Abstract

In 2009, Kong, Wang, and Lee began work on the problem of finding the edge-balanced index sets of complete bipartite graphs $K_{m,n}$ by solving the cases where $n = 1, 2, 3, 4,$ and 5, and also the case where $m = n$. In 2011, Krop and Sikes expanded upon that work by finding $EBI(K_{m,m-2a})$ for odd $m > 5$ and $1 \leq a \leq \frac{m-3}{4}$. In this paper, we provide a general solution to the edge-balanced index set problem for all complete odd bipartite graphs, thereby concluding the problem for this case.

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1 Introduction

1.1 Definitions

For a graph $G = (V, E)$ with vertex set $V$ and edge set $E$, a binary edge-labeling is an injective function $f : E \rightarrow \{0, 1\}$. An edge labeled 1 will be called a 1-edge and an edge labeled 0 will be called a 0-edge. Let $e(1)$ and $e(0)$ represent the number of edges labeled...
1 and 0, respectively. A binary edge-labeling is \textit{edge-friendly} if \(|e(1) - e(0)| \leq 1\). Call the number of 1-edges incident with a vertex \(v\) the 1-degree of \(v\), denoted \(\text{deg}_1(v)\), and the number of 0-edges incident with \(v\) the 0-degree, denoted \(\text{deg}_0(v)\), and note that the degree of \(v\) is \(\text{deg}(v) = \text{deg}_1(v) + \text{deg}_0(v)\). An edge-friendly labeling of \(G\) will induce a (possibly partial) vertex-labeling where a vertex \(v\) will be labeled 1 when \(\text{deg}_1(v) > \text{deg}_0(v)\), labeled 0 when \(\text{deg}_0(v) > \text{deg}_1(v)\), and will be unlabeled when \(\text{deg}_1(v) = \text{deg}_0(v)\). A vertex labeled 1 will be called a 1-vertex and a vertex labeled 0 will be called a 0-vertex. Let \(v(1)\) and \(v(0)\) represent the number of 1-vertices and 0-vertices, respectively. The \textit{edge-balanced index set} of \(G\) is defined as

\[
\text{EBI}(G) = \{|v(1) - v(0)| : \text{over all edge-friendly labelings of } G\}.
\]

A reader interested in the study of graph labelings may find Gallian’s dynamic survey \cite{1} helpful as an introduction to graph labeling problems.

Let \(K_{m,n}\) be a complete bipartite graph with parts of cardinality \(m\) and \(n\) where \(m \geq n\) are positive odd integers. Any edge-friendly labeling of such a complete bipartite graph has \(|e(1) - e(0)| = 1\) and since every vertex has odd degree, every vertex must be labeled either 0 or 1, so \(v(1) + v(0) = m + n\).

### 1.2 History and Motivation

The idea of a balanced labelings was introduced in 1992 by Lee, Liu, and Tan \cite{5}. In 1995, Kong and Lee provided results concerning edge-balanced graphs \cite{2}. In \cite{3}, Kong, Wang, and Lee introduced the problem of finding the \textit{EBI} of complete bipartite graphs by solving the cases where \(n = 1, 2, 3, 4,\) and 5, and also the case where \(m = n\), but left the other cases open. In \cite{4}, Krop and Sikes expanded upon the work of Kong, Wang, and Lee by finding \(\text{EBI}(K_{m,m-2a})\) for odd \(m > 5\) and \(1 \leq a \leq \frac{m-3}{4}\).

In this paper, we conclude the edge-balanced index set problem for complete bipartite graphs \(K_{m,n}\) where \(m \geq n\) are positive odd integers. In particular, we compute the maximal element of \(\text{EBI}(K_{m,n})\) and provide an edge-friendly labeling that maximizes \(|v(1) - v(0)|\). We then show that all smaller elements of the edge-balanced index set can be obtained.

