An analytical approach of the chromaticity correction for compact hadron river

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Abstract. A design of a compact hadron driver for future cancer therapies based on the induction synchrotron concept is given. In order to realize a slow extraction technique in a fast cycling synchrotron, which allows the energy sweep beam scanning, the zero momentum-dispersion \( D(s) \) region and high flat \( D(s) \) region are necessary. The present design lattice which meets both requirements. In this study, details of the lattice parameters is introduced and an analytical solution for chromaticity correction are described and discussed.

1. Introduction
A lattice design of a compact hadron driver for future cancer therapies as shown in Figure 1 has been identified [1]. Whereby the lattice has the two fold symmetry with a circumference of 52.8 m, 2 m long dispersion free straight section, and 3 m long large flat dispersion straight section [1][2]. The lattice parameter is design based on the linear term of motion equation. Hence, the individual common components will be treated as a 3 X 3 matrices as the following equations [3][4].

For drift space,

\[
\text{Matrix Drift, } M_{\text{Drift}} = \begin{bmatrix}
1 & L & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]  

(1)

For focusing magnet,

\[
\text{Matrix Focusing, } M_F = \begin{bmatrix}
\cos[\sqrt{k}L] & 1/\sqrt{k} \sin[\sqrt{k}L] & 0 \\
-\sqrt{k} \sin[\sqrt{k}L] & \cos[\sqrt{k}L] & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]  

(2)
For defocusing magnet,

$$M_D = \begin{bmatrix}
\cosh\left[\sqrt{|k|}L\right] & 1/\sqrt{|k|} \sinh\left[\sqrt{|k|}L\right] & 0 \\
\sqrt{|k|} \sinh\left[\sqrt{|k|}L\right] & \cosh\left[\sqrt{|k|}L\right] & 0 \\
0 & 0 & 1
\end{bmatrix} \tag{3}$$

For bending magnet,

$$M_B = \begin{bmatrix}
\cos\left[\sqrt{k}L\right] & \frac{1}{\sqrt{k}} \sin\left[\sqrt{k}L\right] & \frac{1}{\sqrt{k}} (1 - \cos[\sqrt{k}L]) \\
-\sqrt{k} \sin[\sqrt{k}L] & \cos[\sqrt{k}L] & \sin[\sqrt{k}L] \\
0 & 0 & 1
\end{bmatrix} \tag{4}$$

where:

- \(k\) = \(k\) – value of the related magnets
- \(L\) = Length of the related magnets

For perturbation term, which can be treated as nonlinear term and it is an essential term must be evaluated, so that the beam can beam accelerated inside the accelerator ring with the designed lattice. Therefore, the chromaticity as one of perturbation term caused by the momentum deviation of the particles must be considered and will be discussed at the following section.

**Figure 1.** Outline of the ring and its lattice [1].

A lattice configuration of half ring with two fold symmetry is shown in Figure 2. The configuration is the combination of two cells with a drift space as Cell A-Cell B- Drift Space 4- Cell B-Cell A.
Therefore, the 3 X 3 transfer matrix of the lattice and the Courant-Snyder parameters as the following.

For horizontal axis,

Transfer matrix of Cell $A$, $M_{H\text{cell} \ A} = M_{D1}M_{F1}M_{D2}M_{F2}M_{D3}M_{F3}M_{D4}M_{F4}$  \hspace{1cm} (5)

Transfer matrix of Cell $B$, $M_{H\text{cell} \ B} = M_{D1}M_{F1}M_{D2}M_{F2}M_{D3}M_{F3}M_{D4}M_{F4}$  \hspace{1cm} (6)

Transfer matrix of the lattice, $M_x = M_{H\text{cell} \ A}M_{H\text{cell} \ B}M_{D1}M_{F1}M_{H\text{cell} \ A}$  \hspace{1cm} (7)

\[
M_x = \begin{bmatrix}
M_{x_{11}} & M_{x_{12}} & M_{x_{13}} \\
M_{x_{21}} & M_{x_{22}} & M_{x_{23}} \\
M_{x_{31}} & M_{x_{32}} & M_{x_{33}} \\
\end{bmatrix}
\]  \hspace{1cm} (8)

