Equivalence of light-front and conventional thermal field theory

A. N. Kvinikhidze and B. Blankleider

Department of Physics, Flinders University, Bedford Park, SA 5042, Australia
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Abstract

It is shown that light front thermal field theory is equivalent to conventional thermal field theory. The proof is based on the use of spectral representations, and applies to all Lagrangians for which such equivalence has been proven at zero temperature. It is also pointed out that conventional spectral functions can be used to express light-front finite temperature free propagators. As an application of our approach, we derive the light-front finite temperature spin 1/2 fermion propagator in full Dirac space.

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I. INTRODUCTION.

The light-front (LF) formulation of quantum field theory, where quantization arises from equal LF "time" commutation/anticommutation relations, offers some advantages over the conventional formulation where equal usual times are used for quantization. In this respect, perhaps the three most often quoted advantages of the LF approach are: kinematical boosts, a much less complicated vacuum, and much simpler eigenstates. The last two properties are related and may be especially beneficial for thermal field theory where the main object of interest, the ensemble average, is just a sum over eigenstates.

With such considerations in mind, the problem of formulating thermal field theory on the light front has been addressed in a number of recent papers [1, 2, 3, 4]. In Ref. [2], Alves, Das, and Perez proposed the LF version of the imaginary time scalar particle propagator, and used this to calculate the self-energy loop diagram. The result of this calculation looked so different from the conventional one, that only after Weldon [4] made use of a clever transformation of the integration momentum variable, did it become clear that this difference is illusive.

In the present note we show that the whole LF approach using the imaginary time scalar particle propagator proposed in Ref. [2], is equivalent to the conventional one. Our method is based on spectral representations of Green functions, and can be applied to the case of any Lagrangian for which such equivalence has been shown at zero temperature [5, 6, 7, 8, 9, 10, 11, 12]. In this respect, we note that the spectral function of the Lehmann representation is already well recognized as being very useful for relating different types of Green functions (imaginary time, real time, advanced, retarded, etc.) in the conventional approach. Here we show how to use the conventional spectral function to derive a LF free particle propagator of arbitrary spin in either imaginary or real time formalism. This allows us to derive fermion propagators in the full Dirac index space.

II. GENERAL PROOF OF EQUIVALENCE OF LIGHT FRONT AND CONVENTIONAL THERMAL FIELD THEORY

The problem of calculating the ensemble average \( \langle 0 | e^{-\beta P_L^0} \mathcal{O}_L | 0 \rangle \) in LF quantized field theory can be reduced to the calculation of the fully dressed LF imaginary time propagator \( \Delta^L \) given in coordinate and momentum space by (we use units where \( \hbar = k_B = 1 \))

\[
\Delta^L(\tau, x) = (\text{Tr} e^{-\beta P_L^0})^{-1} \text{Tr} \left\{ e^{-\beta P_L^0 T_\tau} [e^{P_L^0 \tau} \phi(0, x)] e^{-P_L^0 \tau} \bar{\phi}(0) \right\},
\]

(1a)

\[
\Delta^L(i\omega_n, p) = \frac{1}{\sqrt{2}} \int_0^\beta d\tau dx e^{i(\omega_n \tau - p \cdot x)} \Delta^L(\tau, x),
\]

(1b)

where \( \omega_n = 2n\pi T \) for bosons, \( \omega_n = (2n + 1)\pi T \) for fermions, \( x = (x^-, x^\perp) \), \( p = (p^+, p^\perp) \), \( x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3) \), \( p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^3) \), \( p \cdot x = -p^+ x^- + p^\perp x^\perp \), \( \phi(x) \) is a noninteracting particle

\footnote{Note that the correct definition of ensemble average has energy operator \( P_L^0 \) in the exponential \( e^{-\beta P_L^0} \) and not the Hamiltonian \( P_L^+ \) (see the discussion in Ref. [2]).}

\footnote{To distinguish between LF and usual equal time quantization we work in the operator rather than path integral formalism.}
field operator, $\phi(0, x) = \phi(x)|_{x^+ = 0}$, and where the energy operator $P^0_L$ and the operator of some physical quantity $O_L$ are defined in the LF quantized theory, i.e., they depend on interacting field operators whose commutation relations are given on the $x^+ = 0$ hyperplane (see Appendix A for more details). The imaginary time ordering product in Eq. (1a) is defined as

$$
T_I[e^{P^0_L \tau} \phi(0, x)e^{-P^0_L \tau} \bar{\phi}(0)] = \theta(\tau)e^{P^0_L \tau} \phi(0, x)e^{-P^0_L \tau} \bar{\phi}(0) \pm \theta(-\tau)\bar{\phi}(0)e^{P^0_L \tau} \phi(0, x)e^{-P^0_L \tau} \quad (2)
$$

