \tilde{\psi} - \tilde{\psi} \tilde{\mu}_1 + \tilde{\psi} \tilde{\mu}_2 \right]$$

(14)

$$R_{22} = e^{2(\mu_1-\psi)} \left[ \mu''_1 - \mu'_1 \psi' + \mu'_1 \mu'_1 + \mu'_2 \mu'_1 + \nu' \mu'_1 \right] - (\tilde{\nu}) (\tilde{\mu}_1) + \tilde{\nu} + \tilde{\psi} + \tilde{\mu}_2 + (\tilde{\mu}_2)^2 + (\tilde{\psi})^2 - (\tilde{\psi}) (\tilde{\mu}_2) (\tilde{\mu}_1) - (\tilde{\psi}) (\tilde{\mu}_1)$$

(15)

$$R_{33} = e^{2(\mu_2-\psi)} \left[ \mu''_2 + \mu'_2 \mu'_2 - \psi' + \nu' + \mu'_1 \right]$$

Note: The English alphabet indices ($i$, $j$, $k$ . . .) denotes to run (0, 1, 2, 3) that represents $ct$, $u$, $\theta$ and $\phi$ co-ordinates respectively. Metric sign convention is (+, −, −, −).
\[ e^{2(\mu_2 - \mu_1)} \left[ \ddot{\mu}_2 + \dot{\mu}_2 (\dot{\mu}_2 - \ddot{\mu}_1 + \dot{\psi}) \right] \]  
(16)

and \[ R_{12} = \frac{\partial}{\partial \theta} \left( v' + \mu'_2 \right) + (\dot{v}) (v') + (\dot{\mu}_2) (\mu'_2) - (v') (\dot{\psi}) - (\dot{\psi}) (\mu'_1) \]
(17)

3 Solution of the Einstein’s field equation

The objective of the current study is to obtain the line element of space–time geometry outside the ellipsoidal gravitating object. Therefore the Einstein’s field equation outside the gravitating object is simply:

\[ R_{ij} = 0 \]  
(18)

The Eq. (18) can be solved for the metric coefficients \( A, B \) and \( D \). Now re-writing Eqs. (13) and (16) we obtained:

\[ R_{00} = 0 \]

or \[ \frac{\partial}{\partial u} \left( e^{\nu - \psi + \mu_1 + \mu_2} v' \right) + \frac{\partial}{\partial \theta} \left( e^{\nu + \psi - \mu_1 + \mu_2} \dot{v} \right) = 0 \]  
(19)

and

\[ R_{33} = 0 \]

or \[ \frac{\partial}{\partial u} \left( e^{\nu - \psi + \mu_1 + \mu_2} \mu'_2 \right) + \frac{\partial}{\partial \theta} \left( e^{\nu + \psi - \mu_1 + \mu_2} \dot{\mu}_2 \right) = 0 \]  
(20)

The sum of these two Eqs. (19) and (20) \( (R_{00} + R_{33} = 0) \) is:

\[ \frac{\partial}{\partial u} \left[ e^{\mu_1 - \psi} \frac{\partial}{\partial u} (e^{\nu + \mu_2}) \right] + \frac{\partial}{\partial \theta} \left[ e^{\psi - \mu_1} \frac{\partial}{\partial \theta} (e^{\nu + \mu_2}) \right] = 0 \]  
(21)

and the difference of the Eqs. (19) and (20) \( (R_{00} - R_{33} = 0) \) is:

\[ \frac{\partial}{\partial u} \left[ e^{\nu - \psi + \mu_1 + \mu_2} (v' - \mu'_2) \right] + \frac{\partial}{\partial \theta} \left[ e^{\nu + \psi - \mu_1 + \mu_2} (\dot{v} - \dot{\mu}_2) \right] = 0 \]  
(22)

3.1 Solution for the metric coefficient \( A \)

In order to solve for the metric coefficient \( A = e^{\nu} \) we assumed that the space–time line element (4) permits an event horizon. Since there exist a time-like and a space-like Killing vector which implies that \( \partial g_{ij}/\partial t = 0 \) and \( \partial g_{ij}/\partial \phi = 0 \) in static and axially symmetric space–time [6], therefore the event horizon can be defined as a two dimensional null surface that is spanned by the Killing vectors below which the space–time is regular [25, p. 278], [26, p. 124]. So we proceeded with our calculation.
as in [25, p. 278], by which the equation of the event horizon of a static and axially symmetric space–time may be written as: $\mathcal{N}(x^1, x^2) = 0$ where the condition of null of this equation [22, p. 60], [25, p. 278] is:

$$g^{ij} \frac{\partial \mathcal{N}}{\partial x^i} \frac{\partial \mathcal{N}}{\partial x^j} = 0 \quad (23)$$

According to the our present context the Eq. (23) can be written by using the metrics in Eq. (7) as follows:

$$e^{-2\psi}(\mathcal{N}')^2 + e^{-2\mu_1}(\dot{\mathcal{N}})^2 = 0$$

or

$$\Delta(\mathcal{N}')^2 + (\dot{\mathcal{N}})^2 = 0 \quad (24)$$

Now exercising the gauge freedom we supposed that $\Delta$ is a function of $u$ only which is representing the null surface. i.e. $e^{2(\mu_1-\psi)} = \Delta(u)$. So the equation of the null surface is given by,

$$\Delta(u) = 0 \quad (25)$$

Again by the definition of the event horizon, the null surface is spanned by the Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$, so the determinant of the metric of the subspace $(t, \phi)$ must vanish at null surface [25, p. 278], [26, p. 124]. i.e. from Eq. (8) we obtained as:

$$e^{2\beta} = 0 \quad \text{on the surface} \quad \Delta(u) = 0 \quad (26)$$

Therefore with no loss of generality we may assume that [25, p. 279]:

$$e^{\beta} = \Delta^{1/2} f(\theta) \quad (27)$$

which is a separable function of $u$ and $\theta$.

