Theoretical description of the neutron beta decay in the standard model at the level of $10^{-5}$

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In the framework of the Standard Model (SM) a theoretical description of the neutron beta decay is given at the level of $10^{-5}$. The neutron lifetime and correlation coefficients of the neutron beta decay for a polarized neutron, a polarized electron and an unpolarized proton are calculated at the account for i) the radiative corrections (RC) of order $O(\alpha E_e/m_N) \sim 10^{-5}$, i.e. $O(\alpha E_e/m_N)$ RC, to Sirlin’s outer and inner $O(\alpha/\pi)$ RC, where $\alpha$ and $E_e$ are the fine-structure constant and the electron energy, respectively, ii) the outer $O(\alpha E_e/m_N)$ RC, caused by Sirlin’s outer $O(\alpha/\pi)$ RC and the phase-volume of the neutron beta decay, calculated to next-to-leading order in the large nucleon mass $m_N$ expansion, iii) the corrections of order $O(E_e^2/m_N^2) \sim 10^{-5}$, caused by weak magnetism and proton recoil and iv) Wilkinson’s corrections of order $10^{-5}$ (Wilkinson, Nucl. Phys. A 377, 474 (1982)). These corrections define the SM background of the theoretical description of the neutron beta decay at the level of $10^{-5}$, which is required by experimental searches of interactions beyond the SM with experimental uncertainties of a few parts of $10^{-5}$.

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I. INTRODUCTION

A contemporary level of sensitivity of about $10^{-4}$ or even better for experimental investigations of the neutron beta decay [1–3] with a polarized neutron and unpolarized electron and proton [4–7] and with a polarized neutron, a polarized electron and an unpolarized proton [8] demand the theoretical description of the neutron beta decay within the Standard Model (SM) [9, 10] at the level of $10^{-5}$. As has been shown in [11–14] Wilkinson’s corrections [15] provide the SM contributions to the neutron lifetime and correlation coefficients of the neutron beta decay of order $10^{-5}$. Of course, they do not exhaust a complete set of the SM corrections of order $10^{-5}$.

In Refs. [16] and [17] we have calculated radiative corrections (RC) of order $O(\alpha E_e/m_N) \sim 10^{-5}$, i.e. $O(\alpha E_e/m_N)$ RC, where $\alpha$, $E_e$ and $m_N$ are the fine-structure constant [10], the electron energy and the nucleon mass, respectively, to Sirlin’s outer and inner (see [18]) $O(\alpha/\pi)$ RC [19, 20] (see also [21–23]), which are independent of the hadronic structure of the neutron (see [19]) and induced by the hadronic structure of the neutron (see [17]), respectively. In turn, in [24] we have calculated a complete set of corrections of order $O(E_e^2/m_N^2) \sim 10^{-5}$, caused by weak magnetism and proton recoil. Together with Wilkinson’s corrections [15] (see also [11–14]) the corrections, calculated in [16, 17, 24], define the SM background of the theoretical description of the neutron beta decay at the level of $10^{-5}$. In this work we supplement this SM theoretical background of the neutron beta decay by the outer $O(\alpha E_e/m_N) \sim 10^{-5}$ RC, caused by Sirlin’s outer $O(\alpha/\pi)$ RC and the phase-volume of the neutron beta decay, calculated to next-to-leading order (NLO) in the large nucleon mass $m_N$ expansion (see Appendix D).

For the electron-energy and angular distribution of the neutron beta decay for a polarized neutron, a polarized electron and an unpolarized proton we use the most general form, proposed by Jackson et al. [25, 27] and Ebel and
Feldman [28]:
\[
\frac{d^3\lambda_n(E_e, \vec{k}_e, \vec{\kappa}_e, \vec{\zeta}_e, \bar{\zeta}_e)}{dE_e d\Omega_e d\Omega_{\bar{\nu}}^e} = \left( 1 + 3g_A^2 \right) \frac{|G_V|^2}{16\pi^2} \left( E_0 - E_e \right)^2 \sqrt{E_e^2 - m_e^2} E_{\bar{\nu}} F(E_e, Z = 1) \zeta(E_e) \left\{ 1 + b(E_e) \frac{m_e}{E_e} \right\} + a(E_e) \frac{\vec{\zeta}_e \cdot \vec{k}_e}{E_e E_{\bar{\nu}}} + A(E_e) \frac{\vec{\zeta}_e \cdot \vec{\kappa}_e}{E_e E_{\bar{\nu}}} + B(E_e) \frac{\vec{\zeta}_e \cdot \bar{\zeta}_e}{E_e E_{\bar{\nu}}} + K_n(E_e) \frac{(\vec{\zeta}_e \cdot \bar{\zeta}_e)(\vec{k}_e \cdot \bar{\zeta}_e)}{E_e E_{\bar{\nu}}} + Q_n(E_e) \left( \vec{\zeta}_e \cdot \bar{\zeta}_e \right) \left( \bar{\zeta}_e \cdot \vec{\kappa}_e \right) \right. \\
+ D(E_e) \frac{\vec{\zeta}_e \cdot \vec{\kappa}_e}{E_e E_{\bar{\nu}}} + G(E_e) \frac{\vec{\zeta}_e \cdot \vec{\kappa}_e}{E_e E_{\bar{\nu}}} + H(E_e) \frac{\vec{\zeta}_e \cdot \bar{\zeta}_e}{E_e E_{\bar{\nu}}} + N(E_e) \frac{\vec{\zeta}_e \cdot \bar{\zeta}_e}{E_e E_{\bar{\nu}}} + O(E_e) \frac{\vec{\zeta}_e \cdot \vec{\kappa}_e}{E_e E_{\bar{\nu}}} + S(E_e) \left( \vec{\zeta}_e \cdot \bar{\zeta}_e \right) \left( \bar{\zeta}_e \cdot \vec{\kappa}_e \right) \\
+ T(E_e) \left( \vec{\zeta}_e \cdot \vec{\kappa}_e \right) \left( \bar{\zeta}_e \cdot \bar{\zeta}_e \right) + U(E_e) \frac{(\vec{\zeta}_e \cdot \vec{\kappa}_e)(\vec{\zeta}_e \cdot \bar{\zeta}_e)}{E_e E_{\bar{\nu}}} + V(E_e) \frac{\vec{\zeta}_e \cdot \bar{\zeta}_e}{E_e E_{\bar{\nu}}} + W(E_e) \left( \vec{\zeta}_e \cdot \vec{\kappa}_e \right) \left( \bar{\zeta}_e \cdot \bar{\zeta}_e \right),
\]

where we have used the notations in Refs. [11 - 24, 29, 30]. Then, \( g_A \) and \( G_V \) are the axial and vector coupling constants, respectively, and \( \alpha \) is the fine-structure constant. The Cabibbo–Kobayashi–Maskawa (CKM) matrix element \( V_{ud} \) is included in the definition of the vector coupling constant \( G_V \), \( \vec{\zeta}_e \) and \( \bar{\zeta}_e \) are unit spin–polarization vectors of the neutrino and antineutrino, respectively, \( E_0 = (m_e^2 - m_p^2 + m_n^2)/2m_p = 1.2926 \) MeV is the end–point energy of the electron–energy spectrum [1, 2]. In section III we discuss the obtained results and \( F(E_e, Z = 1) \) is the relativistic Fermi function, describing the electron–proton final–state Coulomb interaction, is equal to [32] (see also [15] and a discussion in [12]).

\[
F(E_e, Z = 1) = \left( 1 + \frac{1}{2} \gamma \right) \frac{4(2\gamma)_{\mu\nu} \beta_{\mu\nu}}{T^2(3 + 2\gamma)} \cdot \left[ \Gamma \left( 1 + \gamma + i \frac{\alpha}{\beta} \right) \right]^2,
\]

where \( \beta = k_e / E_e = \sqrt{E_e^2 - m_e^2} / E_e \) is the electron velocity, \( \gamma = \sqrt{1 - \alpha^2} = 1 \), \( r_p = 0.841 \) fm is the electric radius of the proton [33]. The correlation coefficient \( b(E_e) \) is the Fierz interference term [34]. The structure and the value of the Fierz interference term may depend on interactions beyond the SM [34]. An information of a contemporary standard correlation structures in Eq.(1) by Jackson [35, 36] is equal to [32] (see also [15] and a discussion in [12]).

\[
(1)
\]

In section II, we give the analytical expressions for the correlation function \( \zeta(E_e) \), which is responsible for the correct electron-energy spectrum of the neutron beta decay and correct value of the neutron lifetime, and the correlation coefficients \( X(E_e) \) for \( X = a, A, B, \ldots, T \) and \( U \), including i) the \( O(\alpha/\pi) \) and \( O(\alpha m_e/\mu N) \) corrections, caused by weak magnetism and proton recoil, and ii) the \( O(E_e/m_N) \) corrections, caused by weak magnetism and proton recoil, and iii) Wilkinson’s corrections of order of a few parts of \( 10^{-5} \), which we have calculated in Appendices A, B and C. The results, represented in section II, illustrate the SM theoretical description of the neutron beta decay at level of \( 10^{-5} \) with a theoretical accuracy of a few parts of \( 10^{-6} \). In Section III we discuss the obtained results and some problems of the analysis of the contributions of the neutron radiative beta decay. In Appendices A, B, C and D we give detailed calculations of the correlation function \( \zeta(E_e) \) and correlation coefficients \( X(E_e) \) for \( X = a, A, B, \ldots, T \) and \( U \) added in section II. In Appendix E we give the analytical expressions of the correlation function \( \zeta(E_e) \) and correlation coefficients \( X(E_e) \) for \( X = a, A, B, \ldots, U \) as functions of the electron energy \( E_e \) and the axial coupling constant \( g_A \). For the practical applications and numerical analysis the correlation function \( \zeta(E_e) \) and correlation coefficients \( X(E_e) \) for \( X = a, A, B, \ldots, U \) are programmed in [36].

II. CORRELATION FUNCTION AND COEFFICIENTS OF THE ELECTRON-ENERGY AND ANGULAR DISTRIBUTION EQ.(1)

In Appendices A, B, C and D at the level of \( 10^{-5} \) with a theoretical accuracy of a few parts of \( 10^{-6} \) we give a detailed SM calculation of the correlation function \( \zeta(E_e) \) and correlation coefficients in Eq.(1), the correlation...
structures of which are invariant under time-reversal transformation, i.e. $T$-even. According to our analysis carried out in Appendices A, B, C and D, the correlation function $\zeta(E_e)$ and correlation coefficients can be represented in the following form

$$\begin{align*}
\zeta(E_e) &= \zeta(E_e)_{RC} + \zeta(E_e)_{RC-P\beta V} + \zeta(E_e)_{WP} + \zeta(E_e)_{WC}, \\
X(E_e) &= X(E_e)_{RC} + X(E_e)_{RC-P\beta V} + X(E_e)_{WP} + X(E_e)_{WC},
\end{align*}$$

(3)

where $X = a, A, B, K_n, Q_n, G, H, N, K_z, Q_z, S, T$ and $U$. Then, $Y(E_e)$ are the sum of the outer $O(\alpha/\pi)$ RC, calculated to LO in the large nucleon mass $m_N$ expansion, and $O(\alpha E_e/m_N) \sim 10^{-5}$ RC, which are treated as NLO corrections in the large nucleon mass $m_N$ expansion to the outer and inner $O(\alpha/\pi)$ RC and denoted as $Y_{RC-NLO}$ (see Appendix A), ii) the outer $O(\alpha E_e/m_N)$ RC, caused by the outer $O(\alpha/\pi)$ RC and the phase-volume of the neutron beta decay taken to NLO in the large nucleon mass expansion, iii) the sum of the $O(E_e/m_N)$ and $O(E_e^2/m_N^2) \sim 10^{-5}$ corrections, caused by weak magnetism and proton recoil, and iv) Wilkinson’s corrections of order $10^{-5}$. For the practical applications and numerical analysis the analytical expressions of the correlation function $\zeta(E_e)$ and correlation coefficients $X(E_e)$ for $X = a, A, B, K_n, Q_n, G, H, N, K_z, Q_z, S, T$ and $U$ are programmed in [46].

In order to illustrate the SM description of the neutron beta decay at the level $10^{-5}$ we represent the correlation function $\zeta(E_e)$ and correlation coefficients $a(E_e), A(E_e), B(E_e), \ldots, U(E_e)$ with the contributions of the corrections, caused by weak magnetism and proton recoil of order $O(E_e/m_N)$ and $O(E_e^2/m_N^2)$ in Appendix B, and Wilkinson’s corrections in Appendix C, as functions of the variable $E_e/E_0$. According to our calculation in [46], we get

$$\begin{align*}
\zeta(E_e) &= \zeta(E_e)_{RC} + \zeta(E_e)_{RC-P\beta V} - 5.57 \times 10^{-4} \frac{E_0}{E_e} - 3.20 \times 10^{-3} + 9.81 \times 10^{-5} \frac{E_e}{E_0} \\
+ & 7.13 \times 10^{-5} \frac{E_e^2}{E_0^2} - 3.16 \times 10^{-5} \frac{E_e}{\beta^3 E_0}, \\
a(E_e) &= a(E_e)_{RC} + a(E_e)_{RC-P\beta V} - 5.81 \times 10^{-5} \frac{E_0}{E_e} + 3.24 \times 10^{-3} - 9.16 \times 10^{-3} \frac{E_e}{E_0} \\
- & 3.16 \times 10^{-5} \frac{E_0 - E_e}{\beta^3 E_0}, \\
A(E_e) &= A(E_e)_{RC} - 6.71 \times 10^{-5} \frac{E_0}{E_e} - 1.75 \times 10^{-3} \frac{E_e}{E_0}, \\
B(E_e) &= B(E_e)_{RC} + 5.26 \times 10^{-5} \frac{E_0}{E_e} + 3.27 \times 10^{-4} - 3.42 \times 10^{-4} \frac{E_e}{E_0} - 2.87 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
K_n(E_e) &= K_n(E_e)_{RC} + 7.15 \times 10^{-5} \frac{E_0}{E_e} + 1.51 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
Q_n(E_e) &= Q_n(E_e)_{RC-P\beta V} + 3.19 \times 10^{-3} - 7.29 \times 10^{-3} \frac{E_0}{E_e} - 1.97 \times 10^{-5} \frac{E_e^2}{E_0^2} - 3.12 \times 10^{-5} \frac{E_0 - E_e}{\beta^3 E_0}, \\
G(E_e) &= G(E_e)_{RC} - 5.59 \times 10^{-4} \frac{E_0}{E_e} + 3.78 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-4} \frac{E_e^2}{E_0^2}, \\
H(E_e) &= H(E_e)_{RC} + 2.28 \times 10^{-5} \frac{E_e^2}{E_0^2} - 1.28 \times 10^{-3} \frac{E_0}{E_e} + 1.13 \times 10^{-3} - 1.46 \times 10^{-5} \frac{E_e}{E_0}, \\
N(E_e) &= N(E_e)_{RC} + 2.33 \times 10^{-5} \frac{E_e^2}{E_0^2} - 3.21 \times 10^{-4} \frac{E_0}{E_e} + 5.73 \times 10^{-5}, \\
Q_e(E_e) &= Q_e(E_e)_{RC} + Q_e(E_e)_{RC-P\beta V} + 2.24 \times 10^{-5} \frac{E_0}{E_e} + 7.00 \times 10^{-4} + 3.57 \times 10^{-3} \frac{E_e}{E_0} - 1.57 \times 10^{-5} \frac{E_e^2}{E_0^2} \\
+ & 1.04 \times 10^{-4} (1 + \sqrt{1 - \beta^2}) \frac{E_0 - E_e}{\beta^3 E_0}, \\
K_e(E_e) &= K_e(E_e)_{RC} + K_e(E_e)_{RC-P\beta V} + 1.93 \times 10^{-5} \frac{E_0}{E_e} - 6.93 \times 10^{-4} + 9.17 \times 10^{-3} \frac{E_e}{E_0} + 2.95 \times 10^{-5} \frac{E_e^2}{E_0^2} \\
+ & 3.16 \times 10^{-5} (1 + \sqrt{1 - \beta^2}) \frac{E_0 - E_e}{\beta^3 E_0}, \\
S(E_e) &= S(E_e)_{RC} - 2.82 \times 10^{-4},
\end{align*}$$
where the numerical coefficients are calculated at the axial coupling constant \( g_A = 1.2764 \) [5]. The contributions of the corrections of the \( O(\alpha E_e/m_N) \) RC are plotted in [10].

Then, the analytical expressions of \( \zeta(E_e)_{\text{RC}} \) and \( X(E_e)_{\text{RC}} \) for \( X = a, A, \ldots, T \) and \( U \) are given in Appendix A (see Eq.(A-20)) and in [10]. At \( \alpha = 0 \) the correlation function \( \zeta(E_e)_{\text{RC}} \) and the correlation coefficients \( X(E_e)_{\text{RC}} \) for \( X = a, A, \ldots, U \) reduce to their values, calculated to LO in the large nucleon mass \( m_N \) expansion (see Appendix A). The outer RC \( \zeta(E_e)_{\text{RC}-\text{ParV}} \) and \( X(E_e)_{\text{RC}-\text{ParV}} \) are calculated in Appendix D (see Eq.(D-10)).

In addition to the correlation function \( \zeta(E_e) \) and the correlation coefficients in Eq.(4) we give the correlation coefficient \( A^{(3)}(E_e) = A(E_e) + \frac{1}{3} Q_n(E_e) [15] \) that measures the electron (beta) asymmetry of the neutron beta decay [5]:

\[
A^{(3)}(E_e) = A^{(3)}(E_e)_{\text{RC}} + \frac{1}{3} Q_n(E_e)_{\text{RC}-\text{ParV}} - 6.67 \times 10^{-5} \frac{E_0}{E_e} + 9.78 \times 10^{-4} - 4.18 \times 10^{-3} \frac{E_e}{E_0} - 1.04 \times 10^{-5} \frac{E_0 - E_e}{\beta^3 E_0},
\]

(5)
calculated at \( g_A = 1.2764 \) [5], where \( A^{(3)}(E_e)_{\text{RC}} = A(E_e)_{\text{RC}} \), since \( Q_n(E_e)_{\text{RC}} = 0 \) (see Eq.(A-20)). The correlation function \( \zeta(E_e) \) and the correlation coefficients \( X(E_e) \) for \( X = a, A, B, \ldots, U \) in Eq.(4) and \( A^{(5)}(E_e) \) in Eq.(5) describe the neutron beta decay for a polarized neutron, a polarized electron and an unpolarized proton at the level of \( 10^{-5} \) in the framework of the SM with a theoretical accuracy of a few parts of \( 10^{-6} \).

III. DISCUSSION

We have given a SM theoretical description of the neutron beta decay for a polarized neutron, a polarized electron and an unpolarized proton at the level of \( 10^{-5} \) with a theoretical accuracy of a few parts of \( 10^{-6} \). To the well-known \( O(\alpha/\pi) \) RC [19-22] (see also [11 12 14 29 30]) and \( O(\alpha E_e/m_N) \) corrections [17] and [15 22 23] (see also [11 12 14 29 30]) we have added i) the inner \( O(\alpha E_e/m_N) \sim 10^{-5} \) RC [16 17], which are treated as NLO corrections in the large nucleon mass \( m_N \) expansion to Sirlin’s outer and inner \( O(\alpha/\pi) \) RC, calculated to LO in the large nucleon mass \( m_N \) expansion, ii) the outer \( O(\alpha E_e/m_N) \sim 10^{-5} \) RC, induced by Sirlin’s outer \( O(\alpha/\pi) \) RC and the phase-volume of the neutron beta decay, calculated to NLO in the large nucleon mass \( m_N \) expansion, iii) the \( O(E^2_{\beta}/m_N^2) \sim 10^{-5} \) corrections [24], caused by weak magnetism and proton recoil, and iv) Wilkinson’s corrections [15] (see also [11 12 14]) of order \( 10^{-5} \). As has been shown in Eq.(4), all of these corrections define the SM background of the theoretical description of the neutron beta decay at the level of \( 10^{-5} \) with a theoretical accuracy of about a few parts of \( 10^{-6} \) [16].

Having accepted the value of the axial coupling constant \( g_A = 1.2764 \) [5], the correlation function \( \zeta(E_e) \) and correlation coefficients, given in Eq.(4) and Eq.(5), can be used as the SM theoretical background of the neutron beta decay for experimental searches of contributions of interactions beyond the SM with experimental uncertainties of a few parts of \( 10^{-5} \) [17 18 19] (see also [12 24]). Because of Wilkinson’s corrections, induced by the proton recoil in the electron-proton final-state Coulomb interaction (see Eq.(C-8) in Appendix C), and the outer \( O(\alpha E_e/m_N) \) RC (see Eq.(D-10) in Appendix D) the correlation function \( \zeta(E_e) \) and correlation coefficients in Eq.(4) and Eq.(5) are well defined in the experimental electron-energy region \( 0.811 \text{MeV} \leq E_e \leq 1.211 \text{MeV} \) [5].

In Appendix E we give the analytical expressions for the correlation function \( \zeta(E_e) \) and correlation coefficients \( X(E_e) \) for \( X = a, A, B, \ldots, U \) as functions of the electron energy \( E_e \) and the axial coupling constant \( g_A \). These expressions can be used as a SM theoretical background for processing experimental data on the neutron lifetime, the electron-antineutrino angular correlations, and electron and antineutrino asymmetries with experimental uncertainties of about a few parts of \( 10^{-5} \). Such a SM theoretical background and experimental data, obtained with experimental uncertainties of about a few parts of \( 10^{-5} \), should allow to improve the currently available experimental value of the axial coupling constant \( g_A \) [5, 6]. They can be also used for searches of contributions of interactions beyond the SM in experiments with polarized neutrons and electrons [5].

