INSTANTON STRINGS AND HYPERKÄHLER GEOMETRY

Robbert Dijkgraaf

Departments of Mathematics and Physics
University of Amsterdam, 1018 TV Amsterdam

&

Spinoza Institute
University of Utrecht, 3508 TD Utrecht

Abstract

We discuss two-dimensional sigma models on moduli spaces of instantons on $K^3$ surfaces. These $N = (4,4)$ superconformal field theories describe the near-horizon dynamics of the D1-D5-brane system and are dual to string theory on $AdS_3$. We derive a precise map relating the moduli of the $K^3$ type IIB string compactification to the moduli of these conformal field theories and the corresponding classical hyperkähler geometry. We conclude that in the absence of background gauge fields, the metric on the instanton moduli spaces degenerates exactly to the orbifold symmetric product of $K^3$. Turning on a self-dual NS $B$-field deforms this symmetric product to a manifold that is diffeomorphic to the Hilbert scheme. We also comment on the mathematical applications of string duality to the global issues of deformations of hyperkähler manifolds.
1. Introduction

Recently in string theory there has been great interest in a particular class of two-dimensional conformal field theories with $\mathcal{N} = (4,4)$ supersymmetry and central charge $c = 6k$ with $k$ a positive integer. These conformal field theories arise by considering a certain low-energy limit of bound states of D5-branes and D1-branes in type IIB string theory and in various other dual incarnations. They have been instrumental in the microscopic description of the quantum states of five-dimensional black holes as in the ground-breaking computation of Strominger and Vafa [1].

The same sigma models have appeared in the quantization of the six-dimensional world-volume theory of the NS fivebrane, that can be used to describe black holes in matrix theory [2, 3, 4, 5, 6]. In this context one sometimes refers to these models as second-quantized microstrings, little strings or instanton strings.

More recently, in the work of Maldacena on dualities between space-time conformal field theories and supergravity theories on anti-de Sitter spaces, these models were identified with the dual description of six-dimensional string theory on $AdS^3 \times S^3$ [8]. The holographic correspondence between these $AdS$ quantum gravity theories and two-dimensional conformal field theories with their rich algebraic structure, seems to be a fertile proving ground for various conceptional issues in gravity and string theory, see e.g. the recent papers [4, 9, 10, 11, 12] and references therein to the already extensive literature. Finally, there is a fascinating relation between the world-sheet CFT of a string moving in this $AdS^3$ background and the space-time $c = 6k$ CFT that we study in this paper [14].

This particular class of $\mathcal{N} = (4,4)$ SCFT with central charge $6k$ can be identified with sigma models on moduli spaces of instantons on $K3$ or $T^4$ [1]. With the right characteristic classes and compactification, these moduli spaces are smooth, compact $4k$ dimensional hyperkähler manifolds. Independent of the developments in string theory there has been recently mathematical interest in these higher-dimensional compact hyperkähler (or holomorphic symplectic) manifolds and their deformation spaces, following the important work of Mukai on moduli spaces of sheaves on abelian and $K3$ surfaces [15, 16, 17]. See for example [18] for a good review of the recent developments.

In this paper we will see how the insights from string theory and from hyperkähler geometry combine nicely and allow us to make a precise identification between the moduli of the string compactification and those of the (space-time) CFT. Roughly we find two set of canonical identifications:

(1) between the moduli of a special (attractor) family of world-sheet $K3$ sigma model and the classical hyperkähler geometry of the instanton moduli space;

(2) between the moduli of a special family of non-perturbative IIB string theory compactification on $K3$ and the $(4,4)$ superconformal field theory on the instanton moduli space.
We will see in detail how both the metric and $B$-field on $K3$ are used to determine the metric on the compactified instanton moduli space. The RR fields then encode the $B$-field of the sigma model. Furthermore, non-perturbative $U$-dualities of the IIB string compactification translate into $T$-dualities of the hyperkähler sigma model.

The outline of this paper is as follows. In §2 we introduce the D5-brane world-volume theory and its relation to a sigma model on the instanton moduli space. In §3 we will describe in detail the moduli of $K3$ compactifications in string theory. Then in §4 we will use this explicit description to determine the value of these moduli on D-brane bound states that minimalize the BPS mass through the so-called attractor mechanism. We will treat in detail the examples of the D0-D4-brane system and D2-branes in IIA theory (or equivalently D1-D5 and D3 in IIB theory). In §5 we will review some of discussion of hyperkähler manifolds and their deformation spaces in the mathematical literature. Then in §6 we will use these mathematical facts to give an explicit map between the $K3$ moduli and the hyperkähler structures on the instanton moduli space, a map that we will then extend in §7 to string theory moduli and the moduli of the $\mathcal{N} = (4, 4)$ SCFT. We will end with some comments on global issues of these moduli spaces.

2. D5-branes and instanton strings

2.1. Some mathematical preliminaries

In this paper we will be studying sigma models with target space the moduli space $\mathcal{M}$ of instantons on a four-manifold $X$. So, mathematically speaking, we are interested in the quantum cohomology or more generally the Gromov-Witten invariants of $\mathcal{M}$. The classical cohomology of $\mathcal{M}$ is essentially equivalent to the Donaldson invariants of $X$, so the sigma model on $\mathcal{M}$ defines a “stringy” generalization of these invariants — we will refer to them as instanton strings. However, we should hasten to add that for the particular case of a $K3$ or $T^4$ manifold, on which we will mostly concentrate in this paper, the moduli space is a hyperkähler manifold, and the quantum cohomology reduces to the ordinary cohomology. So one should turn in this case to more refined string theory invariants such as the elliptic genus [1, 19].

The quantum cohomology of the moduli space of vector bundles on a Riemann surface is well-known to play an important role in the study of instantons on 4-manifolds that can be written as the product of two Riemann surfaces or, more generally fibrations of this form [20, 21]. If $X$ is of the form $\Sigma \times \Sigma'$, in the adiabatic limit where the volume of $\Sigma'$ is very small compared to $\Sigma$, instantons on $X$ reduce to holomorphic maps of $\Sigma$ into the moduli space of holomorphic vector bundles on $\Sigma'$. In a similar spirit the quantum cohomology of instanton moduli spaces on a four-manifold $X$ should be related
to invariants of Yang-Mills gauge theories on sixfolds (or complex threefolds) of the form \( Y = \Sigma \times X \).

Now, naively, non-abelian quantum gauge theories do not make sense beyond space-time dimension four because they become strongly interacting at short distances, which ruins the renormalizability of the theory. Six-dimensional supersymmetric Yang-Mills can be made sense of as a quantum system in string theory, namely as the world-volume theory of multiple Dirichlet fivebranes \([22]\), with strings providing the natural regulator at short distances \([23]\). At low energies the world-volume theory of \( N \) of these Dirichlet fivebranes reduces to a six-dimensional \( U(N) \) super-Yang-Mills theory. Quantum fivebranes are one of the most mysterious but also mathematically richest objects in string theory, see e.g. \([24, 25]\). Via \( T \)-dualities they are closely related to the study of moduli spaces of stable holomorphic vector bundles on \( Y \), a problem that has very important applications in string theory and that has received some recent mathematical impetus \([26, 27]\).

### 2.2. The D5-brane action

Ignoring fermions and scalar fields, the leading part of the D5-brane world-volume action on a six-manifold \( Y \) embedded in ten-dimensional space-time (we will assume with a trivial normal bundle) is of the form

\[
S = \int_Y \frac{1}{g_s} \text{Tr} \mathcal{F} \wedge \ast \mathcal{F} + \mathcal{C} \wedge \nu'(\mathcal{F}).
\]  

Let us explain the various objects in this Lagrangian: \( g_s \in \mathbb{R}_+ \) is the type IIB string coupling. The covariant field strength \( \mathcal{F} \) is defined as \([28]\)

\[
\mathcal{F} = F - 2\pi i B,
\]  

with \( F \) the usual curvature of the \( U(N) \) connection on the rank \( N \) vector bundle \( \mathcal{E} \) over \( Y \), and \( B \) the background NS tensor field, a harmonic 2-form on \( Y \) (actually, the pull-back to \( Y \) of the ten-dimensional space-time \( B \)-field.) Note that \( B \) is a singlet under \( U(N) \), so it only couples to \( \text{Tr} F \).

The background RR gauge field \( \mathcal{C} \) is the pull-back to \( Y \) of an arbitrary harmonic form of even degree in space-time and it can be decomposed as

\[
\mathcal{C} = \theta + \widetilde{B} + G
\]  

Of course, one of the important lessons we have learned from the success of Seiberg-Witten theory is that in topological applications it can suffice to work with an effective quantum field theory, that only makes sense up to a certain cut-off distance scale.
with $\theta$ a scalar, $\tilde{B}$ the RR 2-form field and $G$ a 4-form with a field strength that satisfies the self-duality constraint in ten dimensions, $dG = *dG$. These RR fields couple to the 5-brane through the generalized Mukai vector $v'$ given by \[29, 30, 31, 32\]

$$v' = \text{Tr} \exp \left( \frac{iF}{2\pi} \right) \wedge \hat{A}(Y)^{1/2}. \quad (2.4)$$

Here the first term is a generalization of the usual Chern character $ch(E)$ including the NS $B$-field. The expression $\hat{A}(Y)$ that figures in the second term is the so-called A-roof genus of the manifold $Y$. It appears in the index theorem of the Dirac operator and can be expressed as a particular combination of Pontryagin classes of $Y$. For a Calabi-Yau threefold (or twofold) the $\hat{A}$ genus equals the Todd genus

$$Td(Y) = 1 + \frac{c_2}{12}, \quad (2.5)$$

and we can write the RR charge vector as

$$v' = \text{Tr} \exp \left( \frac{iF}{2\pi} + B + \frac{c_2}{24} \right). \quad (2.6)$$

Writing out the coupling of the RR gauge fields (on a flat space and ignoring the NS $B$-field) we obtain schematically a combination of the form

$$\int \theta \text{Tr} F^3 + \tilde{B} \wedge \text{Tr} F^2 + G \wedge \text{Tr} F \quad (2.7)$$

From this we see that Yang-Mills instantons, which contribute to the second Chern character $\text{Tr} F^2$, carry charge with respect to $\tilde{B}$ and can therefore be interpreted as bound states with D1-strings. Similarly D3-brane bound states couple to the four-form $C$ and are carried by gauge field configurations with a non-trivial first Chern class or magnetic flux $\text{Tr} F$.

