A FAMILY OF CONFORMING MIXED FINITE ELEMENTS FOR LINEAR ELASTICITY ON TRIANGULAR GRIDS

JUN HU AND SHANGYOU ZHANG

Abstract. This paper presents a family of mixed finite elements on triangular grids for solving the classical Hellinger-Reissner mixed problem of the elasticity equations. In these elements, the matrix-valued stress field is approximated by the full $C^0-P_k$ space enriched by $(k-1) H(\text{div})$ edge bubble functions on each internal edge, while the displacement field by the full discontinuous $P_{k-1}$ vector-valued space, for the polynomial degree $k \geq 3$. The main challenge is to find the correct stress finite element space matching the full $C^{-1}-P_{k-1}$ displacement space. The discrete stability analysis for the inf-sup condition does not rely on the usual Fortin operator, which is difficult to construct. It is done by characterizing the divergence of local stress space which covers the $P_{k-1}$ space of displacement orthogonal to the local rigid-motion. The well-posedness condition and the optimal a priori error estimate are proved for this family of finite elements. Numerical tests are presented to confirm the theoretical results.

Keywords. mixed finite element, symmetric finite element, linear elasticity, triangular grids, inf-sup condition.

AMS subject classifications. 65N30, 73C02.

1. Introduction

It is a challenge to design stable discretizations for the linear elasticity equations based on the Hellinger-Reissner variational principle, in which the stress and displacement are solved simultaneously. This reason lies in, besides the usual discrete K-ellipticity and B-B conditions, there is an additional symmetry constraint on the stress tensor for the problem under consideration. Many methods have been proposed to overcome this difficulty, cf. [3, 6, 7, 26, 28, 30, 31, 32] for earlier works. In a recent work [9], Arnold and Winther designed the first family of mixed finite element methods based on polynomial shape function spaces, which was motivated by a key observation: a discrete exact sequence guarantees the stability of the mixed method. From then on, various stable mixed elements have been constructed, see [2, 4, 5, 11, 17, 19, 23, 30, 33, 34, 31, 32, 12, 13, 20, 21]. Since most of these elements require a local commuting property which implies that the usual Fortin operator can be constructed elementwise, they have many degrees of freedom on each element such that they are difficult to be implemented; while the numerical examples can only be found in [15, 16] so far.

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In a recent paper, a family of conforming mixed finite elements is proposed on rectangular grids for both two and three dimensions. As a result the lowest order elements have 10 plus 4 and 21 plus 6 degrees of freedom on each element for two and three dimensions, respectively, which are two and three dimensional elements due to [25]. These elements were motivated by an observation that conformity of the discrete methods on rectangular meshes can be guaranteed by $H(\text{div})$-conformity of the normal stress and $H^1$-conformity of two corresponding variables for each component of the shear stress. Such an idea was first explored in [24] to design the minimal mixed finite elements on rectangular grids in any dimension. A new explicit constructional proof based on a macro-element technique was proposed to show the discrete inf-sup condition for them. In other words, that constructive proof avoids the discrete exact sequence of [9], which is not possible therein but used nearly everywhere [2, 4, 5, 9, 11].

This paper presents a family of mixed finite elements on triangular grids. In these elements, the matrix-valued stress field is approximated by the full $C^0-P_k$ space enriched by $(k-1)$ $H(\text{div})$ edge bubble functions on each internal edge, while the displacement field by a discontinuous vector-valued $P_k-1$ element for $k \geq 3$. The main difficulty for the discrete stability analysis comes from the discrete inf-sup condition since it is impossible to construct locally the usual Fortin operator (for all $k \geq 3$). To overcome such a difficulty, a new way of proof is particularly proposed to overcome it, characterizing the divergence of local stress space which covers the $P_k-1$ space of displacement orthogonal to the local rigid-motion.

The new family of mixed elements is a simplification of the very first constructed family of stable elements of Arnold-Winther [9]. For the $C^{-1}-P_{k-1}$ displacement field, the stress space of Arnold-Winther is the symmetric $H(\text{div})$-$P_{k+1}$ tensors whose divergence is in $P_{k-1}$, while ours is a subspace of symmetric $H(\text{div})$-$P_k$ tensors. For example, when $k = 3$, the number of degrees of freedom on one Arnold-Winther element is 37 (for stress) plus 12 (for displacement) while ours is 30 plus 12. Computationally, the new element is much simpler as there is no constraints on the polynomial degree deduction of divergence. Mathematically, the new family of mixed element is the simplest one to achieve $P_{k-1}$ approximation for the displacement and $P_k$ approximation for the stress. That is, we eliminate all divergence-free stresses of no approximation power in the Arnold-Winther space. However, we failed to improve the lowest order element in the Arnold-Winther family, $k = 2$. It remains to simplify the Arnold-Winther $k = 2$ element, or to prove its simplicity.

