Finite-state self-similar actions of nilpotent groups

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Abstract

Let $G$ be a finitely generated torsion-free nilpotent group and $\phi : H \to G$ be a surjective homomorphism from a subgroup $H < G$ of finite index with trivial $\phi$-core. For every choice of coset representatives of $H$ in $G$ there is a faithful self-similar action of the group $G$ associated with $(G, \phi)$. We characterize the existence of finite-state self-similar actions for $(G, \phi)$ in terms of the Jordan normal form of $\phi$ viewed as an automorphism of the Lie algebra of $G$.

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1 Introduction and Preliminaries

An action of a group $G$ on the set $X^*$ of words over a finite alphabet $X$ is called self-similar if for every $x \in X$ and $g \in G$ there exist $y \in X$ and $h \in G$ such that $g(xv) = yh(v)$ for all words $v \in X^*$. Self-similar group actions appear naturally in many areas of mathematics and have applications to holomorphic dynamics, fractal geometry, combinatorics, automata theory, etc. (see [7] and the reference therein).

The theory of self-similar group actions can be regarded as the study of positional numeration systems on groups. Bases in these number systems are virtual endomorphisms of groups. A virtual endomorphism of a group $G$ is a homomorphism $\phi : H \to G$ from a subgroup $H < G$ of finite index to $G$. To produce a self-similar action of the group $G$ with “base” $\phi$ we need to choose a set $D$ of coset representatives of $H$ in $G$ called a digit set. Let us enumerate the elements of $D$ by the letters of an alphabet $X$ (here $|X| = [G : H]$ and $D = \{h_x, x \in X\}$). The self-similar action of $G$ on the space $X^*$ associated to the triple $(G, \phi, D)$ is constructed as follows. Every element of the group stabilizes the empty word. For every $x \in X$ and $v \in X^*$ the action of an element $g \in G$ is defined recursively by the rule

$$g(xv) = yh(v) \quad \text{with} \quad h = \phi(h^{-1}_ygh_x),$$

(1)
where \( y \in X \) is the unique letter such that \( h_y^{-1}gh_x \in H \). The constructed action may be not faithful. The kernel of the action does not depend on the choice of the set \( D \) and is equal to the maximal normal \( \phi \)-invariant subgroup of \( G \) called the \( \phi \)-core ([7, Proposition 2.7.5]).

Conversely, one can associate a virtual endomorphism with every faithful self-similar action \((G, X^\phi)\) as follows. The element \( h \) from the definition of self-similar action is called the state of \( g \) at \( x \) and is denoted by \( g|x \) (this element is unique if the action is faithful); iteratively we define the state of \( g \) at every word by the rule \( g|x_1x_2...x_n = g|x_1|x_2...|x_n \). For every letter \( x \in X \) the stabilizer \( St_G(x) \) has finite index in \( G \) and then the map \( \phi_x : St_G(x) \rightarrow G \) given by \( \phi_x(g) = g|x \) is a virtual endomorphism of \( G \). If in addition the action \((G, X^\phi)\) is self-replicating (recurrent), i.e., \( \phi_x \) is surjective and \( G \) acts transitively on \( X \), then \((G, X^\phi)\) can be given by the triple \((G, \phi_x, D)\) for some choice of the digit set \( D \).

As a simple example, consider the group \( \mathbb{Z} \) with the homomorphism \( \phi : 2\mathbb{Z} \rightarrow \mathbb{Z} \), \( \phi(2a) = a \), and choose the digit set \( D = \{0, 1\} \), which is also used as an alphabet with a slight abuse in notations. The associated self-similar action corresponds to the binary number system on \( \mathbb{Z} \). We have \( a|x_1x_2...x_n = b \) and \( a(x_1x_2...x_n) = y_1y_2...y_n \) for \( a, b \in \mathbb{Z} \) if and only if

\[
a = (y_1 - x_1) + 2(y_2 - x_2) + 2^2(y_3 - x_3) + \ldots + 2^n(y_n - x_n) + 2^nb.
\]

In particular, if \( a|_{00...0} = 0 \) then the image \( a(00...0) \) is the usual binary expansion of \( a \).

