ON SPECIAL $p$-BOREL FIXED IDEALS

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Abstract. We define the reduced horseshoe resolution and the notion of conjoined pairs of ideals in order to study the minimal graded free resolution of a class of $p$-Borel ideals and recover Pardue’s regularity formula for them. It will follow from our technique that the graded betti numbers of these ideals do not depend on the characteristic of the base field $k$.

1. Introduction

The study of $p$-Borel fixed ideals is a very interesting and fascinating problem. One could safely argue that in characteristic $p$ very few results are known, in contrast to the case of characteristic zero, where we can describe a minimal graded free resolution of any Borel fixed ideal, determine its regularity, find its graded Betti numbers and more.

It was conjectured in [16] that the regularity $\text{reg}(I)$ of a principal $p$-Borel ideal $I$ is equal to the maximum of some numbers given by a rather complicated formula. In [1] it was proved that $\text{reg}(I)$ is larger than or equal to this maximum, while in [11] the authors prove the opposite inequality (see also [12]).

Another known result is the computation in [9] of the Koszul homology of some special $p$-Borel ideals, which we shall define below, while a more recent result is their CW-resolution given in [14] using algebraic discrete Morse theory. In both papers there are proofs of a formula that gives their regularity, which agrees with Pardue’s formula for principal $p$-Borel ideals.

Here we show how one can use the horseshoe lemma to get the form of the minimal graded free resolution of these special $p$-Borel ideals in an elementary way. Furthermore, we verify Pardue’s regularity formula at the same time. Our idea was born from the observation of the Betti diagrams of several such ideals in MACAULAY 2 [10].

This paper is organized as follows:

In Section 2, we introduce the reduced horseshoe resolution, which will help us deduce the minimal graded free resolution of $R/IJ$ from the minimal

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resolutions of $R/I$ and $R/J$, when the pair $(I, J)$ of ideals in $R$ satisfies certain properties.

In Section 3, we study the minimal graded free resolution over $R$ of a class of ideals, which we call special. Let

$$I = \prod_{j=1}^{s} I_{j}^{[p_j]},$$

where

$$I_{j} = (x_1, x_2, \ldots, x_{\ell_j})^{a_j}$$

and

$$I_{j}^{[p_j]} = (x_1^{p_j}, x_2^{p_j}, \ldots, x_{\ell_j}^{p_j})^{a_j}$$

with

$$n = \ell_1 \geq \ell_2 \geq \ldots \geq \ell_s \geq 1 \quad \text{and} \quad 0 \leq a_j < \frac{p_j+1}{p_j},$$

where the numbers $\frac{p_j+1}{p_j}$ are integers $> 1$ for $j = 1, \ldots, s$.

We call such ideals special. In particular, if $p_j = p^{r_j}$ for $j = 1, 2, \ldots, s$, where $r_s > \ldots > r_2 > r_1 \geq 0$ and $p$ is prime, we call them special $p$-Borel ideals.

In Section 4, we construct a polyhedral cell complex that supports a minimal free resolution of some special ($p$-Borel) ideals.

Finally, in section 5, we examine the iterated mapping cone construction.

2. Reduced Horseshoe Resolution and Conjoined Pairs of Ideals

All ideals in this paper are considered to be monomial ideals. We work over the polynomial ring $R = k[x_1, x_2, \ldots, x_n]$. For small $n$ we may use the letters $a, b, c, d, \ldots$ instead of $x_1, x_2, x_3, x_4, \ldots$.

Let $A \subset B$ be two ideals in $R$ and assume that the minimal graded free resolutions of $A/B$ and $A$ are of the form

$$0 \longrightarrow G_m \xrightarrow{d_m'} G_{m-1} \longrightarrow \cdots \longrightarrow G_2 \xrightarrow{d_2'} G_1 \xrightarrow{d_1'} F_1 \xrightarrow{\epsilon_0'} A/B \longrightarrow 0$$

and

$$0 \longrightarrow F_n \xrightarrow{d_n'} F_{n-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{d_2'} F_1 \xrightarrow{d_1'} R \xrightarrow{\epsilon_0''} R/A \longrightarrow 0$$

Let $F_1 = \bigoplus_{i=1}^{b_i} R(-\alpha_{i,k})$ for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, b_i$, and set

$$m_i(A) = \min\{\alpha_{i,k}| k = 1, 2, \ldots, b_i\}$$

$$M_i(A) = \max\{\alpha_{i,k}| k = 1, 2, \ldots, b_i\}$$

for $i = 1, 2, \ldots, n$. Then the horseshoe lemma associated with the following short exact sequence

$$0 \longrightarrow A/B \xrightarrow{\psi} R/B \xrightarrow{\phi} R/A \longrightarrow 0$$
gives us a free resolution of $R/B$,

\[
\begin{array}{cccccccc}
0 & & d_1 & & d_2 & & d_3 & & \cdots \\
\downarrow \psi_3 & & \downarrow \psi_2 & & \downarrow \psi_1 & & \downarrow \psi_0 & & \cdots \\
\vdots & & \downarrow \phi_3 & & \downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \phi & & \cdots \\
G_3 & \rightarrow & G_2 & \rightarrow & G_1 & \rightarrow & F_1 & \rightarrow & A/B \\
\end{array}
\]

The differential map $\epsilon_0 : F_1 \oplus R \rightarrow R/B$ is defined by $\epsilon_0(x, y) = \psi\epsilon_0(x) + \pi(y)$ for $x \in F_1$ and $y \in R$, while the maps $d_k : G_k \oplus F_{k-1} \rightarrow G_{k-1} \oplus F_{k-1}$ for $k > 1$ are given by the following matrix

\[
d_k = \begin{bmatrix} d_k & \lambda_k \\ 0 & d_k \end{bmatrix},
\]

where the maps $\lambda_k$ are the ones denoted by the dashed arrows in the above commutative diagram. Moreover, the above maps must satisfy the following conditions (see, e.g., [5], p.79-80)

\[
\epsilon''_0 = \phi\pi,
\]

\[
\psi\epsilon_0\lambda_1 + \pi d''_1 = 0 \quad \text{and} \quad d''_{k-1}\lambda_k + \lambda_{k-1}d''_k = 0 \quad \text{for} \quad k > 1.
\]

