ELLIPITC DIFFERENTIAL EQUATIONS WITH MEASURABLE COEFFICIENTS

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Abstract. We prove the unique solvability of second order elliptic equations in non-divergence form in Sobolev spaces. The coefficients of the second order terms are measurable in one variable and VMO in other variables. From this result, we obtain the weak uniqueness of the martingale problem associated with the elliptic equations.

1. Introduction

We study the $L_p$-theory of the elliptic differential equation

$$a^{jk}(x)u_{x^jx^k}(x) + b^j(x)u_{x^j}(x) + c(x)u(x) = f(x) \quad \text{in} \quad \mathbb{R}^d,$$  

(1.1)

where $a^{jk}(x)$ are allowed to be only measurable with respect to one coordinate, say $x^1 \in \mathbb{R}$, where $x = (x^1, x') \in \mathbb{R}^d$, $x' \in \mathbb{R}^{d-1}$.

It is well known that if the coefficients $a^{jk}$ are only measurable, then there could not exist a unique solution to the above equation even in a very generalized sense (see [11, 13]). We are interested in more regular solutions. In 1967 Ural’tseva (see [7] or the original paper [17]) constructed an example of an equation in $\mathbb{R}^d$ for $d \geq 3$ with the coefficients depending only on the first two coordinates for which there is no unique solvability in $W^2_p$ with $p \geq d$ (for any $d \geq 3$ and $p \in (1, d)$ this was known before).

Thus in order to have the unique solvability of the equation in $W^2_p$, we have to impose some (regularity) conditions on the coefficients $a^{jk}$. The most classical case is when $a^{jk}$ are uniformly continuous. We can also have piecewise continuous or VMO coefficients. For details, see [1, 2, 4, 6, 9, 10].

In this paper, we show that there exists a unique solution to the above equation in $W^2_p$, $p \in (2, \infty)$, under the assumption that $a^{jk}(x^1, x')$ are

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measurable in $x^1 \in \mathbb{R}$ and VMO in $x' \in \mathbb{R}^{d-1}$. See Assumptions 2.1 and 2.2 below. If the coefficients $a^{jk}$ are independent of $x' \in \mathbb{R}^{d-1}$ (more generally, uniformly continuous in $x' \in \mathbb{R}^{d-1}$, see Remark 2.6), then the equation is uniquely solvable in $W^2_2$ as well. In addition, we show that one can easily solve the equation with the Dirichlet, Neumann, or oblique derivative boundary condition in a half space, say $\mathbb{R}^d_+ = \{(x^1, x'): x^1 > 0, x' \in \mathbb{R}^{d-1}\}$, using the results for equations in the whole space.

The class of coefficients we are dealing with is considerably more general than those previously known, as long as $p \in [2, \infty)$. It actually contains almost all types of discontinuous coefficients that have been investigated so far. For example, it contains the class of piecewise continuous coefficients investigated in [1, 8, 9]. It also contains VMO coefficients with which elliptic equations were investigated in [1, 2, 6]. Also see the monograph [10], which treats elliptic and parabolic equations with discontinuous coefficients including oblique derivative problems with VMO coefficients. Although, we also slightly touch the oblique derivative problem, we do not say anything about many important issues of equations with VMO coefficients, which are discussed, for instance, in [14, 15, 12].

The highlight of our assumptions on the coefficients $a^{jk}$ would be: no assumptions on the regularity of the coefficients with respect to one variable as far as they are uniformly bounded and elliptic. Having only measurable coefficients (as functions of $x^1 \in \mathbb{R}$), we obtain the $L_2$-estimate for the equation by using the usual Fourier transforms. Based upon this estimate, we establish the $L_p$-estimate, $p \in (2, \infty)$, using the approach initiated by the second author of this paper (for example, see [6]). In this approach we make use of a pointwise estimate of sharp functions of second order derivatives of the solution. As noted in [6], thanks to this method, we do not need any integral representations of the solution nor commutators, which were used, for example, in [1, 2]. Especially, we deal with VMO coefficients in a rather straightforward manner.

One good motivation to consider the above equation in the whole space is to prove weak uniqueness of stochastic processes associated with the elliptic equation. As is shown in [6, 16], we can say that weak uniqueness of the processes holds true once we find a unique solution of the elliptic equation in $W^2_p$, $p \geq d$. More details are in [6, 16].

The paper is organized as follows. In Section 2 we state our main results. The unique solvability of the equation in $W^2_2$ is investigated in Section 3. In Section 4 we present some auxiliary results which are
used in Section 5 where we finally prove the $W^2_p$-estimate, $p \in (2, \infty)$, for the equation.

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2. Main results

We are considering the elliptic differential equation (1.1) where the coefficients $a^j, b^j$, and $c$ satisfy the assumptions below.

**Assumption 2.1.** The coefficients $a^j, b^j$, and $c$ are measurable functions defined on $\mathbb{R}^d$, $a^j = a^{kj}$. There exist positive constants $\delta \in (0, 1)$ and $K$ such that

$$|b^j(x)| \leq K, \quad |c(x)| \leq K,$$

$$\delta |\vartheta|^2 \leq \sum_{j,k=1}^d a^{jk}(x) \vartheta_j \vartheta_k \leq \delta^{-1} |\vartheta|^2$$

for any $x \in \mathbb{R}^d$ and $\vartheta \in \mathbb{R}^d$.

To state another assumption on the coefficients, especially, $a = (a^j)$, we introduce some notations. Let $B'_r(x') = \{y' \in \mathbb{R}^{d-1} : |x' - y'| < r\}$ and $Q_r(x) = Q_r(x^1, x') = (x^1 - r, x^1 + r) \times B'_r(x')$. Denote

$$\text{osc}_{x'}(a, Q_r(x)) = r^{-1} |B'_r|^{-2} \int_{x^1-r}^{x^1+r} \int_{y',z' \in B'_r(x')} |a(t, y') - a(t, z')| dy'dz' dt,$$

$$a^\#(x') = \sup_{x \in \mathbb{R}^d} \sup_{r \leq R} \text{osc}_{x'}(a, Q_r(x)),$$

where $|B'_r|$ is the $d - 1$-dimensional volume of $B'_r(0)$. We write $a \in \text{VMO}_{x'}$ if

$$\lim_{R \to 0} a^\#(x') = 0.$$

We see that $a \in \text{VMO}_{x'}$ if $a$ is independent of $x'$.

**Assumption 2.2.** There is a continuous function $\omega(t)$ defined on $[0, \infty)$ such that $\omega(0) = 0$ and $a^\#(x') \leq \omega(R)$ for all $R \in [0, \infty)$.

**Remark 2.3.** It will be seen from our proofs that in Assumption 2.2 the requirement that $\omega(0) = 0$ can be replaced with $\omega(0) \leq (4N_1)^{-\nu(d+2)}$, where $N_1 = N_1(d, \delta, p)$ and $\nu = \nu(p)$ are the constants entering (5.6).

As usual, we mean by $W^k_p(\mathbb{R}^d)$, $k = 0, 1, \ldots$, the Sobolev spaces on $\mathbb{R}^d$. Set $W^k_p = W^k_p(\mathbb{R}^d)$, $L_p = L_p(\mathbb{R}^d)$, and

$$Lu(x) = a^{jk}(x)u_{x_j x_k}(x) + b^j(x)u_{x^j}(x) + c(x)u(x).$$

Here are our main results.
**Theorem 2.4.** Let $p \in (2, \infty)$. Then there exists a constant $\lambda_0$, depending only on $d$, $\delta$, $K$, $p$, and the function $\omega$, such that, for any $\lambda > \lambda_0$ and $f \in L_p$, there exists a unique $u \in W^2_p$ satisfying $Lu - \lambda u = f$.

Furthermore, there is a constant $N$, depending only on $d$, $\delta$, $K$, $p$, and the function $\omega$, such that, for any $\lambda > \lambda_0$ and $f \in L_p$, 

$$\lambda \|u\|_{L_p} + \sqrt{\lambda} \|u_x\|_{L_p} + \|u_{xx}\|_{L_p} \leq N \|Lu - \lambda u\|_{L_p}.$$  

This theorem obviously covers the case in which the coefficients $a^{jk}$ are independent of $x' \in \mathbb{R}^{d-1}$. However, in that case we can allow $p = 2$, which is detailed in the theorem below. Throughout the paper, we write $N = N(d, \ldots)$ if $N$ is a constant depending only on $d$, $\ldots$.

