CMB Circular and B-mode Polarization from New Interactions

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Standard models describing the radiation transfer of the Cosmic Microwave Background (CMB) through Compton scattering predict that cosmological scalar perturbations at linear order are not able to source V and B polarization modes. In this work we investigate the possibility that such CMB polarization modes are generated even in the presence of linear scalar perturbations only. We provide a general parametrization of the photon-fermion forward-scattering amplitude and compute mixing terms between different CMB polarization modes. We discuss different general extensions of Standard Model interactions which violate discrete symmetries, while preserving CPT. We show that it is possible to source CMB circular polarization by violating parity and charge conjugation symmetries. Instead, B-mode generation is associated to the violation of symmetry for time-reversal. Our results provide a useful tool to constrain new physics using CMB data.

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I. INTRODUCTION

CMB radiation represents a crucial observational tool of modern cosmology. The standard models describing the radiation transfer of the CMB from the recombination epoch until today predict the presence of some level of linear polarization, which has been widely studied and reviewed in the literature (see e.g. Refs. [1–8]). This is the result of the Compton scattering between CMB photons and electrons and gravitational redshift, induced by cosmological perturbations of the metric. Instead, the generation of CMB circular polarization (the so-called V-mode) is usually not considered, because the electron-photon Compton scattering cannot generate it at the classical level.

However, some models have been proposed that can lead to the generation of CMB circular polarization. One possible way is via Faraday conversion of the linear polarization generated at the surface of last scattering by various sources of cosmic birefringence (see e.g. Refs. [9, 10] for a recent review). For instance, in Refs. [11–20] V-mode formation due to magnetic fields is discussed. In Refs. [21–24] V-mode generation due to photon-photon interactions via Heisenberg-Euler interaction is considered. V-mode generation due to interactions coming from extensions of QED is studied, in particular, in Refs. [17–20, 25–26], where Lorentz-violating operators are considered. In Ref. [27] it is shown that a cosmological pseudo-scalar field may generate circular polarization in the CMB, while in Ref. [28] it is shown that V-mode generation can be obtained in axion inflation. Moreover, in Refs. [29, 30] it is shown that forward scattering between CMB photons and neutrinos can source V-modes through Standard Model interactions. Also in Ref. [31] forward scattering between photons and gravitons is shown to lead to circular polarization, under some conditions. In Ref. [32] circular polarization of CMB photons via their Compton scattering with polarized cosmic electrons is considered. In Ref. [33] it is shown that V modes in the CMB may arise from primordial vector and tensor perturbations. In particular, in Refs. [34, 35] the case of chiral gravitational waves is considered.

Despite the fact that CMB circular polarization has not been explored so much up to now, these examples show how its detection might reveal interesting phenomena occurring in the evolution of the universe, thereby displaying
another example of circular polarization and preferred handedness occurring in Nature. For example, it is well known that also living organisms, by interacting with photons, can generate circular polarization (see e.g. Ref. [36]). This may arise either from a particular fundamental interaction of photons with biological molecules or from the scattering of photons with chiral materials. In general, V-mode formation is associated with parity symmetry violation at a certain level. For instance, chiral molecules as left-handed (LH) amino acids are not eigenstates of parity, and for this reason the interaction of light with such molecules may generate V-modes.

Most of the mechanisms to produce V-modes proposed in recent years are based on the forward scattering of CMB photons by a target. In fact, the generation of V-modes depends on the refractive index of the material, which is related to the forward scattering amplitude $M_{for}$ of a fundamental process (see e.g. Ref. [35]). In particular, circular polarization is generated when the refractive index of the LH waves differs from the refractive index of the right-handed (RH) ones. As a result, the existence of a non-vanishing V-mode implies that $M_{for}^{R} \neq M_{for}^{L}$.

In the language of quantum mechanics, the forward scattering amplitude of a beam of radiation $\gamma$ and a target $A$ is given as $M_{for}^{R,L} = \langle \gamma, A | \mathcal{O} | \gamma, A \rangle_{R,L}$, where $| \gamma, A \rangle$ represents the quantum state of the target and of the beam, and $\mathcal{O}$ is the interaction operator. The condition $M_{for}^{R} \neq M_{for}^{L}$ is satisfied either if i) the state of target $| A \rangle$ is not a parity eigenstate, namely $P|A\rangle \neq \pm|A\rangle$ or if ii) $\mathcal{O}$ is not invariant under parity transformation, namely $POP^{-1} \neq \mathcal{O}$. There are several ways in which the first condition can be met. For example, forward scattering of photons with a background of particles can produce V-modes when the power-spectrum of this background violates parity symmetry. Instead, Compton scattering in the presence of a magnetic field is an example of the second condition. Historically, Ref. [35] was the first literature that pointed out the possibility to use the CMB to search for parity-violating interactions.

From the observational point of view, CMB circular polarization is not excluded. As an example, the SPIDER collaboration has recently provided new constraints on the Stokes parameter $V$ at 95 and 150 GHz, by observing angular scales corresponding to $33 < \ell < 307$ [39]. The constraints on the circular polarization power-spectrum $\ell (\ell + 1) C_{\ell}^{V} / (2\pi)$ are reported in a range from $141 \, \mu K^2$ to $255 \, \mu K^2$ at 150 GHz, for a thermal CMB spectrum. Also, in Ref. [10] some interesting detection prospects are discussed.

In this work we will study V-mode polarization generation in the CMB radiation from its direct coupling with linear polarization states induced by the forward scattering of photons with generic fermions at or after the recombination epoch. In particular, we will assume a completely general photon-fermion interaction which may also go beyond QED, but still preserving the CPT symmetry, which up to now is observed to be an exact symmetry of Nature at a fundamental level. In order to do so, we will use a generic parametrization of the photon-fermion scattering amplitude which follows only by the imposition of gauge-invariance (see e.g. Refs. [11,43]). Moreover, we will work in the so-called “quantum Boltzmann equation” formalism (see e.g. Refs. [1,17,22,24,26,29,31,44,45]) for computing the time evolution of CMB polarization. It is possible to show that this formalism is equivalent – at lowest order in scattering kinematics – to the classical radiation transfer, hence it provides a more general framework to work with. We will assume the fermions to be non relativistic to simplify some computations. This assumption is well motivated for Standard Model fermions, as cosmic neutrinos are the only particles which are relativistic soon after the recombination epoch. In particular, we will assume a completely general photon-fermion interaction which may also go beyond polarization states induced by the forward scattering of photons with generic fermions at or after the recombination epoch.

We will show that V-modes can be produced by forward scattering for a generic interaction preserving all the C (charge conjugation), P (parity) and T (time-reversal) discrete symmetries, if the stress tensor of the fermion contains anisotropies. In addition, we will show that V-modes can be sourced also from an interaction violating C and P symmetries, but preserving the CP combination. In this case, together with the anisotropies in the fermionic stress tensor, we need the fermion to interact with the photon only in the L or R-handed state, like the L-handed neutrino in the Standard Model interactions. In particular, this last case confirms and generalizes the results found in Ref. [29]. We will also analyse the cases in which C, T and P symmetries are violated individually, while preserving respectively the combinations CT and PT. We will show that in these cases it is impossible to generate V-modes by forward scattering, but we can have formation of CMB B-modes. In particular, in the case of a generic interaction which violates P, T symmetries it is possible to generate B-modes with no conditions on the fermions the photons interact with, while in the case in which C, T are violated we need the fermion to be L or R-handed. All these conclusions, which represent the main results of this paper, are summarized in Table I.

Thus, our general study shows that some interactions beyond the Standard Model may produce V-modes in the CMB and, at the same time, can provide an additional source of B-modes. Our final Boltzmann equations are expressed in terms of unknown free parameters. Thus, in the future we could use our general approach to put constraints on new physics using CMB data.

The paper is organized as follows. In Sec. I we will introduce a set of equations and useful notations and results that we will use in this work. In Sec. II we will see a generic way to parametrize the photon-fermion scattering amplitude and apply this parametrization in specific cases. In Sec. III we will derive the general form of the forward scattering term in the photon-fermion interaction. In Sec. IV we will study general interactions generating V-mode
polarization in the CMB. In Sec. VII we will see some cases where also CMB B-modes can be sourced. In Sec. VIII we will comment about the difference in the results, by considering the fermion as a Majorana particle, instead of a Dirac particle. Finally, in Sec. VIII we will present our main conclusions.

II. THE TIME EVOLUTION OF STOKES PARAMETERS

The intensity and polarization of CMB anisotropies are completely characterized by a $2 \times 2$ density tensor $^{1}$

$$
\rho_{ij} = \frac{1}{2} \left( \begin{array}{cc}
I + Q & U - iV \\
U + iV & I - Q \end{array} \right),
$$

where $I$, $Q$, $U$, and $V$ are the so-called Stokes parameters, satisfying the inequality $I^2 \geq Q^2 + U^2 + V^2$. The components of the polarization tensor $\rho_{ij}$ satisfy the relations $\rho_{ii} = 1$ and $\rho_{ij} = \rho_{ji}^*$ or, better to say, the $\rho_{ij}$ tensor is Hermitian. Consequently, the diagonal components $\rho_{11}$ and $\rho_{22}$ are real (with $\rho_{11} + \rho_{22} = 1$), while $\rho_{21} = \rho_{12}^*$.

For unpolarized CMB radiation $Q = U = V = 0$, and the parameter $I$ describes the overall radiation intensity. The $Q$ and $U$ Stokes parameters represent the linear polarization of the CMB. In particular, taking two orthogonal $(x, y)$ axes on the polarization plane, the $Q$-mode gives the difference in intensity between CMB photons with polarization vectors along the $x$ and $y$ axes respectively, while the $U$-mode gives the difference in intensity between CMB photons with a polarization vector along axes rotated by 45 degrees with respect to the $x$ and $y$ axes. Finally, the $V$-mode describes the CMB circular polarization or, better to say, it gives the difference in intensity between the two circular polarization modes of CMB radiation.

