Hopf-Galois extensions with central invariants.

Dmitriy Rumynin*

September 22, 1997

Abstract

We study a class of algebra extensions which usually appear in the study of restricted Lie algebras or various quantum objects at roots of unity.

The present paper was inspired by the theory of nonrestricted representations of restricted Lie algebras and the theory of quantum groups at roots of unity where algebras are usually finitely generated modules over their centers. Our objective is to demonstrate that all these theories admit a unifying approach. We use the concept of a Hopf-Galois extension with central invariants to treat these phenomena from a general point of view. We discuss extensions of algebras in Section 1. We define Hopf-Galois extensions with central invariants early in Section 2. Then we study them locally: i.e. we consider localizations at points of the prime spectrum of the subalgebra of invariants. The localizations form a vector bundle of not necessarily commutative algebras on a scheme. The fibers of the bundle are finite-dimensional Frobenius algebras and their irreducible representations coincide with irreducible representations of the Hopf-Galois extension we study. We describe some known examples in which the developed formalism takes place in Section 3. It seems that further attempts to study these extensions should be based on exploring the inherent geometry of the situation. The author is grateful to J. Humphreys for fruitful discussions and the proofreading of the manuscript.

*University of Massachusetts at Amherst
1 Preliminary facts on algebra extensions.

We are working over a ground field $k$ from now on. Concerned with an algebra $U$ (associative with unity), we consider a finitely-generated subalgebra $O$ of $Z$, the center of $U$, such that the module $U_O$ is finitely-generated. Such extensions were first studied in [2]. In this section we collect various technical results needed later.

Under the assumptions we make $U$ is a PI-algebra. One should use [15] as a general reference. The following proposition easily follows from the Hilbert basis theorem.

**Proposition 1** $Z_O$ is a finitely generated module and $Z$ is a finitely generated algebra.

The following is a version of Hilbert’s Nullstellensatz for $U$.

**Theorem 2** A simple $U$-module is finite dimensional. Furthermore, if $\rho$ is a finite dimensional representation then $\rho(Z)$ and $\rho(O)$ are finite extensions of $k$.

The next proposition is similar to [15, Theorem 2.7.4].

**Proposition 3** Simple $U$-modules are isomorphic if and only if the kernels of the corresponding representations are equal.

Proposition 3 shows that assigning the kernel to an irreducible representation is a bijection from the set of simple $U$-modules $\mathcal{Rep}U$ onto the set of primitive ideals. The latter has the structure of a topological space with respect to the Jacobson topology. The central characters $\Omega$ yield a sequence of maps:

$$
\mathcal{Rep}U \cong \text{Prim } U \xrightarrow{\Omega_Z} \text{Max } Z \xrightarrow{\Omega_{Z/O}} \text{Max } O
$$

(1)

We denote the composition $\Omega_{Z/O} \circ \Omega_Z$ by $\Omega_O$. The following lemma is a non-commutative variation of a well-known result in algebraic geometry [2].

**Lemma 4** $\Omega_Z$ and $\Omega_{Z/O}$ are continuous onto maps.
The finitely generated module $U_O$ defines a coherent sheaf of algebras on $\text{Spec } O$. Denoting by $O_{(\eta)}$ the local ring of algebraic functions at $\eta \in \text{Spec } O$, $m_\eta$ its maximal ideal, and $K_\eta$ the quotient field $O_{(\eta)}/m_\eta$, we introduce three algebras:

\begin{align*}
U_{(\eta)} &= U \otimes_O O_{(\eta)} \text{ is an } O_{(\eta)}\text{-algebra;} \\
U_\eta &= U_{(\eta)} \otimes_{O_{(\eta)}} K_\eta \text{ is a } K_\eta\text{-algebra;} \\
U_{[\eta]} &= U/U_\eta \text{ is an } O/\eta\text{-algebra.}
\end{align*}

If the module $U_O$ is projective the sheaf becomes an algebraic vector bundle. The following theorem holds [11, Theorem 4.27]:

**Theorem 5** If $U_O$ is projective then the dimension function $\chi \mapsto \dim_{K_\chi} U_\chi$ is a continuous map from $\text{Spec } O$ to $\mathbb{Z}$.

We would like to understand $O$-bilinear forms $s : U \times U \to O$. Let a $K_\eta$-bilinear form $s_\eta : U_\eta \times U_\eta \to K_\eta$ be the specialization of $s$ to a point $\eta \in \text{Spec } O$. The proof of the following theorem has the same main idea as in the theorem about continuity of the dimension function.

**Theorem 6** Let the module $O U$ be projective. The set $\{ \eta \in \text{Spec } O \mid s_\eta \text{ is non-degenerate} \}$ is open.

**Proof.** $s$ defines an $O$-module map $\hat{s} : O U \to (O U)^*$. Then $s_\eta$ is non-degenerate outside the union of supports of the kernel and the cokernel $\hat{s}$. We will give a precise argument which works for not necessarily finitely generated $O$.

The specialization $s_\eta$ is non-degenerate if and only if $\hat{s}_\eta : K_\eta U_\eta \to (K_\eta U_\eta)^*$ is an isomorphism. It is equivalent the localization $\hat{s}_{(\eta)} : O_{(\eta)} U_{(\eta)} \to (O_{(\eta)} U_{(\eta)})^*$ being an isomorphism. Indeed, the projective module $U_{(\eta)}$ over the local ring $O_{(\eta)}$ must be free which allows us to use the determinant to check whether a map is an isomorphism. Clearly, $\det \hat{s}_{(\eta)} + m_\eta = \det \hat{s}_\eta$. Thus, $\det \hat{s}_{(\eta)}$ is invertible in $O_{(\eta)}$ if and only if $\det \hat{s}_\eta$ is invertible in $K_\eta$.