For vertical axis,

Transfer matrix of Cell $A$, $M_{V\text{cell} \ A} = M_{D1}M_{F1}M_{D2}M_{F2}M_{D3}M_{F3}M_{D4}M_{F4}$  \hspace{1cm} (9)

Transfer matrix of Cell $B$, $M_{V\text{cell} \ B} = M_{D1}M_{F1}M_{D2}M_{F2}M_{D3}M_{F3}M_{D4}M_{F4}$  \hspace{1cm} (10)

Transfer matrix of the lattice, $M_y = M_{V\text{cell} \ A}M_{V\text{cell} \ B}M_{D1}M_{F1}M_{V\text{cell} \ A}$  \hspace{1cm} (11)

\[
M_y = \begin{bmatrix}
M_{y_{11}} & M_{y_{12}} & M_{y_{13}} \\
M_{y_{21}} & M_{y_{22}} & M_{y_{23}} \\
M_{y_{31}} & M_{y_{32}} & M_{y_{33}} \\
\end{bmatrix}
\]  \hspace{1cm} (12)
From Equation (9) and (12), the Courant-Snyder parameters can be derived as:

For horizontal axis,
\[
\beta_x (s) = (M_{x11})^2 \beta_x(0) - 2 M_{x11} M_{x12} \alpha_x(0) + (M_{x12})^2 \gamma_x(0)
\] (13)
\[
\alpha_x (s) = M_{x11} M_{x21} \beta_x(0) + (M_{x12} M_{x21} + M_{x11} M_{x22}) \alpha_x(0) - M_{x12} M_{x22} \gamma_x(0)
\] (14)
\[
\gamma_x (s) = (M_{x21})^2 \beta_x(0) - 2 M_{x21} M_{x22} \alpha_x(0) + (M_{x22})^2 \gamma_x(0)
\] (15)

For vertical axis,
\[
\beta_y (s) = (M_{y11})^2 \beta_y(0) - 2 M_{y11} M_{y12} \alpha_y(0) + (M_{y12})^2 \gamma_y(0)
\] (16)
\[
\alpha_y (s) = M_{y11} M_{y21} \beta_y(0) + (M_{y12} M_{y21} + M_{y11} M_{y22}) \alpha_y(0) - M_{y12} M_{y22} \gamma_y(0)
\] (17)
\[
\gamma_y (s) = (M_{y21})^2 \beta_y(0) - 2 M_{y21} M_{y22} \alpha_y(0) + (M_{y22})^2 \gamma_y(0)
\] (18)

Whereby the initial condition of Courant-Snyder parameters can be identified as:
\[
\beta_{x,y}(0) = \beta_{x,y} (C)
\] (19)
\[
\alpha_{x,y}(0) = \alpha_{x,y} (C)
\] (20)
\[
\gamma_{x,y}(0) = \gamma_{x,y} (C)
\] (21)

where:
\[C = \text{The end position of the ring}\]

For Equation (19), (20) and (21), the beta function along the ring can be identified and it will be used to analysis the chromaticity at the following section.

2. Chromaticity and chromaticity correction
Perturbations of beam are occurred in the beam transportation even in the absence of magnet field and alignment errors. The momentum deviations of particle from the ideal design energy so-called as chromaticity which causes the perturbations in the solutions of the equations of motion [3].
As shown in Figure 3, the lowest order chromatic perturbation is caused by the variation of the focal length of the quadrupoles with energy which the different momentum deviation is occurred. This kind of error is well known from light optics, where a correction of this chromatic aberration by using different kinds of glasses for the lenses in a composite focusing system. For the chromatic perturbations correction in beam optics, the particles with different energies can be separated by the introduction of a dispersion function. Once the particles are separated by energy, different focusing corrections depending on the energy of the particles are introduced. As well-known that higher energy particles are focused less than ideal energy particles and lower energy particles are over focused. For such phenomena correction, a correction magnet: sextupoles with the property as focusing for higher energy particles and defocusing for lower energy particles can be utilized as shown in Figure 4.

