Entropic uncertainty relation based on generalized uncertainty principle

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Abstract

We explore the modification of the entropic formulation of uncertainty principle in quantum mechanics which measures the incompatibility of measurements in terms of Shannon entropy. The deformation in question is the type so called generalized uncertainty principle that is motivated by thought experiments in quantum gravity and string theory and is characterized by a parameter of Planck scale. The corrections are evaluated for small deformation parameters by use of the Gaussian wave function and numerical calculation. As the generalized uncertainty principle has proven to be useful in the study of the quantum nature of black holes, this study would be a step toward introducing an information theory viewpoint to black hole physics.
1 INTRODUCTION

Heisenberg’s Uncertainty Principle (HUP) is one of the most fundamental relations in quantum physics [1]. It is usually stated in Robertson’s form [2],

$$\Delta_\psi A \Delta_\psi B \geq \frac{1}{2} |\langle [A, B] \rangle_\psi|,$$

where $\langle \cdots \rangle_\psi$ is the expectation value with respect to the state $\psi$ and $\Delta_\psi X = \sqrt{\langle X^2 \rangle_\psi - \langle X \rangle_\psi^2}$ is the standard deviation of an observable $X$ with respect to $\psi$. This relation characterizes the complementarity of quantum mechanics; for an incompatible pair of measurements it is impossible to carry out precise measurement simultaneously. It should be noted that the lower bound is state dependent and occasionally the bound becomes meaningless; for example, if the state is an eigenstate of either of the observables. Since uncertainty relation is expected to characterize the incompatibility of measurements, it is thus also important to give a state-independent bound [3]. Along this line, several extensions of the relation have been proposed. Among them, entropic uncertainty relation (EUR) serves a state-independent notion of uncertainty that is characterized by Shannon entropy of the probability distribution as a measure of the imperfection of knowledge. For a finite dimensional case, Maassen and Uffink [4] gave a class of EUR based on a conjecture by Kraus [5], where the most stringent bound is associated with a mutually unbiased pair of observables. For continuum observables, such as the position and momentum which we are interested in, Bialynicki-Birula and Mycielski [6] proposed an EUR based on a mathematical inequality for Fourier transform [7, 8]. Later this uncertainty relation has been refined by taking the resolution of measurement into account [9,10].

On the other hand, it has been suspected that Heisenberg’s uncertainty relation needs to be modified if quantum gravity effect becomes important. In quantum gravity, the space-time itself is subjected to quantum fluctuation and the uncertainty of position measurement is enhanced by further quantum fluctuation. For instance, if we attempt to detect the position of a particle with a light source of ultra high frequency in order to achieve high precision, the energy of the photon will perturb the background geometry and we will not achieve desired accuracy [11]. This argument has been made more precise, through, for example, the analysis of high energy scattering in string theory [12] or the analysis of black holes [13, 14, 15], and the result is encapsulated in the so-called generalized uncertainty principle (GUP). GUP can be regarded as a deformation of HUP with a scale set by the Planck length and is considered to be a phenomenological realization of this extra uncertainty.

It is then natural to ask how EUR should be modified if we consider a quantum gravity effect. In this paper, we explore the correction to EUR for position and momentum observables by taking GUP into account. GUP can be realized as a deformed commutation relation of the position and the momentum which has a Hilbert space representation [19]. With this representation, we may spot the part of wave functions due to GUP in the EUR. To be specified, we start with Gaussian wave functions which saturates EUR bound for the undeformed case. It is thus the deviation from the undeformed case can be considered to well approximate the possible correction terms that come
from the GUP effect.

This paper is organized as follows: In section 2 some overview of EUR is presented. Section 3 introduces GUP and deformed wave functions. In section 4 we consider EUR by use of deformed wave functions and explore the corrections to EUR due to GUP. Finally, conclusion and discussion is given in section 5.

