BI-ORDERING ALMOST-DIRECT PRODUCTS OF (REDUCED) FREE GROUPS

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Abstract. In this paper, we show that almost-direct products of (reduced) free groups admit (reduced) Magnus ordering. Besides the application is already used to to well known groups, we use the main result to give an ordering of some subgroups of the automorphism group of free groups and also to give an ordering of pure monomial braid groups.

1. Introduction

A group $G$ is said to be left-orderable if there exists a strict total ordering $<$ on its elements which is invariant under left-multiplication, i.e. $g < h$ implies $kg < kh$ for all $g, h, k \in G$. If $<$ is also invariant under right-multiplication, then $G$ is said to be bi-orderable.

The motivation of this work is the technique used by Kim and Rolfsen [14] to give an explicit bi-order of pure Artin braid groups, by Yurasovskaya [23], Theodoro de Lima and de Mattos [16] for the case of homotopy string links.

If $H$ and $Q$ are bi-ordered groups, then it is easy to see that the direct product $H \times Q$ is bi-orderable under the lexicographic ordering: $hq < h'q'$ if and only if $q < q'$, or $q = q'$ and $h < h'$. Vinogradov [21] proved that bi-orderability is preserved under free products. We prove here that it also holds for almost-direct product of bi-orderable groups, meaning semi-direct products with an additional condition. Bi-orderability is not quite so well-behaved, being preserved under direct products, but not necessarily even semi-direct products, e.g. the fundamental group of the Klein bottle (see [14, Page 826]).

We note that in this paper we shall consider the semi-direct product written as $\rtimes$, so the lexicographic ordering becomes “eastern”, with comparison starting on the right. For the rest of the text, the terms lexicographic ordering refers to the “eastern” lexicographic ordering as defined here.

An almost-direct product $G = K \rtimes Q$ of two groups $K$ and $Q$ is a semi-direct product of $K$ and $Q$ such that the action of $Q$ on the abelianization of $K$ is trivial.

Let $G$ be a group. If $g, h \in G$ then $[g, h] = ghg^{-1}h^{-1}$ will denote their commutator. The reduced free group (not necessarily finitely generated) $\hat{F}$ is the quotient of the free group $F$ on the free generators $x_1, x_2, \ldots$ obtained by adding the relations $[x_j, gx_jg^{-1}] = 1$ for all $j \in \{1, 2, \ldots\}$ and all $g \in F$.

An almost-direct product of (reduced) free groups is an iterated semi-direct product of (reduced) free groups (not necessarily finitely generated) such that the action of the constituent free groups on the homology of one another is trivial. In symbols, an almost-direct product of free groups (respectively, reduced free groups) is an iterated semidirect
product
\[ G = \times_{i=1}^{\ell} L_{r_i} = L_{r_{\ell}} \times_{\alpha_{\ell}} \left( L_{r_{\ell-1}} \times_{\alpha_{\ell-1}} (\cdots \times_{\alpha_3} (L_{r_2} \times_{\alpha_2} L_{r_1}) \cdots) \right), \]
of free groups (respectively, reduced free groups) in which the action of the group \( \times_{i=1}^{j} L_{r_i} \) on \( H_1(L_{r_i}, \mathbb{Z}) \) is trivial for each \( j \) and \( k \) with \( 1 \leq j < k \leq \ell \), where \( L_{r_i} \) denote the free group \( F_{r_i} \) or the reduced free group \( \widetilde{F}_{r_i} \), respectively. In other words, the automorphisms \( \alpha_k : \times_{i=1}^{k-1} L_{r_i} \rightarrow \text{Aut}(L_{r_k}) \) which determine the iterated semidirect product structure of \( G \) are IA-automorphisms, inducing the identity on the abelianization of \( L_k \).

