OPTIMALITY CONDITIONS INVOLVING THE MITTAG–LEFFLER TEMPERED FRACTIONAL DERIVATIVE

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ABSTRACT. In this work we study problems of the calculus of the variations, where the differential operator is a generalization of the tempered fractional derivative. Different types of necessary conditions required to determine the optimal curves are proved. Problems with additional constraints are also studied. A numerical method is presented, based on discretization of the variational problem. Through several examples, we show the efficiency of the method.

1. Introduction. Fractional calculus (FC) is a generalization of the usual calculus, by considering non-integer order derivatives and integrals [23, 30, 32, 35]. Because of the greater number of parameters involved, the fractional models better fit the observations. To mention a few on them, we refer e.g. biophysics [31], viscoelasticity [12, 17], PID controllers [11], robotics [25, 26], neural systems [27], or blood circulation [13, 14].

Another field where FC has been useful is in calculus of variations (CV). In classical theory of CV, the aim is to find the curves that minimize or maximize a given functional which depends on time, on an unknown curve and on the first derivative of such curve. In the fractional CV, we replace the integer order derivative by a fractional derivative, thus obtaining a more general theory [6, 7, 28, 29].

In [5], variational problems dealing with the tempered fractional derivative were considered. Such kind of fractional operators are obtained from the classical ones (such as the Riemann–Liouville fractional integral and derivative), multiplied by an exponential factor [9, 15, 24, 34]. Also, the usual fractional operators are just a particular case of the tempered fractional ones and in this way they form a more general theory. The objective of this paper is to proceed further, by replacing the
exponential function with the Mittag–Leffler function and so develop even more such theory.

Our next definitions are motivated by the (usual) tempered fractional derivatives \([34]\). They involve the Mittag–Leffler function with one parameter \(\beta\) and so, when we consider the case \(\beta = 1\), we obtain the usual tempered fractional operators. In the following, \(\alpha > 0\), \(\beta > 0\) and \(\lambda \geq 0\) are reals and \(x : [a, b] \rightarrow \mathbb{R}\) is an integrable function.

**Definition 1.1.** Let \(n\) be the integer defined by \(n = [\alpha] + 1\), if \(\alpha \notin \mathbb{N}\), and \(n = \alpha\) otherwise. The left and right Mittag–Leffler tempered Caputo fractional derivatives of \(x\), of order \(\alpha\), are defined as

\[
C_{a^+}^{\alpha, \beta, \lambda} x(t) = \frac{C_D_{a^+}^{\alpha} \left( E_\beta(\lambda t)x(t) \right)}{E_\beta(\lambda t)} = \frac{1}{\Gamma(n - \alpha) E_\beta(\lambda t)} \int_a^t (t - s)^{n-\alpha-1} \left( \frac{d}{ds} \right)^n \left( E_\beta(\lambda s)x(s) \right) ds,
\]

\[
C_{b^+}^{\alpha, \beta, \lambda} x(t) = E_\beta(\lambda t) C_{b^+}^{\alpha} \left( \frac{x(t)}{E_\beta(\lambda t)} \right) = \frac{E_\beta(\lambda t)}{\Gamma(n - \alpha)} \int_t^b (s - t)^{n-\alpha-1} \left( -\frac{d}{ds} \right)^n \left( \frac{x(s)}{E_\beta(\lambda s)} \right) ds,
\]

respectively, where \(C_D_{a^+}^{\alpha}\) and \(C_D_{b^+}^{\alpha}\) denote the left and right Caputo fractional derivatives, respectively (\([16]\)).

The Riemann–Liouville fractional derivatives are defined in a similar way.

**Definition 1.2.** Let \(n = [\alpha] + 1 \in \mathbb{N}\). The left and right Mittag–Leffler tempered Riemann–Liouville fractional derivatives of a function \(x\), of order \(\alpha\), are defined as

\[
D_{a^+}^{\alpha, \beta, \lambda} x(t) = \frac{D_{a^+}^{\alpha} \left( E_\beta(\lambda t)x(t) \right)}{E_\beta(\lambda t)} = \frac{1}{\Gamma(n - \alpha) E_\beta(\lambda t)} \int_a^t (t - s)^{n-\alpha-1} E_\beta(\lambda s)x(s) ds,
\]

\[
D_{b^+}^{\alpha, \beta, \lambda} x(t) = E_\beta(\lambda t) D_{b^+}^{\alpha} \left( \frac{x(t)}{E_\beta(\lambda t)} \right) = \frac{E_\beta(\lambda t)}{\Gamma(n - \alpha)} \left( -\frac{d}{dt} \right)^n \int_t^b (s - t)^{n-\alpha-1} \frac{x(s)}{E_\beta(\lambda s)} ds,
\]

respectively, where \(D_{a^+}^{\alpha}\) and \(D_{b^+}^{\alpha}\) denote the left and right Riemann–Liouville fractional derivatives, respectively (\([16]\)).

**Remark 1.** Notice that, when \(\beta = 1\) we recover the definitions of the tempered Caputo and Riemann-Liouville fractional derivatives, and when \(\lambda = 0\), we obtain the usual (non-tempered) Caputo and Riemann-Liouville fractional derivatives. Hence, the definitions here given and called Mittag–Leffler tempered fractional derivatives may be seen as a generalization of the usual and tempered fractional derivatives.

For this work, we also need the two following two definitions.
Definition 1.3. The left and right tempered Riemann-Liouville fractional integrals of \( x \) are defined as

\[
\mathcal{I}_{a+}^{\alpha,\beta,\lambda} x(t) = \mathcal{I}_{a+}^\alpha (E_\beta(\lambda t)x(t)) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} E_\beta(\lambda s)x(s) \, ds,
\]

\[
\mathcal{I}_{b-}^{\alpha,\beta,\lambda} x(t) = \mathcal{I}_{b-}^\alpha \left( \frac{x(t)}{E_\beta(\lambda t)} \right) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} \frac{x(s)}{E_\beta(\lambda s)} \, ds,
\]

respectively, where \( \mathcal{I}_{a+}^\alpha \) and \( \mathcal{I}_{b-}^\alpha \) are the left and right tempered fractional derivatives. For the sequel, we need an integration by parts formulae involving the left and right tempered fractional derivatives.

