Study of small x behaviour of unintegrated gluon distribution constraining gluon evolution in MD-BFKL equation

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Abstract. In this work we have proposed a modified form of MD-BFKL nonlinear evolution equation imposing a kinematic constraint on the gluon evolution in multi-Regge kinematics. Later we have suggested an analytical solution to our KC imposed MD-BFKL equation and studied nonlinear effect of the transverse momentum dependent (TMD) gluon distribution function at small x ($10^{-6} \lesssim x \lesssim 10^{-2}$). Our results have been compared to various TMDlibuPDF datasets viz. PB-NLO-HERA I+II 2018, ccfm set A0.

1. Introduction
Parton distribution functions (PDFs) are considered as the most significant tool in hadronic collision processes for the calculation of inclusive cross sections [1-5]. The parton distributions of the nucleon cannot be extracted directly from measured structure functions in DIS experiments. They are mainly predicted by using the QCD evolution equations.

At small x the growth of cross sections due to gluon splitting in the DGLAP and BFKL [1] linear evolution equations supposed to violate the Foissart bound. Therefore, the corrections of the higher order QCD, which slows down the infinite growth of parton densities becomes important. In this respect in the past decades two broadly studied nonlinear evolution equations GLR-MQ and BK have been considered as very significant models towards computing shadowing corrections to the original linear DGLAP and BFKL equation. In the recent years, Ruan Shen Yang and Zhu have proposed an unitarized BFKL equation also termed as Modified-BFKL (MD-BFKL) equation incorporating shadowing as well as possible anti-shadowing corrections to the gluon evolution. The key feature of this unitarized BFKL equation is the inclusion of the antishadowing correction terms which actually acts as negative screening effects in the recombination processes. Indeed, the antishadowing effects always coexists with the shadowing effects in the recombination processes as a general conclusion of the momentum conservation. Therefore, antishadowing effects can’t be neglected.

Another important feature of MD-BFKL is that it is consistent to BK equation at the saturation limit where gluon distribution becomes flatter. Before studying the phenomenological aspects of the equation, we have further modified the equation by imposing a kinematic constraint on the gluon evolution which actually sets an infrared cutoff to the real gluon emission term of the equation. Our work is highly motivated towards the small x saturation regime where shadowing correction is dominant one.
2. Formalism

The MD-BFKL equation [6] reads,

\[-x \frac{\partial f(x, k^2)}{\partial x} = \frac{\alpha_s N_c k^2}{\pi} \int_{k_{min}^2}^{\infty} \frac{dk'^2}{k'^2} \left\{ f\left(x, k'^2\right) - f(x, k^2) \frac{|k'^2 - k^2|}{\sqrt{k'^4 + 4k'^4}} \right\}
\]

\[-\frac{36\alpha_s^2}{\pi k^2 R^2} \frac{N_c^2}{N_c^2 - 1} f^2(x, k^2) + \frac{18\alpha_s^2}{\pi k^2 R^2} \frac{N_c^2}{N_c^2 - 1} f^2\left(x, \left(\frac{r}{2}, k^2\right)\right).\]

(1)

where \(f(x, k^2)\) is the unintegrated gluon distribution function, \(x\) and \(k^2\) being the fractional momentum of proton carried by gluon and the transverse momentum of gluon respectively. The quadratic term with the negative sign in the equation is the shadowing correction whereas the positive quadratic term is for the antishadowing correction. The TMD gluon distribution can be interpreted in terms of conventional collinear gluon distribution by the simple relation \(f(x, k^2) = Q^2 \frac{\partial O(x,Q^2)}{\partial Q^2} |_{Q^2=k^2}.\)

In high energy limit, the longitudinal components of the gluon momentum are strongly ordered while there is no ordering on the transverse components of gluon momentum [7]. Since longitudinal momentum factors are very small in small \(x\) region, the gluon virtuality along the chain must be dominated by transverse components of the gluon momentum. Thus, the BFKL kinematics corresponds to

\[x_1 << x_2 << x_3 << \ldots \ll x_n << 1,\]

\[k_1T \sim k_2T \sim k_3T \sim \ldots \sim k_nT,\]

\[k^2 = 2k^+k^- - k_T^2 \approx k_T^2.\]

(2)