We have also to emphasize that for the correct description of the neutron lifetime one has to add the inner radiative corrections \( \Delta R \) and \( \Delta A \) of order \( O(\alpha/\pi) \), defined by the Feynman \( \gamma W \)-box diagrams, to the rates of the neutron beta decay and superallowed nuclear beta decays, which have been calculated to LO in the large nucleon mass \( m_N \) expansion in [15 24]. These corrections are very important for the correct extraction of the value of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element \( V_{ud} \).
Finally we would like to discuss the problem of the removal of infrared divergences for the calculation of the outer RC in the neutron beta decay. Since the virtual photon exchange leads to the dependence of the amplitude of the neutron beta decay on the infrared cut-off \( \mu \), which is an infinitesimal photon mass \( \mu \) in the covariant regularization \([19, 57, 61]\), one has to take into account the contribution of the neutron radiative beta decay \([19, 57, 61]\). For this aim the energy and angular distribution of the rate of the neutron radiative decay should be summed with the energy and angular distribution of the rate of the neutron beta decay \([19, 57, 61]\). In case of the investigation of the electron-energy and angular distribution of the neutron beta decay (see, for example, Eq. (1)), the standard procedure for the calculation of distributions for both the neutron beta decay and neutron radiative beta decay is to integrate, first, over the proton 3-momenta and then over the energy of the antineutrino \([19, 21, 23, 57, 69]\) (see also \([11, 14, 29, 30, 40]\)). In the rest frame of the neutron and after the integration of the proton 3-momentum, the latter appears in the calculation of distributions for both the neutron beta decay and neutron radiative beta decay to integrate, first, over the proton 3-momenta and then over the energy of the antineutrino \([19, 21, 23, 57, 69]\) (see also \([11, 14, 29, 30, 40]\)).

In the rest frame of the neutron and after the integration of the proton 3-momentum, the latter appears in the calculation of distributions for both the neutron beta decay and neutron radiative beta decay to integrate, first, over the proton 3-momenta and then over the energy of the antineutrino \([19, 21, 23, 57, 69]\) (see also \([11, 14, 29, 30, 40]\)). The electron-energy and angular distribution Eq. (1) is usually used for the measurements of the electron (beta) asymmetry, which is characterized by the correlation coefficient \(A^{(3)}(E_e)\) \([5]\). In these measurements the electron asymmetry defines the asymmetry of the emission of decay electrons relative to the neutron spin polarization into solid angles related by the polar angle \(\theta \rightarrow \pi - \theta\) \([5, 20, 23]\) (for the details of the calculation we refer to \([11]\)). The electron-energy and angular distribution Eq. (1) can be also applied to the measurement of the antineutrino asymmetry, which is practically defined by the correlation coefficient \(B(\nu_e)\).

In turn, for the measurements of the electron-antineutrino angular correlations \([77, 79]\) and the proton recoil asymmetry, defined by the correlation coefficient \(C\) \([80]\), one has to use the electron-proton-energy and angular distribution (or the proton-energy and angular distribution) \([65, 69]\). For the calculation of the electron-proton-energy and angular distribution one has to integrate over the antineutrino 3-momentum. Then, having integrated over the electron energy one obtains the proton-energy and angular distribution (for the details of the calculation we refer to \([11]\)). The same procedure should be used for the neutron radiative beta decay \([11]\). In case of the neutron radiative beta decay for the calculation of the electron-proton-photon-energy and angular distribution one deals with direct photon-proton correlations \([65, 69]\). However, as has been shown in \([81]\), the contributions of these correlations do not destroy the radiative corrections, defined by the functions \((\alpha/\pi) g_\alpha(E_e)\) and \((\alpha/\pi) f_\alpha(E_e)\). As has been found in \([81]\), the contributions of the photon-proton correlations in the neutron radiative beta decay to the proton recoil asymmetry \(C\) are of order of \(10^{-4}\). They make the contributions of the radiative corrections to the proton recoil asymmetry \(C\) symmetric with respect to a change \(A_0 \leftrightarrow B_0\), where \(A_0\) and \(B_0\) are the correlation coefficients \(A(E_e)\) and \(B(E_e)\) calculated to LO in the large nucleon mass \(m_N\) expansion \([11, 12]\). They depend on the axial coupling constant only (see also Eq. (A-16) in Appendix A). We are planning to carry out the analysis of the electron-proton-energy and angular distributions of the neutron beta decay at the SM theoretical level of about \(10^{-5}\) in our forthcoming publication.

We would like to note that for practical applications and numerical analysis of the correlation function \(\zeta(E_e)\) and correlation coefficients \(a(E_e), A(E_e), B(E_e), \ldots, U(E_e)\) we have programmed their analytical expressions in \([46]\). We have carried out numerical calculations for the axial coupling constant \(g_A = 1.2764\) \([5]\) and plotted the \(O(\alpha E_e/m_N)\) corrections.

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Appendix A: The electron-energy and angular distribution of the neutron beta decay with the account for the radiative corrections of order $O(\alpha E_e/m_N)$

According to [11, 20, 30], the electron–energy and angular distribution of the neutron beta decay for a polarized neutron, a polarized electron and an unpolarized proton in Eq. [1] is determined by

\[
\frac{d^3\lambda_n(E_e, \vec{k}_e, \vec{k}_\nu, \vec{E}_\nu, \vec{E}_e)}{dE_e d\Omega_e d\Omega_\nu} = (1 + 3g_A^2) \frac{|G_V|^2}{16\pi^5} (E_0 - E_e)^2 \sqrt{E_e^2 - m^2} E_e F(E_e, Z = 1) \times \Phi_n(\vec{k}_e, \vec{k}_\nu) \sum_{\text{pol}} \frac{|M(n \to pe^- \nu_e)|^2}{(1 + 3g_A^2)|G_V|^2 64m_n^2 E_e E_\nu},
\]

(A-1)

where we sum over polarizations of the massive fermions. The function $\Phi_n(\vec{k}_e, \vec{k}_\nu)$ defines the contribution of the phase-volume of the neutron beta decay [11, 24]. It is equal to [11, 24]

\[
\Phi_n(\vec{k}_e, \vec{k}_\nu) = 1 + \frac{E_e}{m_N} \left(1 - \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e E_\nu}\right),
\]

(A-2)

taken to NLO in the large nucleon mass $m_N$ expansion. The amplitude of the neutron beta decay, taking into account the radiative corrections of order $O(\alpha/\pi)$ and $O(\alpha E_e/m_N)$, is defined by [11, 16, 17]

\[
M(n \to pe^- \nu_e) = -2m_e G_V \left\{ (1 + U_1) [\varphi^e_{\text{e}} \varphi_n][\vec{u}_e \gamma^0 (1 - \gamma^5) \nu_e] + g_A (1 + U_2) [\varphi^e_{\text{n}} \bar{\sigma} \varphi_n] [\vec{u}_e \gamma^0 (1 - \gamma^5) \nu_e] + U_3 [\varphi^e_{\text{n}} \varphi_n][\vec{u}_e (1 - \gamma^5) \nu_e] + g_A U_4 [\varphi^e_{\text{e}} \bar{\sigma} \varphi_n] [\vec{u}_e (1 - \gamma^5) \nu_e] + U_5 [\varphi^e_{\text{e}} (\vec{k}_e \cdot \bar{\sigma}) \varphi_n] [\vec{u}_e (1 - \gamma^5) \nu_e] + U_6 [\varphi^e_{\text{n}} (\vec{k}_e \cdot \bar{\sigma}) \varphi_n] [\vec{u}_e (1 - \gamma^5) \nu_e] + U_7 [\varphi^e_{\text{e}} (\vec{k}_e \cdot \bar{\sigma}) \bar{\sigma} \varphi_n] [\vec{u}_e (1 - \gamma^5) \nu_e] + U_8 [\varphi^e_{\text{e}} (\vec{k}_e \cdot \bar{\sigma}) \bar{\sigma} \varphi_n] [\vec{u}_e (1 - \gamma^5) \nu_e].
\]

(A-3)

The functions $U_j$ for $j = 1, 2, \ldots, 8$ are given by

\[
U_1 = \frac{\alpha}{2\pi} \left( \beta \langle \varphi^e_{\text{e}} \rangle + \frac{E_e}{m_N} f_v(E_e) + g_4a(E_e) + \right), \quad U_2 = \frac{\alpha}{2\pi} \left( \beta \langle \varphi^e_{\text{e}} \rangle + \frac{E_e}{m_N} f_A(E_e) + \frac{5}{2} m_N^2 \beta \gamma_n \frac{M_W^2}{m_N^2} + \frac{1}{g_A} \bar{f}_a(E_e) \right), \quad U_3 = \frac{\alpha}{2\pi} \left( -\frac{\sqrt{1 - \beta^2}}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \frac{E_e}{m_N} f_s(E_e) \right), \quad U_4 = \frac{\alpha}{2\pi} \left( -\frac{\sqrt{1 - \beta^2}}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \frac{E_e}{m_N} f_t(E_e) \right), \quad U_5 = \frac{1}{2\pi} \frac{E_e}{m_N} g_s(E_e), \quad U_6 = \frac{1}{2\pi} \frac{E_e}{m_N} g_A(E_e), \quad U_7 = \frac{1}{2\pi} \frac{E_e}{m_N} g_U(E_e), \quad U_8 = \frac{1}{2\pi} \frac{E_e}{m_N} h_A(E_e),
\]

(A-4)

where $\beta = k_e/E_e = \sqrt{1 - m_e^2/E_e^2}$ is the electron velocity, the function $\beta \langle \varphi^e_{\text{e}} \rangle$, where $\mu$ is a covariant infrared cut-off having a meaning of a photon mass [19, 57, 58], was calculated by Sirlin [19] to LO in the large nucleon mass $m_N$ expansion (for the details of the calculation of the function $\beta \langle \varphi^e_{\text{e}} \rangle$, we refer to [11]). It defines so-called outer model-independent radiative corrections [18]. Then, the functions $f_v(E_e)$, $f_A(E_e)$, $f_s(E_e)$, $f_t(E_e)$, $g_v(E_e)$, $g_s(E_e)$, $h_s(E_e)$ and $h_A(E_e)$ determine the inner radiative corrections dependent on the axial coupling constant $g_A$ to Sirlin's outer radiative corrections $O(\alpha/\pi)$, calculated to NLO in the large nucleon mass $m_N$ expansion in [16]. In turn, the functions $g_4a(E_e)$ and $f_4a(E_e)$ describe the inner radiative corrections, caused by the hadronic structure of the neutron and calculated to NLO in the large nucleon mass $m_N$ expansion in [17] as NLO corrections to Sirlin’s inner radiative corrections $O(\alpha/\pi)$, caused by the hadronic structure of the neutron and calculated to LO in the large nucleon mass expansion [19, 20]. The analytical expressions of these functions are equal to [11, 16, 17]

\[
f_{\beta^e}(E_e, \mu) = \frac{3}{4} \frac{\beta \ln m_N^2}{m_e^2} - \frac{11}{8} + \beta \ln \left( \frac{\mu}{m_e} \right) \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 + \frac{1}{2} \beta \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - \frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right),
\]

\[
f_v(E_e) = \frac{1}{4} \frac{\beta \ln m_N^2}{m_e^2} \frac{2 - 3\beta^2}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + (g_A - 1) \left[ -\frac{1}{4} \frac{5}{8} \ln \left( \frac{m_N^2}{m_e^2} \right) + \frac{2 - 5\beta^2}{8\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right],
\]
\[ f_A(E_e) = 1 + \frac{1}{2} \ln \frac{m_N^2}{m_e^2} + \frac{2 - 3\beta^2}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + g_A - \frac{1}{g_A} \left[ \frac{3}{4} \ln \frac{m_N^2}{m_e^2} - \frac{3}{4} \beta \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right], \]
\[ f_S(E_e) = \sqrt{1 - \beta^2} \left\{ - \frac{1}{2} \ln \frac{m_N^2}{m_e^2} + \frac{2E_0 - E_e}{E_e} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + (g_A - 1) \left[ \frac{1}{8} \ln \frac{m_N^2}{m_e^2} - \frac{1}{4\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right] \right\}, \]
\[ f_T(E_e) = \sqrt{1 - \beta^2} \left\{ - \frac{1}{2} \ln \frac{m_N^2}{m_e^2} + \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right\}, \]
\[ g_s(E_e) = (g_A - 1) \sqrt{1 - \beta^2} \ln \left( \frac{1 + \beta}{1 - \beta} \right), \]
\[ h_S(E_e) = \frac{1}{8\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right), \]
\[ g_W(E_e) = - \frac{1}{2} \ln \frac{m_N^2}{m_e^2} + \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) + (g_A - 1) \left[ - \frac{1}{4} \ln \frac{m_N^2}{m_e^2} + \frac{5}{8\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right], \]
\[ h_A(E_e) = - \frac{1}{2} \ln \frac{m_N^2}{m_e^2} + \frac{1}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right), \]

where \( \text{Li}_2(z) \) is the PolyLogarithmic function \[83], and
\[
g_{\text{st}}(E_e) = -G_{\text{st}}^{(V)} \frac{E_0}{m_N} + \left( G_{\text{st}}^{(W)} + F_{\text{st}}^{(W)} \right) \frac{m_N^2}{M_W^2} + F_{\text{st}}^{(V)} \frac{E_e}{m_N} = 0.098 \left( 1 + 0.95 E_e / E_0 \right),
\]
\[
\tilde{f}_{\text{st}}(E_e) = +G_{\text{st}}^{(A)} \frac{E_0}{m_N} - H_{\text{st}}^{(W)} \frac{m_N^2}{M_W^2} - H_{\text{st}}^{(V)} \frac{E_e}{m_N} = 0.057 \left( 1 + E_e / E_0 \right),
\]

where \( E_0 = (m^2_N - m^2_e + m^2_\gamma) / 2m_\gamma = 1.2926 \text{ MeV} \) is the end–point energy of the electron–energy spectrum \[14\], \[22\], calculated for \( m_N = 0.939.9564 \text{ MeV} \), \( m_p = 0.388.2712 \text{ MeV} \) and \( m_e = 0.5110 \text{ MeV} \) \[10\]. Then, \( m_N = (m_N + m_p) / 2 = 938.9188 \text{ MeV} \) and \( M_W = 80.379 \text{ GeV} \) are nucleon and electroweak \( W^- \)-boson masses \[10\], respectively. The structure constants \( G_{\text{st}}^{(V)} \) and so on, calculated in \[17\], are equal to \( G_{\text{st}}^{(V)} = -0.7071 \), \( H_{\text{st}}^{(V)} = 0.67.75 \), \( G_{\text{st}}^{(W)} = 8.94 \), \( G_{\text{st}}^{(A)} = 4.195 \), \( H_{\text{st}}^{(A)} = -0.7071 \), \( H_{\text{st}}^{(W)} = 2.07 \). For the subsequent analysis of radiative corrections we follow \[29\] (see also \[11\]) and represent the function \( f_{\beta^-}(E_e, \mu) \) as follows
\[
f_{\beta^-}(E_e, \mu) = g_n(E_e) + \frac{1 - \beta^2}{2\beta} \ln \frac{1 + \beta}{1 - \beta} - g_{\beta^-}^{(1)}(E_e, \mu),
\]

where \( 2g_n(E_e) \) is Sirlin’s function, defining the outer radiative corrections of order \( O(\alpha/\pi) \) to the neutron lifetime \[19\]. The function \( g_{\beta^-}^{(1)}(E_e, \mu) \) can be removed by the contribution of the neutron radiative beta decay \( n \rightarrow p + e^- + \nu_e + \gamma \) with a real photon \( \gamma \), which should be added, according to Berman \[57\] and Kinosita and Sirlin \[58\] (see also Sirlin \[19\]), for the removal of the dependence of the neutron lifetime on the infrared cut–off. For the detailed calculation of the function \( g_{\beta^-}^{(1)}(E_e, \mu) \) and as well as the function \( g_{\beta^-}^{(1)}(E_e, \omega_{\text{min}}) \), describing the contributions of the neutron radiative beta decay \( n \rightarrow p + e^- + \nu_e + \gamma \) to the neutron lifetime, where \( \omega_{\text{min}} \) is a non-covariant infrared cut-off having a meaning of the photon–energy threshold of the detector, we refer to \[11\]. We adduce here the analytical expressions of these functions for completeness (see \[11\])
\[
g_{\beta^-}^{(1)}(E_e, \omega_{\text{min}}) = \left[ \ln \left( \frac{E_0 - E_e}{\omega_{\text{min}}} \right) - 3 + \frac{1}{3} \left( E_0 - E_e \right) \left( \frac{1}{E_e} + \frac{1}{8} \right) \right] \left[ \frac{1}{2} \ln \frac{1 + \beta}{1 - \beta} \right] - 2 + 1 + \frac{1}{2\beta} \ln \frac{1 + \beta}{1 - \beta} - \frac{1}{4\beta} \ln \frac{1 + \beta}{1 - \beta} - \frac{1}{2} \text{Li}_2 \left( \frac{2\beta}{1 + \beta} \right) - \frac{1}{12} \left( E_0 - E_e \right)^2,
\]

The hermitian conjugate amplitude of the neutron beta decay Eq. (A-3) is equal to
\[
M^\dagger(n \rightarrow p e^- \bar{\nu}_e) = -2m_e G V \left( \left[ 1 + U_1 \right] [\varphi_r^\dagger \varphi_p] [\bar{\nu}_e \gamma^0 (1 - \gamma^5) u_e] + g_A (1 + U_2) [\varphi_r^\dagger \bar{\sigma} \varphi_p] [\bar{\nu}_e \gamma^0 (1 - \gamma^5) u_e] + U_3 [\varphi_r^\dagger \varphi_p] [\bar{\nu}_e (1 + \gamma^5) u_e] - g_A U_4 [\varphi_r^\dagger \bar{\sigma} \varphi_p] [\bar{\nu}_e \gamma^0 (1 + \gamma^5) u_e] + U_5 [\varphi_r^\dagger \bar{\sigma} \bar{\sigma} \varphi_p] [\bar{\nu}_e (1 + \gamma^5) u_e] + U_6 [\varphi_r^\dagger \bar{\sigma} \varphi_p] [\bar{\nu}_e (1 + \gamma^5) u_e] + U_7 [\varphi_r^\dagger \bar{\sigma} \bar{\sigma} \varphi_p] [\bar{\nu}_e \gamma^0 (1 - \gamma^5) u_e] + U_8 [\varphi_r^\dagger \bar{\sigma} \bar{\sigma} \varphi_p] [\bar{\nu}_e \gamma^0 (1 - \gamma^5) u_e] \right).
\]
We use this amplitude for the calculation of the square of absolute value of the amplitude Eq. (A-3), summed over polarizations of massive particles. It is equal to

\[
\sum_{\text{pol}} \frac{|M(n \rightarrow p e^{-i\nu})|^2}{(1 + 3g_A^2)G_F^2|4z|^2} E_{e^0} E_{p^0} \left\{ (1 + 2U_1) \text{tr} \{(1 + \vec{\zeta}_0 \cdot \sigma) \} \text{tr} \{(\vec{k}_e + m_e \gamma^5 \vec{C}_e)\gamma^0 \vec{k}_e \gamma^0 (1 - \gamma^5) \} \right.
\]
\[+ g_A^2 \left( (1 + U_1 + U_2) \text{tr} \{(1 + \vec{\zeta}_0 \cdot \sigma) \} \cdot \text{tr} \{(\vec{k}_e + m_e \gamma^5 \vec{C}_e)\gamma^0 \vec{k}_e \gamma^0 (1 - \gamma^5) \} + g_A^2 \left( (1 + U_1 + U_2) \right) \text{tr} \{(1 + \vec{\zeta}_0 \cdot \sigma) \} \cdot \text{tr} \{(\vec{k}_e + m_e \gamma^5 \vec{C}_e)\gamma^0 \vec{k}_e \gamma^0 (1 - \gamma^5) \} \right.
\]
\[+ U_3 \text{tr} \{(1 + \vec{\zeta}_0 \cdot \sigma) \} \cdot \text{tr} \{(\vec{k}_e + m_e \gamma^5 \vec{C}_e)\gamma^0 \vec{k}_e \gamma^0 (1 - \gamma^5) \} + U_3 \text{tr} \{(1 + \vec{\zeta}_0 \cdot \sigma) \} \cdot \text{tr} \{(\vec{k}_e + m_e \gamma^5 \vec{C}_e)\gamma^0 \vec{k}_e \gamma^0 (1 - \gamma^5) \} - U_4 \text{tr} \{(1 + \vec{\zeta}_0 \cdot \sigma) \} \cdot \text{tr} \{(\vec{k}_e + m_e \gamma^5 \vec{C}_e)\gamma^0 \vec{k}_e \gamma^0 (1 - \gamma^5) \} \right.
\]
\[\left. + U_4 \text{tr} \{(1 + \vec{\zeta}_0 \cdot \sigma) \} \cdot \text{tr} \{(\vec{k}_e + m_e \gamma^5 \vec{C}_e)\gamma^0 \vec{k}_e \gamma^0 (1 - \gamma^5) \} \right) \left. + U_5 \text{tr} \{(1 + \vec{\zeta}_0 \cdot \sigma) \} \cdot \text{tr} \{(\vec{k}_e + m_e \gamma^5 \vec{C}_e)\gamma^0 \vec{k}_e \gamma^0 (1 - \gamma^5) \} + U_5 \text{tr} \{(1 + \vec{\zeta}_0 \cdot \sigma) \} \cdot \text{tr} \{(\vec{k}_e + m_e \gamma^5 \vec{C}_e)\gamma^0 \vec{k}_e \gamma^0 (1 - \gamma^5) \} \right) \right).
\]

\[\left. + U_7 \text{tr} \{(1 + \vec{\zeta}_0 \cdot \sigma) \} \cdot \text{tr} \{(\vec{k}_e + m_e \gamma^5 \vec{C}_e)\gamma^0 \vec{k}_e \gamma^0 (1 - \gamma^5) \} + U_8 \text{tr} \{(1 + \vec{\zeta}_0 \cdot \sigma) \} \cdot \text{tr} \{(\vec{k}_e + m_e \gamma^5 \vec{C}_e)\gamma^0 \vec{k}_e \gamma^0 (1 - \gamma^5) \} - U_9 \text{tr} \{(1 + \vec{\zeta}_0 \cdot \sigma) \} \cdot \text{tr} \{(\vec{k}_e + m_e \gamma^5 \vec{C}_e)\gamma^0 \vec{k}_e \gamma^0 (1 - \gamma^5) \} \right) \right). \] (A-10)