2 Entropic uncertainty relation

Entropic formulation of uncertainty relation is written as

\[ H(A|\psi) + H(B|\psi) \geq c(A, B) \]  

where \( H(A|\psi) \) is Shannon entropy of the probability distribution of the observable \( A \) with respect to a quantum state \( |\psi\rangle \), and \( c(A, B) \) is a state-independent bound. Since Shannon entropy represents ignorance on the distribution, this relation measures an incompatibility of the measurement of \( A \) and \( B \). According to various choices of observables and systems, different types of EUR have been proposed. For a finite dimensional case, we consider observables that have the following spectral decompositions, \( A = \sum_i a_i P_{a_i} \) and \( B = \sum_j b_j P_{b_j} \) where \( P_{a_i} \) and \( P_{b_j} \) are projection operators. The Shannon entropy is defined by \( H(A|\psi) = -\sum_i p_i^{(a)} \log p_i^{(a)} \) with \( p_i^{(a)} = \langle \psi | P_{a_i} | \psi \rangle \) and \( H(B|\psi) \) is also defined in a similar manner. Maassen and Uffink showed that EUR (2.1) holds with \( c(A, B) = -2 \log(\max_{i,j} |\langle a_i | b_j \rangle|) \). It is then the largest lower bound is realized when the inner products of each eigenbasis are \( 1/\sqrt{d} \) with \( d \) being the dimension of the Hilbert space. For observables with continuum spectra, \( \hat{x} \) and \( \hat{p} \), Bialynicki-Birula and Mycielski gave the bound,

\[ \tilde{H}(\hat{x}|\psi) + \tilde{H}(\hat{p}|\psi) \geq \log(e \pi \hbar), \]  

where \( \tilde{H}(\hat{x}|\psi) = -\int dx \rho(x) \log \rho(x) = -\langle \log \rho(x) \rangle_\rho \) and \( \rho(x) = |\psi(x)|^2 \) is the probability density of the position. Similar definition is for \( \tilde{H}(\hat{p}|\psi) \). Later, this bound is refined as

\[ H(x|\psi) + H(p|\psi) \geq \log \left( \frac{\pi \epsilon \hbar}{\delta x \delta p} \right), \]  

where \( \delta x \) and \( \delta p \) are the resolution of the position and the momentum measurement respectively. Similarly, the position (momentum) space is divided into many bins with the width \( \delta x \) (\( \delta p \)), and \( p_i^x \) (\( p_j^p \)) is the probability of finding a particle at the \( i \)-th (\( j \)-th) bin,

\[ p_i^x = \int_{i \delta x}^{(i+1) \delta x} dx \rho(x). \]  

Similarly, \( p_j^p = \int_{j \delta p}^{(j+1) \delta p} dp \rho(p) \) for momentum space. Then the Shannon entropies are calculated via \( H(x|\psi) = -\sum_i p_i^x \log p_i^x \) and \( H(p|\psi) = -\sum_j p_j^p \log p_j^p \). This refined Shannon entropy satisfies the inequality \( H(x|\psi) \geq \tilde{H}(x|\pi) - \ln \delta x \) (the same for \( H(p|\psi) \)), and (2.3) follows. This form nicely

\footnote{Since the Shannon entropy is a convex function, the same inequality holds for a general quantum state \( \hat{\rho} = \sum_i c_i |\psi_i\rangle \langle \psi_i| \) with \( \sum_i c_i = 1. \)
reflects the fact that the actual measurements for position and momentum are restricted by the resolution of measurement apparatus. The bound is not significant if the resolution is classical, namely $\delta x \delta p \gg \hbar$, while it tends to give tight bounds when we consider fine measurements $\delta x \delta p \approx \hbar$. In particular, the bound is saturated by Gaussian wave functions

$$
\psi_0(x) = \left(\frac{\hbar^2}{\pi}\right)^{1/4} e^{-\frac{x^2}{2 \delta x^2}}, \quad \tilde{\psi}_0(p) = \left(\frac{1}{\pi \hbar^2}\right)^{1/4} e^{-\frac{p^2}{2 \delta p^2}},
$$

with fine bin sizes, $\delta x \delta p \ll \hbar$. The difference between the left hand side and the right hand side of (2.3) with respect to the bin size is plotted in Fig. 1. Here, the region $x, p \in [-50, 50]$ is divided into $2N$ bins, and then the bin size is $\delta x, \delta p = 50/N$.