Theorem 1.4. Let \( x, y \in C^n[a, b] \) and \( \alpha > 0 \) an arbitrary real. Then,

\[
\int_a^b x(t) \cdot C^{-\alpha,\beta,\lambda} y(t) \, dt = \int_a^b y(t) \cdot D_{a+}^{\alpha,\beta,\lambda} x(t) \, dt + \left[ \sum_{k=0}^{n-1} \left( \frac{d}{dt} \right)^{n-k-1} \left( E_\beta(\lambda t)y(t) \right) \cdot \left( \frac{d}{dt} \right)^k \mathcal{I}_{b-}^{n-\alpha,\beta,\lambda} x(t) \right]_a^b
\]

and

\[
\int_a^b x(t) \cdot C^{-\alpha,\beta,\lambda} y(t) \, dt = \int_a^b y(t) \cdot D_{b-}^{\alpha,\beta,\lambda} x(t) \, dt - \left[ \sum_{k=0}^{n-1} \left( \frac{d}{dt} \right)^{n-k-1} \left( \frac{y(t)}{E_\beta(\lambda t)} \right) \cdot \left( \frac{d}{dt} \right)^k \mathcal{I}_{a+}^{n-\alpha,\beta,\lambda} x(t) \right]_a^b
\]

Proof. Using the Dirichlet’s formula and then integration by parts, we obtain the next relations:

\[
\int_a^b x(t) \cdot C^{-\alpha,\beta,\lambda} y(t) \, dt
\]

\[
= \int_a^b \int_a^t \frac{x(t)}{\Gamma(n-\alpha) E_\beta(\lambda t)} (t-s)^{n-\alpha-1} \left( \frac{d}{ds} \right)^n \left( E_\beta(\lambda s)y(s) \right) \, ds \, dt
\]

\[
= \int_a^b \left( \frac{d}{dt} \right)^n \left( E_\beta(\lambda t)y(t) \right) \frac{1}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} \frac{x(s)}{E_\beta(\lambda s)} \, ds \, dt
\]

\[
= \int_a^b \left( \frac{d}{dt} \right)^{n-1} \left( E_\beta(\lambda t)y(t) \right) \frac{1}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} \frac{x(s)}{E_\beta(\lambda s)} \, ds \, dt
\]

\[
+ \left[ \left( \frac{d}{dt} \right)^{n-1} \left( E_\beta(\lambda t)y(t) \right) \mathcal{I}_{b-}^{n-\alpha,\beta,\lambda} x(t) \right]_a^b.
\]

Applying twice integration by parts, we obtain consecutively the next two equalities:

\[
\int_a^b \left( \frac{d}{dt} \right)^{n-2} \left( E_\beta(\lambda t)y(t) \right) \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^2 \left( \int_t^b (s-t)^{n-\alpha-1} \frac{x(s)}{E_\beta(\lambda s)} \, ds \right) \, dt
\]

\[
+ \left[ \sum_{k=0}^{n} \left( \frac{d}{dt} \right)^{n-k-1} \left( E_\beta(\lambda t)y(t) \right) \left( \frac{d}{dt} \right)^k \mathcal{I}_{b-}^{n-\alpha,\beta,\lambda} x(t) \right]_a^b.
\]
\[ \begin{align*}
&= \int_{a}^{b} \left( \frac{d}{dt} \right)^{n-3} (E_\beta(\lambda t)) y(t) \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dt} \right)^3 \left( \int_{t}^{b} (s-t)^{n-\alpha-1} \frac{x(s)}{E_\beta(\lambda s)} \, ds \right) \, dt \\
&\quad + \left[ \sum_{k=0}^{2} \left( \frac{d}{dt} \right)^{n-k-1} (E_\beta(\lambda t)) \left( -\frac{d}{dt} \right)^k \frac{1}{\Gamma(n-\alpha-\beta-\lambda)} \int_{a}^{b} x(t) \, dt \right]_a^b.
\end{align*} \]

Repeating this procedure \((n-3)\) more times, we prove the first formula. The second one is proven in a similar way. \(\square\)

The organization of the paper is the following. In Section 2, we present and prove some necessary conditions of optimization, that the candidates to the solution of the variational problem must satisfy. Several problems are considered, namely the fundamental one (Theorem 2.1) or in presence of restrictions, such as isoperimetric or holonomic constraints. Also, we study the case when the Lagrange function contains higher order derivatives, and the Herglotz problem as well. In the following Section 3, we present a numerical method to find approximate solutions of the variational problems. By discretizing the functional and the fractional derivative, we convert the variational problem into a finite optimization one. Then, applying some classical routines, the numerical solution is computed.

2. Necessary conditions of optimality. Consider the functional

\[ \mathcal{J}(x) = \int_{a}^{b} L(t, x(t), C \partial_a^{\alpha, \beta, \lambda} x(t), C \partial_a^{\alpha, \beta, \lambda} x(t)) \, dt, \quad x \in C^1[a, b], \quad \alpha \in (0, 1), \quad (1) \]

where \(L : [a, b] \times \mathbb{R}^3 \to \mathbb{R}\) is continuously differentiable with respect to the second, third and fourth variables. Boundary conditions may be imposed, that is, \(x(a)\) and \(x(b)\) may be fixed reals. To simplify notation, we adopt the next ones:

\[ \partial_i L(a_1, \ldots, a_m) := \frac{\partial L}{\partial a_i}(a_1, \ldots, a_m), \quad \text{if } [x](t) := (t, x(t), C \partial_a^{\alpha, \beta, \lambda} x(t), C \partial_a^{\alpha, \beta, \lambda} x(t)). \]

We seek necessary conditions that every (local) minimizer of the functional should verify. Such type of equations are known in the literature as Euler–Lagrange equations.

2.1. Euler–Lagrange equation. Our first result provides a necessary condition, that allows to determine possible minimizers of the functional (1). Such equations are given by a fractional differential equation dealing with left and right fractional operators [2, 3, 8, 10].

