Modified Congruence Modulo \( n \) with Half The Amount of Residues

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Abstract

We define a new congruence relation on the set of integers, leading to a group similar to the multiplicative group of integers modulo \( n \). It makes use of a symmetry almost omnipresent in modular multiplications and halves the number of residue classes. Using it, we are able to give an elegant description of some results due to Carl Schick, others are reduced to well-known theorems from algebra and number theory. Many concepts from number theory such as quadratic residues and primitive roots are equally applicable. It brings noticeable advantages in studying powers of odd primes, and in particular when studying semiprimes composed of a pair of related primes, e.g. a pair of twin primes. Artin’s primitive root conjecture can be formulated in the new context. Trigonometric polynomials based on chords and related to the new congruence relation lead to new insights into the minimal polynomials of \( 2 \cos(2\pi/n) \) and their relation to cyclotomic polynomials.

1 Introduction

The motivation for this paper comes from the work of Carl Schick \([8]\ [9]\ [10]\). In 2003, Schick found a recurrence relation \([8]\) that yields, for every odd
natural number \( n \), a characteristic cyclic sequence of positive and negative odd integers.

The terms \((q_i)_{i \in \mathbb{N}}\) of this sequence are given by:

\[
q_i = n - 2|q_{i-1}| \text{ with } q_1 = (-1)^{(n+1)/2}.
\]

The following modified recurrence relation yields the absolute value of these terms:

\[
q_i = |n - 2q_{i-1}| \text{ with } q_1 = 1.
\]

In this paper we interpret his findings in a broader context. By introducing a new congruence relation, denoted \(\text{mod}^*\), new insights into the work of Schick and others are gained. Note that the proposed congruence modulo \( n \) requires \( n \) to be odd.

Whereas the standard congruence relation ("mod \( n \)"") yields a least residue system that can be represented by even and odd nonnegative integers smaller than \( n \), the proposed relation leads to a system whose elements can be represented by retaining only odd (or even) representatives. Hence, the size of the residue system is exactly halved. Well known concepts from number theory such as "quadratic residue", "multiplicative group" and "primitive root" can be adapted.

For a particular type of composite numbers, \(\text{mod}^*\) leads to a multiplicative cyclic group. As a result, new insights are gained for Sophie Germain pairs and twin primes.

The paper is organized as follows: First, we recapitulate the standard knowledge in the context of \(\text{mod}^*\). Then we define \(\text{mod}^*\) and adapt well-known concepts to it. In particular, subsection 3.7 adapts Artin's primitive root conjecture in the context of \(\text{mod}^*\) to powers of primes and some composite numbers. Finally, section 4 gives a geometric interpretation of the congruence relation and closely related polynomials are constructed as an application.

## 2 Preliminaries

This section introduces the necessary notation (partially taken from Wikipedia [14]) and summarizes a number of well-known concepts from number theory.
2.1 Multiplicative Group of Integers Modulo $n$

In number theory, the multiplicative group of integers modulo $n$ is well known and often described as follows. The quotient ring $\mathbb{Z}/n\mathbb{Z}$ is defined by the following congruence relation on the ring of integers $\mathbb{Z}$:

$$a \equiv b \pmod{n}$$

$$\downarrow$$

$$a - b \in n\mathbb{Z}$$

where $n\mathbb{Z}$ is the ideal generated by $n$. We denote the group of units (invertible elements) of $\mathbb{Z}/n\mathbb{Z}$ by $(\mathbb{Z}/n\mathbb{Z})^\times$. For simplicity, we also refer to this group by $G_n$.

If $n$ is a power of an odd prime ($n = p^k$), then there is the isomorphism

$$(\mathbb{Z}/p^k\mathbb{Z})^\times \cong C_{\varphi(p^k)},$$

where $C_m$ is the cyclic group of order $m$ and $\varphi$ is Euler’s totient function.

In general, if $n$ is a composite number $n = p_1^{k_1}p_2^{k_2}\cdots p_l^{k_l}$, then the group of units is isomorphic to the direct product of cyclic groups

$$(\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/p_1^{k_1}\mathbb{Z})^\times \times (\mathbb{Z}/p_2^{k_2}\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_l^{k_l}\mathbb{Z})^\times \cong C_{m_1} \times C_{m_2} \times \cdots \times C_{m_l},$$

where $m_i$ equals $\varphi(p_i^{k_i})$.

The order $\lambda(n)$ of the largest cyclic subgroup of the group $G_n$ is given by Carmichael’s function

$$\lambda(n) = \text{lcm}(m_1, m_2, \ldots, m_l).$$

This means that given $n$ and $a^{\lambda(n)} \equiv 1 \pmod{n}$ for any $a$ coprime to $n$, then $\lambda(n)$ is the smallest such exponent.

