THE PROJECTIVE SPACE HAS MAXIMAL VOLUME
AMONG ALL TORIC KähLER-EINSTEIN MANIFOLDS

ROBERT J. BERNMAN, BO BERNDTSSON

Abstract. We prove a conjecture saying that complex projective space has maximal volume (degree) among all toric Kähler-Einstein manifolds of dimension $n$. The proof is inspired by our recent work on sharp Moser-Trudinger and Brezis-Merle type inequalities for the complex Monge-Ampère operator, but is essentially self-contained.

Let $X$ be an $n$–dimensional complex manifold $X$ which is Fano (i.e. its first Chern class $c_1(X)$ is ample/positive). For some time it was expected that top-intersection number $c_1(X)^n$, also called the degree of $X$, is maximal for the $n$–dimensional complex projective space, i.e.

$$c_1(X)^n \leq (n+1)^n,$$

There are now counterexamples to this bound. For example, as shown by Debarre (see page 139 in [5]), even in the case when $X$ is toric (i.e. it admits an effective holomorphic action of the complex torus $\mathbb{C}^n$ with an open dense orbit) there is no universal polynomial bound in $n$ on the $n$–th root of the degree $c_1(X)^n$. However a specific conjecture concerning the toric case (see [13] and references therein), says that the bound above holds when $X$ is Kähler-Einstein (i.e. it admits a Kähler metric $\omega$ with constant Ricci curvature). In this note we will confirm this conjecture:

**Theorem 1.** Let $X$ be an $n$–dimensional smooth toric $F$ variety which admits a Kähler-Einstein metric. Then its first Chern class $c_1(X)$ satisfies the following upper bound

$$c_1(X)^n \leq (n+1)^n.$$

As pointed out in [13] one of the motivations for the bound above in the toric setting is another more general conjecture of Ehrhart in the realm of convex geometry, which can be seen as a variant of Minkowski’s first theorem for non-symmetric convex bodies:

**Conjecture.** (Ehrhart). Let $P$ be an $n$-dimensional convex body which contains precisely one interior lattice point. If the point coincides with the barycenter of $P$ then

$$Vol(P) \leq (n+1)^n/n!$$

The case when $n = 2$ was settled by Ehrhart, as well as the special case of simplices in arbitrary dimensions [8]. As explained in the survey [10] the best general upper bound in Ehrhart’s conjecture, to this date, is $Vol(P) \leq$
\((n+1)^n(1-(n-1)/n)^n(\leq (n+1)^n)\). According to the well-known dictionary between polarized toric varieties \((X, L)\) and convex lattice polytopes \(P\) \([7, 10]\), the previous theorem confirms Ehrhart’s conjecture for lattice polytopes \(P\) of the form
\[(0.2)\quad P = \{x \in \mathbb{R}^n : \langle l_i, x \rangle \leq 1 \}
\]
where \(l_i\) are primitive lattice vectors and such that \(P\) is Delzant, i.e. any vertex of \(P\) meets precisely \(n\) facets and the corresponding \(n\) vectors \(l_i\) generate the lattice \(\mathbb{Z}^n\). In any such polytope the origin 0 is indeed the unique lattice point and, as shown by Wang-Zhou \([15]\), 0 is the barycenter of \(P\) precisely when the corresponding toric Fano manifold \(X\) admits a Kähler-Einstein metric. The cases up to \(n \leq 8\) have previously been confirmed by computer assistance (as announced in \([13]\)), using the classification of Fano polytopes for \(n \leq 8\) \([16]\).

It should also be pointed out that, as shown in \([9]\), Bishop’s volume estimate for Einstein metrics, applied to the unit circle-bundle in the canonical line bundle \(K_X \to X\) translates, to the inequality
\[(0.3)\quad c_1(X)^n \leq (n+1)^n(n+1)/I(X)
\]
where \(I(X)\) is the Fano index of \(X\), i.e. the largest positive integer \(I\) such that \(c_1(X)/I\) is an integral class in the Picard group of \(X\). As is well-known \(I(X) \leq n + 1\) with equality precisely for \(X = \mathbb{P}^n\) (see for example page 245 in \([11]\)) and hence the previous theorem improves on the inequality \((0.3)\) in the case when \(X\) is toric.

