ARCS, VALUATIONS AND THE NASH MAP

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Abstract. This paper gives a map from the set of families of arcs on a variety to the set of valuations on the rational function field of the variety. We characterize a family of arcs which corresponds to a divisorial valuation by this map. We can see that both the Nash map and a certain McKay correspondence are the restrictions of this map. This paper also gives the affirmative answer to the Nash problem for a non-normal variety in a certain category. As a corollary, we get the affirmative answer for a non-normal toric variety.

Keywords: arc space, valuation, toric variety, Nash problem

1. Introduction

In [19], Nash introduces the Nash map which associates a family of arcs through the singularities on a variety (this family is called a Nash component in this paper) to an essential divisor over the variety. In other word, Nash map is a correspondence between the set of certain families of arcs and the set of certain divisorial valuations.

On the other hand, L. Ein, R. Lazarsfeld and M. Mustață ([4]) introduce a map from the set of irreducible cylinders for a non-singular variety to the set of divisorial valuations.

In this paper, we introduce a map from the set of fat arcs to the set of valuations. Here, a fat arc is an arc which does not factor through any proper closed subvarieties. This map is a generalization of Nash map and the map by Ein, Lazarsfeld and Mustață. We can see that some fat arcs correspond to divisorial valuations and the others to non-divisorial valuations. Here, we determine the fat arcs which correspond to divisorial valuations. By this characterization we obtain many examples corresponding to divisorial valuations including Nash components and cylinders in the arc space of a non-singular variety. As a cylinder and a Nash component are of infinite dimension, one may

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have an impression that an arc corresponding to a divisorial valuation should be of infinite dimension. But our characterization gives many finite dimensional families of arcs which correspond to divisorial valuations. Another example is the arc determined by a conjugacy class of a finite group $G$ which gives the quotient variety $X = \mathbb{C}^n/G$ ([3]). The restriction of our map onto a subset of these arcs coincides with the “McKay correspondence” constructed in [11].

This paper also study the Nash problem which asks if the Nash map is bijective. This problem was posed in Nash’s preprint in 1968 (This preprint was published later as [19]). Inspired by this preprint, many people studied the arc spaces of singularities and divisors over the singular varieties (see, Bouvier [1], Gonzalez-Sprinberg [7], Hickel [8], Lejeune-Jalabert [13], [14], [15], Nobile [20], Reguera-Lopez [21]). Then, affirmative answer for the Nash problem is obtained for a minimal 2-dimensional singularity by Reguera-Lopez [21]. For non-minimal 2-dimensional singularities, we do not know the answer of the Nash problem even for a rational double point (Recently the author was informed that a French mathematician proved the affirmative answer for a rational double point). Last year, for a normal toric variety of arbitrary dimension the Nash problem is affirmatively answered but is negatively answered in general in [10]. Though there is a counter example for the Nash problem, it is still an interesting problem to clarify in which category the Nash problem is affirmatively answered. For example, this problem is still open for 2 and 3 dimensional singularities as the counter examples in [10] are normal singularity of dimension greater than or equal to 4. For non-normal singularities, nothing is known about the Nash problem. In this paper, we give the affirmative answer to the Nash problem for a non-normal singularity in a certain category. As a corollary, we obtain that for a non-normal toric variety the Nash problem is affirmative.

This paper is organized as follows: In the second section, we give the basic notions of fat arcs. The map from the set of fat arcs to the set of valuations is given here.

In the third section, we give a characterization of a fat arc which corresponds to a divisorial valuation. Some examples including a cylinder on a non-singular variety are shown. The “McKay correspondence” in [11] also appears as an example.

In the fourth section, we give the basic notions of the arc space of a toric variety, which are used in the following section.

In the fifth section, we define a pretoric variety and prove that the Nash problem is affirmative for a pretoric variety. As a non-normal
toric variety is a pretoric variety, the Nash problem is affirmative for this.

Throughout this paper, the base field is the complex number field \( \mathbb{C} \). A variety is an irreducible reduced scheme of finite type over \( \mathbb{C} \). A valuation is always a discrete valuation and a valuation ring is a discrete valuation ring.

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2. THE VALUATION ASSOCIATED TO A FAT ARC

Definition 2.1. Let \( X \) be a scheme of finite type over \( \mathbb{C} \) and \( K \supset \mathbb{C} \) a field extension. A morphism \( \alpha : \text{Spec} K[[t]] \to X \) is called an arc of \( X \). We denote the closed point of \( \text{Spec} K[[t]] \) by \( 0 \) and the generic point by \( \eta \). The image \( \alpha(0) \) of the closed point is called the center of the arc \( \alpha \). The transcendence degree of \( K \) over \( \mathbb{C} \) is called the dimension of an arc \( \alpha \). Denote the space of arcs of \( X \) by \( X_\infty \).

If \( X \) is an affine variety, the space \( X_\infty \) of arcs of \( X \) is a closed subscheme of \( \text{Spec} \mathbb{C}[x_1, x_2, \ldots] \), where \( \mathbb{C}[x_1, x_2, \ldots] \) is the polynomial ring over \( \mathbb{C} \) with countably infinite number of variables. For a scheme \( X \) of finite type over \( \mathbb{C} \), the arc space \( X_\infty \) is characterized by the following property:

Proposition 2.2. Let \( X \) be a scheme of finite type over \( \mathbb{C} \). Then
\[
\text{Hom}_\mathbb{C}(Y, X_\infty) \simeq \text{Hom}_\mathbb{C}(Y \hat{\times}_\text{Spec} \mathbb{C} \text{Spec} \mathbb{C}[[t]], \text{Spec} \mathbb{C}[[t]]), X)
\]
for an arbitrary \( \mathbb{C} \)-scheme \( Y \), where \( Y \hat{\times}_\text{Spec} \mathbb{C} \text{Spec} \mathbb{C}[[t]] \) means the formal completion of \( Y \times \text{Spec} \mathbb{C} \text{Spec} \mathbb{C}[[t]] \) along the subscheme \( Y \times \text{Spec} \mathbb{C} \{0\} \).

2.3. By thinking of the case \( Y = \text{Spec} K \) for an extension field \( K \) of \( \mathbb{C} \), we see that \( K \)-valued points of \( X_\infty \) correspond to arcs \( \alpha : \text{Spec} K[[t]] \to X \) bijectively. Based on this, we denote the \( K \)-valued point corresponding to an arc \( \alpha : \text{Spec} K[[t]] \to X \) by the same symbol \( \alpha \). The canonical projection \( X_\infty \to X \), \( \alpha \mapsto \alpha(0) \) is denoted by \( \pi_X \). If there is no risk of confusion, we write just \( \pi \).

A morphism \( \varphi : Y \to X \) of varieties induces a canonical morphism \( \varphi_\infty : Y_\infty \to X_\infty \), \( \alpha \mapsto \varphi \circ \alpha \).

The concept “thin” in the following is first introduced in [4].
Definition 2.4. Let $X$ be a variety over $\mathbb{C}$. We say an arc $\alpha : \text{Spec} K[[t]] \to X$ is thin if $\alpha$ factors through a proper closed subvariety of $X$. An arc which is not thin is called a fat arc.

An irreducible subset $C$ in $X_\infty$ is called a thin set if $C$ is contained in $Z_\infty$ for a proper closed subvariety $Z \subset X$. An irreducible subset in $X_\infty$ which is not thin is called a fat set.

In case an irreducible subset $C$ has the generic point $\gamma \in C$ (i.e., the closure $\overline{\gamma}$ contains $C$), $C$ is a fat set if and only if $\gamma$ is a fat arc.

Proposition 2.5. Let $X$ be a variety over $\mathbb{C}$ and $\alpha : \text{Spec} K[[t]] \to X$ an arc. Then, the following hold:

(i) $\alpha$ is a fat arc if and only if the ring homomorphism $\alpha^* : \mathcal{O}_{X,\alpha(0)} \to K[[t]]$ induced from $\alpha$ is injective;

(ii) Assume that $\alpha$ is fat. For an arbitrary proper birational morphism $\varphi : Y \to X$, $\alpha$ is lifted to $Y$.

Proof. As the problems are local, we may assume that $X = \text{Spec} A$ for a $\mathbb{C}$-algebra $A$.

(i). An arc $\alpha : \text{Spec} K[[t]] \to X$ does not factor through any proper closed subvariety of $X$ if and only if the generic point $\eta$ of $\text{Spec} K[[t]]$ is mapped to the generic point of $X$. This is equivalent to the injectivity of $\alpha^* : A \to K[[t]]$ and also equivalent to the injectivity of $\alpha^* : \mathcal{O}_{X,\alpha(0)} \to K[[t]]$.

(ii). Let $\varphi : Y \to X$ be a proper birational morphism. If the image $\alpha(\eta)$ is the generic point of $X$, it is in the open subset on which $\varphi$ is isomorphic. Therefore, the restriction $\text{Spec} K((t)) \to X$ of $\alpha$ is lifted to $Y$. Then, by the criterion of properness, $\alpha$ is lifted to $Y$. \qed

Definition 2.6. Let $\alpha : \text{Spec} K[[t]] \to X$ be a fat arc of a variety $X$ and $\alpha^* : \mathcal{O}_{X,\alpha(0)} \to K[[t]]$ the ring homomorphism induced from $\alpha$. By 2.5, (i), $\alpha^*$ is extended to the injective homomorphism of fields $\alpha^* : K(X) \to K((t))$, where $K(X)$ is the rational function field of $X$. Define a function $v_\alpha : K(X) \setminus \{0\} \to \mathbb{Z}$ by

$$v_\alpha(f) = \text{ord}_t \alpha^*(f).$$

Then, $v_\alpha$ is a valuation of $K(X)$. We call it the valuation corresponding to $\alpha$.

Proposition 2.7 (Upper semicontinuity). For a regular function $f$ on a variety $X$, the map $X_\infty \to \mathbb{Z}_{\geq 0}$, $\alpha \mapsto v_\alpha(f)$ is upper semicontinuous, i.e., for $r \in \mathbb{Z}_{\geq 0}$ the subset $U_r = \{\alpha \in X_\infty \mid v_\alpha(f) \leq r\}$ is open.

Proof. We may assume that $X$ is an affine variety $\text{Spec} A$. Let $X_\infty$ be $\text{Spec} A_\infty$ and let $\Lambda^* : A \to A_\infty[[t]]$ be the ring homomorphism
corresponding to the universal arc \( X_\infty \cong \text{Spec} \mathbb{C}[[t]] \rightarrow X \) of \( X \). If we write \( \Lambda^*(f) = a_0 + a_1 t + a_2 t^2 + \ldots \) with \( a_i \in A_\infty \), then \( X_\infty \setminus U_r \) is the closed subset defined by \( a_0 = a_1 = \ldots = a_r = 0 \).