The order of the group $G_n$ is $|G_n| = \varphi(n)$. If $G_n$ is cyclic, its generators are called primitive roots modulo $n$. Gauss [2] showed that $G_n$ is cyclic (has primitive roots), if and only if $n$ is one of

$$n = 2, 4, p^k \text{ or } 2p^k,$$

where $p$ is an odd prime.
2.2 Artin’s Primitive Root Conjecture

In 1927, Artin formulated his primitive root conjecture \[1\] \[7\]. It states that a given integer \(a\) which is not a perfect square and not \(-1, 0\) or \(1\), is a primitive root modulo infinitely many primes \(p\). If \(N_a(x)\) denotes the number of such primes up to \(x\) for a given integer \(a\), he conjectured an asymptotic formula of the form

\[ N_a(x) \sim A_a \frac{x}{\ln x}, \]

as \(x \to \infty\). For the density of primitive roots \(A_a\) he calculated \(A_{\text{Artin}} \approx 0.3739\), independent of \(a\).

It was later found that \(A_a\) depends on \(a\) (see e.g. Lenstra et al. \[4\]) and it was proven by Heath-Brown \[3\] that one of 2, 3, 5 is a primitive root modulo infinitely many primes.

3 The Congruence Relation mod*

3.1 Definition

In this subsection, the new congruence relation is defined. Using this relation, the number of residue classes is halved compared to the canonical congruence relation defined in equation 3. It is as versatile as mod for multiplication, but it destroys the additive structure of the ring \(\mathbb{Z}\). In section 4, though, an alternate addition operation based on representatives will be defined.

**Definition 1.** Let \(n\) be an odd natural number, and let \(a, b \in \mathbb{Z}\) and coprime to \(n\). Then we define a congruence relation with respect to multiplication as follows:

\[ a \equiv b \pmod* n \]

\[ \uparrow \]

\[ a - b \in n\mathbb{Z} \text{ or } a + b \in n\mathbb{Z}. \]

If we define multiplication of congruence classes as \([a][b] = [ab]\), then we obtain the group \(G'_n\).

Clearly, the relation in definition 1 is an equivalence relation on the set of integers coprime to \(n\): it is reflective, symmetric and transitive. Furthermore, it is compatible with multiplication and therefore a congruence
relation. Note that, due to the loss of the additive structure, this does not define a new quotient ring similar to \( \mathbb{Z}/n\mathbb{Z} \). Rather, it should be interpreted as a “compression” of the equivalence classes in \( G_n \).

With each equivalence class in \( G_n^* \), we can associate a positive representative smaller than \( n \). These representatives form the reduced residue system \( \text{mod}^* n \). For example, if \( n = 9 \), we get the residue system \( \{1, 5, 7\} \) for \( \text{mod}^* \) as opposed to \( \{1, 2, 4, 5, 7, 8\} \) for mod. The representative \( R(a) \) of an integer \( a \) in the reduced residue system mod \( n \), can easily be computed as

\[
R(a) = \begin{cases} 
    a & \text{if } a \text{ is odd,} \\
    n-a & \text{if } a \text{ is even.}
\end{cases}
\]  

(4)

Of course, one could swap “even” and “odd” in the above to obtain only even representatives. For computations, it is often useful to freely use representatives, and obtain the odd (or even) representative in the final step. For example, one would naturally prefer 2 over \( n - 2 \).

Due to the fact that numbers and their additive inverses are considered equivalent, we have the following equality regarding the order of \( G_n^* \):

\[
| G_n^* | = \frac{| G_n |}{2} = \frac{\varphi(n)}{2}
\]

3.2 Alternate Congruence Relation

Throughout the rest of this paper, the following lemma will be useful to answer questions about quadratic residues and primitive roots of \( G_n^* \), if \( n \) is a power of an odd prime. It is essentially a way of converting the congruence relation from definition 1 to the canonical modular congruence relation.

**Lemma 1.** Let \( n \) be a power of an odd prime and let \( a, b \in \mathbb{Z} \) coprime to \( n \). Then the following property holds:

\[
a \equiv b \pmod{n} \quad \Downarrow \quad a^2 \equiv b^2 \pmod{n}
\]

**Proof.** Note that \( a^2 - b^2 = (a - b)(a + b) \). If \( n \) is prime, the result follows from the zero product property in the field \( \mathbb{Z}/n\mathbb{Z} \) and definition 1. If \( n = p^k \), with \( k > 1 \) and \( p \) an odd prime, the property would not hold if both \( a - b \)
and $a + b$ are multiples of a power of $p$ (both are nonzero, since they are coprime to $n$). This implies that $p \mid (a - b)$ and $p \mid (a + b)$, thus $p \mid 2a$, which contradicts the assumption that $a$ is coprime to $n$. \hfill \Box

From the definition of $G_n^*$, it can be seen that

$$G_n^* \cong G_n / \langle -1 \rangle,$$

where $\langle -1 \rangle \subset G_n$ denotes the subgroup generated by $-1$.

