Squares of Fibonacci-Like Numbers

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Abstract
We derive a general recurrence relation for squares of Fibonacci-like numbers. Various properties are developed, including double binomial summation identities.

1 Introduction

The Fibonacci numbers, \( F_j \), and the Lucas numbers, \( L_j \), \( j \in \mathbb{Z} \), are defined by:

\[
F_0 = 0, \quad F_1 = 1, \quad F_j = F_{j-1} + F_{j-2} \quad (j \geq 2), \quad F_{-j} = (-1)^{j-1} F_j
\]

(1.1)

and

\[
L_0 = 2, \quad L_1 = 1, \quad L_j = L_{j-1} + L_{j-2} \quad (j \geq 2), \quad L_{-j} = (-1)^j L_j.
\]

(1.2)

Both \((F_j)_{j \in \mathbb{Z}}\) and \((L_j)_{j \in \mathbb{Z}}\) are examples of a Fibonacci-like sequence. We define a Fibonacci-like sequence, \((G_j)_{j \in \mathbb{Z}}\), as one having the same recurrence relation as the Fibonacci sequence, but with arbitrary initial terms. Thus, given arbitrary integers \(G_0\) and \(G_1\), not both zero, we define

\[
G_j = G_{j-1} + G_{j-2} \quad (j \geq 2); \quad G_{-j} = G_{-j+2} - G_{-j+1}.
\]

(1.3)

and also extend the definition to negative subscripts by writing the recurrence relation as

\[
G_{-j} = G_{-j+2} - G_{-j+1}.
\]

(1.4)

We have [3, equation (1.5)]

\[
G_{-j} = (-1)^j (G_0 L_j - G_j).
\]

(1.5)

The identity (see Brousseau [5, equation (2)])

\[
F_{j-1}^2 + F_{j+2}^2 = 2F_j^2 + 2F_{j+1}^2,
\]

(1.6)

or, more generally,

\[
G_{j-1}^2 + G_{j+2}^2 = 2G_j^2 + 2G_{j+1}^2.
\]

(1.7)
is well known.

Less familiar are identities such as

\[ G_{j+2}^2 + 2G_{j-2}^2 = 3G_{j-1}^2 + 6G_j^2; \]  
(1.8)

\[ 3G_{j+3}^2 + G_{j-3}^2 = 16G_{j+1}^2 + 12G_j^2 \]  
(1.9)

and

\[ F_RF_{R+1}^2 - F_RF_{R-1}^2 = G_R^2 + F_{R+1}^2 - F_{R+1}F_{R-1}G_R^2. \]  
(1.10)

Our aim in writing this paper is to derive the identity

\[ F_{s}F_{m}F_{m-s}G_{j+k}^2 = F_{m-s}F_{m-k}F_{s-k}G_{j}^2 + (-1)^{s+k}F_{k}F_{m}F_{m-k}G_{j+s}^2 \]
\[ - (-1)^{s+k}F_{k}F_{s}F_{s-k}G_{j+m}^2, \]

of which (1.7), (1.8), (1.9) and (1.10) are particular cases, being evaluations at certain \( m, k, j \) and \( s \) choices.

Closed formulas are known for \( \sum_{j=0}^{n} x^j G_j \) and \( \sum_{j=0}^{n} G_{j+k}G_{j-k} \). We will extend these results by providing evaluations for \( \sum_{j=0}^{n} x^j G_j^2 \) and \( \sum_{j=0}^{n} x^j G_j G_{j+s} \) for integers \( n, s \) and \( k \) and arbitrary \( x \).

Finally, we will derive double binomial identities involving the squares of Fibonacci-like numbers.

