ON FIELD THEORETIC GENERALIZATIONS OF A POISSON ALGEBRA$^{*\dagger\ddagger}$

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(Submitted April 1997 —— Accepted July 1997)

A few generalizations of a Poisson algebra to field theory canonically formulated in terms of the polymomentum variables are discussed. A graded Poisson bracket on differential forms and an $(n+1)$-ary bracket on functions are considered. The Poisson bracket on differential forms gives rise to various generalizations of a Gerstenhaber algebra: the noncommutative (in the sense of Loday) and the higher-order (in the sense of the higher order graded Leibniz rule). The $(n+1)$-ary bracket fulfills the properties of the Nambu bracket including the “fundamental identity”, thus leading to the Nambu-Poisson algebra. We point out that in the field theory context the Nambu bracket with a covariant analogue of Hamilton’s function determines a joint evolution of several dynamical variables.

1 Introduction

The purpose of the present paper is to discuss generalizations of a Poisson algebra in field theory which appear within the so-called De Donder–Weyl (DW) canonical theory for fields, the essence of which is briefly recalled below (see also [4] and references therein). The DW theory is in a sense a manifestly space-time symmetric extension of the Hamiltonian formulation to field theory, which belongs to the Lagrangian class of canonical theories known from the calculus of variations (see [3]). All these formulations are referred to as polymomentum here, as their common feature is that variables, called polymomenta, similar to conjugate momenta, are associated to every space-time derivative of fields.

$^{*}$PACS classification: 03.50, 02.40
$^{\dagger}$AMS classification: 70 G 50, 58 F 05, 53 C 80
$^{\ddagger}$Keywords: Classical field theory, De Donder–Weyl theory, Hamiltonian formalism, polysymplectic form, multivector fields, differential forms, differential operators on the exterior algebra, Poisson bracket, Gerstenhaber algebra, graded Loday algebra, noncommutative Gerstenhaber algebra, Nambu bracket, fundamental identity, Schild’s string
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Our interest to the above generalizations is motivated by the question as to whether the DW formulation, or another polymomentum formulation, can provide us with a starting point for a certain “canonical quantization” scheme in field theory. Such a scheme would certainly be of interest because in accordance with the general features of polymomentum formulations it would not assume an explicit distinction between the space and time variables, as the standard Hamiltonian formalism does, and potentially may not suffer from the functional analytic difficulties related to the infinite dimensionality of the latter. A study of analogues of a Poisson algebra in the context of polymomentum formulations seems to be a natural preliminary step towards such a quantization.

The present discussion can be viewed as a continuation of our earlier reports (see [1, 2, 3]). Here we partially review our previous results and partially present some new ones. The latter include the Gerstenhaber algebra structure with respect to the “co-exterior product” in Sect. 2 and a discussion of the Nambu bracket in field theory in Sect. 4.

Let us start by recalling what the DW polymomentum formulation is. In field theory given by the Lagrangian density 

\[ L = L(y^a, \partial_i y^a, x^i) \]

where \( \{y^a\} \) are field variables, \( \{\partial_i y^a\} \) are their space-time derivatives and \( \{x^i\} \) are space-time coordinates, the set of variables 

\[ p^i_a := \partial L/\partial (\partial_i y^a) =: \text{polymomenta} \]

and the quantity 

\[ H := \partial_i y^a p^i_a - L =: \text{the DW Hamiltonian function} \]

can be introduced if 

\[ \det |\partial^2 L/\partial (\partial_i y^a) \partial (\partial_j y^b)| \neq 0. \]

