Two interacting Hofstadter butterflies

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The problem of two interacting particles in a quasiperiodic potential is addressed. Using analytical and numerical methods, we explore the spectral properties and eigenstates structure from the weak to the strong interaction case. More precisely, a semiclassical approach based on noncommutative geometry techniques permits to understand the intricate structure of such a spectrum. An interaction induced localization effect is furthermore emphasized. We discuss the application of our results on a two-dimensional model of two particles in a uniform magnetic field with on-site interaction.

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I. INTRODUCTION

The study of crystal electrons submitted to a magnetic field has been extensive since the early works of Landau and Peierls. These studies have led to deep insights in the physics of electrons in solids (interpretation of the de Haas van Alphen effect, investigation of the Fermi surface...). The number of contributions on the subject between 1950 and 1970 reveals the importance of this subject between 1950 and 1970 reveals the importance of this field. Twenty years ago, Hofstadter numerically computed the spectrum of the Harper model and discovered its fractal structure as a function of the normalized magnetic flux per lattice cell (Fig. 1).

FIG. 1. Hofstadter’s butterfly for rational values of \( \alpha = p/q \) up to \( q = 29 \). For each value of the magnetic flux \( \alpha = p/q \), we generally have \( q \) bands. Near energies equal to \( \pm 4 \) and zero flux, we observe the emergence of Landau levels.

The problem of a two dimensional electron on a periodic lattice has been of special interest in solid state physics during the last fifteen years: superconducting and normal-metal networks. Harper-like models have been used to describe the quantum Hall effect in organic conductors, in Anyon superconductivity and in flux phases for the Hubbard model.

If the lattice is given by the positions of the ions of a metal, the lattice spacing \( a \) is of the order of 1 Å, so that even with the highest magnetic fields that can be produced now, namely \( B \approx 20T \), we get \( \alpha = \gamma/2\pi \approx 0.5 \times 10^{-4} \) which is fairly small and shows that in this situation a “semiclassical” approximation will always be relevant. As a matter of fact, an effective Planck’s constant denoted by \( \gamma \) proportional to the applied magnetic field naturally appears as an adjustable variable of the problem. Therefore the weak magnetic field limit \( \gamma \to 0 \) corresponds to the semiclassical limit \( \hbar \to 0 \). The corresponding classical phase space at \( B = 0 \) is the quasi-momentum space, namely the Brillouin zone of the corresponding lattice. Topologically it is a 2-torus and the appearance of the magnetic field transforms it into a noncommutative 2-torus.

Whenever \( \gamma = 2\pi p/q \) (\( p, q \in \mathbb{N} \)) the lattice Hamiltonian \( H \) recovers some periodicity and Bloch’s theory applies. We shall see then that \( H \) can be represented as a self-adjoint \( \pi \times \pi \) matrix whose entries are periodic functions of the quasi-momentum components. Thus, if \( \gamma \) is close to any rational multiple of \( 2\pi \), it is possible to compute the spectrum using semiclassical methods.

Based on these remarks, many theoretical and mathematical works were published during the last fifteen years using renormalization group analysis and pseudodifferential operators techniques. On the basis of the techniques of noncommutative geometry, another approach was developed in order to reformulate and extend the semiclassical results. The algebraic semiclassical approach is justified by the simplicity of its application and its efficiency, for example in the computation of Landau levels both in Harper-like models and in a model-Hamiltonian on a triangular lattice. The comparison between semiclassical formulae and exact calculations extracted from the various spectra for \( \gamma \in 2\pi\mathbb{Q} \) gives sur-
prisingly accurate agreement even for relatively large $\gamma$’s (namely $\gamma/2\pi \leq 0.2$).

While in the above formulation of the problem of Bloch electrons in a magnetic field the particles are considered on a two-dimensional lattice, it is possible to map it exactly onto a one-dimensional lattice with quasiperiodic potential. The interesting property of such a lattice is the duality between momentum and spatial coordinates pointed out by Aubry and André. This Aubry duality results in a delocalized structure of the eigenstates characterized by an algebraic decay and a multifractal eigenspectrum. This leads to a quasidiffusive wave packet spreading on such a lattice.

Recently, numbers of authors have followed a new path in the study of the combined effect of interaction and disorder. The a priori simple problem of two interacting particles in a random potential has indeed revealed an unsuspectly large interaction induced delocalization effect. However, the opposite effect has been discovered in the case of two particles in a quasiperiodic potential. In this section we devote the second part of the paper to the study of the two-particle Harper problem with on-site interaction on a one-dimensional lattice. These facts have been firmly established by overconvincing numerical and analytical results. It is one of the purposes of this paper to again express these arguments in more details.

We shall present in this work analytical and numerical results derived from the two-particle Harper problem with on-site interaction on a one-dimensional lattice. More precisely we devote the second section to the presentation of the algebraic semiclassical approach on the non interacting problem $U = 0$. The corresponding spectrum is somehow an intricate superposition of two Hofstadter butterflies. The aim of section 3 is to study the small interaction regime where usual perturbation theory can be applied. The evolution of the spectrum as a function of the strength of the interaction will be presented. After building the analytical framework in section 4, we apply it to the computation of the levels in the strong interaction regime. We show that for very large $U$, the spectrum is divided into two parts: one corresponding to the non interacting case and the second one, looking like a Mathieu spectrum corresponding to localized states strongly influenced by the interaction. Based upon Aubry’s duality, it can be proved that all the wave functions are localized in this regime as far as the Mathieu part of the spectrum is concerned. Finally, we discuss in section 5 the problem of two interacting particles on a two-dimensional lattice submitted to a magnetic flux.