**Theorem 2.5.** Let the coefficients $a^{jk}$ be independent of $x' \in \mathbb{R}^{d-1}$. Then there exists a constant $\lambda_0 = \lambda_0(d, \delta, K)$ such that, for any $\lambda > \lambda_0$ and $f \in L_2$, there exists a unique $u \in W^2_2$ satisfying $Lu - \lambda u = f$.

In addition, there is a constant $N = N(d, \delta, K)$ such that, for any $\lambda \geq \lambda_0$ and $u \in W^2_2$, 

$$\lambda \|u\|_{L_2} + \sqrt{\lambda} \|u_x\|_{L_2} + \|u_{xx}\|_{L_2} \leq N \|Lu - \lambda u\|_{L_2}. \tag{2.1}$$

**Remark 2.6.** Theorem 2.4 leads to the weak uniqueness of solutions of stochastic differential equations associated with the operator $L$. For details, see [16, 6]. Theorem 2.5 clearly remains true under the assumption that $a^{jk}(x', x')$ are uniformly continuous as functions of $x' \in \mathbb{R}^{d-1}$ uniformly in $x^1 \in \mathbb{R}$.

Three more results deal with the equation $Lu - \lambda u = f$ in the half space 

$$\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^1 > 0\}.$$ 

Their proofs show the advantage of having the solvability in $\mathbb{R}^d$ of equations whose coefficients are only measurable in one direction. In what follows, we denote by $\dot{W}^2_p(\mathbb{R}^d_+)$ the collection of all $u \in W^2_p(\mathbb{R}^d_+)$ satisfying $u(0, x') \equiv 0$.

**Theorem 2.7.** Let $p \in [2, \infty)$. If $p = 2$, then suppose, additionally, that the assumption in Theorem 2.4 is satisfied. Then there exists a constant $\lambda_0 = \lambda_0(d, \delta, K, p, \omega) \geq 0$ such that, for any $\lambda > \lambda_0$ and $f \in L_p(\mathbb{R}^d_+)$, there exists a unique $u \in W^2_p(\mathbb{R}^d_+)$ satisfying $Lu - \lambda u = f$. 

Furthermore, there is a constant $N = N(d, \delta, K, p, \omega)$ such that, for any $\lambda \geq \lambda_0$ and $u \in W^0_p(\mathbb{R}^d)$,

$$
\lambda \|u\|_{L_p(\mathbb{R}^d)} + \sqrt{\lambda} \|u_x\|_{L_p(\mathbb{R}^d)} + \|u_{xx}\|_{L_p(\mathbb{R}^d)} \leq N \|L u - \lambda u\|_{L_p(\mathbb{R}^d)}.
$$

(2.2)

Proof. We introduce a new operator $\hat{L} v = \hat{\alpha}^{jk} v_{x_j x_k} + \hat{b}^j v_{x_j} + \hat{c} v$ the coefficients of which are as follows. First we view the coefficients $\hat{\alpha}^{jk}$, $\hat{b}^j$, and $\hat{c}$ as functions defined only on $\mathbb{R}^d_+$. Then we define $\hat{\alpha}^{jk}$, $\hat{b}^j$, and $\hat{c}$ to be the odd or even extensions of the original coefficients. Specifically, if $j = k = 1$ or $j, k \in \{2, \ldots, d\}$, then (even extension)

$$
\hat{\alpha}^{jk}(x) = \left\{ \begin{array}{ll}
\hat{\alpha}^{jk}(x^1, x') & \text{if } x^1 \geq 0 \\
\hat{\alpha}^{jk}(-x^1, x') & \text{if } x^1 < 0
\end{array} \right.
$$

If $j = 2, \ldots, d$, then (odd extension)

$$
\hat{a}^{ij}(x) = \hat{a}^{ij}(x) = \left\{ \begin{array}{ll}
\hat{a}^{ij}(x^1, x') & \text{if } x^1 \geq 0 \\
-\hat{a}^{ij}(-x^1, x') & \text{if } x^1 < 0
\end{array} \right.
$$

Similarly, the coefficient $\hat{b}^1(x)$ is the odd extension of $b^1(x)$, and the coefficients $\hat{b}^j(x)$, $j = 2, \ldots, d$, and $\hat{c}(x)$ are the even extensions of $b^j(x)$ and $c(x)$, respectively.

Now we notice that the coefficients of $\hat{L}$ satisfy Assumption 2.1 and 2.2 with $2 \omega$. Then by Theorem 2.4 and 2.5, we can find a constant $\lambda_0 = \lambda_0(d, \delta, K, p, \omega)$ such that, for any $\lambda > \lambda_0$, there exists a unique $u \in W^2_p$ satisfying $\hat{L} u - \lambda u = \hat{f}$, where $\hat{f} \in L_p$ is the odd extension of $f \in L_p(\mathbb{R}^d)$. Obviously, $-u(-x^1, x') \in W^2_p$ also satisfies the same equation, so by uniqueness we have $u(x^1, x') = -u(-x^1, x')$. This implies that $u$, as a function defined on $\mathbb{R}^d_+$, is in the space $W^0_p(\mathbb{R}^d)$. Since $Lu - \lambda u = f$ in $\mathbb{R}^d$, the function $u$ is a solution to the Dirichlet boundary problem.

To prove uniqueness and the estimate (2.2), we use the estimates in Theorem 2.4 and 2.5 and the fact that the odd extension of an element in $W^2_p(\mathbb{R}^d_+)$ is in $W^2_p$. The theorem is proved.

In the same way, only this time taking the even extension of $f$, one gets the solvability of the Neumann problem.

**Theorem 2.8.** Let $p \in [2, \infty)$. If $p = 2$, then suppose, additionally, that the assumption in Theorem 2.6 is satisfied. Then there exists a constant $\lambda_0 = \lambda_0(d, \delta, K, p, \omega) > 0$ such that, for any $\lambda > \lambda_0$ and $f \in L_p(\mathbb{R}^d)$, there exists a unique $u \in W^2_p(\mathbb{R}^d)$ satisfying $Lu - \lambda u = f$ and $u_{x^1} = 0$ on $\partial \mathbb{R}^d_+$. 
Furthermore, there is a constant $N = N(d, \delta, K, p, \omega)$ such that, for any $\lambda \geq \lambda_0$ and $u \in W^2_p(\mathbb{R}^d_+)$ satisfying $u_{x^1} = 0$ on $\partial \mathbb{R}^d_+$,

$$\lambda \|u\|_{L_p(\mathbb{R}^d_+)} + \sqrt{\lambda} \|u_x\|_{L_p(\mathbb{R}^d_+)} + \|u_{xx}\|_{L_p(\mathbb{R}^d_+)} \leq N \|Lu - \lambda u\|_{L_p(\mathbb{R}^d_+)},$$

While the Neumann problem is solved without any effort, oblique derivative problems need some, still simple, manipulations.

Let $\ell$ be a constant vector field $\ell = (\ell^1, \ldots, \ell^d)$, where $\ell^1 > 0$. Set $s = 1 - 1/p$ and recall that $g \in W^s_p(\mathbb{R}^{d-1})$ if

$$\|g\|_{W^s_p(\mathbb{R}^{d-1})} = \|g\|_{L_p(\mathbb{R}^{d-1})} + [g]_s < \infty,$$

where

$$[g]_s^p = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{|g(x') - g(y')|^p}{|x' - y'|^{d+1+sp}} \, dx' \, dy'.$$

**Theorem 2.9.** Let $p \in [2, \infty)$. If $p = 2$, then suppose, additionally, that the assumption in Theorem 2.8 is satisfied. Then there exists a constant $\lambda_0 = \lambda_0(d, \delta, K, p, \omega, \ell) \geq 0$ such that, for any $\lambda > \lambda_0$, $f \in L_p(\mathbb{R}^d_+)$, and $g \in W^{1-1/p}(\mathbb{R}^{d-1})$, there exists a unique $u \in W^2_p(\mathbb{R}^d_+)$ satisfying $Lu - \lambda u = f$ and $\ell^1 u_{x^1} = g$ on $\partial \mathbb{R}^d_+$.

