Reliability Function of General Classical-Quantum Channel

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Abstract – In information theory the reliability function and its bounds, describing the exponential behavior of the error probability, are important quantitative characteristics of the channel performance. From a more general point of view, these bounds provide certain measures of distinguishability of a given set of classical states. In the paper quantum analogs of the random coding and the expurgation lower bounds for the case of pure signal states were introduced. Here we discuss the case of general quantum states, in particular, we prove the expurgation bound conjectured in and find the quantum cutoff rate for arbitrary mixed signal states.

Index Terms – Quantum channel, reliability function, random coding, expurgation.

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I. Introduction

We consider classical-quantum channel with a finite input alphabet \(\{1, \ldots, a\}\) and with arbitrary signal states given by density operators \(S_i\); \(i = 1, \ldots, a\) in a Hilbert space \(\mathcal{H}\). For simplicity of presentation we take \(\mathcal{H}\) finite-dimensional, although with obvious modifications the results hold for a separable \(\mathcal{H}\). The classical channel corresponds to the case of commuting operators \(S_i\), represented by diagonal matrices \(\text{diag}(P(1|\cdot), \ldots, P(b|\cdot))\), where \(P(j|\cdot)\) is the channel transition probability.

Product channel of degree \(n\) acts in the tensor product \(\mathcal{H}^\otimes n = \mathcal{H} \otimes \ldots \otimes \mathcal{H}\) of \(n\) copies of the space \(\mathcal{H}\). Sending a codeword \(w = (i_1, \ldots, i_n), i_k \in \{1, \ldots, a\}\), produces the signal state \(S_w = S_{i_1} \otimes \ldots \otimes S_{i_n}\) in the space \(\mathcal{H}^\otimes n\). A code \((W, X)\) of size \(M\) in \(\mathcal{H}^\otimes n\) is a collection of \(M\) pairs \((w^1, X_1), \ldots, (w^M, X_M)\), where \(W = \{w^1, \ldots, w^M\}\) is a codebook, \(X = \{X_1, \ldots, X_M\}\) is a quantum decision rule, i.e. a collection of positive operators in \(\mathcal{H}^\otimes n\), satisfying \(\sum_{j=1}^M X_j \leq I\). The conditional probability to make a decision in favor of message \(w^k\) provided that codeword \(w^j\) was transmitted is \(\text{Tr} S_{w^j} X_k\), in particular, the probability to make wrong decision is equal to

\[
P_j(W, X) = 1 - \text{Tr} S_{w^j} X_j.
\]

One usually considers the error probabilities

\[
P_{\max}(W, X) = \max_{1 \leq j \leq M} P_j(W, X)
\]

and

\[
\bar{P}(W, X) = \frac{1}{M} \sum_{j=1}^M P_j(W, X).
\]

We shall denote by \(P_e(M, n)\) any of the minimal error probabilities \(\min_{W,X} P_{\max}(W, X), \min_{W,X} \bar{P}(W, X)\). It is known that they are essentially equivalent from the point of view of information theory, see also Sect. 3.

The classical capacity of the classical-quantum channel is defined as the number \(C\) such that \(P_e(2^{nR}, n)\) tends to zero as \(n \to \infty\) for any \(0 \leq R < C\) and does not tend to zero if \(R > C\). Moreover, if \(R < C\) then \(P_e(2^{nR}, n)\)
tends to zero exponentially with \( n \) and we are interested in the logarithmic rate of convergence given by the reliability function

\[
E(R) = - \lim_{n \to \infty} \inf \frac{1}{n} \log P_e(2^n R, n), \quad 0 < R < C. \tag{1}
\]

In the classical information theory [5] there are lower and upper bounds for \( E(R) \), giving important quantitative characteristics of the channel performance. From a more general point of view, these bounds provide certain measures of distinguishability of a given set of classical states. In the paper [3] quantum analogs of the random coding and the expurgation lower bounds were given for the case of pure signal states \( S_i \), represented by rank one density operators. Here we discuss the general case, in particular, we prove the expurgation bound conjectured in [3].