This allows us to write the Euler–Lagrange field equations in the manifestly covariant first order form

\[ \partial_i y^a = \partial H/\partial p^i_a, \quad \partial_i p^i_a = -\partial H/\partial y^a \]

which can naturally be viewed as a multidimensional (or rather “multi-time”) generalization of Hamilton’s canonical equations. They are referred to as the DW Hamiltonian equations. The arena of field dynamics is now the finite dimensional polymomentum phase space of variables 

\[ z^M := (y^a, p^i_a, x^i) \]

and all the space-time variables enter on an equal footing as analogues of the time variable in mechanics. We have argued earlier [1, 2] that the proper analogue of Poisson brackets can be obtained from the following generalization of the symplectic form to the polymomentum phase space which we call the polysymplectic form:

\[ \Omega := dy^a \wedge dp^i_a \wedge \omega_i, \]

where \( \omega := dx^1 \wedge ... \wedge dx^n \) is the volume-form on the space-time manifold, \( \omega_i := \partial_i \mathcal{J} \omega \), and \( \mathcal{J} \) denotes the inner product of a (multi)vector with a form. A possible intrinsic meaning of \( \Omega \) as a representative of a certain equivalence class is discussed in [3].

In the following the variables \( z^v = (y^a, p^i_a) \) and \( x^i \), as well as the corresponding subspaces, are referred to as vertical and horizontal respectively. We also use the notation 

\[ \partial_{M_1} ... M_p := \partial_{M_1} \wedge ... \wedge \partial_{M_p}. \]

### 2 Hamiltonian forms

The usual definition of the Poisson bracket using the symplectic form can be generalized to the DW polymomentum formulation in field theory as follows (see [1, 2] for
more details). Let us consider the map of horizontal forms of degree $p$, $p = 0, ..., n-1$

$$F^p := \frac{1}{(n-p)!} F_{i_1 \cdots i_{n-p}}(z) \partial_{i_1 \cdots i_{n-p}} \omega,$$

to vertical multivectors of degree $(n-p)$,

$$X^{n-p} = \frac{1}{(n-p)!} \left[ X^{a_{i_1 \cdots i_{n-p-1}}} \partial_{a_{i_1 \cdots i_{n-p-1}}} + X^{a_{i_1 \cdots i_{n-p-1}}} \partial_{a_{i_1 \cdots i_{n-p-1}}} + \cdots \right]$$

which is given by

$$X^p \Omega = d^V F^p. \quad (2.1)$$

Here a multivector of degree $q$ is called vertical if it annihilates any horizontal $q$-form, whence the component expression above follows; the higher vertical components of the multivector are omitted as they play no role here. The vertical exterior differential, $d^V$, of a $p$-form is given in components by

$$d^V F^p = \frac{1}{(n-p)!} \partial_v F_{i_1 \cdots i_{n-p}} dz^v \wedge \partial_{i_1 \cdots i_{n-p}} \omega.$$

The horizontal forms (and vertical multivectors) for which the map above exists are called Hamiltonian. The map (2.1) is equivalent to the following relations between the components of a $p$-form $F^p$ and those of the associated $(n-p)$-multivector $X^p$:

$$(n-p) X^{a_{i_1 \cdots i_{n-p-1}}} = \partial_a F^{a_{i_1 \cdots i_{n-p-1}}}, \quad (2.2)$$

$$-(n-p) X^{a_{i_1 \cdots i_{n-p-1}}} \delta^i_j = \partial^a F^{a_{i_1 \cdots i_{n-p-1}}}. \quad (2.3)$$

The latter relation obviously imposes a restriction on forms fulfilling eq. (2.1). Namely, an analysis of its consistency conditions shows that Hamiltonian forms are restricted to the specific polynomials of polymomenta with coefficients depending on the space-time and field variables, i.e. (for $p \neq 0, n$)

$$F^p = \frac{1}{(n-p)!} \sum_{k=0}^{n-p} \sum_{i_1 < \cdots < i_k} \partial_{a_{i_1} \cdots a_{i_k}} f^{a_{i_1} \cdots a_{i_k} i_{k+1} \cdots i_{n-p}}(x^i, y^a) \partial_{i_1 \cdots i_{n-p}} \omega. \quad (2.4)$$

The Poisson bracket of two Hamiltonian forms is now defined as follows:

$$\{ F_1, F_2 \} := (-1)^{n-p} X_{F_1} \omega \Omega. \quad (2.5)$$

It is easy to show that the bracket in (2.5) fulfills the axioms of a graded Lie algebra: graded anticommutativity

$$\{ F_1, F_2 \} = -(-1)^{g_1 g_2} \{ F_2, F_1 \}, \quad (2.6)$$

and the graded Jacobi identity

$$(-1)^{g_1 g_2} \{ F_1, \{ F_2, F_3 \} \} + (-1)^{g_1 g_2} \{ F_2, \{ F_3, F_1 \} \} + (-1)^{g_2 g_3} \{ F_3, \{ F_1, F_2 \} \} = 0, \quad (2.7)$$
where \( g_1 = n - p - 1 \), \( g_2 = n - q - 1 \), \( g_3 = n - r - 1 \) are degrees of the corresponding forms with respect to the bracket operation.