where the upper (lower) sign is for the case of bosons (fermions). Note that although $\tau$ is associated with the usual time, in the sense that it is combined with $P^0_L$ (rather than with $P^-_L$) in the exponent of $e^{P^0_L \tau}$, Eq. (2) nevertheless represents the ordering of the interacting fields with respect to the imaginary LF “time” [see Eq. (3), the discussion below Eq. (3), and the text after Eq. (B5)], in contrast to the usual time ordering in Eq. (3a). It is also important to note that Eqs. (1) define the LF imaginary time formalism exactly, and in the case of scalar particles (only), they define perturbation theory with the free propagator suggested in Ref. [2]. As the perturbation theory for the dressed propagators of Eqs. (1) may not be immediately apparent (the exponents involve the energy operator $P^0_L$ rather than the Hamiltonian $P^-_L$), we outline its derivation in the Appendix B.

In this paper we shall prove that the analytic continuation of the fully dressed LF imaginary time propagator $\Delta^L(i\omega_n, p)$ to real energies, $i\omega_n \to p_0$, is identical to the analytic continuation of the fully dressed conventional imaginary time propagator $\Delta(i\omega_n, p)$, i.e., that $\Delta^L(p_0, p) = \Delta(p_0, p)$, where

$$
\Delta(\tau, x) = (\text{Tr} e^{-\beta P^0_L})^{-1} \text{Tr} \left\{ e^{-\beta P^0_L T_I} e^{P^0_L \tau} \phi(0, x)e^{-P^0_L \tau} \bar{\phi}(0) \right\}, \quad (3a)
$$

$$
\Delta(i\omega_n, p) = \int_0^\beta d\tau dx \ e^{i(\omega_n \tau - p \cdot x)} \Delta(\tau, x), \quad (3b)
$$

$\phi(0, x) = \phi(x)|_{x^0 = 0}$ and $P^0_L$ is the conventional energy operator.

We begin our proof by relating $\Delta^L(i\omega_n, p)$ to the real time LF Green function

$$
\mathcal{G}^L(t, x) = (\text{Tr} e^{-\beta P^0_L})^{-1} \text{Tr} \left\{ e^{-\beta P^0_LT_I} e^{iP^0_L t} \phi(0, x)e^{-iP^0_L t} \bar{\phi}(0) \right\}, \quad (4a)
$$

$$
D^L(p_0, p) = \frac{1}{\sqrt{2}} \int dt dx \ e^{i(p^0_0 - p^0) x} \mathcal{G}^L(t, x), \quad (4b)
$$

where $T_I$ is defined analogously to Eq. (2) as

$$
T_I[e^{P^0_L t} \phi(0, x)e^{-P^0_L t} \bar{\phi}(0)] = \theta(t)e^{iP^0_L t} \phi(0, x)e^{-iP^0_L t} \bar{\phi}(0) \pm \theta(-t)\bar{\phi}(0)e^{iP^0_L t} \phi(0, x)e^{-iP^0_L t}. \quad (5)
$$

This is done by utilizing the Lehmann representation which can be derived for LF Green functions in a way similar to that for conventional Green functions (see Ref. [14] for the conventional case):

$$
D^L(p_0, p) = i \int \frac{dp'_0}{2\pi} \frac{\rho^L(p'_0, p)}{p_0 - p'_0 + i\eta} \pm f(p_0)\rho^L(p_0, p), \quad (6a)
$$

$$
\Delta^L(i\omega_n, p) = - i \int \frac{dp'_0}{2\pi} \frac{\rho^L(p'_0, p)}{i\omega_n - p'_0}, \quad (6b)
$$

$^3$ Hopefully no confusion will arise from our not entirely consistent notation for $\phi(0, x)$ and $\phi(0, x)$; in particular $\phi(0, x) \neq \phi(0, x)|_{x^0 = 0}$. 

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where \( f(p_0) = (e^{\beta p_0} + 1)^{-1} \) is the distribution function for bosons (upper sign) or fermions (lower sign), and the LF spectral function (defined in Appendix C) is, correspondingly,

\[
\rho_L(p_0, p) = \frac{(2\pi)^4}{\text{Tr} e^{-\beta P_L^0}} (1 \mp e^{-\beta p_0}) \sum_{nm} e^{-\beta E_m} \delta^4(p - P_m + P_n) \langle Ln | \phi(0) | Lm \rangle \langle Lm | \tilde{\phi}(0) | Ln \rangle. \tag{7}
\]