Self-similar group actions are closely related to groups generated by automata. Groups generated by finite automata correspond to finite-state self-similar actions of finitely generated groups. Recall that a faithful self-similar action \((G, X^\phi)\) is called finite-state if for every \( g \in G \) the set of its states \( \{g|_v : v \in X^\phi\} \) is finite. Then a finitely generated group has a faithful finite-state self-similar action if and only if it can be generated by a finite automaton. The fundamental question in this theory is what groups possess finite-state self-similar actions, i.e., can be realized by finite automata. This property was proved for free abelian groups \( \mathbb{Z}^n \) ±, Grigorchuk group [2], \( GL_n(\mathbb{Z}) \) [2], lamplighter groups [1], free groups and free products of cyclic groups of order 2 [1], Baumslag-Solitar groups \( B(1, m) \) [1], certain nilpotent groups [2], etc.

The finite-state self-similar actions of \( \mathbb{Z}^n \) can be characterized in term of the associated virtual endomorphism as shown by Nekrashevych and Sidki in [2] (see also [1, Theorem 2.12.1]). A virtual endomorphism of \( \mathbb{Z}^n \) is uniquely extended to a linear operator of \( \mathbb{R}^n \). Then a faithful self-replicating self-similar action of \( \mathbb{Z}^n \) with virtual endomorphism \( \phi \) is finite-state if and only if the spectral radius of \( \phi \) is less than 1. In particular, there is no dependence on the choice of coset representatives.

In this paper we consider self-similar actions of finitely generated torsion-free nilpotent groups. The main goal is to generalize the above mentioned result of Nekrashevych and Sidki. However, self-similar actions of nilpotent groups have a new level of complexity comparing to the actions of abelian groups. For example, a nilpotent group with fixed virtual endomorphism may have a faithful finite-state self-similar action for one choice of coset representatives and be not finite-state for another choice (see example with Heisenberg group in Section [3]). Hence we need to answer two questions: Under what conditions on \( \phi \)
does there exist a finite-state action for \((G, \phi)\)? Under what conditions does every action for \((G, \phi)\) finite-state?

Let \(G\) be a finitely generated torsion-free nilpotent group with surjective virtual endomorphism \(\phi : H \to G\). Since we are interested in faithful self-similar actions of the group \(G\), we assume that \(\phi\)-core is trivial. Then Corollary 1 in [2] implies that \(\phi\) is also injective and hence it is an isomorphism. The same corollary says that if we know that \(\phi\) is an isomorphism then \(\phi\)-core is trivial if and only if the virtual endomorphism \(\phi|_{Z(H)} : Z(H) \to Z(G)\) of the center \(Z(G)\) has trivial core. Since \(Z(G)\) is abelian, by [4], Proposition 2.9.2, the \(\phi|_{Z(H)}\)-core is trivial if and only if no eigenvalue of \(\phi|_{Z(H)}\) is an algebraic integer. Hence one can effectively check when \(\phi\)-core is trivial. By a theorem of Mal’cev (see [3]) there exists the unique real nilpotent Lie group \(L\), Mal’cev completion of \(G\), such that the group \(G\) is a discrete subgroup of \(L\) and the topological space \(L/G\) is compact. Since \(H\) is a subgroup of finite index, the isomorphism \(\phi : H \to G\) lifts to an automorphism of the Lie group \(L\) also denoted by \(\phi\). Let \(\mathcal{L}\) be the Lie algebra of \(L\) and denote the automorphism of \(\mathcal{L}\) induced by \(\phi\) also by \(\phi\). Then the existence of finite-state self-similar action of the group \(G\) can be characterized in terms of the Jordan normal form of \(\phi\).

**Theorem 1.** Let \(G\) be a finitely generated torsion-free nilpotent group. Let \((G, X^*)\) be a faithful self-replicating self-similar action with virtual endomorphism \(\phi\) (associated to some letter \(x \in X\)). If the action \((G, X^*)\) is finite-state then the spectral radius of \(\phi\) is not greater than 1 and for every eigenvalue of modulus 1 the associated Jordan cells in the Jordan normal form of \(\phi\) have size 1. Conversely, if the virtual endomorphism \(\phi\) satisfies the previous condition then there exists a finite-state self-similar action of \(G\) with virtual endomorphism \(\phi\).

