Relativistic tunneling through two "transparent" successive barriers

Massimo Germano
University of Rome "La Sapienza", S.B.A.I. Department (Basic and Applied Sciences for Engineering), via Antonio Scarpa 16, 00161 Rome, Italy
(Dated: September 18, 2019)

In the case of tunneling of relativistic particles, differently from the nonrelativistic case, a limit of "transparent" barrier can also lead to an apparent "superluminal" behavior when considering the phase time. In this limit, the restricting condition of "opaque" barrier of the nonrelativistic case is avoided, nevertheless, the very thin width of a single barrier to obtain this "transparent" limit can result in a problem itself, for probing the effect. A combination of two successive transparent barriers can show an apparent "superluminal" behavior along a macroscopic arbitrary distance "L". Two solutions for energy $E$ above and below the potential square barrier $V$ are found, for both solutions there the apparent superluminal behavior is possible above a threshold of free travelling group velocity (energy) and dependent on the ratio barriers length - free path as function of the ratio group velocity - speed of light.

I. INTRODUCTION

The traversal time of a particle or a wave packet through a forbidden potential barrier has not a unique definition both in nonrelativistic and relativistic case. Different definitions of traversal times have been introduced, the most straightforward being the so called phase time. The phase time is defined, in the nonrelativistic case, to tunneling through a rectangular potential barrier of height $V_0 > E$ and width $\alpha$ the phase time for a wave packet tends, in the limit of "opaque" barrier ($\alpha \gg 1$) to a constant value independent from the width $\alpha$ (Harman effect\(^1\)) so that it could lead to apparent superluminal velocities. Although the interpretation of this apparent superluminal effect is not a subject of the present article, the "opaque" barrier case is a very restricting condition considering the exponential decay of the amplitude through a tunneling process. In the case of relativistic particles\(^2\) through a barrier of width $\alpha$ the phase time has a different expression but it may be still recognized, in this case, a generalized Hartman effect. For example, in the case of two successive barriers\(^3\), the phase time becomes independent, in the limit of "opaque" barrier, both from the width $\alpha$ and from the distance $L$ between the barriers. The limit of opaque barrier gives, by definition, strong constraints for experimental probes, because the wave function amplitude decreases exponentially and this is more effective when more than one barrier is considered. For relativistic particles indeed, differently from the case of nonrelativistic particles, it is possible to consider the opposite limit of "transparent" barrier $k\alpha \ll 1$ that leads, as well, to an apparent superluminal result for the phase time\(^4\). In this article this limit of "transparent" barrier for relativistic particles is applied to a double barrier configuration. The presence of two barriers of width $a$ and distance $L$, in some conditions, leads to a more evident apparent superluminal behavior where the ratio $a/L$ is a key factor.

II. PHASE TIME IN THE APPROXIMATIONS OF "TRANSPARENT BARRIERS" AND RELATIVISTIC PARTICLES

The equations of the momentum outside $(hk)$ and inside $(hq)$ a potential barrier of height $V_0$ of a particle of mass $m$ and energy $E$, are

$$hk = \sqrt{E^2 - m^2c^4}$$

$$hq = \sqrt{m^2c^4 - (V_0 - E)^2}.$$  \hspace{1cm} (1)

To have a proper tunneling, $V_0$ must be in the range $E - mc^2 < V_0 < E + mc^2$ because, below the lower limit the particle has enough energy to propagate over the potential barrier while, above the upper limit, the barrier can become supercritical and spontaneously emit positrons and electrons in the so-called Klein tunneling\(^5\). In the limit of "transparent" barriers $(qa \ll 1)$, the potential satisfies two solutions: for $V_0$ greater than the total energy $E$ we have solution (a)

$$V_0 \approx E + mc^2 - \frac{(hq)^2}{2m} \quad \text{for} \quad V_0 > E \hspace{1cm} (3)$$

and for $V_0$ lesser than the total energy $E$ we have solution (b)

$$V_0 \approx E - mc^2 + \frac{(hq)^2}{2m} \quad \text{for} \quad V_0 < E. \hspace{1cm} (4)$$