Having calculated the traces over the nucleon degrees of freedom and using the properties of the Dirac matrices [31]

\[\gamma^n \gamma^\nu = \gamma^n \gamma^\nu - \gamma^\nu \gamma^n + \gamma^\nu \gamma^\nu + i \epsilon^{\alpha \nu \beta \gamma} \gamma_\beta \gamma^\gamma \] (A-11)

and \(\gamma^\nu \gamma^\nu = 2 \eta^\nu\nu\), where \(\eta^\nu\nu\) is the metric tensor of the Minkowski space–time, \(\epsilon^{\alpha \nu \beta \gamma}\) is the Levi–Civita tensor defined by \(\delta^{\alpha \beta} = 1\) and \(\epsilon^{\alpha \nu \beta \gamma} = -\epsilon^{\nu \beta \gamma \alpha}\) [31], we transcribe the right-hand-side (r.h.s.) of Eq. (A-10) into the form [11] [12] [14]
\( b, a, A, B, \ldots, T \) and
\[
\left( 1 + 3 g_A^2 \right) \sum_{\text{pol.}} \left| \frac{M(n \to p \vec{e}^- \bar{\nu}_e)^2}{(1 + 3 g_A^2) G^2 V^2 4 m_n^2 E_e E_\nu} \right| = \left( \sum_{\text{pol.}} \frac{1 + 2}{1 + 3 g_A^2} U_7 - g_A U_5 \right) \left( 1 + 3 g_A^2 \right) \sum_{\text{pol.}} \left| \frac{M(n \to p \vec{e}^- \bar{\nu}_e)^2}{(1 + 3 g_A^2) G^2 V^2 4 m_n^2 E_e E_\nu} \right|
\]
\[
\chi_\| \cdot \vec{e}_\| - \left[ \left( 1 + 3 g_A^2 \right) \sum_{\text{pol.}} \left( 1 + 3 g_A^2 \right) \sum_{\text{pol.}} \left| \frac{M(n \to p \vec{e}^- \bar{\nu}_e)^2}{(1 + 3 g_A^2) G^2 V^2 4 m_n^2 E_e E_\nu} \right| = \left( \sum_{\text{pol.}} \frac{1 + 2}{1 + 3 g_A^2} U_7 - g_A U_5 \right) \left( 1 + 3 g_A^2 \right) \sum_{\text{pol.}} \left| \frac{M(n \to p \vec{e}^- \bar{\nu}_e)^2}{(1 + 3 g_A^2) G^2 V^2 4 m_n^2 E_e E_\nu} \right|
\]

In terms of the irreducible correlation structures the r.h.s. of Eq. (A.13) is given by

\[
\sum_{\text{pol.}} \left| \frac{M(n \to p \vec{e}^- \bar{\nu}_e)^2}{(1 + 3 g_A^2) G^2 V^2 4 m_n^2 E_e E_\nu} \right| = \zeta(E_e)_{\text{RC}} \left( 1 + b(E_e)_{\text{RC}} \frac{m_e}{E_e} + a(E_e)_{\text{RC}} \frac{\vec{k}_e \cdot \vec{E}_e}{E_e} + A(E_e)_{\text{RC}} \frac{\vec{k}_e \cdot \vec{E}_e}{E_e} + B(E_e)_{\text{RC}} \right)
\]

where the index RC means that these corrections are defined by the outer radiative corrections of order \( O(\alpha/\pi) \), calculated to LO in the large nucleon mass expansion \([12, 13, 19, 21]\), and the inner radiative corrections of order \( O(a E_n/m_X) \) \([16, 17]\). The correlation function \( \zeta(E_e)_{\text{RC}} \) and the correlation coefficients \( \zeta(E_e)_{\text{RC}} X(E_e)_{\text{RC}} \) for \( X = b, a, A, B, \ldots, T \) and \( \nu \) are equal to

\[
\zeta(E_e)_{\text{RC}} = \left( 1 + \frac{2}{1 + 3 g_A^2} U_7 + g_A U_5 \right) \left( 1 + 3 g_A^2 \right) \sum_{\text{pol.}} \left| \frac{M(n \to p \vec{e}^- \bar{\nu}_e)^2}{(1 + 3 g_A^2) G^2 V^2 4 m_n^2 E_e E_\nu} \right|
\]

\[
\zeta(E_e)_{\text{RC}b(E_e)_{\text{RC}}} = 0,
\]
\[
\zeta(E_{RC}a(E_{RC}) = \left( a_0 + \frac{2}{1 + 3g_A^2} (U_1 - g_A^2 U_2) \right) - \frac{2E_e}{1 + 3g_A^2} g_A U_5 \frac{m_e}{E_e} + \frac{2E_e}{1 + 3g_A^2} \left( g_A U_7 + (1 - 2g_A) U_8 \right),
\]
\[
\zeta(E_{RC}A(E_{RC}) = \left( A_0 + \frac{2}{1 + 3g_A^2} (g_A (U_1 + U_2) - 2g_A^2 U_2) \right) + \frac{2E_e}{1 + 3g_A^2} m_e \frac{E_e}{U_5 + 1 + 3g_A^2} \left( U_7 - g_A U_8 \right),
\]
\[
\zeta(E_{RC}B(E_{RC}) = \left( B_0 + \frac{2}{1 + 3g_A^2} (g_A (U_1 + U_2) + 2g_A^2 U_2) \right) + \frac{2E_e}{1 + 3g_A^2} \left( g_A (U_3 + U_4) + 2g_A^2 U_4 + E_6 U_6 \right) \frac{m_e}{E_e} + \frac{2E_e}{1 + 3g_A^2} \left( g_A U_7 - (1 - 2g_A) U_8 \right) \beta^2,
\]
\[
\zeta(E_{RC}K_n(E_{RC}) = \frac{2E_e}{1 + 3g_A^2} \left( (1 - g_A) U_7 + (1 + g_A) U_8 \right),
\]
\[
\zeta(E_{RC}Q_n(E_{RC}) = 0,
\]
\[
\zeta(E_{RC}G(E_{RC}) = -\left( A_0 + \frac{2}{1 + 3g_A^2} (g_A (U_1 + U_2) - 2g_A^2 U_2) \right) - \frac{2E_e}{1 + 3g_A^2} g_A U_5 \frac{m_e}{E_e} - \frac{2E_e}{1 + 3g_A^2} \left( g_A U_7 + (1 + 2g_A) U_8 \right),
\]
\[
\zeta(E_{RC}H(E_{RC}) = -\frac{m_e}{E_e} \left( A_0 + \frac{2}{1 + 3g_A^2} (g_A (U_1 + U_2) - 2g_A^2 U_2) \right) + \frac{2E_e}{1 + 3g_A^2} g_A U_5 \left( -U_3 + g_A^2 U_4 - g_A E_6 U_6 \right) + \frac{2E_e}{1 + 3g_A^2} g_A U_5 \beta^2,
\]
\[
\zeta(E_{RC}N(E_{RC}) = -\frac{m_e}{E_e} \left( A_0 + \frac{2}{1 + 3g_A^2} (g_A (U_1 + U_2) - 2g_A^2 U_2) \right) + \frac{2E_e}{1 + 3g_A^2} \left( -g_A (U_3 + U_4) + 2g_A^2 U_4 + g_A E_6 U_6 \right)
\]
\[
+ \frac{2E_e}{1 + 3g_A^2} g_A U_5 \beta^2 - \frac{1}{3} \frac{2E_e}{1 + 3g_A^2} (1 + g_A) U_6,
\]
\[
\zeta(E_{RC}Q_e(E_{RC}) = -\left( A_0 + \frac{2}{1 + 3g_A^2} (g_A (U_1 + U_2) - 2g_A^2 U_2) \right) - \frac{2E_e}{1 + 3g_A^2} \left( -g_A (U_3 + U_4) + 2g_A^2 U_4 + g_A E_6 U_6 \right)
\]
\[
+ \frac{2}{1 + 3g_A^2} \left( 1 + \frac{m_e}{E_e} \right) \left( g_A U_5 - U_7 + g_A U_8 \right) + \frac{1}{3} \frac{2E_e}{1 + 3g_A^2} (1 + g_A) U_6,
\]
\[
\zeta(E_{RC}K_e(E_{RC}) = -\left( A_0 + \frac{2}{1 + 3g_A^2} (U_1 - g_A U_2) \right) - \frac{2}{1 + 3g_A^2} \left( -U_3 + g_A^2 U_4 - g_A E_6 U_6 \right) - \frac{2E_e}{1 + 3g_A^2} \left( g_A U_5 + g_A U_7 \right)
\]
\[
+ (1 - 2g_A) U_8 \right) \left( 1 + \frac{m_e}{E_e} \right),
\]
\[
\zeta(E_{RC}S(E_{RC}) = -\frac{2}{1 + 3g_A^2} g_A \left( U_3 - U_4 - E_6 U_6 \right) - \frac{2E_e}{1 + 3g_A^2} g_A U_5 - \frac{2E_e}{1 + 3g_A^2} \left( -g_A U_7 + (1 + 2g_A) U_8 \right) \frac{m_e}{E_e},
\]
\[
\zeta(E_{RC}T(E_{RC}) = -\left( B_0 + \frac{2}{1 + 3g_A^2} (g_A (U_1 + U_2) + 2g_A^2 U_2) \right) + \frac{2E_e}{1 + 3g_A^2} \left( g_A U_5 - U_7 + g_A U_8 \right) + \frac{2E_e}{1 + 3g_A^2} \left( 1 - 2g_A \right) U_8),
\]
\[
\zeta(E_{RC}U(E_{RC}) = \frac{2}{1 + 3g_A^2} g_A \left( U_3 - U_4 - E_6 U_6 \right) - \frac{2E_e}{1 + 3g_A^2} g_A U_5 - \frac{2E_e}{1 + 3g_A^2} \left( U_7 - g_A U_8 \right) \frac{m_e}{E_e},
\]
\[
(A-15)
\]
where the correlation coefficients \(a_0, A_0\) and \(B_0\) depend only on the axial coupling constant \(g_A\) \([12]\) (see also \([29]\))
\[
(a_0 = \frac{1 - g_A^2}{1 + 3g_A^2}, \quad A_0 = 2g_A (1 - g_A) \frac{1 + 3g_A}{1 + 3g_A^2}, \quad B_0 = 2g_A (1 + g_A) \frac{1 + 3g_A}{1 + 3g_A^2}. \quad (A-16))
\]

Using the definitions of the functions \(U_j\) for \(j = 1, 2, \ldots, 8\) in Eq. \((A-4)\) and Eq. \((A-7)\) we transcribe Eq. \((A-15)\) as follows
\[
\zeta(E_{RC}) = 1 + \frac{\alpha}{\pi} \left( g_n(E_{RC}) - g^{(1)}_{\beta \gamma}(E_{RC}, \mu) \right) + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} E_e \left( f_V(E_{RC}) + \sqrt{1 - \beta^2} f_s(E_{RC}) + g_A g_V(E_{RC}) \beta^2 \right)
\]
\[
+ (1 + 2g_A) h_A(E_{RC}) \beta^2 + \frac{E_0 - E_e}{E_e} \sqrt{1 - \beta^2} h_s(E_{RC}) + 3g_A^2 f_A(E_{RC}) + 3g_A \sqrt{1 - \beta^2} f_r(E_{RC}) \right) + \frac{3g_A^2}{1 + 3g_A^2},
\]
\[
\zeta(E_{RC}) = \frac{\alpha}{\pi} \frac{m_n^2}{\pi} \frac{M_R}{m_N} + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \left( g_{\alpha}(E_{RC}) + 3g_A g_{\alpha}(E_{RC}) \right),
\]
\[
\zeta(E_{RC})_{rcb} = 0,
\]
\[
\zeta(E_{RC})_{rca} = a_0 \left( 1 + \frac{\alpha}{\pi} \left( g_n(E_{RC}) + \frac{1 - \beta^2}{2\beta} \frac{\alpha}{\pi} \left( 1 + \frac{\beta}{1 - \beta} \right) - g^{(1)}_{\beta \gamma}(E_{RC}, \mu) \right) \right) + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} E_e \left( f_V(E_{RC}) \right)
\]
\[
(A-15)
\]
\(+ g_A \sqrt{1 - \beta^2} g_s(E_c) + g_A g_v(E_c) + (1 - 2g_A) h_A(E_c) - g_A^2 f_A(E_c) + \frac{\alpha}{1 + 3g_A} \left( g_A (E_c) - g_A f_A(E_c) \right), \)

\[\zeta(E_c)_{RCA}(E_c) = A_0 \left( 1 + \frac{\alpha}{\pi} \left( g_A (E_c) + \frac{1 - \beta^2}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - g_A^2(E_c, \mu) \right) \right) + \frac{1}{1 + 3g_A} \frac{\alpha}{\pi} \frac{E_c}{m_N} \left( g_A f_v(E_c) + \sqrt{1 - \beta^2} g_s(E_c) + g_A g_v(E_c) \beta^2 - (1 - 2g_A) h_A(E_c) \beta^2 + g_A (1 + 2g_A) f_A(E_c) + g_A (1 + 2g_A) \sqrt{1 - \beta^2} f_A(E_c) \right) \times f_t(E_c) + \frac{1}{1 + 3g_A} \frac{\alpha}{\pi} \frac{E_c}{m_N} \left( g_A g_s(E_c) + (1 + 2g_A) h_A(E_c) + g_A f_A(E_c) \right) - \frac{\alpha}{1 + 3g_A} \frac{E_c}{m_N} \left( g_A (E_c) - g_A f_A(E_c) \right), \]

\[\zeta(E_c)_{RCA}(E_c) = A_0 \left( 1 + \frac{\alpha}{\pi} \left( g_A (E_c) + \frac{1 - \beta^2}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - g_A^2(E_c, \mu) \right) \right) + \frac{1}{1 + 3g_A} \frac{\alpha}{\pi} \frac{E_c}{m_N} \left( g_A f_v(E_c) + \sqrt{1 - \beta^2} g_s(E_c) + g_A g_v(E_c) \beta^2 - (1 - 2g_A) h_A(E_c) \beta^2 + g_A (1 + 2g_A) f_A(E_c) + g_A (1 + 2g_A) \sqrt{1 - \beta^2} f_A(E_c) \right) \times f_t(E_c) + \frac{1}{1 + 3g_A} \frac{\alpha}{\pi} \frac{E_c}{m_N} \left( g_A g_s(E_c) + (1 + 2g_A) h_A(E_c) + g_A f_A(E_c) \right) - \frac{\alpha}{1 + 3g_A} \frac{E_c}{m_N} \left( g_A (E_c) - g_A f_A(E_c) \right), \]

\[\zeta(E_c)_{RCA}(E_c) = A_0 \left( 1 + \frac{\alpha}{\pi} \left( g_A (E_c) + \frac{1 - \beta^2}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - g_A^2(E_c, \mu) \right) \right) + \frac{1}{1 + 3g_A} \frac{\alpha}{\pi} \frac{E_c}{m_N} \left( g_A f_v(E_c) + \sqrt{1 - \beta^2} g_s(E_c) + g_A g_v(E_c) \beta^2 - (1 - 2g_A) h_A(E_c) \beta^2 + g_A (1 + 2g_A) f_A(E_c) + g_A (1 + 2g_A) \sqrt{1 - \beta^2} f_A(E_c) \right) \times f_t(E_c) + \frac{1}{1 + 3g_A} \frac{\alpha}{\pi} \frac{E_c}{m_N} \left( g_A g_s(E_c) + (1 + 2g_A) h_A(E_c) + g_A f_A(E_c) \right) - \frac{\alpha}{1 + 3g_A} \frac{E_c}{m_N} \left( g_A (E_c) - g_A f_A(E_c) \right), \]
\[ \zeta(E_e)S(E_e)_{RC} = -\frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \left( g_A f_s(E_e) - g_A f_t(E_e) + g_A g_s(E_e) - g_A \frac{E_0 - E_e}{E_e} h_s(E_e) \right) \]

\[ \zeta(E_e)T(E_e)_{RC} = -B_0 \left( 1 + \frac{\alpha}{\pi} \left( g_a(E_e) + \frac{1 - \beta^2}{2\beta} \ln\left( \frac{1 + \beta}{1 - \beta} \right) - \frac{g_0(E_e)}{\mu} \right) \right) - \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \times \left( g_A f_V(E_e) - g_A g_s(E_e) + g_a g_V(E_e) - (1 - 2g_A) h_s(E_e) + g_A(1 + 2g_A) f_A(E_e) \right) \]

\[ \zeta(E_e)U(E_e)_{RC} = \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \left( g_A f_s(E_e) - g_A f_t(E_e) - g_A \frac{E_0 - E_e}{E_e} h_s(E_e) \right) - \sqrt{1 - \beta^2} g_V(E_e) + g_A \sqrt{1 - \beta^2} h_A(E_e) \right) \]

Taking into account the contribution of the neutron radiative beta decay [11, 12, 17, 29, 30] we obtain the correlation function \( \zeta(E_e)_{RC} \) and correlation coefficients \( \zeta(E_e)_{RC} X(E_e)_{RC} \) for \( X = b, a, A, B, \ldots, T \) and \( U \) in Eq. (A-17) in the form:

\[ \zeta(E_e)_{RC} = 1 + \frac{\alpha}{\pi} g_a(E_e) + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \left( f_V(E_e) + \sqrt{1 - \beta^2} f_s(E_e) + g_A g_V(E_e) - (1 + 2g_A) h_A(E_e) \beta^2 + g_A \frac{E_0 - E_e}{E_e} \right) + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \left( g_A f_t(E_e) - g_A f_s(E_e) \right) \]

\[ \zeta(E_e)_{RC} a(E_e)_{RC} = a_0 \left( 1 + \frac{\alpha}{\pi} g_a(E_e) + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \right) \]

\[ \zeta(E_e)_{RC} A(E_e)_{RC} = A_0 \left( 1 + \frac{\alpha}{\pi} g_a(E_e) + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \right) \]

\[ \zeta(E_e)_{RC} B(E_e)_{RC} = B_0 \left( 1 + \frac{\alpha}{\pi} g_a(E_e) + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \right) \]

\[ \zeta(E_e)_{RC} K_a(E_e)_{RC} = \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \left( 1 - g_A \right) g_V(E_e) + (1 + 2g_A) h_A(E_e) \]

\[ \zeta(E_e)_{RC} Q_a(E_e)_{RC} = 0 \]

\[ \zeta(E_e)_{RC} G(E_e)_{RC} = \left( 1 + \frac{\alpha}{\pi} g_a(E_e) + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \right) \]

\[ \zeta(E_e)_{RC} Q_a(E_e)_{RC} = 0 \]
\[-\frac{1}{1 + 3 g_A^2 ζ} \left( \tilde{g}_n(E_n) + 3 g_A \tilde{f}_{st}(E_n) \right),\]

\[\zeta(E_n)_{\text{RC}H(E_n)_{\text{RC}}} = -\frac{m^2}{E_n^2} B_0 \left( 1 + \frac{α}{2} g_n(E_n) + \frac{α}{π} h^{(1)}_n(E_n) \right) - \frac{1}{1 + 3 g_A^2} \frac{E_n}{m_N} \sqrt{1 - β^2} f_n(E_n) + f_S(E_n),\]

\[+ g_{\text{AGS}}(E_n) β^2 + \frac{g_A}{E_n} E_n h_S(E_n) - g^2 A \sqrt{1 - β^2} f_{\text{AGS}}(E_n) + g^2 A \sqrt{1 - β^2} \frac{g_A}{1 + 3 g_A^2} \frac{α}{2} \pi \frac{m_N^2}{M_W^2} \]

\[× \ln \frac{M_W^2}{m_N^2} - \frac{1}{1 + 3 g_A^2} \frac{α}{π} \sqrt{1 - β^2} \left( \tilde{g}_n(E_n) - g_A \tilde{f}_{st}(E_n) \right),\]