If we consider a 6-manifold of topology $\Sigma \times X$ (or a local $X$-fibration) in the adiabatic limit where the volume of the surface $\Sigma$ is much larger than the four-manifold $X$, the SYM theory reduces to a sigma model with target space the moduli space of anti-self-dual connections $F_+ = 0$ on $X$. (Here we assumed that the instanton number is positive, otherwise we should consider self-dual connections or bound states with anti-D1-branes.) If we include a background self-dual harmonic $B$-field, the abelian part of the ASD equation is deformed to

$$F_+ = 2\pi iB. \quad (2.8)$$
The various terms in the D5-brane action (2.1) now acquire a sigma model interpretation. The kinetic term gives a natural metric $g$ on the instanton moduli space which is equivalent to the usual $L^2$-metric on adjoint-valued one-forms on $X$, defined for $\alpha, \beta \in TM$ as

$$g(\alpha, \beta) = \frac{1}{g_s} \int_X \text{Tr}(\alpha \wedge \star \beta), \quad (2.9)$$

where the tangent vectors can be chosen to satisfy $(D\alpha)_+ = D^*\alpha = 0$. From the normalization of the kinetic term we see that with this metric the volume of $\mathcal{M}$ scales with a power of $1/g_s$ and goes to infinity in the weak coupling limit $g_s \to 0$.

If the four-manifold $X$ is hyperkähler (HK) with hyperkähler forms $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$, the instanton moduli space itself is a hyperkähler manifold. In fact, it can be considered as an infinite-dimensional HK quotient with HK moment map

$$\vec{\mu} = F \wedge \vec{\omega}. \quad (2.10)$$

The condition $\mu = 0$ enforces the ASD equation. With $\mathcal{G}$ the infinite-dimensional gauge group, the moduli space of instantons $\mathcal{M}$ can therefore be written as

$$\mathcal{M} = \mu^{-1}(0)/\mathcal{G} \quad (2.11)$$

and this guarantees that it is hyperkähler \cite{33}. From this point of view turning on a self-dual $B$-field on $X$ corresponds to a change in the value of the hyperkähler moment map. In some cases this deformation can resolve certain singularities \cite{34}.

### 2.3. Topological observables and the Donaldson-Mukai map

The coupling of the RR background gauge fields to the D5-brane gauge theory produces natural cohomology classes on the instanton moduli space. In particular, it induces two kinds of natural closed 2-forms on $\mathcal{M}$ that acquire an interpretation as $B$-fields in the sigma model. These two-forms are well-known from the polynomial Donaldson invariants \cite{35} and Witten’s topological field theory realization of these invariants \cite{36}. It is remarkable that in this case we can naturally identify the deformation parameters of quantum cohomology with the standard variables of the four-manifold invariants.

First, for any harmonic two-form on $X$ we have the well-known two-dimensional cohomology class on $\mathcal{M}$ defined by Donaldson. With $\mathcal{B}_I \ (I = 1, \ldots, b_2)$ a basis of $H^2(X, \mathbb{Z})$, these classes take the local form

$$\mathcal{B}_I(\alpha, \beta) = \int_X B \wedge \text{Tr}(\alpha \wedge \beta). \quad (2.12)$$
There is a second natural two-form, namely

\[ \mathcal{B}_0(\alpha, \beta) = \int_X \text{Tr} (F \wedge \alpha \wedge \beta). \quad (2.13) \]

In this way we obtain a natural set of \( b_2 + 1 \) two-forms on the moduli space \( \mathcal{M} \). We will see later that for \( K3 \) surfaces these forms form a basis for \( H^2 \) of the instanton moduli space.

Note that in the case of stable holomorphically vector bundles on a complex surface \( X \), the holomorphic tangent vectors \( \alpha, \beta \) are elements of \( H^{0,1}(X, \text{End} \mathcal{E}) \) and the curvature \( F \) is of type \( (1,1) \). Now we also have a correspondence of Hodge structures: a \( (2,0) \) form \( B_I \) gives a \( (2,0) \) form \( \mathcal{B}_I \) on moduli space etc. The additional two-form \( \mathcal{B}_0 \) is always of type \( (1,1) \).

In topological field theory terms these two-form are operators obtained through the so-called descent formalism \[36\]. The forms \( \mathcal{B}_I \) are the two-form descendents of the BRST invariant operator \( \text{Tr} \phi^2 \); the second form \( \mathcal{B}_0 \) is similarly a four-form descendent of \( \text{Tr} \phi^3 \), an operator that only appears as an independent field in \( U(N) \) gauge theory with \( N > 2 \).

Mathematically, these two-forms are produced using (generalizations of) Donaldson’s \( \mu \) map that associates a cohomology class on the moduli space to a cohomology class on the four-manifold

\[ \mu_2 : H^k(X) \to H^k(\mathcal{M}), \quad (2.14) \]

and which is defined as follows. One considers the universal bundle \( \hat{\mathcal{E}} \) on the product \( X \times \mathcal{M} \). This bundle has a curvature \( \hat{F} \) and a second Chern character \( \text{Tr} \hat{F} \wedge \hat{F} \). For any form \( \mathcal{C} \in H^*(X) \) one now defines a cohomology class on \( \mathcal{M} \) by multiplying with the Chern class and then integrating over the fiber \( X \)

\[ \mu_2(\mathcal{C}) = \int_X \mathcal{C} \wedge \text{Tr} (\hat{F} \wedge \hat{F}). \quad (2.15) \]

With this definition we have

\[ \mathcal{B}_I = \mu_2(B_I). \quad (2.16) \]

Similarly one can define a map \( \mu_3 \) from \( H^k(X) \) to \( H^{k+2}(\mathcal{M}) \) starting from \( \text{Tr} \hat{F}^3 \) and with this map we have \( \mathcal{B}_0 = \mu_3(1) \).

Actually, from the precise coupling \[2.14\] to the generalized Mukai vector \( \nu' \) we see that the relevant map induced by the D5-brane is the generalized Mukai map \[17\]

\[ \mu : H^*(X) \to H^*(\mathcal{M}), \quad (2.17) \]
which does not preserves degrees and which is roughly defined as

$$
\mu(C) = \int_X C \wedge \hat{v}, \quad \hat{v} = e^B \text{ch}(\hat{E}) \hat{A}^{1/2}(X \times M).
$$

(2.18)

We want to stress that this coupling through the Mukai vector gives a natural map from the harmonic RR background fields to the cohomology of the instanton moduli space. We will return to the properties and interpretation of this important map.

3. String theory on K3

Although almost everything discussed in this paper pertains to string compactifications on both four-tori and K3 manifolds, we will mostly concentrate on the latter. We will first recall some facts about string theory on K3. An excellent review of the various applications of K3 surfaces in string theory is given in [37].

3.1. Classical and CFT moduli of K3

Let $X$ be a K3 surface. Recall that $H^2(X, \mathbb{Z})$ equipped with the intersection product is isomorphic to $\Gamma^{3,19}$, the unique even, self-dual lattice of signature $((+)^3, (-)^{19})$. By the global Torelli theorem of K3 surfaces and Yau’s theorem, a hyperkähler structure on a marked K3 surface is uniquely determined by the positive 3-plane $U \subset H^2(X, \mathbb{R})$ spanned by the periods of the three HK forms

$$
U = \langle \omega_1, \omega_2, \omega_3 \rangle
$$

(3.1)

together with the volume $V \in \mathbb{R}_+$. With this metric $U \cong H^2_+(X, \mathbb{R})$, the space of self-dual two-forms on $X$. It will turn out to be convenient to normalize the HK forms $\vec{\omega}$ (unconventionally) such that $\frac{1}{2} \omega_a \wedge \omega_b = \delta_{ab} V/3$.

Since every positive three-plane $U$ can appear as the image under the period map of a HK structure, the moduli space of HK metrics of fixed volume (including singular orbifold metrics, where $-2$ curves are contracted and symmetry enhancing singularities of ADE-type occur) is given by the Grassmannian

$$
\frac{O(3,19)}{O(3) \times O(19)}.
$$

(3.2)

The corresponding moduli space for unmarked K3 surfaces is obtained by further dividing by $O^+(\Gamma^{3,19})$, the component group of the group of diffeomorphism.
If we choose a compatible complex structure, the positive 3-plane $U$ decomposed as the orthogonal sum $P \perp L$, with $P$ the 2-plane spanned by the real and imaginary components of the holomorphic $(2,0)$ form, say $\eta = \omega_2 + i\omega_3$, and $L$ the (oriented) line generated by the real $(1,1)$ Kähler form, say $\omega_1$. The form $\eta$ spans $H^{2,0}(X)$ and its periods take value in the period domain

$$\eta^2 = 0, \quad (\eta + \overline{\eta})^2 > 0.$$  \hspace{1cm} (3.3)

The global Torelli theorem guarantees that the periods of $\eta$ uniquely determine the complex structure on $X$. For a given hyperkähler structure there is a $S^2$ worth of inequivalent compatible complex structures, corresponding to the choices of $P \subset U$.

In conformal field theory one further adds the NS 2-form background field $B \in H^2(X,\mathbb{R}/\mathbb{Z})$. So locally the moduli space of $K3$ CFT’s is

$$\frac{O(3,19)}{O(3) \times O(19)} \times \mathbb{R}^{3,19} \times \mathbb{R}_+.$$  \hspace{1cm} (3.4)

Supersymmetry arguments tell us that this component of the moduli space of $\mathcal{N} = (4,4)$ superconformal field theories with $c = 6$ is locally given by the symmetric space $[38]$

$$\frac{O(4,20)}{O(4) \times O(20)}.$$  \hspace{1cm} (3.5)

A precise analysis of the quantum symmetries shows that moduli space of inequivalent $K3$ sigma model superconformal field theories is obtained as a further quotient by the $T$-duality group $O(\Gamma^{4,20}) = O(4,20,\mathbb{Z})$, with $\Gamma^{4,20}$ the even, self-dual lattice of signature $(4,20)$ $[38]$.