**Theorem 2.1.** Let \(x^*\) be a local minimizer of functional (1). Assume that the maps

\[ t \mapsto \partial_a^{\alpha, \beta, \lambda}(\partial_3 L[x^*](t)) \quad \text{and} \quad t \mapsto \partial_a^{\alpha, \beta, \lambda}(\partial_4 L[x^*](t)) \]

are continuous on the interval \([a, b]\). Then,

\[ \partial_2 L[x^*](t) + \partial_a^{\alpha, \beta, \lambda}(\partial_3 L[x^*](t)) + \partial_a^{\alpha, \beta, \lambda}(\partial_4 L[x^*](t)) = 0, \quad (2) \]

on the interval \([a, b]\). Moreover, if \(x(a)\) in (1) is arbitrary, then

\[ E_\beta(\lambda a) \left[ \frac{1}{\Gamma(1-\alpha-\beta-\lambda)} \partial_3 L[x^*](a) \right] = \frac{1}{E_\beta(\lambda a)} \left[ \frac{1}{\Gamma(1-\alpha-\beta-\lambda)} \partial_4 L[x^*](a) \right]. \]
If $x(b)$ in (1) is arbitrary, then
\[ E_\beta(\lambda b) \mathcal{I}_{b-}^{1-\alpha,\beta,\lambda} \partial_3 L[x^*](b) = \frac{1}{E_\beta(\lambda b)} \mathcal{I}_{a+}^{1-\alpha,\beta,\lambda} \partial_4 L[x^*](b). \]

Proof. Consider an arbitrary function $h \in C^1[a, b]$ and a variation of $x^*$ given by $x^* + \epsilon h$, where $\epsilon$ is a small real. Consider a function $j$, defined in a neighbourhood of zero, given by the rule
\[ j(\epsilon) := \mathcal{J}(x^* + \epsilon h). \]
Since $x^*$ is a minimizer of $\mathcal{J}$, then $\epsilon = 0$ is a minimizer of $j$ and so $j'(0) = 0$. Evaluating $j'(0)$, we obtain
\[ \int_a^b \partial_2 L[x^*](t) h(t) + \partial_3 L[x^*](t) C \mathcal{D}_{a+}^{\alpha,\beta,\lambda} h(t) + \partial_4 L[x^*](t) C \mathcal{D}_{b-}^{\alpha,\beta,\lambda} h(t) \, dt = 0. \]
Applying Theorem 1.4 to the second and third terms of the integrand, we obtain
\[ \int_a^b \left[ \partial_2 L[x^*](t) + \mathcal{D}_{b-}^{\alpha,\beta,\lambda} (\partial_3 L[x^*](t)) + \mathcal{D}_{a+}^{\alpha,\beta,\lambda} (\partial_4 L[x^*](t)) \right] h(t) \, dt 
+ \left[ \frac{1}{E_\beta(\lambda t)} \mathcal{I}_{b-}^{1-\alpha,\beta,\lambda} (\partial_3 L[x^*](t)) \right] h(t) \bigg|_a^b = 0. \] (3)
If we restrict the variations to those for which $h(a) = 0 = h(b)$, taking into consideration the arbitrariness of $h$ in $(a, b)$ and the continuity of the integrand function in (3), we obtain the Euler–Lagrange equation (2). Using this last relation in (3), we get
\[ \left[ \frac{1}{E_\beta(\lambda t)} \mathcal{I}_{b-}^{1-\alpha,\beta,\lambda} (\partial_3 L[x^*](t)) \right] h(t) \bigg|_a^b = 0. \] (4)
If $x(a)$ is arbitrary, then we can take variations such that $h(a) \neq 0$ and $h(b) = 0$, and (4) becomes
\[ E_\beta(\lambda a) \mathcal{I}_{b-}^{1-\alpha,\beta,\lambda} \partial_3 L[x^*](a) = \frac{1}{E_\beta(\lambda a)} \mathcal{I}_{a+}^{1-\alpha,\beta,\lambda} \partial_4 L[x^*](a). \]
On the other hand, if $x(b)$ is arbitrary, considering $h(b) \neq 0$ and $h(a) = 0$, we prove the last necessary condition. \qed

2.2. Isoperimetric problem. For our next result, let $I : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous and differentiable function with respect to the second, third and fourth variables. The following theorem provides a necessary condition of optimization when in presence of an integral constraint.

Theorem 2.2. Let $x^*$ be a solution of the following isoperimetric problem:
\[ \mathcal{J}(x) = \int_a^b L(t, x(t), C \mathcal{D}_{a+}^{\alpha,\beta,\lambda} x(t), C \mathcal{D}_{b-}^{\alpha,\beta,\lambda} x(t)) \, dt \to \min \]
subject to
\[ x(a) = x_a \quad \text{and} \quad x(b) = x_b, \quad x_a, x_b \in \mathbb{R}, \]
and
\[ \mathcal{G}(x) := \int_a^b I(t, x(t), C \mathcal{D}_{a+}^{\alpha,\beta,\lambda} x(t), C \mathcal{D}_{b-}^{\alpha,\beta,\lambda} x(t)) \, dt = K, \quad K \in \mathbb{R}. \] (5)
Assume that the maps
\[ t \mapsto \mathcal{D}_{b-}^{\alpha,\beta,\lambda} (\partial_3 L[x^*](t)), \quad t \mapsto \mathcal{D}_{a+}^{\alpha,\beta,\lambda} (\partial_4 L[x^*](t)), \]
Theorem, there exists a function $h \in \mathbb{R}$. Then, there exist two reals $p_0, p$, not both zero, such that

$$\partial_2 H(x^*)(t) + D_{b-}^{\alpha,\beta,\lambda} (\partial_3 I(x^*) (t)) + D_{a+}^{\alpha,\beta,\lambda} (\partial_4 I(x^*) (t)) = 0 \tag{6}$$

for all $t \in [a, b]$, where function $H$ is defined by $H := p_0 L + p I$.

Proof. First, suppose that there exists some $t_0 \in [a, b]$ such that

$$\partial_2 I(x^*) (t_0) + D_{b-}^{\alpha,\beta,\lambda} (\partial_3 I(x^*) (t_0)) + D_{a+}^{\alpha,\beta,\lambda} (\partial_4 I(x^*) (t_0)) \neq 0. \tag{7}$$

Consider the following functions, defined on a neighbourhood of $(0, 0)$, by

$$j(\epsilon_1, \epsilon_2) := J(x^* + \epsilon_1 h_1 + \epsilon_2 h_2) \quad \text{and} \quad g(\epsilon_1, \epsilon_2) := G(x^* + \epsilon_1 h_1 + \epsilon_2 h_2) - K,$$

where $h_1, h_2 \in C^1[a, b]$ are two arbitrary functions that vanish at the boundaries $t = a$ and $t = b$. If we differentiate $g$, with respect to $\epsilon_2$, we get

$$\frac{\partial g}{\partial \epsilon_2} (0, 0) = \int_a^b \left[ \partial_2 I(x^*) (t) + D_{b-}^{\alpha,\beta,\lambda} (\partial_3 I(x^*) (t)) + D_{a+}^{\alpha,\beta,\lambda} (\partial_4 I(x^*) (t)) \right] h_2 (t) dt.$$

