Online Segregation

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Abstract

A large number of agents from two groups prefer to interact with their own types online and also have preferences over two online platforms. We find that an online platform can be tipped from integrated to segregated without any change in the ratio of the two groups interacting on the platform. Instead, segregation can be triggered by changes in the absolute numbers of both groups, holding the Schelling ratio fixed. In extreme cases, the flight of one group from a platform can be triggered by a change in the group ratio in favor of the group that ends up leaving.

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1 Introduction

Online segregation is a relatively new phenomenon. The rise of the internet, smartphones, and social media has created a world where everyone, but especially young people, are interacting more and more with their peers online. Survey evidence suggests that British teenagers now spend more time online than they do outdoors.\(^1\) The COVID-19 pandemic and associated lockdowns have obviously accelerated this trend. The hope that social networks and online communities would lead to more interconnectedness is now being replaced by the fear that online platforms are actually more polarized and segregated than physical ones (Boyd, 2017). The “white flight” of the 1950s and ’60s (Boustan, 2010) was paralleled in the late 2000s exodus from “ghetto” Myspace to “elite” Facebook (Boyd, 2013); social media usage has been shown to be segregated by race and ethnicity (Duggan et al., 2015).

We introduce a model designed to capture the two main choices facing an individual deciding to interact with other people online: who to interact with and which online platform to do it on? In our setting, individuals care both about who their peers are (types) and the quality of the online platform they use to interact with other people (platform-specific characteristics). We build a simultaneous move game that captures these preferences and show how seemingly innocuous micro-level individual decisions can lead to surprising macro-level population outcomes. We then consider the evolutionary dynamics of a large population interacting in this way indefinitely. In a key departure from much of the existing literature on segregation in physical neighborhoods, there are no congestion effects so everyone can interact on the same online platform if they so wish.\(^2\)

In contrast to many of the segregation tipping point models, our model generates “tipping sets”. One result that emerges from having tipping sets is that online platforms can be tipped from integrated to segregated by a random shock that leaves the ratio of groups on the platform unchanged. In fact, it is even possible that those who left preferred the ratio of types on the platform after the shock while those who remain preferred the ratio before. These are two relatively unique predictions of our model.

\(^1\)See, for example, Frith (2017) and “The average child spends just 7 hours a week outside, but more than twice that playing video games inside”, The Daily Mail, 24 July, 2018.

\(^2\)The assumption of no congestion effects seems reasonable for most online platforms. For example, you probably did not notice that 500,000 new users joined Facebook today. This would not be true for the neighborhood that you live in.
that, to the best of our knowledge, remain untested empirically.

We perform a variety of comparative statics in order to explore what nudges online communities towards being more or less integrated. If online integration is a desired social outcome then the optimal policy in our setting is straightforward: make desirable online communities even more desirable. Improving less desirable online platforms (the online equivalent of gentrification) never leads to integration; in fact it can lead to resegregation. Further, in terms of marginal elasticities, achieving integration through improving desirable platforms is more effective than attempting to reduce intolerance.

The canonical models of segregation were developed by Schelling (1969, 1971, 1978) and focused on physical or face-to-face interactions. Schelling models locational choice, explicitly incorporating preferences over types, in that individuals care about the social composition of their neighborhoods. The various models (Schelling, 1969, 1971, 1978) are differentiated by the definition of a neighborhood. Schelling’s findings are mathematically simple but stark: he shows how a few random shocks can tip an integrated society to a segregated one. Perhaps most surprisingly, a society of tolerant individuals may self-segregate; even when they have no explicit desire to do so. A large body of theoretical work has followed this setup and modelled segregation in a Schelling style framework.3 There is a separate literature on Tiebout sorting (Tiebout, 1956) that our model also speaks to; individuals choose where to live on the basis of neighborhood characteristics.4 Our work fuses Schelling and Tiebout within a dynamic evolutionary framework to explore the question of sorting and segregation in online communities.

The closest papers to our own (in a mathematical sense) are Sethi and Somanathan (2004) and Banzhaf and Walsh (2013). The key difference is that platforms (neighborhoods) do not have capacity constraints in our model. While capacity constraints on locations seems intuitive for physical neighborhoods and maybe even some online platforms, it has the unfortunate consequence that every assignment is a Nash equilibrium when the measure of agents equals the measure of slots - the reason being that every

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3 The Schelling-influenced literature is vast but includes Akerlof (1997), Arrow (1998), Brock and Durlauf (2001), Clark (1991), Ioannides and Seslen (2002), Krugman (1996), Lindbeck et al. (1999), Manski (2000), Pancs and Friend (2007), Rosser (1999), Skyrms and Pemantle (2009), Young (2001), and Zhang (2004b,a, 2011).

4 Tiebout’s work has inspired a large literature on sorting in housing markets including Benabou (1993), Epple and Romer (1991), Epple and Platt (1998), Epple and Sieg (1999), De Bartolome (1990), and Durlauf (1996). Benabou (1993), for example, explores occupational segregation that occurs due to local complementarities in human capital.
slot other than one’s own is occupied so there is nowhere to move to. For this reason, both models are general equilibrium models. Further, both models are static and always have at least two equilibria, and so there is an equilibrium selection problem. By using dynamic equilibrium selection techniques we remove this issue and so can study welfare properties of the selected equilibrium and perform comparative statics on it. Not restricting ourselves to settings with capacity constraints also means that our model better captures most online communities, such as virtual worlds, dating sites, chat-boards, and social media networks.

