RICCI-POSITIVE METRICS ON CONNECTED SUMS OF PROJECTIVE SPACES

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Abstract. It is a well known result of Gromov that all manifolds of a given dimension with positive sectional curvature are subject to a universal bound on the sum of their Betti numbers. On the other hand, there is no such bound for manifolds with positive Ricci curvature: indeed, Perelman constructed positive Ricci metrics on $\#_k \mathbb{C}P^2$. In this paper, we revisit and extend Perelman’s construction to show that $\#_k \mathbb{C}P^n$, $\#_k \mathbb{H}P^n$, and $\#_k \mathbb{O}P^2$ all admit metrics of positive Ricci curvature.

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1. Introduction

Let \( \#_k M \) denote the \( k \)-fold connected sum of \( M \) with itself. Perelman proved the following.

**Theorem 1.1.** [13] For all \( k > 0 \), the manifolds \( \#_k \mathbb{C}P^2 \) admit metrics with positive Ricci curvature.

In this paper, we show that the claim remains true for all complex, quaternionic, and octonionic projective spaces.

**Theorem A.** For all \( k > 0 \) and \( n > 0 \) the manifolds \( \#_k \mathbb{C}P^n \), \( \#_k \mathbb{H}P^n \), and \( \#_k \mathbb{O}P^2 \) admit metrics with positive Ricci curvature.

The main technical tool used in Perelman’s construction is a gluing lemma that gives a sufficient condition on the second fundamental form of the boundary to glue together two Ricci positive Riemannian manifolds along isometric boundaries in a way that preserves positive Ricci curvature.

**Lemma 1.2.** [13, Section 4] Let \(( M^n_i, g_i ) \) with \( i = 1, 2 \) and \( n \geq 3 \) be closed Riemannian manifolds with positive Ricci curvature. Suppose that there is an isometry \( \phi \) between their boundaries such that \( \Pi_1 + \phi^* \Pi_2 \) is positive definite, where \( \Pi_i \) is the second fundamental form of \( \partial M_i \). Then \( M_1 \cup_\phi M_2 \) admits a metric with positive Ricci curvature that agrees with \( g_i \) on \( M_i \) outside of an arbitrarily small tubular neighborhood of \( \partial M_i \).

Thinking of Lemma 1.2 one might want to construct a Ricci positive metric on \( S^n_k \), the sphere with \( k \) disjoint balls removed, so that the boundary \( S^{n-1}_k \) are standard with small extrinsic curvatures. Such a metric would make a universal docking station, to which we may, by Lemma 1.2 attach Ricci positive manifolds with round, convex boundary. Indeed, the bulk of [13] is dedicated to construction such a metric; the following is a direct corollary of Perelman’s construction.

**Proposition 1.3.** For all \( n > 3 \), \( k > 0 \), and \( \rho < 1 \) there exists a metric \( g_{\text{docking}} \) on \( S^n_k \) so that \( \text{Ric}_g \) is positive definite, the metric restricted to each boundary component is round with radius \( \rho \), and \( \Pi = -g_{\text{docking}} \).

We will explain how Proposition 1.3 follows from the work of [13] in Section 4.1. As explained, Proposition 1.3 reduces the problem of taking Ricci positive connected sums of \( M_i \) to finding Ricci positive metrics on \( M^n_i \setminus D^n \) with round, convex boundary.

**Theorem B.** Let \( n \geq 4 \) and \( 1 \leq i \leq k \). Suppose there exists metrics \( g_i \) on \( M^n_i \setminus D^n \) with positive Ricci curvature such that \( \Pi_i \) is positive definite and the metric restricted to the boundary is round. Then there exists a metric on \( \#^k_{i=1} M_i \) with positive Ricci curvature.

A metric \( g_i \) on \( M_i \) satisfying the hypotheses of Theorem A will be called a core metric for \( M \). The combined work of [13] [12] was to construct core metrics for \( \mathbb{C}P^2 \), completing the proof of Theorem 1.1. Perelman’s construction considers the boundary of a normal neighborhood of the embedding.
\( CP^1 \hookrightarrow CP^2 \), which is \( S^3 \), and uses the invariant framing given by the Lie group structure of \( S^3 \) to define a metric on \( CP^2 \setminus D^4 \) analogous to a doubly warped product. Of the construction present in \([13]\), it is only this aspect of the core does not generalize. The body of this paper is therefore dedicated to proving the following.

**Theorem C.** There are metrics \( g_{\text{core}} \) on each of \( CP^n \setminus D^{2n} \), \( HP^n \setminus D^{4n} \), and \( OP^2 \setminus D^{16} \) satisfying the hypotheses of Theorem \([B]\).

Generalizing Perelman’s approach is achieved as follows. Consider in general a projective space \( P^n \) either over \( C, H, \) or \( O \) (where \( n \leq 2 \) for \( O \), see \([10, \text{Corollary 4L.10}]\)), and let \( d \) denote the real dimension of the underlying algebra. The normal neighborhood of the embedding \( P^n \hookrightarrow P^{n+1} \) can be identified with the generalized Hopf fibration \( S^{n(d+1)} \rightarrow P^n \). If \( n(d+1) > 8 \), then this sphere will not admit a global framing, but its tangent space will still admit a global decomposition given by the distribution of fiber tangent spaces. This decomposition is sufficient to define a metric like the doubly warped product belonging to a class which we will call **doubly warped Riemannian submersion metrics**. We study these metrics in general in Section \([2]\) and relate their Ricci curvature to the original Riemannian submersion. We only state this result in the case of totally geodesic fibers, which greatly simplifies the formulas. In Section \([3]\) we apply this theory to the Hopf fibrations to produce metrics on \( P^n \setminus D^{nd} \), ultimately proving Theorem \([C]\).

Doubly warped Riemannian submersion metrics have been studied in generality previously in \([20]\) to show an explicit lower bounds on the dimension of stabilization required to construct Ricci positive metrics on fiber bundles where both fiber and base support metrics with nonnegative Ricci curvature. They have also been used in \([3, \text{Example 3}]\) to construct metrics of nonnegative curvature on connected sums of two projective spaces. A good exposition for such metrics in the context of the Hopf fibrations is present in \([14, \text{Section 1.4.6}]\).

In section \([4]\) we carefully explain how the main technical results of \([13]\) work together to prove Proposition \([1.3]\) and how this proposition allows us prove Theorem \([B]\) and in turn Theorem \([A]\). We further exploit the technical metric constructions of Sections \([3]\) and \([13]\) with some elementary topological constructions to produce further examples of Ricci positive metrics on connected sums. The following theorem summarizes which spaces we have constructed Ricci positive metrics on.

**Theorem A’.** For all \( n, i, j, k, l, \) the following manifolds admit metrics with positive Ricci curvature.

\[
\begin{align*}
(\text{i}) & \quad RP^{2n} \# (\#_j CP^n) \\
(\text{ii}) & \quad RP^{4n} \# (\#_j CP^{2n}) \# (\#_k HP^n) \\
(\text{iii}) & \quad RP^{16} \# (\#_j CP^8) \# (\#_k HP^4) \# (\#_l OP^2) \\
(\text{iv}) & \quad (\#_i S^{2n} \times S^m) \# (\#_j CP^n \times S^m) \\
(\text{v}) & \quad (\#_i S^{4n} \times S^m) \# (\#_j CP^{2n} \times S^m) \# (\#_k HP^n \times S^m) \\
(\text{vi}) & \quad (\#_i S^{16} \times S^m) \# (\#_j CP^8 \times S^m) \# (\#_k HP^4 \times S^m) \# (\#_l OP^2 \times S^m)
\end{align*}
\]
The main technical results of [13] are Theorem C for $\mathbb{C}P^2$, Lemma A.1, Lemma 4.1, and Lemma 1.2 proven respectively on pages 158 (using the entirety of [12]), 159-161, 162, and 163 of [13]. The proof of Theorem 1.1 is explained on pages 157-158 of [13]. Of these results, Lemma 1.2 has attracted the most attention, and has been used in [1] with Hamilton’s work on Ricci flow to prove that the space of Ricci positive metrics on $D^3$ with convex boundary is path connected, and the proof has been carefully studied in [16, Lemma 2.3] and [1, Lemma 2.3]. We refer the reader there for further reading. Lemma 4.1 is proven in [13, Section 3]. The proof is basically complete and what is omitted is a few elementary facts from classical spherical geometry. In this paper we have generalized Perelman’s construction for $\mathbb{C}P^2$ to Theorem C, the construction of the metrics are presented in Sections 2 and 3, and the computations of their curvatures, which for $\mathbb{C}P^2$ were carried out in [12], are the subject of Appendix B.1. The most technical aspect of [13], however, is Lemma A.1, the proof of which is extremely dense. One of our main motivations in writing this paper is to provide a comprehensive study of this proof in Appendix A and to include the necessary curvature computations (omitted from [13]) in Appendix B.2.

Before beginning, let us briefly place the results of this paper in context within the literature. As mentioned, Gromov proved that there is a universal bound on the sum of Betti numbers for manifolds with positive sectional curvature in [8]. This conclusion is false if the hypothesis of positive sectional curvature is weakened to positive Ricci curvature. This was first shown in [15], in which the authors proved a surgery theorem for positive Ricci curvature to show that $\#_k S^n \times S^m$ admits a metric of positive Ricci curvature for all $k$ and for $n, m \geq 2$. This surgery result was strengthened in [18], to allow for surgery with nontrivial framing, which in turn was used in [19] to construct positive Ricci metrics on the group of exotic spheres bounding parallelizable manifolds. Using an algebraic classification of highly connected manifolds the techniques of [19] are used to construct a Ricci positive metric on a specific smooth structure for any highly connected manifold $M^{4k-1}$ in [6]. As mentioned, [4, Example 3] proves that $\mathbb{P}^n \# \mathbb{P}^n$ admits nonnegatively curved metrics, which with a result of [7] proves that these manifolds admit metrics of positive Ricci curvature. And Perelman in [13] prove that $\#_k \mathbb{C}P^2$ admits metrics of positive Ricci curvature, and that also provides a counterexample to the topological stability of Cheeger and Gromoll’s splitting theorem in [5]. Theorem A and Theorem 4.8 provide infinitely many new examples in infinitely many dimensions of metrics with positive Ricci curvature.

We will always assume that manifolds are compact and smooth (possibly with boundary). We will write $(M, g)$ to denote a smooth manifold with smooth Riemannian metric. We will use subscripts on curvatures to indicate the metric they are derived from, i.e. $\text{Ric}_g(X, Y)$ and $K_g(X, Y)$ are the Ricci and sectional curvatures associated to the Levi-Civita connection of $g$. The second fundamental form of a submanifold will be denoted by II, which when that submanifold is the boundary will be considered as $(0, 2)$-form by pairing with outward normal. The boundary is called convex or concave if II is positive or negative definite respectively. When there is no specified submanifold, II refer to the boundary.
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2. Doubly Warped Riemannian Submersion Metrics

We begin in this section by introducing a generalization of doubly warped product metrics using arbitrary fiber bundles in place of products. As warped product metrics, are themselves generalizations of product metrics, in Section 2.1 we study the analogue of product metrics for nontrivial fiber bundles: Riemannian submersion metrics. With this foundation we explain in Section 2.2 how to warp such metrics, and explain the relationship of the Ricci curvature of the doubly warped Riemannian submersion metric to the Ricci curvature of the original Riemannian submersion. Finally, in Section 2.3 we show how doubly warped Riemannian submersion metrics naturally define metrics on fiberwise quotients and suspensions of spherical fibrations, in particular allowing us to construct metrics on disk bundles, which will be our main application of this theory in Section 3.

2.1. Riemannian Submersions. In this section we provide the needed background about Riemannian submersions to define our doubly warped Riemannian submersion metrics. Our notation largely agrees with [2, Chapter 9], which provides a complete account of the subject. We will assume that \( \pi : E \to B \) is a smooth surjective submersion, i.e. a surjective smooth map with differential everywhere onto. We now record some notations and terminology associated to any smooth surjective submersions.

**Definition 2.1.** [2, 9.7 and 9.8] For any smooth surjective submersion \( \pi : (E, g) \to (B, \tilde{g}) \), one may define the following data.

i. The fiber through \( x \in E \), \( F_x := \pi^{-1}(\pi(x)) \);
ii. The fiber metric \( g_x := g|_{F_x} \);
iii. The vertical distribution \( V := \ker d\pi \subseteq TE \);
iv. The horizontal distribution \( H := V^\perp \), where this compliment is taken with respect to \( g \);
v. The vertical and horizontal projections \( V : TE \to V \) and \( H : TE \to H \), are the bundle maps with kernels \( H \) and \( V \), respectively.

Sections of \( V \) and \( H \) are called *vertical* and *horizontal* vector fields respectively.
Figure 1 illustrates these definitions. A smooth surjective submersion \( \pi : (E, g) \to (B, \tilde{g}) \) is called a Riemannian submersion if for all horizontal vectors \( X_i \) we have \( \pi^* \tilde{g}(X_i, X_j) = g(X_i, X_j) \), i.e. if \( d\pi : H_x \to T_{\pi(x)}B \) is an isometry. It is a fact that a Riemannian submersion between two compact manifolds is a smooth fiber bundle with fiber diffeomorphic to a compact manifold \( F \) (see [2, Theorem 9.3]). Thus one can define a metric on the total space of a fibration by specifying the data of Definition 2.1.

**Proposition 2.2.** [2, 9.15] Assume we are given a smooth fiber bundle \( \pi : E \to B \) with fiber \( F \), and that we also specify the following data.

i. A horizontal distribution of planes \( H \);

ii. A base metric \( \tilde{g} \) on \( B \);

iii. A family of fiber metrics \( \hat{g}_b \) on \( F_b \) parameterized by \( b \in B \).

Then there is a unique metric \( g \) on \( E \) that makes \( \pi : (E, g) \to (B, \tilde{g}) \) a Riemannian submersion with horizontal distribution \( H \) and so that \( g \) restricted to \( F_b \) is isometric to \( \hat{g}_b \). Moreover, \( g \) takes the following form

\[
g = \mathcal{H}^* \pi^* \tilde{g} + \mathcal{V}^* \hat{g},
\]

where \( \tilde{g} \) is the symmetric 2-tensor determined by the family of symmetric 2-tensors \( \hat{g}_b \) under the foliation \( E = \cup_{b \in B} F_b \).

In our application, the fibrations we will be considering, the Hopf fibrations, are sphere bundles of Euclidean vector bundles. For such fibrations there are preferred choices in the above construction.

**Proposition 2.3.** [2, 9.60] Given a rank-\( k \) Euclidean vector bundle \( E \) over a Riemannian manifold \( (B, \tilde{g}) \) with bundle metric \( \mu \) and metric connection \( H \). There is a unique metric \( g \) on \( S(E) \) which makes \( \pi : (S(E), g) \to (B, \tilde{g}) \) a Riemannian submersion with horizontal distribution determined by \( H \) and totally geodesic fibers isometric to \( (S^{k-1}, \mu) \) given by

\[
g = \mathcal{H}^* \pi^* \tilde{g} + \mathcal{V}^* \mu.
\]

Where \( \mu \) is the restriction to the unit sphere bundle of the tautological Riemannian metric associated to the bundle metric \( \mu \).

### 2.2. Doubly Warped Riemannian Submersions

In this section we introduce a family of metrics we call doubly warped Riemannian submersion metrics. These metrics generalize doubly warped product metrics, by warping the fiber and base in a nontrivial fiber bundle.

**Definition 2.4.** Suppose we are given a Riemannian submersion \( \pi : (E, g) \to (B, \tilde{g}) \), and two positive functions \( f(t) \) and \( g(t) \) defined for \( t \in I \). The doubly warped Riemannian submersion metric \( \tilde{g} \) on \( I \times E \), is a metric so that \( \text{id} \times \pi : (I \times E, \tilde{g}) \to (I \times B, \tilde{g}) \) is the Riemannian submersion as in Proposition 2.2, specified by the following data.

i. horizontal distribution \( \tilde{H} = TI \oplus H \),
ii. base metric $\tilde{g} = dt^2 + h^2(t)\tilde{g}$,

iii. fiber metrics $\hat{g}_{(t,b)} = f^2(t)\hat{g}_b$,

Let $\mathcal{H} : T(I \times E) \to 0 \oplus H$ be the bundle map with kernel $TI \oplus V$, then $\tilde{g}$ takes the following form

$$\tilde{g} = dt^2 + h^2(t)\mathcal{H}^*\pi^*\tilde{g} + f^2(t)V^*\hat{g}. \quad (1)$$

Of course, it is possible to use Proposition 2.2 to define a more general doubly warped Riemannian submersion metric $\tilde{g}$ on $M \times E$ making $\text{id} \times \pi : (M \times E, \tilde{g}) \to (M \times B, g_M + h^2(x)\tilde{g})$ into a Riemannian submersion for an arbitrary Riemannian manifold $(M, g_M)$ with fibers isometric to $(F, f^2(x)\hat{g})$ for any positive functions $f$ and $h$ defined on $M$. In [20], such metrics were considered on $\mathbb{R}^n \times E$ with $(M, g_M) = (\mathbb{R}^n, dr^2 + r^2 ds_{n-1}^2)$ and functions $f$ and $h$ depending only on $r$. In this paper, however, $\tilde{g}$ and “doubly warped Riemannian submersion metric” will always refer to the case $(M, g_m) = (I, dt^2)$.

Computing the curvature of Riemannian submersions is simplified by assuming that each fiber is a totally geodesic submanifold. We have already indicated that we are concerned primarily in the Hopf fibrations, which by Proposition 2.3, natural admit metrics with totally geodesic fibers. One can also think of $\pi : (\{t\} \times E, \tilde{g}) \to (\{t\} \times B, h^2(t)\tilde{g})$ as the original Riemannian submersion with fiber and base scaled by $f(t)$ and $h(t)$ respectively. If the original had totally geodesic fibers, so does the rescaled metric.

**Corollary 2.5.** If $\pi : (E, g) \to (B, \tilde{g})$ is a Riemannian submersion with totally geodesic fibers, then $\pi : (\{t\} \times E, \tilde{g}) \to (\{t\} \times B, h^2(t)\tilde{g})$ is a Riemannian submersion with totally geodesic fibers for every $t \in I$.

We will use Corollary 2.5 to simplify our curvature computations in Section B.1. The result of these computations is that the Ricci curvatures of $\tilde{g}$ are expressible in terms of the Ricci curvatures
of \((E, g), (B, \hat{g}), \text{ and } (F, \hat{g})\) as well as the derivatives of \(f(t)\) and \(h(t)\). The explicit formulas derived in Section [3.1] is as follows.

**Lemma 2.6.** The Ricci curvatures of \(\tilde{g}\) are as follows.

\[
\begin{align*}
\text{(2)} \quad & \text{Ric}_{\tilde{g}}(\partial_t, \partial_t) = -m \frac{h''(t)}{h(t)} - n \frac{f''(t)}{f(t)}, \\
\text{(3)} \quad & \text{Ric}_{\tilde{g}}(X_i, X_i) = \text{Ric}_{\tilde{g}}(Y_i Y_i) \frac{h^2(t) - f^2(t)}{h^4(t)} - \frac{h''(t)}{h(t)} - (m - 1) \frac{h^2(t)}{h^2(t)} - n \frac{f'(t)h'(t)}{f(t)h(t)} + \text{Ric}_{\tilde{g}}(Y_i Y_i) \frac{f^2(t)}{h^4(t)}, \\
\text{(4)} \quad & \text{Ric}_{\tilde{g}}(V_j, V_j) = \frac{\text{Ric}_{\tilde{g}}(U_j U_j) - (n - 1)f^2(t)}{f^2(t)} - \frac{f''(t)}{f(t)} - m \frac{f'(t)h'(t)}{f(t)h(t)} + (\text{Ric}_{\tilde{g}}(U_j U_j) - \text{Ric}_{\tilde{g}}(U_j U_j)) \frac{f^2(t)}{h^4(t)}. 
\end{align*}
\]

\[
\begin{align*}
\text{Ric}_{\tilde{g}}(X_i, V_j) &= \text{Ric}_{\tilde{g}}(Y_i, U_j) \frac{f(t)}{h^3(t)}, \\
\text{Ric}_{\tilde{g}}(X_i, X_j) &= \frac{1}{h^2(t)} \text{Ric}_{\tilde{g}}(Y_i, Y_j). \\
\text{Ric}_{\tilde{g}}(V_i, V_j) &= \frac{1}{f^2(t)} \text{Ric}_{\tilde{g}}(U_i, U_j). \\
\text{Ric}_{\tilde{g}}(X, \partial_t) &= \text{Ric}_{\tilde{g}}(V, \partial_t) = 0.
\end{align*}
\]

### 2.3. Metrics on Quotients.

Consider the doubly warped product metric \(\tilde{g} = dt^2 + h^2(t)ds_n^2 + f^2(t)ds_m^2\) on \([a, b] \times S^n \times S^m\). One may consider the spaces \(D^{n+1} \times S^m\), \(S^{n+1} \times S^m\), and \(S^{n+m+1}\) as quotients of \([a, b] \times S^n \times S^m\) as follows. The first two are achieved by closing the cylinders \([a, b] \times S^n \times \{x\}\) at one or both ends. The last is achieved by closing the cylinders \([a, b] \times S^n \times \{x\}\) at one end and the cylinders \([a, b] \times \{x\} \times S^m\) at the other end. In order to form smooth metrics on these quotients of \([a, b] \times S^n \times S^m\) in terms of the doubly warped product metric certain conditions on the functions \(h(t)\) and \(f(t)\) must be imposed.

Consider first the metric \(dt^2 + f^2(t)ds_n^2\) on \([a, b] \times S^n\). Take \([a, b] \times S^n\) as spherical coordinates for \(D^{n+1}\), with \(a \times S^n \rightarrow 0\). Converting this metric into cartesian coordinates, imposes the following conditions

\[
\text{(5)} \quad f(\text{even})(a) = 0, \text{ and } f'(a) = 0.
\]

If we assume that \(f\) satisfies [5], then we may consider the metric \(dt^2 + f^2(t)ds_n^2 + h^2(t)ds_m^2\) as defined on \(D^{n+1} \times S^m\). In order for this to be smooth, \(h(t)ds_m^2\) must be smooth along any path through the origin in \(D^{n+1}\). As \(ds_m^2\) is rotationally invariant, this is true if an only if \(h(t)\) is even,

\[1\] A more general formula appears in [20, Proposition 4.2].


i.e. if

\[ h^{(\text{odd})}(a) = 0. \]

Thus \( f(t) \) and \( h(t) \) satisfy (5) and (6) respectively, then the doubly warped product will descend
to a smooth metric on \( D^{n+1} \times S^m \). By similar considerations all of the above quotients can be
realized like this. We summarize this in the following proposition.

**Proposition 2.7.** [14, 1.4.7] Let \((B^m, \hat{g})\) be a Riemannian manifold, and let \( f \) and \( g \) be two
nonnegative functions defined on \([a, b]\) that are positive on \((a, b)\). Then doubly warped product
metric \( \tilde{g} = dt^2 + h^2(t) \hat{g} + f^2(t) ds_n^2 \) descends to a smooth metric on the following spaces under the
following conditions.

(i) \( D^{n+1} \times B \), if

\[
\begin{align*}
  f'(a) &= 1, \quad f^{(\text{even})}(a) = 0, \quad f(b) > 0, \\
  h(a) &> 0, \quad h^{(\text{odd})}(a) = 0, \quad h(b) > 0
\end{align*}
\]

(ii) \( S^{n+1} \times B \), if

\[
\begin{align*}
  f'(a) &= -f'(b) = 1, \quad f^{(\text{even})}(a) = f^{(\text{even})}(b) = 0, \\
  h(a) &> 0, \quad h(b) > 0, \quad h^{(\text{odd})}(a) = h^{(\text{odd})}(b) = 0
\end{align*}
\]

(iii) \( S^{n+m+1} \), if \((B, \hat{g}) = (S^m, ds_n^2)\) and

\[
\begin{align*}
  f'(a) &= 1, \quad f(b) > 0, \quad f^{(\text{even})}(a) = f^{(\text{odd})}(b) = 0, \\
  h(a) &> 0, \quad h'(b) = -1, \quad h^{(\text{odd})}(a) = h^{(\text{even})}(b) = 0
\end{align*}
\]

We now wish to explain how to form fiberwise quotients of nontrivial spherical fibrations. Of the
above three cases, (i) and (ii) generalize to the fiber-wise cone and suspension of spherical fibrations
respectively. Case (iii) only applies to the special case where both spheres can be considered as
fibers, i.e. that the fiberation is trivial. For our application, we will only require the generalization
of case (i).

Assume that we are given a rank-\((k + 1)\) vector bundle \( E \) over a Riemannian manifold \((B, \hat{g})\)
with bundle metric \( \mu \) and connection \( H \). Using Proposition 2.3 this defines a metric \( g \) on \( S(E) \)
making \( \pi : (S(E), g) \to (B, \hat{g}) \) a Riemannian submersion. The following proposition explains how
to form a metric on \( D(E) \).

**Proposition 2.8.** Suppose we are given a Riemannian submersion \((S(E), g) \to (B, \hat{g})\) with \(O(k+1)\)
fiber metrics. Let \( f \) and \( h \) be nonnegative functions defined on \([a, b]\) that are positive on \((a, b)\]
satisfying the following.

\[
\begin{align*}
  f^{(\text{even})}(a) &= 0, \quad f'(a) = 1, \quad h(a) > 0, \quad \text{and} \quad h^{(\text{odd})}(a) = 0.
\end{align*}
\]

Then the doubly warped Riemannian submersion metric \( \tilde{g} = dt^2 + H^* \pi^* \hat{g} + V^* \hat{g} \) on \([a, b] \times S(E)\)
associated to \( g \) descends to a smooth metric on \( D(E) \).
\( F_b \cong S^{k-1} \quad S(E) \quad F_b \cong I \times S^{k-1} \quad I \times S(E) \quad F_b \cong D^k \quad D(E) \cong I \times S(E)/ \sim \)

\[ \begin{array}{ccc}
\pi & \downarrow & \pi \\
\circ & \rightarrow & \circ \\
\downarrow & & \downarrow \\
B & & B \\
& f(t) = h(t) & \\
\end{array} \]

\[ \begin{array}{ccc}
\pi & \downarrow & \pi \\
\circ & \rightarrow & \circ \\
\downarrow & & \downarrow \\
I \times B & & I \times B \\
& h(t) & \\
\end{array} \]

\[ \begin{array}{ccc}
\pi & \downarrow & \pi \\
\circ & \rightarrow & \circ \\
\downarrow & & \downarrow \\
B & & B \\
& \pi t & \\
\end{array} \]

\[ \begin{array}{ccc}
\pi & \downarrow & \pi \\
\circ & \rightarrow & \circ \\
\downarrow & & \downarrow \\
I \times B & & I \times B \\
& \pi t & \\
\end{array} \]

**Figure 3.** The Riemannian submersion \( \pi : (S(E),g) \rightarrow (B,\tilde{g}) \), a standard doubly warped Riemannian submersion metric with respect to this Riemannian submersion, and forming the disk bundle from a quotient of the doubly warped Riemannian submersion metric.