### II. NON INTERACTING MODEL

In his 1930’s study of the electronic diamagnetism of metals, Landau computed the energy spectrum of a free electron subject to a uniform magnetic field. If $B$ is uniform and parallel to one axis, for example axis 3, the kinetic energy is written as:

$$H_L = \hbar^2 \left( \vec{K}_1^2 + \vec{K}_2^2 \right)$$

(1)

with $\vec{K}_\mu = (P\mu - q_e A_\mu)/\hbar$, $\mu = 1, 2$ and $A = (A_1, A_2)$ is the vector potential satisfying $\text{curl}(A) = B$, $q_e$ is the electron charge. Moreover, the quasimomenta $\vec{K}_1, \vec{K}_2$ satisfy

$$[\vec{K}_1, \vec{K}_2] = i q_e B/\hbar$$

(2)

Let us notice that this commutation rule becomes canonical when replacing $\hbar$ by $q_e B/\hbar$. This new effective Planck constant (divided by $2\pi$) is proportional to the magnetic field $B$ and behaves as a varying physical parameter, quite naturally.

The spectrum of $H_L$ is $E_n = E_0 \hbar_{\text{eff}}\omega(2\nu + 1)$ with $E_0 = \hbar^2/2m_e$, $\hbar_{\text{eff}} = q_e B/\hbar$ and $\omega = 1$. Therefore:

$$E_\nu = \hbar \omega_\nu(\nu + 1/2)$$

(3)

where $\omega_\nu = q_e B/m_e$ is the cyclotronic frequency and $\nu$ is the Landau quantum number.

When $B = 0$, the electron energy $E(k)$ for each conduction band is given by Bloch’s theory, where the quasimomentum components $k = (k_1, k_2)$ are defined modulo the reciprocal lattice such that for a simple square lattice in the tight-binding approximation $E(k) = 2E_0 [\cos(k_1a_1) + \cos(k_2a_2)]$ where $a_\mu$ is the vector of the Bravais lattice in the $\mu$-direction. The charge carriers energy is calculated by expanding $E(k)$ near its extremum, denoted by $k_e$, namely:

$$E(k) = E(k_e) + \hbar^2 \left(M^{-1}\right)_{ij} k_i k_j/2 + O(1)$$

(4)

where $M$ stands for the effective mass matrix such that $M^{-1} = D^2 E(k_e)/\hbar^2$.

Thus Landau theory leads to a substitution $k_i \cdot a_i \to \vec{K}_i = \frac{1}{\hbar} (P - q_e A) \cdot a_i$ when an external magnetic field is applied. We have the following commutation rule:

$$[\vec{K}_i, \vec{K}_j] = i q_e B a_i a_j/\hbar = 2i\pi \phi_{ij}/\phi_0 = 2i\pi \alpha = i\gamma$$

(5)

where $\phi_0 = h/q_e$ is the flux quantum, $\phi_{ij}$ is the magnetic flux through the cell generated by $(a_i, a_j)$ and $\alpha = \phi_{ij}/\phi_0$ is the normalized magnetic flux. For a crystal with periodic spacing, the Peierls operator $P(k)$ is represented by an effective Hamiltonian , namely:

$$P(k) = \sum_m h_m(\alpha)e^{im \cdot k}, \ m \in \mathbb{Z}^2$$

(6)

where $h_m(\alpha)$ are smooth functions of $\alpha$. Thus:

$$H_{\text{eff}}(\vec{K}_1, \vec{K}_2) = \sum_m h_m e^{im \cdot \vec{K}}$$

(7)

If several bands intersect the Fermi level, the interband coupling due to the magnetic field is neglected and therefore:
\[ H_{\text{eff}} = 2t \left( \cos \tilde{K}_1 + \cos \tilde{K}_2 \right) \]  

(8)

where \( t \) is physically interpreted as a transfer term corresponding to the required energy for an electron to jump from one site to another (nearest neighbour) of the lattice.

For a wave function \( \psi(n_1, n_2) \) defined on the two-dimensional lattice \( \mathbb{Z}^2 \), the magnetic field effect can be seen through the magnetic translation operators such that:

\[ \langle \mathcal{U}_1 \psi \rangle (n_1, n_2) = e^{-\frac{i}{\hbar} \int f^{(n_1, n_2)}(n_1) A \cdot \vec{d}l} \psi(n_1 - 1, n_2) \]
\[ \langle \mathcal{U}_2 \psi \rangle (n_1, n_2) = e^{-\frac{i}{\hbar} \int f^{(n_1, n_2)}(n_2) A \cdot \vec{d}l} \psi(n_1, n_2 - 1) \]  

(9)

in an appropriate gauge we get:

\[ \langle \mathcal{U}_1 \psi \rangle (n_1, n_2) = \psi(n_1 - 1, n_2) \]
\[ \langle \mathcal{U}_2 \psi \rangle (n_1, n_2) = e^{-i\gamma n_1} \psi(n_1, n_2 - 1) \]  

(10)

Because of the presence of the uniform magnetic field, the magnetic translation operators no longer commute, namely in that case

\[ \mathcal{U}_1 \mathcal{U}_2 = e^{i\gamma} \mathcal{U}_2 \mathcal{U}_1 \]  

(11)

where \( \gamma \) is the normalized magnetic flux per lattice-cell defined by \( \gamma = 2\pi \alpha = 2\pi \phi / \phi_0 \), \( \phi \) being the flux per unit cell and \( \phi_0 = h/\phi_0 \), the flux quantum.

If we set \( \mathcal{U}_1 = \exp(i\tilde{K}_1) \), \( \mathcal{U}_2 = \exp(i\tilde{K}_2) \) using the commutation rule (11), we obtain

\[ [\tilde{K}_1, \tilde{K}_2] = \frac{i q_{B} B_{1} a_{2}}{\hbar} = 2 i \pi \frac{\phi}{\phi_0} = 2 i \pi \alpha = i \gamma \]  

(12)

which corresponds to (5) in the particular case \( i = 1 \) and \( j = 2 \).

Following Harper, the eigenvalue equation is written

\[ E_0 [\psi(n_1 + a, n_2) + \psi(n_1 - a, n_2) + \lambda e^{i q_{B} B_{1} a/\hbar} \psi(n_1, n_2 + a) + \lambda e^{-i q_{B} B_{1} a/\hbar} \psi(n_1, n_2 - a)] = 2E \psi(n_1, n_2) \]  

(13)

\( \lambda \) represents the strength of the quasiperiodic potential.

