IMPACTS OF NOISE ON QUENCHING OF SOME MODELS ARISING IN MEMS TECHNOLOGY

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Abstract. In the current work we study a stochastic parabolic problem. The underlying problem is actually motivated by the study of an idealized electrically actuated MEMS (Micro-Electro-Mechanical System) device in the case of random fluctuations of the potential difference controlling the device. We first present the mathematical model and then we deduce some local existence results. Next for some particular versions of the model, regarding its boundary conditions, we derive quenching results as well as estimations of the probability for such singularity to occur. Additional numerical study of the problem in one dimension follows, investigating the problem further with respect to its quenching behaviour.

1. Introduction

In the present work we investigate the following stochastic semilinear parabolic problem

\[ \frac{\partial u}{\partial t} = \Delta u + \frac{\lambda}{(1-u)^2} + \kappa (1-u) \partial_t W(x,t), \quad \text{in} \quad Q_T := D \times (0,T), \quad T > 0, \quad (1.1a) \]

\[ Bu = \beta_c, \quad \text{on} \quad \Gamma_T := \partial D \times (0,T), \quad (1.1b) \]

\[ 0 \leq u(x,0) = u_0(x) < 1, \quad x \in D, \quad (1.1c) \]

as well as some of its variations rise a mathematical interest. Here \( \lambda \) and \( \kappa \), are given positive constants and \( D \) is a bounded subset of \( \mathbb{R}^d \), \( d = 1, 2, 3 \) with smooth boundary. In addition \( \beta_c \) might be a positive or zero constant whilst the boundary operator \( B \) gives rise to Robin boundary conditions, i.e. \( Bu := \frac{\partial u}{\partial \nu} + \beta u \), for some positive constant \( \beta \).

Remarkably, by setting \( \beta \to \infty \) and \( \beta_c = 0 \) (no string effect at the devise supports, cf. [11], and no external force at it) we obtain Dirichlet boundary conditions. On the other hand, for \( 0 < \beta < \infty \) and \( \beta_c = 0 \) Robin boundary conditions arise (models a string effect in the boundary cf. [11]). Cases for \( \beta_c > 0 \) can also be considered, modelling, additional to the string effect, external forces like in pressure sensors, cf. [45]. The latter consideration, regarding nonhomogeneous boundary conditions, has significant theoretical interest as well. Besides, the term \( \partial_t W(x,t) \) denotes by convention the formal time derivative of the one dimensional real valued Wiener random process \( W(x,t) \) in a complete probability space \( \{\Omega, \mathcal{F}_t, \mathbb{P}\} \) with filtration \( \mathcal{F}_t \) on \( \mathbb{R} \); \( W(x,t) \) is defined rigorously in section 3. Thus \( \kappa (1-u) \partial_t W(x,t) \) represents a multiplicative noise reflecting the fact of the occurrence of possible fluctuations into the physical parameters of the MEMS device, cf. section 2.

Notably towards the limit \( \kappa \to 0^+ \) problem (1.1) is reduced to its deterministic version,

\[ \frac{\partial u}{\partial t} = \Delta u + \frac{\lambda}{(1-u)^2}, \quad \text{in} \quad Q_T, \quad T > 0, \quad (1.2a) \]
\( Bu = 0, \) on \( \Gamma_T, \)
\[Bu = 0, \quad \text{on} \quad \Gamma_T, \quad (1.2b)\]
\[0 \leq u(x,0) = u_0(x) < 1, \quad x \in D, \quad (1.2c)\]
which, for homogeneous boundary conditions, has been extensively studied in \([13, 14, 20, 26, 30]\). For hyperbolic modifications of the deterministic variation of \((1.1)\) an interested reader can check \([15, 21, 28]\). Finally, non-local alterations of parabolic and hyperbolic problems arising in MEMS technology are treated in \([11, 12, 17, 19, 27, 29, 30, 37, 38, 39]\).

Due to the presence of the term \( f(u) := \frac{1}{1 - u^2} \) in \((1.2)\) we have the occurrence of an singular behaviour, called \((finite-time) quenching\), when \( \max_{x \in D} u \to 1 \), which is closely associated with the mechanical phenomenon of \textit{touching down}. Relating to the stochastic problem \((1.1)\), it is worth investigating whether such a problem can perform analogous singular (quenching) behaviour to the deterministic problem \((1.2)\). The main purpose of the current paper is twofold; first to examine the circumstances under which quenching occurs for the stochastic problem \((1.1)\), which is actually a stochastic perturbation of \((1.2)\) derived by a random perturbation of the parameter \( \lambda \), cf. section 2. Secondly, we intend to obtain, using both analytical and numerical approach, estimates of the \textit{probability of quenching} as well as of the \textit{quenching time}, which in that case is a random variable. To the best of our knowledge, this is the very first time in the literature that this second approach is considered in the context of MEMS problems. Apart from its practical importance for MEMS engineers such a consideration has its own theoretical value in the context of singular stochastic PDEs (SPDEs).

The structure of the current work is as follows. In the next section a derivation of the stochastic model \((1.1)\) is presented. In section 3 we provide the main mathematical tools from stochastic calculus used through the manuscript as well as give the concepts of solutions for the stochastic problem \((1.1)\) and its considered variations. Section 4 deals with the local existence of all the underlying versions of \((1.1)\) via Banach’s fixed point theorem. Next, in section 5 we appeal to the key properties of exponential functionals of Brownian motion to derive estimates of quenching time as well as estimates of the quenching probability for stochastic problem \((1.1)\) and some of its variations. As far as we know this is the first time in the literature of SPDEs where such an approach is used for MEMS nonlinearities. A numerical approach delivered in section 6 verifies through various numerical experiments the analytical results of the previous sections for nonhomogeneous conditions. Besides, the numerical approach also provides quenching results for the case of homogeneous boundary conditions which is not treated via the analysis of section 5. The current work closes with discussion of the importance of the obtained results in section 7.

2. The mathematical model

Our main motivation for investigating problem \((1.1)\) is its close connection with the operation of some electrostatic actuated MEMS. By the term "MEMS" we more precisely refer to precision devices which combine both mechanical processes with electrical circuits. MEMS devices range in size from millimetres down to microns, and involve precision mechanical components which can be constructed using semiconductor manufacturing technologies. Indeed, the last decade various electrostatic actuated MEMS have been developed and used in a wide variety of devices applied as sensors and have fluid-mechanical, optical, radio frequency (RF), data-storage, and biotechnology applications. Interesting examples of microdevices of this kind include microphones, temperature sensors, RF switches, resonators, accelerometers, micromirrors, micropumps, microvalves, data-storage devices etc., \([30, 42, 45]\).

The key part of such a electrostatic actuated MEMS device usually consists of an elastic plate (or membrane) suspended above a rigid ground one. Regularly the elastic plate is held fixed at two ends while the other two edges remain free to move, see Figure 4.

When a potential difference \( V \) is applied between the elastic membrane and the rigid ground plate, then a deflection of the membrane towards the plate is observed. Assuming now that the width \( d \) of the gap, between the membrane and the bottom plate, is small compared to the device
length $L$, then the deformation of the elastic membrane $u$, after proper scaling, can be described by
the dimensionless equation

$$\frac{\partial u}{\partial t} = \Delta u + \tilde{\lambda} h(x,t) \left(1 - u\right)^2, \quad x \in D, \ t > 0,$$

(2.1)

see [30, 42, 43]. Here the term $h(x,t)$ describes the varying dielectric properties of the membrane
and for some elastic materials can be taken to be constant; for simplicity henceforth we assume
that $h(x,t) \equiv 1$, although the general case is again considered in section 5. Besides, the parameter
$\lambda$ appears in (2.1) equals to

$$\tilde{\lambda} = \frac{V^2 L^2 \varepsilon_0}{2T \ell^3},$$

and is actually the tuning parameter of the considered MEMS device. Note that $\mathcal{T}$ stands for
the tension of the elastic membrane, $\ell$ is the characteristic width of the gap between the membrane
and the fixed ground plate (electrode), whilst $\varepsilon_0$ is the permittivity of free space. MEMS engineers
are interested in identifying under which conditions the elastic membrane could touch the rigid
plate, a phenomenon is usually called touching down and could lead to the destruction of MEMS
device. Touching down can be described via model (2.1) and occurs when the deformation
$u$ reaches the value 1; such a situation in the mathematical literature is known as quenching (or extinction).

Experimental observations, see [45], show a significant uncertainty regarding the values of $V$
and $\mathcal{T}$. More specifically, $V$ fluctuates around an average value $V_0$ (corresponding to some $\lambda > 0$)
inferring that we end up with the parameter $	ilde{\lambda} = \lambda + \sigma \eta(x,t)$ where $\sigma > 0$ is a coefficient measuring
the intensity of the fluctuation (noise term) $\eta(x,t)$. Naturally, the coefficient $\sigma$ depends on the
deformation $u$ (that is $\sigma \equiv \sigma(u)$), whereas a feasible choice for the noise $\eta(x,t)$ could be a space-
time white noise, i.e. $\eta(x,t) = \partial_t W(x,t)$, and thus we consider $\tilde{\lambda} = \lambda + \sigma(u)\partial_t W(x,t)$. From
the applications point of view it would be compelling to investigate the impact of uncertainty
on the phenomenon of touching down. Accordingly, it would be feasible to choose the diffusion
coefficient $\sigma(u)$ as a power of the difference $1 - u$, i.e. $\sigma(u) = (1 - u)^\vartheta$, measuring the distance to
quenching (touching down). Now choosing $\theta = 3$ we derive
\[
\tilde{\lambda} \frac{1 - u}{(1 - u)^2} = \lambda \frac{1 - u}{(1 - u)^2} + \kappa(1 - u) \partial_t W(x, t),
\]
and thus the above analysis reveals that under some imposed uncertainty model (2.1) can be transformed to (1.1a). Furthermore, it should be noted that the choice $\theta = 3$ leads to a linear type diffusion term for which case the local existence theory is well established, cf [8, 25, 31]. For the case of a model with a general diffusion term $\sigma(u)$ the interested reader can check [25]. Next, in case the two edges of the membrane are attached to a pair of torsional and translational springs, modeling a flexible non ideal support [11, 45], see also Figure 2, then homogeneous boundary conditions of the form (1.1b), with $\beta_c = 0$, are imposed together with the stochastic equation for the deformation $u$ and complemented with initial condition (1.1c).

The case of having $\beta_c > 0$ may arise as well with a configuration where the support or cantilever of MEMS devises might be nonideal and flexible. More specifically, considering the situation in which together with the spring force at the edges of the membrane we also have a significant external force oposite to the spring force, e.g. due to gravity, cf. [45]. The latter consideration would result in a boundary condition of the form $\frac{\partial u}{\partial n} = -\beta u + \beta_c$ where $\beta_c$ stands for this external force. For simplicity and without loss of generality, especially regarding the analysis in section 5 we may take $\beta_c$ to be of the same magnitude as $\beta$. Then we end up with a nonhomogeneous boundary condition of the form $\frac{\partial u}{\partial n} = \beta(1 - u)$ for some $\beta > 0$.

![Figure 2. Schematic representation of a MEMS device with support nonideal and subject to external forces.](image)

Notably, the mathematical model (1.1a), as a stochastic perturbation of (2.1), is build up to capture possible destructions due to the uncertainty in parameter measurements of the MEMS system. Thus, under these circumstances is more realistic compared to (2.1).

3. Preliminaries

The current section is devoted to the introduction of the main mathematical concepts and tools from the area of stochastic calculus that will be used throughout the manuscript. Henceforth, $C, K$ will denote positive constants whose values might change from line to line.

We first consider the complete probability space $\{\Omega, \mathcal{F}_t, \mathbb{P}\}$ with filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. Next take $H := L^2(D)$ and let also $Q \in \mathcal{L}_1(H)$ be a linear non-negative definite and symmetric operator which has an orthonormal basis $\chi_j(x) \in H, j = 1, 2, 3, \ldots$ of eigenfunctions with corresponding eigenvalues $\gamma_j \geq 0, j = 1, 2, 3, \ldots$ such that $\text{Tr}(Q) = \sum_{j=1}^{\infty} \gamma_j < \infty$; that is $Q$ is of trace class.
Then \( W(\cdot, t) \) is a \( Q \)-Wiener process if and only if

\[
W(x, t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \chi_j(x) \beta_j(t), \quad \text{almost surely (a.s.)},
\]

(3.1)

where \( \beta_j(t) \) are independent and identically distributed (i.i.d) \( \mathcal{F}_t \)-Brownian motions and the series converges in \( L^2(\Omega, H) \), cf. [7]. It is worth noting that the eigenfunctions \( \{\chi_j(x)\}_{j=1}^{\infty} \) may be different from the eigenfunctions \( \{\phi_j(x)\}_{j=1}^{\infty} \) of the elliptic operator \( A = -\Delta : D(A) = W^{2,2}(\Omega) \cap W^{1,2}(D) \subset H \rightarrow H \), which is self-adjoint, positive definite with compact inverse. Note that the trace class operator \( Q \) is also a Hilbert-Schmidt operator and then we denote \( Q \in \mathcal{L}_2(H) \).