Furthermore, there is a constant $N = N(d, \delta, K, p, \omega, \ell)$ such that, for any $\lambda \geq \lambda_0$ and $u \in W^2_p(\mathbb{R}^d_+)$,

$$\lambda \|u\|_{L_p(\mathbb{R}^d_+)} + \sqrt{\lambda} \|u_x\|_{L_p(\mathbb{R}^d_+)} + \|u_{xx}\|_{L_p(\mathbb{R}^d_+)} \leq N \left( \|Lu - \lambda u\|_{L_p(\mathbb{R}^d_+)} + (\lambda \lor 1)^{s/2} \|g\|_{L_p(\mathbb{R}^{d-1})} + [g]_s \right), \quad (2.3)$$

where $\lambda \lor 1 = \max\{\lambda, 1\}$, $s = 1 - 1/p$, and $g(x') = \ell^j u_{x^j}(0, x')$.

**Proof.** We can assume that $\ell^1 = 1$. To introduce a new operator

$$L^{\text{vir}} = \hat{a}^{jk} v_{x^j x^k} + \hat{b}^j v_{x^j} + \hat{c} v,$$

we use a linear transformation

$$\varphi(x) = (-x^1, -2\ell' x^1 + x') \quad (\ell' = (\ell^2, \ldots, \ell^d)).$$

Set

$$\hat{a}^{jk}(x) = \begin{cases} a^{jk}(x) & \text{if } x^1 \geq 0 \\ \bar{a}^{jk}(x) & \text{if } x^1 < 0 \end{cases},$$

where

$$\bar{a}^{jk}(x) = \sum_{r,l=1}^d \varphi^r_x \varphi^l_x a^{rl}(\varphi(x)).$$

Also set

$$\hat{b}^j(x) = \begin{cases} b^j(x) & \text{if } x^1 \geq 0 \\ \bar{b}^j(x) & \text{if } x^1 < 0 \end{cases}, \quad \hat{c}(x) = \begin{cases} c(x) & \text{if } x^1 \geq 0 \\ \bar{c}(x) & \text{if } x^1 < 0 \end{cases},$$

and

$$\varphi(x) = \sqrt{\lambda} \|u\|_{L_p(\mathbb{R}^d_+)} + \sqrt{\lambda} \|u_x\|_{L_p(\mathbb{R}^d_+)} + \|u_{xx}\|_{L_p(\mathbb{R}^d_+)} \leq N \sqrt{\lambda \|Lu - \lambda u\|_{L_p(\mathbb{R}^d_+)} + (\lambda \lor 1)^{s/2} \|g\|_{L_p(\mathbb{R}^{d-1})} + [g]_s}. \quad (2.4)$$
where

$$\bar{b}^i(x) = \sum_{r=1}^{d} \varphi_{x^r}^i (\varphi(x)), \quad \bar{c}(x) = c(\varphi(x)).$$

Notice that the coefficients $\hat{a}^{jk}$ satisfy the uniform ellipticity condition with $N\delta$ in place of $\delta$, where $N$ depends only on $\ell$. Also Assumption 2.2 is satisfied with $N\omega$ in place of $\omega$, where $N$ depends only on $\ell$.

After this preparation we are ready to prove the first part of the theorem. Consider a differential equation

$$\hat{L}w - \lambda w = \hat{f}_\lambda$$

in $\mathbb{R}^d$, where $\hat{f}_\lambda$ is defined as follows.

One knows (see, for instance, Theorem 2.9.1 of [18]) that for each $g \in W^2_p(\mathbb{R}^{d-1})$ there is a function $v \in W^2_p(\mathbb{R}^d)$ such that $v = 0$ and $v_{x^1} = g(x')$ on $\partial \mathbb{R}^d_+$ and, for a constant $N$ independent of $g$

$$\|v\|_{W^2_p(\mathbb{R}^d_+)} \leq N\|g\|_{W^2_p(\mathbb{R}^{d-1})}. \tag{2.5}$$

It follows by using dilations that for any $g \in W^2_p(\mathbb{R}^{d-1})$ and $\lambda > 0$, we can find $v \in W^2_p(\mathbb{R}^d)$ satisfying $v = 0$ and $v_{x^1} = g$ on $\partial \mathbb{R}^d_+$, and

$$\lambda\|v\|_{L^p(\mathbb{R}^d_+)} + \sqrt{\lambda}\|v_x\|_{L^p(\mathbb{R}^d_+)} + \|v_{xx}\|_{L^p(\mathbb{R}^d_+)} \leq N \left( \lambda^{s/2} \|g\|_{L^p(\mathbb{R}^{d-1})} + [g]_s \right), \tag{2.6}$$

where $N$ depends only on $d$ and $p$. We take this $v$ and set

$$\hat{f}_\lambda(x) = \begin{cases} f(x) - 2\hat{L}v + 2\lambda v & \text{if } x^1 > 0 \\ f(\varphi(x)) & \text{if } x^1 < 0 \end{cases}. \tag{2.7}$$

Using Theorem 2.4 and 2.5, we find a unique solution $w \in W^2_p$ to (2.4) for $\lambda > \lambda_0$, where $\lambda_0 = \lambda_0(d, \delta, K, p, \omega, \ell)$ is a constant corresponding to the operator $\hat{L}$.

Let $u^+$ be a function on $\mathbb{R}^d_+$ defined by $u^+ = w + 2v$. Also let $u^-$ be a function on

$$\mathbb{R}^d_- = \{ (x^1, x') : x^1 < 0, x' \in \mathbb{R}^{d-1} \}$$

defined by $u^-(x) = u^+(\varphi(x))$. We claim that $w = u^-$ in $\mathbb{R}^d$. This simple fact follows from the uniqueness of solution to the equation

$$\hat{L}u - \lambda u = \hat{f}_\lambda$$

in $\mathbb{R}^d_+$, proved in Theorem 2.7. Indeed, obviously, $w(0, x') = u^-(0, x')$ and it is also easy to check that $\varphi(\varphi(x)) \equiv x$ and

$$\hat{L}u - \lambda u = \hat{f}_\lambda$$

in $\mathbb{R}^d_+$.

Hence on $\partial \mathbb{R}^d_+$

$$w_{x^1} = u^-_{x^1} = (u^+(\varphi))_{x^1} = -u^+_{x^1} - 2 \sum_{j \geq 2} \ell^j u^+_{x^1}.$$
On the other hand, \( w = u^+ - 2v \) on \( \mathbb{R}^d_+ \) and on \( \partial \mathbb{R}^d_+ \)

\[
    w_{x^1} = u^+_{x^1} - 2v_{x^1} = u^+_{x^1} - 2g.
\]

It follows that on \( \partial \mathbb{R}^d_+ \) it holds that \( \ell^j u_{x^j}^+ = g \) and since \( u^+ \in W^2_p(\mathbb{R}^d_+) \) we have proved the existence of the desired solution.

To complete the proof, we now prove only (2.3), which implies uniqueness. Take a \( u \in W^2_p(\mathbb{R}^d_+) \) and set \( g(x') = \ell^j u_{x^j}(0, x') \). Then for each \( \lambda > \lambda_0 = \lambda_0(d, \delta, K, p, \omega, \ell) \), we find an extension \( v \in W^2_p(\mathbb{R}^d_+) \) of \( g \) satisfying \( v = 0 \), \( v_{x^1} = g \) on \( \partial \mathbb{R}^d_+ \), and the estimate (2.6) or (2.5) depending on whether \( \lambda \geq 1 \) or \( 0 < \lambda < 1 \). Define \( w = u - 2v \) in \( \mathbb{R}^d_+ \) and \( w(x) = u(\varphi(x)) \) in \( \mathbb{R}^d_- \). Then \( w(0+, x') = w(0-, x') \) and

\[
    w_{x^1}(0+, x') = u_{x^1}(0, x') - 2g(x') = -u_{x^1}(0, x') - 2 \sum_{j \geq 2} \ell^j u_{x^j}(0, x'),
\]

\[
    w_{x^1}(0-, x') = -u_{x^1}(0, x') - 2 \sum_{j \geq 2} \ell^j u_{x^j}(0, x') = w_{x^1}(0+, x').
\]

It then follows that \( w \) is a function in \( W^2_p \) satisfying \( \hat{L}w - \lambda w = \hat{f}_\lambda \), where \( \hat{f}_\lambda \) is defined as in (2.7) with \( f := Lu - \lambda u \). Hence by Theorem 2.4 and 2.5 we have

\[
    \lambda \|w\|_{L_p} + \sqrt{\lambda} \|w_x\|_{L_p} + \|w_{xx}\|_{L_p} \leq N \|\hat{f}_\lambda\|_{L_p},
\]

where \( N = N(d, \delta, K, p, \omega, \ell) \). This, together with the estimates (2.5) and (2.6) implies (2.3) for \( \lambda > \lambda_0 \). For \( \lambda = \lambda_0 \) we get (2.3) by continuity. \( \square \)

**Remark 2.10.** Let \( \ell(x') = (\ell^1(x'), \ldots, \ell^d(x')) \) be a bounded vector field defined on \( \mathbb{R}^{d-1} \) such that \( \ell(x') \in C^{1-1/p+\varepsilon}(\mathbb{R}^{d-1}), \varepsilon > 0 \), and \( \ell^1(x') \geq \kappa > 0 \). Then using the well-known techniques – freezing coefficients, partition of unity, and the method of continuity, we can replace the constant vector field \( \ell \) by \( \ell(x') \) in the above theorem. Details can be found in [10].

