An analogue of Schur functions for the plane partitions

A. Morozov

ITEP & IITP, Moscow, Russia

ABSTRACT

An attempt is described to extend the notion of Schur functions from Young diagrams to plane partitions. The suggestion is to use the recursion in the partition size, which is easily generalized and deformed. This opens a possibility to obtain Macdonald polynomials by a change of recursion coefficients and taking appropriate limit from three to two dimensions – though details still remain to be worked out. Another perspective is opened by the observation of a rich non-abelian structure, extending that of commuting cut-and-join operators, for which the discovered 3-Schurs are the common eigenfunctions.

1 Introduction

Characters are the central objects in physical applications of group theory, because they describe invariant objects, which take values in numbers, still are often sufficient to capture important properties of correlators, amplitudes and partition functions. Especially interesting from this point of view is reformulation [1] of matrix models in terms of remarkable identity

\[
\langle \text{character} \rangle = \text{character}
\]

where "characters" at the two sides are basically the same Schur functions, only of different arguments – quantum fields at the l.h.s. and couplings (including matrix sizes) at the r.h.s. This result reflects superintegrability (a combination of ordinary KP integrability and Ward-Virasoro constraints, reviewed in [2]) and is closely related to combinatorial treatment of matrix models in [3], see [4] for details and references. In [5], following the general logic of non-linear algebra [6], this relation was extended from matrix to tensor models, where one can find a very nice tensorial analogue of Schur functions, despite most group theory structures are lost. Another extension [7] is to discrete matrix models, where ordinary integrals become Jackson sums and Schur functions are substituted by Macdonald polynomials. However, in this direction one naturally wants to go further – to full-fledged generalization from Young diagrams to plane partitions and from matrix to network models [8], AGT-related to 6d SYM theories. An important step of this kind was made in [9], but fully-3d formulation was not quite achieved. In this paper we make a kind of a complementary attempt – from another side. Hopefully, the two approaches can be unified and provide a much better understanding. In this paper we concentrate on ideally-symmetric 3d extension of Schur functions and only comment on the way to Macdonald deformation.

Schur functions $S_\lambda\{p_k\}$ are labeled by Young diagrams (integer partitions) $\lambda$ and therefore depend on the one-parametric family of time-variables $p_k$, which are the variables in the corresponding partition function. Indeed, the states are $\{m\} = \prod_{k=1}^\infty p_k^{m_k}$, and their generating function is

\[
\sum_{\{m\}} \prod_{k=1}^\infty p_k^{m_k} q^{k m_k} = \prod_{k=1}^\infty \frac{1}{1 - p_k q^k},
\]

so that the number of states is described by

\[
\sum_{\{m\}} \#\{m\} \cdot q^{\sum_k k m_k} = \prod_{k=1}^\infty \frac{1}{1 - q^k}.
\]
Their 3d-extension analogues $S_{\pi}(p_i^{(i)})$ should be labeled by 3d diagrams (plane partitions) $\pi$ and thus depend on the triangular set of time-variables $p_i^{(i)}$ with $i = 1, \ldots, k$, see [10]. Indeed, now the states are $\{|m\rangle\} = \prod_{1 \leq i \leq k} (p_i^{(i)})^{m_i^{(i)}}$, with the generating function

$$
\sum_{\{m\}} \prod_{1 \leq i \leq k} \left( p_i^{(i)} \right)^{m_i^{(i)}} (q^k T^i)^{m_i^{(i)}} = \prod_{1 \leq i \leq k} \frac{1}{1 - p_i^{(i)} q^k T^i}
$$

Then for the number of states we get

$$
\sum_{\{m\}} \#(m) \cdot \prod_{1 \leq i \leq k} (q^k T^i)^{m_i^{(i)}} = \prod_{1 \leq i \leq k} \frac{1}{1 - q^k T^i}
$$

and for $T = 1$ this becomes the well known MacMahon function

$$
\prod_{k=1}^{\infty} \frac{1}{(1 - q^k)^k} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + 86q^7 + \ldots
$$

Explicit definition of Schur functions is usually a two-step process – first one defines Schur polynomials for symmetric representations $[n]$ and then construct generic Schur functions via determinant formulas:

$$
e_{\sum_n \hat{p} n z^n} = \prod_a (1 - x_a z) = \sum_n S_{[n]}(p) z^n
$$

$$
S_{\lambda} = \det_{i,j \leq \lambda} S_{\lambda_i - i + j} = \frac{\det_{a,b \leq \lambda} x_a^b - b + 2}{\Delta(x)}
$$

Miwa transform relates $p$ and $x$ variables:

$$
p_n = \sum_a x_a^n
$$

The true meaning of this procedure is somewhat obscure, it looks intimately related to the structure of fundamental representations, associated with antisymmetrization (determinants) along the rows of the Young diagrams – and thus is not easy to generalize.

Alternative approach makes use of the generalized cut-and-join operators from [10], which are differential operators in $p$ or $x$:

$$\hat{W}_R S_{\lambda}(p) = \psi_R(\lambda) S_{\lambda}(p)
$$

The point is that Schur functions are their common eigenfunctions, and eigenvalues $\psi_R(\lambda)$ are characters of symmetric group. This relation is one of explicit realizations of the Schur-Weyl duality. It is straightforward to deform from Schur to Macdonald polynomials, however, generalization to plane partitions is not so easy, because obscure is the appropriate substitute of the symmetric group.

Thus to proceed we need still another description of Schur functions – which would be generalizable. Such option is provided by the "path integral" (evolution) recursion which involves skew characters

$$
S_{\lambda/\mu}(p + p') = \sum_{\rho: \mu \subseteq \rho \subseteq \lambda} S_{\lambda/\rho}(p) S_{\rho/\mu}(p')
$$

Despite a seeming asymmetry, the r.h.s. is in fact symmetric under the permutation of the sets $\{p\}$ and $\{p'\}$. Here we use as input the definition of skew characters – as linear decompositions of characters themselves, i.e. in this approach we postulate that

$$
S_{[n]/[m]} = S_{[n-m]}, \quad S_{[n,1]/[m]} = S_{[n-m+1]} + S_{[n-m,1]}, \quad \ldots
$$

The point is that these relations can be interpreted as some basic property of Young diagrams – and then the problem is to derive what are the associated characters $S_{\mu}$. Then one can look for appropriate generalization of cut-and-join operators. Our task in this paper is to show how all this works.