For such an operator \( Q \in \mathcal{L}_2(H) \) with \( \text{Tr}(Q) < \infty \), there exists a kernel \( q(x, y) \) such that

\[
(Qu)(x) := \int_D q(x, y)u(y) \, dy, \quad \text{for any } x \in D, u \in H,
\]

see [7, p. 42-43] and [34, Definition 1.64]. The kernel \( q(x, y) \) is also called the covariance function of the \( Q \)-Wiener process \( W(x, t) \).

Let \( X \) be a Banach space with the norm \( \| \cdot \|_X \) we then define the following Hilbert space

\[
\mathcal{L}_2^0(H; X) = \left\{ \psi \in L(H, X) : \sum_{j=1}^{\infty} \|\psi \chi_j^{1/2} \phi_j\|_X^2 = \sum_{j=1}^{\infty} \gamma_j \|\phi_j\|_X^2 < \infty \right\},
\]

with norm \( \|\psi\|_{\mathcal{L}_2^0} = \left( \sum_{j=1}^{\infty} \gamma_j \|\phi_j\|_X^2 \right)^{1/2} \), where \( L(H, X) \) denotes the space of all bounded operators from \( H \) to \( X \). For \( \Psi : [0, T] \rightarrow \mathcal{L}_2^0(H, X) \), the stochastic integral \( \int_0^T \Psi(t) \, dW(t) \) is well defined, [8]. Furthermore we denote by \( \mathcal{L}_2^1(\Omega, H) \) the space of all random variables \( X : \Omega \rightarrow H \) equipped with the norm

\[
\|X(\omega)\|_{\mathcal{L}_2^1(\Omega, H)} := \mathbb{E} \left[ \|X(\omega)\|_H^2 \right]^{1/2} < \infty, \quad \text{for any } \omega \in \Omega,
\]

known also as the space of the mean-square integrable random variables, where \( \mathbb{E}[\cdot] \) stands for the expectation in the probability space \( (\Omega, \mathcal{F}_t, \mathbb{P}) \).

Analogously problem [1.1] is written in the form of an Itô problem as follows

\[
du_t = (\Delta u_t + f(u_t)) \, dt + \sigma(u_t) \, dW_t, \quad \text{in } Q_T, \quad \text{(3.2a)}
\]

\[
0 \leq u_0 \leq 1, \quad \text{almost surely (a.s.),} \quad \text{(3.2b)}
\]

where \( f(u_t) := (1 - u_t^2)^{-2} \) and \( \sigma(u_t) := \kappa(1 - u_t) \).

It can be easily checked that \( f : H \rightarrow H \), satisfies a local Lipschitz condition, i.e. for any \( 0 \leq \rho < 1 \) and \( w_1, w_2 \in B_\rho := \{w \in L^\infty(\Omega) : 0 \leq \|w\|_\infty < \rho\} \) there exists \( C_\rho > 0 \) such that

\[
\|f(w_1) - f(w_2)\|_H \leq C_\rho \|w_1 - w_2\|_H.
\]

Notably, an immediate consequence of (3.3) is the following growth condition

\[
\|f(w)\|_H \leq C_\rho (1 + \|w\|_H) \quad \text{for any } w \in B_\rho.
\]

(3.4)

Besides, \( \sigma : H \rightarrow \mathcal{L}_2^0 \) satisfies a local Lipschitz condition and a linear growth condition as well ([34, Lemma 10.24]), in particular for any \( 0 < \rho_1 < \rho_2 < 1 \) there exists \( K_{\rho_1, \rho_2} > 0 \) such that for any \( w_1, w_2 \in B_\rho, \)

\[
\|\sigma(w_1) - \sigma(w_2)\|_{\mathcal{L}_2^0} \leq K_{\rho_1, \rho_2} \|w_1 - w_2\|_H \quad \text{and} \quad \|\sigma(w)\|_{\mathcal{L}_2^0} \leq K_{\rho} (1 + \|w\|_H).
\]

(3.5)

Then \( u_t = u(\cdot, t) \) can be interpreted as a predictable \( H \)-valued stochastic process. Next recalling that \( A = -\Delta : D(A) = W^{2,2}(\Omega) \cap W^{1,2}(\Omega) \subset H \rightarrow H \) then \( -A \) is a generator of an analytic semi group \( \mathcal{G}(t) = e^{-tA} \) on \( H \).

In the following we introduce some concepts of solutions for problem (3.2) that will be used through the manuscript.
Definition 3.1. A predictable $H$-valued stochastic process $u_t : t \in [0, T]$ such that
\[
\mathbb{P} \left[ \sup_{(x,t) \in D \times [0,T]} |u_t(x)| < 1 \right] = 1,
\]
is called a weak solution of problem (3.2) if for any $v \in \mathcal{D}(A)$ and for any $t \in [0, T]$,
\[
(u_t, v) = (u_0, v) + \int_0^t [- (u_s, Av) + \lambda f(u_s), v)] ds + \int_0^t (\sigma(u_s) dW_s, v), \quad \mathbb{P} \text{- a.s.,} \tag{3.6}
\]
where $(\cdot, \cdot)$ stands for the inner product into Hilbert space $H = L^2(D)$. Note that the stochastic integral $\int_0^t (\sigma(u_s) dW_s, v)$ is well defined, cf. Theorem 2.4 in [7].

Definition 3.2. A predictable $H$-valued stochastic process $u_t : t \in [0, T]$ such that
\[
\mathbb{P} \left[ \sup_{(x,t) \in D \times [0,T]} |u_t(x)| < 1 \right] = 1,
\]
is called a mild solution of (3.2) if for any $t \in [0, T]$, there holds
\[
 u_t = G(t)z_0 + \lambda \int_0^t G(t-s) f(u_s) ds + \int_0^t G(t-s) \sigma(u_s) dW_s, \quad \mathbb{P} \text{- a.s. and a.e. in } D. \tag{3.7}
\]
Besides, the following interesting variation of problem (3.2) is also investigated in the current work
\[
du_t = (g(t) \Delta u_t + \lambda h(x,t) f(u_t)) dt + \kappa(t)(1 - u_t) dW_t, \quad \text{in } Q_T, \tag{3.8a}
\]
\[
0 \leq u_0 < 1, \quad \text{a.s.,} \tag{3.8b}
\]
where $g, \kappa : \mathbb{R}_+ \to \mathbb{R}_+$ and $h : D \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and bounded functions. It is also assumed that $g \in C^1(\mathbb{R}_+)$. Notably, under the given assumptions for $g$, cf. [41], then the Green’s function $G$ associated with the deterministic problem
\[
\zeta_t = g(t) \zeta, \quad \text{in } Q_T,
\]
\[
\mathcal{B}(\zeta) = \beta c, \quad \text{on } \Gamma_T,
\]
\[
0 \leq \zeta(x,0) = \zeta_0(x) < 1, \quad x \in D,
\]
exists and satisfies the growth conditions
\[
\left| \partial_x^m \partial_t G(x,t;y,s) \right| \leq c(t-s)^{-\frac{d+|m|+2\ell}{2}} \exp \left[ -\frac{|x-y|^2}{t-s} \right], \tag{3.9}
\]
where $m = (m_1, ..., m_d) \in \mathbb{N}^N$, $\ell \in \mathbb{N}$ and $|m| + 2\ell \leq 2$, $|m| = \sum_{j=1}^N m_j$. Then we define the corresponding semigroup $\mathcal{E}(t)$ on $H = L^2(D)$ as follows
\[
\mathcal{E}(t)w(x) := \int_D G(x,t;y,0) w(y) dy \quad \text{for any } x \in D \quad \text{and} \quad 0 < t < T, \tag{3.10}
\]
for any $T > 0$. Using estimates (3.9) in conjunction with a standard approach, cf. [7] Lemma 5.1 we then obtain the following key estimate
\[
\int_0^t \|\mathcal{E}(s)\|_{L^2(H)}^2 ds \leq K_1^2 T, \quad 0 < t < T. \tag{3.11}
\]
In a similar manner we define the notion of weak and mild solutions for problem (3.8).
Definition 3.3. A predictable $H$-valued stochastic process $u_t : t \in [0, T]$ such that
\[ \mathbb{P} \left[ \sup_{(x,t) \in D \times [0,T]} |u_t(x)| < 1 \right] = 1, \]
is called a weak solution of problem (3.8) if for any $v \in \mathcal{D}(A)$ and for any $t \in [0, T]$,
\[ (u_t, v) = (u_0, v) + \int_0^t [-g(s)(u_s, Av) + \lambda(h(\cdot, s)f(u_s), v)]ds + \int_0^t (\kappa(t)(1 - u_s)dW_s, v), \quad \mathbb{P} - a.s. \] (3.12)

Definition 3.4. A predictable $H$-valued stochastic process $u_t : t \in [0, T]$ such that
\[ \mathbb{P} \left[ \sup_{(x,t) \in D \times [0,T]} |u_t(x)| < 1 \right] = 1, \]
is called a mild solution of (3.8) if for any $t \in [0, T]$, there holds
\[ u_t = \mathcal{E}(t)u_0 + \lambda \int_0^t \mathcal{E}(t-s)h(\cdot, s)f(u_s)ds + \int_0^t \mathcal{E}(t-s)\tilde{\sigma}(u_s)dW_s, \quad \mathbb{P} - a.s. \text{ and a.e. in } D, \] (3.13)
where $\tilde{\sigma}(u_t) = \kappa(t)(1 - u_t)$ satisfies clearly condition (3.5) for $\kappa(t)$ bounded.

Remark 3.5. Note that any weak (variational) solution is a mild solution under the assumption of the local Lipschitz continuity of $f$, see [22]. Conversely, any regular enough mild solution is also a weak solution, cf. [34]. The weak formulations (3.6) and (3.12) will be used in section 5 for the investigation of the quenching behaviour, whilst in the following section some existence and uniqueness results for mild solutions are presented.

Next we recall that Itô’s formula (see [35, Theorem 5.2 page 88]) entails
\[ F(W_t) - F(W_0) = \int_0^t F'(W_s)dW_s + \frac{1}{2} \int_0^t F''(W_s)ds, \] (3.14)
for any function $F \in C^2(\mathbb{R})$, which in differential form gives
\[ dF(W_t) = F'(W_t)dW_t + \frac{1}{2} F''(W_t)dt. \]

Closing the current section we recall the integration by parts formula for stochastic processes. Indeed, if $X_t$ and $Y_t$ are Itô stochastic processes given by
\[ X_t = X_0 + \int_0^t \Psi_s ds + \int_0^t \Phi_s dW_s \quad \text{and} \quad Y_t = Y_0 + \int_0^t \tilde{\Psi}_s ds + \int_0^t \tilde{\Phi}_s dW_s \]
then
\[ X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X,Y]_t, \quad t \in [0, T] \] (3.15)
where the last term in the above formula is the quadratic variation of $X_t, Y_t$ and is defined as
\[ [X,Y]_t := \int_0^t \tilde{\Phi}_s \tilde{\Phi}_s ds, \] (3.16)
cf. [35, Corollary 7.11 page 119].
4. LOCAL EXISTENCE

In the current we present local existence and uniqueness results for problems (3.2) and (3.8).

Due to conditions (3.3), (3.4) and (3.5) we derive the following local-in-time existence and uniqueness result for problem (3.2).

**Theorem 4.1.** Fix $0 < \rho_0 < 1$ and consider initial data $u_0 \in L^2(\Omega, \mathcal{D}(A))$ such that $\|u_0\|_{L^2(\Omega, \mathcal{D}(A))} < \rho_0$, then there exists $T = T(\rho_0) > 0$ such that problems (3.2) admits a unique mild solution $u_t$ in $[0, T]$. Furthermore, there exists $C_T > 0$ such that

$$\sup_{0 \leq t \leq T} \|u_t\|_{L^2(\Omega, \mathcal{D}(A))} \leq C_T(1 + \|u_0\|_{L^2(\Omega, \mathcal{D}(A))}).$$  \hspace{1cm} (4.1)

**Proof.** The proof is based on Banach’s fixed point theorem and it follows along the same lines with the proof of [25, Theorem 4.4] and thus it is omitted. \hfill \Box

Next the existence and uniqueness for solutions of problem (3.8) can be directly derived by [41], however in the sequel and for the sake of completeness we provide and prove such a result.

**Theorem 4.2.** Suppose that the functions $g(t), \kappa_1(t)$ are bounded in $[0, T]$, and $h(x, t)$ bounded in $D \times [0, T]$ for any $T > 0$. Assume further that $g(t) \in C^1([0, T])$. Then for fixed $0 < \rho_0 < 1$ and initial data $u_0 \in L^2(\Omega, \mathcal{D}(A))$ such that $\|u_0\|_{L^2(\Omega, \mathcal{D}(A))} < \rho_0$, there exists $T = T(\rho_0) > 0$ such that problem (3.8) admits a unique mild solution $u_t$ in $[0, T]$. Furthermore, there exists $C_T > 0$ such that

$$\sup_{0 \leq t \leq T} \|u_t\|_{L^2(\Omega, \mathcal{D}(A))} \leq C_T(1 + \|u_0\|_{L^2(\Omega, \mathcal{D}(A))}).$$  \hspace{1cm} (4.2)

**Proof.** We first note that since $g(t)$ is bounded then $\mathcal{D}(g(t)A) = \mathcal{D}(A) = W^{2, 2}(\Omega) \cap W^{1, 2}(\Omega)$ recalling that $A = -\Delta$.

