GREEN FUNCTION FOR AN ASYMPTOTICALLY STABLE RANDOM WALK IN A HALF SPACE

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Abstract. We consider an asymptotically stable multidimensional random walk \( S(n) = (S_1(n), \ldots, S_d(n)) \). Let \( \tau_x := \min\{n > 0 : x_1 + S_1(n) \leq 0\} \) be the first time the random walk \( S(n) \) leaves the upper half-space. We obtain the asymptotics of \( p_n(x, y) := \mathbb{P}(x + S(n) \in y + \Delta, \tau_x > n) \) as \( n \) tends to infinity, where \( \Delta \) is a fixed cube. From that we obtain the local asymptotics for the Green function \( G(x, y) := \sum_{n=0}^{\infty} p_n(x, y) \), as \( |y| \) and/or \( |x| \) tend to infinity.

1. Introduction, main results and discussion

1.1. Notation and assumptions. Consider a random walk \( \{S(n), n \geq 0\} \) on \( \mathbb{R}^d \), \( d \geq 1 \), where
\[
S(n) = X(1) + \cdots + X(n), \quad n \geq 1
\]
and \( \{X(n), n \geq 1\} \) are independent copies of a random vector \( X = (X_1, \ldots, X_d) \).
For \( x = (x_1, \ldots, x_d) \) in the (non-negative) half space, that is for \( x_1 \geq 0 \), let
\[
\tau_x := \min\{n \geq 1 : x + S(n) \notin \mathbb{H}^+\} = \min\{n \geq 1 : x_1 + S_1(n) \leq 0\}
\]
be the first time the random walk exits the (positive) half space
\[
\mathbb{H}^+ = \{(x_1, \ldots, x_d) : x_1 > 0\}.
\]
When \( x = 0 \) we will omit the subscript and write
\[
\tau := \tau_0 = \min\{n \geq 1 : S(n) \notin \mathbb{H}^+\} = \min\{n \geq 1 : S_1(n) \leq 0\}.
\]
In this paper we study the asymptotic, as \( n \to \infty \), behaviour of the probability
\[
p_n(x, y) := \mathbb{P}(x + S(n) \in y + \Delta, \tau_x > n) \tag{1}
\]
and the Green function
\[
G(x, y) := \sum_{n=0}^{\infty} p_n(x, y).
\]
Here and throughout we denote \( \Delta = [0, 1]^d \) and for \( y = (y_1, \ldots, y_d) \),
\[
y + \Delta = [y_1, y_1 + 1) \times [y_2, y_2 + 1) \times \cdots \times [y_d, y_d + 1).
\]
In this paper we will mostly concentrate on the case when the random walk \( S(n) \) has infinite second moments. More precisely, we shall assume that \( S(n) \) is asymptotically stable of index \( \alpha < 2 \) when we study large deviations for local probabilities and asymptotics for the Green function. The asymptotics for the Green function of walks with finite variances have already been studied in the

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literature: (a) Uchiyama [20] has considered lattice walks in a half space; (b) Duraj et al. [13] have derived asymptotics of Green functions for a wider class of convex cones. It is worth mentioning that the authors of [13] analyse first the case of a half space and use the estimates for the Green function for a half space in the subsequent analysis of convex cones. This fact underlines the importance of the case of half spaces.

We will say that $S(n)$ belongs to the domain of attraction of a multivariate stable law, if

$$\frac{S(n)}{c_n} \xrightarrow{d} \zeta_\alpha,$$

where $\alpha \in (0, 2]$ and $\zeta_\alpha$ has a multivariate stable law of index $\alpha$. Note that we assume that $S(n)$ is already centred. This does not restrict generality, when $\alpha \neq 1$, as one can subtract the mean for $\alpha > 1$ and the centering is not needed for $\alpha < 1$.

Necessary and sufficient conditions for the convergence in (2) are given in [16]. When $\alpha \in (0, 2)$, the convergence will take place if and only if $\tau^\alpha$, $\rho$ with parameter $\alpha$ belongs to the domain of attraction of a positive stable law if

$$\frac{1}{n} \sum_{k=1}^{n} P(S_1(k) > x) \to \sigma(A), \quad x \to \infty,$$

for any measurable $A$ on $\mathbb{S}^{d-1}$. We will write $X \in \mathcal{D}(d, \alpha, \sigma)$ when (2) holds, where $\sigma$ stands for the above measure on the unit sphere. Let $g_{\alpha, \sigma}$ be the density of $\zeta_\alpha$.

For $\alpha \in (0, 2)$ we will assume that $\sigma(\mathbb{S}^{d-1} \cap \mathbb{H}^+) > 0$ and $\sigma(\mathbb{S}^{d-1} \cap \mathbb{H}^-) > 0$, where $\mathbb{H}^- = \mathbb{R}^d \setminus \mathbb{H}^+$. This assumption implies that the first coordinate $X_1$ belongs to the one-dimensional domain of attraction. Moreover, let

$$\mathcal{A} := \{0 < \alpha < 1; |\beta| < 1\} \cup \{1 < \alpha < 2; |\beta| \leq 1\} \cup \{\alpha = 1, \beta = 0\} \cup \{\alpha = 2, \beta = 0\}$$

be a subset in $\mathbb{R}^2$. For $(\alpha, \beta) \in \mathcal{A}$ and a random variable $X_1$ write $X_1 \in \mathcal{D}(\alpha, \beta)$ if the distribution of $X_1$ belongs to the domain of attraction of a stable law with characteristic function

$$G_{\alpha, \beta}(t) := \exp \left\{ -c|t|^\alpha \left( 1 - i \beta \frac{t}{|t|} \tan \frac{\pi \alpha}{2} \right) \right\} = \int_{-\infty}^{+\infty} e^{itu} g_{\alpha, \beta}(u) du, \quad c > 0, \quad (3)$$

and, in addition, $E[X] = 0$ if this moment exists. Then, $X_1 \in \mathcal{D}(\alpha, \beta)$.

We will consider the case when $S_1(n)$ is oscillating, that is when $P(\tau < \infty) = 1$ and $E[\tau] = \infty$. Recall that the random walk $S_1(n)$ oscillates if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} P(S_1(n) > 0) = \sum_{n=1}^{\infty} \frac{1}{n} P(S_1(n) \leq 0) = \infty.$$

Rogozin [17] investigated properties of $\tau$ and demonstrated that the Spitzer condition

$$n^{-1} \sum_{k=1}^{n} P(S_1(k) > 0) \to \rho \in (0, 1) \quad \text{as } n \to \infty \quad (4)$$

holds if and only if $\tau$ belongs to the domain of attraction of a positive stable law with parameter $\rho$. In particular, if $X_1 \in \mathcal{D}(\alpha, \beta)$ then (see, for instance, [23])
Theorem 1. If $X \in \mathcal{D}(d, \alpha, \sigma)$, then there exists a random vector $M_{\alpha, \sigma}$ on $\mathbb{H}^+$ with density $p_{M_{\alpha, \sigma}}(v)$ such that, for all $v \in \mathbb{R}^d$,

$$
\lim_{n \to \infty} \mathbb{P}\left( \frac{S(n)}{c_n} \in u + \Delta \mid \tau > n \right) = \mathbb{P}(M_{\alpha, \sigma} \in u + \Delta) = \int_{u + \Delta} p_{M_{\alpha, \sigma}}(v) dv. \tag{9}
$$

Moreover, for every bounded and continuous function $f$,

$$
\mathbb{E}\left[ f\left( \frac{S(n)}{c_n} \right) \mid \tau > n \right] \to \mathbb{E}[f(M_{\alpha, \sigma})].
$$

condition (1) holds with

$$
\rho := \sigma(\mathbb{H}^+ \cap S^{d-1})
= \int_0^\infty g_{\alpha, \beta}(u) du = \left\{ \begin{array}{ll}
\frac{\alpha}{2}, & \alpha = 1, \\
\frac{\alpha}{2} + \frac{1}{\pi \alpha} \arctan \left( \frac{\beta \tan \frac{\pi \alpha}{2} \sigma}{\sigma} \right), & \text{otherwise}.
\end{array} \right.
\tag{5}
$$

We will further assume that the Spitzer-Doney condition holds

$$
\mathbb{P}(S_1(n) > 0) \to \rho, \quad n \to \infty,
\tag{6}
$$

which is known to be equivalent to (4).

The scaling sequence $\{c_n\}$ can be defined as follows, see [16]. Denote $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Z}_+ := \{1, 2, \ldots\}$ and let $\{c_n, n \geq 1\}$ be a sequence of positive numbers specified by the relation

$$
c_n := \inf \{ u \geq 0 : \mu(u) \leq n^{-1} \},
\tag{7}
$$

where

$$
\mu(u) := \frac{1}{u^2} \int_{-u}^u x^2 \mathbb{P}(|X| \in dx).
$$

It is known (see, for instance, [14] Ch. XVII, §5) that for every $X \in \mathcal{D}(d, \alpha, \sigma)$ the function $\mu(u)$ is regularly varying with index $(-\alpha)$. This implies that $\{c_n, n \geq 1\}$ is a regularly varying sequence with index $\alpha^{-1}$, i.e. there exists a function $l_1(n)$, slowly varying at infinity, such that

$$
c_n = n^{1/\alpha}l_1(n). \tag{8}
$$

Then, convergence (2) holds with this sequence $\{c_n\}$. In addition, the scaled sequence $\frac{S(n)}{c_n}$ converges in distribution, as $n \to \infty$, to the stable law given by (3).

In one-dimensional case the study of asymptotics (1) was initiated in [21], where normal and small deviations of $p_n(0, y)$ were considered. Asymptotics for $p_n(x, y)$ with a general starting point $x$ was studied then in [3] and [10]. Our assumption on $X_1$ is the same as in these papers and we use a similar approach for small and normal deviations. We used a different approach to study large deviations in the multidimensional case. Large deviations seem to be the most complicated part of the present paper.

As the first coordinate plays a distinctive role, we will adopt the following notation. For $X(n)$ we will write $X(n) = (X_1(n), X_{(2, d)}(n))$, where $X_1(n)$ corresponds to the first coordinate and $X_{(2, d)}(n)$ corresponds to the remaining coordinates. Similarly we write $S(n) = (S_1(n), S_{(2, d)}(n))$, $n = 0, 2, \ldots$.

The following conditional limit theorem will be crucial for the rest of this article. The weak convergence in this theorem can be proved similarly to [9]. Existence of the density is shown in Theorem 2 below, but can also be found similarly to Remark 2 in [21].

**Theorem 1.** If $X \in \mathcal{D}(d, \alpha, \sigma)$, then there exists a random vector $M_{\alpha, \sigma}$ on $\mathbb{H}^+$ with density $p_{M_{\alpha, \sigma}}(v)$ such that, for all $v \in \mathbb{R}^d$,
uniformly in $x$ with $x_1 \leq \delta_n c_n$, $\delta_n \to 0$.

Our first result is an analogue of the classical local limit theorem, which is an extension of Theorems 3 and 5 in \[21\] when $x = 0$ and extends \[10\] and \[3\] for arbitrary starting point $x$.

**Theorem 2.** Suppose $X \in D(d, \alpha, \sigma)$. If the distribution of $X$ is non-lattice then, for every $r > 0$, uniformly in $x$ with $x_1 \leq \delta_n c_n$, $\delta_n \to 0$,

$$
sup_{y \in \mathbb{H}^+} \left| c_n^d \mathbb{P}(x + S(n) \in y + r\Delta \mid \tau_x > n) - r^d p_{M_n,\sigma} \left( \frac{y - x}{c_n} \right) \right| \to 0. \quad (10)
$$

If the distribution of $X$ is lattice and if $\mathbb{Z}^d$ is the minimal lattice for $X$ then, uniformly in $x \in \mathbb{H}^+ \cap \mathbb{Z}^d$ with $x_1 \leq \delta_n c_n$, $\delta_n \to 0$,

$$
sup_{y \in \mathbb{H}^+ \cap \mathbb{Z}^d} \left| c_n^d \mathbb{P}(x + S(n) = y \mid \tau_x > n) - p_{M_n,\sigma} \left( \frac{y - x}{c_n} \right) \right| \to 0. \quad (11)
$$

If the ratio $y/c_n$ varies with $n$ in such a way that $y_1/c_n \in (b_1, b_2)$ for some $0 < b_1 < b_2 < \infty$ and $|y_{(2,d)}| = O(c_n)$, we can rewrite \[10\] as

$$
c_n^d \mathbb{P}(S(n) \in y + r\Delta \mid \tau > n) \sim r^d p_{M_n,\sigma}(y/c_n) \quad \text{as } n \to \infty.
$$

However, if $y_1/c_n \to 0$, then, in view of

$$
\lim_{x_1 \to 0} p_{M_n,\sigma}(x) = 0,
$$

relation \[10\] gives only

$$
c_n^d \mathbb{P}(S(n) \in y + \Delta \mid \tau > n) = o(1) \quad \text{as } n \to \infty. \quad (12)
$$

Our next theorem refines \[12\] in the mentioned domain of small deviations, i.e. when $y_1/c_n \to 0$. Let

$$
\tau^+ := \min\{k \geq 1: S(k) \in \mathbb{H}^+\} = \min\{k \geq 1: S_1(k) > 0\}.
$$

Let $\chi^+ := S_1(\tau^+)(\chi^- := -S_1(\tau))$ be the first ascending(descending) ladder height and let $(\chi_n^+|_{n=1}^{\infty}, (\chi_n^-|_{n=1}^{\infty})$ be a sequence of i.i.d. copies of $\chi^+(-\chi^-)$. Let

$$
H(u) := \mathbb{I}\{u > 0\} + \sum_{k=1}^{\infty} \mathbb{P}(\chi_1^+ + \ldots + \chi_k^+ < u) \quad (13)
$$

$$
V(u) := \mathbb{I}\{u \geq 0\} + \sum_{k=1}^{\infty} \mathbb{P}(\chi_1^- + \ldots + \chi_k^- \leq u) \quad (14)
$$

be the renewal function of the ascending (descending) ladder height process. Clearly, $H$ is a left-continuous function.

