LOG MINIMAL MODEL PROGRAM FOR $\overline{M}_g$ : THE SECOND FLIP

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Abstract. We prove an existence theorem for good moduli spaces, and use it to construct the second flip in the log minimal model program for $\overline{M}_g$. In fact, our methods give a uniform, self-contained construction of the first three steps of the log minimal model program for $\overline{M}_g$ and $\overline{M}_{g,n}$.

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1. Introduction

In an effort to understand the canonical model of $\overline{M}_g$, Hassett and Keel introduced the log minimal model program (log MMP) for $\overline{M}_g$. For any $\alpha \in \mathbb{Q} \cap [0, 1]$ such that $K_{\overline{M}_g} + \alpha \delta$ is big, Hassett defined

$$\overline{M}_g(\alpha) := \text{Proj} \bigoplus_{m \geq 0} H^0(\overline{M}_g, [m(K_{\overline{M}_g} + \alpha \delta)])$$

and asked whether the spaces $\overline{M}_g(\alpha)$ admit a modular interpretation [Has05]. In [HH09, HH13], Hassett and Hyeon carried out the first two steps of this program by showing that:

$$\overline{M}_g(\alpha) = \begin{cases} 
\overline{M}_g & \text{if } \alpha \in (9/11, 1] \\
\overline{M}^{ps}_g & \text{if } \alpha \in (7/10, 9/11] \\
\overline{M}_g^c & \text{if } \alpha = 7/10 \\
\overline{M}_g^h & \text{if } \alpha \in (2/3 - \epsilon, 7/10) 
\end{cases}$$

where $\overline{M}^{ps}_g$ is the moduli space of pseudostable curves (see [Sch91]), and $\overline{M}^c_g$ and $\overline{M}^h_g$ are the moduli spaces of c-semistable and h-semistable curves respectively (see [HH09, HH13]). Additional steps of the log MMP for $\overline{M}_g$ are known when $g \leq 5$ [Has05, HL05b, HL10a, Fed12, CMJL12a, CMJL12b, FS13]. In these works, new projective moduli spaces of curves are constructed using Geometric Invariant Theory (GIT). Indeed, one of the most appealing features of the Hassett-Keel program is the way that it ties together different compactifications of $M_g$ obtained by varying the parameters implicit in Gieseker and Mumford’s classical GIT construction of $M_g$ [Mum65, Gie82]. We refer the reader to [Mor09] for detailed discussion of these modified GIT constructions.

In this paper, we develop new techniques for constructing moduli spaces without GIT and apply these to construct the third step of the log MMP, a flip replacing Weierstrass genus two tails by ramphoid cusps. In fact, we give a uniform construction of the first three steps of the log MMP for $\overline{M}_g$, as well as an analogous program for $\overline{M}_{g,n}$. To motivate our approach, let us recall the three-step procedure used to construct $\overline{M}_g$ intrinsically.

1. Prove that the functor of stable curves is a proper, Deligne-Mumford stack $\overline{M}_g$ [DM69].
2. Use the Keel-Mori theorem to show that $\overline{M}_g$ admits a coarse moduli space $\overline{M}_g \to M_g$ [KM97].
3. Prove that a tautological line bundle on $\overline{M}_g$ descends to an ample line bundle on $\overline{M}_g$ [Kol90].

This is now the standard procedure for constructing projective moduli spaces in algebraic geometry. It is indispensable in cases where a global quotient presentation for the relevant moduli problem is not available, or where the associated stability analysis is intractable, and there are good reasons to expect both these issues to arise in further stages of the log
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Unfortunately, this procedure cannot be used to construct the log canonical models $\overline{M}_g(\alpha)$, because the associated moduli stacks $\overline{M}_g(\alpha)$ may include curves with infinite automorphism groups. In other words, the stacks $\overline{M}_g(\alpha)$ may be non-separated and therefore cannot possess a coarse moduli space. The correct fix is to replace the notion of a coarse moduli space by a good moduli space, as defined and developed by Alper [Alp13, Alp12, Alp10b, Alp10a].

In this paper, we develop an intrinsic technique to prove the existence of good moduli spaces without GIT. We prove a general existence theorem (Theorem 4.1) which can be viewed as a generalization of the Keel-Mori theorem [KM97]. This allows us to carry out a modified version of the standard three-step procedure in order to construct moduli interpretations for the log canonical models

$$\overline{M}_{g,n}(\alpha) := \text{Proj} \bigoplus_{m \geq 0} H^0(\overline{M}_{g,n}, [m(K_{\overline{M}_{g,n}} + \alpha \delta + (1 - \alpha)\psi)])$$

Specifically, for $\alpha > 2/3 - \epsilon$, where $\epsilon > 0$ is fixed sufficiently small, we

1. Construct an algebraic stack $\overline{M}_{g,n}(\alpha)$ of $\alpha$-stable curves (Theorem 2.10).
2. Construct a good moduli space $\overline{M}_{g,n}(\alpha) \rightarrow X$ (Theorem 4.27).
3. Show that $K_{\overline{M}_{g,n}(\alpha)} + \alpha \delta + (1 - \alpha)\psi$ on $\overline{M}_{g,n}(\alpha)$ descends to an ample $\mathbb{Q}$-line bundle on $X$, and conclude that $X \simeq \overline{M}_{g,n}(\alpha)$ (Theorem 5.1).

In sum, we obtain the following result.

**Main Theorem.** There exists a diagram

\[
\begin{array}{c}
\overline{M}_{g,n} \xrightarrow{j_1} \overline{M}_{g,n}(\alpha) \xleftarrow{i_1} \overline{M}_{g,n}(\alpha) \xrightarrow{j_2} \overline{M}_{g,n}(\alpha-\epsilon) \xleftarrow{i_2} \overline{M}_{g,n}(\alpha-\epsilon) \xrightarrow{j_3} \overline{M}_{g,n}(\alpha) \xleftarrow{i_3} \overline{M}_{g,n}(\alpha) \\
\downarrow \phi_1 \downarrow \phi_2 \downarrow \phi_3 \downarrow \phi_4 \downarrow \phi_5 \\
\overline{M}_{g,n} \xrightarrow{j_1} \overline{M}_{g,n}(\alpha) \xleftarrow{i_1} \overline{M}_{g,n}(\alpha-\epsilon) \xrightarrow{j_2} \overline{M}_{g,n}(\alpha-\epsilon) \xleftarrow{i_2} \overline{M}_{g,n}(\alpha-\epsilon) \xrightarrow{j_3} \overline{M}_{g,n}(\alpha) \xleftarrow{i_3} \overline{M}_{g,n}(\alpha) \\
\end{array}
\]

where:

1. $\overline{M}_{g,n}(\alpha)$ is the moduli stack of $\alpha$-stable curves, and $i_k^+, i_k^-$ are open immersions of algebraic stacks.
2. The morphisms $\phi_1, \phi_5^-$ are good moduli spaces.
3. The morphisms $j_k^+, j_k^-$ are the projective morphisms induced by $i_k^+, i_k^-$. When $n = 0$, they constitute the steps of the log minimal model program for $\overline{M}_g$. In particular, $j_1^+$ is the first contraction, $j_1^-$ is an isomorphism, $(j_2^+, j_2^-)$ is the first flip, and $(j_3^+, j_3^-)$ is the second flip.
Remark 1.1. Note that the natural divisor for scaling in the pointed case is $K_{\mathcal{X}_{g,n}} + \alpha \delta + (1 - \alpha)\psi = 13\lambda - (2 - \alpha)(\delta - \psi)$ rather than $K_{\mathcal{X}_{g,n}} + \alpha \delta$; see [Smy11b] for a discussion of this point.

Remark 1.2. The parameter $\alpha$ passes through three critical values, namely $9/11, 7/10$ and $2/3$. In the open intervals $(1, 9/11), (9/11, 7/10), (7/10, 2/3)$ and $(2/3, 2/3 - \varepsilon)$, the definition of $\alpha$-stability does not change, and consequently neither do $\overline{M}_g(\alpha)$ or $\overline{M}_g(\alpha)$.

Remark 1.3. The theorem is degenerate in several special cases: For $(g, n) = (1, 1), (1, 2), (2, 0)$, the divisor $K_{\mathcal{X}_{g,n}} + \alpha \delta + (1 - \alpha)\psi$ hits the edge of the effective cone at $9/11, 7/10$, and $2/3$, respectively, and hence the diagram should be taken to terminate at these critical values. Furthermore, when $g = 1$ and $n \geq 3$ or $(g, n) = (3, 0), (3, 1)$, $\alpha$-stability does not change at the threshold value $\alpha = 2/3$, so the morphisms $(i_3^\ast, i_3^{-})$ and $(j_3^\ast, j_3^{-})$ are isomorphisms. Finally, for $(g, n) = (2, 1)$, $j_3^\ast$ is a divisorial contraction and $j_3^{-}$ is an isomorphism.

Remark 1.4. As mentioned above, when $n = 0$ and $\alpha > 7/10 - \varepsilon$, these spaces have been constructed using GIT. In these cases, our definition of $\alpha$-stability agrees with the semistability notions studied in the work of Schubert, Hassett, Hyeon, and Morrison [Sch01, HH09, HH13, HM10].

The key observation underlying our proof of the main theorem is that at each critical value $\alpha_c \in \{9/11, 7/10, 2/3\}$, the inclusions

$$\overline{M}_{g,n}(\alpha_c + \varepsilon) \hookrightarrow \overline{M}_{g,n}(\alpha_c) \hookleftarrow \overline{M}_{g,n}(\alpha_c - \varepsilon)$$

can be locally modeled by an intrinsic variation of GIT problem (Theorem 3.9). It is this feature of the geometry which enables us to verify the hypotheses of Theorem 4.1. We axiomatize this connection between local VGIT and the existence of a good moduli spaces in Theorem 4.2. Roughly speaking, Theorem 4.2 says that if $\mathcal{X}$ is an algebraic stack with a pair of open immersions $\mathcal{X}^+ \hookrightarrow \mathcal{X} \hookleftarrow \mathcal{X}^-$ which can be locally modeled by a VGIT problem, and if the open substack $\mathcal{X}^+$ and the two closed substacks $\mathcal{X}^- \setminus \mathcal{X}^+$ and $\mathcal{X}^+ \setminus \mathcal{X}^-$ each admit good moduli spaces, then $\mathcal{X}$ admits a good moduli space. This paves the way for an inductive construction of good moduli spaces for the stacks $\overline{M}_{g,n}(\alpha)$.

Let us conclude by briefly describing the geometry of the second flip. At $\alpha = 2/3$, the locus of curves with a genus 2 Weierstrass tail (i.e., a genus 2 subcurve nodally attached to the rest of the curve at a Weierstrass point) is flipped to the locus of curves with a ramphoid cusp $(y^2 = x^5)$. See Figure 1. The fibers of $j_3^+$ correspond to varying moduli of Weierstrass tails, while the fibers of $j_3^-$ correspond to varying moduli of ramphoid cusped cripings. Moreover, if $(K, p)$ is a fixed curve of genus $g - 2$, all curves obtained by attaching a Weierstrass genus 2 tail at $p$ or imposing a ramphoid cusp at $p$ are identified in $\overline{M}_g(2/3)$. This can be seen on the level of stacks since, in $\overline{M}_g(2/3)$, all such curves admit an isotrivial specialization to the curve $C_0$, obtained by attaching a rational ramphoid cuspidal tail to $\mathcal{C}$ at $p$. See Figure 2.
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Weierstrass point

Figure 1. Curves with a nodally attached genus 2 Weierstrass tail are flipped to curves with a ramphoid cuspidal ($y^2 = x^5$) singularity.

Figure 2. The curve $C_0$ is the nodal union of a genus $g - 2$ curve $K$ and a rational ramphoid cuspidal tail. All curves obtained by either attaching a Weierstrass genus 2 tail to $K$ at $p$ or imposing a ramphoid cusp on $K$ at $p$ isotrivially specialize to $C_0$. Observe that $\text{Aut}(C_0)$ is not finite.

Outline of the paper. In Section 2, we define the notion of $\alpha$-stability for $n$-pointed curves and prove that they are deformation open conditions. We conclude that $\overline{M}_{g,n}(\alpha)$, the stack of $n$-pointed $\alpha$-stable curves of genus $g$, is algebraic. We also characterize the closed points of $\overline{M}_{g,n}(\alpha_c)$ for each critical value $\alpha_c$. In Section 3, we develop the machinery of local quotient presentations and local variation of GIT, and compute the variation of GIT chambers associated to closed points in $\overline{M}_{g,n}(\alpha_c)$ for each critical value $\alpha_c$. In particular, we show that the inclusions $\overline{M}_{g,n}(\alpha_c + \epsilon) \hookrightarrow \overline{M}_{g,n}(\alpha) \hookrightarrow \overline{M}_{g,n}(\alpha_c - \epsilon)$ are cut out by these chambers. In Section 4, we prove three existence theorems for good moduli spaces, and apply these to give an inductive proof that the stacks $\overline{M}_{g,n}(\alpha)$ admit good moduli spaces. In Section 5, we give a direct proof that the divisor $K_{\overline{M}_{g,n}(\alpha_c)} + \alpha_c \delta + (1 - \alpha_c) \psi$ is nef on $\overline{M}_{g,n}(\alpha_c + \epsilon)$ for each critical value $\alpha_c$, and use this to show that the good moduli spaces of $\overline{M}_{g,n}(\alpha)$ are the corresponding log canonical models $\overline{M}_{g,n}(\alpha)$.

Notation. We work over a fixed algebraically closed field $\mathbb{C}$ of characteristic zero. An $n$-pointed curve $(C, \{p_i\}_{i=1}^n)$ is a connected, reduced, proper 1-dimensional $\mathbb{C}$-scheme $C$ with $n$ distinct smooth marked points $p_i \in C$. A curve $C$ has an $A_k$-singularity at a point $p \in C$ if $\mathcal{O}_{C,p} \cong \mathbb{C}[[x,y]]/(y^2 - x^{k+1})$. An $A_1$- (resp. $A_2$, $A_3$, $A_4$) singularity is also called a node.
(resp. cusp, tacnode, ramphoid cusp). We use the notation $\Delta = \text{Spec } R$ and $\Delta^* = \text{Spec } K$, where $R$ is a discrete valuation ring with fraction field $K$; we set 0, $\eta$ and $\overline{\eta}$ to be the closed point, the generic point and the geometric generic point respectively of $\Delta$.

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2. $\alpha$-stability

In this section, we define $\alpha$-stability (Definition 2.9) and show that it is an open condition (Theorem 2.10). We conclude that $\overline{M}_{g,n}(\alpha)$, the stack of $n$-pointed $\alpha$-stable curves of genus $g$, is an algebraic stack of finite type over $\mathbb{C}$. Moreover, we provide a complete characterization of the closed points of $\overline{M}_{g,n}(\alpha_c)$ for $\alpha_c \in \{2/3, 7/10, 9/11\}$ (Theorem 2.26).

2.1. Definition of $\alpha$-stability. The basic idea is to modify Deligne-Mumford stability by designating certain curve singularities ‘stable,’ and certain subcurves ‘unstable.’ We begin by defining the unstable subcurves associated to the first three steps of the log MMP.

Definition 2.1 (Tails and Bridges).

(1) An elliptic tail is a 1-pointed curve $(E, q)$ of arithmetic genus 1 which admits a finite, surjective, degree two map $\phi: E \to \mathbb{P}^1$ ramified at $q$.

(2) An elliptic bridge is a 2-pointed curve $(E, q_1, q_2)$ of arithmetic genus 1 which admits a finite, surjective, degree two map $\phi: E \to \mathbb{P}^1$ such that $\pi^{-1}(\{\infty\}) = \{q_1 + q_2\}$.

(3) A Weierstrass genus 2 tail (or simply Weierstrass tail) is a 1-pointed curve $(E, q)$ of arithmetic genus 2 which admits a finite, surjective, degree two map $\phi: E \to \mathbb{P}^1$ ramified at $q$.

![Figure 3. An elliptic tail, elliptic bridge, and Weierstrass tail.](image-url)
Remark 2.2. If \((E, q)\) is an elliptic or Weierstrass tail, then \(E\) is irreducible. If \((E, q_1, q_2)\) is an elliptic bridge, then \(E\) is irreducible or \(E\) is the union of two smooth rational curves.

Unfortunately, we cannot describe our \(\alpha\)-stability conditions purely in terms of tails and bridges. As seen in [HH13], one extra layer of combinatorial description is needed, and this is encapsulated in our definition of chains.

Definition 2.3 (Chains). An elliptic chain of length \(r\) is a 2-pointed curve \((E, p_1, p_2)\) which admits a finite, surjective morphism

\[
\gamma: \prod_{i=1}^{r} (E_i, q_{2i-1}, q_{2i}) \hookrightarrow (E, p_1, p_2)
\]

such that:

1. \((E_i, q_{2i-1}, q_{2i})\) is an elliptic bridge for \(i = 1, \ldots, r\).
2. \(\gamma\) is an isomorphism when restricted to \(E_i \setminus \{q_{2i-1}, q_{2i}\}\) for \(i = 1, \ldots, r\).
3. \(\gamma(q_{2i}) = \gamma(q_{2i+1})\) is an \(A_3\)-singularity for \(i = 1, \ldots, r - 1\).
4. \(\gamma(q_1) = p_1, \gamma(q_{2r}) = p_2\).

Remark 2.4. An elliptic chain of length one is simply an elliptic bridge.

Remark 2.5. Our definition of an elliptic chain is similar, but not identical to, the definition of an open tacnodal elliptic chain appearing in [HH13, Definition 2.4]. Whereas open tacnodal elliptic chains are built out of arbitrary curves of arithmetic genus one, our elliptic chains are built out of elliptic bridges. Nevertheless, it is easy to see that our definition of \((7/10 - \epsilon)\)-stability agrees with the definition of \(h\)-semistability in [HH13, Definition 2.7].

Figure 4. An elliptic chain of length 4.

When describing tails and chains as subcurves of a higher-genus curve, it is important to specify the singularities along which the tail or chain is attached. This motivates the following pair of definitions.

Definition 2.6 (Gluing morphism). If \((E, \{q_i\}_{i=1}^m)\) is an \(m\)-pointed curve and \((C, \{p_i\}_{i=1}^n)\) is an \(n\)-pointed curve, a gluing morphism \(\gamma: (E, \{q_i\}_{i=1}^m) \hookrightarrow (C, \{p_i\}_{i=1}^n)\) is a finite morphism \(E \rightarrow C\), which is an open immersion when restricted to \(E \setminus \{q_1, \ldots, q_m\}\). We do not require the points \(\{\gamma(q_i)\}_{i=1}^m\) to be distinct.

Remark 2.7. If \(\gamma: (E, \{q_i\}_{i=1}^m) \hookrightarrow (C, \{p_i\}_{i=1}^n)\) is a gluing morphism, then locally around \(\gamma(q_j)\), \(\gamma\) is simply the normalization of one branch of \(\gamma(q_j) \in C\).
Definition 2.8 (Tails and Chains with Attaching Data). Let \((C, \{p_i\}_{i=1}^n)\) be an \(n\)-pointed curve. We say that

1. \((C, \{p_i\}_{i=1}^n)\) has an \(A_k\)-attached elliptic (resp. Weierstrass) tail if there exists a gluing morphism \(\gamma: (E, q) \hookrightarrow (C, \{p_i\}_{i=1}^n)\) such that
   a. \((E, q)\) is an elliptic (resp. Weierstrass) tail.
   b. \(\gamma(q)\) is an \(A_k\)-singularity of \(C\), or \(k = 1\) and \(\gamma(q)\) is a marked point.

2. \((C, \{p_i\}_{i=1}^n)\) has an \(A_{k_1}/A_{k_2}\)-attached elliptic chain if there exists a gluing morphism \(\gamma: (E, q_1, q_2) \hookrightarrow (C, \{p_i\}_{i=1}^n)\) such that
   a. \((E, q_1, q_2)\) is an elliptic chain.
   b. \(\gamma(q_i)\) is an \(A_{k_i}\)-singularity of \(C\), or \(k_i = 1\) and \(\gamma(q_i)\) is a marked point \((i = 1, 2)\).

Note that this definition entails an essential, systematic abuse of notation: when we say that a curve has an \(A_1\)-attached tail or chain, we always allow the endpoints to be marked points.

We can now define \(\alpha\)-stability.

Definition 2.9 (\(\alpha\)-stability). For \(\alpha \in (2/3-\epsilon, 1]\), we say that an \(n\)-pointed curve \((C, \{p_i\}_{i=1}^n)\) is \(\alpha\)-stable if \(\omega_C(\Sigma_{i=1}^n p_i)\) is ample and:

For \(\alpha \in (9/11, 1)\): \(C\) has only \(A_1\)-singularities.
For \(\alpha = 9/11\): \(C\) has only \(A_1, A_2\)-singularities.
For \(\alpha \in (7/10, 9/11)\): \(C\) has only \(A_1, A_2\)-singularities, and does not contain:
   - \(A_1\)-attached elliptic tails.
For \(\alpha = 7/10\): \(C\) has only \(A_1, A_2, A_3\)-singularities, and does not contain:
   - \(A_1, A_3\)-attached elliptic tails.
   - \(A_4\)-attached elliptic tails.
For \(\alpha \in (2/3, 7/10)\): \(C\) has only \(A_1, A_2, A_3\)-singularities, and does not contain:
   - \(A_1, A_2\)-attached elliptic tails,
   - \(A_1/A_1\)-attached elliptic chains.
For \(\alpha = 2/3\): \(C\) has only \(A_1, A_2, A_3, A_4\)-singularities, and does not contain:
   - \(A_1, A_2, A_3, A_4\)-attached elliptic tails,
   - \(A_1/A_1, A_1/A_4, A_1/A_4\)-attached elliptic chains.
For \(\alpha \in (2/3-\epsilon, 2/3)\): \(C\) has only \(A_2, A_3, A_4\)-singularities, and does not contain:
   - \(A_1, A_2, A_3, A_4\)-attached elliptic tails,
   - \(A_1/A_1, A_1/A_4, A_1/A_4\)-attached elliptic chains,
   - \(A_4\)-attached Weierstrass genus 2 tails.

A family of \(\alpha\)-stable curve is a flat and proper family whose geometric fiber are \(\alpha\)-stable. We let \(\mathcal{M}_{g,n}(\alpha)\) denote the stack of \(n\)-pointed \(\alpha\)-stable curves of arithmetic genus \(g\).

It will be useful to have a uniform way of referring to the singularities allowed and the subcurves excluded at each stage of the log MMP. Thus, for any \(\alpha \in (2/3-\epsilon, 1]\), we use the term \(\alpha\)-stable singularity to refer to any allowed singularity at the given value of \(\alpha\). For example, a \(\frac{7}{10}\)-stable singularity is a node, cusp, or tacnode. Similarly, we use the term \(\alpha\)-unstable subcurve to refer to any excluded subcurve at the given value of \(\alpha\). For
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Figure 5. Curve (A) has an $A_3$-attached elliptic tail. It is never $\alpha$-stable. Curve (B) has an $A_1$-attached Weierstrass tail. It is $\alpha$-stable for $\alpha \geq 2/3$. Curve (C) has an $A_1/A_1$-attached elliptic chain of length 2. It is $\alpha$-stable for $\alpha \geq 7/10$. Curve (D) has an $A_1/A_4$-attached elliptic bridge. It is never $\alpha$-stable.

For example, a $7/10$-unstable subcurve is simply an $A_1$ or $A_3$-attached elliptic tail. With this terminology, we may say that a curve is $\alpha$-stable if it has only $\alpha$-stable singularities and no $\alpha$-unstable subcurves. Furthermore, if $\alpha_c \in \{9/11, 7/10, 2/3\}$ is a critical value, we use the term $\alpha_c$-critical singularity to refer to the newly allowed singularity at $\alpha = \alpha_c$ and $\alpha_c$-critical subcurve to refer to the newly disallowed subcurves at $\alpha = \alpha_c - \epsilon$. Thus, a $7/10$-critical singularity is a tacnode, and a $7/10$-critical subcurve is an elliptic chain with $A_1/A_1$-attaching.

2.2. Deformation openness. In this section, we prove

**Theorem 2.10.** For $\alpha \in (2/3-\epsilon, 1]$, the stack $\overline{M}_{g,n}(\alpha)$ is algebraic and finite-type over Spec $\mathbb{C}$. Furthermore, for each critical value $\alpha_c \in \{2/3, 7/10, 9/11\}$, we have open immersions:

$$\overline{M}_{g,n}(\alpha_c + \epsilon) \hookrightarrow \overline{M}_{g,n}(\alpha_c) \hookleftarrow \overline{M}_{g,n}(\alpha_c - \epsilon).$$

Let $U_{g,n}(A_\infty)$ be the stack of flat, proper families of curves $(\pi: \mathcal{C} \to T, \{\sigma_i\}_{i=1}^n)$, where the sections $\{\sigma_i\}_{i=1}^n$ are distinct and lie in the smooth locus of $\pi$, $\omega_{\mathcal{C}/T}(\Sigma_{i=1}^n \sigma_i)$ is relatively ample, and the geometric fibers of $\pi$ are $n$-pointed curves of arithmetic genus $g$ with only $A$-singularities. Since $U_{g,n}(A_\infty)$ parameterizes canonically polarized curves, $U_{g,n}(A_\infty)$ is algebraic and finite type over $\mathbb{C}$. Let $U_{g,n}(A_\ell) \subset U_{g,n}(A_\infty)$ be the open substack parameterizing curves with at worst $A_1, \ldots, A_\ell$ singularities. We will show that each $\overline{M}_{g,n}(\alpha)$ can be obtained from a suitable $U_{g,n}(A_\ell)$ by excising a finite collection of closed substacks.
Definition 2.11. We let $T^{A_k}, B^{A_{k_1}/A_{k_2}}, W^{A_k}$ denote the following constructible subsets of $U_{g,n}(A_\infty)$:

- $T^{A_k} :=$ Locus of curves containing an $A_k$-attached elliptic tail.
- $B^{A_{k_1}/A_{k_2}} :=$ Locus of curves containing an $A_{k_1}/A_{k_2}$-attached elliptic chain.
- $W^{A_k} :=$ Locus of curves containing an $A_k$-attached Weierstrass tail.

With this notation, we can describe our stability conditions (set-theoretically) as follows:

$$\overline{M}_{g,n}(9/11+\epsilon) = U_{g,n}(A_1)$$
$$\overline{M}_{g,n}(9/11) = U_{g,n}(A_2)$$
$$\overline{M}_{g,n}(9/11-\epsilon) = \overline{M}_{g,n}(9/11) - T^{A_1}$$
$$\overline{M}_{g,n}(7/10) = U_{g,n}(A_3) - \bigcup_{i \in \{1,3\}} T^{A_i}$$
$$\overline{M}_{g,n}(7/10-\epsilon) = \overline{M}_{g,n}(7/10) - B^{A_1/A_1}$$
$$\overline{M}_{g,n}(2/3) = U_{g,n}(A_4) - \bigcup_{i \in \{1,3,4\}} T^{A_i} - \bigcup_{i,j \in \{1,4\}} B^{A_i/A_j}$$
$$\overline{M}_{g,n}(2/3-\epsilon) = \overline{M}_{g,n}(2/3) - W^{A_1}$$

Here, when we write $\overline{M}_{g,n}(9/11) - T^{A_1}$, we mean of course $\overline{M}_{g,n}(9/11) - (T^{A_1} \cap \overline{M}_{g,n}(9/11))$, and similarly for each of the subsequent set-theoretic subtractions.

Now we must show that at each stage the collection of loci $T^{A_k}$ and $B^{A_{k_1}/A_{k_2}}$ that we excise is closed. We break this analysis into two steps: In Lemma 2.16 and Lemma 2.17, we analyze degenerations of $\alpha$-unstable curves, and in Corollaries 2.14 and 2.15, we analyze how the attaching singularities of an $\alpha$-unstable subcurve may degenerate. We combine these results to prove the desired statement in Proposition 2.18.

Definition 2.12. Suppose $C$ is a curve with at worst $A$-singularities. An $A_{2k+1}$-singularity of $C$ lying on two distinct irreducible components is called an outer $A_{2k+1}$-singularity. All other singularities of $C$ are called inner.

Suppose $C \to \Delta$ is a family of curves with at worst $A$-singularities, where $\Delta$ is the spectrum of a DVR. Denote by $C_\pi$ the geometric generic fiber and by $C_0$ the central fiber. We are interested in how the singularities of $C_\pi$ degenerate in $C_0$. Recall from the deformation theory of $A_k$ singularities that an $A_k$-singularity can deform to a collection of $\{A_{k_1}, \ldots, A_{k_r}\}$ singularities if and only if $\sum_{i=1}^r (k_i + 1) \leq k + 1$. In the following proposition, we refine this result for outer singularities.

Proposition 2.13. The limit of an outer singularity is necessarily an outer singularity. Moreover, if an outer $A_{2k+1}$-singularity in $C_0$ is a limit of an $A_{2m+1}$-singularity in $C_\pi$ for some $m < k$, then it must be the limit of a set of outer singularities $\{A_{2m_1+1}, A_{2m_2+1}, \ldots, A_{2m_r+1}\}$.
of $C_\eta$ satisfying $\sum_{i=1}^{r} (2m_i + 2) = 2k + 2$, and necessarily joining the same two components of $C_\eta$. In addition, there exists a simultaneous normalization of $C$ along these generic $A_{2m_i+1}$-singularities.

**Proof.** An outer singularity of $C_\eta$ lies on two irreducible components of $C_\eta$. Therefore, a limit of an outer singularity of $C_\eta$ is necessarily a singularity of $C_0$ lying on two irreducible components of $C_0$, thus an outer singularity of $C_0$.

Suppose that an outer $A_{2k+1}$-singularity $p \in C_0$ is a limit of outer $A_{2m+1}$-singularity of $C_\eta$.

By the deformation theory of $A_{2k+1}$, the local equation of $C$ around $p$ is, after a faithfully flat base change,

$$y^2 = (x - a(t))^{2m+2} \prod_{i=2}^{r} (x - a_i(t))^{k_i}, \quad \text{where } 2m + 2 + \sum_{i=2}^{r} = 2k + 2,$$

with $x = a(t)$ being the equation of the generic $A_{2m+1}$-singularity. If some $k_i$ is odd, then the general fiber of this family is irreducible, contradicting the assumption that the generic $A_{2m+1}$ singularity lies on two irreducible components of $C$. Therefore, all $k_i$’s are even and the local equation of $C$ around $p$ is

$$y^2 = \prod_{i=1}^{r} (x - a_i(t))^{2m+2}. \quad (2.1)$$

It follows that for $m_1 = m, m_2, \ldots, m_r$ satisfying $\sum_{i=1}^{r} (2m_i + 2) = 2k + 2$, the geometric generic fiber $C_\eta$ has $\{A_{2m_1+1}, A_{2m_2+1}, \ldots, A_{2m_r+1}\}$ outer singularities which join the same two irreducible components of $C_\eta$ and whose limit is the $A_{2k+1}$ singularity of $C_0$. Clearly, the normalization of the family defined by (2.1) exists and is a union of two smooth components, flat over $\Delta$. \hfill \Box

Using the previous proposition, we can understand how the attaching singularities of a subcurve may degenerate.

**Corollary 2.14.** Suppose $(\pi: C \to \Delta, \{\sigma_i\}_{i=1}^{n})$ is a family of curves in $\mathcal{U}_{g,n}(A_{\infty})$. Suppose that $\tau$ is a section of $\pi$ such that $\tau(\eta) \in C_\eta$ is a disconnecting $A_{2k-1}$-singularity of the geometric generic fiber. Then $\tau(0) \in C_0$ is also a disconnecting $A_{2k-1}$-singularity.

**Proof.** This follows from Proposition 2.13. Indeed, since $\tau(\eta)$ is a disconnecting singularity it cannot collide with other singularities of $C_\eta$ in $C_0$. Therefore, $\tau(0)$ remains $A_{2k-1}$-singularity. The normalization of $C$ along $\tau$ separates $C$ into two connected components, so $\tau(0)$ is disconnecting. \hfill \Box

**Corollary 2.15.** Suppose $(\pi: C \to \Delta, \{\sigma_i\}_{i=1}^{n})$ is a family of curves in $\mathcal{U}_{g,n}(A_{\infty})$. Suppose that $\tau_1$, $\tau_2$ are sections of $\pi$ such that $\tau_1(\eta), \tau_2(\eta) \in C_\eta$ are $A_{2k_1-1}$ and $A_{2k_2-1}$-singularities of the geometric generic fiber. Suppose also that the normalization of $C_\eta$ along $\tau_1(\eta) \cup \tau_2(\eta)$ consists of two connected components, while the normalization of $C_\eta$ along either $\tau_1(\eta)$ or $\tau_2(\eta)$ individually is connected. Then we have two possible cases for the limits $\tau_1(0), \tau_2(0)$. 

\(\tau_1(0), \tau_2(0)\) are distinct \(A_{2k_1-1}\) and \(A_{2k_2-1}\)-singularities, or
\(\tau_1(0) = \tau_2(0)\) is an \(A_{2k_1+2k_2-1}\)-singularity.

**Proof.** This follows immediately from Proposition 2.13. \(\square\)

**Lemma 2.16** (Limits of tails and bridges).

1. Let \((H \to \Delta, \tau_1)\) be a family in \(\mathcal{U}_{1,1}(A_{\infty})\) whose generic fiber is an elliptic tail. Then the special fiber \((H, p)\) is an elliptic tail.

2. Let \((H \to \Delta, \tau_1, \tau_2)\) be a family in \(\mathcal{U}_{1,2}(A_{\infty})\) whose generic fiber is an elliptic bridge. Then the special fiber \((H, p_1, p_2)\) satisfies one of the following conditions:
   - \((H, p_1, p_2)\) is an elliptic bridge.
   - \((H, p_1, p_2)\) contains an \(A_1\)-attached elliptic tail.

3. Let \((H \to \Delta, \tau_1)\) be a family in \(\mathcal{U}_{2,1}(A_{\infty})\) whose generic fiber is a Weierstrass tail. Then the special fiber \((H, p)\) satisfies one of the following conditions:
   - \((H, p)\) is a Weierstrass tail.
   - \((H, p)\) contains an \(A_1\) or \(A_3\)-attached elliptic tail, or an \(A_1/A_1\)-attached elliptic bridge.

**Proof.** (1) For every \((H, p) \in \mathcal{U}_{1,1}(\infty)\), the curve \(H\) is irreducible, and \(|2p|\) defines a degree 2 map to \(\mathbb{P}^1\) by Riemann-Roch. Hence \(\mathcal{U}_{1,1}(A_{\infty}) = \mathcal{T}^{A_1}\).

For (2), the special fiber \((H, p_1, p_2)\) is a curve of arithmetic genus 1 with \(\omega_H(p_1 + p_2)\) ample. Since \(\omega_H(p_1 + p_2)\) has degree 2, \(H\) has at most 2 components. The possible topological types of \(H\) are listed in the top row of Figure 6. We see immediately that any curve with one of the first three topological types is an elliptic bridge, while any curve with the last topological type contains an \(A_1\)-attached elliptic tail.

Finally, for (3), the special fiber \((H, p)\) is a curve of arithmetic genus 2 with \(\omega_H(p)\) ample and \(h^0(\omega_H(-2p)) \geq 1\) by semicontinuity. Since \(\omega_H(p)\) has degree three, \(H\) has at most three components, and the possible topological types of \(H\) are listed in the bottom three rows of Figure 6. One sees immediately that if \(H\) does not contain an \(A_1\) or \(A_3\)-attached elliptic tail or an \(A_1/A_1\)-attached elliptic bridge, there are only three possibilities for the topological type of \(H\): either \(H\) is irreducible or \(H\) has topological type \((A)\) or \((B)\). However, topological types \((A)\) and \((B)\) do not satisfy \(h^0(\omega_H(-2p)) \geq 1\). Finally, if \((H, p)\) is irreducible, then it must be a Weierstrass tail. Indeed, the linear equivalence \(\omega_H \sim 2p\) follows immediately from the corresponding linear equivalence on the general fiber. \(\square\)

**Lemma 2.17** (Limits of chains). Let \((H \to \Delta, \tau_1, \tau_2)\) be a family in \(\mathcal{U}_{2r-1,2}(A_{\infty})\) whose generic fiber is an elliptic chain of length \(r\). Then the special fiber satisfies one of the following conditions:

(a) \((H, p_1, p_2)\) contains an \(A_1/A_1\)-attached elliptic chain of length \(\leq r\).

(b) \((H, p_1, p_2)\) contains an \(A_1\)-attached elliptic tail.
Figure 6. Topological types of curves in $U_{1,2}(A_\infty)$ and $U_{2,1}(A_\infty)$. For convenience, we have suppressed the data of singularities internal to each component, and we record only the arithmetic genus of each component, and the singularities where two components meet (which are either nodes or tacnodes, as indicated by the picture). Components without a label have arithmetic genus zero.

Proof. We will assume $(H, p_1, p_2)$ contains no $A_1$-attached elliptic tails, and prove that (a) holds. By Lemma 2.16, this assumption implies that if $(E, q_1, q_2)$ is a genus one subcurve of $H$, nodally attached at $q_1$ and $q_2$, and $\omega_E(q_1 + q_2)$ is ample on $E$, then $(E, q_1, q_2)$ is an $A_1/A_1$-attached elliptic bridge.

To begin, let $\gamma_1, \ldots, \gamma_{r-1}$ be sections picking out the tacnodes in the general fiber at which the sequence of elliptic bridges are attached to each other. By Corollary 2.14, the limits $\gamma_1(0), \ldots, \gamma_{r-1}(0)$ remain tacnodes, so the normalization of $\phi: \tilde{\mathcal{H}} \to \mathcal{H}$ along $\gamma_1, \ldots, \gamma_{r-1}$ is well-defined and we obtain $r$ flat families of 2-pointed curves of arithmetic genus 1, i.e. we have

$$\tilde{\mathcal{H}} = \prod_{i=1}^{r}(E_i, \sigma_{2i-1}, \sigma_{2i}),$$

where $\sigma_1 := \tau_1$, $\sigma_{2r} := \tau_2$, and $\phi^{-1}(\gamma_i) = \{\sigma_{2i}, \sigma_{2i+1}\}$. The relative ampleness of $\omega_{\mathcal{H}/\Delta}(\tau_1 + \tau_2)$ implies

1. $\omega_{E_i}(p_1 + 2p_2), \omega_{E_r}(2p_{2r-1} + p_{2r})$ ample on $E_1, E_r$ respectively.
2. $\omega_{E_i}(2p_{2i-1} + 2p_{2i})$ ample on $E_i$ for $i = 2, \ldots, r - 1.$
It follows that for each $1 \leq i \leq r$, either $(E_i, p_{2i-1}, p_{2i})$ is an elliptic bridge or one of the following must hold:

(a) $(E_i, p_{2i-1}, p_{2i}) = ([P^1, p_{2i-1}, q_{2i-1}^2] \cup (E_i', q_{2i-1}, p_{2i})/(q_{2i-1}^2 \sim q_{2i-1}))$, where $(E_i', q_{2i-1}, p_{2i})$ is an elliptic bridge.

(b) $(E_i, p_{2i-1}, p_{2i}) = (E_i', q_{2i-1}, p_{2i})/(q_{2i}^2 \sim q_{2i}^2)$, where $(E_i', q_{2i-1}, p_{2i})$ is an elliptic bridge.

(c) $(E_i, p_{2i-1}, p_{2i}) = ([P^1, p_{2i-1}, q_{2i-1}^2] \cup (E_i', q_{2i-1}, q_{2i}) \cup (P^1, q_{2i}^2, p_{2i})/(q_{2i-1}^2 \sim q_{2i-1}, q_{2i} \sim q_{2i}^2)$, where $(E_i', q_{2i-1}, p_{2i})$ is an elliptic bridge.

In the cases (a), (b), (c) respectively, we say that $E_i$ sprouts on the left, right, or left and right. Note that if $E_1$ or $E_r$ sprouts at all, then $E_1$ or $E_r$ contains an $A_1/A_1$-attached elliptic bridge. Similarly, if $E_i$ sprouts on both the left and right ($2 \leq i \leq r - 1$), then $E_i$ contains an $A_1/A_1$-attached elliptic bridge. Thus, we may assume without loss of generality that $E_1$ and $E_r$ do not sprout and that $E_i$ $(2 \leq i \leq r - 1)$ sprouts on the left or right, but not both. We now observe that any collection $\{E_s, \ldots, E_{s+t}\}$ such that $E_s$ sprouts on the left (or $s = 0$), $E_{s+t}$ sprouts on the right (or $s + t = r$), and $E_k$ does not sprout for $s < k < s + t$, contains an $A_1/A_1$-attached elliptic chain. \hfill \Box

**Proposition 2.18.**

(1) $\mathcal{T}^{A_1} \cup \mathcal{T}^{A_k}$ is closed in $\mathcal{U}_{g,n}(A_\infty)$ for any odd $k$.

(2) $\mathcal{B}^{A_1/A_1}$ is closed in $\mathcal{U}_{g,n}(A_\infty) - \bigcup_{i \in \{1, 3\}} \mathcal{T}^{A_i}$.

(3) $\mathcal{B}^{A_k/A_k}$ and $\mathcal{B}^{A_1/A_k}$ are closed in $\mathcal{U}_{g,n}(A_k) - \mathcal{T}^{A_1} - \mathcal{B}^{A_1/A_1}$ for any even $k$.

(4) $\mathcal{W}^{A_k}$ is closed in $\mathcal{U}_{g,n}(A_k) - \bigcup_{i \in \{1, 3\}} \mathcal{T}^{A_i} - \mathcal{B}^{A_1/A_1}$ for any odd $k$.

**Proof.** The given loci are obviously constructible, so it suffices to show that they are closed under specialization.

For (1), let $\mathcal{C}/\Delta, \{\sigma_i\}_{i=1}^n$ be a family in $\mathcal{U}_{g,n}(A_\infty)$ whose generic fiber lies in $\mathcal{T}^{A_{2m+1}}$. Possibly after a finite base-change, let $\tau$ be the section picking out the attaching $A_{2m+1}$-singularity of the elliptic tail in the generic fiber. By Corollary 2.14, the limit $\tau(0)$ is also $A_{2m+1}$-singularity. Consider the normalization $\widetilde{C} \to \mathcal{C}$ along $\tau$. Let $\mathcal{H} \subseteq \widetilde{C}$ be the component whose generic fiber is an elliptic tail and let $\alpha$ be the preimage of $\tau$ on $\mathcal{H}$. Then $\omega_{\mathcal{H}}(m\alpha)$ is relatively ample. We conclude that either $\omega_{\mathcal{H}_0}(\alpha(0))$ is ample, or $\alpha(0)$ lies on a rational curve attached nodally to the rest of $\mathcal{H}_0$. In the former case, $(\mathcal{H}_0, \alpha(0))$ is an elliptic tail by Lemma 2.16, so $C_0$ contains an elliptic tail with $A_{2m+1}$-attaching, as desired. In the latter case, $\mathcal{H}_0$ contains an $A_1$-attached elliptic tail. We conclude that $C_0 \in \mathcal{T}^{A_{2m+1}} \cup \mathcal{T}^{A_1}$, as desired. The proof for (4) is analogous using Corollary 2.15.

For (2), let $\mathcal{C}/\Delta, \{\sigma_i\}_{i=1}^n$ be a family in $\mathcal{U}_{g,n}(A_\infty)$ whose generic fiber lies in $\mathcal{B}^{A_1/A_1}$. Possibly after a finite base change, let $\tau_1, \tau_2$ be the sections picking out the attaching nodes of an elliptic chain in the general fiber. By Proposition 2.13, $\tau_1(0)$ and $\tau_2(0)$ either remain distinct nodes, or, if the elliptic chain has length 1, can coalesce to form an outer $A_1$-singularity. In either case there exists a normalization of $\mathcal{C}$ along $\tau_1$ and $\tau_2$. Since $C_{\tau}$ becomes
obtained by excising closed substacks from $U$ separating $\tau$. In the latter case, $C_0$ has an elliptic component by Lemma 2.17. In the latter case, $C_0$ has an elliptic component $A_3$-attached to the rest of the curve. This implies that $C_0 \in \mathcal{T}_A^1 \cup \mathcal{T}_A^3$.

The proof of (3) is essentially identical to (2), making use of the observation that in $U_{g,n}(A_k)$, the limit of an $A_k$-singularity must be an $A_k$-singularity.