Figure 1: The difference between the left hand side and the right hand side of (2.3) for Gaussian wave functions (2.5) (with $\hbar = 1$). The horizontal label corresponds to $N = 10 \times n + 1$.

### 3 Generalized uncertainty principle

As described in Sec 1, GUP is a deformation of the standard Heisenberg uncertainty principle. In general, GUP is characterized by the relation,

$$
\Delta x \Delta p \geq \frac{\hbar}{2} (1 + \alpha (\Delta x)^2 + \beta (\Delta p)^2 + \gamma),
$$

where $\alpha$, $\beta$, and $\gamma$ are positive deformation parameters. They are independent of $\Delta x$ and $\Delta p$, but may depend on the expectation values $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$. GUP is motivated by gedanken experiments in quantum gravity, such as loop quantum gravity, string theory, and black hole physics, and in those cases GUP is represented by a restricted relation,

$$
\Delta x \Delta p \geq \frac{\hbar}{2} (1 + \beta (\Delta p)^2 + \gamma),
$$

and the values of the parameters depend on the characteristic scale of underlying theory of gravity. This form can be understood as the result of a standard Robertson uncertainty relation,

$$
\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|,
$$

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in which the canonical commutation relation is deformed,

$$[\hat{x}, \hat{p}] = i\hbar(1 + \beta \hat{p}^2),$$

and $$\gamma = \beta(\hat{p})^2$$ is chosen.

The minimum uncertainty in $$x$$ is $$\Delta x_{\text{min}} = \hbar \sqrt{\beta}$$ which is realized for $$\langle p \rangle = 0$$ states (thus $$\gamma = 0$$). This corresponds to the case with $$\Delta p = \beta^{-1/2}$$ and $$\Delta x_{\text{min}} \Delta p = \hbar$$. Hence there does not exist the eigenstate of the position operator $$\hat{x}$$; not even an approximated normalizable state. We thus work in the momentum space, $$\hat{p} |p\rangle = p |p\rangle$$. The Hilbert space representation is constructed in [19]. On the momentum space wave function $$\psi(p) = \langle p | \psi \rangle$$, the operators are realized as $$\hat{p} \psi(p) = \psi(p)$$ and $$\hat{x} \psi(p) = i\hbar(1 + \beta p^2) \partial_p \psi(p)$$. In order for $$\hat{x}$$ and $$\hat{p}$$ operators to be symmetric, a momentum factor is involved in the definition of scalar product and the completeness condition, that is

$$\langle \psi | \phi \rangle = \int_{-\infty}^{\infty} \frac{dp}{1 + \beta p^2} \psi^*(p) \phi(p),$$

and

$$1 = \int \frac{dp}{1 + \beta p^2} |p\rangle \langle p|.$$

### 3.1 Gaussian wave function in momentum space

We start with the Gaussian wave function in momentum space $$\tilde{\psi}_0(p)$$ in [25] for undeformed ($$\beta = 0$$) case. The expectation value of $$p$$ is $$\langle \hat{p} \rangle = 0$$ and the standard deviation is $$\Delta p = \sqrt{\langle (p - \langle p \rangle)^2 \rangle} = \hbar/\sqrt{2}$$. Now we consider the GUP case and define a “position-space” wave function. Before we proceed, it should be noted that the momentum distribution function is defined as $$\rho(p) = |\psi_0(p)|^2/(1 + \beta p^2)$$ due to the normalization condition $$\langle \psi_0 | \psi_0 \rangle = 1$$ in (3.5). Namely, with GUP, the momentum space wave function is Gaussian but the distribution is not.