The expression for the phase time across two potential barriers of width $a$ separated by a vacuum path of length $L$ is, from Lunardi et. al.\(^6\)

$$\tau_p = \frac{1}{hc^2} \left\{ \left( kL \right) \frac{E}{k^2} - \frac{1}{k^2 q^2 \Gamma^2 + \Delta^2} \right\}. \hspace{1cm} (5)$$

The expressions for $h_1$, $\Gamma$ and $\Delta$ are given in appendix. The approximation for "transparent" barriers $(qa \ll 1)$,
at first order, is given by
\[ \frac{h_{1}}{\Gamma^{2} + \Delta^{2}} \approx \left[ (V_{0} - E)k^{2} \left( \frac{1}{\alpha} - \alpha \right) \right] + \left[ -mc^{2}(k^{2} + q^{2}) \left( \frac{1}{\alpha} + \alpha \right) \right] qa + O(qa)^{2} \]
where \( \alpha \equiv \frac{h}{2mE} \). In the approximation of relativistic particles (\( E \gg mc^{2} \)) and "transparent" barriers (\( qa \ll 1 \)) solutions (a) \[ (1) \] and (b) \[ (3) \] are given in the following:

A. Solution (a) for \( E < V_{0} < E + mc^{2} \)

For this solution with \( V_{0} \) greater than \( E \), \( \alpha \approx \frac{h_{kq}}{2mE} \ll 1 \). Substituting \( V_{0} \) \[ (3) \] into \[ (1) \]
\[ \frac{h_{1}}{\Gamma^{2} + \Delta^{2}} \approx \left[ h^{2}q^{2}k^{2} - mc^{2}q^{2} \left( \frac{1}{\alpha} - \alpha \right) \right] qa \]
that, substituting \( \alpha \approx \frac{h_{kq}}{2mE} \ll 1 \) becomes
\[ \frac{h_{1}}{\Gamma^{2} + \Delta^{2}} \approx \left( -Ekq - \frac{2m^{2}c^{2}E_{q}}{h^{2}} \right) qa; \]
so, the phase time \( \tau_{p} \) \[ (5) \] for this solution becomes
\[ \tau_{p} \approx \left( \frac{L}{c^{2}} + \frac{a}{c^{2}} + \frac{2m^{2}c^{2}a}{h^{2}k^{2}} \right) \frac{E}{h_{k}}. \]
Since the usual phase velocity of the free particle is \( V_{\phi} = \frac{E}{(\hbar k)} \) the final expression for the phase time for this solution is
\[ \tau_{p} \approx \frac{V_{\phi}}{c^{2}} \left[ L + a \left( 1 + \frac{2m^{2}c^{2}}{h^{2}k^{2}} \right) \right]. \]

B. Solution (b) for \( E - mc^{2} < V_{0} < E \)

For this solution with \( E \) greater than \( V_{0} \), \( \alpha \approx \frac{2mc^{2}}{E} \gg 1 \). Substituting \( V_{0} \) \[ (4) \] and \( \alpha \) into \[ (3) \]
\[ \frac{h_{1}}{\Gamma^{2} + \Delta^{2}} \approx \left( -\frac{c^{2}q_{k}h^{2}k^{3}}{E} - \frac{2m^{2}c^{4}q_{k}}{E} \right) qa, \]
finally the expression of the phase time for this solution is
\[ \tau_{p} \approx \frac{V_{\phi}}{c^{2}} \left[ L + \frac{c^{2}}{V_{\phi}^{2}}a \left( 1 + \frac{2m^{2}c^{2}}{h^{2}k^{2}} \right) \right] \]
that is very similar to \[ (10) \] considering that, for relativistic particles, \( V_{\phi} \approx c \).

