THE BLOCK GRAPH OF A FINITE GROUP

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ABSTRACT
In this paper we first define the block graph of a finite group $G$, whose vertices are the prime divisors of $|G|$ and there is an edge between two vertices $p \neq q$ if and only if the principal $p$- and $q$-blocks of $G$ have a nontrivial common complex irreducible character of $G$. Then we determine the block graphs of finite simple groups, which turn out to be complete except those of $J_1$ and $J_4$. Finally, in terms of block graphs with no triangle containing a prime $p$, we obtain a criterion for the $p$-solvability of a finite group which in particular leads to an equivalent condition for the solvability of a finite group.

1. Introduction

Let $G$ be a finite group and $p, q$ two primes. The question of when a $p$-block of $G$ is also a $q$-block of $G$ was studied by Navarro and Willems in [28]. This led to the investigation of block distributions of complex irreducible characters of a finite group with respect to different primes. For instance, Bessenrodt, Malle and Olsson [2] introduced the concept of block separability of characters, and Navarro, Turull and Wolf [27] discussed solvable groups that are block separated. In a series of papers, Bessenrodt and Zhang generally investigated block separations, inclusions and coverings of characters of a finite group; see [3] and [4]. Motivated by their work, we investigate block separations of characters from a graph-theoretical point of view.

Let $\text{Irr}(G)$ be the set of complex irreducible characters of $G$, and denote by $B_p(G)$ the principal $p$-block of $G$ and by $\text{Irr}(B_p(G))$ the set of complex irreducible characters of $G$ contained in $B_p(G)$.

**Definition 1.1:** We construct the block graph $\Delta(G)$ of a finite group $G$ as follows. The vertices are the prime divisors of $|G|$, and there is an edge between two vertices $p \neq q$ if and only if $\text{Irr}(B_p(G)) \cap \text{Irr}(B_q(G)) \neq \{1_G\}$.

By the definition, an equivalent statement of [3, Theorem 4.1] is that the block graph of a finite group consists of isolated vertices if and only if the group is nilpotent. So, for non-nilpotent $\{p, q\}$-groups, their block graphs are always of the form $\bullet \longrightarrow \bullet$. Our first main result is to determine the block graphs of finite non-abelian simple groups, which turn out to be seemingly opposite to the situation for nilpotent groups.
Theorem 1.2: The block graph of a finite non-abelian simple group $S$ is complete except when $S = J_1$ (resp. $J_4$), in which case only the primes $p = 3$ and $q = 5$ (resp. $p = 5$ and $q = 7$) are not adjacent in the block graph of $S$.

Note that the block graphs of alternating groups and sporadic simple groups are known according to [3, Propositions 3.2 and 3.5]. Therefore, in order to prove Theorem 1.2, it remains to determine the block graphs of simple groups of Lie type. Our strategy is to investigate the block distribution of unipotent characters based on the recent results of Kessar and Malle in [22] about Lusztig induction.

Inspired by Brauer’s problem [5] of finding the relations between the properties of the $p$-blocks of characters of a finite group $G$ and structural properties of $G$, we apply Theorem 1.2 to obtain the following criterion for the $p$-solvability of a finite group from a local viewpoint of its block graph.

Theorem 1.3: Let $G$ be a finite group and $p$ a prime divisor of $|G|$. If the block graph of $G$ has no triangle containing $p$, then $G$ is $p$-solvable.

An application of Theorem 1.3 and the celebrated Feit–Thompson theorem leads to an equivalent condition for the solvability of a finite group. This implies that, together with results of Bessenrodt and Zhang [3, Theorem 4.1] and [4, Theorem 2.3], the nilpotency, $p$-nilpotency and solvability of a finite group can be characterized by intersections of principal blocks of some quotient groups.

Theorem 1.4: Let $G$ be a finite group and $S(G)$ the largest normal solvable subgroup of $G$. Then $G$ is solvable if and only if the block graph of $G/S(G)$ has no triangle containing 2.

Proof. It suffices to show the “if” part. Suppose that the block graph of $G/S(G)$ has no triangle containing 2. By Theorem 1.3 and the Feit–Thompson theorem, we have that $G/S(G)$ is solvable, hence $G$ is solvable. ■

2. Characters and blocks

In this section we make some observations on principal blocks of a finite group and its normal subgroups. Our notation of character theory of finite groups mainly follows [21] and [26]. In particular, if $H$ is a subgroup of $G$, then $\chi_H$ denotes the restriction of a character $\chi$ of $G$ to $H$ and $\theta^G$ means the induction of a character $\theta$ of $H$ to $G$. In addition, if $g \in G$, then $H^g = g^{-1}Hg$ and $gH = gHg^{-1}$. Sometimes we use $g^\phi$ to denote $\phi(g)$ for $\phi \in \text{Aut}(G)$. 
**Lemma 2.1:** Let $N$ be a normal subgroup of $G$. If $p$ and $q$ are two distinct primes which are adjacent in $\Delta(G/N)$, then they are adjacent in $\Delta(G)$.

**Proof.** This is clear since $\text{Irr}(B_p(G/N)) \subseteq \text{Irr}(B_p(G))$ by inflation. 

An important tool that we shall make use of is when the principal $p$-block of a normal subgroup is uniquely covered by the principal $p$-block of the top group.

**Lemma 2.2:** Let $N \triangleleft G$ and suppose that $|N|$ is divisible by a prime $p$ such that the principal $p$-block of $G$ is the unique $p$-block of $G$ covering the principal $p$-block of $N$. If for any other prime $r$ dividing $|N|$, the primes $p$ and $r$ are adjacent in $\Delta(N)$, then $p$ and $r$ are adjacent in $\Delta(G)$.

**Proof.** As $B_p(G)$ is the unique block of $G$ covering $B_p(N)$, it follows from [26, Theorem 9.4] that for any $\phi \in B_p(N)$ all the irreducible constituents of $\phi^G$ lie in $B_p(G)$. As $r$ is adjacent to $p$ in $\Delta(N)$, there exists some non-trivial irreducible character $\phi \in B_p(N) \cap B_r(N)$. Furthermore, as $B_r(G)$ covers $B_r(N)$, there is at least one irreducible constituent of $\phi^G$ (which must be non-trivial) lying in $B_r(G)$. By the above observation for $B_p(G)$, it follows that this constituent lies in $B_p(G) \cap B_r(G)$. Thus $p$ and $r$ are adjacent in $\Delta(G)$.

Thus we often want to find primes for which we can produce a unique cover for the principal block of a normal subgroup. The following two results provide criteria to ensure that the principal block of a normal subgroup is uniquely covered by the principal block of the top group.

**Lemma 2.3:** Let $N$ be a normal subgroup of $G$ and $p$ a prime. If $C_G(P) \leq N$ for $P \in \text{Syl}_p(N)$ then $B_p(N)$ is covered by a unique $p$-block of $G$ which must be $B_p(G)$.

**Proof.** This follows from [19, Corollary 2] and the fact that $B_p(G)$ covers $B_p(N)$.

**Lemma 2.4:** Let $N \triangleleft G$ and $\theta \in \text{Irr}(N)$ be such that $\theta^G \in \text{Irr}(G)$. If $b$ is the $p$-block of $N$ containing $\theta$, then $b$ is covered by a unique $p$-block of $G$.

In particular, if $\theta \in B_p(N)$ such that $\theta^G \in \text{Irr}(G)$, then $B_p(N)$ is covered by a unique $p$-block of $G$ which must be $B_p(G)$.

**Proof.** Let $B$ be a $p$-block of $G$ covering $b$. By [26, Theorem 9.4] there exists an irreducible character $\chi$ of $B$ such that $\langle \chi, \theta^G \rangle \neq 0$. Thus as both characters are irreducible, we have $\chi = \theta^G$ and so the block $B$ must be unique.
3. Block graphs of simple groups

In this section we investigate block graphs of finite non-abelian simple groups and prove Theorem 1.2. The block graphs of alternating groups and sporadic simple groups have been determined by Bessenrodt and Zhang [3]. To determine the block graphs of finite simple groups of Lie type, we first establish some lemmas.

Let \( G \) be a connected reductive algebraic group defined over \( \mathbb{F}_q \), with corresponding Frobenius morphism \( F : G \to G \). We call an \( F \)-stable Levi subgroup \( L \leq G \) an \( e \)-split Levi subgroup of \( G \) if it is the centralizer of an \( e \)-torus of \( G \), and a Sylow \( e \)-split Levi subgroup of \( G \) if it is the centralizer of a Sylow \( e \)-torus of \( G \), where \( e \) is a positive integer. In the latter case, we call \( L^F \) a Sylow \( e \)-split Levi subgroup of \( G^F \). For the related terminology, notation and basic facts about connected and finite reductive groups, we refer to [25] and [7].

**Lemma 3.1:** Let \( G \) be a connected reductive algebraic group defined over \( \mathbb{F}_q \), with corresponding Frobenius morphism \( F : G \to G \). Let \( e \) be a positive integer.

(1) A \( G^F \)-conjugate of a Sylow \( e \)-torus of \( G \) is also a Sylow \( e \)-torus of \( G \).

(2) A \( G^F \)-conjugate of a Sylow \( e \)-split Levi subgroup of \( G \) is also a Sylow \( e \)-split Levi subgroup of \( G \).

(3) Sylow \( e \)-split Levi subgroups of \( G \) are minimal among \( e \)-split Levi subgroups of \( G \).

(4) An \( F \)-stable sub-torus of an \( e \)-torus of \( G \) is also an \( e \)-torus.

(5) The Sylow \( e \)-torus \( S \) contained in a Sylow \( e \)-split Levi subgroup \( L = C_G(S) \) of \( G \) is the unique Sylow \( e \)-torus of \( G \) contained in every \( F \)-stable maximal torus of \( L \).

