Gravity as the Square of Gauge Theory

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The BCJ squaring relations provide a simple prescription for the computation of gravity amplitudes in terms of gauge theory ingredients. Unlike the KLT relations, the squaring relations are directly applicable both at tree and loop level. We review the derivation of these relations from on-shell recursion relations, and discuss an off-shell approach to these relations in which the interactions of the gravity Lagrangian arise as the square of the gauge-theory interactions. This article is based on work with Zvi Bern, Tristan Dennen and Yu-tin Huang [Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, Phys. Rev. D 82 (2010), 065003, arXiv:1004.0693 (Ref. 1)] which was presented at String Field Theory and Related Aspects 2010.

§1. From KLT to BCJ

The KLT relations\(^{2,3}\) express tree-level closed-string amplitudes in terms of open-string amplitudes. In the field theory limit \(\alpha' \to 0\), they reduce to relations between gravity and gauge-theory amplitudes. The form of these relations is independent of the particular gauge and gravity theories they relate. In 4 dimensions, for example, they relate Einstein gravity (with an anti-symmetric tensor and dilaton field) to pure Yang-Mills theory, as well as \(\mathcal{N} = 8\) supergravity to \(\mathcal{N} = 4\) super Yang-Mills theory. For more general examples, two distinct gauge theories are needed to construct the amplitudes of the gravity theory. For 3- and 4-point amplitudes, the KLT relations take a reasonably simple form:

\[
\mathcal{M}_3(1, 2, 3) = -iA_3(1, 2, 3)\tilde{A}_3(1, 2, 3),
\]

\[
\mathcal{M}_4(1, 2, 3, 4) = -is_{12}A_4(1, 2, 3, 4)\tilde{A}_4(1, 2, 4, 3).
\] (1.1)

Here, we denote gravity amplitudes by \(\mathcal{M}_n\), and color-ordered amplitudes of the two gauge theories by \(A\) and \(\tilde{A}\), respectively. For very low-point amplitudes, the KLT relations make it reasonably transparent that gravity is, in a certain sense, the “square of gauge theory”. At higher point, however, the KLT relations become more and more unwieldily. The general expression is so involved\(^*\) that we refrain from reproducing it here (see Appendix A of Ref. 5) for one way of writing the all-order expression). Indeed, as they relate (unordered) gravity amplitudes to color-ordered gauge-theory amplitudes, they must necessarily include a complicated sum over permutations of the external lines, which obscures the squaring property of gravity. There are two ways to circumvent the problem: One can make the gravity amplitudes ordered, or the gauge-theory amplitudes unordered. The former approach was pursued by Drummond, Spradlin, Volovich, and Wen in Ref. 6). Though the concept of an “ordered” gravity amplitude is a-priori unphysical, this approach led to an simpler expression for gravity tree amplitudes in \(\mathcal{N} = 8\) supergravity. Throughout

\(^*\) See, however, 4) for a somewhat less cumbersome form of the KLT relations.
According to the BCJ duality, the diagrammatic numerators \( n_i \) can be arranged to satisfy the same Jacobi relation as the color factors \( c_i \).

This paper we will focus on the second approach followed by Bern, Carrasco and Johansson (BCJ) in Ref. 7, in which the full color-dressed (and thus unordered) gauge-theory amplitude was considered. This leads to a surprisingly simple manifestation of gravity as the square of gauge theory. To state the BCJ squaring relations, BCJ instruct us to first manipulate the color-dressed gauge-theory amplitude into a very particular form. We briefly review their approach now.

Consider a gauge-theory amplitude, which we write in the diagrammatic form

\[
\frac{1}{g^{n-2}} A_{\text{tree}}(1, 2, 3, \ldots , n) = \sum_{\text{diags}, i} \frac{n_i c_i}{\prod_{\alpha} s_{\alpha_i}},
\]

where the sum runs over all diagrams \( i \) with only three-point vertices, the \( c_i \) are color factors, the \( n_i \) are kinematic numerators, and the \( s_{\alpha_i} \) are the inverse propagators associated with the channels \( \alpha_i \) of the diagram \( i \). Any gauge-theory amplitude can be put into this form by replacing contact terms with numerator factors canceling propagators, i.e., \( s_{\alpha_i}/s_{\alpha} \) and assigning the contribution to the proper diagram according to the color factor. The value of the color coefficient \( c_i \) of each term is obtained from the diagram \( i \) by dressing each three-point vertex with a structure constant \( \tilde{f}^{abc} = \text{Tr}([T^a, T^b] T^c) \), and dressing each internal line with \( \delta^{ab} \).