**Definition 2.8.** Let \( X \) be a variety. We say that \( D \) is a *divisor over \( X \) if there is a proper birational morphism \( Y \rightarrow X \) from a normal variety \( Y \) to \( X \) and \( D \) is an irreducible divisor on \( Y \).

**Definition 2.9.** A fat arc \( \alpha \) is called a divisorial arc if \( v_\alpha = q \text{val}_D \), where \( q \in \mathbb{N} \) and \( \text{val}_D \) is the valuation defined by an irreducible divisor \( D \) over \( X \). Assume that a fat set has the generic point. Then the fat set is called a divisorial set if the generic point is a divisorial arc.

**Proposition 2.10.** For a fat arc \( \alpha \) of a variety \( X \) of dimension \( n \), the following are equivalent:

(i) \( \alpha \) is divisorial;

(ii) There is a proper birational morphism \( Y \rightarrow X \) from a normal variety \( Y \) such that the center \( \tilde{\alpha}(0) \) of the lifting \( \tilde{\alpha} \) of \( \alpha \) to \( Y \) is the generic point of a divisor on \( Y \);

(iii) Identifying \( K(X) \) with the subfield of \( K((t)) \) by \( \alpha^* \), we denote \( K[[t]] \cap K(X) \) by \( R \) and \( (t) \cap K(X) \) by \( m \); Then the transcendence degree \( \text{trdeg}_\mathbb{C} R/m = n - 1 \).

**Proof.** Assume (i), then \( v_\alpha = q \text{val}_D \) for \( q \in \mathbb{N} \) and a divisor \( D \) on \( Y \), where \( Y \) is normal and there is a proper birational morphism \( Y \rightarrow X \). Let \( \tilde{\alpha} \) is the lifting of \( \alpha \) to \( Y \). Let \( \delta \in Y \) be the generic point of the divisor over \( X \). Let \( \delta \in Y \) be the generic point of \( D \), then, by \( v_\alpha = q \text{val}_D \), we have that

\[
\text{ord}_t \alpha^* f \geq 0 \quad \text{for every } f \in \mathcal{O}_{Y,\delta}
\]

\[
\text{ord}_t \alpha^* f > 0 \quad \text{for every } f \in \mathcal{O}_{Y,\delta}
\]

which means that the field homomorphism \( \alpha^* = \tilde{\alpha}^* : K(X) = K(Y) \rightarrow K((t)) \) gives a local homomorphism \( \mathcal{O}_{Y,\delta} \rightarrow K[[t]] \). Hence, \( \tilde{\alpha}(0) = \delta \) as required in (ii).

Assume (ii), then \( \tilde{\alpha} \) induces an injective local homomorphism \( \mathcal{O}_{Y,\delta} \rightarrow K[[t]] \), where \( \delta \) is the generic point of the divisor over \( X \). Then \( R/m \supseteq \mathcal{O}_{Y,\delta}/m_{Y,\delta} \), where the transcendence degree of the right hand side over \( \mathbb{C} \) is \( n - 1 \). Therefore, \( \text{trdeg}_\mathbb{C} R/m \) must be \( n - 1 \).

Assume (iii), then by Zariski’s local uniformizing theorem ([22], see also [23, II, §14, Theorem 31]) there is a divisor \( D \) over \( X \) such that \( R = \mathcal{O}_{Y,\delta} \), where \( \delta \) is the generic point of \( D \). Since the valuation rings of \( \nu_\alpha \) and \( \text{val}_D \) are the same, it follows that \( v_\alpha = q \text{val}_D \) for some \( q \in \mathbb{N} \), which implies (i).

**Proposition 2.11.** For every divisor \( D \) over a variety \( X \) and every \( q \in \mathbb{N} \), there is a fat arc \( \alpha \in X_\infty \) of \( X \) such that \( v_\alpha = q \text{val}_D \).
Proof. Let $\varphi : Y \rightarrow X$ be a proper birational morphism such that $D$ is a divisor on a normal variety $Y$. Let $\delta$ be the generic point of $D$. Replacing $X$ by an affine open neighborhood of $\varphi(\delta)$, we may assume that $X$ is affine variety $\text{Spec} \, A$. Then, there are injections:

$$A \hookrightarrow \mathcal{O}_{Y,\delta} \hookrightarrow \hat{\mathcal{O}}_{Y,\delta} \simeq K[[t]],$$

where $\hat{\mathcal{O}}$ is the completion by the maximal ideal and $K$ is the residue field of the local ring $\mathcal{O}_{Y,\delta}$. Composing these maps and the homomorphism $K[[t]] \rightarrow K[[t']]$, $t \mapsto t'^q$, we obtain an arc $\alpha' : \text{Spec} \, K[[t']] \rightarrow X = \text{Spec} \, A$. It is easy to see that $v_{\alpha'} = q \text{val}_D$. Then, just take an image of $\alpha' : \text{Spec} \, K[[t']] \rightarrow X$ as a required $\alpha$.

Example 2.12. Let $X = \mathbb{A}^2_\mathbb{C}$ and $\alpha : \text{Spec} \, \mathbb{C}[[t]] \rightarrow X$ the arc defined by the ring homomorphism $\alpha^* : \mathbb{C}[x, y] \rightarrow \mathbb{C}[[t]]$, $x \mapsto t$, $y \mapsto e^t - 1 = t + t^2/2! + t^3/3! + \ldots$. Then $\alpha$ is an arc of $X$ with the center at 0. As $\alpha^*$ is injective, the arc $\alpha$ is fat by Proposition 2.5. But it is not divisorial, because $R/m$ defined in (iii) of Proposition 2.10 is contained in $\mathbb{C}[[t]]/(t) = \mathbb{C}$.

Example 2.13 (A cylinder on a non-singular variety [4]). Let $X$ be a non-singular variety and $C \subset X_{\infty}$ an irreducible cylinder. The paper [4] defines a valuation $\text{val}_C$ of $K(X)$ corresponding to $C$ and proves that $\text{val}_C$ is a divisorial valuation. It is easy to see that this valuation $\text{val}_C$ is the same as our valuation $v_{\gamma}$, where $\gamma$ is the generic point of $C$. In the next section, we will see another proof for the fact that $v_{\gamma}$ is divisorial.

Example 2.14. Let $\varphi : Y \rightarrow X$ be a resolution of the singularities of $X$. Let $E$ be an irreducible divisor on $Y$. Let $\beta$ be the generic point of an irreducible closed subscheme $\pi_Y^{-1}(E) \subset Y_{\infty}$. Then, $\beta$ is the lifting of $\alpha = \varphi_{\infty}(\beta)$ to $Y$ and $\beta(0)$ is the generic point of $E$. Therefore, by Proposition 2.10, $\alpha \in X_{\infty}$ is a divisorial arc. Actually it follows

$$v_{\alpha} = \text{val}_E.$$

To see this, let $v_{\alpha} = q \text{val}_E$ and $\gamma \in X_{\infty}$ be an arc such that $v_{\gamma} = \text{val}_E$ (Proposition 2.11). Then, the lifting $\tilde{\gamma}$ of $\gamma$ to $Y$ should have center $\tilde{\gamma}(0)$ at the generic point of $E$. Therefore $\tilde{\gamma} \in \pi_Y^{-1}(E)$. As $\tilde{\gamma}$ is contained in the closure of $\beta$, it follows that $v_{\alpha}(f) = v_{\beta}(f) \leq v_{\gamma}(f) = v_{\gamma}(f)$ for every $f \in \mathcal{O}_{Y,e}$ by upper semicontinuity, where $e$ is the generic point of $E$. This yields $q = 1$.

Example 2.15 (Nash components, a special case of Example 2.14). Let $X$ be a variety and $\text{Sing} X$ the singular locus of $X$. An irreducible
component \( C \) of \( \pi^{-1}_X(Sing X) \) is called a Nash component if \( C \) is not contained in \( (Sing X)_\infty \). (In [10] a Nash component is called a “good component”.) By [10, Lemma 2.12] every irreducible component of \( \pi^{-1}_X(Sing X) \) is a Nash component if the base field is of characteristic zero. Noting that our base field is \( \mathbb{C} \), let \( \pi^{-1}_X(Sing X) = \bigcup_i C_i \) be the decomposition into Nash components. Let \( \varphi : Y \rightarrow X \) be a resolution such that \( \varphi \) is isomorphic outside of \( Sing X \) and \( \varphi^{-1}(Sing X) \) is a divisor. Let \( \varphi^{-1}(Sing X) = \bigcup_j E_j \) be the decomposition into irreducible components. Then \( \varphi_\infty : \bigcup_j \pi^{-1}_Y(E_j) \rightarrow \bigcup_i C_i \) is bijective outside of \( (Sing X)_\infty \). Hence, for each \( C_i \) there is unique \( E_j \) such that \( \pi^{-1}_Y(E_j) \) is dominant to \( C_i \). Therefore, the generic point \( \beta \) of \( \pi^{-1}_Y(E_j) \) is mapped to the generic point \( \alpha \) of \( C_i \) by the morphism \( \varphi_\infty \). In [19] Nash proved that this \( E_j \) is an essential divisor over \( X \) (for the proof see also [10, Theorem 2.15]). This map 

\[
N : \{ \text{Nash components} \} \rightarrow \{ \text{essential divisors over} \; X \}, \; C_i \mapsto E_j
\]

is called Nash map and Nash problem is the problem if this map is bijective. By the discussion in Example 2.14, it follows that \( v_\alpha = \text{val}_{E_j} \). Hence, a Nash component is divisorial and our correspondence between fat arcs and the valuations gives the Nash map.