**Proof.** As the fiber metric \( \hat{g} \) is \( O(k+1) \) invariant, it must be that \( (F_b,\hat{g}) \cong (S^n,ds_k^2) \). Now, combining Proposition \( 2.3 \) with formula \( 1 \) we see that the doubly warped Riemannian submersion metric takes the form

\[ \tilde{g} = dt^2 + h^2(t)H^*\pi^*\tilde{g} + f^2(t)\nu^*ds_k^2. \]

Fix a point \( b \in B \), and consider \( T(I \times S(E)) \cong TI \times TS(E) \) restricted to \( I \times F_b \cong I \times S^k \). As \( \mathcal{H} \) is invariant under \( O(k+1) \) we see that this bundle is isomorphic to \( TI \oplus T_bB \oplus TS^k \). The 2-form \( \tilde{g} \) restricted to this bundle is therefore isometric to the following.

\[ dt^2 + h^2(t)\tilde{g}_b + f^2(t)ds_k^2. \]

Where \( \tilde{g}_b \) is the metric \( \tilde{g} \) restricted to the point \( b \in B \). By Proposition \( 2.7 \) the assumption on \( f(t) \) at \( t = 0 \) implies that \( dt^2 + f^2(t)ds_k^2 \) defines a smooth metric on \( D^{k+1} \). And for any fixed horizontal vectors \( X,Y \in T_bB \), the function \( h^2(t)\tilde{g}_b(X,Y) \) is even. It follows that the symmetric 2-tensor \( h^2(t)\tilde{g}_b \) is smooth on \( D^{k+1} \). Thus \( \tilde{g} \) descends to a smooth 2-tensor on the bundle \( TB \oplus TD^{k+1} \) over \( I \times D^{k+1} \).

This argument holds for all \( b \), which shows that \( \tilde{g} \) descends to a smooth metric on \( I \times S(E)/ \sim \) where \( \{0\} \times F_x \sim \{0\} \times \{x\} \). The effect of this quotient is to fiber-wise close the cylinder \( I \times S^k \) to \( D^{k+1} \), coinciding therefore with \( D(E) \). \( \square \)

### 3. Construction of the Core

In this section, we present the proof of Theorem \( C \) that there exists a Ricci-positive metric on the punctured projective spaces so that the boundary is a convex, round sphere. In Section \( 3.1 \) we recall that punctured projective spaces can be thought of as disk bundles of the tautological bundle. Thus the work in Section \( 2.3 \) allows us to define doubly warped Riemannian submersion metrics on the punctured projective spaces. In section \( 3.2 \) we explain how this construction agrees
with the definition present in [13]. We conclude with the proof of Theorem C by considering the equations of Lemma 2.6 in the particular case of projective spaces and picking specific choices of warping functions.

3.1. The Form of the Metric. We are now ready to explain how Proposition 2.8 can be applied to produce the core metrics for projective spaces. We first recall some facts about the geometry of projective spaces.

To avoid repetition we will cover all cases simultaneously. Let $\mathbb{P}^n$ be a projective plane over $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ (where if we consider $\mathbb{O} \mathbb{P}^n$, we assume $n \leq 2$ [10, Corollary 4L.10]). Let $d$ be the real dimension of the algebra over which $\mathbb{P}^n$ is defined. The tautological bundle $\gamma_n$ over $\mathbb{P}^n$ is a rank $d$ vector bundle, such that the total space of the associated sphere bundle $S(\gamma_n)$ is diffeomorphic to $S^{d(n+1)-1}$ with fiber $S^{d-1}$. These sphere bundles are called the generalized Hopf fibrations.

Typically, one uses the Hopf fibration to define the metric on the base of the bundle, the Fubini-Study metric $\tilde{g}$ on $\mathbb{P}^n$, by declaring them Riemannian submersions with total space and fibers both round with radius 1. The fact that the Hopf bundles are also the sphere bundles of the tautological bundles, allows us to reverse this relationship. Specifically we may apply Proposition 2.3 to construct new metrics on $S(\gamma_n)$ with respect to a fixed metric on the base and vector bundle data. In particular if we use the bundle data specified by the Hopf fibration we get back $(S^{dn-1}, ds_{dn-1}^2)$.

**Corollary 3.1.** Let $\tilde{g}$ denote the Fubini-Study metric on $\mathbb{P}^{n-1}$, and let $H$ and $V$ the horizontal and vertical distributions determined by the Hopf fibration $(S^{dn-1}, ds_{dn-1}^2) \to (\mathbb{P}^{n-1}, \tilde{g})$. Then

$$ds_{dn-1}^2 = H^* \pi^* \tilde{g} + V^* ds_{d-1}^2.$$  

That the construction in Proposition 2.3 is then compatible with Proposition 2.8 allows us to define metrics on $D(\gamma_n)$. As $\mathbb{P}^n \setminus D^{dn} = D(\gamma_n)$, we can use this to define the core metrics on $\mathbb{P}^n$.

**Definition 3.2.** Let $\tilde{g}$ be the doubly warped Riemannian submersion metric associated to the Hopf fibration $\pi : (S^{nd-1}, ds_{nd-1}^2) \to (\mathbb{P}^{n-1}, \tilde{g})$ and any functions $f$ and $h$ satisfying the hypotheses of Proposition 2.8 on $[0, t_1]$. Then by Proposition 2.8, $\tilde{g}$ descends to a smooth metric on $\mathbb{P}^n \setminus D^{dn}$ taking the following form.

$$\tilde{g} = dt^2 + h^2(t)H^* \pi^* \tilde{g} + f^2(t)V^* ds_{d-1}^2.$$  

The Hopf fibrations are very special in that they are Riemannian submersions for which the base, fiber, and total space are all endowed with Einstein metrics. This greatly simplifies the formulas of Lemma 2.6 as all of the Ricci tensors on the righthand side can be replaced with constants. For $(S^k, r^2 ds_k^2)$, the Einstein constant is $\frac{1}{r^2}(k - 1)$. For $(\mathbb{P}^n, \tilde{g})$ the constants are as follows.

**Proposition 3.3.** [2, Examples 9.81, 9.82, and 9.84] If $\tilde{g}$ is the Fubini-Study metric on $\mathbb{P}^n$ over an algebra of real dimension $d$, then

$$\text{Ric} \tilde{g} = \left(\frac{(n-1)d + 4(d-1)}{11}\right) \tilde{g}.$$  

3.2. **Perelman’s Core.** We now explain how these doubly warped Riemannian submersion metrics are indeed a generalization of the type of metric used in [13] to define metrics on $\mathbb{C}P^2 \setminus D^4$. As in general, $\mathbb{C}P^2 \setminus D^4$ can be identified with the disk bundle of the normal bundle of an embedded $\mathbb{C}P^1 \rightarrow \mathbb{C}P^2$, which is isomorphic as a real vector bundle to the tautological complex line bundle. Recall that the sphere bundle of the tautological line bundle over $\mathbb{C}P^1 \approx S^2$ is the Hopf fibration $S^3 \rightarrow S^2$. Thus we can define a metric on $\mathbb{C}P^2$ by considering metrics on $I \times S^3$.

In this special case, the sphere you are considering, $S^3$, is a Lie group. It therefore admits a globally defined left-invariant, orthonormal frame $X$, $Y$, and $Z$. We may assume that the vector field $Z$ is the image of the vector field $\Theta$ of $S^1$ under the differential of the inclusion of each fiber. Denote the dual covector fields of $X$, $Y$, and $Z$ with respect to $ds^2$ by $dx$, $dy$, and $dz$ respectively. After checking that $dz^2 = \mathcal{V}^* \hat{g}$ and $dx^2 + dy^2 = \mathcal{H}^* \pi^* \hat{g}$, formula (4) becomes $dt^2 + h^2(t)(dx^2 + dy^2) + f^2(t)dz^2$. This is the form in which Perelman provided the metric in [13, Section 2], with $f(t) = \sin t \cos t$ and $h(t) = \frac{1}{100} \cosh \left( \frac{t}{100} \right)$. These particular choices satisfy the requirements of Proposition 2.8 and so this defines a doubly warped Riemannian submersion metric on $\mathbb{C}P^2 \setminus D^4$.

The 7-sphere is also parallelizable, and similarly we may identify three of the frames as coming from the action of $S^3$. Thus we may also use Perelman’s approach to define a metric on $\mathbb{H}P^2$. Beyond $S^7$, there are no further spheres that are parallelizable, thus this approach cannot be used in general. While the tangent bundles of $S^{dn-1}$ are not trivial, they still have a global decomposition as $H \oplus V$ given by the Hopf fibration. And as we have seen in Section 3.1, this decomposition is sufficient to define analogous metrics.

Perelman also used the fact that $S^3$ is a Lie group to compute the Ricci curvature of his metrics. For a Lie group, the curvature tensor of the left-invariant vector fields is

$$ R(X, Y)Z = \frac{1}{4}[[X, Y], Z]. $$

As there are no Lie groups diffeomorphic to $S^{dn-1}$ beyond $S^3$, this necessitates the work done in Section B.1 to compute the Ricci tensor of a doubly warped Riemannian submersion metric. The exact Ricci curvatures computed in [12] are stated as follows.

**Lemma 3.4.** [13, Section 2] Let $X$, $Y$, and $Z$ are the global left-invariant vector fields of $S^3$. If $g = dt^2 + h^2(t)(dx^2 + dy^2) + f^2(t)dz^2$, then

$$ \text{Ric}(\partial_t, \partial_t) = -h''(t)h\frac{f''}{f} - 2\frac{f''}{f}, $$

$$ \text{Ric}(X, X) = \text{Ric}(Y, Y) = 4\frac{h^2 - f^2}{h^4} - h''\frac{h^2}{h^2} - 2\frac{h'^2}{fh} + 2\frac{f^2}{h^3}, $$

$$ \text{Ric}(Z, Z) = -\frac{f''}{f} - 2\frac{f'h'}{fh} + 2\frac{f^2}{h^3} $$

And all other terms in this frame vanish.
As Perelman’s construction agrees with the doubly warped Riemannian submersion, we can check that the formulas in Lemma 3.4 agree with those in Lemma 2.6. Indeed, in the case of $\mathbb{C}P^2 \setminus D^4$ we have $n = 1$, $m = 2$, $\text{Ric}_g = ds_2^3$, $\text{Ric}_\tilde{g} = 0$, and $\text{Ric}_\hat{g} = \frac{1}{2} ds_2^3$. Plugging these values into the formula in Lemma 2.6 agrees with the above.

3.3. Ricci Curvature of the Cores. We are now ready to prove Theorem C.

Proof of Theorem C. Let $\mathbf{P}^n$ denote the projective space over either $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, and let $d$ denote the real dimension of the algebra. By Definition 3.2 we can specify doubly warped Riemannian submersion metrics $\tilde{g}$ on $\mathbf{P}^n \setminus D^nd$ by choosing functions $f(t)$ and $h(t)$ on $[0, t_1]$ that satisfy the hypotheses of Proposition 2.8. We claim that for a certain choice of $f(t)$ and $h(t)$: $\text{Ric}_\tilde{g}$ is positive definite and the boundary is round and convex.

As observed, the underlining Riemannian submersions $\pi(S^{dn-1}, ds_{dn-1}^2) \to (\mathbf{P}^{n-1}, \tilde{g})$ have totally geodesic fibers, and the metrics of the total space, base, and fiber are all Einstein. The Einstein constants of the Fubini-Study metrics are recorded in Proposition 3.3, and the Einstein constant for $(S^k, ds_k^2)$ is $(k - 1)$. In this situation we have the following values.

\begin{align*}
(7) \quad \dim E &= dn - 1, \quad \dim B = d(n - 1), \quad \text{and} \quad \dim F = d - 1. \\
(8) \quad \text{Ric}_g &= [dn - 2]g, \quad \text{Ric}_\tilde{g} = [(n - 2)d + 4(d - 1)]\tilde{g}, \quad \text{and} \quad \text{Ric}_\hat{g} = (d - 2)\hat{g}
\end{align*}

Regardless of the choice of $f(t)$ and $h(t)$, define $t_1$ as the smallest value of $t$ for which $f(t_1) = h(t_1)$. By Corollary 3.1 at $t = t_1$

$$\tilde{g} = h^2(t_1) \left( H^* \pi^* \tilde{g} + V^* ds_{d-1}^2 \right) = h^2(t_1) ds_{d-1}^2.$$ 

Thus the boundary is round regardless of our choice of $h(t)$ and $f(t)$.

Consider the choice $f(t) = \sin(t)$ and $h(t) = \varepsilon < 1$. This choice satisfies the hypotheses of Proposition 2.8 and therefore define metrics on $\mathbf{P}^n \setminus D^nd$. Substituting this choice of $f(t)$ and $h(t)$ into the formulas in Lemma 2.6 and replacing the Ricci curvatures and dimensional constants with those from equations (8) and (7) yields the following.

$$\text{Ric}_\tilde{g}(\partial_t, \partial_t) = (d - 1);$$
$$\text{Ric}_\tilde{g}(X_i, X_i) = [(n - 2)d + 4(d - 1)] \frac{\varepsilon^2 - \sin^2 t}{\varepsilon^4} + (dn - 2) \frac{\sin^2 t}{\varepsilon^4};$$
$$\text{Ric}_\tilde{g}(V_j, V_j) = (d - 2) \tan^2 t + 1 + [(nd - 2) - (d - 2)] \frac{\sin^2 t}{\varepsilon^4}.$$ 

As $\text{Ric}_g = g$ in this case, the off-diagonals of $\text{Ric}_\tilde{g}$ in Lemma 2.6 all vanish. Since $0 \leq t \leq t_1 < \frac{\pi}{2}$, and $\sin(t) \leq \varepsilon$, it is easy to check that the remaining Ricci curvatures are all strictly positive. Thus, for this choice of $f(t)$ and $h(t)$, $\text{Ric}_\tilde{g}$ is positive definite.
By Corollary B.7, we need \( h'(t_1) > 0 \) and \( f'(t_1) > 0 \) in order for the boundary to be convex. For \( h(t) = \varepsilon \) this is not the case. We may replace \( h(t) \) with a function that has \( h'(t_1) > 0 \), and such that \( ||h(t) - \varepsilon||_{C^2} \) is small enough so as not to upset \( \text{Ric}_{\tilde{g}} > 0 \). With this function \( h(t) \), the claim follows. In [13], Perelman chose \( h(t) = \frac{1}{N} \cosh \left( \frac{t}{N} \right) \) and \( N = 100 \). For each \( n \), there exists a choice of \( N \), for which this \( h(t) \) and \( f(t) \) will define \( \tilde{g} \) with convex boundary and positive Ricci curvature.

We have, thus far, omitted \( \mathbb{R}P^n \) from discussion. Since \( \mathbb{R}P^n \) also admits a tautological bundle, one may equally well define a doubly warped Riemannian submersion metrics on \( \mathbb{R}P^n \) as in Section 3.1. But because the fibers of the real Hopf fibration are \( S^0 \), the metric reduces to a warped product metric. In particular, instead of equation (2), such a metric has

\[
\text{Ric}_{\tilde{g}}(\partial_t, \partial_t) = -(n-1) \frac{h''(t)}{h(t)}.
\]

Thus \( h(t) \) must be concave for Ricci curvature to be positive. But in order for \( \tilde{g} \) to be smooth, \( h'(0) = 0 \) by Proposition 2.8 and in order for the boundary to be convex \( h'(t_1) > 0 \) by Lemma B.7. There is no smooth function with \( h'(0) = 0 \), \( h''(t) < 0 \), and \( h'(t_1) > 0 \) for \( 0 < t_1 \). Thus it is not possible to prove Theorem C for \( \mathbb{R}P^n \) using doubly warped Riemannian submersion metrics.\[\square\]

4. Constructions of Connected Sums with Positive Ricci Curvature

In this section we prove assemble the various constructions in Section 3 and Section A in order to form connected sums with positive Ricci curvature. We begin by proving Proposition 1.3 in Section 4.1 by combining the work present in [13], specifically Lemma A.1, Lemma 1.2 and Lemma 4.1 quoted below. Next we show how this with Lemma 1.2 implies Theorem B in Section 4.2. This with Theorem C completes the proof of Theorem A. In Section 4.3, we show how Lemma 4.1 can also be used to define a docking station metric on lens spaces, thus allowing one to generalize Theorem B to include a single lens space. Finally in Section 4.4 we give a topological construction that allows us to form arbitrary connected sums of \( S^m \times \mathbb{P}^n \).

We now summarize the results of [13, Section 3]. While Perelman stated this result only with \( n = 4 \), the constructed metric is a doubly warped product. By increasing the dimension of one of the warped spheres, one gets a metric on \( S^n \) for all \( n \geq 4 \). It is easily seen that all of the curvature conditions transfer to higher dimensions.

**Lemma 4.1.** [13, Section 3] For all \( n \geq 4 \), there is an \( \eta \) such that, for any integer \( k \) and \( 0 < r < \eta \), there is a metric \( g_{\text{ambient}} \) on \( S^n_k \) such that the metric restricted to each boundary component is isometric to \( g_s \) such that

(i) \( K_g > 0 \);
(ii) \( K_{g_s} > 1 \);
(iii) \( |\Pi_{g_s}| < 1 \);
(iv) $g_δ = dφ^2 + f^2(φ)ds_{n-2}^2$ with $φ ∈ [0, πR]$, $sup_φ f = r$, and $0 < r^{n-1}/n < R < 1$.

The proof of Lemma 4.1 occupies the entirety of Section 3 of [13]. The argument is fairly straightforward, relying mostly on careful consideration of classical spherical geometry.

4.1. Constructing the Docking Station. In this section, we explain how Lemma 1.2, Lemma A.1, and Lemma 4.1 fit together to construct the metric of Proposition 1.3.

Proof of Proposition 1.3. As illustrated in Figure 4, we attach $k$ copies of $[0, 1] × S^n$ with the neck metric to $S^n_k$ with the ambient space metric using Lemma 1.2.

Fix $n > 3$, $k > 0$, and $ρ < 1$. Let $S^n_k$ denote $S^n$ minus $k$-disjoint balls. Pick $r < R < 1$, such that $r^{n-1}/n < ρ < R$. By Lemma 4.1, there exists a metric $g_{ambient}$ on $S^n_k$ such that $Ric_{g_{ambient}}$ is positive definite and each boundary component is isometric to $(S^{n-1}_{k}, g_δ)$, where $K_{g_δ} > 1$, $|II_{g_δ}| < 1$, and $g_δ = dφ^2 + f^2(φ)ds_{n-2}^2$ with $φ ∈ [0, πR]$ and $sup_φ f(φ) = r$.

It follows that we may use $g_1 = g_δ$ as the initial data in Lemma A.1. Thus there is a metric $g_{neck}(ρ)$ on $[0, 1] × S^{n-1}$ such that $Ric_{g_{neck}}$ is positive definite, the boundary at 0 is isometric to $(S^{n-1}, g_δ)$ with extrinsic curvatures all identically $-λ$, and the boundary at 1 is isometric to $(S^{n-1}, g_δ)$ and $II_1 > 1$.

Thus there is an isometry $φ$ between the boundary of $([0, 1] × S^n, g_{neck})$ at 1 and any one of the boundary components of $(S^n_k, g_{ambient})$. Both manifolds have positive Ricci curvature, and by construction $φ^* II_1 + II_{g_δ} > 0$. It follows from Lemma 1.2 that there exists a metric $g$ on $S^{n-1} × [0, 1] ∪ φ S^n_k$ that agrees with $g_{neck}$ and $g_{ambient}$ away from a small neighborhood of the image $φ$. In particular, the boundary at 0 remains isometric to $(S^{n-1}, ρ^2/λ^2 ds_{n-1}^2)$ and the remaining disjoint $(k-1)$ boundary components remain isometric to $(S^{n-1}, g_δ)$. Thus we may glue $k$ disjoint copies of $([0, 1] × S^n, g_{neck})$ to each boundary component of $(S^n_k, g_{ambient})$.

The resulting metric $g$ is defined on $S^n_k$. It has positive Ricci curvature and boundary components all isometric to $(S^{n-1}, ρ^2/λ^2 ds_{n-1}^2)$ with extrinsic curvatures $-λ$. Set $g_{docking} = λ^2 g$. The Ricci

![Figure 4. The construction of the docking station.](image-url)
The ambient space

a core

a neck

Figure 5. The construction $\#_{i=1}^k M_i^n$ from the docking station.

Curvature is unaffected by scaling, and each boundary component of the boundary of $(S^n_k, g_{docking})$ is isometric to $(S^{n-1}, \rho^2 ds^2_{n-1})$, and by Lemma B.17 the extrinsic curvatures of the boundary are all $-1$.

The docking station is the most technical aspect of [13] as it involves proving Lemma A.1, which we have included in Section A. One may ask if this construction is really necessary for the proof of Theorem B in Section 4.2. Suppose we tried to use the round $n$-sphere as a docking station. In order to use Lemma 1.2 to attach $M^n \setminus D^n$, if its round boundary has radius $r$, then we must assume that the extrinsic curvatures of the boundary are at least $\cot r$. It is a consequence of [17, Theorem 1], that there is an $r_n > 0$ such that $M$ is contractible if $r < r_n$. So the round sphere cannot be used to construct Ricci positive metrics in this way for arbitrarily large connected sums.

Thus one must endeavor to construct a metric on $S^n$ with arbitrarily many small balls with relatively large second fundamental form. In some sense, the metric constructed in Lemma 4.1 is the easiest such construction (it is a doubly warped product). The issue this raises is that the boundaries of the small balls are not round, they are warped products. On the other hand, for doubly warped Riemannian submersion metrics, there is no way for the boundary to be warped if the fiber bundle is nontrivial. This necessitates Lemma A.1 which proves that it is possible to transition between the two.

4.2. Attaching the Cores. In this section, we explain how Lemma 1.2 and Proposition 1.3 can be used to construct Ricci positive metrics on connected sums of manifolds that admit core metrics. Specifically we prove Theorem B.

2The convexity invariant of [17] is $\Lambda(M \setminus D^n) \geq \cos r$ in this instance, and that $K > \csc^2 r + \cot^2 r$ at the boundary. Thus the hypotheses of Theorem 1 are met for small enough $r$. 

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Proof of Theorem A. As illustrated in Figure 5, we attach the cores \( (M_i^n \setminus D^n, g_i) \) to the docking station.

Suppose we want to take a connect sum of manifolds \( M_i \) with \( 1 \leq i \leq k \). Let \( g_i \) be the metrics on \( M_i \setminus D^n \) with positive Ricci curvature and round, convex boundary \( N_i \). Let \( \rho_i \) be the radius of \( N_i \) and let \( \nu_i = \inf_{S(TN_i)} \Pi g_i \). Pick a small number \( 0 < \rho < 1 \) and define \( s_i \) to be the number such that \( s_i \rho_i = \rho \). Assume that \( \rho \) is so small that \( \frac{\nu_i}{s_i} > 1 \). The manifolds \( (M_i \setminus D^n, s_i^2 g_i) \) have positive Ricci curvature, boundaries isometric to \( (S^{n-1}, \rho^2 ds_{n-1}^2) \), and by Lemma B.17 extrinsic curvatures of the boundaries all greater than \( \frac{\nu_i}{s_i} > 1 \).

By Proposition 1.3 there exists a Ricci positive metric \( g_{\text{docking}} \) on \( S^k_\rho \) with boundaries isometric to \( (S^{n-1}, \rho^2 ds_{n-1}^2) \) and extrinsic curvatures all equal \(-1\). We clearly have isometries \( \phi_i \) between \( N_i \) and the boundary components of \( (S^k_\rho, g_{\text{docking}}) \). By Lemma 1.2 we can glue each of the \( (M_i^n \setminus D^n, s_i^2 g_i) \) to \( (S^k_\rho, g_{\text{docking}}) \) along \( \phi_i \) so that the resulting space admits a Ricci positive metric. The resulting space is clearly diffeomorphic to \( \#_{i=1}^k M_i^n \).

□

With this, we have proven that the connected sums of all complex, quaternionic, and octonionic projective spaces admit metrics with positive Ricci curvature.

Proof of Theorem A. By Theorem C for all \( n > 1 \), \( CP^n \), \( HP^n \), and \( OP^2 \) admits core metrics. Theorem A now follows directly from Theorem B. □

Notice that one may just as well take connected sums between the different projective spaces in their common dimensions. Thus the following is immediate.

**Corollary 4.2.** For all \( n \geq 1 \), \( j \geq 0 \), \( k \geq 0 \), and \( l \geq 0 \), the following manifolds admit metrics with positive Ricci curvature.

\[
\begin{align*}
(1) \quad & \#_j CP^{2n} \# \#_k HP^n \\
(2) \quad & \#_j CP^8 \# \#_k HP^4 \# \#_l OP^2
\end{align*}
\]

4.3. Lens Spaces as Docking Stations. In this section, let \( G \) denote a finite subgroup of the isometry group of \( (S^n, g) \). If \( G \) acts freely, then \( (S^n/G, g) \) is a smooth manifold locally isometric to \( (S^n, g) \) (III Theorem 21.13). In particular, the metric \( g_{\text{ambient}} \) of Lemma 4.1 is a doubly warped product metric and so has a large isometry group: \( O(2) \oplus O(n-1) \). One might hope then that some nontrivial quotients of \( S^n \) might play the role of docking station in Proposition 1.3, allowing us to form new examples of connected sums with positive Ricci curvature. Assume that \( G \) is a finite group acting freely on \( S^n \), then denote by \( (S^n/G)_k \) the quotient manifold with \( k \)-disjoint geodesic balls removed.