## 2 Main identity

**Theorem 1.** If \( j, k, m \) and \( s \) are integers, then

\[ F_{s}F_{m}F_{m-s}G_{j+k}^2 = F_{m-s}F_{m-k}F_{s-k}G_{j}^2 + (-1)^{s+k}F_{k}F_{m}F_{m-k}G_{j+s}^2 \]
\[ - (-1)^{s+k}F_{k}F_{s}F_{s-k}G_{j+m}^2, \]

\( (2.1) \)

**Proof.** Setting \( m = 0 \) in the identity (see Adegoke [3, Theorem 1])

\[ F_{s-k}G_{j+m} = F_{m-k}G_{j+s} + (-1)^{s+k+1}F_{m-s}G_{j+k} \]  
(2.1)

gives

\[ (-1)^{k}F_{s-k}G_{j} = F_{s}G_{j+k} - F_{s}G_{j+s}, \]  
(2.2)

from which, by squaring and re-arranging, we get

\[ 2F_{s}F_{k}G_{j+k}G_{j+s} = F_{s}^2G_{j+k}^2 + F_{k}^2G_{j+s}^2 - F_{s-k}G_{j}^2. \]  
(2.3)

The statement of the theorem then follows by squaring identity (2.1) and using (2.3) to eliminate the cross-term \( G_{j+k}G_{j+s} \) from the right hand side, while making use also of the multiplication formula

\[ 5F_{m}F_{n} = L_{m+n} - (-1)^{n}L_{m-n}. \]  
(2.4)
3 Partial sums and generating function

Lemma 1 ([2, Lemma 2] Partial sum of a \((r+1)\)-term sequence). Let \((X_j)\) be any arbitrary sequence, where \(X_j, j \in \mathbb{Z}\), satisfies a \((r+1)\)-term recurrence relation \(X_j = f_1 X_{j-c_1} + f_2 X_{j-c_2} + \cdots + f_r X_{j-c_r}\), where \(f_1, f_2, \ldots, f_r\) are arbitrary non-vanishing complex functions, not dependent on \(j\), and \(c_1, c_2, \ldots, c_r\) are fixed integers. Then, the following summation identity holds for arbitrary \(x\) and non-negative integer \(n\):

\[
\sum_{j=0}^{n} x^j X_j = \sum_{m=1}^{r} \left\{ x^m f_m \left( \sum_{j=1}^{m} x^{-j} X_j - \sum_{j=n-c_m+1}^{n} x^j X_j \right) \right\} \left/ \left( 1 - \sum_{m=1}^{r} x^m f_m \right) \right.
\]

We note that a special case of Lemma 1 was proved by Zeitlin [7].

Theorem 2. The following identity holds for arbitrary \(x\) and integers \(k\) and \(n\):

\[
x \sum_{j=0}^{n} x^j G_{j+k}^2 = \left( x F_{k+1} F_{k-1} + F_{k+1} F_k - x^2 F_{k-1} F_k \right) S_{G,n}(x)
\]

\[
+ x F_{k-1} F_k \left( x^{n+1} G_n^2 - G_0^2 - G_1^2 + 2G_0 G_1 \right)
\]

\[
+ F_{k+1} F_k \left( x^{n+1} G_{n+1}^2 - G_0^2 \right),
\]

where

\[
S_{G,n}(x) = \sum_{j=0}^{n} x^j G_j^2 = -\left( \frac{(2x^2 + 2x - 1)G_0^2}{x^3 - 2x^2 - 2x + 1} \right) - \frac{(2x^2 - x)G_1^2}{x^3 - 2x^2 - 2x + 1}
\]

\[
+ \frac{x^2 G_2^2}{x^3 - 2x^2 - 2x + 1} + \frac{(2x^2 + 2x - 1)x^{n+1} G_{n+1}^2}{x^3 - 2x^2 - 2x + 1}
\]

\[
+ \frac{(2x - 1)x^{n+2} G_{n+2}^2}{x^3 - 2x^2 - 2x + 1} - \frac{x^{n+3} G_{n+3}^2}{x^3 - 2x^2 - 2x + 1}.
\]

Proof. First, the identity (3.1), derived in Adegoke [1, equation (3.1)], also follows from (1.7) and Lemma 1 with \(X_j = G_j^2\).