### 2 Finding the maximal element of \(\text{EBI}(K_{m,n})\)

\textbf{Lemma 2.1} Let \(K_{m,n}\) be a complete bipartite graph with parts of cardinality \(m\) and \(n\), where \(m \geq n\) are positive odd integers. Then the maximal element of \(\text{EBI}(K_{m,n})\), denoted
max \( EBI(K_{m,n}) \), is

\[
\max EBI(K_{m,n}) = \begin{cases} 
2, & \text{if } n = 1, \\
m + n - 2k - 2, & \text{otherwise},
\end{cases}
\]  

(2.1)

where \( k = \lceil \frac{m-1}{n+1} \rceil \).

\textbf{Proof.}

Let \( K_{m,n} \) be a complete bipartite graph with parts \( A \) and \( B \) of cardinality \( m \geq n \), respectively, where \( m, n \) are positive odd integers. Consider an edge-friendly labeling of this graph such that \( k \) vertices in part \( A \) are labeled 0 and \( m - k \) vertices are labeled 1, and \( j \) vertices in part \( B \) are labeled 0 and \( n - j \) vertices are labeled 1. Without loss of generality, we may assume that \( v(1) \geq v(0) \). Since all vertices are labeled either 0 or 1, the maximal element of \( EBI(K_{m,n}) \) is achieved when \( k \) and \( j \) are minimized, and in this case, \( \max EBI(K_{m,n}) = v(1) - v(0) = (m - k) + (n - j) - (k + j) = m + n - 2(k + j) \).

Since \( K_{m,n} \) has \( mn \) edges, and \( mn \) is odd, any edge-friendly labeling will have either \( e(0) = \frac{mn-1}{2} \) and \( e(1) = \frac{mn+1}{2} \), or else \( e(0) = \frac{mn+1}{2} \) and \( e(1) = \frac{mn-1}{2} \); without loss of generality, we choose an edge-friendly labeling that has the former so that \( e(1) - e(0) = 1 \). In order to minimize \( k \), we want the 0-vertices in part \( A \) to be incident with 0-edges only; that is, we force the 1-edges to be incident with the other \( m - k \) vertices in this part. The number of 1-edges, \( e(1) = \frac{mn+1}{2} \), should be divided among these \( m - k \) vertices, which means the average 1-degree will be \( \frac{e(1)}{m-k} \). For each of these \( m - k \) vertices to be labeled 1, this average 1-degree must be greater than or equal to \( \frac{n+1}{2} \). Then \( k \geq \frac{m-1}{n+1} \), so we let \( k = \lceil \frac{m-1}{n+1} \rceil \), which is greater than or equal to 1 unless \( m = n = 1 \), in which case \( k = 0 \). Similarly, we want \( n - j \) vertices in part \( B \) to be labeled 1, so the average 1-degree, \( \frac{e(1)}{n-j} \), must be greater than or equal to \( \frac{m+1}{2} \). Then \( j \geq \frac{n-1}{m+1} \), so we let \( j = \lceil \frac{n-1}{m+1} \rceil \). Note that \( j = 0 \) if and only if \( n = 1 \) and \( j = 1 \) in all other cases.

Now consider a labeling where \( k < \lceil \frac{m-1}{n+1} \rceil \), say \( k = \lceil \frac{m-1}{n+1} \rceil - a \) for some positive integer \( a \). Counting the 0- and 1-degrees of vertices in part \( A \), we find that \( e(0) \leq kn + (m - k)\frac{n-1}{2} \) and \( e(1) \geq (m - k)\frac{n+1}{2} \), so that

\[
e(1) - e(0) \geq m - \left( \left\lfloor \frac{m-1}{n+1} \right\rfloor - a \right) - \left( \left\lfloor \frac{m-1}{n+1} \right\rfloor - a \right) n
\]

\[
= m - (n + 1) \left\lfloor \frac{m-1}{n+1} \right\rfloor + a(n + 1)
\]  

(2.2)

