EXISTENCE OF SOLUTIONS FOR ANISOTROPIC
CAHN-HILLIARD AND ALLEN-CAHN SYSTEMS IN HIGHER
SPACE DIMENSIONS

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In memory of Alfredo Lorenzi

Abstract. Our aim in this paper is to prove the existence and uniqueness of solutions to Cahn-Hilliard and Allen-Cahn type equations based on a modification of the Ginzburg-Landau free energy proposed in [12] (see also [16]) which takes into account strong anisotropy effects. In particular, the free energy contains a regularization term, called Willmore regularization.

1. Introduction. In [6] and [18], a modified Ginzburg-Landau free energy which takes into account strong anisotropy effects arising during the growth and coarsening of thin films is considered, namely,

$$\Psi_{MGL} = \int_{\Omega} \left( \gamma(n) \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) + \frac{\beta}{2} \omega^2 \right) dx. \quad (1.1)$$

Here, $u$ is the order parameter, $n = \frac{\nabla u}{|\nabla u|}$, $\Omega$ is the domain occupied by the material,

$$F(u) = \frac{1}{4} (u^2 - 1)^2 \quad (1.2)$$

and

$$f(u) = F'(u) = u^3 - u. \quad (1.3)$$

Furthermore, $\gamma(n)$ is a bounded function which describes the anisotropy effects, $G(u) = \omega^2$, $\omega = f(u) - \Delta u$, is called nonlinear Willmore regularization and $\beta$ is a small positive regularization parameter. Such a regularization is relevant, e.g., in determining the equilibrium shape of a crystal in its own liquid matrix, when anisotropy effects are strong. Indeed, in that case, the equilibrium interface may not be a smooth curve and may present facets and corners with slopes of discontinuities (see, e.g., [19]).

The author in [15] proved the well-posedness for a one-dimensional Allen-Cahn system based on (1.1). The analysis in [15] consists in regularizing $\gamma$. Unfortunately, the estimates obtained there are not uniform with respect to the regularization parameter, so that one is not able to pass to the limit.

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We recall that the original Ginzburg-Landau free energy,
\[ \Psi_{GL} = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) dx, \]  
plays a fundamental role in phase separation and transition, see, e.g., [4] and [5].

Actually, in [12], the author proposed another modification of the Ginzburg-Landau free energy which takes into account anisotropy effects, namely,
\[ \Psi_{AMGL} = \int_{\Omega} \left[ \frac{1}{2} |\gamma(n) \nabla u|^2 + F(u) + \frac{\beta}{2} \omega^2 \right] dx. \]  
This model describes dendritic pattern formation for one component melt growth and plays an important role in crystal growth. Compared with (1.1), we will not have to regularize \( \gamma \) when studying the corresponding models. Indeed, obtaining a variational derivative (see below) of (1.1) is not clear without such a regularization.

We considered in [16] the Cahn-Hilliard and Allen-Cahn systems based on (1.5) in one space dimension and proved the existence and uniqueness of solutions. We can note that, in this particular case, \( n \) only takes the values \( \pm 1 \), which makes the analysis easier than in higher space dimensions.

In this article, we now deal with the two-dimensional case and (1.5) can be rewritten in the form
\[ \Psi_{AMGL} = \int_{\Omega} \left[ g(\nabla u) + F(u) + \frac{\beta}{2} \omega^2 \right] dx, \]  
where
\[ g(s) = g(s_1, s_2) = \begin{cases} 
\frac{1}{2} \gamma^2 \left( \frac{s_1}{|s|}, \frac{s_2}{|s|} \right) |s|^2 & \text{if } s = (s_1, s_2) \neq (0, 0), \\
0 & \text{if } s = (s_1, s_2) = (0, 0)
\end{cases} \]  
and
\[ \omega = f(u) - \Delta u, \quad F' = f. \]  

Actually, what follows can also be easily extended to the three-dimensional case (see Remark 1, b) below).

We can write, formally, for a small variation and assuming proper boundary conditions (see Section 3 and Remark 2 below),
\[ D\Psi_{AMGL} = \int_{\Omega} \left[ g'(\nabla u) \cdot \nabla Du + f(u)Du + \beta \omega Du \right] dx \]
\[ = \int_{\Omega} \left[ - \text{div}(g'(\nabla u))Du + f(u)Du + \beta \omega f'(u)Du - \beta \omega \Delta Du \right] dx \]
\[ = \int_{\Omega} \left[ - \text{div}(g'(\nabla u)) + f(u) + \beta \omega f'(u) - \beta \Delta \omega \right] Du dx \]

and the variational derivative of \( \Psi_{AMGL} \) with respect to \( u \) reads
\[ \frac{D\Psi_{AMGL}}{Du} = - \text{div}(g'(\nabla u)) + f(u) + \beta \omega f'(u) - \beta \Delta \omega. \]  

Our aim in this article is to prove the existence and uniqueness of solutions for the Cahn-Hilliard and Allen-Cahn systems associated with the Ginzburg-Landau free energy (1.6), i.e., with (1.9), in higher dimensions. In particular, (1.6) and (1.9) lead to a sixth-order Cahn-Hilliard equation and a fourth-order Allen-Cahn equation.
2. Preliminaries. We assume in what follows that the function \( \gamma \) in (1.5) is of class \( C^2 \).

**Lemma 2.1.** The function \( g \) defined in (1.7) is of class \( C^1 \).

**Proof.** We first prove that \( g \) is a continuous function. We note that, for all \( s = (s_1, s_2) \in \mathbb{R}^2 \),

\[
\left| \frac{s_i}{|s|} \right| \leq 1, \quad |s| = \sqrt{s_1^2 + s_2^2},
\]

which implies that

\[
\gamma^2 \left( \frac{s_1}{|s|}, \frac{s_2}{|s|} \right)
\]

is bounded.

Therefore,

\[
|g(s_1, s_2)| \leq c|s|^2.
\]

For simplicity, we set

\[
U = \frac{s_1}{\sqrt{s_1^2 + s_2^2}} \quad \text{and} \quad V = \frac{s_2}{\sqrt{s_1^2 + s_2^2}}.
\]

We then have, for \((s_1, s_2) \neq (0, 0)\),

\[
\frac{\partial g}{\partial s_1} = \gamma^2(U, V)s_1 + \frac{1}{2} \frac{\partial^2 \gamma^2}{\partial s_1^2}(U, V) \frac{\partial U}{\partial s_1}|s|^2 + \frac{1}{2} \frac{\partial^2 \gamma^2}{\partial s_2^2}(U, V) \frac{\partial V}{\partial s_1}|s|^2
\]

and

\[
\frac{\partial g}{\partial s_2} = \gamma^2(U, V)s_2 + \frac{1}{2} \frac{\partial^2 \gamma^2}{\partial s_1^2}(U, V) \frac{\partial U}{\partial s_2}|s|^2 + \frac{1}{2} \frac{\partial^2 \gamma^2}{\partial s_2^2}(U, V) \frac{\partial V}{\partial s_2}|s|^2
\]

Noting that

\[
\frac{s_i^2}{|s|} \leq |s| \leq |s_i| \quad \text{and} \quad \left| \frac{s_1s_2}{|s|} \right| \leq \frac{1}{2}|s|,
\]

this yields that \( \frac{\partial g}{\partial s_1} \) and \( \frac{\partial g}{\partial s_2} \) are continuous and, consequently, \( g \) is of class \( C^1 \), with

\[
\frac{\partial g}{\partial s_1}(0) = \frac{\partial g}{\partial s_2}(0) = 0.
\]

**Lemma 2.2.** The functions \( \frac{\partial^2 g}{\partial s_i \partial s_j} \), \( i, j = 1, 2 \), belong to \( L^\infty(\mathbb{R}^2) \).