\[\zeta(E_n)_{\text{RC}N(E_n)_{\text{RC}}} = -\frac{m^2}{E_n^2} B_0 \left( 1 + \frac{α}{2} g_n(E_n) + \frac{α}{π} h^{(1)}_n(E_n) \right) - \frac{1}{1 + 3 g_A^2} \frac{E_n}{m_N} \sqrt{1 - β^2} g_{\text{AGS}}(E_n) + g_{\text{AGS}}(E_n) β^2 + \frac{1}{3} \left( 1 - 2 g_A \right) \frac{E_n}{E_n} h_S(E_n) + g_A \left( 1 - 2 g_A \right) \sqrt{1 - β^2} f_{\text{AGS}}(E_n) + g_A \left( 1 - 2 g_A \right) f_{\text{AGS}}(E_n) \]

\[\zeta(E_n)_{\text{RC}Q_4(E_n)_{\text{RC}}} = -\frac{m^2}{E_n^2} B_0 \left( 1 + \frac{α}{2} g_n(E_n) + \frac{α}{π} h^{(1)}_n(E_n) \right) - \frac{1}{1 + 3 g_A^2} \frac{E_n}{m_N} \sqrt{1 - β^2} \left( g_{\text{AGS}}(E_n) + (1 - 2 g_A) \tilde{f}_{st}(E_n) \right),\]

\[\zeta(E_n)_{\text{RC}K_4(E_n)_{\text{RC}}} = -\frac{m^2}{E_n^2} B_0 \left( 1 + \frac{α}{2} g_n(E_n) + \frac{α}{π} h^{(1)}_n(E_n) \right) - \frac{1}{1 + 3 g_A^2} \frac{E_n}{m_N} \left( f_{\text{AGS}}(E_n) - f_{\text{AGS}}(E_n) \right) \]

\[+ g_A \left( 1 + \sqrt{1 - β^2} g_{\text{AGS}}(E_n) + (1 + \sqrt{1 - β^2} g_{\text{AGS}}(E_n) \right) \]

\[+ g_{\text{AGS}}(E_n) - g_A \sqrt{1 - β^2} g_{\text{AGS}}(E_n) \]

\[\zeta(E_n)_{\text{RC}N(E_n)_{\text{RC}}} = -\frac{m^2}{E_n^2} B_0 \left( 1 + \frac{α}{2} g_n(E_n) + \frac{α}{π} h^{(1)}_n(E_n) \right) - \frac{1}{1 + 3 g_A^2} \frac{E_n}{m_N} \sqrt{1 - β^2} \left( g_{\text{AGS}}(E_n) + (1 - 2 g_A) \tilde{f}_{st}(E_n) \right),\]

\[\zeta(E_n)_{\text{RC}S(E_n)_{\text{RC}}} = -\frac{1}{1 + 3 g_A^2} \frac{α}{π} \frac{E_n}{m_N} \left( \tilde{g}_n(E_n) - g_A \tilde{f}_{st}(E_n) + g_{\text{AGS}}(E_n) - g_A \frac{E_0 - E_n}{E_n} \right) + g_A \sqrt{1 - β^2} g_{\text{AGS}}(E_n) \]

\[+ g_{\text{AGS}}(E_n) - g_A \sqrt{1 - β^2} g_{\text{AGS}}(E_n) \]

\[\zeta(E_n)_{\text{RC}T(E_n)_{\text{RC}}} = -\frac{1}{1 + 3 g_A^2} \frac{α}{π} \frac{E_n}{m_N} \left( f_{\text{AGS}}(E_n) - f_{\text{AGS}}(E_n) \right) \]

\[+ g_{\text{AGS}}(E_n) \frac{1}{1 + 2 g_A} h_{\text{AGS}}(E_n) + g_A \frac{1}{1 + 2 g_A} f_{\text{AGS}}(E_n) \]

\[\zeta(E_n)_{\text{RC}U(E_n)_{\text{RC}}} = \frac{1}{1 + 3 g_A^2} \frac{α}{π} \frac{E_n}{m_N} \left( g_{\text{AGS}}(E_n) - g_A \frac{E_0 - E_n}{E_n} \right) + g_{\text{AGS}}(E_n) \frac{1}{1 + 2 g_A} h_{\text{AGS}}(E_n) \]

\[- g_A \sqrt{1 - β^2} g_{\text{AGS}}(E_n) + g_A \sqrt{1 - β^2} h_{\text{AGS}}(E_n) \].

(A-18)

The functions \( g_n(E_n) \) and \( f_{st}(E_n) \) have been calculated by Sirlin [19] and Shann [21] (see also [11] [12] [29]), whereas the function \( h_n^{(1)}(E_n) \) and \( h_n^{(2)}(E_n) \) have been calculated in [12] [14]. They are equal to

\[
\tilde{g}_n(E_n) = \frac{3}{4} \ln \left( \frac{m_N^2}{m_e^2} \right) - 3 + \frac{1}{8} \ln \left( \frac{1 + β}{1 - β} \right) - 2 \left[ \frac{1}{12} \left( \frac{2(E_0 - E_n)}{m_e^2} \right)^2 - \frac{1}{2} - \frac{1}{3} \frac{E_0 - E_n}{E_n} \right] - \frac{2}{β} \text{Li}_2 \left( \frac{2β}{1 + β} \right) \]

\[+ \frac{1}{β} \ln \left( \frac{1 + β}{1 - β} \right) \left[ (1 + β^2) + \frac{1}{12} \frac{(E_0 - E_n)^2}{E_n} \right] - \ln \left( \frac{1 + β}{1 - β} \right),\]
The radiative corrections of order $O(\alpha/\pi)$ and $O(\alpha E_e/m_N)$ to the neutron lifetime and the correlation coefficients of the electron-energy and angular distribution Eq. (1) are given by

\[ \zeta(E_{\text{RC}}) = 1 + \frac{\alpha}{\pi} g_0(E_e) + \zeta(E_{\text{RC-NLO}}), \]

\[ \zeta(E_{\text{RC-NLO}}) = \frac{1}{1 + 3g_A^2} \frac{\alpha E_e}{\pi m_N} f_{\text{fr}}(E_e) + g_A g_E(E_e) \beta^2 + (1 + 2g_A) h_A(E_e) \beta^2 \]

\[ + g_A \frac{E_e - E}{E_e} \sqrt{1 - \beta^2} h_s(E_e) + 3g_A^3 f_A(E_e) + g_A^2 \sqrt{1 - \beta^2} f_T(E_e) \]

\[ + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \left( g_{st}(E_e) - g_{st}f_{st}(E_e) \right) - a_0 \zeta(E_{\text{RC-NLO}}), \]

\[ A(E_{\text{RC}}) = A_0 \left( 1 + \frac{\alpha}{\pi} f_0(E_e) \right) + A(E_{\text{RC-NLO}}), \]

\[ A(E_{\text{RC-NLO}}) = \frac{1}{1 + 3g_A^2} \frac{\alpha E_e}{\pi m_N} \left( g_A f_{\text{fr}}(E_e) + \sqrt{1 - \beta^2} g_s(E_e) + g_A g_E(E_e) (1 - 2g_A) h_A(E_e) \right) \]

\[ + g_A (1 - 2g_A) f_A(E_e) \]

\[ + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \left( g_A \bar{g}_E(E_e) (1 - 2g_A) + g_A \bar{f}_E(E_e) \right) - A_0 \zeta(E_{\text{RC-NLO}}), \]

\[ B(E_{\text{RC}}) = B_0 \left( 1 + \frac{\alpha}{\pi} g_0(E_e) \right) + B(E_{\text{RC-NLO}}), \]

\[ B(E_{\text{RC-NLO}}) = \frac{1}{1 + 3g_A^2} \frac{\alpha E_e}{\pi m_N} \left( g_A f_{\text{fr}}(E_e) + g_A \sqrt{1 - \beta^2} f_{\text{s}}(E_e) + \frac{E_e - E}{E_e} \sqrt{1 - \beta^2} h_s(E_e) \right) \]

\[ + g_A g_E(E_e) \beta^2 + g_A (1 + 2g_A) f_A(E_e) + g_A (1 + 2g_A) \sqrt{1 - \beta^2} f_T(E_e) \]

\[ + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_e}{m_N} \left( g_A \bar{g}_E(E_e) (1 + 2g_A) + g_A \bar{f}_E(E_e) \right) - B_0 \zeta(E_{\text{RC-NLO}}), \]

\[ K_n(E_{\text{RC}}) = K_n(E_e)_{\text{RC-NLO}} = \frac{1}{1 + 3g_A^2} \frac{\alpha E_e}{\pi m_N} \left( (1 - g_A) g_E(E_e) + (1 + 2g_A) h_A(E_e) \right), \]

\[ Q_n(E_{\text{RC}}) = 0, \]

\[ G(E_{\text{RC}}) = - \left( 1 + \frac{\alpha}{\pi} f_0(E_e) \right) + G(E_{\text{RC-NLO}}), \]

\[ G(E_{\text{RC-NLO}}) = - \frac{1}{1 + 3g_A^2} \frac{\alpha E_e}{\pi m_N} \left( g_A \sqrt{1 - \beta^2} g_E(E_e) + g_A (1 - \beta^2) g_E(E_e) + (1 + 2g_A) (1 - \beta^2) f_T(E_e) \right) \]

\[ \times h_A(E_e) - \sqrt{1 - \beta^2} f_{\text{s}}(E_e) - g_A \frac{E_e - E}{E_e} \sqrt{1 - \beta^2} h_s(E_e) - 3g_A^2 \sqrt{1 - \beta^2} f_T(E_e) \right), \]
\[ H(E_\text{rc}) = -\frac{m_e}{E_e} a_0 \left( 1 + \frac{\alpha}{\pi} h_1(E_\text{e}) \right) + H(E_\text{rc-NLO}), \]
\[ H(E_\text{rc-NLO}) = -\frac{1}{1 + 3g_A} \frac{E_e}{m_N} \left( \sqrt{1 - \beta^2} f_V(E_e) + f_S(E_e) + g_{AGS}(E_e) \beta^2 \right) \]
\[ + \frac{g_A}{E_e} \frac{E_0 - E_e}{h_S(E_e) - g_A^2 \sqrt{1 - \beta^2} f_A(E_e) - g_A^2 f_T(E_e)} \right) + \frac{g_A^2}{1 + 3g_A^2} \frac{5}{2} \frac{m_N^2}{M_W^2} \frac{\alpha}{\pi} \left( g_A \frac{h_{st}(E_e)}{E_e} + (1 - 2g_A) f_T(E_e) \right) \]
\[ - \frac{1}{1 + 3g_A^2} \alpha \frac{\beta}{2} \frac{m_N^2}{m_N^2} - \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \left( g_A \frac{h_{st}(E_e)}{E_e} + (1 - 2g_A) f_T(E_e) \right) \right) + \frac{m_e}{E_e} a_0 \zeta(E_\text{rc-NLO}), \]
\[ N(E_\text{rc}) = -\frac{m_e}{E_e} A_0 \left( 1 + \frac{\alpha}{\pi} h_1(E_\text{e}) \right) + N(E_\text{rc-NLO}), \]
\[ N(E_\text{rc-NLO}) = -\frac{1}{1 + 3g_A^2} \frac{e}{\pi} \frac{E_e}{m_N} \left( g_A f_V(E_e) \sqrt{1 - \beta^2} + g_A f_S(E_e) - g_A \frac{h_{st}(E_e)}{E_e} \right) \]
\[ + \frac{1}{3} \left( 1 - 2g_A \right) \frac{E_0 - E_e}{E_e} h_S(E_e) + \frac{1}{3} \left( 1 - 2g_A \right) \frac{E_0 - E_e}{E_e} h_S(E_e) + \frac{g_A}{E_e} \frac{E_0 - E_e}{h_S(E_e) - g_A(1 - 2g_A) f_A(E_e) - g_A(1 - 2g_A) f_T(E_e)} \right) \]
\[ - \sqrt{1 - \beta^2} - \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \left( g_A \frac{h_{st}(E_e)}{E_e} + (1 - 2g_A) f_T(E_e) \right) + \frac{m_e}{E_e} a_0 \zeta(E_\text{rc-NLO}), \]
\[ Q_e(E_\text{rc}) = -a_0 \left( 1 + \frac{\alpha}{\pi} h_2(E_e) \right) + Q_e(E_\text{rc-NLO}), \]
\[ Q_e(E_\text{rc-NLO}) = -\frac{1}{1 + 3g_A^2} \frac{e}{\pi} \frac{E_e}{m_N} \left( g_A f_V(E_e) - g_A f_S(E_e) - g_A(1 + \sqrt{1 - \beta^2}) g_S(E_e) \right) \]
\[ + \frac{1}{3} \left( 1 - 2g_A \right) \frac{E_0 - E_e}{E_e} h_S(E_e) + \frac{1}{3} \left( 1 - 2g_A \right) \frac{E_0 - E_e}{E_e} h_S(E_e) + \frac{g_A}{E_e} \frac{E_0 - E_e}{h_S(E_e) - g_A(1 - 2g_A) f_A(E_e) - g_A(1 - 2g_A) f_T(E_e)} \right) \]
\[ - \sqrt{1 - \beta^2} - \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \left( g_A \frac{h_{st}(E_e)}{E_e} + (1 - 2g_A) f_T(E_e) \right) + \frac{m_e}{E_e} a_0 \zeta(E_\text{rc-NLO}), \]
\[ K_e(E_\text{rc}) = -a_0 \left( 1 + \frac{\alpha}{\pi} h_2(E_e) \right) + K_e(E_\text{rc-NLO}), \]
\[ K_e(E_\text{rc-NLO}) = -\frac{1}{1 + 3g_A^2} \frac{e}{\pi} \frac{E_e}{m_N} \left( f_V(E_e) - f_S(E_e) + g_A \left( 1 + \sqrt{1 - \beta^2} \right) g_S(E_e) \right) \]
\[ + g_A \left( 1 + \sqrt{1 - \beta^2} \right) g_V(E_e) + g_A \left( 1 + \sqrt{1 - \beta^2} \right) h_A(E_e) - g_A \frac{E_0 - E_e}{E_e} h_S(E_e) \]
\[ + \frac{g_A^2}{1 + 3g_A^2} \frac{e}{\pi} \frac{E_e}{m_N} \left( f_V(E_e) - f_S(E_e) + g_A \left( 1 + \sqrt{1 - \beta^2} \right) g_S(E_e) \right) \]
\[ + \frac{g_A^2}{1 + 3g_A^2} \frac{e}{\pi} \frac{E_e}{m_N} \left( f_V(E_e) - f_S(E_e) + g_A \left( 1 + \sqrt{1 - \beta^2} \right) g_S(E_e) \right) \]
\[ - g_A \sqrt{1 - \beta^2} g_V(E_e) + (1 + 2g_A) \sqrt{1 - \beta^2} h_A(E_e) \right) \right), \]
\[ T(E_\text{rc}) = -B_0 \left( 1 + \frac{\alpha}{\pi} f_s(E_e) \right) + T(E_\text{rc-NLO}), \]
\[ T(E_\text{rc-NLO}) = -\frac{1}{1 + 3g_A^2} \frac{e}{\pi} \frac{E_e}{m_N} \left( g_A f_V(E_e) - g_A f_S(E_e) + g_A \left( 1 + 2g_A \right) h_A(E_e) \right) \]
\[ + g_A \left( 1 + 2g_A \right) f_A(E_e) \right) - \frac{1}{1 + 3g_A^2} \frac{e}{\pi} \frac{E_e}{m_N} \left( g_A f_V(E_e) - g_A f_S(E_e) + g_A \left( 1 + 2g_A \right) f_T(E_e) \right) \]
\[ + B_0 \zeta(E_\text{rc-NLO}), \]
\[ U(E_\text{rc}) = U(E_\text{rc-NLO}) = -\frac{1}{1 + 3g_A^2} \frac{e}{\pi} \frac{E_e}{m_N} \left( g_A f_S(E_e) - g_A f_T(E_e) - g_A e - E_e \right) h_S(E_e) \]
\[ - \sqrt{1 - \beta^2} g_V(E_e) + g_A \sqrt{1 - \beta^2} h_A(E_e) \right). \] (A-20)

The correlation function \( \zeta(E_\text{rc}) \) and correlation coefficients \( X(E_\text{rc}) \) contain a complete set of outer radiative corrections of order \( O(\alpha/\pi) \) \[12\,14\,15\,21\], calculated to LO in the large nucleon mass \( m_N \) expansion, and radiative corrections of order \( O(\alpha E_e/m_N) \) \[16\,17\], obtained as NLO corrections in the large nucleon mass \( m_N \) expansion to Sirlin’s outer and inner radiative corrections, calculated to LO in the large nucleon mass \( m_N \) expansion. For \( \alpha = 0 \) the correlation function \( \zeta(E_\text{rc}) \) and the correlation coefficients \( X(E_\text{rc}) \) acquire their expressions, calculated to
LO in the large nucleon mass $m_N$ expansion [11, 2] (see also [29]). We have plotted the $Y(E_e)_{\text{RC-NLO}}$ corrections for $Y = \zeta, a, A, B, \ldots U$ and $A^{(2)}(E_e)$ (see Appendix E) in [46].

Appendix B: The corrections of order $O(E_e/m_N)$ and $O(E_e^2/m_N^2)$, caused by weak magnetism and proton recoil, to next-to-leading and next-to-next-to-leading order in the large nucleon mass $m_N$ expansion

The corrections to the structure function $\zeta(E_e)_{\text{WP}}$ and the correlation coefficients $X(E_e)_{\text{WP}}$ for $X = a, A, B, \ldots T$ and $U$, caused by weak magnetism and proton recoil, we define as follows

$$\zeta(E_e)_{\text{WP}} = \zeta(E_e)_{\text{NLO}} + \zeta(E_e)_{\text{N^2LO}},$$

$$X(E_e)_{\text{WP}} = X(E_e)_{\text{NLO}} + X(E_e)_{\text{N^2LO}}.$$  (B-1)

The corrections $\zeta(E_e)_{\text{NLO}}$ and $X(E_e)_{\text{NLO}}$, which are in principle of order $10^{-3}$ [11, 12, 14] have been calculated in [17] and [11, 12, 14, 15, 22, 23, 29, 30] (see also [24]). The analytical expressions of these corrections are given by

$$\zeta(E_e)_{\text{NLO}} = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ -2g_A \left( g_A + (\kappa + 1) \right) + 10g_A^2 + 4g_A + 2 \right] \frac{E_e}{E_0},$$

$$a(E_e)_{\text{NLO}} = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ 2g_A \left( g_A + (\kappa + 1) \right) - 4g_A \left( 3g_A + (\kappa + 1) \right) \frac{E_e}{E_0} \right] - a_0 \zeta(E_e)_{\text{NLO}},$$

$$A(E_e)_{\text{NLO}} = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ \left( g_A^2 + \kappa g_A - (\kappa + 1) \right) - \left( 5g_A^2 + (3\kappa - 4)g_A - (\kappa + 1) \right) \frac{E_e}{E_0} \right] - A_0 \zeta(E_e)_{\text{NLO}},$$

$$B(E_e)_{\text{NLO}} = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ -2g_A \left( g_A + (\kappa + 1) \right) + \left( 7g_A^2 + (3\kappa + 8)g_A + (\kappa + 1) \right) \frac{E_e}{E_0} \right]$$

$$- \left( g_A^2 + (\kappa + 2)g_A + (\kappa + 1) \right) \frac{m_e^2 E_0}{E_e},$$

$$K_n(E_e)_{\text{NLO}} = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ 5g_A^2 + (\kappa - 4)g_A - (\kappa + 1) \right] \frac{E_e}{E_0} ,$$

$$Q_n(E_e)_{\text{NLO}} = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ \left( g_A^2 + (\kappa + 2)g_A + (\kappa + 1) \right) - \left( 7g_A^2 + (3\kappa + 8)g_A + (\kappa + 1) \right) \frac{E_e}{E_0} \right] ,$$

$$G(E_e)_{\text{NLO}} = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ -2g_A \left( g_A + (\kappa + 1) \right) + \left( 4g_A^2 + 2(\kappa + 1)g_A - 2 \right) \frac{E_e}{E_0} \right]$$

$$+ \left( 4 \right) \frac{m_e E_0}{E_e} \zeta(E_e)_{\text{NLO}},$$

$$N(E_e)_{\text{NLO}} = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ -g_A \left( g_A + (\kappa + 1) \right) + \left( 4g_A^2 + 2(\kappa + 1)g_A - 2 \right) \frac{E_e}{E_0} \right]$$

$$+ \left( \frac{4 \kappa - 2}{3} \right) \frac{E_e}{E_0} \zeta(E_e)_{\text{NLO}},$$

$$Q_e(E_e)_{\text{NLO}} = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ \left( 4g_A^2 + (\kappa - 1)g_A - \frac{2}{3} \right) \frac{E_e}{E_0} \right] + A_0 \frac{m_e E_0}{E_e} \zeta(E_e)_{\text{NLO}},$$

$$K_e(E_e)_{\text{NLO}} = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ -2g_A \left( g_A + (\kappa + 1) \right) + \left( 8g_A^2 + 2(\kappa + 1)g_A + 2 \right) \frac{E_e}{E_0} \right]$$

$$+ \left( 4g_A \left( 3g_A + (\kappa + 1) \right) \frac{E_e}{E_0} \right]$$

$$+ a_0 \zeta(E_e)_{\text{NLO}},$$

$$S(E_e)_{\text{NLO}} = \frac{1}{1 + 3g_A^2} \frac{m_e}{m_N} \left[ -5g_A^2 - (\kappa - 4)g_A + (\kappa + 1) \right] ,$$