II. The capacity and the random coding lower bound

The classical capacity of the channel is given by the formula

\[
C = \max_\pi \left[ H \left( \sum_{i=1}^a \pi_i S_i \right) - \sum_{i=1}^a \pi_i H(S_i) \right], \tag{2}
\]

where \( H(S) = -\text{Tr}S \log S \) is the von Neumann entropy of the state \( S \) and \( \pi = \{\pi_i\} \) are probability distributions on the input alphabet \( \{1, ..., a\} \). This relation was established in [10], [13], using the concept of typical subspace [6]. The proofs in the present paper are direct, making no use of this concept and of the relation (2).

Proposition 1: For any \( \pi \) and \( 0 < s \leq 1 \)

\[
H \left( \sum_{i=1}^a \pi_i S_i \right) - \sum_{i=1}^a \pi_i H(S_i) \geq \frac{1}{s} \mu(\pi, s), \tag{3}
\]

where

\[
\mu(\pi, s) = -\log \text{Tr} \left( \sum_{i=1}^a \pi_i S_i^\dagger S_i^{++} \right)^{1+s}.
\]
Proof. Denote by
\[
H(S, T) = \begin{cases} 
\text{Tr} S (\log S - \log T), & \text{if } \text{supp} S \subseteq \text{supp} T, \\
+\infty, & \text{otherwise}
\end{cases}
\]
\[
H_r(S, T) = -\log \text{Tr} S^{1-r} T^r; \quad 0 \leq r \leq 1,
\]
the relative entropy and the Chernoff-Rényi entropy of the density operators \( S, T \), correspondingly (see [11]). Since
\[
H(S, T) = \frac{d}{dr} \bigg|_{r=0} H_r(S, T),
\]
and
\[
H_r(S, T)
\]
is concave, we have
\[
rH(S, T) \geq H_r(S, T).
\]
Now
\[
H\left( \sum_{i=1}^a \pi_i S_i \right) - \sum_{i=1}^a \pi_i H(S_i) = \sum_{i=1}^a \pi_i H(S_i, \sum_{l=1}^a \pi_l S_l)
\]
\[
\geq -\frac{1}{r} \sum_{i=1}^a \pi_i \log \text{Tr} S_i^{1-r} \left( \sum_{l=1}^a \pi_l S_l \right)^r \geq -\frac{1}{r} \log \text{Tr} \sum_{i=1}^a \pi_i S_i^{1-r} \left( \sum_{l=1}^a \pi_l S_l \right)^r,
\]
by convexity of \(-\log\). By quantum Hölder inequality [12], the argument of \(\log\) is less than or equal to
\[
\left( \text{Tr} \left( \sum_{i=1}^a \pi_i S_i^{1-r} \right) \right)^\frac{p}{r} \left( \text{Tr} \left( \sum_{i=1}^a \pi_i S_i \right) \right)^\frac{q}{r},
\]
if \(p^{-1} + q^{-1} = 1, p > 1\). Putting \(p = \frac{1}{1-r}, q = \frac{1}{r}, s = \frac{r}{1-r}\), and using monotonicity of \(\log\), we obtain (3). \(\quare\)

Assume now that the words in the codebook \(W\) are chosen at random, independently, and with the probability distribution
\[
P\{w = (i_1, ..., i_n)\} = \pi_{i_1} \cdot ... \cdot \pi_{i_n}
\]
for each word. We shall denote expectations with respect to this probability distribution by the symbol \(\mathbb{E}\). In [3] we conjectured the following random coding bound for the error probability
\[
\mathbb{E} \min_X \bar{P}(W, X) \leq c \inf_{0 < s \leq 1} (M - 1)^s \left[ \text{Tr} \left( \sum_{i=1}^a \pi_i S_i^{1+s} \right) \right]^n.
\]
(5)
The bound (5) holds for pure states $S_i$ in which case $S_i^\dagger = S_i$ and $c = 2$. For commuting $S_i$ it reduces to the classical bound of Theorem 5.6.2 [3] with $c = 1$. By putting $M = 2^nR$, it implies the lower bound for the reliability function

