Algebra of linear recurrence relations in arbitrary characteristic

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Preface

The goal of this paper is to present an algebraic approach to the basic results of the theory of linear recurrence relations. This approach is based on the ideas from the theory of representations of one endomorphisms (a special case of which may be better known to the reader as the theory of the Jordan normal form of matrices). The notion of the divided derivatives, an analogue of the divided powers, turned out to be crucial for proving the results in a natural way and in their natural generality. The final form of our methods was influenced by the umbral calculus of G.-C. Rota.

Neither the theory of representation of one endomorphism, nor the theory of divided powers, nor the umbral calculus apply directly to our situation. For each of these theories we need only a modified version of a fragment of it. This is one of the reasons for presenting all proofs from the scratch. Both these fragments and our modifications of them are completely elementary and beautiful by themselves. This is another reason for presenting proofs independent of any advanced sources. Finally, the theory of the linear recurrence relation is an essentially elementary theory, and as such it deserves a self-contained exposition.

The prerequisites for reading this paper are rather modest. Only the familiarity with the most basic notions of the abstract algebra, such as the notions of a commutative ring, of a module over a commutative ring, and of endomorphisms and homomorphisms are needed. No substantial results from the abstract algebra are used. A taste for the abstract algebra and a superficial familiarity with it should be sufficient for the reading of this paper.

The standard expositions of the theory of linear recurrence relations present this theory over algebraically closed fields of characteristic 0, or even only over the field of complex numbers. In contrast, such restrictions are very unnatural from our point of view. The methods of this paper apply equally well to all commutative rings with unit and without zero divisors; no assumptions about the characteristic are needed. Of course, a form of the condition of being algebraically closed is needed. We assume only that all roots of the characteristic polynomial of the linear recurrence relation in question are contained in the ring under consideration (this can be done using any of the standard approaches to the theory also).

Structure of the paper. The main results are stated and proved in Section 7, which depends on all previous ones. Sections 1–6 are independent with only one exception: Section 3 depends on both Section 1 and 2. All references are relegated to the Note bibliographique at the end. The reasons are the same as N. Bourbaki’s reasons for not including any references with the exception of his Notes historiques.

We denote by \( \mathbb{Z} \) the ring of integers and by \( \mathbb{N} \) the set of non-negative integers. We denote by \( \mathbb{k} \) a fixed entire ring, i.e. a commutative ring with a unit without zero divisors and such that its unit is not equal to its zero. There is a canonical ring homomorphism \( \mathbb{Z} \to \mathbb{k} \) taking \( 0, 1 \in \mathbb{Z} \) to the zero and the unit of \( \mathbb{k} \) respectively, making \( \mathbb{k} \) into a \( \mathbb{Z} \)-algebra, and every \( \mathbb{k} \)-module into a \( \mathbb{Z} \)-module. We identify \( 0, 1 \in \mathbb{Z} \) with their images in \( \mathbb{k} \).
1. Divided derivatives of polynomials

Polynomials in two variables. Let $x$ be a variable, and let $\mathbb{k}[x]$ be the $\mathbb{k}$-algebra of polynomials in $x$ with coefficients in $\mathbb{k}$. Let $y$ be some other variable, and let $\mathbb{k}[x,y]$ be the $\mathbb{k}$-algebra of polynomials in two variables $x, y$ with coefficients in $\mathbb{k}$. As is well known, $\mathbb{k}[x,y]$ is canonically isomorphic to the $\mathbb{k}$-algebra $\mathbb{k}[x][y]$ of polynomials in $y$ with coefficients in $\mathbb{k}[x]$. We will identify these two algebras. This allows us to write any polynomial $f(x,y)$ in two variables $x, y$ in the form

$$f(x,y) = \sum_{n=0}^{\infty} g_n(x) y^n.$$  

In fact, this sum is obviously finite. Equivalently, the polynomials $g_n(x)$ are equal to 0 for all sufficient big $n \in \mathbb{N}$. The polynomials $g_n(x)$ are uniquely determined by $f(x,y)$.

The definition of the divided derivatives. Let $p(x) \in \mathbb{k}[x]$. Then $p(x+y) \in \mathbb{k}[x,y]$, and hence $p(x+y)$ has the form

$$p(x+y) = \sum_{n=0}^{\infty} (\delta^n p)(x) y^n,$$

for some polynomials $(\delta^n p)(x)$ uniquely determined by $p(x)$. The sum in (1) is actually finite. Equivalently, $(\delta^n p)(x) = 0$ for all sufficient big $n \in \mathbb{N}$. The coefficient $(\delta^n p)(x)$ in front of $y^n$ in the sum in (1) is called the $n$-th divided derivative of the polynomial $p(x)$. We will also denote $(\delta^n p)(x)$ by $\delta^n p(x)$ or $\delta^n p(x)$.

Operators $\delta^n$. Let $n \in \mathbb{N}$. By assigning $\delta^n p(x) \in \mathbb{k}[x]$ to $p(x) \in \mathbb{k}[x]$ we get a map

$$\delta^n : p(x) \mapsto \delta^n p(x).$$

Clearly, $\delta^n$ is a $\mathbb{k}$-linear operator $\mathbb{k}[x] \to \mathbb{k}[x]$.

After substitution $y = 0$ the equation (1) reduces to

$$p(x) = \delta^0 p(x).$$

Therefore, $\delta^0 = \text{id} = \text{id}_{\mathbb{k}[x]}$.

1.1. Theorem (Leibniz formula). Let $f(x), g(x) \in \mathbb{k}[x]$, and let $n \in \mathbb{N}$. Then

$$\delta^n (f(x)g(x)) = \sum_{i+j = n} \delta^i f(x) \delta^j g(x).$$
Proof. By applying (1) to \( p(x) = f(x) \) and to \( p(x) = g(x) \), we get
\[
f(x+y) = \sum_{i=0}^{\infty} \delta^i f(x) y^i,
g(x+y) = \sum_{j=0}^{\infty} \delta^j g(x) y^j.
\]
By multiplying these two identities, we get
\[
f(x+y)g(x+y) = \sum_{i,j=0}^{\infty} \delta^i f(x) \delta^j g(x) y^{i+j},
\]
and hence
\[
f(x+y)g(x+y) = \sum_{n=0}^{\infty} \left( \sum_{i+j=n} \delta^i f(x) \delta^j g(x) \right) y^n.
\]
The theorem follows. ■

1.2. Corollary (Leibniz formula for \( \delta^1 \)). Let \( f(x), g(x) \in \mathbb{k}[x] \). Then
\[
\delta^1 (f(x)g(x)) = \delta^1 f(x) g(x) + f(x) \delta^1 g(x).
\]
In other terms, \( \delta^1 \) is a derivation of the ring \( \mathbb{k}[x] \). ■

1.3. Lemma. (i) \( \delta^0(1) = 1 \) and \( \delta^n(1) = 0 \) for all \( n \geq 1 \).
(ii) \( \delta^0 x = x, \ \delta^1 x = 1, \) and \( \delta^n x = 0 \) for all \( n \geq 2 \).

Proof. As we noted above, \( \delta^0 = \text{id} \). In particular, \( \delta^0(1) = 1 \). For \( p(x) = 1 \), the formula (1) takes the form
\[
1 = \delta^0(1) y^0 + \sum_{n=1}^{\infty} \delta^n (1) y^n.
\]
Since \( \delta^0(1)y^0 = 1 \cdot y^0 = 1 \), the formula (2) implies that
\[
0 = \sum_{n=1}^{\infty} \delta^n (1) y^n.
\]
It follows that \( \delta^n 1 = 0 \) for \( n \geq 1 \). This proves the part (i) of the lemma. For \( p(x) = x \), the formula (1) takes the form
\[
x + y = \delta^0(x) y^0 + \delta^1(x) y^1 + \sum_{n=2}^{\infty} \delta^n (x) y^n.
\]
It follows that \( \delta^0(x) = x, \ \delta^1(x) = 1, \) and \( \delta^n(x) = 0 \) for all \( n \geq 2 \). This proves the part (ii) of the lemma. ■
1.4. Lemma. \( \delta^i(x^n) = nx^{n-1} \) for all \( n \in \mathbb{N}, \ n \geq 1 \).

