Transport coefficients and nonextensive statistics

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Abstract

We discuss the basic transport phenomena in gases and plasmas obeying the $q$-nonextensive velocity distribution (power-law). Analytical expressions for the thermal conductivity ($K_q$) and viscosity ($\eta_q$) are derived by solving the Boltzmann equation in the relaxation-time approximation. The available experimental results to the ratio $K_q/\eta_q$ constrains the $q$-parameter on the interval $0.74 \leq q \leq 1$. In the extensive limiting case, the standard transport coefficients based on the local Gaussian distribution are recovered, and due to a surprising cancellation, the electric conductivity of a neutral plasma is not modified.

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I. INTRODUCTION

In the last few years, an increasing attention has been paid to possible nonextensive effects in the fields of thermodynamics and statistical mechanics. The main motivation is the lack of a comprehensive treatment including gravitational and Coulombian fields for which the assumed additivity of the entropy present in the standard approach is not valid [1-2]. Inspired on such problems, as well as in the traditional ensemble theory, Tsallis [3] proposed a remarkable $q$-parameterized nonextensive entropic expression which reduces to the extensive Gibbs-Jaynes-Shannon entropy in the limiting case $q = 1$ (see [4] for a regularly updated bibliography on this subject).

Later on, the first attempts exploring the kinetic route associated to this nonextensive approach appeared in the literature[5-10]. The original kinetic derivation advanced by Maxwell [11] was generalized to include power law distributions as required by this enlarged framework. In particular, it was shown that the equilibrium velocity $q$-distribution

$$ f_0(v) = B_q [1 - (1 - q) \frac{mv^2}{2k_BT}]^{1/(1-q)}, $$

is uniquely determined from two simple requirements [3]: (i) isotropy of the velocity space, and (ii) a suitable nonextensive generalization of the Maxwell factorizability condition, or equivalently, the assumption that $F(v) = f(v_x)f(v_y)f(v_z)$. The quantity $B_q$ above denotes the $q$-dependent normalization constant whose expression on the interval $1/3 < q \leq 1$ is

$$ B_q = n(1 - q)^{1/2} \frac{5 - 3q}{2} \frac{3 - q}{2} \frac{\Gamma(\frac{1}{2} + \frac{1}{1-q})}{\Gamma(\frac{1}{1-q})} \left( \frac{m}{2\pi k_B T} \right)^{3/2}, $$

where $n$ is the particle number density, $m$ is the mass and $T$ is the temperature. As expected, in the limiting case $q = 1$, such expressions reduce to the standard Maxwellian ones [5].

More recently, the kinetic foundations of the above distribution were investigated in a deeper level through the generalized Boltzmann’s transport equation [3]

$$ \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = C_q(f), $$

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where $C_q$ is the $q$-nonextensive integral source term measuring the change in $f$ due to collisions. This Boltzmannian like approach incorporated the nonextensive effects using two different ingredients. First, a new functional form to the kinetic local gas entropy, and, second, a nonfactorizable distribution function for the colliding pairs of particles whose physical meaning is quite clear: the Boltzmann chaos molecular hypothesis is not valid in this extended framework. It was also shown that the kinetic version of the Tsallis entropy satisfies an $H_q$-theorem, and more important still, the $q$-parameterized class of power law velocity distributions emerged as the unique nonextensive solution describing the equilibrium states.

In this paper we go one step further by computing the basic nonextensive transport coefficients for dilute gases and plasmas. Due to the intricate form of the new collisional term, we adopt here the relaxation-time model proposed by Bhatnager, Gross and Krook [12] because it permits an exact mathematical treatment. In spite of its limitation, the relaxation-time model (henceforth BGK model) has been proved quite useful from a methodological viewpoint because it yields the correct answer to the problem in a first approximation, and as such, it can guide us to the correct nonextensive results which must rigorously be obtained using the full $q$-transport equation. As we shall see, analytical expressions for the transport coefficients of a dilute gas are readily obtained with basis on the local $q$-nonextensive distribution. In particular, we show that the available experimental results to the ratio $K_q/\eta_q$ constrains the $q$-parameter on the interval $0.74 \leq q \leq 1$. As expected, in the extensive limiting case, the classical expressions of the transport coefficients are readily recovered. However, due to a surprising cancellation, the standard electric conductivity of a dilute neutral plasma is not modified. The analysis presented here is also compared with independent calculations based on rather different approaches [7,10].
II. NONEXTENSIVE TRANSPORT COEFFICIENTS

