Levi-Civita spacetimes have both classical and quantum singularities. The relationship between the two is used here to study and clarify the physical aspects of the enigmatic Levi-Civita spacetimes.

1. Introduction

Classically singular spacetimes may be quantum mechanically nonsingular. Whereas classical singularities are indicated by incomplete geodesics or incomplete paths of bounded acceleration in maximal spacetimes [1, 2], quantum singularities are indicated by quantum wave packets whose behavior is not completely defined by the wave equation and the underlying spacetime. In other words, spacetimes are quantum mechanically singular if boundary conditions need to be introduced at the classical singularity to uniquely specify the quantum wave behavior. Technically, quantum mechanically singular spacetimes are those in which the spatial derivative operator in a wave equation such as the Klein-Gordon equation is not essentially self adjoint on a $C_0^\infty$ domain in $L^2$, a Hilbert space of square integrable functions. Quantum singularities were first considered by Horowitz and Marolf [3] following earlier work by Wald [4]. Certain classically singular spacetimes are quantum mechanically nonsingular (e.g., certain orbifolds and extreme Kaluza-Klein [3]), but other classically singular spacetimes are still singular when probed by quantum wave packets (e.g., Reissner-Nordström, negative mass Schwarzschild and various quasiregular spacetimes [3, 5]). Here we use a physically insightful method known as Weyl’s limit point-limit circle criterion [6] to study the quantum singularities in Levi-Civita spacetimes and gain insight into the meaning of the metric parameters of these enigmatic spacetimes. This conference proceeding is based on [7].
2. Levi-Civita Spacetimes

The metric for a Levi-Civita spacetime \([8]\) has the form

\[
ds^2 = r^{4\sigma} dt^2 - r^{8\sigma^2 - 4\sigma} (dr^2 + dz^2) - \frac{r^{2-4\sigma}}{C^2} d\theta^2
\]

(1)

where \(\sigma\) and \(C\) are real numbers \((C > 0)\). For some parameter values one can interpret the Levi-Civita spacetime as the spacetime of an “infinite line mass”. In fact, after some controversy in the literature (see, e.g. \([9, 10, 11]\)), the following interpretations have become somewhat accepted: \(\sigma = 0\), \(\frac{1}{2}\) locally flat; \(\sigma = 0\), \(C = 1\) Minkowski spacetime; \(\sigma = 0\), \(C \neq 1\) cosmic string spacetime; \(0 < \sigma < \frac{1}{2}\) “infinite line mass” spacetime (modelled by a scalar curvature singularity at \(r = 0\)); \(\sigma = 1/2\) Minkowski spacetime in accelerated coordinates (planar source).

3. Classical and Quantum Singularities

The analysis in \([7]\) uses Weyl’s limit point-limit circle criterion \([6]\) to determine essential self-adjointness of the spatial portion of the Klein-Gordon wave operator on a \(C_0^\infty\) domain in \(L^2\), a Hilbert space of square integrable functions. The conclusions will now be summarized.

If \(\sigma\) is neither zero nor one-half, the Klein-Gordon operator is not essentially self-adjoint, so all \(\sigma \neq 0, \sigma \neq 1/2\) Levi-Civita spacetimes are quantum mechanically singular as well as being classically singular with scalar curvature singularities.

If \(\sigma = 0\) and \(C = 1\), the spacetime is simply Minkowski space. One of the two solutions of the radial Klein-Gordon equation can be rejected because it diverges at a regular point \((r = 0)\) of the spacetime. The operator is therefore quantum mechanically nonsingular (a well known fact, repeated here for completeness).

If \(\sigma = 0\) and \(C \neq 1\), the spacetime is the conical spacetime corresponding to an idealized cosmic string. The cosmic string spacetimes are quantum mechanically singular for azimuthal quantum number \(m\) such that \(|m|C < 1\) and nonsingular if \(|m|C \geq 1\). If arbitrary values of \(m\) are allowed, these spacetimes are quantum mechanically singular in agreement with earlier results \([5]\). These spacetimes are also classically singular with a quasiregular (“disclination”) singularity at \(r = 0\).

If \(\sigma = 1/2\) the classical spacetime is flat and without a classical singularity. This spacetime is also quantum mechanically nonsingular. The Weyl limit point-limit circle techniques used in \([7]\) emphasize the flatness of the spacetime and support a description given in \([9]\) of this spacetime as one given by a cylinder whose radius has tended to infinity.

For the Levi-Civita spacetimes, all that are classically singular are also quantum mechanically singular, and all that are classically nonsingular \((\sigma = 0, C = 1,\) and \(\sigma = 1/2)\) are also quantum mechanically nonsingular. The classically and quantum-mechanically nonsingular spacetimes correspond to isolated values of \(\sigma\), so that (for example) even though the spacetime \(\sigma = 0, C = 1\) is nonsingular, the spacetimes
with $\sigma \to 0$, $C = 1$ are singular. The only discrepancy between classical and quantum singularities are for the $\sigma = 0$, $C \neq 1$ modes with $|m|C \geq 1$, which are quantum mechanically nonsingular in a classically singular spacetime. The physical reason is that the wavefunction for large values of $m$ in a flat space with a quasiregular singularity at $r = 0$ is unable to detect the presence of the singularity because of a repulsive centrifugal potential.

4. Conclusions
The limit point-limit circle criterion that was used in [7] provides physical insight into when quantum singularities are prevented from occurring by potential barriers as well as the true meaning of the $\sigma = 1/2$ case.

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