$$E(R) \geq \max_\pi \sup_{0 < s \leq 1} [\mu(\pi, s) - sR] \equiv E_r(R).$$

This can be calculated explicitly for quantum binary and Gaussian pure state channels [4]. A remarkable feature of the classical case is that there exists the upper bound (the sphere-packing bound) which coincides with $E_r(R)$ for high rates, and thus gives exact expression for $E(R)$. In the quantum case no useful upper bound for $E(R)$ is known yet (see, however, [14] for an incomplete analog of the sphere-packing bound).

We shall prove a general inequality for the error probabilities $P_j(W, X)$, which implies (3) for $s = 1$ with $c = 1$, and will be used in the next Section to obtain the expurgation bound. Moreover, the first part of the argument will be used for alternative operator proof of (5) in case of commuting $S_i$, indicating clearly at which point commutativity comes into play. The proof of (5) in full generality remains open question.

**Lemma:** For any collection $W$ of codewords there is a decision rule $X$ such that

$$P_j(W, X) \leq \text{Tr} \sqrt{S_{w^j}} \sum_{l \neq j} \sqrt{S_{w^l}}, \quad j = 1, \ldots, M. \quad (7)$$

**Proof.** By making a small perturbation of the density operators $S_{w^j}$, we can assume that they are nondegenerate. We choose the following suboptimal decision rule

$$X_j = (\sum_{l=1}^M S_{w^l}^{-r/2})^{-1/2} S_{w^j}^{-r} (\sum_{l=1}^M S_{w^l}^{-r})^{-1/2}, \quad (8)$$

where $r$ is a real parameter, $0 < r \leq 1$. This gives

$$P_j(W, X) = 1 - \text{Tr} S_{w^j} A_j^* A_j, \quad (9)$$

where $A_j = S_{w^j}^{-r/2} (\sum_{l=1}^M S_{w^l}^{-r})^{-1/2}$. Using the Cauchy-Schwarz inequality

$$|\text{Tr} S_{w^j} A_j|^2 \leq \text{Tr} S_{w^j} A_j^* A_j,$$

we obtain

$$P_j(W, X) \leq 2(1 - \text{Tr} S_{w^j} A_j) = 2[1 - \text{Tr} S_{w^j}^{1+r/2} (\sum_{l=1}^M S_{w^l}^{-r})^{-1/2}]. \quad (10)$$
Let $S_{wj} = \sum_{\alpha} \lambda_{j}^{\alpha} |e_{j}^{\alpha}\rangle\langle e_{j}^{\alpha}|$ be the spectral decomposition of the operator $S_{wj}$, then (10) takes the form

$$P_{j}(\mathcal{W}, \mathbf{X}) \leq 2 \sum_{\alpha} \lambda_{j}^{\alpha} |e_{j}^{\alpha}\rangle \left[ I - \left( \frac{\sum_{l=1}^{M} S_{wl}^{r}}{(\lambda_{j}^{\alpha})^{r}} \right)^{-1/2} \right] |e_{j}^{\alpha}\rangle.$$ (11)

Applying the inequality

$$2(1 - x^{-1/2}) \leq (x - 1), \quad x > 0,$$ (12)

we obtain

$$2 \left[ I - \left( \frac{\sum_{l=1}^{M} S_{wl}^{r}}{(\lambda_{j}^{\alpha})^{r}} \right)^{-1/2} \right] \leq \left( \frac{\sum_{l=1}^{M} S_{wl}^{r}}{(\lambda_{j}^{\alpha})^{r}} \right) - I$$ (13)

$$= (\lambda_{j}^{\alpha})^{-r} \sum_{l \neq j} S_{wl}^{r} \sum_{\beta \neq \alpha} \left[ \left( \frac{\lambda_{j}^{\beta}}{\lambda_{j}^{\alpha}} \right)^{r} - 1 \right] |e_{j}^{\beta}\rangle\langle e_{j}^{\beta}|$$ (14)