**Proof of Theorem 2.10.** Proposition 2.18 implies that $\overline{M}_{g,n}(9/11)$ and $\overline{M}_{g,n}(9/11-\epsilon)$ are obtained by excising closed substacks from $U_{g,n}(A_1)$, that $\overline{M}_{g,n}(7/10)$ and $\overline{M}_{g,n}(7/10-\epsilon)$ are obtained by excising closed substacks from $U_{g,n}(A_3)$, and that $\overline{M}_{g,n}(2/3)$ and $\overline{M}_{g,n}(2/3-\epsilon)$ are obtained by excising closed substacks from $U_{g,n}(A_4)$.

### 2.3. Properties of $\alpha$-stability.

In this section, we record several elementary properties of $\alpha$-stability that will be needed in subsequent arguments. Recall that if $(C, \{p_i\}_{i=1}^n)$ is a Deligne-Mumford stable curve and $q \in C$ is a node, then the pointed normalization $(\tilde{C}, \{p_i\}_{i=1}^n, q_1, q_2)$ is Deligne-Mumford stable. The same statement holds for $\alpha$-stable curves.

**Lemma 2.19.** Suppose $(C, \{p_i\}_{i=1}^n)$ is an $\alpha$-stable curve, and $q \in C$ is a node. Then the pointed normalization $(\tilde{C}, \{p_i\}_{i=1}^n, q_1, q_2)$ is $\alpha$-stable.

**Proof.** The follows from the definition of $\alpha$-stability.

Unfortunately, the converse of Lemma 2.19 is false. Nodally gluing two marked points of an $\alpha$-stable curve may fail to preserve $\alpha$-stability if the two marked are both on the same component, or both on rational components. The following lemma says that these are the only problems that can occur.

**Lemma 2.20.**

1. If $(\tilde{C}_1, \{p_i\}_{i=1}^n, q_1)$ and $(\tilde{C}_2, \{p_i\}_{i=1}^n, q_2)$ are $\alpha$-stable curves, then $(\tilde{C}_1, \{p_i\}_{i=1}^n, q_1) \cup (\tilde{C}_2, \{p_i\}_{i=1}^n, q_2)/(q_1 \sim q_2)$ is $\alpha$-stable.

2. If $(\tilde{C}, \{p_i\}_{i=1}^n, q_1, q_2)$ is an $\alpha$-stable curve, then $(\tilde{C}, \{p_i\}_{i=1}^n, q_1, q_2)/(q_1 \sim q_2)$ is $\alpha$-stable provided one of the following conditions hold:
   - $q_1$ and $q_2$ lie on disjoint irreducible components of $\tilde{C}$,
   - $q_1$ and $q_2$ lie on distinct irreducible components of $\tilde{C}$, and at least one of these components is not a smooth rational curve.

**Proof.** Let $C := (\tilde{C}, q_1, q_2)/(q_1 \sim q_2)$, and let $\phi : \tilde{C} \to C$ be the gluing morphism which identifies $q_1, q_2$ to a node $q \in C$. It suffices to show that if $E \subset C$ is an $\alpha$-unstable curve, then $\phi^{-1}(E)$ is an $\alpha$-unstable subcurve of $\tilde{C}$. The key observation is that any $\alpha$-unstable subcurve $E$ has the following property: If $E_1, E_2 \subset E$ are two distinct irreducible components of $E$, then the intersection $E_1 \cap E_2$ never consists of a single node. Furthermore, if one of
Suppose \((\Sigma, C, \omega)\) is that \((\Sigma, q)\) gluing Lemma 2.20 twice: First apply Lemma 2.20(1) to glue \((\Sigma, q)\), \((\Sigma, q)\), \((\Sigma, q)\), \((\Sigma, q)\), follow immediately from Lemma 2.20. For (2), one must apply Proof.

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attached elliptic bridges. From this observation, it follows that no \(\alpha\)-unstable \(E \subset C\) can contain both branches of \(q\). Indeed, the hypotheses of (1) and (2) imply that either the two branches of the node \(q \in C\) lie on distinct irreducible components whose intersection is precisely \(q\), or else that the two branches lie on distinct irreducible components, one of which is irrational. It remains to consider the case where \(E\) is disjoint from \(q\) or contains only one branch of \(q\).

If \(E \subset C\) is disjoint from \(q\), then \(\phi^{-1}\) is an isomorphism in a neighborhood of \(E\) and the statement is clear. If \(E \subset C\) contains only one branch of the node \(q\), then \(q\) must be an attaching point of \(E\). We may assume without loss of generality that \(E\) contains the branch labeled by \(q_1\). Now \(\phi^{-1}(E) \to E\) is an isomorphism away from \(q_1\) and sends \(q_1\) to the node \(q\). Since an \(\alpha\)-unstable curve with nodal attaching is also \(\alpha\)-unstable with marked point attaching, \(\phi^{-1}(E)\) is an \(\alpha\)-unstable subcurve of \(\tilde{C}\). \(\square\)

Our most frequent application of this lemma is to take an \(\alpha\)-stable subcurve and glue in an \(\alpha\)-critical subcurve in some way. We record this special case for future reference.

**Corollary 2.21.**

1. Suppose that \((C, \{p_i\}_{i=1}^n, q_1)\) is \(\frac{2}{n}\)-stable and \((E, q'_1)\) is an elliptic tail. Then \((\tilde{C}, C, \omega, E, \{p_i\}_{i=1}^n)/(q_1 \sim q'_1)\) is \(\frac{3}{11}\)-stable.

2. Suppose \((C, \{p_i\}_{i=1}^n, q_1, q_2)\) is \(\frac{7}{10}\)-stable and \((E, q'_1, q'_2)\) is an elliptic chain. Then \((\tilde{C}, C, \omega, E, \{p_i\}_{i=1}^n)/(q_1 \sim q'_1, q_2 \sim q'_2)\) is \(\frac{7}{10}\)-stable.

3. Suppose \((C_1, \{p_i\}_{i=1}^m, q_1)\) and \((C_2, \{p_i\}_{i=1}^{n-m}, q_2)\) are \(\frac{7}{10}\)-stable and \((E, q'_1, q'_2)\) is an elliptic chain. Then \((\tilde{C}_1 \cup C_2 \cup E, \{p_i\}_{i=1}^n)/(q_1 \sim q'_1, q_2 \sim q'_2)\) is \(\frac{7}{10}\)-stable.

4. Suppose \((C, \{p_i\}_{i=1}^n, q_1)\) is \(\frac{3}{10}\)-stable and \((E, q'_1, q'_2)\) is an elliptic chain. Then \((\tilde{C}, C, \omega, E, \{p_i\}_{i=1}^n)/(q_1 \sim q'_1)\) is \(\frac{3}{10}\)-stable.

5. Suppose that \((C, \{p_i\}_{i=1}^n, q_1)\) is \(\frac{2}{3}\)-stable and \((E, q'_1)\) is a Weierstrass tail. Then \((\tilde{C}, C, \omega, E, \{p_i\}_{i=1}^n)/(q_1 \sim q'_1)\) is \(\frac{2}{3}\)-stable.

**Proof.** (1), (3), (4), (5) follow immediately from from Lemma 2.20. For (2), one must apply Lemma 2.20 twice: First apply Lemma 2.20(1) to glue \(q_1 \sim q'_1\), then apply Lemma 2.20(2) to glue \(q_2 \sim q'_2\), noting that if \(q_2\) and \(q'_2\) do not lie on disjoint irreducible components of \((\tilde{C}, C, \omega, E, \{p_i\}_{i=1}^n, q_2, q'_2)/(q_1 \sim q'_1)\), then \(E\) must be an irreducible genus one curve, so \(q'_2\) does not lie on a smooth rational curve. \(\square\)

Next, we consider a question which does not arise for Deligne-Mumford stable curves: Suppose \((C, \{p_i\}_{i=1}^n)\) is an \(\alpha\)-stable curve, and \(q \in C\) is a non-nodal singularity. When is the pointed normalization \((\tilde{C}, \{p_i\}_{i=1}^n, \{q_i\}_{i=1}^m) (m \in \{1, 2\})\) \(\alpha\)-stable? One obvious obstacle is that \(\omega_{\tilde{C}}(\Sigma_{i=1}^n p_i + \Sigma_{i=1}^m q_i)\) need not be ample. Indeed, one or both of the marked points \(q_i\)
may lie on a smooth $\mathbb{P}^1$ meeting the rest of the curve in a single node. We thus define the stable pointed normalization of $(C, \{p_i\}_{i=1}^n)$ to be the (possibly disconnected) curve obtained by blowing down these semistable $\mathbb{P}^1$'s, i.e., by taking the image of $(\tilde{C}, \{p_i\}_{i=1}^n, \{q_i\}_{i=1}^m)$ under the nef line bundle $\omega_C(S_{i=1}^np_i + \Sigma_{i=1}^mq_i)$. This is well-defined except in several degenerate cases: First, when $(g, n) = (1, 1), (1, 2), (2, 1)$, the stable pointed normalization of a cuspidal, tacnodal, and ramphoid cuspidal curve is a point. In these cases, we regard the stable pointed normalization as being undefined. Second, in the tacnodal case, it can happen that $(\tilde{C}, \{p_i\}_{i=1}^n; \{q_i\}_{i=1}^m)$ has two connected components, one of which is a smooth two pointed $\mathbb{P}^1$. In this case, we define the stable pointed normalization to be the curve obtained by deleting this component and taking the stabilization of the remaining connected component.

In general, the stable pointed normalization of an $\alpha$-stable curve at a non-nodal singularity need not be $\alpha$-stable. Nevertheless, there is one important case where this statement does hold, namely when $\alpha_c$ is a critical value and $q \in C$ is an $\alpha_c$-critical singularity.

**Lemma 2.22.** Let $(C, \{p_i\}_{i=1}^n)$ be an $n$-pointed curve with $\omega_C(\Sigma p_i)$ ample, and suppose $q \in C$ is an $\alpha_c$-critical singularity. Then the stable pointed normalization of $(C, \{p_i\}_{i=1}^n)$ at $q \in C$ is $\alpha_c$-stable if and only if $(C, \{p_i\}_{i=1}^n)$ is $\alpha_c$-stable.

**Proof.** We prove the case when $\alpha_c = 2/3$, and leave the remaining cases to the reader. Let $(C^s, \{p_i\}_{i=1}^n, q_1)$ denote the stable pointed normalization and observe that there is a natural morphism $\gamma: (C^s, \{p_i\}_{i=1}^n) \to (C, \{p_i\}_{i=1}^n)$. Indeed, if taking the pointed normalization does not produce a semistable $\mathbb{P}^1$, then $\gamma$ is just the normalization and $\gamma(q_1)$ is a ramphoid cusp. On the other hand, normalization at $q$ produces a semistable $\mathbb{P}^1$, then $\gamma$ is simply a closed immersion mapping $\gamma(q)$ to the attaching node of this semistable $\mathbb{P}^1$. Note that there is a natural finite map $\gamma: C^s \to C$. This is just the normalization if the pointed normalization is already stable, and is just a closed immersion $C^s/E \hookrightarrow C$ when the pointed normalization has a semistable $\mathbb{P}^1$. In each case, it is trivial to verify that $\omega_{C^s}(\Sigma p_i)$ is ample if and only if $\omega_C(\Sigma p_i + \Sigma q_i)$ is ample, and that $C$ has the appropriate singularities if and only if $C'$ does. Thus, it suffices to verify that $C'$ contains an $\alpha_c$-unstable subcurve if and only if $C$ does. When $\alpha_c = 9/11$, this is trivial, since there are no $\alpha_c$-unstable subcurves. When $\alpha_c = 7/10$, the only $\alpha_c$-unstable curves are elliptic tails, and the statement is easy to check. When $\alpha_c = 2/3$, the only $\alpha_c$-unstable subcurves are elliptic tails and elliptic chains.

First, we show that if $E \subset C^s$ is a $2/3$-unstable subcurve, then $\gamma(E) \subset C$ is also a $2/3$-unstable subcurve. If $E$ does not contain $q_1$, then $\gamma$ is an isomorphism in a neighborhood of $E$, and the statement is obvious. If $E$ contains $q_1$, then $q_1$ must be an attaching point for $E$. Since $\gamma(q_1)$ is either a ramphoid cusp or a node, and elliptic tails or chains with nodal or ramphoid cuspidal attaching are all $2/3$-unstable, $\gamma(E)$ is a $2/3$-unstable subcurve of $C$.

Next, we show that if $E \subset C$ is a $2/3$-unstable subcurve $E$, then $\gamma^{-1}(E) \subset C^s$ is a $2/3$-unstable subcurve. First, we observe that if the ramphoid cusp $q \in C$ lies on a semistable $\mathbb{P}^1$, then $E$ cannot contain this irreducible component. Indeed, $E$ would have to contain the disconnecting node where this $\mathbb{P}^1$ meets the rest of the curve, and this is impossible.
since no $\frac{3}{2}$-unstable subcurve contains a disconnecting node. It follows that $\gamma^{-1}(E)$ maps isomorphically to $E$ except that the marked point $q_1$ maps to a node or ramphoid cusp. But an elliptic tail or chain with nodal or ramphoid cuspidal attaching is also $\frac{3}{2}$-unstable with marked point attaching. \hfill \Box


2.4. $\alpha$-closed curves. In this section, we give an explicit characterization of the closed points of $\overline{\mathcal{M}}_{g,n}(\alpha_c)$ when $\alpha_c$ is a critical value.

Definition 2.23 ($\alpha_c$-atoms).

1. A $\frac{9}{11}$-atom is a 1-pointed curve of arithmetic genus one obtained by gluing $\text{Spec } \mathbb{C}[x, y]/(y^2 - x^3)$ and $\text{Spec } \mathbb{C}[s]$ along $\{x \neq 0\}$ and $\{s \neq 0\}$ via $x = s^{-2}, y = s^{-3}$, and marking the point $s = 0$.

2. A $\frac{7}{10}$-atom is a 2-pointed curve of arithmetic genus one obtained by gluing $\text{Spec } \mathbb{C}[x, y]/(y^2 - x^4)$ and $\text{Spec } \mathbb{C}[s_1] \bigsqcup \text{Spec } \mathbb{C}[s_2]$ along $\{x \neq 0\}$ and $\{s_1 \neq 0\} \bigsqcup \{s_2 \neq 0\}$ via $x = (s_1^{-1}, s_2^{-1}), y = (s_1^{-2}, -s_2^{-2})$, and marking the points $s_1 = 0$ and $s_2 = 0$.

3. A $\frac{2}{3}$-atom is a 3-pointed curve of arithmetic genus two obtained gluing $\text{Spec } \mathbb{C}[x, y]/(y^2 - x^5) \bigsqcup \text{Spec } \mathbb{C}[s]$ along $\{x \neq 0\}$ and $\{s \neq 0\}$ via $x = s^{-2}, y = s^{-5}$, and marking the point $s = 0$.

We will often abuse notation by simply writing $E$ to refer to the $\alpha_c$-atom $(E, q)$ if $\alpha_c \in \{2/3, 9/11\}$ (resp. $(E, q_1, q_2)$ if $\alpha_c = 7/10$).

Remark 2.24. Note that if $E$ is an $\alpha_c$-atom, then $\text{Aut}(E) \cong \mathbb{G}_m$.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{atom.png}
\caption{A $\frac{9}{11}$-atom, $\frac{7}{10}$-atom and $\frac{2}{3}$-atom, respectively.}
\end{figure}

In order to describe the closed points of $\overline{\mathcal{M}}_{g,n}(\alpha_c)$ precisely, we need the following terminology. We say that $C$ admits a decomposition $C = C_1 \cup \ldots \cup C_r$ if $C_1, \ldots, C_r$ are proper subcurves whose union is all of $C$, and either $C_i \cap C_j = \emptyset$ or $C_i$ meets $C_j$ nodally. When $(C, \{p_i\}_{i=1}^n)$ is an $n$-pointed curve, and $C = C_1 \cup \ldots \cup C_r$ is a decomposition of $C$, we always consider $C_i$ as a pointed curve by taking as marked points the subset of $\{p_i\}_{i=1}^n$ supported on $C_i$ and the the attaching points $C_i \cap (C \setminus C_i)$.

Definition 2.25 ($\alpha_c$-closed curves). Let $\alpha_c \in \{2/3, 7/10, 9/11\}$ be a critical value. We say that an $n$-pointed curve $(C, \{p_i\}_{i=1}^n)$ is $\alpha_c$-closed if there is a decomposition $C = K \cup E_1 \cup \ldots \cup E_r$, where

- $E_1, \ldots, E_r$ are $\alpha_c$-atoms.
- $K$ is a $(\alpha_c + \epsilon)$-stable curve containing no $\alpha_c$-critical subcurves.
$\cdot K$ is closed in $\overline{M}_{g,n}(\alpha_c+\epsilon)$.

We call $K$ the core of $(C,\{p_i\}_{i=1}^n)$, and we call the decomposition $C = K \cup E_1 \cup \ldots \cup E_r$ the canonical decomposition of $C$. We allow the possibility that the core is disconnected or empty.

We can now state the main result of this section.

**Theorem 2.26** (Characterization of $\alpha$-closed curves). Let $\alpha_c \in \{9/11, 7/10, 2/3\}$ be a critical value. An $\alpha_c$-stable curve $(C, \{p_i\}_{i=1}^n)$ is a closed point of $\overline{M}_{g,n}(\alpha_c)$ if and only if $(C, \{p_i\}_{i=1}^n)$ is $\alpha_c$-closed.

To prove the above theorem, we need several preliminary lemmas.

**Lemma 2.27.**

(1) Suppose $(E,q)$ is an elliptic tail. Then $(E,q)$ is a closed point of $\overline{M}_{1,1}(9/11)$ if and only if $(E,q)$ is a $\frac{9}{11}$-atom.

(2) Suppose $(E,q_1,q_2)$ is an elliptic bridge. Then $(E,q_1,q_2)$ is a closed point of $\overline{M}_{1,2}(7/10)$ if and only if $(C,q_1,q_2)$ is a $\frac{7}{10}$-atom.

(3) Suppose $(E,q)$ is a Weierstrass tail. Then $(C,q)$ is a closed point of $\overline{M}_{2,1}(2/3)$ if and only if $(C,q)$ is a $\frac{2}{3}$-atom.

**Proof.** We prove case (3) and leave cases (1) and (2) to the reader. First, we show that if $(E,q)$ is any Weierstrass tail, then $(E,q)$ admits an isotrivial specialization to a $\alpha_c$-closed point. Any Weierstrass genus 2 tail can be written as a degree 2 cover of $\mathbb{P}^1$ with equation given by

$$z^2 = x^5 y + a_3 x^3 y^3 + a_2 x^2 y^4 + a_1 xy^5 + a_0 y^6$$

where $a_i \in \mathbb{C}^*$, and the marked point $q$ corresponds to $y = z = 0$. This cover is isomorphic to

$$y^2 = x^5 + \lambda^4 a_3 x^3 z^3 + \lambda^6 a_2 x^2 z^4 + \lambda^8 a_1 x z^5 + \lambda^{10} a_0 z^6$$

for any $\lambda \in \mathbb{C}^*$. Letting $\lambda \to 0$, we obtain an isotrivial specialization of the given Weierstrass tail to the double cover $z^2 = x^5 y$, which is a $\frac{2}{3}$-atom.

Next, we show that if $(E,q)$ is a $\frac{2}{3}$-atom, then $(E,q)$ does not admit any non-trivial isotrivial specializations in $\overline{M}_{2,1}(2/3)$. Let $(\mathcal{E} \to \Delta, \sigma)$ be an isotrivial specialization in $\overline{M}_{2,1}(2/3)$ with generic fiber isomorphic to $(E,q)$. Let $\tau$ be the section of $\mathcal{E} \to \Delta$ which picks out the unique ramphoid cusp of the generic fiber. Since the limit of a ramphoid cusp is a ramphoid cusp in $\overline{M}_{2,1}(2/3)$, $\tau(0)$ is also ramphoid cusp. Now let $\tau: \overline{\mathcal{E}} \to \mathcal{E}$ be the simultaneous normalization of $\mathcal{E}$ along $\tau$, and let $\overline{\tau}$ and $\overline{\sigma}$ be the inverse images of $\tau$ and $\sigma$ respectively. Then $\overline{\mathcal{E}} \to \Delta, \overline{\tau}, \overline{\sigma}$ is an isotrivial specialization of 2-pointed curves of arithmetic genus zero with smooth general fiber. To prove that the original isotrivial specialization is trivial, it suffices to prove that $(\overline{\mathcal{E}} \to \Delta, \overline{\tau}, \overline{\sigma})$ is trivial, i.e. we must show that the special fiber is smooth (equivalently, irreducible). The fact that $\omega_{\mathcal{E}/\Delta}(\sigma)$ is relatively
ample on $E$ implies that $\omega_{\tilde{E}/\Delta}(3\tilde{r} + \tilde{\sigma})$ is relatively ample on $\tilde{E}$, which implies that the special fiber of $\tilde{E}$ is irreducible.

□

Lemma 2.28. Suppose $(C, \{p_i\}_{i=1}^n)$ is a closed point of $\overline{M}_{g,n}(\alpha_c + \varepsilon)$. Then $(C, \{p_i\}_{i=1}^n)$ remains closed in $\overline{M}_{g,n}(\alpha_c)$ if and only if $(C, \{p_i\}_{i=1}^n)$ contains no nodally-attached $\alpha_c$-critical subcurves.

Proof. We prove the case $\alpha_c = 2/3$ and leave the other cases to the reader. First, we show that if $(C, \{p_i\}_{i=1}^n)$ has a nodally-attached Weierstrass tail, then it does not remain closed in $\overline{M}_{g,n}(2/3)$. Let $C = (K, \{p_i\}_{i=1}^n, q_1) \cup (Z, q_1)/(q_1 \sim q_1')$, where $Z$ is a nodally-attached Weierstrass tail. By Lemma 2.27 $(Z, q_1)$ admits an isotrivial specialization to a $2/3$-atom $(E, q_1)$. We may glue this specialization to the trivial family $(K, \{p_i\}_{i=1}^n, q_1) \times \Delta$ to obtain a nontrivial isotrivial specialization of $C \rightarrow (K, \{p_i\}_{i=1}^n, q_1) \cup (E, q_1)/(q_1 \sim q_1')$. By Lemma 2.20 $(K, \{p_i\}_{i=1}^n, q_1) \cup (E, q_1)/(q_1 \sim q_1')$ is $2/3$-stable, so this is a nontrivial isotrivial specialization in $\overline{M}_{g,n}(2/3)$.

Next, we show that if $(C, \{p_i\}_{i=1}^n)$ has no nodally-attached Weierstrass tails, then it remains closed in $\overline{M}_{g,n}(2/3)$. In other words, if there exists a nontrivial isotrivial specialization $(C, \{p_i\}_{i=1}^n) \rightarrow (C_0, \{p_i\}_{i=1}^n)$, then $(C, \{p_i\}_{i=1}^n)$ necessarily contains a nodally-attached Weierstrass tail. To begin, note that if $(C \rightarrow \Delta, \{\sigma_i\}_{i=1}^n)$ is a nontrivial, isotrivial specialization, then the special fiber $C_0$ must contain at least one ramphoid cusp. Otherwise, $(C \rightarrow \Delta, \{\sigma_i\}_{i=1}^n)$ would constitute a nontrivial, isotrivial specialization in $\overline{M}_{g,n}(2/3+\varepsilon)$, contradicting the hypothesis that $(C, \{p_i\}_{i=1}^n)$ is closed in $\overline{M}_{g,n}(2/3+\varepsilon)$. For simplicity, let us assume that the special fiber $C_0$ contains a single ramphoid cusp $q$. Locally around this point, we may write $C$ as

$$z^2 = x^5 + a_3(t)x^3 + a_2(t)x^2 + a_1(t)x + a_0(t)$$

where $a_i(t) \in C[[t]]$. By [CML13, Section 7.6], after possibly a finite base change, there exists a (weighted) blow-up $\phi: \tilde{C} \rightarrow C$ such that the special fiber $\tilde{C}_0$ is isomorphic to the normalization of $C$ at $q$ attached nodally to a curve $T$, where $T$ is defined by an equation $z^2 = x^5 + b_3x^3y^2 + b_2x^2y^3 + b_1xy^4 + b_0y^5$ on $\mathbb{P}(2,2,5)$ for some $[b_3 : b_2 : b_1 : b_0] \in \mathbb{P}(4,6,8,10)$ (depending on the $a_i(t)$) and such that $T$ is attached to $C$ at $[x : y : z] = [1 : 0 : 1]$. Evidently, $T$ is a genus 2 double cover of $\mathbb{P}^1 \cong \mathbb{P}(2,2)$ via the projection $[x : y : z] \mapsto [x : y]$ and $[1 : 0 : 1]$ is a ramification point of this cover. It follows that $\tilde{C}_0$ has a Weierstrass tail.

Now let $\tilde{C} \rightarrow \tilde{C}^s$ be the stabilization morphism associated to a higher power of the line bundle $\omega_{\tilde{E}/\Delta}(\sum, \sigma_i)$. The special fiber of $\tilde{C}^s_0$ is isomorphic to the stable pointed normalization of $C_0$ at $q$, together with a nodally attached Weierstrass tail. By Lemma 2.22 and Corollary 2.21 $(C^s_0, \{p_i\}_{i=1}^n)$ is $\alpha$-stable. Since it contains no ramphoid cusps, it is also $(\alpha_c + \varepsilon)$-stable. By hypothesis, $(C, \{p_i\}_{i=1}^n)$ is closed in $\overline{M}_{g,n}(\alpha + \varepsilon)$, so the family $(C^s \rightarrow \Delta, \{\sigma_i\}_{i=1}^n)$ must be trivial. This implies that the generic fiber $(C, \{p_i\}_{i=1}^n)$ must have a nodally-attached Weierstrass tail.

□
The following lemma says that one can use isotrivial specializations to replace \( \alpha_c \)-critical singularities and subcurves by \( \alpha_c \)-atoms.

**Lemma 2.29.** Let \( (C, \{p_i\}_{i=1}^n) \) be an \( n \)-pointed curve, and let \( E \) be an \( \alpha_c \)-atom.

1. Suppose \( q \in C \) is an \( \alpha_c \)-critical singularity. Then there exists an isotrivial specialization \( C \rightsquigarrow C_0 = \tilde{C} \cup E \) to an \( n \)-pointed curve \( C_0 \) which is the nodal union of \( E \) and the pointed normalization \( \tilde{C} \) of \( C \) at \( q \) along the marked point(s) of \( E \) and the pre-image(s) of \( q \) in \( \tilde{C} \).

2. Suppose \( Z \subset C \) is an \( \alpha_c \)-critical subcurve, i.e. \( C \) is the nodal union of \( K \cup Z \). Then there exists an isotrivial specialization \( C \rightsquigarrow C_0 = K \cup E \) to an \( n \)-pointed curve \( C_0 \) which is the nodal union of \( K \) and \( E \) along the marked point(s) of \( E \) and \( K \cap Z \).

**Proof.** We prove the case \( \alpha_c = 2/3 \), and leave the remaining two cases to the reader. For (1), let \( C \times \Delta \) be the trivial family, let \( \tilde{C} \to C \times \Delta \) be the normalization along \( q \times \Delta \), and let \( \tilde{C}' \to \tilde{C} \) be the blow-up of \( \tilde{C} \) at the point lying over \((q,0)\). Let \( \tau \) denote the strict transform of \( q \times \Delta \) on \( \tilde{C}' \), and note that \( \tau \) passes through a smooth point of the exceptional divisor. A local calculation, as in the proof of Proposition 4.20, shows that we may ‘recrimp,’ i.e. there exists a finite map \( \psi: \tilde{C}' \to \tilde{C} \) such that \( \psi \) is an isomorphism on \( \tilde{C}' - \tau \) and \( \tilde{C}' \) has a ramphoid cusp along \( \psi \circ \tau \). Now \( \tilde{C}' \to \Delta \) is the desired isotrivial specialization. For (2), note that there exists an isotrivial specialization \((Z, q_1) \rightsquigarrow (E, q_1)\) by Lemma 2.27. Gluing this to the trivial family \((K \times \Delta, q_1 \times \Delta)\) gives the desired isotrivial specialization. \( \square \)

**Proof of Theorem 2.26.** We prove the proposition when \( \alpha_c = 2/3 \), and leave the other two cases to the reader. First, we show that if \( (C, \{p_i\}_{i=1}^n) \) is \( \frac{2}{3} \)-closed, then it is a closed point of \( \overline{\mathcal{M}}_{g,n}(2/3) \). Let \( (C \to \Delta, \{\sigma_i\}_{i=1}^n) \) be any isotrivial specialization of \( (C, \{p_i\}_{i=1}^n) \) in \( \overline{\mathcal{M}}_{g,n}(2/3) \); we will show it must be trivial. Let \( C = K \cup E_1 \cup \ldots \cup E_r \) be the canonical decomposition of the generic fiber, and let \( q_i = K \cap E_i \). Each \( q_i \) is a node in the general fiber of \( C \to \Delta \), and by Corollary 2.14 each \( q_i \) remains a node in the special fiber. Possibly after a finite base change, we may normalize along the corresponding sections, and we obtain isotrivial specializations \( \mathcal{K} \) and \( \mathcal{E}_1, \ldots, \mathcal{E}_r \), where \( \mathcal{K} \) is a family in \( \overline{\mathcal{M}}_{g-2r,n+r}(2/3+\epsilon) \) and \( \mathcal{E}_1, \ldots, \mathcal{E}_r \) are families in \( \overline{\mathcal{M}}_{2,1}(2/3) \). (Lemma 2.19 implies that the special fibers of these families remain \((\frac{2}{3}+\epsilon)\)- and \(\frac{2}{3}\)-stable respectively.) Since \( \mathcal{K} \) contains no Weierstrass tails in the general fiber, it is trivial by Lemma 2.28. The families \( \mathcal{E}_1, \ldots, \mathcal{E}_r \) are trivial by Lemma 2.27. It follows that the original family \((C \to \Delta, \{\sigma_i\}_{i=1}^n)\) is trivial, as desired.

Next, we show that if \( (C, \{p_i\}_{i=1}^n) \in \overline{\mathcal{M}}_{g,n}(2/3) \) is a closed point, then \( (C, \{p_i\}_{i=1}^n) \) must be \( \frac{2}{3} \)-closed. First, we claim that every ramphoid cusp of \( C \) must lie on a nodally attached \( \frac{2}{3} \)-atom. Indeed, if \( q \in C \) is a ramphoid cusp which does not lie on a nodally-attached \( \frac{2}{3} \)-atom, then Lemma 2.29 gives an isotrivial specialization \((C, \{p_i\}_{i=1}^n) \rightsquigarrow (C_0, \{p_i\}_{i=1}^n)\) in which \( C_0 \) sprouts a nodally-connected \( \frac{2}{3} \)-atom at \( q \). Note that \((C_0, \{p_i\}_{i=1}^n)\) is \(\frac{2}{3}\)-stable by Lemma 2.22 and Corollary 2.21 so this gives a nontrivial isotrivial specialization in \( \overline{\mathcal{M}}_{g,n}(2/3) \). Second, we claim that \( C \) contains no nodally-connected Weierstrass tails which are not \( \frac{2}{3} \)-atoms. Indeed, if
it does, then Lemma 2.29 gives an isotrivial specialization \((C, \{p_i\}_{i=1}^n) \sim (C_0, \{p_i\}_{i=1}^n)\) which replaces this Weierstrass tail by a \(\frac{2}{3}\)-atom. Note that \((C_0, \{p_i\}_{i=1}^n)\) is \(\frac{2}{3}\)-stable by Lemma 2.19 and Corollary 2.21, so this gives a nontrivial isotrivial specialization in \(\overline{M}_{g,n}(2/3)\). It is now easy to see that \(C\) is \(\alpha\)-closed. Indeed, if \(E_1, \ldots, E_r\) are the nodally attached \(\frac{2}{3}\)-atoms of \(C\), then the complement \(K\) has no ramphoid cusps and no nodally-attached Weierstrass tails. Since \(K\) is \(\frac{2}{3}\)-stable and has no ramphoid cusps, it is \((\frac{2}{3} + \epsilon)\)-stable. Furthermore, \(K\) must be closed in \(\overline{M}_{g,n}(2/3+\epsilon)\), since a nontrivial isotrivial specialization of \(K\) in \(\overline{M}_{g,n}(2/3+\epsilon)\) would induce a nontrivial, isotrivial specialization of \((C, \{p_i\}_{i=1}^n)\) in \(\overline{M}_{g,n}(2/3)\). We conclude that \((C, \{p_i\}_{i=1}^n)\) is \(\frac{2}{3}\)-closed as desired. □

3. Local description of the flips

In this section, we give an étale local description of the inclusions
\[ \overline{M}_{g,n}(\alpha_c + \epsilon) \hookrightarrow \overline{M}_{g,n}(\alpha_c) \hookrightarrow \overline{M}_{g,n}(\alpha_c - \epsilon) \]
at each critical value \(\alpha_c \in \{2/3, 7/10, 9/11\}\). Roughly speaking, our main result says that, étale locally around any closed point of \(\overline{M}_{g,n}(\alpha_c)\), these inclusions are induced by a variation of GIT problem. In Section 3.1 we develop the necessary background material on local quotient presentations and local VGIT in order to state our main result (Theorem 3.9). In Section 3.2 we collect several basic facts concerning local variation of GIT which will be used in subsequent sections. In Section 3.3 we describe explicit coordinates on the formal miniversal deformation space of an \(\alpha_c\)-closed curve. In Section 3.4 we use these coordinates to compute the associated VGIT chambers and thus conclude the proof of Theorem 3.9.

3.1. Local quotient presentations.

**Definition 3.1.** Let \(\mathcal{X}\) be an algebraic stack of finite type over \(\mathbb{C}\), and let \(x \in \mathcal{X}(\mathbb{C})\) be a closed point. We say that \(f: W \to \mathcal{X}\) is a local quotient presentation around \(x\) if

- \(f\) is étale and affine.
- \(G_x\) is reductive.
- \(W = \text{Spec} \, A/G_x\), where \(A\) is a finite type \(\mathbb{C}\)-algebra, and \(G_x\) is the stabilizer of \(x \in \mathcal{X}(\mathbb{C})\).
- There exists a point \(w \in \text{Spec} \, A/G_x\) such that \(f(w) = x\), and \(f\) induces an isomorphism \(G_w \cong G_x\).

We say that \(\mathcal{X}\) admits local quotient presentations if there exist local quotient presentations around all closed points \(x \in \mathcal{X}(\mathbb{C})\). We sometimes write \(f: (W, w) \to \mathcal{X}\) as a local quotient presentation to indicate the chosen preimage of \(x\).

**Proposition 3.2.** [Alp10b, Theorem 3] Let \(\mathcal{X}\) be a normal algebraic stack of finite type over \(\mathbb{C}\), and let \(x \in \mathcal{X}(\mathbb{C})\). If \(\mathcal{X}\) is a quotient stack \([X/G]\) where \(G\) is a connected algebraic group acting on a normal separated scheme \(X\), and \(x \in \mathcal{X}\) has reductive stabilizer, then \(\mathcal{X}\) admits a local quotient presentation around \(x\).
Corollary 3.3. For each $\alpha > 2/3 - \epsilon$, $\overline{M}_{g,n}(\alpha)$ admits local quotient presentations.

We will show that if $\alpha_c$ is a critical value and $f : [\text{Spec } A/G] \to \overline{M}_{g,n}(\alpha_c)$ is a local quotient presentation, there is a canonically defined pair of open immersions

$$[\text{Spec } A/G]^+ \hookrightarrow [\text{Spec } A/G] \hookrightarrow [\text{Spec } A/G]^-$$

defined on the local variation of GIT. To motivate this, recall that if $G$ is a reductive group acting on an affine scheme $X = \text{Spec } A$ with an action $\sigma : G \times X \to X$, there is a natural correspondence between $G$-linearizations of the structure sheaf $\mathcal{O}_X$ and characters $\chi : G \to \mathbb{G}_m = \text{Spec } \mathbb{C}[t]$.

Precisely, a character $\chi$ defines a $G$-linearization $\mathcal{L}$ of the structure sheaf $\mathcal{O}_X$ as follows: the character $\chi$ gives an element $\chi^*(t) \in \Gamma(G, \mathcal{O}^*_G)$ which in turn induces a $G$-linearization $\sigma^* \mathcal{O}_X \to p_2^* \mathcal{O}_X$ defined by $p_2^*(\chi^*(t))^{-1} \in \Gamma(G \times X, \mathcal{O}^*_{G \times X})$. Therefore, we can associate to $\chi$ the semistable loci $X^*_L$ and $X^*_{L-1}$ (cf. [Mum65, Definition 1.7]). The following definition describes explicitly the change in semistable locus as we move from $\chi$ to $\chi^{-1}$ in the character lattice of $G$. See [Tha96] and [DH98] for the general setup of variation of GIT.

Definition 3.4 (VGIT $(+)/(−)$-chambers). Let $G$ be a reductive group acting on an affine scheme $X = \text{Spec } A$. Let $\chi : G \to \mathbb{G}_m$ be a character and set $A_n := \{ f \in A \mid \sigma^*(f) = (\chi^* t)^{-n} f \} = \Gamma(X, \mathcal{L}^\otimes n)^G$. We define the VGIT ideals associated to $\chi$ to be:

$$I^+_\chi := \{ f \in A \mid f \in A_n \text{ for } n > 0 \},$$

$$I^-\chi := \{ f \in A \mid f \in A_n \text{ for } n < 0 \}.$$

The VGIT $(+)$-chamber and $(−)$-chamber associated to $\chi$ are the open subschemes

$$X^+_\chi := X \setminus V(I^+_\chi) \hookrightarrow X, \quad X^-\chi := X \setminus V(I^-\chi) \hookrightarrow X.$$

Remark 3.5. For an alternative characterization of $X^+_\chi$, note that $\chi^{-1}$ defines an action of $G$ on the trivial line bundle $X \times \mathbb{A}^1$ via $g \cdot (x, s) = (g \cdot x, \chi(g)^{-1} \cdot s)$. Then $x \in X^+_\chi$ if and only if the orbit closure $\overline{G \cdot (x, 1)}$ does not intersect the zero section $X \times \{0\}$.

Remark 3.6. Since the open subsets $X^+_\chi, X^-\chi$ are $G$-invariant, we also have stack-theoretic open immersions

$$[X^+_\chi/G] \hookrightarrow [X/G] \hookrightarrow [X^-\chi/G].$$

We will sometimes abuse notation by referring to these open immersions as the VGIT $(+)/(−)$-chambers associated to $\chi$.

The natural inclusions of $(+)/(−)$-chambers induce projective morphisms of GIT quotients.

Proposition 3.7. Let $\mathcal{L}$ be the $G$-linearization of the structure sheaf on $X$ corresponding to a character $\chi$. Then there are natural identifications of $X^+_\chi$ and $X^-\chi$ with the semistable loci
$X_\mathbb{C}^\pm$ and $X_{\mathbb{C}^*}^{\pm 1}$, respectively. There is a commutative diagram

$$
\begin{array}{ccc}
X_\mathbb{C}^+ & \xrightarrow{x} & X \\
\downarrow & & \downarrow \\
X_\mathbb{C}^+//G := \text{Proj} \bigoplus_{d \geq 0} A_d & \xrightarrow{\psi} & \text{Spec} A_0 \\
\downarrow & & \downarrow \\
\text{Proj} \bigoplus_{d \geq 0} A_{-d} =: X_\mathbb{C}^-//G & \xleftarrow{x} & X_\mathbb{C}^-
\end{array}
$$

where $X \to \text{Spec} A_0$, $X_\mathbb{C}^+ \to X_\mathbb{C}^+//G$ and $X_\mathbb{C}^- \to X_\mathbb{C}^-//G$ are GIT quotients. The restriction of $\mathcal{L}$ to $X_\mathbb{C}^+$ (resp. $\mathcal{L}^{-1}$ to $X_\mathbb{C}^-$) descends to line bundle $\mathcal{O}(1)$ on $X_\mathbb{C}^+//G$ (resp. $\mathcal{O}(1)$ on $X_\mathbb{C}^-//G$) relatively ample over $\text{Spec} A_0$. In particular, for every point $x \in X_\mathbb{C}^+ \cup X_\mathbb{C}^-$, the character of $G_x$ corresponding to $\mathcal{L}|_{BG_x}$ is trivial.

**Proof.** This follows immediately from the definitions and [Mum65, Theorem 1.10].

Next, we show how to use the data of a line bundle $\mathcal{L}$ on a stack $\mathcal{X}$ to define $(+)/(-)$-chambers associated to any local quotient presentation of $\mathcal{X}$. In this situation, note that if $x \in \mathcal{X}(\mathbb{C})$ is any point, then there is a natural action of the automorphism group $G_x$ on the fiber $\mathcal{L}|_{BG_x}$, which induces a character $\chi_\mathcal{L}: G_x \to \mathfrak{g}_m$.

**Definition 3.8** ($(+)/(-)$-chambers of a local quotient presentation). Let $\mathcal{X}$ be an algebraic stack of finite type over $\mathbb{C}$ and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Let $x \in \mathcal{X}$ be a closed point with stabilizer $G_x$. If $f: \mathcal{W} = [\text{Spec} A/G_x] \to \mathcal{X}$ is a local quotient presentation around $x$, we define the $(+)/(-)$-chambers of $\mathcal{W}$

$$
\mathcal{W}_\mathcal{L}^+ \leftrightarrow [\text{Spec} A/G_x] \leftrightarrow \mathcal{W}_\mathcal{L}^-
$$

to be the VGIT $(+)/(-)$-chambers associated to the character $\chi_\mathcal{L}: G_x \to \mathfrak{g}_m$.

In our situation, there is a natural line bundle to use in conjunction with the VGIT formalism, namely $\delta - \psi$. Since this line bundle is defined over $\mathcal{M}_{g,n}(\alpha)$ for each $\alpha$, there is an induced character $\chi_{\delta - \psi}: \text{Aut}(C, \{p_i\}_{i=1}^n) \to \mathfrak{g}_m$ for any $\alpha$-stable curve $(C, \{p_i\}_{i=1}^n)$. The main result of this section simply says that the $(+)/(-)$-VGIT chambers associated to $\delta - \psi$ locally cut out the inclusions $\mathcal{M}_{g,n}(\alpha_c + \epsilon) \hookrightarrow \mathcal{M}_{g,n}(\alpha_c) \hookrightarrow \mathcal{M}_{g,n}(\alpha_c - \epsilon)$.

**Theorem 3.9.** Let $(C, \{p_i\}_{i=1}^n)$ be an $\alpha_c$-closed curve. There exists a local quotient presentation $f: \mathcal{W} = [\text{Spec} A/ \text{Aut}(C, \{p_i\}_{i=1}^n)] \to \mathcal{M}_{g,n}(\alpha_c)$ around $(C, \{p_i\}_{i=1}^n)$ such that $f^*(\delta - \psi)$ is the line bundle corresponding to the linearization of $\mathcal{O}_{\text{Spec} A}$ by $\chi_{\delta - \psi}$ and such that there is a Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{W}_{\delta - \psi}^+ & \xrightarrow{c} & \mathcal{W} \xleftarrow{f} \mathcal{W}_{\delta - \psi}^- \\
\downarrow & & \downarrow \\
\mathcal{M}_{g,n}(\alpha_c + \epsilon) & \xleftarrow{c} & \mathcal{M}_{g,n}(\alpha_c) \xrightarrow{f} \mathcal{M}_{g,n}(\alpha_c - \epsilon)
\end{array}
$$

(3.1)
The idea of the proof is to work formally locally. To make this precise, we must introduce some additional notation. If \((C, \{p_1\}_{i=1}^n)\) is an \(\alpha_c\)-closed curve, we let \(T^1(C, \{p_1\}_{i=1}^n)\) be the space of a first-order deformations of \((C, \{p_1\}_{i=1}^n)\), \(\mathcal{C}[T^1(C, \{p_1\}_{i=1}^n)] = \text{Sym}^* T^1(C, \{p_1\}_{i=1}^n)\) and \(\mathcal{C}[[T^1(C, \{p_1\}_{i=1}^n)]]\) the completion of \(\mathcal{C}[T^1(C, \{p_1\}_{i=1}^n)]\) at the origin. Since \(C\) is a local complete intersection with smooth marked points, we may take \(\hat{A} := \mathcal{C}[[T^1(C, \{p_1\}_{i=1}^n)]]\) as the base of a formal miniversal deformation of \((C, \{p_1\}_{i=1}^n)\); thus, we set

\[
\overline{\text{Def}}(C, \{p_1\}_{i=1}^n) := \text{Spf} \mathcal{C}[[T^1(C, \{p_1\}_{i=1}^n)]].
\]

**Definition 3.10** \((I^\pm, I^\mp)\). Let \((C, \{p_i\}_{i=1}^n)\) be an \(\alpha_c\)-closed curve, and let

\[
\overline{\text{Def}}(C, \{p_i\}_{i=1}^n) \rightarrow \overline{\mathcal{M}}_{g,n}(\alpha_c)
\]

be a formal miniversal deformation space for \((C, \{p_i\}_{i=1}^n)\). We define \(I^\pm, I^\mp\) to be the ideals in \(\hat{A} = \mathcal{C}[[T^1(C, \{p_i\}_{i=1}^n)]]\) corresponding to the reduced closed substacks

\[
\mathcal{Z}^+ := \overline{\mathcal{M}}_{g,n}(\alpha_c) \smallsetminus \overline{\mathcal{M}}_{g,n}(\alpha_c + \epsilon) \quad \mathcal{Z}^- := \overline{\mathcal{M}}_{g,n}(\alpha_c) \smallsetminus \overline{\mathcal{M}}_{g,n}(\alpha_c - \epsilon).
\]

**Definition 3.11** \((I^+, I^-)\). If \((C, \{p_i\}_{i=1}^n)\) is an \(\alpha_c\)-closed curve, the affine space

\[
X = \text{Spec} \mathcal{C}[T^1(C, \{p_i\}_{i=1}^n)]
\]

inherits an action of \(G := \text{Aut}(C, \{p_i\}_{i=1}^n)\), and we define the \((+)/(+)\)-VGIT ideals \(I^+\) and \(I^-\) to be the ideals in \(\mathcal{C}[T^1(C, \{p_i\}_{i=1}^n)]\) corresponding to the \(G\)-invariant reduced closed subschemes

\[
X \smallsetminus X^+_{\chi^\delta-\psi} \subseteq X \quad X \smallsetminus X^-_{\chi^\delta-\psi} \subseteq X.
\]

The following proposition says that in order to prove Theorem 3.9, it suffices to show that \(I^\pm_\alpha = I^\pm \hat{A}\) and \(I^\mp_\alpha = I^- \hat{A}\).