Now using this Eq. (27) we proceed to solved Eq. (19) as follows:

$$\frac{\partial}{\partial u} \left[ \Delta^{1/2} \frac{\partial}{\partial u} \left( \Delta^{1/2} \right) \right] = -\frac{1}{f(\theta)} \frac{\partial}{\partial \theta} \left[ \frac{\partial}{\partial \theta} (f(\theta)) \right] = W \quad \text{(a constant)} \quad (28)$$

The above partial differential Eq. (28) of two independent variables are equal and hence separately equal to a constant $W$. Therefore we obtained two ordinary differential equations and they can be solved independently as follows:

$$\frac{d^2}{d\theta^2} \left( f(\theta) \right) + Wf(\theta) = 0 \quad (29)$$

Here if $X = \frac{d}{d\theta}$ then we obtain an auxiliary equation of (29) as:

$$\left( X^2 + W \right) f(\theta) = 0$$
or \( X = \pm i \sqrt{W} \) \hspace{1cm} (30)

where the roots of this auxiliary equation is complex conjugate. So the general solution of the differential Eq. (29) is:

\[
f(\theta) = N \sin \left( \sqrt{W} \theta + \delta \right)
\] (31)

where \( N \) and \( \delta \) are two unknown constants. Since \( e^\beta \) is regular on \( \Delta(u) = 0 \) and on the axis of symmetry \( \theta = 0 \), so in Eq. (31) we have \( \delta = 0 \). In addition to that the horizon \( \Delta(u) = 0 \) is convex and hence by the condition of convexity of the horizon we obtained \( N = W = 1 \) (for the range \( 0 \leq \theta \leq \pi \)). Therefore by maintaining the regularity on the axis of symmetry and convexity of the horizon, we may write as [25, p. 279]:

\[
f(\theta) = \sin(\theta)
\] (32)

Using this result we again solved the differential equation (28) with variable \( u \) as follows (by putting \( W = 1 \)):

\[
\frac{d}{du} \left[ \Delta^{1/2} \frac{d}{du} \left( \Delta^{1/2} \right) \right] = 1
\]

or \( \Delta = u^2 - 2mu + \alpha^2 \) where \( m, \alpha^2 \) are constant of integration. \hspace{1cm} (33)

Therefore from Eq. (27) we obtained as:

\[
e^{2\beta} = \Delta \sin^2 \theta
\] (34)

Now using Eqs. (6) and (9) we obtained from Eq. (34) as follows:

\[
e^{2\nu} = A^2 = \frac{u^2 - 2mu + \alpha^2}{D^2 (u^2 + \eta^2)}
\] (35)

and similarly from Eq. (33) we obtained as:

\[
D^2 = \frac{B^2 (u^2 - 2mu + \alpha^2)}{(u^2 + \eta^2)}
\] (36)

These two Eqs. (35) and (36) establish the relation between the metric coefficients as:

\[
AD = \frac{D}{B} \text{ i.e. } A = \frac{1}{B}
\] (37)

where \( D \neq 0 \).
3.2 Solution for the metric coefficient $D$

The above Eqs. (35) and (37) indicate that if we solved for the metric coefficient $D$ then the other coefficients such as $A$ and $B$ can be found easily. Therefore in order to obtain $D$ we proceed as follows where the required boundary conditions of the metric coefficient may be described as below:

**Boundary conditions:**

(a) If $m = 0$ then $D = 1$ (condition for Minkowski space–time)
(b) If $u \rightarrow \infty$ then $D \rightarrow 1$ (condition for asymptotic flatness)
(c) If $\eta = 0$ then $D = 1$ (condition for spherical symmetry)

Now from Eq. (22) we obtained by using (9), (33) and (34) as:

\[
\frac{\partial}{\partial u} \left[ \Delta \frac{\chi'}{\chi} \right] + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta \frac{\chi'}{\chi} \right] = 0
\]  

(38)

Then substituting the expressions from Eqs. (6), (9) and (37) in the differential equation (38) we obtained:

\[
\Delta D^2 \frac{\partial^2}{\partial u^2} \frac{1}{D^2} + \Delta \left( \frac{\partial}{\partial u} \frac{1}{D^2} \right) \left( \frac{\partial}{\partial u} \frac{1}{D^2} \right) + 2(u - m)D^2 \frac{\partial}{\partial u} \frac{1}{D^2} \\
+ D^2 \frac{\partial^2}{\partial \theta^2} \frac{1}{D^2} + \left( \frac{\partial}{\partial \theta} \frac{1}{D^2} \right) \left( \frac{\partial}{\partial \theta} \frac{1}{D^2} \right) + D^2 \cot \theta \frac{\partial}{\partial \theta} \frac{1}{D^2} \\
+ 2 - \frac{4u(u - m)}{(u^2 + \eta^2)} + \frac{4u^2 \Delta}{(u^2 + \eta^2)^2} - \frac{2\Delta}{(u^2 + \eta^2)} = 0
\]  

