FREE SUBALGEBRAS OF LIE ALGEBRAS CLOSE TO NILPOTENT

ALEXEY BELOV AND ROMAN MIKHAILOV

ABSTRACT. We prove that for every automata algebra of exponential growth, the associated Lie algebra contains a free subalgebra. For \( n \geq 1 \), let \( L_{n+2} \) be a Lie algebra with generator set \( x_1, \ldots, x_{n+2} \) and the following relations: for \( k \leq n \), any commutator of length \( k \) which consists of fewer than \( k \) different symbols from \( \{x_1, \ldots, x_{n+2}\} \) is zero. As an application of this result about automata algebras, we prove that for every \( n \geq 1 \), \( L_{n+2} \) contains a free subalgebra. We also prove the similar result about groups defined by commutator relations. Let \( G_{n+2} \) be a group with \( n+2 \) generators \( y_1, \ldots, y_{n+2} \) and the following relations: for \( k = 3, \ldots, n \), any left-normalized commutator of length \( k \) which consists of fewer than \( k \) different symbols from \( \{y_1, \ldots, y_{n+2}\} \) is trivial. Then the group \( G_{n+2} \) contains a 2-generated free subgroup.

Main technical tool is combinatorics of periodical sequences and period switching.

1. Introduction

Let \( A \) be an associative algebra over a commutative ring with identity, generated by a set \( S \). Denote by \( A^- \) the Lie algebra with the same set of generators \( S \) and operation \( [u, v] = uv - vu \), \( u, v \in A \). In other words \( A^- \) is the Lie subalgebra of \( A^- \) generated by the given set \( S \). The algebra \( A^- \) clearly depends on the choice of the set of generators of \( A \).

For \( n \geq 1 \), let \( L_{n+2} \) be a Lie algebra with generator set \( x_1, \ldots, x_{n+2} \) and following relations: for \( k \leq n \), any commutator of length \( k \) which consists of fewer than \( k \) different symbols from \( \{x_1, \ldots, x_{n+2}\} \) is zero. For example, the trivial commutators in \( L_n \), which correspond to the case \( k = 3 \), are:

\[
[x_i, x_j], \quad i \neq j.
\]

One of the main results of this paper is following:

**Theorem 1.** For every \( n \geq 1 \), \( L_{n+2} \) contains a free Lie subalgebra.

The proof of theorem \( \text{II} \) is based on the theory of monomial algebras. An algebra with basis \( X \) is called monomial if all its defining relations are of the form \( u = 0 \), where \( u \) is a word written in \( X \). Let \( A \) be a finitely generated algebra with generators \( x_1, \ldots, x_n \). The growth function \( V_A(n) \) equal, the dimension of the space generated by words of length \( \leq n \). If \( V_A(n) \) grows exponentially, then \( A \) has exponential growth; if polynomially, then \( A \) has polynomial growth. Intermediate growth is also possible. The polynomial or exponential growth property does not depend on the choice of the generator set.

For \( n \geq 1 \), let \( A_{n+2} \) be the monomial algebra with generators \( x_1, \ldots, x_{n+2} \) and the following relations: \( u(x_1, \ldots, x_{n+2}) = 0 \) if \( |u| = k \) (\( k \leq n \)) and \( u \) consists of fewer than \( k \) symbols from \( \{x_1, \ldots, x_{n+2}\} \). Clearly, the Lie algebra \( A_{n+2}^- \) is a quotient of \( L_{n+2} \). Hence, Theorem II will follow if we will be able to prove that \( A_{n+2}^- \) contains a free subalgebra. The algebra \( A_{n+2} \) has an alternative description, based on the following property: \( u(x_1, \ldots, x_{n+1}) = 0 \) in the algebra \( A_{n+2} \) if the distance between two occurrences of the same letter in \( u(x_1, \ldots, x_{n+1}) \) is less than \( n+1 \).

Consider a super-word \( w = (x_1 \cdots x_{n+1})^\infty \). It is clear that \( w \neq 0 \) and any series of changes \( x_{n+1} \mapsto x_{n+2} \) will not yield zero, since the distance between two occurrences of the same letter is still \( \leq n \).

With the help of above changes it is possible to get \( 2^M \) different non–zero words. It follows that the number of different non–zero words of length \( k \) in the monomial algebra \( A \) is not less
than $2\left\lfloor \frac{k}{n+1} \right\rfloor$, and the algebra $A$ has exponential growth. Algebra $A_{n+2}$ is finitely-presented and hence is an automata \cite{4}. Hence $A_{n+2}^\sim$ contains a 2-generated (and countably generated) free subalgebra, by Theorem \cite{5}.

**Remark.** More complicated proof is needed for the fact that $A_{n+1}^\sim$ also contains a free subalgebra for every $n \geq 3$.