Proof. The case \( n = 1 \) was proved in Lemma 1.3. Suppose that \( n \in \mathbb{N}, \ n \geq 1 \), and we already know that \( \delta^i(x^n) = nx^{n-1} \). By Corollary 1.2,

\[
\begin{align*}
\delta^i(x^{n+1}) &= \delta^i(x \cdot x^n) = \delta^i(x)x^n + x\delta^i(x^n) \\
&= 1 \cdot x^n + x(nx^{n-1}) = x^n + nx^n = (n+1)x^n.
\end{align*}
\]

An application of induction completes the proof. ■

Remark. By Lemma 1.4, the operator \( \delta^i : \mathbb{k}[x] \to \mathbb{k}[x] \) agrees on the powers \( x^n \in \mathbb{k}[x] \) with the operator \( d : f(x) \mapsto f'(x) \) of taking the usual formal derivative. Since both these operators are \( \mathbb{k} \)-linear, \( \delta^i = d \). But if \( i \in \mathbb{N}, \ i \geq 2 \), then the operator \( \delta^i \) is not equal to the operator of taking the \( i \)-th derivative. This immediately follows either from Lemma 1.5 or from Theorem 1.7 below.

Binomial coefficients. Let \( n \in \mathbb{N} \). For \( i \in \mathbb{N}, \ i \leq n \), we define the binomial coefficients \( (i-n|n) \in \mathbb{N} \), by the binomial formula

\[
(3) \quad (x + y)^i = \sum_{n=0}^{i} (i-n|n) x^{i-n} y^n.
\]

Given arbitrary numbers \( a, b \in \mathbb{N} \), we define \( (a \mid b) \) as \( (n-b \mid b) \), where \( n = a + b \). Given arbitrary integers \( a, b \in \mathbb{Z} \), we set \( (a \mid b) = 0 \) if at least one of the numbers \( a, b \) is not in \( \mathbb{N} \).

We prefer the notation \( (a \mid b) \) to the classical one by the typographical reason, and because the new notation helps to bring to the light the fact that we do not use any properties of \( (a \mid b) \) except the above definition.

1.5. Lemma. \( \delta^n(x^i) = (i-n|n) x^{i-n} \).

Proof. It is sufficient to compare (3) with the definition (1) of divided derivatives. ■

The left shift operator. The left shift operator \( \lambda : \mathbb{k}[x] \to \mathbb{k}[x] \) is just the operator of multiplication by the polynomial \( x \):

\[
\lambda p(x) = \lambda(p)(x) = xp(x).
\]
The reasons for calling $\lambda$ the *left shift operator* will be clear later. The main property of the left shift operator and the divided derivatives is the following commutation relation.

1.6. **Theorem.** Let us set $\delta^{-1} = 0$. Then

\[ (4) \quad \delta^n \circ \lambda - \lambda \circ \delta^n = \delta^{n-1}. \]

for all $n \in \mathbb{N}$.

**Proof.** Let $p(x) \in \mathbb{k}[x]$, and let $y$ be a variable different from $x$. By the Leibniz formula from Theorem 1.1,

\[ \delta^n(xp(x)) = \sum_{i+j=n} \delta^i(x) \delta^j(p(x)). \]

But by Lemma 1.3, $\delta^0 x = x$, $\delta^1 x = 1$, and $\delta^i x = 0$ for $i \geq 2$. Therefore

\[ \delta^n(xp(x)) = x \delta^n(p(x)) + \delta^{n+1}(p(x)), \quad \text{or, what is the same,} \]

\[ \delta^n(xp(x)) - x \delta^n(p(x)) = \delta^{n+1}(p(x)). \]

Rewriting the last identity in terms of $\lambda$, we get

\[ \delta^n(\lambda(p)(x)) - \lambda(\delta^n(p)(x)) = \delta^{n+1}(p(x)), \quad \text{i.e.} \]

\[ \delta^n \circ \lambda(p(x)) - \lambda \circ \delta^n(p(x)) = \delta^{n+1}(p(x)). \]

Since $p(x) \in \mathbb{k}[x]$ was arbitrary, this proves the theorem. ■

The following theorem will be not used in the rest of the paper.

1.7. **Theorem (Composition of divided derivatives).** Let $n, m \in \mathbb{N}$. Then

\[ \delta^n \circ \delta^m = (m \mid n) \delta^{n+m}. \]

**Proof.** Let $p(x) \in \mathbb{k}[x]$. Let $u, z$ be two new variables different from both $x$ and $y$. If we apply (1) to $u, z$ in the role of $x, y$ respectively (and use $m$ instead of $n$), we get

\[ p(u+z) = \sum_{m=0}^{\infty} (\delta^m p)(u) z^m. \]

Let us set $u = x + y$ and apply (1) to each polynomial $(\delta^m p)(x+y)$:
(5) \[ p((x+y)+z) = \sum_{m=0}^{\infty} (\delta^m p)(x+y) z^m \]
= \[ \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} \delta^n (\delta^m p)(x) y^n \right) z^m \]
= \[ \sum_{m,n=0}^{\infty} \delta^n \circ \delta^m (p)(x) y^n z^m \]

Alternatively, we can apply (1) to \( y+z \) in the role of \( y \) and then apply (3):

(6) \[ p(x + (y + z)) = \sum_{k=0}^{\infty} \delta^k p(x) (y+z)^k \]
= \[ \sum_{k=0}^{\infty} \delta^k p(x) \left( \sum_{m+n=k} \delta^m p(x) y^m z^n \right) \]
= \[ \sum_{m,n=0}^{\infty} \delta^{m+n} p(x) (m|n) y^m z^n \]
= \[ \sum_{m,n=0}^{\infty} (m|n) \delta^{m+n} p(x) y^m z^n. \]

By the associativity of the addition, \((x+y)+z = x+(y+z)\) and hence
\[ p((x+y)+z) = p(x+(y+z)). \]

By combining this equality with (5) and (6) we conclude that
\[ \delta^n \circ \delta^m (p(x)) = (m|n) (\delta^{m+n} p(x)) \]
for all \( p(x) \in \mathbb{K}[z] \) and \( n, m \in \mathbb{N} \). The theorem follows. ■

2. Sequences and duality

**Sequences.** A sequence of elements of a set \( X \) is defined as a map \( \mathbb{N} \to X \). For a sequence \( s \) we usually denote the value \( s(i), i \in \mathbb{N} \) by \( s_i \) and often call it the \( i \)-th term of \( s \). The set of all sequences of elements of \( X \) will be denoted by \( S_X \). We are, first of all, interested in the case when \( X \) is a \( \mathbb{K} \)-module, and especially in the case when \( X \) is equal to \( \mathbb{K} \) considered as a \( \mathbb{K} \)-module. When it is clear from the context to what set \( X \) the terms of the considered sequences belong, we call the sequences of elements of \( X \) simply sequences.

Let \( M \) be a \( \mathbb{K} \)-module. Then the set \( S_M \) has a canonical structure of a \( \mathbb{K} \)-module. The \( \mathbb{K} \)-module operations on \( S_M \) are the term-wise addition of sequences and the term-wise multiplication of sequences by elements of \( \mathbb{K} \), defined in the following obvious way.
The term-wise sum $r+s$ of sequences $r, s \in S_M$ is defined by $(r+s)_i = r_i + s_i$, and the term-wise product $cs$ of $c \in k$ and $s \in S_M$ is defined by $(cs)_i = cs_i$.

**Modules of homomorphisms.** Let $M', M''$ be $k$-modules. Then the set $\text{Hom}(M', M'')$ of $k$-homomorphisms $M' \to M''$ has a canonical structure of a $k$-module. The addition is defined as the addition of $k$-homomorphisms, and the product $aF$ of an element $a \in k$ and a $k$-homomorphisms $F: M' \to M''$ is defined by $(aF)(m) = aF(m)$, where $m \in M'$. Note that the (obvious) verification of the fact that $aF$ is a $k$-homomorphism uses the commutativity of $k$.

We are mostly interested in the case of $M' = k[x]$, and especially in the case of $M' = k[x]$ and $M'' = k$, where $k[x]$ is considered as a $k$-module by forgetting about the multiplication of elements of $k[x]$, and the ring $k$ is considered as a module over itself.

**For the rest of this section $M$ denotes a fixed $k$-module.**

A pairing between $S_M$ and $k[x]$. Consider a sequence $s \in S_M$ and a polynomial

$$p(x) = \sum_{i=0}^{\infty} c_i x^i \in k[x].$$

Of course, the sum here is actually finite, i.e. $c_i = 0$ for all sufficiently large $i$. Let

$$\langle s, p(x) \rangle = \sum_{i=0}^{\infty} c_i s_i \in M.$$  

Since $c_i = 0$ for all sufficiently large $i$, the sum in the right hand side of this formula is well defined. The map

$$\langle \cdot, \cdot \rangle: S_M \times k[x] \to M$$

defined by

$$\langle \cdot, \cdot \rangle: (s, p(x)) \mapsto \langle s, p(x) \rangle$$

is our pairing between $k[x]$ and $S_M$. Obviously, it is a $k$-bilinear map (and hence indeed deserves to be called a pairing).

The pairing $\langle \cdot, \cdot \rangle$ defines a $k$-linear map

$$D_M: S_M \to \text{Hom}(k[x], M)$$

by the usual rule $D_M(s): p(x) \mapsto \langle s, p(x) \rangle$.  

