Scalability Concept for Predictable Closed-Loop Response of Adaptive Controllers

Simon P. Schatz and Tansel Yucelen

Abstract—We introduce a new concept called scalability to adaptive control in this paper. In particular, we analyze how to scale learning rates of adaptive weight update laws of various adaptive control schemes with respect to given command profiles to achieve a predictable closed-loop response. An illustrative numerical example is provided to demonstrate the proposed concept, which emphasize that it can be an effective tool for validation and verification of adaptive controllers.

I. SCALABLE PERFORMANCE IN MODEL REFERENCE ADAPTIVE CONTROL

In this section, scalability is shown in the standard model reference adaptive control (MRAC) architecture.

A. MRAC Problem Formulation

Consider the uncertain dynamical system given by
\[ \dot{x}(t) = Ax(t) + B\Lambda u(t) + B\Delta(x(t)), \quad x(0) = x_0, \] (1)
where \( x(t) \in \mathbb{R}^n \) is the accessible state vector, \( u(t) \in \mathbb{R}^m \) is the control input vector, \( \Delta(x(t)) : \mathbb{R}^n \to \mathbb{R}^m \) is an uncertainty, \( A \in \mathbb{R}^{n \times n} \) is a known system matrix, \( \Lambda \in \mathbb{R}^{m \times m} \) is an unknown control effectiveness matrix, and \( B \in \mathbb{R}^{n \times m} \) is a known control input matrix. We assume that the pair \((A,B)\) is controllable. Additionally, we assume
\[ \Delta(x(t)) = \Lambda \begin{bmatrix} W_x^T & W_c^T & w_\kappa \end{bmatrix} \omega(t), \] (2)
where \( W_x \in \mathbb{R}^{n \times m} \) represents an uncertainty in the system matrix, \( W_c \in \mathbb{R}^{m \times m} \) represents an uncertainty in the command input matrix, \( \omega = (x(t)^T c(t)^T \kappa)^T \in \mathbb{R}^{n+l+1} \) is a known regressor vector, \( c(t) \in \mathbb{R}^l \) is the uniformly continuous bounded command, \( \kappa \) is a constant, and \( w_\kappa \in \mathbb{R}^m \) represents a constant disturbance. The reference system is given by
\[ \dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0}, \] (3)
where \( x_r(t) \in \mathbb{R}^n \) is the reference model state vector, \( A_r \in \mathbb{R}^{n \times n} \) is the desired Hurwitz system matrix, and \( B_r \in \mathbb{R}^{n \times l} \) is the command input matrix. The control signal \( u(t) \) is given as
\[ u(t) = -K_x x(t) + K_c c(t) - u_{ad}(t), \] (4)
where \( u_{ad}(t) \in \mathbb{R}^m \) is the adaptive control input, \( K_x \in \mathbb{R}^{n \times n} \) is the nominal feedback matrix and \( K_c \in \mathbb{R}^{m \times l} \) is the nominal feedforward matrix chosen such that \( A-BK_x = A_r \) and \( BK_c = B_r \). Using (2) and (4) in (1), yields
\[ \dot{x}(t) = A_r x(t) + B_r c(t) + B\Lambda W^T \omega(t) - B\Lambda u_{ad}(t), \] (5)
where
\[ W \triangleq \begin{bmatrix} W_x^T - \Lambda^* K_x & W_c^T + \Lambda^* K_c & w_\kappa \end{bmatrix}^T, \] (6)
and \( \Lambda^* \triangleq [I_m - \Lambda^{-1}] \). We use the adaptive control law
\[ u_{ad}(t) = \hat{W}^T(t) \omega(t), \] (7)
where \( \hat{W}(t) \in \mathbb{R}^{(n+l+1) \times m} \) is the adaptive weight matrix satisfying the adaptive weight update law
\[ \dot{\hat{W}}(t) = \Gamma \omega(t) e^T(t) P B, \quad \hat{W}(0) = \hat{W}_0, \] (8)
e(t) \triangleq x(t) - x_r(t) is the tracking error, and \( P \in \mathbb{R}^{n \times n} \) is the positive definite solution of the Lyapunov equation
\[ Q + A_r^T P + PA_r = 0, \] (9)
where \( Q \in \mathbb{R}^{n \times n} \) is a positive definite design matrix. Finally, the uncertain dynamical system (1) can now be given as
\[ \dot{x}(t) = A_r x(t) + B_r c(t) - B\Lambda \hat{W}^T(t) \omega(t), \quad x(0) = x_0, \] (10)
where \( \hat{W}(t) \triangleq \hat{W}(t) - W \in \mathbb{R}^{(n+l+1) \times m} \) is the adaptive weight estimation error.

