LOCALIZATION OF $u$-MODULES. IV.
LOCALIZATION ON $\mathbb{P}^{\mathfrak{b}}$

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1. Introduction

1.1. This article is a sequel to [FS]. Given a collection of $m$ finite factorizable sheaves $\{\mathcal{X}_\parallel\}$, we construct here some perverse sheaves over configuration spaces of points on a projective line $\mathbb{P}^{\mathfrak{b}}$ with $m$ additional marked points.

We announce here (with sketch proof) the computation of the cohomology spaces of these sheaves. They turn out to coincide with certain "semiinfinite" Tor spaces of the corresponding $u$-modules introduced by S.Arkhipov. For a precise formulation see Theorem 8.11.

This result is strikingly similar to the following hoped-for picture of affine Lie algebra representation theory, explained to us by A.Beilinson. Let $M_1, M_2$ be two modules over an affine Lie algebra $\hat{\mathfrak{g}}$ on the critical level. One hopes that there is a localization functor which associates to these modules perverse sheaves $\Delta(M_1), \Delta(M_2)$ over the semiinfinite flag space $\hat{G}/\hat{B}^0$. Suppose that $\Delta(M_1)$ and $\Delta(M_2)$ are equivariant with respect to the opposite Borel subgroups of $\hat{G}$. Then the intersection $S$ of their supports is finite dimensional, and one hopes that

$$R^\bullet \Gamma(S, \Delta(M_1) \otimes \Delta(M_2)) = \text{Tor}_{\hat{\mathfrak{g}}}^{\mathfrak{b}}(M_1, M_2)$$

where in the right hand side we have the Feigin (Lie algebra) semiinfinite homology.

As a corollary of Theorem 8.11 we get a description of local systems of conformal blocks in WZW models in genus zero (cf. [MS]) as natural subquotients of some semisimple local systems of geometric origin. In particular, these local systems are semisimple themselves.

In the next paper of the series (joint with R.Bezroukavnikov) we will explain how to glue factorizable sheaves into punctured curves of arbitrary genus. This will give rise to an example of a modular functor. In fact, the cohesive local system described in Chapter 1 of this paper is a genus 0 case of the general construction due to R.Bezroukavnikov.

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1.2. We are grateful to S.Arkhipov and G.Lusztig for the permission to use their unpublished results. Namely, Theorem 2.1 about braiding in the category $\mathcal{C}$ is due to G.Lusztig, and Chapter 2 (semiinfinite homological algebra in $\mathcal{C}$) is an exposition of the results due to S.Arkhipov.

We are also grateful to A.Kirillov, Jr. who explained to us how to handle the conformal blocks of non simply laced Lie algebras.

1.3. Unless specified otherwise, we will keep the notations of [FS]. For $\alpha = \sum_i a_i i \in \mathbb{N}[\mathbb{I}]$ we will use the notation $|\alpha| := \sum_i a_i$.

References to loc.cit. will look like Z.1.1 where Z=I, II or III.

We will keep assumptions of III.1.4 and III.16. In particular, a ”quantization” parameter $\zeta$ will be a primitive $l$-th root of unity where $l$ is a fixed positive number prime to 2, 3.
CHAPTER 1. Gluing over \( \mathbb{P}^k \)

2. Cohesive local system

2.1. Notations. Let \( \alpha \in \mathbb{N}[X] \), \( \alpha = \sum a_\mu \mu \). We denote by \( \text{supp}\alpha \) the subset of \( X \) consisting of all \( \mu \) such that \( a_\mu \neq 0 \). Let \( \pi : J \to X \) be an unfolding of \( \alpha \), that is a map of sets such that \( |\pi^{-1}(\mu)| = a_\mu \) for any \( \mu \in X \). As always, \( \Sigma_\pi \) denotes the group of automorphisms of \( J \) preserving the fibers of \( \pi \).

\( \mathbb{P}^k \) will denote a complex projective line. The \( J \)-th cartesian power \( \mathbb{P}^k \times J \) will be denoted by \( \mathbb{P}^k \times J \). The \( \Sigma_\pi \) acts naturally on \( \mathbb{P}^k \times J \), and the quotient space \( \mathbb{P}^k \times J / \Sigma_\pi \) will be denoted by \( \mathbb{P}_\alpha \).

\( \mathbb{P}_\alpha \) stands for the complement to diagonals in \( \mathbb{P}^k \times J \). We will also use the notation \( \mathbb{P}^k \times J / \Sigma_\pi \) by \( \mathbb{P}^k \times J / \Sigma_\pi \).

The natural projection \( \mathbb{P}^k \times J / \Sigma_\pi \to \mathbb{P}^k \times J / \Sigma_\pi \) will be denoted by \( \pi \), or sometimes by \( \pi_J \).

2.2. Let \( \mathbb{P}^{k_{\epsilon \leftarrow}} \) denote "the" projective line with fixed coordinate \( z \); \( D_{\epsilon} \subset \mathbb{P}^{k_{\epsilon \leftarrow}} \) the open disk of radius \( \epsilon \) centered at \( z = 0 \); \( D := D_1 \). We will also use the notation \( D_{(\epsilon, 1)} \) for the open annulus \( D = D_{\epsilon} \) (bar means the closure).

The definitions of \( D^J, D^*, D^{\alpha}, D^{\alpha_J}, D^{\alpha \alpha_J}, D^{\alpha \alpha}, T D^J, T D^{\alpha J}, T D^{\alpha \alpha}, \) etc., copy the above definitions, with \( D \) replacing \( \mathbb{P}^k \).

2.3. Given a finite set \( K \), let \( \mathbb{P}^{k_{\alpha_k}} \) denote the space of \( K \)-tuples \( (u_k)_{k \in K} \) of algebraic isomorphisms \( \mathbb{P}^{k_{\alpha_k}} \) such that the images \( u_k(D) \) do not intersect.

Given a \( K \)-tuple \( \bar{\alpha} = (\alpha_k) \in \mathbb{N}[X]^K \) such that \( \alpha = \sum_k \alpha_k \), define a space

\[ \mathcal{P}^{\bar{\alpha}} := \mathbb{P}^{k_{\alpha_k}} \times \prod_{\| \in K} T D^{\alpha_k} \]

and an open subspace

\[ \mathcal{P}^{\alpha} := \mathbb{P}^{k_{\alpha_k}} \times \prod_{\| \in K} T D^{\alpha_k} \].
We have an evident ”substitution” map
\[ q_\vec{\alpha} : \mathcal{P}^\vec{\alpha} \longrightarrow \mathcal{T}\mathcal{P}^\alpha \]
which restricts to \( q_{\vec{\alpha}} : \mathcal{P}^{i\vec{\alpha}} \longrightarrow \mathcal{T}\mathcal{P}^{i\alpha} \).

2.3.1. In the same way we define the spaces \( TD^\vec{\alpha}, TD^{o\vec{\alpha}} \).

2.3.2. Suppose that we have an epimorphism \( \xi : L \longrightarrow K \), denote \( L_k := \xi^{-1}(k) \). Assume that each \( \alpha_k \) is in turn decomposed as \( \alpha_k = \sum_{l \in L_k} \alpha_l \), \( \alpha_l \in \mathbb{N}[X]_{\leq} \).

set \( \vec{\alpha}_k = (\alpha_l) \in \mathbb{N}[X]_{\leq}^L \). Set \( \vec{\alpha}_L = (\alpha_l) \in \mathbb{N}[X]_{\leq}^L \).

Let us define spaces
\[ \mathcal{P}^{\vec{\alpha}_L;\vec{\xi}} = \mathcal{P}^\sum_{\parallel \in K} \mathcal{T}\mathcal{D}^{\vec{\alpha}_\parallel} \]
and
\[ \mathcal{P}^{i\vec{\alpha}_L;\vec{\xi}} = \mathcal{P}^\sum_{\parallel \in K} \mathcal{T}\mathcal{D}^{i\vec{\alpha}_\parallel}. \]

We have canonical substitution maps
\[ q_1^{\vec{\alpha}_L;\vec{\xi}} : \mathcal{P}^{\vec{\alpha}_L;\vec{\xi}} \longrightarrow \mathcal{P}^\vec{\alpha} \]
and
\[ q_2^{\vec{\alpha}_L;\vec{\xi}} : \mathcal{P}^{\vec{\alpha}_L;\vec{\xi}} \longrightarrow \mathcal{P}^{\vec{\alpha}_L}. \]

Obviously,
\[ q_{\vec{\alpha}} \circ q_1^{\vec{\alpha}_L;\vec{\xi}} = q_{\vec{\alpha}_L} \circ q_2^{\vec{\alpha}_L;\vec{\xi}}. \]

2.4. Balance function. Consider a function \( n : X \longrightarrow \mathbb{Z}[\frac{x}{\det A}] \) such that
\[ n(\mu + \nu) = n(\mu) + n(\nu) + \mu \cdot \nu \]
It is easy to see that \( n \) can be written in the following form:
\[ n(\mu) = \frac{1}{2} \mu \cdot \mu + \mu \cdot \nu_0 \]
for some \( \nu_0 \in X \). From now on we fix such a function \( n \) and hence the corresponding \( \nu_0 \).
2.5. For an arbitrary $\alpha \in N[X]$, let us define a one-dimensional local system $I_\alpha^D$ on $TD^\alpha$. We will proceed in the same way as in III.3.1.

Pick an unfolding of $\alpha$, $\pi : J \to X$. Define a local system $I_J^D$ on $TD^\alpha$ as follows: its stalk at each point $((\tau_j, x_j))$ where all $x_j$ are real, and all the tangent vectors $\tau_j$ are real and directed to the right, is $B$. Monodromies are:

— $x_i$ moves counterclockwise around $x_j$: monodromy is $\zeta^{-2\pi(i)\cdot \pi(j)}$;
— $\tau_j$ makes a counterclockwise circle: monodromy is $\zeta^{-2\pi(\pi(j))}$.

This local system has an evident $\Sigma_\pi$-equivariant structure, and we define a local system $I_\alpha^D$ as $I_\alpha^D = (\pi_*I_J^D)^{sgn}$ where $\pi : TD^\alpha \to (TD^\alpha)$ is the canonical projection, and $(\bullet)^{sgn}$ denotes the subsheaf of skew $\Sigma_\pi$-invariants.

2.6. We will denote the unique homomorphism $N[X] \to X$ identical on $X$, as $\alpha \mapsto \alpha^\sim$.

2.6.1. **Definition.** An element $\alpha \in N[X]$ is called admissible if $\alpha^\sim \equiv -2\nu_0 \mod lY$.

2.7. We have a canonical “1-jet at 0” map $p_K : \tilde{P}^K \to TP^{lK}$

2.8. **Definition.** A cohesive local system (CLS) (over $\mathbb{P}^{lK}$) is the following collection of data:

(i) for each admissible $\alpha \in N[X]$ a one-dimensional local system $I_\alpha^\sim$ over $TP^{l\alpha}$;
(ii) for each decomposition $\alpha = \sum_{k \in K} \alpha_k$, $\alpha_k \in N[X]$, a factorization isomorphism $\phi_{\tilde{a}} : q_{\tilde{a}}^*I_\alpha^\sim \to \sqrt{k\cdot \pi_K^*I_\alpha^{lK}} \boxtimes I_D^{\alpha_{\tilde{a}}}$

Here $\alpha_K := \sum_{k} \alpha_k^\sim \in N[X]$ (note that $\alpha_K$ is obviously admissible); $\pi_K : TP^{lK} \to TP^{l\alpha}$ is the symmetrization map.