**Proof.** We compute, for instance, \( \frac{\partial^2 g}{\partial s_1 \partial s_2} \) for \((s_1, s_2) \neq (0, 0)\) (the proof is similar for the other second-order derivatives) and have

\[
\frac{\partial^2 g}{\partial s_1 \partial s_2} = \frac{\partial}{\partial s_1} \left( \frac{\partial g}{\partial s_2} \right)
\]

\[
= \frac{\partial}{\partial s_1} \left[ \gamma^2(U, V)s_2 - \frac{1}{2} \frac{\partial^2 \gamma^2}{\partial s_1^2}(U, V) \left( \frac{s_1s_2}{|s|} \right) + \frac{1}{2} \frac{\partial^2 \gamma^2}{\partial s_2^2}(U, V) \left( \frac{s_1^2}{|s|} \right) \right]
\]
\[\begin{align*}
&= \frac{1}{2} \frac{\partial \gamma^2}{\partial s_1}(U, V) \left( \frac{s_3^3}{|s|^3} \right) + \frac{1}{2} \frac{\partial \gamma^2}{\partial s_2}(U, V) \left( \frac{s_1^3}{|s|^3} \right) - \frac{1}{2} \frac{\partial^2 \gamma^2}{\partial s_1 \partial s_2} \left( \frac{s_1 s_2^3}{|s|^4} \right) \\
&\quad - \frac{1}{2} \frac{\partial^2 \gamma^2}{\partial s_2^2}(U, V) \left( \frac{s_1^3 s_2}{|s|^4} \right) + \frac{\partial^2 \gamma^2}{\partial s_1 \partial s_2} (U, V) \left( \frac{s_1^3 s_2^2}{|s|^4} \right).
\end{align*}\]

Noting that

- \( \left( \frac{s_3^3}{|s|^3} \right) \leq 1 \) and \( \left( \frac{s_1^3}{|s|^3} \right) \leq 1 \),
- \( \left( \frac{s_1 s_2^3}{|s|^4} \right) \leq 1 \), \( \left( \frac{s_1^3 s_2}{|s|^4} \right) \leq 1 \), \( \left( \frac{s_1^3 s_2^2}{|s|^4} \right) \leq 1 \),

we finally deduce that

\[\frac{\partial^2 g}{\partial s_1 \partial s_2} \in L^\infty(\mathbb{R}^2).\]

**Lemma 2.3.** The functions \( \frac{\partial g}{\partial s_i}, \ i = 1, 2 \), are Lipschitz continuous.

**Proof.** Let \( u, v \in \mathbb{R}^2 \). We set \( \varphi = \frac{\partial g}{\partial s_i}, \ i = 1, 2 \).

- If \( 0 \notin [u, v] \) (segment), we take \( \delta(s) = u + s(v - u) \) and note that \( \delta(s) \neq 0, \forall s \in [0, 1] \). In that case, we have
  \[\varphi(v) - \varphi(u) = \varphi \circ \delta(1) - \varphi \circ \delta(0) = \int_0^1 \frac{d}{ds} (\varphi \circ \delta)(s) \, ds.\]

It thus follows from Lemma 2.2 that

\[|\varphi(u) - \varphi(v)| \leq c \int_0^1 |\delta'(s)| \, ds = c|v - u|.\]

- If \( 0 \in [u, v] \), we can assume that (up to a rotation and a translation)
  \([u, v] = [-a, a], \text{ with } |a| = \frac{1}{2}|u - v|\).

We then take

\[\delta(s) = \begin{pmatrix} -a \cos(\pi s) \\ a \sin(\pi s) \end{pmatrix},\]

which gives

\(\delta(0) = -a, \ \delta(1) = a, \ |\delta'(s)| = \pi |a|,\)

and have

\[\varphi(v) - \varphi(u) = \varphi(a) - \varphi(-a) = \varphi \circ \delta(1) - \varphi \circ \delta(0) = \int_0^1 \frac{d}{ds} (\varphi \circ \delta)(s) \, ds.\]

It then again follows from Lemma 2.2 that

\[|\varphi(u) - \varphi(v)| \leq c \int_0^1 |\delta'(s)| \, ds = c\pi |a| = \frac{c\pi}{2}|u - v|.\]
Remark 2.4. a) It follows from Lemma 2.3 that \( g' \) is Lipschitz continuous, where \( g' \) denotes the differential of \( g \).

b) We can proceed in a similar way in three space dimensions. In particular, in the second case, we can assume, without loss of generality, that \( u \) and \( v \) belong to the plane \( (O, \vec{i}, \vec{j}, \vec{k}) \) (\( (O, \vec{i}, \vec{j}, \vec{k}) \) being an orthonormal frame) and proceed as in the proof of Lemma 2.3.

Notation and assumptions.

We denote by \((\cdot, \cdot)\) the usual \( L^2 \)-scalar product, with associated norm \( \| \cdot \| \), and we set \( \| \cdot \|_1 = \| (-\Delta)^{-\frac{1}{2}} \cdot \| \), where \((-\Delta)^{-1}\) is the inverse minus Laplace operator associated with periodic boundary conditions and acting on functions with null average.

We set, whenever it makes sense, \( \langle \cdot \rangle = \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} \cdot \, dx \), being understood that, for \( \varphi \in H^{-1}(\Omega) = H_{\text{per}}^{-1}(\Omega)' \), \( \langle \varphi \rangle = \frac{1}{\text{Vol}(\Omega)} \varphi, 1 > H^{-1}(\Omega), H_{\text{per}}^{-1}(\Omega) \), and we note that

\[
\varphi \mapsto (\| \varphi - \langle \varphi \rangle \|^2_{L^2} + \langle \varphi \rangle^2)^{\frac{1}{2}}
\]

is a norm on \( H^{-1}(\Omega) \) which is equivalent to the usual one. Here, \( \Omega = (0, L_1) \times (0, L_2), L_1, L_2 > 0 \).

Throughout the article, the same letter \( c \) (and, sometimes, \( c' \) or \( c'_M \)) when accounting for the dependence on a parameter \( M \) denotes constants which may vary from line to line. Similarly, the same letter \( Q \) denotes monotone increasing (with respect to each argument) functions which may vary from line to line.