For the case of free groups, if \( F_{r_i} \) is freely generated by \( x_{i,p} \), for \( 1 \leq p \leq r_i \), then the group \( G = \times_{i=1}^{\ell} L_{r_i} \) is generated by these elements (for \( 1 \leq i \leq \ell \)) and has defining relations
\[ x_{i,p}^{-1}x_{j,q}x_{i,p} = \alpha_j(\alpha(x_{i,p}))(x_{j,q}) \quad \text{for} \quad 1 \leq i < j \leq \ell, 1 \leq p \leq r_i, 1 \leq q \leq r_j. \]

In the reduced free group case we just need to add the “reduced relations” \([x_{i,p}, g_{x_{i,p}}^{-1}] = 1\) for all \( 1 \leq i \leq \ell \), for all \( p \in \{1, \ldots, r_i\} \) and all \( g \in F_{r_j} \).

Almost-direct products of free groups are a very interesting family of groups and considerable progress has been made in order to understand its properties. This family contains the pure Artin braid groups and the fundamental group of fiber-type arrangements. Falk and Randell [10] used this structure to prove that fiber-type arrangement groups satisfy the celebrated LCS formula, relating the Betti numbers and the ranks of the lower central series quotients of these groups. For the Artin pure braid group, this result was obtained by other means by Kohno [15]. Using the almost-direct product structure of the group \( G \) of a fiber-type arrangement, one can also construct a finite free resolution of the integers over the group ring \( \mathbb{Z}G \) and show that \( G \) is of type \( FL \) [7], show that \( G \) is residually torsion-free nilpotent [11], orderable [14] and [20], etc. We also mention that the structure of almost-direct product of the pure Artin braid group \( P_n \) was used by Linnell and Schick [17] to show that the Artin braid group \( B_n \) fulfills the strong Atiyah conjecture over \( \mathbb{Q} \). More recently, Cohen computed the cohomology ring of almost-direct products of free groups [6]. In this note we shall consider other examples of groups in this family (see Theorem 3 below).

About the importance of the orderability property of a given group is that it implies various structural consequences about that group and derived objects. For instance, left-orderable property of a group implies that it is torsion-free. However, it also implies that its group ring has no zero-divisors, which was a natural open question. Biorderability shows that its group ring embeds in a skew field. In addition, it easily implies that a bi-orderable group has unique roots. More details about orderability may be found in [4].

It is well-known that residually torsion-free nilpotent groups are bi-orderable [20, Proposition 3.16]. However, we want to describe an explicit, calculable ordering of almost-direct product of (reduced) free groups using the reduced Magnus ordering of free groups as stated in our first result.

**Theorem 1.** Let \( G \) be an almost-direct product of (reduced) free groups. Then \( G \) is bi-orderable. The lexicographic order on \( G \), with terms in the free factors compared using the (reduced) Magnus order, is a bi-ordering for \( G \).

We say that a group \( G \) has **generalized torsion** if there exist \( g, h_1, \ldots, h_k \in G \), \( g \neq 1 \), such that \((h_1gh_1^{-1})(h_2gh_2^{-1}) \cdots (h_kgh_k^{-1}) = 1\). The following is an immediate consequence of Theorem 1.
Corollary 2. Let $G$ be an almost-direct product of (reduced) free groups. Hence, $G$ is residually nilpotent and has no generalized torsion, in particular it is torsion-free. Furthermore, the group ring $\mathbb{Z}G$ embeds in a division algebra.

This paper is divided into three sections being the first one the introduction. In Section 2 we recall the Magnus ordering of (reduced) free groups and then we prove Theorem 1. In Section 3 we shall consider some explicit families of almost-direct products of free groups and prove the following theorem.

Theorem 3. Let $n \geq 2$. Let $G$ be one of the following groups:

1. the upper McCool group $C_{b_n}^{+} \leq \text{Aut}(F_n)$,
2. the partial inner automorphism group $I_n \leq \text{Aut}(F_n)$,
3. the fundamental group of the complement of a hypersolvable arrangement $A$, $\pi_1(M_A)$,
4. the fundamental group of the orbit configuration space $F_{Z_r}(C^*, n)$, called the pure monomial braid group and denoted by $P(r, n)$.

Then, we can give a strict total ordering for $G$ which is invariant under multiplication on both sides. Furthermore, $G$ is residually nilpotent, torsion free and the group ring $\mathbb{Z}G$ embeds in a division algebra.

2. Ordering almost-direct products of (reduced) free groups

In this section we prove Theorem 1, showing that almost-direct products of (reduced) free groups admit a bi-order coming from the Magnus ordering of free groups. We start with the following well known result, that is also stated in [23, Lemma 8.10] where a proof was given.

