Raising/Lowering Maps and Modules for the Quantum Affine Algebra $U_q(\hat{sl}_2)$

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Abstract

Let $V$ denote a finite dimensional vector space over an algebraically closed field. Let $U_0, U_1, \ldots, U_d$ denote a sequence of nonzero subspaces whose direct sum is $V$. Let $R : V \to V$ and $L : V \to V$ denote linear transformations with the following properties: for $0 \leq i \leq d$, $RU_i \subseteq U_{i+1}$ and $LU_i \subseteq U_{i-1}$ where $U_{-1} = 0$, $U_{d+1} = 0$; for $0 \leq i \leq d/2$, the restrictions $R^{d-2i}|U_i : U_i \to U_{d-i}$ and $L^{d-2i}|U_{d-i} : U_{d-i} \to U_i$ are bijections; the maps $R$ and $L$ satisfy the cubic $q$-Serre relations where $q$ is nonzero and not a root of unity. Let $K : V \to V$ denote the linear transformation such that $(K - q^{2i-d}I)U_i = 0$ for $0 \leq i \leq d$. We show that there exists a unique $U_q(\hat{sl}_2)$-module structure on $V$ such that each of $R - e_1^{-1}, L - e_0^{-1}, K - K_0, K^{-1} - K_1$ vanish on $V$, where $e_1^{-1}, e_0^{-1}, K_0, K_1$ are Chevalley generators for $U_q(\hat{sl}_2)$. We determine which $U_q(\hat{sl}_2)$-modules arise from our construction.

1 Introduction

A quantum affine algebra is a $q$-analogue of the universal enveloping algebra of a Kac-Moody Lie algebra of affine type. Quantum affine algebras were first introduced and studied by V.G. Drinfeld [6, 7] and M. Jimbo [12, 13] in relation to the Yang-Baxter equation of mathematical physics. Since then quantum affine algebras have played an important role in various areas of mathematics and physics, for example see [2], [14], [15], [16]. In this paper we will be concerned with the quantum affine algebra $U_q(\hat{sl}_2)$.

We study the finite dimensional modules for $U_q(\hat{sl}_2)$. In [4] V. Chari and A. Pressley classified the finite dimensional irreducible $U_q(\hat{sl}_2)$-modules up to isomorphism. These modules were further studied in [5], [17]. However, a complete classification of all finite dimensional $U_q(\hat{sl}_2)$-modules is still unknown. In the present paper we obtain a result that may help in this classification. We now introduce some notation and recall the definition of $U_q(\hat{sl}_2)$.

**Keywords:** Quantum group, quantum affine algebra, affine Lie algebra $\hat{sl}_2$, raising and lowering maps, tridiagonal pair.

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Throughout this paper $\mathbb{K}$ will denote an algebraically closed field. We fix a nonzero scalar $q \in \mathbb{K}$ that is not a root of unity. We will use the following notation:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n = 0, 1, \ldots$$

**Definition 1.1** [4, Definition 2.2] The quantum affine algebra $U_q(\widehat{sl}_2)$ is the unital associative $\mathbb{K}$-algebra with generators $e_i^\pm$, $K_i^{\pm 1}$, $i \in \{0, 1\}$, which satisfy the following relations:

\begin{align*}
K_i K_i^{-1} &= K_i^{-1} K_i = 1, \quad (1) \\
K_0 K_1 &= K_1 K_0, \quad (2) \\
K_i e_i^\pm K_i^{-1} &= q^{\pm 2} e_i^\pm, \quad (3) \\
K_i e_j^\pm K_i^{-1} &= q^{\mp 2} e_j^\pm, \quad i \neq j, \quad (4) \\
e_i^+ e_i^- - e_i^- e_i^+ &= \frac{K_i - K_i^{-1}}{q - q^{-1}}, \quad (5) \\
e_0^+ e_1^- &= e_1^- e_0^+, \quad (6)
\end{align*}

\[(e_i^\pm)^3 e_j^\mp - [3](e_i^\pm)^2 e_j^\pm e_i^\mp + [3]e_i^\pm e_j^\pm (e_i^\pm)^2 - e_j^\pm (e_i^\pm)^3 = 0, \quad i \neq j. \quad (7)\]

We call $e_i^\pm$, $K_i^{\pm 1}$ the Chevalley generators for $U_q(\widehat{sl}_2)$ and refer to (7) as the cubic $q$-Serre relations.

We now give two definitions, state our main result, and then make some comments concerning its significance.

**Definition 1.2** Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. By a decomposition of $V$ we mean a sequence $U_0, U_1, \ldots, U_d$ consisting of nonzero subspaces of $V$ such that $V = \sum_{i=0}^d U_i$ (direct sum). For notational convenience we set $U_{-1} := 0, U_{d+1} := 0$.

**Definition 1.3** Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $U_0, U_1, \ldots, U_d$ denote a decomposition of $V$. Let $K : V \to V$ denote the linear transformation such that, for $0 \leq i \leq d$, $U_i$ is an eigenspace for $K$ with eigenvalue $q^{2i-d}$. We refer to $K$ as the linear transformation that corresponds to the decomposition $U_0, U_1, \ldots, U_d$.

**Note 1.4** With reference to Definition 1.3 we note that $K$ is invertible. Moreover, for $0 \leq i \leq d$, $U_i$ is the eigenspace for $K^{-1}$ with eigenvalue $q^{d-2i}$. We observe that $K^{-1}$ is the linear transformation that corresponds to the decomposition $U_d, U_{d-1}, \ldots, U_0$.

We will be concerned with the following situation.

**Assumption 1.5** Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $U_0, U_1, \ldots, U_d$ denote a decomposition of $V$. Let $K$ denote the linear transformation that corresponds to $U_0, U_1, \ldots, U_d$ as in Definition 1.3. Let $R : V \to V$ and $L : V \to V$ be linear transformations such that
We now state our main result.

Theorem 1.6  Adopt Assumption 1.5. Then there exists a unique $U_q(\widehat{sl}_2)$-module structure on $V$ such that $(R - e_1^1)V = 0$, $(L - e_0^0)V = 0$, $(K - K_0)V = 0$, $(K^{-1} - K_1)V = 0$, where $e_1^1$, $e_0^0$, $K_0$, $K_1$ are Chevalley generators for $U_q(\widehat{sl}_2)$ as in Definition 1.1.

The proof of Theorem 1.6 will take up most of the paper until Section 9. In Sections 10 and 11 we determine which $U_q(\widehat{sl}_2)$-modules arise from the construction of Theorem 1.6.

Remark 1.7 Not all finite dimensional $U_q(\widehat{sl}_2)$-modules arise from the construction of Theorem 1.6. However as we will see, every finite dimensional $U_q(\widehat{sl}_2)$-module is a direct sum of submodules, each of which arises from Theorem 1.6 up to a routine normalization. Thus Theorem 1.6 can be viewed as a step towards the classification of the finite dimensional $U_q(\widehat{sl}_2)$-modules.

Remark 1.8 The proof of Theorem 1.6 involves modifying a construction used in [3] and [10]. The construction originally arose from the study of tridiagonal pairs. According to [3] Definition 1.1 a tridiagonal pair is an ordered pair $(A, A^*)$ of diagonalizable linear transformations on a finite dimensional vector space $V$ such that (i) the eigenspaces of $A$ (resp. $A^*$) can be ordered as $V_0, V_1, \ldots, V_d$ (resp. $V_0^*, V_1^*, \ldots, V_d^*$) with $A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$ (resp. $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$) for $0 \leq i \leq d$; (ii) there are no nonzero proper subspaces of $V$ which are invariant under both $A$ and $A^*$. See [1], [9], [10] for connections between tridiagonal pairs and $U_q(\widehat{sl}_2)$.