**Theorem 3.** Suppose $X \in D(d, \alpha, \sigma)$. If the distribution of $X$ is lattice and if $\mathbb{Z}^d$ is the minimal lattice, then

$$
\mathbb{P}(x + S(n) = y; \tau_x > n) \sim V(x_1) H(y_1) \frac{g \left( 0, \frac{y_2,d - x_2,d}{nc_n} \right)}{nc_n} \quad (15)
$$

uniformly in $x, y \in \mathbb{H}^+ \cap \mathbb{Z}^d$ with $x_1, y_1 \in (0, \delta_n c_n)$ such that $|x - y| \leq A c_n$, where $\delta_n \to 0$ as $n \to \infty$ and $A$ is a fixed constant.
If the distribution of $X$ is non-lattice then

$$P(x + S(n) \in y + \Delta; \tau_x > n) \sim V(x_1) \int_{y_1}^{y_1 + 1} H(u)du \frac{g \left(0, \frac{y_2.d - x_2.d}{c_n} \right)}{nc_n^d}$$

uniformly in $x_1, y_1 \in (0, \delta_n c_n]$ such that $|x - y| \leq Ac_n$, where $\delta_n \to 0$ as $n \to \infty$ and $A$ is a fixed constant.

To obtain the asymptotics for the Green function of $S(n)$ killed at leaving $\mathbb{H}^+$ one has to estimate probabilities of local large deviations for $S(n)$. To this end we assume that there exists a regularly varying $\phi$ of index $-\alpha$ such that

$$P(X \in x + \Delta) \leq \frac{\phi(|x|)}{|x|^d} =: g(|x|).$$

and

$$a_1 \phi(t) \leq P(|X| > t) \leq a_2 \phi(t), \quad t > 0,$$  

for some positive constants $a_1, a_2$. The fact that global assumptions might in general give different asymptotics if the tails are too heavy is known in the multidimensional case since Williamson [22] who constructed a counterexample for the Green functions on the whole space, which has a different asymptotic behaviour without any local assumptions.

We now ready to formulate our bound for local large deviations.

**Theorem 4.** If (15) and (16) hold with some $\alpha < 2$ then

$$p_n(x, y) \leq C_0 g(|x - y|)H(y_1 + 1)V(x_1 + 1).$$

Having asymptotics for $p_n(x, y)$ for all possible ranges (for small, normal and large deviations), one can easily derive asymptotics for the Green function. We start with the lattice case, where we combine Theorem 3 and Theorem 4 to obtain the asymptotics of the Green function near the boundary.

**Theorem 5.** Assume $X \in D(d, \alpha, \sigma)$ with some $\alpha < 2$. Suppose there exists a regularly varying $\phi$ such that (15) and (16) hold.

(i) If the distribution of $X$ is lattice and if $\mathbb{Z}^d$ is minimal for $X$ then we have

$$G(x, y) \sim C \frac{H(y_1)V(x_1)}{|x - y|^d} \int_0^\infty g_{\alpha, \sigma} \left(0, \frac{y_2.d - x_2.d}{|x - y|} \right) t^{d-1}dt$$

for $x_1, y_1 = o(|x - y|)$. In particular, in the isotopic case, that is when the limiting density $\sigma$ is uniform on the unit sphere,

$$G(x, y) \sim C_{\alpha} \frac{H(y_1)V(x_1)}{|x - y|^d}, \quad |x - y| \to \infty,$$

for $x_1, y_1 = o(|x - y|)$.

(ii) If $X$ is non-lattice then

$$G(x, y) \sim C \frac{\int_{y_1}^{y_1 + 1} H(u)du V(x_1)}{|x - y|^d} \int_0^\infty g_{\alpha, \sigma} \left(0, \frac{y_2.d - x_2.d}{|x - y|} \right) t^{d-1}dt$$

for $x_1, y_1 = o(|x - y|)$. In particular, in the isotopic case, that is when the limiting density $\sigma$ is uniform on the unit sphere,

$$G(x, y) \sim C \frac{\int_{y_1}^{y_1 + 1} H(u)du V(x_1)}{|x - y|^d}, \quad |x - y| \to \infty,$$
for \( x_1, y_1 = o(|x - y|) \).

(iii) In addition, there exists a constant \( C \) such that for all \( x, y \in \mathbb{H}^+ \),

\[
G(x, y) \leq C \frac{H(y_1)V(x_1)}{|x - y|^d}.
\] (18)

**Remark 6.** For stable Lévy processes an exact formula (which can be analysed asymptotically) for the Green function \( g(x, y) \) was obtained in [19]. We are not aware of any result of this kind for asymptotically stable random walks when \( \alpha < 2 \).

**Remark 7.** As we have already mentioned, walks with finite variances, which are a particular case of asymptotically stable walks with \( \alpha = 2 \), were considered in [13, Theorem 2] and [20] in the lattice case. In [13] estimates of the behaviour of the Green function relied on [5, 12] and were obtained in a more general situation of convex cones. Using our methods we have obtained asymptotics for the Green function in a half space for all asymptotically stable walks with \( \alpha = 2 \). Specialising this result to the case of finite variances, we can obtain asymptotics under weaker moment conditions than in [13] and stronger than in [20]. The method we use is different from [20] who approached the problem using the potential kernel.

The only difference between \( \alpha < 2 \) and \( \alpha = 2 \) are estimates for local large deviations. In the case \( \alpha = 2 \) one has to take care of the Gaussian component, what leads to very long calculations. For that reason we have decided not to include walks with \( \alpha = 2 \) in the present paper and to consider this case in a separate paper.

**Remark 8.** It seems to be possible to extend the estimates for the Green function for asymptotically isotropic random walk in cones. This will be considered elsewhere and should also allow one to extend the results of [4, 5, 7, 8, 12, 13] to the stable case.

Since exit times from a half space can be considered as exit times for one-dimensional random walks, it is quite natural to use the methods, which are typical for walks on the real line. In the proofs of our results on the asymptotic behaviour of the probabilities we follow this strategy and use a lot methods from [10] and [21]. However it is worth mentioning that additional dimensions cause many additional technical problems, because of the possibility of big jumps of random walk which are still close to the boundary hyperplane \( \{ x : x_1 = 0 \} \). While in the finite variance case one can try to control these jumps by assuming existence of additional moments, this is not possible in the infinite variance case. For these reasons our estimates for \( p_n(x, y) \) can not be considered as a straight forward generalisation of one-dimensional results.

## 2. Preliminary upper bounds

In this section we find bounds for \( \mathbb{P}(\tau_x > n) \). Since \( \tau_x \) is actually a stopping time for the one-dimensional walk \( S_1(k) \), we may apply Lemma 3 from [6], which gives us the following estimate.

**Lemma 9.** Assume that [6] holds. Then there exists \( C_0 \) such that for every \( x \in \mathbb{H}^+ \) one has

\[
\frac{\mathbb{P}(\tau_x > n)}{\mathbb{P}(\tau_0 > n)} \leq C_0 V(x_1), \quad n \geq 1.
\] (19)
Next lemma is an extension of [21, Lemma 20] to the case of half spaces. We will give a proof following a different approach, which relies on Lemma 9. This proof works in one-dimensional case as well, thus simplifying the corresponding arguments of [21].

For \( y = (y_1, \ldots, y_d), z = (z_1, \ldots, z_d) \in \mathbb{R}^d \) we will write \( y \leq z \) if \( y_k \leq z_k \) for all \( 1 \leq k \leq d \).

**Lemma 10.** Assume that \( X \in D(d, \alpha, \sigma) \). Then, there exists \( C > 0 \) such that for all \( x, y \in \mathbb{H}^+ \) and all \( n \geq 1 \) we have

\[
p_n(x, y) \leq \frac{CV(x_1)H(y_1)}{n c_d n}.
\]

Similar result holds for the stopping time \( \tau^+ \):

\[
\mathbf{P}(S(n) \in x + \Delta; \tau^+ > n) \leq \frac{CV(x_1)}{n c_d n}.
\]

**Proof.** We prove the first statement only. The proof of the second estimate requires only notational changes.

Put \( n_1 = \lfloor n/4 \rfloor, n_2 = \lfloor 3n/4 \rfloor - n_1, n_3 = n - \lfloor 3n/4 \rfloor \). We split the probability of interest into 3 parts,

\[
p_n(x, y) = \mathbf{P}(\tau > n_1, x + S(n_1) \in du) \int_{\mathbb{H}^+} \mathbf{P}(\tau > n_2, u + S(n_2) \in dz) \times \mathbf{P}(\tau > n_3, z + S(n_3) \in y + \Delta).
\]

Now we will make use of the time inversion. Let

\[
\tilde{X}(n) = -X(1), \tilde{X}(n-1) = -X(2), \ldots, \tilde{X}(1) = -X(n)
\]

and

\[
\tilde{S}(k) = \tilde{X}(1) + \cdots + \tilde{X}(k) = -X(n) - \cdots - X(n-k+1) = S(n-k) - S(n), \quad k = 1, \ldots, n.
\]

Let \( 1_d = (1, \ldots, 1) \). Then,

\[
\mathbf{P}(z + S(n_3) \in y + \Delta; \tau > n_3) = \mathbf{P}(z + S(n_3) \in y + \Delta; z_1 + \min(S_1(1), \ldots, S_1(n_3)) > 0) = \mathbf{P}(y + 1_d + \tilde{S}(n_3) \in z + (0, 1]^d; z_1 + \min(\tilde{S}_1(n_3 - 1), \ldots, \tilde{S}_1(1)) > \tilde{S}_1(n_3)) \leq \mathbf{P}(y + 1_d + \tilde{S}(n_3) \in z + (0, 1]^d; \tilde{y} + 1_d > n_3).
\]
Then, using the concentration function inequalities, we can continue as follows,

$$\int_{\mathbb{H}^+} P(\tau_u > n_2, u + S(n_2) \in dz) P(\tau_2 > n_3, z + S(n_3) \in y + \Delta)$$

$$\leq \int_{\mathbb{H}^+} P(\tau_u > n_2, u + S(n_2) \in dz) P(y + 1d + \tilde{S}(n_3) \in z + (0, 1]^d, \tilde{\tau}_{y+1d} > n_3)$$

$$\leq \sum_{j_1=0}^{\infty} \sum_{j_2=-\infty}^{\infty} \sum_{j_3=-\infty}^{\infty} P(\tau_u > n_2, u + S(n_2) \in [j_1, j_1 + 1) \times \ldots [j_d, j_d + 1))$$

$$\times P(y + 1d + \tilde{S}(n_3) \in (j_1, \ldots, j_d + 2\Delta, \tilde{\tau}_{y+1d} > n_3)$$

$$\leq \frac{C}{c_n^d} \sum_{j_1=0}^{\infty} \sum_{j_2=-\infty}^{\infty} \sum_{j_3=-\infty}^{\infty} P(y + 1d + \tilde{S}(n_3) \in (j_1, \ldots, j_d + 2\Delta, \tilde{\tau}_{y+1d} > n_3)$$

$$\leq \frac{Cd}{c_n^d} P(\tilde{\tau}_{y+1d} > n_3).$$

As a result we obtain the bound

$$p_n(x, y) \leq \frac{C}{c_n^d} P(\tau_x > n_1) P(\tilde{\tau}_{y+1d} > n_3).$$

Applying Lemma 9 we obtain

$$p_n(x, y) \leq \frac{CH(y_1)}{c_n^d} P(\tau_x > n_1) P(\tilde{\tau}_0 > n_3)$$

$$\leq \frac{CH(y_1) V(x_1)}{c_n^d} P(\tau_0 > n_1) P(\tau_0^+ > n_3).$$

Here recall that one can deduce by Rogozin’s result that [11] holds if and only if there exists a function \( l(n) \), slowly varying at infinity, such that, as \( n \to \infty \),

$$P(\tau > n) \sim \frac{l(n)}{n^{1-p}}, \quad P(\tau^+ > n) \sim \frac{1}{l(\rho) \Gamma(1-\rho) \eta l(n)},$$

$$\quad (21)$$

Then we obtain that \( P(\tau_0 > n) P(\tau_0^+ > n) \sim \frac{c}{n} \) and arrive at the conclusion. □

**Lemma 11.** Assume that the random walk \( S(n) \) is asymptotically stable. Then, there exists \( C > 0 \) such that for \( x, y \in \mathbb{H}^+ \) and all \( n \geq 1 \)

$$p_n(x, y) \leq \frac{CV(x_1) l(n)}{c_n^d n^{1-p}}. \quad (22)$$

**Proof.** For \( n \geq 2 \),

$$p_n(x, y) \leq P(\tau_x > n/2) \sup_{z \in \mathbb{R}^d} P(S_{[n/2]} \in z + \Delta).$$

Applying now a concentration function inequality we obtain

$$p_n(x, y) \leq P(\tau_x > n/2) \frac{C_1}{c_n^d} \leq \frac{CV(x_1) l(n)}{c_n^d n^{1-p}},$$

since

$$P(\tau_x > n) \leq CV(x_1) P(\tau > n) \leq CV(x_1) \frac{l(n)}{n^{1-p}}. \quad \Box$$

Before proving the next lemma recall the following result, see [21] Lemma 13.
Lemma 12. Suppose $X_1 \in D(\alpha, \beta)$. Then, as $u \to \infty$,
\[ H(u) \sim \frac{u^{\alpha\rho}l_2(u)}{\Gamma(1-\alpha\rho)\Gamma(1+\alpha\rho)} \]  
(23)
if $\alpha\rho < 1$, and
\[ H(u) \sim ul_3(u) \]  
(24)
if $\alpha\rho = 1$, where
\[ l_3(u) := \left( \int_0^u P(\chi^+ > y) \, dy \right)^{-1}, \quad u > 0. \]

In addition, there exists a constant $C > 0$ such that, in both cases,
\[ H(cn) \sim Cn \lambda_p(c_n) \]  
(25)
as $n \to \infty$.