The generation and evolution of CMB intensity and polarization can be characterized through the quantum Boltzmann equation $^{4}$

$$
(2\pi)^3 \delta^{(3)}(0) (2k) \frac{d\rho_{ij}(k)}{dt} = i \langle [H_I(t), \mathcal{D}_{ij}(k)] \rangle - \frac{1}{2} \int_{-\infty}^{\infty} dt \langle [H_I(t), [H_I(0), \mathcal{D}_{ij}(k)]] \rangle,
$$

where $\langle \cdots \rangle$ denotes the expectation value of operators, $\mathcal{D}_{ij}(k) = a^\dagger_i(k)a_j(k)$ is the photon number operator, $a^\dagger$ and $a$ are the photon creation and annihilation operators, respectively. The effective interaction Hamiltonian $H_I$ is defined through the expansion of the $S$ matrix up to second order as

$$
S^{(2)} = -i \int_{-\infty}^{\infty} dt H^{(2)}(t),
$$

where $H_I$ is the component of $H^{(2)}$ that describes the Compton scattering between CMB photons and other particles. The first term on the right-hand side of Eq. (2) is the so-called forward scattering term, while the second term is the so-called damping or non-forward scattering term. In this work we will focus on the forward scattering term which is able to generate couplings between different polarization states. When dealing with the standard QED Compton scattering between photons and electrons such a forward scattering term vanishes.

The Stokes parameters can be expanded in terms of a spin-weighted basis as

$$
I(\hat{k}) = \sum_{\ell, m} a^\dagger_{\ell m} Y_{\ell m}(\hat{k}),
$$

$$
V(\hat{k}) = \sum_{\ell, m} a_{\ell m} Y_{\ell m}(\hat{k}),
$$

$$
P^\pm(\hat{k}) = (Q \pm iU)(\hat{k}) = \sum_{\ell, m} a_{\pm 2, \ell m \pm 2} Y_{\ell m}(\hat{k}),
$$

where $\hat{k}$ denotes the photon direction. Moreover, using the spin raising and lowering operators $\hat{\sigma}$ and $\hat{\bar{\sigma}}$ we get

$$
E(\hat{k}) = -\frac{1}{2} \left[ \hat{\bar{\sigma}}^2 P^+(\hat{k}) + \hat{\sigma}^2 P^-(\hat{k}) \right],
$$

$^{1}$ When we refer to the Stokes parameters, we take only the fluctuations over the respective mean value.
where we have introduced the so-called E and B polarization modes. These modes offer an alternative description of CMB linear polarization which, differently from Q and U modes, is invariant under a rotation of the polarization plane. In the following, we will use a description of the radiation transfer both in terms of Q and U modes and E and B modes.

The standard Boltzmann equations in the presence of only linear scalar perturbations are given by \[ I(\ell) = \frac{\partial}{\partial t} \int d^3k \langle a_\ell(\hat{k}) a_\ell(\hat{k})^\dagger \rangle \]

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where \( K \) denotes the Fourier conjugate of \( x \) and we are in a coordinate system where \( K \parallel z \) axis. Here \( \eta \) is the conformal time, the prime denotes differentiation with respect to conformal time, \( \psi \) and \( \phi \) are scalar cosmological (gravitational potential) perturbations, \( v_B \) is the electrons average velocity, \( \mu = \hat{k} \cdot \hat{K} = \cos \theta \), \( P_\ell(\mu) \) is the Legendre polynomial of rank \( \ell \) and \( I^{(S)} + P^{(S)} - P_0^{(S)} \), \( P \) being the strength of the polarization field. The quantities \( I^{(S)}, P^{(S)} \) and \( V^{(S)} \) represent the \( \ell \)-th order terms in the Legendre polynomial expansion of the corresponding modes. Finally, one defines the optical depth \( \tau(\eta) \) as \[ \tau(\eta) = -a(\eta)n_B x_e \sigma_T \]

where \( n_B \) is the electron density, \( x_e \) is the ionization fraction and \( \sigma_T \) is the Thomson cross-section. For more details about the derivation of Eqs. (10), (11) and (12), see Refs. [1], [2], [3].

In particular, it is possible to show that Eq. (10) admits the general integral solution \[ P^{(S)}(\eta_0, K, \mu) = \frac{3}{4}(1 - \mu^2) \int_0^{\eta_0} d\eta e^{iK(\eta_0 - \eta)} [1 - \mu^2] P^{(S)}(\eta, K, \mu) \]

Since scalar perturbations are invariant under rotations and so axially-symmetric around \( z \), we get \( P^+ = P^- \), thus \( U^{(S)} = 0 \) and scalar perturbations source only Q-modes. Moreover, in this case (i.e. for scalar perturbations) the spin raising and lowering operators act like (see e.g. Ref. [46])

\[ \bar{\sigma}_2 P^{(S)} = \sigma_2 P^{(S)} = (1 - \mu^2) P^{(S)}(\eta_0, K, \mu) \]

Therefore, using the definitions (7) and (8) we get

\[ E^{(S)}(\eta_0, K, \mu) = -\frac{3}{4} \int_0^{\eta_0} d\eta e^{-\tau(\eta) \Pi(\eta, K, \mu)} [1 - \mu^2] e^{iK(\eta - \eta_0)} \]

and

\[ B^{(S)}(\eta_0, K, \mu) = 0 \]

\[ f^{(S)} = \frac{1}{2} \int_{-1}^1 d\mu' I(k, \mu') P_\ell(\mu') \]

and an analogous expression for \( V^{(S)} \) and \( P^{(S)} \).

We note that the Boltzmann equations do not coincide between the different references. This is due to the fact that each reference uses his own formal conventions. In our results we have followed the conventions of Ref. [1].

2 We adopt the convention

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This is the well-known result that linear scalar perturbations cannot source B-mode polarization. In fact, it is well known that the B-mode polarization in the CMB is generated mainly by weak gravitational lensing and by tensor perturbations \[7, 8\]. Alternatively, a small amount of B-modes can be generated also by second-order vector and tensor modes sourced by scalar perturbations (see e.g. Refs. \[47, 48\]).

Moreover, since in Eq. (11) the Stokes parameter $V$ has no source, then there is also no $V$-mode generation with linear scalar perturbations only.

From the next section we will start to study the general conditions for generating both $V$ and $B$ modes in the presence of only linear scalar perturbations through photon-fermion forward scattering. We will first write down a very general form for the photon-fermion scattering amplitude (Sec. III), and then we will apply it to the forward scattering contribution in Eq. (2) (in Sec. IV).

### III. GENERAL FORM OF THE PHOTON-FERMION SCATTERING AMPLITUDE

![Fig. 1: Figurative representation of photon-fermion interaction.](image)

We are interested in the Compton scattering of a photon by a fermion (Fig. 1)

$$\gamma(p) + f(q) \rightarrow \gamma(p') + f(q'),$$

where $p(p')$ is the initial (final) momentum of the photon and $q(q')$ is the initial (final) momentum of the fermion. It is possible to construct the invariant amplitude of this process using a general method. The amplitude of such a process can be written in the form \[41–43\]

$$M_{fi} = F_{\lambda\mu}^{s,s'}\epsilon_{\lambda}^{s}\epsilon_{\mu}^{s'},$$

where $\epsilon_{\mu}$ and $\epsilon_{\nu}'$ are the polarization vectors of incoming and outgoing photons and $s, s' = 1, 2$ label the physical transverse polarization of the photons. Gauge-invariance requires $\epsilon_{\mu}\cdot p = \epsilon_{\nu}'\cdot p' = 0$. Moreover, the rank-2 tensor $F^{\mu\nu}$, which is called “Compton tensor”, must satisfy the conserved current condition $p_{\mu}F^{\mu\nu} = p'_{\nu}F^{\mu\nu} = 0$, as a consequence of gauge-invariance. It is possible to provide a general parametrization of $F^{\mu\nu}$ satisfying the previous condition from the linear combination of basis vectors defined below.

We first construct a general form for the Compton tensor $F^{\mu\nu}$ and then study its parity conserving and parity violating aspects. Using the procedure of Refs. \[41, 43\] we can write

$$F^{\mu\nu} = G_0 \left( \hat{e}^{(1)}_{\mu}\hat{e}^{(1)}_{\nu} + \hat{e}^{(2)}_{\mu}\hat{e}^{(2)}_{\nu} \right) + G_1 \left( \hat{e}^{(1)}_{\mu}\hat{e}^{(2)}_{\nu} + \hat{e}^{(2)}_{\mu}\hat{e}^{(1)}_{\nu} \right) + G_2 \left( \hat{e}^{(1)}_{\mu}\hat{e}^{(2)}_{\nu} - \hat{e}^{(2)}_{\mu}\hat{e}^{(1)}_{\nu} \right) + G_3 \left( \hat{e}^{(1)}_{\mu}\hat{e}^{(2)}_{\nu} - \hat{e}^{(2)}_{\mu}\hat{e}^{(1)}_{\nu} \right),$$

where $G_i$ are invariant functions and $e^{(1)}$ and $e^{(2)}$ are two 4-vectors satisfying the orthogonality condition $\hat{e}^{(1)}\cdot\hat{e}^{(2)} = 0$.

