Summary. We present a possible generalization of the exterior differential calculus, based on the operator $d$ such that $d^3 = 0$, but $d^2 \neq 0$. The entities $dx^i$ and $d^2 x^k$ generate an associative algebra; we shall suppose that the products $dx^i dx^k$ are independent of $dx^k dx^i$, while the ternary products will satisfy the relation: $dx^i dx^k dx^m = j dx^k dx^m dx^i = j^2 dx^m dx^i dx^k$, complemented by the relation $dx^i d^2 x^k = j d^2 x^k dx^i$, with $j := e^{2\pi i/3}$.

We shall attribute grade 1 to the differentials $dx^i$ and grade 2 to the "second differentials" $d^2 x^k$; under the associative multiplication law the grades add up modulo 3.

We show how the notion of covariant derivation can be generalized with a 1-form $A$ so that $D\Phi := d\Phi + A\Phi$, and we give the expression in local coordinates of the curvature 3-form defined as $\Omega := d^2 A + d(A^2) + AdA + A^3$.

Finally, the introduction of notions of a scalar product and integration of the $Z_3$-graded exterior forms enables us to define variational principle and to derive the differential equations satisfied by the 3-form $\Omega$. The lagrangian obtained in this way contains the invariants of the ordinary gauge field tensor $F_{ik}$ and its covariant derivatives $D_i F_{km}$. 
1. INTRODUCTION

The models of fundamental interactions and field theories based on the non-commutative geometry have been the object of intense studies in past few years (cf. refs. [1] to [6]). Most of the effort has been concentrated on reproducing the Weinberg-Salam unified theory of electroweak interactions by means of generalized gauge theories developed in non-commutative geometries instead of fibre bundles, and with finite algebra of matrices replacing the infinite algebra of functions on the manifold.

In most developed variants of this approach, the $Z_2$-graded matrix algebras have been used, whose $Z_2$-graded internal differential (defined as a graded commutator with a matrix whose square was equal to 1) was combined with the usual exterior differential forming a $Z_2$-graded Grassmann algebra. In the tensor products of these two algebras, i.e. in the algebra of matrix-valued exterior forms, the $Z_2$-grades of the matrices were added (modulo 2) to the $Z_2$-grades of the exterior forms, defining the $Z_2$-grade of the whole object (cf. refs. [5] and [6]).

Here we would like to generalize this scheme to the case of $Z_3$-grading. The resulting matrix algebra can be represented in the simplest case as the algebra of $3 \times 3$ matrices with entries from an associative algebra over complex numbers, on which a $Z_3$-graded commutator replaces the usual commutation and anti-commutational rules for the $Z_2$-graded $2 \times 2$-matrices.

This leads naturally to the differential $d$ whose square $d^2$ is different from 0, but whose cube does vanish identically, $d^3 = 0$. We show how such matrix algebra appears naturally as the algebra of linear transformations of a $Z_3$-graded generalization of the Grassmann algebra. We also introduce a $Z_3$-graded generalization of exterior algebra of differential forms and show how the notions of connection and curvature can be generalized, too, and how the lagrangians containing the invariants of the curvature and its derivatives can be constructed.

Finally, we briefly discuss the resulting variational principle for the gauge fields, leading to invariant equations of fourth order. Similar theories with even higher order derivatives of the curvature and lagrangians depending on their invariants can be obtained with a similar scheme based on $Z_N$-graded differentials satisfying $d^N = 0$.

2. $Z_3$-GRADED ANALOG OF GRASSMANN ALGEBRA.

The cyclic group $Z_3$ can be represented in the complex plane by means of the cubic roots of 1: let $j := e^{\frac{2\pi i}{3}}$; then one has $j^3 = 1$ and $j + j^2 + 1 = 0$; obviously, $j^n = j^{(n+3)}$.

By analogy with the $Z_2$-graded Grassmann algebras spanned by the set of anti-commuting generators, we may introduce an associative algebra spanned by $N$ generators $\theta^A$, $A, B = 1, 2...N$, whose binary products $\theta^A\theta^B$ will be considered as $N^2$ independent quantities, whereas we shall impose a ternary analog of the
anti-commutation relations:

$$\theta^A \theta^B \theta^C = j \theta^B \theta^C \theta^A = j^2 \theta^C \theta^A \theta^B$$

(1)

A more precise formulation is to say that the algebra in question is the universal algebra defined by the above relations.