One can restate the theorem as follows. Let \(\phi\) be a surjective virtual endomorphism of \(G\) with trivial core. There exists a digit set \(D\) with \(e \in D\) such that the self-similar action associated to \((G, \phi, D)\) is finite-state if and only if the Jordan normal form of \(\phi\) satisfies the condition in the theorem.

**Theorem 2.** Let \(G\) be a finitely generated torsion-free nilpotent group, and let \(\phi\) be a surjective virtual endomorphism of \(G\) with trivial core. Every self-similar action of \((G, \phi)\) is finite-state if and only if the spectral radius of \(\phi\) is less than 1.

In particular, if the Jordan normal form of \(\phi\) satisfies the condition in Theorem 1 and \(\phi\) has an eigenvalue of modulus 1, then the pair \((G, \phi)\) possesses both finite-state and non-finite-state self-similar actions. This situation cannot happen for abelian group \(\mathbb{Z}^n\), because if a virtual endomorphism of \(\mathbb{Z}^n\) has an eigenvalue of modulus 1 then it has a non-trivial core.

## 2 Proof of Theorems 1 and 2

Recall that there exists a bijection between the Lie algebra \(\mathcal{L}\) and the Lie group \(L\) given by the exponential map \(\exp : \mathcal{L} \to L\) with inverse \(\log : L \to \mathcal{L}\). The automorphism \(\phi\) of \(L\)
and $\mathcal{L}$ satisfies
\[ \phi(\log(g)) = \log(\phi(g)) \text{ for all } g \in L. \] (2)

Let $\mathcal{L}_Q \subset \mathcal{L}$ be the set of all linear combinations of vectors from $\log(G)$ over $\mathbb{Q}$. By Theorem 5.1.8 (a) in [4] $\mathcal{L}_Q$ is a Lie algebra over $\mathbb{Q}$ such that $\mathcal{L}_Q \otimes \mathbb{R} = \mathcal{L}$. It is usually said that this defines a rational structure on $L$. It is easy to see that $\phi(\mathcal{L}_Q) \subset \mathcal{L}_Q$. Indeed, since $H$ is of finite index in $G$ it is also a uniform subgroup of $L$, which is by definition commensurable with $G$. Thus by Theorem 5.1.12 in [4] the Lie algebra $\mathcal{L}_Q$ is also equal to the $\mathbb{Q}$-span of vectors from $\log(H)$. Since $\phi(H) \subset G$ we have that $\phi(\mathcal{L}_Q) \subset \mathcal{L}_Q$ by Equation (3). In particular it follows that the matrix of $\phi$ has rational entries in any basis of $L_Q$. Moreover it can be shown using induction on the length of lower central series of $L$ that $[G : H] = \det(\phi^{-1})$.

A self-similar action $(G, X^*)$ is called contracting if there exists a finite set $\mathcal{N} \subset G$ with the property that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all words $v \in X^*$ of length $\geq n$. Every contracting action is finite-state by definition. A self-replicating self-similar action is contracting if and only if the associated virtual endomorphism has spectral radius less than 1 (see [4, Proposition 2.11.11]).

**Proof of sufficiency in Theorem 4.** The assumption on the Jordan normal form of $\phi$ implies the following crucial property: for every $g \in L$ the sequence $\phi^n(g)$ is bounded (i.e., belongs to a compact set).

The Lie algebra $\mathcal{L}$ decomposes in the direct sum $\mathcal{L} = \mathcal{L}_c \oplus \mathcal{L}_r$, where $\mathcal{L}_c$ is a $\phi$-invariant subalgebra such that $\phi|_{\mathcal{L}_c}$ has spectral radius less than 1 (contracting), and the spectrum of $\phi|_{\mathcal{L}/\mathcal{L}_c}$ consists only of numbers of modulus 1. Consider the $\phi$-invariant subgroup $L_c = \exp(\mathcal{L}_c)$ of the Lie group $L$ that corresponds to the subalgebra $\mathcal{L}_c$. One can define $L_c$ directly as a subgroup of all $g \in L$ such that $\phi^n(g) \to 1$ as $n \to \infty$. Define the group $G_c = G \cap L_c$ and its subgroup $H_c = H \cap L_c = \phi^{-1}(G_c) < G_c$ of finite index $[G_c : H_c] = \det(\phi|_{\mathcal{L}_c})^{-1}$. Then $\phi|_{H_c} : H_c \to G_c$ is a contracting isomorphism and every self-similar action of $(G_c, \phi|_{H_c})$ is contracting.

