The criterion power for identifying symmetric models

V G Polosin

Medicine Cybernetics and Computer Science Department, General Medicine Faculty, Penza State University, 40 Krasnay Street, Penza 440026, Russia

E-mail: polosin-vitalij@yandex.ru

Abstract. The paper contains a theoretical material of research of power of entropy-parametrical criterion of the consent that used simultaneously both informative and statistical properties the samples data. As a result of the study, it was shown that the space of the reduced signs of the competing hypothesis is anisotropic. For this reason, in the entropy coefficient and anty-kurtosis space as the position as and the shape of the decision area depends on the direction and distance between the null and competing hypotheses. For various significance levels of the criterion, the dependences of the power of the entropy-parametric criterion on the modulus of the distance between the null and alternative hypotheses are obtained. These dependences are given for various directions in the space of reduced signs of distributions of the competing hypothesis provided that zero hypotheses is Laplace distributions.

1. Statistical measurements in information-measuring and control systems

Currently, the tasks of statistical measurements in information-measuring systems and in industrial automation systems, as a rule, are associated with the measurement and analysis of the distributions probability characteristics of a data large amount [1, 2, 3, 4]. The statistical data samples obtained during the observation period contain information on the structural organization of the object [5, 6]. For complex technical and physical objects, the shape of the accepted hypothetical model of the distribution of the sample set of observations indicates the origin of the influencing factors and the reasons for the scatter of the results [5, 7]. The classic approach of parametric identification of a distribution model is to perform three steps. The first step consists in determining the shape of the hypothetical model is established on the basis of a comparative analysis of the results of experimental studies and the statistical distribution model. The model is accepted subject to compliance with the consent criterion model from the many possible shapes of the models. The second step consists in evaluating its parameters. The third stage consists in assessing the adequacy of the obtained model for experimental observations. The validity of the hypothetical model is established on the basis of a comparative analysis of the results of experimental studies and the statistical distribution model. The model is accepted subject to compliance the consent criterion.

The study of parametric and informational uncertainties in the analysis of statistical distributions allowed develop an entropy-parametric criterion for assessing the validity of symmetric distribution models in the space of signs of the entropy coefficient and anty-kurtosis [7, 8]. The task of testing hypotheses is associated with calculating the probability of the first and second kind of errors based on the assessment of criteria for a sample of results of random values whose exact or approximate distribution is known [9, 10]. In establishing the validity of statistical models, the consequences of a type II errors are quite often more significant. The type II errors consist in accepting the incorrect null
hypothesis $H_0$ provided that competing hypothesis $H_1$ is equitable. Accepting the null hypothesis does not mean that it is true.

This article focuses on the power of the entropy-parametric criterion and the probability of the type II error for choosing the competing hypothesis of symmetric statistical models. The $\beta$ probability of the type II error determined as by the position of the competing hypothesis and as by the position of the region of acceptance of the null hypothesis. In these cases, the priority of the decision-making is assigned to the statistical power of the criterion $(1-\beta)$, that it is equal to the probability of deviation of the null hypothesis in the case when the competing alternative hypothesis is true [11, 12, 13]. The higher the power of the statistical criterion, the less likely it is to make the type II error and the higher the reliability of making the right decision. The power of the criterion is the probability of distinguishing between the alternative hypothesis $H_1$ if the hypothesis is true or the probability of the correct deviation of the null hypothesis $H_0$ [14].

2. The space of entropy–parametric criterion

If the consequences of a type II error are significant during the system’s operation, then the priority of decision-making false with the power of the criterion $(1-\beta)$, this is equal to the probability that the test rejects the null hypothesis ($H_0$) when a specific alternative hypothesis ($H_1$) is true. The calculation of the $\beta$ probability of the occurrence of a type II error is carried out in the space of reduced estimates $\xi'$ and $\eta'$, centered with respect to the position of the competing hypothesis $H_1$. The transposed vector written for the estimates $\xi'$ and $\eta'$ has the form

$$
\begin{bmatrix}
\xi' \\
\eta'
\end{bmatrix}^T = \begin{bmatrix}
k_{1} - k_{d1} \\
S(k_{H1}) \\
\kappa - \kappa_1 \\
S(\chi_{0})
\end{bmatrix}^T.
$$

(1)