In order to construct these two vectors we have to use only the kinematic variables $p, p', q$ and $q'$ and define a system of orthogonal vector basis of the form

$$Q^{\lambda} = (q^{\lambda} + q'^{\lambda}) - \frac{P^{\lambda}}{P^2} (q + q') \cdot P,$$

$$P^{\lambda} = p^{\lambda} + p'^{\lambda},$$

where $P^2 = P_{\mu}P_{\mu}$. This is a pair of orthogonal vectors basis defined as $p, p'$ and also $q, q'$, which in general lie in a plane orthogonal to $P$.
where \( t^\lambda \), for the tree-level contribution to the scattering amplitude, is given by
\[
t^\lambda = q^\lambda - q'^\lambda = p'^\lambda - p^\lambda .
\] (24)

A possible choice of the normalized \( \hat{e}^{(1)} \) and \( \hat{e}^{(2)} \) 4-vectors is given by (see e.g. [41])
\[
\hat{e}^{(1)\lambda} = \frac{N^\lambda}{\sqrt{-N^2}} ,
\] (25)
and
\[
\hat{e}^{(2)\lambda} = \frac{Q^\lambda}{\sqrt{-Q^2}} .
\] (26)

From these definitions it is easy to verify the conserved current condition as
\[
(P_\nu + t^\nu)F^{\mu\nu} = (P_\mu - t^\mu)F^{\mu\nu} = 0 .
\] (27)

In this paper, we are interested in the forward scattering limit in which \( t^\lambda = 0 \) and \( P^2 = 4p^2 = 0 \). Under this condition, \( N^\lambda \) vanishes and the second term in \( Q^\lambda \) becomes singular. Therefore, \( \hat{e}^{(1)\lambda} \) and \( \hat{e}^{(2)\lambda} \) are not well-defined. In order to overcome these problems, we firstly change the normalization in \( \hat{e}^{(2)\lambda} \) as
\[
\hat{e}^{(2)\lambda} = \frac{Q^\lambda}{\sqrt{-4q^2}} = \frac{Q^\lambda}{\sqrt{-4m_f^2}} ,
\] (28)
by noting that the second term in \( Q^\lambda \) does not contribute to the amplitude. Secondly, we introduce a new general variable \( \Delta^\lambda \) instead of \( t^\lambda \) in Eq. (23). In Appendix B it is shown that one can usually find \( \Delta^\lambda \) in terms of kinematic variables, when taking into account loop corrections in Feynman diagrams. In this work, we will assume for simplicity that \( \Delta^0 \gg \Delta^i \). Hence,
\[
\Delta^\lambda \approx (\Delta^0, 0) .
\] (29)

Thus, in the forward scattering limit, we have
\[
N^\lambda = \epsilon^{\lambda\mu\rho}Q_\mu\Delta^0p_\rho
= 4\epsilon^{\lambda\mu\rho}q_\mu\Delta^0p_\rho ,
\] (30)
and \( \hat{e}^{(1)} \) is now defined as
\[
\hat{e}^{(1)i} = \frac{N^i}{\sqrt{-N^2}} \hat{e}^{(1)0} = 0 ,
\] (31)
where \( N^2 \) will stand for the modulus square of the 3-vector
\[
N^i = 4\epsilon^{ijk}q_j\Delta^0p_k ,
\] (32)
that gives
\[
N^2 = 16(\Delta^0)^2|p \times q|^2 .
\] (33)

It is easy to verify that \( F_{\mu\nu} \), with the new definitions of \( \hat{e}^{(1)\lambda} \) and \( \hat{e}^{(2)\lambda} \) in Eqs. (31) and (28), satisfies the conserved current condition.

Before proceeding, it is worth to rewrite the factor of \( G_2 \) in Eq. (20) in a new form for the case of forward scattering. Using the following identity regarding the Levi-Civita tensor [49]
\[
g_{\lambda\mu\nu=\beta\gamma} - g_{\lambda\nu}\epsilon_{\mu\alpha\beta\gamma} + g_{\lambda\sigma}\epsilon_{\mu\nu\alpha\beta} - g_{\lambda\beta}\epsilon_{\mu\nu\alpha\gamma} + g_{\lambda\gamma}\epsilon_{\mu\nu\alpha\beta} = 0 ,
\] (34)
Thus, in the end we have

\[
\hat{e}^{(1)\mu}\hat{e}^{(2)\nu} - \hat{e}^{(2)\mu}\hat{e}^{(1)\nu} = \frac{4}{\sqrt{m_f^2 N^2}} \left( q_\alpha \Delta q_\beta p_\gamma \left( g^{\nu\lambda} \epsilon^{\mu\alpha\beta\gamma} - g^{\mu\lambda} \epsilon^{\nu\alpha\beta\gamma} \right) \right.
\]

\[
= \frac{4}{\sqrt{m_f^2 N^2}} q_\lambda \Delta q_\beta p_\gamma \left( g^{\lambda\alpha} \epsilon^{\mu\nu\beta\gamma} - g^{\beta\lambda} \epsilon^{\mu\nu\alpha\gamma} + g^{\lambda\gamma} \epsilon^{\mu\nu\alpha\beta} \right)
\]

\[
= \frac{4}{\sqrt{m_f^2 N^2}} \left( q^2 \Delta p_\gamma \epsilon^{\mu\nu\beta\gamma} - q \cdot \Delta q_\alpha p_\gamma \epsilon^{\mu\nu\alpha\gamma} + q \cdot p q_\alpha \Delta_\beta \epsilon^{\mu\nu\alpha\beta} \right). \tag{35}
\]

Thus, in the end we have

\[
\hat{e}^{(1)\mu}\hat{e}^{(2)\nu} - \hat{e}^{(2)\mu}\hat{e}^{(1)\nu} = \frac{4}{\sqrt{m_f^2 N^2}} \left( q^2 \Delta p_\gamma \epsilon^{\mu\nu\beta\gamma} - q \cdot \Delta q_\alpha p_\gamma \epsilon^{\mu\nu\alpha\gamma} + q \cdot p q_\alpha \Delta_\beta \epsilon^{\mu\nu\alpha\beta} \right). \tag{36}
\]

The second term on the right-hand side of Eq. (36) is the most interesting one for us. In fact, as it is shown in Appendix B a similar term appears when considering the interaction of a photon with the magnetic moment of a neutrino. In the following, we will only consider this term.

Moreover, it is worth noticing that \(\hat{e}^{(1)}\) is an axial vector and \(\hat{e}^{(2)}\) is a vector. Using this property, it is straightforward to verify that the second and third brackets in Eq. (20) change sign under parity transformation, while the first and fourth brackets remain unchanged. Both these two combinations of the Compton tensor satisfy the crossing symmetry and gauge-invariance. However, \(F^{\mu\nu}\) can be even or odd under parity.

In order to discuss these cases, we first provide the general expression for the coefficients \(G_i\), and then we start from the parity-invariant case by deriving all non-vanishing terms of each \(G_i\) under the parity-invariance condition of the scattering amplitude. The coefficients can be represented in terms of the following bilinear covariant terms \[41, 43\]

\[
G_0 = \bar{u}_r \left[ f_1 + f_2 \mathcal{P} + f_3 \gamma^5 + f_4 \gamma^5 \mathcal{P} \right] u_r, \tag{37}
\]

\[
G_1 = \bar{u}_r \left[ f_5 + f_6 \mathcal{P} + f_7 \gamma^5 + f_8 \gamma^5 \mathcal{P} \right] u_r, \tag{38}
\]

\[
G_2 = \bar{u}_r \left[ f_9 + f_{10} \mathcal{P} + f_{11} \gamma^5 + f_{12} \gamma^5 \mathcal{P} \right] u_r, \tag{39}
\]

\[
G_3 = \bar{u}_r \left[ f_{13} + f_{14} \mathcal{P} + f_{15} \gamma^5 + f_{16} \gamma^5 \mathcal{P} \right] u_r, \tag{40}
\]

where \(u_r\) and \(\bar{u}_r\) are Dirac spinors associated to the fermion; \(r, r'\) label fermion spin, \(\mathcal{P} = P_\mu \gamma^\mu\), \(\gamma^\mu\) and \(\gamma^5\) are Dirac matrices and \(f_i\) are constant coefficients.

The invariant functions \(G_i\) involve four possibilities. One can show that the \(Q\) and \(f\) terms are nothing else than numbers due to the Dirac equation and hence they do not appear in the \(G_i\) invariants. Similarly, all higher powers of the \(\gamma^\mu\) matrices are reduced to the above four possibilities. With the above representation, the time reversal and parity transformations of each bilinear term is evident.

\section*{A. Even-parity amplitude}

In this sub-section, we determine the form of the fermion-photon scattering amplitude with the condition that the amplitude is even under parity transformation. As we have seen, the photon scattering amplitude is represented by \[19\]. Since under parity transformation the polarization vectors change as

\[
(\epsilon_0, \epsilon) \leftrightarrow (\epsilon_0, -\epsilon), \tag{41}
\]

the condition of parity invariance of scattering amplitude \(M_{f_i}\) implies

\[
(F^{00}, F^{0}, F^{ik}) \rightarrow (F^{00}, -F^{0}, F^{ik}). \tag{42}
\]
Using the fact that \( \hat{e}^{(1)} \) and \( \hat{e}^{(2)} \) are a pseudo-vector and a vector, respectively, \( G_0 \) and \( G_3 \) must be scalars and \( G_1 \) and \( G_2 \) must be pseudo-scalars. Consequently, we can obtain the following constraints, as a result of the even-parity condition

\[
f_3 = f_4 = f_5 = f_6 = f_9 = f_{10} = f_{15} = f_{16} = 0 .
\] (43)

Then, we impose the condition of time-reversal invariance. Under time reversal we have

\[
(q_0, q) \leftrightarrow (q'_0, -q') , \quad (p_0, p) \leftrightarrow (p'_0, -p') ,
\] (44)

and

\[
(\epsilon_0, \epsilon) \leftrightarrow (\epsilon^*, -\epsilon^*) .
\] (45)