Further, from eq. (2.4) it is clear that the space of Hamiltonian forms is not closed with respect to the exterior product. However, another graded commutative associative product of horizontal forms can be constructed with respect to which the space of Hamiltonian forms is closed. This product operation \( \bullet \) (let us call it the co-exterior product) is given by the formula

\[
F \bullet G := \ast^{-1}(\ast F \wedge \ast G).
\]  

(2.8)

Clearly, \( \deg(pF \bullet qF) = p + q - n \) and \( pF \bullet qF = (-1)^{(n-p)(n-q)}qF \bullet pF \). Note that only the volume \( n \)-form \( \omega \) is needed in order to define the co-exterior product, not the metric structure. Now, a remarkable fact is that the Poisson bracket defined in (2.5) fulfills the graded Leibniz rule with respect to the co-exterior product

\[
\{ [pF, qF \bullet rF] \} = \{ [pF, qF] \} \bullet rF + (-1)^{(n-q)(n-p-1)}qF \bullet \{ [pF, rF] \}.
\]  

(2.9)

The proof is based on a straightforward componentwise calculation.

Eqs. (2.6), (2.7) and (2.9) lead us to the conclusion that the space of Hamiltonian forms is a Gerstenhaber algebra (see e.g. [4]) with respect to the graded bracket operation \([\cdot, \cdot]\) and the co-exterior product \( \bullet \). An extension of the bracket and the corresponding algebraic structure to arbitrary horizontal forms is discussed in Sect. 3.

Note that the bracket above allows us to write the equations of motion in Poisson bracket formulation (see [1]). For example, the DW Hamiltonian equations (1.1) take the following form:

\[
\ast^{-1} d(y^a \omega_i) = \{ [H, y^a \omega_i] \}, \quad \ast^{-1} d(p^i_a \omega_i) = \{ [H, p^i_a \omega_i] \},
\]  

(2.10)

where \( d \) is the total exterior differential, such that \( d(F^i \omega_i) := \partial_v F^i \frac{\partial x^j}{\partial v_j}dx^j \wedge \omega_i \). Note also that \((n - 1)\) forms \( p^i_a \omega_i \) are canonically conjugate to field variables \( y^a \) in the sense that \( \{ [p^i_a \omega_i, y^b] \} = \delta^b_a \).

3 Non-Hamiltonian forms and a noncommutative Gerstenhaber algebra

Horizontal forms which cannot be mapped to vertical multivectors are called non-Hamiltonian. These forms, however, can be mapped to more general graded differential operators acting on the exterior algebra of forms. Multivectors are just a particular case of the latter. In general, graded differential operators are represented by multivector valued forms. In fact, the map

\[
\tilde{X}_F \downarrow \Omega = d^V \tilde{F}
\]  

(3.1)

always exists if \( \tilde{X}_F \) is taken to be a vertical multivector valued (horizontal) one-form:

\[
\tilde{X} = X^{v_1 ... v_{n-p} k}dx^k \otimes \partial_{v_1 ... v_{n-p}}.
\]

The “interior product” \( \downarrow \) in (3.1) means the substitution of \( \tilde{X} \) into the form \( \Omega \), that is the inner product with the multivector part of \( \tilde{X} \) is supposed to act first and then the
exterior product with the covector part follows. Obviously the degree of the (operation of the substitution of the) operator $\hat{X}$ above is $-(n-p)$.