We may assume that $\epsilon'_0 := \pi|_Ad'_1$, because $\text{Im}(d'_1) = A$. In order to define $\lambda_1$, we choose a basis element $e$ of $F_1$. Then $\pi d'_1(e) = d'_1(e) + B$ and

\[
\psi\epsilon_0\lambda_1(e) = \psi(\pi|_A d''_1(\lambda_1(e))) = d'_1(\lambda_1(e)) + B.
\]

Thus, we need only make sure that $d'_1(e) + d''_1(\lambda_1(e))$ is in $B$. Accordingly, we may define the map $\lambda_1 : F_1 \rightarrow F_1$ such that $\lambda_1(e) = -e$. Hence, the above horseshoe resolution of $R/B$ is certainly not minimal. This leads us to define the reduced horseshoe resolution of $R/B$.

**Definition 1.** Let $A$ and $B$ be two ideals in $R$ as above. Then the complex

\[
\begin{array}{cccccccc}
\cdots & \rightarrow & G_3 \oplus F_3 & \rightarrow & G_2 \oplus F_2 & \rightarrow & G_1 & \rightarrow & R/B \\
\end{array}
\]

is called the reduced horseshoe resolution of \( R/B \) with respect to \( A \).

It is easy to verify that this is a complex. Indeed, note that \( \pi d_1' \lambda_1^{-1} d_1' = -\psi c_0 d_1' = 0 \), because \( c_0 d_1' = 0 \), while \( d_1' \lambda_1^{-1} d_1' [d_2' \lambda_2] = [d_1' \lambda_1^{-1}(d_1' d_2') - d_1' \lambda_1^{-1}(d_1' \lambda_2)] = [0, -d_2' d_2'] = [0, 0] \).

The other relations follow immediately from the fact that \( d_k d_{k+1} = 0 \) for \( k > 1 \).

Remarks 1.

(a) Although the horseshoe resolution of \( R/B \) is not minimal, there is a chance that its reduced horseshoe resolution is. For this to be true, the rank of the free \( R \)-module \( G_1 \) should be equal to the number of the minimal generators of \( B \) and the matrices that represent the maps \( \lambda_k \) for \( k > 1 \) should not have any nonzero constant entry. We are already sure that the matrices that represent the maps \( d_k' \) and \( d_k'' \) do satisfy the latter condition because we started with minimal resolutions of \( R/A \) and \( A/B \).

(b) If \( A = I \) and \( B = IJ \) for some ideal \( I \) and \( J \) in \( R \), we will talk about the reduced horseshoe resolution of \( R/IJ \) with respect to the ordered pair \( (I, J) \). More generally, if \( I_1, ..., I_k \) \( (k > 1) \) are ideals in \( R \), such that we know the minimal graded free resolutions of \( R/I_k \) and \( I_k/I_{k-1}...I_1 \), we will talk about the reduced horseshoe resolution of \( R/(I_k...I_2I_1) \) with respect to the ordered \( k \)-tuple \( (I_k, ..., I_2, I_1) \).

Let \( G(I) \) denote the unique minimal set of monomial generators of a (monomial) ideal \( I \).

Definition 2. We call an ordered pair \( (I, J) \) of monomial ideals \( I \) and \( J \) in \( R \) conjoined, if the following conditions are satisfied:

(i) \( |G(IJ)| = |G(I)||G(J)| \).

(ii) There is a minimal presentation of \( I \),

\[
R^s \overset{\phi}{\longrightarrow} R^t \overset{\psi}{\longrightarrow} I \longrightarrow 0
\]

such that all the entries of the matrix \( \phi \) belong to \( J \).

Example 1. Let \( I = (a^2, b^2, c^2) \) and \( J = (a, b, c) \) in \( R = k[a, b, c] \). Then \( IJ = (a^3, a^2b, ab^2, b^3, a^2c, b^2c, ac^2, bc^2, c^3) \) and a minimal presentation of \( I \) is

\[
\begin{pmatrix}
  b^2 & -c^2 & 0 \\
  -a^2 & 0 & -c^2 \\
  0 & a^2 & b^2
\end{pmatrix}
\]

Hence the pair \( (I, J) \) is conjoined.

The following lemma, which was inspired by lemma 2.2 of [9], gives us a systematic way of constructing conjoined pairs of ideals.
Lemma 3. Let $I$ and $J$ be two monomial ideals in $R = k[x_1, x_2, ..., x_n]$ such that every element of $G(I)$ has degree $d_1$ and every element of $G(J)$ has degree $d_2$. Suppose that $(x_1^{k-1}, ..., x_m^{k-1}) \subset J$ for some integer $k > 1$, where $m = \max\{i | x_i \text{ divides a minimal generator of } I\}$. Then the pair $(I^k, J)$ is conjoined, where $I^k$ is the ideal generated by $\{u^k : u \in G(I)\}$.

Proof. We first show that

$$G(I^k) = \{u^k v : u \in G(I), v \in G(J)\}.$$ 

Certainly, the set on the right hand side above is a generating set of $I^k J$. We need to show that it is minimal. Assume that there are $u \in G(I)$ and $v \in G(J)$ such that $u^k v$ is not a minimal generator. Then, we should have

$$u^k v = u'^k v' w$$

for some monomial $w$. This is impossible, because the above relation together with $\deg(u) = \deg(u')$ and $\deg(v) = \deg(v')$ implies that $\deg(w) = 0$. Since $(x_1^{k-1}, ..., x_m^{k-1}) \subset J$, the degree $d_2$ of every element in $G(J)$ is less than or equal to $k - 1$. Now if $\alpha_i$ and $\alpha'_i$ denote the largest integers such that $x_i^{\alpha_i}$ divides $v$ and $x_i^{\alpha'_i}$ divides $v'$, then $0 \leq |\alpha_i - \alpha'_i| \leq d_2 < k$ for each $i$. Since $k$ divides $|\alpha_i - \alpha'_i|$, we have $\alpha_i = \alpha'_i$ for each $i$ and so $v = v'$. Accordingly, we also have $u = u'$, and so

$$|G(I^k) J| = |G(I^k)| |G(J)|.$$

Now consider a minimal presentation of $I$,

$$R^s \xrightarrow{(a_{i,j})} R^t \xrightarrow{I} 0$$

where all the entries $a_{i,j}$ belong to $(x_1, ..., x_m)$. Then

$$R^s \xrightarrow{(a_{i,j})} R^t \xrightarrow{I^k} 0$$

is a minimal presentation of $I^k$ and all the entries $a_{i,j}^k$ belong to $(x_1^k, ..., x_m^k) \subset J$. Clearly, this does not depend on the characteristic of the base field $k$.