As far as the nonlinear term \( f \) is concerned, we assume more generally that \( f \) is of class \( C^2 \) and that

\[
\begin{align*}
f(0) &= 0, \quad f'(s) \geq -c_0, \quad c_0 \geq 0, \quad s \in \mathbb{R}, \quad (2.1) \\
f(s)s &\geq c_1 F(s) - c_2 \geq -c_3, \quad c_1 > 0, \quad c_2, \quad c_3 \geq 0, \quad s \in \mathbb{R}, \quad (2.2)
\end{align*}
\]

where, here, \( F(s) = \int_0^s f(\tau) \, d\tau \),

\[
\begin{align*}
sf(s)f'(s) - f(s)^2 &\geq -c_4, \quad c_4 \geq 0, \quad s \in \mathbb{R}, \quad (2.3) \\
|f'(s)| &\leq c|f(s)| + c_5, \quad \forall c > 0, \quad c_5 = c_5(c) \geq 0, \quad s \in \mathbb{R}, \quad (2.4) \\
sf''(s) &\geq 0, \quad s \in \mathbb{R}. \quad (2.5)
\end{align*}
\]

Note that these assumptions are satisfied by the cubic nonlinear term (1.3). We can also note that (2.1)-(2.4) are satisfied by polynomials of the form \( f(s) = \sum_{i=1}^{2p+1} a_i s^i, \ a_{2p+1} > 0 \). Assumption (2.5) is, on the contrary, much more restrictive and is needed to obtain dissipative estimates (see below). It is however reasonable, since it is satisfied by the cubic function \( f(s) = s^3 - s \) which is usually considered in the

- If \( u = 0 \) or \( v = 0 \), taking, for instance, \( v = 0 \), then (see the proof of Lemma 2.1)

\[
|\varphi(u) - \varphi(v)| = |\varphi(u)| \leq c|u| = c|u - v|.
\]

In conclusion, we have, in all cases,

\[
|\varphi(u) - \varphi(v)| \leq c|u - v|
\]

and \( \varphi \) is Lipschitz continuous. \( \square \)
Cahn–Hilliard and Allen–Cahn theories (we can further note that this cubic function actually is an approximation of the logarithmic function
\[ f_{\log}(s) = -k_0 s + \frac{k_1}{2} \ln \frac{1 + s}{1 - s}, \quad 0 < k_1 < k_0 \]
(see [4]), which also satisfies (2.5)).

As far as the function \( g \) is concerned, we assume that
\[ \gamma^2(U, V) |s|^2 \geq c_6 |s|^2, \quad c_6 > 0, \quad s \in \mathbb{R}^2 \setminus \{(0, 0)\}. \]
(2.6)
This yields that
\[ g(s) \geq \frac{c_6}{2} |s|^2, \quad s \in \mathbb{R}^2. \]
(2.7)

3. Cahn–Hilliard system. The Cahn–Hilliard equation is an equation of mathematical physics which describes the process of phase separation, by which the two components of a binary alloy spontaneously separate and form a fined-grained structure in which each of the two components appears more or less alternatively. This equation was initially derived as a model for spinodal decomposition in solid materials [4] and has been extended to many other physical systems.

We also note that there is currently a strong interest in the study of sixth-order Cahn–Hilliard equations; these equations also arise in situations such as atomistic models of crystal growth (see [2], [3] and [8]), the description of growing crystalline surfaces with small slopes which undergo faceting (see [17]), oil–water–surfactant mixtures (see [9] and [10]) and mixtures of polymer molecules (see [7]). We refer the reader to [6], [11], [13], [14] and [18] for the mathematical and numerical analysis of such models.

Setting of the problem. Writing the mass conservation, i.e., \( \frac{\partial u}{\partial t} = -\text{div} h \), where \( h \) is the mass flux which is related to the chemical potential \( \mu \) by the constitutive relation \( h = -\nabla \mu \), and that the chemical potential is the variational derivative of (1.6) with respect to \( u \), we end up with the following sixth-order Cahn–Hilliard system, taking, for simplicity, \( \beta = 1 \),
\[ \frac{\partial u}{\partial t} = \Delta \mu, \]
(3.1)
\[ \mu = -\text{div}(g'(\nabla u)) + f(u) + \omega f'(u) - \Delta \omega, \]
(3.2)
\[ \omega = f(u) - \Delta u, \]
(3.3)
together with periodic boundary conditions,
\[ u, \; \mu, \; \omega \text{ are } \Omega - \text{periodic}, \]
(3.4)
and the initial condition
\[ u|_{t=0} = u_0. \]
(3.5)

Remark 3.1. Actually, the Cahn–Hilliard equation usually is endowed with Neumann boundary conditions. In our case, these conditions read
\[ g'(\nabla u) \cdot \nu = \frac{\partial \mu}{\partial \nu} = \frac{\partial \omega}{\partial \nu} = 0 \text{ on } \Gamma, \]
where, here, \( \Omega \) is a regular and bounded domain of \( \mathbb{R}^2 \) with boundary \( \Gamma \) and \( \nu \) denotes the unit outer normal. We also note that
\[ g'(\nabla u) \cdot \nu = \gamma^2 \left( \frac{\nabla u}{|\nabla u|} \right) \frac{\partial u}{\partial \nu} \]
\[ + \frac{1}{|\nabla u|} \left( \frac{\partial \gamma^2}{\partial s_1} \left( \frac{\nabla u}{|\nabla u|} \right) \frac{\partial u}{\partial x_2} - \frac{\partial \gamma^2}{\partial s_2} \left( \frac{\nabla u}{|\nabla u|} \right) \frac{\partial u}{\partial x_1} \right) \left( \frac{\partial u}{\partial x_2} \nu_1 - \frac{\partial u}{\partial x_1} \nu_2 \right). \tag{3.6} \]

In particular, \( \frac{\partial u}{\partial \nu} = 0 \) on \( \Gamma \) does not necessarily imply \( g'(\nabla u) \cdot \nu = 0 \) on \( \Gamma \), contrary to what we had in the one-dimensional case \([16]\) (we need the condition \( \frac{\partial u}{\partial \nu} = 0 \) on \( \Gamma \) for the existence scheme, based on Galerkin approximations, see below). We can note that this holds provided that we have the compatibility relation
\[ s_2 \frac{\partial \gamma^2}{\partial s_1} = s_1 \frac{\partial \gamma^2}{\partial s_2}. \tag{3.7} \]

Unfortunately, (3.7) yields, considering polar coordinates, that \( \gamma(s) = \varphi(|s|) \), i.e., \( \gamma(n) \) is constant. One idea, to handle such boundary conditions could be to consider a different approximation scheme (e.g., a finite differences scheme). This will be addressed elsewhere.

Our main aim in this section is to prove the

**Theorem 3.2.** Assume that (3.18) holds and that \( u_0 \in H^2_{\text{per}}(\Omega) \). Then, (3.1)-(3.5) possesses a unique (variational) solution such that
\[ u \in L^\infty(\mathbb{R}^+; H^2_{\text{per}}(\Omega)) \cap L^2(0,T; H^3_{\text{per}}(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0,T; H^{-1}(\Omega)), \]
\[ \mu \in L^2(0,T; H^1_{\text{per}}(\Omega)) \quad \text{and} \quad \omega \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0,T; H^2_{\text{per}}(\Omega)), \]
for all \( T > 0 \).

