RESONANCE RELATIONS, HOLOMORPHIC TRACE FUNCTIONS
AND HYPERGEOMETRIC SOLUTIONS TO QKZB AND
MACDONALD-RUIJSENAARS EQUATIONS

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Abstract. The resonance relations are identities between coordinates of functions $\psi(\lambda)$ with values in tensor products of representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. We show that the space of hypergeometric solutions of the associated qKZB equations is characterized as the space of functions of Baker-Akhiezer type, satisfying the resonance relations. We give an alternative representation-theoretic construction of this space, using the traces of regularized intertwining operators for the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$, and thus establish the equivalence between hypergeometric and trace function solutions of the qKZB equations.

We define the quantum conformal blocks as distinguished Weyl anti-invariant hypergeometric qKZB solutions with values in a tensor product of finite-dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-modules. We prove that for generic $q$ the dimension of the space of quantum conformal blocks equals the dimension of $\mathcal{U}_q(\mathfrak{sl}_2)$-invariants, and when $q$ is a root of unity is computed by the Verlinde algebra.

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1. Introduction

1.1. The motivation for this paper was to establish correspondence between holomorphic solutions to the quantum Knizhnik-Zamolodchikov-Bernard (qKZB) equations, arising from two different constructions: as trace functions of regularized intertwining operators for the quantum group \( U_q(\mathfrak{sl}_2) \) and as hypergeometric solutions, given in terms of contour integrals.

In this paper we consider the qKZB operators without spectral parameters, which act in the space \( \mathfrak{f} \text{un} \otimes M_{\Lambda}[0] \), where \( \mathfrak{f} \text{un} \) is the space of functions of a complex variable \( \lambda \), and \( M_{\Lambda}[0] \) is the zero weight subspace of a tensor product \( M_{\Lambda_1} \otimes \cdots \otimes M_{\Lambda_n} \) of Verma modules for the quantum group \( U_q(\mathfrak{sl}_2) \). The qKZB operators \( K_j \) are expressed in terms of the dynamical \( R \)-matrix operators \( \mathcal{R}_{\Lambda_j, \Lambda_i} \) and act on \( M_{\Lambda}[0] \)-valued functions of \( \lambda \):

\[
K_j = \left( \mathcal{R}_{\Lambda_{j+1}, \Lambda_j} (\lambda - h^{(j+2, \ldots, n)}) \right)^{-1} \cdots \left( \mathcal{R}_{\Lambda_n, \Lambda_1} (\lambda) \right)^{-1} \Gamma_j \mathcal{R}_{\Lambda_j, \Lambda_{j-1}} (\lambda - h^{(j+1, \ldots, n)}) \cdots \mathcal{R}_{\Lambda_1, \Lambda_j} (\lambda - h^{(2, \ldots, j-1)}). \tag{1.1}
\]

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where $\Gamma_j \psi(\lambda) = \psi(\lambda - h^{(j)})$, and $h^{(i,\ldots,k)}$ must be replaced with $\mu_i + \cdots + \mu_k$ when acting on homogeneous functions $\psi(\lambda)$ with values in $M_{\Lambda_1}[\mu_1] \otimes \cdots \otimes M_{\Lambda_n}[\mu_n]$. Thus defined operators $K_j$ pairwise commute for $j = 1, \ldots, n$, and the corresponding qKZB equations are the equations for common eigenfunctions of operators $K_j$.

A distinguished hypergeometric family of solutions to the general qKZB equations, associated with Felder’s elliptic dynamical $R$-matrix with spectral parameter, was introduced in [FTV1] in terms of contour integrals. Our operators $R_{\Lambda_i,\Lambda_k}(\lambda)$ are the asymptotic limits of this elliptic $R$-matrix, and hence a suitable limit of the integral formulae in [FTV1] yields a family of holomorphic eigenfunctions for the operators $K_j$. Coordinates of these vector-valued eigenfunctions are arranged in the hypergeometric qKZB matrix $\mathbb{H}(\lambda, x; \bar{\Lambda})$, where $x$ is an additional complex parameter. For a fixed $x \in \mathbb{C}$ the matrix can be regarded as an operator $\mathbb{H}(\lambda, x; \bar{\Lambda}) : M_{\bar{\Lambda}}[0] \to \mathfrak{Fun} \otimes M_{\bar{\Lambda}}[0]$, which satisfies the equations

$$K_j \mathbb{H}(\lambda, x; \bar{\Lambda}) = \mathbb{H}(\lambda, x; \bar{\Lambda}) E_j(x), \quad j = 1, \ldots, n, \quad (1.2)$$

for suitable diagonal operators $E_j(x)$. One can show that the hypergeometric construction yields all qKZB solutions of certain type, cf. Theorem 6.9.

1.2. Solutions $\psi(\lambda)$ of the qKZB equations, arising from the hypergeometric construction, satisfy the so-called resonance relations [FV1], which are algebraic identities between values of coordinate functions of $\psi(\lambda)$ at certain values of $\lambda$.

In the simplest case $n = 1$, the resonance relations first appeared in the study of algebraic integrability of Schrödinger operators with the Calogero-Sutherland potential as a system of axioms on a Baker-Akhiezer function $\psi(\lambda, x) = e^{\lambda x} P(\lambda, x)$, where $P(\lambda, x)$ is a monic polynomial in the first variable of degree $m$. In particular, it was shown in [CV] that for generic $x \in \mathbb{C}$ the one-point resonance relations

$$\psi(-\delta, x) = \psi(\delta, x), \quad \delta = 1, \ldots, m,$$

determine such a function uniquely, and that $\psi(\lambda, x)$ is an eigenfunction for the Calogero-Sutherland differential operator in the $x$ variable. This fact has a $q$-difference analogue: a function of the form $\psi(\lambda, x) = q^{\lambda x} P(q^{\lambda x}, x)$, satisfying the same conditions, exists and is unique; moreover this function is an eigenfunction of the Macdonald difference operator in the $x$ variable. In both cases, the function $\psi(\lambda, x)$ also satisfies a difference equation with respect to $\lambda$.

The solutions of the qKZB equations are parameterized by a complex parameter $x$ and a finite set of multi-indices $\vec{m} = (m_1, \ldots, m_n)$. The solutions have the trigonometric quasi-polynomial form

$$\psi(\lambda) = q^{\lambda(x-m)} \left( \sum_{k=0}^{m} q^{2k\lambda} \psi^{(k)} \right), \quad \psi^{(k)} \in M_{\bar{\Lambda}}[0]. \quad (1.3)$$

Here the "level" $m$ is the nonnegative integer, determined by the equation $\Lambda_1 + \cdots + \Lambda_n = 2m$.

In this paper we regard the $n$-point resonance relations of [FV1] as a system of axioms on a vector-valued trigonometric quasi-polynomial function $\psi(\lambda)$ of the form (1.3). A detailed combinatorial description of the $n$-point resonance relations is given in Section 2.

**Theorem.** Let $x, \Lambda_2, \ldots, \Lambda_n \in \mathbb{C}$ be generic. Then for every $\psi^{(0)} \in M_{\bar{\Lambda}}[0]$ there exists a unique function of the form (1.3), satisfying the $n$-point resonance relations.
This is proved in Section 2, see Theorem 2.1.

Using the above theorem, we introduce the fundamental resonance matrix $\Psi(\lambda, x; \vec{\Lambda})$
\[ \Psi(\lambda, x; \vec{\Lambda}) = q^{\lambda(x-m)} \left( \text{Id} + \sum_{k=1}^{m} q^{2k\lambda} \Psi^{(k)} \right), \quad \Psi^{(k)} \in \text{End}(M_{\vec{\lambda}}[0]), \]
whose columns form a basis of the space of functions satisfying the resonance conditions. For a fixed $x \in \mathbb{C}$ the fundamental resonance matrix can be regarded as an operator $\Psi(\lambda, x; \vec{\Lambda}) : M_{\vec{\lambda}}[0] \rightarrow \mathfrak{sl}_2 \otimes M_{\vec{\lambda}}[0]$. Any function of the form (1.3), satisfying the resonance relations, must lie in the image of $\Psi(\lambda, x; \vec{\Lambda})$; in particular, this applies to all hypergeometric qKZB solutions. From this we derive that for generic $x$ the column spaces of $\mathbb{H}(\lambda, x; \vec{\Lambda})$ and $\Psi(\lambda, x; \vec{\Lambda})$ coincide.

1.3. Representation theory of quantum groups gives rise to another family of solutions of the qKZB equations, realized as traces of intertwining operators between Verma modules and their tensor products, see [EV1].

The versions of qKZB operators, considered in [EV1], were associated with the so-called fusion dynamical $R$-matrix, and the corresponding qKZB solutions were meromorphic in $\lambda$. For the simplest case of the quantum group $U_q(\mathfrak{sl}_2)$ and $n = 1$ these meromorphic trace functions were explicitly computed in [EV1], and were related to the corresponding hypergeometric solutions by a simple meromorphic gauge transformation.

In this paper we modify the trace function construction and obtain holomorphic eigenfunctions of the qKZB operators (1.1), arising from Felder’s dynamical $R$-matrix. We show that these holomorphic trace functions are precisely the hypergeometric qKZB solutions, computed using the contour integrals.

The key ingredient in our construction is the notion of a holomorphic intertwining operator $\Phi^\Lambda_{\vec{m}}(\lambda) : M_{\lambda-1} \rightarrow M_{\lambda-\Lambda+2m-1} \otimes M^*_\Lambda$, where $M^*_\Lambda$ denotes the contragredient dual Verma module, see [STV]. These operators holomorphically depend on highest weights $\lambda$ and $\Lambda$. For any $\vec{\Lambda} \in \mathbb{C}^n$ and $\vec{m} \in \mathbb{Z}_{\geq 0}^n$, satisfying the zero weight condition $\Lambda_1 + \ldots + \Lambda_n = 2(m_1 + \ldots + m_n)$, we consider the composition of the holomorphic intertwining operators
\[ \Phi^\Lambda_{\vec{m}}(\lambda) : M_{\lambda-1} \rightarrow M_{\lambda-\Lambda} \otimes M^*_\Lambda \otimes \ldots \otimes M^*_\Lambda, \]
and the corresponding $(M^*_\Lambda \otimes \ldots \otimes M^*_\Lambda)[0]$-valued trace function:
\[ \mathcal{F}_{\vec{m}}(\lambda, x; \vec{\Lambda}) = q^{-\frac{m(m+1)}{2}} (q-q^{-1})^m (q^x - q^{-x}) \left| \left. \text{Tr} \left( \Phi^\Lambda_{\vec{m}}(\lambda) q^{xh} \right) \right|_{M_{\lambda-1}} \right. . \]

In the above formula, the trace is a formal power series in $q^x$, which converges to a $M^*_\Lambda[0]$-valued function $\mathcal{F}_{\vec{m}}(\lambda, x; \vec{\Lambda})$, meromorphic in $x$ and holomorphic in variables $\lambda$ and $\vec{\Lambda}$. Arranging the coordinates of various $\mathcal{F}_{\vec{m}}(\lambda, x; \vec{\Lambda})$ as rows, we obtain the universal trace matrix $\mathcal{F}(\lambda, x; \vec{\Lambda})$. For a fixed $x \in \mathbb{C}$, it can be thought of as an operator $\mathcal{F}(\lambda, x; \vec{\Lambda}) : M_{\Lambda}[0] \rightarrow \mathfrak{sl}_2 \otimes M^*_\Lambda[0]$.

The holomorphic intertwining operators $\Phi^\Lambda_{\vec{m}}(\lambda)$ and their compositions have particularly nice compatibility with inclusions and fusion of Verma modules for $\mathcal{U}_q(\mathfrak{sl}_2)$. That compatibility leads to the following result.
Theorem. Let $x \in \mathbb{C}$, and let $\psi(\lambda)$ be a $M^+_\lambda[0]$-valued function of $\lambda$, which belongs to the image of $F(\lambda, x; \vec{A})$ in $\mathfrak{gl}_n \otimes M^+_\lambda[0]$. Then $\psi(\lambda)$ satisfies the $n$-point resonance relations.

This is proved in Section 3 see Theorem 3.1

By uniqueness of the resonance solutions, the image of $F(\lambda, x; \vec{A})$ is a subspace of the image of $\Psi(\lambda, x; \vec{A})$. In fact, these two spaces coincide, and we have the following relation between $F(\lambda, x; \vec{A})$ and $\Psi(\lambda, x; \vec{A})$.

Theorem. There exists an invertible operator $\Xi_\vec{A} \in \text{End}(M^+_\lambda)$, diagonal with respect to the standard monomial basis, such that

$$\Psi(\lambda, x; \vec{A}) = F(\lambda, x; \vec{A}) \Xi_\vec{A}.$$ 

This is proved in Section 3 see Theorem 3.5

The diagonal operators $\Xi_\vec{A}$ play the role of the operators, relating the actions of the quantum and the dynamical quantum groups, associated with $\mathfrak{gl}_2$. In particular, they provide the gauge equivalence between Drinfeld’s quantum $R$-matrix and the asymptotic $R$-matrix, obtained as the limit of $\mathbb{R}_{\Lambda_1, \Lambda_2}(\lambda)$ as $\lambda \to \infty$, see Theorem 5.2

1.4. We establish the connection between the holomorphic intertwining operators and the so-called dynamical Shapovalov form, which appears, in particular, in the study of the qKZB heat equation and the orthogonality relations, satisfied by the hypergeometric qKZB solutions, see [FY3], [FY4], [TV2]. The dynamical Shapovalov form can be thought of as the bilinear pairing $\mathcal{Q} : M^+_\lambda \otimes M^+_\lambda \to \mathfrak{gl}_n$, defined by $\mathcal{Q}(u^\lambda_{m} \otimes u^\lambda_{m'}) = \delta_{m,m'} \mathcal{Q}^\lambda_{m}(\lambda)$, where

$$\mathcal{Q}^\lambda_{m}(\lambda) = \prod_{j=1}^{n} \frac{[\lambda - \sum_{i=j+1}^{n}(\Lambda_i - 2m_i + k)] [\lambda - \sum_{i=j}^{n}(\Lambda_i - 2m_i) - k]}{[\Lambda_i - k + 1]}. \tag{1.4}$$

We interpret the dynamical Shapovalov form as the ”square norm” of the holomorphic intertwining operators with respect to the inner product $\langle \cdot, \cdot \rangle$ on Verma modules, see [L].

Theorem. Let $\Phi_{\vec{m}}^\lambda(\lambda) : M_{\Lambda-1} \to M_{\mu} \otimes M_\lambda^*$ and $\Phi_{\vec{m'}}^\lambda(\lambda') : M_{\Lambda'-1} \to M_{\mu} \otimes M_\lambda^*$ be the holomorphic intertwining operators as above. Then for any $v \in M_{\Lambda-1}$ and $v' \in M_{\Lambda'-1}$ we have

$$\langle \Phi_{\vec{m}}^\lambda(\lambda)v, \Phi_{\vec{m'}}^\lambda(\lambda')v' \rangle = \delta_{\vec{m},\vec{m'}} \mathcal{Q}_{\vec{m}}^\lambda(\lambda) \langle v, v' \rangle.$$ 

This is proved in Section 3 see Theorem 3.1

The dynamical Shapovalov form appears as the density function for the integrals, participating in the qKZB heat equation and the orthogonality relations for the hypergeometric qKZB solutions. These relations are crucial for the construction in [FY4] and [EV2] of the generalized Fourier transform for the vector-valued spherical functions. In [FY4] the qKZB heat equation and the orthogonality relations were proved for $n$-point hypergeometric solutions at level $m = 1$, and in [EV2] they were established for the one-point meromorphic trace functions, associated with arbitrary quantum groups. We hope that the previous theorem will be instrumental for a uniform general proof of these relations.

1.5. The resonance relations were introduced in [FY1] as the identities satisfied by the elliptic hypergeometric qKZB solutions, and our qKZB matrix inherits the same properties.
Theorem. \([\text{[FV1]}]\) Let \(x \in \mathbb{C}\), and let \(\psi(\lambda)\) be a \(M_{\tilde{A}}[0]\)-valued function of \(\lambda\), which belongs to the image of \(\mathbb{H}(\lambda, x; \tilde{A})\) in \(\mathfrak{g} \mathfrak{u}n \otimes M_{\tilde{A}}[0]\). Then \(\psi(\lambda)\) satisfies the \(n\)-point resonance relations.

The \(M_{\tilde{A}}[0]\)-valued functions \(\mathbb{H}^m(\lambda, x; \tilde{A})\), represented by columns of \(\mathbb{H}(\lambda, x; \tilde{A})\), can therefore be expressed as linear combinations of columns \(\mathcal{F}^m(\lambda, x; \tilde{A})\) of the universal trace matrix. The corresponding transformation operator is closely related to the quantum Knizhnik-Zamolodchikov (qKZ) equations, which we now describe.

For \(x \in \mathbb{C}\) and \(j = 1, \ldots, n\) the qKZ operators \(\mathcal{K}_j(x)\) without spectral parameter, acting in \(M_{\tilde{A}}[0]\), are expressed in terms of the quantum \(R\)-matrix operators \(\mathcal{R}_{\Lambda_1, \Lambda_n} \in \text{End}(M_{\Lambda_1} \otimes M_{\Lambda_n})\):

\[
\mathcal{K}_j(x) = \mathcal{R}_{\Lambda_j, \Lambda_{j+1}}^{-1} \cdots \mathcal{R}_{\Lambda_{j-1}, \Lambda_j}^{-1} \left( q^{xh} \right)_j \mathcal{R}_{\Lambda_1, \Lambda_j} \cdots \mathcal{R}_{\Lambda_{j-1}, \Lambda_j},
\]

The qKZ equations are equations for common eigenvectors of the operators \(\mathcal{K}_j(x)\).

The qKZ operators \(\mathcal{K}_j(x)\) can be regarded as certain limits of the qKZB operators \(\mathbb{K}_j\), restricted to the space of \(M_{\tilde{A}}[0]\)-valued quasi-trigonometric polynomials. The corresponding limits of the qKZB equations become the qKZ equations.

For every level \(\mathbf{m}\) a version of the universal hypergeometric construction \([\text{FTV1]}]\) yields a matrix \(\{\mathcal{H}^m_{\tilde{A}}(x; \tilde{A})\}\), holomorphically depending on \(x\). Regarded as an operator \(\mathcal{H}(x; \tilde{A}) \in \text{End}(M_{\tilde{A}}[0])\), it satisfies

\[
\mathcal{K}_j(x) \mathcal{H}(x; \tilde{A}) = \mathcal{H}(x; \tilde{A}) \mathcal{E}_j(x; \tilde{A}), \quad j = 1, \ldots, n,
\]

for the same operators \(\mathcal{E}_j(x; \tilde{A})\) as in \([\text{1.2]}\). We obtain the following fundamental relations between the universal trace functions and the qKZ and qKZB operators and solutions.

**Theorem.** Let \(n \geq 2\) and let \(x \in \mathbb{C}\) be generic. Then for all \(j = 1, \ldots, n\) we have

\[
\mathbb{K}_j \mathcal{F}(\lambda, x; \tilde{A}) = \mathcal{F}(\lambda, x; \tilde{A}) \mathcal{K}_j(x).
\]

This is proved in Section \([\text{6.3]}\) see Theorem \([\text{6.3]}\)

**Theorem.** Let \(x \in \mathbb{C}\) be generic. Then

\[
\mathbb{H}(\lambda, x; \tilde{A}) = \mathcal{F}(\lambda, x; \tilde{A}) \mathcal{H}(x; \tilde{A}).
\]

This is proved in Section \([\text{4.5]}\) see Theorem \([\text{4.5]}\).

In other words, the universal trace function intertwines the qKZB and qKZ operators, and transforms the hypergeometric solutions of the qKZ equations to the hypergeometric solutions of the qKZB equations. Thus we can regard \(\mathcal{F}(\lambda, x; \tilde{A})\) as the “quantization” operator, reconstructing the qKZB solutions from their asymptotic limit. Note also that the previous theorem gives a formula for the universal trace function in terms of hypergeometric integrals,

\[
\mathcal{F}(\lambda, x; \tilde{A}) = \mathbb{H}(\lambda, x; \tilde{A}) \mathcal{H}(x; \tilde{A})^{-1}.
\]

1.6. The Macdonald-Ruijsemaars operators \(M_{\Theta} : \mathfrak{g} \mathfrak{u}n \otimes M_{\tilde{A}}[0] \rightarrow \mathfrak{g} \mathfrak{u}n \otimes M_{\tilde{A}}[0]\), introduced in \([\text{EV1]}\), are defined for dominant integral highest weights \(\Theta\) by

\[
M_{\Theta} = q^{\Theta_{\tilde{A}}} \sum_{\mu \in \mathfrak{h}^*} \text{Tr} \left| L_{\mathfrak{g}[\mu]} \right| \mathcal{R}_{\Theta, \Lambda_1}(\lambda - h^{(2, \ldots, n)}) \cdots \mathcal{R}_{\Theta, \Lambda_n}(\lambda) \mathcal{T}_{-\mu},
\]

(1.6)
where $T_{-\mu}$ is the shift operator, $T_{-\mu}\psi(\lambda) = \psi(\lambda - \mu)$, and $L_\Theta$ is the irreducible finite-dimensional $U_q(\mathfrak{sl}_2)$-module with highest weight $\Theta$.

The operators $\mathcal{M}_\Theta$ for different $\Theta$ pairwise commute. The Macdonald-Ruijsenaars equations are the equations for common eigenfunctions of operators $\mathcal{M}_\Theta$.

We use the equivalence between the hypergeometric and trace function qKZB solutions to give a new representation-theoretic proof of the vanishing conditions. This is proved in Section 6, see Theorem 6.7.

1.7. Summarying the results, we conclude that for generic $x \in \mathbb{C}$, there exists a distinguished subspace of $M_\Lambda[0]$-valued trigonometric quasi-polynomials, which can be constructed in several equivalent ways: using the resonance relations, the hypergeometric integrals, the holomorphic trace functions, the qKZB equations, or the MR equations, see Theorem [7.1]. We call this subspace the harmonic space $\mathcal{Harm}_{x,\Lambda}$.

For integral dominant $\Lambda$ we obtain the integral version $\mathcal{Harm}_{x,\Lambda} \subset \mathfrak{Fun} \otimes L_\Lambda[0]$ of the harmonic space, which consists of functions with values in the tensor product of finite-dimensional $U_q(\mathfrak{sl}_2)$-modules. The space $\mathcal{Harm}_{x,\Lambda}$ can be characterized as the space of $L_\Lambda[0]$-valued trigonometric quasi-polynomial solutions of qKZB or MR equations.

1.8. When the highest weights $\Lambda$ are nonnegative integers, the Weyl group acts on the space of functions with values in the corresponding tensor product of finite-dimensional modules. The Weyl anti-symmetric solutions of KZB-type equations play an important role in the conformal field theory, see for example [FW].

It was shown in [7.1] that the coordinates of Weyl anti-symmetric hypergeometric qKZB solutions $\psi(\lambda)$ must vanish at the special values of $\lambda$, participating in the resonance relations. These vanishing conditions can be described in terms of the fusion rules for the tensor category of finite-dimensional $\mathfrak{sl}_2$-modules. Namely, for each $\lambda$ and $\tilde{m}$ one constructs a chain of highest weights $\lambda^{(0)}, \ldots, \lambda^{(n)}$, and the vanishing conditions state that $\psi_{\tilde{m}}(\lambda) = 0$ when at least one of the triples $(\lambda^{(i-1)}, \lambda^{(i)}, \Lambda)$ violates the $\mathfrak{sl}_2$-fusion rules. The proof in [7.1] of the vanishing conditions was based on some identities for theta functions entering the hypergeometric integrals.

We use the equivalence between the spaces of hypergeometric solutions and holomorphic trace functions to give a new representation-theoretic proof of the vanishing conditions.

**Theorem.** For $\Theta \in \mathbb{Z}_{\geq 0}$, let $\chi_\Theta(x) = \sum_\mu \dim L_\Theta[\mu] q^{-\mu x}$ denote the character of $L_\Theta$. Then

$$\mathcal{M}_\Theta \mathbb{H}(\lambda, x; \Lambda) = \chi_\Theta(x) \mathbb{H}(\lambda, x; \Lambda).$$  \hspace{1cm} (1.7)

This is proved in Section 6, see Theorem [6.7]. One can show that the hypergeometric construction gives all trigonometric quasi-polynomial solutions to equations (1.7), cf. Theorem [6.11].

When $n = 1$ the Macdonald-Ruijsenaars operators reduce to the ordinary Macdonald operators for $\mathfrak{sl}_2$ with parameters $q$ and $t = q^m$. The Macdonald polynomials $p_\lambda(x)$ are then related to the hypergeometric solution $\mathbb{H}(\lambda, x; 2m)$ by

$$p_\lambda(x) = \frac{(-1)^m q^{m(m+1)/2}}{(q - q^{-1})^{2m+1}} \frac{\Gamma(2m)! \Gamma(x + 1)}{\Gamma(2m+1)!} \mathbb{H}(\lambda + m + 1, x; 2m) - \mathbb{H}(\lambda + m + 1, -x; 2m).$$  \hspace{1cm} (1.8)

Thus the anti-symmetrized qKZB solutions, studied in this paper, can be regarded as vector-valued generalizations of the Macdonald polynomials for $\mathfrak{sl}_2$. The Weyl anti-symmetric solutions of KZB-type equations play an important role in the conformal field theory, see for example [FW].

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Thus the anti-symmetrized qKZB solutions, studied in this paper, can be regarded as vector-valued generalizations of the Macdonald polynomials for $\mathfrak{sl}_2$.
Namely, we prove the Weyl formula for the holomorphic trace functions (Theorem 7.8), which shows that the anti-symmetrized trace functions can be interpreted as traces of regularized intertwining operators between finite-dimensional $U_q(sl_2)$-modules. The violation of the fusion rules as above then implies that the corresponding intertwining operator $\Phi^{\vec{\Lambda}}_{\vec{m}}(\lambda)$ must be the zero operator, and hence the corresponding trace function vanishes.

1.9. The anti-symmetrized hypergeometric qKZB solutions belong to the sum of spaces $\overline{\text{Harm}}_{x,\vec{\Lambda}} + \overline{\text{Harm}}_{-x,\vec{\Lambda}}$. For generic $x$ this sum is direct, because the ”supports” of the corresponding trigonometric quasi-polynomials are the disjoint strings $(x-m, \ldots , x+m)$ and $(-x-m, \ldots , -x+m)$. In particular, the dimension of the space of Weyl anti-symmetric qKZB solutions is the same for all generic $x$.

The hypergeometric qKZB solutions become the true trigonometric polynomials when the parameter $x$ is an integer, and for small integral values of $x$ the spaces $\overline{\text{Harm}}_{x,\vec{\Lambda}}$, $\overline{\text{Harm}}_{-x,\vec{\Lambda}}$ have a nontrivial intersection. The structure of the space of Weyl anti-symmetric qKZB solutions is more subtle in these cases.

We show that for $x=0$ all Weyl anti-symmetrized hypergeometric qKZB solutions vanish identically.

Then we consider the case $x=\pm 1$, and study in detail the corresponding Weyl anti-symmetric solutions $\vartheta^{\vec{m}}(\lambda)$, which we call the quantum conformal blocks. The following result resembles the Macdonald special value identity, and plays an important role in the analysis of functions $\vartheta^{\vec{m}}(\lambda)$.

**Theorem.** Let $\vec{\Lambda} \in \mathbb{Z}^n[2m]$ and $\vec{m} \in \mathbb{Z}^n[m]$. Then $\vartheta^{\vec{m}}(1) = -Q^{\vec{\Lambda}}_{\vec{m}}(1)\vartheta^{(\vec{m})}_\Lambda$, where $Q^{\vec{\Lambda}}_{\vec{m}}(\lambda)$ are the diagonal entries of the matrix of the dynamical Shapovalov form, see (1.4).

This is proved in Section 8, see Theorem 8.5.

1.10. We denote $\text{Conf}_{\vec{\Lambda}}$ the subspace of $\mathfrak{fun} \otimes L_{\vec{\Lambda}}[0]$, spanned by the quantum conformal blocks $\vartheta^{\vec{m}}(\lambda)$. One of our principal results is

**Theorem.** For any $\vec{\Lambda} \in \mathbb{Z}^n$ the dimension of the space $\text{Conf}_{\vec{\Lambda}}$ equals the dimension of $U_q(sl_2)$-invariants in the tensor product $L_{\Lambda_1} \otimes \ldots \otimes L_{\Lambda_n}$.

This is proved in Section 8, see Theorem 8.6.

The proof of the above theorem is constructive in the sense that we explicitly describe the set of indices $\vec{m}$, such that $\vartheta^{\vec{m}}(\lambda)$ is a basis of $\text{Conf}_{\vec{\Lambda}}$. Moreover, we show that for all other $\vec{m}$ the functions $\vartheta^{\vec{m}}(\lambda)$ are identically zero. This property suggests that there may be a correspondence between hypergeometric solutions and Lusztig’s canonical basis, which have similar properties. We plan to investigate this relation in a subsequent paper.

1.11. When the highest weights $\vec{\Lambda}$ are integral, the qKZB and MR equations can be considered independently on each of the cosets $\mathbb{C}/\mathbb{Z}$. Particularly important is the restriction to the lattice $\mathbb{Z} \subset \mathbb{C}$. In that case the operators $\tilde{K}_j$ and $\tilde{M}_\Theta$ may have singularities, and we consider their modified versions $\tilde{\tilde{K}}_j$ and $\tilde{\tilde{M}}_{\Theta}$, which are regular on the lattice. We conjecture that the restrictions of the Weyl anti-symmetrized hypergeometric solutions remain the eigenfunctions of these modified qKZB and Macdonald-Ruijsenaars operators; in some special cases this conjecture was shown to hold, see Section 12 in [FV].
For generic $q$ the restriction of a trigonometric polynomial $\vartheta^\Lambda(\lambda)$ can be uniquely reconstructed from its restriction to $\lambda \in \mathbb{Z}$. However, when $q$ becomes a root of unity, some of the $\vartheta^\Lambda(\lambda)$ may restrict to the zero function on $\mathbb{Z}$. For $q = \exp\left(\frac{i\pi}{\ell}\right)$ with $\ell \in \{3, 4, \ldots\}$, we define the discrete version $\text{Conf}_{\Lambda}^{(\ell)}$ of the conformal block space as the subspace of functions $\psi: \mathbb{Z} \to L_{\Lambda}[0]$, spanned by the restrictions of $\vartheta^\Lambda(\lambda)$.

**Theorem.** Let $\vec{\Lambda} \in \mathbb{Z}^n$ be such that $\Lambda_i \leq \ell - 2$ for $i = 1, \ldots, n$. Then the dimension of the space $\text{Conf}_{\Lambda}^{(\ell)}$ equals the dimension of the Verlinde algebra invariants in the tensor product $L_{\Lambda_1} \otimes \ldots \otimes L_{\Lambda_n}$.

This is proved in Section 5, see Theorem 9.6.

Finally, we conjecture that the spaces $\text{Conf}_{\Lambda}$ and $\text{Conf}_{\Lambda}^{(\ell)}$ are uniquely characterized as the spaces of Weil anti-symmetric trigonometric polynomial solutions of the MR equations $$M_{\Theta}^\Lambda \psi(\lambda) = (\dim_q L_{\Theta}) \psi(\lambda),$$ see Conjecture 8.7 and Conjecture 9.8 for precise formulations.

1.12. The hypergeometric construction for the general $sl_2$ qKZB equations, associated with the elliptic $R$ matrices with spectral parameters, was studied in [FTV1], [FTV2], [FV1], [FV2]. The corresponding trace function construction, based on the intertwining operators for the affine quantum group $U_q(\widehat{sl}_2)$, was developed in [ESV]. It was proved in [ESV] that the constructed trace functions satisfy the general qKZB equations. Conjecturally the elliptic hypergeometric solutions and trace function solutions coincide up to normalization.

We expect that the techniques developed in this paper can be used to establish the desired equivalence of the two constructions. We plan to discuss this subject in a subsequent paper.

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## 2. Resonance relations: the elementary approach

### 2.1. Basic notation

Fix $\eta \in \mathbb{C}$ such that $\text{Im} \eta > 0$, and for any $x \in \mathbb{C}, k \in \mathbb{Z}_{\geq 0}$, denote $$q^x = e^{\pi i nx}, \quad [x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad [x]_k = [x][x + 1] \ldots [x + k - 1], \quad [k]! = [1][2] \ldots [k].$$

In this paper we fix a non-negative integer $m$, and a positive integer $n$. We set $$\mathcal{Z}^n = (\mathbb{Z}_{\geq 0})^n, \quad \mathcal{Z}^n[m] = \{\vec{m} \in \mathcal{Z}^n \mid m_1 + \cdots + m_n = m\}.$$ We denote by $\mathbb{C}^n[2m]$ the hyperplane in $\mathbb{C}^n$, defined by $$\mathbb{C}^n[2m] = \{\vec{\Lambda} \in \mathbb{C}^n \mid \Lambda_1 + \cdots + \Lambda_n = 2m\}.$$ We say that $\vec{\Lambda} \in \mathbb{C}^n[2m]$ is generic, if $\Lambda_2, \ldots, \Lambda_n$ are generic in $\mathbb{C}$.