Hence, invariance of the scattering amplitude \( M_{fi} \) under time reversal yields

\[
(F^{00}, F^{0i}, F^{ik}) \rightarrow (F^{00}, -F^{0i}, F^{ki}) .
\] (46)

Similarly, the relations in Eq. (44) imply

\[
(Q_0, Q) \rightarrow (Q_0, -Q) , \quad (t_0, t) \rightarrow (-t_0, t) , \\
(P_0, P) \rightarrow (P_0, -P) , \quad (N_0, N) \rightarrow (N_0, -N) ,
\] (47)

so that

\[
\left(\hat{e}^{(1,2)}_0, \hat{e}^{(1,2)}_0\right) \rightarrow \left(\hat{e}^{(1,2)}_0, -\hat{e}^{(1,2)}_0\right) .
\] (48)

Thus, invariance under time reversal implies

\[
G_{0,1,3} \rightarrow G_{0,1,3} , \quad G_2 \rightarrow -G_2 ,
\] (49)

and based on the following properties of spinor bilinear terms under a time-reversal transformation

\[
\bar{u}' \gamma^5 \hbar \rightarrow -\bar{u}' \gamma^5 \hbar , \quad \bar{u}' \gamma^5 \bar{P}_u \rightarrow \bar{u}' \gamma^5 \bar{P}_u ,
\] (50)

one can verify the following additional conditions

\[
f_7 = f_{12} = 0 .
\] (51)

Consequently, under parity and time reversal invariance the number of free coefficients is reduced to

\[
G_0 = \bar{u}_r \left[ f_1 + f_2 \bar{P} \right] u_r , \quad G_1 = \bar{u}_r \gamma^5 \bar{P} u_r , \\
G_2 = \bar{u}_r f_1 \gamma^5 u_r , \quad G_3 = \bar{u}_r \left[ f_{13} + f_{14} \bar{P} \right] u_r .
\] (52)

For further investigation, we analyse the transformation under charge conjugation and crossing. The charge conjugation leads to

\[
(\epsilon_0, \epsilon) \leftrightarrow -(\epsilon^*_0, \epsilon^*) .
\] (53)

As a result, invariance of the scattering amplitude \( M_{fi} \) under C transformation leads to

\[
(F^{00}, F^{0i}, F^{ik}) \rightarrow (F^{00}, F^{0i}, F^{ki}) .
\] (54)

On the other hand, the crossing leads to

\[
p \leftrightarrow -p' \quad \text{and} \quad \mu \leftrightarrow \nu ,
\] (55)

then,

\[
\hat{e}^{(1)} \lambda \leftrightarrow \hat{e}^{(1)} \lambda , \quad \hat{e}^{(2)} \lambda \leftrightarrow -\hat{e}^{(2)} \lambda ,
\] (56)

and we find that under charge conjugation and crossing

\[
G_{0,2,3} \rightarrow G_{0,2,3} , \quad G_1 \rightarrow -G_1 .
\] (57)
that is satisfied by the results presented in [42]. Therefore, we can claim that the amplitude will be invariant under CPT and crossing symmetry.

In particular, let us discuss the standard Compton scattering amplitude, which is based on QED. Using the standard Feynman rules, the amplitude of Compton scattering is given by

\[ M_{fi} = -e^2 \epsilon_\mu(p) \epsilon_\nu(p') [\bar{u}(q') Q^{\mu\nu} u(q)] , \]

where

\[ Q^{\mu\nu} = \frac{1}{s-m^2} \gamma^\nu(\gamma p + \gamma q + m) \gamma^\mu + \frac{1}{u-m^2} \gamma^\mu(\gamma q - \gamma p' + m) \gamma^\nu , \]

and the kinematic invariants are

\[ s = (p + q)^2 = (p' + q')^2 = m^2 + 2pq = m^2 + 2p'q' , \]
\[ u = (p - q')^2 . \]

After some straightforward algebra we can find the following values of the coefficients \( f_i \)'s [43]

\[ f_1 = -ma_+ , \quad f_2 = 0 , \quad f_8 = \frac{1}{2}ia_+ , \quad f_{11} = -ma_+ , \quad f_{13} = ma_+ , \quad f_{14} = \frac{1}{2}a_- , \]

where

\[ a_\pm = \frac{1}{s-m^2} \pm \frac{1}{u-m^2} . \]

**B. Odd-parity amplitude**

In this subsection, we impose the odd-parity condition. In this case \( F^{\mu\nu} \) is a pseudo-tensor that under parity operation must transform as

\[ (F^{00}, F^{i0}, F^{ij}) \rightarrow -(F^{00}, -F^{i0}, F^{ij}) . \]

Imposing the odd-parity condition and using the properties of bilinear terms under parity transformation we get

\[ f_1 = f_2 = f_7 = f_8 = f_{11} = f_{12} = f_{13} = f_{14} = 0 . \]

Therefore, those terms that remain after imposing the above condition are

\[ G_0 = \bar{u}_r f_3 \gamma^5 + f_4 \gamma^5 \bar{p}_r u_r , \quad G_1 = \bar{u}_r [f_5 + f_6 \bar{p}] u_r , \]
\[ G_2 = \bar{u}_r [f_9 + f_{10} \bar{p}] u_r , \quad G_3 = \bar{u}_r [f_{15} \gamma^5 + f_{16} \gamma^5 \bar{p}] u_r . \]

Afterwards, imposing the even-time reversal condition, we find

\[ f_3 = f_9 = f_{10} = f_{15} = 0 . \]

Thus, we remain with

\[ G_0 = \bar{u}_r f_4 \gamma^5 \bar{p} u_r , \quad G_1 = \bar{u}_r [f_5 + f_6 \bar{p}] u_r , \quad G_2 = 0 , \quad G_3 = \bar{u}_r f_{16} \gamma^5 \bar{p} u_r . \]

One can show that the resulting amplitude will be odd under charge conjugation. Therefore, the final form of the amplitude is even under CPT transformation. In this case the \( F^{\mu\nu} \) tensor is determined in terms of four free parameters. It is possible to compare our amplitudes with those of Kim and Dass in Ref. [50]. Our results are consistent with the calculation of Kim and Dass which can be found also in Appendix A.
IV. FORWARD SCATTERING TERM

In this section we will provide a general expression for the forward scattering term on the right-hand side of the Boltzmann equation \(2\).

The general form of the interaction Hamiltonian defined in \(3\) can be written as \(1\)

\[
H_I(t) = \int d\mathbf{q} d\mathbf{q}' dp d\mathbf{p}' (2\pi)^3 \delta^3(\mathbf{q} + \mathbf{p}' - \mathbf{q} - \mathbf{p}) \exp \left[ it \left( q^0 + p'^0 - q^0 - p^0 \right) \right] \nonumber \\
\times \left[ \hat{b}_\nu(q') a^\dagger_\nu(p') \bar{u}_\nu(q') F^{\mu\nu}(q, q', p, p') u_\mu(q) e^\dagger_\mu(p) e_\nu(p') a_\nu(p) b_\nu(q) \right] ,
\]

where

\[
d\mathbf{q} = \frac{d^3\mathbf{q}}{(2\pi)^3} m_f , \quad d\mathbf{p} = \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0} .
\]

\(a_\nu\) and \(a^\dagger_\nu\) are photon annihilation and creation operators respectively, which satisfy the canonical commutation relations

\[
[ a_\nu(p), a^\dagger_\nu(p') ] = (2\pi)^3 2p^0 \delta^3(\mathbf{p} - \mathbf{p'}) \delta_{ss'} ,
\]

and \(b^{(r)}\) and \(b^{(r)}\dagger\) are fermion annihilation and creation operators respectively, obeying the canonical commutation relations

\[
\{ b_\nu(q), b^\dagger_\nu(q') \} = (2\pi)^3 \frac{q^0}{m_f} \delta^3(\mathbf{q} - \mathbf{q'}) \delta_{rr'} ,
\]

where \(m_f\) is the fermion mass.

Using Eq. \(65\), the commutation relation in the forward scattering term of Eq. \(2\) becomes

\[
[H_I(0), D_{ij}(k)] = \int d\mathbf{q} d\mathbf{q}' dp d\mathbf{p}' (2\pi)^3 \delta^3(\mathbf{q} + \mathbf{p}' - \mathbf{q} - \mathbf{p}) \bar{u}_\nu(q') F^{\mu\nu}(q, q', p, p') u_\mu(q) e^\dagger_\nu(p) e_\nu(p') \\
\times \left[ \hat{b}_\nu(q') b_\nu(q) a^\dagger_\nu(p') a_j(k) 2p^0 (2\pi)^3 \delta_{ss} \delta^3(\mathbf{p} - \mathbf{k}) - b^\dagger_\nu(q') b_\nu(q) a^\dagger_\nu(k) a_\nu(p) 2p^0 (2\pi)^3 \delta_{ss} \delta^3(\mathbf{p}' - \mathbf{k}) \right] .
\]

After this step, in order to evaluate the forward scattering term we will need to take the expectation value of Eq. \(72\). For this purpose, we provide the following expectation values \(1\)

\[
\langle a^\dagger_m(p') a_n(p) \rangle = 2p^0 (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p'}) \rho_{mm}(\mathbf{p}) ,
\]

and

\[
\langle b^\dagger_m(q') b_n(q) \rangle = \frac{q^0}{m_f} (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q'}) \delta_{m} \frac{1}{2} n_f(q) ,
\]

where \(\rho_{mm}\) is the photon beam polarization matrix and \(n_f\) is the number density of fermions of momentum \(\mathbf{q}\) per unit volume. After using the Dirac delta functions, one can easily perform the integrations over \(\mathbf{p}\), \(\mathbf{p}'\) and \(\mathbf{q}'\) and obtain the limit \(p = p'\) and \(q = q'\) of the integrand, in agreement with the forward scattering condition.

