A note on canonical quantization of fields on a manifold

Ugo Moschella\textsuperscript{1,2} and Richard Schaeffer\textsuperscript{3}
\textsuperscript{1}Università dell’Insubria, 22100 Como, Italia,
\textsuperscript{2}INFN, Sez. di Milano, Italia
\textsuperscript{3}Institut de Physique Théorique, CEA - Saclay, France

April 1, 2009

Abstract

We propose a general construction of quantum states for linear canonical quantum fields on a manifold, which encompasses and generalizes the “standard” procedures existing in textbooks. Our method provides pure and mixed states on the same footing. A large class of examples finds a simple and unified treatment in our approach. Applications discussed here include thermodynamical equilibrium states for Minkowski fields and quantum field theory in the Rindler’s and in the open de Sitter universes. Our approach puts the above examples into perspective and unravels new possibilities for quantization. We call our generalization “extended canonical quantization” as it is suited to attack cases not directly covered by the standard canonical approach.

1 Introduction

Switching from classical to quantum mechanics relies on a set of recipes that, despite their degree of arbitrariness, have proven to successfully describe a large variety of physical phenomena. In the simple case of a mechanical system having a finite number of degrees of freedom, the time-honored procedure of “canonical quantization” essentially amounts to replacing Poisson brackets by commutation relations:

\[ \{q_i, p_j\} = \delta_{ij} \longrightarrow [\hat{Q}_i, \hat{P}_j] = i\hbar \delta_{ij}, \quad i, j = 1, \ldots, N. \]  (1)

In this step \(Q_i\) and \(P_j\) are understood as elements of an abstract Heisenberg algebra. Under suitable technical assumptions the fundamental uniqueness theorem by Stone and von Neumann establishes that there exists only one representation of the commutation relations (1) by operators in a Hilbert space \(H\)

\[ Q_i \rightarrow \hat{Q}_i, \quad P_j \rightarrow \hat{P}_j : \quad [\hat{Q}_i, \hat{P}_j] = i\hbar \delta_{ij} 1_H, \]  (2)
all other representations being unitarily equivalent; $1_{\mathcal{H}}$ denotes the identity operator in the Hilbert space $\mathcal{H}$.

The situation drastically changes when considering systems with infinitely many degrees of freedom. The Stone-Von Neumann theorem fails for infinite systems and there exist uncountably many inequivalent Hilbert space representations of the canonical commutation relations [1, 2]. Therefore, quantizing an infinite system such as a field involves two distinct steps:

1. construction of an infinite dimensional algebra describing the degrees of freedom of the quantum system;

2. construction of a Hilbert space representation of that algebra.

Unfortunately, a complete classification of the possible representations of the canonical commutation relations does not exist and is not foreseen in the near future. This lack of knowledge is especially relevant in curved backgrounds where, generally speaking, the selection of a fundamental state cannot be guided by the same physical principles as in flat space. Indeed, while the CCR’s have a purely kinematical content, the construction and/or the choice of one specific representation in a Hilbert space is always related to dynamics and different dynamical behaviors require inequivalent representations of the CCR’s (see e.g. [1]). This is related to many fundamental issues such as renormalizability, thermodynamical equilibrium and entropy, phase transitions, etc..

The above considerations suggest that it might be useful to re-examine once more the quantum theory of a free Klein-Gordon (KG) field on a curved background. Another reason for doing so is the occurrence [4, 5] of situations where one is forced to stretch the canonical formalism beyond its limits to get a physically meaningful result.

In flat spacetime the “standard” quantization (i.e. Hilbert space realization) of a field is founded on the physical requirement to retain only the positive frequency solutions of the Klein-Gordon equation in order to construct the N-particle excitations of the vacuum. This choice is called the “spectral condition”; in the general setup of QFT it is an independent assumption [6] and produces a construction that is so natural that the non-uniqueness of the ensuing quantization is hardly mentioned in standard textbooks of QFT and often overlooked and forgotten.

The situation is vastly different for fields on curved spacetimes; here a global energy operator may not exist at all and a true spectral condition is in gen-

\footnote{Also in the case of infinite systems, one might want to reserve the name “quantization” only to the first step that consists in replacing a classical Poisson algebra with a CCR or a local observable algebra. This is the viewpoint advocated in the algebraic approach to quantum field theory and there are good reasons to subscribe to this (representation independent) notion of quantization. Indeed, it may be that the algebra which one tries to construct has non-trivial ideals which are missed by proceeding directly to a representation [3]. We adopt however the usage that is (often tacitly) adopted in the vast majority of quantum field theory textbooks to call “quantization” a Hilbert space realization of the field algebra. Indeed, the standard machinery of quantum field theory can be put at work only in a concrete Hilbert space realization (i.e. computing Feynman propagators, constructing perturbative amplitudes, etc.)}
eral not available. What remains is a well-developed formalism for canonical quantization of linear fields\(^2\), based on the introduction of a conserved inner product in the space of classical solutions of the Klein-Gordon equation: given two complex solutions \(\phi_1\) and \(\phi_2\), their inner product is defined as the integral of the conserved current 
\[ j_\mu(x) = i\phi_1^*(x) \nabla_\mu \phi_2(x) - i\phi_2(x) \nabla_\mu \phi_1^*(x) \]
over a three-dimensional Cauchy surface; the necessary assumption is therefore the global hyperbolicity of the spacetime manifold (see e.g. [8]). A canonical quantization is then achieved by finding a splitting of the above space of complex solutions into the direct sum of a subspace where the inner product is positive and of its complex conjugate. The “1-particle” Hilbert space of the theory is finally obtained by completing the chosen positive subspace in the Hilbert topology defined by the inner product. Of course, infinitely many inequivalent theories can be constructed this way: for example, once given a certain quantization, a whole family of inequivalent alternatives can be obtained by means of Bogoliubov’s transformations.

