Generic smooth representations

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Abstract

Let $F$ be a finite extension of $\mathbb{Q}_p$. Here we give a necessary and sufficient condition for an irreducible smooth representation of $GL_n(F)$ to be generic.

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1 Introduction

Let us start by recalling a few facts about the category of smooth representations. The Bernstein decomposition expresses the category of smooth $\mathbb{Q}_p$-valued representations of a $p$-adic reductive group $G$ as the product of certain indecomposable full subcategories, called Bernstein components. Those components are parametrized by the inertial classes. Let me now recall the definition of an inertial class. Let $M$ be a Levi subgroup of some parabolic subgroup of $G$ and let $\rho$ be an irreducible supercuspidal representation of $M$ and consider a set of pairs $(M, \rho)$ as above. We say that two pairs $(M_1, \rho_1)$ and $(M_2, \rho_2)$ are inertially equivalent if and only if there is $g \in G$ and an unramified character $\chi$ of $M_2$ such that:

$$M_2 = M_1^g \text{ and } \rho_2 \simeq \rho_1^g \otimes \chi$$

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where $M_1 := g^{-1}M_1g$ and $\rho_1^g(x) = \rho_1(gxg^{-1}), \forall x \in M_1^g$. An equivalence class of all such pairs will be denoted $[M, \rho]_G$. The set of inertial class equivalences of all such pairs will be denoted by $\mathcal{B}(G)$.

Let $F$ be a finite extension of $\mathbb{Q}_p$ with a finite residue field $k_F$. Let $\mathcal{O}_F$ be its complete discrete valuation ring, let $p$ be the maximal ideal of $\mathcal{O}_F$ with uniformizer $\varpi$, and let $q = |\mathcal{O}_F/\varpi\mathcal{O}_F|$. In this paper we only consider the case $G = GL_n(F)$. Let $E$ be an algebraically closed field of characteristic zero.

Let $\mathcal{R}(G)$ be the category of all smooth $E$-representations of $G$. We denote by $i^G_P : \mathcal{R}(M) \to \mathcal{R}(G)$ the normalized parabolic induction functor, where $P = MN$ is a parabolic subgroup of $G$ with Levi subgroup $M$. Let $\overline{P}$ be the opposite parabolic with respect to $M$. We use the notation $\text{Ind}$ and $c$–$\text{Ind}$ to denote the induction and compact induction respectively.

We are given an inertial class $\Omega := [M, \rho]_G$; where $\rho$ is a supercuspidal representation of $M$ and $D := [M, \rho]_M$. To any inertial class $\Omega$ we may associate a full subcategory $\mathcal{R}^\Omega(G)$ of $\mathcal{R}(G)$, such that $(\pi, V) \in \text{Ob}(\mathcal{R}^\Omega(G))$ if and only if every irreducible $G$-subquotient $\pi_0$ of $\pi$ appear as a composition factor of $i^G_P(\rho \otimes \omega)$ for $\omega$ some unramified character of $M$ and $P$ some parabolic subgroup of $G$ with Levi factor $M$. The category $\mathcal{R}^\Omega(G)$ is called a Bernstein component of $\mathcal{R}(G)$. We will say that a representation $\pi$ is in $\Omega$ if $\pi$ is an object of $\mathcal{R}^\Omega(G)$. According to [Ber84], we have a decomposition:

$$\mathcal{R}(G) = \prod_{\Omega \in \mathcal{B}(G)} \mathcal{R}^\Omega(G)$$

So in order to understand the category $\mathcal{R}(G)$, it is enough to restrict our attention to the components. We may understand those components via the theory of types. This is a way to parametrize all the irreducible representations of $G$ up to inertial equivalence using irreducible representations of compact open subgroups of $G$.

Let $J$ be a compact open subgroup of $G$ and let $\lambda$ be an irreducible representation of $J$. We say that $(J, \lambda)$ is an $\Omega$-type if and only if for every irreducible representation $(\pi, V) \in \text{Ob}(\mathcal{R}^\Omega(G))$, $V$ is generated by the $\lambda$-isotypical component of $V$ as $G$-representation.

Let $\mathcal{R}_\lambda(G)$ be a full subcategory of $\mathcal{R}(G)$ such that $(\pi, V) \in \text{Ob}(\mathcal{R}_\lambda(G))$ if and only if $V$ is generated by $V^\lambda$ (the $\lambda$-isotypical component of $V$) as $G$-representation.
Define $\mathcal{H}(G, \lambda) := \mathcal{H}(G, J, \lambda) := \text{End}_G(c^{-\text{Ind}}_J^G \lambda)$. Then for any $\Omega$-type $(J, \lambda)$, by Theorem 4.2 (ii) [BK98], the functor:

$$
\mathcal{M}_\lambda : \mathcal{R}_\lambda(G) \to \mathcal{H}(G, \lambda) - \text{Mod}
$$

$$
\pi \mapsto \text{Hom}_J(\lambda, \pi) = \text{Hom}_G(c^{-\text{Ind}}_J^G \lambda, \pi)
$$

induces an equivalence of categories. Since $(J, \lambda)$ is an $\Omega$-type, we have $\mathcal{R}_\Omega(G) = \mathcal{R}_\lambda(G)$.