Let us assume plane-wave behaviour in one direction, i.e. we set \( \psi(n_1, n_2) = \int d\beta e^{i \beta n_2} \phi(n_1) \) since the coefficients in the previous equation only involve \( n_1 \):

\[ \psi(n_1, n_2) = e^{i \beta n_2} \phi(n_1) \]

and the eigenvalue becomes:

\[ \phi(n_1 + 1) + \phi(n_1 - 1) + 2 \lambda \cos(2\pi a n_1 + \beta) \phi(n_1) = \mathcal{E} \phi(n_1) \]  

(14)

where we included the additive energy due to the motion in the field direction in the eigenvalue \( \mathcal{E} \) and where we changed the origin of \( n_1 \).

It is possible to characterize the properties of eigenfunctions from (14) by looking at a special regime, namely \( \lambda \ll 1 \). Therefore, the hopping term is dominant and we can treat the quasiperiodic potential part of the eigenvalue equation as a perturbation. It is then easy to see that the solutions are given for \( \lambda = 0 \) by Bloch waves \( \phi_k(n) = \exp(i k n) \) with an energy \( E = 2 \cos k \). For \( 0 < \lambda \ll 1 \), the perturbation theory allows us to perform an expansion of eigenvalues and eigenstates in \( \lambda \) such that:

\[ E(k) = 2 \cos k + \sum_m \lambda^m \epsilon_m(k) \]
\[ \phi_k(n) = e^{i k n} \left( 1 + \sum_m \lambda^m f_m(\gamma n + \beta) \right) \]
\[ = e^{i k n} u_m(\gamma n + \beta) \]  

(15)

Evaluating the first and second order perturbation theory contributions and replacing the expressions (15) in (14) leads to:

\[ \lambda (u_{m+1} + u_{m-1}) + 2 \cos(\gamma m + k) u_m = E(k) u_m \]  

(16)

The previous equation is known as the “almost Mathieu” eigenvalue equation and the argument above is the Aubry duality between momentum and coordinate representations. As far as spectral properties are concerned one can be easily convinced that dealing with Bloch states means that the states are extended. Thanks to this duality, \( \lambda \leftrightarrow 1/\lambda \) between (14) and (16), it is quite natural to get localized states for the almost Mathieu Hamiltonian at small \( \lambda \)’s. More precisely, it has been proved that the spectrum of the almost Mathieu Hamiltonian is pure-point at small \( \lambda \)’s and for almost all \( \beta \)'s, Conversely if \( \lambda \gg 1 \), the almost Mathieu Hamiltonian has purely continuous spectrum for almost all \( \beta \)’s.

Setting \( t = 1 \) in formula (8) and using the magnetic translation operators \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) defined on the two-dimensional square lattice by (10), the previous Harper equation can be written as the action of an effective Hamiltonian such that:

\[ H_{\text{eff}} = \mathcal{U}_1 + \mathcal{U}_1^{-1} + \mathcal{U}_2 + \mathcal{U}_2^{-1} \]  

(17)

In order to study the two interacting particles model on a quasiperiodic lattice we transform the previous eigenvalue equation (14) into (\( \lambda = 1 \))

\[ [2 \cos(\gamma n_1 + \beta_1) + 2 \cos(\gamma n_2 + \beta_2) + U \delta_{n_1, n_2}] \phi_{n_1, n_2} + \phi_{n_1+1, n_2} + \phi_{n_1-1, n_2} + \phi_{n_1, n_2+1} + \phi_{n_1, n_2-1} = E \phi_{n_1, n_2} \]  

(18)

where \( \beta_{1,2} \) are related to the quasimomentum components of the non interacting case. In the following we shall consider \( \beta_{1,2} = \beta \). Here we chose the form of on-site interaction which only influences the symmetric configurations while the antisymmetric ones remain not affected by \( U \). Due to that, we shall only discuss symmetric configurations in the following.
In the most simple case of non interacting particles ($U = 0$), the spectrum can be computed as before and is shown in Fig. 2.

As we pointed out before, $\gamma = 2\pi \alpha$ appears in our problem as an effective Planck constant since the magnetic translation operators $U_1$ and $U_2$ obey canonical commutation rules in $\gamma$. Therefore, we study the semiclassical limit by letting $\gamma \to 0$. It is also possible to perform calculations near a rational value of the magnetic flux, namely $\gamma'' = \gamma - 2\pi p/q \to 0$. The efficiency and the accuracy of our calculations allow to explain some features of the corresponding spectra.

When $\gamma = 0$, corresponding to $B = 0$, we recover the band function $E(k)$, where $k = (k_1, k_2)$. To study the Landau levels, we expand the classical symbol of the Hamiltonian around an extremum of the band function denoted by $k_c$:

$$\mathcal{H}(k) = \mathcal{H}(k_c) + \frac{1}{2} \partial_\mu \partial_\nu \mathcal{H}(k_c)k_\mu k_\nu + \cdots$$  \hspace{1cm} (19)

The quantization of $\mathcal{H}(k)$ consists in replacing the magnetic translation operators by

$$U_j = \exp(i(k_{cj} + \sqrt{\gamma} K_j))$$, \hspace{1cm} \hspace{1cm} (20)

where $k_{cj}$ are the bottom well coordinates and $K_j$ are operators satisfying Heisenberg’s commutation relations $[K_1, K_2] = i$. The quantized of $\mathcal{H}$, denoted by $H$, is written as:

$$H = \sum_m h(m, \gamma)e^{i(m-k_c+\sqrt{\gamma} m \cdot K)}$$

with $m \cdot K = m_1 K_1 + m_2 K_2$. In the weak field limit, one formally expands $H$ in powers of $\sqrt{\gamma}$:

$$H = \sum_m \left\{ h(m, 0)e^{im \cdot k_c} + i\sqrt{\gamma}h(m, 0)e^{im \cdot k_c} m \cdot K \right\}$$

$$+ \gamma \left[ \frac{\partial h}{\partial \gamma}(m, 0)e^{im \cdot k_c} - \frac{1}{2} h(m, 0)e^{im \cdot k_c} (m \cdot K)^2 \right]$$

$$+ O(\gamma^{3/2})$$  \hspace{1cm} (21)

which we rewrite as:

$$H = \mathcal{H}(k_c, 0) + \gamma \left( \frac{\partial \mathcal{H}}{\partial \gamma}(k_c, 0) - \frac{1}{2} \partial_\mu \partial_\nu \mathcal{H}(k_c, 0)K_\mu K_\nu \right)$$

$$+ O(\gamma^{3/2})$$  \hspace{1cm} (22)