**Proof.** Statements (1)–(3) directly follow from their definitions.

For (4), let \( S \) be an \( e \)-torus of \( G \) and \( S_1 \subseteq S \) an \( F \)-stable torus. Denote by \( \Phi_e(x) \) the \( e \)-th cyclotomic polynomial over \( \mathbb{Q} \). By [3] Theorem 13.1], there is a unique “polynomial order” \( P_{S,F}(x) \in \mathbb{Z}[x] \), which is indeed \( \Phi_e(x)^a \) for some \( a \geq 0 \), associated to \( S \) such that \( |S^F| = P_{S,F}(q) \). Furthermore, we have \( P_{S_1,F}(x) | P_{S,F}(x) \) by [3] Proposition 13.2.(ii)]. However, \( \Phi_e(x) \) is irreducible in \( \mathbb{Z}[x] \). Hence \( P_{S_1,F}(x) = \Phi_e(x)^{a_1} \) for some \( a_1 \leq a \), which says that \( S_1 \) is an \( e \)-torus.
We now prove (5). Let $S$ be a Sylow $e$-torus such that

$$L = C_G(S)$$

and

$$|S^F| = \Phi_e(x)^a \quad \text{for some } a \geq 0.$$

Let $T$ be an $F$-stable maximal torus of $L$. Since $T$ is self-centralizing in $L$, and $[S, T] = 1$, it follows that $S \subseteq T$. By [8] Proposition 13.5], $S$ is the unique Sylow $e$-torus in $T$. \qed

**Lemma 3.2:** Let $G$ be a simple algebraic group of simply connected type over $\mathbb{F}_p$ and $F$ a Steinberg endomorphism of $G$ such that $G$ is endowed with an $\mathbb{F}_q$-structure. Assume that $G^F$ is not $3D_4(q)$ and $F$ is not very twisted in the sense of [25] Definition 22.4]. If $e_1 \geq 2$ and $e_2 \geq 2$ are two different integers such that

$$\Phi_{e_i} := \Phi_{e_i}(q) \mid |G^F|,$$

then Sylow $e_1$- and $e_2$-tori of $G^F$ cannot simultaneously be contained in any $F$-stable maximal torus of Sylow $e_i$-split Levi subgroups of $G$.

**Proof.** Let $L_i$ be a Sylow $e_i$-split Levi subgroup of $G^F$ for $i = 1, 2$. We first assume that $G^F$ is a classical group. If $G^F = \text{SL}_n(q)$ and $G^F$ has two Sylow $e_i$-tori lying in the same maximal torus $T$ of $\text{SL}_n(q)$, then they certainly also lie in a common maximal torus of $\text{GL}_n(q)$, namely in $T \text{Z}(\text{GL}_n(q))$. And if $e_i > 1$, then the Sylow $e_i$-tori of $G^F$ are also Sylow $e_i$-tori of $\text{GL}_n(q)$. If $e_1 = 1$, then a Sylow 1-torus of $G^F$ is already a maximal torus of $G^F$ and so certainly does not contain another Sylow $e_2$-torus. The argument for $G^F = \text{SU}_n(q)$ is the same with $e_1 = 1$ replaced by $e_1 = 2$. Thus, we may generally turn to consider the groups $G := G_{\text{SL}}(q), G_{\text{U}}(q), G_{\text{Sp}}(q), G_{\text{SO}_{2n+1}}(q), G_{\text{SO}_{2n}^\pm}(q)$ for some $n$.

Note that by [12] Theorems (2A) and (3D)] there is a unique minimal $1 \leq d_i \leq n$ for each $i$ such that $\phi_{L_i}(q)$, which is $q^{d_i} - 1$ if $G = \text{GL}_n(q), q^{d_i} - (-1)^{d_i}$ if $G = \text{U}_n(q)$, or $q^{d_i} \pm 1$ otherwise, is divisible by $\Phi_{e_i}$ and is a factor of $|G|$. In particular, $\phi_{L_1}(q) \neq \phi_{L_2}(q)$.

For $i = 1, 2$, let $n = a_i d_i + s_i$, where $0 \leq s_i < d_i$. By [12] Theorems (2A) and (3D)], the Sylow $e_i$-split Levi subgroups of $G$ have the form $M_i \times Q_i$, where $M_i$ is a group of the same type as $G$ with rank $s_i$ and $Q_i$ is the direct product of $a_i$ copies of cyclic tori of order $\phi_{L_i}(q)$. In particular, any maximal torus of $L_i$ has the form $T_{M_i} \times Q_i$, where $T_{M_i}$ is a maximal torus of $M_i$. If $d_1 = d_2$ then, since $\phi_{L_1}(q) \neq \phi_{L_2}(q)$, we have

$$\{\phi_{L_1}(q), \phi_{L_2}(q)\} = \{q^{d_1} - 1, q^{d_1} + 1\}.$$
It follows that there is no maximal torus of $L_i$ containing Sylow $e_1$- and $e_2$-tori of $G$ at the same time. So we may assume $d_1 < d_2$. If $d_1 = 1$ then

$$\phi_{L_1}(q) = q + 1$$

by the assumption, hence $d_2 \geq 3$. This means that $d_2 \geq 3$ in any case. Clearly, we have that $\phi_{L_2}(q)$ is not a factor of $\phi_{L_1}(q)^{a_1}$. Since $M_i$ and $G$ are of the same type and the rank of $M_1$ is smaller than $d_2$, it follows that $\phi_{L_2}(q)$ is not a factor of $|M_1|$. By the choices of $\phi_{L_1}(q)$ and $\phi_{L_2}(q)$, this implies that there is no maximal torus of $L_1$ containing Sylow $e_1$- and $e_2$-tori of $G$ at the same time.

We claim that there is also no maximal torus of $L_2$ containing Sylow $e_1$- and $e_2$-tori of $G$ at the same time. Let $s_2 = a'_1 d_1 + s'_1$, where $0 \leq s'_1 < d_1$. If $\phi_{L_1}(q)$ does not divide $\phi_{L_2}(q)$ and is a factor of $|M_2|$, then since $a_1$ is obviously greater than $a'_1$, we conclude that in this case there is no maximal torus of $L_2$ containing Sylow $e_1$- and $e_2$-tori of $G$ at the same time. We now suppose that $\phi_{L_1}(q)$ divides $\phi_{L_2}(q)$. In particular, we have $d_1 \mid d_2$. Observe that for any maximal torus $T_2$ of $L_2$, the exponent of the factor $\phi_{L_1}(q)$ in $|T_2|$ is at most $a_2 + a'_1$, which is smaller than $a_1$. It follows that there is no maximal torus of $L_2$ containing Sylow $e_1$- and $e_2$-tori of $G$ at the same time, as claimed. Hence the lemma holds if $G^F$ is of classical type.

We now assume that $G^F$ is of exceptional type. The structure of Sylow $e_i$-split Levi subgroups of $G^F$ can be found in [7, Table 1] or the library of CHEVIE [14]. So it is easy to see that the lemma holds in this case, and we are done.

**Lemma 3.3:** Let $S$ be a finite simple group of Lie type defined over a finite field of characteristic $p$. Then for any prime divisors $\ell_1, \ell_2$ of $|S|$ different from $p$, the intersection $\text{Irr}(B_{\ell_1}(S)) \cap \text{Irr}(B_{\ell_2}(S))$ has a nontrivial unipotent character of $S$.

**Proof.** By the main theorem of [20], if $S$ is one of $^3D_4(q)$, $^2B_2(q)$, $^2F_4(q)$ or $^2G_2(q)$ and if $\ell$ is a prime dividing $|S|$ with $\ell \neq p$, then the Steinberg character $St$ of $S$ lies in the principal $\ell$-block of $S$. We can then assume that $S$ is not any of those groups in the following. Let $G$ be a simple algebraic group of simply connected type over $\mathbb{F}_p$ and $F$ a Steinberg endomorphism of $G$ with an $\mathbb{F}_q$-structure such that

$$S \cong G^F/Z(G^F).$$

Then $F$ is not very twisted according to [25, Definition 22.4].
Let $e_i := e_{\mathcal{E}_i}(q)$ for $i \in \{1, 2\}$. It follows from [25] Theorem 25.11 that $G$ has a Sylow $e_i$-torus $S_i$, and so the group

$$L_i := C_G(S_i)$$

is a Sylow $e_i$-split Levi subgroup of $G$. Notice that, by [22] Remark 2.2 and Lemma 3.1 (3), the pair $(L_i, 1_{L_i})$ is an $e_i$-Jordan-cuspidal pair for each $i = 1, 2$. Therefore, by [22] Theorem A (a), for $i \in \{1, 2\}$ all irreducible constituents of the Lusztig induction $R_{L_i}^G(1_{L_i^F})$ lie in the principal $\ell_i$-block of $G^F$, and so they lie in the principal $\ell_i$-block of $S$, since the center $Z(G^F)$ is contained in the kernel of every unipotent character of $G^F$. In particular, if $e_1 = e_2$ then the lemma clearly holds since $R_{L_i}^G(1_{L_i^F})$ contains a nontrivial irreducible constituent.

In the following, we assume that $e_1 \neq e_2$.

