Real analyticity of accessory parameters

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Abstract

We consider the problem of the real analytic dependence of the accessory parameters of Liouville theory on the moduli of the problem, for general elliptic singularities. We give a simplified proof of the almost everywhere real analyticity in the case of a single accessory parameter as it occurs e.g. in the sphere topology with four sources or for the torus topology with a single source by using only the general analyticity properties of the solution of the auxiliary equation. We deal then the case of two accessory parameters. We use the obtained result for a single accessory parameter to derive rigorous properties of the projection of the problem on lower dimensional planes. We derive the real analyticity result for two accessory parameters under an assumption of irreducibility.
1 Introduction

In several developments of Liouville theory it is necessary to exploit the nature of the dependence of the accessory parameters on the moduli of the problem. Typical field in which one exploits the real analytic nature of the dependence of the accessory parameters on the moduli, is the proof of the Polyakov relation, relating the accessory parameters to the derivative of the on shell action of Liouville theory with respect to the position of the singularity \[1,2,3,4\] and its extension \[5\] to the moduli of an hyperelliptic surface.

With regard to the proof of the real analytic dependence of the accessory parameters on the position of the singularity we have, for the case of parabolic and finite order singularities the result by Kra \[6\] according to which the accessory parameters are real analytic functions (not analytic functions) of the moduli i.e. of the position of the singularities.

Parabolic singularities (sometimes called punctures) are characterized by the strength of the source given by \[\eta = \frac{1}{2}\] (see eq.(1)), while finite order singularities are elliptic singularities with source strength given by \[\eta = \frac{(1 - 1/n)}{2}, \ n \in \mathbb{Z}^+\].

Kra employs, in presence of only parabolic and finite order singularities, the possibility of using fuchsian mapping techniques.

Such technique is not available in presence of general elliptic singularities.

In presence of a single accessory parameter, e.g. the sphere with the four-sources or the torus with one source, it was proven in \[7,8\] that also for general elliptic singularities, the accessory parameter is a real analytic function of the moduli except for a zero measure region i.e. almost everywhere (a.e.) in the moduli space. In the proof of such a result one exploits only the uniqueness result for the value of the accessory parameter combined with very general analytic properties of the matrix elements of the monodromy matrices. Such analytic properties are a direct consequence of the representation of the monodromy matrices in term of the solutions of the auxiliary differential equation.

In the present paper we give a simplified proof of such a result and explore the general features of the problem when more than one accessory parameter is present, always using only the general analytic properties of the solutions of the auxiliary equation.

To keep the formalism into reasonable complexity we consider the case of two accessory parameters, which is the case e.g. of the sphere with five sources or of the torus with two sources.

We give rigorous results on the projection of the problem on lower dimensional planes which is a necessary step toward the analyticity result. At this stage however it appears that the general analytic properties of the solutions of the auxiliary equation are not sufficient to progress. We show that under an irreducibility assumption one reaches the
final result of the a.e. real analyticity also in presence of two accessory parameters. Such irreducibility properties however should be derived by poking more deeply into the consequences of auxiliary equation.

The paper is organized as follows: In section 2 we give the simplified treatment of the one accessory parameter problem. In section 3 we deal with the two accessory parameter problem giving in subsection 3.1 the results which follow from the general analytic properties of the solutions of the auxiliary equation and in subsection 3.2 the treatment of the irreducible case. In section 4 we summarize the obtained results and we give a discussions of the possible developments. In the Appendix the proof is given of a lemma which is instrumental for all the described developments.

2 The case of a single accessory parameter

We recall that the Liouville conformal field \( \phi \) which satisfies the partial differential equation

\[-\partial_z \partial_{\bar{z}} \phi + e^{\phi} = 2\pi \sum_j \eta_j \delta^2(z - z_j)\]  

(1)

can be expressed in terms of the solutions of the auxiliary ordinary differential equation in the complex plane

\[y'' + Qy = 0\]  

(2)

as

\[e^{-\frac{\phi}{2}} = \frac{1}{\sqrt{2|w_{12}|}} \left[ \kappa^{-2} y_1(z)y_1(z) - \kappa^2 y_2(z)y_2(z) \right]\]  

(3)

where \( y_1, y_2 \) are two independent solution of (2), \( w_{12} \) their wronskian and in the case of the sphere topology

\[Q = \sum_j \eta_j(1 - \eta_j) + \frac{\beta_j}{2(z - z_j)}\]  

(4)

The \( \kappa \) is a real parameter. In the case of the torus and all hyperelliptic surfaces again the solution of eq.(1) can be reduced to the solution of a similar equation [8]. The \( \beta_j \) are the accessory parameters which have to be chosen along with \( \kappa \) as to have the \( \phi \) appearing in eqs.(1, 3) single valued.

The accessory parameters which realize the single valued solution of eq.(1) are unique. This is seen by recalling the uniqueness and existence theorem for the solution of eq.(1) [9, 10, 11, 12, 13] and noticing that

\[e^{\frac{\phi}{2}} \partial_z e^{-\frac{\phi}{2}} = -Q(z) \quad \beta_j = \frac{1}{i\pi} \oint Q(z)dz\]  

(5)
and thus each $\beta_j$ can be recovered from a contour integral in the $z$-plane. It is easily seen that the $\beta_j$ are continuous functions of the moduli $[1, 2]$. 

In addition the $\beta_j$ are subject to Fuchs conditions [7, 8]: e.g. for the sphere with four sources we have three $\beta$’s and two Fuchs conditions and thus one independent accessory parameter to be determined and the same for the torus with one source. In the present section we shall refer for concreteness to the sphere with four sources, but there is no difference in the treatment.

For the location of the sources one can choose $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ and $z_4 = u$ and after choosing a canonical basis around the source at $z = 0$ let $M(C_1)$ be the monodromy around the source at 1 and $M(C_2)$ the monodromy around the source at $u$. After the imposition of single valuedness of the Liouville field around $z = 1$ and $z = u$, i.e. imposition of the $SU(1, 1)$ nature of their monodromies we have that the $SU(1, 1)$ nature of the monodromy at infinity is also assured.

The monodromy matrix around a singularity is given by [7]

$$M = \begin{pmatrix} \tilde{y}_1 & \tilde{y}'_1 \\ \tilde{y}_2 & \tilde{y}'_2 \end{pmatrix} \begin{pmatrix} y'_2 & -y'_1 \\ -y_2 & y_1 \end{pmatrix}. \quad (6)$$

where $y_1, y_2$ are the solution at a point $z$ and $\tilde{y}_1, \tilde{y}_2$ are the same solutions computed at the same point $z$ after encircling the singularity. The solutions $y_j$ at any point in the complex $z$-plane are obtained from the absolutely convergent Cauchy series along a path which keeps at a finite distance of the singularities and as such the $y_j$ are analytic function of the parameters appearing in $Q$ and thus of the free accessory parameter and of the position of the singularity $z_4 = u$.

Thus also the matrix elements of all the monodromy matrices are analytic functions of the free accessory parameter $\beta$ and of $u$.

From (3) we see that $\kappa$ is given by the relative weight of the two canonical solution $y_1, y_2$ at $z = 0$ and thus is fixed by the ratio of $M_{12}$ to $M_{21}$. $M(C_0)$ is diagonal and as $\kappa^2$ is determined by the unique solution of the Liouville equation (1) we must have $M_{12}(C_1) \neq 0$ or $M_{12}(C_2) \neq 0$. Let $M_{12}(C_1) \neq 0$ and thus also $M_{21}(C_1) \neq 0$.

Then we can the form the ratios

$$A(\beta, u) = \frac{M_{12}(C_2)}{M_{12}(C_1)} \quad (7)$$

$$B(\beta, u) = \frac{M_{21}(C_2)}{M_{21}(C_1)} \quad (8)$$

From eq.(6) the matrix elements of $M(C_j)$ are analytic functions of the free accessory parameter $\beta$ and of the modulus $u$. 