Let us now consider a dilute gas whose particles are acted by external forces. In the BGK approximation the $q$-transport equation reads:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = -\frac{f - f_0}{\tau},$$

(4)

where $\mathbf{F}$ is the external force and $m$ is the mass of the particles. In this approximation, $f_0$ is the local equilibrium distribution function and $\tau$ is a relaxation time which is a number of the order of the collision time. We recall that under stationary conditions the distribution function does not depend on time, however, the concentration, and the temperature, are local quantities. The whole system is assumed to be out but close to the local stationary equilibrium state \[13\]. In particular, this means that the resulting nonequilibrium distribution, $f(\mathbf{r}, \mathbf{v})$, is only slightly different from the equilibrium unperturbed stationary distribution, $f_0(\mathbf{r}, \mathbf{v})$, and can be approximated as

$$f(\mathbf{r}, \mathbf{v}) = f_0(\mathbf{v}, \mathbf{r}) + g(\mathbf{v}, \mathbf{r}), \quad |g| \ll f_0.$$  

(5)

In what follows we first analyze the effects of the nonextensive distribution on the thermal conductivity $K_q$. It will be assumed that the heat flux is macroscopically governed by the classical Fourier law

$$\mathbf{q}_e = -K_q \nabla T,$$

(6)

while, microscopically, it is expressed as the usual average value of the perturbed kinetic energy flow \[14,15\]

$$\mathbf{q}_e = \frac{1}{2} m \int v^2 \mathbf{v} gd^3v.$$  

(7)

As one may check, in the absence of external forces, the BGK model leads to

$$g = -\tau \mathbf{v} \cdot \frac{\partial f_0}{\partial \mathbf{r}}.$$  

(8)
Substituting (8) into (7), and rewriting the resulting expression in spherical coordinates

\[(v, \theta, \phi),\]

one obtains

\[q_e = -\frac{1}{2} m \int_0^\infty v^4 \tau dv \int_0^\pi \sin \theta d\theta \int_0^{2\pi} v \left[ v \cdot \frac{\partial f_0}{\partial r} \right] d\phi. \tag{9}\]

Before proceed further, we note that the orthogonality relation

\[\int_0^\pi \int_0^{2\pi} v_i v_j \sin \theta d\theta d\phi = \frac{4\pi}{3} v^2 \delta_{ij} \tag{10}\]

where \(i, j \equiv x, y, z\), may be used to show that the angular integral can be written as

\[\int_0^\pi \sin \theta d\theta \int_0^{2\pi} v \left[ v \cdot \frac{\partial f_0}{\partial r} \right] d\phi = \frac{4\pi}{3} v^2 \frac{\partial f_0}{\partial r}, \tag{11}\]

with equation (8) simplifying to

\[q_e = -\frac{2\pi}{3} m \int_0^\infty v^6 \tau \frac{\partial f_0}{\partial r} dv. \tag{12}\]

As remarked earlier, the concentration \(n\) and the temperature \(T\) in the local \(q\)-distribution are spatially dependent while the pressure remains constant. Thus, as a consequence of the extended virial theorem [6], one may write \(n(r)T(r) = \text{constant}\), and combining that with the spatial gradient of the unperturbed distribution, the heat flow can be rewritten as

\[q_e = -\frac{2\pi}{3} m \tau B_q \int_0^\infty v^8 \left\{ [1 - (1 - q) \frac{mv^2}{2k_B T}]^{q/(1-q)} \frac{mv^2}{2k_B T} \right. \]

\[\left. - \frac{5}{2} \frac{1}{T} [1 - (1 - q) \frac{mv^2}{2k_B T}]^{1/(1-q)} \right\} \frac{\partial T}{\partial r} dv. \tag{13}\]