These isomorphisms must satisfy the following Associativity axiom. In the assumptions of 2.3.3 the equality $\phi_{\tilde{a}L,\xi} \circ q_{\tilde{a}L,\xi}^{l*}(\phi_{\tilde{a}L}) = q_{\tilde{a}L,\xi}^{2*}(\phi_{\tilde{a}L})$ should hold. Here $\phi_{\tilde{a}L,\xi}$ is induced by the evident factorization isomorphisms for local systems on the disk $I_{D,\xi}^{\tilde{a}}$.

Morphisms between CLS’s are defined in the obvious way.
2.9. **Theorem.** Cohesive local systems over $\mathbb{P}^r$ exist. Every two CLS’s are isomorphic. The group of automorphisms of a CLS is $B^*$. 

This theorem is a particular case of a more general theorem, valid for curves of arbitrary genus, to be proved in Part V. We leave the proof in the case of $\mathbb{P}^r$ to the interested reader.

3. **Gluing**

3.1. Let us define an element $\rho \in X$ by the condition $\langle \rho, i \rangle = 1$ for all $i \in I$. From now on we choose a balance function $n$, cf. 2.4, in the form

$$n(\mu) = \frac{1}{2} \mu \cdot \mu + \mu \cdot \rho.$$ 

It has the property that $n(-i') = 0$ for all $i \in I$. Thus, in the notations of loc. cit. we set $\nu_0 = \rho$.

We pick a corresponding CLS $I = \{I_\beta, \beta \in N[X]\}$.

Given $\alpha = \sum a_i i \in N[I]$ and $\bar{\mu} = (\mu_k) \in X^K$, we define an element

$$\alpha_i = \sum a_i \cdot (-i') + \sum_k \mu_k \in N[X]$$ 

where the sum in the right hand side is a formal one. We say that a pair $(\bar{\mu}, \alpha)$ is admissible if $\alpha_i$ is admissible in the sense of the previous section, i.e.

$$\sum_k \mu_k - \alpha \equiv -2\rho \text{ mod } lY.$$ 

Note that given $\bar{\mu}$, there exists $\alpha \in N[I]$ such that $(\bar{\mu}, \alpha)$ is admissible if and only if $\sum_k \mu_k \in Y$; if this holds true, such elements $\alpha$ form an obvious countable set.

We will denote by

$$e : N[I] \longrightarrow N[X]$$ 

a unique homomorphism sending $i \in I$ to $-i' \in X$.

3.2. Let us consider the space $T^\mathbb{P}_\alpha \alpha \in X$; its points are quadruples $((z_k), (\tau_k), (x_j), (\omega_j))$ where $(z_k) \in \mathbb{P}^r$, $\tau_i$ — a non-zero tangent vector to $\mathbb{P}^r$ at $z_k$, $(x_j) \in \mathbb{P}^{(\alpha)}$, $\omega_j$ — a non-zero tangent vector at $x_j$. To a point $z_k$ is assigned a weight $\mu_k$, and to $x_j$ — a weight $-\pi(j)'$. Here $\pi : J \longrightarrow I$ is an unfolding of $\alpha$ (implicit in the notation $(x_j) = (x_j)_{j \in J}$).

We will be interested in some open subspaces:

$$T^\mathbb{P}_\alpha \alpha \in X$$ 

and

$$T^\mathbb{P}_\alpha \alpha \in X.$$
whose points are quadruples \(((z_k), (\tau_k), (x_j), (\omega_j)) \in \dot{T}\mathcal{P}_\mu^\alpha\) with all \(z_k \neq x_j\). We have an obvious symmetrization projection

\[ p^\alpha_\mu : T\mathcal{P}_\mu^\alpha \to \mathcal{T}\mathcal{P}^\alpha_\mu. \]

Define a space

\[ \mathcal{P}_\mu^\alpha = T\mathcal{T}\mathcal{P}^{\langle K \rangle} \times \mathcal{P}^{(\alpha)}; \]

its points are triples \(((z_k), (\tau_k), (x_j))\) where \((z_k), (\tau_k)\) and \((x_j)\) are as above; and to \(z_k\) and \(x_j\) the weights as above are assigned. We have the canonical projection

\[ \dot{T}\mathcal{P}_\mu^\alpha \to \mathcal{P}_\mu^\alpha. \]

We define the open subspaces

\[ \mathcal{P}_\mu^\alpha \subset \mathcal{P}_\mu^{\bullet \alpha} \subset \mathcal{P}_\mu^\alpha. \]

Here the \(\bullet\)-subspace (resp., \(o\)-subspace) consists of all \(((z_k), (\tau_k), (x_j))\) with \(z_k \neq x_j\) for all \(k, j\) (resp., with all \(z_k\) and \(x_j\) distinct).

We define the principal stratification \(\mathcal{S}\) of \(\mathcal{P}_\mu^\alpha\) as the stratification generated by subspaces \(z_k = x_j\) and \(x_j = x_j'\) with \(\pi(j) \neq \pi(j')\). Thus, \(\mathcal{P}_\mu^\alpha\) is the open stratum of \(\mathcal{S}\). As usually, we will denote by the same letter the induced stratifications on subspaces.

The above projection restricts to

\[ T\mathcal{P}_\mu^\alpha \to \mathcal{P}_\mu^\alpha. \]

3.3. Factorization structure.

3.3.1. Suppose we are given \(\vec{\alpha} \in \mathbb{N}[\mathbb{I}]^K\), \(\beta \in \mathbb{N}[\mathbb{I}]\); set \(\alpha := \sum_k \alpha_k\). Define a space

\[ \mathcal{P}_\mu^{\vec{\alpha}, \beta} \subset \tilde{T}\mathcal{K} \times \prod_{\parallel} D^{\alpha_{\parallel}} \times \mathcal{P}^{(\beta)} \]

consisting of all collections \(((u_k), ((x_j^{(k)})_k), (y_j))\) where \((u_k) \in \tilde{T}\mathcal{K}\), \((x_j^{(k)})_k \in D^{\alpha_{\parallel}}\), \((y_j) \in \mathcal{P}^{(\beta)}\), such that

\[ y_j \in \mathbb{P}^{\beta_k} - \bigcup_{\gamma \in K} \tilde{\gamma}(\mathbb{D}) \]

for all \(j\) (the bar means closure).

We have canonical maps

\[ q_{\vec{\alpha}, \beta} : \mathcal{P}_\mu^{\vec{\alpha}, \beta} \to \mathcal{P}_\mu^{\alpha + \beta}; \]

assigning to \(((u_k), ((x_j^{(k)})_k), (y_j))\) a configuration \((u_k(0)), (u_k(\tau)), (u_k(x_j^{(k)})), (y_j))\), where \(\tau\) is the unit tangent vector to \(D\) at 0, and

\[ p_{\vec{\alpha}, \beta} : \mathcal{P}_\mu^{\vec{\alpha}, \beta} \to \prod_{\parallel} D^{\alpha_{\parallel}} \times \mathcal{P}_\mu^{* \beta_{\vec{\alpha}}}; \]

sending \(((u_k), ((x_j^{(k)})_k), (y_j))\) to \(((u_k(0)), (u_k(\tau)), (y_j))\).


3.3.2. Suppose we are given \( \alpha, \beta \in \mathbb{N}[\mathbb{I}] \), \( \gamma \in \mathbb{N}[\mathbb{I}] \); set \( \alpha := \sum_k \alpha_k, \beta := \sum_k \beta_k \).

Define a space \( D^{\alpha, \beta} \) consisting of couples \((D_\varepsilon, (x_j))\) where \( D_\varepsilon \subset D \) is some smaller disk \((0 < \varepsilon < 1)\), and \((x_j) \in D^{\alpha+\beta}\) is a configuration such that \( \alpha \) points dwell inside \( D_\varepsilon \), and \( \beta \) points — outside \( D_\varepsilon \). We have an evident map

\[
q_{\alpha, \beta} : D^{\alpha, \beta} \longrightarrow D^{\alpha+\beta}.
\]

Let us define a space

\[
P_{\alpha, \beta, \gamma} \subset \bar{\mathcal{P}}^K \times \prod_{\| \in \mathcal{K}} D^{\alpha_\|, \beta_\|} \times \mathcal{P}^{(\gamma)}
\]

consisting of all triples \( ((u_k), x, (y_j)) \) where \((u_k) \in \bar{\mathcal{P}}^K, x \in \prod_k D^{\alpha_k, \beta_k}, (y_j) \in \mathcal{P}^{(\gamma)}\) such that

\[
y_j \in \mathbb{P}^{\kappa_k} - \bigcup_{\gamma} \cong_{\gamma}(D).
\]

We have obvious projections

\[
q_{1, \alpha, \beta, \gamma} : P_{\mu}^{\alpha, \beta, \gamma} \longrightarrow P_{\mu}^{\alpha+\beta, \gamma}
\]

and

\[
q_{2, \alpha, \beta, \gamma} : P_{\mu}^{\alpha, \beta, \gamma} \longrightarrow P_{\mu}^{\alpha, \beta+\gamma}
\]

such that

\[
q_{\alpha+\beta, \gamma} \circ q_{1, \alpha, \beta, \gamma} = q_{\alpha, \beta+\gamma} \circ q_{2, \alpha, \beta, \gamma}.
\]

We will denote the last composition by \( q_{\alpha, \beta, \gamma} \).

We have a natural projection

\[
p_{\alpha, \beta, \gamma} : P_{\mu}^{\alpha, \beta, \gamma} \longrightarrow \prod D^{\alpha_\|}_k \times \prod D^{\beta_\|}_k \times \mathcal{P}^{\gamma}_{\mu=\alpha-\beta}.
\]

3.4. Let us consider a local system \( p^{\star \alpha}_{\mu} T^{\alpha}_{\mu} \) over \( T \mathcal{P}^{\alpha}_{\mu} \). By our choice of the balance function \( n \), its monodromies with respect to the rotating of tangent vectors \( \omega_j \) at points \( x_j \) corresponding to negative simple roots, are trivial. Therefore it descends to a unique local system over \( \mathcal{P}^{\alpha}_{\mu} \), to be denoted by \( T^{\alpha}_{\mu} \).

We define a perverse sheaf

\[
T^{\alpha}_{\mu} := |_{\alpha} \mathcal{I}^{\star \alpha}_{\mu} [\dim \mathcal{P}^{\alpha}_{\mu}] \in \mathcal{M}(\mathcal{P}^{\alpha}_{\mu}; S).
\]

3.5. **Factorizable sheaves over \( \mathbb{P}^{\kappa_k} \).** Suppose we are given a \( K \)-tuple of FFS's \( \{X_{\|}\}, X_{\|} \in \mathcal{FS}_{\|}, \| \in \mathcal{K}, \| \in \mathcal{X}/\mathcal{Y} \), where \( \sum_k c_k = 0 \). Let us pick \( \mu = (\mu_k) \geq (\lambda(X_{\|})) \).

Let us call a **factorizable sheaf over \( \mathbb{P}^{\kappa_k} \)** obtained by gluing the sheaves \( X_{\|} \) the following collection of data which we will denote by \( g(\{X_{\|}\}) \).