**Remark 2.11.** A result similar to Theorem 2.9 holds if we replace the boundary condition \( \ell^j u_{x^j} = g \) with \( \ell^j u_{x^j} + \sigma u = g \), where \( \sigma \) is a constant. Indeed, again assuming that \( \ell^1 = 1 \) it is easy to find an infinitely differentiable bounded function \( h(x^1) \) having bounded derivatives and bounded away from zero such that \( h'(0) = -\sigma h(0) \). Then for \( v = u/h \) we have \( \ell^j v_{x^j} = g/h \) on \( \partial \mathbb{R}^d_+ \) and \( Lu - \lambda u = h(\hat{L}v - \lambda v) \), where \( \hat{L} \phi := h^{-1}L(h\phi) \) is an elliptic operator satisfying our hypotheses with a slightly modified \( K \).
3. Proof of Theorem

Thanks to the method of continuity and the denseness of $C_0^\infty(\mathbb{R}^d)$ in $W_2^2$, it suffices to prove the apriori estimate (2.1) for $u \in C_0^\infty(\mathbb{R}^d)$ and $a^{jk}$ that are sufficiently smooth. In addition, on the account of possibly increasing $\lambda_0$ one sees that it suffices to prove (2.1) for $b \equiv 0$, $c \equiv 0$, and $\lambda_0 = 0$. In that case set

$$f = Lu - \lambda u. \quad (3.1)$$

For functions $\phi(x^1, x')$ we denote by $\tilde{\phi}(x^1, \xi)$, $\xi \in \mathbb{R}^{d-1}$, its Fourier transform with respect to $x'$. By taking the Fourier transforms of both sides of (3.1), we obtain

$$a \tilde{u}_{x^1} + i2b \tilde{u}_{x^1} - c \tilde{u} = \tilde{f},$$

$$\tilde{u}_{x^1} + i2b \tilde{u}_{x^1} - \hat{c} \tilde{u} = \tilde{g}, \quad (3.2)$$

where $i = \sqrt{-1}$ and

$$a(x^1) = a^{11}(x^1), \quad b(x^1, \xi) = \sum_{j=2}^d a^{1j}(x^1)\xi^j, \quad \hat{b} = a^{-1}b,$$

$$c(x^1, \xi) = \sum_{j,k=2}^d a^{jk}(x^1)\xi^j\xi^k + \lambda, \quad \hat{c} = a^{-1}c, \quad g = a^{-1}f.$$

Lemma 3.1. We have

$$\delta \leq a = a^{11} \leq \delta^{-1}, \quad |b(x^1, \xi)| \leq \delta^{-1} |\xi|,$$

$$\delta^{-1}(|\xi|^2 + \lambda) \geq c(x^1, \xi) \geq \delta|\xi|^2 + \lambda, \quad (3.3)$$

and

$$a(x^1)c(x^1, \xi) - b^2(x^1, \xi) \geq \delta^2(|\xi|^2 + \lambda).$$

Proof. We prove only the last inequality. From Assumption 2.1, we have

$$\delta(t^2 + |\xi|^2) \leq a(x^1) t^2 + 2 b(x^1, \xi) t + c(x^1, \xi) - \lambda$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^{d-1}$. In particular,

$$\left(a(x^1) - \delta\right) t^2 + 2 b(x^1, \xi) t + c(x^1, \xi) - \delta|\xi|^2 - \lambda \geq 0. $$

This implies that

$$b^2(x^1, \xi) - \left(a(x^1) - \delta\right) \left(c(x^1, \xi) - \delta|\xi|^2 - \lambda\right) \leq 0.$$ 

From this and (3.3) the result follows. \qed
Lemma 3.2. For any $\xi \in \mathbb{R}^d$

$$(|\xi|^2 + \lambda) \int_{\mathbb{R}} |\bar{u}_x|^2 \, dx + (|\xi|^4 + \lambda |\xi|^2 + \lambda^2) \int_{\mathbb{R}} |\bar{u}|^2 \, dx \leq N(\delta) \int_{\mathbb{R}} |\bar{f}|^2 \, dx,$$

(3.4)

$$\int_{\mathbb{R}} |\bar{u}_{x^1}|^2 \, dx \leq N(\delta) \int_{\mathbb{R}} |\bar{f}|^2 \, dx. \quad (3.5)$$

Proof. Estimate (3.4) is a direct consequence of equation (3.2) (allowing one to express $\bar{u}_{x^1}$ through $\tilde{f}$, $\bar{u}_x$, and $\bar{u}$), (3.3), and (3.4).

While proving (3.4) we define a function $\phi(x^1, \xi)$ by $\phi(0, \xi) = 0$ and $\phi_{x^1} = \hat{b}$ and set $\rho = \bar{\theta}\epsilon^\phi$. Then from (3.2) we see that

$$\rho(x^1) + (\hat{b}^2 - i \phi_{x^1})\rho = \bar{g}\epsilon^\phi.$$  

Multiply both sides by $\bar{\theta}$ and integrate the result with respect to $x^1$. Integrating by parts shows that

$$- \int_{\mathbb{R}} |\rho_{x^1}|^2 \, dx + \int_{\mathbb{R}} (\hat{b}^2 - i \phi_{x^1} - \tilde{c})|\bar{u}|^2 \, dx = \int_{\mathbb{R}} \bar{g}\bar{\theta} \, dx.$$  

Taking the real parts of both sides and multiplying by $|\xi|^2 + \lambda$, we have

$$\int_{\mathbb{R}} (|\xi|^2 + \lambda)|\rho_{x^1}|^2 \, dx + \int_{\mathbb{R}} (\tilde{c} - \hat{b}^2)(|\xi|^2 + \lambda)|\bar{u}|^2 \, dx$$  

$$= - \int_{\mathbb{R}} (|\xi|^2 + \lambda)\Re(\bar{g}\bar{\theta}) \, dx.$$  

Note that for any $\epsilon > 0$

$$-(|\xi|^2 + \lambda)\Re(\bar{g}\bar{\theta}) \leq \epsilon(|\xi|^2 + \lambda)^2|\bar{u}|^2 + \epsilon^{-1}|\bar{g}|^2.$$  

From this and Lemma 3.1 we obtain

$$\int_{\mathbb{R}} (|\xi|^2 + \lambda)|\rho_{x^1}|^2 \, dx + \int_{\mathbb{R}} (\delta^4 - \epsilon)(|\xi|^2 + \lambda)^2|\bar{u}|^2 \, dx \leq \epsilon^{-1} \int_{\mathbb{R}} |\bar{g}|^2 \, dx.$$  

By choosing an appropriate $\epsilon > 0$ (e.g. $\epsilon = \delta^4/2$), we arrive at

$$\int_{\mathbb{R}} (|\xi|^2 + \lambda)|\rho_{x^1}|^2 \, dx + \int_{\mathbb{R}} (|\xi|^4 + \lambda |\xi|^2 + \lambda^2)|\bar{u}|^2 \, dx \leq N(\delta) \int_{\mathbb{R}} |\bar{f}|^2 \, dx.$$  

It only remains to observe that in light of (3.3)

$$|\bar{u}_x| = |\rho_{x^1} - i\bar{b}\epsilon^\phi| \leq |\rho_{x^1}| + N(\delta)|\xi||\bar{u}|.$$  

□
Now we can finish the proof of Theorem 2.5. As we pointed out in the beginning of the section we only need to prove (2.1) for \( u \in C_0^\infty(\mathbb{R}^d) \), smooth \( a^{ij}, b \equiv 0, c \equiv 0, \) and \( \lambda_0 = 0. \)

In that case it suffices to add (3.4) and (3.5), integrate over \( \mathbb{R}^d - 1 \) and use Parseval’s identity. The theorem is proved.