Using the division algorithm, we write \( m - 1 = (n + 1)q + r \) where \( 0 \leq r < n + 1 \). If \( r = 0 \), then \( (n + 1) \left\lfloor \frac{m-1}{n+1} \right\rfloor = m - 1 \), and (2.2) implies that \( e(1) - e(0) \geq 1 + a(n+1) > 1 \), since \( a \) and
n are positive integers. Similarly, if $0 < r < n + 1$, then $(n + 1) \left\lceil \frac{m - 1}{n + 1} \right\rceil = m + n - r$, and $\text{(2.2)}$ implies $e(1) - e(0) \geq a + r + n(a - 1) > 1$. In either case, the labeling is not edge-friendly. A similar argument applies to a labeling where $j$ is chosen as 0 instead of 1 when $n \geq 3$.

Therefore, if $m$ is a positive odd integer and $n = 1$, then $m - 1$ is even, $k = \left\lceil \frac{m - 1}{n + 1} \right\rceil = \frac{m - 1}{2}$, and $j = 0$, so that the maximal element of $\text{EBI}(K_{m,1})$ is $m + n - 2(k + j) = 2$. Moreover, for odd integers $m \geq n \geq 3$, we have that $j = 1$ and $\max \text{EBI}(K_{m,n}) = m + n - 2(k + 1)$, where $k = \left\lceil \frac{m - 1}{n + 1} \right\rceil$.

Note that $\max \text{EBI}(K_{m,n})$ is an even integer for positive odd integers $m \geq n$.

3 The $\max \text{EBI}(K_{m,n})$ labeling

Consider a complete bipartite graph $K_{m,n}$ with parts $A$ and $B$ of order $m$ and $n$, respectively, where $m \geq n$ are positive odd integers. Let $k = \left\lceil \frac{m - 1}{n + 1} \right\rceil$ and $j = \left\lceil \frac{n + 1}{m + 1} \right\rceil$. Name the vertices in part $A$ as $\{v_1, \ldots, v_m\}$ and those in part $B$ as $\{u_1, \ldots, u_n\}$. The goal is to provide an edge-friendly labeling such that the vertices $\{v_1, \ldots, v_k\}$ and $u_1$ will be 0-vertices while $\{v_{k+1}, \ldots, v_m\}$ and $\{u_2, \ldots, u_n\}$ will be 1-vertices.

In the case where $n = 1$, we have $k = \frac{m - 1}{2}$ and $j = 0$, so for $1 \leq i \leq \frac{m - 1}{2}$, we label the edge $u_1v_i$ by 0, and for $\frac{m + 1}{2} \leq i \leq m$, we label edge $u_1v_i$ by 1. Then $e(0) = \frac{m - 1}{2}$, $e(1) = \frac{m + 1}{2}$, and the labeling is edge-friendly since $e(1) - e(0) = 1$. Moreover, because $\deg_1(u_1) = \frac{m + 1}{2}$ implies $u_1$ is a 1-vertex along with vertices $v_i$ for $\frac{m + 1}{2} \leq i \leq m$, we have that $v(0) = e(0)$ and $v(1) = e(1) + 1$, which implies $v(1) - v(0) = 2$.

For odd $n \geq 3$, we have that $k = \left\lceil \frac{m - 1}{n + 1} \right\rceil$ and $j = 1$. For each integer $2 \leq i \leq n$ and for each integer $1 \leq i' \leq \frac{m + 1}{2}$, we label edge $u_iv_{s(i,i')}$, by 1, where

$$s(i, i') = \left(\left[(i - 2)\frac{m + 1}{2} + i' - 1\right] \mod (m - k)\right) + k + 1.$$  