(i) For each \( \alpha \in \mathbb{N}[\mathbb{I}] \) such that \( (\mu, \alpha) \) is admissible, a sheaf \( X^{\alpha}_{\mu} \in \mathcal{M}(\mathcal{P}^{\alpha}_{\mu}; S) \).
(ii) For each $\vec{\alpha} = (\alpha_k) \in \mathbb{N}[^I]^K$, $\beta \in \mathbb{N}[^I]$ such that $(\mu, \alpha + \beta)$ is admissible (where $\alpha = \sum \alpha_k$), a factorization isomorphism

$$\phi_{\vec{\alpha}, \beta} : q^*_{\vec{\alpha}, \beta} \mathcal{X}_\mu^{\alpha + \beta} \sim \sqrt{\mu} (\prod_{\bar{i} \in \mathcal{K}} \mathcal{X}_{\mu_{[i]}}^{\alpha_{[i]}}) \boxtimes \mathcal{I}_\mu^{\beta - \vec{\alpha}}.$$

These isomorphisms should satisfy

**Associativity property.** The following two isomorphisms

$$q^*_{\vec{\alpha}, \vec{\beta}, \gamma} \mathcal{X}_\mu^{\alpha + \beta + \gamma} \sim \sqrt{\vec{\alpha}, \vec{\beta}, \gamma} (\prod_{\bar{i} \in \mathcal{K}} \mathcal{X}_{\mu_{[i]}}^{\alpha_{[i]}}) \boxtimes (\prod_{\bar{i} \in \mathcal{K}} \mathcal{I}_{\mu_{[i]} - \alpha_{[i]}}^{\beta_{[i]}}) \boxtimes \mathcal{I}_{\mu_{[i]} - \vec{\alpha}_{[i]} - \vec{\beta}_{[i]}}^{\gamma_{[i]}}$$

are equal:

$$\psi_{\vec{\beta}, \gamma} \circ q^*_{\vec{\alpha}, \vec{\beta}, \gamma} (\phi_{\vec{\alpha}, \beta + \gamma}) = \phi_{\vec{\alpha}, \vec{\beta}} \circ q^*_{\vec{\alpha}, \vec{\beta}, \gamma} (\phi_{\vec{\alpha} + \vec{\beta}, \gamma}).$$

Here $\psi_{\vec{\beta}, \gamma}$ is the factorization isomorphism for $\mathcal{I}$, and $\phi_{\vec{\alpha}, \vec{\beta}}$ is the tensor product of factorization isomorphisms for the sheaves $\mathcal{X}$.  

3.6. **Theorem.** There exists a unique up to a canonical isomorphism factorizable sheaf over $\mathbb{P}^k$ obtained by gluing the sheaves $\{\mathcal{X}_{[i]}\}$.

**Proof** is similar to III.10.3. $\square$
CHAPTER 2. Semiinfinite cohomology.

In this chapter we discuss, following essentially [Ar], the "Semiinfinite homological algebra" in the category $\mathcal{C}$.

4. Semiinfinite functors $\text{Ext}$ and $\text{Tor}$ in $\mathcal{C}$

4.1. Let us call an $\mathfrak{u}$-module $\mathfrak{u}^-$-induced (resp., $\mathfrak{u}^+$-induced) if it is induced from some $\mathfrak{u}^{\geq 0}$ (resp., $\mathfrak{u}^{\leq 0}$)-module.

4.1.1. Lemma. If $M$ is a $\mathfrak{u}^-$-induced, and $N$ is $\mathfrak{u}^+$-induced then $M \otimes_B N$ is $\mathfrak{u}$-projective.

Proof. An induced module has a filtration whose factors are corresponding Verma modules. For Verma modules the claim is easy. $\Box$

4.2. Definition. Let $M^\bullet = \oplus_{\lambda \in X} M^\bullet_\lambda$ be a complex (possibly unbounded) in $\mathcal{C}$. We say that $M^\bullet$ is concave (resp. convex) if it satisfies the properties (a) and (b) below.

(a) There exists $\lambda_0 \in X$ such that for any $\lambda \in X$, if $M^\bullet_\lambda \neq 0$ then $\lambda \geq \lambda_0$ (resp. $\lambda \leq \lambda_0$).
(b) For any $\mu \in X$ the subcomplex $\oplus_{\lambda \leq \mu} M^\bullet_\lambda$ (resp. $\oplus_{\lambda \leq \mu} M^\bullet_\lambda$) is finite.

We will denote the category of concave (resp., convex) complexes by $\mathcal{C}^\uparrow$ (resp., $\mathcal{C}^\downarrow$).

4.3. Let $V \in \mathcal{C}$. We will say that a surjection $\phi : P \rightarrow V$ is good if it satisfies the following properties:

(a) $P$ is $\mathfrak{u}^-$-induced;
(b) Let $\mu \in \text{supp } P$ be an extremal point, that is, there is no $\lambda \in \text{supp } P$ such that $\lambda > \mu$. Then $\mu \not\in \text{supp}(\ker \phi)$.

For any $V$ there exists a good surjection as above. Indeed, denote by $p$ the projection $p : M(0) \rightarrow L(0)$, and take for $\phi$ the map $p \otimes \text{id}_V$.

4.4. Iterating, we can construct a $\mathfrak{u}^-$-induced convex left resolution of $B = L(0)$. Let us pick such a resolution and denote it by $P^\bullet_\downarrow$:

$$\ldots \rightarrow P^\bullet_{-1} \rightarrow P^\bullet_0 \rightarrow L(0) \rightarrow 0$$

We will denote by

$$^\ast : \mathcal{C} \rightarrow \mathcal{C}^{\downarrow \downarrow}$$

the rigidity in $\mathcal{C}$ (see e.g. [AJS], 7.3). We denote by $P^\bullet_\uparrow$ the complex $(P^\bullet_\downarrow)^\ast$. It is a $\mathfrak{u}^-$-induced concave right resolution of $B$. The fact that $P^\bullet_\uparrow$ is $\mathfrak{u}^-$-induced follows since $\mathfrak{u}^-$ is Frobenius (see e.g. [A]).
4.5. In a similar manner, we can construct a $u^+$-induced concave left resolution of $B$. Let us pick such a resolution and denote it $P^\bullet_{\langle}:$

$$\ldots \rightarrow P_{\langle}^{-1} \rightarrow P_{\langle}^0 \rightarrow L(0) \rightarrow 0$$

We denote by $P^\bullet_{\rangle}$ the complex $(P^\bullet_{\langle})^\ast$. It is a $u^+$-induced convex right resolution of $B$.

4.5.1. For $M \in \mathcal{C}$ we denote by $P^\bullet_{\langle}(M)$ (resp., $P^\bullet_{\rangle}(M), P^\bullet_{\downarrow}(M)$) the resolution $P^\bullet_{\langle} \otimes_B M$ (resp., $P^\bullet_{\rangle} \otimes_B M, P^\bullet_{\downarrow} \otimes_B M$) of $M$.

4.6. We denote by $C_{\triangledown}$ the category of $X$-graded right $u$-modules $V = \bigoplus_{\lambda \in X} V_\lambda$ such that

$$K_i|_{V_\lambda} = \zeta^{-(i,\lambda)}$$

(note the change of a sign!), the operators $E_i, F_i$ acting as $E_i : V_\lambda \rightarrow V_{\lambda+i}, F_i : V_\lambda \rightarrow V_{\lambda-i}$. 

4.6.1. Given $M \in \mathcal{C}$, we define $M^\triangledown \in C_{\triangledown}$ as follows: $(M^\triangledown)_\lambda = (M_{-\lambda})^\ast$, $E_i : (M^\triangledown)_\lambda \rightarrow (M^\triangledown)_{\lambda+i}$ is the transpose of $E_i : M_{-\lambda-i} \rightarrow M_{-\lambda}$, similarly, $F_i$ on $M^\triangledown$ is the transpose of $F_i$ on $M$.

This way we get an equivalence

$$\triangledown : C_{\triangledown} \rightarrow C_{\triangledown}$$

Similarly, one defines an equivalence $\triangledown : C_{\triangledown} \rightarrow \mathcal{C}$, and we have an obvious isomorphism $\triangledown \circ \triangledown \cong \text{Id}$.

4.6.2. Given $M \in \mathcal{C}$, we define $sM \in C_{\triangledown}$ as follows: $(sM)_\lambda = M_{\lambda}$; $xg = (sg)x$ for $x \in M, g \in u$ where

$$s : u \rightarrow u^{\text{opp}}$$

is the antipode defined in [AJS], 7.2. This way we get an equivalence

$$s : C \rightarrow C_{\triangledown}$$

One defines an equivalence $s : C_{\triangledown} \rightarrow C$ in a similar manner. The isomorphism of functors $s \circ s \cong \text{Id}$ is constructed in loc. cit., 7.3.

Note that the rigidity $*$ is just the composition

$$* = s \circ \triangledown.$$

4.6.3. We define the categories $C_{\triangledown}^\uparrow$ and $C_{\triangledown}^\downarrow$ in the same way as in [12].

For $V \in C_{\triangledown}$ we define $P^\bullet_{\downarrow}(V)$ as

$$P^\bullet_{\downarrow}(V) = sP^\bullet_{\rangle}(sV);$$

and $P^\bullet_{\rangle}(V), P^\bullet_{\downarrow}(V), P^\bullet_{\langle}(V)$ in a similar way.
4.7. **Definition.** (i) Let $M, N \in \mathcal{C}$. We define
\[
\text{Ext}^\infty_{\mathcal{C}}(M, N) := H^\bullet(\text{Hom}_\mathcal{C}(P^\bullet_\downarrow(M), P^\bullet_\uparrow(N))).
\]
(ii) Let $V \in \mathcal{C}_\nabla, N \in \mathcal{C}$. We define
\[
\text{Tor}^\infty_{\mathcal{C}}(V, N) := H^{-\bullet}(P^\bullet_\uparrow(V) \otimes_\mathcal{C} P^\bullet_\downarrow(N)). \quad \Box
\]
Here we understand $\text{Hom}_\mathcal{C}(P^\bullet_\downarrow(M), P^\bullet_\uparrow(N))$ and $P^\bullet_\uparrow(V) \otimes_\mathcal{C} P^\bullet_\downarrow(N)$ as simple complexes associated with the corresponding double complexes. Note that due to our boundedness properties of weights of our resolutions, these double complexes are bounded. Therefore all $\text{Ext}^\infty_{\mathcal{C}}$ and $\text{Tor}^\infty_{\mathcal{C}}$ spaces are finite dimensional, and are non-zero only for finite number of $i \in \mathbb{Z}$.

4.8. **Lemma.** For $M, N \in \mathcal{C}$ there exist canonical nondegenerate pairings
\[
\text{Ext}^\infty_{\mathcal{C}}(M, N) \otimes \text{Tor}^\infty_{\mathcal{C}}(N^\vee, M) \rightarrow B.
\]

**Proof.** There is an evident non-degenerate pairing
\[
\text{Hom}_\mathcal{C}(M, N) \otimes (N^\vee \otimes_\mathcal{C} M) \rightarrow B.
\]
It follows that the complexes computing Ext and Tor are also canonically dual. \Box

4.9. **Theorem.** (i) Let $M, N \in \mathcal{C}$. Let $R^\bullet_\downarrow(M)$ be a $\mathfrak{u}^+$-induced convex right resolution of $M$, and $R^\bullet_\uparrow(N)$ — a $\mathfrak{u}^-$-induced concave right resolution of $N$. Then there is a canonical isomorphism
\[
\text{Ext}^\infty_{\mathcal{C}}(M, N) \cong H^\bullet(\text{Hom}_\mathcal{C}(R^\bullet_\downarrow(M), R^\bullet_\uparrow(N))).
\]
(ii) Let $V \in \mathcal{C}_\nabla, N \in \mathcal{C}$. Let $R^\bullet_\uparrow(V)$ be a $\mathfrak{u}^-$-induced convex left resolution of $V$, and $R^\bullet_\downarrow(N)$ — a $\mathfrak{u}^+$-induced convex right resolution of $N$ lying in $\mathcal{C}_\downarrow$. Then there is a canonical isomorphism
\[
\text{Tor}^\infty_{\mathcal{C}}(V, N) \cong H^{-\bullet}(R^\bullet_\uparrow(V) \otimes_\mathcal{C} R^\bullet_\downarrow(N)).
\]

**Proof** will occupy the rest of the section.

4.10. **Lemma.** Let $V \in \mathcal{C}_\nabla$; let $R^\bullet_i, i = 1, 2$, be two $\mathfrak{u}^-$-induced convex left resolutions of $V$. There exists a third $\mathfrak{u}^-$-induced convex left resolution $R^\bullet$ of $V$, together with two termwise surjective maps
\[
R^\bullet \rightarrow R^\bullet_i, i = 1, 2,
\]
inducing identity on $V$.