2 The Strategic Setting

2.1 The Model

There is a large population of size \( N \). Every individual is one of two possible types: Group A or Group B, of sizes \( N^A \) and \( N^B \) respectively \((N^A, N^B \geq 2)\). Every individual chooses from two online platforms, \( \ell \) and \( m \), that we refer to as “less desirable” and “more desirable” respectively. (Due to constraints, individuals can only exist on one platform.) Individuals care about platform quality and about what platforms others choose. Since groups are homogeneous, what matters is the number of individuals from each group on a given platform and not on precisely who those individuals are. Letting \( n_{Am} \) \((n_{Bm})\) denote the number of Group A (B) individuals choosing platform \( m \), the relevant strategic information is fully described by the two-dimensional summary statistic that we refer to as the state of play \( \omega = ([\omega]_A, [\omega]_B) = (n_{Am}, n_{Bm}) \). The set of all states is given by \( \Omega := \{0, \ldots, N^A\} \times \{0, \ldots, N^B\} \). The model is closed by specifying utility functions, \( U^A \) and \( U^B \), that a given type receives from choosing a particular platform when the state is \( \omega \). These are given by

\[
\begin{align*}
U^A(m; \omega) &:= (n_{Am} - 1)\gamma_A - n_{Bm}\delta \\
U^A(\ell; \omega) &:= (N^A - n_{Am} - 1)(1 - \gamma_A) - (N^B - n_{Bm})\delta \\
U^B(m; \omega) &:= -n_{Am}\delta + (n_{Bm} - 1)\gamma_B \\
U^B(\ell; \omega) &:= -\left(N^A - n_{Am}\right)\delta + \left(N^B - n_{Bm} - 1\right)(1 - \gamma_B)
\end{align*}
\]
where, the group specific $\gamma$ parameter captures the benefit that an individual receives from choosing the same platform as someone from the same group, while $\delta$ captures the distaste from choosing the same platform as someone from the other group. These two parameters are designed to capture the idea that individuals prefer to interact with people similar to themselves, and not with people of a different political, religious, or ethnic background.

The equations in (1) above completely describe the strategic situation. Note that utility is linearly increasing (decreasing) in the number of those from the same (different) group who locate on the same platform. We assume that both $\gamma_A, \gamma_B \in (1/2, 1)$ so that, while each individual always benefits from choosing the same platform as those in her group, she would strictly prefer to do so on platform $m$. We constrain $\delta$ to be positive, meaning that different types actively dislike each other.

### 2.2 Equilibria

We refer to population behavior at states $(0,0)$, $(0,N^B)$, $(N^A,0)$, and $(N^A,N^B)$, denoted by $\omega_{tt}$, $\omega_{tm}$, $\omega_{mt}$, and $\omega_{mm}$ respectively, as group-symmetric. At states $\omega_{mm}$ and $\omega_{tt}$, everyone is using the same platform so the groups are integrated, while at states $\omega_{tm}$ and $\omega_{mt}$ the groups are segregated. An instance of the model, $G$, is fully parameterised by the 5-element set of parameters $\{N^A, N^B, \gamma_A, \gamma_B, \delta\}$, and from here on we define a given instance of $G$ by its associated parameter set. The following classifies when the various group symmetric states are equilibria.$^5$

**Theorem 1.** Of the group-symmetric states,

1. the fully segregated state $\omega_{mt}$ is always a strict equilibrium.

2. the fully segregated state $\omega_{tm}$ is always a strict equilibrium.

3. the integrated state $\omega_{mm}$ is an equilibrium if and only if

$$\gamma_A \geq \frac{N^B}{N^A - 1} \delta \text{ and } \gamma_B \geq \frac{N^A}{N^B - 1} \delta$$

$^5$It will be shown, once we introduce dynamics, that group-symmetric equilibria are the only serious candidates for long-run behavior.
4. The integrated state $\omega_{\ell\ell}$ is an equilibrium if and only if

$$1 - \gamma_A \geq \frac{N^B}{N^A - 1} \delta \quad \text{and} \quad 1 - \gamma_B \geq \frac{N^A}{N^B - 1} \delta$$

While a formal proof is provided in the Appendix, Theorem 1 is easily understood. Parts 1 and 2 state that both fully segregated outcomes are always strict equilibria. This is intuitive since a deviation from either of these profiles means that an individual goes from locating with everyone in her own group and away from those in the other group, to locating with those in the other group and away from everyone in her own group. Parts 3 and 4 are also straightforward. The entire population locating on the same platform can be an equilibrium only if the total benefit that an individual receives from interacting with those in her own group on that platform exceeds the total distaste from interacting with those of the other group on the same platform (because the alternative, moving to the empty platform, guarantees a pay-off of zero). Since platform $m$ is viewed by everyone as preferable to platform $\ell$, it is immediate that state $\omega_{mm}$ is an equilibrium whenever state $\omega_{\ell\ell}$ is.

Suppose the distaste parameter is large. Then both segregated outcomes are strict equilibria but the integrated outcomes are not. As individuals become more tolerant of each other, i.e., as $\delta$ decreases, the integrated outcomes can emerge as stable outcomes. Consider, for example, the diverging trends of racial tolerance in-person and online (Stephens-Davidowitz, 2014). Survey and interviewed-based measures of racial intolerance in the US document a decline from the 1980s to the late 2000s (Bobo, 2001; Bobo et al., 2012) and this is correlated with rising racial integration in many major cities (Farley et al., 1978, 1993; Glaeser and Vigdor, 2012; Ellen et al., 2012; Baum-Snow and Hartley, 2016). Conversely, racial distaste measured using anonymous behavior on online platforms appears to be more prevalent (Stephens-Davidowitz, 2014) and this has been linked to increased sorting and segregation along racial lines in online communities (Boyd, 2013).

### 2.3 Understanding Individual Behavior

Since individuals in the same group are faced with the same strategic problem, it is useful to partition the set of states, $\Omega$, into a preference map for how individuals from
different groups would behave at each state. To this end we define the following,

\[ \Omega^A_{m>\ell} := \{ \omega \in \Omega \mid U^A(m; \omega) > U^A(\ell; \omega) \} \quad (4) \]
\[ \Omega^B_{m>\ell} := \{ \omega \in \Omega \mid U^B(m; \omega) > U^B(\ell; \omega) \} \quad (5) \]
\[ \Omega^A_{\ell>m} := \{ \omega \in \Omega \mid U^A(\ell; \omega) > U^A(m; \omega) \} \quad (6) \]
\[ \Omega^B_{\ell>m} := \{ \omega \in \Omega \mid U^B(\ell; \omega) > U^B(m; \omega) \} \quad (7) \]

In words, \( \Omega^A_{m>\ell} \) and \( \Omega^B_{m>\ell} \) are the sets of states such that, given the current platform choice of everybody else, a Group A individual and a Group B individual respectively strictly prefer existing on platform \( m \) to platform \( \ell \). Similarly but opposite for \( \Omega^A_{\ell>m} \) and \( \Omega^B_{\ell>m} \).