Denote by $T_i$ the set of $F$-stable maximal tori of $L_i$ for $i = 1, 2$. By Lemmas 3.1 (2) and 3.2, $T_1 \in T_1$ and $T_2 \in T_2$ are not $G^F$-conjugate, and so by [25] Proposition 25.1, their corresponding elements in the Weyl group $W$ of $G$ are not $F$-conjugate. Therefore, we have

$$\langle R_{L_1}^G(1_{L_1^F}), R_{L_2}^G(1_{L_2^F}) \rangle = \sum_{T_1 \in T_1} \sum_{T_2 \in T_2} \frac{|T_1^F||T_2^F|}{|L_1^F||L_2^F|} \langle R_{L_1}^G R_{T_1}^{L_1}(1_{T_1^F}), R_{L_2}^G R_{T_2}^{L_2}(1_{T_2^F}) \rangle$$

$$= \sum_{T_1 \in T_1} \sum_{T_2 \in T_2} \frac{|T_1^F||T_2^F|}{|L_1^F||L_2^F|} \langle R_{T_1}^G(1_{T_1^F}), R_{T_2}^G(1_{T_2^F}) \rangle$$

$$= 0,$$

where the equalities hold by [10] Proposition 12.13, by the transitivity of Lusztig induction, and by [10] Corollary 11.16, respectively. On the other hand, since the Lusztig restriction $^*R_{L_i}^G(1_{G^F}) = 1_{L_i^F}$ by the proof of [10] Corollary 12.7, we have

$$\langle R_{L_i}^G(1_{L_i^F}), 1_{G^F} \rangle_{G^F} = \langle 1_{L_i^F}, ^*R_{L_i}^G(1_{G^F}) \rangle_{L_i^F} = \langle 1_{L_i^F}, 1_{L_i^F} \rangle_{L_i^F} = 1$$

for $i = 1, 2$, and so $R_{L_1}^G(1_{L_1^F})$ and $R_{L_2}^G(1_{L_2^F})$ have a nontrivial common irreducible constituent. Thus the lemma follows.

**Proof of Theorem 1.2** If $S$ is an alternating group, a sporadic simple group or the Tits group, then the theorem holds by [3] Propositions 3.2, 3.5] or by using GAP [13], respectively. So we may assume that $S$ is a simple group of Lie type. By [3] Proposition 3.8], the defining characteristic is adjacent to all other prime divisors of $|S|$. Thus the theorem holds by Lemma 3.3. 

\[ \square \]
4. Centralizers of Sylow subgroups

Here we investigate the centralizer of a Sylow subgroup of a finite simple group $S$ of Lie type in the automorphism group $\text{Aut}(S)$ of $S$, and prove Theorem 4.9.

According to [16, Theorem 2.5.1], $\text{Aut}(S)$ is generated by the inner automorphisms, diagonal automorphisms, field automorphisms and graph automorphisms of $S$. We will identify $\text{Inn}(S)$ with $S$. Also, if we let $G$ be a simple algebraic group of adjoint type over $F_p$, and $F$ a Steinberg endomorphism of $G$ such that $S = O_{p'}^e(G^F)$, then the group $G^F$ is exactly the subgroup of $\text{Aut}(S)$ generated by $S$ and its diagonal automorphisms. To be concise, we will fix this setup of $S$ throughout this section. Furthermore, if $S$ is untwisted, we let $F$ be split Frobenius.

In Table 1, we collect some related data about a finite simple group $S$ of Lie type for later use. The number $d$ is the index $|G^F : S|$, and $T$ is a cyclic subgroup of $S$ which corresponds to a maximal torus of $G^F$. The order of each $T$ in Table 1 is from [18, Tables 6 and 7]. In addition, as listed in [29, Tables 1 and 2] and [30, Table 1], the group $T_e$ is a Sylow $\Phi_e$-subgroup of $T$ with $e$ a regular number of $S$ in the sense of [30], where $\Phi_e := \Phi_e(q)$.

Recall that if $p$ is a prime and $t,n$ are integers greater than 1, then $p$ is a Zsigmondy prime divisor of $t^n - 1$ if $p$ divides $t^n - 1$, but $p$ does not divide $t^m - 1$ for $0 < m < n$. Clearly, $p$ is a Zsigmondy prime divisor of $t^n - 1$ if and only if $p \mid \Phi_n(t)$, but $p$ does not divide $\Phi_m(t)$ for $0 < m < n$. The well-known Zsigmondy’s Theorem asserts that for any $n,t > 1$, $t^n - 1$ has a Zsigmondy prime divisor unless $(n,t) = (6,2)$ or $n = 2$ and $t$ has the form $2^e - 1$ for some integer $e$ (see [33]).

Assume that $|T_e|$ has a Zsigmondy prime $r$. If $S \neq D_n(q)$, then $S$ has a Sylow $r$-subgroup, say $R$, contained in $T_e$. The order of $N_{G^F}(R)/C_{G^F}(R)$ in Table 1 follows from [11, Lemma 2.7], [18, Tables 6 and 7] and the regularity of $e$.

4.1. Simple groups without graph automorphisms. Here we investigate the centralizer of a Sylow subgroup of a simple group $S$ of Lie type in a subgroup of $\text{Aut}(S)$ generated by $S$, its diagonal automorphisms and field automorphisms.

Lemma 4.1: Let $S$ be a finite simple group of Lie type defined over a finite field $\mathbb{F}_q$, where $q = p^f$. Assume that the order $|T_e|$ in Table 1 has a Zsigmondy prime $r$, and let $R \in \text{Syl}_r(S)$. If $\phi$ is a field automorphism of $S$, then $C_{A_1}(R) \leq G^F$, where $A_1 = G^F \langle \phi \rangle$. 
\[
I + b\zeta^\lambda - b = \Phi^2 \quad \text{and} \quad I + b\zeta^\lambda + b + \varepsilon b\zeta^\lambda + \varepsilon = \Phi^2 \quad I + b\zeta^\lambda + b = \Phi^2 \quad \text{(Here) Zsigmondy}
\]

| \(f_g\) | \(f_{\zeta}\) | \(f_6\) | \(f_\zeta\) | \(f_{\zeta}\) | \(f_{(1+u)\zeta}\) | \(f_u\) | \(f_{u\zeta}\) | \(f_{(1+u)\zeta}\) | \(f_{(1+u)\zeta}\) |
|---------|---------|--------|--------|--------|----------------|--------|---------|--------|----------------|
| 6       | 6       | 6      | \(\Phi\) | \(\Phi\) | \(\Phi\)       | \(\Phi\) | \(\Phi\) | \(\Phi\) | \(\Phi\)       |
| 12      | 12      | 12     | \(\Phi\) | \(\Phi\) | \(\Phi\)       | \(\Phi\) | \(\Phi\) | \(\Phi\) | \(\Phi\)       |
| 21      | 21      | 21     | \(\Phi\) | \(\Phi\) | \(\Phi\)       | \(\Phi\) | \(\Phi\) | \(\Phi\) | \(\Phi\)       |
| 18      | 18      | 18     | \(\Phi\) | \(\Phi\) | \(\Phi\)       | \(\Phi\) | \(\Phi\) | \(\Phi\) | \(\Phi\)       |
| 9       | 9       | 9      | \(\Phi\) | \(\Phi\) | \(\Phi\)       | \(\Phi\) | \(\Phi\) | \(\Phi\) | \(\Phi\)       |
| 0       | 0       | 0      | \(\Phi\) | \(\Phi\) | \(\Phi\)       | \(\Phi\) | \(\Phi\) | \(\Phi\) | \(\Phi\)       |
| 8       | 8       | 8      | \(\Phi\) | \(\Phi\) | \(\Phi\)       | \(\Phi\) | \(\Phi\) | \(\Phi\) | \(\Phi\)       |
| 1       | 1       | 1      | \(\Phi\) | \(\Phi\) | \(\Phi\)       | \(\Phi\) | \(\Phi\) | \(\Phi\) | \(\Phi\)       |
| 1       | 1       | 1      | \(\Phi\) | \(\Phi\) | \(\Phi\)       | \(\Phi\) | \(\Phi\) | \(\Phi\) | \(\Phi\)       |

**Table 1.** Data related to a simple group of the type

\(s\)
Proof. Assume that \( 1 \neq \phi' \in \langle \phi \rangle \) is induced by the field automorphism of \( \overline{\mathbb{F}}_q \) sending every element of \( \overline{\mathbb{F}}_q \) to its \( p^k \) with \( 1 \leq k < f \).

First, we assume that \( F \) is split and \( S \neq D_n(q) \). By assumption, we know that \( R \) is cyclic and also a Sylow \( r \)-subgroup of \( G^F \). Write \( R = \langle x \rangle \). Let \( T \subset B \) be an \( F \)-stable maximal torus inside an \( F \)-stable Borel subgroup of \( G \). Let \( h \in G \) be such that \( T^{h^{-1}} \) is \( F \)-stable and \( R \subset T^{h^{-1}} \). Then \( y = x^h \in T \) and

\[
(T^{h^{-1}})^{\phi'} = T^{\phi(h^{-1})}
\]

is \( F \)-stable. Replacing a \( G^F \)-conjugate of \( R \) if necessary, we may assume that \( \phi'g^{-1} \in \mathcal{C}_A(R) \) for some \( g \in G^F \). Then \( x^{\phi'} = x^g \).

Notice that the maximal tori of \( G^F \) containing a Sylow \( r \)-subgroup of \( G^F \) are \( G^F \)-conjugate by the choice of \( R \). Hence there is \( g_1 \in G^F \) such that \( \phi'(h^{-1})g_1h \in N_G(T) \). Since \( F \) is split, we have

\[
N_G(T)/T = (N_G(T)/T)^F,
\]

so by [25, Proposition 23.2], there is some \( n_1 \in N_{G^F}(T) \) inducing the same conjugation action on \( T \) as \( \phi'(h^{-1})g_1h \). However, since \( C_G(T) = T \), it follows that \( t\phi'(h^{-1}) = n_1h^{-1}g_1^{-1} \) for some \( t \in T \). Hence

\[
x^{\phi'} = (y^{h^{-1}})^{\phi'} = (y^{p^k})^{\phi'(h^{-1})} = (y^{p^k})^{n_1h^{-1}g_1^{-1}}.
\]

Comparing \( T^{n_1h^{-1}} \) and \( T^{h^{-1}} \), we deduce that \( t'^n_1h^{-1} = h^{-1}n_2 \) for some \( t' \in T \) and \( n_2 \in G^F \), and so \( x^{\phi'} = (y^{p^k})^{-1}n_2g_1^{-1} \). Thus \( x^{gn_0^{-1}} = x^{p^k} \), where \( n_0 := n_2g_1^{-1} \). This implies \( gn_0^{-1} \in N_{G^F}(R) \). Therefore, the order \( m \) of the automorphism induced by the conjugation of \( gn_0^{-1} \) on \( R \) divides \( |N_{G^F}(R)/C_{G^F}(R)| \) which is \( e \) or \( e/2 \) by Table [I]

Observe that \( x^{p^k} = x \), namely \( x^{p^k} = 1 \). Hence \( r \mid p^{mk} - 1 \), and \( r \mid \Phi_{mk}(p) \) for some integer \( u \geq 1 \). However, we have \( \text{ord}_r(p) = ef \). By [25, Lemma 25.13], we have \( \frac{mk}{u} = ef \) for some integer \( l \geq 0 \). Thus

\[
\frac{mk}{u} \leq mk \leq ek < ef \leq ef^l.
\]

This contradiction shows that \( C_A(R) \leq G^F \).

