Bäcklund Transformations and Loop Group Actions

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Abstract

We construct a local action of the group of rational maps from $S^2$ to $GL(n, C)$ on local solutions of flows of the ZS-AKNS $sl(n, C)$-hierarchy. We show that the actions of simple elements (linear fractional transformations) give local Bäcklund transformations, and we derive a permutability formula from different factorizations of a quadratic element. We prove that the action of simple elements on the vacuum may give either global smooth solutions or solutions with singularities. However, the action of the subgroup of the rational maps that satisfy the $U(n)$-reality condition $g(\bar{\lambda})^*g(\lambda) = I$ on the space of global rapidly decaying solutions of the flows in the $u(n)$-hierarchy is global, and the action of a simple element gives a global Bäcklund transformation. The actions of certain elements in the rational loop group on the vacuum give rise to explicit time periodic multi-solitons (multi-breathers). We show that this theory generalizes the classical Bäcklund theory of the sine-Gordon equation. The group structures of Bäcklund transformations for various hierarchies are determined by their reality conditions. We identify the reality conditions (the group structures) for the $sl(n, R)$, $u(k, n-k)$, KdV, Kupershmidt-Wilson, and Gel’fand-Dikii hierarchies. The actions of linear fractional transformations that satisfies a reality condition, modulo the center of the group of rational maps, gives Bäcklund and Darboux transformations for the hierarchy defined by the reality condition. Since the factorization cannot always be carried out under these reality condition, the action is again local, and Bäcklund transformations only generate local solutions for these hierarchies unless singular solutions are allowed.

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1. **Introduction**

The classical Bäcklund transformations are local geometric transformations, which construct from a given surface of constant Gaussian curvature $-1$ a 2-parameter family of such surfaces. To find such transformations, one needs to solve a system of compatible ordinary differential equations. Since surfaces of Gaussian curvature $-1$ are classically known to be equivalent to local solutions of the sine-Gordon equation

$$q_{xt} = \sin q,$$

(SGE)

this provides a method of deriving new solutions of a partial differential equation from a given solution via the solution of ordinary differential equations (cf. [Da], [Ei]). Most of the known “integrable systems” possess transformations of this type.

Applying Bäcklund transformations $n$ times to a solution of the sine-Gordon equation produces a hierarchy of $2n$-dimensional families of solutions. Moreover, the Bianchi permutability theorem states that the second and higher families can be obtained from the first family through algebraic formulas. This allowed the classical geometers to write down explicit solutions for the sine-Gordon equation and explicit surfaces of curvature $-1$. For example, they applied one Bäcklund transformation to the vacuum solution of the sine-Gordon equation to get the pseudosphere (stationary 1-soliton) and Dini surfaces (1-soliton), applied Bäcklund transformations twice to the vacuum solution to get the K"uen surface (2-soliton), and applied the Bianchi permutability formula with two suitable complex conjugate parameters to get breathers (time periodic solutions).
The sine-Gordon equation is clearly invariant under the Lorentz group, i.e., if \( q \) is a solution of the sine-Gordon equation and \( r \) is a non-zero real number, then \( \hat{q}(x, t) = q(r^{-1}x, rt) \) is again a solution of the sine-Gordon equation (note that we are working in characteristic coordinates). This is called the Lie transformation for \(-1\) curvature surfaces in \( \mathbb{R}^3 \) in the classical surface theory.

What are now called Darboux transformations were discovered by Darboux during his investigation of Liouville metrics. A metric \( ds^2 = A(x, y) (dx^2 + dy^2) \) is Liouville if there is a coordinate system \((u, v)\) such that \( ds^2 \) is of the form

\[
ds^2 = (f(u) - g(v))(du^2 + dv^2)
\]

for some \( f \) and \( g \) of one variable. The classical geometers were interested in such metrics at least in part because Liouville had shown that all geodesics on such surfaces can be obtained by quadratures. The question of deciding whether a metric \( ds^2 \) is Liouville led to the study of the following special second order linear partial differential equation

\[
w_{xy} = (f(x + y) - g(x - y))w.
\]

Darboux was led to look for transformations of Hill’s operators in the process of separating variables in this equation. The original analytic version of Darboux transformation ([Da] v. 2 Chap. 9) is the following: Let \( q \) be a smooth function of one variable, \( k_0 \) a constant, and suppose that \( f \) satisfies \( f'' = (q + k_0)f \). Set

\[
q^\# = f(f^{-1})'' - k_0.
\]

If \( y(x, k) \) is the general solution of the Hills operator with potential \( q \):

\[
y'' = (q + k)y,
\]

then \( z = y' - (f/f')y \) is the general solution of the Hills operator with potential \( q^\# \):

\[
z'' = (q^\# + k)z.
\]

This Darboux theorem gives an algebraic algorithm (without quadrature) to transform general solutions of \( D^2 - q - k \) to those of \( D^2 - q^\# - k \). Next, suppose that we factor

\[
D^2 - q - \lambda_0 = (D + v)(D - v),
\]

In other words, suppose that \( v \) satisfies \( v_x + v^2 = q + \lambda_0 \). (Here \( D = \frac{d}{dx} \).) Choose \( f \) so that \( f'/f = v \). Then

\[
(D - v)(D + v) = D^2 - q^\# - \lambda_0.
\]

Since, if \( q(x, t) \) is a solution of KdV then the Hills operators with potential \( q(\cdot, t) \) are isospectral, it follows that the Darboux transformations of the Hills
operators induce transformations on the space of solutions of KdV. This is a critical observation due to Adler and Moser [AM] and Deift [De].

We give another interpretation of the Darboux transformation. Write the Hills operator \( \frac{d^2}{dx^2} - q - \lambda^2 \) as a first order system \( L_{q,\lambda} = \frac{d}{dx} - \begin{pmatrix} \lambda & q \\ 1 & -\lambda \end{pmatrix} \). Given an eigenfunction of the Hills operator with potential \( q \) and eigenvalue \( k_0 = \alpha^2 \) is the same as given a trivialization of \( L_{q,\alpha} \) (thought as a connection on the line). Then Darboux’ theorem can be reformulated as follows: Given a trivialization of \( L_{q,\alpha} \) for some \( \alpha \in \mathbb{C} \), the map \( q \mapsto q^\# \) transforms the trivialization of \( L_{q,\lambda} \) to that of \( L_{q^\#,\lambda} \) by an algebraic formula. Most of the known integrable systems also possess transformations of this type.

The second author constructed an action of the rational loop group on the space of solutions of harmonic maps from \( R^{1,1} \) to \( SU(n) \) in [U1], and showed that the action of a simple element (i.e., a linear fractional transformation) can be obtained by solving two compatible ordinary differential equations. The starting point of this paper is the realization that this rational loop group action in [U1] can be generalized to solutions of other partial differential equations having Lax pairs and that satisfy the “reality condition” of a compact group. We give an analogous construction of the action of the rational loop group on the space of global solutions of the flows in the AKNS-ZS \( u(n) \)-hierarchy. We will see:

1. The action of a simple element (a linear fractional transformation) corresponds to a global Bäcklund transformation.
2. The Bianchi permutability formula arises from various ways of factoring quadratic elements in the rational loop group into simple elements.
3. The Bäcklund transformations can be computed from solutions of ordinary differential equations given a known solution of the partial differential equation.
4. Once given the trivialization of the Lax pair corresponding to a given solution, the action of a simple element corresponds to a global Darboux transformation and is algebraic.
5. Lie transformations arise as the scaling transformations, which extend the action of the rational loop group to the semi-direct product of the multiplicative group \( R^* \) of non-zero real numbers and the rational loop group.

Since the sine-Gordon equation arises as part of the hierarchy (the \(-1\)-flow for \( su(2) \) with an involution constraint), we can check that we are generalizing the classical theory. The choice of group structure depends on the choice of base point. Hence the group structure is not canonical and was not apparent to the classical geometers.

An interesting observations is that appropriate choices of poles for the rational loop yield time periodic solutions. This gives an insight into the construction of the classical breathers of the sine-Gordon equation ([Da]). There are no simple factors in the rational loop group corresponding to the placement of poles for time periodic solutions. However, there are quadratic elements (product of two simple elements), whose simple factors do not satisfy the algebraic constraints to
preserve sine-Gordon, but which nevertheless generate the well-known breathers (one way to think of them is as the product of two complex conjugate Bäcklund transformations). The product of these quadratic factors generate arbitrarily complicated time periodic solutions.

The sine-Gordon equation also arises as the equation for wave (or harmonic) maps from the Lorentz space $\mathbb{R}^{1,1}$ to $S^2$ (for example, see [P]). Shatah and Strauss proved in [SS] that the classical breather solutions for the sine-Gordon equation produce homoclinic wave maps from $S^1 \times \mathbb{R}$ to $S^2$. Using a simple change of gauge for the Lax pair of the $-1$-flow, the first author proved in [Te] that solutions of the $-1$-flow give rise to wave maps from $\mathbb{R}^{1,1}$ to symmetric spaces. In a forthcoming paper [TU2], we prove that the time periodic $m$-solitons for the $-1$-flow constructed in this paper also give rise to homoclinic wave maps from $S^1 \times \mathbb{R}$ into compact symmetric spaces.

The permutability formula has several useful applications. For example, one of the key ingredients of the study of discrete $-1$ curvature surfaces in $\mathbb{R}^3$ by Bobenko and Pinkall [BP] is the permutability formula for the sine-Gordon equation. Since constant sectional curvature $n$-dimensional submanifolds in Euclidean spaces are given by solutions of the $n$ commuting first flows (cf. [Te], [TU1]), the generalized permutability formula should be useful in the study of discretization of constant curvature submanifolds and soliton equations. We also use the Bianchi permutability formula to write down an explicit formula for $m$-soliton solutions of the $j$-th flow.

Local Bäcklund transformations for the $j$-th flow were constructed by Zakharov and Shabat in [ZS 2], Sattinger and Zurkowski in [SZ 1, 2], by Beals, Deift and Tomei in [BDT], by Gu and Zhou in [GZ] and by Cherndik in [Ch]. Our construction gives a group structure of these transformations, and provide a systematic method of finding such transformations for equations having a Lax pair. The algebraic structure of these transformations also makes many of the mysterious classical results for the sine-Gordon equation apparent.

We give an outline of the method we use to construct Bäcklund transformations and explain how the group structure for these transformations is obtained. Most of the evolution we considered in this paper has a Lax pair with a parameter, i.e., it is given as the condition that a one-parameter family of connections is flat:

$$\left[ \frac{\partial}{\partial x} + A(x, t, \lambda), \frac{\partial}{\partial t} + B(x, t, \lambda) \right] = 0,$$

where $A$ and $B$ are differential operators in $u$ and its derivatives in $x$ and $A, B$ are holomorphic for $\lambda \in C$. The trivialization $E$ of a solution $u$ normalized at $(0, 0)$ is defined to be the solution for the following linear system

$$E_x = EA, \quad E_t = EB, \quad E(0, 0, \lambda) = I.$$

Then $E(x, t, \lambda)$ is holomorphic for $\lambda \in C$. The general view of the specific construction is the Birkhoff factorization theorem. Let $O_\infty$ denote a neighborhood of $\infty$ in $C \cup \{\infty\} = S^2$, $L_+ (GL(n, C))$ the group of holomorphic maps.
from \( C \) to \( GL(n,C) \), (under pointwise multiplication), \( L_-(GL(n,C)) \) the group of holomorphic maps \( h_- \) from \( \mathcal{O}_\infty \) to \( GL(n,C) \) such that \( h_-(\infty) = I \), and \( L(GL(n,C)) \) the group of holomorphic maps from \( \mathcal{O}_\infty \cap C \) to \( GL(n,C) \). The Birkhoff factorization theorem states that the multiplication map

\[
\mu : L_+(GL(n,C)) \times L_-(GL(n,C)) \to L(GL(n,C)), \quad (h_+, h_-) \mapsto h_+ h_-
\]
is one to one and the image is an open dense subset of \( L(GL(n,C)) \). Hence, formally, there is a “dressing action” of \( L_-(GL(n,C)) \) on \( L_+(GL(n,C)) \) defined as follows: given \( h_\pm \in L_\pm(GL(n,C)) \), if \( h_- h_+ \) lies in the image of \( \mu \) then there exists unique \( f_\pm \in L_\pm(GL(n,C)) \) such that \( h_- h_+ = f_+ f_- \). Then the dressing action is defined by \( h_- \sharp h_+ = f_+ \). The Birkhoff factorization is not explicit. Moreover, singularities arise, typically on a codimension two set in the parameter spaces. Because the singular set is closed and its complement is dense, the action is local.

To construct Bäcklund transformation, we choose a linear fractional transformation \( h_- \in L_-(GL(n,C)) \). Since the trivialization \( E(x,t,\lambda) \) of a solution \( u \) of the evolution equation is holomorphic for \( \lambda \in C \), the map \( E(x,t) \in L_+(GL(n,C)) \), where \( E(x,t)(\lambda) = E(x,t,\lambda) \). For each \( (x,t) \), let \( h_- \) acts on \( E(x,t) \). If \( \tilde{E}(x,t) = h_- \sharp E(x,t) \) exists, then a new solution can be obtained from \( \tilde{E}^{-1} \tilde{E}_x \). However, the factorization only involves poles and zeros, and is explicit and algebraic, not abstract. The group structure of these transformations is clearly the one inherited from \( L_-(GL(n,C)) \). However, even these factorizations cannot be always carried out. Hence, in general, it gives rise to a local theory. The only case we obtain a good global theory is when the Lax pair satisfies the \( u(n) \)-reality condition, i.e., \( A \) and \( B \) satisfies:

\[
A(x,t,\bar{\lambda})^* + A(x,t,\lambda) = 0, \quad B(x,t,\bar{\lambda})^* + B(x,t,\lambda) = 0.
\]

Let \( \mathcal{G}, \mathcal{G}_\pm \) denote the Lie algebra of \( L(GL(n,C)) \) and \( L_\pm(GL(n,C)) \) respectively. It was explained in an earlier paper [TU1] that \( (\mathcal{G}, \mathcal{G}_-, \mathcal{G}_+) \) is a Manin-triple, and the \( sl(n,C) \)-hierarchy is a natural hierarchy of flows on the the function space \( C(R, \mathcal{G}_+) \). Moreover, the algebraic and symplectic properties of the \( sl(n,C) \)-hierarchy are determined by this Manin-triple triple. All the hierarchies we considered in this paper are obtained as the restriction of the \( sl(n,C) \)-hierarchy to \( C(R, (\mathcal{G}_+)_{_\theta}) \), where \( (\mathcal{G}_+)_{_\theta} \) is the fixed point set of certain finite order Lie algebra automorphism \( \theta \) of \( \mathcal{G}_+ \). Symplectic structures and the group structure of Bäcklund transformations of these hierarchies only depend on the algebra \( (\mathcal{G}_+)_{_\theta} \). Finite order automorphisms of \( \mathcal{G} \) are not difficult to find. For example, if \( \mathcal{U} \) is a real form of \( sl(n,C) \) defined by a conjugate linear involution \( \sigma \), then \( \sigma \) induces an involution \( \hat{\sigma} \) on \( \mathcal{G} \):

\[
\hat{\sigma}(A)(\lambda) = \sigma(A(\bar{\lambda})).
\]

The restriction of the \( sl(n,C) \)-hierarchy to the subspace \( C(R, (\mathcal{G}_+)_{_\hat{\sigma}}) \) gives the \( \mathcal{U} \)-hierarchy. For example, the second flow in the \( su(2) \)-hierarchy is the focusing
non-linear Schrödinger equation, and the second flow in the $u(1,1)$-hierarchy is
the defocusing non-linear Schrödinger equation. In general, if $\sigma$ is an order $k$
automorphism of $sl(n,C)$, then it induces naturally an order $k$ automorphism $\tilde{\sigma}$
on $G_+$:

$$\tilde{\sigma}(A)(\lambda) = \sigma(A(\alpha^{-1}\lambda)), \quad \text{where} \; \alpha = e^{2\pi i k}.$$ 

For example, the Kupershmidt-Wilson hierarchy is of this type. However, the au-
tomorphisms that give the KdV and Gel’fand-Dikii hierarchies are more difficult
to find. We construct these in section 12 and 14 respectively.

This paper is organized as follows: In section 2, we review the construction
of the ZS-AKNS $sl(n)$-hierarchy of flows. In section 3, we explain various
restriction defined by Lie algebra involutions (reality conditions). We will also
see in this section that the second flow in the $su(2)$-hierarchy is the focusing
non-linear Schrödinger equation, and the second flow in the $u(1,1)$-hierarchy is
the defocusing non-linear Schrödinger equation (i.e., with an opposite sign of the
cubic term). The modified KdV equation is the third flow of the $su(2)$-hierarchy
twisted by the involution $\sigma(x) = -x^t$, and the modified KdV with an opposite
sign of the cubic term is the third flow of the $u(1,1)$-hierarchy twisted by the
involution $\tau(x) = -x^t$. So this group theoretic approach allows several different
generalizations of the two non-linear Schrödinger equations and the two modi-
fied KdV equations depending on the choices of the real form of $sl(n,C)$ and
of various involutions. Although the hierarchies associated to two different real
forms may look similar algebraically, they have quite dissimilar global analytic
behavior. In section 4, we review the construction of local Bäcklund transfor-
mations for the flows in the $sl(n,C)$-hierarchy and give example to show that
the new solutions obtained from applying these transformations to a smooth so-
lution can have singularities. We deal with only the $u(n)$-hierarchy in the next
six sections. In section 5, we construct an action of the rational loop group on
the space of solutions of the $j$-th flow in the $u(n)$-hierarchy and show that the
action of a simple element gives a global Bäcklund transformation. In section 6,
we prove a relation among simple elements of the rational loop group and use
this relation to prove an analogue of the Bianchi permutability formula for the
$j$-th flow. In section 7, we derive $m$-soliton formula in closed form. In section 8,
we show that the scaling transformations of the $j$-th flow extend the action of the
rational loop group to a one-dimensional extension of the rational loop group.
Since the $n$-dimensional system associated to $u(n)$ is given by $n$ commuting first
flows, Bäcklund theory for this system works the same way as for the first flow.
This is explained in section 9. In section 10, we use Bäcklund transformations
and the permutability formula to construct time periodic solutions for the $j$-th
flow and the $-1$-flow. In section 11, we construct Bäcklund transformations for
local solutions of the $u(k,n-k)$-hierarchies of flows. The solutions obtained
this way may be singular. This construction is evidence that the domain of the
inverse scattering transformation for equations whose Lax pairs satisfying the
reality conditions of a non-compact group is in general quite complicated. In
section 12, 13 and 14, we find the reality conditions, algebraic structures and
Bäcklund transformations for the KdV, Kupershmidt-Wilson and Gel’fand-Dikii hierarchies respectively.

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2. The ZS-AKNS $n \times n$ flows

Study of the hierarchy of commuting Hamiltonian flows associated to the non-linear Schrödinger equation began in 1972 with a paper by Zakharov and Shabat ([ZS1]). Ablowitz, Kaup, Newell and Segur generalized these ideas in 1974 [AKNS] to any $2 \times 2$ system, including sine-Gordon and modified KdV. The $n \times n$ case was treated by Zakharov and Shabat in 1979 [ZS2]. Beals and Coifman made these construction analytically rigorous ([BC1,2]). To understand the Bäcklund transformations, only the algebraic description is needed. However, the scattering theory is used in the proofs. We review the construction of these flows and give a simple method to construct local solutions whose reduced wave functions are analytic at infinity.

Let $a = \text{diag}(a_1, \cdots, a_n)$ be a fixed non-zero diagonal matrix in $\text{sl}(n,C)$, and

$$sl(n)_a = \{ y \in sl(n,C) \mid [a, y] = 0 \},$$

$$sl(n)^+_a = \{ y \in sl(n,C) \mid \text{tr}(ay) = 0 \},$$

denote the centralizer of $a$ and its orthogonal complement (with respect to trace) in $sl(n,C)$ respectively. Let $S(R, sl(n)^+_a)$ denote the space of maps in the Schwartz class. To define the $sl(n,C)$-hierarchy of flows, we need part of the scattering theory of Beals and Coifman ([BC1]). Let

$$\Gamma_a = \{ \lambda \in C \mid \text{Re}(\lambda(a_j - a_k)) = 0, 1 \leq j < k \leq n \}.$$

2.1 Theorem ([BC1]). If $u \in S(R, sl(n)^+_a)$, then there exists

$$m : R \times (C \setminus \Gamma_a) \to GL(n, C)$$

such that

(i) $m(x, \lambda)$ is meromorphic for $\lambda \in C \setminus \Gamma_a$, has only poles in $C \setminus \Gamma_a$,

(ii) $m(x, \lambda)$ has an asymptotic expansion at $\lambda = \infty$:

$$m(x, \lambda) \sim I + m_1(x)\lambda^{-1} + m_2(x)\lambda^{-2} + \cdots,$$

(iii) $E(x, \lambda) = m(0, \lambda)^{-1}e^{a\lambda x}m(x, \lambda)$ is holomorphic for $\lambda \in C$.

(iv) $E^{-1}E_x = a\lambda + u$ and $u = [a, m_1]$,

(v) $\lim_{x \to -\infty} m(x, \lambda) = I$. 

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2.2 Remark. A wave function of $u(x)$ is a solution $\psi(x, \lambda)$ of
\[
\psi^{-1}\psi_x = a\lambda + u(x).
\]
This is what in general called a trivialization of the connection
\[
\frac{d}{dx} + a\lambda + u.
\]
For $m$ as in Theorem 2.1, $e^{a\lambda x}m(x, \lambda)$ is a wave function. We call $m$ the global reduced wave function.

Note that if $m$ satisfies condition (ii) and (iii) of Theorem 2.1 then $m$ satisfies condition (iv). This follows from a direct computation because
\[
E^{-1}E_x = m^{-1}am\lambda + m^{-1}m_x = a\lambda + [a, m_1] + \mathcal{O}(\lambda^{-1})
\]
and $E^{-1}E_x$ is holomorphic in $\lambda \in C$ imply that $E^{-1}E_x = a\lambda + [a, m_1]$.

2.3 Definition. For $b \in sl(n, C)$ such that $[b, a] = 0$, let $Q_{b,j}$ denote the coefficient of $\lambda^{-j}$ in the asymptotic expansion of $m^{-1}bm$ at $\lambda = \infty$:
\[
m^{-1}bm \sim Q_{b,0} + Q_{b,1}\lambda^{-1} + Q_{b,2}\lambda^{-2} + \cdots. \tag{2.1}
\]

2.4 Definition. Let $I$ be an open interval of $R$, $\mathcal{O}_\infty$ an open neighborhood of $\infty$ in $S^2 = C \cup \{\infty\}$, $a \in sl(n)$ and $\Gamma_a$ defined as above by $a$. A smooth map $m : I \times (\mathcal{O}_\infty \setminus \Gamma_a) \to GL(n, C)$ is called a local reduced wave function of $u : I \to sl(n)\perp$ if $m$ satisfies conditions (i)-(iv) of Theorem 2.1.

We will show that $Q_{b,j}$ is an operator in $u = [a, m_1]$, and will write it as $Q_{b,j}(u)$. First note that $Q_{b,j}$ satisfies the following recursive formula:
\[
(Q_{b,j}(u))_x + [u, Q_{b,j}(u)] = [Q_{b,j+1}(u), a]. \tag{2.2}
\]
To see this, note that (iii) and (iv) of Theorem 2.1 implies that $\psi(x, \lambda) = e^{a\lambda x}m(x, \lambda)$ satisfies
\[
\psi^{-1}\psi_x = a\lambda + u.
\]
So
\[
\left[ \frac{d}{dx} + a\lambda + u, \psi^{-1}\psi \right] = 0.
\]
But $[b, a] = 0$ implies $\psi^{-1}\psi = m^{-1}bm$. Hence $Q_{b,j}$ satisfies (2.2).

Write
\[
Q_{b,j} = T_{b,j} + P_{b,j} \in sl(n)_a + sl(n)\perp.
\]
Then equation (2.2) gives
\[
P_{b,j} = -\text{ad}(a)^{-1} ((P_{b,j-1})_x + \pi_1([u, Q_{b,j-1}])],
T_{b,j} = -\pi_0([u, P_{b,j-1}]), \tag{2.3}
\]
where $\pi_0$ and $\pi_1$ denote the projection of $sl(n, C)$ onto $sl(n)_a$ and $sl(n)\perp_a$ with respect to $sl(n, C) = sl(n)_a + sl(n)\perp_a$ respectively.

The following theorem is proved by Sattinger [Sa] if $a$ has distinct eigenvalues. Since his proof gives an explicit method to compute the $Q_{b,j}$'s, we will repeat it here.
2.5 Theorem ([Sa]). Let \( a \in \mathfrak{sl}(n) \), and \( m \) a local reduced wave function for \( u : I \to \mathfrak{sl}(n)_{a}^{\perp} \). If \( b \) is a polynomial of \( a \), then the coefficient \( Q_{b,j} \) of \( \lambda^{-j} \) in the asymptotic expansion of \( m^{-1}bm \) is an order \( (j-1) \) polynomial differential operator in \( u \).

PROOF. Since \( b = p(a) \) for some polynomial \( p \) and \( m^{-1}bm = p(m^{-1}am) \), it suffices to prove the Theorem for \( b = a \). It is easy to see that \( Q_{a,1} = u \). We will prove \( Q_{a,j} \) is a polynomial differential operator in \( u \) by induction. Suppose \( Q_{a,i} \) is a polynomial differential operator in \( u \) for \( i \leq j \). Write

\[
Q_{a,i} = P_{a,i} + T_{a,i} \in \mathfrak{sl}(n)_{a}^{\perp} + \mathfrak{sl}(n)_{a}
\]

as before. Using formula (2.3), we see that \( P_{a,i+1} \) is a polynomial differential operator in \( u \). But we can not conclude from formula (2.3) that \( T_{a,i+1} \) is a polynomial differential operator in \( u \). Suppose \( a \) has \( k \) distinct eigenvalues \( c_1, \ldots, c_k \). Then

\[
f(t) = (t-c_1)(t-c_2)\cdots(t-c_k)
\]

is the minimal polynomial of \( a \). So \( f(m^{-1}am) = 0 \), which implies that the formal power series

\[
f(a + Q_{a,1}\lambda^{-1} + Q_{a,2}\lambda^{-2} + \cdots) = 0. \tag{2.4}
\]

Notice that \( f'(a) \) is invertible and \( T_{a,j+1} \) commutes with \( a \). Now compare coefficient of \( \lambda^{-(j+1)} \) in equation (2.4) implies that \( T_{a,j+1} \) can written in terms of \( a, Q_{a,1}, \ldots, Q_{a,j} \). This proves that \( Q_{a,j+1} \) is a polynomial differential operator in \( u \). □

The following Proposition follows from formula (2.3).

2.6 Proposition. Suppose \( u(\cdot,t) \in S(R, \mathfrak{sl}(n)_{a}^{\perp}) \) for all \( t \),

\[
\left[ \frac{\partial}{\partial x} + a\lambda + u, \frac{\partial}{\partial t} + b\lambda^{j} + v_{1}\lambda^{j-1} + \cdots + v_{j} \right] = 0
\]

for some \( v_{1}, \ldots, v_{j} \), and \( \lim_{x \to -\infty} v_{k}(x,t) = 0 \) for all \( 1 \leq k \leq j \). Then \( v_{k} = Q_{b,k}(u) \).

2.7 Definition. The \( j \)-th flow in the \( \mathfrak{sl}(n,C) \)-hierarchy on \( S(R, \mathfrak{sl}(n)_{a}^{\perp}) \) defined by \( b \) is the evolution equation

\[
u_{t} = (Q_{b,j}(u))_{x} + [u, Q_{b,j}(u)] = [Q_{b,j+1}(u), a]. \tag{2.5}
\]

2.8 Example. If \( n = 2 \) and \( a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), then

\[
\mathfrak{sl}(2)_{a}^{\perp} = \left\{ \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \mid q, r \in C \right\}.
\]
For \( u = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \), we have

\[
Q_{a,1}(u) = u = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix},
\]

\[
Q_{a,2}(u) = \begin{pmatrix} -\frac{q r}{2} & -\frac{q r}{2} \\ \frac{r^2}{2} & \frac{r^2}{2} \end{pmatrix},
\]

\[
Q_{a,3}(u) = \begin{pmatrix} -\frac{1}{4}(qr_x - rq_x) & \frac{1}{4}(q_{xx} - 2q^2r) \\ \frac{1}{4}(r_{xx} - 2qr^2) & \frac{1}{4}(qr_x - rq_x) \end{pmatrix},
\]

and the first three flows in the \( sl(2,\mathbb{C}) \)-hierarchy are

\[
q_t = q_x, \quad r_t = r_x,
\]

\[
q_t = -\frac{1}{2}(q_{xx} - 2q^2r), \quad r_t = \frac{1}{2}(r_{xx} - 2qr^2),
\]

\[
q_t = \frac{1}{4}(q_{xxx} - 6qrq_x), \quad r_t = \frac{1}{4}(r_{xxx} - 6qr^2r_x).
\]

As a consequence of the recursive formula (2.2), we have

2.9 Proposition. Let \( \mathcal{O} \) be an open subset of \( \mathbb{R}^2 \). Then \( u : \mathcal{O} \to sl(n)_a^+ \) is a solution of the \( j \)-th flow equation (2.5) if and only if

\[
\left[ \frac{\partial}{\partial x} + (a\lambda + u), \frac{\partial}{\partial t} + (b\lambda^j + Q_{b,1}(u)\lambda^{j-1} + \cdots + Q_{b,j}(u)) \right] = 0.
\]

(This pair of operators is called a Lax pair for the \( j \)-th flow (2.5)).

The following proposition is elementary.

2.10 Proposition. Given two smooth maps \( A, B : \mathbb{R}^2 \to gl(n) \), the following three statements are equivalent:

(i) \( \left\{ \frac{\partial}{\partial x} + A, \frac{\partial}{\partial t} + B \right\} = 0 \),

(ii) \( B_x - A_t + [A, B] = 0 \),

(iii) \( \begin{cases} E_x = EA, \\ E_t = EB, \end{cases} \) is solvable.

It follows from Propositions 2.9 and 2.10 that if \( u \) is a solution of the \( j \)-th flow (2.5) defined by \( b \) then there exists a unique solution \( E(x, t, \lambda) \) for

\[
\begin{cases}
E_x = E(a\lambda + u), \\
E_t = E(b\lambda^j + Q_{b,1}(u)\lambda^{j-1} + \cdots + Q_{b,j}(u)), \\
E(0, 0, \lambda) = I.
\end{cases}
\]

Such \( E \) will be called the trivialization of \( u \) normalized at \((0,0)\). This is a different normalization than found in scattering theory and much of the algebraic
literature. Note that this choice of base point \((0, 0)\) is not canonical. Different choices of base point will not change the definition of Bäcklund transformations which we will construct. However, the group structure of Bäcklund transformations depends on this choice \((0, 0)\). For the rest of this paper we fixed the normalization at \((0, 0)\).

Bäcklund transformations are described algebraically by starting with the solution \(u\), constructing the trivialization \(E\) of \(u\), operating on \(E\) to produce \(\tilde{E}\), and producing a new solution \(\tilde{u}\) from \(\tilde{E}\). So it is important to describe when \(E\) is the trivialization belonging to a solution \(u\). Proposition 2.6 gives such a condition for global solutions, and we give a condition for local solutions next.