The existence of a point $t_0$ fulfilling condition (7) ensures the existence of a function $\tilde{h}_2$ such that

$$\frac{\partial g}{\partial \epsilon_2} (0, 0) \neq 0.$$

Fix $h_2 = \tilde{h}_2$ and variations of type $x^* + \epsilon_1 h_1 + \epsilon_2 \tilde{h}_2$. By the Implicit Function Theorem, there exists a function $\epsilon_2 : (-r, r) \to \mathbb{R}$, ($r > 0$), such that $g(\epsilon_1, \epsilon_2(\epsilon_1)) = 0$. Also, since $\nabla g(0, 0) \neq (0, 0)$ and $(0, 0)$ is a solution of the problem:

$$\text{minimize } j \quad \text{s.t. } g(\epsilon_1, \epsilon_2) = 0,$$

by the Lagrange Multiplier Rule, there exists some $p \in \mathbb{R}$ such that

$$\nabla (j + pg)(0, 0) = (0, 0).$$

In this way, we obtain the condition

$$\int_a^b \left[ \partial_2 H(x^*) (t)h_1 (t) + \partial_3 H(x^*) (t)\partial_2 I(x^*) (t) + \partial_4 H(x^*) (t)\partial_2 I(x^*) (t) \right] dt = 0,$$

where $H = L + pI$. Using the integration by parts formula (Theorem 1.4), and the fact that $h_1(a) = 0 = h_1(b)$, we obtain

$$\int_a^b \left[ \partial_2 H(x^*) (t) + D_{b-}^{\alpha,\beta,\lambda} (\partial_3 H(x^*) (t)) + D_{a+}^{\alpha,\beta,\lambda} (\partial_4 H(x^*) (t)) \right] h_1 (t) dt = 0,$$

and from the arbitrariness of $h_1$ on $(a, b)$, we obtain condition (6), with $p_0 = 1$.

On the other hand, if

$$\partial_2 I(x^*) (t) + D_{b-}^{\alpha,\beta,\lambda} (\partial_3 I(x^*) (t)) + D_{a+}^{\alpha,\beta,\lambda} (\partial_4 I(x^*) (t)) = 0,$$

for all $t \in [a, b]$, then (6) is trivially satisfied, with $p_0 = 0$ and $p = 1$. \qed
2.3. Holonomic constraint. We now deal with a holonomic type problem. Let $g : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function, differentiable with respect to the second and third variables.

**Theorem 2.3.** Let $x^* = (x_1^*, x_2^*)$ be a solution of the following problem:

**minimize** $\mathcal{J}(x_1, x_2) := \int_a^b L(t, x_1(t), x_2(t), C^{\alpha,\beta,\lambda}_a x_1(t), C^{\alpha,\beta,\lambda}_a x_2(t), C^{\alpha,\beta,\lambda}_b x_1(t), C^{\alpha,\beta,\lambda}_b x_2(t)) \, dt$,

subject to $x(a) = x_a$ and $x(b) = x_b$, $x_a, x_b \in \mathbb{R}^2$, and

$$g(t, x_1(t), x_2(t)) = 0. \quad (8)$$

Assume that the maps $t \mapsto D^{\alpha,\beta,\lambda}_b (\partial_t L[x^*](t))$, $t \mapsto D^{\alpha,\beta,\lambda}_b (\partial_b L[x^*](t))$, $t \mapsto D^{\alpha,\beta,\lambda}_a (\partial_t L[x^*](t))$, and $t \mapsto D^{\alpha,\beta,\lambda}_a (\partial_\zeta L[x^*](t))$ are all continuous on $[a, b]$, and

$$\partial_t g(x^*) \neq 0, \quad \forall t \in [a, b].$$

Then, there exists a continuous function $\mu : [a, b] \to \mathbb{R}$ such that

$$\partial_t L[x^*](t) + D^{\alpha,\beta,\lambda}_b (\partial_t L[x^*](t)) + D^{\alpha,\beta,\lambda}_a (\partial_\zeta L[x^*](t)) + \mu(t) \partial_t g(x^*) \neq 0, \quad (9)$$

for all $t \in [a, b]$ and $i = 2, 3, \ldots$.

**Proof.** Consider a variation of $x^*$ of form $(x_1^* + \epsilon h_1, x_2^* + \epsilon h_2)$, where $h_1, h_2 \in C^1[a, b]$ are two arbitrary functions with $h_i(a) = h_i(b) = 0$, for $i = 1, 2$. Inserting this variation in Eq. (8), differentiating it with respect to $\epsilon$, and then setting $\epsilon = 0$, we obtain

$$\partial_2 g(x^*) h_1(t) + \partial_3 g(x^*) h_2(t) = 0, \quad t \in [a, b].$$

Define $\mu : [a, b] \to \mathbb{R}$ as

$$\mu(t) := -\frac{\partial_2 L(x^*) h_1(t) + \partial_3 L(x^*) h_2(t)}{\partial_3 g(x^*)}(t).$$

Thus, the case $i = 3$ in (9) is proven. To prove the other one, observe that

$$\int_a^b \partial_2 L[x^*](t) h_1(t) + \partial_3 L[x^*](t) h_2(t) + \partial_4 L[x^*](t) C^{\alpha,\beta,\lambda}_a h_1(t) + \partial_5 L[x^*](t) C^{\alpha,\beta,\lambda}_b h_2(t) dt = 0,$$

since $x^*$ minimizes $\mathcal{J}$. By Theorem 1.4, we get

$$\int_a^b [\partial_2 L[x^*](t) + D^{\alpha,\beta,\lambda}_b (\partial_4 L[x^*](t)) + D^{\alpha,\beta,\lambda}_a (\partial_5 L[x^*](t))] h_1(t) + [\partial_3 L[x^*](t) + D^{\alpha,\beta,\lambda}_b (\partial_5 L[x^*](t)) + D^{\alpha,\beta,\lambda}_a (\partial_7 L[x^*](t))] h_2(t) dt = 0.$$