This function counts through the integers $\{k+1, k+2, \ldots m\}$ consecutively with wraparound, distributing 1-edges as uniformly as possible among the vertices $\{v_{k+1}, \ldots, v_m\}$ in part $A$ and $\{u_2, \ldots, u_n\}$ in part $B$. At this point, we have labeled $(n - 1)\frac{m + 1}{2}$ edges by 1. Since $e(1) = \frac{mn + 1}{2}$, there are still $e(1) - (n - 1)\frac{m + 1}{2} = \frac{m - n + 2}{2} \geq 1$ edges that need to be labeled 1 to obtain an edge-friendly labeling. So, for each integer $1 \leq i' \leq \frac{m - n + 2}{2}$, we label edge $u_1v_{s(1,i')}$ by 1, where

$$s(1, i') = \left(\left[s\left(n, \frac{m + 1}{2}\right) - k + i' - 1\right] \mod (m - k)\right) + k + 1.$$  

Label all the remaining edges in $K_{m,n}$ by 0.
Counting the number of 1-edges incident with each vertex \( v_i \), where \( k + 1 \leq i \leq m \), we find that \( \deg_1(v_i) \geq \frac{m+1}{2} \), which implies that each vertex in \( \{v_{k+1}, \ldots, v_m\} \) is a 1-vertex. Since \( \deg_1(u_i) = \frac{m+1}{2} \), where \( 2 \leq i \leq n \), each vertex in \( \{u_2, \ldots, u_n\} \) is a 1-vertex as well. Since \( \deg_1(u_1) = \frac{m-n+2}{2} < \frac{m+1}{2} \), vertex \( u_1 \) will be a 0-vertex. Likewise, each of the vertices \( \{v_1, \ldots, v_k\} \) is incident with 0-edges only, so these vertices are 0-vertices. Thus, for this edge-labeling, we have the following counts (by construction):

\[
e(1) = m - n + 2 + (n-1) \frac{m+1}{2} = \frac{mn+1}{2},
\]

\[
e(0) = mn - e(1) = \frac{mn-1}{2},
\]

\[
v(1) = (m-k) + (n-1) = m + n - (k+1),
\]

and \( v(0) = k + 1 \). Then \( e(1) - e(0) = 1 \) and the labeling is edge-friendly. Moreover, under this labeling, \( v(1) - v(0) = m + n - 2(k+1) \in EBI(K_{m,n}) \), and by Lemma 2.1 \( \max EBI(K_{m,n}) = m + n - 2(k+1) \). Thus, we have an edge-friendly labeling of \( K_{m,n} \) for odd integers \( m \geq n \geq 3 \) that attains the maximal value in \( EBI(K_{m,n}) \).

### 4 Smaller indices

The following observation, taken from [3], is helpful in determining elements in the edge-balanced index set for graphs that have all vertices of odd degree: If \( G \) is a graph whose vertices all have odd degree, then the edge-balanced index set of \( G \) contains only even integers. Thus, when searching for terms smaller than \( \max EBI(K_{m,n}) \) in the edge-balanced index set for complete bipartite graphs with both parts of odd order, we need only produce (show there exists) an edge-friendly labeling for which the quantity \( |v(1) - v(0)| \) is even and falls between 0 and \( \max EBI(K_{m,n}) - 2 \), inclusive, if the quantity is realizable.

**Theorem 4.1** Let \( K_{m,n} \) be a complete bipartite graph with parts of cardinality \( m \) and \( n \), where \( m \geq n \) are positive odd integers. Then

\[
EBI(K_{m,n}) = \begin{cases} 
\{2\}, & \text{if } n = 1, \\
\{0, 2, \ldots, \max EBI(K_{m,n})\}, & \text{otherwise},
\end{cases}
\]

(4.1)

where \( \max EBI(K_{m,n}) \) is given by Lemma 2.1.

**Proof.** We consider an edge-friendly labeling of such a complete bipartite graph \( K_{m,n} \) as given the proof of Lemma 2.1 labeled so that \( v(1) - v(0) = \max EBI(K_{m,n}) \) as described in Section 3. If \( n = 1 \), then every edge-friendly labeling of \( K_{m,1} \) has \( |v(1) - v(0)| = \max EBI(K_{m,1}) = 2 \), so \( EBI(K_{m,1}) = \{2\} \).