**Example 2.16.** Here, we use the notation and terminologies of [5]. Let \( M \) be the free abelian group \( \mathbb{Z}^n \) \((n \geq 1)\) and \( N \) its dual \( \text{Hom}_\mathbb{Z}(M, \mathbb{Z}) \). We denote \( M \otimes_\mathbb{Z} \mathbb{R} \) and \( N \otimes_\mathbb{Z} \mathbb{R} \) by \( M_\mathbb{R} \) and \( N_\mathbb{R} \), respectively. The canonical pairing \( \langle \; , \; \rangle : N \times M \rightarrow \mathbb{Z} \) extends to \( \langle \; , \; \rangle : N_\mathbb{R} \times M_\mathbb{R} \rightarrow \mathbb{R} \). A cone in \( N_\mathbb{R} \) generated by \( v_1, \ldots, v_n \in N \) is denoted by \( \langle v_1, \ldots, v_n \rangle \). The group ring \( \mathbb{C}[M] \) is generated by monomials \( x^m \) \((m \in M)\) over \( \mathbb{C} \). Let \( X \) be an affine toric variety defined by a cone \( \sigma \) in \( N \). In [9], for \( v \in \sigma \cap N \) we define

\[
T_\infty(v) = \{ \alpha \in X_\infty \mid \alpha(\eta) \in T, \; \text{ord}_t \alpha^*(x^u) = \langle v, u \rangle \; \text{for} \; u \in M \},
\]

where \( T \) denotes the open orbit and also the torus acting on \( X \). Then, \( T_\infty(v) \) is a divisorial fat set corresponding to \( D_v \), where, for the maximal \( q \in \mathbb{N} \) such that \( v/q \in N \) and \( D_v \) means the divisor \( q(\text{orb}_{R_{\geq 0}^R} v) \). This is proved as follows: The paper [9] proves that \( T_\infty(v) \) is irreducible and locally closed. Let \( \alpha \) be the generic point of \( T_\infty(v) \). Let \( \gamma \in X_\infty \) be a divisorial arc corresponding to \( D_v \) (Proposition 2.11). Then, by the definition of \( T_\infty(v) \), for \( u \in \sigma^\vee \cap M \)

\[
v_\alpha(x^u) = \text{ord}_t \alpha^*(x^u) = \langle v, u \rangle = \text{val}_{D_v}(x^u) = v_\gamma(x^u).
\]

For a general \( f \in \mathbb{C}[\sigma^\vee \cap M] \) we have that

\[
v_\alpha(f) = \text{ord}_t \alpha^*(f) \geq \min_{x^u \in f} \langle v, u \rangle = \text{val}_{D_v}(f) = v_\gamma(f).
\]
Here, $x^u \in f$ means that the coefficient of $x^u$ of $f$ is not zero.

On the other hand, since $\gamma$ belongs to $T_\infty(v)$ whose generic point is $\alpha$, it follows that $v_\alpha(f) \leq v_\gamma(f)$ for every $f \in \mathbb{C}[\sigma^\vee \cap M]$ by upper semicontinuity. Therefore, $v_\alpha = v_\gamma = \text{val}_{D_v}$.

In particular for a primitive $v \in \sigma \cap N$ we obtain the following which will be used later in this paper.

**Lemma 2.17.** Let $X$ be an affine toric variety defined by a cone $\sigma$ in $N$ and $\varphi : Y \rightarrow X$ be an equivariant resolution of the singularities of $X$. Let $D$ be an irreducible invariant divisor on $Y$ corresponding to $v \in \sigma \cap N$. Then the following closures coincide:

$$\varphi_\infty(\pi_1^{-1}(D)) = T_\infty(v).$$

**Proof.** It is sufficient to prove that the generic point of each hand side is contained in the other hand side. Let $\alpha$ be the generic point of $\varphi_\infty(\pi_1^{-1}(D))$. Then, by Example 2.14

$$\text{ord}_t \alpha^*(f) = \text{val}_D(f) = \min_{x^u \in f} \langle v, u \rangle,$$

for a regular function $f$ on $X$. Hence, in particular, $\text{ord}_t \alpha^*(x^u) = \langle v, u \rangle$ for $u \in \sigma^\vee \cap M$, which implies $\alpha \in T_\infty(v)$, as $\sigma^\vee \cap M$ generates the group $M$. Conversely, if $\alpha$ is the generic point of $T_\infty(v)$, $\alpha$ is a divisorial arc corresponding to $D$ by [9, Proposition 5.7]. Let $\tilde{\alpha}$ be the lifting of $\alpha$ to $Y$. Then, by Proposition 2.10, $\tilde{\alpha}(0)$ is the generic point of $D$. Therefore, $\tilde{\alpha} \in \pi_1^{-1}(D)$, which yields that $\alpha = \varphi_\infty(\tilde{\alpha}) \in \varphi_\infty(\pi_1^{-1}(D))$. \qed

### 3. A characterization of a divisorial arc

In this section, we characterize a divisorial arc. For this we start with a simple lemma. We note that this lemma follows immediately from Corollary 1 of [23, VI, §6], when both $R, R'$ in the statement of the lemma are valuation rings.

**Lemma 3.1.** Let $K \supset K'$ be an algebraic extension of fields. Let $R$ and $R'$ be valuation rings in $K$ and $K'$, respectively. Denote the maximal ideals of $R$ and $R'$ by $m$ and $m'$, respectively. Assume that $R$ dominates $R'$. Then the field extension $R/m \supset R'/m'$ is algebraic.

**Proof.** Let $v'$ be a valuation whose valuation ring is $R'$. Let $\overline{f}$ be an arbitrary element of $R/m$ and $f \in R$ an element corresponding to $\overline{f}$. Then, there is an equality

$$a_nf^n + a_{n-1}f^{n-1} + .. + a_0 = 0,$$
with \(a_j \in R'\) \((j = 0, \ldots, n)\). Let \(v'(a_i) = \min_{j=0,\ldots,n} v'(a_j)\). Put \(b_j = a_i^{-1}a_j\) for \(j = 0, \ldots, n\). Note that \(b_j \in R'\) for all \(j\) and \(b_1 = 1\). Then we obtain the equality in \(R'/m':\)

\[
\frac{b_1 f^n}{b_1} + \frac{b_{n-1} f^{n-1}}{b_1} + \ldots + \frac{b_1}{b_1} = 0,
\]

which is an algebraic equation of \(\mathcal{F}\) over \(R'/m'\).

**Lemma 3.2.** Let \(X\) and \(X'\) be varieties of the same dimension with a dominant morphism \(\varphi : X \to X'\). Let \(K \supset K'\) be a field extension. Assume there exists a commutative diagram of arcs

\[
\begin{array}{ccc}
\text{Spec} K[[t]] & \xrightarrow{\alpha} & X \\
\downarrow & & \downarrow \varphi \\
\text{Spec} K'[[t]] & \xrightarrow{\alpha'} & X'.
\end{array}
\]

Then, \(\alpha\) is divisorial if and only if \(\alpha'\) is divisorial.

**Proof.** Let \(R = K[[t]] \cap K(X)\) and \(R' = K'[[t]] \cap K(X')\) and let \(m\) and \(m'\) be the maximal ideals of \(R\) and \(R'\), respectively. Then \(R\) dominates \(R'\). Since \(K(X)\) is algebraic over \(K(X')\), the field extension \(R/m \supset R'/m'\) is algebraic by Lemma 3.1. Therefore, \(\text{trdeg}_C R/m = n - 1\) if and only if \(\text{trdeg}_C R'/m' = n - 1\). Then, by Proposition 2.10 we obtain the assertion.

**Lemma 3.3.** Let \(K \supset K'\) be a field extension and \(f_i = \sum_{j} a_{ij} t^j\) \((i = 1, \ldots, s)\) elements of \(K[[t]]\). Assume that for each \(i\) \((i = 1, \ldots, s)\) there is \(j_i\) such that \(a_{1j_1}, a_{2j_2}, \ldots, a_{sj_s}\) are algebraically independent over \(K'\). Then, by changing the numbering of \(\{i\}\), for each \(i\) we have \(l_i\) such that \(a_{1l_1}, a_{2l_2}, \ldots, a_{sl_s}\) are algebraically independent over \(K'\) and the following hold:

1. \(a_{1j} \in \overline{K'}\) for every \(j < l_1\);
2. \(a_{2j} \in \overline{K'(a_{1j_1})}\) for every \(j < l_2\);
3. \ldots
4. \(a_{sj} \in \overline{K'(a_{1l_1}, \ldots, a_{s-1l_{s-1}})}\) for every \(j < l_s\), where \(\overline{K'}(\ast)\) means the algebraic closure of \(K'(\ast)\) in \(K\).

**Proof.** First we define a partial order \(\leq\) in \(\mathbb{Z}_{\geq 0}^s\) as follows:

\[(b_1, \ldots, b_s) \leq (b'_1, \ldots, b'_s)\text{ if } b_i \leq b'_i \text{ for every } i.\]

Let \((l_1, \ldots, l_s)\) be a minimal element in \(M_s = \{(j_1, \ldots, j_s) \mid a_{1j_1}, a_{2j_2}, \ldots, a_{sj_s}\}\) are algebraically independent over \(K'\).

Let \(A_i = \{a_{ij} \mid j < l_i\}\). Then, every \(a_{ij} \in A_i\) is an element of \(\overline{K'(a_{1l_1}, \ldots, a_{sl_s})}\). Indeed, if \(a_{ij} \in A_i\) was transcendental over \(K'(a_{1l_1}, \ldots, a_{sl_s})\),
then \((l_1, \ldots, l_{i-1}, j, l_{i+1}, \ldots, l_s)\) would be a smaller element than \((l_1, \ldots, l_{i-1}, l_t, l_{i+1}, \ldots, l_s)\) in \(M_s\), which is a contradiction to the minimality of \((l_1, \ldots, l_s)\). Let \(B_i \subset \{a_{1i_1}, \ldots, a_{si_s}\}\) be a minimal subset such that \(A_i \subset K'(B_i)\). We prove by induction on \(s\) that these \(l_t\)'s are required in our lemma.

(I) The case \(s = 1\). By the minimality of \(l_1\), every element \(a_{1j} \in A_1\) is algebraic over \(K'\).

(II) The case \(s \geq 2\). Assume the assertion for \(s - 1\). By changing the numbering of \(\{i\}\), let \(\#B_1 \) attain \(\min \#B_i\). We can see that \((l_2, \ldots, l_s)\) is a minimal element in

\[
M_{s-1} = \{(j_2, \ldots, j_s) \mid a_{2j_2}, a_{3j_3}, \ldots, a_{sj_s}, \text{ are algebraically independent over } K'(a_{1i_1})\}.
\]

Indeed, if \((j_2, \ldots, j_s)\) was a smaller element than \((l_2, \ldots, l_s)\) in \(M_{s-1}\), then \((l_1, j_2, \ldots, j_s)\) would be a smaller element than \((l_1, l_2, \ldots, l_s)\) in \(M_s\), which is a contradiction to the definition of \((l_1, l_2, \ldots, l_s)\).

Then by the hypothesis of induction, we obtain

\[
\begin{align*}
(2) & \quad A_2 \subset K'(a_{1i_1}) , \\
(3) & \quad A_3 \subset K'(a_{1i_1}, a_{2l_2}) , \\
& \quad \cdots \\
(s) & \quad A_s \subset K'(a_{1i_1}, a_{2l_2}, \ldots, a_{s-1l_{s-1}}).
\end{align*}
\]

By this, we can see that \(\#B_2 \leq 1\). Hence, by the minimality of \(\#B_1\), we have \(\#B_1 \leq 1\). If \(\#B_1 = 0\), then the proof is over.