(39)

Now by executing the above boundary condition (c) we put $\eta = 0$ and $D = 1$ in Eq. (39) and obtained as:

\[
\alpha = 0
\]  

(40)

The constant of integration $\alpha = 0$ when $\eta = 0$, indicates that, this is a parameter related to the shape of the gravitating object. Therefore we may consider:

\[
\alpha = \eta
\]  

(41)

Similarly using this boundary condition (c) that $D = 1$ when $\alpha = \eta = 0$ in Schwarzschild limit we obtained from Eq. (35) as:

\[
g_{00} = e^{2\nu} = 1 - \frac{2m}{u}
\]  

(42)

This is exactly same as the time–time component of the Schwarzschild metric [20, p. 284]. Hence we confirmed that the constant of integration $m$ is the half of the Schwarzschild radius ($r_g$) of the gravitating object with mass $M$. That is,
\[ r_g = 2m = \frac{2GM}{c^2} \]  

(43)

where \( G \) is the gravitational constant and \( c \) is the speed of light in vacuum.

Therefore, putting \( \alpha = \eta \) the Eq. (39) can be written as:

\[
\Delta D^2 \frac{\partial^2}{\partial u^2} \frac{1}{D^2} + \Delta \left( \frac{1}{D^2} \right) \left( \frac{\partial^2}{\partial u^2} D^2 \right) + 2(u - m)D^2 \frac{\partial^2}{\partial u^2} \frac{1}{D^2} + \frac{8m\eta^2 u}{(u^2 + \eta^2)^2} = 0
\]  

(44)

In Eq. (44) it is observed that the partial differential equation does not contain any cross terms of variables \( u \) and \( \theta \) and hence this can be solved by separation of variables as calculating in [25, pp. 344–345] and [27]. So we considered,

\[
D = e^{\eta \lambda(u, \theta)}; \quad \text{and}, \quad \lambda(u, \theta) = P(u) + Q(\theta)
\]  

(45)

where \( P(u) \) and \( Q(\theta) \) are respectively functions of \( u \) and \( \theta \) only. Substituting from Eq.(45) into (44) and simplifying we obtained as below:

\[
2\eta \Delta \frac{\partial^2 P}{\partial u^2} + 4(u - m)\eta \left( \frac{\partial P}{\partial u} \right) - \frac{8m\eta^2 u}{(u^2 + \eta^2)^2} = -2\eta \frac{\partial^2 Q}{\partial \theta^2} - 2\eta \cot \theta \left( \frac{\partial Q}{\partial \theta} \right)
\]  

(46)

Since the left hand-side of the above Eq.(46) depend only on variable \( u \), while the right hand-side only depend on the variable \( \theta \) and hence they are separately equal to a constant \( K \) which can be written as below:

\[
2\eta \Delta \frac{\partial^2 P}{\partial u^2} + 4(u - m)\eta \left( \frac{\partial P}{\partial u} \right) - \frac{8m\eta^2 u}{(u^2 + \eta^2)^2} = -2\eta \frac{\partial^2 Q}{\partial \theta^2} - 2\eta \cot \theta \left( \frac{\partial Q}{\partial \theta} \right) = K
\]  

(47)

Therefore, we obtained two ordinary differential equations from Eq.(47) as below:

\[
2\eta \Delta \frac{d^2 P}{du^2} + 4(u - m)\eta \left( \frac{d P}{du} \right) = K + \frac{8m\eta^2 u}{(u^2 + \eta^2)^2}
\]  

(48)

and

\[
2\eta \frac{d^2 Q}{d\theta^2} + 2\eta \cot \theta \left( \frac{\partial Q}{\partial \theta} \right) = -K
\]  

(49)
The solution of Eq. (48) is:

\[ P(u) = \frac{1}{2} \eta \left[ \arctan \left( \frac{u-m}{(\eta^2-m^2)^{1/2}} \right) \left( Km - 2m + 2 \eta a_1 \right) \right. \\
\left. \left( \eta^2 - m^2 \right)^{1/2} \right] - \ln \left( u^2 + n^2 \right) \\
+ \frac{1}{2} (2 + K) \ln \left( u^2 - 2mu + \eta^2 \right) + a_2 \]  

(50)

where \( a_1 \) and \( a_2 \) are constants of integration.

Similarly, the solution of Eq. (49) is:

\[ Q(\theta) = \frac{K}{2 \eta} \ln \left( \sin \theta \right) - a_3 \left[ \ln \left( \cos \left( \theta/2 \right) \right) - \ln \left( \sin \left( \theta/2 \right) \right) \right] + a_4 \]  

(51)

where \( a_3 \) and \( a_4 \) are constants of integration.