Denote by $S_T$ the Banach space of $H$-valued predictable process $u_t : t \in [0, T]$ equipped with the norm

$$\|u_t\|_{S_T} := \sup_{0 \leq t \leq T} \|u_t\|_{L^2(\Omega, \mathcal{D}(A))}.$$  

Now, for any $\rho$, with $0 < \rho_0 < \rho < 1$ we set,

$$S_{\rho, T} := \left\{ u_t \in S_T : \|u_t\|_{S_{\rho, T}} := \sup_{0 \leq t \leq T} \|u_t\|_{L^2(\Omega, \mathcal{D}(A))} \leq \rho < 1 \right\}$$

and we define the operator $\mathcal{M} : S_{\rho, T} \to S_T$ as follows

$$\mathcal{M}(u_t) := \mathcal{E}(t)u_0 + \lambda \int_0^t \mathcal{E}(t-s)f(u_s)ds + \int_0^t \mathcal{E}(t-s)\kappa(t)\sigma(u_s)dW_s.$$  \hspace{1cm} (4.3)

The main idea is to apply Banach’s fixed point theorem to prove existence and uniqueness of the equation $\mathcal{M}(u_t) = u_t$ in $S_{\rho, T}$.

**Step 1:** We first show that $\mathcal{M}$ maps $S_{\rho, T}$ into itself. To this end note that $\mathcal{M}(u_t)$ is a $H$-valued predictable process because $u_0$ is $F_0$-measurable and the stochastic integral is a predictable process. So, it suffices to prove that $\|\mathcal{M}(u_t)\|_{S_{\rho, T}} < \rho$.

By the hypothesis on the initial data $u_0$, and using (3.10) in conjunction with [7, Lemma 5.1] we have regarding the first term in (4.3)

$$\|\mathcal{E}(t)u_0\|_{S_{\rho, T}} \leq \|u_0\|_{S_{\rho, T}} < \rho.$$  \hspace{1cm} (4.4)

Also for $u_t \in S_{\rho, T}$ and by virtue of Sobolev’s inequality there is $\rho_0$ small enough such that

$$\mathbb{E}[\|u_t\|_{\infty}] \leq \rho_0 < 1$$  \hspace{1cm} (4.5)

implying that conditions (3.3)-(3.5) hold true.
where $\kappa$.

In fact for any $0 < t < T$ hence $h_{\gamma} < 0$.

Therefore, we have that for any $0 < t < T$.

Next taking $T := \max_{D \times [0,T]} h(x, t)$ and $C_{\rho_0,T} := N_T C_{\rho_0}$. Next via Itô's isometry, [34, page 322], and by virtue of (3.5) and [34, Exercise 10.7 page 480] we have that for any $0 < t < T$.

$$\left\| \int_0^t E(t-s) f(u_s) ds \right\|_{S_{\rho,T}} \leq \int_0^t \| E(t-s) f(u_s) \|_{S_{\rho,T}} ds \leq N_T \int_0^t \| f(u_s) \|_{S_{\rho,T}} ds \leq N_T C_{\rho_0} (1 + \| u_s \|_{S_{\rho,T}}) ds$$

$$\leq N_T C_{\rho_0} T \left( 1 + \sup_{0 \leq s \leq T} \| u_s \|_{S_{\rho,T}} \right) \leq 2 C_{\rho_0,T} T,$$  

(4.6)

for any $0 < t < T$ where $N_T := \max_{D \times [0,T]} h(x, t)$ and $C_{\rho_0,T} := N_T C_{\rho_0}$. Next via Itô's isometry, [34, page 322], and by virtue of (3.5) and [34, Exercise 10.7 page 480] we have that for any $0 < t < T$.

$$\left\| \int_0^t E(t-s) \kappa(t) \sigma(u_s) dW_s \right\|_{S_{\rho,T}}^2 = \int_0^t \mathbb{E} \left[ \| E(t-s) \kappa(t) \sigma(u_s) \|_{L_2}^2 \right] ds$$

$$\leq (\kappa_T K_{\rho_0})^2 \int_0^t \| E(t-s) \|_{E(H)}^2 ds \left( 1 + \sup_{0 \leq s \leq t} \| u_s \|_{S_{\rho,T}} \right)^2$$  

(4.7)

where $\kappa_T := \max_{[0,T]} \kappa(t)$.

Recalling now the semigroup estimate (3.11) then (4.7) finally reads

$$\left\| \int_0^t E(t-s) \kappa(t) \sigma(u_s) dW_s \right\|_{S_{\rho,T}} \leq 2 \kappa_T T \frac{1}{2}, \quad 0 \leq t \leq T$$  

(4.8)

for $\kappa_T$ a positive constant.

Finally, combining the above relations, (4.4), (4.6) and (4.8) we derive from equation (4.3) that

$$\| M(u_t) \|_{S_{\rho,T}} < \rho_0 + 2 \lambda C_{\rho_0,T} T + 2 \kappa_T \frac{1}{2} \leq \rho_0 + 2 \left( \lambda C_{\rho_0} T + \kappa_T \frac{1}{2} \right).$$  

(4.9)

Next taking $T$ small enough, say smaller than some $T_1 = (\lambda, \rho, \rho_0)$, such that

$$2 \left( \lambda C_{\rho_0,T} T + \kappa_T \frac{1}{2} \right) < \rho - \rho_0, \quad \text{for any} \quad 0 < T < T_1,$$

then (4.9) reads

$$\| M(u_t) \|_{S_{\rho,T}} \leq \rho_0 + \rho - \rho_0 = \rho,$$  

(4.10)

hence $M$ maps $S_{\rho,T}$ to itself.

**Step 2:** Next we show that $M$ is a contraction operator, that is there is a positive constant $0 < \gamma < 1$ such that

$$\| M(u_t) - M(v_t) \|_{S_{\rho,T}} \leq \gamma \| u_t - v_t \|_{S_{\rho,T}}.$$

In fact

$$M(u_t) - M(v_t) = \lambda \int_0^t E(t-s) f(u_s) ds + \int_0^t E(t-s) \kappa(s) (v_s - u_s) dW_s,$$
which implies
\[
\|M(u_t) - M(v_t)\|_{L^2(\Omega,\mathcal{D}(A))}^2 = \left\| \lambda \int_0^t \mathcal{E}(t-s)h(\cdot, s) (f(u_s) - f(v_s)) \, ds \right\|_{L^2(\Omega,\mathcal{D}(A))}^2 \\
+ \left\| \int_0^t \mathcal{E}(t-s)\kappa(s) (v_s - u_s) \, dW_s \right\|_{L^2(\Omega,\mathcal{D}(A))}^2 \\
\leq (\lambda N_T)^2 \left\| \int_0^t \mathcal{E}(t-s) (f(u_s) - f(v_s)) \, ds \right\|_{L^2(\Omega,\mathcal{D}(A))}^2 \\
+ \kappa_T^2 \left\| \int_0^t \mathcal{E}(t-s) (v_s - z_s) \, dW_s \right\|_{L^2(\Omega,\mathcal{D}(A))}^2.
\] (4.11)

The first term in the RHS of (4.11) using (3.3), (3.9) and [7, Lemma 5.1] is estimated as follows
\[
(\lambda N_T)^2 \left\| \int_0^t \mathcal{E}(t-s) (f(u_s) - f(v_s)) \, ds \right\|_{L^2(\Omega,\mathcal{D}(A))}^2 \leq (\lambda N_T C_{\rho,T})^2 \left\| u_t - v_t \right\|_{S_{\rho,T}}^2. 
\] (4.12)

On the other hand, relation (3.5) in conjunction with Itô’s isometry, (3.11) and [34, Exercise 10.7] infers
\[
\kappa_T^2 \left\| \int_0^t \mathcal{E}(t-s) (u_s - v_s) \, dW_s \right\|_{L^2(\Omega,\mathcal{D}(A))}^2 \leq \kappa_T^2 K_T^2 T \left\| u_t - v_t \right\|_{S_{\rho,T}}^2.
\] (4.13)

Now (4.11) by virtue of (4.12) and (4.13) reads
\[
\|M(u_t) - M(v_t)\|_{S_{\rho,T}}^2 \leq (\lambda N_T C_{\rho,T})^2 \left\| u_t - v_t \right\|_{S_{\rho,T}}^2 + \kappa_T^2 K_T^2 T \left\| u_t - v_t \right\|_{S_{\rho,T}}^2 \\
\leq (\lambda^2 N_T^2 C_{\rho,T}^2 + \kappa_T^2 K_T^2) \left\| u_t - v_t \right\|_{S_{\rho,T}}^2 \\
\leq (\lambda^2 N_T^2 C_{\rho,T}^2 + \kappa_T^2 K_T^2) \left\| u_t - v_t \right\|_{S_{\rho,T}}^2 \\
\leq \frac{1}{2} \left\| u_t - v_t \right\|_{S_{\rho,T}}^2.
\]

provided that
\[
T < T_2 := \min \left\{ 1, \frac{1}{2(\lambda^2 N_T^2 C_{\rho,T}^2 + \kappa_T^2 K_T^2)} \right\},
\]
and thus \(M\) is a contraction in \(S_{\rho,T}\) provided that \(T < T_2\).

Consequently, by choosing \(T_0 = \min\{T_1, T_2\}\), we derive that \(M\) has a unique fixed point in \(S_{\rho,T}\) for \(0 < T < T_0\) by Banach’s fixed point theorem and thus problem (3.8) has a unique mild solution in the time interval \([0, T_0]\).

Accordingly, a direct application of Gronwall’s inequality infers estimate (4.1).

\[\square\]

**Remark 4.3.** Due to the obtained regularity, see (4.1), the mild solution provided by Theorem 4.1 is actually a weak solution, cf. [11].

Note that if set \(z = 1 - u\) where \(u\) is the solution of (1.1) then \(z\) satisfies
\[
\frac{\partial z}{\partial t} = \Delta z - \frac{\lambda}{z^2} - \kappa \partial_t W(x,t), \quad \text{in} \quad Q_T,
\] (4.14a)
\[
\mathcal{B}(1-z) = \beta_c \quad \text{on} \quad \Gamma_T,
\] (4.14b)
\[
0 < z_0(x) := z(x,0) = 1 - u_0(x) = \xi(x) \leq 1, \quad x \in D.
\] (4.14c)

In particular, if \(u = 0\) on \(\Gamma_T\) this results in \(z = 1\) for condition (4.14b), or otherwise into \(\frac{\partial z}{\partial u} + \beta z = 0\) if \(u\) satisfies the boundary condition \(\frac{\partial u}{\partial v} = \beta(1-u)\).
Accordingly problem (4.14) can be considered as an Itô equation in the Hilbert space $H = L^2(D)$ and so it can be written by suppressing the dependence on space as follows:

$$dz_t = \left( \Delta z_t - \frac{\lambda}{z_t^2} \right) dt - \kappa z_t dW_t, \quad \text{in} \quad Q_T,$$

(4.15a)

and its local existence and uniqueness stems from Theorem 4.1.

Besides, if $u_t$ satisfies (3.8) then $z_t = 1 - u_t$ solves the following Itô’s problem

$$dz_t = \left( g(t) \Delta z_t - \lambda h(x,t) z_t^{-2} \right) dt - \kappa(t) z_t dW_t, \quad \text{in} \quad Q_T,$$

(4.16a)

and its local existence and uniqueness is guaranteed by Theorem 4.2.

Remarkingy, problems (4.15) and (4.16) are more appropriate for the analysis of the quenching behaviour delivered in the following section.

5. Estimation of Quenching Probability

5.1. The basic model (4.15). In the sequel we will first investigate the quenching behaviour of problem (4.15), whose solution can be expressed as an Itô process as follows

$$z_t = z_0 - \kappa \int_0^t z_s dW_s + \int_0^t \left( \Delta z_s - \frac{\lambda}{z_s^2} \right) ds.$$

(5.1)

Remarkably, the analysis that follows applies to the imposed homogeneous Robin boundary condition $\frac{\partial z_t}{\partial \nu} + \beta z_t = 0$ which corresponds to the situation that a boundary condition (1.1b) is applied for $\beta_c > 0$. The non-homogeneous Robin boundary condition, arises for $\beta_c = 0$ is treated only numerically in section 6.