Next, the following lemma gives a sufficient condition for the minimality of the reduced horseshoe resolution of $R/IJ$ (with respect to $I$). Recall that $m_i(A)$ (resp. $M_i(A)$) is the minimum (resp. maximum) shift in $i$-th homological degree in the minimal graded free resolution of $A$.

Lemma 4. Let $(I, J)$ be a conjoined pair of ideals in $R$. If

$$m_{k+1}(I) > M_1(I) + M_k(J)$$

for $1 \leq k \leq \text{pdim}(I) - 1$, then the reduced horseshoe resolution of $R/IJ$ with respect to $I$ is minimal.
Proof. Let
\[ 0 \to G_m \xrightarrow{d_m''} \cdots \xrightarrow{d_2''} G_2 \xrightarrow{d_1''} G_1 \xrightarrow{d_1'} R \xrightarrow{\epsilon'_0} R/J \to 0 \]
be the minimal graded free resolution of \( R/J \) and let
\[ 0 \to F_n \xrightarrow{d_n''} \cdots \xrightarrow{d_2''} F_2 \xrightarrow{d_1''} F_1 \xrightarrow{d_1'} R \xrightarrow{\epsilon''_0} R/I \to 0 \]
be the minimal graded free resolution of \( R/I \). Since the pair \((I,J)\) is conjoined, there is a minimal presentation of \( I \),
\[ R^s \xrightarrow{\phi} R^t \xrightarrow{\psi} I \to 0, \]
such that all the entries of the matrix that represents \( \phi \) belong to \( J \). Clearly, \( F_1 \cong R^t \). Now, tensoring the above exact sequence with \( R/J \) yields
\[ I/IJ \cong R^t \otimes R/J \cong F_1 \otimes R/J. \]
Therefore, we get the minimal graded free resolution of \( I/IJ \),
\[ 0 \to F_1 \otimes G_m \xrightarrow{d_m''} \cdots \xrightarrow{d_2''} F_1 \otimes G_2 \xrightarrow{d_1''} F_1 \otimes G_1 \xrightarrow{d_1'} F_1 \xrightarrow{\epsilon''_0} I/IJ \to 0 \]
Next, note that for \( j > 1 \) the maps \( \lambda_j \) that appear in the horseshoe lemma,
\[ \lambda_j : F_j \to F_1 \otimes G_j \]
are graded of degree zero. Since
\[ m_j(I) > M_1(I) + M_{j-1}(J), \]
we see that the degree of every basis element in \( F_j \) is larger than the degree of any basis element in \( F_1 \otimes G_j \). Therefore, the matrix that represents \( \lambda_j \) does not have any nonzero constant entry. Finally,
\[ |G(IJ)| = |G(I)||G(J)| = dim(F_1)dim(G_1) = dim(F_1 \otimes G_1), \]
and therefore the reduced horseshoe resolution of \( R/IJ \) with respect to \((I,J)\) is minimal.

Example 2. Consider the conjoined pair \((I,J)\) of the ideals \( I = (a^2, b^2, c^2) \) and \( J = (a, b, c) \) as in example 1. The Betti diagrams for the minimal resolutions of \( R/(a,b,c) \) and \( R/(a^2, b^2, c^2) \) are

\[
\begin{array}{cccc}
\text{total} & 1 & 3 & 3 & 1 \\
0 : & 1 & 1 & . & . \\
1 : & . & 3 & . & . \\
2 : & . & . & 3 & . \\
3 : & . & . & . & 1
\end{array}
\]
In MACAULAY 2 we observe that the Betti diagram for the minimal resolution of $R/J$ is

\[
\begin{array}{cccc}
\text{total} & 1 & 9 & 12 & 4 \\
0 & 1 & . & . & . \\
1 & . & . & . & . \\
2 & . & 9 = 3 \cdot 3 & 12 = 3 \cdot 3 + 3 & 3 = 3 \cdot 1 \\
3 & . & . & . & . \\
\end{array}
\]

The minimal graded free resolution of $R/I$ is of the form

\[
0 \longrightarrow R(-6) \longrightarrow R^{3}(-4) \longrightarrow R^{3}(-2) \longrightarrow R \longrightarrow R/I \longrightarrow 0.
\]

while the minimal graded free resolution of $R/J$ is of the form

\[
0 \longrightarrow R(-3) \longrightarrow R^{3}(-2) \longrightarrow R^{3}(-1) \longrightarrow R \longrightarrow R/J \longrightarrow 0.
\]

If we tensor this with $R^{3}(-2)$, we get the minimal graded free resolution of $I/IJ$,

\[
0 \longrightarrow R^{3}(-5) \longrightarrow R^{3}(-4) \longrightarrow R^{3}(-3) \longrightarrow R^{3}(-2) \longrightarrow I/IJ \longrightarrow 0.
\]

Accordingly the reduced horseshoe resolution of $R/IJ$ with respect to $I$ is

\[
0 \longrightarrow R(-6) \oplus R^{3}(-5) \longrightarrow R^{32}(-4) \longrightarrow R^{3}(-3) \longrightarrow R \longrightarrow R/IJ \longrightarrow 0.
\]

which is minimal.