Remark 1.9 In [3] G. Benkart and P. Terwilliger determine the finite dimensional irreducible modules for the standard Borel subalgebra of $U_q(\widehat{sl}_2)$. The authors adopt Assumption 1.5(i),(ii),(v),(vi) but replace Assumption 1.5(iii),(iv) with the assumption that $V$ is irreducible as a $(K, R, L)$-module. From this assumption they obtain a $U_q(\widehat{sl}_2)$-module structure on $V$ as in Theorem 1.6. The $U_q(\widehat{sl}_2)$-module structure that they obtain is irreducible while the $U_q(\widehat{sl}_2)$-module structure given by our Theorem 1.6 is not necessarily irreducible. As far as we know Theorem 1.6 does not imply the result in [3] nor does the result in [3] imply Theorem 1.6.
Remark 1.10 In the proof of Theorem 1.6 we will need a number of lemmas that are similar to lemmas appearing in [3]. The assumptions in Theorem 1.6 and in [3] are similar except that in [3] the authors assume $V$ is irreducible as a $(K, R, L)$-module and we are not making this assumption. However, in [3] this irreducibility condition is not used in the proofs of the lemmas we require. In such cases we simply cite [3] without proof.

2 Preliminaries

In this section we make a few observations about Assumption 1.5.

Note 2.1 With reference to Assumption 1.5, if we replace $U_i$ by $U_{d-i}$ for $0 \leq i \leq d$, and replace $(K, R, L)$ by $(K^{-1}, L, R)$, then the assumption is still satisfied.

Lemma 2.2 With reference to Assumption 1.5 for $0 \leq i \leq d/2$ and $0 \leq j \leq d - 2i$, the restriction $R^j|_{U_i} : U_i \to U_{i+j}$ is an injection.

Proof: Immediate from Assumption 1.5 (iii).

Lemma 2.3 With reference to Assumption 1.5, the following (i),(ii) hold.

(i) $KR = q^2RK$,

(ii) $KL = q^{-2}LK$.

Proof: Immediate from Assumption 1.5 (i),(ii) and Definition 1.3.

3 An outline of the proof for Theorem 1.6

Our proof of Theorem 1.6 will consume most of the paper from Section 4 to Section 9. Here we sketch an overview of the argument.

We begin by adopting Assumption 1.5. To start the construction of the $U_q(\widehat{sl}_2)$-action on $V$ we require that the linear transformations $R - e_1^-, L - e_0^-, K^{\pm 1} - K_0^{\pm 1}$, and $K^{\pm 1} - K_1^{\pm 1}$ vanish on $V$. This gives the actions of the elements $e_1^-, e_0^-, K_0^{\pm 1}, K_1^{\pm 1}$ on $V$. We define the actions of $e_0^+, e_1^+$ on $V$ as follows. First we prove that $K + R$ is diagonalizable on $V$. Then we show that the set of distinct eigenvalues of $K + R$ on $V$ is \{ $q^{2i-d}$ | $0 \leq i \leq d$ \}. For $0 \leq i \leq d$ we let $V_i$ denote the eigenspace of $K + R$ on $V$ associated with the eigenvalue $q^{2i-d}$. So $V_0, V_1, \ldots, V_d$ is a decomposition of $V$. Next we define the subspaces $W_i$ as follows.

$$W_i = (U_0 + \cdots + U_i) \cap (V_0 + \cdots + V_{d-i}) \quad (0 \leq i \leq d).$$

We show that $W_0, W_1, \ldots, W_d$ is a decomposition of $V$. Then we apply Note 2.1 to the above argument to obtain the following results. $K^{-1} + L$ is diagonalizable on $V$. The set of distinct
eigenvalues of $K^{-1} + L$ on $V$ is $\{q^{d-2i} | 0 \leq i \leq d\}$. For $0 \leq i \leq d$ we let $V_i^*$ denote the eigenspace of $K^{-1} + L$ on $V$ associated with the eigenvalue $q^{d-2i}$. So $V_0^*, V_1^*, \ldots, V_d^*$ is a decomposition of $V$. Next we define the subspaces $W_i^*$ as follows.

\[
W_i^* = (U_i + \cdots + U_d) \cap (V_{d-i}^* + \cdots + V_d^*) \quad (0 \leq i \leq d).
\]

Then $W_0^*, W_1^*, \ldots, W_d^*$ is a decomposition of $V$. Next we define the linear transformation $B : V \rightarrow V$ (resp. $B^* : V \rightarrow V$) such that for $0 \leq i \leq d$, $W_i$ (resp. $W_i^*$) is an eigenspace for $B$ (resp. $B^*$) with eigenvalue $q^{2i-d}$ (resp. $q^{d-2i}$). We let $e_1^+$ act on $V$ as $I - K^{-1}B$ times $q^{-1}(q - q^{-1})^{-2}$. We let $e_0^+$ act on $V$ as $I - KB^*$ times $q^{-1}(q - q^{-1})^{-2}$. We display some relations that are satisfied by $B, B^*, L, R, K, K^{-1}$. Using these relations we argue that the above actions of $e_1^\pm, e_1^\pm, K_0^\pm, K_1^\pm$ satisfy the defining relations for $U_q(\widehat{\mathfrak{s}l}_2)$. In this way we obtain the required action of $U_q(\widehat{\mathfrak{s}l}_2)$ on $V$.

## 4 The linear transformation $A$

In this section we define and discuss a linear transformation that will be useful.

**Definition 4.1** With reference to Assumption 1.5 let $A : V \rightarrow V$ denote the following linear transformation:

\[
A = K + R.
\]

**Lemma 4.2** With reference to Definition 4.1 and Assumption 1.5 the following (i),(ii) hold.

(i) For $0 \leq i \leq d$ the action of $A - q^{2i-d}I$ on $U_i$ coincides with the action of $R$ on $U_i$.

(ii) $(A - q^{2i-d}I)U_i \subseteq U_{i+1}, \quad 0 \leq i \leq d$.

**Proof:** Immediate from Definition 4.1 Definition 4.3 and Assumption 1.5(i). \hfill \Box

**Lemma 4.3** [3, Lemma 4.13] With reference to Definition 4.1 and Assumption 1.5 the following holds. $A$ is diagonalizable on $V$ and the set of distinct eigenvalues of $A$ is $\{q^{2i-d} | 0 \leq i \leq d\}$. Moreover, for $0 \leq i \leq d$, the dimension of the eigenspace for $A$ associated with $q^{2i-d}$ is equal to the dimension of $U_i$.

**Definition 4.4** With reference to Definition 4.1 and Lemma 4.3 for $0 \leq i \leq d$ we let $V_i$ denote the eigenspace for $A$ with eigenvalue $q^{2i-d}$. For notational convenience we set $V_{-1} := 0, V_{d+1} := 0$. We observe that $V_0, V_1, \ldots, V_d$ is a decomposition of $V$.

**Lemma 4.5** Let the decomposition $U_0, U_1, \ldots, U_d$ be as in Assumption 1.5 and let the decomposition $V_0, V_1, \ldots, V_d$ be as in Definition 4.4. Then for $0 \leq i \leq d$ the spaces $U_i, U_{d-i}, V_i, V_{d-i}$ all have the same dimension.
Proof: Immediate from Lemma 4.3 and Assumption 1.5 (iii). □

Definition 4.6 With reference to Lemma 4.5 for \(0 \leq i \leq d\) we let \(\rho_i\) denote the common dimension of \(U_i, U_{d-i}, V_i, V_{d-i}\).

Lemma 4.7 With reference to Definition 4.6 the following (i)–(iii) hold.