**Corollary 4.3.** Suppose that \( G \) is a finite subgroup of \( O(2) \oplus O(n-1) \leq O(n+1) \) and that the action of \( O(n+1) \) on \( S^n \) restricted to \( G \) is free. For any \( n > 3 \), \( k > 0 \), and \( 0 < \rho < 1 \), there is a metric \( g_{\text{lens}} \) on \( (S^n/G)_k \) with positive Ricci curvature and so that each boundary component is isometric to \( (S^{n-1}, \rho^2 ds_{n-1}^2) \) with all extrinsic curvatures equal to \(-1\).
Proof. Start by choosing $0 < r < R < 1$ such that $r^{n-1} < \rho < R$. Use Lemma 4.1 to define a Ricci positive metric $g_{\text{ambient}}$ on $S^n_l$ with $l = |G|k$ whose boundaries are all isometric to $(S^{n-1}, g_\delta)$. As the action of $G$ is free, it is possible to choose these geodesic balls disjoint and invariant under the action of $G$. It follows that the quotient $S^n/G$ has a Ricci positive metric with $k$-disjoint geodesic balls whose boundaries are all isometric to $(S^{n-1}, g_\delta)$ and extrinsic curvatures equal to $-1$. The proof now procedes identically to the proof of Proposition 1.3 in Section 4.1 \( \Box \)

Thus $(S^n/G)_k$ can now replace $S^n_k$ in the construction in the proof of Theorem B in Section 4.2. The effect topologically is taking a connected sum with $S^n/G$. Thus the following is immediate.

**Corollary 4.4.** For all $k \geq 1$ and $n \geq 4$, suppose that $G$ is a finite subgroup of $O(2) \oplus O(n-1) \leq O(n+1)$ and that the action of $O(n+1)$ on $S^n$ restricted to $G$ is free. If there exists Ricci positive metrics $g_i$ on $M^n_i \setminus D^n$ with round, convex boundaries, then the following manifold admits a Ricci positive metric.

$$(S^n/G)\# \left( \#_{i=1}^k M^n_i \right).$$

The main example of such quotients $S^n/G$ are lens spaces. Recall the following definition.

**Definition 4.5.** [10] Example 2.43 For a fixed positive integer $m$, for $1 \leq i \leq n$, let $\ell_i$ be integers with $\gcd(m, \ell_i) = 1$. Let $\theta_j = 2\pi \ell_j/m$. Define an action of $\mathbb{Z}/m\mathbb{Z}$ on $S^{2n-1} \subseteq \mathbb{C}^n$ by the standard action of diag(exp($i\theta_1$), ..., exp($i\theta_n$)) on $\mathbb{C}^n$. Then define the lens space associated to the tuple $(m, \ell_1, \ldots, \ell_n)$ by the quotient of this action

$L(m; \ell_1, \ldots, \ell_n) = S^{2n-1}/(\mathbb{Z}/m\mathbb{Z})$.

In particular, $L(2; 1, \ldots, 1) \cong \mathbb{R}P^{2n-1}$. Notice that under an $\mathbb{R}$-linear isomorphism $\mathbb{C}^n \cong \mathbb{R}^{2n}$, the action of $\mathbb{Z}/m\mathbb{Z}$ defined above acts as a subgroup of $O(2) \oplus O(2n - 2)$. Thus all lens spaces occur as possibilities of $S^{2n-1}/G$ in Corollary 4.4. While $\mathbb{R}P^{2n}$ is not a lens space, we see that $-I \in O(2) \oplus O(n - 2)$ for all $n$ and therefore $\mathbb{R}P^{2n}$ will also occur as $S^{2n}/G$. The following is immediate.

**Corollary 4.6.** For all $k \geq 2$, $n \geq 3$, and $d \geq 4$. If there are core metrics on $M_i$, then the following manifolds admit metrics with positive Ricci curvature

$L(m; \ell_1, \ldots, \ell_n)\# \left( \#_{i=1}^k M^n_i \right)$ and $\mathbb{R}P^d\# \left( \#_{i=1}^k M^d_i \right)$.

It is interesting to note that for a fixed $m$ and $n$, $L(m; \ell_1, \ldots, \ell_n)$ will not be diffeomorphic to $L(m; \ell'_1, \ldots, \ell'_n)$ unless there is a $k \in \mathbb{Z}/m\mathbb{Z}$ and a permutation $\sigma \in S_n$ such that $\ell_i \equiv k\ell'_{\sigma(i)} \mod m$ for all $1 \leq i \leq n$ (see [3]). As it stands, though, we do not have an application to the statement involving $L(m; \ell_1, \ldots, \ell_n)$ in Corollary 4.6. We therefore pose the following questions.

**Question 4.7.** Does there exist an odd dimensional manifold $M^{2n+1}$ that admits a Ricci positive metric on $M^{2n+1} \setminus D^{2n+1}$ with round, convex boundary?
In even dimensions, Theorem C provides the existence of core metrics on various projective spaces. Thus Corollary 4.6 provides new examples in the case of $\mathbb{R}P^{2n}$. In particular this proves the following corollary

**Corollary 4.8.** For all $n$, $i$, $j$, and $k$ the following manifolds admit metrics with positive Ricci curvature.

1. $\mathbb{R}P^{2n} \# (#_j \mathbb{C}P^n)$
2. $\mathbb{R}P^{4n} \# (#_j \mathbb{C}P^{2n}) \# (#_k \mathbb{H}P^n)$
3. $\mathbb{R}P^{16} \# (#_j \mathbb{C}P^8) \# (#_k \mathbb{H}P^4) \# (#_l \mathbb{O}P^2)$

While lens spaces may play the role of the docking station, if they admit core metrics, by Corollary 4.6 we have proven that $L(m; \ell_1, \ldots, \ell_n)$ $\# L(m'; \ell_1', \ldots, \ell_n')$ admits a Ricci positive metric.

As $\pi_1(L(m; \ell_1, \ldots, \ell_n)) = \mathbb{Z}/m\mathbb{Z}$, and if $M^n$ and $N^n$ with $n \geq 3$ are manifolds with nontrivial fundamental groups, then $\pi_1(M \# N)$ is infinite by the Seifert-van Kampen theorem. But Myers' theorem (see [14, Theorem 6.3.3]) implies that a closed manifold with positive Ricci curvature has finite fundamental group. Thus contradicting the assumption that lens spaces have core metrics.

**Corollary 4.9.** For $n \geq 4$, if $|\pi_1(M^n)| > 0$, then there are no Ricci positive metrics on $M^n \setminus D^n$ with round, convex boundaries. In particular, $\mathbb{R}P^n$ and $L(m, \ell_1, \ldots, \ell_k)$ do not admit such metrics.

While we have explained in our remark after the proof of Theorem C that the methods of this paper do not work constructing core metrics on $\mathbb{R}P^n$, the above corollary shows that no such metric exists.

**4.4. Connected Sums of Product Spaces.** In this section, we show how to take these methods slightly further. If we take the cartesian product of $S^m$ with each projective space in the list of manifolds of Corollary 4.2, then the we claim the statement remains true. Specifically we claim the following.

**Corollary 4.10.** For all $i \geq 0$, $j \geq 0$, $k \geq 0$, and $l \geq 0$; all $n \geq 2$; and all $m \geq 3$, the following manifolds admit metrics of positive Ricci curvature.

1. $(#_i S^{2n} \times S^m) \# (#_j \mathbb{C}P^n \times S^m)$
2. $(#_i S^{4n} \times S^m) \# (#_j \mathbb{C}P^{2n} \times S^m) \# (#_k \mathbb{H}P^n \times S^m)$
3. $(#_i S^{16} \times S^m) \# (#_j \mathbb{C}P^8 \times S^m) \# (#_k \mathbb{H}P^4 \times S^m) \# (#_l \mathbb{O}P^2 \times S^m)$

This is a generalization of the following result due to Sha and Yang.

**Theorem 4.11.** [15, Theorem 1] For all $k \geq 1$, $n \geq 2$, and $m \geq 2$, the manifold $\#_k (S^n \times S^m)$ admits a metric with positive Ricci curvature.

The approach in [15] to proving this theorem, was to prove a surgery theorem for Ricci positive manifolds (see [15, Lemma 1] or [19, Theorem 0.3]) under the assumption of an isometrically
embedded standard $S^{n-1} \times D^{m+1}$. Then one can recognize $\#_k(S^n \times S^m)$ as performing surgery on $(k + 1)$ disjoint copies of $S^{n-1} \times D^{m+1}$ inside of $S^{n-1} \times S^{m+1}$, i.e. that

(9) \[ \#_k(S^n \times S^m) \cong \left( S^{n-1} \times \left( S^{m+1} \setminus \bigcup_{i=0}^k D^{m+1} \right) \right) \cup_{\partial} \left( \bigcup_{i=0}^k D^n \times S^m \right). \]

Our approach to proving Corollary 4.10 will rely on an observation like equation (9). Indeed, we will show that one may replace $S^n$ with $M^n_i$ on the lefthand side if one replaces $k$ of the $D^n$ on the righthand side with $D^n \# M^n_i$. While we do not have a surgery theorem present in this paper, we do have the docking station metrics on $S_k^n$, which will allow us to carry out similar constructions as in the proof of Theorem 12 in Section 4.2.

4.4.1. Topological Constructions. The necessary technical result to construct these metrics is the following surgery theorem, which gives a description of connected sums in terms of surgery on an ambient space.

**Proposition 4.12.** For any closed, oriented, and smooth manifolds $N^n_i$.

\[ \#_{i=1}^k(N^n_i \times S^m) \cong \left( S^{n-1} \times \left( S^{m+1} \setminus \bigcup_{i=0}^k D^{m+1} \right) \right) \cup_{\partial} \left( D^n \times S^m \sqcup \bigcup_{i=1}^k (D^n \# N_i) \times S^m \right). \]

This theorem follows from two lemmas. The first lemma establishes the case when $k = 1$. The second lemma establishes the inductive step.

**Lemma 4.13.** Let $N$ be a closed, oriented, and smooth manifold.

\[ N^n \times S^m \cong S^{n-1} \times \left( S^{m+1} \setminus D^{m+1} \cup D^{m+1} \right) \cup_{\partial} \left( D^n \times S^m \cup ((D^n \# N^n) \times S^m) \right). \]
Figure 7. The standard embedding $i : S^{n-1} \times D^{m+1} \hookrightarrow D^{n+m}$.

Proof. As illustrated in Figure 6, we begin by noting that

$$S^{n-1} \times (S^{m+1} \setminus D^{m+1} \sqcup D^{m+1}) \cong S^{n-1} \times I \times S^m \cong (S^n \setminus (D^n \sqcup D^n)) \times S^m.$$  

Thus the righthand side of the equation in the claim becomes

$$(S^n \setminus (D^n \sqcup D^n)) \times S^m \cup \partial ((D^n \sqcup (D^n \# N^n)) \times S^m).$$

Finally, notice that

$$S^n \setminus (D^n \sqcup D^n) \cup \partial (D^n \sqcup D^n \# N^n) \cong N^n.$$

□

Lemma 4.14. Let $N$ and $M$ be closed, oriented, and smooth manifolds. If $f : S^{n-1} \times D^{m+1} \hookrightarrow M$ is nullhomotopic, then

$$(10) \quad M \# (N^n \times S^m) \cong (M \setminus \text{im} f) \cup f|_{\partial} ((D^n \# N^n) \times S^m).$$

Proof. As $f$ is nullhomotopic, we may extend it to an embedding $F : D^{n+m} \hookrightarrow M$. Moreover, there is a coordinate chart $\phi : M \to \mathbb{R}^{n+m}$, such that $\phi \circ F : D^{n+m} \hookrightarrow \mathbb{R}^{n+m}$ is standard. It therefore suffices to prove the claim for $\mathbb{R}^{n+m}$ and the standard embedding of $D^{n+m}$, with $i : S^{n-1} \times D^{m+1} \hookrightarrow D^{n+m}$ given by the inclusion of the normal neighborhood of the obvious inclusion $S^{n-1} \times D^n \hookrightarrow \partial (D^n \times D^n)$ illustrated in Figure 7. Notice also that the compliment of $\text{im} i$ inside $D^{n+m}$ is $\partial D^n \times D^m$.

It is explicit in the righthand side of equation (10) that we are removing $\text{im} i$ from $\mathbb{R}^{n+m}$. It is implicit on the lefthand side that we are removing $D^{n+m}$. The idea of the proof is to show that gluing $N^n \times S^m \setminus D^{n+m}$ into the deleted $D^{n+m}$ can be performed in two steps, where the first step fills in the compliment of $\text{im} i$ so that the second step is transparently equivalent to the righthand side of (10).

We must therefore analyze $(N^n \times S^m) \setminus D^{n+m}$. We claim that as in figure 8

$$(11) \quad N^n \times S^m \setminus D^{n+m} = (N^n \# D^n) \times S^m \cup_{(N^n \# D^n) \times D^n} (N^n \# S^n) \times D^m.$$
To see this, we imagine $N^n$ as $S^n \# N^n$ with $N^n$ only as a small decoration. Indeed, if we write $D^n_+\cup D^n_-$ as the northern and southern hemispheres of $S^n$, then
\begin{equation}
(S^n \times S^m) \setminus D^n_+ \times D^n_- \cong D^n_+ \times S^m \cup D^n_- \times S^m 
\end{equation}
Notice that one may think of $N^n = N^n \# S^n$, where neck is attached to the northern hemisphere, so that $N^n \setminus D^n_- = N^n \# D^n_+$. With this, (11) follows from (12).

Notice also the boundary of $N^n \times S^m \setminus D^{n+m}$ is identified with $S^{n-1} \times D^n \cup S^{n-1} \times S^m \setminus D^{n+m}$ via equation (11) in the same way as $S^{n+m-1}$. We will therefore use the composition of these diffeomorphisms as our attaching map, i.e. the composite
\begin{equation}
\partial(N^n \times S^m \setminus D^{n+m}) \to S^{n-1} \times D^n \cup S^{n-1} \times S^m \setminus D^{n+m} \to \partial(R^{n+m} \setminus D^{n+m}).
\end{equation}

Next we must use (11) to decompose $N^n \times S^m \setminus D^{n+m}$ as a disjoint union of two spaces. We claim that
\begin{equation}
N^n \times S^m \setminus D^{n+m} = (N^n \# D^n) \times S^m \cup \hat{D}^n \times D^m.
\end{equation}
Where $\hat{D}^n$ is an open disk. This is straightforward, again thinking of $N^n = N^n \# S^n$, if we take the compliment of $(N^n \# D^n_+) \times D^n_-$ inside $N^n \times S^m$, we get $\hat{D}^n \times D^m$. This with Equation (11) demonstrates equation (13).

We are now ready to prove the claim. Typically one forms the connected sum $R^{n+m} \# (N^n \times S^m)$ by removing $D^{n+m}$, and then gluing in $N^n \times S^m \setminus D^{n+m}$, as in the lefthand column of Figure 8. Instead, as in the center column of Figure 8 we may glue in two stages: first glue in $\hat{D}^n \times D^m$ and second glue in $N^n \# D^n \times S^m$. In this first stage, the boundary being attached is $\hat{D}^n \times S^{n-1} \to S^{n+m-1}$. Notice that this map extends to the obvious embedding $\hat{D}^n \times D^m \to D^{n+m}$, where the image we have already identified with $D^{n+m} \setminus \text{ini}$. Then, in the second stage we glue in $(N^n \# D^n) \times S^m$. If instead of removing $D^{n+m}$ and then gluing $\hat{D}^n \times D^m$, we simply start in the righthand column of Figure 8 by removing $i(S^{n-1} \times D^{m+1})$ and then glue in $(N^n \# D^n) \times S^m$, this clearly has the exact same effect. This latter approach is exactly the space defined by the righthand side of equation (10), thus proving our claim.

\[\blacksquare\]
\[ M^{n+m} \# (N^n \times S^m) \]
\[ (M^{n+m} \setminus S^{n-1} \times D^{m+1}) \cup_{\partial} (N^n \# D^n \times S^m) \]

\( N^n \times S^m \setminus D^{n+m} \cong (N^n \# D^n) \times S^m \)
\( (N^n \# D^n) \times S^m \)
\( N^n \times S^m \setminus D^{n+m} \)
\( (N^n \# D^n) \times S^m \setminus D^n \times D^m \)
\( (N^n \# D^n) \times S^m \setminus D^n \times D^m \)
\( (N^n \# D^n) \times S^m \setminus S^{n-1} \times D^{m+1} \)

\textbf{Figure 9.} The equivalence of \( M^{n+m} \# (N^n \times S^m) \) and \( M^{n+m} \setminus (S^{n-1} \times D^{m+1}) \cup_{\partial} (N^n \# D^n) \times S^m \).

With this we can now prove our surgery theorem.

\textit{Proof of Proposition 4.12.} As the \( D^{m+1} \) on the righthand-side of the equation in the claim are disjoint, we may perform the surgeries in any order. Specifically, we may first perform surgery on two, and then perform surgery on the rest. Thus the righthand side of the equation becomes

\[ \left( ((S^{n-1} \times (S^{m+1} \setminus (D^{m+1} \sqcup D^{m+1}))) \cup_{\partial} (D^n \times S^m \sqcup (D^n \# N_1) \times S^m)) \setminus \left( \bigsqcup_{i=2}^{k} S^{n-1} \times D^{m+1} \right) \right) \]

\[ \cup_{\partial} \left( \bigsqcup_{i=2}^{k} (D^n \# N_i) \times S^m \right). \]
Figure 10. The implied embedding $S^{n-1} \times D^{m+1} \hookrightarrow N^n \times S^m$ given by the dashed subset.

Figure 11. The nullhomotopy of $S^{n-1} \times D^{m+1} \hookrightarrow N^n \times S^m$.

By Lemma 4.13 this reduces to the following

$$\left( (N_1 \times S^m) \setminus \left( \bigcup_{i=2}^k S^{n-1} \times D^{m+1} \right) \right) \cup \left( \bigcup_{i=2}^k (D^n \# N_i) \times S^m \right).$$

We now would like to claim that Lemma 4.14 applied $(k - 1)$-times proves the claim. This is not immediately obvious. If each of the implied embeddings $S^{n-1} \times D^{m+1} \hookrightarrow N^n \times S^m$ were nullhomotopic, Lemma 4.14 could be applied to any one of them. But in order to guarantee that we can apply Lemma 4.14 in succession to each embedding, we must show that they remain nullhomotopic after performing the other surgeries. It suffices to show that the image of all the nullhomotopies were disjoint. Because then, each could be isotoped to disjoint coordinate charts, for which Lemma 4.14 then applies.

We claim that these nullhomotopies exist and are disjoint. We must begin by giving an explicit description of the implied embeddings. This embedding is traced through Lemma 4.13 in Figure 10 with the dashed regions indicating the embedded $S^{n-1} \times D^{m+1}$. To begin with, the embedding $S^{n-1} \times D^{m+1} \hookrightarrow S^{n-1} \times S^{m+1}$ is the identity in the first factor and the inclusion of a small metric ball in the second factor, for different embeddings picking disjoint balls. After removing two disks from $S^{m+1}$ and transferring the interval to $S^{n-1}$, the embedding of $S^{n-1} \times D^{m+1}$ is just the normal neighborhood of the inclusion of $S^{n-1} \hookrightarrow (I \times S^{n-1})$ inside $(I \times S^{n-1}) \times S^m$. With this we now can describe the nullhomotopy of this embedding as illustrated in Figure 11. First contract the fibers of the disk bundle so that the embeddings are disjoint points in the $S^m$ factor and in the first factor are the inclusions $S^{n-1} \hookrightarrow I \times S^{n-1}$. As we have glued a cap to one end of $I \times S^{n-1}$, this defines
a nullhomotopy of the inclusion $S^{n-1} \hookrightarrow I \times S^{n-1}$. Thus providing a nullhomotopy of the original embeddings, which will remain disjoint in the second factor therefore proving the claim. □

4.4.2. Metric Construction. The result of this section shows that the connected sums in the previous section may be endowed with positive Ricci metrics so long as the spaces $N_i$ have suitable core metrics.

**Proposition 4.15.** Let $n > 2$ and $m \geq 3$. If there exists Ricci positive metrics on $M_i^n \setminus D^n$ with round, convex boundaries, then the following manifold admits a metric with positive Ricci curvature.

$$\#_{i=1}^k (N_i^n \times S^m).$$

**Proof.** We will use the specific construction of $\#_{i=1}^k (M_i \times S^m)$ provided in Proposition 4.12. This theorem decomposes $\#_{i=1}^k (M_i \times S^m)$ as the boundary union of two smooth manifolds:

$$DS := S^{n-1} \times \left( S^{m+1} \setminus \left( \bigcup_{i=0}^k D^{m+1} \right) \right)$$

and

$$Cores := D^n \times S^m \sqcup \bigcup_{i=1}^k (D^n \# M_i) \times S^m.$$

Our approach is to construct metrics on each using Proposition 1.3 and our assumptions about $M_i^n$ to construct metrics on $DS$ and $Cores$ respectively. So that they have positive Ricci curvature, isometric boundaries, and extrinsic curvatures that allow Lemma 1.2 to apply.

To start, we note that there are core metrics on $D^n$. Specifically the round metric of radius $r$ restricted to a geodesic ball $g_r$ suffices

$$g_r = dt^2 + r^2 \sin^2 \left( \frac{t}{r} \right) ds_{n-1}^2.$$  

For ease, we define $M_0^n := D^n$, and rewrite

$$Cores = \bigcup_{i=0}^n (M_i^n \setminus D^n) \times S^m.$$  

We will now define a metric on each of the $(M_i^n \setminus D^n) \times S^m$. Let $g_i$ be the metric that is assumed to exist on $(N^n \setminus D^n)$ in the hypotheses of the proposition, with positive Ricci curvature and round, convex boundary. Consider first the product metric $g_i + \rho ds_m^2$ on $(M_i^n \setminus D^n) \times S^m$. The Ricci tensor of product metrics splits with respect the usual splitting of the tangent space of a product, thus $g_i + \rho ds_m^2$ has positive Ricci curvature. However, the second fundamental form of the boundary restricted to $TS^m$ is zero. To apply Lemma 1.2 we will need the boundary to be convex. This can be achieved by warping the product metric near the boundary.

Take a collar neighborhood $N_{\xi}S^{n-1} = (-\xi, 0) \times S^{n-1}$ inside $N_i^n \setminus D^n$. In these coordinates, the metric splits as $g_i = dt^2 + g_i(t)$ where $g_i(t)$ are metrics on $S^{n-1}$. Define $\chi_\xi(t)$ as follows.

$$\chi_\xi(t) = \begin{cases} 0 & t \leq -\xi \\ e^{-\frac{1}{(t+\xi)^2}} & -\xi < t \leq 0 \end{cases}$$
Notice that \( \chi^{(j)}(\xi) = 0 \), for all \( j \). Thus \( \chi(\xi) \) is smooth, and for all \( \xi > 0 \) and \( j \), there exists a small \( \xi_0 > 0 \), such that \( ||\chi(\xi(t))||_{C^j} < \xi \) for all \( \xi < \xi_0 \). But for all \( \xi > 0 \), \( \chi'(0) > 0 \).

For all \( 0 < \xi < 1 \), \( 0 < \kappa < 1 \), and \( 0 < \rho < 1 \), define a new metric \( \tilde{g}_i(\xi, \kappa, \rho) \) on \( (N^n \setminus D^n) \times S^n \) as follows.

\[
\tilde{g}_i(\xi, \kappa, \rho) = \begin{cases} 
\kappa_i^2 g_i + \rho^2 ds_m^2 & x \in (N^n \setminus \zeta S^{n-1}) \\
\kappa_i^2 (dt^2 + g_i(t)) + \rho^2 (1 + \chi_i(t))ds_m^2 & x \in \zeta S^{n-1}
\end{cases}
\]

Because \( \chi^{(j)}(\xi) = 0 \), this metric is smooth. Fix a small number \( \kappa \) such that each \( \kappa_i \) may be chosen so that \( \kappa_i^2 g_i \) has boundary isometric to \( \kappa^2 ds_m^2 \). Fix \( \rho \), and define \( \tilde{g}_i(\xi) = \tilde{g}_i(\xi, \kappa_i, \rho) \)

It is clear that for all \( \xi > 0 \), there exists a \( \zeta' > 0 \) such that \( ||\tilde{g}_i(\xi) - (\kappa_i^2 g_i + \rho^2 ds_m^2)||_{C^2} < \zeta \) if \( ||\chi(\xi(t))||_{C^2} < \zeta' \). Recall that Ric is a second order operator on metric functions, and so the set of metrics with positive Ricci curvature is open in the space of all metrics with respect to the \( C^2 \) topology. Because Ric is positive definite for \( \kappa_i^2 g_i + \rho^2 ds_m^2 \), it follows that there exists a \( \xi_0 > 0 \) such that \( \tilde{g}_i(\xi) \) will have positive Ricci curvature if \( \xi < \xi_0 \). And because \( \chi'(0) > 0 \), the boundary is convex ([14, Proposition 3.2.1]). Define \( \tilde{g}_i = \tilde{g}_i(\xi) \).

Thus, there exists a Ricci positive metric \( \tilde{g}_i \) on \( (N^n \setminus D^n) \times S^n \) with convex boundary isometric to \( \kappa^2 ds_m^2 + \rho^2 ds_m^2 \). Let \( \nu_i = \inf_{\Sigma(T(S^n \times S^n))} \Pi_i \). We have shown that \( \nu_i > 0 \). Pick a number \( 0 < s < 1 \), such that \( \frac{\nu_i}{s} > 1 \) for all \( i \). Notice that \( ((N^n \# D^n) \times S^n, s^2 \tilde{g}_i) \) has positive Ricci curvature, boundaries isometric to \( (S^{n-1} \times S^m, s^2 \tilde{g}_i(ds_m^2 + (sp)^2 ds_m^2) \) with extrinsic curvatures all at least 1. Define \( g_{\text{cores}} \) to be \( \tilde{g}_i \) on each component of Cores.