By substituting this into (11), we see that for $0 < r \leq 1$

$$P_{j}(\mathcal{W}, \mathbf{X}) \leq \text{Tr} S_{wj}^{1-r} \sum_{l \neq j} S_{wl}^{r},$$ (15)

in particular, for $r = 1/2$ we obtain (7). By continuity argument we can drop the assumption of nondegeneracy of the operators $S_{wj}$.

Corollary: For any collection of states $S_{i}; i = 1, \ldots a$

$$C \geq -\log \min_{\pi} \text{Tr} \left[ \sum_{i=1}^{a} \pi_{i} \sqrt{S_{i}} \right]^{2}.$$ (16)

Proof. Let us apply random coding. Then from (7) using the fact that the words are i.i.d., we find

$$\min_{\mathbf{X}} \hat{P}(\mathcal{W}, \mathbf{X}) \leq (M - 1) \text{Tr} (\sqrt{S_{wj}})^{2}.$$ (17)

The expectation is

$$\text{Tr} (\sqrt{S_{wj}})^{2} = \text{Tr} \left( \sum_{i} \pi_{i} \sqrt{S_{i}} \right)^{n} = \left[ \text{Tr} \left( \sum_{i} \pi_{i} \sqrt{S_{i}} \right)^{2} \right]^{n},$$
which gives (5) with \( s = 1, c = 1 \). Choosing \( M = 2^n R \), we get (16). \( \square \)

Remark: The above proof did not involve the quantum coding theorem (2). On the other hand, by letting \( s = 1 \) in (3) we obtain inequality

\[
H \left( \sum_{i=1}^{a} \pi_i S_i \right) - \sum_{i=1}^{a} \pi_i H(S_i) \geq - \log \text{Tr} \left( \sum_{i=1}^{a} \pi_i S_i \right)^2
\]

which, combined with (2), also gives (16).

The quantity in the right-hand side of (16) is a quantum analog of the cutoff rate widely used in applications of information theory (see [1]). Since it is easier to calculate than the capacity (the minimum over \( \pi \) can be evaluated explicitly), it can be used as a practical lower bound. In particular, consider a quantum-quantum channel \( \Phi \), which is a completely positive trace preserving map of states, and substitute \( S_i = \Phi^\otimes n(T_i) \), where \( T_i \) are the input states in \( \mathcal{H}^\otimes n \) to be optimized after. Then (16) implies a lower bound for the classical capacity of the channel, which might be relevant to the problem of additivity of the capacity with respect to entangled inputs [2], [9], although the problem of additivity of the cutoff rate is itself by no means simple.

Finally let us show how the bound (5) can be obtained for commuting \( S_i \) along these lines. Taking expectation of (11), we get

\[
\mathbb{E} P_j(W, X) \leq \sum_{\alpha} \mathbb{E} \sum_{\alpha} \lambda_j^\alpha \langle e_j^\alpha | \langle 2 \mathbb{E} \left\{ I - \left( \sum_{l=1}^{M} S_{w}^l \right)^{-1/2} / (\lambda_j^\alpha)^{r} \right\} \right| w^j \rangle \langle e_j^\alpha |,
\]

where inside is the conditional expectation with respect to the fixed word \( w^j \). Taking this conditional expectation in (14), we get

\[
\mathbb{E} \left\{ 2 \left[ I - \left( \sum_{l=1}^{M} S_{w}^l \right)^{-1/2} / (\lambda_j^\alpha)^{r} \right] \right| w^j \} \leq (\lambda_j^\alpha)^{-r} (M - 1) E S_{w}^l \right| \sum_{\beta \neq \alpha} \left[ \left( \lambda_j^\beta / \lambda_j^\alpha \right)^{r} - 1 \right] |e_j^\beta \rangle \langle e_j^\beta |.
\]