Lemma 4 ([14, Lemma 1]). Let $G$ and $H$ be bi-ordered groups. Then the lexicographical order on $H \rtimes G$ is a bi-ordering if and only if the action of $G$ on $H$ preserves the order on $H$.

We may conclude the following immediate consequence of this lemma for almost-direct product of bi-orderable groups.

Corollary 5. Let $G = K \rtimes Q$ be an almost-direct product of two bi-ordered groups $K$ and $Q$. Then, the lexicographical order on $G$ is a bi-ordering.

2.1. The Magnus ordering of free groups. In this subsection we follow [14, Section 5]. Let $F_n$ denote the free group of rank $n$, generated by $x_1, \ldots, x_n$. By using a representation of $F_n$ into an appropriated ring, due to Magnus [18], we can produce a bi-ordering with special invariance properties.

Define $\mathbb{Z}[[X_1, X_2, \ldots, X_n]]$ to be the ring of formal power series in the non-commuting variables $X_1, X_2, \ldots, X_n$, and let $\mu: F_n \longrightarrow \mathbb{Z}[[X_1, X_2, \ldots, X_n]]$ be the Magnus map, defined on generators by

$$
\begin{align*}
\mu: \begin{cases} 
x_i \mapsto 1 + X_i \\
x_i^{-1} \mapsto 1 - X_i + X_i^2 - X_i^3 + \cdots
\end{cases}
\end{align*}
$$

Each term of a formal series has a degree, simply the sum of the exponents, and we use the usual notation $O(d)$ to denote the sum of all terms of degree at least $d$. As noted in [18], the subset $G = \{1 + O(1)\}$ of $\mathbb{Z}[[X_1, X_2, \ldots, X_n]]$ is a subgroup. The inverse of an element $1 + W$ is simply $1 - W + W^2 - \cdots$, even when $W \in O(1)$ is itself an infinite series. In [18] is also established the following result.
Theorem 6 (Magnus). The map $\mu : F_n \to G$ is injective.

Because of this, we may omit mention of the map $\mu$ and identify $F_n$ with its image, writing $\tilde{x}_1 = 1 + X_1$, etc.

We now define an ordering on $\mathbb{Z}[[X_1, X_2, \ldots, X_n]]$, following [18, Chapter 5, Exercise 5.6.10] and [3]. First declare $X_1 \prec X_2 \prec \cdots \prec X_n$. If $V, W$ are two distinct series, consider the smallest degree at which they differ, and sort the monomial terms of that degree with variables in lexicographic order, using $\prec$. Compare the coefficients of the first term, when written in the order described, at which $V$ and $W$ differ. Declare $V < W$ precisely when the coefficient of that term in $V$ is smaller than the corresponding coefficient in $W$. We call this the Magnus ordering on $\mathbb{Z}[[X_1, X_2, \ldots, X_n]]$.

We need the following result.

Theorem 7 ([14, Theorem 4]). The Magnus order on $\mathbb{Z}[[X_1, X_2, \ldots, X_n]]$ induces a bi-ordering on $G$ and hence on $F_n$. Moreover, this ordering of $F_n$ is preserved under any $\varphi \in \text{Aut}(F_n)$ which induces the identity on $H_1(F_n) = F_n/[F_n, F_n]$.

Remark 8. We note that Theorem 7 also holds for free groups of infinite rank, see [4, Theorem 3.4].

2.2. The Magnus ordering of reduced free groups. Now, we follow [23, Section 8.3]. Let $\hat{F}_k$ denote the reduced free group obtained as a quotient of a free group of rank $k$. Recall that in the last subsection we considered the Magnus expansion $\mu$ of a free group of rank $k$ into the group of units $U$ of the power series ring $\Lambda$ in non-commutative variables $\{X_1, \ldots, X_k\}$.

Let $\hat{\Lambda}$ denote the quotient ring $\Lambda$ by $\mathcal{R}$, the two-sided ideal of $\Lambda$ generated by monomials with repeats of a variable. We note that $\hat{\Lambda}$ may be described as a polynomial ring in non-commutative variables $\{X_1, \ldots, X_k\}$, such that any monomial with repeat of variable $X_i$, for some $1 \leq i \leq k$, is set to zero.