Lemma 13. There exists a constant $C \in (0, \infty)$ such that, for all $z \in [0, \infty) \times \mathbb{R}^{d-1}$,
\[ \lim_{\varepsilon \downarrow 0} \varepsilon^{-d}P(M_{\alpha,\sigma} \in z + \varepsilon \Delta) \leq C \min\{1, (z_1)^{\alpha\rho}\}. \]

In particular,
\[ \lim_{\varepsilon \downarrow 0} \limsup_{\varepsilon \downarrow 0} \varepsilon^{-d}P(M_{\alpha,\sigma} \in [z, z + \varepsilon]) = 0. \]

Proof. For all $z \in [0, \infty) \times \mathbb{R}^{d-1}$, and all $\varepsilon > 0$ we have
\[ P(M_{\alpha,\sigma} \in z + \varepsilon \Delta) \leq \limsup_{n \to \infty} \lambda_p(S(n) \in c_n z + \varepsilon c_n \Delta | \tau > n) . \]

Applying (22) gives
\[ P(S(n) \in c_n z + \varepsilon c_n \Delta | \tau > n) \leq C \frac{H(c_n)}{nc_n^d \lambda_p(\tau > n)} (\varepsilon c_n)^d . \]

Recalling that $H(x)$ is regularly varying with index $\alpha\rho$ by Lemma 12 and taking into account (23), we get
\[ P(S(n) \in c_n z + \varepsilon c_n \Delta | \tau > n) \leq C\varepsilon^d \min\{1, (z_1 + \varepsilon)^{\alpha\rho}\} \frac{H(c_n)}{n \lambda_p(\tau > n)} \leq C\varepsilon^d \min\{1, (z_1 + \varepsilon)^{\alpha\rho}\}. \]

Consequently,
\[ P(M_{\alpha,\sigma} \in z + \varepsilon \Delta) \leq C\varepsilon^d \min\{1, (z_1 + \varepsilon)^{\alpha\rho}\}. \]  
(26)

This inequality shows that there exists a constant $C \in (0, \infty)$ such that
\[ \limsup_{\varepsilon \downarrow 0} \varepsilon^{-d}P(M_{\alpha,\sigma} \in z + \varepsilon \Delta) \leq C \min\{1, (z_1)^{\alpha\rho}\}, \text{ for all } z \in [0, \infty) \times \mathbb{R}^{d-1}, \]
as desired. \hfill \Box

Corollary 14. Let $X \in D(d, \alpha, \sigma)$. There exists a constant $C \in (0, \infty)$ such that, for all $n \geq 1$ and $x, y \in \mathbb{H}^+$,
\[ p_n(x, y) \leq CV(x_1) H(c_n, y_1) \frac{H(c_n, y_1)}{n c_n^d}. \]  
(27)

Proof. The desired estimates follow from (25) and Lemmata 10 and 11. \hfill \Box
3. Baxter-Spitzer identity

We will need the following multidimensional extension of one-dimensional Baxter-Spitzer identity, see [18, Lemma 3.2].

**Lemma 15.** For \( t \in \mathbb{R}^d \) and \( |s| < 1 \) the following identity

\[
1 + \sum_{n=1}^{\infty} s^n E[e^{i t \cdot S(n)}; \tau_0 > n] = \exp \left\{ \sum_{n=1}^{\infty} s^n E \left[ e^{i t \cdot S(n)}; S_1(n) > 0 \right] \right\}.
\]

We will now follow closely [21]. Put

\( B_n(y) := P(S(n) < y; \tau_x > n) \).

and \( b_n(y) := p_n(0, y) \). Lemma 15 of [21] extends as follows

**Lemma 16.** The sequence of functions \( \{ B_n(y), n \geq 1 \} \) satisfies the recurrence equations

\[
nB_n(y) = P(S(n) < y, S_1(n) > 0) + \sum_{k=1}^{n-1} \int_{\mathbb{R}^d} P(S(k) < y - z, S_1(k) > 0) dB_{n-k}(z) \tag{28}
\]

and

\[
nB_n(y) = P(S(n) < y, S_1(n) > 0) + \sum_{k=1}^{n-1} \int_{\mathbb{R}^d} B_{n-k}(y - z) P(S(k) \in dz, S_1(k) > 0) \tag{29}
\]

The proof is analogous to the proof of Lemma 15 of [21].

To deal with random walks started at an arbitrary point we will prove Lemma 17 below that extends (17) in [10]. Put

\[
p_n^+(0, dy) := P(S(n) \in dy, \tau_x > n).
\]

We will slightly abuse the notation and write

\[
p_n(x, dy) = P(x + S(n) \in dy, \tau_x > n).
\]

**Lemma 17.** For \( x \in \mathbb{H}^+, y \in \mathbb{H}^+ \) we have

\[
p_n(x, dy) = \sum_{k=0}^n \int_{(0, x_1 \wedge y_1) \times \mathbb{R}^{d-1}} p_n^+(0, dz - x) p_{n-k}(0, dy - z) \tag{30}
\]

and for \( x \in \mathbb{H}^+ \cap \mathbb{Z}, y \in \mathbb{H}^+ \cap \mathbb{Z} \)

\[
p_n(x, y) = \sum_{k=0}^n \int_{(0, x_1 \wedge (y_1 + 1)) \times \mathbb{R}^{d-1}} p_n^+(0, dz - x) b_{n-k}(y - z) \tag{31}
\]
Proof. Decomposing the trajectory of the walk at the minimum of the first coordinate and using the duality lemma for random walks, we get

\[
p_n(x, dy) = \sum_{k=0}^{n} \int_{(0, x_1 \wedge y_1) \times \mathbb{R}^{d-1}} P(x + S(k) \in dz, S_1(k) \leq \min_{j \leq k} S_1(j)) \times P(z + S(n-k) \in dy, \tau_0 > n-k)
\]

\[
= \sum_{k=0}^{n} \int_{(0, x_1 \wedge y_1) \times \mathbb{R}^{d-1}} P(x + S(k) \in dz, \tau^+ > k) \times P(z + S(n-k) \in dy, \tau_0 > n-k)
\]

\[
= \sum_{k=0}^{n} \int_{(0, x_1 \wedge y_1) \times \mathbb{R}^{d-1}} p_k^+(0, dz-x)p_{n-k}(0, dy-z).
\]

Integrating (30), we obtain the second equality (31). \qed

4. Probabilities of normal deviations: proof of Theorem 2 for \( x = 0 \).

Let \( H_{y_1}^+ = \{(z_1, z_2, \ldots, z_d) : 0 < z_1 < y_1 \} \). It follows from (28) that

\[
nb_n(y) = P(S(n) \in y + \Delta) + \sum_{k=1}^{n-1} \int_{\mathbb{R}^d} P(S(k) \in y - z + \Delta, S_1(k) > 0)dB_{n-k}(z)
\]

\[
:= R_{\varepsilon_1}^{(0)}(y) + R_{\varepsilon_1}^{(1)}(y) + R_{\varepsilon_1}^{(2)}(y) + R_{\varepsilon_1}^{(3)}(y),
\]

(32)

where, for any fix \( \varepsilon \in (0, 1/2) \) and with a slight abuse of notation,

\[
R_{\varepsilon_1}^{(1)}(y) := \sum_{k=1}^{\varepsilon n} \int_{H_{y_1}^+} P(S(k) \in y - z + \Delta, S_1(k) > 0)dB_{n-k}(z),
\]

\[
R_{\varepsilon_1}^{(2)}(y) := \sum_{k=\varepsilon n}^{(1-\varepsilon)n} \int_{H_{y_1}^+} P(S(k) \in y - z + \Delta, S_1(k) > 0)dB_{n-k}(z),
\]

\[
R_{\varepsilon_1}^{(3)}(y) := P(S_n \in y + \Delta) + \sum_{k=\lceil(1-\varepsilon)n\rceil+1}^{n-1} \int_{\mathbb{R}^d} P(S(k) \in y - z + \Delta, S_1(k) > 0)dB_{n-k}(z)
\]

and

\[
R_{\varepsilon_1}^{(0)}(y) := \sum_{k=1}^{(1-\varepsilon)n} \int_{[y_1, y_1+1) \times \mathbb{R}^{d-1}} P(S(k) \in y - z + \Delta, S_1(k) > 0)dB_{n-k}(z).
\]

Fix some \( t \in \mathbb{Z}^d \). If \( z \in (t + \Delta) \) then

\[
\{S(k) \in y - z + \Delta\} = \{S_j(k) \in [y_j - z_j, y_j - z_j + 1) \text{ for all } j\}
\]

\[
\subseteq \{S_j(k) \in [y_j - t_j - 1, y_j - t_j + 1) \text{ for all } j\}
\]

\[
= \{S(k) \in y - t - 1 + 2\Delta\}.
\]

Consequently,

\[
P(S(k) \in y - z + \Delta, S_1(k) > 0) \leq P(S(k) \in y - t - 1 + 2\Delta, S_1(k) > 0)
\]
for all \( z \in t + \Delta \). Applying this estimate, we conclude that, for every \( A \subset \mathbb{R}^d \),
\[
\int_A \mathbb{P}(S(k) \in y - z + \Delta, S_1(k) > 0)dB_{n-k}(z)
\leq \sum_{t \in \mathbb{Z}^d : (t + \Delta) \cap A \neq \emptyset} \mathbb{P}(S(k) \in y - t - 1 + 2\Delta, S_1(k) > 0)\beta_{n-k}(t).
\]

Combining this estimate with Corollary 14 with \( x = 0 \), we obtain
\[
\int_A \mathbb{P}(S(k) \in y - z + \Delta, S_1(k) > 0)dB_{n-k}(z)
\leq \sum_{t \in \mathbb{Z}^d : (t + \Delta) \cap A \neq \emptyset} H(t_1 \wedge c_{n-k}) \mathbb{P}(S(k) \in y - t - 1 + 2\Delta, S_1(k) > 0).
\]

In order to bound \( R_x^{(0)}(y) \) we apply (33) with \( A = [y_1, y_1 + 1) \times \mathbb{R}^{d-1} \):
\[
R_x^{(0)}(y) 
\leq \sum_{k=1} (1-\varepsilon)n \frac{C}{(n-k)c_n^{d-k}} \sum_{t \in \mathbb{Z}^{d-1} : t_1 \in [y_1-1, y_1+1)} H(t_1 \wedge c_{n-k}) \mathbb{P}(S(k) \in y - t - 1 + 2\Delta, S_1(k) > 0)
\leq C\varepsilon \frac{H(y_1 \wedge c_n)}{nc_n^{d-1}} \sum_{k=1} (1-\varepsilon)n \mathbb{P}(S_1(k) \in (0, 2)).
\]

Noting that \( \mathbb{P}(S_1(k) \in (0, 2)) \to 0 \) as \( k \to \infty \), we get
\[
\frac{c_n^{d}}{H(c_n)} R_x^{(0)}(y) \to 0.
\]

Taking into account (26), we conclude that
\[
\limsup_{n \to \infty} \frac{c_n^{d}}{n\mathbb{P}(\tau > n)} R_x^{(0)}(y) = 0. \tag{34}
\]

Using the Stone theorem we obtain
\[
R_x^{(3)}(y) \leq C \frac{c_n^{d}}{c_n^{d}} \left( 1 + \sum_{k=1}^{\varepsilon n} \mathbb{P}(\tau > k) \right).
\]

Further, by (21),
\[
\sum_{k=0}^{\varepsilon n} \mathbb{P}(\tau > k) \sim \rho^{-1} \varepsilon \rho n \mathbb{P}(\tau > n) \quad \text{as} \quad n \to \infty.
\]

As a result, we obtain
\[
\limsup_{n \to \infty} \frac{c_n^{d}}{n\mathbb{P}(\tau > n)} \sup_{y \in \mathbb{R}^d} R_x^{(3)}(y) \leq C\varepsilon \rho. \tag{35}
\]

Applying (33) with \( A = [0, y_1) \times \mathbb{R}^{d-1} \), we get
\[
R_x^{(1)}(y) \leq \frac{C H(c_n)}{n c_n^{d}} \sum_{k=1}^{\varepsilon n} \mathbb{P}(S_1(k) \in (0, y_1 + 2)) \leq \varepsilon \frac{C H(c_n)}{c_n^{d}}
\]
From this estimate and (25) we deduce

$$\limsup_{n \to \infty} \frac{c_d n}{nP(\tau > n)} \sup_{y \in \mathbb{H}^+} R_{\varepsilon}^{(1)}(y) \leq C\varepsilon. \quad (36)$$

Thus, in the non-lattice case we combine the Stone local limit theorem with the first equality in (9) and obtain, uniformly in $y \in \mathbb{H}^+$,

$$R_{\varepsilon}^{(2)}(y) = \sum_{k=\lfloor c_n \rfloor + 1}^{(1-\varepsilon)n} \frac{1}{c_d n - k} \int_0^{y_1} \int_{\mathbb{R}^{d-1}} g_{\alpha,\sigma} \left( \frac{y - z}{c_{n-k}} \right) dB_k(z) + o \left( \frac{1}{c_d n \varepsilon} \sum_{k=1}^n P(S_1(k) < y_1, \tau > k) \right)$$

$$= \sum_{k=\lfloor c_n \rfloor + 1}^{(1-\varepsilon)n} \frac{P(\tau > k)}{c_d n - k} \int_0^{y_1 / c_k} \int_{\mathbb{R}^{d-1}} g_{\alpha,\sigma} \left( \frac{y - c_k u}{c_{n-k}} \right) P(M_{\alpha,\sigma} \in du)$$

$$+ o \left( \frac{1}{c_d n \varepsilon} \sum_{k=1}^n P(S_1(k) < y_1, \tau > k) + \sum_{k=1}^{n-1} P(\tau > k) \right).$$