The difference between \( \rho^L(p) \) and the conventional spectral function \( \rho(p) \) [see Eqs. (13)] is only in the eigenstates \( |Lm\rangle \) of the LF 4-momentum, \( P_L^0 |Lm\rangle = P_m |Lm\rangle \), which are different from the conventional ones (denoted by \( |m\rangle \) in Ref. [14]). However, in the free case there is no difference between these eigenstates and thus the free LF and conventional spectral functions are identical. The unusual scalar product \( p^0 t - \mathbf{p} \cdot \mathbf{x} \) in the exponent of Eq. (13) can be written in the invariant form \( p^0 t - \mathbf{p} \cdot \mathbf{x} = p \cdot x' \) where \( x' \) is defined by

\[
x'_0 = \frac{x^-}{\sqrt{2}} + t, \quad (x')^3 = -\frac{x^-}{\sqrt{2}}, \quad (x')^1 = x^+, \quad d\mathbf{x} = -\sqrt{2} d^4x'. \tag{8}
\]

Writing

\[
e^{i P_L^0 t} \phi(0, \mathbf{x}) e^{-i P_L^0 t} = \Phi_L(x'), \tag{9}
\]

one finds that \( \Phi_L \) is a LF Heisenberg field operator; i.e., for any four-vector \( a \),

\[
\Phi_L(x + a) = e^{i P_L^a} \Phi_L(x) e^{-i P_L^a}, \tag{10}
\]

with initial condition

\[
\Phi_L(0, \mathbf{x}) = \phi(0, \mathbf{x}). \tag{11}
\]

Given that \( \sqrt{2}(x')^+ = t, T_+ \) ordering in Eq. (1a) implies LF time ordering \( T_+ \), so that Eqs. (4) can be written in the form

\[
D^L(x) = (\text{Tr} e^{-\beta P_L^0})^{-1} \text{Tr} \left\{ e^{-\beta P_L^0 T_+} [\Phi_L(x) \tilde{\Phi}_L(0)] \right\}, \tag{12a}
\]

\[
D^L(p) = \int d^4x e^{ipx} D^L(x), \tag{12b}
\]

with the understanding that \( D^L(p_0, p) = D^L(p) \) and \( D^L(t, x) = D^L(x') \).

The well known relations, corresponding to the LF Eqs. (3), connecting conventional real and imaginary time Green functions via the conventional spectral function \( \rho(p) \) are [14, 15, 16]:

\[
D(p^0, \mathbf{p}) = i \int \frac{dp_0'}{2\pi} \frac{\rho(p_0', \mathbf{p})}{p_0' - p^0 + i\eta} \pm f(p_0) \rho(p_0, \mathbf{p}), \tag{13a}
\]

\[
\Delta(i\omega_n, \mathbf{p}) = -i \int \frac{dp_0'}{2\pi} \frac{\rho(p_0', \mathbf{p})}{i\omega_n - p_0'}. \tag{13b}
\]

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4 We do not show explicit spin indices. For particles with spin, one should consider the field \( \phi \) as a column vector, the field \( \tilde{\phi} \) as a row vector, and quantities like \( \rho^L, D^L, \Delta^L \), etc., as square matrices, in spin index space.

5 Here \( P_L^\mu \) is the operator of the 4-momentum in the LF approach whereas \( P_m^\mu \) is its eigenvalue corresponding to the state \( |Lm\rangle \).
where $D(p^0, p)$ is the conventional real time Green function, defined as
\[
D(t, x) = \langle \text{Tr} e^{-\beta P^0} \rangle^{-1} \text{Tr} \left\{ e^{-\beta P^0 t} T \{ e^{i P^0 t} \phi(0, x) e^{-i P^0 t} \bar{\phi}(0) \} \right\}, \tag{14a}
\]
\[
D(p^0, p) = \int dt \, dx \, e^{i (p^0 t - p \cdot x)} D(t, x). \tag{14b}
\]

For a perturbative treatment the propagator of Eq. (14a) should be written in the interaction representation \cite{17}, with
\[
\Phi(x) \equiv e^{i P^0 t} \phi(0, x) e^{-i P^0 t} = U(0, t) \phi(x) U(t, 0), \tag{15}
\]
\[
D(x) = \langle \text{Tr} e^{-\beta P^0} \rangle^{-1} \text{Tr} \left\{ e^{-\beta P^0} S^{-1} T \{ \phi(x) \bar{\phi}(0) S \} \right\}, \tag{16}
\]
where $P^0_f$ is the free part of $P^0$, $T$ is the usual time ordering operator, $S = U(\infty, -\infty)$, and
\[
U(t_2, t_1) = T \exp \left\{ -i \int_{t_1}^{t_2} dt \int_{-\infty}^{\infty} d^3 x \, \mathcal{P}^0_L(x) \right\}, \tag{17}
\]
$\mathcal{P}^0_L(x)$ being the interaction part of the Hamiltonian density in the interaction picture. We note the unusual appearance (for zero temperature perturbation theory) of the inverse $S$-matrix, $S^{-1}$, the source of doubled degrees of freedom \cite{17}. An analogous LF interaction representation can be written for $D^L$:
\[
\Phi^L(x) \equiv e^{i P^0_L x^+} \phi(0, x) e^{-i P^0_L x^+} = U^L(0, x^+) \phi(x) U^L(x^+, 0), \tag{18}
\]
\[
D^L(x) = \langle \text{Tr} e^{-\beta P^0_L} \rangle^{-1} \text{Tr} \left\{ e^{-\beta P^0_L} S_L^{-1} T_+ \{ \phi(x) \bar{\phi}(0) S_L \} \right\}, \tag{19}
\]
where $P^0_{Lf}$ is the free part of $P^0_L$, $S_L = U_L(\infty, -\infty)$, $P^+_L$ is the LF Hamiltonian, i.e., the minus component of the four-momentum operator, and
\[
U_L(\alpha_2, \alpha_1) = T_+ \exp \left\{ -i \int_{\alpha_1}^{\alpha_2} dx^+ \int_{-\infty}^{\infty} dx \, \mathcal{P}^0_{Lf}(x) \right\}, \tag{20}
\]
$\mathcal{P}^0_{Lf}(x)$ being the interaction part of the LF Hamiltonian density in the interaction representation.