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1. See [11] for a renewed interest to this old viewpoint [12–14]. In this context eq. [11] can be considered as an attempt (not yet fully successful) to build a quantum (topological) field theory, underlying the theory of symmetric polynomials. From another viewpoint, [11] provides a viable deformation of locality to the discrete space of partitions.
2 Schur functions from recursion

Solutions to the equation (11), even graded, do not specify dependence on the "highest" $p$-variables at each step. For example:

$$\Delta S_1 \equiv S_1 \{ p + p' \} - S_1 \{p \} - S_1 \{ p' \} = 0 \implies S_1 = \alpha_1 p_1$$

with arbitrary $\alpha_1$, which can be absorbed into rescaling of $p_1$, thus we put $\alpha_1 = 1$. Next,

$$\Delta S_2 \equiv S_2 \{ p + p' \} - S_2 \{p \} - S_2 \{ p' \} = S_1 \{p\} S_1 \{ p' \} \quad \implies \quad S_2 \{p\} = \frac{\alpha_2 p_2 + p_1^3}{6}$$

Further, in obvious notation:

$$\Delta S_3 \{ p, p' \} = S_2 \otimes S_1 + S_1 \otimes S_2 \implies S_3 \{ p \} = \frac{2\alpha_3 p_3 + 3\alpha_2 p_2 p_1 + p_1^3}{6}$$

and so on. We introduced $\alpha$-parameters so that they are plus-minus unities or zeroes for the true Schur functions – but, as we see, they are not fully restricted by (11).

There can be different ways to impose the further restrictions, which specify $\alpha$’s, the simplest one is to request orthogonality:

$$\hat{S}_R \cdot S_{R'} \sim \delta_{R,R'}$$

where $|R| = |R'|$ and

$$\hat{S}_R := S_R \left\{ k \frac{\partial}{\partial p_k} \right\}$$

The choice of duality transform $p_k \rightarrow k \frac{\partial}{\partial p_k}$ is dictated by the properties of Schur functions, but we will keep it intact for plain partitions – though there is no clear motivation for this. At the same time, this transform is sensitive to $q, t$-transformation from Schur to Macdonald functions.

One of the basic properties of Schur functions is that they are the common eigenfunctions of an infinite commuting $W_Q$ operators (10), of which the simplest is the celebrated cut-and-join

$$\hat{W}_2 = \frac{1}{2} \sum_{a,b=1}^\infty \left( (a+b) p_a p_b \frac{\partial}{\partial p_{a+b}} + ab p_{a+b} \frac{\partial}{\partial p_a \partial p_b} \right) = p_1^2 \frac{\partial}{\partial p_1} + \frac{p_2}{2} \frac{\partial^2}{\partial p_1^2} + \ldots$$

where dots stand for the terms containing $p_n$ or $\partial_n$ with $n \geq 3$, which do not contribute at the level $|R| = 2$.

3 Macdonald recursion

Macdonald polynomials [15] also satisfy (11), but coefficients are no longer unities – they get $q, t$-deformed:

$$\Delta M_\{n\} = \sum_{m=1}^{n-1} \frac{<n>!}{<m>! <n-m>!} \cdot M_\{m\} \otimes M_\{n-m\}$$

where $<n> = \frac{[n](1-t)}{1-q}$ and $[n] = \frac{1-q^n}{1-q}$. For antisymmetric representations Macdonald polynomials coincide with Schur functions, thus

$$\Delta M_\{n\} = \sum_{m=1}^{n-1} M_\{1_m\} \otimes M_\{1_{n-m}\}$$
For more complicated Young diagrams we have:

\[
\Delta M_{[2,1^{n-2}]} = \sum_{i=1}^{n-2} \left( M_{[2,1^{n-2-i}]} \otimes M_{[1]} + M_{[1]} \otimes M_{[2,1^{n-2-i}]} \right) + \sum_{i+j \geq n} (1-t_q)(1-t_q^{i-1}) \cdot M_{[1]} \otimes M_{[1]} \\
\Delta M_{[3,1^{n-3}]} = \sum_{i=1}^{n-3} \left( M_{[3,1^{n-3-i}]} \otimes M_{[1]} + M_{[1]} \otimes M_{[3,1^{n-3-i}]} \right) + \sum_{i+j \geq n-4} (1-t_q)(1-t_q^{i-1}) \cdot M_{[2,1]} \otimes M_{[1]} + \\
+ \frac{(1-q^2^3)(1-q^{i-3})}{(1-q^{i-1})(1-q^{i-2})} \cdot \sum_{i=1}^{n-2} (1-t_q^2) \cdot \left( M_{[2,1^{n-2-i}]} \otimes M_{[1]} + M_{[1]} \otimes M_{[2,1^{n-2-i}]} \right) \\
\ldots
\]

Macdonald polynomials can be obtained by solving these equations, but, like in the case of Schur functions, the p-linear terms are zero-modes of \( \Delta \) and the coefficients in front of them are not fixed by the recursion. To cure this problem one can impose orthogonality restriction, but the rule (17) should also be deformed:

\[
\tilde{M}_R := M_R \left\{ k \cdot \frac{1-q^k}{1-t^k} \right\}
\]

Alternatively one can rescale time-variables and polynomials so that

\[
\mathcal{M}_{[n]} = S_{[n]} \left\{ (1-q)^k \cdot p_k \right\}, \quad \mathcal{M}_{[1^n]} = S_{[1^n]} \left\{ (1-t^{-1})^k \cdot p_k \right\}
\]

get expressed through the Schur functions. Other polynomials, however, remain somewhat more involved – what is natural, because they should somehow carry the information about the third combination like \( \frac{(1-t/q)^k}{1-t^k} \cdot p_k \), somehow mixed with the projection from three dimensions. We make a few more comments on Macdonald case in sec.6 below, but postpone a detailed consideration to a separate publication. The main goal of the present paper is to describe the very idea of lifting from the ordinary to plane partitions.