Consider the following parameters as an example: \( G = (N^A, N^B, \gamma_A, \gamma_B, \delta) = (17, 13, 0.84, 0.95, 0.45) \). Neither inequality in part 4 of Lemma 1 is satisfied so the equilibrium set is \( \{ \omega_{mm}, \omega_{m\ell}, \omega_{\ell m} \} \). Since states \( \omega_{m\ell} \) and \( \omega_{\ell m} \) are always strict equilibria, we use them as reference points. We begin by considering how many individuals of the same type would have to switch platforms for any individual’s best-response to change. Suppose population behavior is given by state \( \omega_{\ell m} \) and, from here, we compute how many Group A individuals are needed to switch to \( m \) in order for platform \( m \) to become the best-response for everyone in Group A. Similarly, we consider how many Group B individuals would have to relocate to platform \( \ell \) for platform \( m \) to become the best-response for everyone in Group A. This can be formalised as follows. Letting \( \mathbb{N} \) denote the natural numbers and \( \mathbb{R} \) denote the real line, for any \( x \in \mathbb{R} \), let \( \lceil x \rceil := \min \{ n \in \mathbb{N} \mid x \leq n \} \). Now define the following,

\[ n^A_{Am} := \min \left\{ N^A, \left\lfloor \frac{(1 - \gamma_A)N^A + 2(\gamma_A - 1) + N^B\delta}{2} \right\rfloor \right\}, \quad (8) \]
\[ n^A_{B\ell} := \min \left\{ N^B, \left\lfloor \frac{(N^A - 1)(1 - \gamma_A) + N^B}{2} \right\rfloor \right\}, \quad (9) \]
\[ n^B_{Bm} := \min \left\{ N^B, \left\lfloor \frac{(1 - \gamma_B)N^B + 2(\gamma_B - 1) + N^A\delta}{2} \right\rfloor \right\}, \quad (10) \]
\[ n^B_{A\ell} := \min \left\{ N^A, \left\lfloor \frac{(N^B - 1)(1 - \gamma_B) + N^A}{2} \right\rfloor \right\}. \quad (11) \]
The integer \( n_{Am}^* \) is the minimum number of Group A players that have to locate at \( m \), when all Group B players are locating at \( m \), in order for \( m \) to be the best-response for those from Group A. If the \( \min \{\cdot, \cdot\} \) operator “kicks in”, it means that all members of a given group switching their platform is not enough for the best-response to change. We will see below that when the \( \min \{\cdot, \cdot\} \) operator kicks in, it means that neither integrated outcome, \( \omega_{mm} \) or \( \omega_{\ell\ell} \), is an equilibrium.

The above is made clearer with a picture of the state space. Since the state space is two-dimensional, it can be depicted as an \((N_A+1) \times (N_B+1)\) grid with, for any state \( \omega \), \([\omega]_A = n_{Am} \) on the horizontal axis and \([\omega]_B = n_{Bm} \) on the vertical axis. Figure 1 shows the state space \( \Omega \) as an \( 18 \times 14 \) lattice, with \([\omega]_A \in \{0, \ldots, 17\} \) on the horizontal-axis, and \([\omega]_B \in \{0, \ldots, 13\} \) on the vertical-axis. Each state is depicted by a circle. The state space \( \Omega \) in Figure 1 is partitioned as \( \{\Omega_{A,\ell} > m, \Omega_{B,\ell} > m, \Omega_{A,m} > \ell \cap \Omega_{B,m} > \ell\} \), with states colour coded according to the element of the partition in which they lie. The set of blue circles is \( \Omega_{A,\ell} > m \), while the red circles denote \( \Omega_{B,\ell} > m \). At states given by hollow circles, the set \( \Omega_{A,m} > \ell \cap \Omega_{B,m} > \ell \), both groups prefer platform \( m \). These sets are defined by \((n_{Am}^*, n_{Bm}^*, n_{Bm}^*, n_{Bm}^*) = (10, 10, 10, 10)\), calculated using equations (8)-(11). That is, \((n_{Am}^*, n_{Bm}^*) = (10, 13)\), \((0, N_B - n_{Bm}^*) = (0, 3)\), \((N_A - n_{Am}^*, 0) = (7, 0)\) and \((N_A, n_{Bm}^*) = (17, 10)\). Equilibrium states are depicted by large circles, with magenta (the combination of red and blue) representing the fully integrated state. For now, ignore the green X and ignore the fact that certain hollow states are shaded magenta.

We have included lines separating the elements of the preference partition. As we will see formally in the next Section, these lines give an intuition for how to define tipping sets. For example, consider the blue line and the state \((10, 13)\). At this state, everybody in the population prefers platform \( m \). However, suppose that one Group A agent relocated to platform \( \ell \) such that population behavior is now given by state \((9, 13)\). At this new state, platform \( \ell \) is preferable for those in Group A. When we introduce best-response based dynamics in the next section, we will see that these two states lead to different outcomes: complete integration and complete segregation.

To be concrete, consider state \((2, 5)\) in Figure 1. At this state, all individuals in the population, and in particular those from Group A, prefer platform \( m \) to \( \ell \). Now suppose that the numbers of both Groups on platform \( m \) are doubled. That is, consider the state \((4, 10)\). The ratio of types on platform \( m \) remains as before, and yet state \((4, 10) \in \Omega_{A,\ell} > m\) so that \( \ell \) is now the preferred venue for Group A members. Thus,
the ratio of two types using platform $m$ has not changed and yet Group $A$ individuals now desire to relocate. Note that this feature can occur even if the ratio on platform $m$ strictly improves for Group $A$; this can be seen by comparing state $(2, 5)$ with the state $(3, 7)$ or the state $(4, 11)$. At both these states, the Schelling ratio on platform $m$ has become more desirable, and yet as a choice it becomes less so.