A key property of the \( \tilde{f}^{abc} \) is that they satisfy the Jacobi identity. Consider, for example, the color factors of the three diagrams illustrated in Fig. 1. They take the schematic form,

\[
c_s \equiv \ldots \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4} \ldots , \quad c_t \equiv \ldots \tilde{f}^{a_1 a_4 b} \tilde{f}^{b a_2 a_3} \ldots , \quad c_u \equiv \ldots \tilde{f}^{a_1 a_3 b} \tilde{f}^{b a_4 a_2} \ldots ,
\]

where the ‘\ldots’ represent factors common to all three diagrams. The color factors then, of course, satisfy the Jacobi identity

\[
c_s + c_t + c_u = 0.
\]

The BCJ conjecture states that numerators \( n_i \) can always be found that satisfy Jacobi relations in one-to-one correspondence with the color Jacobi identities,

\[
c_i + c_j + c_k = 0 \quad \Rightarrow \quad n_i + n_j + n_k = 0.
\]

Here \( i, j \) and \( k \) label three diagrams whose color factors are related by a Jacobi identity. In addition, BCJ duality also requires that the \( n_i \) satisfy the same self-antisymmetry relations as the \( c_i \). That is, if a color factor is anti-symmetric under an
interchange of two legs, the corresponding numerator satisfies the same antisymmetry relations,
\[ c_i \rightarrow -c_i \quad \Rightarrow \quad n_i \rightarrow -n_i. \tag{1.6} \]

While this particular way of expressing gauge-theory amplitudes is interesting for gauge theory in its own right because it leads to new relations among gauge-theory amplitudes,\(^7\)\(^{-10}\) we will focus on its implication for gravity amplitudes. BCJ conjectured in Ref. 7) that gravity tree amplitudes can be constructed directly from the \(n_i\) through “squaring relations”. Consider expressing two color-dressed \(n\)-point gauge-theory amplitudes \(\mathcal{A}_n^{\text{tree}}(1, 2, 3, \ldots, n)\) and \(\mathcal{A}_n^{\text{tree}}(1, 2, 3, \ldots, n)\) in the form (1.2), with the numerators \(n_i\) and \(\tilde{n}_i\) satisfying all duality conditions \(n_i + n_j + n_k = 0\) and \(\tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0\). The conjectured squaring relations state that gravity amplitudes are given simply by
\[ -\frac{i}{(\kappa/2)^{n-2}} \mathcal{M}_n^{\text{tree}}(1, 2, 3, \ldots, n) = \sum_{\text{diags.}} \frac{n_i \tilde{n}_i}{\prod_j s_{\alpha_i}}, \tag{1.7} \]
where the sum runs over the same set of diagrams as in Eq. (1.2). The states appearing in the gravity theory are just direct products of gauge-theory states. The squaring relations (1.7) were explicitly checked in Ref. 7) through eight points and have also been illuminated from the KLT relations in heterotic string theory.\(^9\)

\section{Generalized gauge invariance}

There is a substantial freedom in choosing the gauge-theory numerators \(n_i\). We will call this freedom “generalized gauge invariance”, even though much of the freedom cannot be attributed to conventional gauge invariance. Our proof of the squaring relations will rely on an understanding of the most general form of this freedom at \(n\) points.

Consider a shift of the \(n_i\) in Eq. (1.2),
\[ n_i \rightarrow n_i + \Delta_i. \tag{2.1} \]

The key constraint that the \(\Delta_i\) must satisfy is that they do not alter the value of the amplitude, immediately leading to
\[ \sum_{\text{diags.}} \frac{\Delta_i c_i}{\prod_{\alpha_i} s_{\alpha_i}} = 0. \tag{2.2} \]

Any set of \(\Delta_i\) that satisfies this constraint can be viewed as a valid generalized gauge transformation since it leaves the amplitude invariant. Ordinary gauge transformations, of course, satisfy this property. We may take Eq. (2.2) as the fundamental constraint satisfied by any generalized gauge transformation.