A. Scalar particles

To compare the dressed real time propagators in the conventional and LF formalisms, given in Eq. (16) and Eq. (19) respectively, we first restrict the discussion to the case of scalar particles. For scalar particles $\mathcal{P}^0_{Lf}$ and $\mathcal{P}^0_f$ are the same functions of the free field operators $\phi$ \cite{14}, which means that the perturbation theories for Eq. (16) and Eq. (19) have the same vertices. The inverse $S$-matrix in both Eq. (16) and Eq. (19) leads to free propagators with doubled degrees of freedom, i.e. $2 \times 2$ matrices $\hat{D}^f$ and $\hat{D}^L_f$ whose $(1, 1)$ element is defined
by Eq. (16) and Eq. (19) in the no interaction limit:

\[
\hat{D}_{11}^{I}(x) = D_{11}^{I}(x) = (\text{Tr} e^{-\beta P_0} - 1) \text{Tr} \left\{ e^{-\beta P_0} T\text{r} \phi(x) \phi(0) \right\},
\]

(21a)

\[
\hat{D}_{11}^{L}(t, x) = D_{11}^{L}(t, x) = (\text{Tr} e^{-\beta P_0} - 1) \text{Tr} \left\{ e^{-\beta P_0} T\text{r} \left[ e^{iP_0} \phi(x') \phi(0) \right] \right\},
\]

(21b)

\[
\hat{D}_{11}^{L}(x') = D_{11}^{L}(x') = (\text{Tr} e^{-\beta P_0} - 1) \text{Tr} \left\{ e^{-\beta P_0} T\text{r} \left[ \phi(x') \phi(0) \right] \right\},
\]

(21c)

where \(x'\) is defined in Eq. (8). Similar to the zero temperature case, straightforward calculation of Eqs. (21) shows that \(D_{11}^{L}(p) = D_{11}^{I}(p)\) [see also Eqs. (24)], therefore the full propagators constructed according to Eq. (16) and Eq. (19) are equal to each other, \(D^{L}(p) = D(p)\), which already means the equivalence of real time LF and conventional thermal field theories for the scalar particle case. As we shall now see, this also leads to the identity between \(\Delta^{L}(p_0, p)\) and \(\Delta(p_0, p)\), the analytic continuations of the LF and conventional imaginary time propagators. To define a unique analytic continuation of \(\Delta(i\omega_n, p)\), given for the discrete values \(\omega_n = 2\pi n/\beta\), only two requirements have been needed in the conventional approach: (i) \(|\Delta(i\omega_n, p)| \to 0\) as \(|\omega_n| \to \infty\), and (ii) that \(\Delta(z, p)\) is analytic outside the real axis \([14, 15, 16]\). Placing the same requirements on \(\Delta^{L}(p_0, p)\), one obtains a unique analytic continuation of the LF imaginary time propagator as well. These analytic continuations are provided by Eqs. (6b) and (13b):
For the same models at non-zero temperature one follows a similar procedure: the extra vertices in the LF perturbation theory of Eq. (19) are taken into account by adding the following instantaneous term to the free LF propagator of Eq. (29):

\[ i \frac{\gamma^+}{2p^+} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ldots \tag{23} \]

This turns the LF propagator into the conventional $2 \times 2$ real time propagator. In this way one can prove the equivalence of LF and conventional real time thermal field theories for non-scalar particles described by Lagrangians for which such an equivalence has been shown at zero temperature, as done for example in Refs. [5, 6, 7, 8, 9, 10, 11].\(^7\)

The rest of the proof, showing equivalence of the LF and conventional imaginary time thermal field theories, is based on spectral representations and follows the same procedure presented above for the scalar particle case.