The rest of the paper is organized as follows. In Section 2, we define the weak problem and the finite element method. In section 3, we prove the well-posedness of the finite element problem, i.e. the discrete coerciveness and the discrete inf-sup condition. By which, the optimal order convergence of the new element follows. In Section 4, we provide some numerical results, using $P_3$, $P_4$ and $P_5$ finite elements.

2. THE FAMILY OF FINITE ELEMENTS

Based on the Hellinger-Reissner principle, the linear elasticity problem within a stress-displacement ($\sigma$-$u$) form reads: Find $(\sigma, u) \in \Sigma \times V := H(\text{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$, such that

\[
\begin{align*}
(A\sigma, \tau) + (\text{div} \tau, u) &= 0 & \text{for all } \tau \in \Sigma, \\
(\text{div} \sigma, v) &= (f, v) & \text{for all } v \in V.
\end{align*}
\]
Here the symmetric tensor space for stress \( \Sigma \) and the space for vector displacement \( V \) are, respectively,

\[
H(\text{div}, \Omega, \mathbb{S}) := \left\{ \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \in H(\text{div}, \Omega) \mid \sigma_{12} = \sigma_{21} \right\},
\]

\[
L^2(\Omega, \mathbb{R}^2) := \left\{ (u_1, u_2)^T \mid u_i \in L^2(\Omega) \right\}.
\]

This paper denotes by \( H^k(T, X) \) the Sobolev space consisting of functions with domain \( T \subset \mathbb{R}^2 \), taking values in the finite-dimensional vector space \( X \), and with all derivatives of order at most \( k \) square-integrable. For our purposes, the range space \( X \) will be either \( \mathbb{S}, \mathbb{R}^2 \), or \( \mathbb{R} \). \( \| \cdot \|_{k, T} \) is the norm of \( H^k(T) \). \( \mathbb{S} \) denotes the space of symmetric tensors, \( H(\text{div}, T, \mathbb{S}) \) consists of square-integrable symmetric matrix fields with square-integrable divergence. The \( H(\text{div}) \) norm is defined by

\[
\| \tau \|^2_{H(\text{div}, T)} := \| \tau \|^2_{L^2(T)} + \| \text{div} \tau \|^2_{L^2(T)}.
\]

\( L^2(T, \mathbb{R}^2) \) is the space of vector-valued functions which are square-integrable.

Throughout the paper, the compliance tensor \( A = A(x) : \mathbb{S} \rightarrow \mathbb{S} \), characterizing the properties of the material, is bounded and symmetric positive definite uniformly for \( x \in \Omega \).

This paper deals with a pure displacement problem (2.1) with the homogeneous boundary condition that \( u \equiv 0 \) on \( \partial \Omega \). But the method and the analysis work for mixed boundary value problems and the pure traction problem.

The domain \( \Omega \) is subdivided by a family of quasi-uniform triangular grids \( T_h \) (with the grid size \( h \)). We introduce the finite element space of order \( k \) \((k \geq 3)\) on \( T_h \). The displacement space is the full \( C^{-1} - P_{k-1} \) space

\[
V_h = \{ v \in L^2(\Omega, \mathbb{R}^2) \mid v|_K \in P_{k-1}(K, \mathbb{R}^2) \text{ for all } K \in T_h \}.
\]

The stress space is the full \( C^0 - P_k \) space enriched by \((k-1)\) \( H(\text{div}) \) edge bubble functions on each internal edge. We define the edge bubble functions first. Let \( K \in T_h \) with three edges \( E_i \) and corresponding three barycentric variables \( \lambda_i \). Here \( \lambda_i \) is a linear function which vanishes on edge \( E_i \) and assumes a nodal value 1 at the opposite vertex \( x_i \), see Figure 2.2. Let \( \{ \phi_{i,j}, j = 1, 2, \ldots, k-1 \} \) be the standard \( P_k(K, \mathbb{R}) \) nodal basis functions at \( k-1 \) internal nodes on edge \( E_i \), \( i = 0, 1, 2 \), cf. Figure 2.2 for \( P_3 \) basis functions. With these \((k-1)\) edge bubble functions on each edge, we can define exactly \((k-1)\) stress functions \( \tau_{i,j} \) of zero normal flux:

\[
\tau_{i,j} \cdot n_i = 0, \quad j = 1, 2, \ldots, k-1, \quad i = 0, 1, 2.
\]

In computation, they are defined explicitly as

\[
\tau_{i,j} = \begin{pmatrix} n_{i,2}^2 \phi_{i,j} \\ -n_{i,1} n_{i,2} \phi_{i,j} \\ n_{i,1}^2 \phi_{i,j} \end{pmatrix}.
\]

where \( n_i = (n_{i,1}, n_{i,2}) \) is a unit normal vector on edge \( E_i \), see Figure 2.1.