According to (21),

$$\sum_{k=1}^n P(S^{(1)} < y_1, \tau > k) \leq \sum_{k=1}^n P(\tau > k) \leq CnP(\tau > n).$$

Hence,

$$R_{\varepsilon}^{(2)}(y) = \sum_{k=\lfloor c_n \rfloor + 1}^{(1-\varepsilon)n} \frac{P(\tau > k)}{c_d n - k} \int_0^{x_1 / c_k} \int_{\mathbb{R}^{d-1}} g_{\alpha,\sigma} \left( \frac{x - c_k u}{c_{n-k}} \right) P(M_{\alpha,\sigma} \in du)$$

$$+ o \left( \frac{nP(\tau > n)}{c_d n \varepsilon} \right).$$

Since $c_k$ and $P(\tau > k)$ are regularly varying and $g_{\alpha,\sigma}(x)$ is uniformly continuous in $\mathbb{R}^d$, we let, for brevity, $v = x / c_n$ and continue the previous estimates for $R_{\varepsilon}^{(2)}(y)$ with

$$= \frac{P(\tau > n)}{c_d n} \sum_{k=\lfloor c_n \rfloor + 1}^{(1-\varepsilon)n} \frac{(k/n)^{\alpha-1}}{(1 - k/n)^{1/\alpha}} \int_0^{v^{(1)/(k/n)^{1/\alpha}}} \int_{\mathbb{R}^{d-1}} g_{\alpha,\sigma} \left( \frac{v - (k/n)^{1/\alpha} u}{(1 - k/n)^{1/\alpha}} \right) P(M_{\alpha,\sigma} \in du)$$

$$+ o \left( \frac{nP(\tau > n)}{c_d n \varepsilon} \right)$$

$$= \frac{nP(\tau > n)}{c_d n} f(\varepsilon, 1 - \varepsilon; v) + o \left( \frac{nP(\tau > n)}{c_d n \varepsilon} \right),$$

where, for $0 \leq w_1 \leq w_2 \leq 1$,

$$f(w_1, w_2; v) := \int_{w_1}^{w_2} \frac{t^{\alpha-1} dt}{(1 - t)^{1/\alpha}} \int_0^{v^{(1)/t^{1/\alpha}}} \int_{\mathbb{R}^d} g_{\alpha,\sigma} \left( \frac{v - t^{1/\alpha} u}{(1 - t)^{1/\alpha}} \right) P(M_{\alpha,\sigma} \in du). \quad (37)$$

Observe that, by boundedness of $g_{\alpha,\sigma}(y)$,

$$f(0, \varepsilon; v) \leq C \int_0^\varepsilon t^{\alpha-1} dt \leq C\varepsilon^\alpha.$$
Combining (34) – (38) with representation (32) leads to

$$y \in \mathbb{R}^d$$

for all non-negative integrable function $$\phi$$. Therefore,

$$f(1 - \varepsilon, 1; v) \leq C \int_{1 - \varepsilon}^{1} t^{p-1/\alpha} dt \int_{0}^{v(1)/t^{1/\alpha}} g_{\alpha, \sigma}(v - t^{1/\alpha}u) \, du = \left( z = \frac{v - t^{1/\alpha}u}{(1 - t)^{1/\alpha}} \right)$$

As a result we have

$$\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^+} \left| \frac{c_n^{d}}{nP(\tau > n)} R^{(2)}_\varepsilon(y) - f(0, 1; y/c_n) \right| \leq C\varepsilon^p. \quad (38)$$

Combining (34) – (38) with representation (32) leads to

$$\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^+} \left| \frac{c_n^{d}}{P(\tau > n)} b_n(y) - f(0, 1; y/c_n) \right| \leq C\varepsilon^p. \quad (39)$$

Since $$\varepsilon > 0$$ is arbitrary, it follows that, as $$n \to \infty,$$

$$\frac{c_n^{d}}{P(\tau > n)} b_n(y) - f(0, 1; y/c_n) \to 0 \quad (40)$$

uniformly in $$y \in \mathbb{R}^+$$. Recalling (9), we deduce by integration of (40) and evident transformations that

$$\int_{u + \Delta} f(0, 1; z) \, dz = P(M_{\alpha, \sigma} \in u + r\Delta) \quad (41)$$

for all $$u \in \mathbb{R}^+, r > 0$$. This means, in particular, that the distribution of $$M_{\alpha, \sigma}$$ is absolutely continuous. Furthermore, it is not difficult to see that $$z \mapsto f(0, 1; z)$$ is a continuous mapping. Hence, in view of (11), we may consider $$f(0, 1; z)$$ as a continuous version of the density of the distribution of $$M_{\alpha, \sigma}$$ and let $$p_{M_{\alpha, \sigma}}(z) := f(0, 1; z)$$. This and (40) imply the statement of Theorem 2 for $$\Delta = [0, 1]^d$$. To establish the desired result for arbitrary $$r\Delta, r > 0$$ it suffices to consider the random walk $$S(n)/r$$ and to observe that

$$c'_n := \inf \left\{ u \geq 0 : \frac{1}{u^2} \int_{-u}^{u} x^2 P \left( \frac{|X|}{r} \in dx \right) \right\} = c_n/r^d.$$ 

5. Probabilities of normal deviations when random walks start at an arbitrary starting point

Proof of Theorem 4. Due to the shift invariance in any direction orthogonal to $$(1, 0, \ldots, 0)$$ we may consider, without loss of generality, only the case when the random walk starts at $$x = (x_1, 0, \ldots, 0)$$ with some $$x_1 > 0$$.

As we have already mentioned before, repeating the arguments from [9] one can easily show that $$P(\frac{S(n)}{cn} \in \cdot | \tau > n)$$ and $$P(\frac{S(n)}{cn} \in \cdot | \tau^+ > n)$$ converge weakly. Recall also that the limit of $$P(\frac{S(n)}{cn} \in \cdot | \tau > n)$$ is denoted by $$M_{\alpha, \sigma}$$. 
Fix an arbitrary Borel set \( A \subset \mathbb{H}^+ \). According to Lemma 17,
\[
\mathbb{P} \left( \frac{x + S(n)}{c_n} \in A; \tau > n \right) = \sum_{k=0}^{n} \int_{(0,x_1) \times \mathbb{R}^{d-1}} p_k^+(dz-x) \mathbb{P} \left( \frac{z + S(n-k)}{c_{n-k}} \in A; \tau > n-k \right). \tag{42}
\]
Choose now a sequence \( \{N_n\} \) of integers satisfying \( N_n = o(n) \) and \( \frac{\delta_n c_n}{c_{N_n}} \to 0 \). \tag{43}

We start our analysis of the sum in (42) by noting that
\[
\sum_{k=N_{n+1}}^{n/2} \int_{(0,x_1) \times \mathbb{R}^{d-1}} p_k^+(dz-x) \mathbb{P} \left( \frac{z + S(n-k)}{c_{n-k}} \in A; \tau > n-k \right) \leq \sum_{k=N_{n+1}}^{n/2} \mathbb{P}(S_1(k) \geq -x_1; \tau^+ > k) \mathbb{P}(\tau > n-k) \\
\leq \mathbb{P}(\tau > n/2) \sum_{k=N_{n+1}}^{n/2} \mathbb{P}(S_1(k) \geq -x_1; \tau^+ > k).
\]

Applying the second statement of Lemma 10 to the walk \( S_1(k) \) and recalling that the sequence \( \{c_k\} \) is regularly varying, we obtain
\[
\sum_{k=N_{n+1}}^{n/2} \mathbb{P}(S_1(k) \geq -x_1; \tau^+ > k) \leq \sum_{k=N_{n+1}}^{\infty} \mathbb{P}(S_1(k) \geq -x_1; \tau^+ > k) \leq \sum_{k=N_{n+1}}^{\infty} \frac{C_1 x_1 V(x_1)}{k c_k} \leq \frac{C_2 x_1 V(x_1)}{c_{N_n}} \leq \frac{C_2 \delta_n c_n V(x_1)}{c_{N_n}}. \tag{44}
\]

Taking into account (43), we conclude that
\[
\sum_{k=N_{n+1}}^{\infty} \mathbb{P}(S_1(k) \geq -x_1; \tau^+ > k) = o(V(x_1)) \tag{45}
\]
uniformly in \( x_1 \leq \delta_n c_n \). Consequently,
\[
\sum_{k=N_{n+1}}^{n/2} \int_{(0,x_1) \times \mathbb{R}^{d-1}} p_k^+(dz-x) \mathbb{P} \left( \frac{z + S(n-k)}{c_{n-k}} \in A; \tau > n-k \right) = o(V(x_1) \mathbb{P}(\tau > n)) \tag{46}
\]
uniformly in \( x_1 \leq \delta_n c_n \).
Using once again Lemma 10, we obtain

\[
\sum_{k=n/2}^{n} \int_{(0,x_1] \times \mathbb{R}^{d-1}} p_k^+ (dz - x) P \left( \frac{z + S(n - k)}{c_{n-k}} \in A; \tau > n - k \right)
\]

\[
\leq \sum_{k=n/2}^{n} P(S_1(k) \geq -x_1; \tau^+ > k) P(\tau > n - k)
\]

\[
\leq \sum_{k=n/2}^{n} \frac{C_1 x_1 V(x_1)}{kc_k} P(\tau > n - k) \leq \frac{C_2 x_1 V(x_1)}{nc_n} \sum_{j=0}^{n} P(\tau > j).
\]

Since \(P(\tau > j)\) is also regularly varying, we conclude that

\[
\sum_{k=n/2}^{n} \int_{(0,x_1] \times \mathbb{R}^{d-1}} p_k^+ (dz - x) P \left( \frac{z + S(n - k)}{c_{n-k}} \in A; \tau > n - k \right)
\]

\[
\leq C \frac{x_1 V(x_1)}{c_n} P(\tau > n) = o(V(x_1) P(\tau > n)) (47)
\]

uniformly in \(x_1 \leq \delta_n c_n\).

Choose now a sequence \(\varepsilon_n \to 0\) so that \(c_{N_n} = o(\varepsilon_n c_n)\). By the convergence in the case of start at zero,

\[
P \left( \frac{z + S(n - k)}{c_{n-k}} \in A; \tau > n - k \right) \sim P(M_{\alpha,\sigma} \in A) P(\tau > n)
\]

uniformly in \(k \leq N_n\) and \(z \in B_{\varepsilon_n c_n}(0)\), where \(B_r(y)\) denotes the ball of radius \(r\) with center at \(y\). Therefore,

\[
\sum_{k=0}^{N_n} \int_{(0,x_1] \times \mathbb{R}^{d-1} \cap B_{\varepsilon_n c_n}(0)} p_k^+ (dz - x) P \left( \frac{z + S(n - k)}{c_{n-k}} \in A; \tau > n - k \right)
\]

\[
= [P(M_{\alpha,\sigma} \in A) + o(1)] P(\tau > n) \sum_{k=0}^{N_n} P(S_1(k) \geq -x_1, |S(k)| < \varepsilon_n c_n; \tau^+ > k)
\]

\[
= [P(M_{\alpha,\sigma} \in A) + o(1)] P(\tau > n) \sum_{k=0}^{N_n} P(S_1(k) \geq -x_1; \tau^+ > k)
\]

\[
+ O \left( P(\tau > n) \sum_{k=0}^{N_n} P(S_1(k) \geq -x_1, |S(k)| \geq \varepsilon_n c_n; \tau^+ > k) \right).
\]

By the definition of \(V\),

\[
\sum_{k=0}^{N_n} P(S_1(k) \geq -x_1; \tau^+ > k) = V(x_1) - \sum_{k=N_n+1}^{\infty} P(S_1(k) \geq -x_1; \tau^+ > k).
\]
Using here (45), we obtain

\[
\sum_{k=0}^{N_n} \int_{(0, x_1] \times \mathbb{R}^{d-1} \cap B_{\varepsilon_n c_n}(0)} p_k^+(dz - x) P \left( \frac{z + S(n - k)}{c_{n-k}} \in A; \tau > n - k \right) = \left[ P(M_{\alpha, \sigma} \in A) + o(1) \right] V(x_1) P(\tau > n)
\]

\[+ O \left( P(\tau > n) \sum_{k=0}^{N_n} P(S_1(k) \geq -x_1, |S(k)| \geq \varepsilon_n c_n; \tau^+ > k) \right) \tag{48} \]

uniformly in \( x_1 \leq \delta_n c_n \).