**Proposition 3.12.** Let \(\mathcal{X}\) be a smooth algebraic stack of finite type over \(\text{Spec} \mathbb{C}\) which admits local quotient presentations. Let \(\mathcal{L}\) be a line bundle on \(\mathcal{X}\). Let \(\mathcal{X}^+, \mathcal{X}^-\) be open substacks of \(\mathcal{X}\). Let \(x \in \mathcal{X}\) be a closed point. Let \(\chi: G_x \rightarrow \mathcal{G}_m\) be the character induced from the action of \(G_x\) on the fiber of \(\mathcal{L}\) over \(x\). Let \(T^1(x)\) be the first order deformation space of \(x\), \(\mathcal{C}[T^1(x)] = \text{Sym}^* T^1(x)\) and \(\hat{A} = \mathcal{C}[[T^1(x)]]\) be the completion of \(\mathcal{C}[T^1(x)]\) at the origin. The affine space \(T = \text{Spec} \mathcal{C}[T^1(x)]\) inherits an action of \(G_x\). Let \(I^\pm_\alpha \subseteq \hat{A}\) be the ideals defined by the reduced closed substacks \(\mathcal{Z}^+ = \mathcal{X} \smallsetminus \mathcal{X}^+\) and \(\mathcal{Z}^- = \mathcal{X} \smallsetminus \mathcal{X}^-.\) Let \(I^\pm, I^- \subseteq \mathcal{C}[T^1(x)]\) be the VGIT ideals corresponding to the \(G_x\)-invariant closed subschemes \(T \smallsetminus T^+_\chi\) and \(T \smallsetminus T^-\chi\).

If \(I^{\pm}_\alpha = I^\pm \hat{A}\) and \(I^{\mp}_\alpha = I^- \hat{A}\), then there is a local quotient presentation \(f: \mathcal{W} \rightarrow \mathcal{X}\) around \(x\) such that \(f^*(\mathcal{L})\) is the line bundle corresponding to the linearization of \(\mathcal{O}_{\text{Spec} \hat{A}}\) by the character \(\chi\), and such that there is a Cartesian diagram:

\[
\begin{array}{ccc}
\mathcal{W}^+_\mathcal{L} & \longrightarrow & \mathcal{W} \\
\downarrow & & \downarrow \quad f \\
\mathcal{X}^+ & \longrightarrow & \mathcal{X} \\
\end{array}
\]
The remainder of Section 3 is devoted to the proof of Theorem 3.9. In Section 3.2 we prove basic facts concerning the VGIT chambers defined above and, in particular, we prove Proposition 3.12. In Section 3.3 we construct, for any $\alpha_c$-closed curve $(C, \{p_i\}_{i=1}^n)$, coordinates for $\text{Def}(C, \{p_i\}_{i=1}^n)$ and describe the ideals $I_{Z^+}$ and $I_{Z^-}$. In Section 3.4 we use this coordinate description to compute the VGIT ideals $I^+$ and $I^-$. In particular, Proposition 3.35 shows that $I_{Z^+} = I^+\hat{A}$ and $I_{Z^-} = I^-\hat{A}$, so that Theorem 3.9 follows from Proposition 3.12.

3.2. Preliminary facts about local VGIT. In this section, we collect several basic facts concerning variation of GIT for the action of a reductive group on an affine scheme which will be needed in subsequent sections. In particular, we formulate a version of the Hilbert-Mumford criterion which will be useful for computing the VGIT chambers associated to an $\alpha_c$-closed curve.

Definition 3.13. Recall that given a character $\chi: G \to \mathbb{G}_m$ and a one-parameter subgroup $\lambda: \mathbb{G}_m \to G$, the composition $\chi \circ \lambda: \mathbb{G}_m \to \mathbb{G}_m$ is naturally identified with the integer $n$ such that $(\chi \circ \lambda)^n t = t^n$. We define the pairing of $\chi$ and $\lambda$ as $\langle \chi, \lambda \rangle = n$.

Proposition 3.14 (Affine Hilbert-Mumford criterion). Suppose $G$ is a reductive algebraic group over $\mathbb{C}$ acting on an affine scheme $X = \text{Spec} A$ finite type over $\mathbb{C}$. Let $\chi: G \to \mathbb{G}_m$ be a character. Let $x \in X(\mathbb{C})$. Then $x \notin X^\chi_+$ (resp. $x \notin X^\chi_-$) if and only if there exists a one-parameter subgroup $\lambda: \mathbb{G}_m \to G$ with $\langle \chi, \lambda \rangle > 0$ (resp. $\langle \chi, \lambda \rangle < 0$) such that $\lim_{t \to 0} \lambda(t) \cdot x$ exists.

Proof. Consider the action of $G$ on $X \times \mathbb{A}^1$ induced by $\chi^{-1}$ as in Remark 3.5. Then $x \notin X^\chi_+$ if and only if $G \cdot (x, 1) \cap (X \times \{0\}) \neq \emptyset$. By the Hilbert-Mumford criterion [Mum65, Theorem 2.1], this is equivalent to the existence of a one-parameter subgroup $\lambda: \mathbb{G}_m \to G$ such $\lim_{t \to 0} \lambda(t) \cdot (x, 1) \in X \times \{0\}$. We are done by observing that $\lim_{t \to 0} \lambda(t) \cdot (x, 1) = \lim_{t \to 0} (\lambda(t) \cdot x, t^{\langle \chi, \lambda \rangle}) \in X \times \{0\}$ if and only if $\lim_{t \to 0} \lambda(t) \cdot x$ exists and $\langle \chi, \lambda \rangle > 0$. □

Proposition 3.15. Let $G$ be a reductive group acting on an affine variety $X$ of finite type over $\mathbb{C}$. Let $\chi: G \to \mathbb{G}_m$ be a non-trivial character. Let $\lambda: \mathbb{G}_m \to G$ be a one-parameter subgroup and $x \in X^{-\chi}(\mathbb{C})$ such that $x_0 = \lim_{t \to 0} \lambda(t) \cdot x \in X^G$ is fixed by $G$. Then $\langle \chi, \lambda \rangle > 0$.

Proof. As $x \in X^{-\chi}$, $\langle \chi, \lambda \rangle \geq 0$ by Proposition 3.14. Suppose $\langle \chi, \lambda \rangle = 0$. Considering the action of $G$ on $X \times \mathbb{A}^1$ induced by $\chi$ as in Remark 3.5 then $\lim_{t \to 0} \lambda(t) \cdot (x, 1) = (x_0, 1) \in X^G \times \mathbb{A}^1$. But $X^G$ is contained in the unstable locus $X \setminus X^{-\chi}$ since $\chi$ is a nontrivial linearization. It follows that $G \cdot (x, 1) \cap (X^G \times \{0\}) \neq \emptyset$ which contradicts $x \in X^{-\chi}$. □

Lemma 3.16. Let $G_i$ be reductive algebraic groups acting on affine schemes $X_i$ and $\chi_i: G_i \to \mathbb{G}_m$ be characters for $i = 1, \ldots, n$. Consider the diagonal action of $G = \prod_i G_i$ on $X = \prod_i X_i$. Then $G \cdot x \in X$ is fixed by $G$ for all $x \in X$. □
and the character $\prod_i \chi_i : G \to \mathbb{G}_m$. Then
\[
X \setminus X^+_\chi = \bigcup_{i=1}^n X_1 \times \cdots \times \left( X_i \setminus (X_i^+)^{\chi_i} \right) \times \cdots \times X_n,
\]
\[
X \setminus X^-_\chi = \bigcup_{i=1}^n X_1 \times \cdots \times \left( X_i \setminus (X_i^-)^{\chi_i} \right) \times \cdots \times X_n.
\]

Proof. This follows from Proposition 3.14.

Lemma 3.17. Let $G$ be a reductive algebraic group over $\mathbb{C}$ acting on an affine $X = \text{Spec} \ A$ finite type over $\mathbb{C}$. Let $\chi : G \to \mathbb{G}_m$ be a character. Let $Z \subseteq X$ be a $G$-invariant closed subscheme. Then $Z^+_\chi = X^+_\chi \cap Z$ and $Z^-_\chi = X^-_\chi \cap Z$.

Proof. The follows from Proposition 3.14.

Lemma 3.18. Let $G$ be a reductive group with character $\chi : G \to \mathbb{G}_m$. Suppose $G$ acts on an affine scheme $X = \text{Spec} \ A$ finite type over $\mathbb{C}$. Let $G^\circ$ be the connected component of the identity and $\chi^\circ = \chi|_{G^\circ}$. Then the VGIT chambers $X^+_\chi, X^-_\chi$ for the action of $G$ on $X$ are equal to the VGIT chambers $X^+_\chi^\circ, X^-_\chi^\circ$ for action of $G^\circ$ on $X$.

Proof. This is clear from Proposition 3.14 as any one parameter subgroup $\mathbb{G}_m \to G$ factors through $G^\circ$.

Lemma 3.19. Let $G$ be a reductive group with character $\chi : G \to \mathbb{G}_m$ and $h: \text{Spec} \ A = X \to Y = \text{Spec} \ B$ be a $G$-invariant morphism of affine schemes finite type over $\mathbb{C}$. Assume that
\[
\begin{array}{ccc}
\text{Spec} \ A & \xrightarrow{h} & \text{Spec} \ B \\
\downarrow & & \downarrow \\
\text{Spec} A^G & \longrightarrow & \text{Spec} B^G
\end{array}
\]
is Cartesian. Then $h^{-1}(Y^+_\chi) = X^+_\chi$ and $h^{-1}(Y^-_\chi) = X^-_\chi$.

Proof. We use Proposition 3.14. If $x \notin X^+_\chi$, then there exists $\lambda : \mathbb{G}_m \to G$ with $\langle \chi, \lambda \rangle > 0$ such that $x_0 = \lim_{t \to 0} \lambda(t) \cdot x$ exists. It follows that $h(x_0) = \lim_{t \to 0} \lambda(t) \cdot h(x)$ exists, and so $h(x) \notin Y^+_\chi$. We conclude that $h^{-1}(Y^+_\chi) \subseteq X^+_\chi$. Conversely, suppose $h(x) \notin Y^+_\chi$. Then there exists $\lambda : \mathbb{G}_m \to G$ with $\langle \chi, \lambda \rangle > 0$ such that $\lim_{t \to 0} \lambda(t) \cdot h(x)$ exists. Since $\lim_{t \to 0} \lambda(t) \cdot h(x)$ exists and since $\text{Spec} \ A \to \text{Spec} A^G$ and $\text{Spec} B \to \text{Spec} B^G$ are GIT quotients, there is a
Since the square is Cartesian, the map \( G = \text{Spec} \mathbb{C}[t] \rightarrow \text{Spec} A \) given by \( t \mapsto \lambda(t) \cdot x \) extends to \( \text{Spec} \mathbb{C}[t] \rightarrow \text{Spec} A \). It follows that \( x \not\in X^+ \). We conclude that \( X^+ \subseteq h^{-1}(Y^+) \).

**Lemma 3.20.** Let \( G \) be a reductive group acting on a smooth affine variety \( W = \text{Spec} A \) over \( \mathbb{C} \). Let \( w \in W \) be a fixed point of \( G \). Let \( \chi: G \rightarrow \mathbb{G}_m \) be a character. There is a Zariski-open affine neighborhood \( W' \subseteq W \) containing \( w \) and a \( G \)-invariant étale morphism \( h: W' \rightarrow T = \text{Spec} \mathbb{C}[T_{W,w}] \), where \( T_{W,w} \) is the tangent space at \( w \), such that

\[
h^{-1}(T^+_\chi) = W'_{\chi^+} \quad h^{-1}(T^+_\chi) = W'_{\chi^+}.
\]

**Proof.** The maximal ideal \( m \subseteq A \) of \( w \in W \) is \( G \)-invariant. Since \( G \) is reductive, there exists a splitting \( m/m^2 \hookrightarrow m \) of the surjection \( m \rightarrow m/m^2 \) of \( G \)-representations. The inclusion \( m/m^2 \hookrightarrow m \subseteq A \) induces a morphism on algebras \( \text{Sym}^* m/m^2 ightarrow A \) which is \( G \)-equivariant which in turns gives a \( G \)-equivariant morphism \( h: \text{Spec} A \rightarrow T \) étale at \( w \) in \( W \). By applying Luna’s Fundamental Lemma (see [Lun73]), there exists a \( G \)-invariant open affine \( W' = \text{Spec} A' \subseteq \text{Spec} A \) containing \( w \) such that the diagram

\[
\begin{array}{ccc}
\text{Spec} A' & \longrightarrow & \text{Spec} \mathbb{C}[T_{W,w}] \\
\downarrow & & \downarrow \\
\text{Spec} A'^G & \longrightarrow & \text{Spec} \mathbb{C}[T_{W,w}]^G
\end{array}
\]

is Cartesian with \( \text{Spec} A'^G \rightarrow \text{Spec} \mathbb{C}[T_{W,w}]^G \) étale. From Lemma 3.19, the induced map \( h|_{W'}: W' \rightarrow T \) satisfies \( h|_{W'}^{-1}(T^+_{\chi}) = W^+_{\chi^+} \) and \( h|_{W'}^{-1}(T^+_{\chi}) = W^+_{\chi^+} \).

**Proof of Proposition 3.12.** Let \( f: W = [W/G_x] \rightarrow \mathcal{X} \) be an étale local quotient presentation around \( x \) where \( W = \text{Spec} A \). By Lemma 3.20, after shrinking \( W \), we may assume that there is an induced \( G_x \)-invariant morphism \( h: W \rightarrow T = \text{Spec} \mathbb{C}[T^1(x)] \) such that \( h^{-1}(T^+_\chi) = W^+_\chi \).
and \( h^{-1}(T^+_\chi) = W^+_\chi \). This provides a diagram

\[
\begin{array}{ccc}
\text{Spf } \hat{A} & \xrightarrow{h} & \mathcal{W} = [\text{Spec } A/G_x] \\
& & \xrightarrow{\chi} \mathcal{Y} = [\text{Spec } \mathbb{C}[T^1(x)]/G_x] \\
& & \xrightarrow{\chi'} \mathcal{X}
\end{array}
\]

In particular, \( I^+A \) and \( I^-A \) are the VGIT ideals in \( A \) corresponding to \((+)/(-)\) VGIT chambers. Since \( I^+\hat{A} = I^+_Z \) and \( I^-\hat{A} = I^-_Z \), it follows that the ideals defining \( Z^+, Z^- \) and \( W \setminus W^+_\chi, W \setminus W^-_\chi \) must agree in a Zariski-open neighborhood \( U \subseteq \text{Spec } A \) of \( w \). By shrinking further, we may also assume that the pullback of \( \mathcal{L} \) to \( U \) is trivial. By the shrinking lemma of Lemma 4.6, we may assume that \( U \) is affine scheme such that \( \pi^{-1}(\pi(U)) = U \) where \( \pi: \text{Spec } A \to \text{Spec } A^G \). If we set \( \mathcal{U} = [U/G] \), then the composition \( \mathcal{U} \hookrightarrow \mathcal{W} \to \mathcal{X} \) is a local quotient presentation. By applying Lemma 3.20, \( \mathcal{U}^+ = W^+ \cap \mathcal{U} \) and \( \mathcal{U}^- = W^- \cap \mathcal{U} \) so that in \( \mathcal{U} \) the ideals defining \( Z^+, Z^- \) and \( \mathcal{U} \setminus \mathcal{U}^+, \mathcal{U} \setminus \mathcal{U}^- \) agree. Moreover, the pullback of \( \mathcal{L} \) to \( \mathcal{U} \) is clearly identified with the linearization of \( \mathcal{O}_U \) by \( \chi \). Therefore, \( \mathcal{U} \to \mathcal{X} \) satisfies the desired properties.

□

3.3. Deformation theory of \( \alpha \)-closed curves. Our goal in this section is to describe, for any \( \alpha^- \)-closed curve \((C, \{p_i\}_{i=1}^n)\), coordinates on the formal deformation space \( \hat{\text{Def}}(C, \{p_i\}_{i=1}^n) \) in which the ideals \( I^+_Z \) and \( I^-_Z \) can be described explicitly, and which diagonalize the natural action of \( \text{Aut}(C, \{p_i\}_{i=1}^n) \). In fact, for the purpose of computing the VGIT ideals \( I^- \) and \( I^+ \), it is sufficient to diagonalize the action of the torus \( \mathbb{G}_m^r \subset \text{Aut}(C, \{p_i\}_{i=1}^n) \) induced by automorphisms of the \( \alpha^- \)-atoms of \((C, \{p_i\}_{i=1}^n)\).

To make this precise, recall that if \((C, \{p_i\}_{i=1}^n)\) is an \( \alpha^- \)-closed curve, then \( C \) admits a canonical decomposition \( C = K \cup E_1 \cup \ldots \cup E_r \), where \( K \) is the core and \( E_1, \ldots, E_r \) are \( \alpha^- \)-atoms (Definition 2.25). Note that we always consider \( E_1, \ldots, E_r \) and \( K \) as pointed curves, even if we fail to designate the marked points explicitly. If we let \( \text{Aut}(C, \{p_i\}_{i=1}^n)^o \) denote the connected component of the identity of \( \text{Aut}(C, \{p_i\}_{i=1}^n) \), then there is natural decomposition

\[
\text{Aut}(C, \{p_i\}_{i=1}^n)^o = \text{Aut}(K)^o \oplus \bigoplus_{i=1}^r \text{Aut}(E_i),
\]

and we define \( \text{Aut}(C, \{p_i\}_{i=1}^n)^* = \bigoplus_{i=1}^r \text{Aut}(E_i) = \mathbb{G}_m^r \). In Proposition 3.33, we show that the character \( \chi_{\delta - \psi}: \text{Aut}(C, \{p_i\}_{i=1}^n)^o \to \mathbb{G}_m \) is trivial on \( \text{Aut}(K)^o \) so that the VGIT ideals with respect to \( \text{Aut}(C, \{p_i\}_{i=1}^n)^o \) are identical to those with respect to \( \text{Aut}(C, \{p_i\}_{i=1}^n)^* \).

Happily, it is fairly straightforward to describe the action of \( \text{Aut}(C, \{p_i\}_{i=1}^n)^* \) on \( \hat{\text{Def}}(C, \{p_i\}_{i=1}^n) \). We begin by describing the action of \( \text{Aut}(E) \) on the space of first-order deformations \( T^1(E) \) of a single \( \alpha^- \)-atom \( E \) (Proposition 3.21). After characterizing the combinatorial types of
\(\alpha\)-closed curves (Definition [3.23]), we describe the action of of \(\text{Aut}(C, \{p_i\}_{i=1}^n)^*\) on the first-order deformation space \(T^1(C; \{p_i\}_{i=1}^n)\) for each combinatorial type of an \(\alpha\)-closed curve \((C, \{p_i\}_{i=1}^n)\) (Proposition [3.24]). Finally, we pass from coordinates on the first-order deformation space to coordinates on the formal deformation space \(\text{Def}(C, \{p_i\}_{i=1}^n)\) (Proposition 3.27).

If \((E, q)\) (resp. \((E, q_1, q_2)\)) is an \(\alpha\)-atom with singular point \(\eta \in E\), we may fix an isomorphism \(\text{Aut}(E) \cong G_m = \text{Spec} \mathbb{C}[t]\) and coordinates on \(\hat{O}_{E,\xi}\) and \(\hat{O}_{E,q}\) (resp. \(\hat{O}_{E,q_1}\) and \(\hat{O}_{E,q_2}\)) so that the action of \(\text{Aut}(E)\) is specified by:

- \(\alpha = 9/11\): \(\hat{O}_{E,\xi} \cong \mathbb{C}[x, y]/(y^2 - x^3)\), \(\hat{O}_{E,q} \cong k[[n]]\), and \(G_m\) acts by \(x \mapsto t^{-2} x, y \mapsto t^{-3} y, n \mapsto tn\).

- \(\alpha = 7/10\): \(\hat{O}_{E,\xi} \cong \mathbb{C}[x, y]/(y^2 - x^4)\), \(\hat{O}_{E,q_1} \cong k[[n_1]]\), \(\hat{O}_{E,q_2} \cong k[[n_2]]\), and \(G_m\) acts by \(x \mapsto t^{-1} x, y \mapsto t^{-2} y, n_1 \mapsto tn_1, n_2 \mapsto tn_2\).

- \(\alpha = 2/3\): \(\hat{O}_{E,\xi} \cong \mathbb{C}[x, y]/(y^2 - x^5)\), \(\hat{O}_{E,q} \cong k[[s]]\) and \(G_m\), acts by \(x \mapsto t^{-2} x, y \mapsto t^{-5} y, s \mapsto ts\).

We have an exact sequence of of \(\text{Aut}(E)\)-representations

\[
0 \to \text{Cr}^1(E) \xrightarrow{\alpha} T^1(E) \xrightarrow{\beta} T^1(\hat{O}_{E,\xi}) \to 0
\]

where \(T^1(\hat{O}_{E,\xi})\) denotes the space of first-order deformations of the singularity \(\xi \in E\), and \(\text{Cr}^1(E)\) denotes the space of first-order deformations which induces trivial deformations of \(\xi\). In fact, since the pointed normalization of \(E\) has no non-trivial deformations, we may identity \(\text{Cr}^1(E)\) with the space of crimping deformations, i.e. deformations which fix the pointed normalization and the analytic isomorphism type of the singularity. Note that in the cases \(\alpha = 9/11\) and \(\alpha = 7/10\), \(\text{Cr}^1(E) = 0\), i.e. there is a unique way to impose a cusp on a two-pointed rational curve (resp. tacnode on a pair of two-pointed rational curves).

Proposition 3.21. Let \(E\) be an \(\alpha\)-atom. Fix \(\text{Aut}(E) \cong G_m\) as above.

- \(\alpha = 9/11\): \(T^1(E) \cong T^1(\hat{O}_{E,\xi})\) and there are coordinates on \(T^1(\hat{O}_{E,\xi})\) with weights \(-4, -6\).

- \(\alpha = 7/10\): \(T^1(E) \cong T^1(\hat{O}_{E,\xi})\) and there are coordinates on \(T^1(\hat{O}_{E,\xi})\) with weights \(-2, -3, -4\).

- \(\alpha = 2/3\): \(T^1(E) \cong \text{Cr}^1(E) \oplus T^1(\hat{O}_{E,\xi})\) and there are coordinates on \(\text{Cr}^1(E)\) and \(T^1(\hat{O}_{E,\xi})\) with with weights \(1\) and \(-4, -6, -8, -10\), respectively.

Proof. We prove the case \(\alpha = 2/3\) and leave the other cases to the reader. We have

\(T^1(\hat{O}_{E,\xi}) \cong \mathbb{C}^{\oplus 4}\), Spec \(\mathbb{C}[x, y, \varepsilon]/(y^2 - x^5 - s_1^* \varepsilon x^3 - s_2^* \varepsilon x^2 - s_3^* \varepsilon x - s_0^* \varepsilon^2) \mapsto (s_0^*, s_1^*, s_2^*, s_3^*)\), and \(G_m\) acts by \(s_i^* \mapsto t^{10 - 2i} s_i^*\). Thus, \(G_m\) acts on \(T^1(\hat{O}_{E,\xi})^*\) by \(s_i \mapsto t^{2i - 10} s_i\).
From \cite{vdW10}, Example 1.111, we have
\[ \text{Cr}^1(E) \xrightarrow{\sim} C, \quad \text{Spec} \mathbb{C}[(s + c^* e s^2)^2, (s + c^* e s^2)^5, \varepsilon]/(\varepsilon)^2 \mapsto c^*, \]
and \( G_m \) acts by \( c^* \mapsto t^{-1} c^* \). Thus, \( G_m \) acts on \( \text{Cr}^1(E)^\vee \) by \( c \mapsto tc \). \( \square \)

This proposition immediately implies a description of \( \text{Aut}(C, \{ p_i \}_{i=1}^n) \) on \( T_1^1(C, \{ p_i \}_{i=1}^n) \) for any \( \alpha_c \)-closed curve. But in order to spell this out explicitly however, we must introduce notation for handling the various combinatorial types of \( \alpha_c \)-closed curves. For this, one auxiliary definition is necessary in the case \( \alpha_c = 7/10 \).

**Definition 3.22** (Link). We say that \((E, p_0, p_l)\) is a *link of length* \( l \) if there exists a surjective gluing morphism
\[ \gamma: \prod_{j=1}^l (E_i, q_{j-1}, q_j) \hookrightarrow (E, p_0, p_l) \]
where

1. \((E_j, q_j, q_{j+1})\) is an \( A_3 \)-atom for \( j = 1, \ldots, l \).
2. \( E_j \) meets \( E_{j+1} \) at a node \( q_j \) for \( j = 1, \ldots, l - 1 \).
3. \( \gamma(q_j) = p_j \) for \( j = 0, l \).

![Figure 8. A link of length 3.](image)

**Definition 3.23** (Combinatorial Type of \( \alpha_c \)-closed curve).

- A \( \frac{9}{11} \)-closed curve \((C, \{ p_i \}_{i=1}^n)\) has **combinatorial type**
  (A) If the core is nonempty. In this case, \( C = K \cup E_1 \cup \cdots \cup E_r \) where each \( E_i \) is a \( \frac{9}{11} \)-atom meeting \( K \) at a single node \( q_i \).
  (B) If \((g, n) = (2, 0)\) and \( C = E_1 \cup E_2 \) where \( E_1 \) and \( E_2 \) are \( \frac{9}{11} \)-atoms meeting each other in a single node \( q \in C \).
  (C) If \((g, n) = (1, 1)\) and \( C = E_1 \) is a \( \frac{9}{11} \)-atom.

- A \( \frac{7}{10} \)-closed curve \((C, \{ p_i \}_{i=1}^n)\) has **combinatorial type**
  (A) If the core is nonempty. In this case, we have \( C = K \cup E_1 \cup \cdots \cup E_r \cup E_{r+1} \cup \cdots \cup E_{r+s} \) where
    - For \( i = 1, \ldots, r \), \( E_i = \bigcup_{j=1}^{l_i} E_{i,j} \) is an elliptic link of length \( l_i \) meeting \( K \) at two distinct nodes. In particular, \( E_{i,1} \) meets \( K \) at a node \( q_{i,0} \), \( E_{i,l_i} \) meets \( K \) at a node \( q_{i,l_i} \), and \( E_{i,j} \) meets \( E_{i,j+1} \) at a node \( q_{i,j} \).
• For \( i = r + 1, \ldots, r + s \), \( E_i = \bigcup_{j=1}^{l_i} E_{i,j} \) is a link of length \( l_i \) meeting \( K \) at a single node and terminating in a marked point. In particular, \( E_{i,1} \) meets \( K \) at a node \( q_{i,0} \), and \( E_{i,j} \) meets \( E_{i,j+1} \) at a node \( q_{i,j} \).

(B) If \( n = 2 \) and \((C, p_1, p_2)\) is a link of length \( r \), i.e. \( C = E_1 \cup \ldots \cup E_r \) where each \( E_i \) is a \( \frac{7}{10} \)-atom, \( E_i \) meets \( E_{i+1} \) at a node \( q_i \), and \( p_1 \in E_1, p_2 \in E_r \).

(C) If \( n = 0 \) and \( C \) is a link of length \( r \), whose endpoints are nodally glued. In other words, \( C = E_1 \cup \ldots \cup E_r \), where each \( E_i \) is a \( \frac{7}{10} \)-atom, \( E_i \) meets \( E_{i+1} \) at a node \( q_i \), and \( E_1 \) meets \( E_r \) at a node \( q_0 \).

• A \( \frac{2}{3} \)-closed curve \((C, \{p_i\}_{i=1}^n)\) has combinatorial type

(A) If the core is nonempty. In this case, \( C = K \cup E_1 \cup \ldots \cup E_r \) where each \( E_i \) is a 2/3-atom meeting \( K \) at a single node \( q_i \).

(B) If \((g, n) = (2, 0)\) and \( C = E_1 \cup E_2 \) where \( E_1 \) and \( E_2 \) are 2/3-atoms meeting each other in a single node \( q \in C \).

(C) If \((g, n) = (2, 1)\) and \( C = E_1 \) is a 2/3-atom.

**Proposition 3.24** (Diagonalized Coordinates on \( T^1(C, \{p_i\}_{i=1}^n) \)). Let \((C, \{p_i\}_{i=1}^n)\) be an \( \alpha_c \)-closed curve. Using the notation of Definition 3.23.

- \( \alpha_c = 9/11 \) of Type A. There exist decompositions

\[
\text{Aut}(C, \{p_i\}_{i=1}^n)^* = \prod_{i=1}^r \text{Aut}(E_i)
\]

\[
T^1(C, \{p_i\}_{i=1}^n) = T^1(K) \oplus \bigoplus_{i=1}^r T^1(E_i) \oplus \bigoplus_{i=1}^r T^1(\hat{\mathcal{C}_{C,q_i}})
\]

For \( 1 \leq i \leq r \), let \( t_i \) be the coordinate on \( \text{Aut}(E_i) \cong \mathbb{G}_m \). Let \( g(K) \) be the arithmetic genus of the core \( K \). There are coordinates

- "core" \( k = (k_1, \ldots, k_{3g(K)-3+n+r}) \) on \( T^1(K) \)
- "singularity" \( s_i = (s_{i,0}, s_{i,1}) \) on \( T^1(\hat{\mathcal{C}_{E_i,q_i}}) \) for \( 1 \leq i \leq r \)
- "node" \( n_i \) on \( T^1(\hat{\mathcal{C}_{C,q_i}}) \) for \( 1 \leq i \leq r \)

such that the action of \( \text{Aut}(C, \{p_i\}_{i=1}^n)^* \) on \( T^1(C, \{p_i\}_{i=1}^n) \) is given by

\[
k_i \mapsto k_i, \quad s_{i,0} \mapsto t_i^{-6}s_{i,0}, \quad s_{i,1} \mapsto t_i^{-4}s_{i,1}, \quad n_i \mapsto t_i n_i.
\]

- \( \alpha_c = 9/11 \) of Type B. There are decompositions

\[
\text{Aut}(C)^* = \prod_{i=1}^2 \text{Aut}(E_i) \quad T^1(C) = \bigoplus_{i=1}^2 T^1(E_i) \oplus T^1(\hat{\mathcal{C}_{C,q}})
\]

For \( 1 \leq i \leq 2 \), let \( t_i \) be the coordinate on \( \text{Aut}(E_i) \cong \mathbb{G}_m \). There are coordinates \( s_i = (s_{i,0}, s_{i,1}) \) on \( T^1(E_i) \) with the action \( s_{i,0} \mapsto t_i^{-6}s_{i,0} \) and \( s_{i,1} \mapsto t_i^{-4}s_{i,1} \) and a coordinate \( n \) on \( T^1(\hat{\mathcal{C}_{C,q}}) \) such that \( n \mapsto t_1 t_2 n \).
Figure 9. The left (resp. right) column indicates the combinatorial types of $\frac{7}{10}$-closed (resp. $\frac{2}{3}$-closed) curves. The combinatorial types of $\frac{2}{11}$-closed curves is similar to that of $\alpha = 2/3$. 
There are coordinates $t$ such that the action of $\alpha$ is given by

$$\alpha \cdot t_i = c_i t_i \quad \alpha \cdot n_i = n_i$$

for $i = 1, \ldots, n$. This case is described in Proposition 3.21.

There are decompositions

$$\text{Aut}(C, \{p_i\}^n_{i=1}) = \prod_{i=1}^{r+s} \prod_{j=1}^{l_i} \text{Aut}(E_{i,j})$$

$$T^1(C, \{p_i\}^n_{i=1}) = T^1(K) \oplus \bigoplus_{i=1}^{r+s} \bigoplus_{j=1}^{l_i} T^1(E_{i,j}) \oplus \bigoplus_{i=1}^r T^1(\mathcal{O}_{C,q_i})$$

Let $t_{i,j}$ be the coordinate on $\text{Aut}(E_{i,j}) \cong \mathbb{G}_m$ and let $g(K)$ be the arithmetic genus of $K$. There are coordinates

- "core" $k = (k_i)_{i=1}^{3g(K)-3+n+2r}$ on $T^1(K)$
- "singularity" $s_{i,j} = (s_{i,j,t})_{t=0}^2$ on $T^1(E_{i,j})$ for $1 \leq i \leq r+s$, $1 \leq j \leq l_i$
- "node" $n_{i,j}$ on $T^1(\mathcal{O}_{C,q_i})$ for $1 \leq i \leq r$ such that the action of $\text{Aut}(C, \{p_i\}^n_{i=1})$ is given by

$$k_i \mapsto k_i, \quad s_{i,j,t} \mapsto t_{i,j}^{-4} s_{i,j,t}, \quad n_{i,0} \mapsto t_{i,1} n_{i,0}, \quad n_{i,l_i} \mapsto t_{i,i} n_{i,l_i}, \quad n_{i,j} \mapsto t_{i,j} t_{i,j+1} n_{i,j} (j \neq 0, l_i),$$

- $\alpha_c = 7/10$ of Type B. There are decompositions

$$\text{Aut}(C, p_1, p_2)^* = \prod_{i=1}^r \text{Aut}(E_i)$$

$$T^1(C, p_1, p_2) = \bigoplus_{i=1}^r T^1(E_i) \oplus \bigoplus_{i=1}^{r-1} T^1(\mathcal{O}_{C,q_i})$$

There are coordinates $s_i = (s_{i,0}, s_{i,1}, s_{i,2})$ on $T^1(E_i)$ with an action of $\text{Aut}(E_i) \cong \mathbb{G}_m$ with coordinate $t_i$ as in Proposition 3.21 and coordinates $n_i$ on $T^1(\mathcal{O}_{C,q_i})$ for $i = 1, \ldots, r-1$ with action $n_i \mapsto t_i t_{i+1} n_i$.

- $\alpha_c = 7/10$ of Type C. There are decompositions

$$\text{Aut}(C)^* = \prod_{i=1}^r \text{Aut}(E_i)$$

$$T^1(C) = \bigoplus_{i=1}^r T^1(E_i) \oplus \bigoplus_{i=0}^{r-1} T^1(\mathcal{O}_{C,q_i})$$
There are coordinates $s_i = (s_{i,0}, s_{i,1}, s_{i,2})$ on $T^1(E_i)$ with an action of $\text{Aut}(E_i) \cong \mathbb{G}_m$ with coordinate $t_i$ as in Proposition 3.21 and coordinates $n_i$ on $T^1(\hat{\mathcal{O}}_{C,q})$ for $i = 0, \ldots, r - 1$ with action $n_i \mapsto t_i t_{i-1} n_i$ where $t_{-1} := t_{t-1}$.

• $\alpha_c = 2/3$ of Type A. There exist decompositions

$$\text{Aut}(C, \{p_i\}_{i=1}^n)^* = \prod_{i=1}^r \text{Aut}(E_i)$$

$$T^1(C, \{p_i\}_{i=1}^n) = T^1(K) \oplus \bigoplus_{i=1}^r T^1(E_i) \oplus \bigoplus_{i=1}^r T^1(\hat{\mathcal{O}}_{C,q})$$

For $1 \leq i \leq r$, let $t_i$ be the coordinate on $\text{Aut}(E_i) \cong \mathbb{G}_m$. Let $g(K)$ be the arithmetic genus of the core $K$. There are coordinates

- "core" $k = (k_1, \ldots, k_{3g(K)-3+n+r})$ on $T^1(K)$
- "singularity" $s_i = (s_{i,0}, s_{i,1}, s_{i,2}, s_{i,3})$ on $T^1(\hat{\mathcal{O}}_{E_i,q})$ for $1 \leq i \leq r$
- "crimping" $c_i$ on $C^1(E_i)$ for $1 \leq i \leq r$
- "node" $n_i$ on $T^1(\hat{\mathcal{O}}_{C,q})$ for $1 \leq i \leq r$

such that the action of $\text{Aut}(C, \{p_i\}_{i=1}^n)^*$ on $T^1(C, \{p_i\}_{i=1}^n)$ is given by

$$k_i \mapsto k_i, \quad s_{i,l} \mapsto t_i^{2l-10} s_{i,l}, \quad c_i \mapsto t_i c_i, \quad n_i \mapsto t_i n_i.$$

• $\alpha_c = 2/3$ of Type B. There are decompositions

$$\text{Aut}(C)^* = \prod_{i=1}^2 \text{Aut}(E_i) \quad T^1(C) = \bigoplus_{i=1}^r T^1(E_i) \oplus T^1(\hat{\mathcal{O}}_{C,q})$$

For $1 \leq i \leq 2$, let $t_i$ be the coordinate on $\text{Aut}(E_i) \cong \mathbb{G}_m$. There are coordinates $s_i = (s_{i,0}, s_{i,1}, s_{i,2}, s_{i,3}, c_i)$ on $T^1(E_i)$ with the action $s_{i,l} \mapsto t_i^{2l-10} s_{i,l}, \ c_i \mapsto t_i c_i$ and a coordinate $n$ on $T^1(\hat{\mathcal{O}}_{C,q})$ such that $n \mapsto t_1 t_2 n$.

• $\alpha_c = 2/3$ of Type C. This case is described in Proposition 3.21.

Proof. This follows easily from Proposition 3.21 and the observation that each factor $\text{Aut}(E_i)$ acts trivially on $T^1(K)$. \qed

It is evident that the coordinates of Proposition 3.24 on $T^1(C, \{p_i\}_{i=1}^n)$ diagonalize the natural action of $\text{Aut}(C, \{p_i\}_{i=1}^n)^*$. However, we need slightly more. We need coordinates which diagonalize the natural action of $\text{Aut}(C, \{p_i\}_{i=1}^n)^*$ and which cut out the natural geometrically-defined loci on $\text{Def}(C, \{p_i\}_{i=1}^n) = \text{Spf} \mathbb{C}[T^1(C, \{p_i\}_{i=1}^n)]$. More precisely, the $\{s_i\}$ coordinates should cut out the locus of formal deformations preserving the singularities and the $\{c_i, n_i\}$ coordinates should cut out the locus of formal deformations preserving a Weierstrass tail. This is almost a purely formal statement (see Lemma 3.26 below); however there is one non-trivial geometric input. We must show that the crimping coordinate which defines the locus of ramphoid cuspidal deformations with trivial crimping can be extended
Lemma 3.25. With notation as above, there exist isomorphisms of tangent spaces of these global stacks are naturally identified as deformations of the singularity \( \text{Cr} \) where \( T^{(3.4)} \) 0

(3.3) 

\[ i: T^1_{Z^+,0} \rightarrow T^1_{\mathcal{M}_{2,1}(2/3)_0} = T^1(E) \quad \text{and} \quad j: T^1_{Z^-,0} \rightarrow T^1_{\mathcal{M}_{2,1}(2/3)_0} = T^1(E). \]

On the other hand, recall that we have the exact sequence of \( \text{Aut}(E,q) \)-representations

(3.4)

\[ 0 \to \text{Cr}^1(E) \overset{\alpha}{\rightarrow} T^1(E) \overset{\beta}{\rightarrow} T^1(\hat{\Omega}_{E,\xi}) \to 0 \]

where \( T^1(\hat{\Omega}_{E,\xi}) \) denotes the space of first-order deformations of the singularity \( \xi \in E \), and \( \text{Cr}^1(E) \) denotes the space of first-order crimping deformations. The key point is that the tangent spaces of these global stacks are naturally identified as deformations of the singularity and the crimping respectively.

**Lemma 3.25.** With notation as above, there exist isomorphisms of \( \text{Aut}(E,q) \)-representations

\[
T^1_{Z^-_0} \cong T^1(\hat{\Omega}_{E,\xi}) \\
T^1_{Z^+_0} \cong \text{Cr}^1(E)
\]

inducing a splitting of \( (3.4) \) with \( i = \alpha \) and \( j = \beta^{-1} \).

**Proof.** It suffices to show that the composition

\[
\alpha \circ i: T^1_{Z^-,0} \rightarrow T^1_{\mathcal{M}_{2,1}(2/3)_0} \rightarrow T^1(\hat{\Omega}_{E,\xi})
\]

is an isomorphism, and that the composition

\[
\alpha \circ j: T^1_{Z^+,0} \rightarrow T^1_{\mathcal{M}_{2,1}(2/3)_0} \rightarrow T^1(\hat{\Omega}_{E,\xi})
\]

is zero. The latter follows from the former by transversality of \( T_{Z^-,0} \) and \( T_{Z^+,0} \). To see that \( \alpha \circ i \) is an isomorphism, observe that \( Z^{-} \cong \mathbb{A}^5/\mathbb{G}_m \) with weights \(-4\), \(-6\), \(-8\), \(-10\), where the universal family is given by

\[
(y^2 - x^5 - a_3 x^3 - a_2 x^2 - a_1 x - a_0 \varepsilon, \varepsilon^2) : a_3, \ldots, a_0 \in \mathbb{C}.
\]

On the other hand, there is a natural isomorphism

\[
T^1(\hat{\Omega}_{E,\xi}) = \{ \text{Spec } \mathbb{C}[x,y,\varepsilon]/(y^2 - x^5 - a_3 x^3 - a_2 x^2 - a_1 x - a_0 \varepsilon, \varepsilon^2) : a_3, \ldots, a_0 \in \mathbb{C} \}.
\]

Evidently, \( \alpha \circ i \) is the identity map in the given coordinates. \( \square \)

**Lemma 3.26.** Let \( V \) be a finite-dimensional representation of a torus \( G \), let \( X = \text{Spf } \mathbb{C}[[V]] \), and let \( \mathfrak{m} \subseteq \mathbb{C}[[V]] \) be the maximal ideal. Suppose we are given a collection of \( G \)-invariant formal smooth closed subschemes \( Z_i := \text{Spf } \mathbb{C}[[V]]/I_i,(i = 1, \ldots, r) \) which intersect transversely at 0, and a basis \( x_1, \ldots, x_n \) for \( V \) such that:

1. \( x_1, \ldots, x_n \) diagonalize the action of \( G \).
2. \( I_i/\mathfrak{m}I_i \) is spanned by a subset of \( x_1, \ldots, x_n \).
Then there exist coordinates $X \cong \text{Spf} \mathbb{C}[[x_1',\ldots,x_k']]$ such that

1. $x_1',\ldots,x_n'$ diagonalize the action of $G$.
2. $x_1',\ldots,x_n'$ reduce modulo $m$ to $x_1,\ldots,x_n$.
3. $I_i$ is generated by a subset of $x_1',\ldots,x_n'$.