**Proof.** We will construct $R^\bullet$ inductively, from right to left. Let
\[
R^\bullet_i : \ldots \rightarrow R^0_i \xrightarrow{\xi_i} R^{i-1}_i \xrightarrow{\xi_{i-1}} \ldots \rightarrow V.
\]
First, define $L_0 := R_1^0 \times_V R_0^0$. We denote by $\delta$ the canonical map $L^0 \to V$, and by $q_i^0$ the projections $L^0 \to R_i^0$. Choose a good surjection $\phi_0 : R^0 \to L^0$ and define $p_i^0 := q_i^0 \circ \phi_0 : R_i^0 \to R_i^0$, $\epsilon := \delta \circ \phi_0 : R^0 \to V$.

Set $K_i^{-1} := \ker \epsilon_i; K := \ker \epsilon$. The projections $p_i^0$ induce surjections $p_i^0 : K^{-1} \to K_i^{-1}$. Let us define

$$L^{-1} := \ker((d_1^0 - p_1^0, d_2^0 - p_2^0) : R_1^{-1} \oplus K^{-1} \oplus R_2^{-1} \to K_1^{-1} \oplus K_2^{-1}).$$

We have canonical projections $q_i^{-1} : L^{-1} \to R_i^{-1}$, $\delta_i^{-1} : L^{-1} \to K^{-1}$. Choose a good surjection $\phi_{-1} : R^{-1} \to L^{-1}$ and write $d^{-1} : R^{-1} \to R^0$ for $\delta^{-1} \circ \phi_{-1}$ composed with the inclusion $K^{-1} \hookrightarrow R^0$. We define $p_i^{-1} := q_i^{-1} \circ \phi_{-1}$.

We have just described an induction step, and we can proceed in the same manner. One sees directly that the left $u^-$-induced resolution $R^\bullet$ obtained this way actually lies in $C^\flat_V$.

4.11. Let $N \in C$, and let $R^\bullet$ be a $u^+$-induced convex right resolution of $N$. For $n \geq 0$ let $b_{\geq n}(R^\bullet)$ denote the stupid truncation:

$$0 \to R_0 \to \ldots \to R_n \to 0 \to \ldots$$

For $m \geq n$ we have evident truncation maps $b_{\geq m}(R^\bullet) \to b_{\geq n}(R^\bullet)$.

4.11.1. **Lemma.** Let $R^\bullet \otimes^C V$ be a $u^-$-induced left resolution of a module $V \in C^\flat_V$. We have

$$H^i(R^\bullet \otimes^C V) = \lim_{\leftarrow n} H^i(R^\bullet \otimes^C b_{\leq n} R^\bullet).$$

For every $i \in \mathbb{Z}$ the inverse system

$$\{H^i(R^\bullet \otimes^C b_{\leq n} R^\bullet)\}$$

stabilizes.

**Proof.** All spaces $H^i(R^\bullet \otimes^C b_{\leq n} R^\bullet)$ and only finitely many weight components of $R^\bullet$ and $R^\bullet$ contribute to $H^i$. □

4.12. **Proof of Theorem [4.9].** Let us consider case (ii), and prove that $H^* (R^\bullet \otimes^C R^\bullet(N))$ does not depend, up to a canonical isomorphism, on the choice of a resolution $R^\bullet(V)$. The other independences are proved exactly in the same way.

Let $R^\bullet_i, i = 1, 2$, be two left $u^-$-induced left convex resolutions of $V$. According to Lemma [4.10], there exists a third one, $R^\bullet$, projecting onto $R^\bullet_i$. Let us prove that the projections induce isomorphisms

$$H^*(R^\bullet \otimes^C R^\bullet(N)) \xrightarrow{\sim} H^*(R^\bullet_i \otimes^C R^\bullet(N)).$$

By Lemma [4.11.1], it suffices to prove that

$$H^*(R^\bullet \otimes^C b_{\leq n} R^\bullet(N)) \xrightarrow{\sim} H^*(R^\bullet_i \otimes^C b_{\leq n} R^\bullet(N)).$$
for all $n$. Let $Q_i^\bullet$ be a cone of $R^\bullet \rightarrow R_i^\bullet$. It is an exact $u^-$-induced convex complex bounded from the right. It is enough to check that $H^\bullet(Q_i^\bullet \otimes_C b_{\leq n}R^\bullet_\lambda(N)) = 0$.

Note that for $W \in \mathcal{C}, \mathcal{M} \in \mathcal{C}$ we have canonically

$$W \otimes_C M = (W \otimes sM) \otimes_C B.$$ 

Thus

$$H^\bullet(Q_i^\bullet \otimes_C b_{\leq n}R^\bullet_\lambda(N)) = H^\bullet((Q_i^\bullet \otimes sb_{\leq n}R^\bullet_\lambda(N)) \otimes_C B) = 0,$$

since $(Q_i^\bullet \otimes sb_{\leq n}R^\bullet_\lambda(N))$ is an exact bounded from the right complex, consisting of modules which are tensor products of $u^+$-induced and $u^-$-induced, hence $u$-projective modules (see Lemma 4.1.1).

4.12.1. It remains to show that if $p'$ and $p''$ are two maps between $u^-$-induced convex resolutions of $V, R_1^\bullet \rightarrow R_2^\bullet$, inducing identity on $V$, then the isomorphisms

$$H^\bullet(R_1^\bullet \otimes_C R^\bullet_\lambda(N)) \sim H^\bullet(R_2^\bullet \otimes_C R^\bullet_\lambda(N))$$

induced by $p'$ and $p''$, coincide. Arguing as above, we see that it is enough to prove this with $R^\bullet_\lambda(N)$ replaced by $b_{\leq n}R^\bullet_\lambda(N)$. This in turn is equivalent to showing that two isomorphisms

$$H^\bullet((R_1^\bullet \otimes sb_{\leq n}R^\bullet_\lambda(N)) \otimes_C B) \sim H^\bullet((R_2^\bullet \otimes sb_{\leq n}R^\bullet_\lambda(N)) \otimes_C B)$$

coincide. But $R_1^\bullet \otimes sb_{\leq n}R^\bullet_\lambda(N)$ are complexes of projective $u$-modules, and the morphisms $p' \otimes \text{id}$ and $p'' \otimes \text{id}$ induce the same map on cohomology, hence they are homotopic; therefore they induce homotopic maps after tensor multiplication by $B$.

This completes the proof of the theorem. $\square$

5. SOME CALCULATIONS

We will give a recipe for calculation of $\text{Tor}^\mathcal{C}_{\mathcal{C}^{\vee}}^{\bullet+\bullet}$, which will prove useful for the next chapter.

5.1. Recall that in III.13.2 the duality functor

$$D : \mathcal{C}_{\mathcal{C}^{\vee}} \rightarrow \mathcal{C}^{\vee^{\vee}}$$

has been defined (we identify $\mathcal{C}$ with $\tilde{\mathcal{C}}$ as usually). We will denote objects of $\mathcal{C}_{\mathcal{C}^{\vee}}$ by letters with the subscript $(\bullet)_{\mathcal{C}^{\vee}}$.

Note that $DL(0)_{\mathcal{C}^{\vee}} = L(0)$.

Let us describe duals to Verma modules. For $\lambda \in X$ let us denote by $M^+(\lambda)$ the Verma module with respect to the subalgebra $u^+$ with the lowest weight $\lambda$, that is

$$M^+(\lambda) := \text{Ind}_{u^{\leq 0}}^{u^+} \chi_\lambda$$

where $\chi_\lambda$ is an evident one-dimensional representation of $u^{\leq 0}$ corresponding to the character $\lambda$. 

5.1.1. **Lemma.** We have

$$DM(\lambda)_{\zeta^{-1}} = M^+(\lambda - 2(l - 1)\rho).$$

**Proof** follows from [AJS], Lemma 4.10. □

5.2. Let us denote by $K^\bullet$ a two term complex in $\mathcal{C}$

$$L(0) \to DM(0)_{\zeta^{-1}}$$

concentrated in degrees 0 and 1, the morphism being dual to the canonical projection $M(0)_{\zeta^{-1}} \to L(0)_{\zeta^{-1}}$.

For $n \geq 1$ define a complex

$$K_n^\bullet := b_{\geq 0}(K^\bullet \otimes [1]);$$

it is concentrated in degrees from 0 to $n - 1$. For example, $K_1^\bullet = DM(0)_{\zeta^{-1}}$.

For $n \geq 1$ we will denote by

$$\xi : K_n^\bullet \to K_{n+1}^\bullet$$

the map induced by the embedding $L(0) \to DM(0)_{\zeta^{-1}}$.

We will need the following evident properties of the system $\{K_n^\bullet, \xi_n\}$:

(a) $K_n^\bullet$ is $u^+$-induced;

(b) $K_n^\bullet$ is exact off degrees 0 and $n - 1$; $H^0(K_n^\bullet) = B$. $\xi_n$ induces identity map between $H^0(K_n^\bullet)$ and $H^0(K_{n+1}^\bullet)$.

(c) For a fixed $\mu \in X$ there exists $m \in \mathbb{N}$ such that for any $n$ we have $(b_{\geq m}K_n^\bullet)_{\geq \mu} = 0$. Here for $V = \oplus_{\lambda \in X} V_\lambda \in \mathcal{C}$ we set

$$V_{\geq \mu} := \oplus_{\lambda \geq \mu} V_\lambda.$$

5.3. Let $V \in \mathcal{C}_\forall$; let $R^\bullet_\zeta(V)$ be a $u^-$-induced convex left resolution of $V$. Let $N \in \mathcal{C}$.

5.3.1. **Lemma.** (i) For a fixed $k \in \mathbb{Z}$ the direct system $\{H^k(R^\bullet_\zeta(V) \otimes_\mathcal{C} (K_n^\bullet \otimes N)), \xi_n\}$ stabilizes.