Define the reduced Magnus expansion $\hat{\mu} : \hat{F}_k \to \hat{\Lambda}$ as

$$
\hat{\mu} : \begin{cases} 
\hat{x}_i &\mapsto 1 + X_i \\
\hat{x}_i^{-1} &\mapsto 1 - X_i
\end{cases}
$$

where $\hat{x}_i$, for $1 \leq i \leq k$, are generators of $\hat{F}_k$.

In [23] is also established the following result.

Theorem 9 ([23, Theorem 7.11]). The reduced Magnus expansion $\hat{\mu} : \hat{F}_k \to \hat{\Lambda}$ is injective.

Now we define an ordering on $\hat{\Lambda}$, following [23, Definition 8.11], and we call it the reduced Magnus ordering. Let $f$ and $g$ be polynomials in $\hat{\Lambda}$. We first arrange the monomials within $f$ and $g$ by degree. Let us now assume an alphabetical order on the variables $X_1, \ldots, X_k$, for example $X_1$ is the first letter, $X_2$ is the second letter, etc. Within each degree we arrange the monomials lexicographically. We now compare $f$ and $g$ degree by degree, term by term. We find the first term at which $f$ and $g$ differ and look at its leading coefficients $\varepsilon_f$ and $\varepsilon_g$. We declare that $f > g$ if $\varepsilon_f > \varepsilon_g$.

We note that the reduced Magnus ordering does not define a left-order on $\hat{\Lambda}$. Let $\mathcal{H}$ denote the multiplicative subgroup of elements of $\hat{\Lambda}$ of the form $(1 + \text{higher order terms})$.

The following result will be useful in order to establish an ordering on almost-direct product of reduced free groups.
Theorem 10 ([23, Propositions 8.13 and 8.14]). The reduced Magnus ordering induces a bi-ordering on the subgroup $H$, and hence on $\hat{F}_k$. Moreover, this ordering of $\hat{F}_k$ is preserved under any automorphism of $\hat{F}_k$ that induces the identity on $H_1(\hat{F}_k) = \hat{F}_k/[\hat{F}_k, \hat{F}_k]$.

2.3. Ordering almost-direct products of (reduced) free groups. We start this subsection proving that almost-direct products of (reduced) free groups admits the (reduced) Magnus ordering.

**Proof of Theorem 1.** Recall that an almost-direct product of free groups (respectively, reduced free groups) is an iterated semidirect product
\[ G = \times_{i=1}^\ell L_{r_i} \rtimes_{\alpha_i} (L_{r_{i-1}} \rtimes_{\alpha_{i-1}} (\cdots \rtimes_{\alpha_3} (L_{r_2} \rtimes_{\alpha_2} L_{r_1}) \cdots)) \]
of finitely generated free groups (respectively, reduced free groups) in which the action of the group $\times_{i=1}^\ell L_{r_i}$ on $H_1(L_{r_k}, \mathbb{Z})$ is trivial for each $j$ and $k$ with $1 \leq j < k \leq \ell$, where $L_{r_i}$ denote the free group $F_{r_i}$ (or the reduced free group $\hat{F}_{r_i}$, respectively).

Hence, the result follows from an iterated procedure applying Lemma 4 and Theorem 7 for free groups (or, respectively, Theorem 10 for reduced free groups).

Consider the increasing sequence
\[ L_{r_1} \subseteq L_{r_2} \rtimes_{\alpha_2} L_{r_1} \subseteq \cdots \subseteq L_{r_\ell} \rtimes_{\alpha_\ell} (L_{r_{\ell-1}} \rtimes_{\alpha_{\ell-1}} (\cdots \rtimes_{\alpha_3} (L_{r_2} \rtimes_{\alpha_2} L_{r_1}) \cdots)) \subseteq \cdots \subseteq \times_{i=1}^\infty L_{r_i} \]
where the group $\times_{i=1}^\infty L_{r_i}$ is the direct limit. As doing in [14, Proposition 7] for pure braid groups we may show that the colimit $\times_{i=1}^\infty L_{r_i}$ admits the Magnus ordering.