We define now the stochastic process

$$v_t = e^{\kappa W_t} z_t, \quad 0 \leq t < \tau,$$

(5.2)

cf. [9], where $\tau$ identifies a (random) stopping time, which is actually the quenching time for both $z_t$ and $v_t$. In particular, for any stochastic process satisfying (5.1) there holds

$$\limsup_{t \to \tau} \inf_{x \in D} |z_t(x)| = 0, \quad \text{a.s.} \quad \tau < +\infty.$$

Next using Itô’s formula (3.14) for $F(u) = e^{\kappa u}$ we obtain

$$e^{\kappa W_t} = e^{\kappa W_0} + \kappa \int_0^t e^{\kappa W_s} dW_s + \frac{\kappa^2}{2} \int_0^t e^{\kappa W_s} ds$$

$$= 1 + \kappa \int_0^t e^{\kappa W_s} dW_s + \frac{\kappa^2}{2} \int_0^t e^{\kappa W_s} ds,$$

(5.3)

since $W_0 = 0$, or equivalently

$$d(e^{\kappa W_t}) = \kappa e^{\kappa W_t} dW_t + \frac{\kappa^2}{2} e^{\kappa W_t}.$$

(5.4)

In the sequel, we use for simplicity the notation $z_t(\phi) := \int_D z_t \phi \, dx, \quad t \geq 0,$

for any function $\phi \in C^2(D)$. 
Then problem (5.1), using also second Green’s formula, can be written in a weak formulation as follows

$$z_t(\phi) = z_0(\phi) + \int_0^t \int_D \left[ \frac{\partial z_s}{\partial \nu} \phi - z_s \frac{\partial \phi}{\partial \nu} \right] d\sigma ds + \int_0^t z_s(\Delta \phi) ds$$

$$- \lambda \int_0^t z_s^{-2}(\phi) ds - \kappa \int_0^t z_s(\phi) dW_s,$$

for some test function $\phi$ smooth enough, where

$$z_s^{-2}(\phi) := \int_D z_s^{-2} \phi dx.$$

Next we take as a test function $\phi \in C^2(D)$ satisfying

$$-\Delta \phi = \lambda \phi, \quad x \in D,$$

$$\frac{\partial \phi}{\partial \nu} + \beta \phi = 0, \quad x \in \partial D,$$

normalized as

$$\int_D \phi(x) dx = 1.$$ (5.8)

Note that the principal eigenvalue $\lambda_1$ is positive for $\beta \neq 0$, cf. [3, Theorem 4.3].

In particular the boundary integral in (5.5) thanks to the applied homogeneous Robin-type boundary conditions gives

$$\int_{\partial D} \left[ \frac{\partial z_t}{\partial \nu} \phi - z_t \frac{\partial \phi}{\partial \nu} \right] d\sigma = \int_{\partial D} (-\beta z_t \phi + \beta z_t \phi) d\sigma = 0,$$

and thus the weak formulation (5.5) reduces to

$$z_t(\phi) = z_0(\phi) + \int_0^t z_s(\Delta \phi) ds - \lambda \int_0^t z_s^{-2}(\phi) ds - \kappa \int_0^t z_s(\phi) dW_s.$$ (5.9)

Applying now the integration by parts formula (3.15) to the Itô’s processes defined by (5.1) and (5.3) we have

$$v_t = e^{\kappa W_t} z_t = e^{\kappa W_0} z_0 + \int_0^t e^{\kappa W_s} dz_s + \int_0^t z_s e^{\kappa W_s} + \left[ e^{\kappa W_s}, z_s \right](t),$$

where the quadratic variation is given by

$$[e^{\kappa W_s}, z_s](t) = -\kappa^2 \int_0^t e^{\kappa W_s} z_s ds, \quad t \geq 0,$$ (5.10)

and thus

$$v_t = z_0 + \int_0^t e^{\kappa W_s} dz_s + \int_0^t z_s e^{\kappa W_s} - \kappa^2 \int_0^t e^{\kappa W_s} z_s ds.$$ (5.11)
Next multiplying (5.11) by \( \phi \) and integrating over the domain \( D \) we obtain

\[
v_t(\phi) = z_0(\phi) + \int_0^t e^{\kappa W_s} \left[ \int_D (\Delta z_s - \lambda z_s^{-2}) \phi \, dx \right] \, ds - \kappa \int_0^t e^{\kappa W_s} z_s(\phi) \, dW_s + \kappa \int_0^t e^{\kappa W_s} z_s(\phi) \, dW_s + \frac{\kappa^2}{2} \int_0^t e^{\kappa W_s} z_s(\phi) \, ds - \frac{\kappa^2}{2} \int_0^t z_s(\phi) e^{\kappa W_s} \, ds
\]

\[
= z_0(\phi) + \int_0^t e^{\kappa W_s} \left[ \int_D (\Delta z_s - \lambda z_s^{-2}) \phi \, dx \right] \, ds - \frac{\kappa^2}{2} \int_0^t z_s(\phi) e^{\kappa W_s} \, ds
\]

using also (4.15a) and (5.4) together with second Green’s identity.

Next expressing (5.12) in terms of the \( v_t \), and since \( z_t = v_t e^{-\kappa W_t} \), then thanks to (5.2) we infer

\[
v_t(\phi) = z_0(\phi) - \lambda_1 \int_0^t v_s(\phi) \, ds - \lambda \int_0^t e^{3\kappa W_s} v_s^{-2}(\phi) \, ds - \frac{\kappa^2}{2} \int_0^t v_s(\phi) \, ds
\]

\[
= v_0(\phi) - \left( \lambda_1 + \frac{\kappa^2}{2} \right) \int_0^t v_s(\phi) \, ds - \lambda \int_0^t e^{3\kappa W_s} v_s^{-2}(\phi) \, ds, \quad (5.13)
\]

since \( z_0(\phi) = v_0(\phi) \) due to (5.2).

Then (5.13) implies

\[
\frac{v_t + \varepsilon}{\varepsilon} = \frac{1}{\varepsilon} \left[ - \left( \lambda_1 + \frac{\kappa^2}{2} \right) \int_t^{t+\varepsilon} v_s(\phi) \, ds - \lambda \int_t^{t+\varepsilon} e^{3\kappa W_s} v_s^{-2}(\phi) \, ds \right], \quad (5.14)
\]

and letting \( \varepsilon \to 0 \) in equation (5.14) we derive

\[
\frac{dv_t(\phi)}{dt} = - \left( \lambda_1 + \frac{\kappa^2}{2} \right) v_t(\phi) - \lambda e^{3\kappa W_t} v_t^{-2}(\phi) \quad t > 0, \quad v_0(\phi) > 0, \quad (5.15)
\]

By virtue of Jensen’s inequality, since \( r(s) = s^{-2}, s > 0 \) is convex, and via (5.8) we have

\[
v_t^{-2}(\phi) = \int_D v_t^{-2}(\phi) \, dx \geq \left( \int_D v_t(\phi) \, dx \right)^{-2} = (v_t(\phi))^{-2}
\]

and thus (5.15) leads to the following differential inequality

\[
\frac{dv_t(\phi)}{dt} \leq - \left( \lambda_1 + \frac{\kappa^2}{2} \right) v_t(\phi) - \lambda e^{3\kappa W_t} (v_t(\phi))^{-2}, \quad v_0(\phi) > 0.
\]

By a standard comparison principle we have that \( v_t(\phi) \leq B(t) \) where \( B(t) \) satisfies the following Bernoulli differential equation:

\[
B'(t) = - \left( \lambda_1 + \frac{\kappa^2}{2} \right) B(t) - \lambda e^{3\kappa W_t} B^{-2}(t), \quad B_0 = B(0) = v_0(\phi) > 0,
\]

and is given by

\[
B(t) = e^{-\left( \lambda_1 + \frac{\kappa^2}{2} \right) t} \left[ \frac{B_0^3}{3 \lambda} \int_0^t e^{3 \left( \lambda_1 + \frac{\kappa^2}{2} \right) s + \kappa W_s} \, ds \right]^{1/3}. \quad (5.16)
\]

Next taking into account (5.16) we can define the stopping (quenching) time for \( B(t) \) as

\[
\tau_1 := \inf \left\{ t \geq 0 \left| \int_0^t e^{3 \left( \lambda_1 + \frac{\kappa^2}{2} \right) s + \kappa W_s} \, ds \geq \frac{1}{3 \lambda} B_0^3 \right. \right\}.
\]
and so it follows that $B(t)$ extinests to zero in finite time on the event $\{\tau < +\infty\}$. The fact that $0 \leq v_t(\phi) \leq B(t)$ implies that $\tau_1$ is an uppwer bound of the stopping (quenching) time $\tau$ for $v_t(\phi)$, hence the function

$$t \mapsto \int_D e^{\kappa W_t} z_t(x) \phi(x) \, dx$$

quenches in finite time under the event $\{\tau_1 < +\infty\}$. Using now (5.8) as well as the fact that $t \mapsto e^{\kappa W_t}$ is bounded away from zero on $[0, \tau_1]$, since $\tau_1$ is finite (cf. (5.18) and (5.19) below), then we deduce that the function $t \mapsto \inf_D z_t$ cannot stay away from zero on $[0, \tau_1]$ when $\tau_1 < \infty$. Consequently, $z_t$ also quenches in finite time on the event $\{\tau_1 < +\infty\}$ and $\tau_1$ is an upper bound for the quenching time of $z_t$.

In the sequel we are working towards the estimation of the probability of the event $\{\tau_1 = +\infty\}$, so we have

$$P[\tau_1 = +\infty] = P\left[ \int_0^{t_1} e^{3\kappa W_s + 3(\lambda_1 + \frac{e^2}{2})s} \, ds < \frac{1}{3\lambda} B_0^3, \text{ for all } t > 0 \right] = P\left[ \int_0^{+\infty} e^{3\kappa W_s + 3(\lambda_1 + \frac{e^2}{2})s} \, ds \leq \frac{1}{3\lambda} B_0^3 \right]. \quad (5.17)$$

Then by virtue of the law of the iterated logarithm for $W_t$, cf. [4, 10], that is

$$\lim_{t \to +\infty} \frac{W_t}{t^{1/2} \sqrt{2 \log(\log t)}} = -1, \quad P - a.s., \quad (5.18)$$

and

$$\lim_{t \to +\infty} \frac{W_t}{t^{1/2} \sqrt{2 \log(\log t)}} = +1, \quad P - a.s., \quad (5.19)$$

we deduce that for any sequence $t_n \to +\infty$

$$W_{t_n} \sim \alpha_n t_n^{1/2} \sqrt{2 \log(\log t_n)},$$

with $\alpha_n \in [-1, 1]$, and thus

$$\int_0^{+\infty} e^{3\kappa W_s + 3(\lambda_1 + \frac{e^2}{2})s} \, ds = +\infty.$$

The latter implies that

$$P[\tau_1 = +\infty] = P\left[ \int_0^{+\infty} e^{3\kappa W_s + 3(\lambda_1 + \frac{e^2}{2})s} \, ds \leq \frac{1}{3\lambda} B_0^3 \right] = 0,$$

and hence

$$P[\tau_1 < +\infty] = 1 - P[\tau_1 = +\infty] = 1 - 0 = 1. \quad (5.20)$$

Therefore $B(t)$ and consequently $v_t(\phi)$ quenches a.s. which in turn implies that $z_t(\phi)$ quenches a.s. as well. The latter entails, due also to (5.8), that

$$z_t(\phi) = \int_D z_t(x) \phi(x) \, dx \geq \inf_{x \in D} |z_t(x)|$$

and thus

$$\limsup_{t \to \tau} \inf_{x \in D} |z_t(x)| = 0,$$

for some $\tau \leq \tau_1$ and independently of the initial condition $z_0$ and the parameter value $\lambda$. Thus we have the following result.

**Theorem 5.1.** The weak solution of problem (4.15) quenches in finite time with probability one, i.e. almost surely, regardless the size of its initial condition as well as that of parameter $\lambda$. 


Remark 5.2. The result of Theorem 5.1 shows that the impact of the noise for the dynamics of problem (5.1) is vital. In particular, the presence of the nonlinear term \( f(z) = z^{-2} \) forces the solution towards quenching almost surely. In contrast, for the corresponding deterministic problem, i.e. when \( k = 0 \), and for homogeneous boundary conditions then quenching occurs only either for large initial data or for large values of the parameter \( \lambda \), cf. [13, 26, 30].

5.2. Introducing a regularizing term into model (4.15). A natural question arises is if can modify model (4.15) appropriately so its destructive quenching behaviour can be only limited in a certain range of parameters and so of global-in-time solutions occur as well. To this end we consider a model with a modified nonlinear drift term; indeed the drift term is responsible for the almost surely quenching (cf. Remark 5.2), is now multiplied by \( e^{-3\gamma t} \) for \( \gamma \) some positive constant.