This is quite a different result to Schelling and similar work. It demonstrates that integrated online communities can be tipped to segregated, à la Schelling, but without any change in the ratio of one group to the other or even when the ratio becomes more favourable towards the group that ends up leaving. In other words, we observe a levels effect in preferences in addition to a ratio effect. Thus, the view that an online platform
has become “too A” or “too B” can be triggered by an increase in the absolute numbers of one group or the other, even though relative diversity has remained the same.

3 Evolutionary Dynamics and Tipping Sets

We now suppose the model is the stage game of a repeated interaction. Time is discrete, begins at \( t = 0 \), and continues forever. Each period, one individual is randomly selected (each with equal probability of \( 1/N \)) to update their online platform choice. This one-at-a-time revision protocol is known as *asynchronous learning* (Binmore and Samuelson, 1997; Blume, 2003). When afforded a revision opportunity, we assume that the chosen individual takes a best-response to the current state. This revision protocol and behavioral rule pair makes the standard evolutionary assumption that better-performing actions are no worse represented next period. As time progresses, population behavior is evolving. Our interest is in what happens in the long run. To assist in this we will exploit the fact that our model is a potential game (Shapley and Monderer, 1996). A game is a potential game if the change in each player’s utility from a unilateral deviation can be derived from a common function, referred to as the game’s potential function.

Given that our model is a potential game, we can state the following.

**Theorem 2.** When the revision protocol is asynchronous learning and individuals behave according to myopic best-response, with probability 1 population behavior will come to rest at one of the pure strategy equilibria.

The proof of Theorem 2 is immediate. Since our model is a finite potential game, by Lemma 2.3 in Shapley and Monderer (1996), asynchronous learning with a best-response based updating rule will generate a so-called “finite improvement path” that will terminate at some pure strategy equilibrium.

However, more can be said than Theorem 2. Specifically, the only candidates for long run behavior are the group-symmetric states. Again this can be understood intuitively by referring to Figure 1. From any of the blue states, \( \Omega^{A,\ell \succ m} \), population behavior must drift towards equilibrium \( \omega_{\ell m} = (0,13) \). Similarly but opposite for any of the red states, \( \Omega^{B,\ell \succ m} \), from where population behavior will ultimately come to rest at \( \omega_{m \ell} = (17,0) \). So it remains to consider the evolution from the hollow states. From these, the dynamics are pushing *up and to the right*, and depending on the order that
players are randomly called upon to act, population behavior will ultimately drift to either a blue or red state or to equilibrium $\omega_{mm}$.

Our goal is to define tipping sets. Letting $\omega^t$ denote the population state at time $t \in \mathbb{N}$. We begin by defining the basin of attraction of each equilibrium state following Ellison (2000):

**Definition 1.** For group symmetric equilibrium $\omega^* \in \Omega$, the basin of attraction of $\omega^*$, $D(\omega^*)$, is the set of initial states from which population behavior will converge to $\omega^*$ with probability one. That is,

$$D(\omega^*) = \{ \omega \in \Omega \mid \text{Prob}[\{ \exists t' > 0 \text{ such that } \omega^t = \omega^*, \forall t > t' \mid \omega^0 = \omega \}] = 1 \}. $$

As an example in Figure 1, $D(\omega_{ml})$ is the set of blue states, $D(\omega_{ml})$ is the set of red states, and $D(\omega_{tm})$ is the set of magenta states that are north, east, or north east of the state (10, 10), including the state (10, 10) itself.

The idea of a tipping set then follows naturally. It is the set of outermost states of a particular basin of attraction. That is, the set of states that are closest to those not contained in the basin of attraction, where our notion of close is the taxicab metric ($L^1$ distance) on $\Omega$. Formally, for any pair of states $\omega', \omega''$ we define taxicab distance $\ell: \Omega \times \Omega \rightarrow \mathbb{N}_0$ as $\ell(\omega', \omega'') = |[\omega']_A - [\omega'']_A| + |[\omega']_B - [\omega'']_B|$. We now define a tipping set for an equilibrium.

**Definition 2.** For group symmetric equilibrium $\omega^* \in \Omega$, the Tipping Set of $\omega^*$, $T(\omega^*)$, is defined by

$$T(\omega^*) = \{ \omega \in D(\omega^*) \mid \exists \omega' \not\in D(\omega^*) \text{ with } \ell(\omega, \omega') = 1 \}. $$

Going back again to Figure 1, it is now simple to see the tipping sets. We have that $T(\omega_{tm})$ is the set of blue states that are just north west of the blue line, $T(\omega_{ml})$ is the set of red states just south east of the red line, and $T(\omega_{mm})$ is the set of states that are directly north or directly east of the state (10, 10) including the state (10, 10) itself. Again we emphasize that the ratio of types at each state in a tipping set need not be the same. In fact, as in our example, the ratio can be different at each state in a given tipping set.
4 Equilibrium Selection

While Theorem 2 states that an equilibrium will be reached no matter what state the population begins at, more than one equilibrium may be candidates for the long run outcome. For example, consider state \((5, 5)\) labelled by the green X in Figure 1. From here, there are best-response based paths that lead to \(\Omega^{A, \ell \rightarrow m}\) (blue), others that lead to \(\Omega^{B, \ell \rightarrow m}\) (red), and yet more that lead to equilibrium \(\omega_{mm}\) (magenta). Thus, there is not always “path dependence”.