**Remark 3.3.** We have just proved that if \( b^j = c = 0, \) then
\[
\lambda \|u\|_{L_2} + \sqrt{\lambda} \|u_x\|_{L_2} + \|u_{xx}\|_{L_2} \leq N \|Lu - \lambda u\|_{L_2}
\]
for \( u \in W_2^2 \) and \( \lambda \geq 0, \) where \( N \) depends only on \( \delta. \)

### 4. Auxiliary results

Here we state and prove a series of observations which are needed in the proof of Theorem 2.4. First we introduce some notation. As usual, we set \( B_r(x_0) = \{ x \in \mathbb{R}^d : |x - x_0| < r \} \) and \( B_r = B_r(0). \) By \( |B_r| \) we mean the \( d \)-dimensional volume of \( B_r. \) We denote by \( |u|_0 \) the supremum of \( u \) over \( \mathbb{R}^d. \)

Throughout this section, we assume that \( Lu(x) = L_0u(x) = a^{jk}(x^1)u_{x_jx^k}(x). \)

Our first auxiliary result is the following.

**Lemma 4.1.** There exists \( N = N(d, \delta) \) such that, for any \( u \in W_2^2(B_R) \) with \( u|_{\partial B_R} = 0, \) we have
\[
R^2 \int_{B_R} |u_x|^2 \, dx + \int_{B_R} |u|^2 \, dx \leq N R^4 \int_{B_R} |Lu|^2 \, dx. \tag{4.1}
\]

**Proof.** Assume that (4.1) is true when \( R = 1. \) For a given \( u \in W_2^2(B_R) \) with \( u|_{\partial B_R} = 0, \) we set
\[
L_R = a^{jk}(R^x) \frac{\partial^2}{\partial x_j \partial x^k} \quad \text{and} \quad v(x) = R^{-2}u(Rx).
\]
Then \( v \in W_2^2(B_1) \) and \( L_Rv(x) = (Lu)(Rx) \) in \( B_1. \) Since \( L_R \) satisfies the same ellipticity condition as \( L \) does, we have
\[
\int_{B_R} |u|^2 \, dx = R^{d+4} \int_{B_1} |v|^2 \, dx \leq NR^{d+4} \int_{B_1} |L_Rv|^2 \, dx = NR^4 \int_{B_R} |Lu|^2 \, dx.
\]

Also
\[
\int_{B_R} |u_x|^2 \, dx = R^{d+2} \int_{B_1} |v_x|^2 \, dx
\]
\[ \leq NR^{d+2} \int_{B_1} |L_Rv|^2 dx = NR^2 \int_{B_R} |Lu|^2 dx. \]

This shows that we need only prove the lemma for \( R = 1 \).

In that case we can divide \( L \) by \( a^{11} \) and may assume that \( a^{11} \equiv 1 \).

Then we integrate \( uLu \) over \( B_1 \) using integration by parts to find
\[
\delta \int_{B_1} |u_x|^2 dx \leq \int_{B_1} a^{jk} u_{xj} u_{xk} dx = -\int_{B_1} uLu dx
\leq \left( \int_{B_1} u^2 dx \right)^{1/2} \left( \int_{B_1} (Lu)^2 dx \right)^{1/2}.
\]

We estimate the integral of \( u^2 \) through that of \( |u_x|^2 \) by using Poincaré's inequality and obtain the needed estimate for \( u_x \). This is the only estimate we need to prove since \( u \) is estimated by \( u_x \) again owing to Poincaré's inequality.

The following lemma is almost identical to a theorem in [5]. For completeness, we present here a proof.

**Lemma 4.2.** Let \( 0 < r < R \). There exists \( N = N(d, \delta) \) such that, for \( w \in W^2_2(B_R) \),
\[
\|w\|_{W^2_2(B_r)} \leq N \left( \|Lw - w\|_{L^2(B_R)} + (R - r)^{-2}\|w\|_{L^2(B_R)} \right).
\]

**Proof.** Let
\[
R_0 = r, \quad R_m = r + (R - r) \sum_{k=1}^m \frac{1}{2^k}, \quad m = 1, 2, \ldots,
\]

\[
B_m = \{ x \in \mathbb{R}^d : |x| \leq R_m \}, \quad m = 0, 1, \ldots.
\]

Also let \( \zeta_m \in C_0^\infty(\mathbb{R}^d) \) such that \( \zeta_m(x) = 1 \) in \( B_m \), \( \zeta_m(x) = 0 \) outside of \( B_{m+1} \), and
\[
|\langle \zeta_m \rangle_x|_0 \leq N \frac{2^{m+1}}{(R - r)}, \quad |\langle \zeta_m \rangle_{xx}|_0 \leq N \frac{2^{2m+2}}{(R - r)^2},
\]

where \( N \) depends only on \( d \). To construct them take an infinitely differentiable function \( g(t), t \in (-\infty, \infty) \), such that \( g(t) = 1 \) for \( t \leq 1 \), \( g(t) = 0 \) for \( t \geq 2 \), and \( 0 \leq g \leq 1 \). After this define
\[
\zeta_m(x) = g(2^{m+1}(R - r)^{-1}(|x| - R_m) + 1).
\]
Now we make use of the $L_2$-estimate of $\zeta_m w$, which is from Remark 4.3, as follows.

\[
\|w\|_{L^2(B_m)} \leq \|\zeta_m w\|_{L^2} \leq N \|(L-1)\zeta_m w\|_{L^2} \\
\leq N \|(L-1)w\|_{L^2(B_r)} + N \frac{2^{m+1}}{R-r} \|w_x\|_{L^2(B_{m+1})} + N \frac{2^{2m+2}}{(R-r)^2} \|w\|_{L^2(B_r)},
\]

(4.2)

where $N$ depends only on $d$ and $\delta$. By interpolation inequalities

\[
\|w_x\|_{L^2(B_{m+1})} \leq \varepsilon \|w_{xx}\|_{L^2(B_{m+1})} + N \varepsilon^{-1} \|w\|_{L^2(B_{m+1})},
\]

where $\varepsilon > 0$, and $N$ depends only on $d$ (by using a dilation argument we can take a constant $N$ which does not depend on the radius of $B_{m+1}$). Thus the right hand side of the inequality (4.2) is not greater than

\[
N \|(L-1)w\|_{L^2(B_r)} + \varepsilon \|w_{xx}\|_{L^2(B_{m+1})} + N \varepsilon^{-1} \frac{2^{2m+2}}{(R-r)^2} \|w\|_{L^2(B_r)},
\]

where $0 < \varepsilon < 1$ and $N$ depends only on $d$ and $\delta$. Set

\[ A_m := \|w\|_{L^2(B_m)}, \quad B := \|(L-1)w\|_{L^2(B_r)}, \quad \text{and} \quad C := \|w\|_{L^2(B_r)}. \]

Then from (4.2) and (4.3), we have

\[
\varepsilon^m A_m \leq N \varepsilon^m B + \varepsilon^{m+1} A_{m+1} + N \varepsilon^{m-1} \frac{2^{2m+2}}{(R-r)^2} C.
\]

Choose an $\varepsilon$ such that $0 < 4 \varepsilon < 1$, and notice that $A_m \leq \|w\|_{L^2(B_m)}$. Then we have

\[
\sum_{m=0}^{\infty} \varepsilon^m A_m \leq NB \sum_{m=0}^{\infty} \varepsilon^m + \sum_{m=0}^{\infty} \varepsilon^{m+1} A_{m+1} + N \varepsilon^2 \frac{2^{m+2}}{(R-r)^2} C \sum_{m=0}^{\infty} (4\varepsilon)^{m+1}.
\]