The $\partial \mathcal{H}/\partial \gamma$-term takes into account a possible explicit $\gamma$-dependence of the classical Hamiltonian whereas $\partial_\mu \partial_\nu \mathcal{H}$ represents the inverse effective mass matrix due to the band function curvature. By a unitary transformation, the quadratic term can be written as $\omega (K_1^2 + K_2^2)/2$ where $\omega$ is related to the determinant of the Hessian matrix $\partial_\mu \partial_\nu \mathcal{H}(k_c, 0)$. We recognize here the harmonic oscillator Hamiltonian. For this reason, the energy levels denoted by $E_\nu$ are called “Landau levels” and are equal, to that order in $\gamma$, to $\omega(\nu + 1/2)$ leading to:

$$E_\nu(\gamma) = \mathcal{H}(k_c, 0) + \gamma(2\nu + 1) \left( \frac{\det}{2} D^2 \mathcal{H}(k_c, 0) \right)^{1/2}$$

$$+ \gamma \left( \frac{\partial \mathcal{H}(k_c, 0)}{\partial \gamma} \right) + \cdots + O(\gamma^N)$$ \hspace{1cm} (23)

The formula (23) has been checked numerically on several models. To illustrate it, let us consider the two-particle Harper Hamiltonian on the square lattice (18) near the maximum $k_c = (0, 0)$ of the band function. Using (20) and (22) the quantized Hamiltonian is then expressed as an expansion in powers of $\gamma$:

$$H = 8 - \gamma \left( (K_1^{(1)})^2 + (K_2^{(1)})^2 + (K_1^{(2)})^2 + (K_2^{(2)})^2 \right)$$

$$+ \frac{\gamma^2}{3} \left( (K_1^{(1)})^4 + (K_2^{(1)})^4 + (K_1^{(2)})^4 + (K_2^{(2)})^4 \right)$$

$$+ O(\gamma^3)$$ \hspace{1cm} (24)

where the $K^{(1,2)}$ are quasimoments for particle 1 and 2 respectively. Finally it gives the Landau levels:

$$E_{\nu_1, \nu_2}(\gamma) = 8 - 2\gamma(\nu_1 + \nu_2 + 1)$$

$$+ \gamma^2 \left[ (2\nu_1 + 1)^2 + (2\nu_2 + 1)^2 + 2 \right] / 16$$

$$+ O(\gamma^3)$$ \hspace{1cm} (25)

where $\nu_1$ and $\nu_2$ are the Landau quantum numbers associated with particle 1 and 2 respectively. To check the accuracy of this formula, we compared it to the datas extracted from the numerical spectrum obtained by exact diagonalization. Fig. 3 shows the accuracy of such a semiclassical expansion in the description of the spectrum of the two-particle Harper model when $\gamma \to 0$.  

FIG. 2. Spectrum of the two-particle Harper problem with $U = 0$ obtained for rational values of $\alpha = p/q$ up to $q = 19$. 

Hamiltonian as:

\[ H_{\text{eff}} = 2 \cos(\sqrt{\gamma}K_1) + 2 \cos(\sqrt{\gamma}K_2) \]  

In the semiclassical limit \( \gamma \to 0 \) we expand (26) in a power series around a minimum of potential \( q_N = \pi/\sqrt{\gamma} + 2\pi N/\sqrt{\gamma}, \ N \in \mathbb{Z} \). Keeping only terms up to the second order in \( \gamma \) we end up with a harmonic oscillator. In this approximation and in the continuous case the one-particle wave functions of the unperturbed Hamiltonian are therefore given by:

\[ \psi_\nu(y) = H_\nu\left(\frac{y}{\sqrt{\gamma}}\right) \exp\left(\frac{y^2}{2\gamma}\right) / \sqrt{2^\nu \nu! \sqrt{\pi}} \]  

Here, \( H_\nu(x) \) is a Hermite polynomial, the index \( \nu \) refers to the Landau level, \( y = x - q_N \) in term of the minimum of potential \( q_N \) around which the harmonic approximation has been performed, and \( x \) is the spatial coordinate. This expression is of course valid, provided \( \gamma \) and \( |x - q_N| \ll 1 \), i.e. in the small magnetic field regime, and not too far away from a potential minimum. Extending our expansion to higher powers in \( \gamma \) would allow us to increase the range of validity of this expression. We could indeed write the exact normalized wave functions in an expansion in \( \gamma \) as:

\[ \varphi_\nu(y) = \exp\left(-\frac{y^2}{2\gamma}\right) \left( c_0 H_\nu\left(\frac{y}{\sqrt{\gamma}}\right) + \gamma c_1 H_\nu^{(1)}\left(\frac{y}{\sqrt{\gamma}}\right) + \ldots \right) \]  

For the purpose of discretization, we introduce a continuous variable \( \xi \in \mathbb{R} \) labeling the well, and a discrete one \( l \in \mathbb{Z} \) numbering the sites. Then \( y = \xi - l\sqrt{\gamma} \) since in the chosen representation, the intersite spacing is \( a = \sqrt{\gamma} \). The set \{\( \varphi_\nu \)\} builds a quasiorthogonal basis in the sense that for \( \xi \neq \xi' \), due to the Gaussian envelope of the states we have:

\[ \sum_l \varphi_\nu(\xi - l\sqrt{\gamma})\varphi_{\nu'}(\xi' - l\sqrt{\gamma}) = O(\exp(-1/\gamma))\delta_{\mu,\nu} \]  