### 3.1. A priori estimates.

We first note that, integrating (formally) (3.1) over \( \Omega \), we obtain the conservation of mass, namely,
\[ \langle u(t) \rangle = \langle u_0 \rangle, \quad t \geq 0. \tag{3.8} \]

Multiplying (3.1) by \( (-\Delta)^{-1} \frac{\partial u}{\partial t} \), we have, integrating over \( \Omega \) and by parts,
\[ \| \frac{\partial u}{\partial t} \|^2_{-1} = -((\mu, \frac{\partial u}{\partial t})). \tag{3.9} \]

We then multiply (3.2) by \( \frac{\partial u}{\partial t} \) and integrate over \( \Omega \) to find
\[ ((\mu, \frac{\partial u}{\partial t})) = \int_\Omega g'(\nabla u) \cdot \nabla \frac{\partial u}{\partial t} \, dx + \frac{d}{dt} \int_\Omega F(u) \, dx + ((\omega f'(u) - \Delta \omega, \frac{\partial u}{\partial t})). \tag{3.10} \]

We note that it follows from (3.3) that
\[ ((\omega f'(u), \frac{\partial u}{\partial t})) - ((\Delta \omega, \frac{\partial u}{\partial t})) = \frac{1}{2} \frac{d}{dt} \| \omega \|^2. \tag{3.11} \]

We further have
\[ \int_\Omega g'(\nabla u) \cdot \nabla \frac{\partial u}{\partial t} \, dx = \frac{d}{dt} \int_\Omega g(\nabla u) \, dx. \tag{3.12} \]

We finally deduce from (3.9)-(3.12) that
\[ \frac{d}{dt} \left[ 2 \int_\Omega g(\nabla u) \, dx + 2 \int_\Omega F(u) \, dx + \| \omega \|^2 \right] + 2 \| \frac{\partial u}{\partial t} \|^2_{-1} = 0. \tag{3.13} \]

In particular, (3.13) yields that the free energy decreases along the trajectories, as expected.
We now multiply (3.1) by \((-\Delta)^{-1}\bar{u}\), where \(\bar{u} = u - \langle u \rangle\), and integrate over \(\Omega\). We obtain, owing to (3.8),
\[
\frac{1}{2}\frac{d}{dt} \|\bar{u}\|^2_{-1} = -\langle (\mu, u) \rangle + \text{Vol}(\Omega)\langle \mu \rangle \langle u_0 \rangle,
\]
where, owing to (3.2),
\[
\langle \mu \rangle = \langle f(u) \rangle + \langle \omega f'(u) \rangle.
\]
Multiplying then (3.2) by \(u\) and integrating over \(\Omega\), we have, owing to (3.3),
\[
\langle (\mu, u) \rangle = \int_{\Omega} g'(\nabla u) \cdot \nabla u \, dx + ((f(u), u)) + ((f(u)f'(u), u)) - ((f'(u)\Delta u, u)) - ((\Delta f(u), u)) + \|\Delta u\|^2.
\]
Noting that
\[
((f'(u)\Delta u, u)) = -((f'(u)\nabla u, \nabla u)) - ((uf''(u)\nabla u, \nabla u))
\]
and
\[
((\Delta f(u), u)) = -((f'(u)\nabla u, \nabla u)),
\]
we obtain
\[
\langle (\mu, u) \rangle = \int_{\Omega} g'(\nabla u) \cdot \nabla u \, dx + ((f(u), u)) + \|\omega\|^2 + ((uf''(u)\nabla u, \nabla u))
\]
\[
+ \int_{\Omega} (f(u)f'(u)u - f^2(u)) \, dx
\]
and we finally find, owing to (2.2), (2.3), (2.5) and (3.14),
\[
\frac{d}{dt} \|\bar{u}\|^2_{-1} + c \left[ 2 \int_{\Omega} g(\nabla u) \, dx + 2 \int_{\Omega} F(u) \, dx + \|\omega\|^2 \right] 
\leq 2\text{Vol}(\Omega)\langle \mu \rangle \langle u_0 \rangle + c', \quad c > 0.
\]
Indeed, note that
\[
g'(s) \cdot s = \frac{\partial g}{\partial s_1}(s)s_1 + \frac{\partial g}{\partial s_2}(s)s_2 = \gamma^2(U, V)|s|^2 = 2g(s), \quad s \in \mathbb{R}^2.
\]
We now assume that
\[
|\langle u_0 \rangle| \leq M \quad (\text{hence, } |\langle u(t) \rangle| \leq M, \quad t \geq 0), \quad M \geq 0.
\]
Therefore, owing to (2.4) and (3.15),
\[
|2\text{Vol}(\Omega)\langle u_0 \rangle\langle \mu \rangle| \leq c_M (|\langle f(u) \rangle| + |\langle \omega f'(u) \rangle|)
\leq \frac{c}{2} \left( \int_{\Omega} f(u)^2 \, dx + \int_{\Omega} \omega^2 \, dx \right) + c'_M,
\]
where \(c\) is the constant appearing in (3.17), and we deduce from (3.17) and (3.19) that
\[
\frac{d}{dt} \|\bar{u}\|^2_{-1} + c \left[ 2 \int_{\Omega} g(\nabla u) \, dx + 2 \int_{\Omega} F(u) \, dx + \|\omega\|^2 \right] \leq c'_M, \quad c > 0.
\]
Combining (3.13) and (3.20), we have an inequality of the form
\[
\frac{dE}{dt} + c(E + \|\frac{\partial u}{\partial t}\|^2_{-1}) \leq c'_M, \quad c > 0,
\]
where
\[
E = \|\bar{u}\|^2_{-1} + \langle u \rangle^2 + 2 \int_{\Omega} g(\nabla u) \, dx + 2 \int_{\Omega} F(u) \, dx + \|\omega\|^2.
\]
In particular, we deduce from (3.21) and Gronwall’s Lemma that
\[ E(t) \leq E(0)e^{-ct} + c'M, \quad c > 0, \quad t \geq 0. \]  
(3.23)
Noting that, owing to (2.1),
\[ \|\omega\|^2 \geq \|f(u)\|^2 + \|\Delta u\|^2 - 2c_0\|\nabla u\|^2, \]  
(3.24)
we finally deduce from (2.2), (2.7) and (3.22)-(3.24) that
\[ \|u(t)\|_{H^2_{per}(\Omega)}^2 + \|f(u(t))\|^2 \leq Q(\|u_0\|_{H^2_{per}(\Omega)})e^{-ct} + c'M, \quad c > 0, \quad t \geq 0. \]  
(3.25)
Rewriting (3.1) in the equivalent form
\[ \mu = \langle \mu \rangle - (-\Delta)^{-1} \frac{\partial u}{\partial t}, \]  
(3.26)
we obtain
\[ \|\nabla \mu\| \leq c\|\frac{\partial u}{\partial t}\|^{-1}. \]  
(3.27)
Noting that, proceeding as in (3.19),
\[ |\langle \mu \rangle| \leq c \left( \|u\|^2_{H^2_{per}(\Omega)} + \|f(u)\|^2 + 1 \right), \]  
we finally find
\[ \|\mu\|_{H^1_{per}(\Omega)} \leq c \left( \|\frac{\partial u}{\partial t}\|^{-1} + \|u\|^2_{H^2_{per}(\Omega)} + \|f(u)\|^2 + 1 \right). \]  
(3.28)
Now, owing to (3.2), we have
\[ \Delta \omega = -\text{div}(g'(\nabla u)) + f(u) - \mu + \omega f'(u) \]  
and, owing to (2.4), there holds
\[ \|\Delta \omega\| \leq c \left[ \|\text{div}(g'(\nabla u))\| + \|f(u)\|^2 + \|\omega\|^2 + \|\mu\| + 1 \right], \]  
(3.29)
where
\[ \|\text{div}(g'(\nabla u))\| \leq \sum_{i=1}^{2} \| \frac{\partial}{\partial x_i} (g'(\nabla u))_i \| \leq \|g'(\nabla u)\|_{H^1_{per}(\Omega)^2}. \]  
Recall that \( g' \) is Lipschitz continuous and \( g'(0) = 0 \) and note that (see Lemma 2.1 and Lemma 2.2)
\[ \|g'(\nabla u)\|_{H^1_{per}(\Omega)^2} \leq c\|u\|_{H^2_{per}(\Omega)}. \]  
We thus have, owing to (3.3) and (3.28)-(3.29),
\[ \|\omega\|_{H^2_{per}(\Omega)} \leq c \left( \|\frac{\partial u}{\partial t}\|^{-1} + \|u\|^2_{H^2_{per}(\Omega)} + \|f(u)\|^2 + 1 \right). \]  
(3.30)
We now multiply (3.1) by \( u \) and integrate over \( \Omega \) to get
\[ \frac{1}{2} \frac{d}{dt} \|u\|^2 = -((\nabla \mu, \nabla u)). \]  
(3.31)
Multiplying then (3.2) by \(-\Delta u\) and integrating over \( \Omega \), we obtain, in view of (3.3),
\[ ((\nabla \mu, \nabla u)) = -\int_{\Omega} g'(\nabla u) \cdot \nabla \Delta u \, dx + ((f'(u)\nabla u, \nabla u)) - ((\omega f'(u), \Delta u)) \]  
\[ + ((\Delta f(u), \Delta u)) + \|\nabla \Delta u\|^2. \]  
(3.32)
We note that
\[ |((\omega f'(u), \Delta u))| \leq \|f'(u)\|_{L^\infty(\Omega)} \|\omega\| \|\Delta u\|. \]
where $Q$ is continuous (here, we have used the fact that $H^2_{\text{per}}(\Omega)$ is continuously embedded in $C(\Omega)$, noting that