Now, suppose that \( S = D_n(q) \). If \( n \) is odd, then \( R \) is cyclic. Write \( R = \langle x \rangle \), and let \( \xi \) be an eigenvalue of \( x \) that is a primitive \( |R| \)-th root of unity. Then the eigenvalues of \( x \) are \( \xi^{\pm 1}, \xi^{\pm q}, \ldots, \xi^{\pm q^{n-1}} \). Note that the eigenvalues of \( \phi'(x) \) are \( \xi^{\pm p^k}, \xi^{\pm q^k}, \ldots, \xi^{\pm q^{n-1}p^k} \). Since \( r \) is a Zsigmondy prime of \( \Phi_n \) and \( 1 \leq k < f \), we deduce that \( \xi^{p^k} \neq \xi^{\pm q^i} \) for any \( 0 \leq i \leq n - 1 \). Hence the eigenvalues of \( x \)
and \( \phi'(x) \) are different, and so \( x \) and \( \phi'(x) \) are not conjugate in \( G \). In particular, there does not exist \( g \in G^F \) such that \( \phi'(x) = x^g \). Therefore \( C_{A_1}(R) \leq G^F \).

If \( n \) is even, then \( R \) is the direct product of two cyclic groups of order \( |R|^{1/2} \). We may take \( x \in R \) such that the eigenvalues of \( x \) different from 1 are \( \xi^{\pm 1}, \xi^{\pm q}, \ldots, \xi^{\pm q^2} \), where \( \xi^{q^2+1} = 1 \). Comparing the eigenvalues of \( x \) and \( \phi'(x) \), we deduce that there does not exist \( g \in G^F \) such that \( \phi'(x) = x^g \). Hence \( C_{A_1}(R) \leq G^F \) in this case.

In the following, we assume that \( F \) is twisted. Then \( R \) is cyclic by the choice of \( R \). Write \( R = \langle x \rangle \), and let \( \xi \) be an eigenvalue of \( x \) that is a primitive \( |R| \)-th root of unity.

Suppose that \( S = 2A_n(q) \). If \( n \) is even, then \( r \) is a Zsigmondy prime of \( \Phi_{2(n+1)} \) and the eigenvalues of \( x \) are \( \xi, \xi^{-q}, \ldots, \xi^{-(n-1)} \), and if \( n \) is odd, then \( r \) is a Zsigmondy prime of \( \Phi_{2n} \) and the eigenvalues of \( x \) are \( \xi, \xi^{-q}, \ldots, \xi^{-(n-1)} \) and 1.

With a similar argument on the difference of eigenvalues of \( x \) and \( \phi'(x) \) as for \( S = D_n(q) \), we conclude that \( C_{A_1}(R) \leq G^F \).

If \( S = 2D_n(q) \), then \( r \) is a Zsigmondy prime of \( \Phi_{2n} \) by the choice of \( r \) and the eigenvalues of \( x \) different from 1 are \( \xi^{\pm 1}, \xi^{\pm q}, \ldots, \xi^{\pm q^2} \). As argued above, we obtain \( C_{A_1}(R) \leq G^F \).

We mention here that the argument for \( S = 2D_n(q) \) is also valid for \( S = B_n(q) \).

If \( S = 2B_2(q) \), then \( k \) is odd and the eigenvalues of \( x \) different from 1 are \( \xi^{\pm 1}, \xi^{\pm (q+1)} \) by [32, §4.2.2]. It is easy to see that \( \xi^{2^k} \), which is an eigenvalue of \( \phi'(x) \), is not any one of them. Hence \( x \) and \( \phi'(x) \) have different eigenvalues, and so they are not conjugate in \( G \). In particular, there does not exist \( g \in S = G^F \) such that \( \phi'(x) = x^g \). Thus \( C_{A_1}(R) \leq G^F \).

If \( S = 2G_2(q) \) then, since \( S \leq \Omega_7(q) \) by [32, §4.3], we obtain that \( R \) is a common Sylow \( r \)-subgroup of \( S \) and \( \Omega_7(q) \). Hence we similarly have that \( \phi'(x) \) and \( x^g \) have different eigenvalues and so \( \phi'(x) \neq x^g \) for any \( g \in S \), namely \( C_{A_1}(R) \leq S = G^F \).

If \( S = 2F_4(q) \) then by [32, §4.9], we have \( S \leq GO_{26}(q) \). This implies that the eigenvalues of \( x \), which are primitive \( |R| \)-th roots of unity, are either one or two copies of \( \xi^{\pm 1}, \xi^{\pm q}, \ldots, \xi^{(q^2+1)} \), or \( \xi^{\pm 1}, \xi^{\pm q}, \ldots, \xi^{\pm (q^2+1)}, \eta^{\pm 1}, \eta^{\pm q}, \ldots, \eta^{\pm (q^2+1)} \) for some \( \eta \) different from any \( q \)-power of \( \xi \). However, since the order of \( \phi' \) is odd in this case, it follows that the eigenvalue \( \xi^p \) of \( \phi'(x) \) can not be a \( q \)-power of \( \eta \). As argued above, \( \xi^p \) is not any one of \( \xi^{\pm 1}, \xi^{\pm q}, \ldots, \xi^{(q^2+1)} \), either. Hence \( \phi'(x) \neq x^g \) have different eigenvalues, and so \( C_{A_1}(R) \leq S = G^F \).
If $S = 3D_4(q)$ then $S \leq \Omega_8^+(q^3)$ by [32, §4.6]. Similarly, we have

$$C_{A_1}(R) \leq S = G^F.$$ 

Finally, suppose that $S = 2E_6(q)$. By [31, Fig. 6], maximal subgroups of $S$ containing a Sylow $r$-subgroup of $S$ are all isomorphic to $\text{PSU}_3(q^3).3$ and conjugate in $S$. Let $M$ be one of them such that $N_{G^F}(M)$ contains $C_{G^F}(R)$, and let $M_1$ be the normal subgroup of $M$ such that $M_1 \cong \text{PSU}_3(q^3)$. Clearly, we have $N_{A_1}(M_1) \supseteq N_{A_1}(M)$ and $N_{A_1}(M) \supseteq N_{G^F}(M)$. After replacing a $G^F$-conjugate of $M$, $M_1$ and $R$ if necessary, we may assume that $\alpha$ normalizes $M$ and $M_1$ and

$$C_{A_1}(R) \leq N := N_{A_1}(M) = N_{G^F}(M)\langle \alpha \rangle$$

which is a maximal subgroup of $A_1$. Hence $\alpha$ induces a field automorphism of $M_1$. By the above case where $S = 2A_2(q)$, we have $C_{A_1}(R) = C_N(R) \leq N_{G^F}(M)$. Hence $C_{A_1}(R) \leq G^F$, finishing the proof. 

**Proposition 4.2:** Let $S$ be a finite simple group of Lie type defined over a finite field $\mathbb{F}_q$, where $q = p^f$. If $S$ has no graph automorphism, then $S$ has a Sylow $r$-subgroup $R$ such that $C_A(R) \leq G^F$, where $A = \text{Aut}(S)$.

Moreover, the prime $r$ is explicit in the following sense: if the $|T_e|$ in Table [7] has a Zsigmondy prime, then we let $r$ be this prime; otherwise, either $S = B_3(2) \cong C_3(2), 2A_3(2)$ or $S = A_1(q)$ and $q + 1$ does not have a Zsigmondy prime, in which cases $r$ is 7, 5 or any prime divisor of $2(q - 1)$, respectively.

**Proof.** We first suppose $S = A_1(q) \cong L_2(q)$ so that $|G^F : S| = (2, q - 1)$. If $q + 1$ has a Zsigmondy prime $r$, then by Lemma [4.1], we have $C_A(R) \leq G^F$, where $R \in \text{Syl}_r(S)$. For the case where $q + 1$ does not have a Zsigmondy prime, it follows from Zsigmondy’s Theorem that $q = 2^k - 1$ for some $k \in \mathbb{N}$. (Notice that $q = 2^3 + 1 = 9$ does not occur since otherwise $q + 1 = 10$ has a Zsigmondy prime 5, a contradiction.) By [6, Lemma 2.4(a)], $q = p$ is a Mersenne prime so that $f = 1, A = G^F$, and the conclusion obviously holds.

We now suppose $S \not\cong A_1(q)$. The result follows by Lemma [4.1] if the $|T_e|$ in Table [1] has a Zsigmondy prime. In the remaining cases, we have $S = A_5(2), B_3(2) \cong C_3(2), D_4(2)$, or $2A_3(2)$. (Notice that $2A_2(2)$ is solvable.) Since $S$ has no graph automorphism, we have $S = B_3(2) \cong C_3(2)$ or $2A_3(2)$. Thus, the proposition holds since $A = G^F$ in both cases. 