A key observation is that it is only the algebraic properties of the \(c_i\), and not their explicit values, that enter into the cancellations in Eq. (2.2). This is so because the equation holds for any gauge group. Thus any object that shares the algebraic properties of the \(c_i\) will satisfy a similar constraint. Since the numerators \(n_i, \tilde{n}_i\) of the
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BCJ proposal satisfy exactly the same algebraic properties as the $c_i$, we immediately have

$$\sum_{\text{diags.}} i \frac{\Delta_i n_i}{\prod_{\alpha_i} s_{\alpha_i}} = 0 \quad (2.3)$$

as a key consequence of generalized gauge invariance. This holds for any $\Delta_i$ that satisfies the constraint (2.2). In particular, note that we do not need to require the $\Delta_i$ to satisfy any Jacobi-like relations.\(^*)\) A more direct derivation of this identity was given in Ref. 1).

§3. Derivation of the squaring relations

For simplicity, we consider the case of Einstein gravity obtained from two copies of pure Yang-Mills theory. The direct product of two Yang-Mills theories with $(D-2)$ states each (not counting color), gives $(D-2)^2$ states corresponding to a theory with a graviton, an anti-symmetric tensor and dilaton. At tree level, however, we can restrict ourselves to the pure-graviton sector since the other particles do not enter as intermediate states.

Tree level

We will assume that Yang-Mills tree amplitudes can always be expressed in terms of local numerators that satisfy the BCJ duality, i.e. the Jacobi-like relations (1.5). We will now outline the inductive proof of Ref. 1), which uses on-shell recursion relations with the lower-point amplitudes in the BCJ representation to show that the $n$-point gravity numerator is the square of the $n$-point Yang-Mills numerator in the BCJ representation.

For three points, the squaring relations are trivial: there is only one “diagram” with no propagators, and the relation simply states

$$\frac{-i}{\kappa/2} M_3 = (A_3)^2, \quad (3.1)$$

where $A_3$ is the color-ordered Yang-Mills 3-point amplitude.

For larger numbers of external legs, we proceed inductively. To carry out our derivation of the squaring relations we make use of on-shell recursion relations. These are derived using complex deformations of the external momenta of the amplitude,

$$p_a \to \hat{p}_a(z) = p_a + z q_a \quad a = 1, \ldots, n, \quad \hat{p}_a^2(z) = 0, \quad \sum_{a=1}^{n} q_a = 0. \quad (3.2)$$

Note that both momentum conservation and the on-shell conditions are preserved. To have valid recursion relations we demand that both the gravity and the gauge-

\(^*)\) This insight leads to an “asymmetric” generalization of the squaring relations (1.7) in which only one set of gauge theory numerators (either the $n_i$ or the $\tilde{n}_i$) are required to satisfy the Jacobi-like relations (1.5).\(^1\),\(^2\)
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theory tree amplitude vanish as we take the deformation parameter to infinity:
\[ \mathcal{M}_n(z) \to 0, \quad \hat{A}_n(z) \to 0, \quad \tilde{A}_n(z) \to 0 \quad \text{as} \quad z \to \infty. \quad (3.3) \]

The details of this complex shift (such as the number of shifted lines or the particular choice of \( q_a \)) will not play a role in our analysis, but we note that a large variety of shifts that satisfy (3.3) are known.\(^1\)–\(^3\) The simplest of these are BCFW two-line shifts. At least one BCFW shift exists for any choice of two external lines \( a \) and \( b \), such that both the gauge-theory and the gravity amplitude vanish at large \( z \).\(^4\)

Such shifts are also known to work in \( D \geq 4 \) dimensions. In our analysis we initially pick one arbitrary (but fixed) such shift.

We also pick an arbitrary local choice of gauge-theory numerators \( n_i \) that satisfies the BCJ duality (1.5). The assumption that such a choice exists at all is crucial for the derivation of the squaring relations. As the \( n_i \) are local, their complex deformations \( \hat{n}_i(z) \) under the shift are polynomial in \( z \); in particular \( \hat{n}_i(z) \) has no poles in \( z \).

In Ref. 1) it was shown that the recursion relation that arises from the complex shift leads to the following expression for the gravity amplitude \( \mathcal{M}_n \) in terms of gauge theory numerators:
\[
\frac{1}{(\kappa/2)^{n-2}} \mathcal{M}_n = \sum_{\alpha} \sum_{i} \frac{i}{s_\alpha} \sum_{\alpha\text{-diags.}i} \left[ \hat{n}_i(z_\alpha) \hat{\Delta}_i(z_\alpha) \prod_{\alpha_i} \tilde{s}_{\alpha_i}(z_\alpha) \right. \\
\left. - \frac{\Delta_i^{\alpha} \hat{n}_i(z_\alpha) + \tilde{\Delta}_i^{\alpha} \hat{n}_i(z_\alpha)}{\prod_{\alpha_i} \tilde{s}_{\alpha_i}(z_\alpha)} + \frac{\Delta_i^{\alpha} \tilde{\Delta}_i^{\alpha}}{\prod_{\alpha_i} \tilde{s}_{\alpha_i}(z_\alpha)} \right].
\] (3.4)