4 On 3d recursion and the 3d analogue of Schur functions

Coming back to sec.2 now we have a formalism, which allows straightforward extension from 2d to 3d, i.e. from ordinary to plane partitions. Namely, we can try to study recursion (11) with the Young diagrams \( \mu, \nu, \rho \) substituted by plane partitions, and supplement it by a direct analogue of the rule (17),

\[
\tilde{S}_R := S_R \left\{ k \cdot \frac{\partial}{\partial p_k^{(1)}} \right\}
\]

In what follows we label the three directions by zero, one and two primes. Then the single-row or single-column Young diagrams lie in just one of the three directions, while all other ordinary Young diagrams – in two. For ordinary Schur functions parameters \( \alpha, \beta \) are just plus/minus unities, and orthogonality conditions are easily satisfied.

Emerging at the level \( n \) are the new vectors \( \vec{\alpha}_n(\pi) \) in the \( n \)-dimensional space of \( p_{n,i}^{(i)} \), \( i = 1, \ldots, n \), which describe the 3-Schur functions for all plane partitions \( \pi \) of the size \( |\pi| = n \). They satisfy the most naive recursion rule, which, together with orthogonality, defines their scalar products through those of \( \vec{\alpha} \) at the previous levels.

- Level 2:
  Recursion implies
  \[
  \Delta S_{[2]} = \Delta S'_{[2]} = \Delta S''_{[2]} = S_{[1]} \otimes S_{[1]} \implies S_{[2]} = \vec{\alpha}_2 \vec{p}_2 + p_1^2, \quad S'_{[2]} = \vec{\alpha'}_2 \vec{p}_2 + p_1^2, \quad S''_{[2]} = \vec{\alpha''}_2 \vec{p}_2 + p_1^2
  \]
  with 2-dimensional vectors \( \vec{p}_2 = (p_2^{(1)}, p_2^{(2)}) \) and three \( \vec{\alpha}_2 \).
  Then orthogonality with the standard scalar product
  \[
  \langle p_k^{(i)} | p_l^{(j)} \rangle = k \delta_{k,l} \delta_{i,j}
  \]
and, more generally,

$$
\langle \hat{p}_\alpha | p_{m'} \rangle = \prod_{k} \prod_{i=1}^{k} \langle (y_k^{(i)})^{m_{k,i}} \rangle \langle (y_k^{(i)})^{m'_{k,i}} \rangle = \prod_{k} \prod_{i=1}^{k} \delta_{m_{k,i}, m'_{k,i}}
$$

implies for the three vectors $\alpha_2, \alpha'_2, \alpha''_2$:

$$
\hat{\alpha}_2 \hat{\alpha}_2 = \hat{\alpha}_2 \hat{\alpha}_2 = \hat{\alpha}_2 \hat{\alpha}_2 = -1
$$

(27)

what means that they form a Mercedes star (i.e. are the roots of affine $SL(3)$) and the lengths of the vectors are

$$
\alpha^2_2 = \alpha'^2_2 = \alpha''^2_2 = 2
$$

Thus

$$
\langle S_2 | S_2 \rangle = \frac{\alpha^2_2 + 1}{2} = \frac{3}{2}
$$

(29)

These 3-Schur functions

$$
S^\pm_{[2]} = \frac{p^2 - \frac{1}{\sqrt{2}} \hat{p}_2 \pm \sqrt{3} p_2}{2}, \quad S^0_{[2]} = \frac{\sqrt{2} \hat{p}_2 + p^2}{2}
$$

(30)

are the eigenfunctions of the cut-and-join operator

$$
\hat{W}^0_{[2]} = \frac{p_2}{2} \left( - \frac{\sqrt{3}}{2} \frac{\partial}{\partial p_2} + \frac{\partial^2}{\partial p_1^2} \right) + \left( p^2 - \frac{\hat{p}_2}{\sqrt{2}} \right) \frac{\partial}{\partial p_2} = p^2 \frac{\partial}{\partial p_2} + \frac{p_2}{2} \frac{\partial^2}{\partial p_2^2} - \frac{1}{\sqrt{2}} \left( \hat{p}_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial \hat{p}_2} \right)
$$

(31)

$$
\hat{W}^0_{[2]} S^0_{[2]} = 0, \quad \hat{W}^0_{[2]} S^\pm_{[2]} = \pm \sqrt{3} S^\pm_{[2]}
$$

(32)

while the other two operators

$$
\hat{W}^\pm_{[2]} = p^2 \left( - \frac{1}{2} \frac{\partial}{\partial p_2} + \frac{\sqrt{3}}{2} \frac{\partial}{\partial \hat{p}_2} \right) + -p_2 \pm \sqrt{3} \hat{p}_2 \frac{\partial^2}{\partial p_2^2} - \frac{1}{\sqrt{2}} \left( \hat{p}_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial \hat{p}_2} \right)
$$

(33)

act as raising and lowering generators:

$$
\hat{W}^+_{[2]} S^0_{[2]} = 0, \quad \hat{W}^+_{[2]} S^\pm_{[2]} = - \sqrt{3} S^\pm_{[2]}, \quad \hat{W}^+_{[2]} S^0_{[2]} = \sqrt{3} S^0_{[2]}
$$

(34)

$$
\hat{W}^-_{[2]} S^0_{[2]} = 0, \quad \hat{W}^-_{[2]} S^0_{[2]} = - \sqrt{3} S^0_{[2]}, \quad \hat{W}^-_{[2]} S^\pm_{[2]} = 0
$$

(35)

Two more triples of operators are obtained by $\pm \frac{2\pi}{3}$ rotations in the $(p_2, \hat{p}_2)$ plane. For example,

$$
\hat{W}^0_{2'} = p^2_1 \left( - \frac{1}{2} \frac{\partial}{\partial p_2} + \frac{\sqrt{3}}{2} \frac{\partial}{\partial \hat{p}_2} \right) - p_2 + \sqrt{3} \hat{p}_2 \frac{\partial^2}{\partial p_2^2} + \frac{1}{2\sqrt{2}} \left( (\sqrt{3} p_2 + \hat{p}_2) \frac{\partial}{\partial p_2} + (p_2 - \sqrt{3} \hat{p}_2) \frac{\partial}{\partial \hat{p}_2} \right)
$$

(35)

also has the three 3-Schur functions as its three eigenfunctions:

$$
\hat{W}^+_{2'} S^0_{[2]} = - \sqrt{3} S^0_{[2]}, \quad \hat{W}^+_{2'} S^0_{[2]} = 0, \quad \hat{W}^+_{2'} S^0_{[2]} = \sqrt{3} S^0_{[2]}
$$