Denote by $W$ the vector space on which the representation $\lambda$ is realized. Next, let $(\bar{\lambda}, W^\vee)$ denote the contragradient of $(\lambda, W)$. Then by (2.6) [BK99], the Hecke algebra $\mathcal{H}(G, \lambda) := \text{End}_G(c^{-\text{Ind}}_J^G \lambda)$ can be identified with the space of compactly supported functions $f : G \to \text{End}_E(W^\vee)$ such that $f(j_1, g, j_2) = \bar{\lambda}(j_1) \circ f(g) \circ \bar{\lambda}(j_2)$, with $j_1, j_2 \in \lambda$ and $g \in G$ and the multiplication of two elements $f_1$ and $f_2$ is given by the convolution:

$$
f_1 \ast f_2(g) = \int_G f_1(x) \circ f_2(x^{-1}g) dx
$$

For $u \in \text{End}_E(W^\vee)$, we write $\bar{u} \in \text{End}_E(W)$ for the transpose of $u$ with respect of the canonical pairing between $W$ and $W^\vee$. This gives $(\bar{\lambda}(j))^\vee = \lambda(j)$, for $j \in J$. For $f \in \mathcal{H}(G, \lambda)$, define $\bar{f} \in \mathcal{H}(G, \bar{\lambda})$, by $\bar{f}(g) = f(g^{-1})^\vee$, for all $g \in G$.

Write $\mathfrak{Z}_\Omega$ for the centre of category $\mathcal{R}_\Omega(G)$ and $\mathfrak{Z}_D$ for the centre of category $\mathcal{R}_D(M)$, which is defined the same way as $\mathcal{R}_\Omega(G)$. Recall that the centre of a category is the ring of endomorphisms of the identity functor. For example the centre of the category $\mathcal{H}(G, \lambda) - \text{Mod}$ is $Z(\mathcal{H}(G, \lambda))$, where $Z(\mathcal{H}(G, \lambda))$ is the centre of the ring $\mathcal{H}(G, \lambda)$. We will call $\mathfrak{Z}_\Omega$ a Bernstein centre.

For $G = GL_n(F)$, the types can be constructed in an explicit manner (cf. [BK93], [BK98] and [BK99]) for every Bernstein component. Moreover, Bushnell and Kutzko have shown that $\mathcal{H}(G, \lambda)$ is naturally isomorphic to a tensor product of affine Hecke algebras of type A.

The simplest example of a type is $(I, 1)$, where $I$ is Iwahori subgroup of $G$ and 1 is the trivial representation of $I$. In this case $\Omega = [T, 1]_G$, where $T$ is the subgroup of diagonal matrices and 1 denotes the trivial representation of $T$. We will refer to example as the Iwahori case.

Let $K$ be a maximal compact open subgroup of $G$. In [SZ99] section 6 (just above proposition 2) the authors define irreducible $K$-representations
σ_P(λ), where P is partition valued functions with compact support (cf. section 2 [SZ99]). One has the decomposition:

$$\text{Ind}_J^K \lambda = \bigoplus_P \sigma_P(\lambda)^{\oplus m_{P,\lambda}}$$  \hspace{1cm} (1.1)$$

where the summation runs over partition valued functions with compact support. The integers $m_{P,\lambda}$ are finite and we call them multiplicity of $\sigma_P(\lambda)$.

There is a natural partial ordering, as defined in [SZ99], on the partition valued functions. Let $P_{\text{max}}$ be the maximal partition valued function and let $P_{\text{min}}$ the minimal one. Define $\sigma_{\text{max}}(\lambda) := \sigma_{P_{\text{max}}}(\lambda)$ and $\sigma_{\text{min}}(\lambda) := \sigma_{P_{\text{min}}}(\lambda)$. Both $\sigma_{\text{max}}(\lambda)$ and $\sigma_{\text{min}}(\lambda)$ occur in $\text{Ind}_J^K \lambda$ with multiplicity 1.

In the Iwahori case, $\sigma_{\text{min}}(\lambda)$ is the inflation of Steinberg representation of $GL_n(k_F)$ to $K$ and $\sigma_{\text{max}}(\lambda)$ is the trivial representation. In this simplest case, we have $\Omega = [T, 1]_G$.