We say that $m \in \mathbb{Z}_{\geq 0}$ is $\Lambda$-admissible, if $m - \Lambda \notin \mathbb{Z}_{\geq 0}$. More generally, $\vec{m} \in \mathcal{Z}^n$ is called $\vec{\Lambda}$-admissible, if $m_i - \Lambda_i \notin \mathbb{Z}_{\geq 0}$ for all $i = 1, \ldots, n$. For any $\vec{\Lambda} \in \mathcal{Z}^n$ we denote $$\text{Adm}_{\vec{\Lambda}} = \left\{\vec{m} \in \mathcal{Z}^n \mid \vec{m} \text{ is $\vec{\Lambda}$-admissible}\right\}, \quad \text{Adm}_{\vec{\Lambda}}[m] = \text{Adm}_{\vec{\Lambda}} \cap \mathcal{Z}^n[m].$$ (2.1)
Let \( \mathfrak{Fun} \) denote the space of functions of a complex variable \( \lambda \). For any \( x \in \mathbb{C} \) and \( d \in \mathbb{Z}_{\geq 0} \) we denote \( \mathfrak{Fun}_{x,d} \) the subspace of \( \mathfrak{Fun} \), consisting of the functions of the form
\[
\psi(\lambda) = q^{\lambda(x-m)} \left( \sum_{k=0}^{d} \psi^{(k)} q^{2k\lambda} \right), \quad \psi^{(k)} \in \mathbb{C}.
\]  
(2.2)

We call \( \psi(\lambda) \) of the form (2.2) a \textit{trigonometric quasi-polynomial} of degree \( d \).

2.2. The resonance relations. Let \( V_{n,m} \) denote a complex vector space with a basis \( \{v^{(\bar{m})}\}_{\bar{m} \in \mathbb{Z}^n[m]} \). Any \( V_{n,m} \)-valued function \( \varphi(\lambda) \) of a complex variable \( \lambda \) can be written as
\[
\varphi(\lambda) = \sum_{\bar{m} \in \mathbb{Z}^n[m]} \varphi_{\bar{m}}(\lambda) v^{(\bar{m})},
\]  
(2.3)

where \( \varphi_{\bar{m}}(\lambda) \) are scalar functions; we call them the coordinates of \( \varphi(\lambda) \).

A holomorphic \( V_{n,m} \)-valued function \( \varphi(\lambda) \) is said to satisfy the \( n \)-point resonance relations at level \( m \) with respect to \( \bar{\Lambda} \in \mathbb{C}^n[2m] \), if the following conditions hold for all \( \bar{m} \in \mathbb{Z}^n[m] \).

- **Condition \( C_j(\bar{m}) \) for \( j = 1, \ldots, n-1 \).** For any \( \delta \in \{1, \ldots, m_j\} \) we have
\[
\varphi_{\bar{m}}(-\delta + \sum_{i=j+1}^{n} (\Lambda_i - 2m_i)) = \varphi_{\bar{m}}(\delta + \sum_{i=j+1}^{n} (\Lambda_i - 2m_i)),
\]  
(2.4)

where \( \tau_{j}^{\delta}(\bar{m}) = (m_1, \ldots, m_{j-1}, m_j - \delta, m_{j+1} + \delta, m_{j+2}, \ldots, m_n) \).

- **Condition \( C_n(\bar{m}) \).** For any \( \delta \in \{1, \ldots, m_n\} \) we have
\[
\varphi_{\bar{m}}(-\delta) = \varphi_{\bar{m}}(\delta),
\]  
(2.5)

where \( \tau_{j}^{\delta}(\bar{m}) = (m_1 + \delta, m_2, m_3, \ldots, m_{n-2}, m_{n-1}, m_n - \delta) \).

It is easy to see that for each \( \bar{m} \) the values of \( \varphi_{\bar{m}}(\lambda) \) appear exactly \( m \) times on the left, and \( m \) times on the right side of the equations (2.4), (2.5). Note also that these equations do not directly involve \( \Lambda_1 \). Another useful observation is that if \( \xi(\lambda) \) is an even scalar function, then the resonance relations for \( \varphi(\lambda) \) imply the resonance relations for \( \xi(\lambda) \varphi(\lambda) \).

Example. Let \( n = 1 \). Then the resonance relations are
\[
\varphi_m(-\delta) = \varphi_m(\delta), \quad \delta = 1, \ldots, m. 
\]

Example. Let \( n = 2 \) and \( m = 1 \). Then the resonance relations are
\[
\varphi_{1,0}(-1 + \Lambda_2) = \varphi_{0,1}(-1 + \Lambda_2), \quad \varphi_{0,1}(-1) = \varphi_{1,0}(1).
\]
Theorem 2.1. Let $x \in \mathbb{C}$ and $\Lambda \in \mathbb{C}^n[2m]$ be generic. Then for any vector $\psi^{(0)} = \sum_\vec{m} \psi^{(0)}_\vec{m} \psi^{(\vec{m})}$ there exists a unique function $\psi(\lambda)$ of the form

$$\psi(\lambda) = q^{\lambda(x-m)} \left( \sum_{k=0}^{m} q^{2k\lambda} \psi^{(k)} \right), \quad \psi^{(k)} \in V_{n,m},$$

satisfying the $n$-point resonance relations with respect to $\Lambda$.

Proof. For any $\vec{m}$ we form an $m$-tuple of complex numbers by concatenating the lists of values of $\lambda$, appearing on the left in the resonance relations, associated with $\vec{m}$:

$$(\zeta^{(1)}_m, \ldots, \zeta^{(m)}_m) = (-1 + \sum_{j=2}^{n} (\Lambda_j - 2m_j), \ldots, -m_1 + \sum_{j=2}^{n} (\Lambda_j - 2m_j), \ldots, -1, \ldots, -m_n),$$

and set for each $k = 0, \ldots, m$

$$\xi_{\vec{m},k}(\lambda) = q^{k \lambda} \prod_{i=k+1}^{m} [\lambda - \zeta^{(i)}_m].$$

The coordinates of $\psi(\lambda)$ can be uniquely represented in the form

$$\psi_{\vec{m}}(\lambda) = q^{\lambda x} \sum_{k=0}^{m} c^{(k)}_{\vec{m}} \xi_{\vec{m},k}(\lambda), \quad c^{(k)}_{\vec{m}} \in \mathbb{C},$$

and the initial condition forces $c^{(0)}_{\vec{m}} = (q - q^{-1})^m \psi^{(0)}_{\vec{m}}$. The remaining coefficients $c^{(k)}_{\vec{m}}$ with $k = 1, \ldots, m$ are determined from the resonance relations, which are equivalent to a system of linear equations on $c^{(k)}_{\vec{m}}$. More precisely, the equations \eqref{2.4} and \eqref{2.5} become respectively

$$\sum_{k=0}^{m} c^{(k)}_{\vec{m}} \xi_{\vec{m},k}(\Lambda_i - 2m_i) = \sum_{k=0}^{m} c^{(k)}_{\vec{m}} \xi_{\vec{m},k}(\Lambda_i - 2m_i), \quad i = j + 1,$$

$$\sum_{k=0}^{m} c^{(k)}_{\vec{m}} \xi_{\vec{m},k}(\Lambda_i - 2m_i) = q^{2kx} \sum_{k=0}^{m} c^{(k)}_{\vec{m}'} \xi_{\vec{m}',k}(\delta).$$

Since the number of resonance relations equals the number of variables $c^{(k)}_{\vec{m}}$, it suffices to show that the determinant $D$ of the corresponding matrix is nonzero for generic $x, \Lambda_2, \ldots, \Lambda_n$. It is clear that $D$ is a polynomial in $q^{2x}$; let $D_0$ denote its constant term.

Introduce a partial order $\preceq$ on $\mathbb{Z}^n$ by writing $\vec{m} \preceq \vec{m}'$ if and only if

$$\sum_{i=1}^{j} m_i \geq \sum_{i=1}^{j} m'_i$$

for all $j = 1, \ldots, n$, \hspace{1cm} (2.10)

and list the resonance relations according to the following rules:

1. If $i < j$, then conditions $C_i$ appear before conditions $C_j$.
2. If $\vec{m} \preceq \vec{m}'$, then conditions $C_j(\vec{m})$ appear before conditions $C_j(\vec{m}')$.
3. Conditions $C_j(\vec{m})$ appear in increasing order of $\delta$. 

RESONANCE RELATIONS
Simultaneously we make a compatible ordered list of the undetermined coefficients $c_{m}^{(k)}$, by associating to the equation (2.8) the variable $c_{m}^{(m_1+\cdots+m_{j-1}+i)}$, and similarly for (2.9). Consider the matrix of the system of resonance relations with respect to these orderings of equations and variables, and take its limit as $q \to 0$. It immediately follows from (2.7) that the resulting matrix is upper-triangular, with diagonal entries $q^k \zeta_m^{(k)} \prod_{i=k+1}^{n} (\zeta_m^{(k)} - \zeta_m^{(i)})$.

The constant term $D_0$ is equal to the product of the above entries, which are all nonzero for generic $\Lambda_2, \ldots, \Lambda_n$. Therefore, $D_0 \neq 0$, and hence $D$ does not vanish identically, which implies that generically we have $D \neq 0$.

2.3. The determinant $D$. Computations suggest that the determinant $D$, which appeared in the proof of Theorem 2.1, is always a product of its constant term $D_0$ and several factors of the form $(1 - q^{2(x+k)})$. We formulate

**Conjecture 2.2.** One has $D \neq 0$ unless $x, \Lambda_2, \ldots, \Lambda_n$ satisfy one of the equations
\begin{align}
[x+k]=0, \quad k = -m+2, \ldots, m, \quad (2.11) \\
[\Lambda_{i+1} + \cdots + \Lambda_j - k] = 0, \quad k = 2, \ldots, 2m-2, \quad (2.12)
\end{align}

for some $1 \leq i < j \leq n$.

Equivalently, we expect that up to a constant and a non-vanishing exponential function in $\Lambda$, the determinant $D$ factors into the product of terms appearing on the left in (2.11), (2.12) with certain multiplicities, which can be described combinatorially. This factorization property is reminiscent of some of the determinant formulas arising in representation theory, although the representation-theoretic significance of $D$ is not clear yet.

We illustrate the factorization property of the determinant $D$ - and the argument of the proof of Theorem 2.1 - in another example.

**Example.** Let $n = 2$ and $m = 2$. Then the ordered list of resonance relations for the coordinates of a $V_{n,m}$-valued function $\psi(\lambda)$ is given by
\begin{align*}
\psi_{2,0}(-1 + \Lambda_2) &= \psi_{1,1}(-1 + \Lambda_2), \\
\psi_{2,0}(-2 + \Lambda_2) &= \psi_{0,2}(-2 + \Lambda_2), \\
\psi_{1,1}(-3 + \Lambda_2) &= \psi_{0,2}(-3 + \Lambda_2), \\
\psi_{1,1}(-1) &= \psi_{2,0}(1), \\
\psi_{0,2}(-1) &= \psi_{1,1}(1), \\
\psi_{0,2}(-2) &= \psi_{2,0}(2).
\end{align*}

We look for a trigonometric quasi-polynomial function $\psi(\lambda)$ of the form
\begin{align*}
\begin{pmatrix}
\psi_{0,2}(\lambda) \\
\psi_{1,1}(\lambda) \\
\psi_{2,0}(\lambda)
\end{pmatrix} = q^{\lambda x} \begin{pmatrix}
c_{0,2}^{(0)} \\
c_{1,1}^{(0)} \\
c_{2,0}^{(0)}
\end{pmatrix} \begin{pmatrix}
\lambda + 1 \\
\lambda + 1 \\
\lambda - \Lambda_2 + 1
\end{pmatrix} + c_{0,2}^{(1)} q^{\lambda}[\lambda + 2] + c_{1,1}^{(2)} q^{2\lambda},
\end{align*}

where $c_{0,2}^{(0)}, c_{1,1}^{(0)}, c_{2,0}^{(0)} \in \mathbb{C}$ are the prescribed initial conditions, and $(c_{2,0}^{(1)}, c_{2,0}^{(2)})$ are undetermined coefficients. The resonance relations are linear equations on these variables,
and the determinant $D$ of the corresponding system is given by

$$D = \det \begin{pmatrix}
q^{\Lambda_2 - 1} & q^{2\Lambda_2 - 2} & -q^{\Lambda_2 - 1} & -q^{2\Lambda_2 - 2} & 0 & 0 \\
0 & q^{2\Lambda_2 - 4} & 0 & 0 & 0 & 0 \\
q^{2x + 1} [\Lambda_2 - 3] & 0 & q^{\Lambda_2 - 3} [\Lambda_2 - 2] & q^{2\Lambda_2 - 6} & q^{\Lambda_2 - 3} [\Lambda_2 - 2] & -q^{2\Lambda_2 - 6} \\
0 & 0 & -q^{2x + 1} [2] & q^{-2} & q^{-1} & q^{-2} \\
q^{4x + 2} [\Lambda_2 - 4] & -q^{4x + 4} & 0 & 0 & 0 & 0 \\
\end{pmatrix}.$$  

In the limit $q^{2x} \to 0$ the matrix above becomes upper-triangular, and has determinant $D_0 = q^{4\Lambda_2 - 15} [\Lambda_2 - 2]$. This implies that $D \neq 0$ for generic $x$ and $\Lambda_2$. In fact, one checks that

$$D = q^{4\Lambda_2 - 15} [\Lambda_2 - 2] (1 - q^x)(1 - q^{2(x+1)})^2 (1 - q^{2(x+2)}).$$

Remark 2.3. The existence part of Theorem 2.1 fails when $x$ satisfies (2.11) with $k = 1, \ldots, m$. For $x$ satisfying the remaining conditions (2.11) or $\Lambda_2, \ldots, \Lambda_n$ satisfying (2.12), the existence part is valid, but the uniqueness part fails.

2.4. The fundamental resonance matrix $\Psi(\lambda, x; \tilde{\Lambda})$. For every $\tilde{m}' \in \mathbb{Z}^n [m]$, the initial condition $\psi^{(0)} = \psi^{(\tilde{m}')} \in \mathbb{V}[\mathbb{Z}^n]$ produces a unique function $\Psi^{\tilde{m}'}(\lambda, x; \tilde{\Lambda})$ of the form (2.6), satisfying the resonance relations and meromorphically depending on parameters $x, \tilde{\Lambda}$. Let $\Psi(\lambda, x; \tilde{\Lambda}) = \{ \Psi_{\tilde{m}'}^{\tilde{m}}(\lambda, x; \tilde{\Lambda}) \}_{\tilde{m}, \tilde{m}' \in \mathbb{Z}^n [m]}$ denote the square matrix, formed by arranging the coordinates of $\Psi^{\tilde{m}'}(\lambda, x; \tilde{\Lambda})$ as columns.

We call $\Psi(\lambda, x; \tilde{\Lambda})$ the fundamental resonance matrix at level $m$. For a fixed $x \in \mathbb{C}$ it can be thought of as an operator

$$\Psi(\lambda, x; \tilde{\Lambda}): V_{n, m} \to \mathfrak{Fun} \otimes V_{n, m}, \quad \psi^{(\tilde{m}')} \mapsto \Psi^{\tilde{m}'}(\lambda, x; \tilde{\Lambda}).$$

Example. Let $n = 1$. Then the fundamental resonance matrix reduces to a scalar, and solving the resonance relations using the method of Theorem 2.1 one gets the explicit formula

$$\Psi(\lambda, x; 2m) = C_m q^{\lambda x} \sum_{k=0}^{m} (-q)^{-k} \frac{[m + k]! \ldots [\lambda + k] \ldots [\lambda + m]}{[m - k]!! [k]!! (1 - q^{-2(x+1)}) \ldots (1 - q^{-2(x+k)}),}$$

where $C_m = (q - 1)^m q^{m(m+1)/2}$ is a normalization constant, cf. [EV1].

Example. Let $n = 2$ and $m = 1$. Then the fundamental resonance matrix is given by

$$\Psi(\lambda, x; \tilde{\Lambda}) = \begin{pmatrix}
\Psi_{0, 1}^{0, 1}(\lambda, x; \tilde{\Lambda}) & \Psi_{1, 1}^{0, 1}(\lambda, x; \tilde{\Lambda}) \\
\Psi_{0, 1}^{1, 0}(\lambda, x; \tilde{\Lambda}) & \Psi_{1, 1}^{1, 0}(\lambda, x; \tilde{\Lambda})
\end{pmatrix}$$

$$= q^{\chi(x-1)} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} + \frac{q^{2\Lambda_2 - \Lambda_2}}{x + 1} \begin{pmatrix}
q^{2[-x - \Lambda_2 - 1]} & q^{-x + 1} [2 - \Lambda_2] \\
q^{-x + 1} [\Lambda_2] & [-x - \Lambda_2 + 1]
\end{pmatrix}.$$
Lemma 2.4. Let $\tilde{\Lambda} \in \mathbb{C}^n[2m]$ be generic. Let $\varphi(\lambda) \in \mathfrak{Fun}_{y,d}$ for some $d \in \mathbb{Z}_{\geq 0}$ and generic $y \in \mathbb{C}$. Assume that $\varphi(\lambda)$ satisfies the n-point resonance relations with respect to $\tilde{\Lambda}$. Then $\varphi(\lambda)$ can be represented as

$$\varphi(\lambda) = \sum_{j=0}^{d-m} \Psi(\lambda, y + 2j; \tilde{\Lambda}) w_j, \quad w_j \in V_{n,m}.$$  

In particular, if $d < m$, then $\varphi(\lambda) \equiv 0$.

Proof. We argue by induction on the degree $d$ of the trigonometric quasi-polynomial $\varphi(\lambda)$. If $d < m$, then $\varphi(\lambda)$ has the form (2.1) with $x = y - 2$ and initial condition $\psi^{(0)} = 0$, so the uniqueness part of Theorem 2.1 implies that $\varphi(\lambda) \equiv 0$.

Assume now that $d \geq m$, and consider the function

$$\phi(\lambda) = \varphi(\lambda) - \Psi(\lambda, y; \tilde{\Lambda}) \varphi^{(0)} \in \mathfrak{Fun}_{y+2,d-1}.$$  

It satisfies the resonance relations, and by the induction hypothesis we have

$$\phi(\lambda) = \sum_{j=1}^{d-m} \Psi(\lambda, y + 2j; \tilde{\Lambda}) w_j, \quad w_j \in V_{n,m}.$$  

Then $\varphi(\lambda) = \Psi(\lambda, y; \tilde{\Lambda}) \varphi^{(0)} + \sum_{j=1}^{d-m} \Psi(\lambda, y + 2j; \tilde{\Lambda}) w_j$, which proves the lemma. □

Next we consider the special class of linear difference operators $\mathcal{D}$, acting in the space of $\text{End}(V_{n,m})$-valued functions of a complex variable $x$, which have the form

$$\mathcal{D} = \sum_{\mu} B_\mu(x) T_\mu, \quad B_\mu(x) \in \text{End}(V_{n,m}), \quad \text{(2.16)}$$

where $\mu$ runs over a finite subset of $\mathbb{Z}$, $T_\mu$ is the difference operator acting by $T_\mu \psi(\lambda) = \psi(\lambda - \mu)$ and $B_\mu(x)$ act by right multiplication. In other words, for any $\text{End}(V_{n,m})$ - valued function $\Psi(x)$ we have

$$\mathcal{D} \Psi(x) = \sum_{\mu} \Psi(x + \mu) B_\mu(x). \quad \text{(2.17)}$$

Theorem 2.5. Let $\xi(\lambda) \in \mathbb{C}[q^\lambda, q^{-\lambda}]$ be such that $\xi(\lambda) = \xi(-\lambda)$. Then there exists a unique difference operator $\mathcal{D}_\xi$ as in (2.16), (2.17), such that for any $\lambda \in \mathbb{C}$ we have

$$\mathcal{D}_\xi \Psi(\lambda, x; \tilde{\Lambda}) = \xi(\lambda) \Psi(\lambda, x; \tilde{\Lambda}). \quad \text{(2.18)}$$

Proof. For each $\bar{m}'$ the function $\xi(\lambda) \Psi^{\bar{m}'}(\lambda, x; \tilde{\Lambda})$ satisfies the conditions of Lemma 2.4 and therefore we can write

$$\xi(\lambda) \Psi^{\bar{m}'}(\lambda, x; \tilde{\Lambda}) = \sum_{\mu} \Psi(\lambda, x + \mu; \tilde{\Lambda}) w^{\bar{m}'}_\mu(x), \quad w^{\bar{m}'}_\mu(x, \xi) \in V_{n,m},$$

where $\mu$ runs over some finite set of integers. For each $\mu$ let $B_\mu(x, \xi) \in \text{End}(V_{n,m})$ denote the operator, defined by $B_\mu(x, \xi) \psi^{(0)} = w^{\bar{m}'}_\mu(x, \xi)$. Then the desired equation (2.18) clearly holds if we set $\mathcal{D}_\xi = \sum_{\mu} B_\mu(x, \xi) T_\mu$. To prove the uniqueness of $\mathcal{D}_\xi(x)$, one must show that the only difference operator, annihilating $\Psi(\lambda, x; \tilde{\Lambda})$, is the zero operator. This is easily done by considering the asymptotics $\lambda \to \infty$. □
Example. Let \( n = 2, m = 1 \) and \( \xi(\lambda) = q^\lambda + q^{-\lambda} \). Then (2.18) is illustrated by the identity

\[
\Psi(\lambda, x - 1; \bar{\lambda}) + \Psi(\lambda, x + 1; \bar{\lambda}) \left( 1 - \frac{q^{2-\lambda_2[A]}}{[x][x+1]} + \frac{q^{\lambda_2[2-\lambda_2][x]}{[x][x+1]} \right) = (q^\lambda + q^{-\lambda}) \Psi(\lambda, x; \bar{\lambda})
\]

for the matrix \( \Psi(\lambda, x; \bar{\lambda}) \) given by (2.15).

3. Holomorphic Trace Functions

3.1. Representations of the quantum group \( \mathcal{U}_q(\mathfrak{sl}_2) \). Let \( \mathcal{U}_q(\mathfrak{sl}_2) \) be the algebra with unit 1, generated by \( E, F \) and \( q^r \) for \( x \in \mathbb{C} \), with relations

\[
q^r E = q^{-2r} E q^r, \quad q^r F = q^{-2r} F q^r,
\]

\[
q^{r+s} E q^r = q^{r+s} q^{(x+y)h} F = q^h - q^{-h}.
\]

If \( V \) is a \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module, we say that \( v \in V \) has weight \( \mu \in \mathbb{C} \), if \( q^{r} v = q^{r \mu} v \). The set of all such \( v \in V \) forms the weight subspace \( V[\mu] \). In this paper we consider \( \mathcal{U}_q(\mathfrak{sl}_2) \)-modules which admit a weight subspace decomposition, i.e. split into the direct sum of weight subspaces.

Let \( \lambda \in \mathbb{C} \). Denote by \( M_\lambda \) the Verma \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module, generated by a vector \( v_\lambda \) of weight \( \lambda \). It has a basis \( \{ v^{(0)}_\lambda, v^{(1)}_\lambda, v^{(2)}_\lambda, \ldots \} \), such that

\[
E v^{(m)}_\lambda = [\lambda - m] v^{(m-1)}_\lambda, \quad q^{r} v^{(m)}_\lambda = q^{(\lambda-2m)x} v^{(m)}_\lambda, \quad F v^{(m)}_\lambda = [m+1] v^{(m+1)}_\lambda.
\]

When \( \lambda \in \mathbb{Z}_{>0} \), there exists a \( \mathcal{U}_q(\mathfrak{sl}_2) \)-inclusion

\[
\iota(\lambda) : M_{\lambda-1} \to M_{\lambda-1}, \quad v_{\lambda-1} \mapsto F^h v_{\lambda-1}.
\]

The image of \( \iota(\lambda) \) is then spanned by vectors \( \{ v^{(\lambda)}_{\lambda-1}, v^{(\lambda+1)}_{\lambda-1}, \ldots \} \) and constitutes a proper submodule of \( M_{\lambda-1} \). The corresponding quotient has dimension \( \lambda \) and is denoted by \( L_{\lambda-1} \).

Let \( L^{\lambda}_{\lambda} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C} u^{\lambda}_m \) be the restricted dual of \( M_\lambda \), where \( \{ u^{\lambda}_m \} \) is the dual basis to \( \{ v^{\lambda}_m \} \), determined by \( \langle u_m, v^{m'} \rangle = \delta_{m,m'} \). Define the contravariant \( \mathcal{U}_q(\mathfrak{sl}_2) \)-action on \( M^* \) by

\[
\langle Eu, v \rangle = \langle u, Fv \rangle, \quad \langle Fu, v \rangle = \langle u, Ev \rangle, \quad \langle q^{r} u, v \rangle = \langle u, q^{r} v \rangle
\]

for any \( u \in M^* \) and \( v \in M \). Equivalently, we have explicit formulæ

\[
E u^{\lambda}_m = [m] u^{\lambda-1}_m, \quad q^{r} u^{\lambda}_m = q^{(\lambda-2m)x} u^{\lambda}_m, \quad F u^{\lambda}_m = [\lambda - m] u^{\lambda+1}_m.
\]

We call \( M_\lambda \) the contravariant dual Verma module with highest weight \( \lambda \). When \( \lambda \in \mathbb{Z}_{\geq 0} \), the vectors \( \{ u^{\lambda}_0, \ldots, u^{\lambda}_\lambda \} \) span a submodule \( L^{\lambda}_\lambda \), isomorphic to \( L_\lambda \).

An inner product on a \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module \( V \) is a bilinear form \( \langle \cdot, \cdot \rangle : V \otimes V \to \mathbb{C} \), satisfying

\[
\langle Xv, v' \rangle = \langle v, g(X)v' \rangle, \quad X \in \mathcal{U}_q(\mathfrak{sl}_2), \quad v, v' \in V,
\]

where \( g \) is the algebra anti-involution of \( \mathcal{U}_q(\mathfrak{sl}_2) \), determined by

\[
g(E) = q^{-h} E, \quad g(F) = F q^h, \quad g(q^{r}) = q^{r}.
\]

For any \( \Lambda \in \mathbb{C} \) the Verma module \( M_\Lambda \) admits a unique up to proportionality inner product. We normalize it by the condition \( \langle v_\Lambda, v_\Lambda \rangle = 1 \); then it is given by the explicit formula

\[
\langle v^{(m)}_\Lambda, v^{(m')}_\Lambda \rangle = \delta_{m,m'} q^{-m(\Lambda - m + 1)} \frac{\prod_{k=1}^{m} [\Lambda - k + 1]}{[m]!}.
\]
A tensor product of two $\mathcal{U}_q(\mathfrak{sl}_2)$-modules is equipped with a $\mathcal{U}_q(\mathfrak{sl}_2)$-action by means of the comultiplication $\Delta$, determined by

$$\Delta(E) = E \otimes 1 + q^h \otimes E, \quad \Delta(F) = F \otimes q^{-h} + 1 \otimes F, \quad \Delta(q^\pm) = q^\pm \otimes q^\pm. \quad (3.6)$$

One immediately verifies that if $\mathcal{U}_q(\mathfrak{sl}_2)$-modules $V, W$ are equipped with inner products, then their product gives an inner product form on the $\mathcal{U}_q(\mathfrak{sl}_2)$-module $V \otimes W$.

The comultiplication $\Delta$ is coassociative, and the tensor product of any finite collection of $\mathcal{U}_q(\mathfrak{sl}_2)$-modules also acquires a $\mathcal{U}_q(\mathfrak{sl}_2)$-module structure. For $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ we set

$$M_{\bar{\lambda}} = M_{\lambda_1} \otimes \ldots \otimes M_{\lambda_n}, \quad M^*_{\bar{\lambda}} = M^*_{\lambda_1} \otimes \ldots \otimes M^*_{\lambda_n}.$$  

The modules $M_{\bar{\lambda}}$ and $M^*_{\bar{\lambda}}$ have standard monomial bases $\{v^{(\bar{m})}_\lambda\}$ and $\{u^\lambda_{\bar{m}}\}$ respectively, where $\bar{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and we denote

$$v^{(\bar{m})}_\lambda = v^{(m_1)}_{\lambda_1} \otimes \ldots \otimes v^{(m_n)}_{\lambda_n}, \quad u^\lambda_{\bar{m}} = u^\lambda_{m_1} \otimes \ldots \otimes u^\lambda_{m_n}.$$

Clearly, the vectors $\{u^\lambda_{\bar{m}}\}$ with $\bar{\lambda}$-admissible indices $\bar{m}$ form a basis of the irreducible submodule $L^*_{\bar{\lambda}}$, and similarly the images of vectors $\{v^{(\bar{m})}_\lambda\}$ form a basis of the irreducible quotient $L_{\bar{\lambda}}$.

3.2. The holomorphic intertwining operators and the orthogonality lemma. Let $\lambda, \Lambda \in \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$. Denote by $\Phi^\Lambda_m(\lambda) : M_{\lambda-1} \to M_{\lambda-\Lambda+2m-1} \otimes M^*_{\bar{\Lambda}}$ the intertwining operators determined by

$$\Phi^\Lambda_m(\lambda)v_{\lambda-1} = [-\lambda + \Lambda - 2m + 1]_m \sum_{i=0}^{m} q^{(\lambda-\Lambda+2m-1)(\lambda-1)}_i v^{(i)}_{\lambda+2m-1} \otimes E' u^\lambda_m =$$

$$= \sum_{i=0}^{m} q^{(\lambda-\Lambda+2m-1)(\lambda-1)}_i v^{(i)}_{\lambda+2m-1} \otimes E' u^\lambda_m. \quad (3.7)$$

Matrix elements of the operator $\Phi^\Lambda_m(\lambda)$ holomorphically depend on highest weights $\lambda, \Lambda$. For any $\bar{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ we define the intertwining operator

$$\Phi^\lambda_{\bar{m}}(\Lambda) : M_{\lambda-1} \to M_{\lambda-1} \otimes M^*_{\bar{\Lambda}} \otimes \ldots \otimes M^*_{\lambda_n}$$

as the composition of operators (3.7):

$$\Phi^\lambda_{\bar{m}}(\Lambda) = (\Phi^\lambda_{m_1}(\lambda - \sum_{i=2}^{n} (\lambda_i - 2m_i)) \otimes 1^{n-1}) \ldots (\Phi^\lambda_{m_n}(\lambda - \Lambda_n + 2m_n) \otimes 1) \Phi^\lambda_{m_n}(\Lambda).$$

For $m \in \mathbb{Z}_{\geq 0}$ and generic $\Lambda \in \mathbb{C}$, we denote

$$Q^\Lambda_m = q^{-m(\Lambda-m+1)} \prod_{k=1}^{m} \frac{[m]!}{[k]!} \frac{[m]!}{[\Lambda - k + 1]}, \quad Q^\Lambda_m(\lambda) = [m]! \prod_{k=1}^{m} \frac{[\lambda + k][\lambda - k - \Lambda + 2m]}{[\Lambda - k + 1]}.$$  

It is easy to see that the formula $\langle u^\lambda_{\bar{m}}, u^\lambda_{\bar{m}} \rangle = \delta_{m,m'} Q^\Lambda_m$ defines an inner product on $M^*_{\lambda}$. The bilinear form $Q : M^*_{\lambda} \otimes M^*_{\lambda} \to \mathbb{C}$, defined by $Q(u^\lambda_{\bar{m}}, u^\lambda_{\bar{m'}}) = \delta_{m,m'} Q^\Lambda_m(\lambda)$, is called the dynamical Shapovalov pairing [FV3]. If $\Lambda \in \mathbb{Z}_{\geq 0}$, then $M^*_{\lambda}$ does not admit nonzero inner
Using the Gauss identity for the inner product and dynamical Shapovalov pairing on \( L^*_\Lambda \), we set

\[
Q^\Lambda_{\vec{m}} = \prod_{j=1}^{n} Q^\Lambda_{m_j}, \quad Q^\Lambda_{\vec{m}}(\lambda) = \prod_{j=1}^{n} Q^\Lambda_{m_j}(\lambda - h^{(j+\cdots;n)}),
\]

which yields an inner product and the dynamical Shapovalov pairing on \( M^*_\Lambda \). The following orthogonality lemma establishes a relation between the dynamical Shapovalov pairing and holomorphic intertwining operators \( \Phi^\Lambda_{\vec{m}}(\lambda) \). We will use it in Section 3 to see Theorem 3.6.