This clearly finishes the proof. \hfill \Box

**Remark 4.3.** Using the dilation argument as in the proof of Lemma 4.1, we have

\[
\lambda \|w\|_{L^2(B_r)} + \sqrt{\lambda} \|w_x\|_{L^2(B_r)} + \|w_{xx}\|_{L^2(B_r)} \\
\leq N \left( \|Lw - \lambda w\|_{L^2(B_r)} + (R-r)^{-2} \|w\|_{L^2(B_r)} \right)
\]

for any $\lambda > 0$, where $N$ depends only on $d$ and $\delta$. In particular, by letting $\lambda \to 0$, we have

\[
\|w_{xx}\|_{L^2(B_r)} \leq N \left( \|Lw\|_{L^2(B_r)} + (R-r)^{-2} \|w\|_{L^2(B_r)} \right). \tag{4.4}
\]

In the next few lemmas, we investigate some properties of a solution $h$ of the equation $Lh = 0$. Recall that the coefficients $a^{jk}$ of the operator $L$ do not depend on $x' \in \mathbb{R}^{d-1}$.
Lemma 4.4. Let $\gamma = (\gamma^1, \ldots, \gamma^d)$ be a multi-index such that $\gamma^1 = 0,1,2$. Also let $0 < r < R \leq 4$. If $h$ is a sufficiently smooth function defined on $B_4$ such that $Lh = 0$ in $B_4$, then we have
\[ \int_{B_r} |D^\gamma h|^2 \, dx \leq N \int_{B_R} |h|^2 \, dx, \]
where $N = N(d, \delta, \gamma, R, r)$.

Proof. Set $\gamma' = (0, \gamma^2, \ldots, \gamma^d)$ and notice that
\[ L(D^{\gamma'} h) = 0, \quad \text{that is,} \quad (L - 1)D^{\gamma'} h = -D^{\gamma'} h \quad \text{in} \quad B_4. \]
Then by Lemma 4.2
\[ \|D^{\gamma} h\|_{L_2(B_r)} \leq N \left( \|D^{\gamma'} h\|_{L_2(B_{r_1})} + (r_1 - r)^{-2}\|D^{\gamma'} h\|_{L_2(B_{r_1})} \right), \]
where $r < r_1 < R$. If $|\gamma'| = 0$, then we are done. Otherwise, we can consider a multi-index $\gamma''$ having at least one component less by one than the corresponding component of $\gamma'$. Then, $L(D^{\gamma''} h) = 0$ and
\[ \|D^{\gamma'} h\|_{L_2(B_{r_1})} \leq N \left( \|D^{\gamma''} h\|_{L_2(B_{r_2})} + (r_2 - r_1)^{-2}\|D^{\gamma''} h\|_{L_2(B_{r_2})} \right), \]
where $r < r_1 < r_2 < R$. We repeat this argument as many times as we need. The lemma is proved. $\square$

Denote by $h_x$ a generic derivative $h_{x^j}$, $j = 1, \ldots, d$, and $h_{x'}$ a generic derivative $h_{x^j}$, $j = 2, \ldots, d$. Thus, for example, $h_{xx'}$ can be $h_{x^jx^k}$ where $j \in \{1,2,\ldots,d\}$ and $k \in \{2,\ldots,d\}$.

Lemma 4.5. Let $h$ be a sufficiently smooth function $h$ defined on $B_4$ such that $Lh = 0$ in $B_4$. Then we have
\[ \sup_{B_1} |h_{x^jx^k}|^2 \leq N \int_{B_3} |h|^2 \, dx, \]
where $N = N(d, \delta)$.

Proof. Imagine that we have
\[ \sup_{B_1} |h_{x^j}| \leq N(d, \delta)\|h\|_{L_2(B_{5/2})}. \] \hspace{1cm} (4.5)
Then using the fact that $Lh_{x'} = 0$ we would obtain
\[ \sup_{B_1} |h_{x'x^j}| \leq N\|h_{x'}\|_{L_2(B_{5/2})} \]
and it would only remain to appeal to Lemma 4.4.
Therefore, it suffices to prove (4.3). To do that, we first fix an integer
$k$ such that $k - (d - 1)/2 > 0$. Then due to the Sobolev embedding
theorem, we can find a constant $N$ such that, for each $-1 \leq x^1 \leq 1$,
\[
\sup_{|x'| \leq 1} |h_{x^1, x'}(x^1, x')| \leq N\|h_{x^1, x'}(x^1, \cdot)\|_{W^k_2(B_1^1)}
\]
and
\[
\sup_{|x'| \leq 1} |h_{x^1, x'}(x^1, x')| \leq N\|h_{x^1, x'}(x^1, \cdot)\|_{W^k_2(B_1^1)},
\]
where $B_1^1 = \{x' \in \mathbb{R}^{d-1} : |x'| \leq 1\}$. Set $g$ to be either $h_{x^1, x'}$ or $h_{x^1, x'}$. Then
\[
\int_{-1}^{1} \sup_{|x'| \leq 1} |g(x^1, x')|^2 \, dx^1 \leq N \int_{-1}^{1} \|g(x^1, \cdot)\|^2_{W^k_2(B_1^1)} \, dx^1
\leq N \sum_{|\gamma| \leq k+3, 1 \leq \gamma_1 \leq 2} \|D^\gamma h\|^2_{L^2(B_2)}.
\]
From this and Lemma 4.4, we have
\[
\int_{-1}^{1} \sup_{|x'| \leq 1} |h_{x^1, x'}|^2 \, dx^1 + \int_{-1}^{1} \sup_{|x'| \leq 1} |h_{x^1, x'}|^2 \, dx^1 \leq N\|h\|^2_{L^2(B_{5/2})},
\]
where $N$ depends only on $d$ and $\delta$. Now we notice that, for $x^1, y^1 \in [-1, 1]$,
\[
\sup_{|x'| \leq 1} |h_{x^1, x'}(x^1, x')| - \sup_{|x'| \leq 1} |h_{x^1, x'}(y^1, x')| \
\leq \sup_{|x'| \leq 1} |h_{x^1, x'}(x^1, x') - h_{x^1, x'}(y^1, x')| \leq \int_{x^1}^{y^1} \sup_{|x'| \leq 1} |h_{x^1, x'}(t, x')| \, dt \
\leq |x^1 - y^1|^{1/2} \left( \int_{-1}^{1} \sup_{|x'| \leq 1} |h_{x^1, x'}(t, x')|^2 \, dt \right)^{1/2}.
\]
This and (4.6) imply
\[
\sup_{|x'| \leq 1} |h_{x^1, x'}(x^1, x')| \leq N\|h\|_{L^2(B_{5/2})} |x^1 - y^1|^{1/2} + \sup_{|x'| \leq 1} |h_{x^1, x'}(y^1, x')|.
\]
Take integrals of both sides with respect to $y^1$, and take a supremum
over $x^1$. Then
\[
\sup_{x \in B_1} |h_{x^1, x}| \leq N\|h\|_{L^2(B_{5/2})} + \int_{-1}^{1} \sup_{|x'| \leq 1} |h_{x^1, x}(y^1, x')| \, dy^1 \
\leq N\|h\|_{L^2(B_{5/2})}.
\]
where the last inequality follows from (4.6), and $N$ depends only on $d$ and $\delta$. Similarly, we follow the same steps as above with $h_{x'}x'$ and $h'_{x'}$ in place of $h_{x'^{1}x}$ and $h'_{x'^{1}}$, respectively. Therefore, we have

$$\sup_{x \in B_{1}} |h_{x'}x'(x)| \leq N(d, \delta)\|h\|_{L_{2}(B_{5/2})}.$$ 

Finally, using the fact that $a^{11}h_{x'^{1}x} = -\sum_{j \neq 1 \text{ or } k \neq 1} a^{jk}h_{x'^{j}x'^{k}}$, we finish the proof of (4.5). \hfill \Box

Denote by $(u)_{B_{r}(x_{0})}$ the average value of a function $u$ over $B_{r}(x_{0})$, that is,

$$(u)_{B_{r}(x_{0})} = \frac{1}{|B_{r}|} \int_{B_{r}(x_{0})} u(x) \, dx = \frac{1}{|B_{r}|} \int_{B_{r}(x_{0})} \int_{D} u(x) \, dx.$$

Let $u \in C_{0}^{\infty}(\mathbb{R}^{d})$ and $f := Lu$. Assume that $a^{jk}(x^{1})$ are infinitely differentiable as functions of $x^{1} \in \mathbb{R}$. Then we can find a sufficiently smooth function $h$ defined on $B_{4}$ such that

$$\begin{cases}
Lh = 0 & \text{in } B_{4} \\
h = u & \text{on } \partial B_{4}
\end{cases}.$$

For this solution $h$, we establish the following inequality.