To select between the equilibria we now assume that individuals occasionally choose suboptimal locations. Such “mistakes” are standard in the literature on evolutionary game theory. Specifically, with \(\omega^t\) denoting the population state at time \(t \in \mathbb{N}\), if player \(i\) is afforded an opportunity to revise his choice he chooses platform \(m \in S\) according to the probability distribution \(p^\beta_i(m | \omega^t)\), where for any \(\beta > 0\),

\[
p^\beta_i(m | \omega^t) := \frac{\exp(\beta U_i(m; \omega^t))}{\exp(\beta U_i(m; \omega^t)) + \exp(\beta U_i(\ell; \omega^t))}
\]

and chooses platform \(\ell\) with probability \(1 - p^\beta_i(m | \omega^t)\).

The behavioral rule above is known as logit response. It makes the assumption that the more painful a mistake, the less likely it is to be observed. As \(\beta \downarrow 0\), this approaches uniform randomization over both actions. When \(\beta \uparrow \infty\), the rule approaches best-response. Asynchronous learning coupled with this perturbed best-response describes a stochastic process with a unique invariant probability measure \(\mu^\beta\) on the state space \(\Omega\). A result from Blume (1993) shows that as \(\beta \uparrow \infty\), all the probability mass under \(\mu^\beta\) accumulates on the states that maximize the potential function. That is, the states that maximize the potential are precisely the stochastically stable equilibria (Young, 1993).

To compute the potential at each group-symmetric state, we view the game as one on a fully connected graph with players representing vertices and local interactions represented by edges. The number of edges on any fully connected undirected graph with \(N\) vertices is \(\binom{N}{2}\). Similarly, the number of edges on any fully connected bipartite graph, as is the case with the subgraph connecting all pairs of players from groups \(A\) and \(B\), of size \(N^A\) and \(N^B\) respectively, is \(N^A \times N^B\). The potential at each group-symmetric
state is then given by

\[
\begin{align*}
\rho^*(\omega_{mm}) &= \binom{N^A}{2} \cdot \gamma_A + 0 + \binom{N^B}{2} \cdot \gamma_B \\
\rho^*(\omega_{m\ell}) &= \binom{N^A}{2} \cdot \gamma_A + (N^A \times N^B) \cdot \delta + \binom{N^B}{2} \cdot (1 - \gamma_B) \\
\rho^*(\omega_{\ell m}) &= \binom{N^A}{2} \cdot (1 - \gamma_A) + (N^A \times N^B) \cdot \delta + \binom{N^B}{2} \cdot \gamma_B \\
\rho^*(\omega_{\ell \ell}) &= \binom{N^A}{2} \cdot (1 - \gamma_A) + 0 + \binom{N^B}{2} \cdot (1 - \gamma_B)
\end{align*}
\]  

(13)

where the first term on the right hand side of each equality is the potential due to interactions among Group A individuals, the second due to interactions across the groups, and the third due to interactions among Group B individuals.

Theorem 3 below classifies for what range of the benefit pay-off parameters, \(\gamma_A\) and \(\gamma_B\), each equilibrium is stochastically stable. We simplify the notation by defining the following:

\[
\begin{align*}
\gamma^\rho_A(N^A, N^B, \delta) &:= \frac{N^B}{N^A - 1} \delta + \frac{1}{2} \\
f(N^A, \gamma_A) &:= \binom{N^A}{2} \cdot (2\gamma_A - 1) \\
\gamma^\rho_B(N^A, N^B, \delta) &:= \frac{N^A}{N^B - 1} \delta + \frac{1}{2} \\
f(N^B, \gamma_B) &:= \binom{N^B}{2} \cdot (2\gamma_B - 1)
\end{align*}
\]

**Theorem 3.** The following classifies when each equilibrium is stochastically stable.

1. State \(\omega_{\ell \ell}\) is never stochastically stable.

2. State \(\omega_{mm}\) is stochastically stable if and only if

\[
\gamma_A \geq \gamma^\rho_A \quad \text{and} \quad \gamma_B \geq \gamma^\rho_B
\]

(14)

3. State \(\omega_{m\ell}\) is stochastically stable if and only if

\[
f(N^A, \gamma_A) \geq f(N^B, \gamma_B) \quad \text{and} \quad \gamma_B \leq \gamma^\rho_B
\]

(15)

4. State \(\omega_{\ell m}\) is stochastically stable if and only if

\[
f(N^A, \gamma_A) \leq f(N^B, \gamma_B) \quad \text{and} \quad \gamma_A \leq \gamma^\rho_A
\]

(16)
A proof is provided in the Appendix.

As can be seen from the two expressions in (14) above, the state where all individuals choose platform $m$ is stochastically stable if and only if both benefit payoff parameters, $\gamma_A$ and $\gamma_B$, are sufficiently large. A slight rearranging of the first expression in (14) above yields that one necessary condition for $\omega_{mm}$ to be stochastically stable is that $(N^A - 1)(2\gamma_A - 1) \geq 2N^B\delta$. Note that the term $2\gamma_A - 1$ is simply the difference in benefit that each of two Group $A$ individuals earn from successful coordination with each other on platform $m$ over platform $\ell$ (since $2\gamma_A - 1 = \gamma_A - (1 - \gamma_A)$).

That state $\omega_{\ell\ell}$ is never stochastically stable is intuitive. At both integrated outcomes, the distaste experienced by the presence of those from the other group is equal, but the benefit to coordinating with those in your own group is greater on platform $m$. This confirms that improving the less desirable platform never leads to integration on that platform. The only way integration could occur is if the less desirable platform was improved so much that it became the more desirable platform but then segregation could re-emerge with the two groups switching platform. This is the idea of gentrification simply leading to re-segregation. This result stands in contrast to work on physical neighborhoods by Fernandez and Rogerson (1996) and others who find that neighborhood revitalisation can lead to welfare-enhancing integration. For the purposes of this short paper, we do not delve into the details here but it can be shown that all of the stochastically stable equilibria in Theorem 3 are both Pareto efficient and maximise utilitarian social welfare.