3. Special ($p$-Borel) ideals

Let $p$ be a prime number and let $s, t$ be positive integers with $p$-adic representations

\[
s = \sum a_i p^i \quad \text{and} \quad t = \sum b_i p^i
\]

with $0 \leq a_i, b_i < p$. Then, we define the following order $\prec_p$:

\[
s \prec_p t \iff \left( \frac{t}{s} \right) \not\equiv 0 \mod p
\]

\[
\iff a_i \leq b_i \quad \text{for all} \ i.
\]

If $x_j^i$ is the highest power of $x_j$ that divides a monomial $m$, we write $x_j^i || m$.

**Definition 5.** (see, e.g., [7] or [16]). $I$ is $p$-Borel if for every minimal generator $m$ of $I$ and every $x_j$ such that $x_j^i || m$, then

\[
\left( \frac{x_i}{x_j} \right)^s m
\]

is in $I$ for all $i < j$ and $s \prec_p t$. 
Let \( S = \{ m_1, m_2, \ldots, m_r \} \) be a finite set of monomials. If \( I \) is the smallest \( p \)-Borel fixed ideal such that \( S \) is a subset of \( G(I) \), then we say that \( I \) is generated by \( m_1, m_2, \ldots, m_r \) in the Borel sense and we write
\[
I = \langle m_1, m_2, \ldots, m_r \rangle.
\]
In particular, if \( S = \{ m \} \), then \( I \) is called principal \( p \)-Borel and we write \( I = \langle m \rangle \).

**Example 3.** Let \( R = k[a, b, c] \) with \( \text{char}(k) = 2 \). Then the ideal
\[
I = (a^3, a^2b, ab^2, b^3, a^2c, b^2c, ac^2, bc^2)
\]
is a 2-Borel fixed ideal, minimally generated (in the Borel sense) by \( b^2c \) and \( bc^2 \); that is,
\[
I = \langle b^2c, bc^2 \rangle.
\]
The ideal
\[
J = (a, b, c)(a^2, b^2, c^2)
\]
\[
= (a^3, a^2b, ab^2, b^3, a^2c, b^2c, ac^2, bc^2, c^3)
\]
\[
= \langle c^3 \rangle
\]
is a principal 2-Borel fixed ideal.

The first class of \( p \)-Borel ideals in \( R = k[x_1, \ldots, x_n] \) that were studied were the ones of the form
\[
A = \langle x_\mu^n \rangle,
\]
where \( \mu \) is a positive integer, i.e. the Cohen-Macaulay \( p \)-Borel fixed ideals (see, e.g. [2], [11] and [16]). The basic structure theory of principal \( p \)-Borel ideals was developed in [16], where it was proved that if \( \mu_k = \sum_{k,i} \mu_{ki} p^i \), where \( 0 \leq \mu_{ki} \leq p - 1 \), then
\[
\langle x_1^{\mu_1} \cdots x_n^{\mu_n} \rangle = \prod_{k,i} (x_1^{p^i}, \ldots, x_k^{p^i})^{\mu_{ki}}.
\]
Products like the ones in the above structure of principal Borel that depend on certain values of the \( \mu_k \)'s were studied in [9] and [14]. These results and the observation of the Betti diagrams of several \( p \)-Borel ideals in MACAULAY 2 led us to study the following ideals.

**Definition 6.** A special ideal over \( R \) is an ideal of the form
\[
I = \prod_{j=1}^s I_j^{[p_j]} ,
\]
where
\[
I_j = (x_1, x_2, \ldots, x_{\ell_j})^{a_j} \quad \text{and} \quad I_j^{[p_j]} = (x_1^{p_j}, x_2^{p_j}, \ldots, x_{\ell_j}^{p_j})^{a_j}.
\]
with
\[ n = \ell_1 \geq \ell_2 \geq \ldots \geq \ell_s \geq 1, \]
and
\[ 0 \leq a_j < \frac{p_{j+1}}{p_j} \]
with the numbers \( \frac{p_{j+1}}{p_j} \) being integers \( > 1 \) for all \( j = 1, \ldots, s \). In particular, if \( p_j = p^{r_j} \) for \( j = 1, \ldots, s \) for some prime number \( p \) and some integers \( r_s > \ldots > r_2 > r_1 \geq 0 \), we call it special \( p \)-Borel ideal.

A special \( p \)-Borel ideal is Borel fixed if \( \text{char}(k) = p \). Every \( p \)-Borel Cohen-Macaulay ideal is special, but as it is clear, not every principal \( p \)-Borel ideal is special.

Example 4. Let \( R = \mathbb{k}[a, b, c] \) (\( \text{char}(k) = 2 \)). Then the 2-Borel ideal \((a, b, c)^2(a^4, b^4)\) is special, but not principal, since
\[ (a, b, c)^2(a^4, b^4) = (a^6, a^5b, a^4b^2, a^3b^4, a^2b^5, ab^6, a^5c, a^4bc, ab^4c, b^5c, a^4c^2, b^4c^2) \]
\[ = b^5c, b^4c^2 > \]

The main result in this section is the following

**Theorem 7.** The reduced horseshoe resolution of a special ideal \( I = \prod_{j=1}^{s} I_j^{[p_j]} \) with respect to the ordered \( s \)-tuple \( \left( I_1^{[p_1]}, \ldots, I_s^{[p_s]} \right) \) is minimal.