3.2. The Mukai vector

The moduli space $[33]$ has an obvious interpretation as the Grassmannian parametrizing positive 4-planes $W \subset \mathbb{R}^{4,20}$. This representation follows most naturally by considering a compactification of the type IIA (or IIB) string on $X$. Since the RR fields in IIA theory have even-dimensional curvatures, the lattice of RR charges carried by D-brane bound states (that are point-like in the uncompactified six dimensions) is given by $H^*(X,\mathbb{Z})$. Equivalently, instead of looking at point-like objects, one could look at six-dimensional strings that are obtained by wrapping odd D-branes in type IIB theory, as we will later.

A D-brane state with charge $v \in H^*(X,\mathbb{Z})$ can be described in terms of a vector bundle, or more generally a coherent sheave $\mathcal{E}$ over $X$ with Mukai vector

$$v = ch(\mathcal{E})Td(X)^{1/2} = (r, c_1, r + c_2).$$  \hspace{1cm} (3.6)
Here the Chern character is defined as \( ch(\mathcal{E}) = \text{tr} \exp iF/2\pi \) and \( r \) is the rank of \( \mathcal{E} \), \( ch_2 = -c_2 + \frac{1}{2}c_1^2 \). Note that here we did not use the definition (2.4) of the generalized Mukai vector \( v' \), which includes the effect of the \( B \)-field. In fact the two are simply related as

\[
v' = e^B \wedge v. \tag{3.7}
\]

The point is that because of the \( B \)-field contribution the generalized Mukai vector is generically no longer an integer cohomology class, and we like to fix the charge lattice once and for all and identify it with \( H^*(X, \mathbb{Z}) \). We will instead account for the effect of the \( B \)-field in terms of the moduli of the \( K3 \) sigma model.

It is natural to give \( H^*(X, \mathbb{Z}) \) the Mukai intersection product, defined as \cite{16}

\[
v \cdot v = \int_X (v^2 \wedge v^2 - 2v^0 \wedge v^4), \quad v = (v^0, v^2, v^4), \quad v^i \in H^i(X). \tag{3.8}
\]

We will always identify \( H^*(X, \mathbb{Z}) \cong \Gamma^{4,20} \) with Mukai’s quadratic form. With this definition the moduli space \( \mathcal{M}_v \) of simple sheaves with Mukai vector \( v \) has complex dimension

\[
\dim \mathcal{M}_v = 2 + v \cdot v. \tag{3.9}
\]

### 3.3. A quaternionic formula

The map that associates a real 4-plane \( W \subset H^*(X, \mathbb{R}) \) to the three HK forms \( \vec{\omega} \) and the \( B \)-field can be elegantly formulated as follows. Combine the four 2-forms into a single quaternionic 2-form

\[
b = B + \omega_1 i + \omega_2 j + \omega_3 k. \tag{3.10}
\]

The vector \( b \) should be considered to take value in the quaterionic domain

\[
\{ b \in \mathbb{H}^{3,19} \mid \Im b > 0 \}. \tag{3.11}
\]

With this notation the 4-plane \( W \) can be considered as a quaternionic line in \( H^*(X, \mathbb{R}) \otimes \mathbb{H} \), given in terms of \( b \) as

\[
W = (\exp b) = (1 + b + \frac{1}{2} b \wedge b). \tag{3.12}
\]

Note that \( \exp b \) is a quaternionic null vector in the Mukai inner product. In components the plane \( W \) is spanned by the vectors\cite

\[
(0, \vec{\omega}, B \wedge \vec{\omega}), \quad (1, B, \frac{1}{2} B \wedge B - V). \tag{3.13}
\]

*Here we used the normalization \( 4\vec{\omega} \cdot \vec{\omega} = V \). Note a small discrepancy with the formula of \cite{37, 40} regarding the precise contribution of the \( B \)-field.
Note that the volume $V$ is measured in terms of the string length $\sqrt{\alpha'}$, so the classical limit $\alpha' \to 0$ implies $V \to \infty$.

There are several remarks that might help to understand formula (3.12) better. First, note that $b$ can be written in terms of the complexified Kähler form $\omega = B + i \omega_1$ and the holomorphic $(2,0)$ form $\eta = \omega_2 + i \omega_3$ (that determines the complex structure) as

$$b = \omega + \eta j.$$  
(3.14)

In this notation we see that a mirror symmetry map that interchanges the complex and Kähler structure acts as a simple quaternionic rotation by multiplying by $j$.

Secondly, from the action (2.1) we see that turning on the $B$-field, shifts $F \to F - 2\pi i B$ in the expression of the charge vector

$$v \to e^B \wedge v = v',$$  
(3.15)

which is in line with our formula.

Thirdly, the representation (3.12) can be seen as a quaternionic generalization of an analogous complex representation for the hermitean symmetric domain \cite{11, 12}

$$\frac{O(2,2+s)}{O(2) \times O(2+s)} \cong \{ y \in \mathbb{C}^{1,1+s} \mid \Im y > 0, \ y^{1,0} > 0 \}.$$  
(3.16)

To prove this isomorphism one associates to such a vector $y$ the complex null vector

$$z = (y, 1, -\frac{1}{2} y^2) \in \mathbb{C}^{2,2+s}.$$  
(3.17)

(Here we note the similarity with (3.12).) The corresponding positive two-plane in $\mathbb{R}^{2,2+s}$ is then spanned by $\Re z, \Im z$.

Finally, in the case of a four-torus $T = \mathbb{R}^4/2\pi L$ with lattice $L \cong \mathbb{Z}^4$, we have a similar formula that can be understood along more familiar lines. In the torus case the $T$-duality group $SO(4,4,\mathbb{Z})$ is usually considered to act in the $8_v$ vector representation on the Narain lattice

$$L \oplus L^* \cong \Gamma^{4,4},$$  
(3.18)

equipped with the standard quadratic form

$$v \cdot v = 2(w, k), \quad v = w \oplus k.$$  
(3.19)
Picking a basis $e_\mu$ of $L$ and dual basis $e^\mu$ of $L^*$, the data of a flat metric $G_{\mu\nu}$ and constant $B$-field $B_{\mu\nu}$ determine a positive four-plane in $\mathbb{R}^{4,4}$ spanned by the vectors

$$e_\mu + X_{\mu\nu} e^\nu, \quad X_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}. \quad (3.20)$$

The $T$-duality group acts by fractional linear transformations on the $4 \times 4$ matrix $X_{\mu\nu}$.

However, by triality the group $SO(4,4,\mathbb{Z})$, or more properly its cover $Spin(4,4,\mathbb{Z})$, also acts on the spinor representation $8_s$, that can be canonically identified with IIA string theory RR charge lattice

$$H^{even}(T,\mathbb{Z}) \cong \Lambda^{even} L^* \cong \Gamma^{4,4}. \quad (3.21)$$

Similarly for the type IIB string we have an action in the $8_c$ representation on the lattice $H^{odd}(T,\mathbb{Z})$. In this spinor representation the corresponding positive 4-pane $W$ in $\Lambda^{even}[e^\mu] \cong \mathbb{R}^{4,4}$ is determined as the positive eigenspace of the generalized Hodge $*$-operator

$$* = \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \wedge \gamma_4, \quad (3.22)$$

written in terms of the Clifford algebra generators

$$\gamma_\mu = \frac{\partial}{\partial e^\mu} + X_{mn} e^\mu. \quad (3.23)$$

One now easily checks that $W$ can be represented in the above form (3.12) with $\omega_a$ a basis of self-dual 2-forms. For example, in the above formula it is clear that a shift of $B$ is implemented by conjugating $\gamma_\mu \rightarrow e^{-B} \gamma_\mu e^B$.

4. D-branes and attractive $K3$’s

We will now apply this explicit description of the $K3$ conformal field theory moduli to D-brane bound states. To this end let us consider a general D-brane state labeled with a Mukai vector $v \in \Gamma^{4,20}$, $v^2 > 0$ and $v$ primitive. We will assume that

$$v^2 = 2k - 2, \quad k > 1. \quad (4.1)$$

Such a vector defines an orthocomplementary lattice $v^\perp \subset \Gamma^{4,20}$ of signature $(3, 20)$. The lattice is even but not self-dual (unimodular).

One can use some well-known results in lattice theory\textsuperscript{13, 14} to prove that any two D-brane charge vectors of equal length are related by a $T$-duality transformation

---

* I wish to thank E. Looijenga for a very helpful discussion about this point.
in $O(\Gamma^{4,20}) = O(4,20,\mathbb{Z})$. Indeed, the orthogonal group acts transitively on the set of primitive vectors of a given length in a lattice, if the lattice contains at least 3 hyperbolic lattices $H \cong \Gamma^{1,1}$. Since we have the isomorphism

$$\Gamma^{4,20} \cong 4H \oplus (-2E_8), \quad (4.2)$$

with $E_8$ the root lattice of the corresponding Lie algebra, this is indeed the case. This implies that all lattices $v^\perp$ with fixed $v^2 = 2k-2$ are isomorphic. We sometimes write this lattice as $\Gamma^{3,20}_k$ or just simply $\Gamma^{3,20}$, but note that the integer quadratic form of signature $(3,20)$ depends on the value of $k$.

Furthermore, we can always find a primitive vector $u \in v^\perp$ satisfying

$$u \cdot v = 0, \quad u^2 = -v^2 = 2 - 2k, \quad (4.3)$$

such that the lattice decomposes as

$$\Gamma^{3,20}_k \cong \Gamma^{3,19} \oplus \mathbb{Z} \cdot u. \quad (4.4)$$

Again all choices of such a vector $u$ are related by a stabilizing $T$-duality transformation in $O(v^\perp) = O(3,20,\mathbb{Z})$. To prove this one has to study primitive embeddings of the hyperbolic lattice $(u,v)$ in $\Gamma^{4,20}$. Since this lattice has rank two, such an embedding is unique if the target lattice contains at least four hyperbolic factors, which it indeed does (or equivalently, $\Gamma^{3,20}_k$ should contain at least three hyperbolic factors.)