Notice that $\det(\phi|_{\mathcal{L}/\mathcal{L}_c})$ is a positive number as since $\det(\phi) = \det(\phi|_{\mathcal{L}/\mathcal{L}_c}) \det(\phi|_{\mathcal{L}_c})$, and, at the same time, it is a product of numbers of modulus 1. Hence $\det(\phi|_{\mathcal{L}/\mathcal{L}_c}) \neq 1$ and we get $[G : H] = \det(\phi)^{-1} = \det(\phi|_{\mathcal{L}_c})^{-1} = [G_c : H_c]$.

Take any coset representatives $h_1, h_2, \ldots, h_d$ for $H_c$ in $G_c$. Since $H \cap G_c = H_c$ and $[G : H] = [G_c : H_c]$, the elements $h_1, h_2, \ldots, h_d$ are also coset representatives of $H$ in $G$. Let us consider the associated self-similar action $(G, X^*)$ given by Equation (4). Take any element $g \in G$ and for every word $x_1 x_2 \ldots x_n \in X^*$ consider the state
\[
g|_{x_1 x_2 \ldots x_n} = \phi(h_{y_1}^{-1} \ldots \phi(h_{y_2}^{-1} \phi(h_{y_1}^{-1} g h_{x_1}) h_{x_2}) \ldots h_{x_n})
= \phi(h_{y_1}^{-1} \ldots \phi^{-1}(h_{y_2}^{-1}) \phi^n(h_{y_1}^{-1}) \phi^n(g) \phi^n(h_{x_1}) \phi^{n-1}(h_{x_2}) \ldots \phi(h_{x_n}),
\] (3)

where $y_1 y_2 \ldots y_n = g(x_1 x_2 \ldots x_n)$. The sequence $\phi^n(g)$ is bounded in $L$. The elements $h_{x_1}, h_{x_2}, \ldots, h_{x_n}$ are taken from a finite subset of the $\phi$-invariant subgroup $L_c$ on which $\phi$ is contracting. Then the set of all products of the form $\phi^n(h_{x_1}) \phi^{n-1}(h_{x_2}) \ldots \phi(h_{x_n})$ belongs to a compact subset of $L_c$. Since the product in (3) belongs to the lattice $G$, it assumes a finite number of values. Hence the action $(G, X^*)$ is finite-state. \( \square \)
Proof of necessity in Theorem 3.1. Let $(G, X^*)$ be a finite-state self-similar action with virtual endomorphism $\phi$ associated to the letter $x_1 \in X$, i.e., $\phi = \phi_{x_1}$ and $H = St_G(x_1)$. Let \{h_1 = e, h_2, \ldots, h_d\} be the corresponding set of coset representatives.

Lemma 1. The eigenvalues of $\phi$ have modulus $\leq 1$. Moreover, every eigenvalue of modulus 1 is a root of unity.

Proof. Put $\mathcal{L}^{(0)} = \mathcal{L}$ and let $\mathcal{L}^{(i)} = [\mathcal{L}, \mathcal{L}^{(i-1)}]$ be the lower central series of the Lie algebra $\mathcal{L}$. Since $\phi$ is an automorphism of $\mathcal{L}$ it preserves every term $\mathcal{L}^{(i)}$ and induces an automorphism $\vec{\phi}_i$ on the quotient $\mathcal{L}^{(i)}/\mathcal{L}^{(i+1)}$. The spectrum of $\phi$ is a union of the spectra of $\vec{\phi}_i$. At the same time, every linear map $\vec{\phi}_i$ is a quotient of the tensor product $\vec{\phi}_0 \otimes \vec{\phi}_0 \otimes \cdots \otimes \vec{\phi}_0$ (see [14], Theorem 3.1). Hence it is enough to prove the statement for the automorphism $\vec{\phi}_0$.