The idea of the proof which is inspired by our previous work \([1]\), is that if \(X\) admits a Kähler-Einstein metric \(\omega\) violating the inequality \((0.1)\) (where \(c_1(X)^n/n!\) coincides with the volume of \(\omega\)) then we obtain a remarkably good Moser-Trudinger inequality for \(T^n\)-invariant plurisubharmonic functions on a sufficiently large domain \(\Omega\) in \(\mathbb{C}^n\), equivariantly embedded into \(X\). The contradiction is obtained by showing that the Moser-Trudinger inequality is simply too good to be true. To this end we show that the Moser-Trudinger inequality implies a lower bound on the integrability index of \(T^{n+1}\)-invariant plurisubharmonic functions \(u\) on the product domain \(\Omega' := \Omega \times D\) in \(\mathbb{C}^{n+1}\), where \(D\) is the unit-disc. But the bound is violated by the pluricomplex Green function of \(\Omega'\) with a pole at the origin, which gives the desired contradiction.

**Generalizations.** In fact, our arguments show that the theorem above is valid for any Kähler-Einstein manifold \(X\) which is an \(S^1\)-equivariant compactification of \(\mathbb{C}^n\). However, the symmetry under the full torus simplifies some of the technical aspects of the proof.

Moreover, our method of proof can also be modified to handle the case of possibly singular toric Fano varieties, i.e. \(-K_X\) is an ample \(\mathbb{Q}\)-Cartier divisor (equivalently: the corresponding polytope \(P\) in \((0.2)\) is merely assumed to be rational and not necessarily Delzant). To make the connection with the
Ehrhart conjecture we are then led to extend the result of Wang-Zhou \[15\] to the case of a general Fano varieties using the notation of Kähler-Einstein metrics on Fano manifolds with log-terminal singularities very recently introduced in \[3\]. Even more generally we obtain the existence of Kähler-Einstein metrics on the complex torus $\mathbb{C}^n$ whose boundary behavior is determined by a given convex body $P$. This is a consequence of the following theorem:

**Theorem 2.** Let $P$ be a bounded convex body containing $0$ in its interior. Then $0$ is the barycenter of $P$ if and only if there is a convex function $\phi$ on $\mathbb{R}^n$ such that

$$MA_{\mathbb{R}}(\phi) = e^{-\phi}dx$$

and such that the gradient image of $\phi$ is $P$. Moreover, the solution is unique modulo the action of the group $\mathbb{R}^n$ by translations.

Here $MA_{\mathbb{R}}(\phi)$ denotes the real Monge-Ampère measure of the convex function $\phi$ and $dx$ is the usual Euclidean volume form. Details will appear elsewhere \[2\].

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0.1. **Proof of the theorem.** Assume for a contradiction that $c_1(X)^n > (n + 1)^n$. As is well-known any smooth and compact toric variety may be realized as an equivariant compactification of $\mathbb{C}^n$ with its standard torus action (see for example page 10 in \[7\]). In other words, we may embed $F: \mathbb{C}^n \to X$ as an open dense set in $X$ in such a way that the action of the unit-torus $T^n$ and its complexification is preserved. Let now $\omega$ be an $T^n$–invariant Kähler-Einstein metric on $X$. We can then write

$$F^*\omega = dd^c u := \frac{i}{2\pi} \partial \bar{\partial} u$$

for a smooth $T^n$–invariant function $u$ on $\mathbb{C}^n$ satisfying the Kähler-Einstein equation

$$(dd^c u)^n = Ce^{-u}dV$$

for a positive constant $C$. We may rewrite $C = V_X / \int_{\mathbb{C}^n} e^{-u}dV$, where

$$V_X = \int_{\mathbb{C}^n} (dd^c u)^n$$

which coincides with the top-intersection number $c_1(X)^n$. Let now $\Omega_R$ be the set where $u < R$ and note that the sets $\Omega_R$ exhaust $\mathbb{C}^n$, i.e. $u$ is proper. Indeed, by symmetry $0$ is a critical point for $u$ and since $dd^c u > 0$ $u$ is strictly convex in the logarithmic coordinates $p_i = \log |z_i|^2$ it follows that $u \to \infty$ as $|p| \to \infty$. We can hence fix $R$ sufficiently large so that

$$V_{\Omega_R} := \int_{\Omega_R} (dd^c u)^n > (n + 1)^n.$$
Writing $\Omega := \Omega_R$ and replacing $u$ by $u - R$ we then obtain a $T^n-$invariant smooth plurisubharmonic (psh for short) function $u$ (i.e. $dd^c u \geq 0$) solving the following equation:

$$(dd^c u)^n = V_{\Omega} \frac{e^{-u}}{\int_{\Omega} e^{-u}} dV, \quad u = 0 \text{ on } \partial \Omega$$

on the $T^n-$invariant domain $\Omega$ which is hyperconvex, i.e. it admits a negative continuous psh exhaustion function (namely $u$). We will denote by $\mathcal{H}_0(\Omega)$ the space of all psh functions on $\Omega$ which are continuous up to the boundary, where they are assumed to vanish. Its $T^n-$invariant subspace will be denoted by $\mathcal{H}_0(\Omega)^{T^n}$.