The map $\hat{s}$ is a part of the exact sequence $0 \to O A \to O U \xrightarrow{\hat{s}} (O U)^* \to O B \to 0$. If $s_\eta$ is degenerate for each $\eta$ then we have nothing to prove: the empty set is open. Let us pick $\eta$ with non-degenerate $s_\eta$. Now $O B$ is finitely
generated as a quotient of a finitely generated module. Let $b_1, \ldots, b_n$ be the generators. Since \(O_{(\eta)} = 0\) we can find elements \(x_i \in O\) such that \(x_i b_i = 0\) and \(x_i \notin \eta\). Let \(x = x_1 \cdots x_n\). We have obtained the exact sequence of localizations: \(0 \to OAx^{-1} \to OUX^{-1} \xrightarrow{i} (O^*)^x^{-1} \to OUX^{-1} \sim 0\). The modules \(OUX^{-1}\) and \((O^*)^x^{-1}\) are projective. Thus, the latter sequence is split and \(OAx^{-1}\) is finitely generated. By the argument as above we can find \(y \in O \setminus \eta\) such that \((Ax^{-1})y^{-1} = 0\). This implies that \((sx^{-1})y^{-1}\) is isomorphism. Therefore, \(s\eta\) is non-degenerate for \(\eta\) from the open set \(\{I \mid x \notin I\text{ and } y \notin I\}\). \(\square\)

If \(U_O\) is projective the theory of deformations [7] may be used to get some information about the bundle. Let \(X\) be the points of \(\text{Spec } O\) over a field \(F\).

**Proposition 7** Let \(O^U\) be projective. The set \(\{\chi \in X \mid U_\chi\text{ is a separable } K_\chi\text{-algebra}\}\) is open in the Zariski topology on \(X\).

**Proof.** The algebra \(U_\chi\) is separable if and only if so is \(\tilde{U}_\chi = U_\chi \otimes_{K_\chi} F\). Let \(\eta \in X\) be a point such that \(\tilde{U}_\eta\) is separable. We denote the dimension of \(\tilde{U}_\eta\) over \(F\) by \(n\). By Theorem 5 there exists an open neighborhood \(W\) of \(\eta\) such that for each \(\chi \in W\) the dimension of \(\tilde{U}_\chi\) is \(n\).

Let \(\text{Ass}_n \subseteq (F^n)^* \otimes (F^n)^* \otimes F^n\) consist of all tensors defining a structure of an associative algebra on \(F^n\). Thus we get a map \(\psi : W \to \text{Ass}_n / \text{GL}_n(F)\). The latter may be either a topological space with \(\psi\) being a continuous map or an algebraic variety with a rational map \(\psi\). It is proved in [8] that the set \(V_A = \{B \in \text{Ass}_n \mid A \otimes_F F \cong B \otimes_F F\}\) is open for each separable algebra \(A\). \(V_A / \text{GL}_n(F)\) is open in \(\text{Ass}_n / \text{GL}_n(F)\) since \(V_A\) is saturated. Therefore, the set \(\psi^{-1}(V_{\tilde{U}_\eta} / \text{GL}_n(F))\) is an open neighborhood of \(\eta\) with separable algebras. \(\square\)

We should notice that we have proved a little more. All algebras on some open subset of \(X\) are separable and isomorphic over the algebraic closure. In particular, if \(X\) is irreducible and \(F\) is algebraically closed then all separable algebras of the bundle must be isomorphic. However, we have not proved that the open set is not empty. For instance, if \(H\) is not semisimple and \(U\) is just a tensor product \(O \otimes H\) then the open set on which \(U_\chi\) is separable is empty.

Another important restriction one may impose is \(U\) being prime, i.e. a product of two proper ideals is proper. It implies that both \(Z\) and \(O\) are
domains. Moreover, the set $O^\circ$ of non-zero elements of $O$ is a set of non-zerodivisors of $U$. The localizations with respect to $O^\circ$ behave nicely: $Q(Z)$ is a field and $Q(U)$ is a central $Q(Z)$-algebra. In particular, the dimension of $Q(U)$ over $Q(Z)$ is square. The following theorem may be found in [15, Corollary 5.3.2].

**Theorem 8** If $U$ is prime then the map $\Omega_Z$ (see [1]) is a bijection on the complement of a closed set of smaller dimension.

## 2 Hopf-Galois extensions.

We use the standard notation for the structure maps of a Hopf algebra $H$: $\Delta, \varepsilon, S$ denote the comultiplication, the counity, and the antipode. We write $\Delta(x) = x_1 \otimes x_2$ keeping summation in mind. All tensor products are over $k$ unless otherwise indicated.

### 2.1 Basic properties of Hopf-Galois extensions.

Let $H$ be a finite dimensional Hopf algebra. $U$ is an $H$-comodule algebra if there is a map $\rho : U \longrightarrow U \otimes H$, $\rho(x) = x_0 \otimes x_1$ such that $x_0 \varepsilon(x_1) = x$, $x_0 \otimes \Delta(x_1) = \rho(x_0) \otimes x_1$, and $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in U$. If $O = U^H = \{x \in U$ such that $\rho(x) = x \otimes 1\}$ is a subalgebra of invariants one calls $U \supseteq O$ an $H$-extension. An $H$-extension is an extension with central invariants if $O$ is contained in the center of $U$. An $H$-extension is Hopf-Galois (or specifically $H$-Galois) if the canonical map $\text{can}: U \otimes_O U \longrightarrow U \otimes H$ defined by $\text{can}(x \otimes_O y) = (x \otimes 1)(y_0 \otimes y_1)$ is onto [9, 12].