These functions are periodic in \( \xi \) with period \( 1/\sqrt{\gamma} \). In the semiclassical limit the norm of \( \varphi_\nu \) is:

\[ \|\varphi_\nu\|^2 = \sum_{l=-\infty}^{\infty} |\varphi_\nu(\xi - l\sqrt{\gamma})|^2 = 1/\sqrt{\gamma} \int dy|\varphi_\nu(y)|^2 = 1/\sqrt{\gamma} \]  

Consequently, to get normalized one-particle wave functions on the discrete lattice \( l(\mathbb{Z}) \) we must multiply the \( \varphi \)'s by a factor \( \gamma^{1/4} \). We thus can write the symmetrized two-particle unperturbed wave functions as:

\[ \phi^{\xi,\xi'}_{\nu,\mu}(l, l') = \sqrt{2} \varphi_\nu(\xi - l\sqrt{\gamma})\varphi_{\mu}(\xi' - l'\sqrt{\gamma}) \pm \varphi_{\mu}(\xi - l\sqrt{\gamma})\varphi_\nu(\xi' - l'\sqrt{\gamma})(1 - \delta_{\mu,\nu}(1 - 1/\sqrt{2})) \]  

We are now able to compute the first-order correction for the energy. Because of the exponentially localized character of (28), two particles located on different wells

FIG. 3. Comparison between semiclassical calculations (25) (full curves) and exact numerical spectrum (points) for Landau sublevels in the two-particle Harper model on the square lattice when \( U = 0 \). Datas are extracted in the region of energies corresponding to the maximum \((0, 0)\) of the band function.

III. WEAK INTERACTION REGIME

We present here a simple perturbative treatment that enables to implement the already presented results for the weakly interacting case. The first-order contribution allows to understand the splitting of Landau bands at sufficiently weak interaction, and describes it qualitatively well. It moreover enlightens the mechanism through which interaction affects the system. Using the representation defined by (20), we write the unperturbed Hamiltonian as:

\[ H_{\text{eff}} = 2 \cos(\sqrt{\gamma}K_1) + 2 \cos(\sqrt{\gamma}K_2) \]  

In the semiclassical limit \( \gamma \to 0 \) we expand (26) in a power series around a minimum of potential \( q_N = \pi/\sqrt{\gamma} + 2\pi N/\sqrt{\gamma}, \ N \in \mathbb{Z} \). Keeping only terms up to
have only an exponentially small overlap, and as a consequence do practically not interact. Therefore the first-order interaction induced correction to the energy is non zero only for symmetric wave functions with \( \xi = \xi' \). We have:

\[
\Delta E^{(1)} = U \sum_l \sum_{l'} \left( \phi_{\nu, \mu}^l (l, l') \right)^2 \delta (\xi - \xi' + (l' - l) \sqrt{\gamma})
\]

\[
= U \delta_{\xi, \xi'} \int dy (\varphi_{\mu}(y) \varphi_{\nu}(y))^2 (2 - \delta_{\mu, \nu}) + \text{O}(\exp(-1/\gamma))
\]  

(32)

From (28), the dominant term in the last integral is of order \( O(\sqrt{\gamma}) \) so that we finally have:

\[
\Delta E^{(1)} \sim U \delta_{\xi, \xi'} \sqrt{\gamma}
\]  

(33)

The numerical factor can be estimated from the harmonic approximation (27) which leads to:

\[
\Delta E^{(1)} = U \delta_{\xi, \xi'} \sqrt{\frac{\gamma}{2\pi}} = U \delta_{\xi, \xi'} \sqrt{\alpha}
\]  

(34)

for states with Landau quantum numbers (0,0) and (0,1). This result shows that the interaction primarily acts on two-particle states with high double-site occupancy. In what follows we shall call such states “pair states”. States for which the particles are located around different potential minima practically do not feel the interaction. Therefore, switching on the interaction does not modify most of the spectrum as can be seen on Fig. 4.

From (25) and (34) and for small enough interaction, the shifted part of the spectrum is given by:

\[
E_{\nu_1, \nu_2}(\gamma) \sim 8 + U \sqrt{\frac{\gamma}{2\pi}} - 2\gamma(\nu_1 + \nu_2 + 1)
\]

\[
+ \gamma^2 \left( (2\nu_1 + 1)^2 + (2\nu_2 + 1)^2 + 2 \right) / 16
\]  

(35)

The amazing agreement between the numerically computed spectrum obtained by exact Lanczos diagonalization and (35) is shown in Fig. 5 where \( U = 0.4 \). It is a confirmation of our reasoning: pair states form the shifted part of the spectrum. Because these states are much fewer than states where particles are located in different wells, the shifted spectrum is much less dense. In this sense the interaction splits the butterfly into two parts. One of them is practically not affected by the interaction and corresponds to the states where particles are far from each other. The second one is shifted and relays to the situation where particles form pair states. Here, the interaction results in a global shift of the spectrum. In this way, new states appear in the initial gaps of the non interacting spectrum (see Figs. 4, 6 and Fig. 1(b) in [2]). Direct analysis of eigenstates shows that the corresponding states are exponentially localized [4]. We shall come back to this point later on for the case of strong interaction.