We now turn to defining a metric on DS. By Proposition [1.3] there exists a metric \( g_{\text{docking}} \) on \( S^{n+1} \setminus \bigcup_{i=1}^k D^{n+1} \) with boundary components all isometric to \( (S^n, (sp)^2 ds_m^2) \) and extrinsic curvatures identically \(-1\). Define \( g_{\text{DS}} = (sk)^2 ds_{n-1}^2 + g_{\text{docking}} \). Clearly the boundary is isometric to the boundary of \( (\text{Cores}, g_{\text{cores}}) \). And we see that the second fundamental form of the boundary \( \Pi \) restricted to \( TS^{n-1} \) is zero. As all the extrinsic curvatures of the boundary of \( (\text{Cores}, g_{\text{cores}}) \) are greater than 1, we see that Lemma [1.2] applies. And so there is a smooth metric on \( DS \cup \partial \) Cores with positive Ricci curvature. By Proposition [4.12] we have produced a metric with positive Ricci curvature on \( \#_{i=1}^k (N^n_i \times S^m) \).

\[ \Box \]

In particular, we have shown that complex, quaternionic, and octonionic projective spaces admits core metrics. Notice that in the proof of Proposition [4.15] we have shown that \( S^n \) also admits core metrics. Thus the following is immediate.

**Proof of Corollary [4.10]** By Theorem [1] and the observation about \( (S^n \setminus D^n, g_r) \) in the proof of Proposition [4.15] the spaces \( S^n, CP^n, HP^n, \) and \( OP^2 \) all admit Ricci positive metrics with round, convex boundaries. By Proposition [4.15] the claim follows. \[ \Box \]
Appendix A. Construction of the Neck

In [9], Gromov and Lawson developed a technique to preserve positive scalar curvature under surgery in codimension at least three. In particular, if \( n \geq 3 \), to form positive scalar curvature metrics on connected sums, they showed that for any Riemannian manifold \((M^n, g)\) with positive scalar curvature, it is possible to bend the metric outwards in a small neighborhood of the boundary of \( M \setminus \{pt\} \) to a metric with positive sectional curvature that is isometric to \( dt^2 + \varepsilon ds^2_{n-1} \) near the boundary. The reason this technique fails for positive Ricci curvature is that the second fundamental form of small geodesic balls is very negative. Thus one can see in that bending outward will force the radial derivative of the second fundamental form to be positive, which by the Mainardi-Codazzi equation implies that the Ricci curvature in the radial direction must be negative.

It is reasonable therefore to hope for some local deformation result for positive Ricci curvature under some assumption about the size of the second fundamental form of a small geodesic ball. The following lemma, originally stated in [13, Section 2], claims that is possible to deform the metric \( M \setminus \{pt\} \) to a positive Ricci metric with round, concave boundary under the assumption that the original metric was positively curved at the boundary, with second fundamental form \(|II| < 1\), and such the boundary is isometric to a warped product. See the proof of Proposition 1.3 in Section 4.1 for an explanation of this application.

Lemma A.1. Assume that \( n \geq 3 \), \( 0 < r < R < 1 \), and \( g_1 = d\phi^2 + f_1^2(\phi)ds^2_{n-1} \) is a metric on \( S^n \) with \( \phi \in [0, \pi R] \), \( \sup_\phi f_1(\phi) = r \), and \( K_{g_1} > 1 \). Then for any any \( \rho > 0 \) satisfying \( r^{(n-1)/n} < \rho < R \), there exists a metric \( g = g(\rho) \) defined on \( S^n \times [0,1] \) and constant \( \lambda > 0 \) such that the following are true.

(i) \( \text{Ric}_g \) is positive definite;
(ii) the restriction \( g|_{t=0} \) coincides with \( \frac{\rho^2}{\lambda^2} ds^2_n \) on \( S^n \times \{0\} \);
(iii) the restriction \( g|_{t=1} \) coincides with \( g_1 \) on \( S^n \times \{1\} \);
(iv) the principal curvatures along the boundary \( S^n \times \{0\} \) are equal to \( -\lambda \);
(v) the principal curvatures along the boundary \( S^n \times \{1\} \) are at least 1.

Lemma A.1 was originally proven in [13, Section 2]. The purpose of this appendix is to provide a detailed version of this proof, filling in some of the missing technical details. Indeed, the proof given in [13] is completely accurate, though several assertions go without explanation.

This section is organized into three parts. The first, Section A.1 considers families of warped product metrics and demonstrates the existence of a one parameter family \( \tilde{g}(a,b(a)) \) connecting the metric \( g_1 \) to the round metric of radius \( \rho \), so that the sectional curvatures remain bounded below by 1 in this entire family. The second part, Section A.2, considers metrics on the cylinder \([t_0,t_1] \times S^n\) of the form \( g = \kappa^2(dt^2 + \ell^2 \tilde{g}(h(t),k(t))) \), where \( h \) and \( k \) are essentially parameterizations of the path \((a,b(a))\). The definition of \( h \) and \( k \) includes three independent parameters that determine a family of metrics \( g(t_0,\varepsilon,\delta) \) that will all satisfy the claims of Lemma A.1 at \( t = t_0 \) and \( t = t_1 \).
and will have large sectional curvature in the spherical directions. The third part, Section A.3 is dedicated to showing that the parameters of \( g \) can be chosen so that \( \text{Ric}_g \) is positive definite.

The main theme of this proof, which is entirely due to Perelman, is to define for any interval \([t_0, t_1]\) a metric whose ends are isometric to the desired metrics \( \rho^2 \frac{\lambda_2}{\lambda_1^2} ds_n^2 \) and \( g_1 \) so that the sectional curvatures and extrinsic curvatures of the ends will exhibit the desired behavior as you take the limit as \( t_0 \to \infty \) and \( (t_1 - t_0) \to \infty \). While the details needed to make this work are very technical, the key insight in [13] is the existence of the path of metric \( \tilde{g}(a, b(a)) \), the result of which is that the sectional curvature in the spherical directions are bounded below by \( \frac{1}{t^2} \). This essentially reduces the problem to choosing functions \( h \) and \( k \) that cause the other curvatures decay faster than this.

A.1. A two parameter family of warped product metrics. In this section we begin by defining a two parameter family of warped product metrics \( \tilde{g}(a, b) \) on \( S^n \) that will connect \( \rho^2 \frac{\lambda_2}{\lambda_1^2} ds_n^2 \) to \( g_1 \). In Section A.1.1 we give some background on warped products metrics. Specifically in Lemma A.2 we reparameterize all positively curved warped product metrics so that they are defined on a universal domain \([-\frac{\pi}{2}, \frac{\pi}{2}] \times S^n \). This domain is essential to defining the two parameter family in Section A.1.2. The parameter \( b \) is the waist of the warped product metric, and the parameter \( a \) is the maximum velocity of the parameterization given in Section A.1.1. The metrics \( \tilde{g}(a, b) \) are defined by taking a convex combination of the metric functions for \( r^2 ds_n^2 \) and \( g_1 \) in terms of \( a \) and scaling that metric by \( b \). We conclude with Section A.1.3 in which we show that the one parameter family defined by assuming \( b \) is proportional to a power of \( a \) has sectional curvature larger than 1. Thus showing that \( \rho^2 ds_n^2 \) and \( g_1 \) are in the same path component of the space of all metrics on \( S^n \) with \( K_g > 1 \). This path will be used explicitly to construct a Ricci positive metric on the cylinder in Section A.2.
A.1.1. Renormalizing warped products. We begin by giving some background on warped product metric on $S^n$. Such metrics can be defined as $d\phi^2 + f^2(\phi) ds_{n-1}^2$ on $[0, D] \times S^{n-1}$ for a function $f : [0, D] \to [0, W]$ such that

$$f^{(even)}(0) = f^{(even)}(D) = 0, \quad f'(0) = 1, \quad \text{and} \quad f'(D) = 1.$$  \hspace{1cm} (14)

In this application, we will only be considering $W \leq D$. In this case, we will call $W$ the waist and $D$ the diameter. Let $X$ and $\{\Sigma_i\}_{i=1}^{n-1}$ denote an orthonormal local frame of $S^n$ tangent to $[0, D]$ and $S^{n-1}$ respectively. The sectional curvatures of the warped product metric in these coordinates is given by

$$K(X, \Sigma_i) = -\frac{f''(\phi)}{f(\phi)} \quad \text{and} \quad K(\Sigma_i, \Sigma_j) = \frac{1 - (f'(\phi))^2}{f^2(\phi)}.$$  \hspace{1cm} (15)

By assumption, $g_1$ is a warped product with $D = \pi R$ and $W = r$. The assumption of Lemma A.1 that $g_1$ is a metric on $S^n$ is equivalent to the fact that $f_1$ satisfies equation (14). The assumption that $K_{g_1} > 1$ is equivalent to $f_1$ satisfying certain inequalities determined by equation (15). Notice also that round metrics can be realized as warped products on $[0, \pi \rho] \times S^{n-1}$ as follows

$$\rho^2 ds_n^2 = d\phi^2 + \rho^2 \sin^2 \left( \frac{\phi}{\rho} \right) ds_{n-1}^2.$$  

We wish to define a family of warped product metrics that connect $g_1$ to $\rho^2 ds_n^2$. Because these metrics are defined for $\phi$ in intervals of different lengths, one would have to write down a family of functions defined on a family of intervals. To reduce this complexity, we reparameterize any warped product metric with concave warping function to be defined on the universal domain: $\left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times S^{n-1}$.

**Lemma A.2.** Let $f : [0, D] \to [0, W]$ be a concave function satisfying equation (14), let $g_f = d\phi^2 + f^2(\phi) ds_{n-1}^2$. Then there is a parameterization $\phi : \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \to [0, D]$ such that $g_{f \circ \phi} = A^2(x) dx^2 + W^2 \cos^2 x ds_{n-1}^2$. Where $A(x)$ is a positive function that satisfies

$$A \left( \pm \frac{\pi}{2} \right) = W, \quad A' \left( \pm \frac{\pi}{2} \right) = 0, \quad \text{and} \sup_x A(x) \geq \frac{D}{\pi}.$$  \hspace{1cm} (16)

**Proof.** As $f$ is concave, there is a unique point at which $f$ has a global maximum. This splits $[0, D]$ into two intervals on which $f$ is bijective. On each interval, we may therefore define

$$\phi(x) = f^{-1}(W \cos x).$$

This function is smooth, and by definition $f(\phi(x)) = W \cos x$. It follows then that

$$g_{f \circ \phi} = \left( \frac{d\phi}{dx} \right)^2 dx^2 + W^2 \cos^2 x ds_{n-1}^2.$$  

Therefore $A(x) = \phi'(x)$, which by (17) can be computed explicitly as

$$A(x) = \frac{-W \sin(x)}{f'(\phi(x))}. $$  \hspace{1cm} (18)
The statements in equation (16) about the values of $A$ and $A'$ at 0 and $D$ follow immediately from equation (14).

To see the claim about the supremum of $A(x)$, suppose the contrary that $\phi'(x) < \frac{D}{\pi}$. It follows that the arc-length of $\phi(x)$ is less than $D$ contradicting the fact that $\phi$ is a bijection. $\square$

A.1.2. The two parameters. If $K(X, \Sigma_i) > 0$ for a warped product metric, from equation (15) we see $f'' < 0$. Thus Lemma A.2 applies, in particular to $g_1$ and $\rho^2 ds^2_n$. Let $A_1(x)$ be the function $A(x)$ determined by $g_1$. In this case $D = R$ and $W = r$. For $\rho^2 ds^2_n$, we see that the function $A(x)$ is $\rho$, and $D = W = \rho$. We want use Lemma A.2 to define a two parameter family of metrics $\tilde{g}(a, b)$ that will connect $g_1$ and $\rho^2 ds^2_{n-1}$. To achieve this, we will define functions $A(a, b, x)$ and $B(b, x)$ that interpolate between $\rho$ and $A_1(x)$ and between $\rho \cos x$ and $W \cos x$ respectively. The latter has an obvious candidate, $B(b, x) = b \cos x$ with $b \in [\rho, \rho]$. As we see that $b$ is already naturally associated with the waist, we now endeavor to find a suitable choice for $a$.

If we imagine fixing the waist $b = W$, then we want a way of transitioning from $b^2 ds^2_n$ to $\frac{b}{r^2} g_1$ (two metrics with waist $b$). In this case, we need a function $A(a, b, x)$ that interpolates between $b$ and $\frac{b}{r} A_1(x)$ . We can consider the convex combination of these functions with respect to $\tau \in [0, 1]$, namely

\begin{equation}
\label{eq:19}
A(\tau, b, x) = \tau \frac{b}{r} A_1(x) + (1 - \tau) b.
\end{equation}

We want to replace the parameter $\tau$ with one that has geometric meaning, we will take $a$ to be the maximum velocity of $\tilde{g}(a, b)$ of the parameterization determined by Lemma A.2. We can define $a(\tau) = \sup_x \frac{1}{b} A(\tau, b, x)$. Clearly $a$ is linear in $\tau$. Let $a_1 = \sup_x \frac{A_1(x)}{r}$. Clearly $A(0, b, x) = \frac{b}{r} A_1(x)$ and $A(1, b, x) = b$. Thus $a(0) = a_1$ and $a(1) = 1$. Thus $\tau = \frac{a - 1}{a_1 - 1}$ for $a \in [1, a_1]$, and we may replace $\tau$ with this in equation (19) to get

\begin{equation}
\label{eq:20}
A(a, b, x) = b \left( \frac{a - 1}{a_1 - 1} \frac{A_1(x)}{r} + \left( 1 - \frac{a - 1}{a_1 - 1} \right) \right).
\end{equation}

Which, following Perelman, we will rewrite in terms of a function $\eta(x)$ as follows

\begin{equation}
\label{eq:20}
A(a, b, x) = b \left( \frac{a - 1}{a_1 - 1} \frac{A_1(x)}{r} + 1 \right) = b((a - 1)\eta(x) + 1).
\end{equation}

Definition A.3. Let $a_1 = \sup_x \frac{A_1(x)}{r}$ and $\eta(x)$ be as defined in equation (20). Define for each $(a, b) \in [1, a_1] \times [\rho, \rho]$ a metric $\tilde{g}(a, b)$ on $S^n$ as follows.

\[ \tilde{g}(a, b) = A^2(a, b, x) dx^2 + B^2(b, x) ds^2_{n-1}. \]

Where $A$ and $B$ are defined as follows.

\[ A(a, b, x) = b(\eta(x)(a - 1) + 1), \]

\[ B(b, x) = b \cos x. \]
For a fixed $a$, we see that changing $b$ simply scales the metric. For a fixed $b$ we see that changing $a$ interpolates the velocity of the parameterizations $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [0, b]$ between the constant velocity parameterization and the parameterization determined by $b \frac{\phi_1}{r}$. This behavior is illustrated in Figure 13.

A.1.3. A one parameter family with $K > 1$. By definition $\tilde{g}(1, \rho) = \rho^2 ds_n^2$ and $\tilde{g}(a_1, r) = g_1$, so we have succeeded in finding a two parameter family of metrics that connects our two significant metrics. We now wish to study the curvature of these metrics. In particular, both $\rho^2 ds_n^2$ and $g_1$ have sectional curvature bounded below by 1. This family of metrics $\tilde{g}(a, b)$ remain close to the extreme metrics, so it is reasonable to believe that there is path through this parameter space along which the sectional curvature remains bounded below by 1. If we assume that $b$ is a function of $a$, then we can consider the one parameter family $\tilde{g}(a) = \tilde{g}(a, b(a))$ and ask which choice of $b(a)$ will have $K_{\tilde{g}(a)} > 1$. A first guess at which function will work, would be to take a linear relationship. In the course of trying to bound the curvature below by one, one needs to bound the slope $\frac{a_1 - 1}{r - \rho}$ below by 1, which is false for some choices of $g_1$.

The next simplest relationship is to assume proportionality. Indeed, if $b(a) = \frac{c}{a^{1/\alpha}}$, then

$$\alpha = \frac{\ln a_1}{\ln \rho - \ln r}$$

and $c = \rho$.

To bound the curvature below by 1, one must bound $\alpha$ below by 1. This is implied by the hypotheses of Lemma A.1. To see this, recall the fact that $\sup_x A_1(x) \geq R$, by Lemma A.2 and the assumption that $R > \rho$ in Lemma A.1. Thus $a_1 \geq \frac{R}{r} > \frac{\rho}{r}$, which shows $\alpha > 1$. Note that the proportionality of $b$ to $a$ is equivalent to

$$-\frac{b'(a)}{b(a)} = \frac{1}{\alpha a}.$$
This characterization will be convenient in Section A.2 when we want to consider metrics on the cylinder. If we imagine that \( b(t) \) and \( a(t) \) are functions of interval, then equation (21) is equivalent to

\[
-\frac{b'(t)}{b(t)} = \frac{a'(t)}{\tilde{a}a(1)}.
\]

This will allow us to define functions in a way that is easily relatable to the path \((a, b(a))\), but that will also be convenient for prescribing certain asymptotics and boundary conditions.

We define a one parameter family of metrics with respect to \( a \in [1, a_1] \) as follows

\[
\tilde{g}(a) = \tilde{g}\left(a, \frac{\rho}{a^{1/\alpha}}\right).
\]

The following Lemma is implicit in [13, Section 2], it claims that this one parameter family satisfies the desired curvature bound.

**Lemma A.4.** Let \( \tilde{g}(a) \) be as in (23), then \( K_{\tilde{g}(a)} > 1 \) for all \( a \in [1, a_1] \).

Before proving this lemma we provide the formulas for the sectional curvatures of \( \tilde{g}(a, b) \). These curvatures can be computed directly from Definition A.3 and (ii) of Proposition B.16.

**Corollary A.5.** The sectional curvatures of \( \tilde{g}(a, b) \) are as follows.

\[
K_{\tilde{g}(a,b)}(X, \Sigma) = \frac{1}{b^2} \left( \frac{1}{(1 + \eta(x)(a-1))^2} - \frac{\eta'(x) \tan x(a-1)}{(1 + \eta(x)(a-1))^3} \right)
\]

\[
K_{\tilde{g}(a,b)}(\Sigma_i, \Sigma_j) = \frac{1}{b^2} \left( \frac{1}{\cos^2 x} - \frac{\tan^2 x}{(1 + \eta(x)(a-1))^2} \right)
\]

Before turning to prove Lemma A.4, we state a lemma concerning the function \((1 + (a-1)\eta(x))\). As this function is key to the definition of \( \tilde{g}(a, b) \), it will appear in all but one curvature term. While the following lemma is elementary, it is used repeatedly in the proof of Lemma A.4.
Lemma A.6. For all \( x \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \),

\[
\frac{-1}{a_1 - 1} < \eta(x) \leq 1.
\]

If \( \eta(x) \geq 0 \), then \((1 + (a - 1)\eta(x))\) is nondecreasing with respect to \( a \in [1, a_1] \) and

\[
1 \leq (1 + (a - 1)\eta(x)) \leq a_1.
\]

If \( \eta(x) < 0 \), then \((1 + (a - 1)\eta(x))\) is decreasing with respect to \( a \in [1, a_1] \) and

\[
0 \leq |\sin x| < (1 + (a - 1)\eta(x)) \leq 1.
\]

Moreover, there is a constant \( h > 0 \), depending only on \( g_1 \), such that for all \( a \) and \( x \) we have

\[
(1 + (a - 1)\eta(x)) > ha.
\]

Proof. By the definition of \( \eta(x) \) and \( a_1 \) in equation (20), we see that \( \eta(x) \leq 1 \) (with equality necessarily attained). Considering equation (18) we see that there is a constant \( c > 0 \) depending only on \( g_1 \), such that \( A_1(x) > c \). Applying this to the definition of \( \eta(x) \) in equation (20) we may assume that \( c < r \), and so we see that

\[
\eta(x) > -\frac{r - c}{r} \frac{1}{a_1 - 1}.
\]

This proves the bounds in equation (24).

Assume now that \( \eta(x) \geq 0 \). The function is linear in \( a \), so the fact that it is nondecreasing is clear. By (24) we have \( 0 \leq \eta(x) \leq 1 \), which yields

\[
1 \leq 1 + (a - 1)\eta(x) \leq 1 + (a - 1).
\]

This proves equation (25).

Assume next that \( \eta(x) < 0 \). Again, that the function is decreasing is clear. So for each \( \eta(x) < 0 \) the function has a maximum at \( a = 1 \), implying

\[
1 + (a - 1)\eta(x) \leq 1.
\]

For the lower bound, one must use the definition of \( A_1(x) \) in equation (18) in more detail. By assumption, \( K_{g_1} > 1 \). Looking at the equation for \( K_{g_1}(\Sigma_i, \Sigma_j) \) in equation (15) yields that \( f_1'(\phi(x)) < 1 \). The definition of \( \phi(x) \) in equation (17) for \( f_1 \), forces the sign of \(-r \sin x\) and \( f_1'(\phi(X))\) to agree. Thus we have

\[
A_1(x) = -\frac{r \sin x}{f_1'(\phi(x))} > r|\sin x|.
\]

Rewriting \( 1 + (a - 1)\eta(x) \) in terms of \( A_1(x) \) yields

\[
1 + (a - 1)\eta(x) = 1 + \frac{a - 1}{a_1 - 1} \left( \frac{A_1(x)}{r} - 1 \right) > 1 + \frac{a - 1}{a_1 - 1} (|\sin x| - 1)
\]

That this function is decreasing means it obtains a minimum value at \( a = a_1 \). Substituting \( a = a_1 \) into the righthand side of equation (28), yields the lower bound of equation (26).
To see why equation (27) holds, notice that $1 + (a - 1)\eta(x)$ is linear in $a$ with vertical intercept $1 - \eta(x)$ and slope $\eta(x)$. By (24), the vertical intercept is always nonnegative. Consider now the value at $a_1$. By equation (26), for all $x$ this has a minimum value equal to $d > 0$. It follows then that

$$1 + (a - 1)\eta(x) > \frac{d}{a_1 - 1}a.$$  

Because $\eta(x)$ in equation (20) is entirely determined by $g_1$, so is this constant $h > 0$. 

The geometric meaning of Lemma A.6 is roughly as follows. The function $1 + (a - 1)\eta(x)$, by definition interpolates between 1 and a function $A(x) = r^{-1}A_1(x)$. For a fixed $a$, the function determines the parameterization of the angle of inclination in terms of the variable $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Necessarily, as $a$ increases this will compress the interval to $[0, \pi R]$. The purpose of Lemma A.6 is to keep track of behavior with respect to an individual point $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. This is important as the curvature is determined by this variable as given by equation (15). That the sign of $\eta(x)$ determines the behavior is clear from the formulas, but also has a clear geometric meaning. If we consider equation (20), we can see that the sign of $\eta(x)$ is determined by the difference in slope of $f_1(\phi)$ and $r \cos x$ at the same output values, which by (15) is the sign of $K_{g_1}(\Sigma_i, \Sigma_j) - K_r^2 ds_n^2(\Sigma_i, \Sigma_j)$. If we think carefully about the definition of $\phi(x)$ in Lemma A.2, we can see that $\eta(x)$ being positive or negative corresponds to whether $(1 + (a - 1)\eta(x))$ is locally stretching or compressing the interval as $a$ increases. Thus one expects distinct behaviors, which is seen directly in Lemma A.6.

We now give the proof Lemma A.4. The interested reader can see the penultimate line of Page 160 for Perelman’s statement of the lemma, and read the following two paragraphs for his proof. Our proof is not substantially different, though we include some elementary arguments that were omitted. For instance, the results of A.6 are omitted, which while trivial are not obvious.

Proof of Lemma A.4. Define $L_\Sigma(a) = \ln (K_{\tilde{g}(a)}(\Sigma_i, \Sigma_j))$ and $L_X(a) = \ln (K_{\tilde{g}(a)}(X, \Sigma_i))$. Note that the claim is equivalent to $L_X(a) > 0$ and $L_\Sigma(a) > 0$. Begin by observing that, regardless of the path chosen, $\tilde{g}(1) = \rho^2 ds_n^2$ and $\tilde{g}(a_1) = g_1$. Thus $L_X(a)$ and $L_\Sigma(a)$ are both positive at $a = 1$ and $a_1$. We have suppressed the fact that $L_X(a)$ and $L_\Sigma(a)$ also depend on the parameter $x \in S^n$, but we will consider $x$ fixed in different cases as follows. We begin in Part 1 by considering $L_\Sigma(a)$. Following Lemma A.6 we will consider $\eta(x) \geq 0$ and $\eta(x) < 0$ separately in Part 1.1 and 2 respectively. Then in Part 2 we consider $L_X(a)$, noting that $\tan x\eta'(x)$ plays a role in the sign of $K_{\tilde{g}(a)}(X, \Sigma_i)$ of Corollary A.5. We consider $\tan x\eta'(x) < 0$ and $\tan x\eta'(x) > 0$ in Part 2.1 and 2.2 respectively. In Part 2.2 we must further consider $\eta(x) \geq 0$ and $\eta(x) < 0$ in Part 2.2.1 and 2.2.2 respectively. We will prove that $L_X(a) > 0$ and $L_\Sigma(a) > 0$ in each case.

Part 1: $L_\Sigma(a) > 0$. For $K_{\tilde{g}(a)}(\Sigma_i, \Sigma_j)$, we consider its log derivative, $L_\Sigma'(a)$. The following is immediate from Corollary A.5 and equation (22).

$$L_\Sigma'(a) = \frac{2}{a\alpha a} \left(1 + \frac{\alpha\eta(x)a}{(1 + (a - 1)\eta(x))(1 + (a - 1)\eta(x))^2 - \sin^2 x}\right).$$

Following Lemma A.6 we will consider the cases $\eta(x) \geq 0$ and $\eta(x) < 0$ separately.
**Part 1.1:** $\eta(x) \geq 0$. Assume that $x$ is fixed such that $\eta(x) \geq 0$. We claim that $L'_X(a) \geq 0$ for such values of $x$. Were this the case then $L_{\Sigma}(a) \geq L_{\Sigma}(a_1) > 0$ proving the claim.

To see this, consider equation (25). We see that both factors in the denominator of the large fraction in equation (29) are nonnegative. By assumption, the numerator is nonnegative. Since $\tilde{\alpha}a$ is positive, this shows that $L'_X(a)$ is nonnegative.

**Part 1.2:** $\eta(x) < 0$. Next, fix $x$ so that $\eta(x) < 0$. We claim that either $L'_{\Sigma}(a) \geq 0$ or if $L'_{\Sigma}(a)$ changes sign then it can only change sign from positive to negative. Suppose this were true. Then the first case we conclude again that $L_{\Sigma}(a) \geq L_{\Sigma}(a_1) > 0$. And in the second case we conclude that $L_X(a)$ has no relative minima and so $L_X(a) \geq \min\{L_{\Sigma}(1), L_{\Sigma}(a_1)\} > 0$. Thus proving the claim.