Where $S(k_{H1})$ and $S(\kappa_1)$ are the mean square errors of the entropy coefficient $k_{H1}$ and anti-kurtosis $\kappa_1$ of the competing hypothesis that they calculated in the coordinate space of estimates of the entropy coefficient $k_{H1}$ and anti-kurtosis $\kappa_1$. In the space of the $\xi'$ and $\eta'$ estimates, the position of the sampled data are characterized by random variables as for the distance $r'$ module as and for angular coordinate that is specify the position of the sample data relative to the position of the alternative hypothesis. These distance $r'$ module and angular coordinate are given by

$$
\begin{align}
r' &= \sqrt{(\xi')^2 + (\eta')^2}, \\
\varphi' &= \arctg\left(\frac{\xi'}{\eta'}\right).
\end{align}
$$

(2)

Figure 1 illustrates the position of the null hypothesis $H_0$ in space that is reduced to the mean square errors $S(k_{H0})$ and $S(\kappa_0)$ of the entropy coefficient $k_{H0}$ and anti-kurtosis $\kappa_0$ of the competing hypothesis $H_1$. The transposed vector for the given estimates $\xi$ and $\eta$ of the coordinate space centered with respect to the null hypothesis $H_0$ is given in the form

$$
\begin{bmatrix}
\xi \\
\eta
\end{bmatrix}^T = \begin{bmatrix}
k_{1} - k_{d0} \\
S(k_{0}) \\
\kappa - \kappa_0 \\
S(\chi_{0})
\end{bmatrix}^T.
$$

(3)

Where $S(k_{H0})$ and $S(\kappa_0)$ are the mean square errors of the entropy coefficient $k_{H0}$ and anti-kurtosis $\kappa_0$ of the null hypothesis calculated in the coordinate space of estimates of the entropy coefficient $k_{H0}$ and anti-kurtosis $\kappa_0$ of the null hypothesis respectively. The given coordinates $\xi$ and $\eta$ of the space centered with respect to the null hypothesis and the $\xi'$ and $\eta'$ space coordinates that are centered with respect to the competing hypothesis are related by matrix relations of the form
\[
\begin{bmatrix}
\xi \\
\eta
\end{bmatrix}^T = \begin{bmatrix}
A \xi' \\
B \eta'
\end{bmatrix}^T - \begin{bmatrix}
\xi_0' \\
\eta_0'
\end{bmatrix}^T.
\]

(4)

Figure 1. The space of the competing hypothesis \(H_1\).

Here \(A\) and \(B\) are the ratio of the mean square errors of the characteristics of the models of the null hypothesis \(H_0\) to the alternative hypothesis \(H_1\), recorded for the entropy coefficients and anti-kurtosis respectively. The relationship correspondence vector is given by

\[
\begin{bmatrix}
A \\
B
\end{bmatrix}^T = \begin{bmatrix}
\frac{S(k_{\alpha v})}{S(k_{\alpha o})} \\
\frac{S(\chi_{\alpha v})}{S(\chi_{\alpha o})}
\end{bmatrix}^T.
\]

(5)

The mean square errors of the signs of symmetric distribution models were obtained by approximating the models with an error of 8...10% [15]. Formulas for approximating the root-mean-square errors of the entropy coefficient and anti-kurtosis for hypothesis \(v\) are given using a transposed vector of the form

\[
\begin{bmatrix}
S(k_{\alpha v}) \\
S(\chi_{\alpha v})
\end{bmatrix}^T = \begin{bmatrix}
0.9 \left( \kappa_v k_{\alpha v}^{1.5} \sqrt{n} \right)^{-1} \\
0.1857 n^{-0.5} \kappa_v \sqrt{(\varepsilon_v^2 - 1)^3}
\end{bmatrix}^T.
\]

(6)

Where \(v\) is 1 if it is an alternative hypothesis and 0 if it is a null hypothesis.

In the space of the null hypothesis \(H_0\) with coordinates \(\xi\) and \(\eta\), the decision domain with a confidence probability \((1 - \alpha)\) is given in the form of a circle of radius determined by the significance of the criterion \(\alpha\) [8]. The boundary equation is given by

\[
\xi_0^2 + \eta_0^2 = r_0^2.
\]

(7)

Where \(r_0\) is radius, that is given by as \(r_0 = \sqrt{\ln(\alpha^{-1})}\).