**Corollary**: The cube of any generator must vanish (because in this case the relation (1) amounts to $(\theta^A)^3 = j(\theta^A)^3 = 0$; all the monomials of order 4 or higher are identically null (the proof that follows makes use of the associativity of the postulated product and of the relation 1): (the low braces are there just to indicate to which triple of $\theta$’s the circular permutation is being applied)

$$\theta^A \theta^B \theta^C \theta^D = j \theta^B \theta^C \theta^A \theta^D = j^2 \theta^D \theta^A \theta^B \theta^C = \theta^A \theta^B \theta^C \theta^D = j \theta^A \theta^B \theta^C \theta^D,$$

(2)

therefore, as $1 - j \neq 0$, one has $\theta^A \theta^B \theta^C \theta^D = 0$.

The dimension of this $Z_3$-graded generalization of Grassmann algebra is equal to $N + N^2 + (N^3 - N)/3$; we may also add a ”neutral” element denoted by 1 and commuting with all other generators.

One can note a dissymmetry between the components of this algebra with the grades 1 et 2: as a matter of fact, there are $N$ elements of grade 1, (the $\theta$’s) and $N^2$ elements of grade 2 ($\theta \theta$).

A natural way to re-establish the symmetry is to introduce the set of $N$ “conjugate” generators, $\bar{\theta}^A$, of grade 2, that would satisfy conjugate ternary relations (in which $j$ is replaced by $j^2$):

$$\bar{\theta}^A \bar{\theta}^B \bar{\theta}^C = j^2 \bar{\theta}^B \bar{\theta}^C \bar{\theta}^A$$

(3)

The ternary relation between the $\theta^A$’s can be interpreted as follows:

$$\theta^A \theta^B \theta^C = j \theta^B \theta^C \theta^A$$

which suggests the following relations between the generators $\theta^A$ and $\bar{\theta}^B$:

$$\theta^A \bar{\theta}^B = j \bar{\theta}^B \theta^A, \quad \bar{\theta}^B \theta^A = j^2 \theta^A \bar{\theta}^B.$$  

(4)

The $Z_3$-graded algebra so defined can be naturally divided in three parts, of grade 0, 1 et 2 respectively, with the dimensions of the sub-spaces of grades 1 and 2 being equal: one can write symbolically $A = A_0 + A_1 + A_2$, where

$A_0$ contains: 1, $\theta^A \bar{\theta}^B, \bar{\theta}^A \theta^B \theta^C, \bar{\theta}^A \bar{\theta}^B \bar{\theta}^C, \theta^A \theta^B \bar{\theta}^C \bar{\theta}^D$ and $\theta^A \bar{\theta}^B \theta^C \bar{\theta}^D \bar{E} \bar{F}$;

$A_1$ contains: $\theta^A, \bar{\theta}^B \theta^C, \theta^A \theta^B \bar{\theta}^C, \theta^A \bar{\theta}^B \bar{\theta}^C,$

and $A_2$ contains: $\bar{\theta}^A, \theta^A \theta^B, \bar{\theta}^A \bar{\theta}^B \bar{\theta}^C, \theta^A \theta^B \theta^C \bar{\theta}^D$.

In the case of usual $Z_2$-graded Grassmann algebras the anti-commutation between the generators of the algebra and the assumed associativity imply automatically the fact that *all* grade 0 elements *commute* with the rest of the algebra, while *any two* elements of grade 1 *anti-commute*.

In the case of the $Z_3$-graded generalization such an extension of ternary and binary relations does not follow automatically, and must be imposed explicitly. If
we decide to extend these relations to all elements of the algebra having a well-defined grade (i.e. the monomials in $\theta$'s and $\bar{\theta}$'s, then many additional expressions must vanish, e.g.:

$$\theta A \theta B \bar{\theta} C = \theta B \bar{\theta} C \theta A = \theta B \bar{\theta} C \theta A = \bar{\theta} C \theta A \theta B = 0;$$

because on the one side, $\theta A \bar{\theta} C$ is of grade 0 and commutes with all other elements; at the same time, commuting $\bar{\theta} C$ with $\theta A \theta B$ one gets twice the factor $j^2$, which leads to the overall factor $j^2 \bar{\theta} C \theta A \theta B$; this produces a contradiction which can be solved only by supposing that $\theta A \theta B \bar{\theta} C = 0$.