(ii) We have a canonical isomorphism

$$\text{Tor}^\mathcal{C}_{\frac{\zeta}{\zeta}+\bullet}(V, N) \cong \lim_{\to n} H^{-\bullet}(R^\bullet_\zeta(V) \otimes_\mathcal{C} (K_n^\bullet \otimes N)).$$

**Proof.** (i) is similar to Lemma 4.11.1. (ii) By Theorem 4.3 we can use any $u^+$-induced right convex resolution of $N$ to compute $\text{Tor}^\mathcal{C}_{\frac{\zeta}{\zeta}+\bullet}(V, N)$. Now extend $K_n^\bullet \otimes N$ to a $u^+$-induced convex resolution of $N$ and argue like in the proof of Lemma 4.11.1 again. □
5.4. Recall the notations of 4.4 and take for $R\cdot(V)$ the resolution $P\cdot(V) = P\cdot \otimes V$. Then

$$H\cdot(P\cdot(V) \otimes_C (K_n \otimes N)) = H\cdot(V \otimes_C (P\cdot \otimes K_n \otimes N)).$$

Note that $P\cdot \otimes K_n \otimes N$ is a right bounded complex quasi-isomorphic to $K_n \otimes N$. The terms of $P\cdot \otimes K_n$ are $u$-projective by Lemma 4.1.1, hence the terms of $P\cdot \otimes K_n \otimes N$ are projective by rigidity of $C$. Therefore,

$$H^{-\cdot}(V \otimes_C (P\cdot \otimes K_n \otimes N)) = \text{Tor}_C^\cdot(V, K_n \otimes N).$$

Here $\text{Tor}_C^\cdot(*, *)$ stands for the zeroth weight component of $\text{Tor}_u^\cdot(*, *)$.

Putting all the above together, we get

5.5. **Corollary.** For a fixed $k \in \mathbb{Z}$ the direct system $\{\text{Tor}_k^C(V, K_n \otimes N)\}$ stabilizes.

We have

$$\text{Tor}_C^{k+\cdot}(V, N) = \lim_{\rightarrow n} \text{Tor}_\cdot(V, K_n \otimes N). \quad \Box$$

5.6. Dually, consider complexes $DK_{\zeta-1}$. They form a projective system

$$\{\ldots \rightarrow DK_{n+1, \zeta-1} \rightarrow DK_{n, \zeta-1} \rightarrow \ldots\}$$

These complexes enjoy properties dual to (a) — (c) above.

5.7. **Theorem.** For every $k \in \mathbb{Z}$ we have canonical isomorphisms

$$\text{Tor}_{C, k}^{\cdot+\cdot}(V, N) \cong \lim_{\leftarrow m} \lim_{\rightarrow n} H^{-k}(V \otimes sDK_{m, \zeta-1}) \otimes_C (K_n \otimes N)).$$

Both the inverse and the direct systems actually stabilize.

**Proof** follows from Lemma 5.3.1. We leave details to the reader. $\Box$

Here is an example of calculation of $\text{Tor}_{C, k}^{\cdot+\cdot}$.

5.8. **Lemma.** $\text{Tor}_{C, k}^{\cdot+\cdot}(B, L(2(l-1)\rho)) = B$ in degree 0.

**Proof.** According to Lemma 1.3 it suffices to prove that $\text{Ext}_{C, k}^{\cdot+\cdot}(L(2(l-1)\rho), L(0)) = B$. Choose a $u^+$-induced right convex resolution

$$L(2(l-1)\rho) \rightarrow R_{\cdot}$$

such that

$$R_{\cdot}^0 = DM(2(l-1)\rho)_\zeta = M^+(0),$$

and all the weights in $R_{\cdot}^{\geq 1}$ are $< 2(l-1)\rho$.

Similarly, choose a $u^-$-free right concave resolution

$$L(0) \rightarrow R_{\cdot}$$
such that
\[ R^0_{\rho} = M(2(l-1)\rho) = DM^+(0)_{\zeta-1}, \]
and all the weights of \( R^{\geq 1}_{\rho} \) are > 0. By Theorem 4.9 we have
\[ \Ext^{\geq 1}_{C}(L(2(l-1)\rho), L(0)) = H^\bullet(\Hom_{C}(R^0_{\rho}, R^0_{\rho})). \]
Therefore it is enough to prove that
(a) \( \Hom_{C}(R^0_{\rho}, R^0_{\rho}) = B; \)
(b) \( \Hom_{C}(R^n_{\rho}, R^0_{\rho}) = 0 \) for \((m, n) \neq (0, 0)\).

(a) is evident. Let us prove (b) for, say, \( n > 0 \). \( R^n_{\rho} \) has a filtration with successive quotients of type \( M^+(\lambda), \lambda \leq 0; \) similarly, \( R^n_{\rho} \) has a filtration with successive quotients of type \( DM^+(\mu)_{\zeta-1}, \mu > 0 \). We have \( \Hom_{C}(M^+(\lambda), DM^+(\mu)_{\zeta-1}) = 0 \), therefore \( \Hom_{C}(R^n_{\rho}, R^0_{\rho}) = 0 \). The proof for \( m > 0 \) is similar. Lemma is proven. \( \square \)

**CONFORMAL BLOCKS AND Tor_{C}^{\geq 1}**

5.9. Let \( M \in C \). We have a canonical embedding
\[ \Hom_{C}(B, M) \hookrightarrow M \]
which identifies \( \Hom_{C}(B, M) \) with the maximal trivial subobject of \( M \). Dually, we have a canonical epimorphism
\[ M \twoheadrightarrow \Hom_{C}(M, B)^* \]
which identifies \( \Hom_{C}(M, B)^* \) with the maximal trivial quotient of \( M \). Let us denote by \( \langle M \rangle \) the image of the composition
\[ \Hom_{C}(B, M) \twoheadrightarrow M \twoheadrightarrow \Hom_{C}(M, B)^* \]
Thus, \( \langle M \rangle \) is canonically a subquotient of \( M \).

One sees easily that if \( N \subset M \) is a trivial direct summand of \( M \) which is maximal, i.e. not contained in greater direct summand, then we have a canonical isomorphism \( \langle M \rangle \sim N \). By this reason, we will call \( \langle M \rangle \) the maximal trivial direct summand of \( M \).

5.10. Let
\[ \Delta = \{ \lambda \in X | \langle i, \lambda + \rho \rangle > 0, \text{for all } i \in I; \langle \gamma, \lambda + \rho \rangle < l \} \]
denote the first alcove. Here \( \gamma \in \mathcal{R} \subset \mathcal{Y} \) is the highest coroot.

For \( \lambda_1, \ldots, \lambda_n \in \Delta \) the space of conformal blocks is defined as
\[ \langle L(\lambda_1), \ldots, L(\lambda_n) \rangle := \langle L(\lambda_1) \otimes \ldots \otimes L(\lambda_n) \rangle \]
(see e.g. [A] and Lemma 9.3 below).
5.11. **Corollary.** The space of conformal blocks $\langle L(\lambda_1), \ldots, L(\lambda_n) \rangle$ is canonically a subquotient of $\text{Tor}_\infty^{C_2 + 0}(B, L(\lambda_1) \otimes \cdots \otimes L(\lambda_n) \otimes L(2(l-1)\rho))$.

**Proof** follows easily from the definition of $\langle \bullet \rangle$ and Lemma 5.8. \hfill \square

5.12. Let us consider an example showing that $\langle L(\lambda_1), \ldots, L(\lambda_n) \rangle$ is in general a proper subquotient of $\text{Tor}_\infty^{C_2 + 0}(B, L(\lambda_1) \otimes \cdots \otimes L(\lambda_n) \otimes L(2(l-1)\rho))$.

We leave the following to the reader.

5.12.1. **Exercise.** Let $P(0)$ be the indecomposable projective cover of $L(0)$. We have $\text{Tor}_\infty^{C_2 + 0}(B, P(0)) = B$. \hfill \square

We will construct an example featuring $P(0)$ as a direct summand of $L(\lambda_1) \otimes \cdots \otimes L(\lambda_n)$.

Let us take a root datum of type $sl(2)$; take $l = 5$, $n = 4$, $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = \lambda_4 = 3$ (we have identified $X$ with $\mathbb{Z}$).

In our case $\rho = 1$, so $2(l-1)\rho = 8$. Note that $P(0)$ has highest weight 8, and it is a unique indecomposable projective with the highest weight 8. So, if we are able to find a projective summand of highest weight 0 in $V = L(2) \otimes L(2) \otimes L(3) \otimes L(3)$ then $V \otimes L(8)$ will contain a projective summand of highest weight 8, i.e. $P(0)$.

Let $U_B$ denote the quantum group with divided powers over $B$ (see [L2], 8.1). The algebra $\mathfrak{u}$ lies inside $U_B$. It is wellknown that all irreducibles $L(\lambda)$, $\lambda \in \Delta$, lift to simple $U_B$-modules $\widehat{L}(\lambda)$ and for $\lambda_1, \ldots, \lambda_n \in \Delta$ the $U_B$-module $\widehat{L}(\lambda_1) \otimes \cdots \otimes \widehat{L}(\lambda_n)$ is a direct sum of irreducibles $\widehat{L}(\lambda)$, $\lambda \in \Delta$, and indecomposable projectives $\widehat{P}(\lambda)$, $\lambda \geq 0$ (see, e.g. [A]).

Thus $\widehat{L}(2) \otimes \widehat{L}(2) \otimes \widehat{L}(3) \otimes \widehat{L}(3)$ contains an indecomposable projective summand with the highest weight 10, i.e. $\widehat{P}(8)$. One can check easily that when restricted to $\mathfrak{u}$, $\widehat{P}(8)$ remains projective and contains a summand $P(-2)$. But the highest weight of $P(-2)$ is zero.

We conclude that $L(2) \otimes L(2) \otimes L(3) \otimes L(3) \otimes L(8)$ contains a projective summand $P(0)$, whence

$$\langle L(2), L(2), L(3), L(3) \rangle \neq \text{Tor}_\infty^{C_2 + 0}(B, L(2) \otimes L(2) \otimes L(3) \otimes L(3) \otimes L(8)).$$
CHAPTER 3. Global sections.

6. Braiding and balance in $\mathcal{C}$ and $\mathcal{FS}$

6.1. Let $U_B$ be the quantum group with divided powers, cf [L2], 8.1. Let $\mathcal{R}\mathcal{C}$ be the category of finite dimensional integrable $U_B$-modules defined in [K]IV, §37. It is a rigid braided tensor category. The braiding, i.e. family of isomorphisms

$$\tilde{R}_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V, \quad V, W \in \mathcal{R}\mathcal{C},$$

satisfying the usual constraints, has been defined in [L1], Ch. 32.

6.2. As $\mathfrak{u}$ is a subalgebra of $U_B$, we have the restriction functor preserving $X$-grading

$$\Upsilon : \mathcal{R}\mathcal{C} \longrightarrow \mathcal{C}.$$ 

The following theorem is due to G.Lusztig (private communication).

6.2.1. Theorem. (a) There is a unique braided structure $(R_{V,W}, \theta_V)$ on $\mathcal{C}$ such that the restriction functor $\Upsilon$ commutes with braiding.

(b) Let $V = L(\lambda)$, and let $\mu$ be the highest weight of $W \in \mathcal{C}$, i.e. $W_\mu \neq 0$ and $W_\nu \neq 0$ implies $\nu \leq \mu$. Let $x \in V$, $y \in W_\mu$. Then

$$R_{V,W}(x \otimes y) = \zeta^\lambda y \otimes x;$$

(c) Any braided structure on $\mathcal{C}$ enjoying the property (b) above coincides with that defined in (a). □

6.3. Recall that an automorphism $\tilde{\theta} = \{\tilde{\theta}_V : V \xrightarrow{\sim} V\}$ of the identity functor of $\mathcal{R}\mathcal{C}$ is called balance if for any $V, W \in \mathcal{R}\mathcal{C}$ we have

$$\tilde{R}_{W,V} \circ \tilde{R}_{V,W} = \tilde{\theta}_{V \otimes W} \circ (\tilde{\theta}_V \otimes \tilde{\theta}_W)^{-1}.$$ 

The following proposition is an easy application of the results of [L1], Chapter 32.