$$|f'(u)| \leq Q(|u|) \leq Q(||u||_{L^\infty(\Omega)}),$$

for some monotone increasing and continuous function $Q$) and, proceeding similarly,

$$|\langle (\Delta f(u), \Delta u) \rangle| \leq \|f'(u)\|_{\Delta} \|\nabla \Delta u\|.$$  (3.34)

Finally, noting that it follows from the proof of Lemma 2.1 that $|g'(s)| \leq c|s|$,

$$\left| \int g'(\nabla u) \cdot \nabla \Delta u \, dx \right| \leq \frac{1}{4} \|\nabla \Delta u\|^2 + c \|\nabla u\|^2.$$  (3.35)

It thus follows from (2.1) and (3.31)-(3.35) that

$$\frac{d}{dt} ||u||^2 + c ||u||^2_{H^2_{\text{per}}(\Omega)} \leq Q \left( ||u||_{H^2_{\text{per}}(\Omega)} \right) \left( ||u||^2_{H^2_{\text{per}}(\Omega)} + ||\omega||^2 \right), \quad c > 0,$$  (3.36)

where $Q$ is continuous.

### 3.2. The dissipative semigroup.

#### Proof of Theorem 3.2. a) Uniqueness:

Let $(u_1, \mu_1, \omega_1)$ and $(u_2, \mu_2, \omega_2)$ be two solutions to (3.1)-(3.5) with initial data $u_{1,0}$ and $u_{2,0}$, respectively, such that

$$|\langle u_{i,0} \rangle| \leq M, \quad i = 1, 2.$$  (3.37)

We set $(u, \mu, \omega) = (u_1, \mu_1, \omega_1) - (u_2, \mu_2, \omega_2)$ and $u_0 = u_{1,0} - u_{2,0}$ and have

$$\frac{\partial u}{\partial t} = \Delta u,$$  (3.38)

$$\mu = -\text{div}(g'(\nabla u_1) - g'(\nabla u_2)) + f(u_1) - f(u_2) + \omega_1 f'(u_1) - \omega_2 f'(u_2) - \Delta \omega,$$  (3.39)

$$\omega = f'(u_1) - f'(u_2) - \Delta u,$$  (3.40)

$$u, \mu, \omega \text{ are } \Omega - \text{periodic},$$  (3.41)

$$u|_{t=0} = u_0.$$  (3.42)

We multiply (3.38) by $(-\Delta)^{-1} \bar{u}$, where $\bar{u} = u - \langle u \rangle$, and obtain, integrating over $\Omega$ and by parts,

$$\frac{1}{2} \frac{d}{dt} \|\bar{u}\|^2_{-1} = -\langle (\mu, u) \rangle + \text{Vol}(\Omega) \langle \mu \rangle \langle u \rangle,$$  (3.43)

where, owing to (3.39) and (3.41)

$$\langle \mu \rangle = \langle f(u_1) - f(u_2) \rangle + \langle \omega_1 f'(u_1) - \omega_2 f'(u_2) \rangle.$$  (3.44)

We then multiply (3.39) by $u$ and find, in view of (3.40),

$$\langle (\mu, u) \rangle = \int_{\Omega} (g'(\nabla u_2) - g'(\nabla u_1)) \cdot \nabla u \, dx + \langle f(u_1) - f(u_2), u \rangle + \|\Delta u\|^2$$

$$+ \langle (\omega_1 f'(u_1) - \omega_2 f'(u_2), u) \rangle - \langle (f(u_1) - f(u_2), \Delta u) \rangle.$$  (3.45)

We have, owing to (2.1)