(i) \(\rho_i \neq 0\), \(0 \leq i \leq d\),

(ii) \(\dim(V) = \sum_{i=0}^{d} \rho_i\),

(iii) \(\rho_i = \rho_{d-i}\), \(0 \leq i \leq d\).

Proof: Immediate by Definition 4.6 and since \(U_0, U_1, \ldots, U_d\) is a decomposition of \(V\). □

Lemma 4.8 [3, Lemma 5.2] With reference to Assumption 1.5 and Definition 4.4

\[ U_i + \cdots + U_d = V_i + \cdots + V_d, \quad 0 \leq i \leq d. \]

Lemma 4.9 [3, Lemma 5.3] With reference to Assumption 1.5 and Definition 4.4

\[ (K^{-1} - q^{d-2i})V_i \subseteq V_{i+1}, \quad 0 \leq i \leq d. \]

5 The subspaces \(W_i\)

Definition 5.1 With reference to Assumption 1.5 and Definition 4.4 we define

\[ W_i = (U_0 + \cdots + U_i) \cap (V_0 + \cdots + V_{d-i}), \quad 0 \leq i \leq d. \]

For notational convenience we set \(W_{-1} := 0, W_{d+1} := 0\).

The goal of this section is to prove the following theorem.

Theorem 5.2 With reference to Definition 5.1 the sequence \(W_0, W_1, \ldots, W_d\) is a decomposition of \(V\).

We prove Theorem 5.2 in three steps. First, we show the sum \(\sum_{i=0}^{d} W_i\) is direct. Second, we show \(V = \sum_{i=0}^{d} W_i\). Finally, we show \(W_i \neq 0\) for \(0 \leq i \leq d\).

The following definition and the next few lemmas will be useful in proving the sum \(\sum_{i=0}^{d} W_i\) is direct.
Definition 5.3 With reference to Assumption 1.3 and Definition 4.4 we define

\[ W(i, j) = \left( \sum_{h=0}^{i} U_h \right) \cap \left( \sum_{h=0}^{j} V_h \right), \quad -1 \leq i, j \leq d + 1. \]

With reference to Definition 5.3, note that \( W(i, d - i) = W_i \) for \( 0 \leq i \leq d \).

Lemma 5.4 With reference to Assumption 1.3, Definition 4.1, and Definition 5.3 the following (i), (ii) hold.

(i) \( (A - q^{2j-d}I)W(i, j) \subseteq W(i + 1, j - 1), \quad 0 \leq i, j \leq d, \)

(ii) \( (K^{-1} - q^{d-2i}I)W(i, j) \subseteq W(i - 1, j + 1), \quad 0 \leq i, j \leq d. \)

Proof: (i) Using Definition 5.3 and Definition 4.4 we have \( (A - q^{2j-d}I)W(i, j) \subseteq \sum_{h=0}^{j-1} V_h. \)

Using Definition 5.3 and Lemma 4.2(ii) we have \( (A - q^{2j-d}I)W(i, j) \subseteq \sum_{h=0}^{i+1} U_h. \)

Combining these facts we obtain the desired result.

(ii) Using Definition 5.3 and Lemma 4.3 we have \( (K^{-1} - q^{d-2i}I)W(i, j) \subseteq \sum_{h=0}^{j+1} V_h. \)

Using Definition 5.3 and Note 1.4 we have \( (K^{-1} - q^{d-2i}I)W(i, j) \subseteq \sum_{h=0}^{i-1} U_h. \)

Combining these facts we obtain the desired result. \( \square \)

Lemma 5.5 With reference to Definition 5.3

\[ W(i, d - 1 - i) = 0, \quad 0 \leq i \leq d - 1. \]

Proof: Define \( T = \sum_{i=0}^{d-1} W(i, d - 1 - i). \) It suffices to show \( T = 0. \) By Lemma 5.4(ii) we find \( K^{-1}T \subseteq T. \) Recall that \( K^{-1} \) is diagonalizable on \( V \) and so \( K^{-1} \) is diagonalizable on \( T. \) Also, \( U_j \cap T (0 \leq j \leq d) \) are the eigenspaces for \( K^{-1}|_T. \) Thus \( T = \sum_{j=0}^{d}(U_j \cap T) \) (direct sum).

Suppose, towards a contradiction, \( T \neq 0. \) Then there exists \( j (0 \leq j \leq d) \) such that \( U_j \cap T \neq 0. \) Define \( t := \min\{ j \mid 0 \leq j \leq d, U_j \cap T \neq 0 \} \) and \( r := \max\{ j \mid 0 \leq j \leq d, U_j \cap T \neq 0 \}. \)

Of course \( t \leq r. \) We will now show

\[ r + t \geq d. \] (9)

If \( d/2 < t \) then (9) holds since \( t \leq r. \) So now assume \( 0 \leq t \leq d/2. \) Let \( x \in U_t \cap T \) such that \( x \neq 0. \) By Assumption 1.3(i), we find \( R^{d-2t}x \in U_{d-t}. \) Also, by Assumption 1.3(iii), we find \( R^{d-2t}x \neq 0. \) By Lemma 5.4(i), we find \( AT \subseteq T. \) Using this and Lemma 1.2(i), we find \( R^{d-2t}x \in T. \) So \( R^{d-2t}x \in U_{d-t} \cap T. \) Combining these facts we find \( U_{d-t} \cap T \neq 0. \) This shows \( 9. \) Define \( y := \max\{ j \mid 0 \leq j \leq d - 1, W(j, d - 1 - j) \neq 0 \}. \) By the definition of \( T \) we find \( T \subseteq U_0 + \cdots + U_y \) and so

\[ y \geq r. \] (10)

We will now show

\[ d - y \geq t + 1. \] (11)
By the definition of \( y \) we have \( W(y, d - 1 - y) \neq 0 \). By Definition 5.3 we have \( W(y, d - 1 - y) \subseteq V_0 + \cdots + V_{d-1-y} \). Using these facts and since \( V_0, V_1, \ldots, V_d \) is a decomposition of \( V \) we find \( W(y, d - 1 - y) \nsubseteq V_{d-y} + \cdots + V_d \). Therefore \( T \nsubseteq V_{d-y} + \cdots + V_d \). Using this and Lemma 4.8 we find \( T \nsubseteq V_{d-y} + \cdots + V_d \). Using this and Lemma 4.8 we find \( T \nsubseteq U_{d-y} + \cdots + U_d \) and (11) follows. Adding (9), (10), and (11) we find \( 0 \geq 1 \) for a contradiction. The result follows.

\[ \square \]

**Lemma 5.6** With reference to Definition 5.1, the sum \( \sum_{i=0}^{d} W_i \) is direct.

**Proof:** It suffices to show \( (W_0 + \cdots + W_{i-1}) \cap W_i = 0 \) for \( 1 \leq i \leq d \). Let \( i \) be given. By Definition 5.1, \( W_0 + \cdots + W_{i-1} \subseteq U_0 + \cdots + U_{i-1} \). Also by Definition 5.1, \( W_i \subseteq V_0 + \cdots + V_{d-i} \). By this and Definition 5.3 we find \( (W_0 + \cdots + W_{i-1}) \cap W_i \subseteq W(i-1, d-i) \). But \( W(i-1, d-i) = 0 \) by Lemma 5.5 and so \( (W_0 + \cdots + W_{i-1}) \cap W_i = 0 \).

The following definition and the next few lemmas will be useful in proving \( V = \sum_{i=0}^{d} W_i \).