4.2. Simple groups with a graph automorphism, I. Here we investigate the centralizer of a Sylow subgroup of a simple group $S$ in the automorphism group $A$ of $S$, where $S$ has a graph automorphism and is one of the groups $A_n(q)$ $(n > 1)$, $D_n(q)$ $(n \geq 4)$, $E_6(q)$ or $F_4(q)$ $(q = 2^f)$.

**Lemma 4.3:** Let $S = A_n(q)$, where $q = p^f$. Suppose that $(n, q) \neq (2, 4)$. Let $r$ be a prime divisor of $|S|$ satisfying the following: if the order $|T_{n+1}|$ in Table 7 has a Zsigmondy prime then $r$ is this prime; otherwise $(n+1, q) = (6, 2)$ and $r = 31$. Then $C_A(R) \leq G^F$, where $R \in \text{Syl}_r(S)$.

**Proof.** In order to prove the lemma, it suffices to show that $C_{G^F(\alpha)}(R) \leq G^F$ for any $\alpha \in A = \text{Aut}(S)$.

Notice that $G = \text{PGL}_{n+1}$ and $\alpha$ extends to an automorphism of $G^F$ and an endomorphism of $G$ by [16, Theorem 2.5.1]. By Lemma 4.1, we may assume $\alpha$ is not a field automorphism, so that $n \geq 2$ since $L_2(q)$ has no graph automorphism. Then we have $\alpha = g\sigma_i$ for some $g \in G^F$ and some morphism $\sigma_i$ of $G$ which is the product of the field automorphism such that $(a_{ij}) \mapsto (a_{ij}^{p^f})Z(GL_{n+1})$ with $(a_{ij}) \in \text{GL}_{n+1}$, the inverse-transpose map of matrices, and the conjugation induced by $M_0M_1$, where

$$M_0 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

and $M_1 = \text{diag}(1, -1, 1, -1, \ldots)$ by the proof of [25, Theorem 11.12].

Assume $S \nneq A_5(2)$. Since $R \subseteq T_{n+1} = \langle x \rangle$ lies in a maximal torus of $G$, there is $h \in G$ such that $x^{h^{-1}} = \text{diag}(\lambda, \lambda^q, \ldots, \lambda^{q^n})$, where $\lambda$ is a primitive $\frac{q^{n+1}-1}{q-1}$-th root of unity. Then we have

$$x^\alpha = \text{diag}(\lambda, \lambda^q, \ldots, \lambda^{q^n})^{h\alpha} = (\text{diag}(\lambda, \lambda^q, \ldots, \lambda^{q^n})^{\sigma_i})^{(hg)^{\sigma_i}} = \text{diag}(\lambda^{-p^i q^n}, \ldots, \lambda^{-p^i q}, \lambda^{-p^i})^{M_0M_1(hg)^{\sigma_i}}.$$

We claim that $\lambda^{-p^i} \not\in \{\lambda, \lambda^q, \ldots, \lambda^{q^n}\}$. Otherwise, assume that $\lambda^{-p^i} = \lambda^m$ for some $0 \leq m \leq n$. We have $\lambda^{q^m+p^i} = \lambda^{p^f(p^f-m^{-1}+1)} = 1$. It follows that $\frac{q^{n+1}-1}{q-1} | p^{f-m^{-1}}+1$, and so $i = 0, m = n = 1$, which is a contradiction. Hence the claim holds. This implies that $x^\alpha$ is not conjugate to $x$ in $G$. In particular, $x^\alpha$ is not conjugate to $x$ in $G^F$. 

Now we put $m = (q^n - 1)_{r'}$ and $y = x^m$ so that $R = \langle y \rangle$. As argued above, we have

$$\lambda^{-mp^i} \not\in \{\lambda^m, \lambda^{mq}, \ldots, \lambda^{mq^n}\},$$

and then we may conclude that $y^\alpha$ is not conjugate to $y$ in $G^F$.

Finally, let $S = A_5(2)$. In this case, we may assume $y$ is a generator of $R$ such that $y$ is $G$-conjugate to $\text{diag}(\sigma, \sigma^q, \ldots, \sigma^{q^2}, 1)$, where $\sigma$ is a primitive 31st root of unity. Using a similar argument as above, we get that $C_A(R) \leq G^F$, finishing the proof.  

**Lemma 4.4:** Let $S = D_n(q)$ with $n \geq 4$. Let $r$ be a Zsigmondy prime of $\Phi_n$ as in Table [7] unless $q = 2$ and $n = 6$, in which case let $r = 7$. If $R \in \text{Syl}_r(S)$, then $C_A(R) \leq G^F$.

**Proof.** If $(n, q) = (6, 2)$, then $A = SO^+_{12}(2)$ and the result can be checked directly by [13]. In the following we assume $(n, q) \neq (6, 2)$.

The group $\text{PSL}_n(q)$ embeds into $G^F$ in a natural way. Denote

$$K_n = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

We may choose $\langle x \rangle \leq R$, where $x = \text{diag}(X_1, X_2)$, $X_1 \in \text{PSL}_n(q)$ of order $(q^n - 1)_r$ and $X_2 = K_nX_1^{-tr}K_n$. Let $\Theta$ be the subgroup of $A$ generated by the graph automorphism and the field automorphisms of $S$. In order to prove the lemma, it suffices to show that for any $a_0 \in G^F$ and any $\mu \in \Theta$ of prime order, the product $a_0\mu$ does not centralize $R$.

We first assume $\mu$ is the graph automorphism of $S$ so that by [25, Exercise 20.1 and Example 22.9], it can be induced by the conjugation of

$$g = \begin{pmatrix} I_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-1} \end{pmatrix}$$

which is in $\text{GO}_{2n}(q) \setminus \text{SO}_{2n}(q)$ if $q$ is odd and $\text{SO}_{2n}(q) \setminus \text{SO}_{2n}(q)'$ if $q$ is even.

Assume $a_0 \in G^F$ such that $x^{a_0}g = x$. Notice that $C := C_{\text{GL}_{2n}(q)}(x)$ is a maximal torus of order $(q^n - 1)^2$ and consists of elements having the form $\text{diag}(M_{11}, M_{22})$, where $M_{11} \in T_{1,n}$, $M_{22} \in T_{2,n}$, and $T_{1,n}$ and $T_{2,n}$ are two Singer cycles of $\text{GL}_n(q)$. Furthermore, we have $C \cap \text{CO}_{2n}(q) \subseteq \text{SO}_{2n}(q)$ since
now \( M_{22} = K_nM_{11}^{-1}K_n \). In addition, if \( q \) is even then from the oddness of the order \(|C|\), we know that \( C \cap SO_{2n}(q) \subseteq SO_{2n}(q)' \). However, it follows that \( a_0 \in GO_{2n}(q) \setminus SO_{2n}(q) \) if \( q \) is odd and \( a_0 \in SO_{2n}(q) \setminus SO_{2n}(q)' \) if \( q \) is even, contradicting the choice of \( a_0 \).

We now assume that \( \mu \) is a field automorphism or the product of a field automorphism and the graph automorphism of \( S \). Comparing the eigenvalues of \( x^{a_0\mu} \) and \( x \), we conclude that \( x^{a_0\mu} \neq x \). Thus \( C_A(R) \leq G^F \), and we are done. □

**Lemma 4.5:** Let \( S = E_6(q) \), where \( q = p^f \). Let \( r \) be a Zsigmondy prime of \( \Phi_9 \) as in Table 1 and \( R \in \text{Syl}_r(S) \). Then \( C_A(R) \leq G^F \).

**Proof.** Let \( \alpha \in A \setminus G^F \) and \( A_1 = G^F \langle \alpha \rangle \). In order to prove the lemma, it suffices to show that \( C_{A_1}(R) \leq G^F \) for each \( \alpha \) of prime order \( s \).

By [31, Fig. 6], maximal subgroups of \( S \) containing a Sylow \( r \)-subgroup of \( S \) are all isomorphic to \( \text{PSL}_3(q^2) \cdot 3 \) and conjugate in \( S \). Let \( M \) be one of them such that \( N_{G^F}(M) \) contains \( C_{G^F}(R) \), and let \( M_1 \) be the normal subgroup of \( M \) such that \( M_1 \cong \text{PSL}_3(q^2) \). As in the proof of Lemma 4.1 for the case where \( S = 2E_6(q) \), we may assume that \( \alpha \) normalizes \( M \) and \( M_1 \) and

\[
C_{A_1}(R) \leq N := N_{A_1}(M) = N_{G^F}(M) \langle \alpha \rangle.
\]

If the order \( s \) of \( \alpha \) is odd, then by [15, (7-3)], \( \alpha \) is \( \text{Aut}(S) \)-conjugate to a field automorphism of \( S \). By appropriate replacement, we may assume that \( \alpha \) is a field automorphism of \( S \). Hence \( \alpha \) induces a field automorphism of \( M_1 \). Applying Lemma 4.3 on \( R \) and \( M_1 \), we have \( C_{A_1}(R) = C_N(R) \leq N_{G^F}(M) \).

Finally, suppose that \( s = 2 \). If \( \alpha \) is a field or graph-field automorphism of \( S \) then by [15, (9-1)]

\[
|C_{G^F}(\alpha)|_r = |E_6(q^{f/2})|_r = 1,
\]

and if \( \alpha \) is graph automorphism of \( S \) then by [16, Table 4.5.1] for \( q \) odd and [11, 19.9] for \( q \) even \(|C_{G^F}(\alpha)|_r = |F_4(q)|_r \) or \(|C_4(q)|_r \), which is 1 in both cases. Hence \( \alpha \) induces a non-trivial automorphism of \( M_1 \). Assume that \( g \in G^F \) such that \( g\alpha \) centralizes \( R \). Then \( (R^{g^{-1}})^{g\alpha} = R^{g^{-1}} \), namely \( \alpha \) normalizes \( R^{g^{-1}} \). Since \( |N_{G^F}(R^{g^{-1}})/C_{G^F}(R^{g^{-1}})| \) is odd, there does not exist \( n \in N_{G^F}(R^{g^{-1}}) \) induces the same isomorphism of \( R^{g^{-1}} \) as \( \alpha \). Thus \( C_{A_1}(R) = C_{A_1}(R^{g^{-1}})^g \leq G^F \), finishing the proof. □
Lemma 4.6: Let \( S = F_4(q) \), where \( q = p^f \). Let \( r = 17 \) if \( q = 2 \) or else let \( r \) be a Zsigmondy prime \( r \) of \( \Phi_{12} \) as in Table 1. If \( R \) is a Sylow \( r \)-subgroup of \( S \), then \( C_A(R) \leq S \).