For example, for the right triangle \( K \) shown in Figure 2.2 the two zero-flux \( P_3 \) edge bubbles, on edge \( E_0 \), are

\[
\tau_{0,1} = \begin{pmatrix} \phi_{0,1} & -\phi_{0,1} \\ -\phi_{0,1} & \phi_{0,1} \end{pmatrix}, \quad \tau_{0,2} = \begin{pmatrix} \phi_{0,2} & -\phi_{0,2} \\ -\phi_{0,2} & \phi_{0,2} \end{pmatrix}.
\]

We note that the zero-flux bubbles are no longer zero-flux bubbles under affine transformation, cf. Figure 2.3.
Figure 2.1. A reference triangle and a general triangle with an edge normal vector.

Figure 2.2. The barycentric variables $\lambda_i$ (linear functions) on $K = x_0x_1x_2$, and two edge bubble functions of $P_3$ on edge $E_0$, where $\phi_{0,1} = \tilde{\phi}_{0,1}/\tilde{\phi}_{0,1}(\lambda_1 = \frac{2}{3}, \lambda_2 = \frac{1}{3})$.

Let $E_{h,0}$ be the set of internal edges $E$ of triangulation $\mathcal{T}_h$. For each internal edge $E \in E_{h,0}$, we select randomly one triangle $K$ from the two triangles sharing the edge $E$, say, edge $E_0$ of $K$ is $E$. A global edge bubble stress is defined then by

$$
\tau_{E,j} = \begin{cases} 
\tau_{0,j} & \text{on } K, \\
0 & \text{elsewhere.} 
\end{cases}
$$

The finite element space of order $k$ ($k \geq 3$) for the stress approximation is

$$
\Sigma_h = \left\{ \sigma \in H(\text{div}, \Omega, S) \mid \sigma = \sigma_c + \sigma_b, \ \sigma_c \in H^1(\Omega, S), \right\} 
$$

$$
\left. \sigma_c \right|_K \in P_k(K, S) \ \forall K \in \mathcal{T}_h, \ \sigma_b = \sum_{E \in E_{h,0}} \sum_{j=1}^{k-1} c_{E,j} \tau_{E,j}, 
$$

which is a $H(\text{div})$ bubble enrichment of the $H^1$ space

$$
\tilde{\Sigma}_h = \left\{ \sigma \in H(\text{div}, \Omega, S) \mid \sigma \in H^1(\Omega, S), \ \left. \sigma \right|_K \in P_k(K, S) \ \forall K \in \mathcal{T}_h \right\}.
$$

In computation, we still uses $3 \times \text{dim } P_k$ Lagrange nodal basis locally on each triangle $K$, but globally we break each of $(k - 1)$ zero-flux edge bubble functions into two basis functions. In other words, we break such a zero-flux edge function (a combination of $C^0$-$P_k$ internal-edge basis functions) into two discontinuous basis functions by making it zero on both sides of the edge.
The mixed finite element approximation of Problem (1.1) reads: Find \((\sigma_h, u_h) \in \Sigma_h \times V_h\) such that
\[
\begin{aligned}
(A\sigma, \tau) + (\text{div} \tau, u_h) &= 0 \quad \text{for all } \tau \in \Sigma_h, \\
(\text{div} \sigma, v) &= (f, v) \quad \text{for all } v \in V_h.
\end{aligned}
\]
(2.9)

It follows from the definition of \(V_h (P_{k-1} \text{ polynomials})\) and \(\Sigma_h (P_k \text{ polynomials})\) that
\[
\text{div} \Sigma_h \subset V_h.
\]

This, in turn, leads to a strong divergence-free space:
\[
Z_h := \{ \tau_h \in \Sigma_h \mid (\text{div} \tau_h, v) = 0 \text{ for all } v \in V_h \}
\]
(2.10)
\[
= \{ \tau_h \in \Sigma_h \mid \text{div} \tau_h = 0 \text{ pointwise} \}.
\]

3. Stability and convergence

The convergence of the finite element solutions follows the stability and the standard approximation property. So we consider first the well-posedness of the discrete problem (2.9). By the standard theory, we only need to prove the following two conditions, based on their counterpart at the continuous level.

(1) K-ellipticity. There exists a constant \(C > 0\), independent of the meshsize \(h\) such that
\[
(A\tau, \tau) \geq C\|\tau\|^2_{H(\text{div})} \quad \text{for all } \tau \in Z_h,
\]
(3.1)

where \(Z_h\) is the divergence-free space defined in (2.10).

(2) Discrete B-B condition. There exists a positive constant \(C > 0\) independent of the meshsize \(h\), such that
\[
\inf_{0 \neq v \in V_h} \sup_{0 \neq \tau \in \Sigma_h} \frac{(\text{div} \tau, v)}{\|\tau\|^2_{H(\text{div})}\|v\|^2_{L^2(\Omega)}} \geq C.
\]
(3.2)

It follows from \(\text{div} \Sigma_h \subset V_h\) that \(\text{div} \tau = 0\) for any \(\tau \in Z_h\). This implies the above K-ellipticity condition (3.1).

It remains to show the discrete B-B condition (3.2), in the following two lemmas.