From relations (4) we express the variables \(\xi_0\) and \(\eta_0\) of the curve that bounds the decision-making region in the space of the null hypothesis through the variables and of the same space curve of the alternative hypothesis. Then, we substitute the obtained relations into the boundary equation (7) for the decision-making domain of the null hypothesis \(H_0\). As a result, we obtain the equation of an ellipse bounding the domain of acceptance of the null hypothesis in the coordinate space \(\xi'\) and \(\eta'\) of alternative hypothesis that it is given by

\[
A^{-2} \left( \xi'_0 - \xi'_0 \right)^2 + B^{-2} \left( \eta'_0 - \eta'_0 \right)^2 = r_a^2.
\]

(8)
From the obtained equation (8) it follows that the axes of the ellipse are parallel to the coordinate axes of $O\xi'$ and $O\eta'$ of the space centered with respect to the alternative hypothesis. The values of the semi axes of the ellipse and in the coordinate space of the alternative hypothesis of $\Delta\xi_0'$ and $\Delta\eta_0'$ are proportional to the $r_0$ criterion that it is given by as:

$$
\begin{bmatrix}
\Delta\xi_0' \\
\Delta\eta_0'
\end{bmatrix} = 
\begin{bmatrix}
Ar_0 & Br_0
\end{bmatrix}.
$$

(9)

For small relative displacements of the $H_0$ null and $H_1$ alternative hypotheses relative to each other ($r_0' \ll 1$), we can equate the mean square errors of their estimates. In this case, the curve bounding the region of acceptance of the null hypothesis in the evaluation space $\xi'$ and $\eta'$ centered relative to the alternative hypothesis will take the form of a circle whose radius will be equal to the entropy criterion $r_0$. As the distance between the null and alternative hypotheses increases, the difference between the mean square errors of their estimates increases, that causes compression or extension of the boundaries of the entire decision-making area in accordance with expression (9).

Thus, in spite of the fact that the radius of the circular domains of acceptance of the null and alternative hypotheses in the centered relative to the coordinate spaces of the same hypotheses are equal to the value of the $r_0$ criterion in the space of the alternative hypothesis the region of acceptance of the null hypothesis is deformed with a change in its area. In this case, the total area of the elliptic decision-making region of the null hypothesis in the first quarter of the space of the alternative hypothesis is decreases and this area is increases in the third quarter of the same space. It is shown at figure 1.

3. The power of the entropy–parametric criterion
It should be noted one more feature of the space of the alternative hypothesis, consisting in the fact that the probability of the distribution of the $r'$ distance from the center of the space specified by the position of the alternative hypothesis to the position of sampled data corresponding to the alternative hypothesis is similar to the distribution of $r$ random variables of the distance module from the center of the space of the null hypothesis to positions of the sample of results corresponding to the null hypothesis. In this regard, for a random value of the absolute value of the $r'$ distance in the space of the alternative hypothesis, the two-parameter Weibull distribution [8] is valid. For the given centered estimates of $\xi'$ and $\eta'$ sampled data the normal distribution law is applicable if the estimates is corresponding to the alternative hypothesis.

When comparing the spaces, one more important feature of these spaces should be noted, which consists in the fact that the distribution in the coordinate space $\xi$ and $\eta$ of the possible positions of the samples of results corresponding to the null hypothesis, and the distribution in the coordinate space of $\xi'$ and $\eta'$ of sampled data satisfying the alternative hypothesis, have the same patterns. If we were draw an analogy of spaces, we can write the probability $\Delta P(\xi', \eta')$ of a hypothetical implementation belonging to the alternative hypothesis $H_1$ falling into the elementary region defined by the coordinate segments $[\xi', \xi' + \Delta\xi']$ and $[\eta', \eta' + \Delta\eta']$

$$
\Delta P(\xi', \eta') = (2\pi)^{-1} \exp\left(-0.5\left((\xi')^2 + (\eta')^2\right)\right) \Delta\xi' \Delta\eta'.
$$

(10)

If we summarize the obtained probabilities over the entire elliptic decision-making area on the validity of the null hypothesis that we are write down the integral expression for determining the probability of a type II error when using the entropy criterion $r_0$:

$$
\beta = \int_{-\Delta\eta}^{\Delta\eta} \int_{(\xi' \xi_{0}' \Delta\xi')}^{(\xi' + \Delta\xi', \xi_0')} (2\pi)^{-1} \exp\left(-0.5\left((\xi' + \xi_{0}')^2 + (\eta' + \eta_{0}')^2\right)\right) d\xi' d\eta'.
$$