The above kinematics is referred to as multi-Regge kinematics. Equation (4) implies \(z < 1\) (see figure(1)) and since transverse momenta are of the same order: \(k^2 \approx k_T^2\) it is evident from the fact that \(k_T^2\) has a kinematic cutoff \(k_T^2 < \frac{k_0^2}{z}\) for real gluon emission [8, 9]. The constraint can be written in terms of Heaviside step function as \(\theta\left(\frac{k_0^2}{z} - k_T^2\right)\) and imposed onto the real emission part of the BFKL kernel in equation (1).

\[f(x,k^2) = f^0(x,k^2) + \frac{\alpha_s N_c k^2}{\pi} \int_x^1 \frac{dz}{z} \int_{k_{min}^2}^{\infty} \frac{dk'^2}{k'^2} \left\{ \frac{\theta\left(\frac{k_0^2}{z} - k'^2\right)}{|k'^2 - k^2|} + \frac{f\left(x,\left(\frac{r}{2}, k^2\right)\right)}{\sqrt{k'^4 + 4k'^4}} \right\}.\]

(3)

The kinematic constraint resums a large part of the subleading corrections coming from the choice of scales in the BFKL kernel. The integro-differential form of the KC MD-BFKL equation can be obtained using the identities \(\theta'(y) = \delta(y)\) and \(f(y)\delta(y - a) = f(a)\delta(y - a)\) as,
\[-x \frac{\partial f(x, k^2)}{\partial x} = \frac{\alpha_s N_c k^2}{\pi} \int_{k_{\text{min}}^2}^{k^2} \frac{dk'^2}{k'^2} \left( \frac{\theta(k^2 - k'^2)}{|k'^2 - k^2|} f(x, k'^2) + \frac{\theta(k^2 - k'^2)}{|k'^2 - k^2|} f(x, k'^2)^2 \right) - \frac{f(x, k^2)}{|k^2 - k^2|} \right]
\[+ \frac{f(x, k^2)}{\sqrt{k^4 + 4k'^4}} - \frac{18 \alpha_s^2 N_c^2}{\pi k^2 R^2} \frac{f(x, k^2)}{N_c^2 - 1} f^2(x, k^2) \]

(4)

To simplify the real emission term \( f(x, k^2) \) we have incorporated Regge-like behavior of gluon distribution function. The behavior of structure functions at small-x is well explained in terms of Regge-like ansatz [10]. For small-x, the Regge behaviour of the sea quark and antiquarks distribution is given by \( q_{\text{sea}} \sim x^{-\alpha_P} \) corresponding to a pomeron exchange with an intercept of \( \alpha_P = 1 \). But the valence-quark distribution for small-x given by \( q_{\text{val}}(x) \sim x^{-\alpha_R} \) corresponding to a reggeon exchange with an intercept of \( \alpha_R = 0 \).

At moderate \( Q^2 \), the leading order calculations in \( \ln(1/x) \) with fixed value of \( \alpha_s = 0.2 \) [11,12].

We try to simplify MD-BFKL equation by considering a simple form of Regge like behaviour given as

\[ f(x, k^2) = \rho(k^2)x^{-\lambda}, \]

(5)

where \( \lambda \) is the Regge intercept for gluon distribution function.

Now substituting equation (5) in equation (7) we get

\[-x \frac{\partial f(x, k^2)}{\partial x} = \frac{\alpha_s N_c k^2}{\pi} \int_{k_{\text{min}}^2}^{k^2} \frac{dk'^2}{k'^2} \left( \frac{\theta(k^2 - k'^2)}{|k'^2 - k^2|} f(x, k'^2) + \frac{\theta(k^2 - k'^2)}{|k'^2 - k^2|} f(x, k'^2)^2 \right) - \frac{f(x, k^2)}{|k^2 - k^2|} \right]
\[+ \frac{f(x, k^2)}{\sqrt{k^4 + 4k'^4}} - \frac{18 \alpha_s^2 N_c^2}{\pi k^2 R^2} \frac{f(x, k^2)}{N_c^2 - 1} f^2(x, k^2). \]

(6)

The above equation is our proposed kinematic constraint-imposed MD-BFKL equation. To solve the equation, we expand the gluon distribution \( f(x, k^2) \) around \( k \) at the taking care of BFKL multi-Regge kinematics \( k_T \sim k_R \).
recalling in BFKL multi Regge kinematics which is found to be well behaved at the both integration limits where

\[ \int_{k_{\text{min}}^2}^{k^2} \frac{dk'^2}{k'^2} \left( k'^2 - k^2 \right) \frac{\lambda}{|k'^2 - k^2|} \left( k'^2 - k^2 \right) \frac{\partial f(x, k^2)}{\partial k^2} \]