On the other hand, the left hand side is less or equal than \( 2 I \). If all operators commute, all the matrices are diagonal in the basis \( \{ e_j^\beta \} \) which is the same
for all \( j \), and we can use inequalities \( \min(2, x) \leq 2x^s \) and \( (x + y)^s \leq x^s + y^s \), valid for \( x, y \geq 0 \) and \( 0 \leq s \leq 1 \), to obtain that the left hand side does not exceed

\[
2 \left[ (\lambda_j^\alpha)^{-rs}(M - 1)^s (\mathbb{E} S_{\omega^r})^s + \sum_{\beta \neq \alpha} \left( \frac{\lambda_j^\beta}{\lambda_j^\alpha} \right)^r - 1 \right] ^s [e_j^\beta \langle e_j^\beta \rangle].
\]

Substituting this into (19), we obtain

\[
\mathbb{E} P_j(\mathcal{W}, X) \leq \mathbb{E} \sum_{\alpha} (\lambda_j^\alpha)^{1-rs} \langle e_j^\alpha | 2(M - 1)^s (\mathbb{E} S_{\omega^r})^s | e_j^\alpha \rangle \quad (22)
\]

\[
= 2(M - 1)^s \text{Tr} (\mathbb{E} S_{\omega^r}^{1-rs}) (\mathbb{E} S_{\omega^r}^r)^s. \quad (23)
\]

Choosing \( r = \frac{1}{1+s} \) this gives

\[
\mathbb{E} P_j(\mathcal{W}, X) \leq 2(M - 1)^s \text{Tr} \left( \mathbb{E} S_{\omega^r}^{1+s} \right)^{1+s}, \quad (24)
\]

whence (13) follows.

Moreover, we can omit the factor 2, if we use commutativity from the start, avoid the Cauchy-Schwarz inequality and use \( 1 - x^{-1} \leq x - 1, x > 0 \), instead of (12).

### III. The expurgation lower bound

As it is well known in the classical information theory, for low rates \( R \) codes with high probability of error become to dominate in the random coding ensemble. In order to reduce the influence of choosing such bad codes ingenious expurgation technique has been developed, see [5], Ch. 5.7.

**Theorem:** For arbitrary density operators \( S_i \) the expurgation bound holds:

\[
\min_{\mathcal{W}, X} P_{max}(\mathcal{W}, X) \leq \inf_{s \geq 1} \left( 4(M - 1) \left[ \sum_{i,k=1}^a \pi_i \pi_k (\text{Tr} \sqrt{S_i} \sqrt{S_k})^\frac{1}{s} \right]^n \right)^s. \quad (25)
\]

**Proof.** Using (7) and the inequality \( \sum a_i^r \leq \sum a_i^r, 0 < r \leq 1 \), we obtain for \( s \geq 1 \)
\[(P_j(W,X))^{1/s} \leq \left( \sum_{l \neq j} \text{Tr} \sqrt{S_{wl} \sqrt{S_{wl}}} \right)^{1/s} \leq \sum_{l \neq j} \left( \text{Tr} \sqrt{S_{wj} \sqrt{S_{wl}}} \right)^{1/s}. \quad (26)\]

We again apply the Shannon’s random coding scheme, assuming that the codewords are chosen at random, independently and with the probability distribution (4) for each word. We start with an ensemble of codes with \(M' = 2M - 1\) codewords. Then according to the Lemma from Ch. 5.7 [5] (which is a simple corollary of the central limit theorem) there exists a code in the ensemble of codes with \(M' = 2M - 1\) codewords, for which at least \(M\) codewords satisfy

\[P_j(W,X) \leq \left[ 2E P_j(W,X)^{1/s} \right]^s, \quad (27)\]

for arbitrary \(s \geq 1\) (without loss of generality we can assume that (27) holds for \(j = 1, \ldots, M\)). Then taking into account that \(M' - 1 = 2(M - 1)\), we have from (26)

\[P_j(W,X) \leq \left[ 4(M - 1)E(\text{Tr} \sqrt{S_{wl} \sqrt{S_{wl}}})^{1/s} \right]^s. \quad (28)\]