Assume \(\#B_1 = 1\) and induce a contradiction. Let \(I = \{i \mid \#B_i = 1\}\) and assume \(I = \{1, 2, \ldots, r\}\) \((r \leq s)\).

Letting \(B_i (i \in I)\) play the role of \(B_1\), we carry out the above discussion. Then, we have the inclusion \(A_{\sigma(i)} \subset K'(a_{ilk})\) corresponding to (2) for some \(\sigma(i) \in I\). This map \(\sigma : I \to I\) is bijective. Indeed, if \(\sigma(i) = \sigma(i')\) for \(i \neq i'\), it would follow \(A_{\sigma(i)} \subset K'(a_{ilk}) \cap K'(a_{il'}) = K'\), which contradicts to the minimality of \(\#B_1 = 1\). For every \(i \in I\), take an element \(a_{ilk} \in A_i\) such that \(a_{ilk} \in K'(a_{ilk}) \setminus K'\), where \(i' = \sigma^{-1}(i)\). Then, \(a_{ilk}, a_{ik}, a_{r+1l+1}, \ldots, a_{sl_s}\) are algebraically independent over \(K'\) and \((k_1, k_r, l_{r+1}, \ldots, l_s) < (l_1, l_2, l_{r+1}, \ldots, l_s)\), which is a contradiction.

\begin{lemma}
Let \(K\) be a field. For every element \(f \in K[[t]]\) of order \(d > 0\), there exists an algebraic field extension \(L \supset K\) and an element \(t' \in L[[t]]\) such that \(L[[t]] = L[[t']]\) and \(f = t'^d\) in \(L[[t']]\).
\end{lemma}

\begin{proof}
Let \(f = a_dt^d + a_{d+1}t^{d+1} + \ldots (a_d, a_{d+1}, \ldots \in K)\). Let \(L\) be the extension field of \(K\) by adding the roots of an equation \(X^d - a_d = 0\). Put \(t' = b_1t + b_2t^2 + \ldots\) Then the equation \(a_dt^d + a_{d+1}t^{d+1} + \ldots = (b_1t + b_2t^2 + \ldots)^d\) has a solution \(b_1, b_2, \ldots\) in \(L\).
\end{proof}
Definition 3.5. The extension $L[[t']] \supset K[[t]]$ in Lemma 3.4 is called a canonical extension with respect to $f$.

Theorem 3.6. Let $\alpha : \operatorname{Spec} K[[t]] \longrightarrow X$ be a fat arc of a variety $X$ of dimension $n$ and $e = \alpha(0)$. Let $\alpha^* : \mathcal{O}_{X,e} \longrightarrow K[[t]]$ be the ring homomorphism corresponding to $\alpha$. Let the codimension of the closure $\{e\}$ be $r \geq 2$. Then the following are equivalent:

(i) $\alpha$ is divisorial;
(ii) There exist $x_1, \ldots, x_r \in \mathfrak{m}_{X,e}$ such that, for a canonical extension $L[[t']] \supset K[[t]]$ with respect to $\alpha^*(x_r)$, we have $\alpha^*(x_i) = \sum_{j \geq k_i} a_{ij} t'^j$ with a set of coefficients $a_{1i}, a_{2i}, \ldots, a_{r-1i}$ ($l_i \geq k_i$) which are algebraically independent over $K' = \mathcal{O}_{X,e}/\mathfrak{m}_{X,e}$, where $\mathfrak{m}_{X,e}$ is the maximal ideal of $\mathcal{O}_{X,e}$ and we regard $K'$ as a subfield in $K$ by $\alpha^*$.

Proof. First we may assume that $X$ is affine.

(i) $\Rightarrow$ (ii). By Noether’s normalization lemma, there is a finite dominant morphism from $X$ to a non-singular variety of dimension $n$. Then, by Lemma 3.2 we may assume that $X = \mathbb{A}^n_C = \operatorname{Spec} C[x_1, \ldots, x_r, \ldots, x_n]$ and $\mathfrak{m}_{X,e}$ is generated by $r$-elements $x_1, \ldots, x_r$. Then, $K' = \mathcal{O}_{X,e}/\mathfrak{m}_{X,e} = C(x_{r+1}, \ldots, x_n)$. Let $L[[t']] \supset K[[t]]$ be a canonical extension with respect to $\alpha^*(x_r) = t'^d$. Then, the arc $\alpha' : \operatorname{Spec} L[[t']] \longrightarrow X$ induced from $\alpha$ is lifted as

$$\alpha' : \operatorname{Spec} L[[t']] \longrightarrow X' = \operatorname{Spec} C[x_1, \ldots, x_{r-1}, x_r^\frac{1}{d}, x_{r+1}, \ldots, x_n]$$

and this is divisorial by Lemma 3.2. Hence, there is a divisor $D$ over $X'$ and a local homomorphism $\alpha'' : \mathcal{O}_{Y,\delta} \longrightarrow L[[t']]$, where $Y \supset D$ and $\delta$ is the generic point of $D$. Since $\alpha''(x_r^\frac{1}{d}) = t'$, it follows that $\mathfrak{m}_{Y,\delta} = (x_r^\frac{1}{d})$.

Write $x_i = x_r^\frac{1}{d} u_i$ for a unit $u_i$ in $\mathcal{O}_{Y,\delta}$. Here we note that $a_{ij}$’s are coefficients of $\alpha''(u_i)$. Assume $a_{ij} \in \overline{K'}$ for all $i, j$, where $\overline{K'}$ is the algebraic closure of $K'$ in $L$. Then, $L[[t']] \cap K'(u_1, \ldots, u_{r-1}, x_r^\frac{1}{d}) \subset \overline{K'}[[t']]$ which yields

$$(L[[t']] \cap K(Y')) / ((t') \cap K(Y')) \subset \overline{K'}[[t']] / (t') = \overline{K'}.$$ 

Therefore

$$\text{trdeg}_C(L[[t']] \cap K(Y')) / ((t') \cap K(Y')) = n - r < n - 1,$$

which is a contradiction to Proposition 2.10. Hence, there exists $a_{ij}$ transcendental over $K'$. Let it be $a_{1l_1}$ and assume that $a_{1j} \in \overline{K'}$ for $j < l_1$. Let $R_1 := L[[t']] \cap \overline{K'}(u_1, x_r^\frac{1}{d})$ and $\mathfrak{m}_1$ the maximal ideal. Then,
$L[[t']] / (t') \supset R_1 / m_1 = \overline{K'}(a_{l_1})$. This is proved as follows: Let

$$u'_i = \left( u_1 - \sum_{j<l_1} a_{ij} x_{r_j}^{1/j} \right) / x_{r_1}^{1/j},$$

then $\overline{K'}(u_1, x_{r_1}^{1/j}) = \overline{K'}(u'_1, x_{r_1}^{1/j})$. As $u'_1 (mod m_1) = a_{1j_1}$, it follows that $R_1 / m_1 \supset \overline{K'}(a_{l_1})$. To prove the opposite inclusion, take an element $h(u'_1, x_{r_1}^{1/j}) / g(u'_1, x_{r_1}^{1/j}) \in R_1$, then the leading coefficients of $\alpha^*(h)$ and $\alpha^*(g)$ are in $\overline{K'}[a_{l_1}]$. Therefore the leading coefficient of $\alpha^*(h / g)$ is in $\overline{K'}(a_{l_1})$, which shows that the class $h / g \in R_1 / m_1$ is in $\overline{K'}(a_{l_1})$.

Next, assume that $a_{ij} \in \overline{K'}(a_{l_1})$ for every $i \geq 2$, and $j$. Then,

$$(L[[t']] \cap K(Y)) / ((t') \cap K(Y)) \subset \overline{K'(a_{l_1})[[t']]} / (t') = \overline{K'(a_{l_1})}.$$ 

Therefore

$$\text{trdeg}_C(L[[t']] \cap K(Y)) / ((t') \cap K(Y)) = n - r + 1 < n - 1,$$

which is a contradiction to Proposition 2.10. Hence, there exists $a_{ij}$ ($i \geq 2$) transcendental over $K'(a_{l_1})$. Let it be $a_{2l_2}$. By continuing this procedure, we obtain finally $R_{r-1} = L[[t']] \cap \overline{K'}(u_1, \ldots, u_{r-1}, x_{r_1}^{1/j})$ and $R_{r-1} / m_{r-1} = \overline{K'}(a_{l_1}, \ldots, a_{r-1l_{r-1}})$, with $a_{l_1}, \ldots, a_{r-1l_{r-1}}$ algebraically independent over $K'$.

(ii) $\Rightarrow$ (i). Let $X$ be of dimension $n$. We identify $K(X)$ with the subfield of $K((t))$ by the field homomorphism $\alpha^*$ induced from $\alpha$. Let $R = K[[t]] \cap K(X)$ and $m$ be the maximal ideal of $R$. Then, by Lemma 2.10 it is sufficient to prove that $\text{trdeg}_C R / m \geq n - 1$. As $\text{trdeg}_C K' = n - r$, it is sufficient to prove that $\text{trdeg}_K R / m \geq r - 1$. By $\alpha^*(x_r) = t'^d$, there is a $d$-th root of $x_r$ such that $\alpha^*: K(X) \longrightarrow K((t))$ is lifted as $\tilde{\alpha}^*: K(X)(y) \longrightarrow L((t'))$ with $y \mapsto t'$. Let $\overline{R} = L[[t']] \cap K(X)(y)$ and $\overline{m}$ the maximal ideal of $\overline{R}$. Then, by Lemma 3.1, it is sufficient to prove $\text{trdeg}_K \overline{R} / \overline{m} \geq r - 1$.

By the assumption of (iii) and Lemma 3.3 may assume that

1. $a_{1j} \in \overline{K'}$ for every $j < l_1$;
2. $a_{2j} \in \overline{K'(a_{l_1})}$ for every $j < l_2$;

\ldots

$r - 1$. $a_{r-1j} \in \overline{K'(a_{l_1}, \ldots, a_{r-2l_{r-2}})}$ for every $j < l_{r-1}$.

Then identifying an element of $K(X)(y)$ with the element in $L((t'))$ by $\tilde{\alpha}^*$, we have

$$\frac{x_1 - a_{1k} y^{k_1} - \ldots - a_{1l_1} y^{l_1 - 1}}{y^{l_1}} = a_{1l_1} + a_{1l_1+1} t' + a_{1l_1+2} t'^2 + \ldots.$$
Assume that there exist $\alpha K$ and $\beta R$. The left hand side is in $R$ be an arc defined by $a$.

Example 3.8. Let $\alpha$ and $\beta$ be independent over $C$. Let $\alpha \in \operatorname{Spec} K[[t]]$ be an arc defined by $\alpha^* : \mathbb{C} [x_1, \ldots, x_n] \rightarrow K[[t]], x_i \mapsto a_i t^{j_i}$. Then, by Corollary 3.7 $\alpha$ is a divisorial arc.