Using the previous equation the following Moser-Trudinger type inequality can now be established on $\Omega$ (which has $u$ above as an extremal): there is a positive constant $C$ such that

$$(\text{M-T}) \quad \log \int_{\Omega} e^{-u} dV \leq \frac{1}{V_{\Omega} (n + 1)} \int_{\Omega} (-u)(dd^c u)^n + C$$

for any $u \in \mathcal{H}_0(\Omega)^{T^n}$. The proof may be obtained by repeating the proof of Theorem 1.4 in [1], but for completeness we have given a slightly simplified proof in section 0.2 below, which takes advantage of the full $T^n-$symmetry (while the general argument in [1] only requires $S^1-$symmetry).

Next, from the previous Moser-Trudinger type inequality we deduce another inequality of Brezis-Merle type on the hyperconvex domain $\Omega' = \Omega \times D$ in $\mathbb{C}^{n'}$ where $n' = n + 1$: there is positive constant $A$ such that

$$(\text{B-M);} \quad \int_{\Omega'} e^{-(V_{\Omega'})^{1/n'} u} dV \leq A \left(1 - \int_{\Omega'} (dd^c u)^{n'}\right)^{-1}$$

for any $u \in \mathcal{H}_0(\Omega')^{T^{n'}}$ such that $\int_{\Omega'} (dd^c u)^{n'} < 1$. In particular, since by assumption $V_{\Omega} > (n + 1)^n$, this forces

$$\int_{\Omega'} e^{-n'u} dV \leq A' < \infty$$

for any $u \in \mathcal{H}_0(\Omega')^{T^{n'}}$ such that $\int_{\Omega'} (dd^c u)^{n'} = 1$. More generally, taking limits the previous inequality holds for all $u \in \mathcal{F}(\Omega')^{T^{n'}}$, where $\mathcal{F}(\Omega')$ is Cegrell’s class, which by definition consists of all psh functions $u$ on $\Omega$ which are decreasing limits of elements $u_j$ in $\mathcal{H}_0(\Omega')$ with a uniform upper bound on the Monge-Ampère masses $\int_{\Omega'} (dd^c u)^{n'}$ (see [1] and references therein).

The desired contradiction will now be obtained by exhibiting a function violating the previous inequality. To this end we simply let $u := g$ be the pluricomplex Green function for $\Omega'$ with a pole at 0:

$$(0.5) \quad g(z) := \sup \{ u(z) : u \in (PSH \cap c^0)(\Omega' - \{0\}) : u \leq 0, \ u \leq \log |z|^2 + O(1) \}$$
As is well-known [12] $g$ is continuous up to the boundary on $\Omega'$ a part from a singularity at $z = 0$ and satisfies

$$(dd^c g)^n = \delta_0 \text{ on } \Omega' - \{0\}, \quad g = \log |z|^2 + O(1)$$

In particular $(dd^c g)^n = 1$ and $\int_{\Omega'} e^{-n'g} dV = \infty$. Finally, the proof is concluded by noting that $g$ is $T^n$--invariant. Indeed, since $0$ is invariant under the action of $T^n$ it preserves the convex class of functions where the sup in (0.5) is taken and hence the sup $g$ must be $T^n$--invariant. Alternatively one can also invoke the uniqueness of solutions to (0.4) (in the class of functions with the same regularity properties as $g$).

0.2. Proof of the Moser-Trudinger type inequality (M-T). Let

$$G(u) := \log \int_{\Omega} e^{-u} dV + \frac{1}{V_{\Omega}(n + 1)} \int_{\Omega} u(dd^c u)^n$$

whose Euler-Lagrange equation (i.e. the critical point equation $dG|_u = 0$) is precisely the complex Monge-Ampère equation (0.4). Given $u_0$ and $u_1$ in $\mathcal{H}_0(\Omega)$ there is a unique geodesic $u_t$ connecting them in $\mathcal{H}_0(\Omega)$ which may be defined as the unique solution to the following Dirichlet problem for the Monge-Ampère equation: setting $U(z,t) := u_t(z)$, where now $t$ has been extended to a complex strip $T$ by imposing invariance in the imaginary $t$--direction, we have, for $M := \Omega \times T$, that $U \in C^0(M) \cap PSH(M)$ and