A considerable amount of work has been devoted to the attempt of formulating various alternative prescriptions to select, among the possible representations of a field theory, those which can have a meaningful physical interpretation: here we quote, in view of their importance in semiclassical general relativity, only the local Hadamard condition, ([8, 9, 10] and references therein) and the microlocal spectral condition\(^3\).

However, before facing the problem of selecting a quantization by its physical properties, one should better make sure not to have omitted relevant possibilities; to this scope it is important to construct the most general available quantization scheme. The contribution of the present note is specifically at this level and starts from the rather obvious remark that Bogoliubov transformations are not enough to produce the most general class of canonical theories constructible starting from a given quantization. This observation opens the door to a useful generalization of the canonical formalism. The latter provides access to many additional inequivalent quantizations, some of them having a potentially important physical significance.

One important point in our construction is that it allows for a more flexible use of coordinate systems. The standard formalism of canonical quantization, that we have already briefly summarized, requires the global hyperbolicity of the spacetime manifold. A globally hyperbolic manifold can be foliated by a family of Cauchy surfaces \(\Sigma_t\) where \(t\) is a temporal coordinate; the Klein-Gordon inner product is built by integrating the current associated with two complex solutions over any Cauchy surface and its value does not depend on the chosen surface.

However, it is not uncommon in concrete examples to make use of coordinates that cover only a portion of a certain spacetime manifold and that the spacelike hypersurfaces defined by a condition of the form \(t = \text{constant}\) in the given coordinate system are not Cauchy surfaces for the extended manifold; the patch covered by the coordinate system may or may not be a globally hyperbolic

\(^2\)See e.g. the classic reference book [7]; a more mathematical description of the method can be found in [8] and a recent short account in [9].
manifold in itself.

For example, this situation is encountered in black-hole spacetimes or in the Rindler coordinate system of a wedge of a Minkowski spacetime, and this is well-known and understood [12, 13, 14, 15, 16, 17, 18, 10]. The same phenomenon may also happen in cosmological backgrounds. Depending on the behavior of the scale factor \( a(t) \), the surface \( t = \text{constant} \) may fail to be a Cauchy surface for the maximally extended manifold (while it is a Cauchy surface for the patch covered by the Friedmann-Lemaître-Robertson-Walker coordinates). In this circumstances textbooks suggest the use of the standard canonical formalism [7], but many possible quantizations are lost in this very initial step. We will see how our generalization of the canonical formalism will allow to construct representations that are not attainable by the standard recipes in the above circumstances.

Our construction is useful already in flat spacetime; in Section 3 we show how the KMS quantization of the free scalar field fits in our generalization of the canonical formalism. In Sect. 4 we will give a fresh look to the Rindler model and the Unruh effect and in Section 5 to the open de Sitter model. The common feature shared by our discussion of these well-known examples is that the Minkowski Wightman vacuum and the preferred Euclidean de Sitter vacuum are here reconstructed working solely inside the patches covered by the relevant coordinate systems.

As a conclusion, we summarize our findings in Sect. 6.

2 A general canonical scalar quantum field

Let us consider a real scalar quantum field \( \phi(x) \) on a general background \( \mathcal{M} \). At this initial level of generality it is not necessary to assume any equation of motion for the field \( \phi \). From a mathematical viewpoint, the field is a map

\[
f \rightarrow \phi(f) = \int \phi(x)f(x)dx
\]

from a suitable linear space of test functions, say \( \mathcal{D}(\mathcal{M}) \), to a corresponding field algebra \( \mathcal{F} \). The commutation relations have a purely algebraic content; in particular, for generalized free fields the commutator is a c-number, i.e. a multiple of the identity element of \( \mathcal{F} \):

\[
[\phi(f), \phi(g)] = C(f, g) \mathbf{1} = \int C(x, x') f(x) g(x') \, dx \, dx';
\]

in this equation \( C(x, x') \) is an antisymmetric bidistribution on the manifold \( \mathcal{M} \) which has to vanish coherently with the notion of locality inherent to \( \mathcal{M} \), i.e.

\[
C(x, x') = 0 \quad \text{for } x, x' \in \mathcal{M} \quad \text{“spacelike separated”};
\]

\( dx \) shortly denotes the invariant volume form. The covariant formulation used here supersedes the “equal time” CCR’s mentioned in the Introduction\(^3\).

\(^3\)See [6] for a discussion on this point. For Klein-Gordon fields the covariant commutation relations and the equal time CCR’s are equivalent.
The next step is to realize the field as an operator-valued distribution in a Hilbert space $H$,

$$\phi(f) \rightarrow \hat{\phi}(f), \quad (6)$$

and we know that there are infinitely many inequivalent such representations, having different physical interpretations or no interpretation at all.