$$\langle (f(u_1) - f(u_2), u) \rangle \geq -c_0 \|u\|^2.$$  (3.46)
Furthermore, owing to the continuous embedding $H^2(\Omega) \subset C(\overline{\Omega})$, (3.23) and (3.25),
\[
|((f(u_1) - f(u_2), \Delta u))| \leq \frac{1}{8} \| \Delta u \|^2 + Q(\|u_{1,0}\|_{H^2_{per}(\Omega)}, \|u_{2,0}\|_{H^2_{per}(\Omega)})\|u\|^2
\] (3.47)
and
\[
|((\omega_1 f'(u_1) - \omega_2 f'(u_2), u)| \leq |((\omega_1(f'(u_1) - f'(u_2)), u))| + |((\omega f'(u_2), u))|
\]
\[
\leq |((\omega_1(f(u_1) - f(u_2)), u)| + |((f'(u_2)\Delta u, u))| + |((f'(u_2)(f(u_1) - f(u_2)), u))|
\]
\[
\leq Q(\|u_{1,0}\|_{H^2_{per}(\Omega)}, \|u_{2,0}\|_{H^2_{per}(\Omega)})|\omega_1\|H^2_{per}(\Omega)|\|u\|^2
\]
\[
+ \frac{1}{8} \| \Delta u \|^2 + Q(\|u_{1,0}\|_{H^2_{per}(\Omega)}, \|u_{2,0}\|_{H^2_{per}(\Omega)})\|\omega_1\|H^2_{per}(\Omega) \|u\|^2
\]
\[
\leq \frac{1}{8} \| \Delta u \|^2 + Q(\|u_{1,0}\|_{H^2_{per}(\Omega)}, \|u_{2,0}\|_{H^2_{per}(\Omega)})\|\omega_1\|H^2_{per}(\Omega) + 1)\|u\|^2. \quad (3.48)
\]
Similarly,
\[
|\text{Vol}(\Omega)(u \langle \mu \rangle |) \leq c(\int_\Omega |f(u_1) - f(u_2)| dx + \int_\Omega |\omega_1 f'(u_1) - \omega_2 f'(u_2)| dx | \langle u \rangle |
\]
\[
\leq c(\int_\Omega |f(u_1) - f(u_2)| dx + \int_\Omega |\omega_1 f'(u_1) - f'(u_2)| dx
\]
\[
+ \int_\Omega |f(u_1) - f(u_2)||f'(u_2)| dx + \int_\Omega |\Delta u||f'(u_2)| dx | \langle u \rangle |
\]
\[
\leq Q(\|u_{1,0}\|_{H^2_{per}(\Omega)}, \|u_{2,0}\|_{H^2_{per}(\Omega)})\|u\|\|\langle u \rangle \| + |\langle u \rangle |^2\)
\]
\[
+ \frac{1}{4} \| \Delta u \|^2 + Q(\|u_{1,0}\|_{H^2_{per}(\Omega)}, \|u_{2,0}\|_{H^2_{per}(\Omega)})\|\omega_1\|H^2_{per}(\Omega) \|\langle u \rangle \|
\]
\[
\leq \frac{1}{4} \| \Delta u \|^2 + Q(\|u_{1,0}\|_{H^2_{per}(\Omega)}, \|u_{2,0}\|_{H^2_{per}(\Omega)})\|\omega_1\|H^2_{per}(\Omega) + 1)\|\langle u \rangle \|^2. \quad (3.49)
\]
Recalling that $g'$ is Lipschitz continuous, we then have
\[
|((g'(\nabla u_1) - g'\nabla u_2), \nabla u)| \leq c\|\nabla u\|^2. \quad (3.50)
\]
We finally deduce from (3.43), (3.45)-(3.50) and the interpolation inequality
\[
\|\tilde{u}\| \leq c\|\tilde{u}\|^\frac{1}{2} \|\nabla \tilde{u}\|^\frac{1}{2} \leq c'\|\tilde{u}\|^\frac{1}{2} \|\Delta \tilde{u}\|^\frac{1}{2}
\] (3.51)
that
\[
\frac{d}{dt}(\|\tilde{u}\|^2 + \|u\|^2) + \|\Delta u\|^2 \leq Q(\|u_{1,0}\|_{H^2_{per}(\Omega)}, \|u_{2,0}\|_{H^2_{per}(\Omega)})\|\omega_1\|H^2_{per}(\Omega) + 1)\|\tilde{u}\|^2 + \|\langle u \rangle \|^2. \quad (3.52)
\]
Gronwall’s Lemma then yields, owing to (3.21), (3.25) and (3.30) (written for $(u_1, \mu_1, \omega_1)$),
\[
\|u(t)\|_{H^{-1}(\Omega)} \leq ce^Q(\|u_{1,0}\|_{H^2_{per}(\Omega)}, \|u_{2,0}\|_{H^2_{per}(\Omega)})|t|\|u_0\|_{H^{-1}(\Omega)}, \quad t \geq 0,
\] (3.53)
hence the uniqueness, as well as the continuous dependence with respect to the initial data in the $H^{-1}$-norm.

b) Existence:

The proof of existence is based on a classical Galerkin scheme and the a priori estimates derived in the previous subsection.
A weak (variational) formulation for (3.1)-(3.5) reads
\[
\left((\frac{\partial u}{\partial t}, v)\right) = -((\nabla \mu, \nabla v)), \quad \forall v \in H^1_{\text{per}}(\Omega),
\]
(3.54)
\[
((\mu, v)) = (g'(\nabla u), \nabla v) + ((f(u), v))
+ ((\omega f'(u), v)) + ((\nabla \omega, \nabla v)), \quad \forall v \in H^1_{\text{per}}(\Omega),
\]
(3.55)
\[
((\omega, v)) = ((f(u), v)) + ((\nabla u, \nabla v)), \quad \forall v \in H^1_{\text{per}}(\Omega),
\]
(3.56)
\[
u_{m_{1=0}} = u_0.
\]
(3.57)

We can note that all estimates in Subsection 3.1 follow (formally) from this variational formulation. Indeed, for instance, (3.19) follows from multiplying (3.2) by \(u\) and we can proceed in a similar way for the other estimates obtained by comparison.

Let \(v_0, v_1, \ldots\) be an orthonormal (in \(L^2(\Omega)\)) and orthogonal (in \(H^1_{\text{per}}(\Omega)\)) family associated with the eigenvalues \(0 = \lambda_0 < \lambda_1 \leq \cdots\) of the operator \(-\Delta\) associated with Neumann boundary conditions (note that \(v_0\) is a constant). We set
\[
V_m = \text{Span}\{v_0, v_1, \ldots, v_m\}
\]
and consider the approximated problem
Find \((u_{m}, \mu_{m}, \omega_{m}) : [0, T] \to V_m \times V_m \times V_m\) such that
\[
\left((\frac{\partial u_{m}}{\partial t}, v)\right) = -((\nabla \mu_{m}, \nabla v)), \quad \forall v \in V_m,
\]
(3.58)
\[
((\mu_{m}, v)) = (g'(\nabla u_{m}), \nabla v) + ((\omega f'(u_{m}), v))
+ ((f(u_{m}), v)) + ((\nabla \omega_{m}, \nabla v)), \quad \forall v \in V_m,
\]
(3.59)
\[
((\omega_{m}, v)) = ((f(u_{m}), v)) + ((\nabla u_{m}, \nabla v)), \quad \forall v \in V_m,
\]
(3.60)
\[u_{m_{1=0}} = u_{0,m},
\]
(3.61)
where \(u_{0,m} = P_{m}u_0, P_m\) being the orthogonal projector from \(L^2(\Omega)\) onto \(V_m\).

The existence of a local (in time) solution to (3.58)-(3.61) is standard. Indeed, we have to solve a Lipschitz continuous finite-dimensional system of ODE’s to find \(u_m\), which yields \(\omega_m\) and then \(\mu_m\).

The a priori estimates derived in the previous subsection, which are now justified within the Galerkin approximation, yield that the solution is global and that, up to a subsequence which we do not relabel and owing to classical Aubin-Lions compactness results,
\[
u_m \to u \text{ weak star in } L^\infty(0,T; H^2_{\text{per}}(\Omega)),
\]
strongly in \(C([0,T]; H^{2-\varepsilon}_{\text{per}}(\Omega)), \forall \varepsilon > 0, \text{ and a.e.},
\[
\frac{\partial u_m}{\partial t} \to \frac{\partial u}{\partial t} \text{ weakly in } L^2(0,T; H^{-1}(\Omega)),
\mu_m \to \mu \text{ weakly in } L^2(0,T; H^1_{\text{per}}(\Omega)),
\]
\[
\omega_m \to \omega \text{ weak star in } L^\infty(0,T; L^2(\Omega)) \text{ and weakly in } L^2(0,T; H^2_{\text{per}}(\Omega)),
\]
as \(m \to +\infty, \forall T > 0\).

Note that, owing to (3.21), (3.23) and (3.25), \(u \in L^\infty(\mathbb{R}^+; H^2_{\text{per}}(\Omega))\) and, consequently, \(\omega \in L^\infty(\mathbb{R}^+; L^2(\Omega))\).
As far as the passage to the limit is concerned, the most delicate part is to prove that
\[
\int_0^T \int_\Omega \omega_m f'(u_m) \varphi \, dx \, dt \xrightarrow{m \to +\infty} \int_0^T \int_\Omega \omega f'(u) \varphi \, dx \, dt
\]
and
\[
\int_0^T \int_\Omega g'(\nabla u_m) \cdot \nabla \varphi \, dx \, dt \xrightarrow{m \to +\infty} \int_0^T \int_\Omega g'(\nabla u) \cdot \nabla \varphi \, dx \, dt,
\]
for \(\varphi\) regular enough.