In order to prove this theorem, we will introduce some notation and prove lemma 8 and lemma 9 first. The ideals \( I_j \) are Borel fixed and their minimal graded free resolution is of the form
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & R^{d_{1,j}}(-d_{1,j}) & \longrightarrow & \cdots & \longrightarrow & R^{d_{i,j}}(-d_{i,j}) & \longrightarrow & R/I_j & \longrightarrow & 0 \\
\end{array}
\]
where \( d_{i,j} = a_j + i - 1 \) for \( i = 1, 2, \ldots, \ell_j \) (see, e.g. [8]). Accordingly, the minimal graded free resolution of \( R/I_j^{[p_j]} \) is
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & R^{d_{1,j}}(-c_{1,j}) & \longrightarrow & \cdots & \longrightarrow & R^{d_{1,j}}(-c_{1,j}) & \longrightarrow & R/I_j^{[p_j]} & \longrightarrow & 0 \\
\end{array}
\]
where \( c_{i,j} = p_j d_{i,j} \) for \( i = 1, 2, \ldots, \ell_j \). This does not depend on the characteristic of the base field \( k \). Consider the following free \( R \)-modules \( F_{i,k} \),
\[ F_{i,1} = R^{d_{i,1}}(-(a_1 + i - 1)p_1) \quad (1 \leq i \leq \ell_1) \]
and for \( 1 < k \leq s \),
\[
F_{i,k} = \begin{cases} 
( R^{\beta_{1,k}}(-a_k p_k) \otimes F_{i,k-1} ) \oplus R^{\beta_{1,k-1}}(-(a_k + i - 1)p_k), & \text{if } 1 \leq i \leq \ell_k; \\
R^{\beta_{1,k}}(-a_k p_k) \otimes F_{i,k-1}, & \text{otherwise}. 
\end{cases}
\]
The \( R \)-module \( F_{i,k} \) is the free module that appears in homological degree \( i \) in the minimal graded free resolution of the ideal \( J_k = I_k^{p_{k-1}} I_2^{p_2} I_1^{p_1} \) (\( 1 \leq k \leq s \)).
The degrees of the basis elements of $F_{i,k}$, i.e. the shifts in the minimal free resolution of $J_k$ are the elements of the sets $S_{i,k}$, where for $1 \leq k \leq s$ we set

$$S_{1,k} = \{a_1p_1 + a_2p_2 + \ldots + a_kp_k\},$$

and for $2 \leq i \leq \ell_1 = n$

$$S_{i,k} = \{(a_1+i-1)p_1 + a_2p_2 + \ldots + a_kp_k, (a_2+i-1)p_2 + a_3p_3 + \ldots + a_kp_k, \ldots, (a_k+i-1)p_k\}$$

for $2 \leq i \leq \ell_k - 1$, and $S_{i,k} = S_{i,k-1} + \{a_kp_k\}$ otherwise.

Now we prove the following lemma.

Lemma 8. We have

(a) $(a_k + 1)p_k > a_1p_1 + a_2p_2 + \ldots + a_kp_k$, for $1 \leq k \leq s$.
(b) $(a_k + i - 1)p_k > (a_k-1 + i - 2)p_{k-1} + a_kp_k$ for $2 \leq i \leq \ell_1$ and $k \geq 2$.
(c) For fixed $i, k$ with $1 \leq k \leq s$ and $2 \leq i \leq \ell_k - 1$, the maximum element of $S_{i,k}$ is $c_{i,k} = (a_k + i - 1)p_k$.
(d) For fixed $k$ with $1 \leq k \leq s$, $\max\{S_{i,k} - i | 1 \leq i \leq \ell_1 - 1\}$ is equal to $\max\{S_{i,k} - i | \ell_k \leq i \leq \ell_1 - 1\}$, which is equal to the maximum of the elements

$$(a_1 + \ell_1 - 1)p_1 + a_2p_2 + \ldots + a_kp_k - \ell_1,$$

$$(a_2 + \ell_2 - 1)p_2 + a_3p_3 + \ldots + a_kp_k - \ell_2,$$

$$\ldots,$$

$$(a_k + \ell_k - 1)p_k - \ell_k.$$

Proof.

(a) By induction on $k$. For $k = 1$, the inequality is trivially true. So assume that it is true for $k-1$ for some $k > 1$. Then, by the induction hypothesis and since $a_{k-1} + 1 \leq \frac{p_k}{p_{k-1}}$, we get

$$a_1p_1 + a_2p_2 + \ldots + a_{k-1}p_{k-1} + a_kp_k < (a_{k-1} + 1)p_{k-1} + a_kp_k$$

$$\leq p_k + a_kp_k$$

$$= (a_k + 1)p_k,$$

as desired.

(b) Since $a_{k-1}p_{k-1} < p_k$ and $p_k > p_{k-1}$, for $k > 1$, it follows that for $1 \leq i \leq \ell_1$ we have

$$(a_k + i - 1)p_k = a_kp_k + (i - 2)p_k + p_k$$

$$> a_kp_k + (i - 2)p_{k-1} + a_{k-1}p_{k-1}$$

$$= a_kp_k + (a_{k-1} + i - 2)p_{k-1},$$

as desired.
(c) For \( k = 1 \) this is clearly true, so assume that \( k \geq 2 \). Then it suffices to show that

\[(a_j + i - 1)p_j + a_{j+1}p_{j+1} + \ldots + a_kp_k \leq \sum_{j=1}^{k-1} (a_{j+1} + i - 1)p_{j+1} + a_{j+2}p_{j+2} + \ldots + a_kp_k\]

for \( 1 \leq j \leq k - 1 \). The above inequality is equivalent to

\[(a_j + i - 1)p_j \leq (i - 1)p_{j+1},\]

which is true, since

\[
(a_j + i - 1)p_j = (a_j + 1)p_j + (i - 2)p_j \\
\leq p_{j+1} + (i - 2)p_j \\
\leq p_{j+1} + (i - 2)p_{j+1} \\
= (i - 1)p_{j+1}.
\]

(d) First, the fact that \( \max\{S_{i,k} - i|1 \leq i \leq \ell_1\} \) is equal to \( \max\{S_{i,k} - i|\ell_k \leq i \leq \ell_1\} \) follows from part (c) and

\[
(a_k + i - 1)p_k - i \geq (a_k + i - 2)p_k - (i - 1)
\]

for \( i > 2 \) along with

\[
(a_k + 1)p_k - 2 \geq (a_1p_1 + a_2p_2 + \ldots + a_{k-1}p_{k-1} + a_k) - 1,
\]

which is true from part (a). Now, for \( k = 1 \), our claim is clearly true, so assume that for some \( k \geq 2 \), the maximum element of \( \max\{S_{i,k-1} - i|1 \leq i \leq \ell_k\} \) is equal to the maximum of the elements

\[
(a_1 + \ell_1 - 1)p_1 + a_2p_2 + \ldots + a_{k-1}p_{k-1} - \ell_1, \\
(a_2 + \ell_2 - 1)p_2 + a_3p_3 + \ldots + a_{k-1}p_{k-1} - \ell_2, \\
\vdots, \\
(a_{k-1} + \ell_{k-1} - 1)p_{k-1} - \ell_{k-1}.
\]