2.11 Proposition. Let \(O\) be an open subset of \(R^2\), \(O_\infty\) an open subset of \(S^2\) at \(\infty\), \(a \in \text{sl}(n)\), \(b = p(a)\) for some polynomial \(p\), \(m : O \times (O_\infty \setminus \Gamma_a) \rightarrow GL(n, C)\) a smooth map. Suppose \(m(x, t, \lambda)\) is meromorphic in \(\lambda \in C \setminus \Gamma_a\), the asymptotic expansion of \(m\) at \(\infty\) is

\[
m(x, t, \lambda) \sim I + m_1(x, t)\lambda^{-1} + m_2(x, t)\lambda^{-2} + \cdots,
\]

and

\[
E(x, t, \lambda) = m(0, 0, \lambda)^{-1}e^{a\lambda x + b\lambda^j t}m(x, t, \lambda)
\] (2.8)

is holomorphic in \(\lambda \in C\). Then \(u = [a, m_1] : O \rightarrow sl(n)_a^\perp\) is a solution of the \(j\)-th flow equation (2.5) and \(E\) is the trivialization of \(u\) normalized at \((0, 0)\). (We call \(m\) a reduced wave function of the local solution \(u\)).

PROOF. Since \(E\) is holomorphic in \(\lambda \in C\), \(E^{-1}E_x\) and \(E^{-1}E_t\) are holomorphic in \(\lambda \in C\). But

\[
E^{-1}E_x = m^{-1}am\lambda + m^{-1}m_x,
\]

\[
= (a + [a, m_1]\lambda^{-1})\lambda + O(\lambda^{-1}) = a\lambda + u + O(\lambda^{-1}).
\]

So \(E^{-1}E_x - (a\lambda + u)\) is holomorphic, bounded in \(\lambda \in C\) and tends to zero as \(\lambda \rightarrow \infty\). By Liouville Theorem,

\[
E^{-1}E_x = a\lambda + u.
\]

Note that \(E^{-1}E_x = \Psi^{-1}\Psi_x\) and \(m^{-1}bm = \Psi^{-1}b\Psi\), where \(\Psi(x, t, \lambda) = e^{a\lambda x + b\lambda^j t}m(x, t, \lambda)\). So

\[
[d_x + a\lambda + u, m^{-1}bm] = [d_x + a\lambda + u, \Psi^{-1}b\Psi] = 0.
\]

Then the proof of Theorem 2.5 implies that

\[
m^{-1}bm \sim b + Q_{b, 1}(u)\lambda^{-1} + Q_{b, 2}(u)\lambda^{-2} + \cdots.
\]

A direct computation gives

\[
E^{-1}E_t = m^{-1}bm\lambda^j + m^{-1}m_t = (m^{-1}bm\lambda^j)_+ + O(\lambda^{-1}).
\]
Since $E^{-1}E_t$ is holomorphic in $\lambda \in C$, we get
\[ E^{-1}E_t = b\lambda^j + Q_{b,1}(u)\lambda^{j-1} + \cdots + Q_{b,j}(u). \]

Propositions 2.10 and 2.9 imply that $u$ satisfies the $j$-th flow equation. \[\square\]

The above Proposition gives a simple method to construct a class of local solutions of the $j$-th flow, whose local reduced wave functions are analytic at $\lambda = \infty$. To explain this, we need the Birkhoff Factorization Theorem. Recall that $L_+(GL(n,C))$ is the group of holomorphic maps from $C$ to $GL(n,C)$ (under pointwise multiplication), $L_-(GL(n,C))$ is the group of holomorphic maps $h_-$ from $O_\infty$ to $GL(n,C)$ such that $h_-(\infty) = I$, and $L(GL(n,C))$ is the group of holomorphic maps from $O_\infty \cap C$ to $GL(n,C)$, where $O_\infty$ is an open subset near $\infty$ in $S^2 = C \cup \{\infty\}$.

2.12 Birkhoff Factorization Theorem. The multiplication map
\[ \mu : L_+(GL(n,C)) \times L_-(GL(n,C)) \to L(GL(n,C)), \]
\[ (f_+, f_-) \mapsto f_+ f_-, \]

is a diffeomorphism onto an open dense subset of $L(GL(n,C))$.

Given $f_\pm \in L_\pm(GL(n,C))$, if we can factor
\[ f_- f_+ = \hat{f}_+ \hat{f}_- \in L_+(GL(n,C)) \times L_-(GL(n,C)), \]

then the left dressing action of $f_-$ on $f_+$ (resp. the right dressing action of $f_+$ on $f_-$) is defined by
\[ f_\pm^* f_+ = \hat{f}_+ \quad \text{(resp. } f_-^* f_+ = \hat{f}_-). \]

These dressing actions are only defined locally. The image of $\mu$ is the top Bruhat cell, the singularities in the factorization occur on lower dimensional cells, which have codimension at least two (cf. [PS]).

Let $a \in sl(n,C)$, $b \in sl(n)_{\mathbb{R}}$ (i.e., $[a,b] = 0$), $j > 0$ an integer, and $e_{a,b,j}(x,t)$ the two-parameter subgroup in $L_+(GL(n,C))$ defined by
\[ e_{a,b,j}(x,t)(\lambda) = e^{a\lambda x + b\lambda^j t}. \]

2.13 Proposition. If $f_- \in L_-(GL(n,C))$, then there exists an open neighborhood $\mathcal{O}$ of $(0,0)$ in $\mathbb{R}^2$ such that $f_-^{-1}e_{a,b,j}(x,t)$ can be factored uniquely as
\[ f_-^{-1}e_{a,b,j}(x,t) = E(x,t)m(x,t)^{-1} \in L_+(GL(n,C)) \times L_-(GL(n,C)) \]
for $(x,t) \in \mathcal{O}$. Moreover,
(i) $m(x,t)(\lambda)$ and $E(x,t)(\lambda)$ are smooth in $(x,t) \in \mathcal{O},$
(ii) $u_{f_-}(x,t) = [a, m_1(x,t)] : \mathcal{O} \to sl(n)_{\mathbb{R}}$ is a solution of the $j$-th flow (2.5), where $m_1(x,t)$ is the coefficient of $\lambda^{-1}$ of the expansion of $m(x,t)(\lambda)$ at $\lambda = \infty$ and $E$ is the trivialization of $u_{f_+},$
(iii) $m$ is a local reduced wave function for $u_{f_-}$ and $m(x,t)(\lambda)$ is analytic at $\lambda = \infty.$

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PROOF. Since \( f^{-1}_{e_{a,b,j}}(0,0) = f^{-1} \) lies in the image of the multiplication map \( \mu \). By Birkhoff Theorem 2.12, the image of \( \mu \) is open. So there exists \( O \) such that if \((x,t) \in O \) then \( f^{-1}_{e_{a,b,j}}(x,t) \) can be factored uniquely as
\[
f^{-1}_{e_{a,b,j}}(x,t) = E(x,t)m(x,t)^{-1} \in L_+(GL(n,C)) \times L_-(GL(n,C)).
\]
Since the map \( \mu \) is smooth, (i) follows. The proof of Proposition 2.11 implies (ii) and (iii).

2.14 Remark. The class of local solutions constructed by Proposition 2.13 contains multi-soliton solutions and algebraic geometry solutions. We will prove later that if \( f \in L_-(GL(n,C)) \) is rational then solution \( u_f \) can be given explicitly.

The reduced wave function described using scattering theory typically have only asymptotic expansions at \( \infty \) and are meromorphic off the scattering rays \( \Gamma_a \) described in Theorem 2.1. The relevant factorizations will all extend to cover this case, which is described in detail in [TU1] for \( \Gamma_a = R \) and the \( SU(n) \)-reality condition.

3. Reality conditions

To get the focusing and non-focusing non-linear Schrödinger equations we need impose reality conditions on the \( sl(2,C) \)-hierarchy. We explain reality conditions given by involutions of \( sl(n,C) \). This group theoretic approach allows several different generalizations of the two non-linear Schrödinger equations and two modified KdV equations depending on the choices of the involutions of \( sl(n,C) \).

3.1 Definition. Let \( \mathcal{U} \) denote a real form of \( sl(n,C) \), i.e., \( \mathcal{U} \) is the fix point set of some complex conjugate linear, Lie algebra involution \( \sigma \) of \( sl(n,C) \).

(a) A map \( A \) from \( C \) to \( sl(n,C) \) is said to satisfies the \( \mathcal{U} \)-reality condition if
\[
\sigma(A(\bar{\lambda})) = A(\lambda), \quad \text{for all } \lambda \in C.
\]

(b) A Lax pair \([\frac{\partial}{\partial x} + A(x,t,\lambda), \frac{\partial}{\partial t} + B(x,t,\lambda)] = 0\) is said to satisfy the \( \mathcal{U} \)-reality condition if \( \sigma(A(x,t,\bar{\lambda})) = A(x,t,\lambda) \) and \( \sigma(B(x,t,\bar{\lambda})) = B(x,t,\lambda) \).

It is clear that \( A = \sum_{k \leq n_0} u_k \lambda^j \) satisfies the \( \mathcal{U} \)-reality condition if and only if \( u_k \in \mathcal{U} \) for all \( k \). For example, \( A \) satisfies

(i) \( su(n) \)-reality condition if \( A(\bar{\lambda})^* + A(\lambda) = 0 \) for all \( \lambda \in C \),

(ii) \( su(1,n-1) \)-reality condition if \( A(\bar{\lambda})^*J + JA(\lambda) = 0 \) for all \( \lambda \in C \), where \( J = \text{diag}(1,-1,\ldots,-1) \),

(iii) \( sl(n,R) \)-reality condition if \( A(\bar{\lambda}) = A(\lambda) \).

For \( a \in \mathcal{U} \), let
\[
\mathcal{U}_a = \{ y \in \mathcal{U} \mid [a,y] = 0 \},
\]
\[
\mathcal{U}_a^\perp = \{ y \in \mathcal{U} \mid \text{tr}(ay) = 0 \} = sl(n)^{\perp} \cap \mathcal{U}.
\]
3.2 Proposition. Let \( \mathcal{U} \) be a real form of \( \text{sl}(n,C) \), \( a, b \in \mathcal{U} \) such that \([a, b] = 0\), and \( u \in \mathcal{S}(R, \mathcal{U}_a^\perp) \). Then
\( (1) \ Q_{b,j}(u) \in \mathcal{U} \) for all \( j \),
\( (2) \) the Lax pair of the \( j \)-th flow satisfies the \( \mathcal{U} \)-reality condition,
\( (3) \) the \( j \)-th flow in the \( \text{sl}(n,C) \)-hierarchy leaves \( \mathcal{S}(R, \mathcal{U}_a^\perp) \) invariant.

PROOF. Let \( \sigma \) denote the involution defines \( \mathcal{U} \). Set
\[
A(x,t,\lambda) = a\lambda + u(x,t),
\]
\[
B(x,t,\lambda) = b\lambda^j + Q_{b,1}(u)\lambda^{j-1} + \cdots + Q_{b,j}(u).
\]

It follows from \( a, u \in \mathcal{U} \) that \( A \) satisfies the \( \mathcal{U} \)-reality condition. Since \( \sigma \) is a homomorphism of \( \text{sl}(n,C) \),
\[
\sigma \left( \left[ \frac{\partial}{\partial x} + A(x,t,\bar{\lambda}), \frac{\partial}{\partial t} + B(x,t,\bar{\lambda}) \right] \right) = \left[ \frac{\partial}{\partial x} + A(x,t,\lambda), \frac{\partial}{\partial t} + \sigma(B(x,t,\bar{\lambda})) \right].
\]

Proposition 2.6 implies \( \sigma(B(x,t,\bar{\lambda})) = B(x,t,\lambda) \), which proves (1) and (2). Statement (3) follow from (1). 

3.3 Definition. Let \( \mathcal{U} \) be a real form of \( \text{sl}(n,C) \). The restriction of the \( \text{sl}(n,C) \)-hierarchy of flows to \( \mathcal{S}(R, \mathcal{U}_a^\perp) \) is called the \( \mathcal{U} \)-hierarchy.

3.4 Corollary. Let \( \mathcal{U} \) be the real form of \( \text{sl}(n,C) \) defined by \( \sigma, \hat{\sigma} \) the induced involution on \( \text{SL}(n,C) \), and \( U \) fixed point set of \( \hat{\sigma} \). If \( u \) is a solution of the \( j \)-th flow in the \( \mathcal{U} \)-hierarchy, then the trivialization of \( u \) satisfies the \( \mathcal{U} \)-reality condition: \( \hat{\sigma}(E(x,t,\lambda)) = E(x,t,\lambda) \).

3.5 Examples.
\( (1) \) The \( \text{su}(2) \)-hierarchy. Note that \( \text{su}(2) \) is the fixed point set of the involution \( \sigma(y) = -y^* \) on \( \text{sl}(2,C) \). For \( a = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \), \( \mathcal{U}_a^\perp = \left\{ \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \right\} \ q \in C \). So the space \( \mathcal{S}(R, \mathcal{U}_a^\perp) \) can be identified as \( \mathcal{S}(R, C) \). The first three flows in the \( \text{su}(2) \)-hierarchy are
\[
q_t = q_x,
\]
\[
q_t = \frac{i}{2}(q_{xx} + 2 |q|^2 q),
\]
\[
q_t = -\frac{1}{4}(q_{xxx} + 6 |q|^2 q_x).
\]
Note that the first flow just gives translation, the second flow is the focusing non-linear Schrödinger equation, and the sequence of flows is the hierarchy of commuting flows associated to the non-linear Schrödinger equation.

(2) The $su(n)$-hierarchy. If $a = \text{diag}(a_1, \cdots, a_n) \in su(n)$ has distinct eigenvalues and $b = \text{diag}(b_1, \cdots, b_n) \in su(n)$, then

$$U_a^\perp = \{ (u_{ij}) \in su(n) | u_{ii} = 0 \text{ for all } 1 \leq i \leq n \}.$$  

The first flow in the $su(n)$-hierarchy on $S(R, U_a^\perp)$ defined by $a$ is the translation $u_t = u_x$.

The first flow defined by $b$ ($a, b$ linearly independent) is the $n$-wave equation ([ZMa1, 2]):

$$(u_{ij})_t = \frac{b_i - b_j}{a_i - a_j}(u_{ij})_x + \sum_{k \neq i, j} \left( \frac{b_k - b_j}{a_k - a_j} - \frac{b_i - b_k}{a_i - a_k} \right) u_{ik}u_{kj}, \quad i \neq j. \quad (3.1)$$

(3) The $u(n)$-hierarchy. Let $a = \text{diag}(i, \cdots, i, -i, \cdots, -i)$ be the diagonal matrix with eigenvalues $i, -i$ and multiplicities $k, n-k$ respectively. Then

$$S(R, U_a^\perp) = \left\{ u = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} \bigg| q \in S(R, \mathcal{M}_{k\times(n-k)}) \right\},$$

where $\mathcal{M}_{k\times(n-k)}$ is the space of $k \times (n-k)$ complex matrices and $q^* = \bar{q}^t$. So $S(R, U_a^\perp)$ is naturally identified as $S(R, \mathcal{M}_{k\times(n-k)})$. For $u = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}$, we have

$$Q_{a,0}(u) = a,$$

$$Q_{a,1}(u) = u,$$

$$Q_{a,2}(u) = \begin{pmatrix} \frac{1}{2}qq^* & \frac{1}{2}q_x \\ \frac{1}{2}q_x^* & -\frac{1}{2}q^*q \end{pmatrix}.$$  

So the first three flows on $S(R, \mathcal{M}_{k\times(n-k)})$ in the $su(n)$-hierarchy defined by $a$ are

$$q_t = q_x$$

$$q_t = \frac{i}{2}(q_{xx} + 2qq^*q)$$

$$q_t = -\frac{1}{4}q_{xxx} - \frac{3}{4}(q_sq^*q + qq^*q_x).$$

Note that the second flow is the matrix non-linear Schrödinger equation studied by Fordy and Kulish [FK].
Next we recall the definition of \( u(k, n - k) \). Let \( J = \text{diag}(\epsilon_1, \cdots, \epsilon_n) \) with \( \epsilon_i = 1 \) for \( 1 \leq i \leq k \) and \( \epsilon_j = -1 \) if \( k < j \leq n \), and let

\[
\langle v_1, v_2 \rangle_J = v_1^* J v_2
\]
denote the Hermitian bilinear form on \( C^n \) defined by \( J \). Let \( U(k, n - k) \) denote the group of linear maps of \( C^n \) that preserve \( \langle \cdot, \cdot \rangle_J \), and \( u(k, n - k) \) its Lie algebra. Let \( U(k, n) = \{ g \in GL(n, C) \mid g^* J g = J \} \),

\[
u(k, n) = \{ X \in gl(n, C) \mid X^* J + J X = 0 \}.
\]
The involution that defines \( u(k, n - k) \) is \( \sigma(y) = -J^{-1} y^* J \), and the induced involution on \( U(k, n - k) \) is \( \hat{\sigma}(g) = J^{-1} (g^*)^{-1} J \).

3.6 Example. The \( u(1, 1) \)-hierarchy. Here

\[
u(1, 1) = \{ y \in \text{sl}(2, C) \mid y^* J + J y = 0 \} = \left\{ \begin{pmatrix} i r & q \\ \bar{q} & -i r \end{pmatrix} \right\} r \in R, q \in C ,
\]

where \( J = \text{diag}(1, -1) \). Let \( a = \text{diag}(i, -i) \in u(1, 1) \). Then

\[
u_a^+) = \{ y \in \text{sl}(2, C) \mid y^* J + J y = 0 \} = \left\{ \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} \right\} q \in C .
\]
The second flow in the \( u(1, 1) \)-hierarchy is the defocusing non-linear Schrödinger equation:

\[
q_t = \frac{i}{2} (q_{xx} - 2|q|^2 q).
\]

(3.2)

The classical Bäcklund transformation is a transformation of solutions, not of the \( j \)-th flow, \( j \geq 1 \), but of the \(-1\) flow. As an evolution, the \(-1\) flow is poorly defined. The \((x, t)\) are characteristic coordinates. However, the Bäcklund transformation operates algebraically on solutions. For the physical problem, we would not expect \( u \) to necessarily be in the Schwartz space along the characteristic coordinate \( x \) for all solutions. However, for the solutions we construct in this paper, this is the case.

The \(-1\) flow in the \( su(n) \)-hierarchy defined by \( b \in su(n)_a \) ([Te], [TU1]) is

\[
\begin{cases}
u_t = [a, g^{-1} b g], \\
g^{-1} g_x = u, \quad \lim_{x \to -\infty} g(x, t) = I .
\end{cases}
\]

(3.3)

Its Lax pair is

\[
\frac{\partial}{\partial x} + a \lambda + u, \quad \lambda^{-1} g^{-1} b g = 0 .
\]
Every time we said about the \( j \)-th flows, \( j \geq 1 \), applies to the \(-1\) flow, except that the trivialization \( E(x, t, \lambda) \) has a singularity at \( 0 \in C \), or \( E(x, t, \lambda) \) is holomorphic in \( \lambda \in C - \{0\} \).

We require a further step to connect with the classical theory, since sine-Gordon is the \(-1\) flow in the \( su(2)\)-hierarchy restricted (or twisted) by an involution (this is related to the twisted affine Kac-Moody algebras). This is also referred to in the literature as reduction, which is an unfortunate terminology as reduction has a specific meaning in symplectic geometry. Assume \( \tau \) is a complex conjugate linear, Lie algebra involution of \( sl(n, C) \) and \( \sigma \) a complex linear, Lie algebra involution of \( sl(n, C) \) such that \( \sigma \tau = \tau \sigma \). Then \((\tau, \sigma)\) defines a symmetric space as follows: Let \( \mathcal{U} \) denote the real form defined by \( \tau \). Then \( \sigma(\mathcal{U}) \subset \mathcal{U} \).

Let \( \mathcal{K}, \mathcal{P} \) denote the subgroup corresponding to \( \mathcal{U} \) and \( \mathcal{K} \) respectively. Then \( \mathcal{U}/\mathcal{K} \) is a symmetric space, and \( \mathcal{U} = \mathcal{K} + \mathcal{P} \) is the Cartan decomposition of \( \mathcal{U}/\mathcal{K} \).

**3.7 Definition.** Let \( \mathcal{U} \) be the real form of \( sl(n, C) \) defined by the complex conjugate linear, Lie algebra involution \( \tau \) of \( sl(n, C) \), \( \sigma \) a complex linear, Lie algebra involution of \( sl(n, C) \) such that \( \tau \sigma = \sigma \tau \), and \( \mathcal{U}/\mathcal{K} \) the corresponding symmetric space. We say that \( A(\lambda) \) satisfies the \( \mathcal{U} \)-reality condition twisted by \( \sigma \) or the \( \mathcal{U}/\mathcal{K} \)-reality condition if

\[
\tau(A(\bar{\lambda})) = A(\lambda), \quad \sigma(A(-\lambda)) = A(\lambda).
\]

A direct computation shows that \( A(\lambda) = \sum_j v_j \lambda^j \) satisfies the \( \mathcal{U} \)-reality condition twisted by \( \sigma \) if \( v_j \in \mathcal{K} \) if \( j \) is even, and \( v_j \in \mathcal{P} \) if \( j \) is odd.

If \( u \in \mathcal{U}_{a,\sigma}^+ \), then the recursive formula (2.2) implies that \( Q_{b,j}(u) \in \mathcal{K} \) if \( j \) is odd and is in \( \mathcal{P} \) if \( j \) is even. Since \( a \in \mathcal{P} \), \([Q_{b,j}(u), a]\) is in \( \mathcal{P} \) if \( j \) is even and is in \( \mathcal{K} \) if \( j \) is odd. This proves

**3.8 Theorem ([Te]).** Let \( \tau, \sigma, \mathcal{U}, \mathcal{K}, \mathcal{P} \) be as above, and \( a, b \in \mathcal{P} \) such that \([a, b] = 0\). Let \( \mathcal{U}_{a,\sigma}^+ = \mathcal{K} \cap \mathcal{U}_{a}^+ \). If \( u \in S(R, \mathcal{U}_{a,\sigma}^+) \), then

(i) \( Q_{b,j}(u) \in \mathcal{K} \) if \( j \) is odd and is in \( \mathcal{P} \) if \( j \) is even,
(ii) \([Q_{b,j}(u), a] \in \mathcal{P} \) if \( j \) is even, and is in \( \mathcal{K} \) if \( j \) is odd,
(iii) \( S(R, \mathcal{U}_{a,\sigma}^+) \) is invariant under the odd flows, and is stationary under the even flows.

**3.9 Definition.** The \( j \)-th (\( j \) odd) flow in the \( \mathcal{U} \)-hierarchy twisted by \( \sigma \) defined by \( a, b \) (or the hierarchy associated to the symmetric space \( \mathcal{U}/\mathcal{K} \)) is the \( j \)-th flow in the \( \mathcal{U} \)-hierarchy restricted to \( S(R, \mathcal{U}_{a,\sigma}^+) \):

\[
u_t = (Q_{b,j}(u))_x + [u, Q_{b,j}(u)] = [Q_{b,j+1}(u), a], \quad u : R^2 \to \mathcal{U}_{a,\sigma}^+.
\]
3.10 Corollary. If \( u : R^2 \to U_{a,\sigma}^\perp \) is a solution of the \( j \)-th flow (\( j \) odd) in the \( U \)-hierarchy, then

\[
A(x, t, \lambda) = a\lambda + u(x, t), \\
B(x, t, \lambda) = b\lambda^j + Q_{b,1}(u)\lambda^{j-1} + \cdots + Q_{b,j}(u)
\]

satisfy the \( U \)-reality condition and the \( \sigma \)-reality condition:

\[
\sigma(A(x, t, -\lambda)) = A(x, t, \lambda), \quad \sigma(B(x, t, -\lambda)) = B(x, t, \lambda).
\]

In particular, the trivialization \( E \) of \( u \) satisfies the following reality conditions

\[
\hat{\tau}(E(x, t, \tilde{\lambda})) = E(x, t, \lambda), \quad \hat{\sigma}(E(x, t, -\lambda)) = E(x, t, \lambda).
\]

3.11 Example. The hierarchy associated to \( SU(n)/SO(n) \). Let \( \tau(y) = -y^* \), \( \sigma = -y^t \), and \( a = \text{diag}(i, -i, \cdots, -i) \). Then \( U = su(n) \), and

\[
S(R, U_{a,\sigma}^\perp) = \left\{ \begin{pmatrix} 0 & v \\ -v^t & 0 \end{pmatrix} \bigg| v \in S(R, M_{1\times(n-1)}) \right\},
\]

where \( M_{1\times(n-1)} \) is the space of real \( 1 \times (n-1) \) matrices. The even flows vanishes on \( S(R, U_{a,\sigma}^\perp) \), and the odd flows are extensions of the usual hierarchy of flows for the modified KdV. The third flow twisted by \( \sigma \), written in terms of \( v : R \to M_{1\times(n-1)} \), is the matrix modified KdV equation:

\[
v_t = -\frac{1}{4} \left( v_{xxx} + 3(v_x v^t v + vv^t v_x) \right).
\]

(When \( n = 2 \), \( v = iq \) is a scalar function and the above equation is the classic modified KdV equation: \( q_t = -\frac{1}{4}(q_{xxx} - 6q^2q_x) \).)

3.12 Example. The third flow in the \( sl(n, R) \)-hierarchy twisted by \( \sigma(y) = -y^t \) defined by \( a = \text{diag}(-1, 1, \cdots, 1) \) (the \( SL(n, R)/SO(n) \) hierarchy) is the equation (3.6). When \( n = 2 \), \( SL(2, R)/SO(2) \) is the hyperbolic 2-plane \( H^2 \). The third flow in the \( H^2 \)-hierarchy is the other modified KdV:

\[
q_t = \frac{1}{4}(q_{xxx} + 6q^2q_x).
\]

(Here \( u = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix} \) with \( q \in R \).)
3.13 Example. The $-1$-flow associated to $SU(2)/SO(2) = S^2$. Let $\tau$ and $\sigma$ be the involution in Example 3.11, and $a = \text{diag}(i, -i)$. Then

$$U_{a,\sigma}^\perp = \left\{ \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix} \mid v \in R \right\}.$$  

The $-1$ flow defined by $b = -a/4$ twisted by $\sigma$ is the equation (3.3) for $u = \begin{pmatrix} 0 & q_x \\ -q_x/2 & 0 \end{pmatrix}$ is the Sine-Gordon equation

$$q_{xt} = \sin q.$$  \hfill (3.8)

3.14 Example. The $u(1,1)$-hierarchy twisted by the involution $\tau(y) = -y^t$ is the hierarchy associated to the Lorentzian symmetric space

$$U(1,1)/(U(1,1) \cap O(2, C)).$$

The third flow in this hierarchy is the modified KdV with an opposite sign in the cubic term:

$$q_t = -\frac{1}{4}(q_{xxx} - 6q^2q_x).$$  \hfill (3.9)

(Here $u = \begin{pmatrix} 0 & iq \\ -iq & 0 \end{pmatrix}$ for some real function $q$.)

3.15 Example. The $u(1,n-1)$-hierarchy twisted by $\sigma(y) = -y^t$ is the hierarchy associated to the Lorentzian symmetric space

$$U(1,n-1)/(U(1,n-1) \cap O(n, C)).$$

The third flow in this hierarchy defined by $a = \text{diag}(i, -i, \cdots, -i)$ is a generalization of the other modified KdV (3.9):

$$v_t = -\frac{1}{4}(v_{xxx} - 3(v_xv^tv + vuv^tv_x)), \quad v : R^2 \to M_{1 \times (n-1)}.$$

4. **Bäcklund transformations for the $sl(n,C)$-hierarchy**

The scattering data for the integrable systems we are considering have two parts, discrete data in $C \setminus \Gamma_a$ and continuous data along $\Gamma$. The group which generates the discrete data is intimately connected with Bäcklund transformations. Local Darboux and Bäcklund transformations for the $j$-th flow in the $sl(n,C)$-hierarchy were constructed by many authors (Zakharov and Shabat [ZS 2], Sattinger and Zurkowski [SZ 1, 2], Gu and Zhou [GZ] and Cherdnik [Ch]):
4.1 Theorem. Suppose $u$ is a solution of the $j$-th flow in the $sl(n,C)$-hierarchy that admits a local reduced wave function. Let $E$ denote the trivialization of $u$. Let $\alpha_1, \alpha_2 \in C$, and $V_1, V_2$ complex linear subspace of $C^n$ such that $C^n = V_1 \oplus V_2$. Set $\bar{V}_i(x,t) = E(x,t,\alpha_i)^{-1}(V_i)$. Suppose $\bar{V}_1(x,t) \cap \bar{V}_2(x,t) = 0$ for $(x,t)$ in an open subset $\mathcal{O}$. Then $\bar{u} = u + (\alpha_1 - \alpha_2)[\pi, \tilde{\pi}]$ is a solution of the $j$-th flow on $\mathcal{O}$, where $\tilde{\pi}(x,t)$ is the projection onto $\bar{V}_1(x,t)$ with respect to $C^n = \bar{V}_1(x,t) \oplus \bar{V}_2(x,t)$.

We will reformulate this theorem in terms of the dressing action and give a proof, which will be used in the later sections. Let $L_{\pm}(GL(n,C))$ be as in section 2. By the Birkhoff Factorization Theorem 2.12, there is a local dressing action $\sharp$ of $L_-(GL(n,C))$ on $L_+(GL(n,C))$: Given $g \in L_-(GL(n,C))$ and $f \in L_+(GL(n,C))$, if we can factor

$$gf = \tilde{f}\tilde{g} \in L_+(GL(n,C)) \times L_-(GL(n,C)),$$

then $g\sharp f$ is defined to be $\tilde{f}$. This action $\sharp$ is only defined for $f$ in an open dense subset of $L_+(GL(n,C))$. In certain cases, this factorization can be constructed explicitly. First we choose simple elements (linear fractional transformations) in $L_-(GL(n,C))$. Given constants $\alpha_1, \alpha_2 \in C$ and a linear projection $\pi$ of $C^n$ (i.e., $\pi$ is complex linear and $\pi^2 = \pi$), let $\pi' = I - \pi$ and

$$h_{\alpha_1,\alpha_2,\pi}(\lambda) = \frac{\lambda - (\alpha_1 \pi + \alpha_2 \pi')}{\lambda - \alpha_1} = I + \frac{(\alpha_1 - \alpha_2)}{\lambda - \alpha_1} \pi'.$$

Then $h_{\alpha_1,\alpha_2,\pi} \in L_-(GL(n,C))$ and

$$h_{\alpha_1,\alpha_2,\pi}^{-1}(\lambda) = \frac{\lambda - (\alpha_2 \pi + \alpha_1 \pi')}{\lambda - \alpha_2}.$$

We will call $h_{\alpha_1,\alpha_2,\pi}$ a simple element.

4.2 Proposition. Let $\alpha_1, \alpha_2 \in C$, $\pi$ a projection of $C^n$, $V_1$ and $V_2$ denote the image of $\pi$ and $\pi' = I - \pi$ respectively, and $f \in L_+(GL(n,C))$. If

$$(f(\alpha_1)^{-1}(V_1)) \cap (f(\alpha_2)^{-1}(V_2)) = 0,$$  \hspace{1cm} (4.1)

then $h_{\alpha_1,\alpha_2,\pi}f$ can be factored uniquely as

$$h_{\alpha_1,\alpha_2,\pi}f = \tilde{f}h_{\alpha_1,\alpha_2,\tilde{\pi}} \in L_+(GL(n,C)) \times L_-(GL(n,C)),$$

where $\tilde{\pi}$ is the projection onto $f(\alpha_1)^{-1}(V_1)$ with respect to $C^n = f(\alpha_1)^{-1}(V_1) \oplus f(\alpha_2)^{-1}(V_2)$. 