Proof. The result directly follows from Lemma 4.1 if \( p \) is odd, in which case \( S \) has no graph automorphisms. So we may assume that \( p = 2 \) in the following. Then \( \text{Aut}(S) \) is the semi-direct product of \( S \) and \( \langle \delta \rangle \), where \( \delta \) is the automorphism of \( S \) induced by the graph automorphism of the corresponding Dynkin diagram, and squares to the field automorphism \( x \mapsto x^2 \). Since \( S = F_4(2) \) can be directly checked by the character table of \( \text{Aut}(F_4(2)) \) in [12], we assume \( q > 2 \) in the following.

In order to show that \( C_A(R) \leq S \), it is equivalent to show that \( C_{S\langle \delta^i \rangle}(R) \leq S \) for any \( i \in \mathbb{Z} \). Also, it is equivalent to show that \( C_{S\langle \delta^i \rangle}(R) \leq S \) for any \( \delta^i \) of prime order. If \( f \) is even, then all the automorphisms \( \delta^i \) of prime order are field automorphisms of \( S \). Hence the result follows by applying Lemma 4.1.

Now we suppose that \( f \) is odd. It is easy to see that all the automorphisms \( \delta^i \) of odd prime order are field automorphisms of \( S \). Hence the result also follows by applying Lemma 4.1. So it remains to show that \( C_{S\langle \delta^f \rangle}(R) \leq S \). In this case, we have

\[ |S\langle \delta^f \rangle : S| = 2. \]

Observe that \( C_{S\langle \delta^f \rangle}(R) \) is contained in some proper maximal subgroup of \( S\langle \delta^f \rangle \).

By [31, Fig. 5], there are two \( S \)-conjugacy classes of maximal subgroups containing \( R \). Moreover, all of them are isomorphic to \(^3D_4(q).3\). If \( \delta^f \) normalizes such a maximal subgroup \( M \), then all those maximal subgroups are conjugate in \( S\langle \delta^f \rangle \). Hence we may prove the lemma by applying Lemma 4.1 and a similar argument as in the proof of Lemma 4.5. We now assume that \( \delta^f \) normalizes none of those maximal subgroups. Then all of them are only in the maximal subgroup \( S \) of \( S\langle \delta^f \rangle \). Thus it follows that \( C_{S\langle \delta^f \rangle}(R) \leq S \), finishing the proof.

4.3. Simple groups with a graph automorphism, II. In this subsection we investigate the centralizer of a Sylow subgroup of a simple group \( S \) in the subgroup \( A_0 \) of the automorphism group of \( S \) generated by \( S \) and all of its field automorphisms, where \( S \) is one of the groups \( B_2(q) \) \((q = 2^f)\) or \( G_2(q) \) \((q = 3^f)\).

Lemma 4.7: Let \( S = B_2(q) \cong O_5(q) \), where \( q = 2^f > 2 \). Let \( r \) be a Zsigmondy prime of \( 2^{2f} - 1 \) with respect to \((2, 2f)\) unless \( f = 3 \), in which case let \( r = q - 1 = 7 \). If \( R \in \text{Syl}_r(S) \), then \( C_{A_0}(R) \leq S \).
Proof. We may assume \( f \neq 1 \). In order to prove the lemma, it suffices to show that \( C_{S(\phi)}(R) \leq S \) for any field automorphism \( \phi \) of \( S \) of prime order.

We may first suppose that \( f \neq 3 \). Since the Weyl group of \( S \) is a 2-group and \( r \) is clearly odd, it follows from [7, Proposition 5.2] that \( R \) is \( O_5(q^2) \)-conjugate to

\[
R_1 = \{ \text{diag}(\lambda, \mu, 1, \mu^{-1}, \lambda^{-1}) \mid \lambda, \mu \in \mathbb{F}_{q^2}^* \text{ and have order } (q+1)r \}.
\]

The field automorphism \( \phi \) extends to \( O_5(q^2) \), and we may assume that \( \phi_{R_1}: R_1 \to R_1 \) via \( t \mapsto t^{p^k} \) for some \( 1 \leq k < f \). In order to prove the lemma, it suffices to show that for any \( g \in O_5(q^2) \), there is some \( t \in R_1 \) such that \( t^{g^{-1}} \neq t \), namely, \( t^\phi \neq t^g \).

Let \( u_1 = \text{diag}(\lambda, 1, 1, \lambda^{-1}) \in R_1 \) be of order \( (q+1)_r \). Clearly, \( \lambda^{p^k} \neq 1, \lambda, \lambda^{-1} \).

So \( u_1^\phi \) and \( u_1 \) have different eigenvalues, which means that \( u_1^\phi \) and \( u_1 \) are not conjugate in \( O_5(q^2) \), as wanted.

We now let \( f = 3 \) and so \( r = 7 \). In this case, the Sylow 7-subgroup \( R \) is \( O_5(q) \)-conjugate to

\[
R_1 = \{ \text{diag}(\lambda, \mu, 1, \mu^{-1}, \lambda^{-1}) \mid \lambda, \mu \in \mathbb{F}_q^* \text{ and have order } (q-1)_7 = 7 \}.
\]

Similarly, we may assume that \( \phi_{R_1}: R_1 \to R_1 \) via \( t \mapsto t^2 \) and take \( u_1' = \text{diag}(\lambda', 1, 1, \lambda'^{-1}) \in R_1 \) with \( \lambda' \neq 1 \). Checking eigenvalues of \( u_1^\phi \) and \( u_1' \) again, we conclude that they are not conjugate in \( O_5(q) \), and we are done.

Lemma 4.8: Let \( S = G_2(q) \), where \( q = 3^f \) and \( f > 1 \). Assume that \( \phi \) is a field automorphism of \( S \) of prime order. If \( R \) is a Sylow \( r \)-subgroup of \( S \) for some Zsigmondy prime divisor \( r \) of \( \Phi_2 \), then \( C_{A_0}(R) \leq S \).

Proof. We may assume \( f \neq 1 \). As in Lemma 4.7 it suffices to show that \( C_{S(\phi)}(R) \leq S \) for any field automorphism \( \phi \) of \( S \) of prime order.

Let \( T_2 \) be a Sylow 2-torus of \( G \). Since \( \phi \) is of prime order, we have \( f > 1 \).

So we may assume that \( \phi \) sends \( t \in T_2 \) to \( t^{p^k} \) for some integer \( 1 \leq k < f \). In addition, since the order of the Weyl group of \( S \) is 12 and \( r \geq 5 \), we may assume that \( R \) is a Sylow \( r \)-subgroup of \( T_2^{12} \) by [25, Theorem 25.14 (1)].

In order to show the lemma, it suffices to show that for any \( g \in S \), there is some \( t \in R \) such that \( t^{g^{-1}} \neq t \), namely, \( t^\phi \neq t^g \). We now denote by \( \iota \) an embedding of \( G_2(q) \) into \( H = P\Omega_8^-(q^2) \). Notice that \( H \) has a field automorphism, also denoted by \( \phi \), mapping \( (m_{ij})_{8 \times 8} \in H \) to \( (m_{ij}^{p^{k}})_{8 \times 8} \). Hence it is enough to show that for any \( g \in H \), there is some \( t \in R \) such that \( \iota(t)^\phi \neq \iota(t)^g \).
Clearly, all eigenvalues of elements of $T^F_2$ are in $\mathbb{F}_{q^2}$. There is $h \in H$ such that
\[
\iota(T^F)^h = T_D
\]
\[
= \{ \text{diag}(\lambda, \mu, \lambda^{-1}, 1, 1, \mu^{-1}, \mu^{-1}, \lambda^{-1}) | \lambda, \mu \in \mathbb{F}_{q^2}^* \text{ and have order } q+1 \}. 
\]
Let $t \in R$ be such that
\[
u = \iota(t)^h = \text{diag}(1, \alpha, \alpha^{-1}, 1, 1, \alpha, \alpha^{-1}, 1) \in T_D, 
\]
where $\alpha = \zeta^m$, $m = (q^2 - 1)r'$ and $\zeta$ is a generator of $\mathbb{F}_{q^2}^*$. In particular, $\phi(u)$ is $H$-conjugate to $\phi(\iota(t))$.

If $\phi(u)$ is $H$-conjugate to $u$, then $\phi(u)$ and $u$ have the same eigenvalues. This implies $\alpha^p = \alpha^{-1}$, which is not possible because of the choice of $r$. Hence $\phi(u)$ and $u$ are not conjugate in $H$, and we are done.

**Theorem 4.9:** Let $S$ be a finite simple group of Lie type defined over a finite field $\mathbb{F}_q$, where $q = p^f$. Write $A = \text{Aut}(S)$ and $A_0$ for the subgroup of $A$ generated by $S$ and all of its field automorphisms. Let $r$ be a prime and $R \in \text{Syl}_r(S)$.

Then:

(i) $C_A(R) \leq G^F$ if one of the following occurs:

1. The group $S$ and the prime $r$ are as listed in Table 2 or 1.
2. $S = A_1(q)$ and $q + 1$ does not have a Zsigmondy prime, and $r$ is any prime divisor of $2(q - 1)$.