### 3.3 Relations to Schick’s recurrence relation

The recurrence relation given in equation 2 satisfies

$$q_i \equiv 2^i \pmod{n^*}.$$

Let $\langle 2 \rangle \subseteq G_n^*$ denote the cyclic subgroup generated by 2, then the representatives of the elements of this subgroup correspond to the sequence defined above. Specifically, the sequence consists of the representatives from equation 4 of $\langle 2 \rangle = \langle n - 2 \rangle$, ordered by increasing value of the exponent $i$. Schick denotes the order of $\langle 2 \rangle$ using pes$(n)$. It is the period of the sequence [1] or equivalently [2].

A generalization of Schick’s recurrence relation is possible by the following definition.

**Definition 2.** Let $n$ be an odd natural number and $g$ a positive integer less than and coprime to $n$. Then identical sequences can be generated by

$$q_i \equiv g^i \pmod{n^*}$$

or — where the absolute value is taken $g - 1$ times — by

$$q_{i+1} = |n - |n - \cdots |n - gg_i| \cdots || : q_0 = 1$$

The terms of this sequence correspond to the cyclic subgroup $\langle g \rangle \subseteq G_n^*$. An explicit, non recursive, form of any sequence of the above form can thus be obtained by using mod*. This allows, for example, fast calculation of the terms in such a sequence.
3.4 Applying mod* to Prime Powers

As mentioned in the previous section, applying mod* to an odd number \( n \), the number of congruence classes is halved. This leads to simplifications shown first for powers of an odd prime. Begin by noting that the order of \( G_n^* \) with \( n = p^k \) is given by

\[
| G_n^* | = \frac{\varphi(n)}{2} = \frac{(p-1)p^{k-1}}{2}.
\]

For a prime power \( n \), \( G_n \) is a cyclic group. Below, we show that \( G_n^* \) is also a cyclic group in this case.

**Theorem 1.** Let \( n = p^k \) be the power of an odd prime, then \( G_n^* \) is a cyclic group of order \( \lambda(n)/2 = \varphi(n)/2 \).

**Proof.** By the definition of the Carmichael function, we have for all \( a \in \mathbb{Z} \) coprime to \( n \):

\[
a^{\lambda(n)} \equiv 1 \pmod{n}
\]

This is equivalent to (by lemma 1):

\[
a^{\lambda(n)/2} \equiv 1 \pmod{\text{mod}^* n},
\]

where \( \lambda(n)/2 \) is the smallest such exponent. \( \square \)

Gauss showed that for \( n = p^k \), there are \( \varphi(\varphi(n)) \) primitive roots. In the context of mod*, \( g \) is considered equivalent to its additive inverse \( n-g \). The number of primitive roots is thus \( \varphi(\varphi(n)/2) \), the value of which depends on the parity of \( \varphi(n)/2 \).

**Theorem 2.** Let \( n \) be the power of an odd prime \( n = p^k \), then the average density of primitive roots in \( G_n^* \) is 50% higher than in \( G_n \).

**Proof.** Recall that \( |G_n| = \varphi(n) \) and \( |G_n^*| = \varphi(n)/2 \), thus

\[
\frac{\varphi(|G_n^*|)}{|G_n^*|} = \begin{cases} 2 \frac{\varphi(|G_n|)}{|G_n|} & \text{for } p = 4k + 1, \\ \varphi(|G_n|)/|G_n| & \text{for } p = 4k + 3. \end{cases}
\]

Assuming equal frequencies of the two forms of \( p \) leads in the average to

\[
\frac{\varphi(|G_n^*|)}{|G_n^*|} = f \frac{\varphi(|G_n|)}{|G_n|} : \bar{f} = 1.5.
\]

\( \square \)
Artin’s primitive root conjecture may be adapted to mod*. It then states that any integer \( a > 1 \) which is not a perfect square, is a primitive root mod* infinitely many primes \( p \) and that the density of primitive roots converges to a constant as the number of such primes approaches infinity. For \( a = 2 \) in the context of mod*, Schick found a density of primitive roots \( A_2 \approx 0.561 \approx 1.5 A_{Artin} \), calculated for the primes up to 2,000,000.\(^9\)

3.5 Applying mod* to Odd Composite Numbers

Each odd composite number, that is not a prime power, is composed of at least two distinct odd primes. First, we consider the case where exactly two distinct primes are involved.