We study $U \supseteq O$, a Hopf-Galois extension with central invariants, such that $O$ is a finitely generated algebra throughout this section. Here we list some important properties of such extensions.

**Proposition 9** [9, Theorem 1.7]. $U_O$ is a projective finitely generated module.

**Theorem-Definition 10** [12, 7.2.2; 8.2.4]. The extension is called cleft if the following equivalent conditions hold.

1. $U$ is isomorphic to a crossed product $O \#_\sigma H$ as an algebra.
2. There exists a convolution invertible right $H$-comodule map $\gamma : H \rightarrow U$.

3. There is a linear map from $U$ to $O \otimes H$ which is an isomorphism of left $O$-modules and right $H$-comodules.

Lemma 11 Let $A \subseteq B$ be a cleft $H$-Galois extension such that $A$ is commutative. $A$ is a central subalgebra if and only if $H$ acts trivially on $A$.

**Proof.** Let $\gamma : H \rightarrow U$ be a splitting map (i.e. a convolution invertible right $H$-comodule map). If $A$ is central for every $a \in A, h \in H$ we obtain $h \cdot a = \gamma(h_1)a\gamma^{-1}(h_2) = a\gamma(h_1)\gamma^{-1}(h_2) = a\varepsilon(h)$ Conversely, if the action is trivial we may think of $B$ as a crossed product.

$$ (a\#h)(b\#1) = a(h_1 \cdot b)\sigma(h_2 \otimes 1)\#h_3 = ab\#h = (b\#1)(a\#h) \quad \square $$

We have intentionally not introduced the crossed products. An anxious reader can find them in [12]. The last lemma indicates that we are interested only in crossed products with the trivial action, i.e. in twisted products. In the next section we mostly reformulate the well-known properties of crossed products for twisted products although Theorem 14 has no crossed products counterpart.

### 2.2 Twisted products.

Let $R$ be a commutative $k$-algebra and $\sigma : H \otimes H \rightarrow R$ be a linear map. Let $R[\sigma]H$ be a vector space $R \otimes H$ with an algebra structure defined by the formula

$$ a \otimes g \cdot b \otimes h = ab\sigma(h_1, g_1) \otimes h_2g_2, \quad a, b \in R, g, h \in H \quad (3) $$

It may not be a structure of an associative algebra in general. The following lemma is straightforward [12, Lemma 7.12].

**Lemma 12** The formula $3$ defines a structure of an associative algebra with an identity element $1 \otimes 1$ if and only if for each $g, h, t \in H$

$$ \sigma(h_1 \otimes g_1)\sigma(h_2g_2 \otimes t) = \sigma(g_1 \otimes t_1)\sigma(h \otimes g_2t_2) $$

$$ \sigma(h \otimes 1) = \sigma(1 \otimes h) = \varepsilon(h)1 $$
If $\sigma$ is invertible with respect to the convolution [12] and satisfies the conditions of Lemma 12 we call it a cocycle, although a corresponding cochain complex has been constructed only for cocommutative $H$. We use the term “twisted product” for $R_\sigma[H]$ if $\sigma$ is a cocycle. The following proposition is a reason to introduce cohomological equivalence of cocycles [12, 7.3.4].

**Proposition 13** Two cocycles $\sigma$ and $\tau$ produce isomorphic twisted products if and only if there exists a linear convolution-invertible map $u : H \rightarrow R$ such that $\tau(h \otimes g) = u^{-1}(g_1)u^{-1}(h_1)\sigma(h_2 \otimes g_2)u(h_3g_3)$. Then an isomorphism can be carried out by a map $a \otimes h \mapsto au(h_1) \otimes h_2$.

For any commutative $k$-algebra $R$ we denote $\mathfrak{Gal}_H(R)$ the set of isomorphism classes of twisted products $R_\sigma[H]$. The following theorem is somewhat surprising. The informal intuition behind it is that the action of $H$ is trivial and there is no problem how to extend it.

**Theorem 14** $\mathfrak{Gal}_H$ is a covariant functor from the category of commutative $k$-algebras to the category of sets.

**Proof.** Let $f : R \rightarrow S$ be a morphism of rings. It is clear that a cocycle $\sigma$ with values in $R$ gives rise to a cocycle $f \circ \sigma$ with values in $S$. We define $\mathfrak{Gal}_H(f)(R_\sigma[H]) = S_{f \circ \sigma}[H]$. It is well-defined because equivalent cocycles with values in $R$ have equivalent continuations to $S$. Indeed, if $u : H \rightarrow R$ provides an isomorphism between $R_\sigma[H]$ and $R_\tau[H]$ as in Proposition 13 then $S_{f \circ \sigma}[H]$ and $S_{f \circ \tau}[H]$ are isomorphic through $f \circ u$. Finally, it is apparent that $\mathfrak{Gal}_H(f \circ g) = \mathfrak{Gal}_H(f) \circ \mathfrak{Gal}_H(g)$ for any composition of algebra maps. $\square$

We might go on treating $\mathfrak{Gal}_H$ as a presheaf in etale or Zariski topology to construct the associated sheaf. It would entitle us to consider $H$-Galois extensions (non-commutative torsors) of schemes. But we prefer to stop here for we do not have enough interesting examples to think about.