**FIG. 4.** Spectrum of the two-particle Harper model with on-site interaction at \( U = 0.4 \) up to \( q = 23 \).
IV. STRONG INTERACTION REGIME

The strongly interacting regime needs a special treatment quite analogous to the one presented in section 2. As we will see, Schur’s complement formula can be successfully applied to construct an effective Hamiltonian. The latter is then expanded in a power series in $\gamma$ to deliver highly accurate formulae. From the weakly interacting regime we learned that particles located on different potential minima do not feel each other: for such pairs, the interaction is suppressed by an exponentially small term of order $O(U \exp(-1/\gamma))$. Therefore, this picture remains valid even for large $U$’s, the relevant parameter being the magnetic flux. Pair states on the other hand undergo an energy increase of order $\Delta E \approx U$. Therefore when the strength of the interaction $U > 0$ increases, one part of the spectrum is almost not affected. Another spectral structure appears, initially looking like a shifted butterfly (see Fig. 4 where $U = 0.4$), then evolving to a shifted Mathieu spectrum as the interaction grows bigger and bigger (see Figs. 6, 7 and 8 where $U = 5, 10$ and 20 respectively).
FIG. 7. Spectrum of the two-particle Harper model in the strongly interacting regime $U = 10$ up to $q = 23$.

In this section, we present an analytical approach that allows to understand completely the mechanism driving this evolution of the spectrum. Further details like the splitting of the Landau band $\nu_1 = 0, \nu_2 = 1$ will also be computed, even though the physics is there less transparent (see Fig. 7). We shall concentrate our semiclassical calculation near the band function maximum $k_c = (0,0)$ corresponding to the energy $z \approx U + 4$ in the spectrum. The two-particle Hamiltonian can be expressed in the following way:

$$H_{\text{TIP}} = \sum_{m,n} [2 \cos (\gamma m + \beta) + 2 \cos (\gamma n + \beta)]$$

$$|m \otimes n\rangle \langle m \otimes n| + U \sum_m |m \otimes m\rangle \langle m \otimes m|$$

$$+ \sum_{m \neq n} |m \otimes n\rangle [\langle m \otimes n + 1| + \langle m \otimes n - 1|$$

$$+ \langle m + 1 \otimes n| + \langle m - 1 \otimes n|]$$

The strategy is based on the so-called Schur complement formula. Our Hamiltonian $H_{\text{TIP}}$ is a self-adjoint operator acting on a Hilbert space that can be decomposed as
\( \mathcal{H} = \mathcal{P} \oplus \mathcal{Q} \). Let \( P \) and \( Q \) be the orthogonal projections on each subspace of that decomposition, namely:

\[
P = \sum_m |m \otimes m\rangle \langle m \otimes m|
\]

\[
Q = I - P = \sum_{m \in \mathcal{Q}} |m \otimes n\rangle \langle m \otimes n|
\]

In other words, \( P \) is the eigenprojection on pair states and \( Q \) is its orthogonal. If \( z \) is an eigenvalue of \( H_{\text{TIP}} \) and does not belong to the spectrum of \( QH_{\text{TIP}}Q \) then it is also an eigenvalue of the following effective Hamiltonian

\[
H_{\text{TIP}}^{\text{eff}}(z) = PH_{\text{TIP}}P + PH_{\text{TIP}}Q \frac{1}{z - QH_{\text{TIP}}Q} QH_{\text{TIP}}P
\]

(37)

When \( U \) is large the dominant term in the effective Hamiltonian given by the Schur complement formula (37) corresponds to the pair states. The semiclassical approach we introduced in section 2 remains valid so that \( H_{\text{TIP}}^{\text{eff}}(z) = H_{\text{TIP}}^{\text{eff}}(z_0 + \gamma z_1 + \gamma^2 z_2 + O(\gamma^3)) \). The implicit equation to be solved is then:

\[
H_{\text{TIP}}^{\text{eff}}(z) = z_0 + \gamma z_1 + \gamma^2 z_2 + O(\gamma^3)
\]

(38)

with

\[
H_{\text{TIP}}^{\text{eff}}(z) = H_{\text{TIP}}^{(0)}(z) + \gamma H_{\text{TIP}}^{(1)}(z) + \gamma^2 H_{\text{TIP}}^{(2)}(z) + O(\gamma^3)
\]

(39)

The expansion of the dominant term reads:

\[
P H_{\text{TIP}}P = U + 4 \cos(\sqrt{7}K_2) = U + 4 - 2\gamma K_2^2
\]

(40)

and if we consider \( U \) large, \( z \) is large too so that:

\[
\frac{1}{z - QH_{\text{TIP}}Q} = \frac{1}{z} + \frac{QH_{\text{TIP}}Q + QH_{\text{TIP}}QHQ_{\text{TIP}}Q}{z^2} + O(z^{-4})
\]

(41)

Expressing the different contributions in Schur’s formula and expanding in powers of \( \gamma \) lead to:

\[
H_{\text{TIP}}^{(0)}(z) = 4 + U + \frac{8}{z} + \frac{32}{z^2} + \frac{176}{z^3} + O(z^{-4})
\]

\[
H_{\text{TIP}}^{(1)}(z) = -2(\frac{z^3 + 8z + 64}{z^3}) \left[ K_2^2 + \frac{z^2 + 4z + 34}{z^3 + 8z + 64} K_1^2 \right]
\]

\[
H_{\text{TIP}}^{(2)}(z) = \frac{z^3 + 8z + 256}{z^3} \left[ K_2^4 + \frac{z^2 + 4z + 70}{z^3 + 8z + 256} K_1^4 \right]
\]

\[
+ 2 \frac{z + 8}{z^3} (K_2^2 K_2^2 + K_2^2 K_1^2) - 8 \frac{z + 8}{z^3}
\]

\[
+ 16 \frac{(z + 8)^2}{z^3(z^3 + 8z + 64)} (K_2^2 + K_2^2)
\]

(42)

Finally, we have to solve (38) to get the coefficients \( z_0, z_1 \) and \( z_2 \). The corresponding equations for those coefficients are at most of degree four. We shall give here the equation that \( z_0 \) has to satisfy at the order \( O(z^{-4}) \)

\[
4 + U + \frac{8}{z_0} + \frac{32}{z_0^2} + \frac{176}{z_0^3} = z_0
\]

(43)

In a very similar way used for the computation of \( z_0 \), the analytical expressions of \( z_1 \) and \( z_2 \) can be derived from (38), (42) and (43). The good agreement with the exact numerical spectrum can be seen on Fig. 9 for \( U = 50 \). Here the numerical values for the sublevels are:

- for \( \nu_{1,2} = 0 \), \( z(\gamma) = 54.1597 - 0.2826\gamma + 0.0356\gamma^2 \),
- for \( \nu_{1,2} = (0,1) \), \( z(\gamma) = 54.1597 - 0.8480\gamma + 0.2084\gamma^2 \),
- for \( \nu_{1,2} = (1,1) \), \( z(\gamma) = 54.1597 - 1.4133\gamma + 0.5539\gamma^2 \).