To see this we claim that the large fraction in equation (29) is decreasing with respect to $a$. Indeed, by equation (26), both factors in the denominator are positive and decreasing, and by assumption the numerator is negative and decreasing. Applying the quotient and product rules shows directly that the sign of the derivative of this fraction must be negative. Because $\tilde{\alpha}a$ is positive, we see that $L'_X(a)$ can only change sign from positive to negative.

**Part 2:** $L_X(a) > 0$. Next, for $K_{g_{i}(a)}(X, \Sigma_i)$, we wish to prove that $L_X(a) > 0$. One sees in Corollary A.5 that the sign of $\tan xy'(x)$ will influence the behavior of $L_X(a)$. We will therefore consider $\tan xy'(x) \geq 0$ and $\tan xy'(x) < 0$ separately.

**Part 2.1:** $\tan xy'(x) < 0$. Fix $x$ such that $\tan xy'(x) < 0$. We claim that $L_X(a) \geq 2(\ln a + \ln b(a))$. This is obvious from Corollary A.5 and equation (26).

We claim that $-2(\ln a + \ln b(a))$ is positive. Note that

$$-2 \frac{d}{da}(\ln a + \ln b) = -2 \left( \frac{1}{a} - \frac{1}{\tilde{\alpha}a} \right) = -2 \frac{1}{\tilde{\alpha}a} (\tilde{\alpha} - 1).$$

So, because $\tilde{\alpha} > 1$, $-2(\ln a + \ln b(a))$ is decreasing and therefore bounded below by $-2(\ln a_1 + \ln r)$, where $b(a_1) = r$. Notice that when $\eta(x) = 1$ (which is guaranteed to happen), by Corollary A.5 $K_{g_{i}}(X, \Sigma_i) = \frac{1}{r^2a_1^2}$, but by assumption $K_{g_{i}} > 1$. Taking logarithms shows that $-2(\ln a_1 + \ln r) > 0$, therefore proving that $L_X(a) > 0$ in this case.

**Part 2.2:** $\tan xy'(x) \geq 0$. For the remaining cases, fix $x$ such that $\tan xy'(x) \geq 0$. We will again need to consider the log derivative of $K_{g_{i}(a)}(X, \Sigma_i)$, $L'_X(a)$.

(30) \hspace{1cm} L'_X(a) = \frac{1}{\tilde{\alpha}a} \left( \frac{3}{\eta(x)\tilde{\alpha}a - \eta'(x)\tan \tilde{\alpha}a} \left( \begin{array}{c} \eta(x)\tilde{\alpha}a - \eta'(x)\tan \tilde{\alpha}a \\ 1 + \eta(x) - \eta'(x)\tan x(a - 1) - \frac{3}{1 + \eta(x)}(\eta(x)\tilde{\alpha}a) \end{array} \right) \right).

**Part 2.2.1:** $\eta(x) \geq 0$. Fix $x$ such that $\tan xy'(x) \geq 0$ and $\eta(x) \geq 0$. Equation (30) can be factored as follows.

(31) \hspace{1cm} L'_X(a) = \frac{1}{\tilde{\alpha}a} \left( \frac{1}{1 + \eta(x)(a - 1)} \left( 2(1 - \eta(x)) + 2\eta(x)a(1 - \tilde{\alpha}) - \frac{\eta'(x)\tan \tilde{\alpha}a}{1 + \eta(x) - \eta'(x)\tan x(a - 1)} \right) \right).

We again claim that either $L'_X(a)$ is either nonnegative or if it changes sign then it can only change from positive to negative. Again demonstrating that $L_X(a) > 0$. 

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The two terms in equation (31) outside of the large parentheses are nonnegative by assumption and equation (26). We claim that the function inside the large parentheses is decreasing with a. The terms on the first line of equation (31) are linear in a with slope $2\eta(x)(1 - \tilde{a})$, which by assumption is negative. Thus this term is decreasing. It remains to check that the large fraction is increasing (as it appears with a minus sign). Its derivative is
\[
\frac{d}{da} \left( \frac{\eta'(x) \tan x\tilde{a} a}{1 + (\eta(x) - \tan x\eta'(x))(a - 1)} \right) = \tilde{a} \left( \frac{\eta'(x) \tan x(1 - \eta(x)) + (\eta'(x) \tan x)^2}{(1 + (\eta(x) - \tan x\eta'(x))(a - 1))^2} \right),
\]
which is nonnegative by assumption and equation (35). We conclude that if $L'_X(a) = 0$ and changes sign, then it must change from positive to negative.

Part 2.2.2: $\eta(x) < 0$. Finally, fix $x$ such that $\tan x\eta'(x) \geq 0$ and $\eta(x) < 0$. Equation (30) can be factor as follows.
\[
L'_X(a) = \frac{1}{\tilde{a} a} \left[ 2 - \frac{\tilde{a} a}{1 + \eta(x)(a - 1)} \left( 2\eta(x) + \frac{\eta'(x) \tan x}{1 + (\eta(x) - \tan x\eta'(x))(a - 1)} \right) \right].
\]
We again claim that either $L'_X(a)$ is either nonnegative or if it changes sign then it can only change from positive to negative. Again demonstrating that $L_X(a) > 0$.

If equation (32) has constant sign, then we immediately conclude that $L_X(a)$ is bounded below by the value $a = 1$ or $a = a_1$, which are positive. Thus assume that equation (32) is zero at some point where it changes sign. Because $\tilde{a} a$ is positive, this is only possible if the large bracketed term is zero and changes sign. Notice that the large bracketed term is zero only if the large parenthetical term is positive. The fraction in the large parenthetical term, is the reciprocal of a linear function of $a$ with negative slope, which is therefore increasing. Consider next the fraction multiplying the parenthetical term. It is the quotient of a positive linear functions of $a$ with positive slope by a positive linear function of $a$ with negative slope. Thus this fraction is positive and increasing. Thus the large bracketed term is decreasing as we are subtracting an increasing function from 2, and so can only change from positive to negative.

A.2. The metric on the neck. We now wish to use $\tilde{g}(a, b)$ of Definition A.3 to define a metric $g$ on the cylinder $[0, 1] \times S^n$. We will define this metric on the cylinder so that the metric restricted to each slice $\{t\} \times S^n$ is conformal to $\tilde{g}(h(t), k(t))$, where $h(t)$ and $k(t)$ are some functions on the interval. This choice of metric allows us to use Lemma A.4 to reduce bounding the sectional curvatures in the spherical directions below by bounding the second fundamental form of the slices above. This second fundamental form is proportional to $\partial_t \tilde{g}(h(t), k(t))$. Thus to make this small we want $h'(t)$ and $k'(t)$ to essentially vanish. To achieve this end, we will instead consider defining $g(\kappa, h(t), k(t))$ on $[t_0, t_1] \times S^n$ assuming that $[t_0, t_1]$ is an arbitrarily long interval. After a reparameterization, this will define a metric on $[0, 1] \times S^n$ with the same curvature properties. Thus to prove Lemma A.1 it suffices to construct a metric on $[t_0, t_1] \times S^n$.

We begin in Section A.2.1 by defining the family of metrics $g(\kappa, h(t), k(t))$, referred to in the preceding paragraph, on $[t_0, t_1] \times S^n$, where $h(t)$ and $k(t)$ are functions on $[t_0, t_1]$ and $\kappa > 0$ is a
scaling factor. We then spend some time discussing the necessary assumptions on \( h(t), k(t), \) and \( \kappa \) imposed by assuming that \( g \) satisfies (iii) through (vi) of Lemma A.1 which turns out to be very easily summarized with Proposition A.9. Next in section A.2.2 we find a class of functions \( h(t) \) and \( k(t) \) defined in terms of two new parameters \( \varepsilon \) and \( \delta \) that will satisfy the necessary conditions described in Section A.2.1 and that remain close enough to the special path in Lemma A.4 for suitable choice of parameters. In section A.2.3 we give a precise description of the control we have over \( g \) with our choice of parameters. And finally in Section A.2.4 we explain which choices of parameters will guarantee that \( g \) satisfies Proposition A.9 and therefore (iii) through (vi) of Lemma A.1 and we show how the parameters force \( g \) to inherit large sectional curvature in the spherical directions from Lemma A.4 Going forward we will have produced a three parameter family of metrics \( g(t_0, \varepsilon, \delta) \) on \([0, 1] \times S^n\) that satisfy (ii) through (v) of Lemma A.1 and that has large sectional curvatures which will facilitate in the proof of (i) in Section A.3.

A.2.1. **The form of the metric.** While \( \tilde{g}(a, b) \) of definition A.3 was discussed only for \((a, b) \in [1, a_1] \times [r, \rho]\), it is well defined for all \( a \geq 1 \) and \( b > 0 \). Thus we can consider metrics of the following form on \([t_0, t_1] \times S^n\).

**Definition A.7.** Suppose we are given functions \( h(t) \geq 1 \) and \( k(t) > 0 \) defined on \([t_0, t_1]\) and a constant \( \kappa > 0 \). Define the metric \( g = g(\kappa, h(t), k(t)) \) on \([t_0, t_1] \times S^n\), as follows

\[
g(\kappa, h(t), k(t)) = \kappa^2 \left( dt^2 + t^2 \tilde{g}(h(t), k(t)) \right)
\]

Thus the intrinsic curvature of \( \{t\} \times S^n \) with respect to \( g \) is clearly just \( \frac{1}{t^2 \kappa^2} K_{\tilde{g}(h(t), k(t))} \). If \((h(t), k(t))\) is close to the path \( \left( a, \frac{\rho}{a_1^{1/\alpha}} \right) \), then we can immediately conclude from Lemma A.4 that the intrinsic curvatures are greater than \( \frac{1}{t^2 \kappa^2} \). We will explain in a moment why \((h(t), k(t))\) is not exactly a parameterization of this path. The extrinsic curvature can be computed directly from (i) of Proposition B.16.

**Corollary A.8.** Let \( \Pi_t \) denote the second fundamental form of \((\{t\} \times S^n, g(\kappa, h(t), k(t)))\) with respect to the unit normal \( \partial_t \). The extrinsic curvatures are as follows.

\[
\Pi_t(X, \Sigma_i) = \Pi_t(\Sigma_i, \Sigma_j) = 0,
\]

\[
\Pi_t(X, X) = \frac{1}{\kappa} \left( \frac{1}{t} + \frac{k'(t)}{k(t)} + \frac{\eta(x)h'(t)}{1 + (h(t) - 1)\eta(x)} \right),
\]

\[
\Pi_t(\Sigma_i, \Sigma_i) = \frac{1}{\kappa} \left( \frac{1}{t} + \frac{k'(t)}{k(t)} \right).
\]

We can now explain the motivation for defining \( g(\kappa, h(t), k(t)) \) in this way. Obviously, we want the slices \( \{t\} \times S^n \) to be conformal to \( \tilde{g} \), so we can conclude the intrinsic curvatures are large. The purpose of scaling \( \tilde{g} \) by \( t \) is to correct the following defect. If we considered instead the metric
\[ \kappa^2(dt^2 + \tilde{g}(h(t), k(t))) \]
then the second fundamental form of \( \{t \} \times S^n \) as in (ii) of Proposition B.16 gives
\[ \Pi_t(\Sigma_i, \Sigma_i) = \frac{k'(t)}{\kappa(t)}. \]
Since \( k(t) \) is decreasing from \( \rho \) to \( r \), we see that \( \Pi_t(\Sigma_i, \Sigma_i) < 0 \). Thus the metric \( \kappa^2(dt^2 + \tilde{g}(h(t), k(t))) \) could not satisfy claims (iv) or (v) of Lemma A.1. Considering instead \( g(\kappa, h(t), k(t)) \) then the second fundamental becomes as in equation (35). The limit of this as \( t \) increases can be made to be positive if \( k'(t)/k(t) \) decays more rapidly than \( 1/t \). Since we have the freedom to choose \( [t_0, t_1] \) and the function \( k(t) \), this can be used to make \( \Pi_t \) positive definite for all \( t \). It will satisfy (v) if \( \frac{1}{\kappa(t_1)} > 1 \).

We will consider very carefully which choices of \( h(t) \) and \( k(t) \) should be made, but regardless, to satisfy (iii) of Lemma A.1 we need \( g|_{t=t_1} = g_1 \). We can see from formula (33) that this can only happen if \( h(t_1) = a_1 \) and that we choose our scale to be \( \kappa = \frac{r}{t_1 k(t_1)} \). In order for \( g \) to satisfy (iii) we see from equation (33) that \( h(t_0) = 1 \) and therefore the choice of \( \lambda \) must be
\[ \lambda = \frac{\rho t_1 k(t_1)}{r t_0 k(t_0)}. \]
In our application, we will assume for ease that \( k(t_0) = \rho \), thus \( \lambda = \frac{t_1 k(t_1)}{r t_0} \). Then considering (iv), we see that \( \Pi_t = -\lambda \) if and only if \( k'(t_0) = h'(t_0) = 0 \).

Already with this short discussion we have seen how metrics of the form \( g(\kappa, h(t), k(t)) \) satisfy the claims of Lemma A.1 at \( t = t_0 \) and \( t = t_1 \) under certain assumptions. We summarize this in the following proposition.

**Proposition A.9.** For any functions \( h(t) \) and \( k(t) \) on \([t_0, t_1]\) let \( \kappa = \frac{r}{t_1 k(t_1)} \) and let \( \lambda = \frac{t_1 k(t_1)}{t_0 r} \).

Then the claims of Lemma A.1 for the metric \( g(\kappa, h(t), k(t)) \) reduce to the following conditions on \( h(t) \) and \( k(t) \).

1. \( h(t_0) = 1 \) and \( k(t_0) = \rho \),
2. \( h(t_1) = a_1 \),
3. \( h'(t_0) = k'(t_0) = 0 \),
4. \( -\frac{k'(t_1)}{k(t_1)} < \frac{1}{t_1} \) and \( \kappa < \frac{1}{t_1} \).

**A.2.2. The choice of metric functions.** We still have the freedom to choose \( h(t) \) and \( k(t) \). As mentioned, we wish to choose \( (h(t), k(t)) \) close to the path \( (a, \frac{\rho}{a^{1/a}}) \). Notice that assuming this \( (h(t), k(t)) \) is a parameterization of this path is equivalent to assuming similar to equation (22) that
\[ \frac{h'(t)}{\dot{a} h(t)} = -\frac{k'(t)}{k(t)}. \]

We will therefore define \( h(t) \) and \( k(t) \) in terms of a separable differential equation. We see that this is also amenable to Proposition A.9 as we have the freedom to assume initial conditions to
satisfy (ii) and (iv). This definition will also allow us to have direct control over the magnitude of \( \frac{k'(t_1)}{k(t_1)} \). However, if we take equation (36) as our definition, then \( k(t_1) = r \) and so \( \kappa = \frac{1}{t_1} \). Thus \( \Pi_{t_1}((\Sigma_i, \Sigma_i) < 1 \), and \( g \) will not satisfy (v). To correct this, we will assume that \( k(t_1) = \tilde{r} > r \), forcing \( \kappa < \frac{1}{t_1} \). We will assume then that

\[
\frac{h'(t)}{ah(t)} = \frac{k'(t)}{k(t)} = \beta \Gamma(t).
\]

Where \( \alpha \) and \( \beta \) are some constants that will be determined by the assumptions that \( h(t_1) = a_1 \) and \( k(t_1) = \tilde{r} \), and \( \Gamma(t) \) is some function with the appropriate asymptotics and initial conditions.

We begin by making a choice for \( \Gamma(t) \). According to Proposition [A.9](#A.9), in order for \( g \) to satisfy (iv), we must have that \( \Gamma(t_0) = 0 \). And in order to satisfy (v) we must assume something like \( \Gamma(t) = O\left(\frac{1}{t \ln t}\right) \). This motivates the following definition of the function \( \Gamma(t) \).

\[
\Gamma(t) = \begin{cases} 
\frac{t - t_0}{2t_0^2 \ln(2t_0)} & t_0 \leq t \leq 2t_0 \\
\frac{\ln(2t_0)}{t \ln^2 t} & 2t_0 < t
\end{cases}
\]

The graphs of this function and its derivative appear in Figure 15. One can see that this function is only continuous at \( t = 2t_0 \), so the functions \( h(t) \) and \( k(t) \) will only be once-differentiable. This issue has no bearing on the metric at \( t = t_0 \) and \( t = t_1 \) nor does it affect the smoothness of \( K_g \) is the spherical directions. So we will wait to resolve this issue in Section [A.3](#A.3). The following claim is obvious from Figure 15 and is easy to check.
Lemma A.10. If $t_0 > 2$, then there exists constants $c_{\Gamma}$ and $d_{\Gamma}$ such that $\Gamma(t)$ and $\Gamma'(t)$ satisfy the following bounds for all $t \geq t_0$.

\[
|\Gamma(t)| < \frac{c_{\Gamma} \ln(2t_0)}{t \ln^2 t} \\
|\Gamma'(t)| < \frac{d_{\Gamma} \ln(2t_0)}{t^2 \ln^2 t}
\]

For fixed $[t_0, t_1]$ and $\tilde{r}$, then $\alpha$ and $\beta$ are uniquely determined by equation (37). As we are partly trying to illuminate Perelman’s proof of Lemma A.1, we now introduce two new parameters $\varepsilon$ and $\delta$ in the following way. Assuming that $h(t_1) = a_1$, then the limit of $h(t)$ as $t \to \infty$ must be greater than $a_1$. Therefore there must exist a $\delta > 0$ such that

\[
(39) \quad \int_{t_0}^{\infty} \frac{h'(t)}{h(t)} dt = (1 + \delta) \ln a_1.
\]

Assuming that $k(t_1) = \tilde{r} > r$, we may assume that the limit of $k(t)$ as $t \to \infty$ is greater than $r$. Therefore there must exist a $\varepsilon > 0$ such that

\[
(40) \quad \int_{t_0}^{\infty} \frac{k'(t)}{k(t)} dt = (1 - \varepsilon)(\ln r - \ln \rho).
\]

Conversely if we define $h(t)$ and $k(t)$ by equation (37), then $h(t)$ and $k(t)$ satisfy equations (39) and (40) only if $\alpha$ and $\beta$ are as follows

\[
(41) \quad \beta(t_0, \varepsilon) = \frac{(1 - \varepsilon)(\ln r - \ln \rho)}{\Delta(t_0)} \quad \text{and} \quad \alpha(t_0, \varepsilon, \delta) = \frac{(1 + \delta) \ln a_1}{\beta \Delta(t_0)}.
\]

Where $\Delta(t_0) = \int_{t_0}^{\infty} \Gamma(t) dt$. Clearly $t_0$ and $\delta$ determines the value of $t_1$ by the equation $h(t_1) = a_1$, and then $k(t_1) = \tilde{r}$ is determined additionally by $\varepsilon$. With this we are ready to define our metric in terms of this choice of $h(t)$ and $k(t)$ parameterized by $t_0$, $\varepsilon$, and $\delta$.

We have finally settled on the following choice of metric.

Definition A.11. For any $t_0 > 2$, $0 < \varepsilon < \frac{1}{4}$, and $0 < \delta < \frac{1}{4}$, let $\alpha = \alpha(t_0, \varepsilon, \delta)$ and $\beta = \beta(t_0, \varepsilon)$ be as defined in equation (41), and $h(t)$ and $k(t)$ be as defined by equation (37) with initial conditions $h(t_0) = 1$ and $k(t_0) = \rho$. Then define $g = g(t_0, \varepsilon, \delta) = g\left(\frac{r}{t_1 k(t_1)} h(t), k(t)\right)$, and define $t_1$ to be the unique number so that $h(t_1) = a_1$.

A.2.3. Control. We claim that the metric $g(t_0, \varepsilon, \delta)$ will satisfy Lemma A.1 for suitably chosen parameters. Let us briefly explain how each of these parameters will be used in the proof of Lemma A.1. By choosing $t_0$ larger, one can see from Lemma A.10 that this controls the size of the functions $\frac{h'(t)}{h(t)}$ and $\frac{k'(t)}{k(t)}$. This is made precise in Lemma A.12. By choosing $\delta$ smaller, we make $t_1$ larger relative to $t_0$, essentially controlling the length of the interval. By choosing $\varepsilon$ smaller, we have control over the lower bound on $\tilde{r}$, which is also determined by $t_1$. These last two facts are made precise in Lemma A.13.
Lemma A.12. Let $h(t)$ and $k(t)$ be as in Definition A.11. There are constants $c_1 > 0$ and $c_2 > 0$ that depend only on $g_1$, such that

$$
\left| \frac{h'(t)}{\alpha h(t)} \right| = \left| \frac{k'(t)}{k(t)} \right| < \frac{c_1 \ln(2t_0)}{t \ln^2 t},
$$

$$
\left| \frac{\eta(x)h'(t)}{1 + (h(t) - 1)\eta(x)} \right| < \frac{c_1 \ln(2t_0)}{t \ln^2 t},
$$

$$
\left| \frac{\eta(x)h''(t)}{1 + (h(t) - 1)\eta(x)} \right| < \frac{c_2 \ln(2t_0)}{t^2 \ln^2 t},
$$

$$
\left| \frac{k''(t)}{k(t)} \right| < \frac{c_2 \ln(2t_0)}{t^2 \ln^2 t}.
$$

Proof. The constants $c_1$ and $c_2$ are partially determined by $\alpha$ and $\beta$. It is important to notice that by assuming $t_0 > 2$, $0 < \varepsilon \leq \frac{1}{4}$, and $0 < \delta \leq \frac{1}{4}$, we have assumed that $\alpha$ and $\beta$ are bounded above and below independently of our specific choices $t_0$, $\varepsilon$, and $\delta$.

Inequality (42) follows from Definition A.11 and Lemma A.10. Using the definition of $k(t)$ to compute $\frac{k''(t)}{k(t)}$ yields

$$
\frac{k''(t)}{k(t)} = -\beta^2 \Gamma^2(t) + \beta \Gamma'(t).
$$

By applying Lemma A.10 and noting that $\ln(2t_0) < \ln t$ proves equation (45).

By equation (26) and (25) we have

$$
|\eta(x)| < \max \left\{ 1, \frac{1}{a_1 - 1} \right\}.
$$

And by equation (27), there is a constant $d > 0$ such that $(1 + (h(t) - 1)\eta(x)) > ch(t)$. Thus there is a constant $d > 0$ such that

$$
\left| \frac{\eta(x)}{1 + (h(t) - 1)\eta(x)} \right| < d \left| \frac{1}{h(t)} \right|.
$$

Thus we have

$$
\left| \frac{\eta(x)h'(t)}{1 + (h(t) - 1)\eta(x)} \right| < d \left| \frac{h'(t)}{h(t)} \right| \quad \text{and} \quad \left| \frac{\eta(x)h''(t)}{1 + (h(t) - 1)\eta(x)} \right| < d \left| \frac{h''(t)}{h(t)} \right|.
$$
Applying equations (42) and (45) to equation (46) proves equations (43) and (44). □

The following lemma is a straightforward computation achieved by explicitly solving the separable differential equation (37).

**Lemma A.13.** Let \( h(t), k(t), \) and \( \beta \) be as in Definition A.11. The number \( t_1 \) such that \( h(t_1) = t_1 \) and \( \tilde{r} = k(t_1) \) are expressed in terms of \( t_0, \varepsilon, \) and \( \delta \) as follows.

\[
t_1(t_0, \delta) = \exp \left( 1 + \delta - \frac{4 \ln^2(2t_0)}{\delta} \right),
\]

\[
\tilde{r}(t_0, \delta, \varepsilon) = r \left( \frac{\rho}{r} \right)^{\varepsilon} \exp \left( \frac{\beta \ln(2t_0)}{\ln t_1} \right).
\]

Moreover, for all \( T_1 > 0, \) there are \( T_0 > 0 \) and \( \delta_0 > 0 \) such that \( t_1 > T_1 \) either if \( t_0 > T_0 \) or \( \delta < \delta_0. \) And, for all \( r < r_0 < r_1, \) there are \( 0 < \varepsilon_0 < \varepsilon_1 \) and \( \delta_1 > 0 \) such that \( r_0 < \tilde{r} < r_1 \) if \( \varepsilon_0 < \varepsilon < \varepsilon_1 \) and \( \delta < \delta_1. \)

A.2.4. **Summary of benefits.** Now that we understand the precise relationship of \( t_0, \varepsilon, \) and \( \delta \) to our metric \( g, \) we are ready to summarize the discussion of this section in two corollaries. The first Corollary summarizes the discussion of Section A.2.1. It says that Proposition A.9 will apply to our metric \( g \) if \( \varepsilon \) is fixed and \( \delta \) is chosen small relative to \( \delta. \) The second summarizes the discussion of Section A.2.2. It says that if \( \varepsilon \) and \( \delta \) are small enough, then the path \( (h(t), k(t)) \) is close enough to \( \left( a, \frac{\rho}{a^{1/\alpha}} \right), \) implying that the curvature of \( g \) is relatively large.

**Corollary A.14.** There is an \( \varepsilon_0 > 0, \) and \( \delta(\varepsilon_0) > 0 \) so that \( g(t_0, \varepsilon, \delta) \) satisfies claims (ii), (iii), (iv), and (v) of Lemma A.1 with \( \lambda = \frac{t_1k(t_1)}{t_0r} \) for all \( \varepsilon > \varepsilon_0 \) and \( \delta < \delta_0(\varepsilon_0). \)

**Proof.** By definition \( h(t_0) = 1 \) and \( k(t_0) = \rho, h'(t_0) = k'(t_0) = 0, \) and \( h(t_1) = a_1, \) so by Proposition A.9 \( g \) satisfies (ii) with the specified \( \lambda, \) (iv) with the specified \( \lambda, \) and (iii). It remains to show that we can choose \( \delta_0 \) and \( \varepsilon_0 \) such that \( g \) also satisfies (v).