### 2.3 The structure of the vector bundle of algebras.

We study a vector bundle of finite dimensional algebras on $\text{Spec } O$ defined by a Hopf-Galois extension. Our primary goal is to determine which additional structures the germs and the fibers inherit.
The structure of an $H$-comodule algebra can be extended to $U_{\eta}, U_{(\eta)}, U_{[\eta]}$ (see [3]) for each $\eta$. The following inclusions hold: $U_{H(\eta)}^H \supseteq O/\eta$, $U_{\eta} \supseteq K_{\eta}$, $U_{H(\eta)}^H \supseteq O_{(\eta)}$. If $U$ is prime then $U_{(\eta)} \supseteq O_{(\eta)}$ is a Hopf-Galois extension with central invariants under this action.

Proof. Since $H$ is finite dimensional a coaction of $H$ is equivalent to an action of $H^*$. Let $S$ be a complement in $O$ of a prime ideal $\eta$. $S$ is a central set of invariants in $U$. The algebra $U_{(\eta)}$ is isomorphic to a generalized algebra of quotients $US^{-1}$. Thus, for $h \in H^*$ one can define $h \cdot (xs^{-1}) = (h \cdot x)s^{-1}$.

It is clear that $OS^{-1} \cong O_\eta \subseteq U_{H(\eta)}^H$. On the other hand, given $y \in U_{H(\eta)}^H$ one can find $x \in U, s \in S$ such that $y = xs^{-1}$. Thus, $\varepsilon(h)xs^{-1} = (h \cdot x)s^{-1}$ which implies $\varepsilon(h)x = h \cdot x$ if $S$ is a set of non-zerodivisors of $U$ which holds for prime $U$. It implies that $x \in O$ and, therefore, $y \in O_{(\eta)}$.

Having proved $U_{H(\eta)}^H = O_{(\eta)}$, the extension $U_{(\eta)} \supseteq O_{(\eta)}$ is $H$-Galois because each element of $U_{(\eta)} \otimes H$ can be written with a common denominator as $\sum_i a_i s^{-1} \otimes h_i$. We can find $\sum_i b_i \otimes c_i \in U \otimes O U$ such that $\text{can}(\sum_i b_i \otimes c_i) = \sum_i a_i \otimes h_i$. It is clear that $\text{can}(\sum_i b_i s^{-1} \otimes c_i) = \sum_i a_i s^{-1} \otimes h_i$.

The other two algebras can be realized as quotients of $H$-comodule algebras by ideals generated by invariants: $U_{\eta} \cong U_{(\eta)}/U_{(\eta)}m_{\eta}$ and $U_{[\eta]} \cong U/U_{(\eta)}$. Thus, they admit a canonical extension of $H^*$-action. □

Theorem 16 If the extension $U \supseteq O$ is cleft then the extensions $U_{(\eta)} \supseteq O_{(\eta)}$, $U_{[\eta]} \supseteq O_{[\eta]}$, and $U_{\eta} \supseteq K_{\eta}$ are cleft Hopf-Galois with central invariants.

Proof. Let $\gamma : H \rightarrow U$ be a splitting map. The composition $H \rightarrow O \rightarrow O_{(\eta)}$ splits $U_{(\eta)}$. Thus, $U_{(\eta)} \cong U_{H(\eta)}^H \otimes H$ as $O_{(\eta)}$-modules. On the other hand, changing scalars in $U \cong O \otimes H$ gives $U_{(\eta)} \cong O_{(\eta)} \otimes H$. We emphasize that the subspace $H$ of $U$ is the same in the both cases giving a base of $U$ over two subalgebras $U_{H(\eta)}^H \supseteq O_{(\eta)}$. This proves that $U_{H(\eta)}^H = O_{(\eta)}$ and $U_{(\eta)} \supseteq O_{(\eta)}$ is a cleft $H$-Galois extension. The proofs for $U_{\eta}$ and $U_{[\eta]}$ are similar. □

Another important feature of Hopf-Galois extensions is a Frobenius form [4]. Let $\Lambda$ be a left integral of $H^*$. Given an $H$-Galois extension $A \supseteq B$, we can construct a $B$-bilinear form $s : A \times A \rightarrow B$ as $s(x, y) = x_0 y_0 \Lambda(x_1 y_1)$ for each $x, y \in A$. The form $s$ is non-degenerate according to [4].

Theorem 17 The algebra $U_{\chi}$ is Frobenius for each $\chi \in \text{Spec} \ O$. $U_{\chi}$ is symmetric if $H$ is unimodular with the antipode of order 2.
Proof. The form $s : U \times U \rightarrow O$ is non-degenerate [9]. By the proof of theorem 6 all specializations $s_\eta$ are non-degenerate proving the first statement. It was proved in [13] that $H$ is unimodular with the antipode of order 2 if and only if $\Lambda(xy) = \Lambda(yx)$ for each $x, y \in H$. This implies the second statement. \qed

2.4 Irreducible representations.

Given a simple $U$-module $M$, we get a point $\Omega_O(M) \in \text{Max } O$. Clearly $M$ can be treated as a $U_{\Omega_O(M)}$-module. $U_{\Omega_O(M)}$ being a finite-dimensional Frobenius algebra prompts that the study of blocks and projective covers of simple modules may be interesting.

Definition 18 Two irreducible $U$-modules $M$ and $N$ are said to belong to the same block if $\Omega_O(M) = \Omega_O(N)$ and $M$ and $N$ belong to the same block as $U_{\Omega_O(M)}$-modules.

It is well-known that the action of $Z(U_\chi)$, the center of $U_\chi$, distinguishes the blocks. On the other hand, $\text{rad } Z(U_\chi)$ lies in the kernel of an irreducible representation. These facts amount to the following lemma.