A similar computation can be done near the band function minimum \( k_c = (\pi, \pi) \) corresponding to the energy \( z \approx U - 4 \).

FIG. 9. Comparison between semiclassical calculations (full curves, see text) and exact numerical spectrum (points) for levels in the two-particle Harper model for \( U = 50 \).

The structure of the pair states for \( U \gg 1 \) can be understood in the following way: the diagonal term corresponding to the energy of particles located on the same
site is $4\lambda \cos(\gamma n + \beta) + U$. The transition amplitude on the diagonal $n_{1,2} = n$ is given by the amplitude of the hopping via virtual states with $n_1 - n_2 = \pm 1$ and energy denominator $1/U$. There are two such paths so that the effective amplitude is $V_{\text{eff}} = 2/U$. The same expression can be derived by the Schur formalism (see Sec. IV).

After dividing the Hamiltonian by $V_{\text{eff}}$ we arrive to the eigenfunctions equation in the form of Harper (14) with $\lambda$ replaced by $\lambda_{\text{eff}} = U \gg 1$. Since $\lambda_{\text{eff}} \gg 1$ when $U \gg 1$, the pair states are always within the localized phase of the Harper equation showing exponential localization. In Fig. 10, we show a typical eigenstate of the Mathieu part of the spectrum for $U = 50$ and $\gamma/2\pi = 34/55$. The fact that it is localized confirms the pure-point character of the corresponding spectrum.

Above we showed that in the case of strong interaction, we have $\lambda_{\text{eff}} \gg 1$. This explains the appearance of a pure-point component in the spectrum. However, we think that this pure-point component will even appear for small values of the interaction. Our argument is the following: without interaction, the system obeys Aubry’s duality while the presence of the interaction introduces Aubry’s duality breaking. Indeed, from (18) it is easy to see that the interaction acts in the coordinate space and the symmetry with momentum space disappears when $U \neq 0$. Formally, this argument is not sufficient to prove the existence of pure-point spectrum at arbitrary small $U$. However, the ensemble of numerical datas we have here and in confirms this conjecture.

When $U$ is large, the unshifted part of the spectrum looks very much like the spectrum at $U = 0$. The main difference can be found by looking carefully at the Landau levels (see Fig. 11). The reminiscence of the existence of the interaction is seen through the appearance of a splitting of Landau sublevels. This splitting only exists when Landau quantum numbers are different $\nu_1 \neq \nu_2$ and the two particles are located in the same well. Such a behaviour is illustrated by Fig. 11. The other sublevels are described by the semiclassical formulae obtained in the case $U = 0$ (25). To derive this splitting using semiclassical analysis, we again apply the Schur complement formula. Dealing with the unshifted butterfly leads us to consider as the dominant term $QH_{\text{TIP}}Q$ such that (37) becomes:

$$H_{\text{TIP}}^\text{eff}(z) = QH_{\text{TIP}}Q + QH_{\text{TIP}}P \frac{1}{z - PH_{\text{TIP}}P} PH_{\text{TIP}}Q$$

(44)

Applying the same scheme as before produces an additional shift from the unperturbed energy given in first order in $\gamma$ by:

$$|\delta E(\gamma)| = 4\frac{\gamma}{U + 4}$$

(45)

This shift is valid for the second Landau sublevel ($\nu_1 = 0, \nu_2 = 1$), its accuracy is shown in Fig. 11 and the two splitted subbands are given by: $E(\gamma) = 8 - 4.1666\gamma$ and $E(\gamma) = 8 - 4\gamma$ up to order 1 in $\gamma$.

**FIG. 10.** Semilogplot of $W = |\phi(l, l')|^2$ for a localized state $(E=50.25,-30 \leq \ln W \leq -1)$.

**FIG. 11.** Semiclassical calculations (45) (full curve) and exact numerical spectrum (points) for the splitting of the $\nu_1 = 0, \nu_2 = 1$ Landau sublevel in the two-particle Harper model for $U = 20$. 

10
V. TWO INTERACTING PARTICLES ON A TWO-DIMENSIONAL LATTICE

Even though the studied model was derived from a model of two-dimensional electrons, its effective dimension is 1: as we already pointed out, (18) was derived assuming that the particle propagate as plane-wave in one direction. This assumption, though reasonable in the one-particle model, could be violated by interaction induced quantum interferences in the two-particle case. Therefore the question of the survival of interaction induced localization effect for two interacting particles in two dimensions remains an open problem. In this section we would like to discuss briefly this situation. For two interacting particles moving on a two-dimensional square lattice submitted to a uniform magnetic flux, the eigenvalue equation reads:

\[ e^{i\gamma y_1}\psi_{x_1+1,y_1,x_2,y_2} + e^{-i\gamma y_1}\psi_{x_1-1,y_1,x_2,y_2} + \psi_{x_1,y_1+1,x_2,y_2} + \psi_{x_1,y_1-1,x_2,y_2} + e^{i\gamma y_2}\psi_{x_1,y_1,x_2+1,y_2} + e^{-i\gamma y_2}\psi_{x_1,y_1,x_2-1,y_2} + \psi_{x_1+1,y_1,x_2,y_2+1} + \psi_{x_1+1,y_1,x_2,y_2-1} + \hat{U}\delta_{x_1,x_2}\delta_{y_1,y_2}\psi_{x_1,y_1,x_2,y_2} = E\psi_{x_1,y_1,x_2,y_2} \]  