We have, say, for \(\varphi \in C^2([0,T] \times \bar{\Omega})\) such that \(\varphi(T) = \varphi(0) = 0\),
\[
\int_0^T \int_\Omega \left( \omega_m f'(u_m) - \omega f'(u) \right) \varphi \, dx \, dt = \int_0^T \int_\Omega \left( \omega_m - \omega \right) f'(u) \varphi \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega \omega_m (f'(u_m) - f'(u)) \varphi \, dx \, dt.
\]
(3.62)

The passage to the limit in the first integral in the right-hand side of (3.62) is straightforward, while the passage to the limit in the second one follows from the above convergences which yield, in particular, the inequality
\[
\left| \int_0^T \int_\Omega \omega_m (f'(u_m) - f'(u)) \varphi \, dx \, dt \right| \leq c \|u_m - u\|_{L^2((0,T) \times \Omega)}.
\]

Finally, recalling that \(g'\) is Lipschitz continuous, we have
\[
\left| \int_0^T \int_\Omega (g'(\nabla u_m) - g'(\nabla u)) \cdot \nabla \varphi \, dx \, dt \right| \leq c \|u_m - u\|_{L^2(0,T;H^1_{per}(\Omega))}.
\]

It follows from Theorem 3.2 that we can define the continuous (for the \(H^{-1}\)-norm) semigroup
\[
S(t) : \Phi_M \to \Phi_M, \quad u_0 \mapsto u(t), \quad t \geq 0
\]
(i.e., \(S(0) = Id\) and \(S(t+s) = S(t) \circ S(s), \quad t, s \geq 0\), where
\[
\Phi_M = \{ v \in H^2_{per}(\Omega), \ |\langle v \rangle| \leq M \}, \quad M \geq 0.
\]

We then deduce from (3.25) that \(S(t)\) is dissipative, i.e., it possesses a bounded absorbing set \(B_0 \subset \Phi_M\) (in the sense that, \(\forall B \subset \Phi_M\) bounded, \(\exists t_0 = t_0(B)\) such that \(t \geq t_0 \Rightarrow S(t)B \subset B_0\)).

4. Allen-Cahn system. The Allen-Cahn equation was originally introduced in [1] to describe the motion of antiphase boundaries in crystalline solids.

Assuming the relaxation dynamics \(\frac{\partial u}{\partial t} = -\frac{D\Psi_{AMGL}}{Du}\), we obtain the following Allen-Cahn system, taking again \(\beta = 1\),
\[
\frac{\partial u}{\partial t} - \text{div}(g'(\nabla u)) + f(u) + \omega f'(u) - \Delta \omega = 0,
\]
(4.1)
\[
\omega = f(u) - \Delta u,
\]
(4.2)

\[
together with Dirichlet boundary conditions,
\]
\[
u = \omega = 0 \quad \text{on} \quad \Gamma,
\]
(4.3)

where \(\Gamma\) is the boundary of \(\Omega\) (we assume that \(\Omega\) is a bounded and regular domain of \(\mathbb{R}^2\)), and the initial condition
\[
u|_{t=0} = u_0.
\]
(4.4)
Remark 4.1. We have considered, for simplicity, Dirichlet boundary conditions; actually, the Allen-Cahn equation usually is endowed with such boundary conditions. However, periodic boundary conditions, 
\[ u, \omega \text{ are } \Omega - \text{periodic}, \]
can also be considered (as far as Neumann boundary conditions are concerned, the situation is similar to that in Section 3). In that case, we can obtain the same results by replacing (2.2) by
\[ f(s)s \geq c_1(s^2 + F(s)) - c_2 \geq -c_3, \ c_1 > 0, \ c_2, c_3 \geq 0, \ s \in \mathbb{R}. \]
Note that, in this section, we do not need (2.4).

Our main aim in this section is to prove the

**Theorem 4.2.** We assume that \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \). Then, (4.1)-(4.4) possesses a unique (variational) solution such that
\[ u \in L^\infty(\mathbb{R}^+; H^2(\Omega) \cap H^1_0(\Omega)) \text{ and } \frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega)). \]
Furthermore,
\[ \omega \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H^1_0(\Omega)), \ \forall T > 0. \]
Finally, the associated semigroup is dissipative in \( H^2(\Omega) \cap H^1_0(\Omega) \).

### 4.1. A priori estimates.
We multiply (4.1) by \( \frac{\partial u}{\partial t} \) and have, integrating over \( \Omega \) and by parts,
\[
\| \frac{\partial u}{\partial t} \|^2 + \int_\Omega g'(\nabla u) \cdot \nabla \frac{\partial u}{\partial t} \, dx + \frac{d}{dt} \int_\Omega F(u) \, dx + ((\omega f'(u) - \Delta \omega, \frac{\partial u}{\partial t})) = 0,
\]
which yields, noting that it follows from (4.2) that
\[
((\omega f'(u), \frac{\partial u}{\partial t})) - ((\Delta \omega, \frac{\partial u}{\partial t})) = \frac{1}{2} \frac{d}{dt} \| \omega \|^2
\]
and that
\[
\int_\Omega g'(\nabla u) \cdot \nabla \frac{\partial u}{\partial t} \, dx = \frac{d}{dt} \int_\Omega g(\nabla u) \, dx,
\]
the differential equality
\[
\frac{d}{dt} \left[ 2 \int_\Omega g(\nabla u) \, dx + 2 \int_\Omega F(u) \, dx + \| \omega \|^2 \right] + 2\| \frac{\partial u}{\partial t} \|^2 = 0. \quad (4.5)
\]
In particular, it follows from (4.5) that the energy decreases along the trajectories, as expected.

We then multiply (4.1) by \( u \) and obtain, owing to (4.2),
\[
\frac{1}{2} \frac{d}{dt} \| u \|^2 + \int_\Omega g'(\nabla u) \cdot \nabla u \, dx + ((f(u), u)) + \int_\Omega uf(u)f'(u) \, dx
\]
\[
+ 2((f'(u)\nabla u, \nabla u)) + ((uf''(u)\nabla u, \nabla u)) + \| \Delta u \|^2 = 0,
\]
which yields, owing to (4.2),
\[
\frac{1}{2} \frac{d}{dt} \| u \|^2 + 2 \int_\Omega g(\nabla u) \, dx + ((f(u), u)) + \| \omega \|^2
\]
\[
+ \int_\Omega (uf(u)f'(u) - f(u)^2) \, dx + ((uf''(u)\nabla u, \nabla u)) = 0,
\]
hence, in view of (2.2), (2.3) and (2.5),
\[
\frac{d}{dt} \|u\|^2 + c \left[ 2 \int_\Omega g(\nabla u) \, dx + 2 \int_\Omega F(u) \, dx + \|\omega\|^2 \right] \leq c', \quad c > 0. \tag{4.6}
\]
Summing (4.5) and (4.6), we find an inequality of the form
\[
\frac{dE_1}{dt} + c \left( E_1 + \|\partial_\Omega u\|^2 \right) \leq c', \quad c > 0,
\]
where
\[
E_1 = \|u\|^2 + 2 \int_\Omega g(\nabla u) \, dx + 2 \int_\Omega F(u) \, dx + \|\omega\|^2.
\]
In particular, it follows from (4.7) and Gronwall’s Lemma that
\[
E_1(t) \leq E_1(0)e^{-ct} + c', \quad c > 0, \quad t \geq 0,
\]
hence, in view of (2.2) (which yields that \(\|\omega\|^2 \geq \|\Delta u\|^2 + \|f(u)\|^2 - 2c_0\|\nabla u\|^2\)), (2.7), (4.8) and classical elliptic regularity results,
\[
\|u(t)\|_{H^2(\Omega)} \leq Q(\|u_0\|_{H^2(\Omega)} + e^{-ct} + c', \quad c > 0, \quad t \geq 0. \tag{4.10}
\]
Next, we multiply (4.1) by \(-\Delta u\) to have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 - \int_\Omega g'(\nabla u) \cdot \nabla \Delta u \, dx - \int_\Omega f(u) \Delta u \, dx
\]
\[
- \int_\Omega \omega f'(u) \Delta u \, dx + \int_\Omega \Delta \omega \Delta u \, dx = 0. \tag{4.11}
\]
It thus follows from (4.2) that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 - \int_\Omega g'(\nabla u) \cdot \nabla \Delta u \, dx + \left( (f'(u)\nabla u, \nabla u) \right)
\]
\[
- \left( (\omega f'(u), \Delta u) \right) + \left( \|f(u)\|_{L^2(\Omega)} \right) + \|\nabla \Delta u\|^2 = 0. \tag{4.12}
\]
Now, owing to the continuous embedding \(H^2(\Omega) \subset C(\bar{\Omega})\) and (4.2), there holds
\[
\left| (f'(u)\nabla u, \nabla u) \right| + \left| (\omega f'(u), \Delta u) \right| + \left| (\|f(u)\|_{L^2(\Omega)} \right| \leq Q(\|u\|_{H^2(\Omega)}
\]