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PROOF. It suffices to prove that
\[ \tilde{f} = h_{\alpha_1, \alpha_2, \pi} f h_{\alpha_1, \alpha_2, \tilde{\pi}}^{-1} \]
lies in \( L_+(GL(n, C)) \). Since the right hand side of \( \tilde{f} \) is holomorphic in \( C \setminus \{\alpha_1, \alpha_2\} \) and has simple poles at \( \alpha_1 \) and \( \alpha_2 \), we only need to prove that the residues of \( \tilde{f} \) are zero at both \( \alpha_1 \) and \( \alpha_2 \). But
\[
\begin{align*}
\text{Res} (\tilde{f}, \alpha_1) &= (\alpha_1 - \alpha_2)(I - \pi)f(\alpha_1)\tilde{\pi}, \\
\text{Res} (\tilde{f}, \alpha_2) &= (\alpha_2 - \alpha_1)\pi f(\alpha_2)(I - \tilde{\pi}).
\end{align*}
\]
Since \( \text{Im}(\tilde{\pi}) = f(\alpha_1)^{-1}(V_1) \) and \( \text{Im}(I - \tilde{\pi}) = f(\alpha_2)^{-1}(V_2) \), both residues are zero. 

4.3 Theorem. Let \( u : \mathcal{O}_1 \rightarrow sl(n, C)^\perp \) be a local solution of the \( j \)-th flow (2.5), \( E \) the trivialization of \( u \), \( h_{\alpha_1, \alpha_2, \pi} \) a simple element in \( L_-(GL(n)) \), and \( V_1, V_2 \) denote the image of \( \pi \) and \( I - \pi \) respectively. Assume \( m : \mathcal{O} \times (C \setminus \Gamma) \rightarrow GL(n, C) \) is a local reduced wave function for \( u \). Then there exists an open subset \( \mathcal{O} \subset \mathcal{O}_1 \) such that \( \tilde{V}_1(x, t) \cap \tilde{V}_2(x, t) = 0 \) for \( (x, t) \in \mathcal{O} \), where \( \tilde{V}_i(x, t) = E(x, t, \alpha_i)^{-1}(V_i) \).

Moreover, let \( \tilde{\pi}(x, t) \) denote the projection onto \( \tilde{V}_1(x, t) \) with respect to \( C^n = \tilde{V}_1(x, t) \oplus \tilde{V}_2(x, t) \), then
\[
\begin{align*}
(i) \ \tilde{u} : \mathcal{O} \rightarrow sl(n, C)^\perp \text{ defined by } \tilde{u} = u + (\alpha_1 - \alpha_2)[a, \tilde{\pi}] \text{ is a solution of the } j \text{-th flow with } \tilde{E} \text{ is the trivialization and } \tilde{m} \text{ as a local reduced wave function, where}
\end{align*}
\[
\begin{align*}
\tilde{E}(x, t, \lambda) &= h_{\alpha_1, \alpha_2, \pi} E(x, t) h_{\alpha_1, \alpha_2, \tilde{\pi}(x, t)}^{-1}, \\
\tilde{m}(x, t, \lambda) &= m(x, t, \lambda) h_{\alpha_1, \alpha_2, \tilde{\pi}(x, t)}(\lambda)^{-1},
\end{align*}
\]
(ii) \( \tilde{\pi} \) is the solution of
\[
\begin{cases}
\tilde{\pi}_x = -[u, \tilde{\pi}] - [a, \tilde{\pi}](\alpha_2 + (\alpha_1 - \alpha_2)\tilde{\pi}), \\
\tilde{\pi}_t = -\sum_{k=0}^j [Q_{b, j-k}(u), \tilde{\pi}](\alpha_2 + (\alpha_1 - \alpha_2)\tilde{\pi})^k, \\
\tilde{\pi}^2 = \tilde{\pi}, \quad \tilde{\pi}(0, 0) = \pi.
\end{cases}
\]

PROOF. Since \( \tilde{V}_1(0, 0) \cap \tilde{V}_2(0, 0) = V_1 \cap V_2 = 0 \) and two linear subspaces are in general position is an open condition, there exists an open subset \( \mathcal{O} \) of \( (0, 0) \) in \( \mathcal{O}_1 \) such that \( \tilde{V}_1(x, t) \cap \tilde{V}_2(x, t) = 0 \) for all \( (x, t) \in \mathcal{O} \).

To prove (i), we first note that \( E(x, t) \in L_+(GL(n, C)) \), where \( E(x, t)(\lambda) = E(x, t, \lambda) \). By Proposition 4.2, \( E \) is holomorphic in \( \lambda \in C \). This proves that \( \tilde{m} \) is a local reduced wave function for \( \tilde{u} \).
A simple computation shows that the coefficient of $\lambda^{-1}$ in the asymptotic expansion of $\tilde{m}$ at $\lambda = \infty$ is

$$\tilde{m}_1(x, t) = m_1(x, t) + (\alpha_2 - \alpha_1)\tilde{\pi}'(x, t).$$

Then (i) follows from Proposition 2.11.

To prove (ii), we note that

$$a\lambda + \tilde{u} = \tilde{E}^{-1}\tilde{E}_x$$

$$= (h_{\alpha_1, \alpha_2, \pi}E h_{\alpha_1, \alpha_2, \pi}^{-1})^{-1}(h_{\alpha_1, \alpha_2, \pi}E h_{\alpha_1, \alpha_2, \pi}^{-1})_{x}$$

$$= h_{\alpha_1, \alpha_2, \pi}(E^{-1}E_x)h_{\alpha_1, \alpha_2, \pi}^{-1}.$$ (1)

Multiply the above equation by $(\lambda - \alpha_1)h_{\alpha_1, \alpha_2, \pi}$ on the right to get

$$(a\lambda + \tilde{u})(\lambda - (\alpha_1 \tilde{\pi} + \alpha_2 \tilde{\pi}')) = (\lambda - (\alpha_1 \tilde{\pi} + \alpha_2 \tilde{\pi}'))(a\lambda + u) + (\alpha_1 - \alpha_2)\tilde{\pi}_x.$$ (2)

Compare coefficient of $\lambda^i$ for $i = 0, 1$ to get the ODE for $\tilde{\pi}$ in $x$ variable. Similarly, we have

$$\left(\sum_{k=0}^{j} Q_{b,k}(\tilde{u})\lambda^{j-k}\right)(\lambda - (\alpha_1 \tilde{\pi} + \alpha_2 \tilde{\pi}'))$$

$$= (\lambda - (\alpha_1 \tilde{\pi} + \alpha_2 \tilde{\pi}')) \left(\sum_{k=0}^{j} Q_{b,k}(u)\lambda^{j-k}\right) + (\alpha_1 - \alpha_2)\tilde{\pi}_t.$$ (3)

Compare coefficient of $\lambda^i$ for $0 \leq i \leq j$ to get

$$Q_{b,k}(\tilde{u}) = Q_{b,k}(u) + \sum_{i=1}^{k} [Q_{b,k-i}(u), (\alpha_1 - \alpha_2)\tilde{\pi}]((\alpha_1 \tilde{\pi} + \alpha_2 \tilde{\pi}'))^{i-1},$$

$$(\alpha_1 - \alpha_2)\tilde{\pi}_t = (\alpha_1 \tilde{\pi} + \alpha_2 \tilde{\pi}')Q_{b,j}(u) - Q_{b,j}(\tilde{u})(\alpha_1 \tilde{\pi} + \alpha_2 \tilde{\pi}').$$ (4)

Substitute the first equation for $k = j$ to the second equation to get the ODE for $\tilde{\pi}$ in $t$ variable.  

4.4 Remark. The assumption the $u$ admits a local reduced wave function in Theorem 4.3 is necessary. Without this assumption we can only conclude that

$$\begin{cases} \tilde{E}^{-1}\tilde{E}_x = a\lambda + \tilde{u}, \\ \tilde{E}^{-1}\tilde{E}_t = b\lambda^{j-1} + \cdots + v_j, \end{cases}$$

for some $\tilde{u}$ and $v_1, \cdots, v_j$. In general, it is not clear whether the $v_i$ is equal to $Q_{b,i}(\tilde{u})$. But if $\lim_{x \to -\infty} v_i(x, t) = 0$ for all $t$ then Proposition 2.6 implies that $v_i = Q_{b,i}(\tilde{u})$ for all $1 \leq i \leq j$.

4.5 Definition. Let $u$ be a local solution of the $j$-th flow (2.5) that admits a local reduced wave function, and $h_{\alpha_1, \alpha_2, \pi}$ a simple element. Define $h_{\alpha_1, \alpha_2, \pi} \ast u = \tilde{u}$, where $\tilde{u}$ is the new local solution obtained in Theorem 4.3.

The following example explains why the above constructions only provide local solutions from a global solution of the $j$-th flow in the $sl(n, C)$-hierarchy on $\mathcal{S}(R, sl(n)_{\alpha}^\perp)$. 

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4.6 Example. We apply Theorem 4.3 to the vacuum solution \( u = 0 \) of the \( j \)-th flow (2.6) in the \( \mathfrak{sl}(2,C) \)-hierarchy defined by \( a = \text{diag}(1,-1) \). Let \( \alpha_1, \alpha_2 \in C \), and \( V_1 \) (resp. \( V_2 \)) the subspace spanned by \( v_1 = (c_1,c_2)^t \) (resp. \( v_2 = (d_1,d_2)^t \)). A direct computation implies that \( \tilde{u} \) constructed in Theorem 4.3 is

\[
\tilde{u}(x,t) = \frac{2(\alpha_1 - \alpha_2)}{c_2 d_1 e^{\eta(x,t)} - c_1 d_2 e^{-\eta(x,t)}} \begin{pmatrix} 0 & c_1 d_1 e^{-\xi(x,t)} \\ c_2 d_2 e^{\xi(x,t)} & 0 \end{pmatrix},
\]

where \( \xi = (\alpha_1 + \alpha_2)x + (\alpha_1^2 + \alpha_2^2)t \) and \( \eta = (\alpha_1 - \alpha_2)x + (\alpha_1^2 - \alpha_2^2)t \). Note that \( \tilde{u} \) is not defined at \( (x_0,t_0) \) when \( e^{2((\alpha_1 - \alpha_2)x_0 + (\alpha_1^2 - \alpha_2^2)t_0)} = \frac{c_1 d_2}{c_2 d_1} \). For example, for \( j = 2 \),

(i) if \( \alpha_1 = 2, \alpha_2 = 1, \) and \( v_1 = (1,1)^t, v_2 = (1,2)^t \), then \( \tilde{u}(x,t) \) is singular along the line \( x + 3t = \frac{1}{2} \ln 2 \).

(ii) if \( \alpha_1 = 2, \alpha_2 = 1, \) and \( v_1 = (1,1)^t, v_2 = (-1,2)^t \), then \( \tilde{u}(x,t) \) is smooth on \( \mathbb{R}^2 \) but \( \tilde{u}(x,t) \) goes to infinity when \( x \to \pm \infty \).

(iii) if \( \alpha_1, \alpha_2 \in \mathbb{R} \) satisfying \( |\alpha_1 + \alpha_2| < |\alpha_1 - \alpha_2| \) and \( v_1 = (c_1,c_2)^t, v_2 = (d_1,d_2)^t \) in \( \mathbb{R}^2 \) satisfying \( c_1 d_2 d_1 d_2 < 0 \), then \( \tilde{u}(x,t) \) is smooth on \( \mathbb{R}^2 \) and is rapidly decay in \( x \).

The following theorem explains Bäcklund transformations for the restricted flows.

4.7 Theorem. Let \( U \) be a real form of \( \mathfrak{sl}(n,C) \) defined by \( \tau \), and \( U/K \) the symmetric space defined by \( \sigma \). If \( u \) is a solution of the \( j \)-th flow in the \( U \)-hierarchy (resp. \( U/K \)-hierarchy) and \( h_{\alpha_1,\alpha_2,\pi} \) is a simple element satisfying the \( U \)-reality condition (resp. the \( U/K \)-reality condition), then \( h_{\alpha_1,\alpha_2,\pi} u \) is again a solution of the \( j \)-th flow in the \( U \)- (resp. \( U/K \)-) hierarchy.

PROOF. Let \( E \) be the trivialization of \( u \) normalized at \( (0,0) \). Then \( E \) satisfies the \( U \)-reality condition. Let \( h = h_{\alpha_1,\alpha_2,\pi} \), and \( hE = \tilde{E} \tilde{h} \) as in Theorem 4.3. If \( h \) satisfies the \( U \)-reality condition, then so is \( hE \). But

\[
h(\lambda)E(\lambda) = \tau(h(\lambda)E(\lambda)) = \tau(\tilde{E}(\lambda)\tilde{h}(\lambda)) = \tau(\tilde{E}(\lambda))\tau(h(\lambda)) = \tilde{E}(\lambda)\tilde{h}(\lambda).
\]

Uniqueness of the Birkhoff decomposition implies that \( \tau(\tilde{E}(\lambda)) = \tilde{E}(\lambda) \). Hence \( \tilde{u} \) is a solution of the \( j \)-th flow in the \( U \)-hierarchy. The same proof works for the \( U/K \)-hierarchy.

Next we give some relations among simple elements in \( L_- (GL(n,C)) \).

4.8 Proposition. Let \( h_{\alpha_1,\alpha_2,\pi_1} \) and \( h_{\beta_1,\beta_2,\pi_2} \) be two simple elements. If

\[
\phi = (\alpha_2 + (\alpha_1 - \alpha_2)\pi_1) - (\beta_2 + (\beta_1 - \beta_2)\pi_2)
\]

is invertible, then

\[
h_{\alpha_1,\alpha_2,\pi_1} h_{\beta_1,\beta_2,\pi_2} = h_{\beta_1,\beta_2,\pi_2} h_{\alpha_1,\alpha_2,\pi_1}
\]

if and only if \( \tau_i = \phi \pi_i \phi^{-1} \) for \( i = 1, 2 \).

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PROOF. Set
\[ Y_1 = \alpha_2 + (\alpha_1 - \alpha_2)\tau_1, \quad Y_2 = \beta_2 + (\beta_1 - \beta_2)\pi_2, \]
\[ Z_1 = \alpha_2 + (\alpha_1 - \alpha_2)\pi_1, \quad Z_2 = \beta_2 + (\beta_1 - \beta_2)\tau_2. \]

Then equality (4.4) gives
\[ (\lambda - Y_1)(\lambda - Y_2) = (\lambda - Z_2)(\lambda - Z_1). \]
This holds if and only if
\[ \begin{cases} Y_1 - Z_2 = Z_1 - Y_2, \\ Y_1Y_2 - Z_2Z_1 = 0. \end{cases} \]

Multiply the first equation by \( Y_2 \) on the right and subtract the second equation to get
\[ Z_2 = (Z_1 - Y_2)Y_2(Z_1 - Y_2)^{-1}, \]
where \( Z_1 - Y_2 = \phi \) is invertible by assumption. Multiply the first equation by \( Z_1 \) on the right and subtract the second equation to get
\[ Y_1 = (Z_1 - Y_2)Z_1(Z_1 - Y_2)^{-1}. \]
This finishes the proof.

As a consequence of Proposition 4.8, we obtain an analogue of the Bianchi permutability formula:

**4.9 Corollary.** Let \( h_{\alpha_1,\alpha_2,\pi_1} \) and \( h_{\beta_1,\beta_2,\pi_2} \) be two simple elements such that \( \phi = (\alpha_2 + (\alpha_1 - \alpha_2)\pi_1) - (\beta_2 + (\beta_1 - \beta_2)\pi_2) \) is invertible. Let \( u \) be a local solution of the \( j \)-th flow (2.5), which admits a reduced wave function. Let
\[ u_1 = h_{\alpha_1,\alpha_2,\pi_1} * u = u + (\alpha_1 - \alpha_2)[a, \tilde{\pi}_1], \]
\[ u_2 = h_{\beta_1,\beta_2,\pi_2} * u = u + (\beta_1 - \beta_2)[a, \tilde{\pi}_2], \]

as in Theorem 4.3. Set \( \tilde{\tau}_1 = \tilde{\phi}\tilde{\pi}_1\tilde{\phi}^{-1} \), where
\[ \tilde{\phi} = (\alpha_2 + (\alpha_1 - \alpha_2)\tilde{\pi}_1) - (\beta_2 + (\beta_1 - \beta_2)\tilde{\pi}_2). \]

Then
\[ u_3 = h_{\alpha_1,\alpha_2,\tau_1} * u_2 = u_2 + (\alpha_1 - \alpha_2)[a, \tilde{\tau}_1], \]
\[ = h_{\beta_1,\beta_2,\tau_2} * u_1 = u_1 + (\beta_1 - \beta_2)[a, \tilde{\tau}_2]. \]
5. Bäcklund transformations for the $u(n)$-hierarchy

In this section, we consider Bäcklund transformations for the flows in the $u(n)$-hierarchy, in which the Birkhoff factorization can always be carried out. Let $G^m$ denote the subgroup of rational maps $g \in L_-(GL(n,C))$ such that $g$ satisfies the $U(n)$-reality condition. We obtain an action of $G^m$ on the space of solutions of the $j$-th flow in the $u(n)$-hierarchy on $S(R,U_a^\perp)$. Simple elements generate $G^m$ and the action of these simple elements gives global Bäcklund transformations.

Let $z \in C$, and $\pi$ the Hermitian projection of $C^m$ onto a complex linear subspace $V$, i.e., $\pi^* = \pi$ and $\pi^2 = \pi$. Let $\pi^\perp = I - \pi$ be the Hermitian projection of $C^m$ onto the orthogonal complement $V^\perp$. Let

$$g_{z,\pi} (\lambda) = h_{\bar{z}, z, \pi} (\lambda) = \pi + \frac{\lambda - z}{\lambda - \bar{z}} \pi^\perp.$$  

It is easy to check that $g_{z,\pi}(\bar{\lambda})^* g_{z,\pi}(\lambda) = I$. So $g_{z,\pi} \in G^m_\perp$, and will be called a simple element of $G^m_\perp$. The following theorem was proved by the second author [U1]

5.1 Theorem ([U1]). The set $\{g_{z,\pi} : z \in C \setminus R, \pi$ a Hermitian projection of $C^m\}$ generates $G^m_\perp$.

Let $G_\pm$ denote the subgroup of $f \in L_\pm(GL(n,C))$ such that $f$ satisfies the $U(n)$-reality condition, i.e., $f(\lambda)^* f(\lambda) = I$. If for $f\pm \in G_\pm$ we can factor

$$f_- f_+ = g_+ g_- \in L_+(GL(n,C)) \times L_-(GL(n,C)),$$

then the uniqueness of the Birkhoff Theorem implies that $g_\pm \in G_\pm$. This implies that the dressing action of $G_\perp$ leaves $G_\perp$ invariant. Let $\hat{G}_+$ (resp. $\hat{G}_-$) denote the group of holomorphic map $f : C \setminus \{0\} \rightarrow GL(n,C)$ (resp. $f : O_0 \cup O_\infty \rightarrow GL(n,C)$) satisfying the $U(n)$-reality condition, where $O_0$ and $O_\infty$ are open neighborhood of 0 and $\infty$ in $S^2$ respectively. A similar argument implies that the dressing action of $\hat{G}_-$ leaves $\hat{G}_+$ invariant. We have seen in section 4 that the dressing action of $L_-(GL(n,C))$ on $L_+(GL(n,C))$ is only defined locally. However, we will show that the $U(n)$-reality condition implies that the simple elements act on $G_+$ (resp. $\hat{G}_+$) globally and explicitly. Since simple elements generate $G^m_\perp$, the group $G^m_\perp$ acts globally on $G_+$ (resp. $\hat{G}_+$). We explain these in more detail below.

5.2 Proposition. Let $z \in C$, $\pi$ a Hermitian projection of $C^m$ onto $V$, $g_{z,\pi}$ a simple element of $G^m_\perp$, and $f \in G_+$ (resp. $\hat{G}_+$). Then $g_{z,\pi} f$ can be factored as

$$g_{z,\pi} f = \tilde{f} g_{z,\tilde{\pi}} \in G^m_- \times G_+ \ (\text{resp. } G^m_- \times \hat{G}_+),$$

where $\tilde{\pi}$ is the Hermitian projection of $f(z)^{-1}(V)$. 

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PROOF. Since $g_{z,\pi} = h_{\bar{z},z,\pi}$, it follows from Proposition 4.2 that if $f(\bar{z})^{-1} \cap f(z)(V^\perp) = 0$ then we can factor

$$g_{z,\pi}f = \tilde{f}h_{\bar{z},z,\bar{\pi}},$$

where $\bar{\pi}$ is the projection onto $f(\bar{z})^{-1}(V)$ with respect to $C^m = f(\bar{z})^{-1}(V) \oplus f(z)^{-1}(V^\perp)$. Since $f$ satisfies the $U(n)$-reality condition, $f(\bar{\lambda})^* f(\lambda) = I$. So we have

$$< f(\bar{z})^{-1}(V), f(z)^{-1}(V^\perp) > = < f(z)^*(V), f(z)^{-1}(V^\perp) >$$

$$= < V, V^\perp > = 0,$$  \hspace{1cm} (5.1)

where $< v_1, v_2 > = v_1^* v_2$. Since $< , >$ is positive definite,

$$\left( f(\bar{z})^{-1}(V) \right) \cap \left( f(z)^{-1}(V^\perp) \right) = 0.$$  

So the factorization can also be done. Equation (5.1) also implies that $f(\bar{z})^{-1}(V)$ is perpendicular to $f(z)^{-1}(V^\perp)$. Hence $\bar{\pi}$ is the Hermitian projection of $C^m$ onto $f(\bar{z})^{-1}(V)$. So $h_{\bar{z},z,\bar{\pi}}$ satisfies the $U(n)$-reality condition and $h_{\bar{z},z,\bar{\pi}} = g_{z,\pi}$.  

**5.3 Theorem.** The action $\sharp : G^m_- \times G_+ \to G_+$ (resp. $G^m_- \times \hat{G}_+ \to \hat{G}_+$) is globally defined, where $g^\sharp f = \tilde{f}$ such that $\tilde{f}^{-1} g f \in G^m_-$.  

PROOF. $G^m_-$ is generated by the simple elements. Hence the algorithm for the factorization of the simple elements extends to all of $G^m_-$.  

Henceforth in this section, let $\mathcal{U} = u(n)$, and $\mathcal{M}_{a,b,j}$ denote the space of solutions of the $j$-th flow defined by $b$ in the $u(n)$-hierarchy on $\mathcal{S}(R, \mathcal{U}^\perp_R)$.

The construction of an action of $G^m_-$ on $\mathcal{M}_{a,b,j}$ uses this dressing action. We describe the action for $j \geq 1$. The main difference between $j \geq 1$ and $j = -1$ is the difference between the group $G_+$ and $\hat{G}_+$. The action on $\mathcal{M}_{a,b,j}$ is induced from the “dressing action” $\sharp$ of $G^m_-$ on $G_+$ using trivializations of elements in $\mathcal{M}_{a,b,j}$.

Since the trivialization $E(x,t,\lambda)$ of $u \in \mathcal{M}_{a,b,j}$ satisfies the $U(n)$-reality condition, $E(x,t) \in G_+$, where $E(x,t)(\lambda) = E(x,t,\lambda)$. The following theorem is a consequence of Theorem 4.3 and Proposition 5.2.

**5.4 Theorem.** Let $u \in \mathcal{M}_{a,b,j}$, $E$ the trivialization of $u$, $z \in C \setminus R$, and $\pi$ the projection of $C^m$ onto a complex linear subspace $V$ of $C^m$. For each $(x,t) \in R^2$, set

$$\tilde{V}(x,t) = E(x,t,\bar{z})^*(V),$$

$$\tilde{\pi}(x,t) = \text{the projection of } C^m \text{ onto } \tilde{V}(x,t),$$

$$\tilde{E}(x,t,\lambda) = g_{z,\pi}(\lambda)E(x,t,\lambda)g_{z,\tilde{\pi}(x,t)}(\lambda)^{-1}$$

$$= (\pi + \frac{\lambda - \bar{z}}{\lambda - \tilde{\pi}})E(x,t,\lambda) \left( \tilde{\pi}(x,t) + \frac{\lambda - \bar{z}}{\lambda - \tilde{\pi}} \tilde{\pi}(x,t)^\perp \right).$$

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Then

(i) the smooth map \( \tilde{u} \) from \( R^2 \) to \( U_a^\perp \) defined by \( \tilde{u} = u + (z - \bar{z})[\tilde{\pi}, a] \) is a solution of the \( j \)-th flow equation in the \( u(n) \)-hierarchy and \( \tilde{E} \) is the trivialization of \( \tilde{u} \).

(iv) \( \tilde{\pi} \) satisfies
\[
\begin{align*}
(\tilde{\pi})_x + [az + u, \tilde{\pi}] &= (\bar{z} - z)[\tilde{\pi}, a]\tilde{\pi}, \\
(\tilde{\pi})_t &= \sum_{k=0}^{j} [\tilde{\pi}, Q_{b,j-k}(u)](z + (\bar{z} - z)\tilde{\pi})^k, \\
\tilde{\pi}^* &= \tilde{\pi}, \quad \tilde{\pi}^2 = \tilde{\pi}, \quad \tilde{\pi}(0,0) = \pi,
\end{align*}
\] (5.2)

Next we want to prove that \( \tilde{u} \) in Theorem 5.4 is a solution of the \( j \)-th flow in the \( u(n) \)-hierarchy and \( \tilde{u}(\cdot, t) \) lies in the Schwartz class for all \( t \). To do this, we need a theorem proved in [TU 1] (Theorem 6.6 of [TU1]). We will not repeat the somewhat technical proof in this paper.

5.5 Theorem ([TU 1]). Given \( u \in S(R, U_a^\perp) \), if \( \tilde{\pi} \) is a solution of
\[
(\tilde{\pi})_x + [az + u, \tilde{\pi}] = (\bar{z} - z)[\tilde{\pi}, a]\tilde{\pi},
\] (5.3)
then \([\tilde{\pi}, a]\) is in the Schwartz class, and \( \lim_{x \to \pm\infty} \tilde{\pi}(x, t) \) exists and commutes with \( a \).

As a consequence of Theorems 5.4 and 5.5, we have

5.6 Corollary. The function \( \tilde{u} \) given in Theorem 5.4 lies in \( M_{a,b,j} \), i.e., \( \tilde{u} \) is a solution of the \( j \)-th flow on \( S(R, U_a^\perp) \) defined by \( b \).

To summarize, we have

5.7 Corollary. Let \( u \in M_{a,b,j} \), \( E \) the trivialization of \( u \), \( z \in C \setminus R \), \( V \) a complex linear subspace of \( C^n \), and \( \tilde{\pi}(x,t) \) the Hermitian projection of \( C^n \) onto \( E(x,t,z)^*(V) \). Then \( \tilde{u} = u + (z - \bar{z})[\tilde{\pi}, a] \) is in \( M_{a,b,j} \), \( \tilde{E} \) defined in Theorem 5.4 is the trivialization of \( \tilde{u} \), and \( g_{z,\tilde{\pi}} E(x,t) = \tilde{E}(x,t)g_{z,\tilde{\pi}}(x,t) \in G_+ \times G_{m} \).

5.8 Corollary. If \( u \in M_{a,b,j} \), then system (5.2) is solvable. Moreover, if \( \tilde{\pi} \) is a solution of system (5.2) then \( \tilde{u} = u + (z - \bar{z})[\tilde{\pi}, a] \) is again in \( M_{a,b,j} \).

The above results give methods to construct solutions of the \( j \)-th flow from a given solution. This is done either by an algebraic formula if the trivialization of the given solution is known (Darboux transformation) or by solving two compatible systems of ordinary differential equations (Bäcklund transformation).

Corollary 5.6 is part of the construction of an action of \( G_{m} \) on \( M_{a,b,j} \):
5.9 Theorem. Let $g \in G^m$, $u \in \mathcal{M}_{a,b,j}$, and $E$ the trivialization of $u$. Then:
(i) $gE(x,t)$ can be factored uniquely as
$$gE(x,t) = \tilde{E}(x,t)\tilde{g}(x,t) \in G_+ \times G_-^m$$
with $E(0,0) = I$.
(ii) $E^{-1}\tilde{E}_x = \tilde{A}$, where $\tilde{A}(x,t,\lambda) = a\lambda + \tilde{u}(x,t)$ for some $\tilde{u}(x,t) \in \mathcal{M}_{a,b,j}$.
(iii) $g \ast u = \tilde{u}$ defines an action of $G^m_-$ on $\mathcal{M}_{a,b,j}$.
(iv) $\tilde{E}$ is the trivialization of $\tilde{u}$.

PROOF. To prove uniqueness, we suppose $gE$ has two factorizations:
$$gE(x,t) = E_1(x,t)g_1(x,t) = E_2(x,t)g_2(x,t) \in G_+ \times G_-^m.$$ 
Then
$$E_1^{-1}(x,t,\lambda)E_2(x,t,\lambda) = g_1(x,t)(\lambda)g_2(x,t)^{-1}(\lambda). \quad (5.4)$$
But the left hand side of (5.4) is holomorphic for $\lambda \in C$ and the right hand side is holomorphic at $\lambda = \infty$ for all $(x,t)$. Hence by Liouville Theorem, it must be constant. The right hand side at $\lambda = \infty$ is equal to $I$, which proves the uniqueness.

By Theorem 5.1 the $g_{z,\pi}$’s generate $G^m_+$. To prove the existence of the factorization it suffices to prove that we can factor $g_{z,\pi}E(x,t)$. This is done in Theorem 5.4, since
$$\tilde{E}(x,t) = g_{z,\pi}E(x,t)g_{z,\tilde{\pi}^{-1}(x,t)}$$
can be rewritten as
$$g_{z,\pi}E(x,t) = \tilde{E}(x,t)g_{z,\tilde{\pi}(x,t)} \in G_+ \times G_-^m.$$ 
This completes the proof of (i).

Statement (ii) follows from Corollary 5.6 and the fact that $g_{z,\pi}$’s generate $G^m_-$.

To prove $\ast$ defines an action, we need to prove
$$(gh) \ast u = g \ast (h \ast u)$$
for $g, h \in G^m_-$ and $u \in \mathcal{M}_{a,b,j}$. Let $E$ denote the trivialization of $u$. We factor
$$hE(x,t) = E_1(x,t)h_1(x,t) \in G_+ \times G_-^m,$$
$$gE_1(x,t) = E_2(x,t)g_2(x,t) \in G_+ \times G_-^m.$$ 
Then by definition of $\ast$,
$$\begin{align*}
E_1^{-1}(E_1)x &= A_1, \\
E_2^{-1}(E_2)x &= A_2,
\end{align*}$$

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where
\[
\begin{align*}
A_1(x, t, \lambda) &= a\lambda + (h \ast u)(x, t), \\
A_2(x, t, \lambda) &= a\lambda + (g \ast (h \ast u))(x, t).
\end{align*}
\]

But \((gh)E\) can be factored as
\[
(gh)E = g(hE) = g(E_1h_1) = E_2g_2h_1 = E_2(g_2h_1) \in G_+ \times G_-.
\]
So by definition of *, we have
\[
E_2^{-1}(E_2)_x = a\lambda + (gh) \ast u.
\]
This proves that \((gh) \ast u = g \ast (h \ast u)\).

**5.10 Definition.** The transformation on \(\mathcal{M}_{a,b,j}\) defined by \(u \mapsto g_{z,\pi} \ast u\) is called a Bäcklund transformation for the \(j\)-th flow on \(S(R, U_a^+)}\) defined by \(b\) with parameter \(z\) and initial condition \(\pi\).