Let \( n = pq \) be composed of two distinct odd primes with \( \gcd(p-1, q-1) = 2 \). Then — by switching from mod to mod* — the factor \( C_2 \) is cancelled from the following direct product:

\[
G_n \cong C_2 \times C_{\varphi(n)/2} \\
\downarrow \\
G^*_n \cong C_{\varphi(n)/2},
\]

which shows that \( G^*_n \) is cyclic. A more general definition follows below. It is well known since Gauss, that \( G_n \) is not cyclic for \( n \) composed of two or more odd primes.

Note that for odd composite numbers, the Carmichael function \( \lambda(n) \) is the same for \( G_n \) and \( G^*_n \), in contrast to the situation for prime powers (subsection 3.4).

Definition 3. To distinguish between different cases we define the number

\[
j = \frac{\varphi(n)}{\lambda(n)}
\]

The case \( j = 1 \) is found only for prime powers \( n = p^k \). \( G^*_n \) is a cyclic group of order \( \lambda(n)/2 \).

The case \( j = 2 \) is the main interest of this paper. It is found for odd composite numbers of the form \( n = p_1^{k_1} p_2^{k_2} \) with \( \gcd(\varphi(p_1^{k_1}), \varphi(p_2^{k_2})) = 2 \). \( G^*_n \) is a cyclic group of order \( \lambda(n) = \varphi(n)/2 \).

Cases \( j \geq 4 \) do not bring any noticeable advantages of mod* with respect to the standard mod.
3.6 Cyclicity of $G^*_n$

The well known theorem of Gauss \[2\] that $G_n$ is cyclic exactly for the four forms $n = 2, 4, p^k, 2p^k$ (with $p$ an odd prime and $k$ a natural number), shall now be adapted to $\mod^*$.

**Theorem 3.** Let $G^*_n$ be defined as above (definition \[7\]), then the multiplicative group $G^*_n$ is cyclic (has primitive roots) if and only if $n$ is one of the following:

$$n = \begin{cases} p^k & \text{or} \\ p^kq^l & \text{with } \gcd(\varphi(p^k), \varphi(q^l)) = 2, \end{cases}$$

with $p$ and $q$ distinct odd primes and $k$ and $l$ positive integers.

**Proof.** The two cases follow directly from the definitions of $\varphi(n)$ and $\lambda(n)$ and correspond to the cases $j = 1$ and $j = 2$ (definition \[3\]). \hfill \Box

**Definition 4.** Let $p$ and $q$ be distinct odd primes and $k$ and $l$ positive integers with $\gcd((p^k-1)(q^l-1)) = 2$. Then we denote the product of odd primes $n = p^kq^l$ a cyclic semiprime.

3.7 Cyclic Semiprimes and Artin’s Conjecture

The product of two twin primes, of a pair of Sophie-Germain primes and of many other pairs of primes are cyclic semiprimes.

We demonstrate the case of Sophie Germain prime pairs: it is well known that they are of the form $p_1 = 6k - 1$ and $p_2 = 12k - 1$ with $k \in \mathbb{N}$. (To eliminate in advance all $p_i$ divisible by 5, one can additionally require that $k \equiv 0, 2$ or 4 (mod 5) and so on for 7, 11 . . .)

Because the group $G^*_n$ for $n_{SG} = (6k-1)(12k-1)$ is cyclic in the context of $\mod^*$, we can adapt Artin’s primitive root conjecture and state — if it holds — that a given prime $b$ is a primitive root $\mod^*$ infinitely many cyclic semiprimes $n_{SG}$ of Sophie Germain pairs and that the density of primitive roots approaches a constant value as the number of such pairs approaches infinity.

If $N_b(x)$ denotes the number of Sophie Germain primes less than $x$ for which $b$ is a primitive root (mod*$n_{SG}$), then an asymptotic formula is conjectured of the form

$$N_b(x) \sim A_b \int_2^x \frac{dx}{\ln(x)\ln(2x+1)},$$

9
as $x$ approaches infinity.

Heuristically, the density of primitive roots $A_b$ was calculated to be in the interval $(0.28, 0.47)$ for $x = 10,000,000$ and primes $b < 20$.

### 3.8 Quadratic Residues and their Roots

A difficult problem in number theory, is to find the root of a quadratic residue. Lagrange and Legendre found solutions for specific cases. In general, one has to use algorithms, such as that of Müller [5] or Tonelli-Shanks [15]. Using $\text{mod}^*$, we found a closed-form solution one "level" higher than using the standard modulo.

Level 1: Let $n = p^k$ be an odd prime power with $\varphi(n)/2$ odd. Then every element $b \in G^*_n$ is a quadratic residue, and

\[
x \equiv b^{(\varphi(n)/2+1)/2} \pmod{\star n}
\]

is a root of $b$, a solution of the equation $x^2 \equiv b \pmod{\star n}$. To prove equation 5, square it and remember, that $\varphi(n)/2$ is the size of the cyclic group $G^*_n$. $x$ is itself a quadratic residue. (In the context of the standard modulo the second square root of $b$ would be $n - x$.)