**Definition 5.7** With reference to Assumption 1.5, we define

\[ H_i = \{ v \in U_i \mid R^{d-2i+1} v = 0 \}, \quad 0 \leq i \leq d/2. \]

**Lemma 5.8** With reference to Assumption 1.5, Definition 4.4, and Definition 5.7

\[ H_i = (V_i + \cdots + V_{d-i}) \cap U_i, \quad 0 \leq i \leq d/2. \]

**Proof:** Immediate from Definition 4.4, Definition 5.7, and Lemma 4.2(i).

**Lemma 5.9** With reference to Assumption 1.5 and Definition 5.7

\[ U_i = \min(i,d-i) \sum_{j=0}^{\min(i,d-i)} R^{i-j} H_j \quad (\text{direct sum}), \quad 0 \leq i \leq d. \]

**Proof:** Case 1: \( 0 \leq i \leq d/2 \). The proof is by induction on \( i \). Observe the result holds for \( i = 0 \) since \( U_0 = H_0 \) by Definition 5.7 and Assumption 1.5(i). Next assume \( i \geq 1 \). By induction and Lemma 2.2 we find

\[ RU_{i-1} = \sum_{j=0}^{i-1} R^{i-j} H_j \quad (\text{direct sum}). \]  

(12)

We now show

\[ U_i = RU_{i-1} + H_i \quad (\text{direct sum}). \]  

(13)

Using Assumption 1.5(i) and Definition 5.7 we have \( RU_{i-1} + H_i \subseteq U_i \). We now show \( U_i \subseteq RU_{i-1} + H_i \). Let \( x \in U_i \). By Assumption 1.5(i),(iii) there exists \( y \in U_{i-1} \) such that
Using this we find \( x - Ry \in H_i \). So \( x \in RU_{i-1} + H_i \). We have now shown equality in (13). It remains to show that the sum in (13) is direct. To do this we show \( RU_{i-1} \cap H_i = 0 \). Let \( x \in RU_{i-1} \cap H_i \). By Definition 5.7 we have \( R^{d-2i+1}x = 0 \). Also, there exists \( y \in U_{i-1} \) such that \( x = Ry \). Combining these facts with Assumption 1.5(iii) we find \( y = 0 \) and then \( x = 0 \). We have now shown the sum in (13) is direct and this completes the proof of (13). Combining (12) and (13) we find

\[
U_i = \sum_{j=0}^{i} R^{i-j} H_j \quad \text{(direct sum)}.
\]

Case 2: \( d/2 < i \leq d \). This case follows immediately from Case 1 and Assumption 1.5(iii). \( \blacksquare \)

Recall that \( \text{End}(V) \) is the \( K \)-algebra consisting of all linear transformations from \( V \) to \( V \).

**Definition 5.10** With reference to Definition 4.1, let \( D \) denote the \( K \)-subalgebra of \( \text{End}(V) \) generated by \( A \).

We will be concerned with the following subspace of \( V \). With reference to Definition 5.7 and Definition 5.10, for \( 0 \leq i \leq d/2 \) we define

\[
D_{H_i} = \text{span} \left\{ Xh \mid X \in D, h \in H_i \right\}.
\]

**Lemma 5.11** With reference to Assumption 1.5, Definition 5.7, and Definition 5.10,

\[
D_{H_i} = \sum_{j=0}^{d-2i} R^j H_i \quad \text{(direct sum), \quad 0 \leq i \leq d/2.} \tag{14}
\]

**Proof:** Let \( i \) be given. Define \( \Delta = \sum_{j=0}^{d-2i} R^j H_i \). We first show \( D_{H_i} = \Delta \). Recall \( H_i \subseteq U_i \) by Definition 5.7. By this and Lemma 5.2(i) we find \( \Delta \subseteq D_{H_i} \). We now show \( D_{H_i} \subseteq \Delta \). Since \( D \) is generated by \( A \) and since \( \Delta \) contains \( H_i \) it suffices to show that \( \Delta \) is \( A \)-invariant. We now show \( \Delta \) is \( A \)-invariant. For \( 0 \leq j \leq d - 2i \) and \( h \in H_i \) we show \( AR^j h \in \Delta \). Using Assumption 1.5(i) and Lemma 4.2(i) we find \( AR^j h \in R^j H_i + R^{j+1} H_i \). Recall \( R^{d-2i+1} H_i = 0 \) by Definition 5.7. By these comments we find \( AR^j h \in \Delta \). This completes the proof that \( \Delta \) is \( A \)-invariant and it follows \( D_{H_i} \subseteq \Delta \). We have now shown \( D_{H_i} = \Delta \). It remains to show that the sum \( \sum_{j=0}^{d-2i} R^j H_i \) is direct. This follows since \( U_0, U_1, \ldots, U_d \) is a decomposition of \( V \) and since \( R^j H_i \subseteq U_{i+j} \) (\( 0 \leq j \leq d - 2i \)) by Assumption 1.5(i). \( \blacksquare \)

**Lemma 5.12** With reference to Assumption 1.5 and Definition 5.10,

\[
V = \sum_{i=0}^{d/2} D_{H_i} \quad \text{(direct sum)}. \tag{15}
\]

9
Proof: By Lemma 5.9 and since $U_0, U_1, \ldots, U_d$ is a decomposition of $V$,

$$V = \sum_{i=0}^{d} \sum_{j=0}^{\min(i,d-i)} R^{i-j}H_j \quad (direct \ sum).$$

In this sum we interchange the order of summation and find

$$V = \sum_{i=0}^{d/2} \sum_{j=0}^{d-2i} R^i H_j \quad (direct \ sum).$$

The result now follows by Lemma 5.11.

Lemma 5.13 With reference to Definition 5.1

$$V = \sum_{i=0}^{d} W_i.$$

Proof: Define $V' = \sum_{i=0}^{d} W_i$. We show $V = V'$. By construction $V' \subseteq V$. We now show $V \subseteq V'$. For $0 \leq i \leq d$ we set $j = d - i$ in Lemma 5.4(i) and find $(A - q^{d-2i}I)W_i \subseteq W_{i+1}$. Using this we find $AV' \subseteq V'$. By this and Definition 5.1 we find $DV' \subseteq V'$. By Definition 5.1 and Lemma 5.8 we find $H_j \subseteq W_j$ for $0 \leq j \leq d/2$. Therefore $H_j \subseteq V'$ for $0 \leq j \leq d/2$. By these comments we find $DH_j \subseteq V'$ for $0 \leq j \leq d/2$. Now $V \subseteq V'$ in view of Lemma 5.12. We have now shown $V = V'$.

Corollary 5.14 With reference to Assumption 1.5, Definition 4.4 and Definition 5.1, the following (i)–(iii) hold.

(i) $W_0 + \cdots + W_i = U_0 + \cdots + U_i, \quad 0 \leq i \leq d$,

(ii) $W_i + \cdots + W_d = V_0 + \cdots + V_{d-i}, \quad 0 \leq i \leq d$,

(iii) $\dim(W_i) = \rho_i, \quad 0 \leq i \leq d$.

Proof: (i) Let $i$ be given. Define $\Delta = W_0 + \cdots + W_i$ and $\Gamma = U_0 + \cdots + U_i$. We show $\Delta = \Gamma$. By Definition 5.1 we have $\Delta \subseteq \Gamma$. Thus it suffices to show $\dim(\Delta) = \dim(\Gamma)$. By construction $\dim(\Delta) \leq \dim(\Gamma)$. Suppose, towards a contradiction, that $\dim(\Delta) < \dim(\Gamma)$. Using Definition 4.6 Lemma 5.6 and since $U_0, U_1, \ldots, U_d$ is a decomposition of $V$ we find

$$\sum_{h=0}^{i} \dim(W_h) < \sum_{h=0}^{i} \rho_h. \quad (15)$$

10
By Definition 5.1 we have $\sum_{h=i+1}^{d} W_h \subseteq \sum_{h=0}^{d-i-1} V_h$. Using Definition 4.6, Lemma 4.7(iii), Lemma 5.6 and since $V_0, V_1, \ldots, V_d$ is a decomposition of $V$ we find

$$\sum_{h=i+1}^{d} \dim(W_h) \leq \sum_{h=i+1}^{d} \rho_h. \quad (16)$$

By Lemma 5.6 and Lemma 5.13 we find

$$\dim(V) = \sum_{h=0}^{d} \dim(W_h). \quad (17)$$

Adding (15)–(17) we find $\dim(V) < \sum_{h=0}^{d} \rho_h$. This contradicts Lemma 4.7(ii) and the result follows.