B. Scalability

Now, we assume that the control engineer has found an appropriate adaptive control performance for a certain command history \( c_0(t) \) and a specified learning rate \( \Gamma_0 \), resulting in the adaptive weight update law
\[ \dot{\hat{W}}(t) = \Gamma_0 \omega(t) e^T(t) P B, \quad \hat{W}(0) = \hat{W}_0, \] (11)
for any scaled command profiles \( c(t) = \alpha c_0(t) \) with scalar scaling command coefficients \( \alpha \neq 0 \) given a Lyapunov design matrix \( Q \) it is possible to achieve scaled system responses by choosing \( \Gamma = \Gamma_0 / \alpha^2 \). To show this, we define
\[ z(t) \triangleq x(t)/\alpha, \quad z_0 \triangleq x_0/\alpha, \quad z_r(t) \triangleq x_r(t)/\alpha, \quad z_0 \triangleq x_{r0}/\alpha, \quad e_x(t) \triangleq z(t) - z_r(t) = e(t)/\alpha, \quad \kappa = \alpha \] and
\[ \omega_z(t) \triangleq \alpha \omega(t) = \alpha \left( z(t) - c_0(t) - 1 \right)^T. \] (12)
By applying this transformation to the uncertain dynamical system (1) and the weight update law (5), we have
\[ \dot{z}(t) = A_r z(t) + B_r c_0(t) - B\Lambda \hat{W}^T z_\omega(t), \quad z(0) = z_0, \] (13)
\[ \dot{z}_r(t) = A_r z_r(t) + B_r c_0(t), \quad z_r(0) = z_{r0}, \] (14)
\[ \hat{W}(t) = \Gamma_0 \omega_z(t) e_z^T(t) P B, \quad \hat{W}(0) = \hat{W}_0. \] (15)
Note that the equations (13), (14), and (15) hold for any $\alpha \neq 0$. Further, note that the uncertain system (10) and the reference system (8,14) are scalable in the sense that state histories can be given by a nominal system response scaled by $\alpha$.

II. OTHER MRAC SCHEMES

The scalability notion is applicable to all MRAC based schemes under the assumption that the states are applicable. In particular, in this section it is shown that the $\sigma$-modification and $\epsilon$-modification adaptive control architectures [4, 5], frequency-limited adaptive controllers [1], adaptive control architectures employing closed-loop reference models [6, 7], and command governor-based adaptive controllers [2] can all be modified in order to achieve predictable performances as shown previously.

A. $\sigma$ and $\epsilon$ modification architectures

These robustness modifications have been introduced in order to avoid the phenomena of parameter drift and increase the robustness with respect to unmodeled dynamics. The architectures modify the adaptive weight update law by augmenting it with a “damping-like” term.

In [4] the standard MRAC adaptive weight update law was modified as

$$ \dot{W}(t) = \Gamma(t)e^T(t)PB - \sigma W(t), \quad \dot{W}(0) = \dot{W}_0, $$

where $\sigma > 0$ is a damping coefficient used to “pull” the estimated adaptive weights towards the origin. It was claimed that this $\sigma$-modification prevented the estimated adaptive weight from becoming unbounded. By introducing a scaling factor, as done in Section I-B the $\sigma$-modified adaptive weight update law is given as

$$ \dot{W}(t) = \Gamma_0 \omega_z(t)e^T_z(t)PB - \sigma \dot{W}(t), \quad \dot{W}(0) = \dot{W}_0, $$

where $\Gamma = \Gamma_0/\alpha^2$, $\omega_z(t)$ and $e_z(t)$ are defined in the previous section. It can be readily seen that the adaptive weight response, as before, is invariant to the scaling factor $\alpha$. Furthermore, scalability of the system states and inputs is also evident since the reference system (5) and the uncertain system (10) are not modified.

In [5] the standard MRAC adaptive law was further modified by replacing $\sigma$ in (16) with a time-varying damping coefficient determined by $\sigma_\epsilon ||e(t)||_2$. Therefore, the effect of the modification was determined by the norm of the system’s tracking error. The so-called $\epsilon$-modification adaptive weight update law is given as

$$ \dot{W}(t) = \Gamma(t)e^T(t)PB - \sigma_\epsilon ||e(t)||_2 \dot{W}(t), \quad \dot{W}(0) = \dot{W}_0, $$

where $\sigma_\epsilon > 0$. Similar to (16), by introducing a scaling factor $\alpha$ can be rewritten as

$$ \dot{W}(t) = \Gamma_0 \omega_z(t)e^T_z(t)PB - \sigma \epsilon ||e_z(t)||_2 \dot{W}(t), \quad \dot{W}(0) = \dot{W}_0, $$

where $\sigma_\epsilon = \sigma_0/\alpha$ and, as before, $\Gamma = \Gamma_0/\alpha^2$. Note that, as seen for the $\sigma$-modification case, (18) is invariant to $\alpha$ and, therefore, scalability results.