Proof. Let $x_{i_1},\ldots,x_{i_d}$ be a diagonal basis for $I_i/mI_i$ as a $G$-representation. Consider the surjection

$$I_i \to I_i/mI_i$$

and choose an equivariant section, i.e. choose $x_{i_1}',\ldots,x_{i_d}'$ such that each spans a one-dimensional sub-representation of $G$. By Nakayama’s Lemma, these elements generate $I_i$. Repeating this procedure for each $Z_i$, we obtain $x_{i,j}'$ for $i = 1,\ldots,r$ and $j = 1,\ldots,d_i$. Since the $Z_i$ intersect transversely, these coordinates induce linearly independent elements of $V$, so they may be completed to a diagonal basis, and this gives the necessary coordinate change. \hfill $\square$

**Proposition 3.27** (Explicit Description of $I_{Z^+}, I_{Z^-}$). Let $(C, \{p_i\}_{i=1}^n)$ be an $\alpha_c$-closed curve. There exist coordinates $n_i, s_i, c_i$ (resp. $n_{i,j}, s_{i,j}$) on $\text{Def}(C, \{p_i\}_{i=1}^n)$ such that the action of $\text{Aut}(C, \{p_i\}_{i=1}^n)^*$ on $\text{Def}(C, \{p_i\}_{i=1}^n)$ is given as in Proposition 3.24, and such that the ideals $I_{Z^+}, I_{Z^-}$ are given as follows:

- $\alpha_c = 9/11$, Type A: $I_{Z^+} = \bigcap_{i=1}^r (s_i)$, $I_{Z^-} = \bigcap_{i=1}^r (n_i)$.
- $\alpha_c = 9/11$, Type B: $I_{Z^+} = (s_1) \cap (s_2)$, $I_{Z^-} = (n_1, n_2)$.
- $\alpha_c = 9/11$, Type C: $I_{Z^+} = (s)$, $I_{Z^-} = (0)$.
- $\alpha_c = 7/10$, Type A: $I_{Z^+} = \bigcap_{i,j} (s_{i,j})$, $I_{Z^-} = \bigcap_{i,\mu,\nu \in S} J_{i,\mu,\nu}$ where $S := \{i,\mu,\nu : 1 \leq i \leq r + s, 1 \leq \mu \leq \left[\frac{l}{2}\right], 0 \leq \nu \leq l - 2\mu + 1\}$
  $$J_{i,\mu,\nu} := (n_{i,\mu,\nu}, s_{i,\mu+2}, \ldots, s_{i,\mu+2\nu-2}, n_{i,\mu+2\nu-1}), \text{ for } i = 1,\ldots,r$$
  $$J_{i,\mu,\nu} := (n_{i,\mu,\nu}, s_{i,\mu+2}, \ldots, s_{i,\mu+2\nu-2}), \text{ for } i = r + 1,\ldots,r + s.$$
- $\alpha_c = 7/10$, Type B: $I_{Z^+} = \bigcap_i (s_i)$, $I_{Z^-} = \bigcap_{\mu,\nu \in S} J_{\mu,\nu}$ where $S := \{\mu,\nu : 1 \leq \mu \leq \left[\frac{l}{2}\right], 0 \leq \nu \leq l - 2\mu + 1\}$
  $$J_{\mu,\nu} := (n_\nu, s_\nu+2, \ldots, s_{\nu+2\mu-2}, n_{\nu+2\mu-1}),$$
  and $n_0 := 0$ and $n_l := 0$.
- $\alpha_c = 7/10$, Type C: $I_{Z^+} = \bigcap_i (s_i)$, $I_{Z^-} = \bigcap_{\mu,\nu \in S} J_{\mu,\nu}$ where $S := \{\mu,\nu : 1 \leq \mu \leq \left[\frac{l}{2}\right], 0 \leq \nu \leq l - 1\}$
  $$J_{\mu,\nu} := (n_\nu, s_\nu+2, \ldots, s_{\nu+2\mu-2}, n_{\nu+2\mu-1}),$$
  and the subscripts are taken modulo $l$. 

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• $\alpha_c = 2/3$, Type A: $I_{Z^+} = \bigcap_{i=1}^{r}(s_i), I_{Z^-} = \bigcap_{i=1}^{r}(c_i, n_i)$.
• $\alpha_c = 2/3$, Type B: $I_{Z^+} = (s_1) \cap (s_2), I_{Z^-} = (n, c_1) \cap (n, c_2)$.
• $\alpha_c = 2/3$, Type C: $I_{Z^+} = (s), I_{Z^-} = (c)$.

Proof. We prove the statement when $(C, \{p_i\}_{i=1}^{n})$ is a $2/3$-closed curve of combinatorial type A; the other cases are similar and left to the reader. Let $\hat{\operatorname{Def}}(C, \{p_i\}_{i=1}^{n}) = \operatorname{Spf} \hat{A} \to \overline{\mathcal{M}}_{g,n}(2/3)$ be a miniversal deformation space of $(C, \{p_i\}_{i=1}^{n})$. Formally locally around $(C, \{p_i\}_{i=1}^{n})$, $Z^+$ and $Z^-$ decompose as

$$Z^+ \times_{\overline{\mathcal{M}}_{g,n}(2/3)} \operatorname{Spf} \hat{A} = Z^+_1 \cup \ldots Z^+_r, \quad Z^- \times_{\overline{\mathcal{M}}_{g,n}(2/3)} \operatorname{Spf} \hat{A} = Z^-_1 \cup \ldots Z^-_r$$

where

• $Z^+_i = \operatorname{Spf} \hat{A}/I_{Z^+_i}$ is the locus of deformations which preserve the $i^{th}$ ramphoid cusp.
• $Z^-_i = \operatorname{Spf} \hat{A}/I_{Z^-_i}$ is the locus of deformations which preserve the $i^{th}$ Weierstrass tail.

Since $Z^+_i$ (resp. $Z^-_i$) are smooth, $G$-invariant, formal closed subschemes of $\operatorname{Spf} \hat{A}$, the conormal space of $Z^+_i$ (resp. $Z^-_i$) is canonically identified with $I_{Z^+_i}/m_{\hat{A}}I_{Z^+_i}$ (resp. $I_{Z^-_i}/m_{\hat{A}}I_{Z^-_i}$). Thus, in the notation of Proposition 3.24 we have

$$I_{Z^+_i}/m_{\hat{A}}I_{Z^+_i} \cong \mathcal{T}^1(\hat{\omega}_{E_i, q_i})^\vee,$$

$$I_{Z^-_i}/m_{\hat{A}}I_{Z^-_i} \cong \mathcal{C}^1(E_i)^\vee \oplus \mathcal{T}^1(\hat{\omega}_{E_i, q_i})^\vee$$

where the second isomorphism uses Lemma 3.25 to identify $\mathcal{C}^1(E_i)^\vee$ as the conormal space of the locus of deformations of $E_i$ for which the attaching node remains Weierstrass. As these are linearly independent subspaces of $\mathcal{T}^1(C, \{p_i\}_{i=1}^{n})$, we may apply Lemma 3.26 to the collection of formal closed subschemes $\{Z^+_i, Z^-_i\}$ to obtain coordinates with the required properties. \hfill \Box

3.4. Local VGIT chambers for an $\alpha$-closed curve. In this section, we explicitly compute the VGIT ideals $I^+, I^- \subseteq \mathbb{C}[\mathcal{T}^1(C, \{p_i\}_{i=1}^{n})]$ (Definition 3.11) for any $\alpha_c$-closed curve. The main result (Proposition 3.35) states that the VGIT ideals agree formally locally with the ideals $I_{Z^+}, I_{Z^-}$. By Proposition 3.12, this suffices to establish Theorem 3.9. In order to carry out the computation of $I^+$ and $I^-$, we must do two things: First, we must explicitly identify the character $\chi_{\delta-\psi}: \operatorname{Aut}(C, \{p_i\}_{i=1}^{n}) \to \mathbb{G}_m$ for any $\alpha$-closed curve. Second, we must compute the ideals of positive and negative semi-invariants with respect to this character.

In order to identify the character $\chi_{\delta-\psi}$, it will be useful to characterize the subcurves of an $(\alpha_c+\epsilon)$-stable curve $(C, \{p_i\}_{i=1}^{n})$ which contribute to automorphisms in $\operatorname{Aut}(C, \{p_i\}_{i=1}^{n})^\circ$. We therefore introduce the following definition.
Definition 3.28 (Rosary). We say that \((R, p_1, p_2)\) is a rosary of length \(l\) if there exists a surjective gluing morphism

\[
\gamma: \prod_{i=1}^{l} (R_i; q_{2i-1}, q_{2i}) \hookrightarrow (R, p_1, p_2)
\]

satisfying:

1. \((R_i, q_{2i-1}, q_{2i})\) is a two-pointed smooth rational curve for \(i = 1, \ldots, l\).
2. \(\gamma(q_{2i}) = \gamma(q_{2i+1})\) is an \(A_3\)-singularity for \(i = 1, \ldots, l - 1\).
3. \(\gamma(q_1) = p_1\) and \(\gamma(q_{2l}) = p_2\).

We say that \((R, p_1, p_2)\) is a rosary of length \(l\) in \((C, \{p_i\}_{i=1}^n)\) if \((R, p_1, p_2)\) is a rosary of length \(l\) and there is a gluing morphism \(\gamma: (R, p_1, p_2) \hookrightarrow (C, \{p_i\}_{i=1}^n)\) such that either: (a) \(\gamma(p_j)\) are distinct marked points or nodes for \(j = 1, 2\), or (b) \(\gamma(p_1) = \gamma(p_2)\) is a node or tacnode (in which case \(\gamma\) is surjective).

![Figure 10](image)  
**Figure 10.** Curve A is a rosary of length 3. Curve B contains a rosary of length 4.

Remark 3.29. Our terminology agrees with that of [HH13] Definitions 6.1, 6.3 except that Hassett and Hyeon distinguish between an open and closed rosary. An open (resp. closed) rosary is a rosary such that condition (a) (resp. (b)) above is satisfied.

Remark 3.30. Observe that \(\text{Aut}(R, p_1, p_2) \cong \mathbb{G}_m\).

Lemma 3.31. Suppose \((C, \{p_i\}_{i=1}^n)\) is \((2/3+\epsilon)\)-stable curve such that \(\text{Aut}(C, \{p_i\}_{i=1}^n) \circ \cong \mathbb{G}_m^k\). Then \(C = C_0 \cup R_1 \cup \cdots \cup R_k\) where each \((R_i, p_{i,1}, p_{i,2})\) is a rosary of odd length in \((C, \{p_i\}_{i=1}^n)\).

Proof. We argue inductively on the rank \(k\) of \(\text{Aut}(C, \{p_i\}_{i=1}^n)\). Let \(R_1, \ldots, R_k\) be the rational irreducible components of \(C\) such that there are precisely two points \(q_{2i-1}\) and \(q_{2i}\) on \(R_i\) which are marked points of \(C\), nodes in \(R_i \cap C \setminus R_i\), or tacnodes in \(R_i \cap C \setminus R_i\) lying on a smooth rational irreducible component of \(C \setminus R_i\). If \(k > 0\), one sees that there must be a non-empty
subset \( \{i_1, \ldots, i_t\} \subseteq \{1, \ldots, s\} \) such that \((R, p_{i_1}, p_{i_2}) := (R_{i_1}, q_{2i_1-1}, q_{2i_1}) \cup \cdots \cup (R_{i_t}, q_{2i_t-1}, q_{2i_t})\) is a rosary of length \( l \) in \( C \). Since the rank of the automorphism group of \( C \setminus R \) is \( k - 1 \), we may apply the inductive hypothesis to \( C \setminus R \).

**Definition 3.32.** Let \( E_1, \ldots, E_r \) be the \( \alpha_c \)-atoms of \( (C, \{p_i\}_{i=1}^n) \), and let \( t_i \in \text{Aut}(E_i) \) be the coordinate specified in Proposition 3.24. Let

\[
\chi_{\star} : \text{Aut}(C, \{p_i\}_{i=1}^n)^o \to \mathcal{G}_m = \text{Spec} \mathbb{C}[t]
\]

be the character defined by \( t \mapsto t_1 t_2 \cdots t_r \). Note that \( \chi_{\star} \) is trivial on automorphisms fixing the \( \alpha_c \)-atoms.

The following proposition shows that \( \chi_{\delta - \psi} \) is simply a positive multiple of \( \chi_{\star} \). Since it will be important in Proposition 3.33, we also prove now that the character of \( K_{\mathcal{M}_{g,n}(\alpha_c)} + \alpha_c \delta + (1 - \alpha_c) \psi \) is trivial for \( \alpha_c \)-closed curves.

**Proposition 3.33.** Let \( \alpha_c \in \{9/11, 7/10, 2/3\} \) be a critical value.

1. There exists a positive integer \( N \) such that \( \chi_{\delta - \psi}|_{G^o} = \chi_{\star}^N \) for every \( \alpha_c \)-closed curve \( (C, \{p_i\}_{i=1}^n) \). Specifically,

\[
N = \begin{cases} 
11 & \text{if } \alpha_c = 9/11 \\
10 & \text{if } \alpha_c = 7/10 \\
39 & \text{if } \alpha_c = 2/3
\end{cases}
\]

2. For each \( \alpha_c \)-closed curve, the induced character of \( \text{Aut}(C, \{p_i\}_{i=1}^n)^o \) on the fiber of \( K_{\mathcal{M}_{g,n}(\alpha_c)} + \alpha_c \delta + (1 - \alpha_c) \psi \) is trivial.

**Proof.** We prove the case when \( \alpha_c = 2/3 \) for an \( \alpha_c \)-closed curve \( (C, \{p_i\}_{i=1}^n) \) of type A. The other cases are easier and argued similarly; we refer the reader to [AFS10] for more details and for a more general discussion of how characters are computed for singular curves admitting a \( \mathcal{G}_m \)-action.

We first observe that since \( K_{\mathcal{M}_{g,n}(\alpha_c)} = 13\lambda - 2\delta + \psi \), for any \( \alpha \) we have the identity

\[
K_{\mathcal{M}_{g,n}(\alpha_c)} = \alpha \delta + (1 - \alpha) \psi = 13\alpha - (\alpha - 2)(\delta - \psi).
\]

Let \( C = K \cup E_1 \cup \cdots \cup E_r \) be the canonical decomposition. Then \( G^o = G^* \times \text{Aut}(K)^o \) where \( G^* = \prod_i \text{Aut}(E_i) \). We claim first that \( \text{Aut}(K)^o \) acts trivially on the fiber of \( K_{\mathcal{M}_{g,n}(\alpha_c)} + \psi \) and \( \delta - \psi \) or, equivalently, \( \lambda \) and \( \delta - \psi \). Indeed, if \( \text{Aut}(K)^o \cong \mathcal{G}_m^k \), then by Lemma 3.31 there exist rosaries \((R_1, p_{1,1}, p_{1,2}), \ldots, (R_k, p_{k,1}, p_{k,2})\) of odd length and one-parameter subgroups \( \rho_i : \mathcal{G}_m \to \text{Aut}(R_i, p_{i,1}, p_{i,2}) \subseteq \text{Aut}(K)^o \). It is clear from the character computations of [AFS10, Section 3.2] that the characters of \( \mathcal{G}_m \) induced by \( \rho_i \) of \( \lambda \) and \( \delta - \psi \) is trivial since the contribution of each tacnode to \( \lambda \) and \( \delta - \psi \) is cancelled out by an adjacent tacnode, and the contribution to \( \delta - \psi \) of a node or marked point \( p_{i,1} \) is cancelled out by the contribution of \( p_{i,2} \). Since any automorphism of \( \text{Aut}(K)^o \) can be written as a product of images of the \( \rho_i \)'s, the claim follows. See Remark 3.34 for an alternative argument.
For \( i = 1, \ldots, r \), define the one-parameter subgroup \( \rho_i : \mathbb{C} \rightarrow G^* \) by \( t_i \mapsto t_i \) and \( t_j \mapsto 1 \) for \( j \neq i \). For \( (1) \), from the claim above and since \( \langle \chi_\alpha, \rho_i \rangle = 1 \), it suffices to prove that \( \langle \chi_\alpha, \rho_i \rangle = 39 \). Similarly, to prove \((2)\), it suffices to prove for each \( i \) that \( \langle \chi_{K \circ \alpha, \phi + (1 - \alpha) \psi}, \rho_i \rangle = 0 \). By [AFS10, Section 3.1.3], \( \langle \chi_{\lambda}, \rho_i \rangle = 4 \), \( \langle \chi_\delta, \rho_i \rangle = 39 \) and \( \langle \chi_\psi, \rho_i \rangle = 0 \) so the statement follows from Equation 3.5. \( \Box \)

**Remark 3.34.** For an \( \alpha_c \)-stable curve \( (C, \{p_i\}_{i=1}^n) \), the fact that \( \text{Aut}(K) \) acts trivially on \( \lambda \) and \( \delta - \psi \) follows from purely formal reasons if one assumes inductively that the Main Theorem in Section 4 is true. If \( \alpha_c = 9/11, 7/10 \), the statement is obvious as the core has no infinitesimal automorphisms. For \( \alpha_c = 2/3 \), the core \( K \) is \( \alpha \)-stable for all \( \alpha \in (2/3, 7/10) \) and therefore is an element of \( \overline{\mathcal{M}}_{g-r,n+r}(\alpha) \). Since \( \phi : \overline{\mathcal{M}}_{g-r,n+r}(\alpha) \rightarrow \overline{\mathcal{M}}_{g-r,n+r}(\alpha) \) is a good moduli space with \( \phi^* O(1) = K_{\overline{\mathcal{M}}_{g-r,n+r}(\alpha)} + \alpha \delta + (1 - \alpha) \psi \) for all \( \alpha \in (2/3, 7/10) \), the character of \( \text{Aut}(K)^0 \) on the fiber of \( K_{\overline{\mathcal{M}}_{g-r,n+r}(\alpha)} + \alpha \delta + (1 - \alpha) \psi \) is trivial for all \( \alpha \in (2/3, 7/10) \). Therefore, the characters of \( K_{\overline{\mathcal{M}}_{g-r,n+r}(\alpha)} + \psi \) and \( \delta - \psi \) (and \( \lambda \) as well) are trivial.

Proposition 3.33 and Lemma 3.18 imply that we can compute the VGIT ideals \( I^- \) and \( I^+ \) as the ideals of semi-invariants associated to \( \chi_* \). In the following proposition, we compute these explicitly, and show that they are identical to the ideals \( I_{Z+} \) and \( I_{Z-} \), as described in Proposition 3.27.

**Proposition 3.35** (Description of VGIT ideals). Let \( (C, \{p_i\}_{i=1}^n) \) be an \( \alpha_c \)-closed curve for a critical value \( \alpha_c \in \{2/3, 7/10, 9/11\} \). Then \( I^+ \hat{A} = I_{Z+} \) and \( I^- \hat{A} = I_{Z-} \).

We establish the proposition first in the case of an \( \alpha_c \)-atom, then in the case of a link of \( \alpha_c \)-atoms, and finally for each of the distinct combinatorial types of \( \alpha_c \)-closed curves.

**Proof of Proposition 3.35 in the case of \( \alpha_c \)-atoms.**

**Lemma 3.36.** Let \( E \) be an \( \alpha_c \)-atom. Using the notation of Proposition 3.27 for the action of \( \text{Aut}(E) \) on \( T^1(E) \), we have:

- \( \bullet \alpha_c = 9/11: \ I^+ = \langle s_1, s_0 \rangle, \quad I^- = \langle 0 \rangle. \)
- \( \bullet \alpha_c = 7/10: \ I^+ = \langle s_2, s_1, s_0 \rangle, \quad I^- = \langle 0 \rangle. \)
- \( \bullet \alpha_c = 2/3: \ I^+ = \langle s_3, s_2, s_1, s_0 \rangle, \quad I^- = \langle c \rangle. \)

**Proof.** This is a direct computation from the definitions. The \( I^+ \) (resp. \( I^- \)) ideal is generated by all semi-invariants of negative (resp. positive) weight. \( \square \)

**Proof of Proposition 3.35 in the case of a link of \( \alpha_c \)-atoms.** We handle the special case of a nodally-attached link of \( \frac{10}{7} \)-atoms of length \( l \). Let \( C = K \cup \mathcal{E} \) where \( E = \bigcup_{j=1}^l E_j \) is an elliptic link of length \( l \) meeting \( K \) at two distinct nodes \( q_0, q_1 \in K \) such that \( E_j \) meets \( E_{j+1} \) in a node \( q_j \) for \( j = 1, \ldots, l - 1 \). Using Proposition 3.24, we have \( \text{Aut}(C)^* = \prod_{j=1}^l \text{Aut}(E_j) \cong \mathbb{C}^n_m \) and a decomposition \( T^1(C) = T^1(K) \oplus \bigoplus_{j=1}^l T^1(E_j) \oplus \bigoplus_{j=0}^l T^1(\mathcal{O}_{C,q_j}) \)
with the choice of coordinates \( k = (k_1, \ldots, k_{3g-3l-4}) \) on \( \mathbb{T}^l(K) \), \( s_j = (s_{j,0}, s_{j,1}, s_{j,2}) \) on \( \mathbb{T}^l(E_j) \) for \( j = 1, \ldots, l \), and \( n_j \) on \( \mathbb{T}^l(\hat{O}_{C,a_j}) \) for \( j = 0, \ldots, l \). Moreover, these coordinates can be chosen such that action is given by

\[
\begin{align*}
&k_i \mapsto k_i, \ s_{j,i} \mapsto t_{j}^{-4} s_{j,i}, \ n_0 \mapsto t_1 n_0, \ n_l \mapsto t_l n_l, \ n_j \mapsto t_j t_{j+1} \text{ for } j \neq 0, l.
\end{align*}
\]

**Lemma 3.37.** With the above notation, the vanishing loci of \( I^+ \) and \( I^- \) are

\[
V(I^+) = \bigcup_{j=1}^{l} V(s_j) \quad V(I^-) = \bigcup_{\mu \geq 1} \bigcup_{\nu = 0}^{l-2\mu+1} V_{\mu,\nu}
\]

where \( V_{\mu,\nu} = V(n_\nu, s_{\nu+2}, \ldots, s_{\nu+2l-\nu-1}, n_{\nu+2l-1}) \).

**Remark 3.38.** For instance, \( V_{1,\nu} = V(n_\nu, n_{\nu+1}) \) and \( V_{2,\nu} = V(n_\nu, s_{\nu+2}, n_{\nu+3}) \).

**Proof.** We will use the Hilbert-Mumford criterion of Proposition 3.14. For the \( V(I^+) \) case, suppose \( x \in \text{Def}(C) \) such that for some \( j \), \( s_j(x) = 0 \). Set \( \lambda = (\lambda_i) : G_m \to \mathbb{G}_m \cong \prod_{i=1}^l \text{Aut}(E_i) \) where \( \lambda_i = 1 \) for \( i \neq j \) and \( \lambda_j = 0 \). Then \( \langle \chi, \lambda \rangle = 1 \) and \( \lim_{t \to 0} \lambda(t) \cdot x \) exists so \( x \in V(I^+) \). Conversely, let \( \lambda = (\lambda_i) \) be a one-parameter subgroup with \( \sum_i \lambda_i > 0 \) such that \( \lim_{t \to 0} \lambda(t) \cdot x \) exists. Then for some \( j \), \( \lambda_j > 0 \) which implies that \( s_j(x) = 0 \).

For the \( V(I^-) \) case, the inclusion \( \subseteq \) is easy: suppose that \( x \in V_{\mu,\nu} \) for \( \mu \geq 1 \) and \( \nu = 0, \ldots, l-2\mu+1 \). Set

\[
\lambda = (0, \ldots, 0, -1, 1, -1, \ldots, 1, -1, 0, \ldots, 0)
\]

Then \( \sum_i \lambda_i = -1 \) and \( \lim_{t \to 0} \lambda(t) \cdot x \) exists so \( x \in V(I^-) \). For the \( \supseteq \) inclusion, we will use induction on \( r \). If \( l = 1 \), then \( V(I^-) = V(n_0, n_1) \). For \( l > 1 \), suppose \( x \in V(I^-) \) and \( \lambda = (\lambda_i) : G_m \to G_m' \) is a one-parameter subgroup with \( \sum_{i=1}^l \lambda_i < 0 \) such that \( \lim_{t \to 0} \lambda(t) \cdot x \) exists. If \( \lambda_1 \geq 0 \), then \( \sum_{i=1}^{l-1} \lambda_i < 0 \) so by the induction hypothesis \( x \in V_{\mu,\nu} \) for some \( \mu \geq 1 \) and \( \nu = 0, \ldots, l-2\mu \). If \( \lambda_1 < 0 \), then we immediately conclude that \( n_1(x) = 0 \). If \( \lambda_{l-1} + \lambda_l < 0 \), then \( n_{l-1}(x) = 0 \) so \( x \in V_{1,l-1} \). If \( \lambda_{l-1} + \lambda_l \geq 0 \), then \( \lambda_{l-1} \geq 0 \) so \( s_{l-1}(x) = 0 \). Furthermore, \( \sum_{i=1}^{l-2} \lambda_i < 0 \) so by applying the induction hypothesis and restricting to the locus \( V(n_{l-2}, s_{l-1}, n_{l-1}, s_l, n_l) \), we can conclude either: (1) \( x \in V_{\mu,\nu} \) for \( \mu \geq 1 \) and \( \nu = 0, \ldots, l-2\mu-1 \), or (2) \( x \in V(n_{l-2}, s_{l-2}, \ldots, s_{l-3}) \) for some \( \mu \geq 1 \). In case (2), since \( s_{l-1}(x) = n_l(x) = 0 \), we have \( x \in V_{\mu+1,l-\mu-4} \).

**Remark 3.39.** The chamber \( V(I^+) \) is the closed locus in the deformation space consisting of curves with a tacnode while \( V(I^-) \) consists of curves containing an elliptic chain.

**Proof of Proposition 3.35.** Let \( (C, \{p_i\}_{i=1}^n) \) be an \( \alpha_c \)-closed curve and consider the action of \( \text{Aut}(C, \{p_i\}_{i=1}^n) \) on \( \mathbb{T}^l(C, \{p_i\}_{i=1}^n) \) described in Proposition 3.24. We split the proof into the types of \( \alpha_c \)-closed curves according to Definition 3.23.

- \( \alpha_c \in \{9/11, 2/3\} \) of Type A. By using Lemma 3.16, one may assume that \( r = 1 \) in which case the statement is clear.
• $\alpha_c \in \{9/11, 2/3\}$ of Type B. A simple application of Proposition 3.14 shows that $V(I^+) = (s_1, s_2)$, and $V(I^-) = (n)$ if $\alpha_c = 9/11$ and $V(I^-) = (n, c_1, c_2)$ if $\alpha_c = 2/3$.

• $\alpha_c \in \{9/11, 2/3\}$ of Type C. This is Lemma 3.36.

• $\alpha_c \in \{7/10\}$ of Type A. By Lemma 3.16, it is enough to consider the case when either $r = 1, s = 0$ or $r = 0, s = 1$. The case of $r = 1$ and $s = 0$ is the example worked out in Lemma 3.37, the addition of marked points does not affect the calculation of Lemma 3.37. If $r = 1, s = 0$, the action of $\text{Aut}(C, \{p_i\}_{i=1}^n)$ on $\text{Def}(C, \{p_i\}_{i=1}^n)$ is the same action in Lemma 3.37 restricted to the closed subscheme $V(n_1) = 0$. This case therefore follows from Lemmas 3.17 and 3.37.

• $\alpha_c \in \{7/10\}$ of Type B. The action of $\text{Aut}(C, \{p_i\}_{i=1}^n)$ on $\text{Def}(C, \{p_i\}_{i=1}^n)$ is the same action in Lemma 3.37 restricted to the closed subscheme $V(n_0, n_{r+1}) = 0$ so this case follows from Lemmas 3.17 and 3.37.

• $\alpha_c \in \{7/10\}$ of Type C. This follows from an argument similar to the proof of Lemma 3.37.

Proof of Theorem 3.39. Proposition 3.35 implies that $I_Z^+ = I^+ \hat{A}$ and $I_Z^- = I^- \hat{A}$ so we may apply Proposition 3.12 to conclude the statement of the theorem.

4. Existence of good moduli spaces

In this section, we prove that the stacks $\overline{\mathcal{M}}_{g,n}(\alpha)$ possess good moduli spaces (Theorem 4.27). In Section 4.1, we prove three general existence results for good moduli spaces. The first of these, Theorem 4.1, gives conditions under which one may use a local quotient presentation to construct a good moduli space. As we explain below, this may be considered as an analog of the Keel-Mori theorem [KM97] for algebraic stacks, but in practice the hypotheses of the theorem are much harder to verify than those of the Keel-Mori theorem. Our second existence result, Theorem 4.2, gives one situation in which the hypotheses of Theorem 4.1 are satisfied. It says that if $\mathcal{X}$ is a stack and $\mathcal{X}^+ \hookrightarrow \mathcal{X} \hookleftarrow \mathcal{X}^-$ is a pair of open immersions locally cut out by VGIT, then $\mathcal{X}$ admits a good moduli space if $\mathcal{X}^+, \mathcal{X} \setminus \mathcal{X}^+$, and $\mathcal{X} \setminus \mathcal{X}^-$ do. The third existence result, Theorem 4.3, proves that one can check existence of a good moduli space after passing to a finite cover. These results pave the way for the argument in Section 4.2 which proves the existence of good moduli space $\overline{\mathcal{M}}_{g,n}(\alpha)$ inductively.

4.1. General existence results. In this section, we prove the following three theorems.

Theorem 4.1. Let $\mathcal{X}$ be an algebraic stack of finite type over $\mathbb{C}$. Suppose that:

(1) For every closed point $x \in \mathcal{X}$, there exists a local quotient presentation $f: \mathcal{W} \rightarrow \mathcal{X}$ around $x$ such that:

(a) $f$ is stabilizer preserving at closed points of $\mathcal{W}$. 


(b) \( f \) sends closed points to closed points.

(2) For any \( C \)-point \( x \in X \), the closed substack \( \{x\} \) admits a good moduli space.

Then \( X \) admits a good moduli space.

**Theorem 4.2.** Let \( X \) be an algebraic stack of finite type over \( C \), and let \( L \) be a line bundle on \( X \). Let \( X^+, X^- \subseteq X \) be open substacks, and let \( Z^+ = X \setminus X^+ \) and \( Z^- = X \setminus X^- \) be their reduced complements. Suppose that

1. \( X^+, Z^+, Z^- \) admit good moduli spaces.
2. For all closed points \( x \in Z^+ \cap Z^- \), there exists a local quotient presentation \( W \rightarrow X \) around \( x \) and a Cartesian diagram

\[
\begin{array}{ccc}
W^+_L & \rightarrow & W \\
\downarrow & & \downarrow \\
X^+ & \rightarrow & X \\
\end{array}
\]

where \( W^+_L, W^-_L \) are the VGIT chambers of \( W \) with respect to \( L \).

Then there exist good moduli spaces \( X \rightarrow X \) and \( X^- \rightarrow X^- \) such that \( X^+ \rightarrow X \) and \( X^- \rightarrow X \) are proper and surjective. In particular, if \( X^+ \) is proper over \( C \), then \( X \) and \( X^- \) are also proper over \( C \).

Recall that an algebraic stack \( X \) is called a global quotient stack if \( X \cong [Y/GL_n] \), where \( Y \) is an algebraic space with an action of \( GL_n \).

**Theorem 4.3.** Let \( f : X \rightarrow Y \) be a morphism of algebraic stacks of finite type over \( C \). Suppose that:

1. \( X \rightarrow Y \) is finite and surjective.
2. There exists a good moduli space \( X \rightarrow X \) with \( X \) separated.
3. \( Y \) is a global quotient stack and admits local quotient presentations.

Then there exists a good moduli space \( Y \rightarrow Y \) with \( Y \) separated. Moreover, if \( X \) is proper, so is \( Y \).

Both Theorems 4.2 and 4.3 are proved using Theorem 4.1. In order to motivate the statement of Theorem 4.1 let us give an informal sketch of the proof. If \( X \) admits local quotient presentations, then every closed point \( x \in X \) admits an étale neighborhood of the form

\[
[\text{Spec } A_x/G_x] \rightarrow X,
\]

where \( A_x \) is a finite-type \( C \)-algebra and \( G_x \) is the stabilizer of \( x \). The union \( \bigsqcup_{x \in X} [\text{Spec } A_x/G_x] \) defines an étale cover of \( X \); reducing to a finite subcover, we obtain an atlas \( f : W \rightarrow X \) with the following properties:

1. \( f \) is affine and étale.
(2) \( \mathcal{W} \) admits a good moduli space \( W \).

Indeed, (2) follows simply by taking invariants \( \text{Spec} A_x/G_x \) \( \to \text{Spec} A_x^{G_x} \) and since \( f \) is affine, the fiber product \( \mathcal{R} := \mathcal{W} \times_\mathcal{X} \mathcal{W} \) admits a good moduli space \( R \). We may thus consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{p_1} & \mathcal{W} \\
\downarrow \varphi & & \downarrow \phi \\
R & \xrightarrow{q_1} & W
\end{array}
\]

The crucial question is: can we choose \( f : \mathcal{W} \to \mathcal{X} \) to guarantee that the induced projections \( q_1, q_2 \) are \( \text{étale} \)? If so, then \( R \rightrightarrows W \) defines an \( \text{étale} \) equivalence relation, and the algebraic space quotient \( X \) gives a good moduli space for \( \mathcal{X} \).

If \( \mathcal{X} \) is separated, then we can always do this. Indeed, the condition that \( \mathcal{X} \) is separated implies that the atlas \( f \) may be chosen to be stabilizer preserving\(^1\). Thus, we may take the projections \( \mathcal{R} \rightrightarrows \mathcal{W} \) to be stabilizer preserving and \( \text{étale} \), and this implies that the projections \( R \rightrightarrows W \) are \( \text{étale} \)\(^2\). This leads to a quick proof of the Keel-Mori theorem for separated Deligne-Mumford stacks over an algebraically closed field of characteristic zero. More generally, of course, algebraic stacks are almost never separated so we must find different hypotheses which ensure the projections are \( \text{étale} \). More precisely, can we identify a sufficient set of hypotheses which can be directly verified for geometrically-defined stacks, such as \( \overline{M}_{g,n}(\alpha) \)?

Our result gives at least one plausible answer to these questions. With notation as in Diagram \( \text{[4.2]} \) where \( \mathcal{W} \to W \) and \( \mathcal{R} \to R \) are now good moduli spaces, let us examine what conditions are sufficient to conclude that \( q_1, q_2 \) are \( \text{étale} \). One difficulty is that \( f : \mathcal{W} \to \mathcal{X} \) need not be stabilizer-preserving, since \( \mathcal{X} \) is a non-separated algebraic stack. A second difficulty is that while it is true that if \( \omega \in \mathcal{W} \) is a closed \( \mathbb{C} \)-point with image \( w \in W \), then the formal neighborhood \( \hat{\mathcal{O}}_{W,w} \) can be identified with the invariants \( D^G_w \) of the miniversal deformation space, this is not necessarily true if \( \omega \in \mathcal{W} \) is not a closed point. Thus, in order for \( q_1, q_2 \) to be \( \text{étale} \) at a \( \mathbb{C} \)-point \( r \in R \), or equivalently for the induced maps \( \hat{\mathcal{O}}_{W,w} \to \hat{\mathcal{O}}_{R,r} \) to be isomorphisms, we must manually impose the following conditions: \( p_1, p_2 \) should be stabilizer preserving at \( \rho \) and \( p_1(\rho), p_2(\rho) \) should be closed points, where \( \rho \in \mathcal{R} \) is the unique closed point in the preimage of \( r \in R \). We have now identified two key conditions that will imply that \( R \rightrightarrows W \) is an \( \text{étale} \) equivalence relation:

\(^{1}\)The set of points \( \omega \in W \) where \( f \) is not stabilizer preserving is simply the image of the complement of the open substack \( I_W \subseteq I_\mathcal{X} \times_\mathcal{X} \mathcal{W} \) in \( W \) and therefore is closed since \( I_\mathcal{X} \to \mathcal{X} \) is proper. By removing this locus from \( W, f : \mathcal{W} \to \mathcal{X} \) may be chosen to be stabilizer preserving.

\(^{2}\)To see this, note that if \( r \in R \) is any closed point and \( \rho \in \mathcal{R} \) is its preimage, then \( \hat{\mathcal{O}}_{R,r} \cong D^G_\rho \), where \( D_\rho \) denotes the miniversal formal deformation space of \( \rho \) and \( G_\rho \) is the stabilizer of \( \rho \); similarly \( \hat{\mathcal{O}}_{W,q_1(r)} \cong D^{G_{p_1(\rho)}}_{p_1(\rho)} \). Now \( p_1 \) étale implies \( D_\rho \cong D_{p_1(\rho)} \) and \( p_1 \) stabilizer preserving implies \( G_\rho \cong G_{p_1(\rho)} \), so \( \hat{\mathcal{O}}_{R,r} \cong \hat{\mathcal{O}}_{W,q_1(r)} \), i.e. \( q_1 \) is \( \text{étale} \).
The morphism $f : \mathcal{W} \to \mathcal{X}$ is stabilizer preserving at closed points.

The projections $p_1, p_2 : \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ send closed points to closed points.

Condition (⋆) is precisely hypothesis (1a) of Theorem 4.1. In practice, it is difficult to
directly verify condition (⋆⋆), but it turns out by conditions (1b) and (2),
which are often easier to verify. Appendix A provides various examples of algebraic stacks
highlighting the necessity of conditions (1a), (1b) and (2) in Theorem 4.1.

Section 4.1.2 is devoted to making the above argument precise. Then in Sections 4.1.3 and
4.1.4 we prove Theorems 4.2 and Theorems 4.3 by showing that after suitable reductions,
the hypotheses imply that conditions (1a), (1b) and (2) are satisfied.

4.1.1. Definitions and preparatory material.

Definition 4.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks of finite type over $\mathbb{C}$. We say that

- $f$ sends closed points to closed points if for every closed point $x \in \mathcal{X}$, $f(x) \in \mathcal{Y}$ is closed.
- $f$ is stabilizer preserving at $x \in \mathcal{X}(\mathbb{C})$ if $\text{Aut}_{\mathcal{X}(\mathbb{C})}(x) \to \text{Aut}_{\mathcal{Y}(\mathbb{C})}(f(x))$ is an isomorphism.
- $f$ is stabilizer preserving if $I_{\mathcal{X}} \to I_{\mathcal{Y}} \times_{\mathcal{X}} \mathcal{X}$ is an isomorphism.
- If $\phi : \mathcal{X} \to X$ is a good moduli space, we say that an open substack $\mathcal{U} \subseteq \mathcal{X}$ is
saturated if $\phi^{-1}(\phi(\mathcal{U})) = \mathcal{U}$.

Proposition 4.5. Consider a commutative diagram

$$
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{f} & \mathcal{X} \\
\downarrow{\varphi} & & \downarrow{\phi} \\
\mathcal{W} & \xrightarrow{g} & X
\end{array}
$$

where $f$ is a representable morphism between algebraic stacks of finite type over $\mathbb{C}$. Suppose
$\varphi : \mathcal{W} \to \mathcal{W}$ and $\phi : \mathcal{X} \to X$ are good moduli spaces.

1. Let $w \in \mathcal{W}$ be a closed point. If $f$ is étale and stabilizer preserving at $w$, and $f(w)$
is closed, then $g$ is étale at $\varphi(w)$.
2. Suppose $f$ is separated, étale, sends closed points to closed points, and is stabilizer
preserving at closed points of $\mathcal{W}$. Then $g$ is étale and the above diagram is Cartesian.
3. If $f$ is a separated and étale morphism, then there exists a saturated open substack
$\mathcal{U} \subseteq \mathcal{W}$ such that:
   a. The induced diagram

$$
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{f|_{\mathcal{U}}} & \mathcal{X} \\
\downarrow{\varphi|_{\mathcal{U}}} & & \downarrow{\phi} \\
\varphi(\mathcal{U}) & \xrightarrow{g|_{\varphi(\mathcal{U})}} & X
\end{array}
$$


is Cartesian and \( g|_{\varphi(U)} \) is étale. In particular, \( f|_{U} \) is stabilizer preserving and sends closed points to closed points.

(b) If \( w \in \mathcal{W} \) is a closed point such that \( f \) is stabilizer preserving at \( w \) and \( f(w) \in \mathcal{X} \) is closed, then \( w \in U \).

Proof. \([\text{Alp13, Theorem 5.1}]\) implies part (1) and that \( g \) is étale in (2). The hypotheses in (2) imply that the induced morphism \( \Psi: \mathcal{W} \to W \times_X \mathcal{X} \) is representable, separated, quasi-finite and sends closed points to closed points. \([\text{Alp13, Proposition 6.4}]\) implies that \( \Psi \) is finite. Moreover, since \( f \) and \( g \) are étale, so is \( \Psi \). But since \( \mathcal{W} \) and \( W \times_X \mathcal{X} \) both have \( \mathcal{W} \) as a good moduli space, it follows that a closed point in \( W \times_X \mathcal{X} \) has a unique preimage under \( \Psi \). Therefore, \( \Psi \) is an isomorphism and the diagram is Cartesian. Statement (3) follows from \([\text{Alp10b, Theorem 6.10}]\). \(\square\)

**Lemma 4.6.** Let \( \phi: \mathcal{X} \to X \) be a good moduli space. Let \( x \in \mathcal{X} \) be a closed point and \( U \subseteq \mathcal{X} \) be an open substack containing \( x \). Then exists a saturated open substack \( U_1 \subseteq U \) containing \( x \). Moreover, if \( \mathcal{X} \cong [\text{Spec } A/G] \) with \( G \) reductive, then \( U_1 \) can be chosen to be of the form \([\text{Spec } B/G] \) for a \( G \)-invariant open affine subscheme \( \text{Spec } B \subseteq \text{Spec } A \).

Proof. The substacks \( \{x\} \) and \( \mathcal{X} \setminus U \) are closed and disjoint. By \([\text{Alp13, Theorem 4.16}]\), \( \phi(\{x\}) \) and \( Z := \phi(\mathcal{X} \setminus U) \) are closed and disjoint. Therefore, we take \( U_1 = \phi^{-1}(X \setminus Z) \).

For the second statement, take \( U_1 = \phi^{-1}(U_1) \) for an affine open subscheme \( U_1 \subseteq X \setminus Z \). \(\square\)

4.1.2. Existence via local quotient presentations. In this section, we prove Theorem 4.1.

**Proposition 4.7.** Let \( \mathcal{X} \) be an algebraic stack of finite type over \( \mathbb{C} \). Let \( f: \mathcal{W} \to \mathcal{X} \) be an affine, étale, surjective morphism from an algebraic stack \( \mathcal{W} \) such that:

1. \( \mathcal{W} \) admits a good moduli space \( \varphi: \mathcal{W} \to W \).
2. \( f \) is stabilizer preserving at closed points in \( \mathcal{W} \).
3. The projections \( \mathcal{W} \times_X \mathcal{W} \to \mathcal{W} \) sends closed points to closed points.

Then \( \mathcal{X} \) admits a good moduli space \( \phi: \mathcal{X} \to X \). Furthermore, the induced map \( g: W \to X \) is étale and the following square is Cartesian:

\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{f} & \mathcal{X} \\
\downarrow \varphi & & \downarrow \phi \\
W & \xrightarrow{g} & X
\end{array}
\]

Proof. Set \( \mathcal{X}_1 = \mathcal{W} \), \( X_1 = W \) and \( \mathcal{X}_2 = \mathcal{X}_1 \times_\mathcal{X} \mathcal{X}_1 \) with projections \( p_1, p_2: \mathcal{X}_2 \to \mathcal{X}_1 \). Since \( f \) is affine, there exists a good moduli space \( \phi_2: \mathcal{X}_2 \to X_2 \). The two projections \( p_1, p_2: \mathcal{X}_2 \to \mathcal{X}_1 \) induce two morphisms \( q_1, q_2: X_2 \to X_1 \) such that \( q_i \circ \phi_2 = \phi_1 \circ p_i \) for \( i = 1, 2 \).

By Proposition 4.5 \( q_1 \) and \( q_2 \) are étale, and the induced morphisms \( \mathcal{X}_2 \to X_2 \times_{\mathcal{X}_1, \phi_1} \mathcal{X}_1 \) are isomorphisms.
by Proposition $[4.5]$(2). Similarly, by setting $X_3 = X_1 \times_X X_1 \times_X X_1$, there is a good moduli space $\phi_3: X_3 \to X_3$ where $X_3 = X_2 \times_{q_1, X_1, q_2} X_2$ and an induced diagram

$$
\begin{array}{c}
X_3 \xrightarrow{\phi_3} X_2 \xrightarrow{\phi_2} X_1 \xrightarrow{\phi_1} X \\
\downarrow \quad \downarrow \quad \downarrow \\
X_3 \xrightarrow{\phi_3} X_2 \xrightarrow{\phi_2} X_1
\end{array}
$$

where the appropriate squares are Cartesian. Moreover, by the universality of good moduli spaces, there is an induced identity map $\phi_3: X_3 \to X_3$ giving $X_2 \Rightarrow X_1$ an étale groupoid structure.