Proof. Let $L[[t']] \supset K[[t]]$ be a canonical extension with respect to $\alpha^*(x_r) = \sum_{j \geq d} b_{ij} t^j$. If we write $t = c_1 t' + c_2 t'^2 + \ldots$, then $c_k \in \overline{\mathbb{C}}(b_{ij} | j < d + k)$, where $\overline{\mathbb{C}}$ is the algebraic closure of $\mathbb{C}$ in $L$. If we write $\alpha^*(x_i) = \sum_{j} a_{ij} t^j$, then $a_{id_i} = b_{il_i} c_{d_i} + d_i$, where $d_i \in \overline{\mathbb{C}}(b_{ij} (j < d + l_i), b_{ij} (j < l_i))$. By the assumption on $b_{1l_1}, \ldots, b_{r-1l_{r-1}}$, the coefficients $a_{1l_1}, \ldots, a_{r-1l_{r-1}}$ are algebraically independent over $K'(b_{ij} (j < d + l_i), b_{ij} (j < l_i, i = 1, \ldots, r - 1))$, and therefore algebraically independent over $K'$. Then, we can apply Theorem 3.6.

The following example shows a finite dimensional divisorial arc.

Example 3.8. Let $K = \mathbb{C}(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are algebraically independent over $\mathbb{C}$. Let $\alpha : \operatorname{Spec} K[[t]] \rightarrow \mathbb{A}_\mathbb{C}^n = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$ be an arc defined by $\alpha^* : \mathbb{C} [x_1, \ldots, x_n] \rightarrow K[[t]], x_i \mapsto a_i t^{j_i}$. Then, by Corollary 3.7 $\alpha$ is a divisorial arc. We can also see that the corresponding divisorial valuation is a toric valuation $\mathcal{v}_{D_v}$, where $v = \ldots$
(v_1, \ldots, v_n). To show this, we use the notation and terminologies in [5] (see also Example 2.16). First for a monomial \( x^u (u = (u_1, \ldots, u_n) \in \sigma^v \cap M) \), it follows \( \text{ord} \alpha^*(x^u) = \langle v, u \rangle = \text{val}_{D_n}(x^u) \). Here, \( \sigma \) is the positive octant in \( N_\mathbb{R} \) defining the toric variety \( \mathbb{A}^n_\mathbb{C} \). For an element \( \sum_u b_u x^u \in \mathbb{C}[x_1, \ldots, x_n] \), it is clear that

\[
\text{ord} \alpha^*(\sum_u b_u x^u) \geq \min_{b_u \neq 0} \langle v, u \rangle = \text{val}_{D_n}(\sum_u b_u x^u).
\]

If the equality in (3.8.1) does not hold, then

\[
\sum_u b_u (\prod_{i=1}^n a_i^{u_i}) = 0,
\]

where the sum is over all \( u \) such that \( b_u \neq 0 \) and \( u \) attains \( \min_{b_u \neq 0} \langle v, u \rangle \). This equality gives an algebraic relation of \( a_1, \ldots, a_n \) over \( \mathbb{C} \), which is a contradiction.

**Example 3.9** (a cylinder on a non-singular variety [4]). Let \( X \) be a non-singular variety of dimension \( n \) and \( C \) an irreducible cylinder, i.e., \( C = \psi_m^{-1}(S) \) for an irreducible constructible set \( S \subset X_m \). Here, \( \psi_m : X_\infty \to X_m \) is the morphism of truncation. Then the valuation \( \text{val}_C \) defined in [4] is divisorial.

**Proof.** Let \( \alpha \in C \) be the generic point, then it is sufficient to prove that \( \alpha \) is divisorial. Note that \( \alpha_m = \psi_m(\alpha) \) is the generic point of \( S \). Let the codimension of the center \( e \) of \( \alpha \) be \( r \). Then, by taking a suitable open neighborhood of the center, we may assume that the closure of the center is defined by \( r \)-functions \( x_1, \ldots, x_r \). Let \( \alpha_m^* : \mathcal{O}_{X,e} \to K_m[t]/(t^{m+1}) \) be the ring homomorphism corresponding to \( \alpha_m \). Then \( \alpha_m^* \) can be extended to a local homomorphism \( \alpha_m^* : \widehat{\mathcal{O}_{X,e}} = K'[x_1, \ldots, x_r] \to K_m[t]/(t^{m+1}) \), where \( K' \) is the residue field of \( \mathcal{O}_{X,e} \). Therefore, \( \alpha_m^* \) is determined by \( K' \hookrightarrow K_m \) and the images of \( x_i \)'s. If \( \alpha_m^*(x_i) = \sum_{j=1}^m a_{ij} t^{j} \) (\( a_{ij} \in K_m \)), \( \alpha \) is given by \( \alpha^* : \widehat{\mathcal{O}_{X,e}} \to K[t] \) with the inclusion \( K' \hookrightarrow K_m \hookrightarrow K \) and the images \( \alpha^*(x_i) = \sum_{j=1}^m a_{ij} t^{j} \). Here,

\[
K = K_m(a_{i,m+1}, a_{i,m+2}, \ldots \mid i = 1, \ldots, r)
\]

and \( a_{i,m+1}, a_{i,m+2}, \ldots (i = 1, \ldots, r) \) are algebraically independent over \( K_m \). Hence, by Corollary 3.7 \( \alpha \) is divisorial. \( \square \)

**Example 3.10** (divisorial sets in [3], McKay correspondence in [11]). Let \( d \geq 1 \) be an integer and \( G \) a finite subgroup of \( \text{GL}_n(\mathbb{C}) \) of order \( d \). We fix a primitive \( d \)-th root of unity \( \zeta \in \mathbb{C} \). Let \( X \) be the quotient of \( \mathbb{A}^n_\mathbb{C} \) by the action of \( G \) and \( h : \mathbb{A}^n_\mathbb{C} \to X \) the canonical projection. The following construction of a subset \( (X^0_\infty)[g] \) of \( X_\infty \) corresponding
to a conjugacy class \([g]\) of \(G\) is due to J. Denef and F. Loeser ([3, 2.1]).
We denote the origin of \(\mathbb{A}_C^n\) and its image in \(X\) by 0.
Let \(X_0^0 = (X_\infty \setminus \text{Sing } X_\infty) \cap \pi_\infty^{-1}(0)\).
Let \(\alpha \in X_0^0\) be the arc \(\text{Spec } K[[t]] \to X\) and \(K\) the algebraic closure of \(K\).
We denote the induced arc \(\text{Spec } K[[t]] \to X\) again by \(\alpha\). Then we can lift \(\alpha\) to a morphism \(\tilde{\alpha}\) making the following diagram commutative:
\[
\begin{array}{ccc}
\text{Spec } K[[t^{1/d}]] & \xrightarrow{\tilde{\alpha}} & \mathbb{A}_C^n \\
\downarrow & & \downarrow h \\
\text{Spec } K[[t]] & \xrightarrow{\alpha} & X.
\end{array}
\]
For an element \(g \in G\) let
\[(X_0^0)\alpha = \{\alpha \in X_0^0 \mid \tilde{\alpha}(\zeta t^{1/d}) = g\tilde{\alpha}(t^{1/d})\}.\]
Then \(g\) and \(g'\) are conjugate if and only if \((X_0^0)\alpha = (X_0^0)\alpha'\).
Hence, we can define the subset \((X_0^0)[\alpha] := (X_0^0)\alpha\) for a conjugacy class \([g]\) in \(G\).
We have a decomposition
\[X_0^0 = \coprod_{[g]} (X_0^0)[\alpha],\]
with each \((X_0^0)[\alpha]\) irreducible.
We are going to show that \((X_0^0)[\alpha]\) is a divisorial set. For \(g \in G\)
taking a suitable coordinates system \(x_1, \ldots, x_n\) of \(\mathbb{A}_C^n\), we may assume that the matrix \(g\) is diagonal and the \(i\)-th diagonal coefficient is \(\zeta^{e_i}\) with \(1 \leq e_i \leq d\) (\(i = 1, \ldots, n\)). Then, we have a homomorphism
\[\Lambda : \mathbb{C}[x_1, \ldots, x_n] \to A_\infty[[t^{1/d}]], \quad x_i \mapsto t^{e_i/d} \alpha^*(x_i),\]
where \(\text{Spec } A_\infty = (\mathbb{A}_C^n)_\infty\) and \(\alpha^* : \mathbb{C}[x_1, \ldots, x_n] \to A_\infty[[t]]\) is the ring homomorphism corresponding to the universal arc on \(\mathbb{A}_C^n\). Here, we note that
\[\alpha^*(x_i) = \sum_{j=0}^\infty a_{ij} t^j,\]
where \(\{a_{ij}\}_{1 \leq i \leq n, j \geq 0}\) are algebraically independent over \(\mathbb{C}\).
By restricting \(\Lambda\) to the subring, we obtain a homomorphism \(\lambda^* : \mathbb{C}[x_1, \ldots, x_n]^G \to A_\infty[[t]]\) whose restriction also gives a homomorphism
\[\lambda^* : \mathbb{C}[x_1, \ldots, x_n]^G \to A_\infty[[t]].\]
Let \(K\) be the quotient field of \(A_\infty\). Then, the center of \(\lambda : \text{Spec } K[[t]] \to X\) is 0 and \(\lambda\) factors through the generic point \(\gamma : \text{Spec } \mathbb{C}(\gamma)[[t]] \to X\) of \((X_0^0)[\alpha] ([3, 2.3.4]).\) Therefore, in order to show that \(\gamma\) is divisorial it is sufficient to prove that \(\lambda\) is divisorial. By Lemma 3.2, it is also sufficient to show that \(\lambda' : \text{Spec } K[[t]] \to \mathbb{A}_C^n/\langle g \rangle\) is divisorial. The center of \(\lambda'\) is 0 in \(\mathbb{A}_C^n/\langle g \rangle\) and, for \(i = 1, \ldots, n\), \(x_i^d \in \mathcal{O}_{\mathbb{A}_C^n/\langle g \rangle, 0}\) is mapped to
\[ t^{e_i}(a_{i_0} + a_{i_1}t + a_{i_2}t^2 + \ldots)^d \] by \( \lambda^* \), where \( K = \mathbb{C}(a_{i_0}, a_{i_1}, a_{i_2}, \ldots \mid i = 1, \ldots, n) \) and \( a_{i_j} \)'s are algebraically independent over \( \mathbb{C} \). By Corollary 3.7 \( \lambda \) is divisorial.