The center of \(G_-^a\) is the subgroup of all elements in \(G_-^a\) of the form \(gI\) for some \(g : C \to C\) (here \(I\) is the identity matrix) satisfying \(g(\lambda)^*g(\lambda) = 1\). We show below that the center acts trivially on \(\mathcal{M}_{a,b,j}\).

**5.11 Corollary.** Suppose \(g : C \to C\) is a rational function such that \(g(\lambda)^*g(\lambda) = 1\), i.e., \(gI\) lies in the center of \(G_-^a\). Then \((gI) \ast u = u\) for all \(u \in \mathcal{M}_{a,b,j}\). In particular, if \(g_1, g_2 \in G_-\) such that \(g_1 = gg_2\), then \(g_1 \ast u = g_2 \ast u\).

**PROOF.** Let \(E\) be the trivialization of \(u \in \mathcal{M}_{a,b,j}\) normalized at \((0, 0)\). Since \((gI)E(x, t) = E(x, t)(gI)\), by definition * we have \((gI) \ast u = u\).

The Bäcklund theory for the \(-1\) flow is identical. However, the trivialization \(E\) has singularities at both \(0\) and \(\infty\). Since \(G_-^a\) is always holomorphic on the real axis, we do not need to add an extra condition at \(0\). In this case, the trivialization \(E\) is holomorphic in \(C \setminus \{0\}\). Use the same proof for the positive flows to this case to yield

**5.12 Theorem.** Let \(\mathcal{M}_{a,b,-1}\) denote the space of solutions of the \(-1\)-flow on \(S(R, U_a^{-1})\) defined by \(b\):
\[
\begin{align*}
\begin{cases}
\ u_t = [a, g^{-1}bg], \\
g^{-1}g_x = u, \ \lim_{x \to -\infty} g(x, t) = I.
\end{cases}
\end{align*}
\]
Then the group \(G_-^a\) acts on the space \(\mathcal{M}_{a,b,-1}\). Moreover, let \(u \in \mathcal{M}_{a,b,-1}\), \(E\) the trivialization of \(u\), and \(g_{z,\pi}\) a simple element of \(G_-^a\). Then:

(i) \(g_{z,\pi} \ast u = u + (z - \bar{z})[\pi, a]\), where \(\bar{\pi}(x, t)\) is the projection of \(C^a\) onto
\[
E(x, t, z)^*(\pi(C^a)).
\]

(ii) \(\bar{\pi}\) is the solution to
\[
\begin{align*}
\begin{cases}
\ (\bar{\pi})_x + [az + u, \bar{\pi}] = (\bar{z} - z)[\bar{\pi}, a]\bar{\pi}, \\
\ (\bar{\pi})_t = \frac{1}{|z|^2} ((z - \bar{z})\bar{g}^{-1}bg\bar{\pi} - zg^{-1}bg\bar{\pi} + \bar{z}\bar{g}^{-1}bg), \\
\bar{\pi}^* = \bar{\pi}, \ \bar{\pi}^2 = \bar{\pi}, \ \bar{\pi}(0, 0) = \pi.
\end{cases}
\end{align*}
\]
Now we turn to the twisted case. Given an involution \( \sigma \) of \( SU(n) \), let \( \mathcal{M}_{a,b,j}^{\sigma} \) denote the space of solutions of the \( j \)-th flow on the subspace \( \mathcal{S}(R, \mathcal{U}_{a,j}^{\perp}) \) (here we use the same notations as in section 3). Let \( G_{m,\sigma}^{m} \) denote the subgroup of \( G_{m}^{m} \) of \( g \in G^{m}_{\sigma} \) such that \( \sigma(g(-\lambda)) = g(\lambda) \). Since the trivialization \( E \) of \( u \in \mathcal{M}_{a,b,j}^{\sigma} \) satisfies the same reality condition \( \sigma(E(x,t,\lambda)) = E(x,t,\lambda) \), we obtain:

5.13 Corollary. For \( j = -1 \) or \( j \) a positive integer, then the action of \( G_{m,\sigma}^{m} \) leaves \( \mathcal{M}_{a,b,j}^{\sigma} \) invariant, where \( \mathcal{M}_{a,b,j}^{\sigma} \) is the space of solutions of the \( j \)-th flow in the \( su(n) \)-hierarchy twisted by an involution \( \sigma \) defined by \( b \) on \( \mathcal{S}(R, \mathcal{U}_{a,j}^{\perp}) \).

Use a direct computation to get

5.14 Proposition. Let \( \sigma \) denote the involution on \( SU(n) \) defined by \( \sigma(y) = (y^t)^{-1} \). Then

(i) \( g_{z,\pi} \in G_{m,\sigma}^{m} \) if and only if \( z = -\bar{z} \) and \( \bar{\pi} = \pi \),

(ii) if \( z \in C \) and \( \bar{\pi} = \pi \), then \( g_{z,\pi}g_{-\bar{z},\pi} \in G_{m,\sigma}^{m} \).

5.15 Example. The trivialization of the vacuum solution \( u = 0 \) in \( \mathcal{M}_{a,b,j}^{\sigma} \) is

\[
E(x,t,\lambda) = e^{ax+b\lambda t}.
\]

Suppose \( U \) is a \( n \times k \) matrix such that the columns of \( U \) form a basis of the linear subspace \( V \) of \( C^{m} \). By elementary linear algebra, the Hermitian projection of \( C^{m} \) onto \( V \) is \( \pi = U(U^{*}U)^{-1}U^{*} \). Then Corollary 5.7 implies that

\[
g_{z,\pi} * 0 = (z - \bar{z}) \left[ e^{-a\bar{z}x - b\bar{z}^t t} U(U^{*}e^{a(z - \bar{z})x + b(z - \bar{z})^t t} U)^{-1}U^{*}e^{a\bar{z}x + b\bar{z}^t t} \right], \quad a \tag{5.6}
\]

is in \( \mathcal{M}_{a,b,j} \). These are the 1-solitons for the \( j \)-th flow in the \( su(n) \)-hierarchy. So the space of 1-solitons for the \( j \)-th flow on \( \mathcal{S}(R, \mathcal{U}_{a,j}^{\perp}) \) defined by \( b \in \mathcal{U}_{a} \) is parametrized by the set

\[
\bigcup_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} ((C \setminus R) \times \text{Gr}(k,n)).
\]

Here we use only \( \text{Gr}(k,n) \) with \( k \leq \left\lfloor \frac{n}{2} \right\rfloor \). This is because \( g_{z,\pi} = \lambda - \bar{z} \), \( \lambda - \bar{z} \in \mathcal{U}_{a} \) and Corollary 5.11 implies that \( g_{z,\pi} * 0 = g_{z,\pi} \).

5.16 Example. Let \( a = \text{diag}(-i, i, \ldots, i) \), \( z \in C \setminus R \), and \( \pi \) the Hermitian projection on the subspace spanned by \( (1,v)^{t} = (1,v_{2}, \ldots, v_{n})^{t} \). Then the one-solitons, generated by Bäcklund transformations from the vacuum solution, for the \( j \)-th flow on \( \mathcal{S}(R, \mathcal{U}_{a,j}^{\perp}) \) defined by \( a \) is

\[
B(x,t) = \begin{pmatrix} 0 & B(x,t) \\ -B^{*}(x,t) & 0 \end{pmatrix},
\]

where

\[
B(x,t) = \frac{4 \text{Im}(z)e^{2i(\text{Re}(z)x + \text{Re}(z)^t t)} \bar{v}}{e^{-2(\text{Im}(z)x + \text{Im}(z^t) t)} + e^{2(\text{Im}(z)x + \text{Im}(z^t) t)} \|v\|^{2}}.
\]

We turn now to the classical description of Bäcklund transformation. The classical Bäcklund transformations for the sine-Gordon equation are based on ordinary differential equations:
5.17 Theorem ([Da], [Ei]). Suppose $q$ is a solution of the sine-Gordon equation (3.8), and $s \neq 0$ is a real number. Then the following first order system is solvable for $q^*$:

$$
\begin{align*}
(q^* - q)_x &= 4s \sin\left(\frac{2s \sqrt{q}}{2}\right), \\
(q^* + q)_t &= \frac{1}{s} \sin\left(\frac{2s \sqrt{q}}{2}\right).
\end{align*}
$$

(5.7)

Moreover, $q^*$ is again a solution of the sine-Gordon equation.

5.18 Definition. If $q$ is a solution of the sine-Gordon equation, then given any $c_o \in \mathbb{R}$ there is a unique solution $q^*$ for equation (5.7) such that $q^*(0,0) = c_o$. Then $B_{s,c_o}(q) = q^*$ is a transformation on the space of solutions of the sine-Gordon equation, which is the classical Bäcklund transformation for the sine-Gordon equation.

We now relate the classical Bäcklund transformations and the action of $G_{m,\sigma}$ on the space of solutions of the sine-Gordon equation (i.e., the space $M_{a,b,-1}^\sigma$ with $\sigma, a, b$ defined as in Example 3.13). Note if $s \in \mathbb{R}$, $\tilde{\pi}^* = \tilde{\pi} = (\tilde{\pi})^t$, then by Proposition 5.14, $g_{is,\tilde{\pi}} \in G_{m,\sigma}^\sigma$. Hence $\tilde{\pi}$ is a projection of $C^2$ onto $\left(\begin{array}{c}
\cos\frac{f}{2} \\
\sin\frac{f}{2}
\end{array}\right)$ for some function $f$. In other words,

$$
\tilde{\pi} = \left(
\begin{array}{cc}
\cos^2\frac{f}{2} & \sin\frac{f}{2}\cos\frac{f}{2} \\
\sin\frac{f}{2}\cos\frac{f}{2} & \sin^2\frac{f}{2}
\end{array}\right).
$$

So the first order system (5.5) for $\tilde{\pi}$ becomes

$$
\begin{align*}
(f_x &= \frac{q^*}{2} + 2s \sin f, \\
(f_t &= \frac{1}{2s} \sin(f - q)).
\end{align*}
$$

(5.8)

Write

$$
\tilde{u} = g_{is,\beta} \ast u = \left(
\begin{array}{cc}
0 & \tilde{q}_x/2 \\
-\tilde{q}_x/2 & 0
\end{array}\right).
$$

But $\tilde{u} = u + 2is[\tilde{\pi}, a]$, hence we have $\tilde{q} = 2f - q$. Writing equation (5.8) in terms of $\tilde{q}$, we get

$$
\begin{align*}
(q - \tilde{q})_x &= 4s \sin\left(\frac{2s \sqrt{q}}{2}\right) \\
(q + \tilde{q})_t &= \frac{1}{s} \sin\left(\frac{2s \sqrt{q}}{2}\right),
\end{align*}
$$

which is the classical Bäcklund transformation (5.7) for the sine-Gordon equation. We summarize this computation in the following Proposition:

5.19 Proposition. Let $q$ be a solution of the sine-Gordon equation (3.8), and $0 < c_0 < \pi$. Set

$$
\begin{align*}
u &= \left(
\begin{array}{cc}
0 & \frac{q}{2} \\
-\frac{q}{2} & 0
\end{array}\right), \\
f_o &= \frac{1}{2}(q(0,0) + c_0),
\end{align*}
$$

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and \( \pi = \text{the Hermitian projection onto the complex linear subspace spanned by} (\cos f_0^2, \sin f_0^2)^t, \text{i.e.,} \)

\[
\pi = \begin{pmatrix} \cos^2 f_0^2 & \sin f_0^2 \cos f_0^2 \\ \sin f_0^2 \cos f_0^2 & \sin^2 f_0^2 \end{pmatrix}
\]

Then \( B_{s, c_0}(q) = g_{is, \pi} \ast u. \)

Notice the different roles of the parameter \( s, \) and the parameter \( c_0. \) The location of the pole comes from \( s, \) and the angle of the projection comes from \( c_0. \) The location of the pole is independent of choice of the base point in the trivialization and is canonical, but the angle of the projection changes with the trivialization.

6. Permutability formula

There is a Bianchi permutability theorem for surfaces with Gaussian curvature \(-1\) in \( R^3, \) which gives the following analytical formula for the sine-Gordon equation:

6.1 Theorem ([Da], [Ei]). Suppose \( q_0 \) is a solution of the sine-Gordon equation, \( c_1, c_2 \in R, \) and \( s_1, s_2 \in R \) such that \( s_1^2 \neq s_2^2 \) and \( s_1 s_2 \neq 0. \) Let \( q_i = B_{s_i, c_i}(q_0) \) for \( i = 1, 2. \) Then there exist \( d_1, d_2 \in R, \) which can be constructed algebraically, such that

1. \( B_{s_1, d_1} B_{s_2, c_2} = B_{s_2, d_2} B_{s_1, c_1}, \)
2. let \( q_3 = B_{s_1, d_1} B_{s_2, c_2}(q_0), \) then

\[
\tan \frac{q_3 - q_0}{4} = \frac{s_1 + s_2}{s_1 - s_2} \tan \frac{q_1 - q_2}{4}. \tag{6.1}
\]

This is called the Bianchi permutability formula for the sine-Gordon equation.

The Bianchi permutability formula for the sine-Gordon equation is a consequence of factoring quadratic elements in the rational loop group \( G^m_\mathbb{C} \) in two ways as product of two simple elements. We can also derive an analogue of the Bianchi permutability formula for the \( j \)-th flow using these two different ways of factoring a quadratic elements in \( G^m_\mathbb{C}. \) Note that contrary to the name, the permutability theorem does not mean that the Bäcklund transformations generated by two simple elements are commuting.

Notice that the singularities (both poles and zeroes) of an element in \( G^m_\mathbb{C} \) comes in pairs \((z, \bar{z})\) due to the \( U(n)\)-reality condition. In the \( n \times n \) case, \( n > 2, \) if \( \pi \) is the projection to \( V, \) \( V = V_1 + V_2 \) and \( V_1 \perp V_2, \) then

\[
g_{z, \pi} = \frac{\lambda - \bar{z}}{\lambda - z} g_{z, \pi_1} g_{z, \pi_2},
\]

where \( \pi_i \) is the projection to \( V_i. \) Hence there are infinitely many ways to factor \( g_{z, \pi_1} g_{z, \pi_2} \) as product of simple elements. But we will prove later that a quadratic element with two different singularities factors in exactly two ways.

First we derive some relations among generators of \( G^m_\mathbb{C}. \)
6.2 Theorem. Let \( z_1, z_2 \in C \setminus R \), and \( \pi_1, \pi_2 \) Hermitian projections of \( C^m \). If
\[
\phi = (z_2 - z_1)I + (z_1 - \bar{z}_1)\pi_1 - (z_2 - \bar{z}_2)\pi_2
\] (6.2)
is non-singular, then
(i) \( \tau_i = \phi\pi_i\phi^{-1} \) is a Hermitian projection for \( i = 1, 2 \),
(ii) \( g_{z_2, \tau_2}g_{z_1, \pi_1} = g_{z_1, \tau_1}g_{z_2, \pi_2} \),
(iii) if \( g_{z_2, \tau_2}g_{z_1, \pi_1} = g_{z_1, \tau_1}g_{z_2, \pi_2} \), then \( \pi_1, \pi_2 \) and \( \tau_1, \tau_2 \) are related as in (i).

PROOF. Since \( g_{z, \pi} = h_{z, z, \pi} \), this theorem follows from Proposition 4.8 if we can prove that the \( \tau_i \) is a Hermitian projection. Since \( \tau_i^2 = \tau_i \), we only need to prove that \( \tau_i^* = \tau_i \) for \( i = 1, 2 \). To prove this, we first set up some notations.
Let \( z_i = r_i + \sqrt{-1}s_i \). Set \( \beta_i = \sqrt{-1}(\pi_i - \pi_i^1) \) for \( i = 1, 2 \). Then \( \beta_i \in u(n) \), \( \beta_i^2 = -I \), and
\[
g_{z_i, \pi_i} = \frac{\lambda - r_i + s_i\beta_i}{\lambda - r_i + \sqrt{-1}s_i}.
\]
Set \( y = s_1\beta_1 - s_2\beta_2 \). We claim that \( y^2\beta_i = \beta_i y^2 \). To see this, we note that
\[
y^2 = -(s_1^2 + s_2^2)I - s_1s_2(\beta_1\beta_2 + \beta_2\beta_1).
\]
A direct computation gives
\[
(\beta_1\beta_2 + \beta_2\beta_1)\beta_1 = \beta_1\beta_2\beta_1 - \beta_2,
\]
\[
\beta_1(\beta_1\beta_2 + \beta_2\beta_1) = -\beta_2 + \beta_1\beta_2\beta_1.
\]
So \( y^2 \) and \( \beta_1 \) commute. Similarly, \( y^2 \) and \( \beta_2 \) commute. This proves our claim.
Set \( \xi_i = \sqrt{-1}(\tau_i - \tau_i') \), \( S_i = -r_i + s_i\beta_i \), and \( r = r_2 - r_1 \), where \( \tau_i' = I - \tau_i \). Then
\[
\phi = S_1 - S_2 = rI + y,
\]
\[
\xi_i = (rI + y)^{-1}\beta_i(rI + y).
\]
To prove \( \tau_i \) is a projection is equivalent to prove that \( \xi_i^2 = -I \) and \( \xi_i \in u(n) \).
Since \( \beta_i^2 = -I \) and \( \xi_i \) is conjugate to \( \beta_i \), \( \xi_i^2 = -I \). To prove that \( \xi_i^* = -\xi_i \), we compute directly
\[
\xi_i^* = -((rI + y)^*)^{-1}\beta_i(rI + y)^* = - (rI - y)^{-1}\beta_i(rI - y).
\] (6.3)
But \( (rI - y)(rI + y) = r^2I - y^2 \), which commutes with \( \beta_i \) because both \( I \) and \( y^2 \) commute with \( \beta_i \). So we have \( (r^2 - y^2)\beta_i = \beta_i(r^2 - y^2) \), which implies that
\[
(rI - y)^{-1}\beta_i(rI - y) = (rI + y)\beta_i(rI + y)^{-1}.
\] (6.4)
So \( \xi_i^* = -\xi_i \). Hence \( \tau_i^* = \tau_i \). ■

The following Proposition gives a sufficient condition on \( z_1, z_2 \) so that \( \phi \) defined by formula (6.2) is non-singular.
6.3 Proposition. Let \( z_1 = r_1 + is_1 \), and \( z_2 = r_2 + is_2 \), \( \pi_1, \pi_2 \) Hermitian projections, and \( \phi \) as in formula (6.2). If \( z_1 \neq z_2 \) and \( z_1 \neq \bar{z}_2 \), then \( \phi \) is non-singular.

**PROOF.** Set \( \beta_i = \sqrt{-1} (\pi_i - \pi_i^\dagger) \) as in the proof of Theorem 6.2. Then \( \phi \) given by formula (6.2) can be written as

\[
\phi = -(z_1 - z_2) + 2i(s_1\pi_1 - s_2\pi_2) = -(r_1 - r_2) + (s_1\beta_1 - s_2\beta_2).
\]

Since \( (s_1\beta_1 - s_2\beta_2) \in u(n) \), its eigenvalues are pure imaginary. So if \( r_1 \neq r_2 \), then all eigenvalues of \( -(r_1 - r_2) + (s_1\beta_1 - s_2\beta_2) \) are not zero.

Since \( \beta_i \in u(n) \) and \( \beta_i^2 = -I \), \( \| \beta_i(x) \| = \| x \| \) for all \( x \in C^n \). So

\[
\| (s_1\beta_1 - s_2\beta_2)(x) \| \geq | | s_1 | - | s_2 | | x \|
\]

for all \( x \in C^n \). If \( s_1^2 - s_2^2 \neq 0 \), then \( (s_1\beta_1 - s_2\beta_2) \) is non-singular. Hence all eigenvalues of \( (s_1\beta_1 - s_2\beta_2) \) are non-zero and pure imaginary, which implies that \( -(r_1 - r_2) + (s_1\beta_1 - s_2\beta_2) \) is non-singular. \( \blacksquare \)

Use \( g_{z,\pi}^{-1}(\lambda) = g_{z,\pi}(\overline{\lambda})^* = g_{\bar{z},\pi}(\lambda) \) and Theorem 6.2 to get

6.4 Corollary. Given \( g_{z_1,\pi_1}, g_{z_2,\pi_2} \) in \( G^m \) such that \( z_1 \neq z_2 \) and \( z_1 \neq \bar{z}_2 \), then there exist uniquely Hermitian projections \( \tau_1, \tau_2 \) such that

\[
g_{z_1,\pi_1}g_{z_2,\pi_2} = g_{z_2,\tau_2}g_{z_1,\tau_1},
\]

where \( \tau_i = \phi\pi_i\phi^{-1} \) and \( \phi = (\bar{z}_2 - z_1)I + (z_1 - \bar{z}_1)\pi_1 + (z_2 - \bar{z}_2)\pi_2 \).

The following theorem follows easily from Theorem 6.2 and Proposition 6.3:

6.5 Theorem. Let \( z_1, z_2 \in C \setminus R \) such that \( z_1 \neq z_2 \) and \( z_1 \neq \bar{z}_2 \), and \( \pi_1, \pi_2 \) Hermitian projections of \( C^n \). Let \( u_0 \in M_{a,b,j} \) (\( j = -1 \) or \( j \geq 1 \)), and

\[
u_i = g_{z_i,\pi_1} \ast u_0 = u_0 + (z_i - \bar{z}_i)[\tilde{\pi}_i, a]
\]

for \( i = 1, 2 \) as given in Theorem 5.4. Set

\[
\begin{align*}
\phi &= (z_2 - z_1)I + (z_1 - \bar{z}_1)\pi_1 - (z_2 - \bar{z}_2)\pi_2, \\
\tilde{\phi} &= (z_2 - z_1)I + (z_1 - \bar{z}_1)\tilde{\pi}_1 - (z_2 - \bar{z}_2)\tilde{\pi}_2, \\
\tau_i &= \phi\pi_i\phi^{-1}, \\
\tilde{\tau}_i &= \tilde{\phi}\tilde{\pi}_i\tilde{\phi}^{-1}.
\end{align*}
\]

Then

\[
u_3 = (g_{z_2,\tau_2}g_{z_1,\tau_1}) \ast u_0 = u_0 + (z_1 - \bar{z}_1)[\tilde{\pi}_1, a] + (z_2 - \bar{z}_2)[\tilde{\pi}_2, a] = (g_{z_1,\xi_1}g_{z_2,\pi_2}) \ast u_0 = u_0 + (z_1 - \bar{z}_1)[\tilde{\pi}_1, a] + (z_2 - \bar{z}_2)[\tilde{\pi}_2, a]. (6.5)
\]

As a consequence of Proposition 5.19 and Theorem 6.5, we have

6.6 Proposition. Formula (6.5) for the \(-1\)-flow in the \( su(2)\)-hierarchy twisted by the involution \( \sigma(y) = -y^t \) on \( S(R, U_{a,\sigma}) \) defined by \( b = \frac{a}{4} = \frac{1}{4} \text{ diag}(i, -i) \) is the permutability formula (6.1) for the sine-Gordon equation.
7. **N-soliton formula**

Bäcklund transformations defined in section 6 give an algebraic algorithm to compute the solution \( g^*0 \) of the \( j \)-th flow in the \( su(n) \)-hierarchy for a rational loop \( g \). The procedure is as follows:

(i) Factor \( g = g_{zN,\pi N} \cdots g_{z_1,\pi_1} \) as product of simple elements.

(ii) Apply Theorem 5.4 repeatedly to obtain \( g^*0 \). In other words, we set \( u_0 = 0 \),

\[
E_0(x,t,\lambda) = e^{a\lambda x + b\lambda^j t},
\]

and define \( u_j, \tilde{\pi}_j, E_j \) for \( 1 \leq j \leq N \) by induction as follows:

\[
u_k = u_{k-1} + (z_k - \bar{z}_k)[\tilde{\pi}_k, a],
\]

\[
\tilde{\pi}_k(x,t) = \text{projection onto } E_{k-1}(x,t, z_k)^*(\pi_k(V)),
\]

\[
E_k(x,t,\lambda) = g_{z_k,\pi_k}(\lambda)E_{k-1}(x,t,\lambda)g_{z_k,\tilde{\pi}_k}(x,t)(\lambda)^{-1}.
\]

Then \( g^*0 = u_N \).

Although this algorithm is explicit, it is difficult to write down \( g^*0 \) as a formula in closed form in terms of \( (z_1, \ldots, z_N, \pi_1, \ldots, \pi_N) \). We are motivated by the \( 2 \times 2 \) case considered in the book of Faddeev and Takhtajan [FT] to use the permutability formula to give a formula for \( g^*0 \) in closed form.

To obtain the formula for \( g^*0 \), we first construct local coordinates for \( G^\infty_m \). Note that

\[
g_{z,\pi}(\lambda) = \pi + \frac{\lambda - z}{\lambda - \bar{z}} \pi^\perp
\]

has a simple pole at \( \lambda = \bar{z} \), and is holomorphic but not invertible at \( z \). We will call \( \bar{z} \) a zero of \( g_{z,\pi} \). For \( g \in G^\infty_m \), the zeros and poles occur in pairs \((z_j, \bar{z}_j)\). Since it is more convenient to denote the pole of a simple element as \( z \) in our computation below, we change our notation for simple elements slightly. Set

\[
h_{z,\pi}(\lambda) = I + \frac{z - \bar{z}}{\lambda - \bar{z}} \pi = \pi^\perp + \frac{\lambda - \bar{z}}{\lambda - z} \pi.
\]

Then we have

\[
h_{z,\pi}(\lambda) = g_{\bar{z},\pi^\perp}(\lambda) = \left( \frac{\lambda - \bar{z}}{\lambda - z} \right) g_{z,\pi}.
\]

Since the center of \( G^\infty_m \) acts trivially on \( M_{a,b,j} \) (by Corollary 5.11),

\[
h_{z,\pi} \ast u = g_{z,\pi} \ast u.
\]

**7.1 Definition.** A rational map \( g \in G^\infty_m \) is called regular if \( g \) has only simple poles and all the poles and zeros of \( g \) are distinct.

**7.2 Proposition.** If \( g \) is regular and \( z \) is a simple pole of \( g \), then there exists a unique projection \( \pi \) such that \( gh_{z,\pi}^{-1} \) is holomorphic and non-singular at \( \lambda = z \).
PROOF. Existence follows from Theorem 5.1 and Corollary 6.4. So it remains to prove uniqueness. Assume \( g = h_1 h_{z, \pi_1} = h_2 h_{z, \pi_2} \), where \( h_1, h_2 \) have no zeros and poles at \( z \). So \( h_1(\bar{z}) \) and \( h_2(\bar{z}) \) are non-singular. But
\[
g(\bar{z}) = h_1(\bar{z})\pi_1^\perp = h_2(\bar{z})\pi_2^\perp.
\]
Let \( V_i = \text{Im}(\pi_i) \). Then the above equation implies that
\[
h_1(\bar{z})\pi_1^\perp(V_1) = 0 = h_2(\bar{z})\pi_2^\perp(V_1).
\]
Since \( h_2(\bar{z}) \) is non-singular, \( \pi_2^\perp(V_1) = 0 \). Hence \( V_1 \subset V_2 \). Similarly, \( V_2 \subset V_1 \). This proves that \( \pi_1 = \pi_2 \). 

Given regular \( g \in G^m \) with \( N \) simple poles, there exist uniquely \( \Gamma_g = (z_1, \cdots, z_N, \pi_1, \cdots, \pi_N) \) such that \( g h_{z_k, \pi_k}^{-1} \) is holomorphic at \( z_k \) for all \( 1 \leq k \leq N \). We call \( \Gamma_g \) the singularity data of \( g \).

In the rest of this section, we will derive a formula for \( g * 0 \) in \( M_{a,b,j} \) in terms of the singularity data \( \Gamma_g = (z_1, \cdots, z_N, \pi_1, \cdots, \pi_N) \) of \( g \). Let
\[
e_{a,b,j}(x,t)(\lambda) = e^{a\lambda x + b\lambda^j}
\]
denote the trivialization of the vacuum solution \( u = 0 \) in \( M_{a,b,j} \).

**7.3 Proposition.** Let \( g \in G^m \) be a regular element with singularity data \( \Gamma_g = (z_1, \cdots, z_N, \pi_1, \cdots, \pi_N) \), and \( v_k \) a \( n \times r_k \) matrix of rank \( r_k \) such that \( \pi_k \) is the projection onto the space spanned by columns of \( v_k \). Factor
\[
ge e_{a,b,j}(x,t) = E(x,t)\tilde{g}(x,t) \in G_+ \times G_-
\]
as in Theorem 5.9. Then
\[
\tilde{g}(x,t) = I + \sum_{k=1}^{N} \frac{P_k(x,t)}{\lambda - z_k},
\]
where \( P_k(x,t) \) is an \( n \times n \) matrix of rank \( r_k \) for all \( (x,t) \in R^2 \). Moreover, there exists smooth maps \( \xi_k : R^2 \to M_{n \times r_k} \) such that
(i) \( \xi_k(x,t) \) has rank \( r_k \) for all \( (x,t) \),
(ii) \( \sum_{m=1}^{N} \frac{1}{z_m - \bar{z}_k} \xi_m(x,t)v_m^* e^{\alpha(x_m - z_k)x + b(z_m - \bar{z}_k)t} v_k = e^{-(az_k x + bz_k^j t)} v_k \) \( (7.2) \)
for \( 1 \leq k \leq N \),
(iii) \( P_k(x,t) = \xi_k(x,t) v_k^* e^{az_k x + bz_k^j t} \).
PROOF. It follows from Corollary 5.7 that \( \tilde{g}(x, t) \) is regular, has only simple poles at \( \lambda = z_1, \ldots, z_N \), and is equal to \( I \) at \( \lambda = \infty \). So we can write \( \tilde{g} \) in terms of partial fractions:

\[
\tilde{g}(x, t)(\lambda) = I + \sum_{k=1}^{N} \frac{P_k(x, t)}{\lambda - z_k}
\]

for some \( n \times n \) matrix function \( P_k(x, t) \).

First we claim that the rank of \( P_k(x, t) \) is equal to \( r_k \). By definition of the singularity data, we have \( g = h_k h_{z_k, \pi_k}^{-1} \) for some \( h_k \in G_m^m \) such that \( h_k \) is holomorphic and non-degenerate at \( \lambda = z_k \). So it follows from Theorem 5.9 and Corollary 5.7 that there exist \( \tilde{h}_k(x, t) \) in \( G_m^m \) and projections \( \tilde{\pi}_k(x, t) \) such that

\[
\tilde{g}(x, t) = \tilde{h}_k(x, t) h_{z_k, \tilde{\pi}_k(x, t)}
\]

and \( \tilde{h}_k(x, t) \) is holomorphic and non-singular at \( \lambda = z_k \) for all \( (x, t) \in \mathbb{R}^2 \). Hence the residue of \( \tilde{g}(x, t)(\lambda) \) at \( \lambda = z_k \) is

\[
P_k(x, t) = (z_k - \bar{z}_k) \tilde{h}_k(x, t)(z_k)(\tilde{\pi}_k(x, t)).
\]

So the rank of \( P_k(x, t) \) is equal to that of \( \tilde{\pi}_k(x, t) \), which is \( r_k \). This proves our claim.