III. FREE PROPAGATORS

With the help of spectral functions, we will derive the free propagators of LF thermal field theory, including the spin $1/2$ fermion (spinor) propagator in the entire Dirac index space. In Ref. [2] it was suggested that the fermion propagator be expressed in the spinor subspace projected by $P^+ = \gamma^-\gamma^+ / 2$. Yet there is clear need for the fermion propagator in the entire spinor space, for example, to keep under control the compensation between differences in vertices and propagators with respect to the conventional approach.

Using the eigenstates of the noninteracting system in Eq. (7), one gets the spectral functions of the free scalar and spinor particles:

\[ \rho^{LF}_0(p) = \rho^f_0(p) = 2\pi \epsilon(p_0) \delta(p^2 - m^2) \] (scalar), \hspace{1cm} (24a)

\[ \rho^{LF}_{1/2}(p) = \rho^f_{1/2}(p) = 2\pi \epsilon(p_0)(\bar{p} + m) \delta(p^2 - m^2) \] (spinor), \hspace{1cm} (24b)

where $\epsilon(p_0) = p_0/|p_0|$. Eqs. (24) can also be obtained from Eq. (C4) by using the well known free field commutators/anticommutators. Although the free spectral functions are the same in the LF and conventional approaches, the fermion bare propagators are not:

\[ D^{LF}_{1/2}(p) = \bar{p} + m \left[ \frac{i}{p^2 - m^2 + i\eta} - 2\pi f(|p_0|) \delta(p^2 - m^2) \right] \] (25a)

\[ D^{f}_{1/2}(p) = (\bar{p} + m) \left[ \frac{i}{p^2 - m^2 + i\eta} - 2\pi f(|p_0|) \delta(p^2 - m^2) \right] \] (25b)

where $\bar{p}$ in Eq. (25a) is the on mass shell momentum with components $\bar{p}^- = (p_1^2 + m^2) / 2p^+$, and $\bar{p} = p$, which depend only on $p^+$ and $p^+$. This difference arises because different components of the four-momentum $p$ are fixed in the integrals of Eqs. (6) and (13). A

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\(^6\) This applies only to internal propagators (those not corresponding to an external leg).

\(^7\) In this respect it should be noted that vector particles need a more sophisticated treatment [8, 9], and equivalence may not mean that the dressed LF propagator is equal to the conventional one as in the scalar particle case, or even effectively equal to the conventional one as in the spinor case discussed above.
similar difference arises in the imaginary time formalism where the spinor propagator can be derived using the same spectral function, given by Eq. (24b), and the representation of Eq. (6b):

$$\Delta_{1/2}^{L_f}(i\omega_n, p) = -\frac{\hat{p} + m}{2p_n^+ - m^2 - p_\perp^2}$$ (26)

where $p_n^- = \sqrt{2i(2n+1)\pi T} - p^+ = \sqrt{2p_0^0} - p^+$. When fermions are involved the interaction part of the LF Hamiltonian has an extra term resulting from resolving constraints, whose effect is an instantaneous addition of $\gamma^+ / 2p^+$ to the propagator $\Delta_{1/2}^{L_f}$:

$$\frac{\hat{p} + m}{2p_n^+ - m^2 - p_\perp^2} + \frac{\gamma^+}{2p^+} = \frac{2p^+(\gamma^+p_n^- + \gamma^-p^+ - \gamma_\perp p_\perp + m)}{2p^+(2p_n^+ - m^2 - p_\perp^2)} = \frac{\hat{p}_n + m}{p_n^2 - m^2}$$ (27)

where $p_n = p$. Eq. (27) should be compared with the conventional imaginary time fermion propagator

$$\Delta_{1/2}^{f}(i\omega_n, p) = \left| p \text{ fixed} \right|_{p_0 = i(2n+1)\pi T}^N = \frac{i(2n+1)\pi T \gamma_0 - p \cdot \gamma + m}{[(2n+1)\pi T]^2 + p^2 + m^2}.$$ (28)

The difference between Eq. (27) and Eq. (28) is similar to the scalar particle case in that the replacement $p^0 \rightarrow i\pi(2n+1)T$ in the zero temperature propagator is carried out when different remaining variables $p$ and $p$ are fixed respectively.