Specifically problem (4.15) now is modified to

\[
 dz_t = (\Delta z_t - \lambda e^{-3\gamma t}z_t^{-2})dt - \kappa z_t dW_t, \quad x \in D, \quad t > 0,
\]

(5.21a)

\[
 \frac{\partial z_t}{\partial v} + \beta z_t = 0, \quad x \in \partial D, \quad t > 0, \quad \beta, \kappa, \gamma > 0
\]

(5.21b)

\[
 0 < z_0(x) = z(x, 0) \leq 1.
\]

(5.21c)

In the sequel we proceed similarly as in the proof of Theorem 5.1 so we first set

\[
z_t(\phi) := \int_D z_t \phi \, dx \quad \text{and} \quad z_t^{-2}(\phi) := \int_D z_t^{-2} \phi \, dx
\]

where \( \phi \) solves (5.6)-(5.8) and then by second Green’s identity we obtain

\[
 \Delta z_t(\phi) = z_t(\Delta \phi),
\]

recalling that

\[
z_t(\Delta \phi) := \int_D z_t \Delta \phi \, dx.
\]

Then the weak formulation of (5.21) is :

\[
z_t(\phi) = z_0(\phi) + \int_0^t z_s(\Delta \phi) ds - \int_0^t \lambda e^{-3\gamma s}z_s^{-2}(\phi) ds - \kappa \int_0^t z_s(\phi) dW_s, \quad \mathbb{P} \text{ - a.s.}
\]

(5.22)

We again consider the stochastic process \( v_t = e^{\kappa W_t}z_t \), for \( 0 \leq t < \tau \) with \( \tau \) being the stopping (quenching) time of stochastic process \( z_t \). Next using integration by parts of, see also (3.15) and (3.16), for the stochastic processes

\[
z_t = z_0 - \kappa \int_0^t z_s dW_s + \int_0^t \left( \Delta z_s - \frac{\lambda e^{-3\gamma s}}{z_s^2} \right) ds
\]

and for \( e^{\kappa W_t} \) given by (5.3) we obtain that

\[
v_t(\phi) = v_0(\phi) + \int_0^t e^{\kappa W_s}dz_s(\phi) + \int_0^t z_s(\phi)d(e^{\kappa W_s}) + [e^{\kappa W_s}, z_s(\phi)](t)
\]

(5.23)

where the quadratic variation into (5.23) is given by (5.10).

Therefore, by virtue of (5.21), (5.22) and Itô’s formula, cf. (4.4), we obtain that

\[
v_t(\phi) = v_0(\phi) + \int_0^t v_s(\Delta \phi) ds - \lambda \int_0^t e^{3\kappa W_s}v_s^{-2}(\phi) ds + \frac{\kappa^2}{2} \int_0^t v_s(\phi) ds,
\]

(5.24)

taking also into account that \( z_t = e^{-\kappa W_t}v_t \).

Notably, via (5.24) we deduce that \( v_t(x) = v(x, t) \) is a weak solution of the following random PDE

\[
 \frac{\partial v}{\partial t}(x, t) = \Delta v(x, t) + \left( \gamma - \frac{\kappa^2}{2} \right) v(x, t) + \lambda e^{3\kappa W_t}v^{-2}(x, t), \quad \text{in} \quad Q_T,
\]

(5.25a)
\[
\frac{\partial v(x, t)}{\partial \nu} + \beta v(x, t) = 0, \quad \text{on} \quad \Gamma_T,
\]
\[
v(x, 0) = z_0(x), \quad x \in D.
\]
(5.25b)

Next by virtue of Jensen’s inequality we deduce

Problem (5.25) should be understood trajectorwise, and its local existence, uniqueness and positivity of solution up to eventual quenching time can be derived by [16, Theorem 9, Chapter 7].

Recalling that \( \phi \) solves the eigenvalue problem (5.6)-(5.8) then equation (5.24) is reduced to

\[
v_t(\phi) = v_0(\phi) - (\lambda_1 + \kappa^2/2) \int_0^t v_s(\phi) ds. - \lambda \int_0^t e^{-3(\gamma s - \kappa W_s)} v_s^{-2}(\phi) ds,
\]
(5.26)
or (cf. subsection 5.1) in differential form

\[
\frac{dv_t(\phi)}{dt} = - \left( \lambda_1 + \kappa^2/2 \right) v_t(\phi) - \lambda e^{-3(\gamma t - \kappa W_t)} v_t^{-2}(\phi).
\]

Next by virtue of Jensen’s inequality we deduce

\[
\frac{dv_t(\phi)}{dt} \leq - \left( \lambda_1 + \kappa^2/2 \right) v_t(\phi) - \lambda e^{-3(\gamma t - \kappa W_t)} (v_t(\phi))^{-2}, \quad v_0(\phi) > 0.
\]

By comparison we get \( v_t(\phi) \leq \Psi(t) \) where \( \Psi(t) \) satisfies the following Bernoulli differential equation

\[
\Psi'(t) = - \left( \lambda_1 + \kappa^2/2 \right) \Psi(t) - \lambda e^{-3(\gamma t - \kappa W_t)} \Psi^{-2}(t), \quad \Psi_0 = \Psi(0) = v_0(\phi) > 0,
\]

with solution

\[
\Psi(t) = e^{-\left( \lambda_1 + \kappa^2/2 \right) t} \left[ \Psi_0^3 - 3 \lambda \int_0^t e^{3\left( \lambda_1 - \gamma + \kappa^2/2 \right) s + 3\kappa W_s} ds \right]^{1/3}, \quad 0 \leq t < \tau,
\]

with

\[
\tau_2 := \inf \left\{ t \geq 0 : \int_0^t e^{3\left( \lambda_1 - \gamma + \kappa^2/2 \right) s + 3\kappa W_s} ds \geq \frac{1}{3\lambda} \Psi_0^3 \right\},
\]

being the stopping time of \( \Psi(t) \).

It follows that \( \Psi(t) \) extincts to zero in finite time on the event \( \{ \tau_2 < +\infty \} \). Since \( v_t(\phi) \leq \Psi(t) \) then \( \tau_2 \) is an upper bound for the stopping (extinction) time \( \tau \) of \( v_t(\phi) \), which is also the stopping (quenching) times of \( v_t \) and \( z_t \).

More specifically we have

\[
\mathbb{P} [\tau_2 = +\infty] = \mathbb{P} \left[ \int_0^t e^{3\kappa W_s + 3\left( \lambda_1 - \gamma + \kappa^2/2 \right) s} ds < \frac{1}{3\lambda} \Psi_0^3, \quad \text{for all} \quad t > 0 \right]
\]
\[
= \mathbb{P} \left[ \int_0^{+\infty} e^{3\kappa W_s + 3\left( \lambda_1 - \gamma + \kappa^2/2 \right) s} ds \leq \frac{1}{3\lambda} \Psi_0^3 \right]. \quad (5.27)
\]

Then via the change of variables \( s_1 \mapsto \frac{9\kappa^2}{4} \) and making use of the scaling property of \( W_t \) we obtain

\[
\mathbb{P} [\tau_2 = +\infty] = \mathbb{P} \left[ \frac{4}{9\kappa^2} \int_0^{+\infty} e^{2W_{s_1} + \frac{4}{3\kappa^2} \left( \lambda_1 - \gamma + \kappa^2/2 \right) s} ds_1 \leq \frac{1}{3\lambda} \Psi_0^3 \right]. \quad (5.28)
\]

Setting \( W_s^{(\mu)} := W_s + \mu s \), with \( \mu := \frac{2}{3\kappa^2} \left( \lambda_1 - \gamma + \kappa^2/2 \right) \) then (5.28) reads

\[
\mathbb{P} [\tau_2 = +\infty] = \mathbb{P} \left[ \frac{4}{9\kappa^2} \int_0^{+\infty} e^{2W_{s}^{(\mu)}} ds \leq \frac{1}{3\lambda} \Psi_0^3 \right]. \quad (5.29)
\]

We now distinguish two cases:
(i) We first take $\gamma \geq \lambda_1 + \frac{\kappa^2}{2}$ and thus we have
\[
\int_0^\infty e^{2W_s(\phi)} \, ds = \frac{1}{2Z-\bar{\mu}},
\]
\[\text{cf. see [44, Chapter 6, Corollary 1.2], where } Z-\mu \text{ is a random variable with law } \Gamma(-\mu), \]
\[\text{i.e. } \mathbb{P}(Z-\mu \in dy) = \frac{1}{\Gamma(-\mu)} e^{-y} y^{-\mu-1} \, dy,
\]
where $\Gamma(\cdot)$ is the complete gamma function, cf. [1].

Hence (5.29) entails (see also in [39] formula 1.104(1) page 264)
\[
\mathbb{P}[\tau_2 = +\infty] = \int_0^\infty \frac{1}{\lambda \nu_0^3(\phi)} \left( \frac{g_\kappa y^2}{2} \right)^{-\frac{(\kappa^2 - 2\gamma + 2\lambda_1)}{3\kappa^2}} \exp \left( -\frac{2}{9\kappa^2} y \right) \, dy,
\]
\[\text{(5.30)}\]

hence
\[
\mathbb{P}[\tau_2 < +\infty] = 1 - \mathbb{P}[\tau_2 = +\infty] = \int_0^\infty \frac{1}{\lambda \nu_0^3(\phi)} \left( \frac{g_\kappa y^2}{2} \right)^{-\frac{(\kappa^2 - 2\gamma + 2\lambda_1)}{3\kappa^2}} \exp \left( -\frac{2}{9\kappa^2} y \right) \, dy.
\]

Now since $\tau < \tau_2$ we have that
\[
\mathbb{P}[\tau < +\infty] \geq \int_0^\infty \frac{1}{\lambda \nu_0^3(\phi)} \left( \frac{g_\kappa y^2}{2} \right)^{-\frac{(\kappa^2 - 2\gamma + 2\lambda_1)}{3\kappa^2}} \exp \left( -\frac{2}{9\kappa^2} y \right) \, dy.
\]
\[\text{(5.31)}\]

(ii) Next we assume that $\mu > 0$, i.e. $\gamma < \lambda_1 + \frac{\kappa^2}{2}$. Then using the law of the iterated logarithm, cf. (5.18) and (5.19), for $W_t$, we obtain
\[
\int_0^\infty e^{3\kappa W_s + 3\left(\lambda_1 - \gamma + \frac{\kappa^2}{2}\right)t} \, ds = +\infty,
\]
hence via (5.27) we derive
\[
\mathbb{P}[\tau_2 = +\infty] = \mathbb{P} \left[ \int_0^\infty e^{3\kappa W_s + 3\left(\lambda_1 - \gamma + \frac{\kappa^2}{2}\right)t} \, ds \leq \frac{1}{\lambda \nu_0^3(\phi)} \Psi_0^3 \right] = 0
\]
and thus
\[
\mathbb{P}[\tau_2 < +\infty] = 1 - \mathbb{P}[\tau_2 = +\infty] = 1.
\]

Summarizing the above we have the following result

**Theorem 5.3.**

(i) If $\gamma > \lambda_1 + \frac{\kappa^2}{2}$ then the weak solution of problem (5.21) quenches in finite time with probability bounded below as shown in (5.31).

(ii) In the complementary case when $\gamma < \lambda_1 + \frac{\kappa^2}{2}$ then the weak solution of problem (5.21) quenches in finite time almost surely.

**Remark 5.4.** Let us fix $\gamma$ and $\kappa$ so that $\gamma - \frac{\kappa^2}{2} > 0$. Then Theorem 5.3(ii) entails that quenching behaviour dominates when $\lambda_1$ is big which only occurs when the domain $D$ is rather small.

In Figure 3 an upper bound of the probability of global existence, provided by (5.30), is displayed with respect to the parameter $\lambda$ in Figure 3(a) and with respect to the parameter $a$ in Figure 3(b). In that case an initial condition of the form $z_0(x) = 1 - ax(1 - x)$ is considered. Specifically, in Figure 3(a) we observe a decrease of the probability of global existence, as $\lambda$ increases. Similarly in Figure 3(b) again reducing the minimum of the initial condition results in
decreasing the probability of global existence and this becomes more intensive as \( \lambda \) increases. Besides, in Figure 4 the behaviour of the probability of the global existence, bounded above by the quantity defined in (5.31), is examined with respect to the parameter \( \gamma \) and the noise amplitude \( \kappa \). In particular, the impact of parameter \( \gamma \), i.e. the coefficient of the regularizing term, is displayed in Figure 4(a). Note that the condition \( \gamma > \lambda_1 + \kappa^2 \) should be satisfied (here \( \lambda_1 = \pi^2 \) and \( \kappa = 1 \)); then we observe a peak of the probability at the value \( \gamma = 13.77 \). Moreover in Figure 4(b) the variation of that probability with respect to the parameter \( \kappa \) for various values of the parameter \( \lambda \) is shown. In that case a similar peak is attained at the value \( \kappa = 1.084 \).

5.3. Model (4.16). In the current subsection we investigate the probability of quenching for the solution of problem (4.16) where \( g, \kappa_1 : \mathbb{R}_+ \to \mathbb{R}_+ \) and \( h : D \times \mathbb{R}_+ \to \mathbb{R}_+ \) are continuous functions. Note that from mathematical modelling perspective the function \( g(t) \) represents the dispersion coefficient whilst \( h(x,t) \) describes the varying dielectric properties of the elastic membrane ([14]), cf. section 2.