The classical local Langlands correspondence associates to an irreducible smooth representation of $GL_n$, denoted $\pi$, a Weil-Deligne representation denoted $WD(\pi)$. A consequence of Bernstein-Zelevinsky classification (cf. [BZ77]) of smooth irreducible representations of $GL_n(F)$ and the general form of Weil-Deligne representations (cf. 3.1.3 [Del75]) is that two smooth irreducible $\pi$ and $\pi'$ of $G$ lie in the same Bernstein component if and only if $WD(\pi)|I_F \simeq WD(\pi')|I_F$, where $I_F$ is the inertia subgroup of the absolute Galois group of $F$.

Let $\pi$ be an irreducible smooth generic (i.e. admits a Whittaker model) representation. In my thesis I proved that knowledge of which of the $\sigma_P(\lambda)$’s are contained in $\pi$ allow us to describe completely the monodromy of the associated Weil-Deligne representation. This statement will be made more precise in the Proposition 1.2 below.

It has been observed by Jack Shotton [Sho16, Thm.3.7] that by modifying the proof of [SZ99, Proposition 2 Section 6] and [BcC09, Proposition 6.5.3] in the tempered case, he gets the same result in the generic case. In our notation this result can be stated as follows:

**Proposition 1.2.** Let $\pi$ be an absolutely irreducible generic representation, with semi-simple type $(J, \lambda)$. The following statement are equivalent:

1. $\text{Hom}_K(\sigma_P(\lambda), \pi) \neq 0$ and $\text{Hom}_K(\sigma_{P'}(\lambda), \pi) = 0$, for all partitions valued functions $P'$ such that $P < P'$.
2. \( \pi = i_p^G(Q(\Delta_1) \otimes \ldots \otimes Q(\Delta_k)) \), where \( P \) is the standard parabolic associated to the partition valued function \( \mathcal{P} \), all the segments \( \Delta_i \) are not pairwise linked and \( Q(\Delta_i) \) denotes the Langlands quotient (cf. section 1.2 [Kud94]).

Moreover if \( \sigma_\mathcal{P} \) satisfies the equivalent properties above, it occurs with multiplicity one in \( \pi \).

However in my thesis I use a different method to prove the proposition above. First using the theory of types of Bushnell-Kutzko, I reduce the statement to the Iwahori case. Then, in the Iwahori case, I use the results of Rogawski [Rog85] on modules over Iwahori-Hecke algebra. In this case the proof relies on some easy combinatorics on partitions.

The proposition above suggests that the representation \( \sigma_{\text{min}}(\lambda) \) has a very special role. The main result of this paper is that we can characterize the genericity via \( \sigma_{\text{min}}(\lambda) \). Indeed we will prove the following:

**Theorem 1.3.** Let \( \pi \) be an absolutely irreducible representation in the Bernstein component \( \Omega \), then \( \text{Hom}_K(\sigma_{\text{min}}(\lambda), \pi) \neq 0 \) if and only if \( \pi \) is generic.

The proof of this result illustrates, how we can reduce a statement about irreducible representations of general type to the Iwahori case. It was pointed out to me, recently, by Peter Schneider that the Iwahori case was already treated by [Ree02]. This allows me simplify a little the original proof in my thesis.

In the next section we will prove Theorem 1.3 and then by using a result of our previous work we will deduce that \( \sigma_{\text{min}}(\lambda) \) occurs with multiplicity at most one in an irreducible representation.

## 2 Generic representations

We are given an inertial class \( \Omega = [M, \rho]_G \), where \( \rho \) is a supercuspidal representation of \( M \) and an \( \Omega \)-type \( (J, \lambda) \) with \( J \subset K \) a compact open subgroup of \( G \). Write \( Z_\Omega \) for the centre of Bernstein component of \( \Omega \). Choose a partition valued function \( \mathcal{P}^{\text{min}} \) which is minimal for partial ordering as in [SZ99]. From now on let \( \sigma_{\text{min}}(\lambda) := \sigma_{\mathcal{P}^{\text{min}}}(\lambda) \) with the notations of section 6 in [SZ99].

**Theorem 2.1.** Let \( \pi \) be an absolutely irreducible representation in the Bernstein component \( \Omega \), then \( \text{Hom}_K(\sigma_{\text{min}}(\lambda), \pi) \neq 0 \) if and only if \( \pi \) is generic.
Proof. In this proof \( \sigma := \sigma_{\min}(\lambda) \). Let’s first deal with a particular case before the general case.