**Lemma 3.1.** For any \(v_h \in V_h\), there is a \(\tau_h \in \tilde{\Sigma}_h\) such that, for any polynomial \(p \in P_{k-2}(K, \mathbb{R}^2)\),
\[
\int_K (\text{div} \tau_h - v_h) \cdot p \, dx = 0 \quad \text{and} \quad \|\tau_h\|_{H(\text{div})} \leq C\|v_h\|_{L^2(\Omega)}.
\]
(3.3)

**Proof.** Let \(v_h \in V_h\). By the stability of the continuous formulation, cf. [9], there is a \(\tau \in \Sigma \cap H^1(\Omega, \mathbb{S})\) such that,
\[
\text{div} \tau = v_h \quad \text{and} \quad \|\tau\|_{H^1(\Omega)} \leq C\|v_h\|_{L^2(\Omega)}.
\]
As \(\tau \in H^1(\Omega, \mathbb{S})\), we modify the Scott-Zhang [29] interpolation operator slightly to define a flux preserving interpolation.
\[
I_h : \Sigma \cap H^1(\Omega, \mathbb{S}) \to \Sigma_h \cap H^1(\Omega, \mathbb{S}) = \tilde{\Sigma}_h,
\]
\[
\tau = \begin{pmatrix} \tau_{11} \\ \tau_{12} \end{pmatrix} \mapsto \tau_h = \begin{pmatrix} \tau_{11,h} \\ \tau_{12,h} \end{pmatrix} = I_h \tau.
\]
Here the interpolation is done inside a subspace, the continuous finite element subspace \(\Sigma_h \cap H^1(\Omega, \mathbb{S})\). \(I_h \tau\) is defined by its values at the Lagrange nodes.
At a vertex node \( x_i \), \( I_h \tau(x_i) \) is defined as the nodal value of \( \tau \) at the vertex if \( \tau \) is continuous, but in general, \( I_h \tau(x_i) \) is defined as an average value on an edge at the vertex, as in [29]. After defining the nodal values at vertices of triangles, the nodal values of \( \tau_h \) at the nodes inside each edge are defined by the \( L^2 \)-orthogonal projection on the edge:

\[
\forall p \in P_{k-2}(E, \mathbb{R}), \quad \begin{cases} 
\int_E \tau_{h,11} p \, ds = \int_E \tau_{11} p \, ds, \\
\int_E \tau_{h,12} p \, ds = \int_E \tau_{12} p \, ds, \\
\int_E \tau_{h,22} p \, ds = \int_E \tau_{22} p \, ds,
\end{cases}
\]

(3.4)

where \( E \) is an edge in the triangulation \( T_h \). At the Lagrange nodes inside triangles, \( I_h \tau \) is defined by the \( L^2 \)-orthogonal projection on the triangle:

\[
\int_K \tau_{ij,h} p \, dx = \int_K \tau_{ij} p \, dx \quad \forall p \in P_{k-3}(K, \mathbb{R}),
\]

(3.5)

where \( K \) is an element of \( T_h \). It follows by the stability of the Scott-Zhang operator that

\[
\| I_h \tau \|_{H(\text{div})} \leq C\| \tau \|_{H^1(\Omega)} \leq C\| v_h \|_{L^2(\Omega)}.
\]

By (3.4) and (3.5), we get the partial-divergence matching property of \( I_h \): for any \( p \in P_{k-2}(K, \mathbb{R}^2) \),

\[
\int_K (\text{div} \tau - v_h) \cdot p \, dx = \int_{\partial K} \tau_n \cdot p \, ds - \int_K \tau : \nabla p \, dx - \int_K v_h \cdot p \, dx = 0.
\]

Lemma 3.2. For any \( v_h \in V_h \), if

\[
\int_K v_h \cdot p \, dx = 0 \quad \text{for all } p \in P_{k-2}(K, \mathbb{R}^2),
\]

(3.6)

there is a \( \tau_h \in \Sigma_h \) such that

\[
\text{div} \tau_h = v_h \quad \text{and} \quad \| \tau_h \|_{H(\text{div})} \leq C\| v_h \|_{L^2(\Omega)}.
\]

Proof. We first define the local spaces of bubble stress functions. Let \( x_0, x_1 \) and \( x_2 \) be the three vertices of a triangle \( K \). The referencing mapping is then, cf. Figure 2.1

\[
x = F_K(\hat{x}) = x_0 + (x_1 - x_0, x_2 - x_0) \hat{x}.
\]

Then

\[
\hat{x} = \begin{pmatrix} n_1^T \\ n_2^T \end{pmatrix} (x - x_0),
\]

(3.8)

where

\[
\begin{pmatrix} n_1^T \\ n_2^T \end{pmatrix} = (x_1 - x_0, x_2 - x_0)^{-1}.
\]
Due to the inverse matrix relation, these two vectors $n_1, n_2$ are orthogonal to edges $x_0\delta_2$ and $x_0\delta_1$, respectively. By (3.3), they are coefficients of the barycentric variables:

$$\lambda_1 = n_1 \cdot (x - x_0),$$
$$\lambda_2 = n_2 \cdot (x - x_0),$$
$$\lambda_0 = 1 - \lambda_1 - \lambda_2.$$