The resulting $Z_3$-graded algebra contains only the following products of generators:

$$A_1 = \theta, \{\bar{\theta} \theta\}; \quad A_2 = \bar{\theta}, \{\theta \theta\}; \quad A_0 = \{\theta \bar{\theta}\}, \{\theta \theta \theta\} \quad (5)$$

Let us note that the set of grade 0 (which obviously forms a sub-algebra of the $Z_3$-graded Grassmann algebra) contains the products which could symbolize the only observable combinations of quark fields in quantum chromodynamics based on the $SU(3)$-symmetry.

If we align the basis of our algebra, with all the elements of grade 0 first, next all the elements of grade 1 and finally the elements of grade 2 in a one-column vector, a general linear transformation that would leave these entries in the same order can be symbolized by a matrix whose entries have a definite $Z_3$-grade placed as follows:

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad (6)$$

Note that the position of the three grades did not change in the resulting column; we shall call such an operator a grade 0 matrix. We can introduce two other kinds of matrices that raise all the grades by 1 (resp. by 2), and call them respectively grade 1 and grade 2 matrices, as follows:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad (7)$$

(the numbers 0, 1, 2 symbolize the grades of the respective entries in the matrices).

If we restrict the character of the matrices, admitting only complex-valued matrix elements, then the grades 0, 1 and 2 will reduce themselves to the following three types of $3 \times 3$-block matrices:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \gamma \\ \alpha & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix} \quad (8)$$

representing arbitrary matrices with respective grade 0, 1 and 2.

It is easy to check that these grades add up modulo 3 under the associative
matrix multiplication law.

Let $B,C$ denote two matrices whose grades are $\text{grad}(A) = a$ and $\text{grad}(B) = b$, respectively. We can define the $Z_3$-graded commutator $[B,C]$ as follows:

$$[B,C]_{Z_3} := BC - j^{bc}CB$$

(Note that this $Z_3$-graded commutator does not satisfy the Jacobi identity).

Let $\eta$ be a matrix of grade 0; we can choose for the sake of simplicity

$$\eta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

With the help of the matrix $\eta$ we can define a formal "differential" on the $Z_3$-graded algebra of 3 matrices as follows:

$$dB := [\eta, B]_{Z_3} = \eta B - j^b B \eta$$

It is easy to show that $d(BC) = (dB)C + j^b B (dC)$ and that $d^3 = 0$. The first identity is trivial, whereas the last one follows from the fact that $\eta^3 = Id$ does commute with all the elements of the algebra.

A formal algebraic analogue of connection and curvature forms have been discussed elsewhere (cf. ref. [9] and [10]). Here we shall show how such a $Z_3$-graded exterior differential may be realized on a differential manifold. Some interesting ideas concerning similar techniques may be found in ref.[11].

3. $Z_3$-GRADED EXTERIOR DIFFERENTIAL.

Let $M_n$ be a differentiable manifold of dimension $N$, with local coordinates $x^k$. The variables $x^k$ commute and with respect to the $Z_3$-grading are of grade 0. Our aim now is to define such a linear operation acting on functions of the coordinates $x^k$, and by extension, on a larger algebra of forms that still remains to be defined, that would reproduce the properties of the "algebraic differential" introduced above. To do this, we have to define a linear operation $d$ which acts on the functions of $x^k$; for the definition, it is enough to define its action on the coordinates $x^k$ and on their products.

We postulate that the linear operator $d$ applied to $x^k$ produces a 1-form whose $Z_3$-grade is 1 by definition; when applied two times by iteration, it will produce a new entity, which we shall call a 1-form of grade 2, denoted by $d^2 x^k$. Finally, we require that $d^3 = 0$.

Let $F(M)$ denote the algebra of functions $C^\infty(M)$, over which the $Z_3$-graded algebra generated by the forms $dx^i$ and $d^2 x^k$ behaves as a left module. In other words, we shall be able to multiply the forms $dx^i$, $d^2 x^k$, $dx^i dx^k$, etc. by the functions on the left only; right multiplication will just not be considered here. That is why we will write by definition, e.g.

$$d(x^i x^k) := x^i dx^k + x^k dx^i$$
We shall also assume the following Leibniz rule for the operator $d$ with respect to the multiplication of the $\mathbb{Z}_3$-graded forms: when $d$ crosses a form of grade $p$, and of arbitrary rank, the factor $j^p$ appears as follows:

$$d(\omega \phi) = (d\omega)\phi + j^p \omega (d\phi)$$ \hspace{1cm} (13)

Let us note that in contrast with the $\mathbb{Z}_2$-graded case, the forms are treated as one whole, even when multiplied from the left by an arbitrary function; that means that we can not identify e.g. $(\omega_i dx^i)(\phi_k dx^k)$ with $(\omega_i \phi_k) dx^i dx^k$

This is equivalent with saying that the products of functions by the forms are to be understood in the sense of tensor products, which is associative, but non-commutative.