1. **Simple type case.** If \( \pi \) is supercuspidal, it is generic and there is nothing to prove. So assume that \( \pi \) contains a simple type \((J, \lambda)\) which is not maximal. In this case \( \Omega = [GL_r(F)^e, \omega \otimes \ldots \otimes \omega]_G \) where the tensor product \( \rho := \omega \otimes \ldots \otimes \omega \) is taken \( e \) times and \( \omega \) is a supersupidal representation of \( GL_r(F) \). According to the description of Hecke algebras in section (5.6) of [BK93] there is a support preserving isomorphism of Hecke algebras \( H(GL_L, I_L, 1) \simeq H(G, J, \lambda) \), where \( L \) is a well defined extension of \( F \) (denoted by \( K \) in [BK93]), \( GL_L = GL_e(L) \) with \( I_L \) the Iwahori subgroup of \( GL_L \) and \( K_L \) be a maximal compact subgroup of \( G_L \).

We will recall now the results on supercuspidal representations from chapter 6 of [BK93] and describe the general form of the representation supercuspidal representation \( \omega \) of \( G_0 = GL_r(F) \). The representation \( \omega \) contains a maximal simple type \((J_0, \lambda_0)\). Then there is a finite extension \( \Gamma \) of \( F \) and a uniquely determined representation \( \Lambda_0 \) of \( \Gamma \times J_0 \) such that \( \omega = \text{c-Ind}_{\Gamma \times J_0}^{G_0} \Lambda_0 \) and \( \Lambda_0|_{J_0} = \lambda_0 \). Let \( V = F^n \) the \( F \)-vector space of dimension \( n \) viewed as \( \Gamma \) vector space of dimension \( R = \dim_F V \) and let \( f = R/e \). According to [BK93, Proposition 5.5.14], the extension \( L \) considered in the previous is unramified extension of degree \( f \) of \( \Gamma \). A special case of the support preserving isomorphism in the previous paragraph is the support preserving isomorphism \( \Phi_1 : \mathcal{H}(G_0, J_0, \lambda_0) \simeq \mathcal{H}(L^\times, O_L^\times, 1) \) sending a function supported to \( J_0 \omega_{\Gamma} J_0 = \omega_{\Gamma} J_0 \) to the function supported on \( \omega_{\Gamma} O_L^\times \), where \( O_L \) is the ring of integers of \( L \) and \( \omega_{\Gamma} \) a uniformizer of both \( \Gamma \) and \( L \). Further we observe that the unramified characters of \( G_0 \) are determined by the image of \( \omega_{\Gamma} \) so as are unramified characters of \( L^\times \). Therefore we way and we will identify the unramified characters of \( G_0 \) with the unramified characters of \( L^\times \).

The representation \( \pi \) is a Langlands quotient of the form \( Q(\Delta_1, \ldots, \Delta_s) \) (cf. section 1.2 [Kud94]) such that for \( i < j \) the segment \( \Delta_i \) does not precede \( \Delta_j \). After twisting \( \pi \) by some unramified character we may assume that all the segments are of the form \( \Delta_i = [\omega(\alpha_i), \omega(\alpha_i + e_i - 1)] \), where \( \alpha_i \) is some real number and \( e_i \) an integer such that \( \sum_{i=1}^s e_i = e \). Here the notation \( \omega(\alpha_i) \) means that \( \omega(\alpha_i) := \omega \otimes |\det|^{\alpha_i} \). If \( s = 1 \) then \( \pi \) is generic. Assume that \( s > 1 \).
According to Theorem 7.6.20 in [BK93], the diagram

\[
\begin{array}{c}
\mathcal{H}(G, J, \lambda) \xrightarrow{\Phi} \mathcal{H}(G_L, I_L, 1) \\
\mathcal{H}(M, J_M, \lambda_M) \xrightarrow{\Phi^e \otimes e_1} \mathcal{H}(T_L, T^o_L, 1)
\end{array}
\]

is commutative, where the horizontal arrows are support preserving isomorphisms and \(\lambda_M = \lambda_0 \otimes \ldots \otimes \lambda_0\) (e times), \(J_M = J_0^e\), \(T_L = (L^\times)^e\) and \(T^o_L = (\mathcal{O}_L^\times)^e\). In the book [BK93], the horizontal isomorphisms in the commutative diagram above are given in the other direction. This diagram in turn produces the following commutative diagram:

\[
\begin{array}{c}
\mathcal{R}_\lambda(G) \xrightarrow{\text{Hom}_J(\lambda, \bullet)} \mathcal{H}(G, J, \lambda) - \text{Mod} \xrightarrow{\text{Mod}} \mathcal{H}(G_L, I_L, 1) - \text{Mod} \xrightarrow{T_\lambda} \mathcal{R}_1(G_L) \\
\mathcal{R}_{\lambda_M}(M) \xrightarrow{\text{Hom}_{J_M}(\lambda_M, \lambda_M)} \mathcal{H}(M, J_M, \lambda_M) - \text{Mod} \xrightarrow{\text{Mod}} \mathcal{H}(T_L, T^o_L, 1) - \text{Mod} \xrightarrow{T_1} \mathcal{R}_1(T_L)
\end{array}
\]