Level 2: Let $n = p^k$ be an odd prime power or $n = p^kq^l$ a cyclic semiprime with $j = 2$, with in either case $\varphi(n)/4$ odd. Then $G^*_n$ contains the subset (50%) of all quadratic residues and the coset (50%) of the quadratic non-residues. The subset is a cyclic group of size $\varphi(n)/4$. Let $b$ be a quadratic residue, then

\[
x \equiv b^{(\varphi(n)/4+1)/2} \pmod{\star n}
\]

is a root of $b$. The proof is similar as above for level 1. $x$ is itself a quadratic residue. Multiplying the set of the quadratic residues by a primitive root yields the coset, which is not a cyclic group and which contains all primitive roots and a second root of $b$. (Equation 6 leads with the standard mod only for a few $b$-values to the correct answer.) Examples for level 2 are $n = 13, 77, 605$ and some products of Sophie Germain pairs.

Level 3: Let $n = p^k$ be an odd prime power or $n = p^kq^l$ a cyclic semiprime with $j = 2$, with in either case $\varphi(n)/8$ odd. Then $G^*_n$ contains a subset (25%) of all biquadratic residues, the coset (25%) of the pure quadratic residues, and the cocoset (50%) of all quadratic non-residues. The biquadratic residues are a cyclic group of size $\varphi(n)/8$. Let $b$ be a biquadratic residue, then

\[
x \equiv b^{(\varphi(n)/8+1)/2} \pmod{\star n}
\]
is a root of \( b \). \( x \) is itself a biquadratic residue. Multiplying the subset of the biquadratic residues by an appropriate element of the pure quadratic residues yields the coset, and multiplying the subset united with the coset by a primitive root yields the cocoset. Example for level 3 are \( n = 41, 143 \).

Level 3 yields interesting results. Each squaring halves the number of elements. With the standard modulo, the first squaring divides the number of elements by 4 (a combination of uniting \( a \) with \( n - a \), \( b \) with \( n - b \) and squaring \( a \) and \( b \)). But the overall picture is similar. Therefore, the real advantage of mod* lays in level 2.

One could save one loop in the Tonelli-Shanks algorithm [15] by using mod*, but the starting quadratic residue would have to be odd. It could be an advantage e.g. in the quadratic sieve algorithm [6].

4 Polynomials Related to Odd Integers

We follow some ideas used by Gauss [2] and consider here regular polygons with an odd number of corners. The diagonals of regular polygons are chords of the circumscribed circle. The goal of this section is to describe well known polynomials associated with odd integers in the language of mod* and chords.

4.1 Chords Instead of Trigonometric Functions

Let \( n \) be an odd positive integer. The corners of a regular \( n \)-gon inscribed in a unit circle are the solutions of the equation \( x^n - 1 = 0 \). A single root \( x = 1 \) is real, the others are the roots of unity \( \xi^k \in \mathbb{C} \) where \( \xi = e^{2\pi i/n} (i^2 = -1) \). The polynomial \( x^n - 1 \) is reducible over the real numbers:

\[
x^n - 1 = (x - 1) \sum_{k=0}^{n-1} x^k.
\]

Thus, for the first primitive root of unity \( \xi \):

\[
0 = \sum_{k=0}^{n-1} \xi^k = 1 + \sum_{k=1}^{(n-1)/2} (\xi^k + \xi^{n-k}) = 1 + \sum_{k=1}^{(n-1)/2} 2 \cos \left( \frac{2\pi k}{n} \right) \tag{7}
\]

The terms \( 2 \cos(2\pi k/n) \) can be interpreted as diagonals of a regular \( 2n \)-gon or chords of its circumscribed unit circle, as will be shown in the next subsection.
Up to $k = \lfloor (n - 1)/4 \rfloor$ the sign of terms is positive, it is negative for the rest. The total of all terms is the well known Gauss sum $-1$.

The following definition gives alternate expressions for the chords in equation $7$. This will lead to an easy-to-manipulate representative set of chords.

**Definition 5.** Let $n$ be a positive odd integer. Then a complete representative set of chords related to the number $n$ is defined by

$$s_j = \begin{cases} 2 \cos \left(\pi \frac{j}{n}\right) & \text{if } j \text{ is even} \\ (-1)^{(n-j)/2} 2 \sin \left(\frac{\pi j}{2n}\right) & \text{otherwise} \end{cases}$$

The expression for odd $j$ can be deduced from the equality $\sin(\alpha) = \cos(\pi/2 - \alpha)$.