To see the concrete correspondence \( (X^0_{\infty})[g] \mapsto v_\gamma \), let \( N = \mathbb{Z}^n \) be the lattice for a toric variety \( \mathbb{A}^n_\mathbb{C} \). Then \( N' = N + \frac{1}{d}(e_1, \ldots, e_n)\mathbb{Z} \) is the lattice for a toric variety \( \mathbb{A}^n_\mathbb{C}/(g) \). If we put \( v = \frac{1}{d}(e_1, \ldots, e_n) \in N' \), it follows that

\[
\text{ord}_t \lambda^*(f) = \min_{a^u \in f} \langle v, u \rangle
\]

for a regular function \( f \) on \( \mathbb{A}^n_\mathbb{C}/(g) \). The proof is the same as in Example 3.8. Therefore the valuation \( v_\lambda \) is \( \text{val}_{D_w} \). Hence, the valuation \( v_\gamma = v_1 \) is the restriction of the toric divisorial valuation \( \text{val}_{D_w} \). Consider the case \( G \subset \text{SL}_n(\mathbb{C}) \) and \( \dim X = 3 \). Restricting the map \( [g] \mapsto (X^0_{\infty})[g] \mapsto \text{val}_{D_w} \) onto the subset \( \Gamma^0 \) consisting of \([g]\)'s with \( \sum e_i = d \), we obtain the “McKay correspondence” in [11, Theorem 1.6].

**Corollary 3.11.** Let \( G \) be a finite abelian subgroup of \( \text{GL}_n(\mathbb{C}) \) acting on \( \mathbb{A}^n_\mathbb{C} \). Assume that \( X = \mathbb{A}^n_\mathbb{C}/G \) has an isolated singularity. Then,

\[
\#\{\text{essential divisors over } X\} \leq \#G - 1
\]

**Proof.** Here we also fix a primitive \( d \)-th root of unity \( \zeta \), where \( d = \#G \). As \( G \) is abelian, every conjugacy class consists of only one element of \( G \). Every element \( g \in G \) can be written as a diagonal matrix with the \( i \)-th diagonal coefficient \( \zeta^{e_i} \) with \( 1 \leq e_i \leq d \). We write \( v_g = \frac{1}{d}(e_1, \ldots, e_n) \in N' \), where \( N' \) is the lattice for a toric variety \( X \). First we prove that

\[
(X^0_{\infty})[1] \subset \overline{(X^0_{\infty})[g]}
\]

for some \( g \neq 1 \in G \). Since

\[
X^0_{\infty} = \bigsqcup_{[g]}(X^0_{\infty})[g],
\]

the irreducible components of \( X^0_{\infty} \) are the closures of \((X^0_{\infty})[g] \)'s for some \( g \in G \). If there is an irreducible component \( \overline{(X^0_{\infty})[g]} \) with \( g \neq 1 \), then its generic point is the generic point of \( \varphi_{\infty}(\pi^{-1}_{Y}(D_{v_g})) \) (c.f., Example 2.15), therefore is the generic point of \( T_{\infty}(v_g) \) by Lemma 2.17. The generic point of \((X^0_{\infty})[1]\) belongs to \( T_{\infty}(v_1) \) which is contained in the closure of \( T_{\infty}(v_g) \), because \( v_1 \geq v_g \) ([9, Proposition 4.8], see Proposition 4.5 in this paper). If \((X^0_{\infty})[1]\) is only irreducible component of \( X^0_{\infty} \), then it must be the closure of \( T_{\infty}(v_1) \). Hence, \( T_{\infty}(v_1) \supset T_{\infty}(v_g) \) for \( g \neq 1 \) which is a contradiction to \( v_1 \geq v_g \) ([9, Proposition 4.8]).

Now we obtain that the number of irreducible components of \( X^0_{\infty} \) is less than \( d \). As \( X \) has an isolated singularity at \( 0 \) the components
of $X^0_\omega$ are the Nash components. As Nash map is bijective for a toric variety ([10]), the number of the essential divisors is less than $d$. □

T. Mizutani [16] proved this corollary by an elementary way and gave examples that there are exactly $d - 1$ essential divisors.

4. Essential divisors and the arc space of a toric variety

In this section we summarize the notion of essential divisors and basic properties of the arc space of a toric variety.

When we treat a toric variety, we use the terminologies in 2.16.

**Definition 4.1.** Let $X$ be a variety, $\psi : X_1 \rightarrow X$ a proper birational morphism from a normal variety $X_1$ and $E \subset X_1$ an irreducible exceptional divisor of $\psi$. Let $\varphi : X_2 \rightarrow X$ be another proper birational morphism from a normal variety $X_2$. The birational map $\varphi^{-1} \circ \psi : X_1 \rightarrow X_2$ is defined on a (nonempty) open subset $E^0$ of $E$. The closure of $(\varphi^{-1} \circ \psi)(E^0)$ is well defined. It is called the center of $E$ on $X_2$.

We say that $E$ appears in $\varphi$ (or in $X_2$), if the center of $E$ on $X_2$ is also a divisor. In this case the birational map $\varphi^{-1} \circ \psi : X_1 \rightarrow X_2$ is a local isomorphism at the generic point of $E$ and we denote the birational transform of $E$ on $X_2$ again by $E$. For our purposes $E \subset X_1$ is identified with $E \subset X_2$. Such an equivalence class is called an exceptional divisor over $X$.

**Definition 4.2.** Let $X$ be a variety over $\mathbb{C}$. In this paper, by a resolution of the singularities of $X$ we mean a proper, birational morphism $\varphi : Y \rightarrow X$ with $Y$ non-singular such that $Y \setminus \varphi^{-1}(\text{Sing } X) \rightarrow X \setminus \text{Sing } X$ is an isomorphism. Here, $\text{Sing } X$ is the singular locus of $X$.

**Definition 4.3.** An exceptional divisor $E$ over $X$ is called an essential divisor over $X$ if for every resolution $\varphi : Y \rightarrow X$ the center of $E$ on $Y$ is an irreducible component of $\varphi^{-1}(\text{Sing } X)$.

For a toric variety $X$, an equivariant essential divisor over $X$ is a divisor $E$ over $X$ whose center on every equivariant resolution $\varphi : Y \rightarrow X$ is an irreducible component of $\varphi^{-1}(\text{Sing } X)$.

About essential divisors, we have the following:

**Proposition 4.4 ([10]).** For a toric variety $X$ the notions an essential divisor and an equivariant essential divisor coincide.
Let $\sigma$ be the cone defining an affine toric variety $X$ and $v \in \sigma \cap N$. Then $D_v$ is essential if and only if $v$ is a minimal element in
\[ S = \cup_{\tau < \sigma \text{ singular}} \tau^o \cap N, \]
with respect to the order $\leq_\sigma$ ($v \leq_\sigma v' \iff v' - v \in \sigma$). Here, $\tau^o$ means the relative interior of $\tau$.

In [9], we introduce a locally closed subset $T_\infty(v)$ of the arc space $X_\infty$ of an affine toric variety $X$ as follows (see 2.16):
\[ T_\infty(v) = \{ \alpha \in X_\infty \mid \alpha(\eta) \in T, \text{ ord}_1 \alpha^*(x^u) = \langle v, u \rangle \text{ for } u \in M \}. \]
In order to exhibit the space $X$ explicitly, we denote $T_\infty(v)$ by $T_\infty^X(v)$. The following is obtained in [9].

**Proposition 4.5 ([9]).** Let $X$ be an affine toric variety defined by a cone $\sigma$ in $N$. For $v, v' \in \sigma \cap N$, the relation $v \leq_\sigma v'$ holds if and only if $T_\infty^X(v) \supset T_\infty^X(v')$.

**Lemma 4.6.** Let $X$ be an affine toric variety defined by a cone $\sigma$ in $N$ and $v, v'$ elements in $\sigma \cap N$. Let $\varphi : Y \longrightarrow X$ be an equivariant proper birational morphism in which $D_v$ and $D_{v'}$ appear. If $v \leq_\sigma v'$, then $\varphi(D_v) \supset \varphi(D_{v'})$.

**Proof.** If $v \in \tau^o$, $v' \in \tau'^o$ for faces $\tau, \tau' < \sigma$, then $\varphi(D_v) = \overline{\text{orb} \tau}$ and $\varphi(D_{v'}) = \overline{\text{orb} \tau'}$. By the assumption of the lemma, it follows that $v' = v + v''$ for some $v'' \in \sigma$. As $v' \in \tau'$, $v, v'' \in \tau'$. Hence, $\tau < \tau'$, which yields the assertion of the lemma. \qed

At the end of this section, we prove a technical lemma which is used in the next section.

**Lemma 4.7.** Let $\sigma = \langle e_1, \ldots, e_m \rangle$ be a singular simplicial cone in $N$. Assume that one facet is non-singular, then $v = \sum_{i=1}^m b_i e_i$ ($b_i \geq 1$, $i = 1, \ldots, m$) is not minimal in $S = \cup_{\tau < \sigma \text{ singular}} \tau^o \cap N$.

**Proof.** We may assume that $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_{m-1} = (0, 0, 1, 0, 0)$, and $e_m = (a_1 c - c_1, a_2 c - c_2, \ldots, a_{m-1} c - c_{m-1}, c)$, where $a_i$’s are integers and $0 \leq c_i < c$. Here, we note that $c > 1$, since $\sigma$ is singular. Then, $v' = \frac{c}{c} e_1 + \frac{c}{c} e_2 + \ldots + \frac{c}{c} e_{m-1} + \frac{1}{c} e_m$ is in $S$ and $v' \leq_\sigma v$. \qed

5. the Nash problem for a pretoric variety

**Definition 5.1.** A variety $X$ is called a pretoric variety if

(1) there are a toric variety $Z$ with the torus $T'$ and a finite morphism $\rho : X \longrightarrow Z$ étale on $T'$,
(2) for the normalization $\nu : \overline{X} \rightarrow X$, $\overline{X}$ is a toric variety with the torus $T$ and the composite $\rho \circ \nu : \overline{X} \rightarrow Z$ is the equivariant quotient morphism by the group $N'/N$, where $N$ and $N'$ are the lattice on which the fans of $\overline{X}$ and $Z$ are defined, respectively, and

(3) the subset $\nu^{-1}(\text{Sing } X)$ is an invariant closed set on $\overline{X}$.

We will see two typical examples of a pretoric variety.

5.2 ([6]). When we say “a toric variety”, it means always a normal toric variety. Here, we introduce a not-necessarily normal affine toric variety. A not-necessarily normal affine toric variety is of the form $X_\Gamma = \text{Spec } \mathbb{C}[\Gamma]$ where $\Gamma \subset M = \mathbb{Z}^n$ is a finitely generated semigroup with 0 and $\Gamma$ generates the abelian group $M$. Then, the torus $T = \text{Spec } \mathbb{C}[M]$ acts on $X_\Gamma$. Denote by $K(\Gamma) \subset M_{\mathbb{R}}$, the convex cone which is the convex hull of $\Gamma$ and by $\overline{\Gamma}$ the intersection $K(\Gamma) \cap M$. Then, $X_\Gamma$ is a normal toric variety and the inclusion $\mathbb{C}[\Gamma] \hookrightarrow \mathbb{C}[\overline{\Gamma}]$ induces the equivariant normalization $X_\Gamma \rightarrow X_\Gamma$.