Now, comparing with the Fourier law (3), we obtain an integral expression for the nonextensive thermal conductivity

\[K_q = \frac{\pi \tau m^2}{3k_B T} B_q \int_0^\infty v^8 [1 - (1 - q) \frac{mv^2}{2k_B T}]^{q/(1-q)} dv \]

\[- \frac{5\pi \tau m}{3T} B_q \int_0^\infty v^6 [1 - (1 - q) \frac{mv^2}{2k_B T}]^{1/(1-q)} dv. \tag{14}\]

The above integrals can easily be evaluated. We find
\[ \int_0^\infty v^8 [1 - (1 - q) \frac{m v^2}{2 k_B T}]^{q/(1-q)} dv = \]

\[ \frac{1}{2} \left( \frac{2 k_B T}{m} \right)^{9/2} \frac{105 \pi^{1/2}}{(9 - 7q)(7 - 5q)(5 - 3q)(3 - q)} \left( \frac{1}{1 - q} \right)^{1/2} \frac{\Gamma\left(\frac{1}{1-q}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{1-q}\right)} \]  \tag{15}

and,

\[ \int_0^\infty v^6 [1 - (1 - q) \frac{m v^2}{2 k_B T}]^{1/(1-q)} dv = \]

\[ \frac{1}{2} \left( \frac{2 k_B T}{m} \right)^{7/2} \frac{30 \pi^{1/2}}{(9 - 7q)(7 - 5q)(5 - 3q)(3 - q)} \left( \frac{1}{1 - q} \right)^{1/2} \frac{\Gamma\left(\frac{1}{1-q}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{1-q}\right)} \]  \tag{16}

As one may check, inserting these integrals and the normalization constant \( B_q \) into equation (14), we obtain

\[ K_q = \frac{5}{2} \frac{n \tau k_B^2 T}{m} \left[ \frac{4}{(9 - 7q)(7 - 5q)} \right]. \]  \tag{17}

The nonextensive parameter \( q \) in the above expressions are restricted to positive values with \( q \neq 9/7, 7/5 \). Note also that expression (17) reduces to standard value in the extensive limiting case \cite{15,16}

\[ K_1 = \frac{5}{2} \frac{n \tau k_B^2 T}{m}. \]  \tag{18}

It thus follows that the ratio between the nonextensive thermal conductivity and the extensive value, \( K_q/K_1 \), is dependent on the Tsallis’ thermostatistics through parameter \( q \) (see Fig. 1). Note that for values of \( q \) smaller than unity, the corresponding coefficient \( K_q \) can be much smaller than the standard result (nearly \( K_1/3 \) for \( q \sim 0.7 \)), while for \( q > 1 \) it may increases without limit.

Let us now consider the nonextensive effects on the viscosity coefficient. In this case, the particles of the gas do not move with the same velocity. In a simplified treatment, the particles have constant mean velocity \( u_x \) in the \( x \) direction with the magnitude of \( u_x \) depending only on \( z \), that is, \( u_x = u_x(z) \). Thus, neglecting bulk viscosity, the Navier-Stokes stress reduces to
\[ P_{zx} = -\eta_q \frac{\partial u_x(z)}{\partial z}. \] (19)

A complementary hypothesis is that the collisional state tend to produce a local equilibrium distribution relative to the moving gas with mean velocity \( u_x \). Under such conditions, the stationary \( q \)-distribution becomes

\[ f_0(v_x - u_x(z), v_y, v_z) = B_q \left[ 1 - (1 - q) \frac{mU^2}{2k_BT} \right]^{1/(1-q)}, \] (20)

where \( U_x = v_x - u_x(z), U_y = v_y, U_z = v_z \). Since the resulting distribution \( f \) is stationary but also depends on \( z \) (the direction of the gradient velocity), the BGK equation (4) now reads

\[ v_z \frac{\partial f}{\partial z} = -\frac{f - f_0}{\tau}. \] (21)

The perturbed distribution function obeys

\[ g = -\tau v_z \frac{\partial f_0}{\partial z}, \] (22)

so that

\[ f = f_0 + \tau v_z \frac{\partial f_0}{\partial U_x} \frac{\partial u_x(z)}{\partial z}. \] (23)

In order to calculate the component \( P_{zx} \) of the stress tensor, we consider its kinetic definition

\[ P_{ij} = \int dU U_i U_j f. \] (24)