(ii) Similar to (i).

(iii) By (i), Lemma 5.6 and since $U_0, U_1, \ldots, U_d$ is a decomposition of $V$ we find $\sum_{h=0}^{i} \dim(W_h) = \sum_{h=0}^{i} \rho_h$ for $0 \leq i \leq d$. The result follows. $\square$

**Corollary 5.15** With reference to Definition 5.1

$W_i \neq 0, \quad 0 \leq i \leq d.$

**Proof:** Immediate from Lemma 4.7(i) and Corollary 5.14(iii). $\square$

Combining Lemma 5.6, Lemma 5.13 and Corollary 5.15 we obtain Theorem 5.2.

### 6 Interchanging $R$ and $L$

In this section we use Note 2.1 to obtain results that are analogous to the results in Sections 5 and 6.

**Definition 6.1** With reference to Assumption 1.5 let $A^* : V \to V$ denote the following linear transformation:

$$A^* = K^{-1} + L.$$

**Lemma 6.2** With reference to Definition 6.1 and Assumption 1.5, the following holds. $A^*$ is diagonalizable on $V$ and the set of distinct eigenvalues of $A^*$ is $\{q^{2i-d} | 0 \leq i \leq d \}$. Moreover, for $0 \leq i \leq d$, the dimension of the eigenspace for $A^*$ associated with $q^{d-2i}$ is equal to the dimension of $U_i$.

**Proof:** Apply Note 2.1 to Lemma 4.3 $\square$
Definition 6.3 With reference to Definition 6.1 and Lemma 6.2, for $0 \leq i \leq d$ we let $V_i^*$ denote the eigenspace for $A^*$ with eigenvalue $q^{d-2i}$. For notational convenience we set $V_{-1}^* := 0, V_{d+1}^* := 0$. We observe that $V_0^*, V_1^*, \ldots, V_d^*$ is a decomposition of $V$.

Definition 6.4 With reference to Assumption 1.5 and Definition 6.3 we define

$$W_i^* = (U_i + \cdots + U_d) \cap (V_{d-i}^* + \cdots + V_d^*), \quad 0 \leq i \leq d.$$ 

For notational convenience we set $W_{-1}^* := 0, W_{d+1}^* := 0$.

Theorem 6.5 With reference to Definition 6.4, the sequence $W_0^*, W_1^*, \ldots, W_d^*$ is a decomposition of $V$.

Proof: Apply Note 2.1 to Theorem 5.2 \hfill \Box

7 The linear transformations $B$ and $B^*$

In this section we introduce the linear transformations $B$, $B^*$ and present a number of relations involving $A$, $A^*$, $B$, $B^*$, $K$, $K^{-1}$.

Definition 7.1 With reference to Definition 5.1 and Definition 6.4 we define the following linear transformations.

(i) Let $B : V \to V$ be the linear transformation such that for $0 \leq i \leq d$, $W_i$ is an eigenspace for $B$ with eigenvalue $q^{2i-d}$.

(ii) Let $B^* : V \to V$ be the linear transformation such that for $0 \leq i \leq d$, $W_i^*$ is an eigenspace for $B^*$ with eigenvalue $q^{d-2i}$.

Lemma 7.2 [3, Lemma 7.2] With reference to Definition 4.1, Definition 6.1, and Definition 7.1

\begin{align*}
\frac{qAB - q^{-1}BA}{q - q^{-1}} &= I, \\
\frac{qA^*B^* - q^{-1}B^*A^*}{q - q^{-1}} &= I, \\
\frac{qBA^* - q^{-1}A^*B}{q - q^{-1}} &= I, \\
\frac{qB^*A - q^{-1}AB^*}{q - q^{-1}} &= I.
\end{align*}

12
Lemma 7.3 3 Lemma 9.1 | With reference to Assumption 1.5 and Definition 7.1
\[
\frac{qBK - q^{-1}K^{-1}B}{q - q^{-1}} = I, \tag{22}
\]
\[
\frac{qB^*K - q^{-1}KB^*}{q - q^{-1}} = I. \tag{23}
\]

Lemma 7.4 3 Lemma 10.1 | With reference to Definition 7.1, the following (i),(ii) hold.

(i) \( B^3B^* - [3]B^2B^*B + [3]BB^*B^2 - B^*B^3 = 0, \)
(ii) \( B^*B - [3]BB^*B^2 + [3]BB^* + BB^3 = 0. \)

8 The proof of Theorem 1.6 (existence)

In this section we prove the existence part of Theorem 1.6.

Definition 8.1 With reference to Assumption 1.5 and Definition 7.1 let \( r : V \to V \) and \( l : V \to V \) denote the following linear transformations:
\[
r = \frac{I - KB^*}{q(q - q^{-1})^2}, \quad l = \frac{I - K^{-1}B}{q(q - q^{-1})^2}.
\]

Lemma 8.2 With reference to Definition 8.1, the following (i),(ii) hold.

(i) \( B = K - q(q - q^{-1})^2 Kl, \)
(ii) \( B^* = K^{-1} - q(q - q^{-1})^2 K^{-1}r. \)

Proof: Immediate from Definition 8.1 \( \square \)

Theorem 8.3 With reference to Assumption 1.5 and Definition 8.1, the following (i)–(ix) hold.

(i) \( KK^{-1} = K^{-1}K = I, \)
(ii) \( KR = q^2RK, \quad KL = q^{-2}LK, \)
(iii) \( Kr = q^2rK, \quad Kl = q^{-2}lK, \)
(iv) \( rR = Rr, \quad lL = Ll, \)
(v) \( lR - Rl = \frac{K^{-1} - K}{q - q^{-1}}, \quad rL - LR = \frac{K - K^{-1}}{q - q^{-1}}, \)
(vi) \( R^3L - [3]R^2LR + [3]RLR^2 - LR^3 = 0, \)

13
\( (vi) \quad L^3R - [3]L^2RL + [3]LRL^2 - RL^3 = 0, \)

\( (vii) \quad r^3l - [3]r^2lr + [3]rlr^2 - lr^3 = 0, \)

\( (viii) \quad l^3r - [3]l^2rl + [3]lrl^2 - rl^3 = 0. \)

**Proof:** (i) Immediate from Note 1.4.
(ii) These equations hold by Lemma 2.3.
(iii) Evaluate Lemma 7.3 using Lemma 8.2.
(iv) Evaluate (20), (21) using Definition 4.1, Definition 6.1, and Lemma 8.2 and simplify the result using (ii), (iii) above.
(v) Evaluate (18), (19) using Definition 4.1, Definition 6.1, and Lemma 8.2 and simplify the result using (ii), (iii) above.
(vi), (vii) These relations hold by Assumption 1.5(v), (vi).
(viii), (ix) Evaluate Lemma 7.4 using Lemma 8.2 and simplify the result using (iii) above. \(\blacksquare\)

**Theorem 8.4** With reference to Assumption 1.5 and Definition 8.1, \(V\) supports a \(U_q(\widehat{\mathfrak{sl}}_2)\)-module structure for which the Chevalley generators act as follows:

| generator | action on \(V\) |
|-----------|------------------|
| \(e_0^-\) | \(L\) |
| \(e_1^-\) | \(R\) |
| \(e_1^+\) | \(l\) |
| \(e_0^+\) | \(K\) |
| \(K_1\) | \(K^{-1}\) |
| \(K_0\) | \(K^{-1}\) |
| \(K_1^{-1}\) | \(K\) |

**Proof:** To see that the above action on \(V\) determines a \(U_q(\widehat{\mathfrak{sl}}_2)\)-module, compare the equations in Theorem 8.3 with the defining relations for \(U_q(\widehat{\mathfrak{sl}}_2)\) in Definition 1.1. \(\blacksquare\)

**Proof of Theorem 1.6 (existence):** The existence part of Theorem 1.6 is immediate from Theorem 8.4. \(\blacksquare\)

Note that the \(U_q(\widehat{\mathfrak{sl}}_2)\)-module structure given by Theorem 1.6 is not necessarily irreducible.