(36)

The third "diagonal" operator $\hat{V}^{0\pm}_{[2]}$ is obtained by changing the signs of all the $\sqrt{3}$ in (35). Covariant description of this Cartesian triple, in terms of the triple $\hat{\alpha}_{[2]}, \hat{\alpha}'_{[2]}, \hat{\alpha}''_{[2]}$, is

$$
\hat{V}^{0\pm}_{[2]} = (\hat{\alpha}' - \hat{\alpha}'') \left( p^2_1 - \frac{\hat{p}_2^2}{2} \frac{\partial}{\partial p_2} + \frac{\hat{p}_2}{2} \left( \frac{\partial}{\partial p_2^2} - \frac{\partial}{\partial \hat{p}_2^2} \right) \right)
$$

(37)

where one substitutes three cyclic permutations of vectors. These operators commute at the level two, i.e. on the linear space spanned by $p_2, \hat{p}_2$ and $p^2_1$, but at higher levels additional terms should be taken into account, see (17) below for the next addition. The nine operators, revealed at the level 2, are just a tip of an interesting non-abelian structure, which looks like non-trivial generalization of abelian one in (10).
• Level 3:

Now we have three

\[ \Delta S_{[3]} = S_{[2]} \otimes S_{[1]} + S_{[1]} \otimes S_{[2]} \implies S_{[3]} = \frac{\tilde{\alpha}_3 \tilde{\beta}_3}{3} + \frac{(\tilde{\alpha}_2 \tilde{\beta}_2) p_1}{2} + \frac{p_1^3}{6} \]  

(38)

and three

\[ \Delta S_{[21]} = (S_{[2]} + S_{[2]}) \otimes S_{[1]} + S_{[1]} \otimes (S_{[2]} + S_{[2]}) \implies S_{[21]} = \frac{\tilde{\beta}_3 \tilde{\beta}_3}{3} - \frac{(\tilde{\alpha}_2 \tilde{\beta}_2) p_1}{2} + \frac{p_1^3}{3} \]  

(39)

where the convention is that \( S_3 \) corresponds to a column of length 3, lying in the \( x \) direction, while \( S_{21} \) – to the Young diagram, lying in orthogonal plane \((x', x'')\). The two triples of \( 3d \) vectors, \( \tilde{\alpha}_3, \tilde{\alpha}_3', \tilde{\alpha}_3'' \) and \( \tilde{\beta}_3, \tilde{\beta}_3', \tilde{\beta}_3'' \) are now two triples of \( 3d \) vectors, satisfying orthogonality conditions

\[
\begin{align*}
S_{3} \perp S'_{3} : & \quad \frac{\tilde{\alpha}_3 \tilde{\alpha}_3'}{3} + \frac{\tilde{\alpha}_3 \tilde{\alpha}_3''}{2} + \frac{6}{\sqrt{3}} = 0 \quad \implies \tilde{\alpha}_3 \tilde{\alpha}_3' = 1 \\
S_{21} \perp S'_{21} : & \quad \frac{\tilde{\beta}_3 \tilde{\beta}_3'}{3} + \frac{\tilde{\beta}_3 \tilde{\beta}_3''}{2} + \frac{6}{\sqrt{3}} = 0 \quad \implies \tilde{\beta}_3 \tilde{\beta}_3' = -\frac{1}{2} \\
S_{3} \perp S_{21} : & \quad \frac{\tilde{\alpha}_3 \tilde{\beta}_3}{3} - \frac{\tilde{\alpha}_3 \tilde{\beta}_3}{2} + \frac{6}{\sqrt{3}} = 0 \quad \implies \tilde{\alpha}_3 \tilde{\beta}_3 = 2 \\
S_{3} \perp S'_{21} : & \quad \frac{\tilde{\alpha}_3 \tilde{\beta}_3'}{3} - \frac{\tilde{\alpha}_3 \tilde{\beta}_3''}{2} + \frac{6}{\sqrt{3}} = 0 \quad \implies \tilde{\alpha}_3 \tilde{\beta}_3' = -\frac{5}{2}
\end{align*}
\]

(40)

If we now parameterize the six vectors by

\[
\tilde{\alpha}_3 : (u, 2x, 0), (u, -x, x\sqrt{3}), (u, -x, -x\sqrt{3}), \quad \tilde{\beta}_3 : (v, 2y, 0), (v, -y, y\sqrt{3}), (v, -y, -y\sqrt{3})
\]

(41)

then we get:

\[
\begin{align*}
-2x^2 + u^2 = 1, \quad -2y^2 + v^2 = -\frac{1}{2}, \quad 4xy + uv = 2, \quad -2xy + uv = -\frac{5}{2} \\
\implies xy = \frac{3}{4}, \quad uv = -1, \quad -2x^2u^2 - 2y^2v^2 + \frac{9}{4} + 1 = -\frac{1}{2} \\
\quad 16x^2 \frac{u^2}{2} + 9\frac{u^2}{2} = 30 \implies u^2 = \frac{8}{3}x^2 \\
\implies u^2 = \frac{3}{2}, \quad y^2 = \frac{3}{8}, \quad u^2 = 4, \quad v^2 = \frac{1}{4}
\end{align*}
\]

(42)

and the vector lengths are:

\[
\tilde{\alpha}_3^2 = 4x^2 + u^2 = 6 + 4 = 10, \quad \tilde{\beta}_3^2 = 4y^2 + v^2 = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}
\]

(43)

so that

\[
\begin{align*}
S_{[3]}^0 &= \frac{2P_3 + \sqrt{6}\beta_3}{3} + \frac{\tilde{\beta}_2 p_1}{\sqrt{2}} + \frac{p_1^2}{6}, \quad S_{[3]}^\pm = \frac{4P_3 - \sqrt{6}\beta_3 \pm 3\sqrt{2}\beta_3}{6} + \frac{(\tilde{\beta}_2 \pm \sqrt{2}\beta_2)p_1}{2\sqrt{2}} + \frac{p_1^2}{6}, \\
S_{[21]}^0 &= \frac{-P_3 + \sqrt{6}\beta_3}{6} - \frac{\tilde{\beta}_2 p_1}{\sqrt{2}} + \frac{p_1^2}{3}, \quad S_{[21]}^\pm = \frac{-2P_3 - \sqrt{6}\beta_3 \pm 3\sqrt{2}\beta_3}{12} - \frac{(\tilde{\beta}_2 \pm \sqrt{2}\beta_2)p_1}{2\sqrt{2}} + \frac{p_1^2}{3}
\end{align*}
\]