Note that

\[
S_{F,n}(x) = \sum_{j=0}^{n} x^j F_j^2 = \frac{x(1-x) - (1 - 2x - 2x^2)x^{n+1} F_{n+1}^2}{x^3 - 2x^2 - 2x + 1}
\]

\[
- \frac{(1 - 2x) x^{n+2} F_{n+2}^2 - x^{n+3} F_{n+3}^2}{x^3 - 2x^2 - 2x + 1},
\]

\[
(3.2)
\]

\[
S_{G,n}(1) = \sum_{j=0}^{n} G_j^2 = G_n G_{n+1} - G_0 G_1 + G_0^2,
\]

\[
(3.3)
\]

\[
S_{F,n}(1) = \sum_{j=0}^{n} F_j x^j = \frac{1}{2} F_{n+3}^2 - \frac{1}{2} F_{n+2}^2 - \frac{3}{2} F_{n+1}^2 = F_n F_{n+1},
\]

\[
(3.4)
\]

Setting \(s = 1\) and \(m = -1\) in the identity of Theorem 1 and re-arranging, we have

\[
G_{j+k}^2 = F_{k+1} F_{k-1} G_{j}^2 + F_{k+1} F_k G_{j+1}^2 - F_{k-1} F_k G_{j-1}^2,
\]

\[
(3.5)
\]

which allows us to write

\[
\sum_{j=0}^{n} x^j G_{j+k}^2 = F_{k+1} F_{k-1} \sum_{j=0}^{n} x^j G_j^2 + F_{k+1} F_k \sum_{j=0}^{n} x^j G_{j+1}^2 - F_{k-1} F_k \sum_{j=0}^{n} x^j G_{j-1}^2.
\]

\[
(3.6)
\]
Now,
\[ \sum_{j=0}^{n} x^j G_{j+1}^2 = \sum_{j=1}^{n+1} x^{j-1} G_j^2 = \frac{1}{x} \left( \sum_{j=0}^{n} x^j G_j^2 - G_0^2 + x^{n+1} G_{n+1}^2 \right) \] (3.7)
and
\[ \sum_{j=0}^{n} x^j G_{j-1}^2 = x \sum_{j=1}^{n-1} x^j G_j^2 = x \left( \sum_{j=0}^{n} x^j G_j^2 + \frac{1}{x} G_{j-1}^2 - x^n G_n^2 \right). \] (3.8)

Using (3.1), (3.7) and (3.8) in (3.6) produces the identity of Theorem 2.

Observe that setting \( x = -1 \) in (3.2) makes the right hand side to be an indeterminate form. Application of L'Hospital's rule however provides the evaluation of \( S_{F,n}(-1) \). Thus, we have
\[ 5(-1)^{n-1} S_{F,n}(-1) = 5(-1)^{n-1} \sum_{j=0}^{n} (-1)^j F_j^2 \]
\[ = (n+3)F_{n+3}^2 - (3n+8)F_{n+2}^2 + (n-1)F_{n+1}^2 + (-1)^{n-1}3. \] (3.9)

Setting \( x = 1 \) in the identity of Theorem 2, we have
\[ \sum_{j=0}^{n} G_{j+k}^2 = (F_{k+1}F_{k-1} + F_k^2)(G_nG_{n+1} - G_0G_1 + G_0^2) + F_{k-1}F_k(G_n - G_1 + G_0)(G_n + G_1 - G_0) \]
\[ + F_{k+1}F_k(G_{n+1} - G_0)(G_{n+1} + G_0). \] (3.10)

**Theorem 3** (Generating function of \( F_j^2 \)).
\[ \sum_{j=0}^{\infty} x^j F_j^2 = \frac{x(1-x)}{1-2x-2x^2+x^3}. \] (3.11)

*Proof.* Identity (3.2) as \( n \) approaches infinity; with \( x^n F_n^2 \to 0 \) as \( n \) approaches infinity.