Here, the sum over \( \alpha \) sum goes over residues in the recursion relation at values \( z = z_\alpha \) where an internal propagator \( 1/s_\alpha \) goes on shell, i.e. \( \tilde{s}_\alpha(z_\alpha) = 0 \). The sum over diagrams \( i \) is restricted to diagrams that contain this internal line \( \alpha \). The product \( \prod' \) of propagators goes over all channels \( \alpha_i \) of diagram \( i \), except for the channel \( \alpha \). The \( \Delta_i^{\alpha} \) and \( \tilde{\Delta}_i^{\alpha} \) are kinematic functions that correspond to generalized gauge-transformations within the set of diagrams that contain the channel \( \alpha \). In particular, the \( \Delta_i^{\alpha}, \tilde{\Delta}_i^{\alpha} \) satisfy
\[
\sum_{\alpha\text{-diags.}i} \frac{\Delta_i^{\alpha} c_i}{\prod_{\alpha_i} \tilde{s}_{\alpha_i}(z_\alpha)} = \sum_{\alpha\text{-diags.}i} \frac{\tilde{\Delta}_i^{\alpha} c_i}{\prod_{\alpha_i} \tilde{s}_{\alpha_i}(z_\alpha)} = 0. \quad (3.5)
\]

The cross-terms in (3.4) involving the numerators \( \tilde{\Delta}_i^{\alpha} \hat{n}_i(z_\alpha), \Delta_i^{\alpha} \hat{n}_i(z_\alpha) \) vanish due to the identity (2.3), because the \( \Delta_i^{\alpha}, \tilde{\Delta}_i^{\alpha} \) are generalized gauge transformations and the \( n_i, \tilde{n}_i \) satisfy the BCJ duality. Similarly, it can also be argued\(^1\) that the

\(^1\) Here, both gauge-theory factors are pure Yang-Mills amplitudes and thus \( \tilde{A}_n = A_n \). However, keeping the later generalization to other gravity/gauge-theory pairs in mind, we do not make use of this equality in the following discussion.
last term also vanishes:

\[
\sum_{\alpha\text{-diags}.i} \Delta^\alpha_i \Delta^\alpha_i = 0.
\]  

(3.6)

It follows that

\[
\frac{1}{(\kappa/2)^{n-2}} M_n = \sum_\alpha \frac{i}{s_\alpha} \sum_{\alpha\text{-diags}.i} \hat{n}_i(z_\alpha) \hat{n}_i(z_\alpha) \prod_{i}\hat{s}_\alpha(z_\alpha).
\]  

(3.7)

We recognize this as a sum over residues associated with the complex shift under consideration. Cauchy’s theorem then gives\(^*)

\[
\frac{-i}{(\kappa/2)^{n-2}} M_n \equiv \sum_{\text{diags}.i} \frac{n_i \hat{n}_i}{\prod_{i}s_{\alpha_i}},
\]  

(3.8)

where, as previously stated, the \(n_i\) and \(\hat{n}_i\) are \(n\)-point Yang-Mills numerator in the BCJ representation. This concludes our outline of the derivation\(^1\) of the squaring relations for pure Einstein gravity in arbitrary dimensions \(D \geq 4\).

**Generalization to other gravity/gauge-theory pairs**

The derivation of the squaring relations that was outlined above specifically pertained to pure gravity and pure Yang-Mills theory. However, only a few steps in the derivation actually depend on this specific choice of theories. For a more general gravity/gauge-theory pair, the derivation of Ref. 1) goes through if the following three conditions are satisfied:

1. Every amplitude in the gauge theories can be expressed using local numerators that satisfy the Jacobi-like relations (1.5).
2. There exist “valid” complex shifts of the external momenta, i.e. shifts such that both gauge-theory and gravity tree amplitudes vanish at large \(z\). Such shifts give rise to on-shell recursion relations.
3. Each propagator of every gravity amplitude must develop a pole under at least one of these valid complex shifts.