Lemma 19 1. If $M$ and $N$ belong to the same block then $\Omega_Z(M) = \Omega_Z(N)$.

2. The following two conditions are equivalent for $\chi \in \text{Max } O$:

1) $N$ and $M$ belong to the same block if and only if $\Omega_Z(M) = \Omega_Z(N)$.

2) The composition map $Z \rightarrow Z(U_\chi) \rightarrow Z(U_\chi)/\text{rad } Z(U_\chi)$ is onto.

If $H$ is copointed (i.e. $H^*$ is pointed) then little more is known about simple $U_\chi$-modules. The following fact has been proved in [17, 18]. The second statement follows from the observation that $U^H_\chi = K_\chi$ has a single simple module in the cleft case.

Proposition 20 Let us denote the number of simple $A$-modules $s(A)$. If $H$ is copointed then $s(U_\chi) \leq s(U^H_\chi)s(H)$. If, furthermore, $U \subseteq O$ is cleft then $s(U_\chi)$ divides $s(H)$.

We now assume that the extension $U \supseteq O$ is cleft. Let $M$ and $N$ be two $U_\chi$ modules. According to [10] the spectral sequence $E_2^{p,q} = \text{Ext}^p_H(K, \text{Ext}^q_{U_\chi}(M,N))$ converges to $\text{Ext}^n_{U_\chi}(M,N)$. Since $K_\chi$ is a field the elements of the spectral sequence are trivial unless $q = 0$. We state this as a proposition.
Proposition 21 If $U \supseteq O$ is cleft then for each $U_\chi$-modules $M$ and $N$ we have an isomorphism $\text{Ext}^n_{U_\chi}(M, N) \cong \text{Ext}^n_{U}(\mathbf{k}, \text{Hom}_{K_\chi}(M, N))$ for each $n$.

If there is any reason for the blocks of $U_\chi$-modules to behave uniformly with respect to $\chi$ we should be able to find it looking at the centers of $U_\chi$. We can succeed showing that the dimensions of the centers of different $U_\chi$ are equal under some strong restrictions on $U$. We assume that $H$ is cocommutative and $U \supseteq O$ is cleft for the rest of the section. We call a splitting map $\gamma : H \to U$ equivariant if for each $h, g \in H$.

Lemma 22 The splitting map is equivariant if and only if the corresponding cocycle $\sigma$ possesses the following property $\sigma(a \otimes b) = \sigma(a_1 b S(a_2) \otimes a_3)$ for each $a, b \in H$.

Proof. It is known that $\sigma(h \otimes g) = \gamma(h_1) \gamma(g_1) \gamma^{-1}(h_2 g_2)$. If $\gamma$ is equivariant then

$$\sigma(a_1 b_1 S(a_2) \otimes a_3) = \gamma(a_1 b_1 S(a_4)) \gamma(a_5) \gamma^{-1}(a_2 b_2 S(a_3) a_6) = \gamma(a_1) \gamma(b_1) \gamma^{-1}(a_2 b_2) = \sigma(a \otimes b)$$

If $\sigma$ possesses the property then

$$\gamma(a_1 b S(a_2)) = \gamma(a_1 b S(a_2)) \gamma(a_3) \gamma^{-1}(a_4) = \sigma(a_1 b S(a_4) \otimes a_5) \gamma(a_2 b S(a_3) a_6) \gamma^{-1}(a_7) = \sigma(a_1 b S(a_2) \otimes a_3) \gamma(a_4 b S(a_5) a_6) \gamma^{-1}(a_7) = \sigma(a_1 b S(a_2) \otimes a_3) \gamma(a_2 b S(a_5) a_6) \gamma^{-1}(a_7) = \gamma(a_1) \gamma(b) \gamma^{-1}(a_2)$$

An algebra splitting is a trivial example of an equivariant splitting. It means that the cocycle is $\sigma(x \otimes y) = \varepsilon(x) \varepsilon(y) 1$ and the twisted product is the tensor product in this case. We provide an example of a non-trivial equivariant splitting in Section 3.1.

Definition 23 We call a linear map $\alpha : H \otimes H \to U$ equivariant if for each $a, b \in H$ the following identity holds

$$\alpha(a \otimes b) = \alpha(a_1 b S(a_2) \otimes a_3)$$

Lemma 24 The inverse map under the convolution of an equivariant map is equivariant. The convolution of equivariant maps is equivariant.
Proof. Define $\beta(a \otimes b) = \alpha^{-1}(a_1 b S(a_2) \otimes a_3)$. Since $H$ is cocommutative we yield $\beta * \alpha(a \otimes b) = \beta(a_1 \otimes b_1) \alpha(a_2 \otimes b_2) = \alpha^{-1}(a_1 b_1 S(a_2) \otimes a_3) \alpha(a_4 b_2 S(a_3) \otimes a_6) = \alpha^{-1}(a_1 b_1 S(a_4) \otimes a_5) \alpha(a_2 b_2 S(a_3) \otimes a_6) = \varepsilon(a_1 b_1 S(a_2)) \varepsilon(a_3) = \varepsilon(a) \varepsilon(b)$

Thus, $\beta = \alpha^{-1}$ proving the first statement. If $\alpha$ and $\beta$ are equivariant then

$$\alpha * \beta(a_1 b S(a_2) \otimes a_3) = \alpha(a_1 b_1 S(a_4) \otimes a_5) \beta(a_2 b_2 S(a_3) \otimes a_6) = \alpha(a_1 b_1 S(a_2) \otimes a_3) \alpha(a_4 b_2 S(a_3) \otimes a_6) = \alpha(a_1 \otimes b_1) \beta(a_2 \otimes b_2) = \alpha * \beta(a \otimes b) \quad \square$$

Lemma 25 Let $\tau, \pi : H \times H \rightarrow A$ be two cocycles such that $\tau * \pi^{-1}$ be equivariant. Then $a_i \otimes h^i = \sum_i a_i \otimes h^i$ is central in $A_\pi[H]$ if and only if $a_i \otimes h^i$ is central in $A_\tau[H]$.