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Note that, obviously, \( \langle s, x^i \rangle = s_i \) for every \( s \in \mathcal{S}_M, \ n \in \mathbb{N} \). Therefore
\[
(7) \quad s_i = D_M(s)(x^i)
\]
for every \( i \in \mathbb{N} \) and every \( s \in \mathcal{S}_M \).

2.1. **Theorem (Duality).** The pairing \( \langle \bullet, \bullet \rangle \) is non-degenerate in the sense that the map
\[
D_M: \mathcal{S}_M \rightarrow \text{Hom}(k[x], M)
\]
is an isomorphism of \( k \)-modules.

**Proof.** Note that \( k \)-homomorphism \( F: k[x] \rightarrow M \) is determined by its values \( F(x^i) \) on the monomials \( x^i, i \in \mathbb{N} \) (because every polynomial \( p(x) \in k[x] \) is a finite sum of powers \( x^i, i \in \mathbb{N} \) with coefficients in \( k \)). By (7) the terms \( s_i \) of a sequence \( s \in \mathcal{S}_M \) are equal to the values \( D_M(s)(x^i) \). It follows that all terms of \( s \), and hence the sequence \( s \) are determined by the homomorphism \( D_M(s) \). Therefore, the map \( D_M \) is injective.

In order to prove that \( D_M \) is surjective, let us consider an arbitrary \( k \)-homomorphism \( F: k[x] \rightarrow M \). Let \( s \in \mathcal{S}_M \) be the sequence with \( s_i = F(x^i) \). Then \( k \)-homomorphisms \( F \) and \( D_M(s) \) take the same values at all powers \( x^i \). It follows that \( F = D_M(s) \) (cf. the previous paragraph). Therefore, the map \( D_M \) is surjective. The theorem follows. \( \blacksquare \)

**Dual endomorphisms.** Each \( k \)-endomorphism \( E: k[x] \rightarrow k[x] \) defines its dual endomorphism
\[
E*: \text{Hom}(k[x], M) \rightarrow \text{Hom}(k[x], M)
\]
by the usual formula \( E^*(h) = h \circ E \), for all \( k \)-homomorphisms \( h: k[x] \rightarrow M \). Obviously, if \( E, F \) are two \( k \)-endomorphisms \( k[x] \rightarrow k[x] \), then \( (E \circ F)^* = F^* \circ E^* \).

**Adjoint endomorphisms.** Let \( M \) be a \( k \)-module. Since \( D_M \) is an isomorphism by Theorem 2.1, we can use \( D_M \) to turn the dual map
\[
E*: \text{Hom}(k[x], M) \rightarrow \text{Hom}(k[x], M)
\]
of an endomorphism \( E: k[x] \rightarrow k[x] \) into a map \( \mathcal{S}_M \rightarrow \mathcal{S}_M \). Namely, let
\[
E^\perp = (D_M)^{-1} \circ E^* \circ D_M: \mathcal{S}_M \rightarrow \mathcal{S}_M.
\]
Then \( D_M \circ E^\perp = E^* \circ D_M \). We will call \( E^\perp \) the adjoint endomorphism of \( E \).
For every pair $E, F: \mathbb{k}[x] \to \mathbb{k}[x]$ of $\mathbb{k}$-endomorphisms $(E \circ F)^\perp = F^\perp \circ E^\perp$. This immediately follows from the corresponding property $(E \circ F)^* = F^* \circ E^*$ of dual endomorphisms.

2.2. Lemma. Let $M$ be a $\mathbb{k}$-module, and let $E: \mathbb{k}[x] \to \mathbb{k}[x]$ be a $\mathbb{k}$-endomorphism. The adjoint map $\mathcal{S}_M \to \mathcal{S}_M$ is the unique map $E^\perp$ such that

\[(8) \quad \langle p, E^\perp(s) \rangle = \langle E(p), s \rangle \]

for all $p = p(x) \in \mathbb{k}[x], \; s \in \mathcal{S}_M$.

**Proof.** Let $p = p(x) \in \mathbb{k}[x]$ and let $s \in \mathcal{S}_M$. By the definition of $\mathcal{D}_M$ we have:

\[
\langle p, E^\perp(s) \rangle = \mathcal{D}_M(E^\perp(s))(p) = (\mathcal{D}_M \circ E^\perp(s))(p);
\]

\[
\langle E(p), s \rangle = \mathcal{D}_M(s)(E(p)) = E^*(\mathcal{D}_M(s))(p) = (E^* \circ \mathcal{D}_M(s))(p).
\]

Therefore, (8) is equivalent to $(\mathcal{D}_M \circ E^\perp)(p) = (E^* \circ \mathcal{D}_M(s))(p)$.

It follows that (8) holds for all $p = p(x) \in \mathbb{k}[x], \; s \in \mathcal{S}_M$ if and only if $\mathcal{D}_M \circ E^\perp = E^* \circ \mathcal{D}_M$. The lemma follows. ■

3. Adjoints of the left shift and of divided derivatives

As in the previous section, $M$ denotes a fixed $\mathbb{k}$-module.

The adjoint of the left shift operator. Let $L = \lambda^\perp$, where $\lambda$ is the left shift operator from Section 1. In view of the following lemma call $L$ also the left shift operator.

3.1. Lemma. For every sequence $s \in \mathcal{S}_M$ the terms of the sequence $L(s)$ are $(L(s))_i = s_{i+1}$.

**Proof.** Recall that $s_i = \langle s, x^i \rangle$ for any sequence $s \in \mathcal{S}_M$. Therefore by Lemma 2.2

\[
(L(s))_i = \langle L(s), x^i \rangle = \langle \lambda^\perp(s), x^i \rangle = \langle s, \lambda(x^i) \rangle = \langle s, x^{i+1} \rangle = s_{i+1}.
\]

The lemma follows. ■
3.2. **Corollary.** For every \( n \in \mathbb{N} \) and every sequence \( s \in \mathcal{S}_M \) the terms of the sequence \( L^n(s) \) are \( (L^n(s))_i = s_{i+n} \).

**Proof.** For \( n = 0 \) the corollary is trivial, because \( L^0 = \text{id} \). For \( n \geq 1 \) the corollary follows from Lemma 3.1, if we use an induction by \( n \).  

**The adjoints of the divided derivatives.** Let \( D^n = (\delta^n)^\perp \), where \( n \in \mathbb{N} \) and \( \delta^n \) is the \( n \)-th divided derivative operator from Section 1. Recall (see Section 1) that \( \delta^0: \mathbb{k}[x] \to \mathbb{k}[x] \) is the identity of \( \mathbb{k}[x] \). Therefore \( D^0: \mathcal{S}_M \to \mathcal{S}_M \) is also the identity of \( \mathcal{S}_M \).

Recall that in Theorem 1.6 we also introduced operator \( \delta^{-1} \). Let \( D^{-1} = (\delta^{-1})^\perp \). Since \( \delta^{-1} = 0 \) by the definition, we have \( D^{-1} = 0 \).

The following commutation relations are the most important for our purposes properties of the adjoint operators \( L = \lambda^\perp \) and \( D^n = (\delta^n)^\perp \).

3.3. **Theorem.** For every \( \alpha \in \mathbb{k} \) and every \( n \in \mathbb{N} \)

\[ L \circ D^n - D^n \circ L = D^{n-1} \quad \text{and} \quad (L - \alpha) \circ D^n - D^n \circ (L - \alpha) = D^{n-1}, \]

where we interpret \( \alpha \) as the operator \( \mathcal{S}_M \to \mathcal{S}_M \) of multiplication by \( \alpha \in \mathbb{k} \).

**Proof.** By taking the adjoint identity of the identity (4) from Theorem 1.6, we get \( (\delta^n \circ \lambda)^\perp - (\lambda \circ \delta^n)^\perp = (\delta^{n-1})^\perp \),

and hence \( \lambda^\perp \circ (\delta^n)^\perp - (\delta^n)^\perp \circ \lambda^\perp = (\delta^{n-1})^\perp \).

In view of the definitions of \( L \) and \( D^n \), this implies the first identity of the theorem.

Since \( D^n \) is a \( \mathbb{k} \)-linear operator, we have \( \alpha \circ D^n = D^n \circ \alpha \), where \( \alpha \) is interpreted as the multiplication operator. Clearly, the first identity of the theorem together with \( \alpha \circ D^n = D^n \circ \alpha \) implies the second one.  

**The sequences** \( s(\alpha) \) and \( s(\alpha, n) \). Let \( \alpha \in \mathbb{k} \) and \( n \in \mathbb{N} \). Let us define sequences \( s(\alpha) \) and \( s(\alpha, n) \) by \( s(\alpha)_i = \alpha^i \) and \( s(\alpha, n) = D^n(s(\alpha)) \).
Obviously, \( s(\alpha, 0) = s(\alpha) \). Note that \( s(\alpha) \neq 0 \) even if \( \alpha = 0 \), because \( s(\alpha)_0 = \alpha^0 = 1 \) by the definition for all \( \alpha \in k \).

For explicit formulas for sequences \( s(\alpha, n) \) with \( n \geq 1 \) the reader is referred to Theorem 3.6 below. No such formulas are used in this paper.

3.4. Lemma. Let \( \alpha \in k \) and \( s \in S_M \). Then \( (L-\alpha)(s) = 0 \) if and only if \( s \) has the form \( s = \beta s(\alpha) \), where \( \beta \in k \).

Proof. The condition \( (L-\alpha)(s) \) is equivalent to \( L(s) = \alpha s \). The latter condition holds if and only if \( (L(s))_n = \alpha s_n \) for all \( n \in N \). By Lemma 3.1 \( (L(s))_n = s_{n+1} \). Therefore, \( (L-\alpha)(s) = 0 \) if and only if \( s_{n+1} = \alpha s_n \) for all \( n \in N \). An application of the induction completes the proof. ■

3.5. Lemma. Suppose that \( a \in N \), \( \alpha \in k \), and \( s \in S_M \). If \( n \geq a \), then

\[
(L-\alpha)^a(s(\alpha, n)) = s(\alpha, n-a).
\]

Proof. The lemma is trivial if \( a = 0 \). Let us prove the lemma for \( a = 1 \). In view of the definition of sequences \( s(\alpha, n) \), we need to prove that

\[
(L-\alpha)(D^n(s(\alpha))) = D^{n-1}(s(\alpha)).
\]

By applying the second identity of Theorem 3.3 to \( s(\alpha) \), we get

\[
(L-\alpha) \circ D^n(s(\alpha)) - D^n \circ (L-\alpha)(s(\alpha)) = D^{n-1}(s(\alpha)),
\]

which is equivalent to

\[
(L-\alpha)(D^n(s(\alpha))) - D^n((L-\alpha)(s(\alpha))) = D^{n-1}(s(\alpha)).
\]

Since \( (L-\alpha)(s(\alpha)) = 0 \) by Lemma 3.4, we see that

\[
(L-\alpha)(D^n(s(\alpha))) = D^{n-1}(s(\alpha)),
\]

i.e. \( (L-\alpha)(s(\alpha, n)) = s(\alpha, n-1) \). This proves the lemma for \( a = 1 \). The general case follows from this one by induction. ■

The following theorem is not used in the rest of the paper.
3.6. **Theorem.** Let \( n \in \mathbb{N} \). For every sequence \( s \in \mathcal{S}_M \) the terms of the sequence \( D^n(s) \) are
\[
(D^n(s))_i = (i-n|n)s_{i-n}.
\]
In addition, for every \( \alpha \in k \) the terms of the sequence \( s(\alpha, n) \) are
\[
(s(\alpha, n))_i = (i-n|n)\alpha^{i-n}.
\]

**Proof.** Recall that \( s_i = \langle s, x^i \rangle \) for any sequence \( s \in \mathcal{S}_M \). Together with Lemma 2.2 this fact implies that
\[
(D^n(s))_i = \langle D^n(s), x^i \rangle = \langle (\delta^n)\bot(s), x^i \rangle = \langle s, \delta^n(x^i) \rangle.
\]
Since \( \delta^n(x^i) = (i-n|n)x^{i-n} \) by Lemma 1.5, we have
\[
\langle s, D^n(x^i) \rangle = \langle s, (i-n|n)x^{i-n} \rangle = (i-n|n)\langle s, D^n(x^{i-n}) \rangle = (i-n|n)s_{i-n}.
\]
The first part of the theorem follows. Let us apply the first part to \( s = s(\alpha) \). We get
\[
s(\alpha, n) = (D^n(s(\alpha)))_i = (i-n|n)s(\alpha)_{i-n} = (i-n|n)\alpha^{i-n}.
\]
This proves the second part of the theorem. ■

4. Endomorphisms and their eigenvalues

**Representation of the polynomial algebra defined by an endomorphism.** Let \( x \) be a variable, and let \( k[x] \) be the \( k \)-algebra of polynomials in \( x \) with coefficients in \( k \). Let \( M \) be a \( k \)-module. The \( k \)-endomorphisms \( M \rightarrow M \) form a \( k \)-algebra \( \text{End} M \) with the composition as the multiplication. For every \( k \)-endomorphisms \( E: M \rightarrow M \) and every \( \alpha \in \mathbb{N} \) we will denote by \( E^\alpha \) the \( \alpha \)-fold composition \( E \circ E \circ \ldots \circ E \). As usual, we interpret the 0-fold composition \( E^0 \) as the identity endomorphism \( \text{id} \in \text{End} M \).
For a \( k \)-module endomorphism \( E : M \to M \) and a polynomial

\[
f(x) = c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n \in k[x]
\]

one can define an endomorphism \( f(E) : M \to M \) by the formula

\[
f(E) = c_0 E^n + c_1 E^{n-1} + c_2 E^{n-2} + \ldots + c_n.
\]

The map \( f(x) \mapsto f(E) \) is a homomorphism \( k[x] \to \text{End} M \) of \( k \)-algebras. This follows from the obvious identities \( x^a x^b = x^{a+b} \) and \( E^a \circ E^b = E^{a+b} \). This homomorphism defines a structure of \( k[x] \)-module on \( M \). Of course, this structure depends on \( E \).

We will denote by \( k[E] \) the image of the homomorphism \( f(x) \mapsto f(E) \). Since \( k[x] \) is commutative, the image \( k[E] \) is a commutative subalgebra of \( \text{End} M \).

**Eigenvalues.** Suppose that a \( k \)-module endomorphism \( E : M \to M \) is fixed.

Let \( \alpha \in k \). The kernel \( \text{Ker}(E - \alpha) \) is called the eigenmodule of \( E \) corresponding to \( \alpha \) and is denoted also by \( E_\alpha \). Clearly, \( E_\alpha \) is a \( k \)-submodule of \( M \). An element \( \alpha \in k \) is called an eigenvalue of \( E \) if the kernel \( E_\alpha = \text{Ker}(E - \alpha) \neq 0 \).

The set of elements \( v \in M \) such that \( (E - \alpha)^i(v) = 0 \) for some \( i \in \mathbb{N} \) is called the extended eigenmodule of \( E \) corresponding to \( \alpha \) and is denoted by \( \text{Nil}(\alpha) \). Clearly, \( \text{Nil}(\alpha) \) is a \( k \)-submodule of \( M \).

**4.1. Lemma.** Let \( \alpha \in k \). Then the following statements hold.

(i) The submodules \( E_\alpha \) and \( \text{Nil}(\alpha) \) are \( E \)-invariant.

(ii) \( E_\alpha \) and \( \text{Nil}(\alpha) \) are \( k[x] \)-submodules of \( M \).

(iii) The submodule \( \text{Nil}(\alpha) \) is non-zero if and only if \( \alpha \) is an eigenvalue.

**Proof.** Let us prove (i), (ii) first. Note that \( (E - \alpha)^i \circ E = E \circ (E - \alpha)^i \) for every \( i \in \mathbb{N} \), because \( k[E] \) is a commutative subalgebra of \( \text{End} M \). Therefore, if \( (E - \alpha)^i(v) = 0 \), then

\[
(E - \alpha)^i(E(v)) = ((E - \alpha)^i \circ E)(v) = (E \circ (E - \alpha)^i)(v) = E((E - \alpha)^i(v)) = 0.
\]

In the case \( i = 1 \) this implies that \( E(E_\alpha) \subset E_\alpha \). In general, this implies that

\[
E(\text{Ker}(E - \alpha)^i) \subset \text{Ker}(E - \alpha)^i,
\]

and hence \( E(\text{Nil}(\alpha)) \subset \text{Nil}(\alpha) \). This proves (i), and (ii) immediately follows.
Finally, let us prove (iii). Suppose that \( v \neq 0 \) and \( (E - \alpha)^i(v) = 0 \). Let \( i \) be the smallest integer such that \( (E - \alpha)^i(v) = 0 \). Note that \( i > 0 \) because \( v \neq 0 \). Let \( w = (E - \alpha)^{i-1}(v) \). Then \( w \neq 0 \) and \( (E - \alpha)(w) = 0 \). Therefore \( E_\alpha = \ker (E - \alpha) \neq 0 \). This proves (iii). ■

**Torsion free modules.** A \( \mathbb{k} \)-module \( M \) is called **torsion-free**, if \( \alpha \cdot m = 0 \) implies that either \( \alpha = 0 \), or \( m = 0 \), where \( \alpha \in \mathbb{k} \) and \( m \in M \). Since \( \mathbb{k} \) is assumed to be a ring without zero divisors, \( \mathbb{k}^n \) is a torsion free module for any non-zero \( n \in \mathbb{N} \).

**For the rest of this section we will assume that \( M \) is a torsion-free module.**

**4.2. Lemma.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be distinct eigenvalues of an endomorphism \( E : M \to M \). Let \( \ker_1, \ker_2, \ldots, \ker_n \) be the corresponding eigenmodules, i.e. \( \ker_i = \ker (E - \alpha_i) \) for each \( i = 1, 2, \ldots, n \). Then the sum of these eigenmodules is a direct sum, i.e. an element

\[
v \in \ker_1 + \ker_2 + \ldots + \ker_n
\]

admits only one presentation \( v = v_1 + v_2 + \ldots + v_n \) with \( v_i \in \ker_i \) for all \( i = 1, 2, \ldots, n \).

**Proof.** It is sufficient to prove that if \( v_1 + v_2 + \ldots + v_n = 0 \) and \( v_i \in \ker_i \) for all \( i \), then \( v_1 = v_2 = \ldots = v_n = 0 \). Suppose that \( v_1 + v_2 + \ldots + v_n = 0 \), \( v_i \in \ker_i \) for all \( i \), and not all \( v_i \) are equal to \( 0 \). Consider the maximal integer \( m \) such that

\[
(9) \quad v_1 + v_2 + \ldots + v_m = 0
\]

for some elements \( v_i \in \ker_i \) such that \( v_m \neq 0 \). Note that in this case \( v_i \neq 0 \) also for some \( i \leq m - 1 \), in view of (9). By applying \( E - \alpha_m \) to (9), we get

\[
(E - \alpha_m)(v_1) + \ldots + (E - \alpha_m)(v_{m-1}) + (E - \alpha_m)(v_m) = 0.
\]

Since \( v_i \in \ker_i = \ker(E - \alpha_i) \) and therefore \( E(v_i) = \alpha_i v_i \) for all \( i \), we see that

\[
(10) \quad (\alpha_1 - \alpha_m)v_1 + \ldots + (\alpha_{m-1} - \alpha_m)v_{m-1} + (\alpha_m - \alpha_m)v_m = 0,
\]

\[
(11) \quad (\alpha_1 - \alpha_m)v_1 + \ldots + (\alpha_{m-1} - \alpha_m)v_{m-1} = 0.
\]

Since the eigenvalues \( \alpha_i \) are distinct, \( \alpha_i - \alpha_m \neq 0 \) for \( i \leq m - 1 \). Since our module \( M \) is assumed to be torsion-free, this implies that \( (\alpha_i - \alpha_m)v_i \neq 0 \) if \( i \leq m - 1 \) and \( v_i \neq 0 \). As we noted above, \( v_i \neq 0 \) for some \( i \leq m - 1 \). Therefore, the equality (11) contradicts to the choice of \( m \). This contradiction proves the lemma. ■
4.3. **Lemma.** Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be distinct eigenvalues of an endomorphism $E: M \to M$. Let $\text{Nil}_1, \text{Nil}_2, \ldots, \text{Nil}_n$ be the corresponding extended eigenmodules, i.e. $\text{Nil}_i = \text{Nil}(\alpha_i)$ for $i = 1, 2, \ldots, n$. Then the sum of these extended eigenmodules is a direct sum, i.e. an element $v \in \text{Nil}_1 + \text{Nil}_2 + \ldots + \text{Nil}_n$ admits only one presentation $v = v_1 + v_2 + \ldots + v_n$ with $v_i \in \text{Nil}_i$ for all $i = 1, 2, \ldots, n$.

**Proof.** It is sufficient to prove that if $v_1 + v_2 + \ldots + v_n = 0$ and $v_i \in \text{Nil}_i$ for all $i$, then $v_1 = v_2 = \ldots = v_n = 0$. Suppose that $v_1 + v_2 + \ldots + v_n = 0$, $v_i \in \text{Nil}_i$ for all $i$, and not all $v_i$ are equal to 0. The proof proceeds by replacing, in several steps (no more than $n$), the original elements $v_i$ by new ones in such a way that eventually not only $v_i \in \text{Nil}_i$, but, moreover, $v_i \in \text{Ker}_i = \text{Ker}(E - \alpha_i)$, and still not all $v_i$ are equal to 0. Obviously, this will contradict to Lemma 4.2.

Let $E_i = E - \alpha_i$ for all $i = 1, 2, \ldots, n$. If $i = 1, 2, \ldots, n-1$, or $n$, then $E_i^p(v_i) \neq v_i$ and $E_i^q(v_i) = 0$ for some integer $a \geq 1$. If $v_i \neq 0$, then we define $a_i$ as the largest integer $\alpha \geq 0$ such that $E_i^a(v_i) \neq 0$. Then $E_i^{a_i}(v_i) \neq 0$ and $E_i^{a_i+1}(v_i) = 0$. In particular,

$$(E - \alpha_i)(E_i^{a_i}(v_i)) = E_i(E_i^{a_i}(v_i)) = E_i^{a_i+1}(v_i) = 0,$$

and hence $E_i^{a_i}(v_i) \in \text{Ker}_i$. If $v_i = 0$, then we set $a_i = 0$ and $E_i^{a_i}(v_i) \in \text{Ker}_i$ is still true.

Let us fix an integer $k$ between 1 and $n$. Let $w_i = E_k^{a_k}(v_i)$, where $i = 1, 2, \ldots, n$. By applying $E_k^{a_k}$ to $v_1 + v_2 + \ldots + v_n = 0$, we conclude $w_1 + w_2 + \ldots + w_n = 0$. Note that since the submodules $\text{Nil}_i$ are $E$-invariant by Lemma 4.1, $w_i \in \text{Nil}_i$ for every $i$.

**Claim 1.** If $v_i \neq 0$, then $w_i = 0$.

**Proof of Claim 1.** If $i = k$ and $v_i = v_k \neq 0$, then $w_i = w_k = E_k^{a_k}(v_k) \neq 0$ by the choice of $a_k$. Suppose that $i \neq k$ and $v_i \neq 0$. Then

$$E_i^{a_i}(w_i) = E_i^{a_i}(E_k^{a_k}(v_i)) = E_k^{a_k}(E_i^{a_i}(v_i))$$

But $E_i^{a_i}(v_i) \in \text{Ker}_i$ and $E_i^{a_i}(v_i) \neq 0$ by the choice of $a_i$. Since $E$ acts of $\text{Ker}_i$ as the multiplication by $\alpha_i$, we have

$$E_k^{a_k}(E_i^{a_i}(v_i)) = (E - \alpha_k)^{a_k}(E_i^{a_i}(v_i)) = (\alpha_i - \alpha_k)^{a_k}(E_i^{a_i}(v_i)).$$

Since $\alpha_i \neq \alpha_k$ and $k$ is a ring without zero divisors, $(\alpha_i - \alpha_k)^{a_k} \neq 0$. Since $M$ is a torsion free $k$-module and $E_i^{a_i}(v_i) \neq 0$, this implies that $(\alpha_i - \alpha_k)^{a_k}(E_i^{a_i}(v_i)) \neq 0$. It follows that

$$E_i^{a_i}(w_i) = (\alpha_i - \alpha_k)^{a_k}(E_i^{a_i}(v_i)) \neq 0,$$

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and hence \( w_i \neq 0 \). This completes the proof of the claim. \( \square \)

**Claim 2.** If \( v_i \in \text{Ker}_i \), then \( w_i \in \text{Ker}_i \).

**Proof of Claim 2.** Suppose that \( v_i \in \text{Ker}_i \), i.e. \( E_i(v_i) = 0 \). Since \( E_i = E - \alpha_i \) and \( E_k = E - \alpha_k \) obviously commute, it follows that

\[
E_i(w_i) = E_i(E_k^{a_k}(v_i)) = E_k^{a_k}(E_i(v_i)) = 0.
\]

This proves the claim. \( \square \)

To sum up, we see that by applying \( E_k^{a_k} \) to the equality \( v_1 + v_2 + \ldots + v_n = 0 \) with \( v_i \in \text{Nil}_i \) for all \( i \), we get another equality \( w_1 + w_2 + \ldots + w_n = 0 \) such that for all \( i \):

(i) \( w_i \in \text{Nil}_i \);
(ii) if \( v_i \neq 0 \), then \( w_i \neq 0 \);
(iii) if \( v_i \in \text{Ker}_i \), then \( w_i \in \text{Ker}_i \).

In addition, \( w_k = E_k^{a_k}(v_k) \in \text{Ker}_k \) even if \( v_k \) did not belonged to the eigenmodule \( \text{Ker}_k \). Therefore, we can take \( w_1, w_2, \ldots, w_n \) as the new elements \( v_1, v_2, \ldots, v_n \), increasing the number of elements belonging to the corresponding eigenmodules by an appropriate choice of \( k \) (if some \( v_i \) did not belonged to eigenmodules yet).

It follows that by starting with the equality \( v_1 + v_2 + \ldots + v_n = 0 \) and consecutively applying endomorphisms \( E_k^{a_k} \) for \( k = 1, 2, \ldots, n \), we will eventually prove the equality \( v_1 + v_2 + \ldots + v_n = 0 \) for some new vectors \( v_i \) such that \( v_i \in \text{Ker}_i \) for all \( i \), and still not all \( v_i \) are equal to \( 0 \). The contradiction with Lemma 4.2 completes the proof. \( \blacksquare \)

**4.4. Lemma.** Let \( E : \mathcal{M} \to \mathcal{M} \) be an endomorphism of \( \mathcal{M} \) and let \( \alpha \) be an eigenvalue of \( E \). Suppose that \( v \in \text{Nil}(\alpha) \). Let \( a \geq 0 \) be the largest integer such that \( (E - \alpha)^{a+1}(v) \neq 0 \), and let \( v_i = (E - \alpha)^{i}(v) \) for \( i = 0, 1, \ldots, a \). Then the homomorphism \( k^{a+1} \to \mathcal{M} \) defined by

\[
(x_0, x_1, \ldots, x_a) \mapsto x_0v_0 + x_1v_1 + \ldots + x_av_a
\]

is an isomorphism onto its image. In particular, \( v_0, v_1, \ldots, v_a \) are free generators of a free \( k \)-submodule of \( \mathcal{M} \).

**Proof.** It is sufficient to prove that our homomorphism is injective. In other terms, it is sufficient to prove that if

\[
(12) \quad x_0v_0 + x_1v_1 + \ldots + x_av_a = 0
\]
for some $x_0, x_1, \ldots, x_a \in k$, then $x_0 = x_1 = \ldots = x_a = 0$. Suppose that (12) holds and $x_i \neq 0$ for some $i$. Let $b \in N$ be the minimal integer with the property $x_b \neq 0$. Let us apply $(E-\alpha)^{a-b}$ to (12). Note that if $i > b$, then

$$(E-\alpha)^{a-b}(v_i) = (E-\alpha)^{a-b}(v_i) \neq (E-\alpha)^{a-b+i}(v) = 0$$

because $a-b+i > a$ and $(E-\alpha)^{n}(v) = 0$ for $n > a$ by the choice of $a$. Therefore, the operator $(E-\alpha)^{a-b}$ takes the left hand side of (12) to

$$x_b(E-\alpha)^{a-b}(v_b) = x_b(E-\alpha)^{a-b}(v) = x_b(E-\alpha)^{a-b+b}(v) = x_b(E-\alpha)^{a}(v),$$

and hence the result of application of $(E-\alpha)^{a-b}$ to (12) is

$$(13) \ x_b(E-\alpha)^{a}(v) = 0.$$ 

But $(E-\alpha)^{a}(v) \neq 0$ by the choice of $a$, and $x_b \neq 0$ by the choice of $b$. Since the module $M$ is assumed to be torsion free, these facts together with (13) lead to a contradiction. This contradiction shows that (12) may be true only if $x_i = 0$ for all $i$. ■

5. Torsion modules and a property of free modules

**Torsion modules.** An element $m \in M$ of a $k$-module $M$ is called a torsion element if $x \cdot m = 0$ for some non-zero $x \in k$. A $k$-module $M$ is called a torsion module if every element of $M$ is a torsion element.

**5.1. Lemma.** Let $n \in N$. If $M$ is a $k$-submodule of a $k$-module $N$ and both $M$ and $N$ are isomorphic to $k^n$, then the quotient $N/M$ is a torsion module.

**Proof.** Suppose that $N/M$ is not a torsion module. Then there is an element $v \in N/M$ such that $\alpha \cdot v \neq 0$ if $\alpha \neq 0$. For such a $v$ the map $\alpha \mapsto \alpha \cdot v$ is an injective homomorphism of $k$-modules $k \to N/M$. Let us lift $v \in N/M$ to an element $v_0 \in N$, so $v$ is the image of $v_0$ under the canonical surjection $N \to N/M$. Then the map $\alpha \mapsto \alpha \cdot v_0$ is an injective homomorphism of $k$-modules $k \to N$.

Clearly, if $v_1, v_2, \ldots, v_n$ is a basis of $M$ (which exists because $M$ is isomorphic to $k^n$), then $v_0, v_1, \ldots, v_n$ is a basis of $kv_0 + M$. Therefore, $kv_0 + M$ is a submodule isomorphic to $k^{n+1}$ of the module $N$ isomorphic to $k^n$. In particular, there exist an injective $k$-homomorphism $J : k^{n+1} \to k^n$. 

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Since \( k \) has no zero divisors, it can be embedded into its field of fractions, which we will denote by \( F \). Moreover, the \( k \)-homomorphism \( k^{n+1} \to k^n \) extends to an \( F \)-linear map \( F^{n+1} \to F^n \), which we will denote by \( J_F \).

**Claim.** \( J_F \) is injective.

**Proof of the claim.** Suppose \((y_0, y_1, \ldots, y_n) \in F^{n+1}\) is non-zero and belongs to the kernel of \( J_F \). Since \( F \) is the field of fractions of \( k \), there is an element \( z \in k \) such that \( zy_0, zy_1, \ldots, zy_n \in k \). For such an element \( z \in k \) the \((n+1)\)-tuple \((zy_0, zy_1, \ldots, zy_n)\) belongs to \( k^{n+1} \), and

\[
J(zy_0, zy_1, \ldots, zy_n) = J_F(zy_0, zy_1, \ldots, zy_n) = z J_F(y_0, y_1, \ldots, y_n) = z 0 = 0.
\]

Since \( F \) is the field of fractions of \( k \), \((y_0, y_1, \ldots, y_n) \neq 0\) implies that the \((n+1)\)-tuple \((zy_0, zy_1, \ldots, zy_n) \neq 0\). At the same time this \((n+1)\)-tuple belongs to the kernel of \( J_F \), in contradiction with the injectivity of \( J_F \). The claim follows. \( \square \)

As is well known, for a field \( F \) there are no injective \( F \)-linear maps \( F^{n+1} \to F^n \). The contradiction with the above claim proves that \( N/M \) is indeed a torsion module. \( \blacksquare \)

### 6. Polynomials and their roots

**6.1. Lemma.** Let \( p(x) \in \k[x] \) be a polynomial with leading coefficient 1, and let \( \alpha \in \k \). Then \( \alpha \in \k \) is a root of \( p(x) \) if and only if

\[
(14) \quad p(x) = (x - \alpha)q(x)
\]

for some polynomial \( q(x) \in \k[x] \) with leading coefficient 1. If \( \alpha \) is a root, then \( q(x) \) is uniquely determined by \((14)\).

**Proof.** Suppose that \( p(x) = (x - \alpha)q(x) \) and both \( p(x), q(x) \) have the leading coefficients 1. Then \( \deg q(x) = \deg p(x) - 1 \) and the polynomials \( p(x), q(x) \) have the form

\[
p(x) = x^n + c_1 x^{n-1} + \ldots + a_{n-1} x + c_n,
q(x) = x^{n-1} + d_1 x^{n-2} + \ldots + d_{n-2} x + d_{n-1},
\]

where \( c_1, \ldots, c_n, d_1, \ldots, d_{n-1} \in \k \). Let us compute the product

\[
(x - \alpha)q(x) = (x - \alpha)(x^{n-1} + \ldots + d_{n-2} x + d_{n-1}).
\]
Obviously,

\[(x - \alpha)q(x) = x^n + d_1x^{n-1} + \ldots + d_{n-2}x^2 + d_{n-1}x - \alpha x^{n-1} - \ldots - \alpha d_{n-3}x^2 - \alpha d_{n-2}x - \alpha d_{n-1}.\]

It follows that \(p(x) = (x - \alpha)q(x)\) if and only if

\[
\begin{align*}
    c_1 &= d_1 - \alpha \\
    c_2 &= d_2 - \alpha d_1 \\
    \vdots &= \ldots \\
    c_{n-1} &= d_{n-1} - \alpha d_{n-2} \\
    c_n &= -\alpha d_{n-1},
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
    d_1 &= \alpha + c_1 \\
    d_2 &= \alpha d_1 + c_2 \\
    \vdots &= \ldots \\
    d_{n-1} &= \alpha d_{n-2} + c_{n-1} \\
    \alpha d_{n-1} + c_n &= 0.
\end{align*}
\]