+ \frac{\alpha_s N_c}{\pi} k^2 \left( \int_{k_{\text{min}}^2}^{k^2} \frac{dk'^2}{k'^2} \left( k'^2 - k^2 \right) \frac{\lambda}{|k'^2 - k^2|} - \int_{k_{\text{min}}^2}^{k^2} \frac{dk'^2}{k'^2} \frac{1}{|k'^2 - k^2|} \right)

\[ + \int_{k^2}^{\infty} \frac{dk'^2}{k'^2} \frac{1}{\sqrt{k^4 + 4k'^4}} f(x, k^2) \]

\[ + \frac{18\alpha_s^2 N_c}{\pi k^2 R^2 N_c^2} - 2 \int_{k_{\text{min}}^2}^{k^2} \frac{dk'^2}{k'^2} \frac{1}{\sqrt{k^4 + 4k'^4}} f(x, k^2). \]

(7)

The improper integrals in equation (13) are improper integrals since it blow-up at the integration limit \( k^2 \). To find out the improper integrals we have performed some angular integral prescription and the integral found out to be

\[ \int \frac{dk'^2}{k'^2} \frac{1}{|k'^2 - k^2|} \equiv 2 \int \frac{dk'^2}{k'^2} \frac{1}{\sqrt{k^4 + 4k'^4}} \]

which is found to be well behaved at the both integration limits \( k^2 \) and \( \infty \). Similarly for the second term recalling in BFKL multi Regge kinematics \( k' \sim k \) we can write

\[ \int_{k_{\text{min}}^2}^{k^2} \frac{dk'^2}{k'^2} \left( k'^2 - k^2 \right) \frac{\lambda}{|k'^2 - k^2|} \left( k'^2 - k^2 \right) \frac{1}{|k'^2 - k^2|} \left( k'^2 - k^2 \right) \]

\[ \equiv 2 \int_{k_{\text{min}}^2}^{k^2} \frac{dk'^2}{k'^2} \frac{1}{\sqrt{k^4 + 4k'^4}} \]

(9)

Substituting these integrals in equation (13) we get

\[ -x \frac{\partial f(x, k^2)}{\partial x} = \rho(k^2) \frac{\partial f(x, k^2)}{\partial k^2} + \Phi(k^2) f(x, k^2) - \chi(k^2) f^2(x, k^2) \]

(10)

where

\[ \rho(k^2) = \frac{\alpha_s N_c}{\pi} k^2 \left( \int_{k_{\text{min}}^2}^{k^2} \frac{dk'^2}{k'^2} \frac{k'^2 - k^2}{|k'^2 - k^2|} + \int_{k_{\text{min}}^2}^{k^2} \frac{dk'^2}{k'^2} \left( k'^2 - k^2 \right) \frac{1}{|k'^2 - k^2|} \right) \]

\[ \Phi(k^2) = \frac{\alpha_s N_c}{\pi} k^2 \left( 2 \times \int_{k_{\text{min}}^2}^{k^2} \frac{dk'^2}{k'^2} \frac{1}{\sqrt{k^4 + 4k'^4}} - 2 \times \int_{k_{\text{min}}^2}^{k^2} \frac{dk'^2}{k'^2} \frac{1}{\sqrt{k^4 + 4k'^4}} + \int_{k^2}^{\infty} \frac{dk'^2}{k'^2} \frac{1}{\sqrt{k^4 + 4k'^4}} \right) \]

and

\[ \chi(k^2) = \frac{18\alpha_s^2 N_c^2}{\pi k^2 R^2 N_c^2} - 1. \]
By using an inverse function for $f$ we can recast semi-linear PDE (10) into a linear one, then using method of characteristics we get the solution of (10) as $F(u, v) = 0$, which is any arbitrary differentiable function of $u(x, k^2, f(x, k^2))$ and $v(x, k^2, f(x, k^2))$ such that

$$u(x, k^2, f(x, k^2)) = \frac{e^{-\frac{nk^2}{\lambda}}}{x}.$$  

$$v(x, k^2, f(x, k^2)) = \frac{9\alpha\lambda N_c}{k^3(N_c^2 - 1)R^2} \frac{e^{-\frac{2k^2}{\lambda}}}{k^2f(x, k^2)}$$

(11)
We can get particular solution $f(x, k^2)$ for $x$ and $k^2$ evolution by imposing some initial input distribution to equation (11).