To check that $\Delta: X_2 \to X_1 \times X_1$ is a monomorphism, it suffices to check that there is a unique pre-image of $(x_1, x_1) \in X_1 \times X_1$ where $x_1 \in X_1(\mathbb{C})$. Let $\xi_1 \in X_1$ be the unique closed point in $\phi_1^{-1}(x_1)$. Since $X_1 \to \mathcal{X}$ is stabilizer preserving at $\xi_1$, we can set $G := \text{Aut}_{X_1(\mathbb{C})}(\xi_1) \cong \text{Aut}_{\mathcal{X}(\mathbb{C})}(f(\xi_1))$. There are diagrams

$$
\begin{array}{c}
BG \xrightarrow{} BG \times BG \\
\downarrow \quad \downarrow \\
X_2 \xrightarrow{} X_1 \times X_1
\end{array}
\quad
\begin{array}{c}
X_2 \xrightarrow{(p_1, p_2)} X_1 \times X_1 \\
\downarrow \quad \downarrow \\
X_2 \xrightarrow{\Delta} X_1 \times X_1
\end{array}
$$

where the squares in the left diagram are Cartesian. Suppose $x_2 \in X_2(\mathbb{C})$ is a preimage of $(x_1, x_1)$ under $\Delta: X_2 \to X_1 \times X_1$. Let $\xi_2 \in X_2$ be the unique closed point in $\phi_2^{-1}(x_2)$. Then $(p_1(\xi_2), p_2(\xi_2)) \in X_1 \times X_1$ is closed and is therefore the unique closed point $(\xi_1, \xi_1)$ in the $(\phi_1 \times \phi_1)^{-1}(x_1, x_1)$. But by Cartesianness of the left diagram, $\xi_2$ is the unique point in $X_2$ which maps to $(\xi_1, \xi_1)$ under $\phi_2: X_2 \to X_1 \times X_1$. Therefore, $x_2$ is the unique preimage of $(x_1, x_1)$.

Since $X_2 \times_{q_1, X_1, q_2} X_2 \to X_2$ is an étale equivalence relation, there exists an algebraic space quotient $X$ and induced maps $\phi: \mathcal{X} \to X$ and $X_1 \to X$. Consider

$$
\begin{array}{c}
X_2 \xrightarrow{} X_1 \xrightarrow{} X_1 \\
\downarrow \quad \downarrow \\
X_1 \xrightarrow{} \mathcal{X} \xrightarrow{} X
\end{array}
$$

Since $X_2 \cong X_1 \times X_1$ and $X_2 \cong X_1 \times_X X_1$, the left and outer square above are Cartesian. Since $X_1 \to \mathcal{X}$ is étale and surjective, it follows that the right square is Cartesian. By descent ([Alp13 Prop. 4.7]), $\phi: \mathcal{X} \to X$ is a good moduli space. 

\begin{proposition}
Let $\mathcal{X}$ be an algebraic stack of finite type over $\mathbb{C}$. Let $f: \mathcal{W} \to \mathcal{X}$ be an affine, étale, surjective morphism from an algebraic stack $\mathcal{W}$ such that:

(1) $\mathcal{W}$ admits a good moduli space $\varphi: \mathcal{W} \to \varphi$.
\end{proposition}
(2) \( f \) is stabilizer preserving at closed points of \( W \).
(3) \( f \) sends closed points to closed points.
(4) For any \( \mathbb{C} \)-point \( x \in \mathcal{X} \), the closed substack \( \{ x \} \) admits a good moduli space.

Then \( \mathcal{X} \) admits a good moduli space \( \phi: \mathcal{X} \to X \). Furthermore, the induced map \( g: W \to X \) is \( \acute{e}tale \) and the following square is Cartesian:

\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{f} & \mathcal{X} \\
\downarrow \phi & & \downarrow \phi \\
W & \xrightarrow{g} & X
\end{array}
\]

**Proof.** We simply need to check property (3) in Proposition 4.7. That is, we need to show that projections \( p_1, p_2: R := \mathcal{W} \times_X \mathcal{W} \to \mathcal{W} \) send closed points to closed points. Let \( \rho \in R \) be a closed point and set \( x = f(p_1(\rho)) = f(p_2(\rho)) \in \mathcal{X} \). Let \( Z = \{ x \} \subseteq \mathcal{X} \), \( \mathcal{W}' = \mathcal{W} \times_X Z \) and \( \mathcal{R}' = \mathcal{R} \times_X Z \) with induced maps \( f': \mathcal{W}' \to Z \) and \( p'_1, p'_2: \mathcal{R}' \to \mathcal{W}' \). Consider

\[
\begin{array}{ccc}
\mathcal{R}' & \xrightarrow{p'_1} & \mathcal{W}' \\
\downarrow & & \downarrow f' \\
\mathcal{R} & \xrightarrow{p_1} & \mathcal{W} & \xrightarrow{f} \mathcal{X}
\end{array}
\]

The morphism \( f': \mathcal{W}' \to \mathcal{X}' \) is \( \acute{e}tale \), surjective, affine, sends closed points to closed points and is stabilizer preserving at closed points. Moreover, by hypothesis (1) and (4), there is a diagram

\[
\begin{array}{ccc}
\mathcal{W}' & \xrightarrow{\phi'} & Z \\
\downarrow \phi' & & \downarrow \phi' \\
W' & \xrightarrow{g'} & \text{Spec } k
\end{array}
\]

where \( \phi': \mathcal{W}' \to W' \) and \( \varphi': Z' \to \text{Spec } k \) are good moduli spaces. By Proposition 4.5(2), the above diagram is Cartesian. It follows that the projects \( p'_1, p'_2: \mathcal{R}' \to \mathcal{W}' \) send closed points to closed points as they are the base change of the projections \( W' \times W' \to W' \) by \( \mathcal{W}' \to W' \). Therefore \( p'_1(\rho), p'_2(\rho) \in \mathcal{W}' \) are closed or equivalently \( p_1(\rho), p_2(\rho) \in \mathcal{W} \) are closed.

**Proof of Theorem 4.1.** After taking a disjoint union of finitely many local quotient presentations, there exists an \( \acute{e}tale \), affine and surjective morphism \( f: \mathcal{W} \to \mathcal{X} \) such that (1) \( \mathcal{W} \) admits a good moduli space, (2) \( f \) is both stabilizer preserving at closed points in \( \mathcal{W} \), and (3) \( f \) sends closed points to closed points. The theorem now follows from Proposition 4.8. \qed
4.1.3. **Existence via local VGIT.** In this section, we prove Theorem 4.2. We will need the following lemma on isotrivial specializations.

**Lemma 4.9.** Let $X$ be an algebraic stack of finite type over $\mathbb{C}$, and let $L$ be a line bundle on $X$. Let $X^+, X^- \subseteq X$ be open substacks, and let $Z^+ = X \setminus X^+$, $Z^- = X \setminus X^-$ be their reduced complements. Suppose that for all closed points $x \in X$, there exists a local quotient presentation $f : W \to X$ around $x$ and a Cartesian diagram

\[
W^+ \xrightarrow{f} W \leftarrow W^- \\
X^+ \xrightarrow{f} X \leftarrow X^-
\]

where $W^+ = W^+_L$ and $W^- = W^-_L$ are the VGIT chambers of $W$ with respect to $L$.

1. If $z \in X^+(C) \cap X^-(C)$, then the closure of $z$ in $X$ is contained in $X^+ \cap X^-$. 
2. If $z \in X(C)$ is a closed point, then either $z \in X^+ \cap X^-$ or $z \in Z^+ \cap Z^-$. 

**Proof.** For (1), if the closure of $z$ in $X$ is not contained in $X^+ \cap X^-$, there exists an isotrivial specialization $z \leadsto x$ to a closed point in $X \setminus (X^+ \cap X^-)$. Choose a local quotient presentation $f : W = [W/G_x] \to X$ around $x$ such that (4.3) is Cartesian. Since $f^{-1}(x) \not\subset W^+ \cap W^-$, the character $\chi = L|_{BG_x}$ is non-trivial. Consider points $w \in W$ and $w_0 \in W^G_x$ over $z$ and $x$, respectively. By the Hilbert-Mumford criterion (Mum65, Theorem 2.1), there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \to G_x$ such that $\lim_{t \to 0} \lambda(t) \cdot w = w_0$. As $w \in W^+ \cap W^-$ and $w_0 \in W^G_x$, by applying Proposition 3.15 twice with the characters $\chi$ and $\chi^{-1}$, we see that both $\langle \chi, \lambda \rangle < 0$ and $\langle \chi, \lambda \rangle > 0$, a contradiction.

For (2), choose a local quotient presentation $f : (W, w) \to X$ around $z$ with $W = [W/G_x]$. Let $\chi = L|_{BG_x}$ be the character of $L$. Since $w \in W^G_x$, $w$ can be semistable with respect to $\chi$ if and only if $\chi$ is trivial. It follows that either $w \in W^+ \cap W^-$ in the case $\chi$ is trivial, or $w \not\in W^+ \cup W^-$ in the case $\chi$ is non-trivial. 

**Proof of Theorem 4.2** First, we show that $X$ has a good moduli space by verifying the hypotheses of Theorem 4.1. Let $x_0 \in X$ be any closed point. By Lemma 4.9(2), we have either $x_0 \in X^+ \cap X^-$ or $x_0 \in Z^+ \cap Z^-$. Suppose first that $x_0 \in X^+ \cap X^-$. Since $X^+$ admits a good moduli space, Proposition 4.5(3) implies we may choose a local quotient presentation $f : W \to X^+$ which is stabilizer preserving at closed points and sends closed points to closed points. By applying Lemma 4.6, we may shrink further to assume that $f(W) \subset X^+ \cap X^-$. Then Lemma 4.5(1) implies that the composition $f : W \to X^+ \to X$ still sends closed points to closed points.
Now suppose \(x_0 \in \mathcal{Z}^+ \cap \mathcal{Z}^-\). Choose a local quotient presentation \(f: (\mathcal{W}, w_0) \to \mathcal{X}\) around \(x_0\) inducing a Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{W}^+ & \xrightarrow{f} & \mathcal{W}^- \\
\downarrow & & \downarrow \\
\mathcal{X}^+ & \xrightarrow{f} & \mathcal{X}^-
\end{array}
\]

with \(\mathcal{W}^+ = \mathcal{W}_{\mathcal{E}}^+\) and \(\mathcal{W}^- = \mathcal{W}_{\mathcal{E}}^-\). We claim that, after shrinking suitably, we may assume that \(f\) sends closed points to closed points and is stabilizer preserving at closed points. In proving this claim, we make implicit repeated use of Lemma 3.19 to argue that if \(\mathcal{W}' \subseteq \mathcal{W}\) is an open substack containing \(w_0\), there exists open substack \(\mathcal{W}'' \subseteq \mathcal{W}'\) containing \(w_0\) such that \(\mathcal{W}'' \to \mathcal{X}\) is a local quotient presentation inducing a Cartesian diagram as in (4.4).

Using the hypothesis that \(\mathcal{Z}^+, \mathcal{Z}^-, \mathcal{X}^+\) admit good moduli spaces, we will first show that \(f\) may be chosen to satisfy:

(A) \(f|_{f^{-1}(\mathcal{Z}^+)}\), \(f|_{f^{-1}(\mathcal{Z}^-)}\) is stabilizer preserving and sends closed points to closed points.

(B) \(f|_{\mathcal{W}^+}\) is stabilizer preserving and sends closed points to closed points.

If \(f\) satisfies (A) and (B), then \(f\) must send closed points to closed points and be stabilizer preserving at closed points. Indeed, if \(w \in \mathcal{W}\) is a closed point, then either \(w \in f^{-1}(\mathcal{Z}^+) \cup f^{-1}(\mathcal{Z}^-)\) or \(w \in f^{-1}(\mathcal{X}^+) \cap f^{-1}(\mathcal{X}^-)\). In the former case, (A) immediately implies that \(f\) is stabilizer preserving at \(w\) and \(f(w)\) is closed in \(\mathcal{X}\). In the latter, (B) implies that \(f\) is stabilizer preserving at \(w\) and that \(f(w)\) is closed in \(\mathcal{X}^+\). Since \(f(w) \in \mathcal{X}^+ \cap \mathcal{X}^-\) however, Lemma 4.9(1) implies that \(f(w)\) remains closed in \(\mathcal{X}\). It remains to show that \(f\) can be chosen to satisfy (A) and (B).

For (A), Proposition 4.5(3) implies the existence of an open substack \(Q \subseteq f^{-1}(\mathcal{Z}^+)\) containing \(w_0\) such that \(f|_Q\) is stabilizer preserving and sends closed points to closed points. After shrinking \(\mathcal{W}\) suitably, we may assume \(\mathcal{W} \cap f^{-1}(\mathcal{Z}^+) \subseteq Q\). One argues similarly for \(f|_{f^{-1}(\mathcal{Z}^-)}\).

For (B), Proposition 4.5 implies there exists an open substack \(U \subseteq \mathcal{W}^+\) such that \(f|_U: U \to \mathcal{X}^+\) is stabilizer preserving and sends closed points to closed points; moreover, \(U\) contains all closed points \(w \in \mathcal{W}^+\) such that \(f\) is stabilizer preserving at \(w\) and \(f(w) \in \mathcal{X}^+\) is closed. Let \(U = \mathcal{W}^+ \setminus U\) and let \(\overline{\mathcal{W}}\) be the closure of \(U\) in \(\mathcal{W}\). We claim that \(w_0 \notin \overline{\mathcal{W}}\). Once this is established, we may replace \(\mathcal{W}\) by an appropriate open substack of \(\mathcal{W} \setminus \overline{\mathcal{W}}\) to obtain a local quotient presentation satisfying (B). Suppose, by way of contradiction, that \(w_0 \in \overline{\mathcal{W}}\). Then there exists a specialization diagram

\[
\begin{array}{ccc}
\text{Spec } K = \Delta^* & \xrightarrow{\gamma} & \mathcal{V} \\
\downarrow & & \downarrow \\
\text{Spec } \mathcal{R} = \Delta & \xrightarrow{h} & \mathcal{W}
\end{array}
\]
such that \( h(0) = w_0 \). By Proposition 3.7, there exist good moduli spaces \( W \to W \) and \( W^+ \to W^+ \), and the induced morphism \( W^+ \to W \) is proper. Since the composition \( W^+ \to W^+ \to W \) is universally closed, there exists, after an extension of the fraction field \( K \), a diagram

\[
\begin{array}{c}
\Delta^* \xrightarrow{\tilde{h}} W^+ \xrightarrow{f} W^+
\end{array}
\]

and a lift \( \tilde{h} : \Delta \to W^+ \) that extends \( \Delta^* \to W^+ \) with \( \tilde{w} = \tilde{h}(0) \in W^+ \) closed. There is an isotrivial specialization \( \tilde{w} \sim w_0 \). It follows from Lemma 4.9 that \( \tilde{w} \in f^{-1}(\mathcal{Z}^-) \). By assumption (A), \( f \) is stabilizer preserving at \( \tilde{w} \) and \( f(\tilde{w}) \in W^+ \) is closed so that \( \tilde{w} \in U \).

On the other hand, the generic point of the specialization \( \tilde{h} : \Delta \to W^+ \) lands in \( V \) so that \( \tilde{w} \in V \), a contradiction. Thus, \( w_0 \notin V \) as desired.

We have now shown that \( \mathcal{X} \) satisfies condition (1) in Theorem 4.1 and it remains to verify condition (2). Let \( x \in \mathcal{X}(\mathbb{C}) \). If \( x \in \mathcal{Z}^+ \) (resp. \( x \in \mathcal{Z}^- \)), then \( \{x\} \subseteq \mathcal{Z}^+ \) (resp. \( \{x\} \subseteq \mathcal{Z}^- \)). Therefore, since \( \mathcal{Z}^+ \) (resp. \( \mathcal{Z}^- \)) admits a good moduli space, so does \( \{x\} \). On the other hand, if \( x \in \mathcal{X}^- \cap \mathcal{X}^+ \), then Lemma 4.9(1) implies the closure of \( x \) in \( \mathcal{X} \) is contained in \( \mathcal{X}^+ \). Since \( \mathcal{X}^+ \) admits a good moduli space, so does \( \{x\} \). Now Theorem 4.1 implies that \( \mathcal{X} \) admits a good moduli space \( \mathcal{X} \to X \).

Next, we use Proposition 4.8 to show that \( \mathcal{X}^- \) admits a good moduli space. By taking a disjoint union of local quotient presentations, there exists an étale, affine, stabilizer preserving and surjective morphism \( f : W \to \mathcal{X} \) which sends closed points to closed points. The induced morphism \( f^- : W^- \to \mathcal{X}^- \) is étale, affine, stabilizer preserving and surjective, and we claim that it still sends closed points to closed points.

To see this, let \( w \in W^- \) be a closed point, and let \( \{w\} \subseteq W \) denote the closure of \( w \) in \( W \). We claim that \( f(\{w\}) \) is closed in \( \mathcal{X} \). If not, there exists an isotrivial specialization \( f(w) \sim x_0 \) to a closed point \( x_0 \in \mathcal{X} \) with \( x_0 \notin f(\{w\}) \). Now let \( w \sim w_0 \) be any isotrivial specialization to a closed point \( w_0 \in W \). Since \( f \) sends closed points to closed points, \( f(w) \sim f(w_0) \) is an isotrivial specialization to a closed point \( f(w_0) \in \mathcal{X} \). However, since \( \mathcal{X} \) admits a good moduli space, any \( \mathcal{C} \)-point admits a unique isotrivial specialization to a closed point. Therefore, \( x_0 = f(w_0) \) contradicting \( x_0 \notin f(\{w\}) \). This shows that \( f(\{w\}) \subseteq \mathcal{X} \) is closed, and it follows that \( f(w) \in \mathcal{X}^- \) is closed. Indeed, if there were a specialization \( f(w) \sim x_1 \) in \( \mathcal{X}^- \), then the fact that \( f(\{w\}) \subseteq \mathcal{X}^{-}\) is closed would imply that this specialization lifts to a specialization \( w \sim w_1 \) in \( W \) with \( w_1 \in W^- \), a contradiction since \( w \in W^- \). We conclude that \( f|_{W^-} : W^- \to \mathcal{X}^- \) sends closed points to closed points, and therefore that conditions (1), (2), (3) of Proposition 4.8 are satisfied.

It remains to check that if \( x \in \mathcal{X}^-(\mathbb{C}) \) is any point, then its closure \( \overline{\{x\}} \) in \( \mathcal{X}^-(\mathbb{C}) \) admits a good moduli space. If \( x \in \mathcal{Z}^+ \), then \( \{x\} \) admits a good moduli space since \( \mathcal{X}^+ \) does. If \( x \in \mathcal{X}^- \), then the closure of \( x \) in \( \mathcal{X} \) is contained in \( \mathcal{X}^+ \cap \mathcal{X}^- \) by Lemma 4.9 therefore \( \overline{\{x\}} \)
admits a good moduli space since $\mathcal{X}$ does. By Proposition 4.8 there exists a good moduli space $\mathcal{X}^- \to X^-$.

Finally, we argue that $X^+ \to X$ and $X^- \to X$ are proper and surjective. By taking a disjoint union of local quotient presentations and applying Proposition 4.5(3), there exists an étale, affine, stabilizer preserving and surjective morphism $f : W \to \mathcal{X}$ from an algebraic stack admitting a good moduli space $W \to W$ such that $W = \mathcal{X} \times_X W$. Moreover, if we set $W^+ := f^{-1}(\mathcal{X}^+)$ and $W^- := f^{-1}(\mathcal{X}^-)$, by Proposition 3.7 $W^+$ and $W^-$ admit good moduli spaces $W^+ \to W$ and $W^- \to W$ such that $W^+ \to W$ and $W^- \to W$ are proper and surjective. This gives commutative cubes

The same argument as in the proof that $\mathcal{X}^-$ admits a good moduli space shows that $f|_{W^+} : W^+ \to \mathcal{X}^+$ and $f|_{W^-} : W^- \to \mathcal{X}^-$ send closed points to closed points. By Proposition 4.5(2), the left and right faces are Cartesian squares. Since the top faces are also Cartesian, we have $W^+ = \mathcal{X}^+ \times_X W$ and $W^- = \mathcal{X}^- \times_X W$. In particular, $W^+ \to \mathcal{X}^+ \times_X W$ and $W^- \to \mathcal{X}^- \times_X W$ are good moduli spaces. By uniqueness of good moduli spaces, we have $X^+ \times_X W = W^+$ and $X^- \times_X W = W^-$. Since $W^+ \to W$ and $W^- \to W$ are proper and surjective, $X^+ \to X$ and $X^- \to X$ are proper and surjective by étale descent. □

4.1.4. Existence via finite covers. We now prove Theorem 4.3 by appealing to the following two lemmas:

**Lemma 4.10.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks of finite type over $\mathbb{C}$. Suppose that:

1. $f : \mathcal{X} \to \mathcal{Y}$ is finite, surjective, and stabilizer preserving at closed points.
2. $\mathcal{X}$ admits a good moduli space.
3. $\mathcal{Y}$ admits local quotient presentations.

Then $\mathcal{Y}$ admits a good moduli space.

**Proof.** Let $y \in \mathcal{Y}$ be a closed point and $g : \mathcal{Y}' = [\text{Spec } A/G] \to \mathcal{Y}$ be a local quotient presentation about $y$. Let $y' \in \mathcal{Y}'$ be a preimage of $y$ such that $g$ is stabilizer preserving at
Consider the Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\
\downarrow{g'} & & \downarrow{g} \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}
\end{array}
\]

Observe that for every preimage \(x' \in f'^{-1}(y')\), \(g'\) is stabilizer preserving at \(x'\) and \(g'(x')\) is a closed point of \(\mathcal{X}\). Indeed, the former statement follows from hypothesis (1) together with the fact that \(g\) is stabilizer preserving at \(y'\), and the latter follows simply because \(f(g'(x')) = x\) is closed.

Since \(g\) is affine, \(\mathcal{X}'\) admits a good moduli space. By Proposition 4.5(3), there exists an open substack \(\mathcal{U}' \subseteq \mathcal{X}'\) containing \(f'^{-1}(y')\) such that \(g'|_{\mathcal{U}'}\) is stabilizer preserving and sends closed points to closed points. Then \(\mathcal{V}' = \mathcal{Y}' \setminus f'(\mathcal{X}' \setminus \mathcal{U}')\) is an open substack containing \(y'\) and by Lemma 4.6 after shrinking, we may assume that \(\mathcal{V}'\) is a quotient stack \([\text{Spec } B/\mathbb{G}]\) which is a saturated open substack of \(\mathcal{Y}'\) containing \(y'\). Note that \(f'^{-1}(\mathcal{V}') \subseteq \mathcal{U}'\) and \(f'^{-1}(\mathcal{V}')\) is saturated in \(\mathcal{X}'\). By replacing \(\mathcal{Y}'\) with \(\mathcal{V}'\), we may assume that there is a local quotient presentation \(g \colon \mathcal{Y}' \to \mathcal{Y}\) about \(y\) such that \(g'\) in Diagram (4.5) is stabilizer preserving and sends closed points to closed points. By By Proposition 4.5(2), \(g'\) is the base change of a morphism of algebraic spaces which implies that the projections \(\mathcal{X}' \times \mathcal{X} \mathcal{X}' \to \mathcal{X}'\) are as well. In particular, the projections send closed points to closed points and since \(f\) is finite, the projections \(\mathcal{Y}' \times \mathcal{Y} \mathcal{Y}' \to \mathcal{Y}'\) also send closed points to closed points.

By Proposition 4.7, we conclude that for every closed point \(y \in \mathcal{Y}\), there is a local quotient presentation \(g \colon \mathcal{Y}' \to \mathcal{Y}\) about \(y\) such that \(g(\mathcal{Y}') \subseteq \mathcal{Y}\) admits a good moduli space with the additional property that points that are closed in \(g(\mathcal{Y}')\) are also closed in \(\mathcal{Y}\). It follows from [Alp13, Proposition 7.9] that \(\mathcal{Y}\) admits a good moduli space. \(\square\)

**Lemma 4.11.** Consider a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
\downarrow{g} & & \downarrow{h} \\
X
\end{array}
\]

of algebraic stacks of finite type over \(\mathbb{C}\) where \(X\) is an algebraic space. Suppose that:

1. \(\mathcal{X} \to \mathcal{Y}\) is finite and surjective.
2. \(\mathcal{X} \to X\) is cohomologically affine.
3. \(\mathcal{Y}\) is a global quotient stack.

Then \(\mathcal{Y} \to X\) is cohomologically affine.

**Proof.** We may write \(\mathcal{Y} = [V / \text{GL}_n]\), where \(V\) is an algebraic space by an action of \(\text{GL}_n\). Since \(\mathcal{X} \to \mathcal{Y}\) is affine, \(\mathcal{X}\) is the quotient stack \(\mathcal{X} = [U / G]\) where \(U = \mathcal{X} \times \mathcal{Y} V\). Since \(U \to \mathcal{X}\) is affine and \(\mathcal{X} \to X\) is cohomologically affine, \(U \to X\) is affine by Serre’s criterion. The
morphism $U \to V$ is a finite and surjective so by Chevalley’s theorem, we can conclude that $V \to X$ is affine. Therefore $\mathcal{Y} \to X$ is cohomologically affine. □

**Proof of Theorem 4.3.** Let $Z$ be the scheme-theoretic image of $X \to X \times \mathcal{Y}$. Since $X \to \mathcal{Y}$ is finite and $X$ is separated, $X \to Z$ is finite. By Lemma 4.11, the projection $Z \to X$ is cohomologically affine which implies that $Z$ admits a good moduli space. The composition $Z \to X \times \mathcal{Y} \to \mathcal{Y}$ is finite, surjective and stabilizer preserving. Moreover, $Z$ is a global quotient stack since $\mathcal{Y}$ is. The statement now follows from Lemma 4.10. □

**Remark 4.12.** The hypothesis that $X$ is separated in Theorem 4.3 is necessary. For example, let $X$ be the affine line with the point 0 doubled and let $\mathbb{Z}_2$ act on $X$ by swapping the points at 0 and fixing all other points. Then $X \to [X/\mathbb{Z}_2]$ satisfies the hypotheses but $[X/\mathbb{Z}_2]$ does not admit a good moduli space.

**4.2. Application to $\overline{M}_{g,n}(\alpha)$.** In this section, we apply Theorem 4.2 to prove that the stacks $\overline{M}_{g,n}(\alpha)$ admit good moduli spaces (Theorem 4.27). In order to apply Theorem 4.2, we must show that for each critical value $\alpha_i$, the closed sub stacks

$$\mathcal{S}_{g,n}(\alpha_i) \coloneqq \overline{M}_{g,n}(\alpha_i) \setminus \overline{M}_{g,n}(\alpha_i + \epsilon)$$

$$\mathcal{H}_{g,n}(\alpha_i) \coloneqq \overline{M}_{g,n}(\alpha_i) \setminus \overline{M}_{g,n}(\alpha_i - \epsilon)$$

admit good moduli spaces. We will prove this statement by inductively. Similar to the boundary strata of $\overline{M}_{g,n}$, $\mathcal{H}_{g,n}(\alpha_i)$ can be described (up to a finite cover) as a product of moduli spaces of $\alpha_i$-stable curves of lower genus. Likewise, $\mathcal{S}_{g,n}(\alpha_i)$ can be described (up to a finite cover) as a stacky projective bundles over a moduli space of $\alpha_i$-stable curves of lower genus. We will use induction on $g$ to deduce that these products and projective bundles admit good moduli spaces, and then apply Theorem 4.3 to conclude that $\mathcal{S}_{g,n}(\alpha_i)$ and $\mathcal{H}_{g,n}(\alpha_i)$ admit good moduli spaces.

**4.2.1. Existence for $\mathcal{S}_{g,n}(\alpha_i)$.**

**Lemma 4.13.** We have:

$$\mathcal{S}_{1,1}(9/11) \cong B\mathbb{G}_m$$

$$\mathcal{S}_{1,2}(7/10) \cong B\mathbb{G}_m$$

$$\mathcal{S}_{2,1}(2/3) \cong [\mathbb{A}^1/\mathbb{G}_m], \text{ where } \mathbb{G}_m \text{ acts with weight } 1.$$ 

In particular, the stacks $\mathcal{S}_{1,1}(9/11)$, $\mathcal{S}_{1,2}(7/10)$, $\mathcal{S}_{2,1}(2/3)$ admit good moduli spaces.

**Proof.** The stacks $\mathcal{S}_{1,1}(9/11)$ and $\mathcal{S}_{1,2}(7/10)$ each contain a unique $\mathbb{C}$-point, namely the $\frac{9}{11}$-atom and the $\frac{7}{10}$-atom, and each of these curves have a $\mathbb{G}_m$-automorphism group. The stack $\mathcal{S}_{2,1}(2/3)$ contains two isomorphism classes of curves, namely the $\frac{2}{3}$-atom, and the rational ramphoid cuspidal curve with non-trivial crimping. We construct this stack explicitly as follows: start with the constant family $(\mathbb{P}^1 \times \mathbb{A}^1, \infty \times \mathbb{A}^1)$, let $c$ be a coordinate on $\mathbb{A}^1$, and
$t$ a coordinate on $\mathbb{P}^1 - \infty$. Now let $\mathbb{P}^1 \times \mathbb{A}^1 \to C$ be the map defined by the inclusion of algebras $\mathbb{C}[t^2 + ct^3, t^5] \subset \mathbb{C}[c, t]$ on the complement of the infinity section, and defined as an isomorphism on the complement of the zero section. Then $(C \to \mathbb{A}^1, \infty \times \mathbb{A}^1)$ is a family of rational ramphoid cuspidal curves whose fiber over zero is a $\frac{2}{3}$-atom. Furthermore, $\mathcal{G}_m$ acts on the base and total space of this family by $t \mapsto \lambda^{-1} t, c \mapsto \lambda c$, since the subalgebra $\mathbb{C}[t^2 + ct^3, t^5] \subset \mathbb{C}[c, t]$ is invariant under this action. Thus, the family descends to $[\mathbb{A}^1/\mathcal{G}_m]$ and there is an induced map $[\mathbb{A}^1/\mathcal{G}_m] \to \overline{\mathcal{M}}_{2,1}(2/3)$. This map is a locally-closed immersion by \textup{vdW10} Theorem 1.109], and the image is precisely $\overline{\mathcal{S}}_{2,1}(2/3)$. Thus, $\overline{\mathcal{S}}_{2,1}(2/3) \cong [\mathbb{A}^1/\mathcal{G}_m]$ as desired.

For higher values of $(g, n)$, the key observation is that every curve in $\overline{\mathcal{S}}_{g,n}(\alpha_i)$ can be obtained from an $\alpha_i$-stable curve by ‘sprouting’ an appropriate singularity. We make this precise in the following definition.

**Definition 4.14.** If $(C, p_1)$ is a 1-pointed curve, we say that $C'$ is a (ramphoid) cuspidal sprouting of $(C, p_1)$ if $C'$ contains a (ramphoid) cusp $q \in C'$, and the pointed normalization of $C'$ at $q$ is isomorphic to one of:

(a) $(C, p_1)$.
(b) $(C \cup \mathbb{P}^1, \infty)$ where $C$ and $\mathbb{P}^1$ are glued nodally by identifying $p_1 \sim 0$.

If $(C, p_1, p_2)$ is a 2-pointed curve, we say that $C'$ is a tacnodal sprouting of $(C, p_1, p_2)$ if $C'$ contains a tacnode $q \in C'$, and the pointed normalization of $C'$ at $q$ is isomorphic to one of:

(a) $(C, p_1, p_2)$.
(b) $(C \cup \mathbb{P}^1, p_1, \infty)$ where $C$ and $\mathbb{P}^1$ are glued nodally by identifying $p_2 \sim 0$.
(c) $(C \cup \mathbb{P}^1, p_2, \infty)$ where $C$ and $\mathbb{P}^1$ are glued nodally by identifying $p_1 \sim 0$.
(d) $(C \cup \mathbb{P}^1 \cup \mathbb{P}^1, \infty_1, \infty_2)$ where $C$ is glued nodally to two copies of $\mathbb{P}^1$ along $p_1 \sim 0, p_2 \sim 0$.

In this definition, we allow the possibility that $(C, p_1, p_2) = (C, p_1) \coprod (C, p_2)$ is disconnected, with one marked point on each connected component.

If $(C, p_1)$ is a 1-pointed curve, we say that $C'$ is a one-sided tacnodal sprouting of $(C, p_1)$ if $C'$ contains a tacnode $q \in C'$, and the pointed normalization of $C'$ at $q$ is isomorphic to one of:

(a) $(C, p_1) \coprod (\mathbb{P}^1, 0)$.
(b) $(C \cup \mathbb{P}^1, \infty) \coprod (\mathbb{P}^1, 0)$ where $C$ and $\mathbb{P}^1$ are glued nodally by identifying $p_1 \sim 0$.

**Remark 4.15.** Suppose $C'$ is a cuspidal sprouting, one-sided tacnodal sprouting or ramphoid cuspidal sprouting of $(C, p_1)$ (resp. tacnodal sprouting of $(C, p_1, p_2)$) with $\alpha_i$-critical singularity $q \in C'$. Then $(C, p_1)$ (resp. $(C, p_1, p_2)$) is the stable pointed normalization of $C'$ along $q$. By Lemma 2.22, $C'$ is $\alpha_i$-stable if and only if $(C, p_1)$ (resp. $(C, p_1, p_2)$) is $\alpha_i$-stable.

**Lemma 4.16.** Fix $\alpha_i \in \{9/11, 7/10, 2/3\}$, and suppose $(C, \{p_i\}_{i=1}^n) \in \overline{\mathcal{S}}_{g,n}(\alpha_i)$. 
Proof. If \((C, \{p_i\}_{i=1}^n) \in \mathcal{S}_{g,n}(\alpha_i)\), then \((C, \{p_i\}_{i=1}^n)\) contains an \(\alpha_i\)-critical singularity \(q \in C\). The stable pointed normalization of \((C, \{p_i\}_{i=1}^n)\) along \(q\) is well-defined by our hypothesis on \((g, n)\), and is \(\alpha_i\)-stable by Lemma 2.22.

Lemma 4.16 gives a set-theoretic description of \(\mathcal{S}_{g,n}(\alpha_i)\), and we must now augment this to a stack-theoretic description. This means constructing universal families of cuspidal, tacnodal, and ramphoid cuspidal sproutings. A nearly identical construction was carried out in [Smy11b] for elliptic \(m\)-fold points (in particular, cusps and tacnodes), and for all curve singularities in [vdW10]. The only key difference is that here we allow all branches to sprout \(\mathbb{P}^1\)'s rather than a restricted subset. Therefore, we obtain non-separated, stacky compactifications (rather than Deligne-Mumford compactifications) of the associated crimping stack of the singularity. In what follows, if \(\mathcal{C} \to T\) is any family of curves with a section \(\tau\), we say that \(\mathcal{C}\) has an \(A_k\)-singularity along \(\tau\) if, étale locally on the base, the Henselization of \(\mathcal{C}\) along \(\tau\) is isomorphic to the Henselization of \(T \times \mathbb{C}[x, y]/(y^2 - x^k)\) along the zero section (cf. [vdW10, Definition 1.64]).

**Definition 4.17.** Let \(\text{Sprout}_{g,n}(A_k)\) denote the stack of flat families of curves \((\mathcal{C} \to T, \{\sigma_i\}_{i=1}^{n+1})\) satisfying

1. \((\mathcal{C} \to T, \{\sigma_i\}_{i=1}^{n+1})\) is a \(T\)-point of \(\mathcal{U}_{g,n}(A_k)\).
2. \(\mathcal{C}\) has an \(A_k\)-singularity along \(\sigma_{n+1}\).

The fact that \(\text{Sprout}_{g,n}(A_k)\) is an algebraic stack over \((\text{Schemes}/\mathcal{C})\) is verified in [vdW10]. There are obvious forgetful functors

\[
F_k : \text{Sprout}_{g,n}(A_k) \to \mathcal{U}_{g,n}(A_k),
\]

given by forgetting the section \(\sigma_{n+1}\).

**Proposition 4.18.** \(F_k\) is representable and finite.

Proof. It is clear that \(F_k\) is representable. The fact that \(F_2\) is quasi-finite follows from the observations that a curve \((C, \{p_i\}_{i=1}^n)\) in \(\mathcal{U}_{g,n}(A_k)\) has only a finite number of \(A_k\)-singularities and that for a \(\mathcal{C}\)-point \(x \in \text{Sprout}_{g,n}(A_k)\), the induced map \(\text{Aut}_{\text{Sprout}_{g,n}(A_k)}(x) \to \text{Aut}_{\mathcal{U}_{g,n}(A_k)}(F_k(x))\) on automorphism groups has finite cokernel. To show that \(F_2\) is finite,
it now suffices to verify the valuative criterion for properness: let $\Delta$ be the spectrum of a discrete valuation ring, let $\Delta^*$ denote the spectrum of its fraction field, and suppose we are given a diagram

$$
\begin{align*}
\Delta^* &\longrightarrow \text{Sprout}_{g,n}(A_k) \\
\downarrow F_k &\quad &\downarrow \text{Sprout}_{g,n}(A_k) \\
\Delta &\longrightarrow U_{g,n}(A_k)
\end{align*}
$$

This corresponds to a diagram of families,

$$
\begin{align*}
C_{\Delta^*} &\longrightarrow C \\
\sigma_{n+1} &\longmapsto \sigma_{n+1} \\
\Delta^* &\longrightarrow \Delta
\end{align*}
$$

such that $C_{\Delta^*}$ has a ramphoid cusp along $\sigma_{n+1}$. Since $C \rightarrow \Delta$ is proper, $\sigma_{n+1}$ extends uniquely to a section of $\pi$, and since the limit of an $A_k$-singularity in $U_{g,n}(A_k)$ is necessarily an $A_k$-singularity, $C$ has an $A_k$-singularity along $\sigma_{n+1}$. This induces a unique lift $\Delta \rightarrow \text{Sprout}_{g,n}(A_k)$, cf. [vdW10, Theorem 1.109].}

The stacks $\text{Sprout}_{g,n}(A_k)$ also admit stable pointed normalization functors, given by forgetting the crimping data of the singularity along $\sigma_{n+1}$. To be precise, if $(C \rightarrow T, \{\sigma_i\}_{i=1}^{n+1})$ is a $T$-point of $\text{Sprout}_{g,n}(A_k)$, there exists a commutative diagram

$$
\begin{align*}
\tilde{C} &\quad \phi \\
C^* &\quad \psi \\
\{\tilde{\sigma}_i\}_{i=1}^{n+1} &\quad \{\sigma_i\}_{i=1}^{n+1} \\
C &\quad T \\
\{\sigma_i\}_{i=1}^{n+1} &\quad \{\sigma_i\}_{i=1}^{n+1}
\end{align*}
$$

satisfying:

1. $(\tilde{C} \rightarrow T, \{\tilde{\sigma}_i\}_{i=1}^{n+\bar{k}})$ is a family of $(n+\bar{k})$-pointed curves, where $\bar{k} \in \{1, 2\}$.
2. $\psi$ is the pointed normalization of $C$ along $\sigma_{n+1}$, i.e. $\psi$ is finite and restricts to an isomorphism on the open set $\tilde{C} - \cup_{i=1}^{\bar{k}} \tilde{\sigma}_{n+i}$.
3. $\phi$ is the stabilization of $(\tilde{C}, \{\tilde{\sigma}_i\}_{i=1}^{n+\bar{k}})$, i.e. $\phi$ is the morphism associated to a high multiple of the line-bundle $\omega_{\tilde{C}/T}(\Sigma_{i=1}^{n+\bar{k}} \tilde{\sigma}_i)$.

Remark 4.19. Issues arise when defining the stable pointed normalization for $(g, n)$ small relative to $k$. From now on, we assume $k \in \{2, 3, 4\}$, and that $(g, n) \neq (1, 1), (1, 2), (2, 1)$ when $k = 2, 3, 4$ respectively. This ensures that the stabilization morphism $\phi$ is well-defined. Indeed, under these hypotheses, $\omega_{\tilde{C}}(\Sigma_i \tilde{\sigma}_i)$ will be relatively big and nef, and the only components of fibers of $(\tilde{C}, \{\tilde{\sigma}_i\}_{i=1}^{n+\bar{k}})$ on which $\omega_{\tilde{C}}(\Sigma_i \tilde{\sigma}_i)$ has degree zero will be two-pointed $\mathbb{P}^1$'s.
which meet the rest of the curve in a single node and are marked by one of the sections $\tilde{\sigma}_{n+1}$.

The effect of $\phi$ is simply to blow-down these $\mathbb{P}^1$'s.

Since normalization and stabilization are canonically-defined, the association

$$\langle C \to T, \{\sigma_i\}_{i=1}^n \rangle \to \langle C^s \to T, \{\sigma_i^s\}_{i=1}^{n+1} \rangle$$

is functorial, and we obtain normalization functors:

- $N_2$: Sprout$_{g,n}(A_2) \to U_{g-1,n+1}(A_2)$
- $N_3$: Sprout$_{g,n}(A_3) \to \coprod_{g_1+g_2=g, n_1+n_2=n} (U_{g_1,n_1+1}(A_3) \times U_{g_2,n_2+1}(A_3)) \coprod U_{g-2,n+2}(A_3) \coprod U_{g-1,n+1}(A_3)$
- $N_4$: Sprout$_{g,n}(A_4) \to U_{g-2,n+1}(A_4)$

The connected components of the range of $N_3$ correspond to the different possibilities for the stable pointed normalization of $C$ along $\sigma_{n+1}$. Note that the last case $U_{g-1,n+1}(A_3)$ corresponds to a one-sided tactual nodal sprouting, i.e. one connected component of the pointed normalization of $C$ along $\sigma_{n+1}$ is a family of 2-pointed $\mathbb{P}^1$'s. It is convenient to distinguish these possibilities by defining:

- $\text{Sprout}_{g,n}^{n+1}(A_3) = N_3^{-1}(U_{g-2,n+2}(A_3))$
- $\text{Sprout}_{g,n}^{1,1}(A_3) = N_3^{-1}(U_{g_1,n_1+1}(A_3) \times U_{g_2,n_2+1}(A_3))$
- $\text{Sprout}_{g,n}^{0,2}(A_3) = N_3^{-1}(U_{g-1,n+1}(A_3))$

The following key proposition shows that $N_k$ makes Sprout$_{g,n}(A_k)$ a stacky projective bundle over the moduli stack of pointed normalizations.

We will use the following notation: if $E$ is a locally free sheaf on a stack $\mathcal{X}$, we let $V(E)$ denote the total space of the associated vector bundle, $[V(E)/G_m]$ the quotient stack for the natural action of $G_m$ on the fibers of $V(E)$, and $p: [V(E)/G_m] \to T$ the natural projection.

**Proposition 4.20.** In the following statements, we let $(\pi: \mathcal{C} \to U_{g,n}(A_k), \{\sigma_i\}_{i=1}^n)$ denote the universal family over $U_{g,n}(A_k)$, and $(\pi: \mathcal{C} \to U_{g_1,n_1}(A_k) \times U_{g_2,n_2}(A_k), \{\sigma_i\}_{i=1}^{n_1}, \{\tau_i\}_{i=1}^{n_2})$ the universal family over $U_{g_1,n_1}(A_k) \times U_{g_2,n_2}(A_k)$.

1. Let $E$ be the invertible sheaf on $U_{g-1,n+1}(A_2)$ defined by

$$E := \pi_* (\mathcal{O}_C(-2\sigma_{n+1})/\mathcal{O}_C(-3\sigma_{n+1}))$$

There exists an isomorphism

$$\gamma: [V(E)/G_m] \cong \text{Sprout}_{g,n}(A_2)$$

such that $N_2 \circ \gamma = p$. 

(2) Let $E$ be the locally free sheaf on $\mathcal{U}_{g-2,n+2}(A_3)$ defined by
$$E := \pi_* (\mathcal{O}_C(-\sigma_{n+1})/\mathcal{O}_C(-2\sigma_{n+1}) \oplus \mathcal{O}_C(-\sigma_{n+2})/\mathcal{O}_C(-2\sigma_{n+2}))$$
Then there exists an isomorphism
$$\gamma: [V(E)/G_m] \cong \text{Sprout}_{g,n}^n(A_3)$$
such that $N_3 \circ \gamma = p$.