We will now look at the $K3$ sigma model moduli that minimalize the BPS mass of the corresponding D-brane state. These are the fixed point values of the near-horizon moduli as described by the so-called attractor mechanism [45], see also the extensive analysis of [10] and references therein. In the supergravity limit these are the values of the scalar fields on the D-brane horizon, see e.g. [10]. In the quantum description of the D-brane degrees of freedom these are the relevant values of the $K3$ moduli.

Given a positive 4-plane or polarization $W \subset \mathbb{R}^{4,20}$, the charge vector $v$ decomposes as

$$v = v^4_0 + v^{0,20}_L, \quad v^2 = v^2_L - v^2_R. \quad (4.5)$$

The BPS mass formula is given by

$$m^2 = v^2_L = v^2 - v^2_R, \quad (4.6)$$

\[\text{Note however that the attractive $K3$ surfaces described here differ from the $K3$ surfaces that appear in $K3 \times T^2$ compactifications and that are described in detail in [10]. The latter have much more subtle arithmetic properties than the simple families described here. In particular here we have a unique $T$-equivalency class for given “discriminant” $-v^2$.}\]
and this is clearly minimalized if \( v_R = 0 \), i.e. if the charge vector \( v \) is contained in the 4-plane \( W \). Writing \( W \) as \( U \perp Rv \), we see that the restricted moduli space that preserves the minimal BPS mass is locally given by the Grassmannian of positive 3-planes \( U \) in the complement \( v^\perp \otimes R \cong R^{3,20} \), and can therefore be written as

\[
\frac{O(3, 20)}{O(3) \times O(20)}.
\]

Furthermore, we have the remnant of the \( T \)-duality group \( O(v^\perp) \cong O(3, 20, \mathbb{Z}) \), that preserves the charge vector \( v \) and is therefore an automorphism of the D-brane state. The quantum restricted moduli space is obtained by further quotienting by this \( T \)-duality subgroup.

It might be interesting to consider these attractor moduli in a few explicit special cases, see also the discussion in [32]. We use the notation of IIA theory but the result equally well applies to the IIB theory.

### 4.1. D0-D4-branes

Let us first consider only D0 and D4 branes, no D2-branes [47]. Here we pick a Mukai vector of type \( v = (r, 0, -p) \), with \( r \cdot p > 0 \) and \( r, p \) relative prime. In the gauge theory this is the case of a vector bundle of rank \( r \), \( c_1 = 0 \) and instanton number \( c_2 = r + p \).

The BPS conditions now read (with \( \frac{1}{2} \bar{\omega} \cdot \bar{\omega} = V \), and \( \bar{\zeta} \in \mathbb{R}^3 \) to be determined)

\[
\bar{\zeta} \cdot \bar{\omega} + r B = 0, \tag{4.8}
\]

\[
\bar{\zeta} \cdot \bar{\omega} \wedge B + r \left( \frac{1}{2} B \wedge B - V \right) = -p. \tag{4.9}
\]

These equations can be simply solved by first choosing \( B = -\frac{\bar{\zeta} \cdot \bar{\omega}}{r} \), so that for the given (unrestricted) HK structure \( \bar{\omega} \) the \( B \)-field should be self-dual,

\[
B \in H^2_+(X) \cong \mathbb{R}^3. \tag{4.10}
\]

We have already commented on the interpretation of adding a SD \( B \)-field in terms of shifting the value of HK moment map in §2. Secondly, the overall volume of the \( K3 \) (in units of \( \alpha' \)) should be fixed to be

\[
V = p/r - \frac{1}{2} B^2. \tag{4.11}
\]

So, for small charges an attractive \( K3 \) is typically of the size of the string, and the sigma model is strongly coupled. Note that since \( B \) is self-dual, \( B^2 \) is positive, as is \( V \), so that
consequently we have a bound on the $B$-field of the form $B^2 \leq 2p/r$. Combining the ingredients we see that in this case the restricted moduli space (4.7) decomposes locally as
\[ \frac{O(3, 19)}{O(3) \times O(19)} \times \mathbb{R}^3. \] 

(4.12)

4.2. D2-branes

The example of D2-branes has been studied among others in [48]. Here the Mukai vector is given as $v = (0, q, 0)$ with $q \in H^2(X, \mathbb{Z})$ primitive and satisfying $q \cdot q > 0$. In gauge theory terms we are considering a sheaf localized on a complex curve with $c_1 = q$ and $ch_2 = 0$. Now the BPS conditions read
\[ \zeta \cdot \varpi = q, \quad \zeta \cdot \varpi \wedge B = 0. \] 

(4.13)

The first equation implies that the HK structure must be chosen such that
\[ q \in H^2_+ (X). \] 

(4.14)

This in turn requires that $H^2_+(X) \cap H^2(X, \mathbb{Z})$ should be at least one-dimensional. The total volume of the $K3$ surface is in this case unrestricted. Furthermore, according to the second equation, the $B$-field should satisfy
\[ B \cdot q = 0, \] 

(4.15)

so that $B \in q^\perp \otimes \mathbb{R} \cong \mathbb{R}^{2,19}$. For D2-branes the moduli space (4.7) therefore decomposes in the familiar way as
\[ \frac{O(2, 19)}{O(2) \times O(19)} \times \mathbb{R}^{2,19} \times \mathbb{R}_+. \] 

(4.16)

Equivalently, there must exist a compatible complex structure on $X$ such that $q \in H^{1,1}(X)$ and positive. Therefore the Picard lattice should be at least one-dimensional. That in turn implies that $q$ is Poincaré dual to a homology class $[C]$ of a holomorphic curve $C$ in $X$ of genus $g = 1 + \frac{1}{2} q \cdot q \geq 2$. That is, the $K3$ moduli are such that $q$ can be realized as a supersymmetric two-cycle, a well-known result [48, 49]. Furthermore the $B$-field flux through this curve is required to vanish,
\[ \int_C B = 0. \] 

(4.17)
4.3. General case

After these two examples it is not difficult to write down the formulas for the fixed point moduli for the general D-brane bound state system with \( v = (r, q, p) \) primitive, \( v^2 = 2rp + q^2 > 0 \), since such a vector can be obtained from the previous two examples by simply shifting the \( B \)-field.

For \( r = 0 \) we have D0 and D2-branes. Since we obtain this case from example 4.2 by shifting \( B \) by a 2-form \( B_0 \) satisfying \( B_0 \cdot q = p \), the only difference is now that the total flux of the \( B \)-field satisfies
\[
\int_C B = p. \tag{4.18}
\]

If \( r \neq 0 \) we can write \( v = e^B \wedge v' \) with \( v' = (r, 0, p') = (r, 0, v^2/2r) \) and \( B = q/r \). (We should not worry about \( p' \) not being integer.) So we find that, as in example 4.1, the HK structure can be general and that the \( B \)-field is self-dual up to a shift
\[
B' = B - q/r \in H^2_+(X). \tag{4.19}
\]

In term of the gauge theory this condition insures that the ASD equation \( F + \gamma \gamma = 0 \) makes sense even though \( c_1 \) is non-zero, since the shifted curvature \( F' = F - 2\pi q/r \) satisfies \([\text{Tr } F'] = 0 \). In this case the \( K3 \) volume is given by
\[
V = \frac{v^2}{2r^2} - \frac{1}{2} B' \wedge B'. \tag{4.20}
\]

5. Hyperkähler geometry and moduli of moduli

The D-brane states can be modeled in the BPS or near horizon limit in terms of supersymmetric quantum mechanics on instanton moduli spaces. For example the appropriate cohomology of the moduli space gives the 1/4 BPS states of the D-brane bound state. This quantum mechanical system is best seen as a \( \alpha' \rightarrow 0 \) limit of the corresponding superconformal sigma model that we will discuss in more detail in section 7. We will now turn to a more detailed description of the properties of these instanton moduli spaces.

5.1. Hilbert schemes and moduli spaces of sheaves

If we pick on the \( K3 \) surface \( X \) a compatible complex structure (or holomorphic 2-form \( \eta \)) and Kähler form \( \omega \), the D-brane bound state system can be described in terms of the moduli space of suitable coherent sheaves \[10, 51, 32\]. The moduli space \( \mathcal{M}_v \) of simple
semi-stable (with respect to the polarization $\omega$) torsion-free sheaves $\mathcal{E}$ up to equivalence with Mukai vector $v$ is known to be a HK manifold of dimension $4k$ with

$$k = \frac{1}{2}v^2 + 1.$$  \hfill (5.1)

If the vector $v$ is primitive this moduli space is smooth, compact and simply-connected. Note that if the charge vector $v$ is null, $\mathcal{M}_v$ is a $K3$ surface itself, though not necessarily the same $K3$. This fact is put to good use in the Mukai-Nahm Fourier transform.

As explored by Mukai [15, 16, 17] the moduli spaces $\mathcal{M}_v$ have many beautiful properties, reviewed in for example [18, 52]. First of all $\mathcal{M}_v$ is a simple compact HK space, i.e. it is simply-connected and $h^{2,0} = 1$. So for a given complex structure the holomorphic $(2,0)$ form is unique up to scalars. Any HK manifold can be written as a finite quotient of tori and simple HK manifolds.

String theorists might be surprised to learn that examples of simple compact HK manifolds are very rare in the mathematical literature. Of course in dimension 4 the only example is $K3$. Until very recently, only two examples of simple HK manifolds in dimension $4k > 4$ where known, both constructed by Beauville in 1983 [53]. These are respectively the Hilbert scheme $X^{[k]}$ of zero-dimensional schemes (read, points) of length $k$ of a $K3$ surface $X$, and the generalized Kummer variety, which is obtained by taking the Hilbert scheme $T^{[k+1]}$ of a four-torus $T$ and then quotienting by $T$ (so that one obtains the usual Kummer surface representation of $K3$ for $k = 1$.) An excellent introduction to various aspects of Hilbert schemes of complex surfaces can be found in [54].