It should be noted that all the adaptive control architectures considered in this section are obtained with simple augmentations of the standard MRAC adaptive weight update law. In general, if the augmentation is invariant to the scaling factor $\alpha$ then the modified adaptive control framework will be scalable in the sense introduced in this paper.

B. Frequency-Limited Adaptive Control

The frequency limited adaptive control architecture introduced in [1] employs a gradient based modification term and a low pass filter. It is claimed that the modification term filters high-frequency content out of the adaptive weight update law, allowing for the controller to be tuned with high learning rates in order to enable robust and fast adaptation. The adaptive weight update law is given by

$$ \dot{W}(t) = \Gamma(t)e^T(t)PB - \sigma [\dot{W}(t) - \dot{W}_f(t)], $$

where $\sigma > 0$ is a modification gain and $W_f(t) \in \mathbb{R}^{(n+1) \times m}$ is the low-pass filtered weight estimate of $W(t)$, satisfying

$$ \dot{W}_f(t) = \Gamma_f [\dot{W}(t) - \dot{W}_f(t)], \quad \dot{W}_f(0) = \dot{W}_0, $$

where $\Gamma_f \in \mathbb{R}^{(n+1) \times (n+1)}$ is a positive definite filter gain matrix such that $\lambda_{\text{max}}(\Gamma_f) \leq \gamma_f \max \gamma_f$ and $\gamma_f \max > 0$ is a design parameter.

The adaptive weight update law (21) can incorporate the scaling factor $\alpha$ as

$$ \dot{W}(t) = \Gamma_0 \omega_z(t)e^T_z(t)PB - \sigma [\dot{W}(t) - \dot{W}_f(t)], $$

where $\Gamma = \Gamma_0/\alpha^2$, $e_z(t) = e(t)/\alpha$, and $\omega_z(t) = \omega(t)/\alpha$. Note that once again the adaptive weight update law is invariant with respect to the scaling factor $\alpha$. Therefore, as discussed in the previous section, it can be concluded that a system employing this adaptive control framework will have predictably scalable responses.

C. Reference Model Modification

In [6], [7] the reference model was modified by feeding back the tracking error in order to improve the transient performance of MRAC controllers. Therefore, the uncertain dynamical system (10) and the adaptive weight update law (8) are not changed and can be scaled as shown in Section I-B. However, the reference model is given by

$$ \dot{x}_r(t) = Ax_r(t) + Br e(t) + L e(t), \quad x_r(0) = x_{r0}, $$

where $L \in \mathbb{R}^{n \times n}$ is a positive definite matrix. The scaling factor can then be introduced to the modified reference model by employing, as before, the relations $z_r(t) = x_r(t)/\alpha$, $z_{r0} = x_{r0}/\alpha$, $e_z(t) = e(t)/\alpha$, and $c(t) = \alpha c_0(t)$, resulting in

$$ \dot{z}_r(t) = A_z z_r(t) + B_r c_0(t) + L e_z(t), \quad z_r(0) = z_{r0}. $$

Hence, scalability for adaptive control architectures with modified reference models is obtained.
D. Command Governor Adaptive Control

Here, the scalability notion is applied to the command governor framework for adaptive control [2].

Hence, the overall command is given by

\[ c(t) = c_D(t) + c_g(t), \]  

where \( c_D(t) \in \mathbb{R}^m \) is the bounded, desired tracking command (the original \( c(t) \) from the sections above). The additional command \( c_g(t) = K^{-1}_c \left[ B^TB \right]^{-1} B^T g(t) \in \mathbb{R}^m \), \( \det(K_c) \neq 0 \) is based on a linear system, which is defined as

\[ \dot{\xi}(t) = -\lambda \xi(t) + \lambda e(t), \quad \xi(0) = 0, \]  
\[ g(t) = \lambda \xi(t) + [A_r - \lambda I_n] e(t), \]  

where \( \xi(t) \in \mathbb{R}^n \) denotes the command governor states, \( g(t) \in \mathbb{R}^n \) is the command governor output, and \( \lambda > 0 \) is the command governor gain. Since the additional command is applied on both reference model and nominal controller, the error dynamics of the system do not change and therefore, we have

\[ \dot{e}(t) = A_r e(t) - B A \dot{W}(t) \omega(t), \quad e(0) = x_0 - x_{r0}, \]  

which can be written as

\[ \Lambda \dot{W}(t) \omega(t) = [B^TB]^{-1}B^T \{ A_r e(t) - \dot{e}(t) \}. \]  