At this point, we can fix the Coulomb gauge for the photon polarization vectors, where we have \(e^\mu = (0, \epsilon)\). As a consequence of this gauge-fixing, we are interested in only “latin” components of the Compton tensor \(F^{\mu\nu}\) (thus, latin components of the vector bases \(\hat{e}^{(1)}\) and \(\hat{e}^{(2)}\) to do the contractions in Eq. \(72\)). In particular, using the definitions \(31\) and \(28\) and the result \(36\), the \(F^{ij}\) components in the forward scattering limit can be represented as

\[
\bar{u}_\nu(q') F^{ij} u_\nu(q) = (G_0 + G_3) \hat{e}^{(1)i} \hat{e}^{(1)j} + (G_0 - G_3) \hat{e}^{(2)i} \hat{e}^{(2)j} + G_1 \left( \hat{e}^{(1)i} \hat{e}^{(2)j} + \hat{e}^{(2)i} \hat{e}^{(1)j} \right) + G_2 \left( \hat{e}^{(1)i} \hat{e}^{(2)i} - \hat{e}^{(2)i} \hat{e}^{(1)i} \right) \\
= (G_0 + G_3) \frac{(4\Delta^0)^2}{N^2} (\mathbf{q} \times \mathbf{p})^i (\mathbf{q} \times \mathbf{p})^j + (G_0 - G_3) \frac{q^i q^j}{m_f} + G_1 \frac{4\Delta^0}{m_f^2 N^2} (\mathbf{q} \times \mathbf{p})^i q^j + q^i (\mathbf{q} \times \mathbf{p})^j + G_2 \frac{4 q_0 \Delta^0}{\sqrt{m_f^2 N^2}} (q p_k - q_k p_0) \epsilon^{ijk} .
\]

In the next sections we will study the phenomenological consequences for CMB polarization of the forward scattering term in specific cases.
V. GENERAL CONDITIONS FOR GENERATING CIRCULAR POLARIZATION

In this section, we will give the most general conditions for generating circular polarization from photon-fermion forward scattering. Thus, we will consider specific expressions of the Compton tensor \( G_i \), evaluate Eq. \( (72) \) and study the effects of new interactions on the Stokes parameters.

Since we are searching for effects in the CMB polarization from the recombination epoch onwards, in the computations we will assume the fermions to be highly non-relativistic, i.e. \( q_i \ll q_0 \) and \( q_0 \simeq m_f \).

A. Even-parity amplitude

We start by considering the even-parity terms. The general forms of the \( G_i \) coefficients invariant under time reversal have been derived in the previous section. We have also determined there the coefficients for the QED case.

The coefficients \( G_i \) read

\[
G_0 + G_3 = \bar{u}_r (f_1 + f_2 \rho) u_r , \quad G_1 = \bar{u}_r (f_3 \gamma^5 \rho) u_r , \\
G_2 = \bar{u}_r (f_4 \gamma^5) u_r , \quad G_0 - G_3 = \bar{u}_r (f_5 + f_6 \rho) u_r .
\]

(76)

where \( f_1 = f_1 + f_{15} , \ f_2 = f_2 + f_{14} , \ f_3 = f_8 , \ f_4 = f_{11} , \ f_5 = f_1 - f_{15} \) and \( f_6 = f_2 - f_{14} \). In the non-relativistic limit the bilinear terms read \( \bar{u}_r(q) = 0 \),

\[
\bar{u}_r(q) \gamma^\mu \gamma^5 u_r(q) \approx \begin{cases} 
0 & \mu = 0 \\
\chi^i_r \sigma^i \chi^{r'}_r & \mu = i 
\end{cases}
\]

(77)

where \( \sigma^i \) are Pauli matrices, \( \chi^{r'}_r \) and \( \chi^i_r \) are Weyl spinors,

\[
\bar{u}_r(q) \gamma^5 u_r(q) = 0
\]

and

\[
\bar{u}_r(q) \gamma^\mu u_r(q) = \delta_{rr'} \frac{q^\mu}{m_f} .
\]

(79)

Thus, in this approximation we find

\[
G_0 + G_3 \approx 2f_1 \delta_{rr'} , \quad G_1 = 2f_3 \chi^i_r \rho \cdot \sigma \chi^{r'}_r , \quad G_2 \approx 0 , \quad G_0 - G_3 \approx 2f_5 \delta_{rr'} .
\]

(80)

Using these results, the scattering amplitude is simplified considerably to

\[
M_{fi} = 2f_1 \left( \frac{4 \Delta^0}{N^2} \right) \left( q \times p \right) \cdot e^s (q \times p) \cdot e^{s'} \delta_{rr'} + 2f_3 \left( \frac{q \cdot e^s}{m_f} \right) \left( \frac{q \cdot e^{s'}}{m_f} \right) \delta_{rr'} \\
+ 2f_3 \chi^i_r \rho \cdot \sigma \chi^{r'}_r \frac{4 \Delta^0}{m_f^2 N^2} \left[ (q \times p) \cdot e^s (q \times p) \cdot e^{s'} + (q \cdot e^s)(q \times p) \cdot e^{s'} \right] .
\]

(81)

In this equation the main effects are expected to come from the term multiplying the \( f_5 \). In fact, other terms, containing at least one factor of \( \Delta^0 \), will appear only when considering loop quantum effects. For this reason, in the next steps we will ignore them, since in a perturbation quantum field theory framework they are supposed to be a higher-order effect. Thus, the time evolution of polarization matrix elements is given by (from now on we will explicitly account for spatial dependence in the Boltzmann equations)

\[
\frac{d}{dt} \rho_{ij}(x, k) = \frac{i f_5}{k^0 m_f} \int dq \ n_f(q) (\delta_{is} \rho_{s' j}(x, k) - \delta_{s'j} \rho_{is}(x, k)) (q \cdot e^s) (q \cdot e^{s'}) + \text{standard Compton scattering terms (s.C.s.t.)} .
\]

(82)

Now, expressing Eq. \( (82) \) in terms of the different components, we have

\[
\frac{d}{dt} \rho_{11}^{(1)}(x, k) = \frac{i f_5}{k^0 m_f} \int dq \ n_f(q) (q \cdot e_2) (q \cdot e_1) \left[ \rho_{21}^{(1)}(x, k) - \rho_{12}^{(1)}(x, k) \right] + \text{s.C.s.t.} ,
\]

(83)
\[
\frac{d}{dt} \rho_{22}^{(1)}(x,k) = \frac{i f_5}{k^0 m_f} \int dq \, n_f(x,q) \left[ (q \cdot \epsilon_2)(q \cdot \epsilon_1)(\rho_{22}^{(1)}(x,k) - \rho_{11}^{(1)}(x,k)) + [(q \cdot \epsilon_1)^2 - (q \cdot \epsilon_2)^2] \rho_{12}^{(1)}(x,k) \right] + s.C.s.t. ,
\]

\[
\frac{d}{dt} \rho_{12}^{(1)}(x,k) = \frac{i f_5}{k^0 m_f} \int dq \, n_f(x,q) \left[ (q \cdot \epsilon_2)(q \cdot \epsilon_1)(\rho_{22}^{(1)}(x,k) - \rho_{11}^{(1)}(x,k)) + [(q \cdot \epsilon_1)^2 - (q \cdot \epsilon_2)^2] \rho_{21}^{(1)}(x,k) \right] + s.C.s.t. .
\]

We can also convert the density matrix elements to the normalized Stokes brightness perturbations after changing momentum to the comoving one, \(k_c = a k\), and going to the Fourier space. We find

\[
\frac{d}{d\eta} \tilde{J}^{(S)}(K, k_c) = s.C.s.t. ,
\]

\[
\frac{d}{d\eta} \tilde{Q}^{(S)}(K, k_c) = -2 \frac{a^2(\eta)}{k^0 m_f} \int dq \, n_f(K, q) \left[ (q \cdot \epsilon_1)^2 - (q \cdot \epsilon_2)^2 \right] \tilde{V}^{(S)}(K, k_c) + s.C.s.t. ,
\]

\[
\frac{d}{d\eta} \tilde{V}^{(S)}(K, k_c) = -\frac{a^2(\eta)}{k^0 m_f} \int dq \, n_f(K, q) \left[ -2(q \cdot \epsilon_2)(q \cdot \epsilon_1)Q^{(S)}(K, k_c) + [(q \cdot \epsilon_1)^2 - (q \cdot \epsilon_2)^2]U^{(S)}(K, k_c) \right] + s.C.s.t. .
\]

From the last set of equations we see that the V-modes in the CMB can be generated even with a parity preserving interaction. In particular, it is straightforward to verify that the fermionic number density \(n_f(K, q)\) has to contain anisotropies in order to achieve non-trivial coupling. From the current model of particle physics we know that a fermion can have a parity preserving interaction with a photon only through QED vertices. If we take the value of \(f_5\) for the case of QED, Eq. (61), and we evaluate it in the forward scattering limit, we find that \(f_5 = 0\) \((a_+ = 0)\). Thus, QED does not provide mixing terms among different polarizations, and only a parity preserving theory which goes beyond the standard paradigm could provide some kind of V mode generation.