Therefore using (50) and (51) we obtained from (45) as:

\[ \eta \lambda(u, \theta) = \eta \left( P(u) + Q(\theta) \right) \\
= \frac{1}{2} \eta \left[ \arctan \left( \frac{u}{(\eta^2-m^2)^{1/2}} \right) \left( Km - 2m + 2 \eta a_1 \right) \right. \\
\left. \left( \eta^2 - m^2 \right)^{1/2} \right] + \frac{K}{2} \ln \left( u^2 \right) - \ln \left( 1 + \frac{\eta^2}{u^2} \right) \\
+ \frac{1}{2} (2 + K) \left[ \ln \left( 1 - 2m + \frac{\eta^2}{u^2} \right) \right] + \eta a_2 + \frac{K}{2} \ln \left( \sin \theta \right) \\
- \eta a_3 \left[ \ln \left( \cos \left( \theta/2 \right) \right) - \ln \left( \sin \left( \theta/2 \right) \right) \right] + a_4 \]  

(52)

The constant parameters in Eq. (52) are obtained by applying the above boundary conditions for \( D \) as follows:

(a) \textit{Condition for Minkowski space–time} If \( m = 0 \) then \( D = 1 \) i.e. \( \eta \lambda = 0 \). Therefore from Eq. (52) we obtained as:

\[ \eta a_2 + \eta a_4 = -\frac{1}{2} \eta \left[ \arctan \left( \frac{u}{\eta} \right) \left( 2a_1 \right) + \frac{K}{2} \ln \left( u^2 \right) + \frac{K}{2} \left\{ \ln \left( 1 + \frac{\eta^2}{u^2} \right) \right\} \right] \\
- \frac{K}{2} \ln \left( \sin \theta \right) + \eta a_3 \left[ \ln \left( \cos \left( \theta/2 \right) \right) - \ln \left( \sin \left( \theta/2 \right) \right) \right] \]  

(53)

Substituting this into Eq. (52) we obtained:

\[ \eta \lambda = \frac{1}{2} \eta \left[ \arctan \left( \frac{u}{(\eta^2-m^2)^{1/2}} \right) \left( Km - 2m + 2 \eta a_1 \right) \right. \\
\left. \left( \eta^2 - m^2 \right)^{1/2} \right] - \arctan \left( \frac{u}{(\eta^2)^{1/2}} \right) \left( 2a_1 \right) + \frac{1}{2} (2 + K) \left\{ \ln \left( 1 - 2m + \frac{\eta^2}{u^2} \right) \right\} \\
- \frac{1}{2} (2 + K) \left[ \ln \left( 1 + \frac{\eta^2}{u^2} \right) \right] \]  

(54)
(b) Condition for asymptotic flatness If \( u \to \infty \) then \( D = 1 \) i.e. \( \eta \lambda = 0 \). So the Eq. (54) become:

\[
\arctan \left( \frac{u}{(\eta^2)^{1/2}} \right) (2a_1) = \left[ \arctan \left( \frac{u}{(\eta^2 - m^2)^{1/2}} \right) \frac{(Km - 2m + 2\eta a_1)}{(\eta^2 - m^2)^{1/2}} \right]
\]  

(55)

Again by substituting (55) into Eq. (54) and simplifying we found as below:

\[
\eta \lambda = \frac{1}{2} \left[ \frac{(Km - 2m + 2\eta a_1)}{(\eta^2 - m^2)^{1/2}} \right] \left\{ \arctan \left( \frac{-m}{(\eta^2 - m^2)^{1/2} \left( 1 + \frac{u(u - m)}{(\eta^2 - m^2)} \right)} \right) \right\} 
+ \frac{1}{2} (2 + K) \left\{ \ln \left( 1 - \frac{2m}{u} + \frac{\eta^2}{u^2} \right) \right\} - \frac{1}{2} (2 + K) \left\{ \ln \left( 1 + \frac{\eta^2}{u^2} \right) \right\}
\]  

(56)

(c) Condition for spherical symmetry If \( \eta = 0 \) then \( D = 1 \) i.e. \( \eta \lambda = 0 \). So the Eq. (56) become:

\[
0 = \frac{1}{2} \left[ \frac{(Km - 2m + 2\eta a_1)}{(\eta^2 - m^2)^{1/2}} \right] \left\{ \arctan \left( \frac{-m}{(\eta^2 - m^2)^{1/2} \left( 1 + \frac{u(u - m)}{(\eta^2 - m^2)} \right)} \right) \right\} 
+ \frac{1}{2} (2 + K) \left\{ \ln \left( 1 - \frac{2m}{u} + \frac{\eta^2}{u^2} \right) \right\} - \frac{1}{2} (2 + K) \left\{ \ln \left( 1 + \frac{\eta^2}{u^2} \right) \right\}
\]  

(57)

Here the constant of integration \( a_1 \) can not be a function of \( u \) and the constant \( K \) can not be a function of \((u, \theta)\). Therefore the terms in the above Eq. (57) must be separately zero. This can be achieved by the choice of \( K = -2 \) and \( a_1 = 2m/\eta \).