III. CONDITIONS FOR TRAVERSAL TIME
SUPERLUMINAL BEHAVIOR

For a free relativistic particle, \( c^{2}/V_{g} = V_{\phi} \) where \( V_{\phi} \) is the group velocity or the so called classical velocity of the particle; then, \( \tau_{f} \) could be assumed as the time it would take a free relativistic particle to travel the same path of the example, i.e.
\[ \tau_{f} = \frac{V_{\phi}}{c^{2}} (L + 2a). \]

A. Conditions for solution (a)

So a free particle takes a longer time to travel the distance \( L + 2a \), than the phase time, by an amount \( \Delta t \)
\[ \Delta t \equiv \tau_{f} - \tau_{p} = \frac{V_{\phi}}{c^{2}} \left[ 1 - \frac{2m^{2}c^{2}}{h^{2}k^{2}} \right] = \frac{a}{V_{g}} \left[ 3 - \frac{2c^{2}}{V_{g}^{2}} \right] \]
because \( c^{2}/V_{g}^{2} = (h^{2}k^{2} + m^{2}c^{2})/h^{2}k^{2} \). The tunneling thus is a kind of accelerator of the motion. It must be recalled that equation \[ (14) \] is valid for a relativistic particle with \( V_{g} \approx c \) and it can be seen from \[ (11) \] that the time gain of a tunneling relativistic particle with respect to a free particle begins when the velocity is \( V_{g} > \sqrt{\frac{c}{2}} \) \( \approx 0.82 \) \( c \) and reaches the limit of \( a/V_{g} \) as \( V_{g} \) becomes equal to the limiting value \( c \).

The time gain could be such that the motion could be defined superluminal in the sense considered by the Hartman effect: defining the traversal velocity \( V_{T} \) as the traveled path \( L + 2a \) divided by the phase time \( \tau_{p} \), then
\[ V_{T} = \frac{L + 2a}{\tau_{p}} = \frac{L + 2a}{\tau_{f} - \Delta t} \approx \frac{L + 2a}{\tau_{f}} + \frac{L + 2a}{\tau_{f}} \Delta t \]
that, in terms of free propagating group velocity \( V_{g} \) and of barriers length \( a \), becomes
\[ V_{T} = V_{g} + V_{g} \frac{a}{L + 2a} \left[ 3 - \frac{2c^{2}}{V_{g}^{2}} \right]. \]

Let’s now consider the conditions on \( V_{g} \) and \( a \) such that the traversal velocity \( V_{T} \) tends toward the speed of light in vacuum \( c \). Setting \( V_{T} \rightarrow c \), \( V_{g} = \beta c \) and \( a = \alpha L \), the \[ (10) \] becomes
\[ \alpha = \frac{\beta^{2} - \beta}{2 + 2\beta - 5\beta^{2}}. \]

In figure \[ (11) \] \( \alpha \) vs \( \beta \) is plotted. The curve \( (a) \) shows the values for which \( V_{T} \rightarrow c \). The region on the right of the curve is the region of superluminality. There is no solution for \( \beta = (1 + \sqrt{1})/5 = 0.8633 \) so there is no superluminal effect for \( V_{g} \leq 0.8633c \), whatever be the barrier length \( a \).

Conversely, for \( V_{g} \geq 0.8633c \) there are values of \( \alpha \geq a/L \) for which \( V_{T} \geq c \).
corresponding of equation (17) is, in this case, $\alpha$ so the traversal velocity becomes

$$V_T = V_0 + \frac{a}{L + 2a} \frac{V_0^3}{c^2},$$

and, differently from the previous case, the traversal velocity $V_T$ is always greater than the group velocity $V_g$. Proceeding then like in the case (a), defining the ratios $\alpha = a/L$ and $\beta = V_0/c$ and setting the limit $V_T \rightarrow c$, the corresponding of equation (17) is, in this case,

$$\alpha = \frac{1 - \beta}{\beta^3 + 2\beta - 2}. \tag{20}$$

There is not a possible apparent superluminal behavior of the traversal velocity for $\alpha \leq 0$ thus for $\beta \leq [(9 + \sqrt{105})^{2/3} - 2 \cdot 3^{1/3}] / [3^{2/3}(9 + \sqrt{105})^{1/3}]$ i.e. for $V_g \leq 0.7709c$ while for values above this limit, the apparent superluminal behavior is represented by the space on the right of the curve (b) in Fig. 1.