By Corollary A.8 and equations (42) and (43) of Lemma A.12 there is some \( c > 0 \) such that

\[
|\Pi_{t_1}| > \frac{\tilde{r}}{r} - \frac{c}{\ln t_1} - \frac{c}{\ln^2 t_1}.
\]

If we fix \( \varepsilon_0 > 0, \) then by Lemma A.13 \( \frac{\tilde{r}}{r} \geq 1 + \zeta \) for some \( \zeta(\varepsilon_0) > 0. \) Substituting this into equation (48), we see that \(|\Pi_{t_1}| > 1 \) if \( t_1 \) is chosen large enough so that the remaining terms are bounded by \( \zeta. \) Again, by Lemma A.13 there exists a \( \delta_0(\varepsilon_0) > 0 \) depending on \( \varepsilon_0, \) such that this holds for all \( \delta < \delta_0. \) Thus for all \( \delta < \delta_0, \) \( g \) satisfies (v). □

The above discussion of (v) in the proof of Corollary A.14 is the final consideration in Perelman’s proof of Lemma A.1. In the last paragraph of [13, p. 160], having already fixed \( \varepsilon \) and before scaling by \( \kappa, \) Perelman says, “it remains to choose \( \delta > 0 \) so small, and correspondingly \( t_1 \) so large, that...
normal curvatures of $S^n \times \{t\}$ are
\[ K_{\tilde{g}}(X, \Sigma_i) > \frac{c_s}{t^2}, \text{ and } K_{\tilde{g}}(\Sigma_i, \Sigma_j) > \frac{c_s}{t^2}. \]

Proof. Let $a \in [1, a_1]$ and $\tilde{\alpha} = \frac{\ln a_1}{\ln \rho - \ln r}$ be as in Section A.1. Define $\tilde{b}(a)$
\[ (49) \quad \tilde{b}(a) = \frac{\rho}{a^{1/\tilde{\alpha}}}. \]
By Lemma A.4, $K_{\tilde{g}(a, \tilde{b}(a))} > 1$.

Let $\alpha$ be defined as in equation (41), then the definition of $h(t)$ and $k(t)$ in Definition A.11 is equivalent to
\[ (50) \quad k(t) = \frac{\rho}{h^{1/\alpha}(t)}. \]
Which we may in turn take to define a function $b(a)$ for $a \in [1, a_1]$. It is clear by comparing equation (49) and (50) that $|b(a) - \tilde{b}(a)|_{C^2} \to 0$ as $(\alpha - \tilde{\alpha}) \to 0$, where explicitly
\[ \alpha - \tilde{\alpha} = \left(1 + \frac{\delta}{1 - \varepsilon} - 1\right) \frac{\ln a_1}{\ln \rho - \ln r}. \]
Thus $|b(a) - \tilde{b}(a)|_{C^2} \to 0$ as $(\varepsilon, \delta) \to (0, 0)$. This is also clear graphically in Figure 17 noting that, by Lemma A.13, $\tilde{r} \to r$ as $(\varepsilon, \delta) \to (0, 0)$.

Because $K > 1$ is an open condition on the space of all metrics, the set $U = \{(a, b) : K_{\tilde{g}(a, b)} > 1\}$ is open, and by Lemma A.4, $(a, \tilde{b}(a)) \in U$. It follows that there exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $(a, b(a)) \in U$ for all $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$. Thus $K_{\tilde{g}(h(t), k(t))} > 1$ for all $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$. 

Figure 17. By Lemma A.4, the open set $U$ defined by $U = \{(a, b) : \text{Ric}_{\tilde{g}(a, b)} > 1\}$ contains $(a, \frac{\rho}{a^{1/\tilde{\alpha}}})$. Clearly, if $\tilde{r}$ is close to $r$ then $(h(t), k(t))$ is also contained in $U$. 

Corollary A.15. Let $g(t_0, \varepsilon, \delta)$ be as in Definition A.11, then there exists $\varepsilon_0 > 0$, $\delta_0 > 0$, and $c_s > 0$ such that for all $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$ the ambient sectional curvatures in the spherical of $\{t\} \times S^n$ are as follows.
\[ K_{\tilde{g}(a, \tilde{b}(a))} > \frac{c_s}{t^2}, \text{ and } K_{\tilde{g}(\Sigma_i, \Sigma_j)} > \frac{c_s}{t^2}. \]
Still assuming that $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$. If we combine this fact with Gauss’ formula, Corollary A.8 and equations (42) and (43) of Lemma A.12 we have the following.

$$K_g(X, \Sigma_i) = \left( \frac{t_1 k(t_1)}{r} \right)^2 \left( K_{\tilde{g}(h(t), k(t))} (X, \Sigma_i) - \Pi^2_t (X, \Sigma_i) \right) > \left( \frac{t_1 k(t_1)}{r} \right)^2 \left( \frac{d}{t^2} - \frac{1}{t^2} - \frac{c}{t^2 \ln t} - \frac{c}{t^2 \ln^2 t} \right).$$

$$K_g(\Sigma_i, \Sigma_j) = \left( \frac{t_1 k(t_1)}{r} \right)^2 \left( K_{\tilde{g}(h(t), k(t))} (\Sigma_i, \Sigma_j) - \Pi^2_t (\Sigma_i, \Sigma_j) \right) > \left( \frac{t_1 k(t_1)}{r} \right)^2 \left( \frac{d}{t^2} - \frac{1}{t^2} - \frac{c}{t^2 \ln t} - \frac{c}{t^2 \ln^2 t} \right).$$

There is some $T > 0$ such that the remaining terms $\frac{c}{t^2 \ln t}$ and $\frac{c}{t^2 \ln^2 t}$ bounded above by $\frac{d - 1}{t^2}$ for all $t > T$. Thus if $t_0 > T$ we have proven the claim. \hfill $\square$

In the second full paragraph of [13, p. 160], Perelman suggests “we choose $\varepsilon > 0$ first in such a way that $\left( \frac{2}{r} \right)^\varepsilon g_1$ still has sectional curvatures $> 1.”$ We can see this in our formula (47) that $\tilde{r} \approx \left( \frac{2}{r} \right)^\varepsilon$ if $\delta$ is chosen very small, and by definition $\tilde{g}(h(t_1), k(t_1)) = \frac{r}{r} g_1$. The assumption that the sectional curvatures of $\left( \frac{2}{r} \right)^\varepsilon g_1$ are greater than 1, imply that the proof of Lemma A.4 still applies to the path $(a, b) = (h(t), k(t))$. The proof of Corollary A.15 is essentially the same observation.

### A.3. The curvatures of the neck.

By Corollary A.14 the metric $g(t_0, \varepsilon, \delta)$ can be chosen to satisfy claims [3] through [7] of Lemma A.1. In this section we consider the validity of [8] for suitable choices of $t_0$, $\varepsilon$, and $\delta$. That is, we prove that $\text{Ric}_g$ is positive definite for suitable choices of parameters. The care that was taken in Section A.2.2 was to ensure that the curvature in spherical directions was large, as recorded in Corollary A.15. This one fact will allow us to dominate other sectional curvatures by choosing $t_0$ even larger.

The first difficulty we face in proving something about $\text{Ric}_g$ is that the remaining curvatures are all second order in $h(t)$ and $k(t)$, but $h(t)$ and $k(t)$ are both only once differentiable at $t = 2t_0$. In Section A.3.1, we explain how to apply Perelman’s gluing lemma to resolve this issue, and therefore only need to prove positive Ricci curvature for $t \neq 2t_0$. Once this is resolved, in Section A.3.2 we see directly that the spherical curvatures dominate $\text{Ric}_g(X, X)$ and $\text{Ric}_g(\Sigma_i, \Sigma_i)$, ensuring their positivity. In Section A.3.3 with a little more care, we show that $\text{Ric}_g(T, T)$ is also positive. As the Ricci tensor is not diagonalized in this frame, this does not prove that it is positive definite. There is exactly one nonzero off-diagonal term: $\text{Ric}_g(T, X)$. Thus positive definiteness of $\text{Ric}_g$ reduces to showing that the 2-by-2 sub-matrix spanned by $T$ and $X$ is positive definite. As we will have proven that its trace is positive, we need only show that the determinant is positive. In Section A.3.4 we check that this determinant is dominated by $\text{Ric}_g(X, X)$. We conclude by carefully examining the
dependency of choices of \( t_0, \varepsilon, \) and \( \delta, \) to ensure that there is a metric \( g \) satisfying all of the claims of Lemma \( A.1. \)

A.3.1. Smoothing the neck. In [13, p. 160], when Perelman defines \( h(t) \) and \( k(t), \) he recommends that, “\( \beta \) must be smoothed near \( t = 2t_0. \)” Certainly this works, but then Lemma \( A.12 \) does not follow directly from Lemma \( A.10. \) Instead, we give a slight generalization of Perelman’s gluing lemma, Lemma \( 1.2, \) that can be applied to two manifolds with isometric boundaries, such that the sum of second fundamental forms is zero and both forms have definite signature.

Corollary A.16. Let \( (M_i^n, g_i) \) with \( i = 1, 2 \) and \( n \geq 3 \) be closed Riemannian manifolds with positive Ricci curvature. Suppose that there is an isometry \( \phi \) between their boundaries such that \( \Pi_1 + \phi^* \Pi_2 = 0 \) and \( \Pi_1 \) is positive definite, where \( \Pi_i \) is the second fundamental form of \( \partial M_i. \) Then \( M_1 \cup_\phi M_2 \) admits a metric with positive Ricci curvature that agrees with \( g_i \) on \( M_i \) outside of an arbitrarily small tubular neighborhood of \( \partial M_i. \)

Proof. Choose normal coordinates for tubular neighborhoods of \( \partial M_i = N_i, N_1 \times (-\xi, 0) \) and \( [0, \xi) \times N_2. \) In these coordinates each metric becomes \( g_i = dt^2 + h_i(t) \) where \( g_i(t) \) is a smooth metric on \( N_i. \) We have that \( \Pi_1 = h_1'(0) \) and \( \Pi_2 = -h_2'(0). \) We wish to show that we can bend the metrics outward at the boundary to produce metrics that satisfy the hypotheses of Lemma \( 1.2 \) directly.

We define our bending functions \( \chi_\xi(t) \) to be

\[
\chi_\xi(t) = \begin{cases} 
\exp \left( \frac{-1}{(x + \xi)^2} \right) & -\xi \leq t \leq 0 \\
0 & t < -\xi
\end{cases}
\]

Clearly for all \( k > 0 \) and \( \zeta > 0, \) there exists \( \xi_0 > 0 \) such that \( \|\chi_\xi(t)\|_{C^k} < \zeta \) for all \( \xi < \xi_0. \)

Define the new bent metrics \( \tilde{g}_i(\xi) \) on \( M_i \) as follows.

\[
\tilde{g}_1(\xi) = \begin{cases} 
dt^2 + (1 + \chi_\xi(t))^2 h_1(t) & x \in N_1 \times (-\xi, 0] \\
g_1 & x \not\in N_1 \times (-\xi, 0] 
\end{cases}
\]

\[
\tilde{g}_2(\xi) = \begin{cases} 
dt^2 + (1 + \chi_\xi(-t))^2 h_2(t) & x \in [0, \xi) \times N_2 \\
g_2 & x \not\in [0, \xi) \times N_2 
\end{cases}
\]

It is clear that this defines a smooth metric for all \( \xi > 0. \) It is also clear that, for any \( k > 0 \) and \( \zeta > 0, \) that there exists a \( \xi_0 > 0 \) such that \( \|\tilde{g}_i(\xi) - g_i\|_{C^k} < \zeta \) for all \( \xi < \xi_0. \) Because the set of metrics that satisfy \( \text{Ric}_{\tilde{g}_i} > 0 \) is open in the space of all metrics with respect to the \( C^2 \) topology, it follows that, for some choice of \( \xi > 0, \) \( \text{Ric}_{\tilde{g}_i(\xi)} > 0. \) Fix such a \( \xi, \) and let \( \tilde{g}_i = \tilde{g}_i(\xi). \)

Let \( \tilde{\Pi}_i \) be the second fundamental forms of \( N_i \) with respect to the metrics \( \tilde{g}_i. \) In terms of \( \Pi_i, \) the \( \tilde{\Pi}_i \) are as follows.

\[
\tilde{\Pi}_1 = \chi'_\xi(0) h_1(0) + h_1'(0) = \Pi_1 + \chi'_\xi(0) h_1(0).
\]

\[
\tilde{\Pi}_2 = \chi'_\xi(0) h_2(0) - h_2'(0) = \Pi_2 + \chi'_\xi(0) h_2(0).
\]
By definition $\chi'(0) > 0$ and $h_i(0)$ is positive definite. It follows that $\Pi_1 > \Pi_i$, and so $\Pi_1 + \Pi_2 > \Pi_1 + \Pi_2 = 0$. Thus $\Pi_1 > -\Pi_2$. Because $\text{Ric}_g > 0$, Lemma A.2 applies. \hfill \square

This corollary can be applied to the manifolds $M_1 = [t_0, 2t_0] \times S^n$ and $M_2 = [2t_0, t_1] \times S^n$ at the boundary $t = 2t_0$ both equipped with metrics $g(t_0, \varepsilon, \delta)$. Because $\Pi$ is a first order quantity, and the metric $g(t_0, \varepsilon, \delta)$ is $C^1$, we are exactly in the situation $\Pi_1 + \Pi_2 = 0$. Thus the following is immediate. In essence, it tells us that we may ignore the fact that $g(t_0, \varepsilon, \delta)$ is not smooth. Because if we prove that $\text{Ric}_g$ is positive every except $t = 2t_0$, then we may use Corollary A.16 to smooth the metric there.

**Corollary A.17.** Assume that there is a $T > 0$ such that, for all $t_0 > T$ the metric $g = g(t_0, \varepsilon, \delta)$ defined on $[t_0, t_1] \times S^n$ satisfies claims (i) through (iii) of Lemma A.1 and satisfies claim (i) for all $t \neq 2t_0$. Then for some $t_0 > T$ there exists a smooth metric $\mathring{g}$ on $[t_0, t_1] \times S^n$ that agrees with $g$ outside of an arbitrarily small neighborhood of the set $t = 2t_0$ that satisfies (ii) for all $t$.

**Proof.** We wish to apply Corollary A.16 to $M_1 = [t_0, 2t_0] \times S^n$ and $M_2 = [2t_0, t_1] \times S^n$ with the metric $g$ restricted to both. From Corollary A.8 and the definition of $g$, it is clear that $\Pi_1 + \Pi_2 = 0$. And by assumption, the Ricci curvature is positive. It only remains to show that $\Pi_1$ is positive definite. Combining Corollary A.8, equations (42) and (43) we have

$$
\Pi_{2t_0} > \frac{t_1 k(t_1)}{r} \left( \frac{1}{2t_0} - \frac{c}{2t_0 \ln(2t_0)} \right).
$$

This is positive if $t_0$ is chosen large enough. \hfill \square

A.3.2. The Ricci curvature in the spherical directions is large. We have already found suitable lower bound for $K_g(X, \Sigma_i)$ and $K_g(\Sigma_i, \Sigma_j)$ in Corollary A.15. We must now consider the remaining curvatures of $g$. The following is a direct corollary of (iii) of Proposition B.16 and Lemma B.17. Keep in mind that not all of the formulas are defined at $t = 2t_0$.

**Corollary A.18.** Let $g$ be as in Definition A.14. The sectional curvatures in the time direction are as follows.

\begin{align*}
(51) \quad K_g(T, \Sigma_i) &= - \left( \frac{t_1 k(t_1)}{r} \right)^2 \left( \frac{k''(t)}{k(t)} + \frac{2k'(t)}{tk(t)} \right), \\
(52) \quad K_g(T, X) &= - \left( \frac{t_1 k(t_1)}{r} \right)^2 \left( \frac{k''(t)}{k(t)} + \frac{2k'(t)}{tk(t)} \right) \\
&\quad + \frac{\eta(x) h''(t)}{1 + (h(t) - 1)\eta(x)} + \frac{2\eta(x) h'(t)}{t(1 + (h(t) - 1)\eta(x))} + \frac{k'(t) 2\eta(x) h'(t)}{k(t) 1 + (h(t) - 1)\eta(x)}, \\
R_g(T, \Sigma_i, \Sigma_j, X) &= - \left( \frac{t_1 k(t_1)}{r} \right)^2 \left( \frac{\tan(x) \eta(x) h'(t)}{tk(t)(1 - (h(t) - 1)\eta(x))^2} \right).
\end{align*}

All other Riemannian curvatures of the form $R_g(A, B, B, C)$ vanish.
Investigating Corollary A.18 term by term, we see that each is bounded in absolutely value by \( \frac{c \ln(2t_0)}{t^2 \ln^2 t} \) as in Lemma A.12. Thus the following corollary is immediate.

**Corollary A.19.** There exists \( c_u > 0 \) such that the curvature of \( g \) in the time directions is bounded in absolute value as follows.

\[
|K_g(T, \Sigma_i)| < \frac{c_u \ln(2t_0)}{t^2 \ln^2 t},
\]

\[
|K_g(T, X)| < \frac{c_u \ln(2t_0)}{t^2 \ln^2 t},
\]

\[
|R(T, \Sigma_i, \Sigma_i, X)| < \frac{c_u \ln(2t_0)}{t^2 \ln^2 t}.
\]

Comparing Corollary A.19 with the asymptotics of \( K_g(X, \Sigma_i) \) and \( K_g(\Sigma_i, \Sigma_j) \) in Corollary A.15 shows that \( \text{Ric}_g(X, X) \) and \( \text{Ric}_g(\Sigma_i, \Sigma_i) \) remain as large. In particular, these Ricci curvatures are positive.

**Corollary A.20.** Let \( \varepsilon_0 > 0 \) and \( \delta_0 > 0 \) be as in Corollary A.15. There exists \( T > 0 \) and \( c_l > 0 \) such that for all \( t_0 > T, \varepsilon < \varepsilon_0, \delta < \delta_0 \) we have

\[
\text{Ric}_g(X, X) > \frac{c_l}{t^2},
\]

\[
\text{Ric}_g(\Sigma_i, \Sigma_i) > \frac{c_l}{t^2}.
\]

**A.3.3.** The Ricci curvature in the time direction is positive. It remains to show that \( \text{Ric}_g(T, T) \) is positive and that \( \text{Ric}_g(T, X) \) is dominated by \( \text{Ric}_g(T, T) \) and \( \text{Ric}_g(X, X) \). To achieve both we must bound \( \text{Ric}_g(T, T) \) below. Let us begin by considering equation (51) and (52). For simplicity, let us focus on equation (51) and ignore the scaling factor, we want to show that the following is negative

\[
\frac{k''(t)}{k(t)} + \frac{2k'(t)}{tk(t)}.
\]

Note that \( k''(t) \) changes from positive to negative at \( t = 2t_0 \). For small choices of \( t_0 \), we can see directly from the definition that this curvature will be negative. It is also not immediately clear from the asymptotics in Lemma A.12 that picking \( t_0 \) large will resolve this as the two terms in equation (51) are both proportional to \( \frac{\ln(2t_0)}{t^2 \ln^2 t} \). We must therefore return to the definition. In terms of \( \Gamma(t) \), equation (53) becomes

\[
\beta \left( \beta \Gamma^2(t) - \Gamma'(t) - \frac{2}{t} \Gamma(t) \right).
\]

By Lemma A.10, the \( \Gamma^2(t) \) term has smaller asymptotic behavior, so we must show that \( \Gamma'(t) + \frac{2}{t} \Gamma(t) \) is positive for large enough \( t_0 \).

**Lemma A.21.** There exists a \( T > 0 \) and \( c_n > 0 \), such that for all \( t_0 > T \) we have

\[
\Gamma'(t) + \frac{2}{t} \Gamma(t) > \frac{c_n \ln(2t_0)}{t^2 \ln^2 t}.
\]
Proof. For $t < 2t_0$, we have
\[
\Gamma'(t) + \frac{2}{t} \Gamma(t) = \frac{1}{2t_0^2 \ln(2t_0)} + \frac{1}{t_0^2 \ln(2t_0)} - \frac{1}{tt_0 \ln(2t_0)} > \frac{1}{2t_0^2 \ln(2t_0)}.
\]
Because $\frac{1}{t \ln t}$ is decreasing, we have $\frac{1}{t_0^2 \ln t_0} \geq \frac{1}{t \ln t}$. This proves the claim for $t < 2t_0$.

For $t > 2t_0$, we have
\[
\Gamma'(t) + \frac{2}{t} \Gamma(t) = -\frac{\ln(2t_0)}{t^2 \ln^2 t} - \frac{2 \ln(2t_0)}{t^2 \ln^3 t} + \frac{2 \ln(2t_0)}{t^2 \ln^2 t}
\]
\[
> \frac{\ln(2t_0)}{t^2 \ln^2 t} - \frac{2 \ln(2t_0)}{t^2 \ln^3 t}.
\]
Clearly if $t_0$ is large enough the negative term may be ignored, thus there exists a $T > 0$ for which the claim is true for $t_0 > T$. □

It follows that $K(T, \Sigma_i) > 0$ for $t_0$ large enough (so that $\Gamma(t)$ is sufficiently small). While the first terms of $K(T, X)$ in equation (52) agree with $K(T, \Sigma_i)$, the following two terms that appear have identical asymptotics with opposite sign. One must therefore consider $\text{Ric}(T, T)$ in its entirety, and show that terms from $K(T, \Sigma_i)$ dominate those from $K(T, X)$. Rewriting $\frac{h'(t)}{h(t)}$ in terms of $k'(t) k(t)$, one sees that the coefficients that need to be compared are determined by $n$ and $\alpha$. The following lemma compares the exact coefficients of the terms that dominate $\text{Ric}(T, T)$.

**Lemma A.22.** Let $\alpha$ be as in equation (41), where $\rho$ and $r$ satisfy the hypotheses of Lemma A.1. There exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $\alpha$ satisfies the following for all $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$.

(54) \[
\frac{\alpha h(t)}{1 + (h(t) - 1)\eta(x)} - n < 0.
\]

Proof. We begin by claiming that $\alpha < n$ for $\varepsilon$ and $\delta$ small enough. As observed in the proof of Lemma A.4 by Corollary A.5 and the assumption that $K_{g_i} > 0$ that $\frac{1}{r^2 a_1^2} > 1$. It follows then that $\ln a_1 < -\ln r$. By assumption, $r^{n-1} < \rho^n$. Taking logarithms and solving for $-\ln r$ yields $-\ln r < n(\ln \rho - \ln r)$. Combining these observations yields $\ln a_1 < n(\ln \rho - \ln r)$. Plugging this into the definition of $\alpha$ in equation (41) yields
\[
\alpha < \frac{1 + \delta}{1 - \varepsilon} n.
\]
Thus $\alpha < n$ if $\varepsilon = \delta = 0$. As this is an open condition, there exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $\alpha < n$ for all $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$.

Now we turn to prove inequality (54). If $\eta(x) < 0$, then both terms are negative and the claim is obvious. If $0 \leq \eta(x) \leq 1$, then clearly $\eta(x) h(t) + (1 - \eta(x)) \geq \eta(x) h(t)$. Thus the left-hand side of (54) is bounded above by $\alpha - n$, which is negative if $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$. □
The first paragraph in the proof of Lemma A.22 appears as the penultimate paragraph of [13, p. 160], where Perelman has already assumed that $\varepsilon$ is fixed as in the remark after Corollary A.15 so that $\left(\frac{r}{\varepsilon}\right)^{\varepsilon} g_1$ still has sectional curvatures greater than 1.

We are now ready to prove that $\text{Ric}(T,T)$ is positive for large enough $t_0$. The proof amounts to showing that the coefficient of $-\beta \left(\Gamma'(t) + \frac{2}{t} \Gamma(t)\right)$ in $\text{Ric}(T,T)$ is the left-hand side of (54).

**Lemma A.23.** Let $\varepsilon_0 > 0$ and $\delta_0 > 0$ be as in Lemma A.22. There exists $T > 2$ such that $\text{Ric}(T,T)$ is positive for all $t_0 > T$, $\varepsilon < \varepsilon_0$, and $\delta < \delta_0$. Moreover, there exists $c_T > 0$ such that $\text{Ric}(T,T)$ satisfies the following.

$$\text{Ric}(T,T) > \frac{c_T \ln(2t_0)}{t^2 \ln^2 t}.$$ 

**Proof.** We can compute $\text{Ric}(T,T)$ by adding together equations (52) and (51). We then factor this expression for $\text{Ric}(T,T)$ so that the leading term is transparently $-\beta \left(\Gamma'(t) + \frac{2}{t} \Gamma(t)\right)$, where we notice that

$$\left(\frac{k'(t)}{k(t)}\right)' + \frac{2 k'(t)}{t k(t)} = -\beta \left(\Gamma'(t) + \frac{2}{t} \Gamma(t)\right).$$

We then combine Lemmas A.6, A.12, A.21, and A.22 bound $\text{Ric}(T,T)$ below as follows.

$$\text{Ric}(T,T) = (n - 1) K(T, \Sigma_i) + K(T, X),$$

$$\begin{align*}
= & \left(\frac{\alpha \eta(x) h(t)}{1 + (h(t) - 1) \eta(x)} - n\right) \left(\frac{k'(t)}{k(t)}\right)' + \frac{2 k'(t)}{t k(t)} \\
& - n \left(\frac{k'(t)}{k(t)}\right)^2 - \frac{\eta(x) h(t)}{1 + (h(t) - 1) \eta(x)} \left(2 \frac{k'(t)}{k(t)} \frac{h'(t)}{h(t)} + \left(\frac{h'(t)}{h(t)}\right)^2\right),
\end{align*}$$

$$> \frac{c}{t^2 \ln^2 t} - \frac{c \ln^2 (2t_0)}{t^2 \ln^4 t}.$$ 

To apply Lemma A.22, we must assume that $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$. It is clear that there exists a $T > 2$ for which we may disregard the negative term for all $t_0 > T$. □

**A.3.4. The existence of the neck.** With Corollary A.20 this shows that the diagonals of the Ricci tensor of $g$ are positive. As mentioned in the introduction of Section A.3, in order to prove positive definiteness we must show that the determinant of the 2-by-2 submatrix spanned by $T$ and $X$ is positive, i.e. that $\text{Ric}_g(T,T) \text{Ric}_g(X,X) - \text{Ric}_g^2(T,X)$ is positive. The following claims that this is true for large enough $t_0$. It is a direct corollary of Corollary A.19, Corollary A.20 and Lemma A.23.

**Corollary A.24.** There exists a $T > 0$ such that for all $t_0 > T$ we have

$$\text{Ric}_g(T,T) \text{Ric}_g(X,X) > \text{Ric}_g^2(T,X).$$
Proof. Taking the lower bounds for Corollary [A.20] and Lemma [A.23] on the left-hand side, and the upper bounds from Corollary [A.19] for the right-hand side. The claim reduces to the following inequality
\[
c \ln(2t_0) > \frac{\ln^2(2t_0)}{t^4 \ln^2 t}.
\]
And clearly, there is a \( T > 2 \) for which this is true for all \( t_0 > T \).