Therefore by putting these value \( K = -2 \) and \( a_1 = 2m/\eta \) into Eq. (56) we obtained:

\[
\eta \lambda = 0
\]  

(58)

and hence the metric coefficient \( D \) is not a function, instead a constant which is obtained from Eq. (45), i.e.

\[
D = e^{\eta \lambda} = 1
\]  

(59)

4 Discussion of results

We obtained the value of the metric coefficient \( D \) present in Eq. (59) and consequently the other metric coefficients from Eqs. (5) and (37) are as below:

\[
A^2 = \frac{1}{B^2} = \frac{u^2 - 2mu + \eta^2}{u^2 + \eta^2}
\]  

(60)

and \( D_1 = D_2 = D = 1 \)  

(61)
Using these metric coefficients we obtained the space–time line element for the ellipsoidal shaped gravitating object. The line element can be written from Eqs. (4), (60) and (61) as below:

\begin{align*}
  ds^2 &= \frac{u^2 - 2mu + \eta^2}{u^2 + \eta^2} c^2 dt^2 - \left[ \frac{u^2 + \eta^2 \cos^2 \theta}{u^2 - 2mu + \eta^2} \right] du^2 - \left( u^2 + \eta^2 \cos^2 \theta \right) d\theta^2 \\
  &\quad - \left( u^2 + \eta^2 \right) \sin^2 \theta d\phi^2
\end{align*} \tag{62}

We analyzed some important features of this line element below:

- **Line element in terms of radial distance** The derived line element can be written in terms of a radial distance \((r)\) that is related to \(u\) by the relation [from (2)]:

\begin{equation}
  r^2 = u^2 + \eta^2 \sin^2 \theta \tag{63}
\end{equation}

such that

\begin{equation}
  du = \frac{r \, dr}{\sqrt{r^2 - \eta^2 \sin^2 \theta}} + \frac{\eta^2 \sin \theta \cos \theta d\theta}{\sqrt{r^2 - \eta^2 \sin^2 \theta}} \tag{64}
\end{equation}

Now using Eqs. (63) and (64) we obtained the line element in terms of \((r)\) from Eq. (62) as follows:

\begin{align*}
  ds^2 &= \left[ 1 - \frac{2m(r^2 - \eta^2 \sin^2 \theta)^{1/2}}{(r^2 + \eta^2 \cos^2 \theta)} \right] c^2 dt^2 \\
  &\quad - \left[ \frac{r^2 + \eta^2 \left( \cos^2 \theta - \sin^2 \theta \right)}{r^2 + \eta^2 \cos^2 \theta - 2m \left( r^2 - \eta^2 \sin^2 \theta \right)^{1/2}} \right] \frac{r^2 \, dr^2}{r^2 - \eta^2 \sin^2 \theta} \\
  &\quad - \left[ \frac{\eta^4 \sin^2 \theta \cos^2 \theta}{r^2 - \eta^2 \sin^2 \theta} \right] \frac{r^2 + \eta^2 \cos^2 \theta - 2m \left( r^2 - \eta^2 \sin^2 \theta \right)^{1/2}}{r^2 + \eta^2 \cos^2 \theta - 2m \left( r^2 - \eta^2 \sin^2 \theta \right)^{1/2}} \\
  &\quad \times \left( r^2 + \eta^2 \left( \cos^2 \theta - \sin^2 \theta \right) \right) d\theta^2 \\
  &\quad - \left( r^2 + \eta^2 \cos^2 \theta \right) \sin^2 \theta d\phi^2 \tag{65}
\end{align*}

- **Schwarzschild limit** If we put \(\eta = 0\) in Eq. (65) the line element reduces to the Schwarzschild line element:

\begin{equation}
  ds^2 = \left( 1 - \frac{2m}{r} \right) c^2 dt^2 - \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \tag{66}
\end{equation}

Therefore it can be inferred that the parameter \(\eta\) maintains the ellipsoidal character of the line element.
• **Asymptotic behaviour** Now we rewrite the Eq. (65) in the form as below:

\[
\begin{align*}
    ds^2 &= \left[ 1 - \frac{2m}{r} \left( 1 - \frac{\eta^2}{r^2} \sin^2 \theta \right)^{1/2} \right] c^2 dt^2 \\
    &\quad - \left[ \frac{1 + \frac{\eta^2}{r^2} \left( \cos^2 \theta - \sin^2 \theta \right)}{1 + \frac{\eta^2}{r^2} \cos^2 \theta - \frac{2m}{r} \left( 1 - \frac{\eta^2}{r^2} \sin^2 \theta \right)^{1/2}} \right] \frac{dr^2}{1 - \frac{\eta^2}{r^2} \sin^2 \theta} \\
    &\quad - \left[ 1 + \left\{ \frac{\eta^2 \sin^2 \theta \cos^2 \theta}{r^4 \left( 1 - \frac{\eta^2}{r^2} \sin^2 \theta \right)} \right\} \right] \times r^2 \left( 1 + \frac{\eta^2}{r^2} \left( \cos^2 \theta - \sin^2 \theta \right) \right) d\theta^2 \\
    &\quad - r^2 \left( 1 + \frac{\eta^2}{r^2} \cos^2 \theta \right) \sin^2 \theta d\phi^2
\end{align*}
\]

(67)