5 Comparative Statics

If one accepts stochastic stability as the true selection device, some more natural questions spring to mind. Suppose a planner wanted to induce an integrated outcome, and suppose further that the planner can affect preferences, but that after altering preferences, the planner can do no more. That is, the planner can affect preferences but then must release the stochastic system, letting individual dynamics take over. If the planner’s goal is to guide society towards integration, would she be better off by increasing the benefit parameters, $\gamma_A$ or $\gamma_B$, or by reducing the distaste parameter $\delta$? This section explores these questions. Again, we take the position that the stochastically stable equilibrium will emerge and we consider how payoffs (from the perspective of the
agents) vary with the equilibrium that is selected.

5.1 Varying Group Size

We note the following. The threshold benefit payoff for Group $A$, $\gamma^\star _A$, is decreasing in $N^A$ and increasing in $N^B$ (a similar but opposite statement holds for $\gamma^\star _B$). In the extreme, this means that the integrated outcome, $\omega_{mm}$, can not be stochastically stable if one of the groups is considerably larger than the other. At the margin, the issue is more complicated. If the entire population is located at the integrated outcome, then increasing the size of Group $A$ ($B$) will weakly move the situation towards the segregated outcome that is preferred by Group $A$ ($B$). If population behavior is given by the segregated outcome least preferred by Group $A$ ($B$), then the effect of increasing $N^A$ ($N^B$) is ambiguous (though still beneficial for those in Group $A$). As described above, continually increasing the size of Group $A$ will mean that ultimately their preferred segregated equilibrium will be selected. The issue is whether the integrated outcome is “passed through” along the way. For some parameters the transition from one segregated outcome to the other is instant, while for other parameters the integrated outcome is visited. When this occurs is described in the following theorem.\footnote{We state the theorem only for an increase in the size of Group $A$. Similar but opposite statements occur for increasing the size of Group $B$.}

\textbf{Theorem 4.} Consider the model $\mathcal{G} := (N^A, N^B, \gamma_A, \gamma_B, \delta)$ and a sequence of models $\{\mathcal{G}_k\}_{k=0}^\infty$, where along the sequence, the size of Group $A$ is incrementally increased. That is, for each $k = 0, 1, 2, \ldots$, we define $\mathcal{G}_k := (N^A + k, N^B, \gamma_A, \gamma_B, \delta)$.

1. Suppose the segregated equilibrium $\omega_{ml}$ is stochastically stable for model $\mathcal{G}$. Then, $\omega_{ml}$ is uniquely stochastically stable for all models in $\{\mathcal{G}_k\}_{k=0}^\infty$.

2. Suppose the integrated equilibrium $\omega_{mm}$ is stochastically stable for model $\mathcal{G}$. Then there exists an integer $k \in \mathbb{N}$ such that $\omega_{mm}$ remains stochastically stable for all models in $\{\mathcal{G}_k\}_{k=0}^{k-1}$ and such that $\omega_{ml}$ is stochastically stable for all models in $\{\mathcal{G}_k\}_{k=k}^\infty$.

3. Suppose $\omega_{lm}$ is stochastically stable for model $\mathcal{G}$ so that $f(N^A, \gamma_A) \leq f(N^B, \gamma_B)$ and $\gamma_A \leq \gamma^\star_A$. If, by incrementally increasing $N^A$ we get $\gamma_A > \gamma^\star_A$ while $f(N^A, \gamma_A) \leq f(N^B, \gamma_B)$ then $\omega_{mm}$ becomes stochastically stable and we reduce to case 2. If, by
incrementally increasing $N^A$ we get $f(N^A, \gamma_A) > f(N^B, \gamma_B)$ while $\gamma_A \leq \gamma_A^\rho$ then $\omega_{m\ell}$ becomes stochastically stable and we reduce to case 1.

A proof is provided in the Appendix.

Theorem 4 raises some interesting issues. From the perspective of the individual, an increase in the size of one’s group is always weakly beneficial since either the stochastically stable equilibrium will remain the same or the equilibrium will change to one that yields a higher utility. From the perspective of the planner, things are not so clear. Suppose a planner favors integration and yet the stochastically stable equilibrium is a segregated outcome. One thing the planner could do is to increase the size of the group on platform $\ell$ in the hope that their increased numbers will induce them to locate at $m$ such that integration will occur. However, part 3 of Theorem 4 says that this may not always be possible. Specifically, it may be that by increasing the size of the group at $\ell$ that eventually an immediate flip occurs whereby the other segregated outcome becomes stochastically stable. This result is akin to what is known as “white flight” in the US (Boustan, 2010). White residents leave cities as immigration leads to more integration, fleeing to the suburbs and rural areas. Segregation occurs again, except the segregated outcome is a reversal of the initial segregation. In fact, Theorem 4 shows that unlimited “immigration” of one group always eventually leads to online segregation.

5.2 Varying Payoffs

Again, suppose that one of the segregated outcomes is stochastically stable but the planner would like to see integration occur. Suppose further that the planner has the ability to affect preferences in the following sense. She can increase the benefit that two agents from the same group would earn on platform $m$. For example, adding additional features or making the platform more user-friendly. The other alternative for the planner would be to reduce the common distaste that individuals feel towards those in the other group.

Specifically, suppose the segregated outcome $\omega_{m\ell}$ is stochastically stable and suppose the planner favours integration. Further suppose that the planner has two options: (i) increase the attractiveness of platform $m$ for Group $A$ members by a fixed percentage, or (ii) decrease distaste $\delta$ (i.e., increase tolerance) for all individuals by reducing it by
the same percentage. As Theorem 5 below shows, if integration is the goal, it is always better (in terms of the elasticity of responses) to make the desirable platform even more desirable than to reduce distaste.

**Theorem 5.** Suppose a segregated outcome is stochastically stable but the social planner favors integration. Suppose further that the social planner has the ability to change preferences either by reducing $\delta$ by $x\%$ or by increasing by $x\%$ the benefit that two agents from the group currently located on platform $\ell$ get from coordinating at $m$. Whenever reducing $\delta$ by $x\%$ renders the integrated outcome $\omega_{mm}$ stochastically stable, increasing $\gamma_K$ by $x\%$ will also. But the reverse need not hold.