(44)

and

\[
\langle S_3 | S_3 \rangle = \frac{\alpha_3^2}{3} + \frac{\alpha_3^2}{2} + \frac{1}{6} = \frac{9}{2}, \quad \langle S_{21} | S_{21} \rangle = \frac{\beta_3^2}{3} + \frac{\alpha_3^2}{2} + \frac{2}{3} = \frac{9}{4}
\]

(45)

This is consistent with the most natural form of Cauchy identity for the 3-Schur functions:

\[
\sum_{\pi} \frac{S_{\pi} \{ p \} S_{\pi} \{ p' \}}{S_{\pi} | S_{\pi} >} = \exp \left( \sum_n \frac{\tilde{p}_n \tilde{p'}_n}{n} \right) = \exp \left( p_1 p'_1 + \frac{p_2 p'_2 + \tilde{p}_2 p'_2}{2} + \frac{P_3 + p_3 p'_3 + \tilde{p}_3 p'_3}{3} \right) \ldots
\]

(46)

At the new level we get many new cut-and-join operators, and new terms are revealed in the old ones. More details will be provided elsewhere, here we just mention that operator \( W_{[2]}^p \) is now promoted from 31 to
\[
\mathcal{W}^0_{[2]} = p_1^2 \frac{\partial}{\partial p_2} + \frac{p_2}{2} \frac{\partial^2}{\partial p_1^2} - \frac{1}{\sqrt{2}} \left( p_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial p_2} \right) + \frac{3(\sqrt{3}p_3 + p_2p_1)}{2} \frac{\partial}{\partial p_3} - \frac{3(p_3 + \sqrt{3}p_2p_1)}{2\sqrt{2}} \frac{\partial}{\partial p_3} + \\
\frac{3}{2} \left( \sqrt{3}p_3 - \frac{\hat{P}_3 + \sqrt{3}p_2p_1}{\sqrt{2}} \right) \frac{\partial}{\partial p_3} + \frac{-\sqrt{3}p_3 + p_2p_1}{\sqrt{2}} \frac{\partial^2}{\partial p_2\partial p_1} + \left( p_3 + \frac{-\sqrt{3}p_3 + p_2p_1}{\sqrt{2}} \right) \frac{\partial^2}{\partial p_2\partial p_1}
\]

\hspace{1cm} (47)

and, in addition to (32), it now has six new eigenfunctions (44):

\[
\mathcal{W}^0_{[2]} S^0_{[3]} = 0, \quad \mathcal{W}^0_{[2]} \bar{S}^0_{[3]} = \pm 3\sqrt{2} S^0_{[3]}, \quad \mathcal{W}^0_{[2]} \bar{S}^0_{[21]} = 0, \quad \mathcal{W}^0_{[2]} \bar{S}^0_{[21]} = \pm \frac{3}{2} \frac{\sqrt{3}}{2} S^0_{[21]}
\]

\hspace{1cm} (48)

- Level 4: This time we have two triples, made from [4] and [22] in three different directions (in the case of [22] this is the direction, orthogonal to the plane, to which it belongs), plus a six-plet made from [31], which depends on a pair of directions, plus the first essentially 3d configuration \( \chi \). The total number of relevant vectors in 4d space of \( p_4^{1,2,3,4} \) is 13.

From

\[
\Delta S_4 = S_1 \otimes S_3 + S_2 \otimes S_2 + S_3 \otimes S_3
\]

\[
\Delta' S''_{31} = S_1 \otimes (S'_3 + S_{21}) + S'_2 \otimes S'_2 + S'_2 \otimes S'_2 + S'_2 \otimes S''_2 + (S'_3 + S_{21}) \otimes S_1
\]

\[
\Delta S_{22} = S_1 \otimes S_{21} + S'_2 \otimes S'_2 + S'_2 \otimes S''_2 + S_{21} \otimes S_1
\]

\hspace{1cm} (49)

\[
\Delta S_5 = S_1 \otimes (S_{21} + S'_{21} + S''_{21}) + S_2 \otimes (S'_2 + S''_2) + S'_2 \otimes (S_2 + S'_{21}) + S''_2 \otimes (S_2 + S'_{21}) + (S_{21} + S'_{21} + S''_{21}) \otimes S_1
\]

where \( S_4 \) corresponds to the 1-column Young diagram lying along the \( x \) axis, i.e. belonging to either of the two planes, \( (x, x') \) or \( (x, x'') \), while \( 'S''_{31} \) and \( S_{22} \) are the ordinary Young diagrams lying in the plane \( (x', x'') \), with the leg of length 3 in the former case along \( x' \) and that of length 2 along \( x'' \), so that \( 'S''_{21} = "S''_{31} \), we get:

\[
S_4 = \left( \frac{\alpha_4 p_4}{4} + \frac{\alpha_4 p_4}{3} \right) + \frac{(\alpha_4 + \alpha_3 + \alpha_2 + \alpha_1)p_4}{8} + \frac{(\alpha_2 p_2)}{4} + \frac{p_4}{24}
\]

\[
'S''_{31} = \left( \frac{\beta_3 p_4}{4} + \frac{(\beta_3 + \beta_3 + \beta_2 + \beta_2)}{3} \right) + \frac{(\beta_2 p_2)}{8} \left( (\alpha_2 + 2\alpha_2' + \alpha_2') \right) + \frac{(\alpha_2 + \alpha_2')}{4} p_4 + \frac{p_4}{8}
\]

\[
S_{22} = \left( \frac{\gamma_3 p_4}{4} + \frac{\beta_2 p_2}{3} \right) + \frac{(\alpha_2 p_2)}{8} + \frac{(\alpha_2 p_2)}{4} + \frac{p_4}{12}
\]

\[
S_5 = \left( \frac{\alpha_4 p_4}{4} + \frac{(\alpha_2 p_2)}{4} + \frac{(\alpha_2 p_2)}{4} + \frac{(\alpha_2 p_2)}{4} \right) + \frac{p_4}{24}
\]

(50)

Orthogonality conditions now imply:

\[
S_4 \perp S'_3 : \frac{\alpha_4 p_4}{4} + \frac{\alpha_4 p_4}{3} + \frac{(\alpha_2 p_2)}{3} + \frac{(\alpha_2 p_2)}{3} + 1 + \frac{1}{24} = 0 \quad \alpha_4^{'s} = -1
\]

\[
S_4 \perp S'_{31} : \frac{\alpha_4 p_4}{4} + \frac{\alpha_4 p_4}{3} + \frac{(\alpha_2 p_2)}{3} + \frac{(\alpha_2 p_2)}{3} + 1 + \frac{1}{24} = 0 \quad \alpha_4^{'s} = -1
\]

\[
S_4 \perp S'_{31} : \frac{\alpha_4 p_4}{4} + \frac{\alpha_4 p_4}{3} + \frac{(\alpha_2 p_2)}{3} + \frac{(\alpha_2 p_2)}{3} + 1 + \frac{1}{24} = 0 \quad \alpha_4^{'s} = -1
\]

\[
S_4 \perp S''_{31} : \frac{\alpha_4 p_4}{4} + \frac{\alpha_4 p_4}{3} + \frac{(\alpha_2 p_2)}{3} + \frac{(\alpha_2 p_2)}{3} + 1 + \frac{1}{24} = 0 \quad \alpha_4^{'s} = -1
\]

\[
S_4 \perp S_{22} : \frac{\alpha_4 p_4}{4} + \frac{\alpha_4 p_4}{3} + \frac{(\alpha_2 p_2)}{3} + \frac{(\alpha_2 p_2)}{3} + 1 + \frac{1}{24} = 0 \quad \alpha_4^{'s} = -1
\]

\[
S_4 \perp S_{22} : \frac{\alpha_4 p_4}{4} + \frac{\alpha_4 p_4}{3} + \frac{(\alpha_2 p_2)}{3} + \frac{(\alpha_2 p_2)}{3} + 1 + \frac{1}{24} = 0 \quad \alpha_4^{'s} = -1
\]

\[
S_4 \perp S_5 : \frac{\alpha_4 p_4}{4} + \frac{\alpha_4 p_4}{3} + \frac{(\alpha_2 p_2)}{3} + \frac{(\alpha_2 p_2)}{3} + 1 + \frac{1}{24} = 0 \quad \alpha_4^{'s} = -1
\]
\[ S'_{31} \perp S'_{31} : \quad \frac{\alpha'_{4}}{4} + \frac{(\alpha'_3 + \alpha''_3)(\alpha_3 + \beta''_3)}{3} + \frac{(\alpha''_2 + \alpha'_2)(\alpha''_2 + \alpha'_2)}{16} + \frac{(\alpha''_2 + \alpha'_2)(\alpha''_2 + \alpha'_2)}{4} + \frac{1}{4} = 0 \]
\[ S'_{31} \perp S''_{31} : \quad \frac{\beta''_{4}}{4} + \frac{(\alpha''_3 + \alpha''_3)(\alpha_3 + \beta''_3)}{3} + \frac{2}{3} + \frac{3}{8} = 0 \]
\[ S'_{31} \perp S'''_{31} : \quad \beta''_{4} \frac{\beta''_{4}}{4} + \frac{(\alpha''_3 + \alpha''_3)(\alpha_3 + \beta''_3)}{3} + \frac{2}{3} + \frac{3}{8} = 0 \]
\[ S'_{31} \perp S'''_{31} : \quad \frac{\beta''_{4}}{4} + \frac{(\alpha''_3 + \alpha''_3)(\alpha_3 + \beta''_3)}{3} + \beta''_{4} \frac{\beta''_{4}}{4} + \frac{2}{3} + \frac{3}{8} = 0 \]
\[ S'_{31} \perp S_{22} : \quad \frac{\beta''_{4} \mu''_{4}}{4} - \frac{(\alpha''_2 + \alpha''_2)(\alpha''_2 + \alpha''_2)}{4} + \frac{1}{4} = 0 \]
\[ S'_{31} \perp S_{22} : \quad \frac{\beta''_{4} \mu''_{4}}{4} - \frac{(\alpha''_2 + \alpha''_2)(\alpha''_2 + \alpha''_2)}{4} + \frac{1}{4} = 0 \]
\[ S''_{31} \perp S_{22} : \quad \frac{\beta''_{4} \mu''_{4}}{4} - \frac{(\alpha''_2 + \alpha''_2)(\alpha''_2 + \alpha''_2)}{4} + \frac{1}{4} = 0 \]
\[ S_{22} \perp S_{2} : \quad \gamma_{4} \mu''_{4} \frac{\mu''_{4}}{4} + \frac{(\alpha''_2 + \alpha''_2)^2 + (\alpha''_2 + \alpha''_2)^2 + (\alpha''_2 + \alpha''_2)^2 + (\alpha''_2 + \alpha''_2)^2}{8} + \frac{1}{4} = 0 \]
\[ S_{22} \perp S_{2} : \quad \gamma_{4} \mu''_{4} \frac{\mu''_{4}}{4} + \frac{(\alpha''_2 + \alpha''_2)^2 + (\alpha''_2 + \alpha''_2)^2 + (\alpha''_2 + \alpha''_2)^2 + (\alpha''_2 + \alpha''_2)^2}{8} + \frac{1}{4} = 0 \]

We can try the following ansatz for 13 vectors in 4d space:

\[ \mu_{4} = (0, 0, 0, s^{-1}) \]
\[ \alpha_{4} = (2x, 0, u, 2s), \quad \alpha'_{4} = (-x, x\sqrt{3}, u, 2s), \quad \alpha''_{4} = (-x, -x\sqrt{3}, u, 2s) \]
\[ \beta''_{4} = (0, 2y, v, -3s), \quad \gamma_{4} \mu''_{4} = (0, -2y, v, -3s) \]
\[ \gamma_{4} = (2z, 0, w, 10s), \quad \gamma''_{4} = (-z, z\sqrt{3}, w, 10s) \]

Level 5: We will have six triples, made from [5], [41], [32], [311], [221], [2111] plus two triples of 3d configurations < 2, 1, 1 > ([2] atop [21]) and 1 atop [2, 2]. The total number of vectors will in 5d space is 24.

Level 6: 48 vectors in 6d space.

Level 7: 86 vectors in 7d space – and so on, in accordance with \[ (51) \].
5 Back from plane to ordinary Schur functions

One can consider elimination of extra $p_k^{(i)}$-variables by a kind of projection/contraction onto a 1-dimensional line/direction in $n$-dimensional space. The choice of this direction should be done separately for each $n$. Reduction/projection to ordinary Schur functions is nearly trivial: the relevant line is just the $x$ axis: $p_k^{(i)} = 0$ for $i = 2, \ldots, n$.

- $n = 2$:

  In this case we have three polynomials

  \[ S_2^\perp = S_2^0 = \frac{\sqrt{2}p_2 + p_1^2}{2}, \quad S_2^\pm = \pm \sqrt{\frac{2}{2}}p_2 - \frac{1}{\sqrt{2}}p_2 + p_1^2 \]  

  and we want to eliminate $\tilde{p}_2 = p_2^{(2)}$. Just putting $p_2^{(2)} = 0$ still leaves us with three polynomials, moreover even the resulting $S^\pm$ differ from the ordinary Schur functions

  \[ S_2 = \frac{p_2 + p_1^2}{2} = S_2^+, \quad S_{[1]} = -\frac{p_2 + p_1^2}{2} = S_2^- \]

  by the coefficient in front of $p_2$.

  A viable alternative is to deform the recursion rule:

  \[ \Delta S_2^\pm = S_1 \otimes S_1, \quad \Delta S_2^\perp = 2h^2 \cdot S_1 \otimes S_1 \]  

  what implies

  \[ S_2^\perp = \frac{\tilde{p}_2 + 2h^2p_1^2}{2}, \quad S_2^\pm = \frac{\tilde{p}_2 + p_1^2}{2} \]  

  If we do not deform orthogonality condition, then

  \[ \langle p_2^{(a)} | p_2^{(b)} \rangle = 2\delta^{ab}, \quad \tilde{\alpha}_2^+ \tilde{\alpha}_2^- = -1, \quad \tilde{\alpha}_2^\perp \tilde{\alpha}_2^\pm = -2h^2 \]  

  If we now also preserve the relation

  \[ \tilde{\alpha}_2^+ + \tilde{\alpha}_2^- + \tilde{\alpha}_2^\perp = 0 \]  

  then

  \[ \tilde{\alpha}_2^\perp = (2h, 0), \quad \tilde{\alpha}_2^\pm = (-h, \pm \sqrt{1 + h^2}) \]  

  and we obtain a smooth interpolation between the cases [50] of plain-partition at $h = 1/\sqrt{2}$ and [53] ordinary Schur functions at $h = 0$:

  \[ S_2^\perp = \frac{2h\tilde{p}_2 + 2h^2p_1^2}{2}, \quad S_2^\pm = -\frac{h\tilde{p}_2 \pm \sqrt{1 + h^2}p_2 + p_1^2}{2} \]  

  In the latter case $S_2^\perp$ is naturally vanishing and ”disappears”.

  The $h$-deformation of cut-and-join operator

  \[ \hat{W}_{[2]}^0(h) = p_2^2 \frac{\partial}{\partial p_2} + \frac{p_2}{2} \frac{\partial^2}{\partial p_1^2} - h\left( \tilde{p}_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial \tilde{p}_2} \right) + \ldots \]  

  smoothly interpolates between [51] and [15]. Its eigenfunctions at level 2 are

  \[ S_{[2]}^\lambda(h) \sim (1 - \lambda^2)\tilde{p}_2 + h\lambda p_2 + hp_1^2 \]

  with eigenvalues $\lambda$, which are the three roots of the characteristic equation

  \[ \lambda(\lambda^2 - h^2 - 1) = 0 \]  

  Normalization of eigenfunctions is dictated by the recursion rule, and imposing [54] we obtain [55].
\[ n = 3 : \]

The crucial fact here is that \( \alpha_3 \sim \beta_3^{\prime} + \beta_3^{\prime\prime} \) so that they lie in a plane and can vanish together after projection to orthogonal line. Indeed, according to (11), \( \alpha_3 = (u, 2x, 0) = (2, \sqrt{6}, 0) \) and \( \beta_3^{\prime} + \beta_3^{\prime\prime} = (2v, -2y, 0) = (-1, -\sqrt{3}/2, 0) = -\frac{1}{2} \alpha_3. \)

Deformed recursion

\[
\Delta S_{[3]}^\perp = 2h^2 \left( S_{[2]} \otimes S_{[1]} + S_{[1]} \otimes S_{[2]} \right) \quad \Rightarrow \quad S_{[3]}^\perp = \frac{\alpha_3^+ p_3}{3} + \frac{2h^2 (\alpha_3^+ p_3) p_1}{2} + \frac{4h^4 p_3^2}{6}
\]

\[
\Delta S_{[3]}^\perp = S_{[2]}^\perp \otimes S_{[1]} + S_{[1]} \otimes S_{[2]}^\perp \quad \Rightarrow \quad S_{[3]}^\perp = \frac{\alpha_3^+ p_3}{3} + \frac{(\alpha_3^+ p_3) p_1}{2} + \frac{p_3^2}{6} \quad (63)
\]

and

\[
\Delta S_{[2]}^\perp = (S_{[2]}^\perp + S_{[2]}^-) \otimes S_{[1]} + S_{[1]} \otimes (S_{[2]}^\perp + S_{[2]}^-) \quad \Rightarrow \quad S_{[21]}^\perp = \frac{\beta_3^+ p_3}{3} + \frac{(\beta_3^+ + \beta_3^-) p_1}{2} + \frac{p_3^2}{3}
\]

\[
\Delta S_{[21]}^\perp = (S_{[2]}^\perp + 2h^2 S_{[2]}^\perp) \otimes S_{[1]} + S_{[1]} \otimes (S_{[2]}^\perp + 2h^2 S_{[2]}^-) \quad \Rightarrow \quad S_{21}^\perp = \frac{\beta_3^+ p_3}{3} + \frac{(\beta_3^+ + 2h^2 \beta_3^\perp) p_2}{2} + \frac{p_3^2}{6} \quad (64)
\]

leads to deformed orthogonality conditions:

\[
S_{3}^\perp \perp S_{3}^\perp : \quad \frac{\alpha_3^+ \alpha_3^-}{3} + 2h^2 \frac{\alpha_3^+ \alpha_3^-}{2} + 4h^4 \frac{6}{2\gamma} = 0 \quad \Rightarrow \quad \alpha_3^+ \alpha_3^- = 4h^4 \quad (65)
\]

\[
S_{3}^\perp \perp S_{3}^- : \quad \frac{\alpha_3^+ \alpha_3^-}{3} + \frac{\alpha_3^+ \alpha_3^-}{2} + \frac{6}{\gamma} = 0 \quad \Rightarrow \quad \alpha_3^+ \alpha_3^- = \left[ h^4 = \frac{\gamma}{2} \right] \quad (66)
\]

\[
S_{21}^\perp \perp S_{21}^\perp : \quad \frac{\beta_3^+ \beta_3^-}{3} + \frac{(\beta_3^+ + \beta_3^-) (\beta_3^+ + 2h^2 \beta_3^\perp)}{2} + 2h^2 \frac{6}{\gamma} = 0 \quad \Rightarrow \quad \beta_3^+ \beta_3^- = 2h^2 - 6h^4 \quad \frac{\gamma}{2} \quad (67)
\]

Boxed are the products, surviving at \( h = 0 \), while arrows show how symmetric expressions are restored when the deformation parameter \( 2h^2 = 1 \).

Surviving are also the linear dependencies

\[
(1 - h^2) \alpha_3^+ \alpha_3^- + \beta_3^+ + \beta_3^\perp = 0
\]

and

\[
(3h^2 - 1) \alpha_3^+ + 2h^2 (\beta_3^+ + \beta_3^\perp) = 0
\]

It follows that

\[
(\alpha_3^+)^2 = \frac{(1 + 2h^2)(1 + 3h^2)}{1 - h^2}, \quad (\beta_3^+)^2 = 1 + 3h^4, \quad (\alpha_3^\perp)^2 = 16h^6 \frac{3 - h^2}{3h^2 - 1}, \quad (\beta_3^\perp)^2 = 2h^4 (5 - 3h^2)
\]

Note that these squares are not everywhere positive for real \( h^2 \).
6 Macdonald polynomials from the 3-Schur functions

To consider \( q, t \)-deformations we should choose other lines to project on. At level \( n = 2 \) the appropriate line has the slope \( \theta \) with

\[
\sin(3\theta) = 2\sqrt{2} \cdot \frac{q - t}{\sqrt{(1 - q^2)(1 - t^2)}}
\]  

(68)

The simplest approach to the study of \( q, t \)-deformations is through the algebra of cut-and-join operators. For example,

\[
\hat{W}_{[3]}^0(h) = \frac{p_2}{2} \partial_1^2 + p_1^2 \partial_2 - h \cos(3\theta)(\hat{p}_2 \partial_2 + p_2 \hat{\partial}_2) - \frac{\sin(3\theta)}{\sqrt{2}}(p_2 \partial_2 - 2h^2 \hat{p}_2 \hat{\partial}_2) + \ldots
\]

(69)

interpolates between the 3-Schur operator (63),

\[
\hat{W}_{[3]}^0 = \frac{p_2}{2} \partial_1^2 + p_1^2 \partial_2 - \frac{1}{\sqrt{2}}(\hat{p}_2 \partial_2 + p_2 \hat{\partial}_2) + \ldots
\]

(70)

at \( h = \frac{1}{\sqrt{2}} \) and \( \theta = 0 \) and the differential Macdonald operator

\[
\hat{W}M_{[2]}(q, t) = \frac{p_2}{2} \partial_1^2 + p_1^2 \partial_2 - \frac{\sin(3\theta)}{\sqrt{2}}p_2 \partial_2 + \ldots
\]

(71)

at \( h = 0 \) and \( \theta \) from (68). Like in the case of \( 3 - 2 \) Schur interpolation (61), the eigenfunctions of (69),

\[
S_{\lambda}^0_{[2]}(h, \theta) \sim h \cos(3\theta)(\lambda p_2 + p_1^2) + \left(1 - \frac{\sin(3\theta)}{\sqrt{2}}\lambda - \lambda^2\right) \hat{p}_2
\]

(72)

with \( \lambda \) – the three roots of characteristic equation

\[
\sqrt{2}\lambda(\lambda^2 - 1 - h^2) + \sin(3\theta)\left(\lambda^2(1 - 2h^2) + 2h^2\right) = 0
\]

(73)

interpolate between the 3-Schur functions (60) and Macdonald polynomials – in coordinates which are a further rescaling of \( p_2 \) in (22):

\[
M_{[2]} = M_2^+ = \frac{1}{2} \left( \sqrt{\frac{(1-q)(1+t)}{(1+q)(1-t)}} \cdot p_2 + p_1^2 \right), \quad M_{[11]} = M_2^- = \frac{1}{2} \left( -\sqrt{\frac{(1+q)(1-t)}{(1-q)(1+t)}} \cdot p_2 + p_1^2 \right)
\]

(74)

In these coordinates the two polynomials are orthogonal in the metric (24) – as it should be for the eigenfunctions of the operator (60), which is hermitian in this metric, since \( p_k^1 = k \partial_k \), \( \partial_k^\dagger = \frac{1}{k}p_k \) and \( (p_k \partial_k)^\dagger = \partial_k^\dagger p_k^1 = p_k \partial_k \).

Actually, at level 2 only one combination of \( q \) and \( t \) emerges in the slope (68) – simply because there is only one angle in this case;– but \( q \) and \( t \) get separated at higher levels \( n > 2 \), where projection line in the \( n \)-dimensional space is parameterized by \( n - 1 \) angles. One can also consider a tripe-\( h \) deformation of (60) with three different deformation parameters.

7 Conclusion

In this paper we suggested a way to extend the notion of Schur functions to the case of plane partitions. The definition which allows such generalization is in terms of recursion in the size of the Young diagrams. Postulating it in the most naive form we get a 3d-symmetric analogue of Schur functions, while deformation of the coefficients in the recursion allows to go back from 3d to 2d. Moreover, the freedom in this deformation seems sufficient to provide reductions not only to the ordinary Schur functions, but also to Macdonald polynomials. After such functions are constructed and investigated, one can study their network-model averages and, hopefully, get the corresponding extension of the superintegrability relation (11). Practical realization of this program is, however, quite tedious and will be further developed elsewhere. The present paper just describes the main idea and shows some miracles, confirming that the idea can work.
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