It follows from Corollary 5.7 that \( \tilde{\pi}_k(x, t) \) is the projection onto the space spanned by columns of

\[
\tilde{v}_k(x, t) = e_{a, b, j}(\bar{z})^* (v_k) = e^{-a z_k x - b z_k^t t} v_k.
\]

(7.3)

Since \( g h_{z_k, \pi_k}^{-1} \) is holomorphic at \( \lambda = z_k \),

\[
\tilde{g}(x, t) h_{z_k, \tilde{\pi}_k(x, t)}^{-1} = \left( I + \sum_{j=1}^{N} \frac{P_j(x, t)}{\lambda - z_j} \right) \left( I + \frac{z_k - \bar{z}_k}{\lambda - \bar{z}_k} \tilde{\pi}_k(x, t) \right)
\]

is also holomorphic at \( \lambda = z_k \). So its residue at \( z_k \) is zero. This implies that

\[
P_k(x, t) \tilde{\pi}_k(x, t) \perp = 0.
\]

Therefore the kernel of \( P_k(x, t) \) contains the orthogonal complement of the image of \( \tilde{\pi}_k(x, t) \). But the rank of \( P_k(x, t) \) is \( r_k \), which is the rank of \( \tilde{\pi}_k(x, t) \). So there exists \( n \times r_k \) matrix \( \xi_k(x, t) \) of rank \( r_k \) such that

\[
P_k(x, t) = \xi_k(x, t) \tilde{v}_k(x, t) = \xi_k(x, t) v_k^* e^{a z_k x + b z_k^t t}.
\]

To prove (i) and (iii), it remains to prove that \( \xi_k \) satisfies the linear system (7.2). Write \( \tilde{g}(x, t) = \tilde{h}_k(x, t) h_{z_k, \tilde{\pi}_k(x, t)} \). Since

\[
h_{z_k, \tilde{\pi}_k(x, t)}(\bar{z}_k) = \tilde{\pi}_k(x, t),
\]

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we have
\[ \hat{g}(x, t)(\bar{z}_k)(\tilde{v}_k(x, t)) = 0 \] (7.4)
for all \((x, t)\) and \(1 \leq k \leq N\). Use formulas (7.1) and (7.4) to get
\[ \tilde{v}_k(x, t) + \sum_{m=1}^{N} \frac{\xi_m(x, t)\tilde{v}_m^*(x, t)\tilde{v}_k(x, t)}{\bar{z}_k - z_m} = 0. \] (7.5)
Substitute formula (7.3) in (7.5) to get (7.2).

Suppose \(g \in G_\sim\) is regular and \(\Gamma_g = (z_1, \cdots, z_N, \pi_1, \cdots, \pi_N)\). We compute \(g * 0\) when the rank of \(\pi_k\) is equal to 1 for all \(1 \leq k \leq N\). In this case, \(v_k, \xi_k\) are \(n \times 1\). Set
\[ f_{mk}(x, t) = \frac{v_m^*e^{a(z_m-z_k)x+b(z_m^j-z_k^j)t}v_k}{z_m - \bar{z}_k}. \]
Then equation (7.2) becomes
\[ \sum_{m=1}^{N} \xi_m(x, t)f_{mk}(x, t) = e^{-(a z_k x + b z_k^j t)}v_k, \quad 1 \leq k \leq N. \]
If the matrix \(F(x, t) = (f_{mk}(x, t))\) is non-singular, then we can solve
\[ \xi_k(x, t) = \sum_{m=1}^{N} e^{-(a z_m x + b z_m^j t)}v_m f^{mk}(x, t), \]
where \((f^{mk})\) is the inverse of \(F = (f_{mk})\).

**7.4 Theorem.** Suppose \(g \in G^m\) is regular with singularity data
\[ \Gamma_g = (z_1, \cdots, z_N, \pi_1, \cdots, \pi_N), \]
and \(\pi_k\) is the projection onto \(v_k \in C^n\) for each \(1 \leq k \leq N\). Let \(F(x, t) = (f_{km}(x, t))\) be the \(N \times N\) matrix defined by
\[ f_{mk}(x, t) = \frac{v_m^*e^{a(z_m-z_k)x+b(z_m^j-z_k^j)t}v_k}{z_m - \bar{z}_k}. \]
Suppose \(F(x, t)\) is invertible. Set \(F(x, t)^{-1} = (f^{km}(x, t))\) and
\[ P_k(x, t) = \left( \sum_{m=1}^{N} e^{-(a z_m x + b z_m^j t)}v_m f^{mk}(x, t) \right) \frac{v_k^*e^{a z_k x + b z_k^j t}}{z_k - \bar{z}_k}. \]
Then the $N$-soliton $\tilde{u} = g \ast 0$ of the $j$-th flow on $\mathcal{S}(R, \mathcal{U}_a^\perp)$ defined by $b$ and its trivialization $\tilde{E}(x, t)$ are given below:

$$\tilde{u} = (g \ast 0)(x, t) = \sum_{k=1}^N [P_k(x, t), a],$$

$$\tilde{E}(x, t, \lambda) = g(0)e_{a, b, j}(x, t) \left( I + \sum_{k=1}^N \frac{P_k^*(x, t)}{\lambda - \bar{z}_k} \right).$$

**PROOF.** The formula for $\tilde{E}$ follows from Theorem 5.9. By Theorem 5.9 (ii), $g \ast 0$ is equal to the constant coefficient of the power series expansion of $\tilde{E}^{-1} \tilde{E}_x$ at $\lambda = \infty$. Note that the expansion of $\tilde{g}(x, t, \lambda)$ at $\lambda = \infty$ is

$$\tilde{g}(x, t, \lambda) = I + \sum_{k=1}^N \frac{P_k(x, t)}{\lambda - z_k} = I + \left( \sum_{k=1}^N P_k(x, t) \right) \lambda^{-1} + \cdots.$$

We will omit the variables $x, t$ in the following computation. So we have

$$\tilde{E}^{-1} \tilde{E}_x = \lambda \tilde{g} a \tilde{g}^{-1} - \tilde{g}_x \tilde{g}^{-1} = (I + \sum_{k=1}^N P_k \lambda^{-1} + \cdots) a \lambda (I - \sum_{k=1}^N P_k \lambda^{-1} + \cdots)$$

$$+ (\sum_{k=1}^N (P_k)_x \lambda^{-1} + \cdots)(I - \sum_{k=1}^N P_k \lambda^{-1} + \cdots),$$

which is equal to $a \lambda + \tilde{u}$. But the constant term in equation (7.6) is $\sum_{k=1}^N [P_k, a]$. Hence $\tilde{u}(x, t) = (g \ast 0)(x, t) = \sum_{k=1}^N [P_k(x, t), a]$.

8. **Scaling transformations**

The sine-Gordon equation is clearly invariant under the Lorentz transformations, for example,

8.1 **Proposition ([Da], [Ei]).** If $q$ is a solution of the sine-Gordon equation, then $L_r(q)(x, t) = q(r^{-1}x, rt)$ is also solution of the sine-Gordon equation. ($L_r$ is called a Lie transformation in the classical literature).
It is clear that $L_{r_1 r_2} = L_{r_1} L_{r_2}$. In other words, Lie transformations give an action of the multiplicative group $\mathbb{R}^*$ of non-zero real numbers on the space of solutions of the sine-Gordon equation. The following result relating Bäcklund and Lie transformations is known in classical surface theory (cf. [Da, Ei]).

8.2 Proposition ([Da], [Ei]). Bäcklund transformations and Lie transformations of the sine-Gordon equation are related by the following formula:

$$B_{s,c_o} = L_{r_1}^{-1} B_{1,c_o} L_s.$$

It is known that an analogue of Lie transformations exists for the $j$-th flow, which will be called scaling transformations. We describe these scaling transformations next. If $u \in \mathcal{M}_{a,b,j}$ and $r \in \mathbb{R}^*$, then

$$\hat{u}(x,t) = r^{-1} u(r^{-1} x, r^{-j} t)$$

is again a solution of the $j$-th flow. So $r \ast u = \hat{u}$ defines an action of $\mathbb{R}^*$ on $\mathcal{M}_{a,b,j}$.

The main goal of this section is to explain the relation between the scaling transformation and Bäcklund transformations. In fact, the scaling transformation extends the action of $G_m^-$ to the action of the semi-direct product $\mathbb{R}^* \rtimes G_m^-$ (defined below) on $\mathcal{M}_{a,b,j}$, and Proposition 8.2 follows from the multiplication law of the group $\mathbb{R}^* \times G_m^-$.

First we outline a proof for $r \ast u \in \mathcal{M}_{a,b,j}$ if $u \in \mathcal{M}_{a,b,j}$. Let $E$ be the trivialization of $u$, i.e.,

$$\begin{cases} E^{-1} E_x = a \lambda + u, \\ E^{-1} E_t = b \lambda^j + v_1 \lambda^{j-1} + \cdots + v_j, \end{cases}$$

where $v_i = Q_{b,i}(u)$. Set

$$\hat{E}(x,t,\lambda) = E(r^{-1} x, r^{-j} t, r \lambda).$$

Then

$$\begin{cases} \hat{E}^{-1}(x,t,\lambda) \hat{E}_x(x,t,\lambda) = a \lambda + \hat{u}(x,t), \\ \hat{E}^{-1} \hat{E}_t = b \lambda^j + \hat{v}_1(x,t) \lambda^{j-1} + \cdots + \hat{v}_j(x,t), \end{cases}$$

where $\hat{v}_i(x,t) = r^i v_i(r^{-1} x, r^{-j} t)$ for $1 \leq i \leq j$ and $v_i = Q_{b,i}(u)$. If $b$ is a polynomial in $a$, then $Q_{b,i}(u)$ is a polynomial differential operator. Hence $Q_{b,i}(u)$ vanishes at both $\infty, -\infty$, and the $\hat{v}_i$’s are in the Schwartz class. By Proposition 2.6, we conclude $\hat{u} \in \mathcal{M}_{a,b,j}$. (In fact, this calculation works for either positive or negative $r$, negative $r$ reverses $\pm \infty$).

Next we define a one dimension extension of the group $G_m^-$. 41
8.3 Definition. Let $R^* = \{ r \in R \mid r \neq 0 \}$ denote the multiplicative group, and $R^* \times G^m_-$ the semi-direct product of $R^*$ and $G^m_-$ defined by the homomorphism

$$\rho : R^* \rightarrow \text{Aut}(G^m_-), \quad \rho(r)(g)(\lambda) = g(r\lambda),$$

i.e., the multiplication in $R^* \times G^m_-$ is defined by

$$(r_1, g_1) \cdot (r_2, g_2) = (r_1r_2, g_1(\rho(r_1)(g_2))).$$

8.4 Theorem. Suppose $j \geq 1$ or $j = -1$. Then the action $\ast$ of $G^m_-$ (resp. $G^m_{m,\sigma}$) extends to an action of $R^* \times G^m_-$ (resp. $R^* \times G^m_{m,\sigma}$) on the space $M_{j,a,b}$ (resp. $M^\sigma_{a,b,j}$) by

$$r \ast u(x, t) = r^{-1}u(r^{-1}x, r^{-j}t).$$

PROOF. It is easy to see that $(r_1r_2) \ast u = r_1 \ast (r_2 \ast u)$. Since

$$(r, I) \cdot (1, g)(r^{-1}, I) = (1, \rho(r)(g)),$$

the action $\ast$ extends to an action of $R^+ \times G^m_-$ if

$$(\rho(r)(g)) \ast u = r \ast (g \ast (r^{-1} \ast u)). \quad (8.1)$$

To see this, let $E$ be the trivialization of $u$, and define

$$(r \ast E)(x, t, \lambda) = E(r^{-1}x, r^{-j}t, r\lambda),$$

$$(g \ast E)(x, t, \lambda) = \tilde{E}(x, t, \lambda)$$

for $r \in R^*$ and $g \in G^m_-$, where $\tilde{E}$ is obtained from the factorization

$$gE(x, t) = \tilde{E}(x, t)\tilde{g}(x, t) \in G_+ \times G^-_m$$

as in Theorem 5.9. To prove equation (8.1), it suffices to prove

$$(\rho(r)(g)) \ast E = r \ast (g \ast (r^{-1} \ast E)). \quad (8.2)$$

Write

$$g(\lambda)(r^{-1} \ast E)(x, t, \lambda) = E_1(x, t, \lambda)g_1(x, t, \lambda),$$

$$(\rho(r)(g))(\lambda)E(x, t, \lambda) = E_2(x, t, \lambda)g_2(x, t, \lambda),$$

such that $E_i(x, t) \in G_+$ and $g_i(x, t) \in G^-_m$ for all $(x, t) \in R^2$ and $i = 1, 2$. Note that the second equation gives

$$g(r\lambda)E(x, t, \lambda) = E_2(x, t, \lambda)g_2(x, t, \lambda).$$
By definition, we have
\[ g \ast (r^{-1} \ast E(x, t)) = E_1(x, t), \quad (\rho(r)(g)) \ast E(x, t) = E_2(x, t). \]

Now a direct computation gives
\[
(r \ast (g \ast (r^{-1} \ast E)))(x, t, \lambda) \\
= (g \ast (r^{-1} \ast E))(r^{-1}x, r^{-j}t, r\lambda) \\
= E_1(r^{-1}x, r^{-j}t, r\lambda) \\
= g(r\lambda)(r^{-1} \ast E)(r^{-1}x, r^{-j}t, r\lambda)g_1^{-1}(r^{-1}x, r^{-j}t, r\lambda) \\
= g(r\lambda)E(x, t, \lambda)g_1^{-1}(r^{-1}x, r^{-j}t, r\lambda) \\
= E_2(x, t, \lambda)g_2(x, t, \lambda)g_1^{-1}(r^{-1}x, r^{-j}t, r\lambda) = E_2(x, t, \lambda)g_3(x, t, \lambda).
\]

But \( E_2(x, t) \in G_+ \) and \( g_3(x, t) = g_2(x, t)g_1^{-1}(r^{-1}x, r^{-j}t) \in G_+^m \). So
\[
r \ast (g \ast (r^{-1} \ast E)) = E_2,
\]
which is equal to \((\rho(r)(g)) \ast E\). This proves our claim. \[\Box\]

Since \((r^{-1}, 1)(g_{z, \pi})(r, 1) = (1, g_{rz, \pi})\), we have

**8.5 Corollary.** Suppose \( j \geq 1 \) or \( j = -1 \). If \( u \in \mathcal{M}_{a, b, j} \) (resp. \( \mathcal{M}_{a, b, j}^\sigma \)), then
\[
r^{-1} \ast (g_{z, \pi} \ast (r \ast u)) = g_{rz, \pi} \ast u.
\]

**8.6 Remark.** Corollary 8.5 for the \(-1\)-flow in the \( su(2) \)-hierarchy twisted by \( \sigma(y) = -y^t \) is Proposition 8.2.

9. Bäcklund transformations for \( n \)-dimensional systems

The integrable equations of evolution we have been describing up to this point have two independent variables. The flow of the first variable, regarded as a spatial variable, is used to construct the initial Cauchy data. The second variable is considered to be the time variable, and the flow in this variable is the evolution. In this section, we turn our attention to a family of geometric problems in \( n \) spatial variables, which we shall call \( n \)-dimensional systems. In the applications, the \( n \) variables are on an equal footing, and the flows in each variable is a first flow. The flows commute, and hence the resulting geometric object is always a flat connection on a region of \( \mathbb{R}^n \) with special properties.

These \( n \)-dimensional systems have been discussed in a paper by the first author ([Te]). We give definitions and some of the basic examples. The results on Bäcklund transformations developed in previous sections apply easily to these systems.
9.1 Definition ([Te]). Let $U$ be a rank $n$, semi-simple Lie group, $\mathcal{T}$ a maximal abelian subalgebra of the Lie algebra $\mathcal{U}$, $a_1, \cdots, a_n$ a basis of $\mathcal{T}$, and $\mathcal{T}^\perp$ the orthogonal complement of $\mathcal{T}$ with respect to $(y_1, y_2) = \text{tr}(y_1 y_2)$. The $n$-dimensional system associated to $U$ is the following first order system:
\[
[a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \quad v : R^n \to \mathcal{T}^\perp. \tag{9.1}
\]

9.2 Definition ([Te]). Let $U/K$ be a rank $n$ symmetric space, $\sigma : U \to U$ the corresponding involution, $\mathcal{U} = K + \mathcal{P}$ the Cartan decomposition, $\mathcal{A}$ a maximal abelian subalgebra in $\mathcal{P}$, $a_1, \cdots, a_n$ a basis of $\mathcal{A}$, and $\mathcal{A}^\perp$ the orthogonal complement of $\mathcal{A}$ in $\mathcal{U}$. The $n$-dimensional system associated to $U/K$ is the first order system:
\[
[a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \quad v : R^n \to \mathcal{P} \cap \mathcal{A}^\perp. \tag{9.2}
\]

9.3 Proposition. The following conditions are equivalent:
(i) $v$ is a solution of equation (9.1) (or (9.2))
(ii) $\left[ \frac{\partial}{\partial x_i} + (a_i \lambda + [a_i, v]), \frac{\partial}{\partial x_j} + (a_j \lambda + [a_j, v]) \right] = 0$ for all $i \neq j$.

9.4 Example. Let $U/K = U(n)/O(n)$, and $\mathcal{U} = K + \mathcal{P}$ the Cartan decomposition corresponding to the involution $\sigma(y) = -y^t$. Then $i\mathcal{P}$ is the set of all real symmetric $n \times n$ matrices, and the space $\mathcal{A}$ of all diagonal matrices in $\mathcal{P}$ is a maximal abelian subalgebra in $\mathcal{P}$. Let $e_{ii}$ denote the diagonal matrix such that all entries are zero except the $ii$-th entry is equal to 1. Then $ie_{11}, \cdots, ie_{nn}$ form a basis of $\mathcal{A}$. The space $i(\mathcal{P} \cap \mathcal{A}^\perp)$ is the space of all real symmetric $n \times n$ matrices whose diagonal entries are zero. The $n$-dimensional system (9.2) associated to $U(n)/O(n)$ for $v = iF$ can be written as the system for $F = (f_{ij}) : R^n \to \text{gl}(n, R), \ f_{ij} = f_{ji}, \ f_{ii} = 0 \quad \text{if} \ 1 \leq i \leq n$
\[
\begin{cases}
(f_{ij})_{x_i} + (f_{ij})_{x_j} + \sum_k f_{ik}f_{kj} = 0, & \text{if} \ i \neq j, \\
(f_{ij})_{x_k} = f_{ik}f_{kj}, & \text{if} \ i, j, k \text{ are distinct.} \tag{9.3}
\end{cases}
\]
By Proposition 9.3, $F$ is a solution of system (9.3) if and only if
\[
\left[ \frac{\partial}{\partial x_i} + e_{ii} \lambda + [e_{ii}, F], \ \frac{\partial}{\partial x_j} + e_{jj} \lambda + [e_{jj}, F] \right] = 0
\]
for all $i \neq j$. The $n$-dimensional system (9.3) is the equation for the Levi-Civita connection of an Egoroff metric being flat. Here a metric $ds^2$ on $R^n$ is called an Egoroff metric if it is of the form
\[
ds^2 = \sum_{i=1}^{n} \phi_{x_i} dx_i^2
\]
for some smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$. Set
\[
f_{ij} = \begin{cases} 
\frac{\phi_{x_ix_j}}{2\sqrt{\phi_{x_i}\phi_{x_j}}}, & \text{if } i \neq j, \\
0, & \text{if } i = j.
\end{cases}
\]

It is easy to see that the Levi-Civita connection 1-form for $ds^2$ is
\[
w_{ij} = -f_{ij}(dx_i - dx_j).
\]
The metric $ds^2$ is flat (i.e., $dw = w \wedge w$) if and only if $F = (f_{ij})$ is a solution of system (9.3).

Since the $n$-dimensional system is the system consisting of $n$ commuting first flows, Bäcklund theory developed in section previous sections for the first flow generalizes easily to that of the $n$-dimensional systems (with minor changes).

Let $\mathcal{M}$ denote the space of solutions of the $n$-dimension system (9.1) associated to $u(n)$. Given $v \in \mathcal{M}$, the trivialization $E$ of $v$ is the solution of
\[
\begin{cases}
E^{-1}E_{x_j} = a_j\lambda + [a_j, v], & 1 \leq j \leq n \\
E(0, \lambda) = I.
\end{cases}
\]

Then $E(x, \tilde{\lambda})^*E(x, \lambda) = I$, i.e., $E(x) \in G_m^n$, where $E(x)(\lambda) = E(x, \lambda)$. So the action of $G_m^n$ leaves $\mathcal{M}$ invariant and the action of simple elements give Bäcklund transformations.

Let $\sigma(y) = (y^t)^{-1}$ be the involution of $U(n)$, and $\mathcal{M}^\sigma$ denote the space of solutions of the $n$-dimensional system (9.3) associated to $U(n)/O(n)$. Let $G_m^{n,\sigma}$ denote the subgroup of $g \in G_m^n$ such that $\sigma(g(-\lambda)) = g(\lambda)$. Since the trivialization $E$ of $v \in \mathcal{M}^\sigma$ satisfies the reality condition
\[
\sigma(E(x,t,-\lambda)) = E(x,t,\lambda),
\]
we have $E(x,t) \in G_m^{n,\sigma}$. So the action of $G_m^{n,\sigma}$ leaves $\mathcal{M}^\sigma$ invariant. Hence we obtain a Bäcklund theory for the system (9.3).

In the two theorems below, we write down the analogous Bäcklund transformations and Permutability formula for the $n$-dimensional system (9.1). Given $y \in gl(n)$, we will let $y_*$ denote $y$ with the diagonal entries replaced by zeros.

**9.5 Theorem.** The group $R^* \times G_m^n$ acts on the space $\mathcal{M}$ of solutions of the $n$-dimensional system (9.1) associated to $U(n)$, and the action $*$ is constructed in the same manner as on the spaces of solutions of the first flow. In fact, given $g_{z,\pi} \in G_m^n$ and $v \in \mathcal{M}$, the following initial value problem is solvable for $\tilde{\pi}$ and has a unique solution:
\[
\begin{cases}
(\tilde{\pi})_{x_j} + [a_jz + [a_j, v], \tilde{\pi}] = (\tilde{z} - z)[\tilde{\pi}, a_j]\tilde{\pi}, \\
\tilde{\pi}^* = \tilde{\pi}, \quad \tilde{\pi}^2 = \tilde{\pi}, \quad \tilde{\pi}(0) = \pi.
\end{cases}
\]
Moreover,

(i) $g_{z,\pi} v = v - (z - \bar{z})(\hat{\pi})_*$,

(ii) the trivialization of $g_{z,\pi} v$ is $g_{z,\pi} E g_{z,\bar{z}}^{-1}$, where $E$ is the trivialization of $v$

(iii) $\tilde{\pi}(x)$ is the projection onto the linear subspace $E(x, z)^*(V)$, where $V$ is the image of the projection $\pi$,

(iv) $(r * v)(x) = r^{-1} v(r^{-1} x)$ for $r \in \mathbb{R}^*$.

(v) if $U/K$ is the symmetric space defined by the involution $\sigma$, then the group $\mathbb{R}^* \times G_{m,\sigma}$ leaves the space $\mathcal{M}^*$ of solutions of the $n$-dimensional system (9.2) associated to $U/K$ invariant.

9.6 Theorem. Let $z_1, z_2 \in C \setminus R$ such that $z_1 \neq z_2$ and $z_1 \neq \bar{z}_2$, and $\pi_1, \pi_2$ projections of $C^n$. Let $v_0 \in \mathcal{M}$, and $v_i = g_{z_i,\pi_i} v_0 = v_0 + (z_i - \bar{z}_i)(\hat{\pi}_i)_*$ for $i = 1, 2$ as given in Theorem 9.5. Set

$$\phi = (z_2 - z_1)I + (z_1 - \bar{z}_1)\pi_1 - (z_2 - \bar{z}_2)\pi_2,$$

$$\tilde{\phi} = (z_2 - z_1)I + (z_1 - \bar{z}_1)\tilde{\pi}_1 - (z_2 - \bar{z}_2)\tilde{\pi}_2,$$

$$\tau_i = \phi \pi_i \phi^{-1},$$

$$\tilde{\tau}_i = \tilde{\phi} \tilde{\pi}_i \tilde{\phi}^{-1}.$$

Then $g_{z_2, \tau_2} g_{z_1, \pi_1} = g_{z_1, \tau_1} g_{z_2, \pi_2}$, and

$$v_3 = (g_{z_2, \tau_2} g_{z_1, \pi_1}) * v_0 = v_0 + (z_1 - \bar{z}_1)(\hat{\pi}_1)_* + (z_2 - \bar{z}_2)(\tilde{\pi}_2)_*$$

$$= (g_{z_1, \tau_1} g_{z_2, \pi_2}) * v_0 = v_0 + (z_1 - \bar{z}_1)(\tilde{\tau}_1)_* + (z_2 - \bar{z}_2)(\tilde{\pi}_2)_*. \quad (9.4)$$

10. Time periodic solutions

In this section, we use the action of $G_m^*$ to obtain many solutions of the $j$-th flow that are periodic in time. This is an algebraic calculation, which shows that when the poles are properly placed, the solutions are periodic in time. Multi-solitons will be time periodic if the periods of the component solitons are rationally related. We also show that the classical breather solution of the sine-Gordon equation is obtained from the action of a suitable quadratic element at the vacuum.

10.1 Theorem. Let $j > 1$ be an integer, $a = \text{diag}(ia_1, \ldots, ia_n)$, and $b = \text{diag}(ib_1, \ldots, ib_n)$. If $b_1, \ldots, b_n$ are rational numbers. Then the $j$-th flow equation on $\mathcal{S}(R, U^a_n)$ defined by $b$ has infinitely many $m$-soliton solutions that are periodic in $t$. 

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PROOF. First assume \( j \geq 3 \). Let \( z = \rho e^{2\pi i j} \), \( U \) a constant \( n \times k \) complex matrix of rank \( k \), and \( \pi \) the projection of \( C^n \) onto the linear subspace spanned by the \( k \) columns of \( U \). Then \( z^j = \rho^j \) and formula (5.6) implies that

\[
g_{z,\pi} * 0 = (z - \bar{z}) \left[ e^{-a\bar{z}x - b\rho^j t} U(U^* e^{a(z-\bar{z})x}U)^{-1} U^* e^{a\bar{z}x + b\rho^j t}, a \right].
\]

Since \( b_1, \ldots, b_n \) are rational numbers, there exist \( \tau > 0 \) and integers \( m_r \) such that

\[
\tau = \frac{2\pi m_r}{b_r \rho^j}
\]

for all \( 1 \leq r \leq n \). So \( g_{z,\pi} * 0 \) is periodic in \( t \) with period \( \tau \).

Let \( \rho_1, \ldots, \rho_m \) be distinct rational numbers, and \( z_k = \rho_k e^{2\pi i j} \). Let \( \tau_i \) denote the periods for \( g_{z_k,\pi_i} \). Then it follows from the details in the proofs of Theorems 5.12 and 6.5 that the \( m \)-soliton

\[
(g_{z_1,\pi_1} \cdots g_{z_m,\pi_m}) * 0
\]

is an algebraic function of \( g_{z_1,\pi_1} * 0, \ldots, g_{z_m,\pi_m} * 0 \). Since \( \rho_1, \ldots, \rho_m \) are rational numbers, there exist \( T > 0 \) and integers \( k_i \) such that \( T = k_i \tau_i \). So the \( m \)-soliton is periodic in time with period \( T \).

For \( j = 2 \), let \( z = is \) with \( s \in \mathbb{R} \). Then the 1-soliton

\[
g_{z,\pi} * 0 = 2is \left[ e^{iasx + 2bt} v (v^* e^{2iasx} v)^{-1} v^* e^{iasx - b^2t}, a \right]
\]

is periodic in \( t \). The existence of time-periodic \( m \)-solitons of the second flow can be proved the same way as for the \( j \)-th flow.

10.2 Example. Let \( s \in \mathbb{R} \), and \( c = (c_1, \ldots, c_{n-1})^t \in C^{n-1} \). Recall that the second flow in the \( u(n) \)-hierarchy defined by \( a = b = \text{diag}(i, -i, \ldots, -i) \) (Example 3.5 (3)) is the matrix non-linear Schrödinger equation for:

\[
q_t = -\frac{i}{2}(q_{xx} + 2qq^* q), \quad q : R^2 \to C^{n-1}.
\]

Let \( \pi \) be the Hermitian projection of \( C^n \) onto the complex line spanned by \( (1, c)^t \), where \( c = (c_1, \ldots, c_{n-1}) \). The 1-soliton solution computed in Example 5.16 is

\[
q = g_{is,\pi} * 0 = \frac{4se^{2isx}e^{-2is^2t}}{|c| e^{2sx} + e^{-2sx}}.
\]

This solution is periodic in \( t \).

The same algebra works for the \( -1 \) flow (3.3). Rewrite the \( -1 \) flow in terms of \( g \):

\[
(g^{-1}g_x)_t = [a, g^{-1}ag].
\]
Note that $(x, t)$ are characteristic coordinates. Let
\[ X = x - t, \quad T = x + t \]
be the space-time coordinates. Then (10.1) in \((X, T)\)-coordinate is
\[
(g^{-1} g_T)T - (g^{-1} g_X)X + [g^{-1} g_X, g^{-1} g_T] = [a, g^{-1} ag]. \tag{10.2}
\]
We will obtain solutions periodic in physical time (or space). The trivialization of the vacuum solution for the $-1$-flow (3.3) on $S(R, U_a^\perp)$ defined by $a = \text{diag}(i, \cdots, i, -i, \cdots, -i)$ is
\[
E(\lambda, x, t) = \exp(a(\lambda x + \lambda^{-1}t)).
\]
By formula (5.6), the $1$-soliton $g e^{i \theta_{\pi}} 0$ for the $-1$-flow is a function of
\[
\exp(i \cos \theta (x + t) - \sin \theta (x - t)) = \exp(i \cos \theta X - \sin \theta T).
\]
This proves

**10.3 Theorem.** If $z = e^{i \theta}$ and $a = \text{diag}(i, \cdots, i, -i, \cdots, -i)$, then the $1$-soliton $g_z, \pi * 0$ for the $-1$-flow (10.2) is periodic in time $T$ with period $\frac{2\pi}{\cos \theta}$. A multiple soliton generated by a rational loop with poles at $z_1 = e^{i \theta_1}, \cdots, z_r = e^{i \theta_r}$ will be periodic with period $\tau$ if there exists integers $k_1, \cdots, k_r$ such that
\[
\tau = \frac{2\pi k_j}{\cos \theta_j} \quad \forall 1 \leq j \leq r.
\]

The multi-solitons above satisfy the sine-Gordon equation if the rational loop satisfies $(f(-\lambda)^t)^{-1} = f(\lambda)$, or equivalently $\overline{f(\lambda)} = f(\lambda)$. Now use Theorem 10.3 and Proposition 5.14, with
\[
z_1 = e^{i \theta_1}, \quad z_2 = -e^{-i \theta_1}, \ldots, z_{2k-1} = e^{i \theta_k}, \quad z_{2k} = -e^{-i \theta_k},
\]
to get a $2k$-soliton for the sine-Gordon equation that is periodic in time $T$. To summarize, we have

**10.4 Corollary.** Multiple-breather solutions exists for the sine-Gordon equation.

**10.5 Example.** If $\pi$ is a real symmetric projection (i.e., $\pi^2 = \pi$, $\pi^* = \pi$ and $\overline{\pi} = \pi$), then
\[
(g e^{i \theta_{\pi}} g_{-e^{-i \theta_{\pi}}} ) * 0 = 4 \tan^{-1} \left( \frac{\sin \theta \sin((x + t) \cos \theta)}{\cos \theta \cosh((x - t) \sin \theta)} \right).
\]
This is the classical breather solution for the sine-Gordon equation. Theorem 7.4 gives $m$-breather solutions explicitly.
11. The \( u(k, n-k) \)-hierarchies

We are able to obtain global Bäcklund transformations for flows in the \( su(n) \)-
hierarchy and the group structure of these Bäcklund transformations because the
following three results:

(i) simple elements generate the rational group \( G^m_\pm \),

(ii) we can always do the Birkhoff factorization,

(iii) solutions to the ODE Bäcklund transformations in \( x \)-coordinate lie in the
Schwartz class.