As before, the bracket of two forms can be defined as follows

$$\{[\hat{F}_1, \hat{F}_2]\} := (-1)^{n-p} \hat{X}_1 \, d^p \hat{F}_2.$$  

(3.2)

However, now the bracket above will lack the graded anticommutativity for the graded commutator of operators $\hat{X}_1$ and $\hat{X}_2$ does not vanish unless both are representable by multivectors. Nevertheless, we still arrive here at a very interesting algebraic structure which is related to a noncommutative generalization of Lie algebra introduced by Loday under the name of Leibniz algebra (we use the name “Loday algebra” instead).

Note first that all operators $\hat{X}$ for which eq. (3.1) is fulfilled obey the relation $L_{\hat{X}} \Omega = 0$ (i.e. they are “locally Hamiltonian” in a sense), where $L_{\hat{X}} := [\hat{X}, d^V] := \hat{X} \circ d^V - (-1)^{|\hat{X}|} d^V \circ \hat{X}$ is a generalized Lie derivative; $|\hat{X}|$ denotes the degree of the graded operator $\hat{X}$ and $[\; , \; ]$ is a graded commutator. For any two graded operators a differential bracket operation (an analogue of the Lie bracket of vector fields) can be defined

$$[\hat{X}_1, \hat{X}_2] := [L_{\hat{X}_1}, \hat{X}_2] = L_{\hat{X}_1} \circ \hat{X}_2 - (-1)^{|\hat{X}_2||\hat{X}_1|+1} \hat{X}_2 \circ L_{\hat{X}_1}.$$  

(3.3)

Then it becomes obvious that the bracket in (3.2) is in fact induced by the differential bracket of graded operators associated with forms according to (3.1), i.e.

$$[\hat{X}_F, \hat{X}_G] \, d^V \Omega = -d^V \{[F, G]\}.$$  

(3.4)

Now, the following identities for the bracket (3.2) can be proved:

(i) the left graded Loday identity:

$$\{\{[F, G], K\}\} = \{[F, \{G, K\}\} - (-1)^{(n-F-1)(n-G-1)} \{G, \{F, K\}\}$$  

(3.5)

and (ii) the right graded Leibniz rule:

$$\{[F \land G, K]\} = F \land \{[G, K]\} + (-1)^{G(n-K-1)} \{[F, K]\} \land G.$$  

(3.6)

Proof: Here we omit tildes over graded operators. Capital letters in the exponents of the minus signs denote the exterior degree of the corresponding forms. We also use some easy to reveal properties of the operations introduced above.

First let us prove (3.5):

$$\{\{[F, G], K\}\} = (-1)^{n-(F+G-n-1)} L_{[F, G]} K = -(-1)^{n-(F+G-n+1)} L_{[F, G]} K$$

$$= (-1)^{2n-F-G} [X_F, X_G] \, d^V K = (-1)^{2n-F-G} [L_{X_F}, X_G] d^V K$$

$$= (-1)^{2n-F-G} (L_{X_F} X_G \, d^V K + (-1)^{(n-G)(n-F-1)} X_G \, d^V L_{X_F})$$

$$= \{[F, \{G, K\}]\} - (-1)^{(n-F-1)(n-G-1)} \{[G, \{F, K\}]\}.$$  

(3.7)

In order to prove (3.6) let us construct the operator associated with the exterior product of two forms $F$ and $G$. It obeys

$$X_{F \land G} \Omega = d^V (F \land G) = (-1)^{G(F+1)} G \land d^V F + (-1)^{F} F \land d^V G$$

$$= \{(-1)^{G(F+1)} G \circ X_F + (-1)^{F} F \circ X_G\} \land \Omega,$$
so that

$$X_{F \wedge G} = (-1)^p F \circ X_G + (-1)^{G(F+1)} G \circ X_F$$  \hspace{1cm} (3.8)

whence (3.6) follows. q.e.d.

It is interesting to note that the graded anticommutativity of the bracket is replaced now by a weaker condition which is a consequence of (3.5)

$$\{\{[F,G], K\} = -(1)^{(|F|+1)(|G|+1)} \{\{[G,F], K\}.$$  \hspace{1cm} (3.9)

The structure which appeared here generalizes the known structure of a Gerstenhaber algebra [4]. Namely, the axioms of graded anticommutativity and graded Jacobi identity are weakened to the (left) graded Loday identity, eq. (3.5), and the graded derivation property is valid only in the sense of the right Leibniz rule. The corresponding structure can naturally be called a noncommutative (right) Gerstenhaber algebra.