From the definition of the sets \( S_{i,k} \), we see that \( \max\{S_{i,k} - i|1 \leq i \leq \ell_k\} \) is equal to the maximum of

\[\max\{S_{i,k} - i|1 \leq i \leq \ell_k\},\]

and

\[\max\{S_{i,k-1} - i|\ell_k \leq i \leq \ell_1\} + a_kp_k\]

Now since

\[\max\{S_{i,k} - i|1 \leq i \leq \ell_k\} = (a_k + \ell_k - 1)p_k - \ell_k,\]

and

\[\max\{S_{i,k-1} - i|\ell_k \leq i \leq \ell_1\} + a_kp_k = \max\{S_{i,k-1} - i|1 \leq i \leq \ell_1\} + a_kp_k,\]
from the induction hypothesis, it follows that max\{S_{i,k} - i | 1 \leq i \leq \ell_1\} is equal to the maximum of the elements
\[(a_1 + \ell_1 - 1)p_1 + a_2p_2 + ... + a_{k-1}p_{k-1} + a_kp_k - \ell_1,\]
\[(a_2 + \ell_2 - 1)p_2 + a_3p_3 + ... + a_{k-1}p_{k-1} + a_kp_k - \ell_2,\]
...
\[(a_{k-1} + \ell_{k-1} - 1)p_{k-1} + a_kp_k - \ell_{k-1},\]
\[(a_k + \ell_k - 1)p_k - \ell_k.\]

The proof is now complete.

In order to prove theorem 7 we need the following lemma.

**Lemma 9.** Let \(J_k = I_{k-1}^{p_k} ... I_1^{p_1} (1 \leq k \leq s)\). Then
(a) All elements of \(G(J_k)\) are of equal degree \((1 \leq k \leq s)\).  
(b) \(\left(I_k^{[p_k]}, J_{k-1}\right)\) is a conjoined pair of ideals for \(1 \leq k \leq s\).  
(c) \(m_j(I_k^{[p_k]}) > M_1(I_k^{[p_k]}) + M_{j-1}(J_{k-1})\) for \(2 \leq j \leq \ell_k\) and \(2 \leq k \leq s\).

**Proof.**
(a) This follows by an easy induction and an argument as the one in the proof of Lemma 3.
(b) This follows from the above part, Lemma 3 and part (a) of Lemma 8.
(c) Note that \(m_j(I_k^{[p_k]}) = (a_k + j - 1)p_k\), \(M_1(I_k^{[p_k]}) = a_kp_k\), \(M_{j-1}(J_{k-1}) = (a_{k-1} + j - 2)p_{k-1}\). Thus, the required inequality becomes
\[(a_k + j - 1)p_k > a_kp_k + (a_{k-1} + j - 2)p_{k-1},\]
which is part (b) of Lemma 8.

**Proof of Theorem 7.** Let \(2 \leq k \leq s\) and assume by induction that the minimal graded free resolution of \(R/J_{k-1}\) has been obtained already

\[
0 \rightarrow F_{n,k-1} \xrightarrow{\phi_{n,k-1}} ... \xrightarrow{\phi_{1,k-1}} R \xrightarrow{\phi_{1,k-1}} R/J_{k-1} \rightarrow 0
\]

where the degrees of the basis elements of the free \(R\)-modules \(F_{i,k-1}\) are the elements of \(S_{i,k-1}\), for \(1 \leq i \leq n\). We know that the minimal graded free resolution of \(R/I_k^{[p_k]}\) is of the form

\[
0 \rightarrow R^{\beta_{n, k}} \xrightarrow{\psi_{n,k}} ... \xrightarrow{\psi_{1,k}} R^{\beta_{1,k}} \xrightarrow{\psi_{1,k}} R \xrightarrow{\psi_{1,k}} R/I_k^{[p_k]} \rightarrow 0
\]

From Lemma 9 part (b) above, we see that the pair \((I_k^{[p_k]}, J_{k-1})\) is conjoined. Then Lemma 9 part (c) together with Lemma 4 implies that the reduced
horseshoe resolution of $R/I_k^{[p^r]} J_{k-1}$ is minimal and is of the form

$$0 \longrightarrow F_{n,k} \xrightarrow{\phi_{n,k}} \cdots \xrightarrow{\phi_{1,k}} R \xrightarrow{R/J_k} R \xrightarrow{R/J_k} 0$$

Since our techniques do not depend on the characteristic of the base field, we obtain the following.

**Corollary 10.** The graded betti numbers of a special $p$-Borel ideal do not depend on the characteristic of the base field $k$.

Also, we recover Pardue’s regularity formula.

**Corollary 11.** Let $I = \prod_{j=1}^s I_j^{[p^r_j]}$ be a special $p$-Borel ideal. Then the regularity $reg(I)$ of $I$ is the maximum of

$$a_1 p^r_1 + a_2 p^r_2 + \ldots + a_s p^r_s + (p^r_1 - 1)(\ell_1 - 1),$$

$$a_2 p^r_2 + \ldots + a_s p^r_s + (p^r_2 - 1)(\ell_2 - 1),$$

$$\ldots,$$

$$a_s p^r_s + (p^r_s - 1)(\ell_s - 1).$$

**Proof.** This is immediate from the definition of regularity combined with Lemma 8 part (d), the above proposition and the fact that $reg(I) = reg(S/I) + 1$.

### 4. Cellular resolutions

In $\text{char}(k) = 0$, it is known that the Eliahou-Kervaire resolution of a Borel fixed ideal $I$ is a CW-resolution (see [8]). If $I$ is generated in one degree, then another minimal free resolution of $I$ can be supported on a polyhedral cell complex (see [18]). In $\text{char}(k) = p$, it has been proved in [2] that the minimal free resolution of a Cohen-Macaulay $p$-Borel fixed ideal is a CW-resolution and in [14] that the minimal free resolution of a special $p$-Borel fixed ideal is also a CW-resolution.