$$T(E_e)_{\text{NLO}} = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ 2g_A \left( g_A + (\kappa + 1) \right) - \left( 7g_A^2 + (3\kappa + 8)g_A + (\kappa + 1) \right) \frac{E_e}{E_0} \right] + B_0 \zeta(E_e)_{\text{NLO}},$$

$$U(E_e)_{\text{NLO}} = 0.$$  (B-2)
For the definition of the $O(E^2_c/m_N^2)$ corrections it is convenient to adduce the following expressions, calculated in [11,12,13]:

\[
\begin{align*}
\bar{a}(E_c)_{\text{NLO}} &= \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ 2g_A (g_A + (\kappa + 1)) - 4g_A (3g_A + (\kappa + 1)) E_c \right] E_0, \\
\bar{A}(E_c)_{\text{NLO}} &= \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ \left(g_A^2 + \kappa g_A - (\kappa + 1)\right) - \left(5g_A^2 + (3\kappa - 4)g_A - (\kappa + 1)\right) E_c \right], \\
\bar{B}(E_c)_{\text{NLO}} &= \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[-\left(2g_A (g_A + (\kappa + 1)) + \left(7g_A^2 + (3\kappa + 8)g_A + (\kappa + 1)\right) E_c \right) \right] E_0, \\
&\quad- \left(g_A^2 + (\kappa + 2)g_A + (\kappa + 1)\right) \frac{m_e^2}{E_0^2} E_0, \\
\bar{K}_n(E_c)_{\text{NLO}} &= \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left(5g_A^2 + (\kappa - 4)g_A - (\kappa + 1)\right) E_c \right] E_0, \\
\bar{Q}_n(E_c)_{\text{NLO}} &= \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[\left(g_A^2 + (\kappa + 2)g_A + (\kappa + 1)\right) - \left(7g_A^2 + (\kappa + 8)g_A + (\kappa + 1)\right) E_c \right]. \\
\bar{C}(E_c)_{\text{NLO}} &= \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[\left(2g_A^2 + 2(\kappa + 1)g_A\right) - \left(10g_A^2 + 4(\kappa + 1)g_A + 2\right) E_c \right]
\bar{H}(E_c)_{\text{NLO}} &= \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \frac{m_e}{E_c} \left[-\left(2g_A (g_A + (\kappa + 1)) + \left(4g_A^2 + 2(\kappa + 1)g_A - 2\right) E_c \right) \right], \\
\bar{N}(E_c)_{\text{NLO}} &= \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \frac{m_e}{E_c} \left[-\left(4^3 g_A^2 + \left(\kappa - \frac{1}{3}\right) g_A - \frac{2}{3}(\kappa + 1)\right) \right] \frac{m_e}{E_0}, \\
&\quad+ \left(\frac{16}{3} g_A^2 + \left(\kappa - \frac{16}{3}\right) g_A - \frac{2}{3}(\kappa + 1)\right) \frac{E_c}{E_0}, \\
\bar{Q}_e(E_c)_{\text{NLO}} &= \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[-\left(4^3 g_A^2 + \left(\kappa - \frac{1}{3}\right) g_A - \frac{2}{3}(\kappa + 1)\right) + \left(2g_A^2 + 2(\kappa + 1)g_A\right) \frac{m_e}{E_0} \right] \\
&\quad+ \left(\frac{22}{3} g_A^2 + \left(\kappa - \frac{10}{3}\right) g_A - \frac{2}{3}(\kappa + 1)\right) \frac{E_c}{E_0}, \\
\bar{K}_e(E_c)_{\text{NLO}} &= \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[-2g_A (g_A + (\kappa + 1)) + \left(8g_A^2 + 2(\kappa + 1)g_A + 2\right) \frac{m_e}{E_0} \right] \\
&\quad+ 4g_A (3g_A + (\kappa + 1)) \frac{E_0}{E_c}, \tag{B-3}
\end{align*}
\]

where $\bar{K}_n(E_c)_{\text{NLO}} = K_n(E_c)_{\text{NLO}}$ and $\bar{Q}_n(E_c)_{\text{NLO}} = Q_n(E_c)_{\text{NLO}}$. Using the results obtained in [24], we get the following analytical expressions for the N3LO corrections $\zeta(E_c)_{\text{N3LO}}$ and $X(E_c)_{\text{N3LO}}$ for $X(E_c) = a(E_c), A(E_c), \ldots, U(E_c)$. They are given by

\[
\begin{align*}
\zeta(E_c)_{\text{N3LO}} &= 6 \frac{E^2_c}{m_N^2} \{1 - \frac{1}{3} \frac{E_0}{E_c}\} + \frac{1}{3} \left[1 - 2a_0 \left(1 - \frac{1}{8} \frac{E_0}{E_c}\right) \left(1 - \frac{m_e^2}{E_c^2}\right)\right] + 3 \frac{E_c}{m_N} \zeta(E_c)_{\text{NLO}} - \frac{E_c}{m_N} \bar{a}(E_c)_{\text{NLO}} \\
\times \left(1 - \frac{m_e^2}{E_c^2}\right) - \frac{8}{1 + 3g_A^2} \left(\frac{E_0^2}{M_N^2} + 3g_A^2 \frac{E_0^2}{M_N^2}\right) \frac{E_c}{E_0} \left(1 - \frac{E_c}{E_0}\right), \\
a(E_c)_{\text{N3LO}} &= 12 \frac{E^2_c}{m_N^2} \left\{\left(1 - \frac{1}{8} \frac{E_0}{E_c}\right) + \frac{1}{3} a_0 \left(1 - \frac{1}{4} \frac{E_0}{E_c}\right)\right\} + 3 \frac{E_c}{m_N} \bar{a}(E_c)_{\text{NLO}} - \frac{E_c}{m_N} \zeta(E_c)_{\text{NLO}} \\
- a(E_c)_{\text{NLO}} \zeta(E_c)_{\text{NLO}} + \frac{8}{1 + 3g_A^2} \left(\frac{E_0^2}{M_N^2} + 3g_A^2 \frac{E_0^2}{M_N^2}\right) \frac{E_c}{E_0} \left(1 - \frac{E_c}{E_0}\right), \\
A(E_c)_{\text{N3LO}} &= 3 \frac{E_c}{m_N} \bar{A}(E_c)_{\text{NLO}} - \frac{E_c}{m_N} \bar{K}_n(E_c)_{\text{NLO}} \left(1 - \frac{m_e^2}{E_c^2}\right) - A(E_c)_{\text{NLO}} \zeta(E_c)_{\text{NLO}}, \\
B(E_c)_{\text{N3LO}} &= 6 \frac{E^2_c}{m_N^2} \left(1 - \frac{1}{4} \frac{E_0}{E_c}\right) + 3 \frac{E_c}{m_N} \bar{B}(E_c)_{\text{NLO}} - B(E_c)_{\text{NLO}} \zeta(E_c)_{\text{NLO}} - B_0 \zeta(E_c)_{\text{N3LO}} \\
&\quad- \frac{8g_A}{1 + 3g_A^2} \left(\frac{E_0^2}{M_N^2} + \left(1 + 2g_A\right) \frac{E_0^2}{M_N^2}\right) \frac{E_c}{E_0} \left(1 - \frac{E_c}{E_0}\right). 
\end{align*}
\]
\[ K_n(E_e)_{eNLO} = 3 \frac{E_e}{m_N} K_n(E_e)_{NLO} - 3 \frac{E_e}{m_N} \tilde{A}(E_e)_{NLO}, \]
\[ Q_n(E_e)_{eNLO} = -12 \frac{E_e}{m_N} B_0 \left( 1 - \frac{1}{8} \frac{E_e}{E_0} \right) + 3 \frac{E_e}{m_N} \tilde{Q}_n(E_e)_{NLO} - 3 \frac{E_e}{m_N} \tilde{B}(E_e)_{NLO} - Q_n(E_e)_{eNLO} \zeta(E_e)_{NLO}, \]
\[ G(E_e)_{eNLO} = 6 \frac{E_e}{m_N} \left\{ - \left( 1 - \frac{1}{E_e} \frac{E_0}{E_0} \right) - \frac{2}{3} \left( 1 - \frac{1}{8} \frac{E_e}{E_0} \right) \right\} + 3 \frac{E_e}{m_N} \tilde{G}(E_e)_{NLO} - \frac{E_e}{m_N} \tilde{H}(E_e)_{NLO} \]
\[ - \frac{E_e}{m_N} K_e(E_e)_{NLO} \left( 1 - \frac{m_e}{E_e} \right) - G(E_e)_{NLO} \zeta(E_e)_{NLO} + \zeta(E_e)_{eNLO} + \frac{8}{1 + 3g_A^2} \left( \frac{E_e}{M_V^2} + 3g_A^2 \frac{E_e}{M_A^2} \right) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right), \]
\[ H(E_e)_{eNLO} = -\tilde{H}(E_e)_{NLO} \zeta(E_e)_{eNLO}, \]
\[ N(E_e)_{eNLO} = -\tilde{N}(E_e)_{NLO} \zeta(E_e)_{eNLO}, \]
\[ Q_e(E_e)_{eNLO} = 3 \frac{E_e}{m_N} \tilde{Q}_e(E_e)_{NLO} - Q_e(E_e)_{eNLO} \zeta(E_e)_{NLO}, \]
\[ K_e(E_e)_{eNLO} = 12 \frac{E_e}{m_N} \left\{ \left( 1 - \frac{1}{8} \frac{E_0}{E_e} \right) \left( 1 + \frac{m_e}{E_e} \right) - \frac{1}{2} g_0 \left( 1 - \frac{1}{4} \frac{E_0}{E_e} \right) \right\} + 3 \frac{E_e}{m_N} \tilde{K}_e(E_e)_{NLO} - 3 \frac{E_e}{m_N} \tilde{G}(E_e)_{NLO}, \]
\[ \times \left( 1 + \frac{m_e}{E_e} \right) - K_e(E_e)_{NLO} \zeta(E_e)_{NLO} - \frac{8}{1 + 3g_A^2} \left( \frac{E_e}{M_V^2} + 3g_A^2 \frac{E_e}{M_A^2} \right) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right), \]
\[ S(E_e)_{NLO} = -3 \frac{E_e}{m_N} \tilde{N}(E_e)_{NLO} - S(E_e)_{NLO} \zeta(E_e)_{NLO}, \]
\[ T(E_e)_{eNLO} = \frac{1}{1 + 3g_A^2} \frac{E_e}{2m_N} \left\{ \left( - g_0^2 - 4(g_0 + 1) \right) + (g_0 + 1) \frac{m_e^2}{E_e^2} \right\} \]
\[ + \left( 5g_0^2 + (3g_0 + 11) \right) g_0 + \left( 3 + (1 + 4g_0) \right) \frac{E_e}{E_0} \left( 1 - \frac{2g_0^2 - 26g_0 - 2(1 + 1)(1 + 2)}{E_0} \right) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right), \]
\[ - T_e (E_e)_{NLO} \zeta(E_e)_{NLO} + \frac{8g_0}{1 + 3g_A^2} \left( \frac{E_e}{M_V^2} + (1 + 2g_A) \right) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right), \]
\[ U(E_e)_{eNLO} = \frac{1}{1 + 3g_A^2} \frac{E_e}{2m_N} \left\{ \left( 2g_0 + 1 \right) g_0 + \left( 3g_0 + 1 \right) \right\} - \left( g_0 + 1 \right) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right), \] (B-4)

The terms, proportional to \( E_e^2/M_V^2 \) and \( E_e^2/M_A^2 \) are induced by the vector and axial-vector form factors of the neutron beta decay \(^{24}\) (see also \(^{24}\)). The slope-parameters \( M_V^2 \) and \( M_A^2 \) are related to the charge radius of the proton \( rp = 0.841 \) fm \(^{33}\) and the axial radius \( r_A = 0.635 \) fm of the nucleon \(^{35}\) (see also \(^{56}\)), respectively. This gives \( M_V = \sqrt{1/2}/r_p = 813 \) MeV and \( M_A = \sqrt{1/2}/r_A = 1077 \) MeV. The analytical expressions for the corrections \( \tilde{\zeta}(E_e)_{NLO} \), \( \tilde{X}(E_e)_{NLO} \) and \( \tilde{X}(E_e)_{NLO} \) for \( X = a, A, \ldots, K, S, T, U \) are given in Eqs. (B-2) and (B-3) respectively. We define corrections \( O(E_e^2/m_N^2) \), caused by weak magnetism and proton recoil, at the level of \( 10^{-5} \) with the theoretical accuracy of about \( 10^{-6} \) \(^{36}\). The numerical analysis of relative contributions has been carried out for the axial coupling constant \( g_A = 1.2764 \) \(^{5}\), the value of which agrees well with the recommended value of the axial coupling constant obtained by means of the global analysis of the experimental data on the axial coupling constant by Czarnecki et al. \(^{6}\).

Appendix C: Wilkinson’s corrections of order \( 10^{-5} \) to the neutron beta decay

According to Wilkinson \(^{13}\), the corrections, additional to those calculated in Appendices A and B, should be caused by i) the proton recoil in the electron–proton final–state Coulomb interaction, ii) the finite proton radius, iii) the proton–lepton convolution and iv) the higher–order outer radiative corrections. These corrections to the neutron lifetime and the correlation coefficients \( a(E_e), A(E_e), \ldots, K_e(E_e) \) have been calculated in \(^{11}\) \(^{12}\) \(^{14}\). The contributions of the proton recoil, caused by the phase-volume of the neutron beta decay proportional to \( 1/m_N \) and \( 1/m_N^2 \), which are defined in \(^{13}\) (see also \(^{77}\)) by the function \( S(E_e, E_0, m_N) \), we have taken into account for the calculation of the NLO and \( N^2LO \) corrections in the large nucleon mass \( m_N \) expansion induced by weak magnetism and proton recoil (see Appendix B).
Wilkinson’s corrections, induced by proton recoil in the Coulomb electron–proton final–state interaction

For the calculation of the contribution of the proton recoil in the Fermi function we replace the electron velocity $\beta$ by a velocity of a relative motion of the electron-proton pair \(^{11}\). To NLO in the large nucleon mass $m_N$ expansion a velocity of a relative motion of the electron-proton pair is defined by

$$\beta \rightarrow \vec{v}_{\text{rel}} = \beta - \frac{\vec{k}_p}{m_N}$$

To LO in the large nucleon mass $m_N$ expansion the second term in r.h.s. of Eq.(C-1) vanishes, and a velocity of a relative motion of the electron-proton pair reduces to an electron velocity. As has been shown in \(^{11}\) the Fermi function $F(E_e, Z = 1)$ with a replacement $\beta \rightarrow \vec{v}_{\text{rel}}$, caused by a relative motion of the electron-proton pair in the final state of the neutron beta decay, undergoes the following change (see Appendix H of Ref.\(^{11}\))

$$F(E_e, Z = 1) \xrightarrow{\beta \rightarrow \vec{v}_{\text{rel}}} F(E_e, Z = 1) \left( 1 - \frac{\pi a}{\beta} \frac{E_e}{m_N} - \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} \frac{\vec{k}_e \cdot \vec{k}}{E_e E_0} \right),$$

where we have taken into account the NLO terms in the large nucleon mass expansion. The corrections, caused by the proton recoil in the electron-proton final-state interactions to the neutron lifetime $\zeta(E_e)_{WF}$ and the correlation coefficients $X(E_e)_{WF}$ for $X = a, A, B, \ldots, T$ and $U$ are equal to

$$\zeta(E_e)_{WF} = -\frac{\pi a}{\beta} \frac{E_e}{m_N} - \frac{1}{3} a_0 \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} = -3.16 \times 10^{-5} \frac{E_e}{\beta E_0} + 1.12 \times 10^{-6} \frac{E_0 - E_e}{\beta E_0},$$

$$a(E_e)_{WF} = \frac{1}{3} a_0 \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} = 1.20 \times 10^{-7} \frac{E_0 - E_e}{\beta E_0} - 3.16 \times 10^{-5} \frac{E_0 - E_e}{\beta^3 E_0},$$

$$A(E_e)_{WF} = \frac{1}{3} a_0 A_0 \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} = 1.35 \times 10^{-7} \frac{E_0 - E_e}{\beta E_0},$$

$$B(E_e)_{WF} = \frac{1}{3} a_0 B_0 \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} = -1.11 \times 10^{-6} \frac{E_0 - E_e}{\beta E_0},$$

$$K_n(E_e)_{WF} = -A_0 \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} = 3.78 \times 10^{-6} \frac{E_0 - E_e}{\beta^3 E_0},$$

$$Q_n(E_e)_{WF} = -B_0 \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} = -3.12 \times 10^{-5} \frac{E_0 - E_e}{\beta^3 E_0},$$

$$G(E_e)_{WF} = \frac{1}{3} a_0 (1 - \beta^2) \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} = 1.12 \times 10^{-6} (1 - \beta^2) \frac{E_0 - E_e}{\beta^3 E_0},$$

$$H(E_e)_{WF} = \frac{1}{3} a_0^2 \sqrt{1 - \beta^2} \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} = -1.20 \times 10^{-7} \sqrt{1 - \beta^2} \frac{E_0 - E_e}{\beta E_0},$$

$$N(E_e)_{WF} = -\frac{1}{3} a_0 A_0 \sqrt{1 - \beta^2} \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} = 1.35 \times 10^{-7} \sqrt{1 - \beta^2} \frac{E_0 - E_e}{\beta E_0},$$

$$Q_e(E_e)_{WF} = -\frac{1}{3} a_0 A_0 \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} + \frac{1}{3} B_0 (1 + \sqrt{1 - \beta^2}) \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} =$$

$$= -1.35 \times 10^{-7} \frac{E_0 - E_e}{\beta E_0} + 1.04 \times 10^{-4} (1 + \sqrt{1 - \beta^2}) \frac{E_0 - E_e}{\beta^3 E_0},$$

$$K_e(E_e)_{WF} = -\frac{1}{3} a_0^2 \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} + (1 + \sqrt{1 - \beta^2}) \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} =$$

$$= -1.20 \times 10^{-7} \frac{E_0 - E_e}{\beta E_0} + 3.16 \times 10^{-5} (1 + \sqrt{1 - \beta^2}) \frac{E_0 - E_e}{\beta^3 E_0},$$

$$S(E_e)_{WF} = \sqrt{1 - \beta^2} A_0 \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} = 3.78 \times 10^{-6} \sqrt{1 - \beta^2} \frac{E_0 - E_e}{\beta^3 E_0},$$

$$T(E_e)_{WF} = -\frac{1}{3} a_0 B_0 \frac{\pi a}{\beta^3} \frac{E_0 - E_e}{m_N} = 1.11 \times 10^{-6} \frac{E_0 - E_e}{\beta E_0},$$

$$U(E_e)_{WF} = 0.$$ (C-3)

In comparison with the result, obtained in \(^{12}\), an additional term in $Q_e(E_e)_{WF}$ appears because of the contribution off the correlation coefficient $T(E_e)$. The contributions of Wilkinson’s corrections, caused by proton recoil in the
electron-proton final-state Coulomb interaction, to the electron-energy and angular distribution of the neutron beta decay with correlation structures beyond the standard correlation structures by Jackson et al. [23] and Ebel and Feldman [28] (see Eq. (4)) are given in Appendix F. We would like to emphasize that Wilkinson’s corrections, induced by proton recoil in the final-state Coulomb electron-proton interaction are well defined in the experimental electron-energy region $0.811 \text{ MeV} \leq E_e \leq 1.211 \text{ MeV}$ [27].