The periodicity of trigonometric functions and the symmetry between sine and cosine — mirrored at $\pi/4$ — leads to the congruence relation of this paper. Therefore, chords are related as follows:

$$j \equiv i \pmod{n}, \quad \updownarrow \quad s_j = s_i$$

This opens the way for a third definition of a representative set of the chords related to the number $n$, by alternately selecting odd and even indices, i.e. $s_j$ with $j = \{1, 2, 3, \ldots, (n - 1)/2\}$. As such, the indices are kept small, which is suitable for applying multiple angle functions like $\cos(2\alpha) = 1 - \sin^2(\alpha)$ or more generally Chebyshev polynomials of the first kind: $\cos(n\theta) = T_n(\cos(\theta))$ as given by $[13]$. One can, therefore, express all chords for $j > 1$ by an arbitrary one, for example $s_1 = s$ as calculated by definition $5$. By adapting the Chebyshev polynomials to definition $5$ of the chords one gets the first few chords as

$$s_0 = 2, \quad s_1 = s, \quad s_2 = s^2 - 2, \quad s_3 = s^3 - 3s, \quad s_4 = s^4 - 4s^2 + 2,$$

with the recurrence relation

$$s_j = s \cdot s_{j-1} - s_{j-2},$$

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and the explicit formula

\[ s_j = 2^{1-j} \sum_{h=0}^{\lfloor j/2 \rfloor} (-1)^h \left( \frac{j}{2h} \right) s^{j-2h}(4 - s^2)^h. \]

The corresponding formulae for Chebychev polynomials of the first kind can be found in \[13\] and are easily adapted to chords as defined here. Note also that the signs of \( s_j \) – derived here from the \( n \)-gon – are the opposite of what one would get from Schick’s recurrence relation in equation \[4\] The complete set of chords may be used to construct the polynomials defined in the following theorem.

**Theorem 4.** Let \( P_m \) denote the sum of all chords plus 1, that is

\[ P_m = 1 + \sum_{j=1}^{m} s_j \text{ where } m = \frac{n-1}{2}, \]

then for all odd \( n \), \( P_m = 0 \). Furthermore, if each chord is written in terms of the first, \( s \), one can interpret \( s \) as the independent variable of a polynomial \( P_m(s) \) satisfying \( P_m(s_j) = 0 \).

**Proof.** \( P_m = 0 \) follows from equation \[7\]. \( \Box \)

The first four polynomials are

\[ P_0 = 1, \quad P_1 = s + 1, \quad P_2 = s^2 + s - 1, \quad P_3 = s^3 + s^2 - 2s - 1, \]

with the recurrence relation

\[ P_m = s \cdot P_{m-1} - P_{m-2}, \]

and the explicit formula

\[ P_m = \sum_{k=0}^{m} (-1)^i \binom{i+k}{k} s^k \text{ where } i = \left\lfloor \frac{m-k}{2} \right\rfloor \]

From the recurrence relation one can deduce, that all polynomials are over the integers and monic. Their second and last coefficients have the absolute
value 1. By applying Vieta’s formulae or by comparing with the fundamental theorem of algebra, one gets the sum rule (a Gauss sum)

$$\sum_{j=1}^{m} s_j = -1,$$

and the product rule

$$\prod_{j=1}^{m} s_j = (-1)^{|m/2|} \text{ where } m = \frac{n - 1}{2}.$$

The sum rule holds only for a complete set of chords, whereas the product of all roots of a polynomial factor always has the absolute value 1.

In general, the polynomials associated with the divisors of $n$ divide the polynomial associated with $n$. After canceling out all these factors, a polynomial of degree $\varphi(n)/2$ remains. This polynomial is the minimal polynomial of the chords $s_j$, the monomial of minimal degree for which $s_j$ is a root. In the rest of this paper, it will be denoted by $\Psi_n(s)$.

For example, the polynomial $P_4$, related to the prime power $n = 9$ is, as expected, reducible:

$$P_4 = s^4 + s^3 - 3s^2 - 2s + 1 = (s^3 - 3s + 1)(s + 1).$$

The first factor $(s^3 - 3s + 1) = \Psi_9(s)$ is the minimal polynomial of $P_4$, and in the factor $(s + 1) = P_1$ one recognizes the prime factor 3.