Proof. Let $\alpha = \tau * \pi^{-1}$. $a_i \otimes h^i$ being central in $A_\tau[H]$ provides

$$a_i b \tau((h^i)_1 \otimes g_1) \otimes (h^i)_2 g_2 \otimes g_3 = ba_i \tau(g_1 \otimes (h^i)_1) \otimes g_2 (h^i)_2 \otimes g_3$$

for each $b \in A, g \in H$. Apply $\rho \otimes \Delta$ to this equality.

$$a_i b \tau((h^i)_1 \otimes g_1) \otimes (h^i)_2 g_2 \otimes (h^i)_3 g_3 \otimes g_4 \otimes g_5 = ba_i \tau(g_1 \otimes (h^i)_1) \otimes g_2 (h^i)_2 \otimes g_3 (h^i)_3 \otimes g_4 \otimes g_5$$

Apply $\text{Id} \otimes \text{Id} \otimes (m_H \circ (\text{Id} \otimes S)) \otimes \text{Id}$ where $m_H : H \otimes H \rightarrow H$ is the multiplication.

$$a_i b \tau((h^i)_1 \otimes g_1) \otimes (h^i)_2 g_2 \otimes (h^i)_3 g_3 = ba_i \tau(g_1 \otimes (h^i)_1) \otimes g_2 (h^i)_2 \otimes g_3 (h^i)_3 S(g_4) \otimes g_5$$

Rewrite using cocommutativity of $H$.

$$(h^i)_1 \otimes g_1 \otimes a_i b \tau((h^i)_2 \otimes g_2) \otimes (h^i)_3 g_3 = g_1 (h^i)_1 S(g_2) \otimes g_3 \otimes ba_i \tau(g_4 \otimes (h^i)_2) \otimes g_5 (h^i)_3$$

Apply $(m_A \circ (\alpha \otimes \text{Id})) \otimes \text{Id}.$

$$\alpha((h^i)_1 \otimes g_1) a_i b \tau((h^i)_2 \otimes g_2) \otimes (h^i)_3 g_3 = \alpha(g_1 (h^i)_1 S(g_2) \otimes g_3) ba_i \tau(g_4 \otimes (h^i)_2) \otimes g_5 (h^i)_3$$

Since $A$ is commutative and $\alpha$ is equivariant it may be rewritten as

$$a_i b \pi((h^i)_1 \otimes g_1) \otimes (h^i)_2 g_2 = ba_i \pi(g_1 \otimes (h^i)_1) \otimes g_2 (h^i)_2$$

which proves that $a_i \otimes h^i$ is central in $A_\pi[H]. \quad \square$
Theorem 26 Let us assume that the extension $U \supseteq O$ admits equivariant splitting and $H$ is cocommutative. Then the dimension of $Z(U_\chi)$, the center of $U_\chi$, over $K_\chi$ does not depend on $\chi$.

Proof. Any two fields $K_\chi$ and $K_\eta$ have a common overfield $F$. Let $e_i$ be a basis of $F$ over $K_\chi$. An element of the center of $U_\chi \otimes K_\chi F$ may be written as $\sum a_i \otimes e_i$ with $a_i \in U_\chi$. Since it commutes with elements of $U_\chi \otimes 1$ the equality $Z(U_\chi \otimes K_\chi F) = Z(U_\chi) \otimes K_\chi F$ holds and the dimension of the center does not change with the field extension. Both $U_\chi \otimes K_\chi F \supseteq F$ and $U_\eta \otimes K_\chi F \supseteq F$ inherit equivariant splittings from $U \supseteq O$ and the theorem follows from the lemmas above. \(\blacksquare\)

3 Examples.

A number of interesting examples of central Hopf-Galois extensions arise in study of Hopf algebras. Given a Hopf algebra $U$ with central Hopf subalgebra $O$, we treat $U \supseteq O$ as an $H$-Galois extension with $H=U/U(O \cap \text{ker } \varepsilon)$. The map $\rho: U \rightarrow U \otimes H$ is $\rho(x) = x_1 \otimes \bar{x}_2$. $H$ is finite dimensional if and only if the module $U_O$ is finitely generated.

We discuss one more feature before doing examples. It is the winding action. It describes all $U_\chi$ isomorphic to $H$. This has been shown in [19, Exercise 5.3.4] for the reduced enveloping algebras (see Section 3.2).

Proposition 27 $U_\chi$ is isomorphic to $H$ if and only if $U_\chi$ has a one-dimensional representation.