Using the fact that the words are i.i.d., we find

\[E(\text{Tr} \sqrt{S_{wl} \sqrt{S_{wl}}})^{1/s} = \sum_{i_1, \ldots, i_n, j_1, \ldots, j_n} \pi_{i_1} \cdots \pi_{i_n} \pi_{j_1} \cdots \pi_{j_n} (\text{Tr} \sqrt{S_{i_1} \sqrt{S_{j_1}}})^{1/s} \cdots (\text{Tr} \sqrt{S_{i_n} \sqrt{S_{j_n}}})^{1/s} = \left[ \sum_{i,j} \pi_i \pi_j (\text{Tr} \sqrt{S_i \sqrt{S_j}})^{1/s} \right]^n, \quad (29)\]

whence the theorem follows. □

Again, it is convenient to introduce the function

\[\tilde{\mu}(\pi, s) = -s \log \sum_{i,k=1}^{a} \pi_i \pi_k (\text{Tr} \sqrt{S_i \sqrt{S_k}})^{1/s},\]

then taking \(M = 2^n R\), we obtain the expurgation lower bound for the reliability function

\[E(R) \geq \max_{\pi} \sup_{s \geq 1} (\tilde{\mu}(\pi, s) - sR) \equiv E_{ex}(R),\]
The function $\tilde{\mu}(\pi, s)$ is concave (see Appendix), increasing from the value

$$\tilde{\mu}(\pi, 1) = \mu(\pi, 1) = -\log \text{Tr} \left( \sum_{i=1}^{a} \pi_i \sqrt{S_i} \right)^2$$

for $s = 1$ to

$$\tilde{\mu}(\pi, \infty) = -\sum_{i,k=1}^{a} \pi_i \pi_k \log \text{Tr} \sqrt{S_i \sqrt{S_k}},$$

(which may be infinite).

By introducing

$$E_{ex}(\pi, R) = \sup_{s \geq 1} [\tilde{\mu}(\pi, s) - sR],$$

we can investigate the behavior of $E_{ex}(\pi, R)$ like in the classical case. Namely, for $0 < R \leq \tilde{\mu}'(\pi, 1)$, where $\tilde{\mu}'(\pi, 1) \leq \tilde{\mu}(\pi, 1)$ (see Appendix), the function $E_{ex}(\pi, R)$ is concave, decreasing from

$$E_{ex}(\pi, +0) = \tilde{\mu}(\pi, \infty)$$

(31)

to $E_{ex}(\tilde{\mu}'(\pi, 1)) = \tilde{\mu}(\pi, 1) - \tilde{\mu}'(\pi, 1)$. In the interval $\tilde{\mu}'(\pi, 1) \leq R \leq \tilde{\mu}(\pi, 1)$ it is linear function

$$E_{ex}(\pi, R) = \tilde{\mu}(\pi, 1) - R,$$

and $E_{ex}(\pi, R) = 0$ for $\tilde{\mu}(\pi, 1) \leq R < C$.

Finally, let us evaluate the limiting value $E(+0)$ of the reliability function at zero rate. We remind notation $|A| = \sqrt{A^*A}$ where $A$ is an operator in $\mathcal{H}$.

**Proposition 2:** If $S_i S_k \neq 0$ for any $1 \leq i, k \leq a$ then

$$-\min_{\pi} \sum_{i,k=1}^{a} \pi_i \pi_k \log \text{Tr} \sqrt{S_i \sqrt{S_k}} \leq E(+0)$$

$$\leq -2 \min_{\pi} \sum_{i,k=1}^{a} \pi_i \pi_k \log \text{Tr} \sqrt{S_i \sqrt{S_k}},$$

(32)

If $S_i S_k = 0$ for some $i, k$, then $E(+0) = \infty$.