(ii) $C_{A_0}(R) \leq S$ if one of the following occurs:

3. $S = A_2(4)$ and $r = 7$, $S = B_2(8)$ and $r = 7$, or $S = B_2(2^f)$ ($2^f \neq 2, 8$), and $r$ is a Zsigmondy prime of $2^f + 1$ with respect to $(2, 2f)$.
4. $S = G_2(3^f)$ ($f > 1$), and $r$ is a Zsigmondy prime of $3^f + 1$.

| $S$  | $A_5(2)$ | $B_3(2) \cong C_3(2)$ | $D_6(2)$ | $^2A_3(2) \cong U_4(2)$ | $G_2(3)$ | $F_4(2)$ | $^2F_4(2)'$ |
|------|----------|-----------------------|----------|----------------------|---------|---------|-------------|
| $r$  | 31       | 7                     | 7        | 5                    | 13      | 17      | 13          |

**Proof.** Part (i) follows by Proposition 4.2, Lemmas 4.3–4.6. The case where $S = A_2(4)$ in (3) can be directly checked using [13]. For the other cases of (ii), the result follows by Lemmas 4.7 and 4.8.
We would like to mention here that the $p$-part of the centralizer of a Sylow $p$-subgroup of a simple group $S$ in the automorphism group of $S$ for an odd prime $p$ is known based on [17, Theorem A]. However, our situation has to focus on the $p'$-part of the order of the centralizer, in which case inner automorphisms and graph automorphisms of the simple group cause extra difficulties.

5. Block graphs of almost simple groups

In this section we show that block graphs of almost simple groups have a triangle two of whose vertices can be determined.

We keep the notation in Section 4. Let $\pi$ be a set of primes, and for a finite group $G$, let $\pi(G)$ be the set of prime divisors of $|G|$. We write $\Delta(G)_{\pi}$ for the full subgraph of $\Delta(G)$ whose vertices are those of $\pi(G) \cap \pi$. The main purpose of this section is to prove the following result.

**Theorem 5.1:** Let $S \leq G \leq A$, where $A = \text{Aut}(S)$ and $S$ is a non-abelian simple group. Then for each prime divisor $\ell$ of $|S|$, the subgraph $\Delta(G)_{\pi(S)}$ has a triangle containing $\ell$.

**Proof.** For the case where $S$ is not of Lie type, Theorem 5.1 follows by [2, Proposition 2.1] or by using [13]. For the case where $S$ is a simple group of Lie type, this is done by the subsequent Proposition 5.6.

5.1. Principal blocks and diagonal automorphisms. From now on, we investigate block graphs of almost simple groups with socle isomorphic to a simple group of Lie type. Here we mention a fundamental result.

**Theorem 5.2:** Let $S$ be a finite simple group of Lie type defined over a finite field of characteristic $p$, and let $\ell$ be a prime different from $p$. Denote by $\text{Uch}(S)$ the set of unipotent characters of $S$. Then the restriction map from $\text{Uch}(G^F)$ to $\text{Uch}(S)$ is bijective and keeps their partitions into $\ell$-blocks.

**Proof.** This is [8, Theorem 17.1].

**Lemma 5.3:** Let $S \leq H \leq G \leq A$, where $A = \text{Aut}(S)$ and $S$ is a finite simple group of Lie type over a finite field of characteristic $p$. Then the principal $p$-block of $G$ is the unique $p$-block covering the principal $p$-block of $H$.

**Proof.** Let $P \in \text{Syl}_p(S)$ and $P_H \in \text{Syl}_p(H)$ with $P \subseteq P_H$. By [3, Lemma 2.2], we know that $C_A(P)$ is a $p$-group. Hence $C_G(P_H) \leq C_G(P) \leq C_A(P)$ is a $p$-group. Now the lemma follows by [23, Lemma 2.3].
Lemma 5.4: Let $S \leq G \leq G^F$, where $S$ is a finite simple group of Lie type over a finite field of characteristic $p$. Then $\Delta(G)$ is complete.

Proof. By checking the index $d := |G^F : S|$ in Table 1 case by case, we have $\pi(G) = \pi(S)$. Let $\ell_1, \ell_2 \in \pi(S)$. If $p \nmid \ell_1 \ell_2$, then by Lemma 3.3 there is a nontrivial unipotent character of $S$ lying in both principal $\ell_1$- and $\ell_2$-blocks of $S$. It follows from Theorem 5.2 that there is a nontrivial unipotent character $\chi$ of $G^F$ lying in both principal $\ell_1$- and $\ell_2$-blocks of $G^F$. Since $G^F/S$ is abelian, we have that $G$ is normal in $G^F$. Hence $\chi_G \in \text{Irr}(G)$ is in the principal $\ell_i$-block of $G$ for $i = 1, 2$, namely $\ell_1$ and $\ell_2$ are adjacent in $\Delta(G)$ in this case.

Now we may assume that $\ell_1 = p$. Then by Lemma 5.3 the principal $p$-block of $G$ is the unique $p$-block covering the principal $p$-block of $S$. Thus $\ell_1$ and $\ell_2$ are also adjacent in $\Delta(G)$ by Lemma 2.2 and the completeness of the block graph of $S$, and we are done. $lacksquare$

5.2. Almost simple groups with Lie-type socle. In this subsection, we shall find a triangle in the block graph of an almost simple group according to the results in the previous section.

We first investigate some almost simple groups with socle $S$ which is one of the groups $B_2(2^f)$ or $G_2(3^f)$. The notation and symbols for unipotent characters follow [9]. They make sense due to Theorem 5.2.

Lemma 5.5: Let $S \leq G \leq A$, where $S = B_2(q)$ with $q = 2^f$ or $G_2(q)$ with $q = 3^f$. Let $r$ be a prime as in Theorem 4.9. Then $\Delta(G)|_{(p, r, \ell)}$ is a triangle for any prime divisor $\ell$ of $|S|$ different from $p$ and $r$.

Proof. We first let $S = B_2(q)$ with $q = 2^f$. Note that $B_2(G)$ is the unique 2-block of $G$ covering $B_2(S)$ by Lemma 5.3. We want to show that $B_r(G)$ is the unique $r$-block of $G$ covering $B_r(S)$.

Let $\chi, \chi'$ be the unipotent characters of $S$ corresponding to the symbols

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix},$$

respectively. Let $T$ be a maximal torus of $G$ containing a Sylow 2-torus of $G$ if $f \neq 3$ or a Sylow 1-torus of $G$ if $f = 3$. Since 1, 2 are regular numbers of $S$ by [22, Lemma 3.17], it follows that $T$ is also a Sylow 2-split Levi subgroup of $G$ if $f \neq 3$ or a Sylow 1-split Levi subgroup of $G$ otherwise. Using [14], we know that $\chi$ and $\chi'$ are irreducible constituents of $R^G_T(1_T^F)$. By [22, Theorem A],
both lie in the principal $r$-block of $S$. However, by \cite[Theorem 2.5]{24}, the inertia group of $\chi$ in $G$ is exactly $G \cap A_0$. Hence $\chi$ extends to $G \cap A_0$. By Lemmas \ref{lem:2.3} and \ref{lem:4.7}, $B_r(G \cap A_0)$ is the unique $r$-block of $G \cap A_0$ covering $B_r(S)$. It follows that all extensions of $\chi$ in $G \cap A_0$ lie in $B_r(G \cap A_0)$. Let $\hat{\chi}$ be one of them. Then $\hat{\chi}$ induces irreducibly to $G$, and so by Lemma \ref{lem:2.4}, $B_r(G \cap A_0)$ is the unique $r$-block of $G \cap A_0$ covering $B_r(G \cap A_0)$. It follows that all extensions of $\chi$ in $G \cap A_0$ lie in $B_r(G \cap A_0)$. Let $\hat{\chi}$ be one of them. Then $\hat{\chi}$ induces irreducibly to $G$, and so by Lemma \ref{lem:2.4}, $B_r(G \cap A_0)$ is the unique $r$-block of $G \cap A_0$ covering $B_r(G \cap A_0)$. Thus the lemma follows by Lemma \ref{lem:2.2} and the completeness of the block graph of $S$.

Now let $S = G_2(q)$ with $q = 3^f$. Considering the unipotent characters $\phi'_1, \phi''_1, \phi'_3, \phi''_3$ of $G_2(q)$, we may similarly show that the lemma holds.

**Proposition 5.6:** Let $S \leq G \leq A$, where $A = \text{Aut}(S)$ and $S$ is a simple group of Lie type defined over a finite field of characteristic $p$. Let $r$ be a prime as in Theorem 4.9. Then $\Delta(G)|_{\{p, r, \ell\}}$ is a triangle for any prime divisor $\ell$ of $|S|$ different from $p$ and $r$.

**Proof.** If $S$ is one of the groups $B_2(2^f)$ or $G_2(3^f)$, the proposition holds by Lemma \ref{lem:5.5}. The case where $S = A_2(4)$ can be directly checked using GAP \cite{13}. In the following, we suppose that $S$ is not any one of these groups.

Use the notation in Theorem \ref{thm:4.9} and write $M = G \cap G^F$. Let $R_M \in \text{Syl}_r(M)$ be such that $R \subseteq R_M$. Clearly, we have $C_G(R_M) \leq C_G(R) \leq C_A(R)$. By Theorem \ref{thm:4.9} (i), we have $C_A(R) \leq G^F$. This implies $C_G(R_M) \leq M$, so that $B_r(G)$ is the unique $r$-block of $G$ covering $B_r(M)$.

By Lemma \ref{lem:5.3}, $B_p(G)$ is the unique $p$-block of $G$ covering $B_p(M)$. Therefore, the proposition follows from Lemma \ref{lem:2.2} if $\Delta(M)|_{\{p, r, \ell\}}$ is a triangle for any prime divisor $\ell$ of $|S|$ different from $p$ and $r$. However, this is true by Lemma \ref{lem:5.4}, finishing the proof.