It is not difficult to derive a spectral representation for the $2 \times 2$ propagators of the real time formalism and to write down the analog of Eq. (4) for the corresponding $2 \times 2$ spectral function. Then again we will see that the spectral function of a free particle is identical for the LF and conventional approach. As a result we will obtain the following expression for the LF real time fermion propagator

$$\hat{D}_{1/2}^{L_f}(p) = (\hat{p} + m) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left( \begin{array}{cc} p^2 - m^2 + i\eta \\ -i(p^2 - m^2 - i\eta) \end{array} \right) \delta(p^2 - m^2)$$ (29)

which differs from the conventional one in having $\hat{p}$ as the on mass shell momentum in the numerator of the first term on the right hand side of the equation.
APPENDIX A: LIGHT FRONT QUANTIZATION

As the precise meaning of ”LF quantization” appears to vary in the current literature, here we give the exact sense in which this term is used in the present paper. It is sufficient to consider only the scalar particle case.

We follow the traditional formulation of LF quantization. Starting with a Lagrangian density $\mathcal{L}[^\Phi] \equiv \mathcal{L}(\Phi(x), \partial_\mu \Phi(x))$ and the equations of motion

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial (\partial \Phi/\partial x^\mu)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0,$$

quantization enters through the specification of the initial conditions for these equations. In the case of LF quantization, we specify the initial conditions as

$$\Pi_L(x) = \frac{\partial \mathcal{L}}{\partial (\partial \Phi_L/\partial x^+)}.$$  \hspace{1cm} [\Phi_L(0), \Pi_L(x)]\delta(x^+) = \frac{i}{2}\delta^4(x)  \hspace{1cm} (A2)

where $\Pi_L(x)$ is the momentum conjugate to the field $\Phi_L(x)$, and the commutation relation between $\Pi_L$ and $\Phi_L$ is specified on the LF surface $x^+ = 0$. Note that we have added a subscript $L$ to those solutions of Eq. (A1) whose initial conditions are given by Eq. (A2). The LF initial conditions should be contrasted with those of usual equal time quantization:

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial (\partial \Phi/\partial x^0)}.$$  \hspace{1cm} [\Phi(0), \Pi(x)]\delta(x^0) = i\delta^4(x).  \hspace{1cm} (A3)

The LF four-momentum $P^\mu_L$ is given as an integral on the same surface as that specifying the LF commutation relations:

$$P^\mu_L = \int \delta(x^+) T^{+\mu}[\Phi_L] \, d^4x = \int_{x^+ = 0} T^{+\mu}[\Phi_L] \, dx  \hspace{1cm} (A4)$$

where the energy-momentum tensor $T^{\mu\nu}$ is connected to the Lagrangian in a way that is independent of quantization:

$$T^{\mu\nu}[\Phi] = \frac{\partial \mathcal{L}}{\partial (\partial \Phi/\partial x^\mu)} \frac{\partial \Phi}{\partial x^\nu} - g^{\mu\nu} \mathcal{L}[\Phi].$$

Similarly, the conventional four-momentum is defined by

$$P^\mu = \int \delta(x^0) T^{0\mu}[\Phi] \, d^4x = \int_{x^0 = 0} T^{0\mu}[\Phi] \, dx  \hspace{1cm} (A6)$$

In the free case (no interactions), one can show that the LF and conventional free fields are identical, as are the LF and conventional free momenta:

$$\phi_L(x) = \phi(x), \hspace{1cm} P^\mu_L = P^\mu_f.$$

(A7)

Constructed in this way, both the LF and conventional four-momenta act as generators of space-time translations:

$$[P^\mu_L, \Phi_L(x)] = -i\frac{\partial \Phi_L(x)}{\partial x_\mu},$$

$$[P^\mu, \Phi(x)] = -i\frac{\partial \Phi(x)}{\partial x_\mu}.  \hspace{1cm} (A8a)$$

$$[P^\mu, \Phi(x)] = -i\frac{\partial \Phi(x)}{\partial x_\mu}.  \hspace{1cm} (A8b)$$
We note the universal nature of these commutation relations - they do not depend on the type of quantization used. For the LF energy operator, \( P_L^0 \), and the LF Hamiltonian, \( P_L^- = \frac{1}{\sqrt{2}} (P_L^0 - P_L^3) \), we have that

\[
[P_L^0, \Phi_L(x)] = -i \frac{\partial \Phi_L(x)}{\partial x^0} \mid_{x^3, x^+ \text{ fixed}} = -\frac{i}{\sqrt{2}} \frac{\partial \Phi_L(x)}{\partial x^+} \mid_{x^3, x^+ \text{ fixed}},
\]

\[
[P_L^-, \Phi_L(x)] = -i \frac{\partial \Phi_L(x)}{\partial x^+} \mid_{x^-, x^+ \text{ fixed}} = -\frac{i}{\sqrt{2}} \frac{\partial \Phi_L(x)}{\partial x^0} \mid_{x^-, x^+ \text{ fixed}}.
\]

(A9)

(A10)

It is seen that both \( P^0_L \) and \( P^-_L \) determine evolution in LF time \( x^+ \), the only difference being in the variables which are kept constant.