Nevertheless, we shall show later that such an identification can be done for the forms of maximal degree (i.e. 3), which contain the products of the type $dx^i dx^k dx^m$ or $dx^i d^2 x^m$, whose exterior differentials vanish irrespective of the order of the multiplication; as a matter of fact, it will be easy to show that with the rules of differentiation that we expose below,

$$d((\alpha_i dx^i)(\beta_k dx^k)(\gamma_m dx^m)) = d((\alpha_i \beta_k \gamma_m) dx^i dx^k dx^m)) = 0. \hspace{1cm} (14)$$

With the so established $\mathbb{Z}_3$-graded Leibniz rule, the postulate $d^3 = 0$ suggests in an almost unique way the ternary and binary commutation rules for the differentials $dx^i$ and $d^2 x^k$. To begin with, consider the differentials of a function of the coordinates $x^k$, with the “first differential” $df$ coinciding with the usual one:

$$df := (\partial_i f) dx^i \hspace{1cm}; \hspace{1cm} d^2 f := (\partial_k \partial_i f) dx^k dx^i + (\partial_i f) d^2 x^i \hspace{1cm};$$

$$d^3 f = (\partial_m \partial_k \partial_i f) dx^m dx^k dx^i + (\partial_k \partial_i f) d^2 x^k dx^i + (\partial_i f) d^3 x^i + j(\partial_k \partial_i f) dx^k d^2 x^i; \hspace{1cm} (15)$$

(we remind that the last part of the differential, $(\partial_i f) d^3 x^i$, vanishes by virtue of the postulate $d^3 x^i = 0$).

Supposing that the partial derivatives commute, exchanging the summation indices $i$ et $k$ in the last expression and replacing $1 + j$ by $-j^2$, we arrive at the following two conditions that lead to the vanishing of $d^3 f$:

$$dx^m dx^k dx^i + dx^k dx^i dx^m + dx^i dx^m dx^k = 0 \hspace{1cm}; \hspace{1cm} d^2 x^k dx^i - j^2 dx^i d^2 x^k = 0. \hspace{1cm} (16)$$

which lead in turn to the following choice of relations:

$$dx^i dx^k dx^m = j dx^k dx^m dx^i, \hspace{1cm} \text{and} \hspace{1cm} dx^i d^2 x^k = j d^2 x^k dx^i. \hspace{1cm} (17)$$

By extending these rules to all the expressions with a well-defined grade, and applying the associativity of the $\mathbb{Z}_3$-exterior product, we see that all the expressions of the type $dx^i dx^k dx^m dx^n$ and $dx^i dx^k d^2 x^m$ must vanish, and along with them, also the monomials of higher order that would contain them as factors.
Still, this is not sufficient in order to satisfy the rule $d^3 = 0$ on all the forms spanned by the generators $dx^1$ and $d^2x^k$. It can be proved without much pain that the expressions containing $d^2x^i d^2x^k$ must vanish, too. For example, if we take the particular 1-form $x^i dx^k$ and and apply to it the operator $d$, we get

$$d(x^i dx^k) = dx^i dx^k + x^i d^2x^k; \quad (19)$$

$$d^2(x^i dx^k) = d^2x^i dx^k + (1 + j)dx^i d^2x^k = d^2x^i dx^k - d^2x^k dx^i; \quad (20)$$

which leads to $d^3(x^i dx^k) = d^2x^i d^2x^k - d^2x^k d^2x^i$; then, if we want to keep both the associativity of the "exterior product" and the ternary rule for the entities of grade 2, i.e. $d^2x^i d^2x^k d^2x^m = j^2 d^2x^k d^2x^m d^2x^i$, then the only solution is to impose $d^2x^i d^2x^k = 0$ and to set forward the additional rule declaring that any expression containing four or more operators $d$ must vanish identically.