(3) Let $E$ be the locally free sheaf on $\mathcal{U}_{g_1,n+1}(A_3) \times \mathcal{U}_{g_2,n+1}(A_3)$ defined by
$$E := \pi_* (\mathcal{O}_C(-\sigma_{n+1})/\mathcal{O}_C(-2\sigma_{n+1}) \oplus \mathcal{O}_C(-\tau_{n+1})/\mathcal{O}_C(-2\tau_{n+1}))$$
Then there exists an isomorphism
$$\gamma: [V(E)/G_m] \cong \text{Sprout}_{g_1,n}^{g_2}(A_3)$$
such that $N_3 \circ \gamma = p$.

(4) Let $E$ be the locally free sheaf on $\mathcal{U}_{g-1,n+1}(A_3)$ defined by
$$E := \pi_* (\mathcal{O}_C(-\sigma_{n+1})/\mathcal{O}_C(-2\sigma_{n+1}))$$
Then there exists an isomorphism
$$\gamma: [V(E)/G_m] \cong \text{Sprout}_{g,n}^{0,2}(A_3)$$
such that $N_3 \circ \gamma = p$.

(5) Let $E$ be the locally free sheaf on $\mathcal{U}_{g-2,n+1}(A_4)$ defined by
$$E := \pi_* (\mathcal{O}_C(-2\sigma_{n+1})/\mathcal{O}_C(-4\sigma_{n+1}))$$
There exists an isomorphism
$$\gamma: [V(E)/G_m] \cong \text{Sprout}_{g,n}(A_4)$$
such that $N_4 \circ \gamma = p$.

Proof. We prove the hardest case (5), and leave the others as an exercise to the reader. To construct a map $\gamma: [V(E)/G_m] \rightarrow \text{Sprout}_{g,n}(A_4)$, we start with a family $(\pi: \mathcal{C} \rightarrow X, \{\sigma_i\}_{i=1}^{n+1})$ in $\mathcal{U}_{g-2,n+1}(A_4)$, and construct a family of ramphoid cuspidal sproutings over $[V(E_X)/G_m]$, where
$$E_X := \pi_* (\mathcal{O}_C(-2\sigma_{n+1})/\mathcal{O}_C(-4\sigma_{n+1}))$$
Let $V := V(E_X)$, $p: V \rightarrow X$ the natural projection, and $(\mathcal{C}_V \rightarrow V, \sigma_V)$ the family obtained from $(\mathcal{C} \rightarrow X, \sigma_{n+1})$ by base change along $p$. As the construction is local around $\sigma_{n+1}$, we will not keep track of $\{\sigma_i\}_{i=1}^n$ for the remainder of the argument. If we set $E_V = p^*E_X$, there exists a tautological section $e: \mathcal{O}_V \rightarrow E_V$. Let $Z \subset V$ denote the divisor along which the composition
$$\mathcal{O}_V \rightarrow E_V \rightarrow (\pi_V)_* (\mathcal{O}_{\mathcal{C}_V}(-2\sigma_V)/\mathcal{O}_{\mathcal{C}_V}(-3\sigma_V))$$
we claim that to a ramphoid cusp, and regular subscheme of codimension 2, the exceptional divisor \( E \) of the blow-up is a \( \mathbb{P}^1 \)-bundle over \( \sigma_V(Z) \). In other words, for all \( z \in Z \), we have

\[
\tilde{C}_z = C_z \cup E_z = C_z \cup \mathbb{P}^1
\]

Let \( \tilde{\sigma} \) be the strict transform of \( \sigma_V \) on \( \tilde{C} \), and observe that \( \tilde{\sigma} \) passes through a smooth point of the \( \mathbb{P}^1 \) component in every fiber over \( Z \). We will construct a map \( \tilde{C} \to C' \) which crimps \( \tilde{\sigma} \) to a ramphoid cusp, and \( C' \to X \) will be the desired family of ramphoid cuspidal sproutings.

Setting \( \tilde{\pi} : \tilde{C} \to C_V \to V \) and

\[
\tilde{E} = (\tilde{\pi})_* \left( \mathcal{O}_{\tilde{C}}(-2\tilde{\sigma})/\mathcal{O}_{\tilde{C}}(-4\tilde{\sigma}) \right)
\]

we claim that \( e \) induces a section \( \tilde{e} : \mathcal{O}_V \to \tilde{E} \) with the property that the composition

\[
\mathcal{O}_V \to \tilde{E} \to \tilde{\pi}_* \left( \mathcal{O}_{\tilde{C}}(-2\tilde{\sigma})/\mathcal{O}_{\tilde{C}}(-3\tilde{\sigma}) \right)
\]

is never zero. To see this, let \( U = \text{Spec } R \subset X \) be an open affine along which \( E \) is trivial, and choose local coordinates on \( a, b \) on \( p^{-1}(U) = \text{Spec } R[a, b] \) such that the tautological section \( e \) is given by \( at^2 + bt^3 \), where \( t \) is a local equation for \( \sigma_V \) on \( C_V \). In these coordinates, \( \phi \) is the blow-up along \( a = t = 0 \). Let \( \tilde{a}, \tilde{t} \) be homogenous coordinates for the blow-up and note that on the chart \( \tilde{a} \neq 0 \), \( \tilde{t} : = \tilde{t}/\tilde{a} \) gives a local equation for \( \tilde{\sigma}_V \). In these coordinates, \( \phi \) is given by

\[
(a, b, t') \to (a, b, at')
\]

The section \( at^2 + bt^3 \) pulls back to \( a^3(t'^2 + bt'^3) \), and \( t'^2 + bt'^3 \) is a section of \( \tilde{E} \) over \( p^{-1}(U) \) with the stated property.

We will use \( \tilde{e} \) to construct a map \( \psi : \tilde{C} \to C' \) such that \( C' \) has a ramphoid cusp along \( \psi \circ \tilde{\sigma} \). It is sufficient to define \( \psi \) locally around \( \tilde{\sigma} \), so we may assume \( \tilde{\pi} \) is affine, i.e. \( \tilde{C} := \text{Spec } \tilde{\pi}_* \mathcal{O}_{\tilde{C}} \). We specify a sheaf of \( \mathcal{O}_V \)-subalgebras of \( \tilde{\pi}_* \mathcal{O}_{\tilde{C}} \) as follows: Consider the exact sequence

\[
0 \to \tilde{\pi}_* \mathcal{O}_{\tilde{C}}(-4\tilde{\sigma}) \to \tilde{\pi}_* \mathcal{O}_{\tilde{C}'}(-2\tilde{\sigma}) \to \tilde{E} \to 0
\]

and let \( \mathcal{F} \subset \tilde{\pi}_* \mathcal{O}_{\tilde{C}} \) be the sheaf of \( \mathcal{O}_V \)-subalgebras generated by any inverse image of \( \tilde{e} \) and all functions in \( \tilde{\pi}_* \mathcal{O}_{\tilde{C}}(-4\tilde{\sigma}) \). We let \( \psi : \text{Spec } \mathcal{F} \to C' := \text{Spec } \mathcal{O}_V \mathcal{F} \) be the map corresponding to the inclusion \( \mathcal{F} \subset \tilde{\pi}_* \mathcal{O}_{\tilde{C}} \). By construction, the complete local ring \( \tilde{\mathcal{O}}_{C'_v(\psi \circ \tilde{\sigma})(v)} \subset \tilde{\mathcal{O}}_{\tilde{C}'_v, \tilde{\sigma}(v)} \cong \mathbb{C}[t] \) is of the form \( \mathbb{C}[t^2 + bt^3, t^5] \subset \mathbb{C}[t] \), and this subalgebra is isomorphic to \( \mathbb{C}[x, y]/(y^2 - x^3) \).

Finally, we claim that \( C' \to V \) descends to a family of ramphoid cuspidal sproutings over the quotient stack \( [V/\mathbb{G}_m] \). It suffices to show the subsheaf \( \mathcal{F} \subset \tilde{\pi}_* \mathcal{O}_{\tilde{C}} \) is invariant under the natural action of \( \mathbb{G}_m \) on \( V \). Using the same local coordinates introduced above, the sheaf \( \mathcal{F} \) is generated over the open set \( \text{Spec } R[a, b] \), the algebra is generated by \( t'^2 + bt'^3 \) and \( t^5 \), where \( t' \) is a local equation for \( \tilde{\sigma} \) on \( \tilde{C} \). To see that this subalgebra is \( \mathbb{G}_m \)-invariant, note that the \( \mathbb{G}_m \)-action on \( V = \text{Spec } R[a, b] \) (acting with weight 1 on \( a \) and \( b \)) extends canonically to a \( \mathbb{G}_m \)-action on the blow-up, where \( \mathbb{G}_m \) acts on \( \tilde{a}, \tilde{t} \) with weight 1 and 0 respectively.
\( \mathbb{G}_m \) acts on \( t' = \tilde{t}/\tilde{u} \) with weight -1, so the section \( t'^2 + bt'^3 \) is a semi-invariant. It follows that the algebra generated by \( t'^2 + bt'^3 \) and \( t'^5 \) is \( \mathbb{G}_m \)-invariant as desired. Thus, we obtain a family \( (\mathcal{C}' \to [V/\mathbb{G}_m], \psi \circ \tilde{\sigma}) \) in Sprout_{g,n}(A_4) as desired.

To define an inverse map \( i^{-1}: \text{Sprout}_{g,n}(A_4) \to [V/\mathbb{G}_m] \), we start with a family \( (\mathcal{C} \to X, \sigma) \) in \( \mathcal{U}_{g,n}(A_4) \) such that \( \mathcal{C} \) has an \( A_4 \)-singularity along \( \sigma \). We must construct a map \( X \to [V(\mathcal{E})/\mathbb{G}_m] \). By taking the stable pointed normalization of \( \mathcal{C} \) along \( \sigma \), we obtain a diagram

\[
\begin{array}{ccc}
\mathcal{C}^s & \xrightarrow{\phi} & \tilde{\mathcal{C}} \\
\sigma^* & \downarrow \psi & \downarrow \tilde{\sigma} \\
X & \xleftarrow{\sigma} & \mathcal{C}
\end{array}
\]

satisfying

1. \( (\tilde{\mathcal{C}} \to X, \tilde{\sigma}) \) is a family of \((n+1)\)-pointed curves.
2. \( \psi \) is the pointed normalization of \( \mathcal{C} \) along \( \sigma \), i.e. \( \psi \) is finite and restricts to an isomorphism on the open set \( \tilde{\mathcal{C}} - \tilde{\mathcal{C}} \).
3. \( \phi \) is the stabilization of \( (\tilde{\mathcal{C}}, \tilde{\sigma}) \), i.e. \( \phi \) is the morphism associated to a high multiple of the net line-bundle \( \omega_{\tilde{\mathcal{C}}/X}(\tilde{\sigma}) \).

By Lemma 2.22, \( (\mathcal{C}^s \to X, \sigma^*_1) \) induces a map \( X \to \mathcal{U}_{g-2,n+1}(A_4) \), and we must show that this lifts to define a map \( X \to [V(\mathcal{E})/\mathbb{G}_m] \). To see this, let \( \mathcal{F} \) be the coherent sheaf defined by the following exact sequence

\[
0 \to \pi_* \mathcal{O}_{\mathcal{C}} \cap \pi_* \mathcal{O}_{\mathcal{C}}(-4\tilde{\sigma}) \subset \pi_* \mathcal{O}_{\mathcal{C}} \cap \pi_* \mathcal{O}_{\mathcal{C}}(-2\tilde{\sigma}) \to \mathcal{F} \to 0,
\]

The condition that \( \mathcal{C} \) has a ramphoid cusp along \( \psi \circ \tilde{\sigma} \) implies that \( \mathcal{F} \subset \pi_* \mathcal{O}_{\mathcal{C}}(-2\sigma)/\mathcal{O}_{\mathcal{C}}(-4\sigma) \) is a rank one subbundle. In particular, \( \mathcal{F} \) induces a subbundle of \( \pi_* \mathcal{O}_{\mathcal{C}}(-2\sigma)/\mathcal{O}_{\mathcal{C}}(-4\sigma) \) over the locus of fibers on which \( \phi \) is an isomorphism. A local computation, similar to the one performed in the definition of \( \gamma \), shows that \( \mathcal{F} \) extends to a subsheaf of \( \pi_* \mathcal{O}_{\mathcal{C}}(-2\sigma)/\mathcal{O}_{\mathcal{C}}(-4\sigma) \) over all of \( X \) (though not a subbundle; the inclusion of fibers is zero precisely where \( \phi \) fails to be an isomorphism). The subsheaf \( \mathcal{F} \subset \mathcal{E} \) induces the desired morphism \( X \to [V/\mathbb{G}_m] \). \( \square \)

**Proposition 4.21.** Let \( \alpha_i \in \{9/11, 7/10, 2/3\} \) and suppose that \( \overline{\mathcal{M}}_{g',n'}(\alpha_i) \) admits a proper good moduli space for all \((g',n')\) with \( g' < g \). Then \( \mathcal{S}_{g,n}(\alpha_i) \) admits a proper good moduli space.

**Proof.** Let \( \alpha_i = 9/11 \). By Lemma 4.13 we may assume \((g,n) \neq (1,1)\). By Proposition 4.20, there is a locally free sheaf \( \mathcal{E} \) on \( \overline{\mathcal{M}}_{g-1,n+1}(9/11) \) such that \( [V(\mathcal{E})/\mathbb{G}_m] \) is the base of a universal family of cuspidal sproutings of curves in \( \overline{\mathcal{M}}_{g-1,n+1}(9/11) \). By Lemma 2.22, the
fibers of this family are 9/11-stable so there is an induced map
\[ \Psi: [V(E)/G_m] \to \overline{M}_{g,n}(9/11). \]

By Lemma 4.16 \( \Psi \) maps surjectively onto \( S_{g,n}(9/11) \). Furthermore, \( \Psi \) is finite by Proposition 4.18. By hypothesis, \( \overline{M}_{g-1,n+1}(9/11) \) and therefore \( [V(E)/G_m] \) admits a proper good moduli space. Thus, \( S_{g,n}(9/11) \) admits a proper good moduli space by Theorem 4.3.

Let \( \alpha_i = 7/10 \). By Lemma 4.13 we may assume \( (g,n) \neq (1,2) \). If \( g \geq 2 \), Proposition 4.20(2) provides a locally free sheaf \( E \) on \( \overline{M}_{g-2,n+2}(7/10) \) such that \( [V(E)/G_m] \) is the base of a universal family of tacnodal sproutings of curves in \( \overline{M}_{g-2,n+2}(7/10) \), and there is an induced map \( [V(E)/G_m] \to \overline{M}_{g,n}(7/10) \). Similarly, for every pair of integers \( (i,m) \) such that \( \overline{M}_{g-i,m+1}(7/10) \times \overline{M}_{i,m+1}(7/10) \) is defined, by Proposition 4.20(3), there is a locally free sheaf \( E \) on \( \overline{M}_{g-i,n-m+1}(7/10) \times \overline{M}_{i,m+1}(7/10) \) such that \( [V(E)/G_m] \) is the universal family of tacnodal sproutings. By Lemma 2.22 there are induced maps \( [V(E)/G_m] \to \overline{M}_{g,n}(7/10) \). Finally, Proposition 4.20(4) provides a locally free sheaf on \( \overline{M}_{g-1,n}(7/10) \) such that \( [V(E)/G_m] \) is the base of a universal family of one-sided tacnodal sproutings of curves in \( \overline{M}_{g-1,n}(7/10) \). By Lemma 2.22 there is an induced map \( [V(E)/G_m] \to \overline{M}_{g,n}(7/10) \). The union of the maps \( [V(E)/G_m] \to \overline{M}_{g,n}(7/10) \) cover \( S_{g,n}(7/10) \) by Lemma 4.16. Furthermore, each map is finite by Proposition 4.18. By hypothesis, each of the stacky projective bundles \( [V(E)/G_m] \) admits a proper good moduli space, and therefore so does \( S_{g,n}(7/10) \) by Theorem 4.3.

Let \( \alpha_i = 2/3 \). By Lemma 4.13 we may assume \( (g,n) \neq (2,1) \). By Proposition 4.20(5), there is a locally free sheaf \( E \) on \( \overline{M}_{g-2,n+1}(2/3) \) defined such that \( [V(E)/G_m] \) is the base of a universal family of ramphoid cuspidal sproutings of curves in \( \overline{M}_{g-2,n+1}(9/11) \). By Lemma 2.22, there is an induced map \( \Psi: [V(E)/G_m] \to \overline{M}_{g,n}(9/11) \) which maps surjectively onto \( S_{g,n}(9/11) \) by Lemma 4.16. Furthermore, \( \Psi \) is finite by Proposition 4.18. Thus, \( S_{g,n}(9/11) \) admits a proper good moduli space by Theorem 4.3.

### 4.2.2. Existence for \( \overline{H}_{g,n}(\alpha_i) \)

In this section, we use induction on \( g \) to prove that \( \overline{H}_{g,n}(\alpha_i) \) admits a good moduli space. The base case is handled by the following easy lemma.

**Lemma 4.22.** We have:
\[
\begin{align*}
\overline{H}_{1,1}(9/11) &= [\mathcal{A}^2/G_m], \text{ with weights } -4, -6. \\
\overline{H}_{1,2}(7/10) &= [\mathcal{A}^3/G_m], \text{ with weights } -2, -3, -4. \\
\overline{H}_{2,1}(2/3) &= [\mathcal{A}^4/G_m], \text{ with weights } -4, -6, -8, -10.
\end{align*}
\]
In particular, \( \overline{H}_{1,1}(9/11), \overline{H}_{1,2}(7/10), \overline{H}_{2,1}(2/3) \) each admit a good moduli space.

**Proof.** We describe the case of \( \overline{H}_{2,1}(2/3) \), as the other two are essentially identical. Consider the family of Weierstrass tails over \( \mathcal{A}^4 \) given by:
\[
z^2 = x^5y + a_3x^3y^3 + a_2x^2y^4 + a_1xy^5 + a_0y^6.
\]
where the Weierstrass section is given by \([1, 0, 0]\). Since \(\mathbb{G}_m\) acts on the base and total space of this family by
\[
x \to \lambda^2 x, \quad z \to \lambda^5 z, \quad a_i \to \lambda^{2i-10} a_i,
\]
the family descends to \([\mathbb{A}^4/\mathbb{G}_m]\). One checks that the induced map \([\mathbb{A}^4/\mathbb{G}_m] \to \overline{H}_{2,1}(2/3)\) is an isomorphism.

Lemma 4.22 gives an explicit description of the stack of elliptic tails, elliptic bridges, and Weierstrass tails. In the case \(\alpha_i = 7/10\), we will also need an explicit description of the stack of elliptic chains of length \(r\).

**Lemma 4.23.** Let \(r \geq 1\) be an integer, and let
\[
\mathcal{Z}_{r,2} \subset \overline{\mathcal{M}}_{2r-1,2}(7/10)
\]
denote the closure of the locally-closed substack of elliptic chains of length \(r\). Then \(\mathcal{Z}_{r,2}\) admits a good moduli space.

**Proof.** Lemma 4.22 handles the case \(r = 1\). By induction on \(r\), we may assume that \(\mathcal{Z}_{r-1,2}\) admits a good moduli space. By Proposition 4.20(3), there is a locally free sheaf \(\mathcal{E}\) on \(\mathcal{Z}_{r-1,2} \times \mathcal{H}_{1,2}(7/10)\) such that \([V(\mathcal{E})/\mathbb{G}_m]\) is the base of a universal family of tacnodal sproutings over \(\mathcal{Z}_{r-1,2} \times \mathcal{H}_{1,2}(7/10)\). By Lemma 2.22, there is an induced morphism \(\Psi: [V(\mathcal{E})/\mathbb{G}_m] \to \overline{\mathcal{M}}_{2r-1,2}(7/10)\). The image of \(\Psi\) is \(\mathcal{Z}_{r,2}\), and \(\Psi\) is finite by Proposition 4.18. Since \(\mathcal{Z}_{r-1,2} \times \mathcal{H}_{1,2}(7/10)\) admits a good moduli space, Theorem 4.3 implies that \(\mathcal{Z}_{r,2}\) admits a good moduli space. \(\square\)

For higher \((g, n)\), we can use gluing maps to decompose \(\overline{H}_{g,n}(\alpha_i)\) into products of lower-dimensional moduli spaces.

**Lemma 4.24.** Let \(\alpha_i \in \{9/11, 7/10, 2/3\}\). There exist finite gluing morphisms
\[
\Psi: \overline{\mathcal{M}}_{g_1, n_1+1}(\alpha_i) \times \overline{\mathcal{M}}_{g_2, n_2+1}(\alpha_i) \to \overline{\mathcal{M}}_{g_1+g_2, n_1+n_2}(\alpha_i)
\]
obtained by identifying \((C, \{p_i\}_{i=1}^{n_1+1})\) and \((C', \{p_i'\}_{i=1}^{n_2+1})\) nodally at \(p_{n_1+1} \sim p'_{n_2+1}\).

**Proof.** \(\Psi\) is well-defined by Lemma 2.22. To see that \(\Psi\) is finite, first observe that \(\Psi\) is clearly representable and quasi-finite. Furthermore, since the limit of a disconnecting node is a disconnecting node in \(\overline{\mathcal{M}}_{g,n}(\alpha_i)\) (Corollary 2.14), \(\Psi\) satisfies the valuative criterion for properness. \(\square\)

In the case \(\alpha_i = 7/10\), we will need two additional gluing morphisms.

**Lemma 4.25.** There exist finite gluing morphisms
\[
\overline{\mathcal{M}}_{g,n+2}(7/10) \times \mathcal{Z}_{r,2} \to \overline{\mathcal{M}}_{g+r-1,n+2}(7/10), \quad \mathcal{Z}_{r,2} \to \overline{\mathcal{M}}_{r+1}(7/10),
\]
where the first map is obtained by nodally gluing \((C, \{p_i\}_{i=1}^{n+2})\) and an elliptic chain \((Z, q_1, q_2)\) at \(p_{n+1} \sim q_1\) and \(p_{n+2} \sim q_2\), and the second map is obtained by nodally self-gluing an elliptic chain \((Z, q_1, q_2)\) at \(q_1 \sim q_2\).
Theorem 4.3. Let \( \alpha_i \in \{9/11, 7/10, 2/3\} \) and suppose that \( \overline{M}_{g',n'}(\alpha_i) \) admits a proper good moduli space for all \((g', n')\) satisfying \( g' < g \). Then \( \overline{H}_{g,n}(\alpha_i) \) admits a proper good moduli space.

Proof. Let \( \alpha_i = 9/11 \). By Lemma 4.22, we may assume \((g, n) \neq (1, 1)\). By Lemma 4.24, there exists a finite gluing morphism

\[ \Psi: \overline{M}_{g-1,n+1}(9/11) \times \overline{H}_{1,1}(9/11) \to \overline{M}_{g,n}(9/11), \]

whose image is precisely \( \overline{H}_{g,n}(9/11) \). Now \( \overline{H}_{g,n}(9/11) \) admits a proper good moduli space by Theorem 4.3.

Let \( \alpha_i = 7/10 \). For every \( r \) such that \( \overline{M}_{g-2r,n+2}(7/10) \) (resp. \( \overline{M}_{g-2r-1,n}(7/10) \)) exist, Lemma 4.25 (resp. Lemma 4.24) gives a finite gluing morphisms

\[ \overline{M}_{g-2r,n+2}(7/10) \times Z_{r,2} \to \overline{H}_{g,n}(7/10), \]
\[ \overline{M}_{g-2r-1,n}(7/10) \times Z_{r,2} \to \overline{H}_{g,n}(7/10), \]

which identify \((C, \{p_i\}_{i=1}^{n+2})\) (resp. \((C', \{p_i\}_{i=1}^{n})\)) to \((Z, q_1, q_2)\) at \( p_{n+1} \sim q_1, p_{n+2} \sim q_2 \) (resp. \( p_n \sim q_1 \)). In addition, for every integer \( r \) such that \( \overline{M}_{i,m+1}(7/10) \times \overline{M}_{g-i-2r+1,n-m+1}(7/10) \) exists, Lemma 4.24 gives a finite gluing morphism

\[ \overline{M}_{i,m+1}(7/10) \times \overline{M}_{g-i-2r+1,n-m+1}(7/10) \times Z_{r,2} \to \overline{H}_{g,n}(7/10), \]

which identifies \((C, \{p_i\}_{i=1}^{n+1})\), \((C', \{p_i\}_{i=1}^{n-m+1})\), \((Z, q_1, q_2)\) nodally at \( p_{m+1} \sim q_1, p_{n-m+1} \sim q_2 \). Finally, if \((g, n) = (2r, 0)\), Lemma 4.25 gives a finite gluing morphism

\[ Z_{r,2} \to \overline{H}_{2r}(7/10), \]

which nodally self-glues \((Z, q_1, q_2)\) at \( q_1 \sim q_2 \). The union of these gluing maps covers \( \overline{H}_{g,n}(7/10) \). Thus, \( \overline{H}_{g,n}(7/10) \) admits a proper good moduli space by Theorem 4.3.

Let \( \alpha_i = 2/3 \). By Lemma 4.22, we may assume \((g, n) \neq (2, 1)\). By Lemma 4.24, there exists a finite gluing morphism

\[ \Psi: \overline{M}_{g-2,n+1}(2/3) \times \overline{H}_{2,1}(2/3) \to \overline{M}_{g,n}(2/3), \]

whose image is precisely \( \overline{H}_{g,n}(2/3) \). Now \( \overline{H}_{g,n}(2/3) \) admits a proper good moduli space by Theorem 4.3.

4.2.3. Existence for \( \overline{M}_{g,n}(\alpha) \).
Theorem 4.27. For every $\alpha \in (2/3-\epsilon, 1]$, $\overline{M}_{g,n}(\alpha)$ admits a proper good moduli space. Furthermore, for each critical value $\alpha_c \in \{2/3, 7/10, 9/11\}$, there exists a diagram

$$
\begin{array}{c}
\overline{M}_{g,n}(\alpha_c+\epsilon) \ar[r] & \overline{M}_{g,n}(\alpha_c) \ar[d] & \overline{M}_{g,n}(\alpha_c-\epsilon) \\
X^+ \ar[r] & X & X^-
\end{array}
$$

where $\overline{M}_{g,n}(\alpha_c) \to X$, $\overline{M}_{g,n}(\alpha_c+\epsilon) \to X^+$ and $\overline{M}_{g,n}(\alpha_c-\epsilon) \to X^-$ are good moduli spaces, and $X^+ \to X$ and $X^- \to X$ are proper morphisms of algebraic spaces.

Proof. Fix $\alpha_c \in \{9/11, 7/10, 2/3\}$. Note that $\overline{M}_{0,n}(\alpha_i) = \overline{M}_{0,n}$, so $\overline{M}_{0,n}(\alpha_i)$ admits a proper good moduli space for all $n$. By induction on $g$, we may assume that $\overline{M}_{g',n'}(\alpha_c)$ admits a good moduli space for all $(g',n')$ with $g' < g$. Now Propositions 4.26 and 4.21 imply that $\overline{H}_{g,n}(\alpha_c)$ and $\overline{S}_{g,n}(\alpha_c)$ admit proper good moduli spaces. By induction on $\alpha_c$, we may also assume that $\overline{M}_{g,n}(\alpha_c+\epsilon)$ admits a good moduli space. Appealing to Theorem 3.9, we may apply Theorem 4.2 with $X^+ = \overline{M}_{g,n}(\alpha_c) \smallsetminus \overline{S}_{g,n}(\alpha_c)$ and $X^- = \overline{M}_{g,n}(\alpha_c) \smallsetminus \overline{H}_{g,n}(\alpha_c)$ to conclude that $\overline{M}_{g,n}(\alpha_c)$ and $\overline{M}_{g,n}(\alpha_c-\epsilon)$ admits proper good moduli spaces fitting into the stated diagram. \qed

5. Projectivity of the good moduli space

For $\alpha > 2/3-\epsilon$, let $\phi: \overline{M}_{g,n}(\alpha) \to \overline{M}_{g,n}(\alpha)$ denote the good moduli space of $\overline{M}_{g,n}(\alpha)$, whose existence was established in Theorem 4.27. In this section, we prove that $\overline{M}_{g,n}(\alpha)$ is isomorphic to the log canonical model $\overline{M}_{g,n}(\alpha)$. In particular, $\overline{M}_{g,n}(\alpha)$ is projective.

Theorem 5.1. For $\alpha > 2/3-\epsilon$, we have

- $\overline{M}_{g,n}(\alpha) \cong \text{Proj } \bigoplus_{m \geq 0} H^0(\overline{M}_{g,n}, [m(K_{\overline{M}_{g,n}} + m\alpha \delta + (1-\alpha)\psi)])$, and
- $\phi^*\mathcal{O}_{\overline{M}_{g,n}(\alpha)}(1) \cong K_{\overline{M}_{g,n}(\alpha)} + \alpha \delta + (1-\alpha)\psi$.

The proof proceeds by induction on $\alpha$. Assuming we know the statement of Theorem 5.1 at a critical value $\alpha_c$, Proposition 5.3 allows us to deduce the statement for $\alpha_c - \epsilon$. On the other hand, if we know the statement for $\alpha_c + \epsilon$, Proposition 5.2 allows us to deduce the statement at the next critical value $\alpha_c$, provided that a certain divisor class on $\overline{M}_{g,n}(\alpha_c+\epsilon)$ is known to be nef. In this section, we prove Propositions 5.2 and 5.3. In the next section, we state the requisite positivity result (Theorem 5.4) and prove it in Section 5.4. Together, these three results immediately imply Theorem 5.1.

Proposition 5.2. Fix $\alpha_c \in \{9/11, 7/10, 2/3\}$. Suppose that

- $\overline{M}_{g,n}(\alpha_c) \cong \text{Proj } \bigoplus_{m \geq 0} H^0(\overline{M}_{g,n}, [m(K_{\overline{M}_{g,n}} + \alpha \delta + (1-\alpha)\psi)])$, and
- $\phi^*\mathcal{O}_{\overline{M}_{g,n}(\alpha_c)}(1) \cong K_{\overline{M}_{g,n}(\alpha_c)} + \alpha \delta + (1-\alpha)\psi$.

Then there exists $\epsilon > 0$ sufficiently small such that
\[ \mathcal{M}_{g,n}(\alpha_c - \epsilon) \cong \text{Proj} \bigoplus_{m \geq 0} H^0(\overline{M}_{g,n}, [m(K_{\overline{M}_{g,n}} + (\alpha_c - \epsilon)\delta + (1 - \alpha_c + \epsilon)\psi)]), \] and
\[ \phi^*\mathcal{O}_{\mathcal{M}_{g,n}(\alpha_c - \epsilon)}(1) \cong K_{\overline{M}_{g,n}(\alpha_c - \epsilon)} + (\alpha_c - \epsilon)\delta + (1 - \alpha_c + \epsilon)\psi. \]

**Proof.** By Theorem 4.27, there is a good moduli space \( \phi^- : \mathcal{M}_{g,n}(\alpha_c - \epsilon) \to \overline{M}_{g,n}(\alpha_c - \epsilon) \) and a proper morphism \( \pi : \overline{M}_{g,n}(\alpha_c - \epsilon) \to \overline{M}_{g,n}(\alpha_c) \). We claim that a positive multiple of \( \psi - \delta \) on \( \overline{M}_{g,n}(\alpha_c - \epsilon) \) descends to a line bundle \( \mathcal{N} \) on \( \overline{M}_{g,n}(\alpha_c - \epsilon) \) which is relatively ample over \( \overline{M}_{g,n}(\alpha_c) \).

First, to see that a multiple of \( \psi - \delta \) descends it suffices to check that for every \( \alpha_c \)-stable curve \( (C', \{p'_i\}_{i=1}^n) \), the induced character of \( \text{Aut}(C', \{p'_i\}_{i=1}^n)^\circ \) on \( \delta - \psi \) is trivial; see [Alp13, Theorem 10.3]. Let \( (C', \{p'_i\}_{i=1}^n) \leadsto (C, \{p_i\}_{i=1}^n) \) be the unique isotrivial specialization to a closed \( \alpha_c \)-stable curve. Let \( G = \text{Aut}(C, \{p_i\}_{i=1}^n) \) and \( \chi : G \to \mathbb{G}_m \) be the character induced from the action of \( G \) on the fiber of \( \delta - \psi \). By Theorem 3.9 there is an étale quotient presentation \( f : W = [W/G] \to \overline{M}_{g,n}(\alpha_c) \) about \( (C, \{p_i\}_{i=1}^n) \) with \( W = \text{Spec} \mathcal{A} \) such that \( f^{-1}(\overline{M}_{g,n}(\alpha_c - \epsilon)) = W^-_{\chi} \). Moreover, \( \mathcal{L} = f^*(\delta - \psi) \) is the line bundle on \( W \) corresponding the \( G \)-linearization of \( \mathcal{O}_W \) determined by \( \chi_{\delta - \psi} \). If \( w' \in W^-_{\chi} \) is a point such that \( f(w') = (C', \{p'_i\}_{i=1}^n) \), the character of \( \text{Aut}_{W^-_{\chi}}(w')^\circ \) on \( \mathcal{L} \) is naturally identified with the character of \( \text{Aut}(C', \{p'_i\}_{i=1}^n)^\circ \) on \( \delta - \psi \) but by Proposition 3.7 this character is trivial. We conclude that a positive multiple of \( \psi - \delta \) descends to a line bundle \( \mathcal{N} \) on \( \overline{M}_{g,n}(\alpha_c - \epsilon) \).

To show that \( \mathcal{N} \) is relatively ample over \( \overline{M}_{g,n}(\alpha_c) \), consider the commutative cube

\[
\begin{array}{ccc}
\mathcal{M}_{g,n}(\alpha_c) & \xleftarrow{\phi^-} & \overline{M}_{g,n}(\alpha_c) \\
\downarrow{f} & & \downarrow{f^-} \\
W & \xleftarrow{\phi^*\mathcal{O}} & \overline{W}^- \\
\downarrow{\pi} & & \downarrow{\pi^-} \\
\overline{M}_{g,n}(\alpha_c) & \xleftarrow{f^-} & \overline{M}_{g,n}(\alpha_c - \epsilon)
\end{array}
\]

where \( W = [\text{Spec} \mathcal{A}/G] \to W/\mathcal{G} = \text{Spec} \mathcal{A}^G \) and \( W^-_{\chi} \to \overline{W}^-_{\chi} \to G = \text{Proj} \bigoplus_{d \geq 0} A_d \) are the good moduli spaces as in Proposition 3.7. Since the vertical arrows are good moduli spaces, by Proposition 4.5 and Lemmas 3.19 and 4.6 after shrinking \( W \) by a saturated open substack such that \( f \) sends closed points to closed points and is stabilizer preserving at closed points, we may assume that the left and right face are Cartesian. The argument in the proof of Theorem 4.2 concerning Diagram (5.1) shows that the bottom face is Cartesian.

The restriction of \( \mathcal{L}^\vee \) to \( W^-_{\chi} \) descends to the relative \( \mathcal{O}(1) \) on \( W^-_{\chi}/G \). Therefore, the pullback of \( \mathcal{N} \) on \( \overline{M}_{g,n}(\alpha_c - \epsilon) \) to \( \overline{W}^-_{\chi}/G \) is \( \mathcal{O}(1) \) and, in particular, is relatively ample over \( W/G \). Since the bottom face is Cartesian, it follows by descent that \( \mathcal{N} \) is relatively ample over \( \overline{M}_{g,n}(\alpha_c) \).
Since $\mathcal{O}(1)$ on $\overline{M}_{g,n}(\alpha_c)$ is very ample and $\mathcal{N}$ on $\overline{M}_{g,n}(\alpha_c-\epsilon)$ is relatively ample over $\overline{M}_{g,n}(\alpha_c)$, for $N \gg 0$, $\pi^*\mathcal{O}(N) + \mathcal{N}$ is very ample on $\overline{M}_{g,n}(\alpha_c-\epsilon)$. We compute that
\[
(\phi^-)^*(\pi^*\mathcal{O}(N) + \mathcal{N}) = N(K_{\overline{M}_{g,n}(\alpha_c-\epsilon)} + \psi + \alpha_c(\delta - \psi)) + (\psi - \delta) = N(K_{\overline{M}_{g,n}(\alpha_c-\epsilon)} + \psi + (\alpha_c - \frac{1}{N})(\delta - \psi)) .
\]
Therefore, for $\epsilon > 0$ sufficiently small, $K_{\overline{M}_{g,n}(\alpha_c-\epsilon)} + \psi + (\alpha_c-\epsilon)(\delta - \psi)$ descends to a very ample line bundle on $\overline{M}_{g,n}(\alpha_c-\epsilon)$. The statement follows.

**Proposition 5.3.** Fix $\alpha_c \in \{9/11, 7/10, 2/3\}$. Suppose that

- $\overline{M}_{g,n}(\alpha_c+\epsilon) \cong \text{Proj } \bigoplus_{m \geq 0} H^0(\overline{M}_{g,n}, [m(K_{\overline{M}_{g,n}} + (\alpha_c+\epsilon)\delta + (1 - \alpha_c + \epsilon)\psi)])$, and
- $\phi^*\mathcal{O}_{\overline{M}_{g,n}(\alpha_c+\epsilon)}(1) \cong K_{\overline{M}_{g,n}(\alpha_c+\epsilon)} + (\alpha_c+\epsilon)\delta + (1 - \alpha_c + \epsilon)\psi$.

In addition, suppose that the divisor $K_{\overline{M}_{g,n}(\alpha_c+\epsilon)} + \alpha_c\delta + (1 - \alpha_c)\psi$ is nef on $\overline{M}_{g,n}(\alpha_c+\epsilon)$ and all curves on which it has degree zero are contracted by $\overline{M}_{g,n}(\alpha_c+\epsilon) \to \overline{M}_{g,n}(\alpha_c)$. Then

- $\overline{M}_{g,n}(\alpha_c) \cong \text{Proj } \bigoplus_{m \geq 0} H^0(\overline{M}_{g,n}, [m(K_{\overline{M}_{g,n}} + \alpha_c\delta + (1 - \alpha_c)\psi)])$.
- $\phi^*\mathcal{O}_{\overline{M}_{g,n}(\alpha_c)}(1) \cong K_{\overline{M}_{g,n}(\alpha_c)} + \alpha_c\delta + (1 - \alpha_c)\psi$.

**Proof.** Consider the open immersion of stacks $i^+: \overline{M}_{g,n}(\alpha_c+\epsilon) \to \overline{M}_{g,n}(\alpha_c)$ and the induced map on the moduli space $j: \overline{M}_{g,n}(\alpha_c+\epsilon) \to \overline{M}_{g,n}(\alpha_c)$. The line bundle $K_{\overline{M}_{g,n}(\alpha_c+\epsilon)} + \alpha_c\delta + (1 - \alpha_c)\psi$ has trivial characters on all closed curves with $G_m$-action by Proposition 3.33. It follows that there is a line bundle $L \in \text{Pic}(\overline{M}_{g,n}(\alpha_c))$ such that $\phi^*(L) \cong K_{\overline{M}_{g,n}(\alpha_c)} + \alpha_c\delta + (1 - \alpha_c)\psi$. By Theorem 5.4, $(\phi^- \circ j)^*L \cong K_{\overline{M}_{g,n}(\alpha_c)} + \alpha_c\delta + (1 - \alpha_c)\psi$ is nef on $\overline{M}_{g,n}(\alpha_c+\epsilon)$. In addition, $K_{\overline{M}_{g,n}(\alpha_c+\epsilon)} + \alpha_c\delta + (1 - \alpha_c)\psi$ is clearly big and is finitely generated by [BCHM10]. By a theorem of Zariski, a nef and big finitely generated line bundle is semiample. It follows that $L$ is semiample and is positive on all curves in $\overline{M}_{g,n}(\alpha_c)$. Therefore, $L$ is ample.

By a standard discrepancy computation (see [HH13, Section 2.2]):
\[
\overline{M}_{g,n}(\alpha_c) \cong \text{Proj } R(\overline{M}_{g,n}(\alpha_c), L) \cong \text{Proj } R(\overline{M}_{g,n}(\alpha_c+\epsilon), K_{\overline{M}_{g,n}(\alpha_c+\epsilon)} + \alpha_c\delta + (1 - \alpha_c)\psi) \cong \text{Proj } R(\overline{M}_{g,n}, K_{\overline{M}_{g,n}} + \alpha_c\delta + (1 - \alpha_c)\psi) = \overline{M}_{g}(\alpha_c).
\]

**5.1. Main positivity result.** A well-known result of Cornalba and Harris says that $11\lambda - \delta + \psi$ is nef on $\overline{M}_{g,n}$ and has degree 0 precisely on families of elliptic tails; see [CH88] for the case of $n = 0$. In a similar vein, Cornalba proved in [Cor93] that $12\lambda - \delta + \psi$ is ample on $\overline{M}_{g,n}$ and thus obtained a direct intersection-theoretic proof of projectivity of $\overline{M}_{g,n}$ not relying on GIT or Kollár’s semipositivity techniques from [Ko90]. In the introduction to [Cor93],
Cornalba observes that “...it is hard to see how [these techniques] could be extended to other situations.” In this section, we obtain a generalization of this result for \( \overline{M}_{g,n}(9/11 - \epsilon) \) and \( \overline{M}_{g,n}(7/10 - \epsilon) \), thus showing that Cornalba’s approach can be extended to moduli spaces of cuspidal and tacnodal curves.

**Theorem 5.4.**

(a) \( K_{\overline{M}_{g,n}(9/11 - \epsilon)} + (7/10)\delta + (3/10)\psi \sim 10\lambda - \delta + \psi \) is nef on \( \overline{M}_{g,n}(9/11 - \epsilon) \) and has degree 0 precisely on families of elliptic bridges.
(b) \( K_{\overline{M}_{g,n}(7/10 - \epsilon)} + (2/3)\delta + (1/3)\psi \sim 39/4 \lambda - \delta + \psi \) is nef on \( \overline{M}_{g,n}(7/10 - \epsilon) \) and has degree 0 precisely on families of Weierstrass genus 2 tails.

**Outline of the proof:** We consider a family \((C/B; \{\sigma_i\}_{i=1}^n)\) of \((\alpha_c + \epsilon)\)-stable curves over a smooth and proper curve \(B\). To prove that the divisor in question is non-negative on \(B\), we first break \(C\) into irreducible (but possibly non-normal) components. This is achieved by normalizing along generic outer nodes and outer tacnodes of \(C\) (see Definition 2.12).

Next, for each irreducible component, we normalize along generic inner tacnodes and cusps to arrive at a family of generically at-worst nodal curves. When \(\alpha = 7/10 - \epsilon\), tracking how degrees of the associated divisor classes change under the operation of normalization involves a delicate analysis performed in Section 5.2. For the resulting components of relative arithmetic genus at most 2, we use the standard relations on the moduli stacks of genus \(g \leq 2\) curves together with the Hodge Index Theorem to obtain the requisite positivity. For the irreducible components of relative arithmetic genus greater than 2, we normalize along the generic nodes to arrive at a generically smooth family of pointed curves. If the relative genus is 3 or greater, we apply the Cornalba-Harris inequality. We finish with a case-by-case analysis of the remaining possibilities.

**Preliminaries.** Given a family \(\pi: \mathcal{X} \to B\) of at-worst tacnodal curves curves of arithmetic genus \(g\), we let \(E_{\mathcal{X}/B} := \pi_*\omega_{\mathcal{X}/B}\) be the Hodge bundle. We also consider divisor classes \(\lambda = c_1(\pi_*\omega_{\mathcal{X}/B})\) and \(\kappa := \pi_*(\omega_{\mathcal{X}/B}^2)\). The boundary divisor \(\delta\) can be written as \(\delta = \delta_{irr} + \delta_{red}\), where \(\delta_{red}\) is the divisor of the separating nodes in the family. We have the standard relation \(\kappa = 12\lambda - \delta\).