Applying (25), (26), (27), and (29) onto the uncertain system dynamics (10), using \( G = B \left[ B^TB \right]^{-1} B^T \), we have

\[ \dot{x}(t) = A_r x(t) + B_r c_D(t) + G \{ \lambda \xi(t) - \lambda e(t) - \dot{e}(t) \}, \]  

\[ x(0) = x_0. \]  

In [2] it is shown that \( \lambda \xi(t) - \lambda e(t) - \dot{e}(t) = 0 \) for \( \lambda \to \infty \) and that the overall system is stable.

**Remark 1:** Although the reference model is modified, the closed loop uncertain system still tracks the desired reference model given by

\[ \dot{x}_{r,D}(t) = A_r x_{r,D}(t) + B_r c_D(t), \quad x_{r,D}(0) = x_{r0} \]  

as the last term of (30) is approximately zero for large \( \lambda \).

**Remark 2:** The command governor gain \( \lambda \) can be used to determine a trade off between command governor and adaptive control. Furthermore, note that no adaptive control would be necessary for \( \lambda \to \infty \), which is of no practical relevance. For more information about the command governor refer to [2].

Now, considering scalability, assume there was a \( c_0(t) \) with a certain reference performance and a learning rate \( \Gamma_0 \). Then, applying a command profile \( c_D(t) = \alpha c_0(t) \) and a scaled adaptive gain \( \Gamma = \Gamma_0/\alpha^2 \), scalability can be achieved. Using \( \xi_z(t) = \xi(t)/\alpha, \ g_z(t) = g(t)/\alpha, \) and \( e_z(t) = e(t)/\alpha \), we have

\[ \dot{\xi}_z(t) = -\lambda \xi_z(t) + \lambda e_z(t), \quad \xi_z(0) = 0 \]  
\[ g_z(t) = \lambda \xi_z(t) + [A_r - \lambda I_n] e_z(t). \]  

Hence, \( c_{g,z}(t) = \alpha c_g(t) \) and \( c(t) = \alpha c_0 + \alpha c_g \) holds, which implies that the reference model is also scalable as shown in Section LB. The transformed uncertain system dynamics are given by (using \( f_z(t) \triangleq \lambda \xi_z(t) - \lambda e_z(t) - \dot{e}_z(t) \))

\[ \dot{z}(t) = A_r z(t) + B_r c_0(t) + G f_z(t), \quad z(0) = z_0, \]  

which shows scalability of the uncertain system’s dynamics. Additionally, the invariance of the adaptive weight update law [15] to the scaling factor stays untouched. Consequently, the scalability approach introduced in this paper also holds for the command governor framework.

**REFERENCES**

[1] T. Yucelen and W. M. Haddad, “Low-frequency learning and fast adaptation in model reference adaptive control,” *IEEE Transactions on Automatic Control*, 2013.

[2] T. Yucelen and E. N. Johnson, “A new command governor architecture for transient response shaping,” *International Journal of Adaptive Control and Signal Processing*, 2013.

[3] Z. Dydek, H. Jain, J. Jiang, A. M. Annaswamy, and E. Lavretsky, “Theoretically Verifiable Stability Margins for an Adaptive Controller,” *IEEE Transactions on Automatic Control*, 2006.

[4] P. Ioannou and P. Kokotovic, “Instability analysis and improvement of robustness of adaptive control,” *Automatica*, vol. 20, pp. 583–594, 1984.

[5] K. S. Narendra and A. M. Annaswamy, “A new adaptive law for robust adaptation without persistent excitation,” *IEEE Transactions on Automatic Control*, vol. 32, pp. 134–145, 1987.

[6] E. Lavretsky, “Reference dynamics modification in adaptive controllers for improved transient performance,” *AIAA Guidance, Navigation, and Control Conference*, 2011.

[7] T. E. Gibson, A. M. Annaswamy, and E. Lavretsky, “Adaptive systems with closed-loop reference models: Stability, robustness, and transient performance,” *IEEE Transactions on Automatic Control* (submitted).