With this set of rules we can check that $d^3 = 0$ on all the forms, whatever their grade or degree. Let us show how such calculus works on the example of a 1-form $\omega = \omega_k dx^k$:

$$d(\omega_k dx^k) = (\partial_i \omega_k) dx^i dx^k + \omega_k d^2x^k; \quad (21)$$

$$d^2(\omega_k dx^k) = (\partial_m \partial_i \omega_k) dx^m dx^i dx^k + (\partial_i \omega_k)(d^2x^i dx^k + jdx^i d^2x^k) + \partial_i \omega_k dx^i d^2x^k; \quad (22)$$

after exchanging the summation indices $i$ and $k$ in two last terms and using the fact that $j + 1 = -j^2$ and the commutation relations between $dx^k$ and $d^2x^i$, we can write

$$d^2(\omega_k dx^k) = (\partial_m \partial_i \omega_k) dx^m dx^i dx^k + (\partial_i \omega_k - \partial_k \omega_i) d^2x^i dx^k. \quad (23)$$

where it is interesting to note how the usual anti-symmetric exterior differential appears as a part of the whole expression.

It is also easy to check that

$$\text{Im}(d) \subseteq \text{Ker}(d^2), \quad \text{and} \quad \text{Im}(d^2) \subseteq \text{Ker}(d) \quad (24)$$

4. COVARIANT DIFFERENTIAL, CONNECTION AND CURVATURE.

Let $A$ be an associative algebra with unit element, and let $H$ be a free left module over this algebra. Let $A$ be a $A$-valued 1-form defined on a differential manifold $M$, and let $\Phi$ be a function on the manifold $M$ with values in the module $H$.

We shall introduce the covariant differential as usual:

$$D \Phi := d \Phi + A \Phi; \quad (25)$$

If the module is a free one, any of its elements $\Phi$ can be represented by an appropriate element of the algebra acting on a fixed element of $H$, so that one can
always write $\Phi = B\Phi_o$; then the action of the group of automorphisms of $H$ can be translated as the action of the same group on the algebra $A$.

Let $U$ be a function defined on $M$ with its values in the group of the automorphisms of $H$. The definition of a covariant differential is equivalent with the requirement $DU^{-1}B = U^{-1}DB$; as in the usual case, this leads to the following well-known transformation for the connection 1-form $A$:

$$A \Rightarrow U^{-1}AU + U^{-1}dU;$$  \hspace{1cm} (26)

But here, unlike in the usual theory, the second covariant differential $D^2\Phi$ is not an automorphism: as a matter of fact, we have:

$$D^2\Phi = d(d\Phi + A\Phi) + A(d\Phi + A\Phi) = d^2\Phi + dA\Phi + jAd\Phi + Ad\Phi + A^2\Phi;$$  \hspace{1cm} (27)

the expression containing $d^2\Phi$ and $d\Phi$; whereas $D^3\Phi$ is an automorphism indeed, because it contains only $\Phi$ multiplied on the left by an algebra-valued 3-form:

$$D^3\Phi = d(D^2\Phi) + A(D^2\Phi),$$  \hspace{1cm} (28)

which gives explicitly:

$$d(d^2\Phi + dA\Phi + jAd\Phi + A^2\Phi) + A(d^2\Phi + dA\Phi + jAd\Phi + Ad\Phi + A^2\Phi)$$  \hspace{1cm} (29)

With a direct calculus one observes that all the terms containing $d\Phi$ or $d^2\Phi$ simplify because of the identity $1 + j + j^2 = 0$, leaving only:

$$D^3\Phi = (d^2A + d(A^2) + AdA + A^3)\Phi = (D^2A)\Phi := \Omega\Phi;$$  \hspace{1cm} (30)

Obviously, because $D(U^{-1}\Phi) = U^{-1}(D\Phi)$, one also has:

$$D^3(U^{-1}\Phi) = U^{-1}(D^3\Phi) = U^{-1}\Omega\Phi = U^{-1}\OmegaUU^{-1}\Phi,$$

which proves that the 3-form $\Omega$ transforms as usual, $\Omega \Rightarrow U^{-1}\OmegaUU$ when the connection 1-form transforms according to the law: $A \Rightarrow U^{-1}AU + U^{-1}dU$.

It can be also proved by a direct calculus that the curvature 3-form $\Omega$ does vanish identically for $A = U^{-1}dU$. This computation illustrates very well the technique of the $Z_3$-graded exterior differential calculus introduced above: as a matter of fact, one has

$$d(U^{-1}dU) = dU^{-1}dU + U^{-1}d^2U;$$  \hspace{1cm} (31)

so that the term corresponding to $d^2A$ gives:

$$d^2(U^{-1}dU) = d^2U^{-1}dU + jdU^{-1}d^2U + dU^{-1}d^2U;$$  \hspace{1cm} (32)

next, the term corresponding to $d(A^2) = d(U^{-1}dUdUU^{-1}dU)$ gives

$$dU^{-1}dUU^{-1}dU + U^{-1}dUdUU^{-1}dU + jU^{-1}dUdUU^{-1}dU + jU^{-1}dUU^{-1}d^2U;$$  \hspace{1cm} (33)
whereas

\[ AdA = U^{-1}dUdU^{-1}dU + U^{-1}dUU^{-1}d^2U; \]  

(34)

and finally, the term corresponding to \( A^3 = U^{-1}dUU^{-1}dUU^{-1}dU \) can be written as \(-dUU^{-1}dUU^{-1}dU\) by virtue of the identity \( dUU^{-1} = -UdU^{-1} \) which follows from the Leibniz rule applied to \(UU^{-1} = 1\), i.e. \( d(UU^{-1}) = dUU^{-1} + UdU^{-1} = 0\).

Using this identity whenever possible, and replacing \(1 + j\) by \(-j^2\), we can reduce the whole expression to the following sum of three terms

\[ d^2U^{-1}dU + U^{-1}d^2UU^{-1}dU - j^2U^{-1}dUdU^{-1}dU \]  

(35)

whose vanishing does not at all seem obvious.

However, it is not very difficult to prove that this expression is identically null. First of all, it is enough to prove the vanishing of the expression

\[ d^2U^{-1} + U^{-1}d^2UU^{-1} - j^2U^{-1}dUdU^{-1}, \]

because all the three terms contain the same factor \(dU\) on the right; then, by multiplying on the left by \(U\), we get

\[ Ud^2U^{-1} + d^2UU^{-1} - j^2dUdU^{-1} \]  

(36)

At this point let us note that \(d^2(UU^{-1}) = d^2(Id) = 0\), but then, according to our \(Z_3\)-graded Leibniz rule,

\[ d^2(UU^{-1}) = d(dUU^{-1} + UdU^{-1}) = d^2UU^{-1} + jdUdU^{-1} + UdU^{-1} + Ud^2U^{-1}; \]  

(37)

so that \(Ud^2U^{-1} + d^2UU^{-1} = -djUdU^{-1} - j^2dUdU^{-1}\), and the result can be written as

\[-djUdU^{-1} - j^2dUdU^{-1} = 0 \]  

(38)

because here again the common factor \(-djUdU^{-1}\) is multiplied by \((1 + j + j^2) = 0\), which completes the proof.

5. EXPRESSIONS IN LOCAL COORDINATES.

The curvature 3-form \(\Omega = d^2A + d(A^2) + AdA + A^3\) is of grade 0; therefore it must be decomposed along the elements \(dx^idx^kdx^m\) and \(d^2x^idx^k\). Here is how we can compute its components in a local coordinate system. By definition, \(A = A_i dx^i\), so we have:

\[ dA = \partial_i A_k dx^i dx^k + A_k d^2x^k; \]  

(39)

\[ d^2A = \partial_m \partial_i A_k dx^m dx^i dx^k + \partial_i A_k d^2x^i dx^k + j\partial_i A_k dx^i d^2x^k + \partial_i A_k dx^i d^2x^k; \]  

(40)

After replacing \(1 + j\) by \(-j^2\), and taking into account the relation \(dx^k d^2x^i = jd^2x^i dx^k\), we get:

\[ d^2A = (\partial_m \partial_i A_k)dx^mdx^i dx^k + (\partial_i A_k - \partial_k A_i) d^2x^i dx^k; \]  

(41)
Then, \( d(A^2) + AdA = dAA + jAdA + AdA = dAA - j^2AdA \), \( (42) \)
which leads easily to
\[
(\partial_i A_k A_m) dx^i dx^k dx^m - j^2 (A_m \partial_i A_k) dx^m dx^k dx^i + A_k A_m d^2 x^k dx^m - j^2 A_m A_k dx^m d^2 x^k;
\]
and due to the relations \( dx^m d^2 x^k = j d^2 x^k dx^m \) et \( dx^m dx^i dx^k = jdx^i dx^k dx^m \),
\[
d(A^2) + AdA = (A_m \partial_i A_k - \partial_i A_k A_m) dx^m dx^i dx^k + (A_k A_m - A_m A_k) d^2 x^k dx^m. \tag{44}
\]

Finally, as \( A^3 = A_i A_k A_m dx^i dx^k dx^m \), the curvature 3-form can be written in
local coordinates as follows:
\[
\Omega = d^2 A + d(A^2) + AdA + A^3 = \Omega_{ikm} dx^i dx^k dx^m + F_{ik} d^2 x^i dx^k \tag{45}
\]
where \( \Omega_{ikm} := \partial_i \partial_k A_m + A_i \partial_k A_m - \partial_k A_m A_i + A_i A_k A_m \), \( (46) \)
and \( F_{ik} := \partial_i A_k - \partial_k A_i + A_i A_k - A_k A_i \); \( (47) \)

In \( F_{ik} \) one can easily recognize the 2-form of curvature of the usual gauge theories.