Next we define the random process

\[
M_t := \int_0^t \kappa_1(s) dW_s,
\]
and we set
\[ v_t := e^{Mt} z_t, \quad 0 \leq t < \tau, \]
where again \( \tau \) is the stopping (quenching) time of stochastic process \( z_t \) determined by (4.16).

In the sequel we proceed as in [2]. Itô’s formula implies the semimartingale expansion
\[ e^{Mt} = 1 + \int_0^t \kappa_1(s) e^{Ms} dW_s + \frac{1}{2} \int_0^t \kappa_2(s) e^{Ms} ds. \]

Next by letting \( z_t(\phi) := \int_D z_t \phi dx \) and \( z_t^{-2}(\phi) := \int_D z_t^{-2} \phi dx \), where again \( \phi \in C^2(D) \) solves the eigenvalue problem (5.6)-(5.8), we have
\[ z_t(\phi) = z_0(\phi) + \int_0^t g(s) \Delta z_s(\phi) ds - \lambda \int_0^t h(x, s) z_s^{-2}(\phi) ds - \int_0^t \kappa_1(s) z_s(\phi) dW_s, \]
\( \mathbb{P} \) – a.s. for all \( t \in [0, \tau) \).

Note also that for any fixed \( \phi \), the process \( (z_t(\phi))_{t \in [0, \tau]} \) is also a semimartingale. Moreover using integration by parts formula, cf. (3.15) and (3.16), we get the weak formulation
\[ v_t(\phi) = e^{Mt} z_t(\phi) \]
\[ = e^{Mo} z_0(\phi) + \int_0^t e^{Ms} z_s(\phi) ds - \int_0^t \kappa_1(s) e^{Ms} z_s(\phi) dW_s, \]
where the quadratic variation (see [35, section 7.6, pg. 113]) is given by
\[ [e^{Mt}, z_t(\phi)](t) := - \int_0^t \kappa_2(s) e^{Ms} z_s(\phi) ds. \]

Next (5.35) in conjunction with (5.32), (5.33) and (5.34) yields
\[ v_t(\phi) = z_0(\phi) + \int_0^t e^{Ms} (g(s) \Delta z_s(\phi) - \lambda z_s^{-2}(h(\phi))) ds \]
\[ + \int_0^t e^{Ms} \kappa_1(s) z_s(\phi) dW_s - \int_0^t e^{Ms} \kappa_1(s) z_s(\phi) dW_s \]
\[ + \frac{1}{2} \int_0^t \kappa_2(s) e^{Ms} z_s(\phi) ds - \int_0^t \kappa_2(s) e^{Ms} z_s(\phi) ds \]
\[ = v_0(\phi) + \int_0^t g(s) v_s(\Delta \phi) ds - \lambda \int_0^t e^{3Ms} v_s^{-2}(h(\phi)) ds - \frac{1}{2} \int_0^t \kappa_2(s) v_s(\phi) ds \]
\[ = v_0(\phi) - \lambda_1 \int_0^t g(s) v_s(\phi) ds - \lambda \int_0^t e^{3Ms} v_s^{-2}(h(\phi)) ds - \frac{1}{2} \int_0^t \kappa_2(s) v_s(\phi) ds, \]

where
\[ v_s^{-2}(h(\phi)) := \int_D v_s^{-2}(x) h(x, t) \phi(x) dx, \]
taking also into account that \( z_0(\phi) = v_0(\phi) \) due to (5.32) as well as that \( \Delta v_s(\phi) = v_s(\Delta \phi) = -\lambda_1 v_s(\phi) \) via Green’s second identity.

Equation (5.36) can then be written in differential form as
\[ \frac{dv_t(\phi)}{dt} = - \left( \lambda_1 g(t) + \frac{1}{2} \kappa_2(t) \right) v_t(\phi) - \lambda e^{3Mt} v_t^{-2}(h(\phi)), \]
which by virtue of Jensen’s inequality infers
\[ \frac{dv_t(\phi)}{dt} \leq - \left( \lambda_1 g(t) + \frac{1}{2} \kappa_2(t) \right) v_t(\phi) - \lambda \omega e^{3Mt} (v_t(\phi))^{-2}, \]
where \( \omega := \max_{(x, s) \in D \times [0, \tau]} h(x, s) > 0 \) and by means of a comparison argument we get
\[ v_t(\phi) \leq A(t), \quad 0 \leq t < \tau, \]
where now $A(t)$ denotes the solution of the initial value problem

$$A'(t) = -\left(\lambda_1 g(t) + \frac{1}{2} \kappa_1^2(t)\right) A(t) - \lambda \omega e^{3M_t} A^{-2}(t), \quad 0 < t < \tau, \quad A_0 = A(0) = v_0(\phi) > 0,$$

with solution

$$A(t) = e^{-\left(\lambda_1 K(t) + \frac{1}{2} J(t)\right)} \left[v_0(\phi)^3 - 3\lambda \omega \int_0^t e^{3M_s + 3(\lambda_1 K(s) + \frac{1}{2} J(s))} ds\right]^\frac{1}{3}, \quad (5.38)$$

where $K(t) := \int_0^t g(s) ds$ and $J(t) := \int_0^t \kappa_1^2(s) ds$.

The maximum existence (stopping) time $\tau_3$ of $A(t)$ is then given by

$$\tau_3 := \left\{ t \geq 0 : \int_0^t e^{3M_s + 3(\lambda_1 K(s) + \frac{1}{2} J(s))} ds \geq \frac{1}{3\lambda \omega} v_0^3(\phi) \right\},$$

and actually $A(t)$ quenches in finite time on the event $\{\tau_3 < +\infty\}$. The fact that $0 \leq v_t(\phi) \leq A(t)$ reveals that $\tau_3$ is an upper bound of the stopping (extinction) time $\tau$ for $v_t(\phi)$, hence the function

$$t \mapsto \int_D e^{M_t} z_t(x) \phi(x) \, dx$$

quenches in finite time under the event $\{\tau_3 < +\infty\}$. Using now (5.8) as well as the fact that $t \mapsto e^{M_t}$ is bounded below away from zero (cf. [5.18], [5.19] and the fact that $\kappa_1(t)$ is bounded) on $[0, \tau_3]$, once $\tau_3 < \infty$, then we deduce that the function $t \mapsto \inf_D z_t$ cannot stay away from zero on $[0, \tau_3]$ for $\tau_3 < \infty$. Therefore, $z_t$ quenches in finite time on the event $\{\tau_3 < +\infty\}$ and so $\tau_3$ is an upper bound for the quenching time of $z_t$.

Observe that $M_t = \int_0^t \kappa_1(s) dW_s$ is a continuous martingale and so it can be written as a time-changed Brownian motion $M_t = W_{J(t)}$, where $J(t) = [M](t) = \int_0^t \kappa_1^2(s) ds$ is the quadratic variation of $M_t$, cf. [23, Theorem 4.6 page 174] and [2].

Set $\rho := \frac{1}{3\lambda \omega} v_0^3(\phi)$ then

$$\mathbb{P}(\tau_3 = +\infty) = \mathbb{P}\left(\int_0^t e^{3M_s + 3(\lambda_1 K(s) + \frac{1}{2} J(s))} ds < \frac{1}{3\lambda \omega} v_0^3(\phi), \quad \text{for all} \quad t > 0\right)$$

$$= \mathbb{P}\left(\int_0^\infty e^{3W_{J(s)} + 3(\lambda_1 K(s) + \frac{1}{2} J(s))} ds \leq \rho\right)$$

$$= \mathbb{P}\left(\int_0^\infty \frac{1}{\kappa_1^2(J^{-1}(s_1))} e^{3W_{s_1} + 3(\lambda_1 K(J^{-1}(s_1)) + \frac{1}{2} s_1)} ds_1 \leq \rho\right) \quad (5.39)$$

where $s_1 := J(s)$.

At that point we introduce the assumption that coefficients $g(t)$ and $\kappa_1(t)$ satisfy: there exists some positive constant $C$ such that

$$\frac{1}{\kappa_1^2(t)} e^{3\lambda_1 (K(t) + \frac{1}{2} J(t))} \geq C \quad \text{for any} \quad t \geq 0. \quad (5.40)$$

Then (5.39) via (5.40) reads

$$\mathbb{P}(\tau_3 = +\infty) \leq \mathbb{P}\left(\int_0^\infty e^{3W_{s_1} + \left(-\frac{3}{2} \lambda_1 J^{-1}(s_1) + \frac{1}{2} s_1\right)} ds_1 \leq \frac{\rho}{C}\right)$$

$$= \mathbb{P}\left(\int_0^\infty e^{3W_{s_1} + \frac{1}{2}(1-\lambda_1)s_1} ds_1 \leq \frac{\rho}{C}\right), \quad (5.41)$$
Next we introduce the change of variables $s_2 \mapsto \left(\frac{3}{4}\right)^{\frac{1}{2}} s_1$, and thus again via the scaling property of $W$ then (5.41) entails

$$
P(\tau_3 = +\infty) \leq P\left(\frac{4}{9} \int_0^\infty e^{\frac{3}{4}W_{s_2} + \frac{3}{2}(1-\lambda_1)\frac{1}{2}s_2} ds_2 \leq \frac{\rho}{C}\right) = P\left(\int_0^\infty e^{2\left(\frac{1-\lambda_1}{3}\right)s_2 + 2Ws_2} ds_2 \leq \frac{9\rho}{4C}\right) = P\left(\int_0^{+\infty} e^{2W_s} ds \leq \frac{9\rho}{4C}\right), \tag{5.42}
$$

where $\mu := \frac{1-\lambda_1}{3}$ and $W^{(\mu)} := W + \mu s$.

Next we distinguish the following cases:

(i) Initially we assume that $\mu < 0$, i.e. $\lambda_1 > 1$. Then by virtue of (5.42) and following the same reasoning as in subsection 5.2 we obtain

$$
P(\tau_3 = +\infty) \leq P\left(\frac{1}{2Z_{-\mu}} \leq \frac{9\rho}{4C}\right) = \frac{1}{\Gamma(-\mu)} \int_0^{\frac{2C}{9\rho}} y^{-\mu-1} e^{-y} dy, \tag{5.43}
$$

cf. [41, Corollary 1.2 page 95]. Hence, from (5.43) we derive

$$
P(\tau_3 < +\infty) = 1 - P(\tau_1 = +\infty) \geq 1 - \frac{1}{\Gamma(-\mu)} \int_0^{\frac{2C}{9\rho}} y^{-\mu-1} e^{-y} dy = \frac{1}{\Gamma(-\mu)} \int_0^{\infty} y^{-\mu-1} e^{-y} dy. \tag{5.44}
$$

(ii) In the complimentary case $\mu \geq 0$, i.e. when $\lambda_1 \leq 1$, then via the iterated logarithm law for $W_s$, cf. (5.18) and (5.19), we obtain

$$
\int_0^{+\infty} e^{2W^{(\mu)}_s} ds = +\infty
$$

and thus

$$
P(\tau = +\infty) = P\left(\int_0^{+\infty} e^{2W^{(\mu)}_s} ds \leq \frac{9\rho}{4C}\right) = 0.
$$

The latter implies that

$$
P(\tau < +\infty) = 1 - P(\tau = +\infty) = 1
$$

and so in that case $A(t)$ quenches a.s. independently of the initial condition $v_0$ and the parameter $\lambda$, which also entails that $v_t$ and $z_t$ quench as well.

**Theorem 5.5.** Assume that condition (5.40) holds true for the continuous positive functions $g(t), \kappa_1(t) > 0$. Then:

(i) if $\lambda_1 > 1$ the probability of quenching of the weak solution of problem (4.16) is lower bounded as shown in (5.43),

(ii) whilst for $\lambda_1 \leq 1$ then the weak solution of problem (4.16) quenches in finite time $\tau < \infty$ almost surely.

**Remark 5.6.** Note that in the special case $g(t) = 1, \kappa_1(t) = \kappa =$ constant and $h(x,t) = 1$ then via relation (5.39) we recover the result of Theorem 5.1.

**Remark 5.7.** Remarkably Theorem 5.5 (ii) implies that when the diffusion coefficient $g(t)$ is large, enough ensured by condition (5.40), then quenching behaviour dominates for the case of a big domain $D$. This looks in the counterintuitive to what has been pointed out in Remark 2.4 in the first place, however it is in full agreement with the phenomenon observed in [32] where a strong reaction coefficient, enhanced there by the evolution of underlying domain, fights against the development of a singularity.
Remark 5.8. Note that since $K(t)$ and $J(t)$ are increasing functions we have
\[ e^{3\lambda_1(K(t)+\frac{1}{2}J(t))} \geq e^{3\lambda_1(K(0)+\frac{1}{2}J(0))} = 1, \]
and thus condition (5.40) holds true provided that $\kappa_1(t)$ is bounded above, i.e. $\sup_{(0,\infty)} \kappa_1(t) = L < \infty$. In that case we have that $C = \frac{1}{\kappa_1}$.