**Proof.** The proof is a generalization of that given in [3] for the case of pure states. Note that in this case the left and right hand sides of (32) coincide, giving the exact values of $E(+0)$.
From (31) we see that $E(+0)$ is greater than or equal to the left hand side of (32). On the other hand,

$$P_{\text{max}}(\mathcal{W}, \mathbf{X}) \geq \max_{w \neq w'} \min_{\mathbf{X}} P(\{S_w, S_{w'}\}, \mathbf{X}),$$

where $w, w'$ are arbitrary two codewords from $\mathcal{W}$. The minimal error probability of discrimination between the two equiprobable states $S_w, S_{w'}$ is

$$\min_{\mathbf{X}} P(\{S_w, S_{w'}\}, \mathbf{X}) = \frac{1}{2} \left[ 1 - \frac{1}{2} \text{Tr} |S_w - S_{w'}| \right].$$

(cf. [8]). In [7] the following estimates were established for the trace norm of the difference $S_1 - S_2$ of any two density operators $S_1, S_2$:

$$2(1 - \text{Tr} \sqrt{S_1 \sqrt{S_2}}) \leq \text{Tr} |S_1 - S_2| \leq 2 \sqrt{1 - (\text{Tr} \sqrt{S_1 \sqrt{S_2}})^2}.$$

The proof of the second inequality can be easily modified to obtain

$$\text{Tr} |S_1 - S_2| \leq 2 \sqrt{1 - (\text{Tr} \left| \sqrt{S_1} \sqrt{S_2} \right|)^2},$$

(see also [4]). Therefore we get

$$\min_{\mathbf{X}} P(\{S_w, S_{w'}\}, \mathbf{X}) \geq \frac{1}{2} \left[ 1 - \sqrt{1 - (\text{Tr} \left| \sqrt{S_w} \sqrt{S_{w'}} \right|)^2} \right] \geq \frac{1}{4} (\text{Tr} \left| \sqrt{S_w} \sqrt{S_{w'}} \right|)^2,$$

and

$$P_{\text{max}}(\mathcal{W}, \mathbf{X}) \geq \max_{w \neq w'} \frac{1}{4} (\text{Tr} \left| \sqrt{S_w} \sqrt{S_{w'}} \right|)^2.$$

It follows that

$$E(+0) \leq -\lim_{n \to \infty} \frac{2}{2} \max_{w \neq w'} \log \text{Tr} \left| \sqrt{S_w} \sqrt{S_{w'}} \right|.$$

Repeating argument from the proof of Proposition 3 from [3], we obtain the second inequality in (32). \(\square\)
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APPENDIX

Properties of the functions $\tilde{\mu}(\pi, s), \mu(\pi, s)$.

1. Let us calculate derivatives of the function $\tilde{\mu}(\pi, s)$ with respect to $s$ and show that the first derivative is nonnegative, while the second is nonpositive. Denoting $F_{ik}(s) = (\text{Tr}\sqrt{\sqrt{S_i}S_k})^2$, $F(s) = \sum_{i,k=1}^a \pi_i \pi_k F_{ik}(s)$, we have (with log denoting in what follows natural logarithms)

$$\tilde{\mu}'(\pi, s) = - \log F(s) - sF(s)^{-1}F'(s)$$

$$= F(s)^{-1} \sum_{i,k=1}^a \pi_i \pi_k F_{ik}(s) (\log F_{ik}(s) - \log F(s)).$$

Using the inequality

$$x(\log x - \log y) \geq x - y, \quad x, y > 0 \quad (33)$$

(cf. Proposition 3.16 of [11]), we see that indeed $\tilde{\mu}'(\pi, s) \geq 0$. Taking into account that $F'(1) \leq 0$, we also obtain $\tilde{\mu}'(\pi, 1) \leq \tilde{\mu}(\pi, 1)$.