Also, since

$$\partial_4 L[x^*](t) + D^{\alpha,\beta,\lambda}_b (\partial_5 L[x^*](t)) + D^{\alpha,\beta,\lambda}_a (\partial_7 L[x^*](t)) = \mu(t) \partial_2 g(x^*) h_1(t),$$

$$\partial_5 L[x^*](t) + D^{\alpha,\beta,\lambda}_b (\partial_5 L[x^*](t)) + D^{\alpha,\beta,\lambda}_a (\partial_7 L[x^*](t)) = \mu(t) \partial_3 g(x^*) h_2(t),$$

and

$$\partial_7 L[x^*](t) + D^{\alpha,\beta,\lambda}_b (\partial_5 L[x^*](t)) + D^{\alpha,\beta,\lambda}_a (\partial_7 L[x^*](t)) = \mu(t) \partial_5 g(x^*) h_2(t).$$
we obtain that
\[ \int_a^b \left[ \partial_2 L[x^*](t) + D_{b-}^{\alpha,\beta,\lambda}(\partial_4 L[x^*](t) + D_{a+}^{\alpha,\beta,\lambda}(\partial_b L[x^*](t)) + \mu(t)\partial_2 g[x^*](t)]h_1(t) \right] dt = 0, \]
and by the arbitrariness of \( h_1 \), we conclude the desired remaining formula. \( \square \)

2.4. Higher-order derivatives. Our next result is a generalization of Theorem 2.1, by considering fractional derivatives of arbitrary orders \( \alpha_k > 0 \). Consider a sequence \( \alpha_k \in (k-1,k) \), for \( k = 1, \ldots, n \in \mathbb{N} \). Let \( L : [a,b] \times \mathbb{R}^{2n+1} \to \mathbb{R} \) be a continuous function, and differentiable with respect to the \( k' \)'s variable, for \( k = 2,3,\ldots,2n+2 \).

**Theorem 2.4.** Let \( x^* \in C^n[a,b] \) be a solution of the following problem: minimize
\[
J(x) = \int_a^b L(t,x(t),C_{a+}^{\alpha_1,\beta,\lambda} x(t),\ldots,C_{a+}^{\alpha_n,\beta,\lambda} x(t),\ldots,C_{b-}^{\alpha_n,\beta,\lambda} x(t)) \ dt
\]
subject to
\[
x^{(k)}(a) = x_a^k \quad \text{and} \quad x^{(k)}(b) = x_b^k, \quad \text{where} \ x_a^k, x_b^k \in \mathbb{R}, \ k = 0,1,\ldots,n-1.
\]
Assume that the maps
\[
t \mapsto D_{b-}^{\alpha_k,\beta,\lambda}(\partial_{k+2} L[x^*](t)), \quad k = 1,\ldots,n,
\]
and
\[
t \mapsto D_{a+}^{\alpha_k,\beta,\lambda}(\partial_{k+n+2} L[x^*](t)), \quad k = 1,\ldots,n,
\]
are all continuous on \([a,b] \). Then,
\[
\partial_2 L[x^*](t) + \sum_{k=1}^n \left[ D_{b-}^{\alpha_k,\beta,\lambda}(\partial_{k+2} L[x^*](t)) + D_{a+}^{\alpha_k,\beta,\lambda}(\partial_{k+n+2} L[x^*](t)) \right] = 0, \quad (10)
\]
for all \( t \in [a,b] \).

**Proof.** Consider an arbitrary function \( h \in C^n[a,b] \), with \( h^{(k)}(a) = 0 \) and \( h^{(k)}(b) = 0 \), for all \( k = 0,1,\ldots,n-1 \). Define \( j(\epsilon) := J(x^* + \epsilon h) \), where \( \epsilon \in (-r,r) \), \( r > 0 \). Using the fact that \( j'(0) = 0 \), we obtain
\[
\int_a^b \partial_2 L[x^*](t)h(t) dt + \sum_{k=1}^n \left[ \partial_{k+2} L[x^*](t)D_{a+}^{\alpha_k,\beta,\lambda} h(t) + \partial_{k+n+2} L[x^*](t)D_{b-}^{\alpha_k,\beta,\lambda} h(t) \right] dt = 0.
\]
Applying Theorem 1.4,
\[
\int_a^b \left[ \partial_2 L[x^*](t) + \sum_{k=1}^n \left[ D_{b-}^{\alpha_k,\beta,\lambda}(\partial_{k+2} L[x^*](t)) + D_{a+}^{\alpha_k,\beta,\lambda} (\partial_{k+n+2} L[x^*](t)) \right] \right] h(t) dt = 0.
\]
We obtain condition (10) from the arbitrariness of function \( h \). \( \square \)
2.5. Herglotz problem. The principle of Herglotz was presented in 1930 [22], but only more recently with the works of Georgieva and coauthors [19, 20, 21] it has gained the attention of more researchers (e.g. [1, 4, 36, 37, 38]). The classical Herglotz problem states as follows: determine the trajectories \( x \in C^1[a, b] \) and the corresponding function \( z \in C^1[a, b] \) such that \( z(b) \) attains a minimum value, where the pair \((x, z)\) satisfies the ODE
\[
z'(t) = L(t, x(t), x'(t), z(t)), \quad t \in [a, b],
\]
subject to the boundary conditions
\[
z(a) = z_a, \quad x(a) = x_a, \quad \text{and} \quad x(b) = x_b, \quad z_a, x_a, x_b \in \mathbb{R}.
\]
It is clear that, when \( L \) does not depends on variable \( z \), the Herglotz problem is just a variational problem:
\[
\int_a^b \left[ L(t, x(t), x'(t)) + \frac{z_a}{b-a} \right] dt \to \min, \quad \text{s.t.} \quad x(a) = x_a \quad \text{and} \quad x(b) = x_b.
\]
In our study, we replace the first order derivative \( x' \) by its fractional derivatives \( C_{a_+}^{\alpha, \beta, \lambda} x \) and \( C_{a_-}^{\alpha, \beta, \lambda} x \). In what follows, \( L : [a, b] \times \mathbb{R}^4 \to \mathbb{R} \) is a continuous function, differentiable with respect to the \( k \)'s variable, for \( k = 2, 3, 4, 5 \). The space of admissible functions is \( x \in C^1[a, b] \) such that \( C_{a_+}^{\alpha, \beta, \lambda} x \), \( C_{a_-}^{\alpha, \beta, \lambda} x \in C^1[a, b] \), with fractional order \( \alpha \in (0, 1) \). Also, to simplify the notation, we define
\[
[x, z] := (t, x(t), C_{a_+}^{\alpha, \beta, \lambda} x(t), C_{a_-}^{\alpha, \beta, \lambda} x(t), z(t)).
\]