Let $\lambda$ be an eigenvalue of $\vec{\phi}_0$. Take a basis of $\mathcal{L}/[\mathcal{L}, \mathcal{L}]$ in which $\vec{\phi}_0$ has Jordan normal form, and consider the coordinate of vectors in this basis that corresponds to an eigenvector with eigenvalue $\lambda$. There exists a linear map $\xi : \mathcal{L} \to \mathbb{C}$ such that $\xi([\mathcal{L}, \mathcal{L}]) = 0$ and $\xi(\phi(l)) = \lambda \xi(l)$ for all $l \in \mathcal{L}$. We compose $\xi$ with the logarithmic map $\log : L \to \mathcal{L}$ and denote the composition also by $\xi$. Note that $\log(g_1g_2) = \log(g_1) + \log(g_2) \mod [L, L]$. Thus we have a map $\xi : L \to \mathbb{C}$ such that $\xi(g_1g_2) = \xi(g_1) + \xi(g_2)$ and $\xi(\phi(g)) = \lambda \xi(g)$. The rest of the proof is very similar to the proof of Theorem 2.12.1 in [14], so we only sketch it here.

Since $G$ is a lattice in $L$, there exists $g \in G$ such that $\xi(g) \neq 0$. Let us consider the states $g|_v$ for $v \in X^*$. By Equation (1) we have $g|_x = \phi(h_{g(x)}^{-1}gh_x)$ for every $x \in X$. Then

$$\xi(g|_x) = \lambda \xi(g) + \lambda (\xi(h_x) - \xi(h_{g(x)})) = \lambda \xi(g) + d_x,$$

were $d_x = \lambda (\xi(h_x) - \xi(h_{g(x)}))$.

Suppose $|\lambda| > 1$. Since $\sum_{x \in X} d_x = 0$, it follows that there exits $x_1 \in X$ such that $|\xi(g|_{x_1})| > |\xi(g)|$. Thus we can iteratively construct letters $x_n \in X$ such that $|\xi(g|_{x_1x_2...x_{n+1}})| > |\xi(g|_{x_1x_2...x_n})|$ for each $n$. Hence $g$ is not finite-state, contradiction.

Suppose $|\lambda| = 1$ and $\lambda$ is not a root of unity. As above there is a sequence of letters $x_n \in X$ such that for each $n$ either $|\xi(g|_{x_1x_2...x_{n+1}})| > |\xi(g|_{x_1x_2...x_n})|$ or $\xi(g|_{x_1x_2...x_{n+1}}) = \lambda \xi(g|_{x_1x_2...x_n})$. In either case we have a contradiction with the fact that the action is finite-state.

It is left to prove that Jordan cells for roots of unity have size 1. Let $m$ be an integer number such that $\varepsilon^m = 1$ for every root of unity $\varepsilon$ from the spectrum of $\phi$. Then the spectrum of $\phi^m$ consists of 1 and numbers less than 1 in modulus. The self-similar action $(G, X^*)$ over the alphabet $X$ induces the self-similar action $(G, (X^m)^*)$ over the alphabet $X^m$ of words of length $m$ over $X$. Moreover, since the action $(G, X^*)$ is finite-state then obviously the action $(G, (X^m)^*)$ is also finite-state. Note that the virtual endomorphism $\phi_v$ of the action $(G, (X^m)^*)$ associated to a word $v = x_1...x_m \in X^m$ is the composition $\phi_v = \phi_{x_1} \circ \cdots \circ \phi_{x_m}$ of virtual endomorphisms of the action over $X$. In particular, $\phi_{x_1...x_1} = \phi^m$ and the action $(G, (X^m)^*)$ corresponds to the pair $(G, \phi^m)$. If we know that the size of Jordan cells of $\phi^m$ with eigenvalue 1 have size 1, then the same holds for $\phi$ for roots of unity. Hence we can assume that all roots of unity in the spectrum of $\phi$ are equal to 1.
Suppose there is a Jordan cell of $\phi$ with eigenvalue 1 that has size greater than 1. Then there exist nonzero vectors $v, u \in \mathcal{L}$ such that $\phi(v) = v$ and $\phi(u) = u + v$. Since the matrix of $\phi$ has rational entries, the vectors $v$ and $u$ can be chosen to have rational entries and we assume $v, u \in \mathcal{L}_Q$. By Theorem 5.4.2 from [1], the group $G$ contains a subgroup $G_0$ of finite index such that $\log G_0$ is a lattice in $\mathcal{L}_Q$, i.e., $\log G_0$ is closed under addition and its span over $\mathbb{Q}$ is equal to $\mathcal{L}_Q$. Multiplying $v$ and $u$ by a suitable integer we can assume that they belong to $\log G_0$, and thus $u + nv \in \log G_0$ for all $n \in \mathbb{N}$. Consider the element $g = \exp(u) \in G$. We get