(46)

where \((x_1, y_1, x_2, y_2)\) are integers denoting the positions on the square lattice and \(\hat{U}\) is the on-site interparticle interaction. For \(\hat{U} = 0\), the previous equation can be reduced to the one-dimensional Harper equation we discussed above (18). With interaction, the same equation (18) can be obtained in the ansatz of plane waves propagating in one direction with renormalized interaction \(\hat{U}\). While this plane wave approximation is a standard approach for the one-particle Harper problem, it has to be handled with care in the interacting case. Indeed this plane wave ansatz breaks the symmetry of the original problem (46). This symmetry can be seen in the limit of strong interaction \(U \gg 1\). In this case, there should be two energy bands: one corresponding to the pair states when particles are located on the same site with energy \(E \approx U\) and the other with \(E \approx 1\) for the states in which the two particles avoid each other. In the higher energy band, the eigenvalue equation for the pair states up to the terms of order \(1/U\) has the form:

\[ \frac{2}{U} \left( e^{2\gamma y}\phi_{x+1,y} + e^{-2\gamma y}\phi_{x-1,y} + \phi_{x,y+1} + \phi_{x,y-1} \right) + U\phi_{x,y} = E\phi_{x,y} \]  

(47)

Here the term \(2/U\) represents the transition amplitude for pair states. Its derivation is similar to the case of two interacting particles in the one-dimensional Harper model. Indeed if one keeps \(x_1 = x_2\) then the hopping term is given by \(V_{\text{eff}} = 2/U\) because there are two paths with virtual energy \(U(y_{1,2} \rightarrow y_{1,2} + 1)\) which contribute to the hopping term in the \(y\)-direction. Similarly the hopping in the \(x\)-direction is \(V_{\text{eff}} = 2e^{\pm 2\gamma y}/U\).

This representation shows that the symmetry between the two directions or the Aubry duality is not broken by the interaction. The main reason is that the symmetry of the interaction is invariant under rotations on the square lattice. In the limit of large \(U\), this property can be seen through equation (47). However the symmetry (Aubry’s duality) should also be preserved for small interaction. Due to that, we expect that similarly to the Harper model with \(\lambda = 1\), the interaction will not generate pure-point component in the spectrum. However this conjecture has to be directly checked in further analytical and numerical studies.

VI. CONCLUSIONS

In this paper we have emphasized a localizing effect due to the combined action of an on-site interaction and a quasiperiodic potential. Unlike in the random potential case, extended unperturbed states are localized by the interaction, and this localization occurs at arbitrarily small attractive/repulsive interaction. We successfully identified the mechanism responsible for this effect as a decoupling of a Mathieu-like model from the original two-particle Harper model in the limit of large interaction. Our conjecture is that a similar mechanism will also work for small \(U\) due to an interaction induced breaking of Aubry’s duality. This breaking happens in one-dimensional incommensurate models, however in two-dimensional magnetic models, we expect that the interaction will not break the duality and that a pure-point component in the spectrum will not arise. Further verifications of these conjectures are required.

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1 L.D. Landau, Z. für Phys. 64, 629 (1930).
2 R.E. Peierls, Z. für Phys. 80, 763 (1933).
3 W.J. de Haas, P.M. van Alphen, Communications from the Physical Laboratory of the University of Leiden 208d (1930).
4 P.G. Harper, Proc. Phys. Soc. Lond. A 68, 874 (1955); ibid 879.
5 D.R. Hofstadter, Phys. Rev. B 14, 2239 (1976).
6 B. Pannetier, J. Chaussy, R. Rammal, J.-C. Villegier, Phys. Rev. Lett. 53, 1845 (1984).
7 B. Pannetier, J. Chaussy, R. Rammal, Phys. Scr. 13, 245 (1986).
8 M. Kohmoto, L. Kadanoff, C. Tang, Phys. Rev. Lett. 50, 1870 (1983).
9 Y.H. Chen, F. Wilczek, E. Witten, B.I. Halperin, Int. J. Mod. Phys. B 3, 1001 (1989).
10 R. Rammal, J. Bellissard, Europhys. Lett. 13, 205 (1990).
11 J. Bellissard, in Operator Algebras and Application, Vol. 2, D.E. Evans & M. Takesaki Eds., Cambridge University Press (1988).
12 M. Wilkinson, Proc. Roy. Soc. Lond. A 391, 305 (1984).
13 B. Helffer, J. Sjöstrand, Supplément au Bulletin de la Société Mathématique de France, Tome 116, Fasc. 4, Mémoire 34 (1988).
14 R. Rammal, J. Bellissard, J. Phys. France 51, 1803 (1990).
15 A. Barelli, R. Fleckinger, Phys. Rev. B 46, 11559 (1992).
16 J. Bellissard, C. Kreft, R. Seiler, J. Phys. A 24, 2329 (1991).
17 S. Aubry, G. André, Ann. Isr. Phys. Soc. 3, 133 (1979).
18 T. Geisel, R. Ketzmerick and G. Petschel, Phys. Rev. Lett. 66, 1651 (1991); ibid 67, 3635 (1991); ibid 69, 695 (1992).
19 M. Wilkinson, E.J. Austin, Phys. Rev. B 50, 1420 (1994).
20 D.L. Shepelyansky, Phys. Rev. Lett. 73, 2607 (1994); Y. Imry, Europhys. Lett. 30, 405 (1995).
21 A. Barelli, J. Bellissard, P. Jacquod, D.L. Shepelyansky, to appear in Phys. Rev. Lett. (1996).
22 D.L. Shepelyansky, to appear in Phys. Rev. B (1996).
23 J. Bellissard, R. Lima, D. Testard, Commun. Math. Phys. 88, 207 (1983); Ya.G. Sinai, J. Stat. Phys. 46, 861 (1987); V. Chulaevsky, Ya.G. Sinai, Commun. Math. Phys. 125, 91 (1989); J. Fröhlich, T. Spencer, P. Wittwer, Commun. Math. Phys. 132, 5 (1990).
24 W. Chojnacki, Commun. Math. Phys. 143, 527 (1992).
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