where the horizontal arrows are equivalences of categories, \(T_\lambda = \bullet \otimes \mathcal{H}(G_L, I_L, 1)\) \(c\)-Ind\(_{T^o_L}\) 1 and \(T_1 = \bullet \otimes \mathcal{H}(T_L, T^o_L, 1)\) \(c\)-Ind\(_{T^o_L}\) 1. It follow from this commutative diagram that

\[
\Phi(\text{Hom}_J(\lambda, i^G_E(\rho))) \otimes \mathcal{H}(G_L, I_L, 1) \xrightarrow{\text{c-Ind}\_I_L^G 1} \mathcal{H}(G, J, \lambda) - \text{Mod}
\]

\[
= i^G_E(\Phi^e \otimes \text{Hom}_{J_M}(\lambda_M, \rho)) \otimes \mathcal{H}(T_L, T^o_L, 1) \xrightarrow{\text{c-Ind}\_T^E_L 1}
\]

Observe that the representation \(c\)-Ind\(_{T^o_L}\) 1 is canonically a rank 1 free \(\mathcal{H}(T_L, T^o_L, 1)\)-module. This observation allows to simplify the right hand side.

Since \((J, \lambda)\) is a simple type, \(\lambda_M = \lambda_0 \otimes \ldots \otimes \lambda_0\) (e times), \(J_M = J_0^e\) and \((J_0, \lambda_0)\) is a maximal simple type for the supercuspidal representation \(\omega\), we have:

\[
\text{Hom}_{J_M}(\lambda_M, \rho) = \text{Hom}_{J_M}(\lambda_0 \otimes \ldots \otimes \lambda_0, \omega| J_0 \otimes \ldots \otimes \omega| J_0)
\]

\[
= \text{Hom}_{J_0^e}(\lambda_0 \otimes \ldots \otimes \lambda_0, \lambda_0 \otimes \ldots \otimes \lambda_0) = \text{Hom}_{J_M}(\lambda_M, \lambda_M)
\]

Now notice that \(\text{Hom}_{J_M}(\lambda_M, \lambda_M)\) is the subspace of functions in \(\mathcal{H}(M, J_M, \lambda_M)\) supported on \(J_M\). By our choice of \(\lambda_0, \omega\) and \(\Phi_1\), the support preserving isomorphism \(\Phi^e_1\) maps this space isomorphically onto the space of functions in \(\mathcal{H}(T_L, T^o_L, 1)\) supported on \(T^o_L\). It follows that \(\Phi^e_1(\text{Hom}_{J_M}(\lambda_M, \rho)) = \)
Hom_{T_L}(1, 1). Thus, the representation \( \Phi \otimes \varepsilon_{1} \otimes H(T_{L}, T_{L}, 1) \) is a trivial character of \( T_L \). Then an object \( i_{G} \rho \) in \( \mathcal{R}_{\lambda}(G) \) corresponds to an object \( i_{G} \rho \) in \( \mathcal{R}_{1}(G_L) \).

Let \( F \) be the composition of all the top horizontal arrows. Hence the functor \( F : \mathcal{R}_{\lambda}(G) \rightarrow \mathcal{R}_{1}(G_L) \) from above, is an equivalence of categories. Then

\[
\text{Hom}_{G}(c-\text{Ind}_{K}^{G} \sigma, \pi) = \text{Hom}_{G_{L}}(F(c-\text{Ind}_{K}^{G} \sigma), F(\pi))
\]

We know that \( \pi \) is an irreducible subquotient of \( i_{G}(\omega \otimes \chi_{1} \otimes \ldots \otimes \omega \otimes \chi_{e}) \), where \( \chi_{1}, \ldots, \chi_{e} \) are some unramified characters of \( G_0 \) or \( L^{\times} \). Let \( \rho' = (\omega \otimes \chi_{1}) \otimes \ldots \otimes (\omega \otimes \chi_{e}) = \rho \otimes \chi \), where \( \chi \) is an unramified character of \( M \). According to the page 591 of [BK98] the action of \( H(M, J_{M}, \lambda_{M}) \) on \( \text{Hom}_{J_{M}}(\lambda_{M}, \rho') \) is given by:

\[
f.\phi(w) = \int_{M} \rho'(g).\phi(\check{f}(g^{-1})w)dg
\]