When \(g = 1\), we have \(12\lambda = \delta_{irr}\) and \(\kappa = -\delta_{red}\). On any family of pointed arithmetic genus 1 curves with sections \(\{\sigma_i\}_{i \in \Sigma}\), we have boundary divisor classes \(\delta_{0,S}\) parameterized by \(S \subset \Sigma\). We have \(\delta_{red} := \sum_{S} \delta_{0,S}\) and the following standard relation

\[
(5.2) \quad \psi_i = \lambda + \sum_{S} |S|\delta_{0,S}.
\]

**Lemma 5.5.** Suppose \(C \to B\) is a generically smooth family of curves of genus \(g\).

1. If \(g \geq 1\) and \(\sigma: B \to C\) is a smooth section, then \(\sigma^2 \leq 0\).
2. If \(g = 0\) and \(\sigma, \sigma', \sigma'': B \to C\) are three smooth sections such that \(\sigma\) is disjoint from \(\sigma'\) and \(\sigma''\), then \(\sigma^2 \leq 0\).
Proof. A nodal reduction does not change $\sigma^2$ and passing to the relative minimal model only increases $\sigma^2$. The claim now follows from well-known facts about families of pointed Deligne-Mumford semistable curves.

Lemma 5.6. Suppose $(C \to B; \{\sigma_i\}_{i=1}^n)$ is a generically $(7/10 - \epsilon)$-stable family of $n$-pointed curves. Then $\psi_i < 0$ for some $i$ if and only if $\sigma_i$ lies on a family of $2$-pointed $\mathbb{P}^1$'s attached to the rest of $C$ tacnodally.

Proof. Suppose $\psi_i < 0$. Let $(D; \Sigma)$ be the pointed normalization of the irreducible component on which $\sigma_i$ lies. Since $\psi_i < 0$, the general fiber of $D$ has genus 0 and can be marked only by $\sigma_i$ and one other section, call it $\tau$, by Lemma 5.5. Since $C$ is generically $(7/10 - \epsilon)$-stable, we conclude that $D$ is tacnodally attached to the rest of the curve along $\tau$.

Lemma 5.7. Suppose $C$ is a closed $(7/10 - \epsilon)$-stable curve with a tacnode $p \in C$, then the pointed normalization $(\tilde{C}, p_1, p_2)$ fails to be $(7/10 - \epsilon)$-stable only if one of the $p_i$'s, say $p_1$, lies on an otherwise unmarked $\mathbb{P}^1$ attached to the rest of the curve nodally.

Lemma 5.8. Suppose we have a rational tail marked only by $\sigma$ and attached nodally to the rest of the curve. Then the degree the degree of $\psi - \delta$ does not change under the blow-down of the tail, and the degree of $s\psi - \delta$ decreases if $s > 1$.

Proof. Suppose that the index of the node is $a$. Then desingularizing (i.e. inserting $\ell$ $(-2)$ curves), we obtain a chain of $\ell + 1$ $\mathbb{P}^1$'s with the last marked by $\sigma$. Blowing down each $(-1)$ curve decreases $\psi$ by 1 and decreases $\delta$ by 1.

Finally, we observe that in proving nefness for families of $(\alpha_c + \epsilon)$-stable curves, we can assume that every fiber closed $(\alpha_c + \epsilon)$-stable curves.

Proposition 5.9. Let $\mathcal{M} \to M$ be a good moduli space of an algebraic stack $\mathcal{M}$ of finite type over $\mathbb{C}$. Given a morphism $B \to M$ from a smooth, proper curve $B$ over $\mathbb{C}$, there is a finite morphism $B' \to B$ from a smooth, proper curve over $\mathbb{C}$ and an extension $B' \to \mathcal{M}$ such that the image of every $\mathbb{C}$-point of $B'$ is a closed point of $\mathcal{M}$.

Proof. Let $k(B) \to K'$ be a finite field extension of the fraction field of $B$ such that Spec $k(B) \to M$ lifts to a map Spec $K \to \mathcal{M}$. The image $\xi \in \mathcal{M} \times_{\mathbb{C}} K'$ of the induced map Spec $K' \to \mathcal{M} \times_{\mathbb{C}} K'$ has a unique closed point $\xi_0 \in \mathcal{M} \times_{\mathbb{C}} K'$ in the closure of $\xi$ as $\mathcal{M} \times_{\mathbb{C}} K' \to M \times_{\mathbb{C}} K'$ is a good moduli space. Then after a further finite extension of $K'$, $\xi_0$ is represented by a morphism Spec $K' \to \mathcal{M} \times_{\mathbb{C}} K'$. Let $B'$ be the integral closure of $B$ in $K'$. By a direct limit argument, there exists a finite number of $k$-points $p_1, \ldots, p_n$ of $B'$ such that Spec $K' \to \mathcal{M}$ extends to a map $B' \setminus \{p_1, \ldots, p_n\} \to \mathcal{M}$. Since the image of the generic point is a closed point of $\mathcal{M} \times_{\mathbb{C}} K'$, we can assume that the image of every $\mathbb{C}$-point
in $B' \setminus \{p_1, \ldots, p_n\}$ is a closed point of $\mathcal{M}$. It suffices to find a lift

$$
\text{Spec } \mathbb{C}((x)) \xrightarrow{\text{Spec } \mathbb{C}(p_i)} \mathcal{M} \xrightarrow{\text{Spec } \widehat{\mathcal{O}}_{B',p_i}} \mathcal{M}
$$

after a finite extension of $\mathbb{C}((x))$ for each $i$. Since $\mathcal{M} \to M$ is universally closed, $\mathcal{M} \times_{\mathcal{M}} \text{Spec } \mathbb{C}[[x]]$ is closed. Therefore, if $\eta \in \mathcal{M} \times_{\mathcal{M}} \text{Spec } \mathbb{C}[[x]]$ denotes the image of $\text{Spec } \mathbb{C}((x)) \to \mathcal{M} \times_{\mathcal{M}} \text{Spec } \mathbb{C}[[x]]$, there exists a specialization $\eta \sim x_0$ over the specialization of the generic point to the closed point in $\text{Spec } \mathbb{C}[[x]]$. We can assume $x_0$ is a closed point. After a finite extension of $\mathbb{C}((x))$, the specialization $\eta \sim x_0$ is realized by a morphism $\text{Spec } \mathbb{C}[[x]] \to \mathcal{M}$, which completes the argument. □

**Corollary 5.10.** Suppose a line bundle $\mathcal{L}$ on $\mathcal{M}$ descends to $M$. Then $\mathcal{L}$ is nef on $\mathcal{M}$ if and only if $\mathcal{L}$ has non-negative degree on every curve $f: B \to \mathcal{M} \otimes_k K$ such that all $K$-points of $f(B)$ are closed in $\mathcal{M} \otimes_k K$. □

5.2. $A_3$-prestable normalization. Our first goal is to develop a theory of simultaneous normalization along generic singularities in families of at-worst tacnodal curves. In contrast to the situation for nodal curves, where normalization along a nodal section can always be performed because a node is not allowed to degenerate to a worse singularity, we encounter an additional difficulty of dealing with families where a generic node degenerates to a cusp or a tacnode, or where two nodes can degenerate to a tacnode. The following result, stated in the notation of Section 2.2, describes all possible degenerations of singularities in one-parameter families of tacnodal curves.

**Proposition 5.11.** Suppose $\mathcal{C} \to \Delta$ is a family of at-worst tacnodal curves over $\Delta$, which is the spectrum of a DVR. Denote by $C_{\pi}$ the geometric generic fiber and by $C_0$ the central fiber. The following are the only possible limits in $C_0$ of the singularities of $C_{\pi}$:

1. A limit of a tacnode of $C_{\pi}$ is necessarily a tacnode of $C_0$. Moreover, a limit of an outer tacnode is necessarily an outer tacnode.
2. A limit of a cusp of $C_{\pi}$ is either a cusp or a tacnode of $C_0$.
3. A limit of an inner node of $C_{\pi}$ is either a node, a cusp, or a tacnode of $C_0$.
4. A limit of an outer node of $C_{\pi}$ is either an outer node of $C_0$ or an outer tacnode of $C_0$.
5. Moreover, if an outer tacnode of $C_0$ is a limit of an outer node, it must be a limit of two outer nodes, necessarily joining the same components.

*Proof.* By deformation theory of $A_k$-singularities, a cusp deforms only to a node, a tacnode deforms only either to a cusp, or to a node, or to two nodes. Given this, the result follows directly from Proposition 2.13. □
In the remained of this section, we analyze in detail the following degenerations of the singularities of \( C_\eta \):

(A) An inner node degenerates to a cusp.
(B) An inner node degenerates to a tacnode.
(C) A cusp degenerates to a tacnode.
(D) Two nodes degenerate to a tacnode.

and describe what happens to each of these degenerations under the operation of normalization along the generic singularity. Before we proceed, we introduce terminology and notation that enables our analysis.

Suppose \( C \to B \) is a one-parameter family of curves. A section \( \sigma : B \to C \) is called smooth if \( \sigma(B) \cap \text{Sing}(C/B) = \emptyset \). If \( p \in C \) is a node of its fiber, then by the deformation theory of the node, the local equation of \( C \) at \( p \) is \( xy = t^e \), for some \( e \in \mathbb{Z} \). We call \( e \) the index of the node \( p \), and denote it by \( \text{index}(p) \). In this section, a rational tail (resp. a rational bridge) is an irreducible arithmetic genus 0 component of a curve meeting the rest of the curve in exactly one (resp. two) nodes. If \( (E, p) \subset C_b \) is a rational tail, then the index of \( E \) is defined to be \( \text{index}(p) \). Similarly, if \( (E, p, q) \subset C_b \) is a rational bridge, then the index of \( E \) is defined to be \( \min\{\text{index}(p), \text{index}(q)\} \), and denoted \( \text{index}(E) \). We say that a rational bridge \( E \subset C_b \) is balanced if \( \text{index}(p) = \text{index}(q) \) and \( E \) is adjacent to at most one other rational bridge in \( C_b \). Note that \( E \) is balanced if and only if \( E \) is Q-Cartier in \( C \). For any bridge \( E \subset C \) there exists a unique birational modification of \( C \) along \( E \) which makes \( E \) balanced of index \( \min\{\text{index}(p), \text{index}(q)\} \). If \( E \) is a balanced bridge, we set \( \tilde{E} := \text{index}(E)E \) to be the associated Cartier divisor on \( C \).

Definition 5.12. A pointed family is a one-parameter family \( C \to B \) of curves together with a collection of smooth sections \( \{\sigma_i\}_{i=1}^n \), and

1. Pairs of smooth sections \( \{\eta_i^+, \eta_i^-\}_{i=1}^a \), called inner nodal sections.
2. Smooth sections \( \{\zeta_i\}_{i=1}^b \), called outer nodal sections.
3. Smooth sections \( \{\xi_i\}_{i=1}^c \), called cuspidal sections.
4. Smooth sections \( \{\tau_i\}_{i=1}^d \), called tacnodal sections.

such that the sections are disjoint with the following exceptions: (1) \( \{\eta_i^\pm\}_{i=1}^a \) can meet each other, and (2) \( \zeta_i \) can meet \( \zeta_j \). Furthermore, no more than two sections can meet in any given fiber. We denote the union of all sections of \( C/B \) by \( \Sigma(C/B) \).

The purpose of this definition is to formalize a description of families obtained from families of \((7/10 - \epsilon)\)-stable curve via the process of \( A_\epsilon \)-prestable normalization (see Definition 5.20). The existence of \( A_\epsilon \)-prestable normalization will be proved in Propositions 5.15, 5.16, 5.17, 5.18, 5.19 below.

Given a pointed family \( C \to B \), we make the following definitions:

Definition 5.13. For a cuspidal section \( \xi \), a rational cuspidal bridge associated to \( \xi \) is a rational bridge meeting \( \xi \) and no other sections. We define Cartier divisor \( e(\xi) := \sum \tilde{E} \),
where the sum is taken over all rational cuspidal bridges associated to \( \xi \). For a pair of inner nodal sections \( \{ \eta^+, \eta^- \} \), a rational nodal bridge associated to \( \{ \eta^+, \eta^- \} \) is a rational bridge of \( \mathcal{C}/B \) meeting \( \{ \eta^+, \eta^- \} \) and no other sections. We define a Cartier divisor \( e(\eta^+, \eta^-) := \sum \tilde{F} \), where the sum is taken over all rational nodal bridges associated to \( \{ \eta^+, \eta^- \} \). The sum of all Cartier divisors associated to all the rational cuspidal and nodal bridges of \( \mathcal{C}/B \) is denoted \( E_{\mathcal{C}/B} \).

**Definition 5.14.** With these terminology, we say that a pointed family \( (\mathcal{C}/B; \Sigma(\mathcal{C}/B)) \) is \( A_3 \)-prestable if its fibers have at worst tacnodal singularities, all rational cuspidal and nodal bridges are balanced, and

\[
\omega_{\mathcal{C}/B} \left( \sum_{i=1}^{n} \sigma_i + \sum_{i=1}^{a} (\eta_i^+ + \eta_i^-) + \sum_{i=1}^{b} \xi_i + 2 \sum_{i=1}^{c} \xi_i + 2 \sum_{i=1}^{d} \tau_i + \frac{1}{2} E_{\mathcal{C}/B} \right)
\]

is relatively ample.

Given an \( A_3 \)-prestable family \( \pi : \mathcal{C} \to B \), we introduce the following divisor classes on \( B \):

1. If \( \sigma : B \to \mathcal{C} \) is any section in \( \Sigma(\mathcal{C}/B) \), we set \( \psi_\sigma := \sigma^*(\omega_{\mathcal{C}/B}) = -\pi_*(\sigma^2) \).
2. \( \delta_{\text{sm}} := \sum_{i=1}^{a} \pi_* (\eta_i^+ \cdot \eta_i^-) \).
3. \( \delta_{\text{tn}} := \sum_{i \neq j} \pi_* (\eta_i \cdot \eta_j) + \sum_{i \neq j} \pi_* (\xi_i \cdot \xi_j) \).
4. \( e(n) := -\frac{1}{2} \sum_{E} \pi_* (\tilde{E}^2) \), where the sum is taken over all rational nodal bridges \( E \).
5. \( e(c) := -\frac{1}{2} \sum_{F} \pi_* (\tilde{F}^2) \), where the sum is taken over all rational cuspidal bridges \( F \).
6. \( \psi_{\text{tn}} := \sum_{i=1}^{d} \psi_{\tau_i} \).

In the sequence of the next five propositions, we explain how to “normalize” a family of at-worst tacnodal curves along generic tacnodes, cusps, and nodes. We also explain how to track the change in divisor classes \( \lambda, \delta \) and \( \psi \) under these operations. The case of the tacnode is the most straightforward, as a generic tacnode cannot degenerate to a worse singularities. The analysis in the remaining cases is more delicate. We supply all details in Propositions 5.16 and 5.18 and leave the proofs of Propositions 5.17 and 5.19 to the reader.

**Proposition 5.15** (Normalization along a generic tacnode). Suppose \( \pi : \mathcal{C} \to B \) is a family of curves with a section \( \sigma : B \to \mathcal{C} \) such that \( \sigma(b) \) is an \( A_3 \)-singularity of \( \mathcal{C}_b \) for every closed point \( b \in B \). Denote by \( \nu : \mathcal{C} \to \tilde{C} \) the partial normalization along \( \sigma \). The section \( \sigma \) lifts to sections \( \sigma_{1,2} : B \to \tilde{C} \) and we set \( \psi_{\sigma_i} = \sigma_i^*(\omega_{\tilde{C}/B}) \). Then \( \psi_{\sigma_1} = \psi_{\sigma_2} \), and we have the following formulae

\[
\lambda_{\mathcal{C}/B} = \lambda_{\tilde{C}/B} - \psi_{\sigma_1}, \\
\delta_{\mathcal{C}/B} = \delta_{\tilde{C}/B} - 12 \psi_{\sigma_1}.
\]

**Proof.** The case of arithmetic genus 1 is [Smy11b, Proposition 3.4]. The general case follows by the same argument. \(\square\)
Proposition 5.16 (Type A degeneration). Suppose $C \to B$ is a family with a section $\sigma : B \to C$ such that $\sigma(B)$ is a generic inner node of $C$ degenerating to a cusp over $b \in B$. Denote by $\tilde{C}$ the normalization of $C$ along $\sigma$ and by $\{\sigma_1, \sigma_2\}$ the two preimages of $\sigma$ (which exist after a degree 2 base change). Then $\tilde{C} \to B$ is a flat family with two smooth sections $\{\sigma_i\}_{i=1}^2$ intersecting over $b$, and we have the following formulae:

$$\lambda_{\tilde{C}/B} = \lambda_{\tilde{C}/B} + (\sigma_1 \cdot \sigma_2),$$

$$\delta_{\tilde{C}/B} = \delta_{\tilde{C}/B} - (\psi_{\sigma_1} + \psi_{\sigma_2})\epsilon_{\tilde{C}/B} + 10(\sigma_1 \cdot \sigma_2).$$

Proof. The local equation of $C$ around $\sigma(b)$ is $y^2 = (x - a(t))^2(x + 2a(t))$, where $x = a(t) \in (t)$ is the equation of the generic node for $t \neq 0$. Let $\nu : \tilde{C} \to C$ be the normalization along $\sigma$. Then $\tilde{C}$ has local equation $u^2 = x + 2a(t)$ and $\nu$ is given by $(x, y, t) \mapsto (x, u(x - a(t)), t)$. Clearly, $\tilde{C}$ is smooth in a neighborhood of the preimage of $\sigma(b)$. The preimage of the generic node defined by $u^2 = 3a(t)$. After a finite base change, we can assume that $a(t)$ has a square root so that we obtain two smooth sections $\{\sigma_i\}_{i=1}^2$ satisfying $\nu^{-1}(\sigma) = \sigma_1 \cup \sigma_2$ and intersecting at the point $x = u = t = 0$ lying over $\sigma(b)$.

By duality theory, $\nu^*\omega_{\tilde{C}/B} = \omega_{\tilde{C}/B}(\sigma_1 + \sigma_2)$ and $\mathcal{E}_{\tilde{C}/B} := \tilde{\pi}_*(\omega_{\tilde{C}/B}(\sigma_1 + \sigma_2))$. It follows that $\kappa_{\tilde{C}/B} = \left(\omega_{\tilde{C}/B} + \sigma_1 + \sigma_2\right)^2 = \kappa_{\tilde{C}/B} + \psi_1 + \psi_2 + 2\sigma_1 \cdot \sigma_2$, and by the Grothendieck-Riemann-Roch formula $\lambda_{\tilde{C}/B} = \epsilon_1(\tilde{\pi}_*(\omega_{\tilde{C}/B}(\sigma_1 + \sigma_2))) = \lambda_{\tilde{C}/B} + \sigma_1 \cdot \sigma_2$. \qed

Proposition 5.17 (Type B degeneration). Suppose $C \to B$ is a family with a section $\sigma : B \to C$ such that $\sigma(B)$ is a generic inner node of $C$ degenerating to a tacnode over $b \in B$. Let $\tilde{C}$ be the minimal desingularization of the normalization of $C$ along $\sigma$. Then the preimage of the tacnode is a rational bridge in the fiber of $\tilde{C}_b$. Denote by $\{\sigma_i\}_{i=1}^2$ the two preimages of $\sigma$ (which exist after a degree 2 base change). Then $\tilde{C} \to B$ is a flat family with two smooth sections $\{\sigma_i\}_{i=1}^2$ meeting the rational nodal bridge in the fiber of $\tilde{C}$ over $b \in B$. Denoting by $c(n)$ the index of the rational nodal bridge associated to $\{\sigma_i\}_{i=1}^2$, we have the following formulae:

$$\lambda_{\tilde{C}/B} = \lambda_{\tilde{C}/B} + c(n),$$

$$\delta_{\tilde{C}/B} = \delta_{\tilde{C}/B} - (\psi_1 + \psi_2)\epsilon_{\tilde{C}/B} + 10c(n).$$

Proof. Omitted. \qed

Proposition 5.18 (Type C degeneration). Suppose $C \to B$ is a family with section $\sigma : B \to C$ such that $\sigma(B)$ is a generic cusp of $C$ degenerating to a tacnode over $b \in B$. Let $\mathcal{C}'$ be the normalization of $C$ along $\sigma$ and $\sigma'$ be the reduced preimage of $\sigma$. Let $\tilde{C}$ be the blow-up of $\mathcal{C}'$ along $\sigma'$ and let $\xi$ be the strict transform of $\sigma'$. Then $\tilde{C} \to B$ is a flat family with smooth section $\xi : B \to \tilde{C}$, the preimage of $\sigma(b)$ in $\tilde{C}$ is a rational cuspidal bridge associated to $\xi$. 
Denoting by $e(c)$ the index of this bridge, we have the following formulae:

$$
\lambda_{C/B} = \lambda_{\tilde{C}/B} - \psi_\xi + e(c),
\delta_{C/B} = \delta_{\tilde{C}/B} - 12\psi_\xi + 10e(c).
$$

**Proof.** The local equation of $C$ around $\sigma(b)$ is $y^2 = (x - a(t))^3(x + 3a(t))$, where $x = a(t) \in \mathfrak{t}$ is the equation of the generic cusp for $t \neq 0$. It follows that $C'$ has local equation $u^2 = (x - a(t))(x + 3a(t))$ and $C' \to \tilde{C}$ is given by $(x, y, t) \mapsto (x, u(x - a(t)), t)$. We have $\sigma': x - a(t) = u = 0$, which is not Cartier at $x = u = t = 0$, the preimage of $\sigma(b)$. It follows that $\tilde{C}$ is the blow-up of $C'$ at $x = u = t = 0$ and is smooth along $\xi$. The preimage of $\sigma(b)$ in $\tilde{C}$ is a balanced rational cuspidal bridge $E$ associated to $\xi$, whose index is equal to the valuation of $a(t)$.

By duality theory, $\eta^*\omega_{C/B} = \omega_{\tilde{C}/B}(2\xi + \tilde{E})$ and $E_{C/B} = \tilde{\pi}_*(\omega_{\tilde{C}/B}(2\xi + \tilde{E}))$. Therefore,

$$
\kappa_{C/B} = (\omega_{C/B} + 2\sigma + \tilde{E})^2 = (\omega_{\tilde{C}/B})^2 + 4\xi \cdot \tilde{E} + \tilde{E}^2 = \kappa_{\tilde{C}/B} + 2e(\xi),
$$

and by the Grothendieck-Riemann-Roch formula $\lambda_{C/B} = c_1(\tilde{\pi}_*(\omega_{\tilde{C}/B}(2\xi + \tilde{E}))) = \lambda_{\tilde{C}/B} - \psi_\xi$. The claim follows. \qed

**Proposition 5.19** (Type D degeneration). Suppose $C \to B$ is a family with two generic nodal sections $\sigma_{1,2}$: $B \to C$ degenerating to a tacnode over $b \in B$. Denote by $\tilde{C}$ the normalization of $C$ along $\sigma_1$ and $\sigma_2$, and by $\{\sigma_i^j\}_{i=1}^2$ the two preimages of $\sigma_i$ (which exist after an appropriate finite base change). Then $\tilde{C} \to B$ is a flat family with two pairs of smooth sections $\{\sigma_i^j\}_{i,j=1}^2$ such that $\sigma_i^1$ and $\sigma_i^2$ intersect over $b \in B$ for $j = 1, 2$.

If $B$ is proper, then we have the following formulae:

$$
\lambda_{C/B} = \lambda_{\tilde{C}/B} + \frac{1}{2}(\sigma_1^2 \cdot \sigma_1^1 + \sigma_2^2 \cdot \sigma_2^2),
\delta_{C/B} = \delta_{\tilde{C}/B} - (\psi_1^1 + \psi_1^2 + \psi_2^1 + \psi_2^2)_{\tilde{C}/B} + 4(\sigma_1^1 \cdot \sigma_2^1 + \sigma_2^2 \cdot \sigma_1^2).
$$

**Proof.** Omitted. \qed

**Definition 5.20.** Given an at-worst tacnodal family of curves $(C \to B; \{\sigma_i\}_{i=1}^n)$ such that the geometric generic fiber has $a$ inner nodes, $b$ outer nodes, $c$ cusps, and $d$ tacnodes, Propositions 5.15, 5.16, 5.17, 5.18, 5.19 allow us to obtain a family of $A_3$-prestable curves whose geometric generic fiber is the normalization of the geometric generic fiber of $C$. We call the resulting family marked by the preimages of the generic singularities the $A_3$-prestable normalization of $C/B$ and denote it by $(C^{sn}; \Sigma(C^{sn}))$.

The following is the immediate corollary of Propositions 5.15, 5.16, 5.17, 5.18, and 5.19:
Corollary 5.21. Suppose \( \mathcal{C} \to B \) is a \((7/10-\epsilon)\)-stable family of curves. Let \((\mathcal{C}^{sn}; \Sigma(\mathcal{C}^{sn}))\) be the prestable normalization of \( \mathcal{C} \to B \). Then

\[
(s\lambda - \delta + \psi)_{\mathcal{C}/B} = (s\lambda - \delta + \psi)_{\mathcal{C}^{sn}/B} - (10 - s)(\delta_{sm} + e(n) + e(c)) + (11 - s)\psi_{cusp} + \left(5 - \frac{s}{2}\right)\psi_{tn} + \left(\frac{s}{2} - 4\right)\delta_{tn}
\]

We denote the divisor on the right-hand side by \( D(s) \) and in what follows, we will prove that \( D(s) \geq 0 \) for all families obtained from \((7/10-\epsilon)\)-stable families by \( A_3 \)-prestable normalization.

5.3. Cornalba-Harris and Hodge Index Theorem inequalities.

5.3.1. Cornalba-Harris inequality. In this section, we generalize a well-known Cornalba-Harris result on the positivity of divisor classes for families of Deligne-Mumford curves to the case of tacnodal curves.

Proposition 5.22 (Cornalba-Harris inequality). Suppose \( \mathcal{C} \to B \) is a generically smooth one-parameter family of at-worst tacnodal curves of genus \( g \geq 2 \), over a smooth and proper curve \( B \), with \( \omega_{\mathcal{C}/B} \) relatively nef. Then

\[
\left(8 + \frac{4}{g}\right)\lambda - \delta \geq 0.
\]

Moreover, if the general fiber of \( \mathcal{C}/B \) is non-hyperelliptic, then the inequality is strict.

Proof. If the general fiber of \( \mathcal{C}/B \) is non-hyperelliptic, the result follows from the original argument of Cornalba and Harris. Indeed, if \( C_b \) for some \( b \in B \) is a non-hyperelliptic curve of genus \( g \geq 3 \), then the canonical embedding of \( C_b \) has a stable \( m^{th} \) Hilbert point for some \( m \gg 0 \) by [Mor09, Lemma 14]. Applying verbatim the proof of [CHSS] Proposition 4.3], we obtain

\[
\left(8 + \frac{4}{g} - \frac{2(g-1)}{gm} + \frac{2}{gm(m-1)}\right)\lambda \geq 0.
\]

Now \( \delta \geq 0 \), and if \( \delta = 0 \), then \( \lambda > 0 \) for any non-isotrivial family. We conclude that \((8 + 4/g)\lambda - \delta > 0\).

Suppose now that \( \mathcal{X} \to B \) is a family of at-worst tacnodal curves with a relatively nef dualizing sheaf and a smooth hyperelliptic fiber. First, we construct \( \mathcal{X}/B \) explicitly as a double cover of a family of \((2g+2)\)-pointed curves. To this end, we introduce a notion of \( h \)-semistable \((2g+2)\)-pointed rational curves:

Suppose that \( (\mathcal{Y}/B; \{\sigma_i\}_{i=1}^{2g+2}) \) is a family of \((2g+2)\)-pointed rational curves where \( \sigma_i \) are smooth sections and no more than 4 sections meet at a point. We say that an irreducible component \( E \) in the fiber \( Y_b \) of \( \mathcal{Y}/B \) is an odd bridge if the following conditions hold:

1. \( E \) meets the rest of the fiber \( Y_b \setminus \overline{E} \) in two nodes of equal index,
(2) $E \cdot \sum_{i=1}^{2g+2} \sigma_i = 2$.

(3) the degree of $\sum_{i=1}^{2g+2} \sigma_i$ on each of the connected components of $Y_b \setminus E$ is odd.

If $Y \to Z$ is a blow-down of any collection of odd bridges, we call $Z/B$ an $h$-semistable $(2g+2)$-pointed curve. The image of $\sum_{i=1}^{2g+2} \sigma_i$ in $Z$ will be called the branch locus and denoted by $\Sigma$. Note that while the individual images of $\sigma_i$'s are not Cartier on $Z$ along the image of blown-down odd bridges, the total class of $\Sigma$ is Cartier on $Z$. Any node lying on $\Sigma$ will be called special. We say that a node $p \in Z_b$ is an odd node if the degree of $\Sigma$ on each of the connected component of the normalization of $Y_b$ at $p$. We define by $\delta_{odd}$ the Cartier divisor on $B$ associated to all odd nodes of $Z/B$.

By Lemma 5.23 below, $X$ is a double cover of a $h$-semistable $(2g+2)$-pointed rational curves $(Z/B; \Sigma)$, ramified over $\Sigma$. Let $\delta_{odd}$ be the divisor of odd nodes of $Z/B$ and $\omega := \omega_{Z/B}$. We have the following standard formulae:

$$\lambda_{X/B} = \frac{1}{8} \left( \Sigma^2 + 2\omega\Sigma - \delta_{odd} \right),$$

$$\delta_{X/B} = \Sigma^2 + \omega\Sigma + 2\omega^2 - \frac{3}{2} \delta_{odd}.$$

Suppose $f: \tilde{Z} \to Z$ is a blow-up of the special nodes of $Z$ making the divisor $\Sigma$ Cartier. Then $f^*(\Sigma) = \tilde{\Sigma} + E$ and $f^* \omega = \omega_{\tilde{Z}/B}$. Set $\psi := (\omega \cdot \Sigma)_X$, $\delta_{sm} := \sum_{i\neq j} (\sigma_i \cdot \sigma_j)$, and $e := \frac{-1}{2} \pi_*(E^2)$. Then

$$\lambda_{X/B} = \left( \frac{1}{8} (\psi + 2\delta_{sm} - \delta_{odd}) + \frac{1}{2} e \right)_{\tilde{Z}/B},$$

$$\delta_{X/B} = \left( 2\delta_{sm} + 2\delta_{even} + \frac{1}{2} \delta_{odd} + 5e \right)_{\tilde{Z}/B}.$$

We obtain

$$\left( 8 + \frac{4}{g} \right) \lambda_{X/B} - \delta_{X/B}$$

$$= \left( \left( 1 + \frac{1}{2g} \right) (\psi + 2\delta_{sm} - \delta_{odd}) + \left( 4 + \frac{2}{g} \right) e - (2\delta_{sm} + 2\delta_{even} + \frac{1}{2} \delta_{odd} + 5e) \right)_{\tilde{Z}/B},$$

$$= \left( \frac{2g+1}{2g} \psi + \frac{1}{g} \delta_{sm} + \left( \frac{2}{g} - 1 \right) e - 2\delta_{even} - \left( \frac{3}{2} + \frac{1}{2g} \right) \delta_{odd} \right)_{\tilde{Z}/B}.$$

Multiplying by $2g$, we need to show that on $\tilde{Z}/B$ we have

$$(2g + 1)\psi + 2\delta_{sm} - 4g\delta_{even} - (3g + 1)\delta_{odd} - (2g - 4)e \geq 0.$$
and using the inequality \( 2e \leq \delta_{\text{odd}} \), we obtain the desired claim. \( \square \)

**Lemma 5.23.** Possibly after a finite base change, \( \mathcal{X}/B \) can be represented as a double cover of a family of \( h \)-semistable \((2g + 2)\)-pointed rational curves \((Z; \Sigma)\), ramified over \( \Sigma \).

**Proof.** Let \( U \subset B \) be the locus of smooth fibers. Then all fibers over \( U \) are hyperelliptic and, possibly after a finite base change, we have a birational involution on \( \mathcal{X} \) extending the hyperelliptic involution on these fibers. Because \( \mathcal{X} \) is normal and relatively minimal, this birational involution extends to a biregular involution \( \iota: \mathcal{X} \to \mathcal{X} \). Let \( R \subset \mathcal{X} \) be the locus of \( \iota \)-fixed points. Then \( R \) has no vertical components because all fibers of \( \mathcal{X} \) are reduced. Choose now a sufficiently very ample \( \iota \)-invariant divisor \( H \subset \mathcal{X} \) such that \( H \) avoids \( \text{Sing}(\mathcal{X}/B) \) and intersects \( R \) in finitely many points, all lying over \( U \). After a finite base change, we can assume that \( H = \sum_{i=1}^{2d} h_i \), where \( h_i \) are smooth sections of \( \mathcal{X}/B \). Now \((\mathcal{X}; H)\) is a family of \((7/10 - \epsilon)\)-stable \(2d\)-pointed curves.

Next, let \( \mathcal{P} := \mathcal{X}_U/(\iota) \), let \( \{\sigma_i\}_{i=1}^{2g+2} \) be the branch sections of \( \mathcal{X}_U \to \mathcal{P} \), and let \( \{\tau_i\}_{i=1}^d \) be the image of \( H_U \) in \( \mathcal{P} \). Then \((\mathcal{P}/U; \{\sigma_i\}_{i=1}^{2g+2}, \{\tau_i\}_{i=1}^d)\) is generically a family of \((2g + 2 + d)\)-pointed rational curves. Give \( \{\sigma_i\}_{i=1}^{2g+2} \) weight \( 1/4 \) and \( \{\tau_i\}_{i=1}^d \) weight \( 1 \) and consider the induced complete family \((\overline{\mathcal{P}}/B; \{\sigma_i\}_{i=1}^{2g+2} \times \{\tau_i\}_{i=1}^d)\) in the Hassett’s space of weighted stable pointed curves; see [Has03].

Finally, let \( \overline{\mathcal{P}} \to Z \) be the blow-down of all odd bridges of \( \overline{\mathcal{P}} \), and \( \Sigma \) be the image of \( \sum_{i=1}^{2g+2} \sigma_i \) in \( Z \). Let \( g: \mathcal{X}' \to Z \) be the double cover branched over \( \Sigma \) and set \( H' := g^{-1}(\sum_{i=1}^d \tau_i) \). Then \( \mathcal{X}' \) marked by \( H' \) is also a family of \((7/10 - \epsilon)\)-stable \(d\)-pointed curves. Note that \((\mathcal{X}'; H')\) is isomorphic to \((\mathcal{X}; H)\) over an open subset of \( U \). By properness of \( \overline{M}_{g,2d}(7/10 - \epsilon) \), we conclude that \( \mathcal{X}'/B \) is isomorphic to \( \mathcal{X}/B \). The claim follows. \( \square \)

5.3.2. **Hodge Index Theorem inequalities.** In this section, we apply a method due to Harris [Har84] to obtain inequalities between the \( \psi \) classes and the indices of rational nodal and cuspidal bridges in \( A_3 \)-prestable families. These inequalities are key components of the positivity results obtained in Section 5.4.2.

**Lemma 5.24.** Suppose \( \mathcal{X}/B \) is a pointed family of Gorenstein curves of arithmetic genus \( g \geq 2 \) with a pair of inner nodal sections \( \{\eta^+, \eta^-\} \). Let \( e(\eta^+, \eta^-) \) be the sum of the indices of the rational nodal bridges associated to \( \{\eta^+, \eta^-\} \). Then

\[
\psi_{\eta^+} + \psi_{\eta^-} \geq \frac{2(g-1)}{g+1} \left( e(\eta^+, \eta^-) + e(\eta^+, \eta^-) \right) + \frac{\kappa}{g^2 - 1}.
\]

**Proof.** Consider the three divisor classes \( \langle F, \eta^+ + \eta^- + E, \omega_{\mathcal{X}/B} \rangle \), where \( F \) is the fiber class, and \( E \) is the Cartier divisor of the rational nodal bridges associated to \( \eta^\pm \). Since \( \eta^+ + \eta^- + E + F \) has positive self-intersection, the Hodge Index Theorem implies that the determinant of the
intersection pairing matrix for these three classes is non-negative. It follows that

$$\det \begin{pmatrix} 0 & 2 & 2g-2 \\ 2 & -\psi_+ - \psi_- + 2(\eta^+ \cdot \eta^-) + 2e(\eta^+ + \eta^-) & \psi_+ + \psi_- \\ 2g-2 & \psi_+ + \psi_- & \kappa \end{pmatrix} \geq 0.$$ 

The claim follows by expanding the determinant. □

**Lemma 5.25.** Suppose $\mathcal{X}/B$ is a pointed family of Gorenstein curves of arithmetic genus $g \geq 2$ with a cuspidal section $\xi$. Let $e(\xi)$ be the sum of the indices of the rational cuspidal bridges associated to $\xi$. Then

$$(5.6) \quad \psi_\xi \geq (g-1)e(\xi) + \frac{\kappa}{4g(g-1)}.$$ 

**Proof.** Apply the Hodge Index Theorem to the three classes $\langle F, \xi, 2 \rangle$, where $F$ is the Cartier divisor associated to the rational cuspidal bridges associated to $\xi$. Since $\xi, 2 \rangle$ has positive self-intersection, the determinant of the following matrix is non-negative:

$$\det \begin{pmatrix} 0 & 1 & 2g-2 \\ 1 & -\psi_\xi + \frac{1}{2}e(\xi) & \psi_\xi \\ 2g-2 & \psi_\xi & \kappa \end{pmatrix}$$

The claim follows. □

**Lemma 5.26.** Suppose $(\mathcal{X}/B; \tau)$ is a 1-pointed family of at-worst tacnodal curves of genus 2. Then

$$(5.7) \quad 8\psi_\tau \geq \kappa.$$ 

Moreover, if $\delta_{red} = 0$, then the equality is satisfied if and only if $(\mathcal{X}/B; \tau)$ is a family of Weierstrass tails.

**Proof.** The inequality follows directly from Lemma 5.25 by taking $g = 2$. Moreover, the proof of Lemma 5.25 shows that equality holds if and only if the intersection pairing on the divisors $F, \tau, \omega_{\mathcal{X}/B}$ is degenerate. Since $\omega_{\mathcal{X}/B}$ is relatively nef and $\delta_{red} = 0$, there is a global hyperelliptic involution $\iota: \mathcal{X} \to \mathcal{X}$. Hence $\omega_{\mathcal{X}/B} \equiv \tau + \iota(\tau) + xF$, for some $x \in \mathbb{Z}$. Observe that $\omega_{\mathcal{X}/B} \cdot \tau = \omega_{\mathcal{X}/B} \cdot \iota(\tau)$ and $F \cdot \tau = F \cdot \iota(\tau)$. Since no combination of $\omega$ and $F$ is in the kernel of the intersection pairing, we conclude that

$$\tau^2 = \tau \cdot \iota(\tau).$$

However, the intersection number on the left is negative and the intersection number on the right is positive whenever $\tau \neq \iota(\tau)$. We conclude that the equality holds if only if $\iota(\tau) = \tau$, that is $\tau$ is a Weierstrass section. □

We will also need the variants of the first two lemmas above for the case of relative genus 1 and 0.
Lemma 5.27. Suppose $X/B$ is a pointed family of Gorenstein curves of arithmetic genus 1 with a pair of inner nodal section $\{\eta^+, \eta^-\}$. Assume further that $\{\eta^+, \eta^-\}$ are disjoint from some $N - 2$ smooth sections of $X$. Let $e(\eta^+, \eta^-)$ be the sum of the indices of the rational nodal bridges associated to $\eta^\pm$. Then

$$(\eta^+ \cdot \eta^-) + e(\eta^+, \eta^-) \leq \frac{N}{2(N - 2)}(\psi_{\eta^+} + \psi_{\eta^-}) + \frac{1}{2(N - 2)^2}\delta_{\text{red}}.$$ 

Proof. Let $E$ be the union of the rational nodal bridges associated to $\{\eta^+, \eta^-\}$ and $\Sigma$ be the collection of smooth sections of $X/B$ disjoint from $\{\eta^+, \eta^-\}$. Apply the Hodge Index Theorem to $F, \eta^+ + \eta^- + E, \omega + 2\Sigma$. The determinant of the matrix

$$\begin{vmatrix}
F & \eta^+ + \eta^- + E & \omega + 2\Sigma \\
0 & 2 & \frac{2(N - 2)}{2(N - 2)} \\
\eta^+ + \eta^- + E & -\psi_{\eta^+} - \psi_{\eta^-} + 2(\eta^+ \cdot \eta^-) + 2e(\eta^+, \eta^-) & \psi_{\eta^+} + \psi_{\eta^-} \\
\omega + 2\Sigma & 2(N - 2) & \kappa
\end{vmatrix}$$

is non-negative. Therefore

$$-4\kappa + 8(N - 2)(\psi_{\eta^+} + \psi_{\eta^-}) + 4(N - 2)^2(\psi_{\eta^+} + \psi_{\eta^-}) \geq 8(N - 2)^2((\eta^+ \cdot \eta^-) + e(\eta^+, \eta^-)),$$

which gives the desired inequality using $\kappa = -\delta_{\text{red}}$. \qed

Lemma 5.28. Suppose $X/B$ is a pointed family of Gorenstein curves of arithmetic genus 1 with a cuspidal section $\xi$. Assume further that $\xi$ is disjoint from some $N - 1$ smooth sections of $X$. Let $e(\xi)$ be the sum of the indices of the rational cuspidal bridges associated to $\xi$. Then

$$e(\xi) \leq \frac{2N}{N - 1}\psi_{\xi} + \frac{1}{(N - 1)^2}\delta_{\text{red}}.$$ 

Furthermore, suppose $N = 2$, with $\tau$ being a section disjoint from $\xi$, and $\delta_{\text{red}} = 0$. Then the equality holds if and only if $2\xi \sim 2\tau$.

Proof. Let $E$ be the union of the rational cuspidal bridges associated to $\xi$ and let $\Sigma$ be the collection of smooth sections of $X/B$ disjoint from $\xi$. By the Hodge Index Theorem applied to $<F, \xi + \frac{1}{2}E, \omega + 2\Sigma>$, the determinant of the matrix

$$\begin{vmatrix}
F & \frac{\xi + \frac{1}{2}E}{2(N - 1)} \\
0 & 1 \\
\xi + \frac{1}{2}E & -\psi_{\xi} + \frac{1}{2}e(\xi) & \psi_{\xi} \\
\omega + 2\Sigma & 2(N - 1) & \kappa
\end{vmatrix}$$

is non-negative. Therefore

$$\frac{e(\xi)}{2} \leq \psi_{\xi} + \frac{1}{N - 1}\psi_{\xi} - \frac{1}{2(N - 1)^2}\kappa.$$ 

This gives the desired inequality.

To prove the last assertion observe that because $\delta_{\text{red}} = 0$ and $X/B$ has no elliptic tails, all fibers of $X/B$ are irreducible curves of genus 1. In particular, $\omega = \lambda F$ and it follows from the
existence of the group law on the set of sections of $\mathcal{X}/B$ that there exists a section $\tau'$ such that $2\xi - \tau = \tau'$. Since the intersection pairing matrix on the classes $F, \xi, \tau$ is degenerate, we conclude that some linear combination $(x\xi + y\tau + zF)$ intersects $F, \xi, \tau$ trivially. Intersecting with $\tau$, we obtain $y(\tau \cdot \tau') + z = 0$; and intersecting with $\tau'$, we obtain $y(\tau \cdot \tau') + z = 0$. Hence $\tau^2 = \tau \cdot \tau'$. This leads to a contradiction if $\tau \neq \tau'$.

5.4. Proof of the main positivity result.

5.4.1. Proof of Theorem 5.4(a). Here we prove that $10\lambda - \delta + \psi$ is nef on $\overline{\mathcal{M}}_{g,n}(9/11 - \epsilon)$ and has degree precisely on families of elliptic bridges. Let $\mathcal{C}/B$ be a $(9/11 - \epsilon)$-stable family. Denote by $\tilde{\mathcal{C}}$ the normalization of $\mathcal{C}$ along generic outer nodes of $\mathcal{C}$. Since outer nodes do not specialize to cusps by Proposition 5.11, we have $(10\lambda - \delta + \psi)_{\tilde{\mathcal{C}}/B} = (10\lambda - \delta + \psi)_{\tilde{\mathcal{C}}/B}$.

Moreover, by Lemma 2.19 every connected component of $\tilde{\mathcal{C}}/B$ is a family of generically irreducible $(9/11 - \epsilon)$-stable curves. Let $\tilde{\mathcal{X}}/B$ be an irreducible component of $\tilde{\mathcal{C}}/B$. Denote by $\mathcal{X}$ the $A_3$-prestable normalization of $\tilde{\mathcal{X}}$ along generic cusps and inner nodes. Note that generic cusps do not degenerate but inner nodes might degenerate to cusps. We obtain a family $(\mathcal{X}/B; \{\sigma_i\}_{i=1}^n, \{\eta_{i}^\pm\}_{i=1}^a, \{\xi_i\}_{i=1}^c)$ and by Proposition 5.17

$$(10\lambda - \delta + \psi)_{\tilde{\mathcal{X}}/B} = (10\lambda - \delta + \psi + \psi_{\text{cusp}})_{\mathcal{X}/B}.$$ 

Passing to the relative minimal model of $\mathcal{X}/B$ only decreases the degree of $(10\lambda - \delta + \psi + \psi_{\text{cusp}})_{\mathcal{X}/B}$, hence we can assume that $\omega_{\mathcal{X}/B}$ is relatively nef.