**Theorem 3.1.** Let \( \mu \in \mathbb{C}, \vec{m}, \vec{m}' \in \mathbb{Z}^n \), and let \( \vec{\Lambda} \in \mathbb{C}^n \) be generic. Set \( \lambda = \mu + \sum_{j=1}^{n}(\Lambda_j - 2m_j) \) and \( \lambda' = \mu + \sum_{j=1}^{n}(\Lambda_j - 2m'_j) \). Then for any \( v \in M_{\lambda-1} \) and \( v' \in M_{\lambda'-1} \) we have

\[
\langle \Phi^\Lambda_{\vec{m}}(\lambda) v, \Phi^\Lambda_{\vec{m}'}(\lambda') v' \rangle = \delta_{\vec{m},\vec{m}'} Q^\Lambda_{\vec{m}}(\lambda) \langle v, v' \rangle.
\]

**Proof.** Both sides of (3.9) are trigonometric polynomials in \( q^\mu, q^{\vec{m}} \) and therefore, it suffices to verify the statement when \( \mu \) is generic and \( \Lambda_1, \ldots, \Lambda_n \) are large enough positive integers. Under these assumptions there exists a \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module isomorphism

\[
(M_{\mu-1} \otimes L_{\Lambda_1}) \otimes L_{\Lambda_2} \otimes \cdots L_{\Lambda_n} \cong \bigoplus_{k=0}^{n} M_{\mu-\Lambda_1+2k-1} \otimes L_{\Lambda_2} \otimes \cdots L_{\Lambda_n},
\]

and the direct summands are orthogonal with respect to the contravariant form. Therefore, for \( m_1 \neq m'_1 \) the subspaces \( \Phi^\Lambda_{\vec{m}}(\lambda)(M_{\lambda-1}) \) and \( \Phi^\Lambda_{\vec{m}'}(\lambda')(M_{\lambda'-1}) \) are orthogonal in \( M_{\mu-1} \otimes L^*_\Lambda \). If \( m_1 = m'_1 \), then a similar argument shows that the above subspaces are orthogonal for \( m_2 \neq m'_2 \), etc. Therefore, we obtain the orthogonality relations (3.9) for \( \vec{m} \neq \vec{m}' \).

The desired statement for \( \vec{m} = \vec{m}' \) is easily reduced by induction to the case \( n = 1 \), i.e.

\[
\langle \Phi^\Lambda_{\vec{m}}(\lambda) v, \Phi^\Lambda_{\vec{m}}(\lambda) v' \rangle = Q^\Lambda_{\vec{m}}(\lambda) \langle v, v' \rangle.
\]

(3.10)

The left side of the above equation represents an inner product on \( M_{\lambda-1} \), which by uniqueness must be proportional to the standard one. Therefore, it remains to show that the coefficient of proportionality equals \( Q^\Lambda_{\vec{m}}(\lambda) \). Let \( v = v' = v_{\lambda-1} \), and as before \( \mu = \lambda - \Lambda + 2m \). Then

\[
\langle \Phi^\Lambda_{\vec{m}}(\lambda) v_{\lambda-1}, \Phi^\Lambda_{\vec{m}}(\lambda) v_{\lambda-1} \rangle = \sum_{i=0}^{m} q^{i(-\mu-i)}[-\mu + i + 1]_{m-i}^2 \langle v_{\mu-1}^{(i)}, v_{\mu-1}^{(i)} \rangle (E^\Lambda u_m^\Lambda, E^\Lambda u_m^\Lambda)
\]

\[
= \sum_{i=0}^{m} q^{i(-\mu-i)(\Lambda-m+i+1)} \left( \frac{[m]![-\mu + i + 1]_{m-i}}{[m-i]!} \right) \left( \frac{[\mu-1] \cdots [\mu-i]}{[i]!} \right) \left( \frac{[\Lambda] \cdots [\Lambda-m+i+1]}{[m-i]!} \right)
\]

\[
= q^{-m(\Lambda-m+1)} \left( \frac{[\mu-1] \cdots [\mu-m]^2}{[\Lambda] \cdots [\Lambda-m+1]} \right) \left( \frac{\sum_{i=0}^{m} q^{i(\mu+\Lambda-2m+1)} [-m]_i [\Lambda-m+1]_i}{[i]![-\mu+1]_i} \right).
\]

Using the Gauss identity for the \( q \)-hypergeometric function \( \binom{a}{b} \), we see that

\[
\sum_{i=0}^{m} q^{i(\mu+\Lambda-2m+1)} [-m]_i [\Lambda-m+1]_i = \binom{-m, \Lambda-m+1; 1}{-\mu+1} = q^{m(\Lambda-m+1)} \frac{[-\mu-\Lambda+m]_m}{[-\mu+1]_m},
\]
and therefore we obtain

\[
\langle \Phi_m^\Lambda(\lambda) v_{\lambda-1}, \Phi_m^\Lambda(\lambda) v_{\lambda-1} \rangle = [m]! \frac{[\mu - 1] \cdots [\mu - m][\mu + \Lambda - m] \cdots [\mu + \Lambda - 2m + 1]}{[\Lambda] \cdots [\Lambda - m + 1]} = Q_m^\Lambda(\lambda).
\]

This establishes (3.10) and concludes the proof of the theorem. □

3.3. Two inclusion lemmas. The operators \( \Phi_m^\Lambda(\lambda) \) have nice compatibility properties with respect to the inclusions of Verma modules.

**Lemma 3.2.** Let \( m, \mu \in \mathbb{Z}_{\geq 0} \) and \( \Lambda \in \mathbb{C} \). Then we have a commutative diagram

\[
\begin{array}{ccc}
M_{\mu - 1} \otimes M_\Lambda^* & \xrightarrow{\Phi_m^\Lambda(-\mu + \Lambda - 2m)} & M_{-\mu + \Lambda - 2m - 1} \\
& \downarrow \Phi_m^{\Lambda + \mu}(-\mu + \Lambda - 2m) & \\
M_{-\mu - 1} \otimes M_\Lambda^* & \xleftarrow{\iota(\mu) \otimes 1} & M_{\mu - 1} \otimes M_\Lambda^*. \\
\end{array}
\]

**Proof.** Denote for convenience \( \lambda = -\mu + \Lambda - 2m \). We compute

\[
\Phi_{m+\mu}^\Lambda(\lambda) v_{\lambda-1} = \sum_{i=0}^{m+\mu} q^i(\mu-i) \frac{[\mu + i + 1]_{m+i-1}}{[i]!} F^i v_{\mu-1} \otimes E^i u_{m+\mu}^\Lambda = \sum_{i=\mu}^{m+\mu} q^i(\mu-i) \frac{[\mu + i + 1]_{m+i-1}}{[i]!} F^i v_{\mu-1} \otimes E^i u_{m+\mu}^\Lambda
\]

\[
= \sum_{j=0}^m q^j(\mu-j) \frac{[m]!}{[j!]} F^j v_{\mu-1} \otimes E^j u_{m+\mu}^\Lambda = \sum_{j=0}^m q^j(\mu-j) \frac{[m]!}{[j!]} F^j v_{\mu-1} \otimes E^j u_{m+\mu}^\Lambda = \sum_{j=0}^m q^j(\mu-j) \frac{[m]!}{[j!]} F^j v_{\mu-1} \otimes E^j u_{m+\mu}^\Lambda
\]

This implies that the intertwining operators \( \Phi_m^\Lambda(\lambda) \) and \( (\iota(\mu) \otimes 1) \Phi_{m+\mu}^\Lambda(\lambda) \) agree on the generator \( v_{\lambda-1} \) of the module \( M_{\lambda-1} \), and therefore coincide. □

**Lemma 3.3.** Let \( m, \lambda \in \mathbb{Z}_{\geq 0} \). Then we have a commutative diagram

\[
\begin{array}{ccc}
M_{-\lambda - 1} & \xrightarrow{\iota(\lambda)} & M_{\lambda - 1} \\
\Phi_{m+\lambda}^\Lambda(-\lambda) \downarrow & & \downarrow \Phi_m^\Lambda(\lambda) \\
M_{-\lambda + 2m - 1} \otimes M_\Lambda^* & & \\
\end{array}
\]
Moreover, we have

\[ \Phi_m^\Lambda (\lambda) \Phi^\Lambda v_{\lambda-1} = [-\mu + 1]_m \sum_{i=0}^{m} \frac{q^{(i+1)}_{\mu}}{[i]! [-\mu + 1]_i} \left( \sum_{j=0}^{\lambda} \frac{q^{(\lambda-j)}_j}{[j]! \cdot [m]!} \right) F^{i+j} v_{\mu-1} \otimes F^{\lambda-j} q^{-1} v_{m-i} \]

\[= [-\mu + 1]_m \sum_{i=0}^{m} \frac{q^{(i+1)(\mu-i)}}{[i]! [-\mu + 1]_i} \left( \sum_{j=0}^{\lambda} \frac{[\lambda-j+1]_j [m]!}{[j]! [m-i]!} \right) F^{i+j} v_{\mu-1} \otimes F^{\lambda-j} u_{m-i}^\Lambda \]

\[= [-\mu + 1]_m \sum_{k=0}^{m+\lambda} \frac{q^{(k+1)k}}{[k]!} \left( \sum_{i=0}^{\lambda} \frac{[-k]_i [-m]_i [\Lambda - m + 1]_i}{[i]! [-\mu + 1]_i [\lambda - k + 1]_i} \right) F^k v_{\mu-1} \otimes F^{\lambda-k} u_{m}^\Lambda \]

\[= [-\mu + 1]_m \sum_{k=0}^{m+\lambda} \frac{q^{(k+1)k}}{[k]!} \sum_{i=0}^{\lambda} \frac{[-k]_i [-m]_i [\Lambda - m + 1]_i}{[i]! [-\mu + 1]_i [\lambda - k + 1]_i} \]

where \( _3F_2 \) denotes the generalized q-hypergeometric function. The q-version of Saalschütz’s theorem (see e.g. [S]) states that for any nonnegative integer k we have

\[ _3F_2 \left( \frac{-k, a, b}{c, 1 + a + b - c - k}; 1 \right) = \frac{[c-a]_k [c-b]_k}{[c]_k [c-a-b]_k} \]

and therefore

\[ [\lambda - k + 1]_k _3F_2 \left( \frac{-k, -m, \Lambda - m + 1}{-\mu + 1, \lambda - k + 1}; 1 \right) = [\lambda - k + 1]_k \frac{[-\mu + m + 1]_k [-\mu - \Lambda + m]_k}{[-\mu + 1]_k [-\mu + 2m]_k} \]

\[= (-1)^k [\lambda]_k \frac{[-\mu + m + 1]_k [-\mu - m]_k}{[-\mu + 1]_k [\lambda - k]_k} \]

\[= \frac{[\lambda + m - k]_k [\Lambda - m + k + 1]_k [\Lambda - m + \Lambda + k - \lambda - k]_k}{[\Lambda - m + k + 1]_k [\lambda - k]_k} \]

\[\Phi^\Lambda m (\lambda) \epsilon (\lambda) v_{-\lambda-1} = [-\mu + 1]_m \sum_{k=0}^{m+\lambda} \frac{q^{(k+1)k}}{[k]!} \sum_{i=0}^{\lambda} \frac{[-\mu + m + 1]_k \Phi^k v_{\mu-1} \otimes \Phi^k u_{m+\lambda}^\Lambda =}{[-\mu + 1]_k} \]

\[= \sum_{k=0}^{m+\lambda} \frac{q^{(k+1)k}}{[k]!} \sum_{i=0}^{\lambda} \frac{[-\mu + m + 1]_k \Phi^k v_{\mu-1} \otimes \Phi^k u_{m+\lambda}^\Lambda =}{[-\mu + 1]_k} \]

which shows that the intertwining operators \( \Phi^\Lambda m (\lambda) \epsilon (\lambda) \) and \( \Phi^\Lambda m (\lambda) \epsilon (\lambda) \) agree on the generator \( v_{-\lambda-1} \) of the Verma module \( M_{-\lambda-1} \), and therefore coincide. \qed
3.4. The universal trace matrix $\mathcal{F}(\lambda, x; \vec{\Lambda})$. Let $\vec{\Lambda} \in \mathbb{C}^n[2m]$ for some $m \in \mathbb{Z}_{\geq 0}$. For any $\vec{m} \in \mathbb{Z}^n[m]$ we define the $M^*_{\vec{\Lambda}}[0]$-valued holomorphic trace function $\mathcal{F}_{\vec{m}}(\lambda, x; \vec{\Lambda})$ by

$$\mathcal{F}_{\vec{m}}(\lambda, x; \vec{\Lambda}) = q \frac{m(m+1)}{2} (q - q^{-1})^m (q^x - q^{-x}) \quad \text{Tr}\left|_{M_{\lambda-1}^*} \left( \Phi^\Lambda_{\vec{m}}(\lambda) \ q^{\text{tr}} \right) \right. \quad (3.14)$$

In the above formula, the trace is an infinite power series in $q^{-2x}$, which converges in the domain $|q^{2x}| \gg 1$ to a meromorphic function $\mathcal{F}_{\vec{m}}(\lambda, x; \vec{\Lambda})$.

Using the standard basis $\{ u_{\vec{m}'}^{\Lambda} \}$ of $M^*_{\vec{\Lambda}}[0]$, we define the matrix elements $\{ \mathcal{F}_{\vec{m}}^{\vec{m}'}(\lambda, x; \vec{\Lambda}) \}$ by

$$\mathcal{F}_{\vec{m}}(\lambda, x; \vec{\Lambda}) = \sum_{\vec{m}' \in \mathbb{Z}^n[m]} \mathcal{F}_{\vec{m}}^{\vec{m}'}(\lambda, x; \vec{\Lambda}) u_{\vec{m}'}^{\Lambda},$$

Let $\mathcal{F}(\lambda, x; \vec{\Lambda}) = \{ \mathcal{F}_{\vec{m}}^{\vec{m}'}(\lambda, x; \vec{\Lambda}) \}_{\vec{m}, \vec{m}' \in \mathbb{Z}^n[m]}$ denote the square matrix, formed by arranging coordinates of $\mathcal{F}_{\vec{m}}(\lambda, x; \vec{\Lambda})$ as rows. The columns of this matrix represent vectors

$$\mathcal{F}^{\vec{m}'}(\lambda, x; \vec{\Lambda}) = \sum_{\vec{m} \in \mathbb{Z}^n[m]} \mathcal{F}_{\vec{m}}^{\vec{m}'}(\lambda, x; \vec{\Lambda}) v^{(\vec{m})}_\Lambda.$$

We call $\mathcal{F}(\lambda, x; \vec{\Lambda})$ the universal trace matrix at level $m$. For a fixed $x$ it can be thought of as an operator

$$\mathcal{F}(\lambda, x; \vec{\Lambda}) : M^*_\Lambda[0] \rightarrow \mathfrak{su} \otimes M^*_\Lambda[0], \quad v^{(\vec{m})}_\Lambda \rightarrow \mathcal{F}_{\vec{m}}^{\vec{m}'}(\lambda, x; \vec{\Lambda}).$$

**Theorem 3.4.** Let $\vec{\Lambda} \in \mathbb{C}^n[2m]$, and let $x \in \mathbb{C}$ be generic. Then the vector-valued functions $\{ \mathcal{F}_{\vec{m}}^{\vec{m}'}(\lambda, x; \vec{\Lambda}) \}$ satisfy the $n$-point resonance relations with respect to $\vec{\Lambda}$.

The case of one-point resonance relations was treated in [ES1]. Therefore, we assume that $n > 1$, and prove that the columns of the universal trace matrix satisfy the $n$-point resonance relations.

Let $\vec{m} \in \mathbb{Z}^n$. For $j \in \{1, \ldots, n-1\}$ and $\delta \in \{1, \ldots, m_j\}$, we use Lemma 3.2 and Lemma 3.3 to construct the commutative diagram

$$\begin{array}{ccc}
M_{-\delta - 1} \otimes M^*_{\Lambda_{j+1}} \otimes \cdots \otimes M^*_{\Lambda_n} & \xrightarrow{\delta(\delta) \otimes 1^{n-j}} & M_{-\delta - 1} \otimes M^*_{\Lambda_{j+1}} \otimes \cdots \otimes M^*_{\Lambda_n} \\
\Phi_{m_{j+1}^{\Lambda}}^{\Lambda_{j+1} - 2m_{j+1}} \otimes M^*_{\Lambda_{j+2}} \otimes \cdots \otimes M^*_{\Lambda_n} & \xrightarrow{\delta(\delta) \otimes 1^{n-j}} & M_{-\delta - 1} \otimes M^*_{\Lambda_{j+1}} \otimes \cdots \otimes M^*_{\Lambda_n} \\
M_{-\delta - 1} \otimes M^*_{\Lambda_{j+1}} \otimes \cdots \otimes M^*_{\Lambda_n} & \xrightarrow{\delta(\delta) \otimes 1^{n-j}} & M_{-\delta - 1} \otimes M^*_{\Lambda_{j+1}} \otimes \cdots \otimes M^*_{\Lambda_n} \\
\Phi_{m_{j}^{\Lambda}}^{\Lambda_{j} - 2m_{j}} \otimes M^*_{\Lambda_{j+1}} \otimes \cdots \otimes M^*_{\Lambda_n} & \xrightarrow{\delta(\delta) \otimes 1^{n-j}} & M_{-\delta - 1} \otimes M^*_{\Lambda_{j+1}} \otimes \cdots \otimes M^*_{\Lambda_n} \\
M_{-\delta - 1} \otimes M^*_{\Lambda_{j+1}} \otimes \cdots \otimes M^*_{\Lambda_n} & \xrightarrow{\delta(\delta) \otimes 1^{n-j}} & M_{-\delta - 1} \otimes M^*_{\Lambda_{j+1}} \otimes \cdots \otimes M^*_{\Lambda_n} \\
\Phi_{m_{j-1}^{\Lambda}}^{\Lambda_{j-1} - 2m_{j-1}} \otimes M^*_{\Lambda_{j}} \otimes \cdots \otimes M^*_{\Lambda_n} & \xrightarrow{\delta(\delta) \otimes 1^{n-j}} & M_{-\delta - 1} \otimes M^*_{\Lambda_{j+1}} \otimes \cdots \otimes M^*_{\Lambda_n} \\
M_{-\delta - 1} \otimes M^*_{\Lambda_{j+1}} \otimes \cdots \otimes M^*_{\Lambda_n} & \xrightarrow{\delta(\delta) \otimes 1^{n-j}} & M_{-\delta - 1} \otimes M^*_{\Lambda_{j+1}} \otimes \cdots \otimes M^*_{\Lambda_n}
\end{array}$$
Multiplying by $q^m$ and taking the trace, we prove the resonance relations $(2.4)$. Next, we consider the case $j = n$, and consider the commutative diagram

\[
\begin{array}{ccc}
M_{-\delta-1} & \xrightarrow{i(\delta)} & M_{\delta-1} \\
\Phi^\Lambda_m(-\delta) & \xrightarrow{\otimes} & \Phi^\Lambda_m \otimes M^*_\Lambda_m \\
M_{-\delta-1-(\Lambda+2m)} & \xrightarrow{\otimes} & M^*_\Lambda_m \\
\Phi^\Lambda_m(-\delta) & \xrightarrow{\otimes} & \Phi^\Lambda_m \otimes M^*_\Lambda_m \\
M_{-\delta-1-(\sum^n_{i=2}(\Lambda_i-2m_i))} & \xrightarrow{\otimes} & M^*_\Lambda_m \\
\Phi^\Lambda_m(-\delta) & \xrightarrow{\otimes} & \Phi^\Lambda_m \otimes M^*_\Lambda_m \\
M_{-\delta-1} \otimes M^*_\Lambda_m & \xrightarrow{i(\delta)\otimes 1^n} & M_{\delta-1} \otimes M^*_\Lambda_m \\
\end{array}
\]

In particular, we see that the image of $M_{-\delta-1}$ under $\Phi^\Lambda_m(-\delta) \otimes M^*_\Lambda_m$. Hence the trace function $F_{\tau^\Lambda_m(\vec{m})}(\delta, x; \vec{\Lambda})$ can be computed from the restriction of $\Phi^\Lambda_m(-\delta) \otimes M^*_\Lambda_m$ to the submodule $M_{-\delta-1}$, and therefore is equal to $F_{\vec{m}}(-\delta, x; \vec{\Lambda})$. This proves $(2.5)$ and the theorem.

3.5. **Resonance matrix vs. trace functions.** For any $\vec{\Lambda} \in \mathbb{C}^n$ let $\Xi_{\vec{\Lambda}} \in \text{End}(M_{\vec{\Lambda}})$ denote the invertible operator, acting diagonally in the standard monomial basis $\{v^{(m)}_{\vec{\Lambda}}\}$, with diagonal elements

\[
\Xi^{m}_{\vec{\Lambda}} = q^{\sum^n_{i=1} m_i \sum^i_{j=1}(\Lambda_i-m_i)}. \quad (3.15)
\]

**Theorem 3.5.** Let $x \in \mathbb{C}$ be generic. Then for any $\vec{\Lambda} \in \mathbb{C}^n[2m]$ we have

\[
\Psi(\lambda, x; \vec{\Lambda}) = F(\lambda, x; \vec{\Lambda}) \Xi_{\vec{\Lambda}}. \quad (3.16)
\]

**Proof.** Using the definitions and easy induction on $k \in \mathbb{Z}_{\geq 0}$, we obtain

\[
\Phi^\Lambda_m(\lambda)F^kv_{\lambda-1} = \frac{q^{-m\lambda}}{(q - q^{-1})^m} q^{m(\Lambda-2m)+\frac{m(m+1)}{2}} \left( F^kv_{\lambda-\Lambda+2m-1} \otimes q^{-k(\Lambda-2m)}u^\Lambda_m + O(q^{2\lambda}) \right),
\]

where $O(q^{2\lambda})$ stands for a vector-valued polynomial in $q^{2\lambda}$ of degree at most $m$ and vanishing at zero. Iterating and using the identity

\[
\sum^n_{j=1} \left( m_j \sum^n_{i=j}(\Lambda_i-2m_i) + \frac{m_j(m_j+1)}{2} \right) = -\sum^n_{j=1} \left( m_j \sum^{j-1}_{i=1}(\Lambda_i-m_i) + \frac{\sum^n_{i=1} m_i(1+\sum^n_{i=1} m_i)}{2} + \left( \sum^n_{i=1} m_i \right) \sum^n_{i=1}(\Lambda_i-2m_i) \right),
\]

for every $\vec{m} \in \mathbb{Z}^n[m]$ we get

\[
\Phi^\Lambda_{\vec{m}}(\lambda)F^kv_{\lambda-1} = \frac{q^{-m\lambda+m(m+1)/2}}{(q - q^{-1})^m} \left( \Xi^{\vec{m}}_{\vec{\Lambda}} \right)^{-1} \left( F^kv_{\lambda-1} \otimes v^\Lambda_{\vec{m}} + O(q^{2\lambda}) \right).
\]
Multiplying by $q^{\ell}$ and taking the trace, we obtain
\[
\mathcal{F}_{\ell m}(\lambda, x; \Lambda) = (q^x - q^{-x}) q^{-\lambda_0} (\Xi_\Lambda^{\ell m})^{-1} \sum_{k=0}^{\infty} q^{(\lambda - 2k - 1)x} (u_{m,k}^\lambda + O(q^{2\lambda}))
\]
\[
= q^{\lambda(x-m)} (\Xi_\Lambda^{\ell m})^{-1} (u_{m}^\lambda + O(q^{2\lambda})).
\]
Therefore, we have the expansion
\[
\mathcal{F}(\lambda, x; \Lambda) = q^{\lambda(x-m)} ((\Xi_\Lambda)^{-1} + O(q^{2\lambda})). \tag{3.17}
\]
The two sides of (3.16) are matrix-valued trigonometric quasi-polynomials with the same initial term of the expansion. The columns of both matrices satisfy the resonance relations, and for generic $\Lambda$ the relation (3.16) must hold by the uniqueness property of Theorem 2.1. Since both sides are holomorphic in $\Lambda$, the equality must hold identically. □

**Example.** Let $n = 1$. Then $\Xi_{2m} = 1$, and we have $\mathcal{F}(\lambda, x; 2m) = \Psi(\lambda, x; 2m)$, see (2.14).

**Example.** One computes that for $\Lambda_1 + \Lambda_2 = 2$
\[
\mathcal{F}(\lambda, x) = \begin{pmatrix}
\mathcal{F}_{0,1}^{0,1}(\lambda, x; \Lambda) & \mathcal{F}_{0,1}^{0,0}(\lambda, x; \Lambda) \\
\mathcal{F}_{1,0}^{0,1}(\lambda, x; \Lambda) & \mathcal{F}_{1,0}^{1,0}(\lambda, x; \Lambda)
\end{pmatrix}
\]
\[
= \frac{q^{(\lambda-1)x}}{[x+1]} \begin{pmatrix}
q^{-\Lambda_1}[\lambda + 1] & 0 \\
q^{-\Lambda_2}[\lambda - \Lambda_2 + 1] & q^{2x} \begin{pmatrix}
[\lambda - \Lambda_2 + 1] & -q^{2\lambda-\Lambda_2+1} \Lambda_1 \\
0 & [\lambda - 1]
\end{pmatrix}
\end{pmatrix}.
\]
This matrix, multiplied by $\Xi_\Lambda = \begin{pmatrix} q^{\Lambda_1} & 0 \\
0 & 1 \end{pmatrix}$ on the right, equals the resonance matrix (2.15).

## 4. The Hypergeometric Construction

### 4.1. The hypergeometric pairing
Let $\Lambda \in \mathbb{C}^n$, $m \in \mathbb{Z}_{\geq 0}$ and $\vec{t} = (t_1, \ldots, t_m)$. Denote
\[
\Omega(\vec{t}; \Lambda) = \left(\prod_{i=1}^{m} \prod_{k=1}^{n} \frac{1}{(t_i - q^{2k} t_i - 1)} \right) \left(\prod_{i<j} \frac{(t_i - t_j)^2}{(qt_i - q^{-1} t_j)(q^{-1} t_i - qt_j)} \right).
\]
For any $\vec{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_n) \in \mathbb{Z}^{n+1}[m]$ we introduce the string
\[
\vec{t}_{\vec{\sigma}} = (0, \ldots, 0, q^{-\Lambda_1+2(\sigma_1-1)}, \ldots, q^{-\Lambda_1+2}, q^{-\Lambda_1}, \ldots, q^{-\Lambda_n+2(\sigma_n-1)}, \ldots, q^{-\Lambda_n+2}, q^{-\Lambda_n}),
\]
and for any string $\vec{x} = (x_1, \ldots, x_m)$ define the multiple residue $\text{Res}_{\vec{t}=\vec{x}} A(\vec{t})$ by
\[
\text{Res}_{\vec{t}=\vec{x}} A(\vec{t}) = \text{Res}_{t_1=x_1} \left(\text{Res}_{t_2=x_2} \left(\ldots \text{Res}_{t_m=x_m} A(\vec{t}) \right) \right).
\]
The **hypergeometric pairing** between symmetric polynomials $f(\vec{t}), g(\vec{t})$ is defined by
\[
\mathcal{I}_\Lambda(f, g) = \sum_{\vec{\sigma}} q^{\sigma_0(\sigma_0-1)/2} [\sigma_0]! f(\vec{t}_{\vec{\sigma}})g(\vec{t}_{\vec{\sigma}}) \text{Res}_{\vec{t}=\vec{t}_{\vec{\sigma}}} \frac{\Omega(\vec{t}; \Lambda)}{t_1 \ldots t_m}. \tag{4.1}
\]
It is clear that the hypergeometric pairing is symmetric. The above definition is an algebraic version of a contour integral formula.
Lemma 4.1. In the domain $\text{Re} \Lambda_1 \ll \cdots \ll \text{Re} \Lambda_n \ll 0$ we have

$$
\frac{1}{(2\pi \sqrt{-1})^m} \int_{|t_1|=\cdots=|t_m|=1} \frac{f(\vec{t})}{t_1 \cdots t_m} \Omega(\vec{t}; \vec{\Lambda}) dt_1 \cdots dt_m = \text{m!} \sum_{\sigma \in \mathcal{S}[m]} \frac{q^{\sigma_0(\sigma_0-1)/2}}{|\sigma_0|!} f(\vec{t}_\sigma) \text{Res}_{\vec{t}=\vec{t}_\sigma} \Omega(\vec{t}; \vec{\Lambda})
$$

for any symmetric polynomial $f(\vec{t})$.

Proof. We illustrate the argument for the case $n=1, m=2$; the general case is analogous. Poles of the integrand are determined by the rational expression

$$
\Omega(t_1, t_2; \Lambda_1) = \frac{(t_1 - t_2)^2}{(t_1 - q^{\Lambda_1})(t_2 - q^{\Lambda_1})(t_2 - t_1)(qt_1 - tq_2)(q^{-1}t_1 - qt_2)(t_1 t_2)}.
$$

The condition $\text{Re} \Lambda_1 \ll \cdots \ll 0$ means that $|q^{\Lambda_1}| \gg 1$, and we see that the poles inside the contour $|t_2| = 1$ have first order, and are located at $t_2 = 0, t_2 = q^{-1}t_1$ and $t_2 = q^2t_1$. Applying the residue theorem for the variable $t_2$, and repeating the argument for $t_1$, we get

$$
\frac{1}{(2\pi \sqrt{-1})^2} \int_{|t_1|=|t_2|=1} \frac{f(t_1, t_2) \Omega(t_1, t_2; \Lambda_1)}{t_1 t_2} dt_2 dt_1 =
\text{Res}_{0,0} + \text{Res}_{q^{-\Lambda_1}, 0} + \text{Res}_{0, q^{-\Lambda_1}} + \text{Res}_{q^{-\Lambda_1+2q^{-\Lambda_1}} + \text{Res}_{0, q^2t_1}} + \text{Res}_{q^{-\Lambda_1}, q^2t_1},
$$

where $\text{Res}_{a,b}$ is the shortened notation for $\text{Res}_{t_1=a} \text{Res}_{t_2=b} f(t_1, t_2) \Omega(t_1, t_2; \Lambda_1)$. Using the symmetry of the integrand under the permutation $t_1 \leftrightarrow t_2$, we see that

$$
\text{Res}_{q^{-\Lambda_1}, 0} = \text{Res}_{0, q^{-\Lambda_1}}, \quad \text{Res}_{q^{-\Lambda_1}, q^2t_1} = \text{Res}_{q^{-\Lambda_1+2q^{-\Lambda_1}}}, \quad \text{Res}_{0, q^2t_1} = \frac{q - q^{-1}}{q + q^{-1}} \text{Res}_{0, 0},
$$

and the statement follows. \qed

Example. Let $n = 2$ and $m = 1$. Then the hypergeometric pairing is given by

$$
\mathcal{I}_\Lambda(f, g) = \frac{f(0)g(0)}{q^{\Lambda_1+\Lambda_2}} + \frac{f(q^{-\Lambda_1})g(q^{-\Lambda_1})}{(1 - q^{\Lambda_1-\Lambda_2})(1 - q^{-\Lambda_2})(1 - q^{\Lambda_1+\Lambda_2})} + \frac{f(q^{-\Lambda_2})g(q^{-\Lambda_2})}{(1 - q^{\Lambda_2-\Lambda_1})(1 - q^{-2\Lambda_2})(1 - q^{\Lambda_1+\Lambda_2})}.
$$

4.2. The weight functions. For any $\vec{m} \in \mathcal{Z}^n[m]$ define the weight functions $\omega_{\vec{m}}(\vec{t}; \vec{\Lambda})$ by

$$
\omega_{\vec{m}}(\vec{t}; \vec{\Lambda}) = \frac{[m_1]! \cdots [m_n]!}{[m]!} \sum_{I_1, \ldots, I_n} \left( \prod_{1 \leq k < l \leq n} \prod_{i \in I_k, j \in I_l} g_{i, l} - q^{-1}t_{i, l} \right)
$$

$$
\times \left( \prod_{k=1}^{n} \prod_{i \in I_k} q^{\sum_{l=1}^{k-1} t_{i, l}} q^{m_{i, l}} \prod_{l=k+1}^{n} (q^{\Lambda_{l, i}} - 1) \prod_{t_{i, l}}^{k-1} \prod_{l=1}^{k-1} (q^{\Lambda_{i, t}} - 1) \right).
$$

where the summations are performed over all partitions $\{I_1, \ldots, I_n\}$ of the set $\{1, \ldots, m\}$ into $n$ disjoint subsets, such that $|I_k| = m_k$. One can show that the functions $\omega_{\vec{m}}(\vec{t}; \vec{\Lambda})$ depend polynomially on variables $\vec{t}$, and are invariant under the permutation group $S_m$. 