An interesting candidate gravity/gauge-theory pair are the \(\mathcal{N} = 4\) SYM and \(\mathcal{N} = 8\) supergravity theories in four dimensions. Just as for pure Yang-Mills theory, it remains to be shown that amplitudes with any number of external legs in \(\mathcal{N} = 4\) SYM can be expressed using local numerators satisfying the BCJ duality. Although we expect the duality to work in supersymmetric theories,\(^7\),\(^17\),\(^18\) naively, the conditions (2) and (3) above seem hard to satisfy; while each \(\mathcal{N} = 4\) SYM amplitude with \(n > 4\) external legs admits at least one valid BCFW shift\(^19\),\(^20\) and a variety of valid holomorphic shifts,\(^20\)–\(^22\) the same does not hold for the amplitudes of \(\mathcal{N} = 8\) supergravity.\(^23\) For certain amplitudes, we seem to have no valid BCFW shifts available at all, let alone sufficiently many to satisfy condition (3).

Fortunately, there is a simple fix to this problem: We promote the numerators \(n_i\) to on-shell superfields \(n_i\) and the amplitudes \(A_n, M_n\) to superamplitudes \(\mathfrak{A}_n, \mathfrak{M}_n\),

\(^*)\) This assumes that \(\sum_{\text{diags}.i} \frac{n_i \hat{n}_i}{\prod_{i}s_{\alpha_i}}\) has no pole at \(z = \infty\). One can show\(^1\) that this is indeed the case.
which depend on Grassmann parameters $\eta_{a,A}$ (where $a$ and $A$ denote the particle index and the $SU(N)$ index, respectively). The superamplitudes $A_n$ and $M_n$ are $\eta$-polynomials that encode all $n$-point amplitudes of SYM and supergravity as their coefficients.

At the MHV level, all gauge (gravity) amplitudes are proportional to pure-gluon (pure-graviton) amplitudes. Therefore, the above derivation immediately applies also to the MHV sector of supersymmetric theories. Beyond the MHV level we make use of the results of Refs. 24) and 25), where it was shown that the superamplitudes $A_n$ and $M_n$ vanish under a super-BCFW shift of any two lines $a$ and $b$. We thus have a large number of valid super-BCFW shifts available for the superamplitudes $A_n$ and $M_n$, and conditions (2) and (3) are easily satisfied for this gravity/gauge-theory pair. The remaining analysis carries through without modification, establishing the squaring relations for $\mathcal{N} = 4$ SYM/$\mathcal{N} = 8$ supergravity.

A similar analysis can be repeated for other gravity/gauge-theory pairs by systematically verifying the conditions (1)–(3) above. Whether the KLT relations are valid for a particular gravity/gauge-theory pair is usually addressed using the $\alpha' \to 0$ limit of string theory amplitudes. The three conditions above for the squaring relations, on the other hand, give purely field-theoretic criteria for the validity of “gravity = (gauge theory)×(gauge theory)”.

**Loop level**

It is straight-forward to extend the above derivation of the squaring relations to loop level. Arranging the numerators $n_i(l_1,\ldots,l_L), \tilde{n}_i(l_1,\ldots,l_L)$ of a diagrammatic expansion of the gauge-theory $L$-loop integrand in a way that they satisfy the Jacobi-like relations (1.5), the corresponding gravity integrand takes the form

$$\left(-i\right)^{L+1} \left(\kappa/2\right)^{n+2L-2} M_{L\text{-loop}} = \sum_{\text{diags}, i} \int \prod_{a=1}^{L} d^{D} l_a \frac{n_i \tilde{n}_i}{\left(2\pi\right)^D} \prod_{\alpha_i} s_{\alpha_i}. \quad (3.9)$$

The validity of the generalization of the squaring relations to loop level was argued in Ref. 1) and was explicitly verified in Ref. 12), for example for the 3-loop 4-point amplitudes in $\mathcal{N} = 8$ supergravity.