Example 4.6 shows that all three results fail to be true for the \( sl(n, C) \)- and
\( sl(n, R) \)-hierarchies. We will see that (i)-(iii) again fails for other \( U \)-hierarchies
when \( U \) is the Lie algebra of some non-compact group. We give explicit examples
for the \( u(1,1) \)-hierarchy to explain this phenomenon. However, our computation
in fact works for any real semi-simple Lie algebra.

Let \( J = \text{diag}(\epsilon_1, \cdots, \epsilon_n) \) with \( \epsilon_i = 1 \) for \( 1 \leq i \leq k \) and \( \epsilon_j = -1 \) if \( k < j \leq n \),
and let
\[
\langle v_1, v_2 \rangle_J = v_1^* J v_2
\]
denote the Hermitian bilinear form on \( C^n \) defined by \( J \). Let \( U(k, n-k) \) denote
the group of linear maps of \( C^n \) that preserve \( \langle , \rangle_J \), and \( u(k, n-k) \) its Lie algebra.
Given a linear map \( A : C^n \rightarrow C^n \), let \( A^* J \) denote the adjoint of \( A \), i.e., \( A^* J \) is
defined so that
\[
\langle A(v_1), v_2 \rangle_J = \langle v_1, A^* J (v_2) \rangle_J
\]
for all \( v_1, v_2 \in C^n \). A direct computation shows that
\[
A^* J = J^{-1} A^* J.
\]
A projection \( \pi \) of \( C^n \) is called a \( J \)-projection if \( \pi^* J = \pi \).

It is easy to check that if \( \pi \) is a \( J \)-projection then the simple element \( h_{\bar{z}, z, \pi} \)
satisfies the \( U(k, n-k) \)-reality condition:
\[
f(\bar{\lambda})^{-1} = J^{-1} f(\bar{\lambda})^* J = f(\bar{\lambda})^* J.
\]
Given a global solution \( u \) of the \( j \)-th flow in the \( u(k, n-k) \)-hierarchy, will the
\( U(k, n-k) \)-reality condition prevent the new solution having singularities? Let
\( f = h_{\bar{z}, z, \pi} \). A direct computation gives
\[
\langle f(\bar{z})^{-1}(V_1), f(z)^{-1}(V_2) \rangle_J = \langle f(z)^* J (V_1), f(z)^{-1}(V_2) \rangle_J = \langle V_1, V_2 \rangle_J = 0,
\]
where \( V_1 \) and \( V_2 \) are image of \( \pi \) and \( I - \pi \) respectively. This implies that all
vectors in \( (f(\bar{z})^{-1}(V_1)) \cap (f(z)^{-1}(V_2)) \) are null vectors with respect to \( \langle , \rangle_J \).
Since there are non-zero null vectors with respect to \( <, >_J \), we cannot always
able to do the factorizations in the \( U(k, n-k) \) case. Hence the corresponding
Bäcklund transformations may produce singular solutions. In fact, the following
example shows this does happen.
11.1 Example. Apply Bäcklund transformation to the vacuum solution of the defocusing non-linear Schrödinger equation (3.2) (Example 3.5 (4)) to get solutions

\[ \tilde{u}(x, t) = \frac{-2i(z - \bar{z})}{((c + 1)e^{-\xi(x,t)} - (c - 1)e^{\xi(x,t)})} \begin{pmatrix} 0 & i be^{-\eta(x, t)} \\ -i be^{\eta(x, t)} & 0 \end{pmatrix}, \]

where \( \xi(x, t) = i((z - \bar{z})x + (z^j - \bar{z}^j)t) \) is real and \( \eta(x, t) = i((z + \bar{z})x + (z^j + \bar{z}^j)t) \) is pure imaginary (\( b \in \mathbb{C}, c \) is real, and \( c^2 - 1 = |b|^2 \)). Note that \( \tilde{u} \) blows up at \((x_0, t_0)\) when \( e^{2\xi(x_0, t_0)} = \frac{c + 1}{c - 1} \).

We have seen that the space of pure soliton solutions of the \( j \)-th flow in the \( su(n) \)-hierarchy is the orbit of the group \( G^m \) through the vacuum (recall that \( G^m \) is the group of all rational maps \( g : C \to GL(n, \mathbb{C}) \) satisfying the \( SU(n) \)-reality condition \( g(\lambda)^*g(\lambda) = I \) and \( g(\infty) = I \)). However, Example 11.1 shows that the structure of the space of pure solitons of the \( j \)-th flow in the \( U \)-hierarchy is not clearly understood if \( U \) is the Lie algebra of a non-compact Lie group.

12. The KdV hierarchy

There is a formulation of KdV as a restriction of the third flow. The odd flows in the \( sl(2, \mathbb{R}) \)-hierarchy for \( q, r : \mathbb{R} \to \mathbb{R} \) leaves the submanifold defined by \( r = 1 \) invariant. The KdV equation is the third flow:

\[ q_t = \frac{1}{4}(q_{xxx} - 6qq_x), \] (12.1)

and its Lax pair is

\[ \left[ \frac{\partial}{\partial x} + a \lambda + u, \frac{\partial}{\partial t} + a \lambda^3 + u \lambda^2 + Q_2 \lambda + Q_3 \right] = 0, \]

where

\[ a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}, \]

\[ Q_2 = \begin{pmatrix} -\frac{q^2}{2} & -\frac{q^2}{2} \\ 0 & \frac{q^2}{2} \end{pmatrix}, \quad Q_3 = \begin{pmatrix} \frac{q^2 - 2q^2}{4} & q^2 - 2q^2 \\ -\frac{q^2}{2} & -\frac{q^2}{4} \end{pmatrix}. \]

This Lax pair satisfies the \( sl(2, \mathbb{R}) \)-reality condition. But there is a second reality condition that gives the restriction \( r = 1 \). To see this, let

\[ \phi(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}. \]
Then the Lax pair of KdV satisfies:

\[
\begin{aligned}
A(\bar{\lambda}) &= A(\lambda), \\
\phi(\lambda)^{-1} A(\lambda) \phi(\lambda) &= \phi(-\lambda)^{-1} A(-\lambda) \phi(-\lambda).
\end{aligned}
\] (12.2)

We will call this the KdV reality condition. It is useful to realize that the second condition is another way of saying \( \phi^{-1} A \phi \) is even in \( \lambda \). So the trivialization of a solution of the KdV equation normalized at \( (0, 0) \) satisfies the same reality conditions (12.2), i.e.,

\[
\begin{aligned}
E(x, t, \bar{\lambda}) &= E(x, t, \lambda), \\
\phi(\lambda)^{-1} E(x, t, \lambda) \phi(\lambda) &= \phi(-\lambda)^{-1} E(x, t, -\lambda) \phi(-\lambda).
\end{aligned}
\]

In this section, we prove that Bäcklund transformations of the KdV equation can be obtained in a similar way as before by factoring the product of a degree one rational map and the trivialization of a solution in the opposite order in the loop group. Since the factorization in the loop group of \( SL(2, R) \) can not always be carried out as we have seen in the previous section, the same phenomenon is expected for the KdV equation.

We will show that the Lax pair of all the odd flows in the \( sl(2, R) \)-hierarchy with \( r = 1 \) satisfies the reality conditions (12.2). First, we obtain the following lemma by a direct computation.

**12.1 Lemma.** \( A(\lambda) = a\lambda + \left( \begin{array}{cc} \xi & q \\ r & \eta \end{array} \right) \) satisfies the KdV-reality conditions (12.2) if and only if \( r = 1, q \in R \) and \( \xi = \eta \in R \), i.e., \( A(\lambda) = a\lambda + \left( \begin{array}{cc} \xi & q \\ 1 & \xi \end{array} \right) \).

**12.2 Proposition.** The Lax pairs of the odd flows in the \( sl(2, R) \)-hierarchy for \( u = \left( \begin{array}{cc} 0 & q \\ 1 & 0 \end{array} \right) \) satisfies the KdV-reality conditions (12.2).

**Proof.** Note that the formal power series

\[
Q(\lambda) \sim a + Q_1 \lambda^{-1} + Q_2 \lambda^{-2} + \cdots
\]

satisfies the equation

\[ [d + a\lambda + u, Q(\lambda)] \sim 0. \] (12.3)

Since \( A(\lambda) = a\lambda + u \) satisfies the reality conditions (12.2), we have

\[
\begin{aligned}
\phi(\lambda)^{-1} [d_x + A(\lambda), Q(\lambda)] \phi(\lambda) &= 0 \\
&= [\phi(\lambda)^{-1} (d_x + A(\lambda)) \phi(\lambda), \phi(\lambda)^{-1} Q(\lambda) \phi(\lambda)] \\
&= [d_x + \phi(\lambda)^{-1} A(\lambda) \phi(\lambda), \phi(\lambda)^{-1} Q(\lambda) \phi(\lambda)] \\
&= [d_x + \phi(-\lambda)^{-1} A(-\lambda) \phi(-\lambda), \phi(\lambda)^{-1} Q(\lambda) \phi(\lambda)] \\
&= \phi(-\lambda)^{-1} [d_x + A(-\lambda), \phi(-\lambda) \phi(\lambda)^{-1} Q(\lambda) \phi(\lambda) \phi(-\lambda)^{-1}] \phi(-\lambda).
\end{aligned}
\]
So \(d_x + A(-\lambda), \phi(-\lambda)\phi(\lambda)^{-1}Q(\lambda)\phi(\lambda)\phi(-\lambda)^{-1}\) = 0. It follows from a direct computation that the first two terms of the asymptotic expansion of
\[
\phi(-\lambda)\phi(\lambda)^{-1}Q(\lambda)\phi(\lambda)\phi(-\lambda)^{-1}
\]
is \((-a + \lambda^{-1}u)\). So uniqueness of \(Q\) (Proposition 2.6) implies that
\[
\phi(-\lambda)\phi(\lambda)^{-1}Q(\lambda)\phi(\lambda)\phi(-\lambda)^{-1} = -Q(-\lambda).
\]
In particular, \(\lambda^{2j+1}\phi(-\lambda)\phi(\lambda)^{-1}Q(\lambda)\phi(\lambda)\phi(-\lambda)^{-1}\) is even, i.e., a power series in \(\lambda^2\). This implies that
\[
a\lambda^{2j+1} + u\lambda^{2j} + Q_2\lambda^{2j-1} + \cdots + Q_{2j+1}
\]
satisfies the KdV reality condition (12.2).

There are no linear fractional transformations \(g : C \to GL(2, C)\) satisfying the KdV-reality conditions (12.2). But Corollary 5.11 tells us that rational loops with values in the center of \(GL(n, C)\) act trivially on the space of solutions of the \(j\)-th flow. So the group \(G_{-KdV}\) for constructing Bäcklund transformations of the KdV equation is given as follows:

12.3 Definition. Let \(G_{-KdV}^\infty\) be the group of rational maps \(g : C \to GL(2, C)\) such that \(g(\infty) = I\) and \(g\) satisfies the KdV-reality condition up to center elements. In other words, \(G_{-KdV}^\infty\) is the group of rational maps \(g : C \to GL(2, C)\) such that
(i) \(g(\infty) = I\), and
(ii) there exists some rational function \(f : C \to C\) such that \(fg\) satisfies the KdV-reality condition.

By Corollary 5.11, if \(a\lambda + u\) satisfies the KdV-reality condition and \(g \in G_{-KdV}^\infty\) then both the trivialization \(E\) of \(a\lambda + u\) and \(g\#E\) satisfy the KdV-reality condition.

We need to find the simplest kind of elements in \(G_{-KdV}^\infty\). It follows from Lemma 12.1 that given any \(\xi, k \in R\)
\[
p_{\xi,k}(\lambda) = a\lambda + \left(\begin{array}{c} \xi \\ \frac{\xi^2 - k^2}{1} \end{array}\right) = \left(\begin{array}{cc} \lambda + \xi & \xi^2 - k^2 \\ 1 & -\lambda + \xi \end{array}\right)
\]
satisfies the KdV-reality condition (12.2). (Recall that \(a = \text{diag}(1, -1)\)). So
\[
g(\lambda) = \frac{p_{\xi,k}(\lambda)}{\lambda - k} \in G_{-KdV}^\infty.
\]
The inverse of \(p_{\xi,k}\) is
\[
p_{\xi,k}^{-1}(\lambda) = \frac{p_{-\xi,k}(\lambda)}{\lambda^2 - k^2}.
\]
We call \(\lambda_0\) a zero of \(p_{\xi,k}\) if \(\det(p_{\xi,k}(\lambda_0)) = 0\). The proof of the next Proposition is a direct computation.
12.4 Proposition.
(i) \( k, -k \) are the only zeros of \( p_{\xi,k} \),
(ii) \( p_{\xi,k}(\pm k)(v_{\pm}) = 0 \), where \( v_+ = \begin{pmatrix} k - \xi \\ 1 \end{pmatrix} \) and \( v_- = \begin{pmatrix} -(k + \xi) \\ 1 \end{pmatrix} \),
(iii) \( \text{Im}(p_{\xi,k}(-k)) \) is spanned by \( av_- \) and \( \text{Im}(p_{\xi,k}(k)) \) is spanned by \( av_+ \).

As a consequence we get

12.5 Corollary. Let \( B = \begin{pmatrix} k - \xi & -(k + \xi) \\ 1 & 1 \end{pmatrix} \), and \( Y = -kaBaB^{-1} \). Then \( Y = \begin{pmatrix} \xi & \xi^2 - k^2 \\ 1 & \xi \end{pmatrix} \) and \( a\lambda + Y = p_{\xi,k}(\lambda) \).

We use a method similar to that of the \( su(n) \) and \( sl(n,R) \) hierarchies to construct the Bäcklund transformation for the KdV equation corresponding to \( p_{\xi,k}(\lambda)/(\lambda - k) \). We give an outline here. Let \( q \) be a solution of the KdV, and \( E \) its trivialization at \((0,0)\). First, we take as an Ansatz, that there exists a map \( \tilde{\xi}(x,t) \) such that

\[
\tilde{E}(x,t,\lambda) = p_{\xi,k}(\lambda)E(x,t,\lambda)p_{\xi(x,t),k}^{-1}(\lambda) = \frac{p_{\xi,k}(\lambda)E(x,t,\lambda)p_{-\tilde{\xi}(x,t),k}(\lambda)}{\lambda^2 - k^2} \tag{12.5}
\]

is holomorphic in \( \lambda \in C \). In other word, our Ansatz is that we can factor \( \frac{p_{\xi,k}}{\lambda - k} E(x,t) \) as

\[
\frac{p_{\xi,k}}{\lambda - k} E(x,t) = \tilde{E}(x,t) \frac{p_{\xi(x,t),k}(\lambda)}{\lambda - k} \in G_+ \times G_{KdV}^{-},
\]

where \( G_+ \) is the group of holomorphic maps from \( C \) to \( GL(2,\mathbb{C}) \). Since \( E(x,t) \), \( p_{\xi,k} \) and \( p_{-\tilde{\xi}(x,t),k} \) satisfy the KdV- reality condition (12.2), so is \( \tilde{E}(x,t) \). Since the residues of \( \tilde{E}(x,t,\lambda) \) at \( \lambda = k \) is zero and

\[
\tilde{E}(x,t,\lambda) = \frac{p_{\xi,k}(\lambda)E(x,t,\lambda)p_{-\tilde{\xi}(x,t),k}(\lambda)}{\lambda^2 - k^2},
\]

we get

\[
p_{\xi,k}(k)E(x,t,k)p_{-\tilde{\xi}(x,t),k}(k) = 0.
\]

By Proposition 12.4, we can choose \( \tilde{v}_-(x,t) \) such that \( E(x,t,k)a\tilde{v}_-(x,t) \) is proportional to \( v_+ \). A similar calculation as for the \( sl(n) \)-hierarchy gives the well-known Darboux and Bäcklund transformations for KdV.
12.6 Theorem. Let $q$ be a solution of the KdV equation, and $E$ the trivialization of $q$ normalized at $(x, t) = (0, 0)$. Given $\xi, k \in \mathbb{R}$ with $k \neq 0$, set

$$
\begin{pmatrix}
    f_1(x, t) \\
    f_2(x, t)
\end{pmatrix} = \left( E(x, t, k) \right)^{-1} \begin{pmatrix} k - \xi \\ 1 \end{pmatrix},
$$

$$
\tilde{\xi}(x, t) = k - \frac{f_1(x, t)}{f_2(x, t)},
$$

$$
\tilde{q}(x, t) = -q + 2(\tilde{\xi}^2(x, t) - k^2),
$$

$$
\tilde{E}(x, t, \lambda) = \frac{p_{\xi, k}(\lambda) E(x, t, \lambda) p_{-\tilde{\xi}(x, t), k}(\lambda)}{\lambda^2 - k^2}.
$$

If $f_2$ does not vanish in $\mathcal{O} \subset \mathbb{R}^2$, then $\tilde{q}$ is a solution of the KdV equation defined on $\mathcal{O}$ and $\tilde{E}(x, t, \lambda)$ is the trivialization of $\tilde{q}$.

12.7 Definition. Let $p_{\xi, k} * q$ denote the new solution $\tilde{q}$ obtained in Theorem 12.6.

Using the same method as in previous sections, we compute $\tilde{E}^{-1} \tilde{E}_x$ and $\tilde{E}^{-1} \tilde{E}_t$ to get the usual ordinary differential equations for Bäcklund transformations for KdV:

12.8 Theorem. Let $k \in \mathbb{R}$ be a constant. Then the following first order system for $A : \mathbb{R}^2 \rightarrow \mathbb{R}$ is compatible if and only if $q$ is a solution of KdV:

$$
\begin{cases}
    A_x = q - A^2 + k^2, \\
    A_t = \frac{q_{xx} - 2q^2}{4} - \frac{q_x A}{2} + \frac{q(A^2 + k^2)}{2} - k^2(A^2 - k^2), \\
    A(0, 0) = \xi_0.
\end{cases}
$$

Moreover, if $A$ is a solution of the above system, then the new solution is

$$
\tilde{q} = p_{\xi_0, k} * q = -q + 2(A^2 - k^2).
$$

12.9 Remark. Write $E(x, t, \lambda) = \begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}$. It follows from Theorem 12.6 and a direct computation that

$$
\tilde{\xi} = \frac{f'}{f} \quad \text{and} \quad \tilde{q} = -q + 2\frac{(f')^2}{f^2} - 2k^2 = f(f^{-1})'' - k^2,
$$

where $f = y_1 - (k - \tilde{\xi})z_1$. Since $y_1, z_1$ are solutions of $y'' = qy + k^2y$, so is $f$. In other words, Theorem 12.6 in $x$-variable is the classical Darboux transformation for the Hills operator $\frac{d^2}{dx^2} - q - k^2 = 0$ and the transformation maps a solution $q$ of KdV to a new solution $\tilde{q}$ is the Darboux transformation (cf. [AM], [De]). Our result gives an interpretation of Darboux transformations in terms of the Birkhoff factorization theorem.
12.10 Example. We compute $p_{\xi,k} \ast 0$ for the KdV equation. A direct computation shows that the trivialization of the vacuum solution $q = 0$ of KdV normalized at $(0,0)$ is

$$E_0(x,t,\lambda) = \left( \frac{e^{(\lambda x + \lambda^3 t)}}{\sinh(\lambda x + \lambda^3 t)} e^{-(\lambda x + \lambda^3 t)} \right).$$

Use Theorem 12.6 and a direct computation to get $\tilde{q} = p_{\xi,k} \ast 0 = 2(k^2 - \tilde{\xi}^2)$, where

$$\tilde{\xi}(x,t) = \begin{cases} k \tanh(kx + k^3 t + x_0), & \text{if } b > 0, \\ k \coth(kx + k^3 t + x_0), & \text{if } b < 0, \end{cases}$$

$b = k^2 - \xi^2$ and $x_0 = \frac{1}{2} \ln\left(\frac{|\xi + k|}{|\xi - k|}\right)$. So

$$\tilde{q}(x,t) = \begin{cases} -2k^2 \text{sech}^2(kx + k^3 t + x_0), & \text{if } b > 0, \\ -2k^2 \text{csch}^2(kx + k^3 t + x_0), & \text{if } b < 0. \end{cases}$$

Note that $\tilde{q}$ is a 1-soliton solution if $b > 0$, and $\tilde{q}$ blows up on the line $kx + k^3 + x_0 = 0$ if $b < 0$.

12.11 Remark. The method we discussed above still works when we choose $k = 0$. In fact, set the coefficients of $\lambda$ and $\lambda^2$ in the expansion of

$$p_{\xi,0}(\lambda) E_0(x,t,\lambda) p_{-\tilde{\xi}(x,t),0}$$

in $\lambda$ equal to zero to get $\tilde{\xi}(x,t) = \frac{\xi}{1+\xi x}$ and a rational solution of KdV:

$$\tilde{q}(x,t) = 2\tilde{\xi}^2 = \frac{2\xi^2}{(1+\xi x)^2}.$$ 

Use a computation similar to that of Proposition 4.8 to get:

12.12 Proposition. Given $a_1, a_2, k_1, k_2 \in \mathbb{R}$, if $a_1 - a_2 \neq 0$ then there exist uniquely $\xi_1, \xi_2$ such that

$$p_{\xi_2,k_2} p_{a_1,k_1} = p_{\xi_1,k_1} p_{a_2,k_2}.$$ 

Moreover, $\xi_1 = -a_2 + \frac{k_1^2 - k_2^2}{a_1 - a_2}$ and $\xi_2 = -a_1 + \frac{k_1^2 - k_2^2}{a_1 - a_2}$.

As a consequence, we get the Permutability Formula for the KdV equation:

12.13 Corollary. Suppose $q_0$ is a solution of KdV, and

$$q_i = p_{a_i,k_i} \ast 0 = -q_0 + 2(\xi_i^2 - k_i^2)$$

for $i = 1, 2$. Set $\xi_{12} = -\xi_1 + \frac{k_1^2 - k_2^2}{\xi_1 - \xi_2}$. Then

$$q_{12} = -q_1 + 2(\xi_{12}^2 - k_2^2) = q_0 - 2(\xi_1^2 - k_1^2) + 2(\xi_{12}^2 - k_2^2)$$

is again a solution of KdV.
12.14 Example. Assume $0 < k_1^2 < k_2^2 < \cdots < k_n^2$. Set $b_i = k_i^2 - a_i^2$, and

\[ q_i = p_{a_i, k_i} \ast 0 = 2(\xi_i^2 - k_i^2) \]

for $1 \leq i \leq n$. Given a permutation $i$ of $1, \cdots, n$, define $\xi_{i_1 i_2 \cdots i_n}$ and $q_{i_1 i_2 \cdots i_n}$ by induction:

\[ \xi_{i_1 i_2 \cdots i_r j m} = -\xi_{i_1 i_2 \cdots i_r j} + \frac{k_j^2 - k_m^2}{\xi_{i_1 i_2 \cdots i_r j} - \xi_{i_1 i_2 \cdots i_r m}}, \]

\[ q_{i_1 i_2 \cdots i_r+1} = -q_{i_1 i_2 \cdots i_r} + 2(\xi_{i_1 i_2 \cdots i_r+1}^2 - k_{i_r+1}^2). \]

Use Corollary 12.13 repeatedly, we conclude that $q_{12 \cdots j}$ is a local solution of the KdV for all $1 \leq j \leq n$. In general, these solutions may have singularities. We suspect that if $b_1 > 0$ and $b_i b_{i+1} < 0$ for $1 \leq i \leq n - 1$, then $q_{12 \cdots n}$ is smooth global solution. We can prove this for $n \leq 5$, but do not have a proof for general $n$. It seems fairly clear that these conditions are necessary for the solutions to be non-singular.

13. The Kupershmidt-Wilson hierarchy

The $sl(2, C)$-hierarchy (2.6) leaves the submanifold $q = r$ invariant, and the third flow is the complex modified KdV equation:

\[ q_t = \frac{1}{4}(q_{xxx} - 6q^2 q). \tag{13.1} \]

On this submanifold, the Lax pairs satisfy the following reality condition

\[ \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)^{-1} A(-\lambda) \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) = A(\lambda). \]

Kuperschmidt and Wilson [KW] found a natural $n \times n$ generalization of this restricted hierarchy (KW-hierarchy). First we describe the required reality condition. Let $e_{ij}$ denote the matrix with zero on all entries except the $ij$-th entry is equal to 1, and $\tau \in GL(n)$ the matrix representing the cyclic permutation $(12 \cdots n)$, i.e.,

\[ \tau = e_{21} + e_{32} + \cdots + e_{n,n-1} + e_{1n}. \]

Or equivalently, $\tau(e_i) = e_{i+1}$ for $1 \leq i \leq n$ (here we use the convention that $e_i = e_j$ if $i \equiv j \mod n$).

The reality condition for the $n \times n$ KW-hierarchy is

\[ \tau^{-1} A(\alpha^{-1} \lambda) \tau = A(\lambda), \quad \text{where} \quad \alpha = e^{2\pi i}, \tag{13.2} \]

which is called the KW-reality condition. Since $\tau^n = I$, the order of the automorphism $\text{Ad}(\tau^{-1})$ on $gl(n, C)$ is $n$. Let $G_k$ denote the eigenspace of $\text{Ad}(\tau^{-1})$
corresponding to eigenvalue $\alpha^k$ for $k = 0, 1, \ldots, n - 1$, i.e., $y \in G_k$ if and only if $\tau^{-1}y\tau = \alpha^k y$. Or equivalently, $y = (y_{ij}) \in G_k$ if and only if $y_{i+1,j+1} = \alpha^k y_{ij}$ for all $1 \leq i, j \leq n$. Then

$$gl(n, C) = G_0 + \cdots + G_{n-1}.$$ 

For example, for $n = 3$ we have

$$G_0 = \left\{ \begin{pmatrix} c_1 & c_2 & c_3 \\ c_3 & c_1 & c_2 \\ c_2 & c_3 & c_1 \end{pmatrix} \mid c_i \in C \right\}, \quad G_1 = \left\{ \begin{pmatrix} c_1 & c_2 & c_3 \\ \alpha c_3 & \alpha c_1 & \alpha c_2 \\ \alpha^2 c_2 & \alpha^2 c_3 & \alpha^2 c_1 \end{pmatrix} \mid c_i \in C \right\}.$$ 

$$G_2 = \left\{ \begin{pmatrix} c_1 & c_2 & c_3 \\ \alpha^2 c_3 & \alpha^2 c_1 & \alpha^2 c_2 \\ \alpha c_2 & \alpha c_3 & \alpha c_1 \end{pmatrix} \mid c_i \in C \right\}.$$ 

Because $\text{Ad}(\tau^{-1})$ is a Lie algebra homomorphism, we have

$$[G_i, G_j] \subset G_{i+j}.$$ 

Here $G_i = G_k$ if $i \equiv k \mod n$. A direct computation shows that $A(\lambda) = \sum_{k \leq n_0} u_k \lambda^k$ satisfies the KW-reality condition (13.2) if and only if $u_k \in G_k$ for all $k$.

**13.1 Proposition.** Let $\alpha = e^{2\pi i/n}$, and $a = \text{diag}(1, \alpha, \alpha^2, \ldots, \alpha^{n-1})$. Then the $nk + 1$-th flow in the $sl(n, C)$-hierarchy leaves $S(R, G_0 \cap sl(n)_{a^1})$ invariant, and its Lax pair satisfies the KW-reality condition (13.2).

**PROOF.** Use a proof similar to that of Proposition 12.2 to conclude that $\tau^{-1}Q(\alpha^{-1} \lambda)\tau = \alpha Q(\lambda)$. Hence

$$Q_{a,j}(u) \in G_{1-j}.$$ 

Since $a \in G_1$ and $[G_i, G_1] \subset G_{i+1}$, we obtain $[Q_{kn+2}(u), a] \subset G_{-kn} = G_0$. 

**13.2 Definition.** The KW-equation is the restriction of the $(n + 1)$-th flow in the $sl(n, C)$-hierarchy to $S(R, G_0 \cap sl(n)_{a^1})$

$$u_t = (Q_{a,n+1}(u))_x + [u, Q_{a,n+1}(u)], \quad u : R^2 \rightarrow G_0 \cap sl(n)_{a^1}, \quad (13.3)$$

and the KW-hierarchy consists of restricted 1-st flow, $(n + 1)$-th, $(2n + 1)$-th, $\cdots$ flows in the $sl(n, C)$-hierarchy.

When $n = 2$, $S(R, G_{a^1} \cap G_0)$ is the space of Schwartz class maps from $R$ to $sl(2, C)$ of the form $\begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}$ and the third flow is the complex modified KdV
equation (13.1). For \( n = 3 \), \( S(R, G_0 \cap sl(n)_{a}^\perp) \) is the space of Schwartz class maps from \( R \) to \( sl(3, \mathbb{C}) \) of the form

\[
\begin{pmatrix}
0 & q_2 & q_3 \\
q_3 & 0 & q_2 \\
q_2 & q_3 & 0
\end{pmatrix}.
\]

The fourth flow is the KW-equation, which is of the form

\[
(q_2)_t = P_2(q_2, q_3), \quad (q_3)_t = P_3(q_2, q_3),
\]

where \( P_2, P_3 \) are 4-th order polynomial differential operators. The explicit formulas for \( P_2 \) and \( P_3 \) are long, but they are not difficult to compute (use the method described in the proof of Theorem 2.5). So we will not present them here.

Since the Lax pair of the KW-equation satisfies the KW-reality condition (13.2), the trivialization of a solution of the KW-equation also satisfies the same reality condition. Next we outline our strategy for finding Bäcklund transformations for this equation. We have seen that rational loops with values in the center of \( GL(n, \mathbb{C}) \) do not play any effective role in the factorization (see Corollary 5.11). So the group \( G_{KW}^- \) for constructing Bäcklund transformations is defined as follows:

13.3 Definition. Let \( G_{KW}^- \) denote the group of rational maps \( f : S^2 \to GL(n, \mathbb{C}) \) such that

(i) \( f(\infty) = I \),

(ii) there exists a rational function \( g \) such that \( gf \) satisfies the KW-reality condition.

To construct Bäcklund transformations for the KW-equation, we start with a degree one rational map \( g(\lambda) = \frac{a\lambda + Y}{\lambda - k} \) with \( Y \in G_0 \) as in the KdV case. Note that although \( g \) does not satisfy the KW-reality condition, \( (\lambda - k)g(\lambda) = a\lambda + Y \) does. So \( g \in G_{KW}^- \). Let \( u \) be a local solution of the KW-equation that admits a reduced wave function \( m \), and \( E \) the trivialization of \( u \). Suppose at each \((x, t)\) we can find \( \tilde{Y}(x, t) \in G_0 \) such that

\[
\tilde{E}(x, t, \lambda) = \frac{a\lambda + Y}{\lambda - k} E(x, t, \lambda) \left(\frac{a\lambda + \tilde{Y}(x, t)}{\lambda - k}\right)^{-1}
\]

is holomorphic in \( \lambda \in \mathbb{C} \). Then we can proceed as in the proof of Theorem 4.3 to conclude that \( \tilde{E} \) is the trivialization of some local solution of the \((n + 1)\)-th flow \( \tilde{u} \) in the \( sl(n, \mathbb{C}) \)-hierarchy. But

\[
\tilde{E}(x, t, \lambda) = (a\lambda + Y)E(x, t, \lambda)(a\lambda + \tilde{Y}(x, t))^{-1}.
\]

Since all three terms in the right hand side of (13.4) satisfy the KW-reality condition, \( \tilde{E} \) also satisfies (13.2). Hence \( \tilde{E} \) corresponds to a new solution \( \tilde{u} \) of the
KW-equation. However, in order to prove the expression (13.4) is holomorphic for all $\lambda \in C$, we need to understand the relation between the zeros and kernels of $a\lambda + Y$ and the poles and residues of $(a\lambda + Y)^{-1}$. We do this in the next few Propositions. First we show that $a\lambda + Y$ is determined by a complex number $k$ and a vector $v \in C^n$.