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Proposition 4.2. Let $W = Z^{K}$. STYRKAS AND A. VARCHENKO

The hypergeometric qKZ matrix

Similarly, for any $\vec{m}$, $\vec{m}' \in Z^n[m]$ set

Example. Let $n = 2$ and $m = 1$. Then

$$\omega_{0,1}(t; \vec{A}) = q t (q^{\lambda_1} t - 1),$$
$$\omega_{1,0}(t; \vec{A}) = q t (t - q^{\lambda_2}),$$
$$W_{0,1}(t; \lambda; \vec{A}) = q^{-1} (q^{-\lambda_1+\lambda_2} t - q^{\lambda_2-\lambda_1} (q^{\lambda_1} t - 1),$$
$$W_{1,0}(t; \lambda; \vec{A}) = q^{-1} (-q^{-\lambda_1+\lambda_2} t - q^{\lambda_2-\lambda_1} (t - q^{\lambda_2}),$$
$$W^{0,1}(t; \vec{A}) = q^{-1} (q^{-x+\lambda_1+\lambda_2} t - q^{x+\lambda_2} t - q^{\lambda_1},$$
$$W^{1,0}(t; \vec{A}) = q^{-1} (q^{-x+\lambda_1} t - q^{x+\lambda_1+\lambda_2} (q^{\lambda_2} t - 1).$$

4.3. The hypergeometric qKZ matrix $\mathcal{H}(x; \vec{A})$. Let $x \in \mathbb{C}$ and $\vec{A} \in \mathbb{C}^n$. For $\vec{m}, \vec{m}' \in Z^n[m]$ set

$$\mathcal{H}^{\vec{m}'}_{\vec{m}}(x; \vec{A}) = \mathcal{I}_X \left( (\omega_{\vec{m}'}(\cdot; \vec{A}), W_{\vec{m}'}(\cdot; x; \vec{A}) \right).$$

Proposition 4.2. Let $x \in \mathbb{C}$. Then $\mathcal{H}^{\vec{m}'}_{\vec{m}}(x; \vec{A}) = 0$ unless $\vec{m} \nless \vec{m}'$, and for $\vec{m}' = \vec{m}$ we have

$$\mathcal{H}^{\vec{m}'}_{\vec{m}}(x; \vec{A}) = q^{m(m-1)/2} \prod_{i=1}^{n} [m_i]! \prod_{j=0}^{m_i-1} \frac{q^{x+\sum_{k=1}^{i}(\lambda_k-2m_k)-j} - q^{x+\sum_{k=1}^{i}(\lambda_k-2m_k)-\lambda_1+j}}{q^{\lambda_1-j} - q^{-\lambda_1+j}}.$$  

Proof. One can check that for any $\vec{\sigma} = (\sigma_0, \ldots, \sigma_n) \in Z^{n+1}[m]$ one has

$$\omega_{\vec{m}}(t_{\vec{\sigma}}; \vec{A}) = 0$$

unless $\sigma_0 = 0$ and $\vec{m} \nless (\sigma_1, \ldots, \sigma_n)$. Similarly, for any $\vec{\sigma} = (0, \sigma_1, \ldots, \sigma_n) \in Z^{n+1}[m]$ we have

$$W_{\vec{m}'}(t_{\vec{\sigma}}; x; \vec{A}) = 0$$

unless $\vec{m}' \gtrdot (\sigma_1, \ldots, \sigma_n)$. Therefore, the sum of residues in the definition (4.1) of the hypergeometric pairing is reduced to the sum over the subset of indices $\vec{\sigma} = (0, \sigma_1, \ldots, \sigma_n)$, satisfying $\vec{m} \nless (\sigma_1, \ldots, \sigma_n) \nless \vec{m}'$. The desired vanishing property of $\mathcal{H}^{\vec{m}'}_{\vec{m}}(x; \vec{A})$ follows immediately. Finally, if $\vec{m}' = \vec{m}$, the
only nonzero term in the hypergeometric pairing corresponds to \( \bar{\sigma} = (0, m_1, \ldots, m_n) \), and a straightforward computation of the residue yields (4.3).

For each \( \bar{m}' \in \mathbb{Z}^n \) define the vector \( \mathcal{H}_{\bar{m}'}(x; \ddot{\Lambda}) \in M_{\dot{\Lambda}} \) by

\[
\mathcal{H}_{\bar{m}'}^{m}(x; \ddot{\Lambda}) = \sum_{\bar{m} \in \mathbb{Z}^n \mid m} \mathcal{H}_{\bar{m}}^{m'}(x; \ddot{\Lambda}) v_{\dot{\Lambda}}^{(\bar{m})}. \tag{4.4}
\]

Let \( \mathcal{H}(x; \ddot{\Lambda}) = \{ \mathcal{H}_{\bar{m}}^{m}(x; \ddot{\Lambda}) \} \bar{m}, \bar{m}' \in \mathbb{Z}^n \mid m \) denote the square matrix, formed by vectors \( \mathcal{H}_{\bar{m}'}^{m}(x; \ddot{\Lambda}) \) arranged as columns. We call \( \mathcal{H}(x; \ddot{\Lambda}) \) the hypergeometric qKZ matrix at level \( m \). For any fixed \( x \in \mathbb{C} \) it can be regarded as a weight-preserving operator

\[
\mathcal{H}(x; \ddot{\Lambda}) \in \text{End}(M_{\dot{\Lambda}}), \quad v_{\dot{\Lambda}}^{\bar{m}'} \mapsto \mathcal{H}_{\bar{m}'}^{\bar{m}}(x; \ddot{\Lambda}). \tag{4.5}
\]

Matrix elements of \( \mathcal{H}(x; \ddot{\Lambda}) \) are trigonometric polynomials in \( x \), and are meromorphic in the \( \dddot{\Lambda} \) variables. It follows from Proposition 4.2 that \( \mathcal{H}(x; \dddot{\Lambda}) \) is invertible for generic \( x \in \mathbb{C} \).

**Example.** Let \( n = 2 \) and \( m = 1 \). We have

\[
\mathcal{H}(x; \ddot{\Lambda}) = \begin{pmatrix} \mathcal{H}_{0,1}^{0,1}(x; \ddot{\Lambda}) & \mathcal{H}_{0,1}^{1,1}(x; \ddot{\Lambda}) \\ \mathcal{H}_{1,0}^{0,1}(x; \ddot{\Lambda}) & \mathcal{H}_{1,0}^{1,1}(x; \ddot{\Lambda}) \end{pmatrix} = \begin{pmatrix} q^{\delta - \lambda_1 - \Lambda_2 + \Lambda_1} & 0 \\ q^{\delta - \lambda_1 - \Lambda_2} & q^{-x} \end{pmatrix}. \tag{4.6}
\]

4.4. **The hypergeometric qKZB matrix** \( \mathbb{H}(\lambda, x; \ddot{\Lambda}) \). Let \( \lambda, x \in \mathbb{C} \) and \( \dddot{\Lambda} \in \mathbb{C}^n \). For any \( \bar{m}, \bar{m}' \in \mathbb{Z}^n \mid m \), we set

\[
\mathbb{H}_{\bar{m}}^{\bar{m}'}(\lambda, x; \dddot{\Lambda}) = q^{-x} \mathcal{I}_{\dot{\Lambda}} \left( \mathcal{W}_{\bar{m}}(\, ; \lambda; \dddot{\Lambda}), \mathcal{W}_{\bar{m}'}(\, ; x; \dddot{\Lambda}) \right). \tag{4.7}
\]

For each \( x \in \mathbb{C} \) and \( \bar{m}' \in \mathbb{Z}^n \) define the vector \( \mathbb{H}_{\bar{m}}(\lambda, x; \dddot{\Lambda}) \in \mathfrak{su} \otimes M_{\dot{\Lambda}} \) by

\[
\mathbb{H}_{\bar{m}'}^{\bar{m}}(\lambda, x; \dddot{\Lambda}) = \sum_{\bar{m} \in \mathbb{Z}^n \mid m} \mathbb{H}_{\bar{m}}^{\bar{m}'}(\lambda, x; \dddot{\Lambda}) v_{\dot{\Lambda}}^{(\bar{m})}. \tag{4.8}
\]

Let \( \mathbb{H}(\lambda, x; \ddot{\Lambda}) = \{ \mathbb{H}_{\bar{m}}^{\bar{m}'}(\lambda, x; \dddot{\Lambda}) \} \bar{m}, \bar{m}' \in \mathbb{Z}^n \mid m \) denote the square matrix, formed by vectors \( \mathbb{H}_{\bar{m}}^{\bar{m}'}(\lambda, x; \dddot{\Lambda}) \) arranged as columns. We call \( \mathbb{H}(\lambda, x; \dddot{\Lambda}) \) the hypergeometric qKZB matrix at level \( m \). For any fixed \( x \in \mathbb{C} \) it can be regarded as a weight-preserving operator

\[
\mathbb{H}(\lambda, x; \dddot{\Lambda}) : M_{\dot{\Lambda}} \to \mathfrak{su} \otimes M_{\dot{\Lambda}}, \quad v_{\dot{\Lambda}}^{\bar{m}} \mapsto \mathbb{H}_{\bar{m}}^{\bar{m}'}(\, ; \lambda, x; \dddot{\Lambda}). \tag{4.9}
\]

The following theorem is a version of the more general result in [EV1].

**Theorem 4.3.** Let \( \dddot{\Lambda} \in \mathbb{C}^{n}[2m] \), and let \( x \in \mathbb{C} \), \( \bar{m}' \in \mathbb{Z}^n \mid m \). Then the function \( \mathbb{H}_{\bar{m}'}^{\bar{m}}(\lambda, x; \dddot{\Lambda}) \) satisfies the \( n \)-point resonance relations with respect to \( \dddot{\Lambda} \).

**Proof.** Conditions \( C_j(\bar{m}) \) for \( j < n \) follow from the zero weight condition and the identity

\[
\mathcal{W}_{\bar{m}}(\, ; \n - \sum_{i=1}^j (\Lambda_i - 2m_i) + \sum_{i=j+1}^n (\Lambda_i - 2m_i); \dddot{\Lambda}) = \mathcal{W}_{x_{j+1}^{\bar{m}}}^{m}(\, ; \n - \sum_{i=1}^j (\Lambda_i - 2m_i) + \sum_{i=j+1}^n (\Lambda_i - 2m_i); \dddot{\Lambda}).
\]
Verification of the conditions $C_n(\vec{m})$ is more technical. For complete details, we refer the reader to [FY1], where the resonance relations for the hypergeometric qKZB matrix were established in a more general elliptic case.

The hypergeometric matrix has remarkable symmetries with respect to variables $\lambda, x$.

**Lemma 4.4.** For any $\vec{m}, \vec{m}' \in \mathbb{Z}^n[\mathbf{m}]$ we have
\[
\mathbb{H}_m^n(\lambda, x; \vec{A}) = \mathbb{H}_{\vec{m}'}^\vec{m}(-x, -\lambda; \vec{A}) = \mathbb{H}_{\text{opp}(\vec{m}')}^{\text{opp}(\vec{m})}(-\lambda, -x; \text{opp}(\vec{A})),
\] (4.10)
where $\text{opp}(x_1, \ldots, x_n) = (x_n, \ldots, x_1)$.

**Proof.** Denote $\vec{t}^{-1} = (t_1^{-1}, \ldots, t_n^{-1})$. Straightforward verification shows that
\[
\mathcal{W}_m(\vec{t}; -\lambda; \vec{A}) = t^{\vec{m}} \mathcal{W}_m(\vec{t}^{-1}; \lambda; \vec{A})
\]
\[
\mathcal{W}^{\vec{m}}(\vec{t}; -x; \vec{A}) = t^{\vec{m}} \mathcal{W}^{\vec{m}}(\vec{t}^{-1}; x; \vec{A})
\]
\[
\Omega(\vec{t}; \vec{A}) = t^{-2\vec{m}} \Omega(\vec{t}^{-1}; \vec{A})
\]
and the first of equalities (4.10) is obtained by the change of variables $\vec{t} \mapsto \vec{t}^{-1}$ in integration over the torus $|t_1| \cdots = |t_n| = 1$. The second one follows from the symmetry of the hypergeometric pairing and the identity
\[
\mathcal{W}_m(\vec{t}; \lambda; \vec{A}) = \mathcal{W}_{\text{opp}(\vec{m})}(\vec{t}; \lambda; \text{opp}(\vec{A}))
\]
which implies that $\mathbb{H}_m(\lambda, x; \tilde{\Lambda})$ is a vector-valued function with expansion
\[
\mathbb{H}_m(\lambda, x; \tilde{\Lambda}) = q^{\lambda(x-m)} \left( \sum_{n=0}^{\infty} \mathcal{H}_m(x; \tilde{\Lambda})^n + O(q^{2\lambda}) \right).
\]

The uniqueness property for the solutions of resonance relations and (4.13), (4.14) imply that for generic $x, \tilde{\Lambda}$ we have $\mathcal{F}(\lambda, x; \tilde{\Lambda}) \mathcal{H}(x; \tilde{\Lambda}) = \mathbb{H}(\lambda, x; \tilde{\Lambda})$. Since both sides are meromorphic in $\tilde{\Lambda}$, the desired equality holds identically.

\[\square\]

5. Braiding properties of intertwining operators and the $R$-matrices

5.1. The quantum $R$-matrix. For any $\Lambda_1, \Lambda_2 \in \mathbb{C}$ define $\mathcal{R}_{\Lambda_1, \Lambda_2} \in \text{End}(M_{\Lambda_1} \otimes M_{\Lambda_2})$ by
\[
\mathcal{R}_{\Lambda_1, \Lambda_2} = q^{-\frac{\Lambda_1 \Lambda_2}{2}} \mathcal{R} \big|_{M_{\Lambda_1} \otimes M_{\Lambda_2}},
\]
where
\[
\mathcal{R} = \left( \sum_{k \geq 0} q^{\frac{k(k-1)}{2}} \frac{(q^{-1} - q)^k}{[k]!} E^k \otimes F^k \right)^{h_{\Lambda_1} \cdot h_{\Lambda_2}}.
\]
We refer to the family of operators $\mathcal{R}_{\Lambda_1, \Lambda_2}$ as the quantum $R$-matrix. We define
\[
\mathcal{R}_{\Lambda_1, \Lambda_2} = \mathcal{R}_{\Lambda_2, \Lambda_1} P : M_{\Lambda_1} \otimes M_{\Lambda_2} \rightarrow M_{\Lambda_2} \otimes M_{\Lambda_1},
\]
where $P$ is the permutation of tensor factors, i.e. $P(x \otimes y) = y \otimes x$.

For $m_1, m_2, n_1, n_2 \in \mathbb{Z}_{\geq 0}$ define the matrix elements $(\mathcal{R}_{\Lambda_1, \Lambda_2})^{m_1, m_2}_{n_1, n_2}$ by the relation
\[
\mathcal{R}_{\Lambda_1, \Lambda_2}(v^{(m_1)}_{\Lambda_1} \otimes v^{(m_2)}_{\Lambda_2}) = \sum_{n_1, n_2} (\mathcal{R}_{\Lambda_1, \Lambda_2})^{m_1, m_2}_{n_1, n_2} v^{(n_1)}_{\Lambda_1} \otimes v^{(n_2)}_{\Lambda_2}.
\]

For $\tilde{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ and distinct $i, j \in \{1, \ldots, n\}$, the action of the quantum $R$-matrix in the $i$-th and $j$-th tensor factors yields an operator $\mathcal{R}_{\Lambda_i, \Lambda_j}^{(i,j)} \in \text{End}(M_{\Lambda_i})$, which we will simply denote $\mathcal{R}_{\Lambda_i, \Lambda_j}$; the relevant superscripts can always be reconstructed from the context.

The quantum $R$-matrix has the following properties (for more details see e.g. [LI, CP]):

- The quantum Yang-Baxter equation:
\[
\mathcal{R}_{\Lambda_1, \Lambda_2} \mathcal{R}_{\Lambda_1, \Lambda_3} \mathcal{R}_{\Lambda_2, \Lambda_3} = \mathcal{R}_{\Lambda_2, \Lambda_3} \mathcal{R}_{\Lambda_1, \Lambda_3} \mathcal{R}_{\Lambda_1, \Lambda_2}.
\]

- The weight-preserving property:
\[
(\mathcal{R}_{\Lambda_1, \Lambda_2})^{m_1, m_2}_{n_1, n_2} = 0, \quad \text{if} \ m_1 + m_2 \neq n_1 + n_2.
\]

- The fusion compatibility:
\[
\mathcal{Y}_{\Lambda_1, \Lambda_2} \mathcal{R}_{\Lambda_1 + \Lambda_2, \Lambda_3} = \mathcal{R}_{\Lambda_2, \Lambda_3} \mathcal{R}_{\Lambda_1, \Lambda_3} \mathcal{Y}_{\Lambda_1, \Lambda_2},
\]
\[
\mathcal{Y}_{\Lambda_2, \Lambda_3} \mathcal{R}_{\Lambda_1 + \Lambda_2, \Lambda_3} = \mathcal{R}_{\Lambda_1, \Lambda_3} \mathcal{R}_{\Lambda_1, \Lambda_3} \mathcal{Y}_{\Lambda_2, \Lambda_3},
\]
where linear maps $\mathcal{Y}_{\Lambda', \Lambda''} : M_{\Lambda'} \otimes M_{\Lambda''} \rightarrow M_{\Lambda'} \otimes M_{\Lambda''}$ are defined by
\[
\mathcal{Y}_{\Lambda', \Lambda''}(v^{(m)}_{\Lambda'} \otimes v^{(m')}_{\Lambda''}) = \sum_{m' + m'' = m} q^{m''(\Lambda' - m')} v^{(m')}_{\Lambda'} \otimes v^{(m'')}_{\Lambda''}.
\]

- The rationality property:
\[
(\mathcal{R}_{\Lambda_1, \Lambda_2})^{m_1, m_2}_{n_1, n_2} \in \mathbb{C}(q^{\Lambda_1}, q^{\Lambda_2}).
\]
The vanishing property:

\[(\mathcal{R}_{\Lambda_1,\Lambda_2})^{m_1,m_2}_{n_1,n_2} = 0, \quad \text{if } \Lambda_i \in \{n_i, n_i + 1, \ldots, m_i - 1\} \text{ for } i = 1 \text{ or } 2.\]

The vanishing property implies that for \(\Lambda_1, \Lambda_2 \in \mathbb{Z}_{\geq 0}\) we obtain induced admissible endomorphisms \(\mathcal{R}_{\Lambda_1,\Lambda_2}^{\text{adm}} \in \text{End}(L_{\Lambda_1} \otimes L_{\Lambda_2})\), represented by the matrix elements with \(\Lambda\)-admissible indices. The admissible maps \(\mathcal{Y}_{\Lambda',\Lambda''}^{\text{adm}} : L_{\Lambda'} \otimes L_{\Lambda''} \rightarrow L_{\Lambda'} \otimes L_{\Lambda''}\) are defined as in (5.5) with summation over admissible indices, and an analogue of (5.4) holds for \(\mathcal{R}_{\Lambda_1,\Lambda_2}^{\text{adm}}\) and \(\mathcal{Y}_{\Lambda',\Lambda''}^{\text{adm}}\).

The fusion compatibility property recovers admissible operators for arbitrary \(\Lambda\).

The vanishing property implies that for \(\Lambda\) the quantum dynamical Yang-Baxter equation:

\[\mathcal{R}_{\Lambda,\Lambda} (\lambda) = \lim_{z \to +\infty} \lim_{\tau \to i\infty} R_{\Lambda,\Lambda} (2\eta z, 2\eta \lambda).\]

This, the dynamical \(R\)-matrix is the distinguished family \(\mathbb{R}_{\Lambda_1,\Lambda_2}\) of endomorphisms of tensor products \(M_{\Lambda_1} \otimes M_{\Lambda_2}\) for \(\Lambda_1, \Lambda_2 \in \mathbb{C}\), meromorphically depending on a complex parameter \(\lambda\). We also consider the operators

\[\mathbb{R}_{\Lambda_1,\Lambda_2} (\lambda) = \mathbb{R}_{\Lambda_2,\Lambda_1} (\lambda) P : M_{\Lambda_1} \otimes M_{\Lambda_2} \rightarrow M_{\Lambda_2} \otimes M_{\Lambda_1}.\]

**Remark 5.1.** The operators \(\mathbb{R}_{\Lambda_1,\Lambda_2} (\lambda)\) are isomorphisms of tensor products of modules for a suitable degeneration of the elliptic quantum group \(E_{\tau,\eta}(sl_2)\).

For \(m_1, m_2, n_1, n_2 \in \mathbb{Z}_{\geq 0}\) define the matrix elements \((\mathbb{R}_{\Lambda_1,\Lambda_2} (\lambda))^{m_1,m_2}_{n_1,n_2}\)

\[\mathbb{R}_{\Lambda_1,\Lambda_2} (\lambda) (v_{\Lambda_1}^{(m_1)} \otimes v_{\Lambda_2}^{(m_2)}) = \sum_{n_1,n_2} (\mathbb{R}_{\Lambda_1,\Lambda_2} (\lambda))^{m_1,m_2}_{n_1,n_2} v_{\Lambda_1}^{(n_1)} \otimes v_{\Lambda_2}^{(n_2)}.\]

We use the dynamical notation \(\mathbb{R}_{\Lambda_1,\Lambda_j} (\lambda - h^{(k)})\) for the dynamical \(R\)-matrix acting in \(i\)-th and \(j\)-th tensor factors of \(M_\lambda\), with the convention that \(h^{(k)}\) must be replaced by the weight in \(k\)-th tensor component. For example,

\[\mathbb{R}_{\Lambda_1,\Lambda_3} (\lambda - h^{(2)}) v_{\Lambda_1}^{(m_1)} \otimes v_{\Lambda_2}^{(m_2)} \otimes v_{\Lambda_3}^{(m_3)} = \sum_{n_1,n_3} (\mathbb{R}_{\Lambda_1,\Lambda_3} (\lambda - \Lambda - 2 + 2m_2))^{m_1,m_3}_{n_1,n_3} v_{\Lambda_1}^{(n_1)} \otimes v_{\Lambda_2}^{(m_2)} \otimes v_{\Lambda_3}^{(n_3)}.\]

The dynamical \(R\)-matrix has the following properties, derived from the results in [EV2]:

- the quantum dynamical Yang-Baxter equation:

\[\mathbb{R}_{\Lambda_1,\Lambda_2} (\lambda - h^{(3)}) \mathbb{R}_{\Lambda_1,\Lambda_3} (\lambda) \mathbb{R}_{\Lambda_2,\Lambda_3} (\lambda - h^{(1)}) = \mathbb{R}_{\Lambda_2,\Lambda_3} (\lambda) \mathbb{R}_{\Lambda_1,\Lambda_3} (\lambda - h^{(2)}) \mathbb{R}_{\Lambda_1,\Lambda_2} (\lambda).\]
• The weight-preserving property
\[
(R_{\Lambda_1,\Lambda_2}(\lambda))^{m_1,m_2}_{n_1,n_2} = 0, \quad \text{if } m_1 + m_2 \neq n_1 + n_2.
\]
• The fusion compatibility:
\[
\begin{align*}
\gamma_{\Lambda_1,\Lambda_2} R_{\Lambda_1+\Lambda_2,\Lambda_3}(\lambda) &= R_{\Lambda_2,\Lambda_3}(\lambda) R_{\Lambda_1,\Lambda_3}(\lambda - h^{(2)}) \gamma_{\Lambda_1,\Lambda_2}, \\
\gamma_{\Lambda_2,\Lambda_3} R_{\Lambda_1+\Lambda_2,\Lambda_3}(\lambda) &= R_{\Lambda_1,\Lambda_3}(\lambda - h^{(3)}) R_{\Lambda_2,\Lambda_3}(\lambda) \gamma_{\Lambda_1,\Lambda_2},
\end{align*}
\]
where \(\gamma_{\Lambda',\Lambda''} : M_{\Lambda'} \otimes M_{\Lambda''} \to M_{\Lambda'} \otimes M_{\Lambda''}\) are the linear maps determined by
\[
\gamma_{\Lambda',\Lambda''} v_{\Lambda'}^{(m)} = \sum_{m' + m'' = m} v_{\Lambda'}^{(m')} \otimes v_{\Lambda''}^{(m'')}.
\]
• The rationality property:
\[
(R_{\Lambda_1,\Lambda_2}(\lambda))^{m_1,m_2}_{n_1,n_2} \in \mathbb{C}(q^{\Lambda_1}, q^{\Lambda_2}, q^{2\lambda}).
\]
• The vanishing property:
\[
(R_{\Lambda_1,\Lambda_2}(\lambda))^{m_1,m_2}_{n_1,n_2} = 0 \quad \text{if } \Lambda_i \in \{n_i, n_i + 1, \ldots, m_i - 1\} \text{ for } i = 1 \text{ or } 2.
\]

The vanishing property implies that for \(\Lambda_1, \Lambda_2 \in \mathbb{Z}_{\geq 0}\) we obtain induced admissible endomorphisms \(R_{\Lambda_1,\Lambda_2}^{adm}(\lambda) \in \text{End}(L_{\Lambda_1} \otimes L_{\Lambda_2})\), represented by the matrix elements with \(\Lambda\)-admissible indices. An analogue of (5.7) holds for \(R_{\Lambda_1,\Lambda_2}^{adm}(\lambda)\) and \(\gamma_{\Lambda',\Lambda''}^{adm}\), which are defined as in (5.8) with summation over admissible indices.

The above properties uniquely determine the dynamical \(R\)-matrix from the normalization condition: the fundamental matrix \(R_{1,1}^{adm}(\lambda) \in \text{End}(L_1 \otimes L_1)\) is given by
\[
R_{1,1}^{adm}(\lambda) = \begin{pmatrix}
1 & 0 & 0 \\
0 & q^{-1}[\lambda+1] & -q^{-\lambda-1} \\
0 & [\lambda] & q^{-1}[\lambda-1]
\end{pmatrix},
\]
with respect to the basis \(\{v_1^{(0)} \otimes v_1^{(0)}, v_1^{(0)} \otimes v_1^{(1)}, v_1^{(1)} \otimes v_1^{(0)}, v_1^{(1)} \otimes v_1^{(1)}\}\).

5.3. Gauge equivalence of the \(R\)-matrices. The dynamical \(R\)-matrix \(R_{\Lambda_1,\Lambda_2}(\lambda)\) has a limit as \(\lambda \to -\infty\). We set
\[
R_{\Lambda_1,\Lambda_2} = \lim_{\lambda \to -\infty} R_{\Lambda_1,\Lambda_2}(\lambda).
\]
In other words, the matrix elements \((R_{\Lambda_1,\Lambda_2})^{m_1,m_2}_{n_1,n_2}\) are obtained from the matrix elements \((R_{\Lambda_1,\Lambda_2}(\lambda))^{m_1,m_2}_{n_1,n_2}\) by taking the limit \(q^{2\lambda} \to 0\) of the corresponding rational functions of \(q^{2\lambda}\), regarded as an independent variable. The properties of \(R_{\Lambda_1,\Lambda_2} \in \text{End}(M_{\Lambda_1} \otimes M_{\Lambda_2})\), inherited from the dynamical \(R\)-matrix, are the same as those of \(\mathcal{R}_{\Lambda_1,\Lambda_2}\), but with the modified fusion compatibility property
\[
\begin{align*}
\gamma_{\Lambda_1,\Lambda_2} R_{\Lambda_1+\Lambda_2,\Lambda_3} &= R_{\Lambda_2,\Lambda_3} R_{\Lambda_1,\Lambda_3} \gamma_{\Lambda_1,\Lambda_2}, \\
\gamma_{\Lambda_2,\Lambda_3} R_{\Lambda_1+\Lambda_2,\Lambda_3} &= R_{\Lambda_1,\Lambda_2} R_{\Lambda_1,\Lambda_3} \gamma_{\Lambda_2,\Lambda_3},
\end{align*}
\]
and the normalization condition for $R_{1,1}^{adm} \in \text{End}(L_1 \otimes L_1)$

$$R_{1,1}^{adm} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & q^{-2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  

As in the case of the quantum $R$-matrix, the inductive fusion procedure recovers the matrix elements $(R_{\Lambda_1,\Lambda_2}^{m_1,m_2})_{n_1,n_2}$ for positive integral $\Lambda_1, \Lambda_2$ and admissible $(m_1,m_2), (n_1,n_2)$; the rationality property then determines all the matrix elements for arbitrary $\Lambda_1, \Lambda_2$.

We call the collection of operators $R_{\Lambda_1,\Lambda_2}$ the asymptotic $R$-matrix. Below we show that it is gauge equivalent to the quantum $R$-matrix $\mathcal{R}_{\Lambda_1,\Lambda_2}$, and that the equivalence is given by the diagonal operators $\Xi_{\Lambda_1,\Lambda_2}$ defined by (3.15).

**Proposition 5.2.** For any $\Lambda_1, \Lambda_2 \in \mathbb{C}$ we have

$$\mathcal{R}_{\Lambda_1,\Lambda_2} \Xi_{\Lambda_1,\Lambda_2} = \Xi_{\Lambda_2,\Lambda_1} \mathcal{R}_{\Lambda_1,\Lambda_2}. \quad (5.11)$$

**Proof.** The desired formula is equivalent to an infinite collection of identities between matrix elements of $R$-matrices. Since the matrix elements of both $R$-matrices are rational functions of $q^{\Lambda_1}, q^{\Lambda_2}$, it suffices to check each of these identities for all sufficiently large positive integers $\Lambda_1, \Lambda_2$. To achieve this goal, we establish the admissible version of (5.11) by using double induction in $\Lambda_1$ and $\Lambda_2$.

The base of induction is $\Lambda_1 = \Lambda_2 = 1$, and a straightforward computation settles the case of the gauge equivalence of the fundamental $R$-matrices, cf. the example below.

To prove the inductive step, we start with the following identities:

$$\Xi_{\Lambda_1,\Lambda_2,\Lambda_3} \gamma_{\Lambda_1,\Lambda_2} = \gamma_{\Lambda_1,\Lambda_2} \Xi_{\Lambda_1,\Lambda_2,\Lambda_3},$$

$$\Xi_{\Lambda_1,\Lambda_2,\Lambda_3} \gamma_{\Lambda_2,\Lambda_3} = \gamma_{\Lambda_2,\Lambda_3} \Xi_{\Lambda_1,\Lambda_2,\Lambda_3}, \quad (5.12)$$

which are verified directly from the definitions. Assuming now that that (5.11) is true for pairs $(\Lambda_1, \Lambda_3)$ and $(\Lambda_2, \Lambda_3)$, we get from the explicit formula (3.15) and the weight-preserving properties of the $R$-matrices

$$\mathcal{R}_{\Lambda_1,\Lambda_3} \Xi_{\Lambda_1,\Lambda_3,\Lambda_2} = \Xi_{\Lambda_3,\Lambda_1,\Lambda_2} \mathcal{R}_{\Lambda_1,\Lambda_3},$$

$$\mathcal{R}_{\Lambda_2,\Lambda_3} \Xi_{\Lambda_2,\Lambda_3,\Lambda_1} = \Xi_{\Lambda_1,\Lambda_2,\Lambda_3} \mathcal{R}_{\Lambda_2,\Lambda_3}. \quad (5.13)$$

Since $\gamma_{\Lambda_1,\Lambda_2}$ is injective, the computation

$$\gamma_{\Lambda_1,\Lambda_2} \mathcal{R}_{\Lambda_1,\Lambda_2,\Lambda_3} \Xi_{\Lambda_1,\Lambda_2,\Lambda_3} = \mathcal{R}_{\Lambda_1,\Lambda_3} \mathcal{R}_{\Lambda_2,\Lambda_3} \gamma_{\Lambda_1,\Lambda_2} \Xi_{\Lambda_1,\Lambda_2,\Lambda_3} \quad (\text{due to } (5.4))$$

$$= \mathcal{R}_{\Lambda_1,\Lambda_3} \mathcal{R}_{\Lambda_2,\Lambda_3} \Xi_{\Lambda_1,\Lambda_2,\Lambda_3} \gamma_{\Lambda_1,\Lambda_2} \quad (\text{due to } (5.12))$$

$$= \Xi_{\Lambda_3,\Lambda_1,\Lambda_2} \mathcal{R}_{\Lambda_1,\Lambda_3} \mathcal{R}_{\Lambda_2,\Lambda_3} \gamma_{\Lambda_1,\Lambda_2} \quad (\text{due to } (5.13))$$

$$= \Xi_{\Lambda_3,\Lambda_1,\Lambda_2} \gamma_{\Lambda_1,\Lambda_2} \mathcal{R}_{\Lambda_1,\Lambda_2,\Lambda_3} \quad (\text{due to } (5.10))$$

$$= \gamma_{\Lambda_1,\Lambda_2} \Xi_{\Lambda_1,\Lambda_2,\Lambda_3} \mathcal{R}_{\Lambda_1,\Lambda_2,\Lambda_3} \quad (\text{due to } (5.12))$$

shows that (5.11) holds for the pair $(\Lambda_1 + \Lambda_2, \Lambda_3)$. Note also that all of the above formulae remain valid when the operators are replaced by their admissible versions.

We conclude that the admissible version of (5.11) holds when $\Lambda_1$ is any positive integer and $\Lambda_2 = 1$. A similar inductive argument in (5.11) holds when $\Lambda_1$ is any positive integer and $\Lambda_2 = 1$. A similar inductive argument in $\Lambda_2$ completes the proof. □
Remark 5.3. The operators $\Xi_{\gamma}$ are characterized by the property $\Upsilon^*_{\gamma} = \Xi \gamma_{\lambda}$, where $\Upsilon_{\gamma}$ and $\gamma_{\lambda}$ are inclusions of modules for the quantum group and a suitable version of the dynamical quantum group, respectively. Thus they can be regarded as "intertwining" operators between the two categories of representations.