**§4. An off-shell approach to BCJ duality and the squaring relations**

It is a natural question whether the squaring relations that relate gauge and gravity amplitudes can be made manifest in a Lagrangian approach. In the context of the conventional KLT relations, some progress in this direction was achieved by Bern and Grant.26) The BCJ squaring relations, however, suggest that a more structured approach to this problem is to first rewrite the gauge-theory Lagrangian in a way that the numerators of Feynman diagrams it generates manifestly satisfy the Jacobi-like relations (1.5). If a local Lagrangian of this type could be found, it would enable us to construct a corresponding gravity Lagrangian whose squaring relations with Yang-Mills theory are manifest. Such a construction is indeed possible,1) and we briefly review it here.
Lagrangians with manifest BCJ duality

A Yang-Mills Lagrangian with manifest BCJ duality can only differ from the conventional Yang-Mills Lagrangian by terms that do not affect the amplitudes. The amplitudes are unaffected, for example, by adding total derivative terms or by carrying out field redefinitions. Surprisingly, not only does a Lagrangian with manifest BCJ duality exist, it differs from the conventional Lagrangian by terms whose sum is identically zero by the color Jacobi identity! Although the sum over added terms vanishes, they cause the necessary rearrangements so that the Jacobi-like relations hold for the diagrammatic numerators. Another curious property is that the additional terms are necessarily non-local, at least if we want a covariant Lagrangian without auxiliary fields.

Following Ref. 1), we use an expansion in powers of the gauge field as an ansatz for the BCJ-compatible Lagrangian:

\[ \mathcal{L}_{YM} = \mathcal{L} + \mathcal{L}'_5 + \mathcal{L}'_6 + \ldots \]  

Here, \( \mathcal{L} \) is the usual Yang-Mills Lagrangian, and the \( \mathcal{L}'_n \) with \( n \geq 5 \) are determined to ensure that \( n \)-point amplitudes computed from \( \mathcal{L}_{YM} \) manifestly satisfy the Jacobi-like relations (1.5). In Feynman gauge, a solution for \( \mathcal{L}'_5 \) with this property is given by

\[ \mathcal{L}'_5 = -\frac{1}{2} g^3 f_{a_1 a_2 b} f^{b a_3 c} f_{c a_4 a_5} \times \left( \partial_{[\mu} A_{\nu]}^{a_1} A_{\rho}^{a_2} A_{\sigma}^{a_3} + \partial_{[\mu} A_{\nu]}^{a_3} A_{\rho}^{a_2} A_{\sigma}^{a_1} + \partial_{[\mu} A_{\nu]}^{a_2} A_{\rho}^{a_3} A_{\sigma}^{a_1} \right) \frac{1}{\Box} (A^{a_4 \nu} A^{a_5 \rho}) . \]

As mentioned above, \( \mathcal{L}'_5 \) vanishes by itself by the color-Jacobi identity, but it precisely rearranges the Feynman diagram expansion in a way that the resulting numerators satisfy (1.5). \( \mathcal{L}'_5 \) is, however, not unique. There is a one-parameter family of \( \mathcal{L}'_5 \) that give a BCJ-compatible 5-point amplitude. The corresponding “self-BCJ” operator \( \mathcal{D}_5 \) that can be added to (4.2) without affecting the amplitude was presented in Ref. 1). It corresponds to a “generalized gauge transformation” that preserves the Jacobi-like relations.

Once the BCJ-compatible contributions \( \mathcal{L}'_n \) to the Lagrangian are determined up to a certain order \( n_{\text{max}} \) in the gauge field, it is straight-forward to determine from it a gravity Lagrangian that makes the squaring relations manifest: one first introduces auxiliary fields to convert all higher-point interactions into local cubic interactions. This is necessary, because the squaring relations hold at the level of numerators of diagrams with cubic interactions only. To obtain a gravity Lagrangian, one then strips off color traces, squares the cubic interactions in momentum space, and interprets products of gauge theory physical or auxiliary fields as gravity physical or auxiliary fields, e.g.

\[ A_{\mu}(k) \tilde{A}_{\nu}(k) \rightarrow h_{\mu\nu}(k) . \]  

The result is a gravity Lagrangian that is valid for amplitudes with up to \( n_{\text{max}} \) legs, as discussed in more detail in Ref. 1).

A BCJ-compatible Lagrangian was worked out\(^1\) up to \( \mathcal{L}'_6 \). At the 6-point level, there is a 30-parameter family of self-BCJ interactions that can be added to the
Lagrangian without affecting amplitudes or the Jacobi-like property of diagrammatic numerators. It would be interesting to find a pattern or symmetry principle that singles out a particular choice of self-BCJ terms to be added to the Lagrangian. Such a pattern may then enable us to write down the all-order BCJ-corrected action without having to analyze each n-point level at a time. The “double field theory” approach to gravity of \(^{27}\) (see also \(^{28}\)), with its manifest \(O(D,D)\)-symmetry and “double-Lorentz invariance”, may provide the needed symmetry principle to carry out the Lagrangian analysis of the squaring relations to all orders.

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