We have shown therefore that for large \( t_0 \) and small \( \varepsilon \) and \( \delta \) that the Ricci tensor of \( g \) is positive definite. Combining this with Corollaries [A.14] and [A.17] we are ready to prove Lemma [A.1].

Proof of Lemma [A.1]. By Corollary [A.14] \( g \) will satisfy claims (ii), (iii), and (iv) of Lemma [A.1] for any choice of \( t_0, \varepsilon, \) and \( \delta \).

Consider next the curvature of \( g \). By Corollary [A.17] it suffices to show that there is a \( T > 0 \) and specific choices of \( \varepsilon \) and \( \delta \) for which \( g(t_0, \varepsilon, \delta) \) satisfies (i) for all \( t_0 > T \). So we may disregard the fact that \( g \) is not smooth at \( t = 2t_0 \).

The conclusions of Corollary [A.20], Lemma [A.23] and Corollary [A.24] combined tell us that \( \text{Ric}_g \) is positive definite if \( t_0 \) is large. The hypotheses of these claims require that \( \varepsilon \) and \( \delta \) be chosen small in the sense of Corollary [A.15] so that the sectional curvatures in the spherical directions are bigger than \( \frac{c}{t^2} \) and in the sense of Lemma [A.22] so that \( \text{Ric}_g(T, T) \) is positive. Fix \( \varepsilon \) small enough for the hypotheses of both Corollary [A.15] and Lemma [A.22].

Finally, by Corollary [A.14] \( g \) will satisfy (v) if \( \delta \) is chosen small relative to \( \varepsilon \). Fix \( \delta \) small enough to satisfy Corollary [A.14] as well as Corollary [A.15] and Lemma [A.22]. Thus there is a \( T > 0 \) such that \( \text{Ric}_g \) is positive definite for all \( t_0 > T \). We conclude that there exists a \( t_0 \) so that \( g(t_0, \varepsilon, \delta) \) will, after smoothing, satisfy all of the claims of Lemma [A.1].

□

Appendix B. Curvature Computations

In this appendix we provide the necessary curvature computations to completely determine the Ricci tensor for two classes of metrics: the doubly warped Riemannian submersion metrics of Definition [2.4] and the scaled one parameter family of warped product metrics of Definition [A.11]. The first is covered in Section [B.1] and the second is covered in Section [B.2]. Section [B.1] proves Lemma [2.6], which is a central part of the proof of Theorem [C]. The results of Section [B.2] are quoted throughout the proof of Lemma [A.1] in Section [A].

B.1. Doubly Warped Riemannian Submersion Metrics. In this section we compute \( \text{Ric}_{\tilde{g}} \), where \( \tilde{g} \) is the doubly warped Riemannian submersion metric \( \tilde{g} \) associated to a Riemannian submersion \( \pi : (E, g) \to (B, \tilde{g}) \), in terms of \( \text{Ric}_g \), \( \text{Ric}_{\tilde{g}} \), and \( \text{Ric}_{\hat{g}} \). Specifically we will prove Lemma [2.6].

In Section [B.1.1] we quote the fundamental equations of Riemannian submersions, the O’Neill formulas, for sectional and Ricci curvatures. Briefly in Section [B.1.2] we quote the second fundamental form of time slice with respect to \( \tilde{g} \). We then compute the sectional curvatures of \( \tilde{g} \) in Section [B.1.3].
and conclude by computing the diagonals of $\text{Ric}_{\tilde{g}}$. We conclude Section B.1.4 by computing the rest of the Ricci curvatures.

Let $m$ and $n$ denote the dimensions of $B$ and $F$ respectively. The horizontal vectors $\{X_i\}_{i=1}^m$, vertical vectors $\{V_i\}_{i=1}^n$, and $\partial_t$ will constitute a local orthonormal frame of $\tilde{g}$. The horizontal vectors $\{Y_i\}_{i=1}^m$ and vertical vectors $\{U_i\}_{i=1}^n$ will be the corresponding orthonormal frame on $E$ with respect to $g$. Note that $h(t)X = Y$ and $f(t)V = U$. We will use the notation $\tilde{Y} = d\pi(Y)$. $A$ will denote the $A$-tensor of the Riemannian submersion $\pi: (E, g) \to (B, \tilde{g})$ and $\tilde{A}$ will denote the $A$-tensor of the Riemannian submersion $\text{id} \times \pi: (I \times E, \tilde{g}) \to (I \times B, dt^2 + h^2(t)\tilde{g})$.

### B.1.1. O’Neill’s Formulas

Associated to every Riemannian submersion we define two 2-tensors, the $A$ and $T$ tensor, as local measures of the failure of the metric to be a product.

**Definition B.1.** [2, 9.17 & 9.20] Given a Riemannian submersion $\pi: (E, g) \to (B, \tilde{g})$. We define the $A$ and $T$ tensors for any $X, Y \in TE$ as follows.

\[
A_X Y = H\nabla_{H_X} VY + V\nabla_{H_X} HY,
\]

\[
T_X Y = H\nabla_{V_X} VY + V\nabla_{V_X} HY.
\]

Where $\nabla$ is the Levi-Civita connection of $g$, and $H$ and $V$ are as in Definition 2.1.

The $A$-tensor is the obstruction to $H$ being integrable, and the $T$-tensor is the obstruction to the $F_b$ being totally geodesic (see [2, 9.26]). In particular, when the fibers are totally geodesic submanifolds $T \equiv 0$. The $A$ and $T$ tensors also provide the complete relationship between Riemannian curvature tensors of $(E, g)$, $(B, \tilde{g})$ and $(F, \hat{g})$. The formulas relating these tensors are originally due to O’Neill, and are all referred to as O’Neill’s formulas (see [2, Theorem 9.28]). To begin with we give the O’Neill formulas for sectional curvatures in the case $T \equiv 0$.

**Theorem B.2.** [2, Corollary 9.29] If $X_i$ and $U_i$ are orthonormal horizontal and vertical vector fields in a Riemannian submersion with totally geodesic fibers, then the sectional curvatures take the following form.

\[
K_{\hat{g}}(U_i, U_j) = K_{\tilde{g}}(U_i, U_j),
\]

\[
K_{\hat{g}}(X_i, U_i) = g(A_X U, A_X U),
\]

\[
K_{\hat{g}}(X_i, X_j) = K_{\tilde{g}}(\bar{X}_i, \bar{X}_j) - 3g(A_{\bar{X}_i} X_j, A_{\bar{X}_i} X_j),
\]

where $\bar{X} := d\pi X$.

The statement of O’Neill’s formulas for Ricci curvature involves certain traces of the $A$ and $T$ tensors and their covariant derivative. We introduce the following standard notation.

**Definition B.3.** [2, 9.33] Let $X_i$ and $V_j$ with $1 \leq i \leq n$ and $1 \leq j \leq m$ be an orthonormal frame made up of respectively horizontal and vertical vector fields. Let $E$ stand in for $A$ or $T$, then we
may define the following traces involving the $A$ and $T$ tensors.

\begin{align*}
(55) \quad g(A_{X_i}, A_{X_j}) := \sum_{k=1}^{n} g(A_{X_i} V_k, A_{X_j} V_k) = \sum_{k} g(A_{X_i} X_k, A_{X_j} X_k).
\end{align*}

\begin{align*}
(56) \quad g(A V_i, A V_j) := \sum_{k=1}^{n} g(A_{X_k} V_i, A_{X_k} V_j).
\end{align*}

\begin{align*}
(57) \quad g(T_{X_i}, T_{X_j}) = \sum_{k=1}^{n} g(T_{V_k} X_i, T_{V_k} X_j).
\end{align*}

\begin{align*}
\delta E = - \sum_{i=1}^{m} (\nabla_{X_i} E)_{X_i}.
\end{align*}

\begin{align*}
(58) \quad (\hat{\delta} T)(V_i, V_j) = \sum_{k=1}^{m} g((\nabla_{X_k} T)_{V_i}, X_k).
\end{align*}

\begin{align*}
(59) \quad N = \sum_{i=1}^{n} T_{V_i} V_i.
\end{align*}

With this notation, the O’Neill formulas for Ricci curvature are as follows.

**Theorem B.4.** If $\pi : (E, g) \to (B, \hat{g})$ is a Riemannian submersion with fiber metrics $\hat{g}$, and $X_i$ and $Y_j$ are horizontal and vertical vectors that make up an orthonormal frame for $g$, then $\text{Ric}_g$ can be computed as follows.

\begin{align*}
(60) \quad \text{Ric}_g(V_i, V_j) &= \text{Ric}_{\hat{g}}(V_i, V_j) - g(N, T_{V_i} V_j) + g(A V_i, A V_j) + (\hat{\delta} T)(V_i, V_j),
\end{align*}

\begin{align*}
(61) \quad \text{Ric}_g(X_i, V_j) &= g((\hat{\delta} T)_{V_j}, X_i) + g(\nabla_{V_j} N, X_i) - g((\hat{\delta} A)_{X_i}, V_j) - 2 g(A_{X_i}, T_{V_j}),
\end{align*}

\begin{align*}
(62) \quad \text{Ric}_g(X_i, X_j) &= \text{Ric}_{\hat{g}}(\tilde{X}_i, \tilde{X}_j) - 2 g(A_{X_i}, A_{X_j}) - g(T_{X_i}, T_{X_j}) \\
&\quad + \frac{1}{2} \left( g(\nabla_{X_i} N, X_j) + g(\nabla_{X_j} N, X_i) \right).
\end{align*}

While these formulas are very complicated for general submersion, we can see that when $T \equiv 0$ that \textbf{(59)} is also zero, and so Theorem \textbf{B.4} reduces to the following when the fibers are totally geodesic.

**Corollary B.5.** \textsuperscript{[2, Proposition 9.36]} If $\pi : (E, g) \to (B, \hat{g})$ is a Riemannian submersion with totally geodesic fibers with metric $\hat{g}$, and $X_i$ and $Y_j$ are horizontal and vertical vectors that make up an orthonormal frame for $g$, then $\text{Ric}_g$ can be computed as follows.

\begin{align*}
(63) \quad \text{Ric}_g(U_i, U_j) &= \text{Ric}_{\hat{g}}(U_i, U_j) + g(A U_i, A U_j).
\end{align*}

\begin{align*}
(64) \quad \text{Ric}_g(X_i, U_i) &= -g((\hat{\delta} A)_X U).
\end{align*}

\begin{align*}
(65) \quad \text{Ric}_g(X_i, X_j) &= \text{Ric}_{\hat{g}}(\tilde{X}_i, \tilde{X}_j) - 2 g(A_{X_i}, A_{X_j}).
\end{align*}
In our computations we will need to use both Theorem B.4 and its Corollary B.5. Luckily in our application, the only nonzero values of the $T$ tensor will be $T_i V_i$ and $T_i \partial_t$.

B.1.2. **Extrinsic curvature.** As we aim to use these doubly warped Riemannian submersion metrics to form the metric $g_{core}$ of Theorem C, we also need to know when the boundary of manifolds with such metrics is convex. We take a moment to record the second fundamental form of an embedded time-slice.

**Lemma B.6.** The second fundamental form of $\{t\} \times E, \tilde{g}$ as a submanifold of $([0, t] \times E, \tilde{g})$ is given by

$$(66) \quad \Pi_t = \frac{h'(t)}{h(t)} (h^2(t) \mathcal{H}^* \pi^* \tilde{g}) + \frac{f'(t)}{f(t)} (f^2(t) \mathcal{V}^* \tilde{g}).$$

Where the notation is as in formula (1).

**Proof.** Let $t : I \times E \to I$ denote the projection onto $I$. By [14, Proposition 3.2.1], the second fundamental form $\Pi$ of the level sets of $t$ with respect to the unit normal $\partial_t$ can be computed by

$$\Pi_t = \frac{1}{2} \mathcal{L}_{\partial_t} \tilde{g}.$$

Notice that $\mathcal{L}_{\partial_t} (\mathcal{H}^* \pi^* \tilde{g}) = 0$ and $\mathcal{L}_{\partial_t} (\mathcal{V}^* \tilde{g}) = 0$. The claim is now straightforward to check. □

The concavity of these metrics therefore depends solely on the sign of $f'$ and $h'$.

**Corollary B.7.** Let $\tilde{g}$ be a doubly warped Riemannian submersion metric on $[0, t_1] \times E$. Then the second fundamental form of the boundary at $t = t_1$ is positive definite if and only if $f'(t_1) > 0$ and $h'(t_1) > 0$.

B.1.3. **Sectional Curvature.** In this section we compute $K_{\tilde{g}}$, specifically we claim the following.

**Lemma B.8.** The sectional curvatures of $\tilde{g}$ in the time direction are as follows.

$$(67) \quad K_{\tilde{g}}(\partial_t, X_i) = -\frac{h''(t)}{h(t)},$$

$$(68) \quad K_{\tilde{g}}(\partial_t, V_i) = -\frac{f''(t)}{f(t)}.$$

**Proof.** Because the intervals $I \times \{p\}$ are geodesics, the Mainardi-Codazzi equation reduces to

$$(69) \quad K_{\tilde{g}}(\partial_t, W) = -(\partial_t \Pi_t)(W, W) + \Pi_t^2(W, W).$$

Where $\Pi_t$ is as in Lemma B.6. Taking its derivative we have

$$(70) \quad \partial_t \Pi_t = \left( \frac{h''(t)}{h(t)} + \left( \frac{h'(t)}{h(t)} \right)^2 \right) (h^2(t) \mathcal{H}^* \pi^* \tilde{g}) + \left( \frac{f''(t)}{f(t)} + \left( \frac{f'(t)}{f(t)} \right)^2 \right) (f^2(t) \mathcal{V}^* \tilde{g}).$$
Take the square of equation \((66)\) and combine this with equation \((70)\) into equation \((69)\). This proves formula \((67)\) of the claim as follows.

\[
K_{\tilde{g}}(\partial_t, X) = - \left( \frac{h''(t)}{h(t)} + \left( \frac{h'(t)}{h(t)} \right)^2 \right) + \left( \frac{h'(t)}{h(t)} \right)^2
\]

\[
= - \frac{h''(t)}{h(t)}.
\]

The formula \((68)\) is proved in exactly the same way. \(\square\)

**Lemma B.9.** The sectional curvatures of \(\tilde{g}\) in the spherical directions are as follows.

\[
K_{\tilde{g}}(X_i, X_j) = \frac{1}{h^2(t)} K_g(Y_i, Y_j) - 3 \frac{f'(t)}{h(t)} g(A_i Y_j, A_j Y_j) - \left( \frac{h'(t)}{h(t)} \right)^2
\]

\[
K_{\tilde{g}}(X_i, V_j) = \frac{f^2(t)}{h^4(t)} g(A_i U_j, A_j U_j) - \frac{f'(t) h'(t)}{f(t) h(t)}
\]

\[
K_{\tilde{g}}(V_i, V_j) = \frac{1}{f^2(t)} K_g(U_i, U_j) - \left( \frac{f'(t)}{f(t)} \right)^2
\]

By Lemma [B.6] and Gauss’ formula, we can reduce to considering the Riemannian submersion \((E, h^2(t)\mathcal{H}^* \pi^* \tilde{g} + f^2(t) V^* \tilde{g}) \to (B, h^2(t) \tilde{g})\). We then want to apply O’Neill’s formulas to these submersions to deduce the right-hand side of the equations \((71)\), \((72)\), and \((73)\). But in order to do so, it will be necessary to relate \(\tilde{A}\) and \(A\). We quote the following lemma.

**Lemma B.10.** [20] Lemmas 3.1-3.3] \(\tilde{A}\) and \(A\) are related as follows.

\[
\tilde{A}_X X_j = A_X X_j
\]

\[
\tilde{A}_\partial_i \partial_i = \tilde{A}_h \partial_i = \tilde{A}_h V_i = 0
\]

\[
\tilde{A}_X V_j = \frac{1}{h^2(t)} A_X V_j
\]

**Proof of Lemma B.9** Let \(g_t\) denote \(\tilde{g}\) restricted to \(\{t\} \times E\). It takes the following form

\[
g_t = h^2(t) \mathcal{H}^* \pi^* \tilde{g} + f^2(t) V^* \tilde{g}.
\]

By Gauss’ formula and Lemma [B.6] we have

\[
K_{\tilde{g}}(X_i, X_j) = K_{g_t}(X_i, X_j) - \left( \frac{h'(t)}{h(t)} \right)^2
\]

\[
K_{\tilde{g}}(X_i, V_j) = K_{g_t}(X_i, V_j) - \frac{f'(t) h'(t)}{f(t) h(t)}
\]

\[
K_{\tilde{g}}(V_i, V_j) = K_{g_t}(X_i, X_j) - \left( \frac{f'(t)}{f(t)} \right)^2
\]

As \((E, g_t) \to (B, h^2(t) \tilde{g})\) is a Riemannian submersion with totally geodesic fibers isometric to \((F, f^2(t) \tilde{g})\), we may therefore apply the O’Neill formulas to \(K_{g_t}\) on the righthand side of the above equations.

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\[(77) \quad K_{g}(X_i, X_j) = g_\theta(X_i, X_j) - 3g_\theta(\tilde{X}_i, \tilde{X}_j) - \left( \frac{h'(t)}{h(t)} \right)^2 \]

\[(78) \quad K_{\tilde{g}}(X_i, V_j) = g_\theta(\tilde{A}_X V_j, \tilde{A}_X V_j) - \frac{f'(t)h'(t)}{f(t)h(t)} \]

\[(79) \quad K_{\tilde{g}}(V_i, V_j) = g_\theta(V_i, V_j) - \left( \frac{f'(t)}{f(t)} \right)^2 \]

As $\tilde{g}_t = h^2(t)\tilde{g}$ and $\tilde{g}_t = f^2(t)\tilde{g}$, by Lemma \[B.17\] we have

\[(80) \quad K_{\tilde{g}_t}(\tilde{X}_i, \tilde{X}_j) = \frac{1}{h^2(t)} K_{\tilde{g}}(V_i, V_j) \]

\[(81) \quad K_{\tilde{g}_t}(V_i, V_j) = \frac{1}{f^2(t)} K_{\tilde{g}}(V_i, V_j) \]

By equations (74) and (76) we have $\tilde{A}_X X_j = A_X X_j$ and $\tilde{A}_X V_j = \frac{1}{h^2(t)} A_X V_j$. These vectors are, by Definition \[B.1\], vertical and horizontal respectively. Notice that $\mathcal{H}^* g_\theta = h^2(t)\mathcal{H}^* g$ and $\mathcal{V}^* g_\theta = f^2(t)\mathcal{V}^* g$. Combining these facts with the definition $X_i = \frac{1}{h(t)} V_i$ and $V_i = \frac{1}{f(t)} U_i$ gives

\[(82) \quad g_\theta(\tilde{A}_X X_j, \tilde{A}_X X_j) = \frac{f^2(t)}{h^4(t)} g_\theta(A Y_j, A Y_j) \]

\[(83) \quad g_\theta(\tilde{A}_X V_j, \tilde{A}_X V_j) = \frac{f^2(t)}{h^4(t)} g_\theta(A Y_j, A Y_j) \]

Substituting equations (82) and (80) into equation (77), equations (83) and (64) into equation (78), and equation (81) into equation (79) yields the claim. \[\square\]

With Lemma \[B.9\] we can compute the diagonals of the Ricci tensor.

**Corollary B.11.** Formulas (2), (3), and (4) of Lemma 2.6 are valid.

**Proof.** First consider $\text{Ric}_{\tilde{g}}(\partial_t, \partial_t)$, combining (67) and (68) we have

\[
\text{Ric}_{\tilde{g}}(\partial_t, \partial_t) = \sum_{i=1}^{m} K(\partial_t, X_i) + \sum_{j=1}^{n} K(\partial_t, V_j) = -m \frac{h''}{h} - n \frac{f''}{f}.
\]

This proves equation (2) of the claim.

Next consider $\text{Ric}_{\tilde{g}}(X_i, X_i)$ and $\text{Ric}_{\tilde{g}}(V_i, V_i)$. Using (67), (72), and (74) we have the following, where we use the simplifying notation of (55).

\[
\text{Ric}_{\tilde{g}}(X_i, X_i) = K(X_i, \partial_t) + \sum_{j \neq i} K(X_i, X_j) + \sum_{j=1}^{n} K(X_i, V_j),
\]

\[
= -\frac{h''(t)}{h(t)} + \frac{1}{h^2(t)} \text{Ric}_{\tilde{g}}(Y_j, Y_j) - 2\frac{f^2(t)}{h_4(t)} g(A Y_i, A Y_i) - (m-1) \left( \frac{h'(t)}{h(t)} \right)^2 - n \frac{f'(t)h'(t)}{f(t)h(t)}.
\]
Using (68), (73), and (72) we have the following, where we use the simplifying notation of (56).

\[ \text{Ric}_\tilde{g}(V_i, V_i) = K(V_i, \partial_t) + \sum_{j \neq i} K(V_i, V_j) + \sum_{j=1}^m K(V_i, X_j), \]

(85)

\[ = -\frac{f''(t)}{f(t)} + \frac{1}{f^2(t)} \text{Ric}_\tilde{g}(U_i, U_i) - (n - 1) \left( \frac{f'(t)}{f(t)} \right)^2 + \frac{f^2(t)}{h^4(t)} g(AU_i, AU_i) - m \frac{f'(t)h'(t)}{f(t)h(t)}. \]

Solving for the A-tensor terms in (63) and (65) of Theorem B.3 gives the following.

\[-2g(A_{Y_i}, A_{Y_i}) = \text{Ric}_\tilde{g}(Y_i, Y_i) - \text{Ric}_\tilde{g}(\tilde{Y}_i, \tilde{Y}_i), \text{ and } g(AU_i, AU_i) = \text{Ric}_\tilde{g}(U_i, U_i) = \text{Ric}_\tilde{g}(U_i, U_i). \]

Substituting these into (84) and (85) respectively proves (3) and (4).

B.1.4. Off-diagonals of the Ricci tensor. Lemma B.9 was achieved by using Gauss’ formula to reduce the situation to a Riemannian submersion with totally geodesic fibers, this allows us to compute the diagonals of the Ricci tensor directly. There is no such approach to computing the off-diagonals. We will have to analyze each term in Theorem B.4 to prove the following.

Lemma B.12. If \( \tilde{g} \) is the doubly warped Riemannian submersion metric associated to Riemannian submersion \( \pi : (E, g) \rightarrow (B, \tilde{g}) \) with totally geodesic fibers, then the off-diagonals of \( \text{Ric}_\tilde{g} \) are as follows.

(86) \[ \text{Ric}_\tilde{g}(\partial_t, X_i) = \text{Ric}_\tilde{g}(\partial_t, Y_i) = 0 \]

(87) \[ \text{Ric}_\tilde{g}(X_i, X_j) = \frac{1}{h^2(t)} \text{Ric}_g(Y_i, Y_j) \]

(88) \[ \text{Ric}_\tilde{g}(V_i, V_j) = \frac{1}{f^2(t)} \text{Ric}_g(U_i, U_j) \]

(89) \[ \text{Ric}_\tilde{g}(X_i, V_j) = \frac{f(t)}{h^3(t)} \text{Ric}_g(Y_i, U_j). \]

To begin with, we have yet to mention anything about the T-tensor. As the T-tensor for the original Riemannian submersion \( (E, g) \rightarrow (B, \tilde{g}) \) is identically zero, we will use \( T \) to refer unambiguously to the T-tensor of the Riemannian submersion \( \text{id} \times \pi : (I \times E, \tilde{g}) \rightarrow (I \times B, dt^2 + h^2(t)\tilde{g}). \)

Lemma B.13. The T-tensor of \( \tilde{g} \) is as follows

\[ T_{\partial_t} = T_{X_i} = T_{V_i}X_j = T_{V_i}V_j = 0, \quad T_{V_i}\partial_t = -\frac{f'(t)}{f(t)} V_i, \text{ and } T_{V_i}V_i = \frac{f'(t)}{f(t)} \partial_t. \]

Moreover, the vector \( N \) of (59) is

(90) \[ N = mf'(t) \frac{\partial t}{f(t)}. \]

Proof. It is immediately clear from Definition B.1 that \( T_W = 0 \) for any horizontal vector \( W \), thus \( T_{\partial_t} = T_{X_i} = 0. \)

It is also immediately clear from the definition that \( T_{V_i}V_j = \Pi(V_i, V_j) \) where \( \Pi \) is the second fundamental form of \( F \) embedded in \( I \times E \) with respect to \( \tilde{g} \). Let \( \Pi_x \) denote the second fundamental
form of $F$ embedded in $\{t\} \times E$ with respect to $h^2(t)H^*\pi^*\tilde{g} + f^2(t)V^*\tilde{g}$, and let $\Pi^F_\{t\}$ denote the second fundamental form of $\{t\} \times E$ embedded in $I \times E$ with respect to $\tilde{g}$. It is clear from the definitions of these tensors that $\Pi = \Pi_x + \Pi_t | TF$. But as the original Riemannian submersion has totally geodesic fibers, $\Pi_x = 0$. Thus $\Pi = \Pi_t$, which combined with Lemma [B.6] implies that $T_V V_j = 0$ and $T_V V_i = \frac{f'(t)}{f(t)} \partial_t$.

The final terms we must consider are $T_V \partial_t$ and $T_V X_j$. It is again straightforward from the definition that these are both vertical vectors, and that $\tilde{g}(T_V W, V_j) = -\tilde{g}(T_V V_j, W)$ (see [2, Definition 9.17]). By the above paragraph, we have that $T_V V_j = 0$ and $T_V V_i = \frac{f'(t)}{f(t)} \partial_t$. It follows that $T_V X_j = 0$ and $T_V \partial_t = -\frac{f'(t)}{f(t)} V_i$.

As we want to use Theorem [B.4], we must also consider the various traces of covariant derivatives of $T$ and $A$. We cite the following lemma about those involving $A$.

**Lemma B.14.** [20, Proposition 3.8] With notation as in Definition [B.3] we have

\[(\tilde{\delta} T)(V_i, V_j) = \frac{f^2(t)}{h^2(t)} g((\delta A) X_i, V_j) \]

(91)

\[\tilde{g}((\delta A) \partial_t, V_i) = \tilde{g}((\delta A) \partial_i, X_i) = 0.\]

(92)

As we are only concerned with the off-diagonals of $\text{Ric}_{\tilde{g}}$ in this section, we state that those terms involving covariant derivatives of $T$ in this case, all of which vanish.