Physically these manifolds can be understood as deformations of symmetric products of $K3$ or $T^4$ manifolds. Indeed, for any complex surface $X$ the Hilbert scheme is a smooth manifold that can be obtained as a canonical desingularization

$$X^{[k]} \xrightarrow{\pi} S^k X,$$  \hfill (5.2)

of the symmetric product orbifold $S^k X = X^k/S_k$. The two spaces are closely related: e.g. the cohomology of $X^{[k]}$ coincides with the orbifold cohomology of $S^k X$ [55, 56, 57, 58] and one expects a similar result for the elliptic genus [19]. Note that the definition of the Hilbert scheme requires a choice of complex structure on $X$.

If $X$ is a torus or $K3$ surface, the Hilbert scheme has a canonical $(2,0)$ form, essentially the pull-back of the symmetrization of the corresponding $(2,0)$ form on $X$. However, given a Kähler class $\omega$ on $X$, the Hilbert scheme does not come with a canonical choice of Kähler class. In fact, one has a natural isomorphism

$$H^{1,1}(X^{[k]}) \cong H^{1,1}(X) \oplus \mathbb{C} \cdot u$$  \hfill (5.3)
with \( u \) a class that is Poincaré dual to (twice) the exceptional divisor, the inverse image of the “small diagonal” in \( S^k X \) where at least two points coincide. (In the orbifold cohomology of \( S^k X \) this cohomology class is represented by the ground state of \( \mathbb{Z}_2 \) twisted sector.)

Generically, if \( X \) contains no holomorphic curves, a Kähler class \( \omega \) in \( X \) can be lifted to any combination \( \pi^* S^k \omega - \lambda u \) with \( \lambda > 0 \). There is no natural value for \( \lambda \). In the limit \( \lambda \to 0 \) the Hilbert scheme can be thought to degenerate to the symmetric product. So for \( \lambda = 0 \) we obtain an orbifold metric on the Hilbert scheme in which all the fibers of the projection (5.2) have zero volume. In this sense the symmetric product is a point in the space of HK structures (including orbifolds) on the Hilbert scheme.

There are various other natural constructions of simple HK manifolds. Most relevant are the smooth “instanton moduli spaces” \( \mathcal{M}_v \) of semi-stable torsion-free sheaves on tori or K3 manifolds that we considered before. Remarkably these spaces always turn out to be HK deformations of the Hilbert scheme. (Here there is a technical restriction that \( c_1 \) should be primitive \([59]\), and thus non-zero, that we will ignore.) More precisely, the moduli space \( \mathcal{M}_v \) occurs as a point in the moduli space of HK structures on the Hilbert scheme \( X[k] \) for \( k = \frac{1}{2} v^2 + 1 \). We will see later how the HK structure is precisely determined. (In the physics literature this fact is sometimes stated loosely as an equivalence between \( \mathcal{M}_v \) and a symmetric product.)

Note in this context that the Hilbert scheme itself can be considered as the special moduli space \( \mathcal{M}_v \) of rank one torsion-free sheaves with \( c_1 = 0 \) and \( c_2 = k \) and therefore with Mukai vector

\[
v = (1, 0, 1 - k). \tag{5.4}
\]

In particular, as differentiable manifolds all \( \mathcal{M}_v \) with equal \( v^2 \) (and \( v \) primitive) coincide. Although some of these spaces are birational (certainly not all of them) it is an open problem precisely which birational \( \mathcal{M}_v \) are isomorphic as HK manifolds.

Of course, with the usual identification between anti-self-dual connections and holomorphic bundles, the spaces \( \mathcal{M}_v \) can also be considered as instanton moduli spaces. If we just consider stable holomorphic vector bundles these moduli spaces would in general not be compact. By including coherent sheaves we obtain a natural, smooth compactification that is related, but not equivalent, to the inclusion of ‘point-like’ instantons. For the case of rank one, where no finite-size instantons exist, the torsion-free sheaves only represent point-like instantons and the complicated degenerations where these coincide.

As we discussed in \( \S 2 \), these instanton moduli spaces carry natural hyperkähler metrics induced by the metric on \( K3 \). It might be confusing why the choice of metric on \( K3 \) does not give rise to a unique metric on the moduli space \( \mathcal{M}_v \). The point is precisely that this metric has no canonical extension over the compactification. For example,

---

*I thank D. Huybrechts for particularly useful correspondence concerning this point.*
the Hilbert scheme is a natural compactification of the symmetric product minus the diagonals. The $L^2$-metric defines a metric on the symmetric product, but extending this metric to the Hilbert scheme must assign a volume to the various (symplectic) blow-ups and that introduces a free parameter $\lambda$. We will see in a moment that stringy data, in the form of the self-dual NS $B$-field, will fix these ambiguities for us.

There is an alternative interpretation of this issue in terms of hyperkähler quotients. If the instanton moduli space has a realization as a HK quotient such as the ADHM construction or the infinite-dimensional quotient discussed in §2, the values of the HK moment map, which is an imaginary quaternion, can be associated with the $B$-field on $X$. Remarkably, this $B$-field can also be given a fascinating interpretation as a deformation to a non-commutative manifold, as demonstrated for $\mathbb{R}^4$ in [60, 61].

As we mentioned, up to recently all known constructions of simple hyperkähler manifolds were deformation equivalent to Hilbert schemes. Only recently O’Grady has constructed a canonical desingularization of one of the singular moduli spaces $\mathcal{M}_v$ with $v$ not primitive, that is clearly not related to a Hilbert scheme [22]. It has $k = 5$ and $b_2 \geq 24$ (most likely equal to 24) whereas the Hilbert scheme has $b_2 = 23$. Note that from a physical point of view these singular spaces are moduli spaces of multiple BPS states bound at threshold, and therefore of great interest in studying D-brane dynamics. Also Verbitsky has constructed examples of HK manifolds that do not seem to be related to Hilbert schemes by considering sheaves on simple HK manifolds [63].

5.2. Deformations of hyperkähler manifolds

The local deformation theory of general hyperkähler manifolds and the instanton moduli spaces $\mathcal{M}_v$ in particular is quite well-developed. A good survey can be found in [18], see also [64, 65].

First of all, for any simple $4k$-dimensional HK manifold $Y$ there exists a canonical quadratic form on $H^2(Y, \mathbb{Z})$ constructed by Beauville, that generalizes the intersection form for K3 surfaces. It has rank $b = b_2$ and signature $(3, b - 3)$. Although the construction uses the complex structure and the associated holomorphic $(2, 0)$ form $\eta$, the final result turns out to be independent of any choices and purely topological. Using the Hodge decomposition

$$H^2(Y, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

Beauville’s quadratic form is defined for any $w \in H^2(Y)$ as [58]

$$w \cdot w = \int_Y (\eta \bar{\eta})^{k-1} \left( 2 w^{2,0} \wedge w^{0,2} + k w^{1,1} \wedge w^{1,1} \right).$$

Note that this reduces to the standard intersection form for $k = 1$. The normalization of $\eta$ can be chosen such as to make the quadratic form on $H^2(Y, \mathbb{Z})$ integral and primitive.
By construction the HK 3-plane $U \subset H^2(Y, \mathbb{R})$ obtained by the period map is positive with respect to this quadratic form. In fact, one can prove a local Torelli theorem and use Yau’s theorem to show that locally the moduli space of HK metrics of fixed volume on a general simple HK manifold $Y$ is given by the Grassmannian

$$\frac{O(3, b - 3)}{O(3) \times O(b - 3)}.$$  \hspace{1cm} (5.7)

generalizing the results for $K3$ and $T^4$ with $b = 22$ and 6 respectively.

This result has an obvious lift to the corresponding $\mathcal{N} = (4,4)$ superconformal field theory describing the sigma model on $Y$. After including the $B$-field and the volume, and using general arguments about the holonomy group of the Zamalodchikov metric with $(4,4)$ supersymmetry [56], we see that the moduli space of SCFT’s on the HK manifold $Y$ is locally a quaternionic symmetric space and therefore of the form

$$\frac{O(4, b - 2)}{O(4) \times O(b - 2)}. \hspace{1cm} (5.8)$$

It would be interesting to understand the meaning of the underlying lattice of signature $(4, b - 2)$ in this context. (We will explain the lattice in the particular case of the Mukai moduli spaces in the next section.)

There are however many questions about the global picture, both for the classical geometry and the SCFT: (1) does the moduli space cover everything (if one includes appropriate orbifolds), (2) what is the quotient group, and (3) is the map to the Grassmannian injective, i.e. can there be different HK structures that map to the same period in the above Grassmannian?

For the particular case of the Hilbert scheme $X^{[k]}$ with $k > 1$ of a $K3$ surface $X$ the second cohomology group has rank 23 and can be written as an extension of $H^2(X, \mathbb{Z}) \cong \Gamma_{3,19}$ by the exceptional divisor $u$ \[53\]

$$H^2(X^{[k]}, \mathbb{Z}) \cong \Gamma_{3,19} \oplus \mathbb{Z} \cdot u, \quad u^2 = 2 - 2k < 0. \hspace{1cm} (5.9)$$

Since the moduli spaces $\mathcal{M}_v$ are all HK deformations of the Hilbert scheme, we find quite generally the isomorphism

$$H^2(\mathcal{M}_v, \mathbb{Z}) \cong \Gamma_{3,20}^{3,20}. \hspace{1cm} (5.10)$$

Of course, in the general case there is no canonical choice of the vector $u$, as was the case for the Hilbert scheme. But, as we discussed in §2 all choices of such a vector $u$ are related by a $O(3, 20, \mathbb{Z})$ transformation.
We therefore find that locally the moduli space of HK structures on the Hilbert scheme or the Mukai spaces $\mathcal{M}_v$ is given by

$$\frac{O(3, 20)}{O(3) \times O(20)}.$$  \hspace{1cm} (5.11)

Here we already recognize the form of the moduli space of attractor $K3$ sigma models.