Next, we provide an alternative evaluation of \( \sum_{j=0}^{n} x^j G_{j+k}^2 \), not requiring the initial values \( G_0 \) and \( G_1 \) of the sequence \( (G_r)_{r\in\mathbb{Z}} \).

**Theorem 4.** The following identity holds for arbitrary \( x \neq -1 \) and integers \( k \) and \( n \):
\[ \sum_{j=0}^{n} x^j G_{j+k}^2 = S_{F,n}(x) \left( G_{k+1}^2 + x G_k^2 + 2(1-x)G_k G_{k+1} \right) \]
\[ + (1 - x^{n+1}F_n^2) \left( G_k^2 - 2G_k G_{k+1} \right) + \left( \frac{1 + (-1)^n x^{n+1}}{1 + x} \right) 2G_k G_{k+1}. \]

*Proof.* Squaring the addition formula \( G_{j+k} = F_{j-1}G_k + F_j G_{k+1} \) gives
\[ G_{j+k}^2 = G_k^2 F_{j-1}^2 + G_{k+1}^2 F_j^2 + 2G_k G_{k+1} F_j F_{j-1}. \] (3.12)

But,
\[ F_j F_{j-1} = F_{j-1}(F_{j+1} - F_{j-1}) = F_{j-1}F_{j+1} - F_{j-1}^2 \]
\[ = F_j^2 - F_{j-1}^2 + (-1)^j, \] (3.13)
by Cassini’s identity. Using (3.13) in (3.12), multiplying through by \( x^j \) and summing over \( j \) yields the identity of the theorem.

\[ \square \]
An immediate application of Theorem 4 is to express $\sum_{j=0}^n x^j G_{j+k}^2$ in terms of $\sum_{j=0}^n x^j F_{j+k}^2$. Thus:

$$\sum_{j=0}^n x^j G_{j+k}^2 = (G_1^2 + xG_0^2 + 2(1-x)G_0G_1) \sum_{j=0}^n x^j F_{j+k}^2 + (1 - x^{n+1}F_n^2) (G_0^2 - 2G_0G_1)$$

$$+ \left( \frac{1 + (-1)^n x^{n+1}}{1+x} \right) 2G_0G_1. \quad (3.14)$$

Setting $x = 1$ in the identity of Theorem 4 produces

$$\sum_{j=0}^n G_{j+k}^2 = F_n F_{n+1}(G_{k+1}^2 + G_k^2) + (F_n^2 - 1)(G_{k+1}^2 - G_k^2)$$

$$+ (1 + (-1)^n) G_{k+1} G_k; \quad (3.15)$$

so that

$$\sum_{j=0}^{2n-1} G_{j+k}^2 = F_{2n-1} F_{2n}(G_{k+1}^2 + G_k^2) + (F_{2n-1}^2 - 1)(G_{k+1}^2 - G_k^2) \quad (3.16)$$

and

$$\sum_{j=0}^{2n} G_{j+k}^2 = F_{2n} F_{2n+1}(G_{k+1}^2 + G_k^2) + (F_{2n}^2 - 1)(G_{k+1}^2 - G_k^2)$$

$$+ 2G_{k+1} G_k. \quad (3.17)$$

In particular,

$$\sum_{j=0}^n F_{j+k}^2 = F_n F_{n+1} F_{2k+1} + (F_n^2 - 1) F_{2k} + (1 + (-1)^n) F_k F_{k+1}; \quad (3.18)$$

so that

$$\sum_{j=0}^{2n-1} F_{j+k}^2 = F_{2n} F_{2n+1} F_{2k+1} + (F_{2n-1}^2 - 1) F_{2k}$$

$$\quad (3.19)$$

and

$$\sum_{j=0}^{2n} F_{j+k}^2 = F_{2n} F_{2n+1} F_{2k+1} + (F_{2n}^2 - 1) F_{2k} + 2F_k F_{k+1}. \quad (3.20)$$

### 4 Sums of products

It is convenient to introduce the notation

$$A_n(x; k) = \sum_{j=0}^n x^j G_{j+k}^2, \quad (4.1)$$

with its evaluation as given in Theorem 2. Note that $S_{G,n}(x) = A_n(x; 0)$. 