For odd \( n \geq 3 \), we assume \( v(1) > v(0) \); otherwise, \( v(1) = v(0) \) and we have that \( 0 \in EBI(K_{m,n}) \). The edge-friendly condition requires that parts \( A \) and \( B \) both contain
0-vertices and 1-vertices. Without loss of generality, choose vertices \( x, y \in A \), where \( x \) is a 0-vertex and \( y \) is a 1-vertex, and note that for any such choice, we have \( \deg_0(x), \deg_1(y) \geq \frac{n+1}{2} \).

By the pigeonhole principle, vertices \( x \) and \( y \) are adjacent to at least one common neighbor \( z \) by 0-edge \( xz \) and 1-edge \( yz \). Switching the labels on \( xz \) and \( yz \) either preserves or decreases by 2 the quantity \( v(1) - v(0) \). Moreover, edge-friendliness implies that

\[
\sum_{v \in V} \deg_1(v) - \sum_{v \in V} \deg_0(v) = 2. \tag{4.2}
\]

We claim that for all such choices of vertices \( x, y \in A \), \( \deg_0(x) > \deg_1(y) \). Assume the contrary: for any 0-vertex \( x \) and 1-vertex \( y \) in part \( A \), suppose that

\[
\deg_0(x) \leq \deg_1(y). \tag{4.3}
\]

Let \( V_0 \) and \( V_1 \) represent the sets of 0- and 1-vertices, respectively. Then inequality (4.3) implies that

\[
\sum_{v \in V_1} \deg_1(v) - \sum_{v \in V_0} \deg_0(v) \geq (v(1) - v(0)) \frac{n+1}{2}. \tag{4.4}
\]

Note that the right-hand side of (4.4) is the product of the number of additional 1-vertices (since \( v(1) > v(0) \)) and the minimum of their 1-degrees. Since

\[
\sum_{v \in V_1} \deg(v) - \sum_{v \in V_0} \deg(v) = (v(1) - v(0)) n, \tag{4.5}
\]

from (4.4) it follows that

\[
\sum_{v \in V_1} \deg_0(v) - \sum_{v \in V_0} \deg_1(v) \leq (v(1) - v(0)) \frac{n-1}{2}. \tag{4.6}
\]

Expanding the sums in (4.2) gives the following:

\[
2 = \sum_{v \in V_1} \deg_1(v) + \sum_{v \in V_0} \deg_1(v) - \sum_{v \in V_1} \deg_0(v) - \sum_{v \in V_0} \deg_0(v)
\geq \sum_{v \in V_1} \deg_1(v) - \sum_{v \in V_0} \deg_0(v) + (v(1) - v(0)) \frac{n-1}{2}, \tag{4.7}
\]
by (4.6). Rearranging (4.7) gives

\[
\sum_{v \in V_1} \deg_1(v) - \sum_{v \in V_0} \deg_0(v) \leq 2 - (v(1) - v(0)) \frac{n - 1}{2},
\]

which implies by (4.4) that

\[
(v(1) - v(0)) \left( \frac{n + 1}{2} \right) \leq 2 - (v(1) - v(0)) \frac{n - 1}{2},
\]

or equivalently, \((v(1) - v(0))n \leq 2\). This is a contradiction since \(v(1) > v(0)\) and \(n \geq 3\). Thus, for all 0-vertices \(x\) and 1-vertices \(y\) in \(A\), \(\deg_0(x) > \deg_1(y)\).

We conclude that if we switch the labels on 0-edge \(xz\) and 1-edge \(yz\), where \(z\) is a common neighbor to \(x\) and \(y\), the label on vertex \(z\) will not change, but vertex \(y\) will change from being a 1-vertex to a 0-vertex before vertex \(x\) changes from being a 0-vertex to a 1-vertex. At the moment vertex \(y\) changes its label from 1 to 0, the quantity \(v(1) - v(0)\) is decreased by 2. We may repeat the process on the new edge-friendly labeling, continually decreasing the edge-balanced index by 2, until \(v(1) = v(0)\), at which time we are done. \(\square\)

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