Here we construct a polyhedral cell complex that supports the minimal free resolution of some special ideals. Our main result is the following

**Proposition 12.** There exists a polyhedral cell complex that supports a minimal free resolution of a special ideal $I$ of the form

$$I = (x_1^{p_1}, x_2^{p_2}, \ldots, x_n^{p_n})^{a_1} I_1^{p_2} (x_1^{p_2}, x_2^{p_2}, \ldots, x_n^{p_2}) \cdots (x_1^{p_s}, x_2^{p_s}, \ldots, x_n^{p_s})$$

Before we prove this proposition, we consider a generalized permutohedron ideal. Set $d := a_1$ and recall that $dp_1 < p_2$. Let $u = (dp_1, p_2, 0, \ldots, 0)$ be in $\mathbb{N}^n$. By permuting the coordinates of $u$, we obtain $n(n - 1)$ points in $\mathbb{N}^n$ constituting the vertices of an $(n - 1)$-dimensional generalized permutohedron $\Pi(u)$ (see also, [15]). We label the vertices of $\Pi(u)$ by the monomial generators of

$$K(u) := (x_i^{dp_1} x_j^{p_2} | 1 \leq i, j \leq n, i \neq j),$$
in a natural way and then we label an arbitrary face \( F \) of \( \Pi(u) \) as usual, that is, by the lcm of the monomial labels on all vertices in \( F \). The inequality description of \( \Pi(u) \) is

\[
\Pi(u) = \{(v_1, \ldots, v_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} v_j = dp_1 + p_2 \text{ and } 0 \leq v_i \leq p_2 \text{ for } i = 1, 2, \ldots, n\},
\]

i.e. \( \Pi(u) \) is the intersection of the \((n-1)\)-simplex with the \(n\) half spaces \( \{(v_1, \ldots, v_n) \in \mathbb{R}^n \mid v_i \leq p_2 \} \) (\( i = 1, 2, \ldots, n \)). Since \( K(u) \leq b \) is empty or contractible for all \( b \in \mathbb{Z}^n \), \( \Pi(u) \) supports a free resolution of \( K(u) \). It is easy to see that that this resolution is minimal, since any two comparable faces of the same degree coincide (see [4]). Thus, we have proved the following.

**Lemma 13.** The polyhedral cell complex \( \Pi(u) \) supports a minimal free resolution of \( K(u) \).

We need the following lemma from [18].

**Lemma 14.** Let \( I \) and \( J \) be two monomial ideals in \( R \) such that \( G(I + J) = G(I) \cup G(J) \) set-theoretically. Suppose that

(i) \( X \) and \( Y \) are regular cell complexes in some \( \mathbb{R}^N \) that support a (minimal) free resolution for \( I \) and \( J \), respectively, and

(ii) \( X \cap Y \) is a regular cell complex that supports a (minimal) free resolution for \( I \cap J \).

Then \( X \cup Y \) supports a (minimal) free resolution for \( I + J \).

**Proof of Proposition 12.** It suffices to consider the case \( s = 2 \); the general case is similar. Consider all monomial generators of \((x_1^{p_1}, \ldots, x_n^{p_1})^{d}\) that are not divisible by \( x_j \) and denote the ideal they generate by \( K_j \) (\( 1 \leq j \leq n \)). A minimal free resolution of \( K_j \) is supported on the \((n-2)\)-dimensional complex \( P_j := P_d(x_1^{p_1}, \ldots, \hat{x_j}, \ldots, x_n^{p_1}) \). Multiplying all vertices of \( P_j \) by \( x_j^{p_2} \), we obtain a polyhedral cell complex \( Q_j \) that supports a minimal free resolution of \( x_j^{p_2} K_j \).

Let \( \sigma \subset [n] = \{1, 2, \ldots, n\} \). Replacing the face of \( \Pi(u) \) that lies on the hyperplane \( \{(v_1, \ldots, v_n) \in \mathbb{R}^n \mid v_j = p_2 \} \) by \( Q_j \) for \( j \in \sigma \) gives us a polyhedral cell complex \( \Pi_\sigma \) that supports a minimal free resolution of the ideal

\[
K(u) + \sum_{j \in \sigma} x_j^{p_2} K_j.
\]

The intersection of \( \Pi_\sigma \) with \( x_j^{p_2} P_{a_j} (x_1, \ldots, x_n) \) is \( Q_j \) for \( j \in \sigma \). Applying lemma 14, we glue all these complexes to obtain a polyhedral cell complex
that supports a minimal free resolution of
\[ K(u) + \sum_{j \in \sigma} p_j (x^{p_1}_{r_1}, x^{p_2}_{r_2}, \ldots, x^{p_n}_{r_n})^d. \]

In particular, when \( \sigma = [n] \), we obtain a polyhedral cell complex that supports a minimal free resolution of
\[ (x^{p_1}_{r_1}, x^{p_2}_{r_2}, \ldots, x^{p_n}_{r_n})^d (x^{p_2}_{r_2}, x^{p_3}_{r_3}, \ldots, x^{p_n}_{r_n}). \]

**Remark 1.** It follows from the above proposition that there exists a polyhedral cell complex that supports a minimal free resolution of any Cohen-Macaulay 2-Borel fixed ideal.

**Example 5.** The polyhedral cell that supports a minimal free resolution of \((a, b, c)(a^2, b^2, c^2)\) is

![Polyhedral cell diagram](image)

5. **Mapping cone**

By applying the iterated mapping cone technique (see [6], [13], [17]) as in the case of Borel fixed ideals in characteristic zero, we do not always obtain a minimal resolution in characteristic \( p \). The smallest example that we found in characteristic two using MACAULAY 2 [10] is the following one in three variables.