## 9 The proof of Theorem 1.6 (uniqueness)

In this section we prove the uniqueness part of Theorem 1.6.

The quantum algebra \(U_q(\mathfrak{sl}_2)\) and its finite dimensional modules will be useful in proving uniqueness. We begin by recalling the definition of \(U_q(\mathfrak{sl}_2)\).

**Definition 9.1** [11, Definition 1.1] The quantum algebra \(U_q(\mathfrak{sl}_2)\) is the unital associative \(K\)-algebra with generators \(k, k^{-1}, e, f\) which satisfy the following relations:

\[
kk^{-1} = k^{-1}k = 1, \\
k e = q^2ek, \\
k f = q^{-2}fk, \\
e f - fe = \frac{k - k^{-1}}{q^{-1}},
\]

\(14\)
We now recall the finite dimensional irreducible $U_q(sl_2)$-modules.

**Lemma 9.2** [11, Theorem 2.6] With reference to Definition 9.1, there exists a family

$$V_{\epsilon,d}, \quad \epsilon \in \{-1,1\}, \quad d = 0, 1, 2, \ldots$$

of finite dimensional irreducible $U_q(sl_2)$-modules with the following properties. The module $V_{\epsilon,d}$ has a basis $u_0, u_1, \ldots, u_d$ satisfying:

$$ku_i = \epsilon q^{d-2i}u_i, \quad 0 \leq i \leq d, \quad (24)$$

$$fu_i = [i+1]u_{i+1}, \quad 0 \leq i \leq d-1, \quad f u_d = 0, \quad (25)$$

$$eu_i = \epsilon [d-i+1]u_{i-1}, \quad 1 \leq i \leq d, \quad e u_0 = 0. \quad (26)$$

Moreover, every finite dimensional irreducible $U_q(sl_2)$-module is isomorphic to exactly one of the modules $V_{\epsilon,d}$.

**Remark 9.3** If the characteristic $\text{Char}(K) = 2$ then in Lemma 9.2 we view $\{-1,1\}$ as having a single element.

**Lemma 9.4** [11, Proposition 2.3] Let $V$ denote a finite dimensional $U_q(sl_2)$-module. If $\text{Char}(K) \neq 2$ then the action of $k$ on $V$ is diagonalizable.

**Remark 9.5** [11, p. 19] Assume $\text{Char}(K) = 2$. We display a finite dimensional $U_q(sl_2)$-module on which the action of $k$ is not diagonalizable. Let $k, k^{-1}, e, f$ denote the generators for $U_q(sl_2)$ as in Definition 9.1. Let $k$ and $k^{-1}$ act on the vector space $K^2$ as $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $e$ and $f$ act on $K^2$ as $0$. Then $K^2$ is a finite dimensional $U_q(sl_2)$-module and the action of $k$ on $K^2$ is not diagonalizable.

**Lemma 9.6** [11, Theorem 2.9] Let $V$ denote a finite dimensional $U_q(sl_2)$-module. If the action of $k$ on $V$ is diagonalizable then $V$ is the direct sum of irreducible $U_q(sl_2)$-submodules.

**Lemma 9.7** Let $V$ be a finite dimensional $U_q(sl_2)$-module. Assume the action of $k$ on $V$ is diagonalizable. For $\epsilon \in \{1,-1\}$ and for an integer $d \geq 0$ let $v \in V$ denote an eigenvector for $k$ with eigenvalue $\epsilon q^d$. Then $ev = 0$ if and only if $f^{d+1}v = 0$.

**Proof:** Immediate from Lemma 9.2 and Lemma 9.6

**Lemma 9.8** Let $V$ denote a finite dimensional vector space over $K$. Suppose there are two $U_q(sl_2)$-module structures on $V$. Assume the actions of $k$ on $V$ given by the two module structures agree and are diagonalizable. Assume the actions of $f$ on $V$ given by the two module structures agree. Then the actions of $e$ on $V$ given by the two module structures agree.
Proof: Let $E : V \to V$ (resp. $E' : V \to V$) denote the action of $e$ on $V$ given by the first (resp. second) module structure. We show $(E - E')V = 0$. Using Lemma 9.6 and refering to the first module structure we find $V$ is the direct sum of irreducible $U_q(sl_2)$-submodules. Let $W$ be one the irreducible submodules in this sum. It suffices to show $(E - E')W = 0$. By Lemma 9.2, there exists a nonnegative integer $d$ and $\epsilon \in \{1, -1\}$ such that $W$ is isomorphic to $V_{\epsilon,d}$. Therefore, the eigenvalues for $k$ on $W$ are $eq^{d-2i}$ ($0 \leq i \leq d$), and $\dim(W) = d + 1$. Let $u \in W$ be an eigenvector for $k$ with eigenvalue $eq^d$. By Lemma 9.2, the vectors $u, fu, \ldots, f^d u$ are a basis for $W$. We show $(E - E')f^i u = 0$ for $0 \leq i \leq d$. First assume $i = 0$. Using Lemma 9.2 we find $Eu = 0$. Also by Lemma 9.7 we find $f^{d+1} u = 0$ and so $E'u = 0$ by Lemma 9.7. We have now shown $(E - E')u = 0$. Next let $i \geq 1$. By induction on $i$ we may assume

$$(E - E')f^{i-1} u = 0.$$  
(27)

Using Definition 9.1 and since the actions of $k$ (resp. $f$) on $V$ given by the two module structures agree we find $Ef - fE = E'f - fE'$. Hence

$$f(E - E') = (E - E')f.$$  
(28)

Applying $f$ to (27) and using (28) we find $(E - E')f^i u = 0$. This shows $(E - E')W = 0$ and the result follows. \hfill \square

The following lemma relates $U_q(\hat{sl}_2)$-modules to $U_q(sl_2)$-modules.

**Lemma 9.9** Let $V$ denote a finite dimensional $U_q(\hat{sl}_2)$-module. Then for $i \in \{0, 1\}$, $V$ supports a $U_q(sl_2)$-module structure such that $K_i - k, e_i^+ - e$, and $e_i^- - f$ vanish on $V$.

**Proof:** Immediate from Definition 1.1 and Definition 9.1 \hfill \square

**Proof of Theorem 1.6 (uniqueness):** Suppose there exist two $U_q(\hat{sl}_2)$-module structures on $V$ satisfying the conditions of Theorem 1.4. We show that these two module structures agree. By construction the actions of $e_0^-$ (resp. $e_1^+, K_0, K_1$) on $V$ given by the two $U_q(\hat{sl}_2)$-module structures agree. We now show that the actions of $e_0^+$ on $V$ given by the two $U_q(\hat{sl}_2)$-module structures agree. Note that the actions of $K_0$ on $V$ given by the two $U_q(\hat{sl}_2)$-module structures are diagonalizable. Using Lemma 9.9 $V$ supports two $U_q(sl_2)$-module structures given by the two actions of $K_0, e_0^+, e_0^-$ on $V$. Using Lemma 9.8 we find that the actions of $e_0^+$ on $V$ given by the two $U_q(sl_2)$-module structures agree. Similarly, the actions of $e_1^+$ on $V$ given by the two $U_q(\hat{sl}_2)$-module structures agree. We have now shown the two $U_q(sl_2)$-module structures of $V$ agree. \hfill \square
10  Which $U_q(\widehat{\mathfrak{sl}_2})$-modules arise from Theorem 1.6?