Furthermore,

\[
\sum_{k=0}^{N_n} \int_{(0, x_1] \times \mathbb{R}^{d-1} \cap B_{\varepsilon_n c_n}(0)} p_k^+(dz - x) P \left( \frac{z + S(n - k)}{c_{n-k}} \in A; \tau > n - k \right)
\]

\[\leq \sum_{k=0}^{N_n} P(S_1(k) \geq -x_1, |S(k)| \geq \varepsilon_n c_n; \tau^+ > k) P(\tau > n - k) \]

\[\leq C P(\tau > n) \sum_{k=0}^{N_n} P(S_1(k) \geq -x_1, |S(k)| \geq \varepsilon_n c_n; \tau^+ > k). \tag{49} \]

Having all these estimates one can easily see that it suffices to show that, uniformly in \( x_1 \leq \delta_n c_n \),

\[
\sum_{k=0}^{N_n} P(S_1(k) \geq -x_1, |S(k)| \geq \varepsilon_n c_n; \tau^+ > k) = o(V(x_1)). \tag{50} \]

Indeed, applying (50) to (48) and (49) leads us to the equality

\[
\sum_{k=0}^{N_n} \int_{(0, x_1] \times \mathbb{R}^{d-1}} p_k^+(dz - x) P \left( \frac{z + S(n - k)}{c_{n-k}} \in A; \tau > n - k \right) = \left[ P(M_{\alpha, \sigma} \in A) + o(1) \right] V(x_1) P(\tau > n).
\]

Plugging this and estimates (46), (47) into (42), we get

P \left( \frac{x + S(n)}{c_n} \in A; \tau_x > n \right) = \left[ P(M_{\alpha, \sigma} \in A) + o(1) \right] V(x_1) P(\tau > n)

uniformly in \( x_1 \leq \delta_n c_n \). Recalling that \( P(\tau_x > n) \sim V(x_1) P(\tau > n) \), we have uniform in \( x_1 \leq \delta_n c_n \) convergence

P \left( \frac{x + S(n)}{c_n} \in A; \tau_x > n \right) \to P(M_{\alpha, \sigma} \in A). \]

To prove (51) we fix some \( R \geq 1 \) and notice that

\[
\sum_{k=0}^{N_n} P(S_1(k) \geq -x_1, |S(k)| \geq \varepsilon_n c_n; \tau^+ > k)
\]

\[\leq \sum_{k=0}^{R} P(|S(k)| \geq \varepsilon_n c_n; \tau^+ > k) + \sum_{k=R+1}^{N_n} P(S_1(k) \geq -x_1; \tau^+ > k). \]
Similar to (44),
\[ \sum_{k=R+1}^{N_n} \mathbb{P}(S_1(k) \geq -x_1; \tau^+ > k) \leq c_R \frac{x_1 V(x_1)}{c_R} . \]
Furthermore, using the convergence of measures \( \mathbb{P}(S(n) \in \cdot | \tau^+ > n) \), we have
\[ \sum_{k=0}^{R} \mathbb{P}(|S(k)| \geq \varepsilon_n c_n; \tau^+ > k) = o \left( R \sum_{k=0}^{R} \mathbb{P}(\tau^+ > k) \right) \]
uniformly in \( R \leq N_n \). Fix some \( \gamma > 0 \) and take \( c_R \) such that \( c_R - 1 < x_1 / \gamma \) and \( c_R \geq x_1 / \gamma \). Then
\[ \sum_{k=R+1}^{N_n} \mathbb{P}(S_1(k) \geq -x_1; |S(k)| \geq \varepsilon_n c_n; \tau^+ > k) \leq C \gamma V(x) \]
and, by Lemma 12,
\[ R \mathbb{P}(\tau^+ > R) \leq CV(c_R) \leq C \gamma^{-\alpha(1-\rho)} V(x_1) . \]
Combining these estimates, we conclude that
\[ \lim_{n \to \infty} \sup_{x_1 \leq \delta_n c_n} \frac{1}{V(x_1)} \sum_{k=0}^{N_n} \mathbb{P}(S_1(k) \geq -x_1, |S(k)| \geq \varepsilon_n c_n; \tau^+ > k) \leq C \gamma . \]
Since \( \gamma \) can be chosen arbitrary small we get (50). Thus, the proof of the theorem is complete. □

First proof of Theorem 2. The give a proof in the non-lattice case only. The lattice case is even simpler.
As in the proof of Theorem 1 it suffices to consider the case \( x = (x_1, 0, \ldots, 0) \) with \( x_1 \in (0, \delta_n c_n] \). The case \( x_1 = 0 \) has been considered in Section 4. There we have proven that, uniformly in \( y \in \mathbb{H}^+ \),
\[ b_n(y) \sim \frac{\mathbb{P}(\tau > n)}{c_n^d} p_{M_{a,n}}(y/c_n) . \] (51)
To generalize this relation to the case of positive \( x_1 \), we first notice that, by Lemma 17
\[ p_n(x, y) = \sum_{k=0}^{n} \int_{(0,x_1] \times \mathbb{R}^{d-1}} p^+_k(dz - x) b_{n-k}(y - z) . \] (52)
Fix some \( \gamma \in (0, 1/2) \). The analysis of
\[ \sum_{k=0}^{(1-\gamma)n} \int_{(0,x_1] \times \mathbb{R}^{d-1}} p^+_k(dz - x) b_{n-k}(y - z) \]
is very similar to our proof of Theorem 1. If \( k \leq (1-\gamma)n \) then we have the bound
\[ b_{n-k}(y - z) \leq C(\gamma) \frac{\mathbb{P}(\tau > n)}{c_n^d} . \]
which is an immediate consequence of (51). Using this uniform bound and the local limit theorem (51) directly, and repeating our arguments from the proof of Theorem 1, one can easily obtain

\[
\sup_y \left| P(\tau > n) \sum_{k=0}^{(1-\gamma)n} \int_{[0,x_1] \times \mathbb{R}^{d-1}} p^+_k(dz-x) b_{n-k}(y-z) - p^{M_{n,c}}(y/c_n) \right| \to 0
\]

uniformly in \( x_1 \leq \delta_n c_n \).

For \( k > (1-\gamma)n \) the mentioned above bound for \( b_{n-k} \) is useless, and one needs an additional argument. We first notice that

\[
\int_{[0,x_1] \times \mathbb{R}^{d-1}} p^+_k(dz-x) b_{n-k}(y-z)
\]

\[
\leq \sum_{u \in [0,x_1] \times \mathbb{R}^{d-1} \cap \mathbb{Z}^d} p^+_k(u-x+\Delta)(\max_{z \in u+\Delta} b_{n-k}(y-z))
\]

\[
\leq \sum_{u \in [0,x_1] \times \mathbb{R}^{d-1} \cap \mathbb{Z}^d} p^+_k(u-x+\Delta) P(S(n-k) \in y-u-1+2\Delta; \tau > n-k).
\]

Applying now the second statement of Lemma 10 we get

\[
\int_{[0,x_1] \times \mathbb{R}^{d-1}} p^+_k(dz-x) b_{n-k}(y-z)
\]

\[
\leq \frac{V(x_1)}{kc^d_k} \sum_{u \in [0,x_1] \times \mathbb{R}^{d-1} \cap \mathbb{Z}^d} P(S(n-k) \in y-u-1+2\Delta; \tau > n-k)
\]

\[
\leq \frac{V(x_1)}{kc^d_k} P(\tau > n-k).
\]

Consequently,

\[
\sum_{k=(1-\gamma)n}^{n} \int_{[0,x_1] \times \mathbb{R}^{d-1}} p^+_k(dz-x) b_{n-k}(y-z) \leq \frac{V(x_1)}{nc^d_n} \sum_{j=0}^{\gamma n} P(\tau > j).
\]

Noting now that the regular variation of \( P(\tau > j) \) implies

\[
\sum_{j=0}^{\gamma n} P(\tau > j) \leq C\gamma^{1-\rho} n P(\tau > n),
\]

we conclude that

\[
\lim_{n \to \infty} \frac{c^d_n}{P(\tau > n)} \sum_{k=(1-\gamma)n}^{n} \int_{[0,x_1] \times \mathbb{R}^{d-1}} p^+_k(dz-x) b_{n-k}(y-z) \leq C\gamma^{1-\rho} V(x_1)
\]

for all \( x_1 \leq \delta_n c_n \). Plugging this and (53) into (52) and letting \( \gamma \to 0 \), we get the desired result. \( \square \)

**Second proof of Theorem 2.** If the local assumption (15) holds then we can use an approach similar to that of in [5], see Theorems 5 and 6 there. This approach allows one to avoid considering first the special case \( x = 0 \), as it was done in Section 4. Without loss of generality we may assume that \( x = (x_1,0,\ldots,0) \).
Lemma 12, Combining this bound with the fact that $p$ where $I$ uniformly in $n$

Thus, uniformly in $y \in \mathbb{H}^+$ with $y \leq 2\varepsilon c_n$,

$$c_n^d P(x + S(n) \in y + \Delta | \tau_x > n) \leq C \varepsilon^{\alpha \rho}.$$ Combining this bound with the fact that $p_{M_{n,\sigma}}(z)$ to 0 as $z \to \partial \mathbb{H}^+$, we conclude that

$$\lim_{\varepsilon \to 0} \sup_{y_1 \in \mathbb{H}^+: y_1 \leq 2\varepsilon c_n} \left| c_n^d P(x + S(n) \in y + \Delta | \tau_x > n) - p_{M_{n,\sigma}} \left( \frac{y - x}{c_n} \right) \right| = 0 \quad (54)$$

uniformly in $x$ with $x_1 \leq \delta_n c_n$.

We next consider large values of $y$. More precisely, we assume that $|y| > 3A c_n$ with some $A > 1$. In this case we have, by the Markov property at time $m = [n/2]$, $p_{n}(x, y) = \int_{I(x)} P(x + S(m) \in dz, \tau_x > m) p_{n-m}(z, y)$ $+ \int_{\mathbb{H}^+ \setminus I(x)} P(x + S(m) \in dz, \tau_x > m) p_{n-m}(z, y)$, where $I(x) = \{ z : |z - x| \leq A c_n \}$. If $z \in I(x)$ then $|y - z| > A c_n$ for all sufficiently large values of $n$. Using (75), we have

$$p_{n-m}(z, y) \leq C \frac{n \phi(A c_n)}{c_n^d} \leq C A^{-\alpha} \frac{1}{c_n^d}.$$ Therefore,

$$c_n^d \int_{I(x)} P(x + S(m) \in dz, \tau_x > m) p_{n-m}(z, y) \leq C A^{-\alpha} P(\tau_x > m).$$

Furthermore, using the standard concentration function estimate, we have

$$c_n^d \int_{\mathbb{H}^+ \setminus I(x)} P(x + S(n) \in dz, \tau_x > m) p_{n-m}(z, y) \leq C P(x + S(m) \notin I(x); \tau_x > m) \leq C P(|S(m)| > A c_n; \tau_x > m).$$

Combining these bounds, one gets easily

$$c_n^d P(x + S(n) \in y + \delta | \tau_x > n) \leq C (A^{-\alpha} + P(|S(m)| > A c_n; \tau_x > m))\).$$ Applying now the integral limit theorem, we conclude that

$$\lim_{n \to \infty} \limsup_{y \in \mathbb{H}^+ : |y| > 3A c_n} \left| c_n^d P(x + S(n) \in y + \Delta | \tau_x > n) - p_{M_{n,\sigma}} \left( \frac{y - x}{c_n} \right) \right| = 0$$

uniformly in $x$ with $x_1 \leq \delta_n c_n$.

Thus, it remains to consider $y$ such that $y_1 > 2\varepsilon c_n$ and $|y| \leq 3A c_n$. To analyse this range of values of $y$ we set $m = [(1 - \gamma)n]$ with some $\gamma < 1/2$. Let $B_{c_n}(y)$
denote the ball of radius $\varepsilon c_n$ around $y$. Then, by the Markov property at time $m$, we have

$$p_n(x, y) = \int_{B_{\varepsilon c_n}(y)} \mathbb{P}(x + S(m) \in dz, \tau_x > m)p_{n-m}(z, y)$$

$$+ \int_{\mathbb{H}^+ \setminus B_{\varepsilon c_n}(y)} \mathbb{P}(x + S(m) \in dz, \tau_x > m)p_{n-m}(z, y).$$

Using the large deviations bound (76), one gets easily

$$\sup_{z \in \mathbb{H}^+ \setminus B_{\varepsilon c_n}(y)} p_{n-m}(z, y) \leq C \frac{(n-m)\phi(\varepsilon c_n)}{(\varepsilon c_n)^d} \leq C \gamma \varepsilon^{-d-\alpha} \varepsilon_{n-d}^d.$$ 

Consequently,

$$c_n^d \int_{\mathbb{H}^+ \setminus B_{\varepsilon c_n}(y)} \mathbb{P}(x + S(m) \in dz, \tau_x > m)p_{n-m}(z, y) \leq C \mathbb{P}(\tau_x > n) \gamma \varepsilon^{-d-\alpha}.$$  

(57)

For the integral over $B_{\varepsilon c_n}(y)$ we have

$$\int_{B_{\varepsilon c_n}(y)} \mathbb{P}(x + S(m) \in dz, \tau_x > m)p_{n-m}(z, y)$$

$$= \int_{B_{\varepsilon c_n}(y)} \mathbb{P}(x + S(m) \in dz, \tau_x > m)\mathbb{P}(z + S(n-m) \in y + \Delta)$$

$$+ \int_{B_{\varepsilon c_n}(y)} \mathbb{P}(x + S(m) \in dz, \tau_x > m)\mathbb{P}(z + S(n-m) \in y + \Delta; \tau_x \leq n-m).$$

By the strong Markov property,

$$\mathbb{P}(z + S(n-m) \in y + \Delta; \tau_x \leq n-m)$$

$$= \sum_{k=1}^{n-m-1} \int_{\mathbb{H}^-} \mathbb{P}(z + S(n) \in du, \tau_z = k)\mathbb{P}(u + S(n-m-k) \in y + \Delta).$$

Noting that $|y - u| > \varepsilon c_n$ for all $u \in \mathbb{H}^-$ and using once again (76), we obtain

$$c_n^d \mathbb{P}(z + S(n-m) \in y + \Delta; \tau_x \leq n-m) \leq C \gamma \varepsilon^{-d-\alpha}.$$ 

Consequently,

$$c_n^d \int_{B_{\varepsilon c_n}(y)} \mathbb{P}(x + S(m) \in dz, \tau_x > m)\mathbb{P}(z + S(n-m) \in y + \Delta; \tau_x \leq n-m)$$

$$\leq C \gamma \varepsilon^{-d-\alpha} \mathbb{P}(\tau_x > m).$$

(58)