A proof is provided in the Appendix.

Theorem 5 demonstrates that integration is more elastic with respect to a 1% change in $\gamma_K$ compared to a 1% change in $\delta$. We are, of course, ignoring costs here. Large cost differences could obviously lead to the conclusion that increasing tolerance is a more cost effective way to achieve integration.

6 Conclusion

We have shown in this paper that when individuals care about both the features and the group composition of their online communities, their preferences are reflected in patterns of segregation that are sometimes surprising. A segregation tipping point can occur in our model even when the ratio of the two types using a social network remains unchanged. If integration is the desired social outcome then the optimal policy in our model is clear: make desirable online platforms or social networks even more desirable. Revitalizing less desirable platforms (such as Myspace or Bebo) never leads to integration, it can only lead to resegregation. Further, integration is more elastic in response to improving desirable platforms than attempting to reduce intolerance. Which option is ultimately more effective will depend on relative costs.

Our analysis is abstract enough to permit alternative interpretations. For example, we have alluded to the fact that our model could be used to study physical neighborhoods, especially those where congestion or capacity constraints are not binding. In terms of future work, the most obvious unanswered question is whether empirical evidence exists for our most interesting segregation result: after an influx of newcomers
to an online community, the ratio of $A$ to $B$ remains the same but an increase in the absolute number of $B$s causes the $A$s to move to a different online platform. In other words, in a world of online Tiebout sorting without capacity constraints, do numbers matter more than Schelling’s canonical ratio?
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Appendix for “Online Segregation”

Proof of Lemma 1

Proof. First we note that at any group-symmetric profile, all players from a given group earn the same utility. This is because at any group-symmetric profile, any player from Group K is located with all \( N^K - 1 \) others from his group, and either zero or all of those from the other group. As such we can consider the situation from the perspective of just one agent from each group.

1. At state \( \omega_{m\ell} \), a Group A player earns \( (N^A - 1)\gamma_A \) as he is located with all the members of his own group at \( m \). If he deviates to \( \ell \) he will earn \( -\delta N^B \). Since \( (N^A - 1)\gamma_A > 0 > -\delta N^B \), it is not profitable to deviate.

   At state \( \omega_{m\ell} \), a Group B player earns \( (N^B - 1)(1 - \gamma_B) \) as he is located with all the members of his own group at \( \ell \). If he deviates to \( m \) he will earn \( -\delta N^A \). Since \( (N^B - 1)(1 - \gamma_B) > 0 > -\delta N^A \), it is not profitable to deviate.

2. The proof proceeds the same as that for state \( \omega_{m\ell} \) from part 1 by relabeling A as B and B as A.

3. At state \( \omega_{mm} \), a Group A player earns \( (N^A - 1)\gamma_A - \delta N^B \) as he is located with everyone at \( m \). If he deviates to \( \ell \), he will earn 0. Such a deviation will not be (strictly) profitable provided \( (N^A - 1)\gamma_A - \delta N^B \geq 0 \). Solving this for \( \gamma_A \) yields the first inequality in (2) of part 3.

   At state \( \omega_{mm} \), a Group B player earns \( (N^B - 1)\gamma_B - \delta N^A \) as he is located with everyone at \( m \). If he deviates to \( \ell \), he will earn 0. Such a deviation will not be (strictly) profitable provided \( (N^B - 1)\gamma_B - \delta N^A \geq 0 \). Solving this for \( \gamma_B \) yields the second inequality in (2) of part 3.

4. At state \( \omega_{\ell\ell} \), a Group A player earns \( (N^A - 1)(1 - \gamma_A) - \delta N^B \) as he is located with everyone at \( \ell \). If he deviates to \( m \), he will earn 0. Such a deviation will not be (strictly) profitable provided \( (N^A - 1)(1 - \gamma_A) - \delta N^B \geq 0 \). Solving this for \( \gamma_A \) yields the first inequality in (3) of part 4.

   At state \( \omega_{\ell\ell} \), a Group B player earns \( (N^B - 1)(1 - \gamma_B) - \delta N^A \) as he is located with everyone at \( \ell \). If he deviates to \( m \), he will earn 0. Such a deviation will not
be (strictly) profitable provided 

\((N^B - 1)(1 - \gamma_B) - \delta N^A \geq 0\). Solving this for \(\gamma_B\) yields the second inequality in (3) of part 4.

\[\square\]

**Proof of Theorem 1**

*Proof.* First we note that at any group-symmetric profile, all players from a given group earn the same utility. This is because at any group-symmetric profile, any player from Group \(K\) is located with all \(N^K - 1\) others from his group, and either zero or all of those from the other group. As such we can consider the situation from the perspective of just one agent from each group.

1. At state \(\omega_{m\ell}\), a Group \(A\) player earns \((N^A - 1)\gamma_A\) as he is located with all the members of his own group at \(m\). If he deviates to \(\ell\) he will earn \(-\delta N^B\). Since \((N^A - 1)\gamma_A > 0 > -\delta N^B\), it is not profitable to deviate.

At state \(\omega_{m\ell}\), a Group \(B\) player earns \((N^B - 1)(1 - \gamma_B)\) as he is located with all the members of his own group at \(\ell\). If he deviates to \(m\) he will earn \(-\delta N^A\). Since \((N^B - 1)(1 - \gamma_B) > 0 > -\delta N^A\), it is not profitable to deviate.

2. The proof proceeds the same as that for state \(\omega_{m\ell}\) from part 1 by relabeling \(A\) as \(B\) and \(B\) as \(A\).

3. At state \(\omega_{mm}\), a Group \(A\) player earns \((N^A - 1)\gamma_A - \delta N^B\) as he is located with everyone at \(m\). If he deviates to \(\ell\), he will earn 0. Such a deviation will not be (strictly) profitable provided \((N^A - 1)\gamma_A - \delta N^B \geq 0\). Solving this for \(\gamma_A\) yields the first inequality in (2) of part 3.