The second derivative

$$\tilde{\mu}''(\pi, s) = (sF(s)^2)^{-1} \left[ \left( \sum_{i,k=1}^a \pi_i \pi_k F_{ik}(s) \log F_{ik}(s) \right)^2 
- \sum_{i,k=1}^a \pi_i \pi_k F_{ik}(s) (\log F_{ik}(s))^2 \sum_{i,k=1}^a \pi_i \pi_k F_{ik}(s) \right]$$

is nonpositive by Cauchy-Schwarz inequality.
2. Let us show that \( \mu'(\pi, s) \geq 0 \). Introducing operator valued function

\[
A(s) = \sum_{i=1}^{a} \pi_i S_i^{1+s},
\]

and letting \( G(s) = \text{Tr} A(s)^{1+s} \), we have \( \mu(\pi, s) = -\log G(s) \), so that

\[
\mu'(\pi, s) = -G(s)^{-1}G'(s).
\]

To calculate \( G'(s) \) we use a generalization of formula (3.17) from \([11]\), namely

\[
\frac{d}{ds} \text{Tr} f(s, A(s)) = \text{Tr} f'_s(s, A(s)) + \text{Tr} f'_A(s, A(s)) A'(s).
\]

(34)

We then obtain

\[
\frac{d}{ds} G(s) = -\text{Tr} A(s)^s \sum_{i=1}^{a} \pi_i S_i^{\frac{1}{1+s}} \left[ \log S_i^{\frac{1}{1+s}} - \log A(s) \right].
\]

(35)

From (33) we have

\[
y^s x (\log x - \log y) \geq y^s (x - y), \quad x, y > 0,
\]

(36)

therefore by Proposition 3.16 from \([11]\)

\[
-\frac{d}{ds} G(s) \geq \text{Tr} A(s)^s \sum_{i=1}^{a} \pi_i \left[ S_i^{\frac{1}{1+s}} - A(s) \right] = 0.
\]

In the classical case the function \( \mu(\pi, s) \) is concave in \( s \) \([5]\), Appendix 5B. We conjecture this property holds also in the quantum case, and we postpone this problem to a separate investigation.

3. To compute \( E_r(R) \) according to the definition (3) it is expedient to perform maximization with respect to \( \pi \) first. Maximizing \( \mu(\pi, s) \) is equivalent to minimizing

\[
G(\pi, s) = \text{Tr} \left( \sum_{i=1}^{a} \pi_i S_i^{\frac{1}{1+s}} \right)^{1+s}.
\]

By Proposition 3.1 of \([11]\) this function is convex in \( \pi \), which makes the general criterium of Theorem 4.4.1 from \([3]\) applicable.
From this theorem it follows that probability distribution $\pi$ minimizes $G(\pi, s)$ if and only if there exists a constant $c$ such that

$$\frac{\partial G(\pi, s)}{\partial \pi_j} \geq c, \quad j = 1, \ldots, a,$$

with equality for those $j$, for which $\pi_j > 0$. After some computation, this amounts to

$$\text{Tr} S_j^{1+s} \left( \sum_{i=1}^{a} \pi_i S_i^{1+s} \right)^s \geq \text{Tr} \left( \sum_{i=1}^{a} \pi_i S_i^{1+s} \right)^{1+s}, \quad (37)$$

with the corresponding equalities.

By using this necessary and sufficient condition one shows, as in Example 4 of Sec.5.6 [5], that for two parallel channels 1 and 2

$$\max_{\pi_{12}} \mu_{12}(\pi_{12}, s) = \max_{\pi_1} \mu_1(\pi_1, s) + \max_{\pi_2} \mu_2(\pi_2, s),$$

implying a corresponding additivity property for $E_r(R)$. This gives an answer to a question posed by R. Ahlswede. It is worthwhile to remind that the additivity property does not hold in general for $\bar{\mu}(\pi, s)$ even in the classical case [5], Problem 5.26.

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