**Theorem 5.** The following identity holds for integers \( n, s, k \) and arbitrary \( x \):
\[
2F_s F_k \sum_{j=0}^{n} x^j G_{j+k} G_{j+s} = F_s^2 A_n(x; k) + F_k^2 A_n(x; s) - F_{s-k}^2 S_{G,n}(x) .
\]

**Proof.** Multiply through \((3.3)\) by \(x^j\) and sum over \(j\). \(\square\)

In particular, setting \(x = 1\) in the identity of the theorem and making use of \((3.3)\) and \((3.10)\) produces

\[
2F_s F_k \sum_{j=0}^{n} G_{j+k} G_{j+s} = (F_s^2 (F_{k+1} F_{k-1} + F_k^2) + F_k^2 (F_{s+1} F_{s-1} + F_s^2) - F_{s-k}^2) (G_n G_{n+1} - G_1 G_0 + G_1 G_0) \quad (4.2)
\]

\[
+ F_s F_k (F_s F_{k-1} + F_k F_{s-1}) (G_n - G_1 + G_0) (G_n + G_1 - G_0)
\]

\[
+ F_s F_k (F_s F_{k-1} + F_k F_{s-1}) (G_n + G_0) (G_{n+1} + G_0) .
\]

Alternatively, setting \(x = 1\) in the identity of Theorem 5 and making use of \((3.3)\) and \((3.15)\) gives

\[
2F_s F_k \sum_{j=0}^{n} G_{j+k} G_{j+s} = F_n F_{n+1} \left( F_s^2 (G_{k+1}^2 + G_k^2) + F_k^2 (G_{s+1}^2 + G_s^2) \right)
\]

\[
+ (F_n^2 - 1) \left( F_s^2 (G_{k+1}^2 - G_{k-1}^2) + F_k^2 (G_{s+1}^2 - G_{s-1}^2) \right) \quad (4.3)
\]

\[
+ (1 + (-1)^n) \left( F_s^2 G_{k+1} G_k + F_k^2 G_{s+1} G_s \right)
\]

\[
- F_{s-k}^2 (G_n G_{n+1} - G_1 G_0 + G_0) .
\]

so that

\[
2F_s F_k \sum_{j=0}^{2n-1} G_{j+k} G_{j+s} = F_{2n-1} F_{2n} \left( F_s^2 (G_{k+1}^2 + G_k^2) + F_k^2 (G_{s+1}^2 + G_s^2) \right)
\]

\[
+ (F_{2n-1}^2 - 1) \left( F_s^2 (G_{k+1}^2 - G_{k-1}^2) + F_k^2 (G_{s+1}^2 - G_{s-1}^2) \right) \quad (4.4)
\]

and

\[
2F_s F_k \sum_{j=0}^{2n} G_{j+k} G_{j+s} = F_{2n} F_{2n+1} \left( F_s^2 (G_{k+1}^2 + G_k^2) + F_k^2 (G_{s+1}^2 + G_s^2) \right)
\]

\[
+ (F_{2n}^2 - 1) \left( F_s^2 (G_{k+1}^2 - G_{k-1}^2) + F_k^2 (G_{s+1}^2 - G_{s-1}^2) \right) \quad (4.5)
\]

\[
+ 2 \left( F_s^2 G_{k+1} G_k + F_k^2 G_{s+1} G_s \right)
\]

\[
- F_{s-k}^2 (G_{2n} G_{2n+1} - G_1 G_0 + G_0) .
\]

Identity \((4.2)\) and identities \((4.4)\) and \((4.5)\) subsume Berzsenyi’s results [4].