![Figure 3. Chromaticity focusing errors.](image)

![Figure 4. Chromaticity correction with sextupoles.](image)
The variation of tunes with energy is called the chromaticity and is defined by [3]:

\[ \xi = \frac{\Delta \nu}{\Delta p/p_0} \]  

(22)

where:
- \( \Delta \nu \) = Tune shift
- \( \Delta p/p_0 \) = momentum deviation

Equation (22) is applicable for both circular and open beam lines. Using the definition of the chromaticity and with the derivation, the natural chromaticity at the quadrupoles of the circular accelerators for horizontal and vertical axis can be defined as [3][5]:

\[ \xi_x = -\frac{1}{4\pi} \int_0^C k_x(s)\beta_x(s)ds \]  

(23)

\[ \xi_y = -\frac{1}{4\pi} \int_0^C k_y(s)\beta_y(s)ds \]  

(24)

where:
- \( k_{x,y}(s) \) = \( k \) – value of horizontal or vertical quadrupole magnets as function of position
- \( \beta_{x,y}(s) \) = Horizontal or vertical beta as function of position

To correct both the horizontal and the vertical chromaticity, two different groups of horizontal and vertical sextupoles are required. For chromaticity correction by sextupoles, the required field strength and number of sextupoles need to be optimized. Such of chromaticity correction for individual horizontal and vertical sextupoles can be defined as:

\[ \xi_{sextx} = \frac{1}{4\pi} \int_0^C 2G(s)\eta(s)\beta_x(s)ds \]  

(25)

\[ \xi_{sexty} = \frac{1}{4\pi} \int_0^C G(s)\eta(s)\beta_y(s)ds \]  

(26)

where:
- \( G(s) \) = Strength of sextupoles magnets as function of position
- \( \beta_{x,y}(s) \) = Horizontal or vertical beta as function of position

From Equation (23), (24), (25) and (26), we’re try to find the strength and number of the sextupoles magnets by compensation of the natural chromaticity as defined follow:

\[ \xi_x + \sum_m \xi_{sextx,m} = 0 \]  

(27)

\[ \xi_y + \sum_m \xi_{sexty,m} = 0 \]  

(28)

where:
- \( m \) = Number of sextupole magnets

3. Result and discussion

In general, the approach based on Equation (27) and (28) is used to evaluate the number of sextupoles and its strength and to determine the best combination to correct the chromaticities without causing any sextupoles magnetic design problems. For such evaluation, momentum deviation is assumed as 0.01 and three cases have been evaluated as follow.
Case A
In this lattice, the two sextupoles with the different field strength as G(s) and F(S) are inserted as shown in Figure 5. It is located at the position with the different Courant-Snyder parameters.

![Figure 5](image_url)

**Figure 5.** Case A: Two sextupoles are inserted into the ring.

By considering the sextupoles configuration as shown in “Case A”, the location and required parameters of the Sextupoles are fixed and the field strength can be identified from Equation (24) and (25) as shown in Table 1.

| Sextupole | Length of Sextupoles, l [m] | Initial Position, S[m] | Field Strength, G[m⁻³] |
|-----------|----------------------------|------------------------|------------------------|
| Sextupole 1 | 0.2                        | 6.1512                 | 28.4157                |
| Sextupole 2 | 0.2                        | 18.8536                | -38.3670               |

Table 1. Parameters of sextupoles for Case A.
Figure 6. Horizontal and vertical lattice: Ideal, natural chromaticity and corrected lattice by two sextupoles with $\Delta p/p=0.01$.

From Figure 6, the field strengths of $G_{1,2}(s)$ 28.4157 and -38.3670 can compensate the chromaticity perturbation ideally.

**Case B**

In this lattice, four sextupoles are inserted as shown in Figure 7. Sext1 and Sext3 have same strength and located at the position with the same Courant-Snyder parameters. Also Sext2 and Sext4 have same strength and located at the position with the same Courant-Snyder parameters.
By considering the sextupoles configuration as shown in “Case B”, the locations and parameters of the Sextupoles are fixed and the field strength can be identified from Equation (24) and (25) as shown in Table 2.