B. Odd-parity amplitude

The general form of scattering amplitude for odd-parity was derived in section 111. In that section, we found the general form of coefficients \(G_i\) for the odd-parity case

\[
G_0 = \bar{u}_r(f_4 \gamma^5 P)u_r , \quad G_1 = \bar{u}_r(f_5 + f_6 P)u_r , \quad G_2 = 0 , \quad G_3 = \bar{u}_r(f_{10} \gamma^5 P)u_r .
\]

The amplitude can be constructed using the tensor (74) and replacing the values of the coefficients (91). As in the previous subsection, we focus only on the terms which are expected to give the dominant contributions. Thus, our amplitude reads

\[
M_{fi} = f_p \bar{u}_r(\gamma^5 P)u_r(q) \frac{(q \cdot \epsilon^s)}{m_f} \frac{(q \cdot \epsilon^{s'})}{m_f} ,
\]

where \(f_p = 2(f_4 - f_{10})\). Using this result we can find the time evolution of polarization matrix elements as

\[
\frac{d}{dt} \rho_{ij}(x,k) = i \frac{f_p}{4k^0 m_f} \int dq \, n_f(x,q) \left[ \delta_{is} \rho_{sp'}(x,k) - \delta_{js} \rho_{ip'}(x,k) \right] \bar{u}_r(q) \gamma^5 \bar{k} u_r(q) (q \cdot \epsilon^s) (q \cdot \epsilon^{s'}) + s.C.s.t. .
\]
Therefore, we have
\[ \frac{d}{dt} \rho_{11}^{(1)}(x, k) = -\frac{if_p}{4k^0m_f^2} \int dq n_f(x, q) \bar{u}_r \gamma^5 u_r (q \cdot \epsilon_2) (q \cdot \epsilon_1) \left[ \rho_{21}^{(1)}(x, k) - \rho_{12}^{(1)}(x, k) \right] + \text{s.C.s.t.} \tag{94} \]

\[ \frac{d}{dt} \rho_{12}^{(1)}(x, k) = \frac{d}{dt} \rho_{21}^{(1)}(x, k) = \frac{d}{dt} \rho_{22}^{(1)}(x, k) = \frac{d}{dt} \rho_{11}^{(1)}(x, k) , \tag{95} \]

\[ \frac{d}{dt} \rho_{12}^{(1)}(x, k) = -\frac{if_p}{4k^0m_f^2} \int dq n_f(x, q) \bar{u}_r \gamma^5 u_r \left[ (q \cdot \epsilon_2)(q \cdot \epsilon_1)(\rho_{22}^{(1)}(x, k) - \rho_{11}^{(1)}(x, k)) + [(q \cdot \epsilon_1)^2 - (q \cdot \epsilon_2)^2] \rho_{12}^{(1)}(x, k) \right] + \text{s.C.s.t.} , \tag{96} \]

\[ \frac{d}{dt} \rho_{21}^{(1)}(x, k) = \frac{if_p}{4k^0m_f^2} \int dq n_f(x, q) \bar{u}_r \gamma^5 u_r \left[ (q \cdot \epsilon_2)(q \cdot \epsilon_1)(\rho_{21}^{(1)}(x, k) - \rho_{11}^{(1)}(x, k)) + [(q \cdot \epsilon_1)^2 - (q \cdot \epsilon_2)^2] \rho_{21}^{(1)}(x, k) \right] + \text{s.C.s.t.} . \tag{97} \]

In the non-relativistic limit \( q^0 \approx m_f \) and one can sue Eq. (77). Also in this case we convert the density matrix elements to the normalized Stokes brightness perturbations and go to the Fourier space to obtain

\[ \frac{d}{d\eta} f^{(S)}(K, k_c) = \text{s.C.s.t.} , \tag{98} \]

\[ \frac{d}{d\eta} Q^{(S)}(K, k_c) = \frac{a(\eta)}{f_p} k_c \cdot \chi_r^{\dagger} \sigma \chi_r \int dq n_f(K, q) (q \cdot \epsilon_2)(q \cdot \epsilon_1)V^{(S)}(K, k_c) + \text{s.C.s.t.} , \tag{99} \]

\[ \frac{d}{d\eta} U^{(S)}(K, k_c) = -\frac{a(\eta)}{f_p} k_c \cdot \chi_r^{\dagger} \sigma \chi_r \int dq n_f(K, q) [(q \cdot \epsilon_1)^2 - (q \cdot \epsilon_2)^2] V^{(S)}(K, k_c) + \text{s.C.s.t.} , \tag{100} \]

\[ \frac{d}{d\eta} V^{(S)}(K, k_c) = \frac{a(\eta)}{f_p} k_c \cdot \chi_r^{\dagger} \sigma \chi_r \int dq n_f(K, q) \left[ -2(q \cdot \epsilon_2)(q \cdot \epsilon_1)Q^{(S)}(K, k_c) + [(q \cdot \epsilon_1)^2 - (q \cdot \epsilon_2)^2] U^{(S)}(K, k_c) \right] + \text{s.C.s.t.} . \tag{101} \]

Now, in order to perform the integral over \( q \) we choose the momentum and photon polarization vectors in the following form (see Fig. 2)

\[ \hat{K} = (0, 0, 1) , \tag{102} \]

\[ \hat{k} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) , \tag{103} \]

\[ \hat{q} = (\sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta') , \tag{104} \]

\[ \epsilon_1(k) = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) , \tag{105} \]

\[ \epsilon_2(k) = (-\sin \phi, \cos \phi, 0) . \tag{106} \]
After this, we can expand \( n_f(K, q) \) as

\[
n_f(K, q) = n_f(K, |q|) \sum_{\ell, m} c_{\ell m} Y_{\ell m}(\hat{q}) .
\]

The number density mediated over all the possible fermionic momenta is given by

\[
\bar{n}_f(K) = \sum_{\ell, m} \int d^3 q \frac{d^3 q}{(2\pi)^3} n_f(K, |q|) c_{\ell m} Y_{\ell m}(\hat{q})
= \frac{c_{00}}{(2\pi)^2} \sqrt{\pi} \int d|q||q|^2 n_f(K, |q|) .
\]

We can also define the fermionic anisotropic stress as

\[
\pi_{ij}(K) = m_f \sum_{\ell, m} \int d^3 q n_f(K, |q|) |q|^2 \hat{q}_i \hat{q}_j c_{\ell m} Y_{\ell m}(\hat{q}) ,
\]

that can be written in the form

\[
\pi_{ij}(K) = \pi_f(K) \int d^2 \hat{q} \hat{q}_i \hat{q}_j c_{\ell m} Y_{\ell m}(\hat{q}) ,
\]

where

\[
\pi_f(K) = \int d|q||q|^4 n_f(K, |q|) .
\]

The final result of the forward scattering in the Boltzmann equations reads

\[
\frac{d}{d\eta} Q^{(S)}(K, k_c) = i \sqrt{\frac{2\pi}{15}} \frac{a(\eta) f_p}{16\pi^3 k_c^2 m_f} k_c \cdot \chi_r \sigma \chi_r \pi_f(K) \left[ (c_{22} e^{2i\phi} - c_{22} e^{-2i\phi}) \cos \theta + (c_{21} e^{i\phi} + c_{21} e^{-i\phi}) \sin \theta \right] \times V^{(S)}(K, k_c) + \text{s.C.s.t.} ,
\]

\[
\frac{d}{d\eta} U^{(S)}(K, k_c) = -i \sqrt{\frac{2\pi}{15}} \frac{a(\eta) f_p}{32\pi^3 k_c^2 m_f} k_c \cdot \chi_r \sigma \chi_r \pi_f(K) \left[ \frac{1}{2} (c_{22} e^{2i\phi} + c_{22} e^{-2i\phi}) (\cos 2\theta + 3) + (c_{21} e^{i\phi} - c_{21} e^{-i\phi}) \sin 2\theta + \sqrt{6} c_{20} \sin^2 \theta \right] V^{(S)}(K, k_c) + \text{s.C.s.t.} ,
\]
\[
d\frac{d}{d\eta} V^{(S)}(K, k_c) = \sqrt{\frac{2\pi}{15} \frac{a(\eta)}{32\pi^3 k_0^6 m_f}} k_c \cdot \chi_1^\dagger \sigma \chi_r \pi_f(K) \left\{ -2i \left[ (c_{22} e^{2i\phi} - c_{2-2} e^{-2i\phi}) \cos \theta + (c_{21} e^{i\phi} + c_{-21} e^{-i\phi}) \sin \theta \right] \times Q^{(S)}(K, k_c) \left[ 1 + \frac{1}{2} \left( e^{2i\phi} - e^{-2i\phi} \right) \left( \cos 2\theta + 3 \right) + \left( c_{21} e^{i\phi} - c_{-21} e^{-i\phi} \right) \sin 2\theta + \sqrt{6} c_{20} \sin^2 \theta \right] \times U^{(S)}(K, k_c) \right\} + \text{s.C.s.t. .} \tag{114}
\]

The quantity \( \chi_1^\dagger \sigma \chi_r \) vanishes when we sum over spins if the interacting fermion exists in both left or right-handed states. Thus, looking to this final set of equations, circular polarization in the CMB photons can be generated from a parity violating interaction if two fundamental conditions are satisfied:

1. \( \sum_r \chi_1^\dagger \sigma \chi_r \neq 0 \),
2. \( (c_{22} e^{2i\phi} - c_{2-2} e^{-2i\phi}) \cos \theta \neq 0 \), \( (c_{-21} e^{i\phi} + c_{21} e^{-i\phi}) \sin \theta \neq 0 \).

The first condition implies that the fermion particle should interact only in left or right-handed state. The second condition implies that quadrupolar anisotropies in the stress tensor of the fermion have to appear.

## VI. GENERAL CONDITIONS FOR GENERATING B-MODE POLARIZATION

As we have seen in the previous section, new interactions which are even or odd under party and even under time-reversal can generate V-modes, but are unable to generate B-mode polarization through the forward scattering term. This is due to the fact that in equation (23) the \( G_2 \) coefficient vanishes if the amplitude is even under time-reversal. In this section, we will investigate the case in which the fermion-photon scattering amplitude is odd under time reversal. We will show that this condition may lead to B-mode polarization.