\[
\int_\Omega g'(\nabla u) \cdot \nabla \Delta u \, dx \leq \frac{1}{2} \|\nabla \Delta u\|^2 + c\|\nabla u\|^2,
\]
hence
\[
\frac{d}{dt} \|\nabla u\|^2 + c\|u\|^2_{L^2(\Omega)} \leq Q(\|u\|_{H^2(\Omega)}), \quad c > 0. \tag{4.13}
\]

4.2. The dissipative semigroup.

**Proof of Theorem 4.2. a) Existence:**

The proof of existence of solutions is based on the *a priori* estimates derived in the previous subsection and, e.g., a standard Galerkin scheme.

In particular, it follows from (4.7)-(4.8) and (4.10) that we can construct a sequence of solutions \(u_m\) to a proper approximated problem such that
\[
u_m \to u \text{ weak star in } L^\infty(0, T; H^2(\Omega)),
\]



\[
\frac{\partial u_m}{\partial t} \to \frac{\partial u}{\partial t} \text{ weakly in } L^2(0, T; L^2(\Omega))
\]
and
\[ \omega_m \rightarrow \omega \text{ weak star in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(0, T; H^2(\Omega)), \]
as \( m \rightarrow +\infty, \forall T > 0. \)

The passage to the limit is then standard and can be done as in the previous section.

Finally, it follows from (4.18)-(4.21) and the interpolation inequality
\[ u \in L^\infty(\mathbb{R}^+; H^2(\Omega)) \]
and, consequently,
\[ \omega \in L^\infty(\mathbb{R}^+; L^2(\Omega)). \]

b) **Uniqueness**: Let \( u_1 \) and \( u_2 \) be two solutions to (4.1)-(4.4) with initial data \( u_{1,0} \) and \( u_{2,0}, \) respectively, where \( \omega_1 \) and \( \omega_2 \) are defined from (4.2). We set \( u = u_1 - u_2, \)
\[ \omega = \omega_1 - \omega_2, \quad u_0 = u_{1,0} - u_{2,0} \]
and have
\[ \frac{\partial u}{\partial t} - \text{div}(g'(\nabla u_1) - g'(\nabla u_2)) + f(u_1) - f(u_2) + \omega f'(u_1) - \omega f'(u_2) - \Delta \omega = 0, \quad (4.14) \]
\[ \omega = f(u_1) - f(u_2) - \Delta u, \quad (4.15) \]
\[ u = \omega = 0, \quad \text{on } \Gamma, \quad (4.16) \]
\[ u|_{t=0} = u_0. \quad (4.17) \]

We multiply (4.14) by \( u \) and obtain, integrating over \( \Omega, \)
\[ \frac{1}{2} \frac{d}{dt} \|u\|^2 + \langle g'(\nabla u_1) - g'(\nabla u_2), \nabla u \rangle + \langle f(u_1) - f(u_2), u \rangle \]
\[ + \langle \omega f'(u_1) - \omega f'(u_2), u \rangle - \langle f(u_1) - f(u_2), \Delta u \rangle + \|\Delta u\|^2 = 0. \quad (4.18) \]

We note that, owing to (2.1),
\[ \langle f(u_1) - f(u_2), u \rangle \geq -c_0 \|u\|^2 \]
and that, owing to (4.9), (4.15), the regularity of \( f \) and Poincaré’s inequality,
\[ \|\omega f'(u_1) - \omega f'(u_2), u \| \leq \|\omega f'(u_1), u \| + \|\omega f'(u_2), u \| \]
\[ \leq Q(\|u_1\|_{H^2(\Omega)}, \|u_2\|_{H^2(\Omega)}) \|\omega\| \|u\| + \|\omega f'(u_2), u \| \]
\[ \leq Q(\|u_1\|_{H^2(\Omega)}, \|u_2\|_{H^2(\Omega)}) \|\omega\| \|u\| + \|\nabla u\|^2 \]
\[ \leq \frac{1}{4} \|\Delta u\|^2 + Q(\|u_1\|_{H^2(\Omega)}, \|u_2\|_{H^2(\Omega)}) \|\nabla u\|^2 \quad (4.19) \]
and
\[ \|\omega f'(u_1) - f(u_2), \Delta u \| \leq \frac{1}{4} \|\Delta u\|^2 + Q(\|u_1\|_{H^2(\Omega)}, \|u_2\|_{H^2(\Omega)}) \|u\|^2. \quad (4.20) \]

Recalling that \( g' \) is Lipschitz continuous, we have, similarly,
\[ \|g'(\nabla u_1) - g'(\nabla u_2), \nabla u \| \leq \int_\Omega |g'(\nabla u_1) - g'(\nabla u_2)| \|\nabla u\| dx \leq c \|\nabla u\|^2. \quad (4.21) \]

We finally deduce from (4.18)-(4.21) and the interpolation inequality
\[ \|\nabla u\| \leq c \|u\|^{\frac{1}{2}} \|\Delta u\|^{\frac{1}{2}} \]
that
\[ \frac{d}{dt} \|u\|^2 + \|\Delta u\|^2 \leq Q(\|u_1\|_{H^2(\Omega)}, \|u_2\|_{H^2(\Omega)}) \|u\|^2. \quad (4.22) \]

Gronwall’s Lemma then yields
\[ \|u_1(t) - u_2(t)\| \leq c e^{Q(\|u_1\|_{H^2(\Omega)}, \|u_2\|_{H^2(\Omega)}) t} \|u_0\|, \quad t \geq 0, \quad (4.23) \]
hence the uniqueness, as well as the continuous dependence with respect to the initial data in the $L^2$-norm.

It follows from the above that we can define the continuous (for the $L^2$-norm) semigroup

$$S(t) : \Phi \rightarrow \Phi, \quad u_0 \mapsto u(t), \quad t \geq 0,$$

where $\Phi = H^2(\Omega) \cap H^1_0(\Omega)$. Finally, the dissipativity of $S(t)$ follows from (4.10).

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