As the LF four-momentum \( P^\mu_L \) is defined on the LF hyperplane \( x^+ = 0 \), as indicated by Eq. (A3), it follows that it is a functional of \( \Phi_L(0, \underline{x}) = \phi(0, \underline{x}) \). With the same being true of the free momentum operator \( P^\mu_f \), it follows that \( P^0_L(\tau) \equiv e^{\tau P^0_L} (P^0_L - P^0_f) e^{-\tau P^0_L} \) depends on \( \phi_f(\underline{x}) = e^{P^0_f \tau} \phi(0, \underline{x}) e^{-P^0_f \tau} \). This observation is at the heart of LF imaginary time perturbation theory, discussed in Appendix B.

**APPENDIX B: LF IMAGINARY TIME PERTURBATION THEORY**

Here we give a brief derivation of perturbation theory for the fully dressed LF imaginary time propagator \( \Delta_L^\tau \) given in Eqs. (1). The perturbation theory is based on the following operator exponent expansion

\[
\mathcal{W}(\tau_1, \tau_2) \equiv e^{P^0_L \tau_1} e^{P^0_L (\tau_2 - \tau_1)} e^{-P^0_L \tau_2} = T_\tau \exp \left[ \int_{\tau_1}^{\tau_2} P^0_{LI}(\tau) d\tau \right]
\]

(B1)

where \( P^0_L \) and \( P^0_f \) are the zero components of the LF four-momentum operator in the interacting and free case, respectively, \( P^0_{LI}(\tau) = e^{P^\tau_L}(P^0_L - P^0_f) e^{-P^\tau_L} \), and \( T_\tau \) is the ordering operator, as in Eq. (2), which orders \( \tau \)-dependent quantities with respect to \( \tau \), so that

\[
T_\tau [P^0_{LI}(\tau) P^0_{LI}(\tau')] = \theta(\tau - \tau') P^0_{LI}(\tau) P^0_{LI}(\tau') + \theta(\tau' - \tau) P^0_{LI}(\tau') P^0_{LI}(\tau).
\]

(B2)

Note that both \( P^0_L \) and \( P^0_f \) are expressed in terms of the free field operators \( \phi(0, \underline{x}) \) on the LF hyperplane \( x^+ = 0 \). Then by analogy with textbook derivations of perturbation theory in the imaginary time formalism [14, 15, 16], the Green function \( \Delta^L_\tau \) of Eq. (1a) can be written in the form

\[
(\text{Tr} e^{-\beta P^0_L}) \Delta^L_\tau(\tau, \underline{x}) = \text{Tr} \left\{ e^{-\beta P^\tau_L} T_\tau [e^{P^\tau_L \phi(0, \underline{x})} e^{-P^\tau_L \phi(0)}] \right\}
\]

\[
= \text{Tr} \left\{ e^{-\beta P^\tau_f} T_\tau \left[ e^{\int_0^{\tau} P^0_{LI}(\tau') d\tau'} e^{P^\tau_f \phi(0, \underline{x})} e^{-P^\tau_f \phi(0)} \right] \right\}
\]

(B3)

where \( T_\tau \) orders operators \( P^0_{LI}(\tau') \), \( \phi_f(\underline{x}) = e^{P^\tau_f} \phi(0, \underline{x}) e^{-P^\tau_f} \) and \( \phi(0) = \phi(0) \) with respect to \( \tau' \), \( \tau \) and 0 according Eq. (2). It is important to note that as a result of the definition of Eq. (1a), the interaction part of the energy operator, \( P^0_{LI}(\tau') \), depends on \( \phi_f(\underline{x}) = e^{P^\tau_f} \phi(0, \underline{x}) e^{-P^\tau_f} \), the free field operators on the \( x^+ = 0 \) LF hyperplane \( \phi(0, \underline{x}) = \phi(0, \underline{x}) \), shifted by imaginary usual time. In the analogous conventional expression given in Refs.
the interaction part of the energy depends on the free field operators on the hyperplane \( t = 0 \) (not \( x^+ = 0 \) shifted by imaginary usual time. This difference is reflected in the difference of the imaginary time propagators of the corresponding perturbation theories. Indeed using Wick’s theorem in Eq. [33] one ends up with imaginary time perturbation theory with propagators

\[
\Delta^{LF} (\tau, \vec{p}) = (\text{Tr} e^{-\beta P^0})^{-1} \text{Tr} \left\{ e^{-\beta P^0} T_\tau [e^{P^0 T_\tau} \phi(0, \vec{p}) e^{-P^0 T_\tau} \bar{\phi}(0)] \right\} \quad (B4)
\]

\[
\Delta^{LF} (i\omega_n, p) = \frac{1}{\sqrt{2}} \int_0^\beta d\tau d\vec{x} e^{i(\omega_n \tau - p \cdot \vec{x})} \Delta^{LF} (\tau, \vec{x}) \quad (B5)
\]