(11)
It is convenient to set the coordinates and positions of the null hypothesis in the space of the alternative hypothesis in the polar coordinates of the radius vector module and position angle:

\[
\begin{bmatrix}
\xi_0' \\
\eta_0'
\end{bmatrix} = \mathcal{K}_0' \begin{bmatrix}
\sin(\phi_0') \\
\cos(\phi_0')
\end{bmatrix}. 
\] (12)

If in calculating the integral expression (11) Cartesian coordinates is replaced the polar coordinates (12) that it is the dependence of the type II error as the change in the distance and as the angle of the position of the null hypothesis in the space of the alternative hypothesis. The probability of error of the type II error corresponds to the probability of the statement that the null hypothesis \( H_0 \) is true by the right alternative hypothesis \( H_1 \). The probability of the type II error is calculated for the \( \alpha \) given probability of the appearance of the type I error.

In practice, to reveal the effect the power of the criterion is used that it is equal to the probability of the opposite event \( 1 - \beta \) instead of the probability of a type II error. The power of the criterion consists in the assertion that the false null hypothesis is correctly rejected. Power is one of the main characteristics of a statistical criterion, which is associated with the number of sample results, the level of significance of the criterion, the probability of type II error [10, 14]. During planning of studies, the apriority power analysis is very useful since it allows you to indicate a measure of the distance between the \( H_0 \) null hypothesis and the \( H_1 \) alternative hypothesis that it is necessary to detect the effect. If the effect is less than expected, then an assessment of the actual power of the observed effect is necessary in order to recognize the result as adequate. If the power is below the required level, then the result is statistically insignificant.

Graphs of the dependence of the power of the entropy-parametric criterion on the modulus of the distance between the alternative and null hypotheses at different levels of significance of the criterion are given in figure 2.

![Figure 2](image)

**Figure 2.** Dependences of the power of the entropy–parametric criterion.

Figure 2 shows graphs of the dependence of the power \( (1-\beta) \) of the entropy–parametric criterion on the modulus of the distance between the alternative and null hypotheses at various significance levels of the \( \alpha \) criterion. It is given for decision-making if zero hypothesis is Laplace distribution. The graphs are given for the null hypothesis of the Laplace distribution, where the numbers 1, 2, 3, 4, 5, 6, 7 indicate the curves at significance levels of 0.1%, 1%, 2%, 5%, 20%, 50%, respectively. All the curves have calculated for the \( \phi_0 \) angle is zero by number 1 and for the angle is \( \pi \) by number 2.
4. Results and discussion
During the research conducting was decided there is a real effect, provided that the statistical power of at least 0.8. In this case, the probability is corresponding to 80% for the statement: "The false null hypothesis is correctly rejected". Figure 2 shows the dependency graphs for the $\beta$ probability of the occurrence of type II error as the using the $r_\alpha$ entropy criterion on the $r'_0$ distance between the hypotheses that were defined in the space of the alternative hypothesis.

From the analysis of the graphs in figure 2, we see that the steepest decrease in the probability $\beta$ of the occurrence of a second-type error from maximum to minimum occurs with increasing distance of $r'_0$ at angles of $\varphi_0$ equal to zero. The gentlest decrease in the probability $\beta$ occurs at angles of $\varphi_0$ that is equal to $\pi$. Despite the pronounced dependence is the rate of decrease in probability $\beta$ by increasing distance between hypotheses if the angle of the null hypothesis is change on relative to the alternative hypothesis, there are characteristic distances between hypotheses for a $\alpha$ level of significance of the entropy criterion, such that they are the probability of a type II error practically does not depend on the angle of the hypothesis position $H_0$.

Based on the analysis of the reduced space centered with respect to the alternative hypothesis $H_1$, an expression is obtained for calculating the probability of the occurrence of the type II error. In particular, the criterion power anisotropy was established in the space of the given estimates of $\xi'$ and $\eta'$ of the entropy coefficient and anty-kurtosis centered relative to the alternative hypothesis. Values of the power criterion of $(1−\beta(r'_0))$ were determined such that the anisotropic properties of the reduced space are neglected.

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