**Corollary 6.** The following identities hold for integer \(n\) and arbitrary \(x\):

\[
\sum_{j=0}^{n} x^j G_{j+1} G_{j-2} = (1 - x) S_{G,n}(x) + x^{n+1} G_n^2 - (G_1 - G_0)^2 , \quad (4.6)
\]

\[
2x \sum_{j=0}^{n} x^j G_j G_{j-1} = (1 - x^2) S_{G,n}(x) + x^{n+1} G_{n+1}^2 + x^{n+2} G_n^2 - x(G_1 - G_0)^2 - G_0^2 . \quad (4.7)
\]
In particular, we have
\[
\sum_{j=0}^{n} G_{j+1}G_{j-2} = (G_n - G_1 + G_0)(G_n + G_1 - G_0) \quad (4.8)
\]
and
\[
2 \sum_{j=0}^{n} G_jG_{j-1} = G_{n+1}G_{n-1} + (G_n - G_0)(G_n + G_0) + (G_1 - G_0)(2G_0 - G_1). \quad (4.9)
\]

**Theorem 7.** The following identity holds for integers \( k \) and \( n \) and arbitrary \( x \):
\[
(-1)^k 2x \sum_{j=0}^{n} x^j G_{j+k}G_{j-k} = (xL_k^2 - (1 + x^2)F_k^2 - 2xF_{k-1}F_{k+1})S_n(x)
\]
\[
+ xF_k^2 \left( x^{n+1}G_n^2 - G_0^2 - G_1^2 + 2G_0G_1 \right)
\]
\[
- F_k^2 \left( x^{n+1}G_{n+1}^2 - G_0^2 \right).
\]

In particular, we have
\[
(-1)^k 2 \sum_{j=0}^{n} G_{j+k}G_{j-k} = (L_k^2 - 2F_{k-1}F_{k+1} - 2F_k^2)(G_nG_{n+1} - G_0G_1 + G_0^2)
\]
\[
- F_k^2 \left( G_{n-1}G_{n+2} - 2G_0G_1 + G_1^2 \right). \quad (4.10)
\]

**Proof.** Setting \( r = 0 \) in the identity (see Adegoke [3, equation (2.12)])
\[
F_{2k}G_{j+r} = F_{r+k}G_{j+k} - F_{r-k}G_{j-k}
\]
gives
\[
L_kG_j = G_{j+k} + (-1)^k G_{j-k}, \quad (4.11)
\]
from which by squaring, we get
\[
L_k^2G_j^2 = G_{j+k}^2 + G_{j-k}^2 + 2(-1)^kG_{j+k}G_{j-k}. \quad (4.12)
\]
Multiply (4.13) by \( x^{j+1} \), re-arrange and sum over \( j \) to obtain
\[
(-1)^k 2x \sum_{j=0}^{n} x^j G_{j+k}G_{j-k} = xL_k^2 \sum_{j=0}^{n} x^j G_j^2 - x \sum_{j=0}^{n} x^j G_{j+k} - x \sum_{j=0}^{n} x^j G_{j-k},
\]
from which the result follows by using the identity of Theorem 2 to evaluate the last two sums on the right hand side. \( \square \)

**Corollary 8** (Generating function of \( G_{j+k}G_{j-k} \)).
\[
(-1)^k 2x \sum_{j=0}^{\infty} x^j G_{j+k}G_{j-k}
\]
\[
= \frac{(xL_k^2 - (1 + x^2)F_k^2 - 2xF_{k+1}F_{k-1})(x^2G_2^2 - (2x^2 + 2x - 1)G_0^2 - (2x^2 - x)G_1^2)}{x^3 - 2x^2 - 2x + 1}
\]
\[
- xF_k^2(G_1 - G_0)^2 + F_k^2G_0^2.
\]