For a normalized state $$\langle \psi | \psi \rangle = 1$$, the corresponding momentum space Gaussian wave function with GUP reads

$$\tilde{\psi}_\beta(p) = \langle p | \psi \rangle = F(\beta)e^{-\frac{\sigma^2}{2\beta}}$$,

where $$\sigma$$ is a parameter for the width (more precisely, the half width is given by $$2\sqrt{2\ln 2\hbar/\sigma}$$), and $$F(\beta)$$ is the normalization factor,

$$F(\beta) = e^{-\frac{\sigma^2}{2\beta} \pi^{-1/2} \beta^{1/4}} \left[ \text{erfc} \left( \frac{\sigma}{\hbar \sqrt{\beta}} \right) \right]^{-1/2},$$

and $$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} dt e^{-t^2}$$ is the complementary error function. Note that $$F(\beta)$$ is monotonically increasing in $$\beta$$ and $$F(0) = (\sigma^2/\pi \hbar^2)^{1/4}$$. We then define the probability distribution of momentum as

$$\rho_\beta(p) = \frac{|\tilde{\psi}_\beta(p)|^2}{1 + \beta p^2}, \quad \int_{-\infty}^{\infty} dp \rho_\beta(p)^2 = 1.$$

With this probability distribution, the average $$\langle p \rangle = 0$$ remains but the deviation $$\Delta p$$ is altered. This will affect the position uncertainty as we shall see later.
3.2 Pseudo-position space wave function

In [19], a pseudo-position eigenstate $|\psi^\text{ML}_\xi\rangle$ is constructed and its momentum space representation is

$$\psi^\text{ML}_\xi(p) = \sqrt{\frac{2\sqrt{\beta}}{\pi}}(1 + \beta p^2)^{-1/2} \exp \left[ -i \frac{\xi \arctan(\sqrt{\beta}p)}{\hbar\sqrt{\beta}} \right].$$  (3.10)

This state has minimum uncertainty, $\langle \hat{x} \rangle = \xi$ and $\Delta x = \Delta x_{\text{min}} = \hbar\sqrt{\beta}$. By using this, a position space wave function is obtained from a momentum space one as

$$\psi(\xi) = \frac{1}{\sqrt{2\beta}} \int_{-\infty}^{\infty} \tilde{\psi}_{\beta}(p) dp \left( \frac{1 + \beta p^2}{\hbar^2} \right)^{3/2} \exp \left[ i \frac{\xi \arctan(\sqrt{\beta}p)}{\hbar\sqrt{\beta}} \right].$$  (3.11)

For a suitably normalized momentum space wave function $\int \frac{dp}{1 + \beta^2 p^2} |\tilde{\psi}(p)|^2 = 1$, this $\psi(\xi)$ is not normalized, instead

$$\int_{-\infty}^{\infty} d\xi |\psi(\xi)|^2 = 4\sqrt{\beta} \int_{-\infty}^{\infty} |\tilde{\psi}(p)|^2 dp \left( \frac{1 + \beta p^2}{\hbar^2} \right)^2. \quad (3.12)$$

Namely, $\xi$ integration (or the completeness of $|\psi^\text{ML}_\xi\rangle$) needs to involve a $p$ dependent factor $(4\sqrt{\beta})^{-1}(1 + \beta p^2)$. In order to define the probability distribution, we include this factor and use, instead of (3.11), the following pseudo-position space wave function:

$$\psi_{\beta}(\xi) = \frac{1}{\sqrt{2\pi h}} \int_{-\infty}^{\infty} \tilde{\psi}_{\beta}(p) dp \left( \frac{1 + \beta p^2}{\hbar^2} \right)^{3/2} \exp \left[ i \frac{\xi \arctan(\sqrt{\beta}p)}{\hbar\sqrt{\beta}} \right].$$  (3.13)

We define the probability distribution in $\xi$ space as

$$\rho_{\beta}(\xi) = |\psi_{\beta}(\xi)|^2, \quad \int_{-\infty}^{\infty} d\xi \rho_{\beta}(\xi) = 1.$$  (3.14)

We next examine the lower bound of the uncertainty in the pseudo-position space $\xi$ for the Gaussian wave function. In the momentum space, the deviation for $\tilde{\psi}_{\beta}$ is

$$\Delta p = \frac{1}{\sqrt{\beta}} \sqrt{1 - \frac{\hbar \sqrt{\pi}}{\sigma} F(\beta)^2}. \quad (3.15)$$

From GUP, the uncertainty for Gaussian wave function reads $\Delta x_{\text{min, Gaussian}} = \frac{\hbar}{2\Delta p}(1 + \beta(\Delta p)^2)$. $\beta$ dependence of the minimum value, while the inequality is saturated, is shown in Fig. 2.