$$\phi^n(g) = \phi^n(\exp(u)) = \exp(\phi^n(u)) = \exp(u + nv) \in G_0 \subset G,$$

and hence $\phi^n-1(g) \in \phi^{-1}(G) = H = \text{St}_G(x_1)$ for all $n \geq 1$. Then the element $g$ fixes the word $x_1x_1 \ldots x_1$ ($n$ times) and has the state $g|_{x_1x_1 \ldots x_1} = \phi^n(g) = \exp(u + nv)$ for all $n \geq 1$. Since all elements $u + nv$ are different, the element $g$ is not finite-state. We got a contradiction. \qed

**Proof of Theorem 3.** If the spectral radius of $\phi$ is less than 1, then the action is contracting and thus finite-state.

For the converse, it is sufficient to prove that if the Jordan normal form of $\phi$ satisfies item 1 of the theorem and the spectrum of $\phi$ contains a root of unity then there exists a non-finite-state action for $(G, \phi)$. As in the previous proof, we can assume that all roots of unity from the spectrum are equal to 1, and we find an element $h \in H$ such that $\phi(h) = h$. The virtual endomorphism $\phi|_Z(H) : Z(H) \to Z(G)$ has spectral radius less than 1. Let us choose a set of coset representatives $D = \{h_1 = e, h_2, \ldots , h_d\}$ for $Z(H)$ in $Z(G)$. If every element $g \in Z(G)$ can be expressed as a product

$$g = h_{i_1}\phi^{-1}(h_{i_2}\phi^{-1}(\ldots \phi^{-1}(h_{i_n}) \ldots )) = h_{i_1}\phi^{-1}(h_{i_2})\phi^{-2}(h_{i_3}) \ldots \phi^{-n+1}(h_{i_n}) \quad (4)$$

for $h_{i_j} \in D$, then we take $k > 1$ such that the set $D^k = \{h_1^k = e, h_2^k, \ldots , h_d^k\}$ consists of coset representatives for $Z(H)$ in $Z(G)$ and replace $D$ by $D^k$. In this case every product in (4) belongs to a proper subgroup $Z(G)^k$ of $Z(G)$. We complete $D$ to the set of coset representatives of $H$ in $G$ by elements $h_{d+1}, \ldots , h_d$. Replace the coset representative $h_1 = e$ by the element $h \in H$. Let us prove that the associated self-similar action of the group $G$ is not finite-state.

Take an element $g \in Z(G)$ that cannot be expressed in the form (4), and consider the state of $g$ at the word $x_1x_1 \ldots x_1$ ($n$ times):

$$g|_{x_1x_1 \ldots x_1} = \phi(h^{-1}_{y_1} \ldots \phi(h^{-1}_{y_2} \phi(h^{-1}_{y_1} g h)) \ldots h) = \phi(h^{-1}_{y_1} \ldots \phi(h^{-1}_{y_2} \phi(h^{-1}_{y_1} g)) \ldots ) h^n,$$