Proof. The direct implication is apparent. Let $U_\chi$ have a one-dimensional representation $\alpha: U_\chi \rightarrow k$ for some $\chi$. This particularly means $k = K_\chi$. The composition $\tilde{\alpha}: U \rightarrow U_\chi \xrightarrow{\alpha} k$ is a one-dimensional representation of $U$. We define $\hat{\alpha}: U \rightarrow U$ as $\hat{\alpha}(x) = \tilde{\alpha}(x_1)x_2$. Clearly, $U/I \cong U/\hat{\alpha}(I)$ for each ideal $I$. $\hat{\alpha}(x) = \tilde{\alpha}(x_1)\varepsilon(x_2) = \varepsilon(\hat{\alpha}(x))$ proving that $\hat{\alpha}(\text{ker } \tilde{\alpha}) = \text{ker } \varepsilon$. Finally, $\hat{\alpha}(O) = O$ since $O$ is a Hopf subalgebra and, therefore, $\hat{\alpha}(U(O \cap \text{ker } \tilde{\alpha})) = U(O \cap \text{ker } \varepsilon)$.
3.1 Groups.

Let $H$ be a central subgroup of a group $G$. Then any set splitting $\tilde{\gamma} : G/H \longrightarrow G$ gives rise to a coalgebra splitting $\gamma : \mathbb{k}G/H \longrightarrow \mathbb{k}G$. This example fits precisely to the setup of the present paper if $H$ is finitely generated of finite index. Let us further assume that $G$ is a free Abelian group and $H$ is proper. Then there exists no Hopf algebra splitting $\gamma : \mathbb{k}G/H \longrightarrow \mathbb{k}G$ for $G/H$ must be a subgroup of $G$ otherwise. This provides an example of a non-trivial equivariant splitting because $\gamma(h_1gS(h_2)) = \gamma(gh_1S(h_2)) = \gamma(g)\varepsilon(h) = \gamma(g)\gamma(h_1)\gamma^{-1}(h_2) = \gamma(h_1)\gamma(g)\gamma^{-1}(h_2)$.

3.2 Restricted Lie algebras.

Let $L$ be a finite dimensional restricted Lie algebra over an algebraically closed field $\mathbb{k}$ of characteristic $p$. The universal enveloping algebra $U(L)$ has a central Hopf subalgebra $O$ generated by $x^p - x^{[p]}$ for $x \in L$. The quotient $U(L)/U(L)(\ker\varepsilon \cap O)$ is the restricted enveloping algebra $\mathfrak{u}(L)$.

For any $\chi \in L^*$ the reduced enveloping algebra $U^\chi$ is a quotient of $U(L)$ by the ideal generated by $x^p - x^{[p]} - \chi(x)^p 1_{U(L)}$ for all $x \in L$ [19, 2]. We should notice that the standard notation $U^\chi$ for the reduced enveloping algebra has already been used for a different object in the present paper. We will see pretty soon that our algebras $U^\chi$ are not far from the reduced enveloping algebras. The map $x \mapsto x^p - x^{[p]}$ can be extended to a semilinear isomorphism between the symmetric algebra $S(L)$ and $O$. This defines an isomorphism between Spec$O$ and the Frobenius shift $L^* (1)$. The fiber algebras at the closed points are precisely the reduced enveloping algebras we have just defined.

Lemma 28 $U^\chi$ is isomorphic to $U^\chi^\ast$. If we think of $U^\chi$ as a bundle of algebras over $L^*$ then the isomorphism is a pull-back along the map $L^* \longrightarrow L^*(1) \cong$ Spec$O$.

Furthermore, $U(L) \supseteq O$ is a cleft $u(L)$-Galois extension and, therefore, all algebras $U^\chi$ are twisted products. Its being cleft follows from $u(L)$ being pointed but we can also construct the splitting explicitly.

Proposition 29 Let us choose an ordered basis $e_i$ of $L$. It gives rise to PBW-bases on $U(L)$ and $u(L)$. Denoting their elements as $e^\alpha$ we can construct a splitting map $\gamma : u(L) \longrightarrow U(L)$ by $\gamma(e^\alpha) = e^\alpha$.
Proof. It is apparent that \( \gamma \) is a linear splitting as well as a coalgebra map. 

As an application of this fact we may give an explicit formula for multiplication in \( U_\chi \). Having chosen an ordered basis of \( L \), we identify \( U_\chi \) and \( u(L) \) as vector spaces using a PBW-basis in both. We use the standard notation for the Hopf algebra structure maps of \( u(L) \); the multiplication of \( U_\chi \) is denoted \( \circ \). We think of \( \chi \) as a linear map from \( O \) to \( k \). The map \( \gamma : u(L) \to U(L) \) is constructed in Proposition 29. The following proposition easily follows from the discussion above as well as the observation that \( \gamma^{-1}(x) = S(\gamma(x)) \).

Proposition 30 The element \( \sigma(x \otimes y) = \gamma(x_1)\gamma(y_1)S(\gamma(x_2y_2)) \) belongs to \( O \) for each \( x, y \in u(L) \). The multiplication of \( U_\chi \) can be written as \( x \circ y = \chi(\sigma(x_1 \otimes y_1))x_2y_2 \).

The case of \( L = sl_2(k) \) is the easiest to visualize. We assume that \( p > 2 \). Let \( e, f, h \) be the standard basis of \( sl_2(k) \). Following [16], we denote \( x = e^p, y = f^p, z = h^p - h, t = (h + 1)^2 - 4ef \). Then \( O \) is isomorphic to the polynomial algebra \( k[x, y, z] \) and \( Z \) is isomorphic to \( k[x, y, z, t]/(t^p - 2t^{(p+1)/2} + t - (z^2 - 4xy)) \).