4 Generalized Uncertainty correction to entropic uncertainty relation

In this section, we explore corrections to the entropic uncertainty relation due to GUP for the case of the Gaussian wave function.

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2 As we discuss in Section 4.1.1, this wave function is related to a wave function in an auxiliary momentum space.
The entropic uncertainty relation is based on the following probability distributions and the Shannon entropies,

\[
H(p|\psi) = -\sum_i p_i \ln p_i, \quad p_i = \int_{i}^{i+\delta p} dp \rho_\beta(p),
\]

\[
H(\xi|\psi) = -\sum_i q_i \ln q_i, \quad q_i = \int_{i}^{i+\delta \xi} d\xi \rho_\beta(\xi),
\]

where \(\rho_\beta(p)\) and \(\rho_\beta(\xi)\) are given by (3.9) and (3.14) respectively. For sufficiently small \(\beta\), the sum of these entropies will be bounded by the standard bound with a correction term depending on \(\beta\),

\[
H(\xi|\psi) + H(p|\psi) \geq \log \left( \frac{\pi e h}{\delta x \delta p} \right) + K(\beta; \delta \xi, \delta p).
\]

The right hand side is expected to be a state independent bound. Since in the undeformed case \((\beta = 0)\) the Gaussian wave function is to saturate the bound for sufficiently small bin sizes, we expect that the difference of the left hand side from the undeformed result approximates the correction function \(K(\beta; \delta \xi, \delta p)\),

\[
H_{\text{tot}} - H_{\text{tot}}|_{\beta=0} \approx K(\beta; \delta \xi, \delta p),
\]

where \(H_{\text{tot}} = H(\xi|\psi) + H(p|\psi)\). As we have seen, the difference is almost saturated around \(\delta \xi = \delta p = 0.5\) for the undeformed case, we may fit the correction function for small values of \(\beta\).

These \(\beta\)-deformed Shannon entropy of the position and the momentum are plotted for various \(\beta\) and bin sizes in Fig. 3. Here, the interval from \(-50\) to \(50\) is divided into \(2N\) bins. Thus, the bin size \(\delta \xi\) and \(\delta p\) is \(50/N\).

To see the correction due to \(\beta\), we plot the difference of Shannon entropies between \(\beta\)-deformed and undeformed cases. \(H(\xi|\psi) - H(\xi|\psi)|_{\beta=0}, \ H(p|\psi) - H(p|\psi)|_{\beta=0}\), and \(H_{\text{tot}} - H_{\text{tot}}|_{\beta=0}\) are shown in Fig. 4, Fig. 5 and Fig. 6 respectively. The horizontal axis represents \(\beta/25\); namely plot region is for \(0 \leq \beta \leq 2.0\). The bin size is chosen as \(\delta x = \delta p = \frac{50}{N}\) and \(N = 10, 20, \ldots, 100\) are shown. For small bin size, \(N = 80, 90,\) and \(100\), the \(\beta\) dependence tends to be convergent (we have checked up to \(N = 300\) that the \(N = 100\) line is sufficiently stable for \(\beta \leq 2.0\)).
The $\beta$-deformed Shannon entropy. Left is $H(\xi|\psi)$ and the right is $H(p|\psi)$. $N$ is from 10 to 100 by step 10. $\beta$ is from $1/25 = 0.04$ to $50/25 = 2$ by step 0.04.

Figure 4: $H(\xi|\psi) - H(\xi|\psi)|_{\beta=0}$ against $\beta$ for $N = 10$ (dark blue) to $N = 100$ (red). The horizontal axis is $\beta/25$.