Theorem 1.6 gives a way to construct finite dimensional $U_q(\widehat{\mathfrak{sl}_2})$-modules. Not all finite dimensional $U_q(\widehat{\mathfrak{sl}_2})$-modules arise from this construction; in this section we determine which ones do.

**Definition 10.1** Let $V$ denote a finite dimensional $U_q(\widehat{\mathfrak{sl}_2})$-module. Let $d$ denote a non-negative integer. We say $V$ is basic of diameter $d$ whenever there exists a decomposition $U_0, U_1, \ldots, U_d$ of $V$ and linear transformations $R : V \to V$ and $L : V \to V$ satisfying Assumption 1.5 such that the given $U_q(\widehat{\mathfrak{sl}_2})$-module structure on $V$ agrees with the $U_q(\widehat{\mathfrak{sl}_2})$-module structure on $V$ given by Theorem 1.6.

Our goal for this section is to determine which $U_q(\widehat{\mathfrak{sl}_2})$-modules are basic. We begin with a lemma.

**Lemma 10.2** Let $V$ denote a finite dimensional $U_q(\widehat{\mathfrak{sl}_2})$-module. If $\text{Char}(\mathbb{K}) \neq 2$ then the actions of $K_0$ and $K_1$ on $V$ are diagonalizable.

*Proof:* For $i \in \{0, 1\}$, view $V$ as a $U_q(\mathfrak{sl}_2)$-module under the action of $K_i, e_i^+, e_i^-$ as in Lemma 9.3. The result now follows immediately by Lemma 9.3. $\square$

**Remark 10.3** Assume $\text{Char}(\mathbb{K}) = 2$. We display a finite dimensional $U_q(\widehat{\mathfrak{sl}_2})$-module on which the actions of $K_0$ and $K_1$ are not diagonalizable. Let $e_i^\pm, K_i^\pm, i \in \{0, 1\}$, denote the Chevalley generators for $U_q(\widehat{\mathfrak{sl}_2})$ as in Definition 1.1. Let $K_0^\pm$ and $K_1^\pm$ act on the vector space $\mathbb{K}^2$ as $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $e_0^\pm$ and $e_1^\pm$ act on $\mathbb{K}^2$ as 0. Then $\mathbb{K}^2$ is a finite dimensional $U_q(\widehat{\mathfrak{sl}_2})$-module and the actions of $K_0$ and $K_1$ on $\mathbb{K}^2$ are not diagonalizable.

**Theorem 10.4** Let $d$ denote a nonnegative integer and let $V$ denote a finite dimensional $U_q(\widehat{\mathfrak{sl}_2})$-module. With reference to Definition 10.1 the following are equivalent.

(i) $V$ is basic of diameter $d$.

(ii) $(K_0 K_1 - I)V = 0$, the action of $K_0$ on $V$ is diagonalizable, and the set of distinct eigenvalues for $K_0$ on $V$ is $\{q^{2i-d} | 0 \leq i \leq d\}$.

*Proof:* (i) $\Rightarrow$ (ii): Let $U_0, U_1, \ldots, U_d$ be the decomposition of $V$ from Definition 10.1. Let $K : V \to V$ be the linear transformation that corresponds to this decomposition as in Definition 1.3. By Definition 10.1 and Theorem 1.6 we find $K - K_0$ and $K^{-1} - K_1$ vanish on $V$. The result follows.

(ii) $\Rightarrow$ (i): For $0 \leq i \leq d$ let $U_i \subseteq V$ be the eigenspace for the action of $K_0$ on $V$ with eigenvalue $q^{2i-d}$. Note that the sequence $U_0, U_1, \ldots, U_d$ is a decomposition of $V$. Define linear transformations $R : V \to V$ and $L : V \to V$ by $(R - e_1)V = 0$ and $(L - e_0)V = 0$.  

17
We now check that $R$ and $L$ satisfy Assumption 1.5(i)–(vi). Using (3) and (4), it is routine to check that $R$ and $L$ satisfy Assumption 1.5(i),(ii). Next we verify that $R$ satisfies Assumption 1.5(iii). View $V$ as a $U_q(\hat{sl}_2)$-module under the action of $K_1, e_1^+, e_1^-$ as in Lemma 9.3. Since $(K_0 K_1 - I) V = 0$ and the action of $K_0$ on $V$ is diagonalizable we find that the action of $K_1$ on $V$ is diagonalizable. Thus, by Lemma 9.6 $V$ is a direct sum of irreducible $U_q(\hat{sl}_2)$-submodules. Let $W$ be one of the irreducible submodules in this sum. Note that $RW$ agrees with the $V$-module structure on $V$ given by Theorem 1.6. Using the uniqueness statement in Theorem 1.6 it suffices to show each of $R, L$ satisfy Assumption 1.5(iii). The proof that $L$ satisfies Assumption 1.5(iv) is similar. By (7) we find $R$ and $L$ satisfy Assumptions 1.5(v),(vi). We have now shown that $R$ and $L$ satisfy Assumption 1.5(i)–(vi) and so Theorem 1.6 applies. It remains to show that the given $U_q(\hat{sl}_2)$-module structure on $V$ agrees with the $U_q(\hat{sl}_2)$-module structure on $V$ given by Theorem 1.6. Using the uniqueness statement in Theorem 1.6 it suffices to show each of $R - e_1^-, L - e_0^-, K - K_0, K^{-1} - K_1$ vanish on $V$, where $K : V \rightarrow V$ is the linear transformation that corresponds to the decomposition $U_0, U_1, \ldots, U_d$ as in Definition 1.3. The first two equations were mentioned earlier. Using Definition 1.3 we find $(K - K_0) V = 0$. Since $(K_0 K_1 - I) V = 0$ we find $(K^{-1} - K_1) V = 0$. We have now shown that the given $U_q(\hat{sl}_2)$-module structure on $V$ agrees with the $U_q(\hat{sl}_2)$-module structure on $V$ given by Theorem 1.6.

11 The relationship between general $U_q(\hat{sl}_2)$-modules and basic $U_q(\hat{sl}_2)$-modules

Throughout this section $V$ will denote a nonzero finite dimensional $U_q(\hat{sl}_2)$-module (not necessarily irreducible) on which the actions of $K_0$ and $K_1$ are diagonalizable (see Lemma 10.2).

In this section we will show, roughly speaking, that $V$ is made up of basic $U_q(\hat{sl}_2)$-modules. We will use the following definition.

**Definition 11.1** For $\epsilon_0, \epsilon_1 \in \{1, -1\}$ we define

$$V_{\text{even}}^{(\epsilon_0, \epsilon_1)} = \text{span} \{ v \in V \mid K_0 v = \epsilon_0 q^i v, K_1 v = \epsilon_1 q^{-i} v, i \in \mathbb{Z}, i \text{ even} \},$$

$$V_{\text{odd}}^{(\epsilon_0, \epsilon_1)} = \text{span} \{ v \in V \mid K_0 v = \epsilon_0 q^i v, K_1 v = \epsilon_1 q^{-i} v, i \in \mathbb{Z}, i \text{ odd} \}.$$  

**Theorem 11.2** With reference to Definition 11.1,

$$V = \sum_{(\epsilon_0, \epsilon_1)} \sum_{\sigma} V_{\sigma}^{(\epsilon_0, \epsilon_1)} \quad \text{(direct sum of $U_q(\hat{sl}_2)$-modules)}, \quad (29)$$

where the first sum is over all ordered pairs $(\epsilon_0, \epsilon_1)$ with $\epsilon_0, \epsilon_1 \in \{1, -1\}$, and the second sum is over all $\sigma \in \{\text{even, odd}\}$. 

