The unbalanced gyroscope stability investigation

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Abstract. In this paper the stability of motion of an unbalanced gyroscope with a flexible shaft in gimbal suspension is considered. First, we consider linear Hamiltonian systems of differential equations. A solution stability criterion is obtained in one case. Next, we explore the gyroscope motion. It consists of an internal and external framework. In the normal position, the axes of the frames and the rotor are mutually perpendicular and intersect at a point that is a fixed point of suspension. Friction in the bearings of the rotor and the frame is missing. The rotor motion system equations and gyroscope frames are considered in the first approximation. The case is considered when the characteristic equation corresponding to the motion equations in the first approximation has eigenvalues with a zero-real part. The stability criterion is obtained.

Introduction
Let us consider the unbalanced gyroscope motion stability problem with a flexible shaft in the cardan suspension. Kardanov suspension consists of internal and external frames. The axes of the frame and the rotor in the normal position are mutually perpendicular and intersect at one point, which is a fixed point of suspension. Suppose that there is no friction in the rotor bearings and frames. The system of equations of motion of the gyroscope and rotor framework is considered in the first approximation. Gyro is used in the design and construction of tunnels, mines. The gyroscopic effect is taken into account in the manufacture of prefabricated building structures. Gyroscopes are used to control the stability of boom tower cranes. Gyroscopes are used for mounting heavy concrete structures. Gyroscopic effects are taken into account when processing surfaces with a rotating tool. Hamiltonian systems are of great interest for statistical mechanics and fluid mechanics. Hamiltonian systems often arise in filtration problems in the production of building materials. In the design of underground and hydraulic structures, this problem arises. Let us consider a linear Hamiltonian system

\[
\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}
\]

with the following Hamiltonian (called quadratic):

\[
H(x,y) = 1/2 \, x^T A x + x^T B y + 1/2 \, y^T C y, \tag{1}
\]

where both \(x\) and \(y\) are \(n\)-dimensional column-vectors of conjugate Hamiltonian variables; \(A\), \(B\), \(C\) are \(n\)-by-\(n\) matrices, \(A\) and \(C\) are symmetric matrices (all matrices are real).

The system of equations can also be written as follows:
\[
\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = V \begin{pmatrix} x \\ y \end{pmatrix},
\]

Where

\[
V = \begin{pmatrix} B^T & C \\ -A & -B \end{pmatrix},
\]

(2)

Matrix \(V\) is called Hamiltonian matrix.

Let us consider some properties of Hamiltonian matrices. It is known [1-3] that the set of eigenvalues of the matrix \(V\) consists of pairs \((\lambda, \lambda')\) of opposite numbers: \(\lambda' = -\lambda\). Non-zero opposite eigenvalues \(\lambda, -\lambda\) have the same multiplicity and form Jordan blocks the same order. The zero eigenvalue has always an even multiplicity [1-4]. The eigenvalues of the Hamiltonian matrix \(V\) will be found [1-4] by solving the characteristic equation (\(E_{2n}\) is the \(2n\)-by-\(2n\) identity matrix):

\[
\det(V - \lambda E_{2n}) = 0
\]

The polynomial on the left side of this equation (called the characteristic polynomial) is the polynomial of \(\lambda\).

**Some properties of linear Hamiltonian systems**

*Theorem 1.* Let \(V\) be Hamiltonian matrix (2), \(\det C \neq 0\). Then

\[
V - \lambda E_{2n} \sim \begin{pmatrix} E_n & O_n \\ O_n & Q(\lambda) \end{pmatrix},
\]

(3)

\[
Q(\lambda) = \lambda^2 C^{-1} + \lambda(BC^{-1} - C^{-1}B^T) + A - BC^{-1}B^T,
\]

(4)

where \(E_n, O_n\) are \(n\)-by-\(n\) identity and zero matrices.