**6. Triangles and $p$-solvability**

In this section we prove Theorem \ref{thm:1.3}, starting with a straightforward observation that will be used for a reduction of the proof.

**Lemma 6.1:** Let $G = H \wr K = (H_1 \times \cdots \times H_t) \times K$ with $K \leq \text{Sym}(t)$. Furthermore, let $1 \neq L_i \leq H_i$ and set $L = L_1 \times \cdots \times L_t$. Then

$$C_G(L) \leq H_1 \times \cdots \times H_t.$$
Proof. Choose an element \((\bar{h}, \sigma) \in C_G(L)\) with \(\bar{h} \in H_1 \times \cdots \times H_t\) and \(\sigma \in K\) and assume there exist \(1 \leq r \neq s \leq t\) with \(\sigma(r) = s\). Consider an element \((l_1, \ldots, l_t) \in L\) such that \(l_i = 1\) if and only if \(i \neq r\). Then

\[(l_1, \ldots, l_t)^{(\bar{h}, \sigma)} = (l_1^{h_1}, \ldots, l_t^{h_t})^\sigma,\]

where \(l_i^{h_i} = 1\) if and only if \(i \neq r\). By writing \((l_1^{h_1}, \ldots, l_t^{h_t})^\sigma = (l'_1, \ldots, l'_t)\), we observe that \(l'_i = 1\) if and only if \(i \neq s\). Thus \(r = s\). In particular, \(\sigma\) must be trivial and the result follows.

Proof of Theorem 1.3 Assume that \(G\) is a minimal counterexample to Theorem 1.3. We deduce a contradiction by the following steps.

**Step 1.** The group \(G\) has a unique minimal normal subgroup which must be non-abelian and have order divisible by \(p\).

Assume that \(G\) has two distinct minimal normal subgroups \(N_1\) and \(N_2\). If \(\Delta(G/N_i)\) has a triangle containing \(p\), then so does \(\Delta(G)\) by Lemma 2.1. Thus, by the minimality of \(|G|\), it follows that both \(G/N_1\) and \(G/N_2\) are \(p\)-solvable. Since \(G\) embeds into the direct product \(G/N_1 \times G/N_2\), we have that \(G\) is \(p\)-solvable, giving a contradiction. So \(G\) must have a unique minimal normal subgroup. Moreover, it must be non-abelian and have order divisible by \(p\), otherwise \(G/N\) and \(N\) are both \(p\)-solvable.

We now assume the direct product of \(t\) copies of \(S\), simply denoted by \(S^t\), is the unique minimal normal subgroup of \(G\), where \(t \in \mathbb{N}\) and \(S\) is a non-abelian simple group. Since \(G = G/C_G(S^t)\) can be viewed as a subgroup of \(\text{Aut}(S^t)\), we may suppose \(S^t \leq G \leq \text{Aut}(S) \wr \text{Sym}(t)\). Write \(A = \text{Aut}(S)\) and \(M = A^t \cap G\).

**Step 2.** The subgraph \(\Delta(M)|_{\pi(S)}\) has no triangle containing \(p\).

Let \(q\) be a prime divisor of \(|S|\) and \(Q \in \text{Syl}_q(M)\). As \(S^t \leq M\) it follows that there exists \(Q_0 \in \text{Syl}_q(S^t)\) such that \(Q_0 \leq Q\). Therefore \(C_G(Q) \leq C_G(Q_0)\) which, by Lemma 2.1, is contained in \(A^t\). Thus \(C_G(Q) \leq M\). Therefore, by Lemma 2.3 for any prime divisor \(q\) of \(|S|\) the principal \(q\)-block of \(G\) is the unique \(q\)-block covering the principal \(q\)-block of \(M\). Now, applying Lemma 2.2 we get that \(\Delta(M)|_{\pi(S)}\) is a subgraph of \(\Delta(G)\). In particular, the block graph \(\Delta(M)|_{\pi(S)}\) has no triangle containing \(p\).

Now let \(S\) be the first factor of \(S^t\). Since \(S\) is normal in \(A\) and also in \(A^t\), it follows that \(S\) is normal in \(M\). Hence \(C_M(S) \leq M\). Write \(\overline{M} = M/C_M(S)\).
STEP 3. The subgraph $\Delta(M)|_{\pi(S)}$ has no triangle containing $p$.

By Lemma 2.1, the block subgraph $\Delta(M)|_{\pi(S)}$ is a subgraph of $\Delta(M)|_{\pi(S)}$. Hence, the subgraph $\Delta(M)|_{\pi(S)}$ has no triangle containing $p$.

Notice that $M$ is almost simple with socle isomorphic to $S$. We finally get a contradiction by applying Theorem 5.1.

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References

[1] M. Aschbacher and G. M. Seitz, *Involutions in Chevalley groups over fields of even order*, Nagoya Mathematical Journal 63 (1976), 1–91.

[2] C. Bessenrodt, G. Malle and J. B. Olsson, *Separating characters by blocks*, Journal of the London Mathematical Society 73 (2006), 493–505.

[3] C. Bessenrodt and J. Zhang, *Block separations and inclusions*, Advances in Mathematics 218 (2008), 485–495.

[4] C. Bessenrodt and J. Zhang, *Character separation and principal covering*, Journal of Algebra 327 (2011), 170–185.

[5] R. Brauer, *Blocks of characters and structure of finite groups*, Bulletin of the American Mathematical Society 1 (1979), 21–62.

[6] P. Brockhaus and G. O. Michler, *Finite simple groups of Lie type have non-principal $p$-blocks, $p \neq 2$*, Journal of Algebra 94 (1985), 113–125.

[7] M. Broué, G. Malle and J. Michel, *Generic blocks of finite reductive groups*, Astérisque 212 (1993), 7–92.

[8] M. Cabanes and M. Enguehard, *Representation Theory of Finite Reductive Groups*, New Mathematical Monographs, Vol. 1, Cambridge University Press, Cambridge, 2004.

[9] R. W. Carter, *Finite Groups of Lie Type*, Pure and Applied Mathematics (New York), John Wiley, New York, 1985.

[10] F. Digne and J. Michel, *Representations of Finite Groups of Lie Type*, London Mathematical Society Student Texts, Vol. 21, Cambridge University Press, Cambridge, 1991.

[11] O. Dudas and R. Rouquier, *Coxeter orbits and Brauer trees III*, Journal of the American Mathematical Society 27 (2014), 1117–1145.

[12] P. Fong and B. Srinivasan, *Generalized Harish-Chandra theory for unipotent characters of finite classical groups*, Journal of Algebra 104 (1986), 301–309.

[13] The GAP Group, *GAP – Groups, Algorithms, and Programming*, 4.8.6, 
http://www.gap-system.org
[14] M. Geck, G. Hiss, F. Lübeck, G. Malle and G. Pfeiffer, CHEVIE—A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras, Applicable Algebra in Engineering, Communication Computing 7 (1996), 175–210.

[15] D. Gorenstein and R. Lyons, The local structure of finite groups of characteristic 2 type, Memoirs of the American Mathematical Society 42 (1983).

[16] D. Gorenstein, R. Lyons and R. Solomon, The Classification of the Finite Simple Groups. Number 3. Part I. Chapter A, Mathematical Surveys and Monographs, 40, American Mathematical Society, Providence, RI, 1998.

[17] F. Gross, Automorphisms which centralize a Sylow $p$-subgroup, Journal of Algebra 77 (1982), 202–233.

[18] R. Guralnick and G. Malle, Products of conjugacy classes and fixed point spaces, Journal of the American Mathematical Society 25 (2012), 77–121.

[19] M. Harris and R. Knörr, Brauer correspondence for covering blocks of finite groups, Communications in Algebra 13 (1985), 1213–1218.

[20] G. Hiss, Principal blocks and the Steinberg character, Algebra Colloquium 17 (2010), 361–364.

[21] I. M. Isaacs, Character Theory of Finite Groups, Dover, New York, 1994.

[22] R. Kessar and G. Malle, Lusztig induction and $\ell$-blocks of finite reductive groups, Pacific Journal of Mathematics 279 (2015), 269–298.

[23] Y. Liu and J. Zhang, Small intersections of principal blocks, Journal of Algebra 472 (2017), 214–225.

[24] G. Malle, Extensions of unipotent characters and the inductive McKay condition, Journal of Algebra 320 (2008), 2963–2980.

[25] G. Malle and D. Testerman, Linear Algebraic Groups and Finite Groups of Lie Type, Cambridge Studies in Advanced Mathematics, Vol. 133, Cambridge University Press, Cambridge, 2011.

[26] G. Navarro, Characters and Blocks of Finite Groups, London Mathematical Society Lecture Note Series, Vol. 250, Cambridge University Press, Cambridge, 1998.

[27] G. Navarro, A. Turull and T. R. Wolf, Block separation in solvable groups, Archiv der Mathematik 85 (2005), 293–296.

[28] G. Navarro and W. Willems, When is a $p$-block a $q$-block?, Proceedings of the American Mathematical Society 125 (1997), 1589–1591.

[29] B. Späth, The McKay conjecture for exceptional groups and odd primes, Mathematische Zeitschrift 261 (2009), 571–595.

[30] B. Späth, Sylow $d$-tori of classical groups and the McKay conjecture. I, Journal of Algebra 323 (2010), 2469–2493.

[31] T. Weigel, Generation of exceptional groups of Lie-type, Geometriæ Dedicata 41 (1992), 63–87.

[32] R. A. Wilson, The Finite Simple Groups, Graduate Texts in Mathematics, Vol. 251, Springer, London, 2009.

[33] K. Zsigmondy, Zur Theorie der Potenzreste, Monatshefte für Mathematik und Physik 3 (1892), 265–284.