IV. CONCLUSIONS

For relativistic particles passing through a two forbidden barriers of width $a$ and distance $L$ in regime of "transparent" barrier approximation, $ka \ll 1$ an apparent superluminal behavior is found defining the traversal velocity as path divided by phase time. In the two cases of energy slightly above and under the barrier potential, heights and conditions for superluminal behavior are found with the former case more favorable than the latter, with equivalent conditions, depending, the gain in time, on the energy of the particle and proportional to the width $a$ of the barriers.

Appendix

$$\Gamma \equiv 8\alpha^2 \cosh(2qa) - 4(1 + \alpha^2)^2 \sin^2(kL) \sinh^2(qa). \tag{A.1}$$

$$\Delta \equiv 4\alpha(1 - \alpha^2) \sinh(2qa) + 2(1 + \alpha^2)^2 \sin(2KL) \sinh^2(qa). \tag{A.2}$$

$$h_1 \equiv \Delta \{2(1 + \alpha^2)[(1 + \alpha^2) E q^2(2kL) \sin(2kL) + 4\alpha^2mc^2(k^2 + q^2) \cos(2KL)] \sinh^2(qa) - 4\alpha^2mc^2(k^2 + q^2) \} +$$

$$+ \Delta \{2(1 + \alpha^2) E q^2(2kL) \cos(2kL) \cosh(2qa) + 2(1 + \alpha^2)$$

$$[(1 + \alpha^2) E q^2(2kL) \sin(2kL) + 4\alpha^2mc^2(k^2 + q^2) \sin(2kL)] \sinh^2(qa) + [4\alpha(1 - 3\alpha^2) mc^2(k^2 + q^2) - (1 + \alpha^2)^2 k^2(2qa)(E - V_0) \sin(2kL)] \sinh(2qa) \}$$

B. Conditions for solution (b)

In the case in which $E > V_0 > E - mc^2$, the gain in time is

$$\Delta t \equiv t_f - t_p = \frac{aV_0}{c^2} \tag{18}$$

so the traversal velocity becomes

$$V_T = V_0 + \frac{a}{L + 2a} \frac{V_0^3}{c^2}, \tag{19}$$

References

[1] T.E. Hartman, J. Appl. Phys. 33, 3427 (1962).
[2] M. Büttiker and R. Landauer, Phys. Rev. Lett. 49, 1739 (1982).
[3] M. Büttiker, Phys. Rev. B 27, 6178 (1983).
[4] E.H. Hauge and A.M. Støøneng, Rev. Mod. Phys. 61, 917 (1989).
[5] H.G. Winful, Phys. Rev. A 70, 052112 (2004).
[6] Chun-Fang Li and Xi Chen, Ann. Phys. 11, 916 (2002).
[7] J.T. Lunardi and L.A. Manzoni, Phys. Rev. A 76, 042111 (2007).
[8] M. Germano, Int. J. Theor. Phys. 54 (2), 435 (2015).
[9] N. Dombe, P. Kennedy and A. Calogeroacos Phys. Rev. Lett. 85, 1787 (2000).
[10] J.R. Fletcher, J. Phys. C. 18, L55 (1985).
[11] D. Sokolovski and L. Baskin, Phys. Rev. A. 36, 4604 (1987).
[12] J. Jakiel, V.S. Olkhovsky and E. Recami, Phys. Lett. A. 248, 156 (1998).
[13] H.G. Winful, Phys. Rev. Lett. 91, 260401 (2003).