where \( f \in H(M, J_{M}, \lambda_{M}) \), \( \phi \in \text{Hom}_{J_{M}}(\lambda_{M}, \rho') \) and \( w \) is a vector in the underlying vector space of \( \lambda_{M} \). We want to understand the compatibility of this action with twisting and support preserving isomorphisms of Hecke algebras. Since \( f \) has a compact support, without loss of generality we may assume that \( f \) is supported on \( J_{M}.m.J_{M} \) for some \( m \in M \). The element \( m \) is block diagonal matrix with \( e \) blocs. Without loss of generality we may assume that each bloc is some power of the uniformizer \( \varpi_{T} \). For convenience we assume that \( \int_{0}^{j_{1}} dj = 1 \), this implies that \( \int_{J_{M}} dj = 1 \) and \( \int_{T_{L}} dt = 1 \). Then,

\[
f.\phi(w) = \int_{J_{M}.m.J_{M}} \rho'(g).\phi(f(g)^{\vee}w)dg
\]

\[
= \int_{j_{1} \in J_{M}} \int_{j_{2} \in J_{M}} \rho'(j_{1}.m.j_{2}).\phi(f(j_{1}.m.j_{2})^{\vee}w)dj_{1}dj_{2}
\]

By definition we have:

\[
f(j_{1}.m.j_{2})^{\vee} = (\check{\lambda}_{M}(j_{1}).f(m).\check{\lambda}_{M}(j_{2}))^{\vee} = \lambda_{M}(j_{2}^{-1}).f(m)^{\vee}.\lambda_{M}(j_{1}^{-1})
\]

Moreover \( \phi \) is \( J_{M} \)-equivariant, thus

\[
\phi(\lambda_{M}(j_{2}^{-1}).f(m)^{\vee}.\lambda_{M}(j_{1}^{-1})) = \rho'(j_{2}^{-1}).\phi(f(m)^{\vee}.\lambda_{M}(j_{1}^{-1}))
\]
This simplifies the integral:

\[ \int_{j_1 \in J_M} \rho(j_1).\rho'(m).\phi(f(m)\chi_M(j_1^{-1})w) dj_1 = \int_{j_1 \in J_M} \rho(j_1).\rho'(m).\phi(f(j_1.m)\chi_M w) dg \]

\[ = \int_{g \in M} 1_{J_M}(g)\rho(g).\rho'(m).\phi(f(g.m)\chi_M w) dg, \]

where \(1_{J_M}\) is the characteristic function of \(J_M\). Each bloc in the matrix \(m\) normalizes \(J_0\), hence \(m\) normalizes \(J_M\). The group \(M\) is reductive, hence unimodular. In the integral above we make a change of variables \(h = m^{-1}g\), this change of variables does not affect the Haar measure \(dg\). We write the integral above as

\[ \int_{h \in M} 1_{J_M}(hm^{-1})\rho(mh^{-1}).\rho'(m).\phi(f(m.h)\chi_M w) dg \]

\[ = \int_{j \in m^{-1}J_M} \rho'(m).\phi(f(m)\chi_M w) dj = \int_{j \in m^{-1}J_M} \rho'(m).\phi(\chi_M(j^{-1})f(m)\chi_M w) dj \]

\[ = \rho'(m).\phi(f(m)\chi_M w) = \chi(m).\rho(m).\phi(f(m)\chi_M w) \]

in order to get the last equality we used \(J_M\)-equivariance of \(\phi\), the fact that \(m\) normalizes \(J_M\) and the normalization of the Haar measure. Observe that for a more general \(f\) we will not get a simple multiplication by a character, but a sum of integrals of the form above. The expression above is compatible with the support preserving isomorphism \(\Phi_1^{\otimes e}\), in a sense that:

\[ \Phi_1^{\otimes e}(\chi(m).\rho(m).\phi(f(m)\chi_M w)) = \chi(m).\Phi_1^{\otimes e}(\phi)(\Phi_1^{\otimes e}(f)(m)\chi_M w), \]

where \(m\) is naturally seen as an element of \(T_L\) because its diagonal blocs are some powers of the uniformizer \(\varpi_T\) and \(\chi\) is seen as unramified character of \(T_L\). This is, of course, compatible with the same computation of the integral replacing \(H(M, J_M, \lambda_M)\) by \(H(T_L, T_L', 1)\) and \(\text{Hom}_{J_M}(\lambda_M, \rho')\) by \(\text{Hom}_{T_L}(1, \chi)\).