Additionally, using the substitution $s = 2x$, one obtains the polynomials given by Schick [8] and, after factoring, the minimal polynomials of $\cos(2\pi/n)$ as in [12]. Given the minimal polynomial $\Psi_n(s)$, the cyclotomic polynomial $\Phi_n(x)$ is obtained through the substitution $s = x + 1/x$: $\Phi_n(x) = x^{\varphi(n)/2} \cdot \Psi_n\left(x + \frac{1}{x}\right)$. (9)

D. Surowski and P. McCombs [11] give the following formula for the minimal polynomial of $2\cos(2\pi/p)$ for an odd prime $p = 2m + 1$:

$$\Psi_p(s) = \sum_{i=0}^{m} (-1)^i \sigma_i s^{m-i} \text{ where } \begin{cases} \sigma_{2k} = (-1)^k \binom{m-k}{k} \\ \sigma_{2k-1} = (-1)^k \binom{m-k}{k-1} \end{cases}.$$ 

One can verify that this somewhat complex expression yields the same polynomials as the the explicit formula for $\Psi_p(s) = P_m(s)$ from equation [8].
4.2 Geometric Interpretation of the Chords

Figure 1: Chords associated with $n = 11$.

Figure 1 shows the upper half of a polygon with 11 corners inscribed in a unit circle. Additionally, the corners of an 22-gon are marked. Five chords $s_1, s_3, s_5, s_7$ and $s_9$ are associated with the number 11. They are diagonals or a side of the 22-gon and may be drawn at several positions. Here they are drawn to demonstrate the sum rule. The angles between two intersecting chords is the constant $\arcsin(s_1) = \pi/n$. The chords $s_1, s_5$ and $s_9$ have a negative sign (definition 5). The green lines demonstrate that $-s_9 + s_7 - s_5 + s_3 - s_1 = -1$, the Gauss sum.

It would also be possible to demonstrate in the figure

* The equality $s_j = s_{n-j}$, by mirroring around the line $y = x$.
* The recurrence relation $s_j = s \cdot s_{j-1} - s_{j-2}$.
* The multiplication of chords, up to demonstrating the product rule.
4.3 Chord Arithmetic

The chords $s_j$ of definition 5 combine sine and cosine functions, use their periodicity and symmetry, and map these properties to the index number $j$. This subsection demonstrates that arithmetic of chords becomes an arithmetic of index numbers.

Let $n$ be an odd positive integer and $s_j$ a chord, then the following equation holds:

$$s_i s_j = s_{i+j} + s_{i-j}, \quad (10)$$

where an additional chord $s_0 = 2$ for the case $i = j$ has to be defined. It is the neutral additive element, the diameter of the unit circle. Equation $10$ follows readily from the equation

$$2 \sin(\alpha) \sin(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta),$$

and other, similar, formulae for multiplication of sine and cosine functions. To recognize equal chords one has to find — at any arbitrary step in the process — appropriate representatives of the index numbers mod $n$.

The analog of equation $10$ for Chebyshev polynomials of the first kind is

$$2 T_i T_j = T_{i+j} + T_{i-j}. \quad (10)$$

Example use. The use of the above rule is demonstrated for two examples, $P_3$ and $P_6$. In accordance with the fundamental theorem of algebra, all coefficients of the polynomial $P_m$, except the first one, are composed of products and sums of chords. Products of chords may be transformed into sums of chords by equation $10$.

$$P_3 = \prod_{j=1}^{3} (s - s_j) = s^3 - (s_1 + s_2 + s_3)s^2 + (s_1 s_2 + s_1 s_3 + s_2 s_3)s - s_1 s_2 s_3$$

For the Gauss sum $s_1 + s_2 + s_3$ the result is known, it is $-1$. Applying $10$ to the third term and choosing the appropriate representative $j \in \{1, 2, 3\}$, yields:

$$s_1 s_2 + s_1 s_3 + s_2 s_3 = s_3 + s_1 + s_4 + s_2 + s_5 + s_1 = 2(s_1 + s_2 + s_3) = -2.$$
To the last term $s_1s_2s_3$, equation [10] is applied sequentially.

$$s_1s_2s_3 = (s_3 + s_1)s_3 = s_6 + s_0 + s_4 + s_2 = s_1 + 2 + s_3 + s_2 = 1.$$  

Introducing these results into the polynomial yields

$$P_3 = s^3 + s^2 - 2s + 1.$$  

Polynomials $P_m$ for primes or prime powers of the form $n = 4k + 1$ can be factored as demonstrated here for $n = 13$:

$$P_6 = s^6 + s^5 - 5s^4 + 4s^3 + 6s^2 + 3s - 1 = (s^3 + c_1s^2 - s - 1 - c_1)(s^3 + c_2s^2 - s - 1 - c_2),$$

where $c_1 = s_1 + s_3 + s_9 = 1 - \sqrt{13}/2$ and $c_2 = s_5 + s_7 + s_{11} = 1 + \sqrt{13}/2$.

In this example the representatives are chosen differently, namely $j \in \{1, 3, 5, 7, 9, 11\}$. The numbers 1, 3, and 9 are the quadratic residues in $G^*_n$. The value of $c_1$ can be deduced from the Gauss sum [2] of all quadratic residues in $G^*_n$, that is $\sqrt{13}$.