**Lemma 4.6.** There exists a constant $N = N(d, \delta)$ such that

$$\sup_{B_{1}} |h_{xx'}|^{2} \leq N \int_{B_{4}} |f|^{2} \, dx + N \int_{B_{4}} |u_{xx}|^{2} \, dx.$$

**Proof.** Define

$$\tilde{u} := u - u_{B_{4}} - (u_{x^{1}})_{B_{4}}x^{1} \quad \text{in } B_{4},$$

$$\tilde{h} := h - u_{B_{4}} - (u_{x^{1}})_{B_{4}}x^{1} \quad \text{in } B_{4}.$$

Then

$$L\tilde{u} = f, \quad L\tilde{h} = 0 \quad \text{in } B_{4} \quad \text{and} \quad \tilde{h} = \tilde{u} \quad \text{on } \partial B_{4}.$$ 

By Lemma 4.5 we see that

$$\sup_{B_{1}} |h_{xx'}|^{2} = \sup_{B_{1}} |\tilde{h}_{xx'}|^{2} \leq N \int_{B_{3}} |\tilde{h}|^{2} \, dx.$$

Let $\eta$ be a function in $C_{0}^{\infty}(\mathbb{R}^{d})$ such that $\eta(x) = 0$ in $B_{3}$ and $\eta(x) = 1$ at $\partial B_{4}$. Then $\tilde{h} - \eta \tilde{u} \in W_{2}^{2}(B_{4})$ and $\tilde{h} - \eta \tilde{u} = 0$ on $\partial B_{4}$. Therefore, by Lemma 4.1

$$\int_{B_{3}} |\tilde{h}|^{2} \, dx = \int_{B_{3}} |\tilde{h} - \eta \tilde{u}|^{2} \, dx \leq N(d, \delta) \int_{B_{4}} |L(\eta \tilde{u})|^{2} \, dx.$$
Note that
\[
L(\eta \tilde{u}) = \eta Lu + 2a^{ij} \eta x_i \tilde{u}_{x_j} + \tilde{u} L \eta
\]
\[
= \eta f + 2a^{ij} \eta x_i (u_{x_j} - (u_{x_j})_B) + (u - u_{B_4} - (u_{x_i})_B x_i)L \eta.
\]

Hence we have
\[
\int_{B_4} |L(\eta \tilde{u})|^2 \, dx \leq N \int_{B_4} (|f|^2 + |u_{x_j} - (u_{x_j})_B|^2) \, dx
\]
\[
+ N \int_{B_4} |u - u_{B_4} - (u_{x_i})_B x_i|^2 \, dx \leq N \int_{B_4} |f|^2 \, dx + N \int_{B_4} |u_{xj}|^2 \, dx,
\]
where the last inequality follows from Lemmas 3.1 and 3.2 in \([6]\), and \(N\) depends only on \(d\) and \(\delta\). The lemma is proved.

**Lemma 4.7.** Let \(\kappa \geq 4\), and \(r > 0\). Also let \(a^{jk}(x^j)\) be infinitely differentiable. For a given \(u \in C^\infty_0(\mathbb{R}^d)\), we find a smooth function \(h\) defined on \(B_{\kappa r}\) such that \(Lh = 0\) in \(B_{\kappa r}\) and \(h = u\) on \(\partial B_{\kappa r}\). Then there exists a constant \(N = N(d, \delta)\) such that
\[
\int_{B_r} |h_{x^k} - (h_{x^k})_{B_r}|^2 \, dx \leq N \kappa^{-2} \left[ (|Lu|^2)_{B_{\kappa r}} + (|u_{x^j}|^2)_{B_{\kappa r}} \right]. \quad (4.7)
\]

**Proof.** Using the dilation argument as in the proof of Lemma 4.1, we see that we need to prove only the case \(r = 1\). In that case we first observe that by using the same dilation argument and Lemma 4.6 we have
\[
\sup_{B_{\kappa/4}} |h_{x^k}|^2 \leq N \kappa^{-2} \left[ (|Lu|^2)_{B_{\kappa r}} + (|u_{x^j}|^2)_{B_{\kappa r}} \right],
\]
where \(N\) depends only on \(d\) and \(\delta\). Now we need only observe that \(\kappa/4 \geq 1, \ r = 1\), and the left hand side of the inequality (4.7) is not greater than a constant times \(\sup_{B_1} |h_{x^k}|^2\). The lemma is proved.

Using the results obtained above, we will finally arrive at

**Lemma 4.8.** There exists a constant \(N = N(d, \delta)\) such that, for any \(\kappa \geq 4, \ r > 0, \) and \(u \in C^\infty_0(\mathbb{R}^d)\), we have
\[
\int_{B_r} |u_{xx} - (u_{xx})_{B_r}|^2 \, dx \leq N \kappa^{-d} (|Lu|^2)_{B_{\kappa r}} + N \kappa^{-2} (|u_{x^j}|^2)_{B_{\kappa r}}. \quad (4.8)
\]

**Proof.** We can assume that \(a^{jk}(x^j)\) are infinitely differentiable. In that case, we find a sufficiently smooth \(h\) defined on \(B_{\kappa r}\) such that \(Lh = 0\) in \(B_{\kappa r}\) and \(h = u\) on \(\partial B_{\kappa r}\). Note that \(L(u - h) = Lu\) in \(B_{\kappa r}\) and \(u - h = 0\) on \(\partial B_{\kappa r}\). From Lemma 4.7 we have
\[
\int_{B_r} |h_{xx} - (h_{xx})_{B_r}|^2 \, dx \leq N \kappa^{-2} \left[ (|Lu|^2)_{B_{\kappa r}} + (|u_{x^j}|^2)_{B_{\kappa r}} \right]. \quad (4.9)
\]
On the other hand, from estimate (4.4) we have
\[
\int_{B_r} |u_{xx'} - h_{xx'}|^2 \, dx \leq N \left( \int_{B_{kr}} |Lu|^2 \, dx + r^{-4}(\kappa - 1)^{-4} \int_{B_{kr}} |u - h|^2 \, dx \right).
\]
Moreover, by Lemma 4.1
\[
\int_{B_{kr}} |u_{xx'} - h_{xx'}|^2 \, dx \leq N \kappa^4 (|Lu|_{B_{kr}}^2).
\]
This and (4.9) prove the inequality (4.8) with \( (h_{xx'})_{B_r} \) in place of \( (u_{xx'})_{B_r} \). Now we need only notice that
\[
\int_{B_r} |u_{xx'} - (u_{xx'})_{B_r}|^2 \, dx \leq \int_{B_r} |u_{xx'} - (h_{xx'})_{B_r}|^2 \, dx.
\]
The lemma is proved. \( \square \)

5. Proof of Theorem 2.4

In this section we suppose that all assumption of Theorem 2.4 are satisfied. Recall that
\[
Lu(x) = \alpha^j(x)u_{xj,k}(x) + b^j(x)u_{xj}(x) + c(x)u(x),
\]
\[
L_0 u(x) = \alpha^j(x)u_{xj,k}(x).
\]
We use the maximal and sharp functions given by
\[
Mg(x) = \sup_{r > 0} \int_{B_r(x)} |g(y)| \, dy,
\]
\[
g^\#(x) = \sup_{r > 0} \int_{B_r(x)} |g(y) - (g)_{B_r(x)}| \, dy.
\]