By the Stone local limit theorem,

$$\int_{B_{\varepsilon c_n}(y)} \mathbb{P}(x + S(m) \in dz, \tau_x > m)\mathbb{P}(z + S(n-m) \in y + \Delta)$$

$$= \frac{1 + o(1)}{c_{n-m}^d} \int_{B_{\varepsilon c_n}(y)} \mathbb{P}(x + S(m) \in dz, \tau_x > m)g_{\alpha, \sigma} \left( \frac{y - z}{c_{n-m}} \right).$$
Recalling that $g_{\alpha,\sigma}$ is bounded, we then get

$$
\frac{c_n^d}{P(\tau_x > m)} \int_{B_{2c_n}(y)} P(x + S(m) \in dz, \tau_x > m) P(z + S(n - m) \in y + \Delta) = \gamma^{-d/\alpha} E \left[ g_{\alpha,\sigma} \left( \frac{y - x - S(m)}{c_{n-m}} \right) I \{ x + S(m) \in B_{c_n}(y) \} | \tau_x > m \right] + o(1).
$$

(59)

Applying now the integral limit theorem, we infer that

$$
E \left[ g_{\alpha,\sigma} \left( \frac{y - x - S(m)}{c_{n-m}} \right) I \{ x + S(m) \in B_{c_n}(y) \} | \tau_x > m \right]
= \gamma^{-d/\alpha} \left[ g_{\alpha,\sigma} \left( \frac{y - x}{\gamma^{1/\alpha} c_n} - \frac{(1 - \gamma)^{1/\alpha}}{\gamma^{1/\alpha}} M_{\alpha,\sigma} \right) I \{ M_{\alpha,\sigma} \in B_{\varepsilon(1-\gamma)^{-1/\alpha}} \left( \frac{y - x}{c_n} \right) \} \right] + o(1)
$$

$$
= \gamma^{-d/\alpha} \left[ g_{\alpha,\sigma} \left( \frac{y - x}{\gamma^{1/\alpha} c_n} - \frac{(1 - \gamma)^{1/\alpha}}{\gamma^{1/\alpha}} M_{\alpha,\sigma} \right) \right] I \{ M_{\alpha,\sigma} \notin B_{\varepsilon(1-\gamma)^{-1/\alpha}} \left( \frac{y - x}{c_n} \right) \} + o(1).
$$

(60)

Finally, we notice that, uniformly in $w \in \mathbb{H}^+$,

$$
E \left[ g_{\alpha,\sigma} \left( \frac{w}{\gamma^{1/\alpha} c_n} - \frac{(1 - \gamma)^{1/\alpha}}{\gamma^{1/\alpha}} M_{\alpha,\sigma} \right) \right] = \int_{\mathbb{R}^d} g_{\alpha,\sigma} \left( \frac{w}{\gamma^{1/\alpha} c_n} - \frac{(1 - \gamma)^{1/\alpha}}{\gamma^{1/\alpha}} u \right) p_{M_{\alpha,\sigma}}(u) du
$$

$$
= \gamma^{d/\alpha} \int_{\mathbb{R}^d} g_{\alpha,\sigma}(v) p_{M_{\alpha,\sigma}} \left( \frac{w - \gamma^{1/\alpha} u}{(1 - \gamma)^{1/\alpha}} \right) dv
$$

$$
= \gamma^{d/\alpha} \left( p_{M_{\alpha,\sigma}}(w) + o(1) \right) \text{ as } \gamma \to 0.
$$

(61)

Combining (58)–(61), we conclude that

$$
\lim_{\gamma \to 0} \limsup_{n \to \infty} \left| \frac{c_n^d}{P(\tau_x > m)} \int_{B_{2c_n}(y)} P(x + S(m) \in dz, \tau_x > m) p_{n-m}(z, y) \right.
$$

$$
- p_{M_{\alpha,\sigma}} \left( \frac{y - x}{c_n} \right) \right| = 0.
$$

From this relation and from (57), we get

$$
\sup_{y \in \mathbb{H}^+: y \leq 2c_n, |y| \leq 3Ac_n} \left| \frac{c_n^d}{P(x + S(n) \in y + \Delta | \tau_x > n)} - p_{M_{\alpha,\sigma}} \left( \frac{y - x}{c_n} \right) \right| \to 0
$$

uniformly in $x$ with $x_1 \leq \delta_n c_n$. This completes the proof of the theorem. □
6. Probabilities of Small Deviations When Random Walk Starts at the Origin

**Proposition 18.** Suppose $X \in D(d, \alpha, \sigma)$ and the distribution of $X$ is non-lattice. Then

$$c_n^d P(S(n) \in y + \Delta \mid \tau > n) \sim g_{\alpha, \sigma} \left(0, \frac{y_{(2,d)}}{c_n} \right) \frac{f_{y_1 + \Delta} H(u) du}{n P(\tau > n)}, \quad \text{as } n \to \infty \quad (62)$$

uniformly in $y_1 \in (0, \delta_n c_n]$, where $\delta_n \to 0$ as $n \to \infty$.

**Proof.** First, using once again (33), we get

$$R_x^{(0)}(y) + R_x^{(1)}(y) + R_x^{(2)}(y) \leq C \frac{H(y_1)}{nc_n^d} \sum_{k=1}^{(1-\varepsilon)n} \frac{P(S_1(k) \in (0, y_1 + 2))}{c_k} \leq C(y_1 + 3) \frac{n}{c_n} \leq C \delta_n n.$$

When $\alpha \in (1, 2]$, using the Stone theorem, we proceed as follows

$$\sum_{k=1}^{(1-\varepsilon)n} P(S_1(k) \in (0, y_1 + 2)) \leq C(y_1 + 3) \frac{n}{c_n} \leq C \delta_n n.$$

Now we will consider the case $\alpha \leq 1$. Fix $\beta > 0$ and notice that $1/\alpha - 1 + \beta > 0$ for any $\alpha \leq 1$. Since $c_n$ is regularly varying of index $1/\alpha$, by Potter’s bounds, there exists $C > 0$, such that for $k \leq n$,

$$\frac{c_n}{c_k} \leq C \left( \frac{n}{k} \right)^{1/\alpha + \beta}.$$

Then, for the sequence $\gamma_n = \delta_n^{1/(\alpha - 1 + \beta)} \to 0$,

$$\sum_{k=1}^{(1-\varepsilon)n} P(S_1(k) \in (0, y_1 + 2)) \leq \gamma_n n + \sum_{k=\gamma_n}^{(1-\varepsilon)n} P(S_1(k) \in (0, y_1 + 2))$$

$$\leq \gamma_n n + C \sum_{k=\gamma_n}^{(1-\varepsilon)n} \frac{y_1 + 2}{c_k} \leq \gamma_n n + C \frac{y_1 + 2}{c_n} \sum_{k=\gamma_n}^{(1-\varepsilon)n} \left( \frac{n}{k} \right)^{1/\alpha + \beta}$$

$$\leq \gamma_n n + C \left( \frac{y_1 + 2}{c_n} \right) \left( \frac{n}{\gamma_n n} \right)^{1/\alpha + \beta - 1} \leq \gamma_n n + C n \delta_n \left( \frac{1}{\gamma_n} \right)^{1/\alpha + \beta - 1} \leq C n (\gamma_n + \delta_n^{1/2}).$$

Therefore,

$$R_x^{(0)}(y) + R_x^{(1)}(y) + R_x^{(2)}(y) \leq C (\gamma_n + \delta_n^{1/2}) \frac{H(y_1)}{c_n^d},$$

and, as a result, for any fixed $\varepsilon > 0$,

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^d : 0 < y_1 \leq \delta_n c_n} \frac{c_n^d}{H(y_1)} \left( R_x^{(0)}(y) + R_x^{(1)}(y) + R_x^{(2)}(y) \right) = 0. \quad (63)$$
where $\Delta$ holds for any fixed $\varepsilon > 0$ there exists a sequence $\epsilon_n \downarrow 0$, such that (63) is still true. Moreover, it will be true for any $\epsilon_n' \downarrow 0$ such that $\epsilon_n' \geq \epsilon_n$. We will assume now that $\varepsilon = \epsilon_n$. It is clear and will be used in the subsequent proof that we can increase $\epsilon_n$ and (63) will hold as long as $\epsilon_n < 1/2$.

Now we represent

$$R_{\varepsilon}^{(3)}(y) = R_{\varepsilon}^{(4)}(y) + R_{\varepsilon}^{(5)}(y) + R_{\varepsilon}^{(6)}(y),$$

where

$$R_{\varepsilon}^{(4)}(y) = \mathbf{P}(S(n) \in y + \Delta) + \sum_{k=1}^{\epsilon_n} \int_{z_1 \in (0,y_1], |z^{(2,d)}| \leq \epsilon_n^{1/3} c_n} \mathbf{P}(S(n-k) \in y-z+\Delta) dB_k(z)$$

$$R_{\varepsilon}^{(5)}(y) = \sum_{k=1}^{\epsilon_n} \int_{z_1 \in (y_1,y_1+1], |z^{(2,d)}| \leq \epsilon_n^{1/3} c_n} \mathbf{P}(S(n-k) \in y-z+\Delta) dB_k(z)$$

$$R_{\varepsilon}^{(6)}(y) = \sum_{k=1}^{\epsilon_n} \int_{z \in \mathbb{R}^d, |z^{(2,d)}| > \epsilon_n^{1/3} c_n} \mathbf{P}(S(n-k) \in y-z+\Delta, S(n-k)_1 > 0) dB_k(z).$$

Let $\epsilon_n$ an arbitrary converging to zero sequence of positive numbers. First, by the Stone local limit theorem,

$$\int_{0 < z_1 < y_1, |z^{(2,d)}| \leq \epsilon_n^{1/3} c_n} \mathbf{P}(S(n-k) \in y-z+\Delta) dB_k(z)$$

$$= \frac{g_{\alpha,\sigma}(0, \frac{y^{(2,d)}}{c_n n-k}) + \Delta_1(n-k,y)}{c_n^{d-k}} \int_{0 < z_1 < y_1, |z^{(2,d)}| \leq \epsilon_n^{1/3} c_n} dB_k(z),$$

where $\Delta_1(n-k,y) \to 0$ uniformly in $z$ such that $z_1 \in (0,\delta_n c_n)$ and $k \in [1,\epsilon_n n]$. Therefore,

$$R_{\varepsilon}^{(4)}(y) = \left( \frac{g_{\alpha,\sigma}(0, \frac{y^{(2,d)}}{c_n n-k}) + \Delta_1(n,y)}{c_n^{d-k}} \right) \left( \frac{1}{c_n^{d-n}} + \sum_{k=1}^{\epsilon_n n} \frac{1}{c_n^{d-k}} \int_{0 < z_1 < y_1, |z^{(2,d)}| \leq \epsilon_n^{1/3} c_n} dB_k(z) \right)$$

$$= \frac{g_{\alpha,\sigma}(0, \frac{y^{(2,d)}}{c_n n}) + \Delta_1(n,y)}{c_n^{d-n}} \left( 1 + \sum_{k=1}^{\epsilon_n n} \int_{0 < z_1 < y_1, |z^{(2,d)}| \leq \epsilon_n^{1/3} c_n} dB_k(z) \right). \tag{64}$$

where $\Delta_1(n,y) \to 0$ uniformly in $y$ with $y_1 \in (0,\delta_n c_n)$ and $k \in [1,\epsilon_n n]$. Now we represent

$$1 + \sum_{k=1}^{\epsilon_n n} \int_{0 < z_1 < y_1, |z^{(2,d)}| \leq \epsilon_n^{1/3} c_n} dB_k(z) = H(x_1) - \sum_{k=\epsilon_n n}^{\infty} \mathbf{P}(S_1(k), \tau > k)$$

$$- \sum_{k=1}^{\epsilon_n n} \int_{0 < z_1 < y_1, |z^{(2,d)}| > \epsilon_n^{1/3} c_n} dB_k(z). \tag{65}$$
For any fixed $\varepsilon > 0$, by Corollary 22 of [21],

$$\sum_{k=\varepsilon_n}^{\infty} P(S_1(k) < x_1, \tau > k) \leq C y_1 H(y_1) \sum_{k=\varepsilon_n}^{\infty} \frac{1}{k c_k} \leq C \frac{y_1 H(y_1)}{\varepsilon_n} = o(H(y_1))$$

uniformly in $y$ with $y_1 \leq \delta_n c_n$. Hence, this bound holds for some sequence $\varepsilon_n \downarrow 0$. Increasing the original sequence $\varepsilon_n$ if needed we obtain

$$\sum_{k=\varepsilon_n}^{\infty} P(S_1(k) < y_1, \tau > k) = o(H(y_1)) \quad (66)$$

uniformly in $y$ with $y_1 \leq \delta_n c_n$.