With them, we define the $H(\text{div}, K, S)$ bubble functions

$$(3.9) \quad \Sigma_{K,b} = \text{span}\{\lambda_2 \lambda_0 p_1 n_1^T n_1, \lambda_2 \lambda_1 p_2 n_2^T n_2, \lambda_1 \lambda_0 p_0 n_0^T n_0\},$$

where $p_1, p_2$ and $p_0 \in P_{k-2}(K, \mathbb{R})$, and

$$n_1^T = \begin{pmatrix} -n_{12} \\ n_{11} \end{pmatrix}, \quad \text{if} \quad n_1 = \begin{pmatrix} n_{11} \\ n_{12} \end{pmatrix},$$
$$n_0 = -n_1 - n_2.$$

Note that $\tau_b \cdot n_j = 0$ on the three edges ($\lambda_j = 0$), for all $\tau_h \in \Sigma_{K,b}$. Thus, the match of $\text{div} \tau_h = v_h$ is done locally on $K$, independently of the matching on next element.

We begin to prove the lemma. Let $v_h \in V_h$ satisfying (3.6). We show there is a local $\tau \in \Sigma_{K,b}$ such that $\text{div} \tau = v_h$, on each element $K$. As $v_h$ satisfies (3.6), $v_h|_K \in V_{K,\perp}$ where $V_{K,\perp}$ is the rigid-motion free space

$$V_{K,\perp} = \{v_h \in P_{k-1}(K, \mathbb{R}^2) \mid \int_K v_h \cdot \left(\begin{array}{c} a-b y \\ c + b x \end{array}\right) \, dx = 0, \forall a, b, c \in \mathbb{R}\}.$$

We prove $\text{div} \Sigma_{K,b} = V_{K,\perp}$ next. By definition, $\text{div} \Sigma_{K,b} \subset V_{K,\perp}$. If $\text{div} \Sigma_{K,b} \neq V_{K,\perp}$, there is a $v_h \in V_{K,\perp}$ orthogonal to $\text{div} \Sigma_{K,b}$, i.e.,

$$\int_K \text{div} \cdot v_h \, dx = \int_K \tau \cdot \epsilon(v_h) \, dx = 0 \quad \forall \tau \in \Sigma_{K,b},$$

where $\epsilon(v_h)$ is the symmetric gradient, $(\nabla v_h + \nabla^T v_h)/2$. We show next $v_h = 0$.

Let $\{m_i, i = 0, 1, 2\}$ be the dual basis, under $\mathbb{R}^4$ inner-product, of

$$n_i = n_1^T n_i^T, \quad i = 1, 2, 0,$$

i.e.

$$m_j \cdot n_i = \begin{pmatrix} m_{j1} & m_{j2} \\ m_{j2} & m_{j3} \end{pmatrix} \cdot \begin{pmatrix} n_{11}^2 \\ -n_{11} n_{12} \\ -n_{12} n_{11} \\ n_{12}^2 \end{pmatrix} = m_{j1} n_{11}^2 - 2m_{j2} n_{11} n_{12} + m_{j3} n_{12}^2 = \delta_{ij}.$$

As noted above, $\{n_i\}$ are three normal vectors of element $K$. In Lemma 3.2 below, we shall prove that the three matrices $n_i, i = 0, 1, 2$, are linearly independent. Hence the above equation has a unique solution $m_j$. Therefore we have a unique expansion, as $\epsilon(v_h) \in P_{k-2}(K, S)$,

$$\epsilon(v_h) = q_1 m_1 + q_2 m_2 + q_0 m_3, \quad \text{for some} \quad q_1, q_2, q_0 \in P_{k-2}(K, \mathbb{R}).$$

Selecting $\tau_1 = \lambda_2 \lambda_0 q_1 n_1^T n_1^T \in \Sigma_{K,b}$, we have

$$0 = \int_K \tau_1 \epsilon(v_h) \, dx = \int_K \lambda_2 \lambda_0 q_1^2 (x) \, dx.$$
As $\lambda_2\lambda_0 > 0$ on $K$, we conclude that $q_1 = 0$. Similarly, $q_2$ and $q_0$ are zero, and $v_h = 0$.

As we assume $k \geq 3$, the condition \[ (3.7) \] that $v_h$ is orthogonal to $P_{k-2}(K, \mathbb{R}^2)$ implies that $v_h \in V_{h,\perp} = \text{div} \Sigma_{K,b}$. The selection of $\tau_h$, locally on element $K$, is made by

$$\|\tau_h\|_{L^2(\Omega)} = \min\{\|\tau\|_{L^2(\Omega)} \mid \text{div} \tau = v_h, \tau \in \Sigma_{K,b}\}.$$  

The boundedness of div operator in \[ (3.7) \] follows the scaling argument with affine mappings. Thus \[ (3.7) \] holds.