If we add another block row (column) in the block matrix to any block row (column) multiplied on the left (right) by the matrix of the corresponding size, then we will get an equivalent matrix, its determinant and rank won’t change [4]. We multiply the second block column of the matrix \(V - \lambda E\) on the right side by the matrix \(C^{-1}(\lambda E_n - B)\) and we’ll add it to the first column, we get:

\[
V - \lambda E_{2n} = \begin{pmatrix} B^T - \lambda E_n & C \\ -A & -B - \lambda E_n \end{pmatrix}
\sim
\begin{pmatrix} O_n & Q(\lambda) \\ -(\lambda^2 C^{-1} + \lambda(BC^{-1} - C^{-1}B^T) + A - BC^{-1}B^T) & -B - \lambda E_n \end{pmatrix}
\]

We subtract the first block row from the second block row multiplied by the matrix \(-(B + \lambda E_n)C^{-1}\) on the left. We multiply the second column on the left by \(C^{-1}\) and then interchange the first and second columns, and get the matrix (3).

*Corollary 1.* Let \(V\) be the Hamiltonian matrix (2), \(\det C \neq 0\). Then its characteristic equation looks like

\[
\det(\lambda^2 C^{-1} + \lambda(BC^{-1} - C^{-1}B^T) + A - BC^{-1}B^T) = 0.
\]

*Corollary 2.* Let \(V\) be the Hamiltonian matrix (2), \(\det C \neq 0\). Then

\[
\det V = (-1)^{n+1} \det C(A - BC^{-1}B^T).
\]

If \(C = E\)

\[
\det V = (-1)^{n+1} \det(A - BB^T).
\]

*Corollary 3.* If \(V\) is the Hamiltonian matrix (2), \(\det C \neq 0\), then eigenvalue \(\lambda_0\) forms \(m = \text{def}Q(\lambda_0)\) of Jordan blocks.
Corollary 4. If eigenvalue $\lambda_0$ of the Hamiltonian matrix forms $m$ of the Jordan blocks in a Jordan normal form.

If multiplicity of eigenvalue $\lambda_0$ $k \leq 3$ or $k - m = \{0, 1, k-1\}$, Jordan block orders corresponding to $\lambda_0$ can be determined. Indeed, in these cases, the decomposition of $k$ numbers into $m$ blocks will be one-valued (to an accuracy of permutation).

Note that the corollary conditions are always fulfilled for the Hamiltonian matrix not higher than the sixth order.

Theorem 2. Let $V$ be the Hamiltonian matrix (2), submatrix $C$ is positively defined, $CB=B^T C$. Then non-zero eigenvalues form blocks of the first order in the Jordan normal form, zero eigenvalue (if any) forms blocks of the second order.

Proof. In this case $Q(\lambda) = \lambda^2 C^{-1} + A - BC^{-1}B^T$. Matrix $A - BC^{-1}B^T$ is symmetric, matrix $C^{-1}$ is symmetric, positively defined. Consequently, there is [5] a nondegenerate matrix $T$ such that $C^{-1} = T^T, A - BC^{-1}B^T = T^T \Lambda T$, where $\Lambda$ is diagonal matrix. By this means, matrix $Q(\lambda)$ can be conceived of as $Q(\lambda) = T^T (\lambda^2 E + \Lambda) T$.

Let us find the greatest common divisors $d_k(\lambda)$ of all minors of order $k (k=1, n)$ of the matrix $\lambda^2 E + \Lambda$. We get the invariant factors from the formula [5]:

$$e_i(\lambda) = d_i(\lambda), \quad e_i(\lambda) = \frac{d_k(\lambda)}{d_{k-1}(\lambda)}, \quad k = 2, n.$$  

Thus, the invariant factors other than 1 are decomposed into the product of multipliers of the form $(\lambda^2 + \lambda_i)$. Every invariant factor has all $\lambda_i$ different. Using invariant factors, we determine the size of Jordan blocks [5].

Theorem 3. Let us consider the Hamiltonian matrix (2) of the $2n$ order with a positively determined submatrix $C$. Let the matrix $V$ has a non-zero $n$-fold eigenvalue $\lambda_0$. In order for the matrix $V$ to be similar to a diagonal matrix, it is necessary and sufficient

$$CB=B^T C \quad (5)$$

Proof. The Hamiltonian matrix $V$ with eigenvalue $\lambda_0$ of $n$ multiplicity is similar to a diagonal matrix if and only if it forms $n$ Jordan blocks. The latter occurs if and only if $Q(\lambda_0) = O$.