In the following, we will restrict our attention to generalized free fields, i.e. fields whose truncated $n$-point functions vanish for $n > 2$ (and the one-point function vanishes as well). The quantum theory of generalized free fields is therefore completed encoded in the knowledge of a positive semi-definite\(^4\) two-point function $W(x, x')$, a distribution whose interpretation is that of being the two-point “vacuum” expectation value of the field:

$$W(x, x') \equiv \langle \Omega, \hat{\phi}(x)\hat{\phi}(x')\Omega \rangle. \quad (7)$$

The following is the crucial property that $W(x, x')$ has to satisfy to induce a representation of the commutation rules (4): $W(x, x')$ must realize a splitting of the commutator $C(x, x')$ by solving the following fundamental functional equation:

$$C(x, x') = W(x, x') - W(x', x). \quad (8)$$

Given a $W$ satisfying (8), a Hilbert space representation of the field algebra (4) can be constructed explicitly. The one-particle space $H^{(1)}$ is obtained by the standard Hilbert space completion the space of test function $D(M)$ w.r.t. the positive semi-definite pre-Hilbert product provided by the two-point function (see e.g. [6]):

$$\langle f, g \rangle = W(f^*, g) = \int W(x, x') f^*(x') g(x') dx dx'. \quad (9)$$

The full Hilbert space of the theory is the symmetric Fock space $\mathcal{H} = F_s(H^{(1)})$. Each field operator $\hat{\phi}(f)$ can be decomposed into “creation” and “annihilation” operators $\hat{\phi}(f) = \hat{\phi}^+(f) + \hat{\phi}^-(f)$ defined by their action on the dense subset of $F_s(H^{(1)})$ of elements the form $h = (h_0, h_1, \ldots, h_n, 0, 0, 0, \ldots)$:

$$\left(\hat{\phi}^- (f) h\right)_{n} (x_1, \ldots, x_n) = \sqrt{n+1} \int W(x, x') f(x) h_{n+1} (x', x_1, \ldots, x_n) dx dx', \quad (10)$$

$$\left(\hat{\phi}^+ (f) h\right)_{n} (x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(x_k) h_{n-1} (x_1, \ldots, \hat{x}_k, \ldots, x_n). \quad (11)$$

Because of (8) these formulae imply the commutation relations (4).

---

\(^4\)This hypothesis should however be relaxed to deal with local and covariant gauge quantum field theories [1]
2.1 Klein-Gordon fields

We now proceed to describe the construction of two-point functions $\mathcal{W}$ that fulfill the above requirements by further restricting our attention to Klein-Gordon fields.

The most general two-point function will be shortly seen to be obtainable beyond the scheme of canonical quantization, but let us follow at first the standard recipes [7] and construct a family of complex classical solutions $\{u_i(x)\}$ of the Klein-Gordon equation; $\{u_i(x)\}$ has to be a complete (see e.g. [18]) and orthonormal set in the following sense:

\begin{align}
(u_i, u_j) &= \delta_{ij}, \quad (u_i^*, u_j^*) = -\delta_{ij}, \quad (u_i, u_j^*) = 0, \\
(12)
\end{align}

where $(u, v)$ denotes the Klein-Gordon inner product on a globally hyperbolic manifold $\mathcal{M}$.

The standard canonical quantization of the Klein-Gordon field corresponding to the set $\{u_i(x)\}$ is then achieved in the following two steps: first step, write the formal expansion of the field $\phi(x)$ in terms of the elements of a CCR algebra

\begin{align}
\phi(x) &= \sum [u_i(x)a_i + u_i^*(x)a_i^\dagger] \\
(13)
\end{align}

in terms of the elements of a CCR algebra

\begin{align}
[a_i, a_j^\dagger] &= \delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0; \\
(14)
\end{align}

second step, construct the corresponding Fock representation $\phi(x) \to \tilde{\phi}(x)$ which is fully characterized by the annihilation conditions

\begin{align}
\hat{a}_i|\Omega\rangle = 0, \quad \forall i. \\
(15)
\end{align}

While the first step just encodes the covariant (unequal time) commutation relations

\begin{align}
[\phi(x), \phi(y)] = C(x, y) = \sum [u_i(x)u_i^*(y) - u_i(y)u_i^*(x)]; \\
(16)
\end{align}

the Fock space construction in the second step is completely equivalent to the assignment of the two point vacuum expectation value

\begin{align}
\mathcal{W}(x, x') = \langle \Omega, \tilde{\phi}(x)\tilde{\phi}(y)|\Omega\rangle = \sum u_i(x)u_i^*(y). \\
(17)
\end{align}

---

5There are mathematical problems in this very initial step because the space of complex solutions of the Klein-Gordon equation is not an Hilbert space but it has just an indefinite metric induced by the Klein-Gordon product; therefore one is not entitled to speak of a “basis” unless some Hilbert topology is added (but extra information is needed). This problem has been circumvented and solved [8, 10] by studying the possible Hilbert topologies that one can give to the space of real classical solutions of the Klein-Gordon equation for a description of these results. Here we are interested in the heuristics, but there is a relation of our findings with the aforementioned construction. This will be studied elsewhere.
Note that this two-point function is the most simple solution for the split equation $(8)$, after the covariant commutator $C(x, y)$ has been expanded in the basis of modes $u_i(x)$ as in Eq. (16); the permuted function is simply

$$W(x, x') = W(x', x) = \langle \psi_0, \hat{\phi}(x') \hat{\phi}(x)\psi_0 \rangle = \sum u_i^*(x) u_i(x').$$  (18)

There are however infinitely many other solutions of the functional equation $(8)$ giving rise to (possibly) inequivalent canonical quantizations. We will now show how to construct many of them by using the complete set $(12)$; to this end, it is useful to begin the discussion by reviewing the standard theory of Bogoliubov transformations.