The notion of order-respecting for the product of abelianiziation maps from Artin pure braid groups $P_n$ given in [14, Proposition 9] may be extended to almost-direct products of (reduced) free groups. We say that a mapping $x \longrightarrow x'$ of ordered sets is order-respecting if $x < y$ implies $x' \leq y'$.

**Proposition 11.** With the Magnus ordering (respectively, reduced Magnus ordering) on an almost-direct product of free groups (respectively, reduced free groups)
\[ \times_{i=1}^\ell L_{r_i} = L_{r_\ell} \rtimes_{\alpha_\ell} (L_{r_{\ell-1}} \rtimes_{\alpha_{\ell-1}} (\cdots \rtimes_{\alpha_3} (L_{r_2} \rtimes_{\alpha_2} L_{r_1}) \cdots)) \]
of finitely generated free groups (respectively, reduced free groups), where $L_{r_i}$ denote the free group $F_{r_i}$ or the reduced free group $\hat{F}_{r_i}$, respectively, and lexicographic ordering on the product of infinite cyclic groups, the product of the abelianization maps $ab_1 \times ab_2 \times \cdots \times ab_{n-1}$, as a mapping
\[ \times_{i=1}^\ell L_{r_i} = L_{r_\ell} \rtimes_{\alpha_\ell} (L_{r_{\ell-1}} \rtimes_{\alpha_{\ell-1}} (\cdots \rtimes_{\alpha_3} (L_{r_2} \rtimes_{\alpha_2} L_{r_1}) \cdots)) \longrightarrow \mathbb{Z}^{r_\ell} \times \mathbb{Z}^{r_{\ell-1}} \times \cdots \times \mathbb{Z}^{r_2} \times \mathbb{Z}^{r_1}, \]
is order-respecting. Moreover, it is compatible with the retraction $\rho$: $\times_{i=1}^\ell L_{r_i} \longrightarrow \times_{i=1}^{\ell-1} L_{r_i}$ in that the following diagram commutes:
\[
\begin{array}{ccc}
\times_{i=1}^\ell L_{r_i} & \longrightarrow & \mathbb{Z}^{r_\ell} \times \mathbb{Z}^{r_{\ell-1}} \times \cdots \times \mathbb{Z}^{r_2} \times \mathbb{Z}^{r_1} \\
\rho \downarrow & & \downarrow \text{projection} \\
\times_{i=1}^{\ell-1} L_{r_i} & \longrightarrow & \mathbb{Z}^{r_{\ell-1}} \times \mathbb{Z}^{r_{\ell-2}} \times \cdots \times \mathbb{Z}^{r_2} \times \mathbb{Z}^{r_1}
\end{array}
\]

**Proof.** The abelianization of the free group $F_{r_i}$ and the reduced free group $\hat{F}_{r_i}$ is the free abelian group of rank $r_i$, which we may identify with $r_i$-tuples of integers, $\mathbb{Z}^{r_i}$, and order lexicographically. In this setting the abelianization may be realized as the map $ab_{r_i}(w) = (q_1, \ldots, q_{r_i})$ where $q_j$ is the exponent sum of $x_j$ in $w$. Then, it is clear from the definition of the Magnus ordering of $F_{r_i}$ that $ab_{r_i}$ is order preserving. \(\square\)
3. Applications

It is known that all the groups in the following list admit a bi-ordering. In this paper, we may give another proof for them, by applying Theorem 1, since they are almost-direct product of (reduced) free groups and so they admit an explicit order coming from the (reduced) Magnus ordering of (reduced) free groups.

- Pure Artin braid groups. The result was proved by Kim and Rolfsen [14] and it is the main motivation of this work following here the idea given by them.
- For fibre-type hyperplane arrangements it was proved, independently, by Kim and Rolfsen [14, Theorem 19] and Paris [20, Theorem 3.15 and Proposition 3.16].
- A certain subgroup $\mathbb{K}_n$ of pure braid groups on closed orientable surfaces of genus $g \geq 1$ admits a bi-order, for free groups of infinite rank, see the paper of González-Meneses [12]. As a consequence was possible to give an order in the pure braid group on closed orientable surfaces of genus $g \geq 1$ (that is not an almost-direct product) [12].
- Homotopy string links over the disk admits a bi-order [23].
- A subgroup $\tilde{\mathbb{K}}_n$ of homotopy string links on closed orientable surfaces of genus $g \geq 1$ admits a bi-order, for reduced free groups [16]. Using this was possible to give an order in the group of homotopy string links on closed orientable surfaces of genus $g \geq 1$ (that is not an almost-direct product) [16].