Example 5.3. A not-necessarily normal toric variety is a pretoric variety. This is proved as follows: Let $X = \text{Spec } \mathbb{C}[\Gamma]$ be a not-necessarily normal toric variety of dimension $n$ and $\overline{X} = \text{Spec } \mathbb{C}[\sigma^\vee \cap M]$ the normalization of $X$. Subdivide $\sigma^\vee$ into simplicial cones without adding any 1-dimensional cones. Let $\tau_1, \tau_2, ..., \tau_s$ be the $n$-dimensional simplicial cones which are obtained by this subdivision. We can take generators $e_1^{(i)}, ..., e_n^{(i)}$ of $\tau_i$ in $\Gamma$. Define $M_i = \oplus_{j=1}^n \mathbb{Z}e_j^{(i)}$, then $M_i$ is a subgroup of $M$ of finite index. Let $M'$ be the intersection $\bigcap_{i=1}^s M_i$. Then, $M'$ is a subgroup of $M$ of finite index. It follows that $\sigma^\vee \cap M' \subset \Gamma$. Indeed, an arbitrary element $u \in \sigma^\vee \cap M'$ is contained in $\tau_i \cap M_i$ for some $i$. Then, by the definition of $M_i$, we have that $u = \sum_{j=1}^n a_j e_j^{(i)}$ with $a_j \in \mathbb{Z}_{\geq 0}$. As $e_j^{(i)}$'s are in $\Gamma$, it follows that $u \in \Gamma$. By this inclusion $\sigma^\vee \cap M' \subset \Gamma$ we obtain a finite morphism $\rho : X \rightarrow Z = \text{Spec } \mathbb{C}[\sigma^\vee \cap M']$. The other conditions for a pretoric variety follows immediately.

The following is an example of a pretoric variety without a toric action.

Example 5.4. Let $\overline{X}$ be Spec $\mathbb{C}[x, y]$ and $X$ be Spec $\mathbb{C}[x, y^3, y^4]$, then $X$ is a non-normal toric variety with the normalization $\nu : \overline{X} \rightarrow X$. Therefore we have a diagram $\overline{X} \xrightarrow{\nu} X \xrightarrow{\rho} Z$ as in Definition 5.1. Let $X_0$ be Spec $\mathbb{C}[x, y + y^2, y^3, y^4]$, then $X_0$ is a pretoric variety with the diagram: $\overline{X} \rightarrow X_0 \rightarrow Z$. By the definition, $X_0$ does not admit a toric action.
Definition 5.5. Let $X$ be an affine pretoric variety and $\overline{X} \xrightarrow{\nu} X \xrightarrow{\rho} Z$ the diagram as in the definition 5.1. Let $T$ and $T'$ be the tori of $\overline{X}$ and $Z$, respectively. Let $\overline{X}$ and $Z$ be defined by a cone $\sigma$ in $N$ and $\sigma$ in $N'$. The dual of $N, N'$ is denoted by $M, M'$. For $v \in \sigma \cap N \subset \sigma \cap N'$ the subsets $T^X_{\infty}(v)$ and $T^Z_{\infty}(v)$ are defined, since $\overline{X}$ and $Z$ are toric varieties. Here, we define $T^X_{\infty}(v)$ for a pretoric variety $X$ as follows:

$$T^X_{\infty}(v) = \{ \alpha \in X_{\infty} \mid \alpha(\eta) \in \rho^{-1}(T') = \nu(T), \text{ ord } \alpha^*(x^u) = \langle v, u \rangle \text{ for } u \in M \},$$

Lemma 5.6. Let $X$ be an affine pretoric variety. Under the same notation as in the previous definition, for $v \in \sigma \cap N$

(i) the morphism $\nu_{\infty}$ gives a bijection $T^X_{\infty}(v) \longrightarrow T^X_{\infty}(v)$ and

(ii) the morphism $(\rho \circ \nu)_{\infty}$ gives a surjection $T^X_{\infty}(v) \longrightarrow T^Z_{\infty}(v)$.

Proof. By the definition of $T^X_{\infty}(v)$'s it is clear that the images of these morphisms are in the target sets. For the proof of (i), take an arc $\alpha \in T^X_{\infty}(v)$, then $\alpha(\eta) \in \nu(T) \simeq T$ and therefore $\alpha$ is lifted uniquely to $\overline{X}$, as $\nu$ is proper. For the proof of (ii), take an arc $\alpha \in T^Z_{\infty}(v)$. Then $\alpha$ corresponds to a ring homomorphism $\alpha^*: \mathbb{C}[\sigma^\vee \cap M'] \longrightarrow K[[t]]$. Let $\overline{K}$ be the algebraic closure of $K$ and denote the composite of $\alpha^*$ and the canonical inclusion $K[[t]] \hookrightarrow \overline{K}[[t]]$ by again $\alpha^*$. Then, by Denef and Loeser [3], we obtain the following commutative diagram for $d = \#N'/N$:

$$
\begin{array}{ccc}
\mathbb{C}[\sigma^\vee \cap M] & \xrightarrow{\alpha^*} & \overline{K}[[t^{1/d}]] \\
\cup & & \cup \\
\mathbb{C}[\sigma^\vee \cap M'] & \xrightarrow{\alpha^*} & \overline{K}[[t]].
\end{array}
$$

As $v \in \sigma \cap N \subset \sigma \cap N'$, it follows that $\langle v, u \rangle$ is a non-negative integer for $u \in \sigma^\vee \cap M$. For $u \in \sigma^\vee \cap M$, let $\tilde{\alpha}^*(x^u) = \sum_{i=0}^{\infty} a_i t^i/d$. By considering $\alpha^*(x^{du}) = \tilde{\alpha}^*(x^u)^d$, we obtain that $a_i = 0$ for $i \not\equiv 0 (\text{mod } d)$, which means $\tilde{\alpha}^*(x^u) \in \overline{K}[[t]]$. This gives the surjectivity of the morphism in (ii).

Lemma 5.7. Let $X$ be an affine pretoric variety. Under the same notation as in Definition 5.5, let $v, v' \in \sigma \cap N$. Then, $v \leq_v v'$ if and only if $T^X_{\infty}(v) \supset T^X_{\infty}(v')$.

Proof. If $v \leq_v v'$, then by Proposition 4.5 it follows that $T^X_{\infty}(v) \supset T^X_{\infty}(v')$. Hence, by Lemma 5.6, (i), we obtain $T^X_{\infty}(v) \supset T^X_{\infty}(v')$. Conversely, if $T^X_{\infty}(v) \supset T^X_{\infty}(v')$, then by Lemma 5.6, (ii), we have that $T^Z_{\infty}(v) \supset T^Z_{\infty}(v')$. Hence, by Proposition 4.5, it follows that $v \leq_v v'$. \qed
5.8. Let $X$ be an affine pretoric variety and $\mathbf{X} \xrightarrow{\nu} X \xrightarrow{\rho} Z$ the diagram as in the definition 5.1. Let $T$ and $T'$ be the tori of $\mathbf{X}$ and $Z$, respectively. Let $\mathbf{X}$ and $Z$ be defined by a cone $\sigma$ in $N$ and $\sigma$ in $N'$, respectively. The dual of $N, N'$ is denoted by $M, M'$.

As $\nu^{-1}(\text{Sing } X)$ is an invariant closed set, an irreducible component of $\nu^{-1}(\text{Sing } X) \backslash \text{Sing } \mathbf{X}$ is written by $\text{orb}_i$ for some non-singular face $\tau_i < \sigma (i = 1, \ldots, r)$. Let $e_i$ be the barycenter of $\tau_i$, i.e., $e_i$ is the sum of the generators of $\tau_i$ in $N$.

Let $v_j (j = 1, \ldots, s)$ be the minimal elements of $S = \bigcup_{\tau < \sigma: \text{singular}} \tau^o \cap N$, i.e., $D_{v_j} (j = 1, \ldots, s)$ are the essential divisors over $\mathbf{X}$ (see Proposition 4.4). We consider the minimal elements of $\{e_i, v_j\}_{j=1}^{r}$ and obtain the following:

**Lemma 5.9.** Each $e_i$ ($i = 1, \ldots, r$) is minimal among $\{e_i, v_j\}_{j=1}^{r}$ with respect to the order $\leq_{\sigma}$.

**Proof.** Let $\varphi : Y \longrightarrow \mathbf{X}$ be an equivariant resolution in which $D_{e_i}$'s and $D_{v_j}$'s appear. Then, $\varphi(D_{e_i}) = \overline{\text{orb}_i}$ is an irreducible component of $\nu^{-1}(\text{Sing } X) \backslash \text{Sing } \mathbf{X}$. Therefore, $\varphi(D_{e_i}) \not\subset \varphi(D_{e_k})$ for $k \neq i$. On the other hand, as $\varphi(D_{v_j}) \subset \text{Sing } \mathbf{X}$, it follows that $\varphi(D_{e_i}) \not\subset \varphi(D_{v_j})$. Hence, by Lemma 4.6, we obtain that $e_k \not\preceq e_i \ (k \neq i)$ and $v_j \not\preceq e_i \ (j = 1, \ldots, s)$. Thus, $e_i$ is minimal.

**Lemma 5.10.** Let $\{e_i, v_j\}_{j=1}^{r}$ ($w \leq s$) be the set of minimal elements in $\{e_i, v_j\}_{j=1}^{s}$. Then, there is the inclusion

$$\{\text{essential divisors over } X\} \subset \{D_{e_i}, D_{v_j}\}_{j=1}^{r}.$$

**Proof.** Let $D$ be an essential divisor over $X$. Let $\varphi : Y \longrightarrow X$ be a resolution on which $D$ appears, then $\varphi$ factors through the normalization:

$$Y \xrightarrow{\psi} \mathbf{X} \xrightarrow{\nu} X.$$

Here, we may assume that $\psi$ is an equivariant morphism. Then, we can put $D = D_v$ for some $v \in \sigma \cap N$. As $\varphi(D_v) \subset \text{Sing } X$, we have $\psi(D_v) \subset \text{Sing } \mathbf{X}$ or $\psi(D_v) \subset \overline{\text{orb}_i}$.