3. Result and Discussion
We have studied both Bjorken $x$ and transverse momentum $k_T^2$ dependent gluon distribution in the kinematic region of small $x$ i.e. $10^{-6} \leq x \leq 10^{-2}$ and moderate $k_T^2$ i.e. $1 \leq k_T^2 \leq 10^2$. The ultraviolet cutoff $k_{\text{min}}^2$ have been chosen in such a way that it lies in the boundary of the perturbative and nonperturbative QCD phase space. We have plotted for $x$ evolution constraining $k_T^2$ at some scale show in figure 1. Similarly, for $k_T^2$ evolution we have plotted gluon distribution constraining $x$ as shown in figure 2. We compared our result with TMDlib Datasets viz. PB-NLO-HERAI+II-2018 and ccfm-set-A0. The input gluon distribution in our calculation is taken from PB-NLO-HERAI+II-2018. The PB-NLO-HERAI+II-2018 parameterization includes collinear and TMD parton densities obtained from fits to the HERA precision cross-section measurement. The PDFs are evolved by NLO DGLAP evolution using a recently developed theory called parton branching (PB) method which allows the calculation of both collinear and TMD...
densities simultaneously. On the other hand, the other parameterization ccfm-set-A0 includes uPDF sets extracted from CCFM evolution taking starting distribution from GBW parametrization.

Our results are obtained for two different form of shadowing: conventional R= 5 GeV\(^{-1}\) (order of proton radius) where gluons are spread throughout the nucleus and extreme R= 2 GeV\(^{-1}\) where gluons are expected to concentrated in hotspots within the proton disk. From the figures 1 and 2 it is clear that shadowing correction are more prominent when gluons are concentrated in hotspots within proton.

The \(k^2\) evolution in figure 1 is studied corresponding to four different values of \(x\) as indicated in the figure. The input distribution is taken at \(k^2\) from PB-NLO-HERAI+II-2018 dataset. Our evolution shows a similar growth as datasets for all \(x\) at conventional shadowing condition R= 5 GeV\(^{-1}\). The extreme shadowing form R= 2 GeV\(^{-1}\) seems to overestimate the dataset particularly at very small-x values. It is also observed that the growth of \(f(x,k^2)\) is almost linear for the entire kinematic range of \(k^2\). This is expected since the net shadowing term which has \(1/k^2\) dependence, which suppresses the contribution from the shadowing term at large \(k^2\).

The \(x\) evolution of unintegrated gluon distribution \(f(x,k^2)\) is shown in figure 2 for different values of \(k^2\) ranging from 4 GeV\(^2\) to 64 GeV\(^2\). The input is taken at higher \(x\) value \(x=0.01\) and then evolved down to smaller \(x\) value up to \(x=0.000001\). We observed that the singular \(x^{-4}\) growth of the gluon is eventually suppressed by the net shadowing effect. However, it is hard to establish the existence of shadowing for \(x>0.001\). The obvious distinction between the two form of shadowing R= 5 GeV\(^{-1}\) (conventional) and R= 2 GeV\(^{-1}\) ("hotspot") is also observed towards small-x (\(x<0.0001\)).

4. Conclusion
In this work, we have modified the MD-BFKL equation by imposing a kinematic constraint on the real gluon emission term due to the strong ordering of their transverse momentum, on the BFKL kernel of MD-BFKL equation. We further simplified the KC improved MD-BFKL equation by considering Regge like behaviour of gluon distribution function. Then, we solved our KC imposed MD-BFKL equation panalytically. Through our phenomenology we checked the validity of Regge ansatz on gluons and we compared our results with various TMDlib experimental data fits by various global collaborations. We have seen a good consistency between our theoretical prediction and the various data fits for both \(x\) and \(k^2\) evolution. Finally, from this work, we conclude owing to the fact that the nonlinear effects play significant roles at small-x, our solution of kinematic constraint-imposed MD-BFKL equation can be a reliable nonlinear evolution equation for the prediction of gluon distribution particularly at very small-x (\(\leq 10^{-3}\)) i.e. in the vicinity saturation region.

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