In the following subsections we exhibit other groups with the structure of an almost-direct product of (reduced) free groups that, as far as we know, it was not stated in the literature that they admit a bi-ordering. So, applying Theorem 1 we may conclude the proof of Theorem 3.

3.1. Subgroups of the automorphism group of free groups. In this subsection we prove the first two items of Theorem 3.

The automorphism group of a free group is one of the most interesting groups in combinatorial group theory. This group and many of their subgroups were deeply studied however many questions still remain open. In this section we shall prove that some of its subgroups are bi-ordered, and we may use the explicit order coming from the Magnus order for free groups described in Subsection 2.2.

Let $F_n$ be a free group of rank $n \geq 2$ generated by $n$ letters $\{x_1, x_2, \ldots, x_n\}$ and $\text{Aut}(F_n)$ be its automorphism group. For any two elements $a$ and $b$ of a group $G$ its commutator is given by $[a, b] = a^{-1}b^{-1}ab$ and let $[G, G]$ denote the commutator subgroup of $G$. The group $\text{Aut}(F_n/[F_n, F_n])$ is isomorphic to the general linear group $GL_n(\mathbb{Z})$ over the ring of integers. The kernel of the natural map

$$\text{Aut}(F_n) \longrightarrow GL_n(\mathbb{Z})$$

consists of automorphisms acting identically modulo the commutator subgroup $[F_n, F_n]$ is called the $IA$-automorphism group and is denoted by $IA_n$, see [18]. Therefore, the following short exact sequence holds

$$1 \longrightarrow IA_n \longrightarrow \text{Aut}(F_n) \longrightarrow GL_n(\mathbb{Z}) \longrightarrow 1.$$ 

Nielsen (for $n \leq 3$) and Magnus (for all $n$) showed that the group $IA_n$ is generated by automorphisms

$$\varepsilon_{ijk} : \begin{cases} x_i \mapsto x_i[x_j, x_k] & \text{with } i \neq j, k, j > i, \\ x_l \mapsto x_l & \text{with } l \neq i, \end{cases} \quad \text{and} \quad \varepsilon_{ij} : \begin{cases} x_i \mapsto x_i^{-1}x_ix_j & \text{with } i \neq j, \\ x_l \mapsto x_l & \text{with } l \neq i, \end{cases}$$

Let us consider the subgroup of $IA_n$ generated by the automorphisms $\varepsilon_{ij}$, for $1 \leq i \neq j \leq n$. It is called the basis-conjugating automorphism group and is denoted by $Cb_n$. As mentioned in the introduction of [8] this subgroup has topological interpretations as the pure of motions of $n$ unlinked circles in $S^3$ and so it is known as the “group of loops” or “loop braid group”, and it is also known under the name of the pure braid-permutation group. A presentation for $Cb_n$ was obtained by McCool [19], and it is also listed in [8, Theorem 1.1] and [1, equations (1)-(3)].

The subgroup of $Cb_n$ generated by the $\varepsilon_{ij}$, for $1 \leq i < j \leq n$, denoted by $Cb^+_n$ here, is called the upper triangular McCool groups or just upper McCool groups. By [8, Theorem 1.2] we have the decomposition of $Cb^+_n$ described as an almost-direct product of free groups $Cb^+_n = \times_{j=2}^n F_{j-1}$.

Now, following [1, Section 2], in the group $Cb_n$ we consider the elements

$$c_{ni} = \varepsilon_{1i} \varepsilon_{2i} \cdots \varepsilon_{ni}, \quad i = 1, \ldots, n,$$

where $\varepsilon_{ii} = 1$, by definition. The element $c_{ni}$ is an inner automorphism of $F_n$ which is a conjugation by an element $x_i$:

$$c_{ni} : x_k \mapsto x_i^{-1} x_k x_i, \quad k = 1, \ldots, n.$$

Define a subgroup $H_n \leq Cb_n$, $n \geq 2$, which is generated by elements $c_{ni}$, for $i = 1, \ldots, n$.