Let us prove its necessity. Let $Q(\lambda_0) = O$. What this means is

$$\lambda_0^2 C^{-1} + \lambda_0 (BC^{-1} - C^{-1}B^T) + A - BC^{-1}B^T = O$$

Matrix $\lambda_0^2 C^{-1} + A - BC^{-1}B^T$ is symmetric, and matrix $BC^{-1} - C^{-1}B^T$ is skew-symmetric. From the uniqueness of the decomposition of the zero matrix into the symmetric and skew-symmetric parts, it follows that

$$\lambda_0^2 C^{-1} + A - BC^{-1}B^T = O, \quad BC^{-1} - C^{-1}B^T = O.$$  

Having multiplied the last equality on the left and right by the matrix $C$, we get equality (5).

Sufficiency is proven in Theorem 2.

Corollary 5. Let us consider the linear canonical system of differential equations with Hamiltonian (1), where the matrix $C$ is positively defined. Let the characteristic equation has a non-zero $n$-fold pure imaginary root. The general solution of this system is stable (according to Lyapunov) if $CB=B^T C$.

Remark. The submatrix positive definiteness condition $C$ means that the Hamiltonian is a positively defined quadratic form relative to the generalized momenta.

On the unbalanced gyroscope motion stability with a flexible shaft in a gimbal
Let us consider the motion of an unbalanced gyroscope with a flexible shaft in a gimbal. The Kardanov suspension consists of internal and external frames. The axes of the frame and the rotor in the normal position are mutually perpendicular and intersect at one point, which is a fixed point of suspension. If suppose that there is no friction in the rotor bearings and frames. Let the framework inertia principal axes and the gyroscope rotor coincide and be directed along the axes of rotation in the normal position.

Let the rotation of the frame be characterized by the angles $\alpha$ and $\beta$. In order to take into account the fact that the rotor turns around the transverse axes, we introduce the angles $\alpha_1$ and $\beta_1$. Let $A = A_2 + A_3$ be the total inertia moment of the frames relative to the outer axis of suspension; $c$ is the coefficient of rigidity of the shaft; $c_1$ and $c_2$ are coefficients of possible connections of the frames; $A_1$, $B_1$, $C_1$ are the main inertia moments of the rotor; $A_2$, $B_2$, $C_2$ are the main inertia moments of the inner frame; $A_3$, $B_3$, $C_3$ are the main inertia moments of the outer frame. Then the equations of motion of the gyroscope frames at a first approximation will be written in the form of (6):

$$
\begin{align*}
A \frac{d^2 \alpha}{dt^2} + c_1 \alpha + c(\alpha - \alpha_1) &= 0 \\
B_2 \frac{d^2 \beta}{dt^2} + c_2 \beta + c(\beta - \beta_1) &= 0
\end{align*}
$$

The homogeneous part of the corresponding equations for the rotor looks like [6-10]:

$$
\begin{align*}
A_1 \frac{d^2 \alpha_1}{dt^2} + H \frac{d \beta_1}{dt} + c(\alpha_1 - \alpha) &= 0 \\
A_1 \frac{d^2 \beta_1}{dt^2} - H \frac{d \alpha_1}{dt} + c(\beta_1 - \beta) &= 0,
\end{align*}
$$

where $H$ is angular momentum.