**Theorem 2.5.** Let \((x^*, z^*)\) be a solution of the following Herglotz problem:
\[
z(b) \to \min
\]
subject to
\[
z'(t) = L(t, x(t), C_{a_+}^{\alpha, \beta, \lambda} x(t), C_{a_-}^{\alpha, \beta, \lambda} x(t), z(t)), \quad t \in [a, b]
\]
and
\[
z(a) = z_a, \quad x(a) = x_a, \quad \text{and} \quad x(b) = x_b, \quad z_a, x_a, x_b \in \mathbb{R}.
\]
Assume that the maps
\[
t \mapsto C_{a_-}^{\alpha, \beta, \lambda} \left( \lambda(t) \partial_3 L[x^*, z^*](t) \right) \quad \text{and} \quad t \mapsto C_{a_+}^{\alpha, \beta, \lambda} \left( \lambda(t) \partial_4 L[x^*, z^*](t) \right)
\]
are continuous on \([a, b]\), where \( \lambda : [a, b] \to \mathbb{R} \) is the function
\[
\lambda(t) := \exp \left( -\int_a^t \partial_5 L[x^*, z^*](\tau) d\tau \right).
\]
Then,
\[
\lambda(t) \partial_2 L[x^*, z^*](t) + C_{a_-}^{\alpha, \beta, \lambda} \left( \lambda(t) \partial_3 L[x^*, z^*](t) \right) + C_{a_+}^{\alpha, \beta, \lambda} \left( \lambda(t) \partial_4 L[x^*, z^*](t) \right) = 0,
\]
for all \( t \in [a, b] \).

**Proof.** Consider a variation of \( x^* \) of form \( x^* + \epsilon h \), where \( h \in C^1[a, b] \) with \( h(a) = h(b) = 0 \). The corresponding variation of \( z^* \) is given by
\[
\theta(t) = \frac{d}{d\epsilon} z^*[x^* + \epsilon h, t]_{\epsilon=0}.
\]
of each one of these subintervals by interval $[a,b]$. Since we only consider functionals depending on the left fractional derivative, but with similar arguments we could use the same routine for functionals also depending on the right fractional derivative. We follow the approach used in [5], where an Euler-type method was derived for variational problems involving tempered Caputo derivatives, that is, problems similar to the ones here but where the Mittag-Leffler is replaced with the exponential function. We must then approximate the integral and the fractional derivative in (1).

In order to discretize the functional (1), we first consider a partition of the time interval $[a, b]$ into $N$ subintervals of equal size $\tau = \frac{b-a}{N}$, and we define the midpoints of each one of these subintervals by

$$t_{i+\frac{1}{2}} = a + \left( i + \frac{1}{2} \right) \tau, \quad i = 0, 1, \ldots, N - 1.$$
Using the midpoint quadrature rule to approximate the integral in (1), we obtain:

\[
\mathcal{J}(x) = \int_a^b L(t, x(t), C^\alpha_{a+} x(t), C^\alpha_{b-} x(t)) \, dt \\
\approx \tau \sum_{i=0}^{N-1} L \left( t_{i+\frac{1}{2}}, x(t_{i+\frac{1}{2}}), C^\alpha_{a+} x(t_{i+\frac{1}{2}}), C^\alpha_{b-} x(t_{i+\frac{1}{2}}) \right). \tag{11}
\]

We now proceed with the discretizations of the fractional derivatives. We restrict ourselves to the case where 0 < \alpha < 1.

In this case, in [39], the following modified L1 approximation for the Caputo derivative of a function x was provided:

\[
C^\alpha_{a+} x(t_{i+\frac{1}{2}}) \approx \tau^{-\alpha} \left[ b^{(\alpha)}_0 x \left( t_{i+\frac{1}{2}} \right) \right] - \sum_{k=1}^i \left( b^{(\alpha)}_{i-k} - b^{(\alpha)}_{i-k+1} \right) x \left( t_{i-k} \right) - \left( b^{(\alpha)}_i - B^{(\alpha)}_i \right) x \left( t_{\frac{1}{2}} \right) = C^\alpha_{a+} x_{i+\frac{1}{2}}, \tag{12}
\]

where the coefficients \( b^{(\alpha)}_i \) and \( B^{(\alpha)}_i \) are given by:

\[
b^{(\alpha)}_i = \frac{1}{\Gamma(2-\alpha)} \left[ (i+1)^{1-\alpha} - i^{1-\alpha} \right], \\
B^{(\alpha)}_i = \frac{2}{\Gamma(2-\alpha)} \left[ \left( i + \frac{1}{2} \right)^{1-\alpha} - i^{1-\alpha} \right].
\]

In Lemma 3.1 of [39] it was also proved that, if \( x \in C^2([a, b]) \), then

\[
\left| C^\alpha_{a+} x(t_{i+\frac{1}{2}}) - C^\alpha_{a+} x_{i+\frac{1}{2}} \right| \leq c \tau^{2-\alpha}, \quad i = 0, 1, \ldots, N-1, \tag{13}
\]

for some positive constant c which is independent of N.

In what follows, we proceed similarly as in [39] to obtain equally accurate approximations for \( C^\alpha_{b-} x(t_{i+\frac{1}{2}}) \), \( i = 0, 1, \ldots, N-1 \). Naturally, since the derivation of the approximation formula is analogous, estimate (13) also holds in this case.

For 0 < \alpha < 1, we have:

\[
C^\alpha_{b-} x(t_{i+\frac{1}{2}}) = \frac{-1}{\Gamma(1-\alpha)} \int_{t_{i+\frac{1}{2}}}^{b} \left( s - t_{i+\frac{1}{2}} \right)^{-\alpha} x'(s) \, ds \\
= \frac{-1}{\Gamma(1-\alpha)} \sum_{j=i+1}^{N-1} \int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \left( s - t_{i+\frac{1}{2}} \right)^{-\alpha} x'(s) \, ds \\
- \frac{1}{\Gamma(1-\alpha)} \int_{t_{N-\frac{1}{2}}}^{b} \left( s - t_{i+\frac{1}{2}} \right)^{-\alpha} x'(s) \, ds \\
\approx \frac{-1}{\Gamma(1-\alpha)} \sum_{j=i+1}^{N-1} \int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \left( s - t_{i+\frac{1}{2}} \right)^{-\alpha} \frac{x(t_{j+\frac{1}{2}}) - x(t_{j-\frac{1}{2}})}{\tau} \, ds \\
- \frac{1}{\Gamma(1-\alpha)} \int_{t_{N-\frac{1}{2}}}^{b} \left( s - t_{i+\frac{1}{2}} \right)^{-\alpha} \frac{x(b) - x(t_{N-\frac{1}{2}})}{s - t_{i+\frac{1}{2}}} \, ds.
\]