**13.4 Proposition.** Let $Y \in G_0$, and $f(\lambda) = a\lambda + Y$ (so $f$ satisfies the KW-reality condition (13.2)). Then

(i) there is a constant $k$ such that $\det(f(\lambda)) = (-1)^{n+1}(\lambda^n - k^n)$,
(ii) if $f(k)(v) = 0$, then $f(\alpha^j k)(\tau^{-j}(v)) = 0$,
(iii) $\tau^{-j}(v)$ are eigenvector of $a^{-1}Y$ with eigenvalues $-\alpha^j k$ for $1 \leq j \leq n - 1$,
(iv) if $v, \tau^{-1}(v), \ldots, \tau^{-(n-1)}(v)$ are linearly independent, then $Y = -kaBaB^{-1}$, where $B$ is the matrix whose $j$-th column is $\tau^{j-1}(v)$ for $1 \leq j \leq n$,
(v) $\det(Y) = (-k)^n$.

**PROOF.** Since $\tau^{-1}f(\alpha^{-1}\lambda)\tau = f(\lambda)$, $\deg(f(\alpha^{-1}\lambda)) = \deg(f(\lambda))$. Hence $\det(f(\lambda))$ is a polynomial in $\lambda^n$. But the leading term of $\det(f(\lambda))$ is $\alpha^{-\frac{n(n-1)}{2}}\lambda^n$, which is equal to $(-1)^{n+1}\lambda^n$. This proves (i). The rest of the Proposition follows from elementary linear algebra. 

**13.5 Definition.** Let $B$ denote the map

$$B : C^n \to gl(n, C) \quad \text{defined by}$$

$$v \mapsto B(v) = (v, \tau^{-1}(v), \ldots, \tau^{-(n-1)}(v)),$$

i.e., the $i$-th column of $B(v)$ is $\tau^{-(i-1)}(v)$ for $1 \leq i \leq n$. In other words,

$$B(v) = B \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ v_2 & v_3 & \cdots & v_1 \\ \vdots & \vdots & \ddots & \vdots \\ v_n & v_1 & \cdots & v_{n-1} \end{pmatrix}.$$ 

**13.6 Definition.** Given $v \in C^n$ and $k \in C$, if $B(v)$ is non-singular, we define

$$p_{v,k}(\lambda) = a\lambda - kaB(v)aB(v)^{-1},$$

where $B(v)$ is the operator defined by (13.5) (or (13.6)).

As a consequence of Proposition 13.4 we have

**13.7 Corollary.** Suppose $Y \in G_0$ and $a^{-1}Yv = -kv$ for some non-zero vector $v$. Then $f(\lambda) = a\lambda + Y = p_{v,k}(\lambda)$. Or equivalently, if $f(\lambda) = a\lambda + Y$ satisfies the KW-reality condition (13.2) and $f(k)(v) = 0$ then $f = p_{v,k}$. 

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13.8 Proposition. Given \( v \in C^n \) and \( k \in C \), if \( B(v) \) is non-singular, then
(i) \( p_{v,k}(\lambda) \) satisfies the reality condition (13.2),
(ii) \( p_{v,k}(\alpha^j k)\tau^{-i}v = 0 \) for \( 0 \leq i \leq (n - 1) \).

PROOF. Set \( Y = -kaB(v)aB(v)^{-1} \). To prove (i), it suffices to prove \( \tau^{-1}Y\tau = Y \). Note that \( a^{-1}Yv_j = -k\alpha^jv_j \), where \( v_j = \tau^{-i}v \). Since \( \tau v_j = v_{j-1} \) and \( \tau^{-1}a\tau = \alpha a \), we get
\[
a^{-1}\tau^{-1}Y\tau v_j = a^{-1}\tau^{-1}a(a^{-1}Y)\tau v_j = a^{-1}\tau^{-1}a(a^{-1}Y)v_{j-1} = -k\alpha^j v_j.
\]
This proves that \( a^{-1}Y \) and \( a^{-1}\tau^{-1}Y\tau \) have the same eigenvalues and eigenvectors. Hence \( a^{-1}Y = a^{-1}\tau^{-1}Y\tau \), which implies \( Y = \tau^{-1}Y\tau \). This proves (i).

By definition of \( Y \), \( (ak + Y)v = 0 \). Since \( p_{v,k}(\lambda) = a\lambda + Y \) satisfies (13.2), statement (ii) follows from Proposition 13.4.

13.9 Proposition. Suppose \( p(\lambda) = a\lambda + Y \) satisfies the KW-reality condition (13.2) and \( k \) is a zero of \( p(\lambda) \), i.e., \( \det(p(k)) = 0 \). Then
\[
p(\lambda)^{-1} = \frac{(\lambda + \alpha a^{-1}Y)(\lambda + \alpha^2 a^{-1}Y) \cdots (\lambda + \alpha^{n-1}a^{-1}Y)a^{-1}}{\lambda^n - k^n}.
\]

PROOF. It follows from Proposition 13.4 that the eigenvalues of \( a^{-1}Y \) are \(-k, -\alpha k, \ldots, -\alpha^{n-1}k\). So \( (a^{-1}Y)^n = (-k)^n \). But
\[
(\lambda + z)(\lambda + \alpha z) \cdots (\lambda + \alpha^{n-1}z) = \lambda^n + (-1)^{n+1}z^n.
\]
Hence
\[
(\lambda + a^{-1}Y) \cdots (\lambda + \alpha^{n-1}a^{-1}Y) = \lambda^n + (-1)^{n+1}(a^{-1}Y)^n = \lambda^n - k^n,
\]
which finishes the proof.

Next we factor \( p_{v,k}^{-1} \) as the product of simple elements. Note that
\[
(\lambda + \alpha a^{-1}Y) \cdots (\lambda + \alpha^{n-1}a^{-1}Y)a^{-1}
= (\lambda + \alpha a^{-1}Y) \cdots (a^{n-1}Y + \alpha^{n-1}a^{-1}Ya^{n-1})
= (\lambda + \alpha a^{-1}Y) \cdots a^{n-2}(a\lambda + \alpha^{n-1}a^{-(n-1)}Ya^{n-1})
= \ldots
= (a\lambda + \alpha a^{-1}Ya)(a\lambda + \alpha^2 a^{-2}Ya^2) \cdots (a\lambda + \alpha^{n-1}a^{-(n-1)}Ya^{n-1}).
\]

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Now suppose $k$ is a zero of $a\lambda + Y$ and $a^{-1}Yv = -kv$. By Corollary 13.7, $a\lambda + Y = p_{k,v}(\lambda)$. Set $p_{j}(\lambda) = a\lambda + \alpha^j a^{-j}Ya^j$ for $1 \leq j \leq n - 1$. Then

$$p_{j}(\alpha^j k)(a^{-j}v) = \alpha^j ka(a^{-j}v) + \alpha^j a^{-j}Ya^j(a^{-j}v)$$
$$= k\alpha^j a^{1-j}v + \alpha^j a^{-j}Yv$$
$$= k\alpha^j a^{1-j}v + \alpha^j a^{1-j}(a^{-1}Y)v = 0.$$

By Corollary 13.7 again, $p_{j} = p_{a^{-j}v,\alpha^j k}$. As a consequence of this computation and Proposition 13.9, we see that for $p(\lambda) = p_{v,k}(\lambda)$, formula (13.7) can be written as

$$p_{v,k}(\lambda)^{-1} = \frac{p_{a^{-1}v,ak}(\lambda)p_{a^{-2}v,a^2k}(\lambda)\cdots p_{a^{-(n-1)}v,\alpha^{(n-1)}k}(\lambda)}{\lambda^n - k^n}. \quad (13.8)$$

**13.10 Proposition.** Suppose the entries of $p, h : C \to GL(n, C)$ are polynomial such that $p(\lambda)h(\lambda) = f(\lambda)I$ for some polynomial $f : C \to C$. If $f(k) = 0$ for some $k \in C$, then

$$\text{Im}(h(k)) \subset \text{Ker}(p(k)).$$

**PROOF.** Since $p(k)h(k) = 0$, $\text{Im}(h(k)) \subset \text{Ker}(p(k))$. □

As a consequence of Propositions 13.9 and 13.10, we have

**13.11 Corollary.** Write $p_{v,k}(\lambda)^{-1} = \frac{h(\lambda)}{\lambda^n - k^n}$, where $h(\lambda)$ is the degree $(n - 1)$ polynomial in the numerator of the formula (13.8). Then the image of $h(\alpha^j k)$ is the one dimensional space spanned by $\tau^{-i}v$ for $0 \leq i \leq (n - 1)$.

Now we are ready to construct Bäcklund and Darboux transformations.

**13.12 Theorem.** Let $v \in C^n$, $k \in C$ non-zero, $u : \mathcal{O}_1 \to \mathcal{G}_0 \cap sl(n, C)$ a solution of the KW-equation, and $E$ the trivialization of $u$ normalized at $(0,0)$. Let $\tilde{v}(x,t) = E(x,t,k)^{-1}v$, and $B$ the operator from $C^n$ to $gl(n)$ defined by formula (13.5) (or (13.6)). If $B(v)$ is non-singular, then there exists an open subset $\mathcal{O}$ of $\mathcal{O}_1$ such that $B(\tilde{v}(x,t))$ is non-singular for all $(x,t) \in \mathcal{O}$. Moreover, (i) $\tilde{u} = au^{-1} + [\tilde{Y}, a]a^{-1}$ is again a solution of the KW-equation defined on $\mathcal{O}$, where

$$\tilde{Y}(x,t) = -kaB(\tilde{v}(x,t))aB^{-1}\tilde{v}(x,t),$$

(ii) $\tilde{E}(x,t,\lambda) = p_{v,k}(\lambda)E(x,t,\lambda)p_{\tilde{v}(x,t),k}(\lambda)^{-1}$ is the trivialization of $\tilde{u}$ normalized at $(0,0)$,

(iii) $Y$ is a solution of

$$\begin{cases}
Y_x = Yu - (au^{-1} + [Y, a]a^{-1})Y, \\
Y_t = YQ_{a,n+1}(u) - Q_{a,n+1}(au^{-1} + [Y, a]a^{-1})Y, \\
\tau^{-1}Y_\tau = Y,
\end{cases}$$

where $Q_{a,n+1}$ is the polynomial differential operator defined in the $sl(n, C)$-hierarchy.
PROOF. First we prove that \( \tilde{E} \) is holomorphic for \( \lambda \in \mathbb{C} \). It follows from formula (13.8) that \( \tilde{E} \) is holomorphic for \( \lambda \in \mathbb{C} \setminus \{ k, \alpha k, \ldots, \alpha^{n-1} k \} \), and has possible simple poles at \( \lambda = \alpha^i k \) for \( i = 0, 1, \ldots, (n-1) \). We claim that the residue of \( \tilde{E}(x, t, \lambda) \) at \( \lambda = \alpha^i k \) is zero. To see this we use formula (13.8) to write
\[
p_{\tilde{v}(x, t), k}(\lambda) = \frac{\tilde{h}(x, t, \lambda)}{\lambda^n - k^n}.
\]
For \( 0 \leq i \leq (n-1) \), set
\[
f_i(\lambda) = (\lambda - k) \cdots (\lambda - \alpha^{i-1} k)(\lambda - \alpha^{i+1} k) \cdots (\lambda - \alpha^{n-1} k),
\]
i.e., \( f_i(\lambda) = \frac{\lambda^n - k^n}{\lambda - \alpha^i k} \). The residue of \( \tilde{E}(x, t, \lambda) \) at \( \lambda = \alpha^i k \) is equal to
\[
p_{v, k}(\alpha^i k) E(x, t, \alpha^i k) \tilde{h}(x, t, \alpha^i k) / f_i(\alpha^i k).
\]
But definition of \( \tilde{v}(x, t) \) implies \( E(x, t, k)(\tilde{v}(x, t)) = v \). Since \( E \) satisfies the reality condition (13.2), we have
\[
E(x, t, \alpha^i k) = \tau^{-i} E(x, t, k) \tau^i.
\]
By Corollary 13.11, the image of \( \tilde{h}(x, t, \alpha^i k) \) is the space spanned by \( \tau^{-i} \tilde{v}(x, t) \). So the image of
\[
p_{v, k}(\alpha^i k) E(x, t, \alpha^i k) \tilde{h}(x, t, \alpha^i k)
\]
is spanned by
\[
p_{v, k}(\alpha^i k) \tau^{-i} E(x, t, k) \tau^i (\tau^{-i} \tilde{v}(x, t)) = p_{v, k}(\alpha^i k) \tau^{-i} v,
\]
which is zero as follows from Proposition 13.8 (ii). This proves that the residue of \( \tilde{E} \) is zero at \( \alpha^i k \). Hence \( \tilde{E}(x, t, \lambda) \) is holomorphic for \( \lambda \in \mathbb{C} \).

The rest of the theorem can be proved exactly the same as Theorem 4.3.

Relations among simple elements can be obtained by a direct computation as in Proposition 4.8:

13.13 Proposition. Let \( Y, Z \in \mathcal{G}_0 \) such that \( (Y - Z) \) is non-degenerate. Set
\[
\tilde{Y} = a(Y - Z)a^{-1}Z(Y - Z)^{-1},
\]
\[
\tilde{Z} = a(Y - Z)a^{-1}Y(Y - Z)^{-1}.
\]
Then
\[(i) \ \tilde{Y}, \tilde{Z} \in \mathcal{G}_0,\]
\[(ii) \ (a \lambda + \tilde{Y})(a \lambda + Y) = (a \lambda + \tilde{Z})(a \lambda + Z).\]
The dimension of \( G_0 \cap sl(n)_{\perp} \) is \((n - 1)\). So the KW-equation (13.3), its Bäcklund transformations and permutability formula should be expressed in terms of \((n - 1)\) independent functions. Since an element in \( G_0 \) is determined by its first row, we identify the space \( \mathcal{M}_{1 \times n} \) of \( 1 \times n \) complex matrices as \( G_0 \) via the linear isomorphism:

\[
\zeta : \mathcal{M}_{1 \times n} \to G_0
\]

\[v = (v_1, \ldots, v_n) \mapsto \zeta(v), \text{ where } (\zeta(v))_{ij} = v_{j-i+1}.\]  

(Again \( v_i = v_j \) if \( i \equiv j \mod n \)).

Let \((0, q_2, \ldots, q_n)\) denote the first row of \( u \in S(R, G_0 \cap sl(n)_{\perp}) \), i.e., \( u = \zeta((0, q_2, \ldots, q_n)) \). Let \((L_1(q), \ldots, L_n(q))\) denote the first row of \( Q_{a,n+1}(u) \). It follows from Theorem 2.5 that each \( L_j(q) \) is an order \( n \) polynomial differential operator in \( q_2, \ldots, q_n \). Since \( Q_{a,n+1}(u) \in G_0 \), we can write

\[Q_{a,n+1}(u) = \zeta(L_1(q), \ldots, L_n(q)).\]

A direct computation implies that the KW-equation (13.3) written in terms of \( q \) is

\[
\frac{\partial q_j}{\partial t} = \frac{\partial}{\partial x} L_j(q) + \sum_{i=1}^{n} q_i L_{j-i+1}(q) - L_i(q) q_{j-i+1} = \frac{\partial}{\partial x} L_j(q),
\]

i.e.,

\[
\frac{\partial q_j}{\partial t} = \frac{\partial}{\partial x}(L_j(q)), \quad 2 \leq j \leq n. \tag{13.10}
\]

The Darboux transformation for the KW equation in Theorem 13.12 written in terms of \( q \) gives:

**13.14 Corollary.** Let \( q = (0, q_2, \ldots, q_n) \) be a solution of the \( n \times n \) KW-equation (13.10), and \( E \) the trivialization normalized at \((0, 0)\). Given \( v \in C^n \) and \( k \in C \), let \( \tilde{v}(x, t) = E(x, t, k)^{-1}(v) \), and \( B \) the operator defined by the formula (13.6). Let \( y(x, t) = (y_1(x, t), y_2(x, t), \ldots, y_n(x, t)) \) denote the first row of the matrix

\[-kaB(\tilde{v}(x, t))aB(\tilde{v}(x, t))^{-1},\]

and \( \tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_n) \) is again a solution of the KW-equation (13.10). Then \( \tilde{q} = (0, \tilde{q}_2, \ldots, \tilde{q}_n) \) is again a solution of the KW-equation (13.10).

The ODE version of Bäcklund transformations for the KW-equation in Theorem 13.12 written in terms of \( q \) gives:
13.15 Corollary. Suppose \( q = (0, q_2, \ldots, q_n) \) is a solution of the KW-equation (13.10). Then the following systems for \( y = (y_1, \ldots, y_n) \) are compatible:

\[
\begin{align*}
(y_j)_x &= \sum_{i=1}^{\tilde{n}} (1 - \alpha^{i-j})y_i(q_{j+1-i} - y_{j+1-i}), \\
(y_j)_t &= \sum_{i=1}^{\tilde{n}} L_{j+1-i}(q)y_i - L_i((q - y)a^{-1} + y)y_{j+1-i}.
\end{align*}
\]

(BT\(_q\)\(_KW\))

Moreover, if \( y(x, t) \) is a solution of \( BT_q^{KW} \), then

(i) \( \tilde{q} = (q - y)a^{-1} + y \) is again a solution of the KW-equation,

(ii) \( \det(\zeta(y(x, t))) \) is a constant.

As a consequence of Proposition 13.13, we have

13.16 Corollary. Suppose \( q = (0, q_2, \ldots, q_n) \) is a solution of (13.10), \( \xi, \eta \) are solutions of \( BT_q^{KW} \), and

\[
q' = (q - \xi)a^{-1} + \xi, \quad q'' = (q - \eta)a^{-1} + \eta
\]

are the corresponding new solutions of the KW-equation. Assume \( \det(\zeta(\xi - \eta)) \neq 0 \), where \( \zeta \) is the operator defined by formula (13.9). Set

\[
\begin{align*}
\tilde{\xi} &= (\xi - \eta)a^{-1}\zeta(\eta)a^{-1}(\zeta(\xi - \eta))^{-1}, \\
\tilde{\eta} &= (\xi - \eta)a^{-1}\zeta(\xi)a^{-1}(\zeta(\xi - \eta))^{-1}.
\end{align*}
\]

Then \( \tilde{\xi} \) is a solution of \( BT_q^{KW} \), \( \tilde{\eta} \) is a solution of \( BT_q^{KW} \), and

\[
\begin{align*}
\tilde{q} &= q'a^{-1} + \tilde{\xi}(I - a^{-1}) = (\xi + (q - \xi)a^{-1})a^{-1} + \tilde{\xi}(I - a^{-1}) \\
&= q''a^{-1} + \tilde{\eta}(I - a^{-1}) = (\eta + (q - \eta)a^{-1})a^{-1} + \tilde{\eta}(I - a^{-1})
\end{align*}
\]

(13.12)

is again a solution of the KW-equation.

13.17 Corollary. Let \( q, q', q'' \) and \( \tilde{q} \) be as in Corollary 13.16. Then \( \tilde{q} \) is an algebraic function of \( q, q', q'' \).

PROOF. Use formulas (13.11) and (13.12) to write \( \tilde{q} \) in terms of \( q, \xi \) and \( \eta \). Corollary 13.14 implies

\[
\begin{align*}
\xi_j &= \frac{q_j' - \alpha^{1-j}q_j}{1 - \alpha^{1-j}}, \quad \eta_j = \frac{q_j'' - \alpha^{1-j}q_j}{1 - \alpha^{1-j}}, \quad 2 \leq j \leq n.
\end{align*}
\]

So \( \xi_2, \ldots, \xi_n, \eta_2, \ldots, \eta_n \) are algebraic functions of \( q, q', q'' \). Corollary 13.15 (ii) implies that \( \det(\zeta(\xi)) = c_1 \) and \( \det(\zeta(\eta)) = c_2 \) are constant. But \( \det(\zeta(\xi)) \) (resp. \( \det(\zeta(\eta)) \)) is a degree \( n \) polynomial in \( \xi_1 \) (resp. \( \eta_1 \)). This implies that \( \xi_1 \) (resp. \( \eta_1 \)) can be written as an algebraic function of \( \xi_2, \ldots, \xi_n \) (resp. \( \eta_2, \ldots, \eta_n \)). Hence \( \tilde{q} \) is an algebraic function of \( q, q' \) and \( q'' \).
14. The Gel’fand-Dikii Hierarchy

The Gel’fand-Dikii ($GD_n$-) hierarchy is a hierarchy of flows on the space $P_n$ of $n$th order scalar differential operators

$$L = D^n - (p_1 D^{n-2} + p_2 D^{n-3} + \cdots + p_{n-1}),$$

where $D = \frac{d}{dx}$ and $p_i \in S(R, C)$. Flows in this hierarchy are given by

$$\frac{\partial L}{\partial t} = [L^{j/n}+, L],$$

where $L^{j/n}+$ is the differential operator part of the pseudo-differential operator $L^{k/n}$. The spectral problem $Ly_1 = \lambda^n y_1$ is equivalent to the spectral problem of the following first order system for $y = (y_1, \cdots, y_n)$:

$$\frac{d}{dx}(y_1, \cdots, y_n) = (y_1, \cdots, y_n)(e_1n \lambda^n + b + v),$$

$$= (y_1, \cdots, y_n) \begin{pmatrix} 0 & \cdots & p_{n-1} + \lambda^n \\ 1 & 0 & \cdots & p_{n-2} \\ 0 & 1 & 0 & \cdots & p_{n-3} \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{14.1}$$

Here

$$b = e_{21} + \cdots + e_{nn-1} = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}; \tag{14.2}$$

$$v = \begin{pmatrix} 0 & 0 & \cdots & p_{n-1} \\ 0 & 0 & \cdots & p_{n-2} \\ \vdots \\ 0 & 0 & \cdots & p_1 \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and $e_{ij}$ is the $n \times n$ matrix whose $ij$-th entry is 1 and all other entries are 0. The $GD_n$-hierarchy gives rise to a hierarchy on the space $\mathcal{M}_n$ of all $v : R \rightarrow sl(n, C)$ of the form (14.1) with $p_i \in S(R, C)$ for $1 \leq i \leq n - 1$. But unlike all the hierarchies we have discussed in previous sections, when $n \geq 3$, $\mathcal{M}_n$ is not determined by a reality condition and the corresponding flows on $\mathcal{M}_n$ are not the restriction of the flows in the $sl(n, C)$-hierarchy to $\mathcal{M}_n$. Drinfeld and Sokolov [DS1, 2] gave a description of the symplectic structures and the flows on $\mathcal{M}_n$ using a symplectic quotient.
When \( n = 2 \), the \( GD_2 \)-hierarchy is the complex KdV hierarchy. In section 12, we saw that this hierarchy is obtained by restricting the \( sl(2,C) \)-hierarchy to the submanifold that is defined by the reality condition:

\[
\phi(\lambda)^{-1}A(\lambda)\phi(\lambda) = \phi^{-1}(-\lambda)A(-\lambda)\phi(-\lambda), \quad \phi(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.
\]  
(14.3)

In other words, \( A(\lambda) \) lies in the Lie subalgebra of fixed points of the involution \( \theta \) defined by

\[
\theta = \text{Ad}(\phi) \circ \tau \circ \text{Ad}(\phi)^{-1},
\]

where \( \tau(A(\lambda)) = A(-\lambda) \). In fact, we showed that the odd flows in the \( sl(2,C) \)-hierarchy leaves invariant the space \( S_\phi \) of all \( u : R \to sl(2,C) \) such that \( A(\lambda) = a\lambda + u \) satisfies the reality condition (14.3). The \( GD_2 \)-hierarchy is the \( sl(2,C) \)-hierarchy restricted to \( S_\phi \). The main purpose of this section is to generalize this construction to the \( GD_n \)-hierarchy. However, we need to use a different \( gl(n) \)-valued first order linear operator \( \frac{d}{dx} + A(\lambda,x) \) than the one given by the formula (14.1). Here \( A(\cdot,x) \) is fixed by certain order \( n \) Lie algebra homomorphism \( \sigma_n \):

\[
\sigma_n = \text{Ad}(\phi_n) \circ \tau_n \circ \text{Ad}(\phi_n)^{-1}.
\]

To motivate the choice of \( \phi_n \), we first explain the relation between

\[
\phi_2(\lambda) = \phi(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}
\]

and the phase space of the \( GD_2 \)-hierarchy. The vacuum \( L_0 = D^2 - \lambda^2 \) corresponds to

\[
\frac{d}{dx} + A_0(\lambda), \quad \text{where } A_0(\lambda) = \begin{pmatrix} 0 & \lambda^2 \\ 1 & 0 \end{pmatrix}.
\]

\( A_0(\lambda) \) can be diagonalized by \( V(\lambda) = \begin{pmatrix} 1 & \lambda \\ 1 & -\lambda \end{pmatrix} \):

\[
V(\lambda) \begin{pmatrix} 0 & \lambda^2 \\ 1 & 0 \end{pmatrix} V(\lambda)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda.
\]

Now factor \( V(\lambda) = \phi_-(\lambda)\phi_+(\lambda) \) so that \( \phi_-(\lambda) \) is lower-triangular and \( \phi_+(\lambda) \) is upper-triangular with 1 on the diagonal:

\[
V(\lambda) = \phi_-(\lambda)\phi_+(\lambda) = \begin{pmatrix} 1 & 0 \\ 1 & -2\lambda \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.
\]

Note that \( \phi_+ \) is exactly the \( \phi \) used to define the reality condition for the KdV-hierarchy. This gives us a hint for the choice of the \( GD_n \)-reality condition. We
proceed as follows: The vacuum for the $GD_n$-hierarchy is the order $n$ operator $L_0 = D^n - \lambda^n$, and the corresponding first order system is $\frac{d}{dx} + A_0(\lambda)$, where $A_0(\lambda) = b + e_{1n}\lambda^n$. It is easy to check that

$$V(\lambda) = \begin{pmatrix} 1 & \lambda & \cdots & \lambda^{n-1} \\ 1 & \alpha \lambda & \cdots & (\alpha\lambda)^{n-1} \\ 1 & \alpha^2 \lambda & \cdots & (\alpha^2\lambda)^{n-1} \\ \vdots \\ 1 & \alpha^{n-1}\lambda & \cdots & (\alpha^{n-1}\lambda)^{n-1} \end{pmatrix} = ((\alpha^{i-1}\lambda)^{j-1})$$

diagonalizes $A_0(\lambda)$, where $\alpha = e^{\frac{2\pi i}{n}}$. In fact,

$$V(\lambda)A_0(\lambda)V(\lambda)^{-1} = a\lambda, \quad a = \text{diag}(1, \alpha, \ldots, \alpha^{n-1}).$$

### 14.1 Proposition. $V(\lambda)$ can be factored uniquely as

$$V(\lambda) = \phi_n^{-}(\lambda)\phi_n(\lambda),$$

where $\phi_n^{-}$ is lower-triangular and $\phi_n$ is upper-triangular with 1’s on the diagonal.

**PROOF.** It is an elementary result in linear algebra that the factorization of $V = \phi_n^{-}\phi_n$ can be carried out using the Gaussian elimination if all the principal $k \times k$ minors $\Delta_k$ of $V = (v_{ij})$ are non-zero. But

$$\Delta_k = \det((v_{ij})_{1 \leq i,j \leq k}) = \lambda^k \prod_{0 \leq i < j \leq k-1} (\alpha^j - \alpha^i),$$

which is not zero for $\lambda \neq 0$.

For example, we use Gaussian elimination to factor $V$ and get

$$\phi_3(\lambda) = \begin{pmatrix} 1 & \lambda & \lambda^2 \\ 0 & 1 & (1 + \alpha)\lambda \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha = e^{\frac{2\pi i}{3}}$$

$$\phi_4(\lambda) = \begin{pmatrix} 1 & \lambda & \lambda^2 & \lambda^3 \\ 0 & 1 & (1 + \alpha)\lambda & (1 + \alpha + \alpha^2)\lambda^2 \\ 0 & 0 & 1 & (1 + \alpha + \alpha^2)\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha = e^{\frac{2\pi i}{4}} = i.$$

However, it is difficult to write down an explicit formula for $\phi_n$ using Gaussian elimination. In order to do this, we need first prove some properties of $\phi_n$. Let $G(k)$ denote the subspace spanned by

$$\{e_{i,i+k} \mid 1 \leq i, i + k \leq n\}.$$

Then

$$G(k)G(m) \subset G(k + m). \quad (14.4)$$

Let $N_+ = \sum_{i=1}^{n-1} G(i)$ denote the subalgebra of strictly upper triangular matrices, $B_+ = \sum_{i=0}^{n} G(i)$ the subalgebra of upper triangular matrices, and $N_+, B_+$ the Lie group associated to $N_+, B_+$ respectively.
14.2 Proposition. Let \( \phi_n(\lambda) \) be the polynomial obtained in Proposition 14.1. Then

(i) \( \phi_n = I + \sum_{i=1}^{n-1} f_i \lambda^i \) for some constant \( f_i \in \mathcal{N}_+ \),
(ii) \( \phi_n(\lambda)(e_{1n} \lambda^n + b)\phi_n(\lambda)^{-1} = a\lambda + b. \)

PROOF. The Gaussian elimination proves (i).

Since \( V(\lambda)(e_{1n} \lambda^n + b)V(\lambda)^{-1} = a\lambda + b \) and \( V(\lambda) = \phi_-(\lambda)\phi_n(\lambda) \),

\[
\phi_n(\lambda)(e_{1n} \lambda^n + b)\phi_n(\lambda)^{-1} = \phi_-(\lambda)^{-1}a\lambda\phi_-(\lambda).
\]

Note that the left hand side lies in \( \sum_{k \geq -1} \mathcal{G}(k) \) and the right hand side lies in \( \sum_{k \leq 0} \mathcal{G}(k) \). Moreover, the \( \mathcal{G}_0 \)-component of the right hand side is \( a\lambda \) and the \( \mathcal{G}_{-1} \)-component of the left hand side is \( b \). This proves (ii).

In the following, we use Proposition 14.2 (ii) to get an explicit formula for \( \phi_n \). We need a Lemma, which is proved by a direct computation and (14.4).

14.3 Lemma. Let \( b = e_{21} + e_{32} + \cdots + e_{nn-1} \). Then

1. \([b, \mathcal{G}(i)] \subset \mathcal{G}(i-1) \) and \( \text{ad}(b) \) is injective on \( \mathcal{N}_+ \),
2. \([b, x] \in \mathcal{G}(i-1) \) if and only if \( x \in \mathcal{G}(i) \),
3. if \( [b, x] = \sum_{k=1}^{n-i+1} c_k e_{k,k+i-1} \in \mathcal{G}(i-1) \) for \( i \geq 1 \), then \( \sum_{k=1}^{n-i+1} c_k = 0 \) and \( x = -\sum_{k=1}^{n-i} (\sum_{j=1}^{k} c_j) e_{k,k+i} \).