Following standard lines we see that (for details see Ref. [16])

\[ P_{zx} = m \int d^3U \tau \frac{\partial f_0}{\partial U_x} U_x^2 \frac{\partial u_x(z)}{\partial z}, \] (25)

and comparing this result with the macroscopic equation (19) we find

\[ \eta_q = -m \tau \int d^3U \frac{\partial f_0}{\partial U_x} U_x^2 U_x, \] (26)

where
\[ \frac{\partial f_0}{\partial U_x} = -\frac{B_q}{k_BT}[1 - (1 - q)\frac{mU^2}{2k_BT}]^{q/(1-q)}mU_x \]  

(27)

Replacing (27) into (26) it follows that

\[ \eta_q = m^2\tau B_q \int \int \int dU_x dU_y dU_z [1 - (1 - q)\frac{m}{2k_BT}(U_x^2 + U_y^2 + U_z^2)]^{q/(1-q)}U_x^2U_y^2, \]  

(28)

and performing the elementary integrals

\[ \eta_q = m^2\tau B_q \left( \frac{2k_BT}{m} \right)^{7/2} \frac{2\pi^{3/2}}{(7-5q)(5-3q)(3-q)} \left( \frac{1}{1-q} \right)^{1/2} \frac{\Gamma(\frac{1}{1-q})}{\Gamma(\frac{1}{2} + \frac{1}{1-q})}. \]  

(29)

Finally, substituting the expression of \( B_q \) and cancelling out the common factors, the viscosity coefficient assumes the rather simple form

\[ \eta_q = \frac{2}{(7-5q)}n\tau k_BT. \]  

(30)

This expression is valid only for \( q \neq 7/5 \), and as expected, in the limit \( q \to 1 \) it reduces to the standard extensive result

\[ \eta_1 = n\tau k_BT. \]  

(31)

In Figure 2 we show the ratio \( \eta_q/\eta_1 \) as a function of the nonextensive parameter. This plot is similar to what happens with the thermal conductivity (see Fig.1), and as expected, only for \( q = 1 \) the standard Maxwellian result is recovered. Note also that combining equations (17) and (30) we also obtain a quite simple expression to the dimensionless ratio, \( \Lambda \), involving the transport coefficients, \( K_q \) and \( \eta_q \), namely

\[ \Lambda = \frac{mK_q}{k_B\eta_q} = \frac{5}{9-7q}. \]  

(32)

Such a quantity (sometimes called Eucken’s ratio) plays an important experimental role because the unknown collision time appearing in both transport coefficients cancels out in this ratio. In Figure 3, we plot \( \Lambda \) as a function of \( q \). Note that \( \Lambda \) does not depend on the temperature, and for \( q = 1 \) it reduces to the extensive result \( \Lambda = 2.5 \). As widely known, this pure number lies experimentally within the range \( 1.3 \leq \Lambda \leq 2.5 \).
Therefore, the allowed values to the nonextensive parameter is restricted on the interval \(0.74 \leq q \leq 1\), thereby showing that the Maxwellian result is only marginally compatible with such measurements.

It is worth mentioning that the departure from standard Maxwellian prediction is usually taken as an indication that the energy of the molecules must include another forms than mere kinetic energy of translation, or equivalently, an anomalous specific heat [18]. However, as we have seen, a possible alternative explanation (at least for monatomic gases) is provided by the existence of nonextensive effects associated to Tsallis' thermostatistics.

For completeness, we now investigate the possible nonextensive effects on the electric conductivity of a dilute neutral plasma. As before, we consider the BGK approximation assuming that the external electric field is uniform and points to the positive \(z\) direction, that is, \(\mathbf{E} = E \mathbf{\hat{z}}\). The relevant macroscopic equation is now the Ohm’s law

\[
\mathbf{j}_z = \sigma_q E, \tag{33}
\]

where \(\sigma_q\) is the electric conductivity. Microscopically, the electric current density is defined as the statistical average [14,15]

\[
\mathbf{j}_n = e \int d^3v f \mathbf{v}_n, \tag{34}
\]

where the subscript \(n\) means that the charge flow is directed along the normal to the corresponding surface element. Note that only the component \(j_z\) does not vanish since for an homogeneous and isotropic medium the current \(\mathbf{j}\) is parallel to the electric field. As one may check, from equations (1), (5) and (34) it follows that

\[
\mathbf{j}_z = \frac{e^2 \tau E B_q}{k_B T} \int d^3v [1 - (1 - q) \frac{m v^2}{2 k_B T}]^{q/(1-q)} v_z^2. \tag{35}
\]