where $g(x_1x_1 \ldots x_1) = y_1y_2 \ldots y_n$. All elements $h_{y_i}$ are taken from the set $\{h, h_2, \ldots , h_d\}$. The elements $\phi(h^{-1}_{y_1} \ldots \phi(h^{-1}_{y_2} \phi(h^{-1}_{y_1} g)) \ldots )$ with all $h_{y_k} \in \{h_2, \ldots , h_d\}$ belongs to the center $Z(G)$. We can move every element $h_{y_i}$ that is equal to $h$ to the right and include in the power $h^n$. Hence we can write the state as

$$g|_{x_1x_1 \ldots x_1} = \phi(h^{-1}_{y_1} \ldots \phi(h^{-1}_{y_2} \phi(h^{-1}_{y_1} g)) \ldots ) h^n, \quad (5)$$
with all $h_{y_i} \in \{e, h_2, \ldots, h_d\}$ (here we replace every $h_{y_i} = h$ by $h_{y_i} = e$), and $m$ is equal to the number of letters $y_i$ not equal to $x_1$. As above, all the products $\phi(h_{y_1}^{-1}) \cdots \phi^{n-1}(h_{y_2})\phi^n(h_{y_1}^{-1})\phi^n(g)$ belong to a compact subset of $L$, but also belong to the lattice $G$. Hence these products assume a finite number of values. Let us analyze the values of $m$.

Notice that $g$ cannot stabilize the sequence $x_1 x_1 \ldots$. Indeed, if $g(x_1) = x_1$ then $g|_{x_1} = \phi(h^{-1}gh) = \phi(g) \in Z(G)$. Hence, if $g(x_1 x_1 \ldots) = x_1 x_1 \ldots$ then $\phi^n(g) \in Z(G)$ for all $n \geq 1$. It implies that there exists a non-trivial normal $\phi$-invariant subgroup of $Z(G)$ and we get a contradiction with the faithfulness of the action. Suppose $g$ changes only finitely many letters in the sequence $x_1 x_1 \ldots$. Then $g|_{x_1 x_1 \ldots x_1}$ and $g|_{x_1 x_1 \ldots x_1} h^{-m}$ stabilize $x_1 x_1 \ldots$ for long enough word $x_1 x_1 \ldots x_1$. At the same time

$$g|_{x_1 x_1 \ldots x_1} h^{-m} = \phi(h_{y_1}^{-1} \cdots \phi(h_{y_2}^{-1} \phi(h_{y_1}^{-1} g)) \cdots) \in Z(G).$$

and we get that this element should be trivial. However in this case $g$ can be expressed in the form (3), we get a contradiction with the choice of $g$.

Since $g$ changes infinitely many letters in the sequence $x_1 x_1 \ldots$, the number $m$ in Equation (3) goes to infinity as the length of $x_1 x_1 \ldots x_1$ goes to infinity. The elements $h^m$ are all different, and hence $g$ has infinitely many states. \hfill \Box

### 3 Example

Consider the discrete Heisenberg group

$$G = \left\{ (x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

its subgroup $H = \{(x, 2y, 2z) : x, y, z \in \mathbb{Z}\}$, and the isomorphism $\phi : H \to G$ given by $\phi(x, y, z) = (x, y/2, z/2)$. One can directly check that the $\phi$-core($H$) is trivial, and every self-similar action for the pair $(G, \phi)$ is faithful (we can also notice that $\phi|_{Z(H)} : Z(H) \to Z(G)$ leads to the faithful self-similar action of $(Z(G), \phi|_{Z(H)})$, and hence the same holds for $(G, \phi)$ by [2, Corollary 1]).

The matrix of $\phi$ is diagonal with eigenvalues $1, \frac{1}{2}, \frac{1}{2}$, and Theorems [4] and [2] imply that there exist both finite-state and non-finite-state self-similar actions for the pair $(G, \phi)$. First, let us construct a finite-state action. Choose coset representatives $D = \{(0, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1)\}$ and consider the associated self-similar action $(G, X^\star)$ over the alphabet $X = \{1, 2, 3, 4\}$ given by Equation (1). The action of the generators $a = (1, 0, 0)$ and $b = (0, 1, 0)$ of the group satisfies the following recursions:

$$a(1v) = 1a(v) \quad a(2v) = 4a(v) \quad a(3v) = 3a(v) \quad a(4v) = 2(b^{-1}ab)(v)$$

$$b(1v) = 2v \quad b(2v) = 1b(v) \quad b(3v) = 4v \quad b(4v) = 3b(v)$$
The elements $a$ and $b$ are finite-state, namely the states of $a$ are $a, b^{-1}ab, b^{-2}ab^2$ and the states of $b$ are $e, b$. Hence the action $(G, X^*)$ is finite-state.