Let us consider now an analogue of the left Leibniz rule with respect to the exterior product. One cannot expect the graded Leibniz rule to be fulfilled here because multivector valued forms are not graded derivations but rather higher-order graded differential operators on the exterior algebra which are composed from derivations given, according to the Frölicher-Nijenhuis theorem [7], by vectors and vector valued forms. Consequently, an analogue of the Leibniz rule for higher-order differential operators will appear here. This property is similar to the “second order Leibniz rule” for the operator of second derivative

$$(abc)^n = (ab)^n c + (ac)^n b + (bc)^n a - a^n bc - ab^n c - abc^n.$$  \hspace{1cm} (3.10)

To formulate the higher-order graded analogue of the property above let us recall Koszul’s characterization of higher-order graded differential operators on (graded) commutative algebras [8]. Given an operator $D$ on the algebra $\Lambda^*$ one can construct a set of $r$-linear maps associated with it, $\Phi^r_D : \otimes^r \Lambda^* \rightarrow \Lambda^*$, given by

$$\Phi^r_D(F_1, ..., F_r) := m \circ (D \otimes 1) \lambda^r(F_1 \otimes ... \otimes F_r)$$  \hspace{1cm} (3.11)

for all $F_1, ..., F_r$ in $\Lambda^*$. Here $m$ is the multiplication map in $\Lambda^*$, $m(F_1 \otimes F_2) := F_1 \wedge F_2$, and $\lambda^r$ is a linear map $\otimes^r \Lambda^* \rightarrow \Lambda^* \otimes \Lambda^*$ given in terms of the map $\lambda : \Lambda^* \rightarrow \Lambda^* \otimes \Lambda^*$:

$$\lambda(F) := F \otimes 1 - 1 \otimes F$$ as follows: $\lambda^r(F_1 \otimes ... \otimes F_r) := \lambda(F_1) \wedge ... \wedge \lambda(F_r)$. The graded differential operator $D$ is said to be of $r$-th order iff $\Phi^{r+1}_D = 0$ identically. For example, the identity (3.10) can compactly be written as $\Phi^n_3(a, b, c) = 0$.

Now, the higher-order Leibniz rule for the (left) bracket with a $p$-form $F$ can be written as follows:

$$\Phi^n_{\{F, \ldots \}}(F_1, ..., F_{n-p+1}) = 0.$$  \hspace{1cm} (3.12)

The simplest non-trivial generalization corresponds to $p = (n-2)$. In this case the following (left) second-order graded Leibniz rule is obtained (cf. (3.10))

$$\{\{\{F, F \wedge F \wedge F\} = \{\{\{F, F \wedge F\} \wedge F + (1)^{(q+r+s)} \{\{F, F \wedge F\} \wedge \hat{F}$$

$$+ (1)^{s(r+s)} \{\{F, F \wedge F\} \wedge F - \{\{F, F\} \wedge \hat{F} \wedge \hat{F} + (1)^{s(r+s)} \{\{F, F\} \wedge \hat{F} \wedge \hat{F} \}$$

$$- (1)^{q(r+s)} \{\{F, F\} \wedge \hat{F} \wedge \hat{F} - (1)^{s(q+r)} \{\{F, \hat{F}\} \wedge \hat{F} \wedge \hat{F}.$$  \hspace{1cm} (3.13)
Thus, in addition to the structure of a noncommutative Gerstenhaber algebra we have here a higher-order generalization of the left graded Leibniz rule. This is another feature of field theory in the polymomentum formulation: the Poisson bracket can act as a higher order (algebraic) differential operator on the algebra of dynamical variables (here, the exterior algebra of forms), not only as a first order differentiation like in mechanics. The structure given by the replacement of the graded derivation property of the bracket by the higher-order (left) graded Leibniz rule, eq. (3.12), can be naturally referred to as a higher-order (left) Gerstenhaber algebra.