**Example 6.** Let \( R = k[a, b, c] \) (\( \text{char}(k) = 2 \)),
\[ I = (a^3, a^2b, ab^2, b^3, a^2c, b^2c, ac^2, bc^2), \]
and
\[ I' = (a^3, a^2b, ab^2, b^3, a^2c, b^2c, ac^2). \]
Starting with the ideal \((a^3)\) and adding the monomial generators of \(I'\) one at a time in the order that appears above, the iterated mapping cone gives us a minimal free resolution of \(I'\). However, if \(f\) is the map from the resolution of \(R/(I' : bc^2)\) to the resolution of \(R/I'\) induced by multiplication by \(bc^2\), then the mapping cone of \(f\) does not give us a minimal free resolution of \(R/I\). This is clear from the following Betti diagrams of \(I'\) and \(I\), since \(\beta_{2,5}(I') = 1\), while \(\beta_{2,5}(I) = 0\).

\[
\begin{array}{cccccc}
\text{total} & 1 & 7 & 9 & 3 \\
0 & 1 & . & . & . \\
1 & . & . & . & . \\
2 & . & 7 & 8 & 2 \\
3 & . & . & 1 & 1 \\
\end{array}
\quad \begin{array}{cccccc}
\text{total} & 1 & 8 & 10 & 3 \\
0 & 1 & . & . & . \\
1 & . & . & . & . \\
2 & . & 8 & 10 & 2 \\
3 & . & . & . & 1 \\
\end{array}
\]

As the proof of the following proposition shows, the ordering of the monomial generators of the above ideal is not important.

**Proposition 15.** There exists a \(p\)-Borel fixed ideal \(I\) such that for any ordering \(m_1 \succ m_2 \succ \ldots \succ m_r\) of its minimal generators, there is some \(i\) with \(2 \leq i \leq r\), such that the mapping cone of multiplication by \(m_i\) from a minimal resolution of \(R/(m_1, \ldots, m_{i-1}) : m_i\) to a minimal resolution of \(R/(m_1, \ldots, m_{i-1})\) is not a minimal free resolution of \(R/(m_1, \ldots, m_{i-1})\).

**Proof.** Let \(R = \mathbb{k}[a, b, c] \) with \(\text{char}(\mathbb{k}) = 2\) and consider the ideal

\[
I = (a^3, a^2b, ab^2, b^3, a^2c, b^2c, ac^2, bc^2),
\]

The minimal cellular resolution of \(I\) consists of two triangles with vertices in \(\{a^3, a^2b, a^2c\}\) and \(\{ab^2, b^3, b^2c\}\), and the hexagon \(\Pi(1, 2, 0)\) with vertices in \(\{a^2b, ab^2, a^2c, b^2c, ac^2, bc^2\}\).

It suffices to check whether we can get the above hexagon by an iterated method, i.e. by adding one monomial at a time in a suitable order and by considering each time the minimal cellular resolution of the corresponding ideal that we get.

This is impossible. Indeed, let \(I(S)\) be the ideal generated by be a 5-element subset \(S\) of the vertices of the 6-gon \(\{a^2b, ab^2, a^2c, b^2c, ac^2, bc^2\}\). By considering cases, it is easy to see that the total betti number \(\beta_5(I(S))\) is non-zero. That is, the minimal free resolution of \(I(S)\) is supported on a 2-dimensional polygon, i.e. there is an edge that connects two vertices, which are not connected in \(\Pi(1, 2, 0)\). For example, if \(S = \{a^2b, ab^2, a^2c, b^2c, ac^2\}\), we see that there is an edge between \(ac^2\) and \(b^2c\), which is denoted with a dashed edge in the above figure. This means that when we add \(bc^2\) to \(S\), we must erase that edge in order to obtain \(\Pi(1, 2, 0)\), which supports a minimal free resolution of the ideal generated by the new set \(S \cup \{bc^2\}\).

**Remark 2.** This means that there is a lifting map \(\lambda_i\) in the mapping cone of the map \(f\) from example 6 with a non-zero constant entry in the matrix that
represents it. Thus, there is a multigraded free module $R(-e)$ that appears in the $i$-th homological degree in both resolutions of $R/I'$ and $R/(I' : bc^2)$. If we cancel (by a change of basis) the two copies of $R(-e)$ that appear in the mapping cone, we obtain a minimal free resolution of $R/I$.

However, this does not mean that every time we have a copy of $R(-e)$ in the same homological degree, we could obtain a minimal free resolution by cancelling it. One of the smallest examples in characteristic two that we found using MACAULAY 2 [10] is the following one in five variables.

**Example 7.** Let $R = k[a, b, c, d, e]$ with $\text{char}(k) = 2$ and let $B$ be the 2-Borel fixed ideal

$$B = \langle ace^2, b^2e^2, bcd^2, bc^2d \rangle.$$ 

Then

$$B : (bce^2) = (b, a, d^2, cd, e^2)$$

and the 2-Borel fixed ideal $A = (B, bce^2) = \langle bc^2d, bce^2 \rangle$ has 30 generators. The Betti diagrams of $B$ and $A$ are

|       | total : | 1  | 29 | 78 | 83 | 41 | 8  |
|-------|---------|----|----|----|----|----|----|
| 0     |         | 1  | .  | .  | .  | .  | .  |
| 1     |         | .  | .  | .  | .  | .  | .  |
| 2     |         | .  | .  | .  | .  | .  | .  |
| 3     |         | .  | .  | .  | .  | .  | .  |
| 4     |         | .  | .  | .  | .  | .  | .  |
| 5     |         | .  | .  | .  | 1  | 5  | 3  |

|       | total : | 1  | 30 | 83 | 91 | 46 | 9  |
|-------|---------|----|----|----|----|----|----|
| 0     |         | 1  | .  | .  | .  | .  | .  |
| 1     |         | .  | .  | .  | .  | .  | .  |
| 2     |         | .  | .  | .  | .  | .  | .  |
| 3     |         | .  | .  | .  | .  | .  | .  |
| 4     |         | .  | .  | .  | .  | .  | .  |
| 5     |         | .  | .  | .  | 1  | 5  | 3  |

We note that one copy of each of $R(-(1, 2, 1, 0, 2)), R(-(0, 2, 2, 2, 2))$ and $R(-(1, 2, 2, 2, 2))$ appears in homological degrees 2, 4 and 5, respectively, in
the resolutions of $R/B$ and $R/(B : bce^2)$, but the two copies of $R(-(1, 2, 1, 0, 2))$ that appear in the mapping cone cannot be cancelled.

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