**Lemma B.15.** Let $i \neq j$, then with the notation as in Definition [B.3] we have

\[(\tilde{\delta} T)(V_i, V_j) = \tilde{g}((\delta T)V_i, \partial_t) = \tilde{g}((\delta T)V_i, X_j) = 0\]

Proof. Applying the definition of $\tilde{\delta}$ of equation (58) we have

\[\hat{g}((\delta T)V_i, V_j) = \sum_{k=1}^m \tilde{g}((\nabla_{X_k} T)V_i, V_j) + \tilde{g}((\nabla_{\partial_t} T)V_i, \partial_t).\]

(93)

Each of the terms in equation (93) involving $X_i$ we can expand as follows

\[\tilde{g}((\nabla_{X_k} T)V_i, X_k) = \tilde{g}((\nabla_{X_k} (T V_i))V_j, X_k) - \tilde{g}(T \nabla_{X_k} V_i, V_j, X_k) - \tilde{g}(T V_i \nabla_{X_k} V_j, X_k).
\]

(94)

It is straightforward to show that $\nabla_{X_k} V_i$ will be 0. By Lemma [B.13], $T_V V_j = 0$ if $i \neq j$. Thus each $T$ term in equation (94) is 0, and so each term involving $X_i$ in (93) is 0.

It remains to consider the term involving $\partial_t$, expanding this term we have

\[\tilde{g}((\nabla_{\partial_t} T)V_i, \partial_t) = \tilde{g}((\nabla_{\partial_t} (T V_i)), \partial_t) - \tilde{g}(T \nabla_{\partial_t} V_i, V_j, \partial_t) - \tilde{g}(T_V \nabla_{\partial_t} V_j, \partial_t).
\]

(95)

It is easy to see that $\nabla_{\partial_t} V_i$ is in the span of $V_i$, so by Lemma [B.13] each $T$ term in equation (95) is zero. This proves that (93) is zero.

---

3 As $\{t\} \times E$ is a hypersurface in $I \times E$, we have heretofore considered $\Pi_t$ as a (2,0)-tensor, here we have replaced it with the corresponding (2,1)-tensor.
Next apply definition of $\hat{\delta}$ of (57).
\begin{equation}
(96) \quad \tilde{g}((\delta T)V_i, X_j) = \sum_{k=1}^n \tilde{g}((\nabla_{V_k} T)V_k, X_j).
\end{equation}

Each term in equation (96) can be expanded as follows
\begin{equation}
(97) \quad \tilde{g}((\nabla_{V_k} T)V_k, X_j) = \tilde{g}((\nabla_{V_k} (TV_k V_i), X_j) - \tilde{g}(T\nabla_{V_k} V_k, X_j) - \tilde{g}(TV_k \nabla_{V_k} V_i, X_j).
\end{equation}

It is easy to check that $\nabla_{V_k} V_k$ is either 0 or in the span of $\partial_t$. Thus the last two terms of equation (97) are 0, as the $T$ term will be vertical. The first term in equation (97) is proportional to $\tilde{g}(\nabla_{V_i} \partial_t, X_j)$, it is easy to check that this is 0. Thus equation (96) is 0. The proof for $\partial_t$ is identical. \qed

With this we are ready to prove Lemma B.12

**Proof of Lemma B.12**. We begin by showing that Theorem B.4 reduces to Theorem B.5 when the metric in question is a doubly warped Riemannian submersion.

Let us begin by considering $\text{Ric}_{\tilde{g}}(V_i, V_j)$. By Lemma B.15, $(\delta T)(V_i, V_j) = 0$. The one other term in equation (60) involving $T$ is $\tilde{g}(N, TV_i V_j)$, which by Lemma B.13 is zero if $i \neq j$. Thus equation (60) reduces to equation (63).

Next consider $\text{Ric}_{\tilde{g}}(X_i, V_j)$. By Lemma B.15, $\tilde{g}(\delta T)V_i, X_j) = 0$. The two other terms involving $T$ in equation (61) are $\tilde{g}(\nabla_{V_i} N, X_j)$ and $2\tilde{g}(\tilde{\partial}_i, TV_i)$. By Lemma B.13, $\tilde{g}(\nabla_{V_i} N, X_j)$ is proportional to $\tilde{g}(\nabla_{V_i} \partial_t, X_j)$, which is easily seen to be zero. And using the first characterization of definition (55), we have
\begin{equation}
2\tilde{g}(\tilde{\partial}_i, TV_i) = \sum_{k=1}^m \tilde{g}(\tilde{\partial}_i X_k, TV_i X_k) + \tilde{g}(\tilde{\partial}_i \partial_t, TV_i \partial_t).
\end{equation}

Each term involving $X_k$ is zero, since $TV_i X_k = 0$ by Lemma B.13. The term involving $\partial_t$ is also 0, since $\tilde{\partial}_i \partial_t = 0$ by equation (75). The proof for $\partial_t$ is identical, thus equation (61) reduces to (64).

Finally consider, $\text{Ric}_{\tilde{g}}(X_i, X_j)$. The two terms involving $T$ in equation (62) are $\tilde{g}(TX_i, X_j)$ and $1/2(\tilde{g}(\nabla_{X_i} N, X_j) + \tilde{g}(\nabla_{X_j} N, X_i))$. By Lemma B.13, $\tilde{g}(TX_i, X_j) = 0$ (even if $X_i$ is replaced by $\partial_t$). The remaining term reduces to a linear combination of $\tilde{g}(\nabla_{X_i} \partial_t, X_j)$ and $\tilde{g}(\nabla_{X_j} \partial_t, X_j)$, it is easy to check that these vanish (even if $X_i$ is replaced with $\partial_t$). This proves that equation (62) reduces to (65).

So it suffices to consider Corollary B.5 to compute these curvatures of $\tilde{g}$.

For $\text{Ric}_{\tilde{g}}(X_i, V_j)$ apply (64) to $\tilde{g}$, then (91) and the fact that $X_i = 1/h(t)V_i$ and $V_i = 1/f(t)U_i$ implies that $\text{Ric}_{\tilde{g}}(X_i, V_i) = f(t)/h^3(t)\tilde{g}((\delta A)Y_i, U_j)$. Applying (64) to $g$, we deduce equation (89).

Next consider $\text{Ric}_{\tilde{g}}(X_i, \partial_t)$ and $\text{Ric}_{\tilde{g}}(V_i, \partial_t)$. Both $\partial_t$ and $X$ are horizontal vector fields of $\tilde{g}$, so by (65) we have
\begin{equation}
\text{Ric}_{\tilde{g}}(X, \partial_t) = \text{Ric}_{\tilde{g}}(\tilde{X}, \partial_t) - 2\tilde{g}(A_X \partial_t, A_X \partial_t).
\end{equation}

By (92), the $A$-tensor term vanishes. From definition 2.4, $\tilde{g} = dt^2 + h^2(t)\tilde{g}$ is a warped product metric. The Ricci curvature $\text{Ric}_{\tilde{g}}(X, \partial_t)$ is zero for warped products (see [14, 4.2.3]). Thus
\( \text{Ric}_g(X_i, \partial_t) = 0 \). For \( \text{Ric}_g(\partial_t, V) \), combine (64) with (92) to see that it vanishes. This proves equation (86) of the claim.

Finally consider \( \text{Ric}_g(X_i, X_j) \) and \( \text{Ric}_g(V_i, V_j) \). It is easy to check using Gauss’ equation that \( \text{Ric}_g(V_i, V_j) = \frac{1}{f^2(t)} \text{Ric}_g(U_i, U_j) \), and by equation (76) and definition \( \tilde{g}(\tilde{A}V_i, \tilde{A}V_j) = \frac{1}{f^2(t)} g(AU_i, AU_j) \). Combining these into equation (63) proves (88). Similarly for \( \text{Ric}_g(X_i, X_j) \), it is easy to see that \( \text{Ric}_g(X_i, X_j) = \frac{1}{h^2(t)} \text{Ric}_g(Y_i, Y_j) \), and by equation (76) \( \tilde{g}(\tilde{A}X_i, \tilde{A}X_j) = \frac{1}{h^2(t)} g(AY_i, AY_j) \). Combining these into equation (65) proves formula (87).

This proves Lemma 2.6.

Proof of Lemma 2.6. Equations (2), (3), and (4) follow from Corollary B.11, and the remaining equations follow from Lemma B.12.

B.2. Scaled One-parameter Families of Warped Products. In this section we will use \( g \) to denote a metric on \( [t_0, t_1] \times [x_0, x_1] \times S^n \) of the form \( g = dt^2 + g_t \), and \( g_t \) is a one parameter family of warped product metrics on \( [x_0, x_1] \times S^n \) of the form \( g_t = A(t, x)dx^2 + B(t, x)ds_n^2 \). Let \( T, X, \) and \( \Sigma_t \) denote a local orthonormal frame for \( g \) tangent to \( [t_0, t_1] \times [x_0, x_1] \times S^n \) respectively. The purpose of this section is to prove the following for such metrics \( g \).

Proposition B.16. The curvatures of \( g \) are as follows.

(i) The sectional curvature of \( g(t) \) on \( \{t\} \times [x_0, x_1] \times S^n \) is as follows.

\[
K_{g_t}(X, \Sigma_t) = \frac{A_x B_x}{A^2 B} - \frac{B_{xx}}{AB^2}, \quad \text{and} \quad K_{g_t}(\Sigma_i, \Sigma_j) = \frac{1}{B^2} - \frac{B^2_x}{A^2 B^2}.
\]

Moreover \( \text{Ric}_{g(t)} \) is diagonalized in this frame.

(ii) The second fundamental form \( \Pi_t \) of the submanifold \( \{t\} \times [x_0, x_1] \times S^n \) inside of \( [t_0, t_1] \times [x_0, x_1] \times S^n \) with respect to the normal vector \( T \) is as follows.

\[
\Pi_t(X, X) = \frac{A_t}{A}, \quad \Pi_t(X, \Sigma_i) = 0, \quad \text{and} \quad \Pi_t(\Sigma_i, \Sigma_j) = \frac{B_t}{B} \delta_{ij}.
\]

(iii) Combining (1) and (2), the sectional curvatures of \( g \) not involving \( T \) are

\[
K_g(X, \Sigma_j) = -\frac{A_t B_t}{AB} + \frac{A_x B_x}{A^2 B} - \frac{B_{xx}}{BA^2}, \quad \text{and} \quad K_g(\Sigma_i, \Sigma_j) = \frac{1}{B^2} - \frac{B^2_x}{A^2 B^2} - \frac{B^2_t}{B^2}.
\]

The remaining sectional curvatures are

\[
K_g(T, X) = -\frac{A_t}{A}, \quad \text{and} \quad K_g(T, \Sigma_i) = -\frac{B_t}{B}.
\]

And the Ricci tensor has one off-diagonal term

\[
\text{Ric}_g(X, T) = nR_g(X, \Sigma_i, \Sigma_i, T) = n \left( \frac{A_t B_x}{BA^2} - \frac{B_{xt}}{AB} \right).
\]
The proof of Proposition B.16 is presented in the following sections. In Section B.2.1 we compute the second fundamental forms relevant to the computation, in particular proving (ii). Next in Section B.2.2 we compute the intrinsic sectional curvatures of the time slice \( \{ t \} \times [x_0, x_1] \times S^n \) proving part (i). In Section B.2.3 the sectional curvatures of \( g \) are computed proving most of (iii). And finally concluding in Section B.2.4 by proving the part of (iii) concerning the off-diagonals of the Ricci tensor.

As our main application is the metric of Definition A.11 which is of the form \( \kappa^2 g \), we quote the following is general fact about scaling metrics.

**Lemma B.17.** Let \( A', B', C', \) and \( D' \) be the orthonormal vector fields of \( \kappa^2 g \) corresponding to the orthonormal vector fields \( A, B, C, \) and \( D \) of \( g \). Let \( II' \) and \( II \) be second fundamental forms of the same embedded hypersurface with respect to \( \kappa^2 g \) and \( g \) respectively.

\[
\mathcal{R}_{\kappa^2 g}(A', B', C', D') = \frac{1}{\kappa^2} \mathcal{R}_g(A, B, C, D) \quad \text{and} \quad II'(A', B') = \frac{1}{\kappa} II(A, B).
\]

**Proof.** This is easily proven from definitions. For a reference, see the more general fact about conformal scaling: [2, Theorem 1.159]. When the conformal factor is constant, the formulas reduce to the above. \( \square \)

**B.2.1. Extrinsic Curvature.** In this short section we record the second fundamental forms relevant to our curvature computations. Obviously we will consider the second fundamental form of \( \{ t \} \times [x_0, x_1] \times S^n \) inside of \( [t_0, t_1] \times [x_0, x_1] \times S^n \). Notice that \( g_t \) is a warped product metric on \( [x_0, x_1] \times S^n \), so it will also be necessary to consider the second fundamental form of \( \{ t \} \times \{ x \} \times S^n \) inside of \( \{ t \} \times [x_0, x_1] \times S^n \) with respect to \( g_t \).

**Lemma B.18.** Let \( II_t \) be the second fundamental form of \( \{ t \} \times [x_0, x_1] \times S^n \) embedded in \( [t_0, t_1] \times [x_0, x_1] \times S^n \) with respect to \( g \) and the unit normal \( T \). Then

\[
II_t = \frac{A_t}{A} (A^2 dx^2) + \frac{B_t}{B} (B^2 ds_n^2).
\]

**Proof.** Because the intervals \( [t_0, t_1] \times \{ x \} \times \{ p \} \) are geodesics with respect to \( g \), by [14, Proposition 3.2.1] we have that the second fundamental form of \( \{ t \} \times [x_0, x_1] \times S^n \) is

\[
\Pi_t = \frac{1}{2} \partial_t g = \frac{A_t}{A} (A^2 dx^2) + \frac{B_t}{B} (B^2 ds_n^2).
\]

As \( dx^2 \) and \( ds_n^2 \) are invariant with respect to \( T \). \( \square \)

**Lemma B.19.** Let \( II_x \) be the second fundamental form of \( \{ t \} \times \{ x \} \times S^n \) embedded in \( \{ t \} \times [x_0, x_1] \times S^n \) with respect to \( g_t \) and the unit normal \( X \). Then

\[
II_x = \frac{A_x}{A^2} (A^2 dx^2) + \frac{B_x}{AB} (B^2 ds_n^2)
\]
Proof. Note that \( g \) restricted to \( \{t\} \times [x_0, x_1] \times S^n \) is \( A^2(t, x)dx^2 + B^2(t, x)ds_n^2 \). Because \( X = \frac{\partial_x}{A} \) is a unit vector, applying \cite{[14]} Proposition 3.2.1 we have
\[
\Pi_x = \frac{1}{2} \mathcal{L}_X g_t = \frac{A_x}{A^2} (A^2 dx^2) + \frac{B_x}{AB} ds_n^2.
\]
As \( dx^2 \) and \( ds_n^2 \) are invariant with respect to \( X \).

B.2.2. The intrinsic curvatures of a time slice. Next we compute the curvatures in the spherical directions, \( X \) and \( \Sigma \). To do this, we consider the restricted metric \( g(t) \) on the submanifolds \( \{t\} \times [x_0, x_1] \times S^n \), and compute its intrinsic curvature.

**Lemma B.20.** The curvatures \( g_t \) involving \( X \) are as follows.
\[
K_{g_t}(\Sigma, X_t) = \frac{A_x B_x}{A^3 B} - \frac{B_{xx}}{A^2 B}
\]

Proof. By \cite{[14]} Proposition 3.2.11 we have
\[
K_{g_t}(X, W) = \Pi^2_x (W, W) - (\mathcal{L}_X \Pi_x)(W, W).
\]
(100)

Where \( \Pi_x \) as in (99). We compute the derivative of \( \Pi_x \) as follows
\[
\mathcal{L}_X \Pi_x = \left( \frac{A_{xx} A^2 - 2 A_x^2 A}{A^5} + \frac{A_x}{A^4} \right) (A^2 dx^2) + \left( \frac{B_{xx} AB - B_x^2 A - B_x B_x}{A^3 B^2} + \frac{2 B_x^2}{A^2 B^2} \right) (B^2 ds_n^2)
\]
(101)

Taking the square of equation (99) and substituting it with (101) into (100) yields the following
\[
K_{g_t}(X, -) = \frac{A_x^2}{A^4} (A^2 dx^2) + \frac{B_x^2}{A^2 B^2} (B^2 ds_n^2) - \frac{A_{xx}}{A^3} (A^2 dx^2) - \left( \frac{B_{xx}}{A^2 B} + \frac{B_x^2}{A^2 B^2} - \frac{A_x B_x}{A^3 B} \right) (B^2 ds_n^2)
\]
\[
= \left( \frac{A_x^2}{A^4} - \frac{A_{xx}}{A^3} \right) (A^2 dx^2) + \left( - \frac{B_{xx}}{A^2 B} + \frac{A_x B_x}{A^3 B} \right) (B^2 ds_n^2)
\]
The claim follows.

Finally we must compute those curvatures of \( g_t \) in the spherical directions.

**Lemma B.21.** The curvatures of \( g_t \) not involving \( X \) are as follows.
\[
K_{g_t}(\Sigma_i, \Sigma_j) = \frac{1}{B^2} - \frac{B_x^2}{A^2 B^2}.
\]

Proof. Let \( g_x \) denote the metric \( g_t \) restricted to \( \{t\} \times \{x\} \times S^n \) inside of \( \{t\} \times [x_0, x_1] \times S^n \). Notice that \( g_x = B^2(t, x)ds_n^2 \) is round with radius \( B \), so \( K_{g_x}(\Sigma_i, \Sigma_j) = \frac{1}{B^2} \). By Gauss' formula and equation (99) we have
\[
K_{g_t}(\Sigma_i, \Sigma_j) = K_{g_x}(\Sigma_i, \Sigma_j) - \Pi_x(\Sigma_i, \Sigma_i) \Pi_x(\Sigma_j, \Sigma_j) = \frac{1}{B^2} - \frac{B_x^2}{A^2 B^2}.
\]
\]
The sectional curvatures. We begin by computing the sectional curvatures involving $T$.

**Lemma B.22.** The sectional curvatures of $g$ involving $T$ are as follows.

$$K_g(T, X) = -\frac{A_t}{A} \text{ and } K_g(T, \Sigma) = -\frac{B_t}{B}.$$  

*Proof.* Because the intervals $[t_0, t_1] \times \{x\} \times \{p\}$ are geodesics of $g$, the Mainardi-Codazzi equations reduce to

\begin{equation}
(102)
K(T, W) = \Pi_t^2(W, W) - (\partial_t \Pi_t)(W, W),
\end{equation}

where $W$ is any vector normal to $T$. Computing the derivative of $\Pi_t$ we have

\begin{equation}
(103)
\partial_t \Pi_t = \frac{A_t A - A_t^2}{A^2} dx^2 + \frac{2A_t^2}{B^2} (B^2 ds_n^{-1}) + \frac{2B_t^2}{B} (B^2 ds_n^{-1}).
\end{equation}

Taking the square of equation (98) and substituting that with (103) into (102) yields the following:

$$K(T, -) = \frac{A_t^2}{A^2} (A^2 dx^2) + \frac{B_t^2}{B^2} (B^2 ds_n^{-1}) - \frac{A_t A + A_t^2}{A^2} (A^2 dx^2) - \frac{B_t B + B_t^2}{B^2} (B^2 ds_n^{-1}),$$

$$= -\frac{A_t}{A} (A^2 dx^2) - \frac{B_t}{B} (B^2 ds_n^{-1}).$$

The claim follows. \qed

Having determined $K_{g_t}$ in Lemmas B.20 and B.21 and $\Pi_t$ in equation (98), the following is a direct consequence of Gauss’ formula.

**Corollary B.23.** The curvatures of $g$ not involving $T$ are as follows.

$$K_g(X, \Sigma_i) = \frac{B_x A_x - B_x x}{A^2 B} - \frac{B_t A_t}{A B}, \text{ and } K_g(\Sigma_i, \Sigma_j) = \frac{1}{B^2} - \frac{B_t^2}{B^2 A^2} - \frac{B_t^2}{B^2 A^2}.$$  

B.2.4. Off-diagonals of the Ricci tensor. In this section we consider those curvatures of the form $R_g(A, B, B, C)$. Choose local coordinates $\sigma_i$ for $S^n$ so that $S_i := \partial_{\sigma_i}$ is an orthonormal basis of $(S^n, ds_n^2)$ at one point. We may assume that $\frac{S_i}{B} = \Sigma_i$ at this point. We will use $S_i$ along with $\partial_x$ and $\partial_t$ as a local frame. Because $\partial_x$ and $\partial_t$ are global and $g$ is homogeneous in the $S^n$ factor, it suffices to perform computations in these coordinates.

**Lemma B.24.** In this frame, the off-diagonals of $\text{Ric}_g$ are zero except for $\text{Ric}_g(X, T) = n \text{R}_g(X, \Sigma_i, \Sigma_i, T)$, where

$$R_g(X, \Sigma_i, \Sigma_i, T) = \frac{B_x A_x}{AB} - \frac{A_t B_t x}{A^2 B}.$$  

*Proof.* As $\text{Ric}(A, B) = \sum_{C_i} R(A, C_i, C_i, B)$, the off-diagonal terms of $\text{Ric}_g$ are determined by $R(T, \Sigma_i, \Sigma_i, X)$, $R_g(T, \Sigma_i, \Sigma_i, \Sigma_j)$, $R_g(X, \Sigma_i, \Sigma_i, \Sigma_j)$, and $R(\Sigma_i, \Sigma_j, \Sigma_j, \Sigma_k)$.

First, consider $R_g(\Sigma_i, \Sigma_j, \Sigma_j, \Sigma_k)$. By applying Gauss’ formula twice using $\Pi_x$ followed by $\Pi_t$, one sees that the terms involving $\Pi_t$ and $\Pi_x$ vanish, thus reducing to $R_g(\Sigma_i, \Sigma_j, \Sigma_j, \Sigma_k)$ =
Consider next $R_g(T, \Sigma_i, \Sigma_j, \Sigma_j)$ and $R_g(X, \Sigma_i, \Sigma_i, \Sigma_j)$. Let $\Pi'_x$ and $g'_x$ be the second fundamental form and metric of $[t_0, t_1] \times \{x\} \times S^n$ with respect to $g$. By applying Gauss’ formula with respect to $\Pi'_x$ and $g'_x$, one can check that $R_g(T, \Sigma_i, \Sigma_i, \Sigma_j) = R_{g'_x}(T, \Sigma_i, \Sigma_i, \Sigma_j)$ and $R_g(X, \Sigma_i, \Sigma_i, \Sigma_j) = R_{g'_x}(X, \Sigma_i, \Sigma_i, \Sigma_j)$. Both the metric $g_t$ and $g'_x$ are warped product metrics. This curvature is known to vanish in these coordinates \[14, \text{Section 4.2.3}\].

We can compute $R_g(\partial_x, S_i, S_i, T)$ in these coordinates as follows.

\begin{equation}
R_g(\partial_x, S_i, S_i, T) = \frac{1}{2} (\partial_\sigma \partial_\sigma g tx + \partial_\sigma \partial_\sigma g tx - \partial_\sigma \partial_\sigma g tx - \partial_\sigma \partial_\sigma g tx) + g ab \left( \Gamma_{\sigma \sigma \sigma}^a \Gamma_{\sigma x}^b - \Gamma_{\sigma x}^a \Gamma_{\sigma x}^b \right).
\end{equation}

First consider the second derivatives of the metric. Of those metric functions being considered, only $g_{\sigma \sigma \sigma}$ is nonzero. And the desired second derivative is as follows.

\begin{equation}
\partial_t \partial_x g_{\sigma \sigma \sigma} = \partial_t \partial_x (B^2) = 2 \partial_t (B x B) = 2 (B x B + B x B).
\end{equation}

Second, considering the metric functions being summed against in the Christoffel symbol term, the function is nonzero only if $i = j$. Consider the second Christoffel identity.

$$\Gamma_{ab}^c = \frac{1}{2} g^{cd} (\partial_d g_{ab} - \partial_d g_{ab} + \partial_d g_{cd}).$$

If all three indices, $a$, $b$, and $c$ are distinct, then all three metric functions will vanish. Thus, in equation \[104\] the only indices that could possibly give a nonzero summand are $a = b = t$, $a = b = x$, and $a = b = \sigma_i$. Within these, only 9 need to be computed. Of those 9, only 5 are nonzero. These are as follows.

$$\Gamma_{\sigma \sigma \sigma}^t = -B t B, \quad \Gamma_{\sigma \sigma \sigma}^x = -B x A, \quad \Gamma_{\sigma \sigma \sigma}^x = B x A, \quad \Gamma_{\sigma \sigma \sigma}^t = B x B, \quad \Gamma_{\sigma \sigma \sigma}^t = B x B.$$

Finally, when written down, only the cases $a = b = x$ and $a = b = \sigma_i$ are nonzero. These are as follows.

\begin{equation}
g_{xx} (\Gamma_{\sigma \sigma \sigma}^x \Gamma_{\sigma x}^x - 0) = A^2 \left( \frac{B x A}{A^2} \right) = -A t B x \frac{A}{A^2}
\end{equation}

\begin{equation}g_{\sigma \sigma \sigma} (0 - \Gamma_{\sigma \sigma \sigma}^x \Gamma_{\sigma \sigma \sigma}^x) = -B^2 \left( \frac{B x B}{B} \right) = -B t B x.
\end{equation}

Thus the only nonzero terms of \[104\] are those computed in equations \[105\], \[106\], and \[107\]. Combining these yields the following.

$$R_g(\partial_x, S_i, S_i, T) = \frac{1}{2} (2 (B x B + B x B)) + \frac{-A t B x B}{B} - B t B x = B x B - \frac{A t B x B}{B}.$$

Finally, using the fact that $R_g$ is a tensor, we see that

$$R_g(X, \Sigma_i, \Sigma_i, T) = R_g \left( \frac{\partial_x}{A}, S_i, S_i, T \right) = \frac{1}{AB} R_g(\partial_x, S_i, S_i, T) = \frac{B x B}{A B} - \frac{A t B x}{A^2 B}.$$

\[\square\]
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