A Bogoliubov transformation amounts to the construction of a second complete system $\{v_i(x)\}$ by the specification of two complex operators (matrices) $a_{ij}$ and $b_{ij}$ such that

$$v_i(x) = a_{ij} u_j(x) + b_{ij} u_j^*(x),$$  (19)

$$u_j(x) = v_i(x) a_{ij}^* - v_i^*(x) b_{ij}. $$  (20)

By composing the direct and inverse transformations it follows that $a$ and $b$ must satisfy the following conditions:

$$a_{il} a_{jl}^* - b_{il} b_{jl}^* = \delta_{ij}, \quad a_{il} b_{jl} - b_{il} a_{jl} = 0,$$  (21)

$$a_{il}^* a_{lj} - b_{il}^* b_{lj} = \delta_{ij}, \quad a_{il}^* b_{lj} - b_{il} a_{lj}^* = 0. $$  (22)

The standard Fock quantization based on the system $\{v_i(x)\}$ is then encoded in the two-point function

$$W_{a,b}(x, x') = \sum v_i(x) v_i^*(y) = \sum [a_{ij} a_{il}^* u_j(x) u_l^*(y) + b_{ij} b_{il} u_j^*(x) u_l(x)]$$

interpreted as the two-point vacuum expectation value of the quantum field. Positive definiteness of $(23)$ is evident.

When $b$ is a Hilbert-Schmidt operator this quantization turns out to be unitarily equivalent to the Fock quantization $(17)$. Otherwise $(17)$ and $(23)$ give rise to inequivalent quantizations. The commutator must however be independent of the choice of $a$ and $b$; condition $(22)$ precisely implies that this is true:

$$W_{a,b}(x, x') - W_{a,b}(x', x) = \sum [v_i(x) v_i^*(x') - v_i(x') v_i^*(x)] =$$

$$= \sum [u_i(x) u_i^*(x') - u_i(x') u_i^*(x)] = C(x, x'). $$  (24)

At this point in most textbooks the story about canonical quantization comes to an end. There is however is much room left. We show in this paper that new representations can be produced by enlarging the family of two-point functions displayed in $(23)$. Consider indeed two hermitian matrices $A$ and $B$ and a complex matrix $C$ and construct the general quadratic form

$$Q(x, x') = \sum |A_{ij} u_i(x) u_j^*(x') + B_{ij} u_i^*(x) u_j(x')| +$$

7
Now we ask that $Q(x, y)$ be a solution of Eq. (8); by imposing the commutation relations (16) we get the following conditions on the operators $A, B$ and $C$

$$A_{ij} - B_{ji} = \delta_{ij}, \quad C_{ij} - C_{ji} = 0.$$  

(26)

In the end we obtain the most general expression for a canonical two-point function solving the Klein-Gordon equation:

$$W(x, y) = \sum [\delta_{ij} + B_{ji}] u_i(x) u_j^*(x') + \sum B_{ij} u_i^*(x) u_j(x') + \text{Re} \sum C_{ij} [u_i(x) u_j(x') + u_i(x') u_j(x)] + S(x, x').$$  

(27)

Only the first diagonal term at the RHS contributes to the commutator. The other terms altogether constitute the most general combination of the modes (12) so that the total contribution to the commutator vanish. Eq. (27) provides a considerable enlargement of the family of possible quantizations as compared to the subset (23) provided by the standard canonical quantization rules plus Bogoliubov transformations. We stress once more that however “canonicity” is preserved in the sense the commutator is always the same and does not depend on the operators $B$ and $C$. For example, when $\mathcal{M}$ is Minkowski space, Eq. (27) is the most general superposition of positive and negative energy modes that preserves the standard equal-time CCR’s.

Eq. (27) reduces to a Bogoliubov transformation of the reference theory only in the special case (23). These states are pure states. The states that we have added in the enlarged canonical formalism are in general mixed states: the representation of the field algebra is not irreducible.

As it will made clear in the discussion of concrete examples, simple but important examples of mixed states are provided by the following family of models:

$$W_{a,b}(x, x') = \sum [a_{ij} a_{il}^* u_j(x) u_l^*(x') + b_{ij} b_{il}^* u_j^*(x) u_l(x)],$$  

(28)

with $a_{ij} a_{il}^* - b_{ij} b_{il}^* = \delta_{jl}$.

There is even place for a further generalization: the term $S(x, x')$, which we have not yet commented. This is a bisolution of the Klein-Gordon equation that is not “square-integrable” (even in a generalized sense). It is of classical nature and symmetric in the exchange of $x$ and $x'$. Quantum constraints do not generally forbid the existence of such a contribution. Its introduction may be necessary to access to degrees of freedom which cannot be described in terms of the $L^2$ modes (12). This important extension to non-$L^2$ “classical” modes deserves a thorough examination [19] is incidentally mentioned here.

We now pass to discussing a few examples where the virtues of our extension of the canonical formalism will be made clear.
3 KMS quantization

A toy example in this class can be constructed in flat spacetime starting from the standard plane wave solution to (12):

\[ u_k(x) = u_k(t, x) = \frac{1}{\sqrt{2\omega(2\pi)^3}} \exp(-i\omega t + ik \cdot x), \quad \omega = \sqrt{k^2 + m^2}. \] (29)

The reference two-point function w.r.t. this set of modes satisfies positivity of the spectrum of the energy operator in every Lorentz frame [6]:

\[ \mathcal{W}(x, x') = \int u_k(x) u^*_k(x') dk = \frac{1}{(2\pi)^3} \int e^{-ik(x-x')} \theta(k^0) \delta(k^2 - m^2) dk. \] (30)

The field \( \phi \) and its conjugate \( \pi(t, x) = \partial_t \phi(t, x) \) can be reconstructed as in (11); CCR’s hold literally i.e. \([\phi(t, x), \pi(t, y)] = i\delta(x - y)\). In the toy example that follows we consider the diagonal operators

\[ a_{kk'} = \sqrt{\frac{e^{\beta/2}}{2 \sinh(\beta/2)}} \delta_{kk'}, \quad b_{kk'} = \sqrt{\frac{e^{-\beta/2}}{2 \sinh(\beta/2)}} \delta_{kk'}. \] (31)

depending on a constant \( \beta \). Proceeding as in (23) by Bogoliubov transformations one obtains noncovariant canonical quantizations of the Klein-Gordon field. On the other hand \( \mathcal{W}_{a,b} \) constructed as in (28) provides a covariant canonical quantization that cannot be obtained by Bogoliubov transformations. Consequently the two-point function \( \mathcal{W}_{a,b} \) cannot satisfy the spectral condition [6] and, as anticipated, states with negative energy are now present in the Hilbert space of the model:

\[ \mathcal{W}_{a,b}(k) = \left[ \frac{1}{1 - e^{-\beta}} \theta(k^0) + \frac{1}{e^{\beta} - 1} \theta(-k^0) \right] \delta(k^2 - m^2). \] (32)