### A. Even-parity and odd-time reversal amplitude

As we discussed in section III after imposing the even-parity condition the \( G_2 \) coefficient is restricted to be (see Eq. (19) and (20))

\[
G_2 = \bar{u}_r \left( f_{11} \gamma^9 + f_{12} \gamma^5 P \right) u_r . \tag{115}
\]

After imposing the odd-time reversal condition, the only non-zero coefficients are

\[
G_1 = \bar{u}_r \left( f_7 \gamma^5 \right) u_r \quad \text{and} \quad G_2 = \bar{u}_r \left( f_{12} \gamma^5 P \right) u_r . \tag{116}
\]

Moreover, we impose the odd charge conjugation condition, so that the amplitude is even under CPT. As a result we get \( f_7 = 0 \). Finally, using Eqs. (14), in the non-relativistic limit the scattering amplitude is reduced to

\[
M_{f_1} \approx -8 f_{12} \chi_1^\dagger \sigma \chi_r \Delta^0 \sqrt{\frac{m_f}{N^2}} \cdot \left( \epsilon^s \times \epsilon^{s'} \right) . \tag{117}
\]

The only term which survives multiplies a factor of \( \Delta^0 \). Hence, the corresponding effect will be a loop quantum effect. The time evolution of the brightness Stokes parameters is given by

\[
\frac{d}{d\eta} Q^{(S)}(K, k_c) \approx -a(\eta) f_{12} \bar{k}_c \cdot \chi_1^\dagger \sigma \chi_r \int dq n_f(K, q) \frac{m_f \cdot (\epsilon_1 \times \epsilon_2)}{|q| \sin \psi} U^{(S)}(K, k_c) + \text{s.C.s.t.} , \tag{118}
\]

and

\[
\frac{d}{d\eta} U^{(S)}(K, k_c) \approx a(\eta) f_{12} \bar{k}_c \cdot \chi_1^\dagger \sigma \chi_r \int dq n_f(K, q) \frac{m_f \cdot (\epsilon_1 \times \epsilon_2)}{|q| \sin \psi} Q^{(S)}(K, k_c) + \text{s.C.s.t.} , \tag{119}
\]

and hence,

\[
\frac{d}{d\eta} P^{\pm(S)} + iK \mu P^{\pm(S)} = \mp i\alpha' P^{\pm(S)} + \text{s.C.s.t.} , \tag{120}
\]
Then, using Eqs. (7), (8) and (15) we get the following expressions for the E and B modes:

\[
\alpha'(\eta) = a(\eta) f_{12} \hat{k}_c \cdot \chi_1^r \hat{k}_c \cdot (\epsilon_1 \times \epsilon_2) \int dq n_f(K, q) \frac{m_f}{|q|} \frac{1}{\sin \psi}
\]

with

\[
\alpha(\eta) = - \int_{\eta}^{\eta_0} \alpha'(\eta') \, d\eta'
\]

and

\[
\sin \psi = [(\sin \theta \sin \theta' \sin(\phi - \phi'))^2 + (\cos \phi \cos \theta' \sin \theta - \cos \theta \cos \phi' \sin \theta')^2 + (\cos \theta' \sin \phi - \cos \theta \sin \phi')^2]^{1/2}.
\]

As a result, Eq. (120) can be rewritten as

\[
\frac{d}{d\eta} \left[ P_{\pm}(S) e^{iK\eta \pm i\alpha(\eta) - \tau(\eta)} \right] = e^{iK\eta \pm i\alpha(\eta) - \tau(\eta)} \left( \frac{1}{2} \begin{pmatrix} \tau' \sin \eta \end{pmatrix} [1 - P_2(\mu)] \Pi \right),
\]

where again \( \Pi = f^{(2)}(S) + P^{(2)}(S) - P^{(0)}(S) \). Integrating the last equation gives the general solution

\[
P_{\pm}(S)(\eta_0, K, \mu) = \frac{3}{4} (1 - \mu^2) \int_0^{\eta_0} d\eta e^{iK(\eta - \eta_0)\mu \pm i\alpha(\eta) - \tau(\eta)} \tau'(\eta) \Pi(\eta, K).
\]

Then, using Eqs. (7), (8) and (15) we get the following expressions for the E and B modes

\[
E^{(S)}(\eta_0, K, \mu) = -\frac{3}{4} \int_0^{\eta_0} d\eta g(\eta) \Pi(\eta, K) \partial_\mu^2 \left[ (1 - \mu^2)^2 e^{iK(\eta - \eta_0)\mu} \cos \alpha(\eta) \right],
\]

\[
B^{(S)}(\eta_0, K, \mu) = -\frac{3}{4} \int_0^{\eta_0} d\eta g(\eta) \Pi(\eta, K) \partial_\mu^2 \left[ (1 - \mu^2)^2 e^{iK(\eta - \eta_0)\mu} \sin \alpha(\eta) \right],
\]

where \( g(\eta) = \tau' e^{-\tau} \) is the so-called visibility function.

Also in this case we need the fermion to be left or right-handed, otherwise \( \alpha = 0 \) since \( \chi_1^r \sigma \chi_r = 0 \). Anyway, in this case the angular integral inside the definition of \( \alpha \), Eq. (121), is not equal to 0 if \( n_f(K, q) \) is isotropic. Thus, we do not have to impose any particular condition to the fermionic stress tensor.

B. Odd-parity and Odd-time-reversal amplitude

The expressions of the coefficients \( G_i \)'s under odd-parity condition have been presented in Eq. (158). Hence, \( G_2 \) is restricted to

\[
G_2 = \bar{u}_r (f_9 + f_{10} \bar{P}) u_r.
\]

Then applying the odd-time-reversal condition on \( G_i \), we get

\[
f_4 = f_5 = f_6 = f_{16} = 0.
\]

Therefore,

\[
G_0 = \bar{u}_r (f_3 \gamma^5) u_r , \quad G_1 = 0 , \quad G_2 = \bar{u}_r (f_9 + f_{10} \bar{P}) u_r , \quad G_3 = \bar{u}_r (f_{15} \gamma^5) u_r , \quad G_4 = \bar{u}_r (f_{16} \gamma^5) u_r ,
\]

which are all even under charge conjugation. Hence, the final form of amplitude will be even under CPT. In the non-relativistic limit, the \( G_0 \) and \( G_3 \) coefficients vanish and

\[
G_2 \approx \bar{u}_r f_9 u_r.
\]

Thus, the final form of the amplitude is simplified to

\[
M_{fi} \approx 2 f_9 \Delta^0 \sqrt{\frac{m_f^2}{N_f^2}} \mathbf{P} \cdot (\mathbf{e} \times \mathbf{e}'.
\]

The corresponding E-mode and B-mode polarizations are derived using the same method that we used to derive Eqs. (126) and (127). The only difference is that the parameter $\alpha'(\eta)$ changes into the following form:

$$\alpha'(\eta) = \frac{a^2(\eta) f_q}{2k_c^2} \langle \epsilon_1 \times \epsilon_2 \rangle \int dq n_f(K, q) \frac{1}{|q|} m_f \sin \psi .$$

(133)

As a result, in this case there is no restriction on the handedness of the fermion. In fact, $\alpha'$ can be different from zero if the fermion interacts both in the left and right-handed states. Moreover, also in this case we do not have to impose any particular condition in the fermion stress tensor since we do not need anisotropies for providing a value different from zero to the angular integral contained in the $\alpha'$ expression.

| Symmetries broken | V-mode formation | B-mode formation |
|-------------------|-----------------|-----------------|
| All preserved     | Anisotropies in $n_f(K, q)$ | / |
| C & P             | Anisotropies in $n_f(K, q)$ | Only R or L handed fermion |
| C & T             | / | Only R or L handed fermion |
| P & T             | / | No conditions |

TABLE I: In this table we summarize the conditions one needs to impose on the fermion to produce V and B modes from fermion-photon forward scattering in the different cases analyzed.

VII. MAJORANA FERMIONS

In the previous sections we assumed the fermion to be a Dirac spinor. In this section, we will analyze what changes when the interacting fermion is a Majorana spinor, instead of a Dirac spinor. Analogous considerations have already been made in Ref. [30] for the case in which the fermion is a neutrino.

A Majorana fermion is a particle which coincides with its own antiparticle and hence it has no electric charge [53–55]. The Majorana spinor is defined as

$$\psi_M = \gamma^0 C \psi_M^* ,$$

(134)

where $C$ is the charge conjugation operator. The properties of Majorana bilinear terms under parity, charge conjugation and time reversal transformations have been summarized in Refs. [53–55]. The Majorana condition implies $\psi_M = \psi_M^\dagger$. As a result, a Majorana spinor transforms under charge conjugation as

$$C^{-1} \psi_M C = \psi_M .$$

(135)

Thus, in general we can write

$$C^{-1} (\bar{\psi}_MA \psi_M) C = \bar{\psi}_M A \psi_M ,$$

(136)

that for $A = \gamma^\mu$ becomes

$$\bar{\psi}_M \gamma^\mu \psi_M = 0 .$$

(137)

However, one can show that the transformations of the other Majorana bilinear terms under $P$, $T$ and $C$ are the same as Dirac bilinear terms. It was discussed in Ref. [56] that the Compton scattering amplitude for Majorana fermions is given by

$$M_{fi} = \bar{u}_r(q') \epsilon^s_{\mu} \left[ F^{\mu\nu}(q, q', p, p') + C (F^{\rho\nu}(-q', -q, p, p'))^T C^{-1} \right] \epsilon^s_{\nu} u_r(q) ,$$

(138)

$F^{\mu\nu}$ being as in Eq. (20). Now, if in general

$$C (F^{\rho\nu}(-q', -q, p, p'))^T C^{-1} = -F^{\mu\nu}(q, q', p, p') ,$$

(139)
we find that $M_{fi}^M = 0$ identically. However, if

$$C \left( F^\mu_\nu(-q', -q, p, p') \right)^T C^{-1} = F^\mu_\nu(q, q', p, p') , \quad (140)$$

then the scattering amplitude becomes

$$M_{fi}^M = 2M_{fi}^D . \quad (141)$$

Thus, when the Compton tensor $F^\mu_\nu$ transforms like a pseudo-tensor under C, we get no fermion-photon forward scattering mixing. On the contrary, when $F^\mu_\nu$ is invariant under C, we get the same coupling as discussed in the previous sections, but with an additional factor of 2 with respect to the Dirac fermion case.