Example. Consider the level $m = 1$ weight subspace, spanned by $\{v^{(0)}_{\Lambda_1} \otimes v^{(1)}_{\Lambda_2}, v^{(1)}_{\Lambda_1} \otimes v^{(0)}_{\Lambda_2}\}$. The dynamical $R$-matrix and its asymptotic version are given by

$$\mathcal{R}_{\Lambda_1, \Lambda_2}(\lambda) = \left( \begin{array}{cc} q^{-\Lambda_1} - 1 & q^{-\Lambda_2} - 1 \\ 0 & q^{-\Lambda_1} \end{array} \right), \quad \mathcal{R}_{\Lambda_1, \Lambda_2}(\lambda) = \left( \begin{array}{cc} q^{-\Lambda_1} & 0 \\ 1 - q^{-\Lambda_1} & 1 \end{array} \right),$$

and the quantum $R$-matrix is given by

$$\mathcal{R}_{\Lambda_1, \Lambda_2} = \left( \begin{array}{cc} q^{-\Lambda_1} & q^{-\Lambda_2}(q^{\Lambda_1} - q^{\Lambda_1}) \\ 0 & q^{-\Lambda_2} \end{array} \right).$$ (5.14)

The gauge equivalence between $\mathcal{R}_{\Lambda_1, \Lambda_2}$ and $\mathcal{R}_{\Lambda_1, \Lambda_2}$ is illustrated by the matrix identity

$$\left( \begin{array}{cc} q^{-\Lambda_1} & q^{-\Lambda_2}(q^{\Lambda_1} - q^{\Lambda_1}) \\ 0 & q^{-\Lambda_2} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & q^{\Lambda_2} \end{array} \right) = \left( \begin{array}{cc} q^{\Lambda_1} & 0 \\ 0 & q^{\Lambda_2} \end{array} \right) \left( \begin{array}{cc} q^{-\Lambda_1} & 0 \\ 1 - q^{-\Lambda_1} & 1 \end{array} \right).$$

5.4. The fusion lemma. Let $\pi_{\Lambda_1, \Lambda_2} : M^*_{\Lambda_1} \otimes M^*_{\Lambda_2} \rightarrow M^*_{\Lambda_1 + \Lambda_2}$ be the unique $U_q(\mathfrak{sl}_2)$-module surjection, normalized by $\pi_{\Lambda_1, \Lambda_2}(u_{\Lambda_1} \otimes u_{\Lambda_2}) = u_{\Lambda_1 + \Lambda_2}$. More explicitly, for $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ we have

$$\pi_{\Lambda_1, \Lambda_2}(u_{m_1}^{\Lambda_1} \otimes u_{m_2}^{\Lambda_2}) = q^{(\Lambda_1 - m_1)m_2} u_{m_1 + m_2}^{\Lambda_1 + \Lambda_2}. \quad (5.15)$$

Lemma 5.4. For any $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ we have a commutative diagram

$$\begin{array}{ccc}
M_{\lambda - \Lambda_1 - \Lambda_2 + 2m_1 + 2m_2 - 1} \otimes M^*_{\Lambda_1} \otimes M^*_{\Lambda_2} & \xrightarrow{\Phi_{\Lambda_1, \Lambda_2}(\lambda)} & M_{\lambda - \Lambda_1 - \Lambda_2 + 2m_1 + 2m_2 - 1} \otimes M^*_{\Lambda_1 + \Lambda_2}
\end{array}$$

(5.16)

Proof. Set $\mu = \lambda - \Lambda_2 + 2m_2$, $\nu = \lambda - \Lambda_1 - \Lambda_2 + 2m_1 + 2m_2$. We compute

$$\Phi_{m_1, m_2}(\lambda) v_{\lambda - 1} = (\Phi_{m_1}(\mu) \otimes 1) \left( \sum_{i=0}^{m_2} q^{i(\mu - i)} \frac{[\mu + i + 1]_{m_2-i}}{[i]!} F^i v_{\mu - 1} \otimes E^i u_{m_2}^{\Lambda_2} \right)$$

$$= \sum_{i=0}^{m_2} q^{i(\mu - i)} \frac{[\mu + i + 1]_{m_2-i}}{[i]!} \Delta(F^i) \left( \sum_{j=0}^{m_1} q^{j(\mu - j)} \frac{[\mu + j + 1]_{m_1-j}}{[j]!} \right) \otimes E^i u_{m_2}^{\Lambda_2}$$

$$= v_{\nu - 1} \otimes [\nu + 1]_{m_1} \left( \sum_{i=0}^{m_2} q^{i(\mu - i)} \frac{[\mu + i + 1]_{m_2-i}}{[i]!} F^i v_{m_1}^{\Lambda_1} \otimes E^i u_{m_2}^{\Lambda_2} \right) + \ldots,$$

where we omitted the terms involving $F^k v_{\nu - 1}$ with $k > 0$. We now study the leading term. Denote for brevity

$$u = [\nu + 1]_{m_1} [-\mu + 1]_{m_2} \sum_{i=0}^{m_2} q^{i(\mu - i)} \frac{[\mu + i + 1]}{[i]!} F^i u_{m_1}^{\Lambda_1} \otimes E^i u_{m_2}^{\Lambda_2}.$$
Explicit formula (5.15) for the map $\pi_{\Lambda_1, \Lambda_2}$ yields
\[
\pi_{\Lambda_1, \Lambda_2}(F^i u_{m_1}^{\Lambda_1} \otimes E^i u_{m_2}^{\Lambda_2}) = q^{(\Lambda_1 - m_1 - 1)(m_2 - 1)} [-m_2]_i [m_1 - \Lambda_1]_i u_{m_1 + m_2}^{\Lambda_1 + \Lambda_2},
\]
and combining it with the summation formula
\[
\sum_{i=0}^{m_2} q^{i(-m_2 + m_1 - \Lambda_1 + \mu)} [-m_2]_i [m_1 - \Lambda_1]_i = q^{m_2(m_1 - \Lambda_1)} \frac{[-\mu + \Lambda_1 - m_1 + 1]_{m_2}}{[-\mu + 1]_{m_2}},
\]
which is a special case of the $q$-hypergeometric Gauss identity, we conclude that
\[
\pi_{\Lambda_1, \Lambda_2}(u) = [-\nu + 1]_{m_1} [-\mu + \Lambda_1 - m_1 + 1]_{m_2} u_{m_1 + m_2}^{\Lambda_1 + \Lambda_2} = [-\nu + 1]_{m_1 + m_2} u_{m_1 + m_2}^{\Lambda_1 + \Lambda_2}.
\]
Now (5.17) yields
\[
(1 \otimes \pi_{\Lambda_1, \Lambda_2}) \Phi_{m_1, m_2}^{\Lambda_1, \Lambda_2}(\lambda) v_{\lambda - 1} = [-\nu + 1]_{m_1 + m_2} v_{\nu - 1} \otimes u_{m_1 + m_2}^{\Lambda_1 + \Lambda_2} + \ldots,
\]
which shows that the operators $(1 \otimes \pi_{\Lambda_1, \Lambda_2}) \Phi_{m_1, m_2}^{\Lambda_1, \Lambda_2}(\lambda)$ and $\Phi_{m_1 + m_2}^{\Lambda_1 + \Lambda_2}(\lambda)$ have the same leading terms. For generic $\lambda$ the leading term uniquely determines the intertwining operator, proving the equality of all other coefficients in the expansions of both operators. Since all these coefficients are holomorphic in $\lambda$, we conclude that (5.16) holds for every value of $\lambda$. \qed

Remark 5.5. Lemma 5.4 shows that for $\mathcal{U}_q(\mathfrak{sl}_2)$ our regularized intertwining operators $\Phi_m^\Lambda(\lambda)$ can be constructed inductively using fusion. We start with $\Phi^\Lambda_0(\lambda)$ and $\Phi^\Lambda_1(\lambda)$, which determine the operators associated with the fundamental representation $L^*_\Lambda$. For nonnegative integers $\Lambda_1, \Lambda_2$ the operators associated with $L^*_\Lambda_1$ and $L^*_\Lambda_2$ yield the operators associated with $L^*_{\Lambda_1 + \Lambda_2}$, which determines $\Phi_m^\Lambda(\lambda)$ for all large integers $\Lambda$, and hence also for arbitrary $\Lambda \in \mathbb{C}$.

It is an open problem to construct similar regularized (holomorphic) intertwining operators for higher rank quantum groups. Lemma 5.4 suggests that in the higher rank case it suffices to consider the fundamental representation, and then extend the operators to other representations by using fusion. \qed

5.5. The braiding lemma. Let $\mathcal{R}_{\Lambda_1, \Lambda_2}^*: M^*_\Lambda_1 \otimes M^*_\Lambda_2 \rightarrow M^*_\Lambda_1 \otimes M^*_\Lambda_2$ be the dual linear map to $\mathcal{R}_{\Lambda_1, \Lambda_2}$. In other words, we have
\[
\mathcal{R}_{\Lambda_1, \Lambda_2}^* u_{m_1}^{\Lambda_1} \otimes u_{m_2}^{\Lambda_2} = \sum_{n_1, n_2} (\mathcal{R}_{\Lambda_1, \Lambda_2})_{m_1, m_2}^{n_1, n_2} u_{n_1}^{\Lambda_1} \otimes u_{n_2}^{\Lambda_2},
\]
where $(\mathcal{R}_{\Lambda_1, \Lambda_2})_{m_1, m_2}^{n_1, n_2}$ are defined in (5.3). Equivalently, $\mathcal{R}_{\Lambda_1, \Lambda_2}^* = q^{-\frac{\Lambda_1 \Lambda_2}{2}} \mathcal{R}' |_{M^*_\Lambda_1 \otimes M^*_\Lambda_2}$, where
\[
\mathcal{R}' = q^{-\frac{b\Lambda}{2} h} \left( \sum_{k \geq 0} q^{-k(k-1)/2} \left( \frac{q^{-1} - q}{{k!}^2} \right) F^k \otimes E^k \right).
\]
The element \( R' \), defined in (5.19), has the property that
\[
R' \Delta(X) = \Delta^{op}(X) R', \quad X \in \mathcal{U}_q(\mathfrak{sl}_2),
\]
where \( \Delta^{op}(X) = P \Delta(X) \). Therefore, we obtain \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module isomorphisms
\[
\tilde{R}^*_{\Lambda_1, \Lambda_2} = P R^*_{\Lambda_1, \Lambda_2} : M_{\Lambda_1} \otimes M_{\Lambda_2} \to M_{\Lambda_2} \otimes M_{\Lambda_1},
\]
with respect to the \( \mathcal{U}_q(\mathfrak{sl}_2) \)-action in tensor products, defined using the comultiplication (3.6). The compositions \( \tilde{R}^*_{\Lambda_1, \Lambda_2} \Phi^{\Lambda_1, \Lambda_2}_m (\lambda) : M_{\Lambda_1} \to M_{\Lambda_1 - \Lambda_2 + 2m_1 + 2m_2} \otimes M_{\Lambda_2} \otimes M_{\Lambda_1} \) are \( \mathcal{U}_q(\mathfrak{sl}_2) \)-intertwining operators, and can be written as linear combinations of operators \( \Phi^{\Lambda_2, \Lambda_1}_{n_2, n_1} (\lambda) \). We now show that the corresponding connection matrix is precisely equal to \( R_{\Lambda_1, \Lambda_2} (\lambda) \).

**Lemma 5.6.** For any \( \Lambda_1, \Lambda_2 \in \mathbb{C} \) and \( m_1, m_2 \in \mathbb{Z}_{\geq 0} \) we have
\[
\tilde{R}^*_{\Lambda_1, \Lambda_2} \Phi^{\Lambda_1, \Lambda_2}_m (\lambda) = \sum_{n_1, n_2} (\mathcal{R}_{\Lambda_1, \Lambda_2} (\lambda))_{n_1, n_2}^{n_1, n_2} \Phi^{\Lambda_2, \Lambda_1}_{n_2, n_1} (\lambda). \quad (5.20)
\]

**Proof.** It is convenient to rewrite (5.20) using the formal generating functions
\[
\phi_{\Lambda_1} (\lambda) = \sum_m \phi_{\Lambda_1}^m (\lambda) \otimes v_{\Lambda_1}^m. \quad (5.21)
\]
These generating functions have a useful property
\[
\pi_{\Lambda_1, \Lambda_2} \Phi^{\Lambda_1, \Lambda_2}_m (\lambda) = \chi_{\Lambda_1, \Lambda_2} \Phi^{\Lambda_1 + \Lambda_2}_m (\lambda), \quad (5.22)
\]
established by the following computation based on Lemma 5.3:
\[
\sum_{m_1, m_2} \pi_{\Lambda_1, \Lambda_2} \Phi^{\Lambda_1, \Lambda_2}_m (\lambda) \otimes v_{\Lambda_1}^{m_1} \otimes v_{\Lambda_2}^{m_2} = \sum_{m_1, m_2} \Phi^{\Lambda_1 + \Lambda_2}_{m_1 + m_2} (\lambda) \otimes v_{\Lambda_1}^{m_1} \otimes v_{\Lambda_2}^{m_2}
\]
\[
= \sum_m \sum_{m_1 + m_2 = m} \Phi^{\Lambda_1 + \Lambda_2}_m (\lambda) \otimes v_{\Lambda_1}^{m_1} \otimes v_{\Lambda_2}^{m_2} = \sum_m \Phi^{\Lambda_1 + \Lambda_2}_m (\lambda) \otimes \gamma_{\Lambda_1, \Lambda_2} v_{\Lambda_1 + \Lambda_2}^{(m)}.
\]
Returning to the desired relation (5.20), we see that it is equivalent to
\[
\tilde{R}^*_{\Lambda_1, \Lambda_2} \Phi^{\Lambda_1, \Lambda_2}_m (\lambda) = \mathcal{R}_{\Lambda_1, \Lambda_2} (\lambda) \Phi^{\Lambda_2, \Lambda_1}_m (\lambda), \quad (5.23)
\]
which we prove for \( \Lambda_1, \Lambda_2 \in \mathbb{Z}_{\geq 0} \) using double induction. Assuming that (5.23) holds for pairs \( (\Lambda_1, \Lambda_3) \) and \( (\Lambda_2, \Lambda_3) \), and using the fusion compatibility
\[
\mathcal{R}^*_{\Lambda_1 + \Lambda_2, \Lambda_3} \pi_{\Lambda_1, \Lambda_2} = \pi_{\Lambda_1, \Lambda_2} \mathcal{R}^*_{\Lambda_1, \Lambda_3} \mathcal{R}^*_{\Lambda_2, \Lambda_3}, \quad (5.24)
\]
we compute
\[
\gamma_{\Lambda_1, \Lambda_2} \tilde{R}^*_{\Lambda_1, \Lambda_2, \Lambda_3} \Phi^{\Lambda_1 + \Lambda_2, \Lambda_3}_m (\lambda) = \mathcal{R}^*_{\Lambda_1, \Lambda_2, \Lambda_3} \pi_{\Lambda_1, \Lambda_2} \Phi^{\Lambda_1 + \Lambda_2, \Lambda_3}_m (\lambda) \quad (due to (5.22))
\]
\[
= \pi_{\Lambda_1, \Lambda_2} \mathcal{R}^*_{\Lambda_1, \Lambda_3} \mathcal{R}^*_{\Lambda_2, \Lambda_3} \Phi^{\Lambda_1 + \Lambda_2, \Lambda_3}_m (\lambda) \quad (due to (5.24))
\]
\[
= \pi_{\Lambda_1, \Lambda_2} \mathcal{R}_{\Lambda_3, \Lambda_1} (\lambda - h^{(2)}) \mathcal{R}_{\Lambda_3, \Lambda_2} (\lambda) \Phi^{\Lambda_1, \Lambda_2}_m (\lambda) \quad (by assumption)
\]
\[
= \mathcal{R}_{\Lambda_3, \Lambda_1} (\lambda - h^{(2)}) \mathcal{R}_{\Lambda_3, \Lambda_2} (\lambda) \gamma_{\Lambda_1, \Lambda_2} \Phi^{\Lambda_1, \Lambda_2 + \Lambda_3}_m (\lambda) \quad (due to (5.22))
\]
\[
= \gamma_{\Lambda_1, \Lambda_2} \mathcal{R}_{\Lambda_3, \Lambda_1 + \Lambda_2} (\lambda) \Phi^{\Lambda_1, \Lambda_2 + \Lambda_3}_m (\lambda), \quad (due to (5.24))
\]
and since $\gamma_{\Lambda_1,\Lambda_2}$ is injective, we see that the braiding relation is also valid for $(\Lambda_1 + \Lambda_2, \Lambda)$. Similarly, if (5.23) is satisfied for pairs $(\Lambda_1, \Lambda_2)$ and $(\Lambda_1, \Lambda_3)$, then it holds for $(\Lambda_1, \Lambda_2 + \Lambda_3)$.

To complete the proof, we note that braiding relation holds in the admissible case for $\Lambda_1 = \Lambda_2 = 1$ (which is the result of direct computation, see example below), and by induction for arbitrary $\Lambda_1, \Lambda_2 \in \mathbb{Z}_{>0}$. In the general case, (5.23) is equivalent to a family of equalities of rational functions, which are now established for $\Lambda_1, \Lambda_2$ in the positive integral cone, and therefore must hold identically. \hfill \Box

Example. Let $m = 1$. Then (5.23) reduces to the matrix identity

$$
\begin{pmatrix}
-\rho + \rho_2 & q^{-\rho + \rho_1+1} & q^{-\rho - \rho_1} \\
0 & -\rho + \rho_1 & q^{-\rho - \rho_1}
\end{pmatrix}
\begin{pmatrix}
q^{-\rho} & q^{-\rho} \\
0 & q^{-\rho}
\end{pmatrix}
= \begin{pmatrix}
-q^{-\rho} & q^{-\rho} \\
-q^{-\rho} & q^{-\rho}
\end{pmatrix}
= \begin{pmatrix}
-q^{-\rho} & q^{-\rho} \\
-q^{-\rho} & q^{-\rho}
\end{pmatrix}.
$$

6. Difference equations satisfied by the trace functions

6.1. The qKZ and qKZB operators. For any $x \in \mathbb{C}$ define $(q^{x^h})_j \in \text{End}(M_\chi)$ by

$$(q^{x^h})_j = q^{(\Lambda_j - 2m_j)x}v^{(\bar{m})}_\chi,$$

and $\Gamma_j \in \text{End} (\mathfrak{师} \otimes M_\chi)$ by

$$\Gamma_j \left( \xi(\lambda) \otimes v^{(\bar{m})}_\chi \right) = \xi(\lambda - \Lambda_j + 2k_1) \otimes v^{(\bar{m})}_\chi,$$

for any $\bar{m} \in \mathbb{Z}^n$ and $\xi \in \mathfrak{师}$. We also set

$$(q^{x^h})_\circ = P_\circ (q^{x^h})_1, \quad \Gamma_\circ = P_\circ \Gamma_1,$$

where $P_\circ$ is the permutation map, defined by $P_\circ (x_1 \otimes x_2 \otimes \ldots \otimes x_n) = x_2 \otimes \ldots \otimes x_n \otimes x_1$.

For any $x \in \mathbb{C}$ we define the qKZ operators by

$$K_j(x) = (\tilde{\mathcal{K}}_{\Lambda_j, \Lambda_j+1})^{-1} \ldots (\tilde{\mathcal{K}}_{\Lambda_j, \Lambda_n})^{-1} (q^{x^h})_\circ \tilde{\mathcal{K}}_{\Lambda_1, \Lambda_j} \ldots \tilde{\mathcal{K}}_{\Lambda_{j-1}, \Lambda_j}, \quad j = 1, \ldots, n.$$

We introduce the qKZB operators $\mathbb{K}_j$, acting in $\mathfrak{师} \otimes M_\chi$, by

$$\mathbb{K}_j \psi(\lambda) = (\tilde{\mathcal{K}}_{\Lambda_j, \Lambda_j+1}(\lambda - h^{(j+2, \ldots, n)})^{-1} \ldots (\tilde{\mathcal{K}}_{\Lambda_j, \Lambda_n}(\lambda))^{-1} \Gamma_\circ \tilde{\mathcal{K}}_{\Lambda_1, \Lambda_j}(\lambda - h^{(j+1, \ldots, n)}) \ldots \tilde{\mathcal{K}}_{\Lambda_{j-1}, \Lambda_j}(\lambda - h^{(j+1, \ldots, n)}) \psi(\lambda).$$

One of the fundamental properties of the universal trace operator $F(\lambda, x; \tilde{\chi})$ is that it intertwines the qKZ and qKZB operators.

Theorem 6.1. Let $\tilde{\chi} \in \mathbb{C}^n[2m]$, and let $x \in \mathbb{C}$. Then

$$\mathbb{K}_j F(\lambda, x; \tilde{\chi}) = F(\lambda, x; \tilde{\chi}) K_j(x), \quad j = 1, \ldots, n. \quad (6.1)$$
Proof. For any \( j \in \{1, \ldots, n-1\} \), let \( \sigma_j \) denote the permutation of a set of \( n \) elements, transposing elements in positions \( j \) and \( j+1 \). First, we observe that
\[
\mathcal{R}_{\Lambda_j, \Lambda_{j+1}} (\lambda - h^{(j+2, \ldots, n)}) \mathcal{F}(\lambda, x; \vec{\Lambda}) = \mathcal{F}(\lambda, x; \sigma_j(\vec{\Lambda})) \mathcal{R}_{\Lambda_j, \Lambda_{j+1}}.
\]
(6.2)
Indeed, this is equivalent to
\[
\mathcal{R}_{\Lambda_j, \Lambda_{j+1}} (\lambda - h^{(j+2, \ldots, n)}) \left| \begin{array}{c}
\Phi^\Lambda(\lambda) q^{\tau h} = \mathcal{R}_{\Lambda_j, \Lambda_{j+1}}^* \left| \begin{array}{c}
\Phi^{\sigma_j(\vec{\Lambda})}(\lambda) q^{\tau h},
\end{array} \right.
\end{array} \right.
\]
where \( \mathcal{R}_{\Lambda_j, \Lambda_{j+1}} (\lambda - h^{(j+1, \ldots, n)}) \) and \( \mathcal{R}_{\Lambda_j+1, \Lambda}^* \) act respectively in \( M_\Lambda \) and \( M_\Lambda^* \). Moving both \( R \)-matrices inside the trace and invoking Lemma 5.6, we establish (6.2).

Let \( \sigma_\circ = \sigma_{n-1} \ldots \sigma_1 \) denote the cyclic permutation, moving the first element to the last position. Our second observation is that
\[
\Gamma_\circ \mathcal{F}(\lambda, x; \vec{\Lambda}) = \mathcal{F}(\lambda, x; \sigma_\circ(\vec{\Lambda})) (q^{\tau h})_\circ.
\]
(6.3)
Indeed, using the cyclic property of trace, we compute:
\[
\mathcal{F}_{m_1, m_2, \ldots, m_n} (\lambda - \Lambda_1 + 2m_1, x; \vec{\Lambda}) = \text{Tr}_{M_{\Lambda_1+2m_1-1}} \left( (\Phi^{\Lambda_1}_{m_1}(\lambda) \otimes \mathbb{1}^{n-1}) \Phi^{\Lambda_2, \ldots, \Lambda_n}_{m_2, \ldots, m_n} (\lambda - \Lambda_1 + 2m_1) q^{\tau h} \right)
\]
\[
= P_\circ \text{Tr}_{M_{\Lambda-1}} \left( (\Phi^{\Lambda_2, \ldots, \Lambda_n}_{m_2, \ldots, m_n} (\lambda - \Lambda_1 + 2m_1) q^{\tau h} \otimes \mathbb{1}) \Phi^{\Lambda_1}_{m_1}(\lambda) \right)
\]
\[
= P_\circ \text{Tr}_{M_{\Lambda-1}} \left( (q^{\tau h})_{\Lambda_1} \Phi^{\Lambda_2, \ldots, \Lambda_n}_{m_2, \ldots, m_n, m_1}(\lambda) q^{\tau h} \right)
\]
\[
= (q^{\tau h})_\circ \mathcal{F}_{m_2, \ldots, m_n, m_1} (\lambda, x; \sigma_\circ(\vec{\Lambda})),
\]
which implies (6.3). The theorem is now proved by repeatedly applying (6.2) and (6.3). \( \square \)

### 6.2. The qKZ equations.

For a fixed \( x \in \mathbb{C} \) the operators \( \mathcal{K}_j(x) \) pairwise commute, and the problem of finding their common eigenvectors yields the qKZ equations on \( \phi \in M_\Lambda^* \):
\[
\mathcal{K}_j(x) \phi = \varepsilon_j \phi, \quad j = 1, \ldots, n.
\]
(6.4)
The qKZ operators are triangular in a suitable basis, and their eigenvalues \( \varepsilon_j \) can be described explicitly. For any \( \vec{m} \in \mathbb{Z}^n \) set
\[
\varepsilon_j^{\vec{m}}(x; \vec{\Lambda}) = q^{(\Lambda_j - 2m_j)x + \sum_{i=j+1}^{n} (m_i - 2m_j) - \sum_{i=1}^{j-1} (m_i - 2m_j)}. \quad (6.5)
\]

**Lemma 6.2.** Let \( x \in \mathbb{C} \) and \( \vec{\Lambda} \in \mathbb{C}^n \). Suppose \( \varphi \in M_\Lambda^* \) is a nonzero solution of the system (6.4). Then there exists \( \vec{m} \in \mathbb{Z}^n \) such that \( \varepsilon_j = \varepsilon_j^{\vec{m}}(x; \vec{\Lambda}) \) for all \( j = 1, \ldots, n \).

**Proof.** The qKZ operators are weight-preserving, and we may assume that \( \varphi \) is a linear combination of \( v^{(\vec{m})}_\Lambda \) with \( \vec{m} \in \mathbb{Z}^n[m] \) for some fixed \( m \in \mathbb{Z}_{\geq 0} \). Let \( \vec{m} \) be minimal in the sense of partial ordering \( \preceq \) given by (2.10), such that \( v^{(\vec{m})}_\Lambda \) occurs in the expansion of \( \varphi \) with nonzero coefficient; we may assume that this coefficient is equal to 1. Triangularity of the universal \( R \)-matrix implies that for any \( \vec{k} \)
\[
\mathcal{R}_{\Lambda_i, \Lambda_{i+1}} v^{(\vec{k})}_\Lambda = q^{-k_i\Lambda_i k_{i+1}} v^{(\vec{k})}_\Lambda + \text{ higher order terms,} \quad i = 1, \ldots, j - 1,
\]
\[
\mathcal{R}_{\Lambda_{j-1}, \Lambda_j}^{-1} v^{(\vec{k})}_\Lambda = q^{k_j\Lambda_j - 2k_{j+1}} v^{(\vec{k})}_\Lambda + \text{ higher order terms,} \quad i = j + 1, \ldots, n,
\]
where “higher order terms” stands for some linear combinations of \( v^{(l)}_\Lambda \) with \( l \gg k \). Therefore,
\[
\mathcal{K}_j(x) v^{(\vec{m}')}_\Lambda = \varepsilon_j^{\vec{m}'}(x; \vec{\Lambda}) v^{(\vec{m})}_\Lambda + \text{higher order terms.}
\]
Comparing coefficients before \( v^{(\vec{m})}_\Lambda \) in (6.4), we obtain the desired statement. \( \square \)

**Theorem 6.3.** Let \( x \in \mathbb{C} \) and \( \vec{\Lambda} \in \mathbb{C}^n \). Then for any \( \vec{m}' \in \mathbb{Z}^n \) we have
\[
\mathcal{K}_j(x) \mathcal{H}^{\vec{m}'}(x; \vec{\Lambda}) = \varepsilon_j^{\vec{m}'}(x; \vec{\Lambda}) \mathcal{H}^{\vec{m}'}(x; \vec{\Lambda}), \quad j = 1, \ldots, n,
\]
where \( \varepsilon_j^{\vec{m}'}(x; \vec{\Lambda}) \) are as in (6.5).

**Proof.** This statement follows from the more general results in [TV1], see Theorem 6.2 and subsequent remarks there. \( \square \)

**Example.** Let \( n = 2 \) and \( m = 1 \). The qKZ equations for \( j = 1 \) reduce to the identity
\[
\begin{pmatrix}
q^{(1-x)\Lambda_1} & 0 \\
q^{-x(1-q^2)\Lambda_1 + \Lambda_2} & q^{x(2-\Lambda_1) + \Lambda_2}
\end{pmatrix} \mathcal{H}(x; \vec{\Lambda}) = \mathcal{H}(x; \vec{\Lambda}) \begin{pmatrix}
q^{(1-x)\Lambda_1} & 0 \\
q^{-x(2-\Lambda_1) + \Lambda_2} & q^{x(2-\Lambda_1) + \Lambda_2}
\end{pmatrix},
\]
for the hypergeometric qKZ matrix \( \mathcal{H}(x; \vec{\Lambda}) \), described in (4.6). \( \square \)

### 6.3. The qKZB equations

If \( \vec{\Lambda} \in \mathbb{C}^n[2\mathbb{m}] \), then the weight subspace \( M_{\vec{\Lambda}}[0] \neq 0 \), and the qKZB operators restricted to \( M_{\vec{\Lambda}}[0] \) pairwise commute. The problem of finding their common eigenfunctions yields the qKZB equations on \( \varphi \in \mathfrak{Fun} M_{\vec{\Lambda}}[0] \):
\[
\mathbb{K}_j \varphi(\lambda) = \varepsilon_j \varphi(\lambda), \quad j = 1, \ldots, n.
\]
(6.6)

In this paper we study solutions for (6.6), which are trigonometric quasi-polynomials. The following lemma shows that they only exist when the eigenvalues \( \varepsilon_j \) are given by the formula (6.5) for the qKZ eigenvalues.

**Lemma 6.4.** Let \( \vec{\Lambda} \in \mathbb{C}^n[2\mathbb{m}] \). Suppose \( \psi(\lambda) \) is a nonzero \( M_{\vec{\Lambda}}[0] \)-valued trigonometric quasi-polynomial solution of the system (6.6). Then there exists \( x \in \mathbb{C} \) and \( \vec{m} \in \mathbb{Z}^n[\mathbb{m}] \) such that \( \varepsilon_j = \varepsilon_j^{\vec{m}}(x; \vec{\Lambda}) \) for all \( j = 1, \ldots, n \).

**Proof.** The idea of the proof is to observe is that the principal term of the expansion of \( \psi(\lambda) \) satisfies a version of the qKZ equations, and use an analogue of Lemma 6.2. For \( y \in \mathbb{C} \) define the operators \( \mathbb{K}_j(y) \in \text{End}(M_{\vec{\Lambda}}) \), associated with the asymptotic \( R \)-matrix \( R_{\Lambda_1,\Lambda_2} \) by
\[
\mathbb{K}_j(y) = (\check{R}_{\Lambda_j,\Lambda_{j+1}})^{-1} \cdots (\check{R}_{\Lambda_j,\Lambda_n})^{-1} \left(q^y\right)_\circ(y) \check{R}_{\Lambda_1,\Lambda_j} \cdots \check{R}_{\Lambda_{j-1},\Lambda_j}.
\]
Write the trigonometric quasi-polynomial \( \psi(\lambda) \) as
\[
\psi(\lambda) = q^y \left( \sum_{k=0}^d q^{2k\lambda} \psi^{(k)} \right), \quad \psi^{(k)} \in M_{\vec{\Lambda}}[0], \quad \psi^{(0)} \neq 0,
\]
for appropriate \( y \in \mathbb{C} \) and \( d \in \mathbb{Z}_{\geq 0} \). Then the leading coefficient \( \psi^{(0)} \) satisfies the equations
\[
\mathbb{K}_j(y) \psi^{(0)} = \varepsilon_j \psi^{(0)}, \quad j = 1, \ldots, n.
\]
The operators $\mathbf{R}_{\Lambda_1, \Lambda_2}$ are triangular, and more precisely

$$\mathbf{R}_{\Lambda_1, \Lambda_2}^{(k)} = q^{-2k(\Lambda_1 \cdot \Lambda_i - \Lambda_i \cdot \Lambda_1)} q^{(k)} + \text{higher order terms, } \quad i = 1, \ldots, j - 1,$$

$$\mathbf{R}_{\Lambda_2, \Lambda_1}^{(k)} = q^{2k(\Lambda_2 \cdot \Lambda_i - \Lambda_i \cdot \Lambda_2)} q^{(k)} + \text{higher order terms, } \quad i = j + 1, \ldots, n.$$ 

The argument as in the proof of Lemma 6.2 implies that there exists $\bar{m} \in \mathbb{Z}^n[\mathbf{m}]$, such that

$$\varepsilon_j = q^{(\Lambda_j - 2m_j) + 2\sum_{i=j+1}^n m_j(\Lambda_i - m_i) - 2\sum_{i=1}^{j-1} m_i(\Lambda_i - m_i)}.$$

(6.7)

We now set $x = y + \mathbf{m}$, and from the identity

$$(\Lambda_j - 2m_j) \sum_{i=1}^n m_i + 2 \sum_{i=j+1}^n m_j(\Lambda_i - m_i) - 2 \sum_{i=1}^{j-1} m_i(\Lambda_i - m_i) =$$

$$\sum_{i=j+1}^n (\Lambda_i m_j + \Lambda_j m_i - 2m_i m_j) - \sum_{i=1}^{j-1} (\Lambda_i m_j + \Lambda_j m_i - 2m_i m_j) + m_j \sum_{i=1}^n (\Lambda_i - 2m_i)$$

and the zero weight condition we get the equivalence between (6.7) and (6.5).