Wilkinson’s corrections, induced by i) the finite proton–radius $r_p$, ii) the lepton–nucleon convolution and iii) the higher–order outer radiative corrections

The corrections under consideration, caused by i) the finite proton–radius $r_p$, ii) the lepton–nucleon convolution and iii) the higher–order outer radiative corrections, are defined by the functions $L(E_e, Z = 1)$, $C(E_e, Z = 1)$ and $J(Z = 1)$, respectively [15] (see also [12]). According to Wilkinson [15], the contribution of $J(Z = 1)$ is equal to $J(Z = 1) = 1 + 3.92 \times 10^{-3}$ (see also [12]). The corrections $L(E_e, Z = 1)$ and $C(E_e, Z = 1)$, adapted for the neutron beta decay, are determined by [12]:

\[
L(E_e, Z = 1) = 1 + \frac{13}{60} \alpha^2 - \alpha r_p E_e \left(1 - \frac{1}{2} \frac{m_e^2}{E_e^2}\right) = 1 + 1.15 \times 10^{-5} - 4.02 \times 10^{-5} \frac{E_e}{E_0},
\]

\[
C(E_e, Z = 1) = 1 + \left[-\frac{9}{20} \alpha^2 + \frac{1}{5} m_e r_p + \frac{1}{E_0^2} \frac{E_e}{E_0} + \left(\frac{1}{5} \frac{1}{3} \frac{g_A^2}{r_p} \frac{1}{1 + \frac{3g_A^2}{E_e}}\right) \frac{E_e}{E_0} + \frac{2}{15} \frac{m_e}{E_0} \frac{r_p}{E_0} \frac{1}{1 + \frac{3g_A^2}{E_e}} \frac{m_e}{E_e}\right],
\]

\[
+ \left[-\frac{3}{5} \alpha r_p E_0 + \frac{2}{3} \frac{m_e^2}{E_0^2} r_p + \frac{1}{5} \alpha r_p E_0 - \frac{2}{15} \frac{E_0 r_p^2}{E_0} \frac{1}{1 + \frac{3g_A^2}{E_e}} \frac{E_e}{E_0} + \frac{2}{15} \frac{m_e}{E_0} \frac{r_p}{E_0} \frac{1}{1 + \frac{3g_A^2}{E_e}} \frac{m_e}{E_e}\right] E_0^2 \frac{E_e}{E_0}^2 = 1 - 2.78 \times 10^{-5} - 1.24 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2}. \tag{C-4}
\]

For the calculation of the numerical values of the constant terms and coefficients in front of the powers of $E_e/E_0$ we use $r_p = 0.841 \text{ fm}$ [33]. Following Wilkinson [15] we define Wilkinson’s correction, caused by i) the finite proton–radius $r_p$, ii) the lepton–nucleon convolution and iii) the higher–order outer radiative corrections, as follows

\[
L(E_e, Z = 1)C(E_e, Z = 1)J(Z = 1) = 1 + \delta L(E_e, Z = 1) + \delta C(E_e, Z = 1) + \delta J(Z = 1) = 1 + \zeta(E_e)_{\text{WR}}, \tag{C-5}
\]

where $\zeta(E_e)_{\text{WR}}$ is equal to

\[
\zeta(E_e)_{\text{WR}} = 3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2}. \tag{C-6}
\]

The contributions of Wilkinson’s corrections, caused by i) the finite proton–radius $r_p$, ii) the lepton–nucleon convolution and iii) the higher–order outer radiative corrections, to the correlation coefficients are equal to

\[
a(E_e)_{\text{WR}} = -a_0 \zeta(E_e)_{\text{WR}} = -a_0 \left(3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2}\right),
\]

\[
A(E_e)_{\text{WR}} = -A_0 \zeta(E_e)_{\text{WR}} = -A_0 \left(3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2}\right),
\]

\[
B(E_e)_{\text{WR}} = -B_0 \zeta(E_e)_{\text{WR}} = -B_0 \left(3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2}\right),
\]

\[
K_a(E_e)_{\text{WR}} = Q_a(E_e)_{\text{WR}} = 0,
\]

\[
G(E_e)_{\text{WR}} = \zeta(E_e)_{\text{WR}} = 3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2},
\]

\[
H(E_e)_{\text{WR}} = \frac{m_e}{E_e} a_0 \zeta(E_e)_{\text{WR}} = \frac{m_e}{E_e} a_0 \left(3.76 \times 10^{-4} \frac{E_0}{E_e} - 5.26 \times 10^{-5} - 1.26 \times 10^{-5} \frac{E_e}{E_0}\right),
\]

\[
N(E_e)_{\text{WR}} = \frac{m_e}{E_e} A_0 \zeta(E_e)_{\text{WR}} = \frac{m_e}{E_e} A_0 \left(3.76 \times 10^{-4} \frac{E_0}{E_e} - 5.26 \times 10^{-5} - 1.26 \times 10^{-5} \frac{E_e}{E_0}\right),
\]

\[
Q_e(E_e)_{\text{WR}} = A_0 \zeta(E_e)_{\text{WR}} = A_0 \left(3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2}\right),
\]

\[
K_e(E_e)_{\text{WR}} = a_0 \zeta(E_e)_{\text{WR}} = a_0 \left(3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2}\right),
\]
\[ S(E_e)_{\text{WR}} = U(E_e)_{\text{WR}} = 0, \]
\[ T(E_e)_{\text{WR}} = B_0 \zeta(E_e)_{\text{WR}} = B_0 \left( 3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2} \right). \]  

Now we may obtain the total contributions of Wilkinson’s corrections to the correlation function \( \zeta(E_e) \) and the correlation coefficients. We would like to emphasize that the correction \( \zeta(E_e)_{\text{WR}} \) does not depend practically on the value of the axial coupling constant \( g_A \).

**Wilkinson’s corrections to the neutron beta decay**

Summing the contributions in Eqs. (C-3), (C-6) and (C-7) we define total Wilkinson’s corrections to the neutron lifetime and the correlation coefficients of the neutron beta decay

\[ \zeta(E_e)_{\text{WC}} = 3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2} - \frac{\pi \alpha}{\beta} \frac{E_e}{m_N}, \]
\[ a(E_e)_{\text{WC}} = -3.76 \times 10^{-4} a_0 - \frac{\pi \alpha}{\beta^3} \frac{E_0 - E_e}{m_N}, \]
\[ A(E_e)_{\text{WC}} = -3.76 \times 10^{-4} A_0, \]
\[ B(E_e)_{\text{WC}} = -B_0 \left( 3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2} \right), \]
\[ K_e(E_e)_{\text{WC}} = 0, \]
\[ Q_e(E_e)_{\text{WC}} = -B_0 \frac{\beta \alpha}{\beta^3} \frac{E_0 - E_e}{m_N}, \]
\[ G(E_e)_{\text{WC}} = 3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2}, \]
\[ H(E_e)_{\text{WC}} = N(E_e)_{\text{WC}} = 0, \]
\[ Q_e(E_e)_{\text{WC}} = 3.76 \times 10^{-4} A_0 + \frac{1}{3} B_0 \left( 1 + \sqrt{1 - \beta^2} \right) \frac{\pi \alpha}{\beta^3} \frac{E_0 - E_e}{m_N}, \]
\[ K_e(E_e)_{\text{WC}} = 3.76 \times 10^{-4} a_0 + \left( 1 + \sqrt{1 - \beta^2} \right) \frac{\pi \alpha}{\beta^3} \frac{E_0 - E_e}{m_N}, \]
\[ S(E_e)_{\text{WC}} = U(E_e)_{\text{WC}} = 0, \]
\[ T(E_e)_{\text{WC}} = B_0 \left( 3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_e}{E_0} - 1.26 \times 10^{-5} \frac{E_e^2}{E_0^2} \right). \]  

Wilkinson’s corrections in Eq. (C-8) are calculated at the account for the contributions, which are not smaller than a few parts of \( 10^{-5} \) with a theoretical accuracy of about a few parts of \( 10^{-6} \) in the experimental electron-energy region \( 0.811 \text{ MeV} \leq E_e \leq 1.211 \text{ MeV} \).

**Appendix D: Radiative corrections of order \( O(\alpha E_e/m_N) \), induced by Sirlin’s outer radiative corrections of order \( O(\alpha/\pi) \) and the phase-volume of the neutron beta decay**

In this Appendix we analyze the contributions of the radiative corrections \( O(\alpha E_e/m_N) \), induced by Sirlin’s outer radiative corrections of order \( O(\alpha/\pi) \) and the phase-volume of the neutron beta decay taken to NLO in the large nucleon mass \( m_N \) expansion. For this aim we rewrite Eq. (A-13) keeping the contributions of the \( O(\alpha/\pi) \) corrections only. We get

\[ \sum_{\text{pol.}} \frac{|M(n \to pe^{-}\bar{\nu}_e)|^2}{(1 + 3g_A^2)|G_V|^2 64m_e^2 E_e E_\nu} = \left[ \left( 1 + \frac{\alpha}{\pi} f_\beta(E_e, \mu) \right) + B_0 \left( 1 + \frac{\alpha}{\pi} f_\beta(E_e, \mu) \right) \frac{\bar{\zeta}_n \cdot \bar{\zeta}_\nu}{E_\nu} \right] \left( 1 - \frac{m_e}{E_e} \zeta_0 \right) \]
\[ + a_0 \left( 1 + \frac{\alpha}{\pi} f_\beta(E_e, \mu) \right) \frac{\bar{\zeta}_\nu}{E_\nu} + A_0 \left( 1 + \frac{\alpha}{\pi} f_\beta(E_e, \mu) \right) \zeta_n \cdot \left( \frac{\bar{\zeta}_\nu}{E_\nu} - \frac{m_e}{E_e} \zeta_0 \right) + \left[ - \frac{\alpha}{\pi} \sqrt{1 - \beta^2} \ell \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right] \]
\[ - B_0 \frac{\beta \alpha}{2 \beta^3} \ell \ln \left( \frac{1 + \beta}{1 - \beta} \right) \left[ \frac{m_e}{E_e} + \left( A_0 \frac{\beta \alpha}{2 \beta^3} \ell \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right) \right] \zeta_n + a_0 \frac{\beta \alpha}{2 \beta^3} \ell \ln \left( \frac{1 + \beta}{1 - \beta} \right) \zeta_n \bar{\zeta}_n + \ldots, \]  

\[ \text{(D-1)} \]
where the ellipsis denotes the contributions, which are not important for the aim of this Appendix. In terms of irreducible correlation structures and using Eq. (A-7) we transcribe the r.h.s. of Eq. (D-1) into the form

\[
\sum_{\text{pol}} \frac{|M(n \to p e^{-}\bar{\nu}_e)|^2}{(1 + 3g_3^2)|G_V|^2|G_A|^2} = 1 + \frac{\alpha}{\pi} \left( g_n(E_e) - g_{\beta\gamma}(E_e, \mu) \right) + a_0 \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) + \frac{\sqrt{1 - \beta^2}}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right) \right]
\]

\[
- g_{\beta\gamma}(E_e, \mu) \right] \frac{\bar{\xi}_n \cdot \bar{\xi}_e}{E_e} + A_0 \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) + \frac{1 - \beta^2}{2\beta} \ln \left( 1 + \frac{1 + \beta}{1 - \beta} \right) \right) \right] \frac{\bar{\xi}_n \cdot \bar{\xi}_e}{E_e} + B_0 \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) \right) \right] \frac{\bar{\xi}_n \cdot \bar{\xi}_e}{E_e}
\]

\(
\beta = 2 \ln \left( \frac{1 + \beta}{1 - \beta} \right) - g_{\beta\gamma}(E_e, \mu) \right] \frac{\bar{\xi}_n \cdot \bar{\xi}_e}{E_e} - A_0 \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) \right) \right] \frac{\bar{\xi}_n \cdot \bar{\xi}_e}{E_e}
\]

\[
\Phi_n(\bar{\xi}_e, \bar{\xi}_e) \sum_{\text{pol}} \frac{|M(n \to p e^{-}\bar{\nu}_e)|^2}{(1 + 3g_3^2)|G_V|^2|G_A|^2} = \left[ 1 + 3 \frac{E_e}{m_N} \left( 1 - \frac{\bar{\xi}_n \cdot \bar{\xi}_e}{E_e} \right) \right] \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) - g_{\beta\gamma}(E_e, \mu) \right) \right]
\]

\[
+ a_0 \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) + \frac{\sqrt{1 - \beta^2}}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - g_{\beta\gamma}(E_e, \mu) \right) \right] \frac{\bar{\xi}_n \cdot \bar{\xi}_e}{E_e} + A_0 \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) + \frac{1 - \beta^2}{2\beta} \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right) \right] \frac{\bar{\xi}_n \cdot \bar{\xi}_e}{E_e}
\]

\[
\beta = 2 \ln \left( \frac{1 + \beta}{1 - \beta} \right) - g_{\beta\gamma}(E_e, \mu) \right] \frac{\bar{\xi}_n \cdot \bar{\xi}_e}{E_e} - A_0 \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) \right) \right] \frac{\bar{\xi}_n \cdot \bar{\xi}_e}{E_e}
\]
In order to remove the dependence of the electron-energy and angular distribution of the neutron beta decay on the infrared cut-off \( \mu \) we have to take into account the contribution of the neutron radiative beta decay \( n \rightarrow p + e^- + \nu_e + \gamma \), where \( \gamma \) is a real photon. It is well-known \([57] [61]\) (see also \([19, 21]\) and \([11, 12, 14]\) that the contribution of the neutron radiative beta decay is extremely needed for cancellation of the infrared divergences in the radiative corrections of order \( O(\alpha/\pi) \), caused by one-virtual photon exchanges.

For the removal of the infrared dependence we use the following electron-photon-energy and angular distribution, calculated in \([30]\) (see Eq.B-14 in Ref.\([30]\)):

\[
\begin{align*}
\frac{d^6 \lambda_{\beta}(E_e, \omega, \vec{k}_e, \vec{k}_\nu, \vec{\xi}_e, \vec{\xi}_\nu)}{d\omega dE_e d\vec{k}_e d\vec{k}_\nu d\vec{\xi}_e d\vec{\xi}_\nu} &= (1 + 3 g_\lambda^2) \frac{\alpha}{\pi} \frac{|G|}{E_e} \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) \left( E_0 - E_e - \omega \right) \Phi_{\nu/\gamma}(\vec{k}_e, \vec{k}_\nu, \omega) \\
&\times \left\{ \frac{\omega^2}{E_e^2} + (1 - \frac{1}{\beta^2}) \left[ \frac{1}{\beta^2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + \frac{\omega^2}{E_e^2} \left[ \frac{1}{\beta^2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + \frac{\omega^2}{E_e^2} \left[ \frac{1}{\beta^2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] \right\} \\
&\times \left\{ \frac{1}{\beta^2 E_e^2} \left[ \frac{1}{\beta^2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + (1 + \frac{1}{\beta^2}) \left[ \frac{1}{\beta^2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] + \frac{1}{\beta^2 E_e^2} \left[ \frac{1}{\beta^2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right] \right\} + \frac{1}{\beta^2 E_e^2} \left[ \frac{1}{\beta^2} \ln \left( \frac{1 + \beta}{1 - \beta} \right) - 2 \right]
\end{align*}
\]

where \( \Phi_{\nu/\gamma}(\vec{k}_e, \vec{k}_\nu, \omega) \) is the contribution of the phase-volume of the neutron radiative beta decay \([82]\)

\[
\Phi_{\nu/\gamma}(\vec{k}_e, \vec{k}_\nu, \omega) = 1 + 3 \frac{E_e}{m_N} \left( 1 - \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e E_\nu} \right) + O \left( \frac{\omega}{m_N} \right)
\]

The contribution of the phase-volume Eq.\([D-6]\) is the rest of the expression, calculated to NLO in the large nucleon mass \( m_N \) expansion and the integration over the directions of the photon 3-momentum \([82]\). As has been shown in \([82]\), the contributions of the terms \( O(\omega/m_N) \) are of order of \( 10^{-6} \) and even smaller. So, the \( O(\alpha E_e/m_N) \) corrections can be induced only by the second term in Eq.\([D-6]\).

Having integrated over \( \omega \) we arrive at the following expression

\[
\begin{align*}
\frac{d^6 \lambda_{\beta}(E_e, \vec{k}_e, \vec{k}_\nu, \vec{\xi}_e, \vec{\xi}_\nu)}{dE_e d\vec{k}_e d\vec{k}_\nu d\vec{\xi}_e d\vec{\xi}_\nu} &= (1 + 3 g_\lambda^2) \frac{\alpha}{\pi} \frac{|G|}{E_e} \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) \left( E_0 - E_e \right) \left[ 1 + 3 \frac{E_e}{m_N} \left( 1 - \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e E_\nu} \right) \right] \\
&\times \left\{ \frac{g_{\beta}(E_e, \mu)}{E_e E_\nu} g_{\beta}(E_e, \mu) + A_0 \frac{\vec{\xi}_e \cdot \vec{k}_\nu}{E_e E_\nu} g_{\beta}(E_e, \mu) + B_0 \frac{\vec{\xi}_e \cdot \vec{k}_\nu}{E_e E_\nu} g_{\beta}(E_e, \mu) + (-1) \frac{\vec{\xi}_e \cdot \vec{k}_\nu}{E_e E_\nu} g_{\beta}(E_e, \mu) \right\} \\
&+ (-1) \frac{m_e}{E_e} \frac{\vec{\xi}_e \cdot \vec{k}_\nu}{E_e E_\nu} g_{\beta}(E_e, \mu) + (-1) \frac{m_e}{E_e} \frac{\vec{\xi}_e \cdot \vec{k}_\nu}{E_e E_\nu} g_{\beta}(E_e, \mu) + (-1) A_0 \frac{\vec{\xi}_e \cdot \vec{k}_\nu}{E_e E_\nu} g_{\beta}(E_e, \mu) \\
&+ (-1) A_0 \frac{\vec{\xi}_e \cdot \vec{k}_\nu}{E_e E_\nu} g_{\beta}(E_e, \mu) + (-1) \frac{\vec{\xi}_e \cdot \vec{k}_\nu}{E_e E_\nu} g_{\beta}(E_e, \mu) \right\}.
\end{align*}
\]
The functions $g_{\beta n}^{(j)}(E_e, \mu)$ for $j = 1, 2, 3, 4$ have been calculated in [11, 12, 14, 29]. As has been shown in [11, 12, 14, 29], the difference between functions $g_{\beta n}^{(j)}(E_e, \mu) - g_{\beta n}^{(j)}(E_e, \mu)$ for $j = 2, 3, 4$ does not depend on the regularization, i.e.

$$\lim_{\mu \to 0} \left( g_{\beta n}^{(j)}(E_e, \mu) - g_{\beta n}^{(j)}(E_e, \mu) \right) = \lim_{\omega_{\text{min}} \to 0} \left( g_{\beta n}^{(1)}(E_e, \omega_{\text{min}}) - g_{\beta n}^{(j)}(E_e, \omega_{\text{min}}) \right),$$

where $\omega_{\text{min}}$ is a non-covariant infrared cut-off, which can be also treated as the photon–energy threshold of the detector [11, 12, 14, 29]. Summing up Eqs. (D-4) and (D-7), we obtain the total electron-energy and angular distribution of the neutron beta decay

$$\frac{d^5 \lambda_n(E_e, \vec{k}_e, \vec{k}_\nu, \vec{\xi}_n, \vec{\xi}_n)}{dE_e d\Omega_e d\Omega_\nu} = (1 + 3g_\lambda^2) \frac{|G_V|^2}{16\pi^3} (E_0 - E_e)^2 \sqrt{E_0^2 - m_0^2 E_e} F(E_e, Z = 1) \left[ 1 + 3 \frac{E_e}{m_N} \left( 1 - \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e} \right) \right]$$

$$\left\{ \left[ 1 + \frac{\alpha}{\pi} g_n(E_e) \right] + a_n \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) + f_n(E_e) \right) \right] \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e} + A_0 \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) + f_n(E_e) \right) \right] \frac{\vec{\xi}_n \cdot \vec{k}_\nu}{E_e} \right\} \frac{E_e}{E_0}$$

$$+ B_0 \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) \right) \right] \frac{\vec{\xi}_n \cdot \vec{k}_\nu}{E_0} - \frac{m_e}{E_e} A_0 \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) + h_n^{(1)}(E_e) \right) \right] \frac{\vec{\xi}_n \cdot \vec{\xi}_n - A_0}{E_e} - A_0 \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) + h_n^{(2)}(E_e) \right) \right] \frac{\vec{\xi}_n \cdot \vec{\xi}_n}{E_e + m_e}$$

$$- a_0 \left[ 1 + \frac{\alpha}{\pi} \left( g_n(E_e) + h_n^{(2)}(E_e) \right) \right] \frac{\vec{\xi}_n \cdot \vec{\xi}_n}{E_e + m_e}.$$ 

The detailed calculation of the functions $(\alpha/\pi)\tilde{g}_n(E_e)$ [19], $(\alpha/\pi) f_n(E_e)$ [21], $(\alpha/\pi) h_n^{(1)}(E_e)$ and $(\alpha/\pi) h_n^{(2)}(E_e)$ one can find in [11, 12, 14, 29, 30]. The behaviour of the functions $(\alpha/\pi)\tilde{g}_n(E_e), (\alpha/\pi) f_n(E_e), (\alpha/\pi) h_n^{(1)}(E_e)$ and $(\alpha/\pi) h_n^{(2)}(E_e)$ multiplied by the factor $3(E_e/m_N)$, caused by the phase-volume of the neutron beta decay, is shown in Fig. 1.

FIG. 1: The outer radiative corrections of order $O(\alpha E_e/m_N)$ in the electron-energy region $m_e \leq E_e < E_0$, induced by the outer radiative corrections of order $O(\alpha/\pi)$ and the phase-volume of the neutron beta decay, calculated to NLO in the large nucleon mass $m_N$ expansion.

One can see that in the electron-energy region $m_e \leq E_e < E_0$ the functions $3(\alpha/\pi)(E_e/m_N)\tilde{g}_n(E_e)$, $3(\alpha/\pi)(E_e/m_N) h_n^{(1)}(E_e)$ and $3(\alpha/\pi)(E_e/m_N) h_n^{(2)}(E_e)$ are of order of a few parts of $10^{-5}$, whereas the function $3(\alpha/\pi)(E_e/m_N) f_n(E_e)$ is of order of a few parts of $10^{-6}$. However, the functions $3(\alpha/\pi)(E_e/m_N) h_n^{(1)}(E_e)$ for $j = 1, 2$ are multiplied by either $(m_e/E_e)X_0$ or $X_0$, where $X_0 = a_0, A_0 \sim -0.1$. As a result, the values of the functions

$$3 \times 10^5$$

$$3 \times 10^6$$
3(α/π)(mc/mN)XgAh3(Ec) and 3(α/π)(Ec/mN)XgAh3(Ec) become of order of a few parts of 10^-6. Hence, the outer radiative corrections of order O(αEc/mN) are defined by the function 3(α/π)(Ec/mN)gα(Ec) only.