4 The Nambu-type bracket

In this section we show that the polysymplectic form can also be used for defining the Nambu-type analogue of the Poisson bracket. Here we only outline the idea in the particular case of field theory in two dimensions, so that the polysymplectic form $\Omega$ is a three-form now. We can choose either the form (1.2) corresponding to the DW canonical theory, or another appropriate non-degenerate closed three-form (see e.g. eq. (4.9) below) which will thus correspond to a certain canonical theory from the Lepagean class of theories (cf. e.g. [8]).

The polysymplectic form maps a function $F = F(y, p, x)$ of the polymomentum phase space variables to a bivector field $X_F$

$$X_F \mathcal{J} \Omega = dF.$$  

(4.1)

Then the bracket of three functions can be defined as

$$\{F, G, K\} := X_F \mathcal{J} (dG \wedge dK).$$

(4.2)

This bracket is antisymmetric in all three arguments and satisfies the Leibniz rule

$$\{F, G, K \cdot L\} = \{F, G, K\} \cdot L + \{F, G, L\} \cdot K.$$  

(4.3)

Let us consider the analogue of the Jacobi identity for the bracket above. Note first that the Nambu bracket (4.2) can be related to the binary bracket of a function with a one-form (cf. Sect. 2)

$$\{F, G\} = X_F \mathcal{J} (dG \wedge dK) = X_F \mathcal{J} d(GdK) = \{[F, GdK]\}.$$  

(4.4)

Then the Jacobi identity for the binary bracket, which is easy to prove, can be used for deriving the analogue of the Jacobi identity for the Nambu bracket:

$$\{\{F, G, H\}K, L\} = \{[F, GdH], KdL]\} = \{[F, GdH], KdL]\} = \{[F, GdH], KdL]\},$$

(4.5)

where the bracket of two one-forms $GdH$ and $KdL$ is given by (cf. Sect. 3)

$$\{[GdH, KdL]\} := -X_{GdH} \mathcal{J} d(KdL) = -L_{X_{GdH}} (KdL)$$

where the bracket of two one-forms $GdH$ and $KdL$ is given by (cf. Sect. 3)
and $X_{GdH} \omega = d(GdH)$. Now, let us express the first term in (4.5) in terms of the brackets of functions with one-forms:

$$\{ [F, \{GdH, KdL\}] \} = X_F \omega (dK \wedge dL)$$

$$= X_F \omega (dGdH, KdL) - \{ [F, \{GdH, K\}] dL \} - \{ [F, \{GdH, L\}] dK \}.$$

Rewriting the result in terms of the Nambu bracket according to (4.4) and using the antisymmetry properties of the latter we obtain

$$\{ [G, H, F, K, L] + \{ F, [G, H, K], L \} + \{ F, K, [G, H, L] \} = \{ [G, H, F, K, L] \}.$$ (4.6)

The identity above is known as the “fundamental identity” for the Nambu bracket which plays the role of an analogue of the Jacobi identity for the latter. It shows that the Nambu bracket introduced above endows the space of functions on the polymomentum phase space with the structure of what could be called a Nambu-Poisson algebra.

In order to write the equations of motion of dynamical variables in terms of the Nambu bracket let us first observe that given an appropriate polymomentum Hamiltonian function $H$ we can associate with it the “Hamiltonian flow” which is a distribution of two-planes given by the equation

$$\partial(z^M, z^N) / \partial(y^1, y^2) = X^N_H(y, p, x).$$ (4.7)

In the particular case when $H$ is chosen to be the DW Hamiltonian function and $\Omega$ is given by (1.2) this equation reproduces the De Donder–Weyl Hamiltonian form of field equations, eq. (1.1). From (4.7) it follows that the bracket with $H$ determines joint equations of motion of two dynamical variables:

$$\frac{d(F, G)}{d(x^1, x^2)} = \{ H, F, G \} + \frac{\partial(F, G)}{\partial(x^1, x^2)},$$ (4.8)

where the “total Jacobian” is introduced:

$$\frac{d(F, G)}{d(x^1, x^2)} := \frac{1}{2} \frac{\partial(F, G)}{\partial(z^M, z^N)} \frac{\partial(z^M, z^N)}{\partial(x^1, x^2)}.$$ (4.9)

As we have mentioned above Nambu brackets can be defined within the polymomentum canonical theories different from the De Donder–Weyl theory. All these theories are known to follow from the general Lepage framework in the calculus of variations of multiple integral problems (see e.g. [8]). The difference between them results after all in a different polysymplectic form and a different definition of the polymomenta and the analogue of the canonical Hamilton’s function $H$.