Alternatively, if $\kappa_1(t)$ gets unbounded as $t \to \infty$ but satisfies the growth condition
\[ \frac{d\kappa_1^2(t)}{dt} \leq \beta \kappa_1^2(t), \quad t > 0, \quad \text{for some } \beta > 0, \]
then by virtue of L’Hôpital’s rule we can show that
\[ \lim_{t \to \infty} \frac{e^{J(t)}}{\kappa_1^2(t)} = \infty \]
and then using again the monotonicity of $K(t)$ we derive (5.40) with $C = 1$.

In relation to applications it is of particular interest to simulate the stochastic process describing the operation of MEMS device and so to investigate under which circumstances it quenches. For that purpose in the following section we present such a numerical algorithm together with various related simulations for problem (1.1).

6. Numerical Solution

6.1. Finite Elements approximation. In the current section we present a numerical study of problem (1.1) in the one-dimensional case. For that purpose we apply a finite element semi-implicit Euler in time scheme, cf. [34]. The considered noise term is a multiplicative one and of the form $\sigma(u)\,dW_t$ for $\sigma(u) = \kappa(1-u)$ with $\kappa > 0$. We also assume homogeneous Dirichlet boundary conditions at the points $x = 0,1$, although some of the presented numerical experiments also concern homogeneous and nonhomogeneous Robin boundary conditions. A homogeneous Dirichlet boundary condition $u(0,t) = u(1,t) = 0$ corresponds in having $z = 1$ at those points. Remarkably, this is a case is not actually covered by the analysis in section 5.

We apply a discretization in $[0,T] \times [0,1]$, $0 \leq t \leq T$, $0 \leq x \leq 1$ with $t_n = n\delta t$, $\delta t = [T/N]$ for $N$ the number of time steps and we also introduce the grid points in $[0,1]$, $x_j = j\delta x$, for $\delta x = 1/M$ and $j = 0,1, \ldots, M$.

Then we proceed with a finite element approximation for problem (1.1). Let $\Phi_j$, $j = 1, \ldots, M-1$, denote the standard linear $B-$ splines on the interval $[0,1]$
\[ \Phi_j = \begin{cases} \frac{y-y_{j-1}}{y_j-y_{j-1}}, & y_{j-1} \leq y \leq y_j, \\ \frac{y_{j+1}-y}{y_{j+1}-y_j}, & y_j \leq y \leq y_{j+1}, \\ 0, & \text{elsewhere in } [0,1], \end{cases} \]
for $j = 1,2, \ldots, M-1$. We then set $u(x,t) = \sum_{j=1}^{M-1} a_{uj}(t) \Phi_j(x)$, $t \geq 0$, $0 \leq x \leq 1$.

Substituting the later expression for $u$ into equation (1.1a) and applying the standard Galerkin method, i.e. multiplying with $\Phi_i$, for $i = 1,2, \ldots, M-1$ and integrating over $[0,1]$, we obtain a system of equations for the $a_{uj}$’s as follows
\[
\sum_{j=1}^{M-1} \dot{a}_{uj}(t) \langle \Phi_j(x), \Phi_i(x) \rangle = - \sum_{j=1}^{M-1} a_{xj}(t) \langle \Phi_j'(x), \Phi_i'(y) \rangle \\
+ \left\langle F \left( \sum_{j=1}^{M-1} a_{uj}(t) \Phi_j(x) \right), \Phi_i(x) \right\rangle, \\
+ \left\langle \sigma \left( \sum_{j=1}^{M-1} a_{uj}(t) \Phi_j(x) \right) dW(x,t), \Phi_i(x) \right\rangle,
\]
(6.2)
where \( <f,g> := \int_0^1 f(x)g(x)dx \) and \( i = 1, 2, \ldots, M - 1 \), and in our case \( F(s) = \frac{\lambda}{(1-s)^2} \), \( \sigma(s) = \kappa(1-s) \).

Setting \( a_u = [a_{u1}, a_{u2}, \ldots, a_{uM-1}]^T \) the system of equations for the \( a_u \)'s take the form
\[
A\hat{a}_u(t) = -BA_u(t) + b(t) + b_s(t),
\]
for
\[
b(u) = \left\{ \left\langle F \left( \sum_{j=1}^{M-1} a_{u_j}(t) \Phi_j(x) \right) , \Phi_i(x) \right\rangle \right\}_i,
\]
\[
b_s(u, \Delta W_t) = \left\{ \left\langle \sigma \left( \sum_{j=1}^{M-1} a_{u_j}(t) \Phi_j(x) \right) \Delta W(x,t) , \Phi_i(x) \right\rangle \right\}_i,
\]
the latter coming from the corresponding Itô integral, and \( dW_t \simeq \Delta W_h(x,t) = W_h(t + \delta t, x) - W_h(t, x) \) for \( W_h(t) \) the finite sum giving the discrete approximation of \( W(t) \).

Moreover we approximate the space-time white noise by taking
\[
W_h(t + \delta t) - W_h(t) = \sum_{n=0}^{m-1} W_h(t + t_{n+1}) - W_h(t + t_n).
\]

Moreover we approximate the space-time white noise by taking
\[
W_h(t^{n+1}) - W_h(t^n) = \sqrt{\delta t_r} \sum_{j=1}^{M-1} \sqrt{q_j} \chi_j \xi_j^n,
\]
where \( \xi_j^n := (\beta_j(t_{n+1}) - \beta_j(t_n))/\sqrt{\delta t_r} \) and \( \xi_j^n \sim N(0,1) \) are i.i.d. random variables for i.i.d. Brownian motions \( \beta_j(t) \). Also the eigenfunctions \( \chi_j = \chi_j(x) = \sqrt{2} \sin(j\pi x), j \in \mathbb{N}^+ \) are taken as a basis of \( L^2(0,1) \) and \( q_j \) are chosen to be
\[
\xi_j = \left\{ \begin{array}{ll}
  l^{-(2r+1+\epsilon)} & j = 2l + 1, j = 2l, \\
  0 & j = 1,
\end{array} \right. \tag{6.3}
\]
for \( l \in \mathbb{N} \), \( r \) being the regularity parameter, \( 0 < \epsilon < 1 \) to obtain an \( H^r_\delta(0,1) \)-valued process.

We then apply a semi-implicit Euler method in time by taking
\[
A\hat{a}_u(t_n) \simeq A \left( a_{u+1} - a_u \right) / (\delta t) = -BA_{u+1} + b(u^n) + b_s(u^n)
\]
or
\[
(A + \delta t B) a_{u+1} = a_u + \delta t b(u^n) + \delta t b_s(u^n, \Delta W^n_h)
\]
with the \((M-1) \times (M-1)\) matrices \( A, B \) having the form
\[
A = \delta x \begin{bmatrix}
  2/3 & 1/2 & 0 & \cdots & 0 \\
  1/2 & 1/3 & 0 & \cdots & 0 \\
  & \ddots & \ddots & \ddots & \ddots \\
  0 & \cdots & \frac{1}{3} & \cdots & 0
\end{bmatrix},
\]
\[
B = \frac{1}{\delta y} \begin{bmatrix}
  2 & -1 & 0 & \cdots & 0 \\
  -1 & 2 & -1 & \cdots & 0 \\
  & \ddots & \ddots & \ddots & \ddots \\
  0 & \cdots & 0 & 0 & -1
\end{bmatrix},
\]
\[
b^n = b(u^n) = \left\{ \left\langle F \left( \sum_{j=0}^{M-1} a_{u_j}(t_n) \Phi_j(x) \right) , \Phi_i(x) \right\rangle \right\}_i,
\]
\[
b_s^n = b_s(u^n, \Delta W^n_h) = \left\{ \left\langle \sigma \left( \sum_{j=0}^{M} a_{u_j}(t_n) \Phi_j(x) \right) \Delta W^n_h , \Phi_i(x) \right\rangle \right\}_i,
\]
for \( a_{u_j} = a_{u_j}(t_n), i = 1 \ldots, M - 1. \)
Finally the corresponding algebraic system for the \( a_n \)'s after some manipulation becomes

\[
a_{n+1} = (A + \delta t B)^{-1} \left[ a_n + \delta t b^n + \delta t b_s^n \right],
\]

for \( a_1 \) being determined by the initial condition.

6.2. Simulations. Initially we present a realization of the numerical solution of problem (1.1) in Figure 5(a) for \( \lambda = 1, \kappa = 1, M = 102, N = 10e4, r = 0.1 \) and initial condition \( u(x, 0) = c x (1 - x) \) for \( c = 0.1 \) and homogeneous Dirichlet boundary conditions \( (\beta \to \infty, \beta_c = 0) \). By this performed realization the occurrence of quenching is evident. For a different realization but for the same parameters in Figure 5(b) the maximum of the solution at each time step is plotted and again a similar quenching behaviour is observed. Next in Figure 6(a) we observe the quenching behaviour of five realisations of the numerical solution of problem (1.1) for \( \lambda = 2 \). In an extra realization depicted in Figure 6(b) the spatial distribution of the numerical solution at different time instants can be seen.

An interesting direction worth investigating is the derivation of estimates of the probability of quenching in a specific time interval \( [0, T] \) for some \( T > 0 \). It is known, cf. \[25\], that for imposed
Dirichlet boundary conditions, then the solution $u$ will eventually quench in some finite time $T_q$ for large enough values of the parameter $\lambda$ or big enough initial data.

From the application point of view an estimate of the probability that $T_q < T$ would be useful with respect to various values of the parameter $\lambda$.

In Table (T1) the results of such a numerical experiment are presented. In particular, implementing $N_R$ realizations then in the first column we print out the values of $\lambda$ considered, while the second column contains the number of times that the solution quenched before the time $T$, whilst in the last two columns the mean $m(T_q)$ and the variance $Var(T_q)$ of the quenching time respectively are given. The rest of the parameters were taken to be the same as in the previous simulations but with $\kappa = 0.1$.

By the results in Table (T1) we observe that in a finite time interval the stochastic problem performs a dynamic behaviour which resembles that of the deterministic one. Specifically, increasing the value of $\lambda$ initially we have no quenching in this time interval while after $\lambda > \lambda^* > 1$ we have quenching almost surely at a time $T_q$ with mean and variance decreasing with $\lambda$.

### Table (T1)

Realizations of the numerical solution of problem (1.1) for $N_R = 1000$ in the time interval $[0, 10]$.

| $\lambda$ | Quenching times | $m(T_q)$ | $\sigma^2(T_q)$ |
|-----------|----------------|----------|---------------|
| 0.5       | 0              | -        | -             |
| 1         | 0              | -        | -             |
| 1.5       | 1000           | 1.4642   | 0.0071        |
| 2         | 1000           | 0.3542   | 3.7852e-05    |
| 2.5       | 1000           | 0.2184   | 4.2468e-06    |

Additionally, we perform another experiment for simulation time $T = 1$ and $\lambda = 1.65$, chosen in a $\lambda$-range where the occurrence of quenching is not definite, and for a larger number of realizations $N_R = 10^4$, whilst the rest of the parameter values being kept the same as in Table (T1). Then we obtain a numerical estimation for the probability of quenching equal to 0.3464 with $m(T_q) = 0.3380$ and $Var(T_q) = 0.2157$.

Next we consider the case of nonhomogeneous boundary conditions of the form (1.1b) or equivalently (4.14b) with $\beta = \beta_c$, since such a case is of particular interest in the light of the quenching results of section 5. A simulation implementing the previously described numerical algorithm for this particular case is presented in Figure 7(a). The presented realization is for problem (1.1) and the parameters used here are $\lambda = 0.3$, $k = 1$, $\beta = \beta_c = 1$. Also, in Figure 7(b) the quenching of $||u(\cdot, t)||_{\infty}$ for one realization is depicted.

Similarly in the next set of graphs in Figure 8(a) we display the quenching behaviour of five realisations of the numerical solution of problem (1.1) for $\lambda = 0.3$. In an extra realization provided by Figure 8(b) the spatial distribution of the numerical solution at different time instants is presented.

Additionally in the following Table (T2) we present the results of such a numerical experiment. Indeed, implementing $N_R$ realizations we derive analogous results as in Table (T1).

### Table (T2)

Realizations of the numerical solution of problem (1.1) in the case of nonhomogeneous Robin boundary conditions for $N_R = 1000$ in the time interval $[0, 1]$.

| $\lambda$ | Quenching times | $m(T_q)$ | $\sigma^2(T_q)$ |
|-----------|----------------|----------|---------------|
| 0.2       | 0              | -        | -             |
| 0.4       | 0              | -        | -             |
| 0.6       | 0              | -        | -             |
| 0.8       | 1000           | 0.75945  | 0.0014        |
| 1         | 1000           | 0.5547   | 5.41553e-04   |
Figure 7. (a) Realisation of the numerical solution of problem (1.1) for $\lambda = 0.3$, $\kappa = 1$, $M = 102$, $N = 10e4$, $r = 0.1$, initial condition $u(x,0) = cx(1-x)$ for $c = 0.1$ and with $\beta = \beta_0 = 1$ in the nonhomogeneous boundary condition. (b) Plot of $\|u(\cdot,t)\|_{\infty}$. The quenching behaviour is apparent.