Theorem 6.5. Let $\bar{\Lambda} \in \mathbb{C}^n[2\mathbf{m}]$. Then for $\bar{m} \in \mathbb{Z}^n[\mathbf{m}]$ we have

$$\mathbb{K}_j \mathbb{H}^{\bar{m}}(\lambda, x; \bar{\Lambda}) = \varepsilon_j^{\bar{m}}(x; \bar{\Lambda}) \mathbb{H}^{\bar{m}}(\lambda, x; \bar{\Lambda}), \quad j = 1, \ldots, n.$$

where $\varepsilon_j^{\bar{m}}(x; \bar{\Lambda})$ are as in (6.5).

Proof. The statement follows from Theorem 31 in [FTV1].

Remark 6.6. Using our Theorem 6.5 and Theorem 6.4 one can immediately derive Theorem 6.5 from the somewhat easier Theorem 6.3 derived from the results in [TV1].

Example. Let $n = 2$ and $\mathbf{m} = 1$. The qKZB equations (1.2) reduce to the matrix identity, satisfied when $\Lambda_1 + \Lambda_2 = 2$:

$$\begin{pmatrix} q^{\lambda_1 \frac{[\lambda+1]}{[\lambda-A_1+1]}} & -q^{\lambda_1 \frac{[\lambda]}{[\lambda-A_1+1]}} \\ q^{-\lambda_1 + \lambda_2 - \lambda_1 \frac{[\lambda]}{[\lambda-A_1+1]}} & q^{\lambda_2 \frac{[\lambda]}{[\lambda-A_1+1]}} \end{pmatrix} = \begin{pmatrix} \mathbb{H}_{0,1}^{\lambda_1}(\lambda - \Lambda_1, x; \bar{\Lambda}) & \mathbb{H}_{0,1}^{\lambda_1}(\lambda - \Lambda_1 + 2, x; \bar{\Lambda}) \\ \mathbb{H}_{0,1}^{\lambda_1}(\lambda - \Lambda_1 + 2, x; \bar{\Lambda}) & \mathbb{H}_{0,1}^{\lambda_1}(\lambda - \Lambda_1, x; \bar{\Lambda}) \end{pmatrix} = \begin{pmatrix} 0 & q^{(1-x)\Lambda_1} \\ 0 & q^{x(2-\Lambda_1)+\Lambda_2} \end{pmatrix}. $$

6.4. The Macdonald-Ruijsenaars equations. For every $\Theta \in \mathbb{Z}_{\geq 0}$ define the Macdonald-Ruijsenaars (MR) operators $\mathbb{M}_\Theta : \mathfrak{f} \mathfrak{u} \mathfrak{n} \otimes M_\Lambda[0] \rightarrow \mathfrak{f} \mathfrak{u} \mathfrak{n} \otimes M_\Lambda[0]$ by

$$\mathbb{M}_\Theta \psi(\lambda) = q^{\Theta \mathbf{m}_\Theta} \sum_{\mu \in \mathbf{h}^*} \text{Tr} \left| L_{\Theta, \mu} \right| \mathbb{R}_{\Theta, \Lambda_1}(\lambda - h^{(2, \ldots, n)}) \ldots \mathbb{R}_{\Theta, \Lambda_n}(\lambda) \psi(\lambda - \mu).$$

In particular, it follows from this definition that the operator $\mathbb{M}_\Theta = 0$ is the identity operator. The operators $\mathbb{M}_\Theta$ for various $\Theta \in \mathbb{Z}$ pairwise commute [EV1]. The problem of finding their common eigenfunctions yields the Macdonald-Ruijsenaars equations on $\psi \in \mathfrak{f} \mathfrak{u} \mathfrak{n} \otimes M_\Lambda[0]$:

$$\mathbb{M}_\Theta \psi(\lambda) = \mathcal{X}_\Theta \psi(\lambda), \quad \Theta \in \mathbb{Z}_{\geq 0}, \mathcal{X}_\Theta \in \mathbb{C}. \quad (6.8)$$
For any $\Theta \in \mathbb{Z}_{\geq 0}$ define the character $\chi_{\Theta}(x)$ of the irreducible $\mathcal{U}_q(sl_2)$-module $L_\Theta$ by
\[\chi_{\Theta}(x) = \sum_{\mu} \dim L_\Theta(\mu) q^{-\mu_x}\.\]

**Theorem 6.7.** Let $x \in \mathbb{C}$, and let $\bar{\Lambda} \in \mathbb{C}^n[2m]$. Then for every $\Theta \in \mathbb{Z}_{\geq 0}$ one has
\[M_\Theta \mathcal{H}(\lambda, x; \bar{\Lambda}) = \chi_{\Theta}(x) \mathcal{H}(\lambda, x; \bar{\Lambda})\].

**Proof.** It suffices to prove that in the trace function convergence domain $|q^{2x}| \gg 1$ we have
\[M_\Theta \mathcal{F}(\lambda, x; \bar{\Lambda}) = \chi_{\Theta}(x) \mathcal{F}(\lambda, x; \bar{\Lambda})\].

Indeed, Theorem 4.5 then shows that (6.9) holds when $|q^{2x}| \gg 1$, and since $\mathcal{H}(\lambda, x; \bar{\Lambda})$ is a trigonometric quasi-polynomial in $x$, the desired equations are valid for all $x \in \mathbb{C}$.

Consider the operators $\Phi(\lambda) : M_\lambda \otimes M_\Lambda \to \bigoplus \mu M_{\lambda-\mu} \otimes M_\mu$, which are dualized versions of the operators (5.21), obtained by composing $\Phi(\lambda)$ with the evaluation pairing $M_\lambda \otimes M_\Lambda \to \mathbb{C}$. If $\Theta \in \mathbb{Z}_{\geq 0}$, the admissible operator $\Phi(\lambda) : M_{\lambda-\mu} \otimes L_\Theta \to \bigoplus \mu M_{\lambda-\mu} \otimes L_\Theta$ is defined in the same way. We form the following commutative diagram:

\[
\begin{array}{cccccc}
M_{\lambda-1} \otimes L_\Theta \otimes M_\Lambda[0] & \xrightarrow{\Phi(\lambda)} & M_{\lambda-1} \otimes L_\Theta \otimes M_\Lambda[0] & \xrightarrow{q^{2x} \otimes 1 \otimes \text{inclusion}} & M_{\lambda-1} \otimes L_\Theta \otimes M_\Lambda[0] \\
M_{\lambda-1} \otimes L_\Theta \otimes M_\Lambda & \xrightarrow{\Phi(\lambda)} & M_{\lambda-1} \otimes L_\Theta \otimes M_\Lambda & \xrightarrow{\text{projection}} & M_{\lambda-1} \otimes L_\Theta \otimes M_\Lambda \\
M_{\lambda-h^{(n)}} \otimes L_\Theta \otimes M_\Lambda & \xrightarrow{\Phi(\lambda-h^{(n)})} & M_{\lambda-h^{(n)}} \otimes L_\Theta \otimes M_\Lambda & \xrightarrow{\text{projection}} & M_{\lambda-h^{(n)}} \otimes L_\Theta \otimes M_\Lambda \\
M_{\lambda-h^{(2\ldots,n-1)}} \otimes L_\Theta \otimes M_\Lambda & \xrightarrow{\Phi(\lambda-h^{(2\ldots,n-1)})} & M_{\lambda-h^{(2\ldots,n-1)}} \otimes L_\Theta \otimes M_\Lambda & \xrightarrow{\text{projection}} & M_{\lambda-h^{(2\ldots,n-1)}} \otimes L_\Theta \otimes M_\Lambda \\
M_{\lambda-h^{(1\ldots,n-1)}} \otimes L_\Theta \otimes M_\Lambda & \xrightarrow{\Phi(\lambda-h^{(1\ldots,n-1)})} & M_{\lambda-h^{(1\ldots,n-1)}} \otimes L_\Theta \otimes M_\Lambda & \xrightarrow{\text{projection}} & M_{\lambda-h^{(1\ldots,n-1)}} \otimes L_\Theta \otimes M_\Lambda \\
M_{\lambda-1} \otimes L_\Theta \otimes M_\Lambda[0] & \xrightarrow{\Phi(\lambda)} & M_{\lambda-1} \otimes L_\Theta \otimes M_\Lambda[0] & \xrightarrow{q^{2x} \otimes 1 \otimes \text{inclusion}} & M_{\lambda-1} \otimes L_\Theta \otimes M_\Lambda[0] \\
\end{array}
\]

where we used the standard dynamical notation $h^{(i\ldots,j)}$ with $L_\Theta$ labeled by 0, and the summation over weights is implicit, for example $M_{\lambda-h^{(0)}} \otimes L_\Theta$ means $\bigoplus \mu M_{\lambda-\mu} \otimes L_\Theta[\mu]$.

Commutativity of the top square of the diagram is just a reformulation of the weight-preserving property of the intertwining operators $\Phi(\lambda)$. Similarly, the squares in the middle commute due to the braiding relation (5.23). Commutativity of the bottom square is obvious.

Let $A$ and $B$ denote respectively the compositions of operators in the left and right vertical columns in the above diagram, multiplied by the scalar $q^{-\frac{m(m+1)}{2}}(q-q^{-1})^m(q^x-q^{-x})$, which appear in the definition of the trace function $\mathcal{F}(\lambda, x; \bar{\Lambda})$. We observe that
On the other hand, it is known (see [EV1, STV]) that for generic $\lambda$ the operator $\Phi_\Theta(\lambda)$ is a linear isomorphism, and therefore the two traces above must be the same. Hence

$$M_\Theta F(\lambda, x; \vec{\Lambda}) = q^{-m_\Theta} M_{\Theta^+_1 \cdots \cdots} R_{\Theta,\Lambda_n} q^{-xh}.$$

Finally, using the triangularity of the quantum $R$-matrix, one easily shows that

$$q^{m_\Theta} \text{Tr} \bigg|_{L_{\Theta^+_1 \cdots \cdots} \cdots} R_{\Theta,\Lambda_1} \cdots R_{\Theta,\Lambda_n} q^{-xh} = \chi_\Theta(x) \text{Id}_{M_{\lambda}^1},$$

which establishes (6.11), and thus completes the proof of the theorem. \hfill \Box

**Example.** Let $n = 2, m = 1$, and let $\Theta = 1$. Then we have

$$M_1 = \begin{pmatrix} \frac{|\lambda-A_2|}{|\lambda-A_2+2|} & -\frac{|\lambda|}{|\lambda-A_2+2|} \\ 0 & \frac{|\lambda+A_1|}{|\lambda-A_2|} \end{pmatrix} T_{+1} + \begin{pmatrix} 0 & 1 \\ -\frac{|\lambda+A_1|}{|\lambda-A_2|} & \frac{|\lambda-A_2+1|}{|\lambda-A_2|} \end{pmatrix} T_{-1},$$

and one can check that the hypergeometric qKZB matrix (4.11) satisfies the equation

$$M_1 H(\lambda, x; \vec{A}) = (q^x + q^{-x}) H(\lambda, x; \vec{A}).$$

**Remark** 6.8. The operators $M_\Theta$ for $\Theta = 2, 3 \ldots$ can be expressed in terms of $M_1$, since for any $\Theta$ we have $M_\Theta M_1 = M_{\Theta^+_1} + M_{\Theta^{-1}}$. More precisely, let $p_n(t)$ be the $n$-th Chebyshev polynomial of the second kind, so that

$$p_n(\cos \alpha) = \frac{\sin(n+1)\alpha}{\sin \alpha}.$$ 

Then $M_\Theta = p_\Theta(M_1/2)$, for example $M_2 = (M_1)^2 - 1$, $M_3 = (M_1)^3 - 2 M_1$, etc. \hfill \Box

### 6.5. Completeness of the hypergeometric solutions

In this subsection we consider the qKZB and MR equations in the space of the formal (without any convergence assumptions) expressions of the form

$$\psi(\lambda) = q^{\lambda(x-m)} \left( \sum_{j=0}^{\infty} \psi^{(j)} q^{2j\lambda} \right), \quad \psi^{(j)} \in M_{\lambda}^1[0], \quad \psi^{(0)} \neq 0. \quad (6.11)$$

The qKZB and MR operators admit similar power series expansions, and therefore act in the above space. We refer to their eigenfunctions as formal solutions of the corresponding equations. We show that generically all formal solutions are in fact trigonometric quasi-polynomials, and come from the hypergeometric construction.
Theorem 6.9. Let \( x \in \mathbb{C} \) and \( \tilde{\Lambda} \in \mathbb{C}^n [2m] \) be generic. Suppose that \( \psi(\lambda) \) as in (6.11) is a formal solution of the qKZB equations. Then there exists \( \tilde{m} \in \mathbb{Z}^n [m] \) such that the corresponding eigenvalues are equal to \( \varepsilon_j^{\tilde{m}} (x; \tilde{\Lambda}) \), and \( \psi(\lambda) \) is proportional to \( \mathbb{H}^{\tilde{m}}(\lambda, x; \tilde{\Lambda}) \).

In particular, \( \psi(j) = 0 \) for \( j > m \), and \( \psi(m) \neq 0 \).

Proof. Let \( \tilde{m} \) be minimal such that \( v_j^{(\tilde{m})} \) occurs in the expansion of \( \psi(0) \) with nonzero coefficient. Arguing as in the proof of Lemma 6.4 we see that the qKZB eigenvalues are given by \( \varepsilon_j = \varepsilon_j^{\tilde{m}} (x; \tilde{\Lambda}) \). Subtracting from \( \psi(\lambda) \) a suitable multiple of \( \mathbb{H}^{\tilde{m}}(\lambda, x; \tilde{\Lambda}) \), we get a function \( \varphi(\lambda) \) satisfying
\[
\mathbb{K}_j \varphi(\lambda) = \varepsilon_j^{\tilde{m}} (x; \tilde{\Lambda}) \varphi(\lambda),
\]
such that the expansion of \( \varphi(\lambda) \) does not contain the term \( q^{\lambda(x-m)} \varphi^{(\tilde{m})}_{\tilde{\Lambda}} \); we claim that \( \varphi(\lambda) \equiv 0 \). Indeed, otherwise the expansion of \( \varphi(\lambda) \) would begin with a nonzero multiple of \( q^{\lambda(x-m+2k)} \varphi^{(\tilde{m})}_{\tilde{\Lambda}} \) for some \( \tilde{m}' \in \mathbb{Z}^n [m] \) and \( k \in \mathbb{Z}_{\geq 0} \), such that either \( k > 0 \) or \( \tilde{m}' > \tilde{m} \). We would then get
\[
\varepsilon_j^{\tilde{m}} (x; \tilde{\Lambda}) = \varepsilon_j^{\tilde{m}'} (x + 2k; \tilde{\Lambda}), \quad j = 1, \ldots, n,
\]
which is impossible for generic \( x \). Thus \( \varphi(\lambda) \equiv 0 \), and \( \psi(\lambda) \) is a multiple of \( \mathbb{H}^{\tilde{m}}(\lambda, x; \tilde{\Lambda}) \). \( \square \)

Remark 6.10. The key fact used in the proof is the simplicity of the spectrum of the qKZB system (i.e. that (6.12) holds only when \( k = 0, m' = \tilde{m} \)), which for special \( \tilde{\Lambda} \in \mathbb{C}^n [2m] \) needs not be true. Nevertheless, the argument as above shows that if (6.12) has a finite number of solutions, then \( \psi(\lambda) \) is a finite linear combination of corresponding \( \mathbb{H}^{\tilde{m}'}(\lambda, x + 2k; \tilde{\Lambda}) \), and in particular is a trigonometric quasi-polynomial. The only case when the spectrum becomes infinitely degenerate is when \( \tilde{\Lambda} = 2\tilde{m} \), in which case \( \varepsilon_j^{\tilde{m}} (x; \tilde{\Lambda}) \) are the same for all \( x \), and there exist infinite formal power series solutions, for example \( \psi(\lambda) = \sum_{k=0}^{\infty} \mathbb{H}^{\tilde{m}}(\lambda, x + 2k; \tilde{\Lambda}) \).

Theorem 6.11. Let \( x \in \mathbb{C} \) and \( \tilde{\Lambda} \in \mathbb{C}^n [2m] \) be generic. Suppose that \( \psi(\lambda) \) as in (6.11) is a solution of the MR equations (6.8). Then the corresponding eigenvalues are equal to \( \chi_\Theta(x) \), and \( \psi(\lambda) \) is a linear combination of \( \mathbb{H}^{\tilde{m}}(\lambda, x; \tilde{\Lambda}) \) with \( \tilde{m} \in \mathbb{Z}^n [m] \).

In particular, \( \psi(j) = 0 \) for \( j > m \), and \( \psi(m) \neq 0 \).

Proof. Let \( M_\Theta \) denote the limit as \( q^{2\lambda} \to 0 \) of the Macdonald-Ruijsenaars operator \( M_\Theta \). It is easy to derive from the triangularity of \( R_{\Lambda_1, \Lambda_2} \)
\[
\text{Tr} |_{L_\Theta[\mu]} R_{\Theta, \Lambda_1} \cdots R_{\Theta, \Lambda_n} = q^{-m(\Theta + \mu)} \dim L_\Theta[\mu],
\]
and we obtain the explicit formula for \( M_\Theta \):
\[
M_\Theta = \sum_{\mu} q^{-m \mu} \dim L_\Theta[\mu] \ T_{-\mu}.
\]

(6.13)

The leading term \( q^{\lambda(x-m)} \psi(0) \) of the expansion of \( \psi(\lambda) \) satisfies the equations
\[
M_\Theta \left( q^{\lambda(x-m)} \psi(0) \right) = \chi_\Theta \left( q^{\lambda(x-m)} \psi(0) \right),
\]
and a straightforward computation using (6.13) shows that \( \chi_\Theta = \chi_\Theta(x) \).
For generic \( x \) a formal solution of the MR equations is uniquely determined by the initial term of its expansion, and therefore the dimension of the solution space is at most \( \dim M_{\Lambda}[0] = \# \mathbb{Z}^n[\mathbf{m}] \), cf. Lemma 5.4 in [EV1]. On the other hand, Theorem 6.7 shows that \( \mathbb{H}^{\tilde{m}}(\lambda, x; \tilde{\Lambda}) \) satisfy the desired MR equations. Comparing the dimensions, we see that the linearly independent functions \( \{\mathbb{H}^{\tilde{m}}(\lambda, x; \tilde{\Lambda})\}_{\tilde{m} \in \mathbb{Z}^n[\mathbf{m}]} \) form a basis of the space of formal solutions. \( \square \)

7. The harmonic space

7.1. The harmonic space: generic highest weights. In this section we assume that \( \tilde{\Lambda} \in \mathbb{C}^n[2\mathbf{m}] \) is generic. For each \( x \in \mathbb{C} \) we denote \( \mathcal{H}_{x, \tilde{\Lambda}} \) the image in \( \mathfrak{Fun} \otimes M_{\Lambda}[0] \) of the operator \( \mathbb{H}(\cdot, x; \tilde{\Lambda}) \), defined as in (4.9); in other words,

\[
\mathcal{H}_{x, \tilde{\Lambda}} = \operatorname{Span} \left\{ \mathbb{H}^{\tilde{m}'}(\cdot, x; \tilde{\Lambda}) \mid \tilde{m}' \in \mathbb{Z}^n[\mathbf{m}] \right\}.
\]

(7.1)

It is clear that \( \mathcal{H}_{x, \tilde{\Lambda}} \) is a subspace of \( \mathfrak{Fun}_{x, \mathbf{m}} \otimes M_{\Lambda} \). Summarizing the results of the previous sections, we obtain

**Theorem 7.1.** Let \( x \in \mathbb{C} \) be generic. Then \( \mathcal{H}_{x, \tilde{\Lambda}} \) has dimension

\[
\dim \mathcal{H}_{x, \tilde{\Lambda}} = \dim M_{\Lambda}[0] = \begin{bmatrix} m + n - 1 \\ m \end{bmatrix},
\]

and admits the following descriptions:

1. \( \mathcal{H}_{x, \tilde{\Lambda}} = \operatorname{Span} \left\{ \mathcal{F}^{\tilde{m}'}(\cdot, x; \tilde{\Lambda}) \mid \tilde{m}' \in \mathbb{Z}^n[\mathbf{m}] \right\} \).
2. \( \mathcal{H}_{x, \tilde{\Lambda}} = \operatorname{Span} \left\{ \mathcal{G}^{\tilde{m}'}(\cdot, x; \tilde{\Lambda}) \mid \tilde{m}' \in \mathbb{Z}^n[\mathbf{m}] \right\} \).
3. \( \mathcal{H}_{x, \tilde{\Lambda}} = \operatorname{Span} \left\{ \psi \in \mathfrak{Fun}_{x, \mathbf{m}} \otimes M_{\Lambda}[0] \mid \psi \text{ is a solution of the MR equations} \right\} \).

Moreover, if \( n > 1 \), then we also have

4. \( \mathcal{H}_{x, \tilde{\Lambda}} = \operatorname{Span} \left\{ \psi \in \mathfrak{Fun}_{x, \mathbf{m}} \otimes M_{\Lambda}[0] \mid \psi \text{ is a solution of the qKZB equations} \right\} \).

**Proof.** Theorem 4.3 implies (1). Theorem 3.3 implies (2). Theorem 6.7 and Theorem 6.5 show that \( \mathcal{H}_{x, \tilde{\Lambda}} \) is contained in the right hand sides of (3) and (4). The opposite inclusions in (3) and (4) follow from Theorem 6.11 and Theorem 6.9. \( \square \)

7.2. The harmonic space: integral highest weights. In this subsection we assume that \( \tilde{\Lambda} \in \mathbb{Z}^n[2\mathbf{m}] \). Then the natural projection from \( M_{\Lambda} \) to \( L_{\Lambda} \) gives rise to a map from \( \mathfrak{Fun} \otimes M_{\Lambda}[0] \) to \( \mathfrak{Fun} \otimes L_{\Lambda}[0] \):

\[
\varphi(\lambda) = \sum_{\tilde{m} \in \mathbb{Z}^n[\mathbf{m}]} \varphi_{\tilde{m}}(\lambda) \psi_{\tilde{\Lambda}}^{\tilde{m}}, \quad \mapsto \quad \overline{\varphi}(\lambda) = \sum_{\tilde{m} \in \Adm_{\tilde{\Lambda}[0]}[\mathbf{m}]} \varphi_{\tilde{m}}(\lambda) \psi_{\tilde{\Lambda}}^{\tilde{m}}.
\]

For generic \( x \in \mathbb{C} \) and \( \tilde{m}' \in \mathbb{Z}^n[\mathbf{m}] \) this yields \( \overline{\mathcal{F}}^{\tilde{m}'}(\lambda, x; \tilde{\Lambda}), \overline{\mathcal{G}}^{\tilde{m}'}(\lambda, x; \tilde{\Lambda}) \in \mathfrak{Fun}_{x, \mathbf{m}} \otimes L_{\Lambda}[0] \). The matrix elements of the hypergeometric matrices have poles for integral values of \( \Lambda_{i} \). However, \( \mathbb{H}^{\tilde{m}'}(\lambda, x; \tilde{\Lambda}) \) and \( \mathcal{H}_{\tilde{m}}^{\tilde{m}'}(\lambda; \tilde{\Lambda}) \) are regular at \( \tilde{\Lambda} \in \mathbb{Z}^n[2\mathbf{m}] \), provided that at least one
of the indices \( \vec{m}, \vec{m}' \) is \( \vec{\Lambda} \)-admissible, see [MV]. Therefore, vectors \( \overline{\mathcal{H}}^{\vec{m}'}(\lambda; \vec{\Lambda}) \in L_{\vec{\Lambda}}[0] \) and functions \( \overline{\mathcal{H}}^{\vec{m}'}(\lambda, x; \vec{\Lambda}) \in \mathfrak{Fun} \otimes L_{\vec{\Lambda}}[0] \) are well-defined for any \( \vec{m}' \in \mathbb{Z}^n[\vec{m}] \).

Let \( \overline{\mathcal{H}}(x; \vec{\Lambda}) = \{ \mathcal{H}^{\vec{m}'}(x; \vec{\Lambda}) \}_{\vec{m}, \vec{m}' \in \text{Adm}_{\vec{m}}[\vec{m}]} \) denote the submatrix of the hypergeometric qKZ matrix, corresponding to the \( \vec{\Lambda} \)-admissible indices, and similarly for \( \overline{\mathcal{H}}(\lambda, x; \vec{\Lambda}), \overline{\mathcal{H}}(\lambda, x; \vec{\Lambda}) \) and \( \overline{\mathcal{H}}(\lambda, x; \vec{\Lambda}) \). For any fixed \( x \in \mathbb{C} \) these matrices can be regarded as operators

\[
\overline{\mathcal{H}}(x; \vec{\Lambda}) : L_{\vec{\Lambda}}[0] \rightarrow L_{\vec{\Lambda}}[0],
\]

\[
\overline{\mathcal{H}}(\lambda, x; \vec{\Lambda}), \overline{\mathcal{H}}(\lambda, x; \vec{\Lambda}), \overline{\mathcal{H}}(\lambda, x; \vec{\Lambda}) : L_{\vec{\Lambda}}[0] \rightarrow \mathfrak{Fun} \otimes L_{\vec{\Lambda}}[0].
\]

For each \( x \in \mathbb{C} \) we denote \( \overline{\text{Harm}}_{x, \vec{\Lambda}} \) the image in \( \mathfrak{Fun} \otimes L_{\vec{\Lambda}}[0] \) of the operator \( \overline{\mathcal{H}}(\cdot, x; \vec{\Lambda}) \), i.e.

\[
\overline{\text{Harm}}_{x, \vec{\Lambda}} = \text{Span} \left\{ \overline{\mathcal{H}}^{\vec{m}'}(\cdot, x; \vec{\Lambda}) \mid \vec{m}' \in \mathbb{Z}^n[\vec{m}] \right\}.
\]

**Theorem 7.2.** Let \( x \in \mathbb{C} \) be generic. Then \( \overline{\text{Harm}}_{x, \vec{\Lambda}} \) admits the following descriptions:

1. \( \overline{\text{Harm}}_{x, \vec{\Lambda}} = \text{Span} \left\{ \overline{\mathcal{H}}^{\vec{m}'}(\cdot, x; \vec{\Lambda}) \mid \vec{m}' \in \text{Adm}_{\vec{m}}[\vec{m}] \right\} \).

2. \( \overline{\text{Harm}}_{x, \vec{\Lambda}} = \text{Span} \left\{ \mathcal{H}^{\vec{m}'}(\cdot, x; \vec{\Lambda}) \mid \vec{m}' \in \text{Adm}_{\vec{m}}[\vec{m}] \right\} \).

3. \( \overline{\text{Harm}}_{x, \vec{\Lambda}} = \text{Span} \left\{ \psi \in \mathfrak{Fun}_{x, \vec{m}} \otimes L_{\vec{\Lambda}}[0] \mid \psi \text{ is a solution of the MR equations} \right\} \).

Moreover, if \( n > 1 \) and at least one of \( \Lambda_i \) is odd, then we also have

4. \( \overline{\text{Harm}}_{x, \vec{\Lambda}} = \text{Span} \left\{ \psi \in \mathfrak{Fun}_{x, \vec{m}} \otimes L_{\vec{\Lambda}}[0] \mid \psi \text{ is a solution of the qKZB equations} \right\} \).

The dimension of the space \( \overline{\text{Harm}}_{x, \vec{\Lambda}} \) is equal to the cardinality of \( \text{Adm}_{\vec{m}}[\vec{m}] \).

**Proof.** The argument is an obvious "admissible" modification of the proof of Theorem 7.1. The assumption that one of \( \Lambda_i \) is odd guarantees that \( \vec{\Lambda} \neq 2\vec{m} \), see Remark 6.10.

**Remark 7.3.** It is plausible that even for special values of \( x \) the space \( \overline{\text{Harm}}_{x, \vec{\Lambda}} \) contains all trigonometric quasi-polynomial solutions of the corresponding qKZB and MR equations.

### 7.3. The Weyl reflection

We keep the assumption \( \vec{\Lambda} \in \mathbb{Z}^n[2\vec{m}] \). Define the linear map

\[
S : L_{\vec{\Lambda}}[0] \rightarrow L_{\vec{\Lambda}}[0],
\]

\[
v^{(\vec{m})}_\Lambda \mapsto v^{(\vec{m})}_{\vec{\Lambda}} - \vec{\Lambda} - \vec{m},
\]

and let \( S \) denote the composition of \( S \) with the involution \( \varphi(\lambda) \mapsto \varphi(-\lambda) \) of the space \( \mathfrak{Fun}:

\[
S : \mathfrak{Fun} \otimes L_{\vec{\Lambda}}[0] \rightarrow \mathfrak{Fun} \otimes L_{\vec{\Lambda}}[0],
\]

\[
\xi(\lambda) \otimes v^{(\vec{m})}_\Lambda \mapsto \xi(-\lambda) \otimes v^{(\vec{m})}_{\vec{\Lambda}} - \vec{\Lambda}.
\]

The involution \( S \) is called the Weyl reflection, and defines an action of the Weyl group in the space of \( L_{\vec{\Lambda}}[0] \)-valued functions. The following lemma is similar to Theorem 42 in [FV1].

**Lemma 7.4.** The Weyl reflection \( S \) commutes with the qKZB operators \( \Pi_j \) and the Macdonald-Ruijsenaars operators \( \mathcal{M}_{ij} \).

**Proof.** It is easy to prove by induction (see also Theorem 41 in [FV1]) that for any \( \Lambda_1, \Lambda_2 \)

\[
\mathbb{R}_{\Lambda_1, \Lambda_2}(-\lambda) (s_{\Lambda_1} \otimes s_{\Lambda_2}) = (s_{\Lambda_1} \otimes s_{\Lambda_2}) \mathbb{R}_{\Lambda_1, \Lambda_2}(\lambda),
\]
where \( s_\Lambda \in \text{End}(L_\Lambda) \) is defined by \( s_\Lambda v_\lambda^{(m)} = v_\lambda^{(\Lambda-m)} \). Since \( S = s_\Lambda \otimes \ldots \otimes s_{\Lambda_n} \), the desired statement easily follows from the definitions and the relation \( \Gamma_j S = S \Gamma_j \). \( \square \)

**Theorem 7.5.** For any \( x \in \mathbb{C} \) we have

\[
S \mathbb{H}(\lambda, x; \Lambda) = \mathbb{H}(\lambda, -x; \Lambda) S.
\]

**Proof.** We need to show that for any \( \vec{m} \in \text{Adm}_\Lambda \) the following holds:

\[
S \mathbb{H}^{\vec{m}}(\lambda, x; \Lambda) = \mathbb{H}^{{\vec{m}}-\vec{m}}(\lambda, -x; \Lambda).
\]  

(7.3)

Assume that \( x \in \mathbb{C} \) is generic. It is clear that \( S (\mathfrak{Fun}_{x,m} \otimes L_\Lambda[0]) \subset \mathfrak{Fun}_{-x,m} \otimes L_\Lambda[0] \). Since the subspace \( \mathfrak{Harm}_{x,\Lambda} \subset \mathfrak{Fun}_{x,m} \) is characterized as the joint eigenspace for the MR operators, Lemma 7.3 implies that \( S \mathfrak{Harm}_{x,\Lambda} \subset \mathfrak{Harm}_{-x,\Lambda} \). Therefore, \( S \mathbb{H}^{\vec{m}}(\lambda, x; \Lambda) \) can be written as a linear combination of \( \mathbb{H}^{\vec{m}'}(\lambda, -x; \Lambda) \) with \( \vec{m}' \in \text{Adm}_\Lambda \). Applying Lemma 7.4 again, we conclude that \( S \mathbb{H}^{\vec{m}}(\lambda, x; \Lambda) \) must satisfy the qKZB equation with eigenvalues

\[
epsilon_j = \epsilon_j^{\vec{m}}(x; \Lambda) = \epsilon_j^{{\vec{m}}-\vec{m}}(-x; \Lambda).
\]

It now follows that \( S \mathbb{H}^{\vec{m}}(\lambda, x; \Lambda) = C \mathbb{H}^{{\vec{m}}-\vec{m}}(\lambda, -x; \Lambda) \) for some scalar \( C \), and comparing the principal terms of both sides, we get \( C = 1 \). We conclude that (7.3) is valid for generic \( x \in \mathbb{C} \), and since both sides are holomorphic in \( x \), the desired equality holds identically. \( \square \)

### 7.4. The generalized Weyl character formula.

**Lemma 7.6.** For \( \vec{m} \in \text{Adm}_\Lambda \) we have \( \Phi_\Lambda^{\vec{m}}(\lambda)(M_{\Lambda-1}) \subset M_{\Lambda-1-\sum_{i=1}^{n}(\lambda_i-2m_i)} \otimes L_\Lambda^* \).