6.4. Proposition. The category $\mathcal{R}\mathcal{C}$ admits a unique balance $\tilde{\theta}$ such that

— if $\tilde{L}(\lambda)$ is an irreducible in $\mathcal{R}\mathcal{C}$ with the highest weight $\lambda$, then $\tilde{\theta}$ acts on $\tilde{L}(\lambda)$ as multiplication by $\zeta^{n(\lambda)}$. □

Here $n(\lambda)$ denotes the function introduced in [3.1].

Similarly to 6.2.1, one can prove
6.5. **Theorem.** (a) There is a unique balance \( \theta \) on \( C \) such that \( \Upsilon \) commutes with balance;
(b) \( \theta_{\lambda(\lambda)} = \zeta^{n(\lambda)} \);
(c) If \( \theta' \) is a balance in \( C \) having property (b), then \( \theta = \theta' \). □

6.6. According to Deligne’s ideology, [DI], the gluing construction of 3.6 provides the category \( \mathcal{FS} \) with the balance \( \theta^{FS} \). Recall that the braiding \( R^{FS} \) has been defined in III.11.11 (see also 11.4). It follows easily from the definitions that \( (\Phi(R^{FS}), \Phi(\theta^{FS})) \) satisfy the properties (b)(i) and (ii) above. Therefore, we have \( (\Phi(R^{FS}), \Phi(\theta^{FS})) = (R, \theta) \), i.e. \( \Phi \) is an equivalence of braided balanced categories.

7. **Global sections over \( \mathcal{A}(\mathcal{K}) \)**

7.1. Let \( K \) be a finite non-empty set, \( |K| = n \), and let \( \{X_i\} \) be a \( K \)-tuple of finite gactorizable sheaves. Let \( \lambda_k := \lambda(X_i) \) and \( \lambda = \sum_k \lambda_k \); let \( \alpha \in \mathbb{N}[\pi] \). Consider the sheaf \( \mathcal{X}^\alpha(\mathcal{K}) \) over \( \mathcal{A}_X^\alpha(\mathcal{K}) \) obtained by gluing \( \{X_i\} \), cf. III.10.3. Thus
\[
\mathcal{X}^\alpha(\mathcal{K}) = \{X_i\}_\alpha
\]
in the notations of loc.cit.
We will denote by \( \eta \), or sometimes by \( \eta^\alpha \), or \( \eta^\alpha_K \) the projection \( \mathcal{A}^\alpha(\mathcal{K}) \to \mathcal{O}(\mathcal{K}) \). We are going to describe \( R\eta_*\mathcal{X}^\alpha(\mathcal{K})[-\backslash] \). Note that it is an element of \( D(\mathcal{O}(\mathcal{K})) \) which is smooth, i.e. its cohomology sheaves are local systems.

7.2. Let \( V_1, \ldots, V_n \in \mathcal{C} \). Recall (see II.3) that \( C^\bullet_u(V_1 \otimes \ldots \otimes V_n) \) denotes the Hochschild complex of the \( u^- \)-module \( V_1 \otimes \ldots \otimes V_n \). It is naturally \( X \)-graded, and its \( \lambda \)-component is denoted by the subscript \( (\bullet)_\lambda \) as usually.
Let us consider a homotopy point \( z = (z_1, \ldots, z_n) \in \mathcal{O}(\mathcal{K}) \) where all \( z_i \) are real, \( z_1 < \ldots < z_n \). Choose a bijection \( K \to [n] \). We want to describe a stalk \( R\eta_*\mathcal{X}^\alpha(\mathcal{K})_\mathbf{z}[-\backslash] \).

The following theorem generalizes Theorem II.8.23. The proof is similar to loc. cit, cf. III.12.16, and will appear later.

7.3. **Theorem.** There is a canonical isomorphism, natural in \( \mathcal{X}_\mathbf{i} \),
\[
R\eta_*\mathcal{X}^\alpha(\mathcal{K})_\mathbf{z}[-\backslash] \cong C^\bullet_u(\Phi(\mathcal{X}_{\infty}) \otimes \ldots \otimes \Phi(\mathcal{X}_\mathbf{i}))_{\lambda-\alpha}. □
\]

7.4. The group \( \pi_1(\mathcal{O}(\mathcal{K}); z) \) is generated by counterclockwise loops of \( z_{k+1} \) around \( z_k \), \( \sigma_i, k = 1, \ldots, n-1 \). Let \( \sigma_k \) act on \( \Phi(\mathcal{X}_{\infty}) \otimes \ldots \otimes \Phi(\mathcal{X}_\mathbf{i}) \) as
\[
\text{id} \otimes \ldots \otimes R_{\Phi(\mathcal{X}_{\infty})_{\mathbf{i}+\infty}, \Phi(\mathcal{X}_\mathbf{i})_{\mathbf{i}+\infty}} \circ R_{\Phi(\mathcal{X}_\mathbf{i}), \Phi(\mathcal{X}_\mathbf{i})_{\mathbf{i}+\infty}} \otimes \ldots \otimes \text{id}.
\]
This defines an action of \( \pi_1(\mathcal{O}(\mathcal{K}); z) \) on \( \Phi(\mathcal{X}_{\infty}) \otimes \ldots \otimes \Phi(\mathcal{X}_\mathbf{i}) \), whence we get an action of this group on \( C^\bullet_u(\Phi(\mathcal{X}_{\infty}) \otimes \ldots \otimes \Phi(\mathcal{X}_\mathbf{i})) \) respecting the \( X \)-grading. Therefore we get a complex of local systems over \( \mathcal{O}(\mathcal{K}) \); let us denote it \( C^\bullet_u(\Phi(\mathcal{X}_{\infty}) \otimes \ldots \otimes \Phi(\mathcal{X}_\mathbf{i}))^\circ \).
7.5. **Theorem.** There is a canonical isomorphism in \( \mathcal{D}(\mathcal{O}(\mathcal{K})) \)

\[
R\eta_* \mathcal{X}^{\alpha}(\mathcal{K})[-\cdot] \simto \mathcal{C}_u^\bullet(\Phi(\mathcal{X}_\infty) \otimes \ldots \otimes \Phi(\mathcal{X}_\lambda))_{\lambda - \alpha}.
\]

**Proof** follows from 6.6 and Theorem 7.3. \( \square \)

7.6. **Corollary.** Set \( \lambda_\infty := \alpha + 2(l - 1)\rho - \lambda \). There is a canonical isomorphism in \( \mathcal{D}(\mathcal{O}(\mathcal{K})) \)

\[
R\eta_* \mathcal{X}^{\alpha}(\mathcal{K})[-\cdot] \simto \mathcal{C}_u^\bullet(\Phi(\mathcal{X}_\infty) \otimes \ldots \otimes \Phi(\mathcal{X}_\lambda) \otimes \mathcal{D}\mathcal{M}(\lambda_\infty)_{\zeta^{-\infty}})_{\lambda - \alpha}.
\]

**Proof.** By Shapiro’s lemma, we have a canonical morphism of complexes which is a quasiisomorphism

\[
\mathcal{C}_u^\bullet(\Phi(\mathcal{X}_\infty) \otimes \ldots \otimes \Phi(\mathcal{X}_\lambda))_{\lambda - \alpha} \longrightarrow \mathcal{C}_u^\bullet(\Phi(\mathcal{X}_\infty) \otimes \ldots \otimes \Phi(\mathcal{X}_\lambda) \otimes \mathcal{M}^+(\alpha - \lambda))_{\lambda - \alpha}.
\]

By Lemma 5.1.1, \( \mathcal{M}^+(\alpha - \lambda) = \mathcal{D}\mathcal{M}(\lambda_\infty)_{\zeta^{-1}} \). \( \square \)

8. **Global sections over \( \mathcal{P} \)**

8.1. Let \( J \) be a finite set, \( |J| = m \), and \( \{\mathcal{X}_i\} \) a \( J \)-tuple of finite factorizable sheaves. Set 

\[
\mu_j := \lambda_\mu(\mathcal{X}_i), \quad \vec{\mu} = (\mu_j) \in \mathcal{X}^J. \quad \text{Let } \alpha \in \mathbb{N}[\mathcal{I}] \text{ be such that } (\vec{\mu}, \alpha) \text{ is admissible, cf. 3.1. Let } \mathcal{X}^\alpha_{\vec{\mu}} \text{ be the preverse sheaf on } \mathcal{P}^\alpha_{\vec{\mu}} \text{ obtained by gluing the sheaves } \mathcal{X}_i, \text{ cf. 3.5 and 3.6.}
\]

Note that the group \( \text{PGL}_2(\mathbb{C}) = \text{Aut}(\mathbb{P}^J) \) operates naturally on \( \mathcal{P}^\alpha_{\vec{\mu}} \) and the sheaf \( \mathcal{X}^\alpha_{\vec{\mu}} \) is equivariant with respect to this action.

Let

\[
\vec{\eta} : \mathcal{P}^\alpha_{\vec{\mu}} \longrightarrow \mathcal{T}\mathcal{P}^{1,J}
\]

denote the natural projection; we will denote this map also by \( \vec{\eta}_1 \) or \( \vec{\eta}_J \). Note that \( \vec{\eta} \) commutes with the natural action of \( \text{PGL}_2(\mathbb{C}) \) on these spaces. Therefore \( R\vec{\eta}_* \mathcal{X}^\alpha_{\vec{\mu}} \) is a smooth \( \text{PGL}_2(\mathbb{C}) \)-equivariant complex on \( \mathcal{T}\mathcal{P}^{1,J} \). Our aim in this section will be to compute this complex algebraically.

Note that \( R\vec{\eta}_* \mathcal{X}^\alpha_{\vec{\mu}} \) descends uniquely to the quotient

\[
\mathcal{T}\mathcal{P}^{1,J} :\!\!: = \mathcal{T}\mathcal{P}^{1,J} / \text{PGL}_\infty(\mathbb{C})
\]

8.2. Let us pick a bijection \( J \simto [m] \). Let \( \mathcal{Z} \) be a contractible real submanifold of \( \mathcal{T}\mathcal{P}^{1,J} \) defined in [KL]II, 13.1 (under the name \( \mathcal{Y}_a \)). Its points are configurations \( (z_1, \tau_1, \ldots, z_m, \tau_m) \) such that \( z_j \in \mathbb{P}^J(\mathbb{R}) = \mathcal{S} \subset \mathbb{P}^J(\mathbb{C}) \); the points \( z_j \) lie on \( \mathcal{S} \) in this cyclic order; they orient \( \mathcal{S} \) in the same way as \( (0, 1, \infty) \) does; the tangent vectors \( \tau_j \) are real and compatible with this orientation.
8.3. **Definition.** An $m$-tuple of weights $\vec{\mu} \in X^m$ is called *positive* if
\[
\sum_{j=1}^{m} \mu_j + (1 - l)2\rho \in \mathbb{N}[I] \subset X.
\]
If this is so, we will denote
\[
\alpha(\vec{\mu}) := \sum_{j=1}^{m} \mu_j + (1 - l)2\rho \square
\]

8.4. **Theorem.** Let $X_\infty, \ldots, X_\mathfrak{s} \in \mathcal{FS}$. Let $\vec{\mu}$ be a positive $m$-tuple of weights, $\mu_j \geq \lambda(X_i)$, and let $\alpha = \alpha(\vec{\mu})$. Let $X^a_\vec{\mu}$ be the sheaf on $P^a_\vec{\mu}$ obtained by gluing the sheaves $X_i$. There is a canonical isomorphism
\[
R^\bullet \eta_* X^a_\vec{\mu}[-\infty]_Z \cong \text{Tor}_{\bullet}^C(B, \Phi(X_\infty) \otimes \ldots \otimes \Phi(X_\mathfrak{s})).
\]

**Proof** is sketched in the next few subsections.

8.5. **Two-sided Čech resolutions.** The idea of the construction below is inspired by [B], p. 40.