At state \(\omega_{mm}\), a Group \(B\) player earns \((N^B - 1)\gamma_B - \delta N^A\) as he is located with everyone at \(m\). If he deviates to \(\ell\), he will earn 0. Such a deviation will not be (strictly) profitable provided \((N^B - 1)\gamma_B - \delta N^A \geq 0\). Solving this for \(\gamma_B\) yields the second inequality in (2) of part 3.

4. At state \(\omega_{\ell\ell}\), a Group \(A\) player earns \((N^A - 1)(1 - \gamma_A) - \delta N^B\) as he is located with everyone at \(\ell\). If he deviates to \(m\), he will earn 0. Such a deviation will not be (strictly) profitable provided \((N^A - 1)(1 - \gamma_A) - \delta N^B \geq 0\). Solving this for \(\gamma_A\) yields the first inequality in (3) of part 4.
At state $\omega_{\ell \ell}$, a Group $B$ player earns $(N_B^B - 1)(1 - \gamma_B) - \delta N_A$ as he is located with everyone at $\ell$. If he deviates to $m$, he will earn 0. Such a deviation will not be (strictly) profitable provided $(N_B^B - 1)(1 - \gamma_B) - \delta N_A \geq 0$. Solving this for $\gamma_B$ yields the second inequality in (3) of part 4.

\[
\square
\]

**Proof of Theorem 3**

\textit{Proof.} 1. By comparing the first and fourth equations in (13) and observing that both $\gamma_A > \frac{1}{2}$ and $\gamma_B > \frac{1}{2}$, it is immediate that the potential at $\omega_{mm}$ is always strictly greater than at $\omega_{\ell \ell}$. As such $\omega_{\ell \ell}$ is never stochastically stable.

2. Given that $\omega_{\ell \ell}$ is never stochastically stable, parts 2, 3, and 4 all follow from comparison of the first three equations in (13). We prove only part 2 as the rest follow in a similar manner.

For $\omega_{mm}$ to be stochastically stable we need that both

\[
\rho^*(\omega_{mm}) \geq \rho^*(\omega_{m \ell}) \quad \text{and} \quad \rho^*(\omega_{mm}) \geq \rho^*(\omega_{\ell m}).
\]

Considering the first inequality, from taking the first and second expressions in (13), we get that

\[
\left(\frac{N_A^A}{2}\right) \cdot \gamma_A + \left(\frac{N_B^B}{2}\right) \cdot \gamma_B \geq \left(\frac{N_A^A}{2}\right) \cdot \gamma_A + (N_A^A \times N_B^B) \cdot \delta + \left(\frac{N_B^B}{2}\right) \cdot (1 - \gamma_B),
\]

which after some rearranging yields the first inequality in (14). Considering the second inequality, and taking the first and third expressions in (13), we get that

\[
\left(\frac{N_A^A}{2}\right) \cdot \gamma_A + \left(\frac{N_B^B}{2}\right) \cdot \gamma_B \geq \left(\frac{N_A^A}{2}\right) \cdot (1 - \gamma_A) + (N_A^A \times N_B^B) \cdot \delta + \left(\frac{N_B^B}{2}\right) \cdot \gamma_B,
\]

which after some rearranging yields the second inequality in (14).

\[
\square
\]
**Proof of Theorem 4**

*Proof.* 1. It is clear from the expressions in (13) that $\rho^*(\omega_{m\ell})$ increases more than $\rho^*(\omega_{mm})$ and $\rho^*(\omega_{\ell m})$ for any incremental increase in $N^A$.

2. Given $\omega_{mm}$ is stochastically stable, we know that $\gamma_A \geq \gamma_A^*$ and $\gamma_B \geq \gamma_B^*$. Furthermore, we have that $\gamma_A^*$ is decreasing in $N^A$, so the first inequality will always hold. Eventually, for some $\hat{k}$, we have $\gamma_B > \gamma_B^*(N^A + \hat{k}, N^B, \delta)$. Given that $\gamma_A$ is still large enough, it cannot be that $\omega_{\ell m}$ is stochastically stable by the second inequality in (16), and so $\omega_{m\ell}$ must be stochastically stable. From here we are reduced to case 1, so $\omega_{m\ell}$ remains stochastically stable.

3. Suppose $\omega_{\ell m}$ is stochastically stable, meaning both inequalities in (16) hold. Eventually, one of these inequalities will cease to bind. If the first inequality breaks, then we are reduced to case 2. If the second inequality breaks, then we are reduced to case 1.

\[\square\]

**Proof of Theorem 5**

*Proof.* Without loss of generality we suppose that $\omega_{\ell m}$ is stochastically stable so that the first inequality in (14) does not hold. In the hope of making this inequality bind, the planner can make one of the following two changes:

\[
\begin{align*}
\gamma_A &\mapsto \gamma^*_A = (1 + x)\gamma_A \\
\delta &\mapsto \delta^* = (1 - x)\delta
\end{align*}
\]

It is immediate that increasing $\gamma_A$ to $\gamma^*_A = (1 + x)\gamma_A$ is an $x\%$ increase in the range of values for $\gamma_A$. We now show that reducing $\delta$ to $\delta^* = (1 - x)\delta$ does not induce as large an expansion of values for $\gamma_A$. A reduction in $x\%$ of $\delta$ to $\delta^*$ changes the threshold value
\( \gamma_A^{\rho^*}(N^A, N^B, \delta^*). \) We have that

\[
\frac{\gamma_A^{\rho^*}(N^A, N^B, \delta^*)}{\gamma_A^{\rho^*}(N^A, N^B, \delta)} = \frac{\frac{N^B}{N^A-1} \delta^* + \frac{1}{2}}{\frac{N^B}{N^A-1} \delta + \frac{1}{2}} = \frac{\frac{N^B}{N^A-1} (1 - x) \delta + \frac{1}{2}}{\frac{N^B}{N^A-1} \delta + \frac{1}{2}}
\]

which is a reduction in distaste that is strictly less than \( x\% \). \( \square \)