The systems of equations (6) and (7) can be written in the following matrix form:

$$
G \frac{d^2 w}{dt^2} + P \frac{dw}{dt} - Dw = 0,
$$

where

$$
w = \begin{pmatrix} \alpha \\ \beta \\ \alpha_1 \\ \beta_1 \end{pmatrix}; \quad G = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{pmatrix};
$$

$$
P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H \\ 0 & 0 & -H & 0 \end{pmatrix}; \quad D = \begin{pmatrix} -(c + c_1) & 0 & c & 0 \\ 0 & -(c + c_2) & 0 & c \\ c & 0 & -c & 0 \\ 0 & c & 0 & -c \end{pmatrix}.
$$

Let us introduce $\lambda$-matrix $Q(\lambda) = \lambda^2 G + \lambda P - D$. We will make up the characteristic equation for the matrix equation:

$$
\det Q(\lambda) = \begin{vmatrix} A\lambda^2 + c + c_1 & 0 & -c & 0 \\ 0 & B_2\lambda^2 + c + c_2 & 0 & -c \\ -c & 0 & A_1\lambda^2 + c & H\lambda \\ 0 & -c & -H\lambda & A_1\lambda^2 + c \end{vmatrix} = 0.
$$
Let us consider the stability issue of the differential equations obtained homogeneous linear system solution provided that the characteristic equation has no roots with a non-zero real part.

Let us consider the following minor of the third order of the matrix $Q(\lambda)$:

\[
\begin{vmatrix}
0 & -c & 0 \\
B_2\lambda^2 + c + c_2 & 0 & -c \\
0 & A_\lambda^2 + c & H\lambda
\end{vmatrix} = cH\lambda(B_2\lambda^2 + c + c_2).
\]

It is non-zero if the root of the characteristic equation meets the relations:

\[
\lambda^2 \neq -\frac{c + c_2}{B_2}, \quad \lambda \neq 0.
\]

If $\lambda^2 = -\frac{c + c_2}{B_2}$, the minor at the intersection of the first, second, fourth rows and the last three columns is non-zero:

\[
\begin{vmatrix}
0 & -c & 0 \\
B_2\lambda^2 + c + c_2 & 0 & -c \\
- c & -H\lambda & A_\lambda^2 + c
\end{vmatrix} = -c^3 \neq 0.
\]

Therefore, the rank of the matrix $Q(\lambda_0)$ is equal to 3, where $\lambda_0$ is an arbitrary non-zero root of the characteristic equation. The number of Jordan blocks \[1\] corresponding to the $\lambda_0$ is equal to def $Q(\lambda_0) = 1$. Thus, any non-zero multiple root of the characteristic equation forms a Jordan block above the first order. In this case, we’ll get an unstable motion.

Let us consider now

\[
Q(0) = \det(-D) = \begin{vmatrix}
c + c_1 & 0 & -c & 0 \\
0 & c + c_2 & 0 & -c \\
- c & 0 & c & 0 \\
0 & -c & 0 & c
\end{vmatrix} = c_1c_2c^2.
\]

The zero root of the characteristic equation is obtained only if $c_1 = 0$ or $c_2 = 0$. In this case $\text{rg } Q(0) = 2$ if $c_1 = c_2 = 0$; $\text{rg } Q(0) = 3$ if $c_1 \neq 0$ or $c_2 \neq 0$. Therefore, two Jordan blocks correspond to the zero root (which is always multiplicity 2 \[11\]) when $c_1 = c_2 = 0$ and one Jordan block when $c_1 \neq 0$ or $c_2 \neq 0$. In the latter case, the motion is unstable.

**Corollary 6.** Let the characteristic equation of the obtained system (6), (7) has no eigenvalues with a non-zero real part, $c_1$ and $c_2$ are coefficients of possible connections of the frames. Motion is stable if and only if the characteristic equation (8) has no multiple non-zero roots or zero roots when $c_1 \neq 0$ or $c_2 \neq 0$.

**Summary**

Thus, we can assume the following. If we consider the equations of motion of the gyroscope frames (6) and rotor motion (7), we obtain stability criterion. With this method we can investigate the underground structures stability \[12-18\]. We obtain necessary and sufficient conditions for transformation the Hamiltonian matrix to the diagonal form in the case of two nonzero eigenvalues.

We obtain new properties of linear Hamiltonian systems of differential equations. The Hamiltonian systems have an important role in optimal control problems, multivariable and large-scale systems, scattering theory, estimation, detection and transportation \[18-20\].

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