Fix a large positive number $A$ and let

$$c^-(y) := \inf\{ k \geq 1 : c_k > y \}.$$

Note that $c_{c^{-}(y)} > y$ and $c_{c^{-}(y)-1} \leq y$. Also note that since $\mu(y)$ is regularly varying, $c^{-}(y) \sim 1/\mu(y)$ as $y \to \infty$. Then,

$$\sum_{k=\varepsilon_n}^{\infty} \int_{0 < z_1 < y_1 \cdot z^{(2,d)}} dB_k(z) \leq R_{\varepsilon_n}^{(4,1)}(y) + R_{\varepsilon_n}^{(4,2)}(y),$$

where

$$R_{\varepsilon_n}^{(4,1)}(y) = \sum_{k=1}^{c^{-}(A y_1)-1} P(S_1(k) < y_1, |S_{2,d}(k)| > \varepsilon_n^{1/3} c_n, \tau > k)$$

and

$$R_{\varepsilon_n}^{(4,2)}(y) = \sum_{k=c^{-}(A y_1)}^{\varepsilon_n} P(S_1(k) < y_1, |S_{2,d}(k)| > \varepsilon_n^{1/3} c_n, \tau > k).$$

We can estimate $R_{\varepsilon_n}^{(4,2)}(x)$ similarly to the above

$$R_{\varepsilon_n}^{(4,2)}(y) \leq \sum_{k=c^{-}(A y_1)}^{\infty} P(S_1(k) < y_1, \tau > k) \leq C(y_1 + 1) H(y_1) \sum_{k=c^{-}(A y_1)}^{\infty} \frac{1}{k c_k} \leq C \frac{(y_1 + 1) H(y_1)}{A(y_1 + 1)} = \frac{C}{A} H(y_1).$$

Clearly, taking $A$ sufficiently large we can make this term much smaller than $H(y_1)$. By Theorem [4]

$$P(S_1(k) < y_1, |S_{2,d}(k)| > \varepsilon_n^{1/3} c_n | \tau > k) \leq P(|S_{2,d}(k)| > \varepsilon_n^{1/3} c_n | \tau > k) \to 0,$$

uniformly in $k \leq \varepsilon_n$. Combining this with (65), we conclude that

$$R_{\varepsilon_n}^{(4,1)}(y) = o \left( \sum_{k=1}^{c^{-}(A y_1)-1} P(\tau > k) \right) = o \left( \sum_{k=1}^{c^{-}(A y_1)-1} \frac{H(c_k)}{k} \right) = o \left( H(y_1) \right)$$
uniformly in $y$ with $y_1 \leq \delta_n c_n$. Hence,

$$\int_{0 < z_1 < y_1, |z|^{2,d} > \varepsilon_n^{1/3} c_n} dB_k(z) = o(H(y_1))$$

(67)

uniformly in $y$ with $y_1 \leq \delta_n c_n$. Combining (64), (65), (66) and (67) we obtain that

$$R^{(4)}_{\varepsilon_n}(y) = g_{\alpha, \sigma} \left(0, \frac{y_1(n)}{c_n} \right) + \Delta_2(n, y) \frac{\varepsilon_n}{c_n} H(y_1),$$

(68)

where $\Delta_2(n, y) \to 0$ uniformly in $y$ such that $y_1 \in (0, \delta_n c_n)$. Using (67) and the Stone local limit theorem, one can easily conclude that

$$R^{(6)}_{\varepsilon_n}(y) \leq C \frac{\varepsilon_n}{c_n} \sum_{k=1}^n \int_{0 < z_1 < y_1, |z|^{2,d} > \varepsilon_n^{1/3} c_n} dB_k(z) = o \left( \frac{H(y_1)}{c_n} \right).$$

(69)

uniformly in $y$ such that $y_1 \in (0, \delta_n c_n)$.

Analysis of $R^{(5)}_{\varepsilon}(y)$ is very similar to that of $R^{(4)}_{\varepsilon}(y)$. First we make use of the Stone theorem,

$$R^{(5)}_{\varepsilon}(y) = \frac{g_{\alpha, \sigma} \left(0, \frac{y_1(n)}{c_n} \right) + \Delta_2(n, y) \frac{\varepsilon_n}{c_n} (z_1 - y_1) dB_k(z)}{\sum_{k=1}^n \int_{z_1 \in (y_1, y_1+1), |z|^{2,d} \leq \varepsilon_n^{1/3} c_n} dB_k(z)},$$

where $\Delta_2(n, y) \to 0$ uniformly in $y$ with $y_1 \in (0, \delta_n c_n)$. Then, using the same arguments as above, we obtain

$$R^{(5)}_{\varepsilon}(y) = \frac{g_{\alpha, \sigma} \left(0, \frac{y_1(n)}{c_n} \right) + \Delta_4(n, y) \frac{\varepsilon_n}{c_n} (1 + z_1 - y_1) dB_k(z)}{\sum_{k=1}^n \int_{z_1 \in (y_1, y_1+1)} dB_k(z)},$$

where $\Delta_4(n, y) \to 0$ uniformly in $y$ with $y_1 \in (0, \delta_n c_n)$. Integrating by parts we can complete the proof now.

\[\square\]

7. Probabilities of small deviations when random walks start at an arbitrary starting point

Proof of Theorem 3 in the lattice case. For the lattice distribution we have from [40]

$$p_n(x, y) = P(x + S(n) = y, \tau_x > n) = \sum_{k=0}^n \sum_{z_1 \leq \cdots \leq z_d} p_k^+ (0, z - x) b_{n-k} (y - z).$$

Let $N_n$ be the sequence of integers, which was constructed in the proof of Theorem 1. We shall use the representation

$$p_n(x, y) = P_1(x, y) + P_2(x, y) + P_3(x, y),$$
where

\[ P_1(x, y) := \sum_{k=0}^{N_n} \sum_{z_1=1}^{x_1 \land y_1} \sum_{z_2, \ldots, z_d} p_k^+(0, z-x) b_{n-k}(y-z), \]

\[ P_2(x, y) := \sum_{k=N_n+1}^{n-N_n} \sum_{z_1=1}^{x_1 \land y_1} \sum_{z_2, \ldots, z_d} p_k^+(0, z-x) b_{n-k}(y-z), \]

\[ P_3(x, y) := \sum_{k=N_n}^{n} \sum_{z_1=1}^{x_1 \land y_1} \sum_{z_2, \ldots, z_d} p_k^+(0, z-x) b_{n-k}(y-z). \]

To estimate \( P_2(x, y) \) we shall proceed as in the analysis of normal deviations. Note that by Lemma 10

\[ b_{n-k}(y-z) \leq C \frac{H(y_1 - z_1)}{(n-k)c_{n-k}}. \]

Then

\[ P_2(x, y) \leq C \frac{H(y_1)}{N_n c_{n-k}^d} \sum_{k=N_n+1}^{n} \sum_{z_1=1}^{x_1 \land y_1} \sum_{z_2, \ldots, z_d} p_k^+(0, z-x) \]

\[ \leq C \frac{H(y_1)}{N_n c_{n-k}^d} \sum_{k=N_n+1}^{n} \sum_{z_1=1}^{x_1 \land y_1} \sum_{z_2, \ldots, z_d} p_k^+(0, z-x) \]

Using now \( 45 \) and increasing, if needed \( N_n \), we conclude that

\[ P_2(x, y) = o \left( \frac{H(y_1) V(x_1)}{N_n c_{n-k}^d} \right) = o \left( \frac{H(y_1) V(x_1)}{n c_{n-k}^d} \right). \]  

(70)

Now we will consider the first term \( P_1(x, y) \). Let \( \varepsilon_n \downarrow 0 \) be the sequence, which we have defined in the proof of Theorem 1. We will need the following sets

\[ A_1(x, y) = \{ z : |x-z| \leq \varepsilon_n c_n, z_1 \in (0, x_1 \land y_1] \} \]

\[ C_1(x, y) = \{ z : |x-z| > \varepsilon_n c_n, z_1 \in (0, x_1 \land y_1] \} \]

Applying now the asymptotics for small deviations of walks starting at zero, we get

\[ \sum_{k=0}^{N_n} \sum_{z \in A_1(x, y)} p_k^+(0, z-x) b_{n-k}(y-z) \]

\[ \approx \frac{g_{n, \sigma}}{n c_{n-k}^d} \sum_{k=0}^{N_n} \sum_{z \in A_1(x, y)} p_k^+(0, z-x) H(y_1 - z_1). \]

Next we note that

\[ \sum_{k=0}^{N_n} \sum_{z \in C_1(x, y)} p_k^+(0, z-x) b_{n-k}(y-z) \]

\[ \leq C \frac{H(y_1)}{n c_{n-k}^d} \sum_{k=0}^{N_n} \mathbb{P}(|S(k)| \geq \varepsilon_n c_n, S_1(k) > x_1, \tau^+ > k) \]
and
\[ \sum_{k=0}^{N_n} \sum_{z \in C_1(x, y)} p_k^+(0, z - x)H(y_1 - z_1) \leq H(y_1) \sum_{k=0}^{N_n} \mathbb{P}(\{|S(k)| \geq \varepsilon nc_n, S_1(k) > -x_1, \tau^+ > k\}). \]

Taking into account (50), we obtain
\[ P_1(x, y) \sim \frac{g_{\alpha, \sigma}(0, \frac{y_2 - x_2 n}{c_n})}{nc_n^d} \int_{\mathbb{R}^d} (V(x_1 - z_1) - V(x_1 - z_1 - 1))H(y_1 - z_1), \quad (71) \]
where we also replaced \( \sum_{0}^{N_n} \) by \( \sum_{0}^{\infty} \) using the arguments in the proof of Proposition 11 in [10]. Analogous arguments give us
\[ P_3(x, y) \sim \frac{g_{\alpha, \sigma}(0, \frac{y_2 - x_2 n}{c_n})}{nc_n^d} \int_{\mathbb{R}^d} V(x_1 - z_1)(H(y_1 - z_1) - H(y_1 - z_1 - 1)). \]

Then, the arguments at the end of the proof of Proposition 11 in [10] give
\[ P_1(x, y) + P_3(x, y) \sim V(x_1)H(y_1) \frac{g(0, \frac{y_2 - x_2 n}{c_n})}{nc_n^d}. \]

\[ \square \]

Proof of Theorem 3 in the non-lattice case. The proof is very similar to the proof in the lattice case.

For the non-lattice distribution we will make use of (31). We split the sum as follows,
\[ p_n(x, y) = P_1(x, y) + P_2(x, y) + P_3(x, y), \]
where
\[ P_1(x, y) := \sum_{k=0}^{N_n} \int_{(0, x_1 \wedge (y_1 + 1)] \times \mathbb{R}^{d-1}} p_k^+(0, dz - x) b_{n-k}(y - z), \]
\[ P_2(x, y) := \sum_{k=N_n+1}^{n-N_n-1} \int_{(0, x_1 \wedge (y_1 + 1)] \times \mathbb{R}^{d-1}} p_k^+(0, dz - x) b_{n-k}(y - z), \]
\[ P_3(x, y) := \sum_{k=N_n+1}^{n-N_n} \int_{(0, x_1 \wedge (y_1 + 1)] \times \mathbb{R}^{d-1}} p_k^+(0, dz - x) b_{n-k}(y - z). \]

There is virtually no difference in estimates for \( P_2(x, y) \). So repeating the same arguments we obtain
\[ P_2(x, y) = o \left( \frac{H(y_1)V(x_1)}{nc_n^d} \right). \quad (72) \]

Now we will consider the first term \( P_1(x, y) \). We will need the following sets
\[ A_1(x, y) = \{ z : |x - z| \leq \varepsilon nc_n, z_1 \in (0, x_1 \wedge y_1) \} \]
\[ C_1(x, y) = \{ z : |x - z| > \varepsilon nc_n, z_1 \in (0, x_1 \wedge y_1) \} \]
Now we have,
\[
\sum_{k=0}^{N_n} \int_{A_1(x,y)} p_k^+(0,dz-x)b_{n-k}(y-z) \sim \frac{g_{a,\sigma}(0,\frac{y_2.d-y_1.d}{n\epsilon_n})}{nc_n^d} \sum_{k=0}^{N_n} \int_{A_1(x,y)} p_k^+(0,dz-x) \int_{y_1-z_1}^{y_1-z_1+1} H(u)du.
\]

Now note that by (50)
\[
\sum_{k=0}^{\epsilon_n n} \int_{C_1(x,y)} p_k^+(0,dz-x)H(y_1-z_1) \leq H(y_1) \sum_{k=0}^{\epsilon_n n} \mathbb{P}(|S(k)| \geq \delta c_n, S_1(k) > -x_1, \tau^+ > k) = o(H(y_1)V(x_1)),
\]
provided that \(\gamma_n\) and \(\epsilon_n\) converges to 0 sufficiently slowly. As a result,
\[
P_1(x,y) \sim \frac{g_{a,\sigma}(0,\frac{y_2.d-y_1.d}{c_n})}{nc_n^d} \int_{(0,x_1 \land (y_1+1))} V(x_1-dz_1) \int_{y_1-z_1}^{y_1-z_1+1} H(u)du,
\]
where we also replaced \(\sum_{k=0}^{\epsilon_n n}\) by \(\sum_{k=0}^{\infty}\) using similar arguments. Analogous arguments give us
\[
P_3(x,y) \sim \frac{g_{a,\sigma}(0,\frac{y_2.d-y_1.d}{c_n})}{nc_n^d} \int_{(0,x_1 \land (y_1+1))} V(x_1-z_1)H(y_1-z_1+\Delta)dz_1.
\]

Now note that integration by parts of the first integral gives
\[
\int_{(0,x_1 \land (y_1+1))} V(x_1-dz_1) \int_{y_1-z_1}^{y_1-z_1+1} H(u)du + \int_{(0,x_1 \land (y_1+1))} V(x_1-z_1)H(y_1-z_1+\Delta)dz_1 = V(x_1) \int_{y_1}^{y_1+1} H(u)du.
\]
As a result,
\[
P_1(x,y) + P_3(x,y) \sim V(x_1) \int_{y_1}^{y_1+1} H(u)du \frac{g(0,\frac{y_2.d-y_1.d}{c_n})}{nc_n^d}.
\]

8. Probabilities of large deviations when random walk starts at the origin

We will need the following large deviations estimates.

**Proposition 19.** Let \(X \sim \mathcal{D}(d,\alpha,\sigma)\) with some \(\alpha < 2\). Suppose that there exists a regularly varying \(\varphi\) such that the upper bound in (10) holds. Then, there exists constant \(C_H\) such that for \(|x| \geq c_n\) we have
\[
\mathbb{P}(S(n) \in x + \Delta) \leq C_H \frac{1}{c_n^{\alpha+1}}(\varphi(|x|)).
\]

If, in addition, (15) holds, then
\[
\mathbb{P}(S(n) \in x + \Delta) \leq C_H n \frac{\varphi(|x|)}{|x|^d}.
\]
This result is proved in [1, Theorem 2.6] in the lattice case. We omit the proof of non-lattice case, as it can be done very similarly to [1, Theorem 2.6].