**Case 1.** $\psi(D_v) \not\subset \text{Sing } \mathbf{X}$ and $\psi(D_v) \subset \overline{\text{orb}_i}$.

In this case we show that $\psi(D_v) = \overline{\text{orb}_i}$ and $v = e_i$. Let $\psi(D_v) = \overline{\text{orb}_\gamma}$, then $\gamma$ is a non-singular face of $\sigma$ such that $\tau_i < \gamma$. Let $\psi_0 : Y_0 \longrightarrow \mathbf{X}$ be an equivariant resolution of $\mathbf{X}$. As $\psi_0$ is isomorphic away from $\text{Sing } \mathbf{X}$, $\text{orb}_\gamma, \text{orb}_i$ are still on $Y_0$. Then, take the blow up $\psi_1 : Y_1 \longrightarrow Y_0$ with the center $\overline{\text{orb}_i}$, then $Y_1$ is again non-singular. Here, $\psi_1$ corresponds to the star-shaped subdivision $\Sigma_1$ of the fan $\Sigma_0$ of $Y_0$ by $e_i$. Take a cone $\gamma'$ in $\Sigma_1$ such that $v \in \gamma'$. Here, we note that
\langle e_i \rangle < \gamma'$, because, $v$ is in the relative interior of $\gamma$ and the subdivision is star shaped with the center $e_i \in \gamma$. Thus, the center of $D_v$ on $Y_1$ is $\text{orb}\gamma'$ which is contained in $D_{e_i}$. Therefore, the center of $D_v$ can be a component of $(\nu \circ \psi \circ \psi_1)^{-1}(\text{Sing} X)$ only if $\text{orb}\gamma' = D_{e_i}$, in which case $\gamma' = \langle e_i \rangle$. This implies $v = e_i$.

**Case 2.** $\psi(D_v) \subset \text{Sing} \overline{X}$.

In this case, we prove that $v = v_j$ ($j = 1, \ldots, w$). First, $D_v$ must be an essential divisor over $\overline{X}$. Because, if $D_v$ is not an essential divisor over $\overline{X}$, there is a resolution $\psi' : Y' \rightarrow \overline{X}$ such that the center of $D_v$ on $Y'$ is not an irreducible component of $\psi'^{-1}(\text{Sing} \overline{X})$. Therefore, the center of $D_v$ cannot be an irreducible component of $(\nu \circ \psi')^{-1}(\text{Sing} \overline{X})$. Now, for the lemma, it is sufficient to prove that $D_{v_j}$ ($j > w$) is not an essential divisor over $X$. For this, we construct an equivariant birational morphism $\psi'' : Y'' \rightarrow \overline{X}$ such that $\nu \circ \psi''$ is a resolution of $X$ and the center of $D_{v_j}$ on $Y''$ is not an irreducible component of $(\nu \circ \psi'')^{-1}(\text{Sing} \overline{X})$.

As $v_j$ ($j > w$) is not minimal in $\{e_i, v_j\}_{i=1, \ldots, r}$, there is an element $e_i$ such that $e_i \leq \sigma v_j$. Then, there is an element $e_i' \in \sigma \cap N$ such that $v_j = e_i + e_i'$. Here, $e_i'$ is in the relative interior of a non-singular face $\tau$ of $\sigma$. This is proved as follows: As $v_j \in S$ (see, 5.8), $v_j \in \gamma^0$ for some singular face $\gamma < \sigma$. If $e_i'$ is in the relative interior of $\gamma$ or $e_i'$ is in the relative interior of a singular face of $\gamma$, then it contradicts to the minimality of $v_j$ in $S$.

Let $\tau = \langle f_1, \ldots, f_m \rangle$. Consider the cone $\delta = \langle e_i, f_1, \ldots, f_m \rangle$. If $\delta$ is singular, then, by Lemma 4.7, $v_j$ is not minimal in $S = \bigcup_{\tau' \subset \delta \text{ singular}} \tau'' \cap N$, therefore by [10, Lemma 3.15], we have a non-singular subdivision $\Delta$ of $\delta$ in which $\langle v_j \rangle$ does not appear as a one-dimensional cone and every non-singular face of $\delta$ does not change. Here, $\Delta$ is obtained by successive star-shaped subdivision by centers $\lambda_1, \ldots, \lambda_l$.

Now, we construct a subdivision of $\sigma$.

**Step 1.** Take the star-shaped subdivision $\Sigma_1$ of $\sigma$ by $e_i$. Here, the cone $\delta$ appears in $\Sigma_1$. Note that the morphism corresponding to this subdivision is isomorphic outside of $\text{orb}\tau_i$.

**Step 2.** If $\Sigma_1$ is simplicial, then put $\Sigma_2 = \Sigma_1$. If $\Sigma_1$ is not simplicial, then take a one-dimensional face $\langle \lambda \rangle$ of a minimal dimensional non-simplicial cone and make the star-shaped subdivision of $\Sigma_1$ by $\lambda$. Then, simplicial cones in $\Sigma_1$ do not change. Continuing this procedure, we obtain a simplicial subdivision $\Sigma_2$.

**Step 3.** If $\delta$ is non-singular, then put $\Sigma_3 = \Sigma_2$. If $\delta$ is singular, take the successive star-shaped subdivisions $\Sigma_3$ of $\Sigma_2$ with the centers $\lambda_1, \ldots, \lambda_l$. Then, the cone $\delta$ in $\Sigma_2$ is replaced by the fan $\Delta$. Note that
this subdivision does not change non-singular cones of $\Sigma_2$, therefore the morphism corresponding to this subdivision is isomorphic outside of the singularities.

**Step 4.** If $\Sigma_3$ is singular, take a cone $\lambda = \langle p_1, \ldots, p_t \rangle \in \Sigma_2$ with the maximal multiplicity. The multiplicity is $\text{vol } P_{\lambda} = \{\sum_{i=1}^t c_i p_i \mid 0 \leq c_i < 1\}$. Since $\text{vol } P_{\lambda} > 1$, there is a non-zero element $n' \in P_{\lambda} \cap N$. Take the star-shaped subdivision with the center $n'$.

Then, the multiplicities of new cones on $\lambda$ become less than $\lambda$ and all non-singular cones of $\Sigma_3$ are unchanged. Continuing this procedure, we finally obtain a non-singular subdivision $\Sigma_4$.

This subdivision $\Sigma_4$ gives a birational morphism $\psi'' : Y'' \rightarrow X$ which is isomorphic outside of $\text{orb} \tau_i \cup \text{Sing } X$. Therefore $\nu \circ \psi'' : Y'' \rightarrow X$ is a resolution of singularities of $X$. As $\langle v_j \rangle$ does not appear in $\Sigma_4$ as a cone, the center of $D_{v_j}$ on $Y''$ is contained in $D_{e_i}$ or some exceptional divisor $D$ on $Y''$. Thus, $D_{v_j}$ ($j > w$) is not an essential divisor over $X$. □

**Theorem 5.11.** Let $X$ be an affine pretoric variety. Then the Nash map

$$
\mathcal{N} : \{\text{Nash components in } \pi_X^{-1}(\text{Sing } X)\} \rightarrow \{\text{essential divisors over } X\}
$$

is bijective.

**Proof.** Let $\varphi : Y \rightarrow X$ be a resolution in which $D_{e_i}$ ($i = 1, \ldots, r$) and $D_{v_j}$ ($j = 1, \ldots, w$) appear. By Lemma 5.10, an essential divisor over $X$ is $D_{e_i}$ ($i = 1, \ldots, r$) or $D_{v_j}$ ($j = 1, \ldots, w$). Then, by the discussion in Example 2.15, the union of the Nash components is

$$
\bigcup_{i=1}^r \varphi_{\infty} \pi_Y^{-1}(D_{e_i}) \cup \bigcup_{j=1}^w \varphi_{\infty} \pi_Y^{-1}(D_{v_j}).
$$

By Lemma 2.17 and Lemma 5.6, it follows that

$$
\overline{\varphi_{\infty} \pi_Y^{-1}(D_{e_i})} = T_{\infty_X}^X(e_i)
$$

$$
\overline{\varphi_{\infty} \pi_Y^{-1}(D_{v_j})} = T_{\infty_X}^X(v_j).
$$

Here, each closed set does not contain any other set by Lemma 5.7, because each element of $\{e_i, v_j \}_{i=1}^r \cup \{v_j \}_{j=1}^w$ is minimal among them. Hence, the number of the Nash components is $r + w$, while the number of the essential divisors is less than or equal to $r + w$ by Lemma 5.10. Since the Nash map is injective, the both numbers must be $r + w$ and the Nash map is bijective. □

**Corollary 5.12.** If $X$ is a non-normal toric variety, then the Nash map for $X$ is bijective.
For a general non-normal variety, we have a counter example to the Nash problem. It is obtained from the counter example in [10].

Example 5.13. Let $X$ be a non-normal variety with the normalization $\nu : \overline{X} \to X$ such that $X$ is the hypersurface of $\mathbb{C}^5$ defined by $x_1^3 + x_3^3 + x_4^3 + x_5^6 = 0$. Then the Nash problem is negative for $X$.

Indeed, let $\psi_1 : Y_1 \to \overline{X}$ be the blow-up at $0$, then $Y_1$ has an isolated singularity $P$. Then, let $\psi_2 : Y_2 \to Y_1$ be the blow-up at $P$. Let $E_1 \subset Y_2$ be the proper transform of the exceptional divisor of $\psi_1$ and $E_2$ the exceptional divisor of $\psi_2$. In [10, §4] we proved that $E_2$ is not ruled and

$$(\psi_1\psi_2)_\infty(\pi_{Y_2}^{-1}(E_2)) \subset (\psi_1\psi_2)_\infty(\pi_{Y_2}^{-1}(E_1)).$$

Therefore, it follows that $E_2$ is an essential divisor over $X$ and

$$(\nu\psi_1\psi_2)_\infty(\pi_{Y_2}^{-1}(E_2)) \subset (\nu\psi_1\psi_2)_\infty(\pi_{Y_2}^{-1}(E_1)),$$

which shows that there is no Nash component corresponding to $E_2$ (see 2.15)

To construct such an example concretely, we can define $X$ as follows: Let $e_1, ..., e_5$ be a basis of $M = \mathbb{Z}^5$ and $\Gamma$ the subsemigroup of $M$ generated by $e_1, ..., e_4, 2e_5, 3e_5$. Then, the canonical morphism $\nu : \mathbb{C}^5 \to \text{Spec} \mathbb{C}[\Gamma]$ induced from the injection $\Gamma \hookrightarrow \bigoplus_{i=1}^5 \mathbb{Z}_{\geq 0} e_i$ is the normalization of a non-normal toric variety $\text{Spec} \mathbb{C}[\Gamma]$. It is sufficient to let $X$ be the image $\nu(\overline{X})$.

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