7
5 Double binomial sums

Lemma 2 ([2, Lemma 5]). Let \((X_r)\) be any arbitrary sequence, \(X_r\) satisfying a four-term recurrence relation \(hX_r = f_1X_{r-a} + f_2X_{r-b} + f_3X_{r-c}\), where \(h, f_1, f_2\) and \(f_3\) are arbitrary non-vanishing functions and \(a, b\) and \(c\) are integers. Then, the following identities hold:

\[
\sum_{j=0}^{n} \sum_{i=0}^{j} \binom{n}{j} \binom{j}{i} f_3^{n-j} f_2^{n+j-i} f_1^{i} X_{r-cn+(c-b)j+(b-a)i} = h^n f_2^n X_r, \quad (5.1)
\]

\[
\sum_{j=0}^{n} \sum_{i=0}^{j} \binom{n}{j} \binom{j}{i} f_2^{n-j} f_3^{n+j-i} f_1^{i} X_{r-bn+(b-c)j+(c-a)i} = h^n f_3^n X_r, \quad (5.2)
\]

\[
\sum_{j=0}^{n} \sum_{i=0}^{j} \binom{n}{j} \binom{j}{i} f_1^{n-j} f_3^{n+j-i} f_2^{i} X_{r-an+(a-c)j+(c-b)i} = h^n f_3^n X_r, \quad (5.3)
\]

\[
\sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^i \binom{n}{j} \binom{j}{i} h^i f_3^{n-j} f_2^i X_{r-(c-a)n+(c-b)j+bi} = (-h)^n X_r, \quad (5.4)
\]

\[
\sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^i \binom{n}{j} \binom{j}{i} h^i f_3^{n-j} f_1^i X_{r-(c-b)n+(c-a)j+ai} = (-h)^n X_r \quad (5.5)
\]

and

\[
\sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^i \binom{n}{j} \binom{j}{i} h^i f_2^{n-j} f_1^i X_{r-(b-c)n+(b-a)j+ai} = (-h)^n X_r. \quad (5.6)
\]

Theorem 9. The following identities hold for non-negative integer \(n\) and integers \(s, k, m, r\):

\[
\sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^{i+(s+k+1)j} \binom{n}{j} \binom{j}{i} F_s^{n-j+i} F_k^{n+j} F_m^{2n-i} F_m^{n+j-i} G_r^{2} = (F_m F_k F_m^{2} F_m^{s} F_s^{s-k})^n G_r^{2}, \quad (5.7)
\]

\[
\sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^{j+(s+k)(i+j)} \binom{n}{j} \binom{j}{i} F_s^{n+j} F_k^{n-j+i} F_m^{2n-i} F_m^{n+j-i} G_r^{2} = (-1)^{(s+k-1)n} (F_m F_s F_m^{2} F_s^{m-k} F_s^{s-k})^n G_r^{2}, \quad (5.8)
\]

\[
\sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^{(s+k)(i+j)+i} \binom{n}{j} \binom{j}{i} F_s^{n+j} F_k^{n-j+i} F_m^{2n-i} F_m^{n+j-i} G_r^{2} = (-1)^{(s+k)n} (F_m F_s F_m^{2} F_s^{m-k} F_s^{s})^n G_r^{2}. \quad (5.9)
\]

Proof. Change \(j\) to \(r\) and re-arrange the identity of Theorem 1 as

\[
F_m^{s} F_m^{s-k} F_s^{s-k} G_r^{2} = (-1)^{s+k} F_k^{s} F_s^{s-k} G_r^{2} + (-1)^{s+k+1} F_k^{s} F_m^{s-k} G_r^{2} + F_s^{s} F_m^{s-k} G_r^{2}. \]

8
In Lemma 2 with $X_r = G_r^2$, set $h = F_{m-s}F_{m-k}F_{s-k}$, $f_1 = (-1)^{s+k}F_kF_sF_{s-k}$, $f_2 = (-1)^{s+k+1}F_kF_mF_{m-k}$, $f_3 = F_sF_mF_{m-s}G_{r+k}^2$, $a = -m$, $b = -s$ and $c = -k$ in identities (5.1) – (5.6).

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Concerned with sequences: