Leading the non-cancelling IR divergences in massless gauge theories: Abelian case

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Abstract. We study the cancellation of both soft and collinear infrared divergences at next-to-leading order correction in a process where a massless electron is scattered off of a static point charge. We show that a self-consistent application of the KLN theorem requires including all initial and final degenerate states, which require including diagrams with an arbitrary number of disconnected photons forming a divergent series. We improve on previous work by rearranging the series in a controlled way and exploit the Monotone Convergence Theorem to prove the uniqueness of our rearranged result. This rearrangement yields a factorization of the infinite contribution from the initial state soft photons, that then cancels in the physically observable cross section.

1. Introduction

The infra-red (IR) catastrophe in purely massless gauge theories are known in two forms: soft, which is due to the massless nature of the radiation and appear when a soft (the word soft means that the photon energy is less than some experimental energy resolution $\Delta$) massless particle is absorbed or emitted from the radiator, and collinear, which comes when the radiating particle is considered to be massless and usually happens when a particle is emitted or absorbed or emitted collinearly with the radiator.

The first approach towards eliminating the IR divergences was given by Bloch and Nordsieck (BN) [1, 2]. Their idea is, briefly, that it is impossible to specify a final state of a scattering process involving a charged particle because an arbitrary number of soft photons can be emitted without being detected. The BN cancellation requires adding both the virtual and the real emission corrections. The BN theorem has long been known for the success in eliminating the soft IR divergences. However, the appearance of quantum chromodynamics (QCD) introduces the collinear IR divergences where one should deal with light quarks. A more general theorem has been introduced to solve the mass singularities in gauge theories by Kinoshita [3], Lee and Nauenberg [4] (KLN). The KLN theorem states, in a modern language, that one should sum over both the initial and final degenerate indistinguishable states to get rid of the mass singularities. A combination of the BN theorem and the KLN theorem to eliminate the soft and collinear IR divergences respectively produce an IR finite formula for the cross section. However, as is shown by M. Lavelle and D. McMullan (LM) [5], such a treatment is inconsistent, where it breaks the time reversal symmetry and one should stick with the more general KLN theorem. A naive attempt to solve the problem by including the initial state absorption of a soft photon will spoil...
time how to rigorously perform the delicate rearrangement of the resultant formally divergent series. The result ...

Following [9, 10], we consider a general form of our process with \( m \) incoming soft photons and \( n \) outgoing processes to \( d\sigma \) collinear processes. Adding these additional two pro-

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collinear divergences. The BN theorem requires first including the tree level Fig. 1 (a) and NLO vertex (b) as well as the emission of a soft photon from either the incoming or the outgoing electrons (c) and (d). The contribution from these processes to the differential cross section is given by

\[ d\sigma_{BN} = d\sigma_0 \left\{ 1 + \frac{\alpha_e}{\pi^2} \left[ \log \left( \frac{E^2}{\Delta^2} \right) \left( 1 - \log \left( \frac{-q^2}{m_e^2} \right) \right) + \frac{3}{2} \log \left( \frac{-q^2}{m_e^2} \right) + O(1) \right] \right\} , \tag{1} \]

where \( \alpha_e \) is the QED coupling constant and \( O(1) \) are terms that are finite at zero photon and electron masses. We note that Eq. (1) is free of soft IR divergences, but it diverges logarithmically as we set \( m_e \to 0 \). We now use the KLN theorem to remove the potentially large logs, \( \log(-q^2/m_e^2) \), in the high energy limit \( -(q^2 \gg m^2) \) by summing over the indistinguishable hard collinear final emission Fig. 1 (d) and initial absorption (e) processes. We note that the remaining amplitudes where a hard photon is taking part in the scattering process can be distinguished from either the incoming or the outgoing electrons [12]. One can easily see that the LN result can be obtained by replacing the \( m_e^2 \) by \( \delta^2 E^2 [4] \), and the result is free of any IR divergences. Here we point out that we used a combination of the BN and KLN theorems for such a cancellation, which is clearly inconsistent [5], and to stay in the spirit of the KLN theorem one should also add the absorption of a soft photon as a degenerate initial state, which spoils the finiteness of the cross section at NLO.

3. The role of the disconnected diagrams

The disconnected diagrams were first introduced in [4] to provide the cancellation of the mass singularities from the diagrams neglected above when the photon is hard and collinear in such a way that it can be distinguished, and have been used later by De. Calan [7] for his simplified model and T. Muta in QCD [8]. However, they seemed to include an asymmetric number of diagrams from the initial and final states, where we will follow their lead to check the inconsistency of their treatment. We first include the contributions from the interference between the disconnected diagram shown in Fig. 1 (g) and the absorption-emission diagrams such as the one in (h). We note that the Feynman rule for the disconnected photon is \((2\pi)^3 2\omega k_1 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \delta_{\epsilon_1 \epsilon_2}\) where \( \omega \) is the energy of the photon of momentum \( k \), \( \epsilon_1 \) and \( \epsilon_2 \) are the polarization vectors for the incoming photon of momentum \( k_1 \) and the outgoing photon of momentum \( k_2 \) respectively.

![Figure 2. The absorption and emission with a disconnected photon.](image)

It can be easily shown that including the contributions from the interference of the disconnected diagram Fig. 1 (g) with the absorption-emission diagrams Fig. 1 (h) is not sufficient for the cancellation, even adding the contributions from the absorption or emission with one disconnected photon, such as the ones in Fig. 2, keeps the cross section IR divergent and raises the question of adding even more disconnected photons to the former processes, while the most important question becomes; is there a full systematic method to apply the KLN
theorem properly? We show here that the resolution is to include all possible contributions from degenerate initial and final states, including the contributions from the disconnected diagrams with an arbitrary number of disconnected soft photons.

4. IASZ treatment
Following [10, 11], we consider a general form of our process with \(m\) incoming soft photons and \(n\) outgoing soft photons:

\[
e^{-} + m\gamma \text{(soft)} \rightarrow e^{-} + n\gamma \text{(soft)},
\]

with an amplitude \(M_{mn}\). The transition probability for the process becomes

\[
P_{mn} = \frac{1}{m!n!} \sum_{i,f} |M_{mn}|^2,
\]

where \(P_{mn}\) contains contributions from both fully and partially connected cut diagrams and the sum over initial \((i)\) and final \((f)\) states exist. The total LN probability will be

\[
P = \sum_{m,n=0}^{\infty} P_{mn},
\]

where the KLN theorem ensures that the quantity \(P\) is free of the IR (soft or collinear) divergences.

It was shown in [10, 11] that any cut diagram from \(P_{mn}\) at NLO correction can be constructed from four probabilities: \(P_{00}\), which is the cut diagram with no real emission or absorption of soft photons, this may include the leading term, the vertex correction, the vacuum polarization, etc.; \(P_{10}\), and \(P_{01}\), which includes all cut diagrams with one soft photon in the initial or the final states respectively; \(P_{11}\), which includes all cut diagrams from the interference between the disconnected diagram Fig. 1 (g) and the absorption-emission diagrams like Fig. 1 (h). The fully connected cut diagrams are given by any of the previous basic probabilities, while the partially connected ones are those probabilities multiplied by a number of \(\delta\) functions according to the number of disconnected photons. So we can construct \(P_{mn}\) by splitting each cut diagram up into connected and disconnected parts.

We define the disconnected part by the function \(D(m-i,n-j)\) that describes the number of \(m-i\) incoming and \(n-j\) outgoing soft photons that can be joined together and become disconnected from the electron line. By definition \(D(0,0) = 1\) and \(D(a,b) = 0\) for \(a \neq b\). One may show that the transition probability for the general process at NLO is given by [10, 11]

\[
P_{mn} = \frac{D(m,n)}{m!n!} P_{00} + \sum_{i=0}^{m} \frac{D(m-i,n-i-1)}{(m-i)!(n-i-1)!} P_{01} + \sum_{i=0}^{n} \frac{D(m-i-1,n-i)}{(m-i-1)!(n-i)!} P_{10} + \sum_{i=0}^{\min(m,n)} \frac{D(m-i-1,n-i-1)}{(m-i-1)!(n-i-1)!} \tilde{P}_{11},
\]

Figure 3. A generic cut diagram with \(m\) incoming and \(n\) outgoing soft photons that is partially disconnected. (Note that the blobs include the possibility of connecting incoming photon lines from the left with conjugate incoming photon lines from the right.)
IASZ were able to rearrange the full series by using the following identity
\[ \mathcal{D}(m, n) = \sum_{i=0}^{n} \mathcal{D}(m - i, n - i) - \sum_{i=0}^{n} \mathcal{D}(m - i - 1, n - i - 1) \]  
(6)
to give the following IR finite probability
\[ \mathcal{P} = \sum_{m, n} \sum_{i} \frac{\mathcal{D}(m - i, n - i)}{(m - i)! (n - i)!} \left[ \mathcal{P}_{00} + \mathcal{P}_{01} + \mathcal{P}_{10} + (\tilde{\mathcal{P}}_{11} - \mathcal{P}_{00}) \right]. \]  
(7)

However, one can easily see that the probability \( \mathcal{P} \) tends to zero once we send it back to tree level.

One may show that the series can be rearranged in different ways. However, we are looking for a rearrangement with special features: to be IR safe and to keep the tree level contribution finite. We give here the rearrangement that satisfies these criteria and rigorously prove that our result is unique. In order to give such a proof we manipulate Eq. (5) in a controlled way by introducing a convergence factor that becomes small for large \( i \): we take
\[ \mathcal{D}(m - i, n - j) \to \mathcal{D}(m - i, n - j) e^{-(i+j)/\Lambda} \]  
(8)
with \( \Lambda \gg 1 \). The convergent factor \( \Lambda \) allows us to sum up to a finite number of disconnected soft photons \( N \). Since \( \mathcal{D}(m, n) = 0 \) for \( m \neq n \), we may simplify our manipulations by replacing the double sum over \( m \) and \( n \) with a single sum over \( n \). With the above convergence factor we are guaranteed that \( \mathcal{P} = \lim_{\Lambda \to \infty} \lim_{N \to \infty} \mathcal{P}_N(\Lambda) \) converges. Now we rearrange the partial sum \( \mathcal{P}_N(\Lambda) \) up to \( N \) soft photons to find
\[ \mathcal{P}_N(\Lambda) = \sum_{n=0}^{N} \frac{\mathcal{D}(n, n)}{(n!)^2} \left[ \mathcal{P}_{00} + e^{-\frac{1}{\Lambda}} \mathcal{P}_{01} \right] + \sum_{n=1}^{N} \sum_{i=1}^{n} \frac{\mathcal{D}(n - i, n - i)}{((n - i)!)^2} \left[ e^{-\frac{2i+1}{\Lambda}} \mathcal{P}_{01} + e^{-\frac{2i-1}{\Lambda}} \mathcal{P}_{10} + e^{-\frac{2i}{\Lambda}} \tilde{\mathcal{P}}_{11} \right]. \]  
(9)

One can easily show, using the same convergent factor, that the IASZ result is very different from Eq. (9) after swapping the limit in the full sum. So we check the possibility of switching the limits in order to prove the uniqueness of our result. We exploit the Monotone Convergence Theorem to justify that our result converges to the same value under the interchanging limits procedure by proving that the partial sum \( \mathcal{P}_N(\Lambda) \)
1) monotonically increase in \( N \) for each \( \Lambda \) and 2) monotonically increase in \( \Lambda \) for each \( N \) [13]. We use the fact that \( \mathcal{P}_{01} + \mathcal{P}_{10} = -\tilde{\mathcal{P}}_{11} \) [8] to simplify Eq. (9). Then we have
\[ \mathcal{P}_N(\Lambda) = \sum_{n=0}^{N} \frac{\mathcal{D}(n, n)}{(n!)^2} \left[ \mathcal{P}_{00} + e^{-\frac{1}{\Lambda}} \mathcal{P}_{01} \right] + \sum_{n=1}^{N} \sum_{i=1}^{n} \frac{\mathcal{D}(n - i, n - i)}{((n - i)!)^2} 2\mathcal{P}_{01} e^{-\frac{2i}{\Lambda}} \left[ \cosh \left( \frac{1}{\Lambda} \right) - 1 \right]. \]  
(10)

Since \( \mathcal{P}_{00}, \mathcal{P}_{01}, \Lambda, \) and \( \mathcal{D}(n, n) \) are all strictly positive, Eq. (10) clearly increases monotonically in \( N \) for fixed \( \Lambda \). To show that \( \mathcal{P}_N(\Lambda) \) increases monotonically in \( \Lambda \) for fixed \( N \), we take the derivative with respect to \( \Lambda \):
\[ \frac{d\mathcal{P}_N(\Lambda)}{d\Lambda} = \sum_{n=0}^{N} \frac{\mathcal{D}(n, n)}{(n!)^2} \left[ \frac{1}{\Lambda^2} e^{-\frac{1}{\Lambda}} \mathcal{P}_{01} \right] + \mathcal{O}\left( \frac{1}{\Lambda^3} \right). \]  
(11)
Although one finds that the higher order in $1/\Lambda$ correction term is negative, for any $N$ we can find a $\Lambda$ large enough such that the first term, which is strictly positive, dominates. We have thus proved that we may exchange limits for our rearranged formula Eq. (10), and we may evaluate the $\Lambda \to \infty$ limit first, yielding our main result:

$$P = (P_{00} + P_{01}) \sum_{n=0}^{\infty} \frac{D(n,n)}{(n!)^2}. \quad (12)$$

We note that the contribution from the emission or absorption of a hard photon may also have an arbitrary number of disconnected soft photons and remains degenerate with our state, in such a way that none of the disconnected photons can be attached to the hard photon. Let us call the probabilities from the contribution of the absorption of hard and collinear photons $P_{10}^{hc}$ and the contribution from the emission of hard and collinear photons $P_{01}^{hc}$, then the contribution from the diagrams where a hard photon is taking part with an infinite number of disconnected photons becomes

$$P^{hc} = (P_{10}^{hc} + P_{01}^{hc}) \sum_{n=0}^{\infty} \frac{D(n,n)}{(n!)^2}. \quad (13)$$

Finally, the total probability contributions from all the initial and final degenerate states will be

$$P_{tot} = (P_{00} + P_{01} + P_{10}^{hc} + P_{01}^{hc}) \sum_{n=0}^{\infty} \frac{D(n,n)}{(n!)^2}. \quad (14)$$

We find that all the soft initial state physics of the infinite number of undetectably soft photons completely factorizes. When the cross section is computed, one simply divides out by this unobserved infinity. We have thus rendered all soft and collinear IR divergences harmless. One can now apply Eq. (14) to find the complete NLO differential cross section to the Rutherford scattering [14]

$$d\sigma_{NLO} = d\sigma_0 \left\{ 1 + \frac{\alpha}{\pi} \left[ \log \left( \frac{-q^2}{\delta^2 E^2} \right) \left( \log \left( \frac{E^2}{\Delta^2} \right) + \frac{3}{2} \right) + \frac{2}{3} \log \left( \frac{-q^2}{\mu_{\overline{MS}}^2} \right) \right. 
+ \frac{\pi^2}{2} \left( \left( \frac{4E^2}{-q^2} \right) - 1 \right)^{-\frac{1}{2}} - \frac{8}{3} \right] + \frac{5}{36} + O(m^2, \delta^2) \right\} \quad (15)$$

5. Conclusions
The application of the KLN theorem requires including all degenerate initial and final states to achieve the IR finiteness of the cross section. These degenerate states include disconnected diagrams with an arbitrary number of disconnected photons which form a divergent series of diagrams at the transition probability level. We performed a rearrangement of this formally divergent series under the control of a convergent factor, which helped us to check the validity and the uniqueness of our rearrangement through the application of the Monotone Convergence Theorem. Our rearrangement shows that we only need to sum up to a finite number of diagrams, where the disconnected part remains factorized, and more important that the sum of these diagrams is IR safe and leads to the tree level contribution when we take the limit of the coupling constant to 0. Therefore, we arrived at the extremely non-trivial result that for NLO Rutherford scattering the summation over all indistinguishable initial and final states is equivalent to the summation over only the initial hard collinear and final soft, hard collinear, and soft and collinear degenerate states.
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