In the next example the above toy model is generalized to non-constant but yet diagonal matrices. Let us consider in particular the operators

\[ a_{kk'} = \sqrt{\frac{e^{\beta \omega/2}}{2 \sinh(\beta \omega/2)}} \delta_{kk'}, \quad b_{kk'} = \sqrt{\frac{e^{-\beta \omega}}{2 \sinh(\beta \omega/2)}} \delta_{kk'}. \] (33)

Here Bogoliubov transformations (23) give an otherwise uninteresting canonical Klein-Gordon quantum field theory. On the contrary, the two-point function

\[ \mathcal{W}_{\beta}(x, x') = \frac{1}{(2\pi)^3} \int e^{-ik(x-x')} \left[ \frac{\theta(k^0)}{1 - e^{-\beta k^0}} + \frac{\theta(-k^0)}{e^{-\beta k^0} - 1} \right] \delta(k^2 - m^2) dk, \] (34)

that is a special instance of the family of models exhibited in Eq. (28), is of fundamental importance in quantum field theory as it provides the well-known Kubo-Martin-Schwinger (KMS) thermal representation of the Klein-Gordon field at inverse temperature \( \beta \) [7, 20, 21]. This quantization can of course
can be obtained by a variety of other means. To compare our construction to the literature, we see a point of contact with the so-called thermofield theory [22, 17] where the KMS representation is also obtained in an approach inspired from canonical quantization in a fundamental state rather than by a statistical average as usual [20]. However, there is a difference in that we do not need to introduce any doubling of the degrees of freedoms by means of an auxiliary “dummy” space but we insist in representing one and the same field algebra. Indeed the so called doubling of the degrees of freedoms used in thermofield theory is an artifact of the momentum space representation used to implement the x-space CCR’s; these momentum space deformations however are a very useful and convenient mathematical tool to perform practical calculations. The KMS construction thus is seen to be an example encompassed by the construction (27), which of course is much more general.

4 Rindler space

In the following important example we will apply our method to revisit the widely studied Rindler spacetime and the Unruh effect [13]. To keep the discussion at the simplest level, but still rigorous and general, we will consider the two-dimensional massive Klein-Gordon field. Indeed, the massless case, which is usually discussed in textbooks (see e.g. [7]), is very special because of its conformal invariance. Also, the massless Klein-Gordon theory in two-dimensions has an infrared behavior that renders the (local and covariant) canonical two-point function not positive-definite and the linear space of states of the model includes necessarily negative-norm unphysical states [23, 24]. The general dimensional case easily follows from the two-dimensional massive case.

The two-dimensional Rindler spacetime can be identified with (say) the right wedge of the two-dimensional Minkowski spacetime w.r.t. a chosen origin. The relevant coordinate system is constructed from the action of the Lorentz boosts that leave the Rindler wedge invariant (which are of course isometries of the wedge):

\[
\begin{align*}
  x^0 &= e^\xi \sinh \eta, \quad x^1 = e^\xi \cosh \eta, \\
  ds^2 &= e^{2\xi}(d\eta^2 - d\xi^2).
\end{align*}
\]

The variable \(\eta\) is interpreted as the Rindler time coordinate. With the help of these coordinates the massive Klein-Gordon equation is written as follows:

\[
\partial^2_\eta \phi - \partial^2_\xi \phi + m^2 e^{2\xi} \phi = 0. \tag{37}
\]

Let us consider factorized solutions of the form \(u(\eta, \xi) = e^{-i\omega \eta} F_\omega(\xi)\), which are of positive frequency \(\omega > 0\) w.r.t. the Rindler time \(\eta\); the factor \(F_\omega(\xi)\) obeys the modified Bessel equation:

\[
-\partial^2_\xi F + m^2 e^{2\xi} F = \omega^2 F. \tag{38}
\]
The solution that behaves well at infinity is the Bessel-Macdonald function $K_{i\omega}(me^\xi)$ [25]; therefore, a convenient system solving (12) for the massive Klein-Gordon equation in the Rindler universe can be written as follows:

\[
\begin{align*}
    u_\omega(\eta, \xi) &= \frac{\sqrt{\sinh \pi \omega}}{\pi} e^{-i\omega \eta} K_{i\omega}(me^\xi) \\
    u^*_\omega(\eta, \xi) &= \frac{\sqrt{\sinh \pi \omega}}{\pi} e^{i\omega \eta} K_{i\omega}(me^\xi)
\end{align*}
\]

\[\omega > 0. \tag{39}\]

A spacelike surface that may be used to compute the normalization in (39) is for instance the half-line $\eta = \eta_0 (\xi \in \mathbb{R})$. The result does not depend on the choice of one particular half line because they all share the same origin. In doing this we are applying the standard canonical formalism in the Rindler wedge.

Of course the system (39) is not enough to perform canonical quantization on the whole Minkowski spacetime. In the original approach [13, 7] the system (39) is supplemented by an analogous family of “left” modes. The so completed system can be used to put the machinery of Bogoliubov transformations at work and recover the standard Wightman ground state [13, 7]. In the end, the Unruh effect is exhibited by restriction of Wightman vacuum to the Rindler wedge (in the coordinate system (35)). This fact is however general and model independent: restricting a Wightman quantum field theory to a wedge always gives rise to a KMS state [14, 15].