**Proof.** The explicit definition (7.4) shows that the image of \( \Phi_\Lambda^{\vec{m}}(\lambda) \) lies in the submodule \( M_{\Lambda-\Lambda+2m-1} \otimes L_\Lambda^* \), settling the case \( n = 1 \). Obvious induction extents it for \( n > 1 \). \( \square \)

**Lemma 7.7.** Let \( \lambda \in \mathbb{Z}_{>0} \). Let \( \Lambda \in \mathbb{Z}^n[2m] \) and \( \vec{m} \in \text{Adm}_\Lambda[m] \). Then \( \Phi_\Lambda^{\vec{m}}(\lambda) \) induces an operator

\[
\iota^\Lambda \Phi_\Lambda^{\vec{m}}(\lambda) : L_{\Lambda-1} \rightarrow L_{\Lambda-1} \otimes L_\Lambda^* \otimes \ldots \otimes L_\Lambda^*.
\]

(7.4)

**Proof.** Lemma 7.2 and Lemma 7.3 imply that for \( \Lambda \in \mathbb{Z}_{\geq 0} \) and \( m \in \{0,1,\ldots,\Lambda\} \) we have

\[
\Phi_\Lambda^{\vec{m}}(\lambda) \circ \iota(\lambda) = (\iota(\lambda - \Lambda + 2m) \otimes 1) \circ \Phi_\Lambda^{\vec{m}}(\lambda)(-\lambda), \quad \text{if } \lambda - \Lambda + 2m \geq 0,
\]

\[
(\iota(-\Lambda + \Lambda - 2m) \otimes 1) \circ \Phi_\Lambda^{\vec{m}}(\lambda) \circ \iota(\lambda) = \Phi_\Lambda^{\vec{m}}(\lambda)(-\lambda), \quad \text{if } \lambda - \Lambda + 2m \leq 0.
\]

(7.5)
Therefore, we can form a commutative diagram

\[
\begin{array}{ccc}
M_{\lambda-1} & \xrightarrow{\iota(\lambda)} & M_{\lambda-1} \\
\Phi_{\lambda-n}^n(-\lambda) & \downarrow & \Phi_{\lambda-n}^n(\lambda) \\
M_{\lambda+\Lambda-2m-1} \otimes L_{\Lambda_n}^* & \xrightarrow{\iota((\lambda+\Lambda+2m)) \otimes 1^1} & M_{\lambda+\Lambda+2m-1} \otimes L_{\Lambda_n}^* \\
\Phi_{\lambda-m}^1(-\lambda) & \downarrow & \Phi_{\lambda-m}^1(\lambda) \\
M_{\lambda-\Lambda+1+2m-1} \otimes L_{\Lambda_2}^* \otimes \ldots \otimes L_{\Lambda_n}^* & \xrightarrow{\iota((\lambda+\Lambda+2m)) \otimes 1^n} & M_{\lambda+\Lambda-2m-1} \otimes L_{\Lambda_2}^* \otimes \ldots \otimes L_{\Lambda_n}^* \\
\end{array}
\]

where the directions of the horizontal arrows are determined by the positivity/negativity of the corresponding highest weights, and each square is commutative due to (7.5). Thus we obtain \( \Phi_{\lambda-m}^\Lambda(\lambda)M_{\lambda-1} \subset M_{\lambda-1} \otimes L_{\Lambda}^*[0] \), and the desired statement follows.

Let \( \lambda \in \mathbb{Z}_{\geq 0} \). Define the trace functions \( \mathcal{F}_m(\lambda, x; \bar{\Lambda}) \), associated with \( \Phi_{\lambda-m}^\Lambda(\lambda) \), by analogy with (3.14):

\[
\mathcal{F}_m(\lambda, x; \bar{\Lambda}) = q^{m-m + 1}(q - q^{-1})^m(q^x - q^{-x}) \operatorname{Tr}_{L_{\Lambda-1}} \left( \Phi_{\lambda-m}^\Lambda(\lambda) q^{xh} \right).
\]

The coordinates \( \left\{ \mathcal{F}_{m, m'}^\Lambda(\lambda, x; \bar{\Lambda}) \right\}_{m, m' \in \operatorname{Adm}_\Lambda[\mathbf{m}]} \) of the \( L_{\Lambda}^*[0] \)-valued functions \( \mathcal{F}_m(\lambda, x; \bar{\Lambda}) \) form a matrix, which can be regarded as a linear map \( \mathcal{F}(\lambda, x; \bar{\Lambda}) : L_{\Lambda}^*[0] \rightarrow \mathfrak{fun} \otimes L_{\Lambda}^*[0] \).

**Theorem 7.8.** Let \( \lambda \in \mathbb{Z}_{\geq 0} \), \( m \in \mathbb{Z}_{\geq 0} \) and \( \bar{\Lambda} \in \mathbb{Z}[2\mathbf{m}] \). Then for any \( x \in \mathbb{C} \) we have

\[
\mathcal{F}(\lambda, x; \bar{\Lambda}) = \mathcal{F}(\lambda, x; \bar{\Lambda}) - \mathcal{S} \mathcal{F}(\lambda, x; \bar{\Lambda}).
\]

**Proof.** The desired statement is equivalent to a family of relations for \( m \in \operatorname{Adm}_\Lambda \):

\[
\mathcal{F}_m(\lambda, x; \bar{\Lambda}) = \mathcal{F}_{\lambda-m}(\lambda, x; \bar{\Lambda}) - \mathcal{F}_{\Lambda-m}(\lambda, x; \bar{\Lambda}).
\]

We see from the commutative diagram in the proof of Lemma 7.7 that

\[
\Phi_{\lambda-m}^\Lambda(\lambda) \circ \iota(\lambda) = (\iota(\lambda) \otimes 1^n) \circ \Phi_{\lambda-m}^\Lambda(\lambda),
\]

or equivalently that the restriction of \( \Phi_{\lambda-m}^\Lambda(\lambda) \) to the submodule \( \iota(\lambda)(M_{\lambda-1}) \subset M_{\lambda-1} \) is equal to \( (\iota(\lambda) \otimes 1^n) \circ \Phi_{\lambda-m}^\Lambda(\lambda) \). Therefore,

\[
\operatorname{Tr}_{L_{\Lambda-1}} \left( \Phi_{\lambda-m}^\Lambda(\lambda) q^{xh} \right) = \operatorname{Tr}_{M_{\lambda-1}} \left( \Phi_{\lambda-m}^\Lambda(\lambda) q^{xh} \right) - \operatorname{Tr}_{M_{\lambda-1}} \left( \Phi_{\lambda-m}^\Lambda(\lambda) q^{xh} \right) 
\]

\[
= \operatorname{Tr}_{M_{\lambda-1}} \left( \Phi_{\lambda-m}^\Lambda(\lambda) q^{xh} \right) - \operatorname{Tr}_{M_{\lambda-1}} \left( \Phi_{\lambda-m}^\Lambda(\lambda) q^{xh} \right),
\]

which implies (7.7), and thus proves the theorem. \( \square \)
Theorem 7.8 can be regarded as a generalization of the Weyl character formula. It also allows to extend the definition of $\mathcal{F}(\lambda, x; \vec{A})$ from the positive integral cone $\lambda \in \mathbb{Z}_{>0}$ to arbitrary values of $\lambda$.

8. TOWARDS QUANTUM CONFORMAL BLOCKS

8.1. Fusion rules for $\mathfrak{sl}_2$ and the Grothendieck ring. Let symbols $\binom{\lambda \mu}{\nu}$ be defined by

$$\binom{\lambda \mu}{\nu} = \begin{cases} 1, & \lambda, \mu, \nu, \frac{\lambda + \mu + \nu}{2} \in \mathbb{Z}_{\geq 0} \text{ and } \lambda + \mu \geq \nu, \lambda + \nu \geq \mu, \mu + \nu \geq \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

These numbers arise in representation theory as the fusion rules for finite-dimensional complex representations of $\mathfrak{sl}_2$, or the associated quantum group $U_q(\mathfrak{sl}_2)$ with generic $q$. Namely, for $\lambda, \mu, \nu \in \mathbb{Z}_{\geq 0}$ we have the equivalent definition

$$\binom{\lambda \mu}{\nu} = \text{multiplicity of } L_\nu \text{ in the composition series of } L_\lambda \otimes L_\mu.$$ 

We say that a triple $(\lambda, \mu, \nu)$ violates the $\mathfrak{sl}_2$ fusion rules, if $\binom{\lambda \mu}{\nu} = 0$.

The Grothendieck ring $\mathfrak{Gr}$, associated with the tensor category of finite-dimensional representations, is the ring with generators $\{L_\lambda\}_{\lambda \in \mathbb{Z}_{\geq 0}}$ and multiplication

$$[L_\lambda] \cdot [L_\mu] = \sum_\nu \binom{\lambda \mu}{\nu} [L_\nu].$$

The ring $\mathfrak{Gr}$ is commutative, associative, and has the unit element $[L_0]$.

We will need a combinatorial description of multiple products in $\mathfrak{Gr}$. For $\mu, \nu \in \mathbb{Z}$ define $\text{Path}_{\vec{A}}[\mu \rightsquigarrow \nu]$ to be the subset of $\text{Adm}_{\vec{A}}$, consisting of $\vec{m}$ satisfying

$$\mu - \sum_{i=1}^{n} (\Lambda_i - 2m_i) = \nu, \quad \mu - \sum_{i=j}^{n} (\Lambda_i - 2m_i) \geq m_j \quad \text{for all } j = 1, \ldots, n, \quad (8.1)$$

A representation-theoretic interpretation of $\text{Path}_{\vec{A}}[\mu \rightsquigarrow \nu]$ is as follows. For $\vec{m} \in \mathbb{Z}^n$ denote

$$\mu_j = \mu - \sum_{i=j+1}^{n} (\Lambda_i - 2m_i), \quad j = 0, \ldots, n,$$

so that, in particular, $\mu_n = \mu$ and $\mu_0 = \nu$. Then $\vec{m} \in \text{Path}_{\vec{A}}[\mu \rightsquigarrow \nu]$ if and only if the triples $(\mu_j, \Lambda_j, \mu_{j-1})$ do not violate the $\mathfrak{sl}_2$ fusion rules for all $j = 1, \ldots, n$.

**Lemma 8.1.** Let $\vec{A} \in \mathbb{Z}^n$. Then in the Grothendieck ring $\mathfrak{Gr}$ we have

$$[L_{\Lambda_1}] \cdot \ldots \cdot [L_{\Lambda_n}] = \sum_\nu \# \text{Path}_{\vec{A}}[\nu \rightsquigarrow 0] [L_\nu].$$

**Proof.** By induction on $n$. \qed

In particular, $\# \text{Path}_{\vec{A}}[0 \rightsquigarrow 0]$ equals the dimension of the subspace $(L_{\Lambda_1} \otimes \ldots \otimes L_{\Lambda_n})^{U_q(\mathfrak{sl}_2)}$ of $U_q(\mathfrak{sl}_2)$-invariants in the tensor product $L_{\Lambda_1} \otimes \ldots \otimes L_{\Lambda_n}$.
Example. Let \( n = 4 \) and \( \Lambda_1 = \cdots \Lambda_4 = 1 \). Then
\[
\text{Path}_\Lambda[0 \sim 0] = \{(0,0,1,1), (0,1,0,1)\},
\text{Path}_\Lambda[2 \sim 0] = \{(0,0,0,1), (0,0,1,0), (0,1,0,0)\},
\text{Path}_\Lambda[4 \sim 0] = \{(0,0,0,0)\},
\]
and in the Grothendieck ring we have
\[
[L_1] \cdot [L_1] \cdot [L_1] \cdot [L_1] = 2[L_0] + 3[L_2] + [L_4].
\]

8.2. Weyl anti-symmetric functions and vanishing conditions. In this section we assume \( \tilde{\Lambda} \in \mathbb{Z}^n[2\mathbb{m}] \).

Weyl anti-symmetric elements of the harmonic space satisfy the so-called vanishing conditions, which can be elegantly formulated in terms of the fusion rules for finite-dimensional representations.

Lemma 8.2. Let \( \delta \in \mathbb{Z}_{\geq 0}, \tilde{m} \in \text{Adm}_\Lambda[\mathbb{m}] \) be such that \( \tilde{m} \notin \text{Path}_\Lambda[\delta - 1 \sim \delta - 1] \). Then for any \( x \in \mathbb{C} \) and \( \tilde{m} \in \text{Adm}_\Lambda[\mathbb{m}] \) we have

\[
\mathbb{H}^\delta_{\tilde{m}}(\delta, x; \tilde{\Lambda}) = \mathbb{H}^\delta_{\Lambda - \tilde{m}}(-\delta, x; \tilde{\Lambda}).
\]

Proof. The operator \( \Phi^\delta_{\tilde{m}}(\delta) \), defined as in (7.3), vanishes under the given assumptions on \( \delta \) and \( \tilde{m} \). Therefore, \( \mathcal{F}_{\tilde{m}}(\delta, x; \tilde{\Lambda}) = 0 \), and the generalized Weyl formula (7.7) yields

\[
\mathcal{F}_{\tilde{m}}(\delta, x; \tilde{\Lambda}) = \mathcal{F}_{\Lambda - \tilde{m}}(-\delta, x; \tilde{\Lambda}),
\]

where the equality holds for all \( x \in \mathbb{C} \), such that both sides are well-defined. Theorem 7.2 now implies that the equation (8.2) is valid for generic \( x \). Since \( \mathbb{H}(\lambda, x; \tilde{\Lambda}) \) is holomorphic in \( x \), we see that (8.2) holds for all \( x \in \mathbb{C} \). □

Theorem 8.3. Suppose that \( \psi(\lambda) \in \mathfrak{Hy} \text{arm}_\Lambda \) is Weyl anti-symmetric. Then for any \( \delta \in \mathbb{Z}_{\geq 0} \)
and \( \tilde{m} \in \text{Adm}_\Lambda \) such that \( \tilde{m} \notin \text{Path}_\Lambda[\delta - 1 \sim \delta - 1] \), we have

\[
\psi_{\tilde{m}}(\delta) = \psi_{\Lambda - \tilde{m}}(-\delta) = 0.
\]

In particular, we have \( \psi_{\tilde{m}}(0) = 0 \) for every \( \tilde{m} \in \text{Adm}_\Lambda \).

Proof. It follows from the Lemma 8.2 that the desired equations (8.3) are valid for functions
of the form \( \psi(\lambda) = \mathbb{H}^{\tilde{m}}(\delta, x; \tilde{\Lambda}) - \mathbb{H}^{\tilde{m}}(\delta, x; \tilde{\Lambda}) \) with \( \tilde{m} \in \text{Adm}_\Lambda \). On the other hand, such \( \psi(\lambda) \) span the subspace of Weyl anti-symmetric functions, which establishes (8.3) in the general case. Finally, for \( \delta = 0 \) the inequality (8.1), corresponding to \( j = n \), is violated for all \( \tilde{m} \), and we get the last statement. □

Equations (8.3) appeared in [FV1] for the elliptic Weyl anti-symmetric hypergeometric qKZB solutions, and are called the vanishing conditions. The proof in [FV1] was based on the detailed analysis of the combinatorics of resonance relations. Our representation-theoretic argument might be useful in the study of higher rank analogues of this phenomenon.

Example. Let \( n = 4, \ m = 2, \) and \( \Lambda_1 = \cdots = \Lambda_4 = 1 \). Then for any Weyl anti-symmetric function \( \psi(\lambda) \) the vanishing conditions \( \psi_{\tilde{m}}(\delta) = 0 \) hold for the following \( \tilde{m} \) and \( \delta \):
As a consequence, we obtain the Weyl symmetry of some of the hypergeometric $q$KZB solutions.

**Theorem 8.4.** Let $\delta \in \mathbb{Z}_{\geq 0}$, $\vec{m} \in \text{Adm}_{\lambda}$ be such that $\vec{m} \notin \text{Path}_{\lambda}[^{\delta - 1 \sim \delta - 1}]$. Then
\[
\mathbb{H}^{\vec{m}}(\lambda, -\delta; \vec{\lambda}) = \mathbb{H}^{\vec{m}}(\lambda, -\delta; \vec{\lambda}).
\]

In particular, for any $\vec{m} \in \text{Adm}_{\lambda}$ the function $\mathbb{H}^{\vec{m}}(\lambda, 0; \vec{\lambda})$ is Weyl symmetric.

**Proof.** Observe that $\vec{m} \in \text{Path}[\mu \sim \nu]$ if and only if $\text{opp}(\vec{\lambda} - \vec{m}) \in \text{Path}_{\text{opp}(\vec{\lambda})}[\nu \sim \mu]$. Using the symmetry (4.10) and Lemma 8.2, for any $\lambda \in \mathbb{C}$ and $\vec{m}' \in \text{Adm}_{\lambda}$ we get
\[
\mathbb{H}^{\vec{m}}(\lambda, -\delta; \vec{\lambda}) = \mathbb{H}^{\text{opp}(\vec{m}')}(-\delta, \lambda; \text{opp}(\vec{\lambda})) = \mathbb{H}^{\text{opp}(\vec{m}')}(-\delta, \lambda; \text{opp}(\vec{\lambda})) = \mathbb{H}^{\vec{m}'}(\lambda, \delta; \vec{\lambda}).
\]

Therefore, $\mathbb{H}^{\vec{m}}(\lambda, \delta; \vec{\lambda}) = \mathbb{H}^{-\vec{m}}(\lambda, -\delta; \vec{\lambda})$, and the desired statement follows from (7.3). □

### 8.3. The special value identity.

For any $\vec{m} \in \mathbb{Z}^{n}[m]$ denote
\[
\vartheta^{\vec{m}}(\lambda) = \frac{\mathbb{H}^{\vec{m}}(\lambda, -1; \vec{\lambda}) - \mathbb{H}^{-\vec{m}}(\lambda, 1; \vec{\lambda})}{(q - q^{-1})^{m+1}}.
\]

According to (7.3), the functions $\vartheta^{\vec{m}}(\lambda)$ are Weyl anti-symmetric. The following important result, expressing the special value $\vartheta^{\vec{m}}(1)$ as a product, is somewhat similar to the Macdonald special value identity, though the precise connection between them is unclear.

**Theorem 8.5.** Let $\vec{\lambda} \in \mathbb{Z}^{n}[2m]$ and $\vec{m} \in \mathbb{Z}^{n}[m]$. Then
\[
\vartheta^{\vec{m}}(1) = -\mathcal{Q}^{\vec{m}}(1; \vec{\lambda}) \mathcal{P}^{\vec{m}}(1; \vec{\lambda}).
\]

**Proof.** Let the matrix elements $\{\beta^{\vec{m}}\}$ be defined by
\[
\Phi^{\vec{m}}(1)v_0 = v_0 \otimes \sum_{\vec{m}'} \beta^{\vec{m}} \mathcal{H}^{\vec{m}'}(1, \vec{\lambda}),
\]

where the omitted terms involving $v_0^{(k)}$ with $k > 0$. Then we have
\[
\mathcal{F}^{\vec{m}}(1, x; \vec{\lambda}) = q^{\frac{m(m+1)}{2}}(q - q^{-1})^{m}q^{-x} \sum_{\vec{m}'} \beta^{\vec{m}'} u^{\vec{m}'}_{\vec{m}'}.
\]

Our next goal is to compute the matrix elements $\mathcal{H}^{\vec{m}'}(1; -\lambda, \vec{\lambda})$. We have
\[
(\frac{\mathcal{H}^{\vec{m}'}(1; -\lambda, \vec{\lambda})}{q})^{-1} = \lim_{q^{n} \to 0} q^{\lambda(m+1)} \mathcal{H}^{\vec{m}'}(1; -\lambda, \vec{\lambda}) = \lim_{q^{n} \to 0} q^{\lambda(m+1)} \mathcal{H}^{\vec{m}'}(1; 1, -\lambda, \vec{\lambda})
\]
\[
= \lim_{q^{n} \to -\infty} q^{-x(m+1)} \mathcal{H}^{\vec{m}'}(1, x; \vec{\lambda}) = \lim_{q^{n} \to -\infty} \left( q^{-x(m+1)} \sum_{k} \mathcal{F}^{\vec{m}'}(1, x; \vec{\lambda}) \mathcal{H}^{\vec{m}'}(k; x; \vec{\lambda}) \right)
\]
\[
= \sum_{k} \left( q^{-x} \mathcal{F}^{\vec{m}'}(1, x; \vec{\lambda}) \right) \left( q^{-x} \mathcal{H}^{\vec{m}'}(x; \vec{\lambda}) \right).
\]
Observe now that $\lim_{q^{-r} \rightarrow 0} s^{-nr} F(x; \vec{\lambda})$ is a diagonal matrix. Indeed, $F_{\vec{k}}(x; \vec{\lambda}) = 0$ unless $\vec{k} \ll \vec{m}$ due to Proposition 4.2. If $\vec{k} \ll \vec{m}$, but $\vec{k} \neq \vec{m}$, then $F_{\vec{m}}(x; \vec{\lambda}) = 0$, and we compute

$$\lim_{q^{-r} \rightarrow 0} s^{-nr} F_{\vec{m}}(x; \vec{\lambda}) = \lim_{q^{-r} \rightarrow 0} s^{-nr} F_{\vec{m}}(x; \vec{\lambda}) = \lim_{q^{-r} \rightarrow 0} s^{-nr} F_{\vec{m}}(x; \vec{\lambda}) = \lim_{q^{-r} \rightarrow 0} s^{-nr} F_{\vec{m}}(x; \vec{\lambda}) = \lim_{q^{-r} \rightarrow 0} s^{-nr} F_{\vec{m}}(x; \vec{\lambda}) = 0.$$

The only remaining entries $\lim_{q^{-r} \rightarrow 0} s^{-nr} F_{\vec{m}}(x; \vec{\lambda})$ are easily computed using 4.3:

$$\lim_{q^{-r} \rightarrow 0} s^{-nr} F_{\vec{m}}(x; \vec{\lambda}) = \frac{q^{m(m-1)/2}}{(q-q^{-1})^m} \prod_{i=1}^n q^{m_i \sum_{k=1}^{\infty} (\Lambda_k - 2m_k) - \frac{m(m_i-1)}{2} [m_i]}. $$

It is also clear that $\lim_{q^{-r} \rightarrow 0} s^{-nr} F_{\vec{m}}(1, \vec{\lambda}) = \frac{q^{m(m+1)/2}}{(q-q^{-1})^m} \beta_{\vec{m}}^{\vec{m}'}$, and thus (8.5) yields

$$F_{\vec{m}}(1, \vec{\lambda}) = \sum_{i=1}^n q^{m_i \sum_{k=1}^{\infty} (\Lambda_k - 2m_k) - \frac{m(m_i-1)}{2} [m_i]} \beta_{\vec{m}'}^{\vec{m}} = q^{m(m+1)/2} Q_{\vec{m}}^{\vec{m}}.$$ 

Finally, we compute using Lemma 3.1

$$\vartheta^{\vec{m}}(1) = \frac{q}{(q-q^{-1})^{m+1}} \sum_{\vec{m}} s^{-nr} F_{\vec{m}}(1, \vec{\lambda}) = -\sum_{\vec{m}} \beta_{\vec{m}}^{\vec{m}'} \beta_{\vec{m}'}^{\vec{m}} Q_{\vec{m}}^{\vec{m}} \beta_{\vec{m}'}^{\vec{m}}$$

$$= -\langle \sum_{\vec{m}} \beta_{\vec{m}}^{\vec{m}'} u_{\vec{m}}^{\vec{m}'} \sum_{\vec{m}} \beta_{\vec{m}'}^{\vec{m}} u_{\vec{m}}^{\vec{m}'} \rangle = -\langle \Phi_{\vec{m}}^{\vec{m}'}(1) v_0, \Phi_{\vec{m}'}^{\vec{m}'}(1) v_0 \rangle = -Q_{\vec{m}}^{\vec{m}'}(1; \vec{\lambda}) v_\lambda^{(\vec{m})}. $$

\[ \square \]

8.4. The space $\text{Conf}_{\vec{\lambda}}$. In this section we propose a quantum analogue of the space of conformal blocks on the torus in conformal field theory.

Let $\vec{\lambda} \in \mathcal{Z}^n[2m]$. Define the space $\text{Conf}_{\vec{\lambda}}$ by

$$\text{Conf}_{\vec{\lambda}} = \text{Span} \left\{ \vartheta(\lambda) \in \mathcal{H}_{\text{atm}}_{\vec{\lambda}}^{(-1, \vec{\lambda})} + \mathcal{H}_{\text{atm}}_{\vec{\lambda}}^{(1, \vec{\lambda})} \mid \mathcal{S}_{\vec{\lambda}} \vartheta(\lambda) = -\vartheta(\lambda) \right\}. $$

**Theorem 8.6.** Let $\vec{\lambda} \in \mathcal{Z}^n$. Then $\{ \vartheta^{\vec{m}}(\lambda) \}_{\vec{m} \in \text{Adm}_{\vec{\lambda}}}^{\vec{m}}$ is a basis of $\text{Conf}_{\vec{\lambda}}$. In particular,

$$\dim \text{Conf}_{\vec{\lambda}} = \dim (L_{\Lambda_1} \otimes \ldots \otimes L_{\Lambda_n})^{U_q(sl_2)}.$$ 

**Proof.** It is clear from the definitions that $\{ \vartheta^{\vec{m}}(\lambda) \}_{\vec{m} \in \text{Adm}_{\vec{\lambda}}}^{\vec{m}}$ is a spanning set for $\text{Conf}_{\vec{\lambda}}$. Moreover, Theorem 8.4 implies that $\vartheta^{\vec{m}}(\lambda) \equiv 0$ if $\vec{m} \notin \text{Adm}_{\vec{\lambda}}$. Therefore, we only need to check that the functions $\{ \vartheta^{\vec{m}}(\lambda) \}_{\vec{m} \in \text{Path}_{\vec{\lambda}}^{(0 \leadsto 0)}}$ are linearly independent.

Using the explicit formulæ, one checks that $Q_{\vec{m}}^{\vec{m}'}(1; \vec{\lambda}) \neq 0$ when $\vec{m} \notin \text{Path}_{\vec{\lambda}}^{(0 \leadsto 0)}$, and Theorem 8.5 implies that $\{ \vartheta^{\vec{m}}(1) \}_{\vec{m} \in \text{Path}_{\vec{\lambda}}^{(0 \leadsto 0)}}$ are linearly independent vectors. In particular, $\{ \vartheta^{\vec{m}}(\lambda) \}_{\vec{m} \in \text{Path}_{\vec{\lambda}}^{(0 \leadsto 0)}}$ are linearly independent functions, which completes the proof. \[ \square \]
Example. Let \( n = 4, \ m = 2, \) and \( \Lambda_1 = \cdots = \Lambda_4 = 1. \) Then the space \( \text{Conf}(\vec{A}) \) is spanned by two functions \( \vartheta^{(0,0,1,1)}(\lambda) \) and \( \vartheta^{(0,1,0,1)}(\lambda) \), described in the table below:

| \( \vec{m} \) | \( \vartheta^{(0,0,1,1)}(\lambda) \) | \( \vartheta^{(0,1,0,1)}(\lambda) \) |
|--------------|----------------|----------------|
| \( (0,0,1,1) \) | \(-2[\lambda][\lambda + 1][\lambda + 2] \) | \( 0 \) |
| \( (0,1,0,1) \) | \( [\lambda - 1][\lambda][\lambda + 1] \) | \( -[\lambda][\lambda + 1]^2 \) |
| \( (0,1,1,0) \) | \( [\lambda - 1][\lambda][\lambda + 1] \) | \( [\lambda - 1][\lambda][\lambda + 1] \) |
| \( (1,0,0,1) \) | \( [\lambda - 1][\lambda][\lambda + 1] \) | \( -[\lambda - 1]^2[\lambda] \) |
| \( (1,0,1,0) \) | \(-2[\lambda][\lambda - 1][\lambda - 2] \) | \( 0 \) |

In conclusion, we note that according to Theorem 6.7, the conformal blocks \( \vartheta^{\vec{m}}(\lambda) \) satisfy the MR equations (6.5) with eigenvalues

\[ \chi_{\Theta} = \dim_q L_{\Theta} = \sum_\mu \dim L_{\Theta}[\mu] \ q^\mu. \]

We conjecture that there are no other trigonometric polynomial solutions.

**Conjecture 8.7.** Let \( \vec{A} \in \mathbb{Z}^n[2\mathbb{m}], \) and let \( \vartheta(\lambda) \) be a Weyl anti-symmetric \( L_{\vec{A}}[0] \)-valued trigonometric polynomial, satisfying \( M_{\Theta} \vartheta(\lambda) = (\dim_q L_{\Theta}) \ \vartheta(\lambda) \). Then \( \vartheta(\lambda) \in \text{Conf}_{\vec{A}} \).

In other words, the space \( \text{Conf}_{\vec{A}} \) should be characterized as the distinguished eigenspace of the MR operators. Similarly, we expect that \( \text{Conf}_{\vec{A}} \) can be described as the span of trigonometric polynomial solutions of the qKZB equations of trigonometric degree at most \( m + 1 \).

### 8.5. Two-point quantum conformal blocks.

In this subsection we assume that \( n = 2 \). In this case the Macdonald-Ruijsenaars operator \( M_1 \) can be written out explicitly.

**Lemma 8.8.** The Macdonald-Ruijsenaars operator \( M_1 \) is given by

\[ M_1 \psi(\lambda) = A_+(\lambda)\psi(\lambda + 1) + A_-(\lambda)\psi(\lambda - 1), \]

where \( A_{\pm}(\lambda) \in \text{End}(L_{\vec{A}}[0]) \) are represented by matrices with matrix elements

\[
\begin{align*}
(A_+(\lambda))_{m_1,m_2}^{m_1,m_2} &= \frac{[\lambda - m_1][\lambda - \Lambda_2 + m_2]}{[\lambda - \Lambda_2 + 2m_2]}, & (A_+(\lambda))_{m_1+1,m_2-1}^{m_1+1,m_2-1} &= \frac{[m_2][m_1 - \Lambda_1]}{[\lambda - \Lambda_2 + 2m_2]}, \\
(A_-\lambda))_{m_1,m_2}^{m_1,m_2} &= \frac{[\lambda + m_2][\lambda + \Lambda_1 - m_1]}{[\lambda - \Lambda_2 + 2m_2]}, & (A_-\lambda))_{m_1-1,m_2+1}^{m_1-1,m_2+1} &= \frac{[m_1][m_2 - \Lambda_2]}{[\lambda - \Lambda_2 + 2m_2]},
\end{align*}
\]

and all other matrix elements of \( A_{\pm}(\lambda) \) are zero.

**Proof.** It is clear from the definition that the only nonzero matrix elements of \( A_+(\lambda) \) are

\[
\begin{align*}
(A_+(\lambda))_{m_1,m_2}^{m_1,m_2} &= q^{m_1+m_2} (\mathbb{R}_{1,1}(\lambda - \Lambda + 2m_2))_{m_1,m_2}^{m_1,m_2} \ (\mathbb{R}_{1,0}(\lambda))_{m_1,m_2}^{m_1,m_2}, \\
(A_+(\lambda))_{m_1+1,m_2-1}^{m_1+1,m_2-1} &= q^{m_1+m_2} (\mathbb{R}_{1,1}(\lambda - \Lambda + 2m_2))_{m_1+1,m_2-1}^{m_1+1,m_2-1} \ (\mathbb{R}_{1,0}(\lambda))_{0,m_2}^{0,m_2-1}.
\end{align*}
\]
From the computations of the dynamical $R$-matrix $R_{1,\lambda}(\lambda)$ in [FTV2], we obtain
\[
(R_{1,\lambda}(\lambda))_{0,m}^{0,m} = \frac{q^{-m}[\lambda + m]}{[\lambda]}, \quad (R_{1,\lambda}(\lambda))_{1,m}^{1,m-1} = \frac{-q^{-\lambda-m}[m]}{[\lambda]},
\]
\[
(R_{1,\lambda}(\lambda))_{1,m}^{0,m+1} = \frac{q^{\lambda-m}[\lambda - m]}{[\lambda]}, \quad (R_{1,\lambda}(\lambda))_{1,m}^{1,m} = \frac{-q^{\Lambda+m}[-\lambda + \lambda + m]}{[\lambda]},
\]
which implies the desired statement for $A_+(\lambda)$. The case of $A_-(\lambda)$ is completely analogous. □

**Example.** Let $m = 1$. Then the MR operator $M_1$ is given by
\[
M_1 = \begin{pmatrix} \frac{-\lambda-\lambda+1}{\lambda-\lambda+2} & \frac{[\lambda_1]}{[\lambda]} & 0 \\ 0 & 0 & 0 \\ \frac{\lambda-1}{\lambda-\lambda+2} & \frac{[\lambda]}{[\lambda]} & \frac{\lambda\lambda+1-2}{\lambda}\end{pmatrix} T_1 + \begin{pmatrix} \frac{[\lambda]}{[\lambda]} & 0 \\ 0 & 0 \\ \frac{-\lambda-\lambda+1}{\lambda-\lambda+2} & \frac{\lambda\lambda+2}{\lambda}\end{pmatrix} T_{-1}.
\]

**Example.** Let $m = 2$. Then the MR operator $M_1$ is given by
\[
M_1 = \begin{pmatrix} \frac{-\lambda-\lambda+2}{\lambda-\lambda+4} & \frac{[\lambda_1]}{\lambda-\lambda+2} & 0 \\ 0 & 0 & 0 \\ \frac{\lambda-1}{\lambda-\lambda+2} & \frac{[\lambda]}{[\lambda]} & \frac{-\lambda-\lambda+2}{\lambda}\end{pmatrix} T_1 + \begin{pmatrix} \frac{[\lambda]}{[\lambda]} & 0 \\ 0 & 0 \\ \frac{-\lambda-\lambda+1}{\lambda-\lambda+2} & \frac{\lambda\lambda+2}{\lambda}\end{pmatrix} T_{-1}.
\]

The following gives an explicit description of the two-point conformal block spaces.