Theorem 5.1. Let \( \mu, \nu \in (1, \infty), \ 1/\mu + 1/\nu = 1, \) and \( R \in (0, \infty) \). Then there exists a constant \( N = N(d, \delta, \mu) \) such that, for any \( u \in C_0^\infty(B_R) \), we have
\[
(u_{xx'})^\# \leq N \left( a_R^\#(x') \right)^\alpha \left[ M(|u_{xx}|^{2\mu}) \right]^\beta
+ N \left[ M(|L_0 u|^2) \right]^{1/(d+2)} \left[ M(|u_{xx}|^2) \right]^{d/(2d+4)}, \quad (5.1)
\]
where \( \alpha = \nu^{-1}(d + 2)^{-1}, \ \beta = 2^{-1}\mu^{-1}. \)
Proof. Fix $\kappa \geq 4$, $r \in (0, \infty)$, and $x_0 = (x_0^1, x_0^2) \in \mathbb{R}^d$. Introduce
\[
\tilde{a}^{jk}(x^1) = \frac{1}{|B_{\kappa r}|} \int_{B_{\kappa r}(x_0^1)} a^{jk}(x^1, y') \, dy' \quad \text{if} \quad \kappa r < R,
\]
\[
\tilde{a}^{jk}(x^1) = \frac{1}{|B_{R}|} \int_{B_{R}(x_0^1)} a^{jk}(x^1, y') \, dy' \quad \text{if} \quad \kappa r \geq R,
\]
\[
A = M(|L_0 u|^2)(x_0), \quad B = M(|u_{xx}|^2)(x_0),
\]
\[
C = (M(|u_{xx}|^{2\mu})(x_0))^{1/\mu}.
\]
Set $\tilde{L}_0 u = \tilde{a}^{jk}(x^1) u_{x^j x^k}$. Then Lemma 4.8 along with the fact that $\kappa \geq 4$ allows us to obtain
\[
\int_{B_{\kappa r}(x_0)} |\tilde{L}_0 u|^2 \, dx \leq 2 \int_{B_{\kappa r}(x_0)} |\tilde{L}_0 u - L_0 u|^2 \, dx + 2 \int_{B_{\kappa r}(x_0)} |L_0 u|^2 \, dx
\]
\[
(5.2)
\]
and
\[
\int_{B_{\kappa r}(x_0)} |\tilde{L}_0 u - L_0 u|^2 \, dx = \int_{B_{\kappa r}(x_0) \cap B_R} |\tilde{L}_0 u - L_0 u|^2 \, dx
\]
\[
\leq \left( \int_{B_{\kappa r}(x_0) \cap B_R} |\tilde{a} - a|^{2\nu} \, dx \right)^{1/\nu} \left( \int_{B_{\kappa r}(x_0)} |u_{xx}|^{2\mu} \, dx \right)^{1/\mu} := I^{1/\nu} J^{1/\mu}.
\]
If $\kappa r < R$, we have
\[
I \leq N \int_{x_0^1 - \kappa r}^{x_0^1 + \kappa r} \int_{B_{\kappa r}(x_0^1)} |\tilde{a}(x^1) - a(x^1, x')| \, dx' \, dx^1
\]
\[
\leq N(\kappa r)^d a_{\kappa r}^\#(x') \leq N(\kappa r)^d a_R^\#(x').
\]
In case $\kappa r \geq R$
\[
I \leq N \int_{-R}^{R} \int_{B_R^i} |\tilde{a}(x^1) - a(x^1, x')| \, dx' \, dx^1
\]
\[
\leq N R^d a_R^\#(x') \leq N(\kappa r)^d a_R^\#(x').
\]
Hence
\[
\int_{B_{\kappa r}(x_0)} |L_0 u - L_0 u|^2 \, dx \leq N(\kappa r)^d \nu \left( a_R^\#(x') \right)^{1/\nu} \left( \int_{B_{\kappa r}(x_0)} |u_{xx}|^{2\mu} \, dx \right)^{1/\mu}.
\]
From this and \((5.3)\) it follows that
\[
\left( |L_0 u|^2 \right)_{B_{N\kappa}(x_0)} \leq N \left[ \left( a_R^{\#(x')} \right)^{1/\nu} \left( |u_{xx}|^2 \right)^{1/\mu} \right]_{B_{N\kappa}(x_0)} + \left( |L_0 u|^2 \right)_{B_{N\kappa}(x_0)}.
\]
This and \((5.2)\) allow us to have
\[
\int_{B_r(x_0)} |u_{xx'} - (u_{xx'})_{B_r(x_0)}|^2 \, dx \leq N \kappa^d \left( a_R^{\#(x')} \right)^{1/\nu} \left( |u_{xx}|^2 \right)^{1/\mu}
\]
\[
+ N \kappa^d \left( |L_0 u|^2 \right)_{B_{N\kappa}(x_0)} + N \kappa^{-2} \left( |u_{xx}|^2 \right)_{B_{N\kappa}(x_0)}
\]
\[
\leq N \kappa^d \left( a_R^{\#(x')} \right)^{1/\nu} C + N \kappa^d A + N \kappa^{-2} B,
\]
for all \( r > 0 \) and \( \kappa \geq 4 \). In addition, the above inequality is also true for \( 0 < \kappa < 4 \) since then
\[
\int_{B_r(x_0)} |u_{xx'} - (u_{xx'})_{B_r(x_0)}|^2 \, dx \leq \int_{B_r(x_0)} |u_{xx'}|^2 \, dx \leq B \leq 16 \kappa^{-2} B.
\]
By taking the supremum with respect to \( r > 0 \), and then minimizing with respect to \( \kappa > 0 \), we have
\[
\left[ u_{xx'}^{\#}(x_0) \right]^2 \leq N \left( \left( a_R^{\#(x')} \right)^{1/\nu} C + A \right)^{\frac{2}{d+2}} B^{\frac{d}{d+2}}
\]
\[
\leq N \left( a_R^{\#(x')} \right)^{\frac{2}{d+2}} C^{\frac{2}{d+2}} B^{\frac{d}{d+2}} + N A^{\frac{2}{d+2}} B^{\frac{d}{d+2}},
\]
where \( N = N(d, \delta, \mu) \). Notice that \( B \leq C \). Thus by replacing \( B \) with \( C \) in the first term on the right we finish the proof. \( \square \)

**Corollary 5.2.** For \( p > 2 \), there exist constants \( R = R(d, \delta, p, \omega) \) and \( N = N(d, \delta, p) \) such that, for any \( u \in C_0^\infty(B_R) \), we have
\[
\| u_{xx} \|_{L_p} \leq N \| L_0 u \|_{L_p}, \tag{5.4}
\]

**Proof.** Choose real numbers \( \mu > 1 \) such that \( p > 2\mu \). Then we use the inequality \((5.1)\) along with the Fefferman-Stein theorem on sharp functions and the Hardy-Littlewood maximal function theorem. We also use Hölder’s inequality to have (note that \( p/2\mu > 1 \) and \( p/2 > 1 \))
\[
\| u_{xx} \|_{L_p} \leq N \left( a_R^{\#(x')} \right)^{\frac{1}{d+2}} \| u_{xx} \|_{L_p} + N \| L_0 u \|_{L_p} \| u_{xx} \|_{L_p} \tag{5.5}
\]
where \( 1/\mu + 1/\nu = 1 \), and \( N \) depends only on \( d, \delta \), and \( p \). Since
\[
u_{x_1 x_1} = L_0 u - \sum_{j \neq 1 \lor k \neq 1} \frac{a_{jk}}{a_{11}} u_{x_j x_k},
\]
by using (5.5) we arrive at
\[ \|u_{xx}\|_{L^p} \leq N_1 \left( a_R^\#(x') \right)^{\frac{1}{(d+2)}} \|u_{xx}\|_{L^p} + N \|L_0 u\|_{L^p} + N \|L_0 u\|_{L^{\frac{d+2}{2}}}^2 \|u_{xx}\|_{L^{\frac{d+2}{2}}}. \]

(5.6)

We now invoke Assumption 2.2 by which we can choose a sufficiently small \( R \) such that
\[ N_1 \left( a_R^\#(x') \right)^{\frac{1}{(d+2)}} \leq 1/2. \]

Then we have
\[ \frac{1}{2} \|u_{xx}\|_{L^p} \leq N \|L_0 u\|_{L^p} + N \|L_0 u\|_{L^{\frac{d+2}{2}}}^2 \|u_{xx}\|_{L^{\frac{d+2}{2}}}, \]

which implies (5.4). \( \square \)

**Proof of Theorem 2.4.** Since we have an \( L_p \)-estimate for functions with small compact support, we can just follow the standard argument, which can be found in [6]. \( \square \)

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