Using the definition (7) of \( c_n \), we obtain from Corollary 14 the following upper bound. (Recall that \( g(r) = \frac{\phi(|r|)}{r} \).)

**Lemma 20.** For any \( A > 1 \) there exists \( c_A \) such that

\[
b_n(x) \leq c_A H(x_1) g(|x|),
\]

for \( x \) with \( c_n \leq |x| \leq Ac_n \).

The main goal of this section is to obtain an upper bound for \( b_n(x) \) in the case \( |x| > Ac_n \). We now obtain a bound, which will be valid also for \( |x| > Ac_n \).

**Lemma 21.** Suppose that \( X \) is asymptotically stable with \( \alpha \in (0, 2) \). If (15) holds then there exists \( \gamma > 0 \) such that for all \( y \) with \( c_n \leq |y| \leq Ac_n \) we have

\[
b_n(y) \leq \gamma H(y_1 + 1) g(|y|).
\]

**Proof.** We will first introduce some constants and sequences that will be used throughout the proof. Set

\[
\rho_n = \frac{1}{n} \sum_{k=1}^{n} P(S_1(k) > 0).
\]

Fix \( \delta \in (0, 1) \) such that

\[
e^\delta \sup_{n \geq 1} \rho_n + \delta e^\delta < 1
\]

and let \( \tilde{A} \) be such that

\[
\int_{|y - z| > c_n, |y| > |y|/2} g(|y|) \frac{g(|z|)}{g(|y|)} dz \\
+ \int_{|y - z| > c_n, |z| > |y|/2} g(|y|) \frac{g(|z|)}{g(|y|)} dz \leq e^\delta
\]

(80)

for \( y \) with \( |y| > 1 \) and \( k \geq 1 \). Let \( A \) be such that

\[
\sup_{n \geq 1} \sup_{y, z : |y| > Ac_n, |z| \leq \tilde{A}c_n + 1} \frac{g(|y - z|)}{g(|y|)} \leq e^\delta.
\]

(81)

By Lemma 21 there exist \( c_A > 1 \) such that (80) holds for \( y \) with \( c_n \leq |y| \leq Ac_n \).

The proof will be done by induction. We will inductively construct an increasing sequence \( \gamma_n \) such that

\[
b_n(y) \leq \gamma_n H(y_1) g(|y|)
\]

(82)

for \( y \) with \( |y| > c_n \) and \( n \geq 1 \). Then we will show that \( \sup_{n} \gamma_n < \infty \). We put \( \gamma_1 = c_A \) and then the base of induction \( n = 1 \) is immediate. Since \( \gamma_n \) will be increasing, it follows from the definition of \( A \) that (82) holds for \( y \) such that \( |c_n| < y \leq Ac_n \). Hence, we will consider only \( y \) with \( |y| > c_n \).
Using first the local large deviations bound (75) and then the regular variation assumption we obtain
\[ R_n(y + \Delta) = P(S(n) \in y + \Delta) + \sum_{k=1}^{n-1} \int_{R^d} P(S(k) \in y - z + \Delta, S_1(k) > 0) dB_{n-k}(z) =: R^{(1)}(y) + R^{(2)}(y) + R^{(3)}(y), \]  
where
\[ R^{(1)}(y) = P(S(n) \in y + \Delta) + \sum_{k=1}^{n-1} \int_{|z| \leq |y|/2} P(S(k) \in y - z + \Delta, S_1(k) > 0) dB_{n-k}(z), \]
\[ R^{(2)}(y) = \sum_{k=1}^{n-1} \int_{|z| > |y|/2, |y - z| \leq \tilde{A}_c} P(S(k) \in y - z + \Delta, S_1(k) > 0) dB_{n-k}(z), \]
\[ R^{(3)}(y) = \sum_{k=1}^{n-1} \int_{|z| > |y|/2, |y - z| > \tilde{A}_c} P(S(k) \in y - z + \Delta, S_1(k) > 0) dB_{n-k}(z). \]

Using first the local large deviations bound (75) and then the regular variation of \( g \), we get
\[ R^{(1)}(y) \leq C_H n g(|y|) + C_H \sum_{k=1}^{n-1} \int_{|z| \leq |y|/2, 0 \leq z_1 \leq y_{1} + 1} k g(|y - z|) dB_{n-k}(z) \]
\[ \leq C_H n g(|y|) + C_H g(|y|) \sum_{k=1}^{n-1} k \int_{|z| \leq |y|/2, 0 \leq z_1 \leq y_{1} + 1} dB_{n-k}(z) \]
\[ \leq (C_H + 1) C_H n g(|y|) H(y_{1} + 1). \]  
Second, integrating by parts and using then the definition (81) of \( A \) and the induction assumption we obtain
\[ R^{(2)}(y) \leq \sum_{k=1}^{n-1} \int_{|y| > |z|/2, |y - z| \leq \tilde{A}_c, 0 \leq z_1 \leq y_{1} + 1} P(S(k) \in y - z + \Delta, S_1(k) > 0) dB_{n-k}(z) \]
\[ \leq \sum_{k=1}^{n-1} \int_{|y - z| > |z|/2 - 1, |z_1| \leq \tilde{A}_c + 1, 0 \leq z_1 \leq y_{1} + 1} b_{n-k}(y - z) P(S(k) \in d z, S_1(k) > 0) \]
\[ \leq \sum_{k=1}^{n-1} \int_{|y - z| > |z|/2 - 1, |z_1| \leq \tilde{A}_c + 1, 0 \leq z_1 \leq y_{1} + 1} \gamma_{n-1} g(|y - z|) H(y_{1} - z_1 + 1) P(S(k) \in d z, S_1(k) > 0) \]
\[ \leq e^{\delta} \gamma_{n-1} g(|y|) H(y_{1} + 1) \sum_{k=1}^{n-1} P(S_1(k) > 0) \]
\[ = e^{\delta} (n - 1) \gamma_{n-1} \rho_{n-1} g(|y|) H(y_{1} + 1). \]  
Assume now that we have already constructed the elements \( \gamma_k \) for \( k \leq n - 1 \). We shall construct the next value \( \gamma_n \). It follows from (23) that
\[ nb_n(y + \Delta) = P(S(n) \in y + \Delta) \]
Third, using the induction assumption and (76),

$$R^{(3)}(y) \leq C_H \gamma_{n-1} \sum_{k=1}^{n-1} k \int_{[z] > |y|/2, |x-z| > A_k, 0 \leq z \leq y_1 + 1} g(|y-z|)g(|z|)H(z)dz$$

$$\leq C_H \gamma_{n-1} H(y_1 + 1) \sum_{k=1}^{n-1} k \int_{[z] > |y|/2, |y-z| > A_k, 0 \leq z \leq y_1 + 1} g(|y-z|)g(|z|)dz.$$

We can estimate the integral as follows,

$$\int_{|z| > \frac{y-z}{2}} g(|y-z|)g(|z|)dz \leq \frac{\delta}{C_H k} g(|y|),$$

using the definition (80) of $\tilde{A}$. Hence,

$$R^{(3)}(y) \leq C_H \gamma_{n-1} H(y_1 + 1) \frac{\delta}{C_H} g(|y|)(n-1) \leq \delta \gamma_{n-1} e^\delta H(y_1 + 1) g(|y|)(n-1). \quad (86)$$

Combining (84), (85) and (86) we obtain that

$$b_n(y) \leq ((C_g + 1)C_H + \gamma_{n-1}(e^\delta \rho_{n-1} + \delta e^\delta))g(|y|)H(y_1 + 1).$$

Then, for

$$\gamma_n := \max(\gamma_{n-1}, (C_g + 1)C_H + \gamma_{n-1}(e^\delta \rho_{n-1} + \delta e^\delta))$$

the inequality (82) holds. Then, using the definition (79) of $\delta$ it is not difficult to show that

$$\limsup_{n \to \infty} \gamma_n < \infty.$$

Hence, the statement of the lemma holds with

$$\gamma := \sup_n \gamma_n < \infty.$$

□

9. Probabilities of large deviations when random walks start at an arbitrary starting point

Proof of Theorem 4. Let

$$A(x, y) = \left\{ z : |y-z| \geq \frac{1}{2}|y-x|, z_1 \in (0, x_1 \wedge (y_1 + 1)) \right\}$$

$$C(x, y) = \left\{ z : |x-z| \geq \frac{1}{2}|y-x|, z_1 \in (0, x_1 \wedge (y_1 + 1)) \right\}.$$

If $|y-z| \geq \frac{1}{2}|y-x|$ then, by Lemma 21

$$b_{n-k}(y-z) \leq \gamma_0 H(y_1 - z_1 + 1)g(|y-z|) \leq \gamma_0 H(y_1 - z_1 + 1)g\left(\frac{|x-y|}{2}\right).$$
This implies that
\[ n \sum_{k=0}^{n} \int_{A(x,y)} p_k^+(0, dz - x) b_{n-k}(y - z) \]
\[ \leq \gamma_0 H(y_1 + 1) g \left( \frac{|x-y|}{2} \right) \sum_{k=0}^{n} P(x + S(k) \in A(x, y), \tau^+ > k) \]
\[ \leq \gamma_0 H(y_1 + 1) g \left( \frac{|x-y|}{2} \right) \sum_{k=0}^{n} P(x_1 + S_1(k) \in (0, x_1 \wedge (y_1 + 1)], \tau^+ > k) \]
\[ \leq \gamma_0 H(y_1 + 1) g \left( \frac{|x-y|}{2} \right) V(x_1). \]

By the same argument,
\[ n \sum_{k=0}^{n} \int_{C(x,y)} p_k^+(0, dz - x) b_{n-k}(y - z) \]
\[ \leq n \sum_{k=0}^{n} \sum_{v \in \mathbb{Z}^d : (v+\Delta) \cap C(x,y) \neq \emptyset} P(S(k) \in v - x + \Delta, \tau^+ > k) \max_{z \in v+\Delta} b_{n-k}(y - z) \]
\[ \leq c2^d V(x_1 + 1) g \left( \frac{|x-y|}{2} - 1 \right) H(y_1 + 1). \]

These estimates give the desired bound. □

10. ASYMPTOTICS FOR THE GREEN FUNCTION NEAR THE BOUNDARY

Proof of Theorem 5. We consider the lattice case only. Fix \( A > 0 \). Then
\[ G(x, y) = G_1(x, y) + G_2, (x, y) := \sum_{n: A \leq |x-y|} p_n(x, y) + \sum_{n: A \geq |x-y|} p_n(x, y). \]

Using Theorem 4 we obtain
\[ G_1(x, y) \leq CH(y_1) V(x_1) \sum_{n: A \leq |x-y|} g(|x-y|) \]
\[ \leq CH(y_1) V(x_1) \frac{g(|x-y|)}{\mu(|x-y|/A)} \]
\[ \leq \frac{C}{A^\alpha} H(y_1) V(x_1) |x-y|^d. \]

For the second term make use of Theorem 4
\[ G_2(x, y) \sim H(y_1) V(x_1) \sum_{n: A \geq |x-y|} \frac{g_{\alpha,c}(0, \frac{y_2 - x_2 \cdot d}{cn})}{nc^n}. \]
We will now analyse the series. Using the regular variation of $c_n$ we can write it as
\[
\sum_{n: A c_n \geq |x-y|} g_{\alpha,\sigma}(0, \frac{y_2 d - x_2 d}{c_n}) \sim \sum_{n: n \geq A^{-\alpha}/\mu(|x-y|)} g_{\alpha,\sigma}(0, \frac{y_2 d - x_2 d}{c_n})
\]
\[
\sim \int_{A^{-\alpha}/\mu(|x-y|)}^{\infty} g_{\alpha,\sigma}(0, \frac{y_2 d - x_2 d}{c_n}) \frac{tc_d}{\mu([t])} \, dt
\]
\[
\sim \int_0^{\mu(|x-y|) A^{\alpha}} g_{\alpha,\sigma}(0, \frac{y_2 d - x_2 d}{c_n}) \frac{z^{d/\alpha}}{z^{d/\alpha-1}} \, dz
\]
\[
\sim \frac{1}{|x-y|^d} \int_0^{A^{\alpha}} g_{\alpha,\sigma} \left(0, \frac{y_2 d - x_2 d}{|x-y|} z^{1/\alpha} \right) z^{d/\alpha-1} dz,
\]
as $|x-y| \to \infty$. Here we used the fact that since $c_n$ is regularly varying of index $1/\alpha$, for a fixed $z > 0$,
\[
c(1/\mu(|x-y|)z)) \sim z^{-1/\alpha} c((1/\mu(|x-y|))) \sim z^{-1/\alpha} |x-y|,
\]
as $|x-y| \to \infty$. Thus, in the general case,
\[
G_2(x, y) \sim C \frac{H(y_1) V(x_1)}{|x-y|^d} \int_0^{A^{\alpha}} g_{\alpha,\sigma} \left(0, \frac{y_2 d - x_2 d}{|x-y|} z^{1/\alpha} \right) z^{d/\alpha-1} dz.
\]
Letting $A \to \infty$ and substituting $z^{1/\alpha} = t$ we arrive at the conclusion. Noting that in the isotropic case the ratio $\frac{y_2 d - x_2 d}{|x-y|}$ belongs asymptotically to the unit sphere, we obtain the result in this case as well.

To obtain the upper bound in the analysis of the second term we make use of Lemma 10 instead of Theorem 8.

\[
\square
\]

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