For instance, in the case of the Schild (or Nambu-Goto) string a canonical theory can be developed which is based on the following analogue of the polysymplectic form (cf. [11], see also [12])

$$\Omega_C := dp_ab \wedge dy^a \wedge dy^b.$$ (4.9)
Here, new polymomenta $p_{ab}$ are given by $p_{ab} := \partial L/\partial v^{ab}$, where the jacobian $v^{ab} := \partial(y^a, y^b)/\partial(x^1, x^2)$ naturally plays the role of a two-dimensional analogue of “generalized velocities”. The analogue of the Hamiltonian function, $H_C$, which is now a function of new polymomentum phase space variables $(y^a, p_{ab}, x^i)$, is naturally defined as follows: $H_C = p_{ab}v^{ab} - L$. For the Schild string with $L = \frac{1}{2}v^{ab}v^{ab}$ (for the seek of simplicity we put the string tension $T = 1$) we obtain $H_C = \frac{1}{2}p_{ab}p^{ab}$. The form $\Omega_C$ maps functions to bivectors and gives rise to the Nambu bracket of three functions as in (4.2). The equations of motion written in terms of the Nambu bracket assume the form

$$\frac{d(y^a, y^b)}{d(x^1, x^2)} = \{H_C, y^a, y^b\} = p^{ab}, \quad \frac{d(p_{ab}, y^b)}{d(x^1, x^2)} = \{H_C, p_{ab}, y^b\} = 0. \quad (4.10)$$

It is easy to show that these equations are equivalent to the equations of motion of the Schild string (see e.g. [12]).

In general, in $n$-dimensional field theory an $(n+1)$-ary Nambu bracket can be defined. In mechanics ($n = 1$) this bracket reduces to the usual binary Poisson bracket. In this sense the Nambu bracket is a generalization of the Poisson bracket to field theory (within the polymomentum formulations). In this case, the Nambu bracket with a properly defined polymomentum analogue of Hamilton’s canonical function determines a joint evolution of $n$ dynamical variables, contrary to the common point of view in generalized Nambu mechanics in which the evolution of a dynamical variable is supposed to be given by the bracket with several “Hamiltonians” (see e.g. [3, 10]).

5 Conclusion

When trying to extend the notion of the Poisson bracket to the polymomentum formulations in field theory, we immediately face the problem related to the asymmetry between the number of field variables and the polymomenta. One way out is to define a bracket on differential forms instead of functions. Another way is to consider an $(n+1)$-ary bracket of the Nambu type instead of a binary bracket. In these two approaches the role of dynamical variables in field theory is played by forms and functions, respectively (not functionals as in the standard Hamiltonian formalism). Here we have considered the algebraic structures which appear within both of these approaches. We have shown that on a certain subspace of forms, called Hamiltonian, the bracket of forms gives rise to a Gerstenhaber algebra, whereas on arbitrary horizontal forms its generalizations – noncommutative (in the sense of Loday) and higher-order (in the sense of the higher order Leibniz property of the bracket) – appear. On the other hand, the $(n+1)$-ary bracket of functions fulfills all the properties of the Nambu bracket including the famous fundamental identity [10], thus leading to a Nambu-Poisson algebra. All the abovementioned structures reduce to the familiar Poisson algebra when $n = 1$, that is in mechanics. In this sense they all are generalizations of a Poisson algebra to field theory. An interesting problem for future research would be the development of a quantization scheme in field theory based on the structures discussed above.
Acknowledgements. I would like to thank B. De Witt and G. Sardanashvily for useful discussions at the Bialowieża’96 Workshop. I am also grateful to C. Roger for drawing my attention to Loday’s paper [5].

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