**Table 2. Parameters of sextupoles for Case B.**

| Sextupole  | Length of Sextupoles, $l$ [m] | Initial Position, $S$ [m] | Field Strength, $G_i$ [m$^{-3}$] |
|------------|------------------------------|---------------------------|----------------------------------|
| Sextupole 1 | 0.2                          | 6.1512                    | 14.0728                          |
| Sextupole 2 | 0.2                          | 18.8536                   | -19.1835                         |
| Sextupole 3 | 0.2                          | 34.0560                   | 14.0728                          |
| Sextupole 4 | 0.2                          | 45.7584                   | -19.1835                         |

**Figure 7.** Case B: Four sextupoles are inserted into the ring.
From Figure 8, the field strengths of four Sextupoles with $G_{1,3}(s)$ as 14.0728 and $G_{2,4}(s)$ as -19.1835 can compensate the chromaticity perturbation ideally.

**Case C**

In this lattice, the four sextupoles are inserted as shown in Figure 9. Sext1 and Sext3 have same strength and located at the position with the same Courant-Snyder parameters. Also Sext2 and Sext4 have same strength and located at the position with the same Courant-Snyder parameters.
By considering the sextupole configuration as shown in “Case C”, the locations and parameters of the sextupoles are fixed and the field strength can be identified from Equation (24) and (25) as shown in Table 3.

**Table 3. Parameters of sextupoles for Case C**

| Sextupole | Length of Sextupoles, l [m] | Initial Position, S[m] | Field Strength , Gi[m³] |
|-----------|-----------------------------|------------------------|--------------------------|
| Sextupole 1 | 0.2                         | 5.3012                 | 0.74776                  |
| Sextupole 2 | 0.2                         | 24.3584                | -0.67603                 |
| Sextupole 3 | 0.2                         | 32.0060                | 0.74776                  |
| Sextupole 4 | 0.2                         | 44.4084                | -0.67603                 |
From Figure 10, the field strengths of four sextupoles with \( G_{1,3}(s) \) as 0.74776 and \( G_{2,4}(s) \) as -0.67603 can compensate the chromaticity perturbation ideally.

For three cases, we can summarize in Table 4. We can realized that more gentle correction uses two groups of sextupoles with individual magnets distributed more evenly around the circular accelerator and the total required sextupole strength is spread over all sextupoles and the strength could be reduced. These features are desired because the smaller strength will make the sextupole magnet much smaller.

**Table 4. Comparison of Case A, B and C.**

| Case | Types of Sextupoles | Number of Sextupoles | \( G_1(s)[m^{-3}] \) | \( G_2(s)[m^{-3}] \) | \( \Delta \varepsilon_x \) | \( \Delta \varepsilon_y \) |
|------|---------------------|----------------------|---------------------|---------------------|-----------------|-----------------|
| A    | 2                   | 2                    | 28.1457             | -38.367             | 0               | 0               |
| B    | 2                   | 4                    | 14.0728             | -19.1835            | 0               | 0               |
| C    | 2                   | 4                    | 0.74776             | -0.67603            | 0               | 0               |

According Table 4, we noticed that three cases able to compensate the natural chromaticities perfectly. In order to compensate the natural chromaticity effectively, two or more sextupoles with different strength and Courant-Snyder parameters are required. The number of sextupoles plays important role as well. With the increment of the number of sextupoles, it will decrease the required
strength of the sextupoles. Meanwhile, case B and case C are used to evaluate the effectiveness of the sextupoles location. As obvious, the location is important and the larger Courant-Snyder parameters is a major compensator for the chromaticity correction. Hence, case C is an ideal configuration of natural chromaticity correction by using sextupoles.

4. Conclusion
In this paper, approach of natural chromaticity correction by sextupoles is introduced. The number of the sextupoles have been optimized and the strength is identified. Case C is an ideal case with 4 sextupoles and two field strengths and able to compensate the chromaticity perfectly. In near future, the nonlinear perturbation term of the Closed Orbit Distortion (COD) must be considered and evaluated.

5. References
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