The $\beta$ dependence appears in an opposite way between the difference of Shannon entropy for the position operator and that for the momentum operator. The pseudo-position wave function $\psi_\beta$ increases uncertainty in position as $\beta$ gets bigger, since its minimum position uncertainty $\Delta x_{\text{min, Gaussian}}$ is an increasing function of $\beta$ as shown in Fig. 2. On the other hand, in the momentum distribution function $\rho_\beta(p) = |\psi_0(p)|^2/(1 + \beta p^2)$, the $\beta$ dependence is only in the factor $1/(1 + \beta p^2)$. As $\beta$ increases, this factor tends to squeeze up $\rho(p)$ near $p = 0$, and the uncertainty gets smaller. Therefore, it is natural that the Shannon entropy for the position operator is an increasing function of $\beta$ while that for the momentum operator is a decreasing one. The sum, $H_{\text{tot}} = H(\xi|\psi) + H(p|\psi)$, is a monotonically increasing function of $\beta$. This sum is to measure the incompatibility of the measurements of $x$ and $p$, and should be related to $\Delta x\Delta p$. For the $\beta$-deformed Gaussian wave functions, $\Delta x\Delta p = \frac{\hbar^2\sqrt{\pi}}{2\sigma} F(\beta)^2$. $F(\beta)$ is a monotonically increasing

Figure 5: $H(p|\psi) - H(p|\psi)|_{\beta=0}$ against $\beta$ for $N = 10$ (dark blue) to $N = 100$ (red). The horizontal axis is $\beta/25$. 

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Figure 6: \( H_{\text{tot}} - H_{\text{tot}}|_{\beta=0} \) against \( \beta \) for \( N = 10 \) (dark blue) to \( N = 100 \) (red). The horizontal axis is \( \beta/25 \).

function of \( \beta \) and so is \( \Delta x \Delta p \). We thus again observe that the \( \beta \) dependence is consistent with what we anticipate from the variance type GUP observation.

4.1 Evaluation of the correction term

Having obtained the results for some values of \( \beta \), we will fit the points \( (N = 100 \text{ case}) \) by a function of \( \beta \). Since the plot suggests a concave function, we may use a power function of \( \beta \), \( \beta^b \), and the result is

\[
K(\beta; 0.5, 0.5) \sim \begin{cases} 
\beta^{0.772054} & (\beta \leq 1.0) \\
\beta^{0.654778} & (\beta \leq 2.0) 
\end{cases}.
\]

(4.5)

It turns out that, even for such small regions for \( \beta \), the fitting by a polynomial of \( \beta \) (to tenth order) does not give a stable result.

We have also tried to fit a result with even larger \( \beta \); \( 0 \leq \beta \leq 10 \) with \( N = 300 \) and the bin size 0.5 (namely the integration region is taken \( \xi, p \in [-150, 150] \)). In this case, the fitting of a type \( a\beta \ln \beta + b(\beta^c) \) turns out to be fairly good,

\[
K(\beta; 0.5, 0.5) \simeq -0.204103\beta \ln \beta + 0.32293\beta^{1.24049}.
\]

(4.6)

These fittings are shown in Fig. 7. However, for large values of \( \beta \) (say \( \beta \geq 3.0 \)) it would be necessary to use finer bin sizes to saturate the bound since the data points may not be sufficiently close to the saturation of the bound.

Figure 7: Fitting for the correction term \( K(\beta; 0.5, 0.5) \): (Left) by \( \beta^{0.654778} \) for \( \beta \leq 2.0 \). (Right) \(-0.204103\beta \ln \beta + 0.32293\beta^{1.24049} \) against the data points \( \beta \leq 10 \).
4.1.1 Using auxiliary momentum variable

In the paper [24], the entropic uncertainty relation has been investigated by use of an auxiliary momentum variable

\[ p = \frac{1}{\sqrt{\beta}} \tan(\sqrt{\beta}q), \tag{4.7} \]

where \( x \) and \( q \) satisfy the standard Heisenberg algebra, \([x, q] = i\hbar\). Note that \( \tilde{\psi}(q) \) is defined on a finite interval \(-q_0 \leq q \leq q_0\) with \( q_0 = \pi/2\sqrt{\beta} \). Thus, the wave function \( \psi(x) \) and \( \tilde{\psi}(q) \) form the standard Fourier pair,

\[ \psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-q_0}^{q_0} dq \psi(q) e^{i\frac{qx}{\hbar}}. \tag{4.8} \]