We treat points of \( \text{Max } O \) as elements of \( L^*(1) \) as well as triples \((x, y, z)\). Points of \( \text{Max } Z \) are quadruples \((x, y, z, t)\) satisfying the equation:

\[
t^p - 2t^{(p+1)/2} + t = (z^2 - 4xy)
\]

One may observe three types of points in \( L^*(1) \). If \( \chi = (x, y, z) \) satisfies \( z^2 - 4xy \neq 0 \) then the equation 4 on \( t \) has \( p \) distinct roots and \( U_\chi \) is a direct sum of \( p \) copies of \( M_p(k) \). Over the cone \( z^2 - 4xy = 0 \) the equation 4 has 1 single root 0 and \( \frac{p-1}{2} \) double roots \( t^2 \mod p, i = 1, \ldots, \frac{p-1}{2} \). A point \((x, y, z, t)\) is singular if and only if \( x = y = z = 0 \) and \( t \neq 0 \) as easy to see by a direct computation. The point \((0, 0, 0, 0)\) corresponds to the Steinberg module for \( u(sl_2(k)) \). A singular point \((0, 0, 0, i)\) corresponds to the block consisting of 2 restricted \( sl_2(k) \)-modules of dimensions \( i \) and \( p - i \). For remaining \( \chi \) of the cone all \( U_\chi \) are isomorphic. This algebra has \( \frac{p+1}{2} \) simple modules of dimension \( p \) [3]. One of them is projective; the others comprise distinct blocks and have double projective covers.

One may consult [3, 8, 8] for the case of any classical semisimple Lie algebra. However, the precise details are unknown in the general case. Another case which may be treated explicitly is one of a completely solvable Lie algebra \( L \) [13, 8].
3.3 Quantum groups.

The quantum enveloping algebra $U_q(g)$ of a semisimple Lie algebra $g$ is generated by the elements $E_\alpha, F_\alpha, K_i$. If $q$ is a primitive $l$-th root of unity then $O$ is the subalgebra generated by $E_\alpha^l, F_\alpha^l, K_i^l$. If $n$ is the dimension of $g$ and $r$ is the rank we get a bundle of algebras on $\mathbb{C}^{n-r} \times \mathbb{C}^r$. This resembles the case of restricted Lie algebras. The construction has been developed in [3].

3.4 Quantum linear groups.

The following example has been worked out for any complex semisimple group in [4]. We follow [14] with an elementary approach to $SL_n$. The quantum linear group $SL_q(n)$ is a non-commutative $\mathbb{C}$-algebra generated by $X_{ij}, \ i, j = 1, \ldots, n$. The following are defining relations:

\[
X_{ri}X_{rj} = q^{-1}X_{rj}X_{ri} \text{ if } i < j
\]
\[
X_{ri}X_{si} = q^{-1}X_{si}X_{ri} \text{ if } r < s
\]
\[
X_{ri}X_{sj} = X_{sj}X_{ri} \text{ if } i > j \text{ and } r < s
\]
\[
X_{ri}X_{sj} - X_{sj}X_{ri} = (q^{-1} - q)X_{si}X_{rj} \text{ if } i < j \text{ and } r < s
\]
\[
\sum_{\sigma \in S_n} (-q)^{-l(\sigma)}X_{1\sigma(1)} \cdots X_{n\sigma(n)} = 1
\]

$l(\sigma)$ denotes the length of a permutation $\sigma$ which is the minimal number of transpositions required to represent $\sigma$. $SL_q(n)$ has a structure of a Hopf algebra with the comultiplication $\Delta(X_{ij}) = \sum_k X_{ik} \otimes X_{kj}$ and the counity $\varepsilon(X_{ij}) = \delta_{ij}$. The definition of the antipode requires algebraic complements; the inquisitive reader may refer to [14].

If $q$ is an $m$-th root of unity the subalgebra $O$ generated by $X_{ij}^m$ is central. Furthermore, it is a Hopf subalgebra isomorphic to $\mathbb{C}[SL_n]$. Thus, following the construction of the present paper one gets a bundle of algebras on $SL_n(\mathbb{C})$. If $q$ is a primitive root and $m$ is odd then there are at most $(n!)^2$ non-isomorphic fiber algebras in this bundle. Subsets over which the fiber algebras are isomorphic are parametrized by pairs $(w_1, w_2) \in S_n \times S_n$ in [4]. We think of $S_n$ as a subset of $SL_n(\mathbb{C})$ using a splitting of the map $N \rightarrow S_n$ where $N$ is the normalizer of the subgroup of diagonal matrices. If $B_+$ and $B_-$ are the subgroups of upper and lower triangular matrices then the subset corresponding to $(w_1, w_2)$ is $(B_+w_1B_+) \cap (B_-w_2B_-)$. 

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3.5 Subvarieties

Restricting the bundle of algebras to a locally closed subset of Spec $O$ provides an example of a Hopf-Galois extension of not necessarily affine spaces. For instance, a restriction to nilpotent orbits of a modular semisimple Lie algebra of the bundle in Example 3.2 has been studied in [6].

3.6 Quantum commutative algebras.

The following idea has been utilized in [1] to get examples of central Hopf-Galois extensions. Let $(H, \langle \cdot, \cdot \rangle)$ be a finite-dimensional coquasitriangular Hopf algebra. We consider an $H$-Galois extension $U \supseteq O$ such that $U$ is quantum commutative; i.e. $xy = y_0x_0\langle x_1, y_1 \rangle$ for each $x, y \in U$. These conditions imply that the subalgebra $O$ is central. Indeed, $\langle 1, h \rangle = \varepsilon(h)$ for every $h \in H$ [12, 10.1.8]. Given $x \in U, y \in U$, we have $xy = y_0x_0\langle 1, y_1 \rangle = yx$.

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Address
Department of Mathematics & Statistics,
LGRT, UMass, Amherst, MA, 01003.
E-mail: rumynin@math.umass.edu