We are now going to show how our formalism allows for a direct construction of the Wightman vacuum solely within the right Rindler wedge in terms of the “right” modes (39) alone, avoiding the need of extending the system to the left wedge.

In the first step, insertion of the modes (39) in Eq. (27) (with $S(x, y) = 0$) provides a huge family of mathematically admissible two-point functions (and therefore states) for the massive Rindler Klein-Gordon field, all of them sharing the same commutator $C(x, y)$ and, a fortiori, the canonical equal time commutation relations.

In the second step, we select those theories in which the wedge-preserving Lorentz boosts $\eta \to \eta + a$ are unbroken symmetries; this condition imposes the following restrictions on (27):

\[B_{\omega, \omega'} = b(\omega) \delta_{\omega, \omega'}, \quad C_{\omega, \omega'} = 0. \tag{40}\]

At this point, we have constructed a family of states parameterized by an arbitrary function $b(\omega)$; they are associated to the following two-point functions:

\[
W_b(x, y) = \frac{1}{\pi^2} \int_0^\infty \left[ (b(\omega) + 1)e^{-i\omega(\eta-\eta')} + b(\omega)e^{i\omega(\eta-\eta')} K_{i\omega}(me^\xi)K_{i\omega}(me^{\xi'}) \sinh \pi \omega \right] d\omega \tag{41}\]

The function $b(\omega)$ should be such that the integral in (41) converges in the sense of distributions. The choice $b = 0$ reproduces the Fulling vacuum [26, 27] for the Rindler’s Klein-Gordon field. Taking inspiration from the examples of the previous section, it is now useful to introduce a function $\gamma(\omega)$ such that

\[b(\omega) = \frac{e^{-\frac{1}{2}\gamma(\omega)}}{2 \sinh(\gamma(\omega)/2)}. \tag{42}\]
so that the two-point function is rewritten as follows:

\[
\mathcal{W}_\gamma(x, y) = \frac{1}{\pi^2} \int_0^\infty \left[ \frac{e^{-i\omega(\eta - \eta')} + e^{i\omega(\eta - \eta')}}{1 - e^{-\gamma(\omega)}} \right] K_{i\omega}(m\xi)K_{i\omega}(m\xi') \sinh \pi \omega \ d\omega;
\]

in this parametrization the Fulling vacuum corresponds to the choice \( \gamma = \infty \).

It possible to find a choice of \( \gamma(\omega) \) such that the corresponding quantum field theory is fully Poincaré invariant. Hence we study the variation of (43) w.r.t. infinitesimal space translations \( e^{i\xi} \delta \eta = -\epsilon \sinh \eta \) and \( e^{i\xi} \delta \xi = \epsilon \cosh \eta \) that map the wedge into itself. Imposing vanishing of the variation and the absence of negative Minkowskian energies we get the unique solution \( \gamma(\omega) = 2\pi \omega \). In this case the two-point function \( \mathcal{W}_{2\pi} \) can be explicitly identified:

\[
\mathcal{W}_{2\pi}(x, x') = \frac{1}{\pi^2} \int_0^\infty K_{i\omega}(m\xi)K_{i\omega}(m\xi') \cosh \omega(\pi - i\eta - i\eta') d\omega =
\]

\[
= \frac{1}{2\pi} K_0 \left( m\sqrt{e^{2\xi} + e^{2\xi'} - 2e^{\xi + \xi'} \cosh(\eta - \eta')} \right) = \frac{1}{2\pi} K_0 \left( m\sqrt{-(x - x')^2} \right). \tag{44}
\]

We have recovered the standard Poincaré invariant quantization of a massive Klein-Gordon field. Unruh’s interpretation follows. The value of our “extended canonical quantization” appears here clearly, as it may work in situations where the extension to a larger manifold is not as obvious as in the Rindler case.

A remarkable difference between the system (29) and the system (39) is that the modes (39) cannot be distinguished from their complex conjugates by their behavior at imaginary infinity; both, indeed, have the same analyticity properties in the imaginary time variable simply because the time coordinate \( \eta \) is periodic in the imaginary part. This is one sort of circumstance where our generalization of the canonical scheme proves to be useful. Similar remarks apply to the previously displayed KMS quantization. Note also that there is no Poincaré invariant quantum field theory in the class of models that can be obtained by standard Bogoliubov transformations (23) of the Rindler modes (39) \textit{within the Rindler wedge}.

There are other theories having a special status in the family (41), which we recall is already a subset of the general family (27). The most noticeable example is the one-parameter family of states identified by the choice \( \gamma(\omega) = \beta \omega, \quad \beta > 0. \) \tag{45}

Let us write the corresponding two-point function explicitly:

\[
\mathcal{W}_\beta(x, y) = \frac{1}{\pi^2} \int_0^\infty \left[ \frac{e^{-i\omega(\eta - \eta')} + e^{i\omega(\eta - \eta')}}{1 - e^{-\beta \omega}} \right] K_{i\omega}(m\xi)K_{i\omega}(m\xi') \sinh \pi \omega \ d\omega.
\]

Since \( K_{i\omega} = K_{-i\omega} \) and since \( |K_{i\omega}(\rho)K_{i\omega}(\rho')\sinh \pi \omega| \) is bounded at infinity in the \( \omega \) variable, one can immediately check that \( \mathcal{W}_\beta(x, y) \) verifies the KMS analyticity and periodicity properties in imaginary time \([2, 21]\) at inverse temperature.
These states are precisely the KMS states in Rindler space, which have been introduced and characterized in [16, 17]. The special value $\beta = 2\pi$ has also been identified [16] with the restriction to the wedge of the Wightman vacuum on the basis of the Bisognano-Wichmann and Reeh-Schlieder theorems. Our proof follows just by enforcing the requirement that the wedge preserving translation be an exact symmetry.