Applying the arguments leading to Eq. [8] to the imaginary time \( t = i\tau \), one gets

\[
e^{P^0 T_\tau} \phi(0, \vec{p}) e^{-P^0 T_\tau} = \phi(z), \quad \text{where } z \text{ is a complex coordinate space four-vector with purely imaginary LF time } z^+ = -i\tau/\sqrt{2} \text{ and real spatial components } z^3 = -x^3/\sqrt{2} \text{ and } z^\perp = x^\perp.
\]

This suggests that the propagator of Eq. [B4] corresponds to imaginary LF time ordering, in contrast to the usual time ordering in the conventional approach. Direct calculation leads to the scalar particle imaginary time propagator suggested in Ref. [2],

\[
\Delta_0^{LF} (i\omega_n, p) = -\frac{1}{2p^0 p^+ - m^2 - p^\perp}, \quad (B6)
\]

with \( p^- = \sqrt{2}i\pi 2nT - p^+ = \sqrt{2}p_0^0 - p^+ \), and the spinor particle propagator of Eq. [26] derived above. Even in the scalar particles case, the difference between the imaginary time formalisms in the two quantisations is more than a simple change of variables: although the energy variable is purely imaginary in both Eq. [B6] and the conventional propagator, \( p^3 \) is real in the conventional approach but complex in the LF one. Note also that in the LF dynamics the interaction does not affect \( P^+_L \). Only the operator \( P^-_L \) acquires an interaction part, therefore the interaction part of the energy operator is \( P^0_{LI} = P^+_L/\sqrt{2} \) and in the case of scalar particles (only) it is related to the interaction Lagrangian (without coupling with derivatives) by a factor of \( \sqrt{2} \): \( P^0_{LI} = -L_I/\sqrt{2} \)

**APPENDIX C: LIGHT-FRONT SPECTRAL FUNCTION**

Here we introduce the LF spectral function \( \rho^L \) used in Eqs. [3], and specify some of its properties by exploiting the analogy with the well known spectral function \( \rho \) of conventional thermal field theory [14]. In coordinate space, one can use Eq. [12a], defining the real time LF propagator \( D^L(x) \), to write

\[
D^L(x) = \theta(x^+)D^L>(x) + \theta(-x^+)D^L<(x) = \theta(x^+)\rho^L(x) + D^L<(x) \quad (C1)
\]

where

\[
D^L>(x) = (\text{Tr} e^{-\beta P^0_L})^{-1} \text{Tr} \left\{ e^{-\beta P^0_L} \Phi_L(x) \bar{\Phi}_L(0) \right\}, \quad (C2)
\]

\[
D^L<(x) = (\text{Tr} e^{-\beta P^0_L})^{-1} \text{Tr} \left\{ e^{-\beta P^0_L} \bar{\Phi}_L(0) \Phi_L(x) \right\}, \quad (C3)
\]

and

\[
\rho^L(x) = D^L>(x) + D^L<(x) = (\text{Tr} e^{-\beta P^0_L})^{-1} \text{Tr} \left\{ e^{-\beta P^0_L} [\Phi_L(x), \bar{\Phi}_L(0)]_\mp \right\}, \quad (C4)
\]
is the LF spectral function defined as the ensemble average of the commutator / anticommutator \([\Phi_L(x), \Phi_L(0)]_\mp\). The Kubo-Martin-Schwinger relation is the same as in Ref. [14],

\[
D^{<}(x) = D^{>}(x^0 - i\beta, \mathbf{x}).
\]  

(C5)

It is easy to express the above equations in momentum space thereby obtaining the following relations:

\[
\rho^L(p) = D^{>}(p) \mp D^{<}(p),
\]  

(C6)

\[
D^{<}(p) = \pm e^{-\beta p^0} D^{>}(p),
\]  

(C7)

\[
D^{>}(p) = [1 \pm f(p^0)] \rho^L(p),
\]  

(C8)

\[
D^{<}(p) = \pm f(p^0) \rho^L(p),
\]  

(C9)

which lead to Eq. (6a). The \(x^+\) ordering in Eq. (C1) (instead of conventional \(x^0\) ordering) is reflected in that \(p\) is fixed (not \(p\)) in \(\rho^L(p)\) in the spectral energy integral of Eq. (6a). To derive Eq. (6b), one can easily verify that

\[
\Delta^L(\tau, \mathbf{z}) = \mathcal{D}^{>}(0\tau, \mathbf{z}) = D^{>}(\mathbf{z})
\]  

(C10)

with \(z^+ = -\frac{\tau}{\sqrt{2}}, z^3 = -\frac{x^-}{\sqrt{2}}, \) and \(z^\perp = x^\perp\), if \(\tau\) is in the interval \([0, \beta]\).

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