Then by the equivalence of categories described above, \(F(\pi)\) is an irreducible subquotient of \(F(\psi^G((\omega \otimes \chi_1) \otimes \ldots (\omega \otimes \chi_e))) = i_{B_L}(\chi_1 \otimes \ldots \otimes \chi_e).\)

Let now \(\Delta = [\omega(\alpha), \omega(\alpha + e - 1)]\), a segment in \(G\), where \(\alpha\) is a real number. Then the commutative diagram above shows that the \(G\)-representation
\[i^G_P(\Delta)\] corresponds to \(G_L\)-representation \(F(i^G_P(\Delta)) = i^{GL}_{BL}(\Delta_L)\), where \(\Delta_L = [1(\alpha), 1(\alpha + e - 1)]\) is a segment in \(G_L\) and 1 is the trivial character of \(L^\times\). We know that \(i^G_P(\Delta)\) admits a unique irreducible quotient \(Q(\Delta)\), so the \(G\)-representation \(Q(\Delta)\) corresponds to the \(G_L\) representation \(F(Q(\Delta)) = Q(\Delta_L)\).

The similar argument works with multiple segments. Therefore \(F(\pi) = Q(\Delta'_1, \ldots, \Delta'_s)\), where \(\Delta'_i = [1(\alpha_i), 1(\alpha_i + e_i - 1)]\) for all \(i\).

Since the isomorphisms of Hecke algebras are support preserving, we also have the following commutative diagram:

\[
\begin{array}{cccccc}
\mathcal{R}_\lambda(G)^{\text{Hom}_J(\lambda, \ast)} & \longrightarrow & \mathcal{H}(G, J, \lambda) - \text{Mod} & \longrightarrow & \mathcal{H}(G_L, I_L, 1) - \text{Mod} & \longrightarrow & \mathcal{R}_1(G_L) \\
c{-\text{Ind}}^G_K & \uparrow & \uparrow & & \uparrow & \uparrow & c{-\text{Ind}}^{GL}_{KL} \\
\mathcal{R}_\lambda(K)^{\text{Hom}_J(\lambda, \ast)} & \longrightarrow & \mathcal{H}(K, J, \lambda) - \text{Mod} & \longrightarrow & \mathcal{H}(K_L, I_L, 1) - \text{Mod} & \longrightarrow & \mathcal{T}_K \mathcal{R}_1(K_L)
\end{array}
\]

where \(T_{KL} = \ast \otimes_{\mathcal{H}(K_L, I_L, 1)} c{-\text{Ind}}^K_{I_L} 1\).

If we denote the composition of all the top horizontal arrow by \(F\) and the composition of all the bottom horizontal arrow by \(F_K\), then \(F(c{-\text{Ind}}^G_K \sigma) = c{-\text{Ind}}^{GL}_{KL} F_K(\sigma)\). The same argument that computes \(F(\pi)\) shows that we also have \(F_K(\sigma) = \sigma_{\text{p-min}}(\text{trivial}) = st\), where \(st\) denotes the inflation of Steinberg representation of \(GL_n\) over a finite field. Alltogether we have:

\[
\text{Hom}_G(c{-\text{Ind}}^G_K \sigma, \pi) = \text{Hom}_{GL}(F(c{-\text{Ind}}^G_K \sigma), F(\pi))
\]

\[
= \text{Hom}_{GL}(c{-\text{Ind}}^{GL}_{KL} F_K(\sigma), Q(\Delta'_1, \ldots, \Delta'_s))
\]

\[
= \text{Hom}_{KL}(F_K(\sigma), Q(\Delta'_1, \ldots, \Delta'_s)|K_L)
\]

Observe that \(Q(\Delta'_1, \ldots, \Delta'_s)\) is generic if and only if \(\pi\) is generic and \(\text{Hom}_{KL}(st, Q(\Delta'_1, \ldots, \Delta'_s)|K_L) \neq 0\) if and only if \(\text{Hom}_{KL}(\sigma, \pi|K) \neq 0\). So we are reduced to consider the case when \((J, \lambda) = (I, 1)\). However this was proven in section 7.2 [Ree02].

2. **Semi-simple type case (general case).** Let now \(\lambda\) be some general semi-simple type. The second part of Main Theorem of section 8 in [BK98] gives a support preserving Hecke algebra isomorphism \(j : \mathcal{H}(\overline{M}, \lambda_M) \rightarrow \mathcal{H}(G, \lambda)(\text{here } \overline{M} \text{ is a unique Levi subgroup of } G \text{ which contains the } N_G(M)\text{-stabilizer of the inertia class } D \text{ and is minimal for this property}), and the section 1.5 gives a tensor product decomposition \(\mathcal{H}(\overline{M}, \lambda_M) = \mathcal{H}_1 \otimes_{\mathcal{Q}_p} \ldots \otimes_{\mathcal{Q}_p}\)
$\mathcal{H}_s$, where $\mathcal{H}_i = \mathcal{H}(G_i, J_i, \lambda_i)$ is an affine Hecke algebras of type A and $(J_i, \lambda_i)$ is some simple type with $G_i$ some general linear group over a $p$-adic field.