**Lemma 3.3.** Let $v_1 \in \mathbb{R}^2$ and $v_2 \in \mathbb{R}^2$, and $v_0 = v_1 + v_2$. Suppose that $v_1$ and $v_2$ are linearly independent. Then three matrices $v_1 v_1^T$, $v_2 v_2^T$, and $v_0 v_0^T$ are linearly independent.

**Proof.** Let $v_1 = (a_1, a_2)^T$ and $v_2 = (b_1, b_2)^2$, which leads to $v_0 = (a_1 + b_1, a_2 + b_2)^T$.

Hence

$$v_1 v_1^T = \begin{pmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{pmatrix}, \quad v_2 v_2^T = \begin{pmatrix} b_1^2 & b_1 b_2 \\ b_1 b_2 & b_2^2 \end{pmatrix}$$

and

$$v_0 v_0^T = \begin{pmatrix} (a_1 + b_1)^2 & (a_1 + b_1)(a_2 + b_2) \\ (a_1 + b_1)(a_2 + b_2) & (a_2 + b_2)^2 \end{pmatrix}.$$  

To prove the desired result, it suffices to show that the rank of the matrix

$$\begin{pmatrix} a_1^2 & b_1^2 \\ a_2^2 & b_2^2 \\ a_1 a_2 & b_1 b_2 \\ (a_1 + b_1)^2 & (a_2 + b_2)^2 \end{pmatrix},$$

is three. A direct calculation finds that the determinant of the above matrix is $$(a_1 b_2 - a_2 b_1)^3.$$ Since $v_1$ and $v_2$ are linearly independent, we have

$$a_1 b_2 - a_2 b_1 \neq 0,$$

which completes the proof.

**Remark 3.1.** The lemma \[ (3.7) \] can be proved differently, by counting the dimension of vector spaces. Due to the linearly independent vectors (in matrix form),

$$\dim \Sigma_{K,b} = 3 \dim P_{k-2}^2 = \frac{3}{2} k^2 - \frac{3}{2} k.$$  

If we can show that the div-free bubbles of $\Sigma_{K,b}$ must be the bubble Airy functions, namely,

$$\text{div} \tau_h = 0 \Rightarrow \tau_h = \begin{pmatrix} \frac{\partial^2 w_h}{\partial x^2} \\ \frac{\partial^2 w_h}{\partial y^2} \\ -\frac{\partial^2 w_h}{\partial x \partial y} \end{pmatrix},$$

where $w_h = w_K b_K^2$ for some $w_K \in P_{k-4}(K, \mathbb{R})$, where $b_K = \lambda_0 \lambda_1 \lambda_2$ is the element $P_3$ bubble, then we would get

$$\dim \text{div} \Sigma_{K,b} = \dim \Sigma_{K,b} - \dim P_{k-4} = k^2 + k - 3 = 2 \dim P_{k-1}^2 - 3 = \dim V_{K,\perp}.$$  

As $\dim \Sigma_{K,b} \subset V_{K,\perp}$, the dimension counting will prove $\dim \Sigma_{K,b} = V_{K,\perp}$.
We are going to prove (3.10). Since \( w_h \) can be selected up to a linear function, we start to take \( w_h \) such that it vanishes at three vertices of element \( K \). Since \( \tau_h \in \Sigma_{K,b} \), it follows that
\[
\frac{\partial}{\partial n_i^+} \frac{\partial w_h}{\partial x} |_{E_i} = \frac{\partial}{\partial n_i^+} \frac{\partial w_h}{\partial y} |_{E_i} = 0, \quad i = 0, 1, 2.
\]
This implies that
\[
\frac{\partial^2 w_h}{\partial (n_i^+)^2} |_{E_i} = \frac{\partial^2 w_h}{\partial n_i^+ \partial n_i} |_{E_i} = 0, \quad i = 0, 1, 2.
\]
Hence \( \frac{\partial w_h}{\partial n_i^+} \) is a constant on \( E_i \). Since \( w_h \) vanishes on three vertices of \( K \), this indicates that \( w_h \) vanishes on \( E_i \), which implies that \( \frac{\partial w_h}{\partial n_i^+} = 0 \) on \( E_i \). Consequently, \( \nabla w_h \) vanishes on \( \partial E_i \). By (3.11), \( \frac{\partial w_h}{\partial n_i^+} \) is a constant on \( E_i \). This implies that \( \frac{\partial w_h}{\partial n_i^+} = 0 \) on \( E_i \), which completes the proof of (3.10).

We are in the position to show the well-posedness of the discrete problem.

**Lemma 3.4.** For the discrete problem (2.9), the K-ellipticity (3.1) and the discrete B-B condition (3.2) hold uniformly. Consequently, the discrete mixed problem (2.9) has a unique solution \((\sigma_h, u_h) \in \Sigma_h \times V_h\).