Computing the integrals above, after some algebraic manipulations, we achieve

\[
C^\alpha_{b-} x(t_{i+\frac{1}{2}}) \approx \tau^{-\alpha} \left[ d^{(\alpha)}_i x(b) - \left( d^{(\alpha)}_i - D^{(\alpha)}_{N-i} \right) x \left( t_{N-\frac{1}{2}} \right) - D^{(\alpha)}_1 x \left( t_{i+\frac{1}{2}} \right) \\
+ \sum_{j=i+1}^{N-1} \left( D^{(\alpha)}_j - D^{(\alpha)}_{j-1} \right) x(t_{j+\frac{1}{2}}) \right], \quad i = N-1, N-2, \ldots, 0, \tag{14}
\]
where the coefficients $d_i^{(a)}$ and $D_i^{(a)}$ are given by:
\[
d_i^{(a)} = \frac{2}{\Gamma(2-a)} \left[ (N - i - 1)^{1-a} - \left( N - i - \frac{1}{2} \right)^{1-a} \right],
\]
\[
D_i^{(a)} = \frac{1}{\Gamma(2-a)} \left[ (i - 1)^{1-a} - i^{1-a} \right].
\]

In order to approximate the left and right Mittag–Leffler tempered Caputo derivatives at $t = t_{i+1}^*$, we take into account the relationship given in Definition 1.1, and we use the modified L1 approximation formulas (12) and (14) to obtain:
\[
C_{D_a^+}^{\alpha, \beta, \lambda} x(t_{i+1}^*) = \left( E_\beta \left( \lambda t_{i+1}^* \right) \right)^{-1} C_{D_a^+}^{\alpha} \left( E_\beta \left( \lambda t_{i+1}^* \right) \right) x(t_{i+1}^*)
\]
\[
\approx \left( E_\beta \left( \lambda t_{i+1}^* \right) \right)^{-1} \tau^{-a} \left[ b_0^{(a)} E_\beta(\lambda t_{i+1}^*) x(t_{i+1}^*) - \sum_{k=1}^{i} \left( b_i^{(a)} - b_i^{(a)} \right) \right]
\]
\[
\times E_\beta \left( \lambda t_{k-\frac{1}{2}} \right) x(x_{k-\frac{1}{2}}) - \left( b_i^{(a)} - B_i^{(a)} \right) E_\beta \left( \lambda t_{k-\frac{1}{2}} \right) x(x_{k-\frac{1}{2}}) - B_i^{(a)} E_\beta(\lambda a) x(a),
\]
\[
C_{D_{b-}^+}^{\alpha, \beta, \lambda} x(t_{i-1}^*) = E_\beta \left( \lambda t_{i-1}^* \right) C_{D_{b-}^+}^{\alpha} \left( E_\beta \left( \lambda t_{i-1}^* \right) \right)^{-1} x(t_{i-1}^*)
\]
\[
\approx E_\beta \left( \lambda t_{i-1}^* \right) \tau^{-a} \left[ d_i^{(a)}(E_\beta(\lambda b))^{-1} x(b) + \sum_{k=1}^{N-1} \left( D_k^{(a)} - D_k^{(a)} \right) \right]
\]
\[
\left( E_\beta \left( \lambda t_{N-\frac{1}{2}} \right) \right)^{-1} x(t_{N-\frac{1}{2}}) - D_{i}^{(a)}(E_\beta(\lambda t_{i+1}^*))^{-1} x(t_{i+1}^*).\]

It should be noticed that error estimates as the one in (13) will also hold for these left and right Mittag–Leffler tempered Caputo fractional derivatives of $x$ as long as functions $E_\beta(\lambda t)x(t)$ and $x(t)$ belong to $C^2[a, b]$, respectively.

Inserting (15) and (16) in (11), we obtain
\[
\tilde{J}(x) \approx \tau \sum_{i=0}^{N-1} L \left( t_{i+\frac{1}{2}}, x_{i+\frac{1}{2}}, \Lambda_i^{(+)}, \Lambda_i^{(-)} \right) := \tilde{J}(x) = \tau \Psi \left( x_{\frac{1}{2}}, x_{\frac{1}{2}}, \ldots, x_{N-\frac{1}{2}} \right),
\]
where $x_{i+\frac{1}{2}}$ denotes an approximate value of $x(t_{i+\frac{1}{2}})$, and
\[
\Lambda_i^{(+)} = \left( E_\beta \left( \lambda t_{i+\frac{1}{2}} \right) \right)^{-1} \tau^{-a} \left[ b_0^{(a)} E_\beta \left( \lambda t_{i+\frac{1}{2}} \right) x_{i+\frac{1}{2}} - \sum_{k=1}^{i} \left( b_i^{(a)} - b_i^{(a)} \right) \right]
\]
\[
\times E_\beta \left( \lambda t_{k-\frac{1}{2}} \right) x_{k-\frac{1}{2}} - \left( b_i^{(a)} - B_i^{(a)} \right) E_\beta \left( \lambda t_{k-\frac{1}{2}} \right) x_{k-\frac{1}{2}} - B_i^{(a)} E_\beta(\lambda a) x(a),
\]
and
\[
\Lambda_i^{(-)} = \tau^{-a} E_\beta \left( \lambda t_{i+\frac{1}{2}} \right) \left[ d_i^{(a)}(E_\beta(\lambda b))^{-1} x(b) + \sum_{k=i+1}^{N-1} \left( D_k^{(a)} - D_{k-1}^{(a)} \right) \right]
\]
\[
\left( E_\beta \left( \lambda t_{N-\frac{1}{2}} \right) \right)^{-1} x_{N-\frac{1}{2}} - D_{i}^{(a)}(E_\beta(\lambda t_{i+1}^*))^{-1} x_{i+\frac{1}{2}},
\]
i = 0, 1, \ldots, N - 1.
Hence, the discrete solution of the variational problem corresponds to the solution of the system of equations

\[
\frac{\partial \Psi}{\partial x_i^{\frac{1}{2}}} = 0, \quad i = 0, 1, \ldots, N - 1. \quad (17)
\]