Let $M$ be a Levi subgroup of $P = MN$, then $K \cap M = \prod_{i=1}^{s} K_i$, where $K_i$ is a maximal compact subgroup of $i$-th factor in $M$. By definition, see the end of section 6 in [SZ99], the restriction of $K$-representation $\sigma$ to $K \cap N$ is trivial, and $\sigma|K \cap M \simeq \sigma_1 \otimes \ldots \otimes \sigma_s$ where $\sigma_i := \sigma_{P_{\min}}(\lambda_i)$ with obvious notations.

According to Theorem (8.5.1) in [BK93] the irreducible representation $\pi$ is of the form

$$\pi \simeq \pi_1 \times \ldots \times \pi_s,$$

such that $\pi_i$ is irreducible representation of $G_i$ and contains the simple type $(J_i, \lambda_i)$. Moreover the supersupidal support of $\pi_i$ is disjoint from supersupidal support of $\pi_j$ for $i \neq j$. Then

$$\text{Hom}_K(\sigma, \pi) = \text{Hom}_K(\sigma, \text{Ind}^K_{K \cap P}(\pi_1|K_1 \otimes \ldots \otimes \pi_s|K_s))$$

$$= \text{Hom}_{K \cap P}(\sigma_{K \cap M}, \pi_1|K_1 \otimes \ldots \otimes \pi_s|K_s)$$

$$= \text{Hom}_{K \cap M}(\sigma_1 \otimes \ldots \otimes \sigma_s, \pi_1|K_1 \otimes \ldots \otimes \pi_s|K_s)$$

is non zero if and only $\text{Hom}_K(\sigma, \pi)|K_i$ is non zero for all $i$. By simple type case $\pi_i$ is a generalized steinberg representation. It follows that $\pi$ is generic.

**Lemma 2.2.** We have $\dim \text{Hom}_K(\sigma_{\min}(\lambda), \pi) = 1$, for $\pi$ an irreducible generic representation of $G$ in $\Omega$.

**Proof.** Let $x \in m\text{-Spec } \mathfrak{z}_\Omega$ a maximal ideal defined by $\pi$. Since $\pi$ is generic we have that $\text{Hom}_K(\sigma_{\min}(\lambda), \pi) \neq 0$ by Proposition [2.1]. It follows that we have $c\text{–Ind}^G_K \sigma_{\min}(\lambda) \otimes \mathfrak{z}_\Omega \kappa(x) \rightarrow \pi$. Since the functor $\text{Hom}_K(\sigma_{\min}(\lambda), )$ is exact, we have $\text{Hom}_K(\sigma_{\min}(\lambda), c\text{–Ind}^G_K \sigma_{\min}(\lambda) \otimes \mathfrak{z}_\Omega \kappa(x)) \rightarrow \text{Hom}_K(\sigma_{\min}(\lambda), \pi)$. Moreover by Frobenius reciprocity we have that

$$\text{Hom}_K(\sigma_{\min}(\lambda), c\text{–Ind}^G_K \sigma_{\min}(\lambda) \otimes \mathfrak{z}_\Omega \kappa(x))$$

$$= \text{Hom}_G(c\text{–Ind}^G_K \sigma_{\min}(\lambda), c\text{–Ind}^G_K \sigma_{\min}(\lambda) \otimes \mathfrak{z}_\Omega \kappa(x))$$

and then by Lemma 5.2 [Pyv18]:

$$\text{Hom}_K(\sigma_{\min}(\lambda), c\text{–Ind}^G_K \sigma_{\min}(\lambda) \otimes \mathfrak{z}_\Omega \kappa(x))$$
\[ \simeq \text{Hom}_K(\sigma_{\text{min}}(\lambda), c\text{-Ind}_K^G \sigma_{\text{min}}(\lambda)) \otimes \mathbb{Z}_\Omega \kappa(x) \]

Moreover by Corollary 7.2 [Pyv18], \( \text{Hom}_K(\sigma_{\text{min}}(\lambda), c\text{-Ind}_K^G \sigma_{\text{min}}(\lambda)) \simeq \mathbb{Z}_\Omega \). Hence we have a surjective map of \( \kappa(x) \)-vector spaces:

\[ \kappa(x) \twoheadrightarrow \text{Hom}_K(\sigma_{\text{min}}(\lambda), \pi) \]

Then \( 1 \geq \dim \text{Hom}_K(\sigma_{\text{min}}(\lambda), \pi) \) and this space is non-zero, hence it must be one-dimensional.

\[ \square \]

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