**Proof.** The K-ellipticity immediately follows from the fact that \( \text{div} \Sigma_h \subset V_h \). Therefore we only need to prove the discrete B-B condition (3.2). For any \( v_h \in V_h \), it follows from Lemma 3.1 that there exists a \( \tau_1 \in \Sigma_h \) such that, for any polynomial \( p \in P_{k-2}(K, \mathbb{R}^2) \),
\[
\int_K (\text{div} \tau_1 - v_h) \cdot p \, dx = 0 \quad \text{and} \quad \|\tau_1\|_{H(\text{div})} \leq C \|v_h\|_{L^2(\Omega)}.
\]
Then it follows from Lemma 3.2 that there is a \( \tau_2 \in \Sigma_h \) such that
\[
\text{div} \tau_2 = v_h - \text{div} \tau_1 \quad \text{and} \quad \|\tau_2\|_{H(\text{div})} \leq C \|\text{div} \tau_1 - v_h\|_{L^2(\Omega)},
\]
Let \( \tau = \tau_1 + \tau_2 \). This implies that
\[
\text{div} \tau = v_h \quad \text{and} \quad \|\tau\|_{H(\text{div})} \leq C \|v_h\|_{L^2(\Omega)},
\]
this proves the discrete B-B condition (3.2).

**Theorem 3.1.** Let \((\sigma, u) \in \Sigma \times V\) be the exact solution of problem (2.1) and \((\tau_h, u_h) \in \Sigma_h \times V_h\) the finite element solution of (2.9). Then, for \( k \geq 3 \),
\[
\|\sigma - \sigma_h\|_{H(\text{div})} + \|u - u_h\|_{L^2(\Omega)} \leq Ch^k (\|\sigma\|_{H^{k+1}(\Omega)} + \|u\|_{H^k(\Omega)}).
\]
**Proof.** The stability of the elements and the standard theory of mixed finite element methods [13] [14] give the following quasi-optimal error estimate immediately
\[
\|\sigma - \sigma_h\|_{H(\text{div})} + \|u - u_h\|_{L^2(\Omega)} \leq C \inf_{\tau_h \in \Sigma_h, u_h \in V_h} (\|\sigma - \tau_h\|_{H(\text{div})} + \|u - u_h\|_{L^2(\Omega)}).
\]
Let \( P_h \) denote the local \( L^2 \) projection operator, or triangle-wise interpolation operator, from \( V \) to \( V_h \), satisfying the error estimate
\[
\|v - P_h v\|_{L^2(\Omega)} \leq Ch^k \|v\|_{H^k(\Omega)} \quad \text{for any} \ v \in H^k(\Omega, \mathbb{R}^2).
\]
Choosing $\tau_h = I_h \sigma \in \Sigma_h$ where $I_h$ is defined in (3.4) and (3.5), we have [29], as $I_h$ preserves symmetric $P_k$ functions locally,

$$\|\sigma - \tau_h\|_{L^2(\Omega)} + h |\sigma - \tau_h|_{\text{div}} \leq Ch^{k+1} \|\sigma\|_{H^{k+1}(\Omega)}.$$  

(3.18)

Let $v_h = P_h v$ and $\tau_h = I_h \sigma$ in (3.16), by (3.17) and (3.18), we obtain (3.15).

4. Numerical tests

We compute a 2D pure displacement problem on the unit square $\Omega = [0, 1]^2$ with a homogeneous boundary condition that $u \equiv 0$ on $\partial \Omega$. In the computation, we let $A\sigma = \frac{1}{2\mu} \left( \sigma - \frac{\lambda}{2\mu + n\lambda} \text{tr}(\sigma) \delta \right), \quad n = 2,$

where $\delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $\mu = 1/2$ and $\lambda = 1$ are the Lamé constants. Let the exact solution be

$$u = \begin{pmatrix} e^{x-y}x(1-x)y(1-y) \\ \sin(\pi x) \sin(\pi y) \end{pmatrix}.$$

The true stress function $\sigma$ and the load function $f$ are defined by the equations in (2.1), for the given solution $u$.

In the computation, the level one grid consists of two right triangles, obtained by cutting the unit square with a north-east line. Each grid is refined into a half-sized grid uniformly, to get a higher level grid. In all the computation, the discrete systems of equations are solved by Matlab backslash solver.

Table 4.1. The errors $e_h = I_h u - u_h$ and $\epsilon_h = I_h \sigma - \sigma_h$, and the order of convergence, by the $P_3$ element, for (4.1).

| $h^n$ | $\|e_h\|_{L^2(\Omega)}$ | $\|\epsilon_h\|_{L^2(\Omega)}$ | $\|\text{div}(\epsilon_h)\|_{L^2(\Omega)}$ | $\text{dim } V_h$ | $\text{dim } \Sigma_h$ |
|------|---------------------|---------------------|---------------------|----------------|---------------- |
| 1    | 0.786096 0.000     | 4.4568 0.000       | 0.0179 0.000       | 24            | 50            |
| 2    | 0.147520 2.41      | 0.9877 2.17        | 0.0222 1.81        | 96            | 163           |
| 3    | 0.011117 3.73      | 0.1336 2.89        | 0.0035 2.68        | 1536          | 2277          |
| 4    | 0.000727 3.93      | 0.0170 2.97        | 0.0005 2.89        | 6144          | 8675          |
| 5    | 0.000046 3.98      | 0.0021 2.99        | 0.0001 2.97        | 24            | 50            |