As result, the correlation function ζ(Ec) and the correlation coefficients α(Ec), Qα(Ec), Qε(Ec) and Kε(Ec) acquire the following outer or model-independent O(αEc/mN) radiative corrections:

\[
\begin{align*}
\zeta(Ec)_{\text{RC-PHV}} &= 3 \frac{\alpha}{\pi} \frac{E_c}{m_N} g_\alpha(E_c), \\
\alpha(Ec)_{\text{RC-PHV}} &= -3 \frac{\alpha}{\pi} \frac{E_c}{m_N} g_\alpha(E_c), \\
Q_\alpha(Ec)_{\text{RC-PHV}} &= -3 B_0 \frac{\alpha}{\pi} \frac{E_c}{m_N} g_\alpha(E_c), \\
Q_\epsilon(Ec)_{\text{RC-PHV}} &= B_0 \frac{\alpha}{\pi} \frac{E_c}{m_N} g_\alpha(E_c) \left(1 + \frac{m_\epsilon}{E_c}\right),\\
K_\epsilon(Ec)_{\text{RC-PHV}} &= 3 \frac{\alpha}{\pi} \frac{E_c}{m_N} g_\alpha(E_c) \left(1 + \frac{m_\epsilon}{E_c}\right).
\end{align*}
\]

These radiative corrections are of order of a few parts of 10^-5 in the experimental electron-energy region 0.811 MeV ≤ E_c ≤ 1.211 MeV. They are plotted in [10]. The analytical expression of the function gα(Ec) is given in Eq. (A-19). We have to notice that there is the contribution, proportional to 3(α/π)(Ec/mN)gα(Ec), to the electron-energy and angular distribution of the neutron beta decay with the correlation structure beyond the standard correlation structures in Eq. (1) (see the last term in Eq. (F-2)).

Appendix E: Analytical expressions for the correlation function ζ(Ec) and correlation coefficients α(Ec), Qα(Ec), Kα(Ec) and A(α)(Ec)

In this Appendix we give the analytical expressions for the correlation function ζ(Ec) and the correlation coefficients X(Ec) for X = α, A, B, . . . , U and also for the correlation coefficient A(α)(Ec) = A(Ec) + \frac{1}{n} Qα(Ec) as functions of the electron energy Ec and the axial coupling constant gA. The correlation function ζ(Ec) and correlation coefficients are calculated with a theoretical accuracy of about 10^-6.

For the correlation function ζ(Ec) we obtain the following expression

\[
\zeta(Ec) = 1 + \frac{\alpha}{\pi} g_\alpha(Ec) + \frac{1}{1 + 3g^2_A} \frac{E_0}{m_N} \left[- 2g_A \left(g_A + (\kappa + 1)\right) + \left(10g_A^2 + 4(\kappa + 1)g_A + 2\right) \frac{E_c}{E_0}\right]
\]

Here ̃gα(Ec) is Sirlin’s function [19] (see also Eq. (A-19). Then, the term proportional to E_0/m_N defines the well-known corrections O(E_0/m_N), caused by weak magnetism and proton recoil (see, for example, [11] [23]). The corrections ̃ζ(Ec)_{RC-NLO}, ̃ζ(Ec)_{RC-PHV}, ̃ζ(Ec)_{NLO} and ̃ζ(Ec)_{WC} are defined by the O(E_0/m_N) inner and outer radiative corrections, the O(E_0^2/m_N^2) corrections, caused by weak magnetism and proton recoil, and Wilkinson’s corrections, respectively. They are equal to

\[
\begin{align*}
\zeta(Ec)_{\text{RC-NLO}} &= \frac{1}{1 + 3g^2_A} \frac{\alpha}{\pi} \frac{E_c}{m_N} \left(f_\nu(Ec) + \sqrt{1 - \beta^2} f_S(Ec) + g_A g_\nu(Ec) \beta^2 \right) + g_A \frac{E_0 - E_c}{E_c} \sqrt{1 - \beta^2} h_S(Ec) + 3g_A f_a(Ec) + 3g_A^2 \sqrt{1 - \beta^2} f_\tau(Ec) + \frac{3g_A^2}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{5}{2} \frac{m_N^2}{M_R^2} \frac{m_N^2}{m_N}
\end{align*}
\]

\[
\zeta(Ec)_{\text{RC-PHV}} = 3 \frac{\alpha}{\pi} \frac{E_c}{m_N} g_\alpha(Ec),
\]

\[
\zeta(Ec)_{\text{NLO}} = \frac{E_c}{m_N} \zeta(Ec)_{\text{NLO}} - \frac{E_c}{m_N} \alpha(Ec)_{\text{NLO}}
\]

\[
\times \left(1 - \frac{m_N^2}{E_c^2}\right) = \frac{8}{1 + 3g_A} \left(\frac{E_0^2}{M_R^2} + 3g_A E_0^2 M_A^2\right) \frac{E_c}{E_0} \left(1 - \frac{E_c}{E_0}\right),
\]

\[
\zeta(Ec)_{\text{WC}} = 3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_c}{E_0} - 1.26 \times 10^{-5} \frac{E_0^2}{E_0^2} \frac{\pi \alpha}{\beta} \frac{E_c}{m_N}.
\]

The terms, proportional to E_0^2/M_R^2 and E_0^2/M_A^2, appear from the contributions of the vector and axial-vector form-factors of the neutron beta decay. For numerical analysis we use M_V = 813 MeV and M_A = 1077 MeV [24]. The first three terms in ζ(Ec)_WC do not depend practically on the axial coupling constant gA.
Of course, the correlation function $\zeta(E_c)$ in Eq. (E-1) should be supplemented by the inner radiative corrections $\Delta^V_{\mu}$ and $\Delta^R_{\mu}$ of order $O(\alpha/\pi)$, caused by the Feynman $\gamma W^-$-box diagrams and calculated in [38, 59].

For the correlation coefficient $a(E_c)$ we obtain the following expression

$$a(E_c) = a_0 \left\{ 1 + \frac{\alpha}{\pi} f_n(E_c) + \frac{1}{(1 - g_A^2)(1 + 3g_A^2)} \frac{E_0}{m_N} \left( a_1 + a_2 \frac{E_c}{E_0} + a_3 \frac{E_c^2}{E_0^2} \right) \right\}$$

$$+ a(E_c)_{RC-NLO} + a(E_c)_{RC-\phi V} + a(E_c)_{NLO} + a(E_c)_{WC},$$

where the function $f_n(E_c)$ has been calculated by Shann [21] (see also Eq. (A-19)) and the coefficients $a_1$, $a_2$ and $a_3$ are equal to $[11]

$$a_1 = 4g_A(g_A^2 + 1) \left( g_A + (\kappa + 1) \right)$$

$$a_2 = -26g_A^4 - 8(\kappa + 1)g_A^3 - 20g_A^2 - 8(\kappa + 1)g_A - 2$$

$$a_3 = -2g_A(g_A^2 - 1) \left( g_A + (\kappa + 1) \right) \frac{m_e^2}{E_0^2}$$

The corrections $a(E_c)_{RC-NLO}$, $a(E_c)_{RC-\phi V}$, $a(E_c)_{NLO}$ and $a(E_c)_{WC}$, defined by the $O(\alpha E_c/m_N)$ inner and outer radiative corrections, the $O(E_c^2/m_N^2)$ corrections, caused by weak magnetism and proton recoil, and Wilkinson’s corrections, respectively, are given by

$$a(E_c)_{RC-NLO} = \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_c}{m_N} \left( f_{\gamma V}(E_c) + g_A \sqrt{1 - \beta^2 gs(E_c)} + g_A g_{\gamma V}(E_c) + (1 - 2g_A) h_{\gamma}(E_c) - g_A^2 f_{\gamma A}(E_c) \right)$$

$$- \frac{e}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{5 m_N^2}{2 M_W^2} \ln \frac{M_W^2}{m_N^2} + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \left( g_{\gamma A}(E_c) - g_A f_{\gamma A}(E_c) \right) - a_0 \zeta(E_c)_{RC-NLO},$$

$$a(E_c)_{RC-\phi V} = -3 \frac{\alpha}{\pi} \frac{E_c}{m_N} g_n(E_c),$$

$$a(E_c)_{NLO} = 12 \frac{E_c^3}{m_N^2} \left( 1 - \frac{1 - 1}{4} \frac{E_0}{E_n} \right) + \frac{1}{4} a_0 \left( 1 - \frac{1}{4} \frac{E_0}{E_n} \right) \right\} + 3 \frac{E_c}{m_N} a(E_c)_{NLO} - \frac{E_c}{m_N} \zeta(E_c)_{NLO}$$

$$- a(E_c)_{NLO} \zeta(E_c)_{NLO} + \frac{8}{1 + 3g_A^2} \left( E_c^2 M_W^2 + 3g_A^2 E_0^2 \frac{M_W^2}{M_A^2} \right) \frac{E_c}{E_0} \left( 1 - \frac{E_c}{E_0} \right),$$

$$a(E_c)_{WC} = -3.76 \times 10^{-4} a_0 - \frac{\alpha}{\beta^2} \frac{E_0}{m_N} - E_c - E_n$$

For the correlation coefficient $A(E_c)$ we obtain the following expression

$$A(E_c) = A_0 \left\{ 1 + \frac{\alpha}{\pi} f_n(E_c) + \frac{1}{2g_A(1 - g_A)(1 + 3g_A^2)} \frac{E_0}{m_N} \left( A_1 + A_2 \frac{E_c}{E_0} + A_3 \frac{E_c^2}{E_0^2} \right) \right\}$$

$$+ A(E_c)_{RC-NLO} + A(E_c)_{NLO} + A(E_c)_{WC},$$

where the function $f_n(E_c)$ has been calculated by Shann [21] (see also Eq. (A-19)) and the coefficients $A_1$, $A_2$ and $A_3$ are given by $[11]

$$A_1 = -g_A + \kappa g_A^2 + (\kappa + 2) g_A^3 + \kappa g_A - (\kappa + 1),$$

$$A_2 = 5g_A^4 - \kappa g_A^3 - (5\kappa + 6) g_A^2 - 3\kappa g_A + (\kappa + 1),$$

$$A_3 = -4g_A^2 g_A (1 - \kappa) \left( g_A + (\kappa + 1) \right) \frac{m_e^2}{E_0^2}$$

The corrections $A(E_c)_{RC-NLO}$, $A(E_c)_{NLO}$ and $A(E_c)_{WC}$, defined by the $O(\alpha E_c/m_N)$ inner radiative corrections, the $O(E_c^2/m_N^2)$ corrections, caused by weak magnetism and proton recoil, and Wilkinson’s corrections, respectively, are equal to

$$A(E_c)_{RC-NLO} = \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_c}{m_N} \left( g_A f_{\gamma V}(E_c) + \sqrt{1 - \beta^2 gs(E_c)} + g_{\gamma V}(E_c) - g_A h_{\gamma}(E_c) + g_A (1 - 2g_A) f_{\gamma A}(E_c) \right)$$

$$+ \frac{1}{1 + 3g_A^2} g_A (1 - 2g_A) \frac{\alpha}{\pi} \frac{5 m_N^2}{2 M_W^2} \ln \frac{M_W^2}{m_N^2} + \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \left( g_A g_{\gamma A}(E_c) + (1 - 2g_A) f_{\gamma A}(E_c) \right) - A_0 \zeta(E_c)_{RC-NLO},$$
\[ A(\varepsilon)_{N^2\text{LO}} = 3 \frac{E_0}{m_N} \tilde{A}(\varepsilon)_{\text{NLO}} - \frac{E_0}{m_N} K_n(\varepsilon)_{\text{NLO}} \left( 1 - \frac{m_e^2}{E_0^2} \right) - A(\varepsilon)_{\text{NLO}} \zeta(\varepsilon)_{\text{NLO}} \]

\[ A(\varepsilon)_{\text{WC}} = -3.76 \times 10^{-4} A_0. \quad (E-8) \]

For the correlation coefficient \(B(\varepsilon_c)\) we obtain the following expression

\[ B(\varepsilon_c) = B_0 \left\{ 1 + \frac{1}{2g_A(1+g_A)(1+3g_A^2)} \frac{E_0}{m_N} \left( B_1 + B_2 \frac{E_c}{E_0} + B_3 \frac{E_0}{E_c} \right) \right\} + B(\varepsilon)_{\text{RC-NLO}} + B(\varepsilon)_{\text{N^2LO}} + B(\varepsilon)_{\text{WC}}, \]

where the coefficients \(B_1, B_2\) and \(B_3\) are given by

\[ B_1 = -2g_A(1-g_A)^2 \left( g_A + (\kappa + 1) \right), \]

\[ B_2 = g_A^2 + (\kappa - 4) g_A^3 - (5\kappa + 2) g_A^2 + (3\kappa + 4) g_A + (\kappa + 1), \]

\[ B_3 = (g_A^2 - 1)(1 + g_A) \left( g_A + (\kappa + 1) \right) \frac{m_e^2}{E_0^2}. \quad (E-9) \]

The corrections \(B(\varepsilon)_{\text{RC-NLO}}, B(\varepsilon)_{\text{N^2LO}}\) and \(B(\varepsilon)_{\text{WC}}\), defined by the \(O(\alpha E_c/m_N)\) inner radiative corrections, the \(O(E_c^2/m_N^2)\) corrections, caused by weak magnetism and proton recoil, and Wilkinson’s corrections, respectively, are equal to

\[ B(\varepsilon)_{\text{RC-NLO}} = \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_{\text{c}}}{m_N} \left( g_A f_V(\varepsilon_c) + g_A \sqrt{1 - \beta^2} f_S(\varepsilon_c) + \frac{E_0 - E_c}{E_c} \sqrt{1 - \beta^2} h_S(\varepsilon_c) + g_A g_V(\varepsilon_c) \beta^2 \right) \]

\[ - 2g_A (1 - g_A)^2 \left( g_A + (\kappa + 1) \right) \frac{E_{\text{c}}}{m_N} \left( g_A f_V(\varepsilon_c) + g_A \sqrt{1 - \beta^2} f_S(\varepsilon_c) + \frac{E_0 - E_c}{E_c} \sqrt{1 - \beta^2} h_S(\varepsilon_c) + g_A g_V(\varepsilon_c) \beta^2 \right) \]

\[ \times \frac{\alpha}{\pi} \frac{5}{2} \frac{m_N^2}{M_{\text{W}}^2} \frac{M_{\text{W}}^2}{m_N} \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \left( g_A g_{\text{est}}(\varepsilon_c) + (1 + 2g_A) f_{\text{est}}(\varepsilon_c) \right) - B_0 \zeta(\varepsilon)_{\text{RC-NLO}}, \]

\[ B(\varepsilon)_{\text{N^2LO}} = 6 \frac{E_{\text{c}}^2}{m_N^2} B_0 \left( 1 - \frac{E_0}{4E_c} \right) + 3 \frac{E_0}{m_N} B(\varepsilon)_{\text{NLO}} - B(\varepsilon)_{\text{NLO}} \zeta(\varepsilon)_{\text{NLO}} - B_0 \zeta(\varepsilon)_{\text{N^2LO}} \]

\[ - \frac{8g_A}{1 + 3g_A^2} \left( \frac{E_{\text{c}}^2}{M_{\text{W}}^2} + (1 + 2g_A) \frac{E_{\text{c}}^2}{M_{\text{W}}^2} \right) \frac{E_c}{E_0} \left( 1 - \frac{E_0}{E_c} \right), \]

\[ B(\varepsilon)_{\text{WC}} = B_0 \left( 3.76 \times 10^{-4} - 5.26 \times 10^{-5} \frac{E_c}{E_0} - 1.26 \times 10^{-5} \frac{E_c^2}{E_0^2} \right) + \frac{1}{3} a_0 B_0 \frac{\pi \alpha}{\beta^2} \frac{E_0 - E_c}{m_N}. \quad (E-11) \]

The coefficients of Wilkinson’s term in the parentheses do not practically depend on the axial coupling constant \(g_A\).

For the correlation coefficient \(K_n(\varepsilon)\) we obtain the following expressions

\[ K_n(\varepsilon) = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left( 5g_A^2 + (\kappa - 4) g_A + (\kappa + 1) \right) + K_n(\varepsilon)_{\text{RC}} + K_n(\varepsilon)_{\text{N^2LO}} + K_n(\varepsilon)_{\text{WC}}. \quad (E-12) \]

The corrections \(K_n(\varepsilon)_{\text{RC-NLO}}, K_n(\varepsilon)_{\text{N^2LO}}\) and \(K_n(\varepsilon)_{\text{WC}},\) defined by the \(O(\alpha E_c/m_N)\) inner radiative corrections, the \(O(E_c^2/m_N^2)\) corrections, caused by weak magnetism and proton recoil, and Wilkinson’s corrections, respectively, are equal to

\[ K_n(\varepsilon)_{\text{RC-NLO}} = \frac{1}{1 + 3g_A^2} \frac{\alpha}{\pi} \frac{E_0}{m_N} \left( (1 - g_A) g_V(\varepsilon_c) + (1 + 2g_A) h_A(\varepsilon_c) \right), \]

\[ K_n(\varepsilon)_{\text{N^2LO}} = 3 \frac{E_0}{m_N} K_n(\varepsilon)_{\text{NLO}} - 3 \frac{E_0}{m_N} \tilde{A}(\varepsilon)_{\text{NLO}}. \]

\[ K_n(\varepsilon)_{\text{WC}} = -A_0 \frac{\pi \alpha}{\beta^2} \frac{E_0 - E_c}{m_N}. \quad (E-13) \]

For the correlation coefficient \(Q_n(\varepsilon)\) we obtain the following expression

\[ Q_n(\varepsilon) = \frac{1}{1 + 3g_A^2} \frac{E_0}{m_N} \left[ (g_A^2 + (\kappa + 2)g_A + (\kappa + 1)) - (7g_A^2 + (\kappa + 8)g_A + (\kappa + 1)) \frac{E_c}{E_0} \right] + Q_n(\varepsilon)_{\text{RC-NLO}} + Q_n(\varepsilon)_{\text{RC-PeV}} + Q_n(\varepsilon)_{\text{N^2LO}} + Q_n(\varepsilon)_{\text{WC}}. \quad (E-14) \]
where the corrections $Q_n(E_c)_{\text{RC-NLO}}, Q_n(E_c)_{\text{RC-PbV}}, Q_n(E_c)_{\text{N2LO}}$ and $Q_n(E_c)_{\text{WC}}$, defined by the $O(\alpha E_c/m_N)$ inner and outer radiative corrections, the $O(E^2_c/m_N^2)$ corrections, caused by weak magnetism and proton recoil, and Wilkinson’s corrections, respectively, are equal to

$$Q_n(E_c)_{\text{RC-NLO}} = 0,$$

$$Q_n(E_c)_{\text{RC-PbV}} = -3B_0 \frac{\alpha}{\pi} \frac{E_c}{m_N} \tilde{g}_n(E_c),$$

$$Q_n(E_c)_{\text{N2LO}} = -12E_c^2 \frac{m_N}{B_0} \left(1 - \frac{1}{8} \frac{E_n}{E_c}\right) + 3 \frac{E_c}{m_N} \tilde{Q}_n(E_c)_{\text{NLO}} - 3 \frac{E_c}{m_N} \tilde{B}(E_c)_{\text{NLO}} - Q_n(E_c)_{\text{NLO}} \zeta(E_c)_{\text{NLO}},$$

$$Q_n(E_c)_{\text{WC}} = -B_0 \frac{\pi \alpha}{\beta^2} \frac{E_0 - E_c}{m_N}.$$

Now we are able to give the analytical expression for the correlation coefficient $A^{(3)}(E_c)$ defined by [15]

$$A^{(3)}(E_c) = A(E_c) + \frac{1}{3} Q_n(E_c).$$

This correlation coefficient is responsible for the electron (beta) asymmetry in the neutron beta decay [5] (see also [11]). The correlation coefficient $A^{(3)}(E_c)$ we give in the following form [15] (see also [11])

$$A^{(3)}(E_c) = A_0 \left\{ 1 + \frac{\alpha}{\pi} f_n(E_c) + \frac{g_0 + \kappa + 1}{g_0(1 - g_0)(1 + 3g_0^2)} \frac{E_0}{m_N} \left(A_1^{(3)} + A_2^{(3)} \frac{E_0}{E_c} + A_3^{(3)} \frac{E_0}{E_c} \right) \right\} + A^{(3)}(E_c)_{\text{RC-NLO}} + A^{(3)}(E_c)_{\text{RC-PbV}} + A^{(3)}(E_c)_{\text{N2LO}} + A^{(3)}(E_c)_{\text{WC}},$$

(E-17)

The function $f_n(E_c)$ is given in Eq.(A-10). The coefficients $A_1^{(3)}, A_2^{(3)}$ and $A_3^{(3)}$ are equal to [15] (see also [11])

$$A_1^{(3)} = g_0^2 + \frac{2}{3} g_0 - \frac{1}{3},$$

$$A_2^{(3)} = -g_0^3 - 3g_0^2 - \frac{5}{3} g_0 + \frac{1}{3},$$

$$A_3^{(3)} = 2g_0^2(1 - g_0) \frac{m_N^2}{E_0}$$

(E-18)

and the terms $A^{(3)}(E_c)_{\text{RC-NLO}}, A^{(3)}(E_c)_{\text{RC-PbV}}, A^{(3)}(E_c)_{\text{N2LO}}$ and $A^{(3)}(E_c)_{\text{WC}}$, defined by the radiative corrections $O(\alpha E_c/m_N)$, the corrections $O(E^2_c/m_N^2)$, caused by weak magnetism and proton recoil, and Wilkinson’s corrections, respectively, are equal to

$$A^{(3)}(E_c)_{\text{RC-NLO}} = \frac{1}{1 + 3g_0^2} \frac{E_0}{m_N} \left( g_A f_V(E_c) + \sqrt{1 - \beta^2} g_0 f_S(E_c) + g_V(E_c) - g_A h_A(E_c) + g_A(1 - 2g_A) f_A(E_c) \right) + \frac{1}{1 + 3g_0^2} \frac{g_0(1 - 2g_A)}{\pi} \frac{m_N^2}{M_W^2} \frac{E_0}{m_N} \left( g_A \tilde{g}_A(E_c) + (1 - 2g_A) \tilde{f}_A(E_c) \right) - A_0 \zeta(E_c)_{\text{RC-NLO}},$$

$$A^{(3)}(E_c)_{\text{RC-PbV}} = \frac{1}{3} Q_n(E_c)_{\text{RC-PbV}} = -B_0 \frac{\alpha}{\pi} \frac{E_c}{m_N} \tilde{g}_n(E_c),$$

$$A^{(3)}(E_c)_{\text{N2LO}} = A(\zeta(E_c))_{\text{N2LO}} + \frac{1}{2} Q_n(E_c)_{\text{N2LO}},$$

$$A^{(3)}(E_c)_{\text{WC}} = 3.76 \times 10^{-4} A_0 - \frac{3}{3} B_0 \frac{\pi \alpha}{\beta^2} \frac{E_0 - E_c}{m_N}.$$

(E-19)

The functions $\zeta(E_c)_{\text{NLO}}, a(E_c)_{\text{NLO}}, A(E_c)_{\text{NLO}}, B(E_c)_{\text{NLO}}$ and $Q_n(E_c)_{\text{NLO}}$ are given in Eq.(B-2).