VIII. CONCLUSIONS

In the standard lore circular and B-mode polarization of CMB photons cannot be generated via Compton scattering with electrons from linear scalar perturbations. In this work we studied V and B modes generation in the CMB due to the forward scattering with a generic fermion in the presence of just linear scalar perturbations. We assumed interactions which may also go beyond the Standard Model of particle physics, keeping only gauge-invariance and the preservation of CPT symmetry. We derived various sets of Boltzmann equations describing the radiation transfer of CMB polarization. These equations are valid only in the non-relativistic limit, i.e. if we assume the mass of the fermion to provide the dominant contribution to its energy. This assumption is well-motivated in the case of the CMB, since we are interested in a low-energy effect which arises after the recombination epoch. Our final results are qualitatively summarized in table III. We can have V-mode production both preserving all the discrete symmetries and breaking the C and P symmetries. Instead, B-modes may arise only from the breaking of the T symmetry. Since our results are expressed in terms of free parameters, they offer a viable tool to put constraints on fundamental physics properties beyond the standard paradigms. We leave this intriguing and interesting possibility for future research.

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Appendix A: Comparison of our amplitude with the Kim and Dass’s amplitude

In Ref. 50 Kim and Dass calculated the parity violating part of Compton amplitude using the procedure of Ref. 57. They constructed $F^\mu_\nu$ using the minimal pseudo-tensors violating parity. The general parity violating amplitude is defined as 50

$$F^\mu_\nu = \sum_i L^i_{\mu\nu} A_i(x, y) , \quad (A1)$$

where $x = p \cdot q = p' \cdot q'$ and $y = p \cdot q' = p' \cdot q$. We change the kinematic variables defined in 50 to synchronize their notation with the notation of this paper. Moreover, we define a new variable $Q'$ as

$$Q' = q + q' , \quad (A2)$$

and remove the factor 1/2 adopted in 50 for kinematic variables. Hence, based on our notation, the $L^i_{\mu\nu}$ tensors defined in 50 are reconstructed as

$$L^1_{\mu\nu} = Q' \cdot P \epsilon_{\mu\nu\alpha\beta} Q'^\alpha P^\beta + Q'_\mu N_\nu + N_\mu Q'_\nu , \quad (A3)$$

$$L^2_{\mu\nu} = -P \left( Q'_\mu N_\nu + Q'_\nu N_\mu \right) + Q' \cdot P \left( \gamma_\mu N_\nu + \gamma_\nu N_\mu \right) , \quad (A4)$$
\[ \mathcal{L}_{\mu\nu}^3 = \gamma^5 \hat{P} \left( P^2 g_{\mu\nu} - P_{\mu} P_{\nu} + t_{\mu} t_{\nu} \right) , \]  
(A5)

\[ \mathcal{L}_{\mu\nu}^4 = -P^2 (Q'_{\mu} N_{\nu} + Q'_{\nu} N_{\mu}) + Q' \cdot P (N_{\mu} P_{\nu} + N_{\nu} P_{\mu}) , \]  
(A6)

\[ \mathcal{L}_{\mu\nu}^5 = \gamma^5 \hat{P} \left[ P^2 Q'_{\mu} Q'_{\nu} + (Q' \cdot P)^2 g_{\mu\nu} - Q' \cdot P (Q'_{\mu} P_{\nu} + Q'_{\nu} P_{\mu}) \right] . \]  
(A7)

We can express the \( G_i \) coefficients defined in \[(20)\] in terms of the \( A_i \) coefficients. The results are

\[
G_0 = \frac{1}{P^2} \left\{ 2 \epsilon_{\mu\nu\rho\sigma} P^\rho P^\mu Q'' t^\nu Q'^\sigma \times \left[ (A_1 - A_1 P^2) ((P \cdot Q)^2 - P^2(Q'\cdot Q'^2) + 2Q^2 ((P \cdot Q')^2 - P^2Q'^2) + 2P^2 (P \cdot Q)(Q' \cdot Q') (P \cdot Q') + 2A_2 P^2 (P \cdot Q') (P \cdot Q'^2) - 2P^2 (P \cdot Q') (P \cdot Q'^2) + 2Q^2 ((P \cdot Q')^2 - P^2Q'^2) ) \right] + \gamma^5 \hat{P} \left[ (P \cdot Q')^2 (A_5 t^2 (P \cdot Q)^2 - A_5 Q^2 - A_3) \right] + P^2 t^2 (A_5 Q'^2 + A_3) (P \cdot Q)^2 - P^2 t^2 (Q'^2 (A_5 Q'^2 + A_3) - A_5 (Q' \cdot Q'^2)^2 + Q^2 (A_5 Q'^2 + A_3) ) - 2A_5 P^2 t^2 (P \cdot Q')(Q' \cdot P') \right] - 2A_2 P^2 (P \cdot Q') Q' \epsilon_{\mu\nu\rho\sigma} P^\rho Q'' t^\nu Q'^\sigma \right\} ,
(A8)

\[
G_1 = \frac{2}{P^2} \left\{ t^2 \left[ (A_4 P^2 - A_1) ((P \cdot Q)^2 - (P \cdot Q')^2 - 2P^2Q'^2) + 2P^2 (P \cdot Q')(Q' \cdot P')(P \cdot Q') + 2A_2 P^2 (P \cdot Q')(P \cdot Q') ((P \cdot Q)^2 - P^2Q'^2) + A_2 P^2 Q(P \cdot Q') (P \cdot Q' ) \right] - 2A_5 P^2 t^2 (P \cdot Q')(Q' \cdot P') \right] \} ,
(A9)

\[
G_2 = 0 ,
(A10)

\[
G_3 = \frac{1}{P^2} \left\{ -2 \epsilon_{\mu\nu\rho\sigma} P^\rho P^\mu Q'' t^\nu Q'^\sigma \left[ (P P^2 (A_2 \hat{P} + A_4 P^2 - A_1) ((P \cdot Q)^2 - (P \cdot Q')^2) + 2Q^2 ((P \cdot Q')^2 - P^2Q'^2) ) + (A_1 - A_4 P^2)(P \cdot Q') \right] + P^2 \left[ 2A_2 t^2 \gamma^5 \hat{P} ((P \cdot Q)^2 - P^2Q'^2) + \gamma^5 \hat{P} \left[ (P \cdot Q)^2 (A_5 t^2 (P \cdot Q)^2 + A_5 Q^2 + A_3) \right] - P^2 \left[ -t^2 (A_5 Q'^2 + A_3) (P \cdot Q)^2 + P^2 t^2 (Q'^2 (A_5 Q'^2 + A_3) - A_5 (Q' \cdot Q'^2)^2 + Q^2 (A_5 Q'^2 + A_3) ) - 2A_5 P^2 t^2 (P \cdot Q')(Q' \cdot P') \right] + 2A_2 P^2 (P \cdot Q') Q' \epsilon_{\mu\nu\rho\sigma} P^\rho Q'' t^\nu Q'^\sigma \right\} ,
(A11)

As one can see, \( G_2 = 0 \) is consistent with what had been found in Eq. \[(57)\].

**Appendix B: Interaction of photons with neutrino magnetic moment**

To be able to consistently define the basis vector \( \hat{c}^{(1)\lambda} \) in the forward scattering limit, Eq. \[(24)\], we introduced a new variable \( \Delta^\lambda \), which takes the place of \( t^4 \) in the general definition \[(23)\], and claimed that the subsequent new terms in the Compton tensor may arise from loop corrections in the Feynman diagrams. Here, we consider an explicit example and compare the Compton tensor of this example with the general forward scattering Compton tensor derived at the end of Sec. \[(17)\].

If the neutrino has a magnetic moment, its interaction with photon is characterized by the following effective Hamiltonian \[(55, 61)\] (for a recent review, see Ref. \[(62)\])

\[
\mathcal{H} \sim \mu_{e} \mu \bar{\nu}_{\nu} (q') \sigma_{\alpha\beta} u_{\nu}(q') \mathcal{F}^{\alpha\beta} ,
(B1)
\]

where \( \mathcal{F}^{\alpha\beta} \) is the field strength of the photon, \( \mu_{e} \) is the magnetic moment of the electron and \( \mu \) is the magnetic moment of the neutrino. From Eq. \[(11)\] we can derive the forward scattering amplitude in the following form

\[
M_{fi} = F^{\lambda \tau} \epsilon^{'*}_{\lambda} \epsilon^{'*}_{\tau} ,
(B2)
\]

where

\[
F^{\lambda \tau} \sim (\mu_{e} \mu) \epsilon^{\lambda \tau \alpha\beta} p_{\alpha} q_{\beta} ,
(B3)\]
$p$ being the photon momentum and $q$ the neutrino momentum.

It is immediate to verify that this term is equivalent in form to the second term of Eq. (30), which contains the quantity $\Delta^\lambda$. This simple example shows that effectively the new definition of $\tilde{c}^{(1)}\lambda$ is sensitive to loop quantum effects and provides a more general expression for the Compton tensor that, as discussed in Sec. VI may cause B-mode generation in the CMB.

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