First, we use the $P_3$ finite element, $k = 3$ in (2.1) and (2.7), i.e., the $P_3$ stress element and $P_2$ displacement element. In Table 4.1 the errors and the convergence order in various norms are listed for the true solution (4.1). Here $I_h$ is the usual nodal interpolation operator, similar to the averaging interpolation operator $I_h$ used in the analysis, (3.4)–(3.5). An order 3 convergence is observed for both displacement and stress, see Table 4.1 as shown in the theorem. However, there is a superconvergence for the displacement in $L^2$-norm and another superconvergence for the stress in $L^2$-norm, for $P_3$ finite elements. This is observed from Table 4.1 not proved yet. For better observing this property, we plot the finite element solution $(\sigma_h)_{11}$ and its error, on level 4 grid, in Figure 4.1. We also plot the finite element solution $(u_h)_{11}$ and its error, on level 4 grid, in Figure 4.2. It is apparent that there is at least one superconvergent point on each triangle, for the $P_3$ solutions, but not for the $P_3$ solutions.
In the second computation, we use $P_3$ finite elements, i.e., $k = 4$ in (2.4) and (2.7). The data are listed in Table 4.2. This time, the order of convergence is exactly as proved in the theorem, order 4 in all norms. That is, there is no superconvergence. This can also be seen from Figure 4.1 and 4.2 that $P_4$ solutions have no zero point (superconvergent point) each element.

In the last computation, we use $P_5$ finite elements, i.e., $k = 5$ in (2.4) and (2.7). The data are listed in Table 4.3. The order of convergence in $H(\text{div})$ norm is as proved in the theorem, order 5. But again, as $P_3$ elements, $P_5$ elements have a superconvergence in $L^2$ for both displacement and stress. We note that the computer accuracy limits the order of convergence on the highest level.

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Table 4.2. The errors $e_h = I_h u - u_h$ and $e_h = I_h \sigma - \sigma_h$, and the order of convergence, by the $P_4$ element ($k = 4$ in (2.4) and (2.7)), for (4.1).

|   | $\|e_h\|_{L^2(\Omega)}$ | $h^n$ | $\|e_h\|_{L^2(\Omega)}$ | $h^n$ | $\|\text{div}(e_h)\|_{L^2(\Omega)}$ | $h^n$ | $\text{dim } V_h$ | $\text{dim } \Sigma_h$ |
|---|--------------------------|-------|--------------------------|-------|---------------------------------|-------|------------------|------------------|
| 1 | 0.2480831                | 0.00  | 2.620225                 | 0.00  | 0.048588                        | 0.00  | 40               | 78               |
| 2 | 0.0135229                | 4.20  | 0.163156                 | 4.01  | 0.003620                        | 3.75  | 160              | 267              |
| 3 | 0.0011529                | 3.55  | 0.010783                 | 3.92  | 0.000243                        | 3.90  | 640              | 987              |
| 4 | 0.0000786                | 3.88  | 0.000684                 | 3.98  | 0.000015                        | 3.99  | 2560             | 3795             |
| 5 | 0.0000050                | 3.97  | 0.000043                 | 3.99  | 0.000001                        | 4.00  | 10240            | 14883            |

Figure 4.2. The solution of $(u_h)_1$ and the error by $P_3$ finite element on level 4. The error (bottom) for $(u_h)_1$ by $P_4$ finite element on level 3.

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Table 4.3. The errors $e_h = I_h u - u_h$ and $e_h = I_h \sigma - \sigma_h$, and the order of convergence, by the $P_5$ element ($k = 5$ in (2.4) and (2.7)), for (4.1).

| $\|e_h\|_{L^2(\Omega)}$ | $h^n$ | $\|\epsilon_h\|_{L^2(\Omega)}$ | $h^n$ | $\|\text{div}(\epsilon_h)\|_{L^2(\Omega)}$ | $h^n$ | $\dim V_h$ | $\dim \Sigma_h$ |
|--------------------------|-------|---------------------------------|-------|--------------------------------|-------|-----------|-----------|
| 1                        | 0.0199263 0.00 | 0.285759 0.00 | 0.006086 0.00 | 60 | 112 |
| 2                        | 0.0022430 3.15 | 0.024032 3.57 | 0.000509 3.58 | 240 | 395 |
| 3                        | 0.0000426 5.72 | 0.000025 4.97 | 0.00001 4.93 | 3840 | 5747 |
| 4                        | 0.0000007 5.92 | 0.000000 4.98 | 0.000000 4.98 | 15360 | 22627 |

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LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China. hujun@math.pku.edu.cn

Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA. szhang@udel.edu