18
Proof: Using (2) and Definition 11.1, we find $V_{e_{n,-1}}^{(e_{0},e_{1})}$ and $V_{e_{n,-1}}^{(e_{0},e_{1})}$ are invariant under the action of $K_{0}$ and $K_{1}$. Using (3) and (4) we find $V_{e_{n,-1}}^{(e_{0},e_{1})}$ and $V_{e_{n,-1}}^{(e_{0},e_{1})}$ are invariant under the action of $e_{0}^{\pm 1}, e_{1}^{\pm 1}$. Thus the subspaces on the right hand side of (29) are $U_{q}(\hat{A}_{2})$-submodules of $V$. We now show the sum on the right hand side of (29) equals $V$. Recall that the actions of $K_{0}$ and $K_{1}$ on $V$ are both diagonalizable. Using this and (2) we find that the actions of $K_{0}$ and $K_{1}$ on $V$ are simultaneously diagonalizable. Thus $V$ is the direct sum of common eigenspaces for the actions of $K_{0}$ and $K_{1}$ on $V$. It remains to show that any common eigenvector for the actions of $K_{0}$ and $K_{1}$ on $V$ is in one of the subspaces on the right hand side of (29). Let $v$ denote a common eigenvector for the action of $K_{0}$ and $K_{1}$ on $V$. By construction there exist $\alpha, \beta \in \mathbb{K}$ such that $K_{0}v = \alpha v$ and $K_{1}v = \beta v$. Using Lemma 9.9, Lemma 9.6, and Lemma 9.2 we find that there exists an $\epsilon_{0} \in \{1, -1\}$ and an integer $i$ such that $\alpha = \epsilon_{0}q^{i}$. For every $m \in \mathbb{Z}$ define $T_{m} := \{ x \in V \mid K_{0}x = \epsilon_{0}q^{i+2m}x \}$ and $K_{1}x = \beta q^{2m}x \}$, and define $T := \sum_{m \in \mathbb{Z}} T_{m}$. Observe $K_{0}K_{1} - \epsilon_{0}q^{i}\beta I$ vanishes on $T$. Using (3) and (4) we find $T$ is a $U_{q}(\hat{A}_{2})$-module. Also, $0 \neq v \in T$ and so $T$ is not the zero module. Let $W$ denote an irreducible $U_{q}(\hat{A}_{2})$-module contained in $T$. By [4, Proposition 3.2], there exists an $\epsilon \in \{1, -1\}$ such that $K_{0}v = \epsilon v$ and $K_{1}v = \epsilon v$. So we find $\epsilon = \epsilon_{0}q^{i}\beta$. Define $\epsilon_{1} = \epsilon_{0}^{-1}$. Then $\epsilon_{1} \in \{1, -1\}$ and $\beta = \epsilon_{1}q^{-i}$. We have now shown $K_{0}v = \epsilon_{0}q^{i}v$ and $K_{1}v = \epsilon_{1}q^{-i}v$. Therefore $v \in V_{e_{n,-1}}^{(e_{0},e_{1})}$ if $i$ is even or $v \in V_{e_{n,-1}}^{(e_{0},e_{1})}$ if $i$ is odd. This shows that the sum on the right hand side of (29) equals $V$. By Definition 11.1 the sum in (29) is direct.

Lemma 11.3 With reference to Definition 11.1, Definition 11.1, and Theorem 11.2, the following are equivalent.

(i) $V = V_{e_{n,-1}}^{(1,1)}$.

(ii) $V$ is basic of even diameter.

(iii) The spaces $V_{e_{n,-1}}^{(1,1)}$, $V_{e_{n,-1}}^{(1,1)}$, $V_{e_{n,-1}}^{(1,1)}$, $V_{e_{n,-1}}^{(1,1)}$, $V_{e_{n,-1}}^{(1,1)}$, $V_{e_{n,-1}}^{(1,1)}$, $V_{e_{n,-1}}^{(1,1)}$, $V_{e_{n,-1}}^{(1,1)}$ are all zero.

Proof: (i)$\Rightarrow$(ii): We use Theorem 10.4. By Definition 11.1 we find $K_{0}K_{1} - I$ vanishes on $V$. Recall that the action of $K_{0}$ on $V$ is diagonalizable. Using Lemma 9.9, Lemma 9.6, and Lemma 9.2 we find that there exists a nonnegative integer $d$ such that the set of distinct eigenvalues for the action of $K_{0}$ on $V$ is $\{q^{2i-d} \mid 0 \leq i \leq d\}$. So by Theorem 10.4 $V$ is basic of diameter $d$. By Definition 11.1 $d$ is even.

(ii)$\Rightarrow$(i): Immediate from Theorem 10.4 and Definition 11.1.

(i)$\Leftrightarrow$(iii): Immediate from Theorem 11.2.

Lemma 11.4 With reference to Definition 11.1, Definition 11.1, and Theorem 11.2, the following are equivalent.

(i) $V = V_{e_{n,-1}}^{(1,1)}$.
(ii) \( V \) is basic of odd diameter.

(iii) The spaces \( V_{\text{odd}}^{(-1,1)}, V_{\text{odd}}^{(-1,1)}, V_{\text{even}}^{(-1,1)}, V_{\text{even}}^{(-1,1)}, V_{\text{even}}^{(-1,1)}, V_{\text{even}}^{(-1,1)} \) are all zero.

Proof: Similar to the proof of Lemma 11.3

Refering to (29), even though the six submodules \( V_{\text{odd}}^{(-1,1)}, V_{\text{even}}^{(-1,1)}, V_{\text{odd}}^{(-1,1)}, V_{\text{even}}^{(-1,1)}, V_{\text{odd}}^{(-1,1)}, V_{\text{even}}^{(-1,1)} \) are not basic they become basic after a routine normalization. This is explained in the following lemma, definition, and remark.

Lemma 11.5 [4, Prop. 3.3] For any choice of scalars \( \epsilon_0, \epsilon_1 \) from \( \{1, -1\} \), there exists a \( \mathbb{K} \)-algebra automorphism of \( U_q(\widehat{\mathfrak{sl}}_2) \) such that

\[
K_i \rightarrow \epsilon_i K_i, \quad e_i^+ \rightarrow \epsilon_i e_i^+, \quad e_i^- \rightarrow \epsilon_i e_i^-,
\]

for \( i \in \{0, 1\} \). We refer to the above automorphism as \( \tau(\epsilon_0, \epsilon_1) \).

Definition 11.6 Let \( W \) denote a \( U_q(\widehat{\mathfrak{sl}}_2) \)-module. Let \( \tau \) be an automorphism of \( U_q(\widehat{\mathfrak{sl}}_2) \). We define a new \( U_q(\widehat{\mathfrak{sl}}_2) \)-module structure on \( W \) as follows. For \( x \in U_q(\widehat{\mathfrak{sl}}_2) \) and \( w \in W \) define \( x.w \) (new action) = \( \tau(x).w \) (original action). We refer to this new \( U_q(\widehat{\mathfrak{sl}}_2) \)-module structure as \( W \) twisted via \( \tau \).

Remark 11.7 With reference to (29) each submodule \( V_{\sigma}^{(\epsilon_0, \epsilon_1)} \) becomes basic upon twisting via \( \tau(\epsilon_0, \epsilon_1) \).

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