We know that the expression \( F_{ik} \) is covariant with respect to the gauge transformations; on the other hand, the 3-form \( \Omega \) is also covariant; therefore, the local expression \( \Omega_{ijk} \) must be covariant, too. As a matter of fact, it can be expressed as a combination of covariant derivatives of the 2-form \( F_{ik} \).

In order to find the covariant expression of \( \Omega_{ikm} \), it suffices to recall that due to the particular symmetry of the ternary exterior product \( dx^i dx^k dx^m \), we can replace \( \Omega_{ikm} \) by \( \frac{1}{3}(\Omega_{ikm} + j^2 \Omega_{kmi} + j \Omega_{mik}) \) and analyze the abelian case, when this expression reduces itself to \( \Omega_{ikm} = \partial_i \partial_k A_m \).

Substituting for \( \partial_i \partial_k A_m \) the equivalent expression \( \frac{1}{3}(\partial_i \partial_k A_m + j^2 \partial_k \partial_m A_i + j \partial_m \partial_i A_k) \)
and then \( \frac{1}{3}(j(\partial_k \partial_m A_i - \partial_i \partial_m A_k) + j(\partial_m \partial_i A_k - \partial_i \partial_k A_m)) \), because \( 1 = -j - j^2 \),
we can easily recognize \( \frac{1}{3}(j\partial_i[\partial_m A_k - \partial_k A_m] + j^2 \partial_k[\partial_m A_i - \partial_i A_k]) \);
which in a general non-abelian case must lead to the following expression:
\[
\Omega_{ikm} = \frac{1}{3}[j D_i F{mn} + j^2 D_k F{mi}], \tag{48}
\]
or, equivalently,
\[
\Omega_{ikm} = -\frac{1}{6}[D_i F_{mn} + D_k F_{mi}] + \frac{j\sqrt{3}}{6}[D_i F_{mn} - D_k F_{mi}] \tag{49}
\]

6. DUALITY, INTEGRATION, VARIATIONAL PRINCIPLE.

The natural symmetry between \( j \) et \( j^2 \), which leads to the possibility of choosing one of these two complex numbers as the generator of the group \( \mathbb{Z}_3 \), and
simultaneous interchanging the rôles between the grades 1 and 2, suggests that we could extend the notion of complex conjugation \( j \Rightarrow (j)^* := j^2 \), with \(((j)^*)^* = j\), to the algebra of \( \mathbb{Z}_3 \)-graded exterior forms and the operator \( d \) itself.

It does not seem reasonable to use the "second differentials" \( d^2x^i \) as the objects conjugate to the "first differentials" \( dx^i \), because the rules of \( \mathbb{Z}_3 \)-graded exterior differentiation we have imposed break the symmetry between these two kinds of differentials: remember that the products \( dx^i dx^k \) and \( dx^i dx^k dx^m \) are admitted, while we require that \( d^2x^i d^2x^k \) and \( d^2x^i d^2x^k d^2x^m \) must vanish.

This suggests the introduction of a "conjugate" differential \( \delta \) of grade 2, the image of the differential \( d \) under the conjugation \( \ast \), satisfying the following conjugate relations:

\[
\delta x^i \delta x^k \delta x^m = j^2 \delta x^k \delta x^m \delta x^i, \quad \delta x^i \delta^2 x^k = j^2 \delta^2 x^k \delta x^i.
\] (50)

One notes that \( \delta^2 x^k \) is of grade 1 \((2+2 = 4 = 1 \pmod{3})\).

All the relations existing between the operator \( d \) and the exterior forms generated by \( dx^i \) and \( d^2x^k \) are faithfully reproduced under the conjugation \( \ast \) if we consider the \( \mathbb{Z}_3 \)-graded algebra generated by the entities \( \delta x^i \) and \( \delta^2 x^k \) as a right module over the algebra of functions \( F(M) \), with the operator \( \delta \) acting on the right on this module.