Let us change the coset representatives and choose $D' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1)\}$. Then the action of the generators of the group $G$ satisfies the recursions

$$
\begin{align*}
    a(1v) &= 1a(v) & a(2v) &= 4a(v) & a(3v) &= 3a(v) & a(4v) &= 2(b^{-1}ab)(v) \\
    b(1v) &= 2a(v) & b(2v) &= 1(a^{-1}b)(v) & b(3v) &= 4b(v) & b(4v) &= 3b(v)
\end{align*}
$$

In order to see that this action is not finite-state, let us look at the action of the element $c = (0, 0, 1) = a^{-1}b^{-1}ab$:

$$
\begin{align*}
    c(1v) &= 3(a^2b^{-1}a^{-1}b)(v) & c(2v) &= 4c(v) & c(3v) &= 1a^{-1}(v) & c(4v) &= 2c^2(v)
\end{align*}
$$

It is not difficult to deduce that all powers $c^n$ for $n \in \mathbb{N}$ are the states of $c$ and hence the element $c$ is not finite-state.

**Remark about growth of Schreier graphs.** Let $G$ be a group with a finite generating set $S$ and acting on a set $M$. The (simplicial) graph $\Gamma(G, S, M)$ of the action is the graph with the set of vertices $M$ and two points $u, v \in M$ are connected by an edge if $s(u) = v$ or $s(v) = u$ for some $s \in S$. The connected component of $\Gamma(G, S, M)$ around a point $w \in M$ is called the orbital Schreier graph $\Gamma_w(G, S)$. The graph $\Gamma_w(G, S)$ is the Schreier coset graph of $G$ with respect to the stabilizer $St_G(w)$.

Every self-similar action $(G, X^*)$ naturally extends to the action of $G$ on the space $X^\infty$ of left-infinite words $x_1x_2\ldots$ over the alphabet $X$. We get an uncountable family of orbital Schreier graphs $\Gamma_w$ for $w \in X^\infty$ of the action $(G, X^\infty)$. For every contracting self-similar action $(G, X^\infty)$ all orbital Schreier graphs $\Gamma_w$ have polynomial growth (see [1, Proposition 2.13.8]). Since nilpotent groups have polynomial growth, all their Schreier graphs also have polynomial growth. Hence the constructed above self-similar action of the Heisenberg group provides an example of a self-replicating finite-state and non-contracting self-similar action with orbital Schreier graphs of polynomial growth. The following problem seems to be interesting: characterize finitely-generated finite-state self-similar groups (i.e., groups generated by finite automata), whose all orbital Schreier graphs $\Gamma_w$ have polynomial growth. More generally, given a finitely generated group $G$ and a nested sequence $\{H_n\}_{n \geq 1}$ of subgroups of finite index in $G$ with trivial intersection $\cap_{n \geq 1} H_n = \{e\}$, consider the action of $G$ on the coset tree of $\{H_n\}_{n \geq 1}$. In what cases the orbital Schreier graphs of the action of $G$ on the boundary of the coset tree have polynomial growth?

We also want to remark the following property of the finite-state self-similar action of the Heisenberg group constructed above. The action is not free on the orbits of points from $\{1, 3\}^\infty$, the stabilizer $St_G(w)$ for $w \in \{1, 3\}^\infty$ is the infinite cyclic group, and the corresponding Schreier graphs $\Gamma_w$ have polynomial growth of degree 3. The action on the other orbits is free, and the corresponding Schreier graphs $\Gamma_w$ have polynomial growth of degree 4. Hence we get an example of a finite-state self-similar group with transitive action on $X^n$ for all $n \in \mathbb{N}$, which has orbital Schreier graphs $\Gamma_w$ with different growth.
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