In a similar way, we define subgroups $H_k \leq Cb_n$, $k = 2, 3, \ldots, n - 1$

$$H_k = \langle c_{k1}, c_{k2}, \ldots, c_{kk} \rangle.$$

The group $H_k$ is the inner automorphism group of a group $F_k = \langle x_1, x_2, \ldots, x_k \rangle$. We define the group $I_n$ called the partial inner automorphism group given by

$$I_n = \langle H_2, H_3, \ldots, H_n \rangle \leq Cb_n.$$

Follows from [1, Theorem 1] that $I_n = \times_{j=2}^n F_j$ is an almost-direct product of free groups.

As a consequence, from Theorem 1, we have that the lexicographic ordering on the upper McCool group $Cb^+_n$ and on the partial inner automorphism group $I_n$ is bi-invariant, in which each free group was given the Magnus ordering.

**Remark 12.** Is it possible to order the group of basis conjugating automorphisms of a free group $Cb_n$ since it is a residually torsion free nilpotent group [2, P. 11]. For the case of 3 strands we may give an explicit order coming from the Magnus order of free groups because $Cb_3$ is an almost-direct product of two free groups of rank $3$, see [1, Proposition 2].

### 3.2. Hypersolvable arrangements.

In this section, following [13], we consider the hypersolvable arrangements, a class which contains the fiber type arrangements (also called supersolvable arrangements).

Let $V$ be a finite dimensional complex vector space. A finite collection of linear hyperplanes $\mathcal{A}$ contained in $V$ will be called a complex arrangement in $V$. Let $M_\mathcal{A} = V \setminus \bigcup_{H \in \mathcal{A}} H$ denote the complement of $\mathcal{A}$. We define the rank of an arrangement as

$$rk \mathcal{A} = codim_V (\cap_{H \in \mathcal{A}} H).$$

The hypersolvable arrangements are a combinatorial generalization of the fiber-type (supersolvable) class. Now we describe the inductive steps given in [13, Section 1] to define it. In what follows we shall consider combinatorial conditions on an arrangement pair $(\mathcal{A}, \mathcal{B})$, $\mathcal{B} \subseteq \mathcal{A}$. Set $\overline{\mathcal{B}} = \mathcal{A} \setminus \mathcal{B}$ and denote the elements of $\overline{\mathcal{B}}$ by $\alpha, \beta, \gamma, \ldots$, and the elements of $\mathcal{B}$ by $a, b, c, \ldots$. Now we give a some definitions necessary to introduce the concept of hypersolvable arrangements.
DEFINITION 13. The arrangement $\mathcal{B}$ is closed in $\mathcal{A}$ if $rk\{\alpha, \beta, \gamma\} = 3$ for any $\alpha, \beta \in \mathcal{B}$, $\alpha \neq \beta$, and any $c \in \overline{\mathcal{B}}$.

DEFINITION 14. We say that $\mathcal{B}$ is complete in $\mathcal{A}$ if given any $a, b \in \overline{\mathcal{B}}$, $a \neq b$, there exists $\gamma \in \mathcal{B}$ such that $rk\{a, b, \gamma\} = 2$.

LEMMA 15 ([13, Lemma 1.3]). If $\mathcal{B}$ is closed and complete in $\mathcal{A}$ then

1. $rk\mathcal{A} - rk\mathcal{B} \leq 1$.
2. The element $\gamma \in \mathcal{B}$ in Definition 14 is uniquely determined by $a$ and $b$.

If $\mathcal{B}$ is closed and complete in $\mathcal{A}$ we shall put $\gamma := f(a, b)$ in Definition 14.

DEFINITION 16. The arrangement $\mathcal{B}$ is solvable in $\mathcal{A}$ if it is closed and complete in $\mathcal{A}$ and if for any distinct elements $a, b, c \in \mathcal{B}$ such that the elements $f(a, b), f(b, c), f(a, c)$ of $\mathcal{B}$ defined in Lemma 15(2) are distinct one has that $rk\{f(a, b), f(b, c), f(a, c)\} = 2$.