**Theorem 8.9.** Let $\lambda_1, \lambda_2 \in \mathbb{C}_{\geq 0}$ be such that $\lambda_1 + \lambda_2 = 2m$.

1. If $\lambda_1 \neq \lambda_2$, then $\text{Conf}_{\lambda} = 0$.
2. If $\lambda_1 = \lambda_2 = m$, then $\text{Conf}_{\lambda} = \mathbb{C} \vartheta^{(0,m)}$, where
\[
\vartheta^{(0,m)}(\lambda) = \sum_{j=0}^{m} (-1)^{j+1} \prod_{k=j-m}^{j} [\lambda - k] F^j v_m \otimes F^{m-j} v_m.
\]

**Proof.** The dimensions of the spaces $\text{Conf}_{\lambda}$ are computed by Theorem 8.6, and it remains to describe it explicitly for the case $\lambda_1 = \lambda_2 = m$. The vanishing conditions of Theorem 8.3 for the function $\psi(\lambda) = \vartheta^{(0,m)}(\lambda)$ are given by
\[
\psi_{m_1,m_2}(\delta) = 0, \quad \delta = -m_2, \ldots, m_1.
\]

Therefore, $\psi_{m_1,m_2}(\lambda)$ must be divisible by $\prod_{k=-m_2}^{m_1} [\lambda - k]$ as Laurent polynomials in $q^\lambda$, and comparing the degrees we see that the ratios are in fact constants, i.e. we can write
\[
\psi(\lambda) = \sum_{j=0}^{m} c_j \prod_{k=j-m}^{j} [\lambda - k] F^j v_m \otimes F^{m-j} v_m
\]
for some $c_j \in \mathbb{C}$. From Theorem 8.5 we get $\psi(0) = -[m]! [m+1]! v^{(0)}_m \otimes v^{(m)}_m = -[m+1]! v_m \otimes F^m v_m$, which gives $c_0 = -1$. The Macdonald-Ruijsenaars equation $M_1 \vartheta(\psi(\lambda) = (q + q^{-1}) \psi(\lambda)$ yields recurrent relations on $c_j$. Using the explicit description of the operator $M_1$ from Lemma 8.8, they are simplified to $c_j = -c_{j-1}$ for $j = 1, \ldots, n$. □
9. Conformal blocks at roots of unity and the Verlinde algebra

9.1. Representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ at roots of unity and the Verlinde algebra. In this section we drop the assumptions $\text{Im} \eta > 0$ and consider the case $\eta = \frac{1}{r}$ for some positive integer $r \geq 3$. In other words, $q = \exp(\frac{\pi i}{r})$ becomes a primitive $2r$-th root of unity.

We define the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ in the same way as in (8.1) and (3.6). We have the infinite-dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-modules $M_\lambda$ and $M_\lambda^*$ defined as in (3.2), (3.5), and for $\lambda \in \mathbb{Z}_{\geq 0}$ the finite-dimensional modules $L_\lambda, L_\lambda^*$ are defined as before. The modules $L_\lambda, L_\lambda^*$ are irreducible if and only if $\lambda \in \{0, 1, \ldots, \ell - 1\}$, in which case we have $L_\lambda \cong L_\lambda^*$.

When $q$ is a root of unity, the tensor products $L_\lambda \otimes L_\mu$ need not be completely reducible. Nevertheless, there exists a semisimple truncated tensor category, associated with $\mathcal{U}_q(\mathfrak{sl}_2)$-modules $\{L_0, \ldots, L_{\ell-2}\}$, with the fusion rules given by

$$\begin{pmatrix} \lambda & \mu \\ \nu \end{pmatrix}_\ell = \begin{cases} 1, & \lambda, \mu, \nu, \frac{\lambda+\mu+\nu}{2} \in \{0, 1, \ldots, \ell - 2\} \text{ and } \lambda + \mu \geq \nu, \lambda + \nu \geq \mu, \mu + \nu \geq \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

For $\lambda, \mu, \nu \in \{0, 1, \ldots, \ell - 2\}$ we have the equivalent definition

$$\begin{pmatrix} \lambda & \mu \\ \nu \end{pmatrix}_\ell = \text{multiplicity of } L_\nu \text{ as a direct summand of } L_\lambda \otimes L_\mu.$$

These numbers differ from the composition series multiplicities, since $L_\nu$ may occur as Jordan-Hölder constituents in reducible but indecomposable tilting modules. We say that a triple $(\lambda, \mu, \nu)$ violates the $\ell$-truncated $\mathfrak{sl}_2$ fusion rules, if $(\lambda, \mu, \nu)_\ell = 0$.

The associated Verlinde algebra $\text{Ver}_\ell(\mathfrak{sl}_2)$ is the algebra with generators $\{L_\lambda\}_{\lambda \in \{0, 1, \ldots, \ell-2\}}$ and multiplication

$$[L_\lambda] \cdot [L_\mu] = \sum_\nu \begin{pmatrix} \lambda & \mu \\ \nu \end{pmatrix}_\ell [L_\nu].$$

The algebra $\text{Ver}_\ell(\mathfrak{sl}_2)$ is commutative, associative, and has the unit element $[L_0]$.

**Example.** Let $\ell = 3$. Then $\text{Ver}_\ell(\mathfrak{sl}_2)$ is generated by $[L_0], [L_1]$ with multiplication

$$[L_0] \cdot [L_0] = [L_0], \quad [L_0] \cdot [L_1] = [L_1], \quad [L_1] \cdot [L_1] = [L_0].$$

In other words, the Verlinde algebra for $\ell = 3$ is isomorphic to the group algebra $\mathbb{C}[\mathbb{Z}_2]$. □

For $\mu, \nu \in \mathbb{Z}$ denote $\text{Path}_\Lambda^{(\ell)}[\mu \rightsquigarrow \nu]$ the subset of $\text{Path}_\Lambda[\mu \rightsquigarrow \nu]$, consisting of $\vec{m}$, satisfying in addition to (8.1) the extra conditions

$$\mu - \sum_{i=j+1}^{n} (\Lambda_i - 2m_i) \leq \ell - m_j - 2 \quad \text{for all } j = 1, \ldots, n. \quad (9.1)$$

**Lemma 9.1.** Let $\Lambda_1, \ldots, \Lambda_n \in \{0, 1, \ldots, \ell - 2\}$. Then in the Verlinde algebra we have

$$[L_{\Lambda_1}] \cdot \ldots \cdot [L_{\Lambda_n}] = \sum_\nu \# \text{Path}_\Lambda^{(\ell)}[\nu \rightsquigarrow 0] [L_\nu].$$

**Proof.** By induction on $n$. □
Denote by \((L_{\Lambda_1} \otimes \ldots \otimes L_{\Lambda_n})^{\text{Ver}(\mathfrak{sl}_2)}\) the sum of all direct summands of \(L_{\Lambda_1} \otimes \ldots \otimes L_{\Lambda_n}\), isomorphic to \(L_0 = \mathbb{C}\). Then, in particular, \(\dim(L_{\Lambda_1} \otimes \ldots \otimes L_{\Lambda_n})^{\text{Ver}(\mathfrak{sl}_2)} = \# \text{Path}_\Lambda [0 \leadsto 0]\). The remaining part of this section is devoted to proving an analogue of Theorem 8.6: for metric matrices and holomorphic intertwining operators remain well-defined when \(q\) becomes a root of unity.

Lemma 9.2. Let \(\vec{m}, \vec{m}' \in \mathbb{Z}^n[m]\). For \(\lambda, x \in \mathbb{C}\) and \(\vec{A} \in \mathbb{C}^n\) consider the expression

\[
H = q^{-\lambda x + mx + m\lambda} \left( \prod_{j=1}^{n} \prod_{k=1}^{\min(m_j, m'_j)} (1 - q^{2(\Lambda_j - k)}) \right) \mathfrak{H}_{\vec{m}} \left( \lambda, x; \vec{A} \right).
\]

Then \(H\) is a Laurent polynomial in \(q^2, q^{2x}, q^{2\lambda}, q^{A_1}, \ldots, q^{A_n}\).

Proof. It is easy to see that each residue term in the definition of the hypergeometric pairing has the desired representation. We leave details to the reader. \(\square\)

Lemma 9.3. Let \(\Lambda \in \mathbb{Z}\). Then the matrix elements of the operator \((q - q^{-1})^m \Phi^\Lambda_m(\lambda)\) with respect to bases \(\{v^{(k)}_{\lambda-1}\}\) and \(\{v^{(k)}_{\lambda-\Lambda+2m-1} \otimes u^\Lambda_i\}\) are Laurent polynomials in \(q, q^{2\lambda}\).

Proof. Denote for convenience \(\mu = \lambda - \Lambda + 2m\). Using (3.7), we obtain

\[
\Phi^\Lambda_m(\lambda) v^{(k)}_{\lambda-1} = \left( \sum_{j=0}^{k} \frac{q^{j(k-j)}}{j!(k-j)!} F_j \otimes F_{k-j} q^{-j}\right) \left( \sum_{i=0}^{m} \frac{q^{i(\mu-i)}}{i!} [-\mu + i + 1]_{m-i} F_i v_{\mu-1} \otimes E_i u^\Lambda_i \right)
\]

\[
= \sum_{l=0}^{m+k} C^k_{l} (\mu, \Lambda, m) v^{(l)}_{\mu-1} \otimes u^\Lambda_{m-l+k},
\]

where we denoted

\[
C^k_{l}(\mu, \Lambda, m) = \sum_{i+j=l} \frac{q^{j(k-j-\Lambda+2m-2i+i(\mu-i))}}{j!} \left[ m \atop i \right] \left[ \Lambda - m + i \atop k - j \right] [-\mu + i + 1]_{m-i}.
\]

Treating \(q\) as a formal variable, we see that \((q - q^{-1})^m [-\mu + i + 1]_{m-i}\) are Laurent polynomials in \(q, q^\mu\). Also, for any \(a \in \mathbb{Z}, b \in \mathbb{Z}_{>0}\) the binomial coefficient \(\left[ \begin{array}{c} a \\ b \end{array} \right]\) is a Laurent polynomial in \(q\). The desired matrix elements are linear combinations of \((q - q^{-1})^m C^k_{l}(\mu, \Lambda, m)\), and therefore are well-defined for any nonzero \(q \in \mathbb{C}\). \(\square\)
9.3. Weyl symmetry and vanishing conditions at roots of unity. For \( \mu \in \{0, \ldots, \ell - 1\} \) there exists a proper inclusion of \( \mathcal{U}_q(\mathfrak{sl}_2) \)-modules \( j(\mu) : L_{\ell - \mu - 1} \to L_{\ell + \mu - 1} \), uniquely determined by \( j(\mu)(v_{\ell - \mu - 1}) = F^\mu v_{\ell + \mu - 1} \).

**Lemma 9.4.** Let \( \mu, \Lambda \in \{0, 1, \ldots, \ell - 1\} \), and let \( m \in \mathbb{Z}_{\geq 0} \) be such that \( m + \mu \leq \Lambda \). Then we have commutative diagrams

\[
\begin{array}{ccc}
L_{\ell - \mu - 1} \otimes L^*_\Lambda & \xrightarrow{j(\mu) \otimes 1} & L_{\ell + \mu - 1} \otimes L^*_\Lambda \\
\Phi^\Lambda_m(\ell - \mu + \Lambda - 2m) & & \Phi^\Lambda_{m + \mu}(\ell - \mu + \Lambda - 2m) \\
L_{\ell - \mu - 1} & \xrightarrow{j(\mu)} & L_{\ell + \mu - 1} \\
\Phi^\Lambda_{m + \mu}(\ell - \mu) & & \Phi^\Lambda_{m + \mu}(\ell - \mu) \\
L_{\ell - \Lambda + 2m + \mu - 1} \otimes L^*_\Lambda & & L_{\ell - \Lambda + 2m - 1} \otimes L^*_\Lambda
\end{array}
\]

**Proof.** Similar to proofs of Lemma 3.2 and Lemma 3.3; we skip further details.

**Lemma 9.5.** Let \( \delta \in \{0, \ldots, \ell - 1\} \), \( \vec{m} \in \text{Path}_\Lambda[\delta - 1 \sim \delta - 1] \) be such that at least one of the inequalities (9.1) is violated. Then for any \( x \in \mathbb{C} \) and \( \vec{m}' \in \text{Adm}_\Lambda[\vec{m}] \) we have

\[
\mathbb{H}\vec{m} (\delta, x; \vec{\Lambda}) = \mathbb{H}\vec{m}' (2\ell - \delta, x; \vec{\Lambda}).
\]

**Proof.** Using the operators \( '\Phi^\Lambda_{\vec{m}}(\delta) \), defined as in (7.4), we form the commutative diagram

\[
\begin{array}{ccc}
L_{\ell - \delta - 1} & \xrightarrow{j(\lambda)} & L_{2\ell - \delta - 1} \\
\Phi^\Lambda_{\vec{m}}(\lambda) & & \Phi^\Lambda_{\lambda - m - \ell}(2\ell - \lambda) \\
L_{\delta + \Lambda - 2m + 1 - 1} \otimes L^*_\Lambda & \xrightarrow{j(\lambda + \Lambda - 2m + 1 - 1) \otimes 1} & L_{2\ell - \delta - \Lambda + 2m - 1 - 1} \otimes L^*_\Lambda \\
\Phi^\Lambda_{\vec{m}}(\lambda + \Lambda - 2m + 1) \otimes 1^{n - 1} & & \Phi^\Lambda_{\lambda - m - \ell}(2\ell - \lambda - 2m + 1) \otimes 1^{n - 1} \\
L_{\delta} \otimes L^*_\Lambda \otimes \cdots \otimes L^*_\Lambda & \xrightarrow{j(\lambda) \otimes 1^{n}} & L_{2\ell - \delta - 1} \otimes L^*_\Lambda \otimes \cdots \otimes L^*_\Lambda
\end{array}
\]

Multiplying by \( q^{x\mu} \) and taking the trace, we obtain \( \mathcal{F}_{\vec{m}}(\delta, x; \vec{\Lambda}) = \mathcal{F}_{\vec{\Lambda} - \vec{m}}(2\ell - \delta, x; \vec{\Lambda}) \), which can be rewritten using the Weyl formula (7.6) as

\[
\mathcal{F}_{\vec{m}}(\delta, x; \vec{\Lambda}) - \mathcal{F}_{\vec{\Lambda} - \vec{m}}(-\delta, x; \vec{\Lambda}) = \mathcal{F}_{\vec{\Lambda} - \vec{m}}(2\ell - \delta, x; \vec{\Lambda}) - \mathcal{F}_{\vec{m}}(\delta - 2\ell, x; \vec{\Lambda}).
\]

Using the quasi-periodicity \( \mathcal{F}(\lambda + 2\ell, x; \vec{\Lambda}) = e^{2\pi i x} \mathcal{F}(\lambda, x; \vec{\Lambda}) \), we obtain

\[
\mathcal{F}_{\vec{m}}(\delta, x; \vec{\Lambda}) = \mathcal{F}_{\vec{\Lambda} - \vec{m}}(2\ell - \delta, x; \vec{\Lambda}).
\]

The desired equation (9.4) for the hypergeometric qKZB matrix now follows from the admissible version of Theorem 7.2.
9.4. Conformal blocks at roots of unity. In this subsection we assume that $\Lambda \in \mathbb{Z}^n[2m]$ is such that $\Lambda_j \in \{0, 1, \ldots, \ell - 1\}$ for all $j = 1, \ldots, n$.

To each function $\varphi : \mathbb{C} \to L_\Lambda[0]$ we associate its restriction $\tilde{\varphi} : \mathbb{Z} \to L_\Lambda[0]$ to the integral lattice. We refer to $\tilde{\varphi}$ as the sequence, corresponding to the function $\varphi$.

Define the space $\text{Conf}_\Lambda^{(t)}$ of discrete conformal blocks to be the space of sequences, corresponding to functions $\tilde{\varphi}(\lambda) \in \text{Conf}_\Lambda^{(t)}$. Due to the quasi-periodicity $\tilde{\varphi}(\lambda + \ell) = (-1)^{m+1} \tilde{\varphi}(\lambda)$, a discrete conformal block is completely determined by its values at $\lambda = 0, 1, \ldots, \ell - 1$, and thus $\text{Conf}_\Lambda^{(t)}$ can be thought of as a subspace of $(L_\Lambda[0])^\ell$.

**Theorem 9.6.** The set $\{\tilde{\varphi}^m(\lambda)\}_{\tilde{m} \in \text{Path}_\Lambda^{(0 \to 0)}}$ is a basis of $\text{Conf}_\Lambda^{(t)}$. In particular,

$$\dim \text{Conf}_\Lambda^{(t)} = \dim (L_{\Lambda_1} \otimes \ldots \otimes L_{\Lambda_n})^{\text{Ver}(\mathbb{Z}_2)}.$$

**Proof.** The argument is similar to that of Theorem 8.6. If $\tilde{m} \notin \text{Path}_\Lambda^{(0 \to 0)}$, then we claim that $\tilde{\varphi}^m(\lambda) \equiv 0$. Indeed, for $\tilde{m} \notin \text{Path}_\Lambda$ this follows from the argument in the proof of Theorem 8.6. Assume now that $\tilde{m} \in \text{Path}_\Lambda$, or in other words that the inequalities (8.1) hold, but some of (9.1) fail. Using Lemma 9.5 and the symmetry of $H(\lambda, x; \Lambda, \bar{\Lambda})$, for any $\tilde{m}'$ we compute

$$H_{\Lambda}^{\tilde{m}'}(\lambda, -1; \bar{\Lambda}) = H_{\Lambda}^{\tilde{m}'}(1, -\lambda; \bar{\Lambda}) = H_{\Lambda}^{\tilde{m}'}(2\ell - 1, -\lambda; \bar{\Lambda}) = e^{-2\pi i \lambda} H_{\Lambda}^{\tilde{m}'}(-1, -\lambda; \bar{\Lambda}) = e^{-2\pi i \lambda} H_{\Lambda}^{\tilde{m}'}(\lambda, 1; \bar{\Lambda}),$$

and for $\lambda \in \mathbb{Z}$ we obtain $H_{\Lambda}^{\tilde{m}'}(\lambda, -1; \bar{\Lambda}) = H_{\Lambda}^{\tilde{m}'}(\lambda, 1; \bar{\Lambda})$, which means that $\tilde{\varphi}^m(\lambda) = 0$.

It remains to verify the linear independence of $\{\tilde{\varphi}^m(\lambda)\}_{\tilde{m} \in \text{Path}_\Lambda^{(0 \to 0)}}$. As in the proof of Theorem 8.6, one immediately checks from the definitions that $\tilde{\varphi}^m(1) = -Q_{\Lambda}^\lambda(1; \bar{\Lambda}) v^{(\tilde{m})} \neq 0$ when $\tilde{m} \in \text{Path}_\Lambda^{(0 \to 0)}$. This proves the desired linear independence, and concludes the proof of the theorem.

**Example.** Let $n = 4$, $m = 2$, and $\Lambda_1 = \cdots = \Lambda_4 = 1$. Then $\text{Conf}_\Lambda$ is two-dimensional, as described in the example in Section 8. However, the values $\tilde{\varphi}^{(0,0,1,1)}(\lambda)$ for all $\lambda \in \mathbb{Z}$ are easily seen to be zero when $q = e^{\pi i / 3}$. Therefore, $\text{Conf}_\Lambda^{(2 \to 2)}$ in this case is one-dimensional, and is spanned by the restriction of $\tilde{\varphi}^{(0,0,1,1)}(\lambda)$ to $\mathbb{Z}$.

On the other hand, in the Verlinde algebra, corresponding to $\ell = 3$, we compute

$$[L_1] \cdot [L_1] \cdot [L_1] \cdot [L_1] = [L_0],$$

and the space of Verlinde algebra invariants in this case also has dimension one.

9.5. The qKZB and Macdonald-Ruijsenaars operators on the lattice. The discrete conformal block space $\text{Conf}_\Lambda^{(t)}$ should admit a characterization as the space of solutions of the qKZB and MR equations, restricted to the integral lattice. The Macdonald-Ruijsenaars difference operators involve only integral shifts in the $\lambda$ variable, and the same is true for the qKZB operators because the highest weights $\Lambda$ are integers. Therefore, we can consider the restriction of these operators to cosets $\mathbb{C}/\mathbb{Z}$, i.e. regard them as difference operators on the lattice of the form $\varepsilon + \mathbb{Z}$ with $\varepsilon \in \mathbb{C}$. 

\[\square\]
However, in the most important case $\varepsilon = 0$, the coefficients of operators $\mathbb{K}_j$ and $\mathbb{M}_\Theta$ have poles for small integral values of $\lambda$, and in order to get well-defined difference operators we introduce their modified versions $\tilde{\mathbb{K}}_j$ and $\tilde{\mathbb{M}}_\Theta$ by the following procedure.

Write the Macdonald-Ruijsenaars operator as $\mathbb{M}_\Theta = \sum_{\mu \in \mathbb{Z}} A_\mu(\lambda) T_\mu$ for some meromorphic in $\lambda$ coefficients $A_\mu(\lambda) \in \text{End}(L_\lambda[0])$ with matrix elements $(A_\mu(\lambda))_{m}^{\bar{m}}$, and for each $\delta \in \mathbb{Z}$ define the reduced coefficients $\bar{A}_\mu(\lambda) \in \text{End}(L_\lambda[0])$ with matrix elements

$$(\bar{A}_\mu(\delta))_m^{\bar{m}} = \begin{cases} (A(\delta))_m^{\bar{m}}, & \text{if } (A(\lambda))_m^{\bar{m}} \text{ is well-defined at } \lambda = \delta, \\ 0, & \text{if } (A(\lambda))_m^{\bar{m}} \text{ has a pole at } \lambda = \delta. \end{cases}$$

This reduction procedure makes sense both when $q$ is generic, and when $q$ is a root of unity; in the latter case the matrices $\bar{A}_\mu(\delta)$ are $\ell$-periodic. For each $\varphi : \mathbb{Z} \to L_{\lambda}[0]$ we now set

$$\tilde{\mathbb{M}}_\Theta \varphi(\delta) = \sum_{\mu \in \mathbb{Z}} \bar{A}_\mu(\delta) \varphi(\delta + \mu).$$

The reduced coefficients of $\mathbb{K}_j$ and the modified operator $\tilde{\mathbb{K}}_j$ are defined in a similar fashion.

The above reduction removes the singularities of the qKZB and MR operators in a somewhat $ad\ hoc$ fashion, and in general the eigenfunctions of $\mathbb{K}_j$ and $\mathbb{M}_\Theta$ need not restrict on the lattice to eigensequences of $\tilde{\mathbb{K}}_j$ and $\tilde{\mathbb{M}}_\Theta$. However, the special values of $\lambda$, affected by the reduction procedure, are precisely the values participating in the resonance relations and the vanishing conditions, and it seems plausible that the order of vanishing of $\bar{\vartheta}^{\bar{m}}(\lambda)$ at those points is higher than the order of the removed poles of $\mathbb{K}_j$ and $\mathbb{M}_\Theta$. The latter property would imply that the restrictions $\tilde{\vartheta}^{\bar{m}}(\lambda)$ remain the eigenfunctions of the modified operators:

**Conjecture 9.7.** The sequences $\tilde{\vartheta}^{\bar{m}}(\lambda)$ satisfy the equations

$$\tilde{\mathbb{K}}_j \tilde{\vartheta}(\delta) = \varepsilon^{\bar{m}} (-1; \bar{\Lambda}) \tilde{\vartheta}(\delta), \quad \tilde{\mathbb{M}}_\Theta \tilde{\vartheta}(\delta) = (\dim_q L_\Theta) \tilde{\vartheta}(\delta). \quad (9.5)$$

The results of Section 12 in [FV1] imply that this conjecture holds for the operators $\tilde{\mathbb{K}}_j$ in the case when $\Lambda_1 = \cdots = \Lambda_n = 1$.

Finally, we formulate our last conjecture, which is similar to Conjecture 5.7.

**Conjecture 9.8.** Let $q = \exp\left(\frac{\pi i}{\lambda}\right)$ with $\ell \geq 3$. Let $\tilde{\vartheta} : \mathbb{Z} \to L_{\lambda}[0]$ be a solution of the equations $\tilde{\mathbb{M}}_\Theta \tilde{\vartheta}(\delta) = (\dim_q L_\Theta) \tilde{\vartheta}(\delta)$ such that $\tilde{\vartheta}(\delta + \ell) = (-1)^{m+1} \tilde{\vartheta}(\delta)$. Then $\tilde{\vartheta}(\delta) \in \text{Conf}_{\lambda}^{(\ell)}$.

We illustrate it on the simplest example.

**Example.** Let $n = 2, m = 1$ and $\Lambda_1 = \Lambda_2 = 1$. Suppose that a vector-valued sequence $\varphi(\delta) = (\varphi_0(\delta), \varphi_1(\delta))$, $\delta \in \mathbb{Z}$, satisfies the reduced MR equations. Using the explicit formula for the Macdonald operator in Lemma 8.8 we get

$$M_1 = \begin{pmatrix} |\lambda| & -1 \lambda \lambda^{\lambda-1} \\ \lambda^{\lambda-1} & 0 \lambda \lambda^{\lambda-1} \end{pmatrix} T_1 + \begin{pmatrix} \lambda^{\lambda+1} & 0 \\ \lambda \lambda^{\lambda-1} & 0 \lambda \lambda^{\lambda-1} \end{pmatrix} T_{-1},$$

where $\lambda \in \mathbb{C}$.
and from the definitions we see that
\[ \tilde{M}_1 \varphi(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_{0,1}(1) \\ \varphi_{1,0}(1) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_{0,1}(-1) \\ \varphi_{1,0}(-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]
\[ \tilde{M}_1 \varphi(1) = \begin{pmatrix} \frac{2}{\pi i} & -\frac{1}{\pi i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_{0,1}(2) \\ \varphi_{1,0}(2) \end{pmatrix} + \begin{pmatrix} [2] & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_{0,1}(0) \\ \varphi_{1,0}(0) \end{pmatrix} = \begin{pmatrix} \frac{\varphi_{0,1}(2) - \varphi_{1,0}(2)}{[2]} + [2] \varphi_{0,1}(0) \\ 0 \end{pmatrix}, \]
\[ \tilde{M}_1 \varphi(-1) = \begin{pmatrix} 0 & 0 \\ 0 & [2] \end{pmatrix} \begin{pmatrix} \varphi_{0,1}(0) \\ \varphi_{1,0}(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\frac{[2]}{[2]} & 0 \end{pmatrix} \begin{pmatrix} \varphi_{0,1}(-2) \\ \varphi_{1,0}(-2) \end{pmatrix} = \begin{pmatrix} 2 \varphi_{0,1}(0) - \frac{\varphi_{0,1}(-2) - \varphi_{1,0}(-2)}{[2]} \end{pmatrix}. \]

The equation \( \tilde{M}_1 \varphi(\delta) = [2] \varphi(\delta) \) now implies the vanishing conditions
\[ \varphi_{0,1}(0) = \varphi_{1,0}(0) = \varphi_{0,1}(-1) = \varphi_{1,0}(1) = 0. \]

Let \( \ell = 4 \). The quasi-periodicity shows that our sequence must have the form
\[
\begin{array}{cccccccccccc}
\delta & \ldots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \ldots \\
\varphi_{0,1}(\delta) & \ldots & 0 & 0 & \alpha_1 & \alpha_2 & 0 & 0 & \alpha_1 & \alpha_2 & 0 & 0 & \alpha_1 & \ldots \\
\varphi_{1,0}(\delta) & \ldots & \beta_2 & 0 & 0 & \beta_1 & \beta_2 & 0 & 0 & \beta_1 & \beta_2 & 0 & 0 & \ldots \\
\end{array}
\]
for some \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C} \). Since \( q = \exp\left(\frac{\pi i}{\ell}\right) \), we get \( [2] = \sqrt{2} \) and \( [3] = 1 \), and it is easy to see the Macdonald-Ruijsenaars equation \( \tilde{M}_1 \varphi(\delta) = [2] \varphi(\delta) \) simplifies to
\[
\begin{array}{ccc}
\delta = 1 & \delta = 2 & \delta = 3 \\
2\alpha_1 = \alpha_2 - \beta_1 & 2\alpha_2 = \alpha_1 - \beta_2 & 2\beta_2 = \beta_1 - \alpha_2 \\
\end{array}
\]

These equations have a unique up to proportionality solution, given by \( \alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = -1 \), which coincides with the discrete conformal block \( \tilde{\varphi}^{(0,1)}(\delta) \). Therefore, our Conjecture 9.8 holds in this case.

\[ \square \]

References

[CP] V. Chari, A. Pressley, A guide to quantum groups. Cambridge University Press, Cambridge, 1994.
[CV] O. Chalykh, A. Veselov, Commutative rings of partial differential operators and Lie algebras. Comm. Math. Phys. 126 (1990), no. 3, 597–611.
[D] V. Drinfeld, Almost cocommutative Hopf algebras. Leningrad Math. J. 1 (1990), no. 2, 321–342.
[ESV] P. Etingof, O. Schiffmann, A. Varchenko, Traces of intertwiners for quantum groups and difference equations. Lett. Math. Phys. 62 (2002), no. 2, 143–158.
[ES1] P. Etingof, K. Styrkas, Algebraic integrability of Macdonald operators and representations of quantum groups. Compositio Math. 114 (1998), no. 2, 125–152.
[ES2] P. Etingof, K. Styrkas, Algebraic integrability of Macdonald operators and representations of quantum groups. Compositio Math. 114 (1998), no. 2, 125–152.
[EV1] P. Etingof, A. Varchenko, Traces of intertwiners for quantum groups and difference equations. Duke Math. J. 104 (2000), no. 3, 391–432.
[EV2] P. Etingof, A. Varchenko, Orthogonality and the QKZB-heat equation for traces of \( U_q(\mathfrak{g}) \)-intertwiners. Duke Math. J. 128 (2005), no. 1, 83–117.
[FR] I. Frenkel, N. Reshetikhin, Quantum affine algebras and holonomic difference equations. Comm. Math. Phys. 146 (1992), no. 1, 1–60.
[F] G. Felder, Conformal field theory and integrable systems associated to elliptic curves. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 1247–1255, Birkhäuser, Basel, 1995.
[FTV1] G. Felder, V. Tarasov, A. Varchenko, Monodromy of solutions of the elliptic quantum Knizhnik-Zamolodchikov-Bernard difference equations. Internat. J. Math. 10 (1999), no. 8, 943–975.
[FTV2] G. Felder, V. Tarasov, A. Varchenko, Solutions of the elliptic qKZB equations and Bethe ansatz. I. Topics in singularity theory, 45–75, Amer. Math. Soc. Transl. Ser. 2, 180, Amer. Math. Soc., Providence, RI, 1997.

[TV2] V. Tarasov, A. Varchenko, Small elliptic quantum group $e_{\tau,\gamma}(sl_N)$. Mosc. Math. J. 1 (2001), no. 2, 243–286, 303–304.

[FTV1] G. Felder, A. Varchenko, Resonance relations for solutions of the elliptic QKZB equations, fusion rules, and eigenvectors of transfer matrices of restricted interaction-round-a-face models. Commun. Contemp. Math. 1 (1999), no. 3, 335–403.

[FV2] G. Felder, A. Varchenko, On representations of the elliptic quantum group $E_{\tau,\eta}(sl_2)$. Comm. Math. Phys. 181 (1996), no. 3, 741–761.

[FV3] G. Felder, A. Varchenko, The $q$-deformed Knizhnik-Zamolodchikov-Bernard heat equation. Comm. Math. Phys. 221 (2001), no. 3, 549–571.

[L] G. Lusztig, Introduction to quantum groups. Progress in Mathematics, 110. Birkhäuser Boston, 1993.

[FV4] G. Felder, A. Varchenko, $q$-deformed KZB heat equation: completeness, modular properties and $SL(3,\mathbb{Z})$. Adv. Math. 171 (2002), no. 2, 228–275.

[STV] K. Styrkas, V. Tarasov, A. Varchenko, How to regularize singular vectors and kill the dynamical Weyl group. Adv. Math. 185 (2004), no. 1, 91–135.

[TV1] V. Tarasov, A. Varchenko, Geometry of $q$-hypergeometric functions, quantum affine algebras and elliptic quantum groups, Astérisque 246 (1997) 1–135.