5 Open de Sitter space

This model may be described by the metric

$$ds^2 = dt^2 - \sinh^2 t \frac{(dx_1^2 + dx_2^2)}{r^2}, \quad t > 0, \quad r > 0, \quad x_1, x_2 \in \mathbb{R}.$$  (47)

Such a metric defines an instance of a Lemaître-Friedmann hyperbolic space [28] on the de Sitter hyperboloid, which we call here “open de Sitter space”. The coordinate system used in (47) describes only part of a larger manifold and, correspondingly, the spacelike surfaces $t=\text{const}$ are not complete Cauchy surfaces for the complete manifold i.e. for the de Sitter universe itself.

Consider now the massive Klein-Gordon equation in the open de Sitter universe. A system of function solving (12) is the following:

$$u_{iq,k}(x) = -\frac{iq}{(2\pi)^2} \sqrt{\frac{\pi}{2 \sinh \pi q}} \frac{P_{\frac{1}{2}+i\nu}(\cosh t)}{\sinh t} \left( \frac{(x-k)^2}{2r} + \frac{r}{2} \right)^{iq-1}. \quad (48)$$

$P$ is the associate Legendre function of the first kind [25]; the parameter $\nu$ is related to the mass by $m^2 = \frac{9}{4} + \nu^2$. We then apply the most general quantization scheme given in Eq. (27) to find in that class the so-called “Euclidean” [29, 30] fully de Sitter invariant theory. For $m^2 > 2$ the following operators

$$B_{qkq'k'} = e^{-\pi q \delta_{qq'} \delta_{kk'}} \frac{\Gamma \left( \frac{3}{2} - i\nu - iq \right) \Gamma \left( \frac{3}{2} + i\nu - iq \right)}{2 \sinh \pi q \Gamma \left( \frac{5}{2} - i\nu \right) \Gamma \left( \frac{5}{2} + i\nu \right)}$$

give the answer we seek:

$$\mathcal{W}(x, x') = \int_0^\infty dq \int dk \frac{e^{\pi q u_{iq,k}(x)u_{iq,k}(x')}}{2 \sinh \pi q} + \frac{e^{-\pi q u^*_iq,k(x)u^*_iq,k(x')}}{2 \sinh \pi q} +$$

$$+ 2\text{Re} \int_0^\infty dq \frac{\Gamma \left( \frac{1}{2} - i\nu - iq \right) \Gamma \left( \frac{1}{2} + i\nu - iq \right)}{2 \sinh \pi q \Gamma \left( \frac{3}{2} - i\nu \right) \Gamma \left( \frac{3}{2} + i\nu \right)} \int dk u_{iq,k}(x)u_{iq,k}(x') =$$

$$\frac{\Gamma \left( \frac{3}{2} + i\nu \right) \Gamma \left( \frac{3}{2} - i\nu \right)}{8\pi^2} \left( \zeta^2 - 1 \right)^{-\frac{1}{2}} P_{-\frac{1}{2}+i\nu}(-\zeta). \quad (49)$$

where $\zeta = (2rr')^{-\frac{1}{2}} \sinh t \sinh t' [(x-x')^2 + r^2 + r'^2] - \cosh t \cosh t'$. For $m^2 < 2$ the Euclidean vacuum can be recovered by adding a contribution $S(x, x')$ which is classical and hence need not be square integrable as discussed after our general
formula for quantization Eq. (27); here is an example where the $L^2$ canonical quantization fails. The implications of this novelty in quantum field theory and statistical physics deserve a full specific discussion and will be explored elsewhere [19].

We see once again that our quantization scheme allows to work solely within the known coordinate patch; no extension of the “physical” space to “elsewhere” has been used. Performing such an extension [4, 5] indeed yields the same result, but with much more effort. Furthermore, in cases where the geometry is not so well understood as in the de Sitter case, the way to describe the extension (if any) may be not under control: the power of our generalization of canonical quantization in curved space-times appears here clearly.

6 Summary and concluding remarks

There is much more flexibility in canonical quantization than is usually believed. The simple but very general modification of the standard formalism that we have described in Sect. 2 opens a vast class of new possibilities for constructing canonical fields by means of the Fock construction. The modification amounts to considering the most general quadratic combination of a given set of modes that is compatible with a given commutator function, or, equivalently, solves the fundamental split Equation (8).

Our scheme produces for instance an original and simple construction of the thermal equilibrium states, as long as a wealth of other similar unexplored possibilities: pure states and mixed as well are encompassed in our construction.

Another example which has even more interesting features is quantum field theory in the Rindler wedge. Here, the important characteristic of our approach is its ability to reconstruct the standard Poincaré invariant vacuum working solely within the Rindler wedge. Only local invariance is required to get the globally invariant vacuum, with no need to consider analytic continuation to (or from) whatever “external” complementary space which may be introduced.

We have briefly discussed the de Sitter case in our last example. Also in this case the coordinate system gives access only to a portion of the manifold; while canonical quantization in these coordinates (plus Bogoliubov transformations) produces theories that are not de Sitter invariant, application of our procedure gives rise to the preferred de Sitter invariant theory [29, 30]. When the mass is lower than a critical value there are also non-standard non-square integrable modes which come into the play. These modes were known to exist, but their physical relevance was quite uncertain. We have seen here that such contributions are not at variance with the principles of quantum mechanics since they do not contribute to the commutator. This comforts their relevance, and is another novel feature which will be discussed in detail in a further work [19].