Figure 8. (a) Realisation of the $\|u(\cdot,t)\|_{\infty}$ of the numerical solution of problem (1.1) for $\lambda = 2$, $\kappa = 1$, $M = 102$, $N = 10e4$, $r = 0.1$ and initial condition $u(x,0) = cx(1-x)$ for $c = 0.1$. (b) Plot of $u(x,t_i)$ from a different realization with the same values of the parameters at five time instants.

We notice a transition of the behaviour of the solution $u$ around the value $\lambda \sim 0.7$. So, in the next table, Table (T3), we focus around this value and point out a gradual increase of the number of quenching results as the parameter $\lambda$ increases.

| $\lambda$ | Quenching times $m(T_q)$ | $\text{Var}(T_q)$ |
|-----------|--------------------------|------------------|
| 0.6       | 0                        | -                |
| 0.65      | 85                       | 0.0841           | 0.0762           |
| 0.675     | 594                      | 0.5777           | 0.2284           |
| 0.7       | 877                      | 0.8227           | 0.0958           |
| 0.75      | 1000                     | 0.8332           | 0.0016           |

In the next set of experiments we solve numerically problem (4.16). We choose the diffusion coefficient to be of the form $g = g(t) = c_0 + c_1 \cos(\omega t)$, with $c_0 = 1$, $c_1 = 0.1$, $\omega = 10$. We also
consider a potential in the source term of the form $h(x) = x^b$, for $b = \frac{1}{2}$. The results of these experiments are demonstrated in Table (T4).

**Table (T4)**

Realizations of the numerical solution of problem (4.16) in the case of nonhomogeneous Robin boundary conditions for $N_R = 1000$ in the time interval $[0, 1]$.

| $\lambda$ | Quenching times | $m(T_q)$ | $Var(T_q)$ |
|---|---|---|---|
| 0.6 | 0 | - | - |
| 0.8 | 0 | - | - |
| 1 | 776 | 0.7348 | 0.1568 |
| 1.2 | 1000 | 0.7432 | 0.0011 |
| 1.4 | 1000 | 0.6029 | 5.2540e-04 |

Moreover focusing again around the value $\lambda \sim 1$ we can observe the transitional behaviour of the system in Table (T5) for $T = 1$.

**Table (T5)**

Realizations of the numerical solution of problem (4.16) in the case of nonhomogeneous Robin boundary conditions for $N_R = 1000$ in the time interval $[0, 1]$.

| $\lambda$ | Quenching times | $m(T_q)$ | $Var(T_q)$ |
|---|---|---|---|
| 0.9 | 0 | 0 | 0 |
| 0.95 | 35 | 0.0347 | 0.0333 |
| 0.97 | 191 | 0.1883 | 0.1505 |
| 0.99 | 431 | 0.4198 | 0.2331 |
| 0.995 | 502 | 0.4870 | 0.2358 |
| 1.1 | 993 | 0.8442 | 0.0068 |

7. Discussion

In the current work we demonstrate an investigation of a $d$-dimensional, $d = 1, 2, 3$, stochastic parabolic problem related to the modelling of an electrostatic MEMS device part of which is a membrane-rigid plate system. Firstly, the basic stochastic model together is presented. Later, local existence and uniqueness of the basic stochastic $u$-problem (1.1), as well as of its main variations, and for general boundary conditions is established via Banach’s Fixed point theorem.

Next, and for a certain form of boundary conditions (cf. equation (4.14b)) it is shown that the solution of $z$-problem (4.14) quenches almost surely regardless the chosen initial condition or the value of the tuning parameter $\lambda$. This is actually a striking and counterintuitive result; indeed in almost every case quenching for the corresponding deterministic problem occurs only if the parameter $\lambda$ or the initial data are large enough. To the best of our knowledge, this the first result of such kind is derived in the context of semilinear SPDEs related to MEMS.

Besides, adding a regularizing term into equation (4.14a), in the form of a modified nonlinear drift term, changes the dynamics of solution $z = 1 - u$ and we then obtain a dynamical behaviour resembles that of the deterministic problem. Moreover, in this particular case a lower estimate of the quenching probability is provided by formula (5.31).

The case of including time dependent coefficients related to dispersion and varying dielectric properties in the equation is tackled by similar analysis method. Again a lower bound for the quenching probability or quenching almost surely are derived, depending on the size of the first eigenvalue of corresponding eigenvalue problem.

We end our investigation by the implementation of an finite element numerical method, for the solution of the stochastic time-dependent problem in the one-dimensional case. We also provide a series of numerical experiments initially for the case of homogeneous Dirichlet boundary conditions (for the $u$-problem) and next for nonhomogeneous Robin conditions. In each case we present various results estimating the quenching times in a specific time interval $[0, T]$, which are of particular interest for MEMS practitioners.
References

[1] M. Abramowitz and I.A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, volume 55. Dover Publications, New York, 1972. 9th Edition.

[2] A. Alvarez, J. Alfredo López-Mimbela and N. Privault, Blowup estimates for a family of semilinear SPDEs with time-dependent coefficients, Differ. Equ. Appl., 7 (2) (2015), 201–219.

[3] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976), 620–709.

[4] M.A. Arcones, On the law of the iterated logarithm for Gaussian processes, J. Theor. Probab. 8, (1995), 877–903.

[5] A.N. Borodin, & P. Salminen, Handbook of Brownian motion—facts and formulae. Second edition. Probability and its Applications. Birkhäuser Verlag, Basel, 2002.

[6] D. Conus, M. Joseph & D. Khoshnevisan, Correlation-length bounds, and estimates for intermittent islands in parabolic SPDEs, Electron. J. Probab. 17 (2012), 1–15. ISSN: 1083-6489 DOI: 10.1214/EJP.v17-2429.

[7] P-L Chow, Stochastic Partial Differential Equations, Chapman and Hall/CRC, 2007.

[8] G. Da Prato & J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge, 1992.

[9] M. Dozzi & J. A. López-Mimbela, Finite-time blowup and existence of global positive solutions of a semi-linear SPDE, Stoch. Proc. Applications 120, (2010), 767–776.

[10] M. Dozzi, E.T. Kolkovska, & J. A. López-Mimbela, Finite-time blowup and existence of global positive solutions of a semi-linear stochastic partial differential equation with fractional noise, Modern stochastics and applications, 95–108, Springer Optim. Appl., 90, Springer, Cham, 2014.

[11] O. Drosinou, N. I. Kavallaris and C.V. Nikolopoulos, A study of a nonlocal problem with Robin boundary conditions arising from MEMS technology, arXiv:1906.12093v1.

[12] G. K. Duong & H. Zaag, Profile of a touch-down solution to a nonlocal MEMS model, Math. Mod. Meth. Appl. Sciences 29 (7) (2019), 1279–1348.

[13] P. Esposito, N. Ghoussoub, Y. Guo, Mathematical analysis of partial differential equations modeling electrostatic MEMS, Courant Lecture Notes in Mathematics, 20. Courant Institute of Mathematical Sciences, New York, American Mathematical Society, Providence, RI, 2010.

[14] G. Flores, G. Mercado, J. A. Pelesko & N. Smyth, Analysis of the dynamics and touchdown in a model of electrostatic MEMS, SIAM J. Appl. Math., 67 (2006/07), 434–446.

[15] G. Flores, Dynamics of a damped wave equation arising from MEMS, SIAM J. Appl. Math., 74 (2014), 1025–1035.

[16] A. Friedman, Partial Differential Equations of Parabolic Type, 1983, Prentice-Hall Inc.

[17] J.-S. Guo, B. Hu & C.-J. Wang, A nonlocal quenching problem arising in micro-electro mechanical systems, Quarterly Appl. Math., 67 (2009), 725–734.

[18] J.-S. Guo and N.I. Kavallaris, On a nonlocal parabolic problem arising in electrostatic MEMS control, Discrete Contin. Dyn. Syst., 32 (2012), 1723–1746.

[19] J.-S. Guo, N.I. Kavallaris, C.-J. Wang & C.-Y. Yu, The bifurcation diagram of a micro-electro mechanical system with Robin boundary condition, preprint.

[20] N. Ghoussoub & Y. Guo, On the partial differential equations of electrostatic MEMS devices II: dynamic case, Nonlinear Diff. Eqns. Appl., 15 (2008) 115–145.

[21] Y. Guo, Dynamical solutions of singular wave equations modeling electrostatic MEMS, SIAM J. Appl. Dyn. Syst., 9 (2010), 1135–1163.

[22] I. Gyöngy & C. Rovira, On $L^p$-solutions of semilinear stochastic partial differential equations Stochastic Process. Appl., 90(1) (2000), 83–108.

[23] I. Karatzas & S. Shreve, Brownian motion and stochastic calculus, Vol. 113 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2nd edition, 1991.

[24] N. I. Kavallaris, Explosive solutions of a stochastic non-local reaction-diffusion equation arising in shear band formation, Math. Methods Appl. Sci. 38(16) (2015), 3564–3574.

[25] N. I. Kavallaris, Quenching solutions of a stochastic parabolic problem arising in electrostatic MEMS control, Math. Methods Appl. Sci. 41 (3) (2018), 1074–1082.

[26] N.I. Kavallaris, T. Miyasita and T. Suzuki, Touchdown and related problems in electrostatic MEMS device equation, Nonlinear Diff. Eqns. Appl., 15 (2008), 363–385.

[27] N. I. Kavallaris, A. A. Lacey, C. V. Nikolopoulos and D. E. Tzanetis, A hyperbolic non-local problem modelling MEMS technology, Rocky Mountain J. Math., 41 (2011), 505–534.

[28] N. I. Kavallaris, A. A. Lacey, C. V. Nikolopoulos and D. E. Tzanetis, On the quenching behaviour of a semilinear wave equation modelling MEMS technology, Discrete Contin. Dyn. Syst., 35 (2015), 1009–1037.

[29] N.I. Kavallaris, A.A. Lacey and C.V. Nikolopoulos, On the quenching of a nonlocal parabolic problem arising in electrostatic MEMS control, Nonlinear Analysis, 138 (2016), 189–206.
[30] N.I. Kavallaris & T. Suzuki, *Non-Local Partial Differential Equations for Engineering and Biology: Mathematical Modeling and Analysis*, Mathematics for Industry Vol. 31 Springer Nature 2018.

[31] N.I. Kavallaris & Y. Yan, *Finite-time blow-up of a non-local stochastic parabolic problem*, Stoch. Proc. Applications, **130**(9), (2020), 5605–5635 doi.org/10.1016/j.spa.2020.04.002.

[32] N.I. Kavallaris, R. Barreira & A. Madzvamuse, *Dynamics of shadow system of a singular gierer-meinhardt system on an evolving domain*, Jour. Nonl. Science **31**(5), (2021) DOI :10.1007/s00332-020-09664-3.

[33] J. López-Mimbela & Pérez, *Global and nonglobal solutions of a system of nonautonomous semilinear equations with ultracontractive Lévy generators*, J. Math.Anal.Appl. **423** (2015) 720–733.

[34] G.J. Lord , C.E. Powell & T. Shardlow *An Introduction to Computational Stochastic PDEs*, Cambridge University Press: Cambridge, UK, 2014.

[35] V. Mackevičius, *Introduction to Stochastic Analysis: Integrals and Differential Equations*, Wiley 2011.

[36] H. Matsumoto & M. Yor, *Exponential functionals of Brownian motion, I: Probability laws at fixed time*, Prob. Surveys **2** (2005), 312–347.

[37] T. Miyasita, *On a nonlocal biharmonic MEMS equation with the Navier boundary condition*, Sci. Math. Jpn. **80**(2) (2017), 189–208.

[38] T. Miyasita, *Convergence of solutions of a nonlocal biharmonic MEMS equation with the fringing field*, J. Math. Anal. Appl. **454**(1) (2017), 265–284.

[39] T. Miyasita, *Global existence of radial solutions of a hyperbolic MEMS equation with nonlocal term*, Differ. Equ. Appl. **7**(2) (2015), 169–186.

[40] P. Salminen & M. Yor, *Properties of perpetual integral functionals of Brownian motion with drift*, Ann. Inst. H. Poincaré Probab. Statist., **39**(4) (2003), 737–742.

[41] M. Sanz-Solé & P.-A. Vuillermot, *Equivalence and Hölder-Sobolev regularity of solutions for a class of non-autonomous stochastic partial differential equations*, Ann. Inst. H. Poincaré Probab. Statist., **39**(4) (2003), 737–742.

[42] J.A. Pelesko, D.H. Bernstein, *Modeling MEMS and NEMS*, Chapman Hall and CRC Press, 2002.

[43] J.A. Pelesko & A.A. Triolo, *Nonlocal problems in MEMS device control*, J. Eng. Math. **41** (2001) 345–366.

[44] M. Yor, *On some exponential functionals of Brownian motion*, Adv. Appl. Probab. **24** (1992), 509–531.

[45] M. Younis, *MEMS Linear and Nonlinear Statics and Dynamics*, Springer, New York, 2011.

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