The first equation is given by

\[
\partial_2 L \left( t_{\frac{i}{2}}, x_{\frac{i}{2}}, \Lambda_0^{(1)}, \Lambda_0^{(-)} \right) + \tau^{-\alpha} \partial_3 L \left( t_{\frac{i}{2}}, x_{\frac{i}{2}}, \Lambda_0^{(0)}, \Lambda_0^{(-)} \right) B_0^{(a)} - \tau^{-\alpha} \partial_3 L \left( t_{\frac{i}{2}}, x_{\frac{i}{2}}, \Lambda_0^{(0)}, \Lambda_0^{(-)} \right) D_1^{(a)} + \tau^{-\alpha} \sum_{j=1}^{N-1} \partial_3 L \left( t_{\frac{j}{2}}, x_{\frac{j}{2}}, \Lambda_j^{(1)}, \Lambda_j^{(-)} \right) \left( -b_j^{(a)} + B_j^{(a)} \right) \times E_\beta \left( \lambda t_{\frac{j}{2}} \right) \left( E_\beta \left( \lambda t_{\frac{j}{2}} \right) \right)^{-1} = 0; \quad (18)
\]

The last equation writes:

\[
\partial_2 L \left( t_{N-\frac{i}{2}}, x_{N-\frac{i}{2}}, \Lambda_{N-1}^{(1)}, \Lambda_{N-1}^{(-)} \right) + \tau^{-\alpha} \partial_3 L \left( t_{N-\frac{i}{2}}, x_{N-\frac{i}{2}}, \Lambda_{N-1}^{(0)}, \Lambda_{N-1}^{(-)} \right) b_0^{(a)} - \tau^{-\alpha} \partial_3 L \left( t_{N-\frac{i}{2}}, x_{N-\frac{i}{2}}, \Lambda_{N-1}^{(0)}, \Lambda_{N-1}^{(-)} \right) d_0^{(a)} = 0,
\]

and the remaining \(N-2\) equations are

\[
\partial_2 L \left( t_{m+\frac{i}{2}}, x_{m+\frac{i}{2}}, \Lambda_m^{(1)}, \Lambda_m^{(-)} \right) + \tau^{-\alpha} \partial_3 L \left( t_{m+\frac{i}{2}}, x_{m+\frac{i}{2}}, \Lambda_m^{(0)}, \Lambda_m^{(-)} \right) b_0^{(a)} - \tau^{-\alpha} \partial_3 L \left( t_{m+\frac{i}{2}}, x_{m+\frac{i}{2}}, \Lambda_m^{(0)}, \Lambda_m^{(-)} \right) D_1^{(a)} + \tau^{-\alpha} \sum_{l=m+1}^{N-1} \partial_3 L \left( t_{l+\frac{i}{2}}, x_{l+\frac{i}{2}}, \Lambda_l^{(1)}, \Lambda_l^{(-)} \right) \left( b_l^{(a)} - b_{l-1}^{(a)} \right) \times E_\beta \left( \lambda t_{l+\frac{i}{2}} \right) \left( E_\beta \left( \lambda t_{l+\frac{i}{2}} \right) \right)^{-1} + \tau^{-\alpha} \partial_3 L \left( t_{l+\frac{i}{2}}, x_{l+\frac{i}{2}}, \Lambda_l^{(0)}, \Lambda_l^{(-)} \right) D_1^{(a)} + \tau^{-\alpha} \sum_{k=0}^{m-1} \partial_3 L \left( t_{k+\frac{i}{2}}, x_{k+\frac{i}{2}}, \Lambda_k^{(1)}, \Lambda_k^{(-)} \right) \times E_\beta \left( \lambda t_{k+\frac{i}{2}} \right) \left( E_\beta \left( \lambda t_{k+\frac{i}{2}} \right) \right)^{-1} = 0, \quad (20)
\]

\(m = 1, 2, \ldots, N - 2\). Hence, the discrete solution of the variational problem, \(x_{\frac{i}{2}}, x_{\frac{i}{2}}, \ldots, x_{\frac{i}{2}}\), is the solution of the system of \(N\) equations (18)-(20), which can be solved in any mathematical software. In particular, if the Matlab is chosen, a code for the computation of the Mittag-Leffler function is available at \(https://www.mathworks.com/matlabcentral/fileexchange/48154-the-mittag-leffler\)-function (see also [18]). If the user intends to use the \textit{Mathematica} software, then in order to compute \(E_\beta(z)\) the command \textit{MittagLefflerE}[\beta, z] can be used.
For the numerical approximation of the isoperimetric problem, we proceed similarly by using the same numerical approach, but where $L$ is replaced with the function $H = L + pI$, where $p$ is a real. Since in this case, the resulting system of the form (17) is a system of $N$ equations on the $(N + 1)$ unknowns $x_{\frac{1}{2}}, x_{\frac{3}{2}}, \ldots, x_{\frac{N-1}{2}}, p$, the extra needed equation is obtained through the discretization of (5). Using the same quadrature formulas and the same approximation for the Caputo derivatives as before, we obtain:

$$\tau \sum_{i=0}^{N-1} I(t_{i+\frac{1}{2}}, x_{i+\frac{1}{2}}, A_i) = K.$$  

4. Example. Since numerical results for the particular cases when $\lambda = 0$ or $\beta = 1$ are already available in the literature, in this section we present one example with $\lambda \neq 0$ and $\beta \neq 1$, with a known analytical solution.

Let us consider the following problem:

$$J = \int_0^1 \left( x(t) - \frac{t^2}{E_{0,5}(t)} \right)^2 + \left( C D_{0+}^{0.5,0.5,1} x(t) - \frac{2}{E_{0,5}(t) \Gamma\left(\frac{5}{2}\right) t^{\frac{3}{2}}} \right)^2 dt \to \min$$

subject to the conditions

$$x(0) = 0 \quad \text{and} \quad x(1) = \frac{1}{E_{0,5}(1)}.$$  

It is easy to see that the function $x^*(t) = \frac{t^2}{E_{0,5}(t)}$ satisfies the Euler-Lagrange equation (2). Also, since $J \geq 0$ and $J(x^*) = 0$, we conclude that $x^*$ is in fact a solution to the variational problem.

In Table 1 we present, for this example, the maximum of the absolute error:

$$E = \max_{i=0,\ldots,N-1} \left| x_{i+\frac{1}{2}} - x^*\left(t_{i+\frac{1}{2}}\right) \right|$$

for several values of $N$. We observe that the error decreases when $N$ increases, as it is expected.

| $N$ | 5  | 10 | 20 | 40 |
|-----|----|----|----|----|
| $E$ | $8.81 \times 10^{-3}$ | $3.61 \times 10^{-3}$ | $1.40 \times 10^{-3}$ | $5.24 \times 10^{-4}$ |

Table 1. Maximum of the committed absolute error in the approximation of the solution.

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