Here if the value of \( r \) is sufficiently large such that the terms with \( r^2 \) or higher powers of \( r \) in the denominator can be neglected (or put to zero), then we obtain from Eq. (67) as:

\[
\begin{align*}
    ds^2 &= \left[ 1 - \frac{2m}{r} \right] c^2 dt^2 - \left[ \frac{1}{1 - \frac{2m}{r}} \right] dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2
\end{align*}
\]

(68)

This is exactly same as the line element of the spherically symmetric space–time. From this result it is clear that at sufficiently large distance the space–time geometry due to an ellipsoidal object behaves like that due to a spherically symmetric object. Further, at \( r \rightarrow \infty \) the line element (67) reduces to the Minkowski metric which indicates that the space–time at large distance from the source is flat. So the derived line element satisfies the condition of asymptotic flatness. In addition to that if \( m = 0 \) then the line element (62) reduces to the Minkowski space–time element in ellipsoidal coordinates as shown in Eq. (3). Hence it can be inferred that the parameter \( m \) contains the total mass of the gravitating object.

• **Physical interpretation of \( \eta \)** In Sect. 2 we defined \( \eta \) as the linear eccentricity of the gravitating object which ultimately means that if the shape of the object is spherical then this parameter is zero. This definition of \( \eta \) can be further interpreted in terms of mass of the gravitating object as follows:

It is obvious that the total volume \( (V) \) and the total mass \( (M) \) of an ellipsoidal oblate object with semi-major axis \( (a) \) and semi-minor axis \( (b) \) and having mass density \( \rho \) are:

\[
    V = \frac{4}{3} \pi a^2 b \quad \text{and} \quad M = \rho V
\]

(69)
Therefore if $\eta$ is the linear eccentricity of the ellipsoidal shape object then its semi-major axis is $a = \sqrt{b^2 + \eta^2}$ and the total mass is:

$$M = \frac{4}{3}\pi \rho b \left( b^2 + \eta^2 \right) = M_b \left( 1 + \frac{\eta^2}{b^2} \right)$$  \hspace{1cm} (70)

where $M_b = \frac{4}{3}\pi \rho b^3$ is the mass of the sphere with radius $b$. Further simplifying Eq. (70) we obtained:

$$\eta^2 = \frac{b^2 (M - M_b)}{M_b} = \frac{b^2 \Omega}{M_b}$$  \hspace{1cm} (71)

in which $\Omega = M - M_b$ indicates the mass that is spanned outside the sphere of radius $b$. Therefore the parameter $\eta$ is also related to the mass of the gravitating object.

- **Singularity and horizons**

  The singularity of the space–time is the point at which the line element becomes undefined. In the present context the space–time horizon can be studied from Eq. (25) i.e. $\Delta = 0$ which can be written as below in terms of $r$:

$$r^2 + \eta^2 \cos^2 \theta - 2m \left( r^2 - \eta^2 \sin^2 \theta \right)^{1/2} = 0$$  \hspace{1cm} (72)

The roots of this equation are as presented is a single equation below:

$$r = \pm \sqrt{2m^2 - \eta^2 \cos^2 \theta \pm 2m \sqrt{m^2 - \eta^2}}$$  \hspace{1cm} (73)

From Eq. (73) it can be inferred that the space–time geometry due to an ellipsoidal object possesses two event horizons where both of them are non-spherical. Additionally the gravitational singularities can be identified from the study of curvature invariants such as Kretschmann scalar which is expressed as $\mathcal{K} = R_{ijkl} R^{ijkl}$ [6, 13]. The Kretschmann scalar for the metric (62) has been calculated here, which is:

$$\mathcal{K} = \frac{F(u, \theta)}{(u^2 + \eta^2)^4 (u^2 + \eta^2 \cos^2 \theta)^6 (u^2 - 2mu + \eta^2)}$$  \hspace{1cm} (74)

where

$$F(u, \theta) = \left[ 4m^2 u^2 (u^2 - 2mu + \eta^2)(\eta^4 - u^4)(u^2 + \eta^2 \cos^2 \theta)^2 
+ 16m^2 u^6 (\eta^2 + u^2)^2 (u^2 - 2mu + \eta^2)(u^2 + \eta^2 \cos^2 \theta)^2 
+ 4m^2 u^2 (\eta^2 - u^2)^2 (u^2 - 2mu + \eta^2)(u^2 + \eta^2 \cos^2 \theta)^4 
+ 4m^2 u^2 (u^2 - 2mu + \eta^2)(u^2 + \eta^2 \cos^2 \theta)^2 \left( \eta^2 - u^2 \right) u^2 
+ \eta^2 (3\eta^2 + u^2) \cos^2 \theta \right]^2 + (u^2 + \eta^2)^4 (u^2 - 2mu + \eta^2)$$
This reduces to the Schwarzschild value \( K = 48m^2/u^6 \) when \( \eta = 0 \). In the present context the singularity appears at \( u^2 = -\eta^2, u^2 = -\eta^2 \cos^2 \theta \) and \( \Delta = u^2 - 2mu + \eta^2 = 0 \). On the other hand \( \Delta = 0 \) is also the horizon according to the metric \( (62) \) and there are no additional horizons outside it, which indicates that for all \( \eta \neq 0 \) the singularities appeared are naked.