In terms of the physical momentum \( p \), this relation reads

\[ \psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp \psi(p) \exp \left[ \frac{x \arctan(\sqrt{\beta}p)}{\hbar\sqrt{\beta}} \right], \tag{4.9} \]

where \( \psi(p) = \psi(q)|_{q=\beta^{-1/2}\arctan(\sqrt{\beta}p)} \). This is nothing but the modified wave function in the pseudo-position space (3.13); namely, the wave function in the pseudo-position space can be viewed as the conventional Fourier transform of the auxiliary momentum space wave function. We thus rename the variable \( x \) as \( \xi \) and look at the correction term.

Since \( \psi(\xi) \) and \( \psi(q) \) are usual Fourier pair, the Shannon entropies based on the probability distributions from these wave functions satisfy Bialynicki-Birula and Mycielski type entropic uncertainty relation

\[ \tilde{H}(\xi|\psi) + \tilde{H}(q|\psi) \geq \log(\pi e). \tag{4.10} \]

where \( \tilde{H}(q|\psi) = -\langle \log \tilde{\rho}(q) \rangle_{\tilde{\rho}(q)} \) is calculated by use of a probability distribution \( \tilde{\rho}(q) \). We may relate this with the distribution in \( p \) space by \( \rho(p)dp = \tilde{\rho}(q)dq \), which implies \( \tilde{\rho}(q) = (1 + \beta p^2)\rho(p) \). Here \( \rho(p)dp \) and \( \tilde{\rho}(q)dq \) are normalized in the intervals \(-\infty < p < \infty\) and \(-q_0 \leq q \leq q_0\) respectively. Thus, the Shannon entropy for \( \rho(p) \) can be evaluated as

\[ \tilde{H}(p|\psi) = -\int_{-\infty}^{\infty} dp \rho(p) \log \rho(p) = -\int_{-\infty}^{\infty} dp \rho(p) (\log \tilde{\rho}(q) - \log(1 + \beta p^2)) \]

\[ = \tilde{H}(q|\psi) + \langle \log(1 + \beta p^2) \rangle_{\rho(p)}. \tag{4.11} \]

By introducing bins for \( \xi \) and \( p \) spaces, we obtain the refined entropic uncertainty relation with a correction term \[25,\]

\[ H(\xi|\psi) + H(p|\psi) \geq \log \left( \frac{\pi e\hbar}{\delta x \delta p} \right) + \langle \log(1 + \beta p^2) \rangle_{\rho}. \tag{4.12} \]

\[ ^{3}\text{It should be noted that this form is different from the wave function in the position space of minimum uncertainty (3.11) originally proposed by Kempf et al. [19].}\]
We may identify the extra term on the right hand side as the correction term $K$

$$K(\beta) = \alpha \langle \log (1 + \beta p^2) \rangle_\rho, \quad (4.13)$$

where $\alpha$ is an overall normalization factor. Note that the correction term here is state dependent and is not exactly what we seek for. It also does not depend on the bin size. However, we may expect that the correction term evaluated with the Gaussian wave function. By tuning $\alpha$, we fit the numerical data of the Shannon entropy by $K(\beta)$ numerically in Fig. 8. As seen, it provides a rather nice correction term.

![Figure 8](image)

Figure 8: Fitting for the correction term $K(\beta)$ (4.13) with $\alpha = 1.39$ for $\beta \leq 2.0$ (Left) and with $\alpha = 1.48$ for $\beta \leq 10$ (Right). Here $\sigma = \hbar = 1$.