**Application to minimal polynomials.** Another application of equation [10] is the explicit construction of the minimal polynomial of $2 \cos(2\pi/q_k)$ where $q$ is an odd prime. Let $p$ be any divisor of $n$, an odd integer. Consider the expression $P_{(n-p)/2} - P_{(n-p)/2-1}$:

$$P_{(n-p)/2} - P_{(n-p)/2-1} = 1 + \sum_{i=0}^{(n-p)/2} s_i - \left(1 + \sum_{i=0}^{(n-p)/2-1} s_i\right) = s_{(n-p)/2}. \quad (11)$$

Multiplication by $P_{(p-1)/2}$ yields:

$$P_{(p-1)/2}(P_{(n-p)/2} - P_{(n-p)/2-1})$$

$$= s_{(n-p)/2} \left(1 + \sum_{i=1}^{(p-1)/2} s_i\right)$$

$$= s_{(n-p)/2} + \sum_{i=1}^{(p-1)/2} s_{(n-p)/2+i} + \sum_{i=1}^{(p-1)/2} s_{(n-p)/2-i}$$

$$= \sum_{i=0}^{(n-1)/2} s_i$$

$$= P_{(n-1)/2} - P_{(n-1)/2-p}.$$
Thus, we have the formula:

\[ P_{(n-1)/2} = P_{(p-1)/2}(P_{(n-p)/2} - P_{(n-p)/2-1}) + P_{(n-1)/2-p}. \]

Since the number \( n - 2p \) is divisible by \( p \) we can apply the above formula repeatedly, expanding the last term:

\[ P_{(n-1)/2-p} = P_{(p-1)/2}(P_{(n-3p)/2} - P_{(n-3p)/2-1}) + P_{(n-1)/2-2p} \]
\[ P_{(n-1)/2-2p} = P_{(p-1)/2}(P_{(n-5p)/2} - P_{(n-5p)/2-1}) + P_{(n-1)/2-3p} \]

\[ \cdots \]

We can do this \( i \) times, as long as \( i < (n - 1)/(2p) \). The summation thus stops at \( \lfloor (n - 1)/(2p) \rfloor \). As such, we get the following formula:

\[ P_{(n-1)/2} = P_{(p-1)/2} \left( 1 + \sum_{2|k}^{\lfloor (n-1)/(2p) \rfloor} (P_{(k-p)/2} - P_{(k-p)/2-1}) \right). \]

In accordance with equation 11, one could also express each difference of polynomials as a chord. For the case \( n = q^k \), with \( q \) prime, one easily finds the minimal polynomial by taking \( p = q^{k-1} \) in the above:

\[ \Psi_{q^k} = 1 + \sum_{2|l}^{\lfloor (q^k-1)/(2q^{k-1}) \rfloor} (P_{(q^k-1q^{k-1})/2} - P_{(q^k-1q^{k-1})/2-1}) \]

\[ = 1 + \sum_{2|l}^{\lfloor q/2 \rfloor} (P_{(q^k-1q^{k-1})/2} - P_{(q^k-1q^{k-1})/2-1}). \]

Substituting equation 8 yields an explicit formula for the minimal polynomial of a prime power. As an example, consider \( n = 27 \). The minimal polynomial is given by:

\[ \Psi_{27} = P_9(s) - P_8(s) + 1 = s^9 - 9s^7 + 27s^5 - 30s^3 + 9s + 1. \]

Using equation 9, the cyclotomic polynomial can be deduced:

\[ \Phi_{27}(x) = x^9 \cdot \Psi_{27} \left( x + \frac{1}{x} \right) = x^9(x^9 + x^{-9} + 1) = x^{18} + x^9 + 1. \]
5 Conclusions

We have defined a new congruence relation that can be used to study the behavior of the sequence given by Schick and other, similar, sequences.

By defining mod\(^\ast\), we can simplify several aspects of Schick’s work. Furthermore, the multiplicative group of integers mod\(^\ast\) \( n \) also has a number of properties that are interesting by themselves. In particular, mod\(^\ast\) yields relatively more quadratic residues and the process of finding square roots is simplified. For some special composite numbers, the multiplicative group of integers mod\(^\ast\) \( n \) is cyclic. In this case, one can define ”primitive roots” and adapt Artin’s primitive root conjecture. Examples of such composites are Sophie Germain and twin prime pairs.

Finally, polynomials related to odd integers and mod\(^\ast\) are introduced. These lead to a simple expression for the minimal polynomial of \( 2 \cos(2\pi/n) \), where \( n \) is an odd prime. More complicated (but explicit) expressions are obtained for \( n \) a prime power.

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