- **Multipole moments** The Geroch multipole moments [28] for the presently obtained line element \( (62) \) can be calculated by writing the Eq. \((65)\) in prolate spheroidal coordinates \((t, \sigma, \tau, \phi)\).\(^3\) This is achieved by performing coordinate transformation using the relation below \([6]\):

\[
t = t, \quad \sigma = \frac{r}{m} - 1, \quad \tau = \cos \theta, \quad \text{and} \quad \phi = \phi \quad (76)
\]

Accordingly the line element \((60)\) in prolate spheroidal coordinate system (as mentioned by \([6]\)) can be written as:

\[
ds^2 = \left[ 1 - \frac{2 \left\{ (\sigma + 1)^2 - \frac{\eta^2}{m^2} (1 - \tau^2) \right\}^{1/2}}{(\sigma + 1)^2 + \frac{\eta^2}{m^2} \tau^2} \right] c^2 dt^2
\]

\[
- \left[ \frac{(\sigma + 1)^2 + \frac{\eta^2}{m^2} (2\tau^2 - 1)}{(\sigma + 1)^2} \times \frac{m^2(\sigma+1)^2}{(\sigma+1)^2 - \frac{\eta^2}{m^2} (1 - \tau^2)} \right] d\sigma^2
\]

\[
- \left[ 1 + \frac{\eta^2 (1 - \tau^2) \tau^2}{m^4(\sigma+1)^2 - \eta^4 (1 - \tau^2)^2} \right] \frac{m^2(\sigma+1)^2}{(\sigma+1)^2 - \frac{\eta^2}{m^2} (1 - \tau^2)}^{1/2} \right] \right] d\tau^2
\]

\[
- m^2 \left\{ (\sigma + 1)^2 + \frac{\eta^2}{m^2} \tau^2 \right\} (1 - \tau^2) d\phi^2 \quad (77)
\]

\(^3\) We denoted the prolate coordinates by \((t, \sigma, \tau, \phi)\) instead of \((t, x, y, \phi)\) used in \([6]\) in order to avoid confusion with the Cartesian coordinates in Eq. \((2)\).
The Ernst potential $\xi(\sigma, \tau)$ for such static axially symmetric space–time is defined as

$$\xi(\sigma, \tau) = \frac{1 - e^{2v}}{1 + e^{2v}}$$  \hspace{1cm} (78)

from which the Geroch multipole moments can be calculated on the axis of symmetry, i.e. at $\tau = 1$ [6, 17]. For that it is necessary to change the coordinate $\sigma$ by using the relation $\sigma = \frac{\tilde{z}}{m} = \frac{1}{\xi m}$ and then the multipole moments can be determined as follows:

$$M_l = m_l + d_l$$  \hspace{1cm} (79)

where the recurrence formula for $m_l$ are [17, pp. 828–830]:

$$m_l = -\frac{h_{l+1}}{(l + 1)!} \bigg|_{\tau=1, \tilde{z}=0}$$  \hspace{1cm} (80)

$$h_1 = \frac{dv}{d\tilde{z}}$$  \hspace{1cm} (81)

and $h_l = \frac{dh_{l-1}}{d\tilde{z}} + 2\xi h_1 h_{l-1}$, for all $l \geq 2$  \hspace{1cm} (82)

The additional term $d_l$ in Eq. (79) can be calculated from Geroch definition of multipole moments which is zero for all $l < 4$ and its expression for $l \geq 4$ is shown in [6]. Following the same procedure, we can obtain the monopole and quadrupole moments for the line element (62) as below:

$$M_0 = m$$  \hspace{1cm} (83)

$$M_2 = -m \eta^2$$  \hspace{1cm} (84)

Here the monopole moment describes the total mass and the quadrupole moment describes the deformation of the object from the spherical symmetry. This is an important characteristic of the deformed Schwarzschild object.

- **Comparison with some existing solutions** In 2011 Nikouravan [19] claimed a solution to the space–time produced by the ellipsoidal objects which is written as follows:

$$ds^2 = \left(1 - \frac{2m}{u}\right) c^2 dt^2 - \frac{1}{\left(1 - \frac{2m}{u}\right)} \left[ \frac{u^2 + \eta^2 \cos^2 \theta}{u^2 + \eta^2} \right] du^2$$

$$- \left( u^2 + \eta^2 \cos^2 \theta \right) d\theta^2 - \left( u^2 + \eta^2 \right) \sin^2 \theta d\phi^2$$  \hspace{1cm} (85)

This line element (85) does not coincide with the one we obtained in the present paper [Eq. (62)]. But it is to be noted that Eq. (85) also reduces to Schwarzschild line element at sufficiently large value of $u$ and possesses asymptotically flat behaviour.
The observed differences between the Eqs. (62) and (85) may be due to the assumptions made and (or) procedures followed during the derivation of the two line elements. However, using the above mentioned procedure, if we calculate the Geroch quadrupole moment \( M_2 \) for the Nikouravan [19] metric [corresponding to Eq. (85)], we find the term \( M_2 \) becomes zero which is unacceptable. And this is a remarkable difference when we calculate the value of the same term \( M_2 \) based on the present work [viz. metric (62)]. Since the Geroch quadrupole moment \( M_2 \) is zero for the Nikouravan [19] metric, we think that this is a sufficient condition to state that this metric does not represent the space–time geometry produced by an ellipsoidal shaped object. Nevertheless, further investigation is required to understand the actual cause for such result.

On the other hand, the \( q \)-metric which is the simplest form of the line elements based on the quadrupole parameter is written as below [13]:

\[
\text{ds}^2 = \left(1 - \frac{2m}{r}\right)^{1+q} c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-q} \left[ \left(1 + \frac{m^2 \sin^2 \theta}{r^2 - 2mr}\right)^{-q(2+q)} \right. \\
+ \frac{dr^2}{1 - 2m/r} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] 
\]

(86)

Here it is obvious that the line element in Eq. (62) will be different from (86) as the former is based on linear eccentricity (\( \eta \)) while the latter is based on quadrupole parameter (\( q \)). The line element as in Eq. (86) was suggested by Quevedo with the interpretation that \( q \) represents the dimensionless quadrupole moment parameter. However no exact and explicit mathematical expression for \( q \) was provided at that time [13]. Till today other investigators, e.g. Boshkayev et al. [29], Mejía et al. [30], Neznamov and Shemarulin [31], Faraji [32] and others are using the same line element.

Additionally, the Geroch monopole and quadrupole moments of the metric (86) can be written as [13]:

\[
M_0 = (1 + q)m
\]

(87)

\[
M_2 = -\frac{m^3}{3} q(1 + q)(2 + q)
\]

(88)

The allowed values of the parameter \( q \) are \(-1 < q \) in order to avoid the negative total mass and it describes the deformation of the object which is positive, negative and zero for oblate, prolate and spherical shape of the object respectively. Redefining \( M_2 = pM_0^3 \) as mentioned in [33] we obtained from Eqs. (87) and (88) as below [30]:

\[
q = \frac{1}{\sqrt{1 + 3p}} - 1, \quad \text{and} \quad m = M_0\sqrt{1 + 3p}
\]

(89)

Hence the values of the parameter \( p \) are restricted by the inequality \(-\frac{1}{3} < p \) in order to have real values of \( q \).
A comparison with the multipole moments due to $q$-metric [Eqs. (87), (88)] and those of the metric (62) suggests that the redefinition of the parameter $q$ can lead to equivalent values of monopole and quadrupole moments for both the line elements [(62) and (86)]. In addition to that both the line elements with $\eta^2 > 0$ (or $\eta^2 < 0$) and $q > 0$ (or $q < 0$) respectively will represent the exterior gravitational field for a static oblate (or prolate) mass. However, the $q$-metric cannot be used to represent the gravitational field due to the objects having $p \leq -\frac{1}{3}$ (that includes the Sun, the Earth etc.) as mentioned in [16]. Therefore either the $q$-metric approach should be considered only for a set of limited number of celestial objects or further investigation is required to properly understand the parameter $q$. On the other hand, $\eta$ (introduced in the present work) is an easy to understand parameter, which represents the linear eccentricity of the gravitational object and is written as $\eta = \sqrt{a^2 - b^2}$, where $a$ and $b$ denote the equatorial and the polar radius of the celestial objects respectively. Therefore the parameter $\eta$ can be measured from observation directly without any restrictions. This indicates that the metric (62) with its $\eta$ parameter, has notable advantages in astrophysical applications over the $q$-metric. Apart from that, both the $q$-parameter line element [13, 34] and the line element in present case (62) possess naked singularities.

Therefore from the above discussions, we conclude that the line element (62) also describes the gravitational field of mass with quadrupole moment. Additionally, it is also to be noted that the present line element would be easier to be calculated as compared to the line element with $q$-parameter (86).

5 Conclusion

In this paper we solved the Einstein’s field equation in order to obtain the line element outside the static, ellipsoidal object. Consequently we obtained a line element for the said space–time geometry which is characterized by the parameter $\eta$ called linear eccentricity. It is observed that this line element possesses asymptotically flat behaviour which means that at spatial infinity the line element reduces to the Minkowski space–time (i.e. the gravitational field at large distance is negligible). In addition to that the derived line element takes the form of the Schwarzschild line element when we neglect terms in the order of $r^{-2}$ or higher in the expression for metric elements at large distances. In other words, the gravitational fields behave like that of the spherical symmetric objects in that particular distance. Notably the derived line element (62) possesses non-zero quadrupole moment which is an important properties of a deformed Schwarzschild object. Further the space–time characterized by the parameter $\eta$ also exhibits naked singularities similar to as observed for line element with quadruple parameter $q$ [6, 13].

The line element derived in the present work consists of physically relevant characters and therefore we conclude that this represents most effectively the gravitational field produced by the static ellipsoidal objects with linear eccentricity $\eta$. Additionally

---

\[4\] The notations $\gamma$ and $q$ used in reference [16] are respectively denoted by $(1 + q)$ and $p$ in the present work.
as the derived line element contains easily measurable parameters, so it will be suitable in various theoretical studies as well as experimental measurements.

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Data availability statement Since this is a theoretical work data availability is not applicable and no data has been used or analysed throughout the work.

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