If $g \geq 3$, then $10\lambda - \delta > 0$ by the Cornalba-Harris inequality (Proposition 5.22). Since we also have $\psi, \psi_{\text{cusp}} \geq 0$, we conclude that $(10\lambda - \delta + \psi)_{\tilde{\mathcal{X}}/B} > 0$.

If $g = 2$, then $10\lambda - \delta \geq 0$ and because there must be at least one marked section, we also have $\psi > 0$. We conclude that $(10\lambda - \delta + \psi)_{\tilde{\mathcal{X}}/B} > 0$.

If $g = 1$, then $\lambda = \delta_{\text{irr}}/12$. Let $N = n + 2a + c$ be the total number of sections. Then we have $\psi = N\delta_{\text{irr}}/12 + \sum S|S\delta_{0,S} \geq N\delta_{\text{irr}}/12 + 2\delta_{\text{red}}$. If $N \geq 3$, we obtain $10\lambda - \delta + \psi \geq 10\delta_{\text{irr}}/12 + N\delta_{\text{irr}}/12 + 2\delta_{\text{red}} - (\delta_{\text{irr}} + \delta_{\text{red}}) > 0$. If $N = 2$, then $10\lambda - \delta + \psi \geq \delta_{\text{red}} > 0$ and $\psi_{\text{cusp}} \geq 0$. We conclude that $10\lambda - \delta + \psi + \psi_{\text{cusp}} \geq 0$ with the equality holding if and only if there are no cuspidal sections and $\delta_{\text{red}} = 0$. This holds if and only if $\mathcal{X}/B$ is an elliptic bridge of the original family.

If $g = 0$, then all fibers of $\mathcal{X}/B$ are in fact at worst nodal and $(10\lambda - \delta + \psi + \psi_{\text{cusp}})_{\mathcal{X}/B} = (\psi - \delta) + \psi_{\text{cusp}}$. On the resulting family of genus 0 curves sections $\eta_i^+$ and $\eta_i^-$ can intersect. Blowing-up these points of intersection and setting $\delta_{\text{sm}} := \sum \delta_{\text{(sm)}} \delta_{(\eta_i^+, \eta_i^-)}$, we obtain $\psi - \delta + \psi_{\text{cusp}} = \psi - \delta - \delta_{\text{sm}} + \psi_{\text{cusp}}$. If $a = 0$, then $\delta_{\text{sm}} = 0$ and we are done because $\psi - \delta > 0$. If $a \geq 2$, then by Lemma 5.33 below, $\psi_{\text{in}} \geq 4\delta_{\text{sm}}$. Also, $3\psi/4 \geq \delta$. It follows that $\psi > \delta + \delta_{\text{sm}}$ and so we are done. Finally, if $a = 1$, then $\mathcal{X}/B$ is an $A_3$-prestable normalization of a generically nodal arithmetic genus 1 along the generic node and the proof in the case of $g = 1$ goes through without any modifications.
5.4.2. Proof of Theorem 5.4(b). In the remaining part of the paper, we prove Theorem 5.4(b). Let $C \to B$ be a $(7/10-\epsilon)$-stable non-locally isotrivial family of curves. We deal first with the case when the generic fiber $C_\eta$ has a rosary (see Definition 5.28). Note that because $C/B$ is non-locally isotrivial, the rosary cannot be closed in the sense of [HH13, Definitions 6.3].

5.4.3. Generic rosaries. Consider the maximal (in length) rosary $P := P_1 \cup P_2 \cup \cdots \cup P_k$ of the geometric generic fiber $C_\eta$. After a finite étale base change that kills the monodromy of tacnodes, we can assume that there are tacnodal sections $\{\tau_j: B \to C\}$ such that $\{\tau_j(b)\}$ are the tacnodes of $C_b$ for every $b \in B$ and $\tau_j(\eta) = P_j \cap P_{j+1}$.

Let $R := C_\eta \setminus P$ be the rest of $C_\eta$. For $i = 1, k$, the component $P_i$ intersects $R$ in a unique outer node or an outer tacnode $P_i \cap R$, so the limit of $P_i \cap R$ is still a node or a tacnode in every fiber by Proposition 5.11. Let $\mathcal{C} = \mathcal{R} \cup \mathcal{P}$ be the resulting decomposition of $C$ into components such that the geometric generic fiber of $\mathcal{P}$ and $\mathcal{R}$ is $P$ and $R$, respectively. Let $\sigma_p: B \to \mathcal{C}$ be the section defined by $R \cap P_1$, $\sigma_q: B \to \mathcal{C}$ be the section defined by $R \cap P_k$, and $\tau_i: B \to \mathcal{C}$, for $i = 1, \ldots, k-1$ be the section defined by $P_i \cap P_{i+1}$. We mark $\mathcal{R}$ by the sections of $\mathcal{C}$ and by $\sigma_p, \sigma_q$, and consider the following cases:

Case I: $P$ is $A_1/A_1$-attached to the rest of the curve. In this case, $k$ must be odd and $\mathcal{R}$ must be $(7/10-\epsilon)$-stable. Furthermore, $\mathcal{P}$ is locally isotrivial and because the number of the tacnodal section $\tau_i$ is even, one immediately computes Proposition 5.15 that $(39/4\lambda - \delta + \psi)_{\mathcal{C}/B} = (39/4\lambda - \delta + \psi)_{\mathcal{R}/B}$. Thus we reduce to proving Theorem 5.4(b) for $\mathcal{R}$.

Case II: $P$ is $A_1/A_3$-attached to the rest of the curve. Suppose $\sigma_p$ is a nodal section and $\sigma_q$ is a tacnodal section. By the maximality assumption of $P$, the point $q$ does not lie on a 2-pointed rational component of $R$. It follows by Lemma 5.5 that $(\psi)_\mathcal{R} \geq 0$. By Proposition 5.15

$(39/4\lambda - \delta + \psi)_{\mathcal{C}/B} = (39/4\lambda - \delta + \psi)_{\mathcal{R}/B} + \frac{5}{4}(\psi_q)_{\mathcal{R}/B} + (\psi_p)_{\mathcal{R}/B} + \frac{9}{4} \sum_{i=1}^{k-1} (\psi_{\tau_i})_{\mathcal{P}_i}$.

Observe that $\psi_{\tau_{i-1}} = -\psi_q, \psi_p = -\psi_{\tau_i}$, and $\psi_{\tau_{i-1}} = -\psi_{\tau_i}$ for $i = 2, \ldots, k-1$. If $k$ is odd, then $\sum_{i=1}^{k-1} (\psi_{\tau_i})_{\mathcal{P}_i} = 0$ and $\psi_p = -\psi_q$. We thus obtain:

$(39/4\lambda - \delta + \psi)_{\mathcal{C}/B} = (39/4\lambda - \delta + \psi)_{\mathcal{R}/B} + \frac{1}{4}(\psi_q)_{\mathcal{R}/B} \geq (39/4\lambda - \delta + \psi)_{\mathcal{R}/B}$.

Noting that if $\psi_q$ is zero, then $\mathcal{P}$ is a trivial family, we reduce to proving Theorem 5.4(b) for $\mathcal{R}$.

If $k$ is even, then $\psi_p = \psi_q$ and $\sum_{i=1}^{k-1} (\psi_{\tau_i})_{\mathcal{P}_i} + \psi_q = 0$, so that $(39/4\lambda - \delta + \psi)_{\mathcal{C}/B} = (39/4\lambda - \delta + \psi)_{\mathcal{R}/B}$. Furthermore, we observe that the family $\mathcal{P}$ is locally isotrivial and the crimping at $q$ is constant. Thus we reduce to proving Theorem 5.4(b) for $\mathcal{R}$.

Case III: $P$ is $A_3/A_3$-attached to the rest of the curve. By the maximality assumption on $P$, neither $p$ nor $q$ lies on a 2-pointed rational component of $R$. It follows by Lemma 5.5 that $(\psi_p)_{\mathcal{R}}, (\psi_q)_{\mathcal{R}} \geq 0$. However, $\psi_p = (-1)^k \psi_q$. Therefore, either $\psi_p = \psi_q = 0$, in which case $\mathcal{P}$ is a trivial family, or $k$ is even and $\psi_p = \psi_q > 0$. In either case,

$(39/4\lambda - \delta + \psi)_{\mathcal{C}/B} = (39/4\lambda - \delta + \psi)_{\mathcal{R}/B} + \frac{1}{4}(\psi_q)_{\mathcal{R}/B} \geq (39/4\lambda - \delta + \psi)_{\mathcal{R}/B}$.
and the inequality is strict if \( P \) is not trivial.

5.4.4. Disentangling irreducible components. Let \( C/B \) be a \((7/10-\epsilon)\)-stable family. By Section 5.4.3 we can assume that \( C/B \) has no generic rosaries. We denote by \( \tilde{C} \) the normalization of \( C \) along generic outer tacnodes and generic outer nodes of \( C \), which exists by Section 5.2. We mark \( \tilde{C} \) by the sections of \( C \), as well as by sections \( \{\zeta_i^\pm\} \) which are preimages of the outer nodes of \( C \) and by sections \( \{\tau_i^\pm\} \) which are preimages of the outer tacnodes of \( C \). We call \( \{\zeta_i^\pm\} \) outer nodal sections and \( \{\tau_i^\pm\} \) outer tacnodal sections. Note that \( \tilde{C} \) is a disconnected union of irreducible components.

Let \( \tilde{X} \) be an irreducible component of \( \tilde{C} \). We normalize \( \tilde{X} \) along its generic cusps and tacnodes to arrive at a family \( X \) with at worst nodal generic singularities. Let \( a \) be the number of generic nodes of \( X \) and \( g \) be the geometric genus of the general fiber of \( X \). Our further analysis breaks down according to the following possibilities:

- **(S1)** \( g \geq 3 \).
- **(S2)** \( g = 2 \), \((g, a) = (1, 1)\), \((g, a) = (0, 2)\).
- **(S3)** \( g = 1 \), \((g, a) = (0, 1)\).
- **(S4)** \( g = 0 \), \( a \geq 3 \).

In the cases (S1) and (S4) we perform \( A_3 \)-prestable normalization of \( X \) along all generic inner nodes. In the case (S2) (resp. (S3)) we observe that \( X/B \) is a family of arithmetic genus 2 (resp. 1) and with at worst nodal generic fiber. In each case, our goal is to prove that \( D(39/4) \geq 0 \), with the inequality holding only for families of Weierstrass tails. (Recall that \( D(s) \) was defined right after Corollary 5.21.) The cases (S1)-(S4) is considered in Sections 5.4.5, 5.4.6, 5.4.7, and 5.4.8 respectively.

5.4.5. Geometric genus \( g \geq 3 \). Let \( (X/B; \Sigma) \) be an \( A_3 \)-prestable normalization of an irreducible component of \( C/B \) such that the general fiber of \( X/B \) is smooth of genus \( g \geq 3 \). The main result of this section is:

**Proposition 5.29.** For any \( s \in [10, 39/4] \), the following holds on \( X/B \):

\[
M(s) := (s\lambda - \delta + \psi) - (10 - s)(\delta_{sm} + e(n) + e(c)) + (11 - s)\psi_{cusp} + (5 - s/2)\psi_{tn} \geq 0.
\]

**Corollary 5.30.** \( D(39/4) > 0 \).

**Proof.** This follows from \( D(s) = M(s) + (\frac{s}{2} - 4)\delta_{tn} \) and \( \delta_{tn} \geq 0 \). \( \square \)

**Proof of Proposition 5.29.** We proceed in two stages. First, we pass to the relative minimal model of \( X \), using the following observation:

**Claim.** The degree of the line bundle \( M(s) \) is not increasing under successive blow-down of \((-1)\)-curves in the fiber of \( X \).
Proof. If the \((-1)\)-curve of index \(e\) is adjacent to no rational nodal or cuspidal bridges, then it is either marked by 2 sections or is marked by a cuspidal or tacnodal section. In either case, the degree of \(\delta\) decreases exactly by \(e\) but the degree of \(\psi + (11 - s)\psi_{\text{cusp}} + (5/2)\psi_{\text{tn}}\) decreases at least by \(e\). If the \((-1)\)-curve is adjacent to a rational cuspidal bridge \(E\) of index \(e\), then the \((-1)\)-curve is marked by at least one section. Blowing down first the \((-1)\)-curve, followed by blowing down \(E\) decreases \(\delta\) by \(2e\), decreases \(e(c)\) by \(e\), and decreases \(\psi + (11 - s)\psi_{\text{cusp}}\) by at least \((13 - s)e\). The claim follows from \((13 - s)e - 2e - e(10 - s)/4 = e(34/4 - 3s/4) > 0\). Analysis of the case where the \((-1)\)-curve is adjacent to a rational nodal bridge is analogous.

By the claim above, we can assume that \(\mathcal{X}/B\) has a relatively nef dualizing sheaf. Applying the Cornalba-Harris inequality from Proposition 5.22, we have \(8 + \frac{3}{8} \lambda - \delta \geq 0\). Because \(\delta \geq 0\), we also have \(s\lambda - \delta \geq 0\) for any \(s \geq 8 + 4/3 = 28/3\). In particular, \(\kappa = 12\lambda - \delta \geq 0\).

Since \(g \geq 3\) and \(\kappa \geq 0\), the inequalities of Lemmas 5.24 and 5.25 give

\[
\delta_{\text{sm}} + e(n) + e(c) \leq \psi_{\text{in}} + \psi_{\text{cusp}}.
\]

Combining this with the inequality \(s\lambda - \delta \geq 0\) for \(s \geq 39/4\), we obtain

\[
(s\lambda - \delta + \psi)\psi_{\text{in}} - (10 - s)(\delta_{\text{sm}} + e(n) + e(c)) \geq s\lambda - \delta + \frac{3}{4} \psi \geq s\lambda - \delta \geq 0.
\]

Together with \(\psi_{\text{cusp}}, \psi_{\text{tn}} \geq 0\), this gives the desired result \(s \in [39/4, 10]\). Moreover, the equality can hold if and only if \(\lambda = \delta = \psi = 0\), that is when \(\mathcal{X}/B\) is a trivial family.

5.4.6. Arithmetic genus 2. Let \(\mathcal{X}/B\) be an irreducible \(A_3\)-prestable family of relative arithmetic genus 2 with a generic inner nodes. First, we observe that by [Ko90 Theorem 4.3] we have \(\lambda_{\mathcal{X}/B} \geq 0\).

Proposition 5.31.

\[M := (39/4)\lambda - \delta + \psi - (1/4)(\delta_{\text{sm}} + e(n) + e(c)) + (5/4)\psi_{\text{cusp}} + (1/8)\psi_{\text{tn}} \geq 0.\]

Corollary 5.32. \(D(39/4) > 0\).

Proof of Proposition 5.31: Pass to the relative minimal model. Clearly, \(\delta_{\text{red}} \geq 0\). We have the relation \(10\lambda = \delta_{\text{irr}} + 2\delta_{\text{red}}\), which gives \(\delta \leq 10\lambda\), so that \(\kappa = 12\lambda - \delta \geq 2\lambda \geq 0\). By Lemmas 5.24 and 5.25, we have the inequalities

\[
\psi_{\text{in}} \geq \frac{7}{10}(\delta_{\text{sm}} + e(n)) + \frac{a\kappa}{3},
\]

and

\[
\psi_{\text{cusp}} \geq \frac{1}{4} e(c) + \frac{c\kappa}{8}.
\]

The family \(\mathcal{X}/B\) has at least one smooth section \(\tau\). By Lemma 5.25, \(\psi_{\tau} \geq \kappa/8\). Putting these inequalities together, we obtain:

\[
(39/4)\lambda - \delta + \psi - (1/4)(\delta_{\text{sm}} + e(n) + e(c)) + (5/4)\psi_{\text{cusp}} + (1/8)\psi_{\text{tn}} \geq -\lambda/4 + \psi_{\tau}.
\]
But $\psi_\tau \geq (12\lambda - \delta)/8$ or $8\psi_\tau \geq 12\lambda - \delta \geq 2\lambda$, so we have established that $M \geq 0$. By the above, the equality is achieved only if $\delta_{\text{red}} = 0$, there is exactly one marked section $\tau$, and the equality is achieved in Lemma 5.26. The last condition implies that $(X/B; \tau)$ is a family of genus 2 Weierstrass tails.

5.4.7. Arithmetic genus 1. Let $(X/B; \Sigma)$ be an irreducible $A_3$-prestable family of relative arithmetic genus 1 with $\ell$ generic inner nodes, where as before

\[ \Sigma = \{ \{ \sigma_i \}_{i=1}^k, \{ \eta_i^+, \eta_i^- \}_{i=1}^a, \{ \zeta_i \}_{i=1}^b, \{ \xi_i \}_{i=1}^c, \{ \tau_i \}_{i=1}^d \}. \]

(Here, either $\ell = 0$ or $a = 0$.) We let $N := k + 2a + b + c + d$ be the total number of marked sections. Note that $N \geq 2$ since $X/B$ cannot have elliptic tails.

First, we observe that $\lambda \geq 0$. Indeed, if the generic fiber is smooth, this follows from $\lambda = \delta_{\text{irr}}/12 \geq 0$, and if the generic fiber is nodal, this follows from Proposition 5.16.

By blowing up the points of $\eta_i^+ \cap \eta_j^+$ for $i \neq j$, and $\zeta_i \cap \zeta_j$ for $i \neq j$, we assume that no sections from $\Sigma$ intersect with the only exception that an inner nodal section $\eta_i^+$ can intersect $\eta_i^-$, for all $1 \leq i \leq a$. We then introduce the following divisor class on $B$:

\[ \delta^{\text{in}}_2 := \sum_{i \neq j} \delta_{0; \eta_i^+, \eta_j^+} + \sum_{i \neq j} \delta_{0; \zeta, \zeta}. \]

Note that the only $(-1)$-curves on $X$ marked by a single section from $\Sigma$ are either marked by cuspidal or tacnodal sections, or meet rational nodal or cuspidal bridges. We denote by $X_{\text{min}}/B$ the successive blow-down of all such $(-1)$-curves on $X/B$. Introducing auxiliary divisor classes on $B$ by the formula

\[ \delta_{\text{aux}} := \sum_{i=1}^c \sum_{\xi \in S, |S|=2} \delta_{0; \xi} + \sum_{i=1}^a \sum_{\eta_i^+, \eta_i^- \in S, |S|=3} \delta_{0; \eta_i^+, \eta_i^-, \xi}, \]

we obtain the inequality

\[ D(39/4)X/B \geq \left( \frac{39}{4} \lambda + \psi + \frac{5}{4} \psi_{\text{cusp}} + \frac{1}{8} \psi_{\text{tn}} - \delta - \frac{1}{4}(\delta_{\text{sm}} + e(n) + e(c) + \delta_{\text{aux}}) - \frac{1}{8} \delta^{\text{in}}_2 \right) X_{\text{min}}/B. \]

Moreover, the inequality is strict if there is a $(-1)$-curve in the fiber of $X/B$ marked by a single cuspidal or tacnodal section.

Applying Lemma 5.27 we obtain:

\[ e(n) + \delta_{\text{sm}} = \sum_{i=1}^a \left( (\eta_i^+ \cdot \eta_i^-) + e(\eta_i^+, \eta_i^-) \right) \leq \frac{N}{2(N-2)} \sum_{i=1}^a (\psi_{\eta_i^+}^+ + \psi_{\eta_i^-}^-) + \frac{a}{2(N-2)^2} \delta_{\text{red}}, \]

and applying Lemma 5.28 we obtain

\[ e(c) = \sum_{i=1}^c e(\xi_i) \leq \frac{2N}{N-1} \sum_{i=1}^c \psi_{\xi_i} + \frac{c}{(N-1)^2} \delta_{\text{red}}. \]
Therefore
\[
\frac{1}{4} \left( \delta_{sm} + e(n) + e(c) \right) \leq \frac{N}{8(N-2)} \sum_{i=1}^{a} (\psi_{\eta_i}^+ + \psi_{\eta_i}^-) + \frac{N}{2(N-1)} \psi_{\text{cusp}} + \left( \frac{a}{8(N-2)^2} + \frac{c}{4(N-1)^2} \right) \delta_{\text{red}}.
\]

Case of \( N \geq 4 \): Noting that \( 2a + c \leq N \), we obtain
\[
\frac{1}{4} \left( \delta_{sm} + e(n) + e(c) \right) \leq \frac{1}{4} \sum_{i=1}^{a} (\psi_{\eta_i}^+ + \psi_{\eta_i}^-) + \frac{2}{3} \sum_{i=1}^{c} \psi_{\xi_i} + \frac{1}{4} \delta_{\text{red}},
\]
with the equality holding if and only if all the quantities in question are 0.

Putting this together, and using Equation (5.2), we obtain
\[
\frac{39}{4} \lambda + \psi + \frac{5}{4} \sum_{i=1}^{c} \psi_{\xi_i} + \frac{1}{8} \sum_{i=1}^{d} \psi_{\tau_i} - \delta - \frac{1}{4} \left( \delta_{sm} + e(n) + e(c) + \delta_{\text{aux}} \right) - \frac{1}{8} \delta_{\text{tn}} \geq \frac{7}{12} \psi_{\text{cusp}} + \frac{4N - 2a - 9}{48} \delta_{\text{irr}} + \sum_{S \subset \Sigma} f(S) \delta_{0:S},
\]
where
\[
f(S) \geq \begin{cases} 
|S| - 1 - 1/4 - |S|/4 - 1/8 & \text{if } \delta_{0:S} \text{ is not auxiliary} \\
|S| - 1 - 1/4 - 1/4 & \text{if } \delta_{0:S} \text{ is auxiliary and } |S| = 2 \\
|S| - 1 - 1/4 - |S|/4 - 1/4 & \text{if } \delta_{0:S} \text{ is auxiliary and } |S| = 3
\end{cases}
\]

Since \( 4N - 2a > 9 \) and \( f(S) > 0 \), we conclude that \( D(39/4)_{\mathcal{X}/B} > 0 \).

Case of \( N = 3 \): We can assume that \( a = 0 \), as otherwise \( \mathcal{X} \) can be realized as the normalization along a generic inner node of an arithmetic genus 2 family.

Lemma 5.28 gives \( e(c)/4 \leq \frac{3}{4} \psi_{\text{cusp}} + \frac{5}{4} \delta_{\text{red}} \). Therefore,
\[
\frac{39}{4} \lambda + \psi + \frac{5}{4} \psi_{\text{cusp}} + \frac{1}{8} \psi_{\text{tn}} - \delta - \frac{1}{4} (e(c) + \delta_{\text{aux}}) - \frac{1}{8} \delta_{\text{tn}} \geq \frac{1}{2} \psi_{\text{cusp}} + \frac{3}{48} \delta_{\text{irr}} + \sum_{S \subset \Sigma} f(S) \delta_{0:S},
\]
where
\[
f(S) \geq \begin{cases} 
|S| - 1 - 1/4 - |S|/4 - 1/8 & \text{if } S \text{ is not auxiliary} \\
|S| - 1 - 1/4 - 1/4 & \text{if } S \text{ is auxiliary}
\end{cases}
\]

Since \( f(S) > 0 \) in every case, \( D(39/4)_{\mathcal{X}/B} > 0 \).

Case of \( N = 2 \): Since \( \mathcal{X}/B \) has neither elliptic tails nor elliptic bridges (nodally or tacnoda\-lly attached), there is only one possibility to consider: \( \Sigma = \{ \xi, \tau \} \), where \( \xi \) is a cuspidal section and \( \tau \) is either an outer nodal or tacnodal section. We have \( \delta_{\text{red}} = 0 \) and using the inequality \( e(c) \leq 4\psi_{\xi} \) from Lemma 5.28 we obtain:
\[
D(39/4) \geq \frac{39}{4} \lambda - \delta + \psi + \frac{5}{4} \psi_{\xi} + \frac{1}{8} \psi_{\text{tn}} - \frac{1}{4} e(c) \geq \frac{39}{48} \delta_{\text{irr}} - \delta_{\text{irr}} + \frac{8}{48} \delta_{\text{irr}} + \frac{1}{48} \delta_{\text{irr}} = 0.
\]
Moreover, the equality holds only if $\psi_{\text{tn}} = 0$, that is the attaching section is nodal, and the equality holds in Lemma 5.28. It follows that $2\xi \sim 2\tau$. This implies that $\mathcal{X}/B$ is the normalization of the generically cuspidal family of Weierstrass genus 2 tails.

**5.4.8. Genus 0 case.** Let $(\mathcal{X}/B; \Sigma(\mathcal{C}/B))$ be a genus 0 component of the $A_3$-prestable normalization of $\mathcal{C}/B$, where

$$
\Sigma(\mathcal{C}/B) = \{\{\sigma_i\}_{i=1}^k, \{\eta_i^+, \eta_i^-\}_{i=1}^a, \{\zeta_i\}_{i=1}^b, \{\xi_i\}_{i=1}^c, \{\tau_i\}_{i=1}^d\}
$$

are marked, inner nodal, outer nodal, cuspidal, and tacnodal sections, respectively. We let $N = k + 2a + b + c + d$ be the total number of sections. Blow-up all the points of intersection of all sections from $\Sigma(\mathcal{C}/B)$ and set

$$
\delta_{\text{sm}} := \sum_{i=1}^a \delta_{\{\eta_i^+, \eta_i^-\}}, \quad \delta_{\text{in}} := \sum_{i \neq j} \delta_{\{\eta_i^+, \eta_j^-\}}, \quad \delta_{\text{on}} := \sum_{i \neq j} \delta_{\{\zeta_i, \zeta_j\}},
$$

$$
\delta_{3NB} := \sum_{i=1}^a \sum_{\alpha \neq \eta_i^+, \eta_i^-} \delta_{\{\eta_i^+, \eta_i^-, \alpha\}}, \quad \delta_{2} := \delta_{\text{in}} + \delta_{\text{on}}, \quad \delta_{CT} := \sum_{i=1}^c \sum_{\alpha \neq \xi_i} \delta_{\{\xi_i, \alpha\}}.
$$

Our goal is to prove that

$$
D(39/4) = \psi + \frac{5}{4}\psi_{\text{cusp}} + \frac{1}{8}\psi_{\text{tn}} - \delta - \frac{5}{4}\delta_{\text{sm}} - \frac{1}{4}(e(n) + e(c)) - \frac{1}{8}\delta_{2} \geq 0.
$$

Unfortunately, the family $\mathcal{X}/B$ might not be Deligne-Mumford stable. To make book-keeping easier, we blow-down all $(-1)$-curves in the fibers to arrive at a Deligne-Mumford stable model of $\mathcal{X}/B$. To prove that $D(39/4)$ was non-negative on the original family, we observe that blowing down a $(-1)$-curve not meeting either a nodal or a cuspidal bridge does not change the degree of $D(39/4)$, blowing down a $(-1)$-curve of index $e$ meeting a rational nodal bridge increases the degree of $D(39/4)$ by $e/4$, but also increases the degree of $\delta_{3NB}$ by $e$; similarly, blowing down a $(-1)$-curve of index $e$ meeting a rational cuspidal bridge increases the degree of $D(39/4)$ by $e/4$, but also increases the degree of $\delta_{CT}$ by $e$.

Therefore, we reduce to proving that

$$
\psi + \frac{5}{4}\psi_{\text{cusp}} + \frac{1}{8}\psi_{\text{tn}} - \delta - \frac{5}{4}\delta_{\text{sm}} - \frac{1}{4}(\delta_{3NB} + \delta_{CT} + e(n) + e(c)) - \frac{1}{8}\delta_{2} \geq 0.
$$

On $\overline{M}_{0,N}$, we have the following standard relation:

$$
(5.8) \quad \psi = \sum_{r \geq 2} \frac{r(N - r)}{N - 1} \delta_r.
$$
Suppose $a = 0$: Then $\delta_{sm} = \delta_2 = e(n) = \delta_{NB}^3 = 0$. Note that $N \geq 4$ because the family is non-trivial. If $N \geq 5$, then using the inequality $2e(c) \leq \delta$, we reduce to proving
\[
\psi + \frac{5}{4}\psi_{cusp} + \frac{1}{8}\psi_{tn} - \frac{9}{8}\delta - \frac{1}{4}\delta_{2}^{CT} - \frac{1}{8}\delta_{2}^t > 0.
\]
This follows from the standard relation (5.8):
\[
\psi \geq \frac{3}{2} \sum_{r \geq 2} \delta_r > \frac{11}{8}\delta_2 + \frac{9}{8} \sum_{r \geq 3} \delta_r > \frac{9}{8}\delta + \frac{1}{4}\delta_{2}^{CT} + \frac{1}{8}\delta_{2}^t.
\]

If $N = 4$, then in addition, $e(c) = 0$ and by the standard relation $\psi = \frac{4}{3}\delta > \frac{5}{4}\delta \geq \delta + \frac{1}{4}\delta_{2}^{CT} + \frac{1}{8}\delta_{2}^t$, as required.

From now on, we can assume that $a \geq 1$. We will need the following auxiliary result:

**Lemma 5.33.** Let $\psi_{in} := \sum_{i=1}^{a} \psi_{\eta_i^+} + \psi_{\eta_i^-}$. If $a \geq 2$, then
\[
\psi_{in} \geq 4\delta_{sm} + 2 \sum_{i \neq j} \delta_{\{\eta_i^+, \eta_j^\pm\}}.
\]

**Proof.** This follows by considering the following relation on $\overline{M}_{0,N}$:
\[
\sum_{i \neq j} (\psi_{\eta_i^+} + \psi_{\eta_j^+} - \sum_{\eta_i^+ \in I, \eta_j^\pm \in J} \delta_{I,J}) - (a - 1) \sum_{i=1}^{a} (\psi_{\eta_i^+} + \psi_{\eta_i^-} - \sum_{\eta_i^+ \in I, \eta_i^- \in J} \delta_{I,J}) = 0.
\]

Note that the cases of $a = 1, 2$ have been considered in Sections 5.4.7 and 5.4.6. If $a = 3$ and $N = 6$, then
\[
\psi_{cusp} = \psi_{tn} = \delta_{NB}^3 = \delta_{2}^{CT} = e(c) = \delta_{2}^t = 0,
\]
and we reduce to proving the inequality
\[
\psi - \delta - \frac{5}{4}\delta_{sm} - \frac{1}{4}e(n) > 0.
\]
Combining the standard relation (5.8), Lemma 5.33 and the inequality $e(n) \leq 2\delta_2$ on $\overline{M}_{0,6}$, we obtain the requisite inequality.

It remains to consider the case of $N \geq 7$. We combine the inequality of Lemma 5.33 with the standard relation (5.8), and the obvious $\psi - \psi_{in} \geq 0$, to obtain the inequality
\[
3(\psi - \sum_{r \geq 2} r(N - r)\delta_r/(N - 1)) + (\psi_{in} - 4\delta_{sm} - 2 \sum_{i \neq j} \delta_{\{\eta_i^+, \eta_j^\pm\}}) + (\psi - \psi_{in}) \geq 0.
\]
This can be rewritten as
\[
4\psi \geq \left[ \frac{6(N - 2)}{N - 1} + 4 \right] \delta_{sm} + \left[ \frac{6(N - 2)}{N - 1} \right] (\delta_2 - \delta_{sm}) + \left[ \frac{9(N - 2)}{N - 1} - 2 \right] \sum_{r \geq 3} \delta_r + 2 \sum_{r \geq 3} \delta_r.
\]
From here, we use the inequality \( e(n) + e(c) \leq 2 \sum_{r \geq 3} \delta_r \) to arrive at

\[
4\psi \geq \left[ \frac{6(N-2)}{N-1} + 4 \right] \delta_{sm} + \left[ \frac{6(N-2)}{N-1} \right] (\delta_2 - \delta_{sm}) + \left[ \frac{9(N-2)}{N-1} - 2 \right] \sum_{r \geq 3} \delta_r + e(n) + e(c).
\]

Using \( N \geq 7 \), we finally get

\[
\psi \geq \frac{9}{4} \delta_{sm} + \frac{5}{4} (\delta_2 - \delta_{sm}) + \delta + \frac{1}{4} (e(n) + e(c)).
\]

The equality can be achieved only if \( N = 7 \) and if \( \psi - \psi_{in} = 0 \), meaning that all sections are inner nodal. These two conditions cannot hold at the same time, so the inequality is strict. This completes the proof of Theorem 5.4 (b).

APPENDIX A.

In this appendix, we give examples of algebraic stacks including moduli stacks of curves which fail to have a good moduli space owing to a failure of conditions (1a), (1b), and (2) in Theorem 4.1. Note that there is an obviously necessary topological condition for a stack to admit a good moduli space, namely that every \( C \)-point has a unique isotrivial specialization to a closed point, and each of our examples satisfies this condition. The purpose of these examples is to illustrate the more subtle kinds of stacky behavior that can obstruct the existence of good moduli spaces.

Failure of condition (1a) in Theorem 4.1

Example A.1. Let \( \mathcal{X} = [X/\mathbb{Z}_2] \) be the quotient stack where \( X \) is the non-separated affine line and \( \mathbb{Z}_2 \) acts on \( X \) by swapping the origins and fixing all other points. The algebraic stack clearly satisfies condition (1b) and (2). Then there is an étale, affine morphism \( \mathbb{A}^1 \rightarrow \mathcal{X} \) which is stabilizer preserving at the origin but is not stabilizer preserving in an open neighborhood. The algebraic stack \( \mathcal{X} \) does not admit a good moduli space.

While the above example may appear entirely pathological, we now provide two natural moduli stacks similar to this example.

Example A.2. Consider the Deligne-Mumford locus \( \mathcal{X} \subseteq [\text{Sym}^4 \mathbb{P}^1/\text{PGL}_2] \) of unordered tuples \( (p_1, p_2, p_3, p_4) \) where at least three points are distinct. Consider the family \( (0, 1, \lambda, \infty) \) with \( \lambda \in \mathbb{P}^1 \). When \( \lambda \not\in \{0, 1, \infty\} \), \( \text{Aut}(0, 1, \lambda, \infty) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \); indeed, if \( \sigma \in \text{PGL}_2 \) is the unique element such that \( \sigma(0) = \infty, \sigma(\infty) = 0 \) and \( \sigma(1) = \lambda \), then \( \sigma([x,y]) = [y, \lambda x] \) so that \( \sigma(\lambda) = 1 \) and therefore \( \sigma \in \text{Aut}(0, 1, \lambda, \infty) \). Similarly, there is an element \( \tau \) which acts via \( 0 \leftrightarrow 1 \), \( \lambda \leftrightarrow \infty \) and an element \( \alpha \) which acts via \( 0 \leftrightarrow \lambda, 1 \leftrightarrow \infty \). However, if \( \lambda \in \{0, 1, \infty\} \), \( \text{Aut}(0, 1, \lambda, \infty) \cong \mathbb{Z}/2\mathbb{Z} \).

Therefore if \( x = (0, 1, \infty, \infty) \), any étale morphism \( f: [\text{Spec} A/\mathbb{Z}_2] \rightarrow \mathcal{X} \), where \( \text{Spec} A \) is a \( \mathbb{Z}_2 \)-equivariant algebraization of the deformation space of \( x \), will be stabilizer preserving at \( x \) but not in any open neighborhood. This failure of condition (1a) here is due to the fact
that automorphisms of the generic fiber to not extend to the special fiber. The algebraic
stack $X$ does not admit a good moduli space but we note that if one enlarges the stack
$X \subseteq [(\text{Sym}^4 \mathbb{P}^1)^{ss}/\text{PGL}_2]$ to include the point $(0, 0, \infty, \infty)$, there does exist a good moduli
space.

**Example A.3.** Let $U_2$ be the stack of all reduced, connected curves of genus 2, and let
$[C] \in U_2$ denote a cuspidal curve whose pointed normalization is a generic 1-pointed smooth
elliptic curve $(E, p)$. We will show that any Deligne-Mumford open neighborhood $M \subset U_2$
of $[C]$ is non-separated and fails to satisfy condition (1a).

Note that $\text{Aut}(C) = \text{Aut}(E, p) = \mathbb{Z}/2\mathbb{Z}$. Thus, to show that no étale neighborhood
$[\text{Def}(C)/\text{Aut}(C)] \to M$ can be stabilizer preserving where $\text{Def}(C) = \text{Spec} \mathbb{A}$ is an $\text{Aut}(C)$-equivariant algebraized
miniversal deformation space, it is sufficient to exhibit a family $C \to C'$ whose special fiber is
$C$, and whose generic fiber has automorphism group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. To do this, let $C'$ be the
curve obtained by nodally gluing two identical copies of $(E, p)$ along their respective marked
points. Then $C'$ admits an involution swapping the two components, and a corresponding
degree 2 map $C' \to E$ ramified over the single point $p$. We may smooth $C'$ to a family $C' \to \Delta$
of smooth double covers of $E$, simply by separating the ramification points. By [Smy11a,
Lemma 2.12], there exists a birational contraction $C' \to C$ contracting one of the two copies
of $E$ in the central fiber to a cusp. The family $C \to \Delta$ now has the desired properties; the
generic fiber has both a hyperelliptic and bielliptic involution while the central fiber is $C$.

**Failure of condition (1b) in Theorem 4.1**

**Example A.4.** Let $X = [\mathbb{A}^2 \setminus 0/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts via $t \cdot (x, y) = (x, ty)$. Let $U = \{y \neq 0\} = [\text{Spec} \mathbb{C}[x, y]/\mathbb{G}_m] \subseteq X$. Observe that the point $(0, 1)$ is closed in $U$ and $X$. Then
the open immersion $f: U \to X$ has the property that $f(0, 1) \in X$ is closed but for $x \neq 0,$
$(x, 1) \in U$ is closed but $f(x, 1) \in X$ is not closed. There is no open neighborhood of $(0, 1)$
in $X$ which admits a good moduli space.

**Example A.5.** Let $M = \overline{M}_g \cup M^1 \cup M^2$, where $M^1$ consists of all curves of arithmetic
 genus $g$ with a single cusp and smooth normalization, and $M^2$ consist of all curves of the
form $D \cup E_0$, where $D$ is a smooth curve of genus $g-1$ and $E_0$ is a rational cuspidal curve
attached to $C$ nodally.

We observe that $M$ has the following property: If $C = D \cup E$, where $D$ is a curve of
 genus $g-1$ and $E$ is a curve an elliptic tail, then $[C] \in M$ is a closed point if and only if $D$
is singular. Indeed, if $D$ is smooth, then $C$ admits an isotrivial specialization to $D \cup E_0$, where
$E_0$ is a rational cuspidal tail.

Now consider any curve of the form $C = D \cup E$ where $D$ is a singular curve of genus $g-1$
and $E$ is a smooth elliptic tail, and, for simplicity, assume that $D$ has no automorphisms. We
claim that there is no étale neighborhood of the form $[\text{Def}(C)/\text{Aut}(C)] \to M$, which sends
closed points to closed points. Indeed, curves of the form $D' \cup E$ where $D'$ is smooth will
appear in any such neighborhood and will obviously be closed in $[\text{Def}(C)/\text{Aut}(C)]$ (since this is a Deligne-Mumford stack), but are not closed in $\mathcal{M}$.

Failure of condition (2) in Theorem 4.1

Example A.6. Let $\mathcal{X} = [C/\mathbb{G}_m]$ where $C$ is the nodal cubic curve with the $\mathbb{G}_m$-action given by multiplication. Observe that $\mathcal{X}$ is an algebraic stack with two points–one open and one closed. But $\mathcal{X}$ does not admit a good moduli space; if it did, $\mathcal{X}$ would necessarily be cohomologically affine and consequently $C$ would be affine, a contradiction. However, there is an étale and affine morphism (but not finite) morphism $\mathcal{W} = [\text{Spec}(k[x,y]/xy)/\mathbb{G}_m] \to \mathcal{X}$ where $\mathbb{G}_m = \text{Spec}(k[t])$, acts on $\text{Spec}k[x,y]/xy$ via $t \cdot (x,y) = (tx, t^{-1}y)$ which is stabilizer preserving and sends closed points to closed points; however, the two projections $\mathcal{W} \times_X \mathcal{W} \to \mathcal{W}$ do not send closed points to closed points.

To realize this étale local presentation concretely, we may express $C = X/\mathbb{Z}_2$ where $X$ is the union of two $\mathbb{P}^1$’s with coordinates $[x_1,y_1]$ and $[x_2,y_2]$ glued via nodes at $0_1 = 0_2$ and $\infty_1 = \infty_2$ by the action of $\mathbb{Z}/2\mathbb{Z}$ where $-1$ acts via $[x_1,y_1] \leftrightarrow [y_2,x_2]$. There is a $\mathbb{G}_m$-action on $X$ given by $t \cdot [x_1,y_1] = [tx_1,y_1]$ and $t \cdot [x_2,y_2] = [x_1,ty_1]$ which descends to an action on $C$. We therefore have a Cartesian diagram

\[
\begin{array}{ccc}
\mathbb{Z}_2 \times [X/\mathbb{G}_m] & \xrightarrow{p_1} &[X/\mathbb{G}_m] \\
\downarrow & & \downarrow \\
[X/\mathbb{G}_m] & \xrightarrow{p_2} &[C/\mathbb{G}_m]
\end{array}
\]

Let $p = \infty_2 = 0_1, q = \infty_1 = 0_2 \in X$. If we let $\mathcal{W} = [(X \setminus \{p\})/\mathbb{G}_m]$ and consider $f : \mathcal{W} \to \mathcal{X}$, we obtain a Cartesian diagram

\[
\begin{array}{ccc}
[(X \setminus \{p\})/\mathbb{G}_m] \coprod [(X \setminus \{p, q\})/\mathbb{G}_m] & \xrightarrow{p_1} & [(X \setminus \{p\})/\mathbb{G}_m] \\
\downarrow & & \downarrow \\
[(X \setminus \{p\})/\mathbb{G}_m] & \xrightarrow{f} & [C/\mathbb{G}_m]
\end{array}
\]

But $[(X \setminus \{p, q\})/\mathbb{G}_m] \cong \text{Spec} k \coprod \text{Spec} k$ and the projections $p_1, p_2 : \mathcal{W} \times_X \mathcal{W} \to \mathcal{W}$ correspond to the inclusion of the two open points into $\mathcal{W}$ which clearly doesn’t send closed points to closed points.

Example A.7. Let $C$ be the Deligne-Mumford semistable curve $D \cup E$, obtained by gluing a copy of $E := \mathbb{P}^1$ to a smooth genus $g - 1$ curve $D$ at two points $p, q$. For simplicity, let us assume that $\text{Aut}(D,p,q) = 0$, so $\text{Aut}(C) = \mathbb{G}_m$. Let $\mathcal{M}_{gss}^{ss,1}$ be the algebraic stack of Deligne-Mumford semistable curves $F$ where any rational subcurve connected to $F$ at only two points is smooth.

We will show that $\mathcal{M}_{gss}^{ss,1}$ fails condition (2), i.e. the closed substack $\mathcal{Z} := \{[C]\}$ fails to admit a good moduli space. It is easy to see that there is a unique isomorphism class of curves which isotrivially specializes to $C$, namely the nodal curve $C'$ obtained by gluing $D$
at \( p \) and \( q \). Thus, \( \{[C]\} \) has two points – one open and one closed. We will show that \( \{[C]\} \) is isomorphic to the example given in Example A.6 of the quotient stack \([X/G_m]\) of the nodal cubic \( X \) modulo \( G_m \); \([X/G_m]\) does not admit a good moduli space because \( X \) is not affine.

To prove this, let us start by considering the constant family \( D \times \mathbb{P}^1 \) with two constant sections corresponding to \( p, q \in D \). Blowing up this family at \( (p, 0) \) and \( (q, \infty) \), taking the strict transforms of the sections, and then identifying them nodally, we obtain a flat family of curves whose fibers are \( C \) over 0 and \( \infty \) and \( C' \) over every other point of \( \mathbb{P}^1 \). The corresponding map \([\mathbb{P}^1/G_m] \to \{[C]\}\) is easily seen to factor through \([X/G_m]\), and the corresponding map \([X/G_m] \to \{[C]\}\) is an isomorphism.

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