These equalities allow to compute the coefficients \(d_1, d_2, \ldots, d_{n-1}\) in terms of the coefficients \(c_1, c_2, \ldots, c_{n-1}\). Namely, \(d_1 = \alpha + c_1\) and

\[
    d_i = \alpha^i + c_1\alpha^{i-1} + c_2\alpha^{i-2} + \ldots + c_{i-1}\alpha + c_i
\]

for \(2 \leq i \leq n-1\). Therefore, the last equality \(\alpha d_{n-1} + c_n = 0\) holds if and only if

\[
    \alpha(\alpha^{n-1} + c_1\alpha^{n-2} + \ldots + c_{n-1}\alpha) + c_n = 0,
\]

i.e. if and only if \(p(\alpha) = 0\). The lemma follows. \(\blacksquare\)

6.2. **Corollary.** Let \(p(x) \in k[x]\) be a polynomial with leading coefficient \(1\), and let \(\alpha \in k\). Then there is a number \(m \in \mathbb{N}\) and a polynomial \(r(x) \in k[x]\) such that \(p(x) = (x - \alpha)^m r(x)\) and \(\alpha\) is not a root of \(r(x)\). The number \(m\) and the polynomial \(r(x)\) are uniquely determined by \(p(x)\) and \(\alpha\).

**Proof.** If \(p(\alpha) \neq 0\), then, obviously, \(m = 0\) and \(r(x) = p(x)\). If \(p(\alpha) = 0\), we can apply Lemma 6.1. If \(q(\alpha) \neq 0\), then \(m = 1\), \(r(x) = q(x)\) and we are done. If \(q(\alpha) = 0\), then we can apply Lemma 6.1 again. Eventually we will get a presentation \(p(x) = (x - \alpha)^m r(x)\) such that \(r(\alpha) \neq 0\). By consecutively applying the uniqueness part of Lemma 6.1, we
see that \((x-\alpha)^{m-1}\tau(x)\), \((x-\alpha)^{m-2}\tau(x)\), \ldots, and, eventually, \(m\) and \(\tau(x)\) are uniquely determined by \(p(x)\) and \(\alpha\). ■

**The multiplicity of a root.** We will denote by \(\deg p(x)\) the degree of the polynomial \(p(x)\). If \(\alpha\) is a root of \(p(x)\), then the number \(m\) from Corollary 6.2 is called the **multiplicity** of the root \(\alpha\).

**6.3. Corollary.** Let \(p(x) \in \mathbb{k}[x]\) be a polynomial with leading coefficient \(1\). The number \(k\) of distinct roots of \(p(x)\) is finite and \(k \leq \deg p(x)\). If \(\alpha_1, \alpha_2, \ldots, \alpha_k\) is the list of all distinct roots of \(p(x)\), and if \(\mu_1, \mu_2, \ldots, \mu_k\) are the respective multiplicities of these roots, then

\[ p(x) = (x - \alpha_1)^{\mu_1} (x - \alpha_2)^{\mu_2} \ldots (x - \alpha_k)^{\mu_k} r(x), \]

where \(r(x) \in \mathbb{k}[x]\) has no roots in \(\mathbb{k}\). The polynomial \(r(x)\) is uniquely determined by \(p(x)\).

**Proof.** By consecutively applying the existence part of the Corollary 6.2, we see that a factorization of the form (15) exists. Similarly, the uniqueness of \(r(x)\) follows from the uniqueness part of Corollary 6.2. ■

**Polynomials with all roots in \(\mathbb{k}\).** Again, let \(p(x) \in \mathbb{k}[x]\) be a polynomial with leading coefficient \(1\). We say that \(p(x)\) **has all roots in \(\mathbb{k}\)**, if in the factorization (15) the polynomial \(r(x) = 1\). In other words, \(p(x)\) **has all roots in \(\mathbb{k}\)**, if \(p(x)\) has the form

\[ p(x) = (x - \alpha_1)^{\mu_1} (x - \alpha_2)^{\mu_2} \ldots (x - \alpha_k)^{\mu_k}, \]

for some \(\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{k}\) and some non-zero \(\mu_1, \mu_2, \ldots, \mu_k \in \mathbb{N}\). Obviously, then

\[ \deg p(x) = \mu_1 + \mu_2 + \ldots + \mu_k. \]

**7. The main theorems**

Let us fix for the rest of this section a polynomial

\[ p(x) = x^n + c_1 x^{n-1} + \ldots + c_{n-1} x + c_n \in \mathbb{k}[x] \]

with the leading coefficient \(1\).

Consider the left shift operator \(L: \mathcal{G}_\mathbb{k} \to \mathcal{G}_\mathbb{k}\) from Section 2. As it was explained in Section 4, the operator \(L\) defines a homomorphism of \(\mathbb{k}\)-algebras \(\mathbb{k}[x] \to \text{End} \mathcal{G}_\mathbb{k}\) by the rule \(f(x) \mapsto f(L)\). We are interested in the kernel \(\ker p(L)\).
7.1. Lemma. A sequence \( s \in \mathcal{S}_k \) belongs to \( \ker p(L) \) if and only if

\[
\tag{16}
\sum_{i} c_i s_{i-1} + \ldots + c_{n-1} s_{i-n+1} + c_n s_{i-n} = 0
\]

for all \( i \in \mathbb{N}, \ i \geq n. \)

**Proof.** Let us compute the terms of \( p(L)(s) \) using Corollary 3.2 at the last step:

\[
(p(L)(s))_i = (L^n + c_1 L^{n-1} + \ldots + c_n)(s)_i = (L^n(s))_i + c_1 (L^{n-1}(s))_i + \ldots + c_n (s)_i.
\]

This calculation shows that \( s \in \ker f(L) \) if and only if

\[
\tag{17}
\sum_{i} c_i s_{i+n} + \ldots + c_{n-1} s_{i+1} + c_n s_i = 0
\]

for all integers \( i \geq 0. \) Clearly, (17) holds for all integers \( i \geq 0 \) if and only if (16) holds for all integers \( i \geq n. \) The lemma follows. ■

**Remark.** Classically, a sequence \( s \in \mathcal{S}_k \) is called **recurrent** if its terms satisfy (16) for all \( i \in \mathbb{N}, \ i \geq n, \) and the equation (16) is called a **linear recurrence relation.** This explains the title of the paper.

7.2. Lemma. The map \( F: \ker p(L) \to \mathbb{k}^n \) defined by

\[
F: s \mapsto (s_0, s_1, \ldots, s_{n-1}) \in \mathbb{k}^n
\]

is an isomorphism. In particular, \( \ker p(L) \) is a free \( \mathbb{k} \)-module of rank \( n. \)

**Proof.** By Lemma 7.1 the kernel \( \ker p(L) \) is equal to the \( \mathbb{k} \)-submodule of \( \mathcal{S}_k \) consisting of sequences \( s \) satisfying the relation (16) for all \( i \in \mathbb{N}, \ i \geq n. \) Clearly, (16) allows to compute each term \( s_i, i \geq n \) of \( s \) as the linear combination of \( n \) immediately preceding terms \( s_{i-1}, s_{i-2}, \ldots, s_{i-n} \) of \( s \) with coefficients \( c_1, c_2, \ldots, c_n \) independent of \( i. \) Therefore, such a sequence \( s \) is determined by its first \( n \) terms \( s_0, s_1, \ldots, s_{n-1}. \) Moreover, these \( n \) terms can be prescribed arbitrarily. The lemma follows. ■

7.3. Theorem. Suppose that \( \alpha \in \mathbb{k} \) is a root of \( p(x) \) of multiplicity \( n. \) Then \( p(L)(s(\alpha, a)) = 0 \) for each \( a \in \mathbb{N}, \ 0 \leq a \leq n-1, \) where \( s(\alpha, a) \) are the sequences defined in Section 3, the paragraph immediately preceding Lemma 3.4.
Proof. By Corollary 6.2, \( p(x) \) has the form \( p(x) = (x - \alpha)^{\mu} q(x) \). Therefore
\[
(18) \quad p(L) = (L - \alpha)^{\mu} q(L) = q(L)(L - \alpha)^{\mu}.
\]
If \( a \leq \mu - 1 \), then \( \mu - 1 - a \geq 0 \), and we can present \( (L - \alpha)^{\mu} \) as the following product:
\[
(19) \quad (L - \alpha)^{\mu} = (L - \alpha)^{\mu - 1 - a}(L - \alpha)(L - \alpha)^{a}.
\]
By Lemma 3.5
\[
(20) \quad (L - \alpha)^{a} (s(\alpha, a)) = s(\alpha, a - a) = s(\alpha, 0) = s(\alpha),
\]
and by Lemma 3.4,
\[
(21) \quad (L - \alpha) (s(\alpha)) = 0.
\]
By combining (19), (20), and (21), we see that
\[
(18) \quad (L - \alpha)^{\mu} (s(\alpha, a)) = 0.
\]
By combining the last equality with (18), we get
\[
p(L)(s(\alpha, a)) = q(L)(L - \alpha)^{\mu} (s(\alpha, a)) = q(L)(0) = 0.
\]
The theorem follows. ■

7.4. Theorem. Suppose that \( p(x) \) has all roots in \( \mathbb{k} \). Let \( k \) be the number of distinct roots of \( p(x) \), let \( \alpha_1, \alpha_2, \ldots, \alpha_k \) be these roots, and let \( \mu_1, \mu_2, \ldots, \mu_k \) be, respectively, the multiplicities of these roots. Then sequences \( s(\alpha_u, a) \), where \( 1 \leq u \leq k \) and \( 0 \leq a \leq \mu_u - 1 \), are free generators of a free \( \mathbb{k} \)-submodule of \( \text{Ker} f(L) \).