14.4 Proposition. Suppose \( \phi_n(\lambda) = I + f_1 \lambda + \cdots + f_{n-1} \lambda^n \) is a \( \mathcal{N}_+ \)-valued map of degree \( n - 1 \) in \( \lambda \). Then

\[
\phi_n(\lambda)(e_{1n} \lambda^n + b) = (a\lambda + b)\phi_n(\lambda) \tag{14.5}
\]

if and only if

\[
f_i = (1 + \alpha + \cdots + \alpha^{i-1})^{-1} \Lambda^i, \quad \text{where} \tag{14.6}
\]

\[
\Lambda = \sum_{i=0}^{n-1} (1 + \alpha + \cdots + \alpha^{i-1}) e_{i,i+1}. \tag{14.7}
\]

The verification that this formula gives a solution is quite tedious. However, it helps to know there is a unique solution \( \phi_n \) for equation (14.5). It is also helpful to note that, after doing the computation for \( n = 3, 4 \) by the Gaussian elimination, that \( f_i \in \mathcal{G}(i) \) and \( f_i f_j = f_j f_i \). Hence we expect \( f_i = c_i f_i^1 \) for some constant \( c_i \).

PROOF. Compare coefficients of \( \lambda^i \) in equation (14.5)

\[
(I + f_1 \lambda + \cdots + f_{n-1} \lambda^{n-1})(e_{1n} \lambda^n + b) = (a\lambda + b)(I + f_1 \lambda + \cdots + f_{n-1} \lambda^{n-1})
\]

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for \( 1 \leq i \leq n \) to get

\[
\begin{cases}
    f_1 b = bf_1 + a, \\
    f_i b = bf_i + f_{i-1}, & \text{if } 2 \leq i \leq n-1, \\
    e_{1n} = af_{n-1}.
\end{cases}
\]  

(14.8)

Since \( a = \sum_{i=1}^{n} \alpha^{i-1} e_{ii} \) and \( \sum_{i=0}^{n-1} \alpha^i = 0 \) and the first equation of system (14.8) is \([b, f_1] = -a\), Lemma 14.3 (3) implies that

\[f_1 = \Lambda = \sum_{i=1}^{n-1} (1 + \alpha + \cdots + \alpha^{i-1}) e_{i,i+1} \in G(1).\]

In particular \( f_1 \in G(1) \). The second to the \((n-1)\)-th equation of (14.8) is \([f_i, b] = f_{i-1}\). So Lemma 14.3 (2) and induction imply that \( f_i \in G(i) \) for \( 1 \leq i \leq n - 1 \).

Since \( \operatorname{ad}(b) \) is injective on \( N_+ \), system (14.8) has at most one solution.

Proposition 14.2 shows that the overdetermined system (14.8) has a solution. So system (14.8) has a unique solution.

It remains to prove the formula for \( f_j \). We need the following simple equalities:

\[
\begin{align*}
\Lambda b - b\Lambda &= a, \quad (14.9) \\
\Lambda a &= \alpha a \Lambda. \quad (14.10)
\end{align*}
\]

(They can be proved by a direct computation.) Next we claim that

\[
\Lambda^k b - b\Lambda^k = (1 + \alpha + \cdots + \alpha^{k-1}) a \Lambda^{k-1}
\]

(14.11)
is true for \( 1 \leq k \leq n - 1 \). This equality implies that \( f_k = (1 + \alpha + \cdots + \alpha^{k-1})^{-1} \Lambda^k \) solves system (14.8). We use induction to prove equality (14.11). When \( k = 1 \), equality (14.11) is (14.9). Now suppose to prove equality (14.11) is true for \( k \). Then

\[
\begin{align*}
\Lambda^{k+1} b - b\Lambda^{k+1} &= \Lambda(\Lambda^k b) - b\Lambda^{k+1} \\
&= \Lambda \left( b\Lambda^k + \left( \sum_{i=0}^{k-1} \alpha^i \right) a\Lambda^{k-1} \right) - b\Lambda^{k+1} \\
&= (b\Lambda + a)\Lambda^k + \left( \sum_{i=0}^{k-1} \alpha^i \right) \Lambda a\Lambda^{k-1} - b\Lambda^{k+1} \\
&= b\Lambda^{k+1} + a\Lambda^k + \left( \sum_{i=0}^{k-1} \alpha^i \right) \alpha a\Lambda^{k-1} - b\Lambda^{k+1} \\
&= (1 + \alpha + \cdots + \alpha^k) a \Lambda^k.
\end{align*}
\]

This completes the proof. \( \blacksquare \)

Since \( f_i = c_i \Lambda^i \) with \( c_i = (\sum_{k=0}^{i-1} \alpha^k)^{-1} \), we have
14.5 Corollary. Let $f_i \in \mathcal{G}(i)$ be as in Proposition 14.4. Then $f_if_j = fjf_i$ and $f_{1,i+1} = 1$ for all $1 \leq i, j \leq n - 1$.

Next consider the $GD_n$-reality condition:

$$
\phi_n(\lambda)^{-1}A(\lambda)\phi_n(\lambda) = \phi_n(\alpha\lambda)^{-1}A(\alpha\lambda)\phi_n(\alpha\lambda), \quad \alpha = e^{\frac{2\pi i}{n}}. \quad (14.12)
$$

The following statements are easily seen to be equivalent:

(i) $A$ satisfies the $GD_n$-reality condition,
(ii) all entries of $\phi_n(\lambda)^{-1}A(\lambda)\phi_n(\lambda)$ are polynomial in $\lambda^n$,
(iii) $A$ is a fixed point of the order $n$ automorphism

$$
\text{Ad}(\phi_n) \circ \tau_n \circ \text{Ad}(\phi_n)^{-1},
$$

where $\tau_n(\lambda) = A(\alpha\lambda)$.

The first step in the construction of Bäcklund transformations is to determine the condition for $A\lambda + Y$ to satisfy the $GD_n$-reality condition. We need two Lemmas:

14.6 Lemma. $\phi_n(\lambda)^{-1} = I + g_1\lambda + \cdots + g_{n-1}\lambda^{n-1}$, where $g_i \in \mathcal{G}(i)$ are constant.

PROOF. Note that

$$(I + g_1\lambda + \cdots + g_{n-1}\lambda^{n-1})(I + f_1\lambda + \cdots + f_{n-1}\lambda^{n-1}) = I \quad (14.13)$$

holds if and only if the coefficients of $\lambda^j$ are zero for $1 \leq j \leq 2(n - 1)$. Since $\mathcal{G}(i)\mathcal{G}(j) \subset \mathcal{G}(i+j)$ and $\mathcal{G}(m) = 0$ if $m > n$, the coefficient of $\lambda^j$ in (14.13) is zero for all $j \geq n$. The coefficient of $\lambda$ in (14.13) is zero implies that $g_1 = -f_1 \in \mathcal{G}(1)$. The coefficient of $\lambda^j$ is zero implies that

$$-g_j = f_j + \sum_{i=1}^{j-1} g_if_{j-i}. \quad (14.14)$$

By induction on $j$, we conclude that $g_j \in \mathcal{G}(j)$.

14.7 Lemma. Let $\Lambda = \sum_{i=1}^{n-1} (1 + \alpha + \cdots + \alpha^{i-1})e_{i,i+1} \in \mathcal{G}(1)$ be as in Proposition 14.4. Then the centralizer

$$
\text{gl}(n)\Lambda = \{Z \in \text{gl}(n) \mid Z\Lambda = \Lambda Z\} = \sum_{i=0}^{n-1} C\Lambda^i.
$$
PROOF. Let $c_i = 1 + \alpha + \cdots + \alpha^{i-1}$. Write $Z = \sum_{1 \leq i, j \leq n} z_{ij} e_{ij}$. Then $Z \Lambda = \Lambda Z$ if and only if $z_{i,j-1} c_{j-1} = c_i z_{i+1,j}$. Hence

$$z_{i,i+j} = \left( \prod_{k=1}^{i-1} \frac{c_k + j}{c_k} \right) z_{1,1+j}.$$ 

So the dimension of $gl(n)_{\Lambda}$ is $n$. But $I, \Lambda, \cdots, \Lambda^{n-1} \in gl(n)_{\Lambda}$ and are linearly independent. Hence $gl(n)_{\Lambda} = \sum_{i=0}^{n-1} C \Lambda^i$. 

14.8 Proposition. $A(\lambda) = a \lambda + Y$ satisfies the $GD_n$-reality condition if and only if

$$Y = b + \sum_{i=0}^{n-1} y_i f_i,$$

where $f_i$’s are defined in Proposition 14.4, $f_0 = I$ and $b = e_{21} + e_{32} + \cdots + e_{nn-1}$.

PROOF. Suppose $a \lambda + Y$ satisfies the $GD_n$-reality condition. By Proposition 14.6, $\phi_n(\lambda)^{-1}(a \lambda + Y)\phi_n(\lambda)$ is a polynomial in $\lambda$ with degree $\leq 2n - 1$ with constant term $Y$. But the $GD_n$-reality condition implies that it is a polynomial in $\lambda^n$. So

$$\phi_n(\lambda)^{-1}(a \lambda + Y)\phi_n(\lambda) = C_0 \lambda^n + Y$$

for some $C_0 \in gl(n)$. Write $Y = b + Z$. It follows from Proposition 14.4 that we have

$$\phi_n(\lambda)^{-1}(a \lambda + Y)\phi_n(\lambda) = \phi_n(\lambda)^{-1}(a \lambda + b + Z)\phi_n(\lambda)$$

$$= e_{1n} \lambda^n + b + \phi_n(\lambda)^{-1} Z \phi_n(\lambda).$$

Hence

$$\phi_n(\lambda)^{-1} Z \phi_n(\lambda) = C \lambda^n + Z,$$

where $C = C_0 - e_{1n}$. So $Z \phi_n(\lambda) = \phi_n(\lambda)(C \lambda^n + Z)$, i.e.,

$$Z(I + f_1 \lambda + \cdots + f_{n-1} \lambda^{n-1}) = (I + f_1 \lambda + \cdots + f_{n-1} \lambda^{n-1})(C \lambda^n + Z).$$

Because the left hand side has degree $n - 1$ in $\lambda$, the coefficient of $\lambda^n$ of the right hand side is zero. This implies that $C = 0$. So we have

$$\phi_n(\lambda)^{-1} Z \phi_n(\lambda) = Z,$$

i.e.,

$$Z(I + f_1 \lambda + \cdots + f_{n-1} \lambda^{n-1}) = (I + f_1 \lambda + \cdots + f_{n-1} \lambda^{n-1})Z.$$

Compare coefficient of $\lambda^i$ in the above equation to get $f_j Z = Z f_j$ for all $1 \leq j \leq n - 1$. But recall that $f_i = c_i \Lambda^i$ for some non-zero constant $c_i$. Hence $Z \Lambda = \Lambda Z$.

By Lemma 14.7, $Z = \sum_{i=0}^{n-1} y_i f_i$ for some constant $y_0, y_1, \cdots, y_{n-1}$. 

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For example, for \( n = 3, 4 \), \( a\lambda + Y_n \) satisfies the \( GD_n \)-reality condition if and only if

\[
Y_3 = \begin{pmatrix} y_0 & y_1 & y_2 \\ 1 & y_0 & (1 + \alpha)y_1 \\ 0 & 1 & y_0 \end{pmatrix}, \quad \alpha = e^{2\pi i \frac{3}{4}},
\]

\[
Y_4 = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ 1 & y_0 & (1 + \alpha)y_1 & (1 + \alpha + \alpha^2)y_2 \\ 0 & 1 & y_0 & (1 + \alpha + \alpha^2)y_1 \\ 0 & 0 & 1 & y_0 \end{pmatrix}, \quad \alpha = e^{2\pi i \frac{1}{4}}.
\]

**14.9 Definition.** For \( y = (y_0, \cdots, y_{n-1}) \), we set

\[
Y_y = b + \sum_{i=0}^{n} y_i f_i,
\]

\[
f_y(\lambda) = a\lambda + Y_y,
\]

where \( f_0 = I \) and \( f_i \)’s are given in Proposition 14.4.

**14.10 Definition.** Let \( S_{\phi_n} \) denote the space of Schwartz maps \( u : \mathbb{R} \to sl(n,a) \) such that \( A(\lambda) = a\lambda + b + u \) satisfies the \( GD_n \)-reality condition, where \( b = e_{21} + \cdots + e_{nn-1} \). In other words, \( S_{\phi_n} \) is the space of \( u = Y_q \) for some smooth Schwartz map \( q = (0, q_1, \cdots, q_{n-1}) \).

**14.11 Proposition.** Given \( u : \mathbb{R} \to sl(n,a) \), if \( a\lambda + u \) satisfies the \( GD_n \)-reality condition, then the Lax pair of the \((nj+1)\)-th flow in the \( sl(n, C)\)-hierarchy satisfies the \( GD_n \)-reality condition.

**PROOF.** The proof is similar to that of Proposition 12.2 except we replace \(-\lambda\) by \(\alpha\lambda\), where \(\alpha = e^{2\pi i \frac{1}{n}}\). So we have

\[
\phi_n(\lambda)\phi_n(\lambda)^{-1}Q(\lambda)\phi_n(\lambda)\phi_n(\alpha\lambda)^{-1} = \alpha Q(\alpha\lambda),
\]

i.e., \(\phi_n(\lambda)^{-1}Q(\lambda)\phi_n(\lambda) = \alpha\phi_n(\alpha\lambda)^{-1}Q(\alpha\lambda)\phi_n(\alpha\lambda)\). This implies that

\[
\lambda^{nj+1}\phi_n(\lambda)^{-1}Q(\lambda)\phi_n(\lambda)
\]

is a power series in \(\lambda^n\). In particular,

\[
\phi_n(\lambda)^{-1}(\lambda^{nj+1}Q(\lambda))_{+}\phi_n(\lambda)
\]

\[
= \phi_n(\lambda)(a\lambda^{nj+1} + u\lambda^{nj+1} + Q_2\lambda^{nj-1} + \cdots + Q_{nj+1})\phi_n(\lambda)
\]

is a polynomial in \(\lambda^n\).

As a consequence we have
14.12 Corollary. The \( jn + 1 \)-th flow in the \( sl(n,C) \)-hierarchy leaves the submanifold \( S_{\phi_n} \) invariant for all \( j \geq 0 \).

The first order \( n \times n \) system

\[
\frac{d}{dx} (y_1, \ldots, y_n) = (y_1, \ldots, y_n)(a\lambda + Y_q)
\]

is equivalent to a unique order \( n \) differential operator of \( y_1 \). Hence \( S_{\phi_n} \) is isomorphic to the phase space \( \mathcal{P}_n \) of the \( GD_n \)-hierarchy, and the restriction of the \( sl(n,C) \)-hierarchy to \( S_{\phi_n} \) corresponds to the \( GD_n \)-hierarchy. It will still be called the \( GD_n \)-hierarchy.

14.13 Example. \( S_{\phi_3} \) is the space of \( u = q_1 f_1 + q_2 f_2 \). A direct computation shows that the third order scalar differential operator corresponding to the first order system \( \frac{du}{dx} - y(a\lambda + u) = 0 \) is

\[
D^3 - ((1 - \alpha^2)q_1 D + ((q_1)_x + q_2)) = \lambda^3.
\]

Hence

\[
p_1 = (1 - \alpha^2)q_1, \quad p_2 = (q_1)_x + q_2
\]

defines a linear isomorphism from \( S_{\phi_3} \) to \( \mathcal{P}_3 \), and the inverse is given by

\[
q_1 = \frac{p_1}{1 - \alpha^2}, \quad q_2 = p_2 - \frac{(p_1)_x}{1 - \alpha^2}.
\]

The general outline for constructing Bäcklund and Darboux transformations is the same as in our previous examples. But the computations are quite involved. We give an admittedly rather brief description of how the construction goes.

14.14 Definition. Let \( G_{GD}^- \) denote the group of rational maps \( f : S^2 \rightarrow GL(n,C) \) such that

(i) \( f(\infty) = I \),

(ii) there exists a rational map \( g : C \rightarrow C \) such that \( gf \) satisfies the \( GD_n \)-reality condition.

\( G_{GD}^- \) acts on the space of local solutions of the \( (n+1) \)-th flow in the \( GD_n \)-hierarchy, and the action of a linear fractional map in \( G_{GD}^- \) gives a Bäcklund transformation. To construct Bäcklund transformations for the \( GD_n \)-equation, we start with a degree one rational map \( \theta_y(\lambda) = \frac{a\lambda + Y_y}{\lambda - k} \), where \(-k\) is an \( n \)-th root of \( \det(Y_y) \) (i.e., \( \det(Y_y) = (-k)^n \)). Note that although \( \theta_y \) does not satisfy the \( GD_n \)-reality condition, \( (\lambda - k)\theta_y(\lambda) = a\lambda + Y_y \) does. So \( \theta_y \in G_{GD}^- \).

14.15 Proposition. \( f_y(\lambda) = a\lambda + Y_y \) has the following properties:

(i) \( \det(f_y(\lambda)) = (-1)^n(\lambda^n - k^n) \), where \( \det(Y_y) = (-k)^n \).

(ii) \( f_y(\lambda)^{-1} = \frac{h(\lambda)}{\lambda^n - k^n} \) for some \( gl(n) \)-valued polynomial of degree \( (n-1) \).
PROOF. Since \( f_y(\lambda) \) satisfies the \( GD_n \)-reality condition,
\[
\det(f_y(\lambda)) = \det(f_y(\alpha \lambda))
\]
for \( \alpha = e^{2\pi i/n} \). So \( \det(f_y(\lambda)) \) is a polynomial in \( \lambda^n \). But \( \det(f_y(\lambda)) \) is of degree \( n \) in \( \lambda \) whose leading term is
\[
\prod_{i=0}^{n-1} \alpha^i = \alpha^{\frac{n(n-1)}{2}} = (-1)^{n+1}
\]
and the constant term is \( \det(Y_y) \). So \( \det(f_y(\lambda)) = (-1)^{n+1}(\lambda^n - k^n) \) for some \( k \in C \). This proves (i). Statement (ii) follows from the Cramer’s rule.  

Let \( u \) be a local solution of \( n + 1 \)-th flow in the \( GD_n \)-hierarchy that admits a reduced wave function \( m \), and \( \tilde{E} \) the trivialization of \( u \). Suppose at each \((x, t)\) we can find \( \tilde{y}(x, t) \) such that
(i) \( \det(Y_{\tilde{y}(x, t)}) = (-k)^n \), and
(ii) \( \tilde{E}(x, t, \lambda) = \frac{a\lambda + Y_y}{\lambda - k} E(x, t, \lambda) \left( \frac{a\lambda + Y_y(x, t)}{\lambda - k} \right)^{-1} \) is holomorphic in \( \lambda \in C \).

Then we can proceed the same way as in the proof of Theorem 4.3 to conclude that \( \tilde{E} \) is the trivialization of some local solution of the \((n + 1)\)-th flow \( \tilde{u} \) in the \( sl(n, C) \)-hierarchy. First notice that the denominators in \( \tilde{E} \) can be canceled. So we get
\[
\tilde{E}(x, t, \lambda) = (a\lambda + Y_y)E(x, t, \lambda)(a\lambda + Y_{\tilde{y}(x, t)})^{-1}.
\]  
(14.14)

Since all three terms in the right hand side of (13.4) satisfy the \( GD_n \)-reality condition, \( \tilde{E} \) also satisfies the \( GD_n \)-reality condition. Hence \( \tilde{E} \) corresponds to a new solution \( \tilde{u} \) of the \((n + 1)\)-th flow in the \( GD_n \)-hierarchy. Proposition 13.10 implies that the image of \( (ak + Y_y)^{-1} \) is the kernel of \( (ak + Y_y) \). But formula (14.14) is holomorphic in \( \lambda \in C \) implies that the residue of the right hand side at \( \lambda = k \) is zero. Hence
\[
\text{Ker}(ak + Y_y) = E(x, t, k)(\text{Im}(ak + Y_{\tilde{y}(x, t)})^{-1}) = E(x, t, k)(\text{Ker}(ak + Y_{\tilde{y}(x, t)})).
\]

Therefore, we need to find the relation between the zeros and kernels of \( a\lambda + Y_y \) and \( y \). We do this in the following few Propositions.

14.16 Definition. Let \( C : C^n \to gl(n) \) denote the map defined by \( C(v) = \) the matrix whose first column is \( v \) and whose \( i + 1 \)-th column is \( \phi_n(\alpha^i k)\phi_n(k)^{-1}(v) \) for \( 1 \leq i \leq n - 1 \).

Let \( \ell_n : C^n \to C \) denote the projection onto the \( n \)-th coordinate.

14.17 Proposition. Suppose \( \det(f_y(\lambda)) = (-1)^{n+1}(\lambda^n - k^n) \) and \( 0 \neq v \in C^n \) such that \( f_y(k)(v) = 0 \). Then
(i) \( f(\alpha^j k)v_j = 0 \), where \( v_j = \phi_n(\alpha^j k)\phi_n(k)^{-1}v \) for \( 1 \leq j \leq n - 1 \),
(ii) if \( k \neq 0 \), then \( C(v) \) is non-singular and \( Y_y = -kaC(v)aC(v)^{-1} \).
PROOF. Since $\phi_n(k)^{-1} f(k) \phi_n(k) = \phi(ak)^{-1} f(ak) \phi_n(ak)$, (i) follows.
Use (i), we get $(\alpha^j ka + Y_y) v_j = 0$, so $Y_y v_j = -\alpha^j k a v_j$. Write this in terms of matrix to get $Y_y C(c) = -ka C(v) a$, which proves (ii).

Let $k \in C$, and $v \in C^n$ a non-zero vector. Set
\[ h_{k,v}(\lambda) = a\lambda - ka C(v) a C(v)^{-1}. \] (14.15)
The above Proposition says that if $\text{det}(Y_y) = (-k)^n$ and $(ak + Y_y)v = 0$, then
\[ f_y(\lambda) = h_{k,v}(\lambda). \]
Now given any $k \in C$ and $v \in C^n$, does $h_{k,v}$ satisfies the $GD_n$-reality condition? We will answer this next.

Let $Y_n$ denote the set of all $y = (y_0, y_1, \cdots, y_{n-1}) \in C^n$ such that
\[ \text{det}(Y_y) = \text{det}(b + \sum_{i=0}^{n-1} y_i f_i) \neq 0, \]
and
\[ \Delta = \{ k \in C \mid 0 \leq \text{arg}(k) < \frac{2\pi}{n}, k \neq 0 \}, \]
\[ V_n = \{ (r_1, \cdots, r_{n-1}, 1)^t \mid r_i \in C \}. \]

14.18 Proposition. Let $K_n : Y_n \to \Delta \times V_n$ be the map defined by $K_n(y) = (k, v)$, where $k \in \Delta$ such that $\text{det}(Y_y) = (-k)^n$ and $(ak + Y_y)v = 0$. Then
(i) $K_n$ is bijective,
(ii) both $K_n$ and $K_n^{-1}$ are algebraic maps.

PROOF. Proposition 14.17 implies that $K_n$ is one to one.

Let $k \in C$ be non-zero, and $v = (r_1, \cdots, r_{n-1}, 1)^t$. To prove $K_n$ is onto is equivalent to prove the following linear system has a non-zero solution $y$:
\[ (ak + b + \sum_{i=0}^{n-1} y_i f_i)v = 0. \] (14.16)
Write $f_i = \sum_{k=1}^{n-i} c_{k,k+i} e_{k,k+i}$. We claim that $c_{k,k+i} \neq 0$ for all $1 \leq k \leq n-i$. To see this, we recall that $f_i = s_i^{-1} \Lambda^i$ and $\Lambda = \sum_{i=1}^{n-1} s_i e_{i,i+1}$, where $s_i = \sum_{k=0}^{i-1} \alpha^k$.
Since $s_1, \cdots, s_{n-1}$ are non-zero, our claim is proved. System (14.16) in matrix form is
\[
\begin{pmatrix}
  k + y_0 & c_{11} y_1 & \cdots & c_{1n} y_{n-1} \\
  1 & \alpha k + y_0 \\
  0 & 1 \\
  1 & \alpha^{n-2} k + y_0 & c_{n-1,n} y_1 & c_{n-1,n+1} y_2 & \cdots & c_{n-1,n-1} y_{n-2} & c_{n-1,n} y_{n-1}
\end{pmatrix}
\begin{pmatrix}
  r_1 \\
  r_2 \\
  \vdots \\
  r_{n-1} \\
  0
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{pmatrix}. \] (14.17)
The last equation in (14.17) implies \( y_0 = -(r_{n-1} + \alpha^{n-1} k) \). Substitute \( y_0 \) to the \((n-1)\)-th equation to get
\[
y_1 = -\frac{1}{c_{n-1,n}} (r_{n-2} + (\alpha^{n-2} k - r_{n-1} - \alpha^{n-1} k) r_{n-1}).
\]
The \((n-j)\)-th equation gives a recursive formula for \( y_j \) in terms of \( y_1, \ldots, y_{j-1} \). This is due to the fact that the matrix is the sum of \( b = e_{21} + e_{32} + \cdots + e_{nn-1} \) and an upper triangular matrix. So \( y \) is solved explicitly in terms of \( k, v \).

Using the same kind of arguments as for the KdV- and KW-hierarchies we get local Darboux and Bäcklund transformations. We give an outline of the results. Let \( q = (0, q_1, \ldots, q_{n-1}) \), and
\[
u = Y_q = b + \sum_{i=1}^{n-1} q_i f_i\]
a solution of the \( n + 1 \)-th flow in the \( GD_n \)-hierarchy on \( S_{\phi_n} \), and \( E \) the trivialization of \( u \) normalized at \((0,0)\). Let \((k, v) \in \Delta \times \bar{V}_n \). Set
\[
\tilde{v}(x, t) = E(x, t, k)^{-1}(v).
\]
Since \( \tilde{v}(0,0) = v \), there exists an open neighborhood \( \mathcal{O} \) of \((0,0)\) in \( R^2 \) such that \( \ell_n(\tilde{v}(x, t)) \neq 0 \) for all \((x, t) \in \mathcal{O} \). Set
\[
\tilde{y}(x, t) = K_n^{-1} \left( k, \frac{\tilde{v}(x, t)}{\ell_n(\tilde{v}(x, t))} \right).
\]
Then
(i) \( \tilde{u} = au^{-1} + [Y_{\tilde{y}(x,t)}, a] a^{-1} \) is again a solution of the \( n + 1 \)-th flow in the \( GD_n \)-hierarchy. In other words, \( \tilde{u} = Y_{\tilde{q}} \), where \( \tilde{q} = (0, \tilde{q}_1, \ldots, \tilde{q}_{n-1}) \) and
\[
\tilde{q}_i = \alpha^{-i} q_i + (1 - \alpha^{-i}) y_i, \quad 1 \leq i \leq n - 1.
\]
Let
\[
\theta_{k,v} = \frac{h_{k,v}}{\lambda - k}, \quad \tilde{u} = \theta_{k,v} \ast u, \quad \tilde{q} = \theta_{k,v} \ast q,
\]
where \( h_{k,v} \) is defined by formula (14.15) \((h_{k,v} = f_y \text{ if } y = K_n^{-1}(k, v))\). This gives the Darboux transformation for the \( (n+1) \)-th flow in the \( GD_n \)-hierarchy.
(ii) \( Y_{\tilde{y}(x,t)} \) is a solution of
\[
\begin{align*}
Y_x &= Yu - (au^{-1} + [Y, a] a^{-1}) Y, \\
Y_t &= Y Q_{a,n+1}(u) - Q_{a,n+1}(au^{-1} + [Y, a] a^{-1}) Y,
\end{align*}
\]
where \( Q_{a,n+1} \) is the polynomial differential operator defined in the \( sl(n, C) \)-hierarchy. This gives the Bäcklund transformation.
14.19 Example. The trivialization $E_0$ of the vacuum $u = 0$ is

$$E_0(x, t, \lambda) = e^{a(\lambda x + b) + a\lambda^{n+1}t},$$

where $b = e_{21} + \cdots + e_{n,n-1}$. Given $(k, v) \in \Delta \times V_n$, set

$$\tilde{v}(x, t) = e^{-a(kx + b) - ak^{n+1}t}v,$$

$$\tilde{y}(x, t) = K_n^{-1}\left(k, \frac{\tilde{v}(x, t)}{\ell_n(\tilde{v}(x, t))}\right),$$

$$q_j = (1 - \alpha^{-j})\tilde{y}_j.$$  
Then $u = Y_q$ is a solution of the $j$-th flow in the $GD_n$-hierarchy.

We obtain the following relation among simple elements:

14.20 Proposition. Let $(k_1, v_1), (k_2, v_2) \in \Delta \times V_n$ such that $k_1^n \neq k_2^n$. Then

$$h_{k_2, \xi_2}h_{k_1, v_1} = h_{k_1, \xi_1}h_{k_2, v_2}$$

if and only if $\xi_1$ is parallel to $h_{k_2, v_2}(k_1)(v_1)$ and $\xi_2$ is parallel to $h_{v_1, k_1}(k_2)(v_2)$.

PROOF. Write $h_{k_i, v_i} = a\lambda + Y_i$ and $h_{k_i, \xi_i} = a\lambda + Z_i$. If

$$(a\lambda + Z_2)(a\lambda + Y_1) = (a\lambda + Z_1)(a\lambda + Y_2),$$

then set $\lambda = k_2$ to get

$$(ak_2 + Z_2)(ak_1 + Y_1)(v_2) = (ak_2 + Z_1)(ak_2 + Y_2)(v_2).$$

But the right hand side is zero by definition of $Y_{v,k}$. So $(ak_2 + Y_1)(v_2)$ lies in the kernel of $(ak_2 + Z_2)$, which is $C\xi_2$. This implies that $\xi_2$ is parallel to $(ak_2 + Y_1)(v_2)$. Similarly, $\xi_1$ is parallel to $(ak_1 + Y_2)(v_1)$. ■

We give the Permutability formula next. Let $u_0 = Y_{q(0)}$ be a solution of the $(n + 1)$-th flow in the $GD_n$-hierarchy, and $E(x, t, \lambda)$ the trivialization of $u_0$ normalized at $(0, 0)$. Let $(k_1, v_1), (k_2, v_2) \in \Delta \times V_n$ such that $k_1^n \neq k_2^n$, and

$$u_i = \theta_{k_i, v_i} \ast u_0, \quad q^{(i)} = \theta_{k_i, v_i} \ast q^{(0)}.$$

Set

$$\tilde{v}_i(x, t) = E(x, t, k_i)^{-1}(v_i), \quad i = 1, 2,$$

$$\xi_2(x, t) = h_{k_1, \tilde{v}_1(x, t)}(k_2)(\tilde{v}_2(x, t)),$$

$$y^{(i)}(x, t) = K_n^{-1}\left(k_i, \frac{\tilde{v}_i(x, t)}{\ell_n(\tilde{v}_i(x, t))}\right) \quad i = 1, 2,$$

$$q_j^{(i)} = \alpha^{-j}q_j^{(0)} + (1 - \alpha^{-j})y_j^{(i)}, \quad i = 1, 2, \text{ and } 1 \leq j \leq n-1$$

$$\tilde{y}^{(2)}(x, t) = K_n^{-1}\left(k_2, \frac{\xi_2(x, t)}{\ell_n(\xi_2(x, t))}\right).$$

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Set
\[ q_j^{(3)} = \alpha^{-j} q_j^{(1)} + (1 - \alpha^{-j}) \tilde{y}_j^{(2)}. \]

Then
(i) \( u_3 = Y q_j^{(3)} \) is a solution of the \( GD_n \)-hierarchy,
(ii) \( u_3 = (\theta_{k2, \xi2(0,0)} \theta_{k1, v_1}) * u_0. \)
(iii) since \( y^{(i)} \) can be written as an algebraic function of \( q^{(0)} \) and \( q^{(i)} \), \( q^{(3)} \) can be written as an algebraic function of \( q^{(0)}, q^{(1)}, q^{(2)} \); this is the permutability formula for the \( GD_n \)-hierarchy.
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