For the experimental analysis of the antineutrino asymmetry in the neutron beta decay one has to use Eqs.(27) and (28) in Ref.[11] and the correlation coefficients $a(E_c), A(E_c), B(E_c), K_n(E_c)$ and $Q_n(E_c)$ given in this Appendix. For the account for the contribution of the Fierz interference term $b$ in the antineutrino asymmetry one may use Eqs.(19) and (20) in Ref.[10], where the correlation coefficients $X(E_c)$ are replaced by $X(E_c)/(1 + b m_e/E_c)$ for $X(E_c) = a(E_c), A(E_c), B(E_c), K_n(E_c)$ and $Q_n(E_c)$, respectively.

For the correlation coefficients $G(E_c), H(E_c), N(E_c), Q_n(E_c), K_n(E_c), S(E_c), T(E_c)$ and $U(E_c)$ as functions of
the electron energy $E_e$ and the axial coupling constant $g_A$ we give the analytical expressions in the following form

$$G(E_e) = \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) + G(E_e)_{\text{NLO}} + G(E_e)_{\text{RC–NLO}} + G(E_e)_{\text{N2LO}} + G(E_e)_{\text{WC}},$$

$$H(E_e) = \frac{m_e}{E_e} a_0 \left(1 + \frac{\alpha}{\pi} h_n^{(1)}(E_e)\right) + H(E_e)_{\text{NLO}} + H(E_e)_{\text{RC–NLO}} + H(E_e)_{\text{N2LO}} + H(E_e)_{\text{WC}},$$

$$N(E_e) = \frac{m_e}{E_e} A_0 \left(1 + \frac{\alpha}{\pi} h_n^{(1)}(E_e)\right) + N(E_e)_{\text{NLO}} + N(E_e)_{\text{RC–NLO}} + N(E_e)_{\text{N2LO}} + N(E_e)_{\text{WC}},$$

$$Q_e(E_e) = -a_0 \left(1 + \frac{\alpha}{\pi} h_n^{(2)}(E_e)\right) + Q_e(E_e)_{\text{NLO}} + Q_e(E_e)_{\text{RC–NLO}} + Q_e(E_e)_{\text{RC–PAV}} + Q_e(E_e)_{\text{N2LO}} + Q_e(E_e)_{\text{WC}},$$

$$K_e(E_e) = -a_0 \left(1 + \frac{\alpha}{\pi} h_n^{(2)}(E_e)\right) + K_e(E_e)_{\text{NLO}} + K_e(E_e)_{\text{RC–NLO}} + K_e(E_e)_{\text{RC–PAV}} + K_e(E_e)_{\text{N2LO}} + K_e(E_e)_{\text{WC}},$$

$$S_e(E_e) = S(E_e)_{\text{NLO}} + S(E_e)_{\text{RC–NLO}} + S(E_e)_{\text{N2LO}},$$

$$T(E_e) = -B_0 \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) + T(E_e)_{\text{NLO}} + T(E_e)_{\text{RC–NLO}} + T(E_e)_{\text{N2LO}} + T(E_e)_{\text{WC}},$$

$$U(E_e) = U(E_e)_{\text{RC–NLO}} + U(E_e)_{\text{N2LO}}.$$

(E-20)

The analytical expressions of the functions $f_n(E_e)$, $h_n^{(1)}(E_e)$ and $h_n^{(2)}(E_e)$ are given in Eq. (A-19). The corrections $X(E_e)_{\text{NLO}}, X(E_e)_{\text{RC–NLO}}, X(E_e)_{\text{RC–PAV}}, X(E_e)_{\text{N2LO}}$ and $X(E_e)_{\text{WC}}$ for $X = G, H, N, Q_e, K_e, S, T, U$ are adduced in Eq. (B-2), Eq. (A-20), Eq. (A-10), Eq. (B-4) and Eq. (C-8), respectively.

For the practical applications and numerical analysis the analytical expressions of the correlation function $\zeta(E_e)$ and correlation coefficients $X(E_e)$ for $X = a, A, B, \ldots, U$ and $A^{(3)}(E_e)$ are programmed in [66].

### Appendix F: Contributions to the electron-energy and angular distribution of the neutron beta decay with correlation structures beyond Eq. [1]

The electron-energy and angular distribution of the neutron beta decay Eq. [1], supplemented by the contributions with correlation structures beyond the standard ones, takes the form

$$d^3\lambda_n(E_e, \vec{k}_e, \vec{p}_\nu, \vec{\xi}_n, \vec{\zeta}_e) = \left[1 + 3g_A^2\right] \left(\frac{G_V^2}{16\pi^5}\right) \left(E_0 - E_e\right)^2 \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) \zeta(E_e) \left\{1 + b(E_e) \frac{m_e}{E_e}ight.$$

$$+ a(E_e) \frac{\vec{k}_e \cdot \vec{p}_\nu}{E_e E_{\nu}} + A(E_e) \frac{\vec{\xi}_n \cdot \vec{k}_e}{E_e} + B(E_e) \frac{\vec{\xi}_n \cdot \vec{p}_\nu}{E_{\nu}} + K_n(E_e) \frac{\vec{\xi}_n \cdot \vec{k}_e}{E_e^2 E_{\nu}} + Q_n(E_e) \frac{\vec{\xi}_n \cdot \vec{p}_\nu}{E_{\nu} E_e^2}$$

$$+ D(E_e) \frac{\vec{\xi}_n \cdot \vec{\xi}_n \cdot (\vec{k}_e \times \vec{\zeta}_e)}{E_e E_{\nu}} + G(E_e) \frac{\vec{\xi}_e \cdot \vec{k}_e}{E_e} + H(E_e) \frac{\vec{\xi}_e \cdot \vec{p}_\nu}{E_{\nu}} + N(E_e) \frac{\vec{\xi}_n \cdot \vec{\xi}_e}{E_e E_{\nu}} + Q_e(E_e) \frac{\vec{\xi}_n \cdot \vec{\xi}_e}{E_e E_{\nu}} \left(\vec{\xi}_n \cdot \vec{k}_e \right) \frac{\vec{\xi}_e \cdot \vec{\zeta}_e}{E_{\nu}}$$

$$+ K_e(E_e) \frac{\vec{\xi}_e \cdot \vec{k}_e}{(E_e + m_e) E_{\nu}} + R(E_e) \frac{\vec{\xi}_n \cdot (\vec{k}_e \times \vec{\zeta}_e)}{E_e E_{\nu}} + L(E_e) \frac{\vec{\xi}_e \cdot (\vec{k}_e \times \vec{\zeta}_e)}{E_e E_{\nu}} + S(E_e) \frac{\vec{\xi}_n \cdot \vec{\xi}_e}{E_e E_{\nu}} \left(\vec{\xi}_n \cdot \vec{k}_e \right) \frac{\vec{\xi}_e \cdot \vec{\zeta}_e}{E_{\nu}}$$

$$+ T(E_e) \frac{\vec{\xi}_n \cdot \vec{\xi}_n \cdot (\vec{k}_e \times \vec{\zeta}_e)}{E_e E_{\nu}} + U(E_e) \frac{\vec{\xi}_n \cdot \vec{\xi}_e \cdot \vec{\zeta}_e}{E_e E_{\nu}} + V(E_e) \frac{\vec{\xi}_n \cdot \vec{\xi}_e \cdot \vec{\zeta}_e}{E_e E_{\nu}} + W(E_e) \frac{\vec{\xi}_n \cdot \vec{\xi}_e \cdot \vec{\zeta}_e}{E_e E_{\nu}} \left(\vec{\xi}_n \cdot \vec{k}_e \right) \frac{\vec{\xi}_e \cdot \vec{\zeta}_e}{E_{\nu}} \right\}$$

$$+ \frac{d^3\lambda_n(E_e, \vec{k}_e, \vec{p}_\nu, \vec{\xi}_n, \vec{\zeta}_e)}{dE_e d\Omega_e d\Omega_\nu} \left|_{\text{RC–NLO}} + \frac{d^3\lambda_n(E_e, \vec{k}_e, \vec{p}_\nu, \vec{\xi}_n, \vec{\zeta}_e)}{dE_e d\Omega_e d\Omega_\nu} \left|_{\text{NLO}} + \sum_{m=1}^{4} \frac{d^3\lambda_n^{(m)}(E_e, \vec{k}_e, \vec{p}_\nu, \vec{\xi}_n, \vec{\zeta}_e)}{dE_e d\Omega_e d\Omega_\nu} \left|_{\text{N2LO}} \right\}$$

(F-1)

where the last seven terms are defined by the following expressions (see also [24]):

$$\left\{ \frac{2E_e}{1 + 3g_A^2} \left(2g_A U_B - (1 - g_A) U_T - (1 + g_A) U_8\right) \left(\vec{\xi}_n \cdot \vec{k}_e \right) \left(\vec{\xi}_e \cdot \vec{\zeta}_e\right) \frac{\vec{k}_e \cdot \vec{\zeta}_e}{(E_e + m_e) E_e^2 E_{\nu}^2} - \frac{2E_e}{1 + 3g_A^2} \left(1 + g_A\right) U_6 \right\} \left(\vec{\xi}_n \cdot \vec{\xi}_e \right) \left(\vec{\xi}_e \cdot \vec{\zeta}_e\right) \frac{\vec{\xi}_e \cdot \vec{\zeta}_e}{3 (E_e + m_e) E_e E_{\nu}}$$

$$+ 3B_0 \frac{\alpha}{\pi} \frac{E_e}{m_N} g_n(E_e) \left(\vec{\xi}_n \cdot \vec{\xi}_e \right) \left(\vec{\xi}_e \cdot \vec{\xi}_e\right) \frac{\vec{\xi}_e \cdot \vec{\zeta}_e}{E_e^2 E_{\nu}^2} \right\},$$

(F-2)
where the functions $U_5, U_6, U_7$ and $U_8$ are given in Eq. [A4], and

$$
\frac{d^5\lambda_n(E, \vec{k}_\perp, \vec{p}_\perp, \vec{c}_n, \vec{c}_\perp)}{dE d\vec{k}_\perp d\vec{p}_\perp d\vec{c}_n d\vec{c}_\perp} \bigg|_{\text{NLO}} = (1 + 3g_A^2) \frac{|G_V|^2}{16\pi^5} (E_0 - E_c)^2 \sqrt{E_c^2 - m_e^2} E_c F(E_c, Z = 1) \zeta(E_c)
$$

$$
\times \frac{E_c}{m_N} \left\{ -3 \left( 1 - g_A^2 \right) \left( \frac{E_0 - E_c}{E_0} \right)^2 - 1 \cdot \frac{k^2}{E_c E_0} \right\} + 3 \left( 1 - g_A^2 \right) \left( \frac{E_0 - E_c}{E_0} \right)^2 \left( 1 - \frac{m_e}{E_c} \right) \left( \vec{c}_\perp \cdot \vec{k}_\perp \right) m_e
$$

$$
+ 3 \cdot \frac{1 - g_A^2}{1 + g_A^2} \left( \frac{(\vec{c}_n \cdot \vec{k}_\perp)(\vec{c}_n \cdot \vec{k}_\perp)}{(E_n + m_e) E_c^2 E_0^2} - \frac{1}{3} \left( 1 - \frac{m_e}{E_c} \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right) \right) \right\}
$$

\hspace{1cm} (F-3)

and

$$
\frac{d^5\lambda_n^{(1)}(E, \vec{k}_\perp, \vec{p}_\perp, \vec{c}_n, \vec{c}_\perp)}{dE d\vec{k}_\perp d\vec{p}_\perp d\vec{c}_n d\vec{c}_\perp} \bigg|_{\text{NLO}} = (1 + 3g_A^2) \frac{|G_V|^2}{16\pi^5} (E_0 - E_c)^2 \sqrt{E_c^2 - m_e^2} E_c F(E_c, Z = 1) \zeta(E_c)
$$

$$
\times \left\{ \frac{E_0^2}{2m_N^2} \left\{ \frac{-k}{1 + 3g_A^2} (g_A + 2\gamma + 1) \right\} + \frac{1}{1 + 3g_A^2} \left( \vec{c}_n \cdot \vec{k}_\perp \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right) \right\} + \frac{1}{1 + 3g_A^2} \left( \vec{c}_n \cdot \vec{k}_\perp \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right)
$$

$$
\left\{ \frac{8g_A}{1 + 3g_A^2} \left( \frac{E_0^2}{M_A^2} \right) + (1 + 2g_A) \left( \frac{E_0^2}{M_A^2} \right) \right\} \left( \vec{c}_n \cdot \vec{k}_\perp \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right)
$$

$$
- \frac{8g_A}{1 + 3g_A^2} \left( \frac{E_0^2}{M_A^2} \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right)
$$

\hspace{1cm} (F-4)

and

$$
\frac{d^5\lambda_n^{(2)}(E, \vec{k}_\perp, \vec{p}_\perp, \vec{c}_n, \vec{c}_\perp)}{dE d\vec{k}_\perp d\vec{p}_\perp d\vec{c}_n d\vec{c}_\perp} \bigg|_{\text{NLO}} = (1 + 3g_A^2) \frac{|G_V|^2}{16\pi^5} (E_0 - E_c)^2 \sqrt{E_c^2 - m_e^2} E_c F(E_c, Z = 1) \zeta(E_c)
$$

$$
\times 6 \frac{E_0^2}{m_N^2} \left\{ \left( 1 - \frac{1 - g_A^2}{1 + 3g_A^2} \left( \frac{1}{E_0} - \frac{1}{E_c} \right) \left( \frac{E_0 - E_c}{E_0} \right)^2 - \frac{1}{3} \frac{k^2}{E_c E_0} \right) \right\} + 2 g_A \left( 1 - g_A \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right)
$$

$$
\left\{ \left( \frac{E_0}{E_c} \right)^2 \left( \vec{c}_n \cdot \vec{k}_\perp \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right) \right\} + 2 g_A \left( 1 - g_A \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right)
$$

$$
\left\{ \left( \vec{c}_n \cdot \vec{k}_\perp \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right) \right\} + 4 g_A \left( 1 + g_A \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right)
$$

\hspace{1cm} (F-5)

and

$$
\frac{d^5\lambda_n^{(3)}(E, \vec{k}_\perp, \vec{p}_\perp, \vec{c}_n, \vec{c}_\perp)}{dE d\vec{k}_\perp d\vec{p}_\perp d\vec{c}_n d\vec{c}_\perp} \bigg|_{\text{NLO}} = (1 + 3g_A^2) \frac{|G_V|^2}{16\pi^5} (E_0 - E_c)^2 \sqrt{E_c^2 - m_e^2} E_c F(E_c, Z = 1) \zeta(E_c)
$$

$$
\times \frac{E_0^2}{2m_N^2} \left\{ \left( g_A + (\kappa + 1)^2 \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right) \right\} + \frac{1}{1 + 3g_A^2} \left( \vec{c}_n \cdot \vec{k}_\perp \right) \left( \vec{c}_n \cdot \vec{k}_\perp \right)
$$

\hspace{1cm} (F-6)
\begin{align}
\times \left( \frac{\xi_e \cdot \vec{k}_e}{E_e E_\nu^2} - \frac{1}{3} \frac{\xi_e \cdot \vec{k}_e}{E_e} \right) + \frac{1}{1 + 3g_A^2} \left( \frac{g_A^2}{E_e E_\nu^2} \left( \frac{\xi_e \cdot \vec{k}_e}{E_e} \right)^2 \right) \frac{\xi_e \cdot \vec{k}_e}{E_e + m_e} + \frac{1}{1 + 3g_A^2} \left[ (g_A + (\kappa + 1)) \frac{E_\nu^2}{E_0^2} + (g_A + (\kappa + 1)) \frac{E_e E_\nu}{E_0^2} + g_A (g_A + (\kappa + 1)) \frac{E_\nu}{E_0} \right] \\
\times \left( \frac{\xi_e \cdot \vec{k}_e}{E_e} \right) + \frac{1}{1 + 3g_A^2} \left[ g_A (g_A + (\kappa + 1)) \frac{E_\nu}{E_0} + (g_A + (\kappa + 1)) \frac{E_e E_\nu}{E_0^2} \right] \\
+ (\kappa + 1) \frac{E_\nu^2}{E_0^2} - (\kappa + 2) (g_A + (\kappa + 1)) \frac{E_e E_\nu}{E_0^2} \left( 1 + \frac{m_e}{E_e} \right) \left( \frac{\xi_e \cdot \vec{k}_e}{E_e} \right) \frac{\xi_e \cdot \vec{k}_e}{E_e + m_e} + \frac{1}{1 + 3g_A^2} \left[ \kappa (g_A - (\kappa + 1)) \frac{m_e}{E_\nu} - g_A (g_A - (\kappa + 1)) \frac{E_e}{E_\nu} + (\kappa + 1) (g_A - (\kappa + 1)) \frac{E_\nu}{E_0} \right] \\
+ (\kappa + 1) (3g_A + (\kappa + 1)) \frac{E_e E_\nu}{E_0^2} \left( \frac{\xi_e \cdot \vec{k}_e}{E_e} \right) \frac{\xi_e \cdot \vec{k}_e}{E_e + m_e} \right) \quad (F-6)
\end{align}

and

\begin{align}
\frac{d^3 \lambda_n}{dE_e d\Omega_e d\Omega_\nu} & \bigg|_{NLO} = (1 + 3g_A^2) \frac{|G_V|^2}{16\pi^2} (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) \xi_e \\
\times \frac{3E_e}{m_N} \left\{ - \frac{E_e}{E_\nu} \left( \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e^2 E_\nu^2} - \frac{1}{3} \frac{k_e^2}{E_e} \right) - \frac{\xi_e \cdot \vec{k}_e}{E_e} \right\} - \frac{E_e}{m_N} \left\{ - \frac{E_e}{E_\nu} \left( \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e^2 E_\nu^2} - \frac{1}{3} \frac{k_e^2}{E_e} \right) - \frac{\xi_e \cdot \vec{k}_e}{E_e} \right\} \\
- \frac{E_e}{m_N} \left\{ - \frac{E_e}{E_\nu} \left( \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e^2 E_\nu^2} - \frac{1}{3} \frac{k_e^2}{E_e} \right) - \frac{\xi_e \cdot \vec{k}_e}{E_e} \right\} + \frac{3E_e}{m_N} \left( 1 - \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e E_\nu} \right) \frac{d^3 \lambda_n}{dE_e d\Omega_e d\Omega_\nu} \right|_{NLO} \quad (F-7)
\end{align}

where \( \vec{k}_0(E_e)_{NLO} \) and \( \xi_e(E_e)_{NLO} \) coincide with \( K_n(E_e)_{NLO} \) and \( Q_n(E_e)_{NLO} \), which are given in Eq. (B-2), whereas \( H(E_e) \), \( Q_n(E_e) \) and \( K_n(E_e) \) are defined by the expressions in Eq. (B-3). The last term in Eq. (F-1) is equal to

\begin{align}
\frac{d^3 \lambda_n}{dE_e d\Omega_e d\Omega_\nu} \bigg|_{WC} = (1 + 3g_A^2) \frac{|G_V|^2}{16\pi^2} (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) \\
\times \left\{ - B_0 \frac{\pi \alpha}{\beta^3} \frac{E_0 - E_e}{m_N} \left( \frac{\xi_e \cdot \vec{k}_e}{E_e E_\nu^2} - \frac{1}{3} \frac{k_e^2}{E_e^2} \right) \right\} \quad (F-8)
\end{align}

These contributions to the electron-energy and angular distribution of the neutron beta decay vanish after the integration over the directions of the antineutrino 3-momentum \( \vec{k}_\nu \). Because of the contributions of Wilkinson’s corrections, caused by the proton recoil in the electron-proton final-state Coulomb interaction, and the \( O(\alpha E_e/m_N) \) outer radiative corrections (see Eq. (D-10)) the electron-energy and angular distributions in Eqs. (F-2) - (F-8) are well defined in the experimental electron-energy region 0.811 MeV \( \leq E_e \leq 1.211 \text{ MeV} \).
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