Though $K(\beta)$ can only be evaluated numerically, there is an analytic expression for a correction term. From Jensen’s inequality, we have $\log (1 + \beta \langle p^2 \rangle_\rho) \geq \langle \log(1 + \beta p^2) \rangle_\rho$. The left hand side, $\tilde{K}(\beta) = \alpha \log (1 + \beta \langle p^2 \rangle_\rho)$, can be evaluated analytically as

$$\tilde{K}(\beta) = \alpha \log \left[ 1 + \frac{1}{\sqrt{\beta}} \sqrt{-1 + \frac{\hbar \sqrt{\pi} F(\beta)^2}{\sigma}} \right], \quad (4.14)$$

where a normalization factor $\alpha$ is introduced again. Since $\tilde{K}(\beta) \geq K(\beta)$, this correction term would overestimate the actual bound, but it turns out that $\tilde{K}(\beta)$ fits the numerical result nicely, especially for small $\beta$, as shown in Fig. 9.

![Figure 9](image)

Figure 9: Fitting for the correction term $\tilde{K}(\beta)$ (4.14) with $\alpha = 1.161$ for $\beta \leq 2.0$ (Left) and with $\alpha = 1.106$ for $\beta \leq 10$ (Right). Here $\sigma = \hbar = 1$. 

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5 Conclusion and discussion

In this paper, we explore the correction to the entropic uncertainty relation due to generalized uncertainty principle for small value of the deformation parameter $\beta$. The Gaussian wave function is used since it tends to saturate the standard bound of entropic uncertainty relation for sufficiently fine bin sizes. The correction due to GUP is represented as a deformation of the wave functions, and the Shannon entropies for these $\beta$ deformed probability distributions are calculated numerically. We first fit the result by some simple functions of $\beta$; for small $\beta$, we obtain $K(\beta; \delta \xi, \delta p) \sim \beta^{0.7}$ as an approximated correction term. a state dependent correction term based on an auxiliary momentum variable $[24, 25]$, for the Gaussian wave function.

We have used a pair of Gaussian wave functions in position and momentum representation. In the undeformed case ($\beta = 0$), the EUR saturates the bound $[2,3]$ for sufficiently fine bin sizes. The deformation alters both momentum and position (or more precisely, pseudo-position) probability distributions. The behavior of the Shannon entropies for these distributions are consistent with consideration based on GUP; namely, the correction term $K(\beta; \delta \xi, \delta p)$ appears as a monotonically increasing function of $\beta$, which is anticipated from $\Delta x \Delta p$ in GUP. We also consider an auxiliary momentum variable $q$ that satisfies the standard Heisenberg algebra with $x$ $[24]$ and then the wave functions are related by the usual Fourier transform. Putting $q$ back to the physical momentum, we find that the relation between the wave functions of physical momentum and pseudo-position space is naturally reproduced. This procedure also provides a state-dependent correction term. This correction term is evaluated with Gaussian wave function, we find that this correction term fits the numerical result very well.

So far, the corrections to the bound is investigated numerically. It is obviously important to explore the analytic form of the correction to understand the property of quantum gravity through entropic entropy. For example, in a finite dimensional system, an explicit map between a variance type uncertainty relation and an entropic type of it have been constructed $[21]$. It is then intriguing to investigate a similar type of mappings in our case$^4$. Out method and result serve a first hint toward this direction.

GUP implies various modification of the consequence of standard theory of gravity: for example the thermodynamic properties of black holes $[14, 15, 16, 17, 18]$, inflationary cosmology $[22]$, and the entropic bound $[23]$. It should be interesting to reformulate these kinds of modification in terms of the deformed entropic uncertainty relation.

At the final stage of the first version of this work, we noticed a paper on entropic uncertainty relation with a minimal length $[24]$ which addresses a similar question but uses a different approach and another paper $[25]$ appeared later to to analyze EUR based on $[24]$. In Sec. 4.1.1, we discuss their formulation in our setup. A related work $[26]$ showed up too, which deals with the optimal bound for both variance and entropic uncertainty relations in the presence of minimal length. Compared to ours, their paper argues more detailed bound for the sum of entropy functions, but

$^4$We are informed that there has been a work that demonstrates an explicit connection between the entropic and variance based uncertainty relations $[20]$. 

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they use a special binning for $p$ space (we take $p$ as physical momentum and set up a natural binning for $p$).

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