Let $P$ be a topological space, $\mathcal{U} = \{U_i\}_{i = \infty, \ldots, N}$ an open covering of $P$. Let $j_{i_0i_1\ldots i_a}$ denote the embedding
\[
U_{i_0} \cap \ldots \cap U_{i_a} \hookrightarrow P.
\]
Given a sheaf $\mathcal{F}$ on $P$, we have a canonical morphism
\[
\mathcal{F} \rightarrow \check{\mathcal{C}}(\mathcal{U}; \mathcal{F}) \tag{1}
\]
where
\[
\check{\mathcal{C}}(\mathcal{U}; \mathcal{F}) = \oplus_{i_0 < \ldots < i_a} |j_{i_0i_1\ldots i_a}| \check{\mathcal{C}}(U; \mathcal{F}),
\]
the differential being the usual Čech one.

Dually, we define a morphism
\[
\check{\mathcal{C}}(\mathcal{U}; \mathcal{F}) \rightarrow \mathcal{F} \tag{2}
\]
where
\[
\check{\mathcal{C}}(\mathcal{U}; \mathcal{F}) = \oplus_{i_0 < \ldots < i_a} |j_{i_0i_1\ldots i_a}| \check{\mathcal{C}}(U; \mathcal{F}).
\]
If $\mathcal{F}$ is injective then the arrows (1) and (2) are quasiisomorphisms.

Suppose we have a second open covering of $P$, $\mathcal{V} = \{V_i\}_{i = \infty, \ldots, N}$. Let us define sheaves
\[
\check{\mathcal{C}}^a_b(\mathcal{U}, \mathcal{V}; \mathcal{F}) := \check{\mathcal{C}}(\mathcal{V}; \check{\mathcal{C}}(\mathcal{U}; \mathcal{F}));
\]
they form a bicomplex. Let us consider the associated simple complex \( \check{C}^\bullet(U, V; F) \), i.e.
\[
\check{C}^\bullet(U, V; F) = \oplus_{i=0}^{-N} \check{C}^i(U, V; F).
\]
It is a complex concentrated in degrees from \(-N\) to \(N\). We have canonical morphisms
\[
\mathcal{F} \longrightarrow \check{C}^\bullet(U; F) \longleftarrow \check{C}^\bullet(U, V; F)
\]
If \( \mathcal{F} \) is injective then both arrows are quasiisomorphisms, and the above functors are exact on injective sheaves. Therefore, they pass to derived categories, and we get a functor
\[
K \mapsto \check{C}^\bullet(U, V; K)
\]
from the bounded derived category \( \mathcal{D}(\mathcal{P}) \) to the bounded filtered derived category \( \mathcal{DF}(\mathcal{P}) \). This implies

8.6. **Lemma.** Suppose that \( K \in \mathcal{D}(\mathcal{P}) \) is such that \( R^i\Gamma(P; \check{C}^\bullet(U, V; K)) = 0 \) for all \( a, b \) and all \( i \neq 0 \). Then we have a canonical isomorphism in \( \mathcal{D}(\mathcal{P}) \),
\[
R\Gamma(P; K) \cong R\Gamma(P; \check{C}^\bullet(U, V; K)).
\]

8.7. Returning to the assumptions of theorem \( 8.4 \), let us pick a point \( z = (z_1, \tau_1, \ldots, z_m, \tau_m) \) \( T^\mathcal{P} \) such that \( z_j \) are real numbers \( z_1 < \ldots < z_m \) and tangent vectors are directed to the right.

By definition, we have canonically
\[
R\bar{\eta}_\ast(P^\alpha_{\vec{\mu}}; X^\alpha_{\vec{\mu}}) \cong R\Gamma(P^\alpha; K)
\]
where
\[
K := X^\alpha_{\vec{\mu}}|_{\bar{\eta}^{-\infty}(\mathcal{P})} - \bar{\mu}.
\]
Let us pick \( N \geq |\alpha| \) and reals \( p_1, \ldots, p_N, q_1, \ldots, q_N \) such that
\[
p_1 < \ldots < p_N < z_1 < \ldots < z_m < q_N < \ldots < q_1.
\]
Let us define two open coverings \( U = \{U_i\}_{i=\infty}^N \) and \( V = \{V_j\}_{j=\infty}^m \) of the space \( \mathcal{P}^\alpha \) where
\[
U_i = \mathcal{P}^\alpha - \bigcup \{U_j = \infty\}; \quad V_j = \mathcal{P}^\alpha - \bigcup \{V_j = \infty\},
\]
where \( t_k \) denote the standard coordinates.

8.8. **Lemma.** (i) We have
\[
R^i\Gamma(P^\alpha; \check{C}^\bullet(U, V; K)) = 0
\]
for all \( a, b \) and all \( i \neq 0 \).
(ii) We have canonical isomorphism
\[
R^0\Gamma(P^\alpha; \check{C}^\bullet(U, V; K)) \cong \int \mathbb{D}K^\bullet_{N, \zeta} \otimes \Phi(X^\infty) \otimes \ldots \otimes \Phi(X^\zeta),
\]
in the notations of \( 5.7 \).
Proof (sketch). We should regard the computation of $R\Gamma(P^\alpha; \tilde{C}^\bullet(U, V; K))$ as the computation of global sections over $P^\alpha$ of a sheaf obtained by gluing $A_i$ into points $z_j$, the Verma sheaves $\mathcal{M}(t)$ or irreducibles $\mathcal{L}(t)$ into the points $p_j$, and dual sheaves $D\mathcal{M}(t)_{\zeta^{-\infty}}$ or $D\mathcal{L}(t)_{\zeta^{-\infty}}$ into the points $q_j$.

Using $\text{PGL}_2(\mathbb{R})$-invariance, we can move one of the points $p_j$ to infinity. Then, the desired global sections are reduced to global sections over an affine space $\mathcal{A}^\alpha$, which are calculated by means of Theorem 7.3.

Note that in our situation all the sheaves $\tilde{C}^\bullet(U, V; K)$ actually belong to the abelian category $\mathcal{M}(P^\alpha)$ of perverse sheaves. So $\tilde{C}^\bullet(U, V; K)$ is a resolution of $K$ in $\mathcal{M}(P^\alpha)$. □

8.9. The conclusion of 8.4 follow from the previous lemma and Theorem 5.7. □

8.10. The group $\pi_1(\text{TP}^\text{om}, Z)$ operates on the spaces Tor$^\mathcal{C}_{\zeta^+}^\bullet(B, \Phi(\mathcal{X}_\infty) \otimes \ldots \otimes \Phi(\mathcal{X}_\vartheta))$ via its action on the object $\Phi(\mathcal{X}_\infty) \otimes \ldots \otimes \Phi(\mathcal{X}_\vartheta)$ induced by the braiding and balance in $\mathcal{C}$. Let us denote by

$$\text{Tor}^\mathcal{C}_{\zeta^+}^\bullet(B, \Phi(\mathcal{X}_\infty) \otimes \ldots \otimes \Phi(\mathcal{X}_\vartheta))$$

the corresponding local system on $\text{TP}^\text{om}$.

8.11. Theorem. There is a canonical isomorphism of local systems on $\text{TP}^\text{om}$:

$$R^{•-2m}\eta_*\mathcal{X}_\mu^\alpha \cong \text{Tor}^\mathcal{C}_{\zeta^+}^\bullet(B, \Phi(\mathcal{X}_\infty) \otimes \ldots \otimes \Phi(\mathcal{X}_\vartheta))$$

Proof follows immediately from 6.6 and Theorem 8.4. □

9. Application to conformal blocks

9.1. In applications to conformal blocks we will encounter the roots of unity $\zeta$ of not necessarily odd degree $l$. So we have to generalize all the above considerations to the case of arbitrary $l$.

The definitions of the categories $\mathcal{C}$ and $\mathcal{FS}$ do not change (for the category $\mathcal{C}$ the reader may consult [AP], §3). The construction of the functor $\Phi : \mathcal{FS} \rightarrow \mathcal{C}$ and the proof that $\Phi$ is an equivalence repeats the one in III word for word.

Here we list the only minor changes (say, in the definition of the Steinberg module) following [L1] and [AP].

9.1.1. So suppose $\zeta$ is a primitive root of unity of an even degree $l$.

We define $\ell := \frac{l}{2}$. For the sake of unification of notations, in case $l$ is odd we define $\ell := l$. For $i \in I$ we define $\ell_i := \frac{\ell}{(\ell, d_i)}$ where $(\ell, d_i)$ stands for the greatest common divisor of $\ell$ and $d_i$. 
For a coroot $\alpha \in R \subset Y$ we can find an element $w$ of the Weyl group $W$ and a simple coroot $i \in Y$ such that $w(i) = \alpha$ (notations of [L1], 2.3). We define $\ell_\alpha := \frac{\ell}{(\ell, d_i)}$, and the result does not depend on a choice of $i$ and $w$.

We define $\gamma_0 \in R$ to be the highest coroot, and $\beta_0 \in R$ to be the coroot dual to the highest root. Note that $\gamma_0 = \beta_0$ iff our root datum is simply laced.

9.1.2. We define $Y_\ell^* := \{ \lambda \in X | \lambda \cdot \mu \in \ell \mathbb{Z} \text{ for any } \mu \in X \}$

One should replace the congruence modulo $lY$ in the Definition 2.6.1 and in 3.1 by the congruence modulo $Y_\ell^*$.

We define $\rho_\ell \in X$ as the unique element such that $\langle i, \rho_\ell \rangle = \ell - 1$ for any $i \in I$.

Then the Steinberg module $L(\rho_\ell)$ is irreducible projective in $C$ (see [AP] 3.14).

Note also that $\rho_\ell$ is the highest weight of $u^+$.

One has to replace all the occurrences of $(l - 1)2\rho$ in the above sections by $2\rho_\ell$.

In particular, the new formulations of the Definition 8.3 and the Theorem 8.4 force us to make the following changes in 3.1 and 3.4.

In 3.1 we choose a balance function $n$ in the form

$$n(\mu) = \frac{1}{2} \mu \cdot \mu - \mu \cdot \rho_\ell$$

In other words, we set $\nu_0 = -\rho_\ell$. This balance function does not necessarily have the property that $n(-i') \equiv 0 \mod l$. It is only true that $n(-i') \equiv 0 \mod \ell$.

We say that a pair $(\vec{\mu}, \alpha)$ is admissible if $\sum_k \mu_k - \alpha \equiv 2\rho_\ell \mod Y_\ell^*$.