If $U$ is the positive 3-plane in $H^2(X^{[k]}, \mathbb{R})$ determined by the HK structure, then $U$ induces a polarization of the divisor $u$,

$$u = u_{L}^{3,0} + u_{R}^{0,20}. \hspace{1cm} (5.12)$$

Now the condition that the Hilbert scheme degenerates to the orbifold metric of the symmetric product can be expressed by the fact that $U$ is orthogonal to $u$, or equivalently

$$u_L = 0. \hspace{1cm} (5.13)$$

In heterotic CFT language we have a chiral vertex operator of weight $(0, k - 1)$. The appearance of this singularity is very similar to the $A_1$ type singularities in $K3$ associated to vanishing of $-2$ curves.

6. Attractor $K3$'s and instanton moduli spaces

We will now proceed to indicate more precisely how the moduli of the attractor $K3$ string compactification are related to the moduli of the corresponding D-brane moduli space $\mathcal{M}_v$. Hereto we make use of the following important result. Consider a primitive Mukai vector $v \in H^*(X, \mathbb{Z})$ with $v^2 > 0$. Then the Mukai map (2.18) restricted to $v^\perp$ gives an isomorphism of lattices\cite{17, 59}:

$$\mu : v^\perp \cong H^2(\mathcal{M}_v, \mathbb{Z}). \hspace{1cm} (6.1)$$

That is, we can canonically identify the restriction of the Mukai form to the orthocomplement of the RR charge vector $v$ with Beauville’s “intersection form” on the second cohomology of the D-brane moduli space $\mathcal{M}_v$. If $v$ is null, so that $\mathcal{M}_v$ itself is a $K3$ space,\footnote{In\cite{59} this isomorphism is only proved for the case $c_1$ primitive and therefore non-zero, but it is mentioned that this most likely also holds for the more general case, at least when semi-stability implies stability.}
surface, the identity is of the form $H^2(\mathcal{M}_v) \cong v^\perp/v$, an important ingredient in the Mukai-Nahm transform a.k.a. $T$-duality.

In fact, there is an isomorphism at the level of Hodge structures \cite{59} that can be written as

\begin{align}
H^{2,0}(\mathcal{M}_v) & \cong H^{2,0}(X) \\
H^{1,1}(\mathcal{M}_v) & \cong \left( H^0(X) \oplus H^{1,1}(X) \oplus H^4(X) \right) \cap v^\perp. \tag{6.2}
\end{align}

Stated otherwise, the positive two-plane $P$ in $H^2(\mathcal{M}_v, \mathbb{R})$ spanned by the real and imaginary components of the holomorphic $(2,0)$ form on $\mathcal{M}_v$, coincides with the similar plane in $H^*(X, \mathbb{R})$ spanned by the holomorphic $(2,0)$ form on $X$. So the complex structure of $\mathcal{M}_v$ is directly determined by the complex structure of the attractor $K3$ — a fact we already understood.

The missing ingredient is the matching of the Kähler forms. As we mentioned before, there is no a priori relation between the Kähler form on the $K3$ surface and the Kähler form on the moduli space. This is not a surprise, since a simple counting of deformation moduli tells us that the $B$-field on the $K3$ surface should contribute to the determination of the Kähler form on the moduli space.

With our detailed description of the moduli of the string compactification and the instanton moduli space this identification is now straightforward. Recall that for a given Mukai or RR charge vector $v$ the attractor moduli of the conformal field theory on $K3$ determine a positive 3-plane $U$ in $v^\perp \otimes \mathbb{R} \cong \mathbb{R}^{3,20}$. Since we can canonically identify $v^\perp \cong H^2(\mathcal{M}_v, \mathbb{Z})$ including the Hodge structures, we see that the 3-plane $U$ can also be identified with a three-plane in $H^2(\mathcal{M}_v, \mathbb{R})$. We claim $U$ is the image under the period map of the HK structure on $\mathcal{M}_v$. So we have established a completely canonical relation between the sigma model (including the $B$-field) on $K3$ and the classical hyperkähler geometry on $\mathcal{M}_v$.

Let us illustrate this discussion with the two concrete examples we studied before.

6.1. D0-D4 branes

The D0-D4 (or D1-D5) bound state system has charge vector $v = (r, 0, -p)$, $r, p > 0$ and coprime. In this case $v^\perp$ is spanned by $H^2(X, \mathbb{Z})$ and the vector $u = (r, 0, p)$, which is primitive and satisfies

\[ u \cdot v = 0, \quad u^2 = -v^2 = -2rp < 0. \tag{6.3} \]

When viewed as an element in $H^2(\mathcal{M}_v)$ the vector $u$ is of type $(1,1)$ and Poincaré dual to the exceptional divisor in the Hilbert scheme. So the “period” $\omega \cdot u$ of a Kähler form
ω along u determines to which extent the Hilbert scheme is deformed away from the symmetric product.

Now, according to example (1) of §4, the restricted deformation moduli in this case corresponded to a general HK structure on X together with a self-dual B-field and volume fixed at \( V = r/p - \frac{1}{2}B^2 \). One easily sees that for \( B = 0 \) the 3-plane \( U \subset v^\perp \otimes \mathbb{R} \cong \mathbb{R}^{3,20} \) lies entirely in \( H^2(X) \). Therefore we can draw the conclusion that for zero B-field (but general metric) on the K3 manifold, the metric on the instanton moduli space determined by the string theory compactification is of the orbifold type that corresponds to a symmetric product, justifying the earlier remarks in the string literature that the D0-D4 system is described by quantum mechanics on the symmetric product [47]. This orbifold quantum mechanics system should be properly understood as a limit of the corresponding SCFT that we will discuss in the next section.

For a general non-vanishing self-dual B-field the corresponding 3-plane is spanned by the three vectors
\[
\left(1, B + \frac{\bar{\omega} \cdot \omega}{\omega_a \cdot B} \omega_a, p/r\right)
\]
which all have a non-vanishing inner product with the exceptional divisor \( u \).

6.2. D2-branes

The D2-brane system has \( v = (0, q, 0) \). In this case the matching of moduli is even more involved. The lattice \( v^\perp \) can be written as the direct sum of \( \Gamma^{1,1} \cong H^0(X) \oplus H^4(X) \) and the orthocomplement \( q^\perp \) in \( H^2(X) \). The restricted moduli required \( q \) to be self-dual and \( q \cdot B = 0 \). The possible metrics on \( X \) are restricted to a \( 2 \times 19 + 1 \) dimensional family. To obtain the full set of \( 3 \times 20 \) deformations of \( M_v \) we now have to include all \( 2 + 19 \) components of the B-field. It would be interesting to understand this better from the point of the D2-brane gauge theory.

7. The D1-D5-brane system and instanton strings

The case of the six-dimensional strings obtained by wrapping odd D-branes in type IIB string theory compactified on a K3 surface is even more interesting, since we will now find a correspondence between on the one hand (non-perturbative) string theory on an attractor K3 and on the other hand a \( c = 6k \mathcal{N} = (4, 4) \) superconformal field theory.

7.1. Type IIB on K3

In the type IIB case the Ramond-Ramond fields are even-dimensional forms and take
value in $H^*(X,\mathbb{R}) \cong \mathbb{R}^{4,20}$. Together with the string coupling constant $g_s \in \mathbb{R}^+$ this gives a moduli space that is locally of the form

$$\frac{O(4,20)}{O(4) \times O(20)} \times \mathbb{R}^{4,20} \times \mathbb{R}_+.$$  

(7.1)

Supergravity arguments indicate that the full moduli space is actually the symmetric space

$$\frac{O(5,21)}{O(5) \times O(21)}.$$  

(7.2)

The full $U$-duality automorphism group is $O(\Gamma^{5,21})$, where $\Gamma^{5,21}$ is the even, self-dual lattice of signature $(5,21)$. It contains the perturbative $T$-duality subgroup $O(\Gamma^{4,20})$, that has an interpretation on the level of the sigma model, but it has also extra non-perturbative symmetries.

The occurrence of the type IIB moduli space and the lattice $\Gamma^{5,21}$ can be explained in terms of the spectrum of six-dimensional strings. The superstring contains besides fundamental strings also their magnetic duals, the Neveu-Schwarz 5-branes. Furthermore there are now odd-dimensional D-branes of dimension 1, 3, and 5. Every string or brane can be wrapped around an even-dimensional cycle of the $K3$ manifold to give a string in six dimensions. All in all this gives a rank 26 lattice of strings that is isomorphic to $\Gamma^{5,21}$. This lattice can be considered as the direct sum of the lattice $H^*(X,\mathbb{Z}) \cong \Gamma^{4,20}$ of RR charges and an extra copy of $H^0(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) \cong \Gamma^{1,1}$ labeling the fundamental strings and NS5-branes. The two copies of $\Gamma^{1,1}$ are permuted by type IIB $S$-duality, that interchanges strings with D1-branes and NS 5-branes with D5-branes (D3-branes are self-dual). In fact, it is an elegant result that the full $U$-duality group $O(\Gamma^{5,21})$ is generated by the $\mathbb{Z}_2$ $S$-duality together with the $T$-duality group $O(\Gamma^{4,20})$ [37].

There are again formulas that express how the standard sigma moduli of $K3$, combined into the quaternionic 2-form $b$, together with the Ramond-Ramond gauge fields

$$C = (\theta, \tilde{B}, G) \in H^*(X,\mathbb{R})$$  

(7.3)

and the string coupling $g_s \in \mathbb{R}_+$, determine a positive 5-plane $Z$ in $\mathbb{R}^{5,21}$. Writing the charge lattice in terms of Ramond and Neveu-Schwarz charges as $\Gamma^{5,21} = \Gamma^{4,20}_R \oplus \Gamma^{1,1}_{NS}$, the 5-plane is spanned by the vectors [37]

$$(\exp b, 0, C \cdot \exp b), \quad (C, 1, 1/g_s).$$  

(7.4)
A six-dimensional string with charge vector $v \in \Gamma_{5,21}$ can be described in the decoupling or near-horizon limit as a two-dimensional $\mathcal{N} = (4,4)$ superconformal field theory. All primitive vectors $v$ of equal length are equivalent under $U$-duality, and we can choose the vector $v$ to lie in the RR lattice $H^*(X,\mathbb{Z}) \cong \Gamma_{4,20}$. In that case the SCFT should be identified with the sigma model on the Mukai moduli space $\mathcal{M}_v$. The attractor moduli of the type IIB $K3$ compactifications should now be matched with the HK metric and $B$-field of $\mathcal{M}_v$.

We see directly that the local structure of the moduli spaces coincides. Picking a charge vector $v \in \Gamma_{5,21}$ with $v^2 = 2k - 2 > 0$ leads to an attractor moduli space that describes positive 4-planes $W$ in $v^\perp \otimes \mathbb{R} \cong \mathbb{R}^{4,21}$ and this is given by the Grassmannian

$$\frac{O(4,21)}{O(4) \times O(21)}. \quad (7.5)$$

There is a stabiliser $U$-duality subgroup $O(\Gamma_{4,21}^{4,21})$ with

$$\Gamma_{k}^{4,21} \cong \Gamma_{k}^{3,20} \oplus \Gamma_{1,1}^{1,1}. \quad (7.6)$$

If we have only D-brane charges, i.e. if the (primitive) charge vector $v$ is of the form

$$v = (Q_5, Q_3, -Q_1) \in H^*(X,\mathbb{Z}), \quad (7.7)$$

the attractor moduli are described completely analogously to example 4.2. The fixed point conditions read (with $\zeta \in \mathbb{H}$ a quaternion to be determined)

$$\Re(\zeta b) = v, \quad b \cdot C = 0. \quad (7.8)$$

These equations have an obvious solution. First of all the 4-plane $W \subset H^*(X,\mathbb{R})$ determined by the “quaternionic” Kähler form $b$ should contain the vector $v$. This condition is equivalent to the fact that the $K3$ sigma model is attractive for the D-brane charge vector $v$.

Secondly, the RR fields $C$ should satisfy

$$v \cdot C = Q_1 \theta + Q_3 \cdot \tilde{B} - Q_5 G = 0. \quad (7.9)$$

So the total flux of RR fields through the collection of D-branes should vanish. The RR fields therefore lie in $v^\perp \otimes \mathbb{R} \cong \mathbb{R}^{3,20}$ (modulo shifts). In this case the string coupling
constant $g_s$ is not fixed. This description corresponds to the decomposition of\((7.3)\) as

$$\frac{O(3,20)}{O(3) \times O(20)} \times R^{3,20} \times R_+ \quad (7.10)$$

In this form we immediately recognize the moduli of the CFT on $\mathcal{M}_v$. We already saw in the previous section how the sigma model moduli of $K3$ relate to the HK structure on $\mathcal{M}_v$ (the first factor). So this identification works just as well in the CFT. In particular, in the absence of NS $B$-fields, the HK metric is of the symmetric product orbifold form. Now the IIB string RR fields $\mathcal{C}$ can be related to the sigma model $B$-field, using the identification $H^2(\mathcal{M}_v) \cong \nu^\perp$ (the second factor). Finally, the inverse string coupling is identified with the volume or equivalently $\alpha'$ of the sigma model (the last factor).

These identifications can also be understood physically from the D-brane gauge theory point of view as we discussed in §2. We have already remarked that the string coupling plays the role of $\alpha'$ for the $c = 6k$ CFT, so that the weakly coupled IIB string regime coincides with sigma model perturbation theory of the instanton string. Note that in the absence of D3-branes and with $B$ and $\mathcal{C}$ set to zero the volume of the $K3$ surface is fixed to be $Q_1/Q_5$.

The fact that the RR field $\tilde{B}$ becomes a sigma model $B$-field on the instanton string is not surprising, since that string is essentially the D-string and $S$-duality tells us that the D-string couples to $\tilde{B}$ just as the fundamental string couples to $B$. Also, from the D5-brane action we have derived that the RR gauge fields produce two-form $B$-fields for the instanton string. More precisely, the RR background field $\tilde{B}$ induces the forms $\mathcal{B}_I$ of \((2.12)\) which can be identified with the usual cohomology classes of Donaldson theory, the descendents of $\text{Tr} F^2$. Similarly the $\theta$-angle produces an extra field of the form $\mathcal{B}_0$ \((2.13)\), a descendent of $\text{Tr} F^3$. Furthermore the Mukai map from the RR charge lattice to the second cohomology of the instanton moduli space is exactly induced by the D5-brane couplings.

We now want to understand in string theory terms the condition \((7.9)\) that the total RR flux through the D-branes should be zero, which forces a non-zero coupling to the 4-form field $\mathcal{C}$. One way to explain this constraint is that in the reduction from the six-dimensional SYM theory to the two-dimensional sigma model we also obtain a two-dimensional Yang-Mills field with curvature $f$. In general there can be a FI term coupling to the flux of the $U(1)$ curvature $\text{Tr} f$. Working through the reduction we see that this FI term is induced by the RR background fields as

$$\int_\Sigma (\nu \cdot \mathcal{C}) \, \text{Tr} f \quad (7.11)$$

So in order to avoid this term we have to choose the combination of $\mathcal{C}$ such that the total flux vanishes.
7.3. Global issues and u-duality

Let us finally comment on some of the global issues of the various moduli spaces that we have discussed so far. Let us begin with the map between the $K3$ CFT moduli and the classical HK geometry of the instanton moduli space $\mathcal{M}_v$. Starting from the $T$-duality of the $K3$ sigma model, one derives that the global moduli space of the perturbative D-brane system, that is obtained in the $g_s \rightarrow 0$ limit, is given by the Narain space

$$O(\Gamma^{3,20}_k)/O(3,20) \backslash O(3) \times O(20).$$

(7.12)

We have argued that this moduli space should correspond to the space of HK metrics of fixed (in fact, very large) volume on the (unmarked) Mukai moduli space $\mathcal{M}_v$. The space (7.12) is the classical HK period domain and it is known that the moduli space of HK structures maps finitely into this period domain. It is further believed that this map is generically injective [67]. String theory clearly seems to suggest that the map is surjective if one includes singular orbifold metrics. (We will momentarily return to the injectivity."

Similarly there is a statement for the D-brane conformal field theory, where $U$-duality now implies that the global form of the space of deformations is

$$O(\Gamma^{4,21}_k)/O(4,21) \backslash O(4) \times O(21).$$

(7.13)

Here the $U$-duality stabilizer group $O(v^\perp) = O(4,21,\mathbb{Z})$ has an interpretation as a $T$-duality of the $\mathcal{N} = (4,4)$ SCFT.

When we interpret this sigma model as a DLCQ of the six-dimensional little string theory on $K3 \times \mathbb{R}^{1,1}$, the four extra moduli compared to the usual 80 $K3$ moduli are interpreted as a (quaternionic) string coupling constant [3, 5]. (In matrix theory these same couplings correspond to the components $C_{\mu \nu -}$ of the background 3-form gauge field of 11-dimensional M-theory [3].) The group $O(4,21,\mathbb{Z})$ interchanges this string coupling with the geometric $K3$ moduli, and therefore can be considered as a “little $u$-duality” of the six-dimensional string theory. It is nice to see in this concrete example how intrinsic non-geometric objects like the string coupling are treated on equal footing with the geometric ones. It would be very interesting to find an “m-theory” interpretation of this duality [68].

There is however a more interesting point with possible mathematical implications. All D-branes with Mukai vectors of equal length are related by dualities. Therefore the corresponding SCFT’s should be isomorphic, after a possible shift in their moduli. We know that this particular component of the moduli space of $c = 6k \mathcal{N} = (4,4)$ SCFT’s takes the form (7.13) and we can find the particular point a certain D-brane configuration is mapped to by the period map that we discussed in detail.
At the level of classical HK structures the global structure of the moduli space is actually not known. Taking the classical, large volume limit of the SCFT moduli space, string theory seems to arrive at the \((7.12)\) as the moduli space of HK structures of fixed volume — a very strong statement, indicating a global Torelli theorem for the Hilbert scheme. In general it is not known if every two Mukai spaces with the same period point are isomorphic as HK manifolds. However, it is known that these spaces are birationally equivalent \([18]\). Furthermore, as soon as one deforms the complex structure away from the point where the spaces have an interpretation as moduli spaces of sheaves on \(K3\), the deformed spaces become isomorphic as HK manifolds. Therefore the moduli space of HK structures seems to be a priori non-Hausdorff with non-separated points.

Can we have a non-Hausdorff moduli space of SCFT’s? The general idea of conformal perturbation theory, where the neighbourhood of a CFT is parametrized by the exactly marginal operators, combined with the \(\mathcal{N} = (4,4)\) supersymmetry suggests that it is not the case. (See however \([39]\), although that example is of a somewhat different nature because of the time-like compactification.). It seems difficult to give a completely rigorous argument that conformal perturbation theory works in this case and that the moduli space of \(\mathcal{N} = (4,4)\) SCFT is Hausdorff, although everything in string theory and supergravity indicates that it is. In the work of Aspinwall and Morrison on \(K3\) compactifications \([39]\), which underlies much what has been said here, it was an input that the moduli space was Hausdorff. Therefore it is not a surprise that it also comes out in the form of the nice arithmetic quotients \((7.12)\) and \((7.13)\). Of course there are many examples of equivalent string compactifications on manifolds that are only birational \([70]\), and therefore an other possibility is the CFT moduli space is better behaved than the classical geometries. This point clearly deserves further study.

Acknowledgements

I would like to acknowledge useful discussions with S. Agnihotri, J. de Boer, L. Göttscbe, B. Fantechi, C. Hofman, M. Kontsevich, E. Looijenga, E. Martinec, G. Moore, J.-S. Park, E. Verlinde, H. Verlinde, and in particular D. Huybrechts.

References

[1] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, Phys. Lett. B379 (1996) 99–104, hep-th/9601029.

[2] R. Dijkgraaf, E. Verlinde and H. Verlinde, *BPS spectrum of the five-brane and black hole entropy*, Nucl. Phys. B486 (1997) 77–88, hep-th/9603121; *BPS quantization of the five-brane*, Nucl. Phys. B486 (1997) 89–113, hep-th/9604053.
[3] R. Dijkgraaf, E. Verlinde, and H. Verlinde, *5D Black holes and matrix strings*, Nucl. Phys. **B506** (1997) 121–142, hep-th/9704018.

[4] O. Aharony, M. Berkooz, S. Kachru, N. Seiberg, and E. Silverstein, *Matrix description of interacting theories in six dimensions*, Adv. Theor. Math. Phys. **1** (1998) 148–157, hep-th/9707079.

[5] E. Witten, *On the conformal field theory of the Higgs branch*, J. High Energy Phys. **07** (1997) 003, hep-th/9707093.

[6] O. Aharony, M. Berkooz, N. Seiberg, *Light-cone description of (2,0) superconformal theories in six dimensions*, Adv. Theor. Math. Phys. **2** (1998) 119–153, hep-th/9712117.

[7] R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Notes on matrix and micro strings*, hep-th/9709107.

[8] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, hep-th/9711200.

[9] A. Strominger, *Black hole entropy from near-horizon microstates*, hep-th/9712251.

[10] J. Maldacena and A. Strominger, *AdS$^3$ black holes and a stringy exclusion principle*, hep-th/9804083.

[11] E. Martinec, *Matrix models of AdS gravity*, hep-th/9804111; *Conformal field theory, geometry, and entropy*, hep-th/9809021.

[12] J. de Boer, *Six-dimensional supergravity on $S^3 \times AdS_3$ and 2d conformal field theory*, hep-th/9806104.

[13] T. Banks, M.R. Douglas, G.T. Horowitz, and E. Martinec, *AdS dynamics from conformal field theory*, hep-th/9808016.

[14] A. Giveon, D. Kutasov, and N. Seiberg, *Comments on string theory on AdS$_3$*, hep-th/9806194.

[15] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984) 101–116.

[16] S. Mukai, *On the moduli space of bundles on K3 surfaces I*, in M.F. Atiyah et al. Eds., *Vector Bundles on Algebraic Varieties* (Oxford, 1987).

[17] S. Mukai, *Moduli of vector bundles on K3 surfaces and symplectic manifolds*, Sugaku Expositions, **1/2** (1988) 139–174.
[18] D. Huybrechts, *Compact hyperkähler manifolds: basic results*, alg-geom/9705025.

[19] R. Dijkgraaf, G. Moore, E. Verlinde, and H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, Commun. Math. Phys. **185** (1997) 197–209, hep-th/9608096.

[20] S. Dostoglou and D. Salomon, *Self-dual instantons and holomorphic curves*, Ann. Math. (2) **139** (1994), 581–640.

[21] M. Bershadsky, A. Johansen, V. Sadov, and C. Vafa, *Topological reduction of 4D SYM to 2D σ-models*, Nucl. Phys. **B448** (1995) 166–186, hep-th/9501096.

[22] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, Phys. Rev. Lett. **75** (1995) 4724–4727, hep-th/9501017.

[23] N. Seiberg, *Matrix description of M-theory on T^5 and T^5/Z_2*, Phys. Lett. **B408** (1997) 98–104, hep-th/9705221.

[24] E. Witten, *Five-brane effective action in M-theory*, J. Geom. Phys. **22** (1997) 103–133, hep-th/9610234.

[25] R. Dijkgraaf, *The mathematics of fivebranes*, in *Proceedings of the ICM Berlin 1998*, Doc. Math. III (1998) 133–142, hep-th/9810157.

[26] S.K. Donaldson and R.P. Thomas, *Gauge theory in higher dimensions*, Oxford preprint 1996.

[27] R.P. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K3 fibrations*, math.AG/9806111.

[28] E. Witten, *Bound states of strings and p-branes*, Nucl. Phys. **B460** (1996) 335–350, hep-th/9510136.

[29] M. Li, *Boundary states of D-branes and Dy-strings*, Nucl. Phys. **B460** (1996) 351, hep-th/9510161.

[30] M. Douglas, *Branes within branes*, hep-th/9512077.

[31] M. Green, J. Harvey, and G. Moore, *I-brane inflow and anomalous couplings on D-branes*, Class. Quant. Grav. **14** (1997) 47–52, hep-th/9605033.

[32] J.A. Harvey and G. Moore, *On the algebra of BPS states*, Commun. Math. Phys. **197** (1998) 489–519, hep-th/9609017.
[33] N.J. Hitchin, A. Karlhede, U. Lindström, and M. Rocek, Hyperkähler metrics and supersymmetry, Commun. Math. Phys. 108 (1987) 535–589.

[34] H. Nakajima, Resolutions of moduli spaces of ideal instantons on $\mathbb{R}^4$, in Topology, Geometry and Field Theory, (World Scientific, 1994).

[35] S. Donaldson and P. Kronheimer, The Geometry of Four-Manifolds (Oxford, 1990).

[36] E. Witten, Topological quantum field theory, Commun. Math. Phys. 117 (1988) 353.

[37] P. Aspinwall, K3 surfaces and string duality, TASI 96 lectures, hep-th/9611137.

[38] N. Seiberg, Observations on the moduli space of superconformal field theories, Nucl. Phys. B303 (1988) 286–304.

[39] P.S. Aspinwall and D.R. Morrison, String theory on K3 surfaces, in B. Greene and S.-T. Yau Eds. Mirror Symmetry II (International Press, 1996), hep-th/9404151.

[40] G. Moore, Arithmetic and attractors, hep-th/9807087.

[41] R. E. Borcherds, Automorphic forms on $O_{s+2,2}(R)$ and infinite products, Invent. Math. 120 (1995) 161.

[42] J. Harvey and G. Moore, Algebras, BPS states, and strings, Nucl. Phys. B463 (1996) 315–368, hep-th/9510182.

[43] E. Looijenga, C. Peters, Torelli theorems for Kähler K3 surfaces, Comp. Math. 42 (1980/81), no. 2, 145–186.

[44] V.V. Nikulin, Integral symmetric bilinear forms and some of their applications, Math. Izvestija 14 (1980) 103–167.

[45] S. Ferrara, R. Kallosh, and A. Strominger, $N = 2$ Extremal black holes, Phys. Rev. D52 (1995) 5412–5416, hep-th/9508072.

[46] L. Andrianopoli, R. D’Auria, S. Ferrara, and M. Lledo, Horizon geometry, duality and fixed scalars in six dimensions, hep-th/9802147.

[47] C. Vafa, Gas of D-branes and Hagedorn density of BPS states, Nucl. Phys. B463 (1996) 415, hep-th/9511020. Instantons on D-branes, Nucl. Phys. B463 (1996) 435–442, hep-th/9512078.

[48] M. Bershadsky, V. Sadov, and C. Vafa, D-strings on D-manifolds, Nucl. Phys. B463 (1996) 398–414, hep-th/9510224.
[49] S.-T. Yau and E. Zaslow, BPS states, string duality, and nodal curves on K3, Nucl. Phys. B471 (1996) 503–512, [hep-th/9512121].

[50] M. Bershadsky, V. Sadov, and C. Vafa, D-Branes and topological field theories, Nucl. Phys. B463 (1996) 166, [hep-th/9511222].

[51] D.R. Morrison, The geometry underlying mirror symmetry, [alg-geom/9608006].

[52] D. Huybrechts and M. Lehn, The Geometry of Moduli Spaces of Sheaves, Aspects of Mathematics E31 (Vieweg Verlag, 1997).

[53] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, J. Diff. Geom. 18 (1983) 755–782.

[54] H. Nakajima, Lectures on Hilbert schemes of points on surfaces, Tokyo preprint 1996.

[55] L. Göttsche, The Betti numbers of the Hilbert scheme of points on a smooth projective surface, Math. Ann. 286 (1990) 193–207; Hilbert Schemes of Zero-dimensional Subschemes of Smooth Varieties, Lecture Notes in Mathematics 1572, Springer-Verlag, 1994.

[56] L. Göttsche and W. Soergel, Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces, Math. Ann. 296 (1993), 235–245.

[57] J. Cheah, On the cohomology of Hilbert schemes of points, J. Alg. Geom. 5 (1996), 479–511.

[58] F. Hirzebruch and T. Höfer, On the Euler number of an orbifold, Math. Ann. 286 (1990) 255.

[59] K. O’Grady, The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface, J. Alg. Geom. 6 (1997) 599–644, [alg-geom/9510001].

[60] N. Nekrasov and A. Schwarz, Instantons on noncommutative $\mathbb{R}^4$ and (2,0) superconformal six-dimensional theory, [hep-th/9802068].

[61] A. Astashkevich, N. Nekrasov, and A. Schwarz, On noncommutative Nahm transform, [hep-th/9810147].

[62] K.G. O’Grady, Desingularized moduli spaces of sheaves on a K3, I & II, [alg-geom/9705009, math.AG/9805099].

[63] M. Verbitsky, Hyperholomorphic sheaves and new examples of hyperkähler manifolds, [alg-geom/9712012].
[64] D. Huybrechts, *Birational symplectic manifolds and their deformations*, J. Diff. Geom. 45 (1997) 488–513, alg-geom/9601015.

[65] M. Verbitsky, *Cohomology of compact hyperkähler manifolds*, GAFA 6(4) (1996) 601–612, alg-geom/9403006.

[66] S. Cecotti, *N=2 Landau-Ginzburg vs. Calabi-Yau sigma model: nonperturbative aspects*, Int. J. Mod. Phys. A6 (1991) 1749–1814.

[67] D. Huybrechts, private communication.

[68] A. Losev, G. Moore, and S.L. Shatashvili, *M & m*, Nucl. Phys. B522 (1998) 105–124, hep-th/9707250.

[69] G. Moore, *Finite in all directions*, hep-th/9305139.

[70] P.S. Aspinwall, B.R. Greene, and D.R. Morrison, *Multiple mirror manifolds and topology change in string theory*, Phys. Lett. B303 (1993) 249–259, hep-th/9301043.