Proof. By Theorem 7.3, all these sequences belong to \( \text{Ker} p(L) \). In particular, they are generators of a \( \mathbb{k} \)-submodule of \( \text{Ker} p(L) \). Let us prove that they are free generators.

Let \( 1 \leq u \leq k \). By Lemma 3.5,
\[
(22) \quad (L - \alpha_u)^{\mu_u - 1} (s(\alpha_u, \mu_u - 1)) = s(\alpha_u, (\mu_u - 1) - (\mu_u - 1)) = s(\alpha_u, 0) = s(\alpha_u).
\]
By Lemma 3.4, \( (L - \alpha_u)(s(\alpha_u)) = 0 \) and hence
\[
(L - \alpha_u)^{\mu_u} s(\alpha_u, \mu_u - 1) = (L - \alpha_u)(s(\alpha)) = 0.
\]
In particular, \( s(\alpha_u, \mu_u - 1) \) belongs to the extended eigenmodule \( \text{Nil}(\alpha_u) \).
In addition, (22) together with the fact that \( s(\alpha_u) \neq 0 \) implies that \( \mu_u - 1 \) is the largest integer \( a \) such that

\[
(L - \alpha_u)^a(s(\alpha_u, \mu_u - 1)) \neq 0.
\]

Lemma 4.4 implies that the sequences \( (L - \alpha_u)^a(s(\alpha, \mu_u)) \) for \( a = 0, 1, \ldots, \mu_u - 1 \) form a basis of a free submodule of \( \text{Nil}(\alpha_u) \subset \text{Ker} p(L) \). Since

\[
(L - \alpha_u)^a(s(\alpha_u, \mu_u)) = s(\alpha_u, \mu_u - a)
\]

by Lemma 3.5, this implies that the sequences \( s(\alpha_u, a) \) for \( a = 0, 1, \ldots, \mu_u - 1 \) form a basis of a free submodule of \( \text{Nil}(\alpha_u) \subset \text{Ker} p(L) \). By combining this result with Lemma 4.3, we see that the sequences \( s(\alpha_u, a) \) from the theorem form a basis of a free submodule of \( \text{Ker} p(L) \). This completes the proof. ■

**7.5. Theorem.** Let \( S \subset \text{Ker} p(L) \) be the free \( k \)-module generated by the sequences \( s(\alpha_u, a) \) from Theorem 7.4. Then the quotient \( k \)-module \( (\text{Ker} p(L))/S \) is a torsion module.

**Proof.** Let \( n = \deg p(x) \). By Lemma 7.2, \( \text{Ker} p(L) \) is a free module of rank \( n \), i.e. is isomorphic to \( k^n \). Since \( n = \mu_1 + \ldots + \mu_k \), we have exactly \( n \) sequences \( s(\alpha_u, a) \). By Theorem 7.4, they are free generators of \( S \). In particular, \( S \) is also isomorphic to \( k^n \). It remains to apply Lemma 5.1. ■

**7.6. Corollary.** If \( k \) is a field, then \( S = \text{Ker} p(L) \).

**Proof.** A torsion module over a field is equal to 0. ■

**Note bibliographique**

This note is concerned only with the works which influenced the present paper. The author did not attempted to write even an incomplete account of the history of the theory of linear recurrence relations.

The theory of recurrent sequences is an obligatory topic for any introduction to combinatorics. But all too often the proofs are presented only for Fibonacci numbers, even if the general case is discussed. The author stumbled upon this tradition in P. Cameron’s textbook [C]. The discussion of the general case in [C] is limited by the following.
In this case, suppose that $\alpha$ is a root of the characteristic equation with multiplicity $d$. Then it can be verified that the the $d$ functions $\alpha^n$, $n\alpha^n$, $\ldots$, $n^{d-1}\alpha^n$ are all solutions of the recurrence relation. Doing this for every root, we again find enough independent solutions that $k$ initial values can be fitted.

The justification of this is the fact that the solutions claimed can be substituted in the recurrence relation and its truth verified.

This discussion ignores at least two significant issues. First, the claim that the sequences $n^{i}\alpha^n$ provide solutions cannot be justified by the substitution of them in the recurrence relations and a routine verification simply because they are solutions only if $i \leq d-1$. Second, one needs to prove that these solutions are linearly independent.

The standard approach to the general case is based on the theory of generating functions and the partial fractions expansion of rational functions. This method is elegantly presented in Chapter 3 of M. Hall’s classical book [H]. A recent presentation of this method can be found in Section 3.1 of M. Aigner’s book [A]. When this approach is chosen, the existence of partial fraction expansions is simply quoted as a tool external to the theory of linear recurrence relations.

The theory of partial fractions is an application of the theory of modules over principal entire rings. See N. Bourbaki [B], Section 2.3, or S. Lang [L], Section IV.5. Hence the theory of linear recurrence relations is also an application of the theory of modules. Once this is realized, it is only natural to use the theory of modules directly, without using the generating functions and partial fractions as an intermediary.

Another application of the same part of the theory of modules is the theory of the Jordan normal form. See N. Bourbaki [B], Section 5, or S. Lang, Chapter XIV. Of course, the theory of the Jordan normal form precedes the theory of modules and is usually presented without any references to the latter.

It is only natural to adapt directly the standard arguments from the theory of the Jordan normal form to the theory of linear recurrence relation. This is done in Section 4 of the present paper, which was heavily influenced by I.M. Gelfand’s classics [G].

The material of Sections 2 and 3 took its present form under the influence of G.-C. Rota’s ideas about the umbral calculus. See, for example, [RR].

The definition of divided derivatives was motivated by the desire to prove the main results without any restrictions on the characteristic of the base ring $\mathbb{k}$ and at the same time to avoid brute force calculations with binomial coefficients. After this work

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*For the purposes of this paper, any calculation with binomial coefficients using the well known expression in terms of factorials is a brute force calculation.*
was completed, the author came across the paper [D] by J. Dieudonné, from which he learned that this notion was first introduced by H. Hasse, F.K. Schmidt, and O. Teichmüller in 1936, with applications to various questions of algebra in mind. See the references in [D]. This was a pleasant surprise, especially because a major part of the author’s work is devoted to Teichmüller modular groups and Teichmüller spaces.

Bibliographie

[A] M. Aigner, *A course in enumeration*, Springer-Verlag, Berlin, Heidelberg, 2007.

[B] N. Bourbaki, *Algèbre, Chapitre VII, Modules sur les anneaux principaux*, Masson, Paris, 1981 (édition originale). Springer-Verlag, Berlin, Heidelberg, 2007 (réimpression inchangée).

[C] P.J. Cameron, *Combinatorics: topics, techniques, algorithms*. Cambridge University Press, Cambridge, 1994.

[D] J. Dieudonné, *Le calcul différentiel dans les corps de caractéristique p > 0*, Proceedings of the International Congress of Mathematicians, 1954, V. 1, 240–252.

[G] I.M. Gelfand, *Lectures on linear algebra* (Russian), 4th edition, “Nauka”, Moscow, 1971. 271 pp.

[Ge] A.O. Gel’fond, *The calculus of finite differences* (in Russian), Gos. izd-vo tekhniko-teoret. lit-ry, Moscow, 1952. 479 pp.

[H] M. Hall, Jr., *Combinatorial theory*, Blaisdell Publishing Co., Waltham (Massachusetts) - Toronto - London, 1967 (Original edition). Wiley, Hoboken, NJ, 1986 (Second edition).

[L] S. Lang, *Algebra*, Revised 3rd edition, Springer-Verlag, New York, 2002.

[RR] S.M. Roman, G.-C. Rota, *The umbral calculus*, Advances in Mathematics, V. 27, No. 2 (1978), 95–188.

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