The rules \( d^3 = 0 \) and \( \delta^3 = 0 \) suggest their natural extension:

\[
d\delta = \delta d = 0 \tag{51}\]

We would like to be able to form quadratic expressions that could define a scalar product; to do this, we should postulate that the algebra generated by the elements \( dx^i, d^2x^k \) and its conjugate algebra generated by the elements \( \delta x^i, \delta x^k \) commute with each other.

Then, we can define scalar products for the forms of maximal degree 3: \( \langle \omega | \phi \rangle := \ast \omega \phi \), and integrating this result with respect to the usual volume element defined on the manifold \( M \), which gives explicitly:

\[
\int \bar{\omega}_{ikm} \phi_{prs} < \delta x^i \delta x^k \delta x^m | dx^p dx^r dx^s >, \int \bar{\psi}_{ik} \chi_{mn} < \delta x^i \delta^2 x^k | d^2x^m dx^n > \tag{52}\]

What remains now is to determine the scalar products of the basis of forms; in order to assure the hermiticity of the product, one can always choose an "orthonormal" basis in which we should have:

\[
< \delta x^i \delta x^k \delta x^m | dx^p dx^r dx^s >= \delta^m_p \delta^k_i \delta^l_r \text{ et } < \delta x^i \delta^2 x^k | d^2x^m dx^n >= \mu \delta^m_i \delta^k_l. \tag{53}\]

Here the first scalar product is normed to 1, and \( \mu \) is the ratio between the two types of "elementary volume".

We shall consider the two types of forms of degree 3, \( dx^i dx^k dx^m \) and \( d^2x^p dx^r \), as being mutually orthogonal.

Because in both cases the differentials of the volume forms are identically null, we can formally apply Stokes’ formula with a vanishing contribution of the
"surface term", which enables us to derive the classical Euler-Lagrange equations. With the Lagrangean density defined on the manifold as $\langle \Omega | \Omega \rangle$ we get:

$$L = \left[ \frac{4}{3} D_i F_{mk} D^i F^{mk} - \frac{2}{3} D_k F_{mi} D^i F^{mk} \right] + \mu [4 F_{ik} F^{ik}]$$  

(54)

(We suppose here that the manifold $M$ is flat, and we raise and lower the indices by means of the metric tensor $\delta_{ik}$ and its inverse $\delta^{kl}$).

The corresponding Euler-Lagrange are then easily deduced:

$$D^m D^i D_i F_{mk} - D^i D^k D_m F_{ik} + \frac{3\mu}{4} (D^i F_{ik}) = 0$$  

(55)

In the abelian case, and with the usual choice of gauge ($\partial^i A_i = 0$), these equations reduce themselves to:

$$\triangle \triangle A_k + \frac{3\mu}{4} \triangle A_k = 0$$  

(56)

It is worthwhile to note that this scheme can be readily generalized to a $Z_N$-graded case, with an exterior differential operator satisfying $d^N = 0$, leading to the invariants built with the higher-order covariant derivatives of the usual curvature tensor, leading to the equations containing higher powers of the Laplace-Beltrami operator.

The equations of this type have been studied before ([L.Vainerman, (1974)]) In quantum field theory, they appear in the perturbative developments of non-linear effective Lagrangeans (the so-called Schwinger terms). In such expansions, the consecutive powers of the Laplacian (or the d’Alembertian) operator appear, with the coefficients depending on the particular choice of Lagrangean. It would be interesting to compare the coefficients appearing in these theories with the coefficients depending on the choice of normalization of our various "volume elements" of higher orders, which appear in the above model.

Nevertheless, from the mathematical point of view it seems that a theory based on the $Z_3$-grading is particularly interesting because of its exceptional character related to the fact that the group $S_3$ is the last of the permutation groups that has a faithful representation in complex plane, which could lead to some special applications in physics, e.g. a better description of the quark fields (cf.[R.Kerner, 1992], [B. Le Roy, 1995], and [ V.Abramov, R.Kerner, B.Le Roy (1995)].).

Another possible application of this formalism may be a new description of differential operators on Riemannian manifolds. Consider a linear operation $d$ that produces a symmetric 2-form from a given 1-form,

$$(d\xi)_{ij} = \partial_i \xi_j + \partial_j \xi_i$$

The complex thus includes 1-forms and symmetric 2-forms (metrics). Now, let $d$ of a metric be defined as its Christoffel connection, and $d$ of the latter as Riemannian curvature. Then we have $d^2 \neq 0$, but $d^3 = 0$.

The applications to gauge theories with well defined non-abelian gauge groups will be the object of our forthcoming articles.
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