9.1.3. The last change concerns the definition of the first alcove in 5.10.

The corrected definition reads as follows:

if $\ell_i = \ell$ for any $i \in I$, then

$$\Delta_l = \{ \lambda \in X | \langle i, \lambda + \rho \rangle > 0, \text{ for all } i \in I; \langle \gamma_0, \lambda + \rho \rangle < \ell \};$$

if not, then

$$\Delta_l = \{ \lambda \in X | \langle i, \lambda + \rho \rangle > 0, \text{ for all } i \in I; \langle \beta_0, \lambda + \rho \rangle < \ell \beta_0 \}$$

9.2. Let $\hat{\mathfrak{g}}$ denote the affine Lie algebra associated with $\mathfrak{g}$:

$$0 \longrightarrow \mathbb{C} \longrightarrow \hat{\mathfrak{g}} \longrightarrow \mathfrak{g}((\epsilon)) \longrightarrow \mathfrak{c}.$$ 

Let $\hat{O}_\kappa$ be the category of integrable $\hat{\mathfrak{g}}$-modules with the central charge $\kappa - \hat{\mathfrak{h}}$ where $\hat{\mathfrak{h}}$ stands for the dual Coxeter number of $\mathfrak{g}$. It is a semisimple balanced braided rigid tensor category (see e.g. [MS] or [F]).
Let $O_{-\kappa}$ be the category of $\mathfrak{g}$-integrable $\hat{\mathfrak{g}}$-modules of finite length with the central charge $-\kappa - \hat{h}$. It is a balanced braided rigid tensor (bbrt) category (see [KL]). Let $O_{-\kappa}$ be the semisimple subcategory of $O_{-\kappa}$ formed by direct sums of simple $\mathfrak{g}$-modules with highest weights in the alcove $\nabla_{\kappa}$:

$$\nabla_{\kappa} := \{ \lambda \in X | \langle i, \lambda + \rho \rangle > 0, \text{ for all } i \in I; \langle \beta_0, \lambda + \rho \rangle < \kappa \}$$

The bbrt structure on $O_{-\kappa}$ induces the one on $\tilde{O}_{-\kappa}$, and one can construct an equivalence $\tilde{O}_{-\kappa} \sim \tilde{O}_{-\kappa}$ respecting bbrt structure (see [F]). D.Kazhdan and G.Lusztig have constructed an equivalence $O_{-\kappa} \sim \pi C_{\zeta}$ (notations of 6.1) respecting bbrt structure (see [KL] and [L3]). Here $\zeta = \exp(\pi \sqrt{-1} d\kappa)$ where $d = \max_{i \in I} d_i$. Thus $l = 2d\kappa$, and $\ell = d\kappa$.

Note that the alcoves $\nabla_{\kappa}$ and $\Delta_{\ell}$ (see [L1.3]) coincide.

The Kazhdan-Lusztig equivalence induces an equivalence $\tilde{O}_{-\kappa} \sim \tilde{O}_{\zeta}$ where $\tilde{O}_{\zeta}$ is the semisimple subcategory of $\pi C_{\zeta}$ formed by direct sums of simple $U_B$-modules $L(\lambda)$ with $\lambda \in \Delta$ (see [A] and [AP]). The bbrt structure on $\pi C_{\zeta}$ induces the one on $\tilde{O}_{\zeta}$, and the last equivalence respects bbrt structure. We denote the composition of the above equivalences by $\phi : \tilde{O}_{-\kappa} \sim \tilde{O}_{\zeta}$.

Given any bbrt category $\mathcal{B}$ and objects $L_1, \ldots, L_m \in \mathcal{B}$ we obtain a local system $\text{Hom}_{\mathcal{B}}(1, L_1 \otimes \ldots \otimes L_m)^{\circ}$ on $TP^\otimes$ with monodromies induced by the action of braiding and balance on $L_1 \otimes \ldots \otimes L_m$.

Here and below we write a superscript $X^{\circ}$ to denote a local system over $TP^\otimes$ with the fiber at a standard real point $z_1 < \ldots < z_m$ with tangent vectors looking to the right, equal to $X$.

Thus, given $L_1, \ldots, L_m \in \tilde{O}_{\kappa}$, the local system

$$\text{Hom}_{\tilde{O}_{\kappa}}(1, L_1 \otimes \ldots \otimes L_m)^{\circ}$$

called local system of conformal blocks is isomorphic to the local system $\text{Hom}_{\tilde{O}_{\zeta}}(1, \phi(L_1) \otimes \ldots \otimes \phi(L_m))^{\circ}$. Here $\otimes$ will denote the tensor product in "tilded" categories.

To unburden the notations we leave out the subscript $\zeta$ in $\pi C_{\zeta}$ from now on.

For an object $X \in \pi C$ let us define a vector space $\langle X \rangle_{\pi C}$ in the same manner as in [5.3], i.e. as an image of the canonical map from the maximal trivial subobject of $X$ to the maximal trivial quotient of $X$. Given $X_1, \ldots, X_m \in \pi C$, we denote

$$\langle X_1, \ldots, X_m \rangle := \langle X_1 \otimes \ldots \otimes X_m \rangle_{\pi C}.$$
9.2.1. Lemma. We have an isomorphism of local systems
\[ \text{Hom}_{\hat{\mathcal{O}}}(1, \phi(L_1) \otimes \ldots \otimes \phi(L_m)) \cong \langle \phi(L_1), \ldots, \phi(L_m) \rangle_{\hat{\mathcal{O}}} \]

Proof. Follows from [A]. ◆

9.3. Lemma. The restriction functor \( \Upsilon : \hat{\mathcal{C}} \rightarrow \mathcal{C} \) (cf. 6.2) induces isomorphism
\[ \langle \phi(L_1), \ldots, \phi(L_m) \rangle_{\hat{\mathcal{C}}} \simarrow \langle \Upsilon \phi(L_1), \ldots, \Upsilon \phi(L_m) \rangle_{\mathcal{C}}. \]

Proof. We must prove that if \( \lambda_1, \ldots, \lambda_m \in \Delta \), \( \hat{L}(\lambda_1), \ldots, \hat{L}(\lambda_m) \) are corresponding simples in \( \hat{\mathcal{C}} \), and \( L(\lambda_i) = \Upsilon L(\lambda_i) \) — the corresponding simples in \( \mathcal{C} \), then the maximal trivial direct summand of \( \hat{L}(\lambda_1) \otimes \ldots \otimes \hat{L}(\lambda_m) \) in \( \hat{\mathcal{C}} \) maps isomorphically to the maximal trivial direct summand of \( L(\lambda_1) \otimes \ldots \otimes L(\lambda_m) \) in \( \mathcal{C} \).

According to [A], [AP], \( \hat{L}(\lambda_1) \otimes \ldots \otimes \hat{L}(\lambda_m) \) is a direct sum of a module \( \hat{O}_{\lambda} \) and a negligible module \( N \in \hat{\mathcal{C}} \). Here negligible means that any endomorphism of \( N \) has quantum trace zero (see loc. cit.). Moreover, it is proven in loc. cit. that \( N \) is a direct summand of \( W \otimes M \) for some \( M \in \hat{\mathcal{C}} \) where \( W = \oplus_{\omega \in \Omega} \hat{L}(\omega) \),
\[ \Omega = \{ \omega \in X | \langle i, \omega + \rho \rangle > 0 \text{ for all } i \in I; \langle \beta_0, \omega + \rho \rangle = \kappa \} \]
being the affine wall of the first alcove. By loc. cit., \( W \) is negligible. Since \( \Upsilon \hat{L}(\omega) = L(\omega), \omega \in \Omega \) and since \( \Upsilon \) commutes with braiding, balance and rigidity, we see that the modules \( L(\omega) \) are negligible in \( \mathcal{C} \). Hence \( \Upsilon W \) is negligible, and \( \Upsilon W \otimes \Upsilon M \) is negligible, and finally \( \Upsilon N \) is negligible. This implies that \( \Upsilon N \) cannot have trivial summands (since \( L(0) \)) is not negligible).

We conclude that
\[ \langle \Upsilon \hat{L}(\lambda_1) \otimes \ldots \otimes \hat{L}(\lambda_m) \rangle_{\mathcal{C}} = \langle \Upsilon \hat{L}(\lambda_1) \otimes \ldots \otimes \Upsilon \hat{L}(\lambda_m) \rangle_{\mathcal{C}} = \langle \hat{L}(\lambda_1) \otimes \ldots \otimes \hat{L}(\lambda_m) \rangle_{\hat{\mathcal{C}}} \]

9.4. Corollary 5.11 implies that the local system
\[ \langle \Upsilon \phi(L_1), \ldots, \Upsilon \phi(L_m) \rangle_{\mathcal{C}} \]
is canonically a subquotient of the local system
\[ \text{Tor}_{\hat{\mathcal{C}}}^{1+0}(B, \Upsilon \phi(L_1) \otimes \ldots \otimes \Upsilon \phi(L_m) \otimes L(2\rho)) \]
(the action of monodromy being induced by braiding and balance on the first \( m \) factors).
9.5. Let us fix a point $\infty \in \mathbb{P}^d$ and a nonzero tangent vector $v \in T_\infty \mathbb{P}^d$. This defines an open subset 

$$T A^\oplus \subset T \mathbb{P}^d$$

and the locally closed embedding

$$\xi : T A^\oplus \hookrightarrow T \mathbb{P}^{d+\infty}.$$ 

Given $\lambda_1, \ldots, \lambda_m \in \Delta$, we consider the integrable $\hat{g}$-modules $\hat{L}(\lambda_1), \ldots, \hat{L}(\lambda_m)$ of central charge $\kappa - \tilde{h}$.

Suppose that $\lambda_1 + \ldots + \lambda_m = \alpha \in \mathbb{N}[\mathbb{L}] \subset \mathbb{X}$.

We define $\lambda_\infty := 2\rho_\ell$, and $\tilde{\lambda} := (\lambda_1, \ldots, \lambda_m, \lambda_\infty)$. Note that $\tilde{\lambda}$ is positive and $\alpha = \alpha(\tilde{\lambda})$, in the notations of \ref{sec:8.3}.

Denote by $X^\alpha_\lambda$ the sheaf on $P^\alpha_\lambda$ obtained by gluing $L(\lambda_\infty), \ldots, L(\lambda_\infty), L(\lambda_\infty)$. Note that

$$X^\alpha_\lambda = |_j T^\alpha_\lambda$$

where $j : P^{d\alpha}_\lambda \hookrightarrow P^\alpha_\lambda$.

Consider the local system of conformal blocks

$$\text{Hom}_{\mathcal{O}_\kappa}(1, \hat{L}(\lambda_1) \otimes \ldots \otimes \hat{L}(\lambda_m))^\circ.$$

If $\sum_{i=1}^m \lambda_i \not\in \mathbb{N}[\mathbb{L}] \subset \mathbb{X}$ then it vanishes by the above comparison with its ”quantum group” incarnation.

9.6. **Theorem.** Suppose that $\sum_{i=1}^m \lambda_i = \alpha \in \mathbb{N}[\mathbb{L}]$. Then the local system of conformal blocks restricted to $T A^\oplus$ is isomorphic to a canonical subquotient of a ”geometric” local system

$$\xi^* R^{-2m-2j_{m+1}^\alpha} j^*_\infty I^\alpha_\lambda.$$ 

**Proof.** This follows from Theorem \ref{sec:8.11} and the previous discussion. \square

9.7. **Corollary.** The above local system of conformal blocks is semisimple. It is a direct summand of the geometric local system above.

**Proof.** The geometric system is semisimple by Decomposition theorem, \cite{BBD}, Théorème 6.2.5. \square

9.8. Example \ref{sec:5.12} shows that in general a local system of conformal blocks is a proper direct summand of the corresponding geometric system.
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