3
The $SU(1, 1)$ nature of the monodromy i.e. the possibility of choosing a $\kappa$ such that all monodromies become $SU(1, 1)$ implies

$$A(\beta, u) = \bar{B}(\bar{\beta}, \bar{u}). \tag{9}$$

Once eq. (9) is satisfied, by exploiting the freedom on the real parameter $\kappa$ we can obtain $M_{21}(C_1) = \overline{M_{12}(C_1)}$ and $M_{21}(C_2) = \overline{M_{12}(C_2)}$. On the other hand the monodromies $M(C_j)$ are by construction $SL(2, C)$ and thus we have

$$M_{11}(C_j)M_{22}(C_j) = 1 + M_{12}(C_j)M_{21}(C_j) > 1. \tag{10}$$

Being the singularity elliptic we have $M_{11}(C_j) + M_{22}(C_j) = -2 \cos \alpha_j = \text{real}$, giving $M(C_j) \in SU(1, 1)$ which is the necessary and sufficient condition for having a single valued $\phi$. We recall now that $M(C_0)$ is already $SU(1, 1)$, $M(C_1)$ and $M(C_2)$ become $SU(1, 1)$ using eq. (9) and $M(C_\infty)$ becomes $SU(1, 1)$ as a consequence.

Thus the satisfaction of (9) is both necessary and sufficient for the single-valuedness of $\phi$ all over the plane.

It is useful now to proceed with the method of polarization i.e. to consider $\beta$ and $\bar{\beta}$, and $u$ and $\bar{u}$ as independent complex variables. Then eq. (9) becomes the system

$$A(\beta, u) = \bar{B}(\beta^c, \bar{u})$$
$$B(\beta, u) = \bar{A}(\beta^c, \bar{u}) \tag{11}$$

with the proviso that at the end we shall be interested into the self-conjugate (s.c.) solutions of eq. (11) i.e. in the solution for which at $u = \bar{u}$ we have $\beta^c = \bar{\beta}$.

We are interested in the analytic behavior of $\beta$ in a neighborhood of $u_0$ and to simplify the notation we shall from now on denote with $u$ the difference $u - u_0$ and with $\beta$ the difference $\beta - \beta(u_0)$. In addition we shall denote with $A(\beta, u)$ the difference $A(\beta, u) - A(\beta(u_0), u_0)$ and the same for $B$ so that now $A(0, 0) = 0$, $B(0, 0) = 0$.

We notice that due to the structure of (11) if $\beta, \beta^c$ is a solution at $u, \bar{u}$ and $\bar{\beta}$ is a solution at $\bar{u}$, $\bar{\beta}$ in particular if the solution of (11) at $u = \bar{u} = 0$ is unique it is self-conjugate and thus due to uniqueness of the physical solution $\beta = \beta^c = 0$.

Before entering the details we outline the structure of the proof of the real analyticity of $\beta(u)$. If for $u = \bar{u} = 0$ the origin $\beta = \beta^c = 0$ is an isolated solution of the polarized system (11), then as we shall see the real analyticity of $\beta$ easily follows. What we shall prove is that if the origin $\beta = \beta^c = 0$ is not an isolated solution then we can construct a solution with $\beta^c = \bar{\beta} \neq 0$ thus violating the uniqueness theorem.

We come now to the details. $A(\beta, 0)$ and/or $B(\beta, 0)$ depend on $\beta$ otherwise the system (11) would not determine $\beta = \beta^c = 0$ for $u = \bar{u} = 0$. 

4
Using Weierstrass preparation theorem (WPT) \cite{14, 15, 16} we can rewrite system (11) as

\[
P_1(\beta^c|\beta, u, u^c) = 0 \\
P_2(\beta^c|\beta, u, u^c) = 0.
\]  

(12)

Necessary and sufficient condition for the two polynomials (12) to have a common root \(\beta_c\) is the vanishing of their resultant \cite{14, 17} \[
R(P_1, P_2) \equiv h(\beta, u, u^c) = 0.
\]  

(13)

In particular from the existence result we know \[
h(0, 0, 0) = 0.
\]  

(14)

We exploit again Weierstrass preparation theorem applied to (13) writing \[
h(\beta, u, u^c) = w(\beta, u, u^c)P(\beta|u, u^c) = 0
\]  

(15)

being \(w(\beta, u, u^c)\) a unit, i.e. an analytic function non zero in a neighborhood of \(\beta = u = u^c = 0\). However to apply the WPT to \(h\) we must have that \(h(\beta, 0, 0) \neq 0\). We shall prove that if \(h(\beta, 0, 0) \equiv 0\), which means that we can solve eq. (11) for \(u = u^c = 0\) for any \(\beta\) in a neighborhood of zero, then we have s.c. solutions of eq. (11) at \(u = u^c = 0\) for \(\beta \neq 0\) different from zero, which goes against the uniqueness theorem. Reducing the dependence of \(\beta\) on \(u, u^c\) through the non trivial relation \(P(\beta|u, u^c) = 0\) is the major step in analyzing the analytic properties of such a dependence \cite{7}.

As we mentioned above we prove now that if \(h(\beta|0, 0) \equiv 0\) then we have s.c. solutions of eq. (11) at \(u = u^c = 0\) for \(\beta\) different from zero thus violating the uniqueness result.

We consider first the case in which the analytic function of \(\beta\), \(A(\beta, 0)\) is of order 1 at \(\beta = 0\) i.e. \(A(\beta, 0) = \beta a(\beta, 0)\) with \(a(0, 0) \neq 0\). Then the system (11) goes over to \[
\beta a(\beta, 0) = \beta^c \bar{b}(\beta^c, 0) \\
\beta b(\beta, 0) = \beta^c \bar{a}(\beta^c, 0).
\]  

(16)  

(17)

Multiplying we obtain for the solutions of the system \[
a(\beta, 0)\bar{a}(\beta^c, 0) = b(\beta, 0)\bar{b}(\beta^c, 0).
\]  

(18)

We notice that eq. (16) given \(\beta\) near \(\beta = 0\) can be solved in the form of the convergent series \[
\beta^c = \frac{a(0, 0)}{b(0, 0)}\beta + c_2\beta^2 + \ldots
\]  

(19)
and similarly for eq. (17) given $\beta^c$. This is the result of the implicit function theorem \cite{14, 15}. In the limit $\beta \to 0$ we have
\begin{equation}
\frac{\beta^c}{\beta} \to \frac{a(0,0)}{b(0,0)} = \frac{b(0,0)}{a(0,0)} .
\tag{20}
\end{equation}

An other consequence of the system \cite{16, 17} is
\begin{equation}
\beta^2 a(\beta,0)b(\beta,0) = (\beta^c)^2 \bar{a}(\beta^c,0)\bar{b}(\beta^c,0) .
\tag{21}
\end{equation}

We look for a s.c. solution of (21) i.e. a solution with $\beta^c = \bar{\beta}$ or more explicitly $\beta = \rho e^{i\alpha(\rho)}$, $\beta^c = \rho e^{-i\alpha(\rho)}$ where the unknown is $\alpha(\rho)$, with boundary condition \cite{20}.

The function $\alpha(\rho)$ is given by the vanishing of the real function
\begin{equation}
f(\rho, \alpha) = i(e^{2i\alpha(\rho)}a(\rho e^{i\alpha(\rho)},0)b(\rho e^{i\alpha(\rho)},0) - e^{-2i\alpha(\rho)}\bar{a}(\rho e^{-i\alpha(\rho)},0)\bar{b}(\rho e^{-i\alpha(\rho)},0)) .
\tag{22}
\end{equation}

For solving the equation $f(\rho, \alpha) = 0$ in the neighborhood of $\rho = 0$ we use the real implicit function theorem. We have from \cite{20, 18}
\begin{equation}
f(0, \alpha(0)) = 0
\tag{23}
\end{equation}

and
\begin{equation}
\left. \frac{\partial f(\rho, \alpha)}{\partial \alpha} \right|_{\rho=0} = -2(e^{2i\alpha(0)}a(0,0)b(0,0) + e^{-2i\alpha(0)}\bar{a}(0,0)\bar{b}(0,0)) = -4e^{2i\alpha(0)}a(0,0)b(0,0) \neq 0 .
\tag{24}
\end{equation}

Thus chosen $\alpha(0)$ satisfying \cite{20}, $\alpha(\rho)$ exists and is unique around $\rho = 0$. We show now that $\beta = \rho e^{i\alpha(\rho)}$, $\beta^c = \rho e^{-i\alpha(\rho)}$ is a solution of the system \cite{16, 17}. Always working in a neighborhood of the origin we know that given any $\beta$, and in particular $\beta = e^{i\alpha(\rho)}\rho$, we have a unique $\beta^c$ which solves \cite{16, 17}. On the other hand given $\beta$, the solutions $\beta^c$ of eq. (21) with boundary conditions $\beta^c/\beta = a(0,0)/\bar{b}(0,0)$ is unique. In fact setting $\beta^c = \Omega \bar{\beta}$, $\Omega(0) = 1$ to fulfill the boundary conditions we have with
\begin{equation}
F = \Omega^2 \bar{a}(\Omega \bar{\beta},0)\bar{b}(\Omega \bar{\beta},0) - \bar{a}(\bar{\beta},0)\bar{b}(\bar{\beta},0)
\tag{25}
\end{equation}

\begin{equation}
F(0) = 0, \quad \left. \frac{\partial F}{\partial \Omega} \right|_{\Omega=1, \beta=0} = 2\bar{a}(0,0)\bar{b}(0,0) \neq 0
\tag{26}
\end{equation}

and using the implicit function theorem we have that $\Omega \equiv 1$ is the unique solution of eq. (25) in a neighborhood of $\rho = 0$. Thus being the solution of eq. (21) for $\beta = \rho e^{i\alpha(\rho)}$ unique, this is also the unique solution of eq. (11) at $u = u^c = 0$.

In conclusion we found a non zero s.c. to the original system (11) at $u = u^c = 0$ and this goes against the uniqueness theorem for the solution of the Liouville equation which implies the uniqueness of the accessory parameters.
We treat now the case in which \( A \) and \( B \) are of higher order at the origin. We have
\[
\beta^m a(\beta, 0) = (\beta^c)^m b(\beta^c, 0) \quad (27)
\]
\[
\beta^n b(\beta, 0) = (\beta^c)^n \bar{a}(\beta^c, 0). \quad (28)
\]
Then \( n = m \). We have also for a solution of eqs. (27, 28) the equation
\[
\beta^{2m} a(\beta, 0) b(\beta, 0) = (\beta^c)^{2m} \bar{a}(\beta^c, 0) \bar{b}(\beta^c, 0). \quad (29)
\]
and also for \( \beta \to 0 \)
\[
\left( \frac{\beta^c}{\beta} \right)^m \to \frac{a(0, 0)}{b(0, 0)} = \frac{b(0, 0)}{a(0, 0)} \quad (30)
\]
The root of eq. (30) is not a choice but is given by the hypothesis of existence of a solution, which we want to disprove. Around such a root
\[
\left( \frac{a(\beta, 0)}{b(\beta^c, 0)} \right)^\frac{1}{m} \quad (31)
\]
is an analytic function both of \( \beta \) and \( \beta^c \), being \( a \) and \( b \) units and thus under such boundary condition \( \beta^c \) is the unique solution of the system (27, 28).

Again we solve as previously the equation
\[
\beta^{2m} a(\beta, 0) b(\beta, 0) = (\bar{\beta})^{2m} \bar{a}(\bar{\beta}, 0) \bar{b}(\bar{\beta}, 0) \quad (32)
\]
with \( \beta = \rho e^{i\alpha(\rho)} \), \( \bar{\beta} = \rho e^{-i\alpha(\rho)} \) and with
\[
e^{-2i\alpha(0)} = \left( \frac{a(0, 0)}{b(0, 0)} \right)^\frac{1}{m}. \quad (33)
\]
Again given \( \beta = \rho e^{i\alpha(\rho)} \) the solution of eq. (29) is unique and thus the unique solution of the system (27, 28) with our \( \alpha(\rho) \) is given by \( \beta^c = \rho e^{-i\alpha(\rho)} \) and we obtained a non zero s.c. solution of the initial system (11) thus violating the uniqueness theorem.

We have reached the conclusion that \( h(\beta, 0, 0) \not\equiv 0 \) and thus we can apply the WPT giving \( \beta \) as solution of \( P(\beta|u, u^c) = 0 \), with the \( P \) appearing in (15).

For completeness we recall the proof of how from \( h(\beta|0, 0) \not\equiv 0 \) the a.e. real analyticity of \( \beta \) follows.

From \( h(\beta|0, 0) \not\equiv 0 \) we derived the equivalent W-polynomial \( P(\beta|u, u^c) \). We notice that as a rule \( P(\beta|u, u^c) = 0 \) in addition to the physical solution \( \beta(u, \bar{u}) \) will have other solutions for \( u^c \neq \bar{u} \). We decompose \( P \) in irreducible components
\[
P(\beta|u, u^c) = P_1(\beta|u, u^c) \ldots P_n(\beta|u, u^c). \quad (34)
\]
We know that the discriminant \(D_j(u, u^c)\) of \(P_j\) is an analytic function of \(u, u^c\) which is not identically zero and thus vanishes on a thin \(\text{set } S\) of \(u, u^c\) which as such is of zero 4-dimensional measure in \(u, u^c\). Then we have that except for such a set all the solutions of \(P_j = 0\) are distinct and thus analytic function of \(u, u^c\) and in particular the physical solution, obtained setting \(u^c = \bar{u}\), is a real analytic function of \(u\). The subset \(S'\) of \(S\) with \(u^c = \bar{u}\) has zero 2-dimensional measure.

3 The case of two accessory parameters

3.1 General results

We come now to the more complicated case of two accessory parameters. Typical examples are the sphere with five sources and the torus with two sources. This time we need three monodromies to determine the \(\beta_1\) and \(\beta_2\). We shall be interested in the dependence of the accessory parameters on a single modulus which we shall call \(u\), e.g. the position of the third source in the problem of the sphere with five sources as the same treatment can be repeated for the other moduli. Moreover after a shift we shall work in the neighborhood of \(u = 0\). This covers the general case.

Following the discussion given at the beginning of the previous section we define, with \(M_{12}(C_1) \neq 0\) and thus \(M_{21}(C_1) \neq 0\)

\[
A(\beta_1, \beta_2, u) = \frac{M_{12}(C_2)}{M_{12}(C_1)} \quad (35)
\]

\[
B(\beta_1, \beta_2, u) = \frac{M_{21}(C_2)}{M_{21}(C_1)} \quad (36)
\]

\[
C(\beta_1, \beta_2, u) = \frac{M_{12}(C_3)}{M_{12}(C_1)} \quad (37)
\]

\[
D(\beta_1, \beta_2, u) = \frac{M_{21}(C_3)}{M_{21}(C_1)}. \quad (38)
\]

The \(SU(1, 1)\) relations for the monodromy matrices, after performing polarization, form two systems of equations and with the already discussed shift in the \(\beta_j\) the \(u\) and the


\[ A(\beta_1, \beta_2, u) = \bar{B}(\beta_1^c, \beta_2^c, u^c) \quad I \]

\[ B(\beta_1, \beta_2, u) = \bar{A}(\beta_1^c, \beta_2^c, u^c) \quad I^c \]

\[ C(\beta_1, \beta_2, u) = \bar{D}(\beta_1^c, \beta_2^c, u^c) \quad II \]

\[ D(\beta_1, \beta_2, u) = \bar{C}(\beta_1^c, \beta_2^c, u^c) \quad II^c. \]

(39)

with \( A(0, 0, 0) = 0 \) and the same for \( B, C, D \). We notice again the self-conjugate structure of the system (39): If at \( u, u^c, \beta_1, \beta_1^c, \beta_2, \beta_2^c \) is a solution of (39) then \( \bar{\beta}_1^c, \bar{\beta}_1, \bar{\beta}_2^c, \bar{\beta}_2 \) is a solution of (39) at \( \bar{u}, \bar{u}^c \).

The uniqueness theorem on the solution of Liouville equation tells us that the unique s.c. solution to (39) at \( u = u^c = 0 \) i.e. to

\[ A(\beta_1, \beta_2, 0) = \bar{B}(\bar{\beta}_1, \bar{\beta}_2, 0) \]

\[ C(\beta_1, \beta_2, 0) = \bar{D}(\bar{\beta}_1, \bar{\beta}_2, 0) \]

(40)

is \( \beta_1 = \beta_2 = 0 \). It will be useful, by performing a linear invertible transformation

\[ \beta_j \rightarrow a_{j1} \beta_1 + a_{j2} \beta_2, \quad \beta_j^c \rightarrow \bar{a}_{j1} \beta_1^c + \bar{a}_{j2} \beta_2^c, \]

(41)

to render \( A(\beta_1, \beta_2, 0) \) regular in both variables and the same for \( B, C, D \) by a single transformation [16] without loosing the s.c. property of the system. It means that if the order of \( A(\beta_1, \beta_2, 0) \) is e.g. 2 than after the transformation \( \beta_1^c \) appears with coefficient \( a_1(\beta_1, \beta_2) \) with \( a_1(0, 0) \) different from zero and the same for \( \beta_2, \) i.e. \( a_2(0, 0) \neq 0 \).

We want to deal here with the general situation when the order of the system \( I, I^c \) and \( II, II^c \) at \( u = u^c = 0 \) is arbitrary.

Given the \( I, I^c, II, II^c \) at \( u = u^c = 0 \) if they all do not depend on the \( \beta \)’s we can set \( \beta_1^c = \bar{\beta}_1 \neq 0, \beta_2^c = \bar{\beta}_2 = 0 \) to have a violation of the uniqueness theorem. From the structure of the equations we see that if the system depends on \( \beta_1 \) it depends also on \( \beta_1^c \).

If at this point it does not depend on \( \beta_2 \) and \( \beta_2^c \) we can set \( \beta_1^c = \bar{\beta}_1 = 0 \) and \( \beta_2^c = \bar{\beta}_2 \neq 0 \) to have a violation of the uniqueness theorem. Thus (39) depends on all the \( \beta \)’s. As we chose a regular set of variable we can eliminate \( \beta_2^c \). In fact as \( \bar{B}(0, \beta_2^c, 0) \) depends on \( \beta_2^c \) we have for some \( \varepsilon \)

\[ |\bar{B}(0, \beta_2^c, 0)| > \eta \quad \text{for} \quad |\beta_2^c| = \varepsilon \]

(42)

and then

\[ |A(\beta_1, \beta_2, u) - \bar{B}(\beta_1^c, \beta_2^c, u^c)| > \eta/2 \]

(43)
for $\Delta \times \{|\beta^c_2| = \varepsilon\}$ where $\Delta = \{|\beta_1|^2 + |\beta^c_1|^2 + |\beta_2|^2 + |u|^2 + |u^c|^2 < \varepsilon_1\}$. This is a sufficient condition for the projection of the system (39) on the hyperplane $\beta_1, \beta^c_1, \beta_2, u, u^c$ [14]. The projection is an analytic variety given by the vanishing of a finite set of analytic functions
\begin{align*}
f_j(\beta_1, \beta^c_1, \beta_2, u, u^c) &= 0.
\end{align*}
Their number is given by $N = (m + 3)(m + 2)(m + 1)/3! - 1$ [14], being $m$ the order of the Weierstrass polynomial for $A(\beta_1, \beta_2, u) - \bar{B}(\beta^c_1, \beta^c_2, u^c)$ related to the variable $\beta^c_2$. For $m = 1$ such a number is 3 as expected. The number $N$ may depend on the chosen domain but we shall be interested in the germ [14, 15] of the variety i.e. in an arbitrary non zero neighborhood of the origin.

It is important to recall that the vanishing of the functions (44) is both a necessary and sufficient condition for having above such a point a solution of the system (39). Obviously $f_j(0, 0, 0, 0, 0) = 0$.

We now investigate the nature of the $f_j(\beta_1, \beta^c_1, \beta_2, 0, 0)$. In the remainder of this section we shall be interested in the variety at $V$ at $u = u^c = 0$ which we call $V_0$ and thus we shall omit the last argument in $A, B, C, D$ understanding that $u = u^c = 0$ and the same in the $f_j$.

If all $f_j(0, 0, \beta_2)$ do not depend on $\beta_2$ we have for any $\beta_2$ and $\beta^c_1 = \beta_1 = 0$ at least a $\beta^c_2$ which solves system (39). In particular for any $\beta_2$ exists at least a $\beta^c_2$ which solves
\begin{align*}
A(0, \beta_2) &= \bar{B}(0, \beta^c_2)
B(0, \beta_2) &= \bar{A}(0, \beta^c_2)
\end{align*}
which is the one-accessory parameter problem we have already solved in section 2. From the projection theorem we know that given $\beta_2$ we have at least one solution of (15) which is also a solution of the the same system with $C, D$ replacing $A, B$. Then following the procedure of section 2 we reach a s.c. solution i.e. a solution with $\beta^c_2 = \bar{\beta}_2$ violating the uniqueness theorem.

The conclusion is that $f_j(0, 0, \beta_2) \neq 0$ i.e. for some $j$ the $f_j$ is not identically zero.

We consider now the dependence of $f_j(0, \beta^c_1, 0)$ on $\beta^c_1$. If $f_j(0, \beta^c_1, 0) \equiv 0$ then we have solutions for $\beta_1 = \beta_2 = 0$ and any $\beta^c_1$. The consequence is that for any $\beta^c_1$ we have at least a $\beta^c_2$ such that
\begin{align*}
A(0, 0) &= 0 = \bar{B}(\beta^c_1, \beta^c_2)
B(0, 0) &= 0 = \bar{A}(\beta^c_1, \beta^c_2)
C(0, 0) &= 0 = \bar{D}(\beta^c_1, \beta^c_2)
D(0, 0) &= 0 = \bar{C}(\beta^c_1, \beta^c_2)
\end{align*}
(46)
or for any $x$ we have a $y$ such that $A(x, y) = B(x, y) = C(x, y) = D(x, y) = 0$ giving rise to a non zero solution of

$$
A(x, y) = \bar{B}(\bar{x}, \bar{y}) \\
C(x, y) = \bar{D}(\bar{x}, \bar{y})
$$

(47)

violating again the uniqueness theorem.

Thus we reach the conclusion that $f_j(0, \beta^c_1, 0) \neq 0$.

Using $f_j(0, \beta^c_1, 0) \neq 0$ we can project out $\beta_2$ and thus reach the projected analytic variety $g_j^{(1)}(\beta_1, \beta^c_1) = 0$ which are the necessary and sufficient condition for having a solution of the system $f_j(\beta_1, \beta^c_1, \beta_2) = 0$ and thus a point of the variety $V_0$ above $\beta_1, \beta^c_1$. Notice that due to the s.c. structure of $g$ we have also $\bar{g}_j^{(1)}(\beta^c_1, \beta_1) = 0$. We could also have $g_j^{(1)}(\beta_1, \beta^c_1) \equiv 0$ which means that the projection of the variety on the plane $\beta_1, \beta^c_1$ is a whole open neighborhood of $\beta_1 = \beta^c_1 = 0$.

Similarly using $f_j(0, \beta^c_1, 0) \neq 0$ we can project out the variable $\beta^c_1$ and reach the projected analytic variety $k_j(\beta_1, \beta_2) = 0$ which is the necessary and sufficient condition for having a point of the variety $f_j(\beta_1, \beta^c_1, \beta_2) = 0$ above $\beta_1, \beta_2$ and thus a point of the original variety $V_0$ above $\beta_1, \beta_2$.

We further remark that if $\beta_1, \beta^c_1, \beta_2, \beta^c_2$ is a point of $V_0$ then both $g_j^{(1)}(\beta_1, \beta^c_1) = 0$ and $k_j(\beta_1, \beta_2) = 0$ have to be satisfied. The reverse however is not necessarily true, i.e. if $\beta_1, \beta^c_1, \beta_2$ are such that $k_j(\beta_1, \beta_2) = 0$ and $g_j^{(1)}(\beta_1, \beta^c_1) = 0$ is not granted that above $\beta_1, \beta^c_1, \beta_2$ we have a point of the variety because $g_j^{(1)}(\beta_1, \beta^c_1) = 0$ does not assure that above $\beta_1, \beta^c_1$ we have a point of the variety with the chosen $\beta_2$.

All these constraints $g_j^{(k)}(\beta_k, \beta^c_k) = 0$, $k_j(\beta_1, \beta_2) = 0$, $k_j^{(2)}(\beta^c_1, \beta^c_2) = 0$ are necessary and sufficient conditions for having above the given a pair of $\beta$’s at least one solution of $g$ at $u = u^c = 0$.

We remark that if $g_j^{(1)}(\beta_1, \beta^c_1) \neq 0$ then also $k_j(\beta_1, \beta_2) \neq 0$. In fact if $k_j(\beta_1, \beta_2) \equiv 0$ we have a point the variety above $\beta_1 = 0$ and any $\beta_2$. But below such point we must have $g_j^{(1)}(0, \beta^c_1) = 0$ i.e. $\beta^c_1 = 0$ if $g_j^{(1)}(\beta_1, \beta^c_1) \neq 0$. Thus we are reduced to the problem $\beta_1 = \beta^c_1 = 0$ with the assurance that we have a solution of the system $g$ for any $\beta_2$.

This is the one-$\beta$ problem that we have already solved.

Vice-versa if $k_j(\beta_1, \beta_2) \neq 0$ and $g_j^{(1)}(\beta_1, \beta^c_1) \equiv 0$, for $\beta_1 = 0$ and any $\beta^c_1$ we have a point on the variety $V_0$. But below such point we must have also $\beta_2 = 0$ due to $k_j(\beta_1, \beta_2) \neq 0$, and thus solutions with $\beta_1 = \beta_2 = 0$ and any $\beta^c_1$. Explicitly for any $\beta^c_1$ we have a $\beta^c_2$ which satisfies the system $(46)$ and thus we have

$$
0 = A(\beta^c_1, \beta^c_2) = \bar{B}(\beta^c_1, \beta^c_2) \\
0 = C(\beta^c_1, \beta^c_2) = \bar{D}(\beta^c_1, \beta^c_2)
$$

(48)

(49)
which is a s.c. solution of the system \((39)\).

We conclude that the relations \(g_j(1)(\beta_1, \beta_1^c) \neq 0\) and \(k_j(\beta_1, \beta_2) \neq 0\) are equivalent.

We can perform the same reasoning eliminating the variable \(\beta_2\) to reach a variety given by

\[
f_j^c(\beta_1, \beta_1^c, \beta_2^c) = f_j(\beta_1, \beta_1^c, \beta_2^c) = 0
\]

with the result

\[
f_j^c(0, 0, \beta_2^c) \neq 0 \quad \text{and} \quad f_j^c(\beta_1, 0, 0) \neq 0
\]

from which the projections \(g_j^{(1)}(\beta_1, \beta_1^c) = 0\) and \(k_j^c(\beta_1^c, \beta_2^c) = 0\) can be performed.

Moreover due to the regular choice of variables there is no qualitative distinction between the index 1 and 2 we have also the projections

\[
g_j^{(2)}(\beta_2, \beta_2^c) = 0.
\]

We have reached the result that the conditions which express the non trivial nature of the related projections

\[
g_j^{(1)}(\beta_1, \beta_1^c) \neq 0, \quad k_j(\beta_1, \beta_2) \neq 0, \quad k_j^c(\beta_1^c, \beta_2^c) \neq 0, \quad g_j^{(2)}(\beta_2, \beta_2^c) \neq 0
\]

are all equivalent. Furthermore we notice that due to the s.c. nature of the system \((39)\) we have for the solutions of \(g_j^{(1)}(\beta_1, \beta_1^c) = 0\) the validity of \(g^{(1)}(\beta_1^c, \beta_1) = 0\) and the same with 1 replaced by 2 and \(k_j^c(\beta_1^c, \beta_2^c) = k_j(\beta_1^c, \beta_2^c)\). From now on we shall drop the upper index (1) in \(g_j^{(1)}\).

1) Let us consider first the case in which \(g_j(\beta_1, \beta_1^c) \neq 0\) with the consequences of eq.\((53)\).

In this case \(\beta_1^c\) is driven by \(\beta_1\) and we have a variety of complex dimension 1. We recall that \(f_j(0, 0, \beta_2) \neq 0\). Then a general results \([14]\) tells us that the variety \(V_0\) is given by a union of branches of a W-type variety i.e. of

\[
P_1(\beta_1^c | \beta_1) = 0
\]

\[
P_2(\beta_2 | \beta_1) = 0
\]

\[
P_2^c(\beta_1^c | \beta_1) = 0
\]

where the \(P\)'s are Weierstrass polynomial with discriminant not identically zero, thus W-polynomials with simple sheets. We can also write

\[
P(\beta_2 | \beta_1) = 0
\]

\[
P^c(\beta_2^c | \beta_1^c) = 0
\]

where in the last the \(\beta_1^c\) is driven by \((54)\).
We recall that due to the s.c. structure we have also the validity of

\[
\bar{P}_1(\beta_1|\beta_1^c) = 0 \quad (59)
\]

\[
\bar{P}_2(\beta_1^c|\beta_1^c) = 0 \quad (60)
\]

In the present case i.e. \( g_j(\beta_1, \beta_1^c) \neq 0 \) we can proceed to the further projection \( h_l(\beta_1|u, u^c) = 0 \) and if \( h_l(\beta_1|0, 0) \neq 0 \) we can apply the procedure of [7, 8] summarized at the end of section 2 of the present paper to prove that \( \beta_1 \) is a real analytic function of \( u \).

2) Let us consider now the case in which \( g_j(\beta_1, \beta_1^c) \equiv 0 \)

The result \( f_j(0, 0, \beta_2) \neq 0 \) combined with the absence of constraints between \( \beta_1 \) and \( \beta_1^c \) makes \( V_0 \) a variety of complex dimension 2 and thus given by the union of a number of branches of the W-type variety

\[
P_2(\beta_2|\beta_1, \beta_1^c) = 0 \quad (61)
\]

\[
\bar{P}_2(\beta_2^c|\beta_1^c, \beta_1) = 0 \quad (62)
\]

We notice that if we prove that \( (0, 0, 0, 0) \) is an isolated solution of \( (39) \) at \( u = u^c = 0 \), in the sense that for some \( \varepsilon \) the origin is the only solution with \( |\beta_j| < \varepsilon, |\beta_j^c| < \varepsilon \) then we have both \( g_j(\beta_1, \beta_1^c) \neq 0 \) and \( h_l(\beta_1|0, 0) \neq 0 \) and this is enough to reach the a.e. real analyticity of \( \beta_1 \) as a function of \( u \).

In fact choosing as domain \( H = \{|\beta_j| < \varepsilon, |\beta_j^c| < \varepsilon\} \) the solutions of \( h_l(\beta_1|0, 0) = 0 \) are exactly the projection of the solutions of the system (39) lying in \( H \) and thus if there are no solutions in \( H \) except the origin, the only projection is \( \beta_1 = 0 \), i.e. the only solution of \( h_l(\beta_1|0, 0) = 0 \) is \( \beta_1 = 0 \). We recall that the projections on lower dimensional planes and thus the \( g_j \) and \( h_l \) depend on the chosen initial domain and we are concerned with a neighborhood of the origin.

Summarizing, in this section we proved that we can always project the variety \( V_0 \) on the planes \( (\beta_1, \beta_1^c), (\beta_2, \beta_2^c), (\beta_1, \beta_2), (\beta_1^c, \beta_2^c) \). These projections are all non trivial if any of the relations of eq. (39) are satisfied. If the further projection \( h_l(\beta_1|u, u^c) = 0 \) is non trivial we have that the \( \beta_1 \) is a real analytic function of \( u \). Furthermore such a result is always true if one proves that the origin is an isolated point of \( V_0 \).

The above scheme worked perfectly in the case of one accessory parameter; it is however too general for the two parameter case, in the sense that we cannot prove, without further information, that the projection \( h_l(\beta_1|0, 0) \) is non trivial. Such information should be provided by a more detailed exploitation of the consequences of the auxiliary equation (2) or even from (1). Here below as an illustration we deal with the irreducible case.
3.2 The irreducible case

As we saw in the previous subsection we have to examine the two possibilities \( g_j \neq 0 \) and 
\( g_j \equiv 0 \).

1. \( g_j \neq 0 \).

We shall now work under the assumption that the variety \( V_0 \) is irreducible.

The variety \( g_j = 0 \) is the projection of \( V_0 \) on the plane \( \beta_1, \beta_1^c \) and due to the irreducibility of \( V_0 \) the projection \( g_j = 0 \) is irreducible. From the properties of (53) we see that in this case the dimension of \( V_0 \), i.e. the dimension of the associated manifold \( V_0^- \) is 1.

Then the variety \( V_0 \) is described by a union of branches of the W-type variety [14]

\[
\begin{align*}
P_g(\beta_1 | \beta_1) &= 0 \quad (63) \\
P_2(\beta_2 | \beta_1) &= 0 \quad (64) \\
P_2^c(\beta_2^c | \beta_1) &= 0 . \quad (65)
\end{align*}
\]

The \( P_g, P_2, P_2^c \) are irreducible as a consequence of the irreducibility of \( V_0 \); here \( P_g \) plays the role of the \( g \) of the Appendix.

The system (63) can also be rewritten as

\[
\begin{align*}
P_2(\beta_2 | \beta_1) &= 0 \quad (66) \\
\bar{P}_2(\beta_2 | \beta_1^c) &= 0 \quad (67)
\end{align*}
\]

where \( \beta_1^c \) is driven by (63). We recall that in addition to (63) we have by conjugacy the validity of

\[
\bar{P}_g(\beta_1 | \beta_1^c) = 0 . \quad (68)
\]

In the Appendix the following result is proven:

If \( g(\beta_1, \beta_1^c) \) is irreducible and for every \( \beta_1 \) there exists a \( \beta_1^c \) solution of \( g(\beta_1, \beta_1^c) = \bar{g}(\beta_1^c, \beta_1) = 0 \), then there exist \( \beta_1 \), with \( |\beta_1| \) as small as we like, such that \( \beta_1^c = \bar{\beta}_1 \) is solution of the previous system.

A point on \( V_0 \) is given by a compatible pair of \( \beta_1, \beta_1^c \) i.e. a solution of (63) and (68) and by the Puiseux series [14 [18]

\[
\begin{align*}
\beta_2 &= a_1 \beta_1^{\frac{1}{m}} e^{\frac{2\pi i H}{m}} + a_2 \beta_1^{\frac{2}{m}} e^{\frac{4\pi i H}{m}} + \ldots \quad (69) \\
\beta_2^c &= \bar{a}_1(\beta_1^c)^{\frac{1}{m}} e^{\frac{2\pi i K}{m}} + \bar{a}_2(\beta_1^c)^{\frac{2}{m}} e^{\frac{4\pi i K}{m}} + \ldots \quad (70)
\end{align*}
\]

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where the integers $H, K$ characterize the branch. We know by conjugacy that to the point $\beta_1, \beta_1^c, \beta_2, \beta_2^c$ there correspond the point $\beta_1^c, \beta_1, \beta_2^c, \beta_2$ which due to the irreducibility assumption has to belong to the above branch.

The values of $\beta_2, \beta_2^c$ belonging to such a branch, relative to a given compatible pair $\beta_1, \beta_1^c$ are

$$
\beta_2 = a_1 \beta_1^\frac{1}{m} e^{\frac{2\pi i H}{m}} e^{\frac{2\pi i n}{m}} + a_2 \beta_1^\frac{1}{m} e^{\frac{4\pi i H}{m}} e^{\frac{4\pi i n}{m}} + \ldots \tag{71}
$$

$$
\beta_2^c = \bar{a}_1(\beta_1^c)\frac{1}{m} e^{\frac{2\pi i K}{m}} e^{\frac{2\pi i n}{m}} + \bar{a}_2(\beta_1^c)\frac{1}{m} e^{\frac{4\pi i K}{m}} e^{\frac{4\pi i n}{m}} + \ldots \tag{72}
$$

where $n$ is the order of the $g$. This is due to the fact that under $\beta_1 \rightarrow \beta_1 e^{2\pi i n}$ both $\beta_1$ and $\beta_1^c$ are unchanged. By conjugacy we must have that substituting in (69), $\beta_1 \rightarrow \beta_1^c$ $\beta_1^c \rightarrow \bar{\beta}_1$ we find for some $N, \bar{\beta}_2^c$ i.e.

$$
a_1(\beta_1^c)\frac{1}{m} e^{\frac{2\pi i H}{m}} e^{\frac{2\pi i n}{m}} = (\bar{a}_1(\beta_1)^\frac{1}{m} e^{\frac{2\pi i K}{m}}) \tag{73}
$$

i.e.

$$
H + K + nN = mZ \tag{74}
$$

Thus $H + K$ has to belong to the $(m, n)$ ideal. We ask now, whether under such a restriction on $H + K$ we find a s.c. point on $V_0$ i.e. $\beta_2^c = \bar{\beta}_2$ with $\beta_1^c = \bar{\beta}_1$. From (71,72) the problem is to find an $N_1$ such that

$$
H + K + 2nN_1 = mZ_1 \tag{75}
$$

Combining with the above

$$
n(2N_1 - N) = mZ_2 \tag{76}
$$

This equation in which $N_1$ and $Z_2$ are free is easily solved for $m$ odd and when $m$ is even if also $N$ is even. To deal with the case $m$ even and $N$ odd, we go over to the sheet obtained by rotating $\beta_1$ by $e^{i\pi n}$ as done in the Appendix for even $n$. Then eqs. (71,72) become

$$
\beta_2 = a_1 \beta_1^\frac{1}{m} e^{\frac{2\pi i H}{m}} e^{i\frac{\pi n}{m}} e^{\frac{2\pi i n}{m}} + a_2 \beta_1^\frac{1}{m} e^{\frac{4\pi i H}{m}} e^{\frac{4\pi i n}{m}} + \ldots \tag{77}
$$

$$
\beta_2^c = \bar{a}_1(\beta_1^c)\frac{1}{m} e^{\frac{2\pi i K}{m}} e^{\frac{2\pi i n}{m}} + \bar{a}_2(\beta_1^c)\frac{1}{m} e^{\frac{4\pi i K}{m}} e^{\frac{4\pi i n}{m}} + \ldots \tag{78}
$$

which is like increasing $H$ and $K$ by $\frac{n}{2}$. Then eq. (74) becomes

$$
H + K + n + nN = mZ \tag{79}
$$

which makes the new $N$ even ad thus eq. (75) now soluble.
2. \( g_j \equiv 0 \).

The variety is given by a branch of the W-type variety\(^\star\) 

\[
P(\beta_2 | \beta_1, \beta_1^c) = 0
\]

\[
P^c(\beta_2^c | \beta_1, \beta_1^c) = P^c(\beta_2^c | \beta_1, \beta_1) = 0
\] (80)

where the \( P \) are irreducible polynomials. In the present case we have no constraint on \( \beta_1 \) and \( \beta_1^c \). We set \( \beta_1 = vx, \beta_1^c = \bar{vx}, \) with \(|v| = 1\).

Notice that for \( x = \text{real} \) we have \( \beta_1^c = \bar{\beta}_1 \). If for some \( v \), \( q(\beta_2, x) \equiv P(\beta_2 | vx, \bar{vx}) \) is irreducible we shall see right below that applying techniques similar to those developed in the Appendix we find a solution of system (39), with \( \beta_1^c = \bar{\beta}_1 \) and \( \beta_2^c = \bar{\beta}_2 \).

We know that \( P(\beta_2 | \beta_1, \beta_1^c) \) is irreducible, but from this it does not follow necessarily that there exists a \( v \) for which \( q(\beta_2, x) \) is irreducible. The irreducibility of the subvariety \( V^e \) given by (39) with \( \beta_1 = vx, \beta_1^c = \bar{vx} \) is an independent assumption.

We come now to the proof of the existence of a s.c. solution. We have the two equations

\[
P(\beta_2 | vx, \bar{vx}) = 0
\]

\[
\bar{P}(\beta_2^c | \bar{vx}, vx) = 0
\] (81)

and all solutions to the previous are given by the Puiseux series

\[
\beta_2 = c_1 x^{\frac{1}{m}} + c_2 x^{\frac{2}{m}} + ... \\
\beta_2^c = \bar{c}_1 x^{\frac{1}{m^c}} + \bar{c}_2 x^{\frac{2}{m^c}} + ... 
\] (82)

As these equations are identically satisfied we can send \( x^{\frac{1}{m}} \rightarrow e^{\frac{2\pi i}{m}} x^{\frac{1}{m}} \) and still have a solution. If for \( x \) there is a point \( vx, \bar{vx}, \beta_2, \beta_2^c \) on \( V^e \), i.e. solutions of (39) at \( u = u^c = 0 \), then there exist \( h \) and \( k \) such that

\[
\beta_2 = c_1 x^{\frac{1}{m}} e^{\frac{2\pi i h}{m}} + c_2 (x^{\frac{1}{m}} e^{\frac{2\pi i h}{m}})^2 + ... \\
\beta_2^c = \bar{c}_1 x^{\frac{1}{m^c}} e^{\frac{2\pi i k}{m^c}} + \bar{c}_2 (x^{\frac{1}{m^c}} e^{\frac{2\pi i k}{m^c}})^2 + ... 
\] (83)

where here by \( x^{\frac{1}{m}} \) the principal value is understood.

Such formulae must satisfy the conjugacy condition, i.e. that if we send \( \beta_1 \) in \( \beta_1^c \) (and \( \beta_1^c \) in \( \beta_1 \)) we must have as solutions \( \beta_2^c \) and \( \beta_2 \). This means that for some \( N \)

\[
c_1 x^{\frac{1}{m}} e^{\frac{2\pi i h}{m}} e^{\frac{2\pi i N}{m}} + c_2 (x^{\frac{1}{m}} e^{\frac{2\pi i h}{m}} e^{\frac{2\pi i N}{m}})^2 + ... \\
\bar{c}_1 x^{\frac{1}{m^c}} e^{\frac{2\pi i k}{m^c}} e^{\frac{2\pi i N}{m^c}} + \bar{c}_2 (x^{\frac{1}{m^c}} e^{\frac{2\pi i k}{m^c}} e^{\frac{2\pi i N}{m^c}})^2 + ... 
\] (84)

give respectively \( \beta_2^c \) and \( \beta_2 \) for real \( x \). I.e.

\[
c_1 x^{\frac{1}{m}} e^{\frac{2\pi i h}{m}} e^{\frac{2\pi i N}{m}} = \bar{c}_1 x^{\frac{1}{m^c}} e^{\frac{2\pi i k}{m^c}} e^{\frac{2\pi i N}{m^c}}
\] (85)
giving
\[ h + k + N = mZ . \] (86)

Given such a constraint on \( h + k \) we look for a s.c. solution, i.e. for an \( N_1 \) such that
\[ \frac{1}{c_1} x_1^m e^{\frac{2\pi i h}{m}} e^{\frac{2\pi i N_1}{m}} = \bar{c}_1 x_1^m e^{\frac{2\pi i k}{m}} e^{\frac{2\pi i N_1}{m}} \] (87)
requiring
\[ 2N_1 + h + k = 2N_1 - N + mZ = mZ_1 . \] (88)

For \( m \) odd or \( m \) even and \( N \) even this equation is always soluble. If \( m \) is even and \( N \) odd we work in (83) on the real negative \( x \) axis as done in the Appendix.

\[ \beta_2 = c_1 x_1^\frac{1}{m} e^{\frac{2\pi i h}{m}} e^{\frac{2\pi i}{m}} + \ldots \] (89)
\[ \beta_2' = \bar{c}_1 x_1^\frac{1}{m} e^{\frac{2\pi i k}{m}} e^{\frac{2\pi i}{m}} + \ldots \] (90)

which amounts to increasing \( h \) and \( k \) by \( 1/2 \). Then the conjugacy condition imposes
\[ h + k + 1 + N = mZ \] (91)

and now \( N \) is even, which allows the above equation to be solved.

4 Conclusions

In several developments and applications of Liouville theory one needs to exploit the real analytic nature of the dependence of the accessory parameters on the moduli of the problem. Real analyticity is a necessary requirement to extract e.g. the Polyakov relation [1, 2, 3, 4] and its extension to the moduli space [5]. Kra [6] proved using fuchsian mapping techniques that, for parabolic singularities (punctures) and finite order singularities, the accessory parameters are real analytic functions of the moduli. Such a technique is not available in the case of general elliptic singularities. On the other hand in most of applications one deals with general elliptic singularities.

In the case of one independent accessory parameter like the sphere topology with four sources or the torus with one source it was proven [7, 8] that the accessory parameters are real analytic functions except for a zero measure set in the moduli space. A weaker result (real analyticity on an everywhere dense set) had been proven in [11, 12].

In section 2 we give a simplified version of the proof of such a.e. real analyticity. Such a result follows only from the general analytic properties of the solutions of the auxiliary equation and the uniqueness theorem for the solution of Liouville equation [9, 10, 11, 12, 13].
One naturally asks whether the general analytic properties of the solutions of the auxiliary equation are again sufficient to establish the real analyticity of the accessory parameter when the independent parameters are more than one in number.

Thus in section 3 we considered the extension to the case of two independent accessory parameters. Typical examples are the sphere the five sources and the torus with two sources.

Again the aim is to extract all possible information from the general analytic properties of the solutions of the auxiliary equation. The main result obtained in section 3 is that, using the results obtained in the one-parameter case, one can project the problem on a two dimensional (complex) plane; however this is not enough for the general proof of real analyticity. For this reason we examined in subsection 3.2 the irreducible case.

We have two possibilities where the dimension of the variety is 1 or 2. For dimension 1 irreducibility of $V_0$ is sufficient for reaching the final result i.e. the proof of the real analytic dependence of the two accessory parameters on the moduli of the problem, e.g. the position of the sources and/or the moduli of the higher genus surface.

For dimension 2 we need the irreducibility of the variety $V_v$. After that again one proves the real analytic nature of the dependence of the accessory parameters.

Obviously the irreducibility of the manifolds $V_0$ or $V_v$ should be proven by poking more deeply into the consequences of the auxiliary equation (2) or by other procedures. We recall that irreducibility of a variety is equivalent to the connectedness property of the related manifold.

The question of the dependence of the accessory parameters on the moduli, like the position of the sources, is not a trivial one when we have more than one accessory parameter. One could think to extract the nature of the dependence from the constructive procedure for finding the solution of the Liouville equation (1) but this is not simple. The reason is that while the dependence of the real field $\phi(z)$ on $z$ can be easily shown to be $C^\infty$ and actually real analytic except at the sources, its dependence on the positions of the sources is highly non trivial. The constructive proof of the field $\phi$ goes through an iterative [9, 10] or minimization [11, 12, 13] procedure where is not easy to follow the nature of the dependence on the position of the sources.

In the present investigation we exploited only the very general analytic properties of the solutions of the auxiliary equations. For two or more accessory parameters such properties do not appear to be sufficient for proving the real analyticity of the dependence of the accessory parameters on the moduli on the problem and probably one has to develop more deeply the consequences of the auxiliary equation. Proving the irreducibility of the variety $V_0$ and $V_v$ would solve the problem. On the other hand it is remarkable that the general
analytic properties of the solutions of the auxiliary equation are sufficient for providing
the complete proof of the real analytic nature of the accessory parameter when we have
a single accessory parameter.

Appendix

We prove here the following result:

Lemma: If \( g(\beta_1, \beta_1^c) \) is irreducible in \( \Delta = \{ |\beta_1| < \eta, |\beta_1^c| < \eta \} \) and for every \( \beta_1 \) in \( \Delta \) there
exists a \( \beta_1^c \) solution of \( g(\beta_1, \beta_1^c) = \bar{g}(\beta_1^c, \beta_1) = 0 \), then there exist \( \beta_1 \), with |\( \beta_1 \)| as small as
we like, such that \( \beta_1^c = \bar{\beta}_1 \) is solution of the previous system.

Proof: We can always work with \( g(\beta_1, \beta_1^c) \) in regular form. Due to irreducibility all
solutions of \( g(\beta_1, \beta_1^c) = 0 \) are given by the Puiseux series \[14, 18\]

\[
\beta_1^c = B_h \beta_1^n + B_{h+1} \beta_1^{n+1} + \ldots
\] (92)

where \( n \) is the order of the W-polynomial associated to \( g \), \( \beta_1^c \) is one choice for the \( n \)-th
root and thus we have \( n \) solutions. Similarly all solutions of \( \bar{g}(\beta_1^c, \beta_1) = 0 \) are
\[
\beta_1 = \bar{B}_h (\beta_1^c)^n + \bar{B}_{h+1} (\beta_1^c)^{n+1} + \ldots
\] (93)

Having a common solution implies that for some \( l \) and \( s \) we have
\[
\beta_1^c = B_h e^{\frac{2\pi i hl}{n}} \beta_1^n + B_{h+1} e^{\frac{2\pi i (h+1)l}{n}} \beta_1^{n+1} + \ldots
\] (94)
\[
\beta_1 = \bar{B}_h e^{\frac{2\pi i hl}{n}} (\beta_1^c)^n + \bar{B}_{h+1} e^{\frac{2\pi i (h+1)l}{n}} (\beta_1^c)^{n+1} + \ldots
\] (95)

where by \( \beta_1^c \) we understood the principal value. Consistency for small \( \beta_1 \) implies
\[
\beta_1 \beta_1^c = B_h \bar{B}_h e^{\frac{2\pi i (h+l)}{n}} (\beta_1 \beta_1^c)^n
\] (96)
i.e. \( h = n \) and \( B_h \bar{B}_h = 1 \). After multiplying \( \beta_1 \) by a \( v, |v| = 1 \) and \( \beta_1^c \) by \( \bar{v} \) we can rewrite
the equations (92,93) as
\[
\beta_1^c = \beta_1 (1 + c_1 \beta_1^n + c_2 \beta_1^{n+1} + \ldots
\]
\[
\beta_1 = \beta_1^c (1 + \bar{c}_1 (\beta_1)^n + \bar{c}_2 (\beta_1)^{n+1} + \ldots
\] (97)
and we have two choices for \( v \) differing by sign. Having a common solution now means,
that for some \( h \) and \( k \) we have
\[
\beta_1^c = \beta_1 [1 + c_1 \beta_1^n e^{\frac{2\pi i h}{n}} + c_2 (\beta_1^n e^{\frac{2\pi i h}{n}})^2 + \ldots]
\]
\[
\beta_1 = \beta_1^c [1 + \bar{c}_1 (\beta_1)^n e^{\frac{2\pi i h}{n}} + \bar{c}_2 ((\beta_1)^n e^{\frac{2\pi i h}{n}})^2 + \ldots]
\] (98)

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which implies
\[-c_1 e^{\frac{2\pi i h}{n}} = \bar{c}_1 e^{\frac{2\pi i k}{n}}\] (99)
i.e. for the argument \(\varphi\) of \(c_1\)
\[\varphi + \frac{\pi}{2} = \frac{\pi}{n} (k - h) + \pi M, \quad M = 0, 1.\] (100)
In order to reach the s.c. solution we go around the origin in \(\beta_1, N_1\) times as to have
\[\frac{c_1 e^{\frac{2\pi i (h + N_1)}{n}}}{c_1 e^{\frac{2\pi i (k + N_1)}{n}}} = \frac{\bar{c}_1 e^{\frac{2\pi i (h + N_1)}{n}}}{\bar{c}_1 e^{\frac{2\pi i (k + N_1)}{n}}}\] (101)
i.e.
\[2N_1 = -(h + k) + nZ, \quad Z = \text{integer}.\] (102)
We distinguish the two cases \(n = \text{even} \) and \(n = \text{odd} \).
1) \(n = \text{even}\).
For \(h + k\) even this equation is always soluble by \(2N_1 = -(h + k), Z = 0\).
For \(h + k\) even we have also the solution \(2N_1 = -(h + k) + n\) for \(Z = 1\), which is obtained from the previous by \(N_1 \to N_1 + n/2\).
For \(h + k\) odd we perform the s.c. transformation \(\beta_1 = e^{i\alpha} \beta'_1, \beta^c_1 = e^{-i\alpha} \beta'^c_1\) with \(\alpha = \pi\).
The equations become renaming \(\beta'_1, \beta'^c_1\) again as \(\beta_1, \beta^c_1\)
\[\beta'_1 = \beta_1 [1 + c_1 e^{\frac{in}{\pi}} \beta_1^{1/2} e^{\frac{2\pi i h'}{n}} + \ldots]\] (103)
\[\beta_1 = \beta^c_1 [1 + \bar{c}_1 e^{-\frac{in}{\pi}} (\beta'^c_1)^{1/2} e^{\frac{2\pi i k'}{n}} + \ldots]\] (104)
where we must allow for different \(k', h'\). We have now the compatibility restriction
\[-c_1 e^{\frac{ix}{n}} e^{\frac{2\pi i h'}{n}} = \bar{c}_1 e^{-\frac{ix}{n}} e^{\frac{2\pi i k'}{n}}\] (105)
i.e.
\[2\varphi + \pi + \pi \left(\frac{2h' + 1}{n}\right) = \pi \left(\frac{2k' - 1}{n}\right) + 2M\pi\] (106)
being \(M\) an integer, i.e.
\[\varphi = \pi \left(- \frac{1}{2} + M + \frac{(k' - h' - 1)}{n}\right)\] (107)
while we had before
\[\varphi = \pi \left(- \frac{1}{2} + M + \frac{k - h}{n}\right)\] (108)
The s.c. requirement becomes now
\[e^{\frac{2\pi i (h' + N_1)}{n}} = e^{\frac{2\pi i (k' + N_1)}{n}}\] (109)
which having changed the parity of $h + k$ is now soluble.

Again we have two solution, one with $N_1$ and the other with $N_1 \rightarrow N_1 + n/2$.

As we shall see below for $n$ even, which we are dealing with, the case $h + k$ even will give rise to s.c. solutions near the positive real axis of $\beta_1$ while the case $h + k$ odd will give rise to s.c. solutions near the negative real axis.

2) $n = \text{odd}$.

For $h + k$ even eq. (102) can be solved by

$$2N_1 = -(h + k), \ Z = 0.$$ 

For $h + k$ odd eq. (102) can be solved by

$$2N_1 = -(h + k) + n, \ Z = 1.$$ 

Summing up, renaming the $c_k$ we have that our solution obeys

$$\beta^c_1 = \beta_1[1 + c_1\beta^c_1 + c_2\beta^c_1 + \ldots] \quad (111)$$

$$\beta_1 = \beta^c_1[1 + \bar{c}_1(\beta^c_1)^n + \bar{c}_2(\beta^c_1)^{2n} + \ldots]. \quad (112)$$

In order to find a s.c. solution we solve in $\alpha(\rho)$ the above equation with $\beta_1 = \rho e^{i\alpha(\rho)}$ and $\beta^c_1 = \rho e^{-i\alpha(\rho)}$. The variable $\alpha$ is determined by the vanishing of the real function

$$i \left( e^{2i\alpha}(1 + c_1\rho^c_1 e^{i\frac{\beta^c_1}{n}} + c_2\rho^c_1 e^{i\frac{2\beta^c_1}{n}} + \ldots) - e^{-2i\alpha}(1 + \bar{c}_1\rho^c_1 e^{-i\frac{\beta^c_1}{n}} + \bar{c}_2\rho^c_1 e^{-i\frac{2\beta^c_1}{n}} + \ldots) \right)$$

which is soluble due to the implicit real function theorem. Due to the s.c. structure of our problem and using one of the two choices for $v$ we have also

$$\rho e^{i\alpha} = \rho e^{-i\alpha}(1 + \bar{c}_1\rho^c_1 e^{-i\frac{\beta^c_1}{n}} + \bar{c}_2\rho^c_1 e^{-i\frac{2\beta^c_1}{n}} + \ldots). \quad (113)$$

Taking the complex conjugate of eq. (113) and multiplying it by eq. (111) we obtain

$$\rho^2(1 + c_1\rho^c e^{i\frac{\beta^c_1}{n}} + c_2\rho^c e^{i\frac{2\beta^c_1}{n}} + \ldots) = \rho^2(1 + c_1\rho^c e^{-i\frac{\beta^c_1}{n}} + c_2\rho^c e^{-i\frac{2\beta^c_1}{n}} + \ldots) \quad (114)$$

with the unique solution for small $\rho$, $\rho_c = \rho$. Thus we constructed a s.c. solution to $g(\beta_1, \beta^c_1) = \bar{g}(\beta^c_1, \beta_1) = 0$ with $\text{Re} \beta_1 > 0$ i.e. near the positive real axis.

For even $n$ we have in addition to the solution with $N_1$ the solution with $N_1 + n/2$ which gives the equations

$$\beta^c_1 = \beta_1[1 - c_1\beta_1 + c_2\beta_1^2 - \ldots] \quad (116)$$

$$\beta_1 = \beta^c_1[1 - \bar{c}_1(\beta^c_1)^n + \bar{c}_2(\beta^c_1)^{2n} - \ldots] \quad (117)$$
and thus an other s.c. solution which according to the discussion given above has $\text{Re}\beta_1 > 0$, for $h + k$ even and $\text{Re}\beta_1 < 0$ for $h + k$ odd.

For odd $n$ we can rotate by the integer number of times $\beta_1 \rightarrow \beta_1 e^{i(n+1)\pi} = -\beta_1 e^{i\pi n}$ and we have the solution

\[
\beta'_1 = \beta_1 [1 - c_1(-\beta_1)^\frac{1}{n} + c_2(-\beta_1)^\frac{2}{n} - \ldots]
\]

\[
\beta_1 = \beta'_1 [1 - \bar{c}_1(-\beta'_1)^\frac{1}{n} + \bar{c}_2(-\beta'_1)^\frac{2}{n} - \ldots].
\]

Define $\beta'_1 = -\beta_1$, $\beta'_1 = -\beta'_1$. Given the s.c. structure of the coefficients we can now construct a s.c. solution $\beta'_1 = \beta'_1$ with $\text{Re}\beta'_1 > 0$, i.e. $\text{Re}\beta_1 < 0$.

Summarizing for $n$ even we have either two s.c. solutions near the positive real axis or two s.c. solution near the negative real axis, while for $n$ odd we have always one s.c. solution near the positive real axis and one near the negative real axis.
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