Performing the integration and comparing to the Ohm’s law the resulting electric conductivity is

\[
\sigma_q = \frac{e^2 \tau B_q}{k_B T} \pi^{3/2} \left( \frac{2 k_B T}{m} \right)^{5/2} \frac{2}{(3-q)(5-3q)} \left( \frac{1}{1-q} \right)^{1/2} \frac{\Gamma \left( \frac{1}{1-q} \right)}{\Gamma \left( \frac{1}{2} + \frac{1}{1-q} \right)}. \tag{36}
\]
or still, inserting the value of $B_q$ (see Eq. (2))

$$\sigma_q = \frac{n e^2 \tau^2}{m}. \quad (37)$$

Therefore, within the BGK approximation, the above cancellation results that the electric conductivity does not depend on the nonextensive $q$-parameter.

III. CONCLUSION

In this paper we discussed some possible nonextensive effects on the transport phenomena in gases and plasmas. It was shown that the theoretical analysis based on the relaxation-time approximation lead to simple analytic expressions to the heat conduction and viscosity coefficients, while the electric conductivity was not modified.

It is interesting to compare our results (and the overall approach) with independent analyzes and some previous expressions to nonextensive transport coefficients appearing in the literature [7,10]. We first notice that our main results are quite different from those obtained by Boghosian [7]. Probably, the basic reason comes from the fact that in his paper it was advocated a different choice to the BGK operator, namely, $C_q = -\tau^{-1}(f^q - f_0^q)$. Note that it depends nonlinearly on the nonextensive parameter and should be compared with the standard BGK operator assumed in the present paper (see our equation (3)).

In the treatment adopted here, the nonextensive effects appear only implicitly through the power law equilibrium distribution. In addition, Boghosian (and also the authors of Ref. [10]) assumed that the pressure in the perturbed gas does not remain constant thereby giving rise to an anomalous heat conduction ($T_q$ in their notation). In this concern, our approach may be considered more conservative. As a matter of fact, using the extended virial theorem [3], we have assumed constant pressure as usually done in the BGK approximation so that the anomalous heat conduction coefficient, $T_q$, is absent. Another crucial difference among these works is that three kinds of averaging methods has been considered. Boghosian [7] and Potiguar et al. [10] employed unnormalized and normalized $q$-expectations values,
respectively, while we have computed the averages in the usual manner. In this way, it is not surprising that although obtaining coefficients $K_q$ and $\eta_q$ proportional to the temperature, the behavior of the dimensionless ratio $\Lambda \propto K_q/\eta_q$ as a function of the $q$ parameter is quite different (compare our equation (32) with equations (83) and (40) in the quoted papers). Since this ratio can be measured in laboratory, hopefully, these different approaches, as well as their basic underlying assumptions will be experimentally confronted in the near future.

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FIG. 1. Nonextensive thermal conductivity. We show the dimensionless ratio $K_q/K_1$ as a function of the $q$ parameter. The normalizing quantity $K_1$ corresponds to the Maxwellian expression obtained in the limit $q = 1$. Note that for $q \to 1.3$ the ratio $K_q/K_1$ increases with no limit.

FIG. 2. Nonextensive viscosity coefficient. The plot shows the dimensionless ratio $\eta_q/\eta_1$, where $\eta_1$ denotes the Maxwellian result, as a function of the nonextensive parameter $q$. For $q \to 1.4$, this ratio goes to infinity.
FIG. 3. The $\Lambda$ parameter. The experimental dimensionless Eucken’s ratio, $\Lambda = mK_0/k_B\eta_q$, is shown as a function of the nonextensive parameter $q$. The shadowed region shows the measured interval $1.3 \leq \Lambda \leq 2.5$, for which the $q$ parameter is restricted to $0.74 \leq q \leq 1$. Note that the Maxwellian prediction is only marginally compatible with such experiments.