$$(dd^c U)^{n+1} = 0, \quad \text{in } M$$

and on $\partial M$ the function $U$ coincides with the boundary data determined by $u_0$. Alternatively, $U$ can be directly defined as the sup over all $V$ in $C^0(M) \cap PSH(M)$ restricting to the given boundary data on the boundary. In particular, if $u_0$ and $u_1$ are in $\mathcal{H}_0(\Omega)^{T^n}$ then so is $u_t$. In fact, in the $T^n$--invariant case $u_t$ may alternatively be obtained by Legendre transform considerations (compare [14]). Indeed, letting $p_i := \log |z_i|^2$ we can identify $u(z)$ with a convex function on a domain in $\mathbb{R}^n$ (with coordinates $p_i$) that we, abusing notation slightly, write as $u(p)$. Moreover, we can extend $u(p)$ to become a smooth convex function on all of $\mathbb{R}^n$ such that $u(p) = C \max_i \{p_i\} + O(1)$ as $p \to \infty$ and by a simple approximation argument we may as well assume that $u_t(p)$ is smooth and strictly convex. Then $u_t(z)$ may be obtained by connecting the Legendre transforms $u_t^*$ of $u_0(p)$ and $u_1(p)$ by an affine curve and then taking the Legendre transform again, i.e. $u_t(x) = (u_0^*(1-t) + u_1^* t)^*$. In particular $u_t$ is always smooth in the $T^n$--invariant case if its end points are smooth and strictly convex, which simplifies some of the technical points of the proof in [1].

As shown in [14] the functional

$$t \mapsto \log \int_{\Omega} e^{-u_t} dV$$

is concave along any geodesic as long as the domain $\Omega$ is $S^1$--invariant. In the $T^n$--invariant case this fact can also be deduced from the Prekopa-Leindler...
inequality for convex functions on $\mathbb{R}^n$. Indeed, since $u(p)$ is convex in $p$ and vanishes on $\partial \Omega$ we may extend it to a convex function on all of $\mathbb{R}^n$ by letting it be equal to $\infty$ on the complement of $\Omega$. Then the required concavity follows from the Prekopa-Leindler inequality which says that $-\log \int_{\mathbb{R}^n} e^{-u} dp_1 \wedge \cdots \wedge dp_n$ is convex in $t$ if $v$ is convex in $(p, t)$ (take $u_t(p) = u_t(p) + \sum_i p_i$).

Next, we note that $E(u) := \int_\Omega u(dd^c u)^n$ is affine along a geodesic $u_t$. Indeed, letting $t$ be complex a direct calculation gives

\[ dd^c E(u_t) = \int_\Omega (dd^c u)^{n+1} \]

which, by definition, vanishing if $u_t$ is a geodesic. All in all this means that $\mathcal{G}(u_t)$ is concave along a geodesic. Letting now $u$ be an arbitrary element in $\mathcal{H}_0(\Omega)^T$ we take $u_t$ to be the geodesic connecting the solution $u_0$ of equation (0.4) (obtained from the Kähler-Einstein metric on $X$) and $u_1 = u$. Then $\mathcal{G}(u_t)$ has a critical point at $t = 0$, i.e. its right derivative vanishes for $t = 0$ and hence by concavity $\mathcal{G}(u_1) \leq \mathcal{G}(u_0)$ which concludes the proof of the M-T inequality with $C = \mathcal{G}(u_0)$.

0.3. **Proof of the Brezis-Merle type inequality (B-M).** Let $u(z, t)$ be an element in $\mathcal{H}_0(\Omega \times D)^{T_{n+1}}$. Applying the M-T inequality established above on $\Omega$ for $t$ fixed gives

\[ \int_\Omega e^{-u_t} dV \leq \exp\left(-\frac{1}{V_\Omega} \frac{1}{n+1} E(u_t)\right) \]

By 0.7 $E(u_t)$ is psh for $t \in D$, vanishing on the boundary and hence applying the one-dimensional Brezis-Merle inequality on the unit-disc:

\[ \int_D e^{-v} dV \leq A(1 - \int_\Omega dd^c v)^{-1}, \]

for $v \in \mathcal{H}_0(D)$ such that $\int_\Omega dd^c v < 1$, gives

\[ \int_{\Omega \times D} e^{-u} dV \leq A(1 - \frac{1}{V_\Omega} \int_{\Omega \times D} (dd^c u)^{n+1})^{-1} \]

so that rescaling $u$ concludes the proof. Note that the one-dimensional Brezis-Merle inequality used above is a simple consequence of Green’s formula. In fact, we only need the basic fact that $\int_D e^{-v} < \infty$ if $v$ is subharmonic and bounded on $D$ and $\int_D dd^c v < 1$.

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E-mail address: robertb@chalmers.se, bob@chalmers.se  
Current address: Mathematical Sciences - Chalmers University of Technology and University of Gothenburg - SE-412 96 Gothenburg, Sweden