We are now ready for the definition of the main object of this section.

DEFINITION 17. We say that $\mathcal{A}$ is hypersolvable if there exists a sequence of arrangements $\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_i \subset \mathcal{A}_{i+1} \subset \cdots \subset \mathcal{A}_\ell$ with $rk\mathcal{A}_i = 1$, $\mathcal{A}_\ell = \mathcal{A}$ such that $\mathcal{A}_i$ is solvable in $\mathcal{A}_{i+1}$, for $i = 1, \ldots, \ell - 1$. Such a sequence will be called a composition series of $\mathcal{A}$.

We note that, from [13, Proposition 1.10], if $\mathcal{A}$ is a fiber-type arrangement (supersolvable) then it has a composition series as in the last definition, so the class of fiber-type arrangements is inside in the class of hypersolvable ones. From [10] we know that the fundamental group of the complement of a fiber-type arrangement is an almost-direct product of free groups, and it admits a bi-ordering as proved independently in [14] and [20].

Let $\mathcal{A}$ be a hypersolvable arrangement. Follows from [13, Theorem C, item (i)] that the fundamental group of the complement of $\mathcal{A}$, $\pi_1(M_{\mathcal{A}})$, is an iterated almost-direct product of free groups, the ranks of the free groups are described in [13, Corollary 4.4]. Hence, from Theorem 1 we conclude that the lexicographic ordering on $\pi_1(M_{\mathcal{A}})$ is bi-invariant, in which each free group was given the Magnus ordering.

In this way, we proved the third item of Theorem 3 and extend the result obtained in [14] and [20] of bi-ordering the fundamental group of the complement of a fiber-type arrangement (supersolvable) to the class of hypersolvable arrangements.

3.3. Pure monomial braid groups and orbit configuration spaces. In this subsection we prove the fourth item of Theorem 3.

Let $M$ be a manifold and let $G$ be a discrete group that acts properly discontinuously on $M$. Let

$$F_G(M, n) = \{(x_1, \ldots, x_n) \in M^n \mid G \cdot x_i \cap G \cdot x_j = \emptyset \text{ if } i \neq j\}$$

denote the orbit configuration space of $M$ under the action of $G$, which consists of all ordered $n$-tuples of points in $M$ which lies in distinct orbits. These spaces were introduced in [22] and have special features in their loop space homology and features of certain Lie algebras.

Let $Q^G_n$ denote the union of $n$ distinct orbits, $G \cdot x_1, \ldots, G \cdot x_n$, in $M$. From [22, Theorem 2.2], there are fibrations $F_G(M, n) \rightarrow F_G(M, i)$ with fiber over the point $(p_1, p_2, \ldots, p_i)$ in $F_G(M, i)$ given by $F_G(M \setminus Q^G_{n-i}, n - i)$. This result is a natural generalization to orbit configuration spaces of the well known Fadell-Neuwirth theorem for configuration spaces [9, Theorem 3].

The orbit configuration space $F_G(\mathbb{C}^*, n)$, where $G = \mathbb{Z}/r\mathbb{Z}$, was discussed in [5]. It is the complement of the reflection arrangement associated to the (full) monomial group $G(r, n)$, the complex reflection group isomorphic to the wreath product of the symmetric
group $S_n$ and $G = \mathbb{Z}/r\mathbb{Z}$. Its fundamental group, denoted by $P(r, n) = \pi_1(F_G(C^*, n))$, is called the *pure monomial braid group*. The special case $r = 2$ is a fiber type hyperplane arrangement, its fundamental group considered separately in [5, Theorem 1.4.3] was called the *Brieskorn generalized pure braid group* (see also [5, Remark 2.2.5]).

Follows from [5, Theorem 2.1.3 (item 4) and Proposition 2.2.2] that the pure monomial braid groups $P(r, n) = \bigoplus_{j=1}^{n} F_{r(j-1)+1}$ are almost-direct products of free groups. Hence, from Theorem 1 we conclude that the lexicographic ordering on $P(r, n)$ is bi-invariant, in which each free group was given the Magnus ordering.

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