In all these cases, our procedure allows the construction of the same fundamental state in various coordinate systems also where the usual canonical quantization is not able of doing so. We expect such kind of invariance to exist on general grounds, reflecting the requirement that the description of a given
physical state be possible in local system of coordinates. To achieve this goal our generalization of the standard canonical quantization procedure seems relevant and opens the way for a general study of these invariance properties. We have insisted in using the words “canonical quantization” because we are always looking for Fock representations of the canonical commutation relations. Our formalism goes however beyond the standard formalism in that it produces pure and mixed states on an equal footing; also it allows negative energy states always within the limits of canonicity; non-$L^2$ contributions are also allowed (these contributions are of classical nature). That is why it might be called “extended canonical quantum field theory”. We leave to further study the question whether this extension encompasses all the possible Fock representations of the Klein-Gordon field algebra.

References

[1] F. Strocchi. *Elements of quantum mechanics of infinite systems*. World Scientific, Singapore, 1985.

[2] R. Haag. *Local Quantum Physics. Fields, Particles, Algebras*. Springer-Verlag, Berlin Heidelberg New York, second edition, 1996.

[3] D. Buchholz and H. Grundling. The resolvent algebra: A new approach to canonical quantum systems. 2007. arXiv math.OA 0705.1988.

[4] M. Sasaki, T. Tanaka, and K. Yamamoto. Euclidean vacuum mode functions for a scalar field on open de Sitter space. *Phys. Rev.*, D51:2979–2995, 1995.

[5] U. Moschella and R. Schaeffer. Quantum fluctuations in the open universe. *Phys. Rev.*, D57:2147–2151, 1998.

[6] R. F. Streater and A. S. Wightman. *PCT, spin and statistics, and all that*. Addison-Wesley, Redwood, 1989.

[7] N. D. Birrell and P. C. W. Davies. *Quantum fields in curved space*. Cambridge University Press, Cambridge, 1982.

[8] R. M. Wald. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*. Chicago University Press, Chicago, 1994.

[9] B. S. Kay. Quantum field theory in curved spacetime. In J.-P. Fr
toise, G. Naber, and T.S. Tsou, editors, *Encyclopedia of Mathematical Physics*, pages 202–212. Academic (Elsevier), 2006.

[10] B. S. Kay and R. M. Wald. Theorems on the Uniqueness and Thermal Properties of Stationary, Nonsingular, Quasifree States on Space-Times with a Bifurcate Killing Horizon. *Phys. Rept.*, 207:49–136, 1991.
[11] M. J. Radzikowski. Micro-local approach to the Hadamard condition in quantum field theory on curved space-time. *Commun. Math. Phys.*, 179:529–553, 1996.

[12] S. W. Hawking. Particle Creation by Black Holes. *Commun. Math. Phys.*, 43:199–220, 1975.

[13] W. G. Unruh. Notes on black hole evaporation. *Phys. Rev.*, D14:870, 1976.

[14] G. L. Sewell. Quantum fields on manifolds: PCT and gravitationally induced thermal states. *Ann. Phys.*, 141:201–224, 1982.

[15] J. J. Bisognano and E. H. Wichmann. On the Duality Condition for Quantum Fields. *J. Math. Phys.*, 17:303–321, 1976.

[16] B. S. Kay. The double wedge algebra for quantum fields on Schwarzschild and Minkowski space-times. *Commun. Math. Phys.*, 100:57, 1985.

[17] B. S. Kay. Purification of KMS states. *Helv. Phys. Acta*, 58:1030, 1985.

[18] S. A. Fulling and S. N. M. Ruijsenaars. Temperature, periodicity and horizons. *Phys. Rept.*, 152(3):135–176, 1987.

[19] U. Moschella and R. Schaeffer. In preparation.

[20] J. I. Kapusta. *Finite Temperature Field Theory*. Cambridge University Press, Cambridge, 1989.

[21] J. Bros and D. Buchholz. Particles and propagators in relativistic thermo field theory. *Z. Phys.*, C55:509–514, 1992.

[22] H. Umezawa, H. Matsumoto, and M. Tachiki. *Thermo Field Dynamics and Condensed States*. North-Holland, Amsterdam, 1982.

[23] B. Klaiber. The Thirring model. In *Boulder 1967, Lectures In Theoretical Physics Vol. Xa - Quantum Theory and Statistical Physics*, New York 1968, 141-176.

[24] G. Morchio, D. Pierotti, and F. Strocchi. Infrared and vacuum structure in two-dimensional local quantum field theory models: the massless scalar field. *J. Math. Phys.*, 31:1467, 1990.

[25] A. Erdélyi. *The Bateman manuscript project. Higher Transcendental Functions*, volume I,II. McGraw-Hill, New York, 1953.

[26] S. A. Fulling. Nonuniqueness of canonical field quantization in Riemannian space-time. *Phys. Rev.*, D7:2850–2862, 1973.

[27] S. A. Fulling. Alternative Vacuum States in Static Space-Times with Horizons. *J. Phys.*, A10:917–951, 1977.
[28] U. Moschella and R. Schaeffer. Quantum Theory on Lobatchevski Spaces. *Class. Quant. Grav.*, 24:3571–3602, 2007.

[29] G. W. Gibbons and S. W. Hawking. Cosmological Event Horizons, Thermodynamics, and Particle Creation. *Phys. Rev.*, D15:2738–2751, 1977.

[30] J. Bros and U. Moschella. Two-point Functions and Quantum Fields in de Sitter Universe. *Rev. Math. Phys.*, 8:327–392, 1996.