The similar situation takes place in the case of groups. As a natural group-theoretical analog of Theorem \cite{4} we have the following:

**Theorem 2.** For $n \geq 1$, let $G_{n+2}$ be a group with $n+2$ generators $y_1, \ldots, y_{n+2}$ and the following relations: for $k = 3, \ldots, n$, any left-normalized commutator of length $k$ which consists of fewer than $k$ different symbols from $\{y_1, \ldots, y_{n+2}\}$ is trivial. Then the group $G_{n+2}$ contains a 2-generated free subgroup.

Note that the groups $G_{n+2}$ are related to the construction of amenable groups with fast Følner function from \cite{5}.

It is a pleasure for us to thank Mikhail Gromov for posing the problem and his helpful comments and Louis Rowen for useful discussions and suggestions. The authors were supported by the Israel Science Foundation grant No. 1178/06. The research of the second author was partially supported by Russian Science Support Foundation.

2. THE PERIODICITY

The order $a_1 \prec a_2 \prec \cdots \prec a_n$ induces a lexicographical order on the set of all words. Two words are incompatible with respect to this order, if one is initial in the of other. By $|v|$ will be denoted the length of a word $v$. By $u \subset v$ will be denoted the occurrence of a word $u$ in a word $v$. A word $u$ is called cyclic, if for some $k > 1$, $u = v^k$; otherwise it is called noncyclic or nonperiodic. If $W = u^k r$, where $r$ is an initial segment in $u$, then $W$ is called quasiperiodic of order $|u|$. In this case $W$ is a subword of $u^\infty$ (see next paragraph). Words $u$ and $v$ are called cyclically conjugate, if, for some words $c$ and $d$, $u = cd$ and $v = dc$. The cyclic conjugacy relation is an equivalence relation.

A superword is a word which is infinite in both directions. A word which is infinite to the left, is called a left superword; a word which is infinite to the right, is called a right superword. By $u^\infty$ will be denoted a superword with the period $u$, by $u^\infty/2$ will be denoted a right (left) superword, which begins (terminates) with the word $u$.

Since it will clear from the context, which superword is under consideration, left or right, we do not introduce special notation. The notation $u^\infty/2 \cdot s \cdot v^\infty/2$ means, for example, that $u^\infty/2$ is a left superword and $v^\infty/2$ is a right one.

Right superwords (unlike finite words, for which incomparable elements exist) constitute a linear ordered set with respect to the left lexicographic ordering; the same is true for left superwords with respect to the right lexicographic ordering.

### 2.1. Periodic superwords. Subwords of $u^\infty$

By $u$ will be denoted a non-cyclical word. We recall some propositions from \cite{4}. By $A_{u^\infty}$ we denote an algebra whose defining relations has following form: $s = 0$ where $s$ is a word which is not a subword of $u^\infty$.

**Proposition 1.** Each two subwords in $u^\infty$ of length $N|u|$ are cyclically conjugate; they coincide only when the distance between their first letters is divisible by the period.

**Proposition 2.** a) The beginning subword of length $|u| - 1$ uniquely defines the word from $A_{u^\infty}$.

If the initial subwords of length $|u| - 1$ in two subwords $v$ and $v'$ coincide ($v$ and $v'$ are subwords in the superword $u^\infty$), then one of them is a subword in another. If $|c| \geq |u|$ and $d_1$ and $d_2$ are lexicographically comparable, then at least one of the words $cd_1, cd_2$ is not a subword in $u^\infty$. 
b) The positions of the occurrences of a word \( v \) of length \( \geq |u| \) in \( u^\infty \) differs by a period multiple.

c) If \(|v| \geq |u|\) and \( v^2 \subset u^\infty \), then \( v \) is cyclically conjugate to a power of \( u \). Therefore, nonnilpotent words in \( A_u^\infty \) are exactly those words, which are cyclically conjugate to words of the form \( u^k \).

**Proposition 3.** If \( uW = Wr \), then \( uW \) is a subword of \( u^\infty \) and \( W = u^n r \), where \( r \) is an initial segment in \( u \).

**Remark.** The periodicity of an infinite word means its invariance with respect to a shift. In the one-sided infinite case a pre-period appears; in the finite case there appear effects related to the truncation. This, together with superword technique is the essence of a great many combinatorial arguments (see Proposition 3) especially Bernside type problems. Proofs of the Shestakov hypothesis (nilpotency of subalgebra of \( n \times n \) matrix algebra with all words of length \( \leq n \) are nilpotent), of the Shirshov height theorem (the normal basis of associative affine PI-algebra \( A \) contains only piece-wise periodic words, number of periodic parts is less than \( h(A) \)), and length of each period is \( \leq n - \) maximal dimension of matrix algebra satisfying all identities of \( A \) is a coincidence theorem of the nilradical and the Jacobson radical in a monomial algebra, are examples [4], [1], [3].

**Lemma 1** (on overlapping). If a subword of length \( m + n - 1 \) occurs simultaneously in two periodic words of periods \( m \) and \( n \), then they are the same, up to a shift.

**Lemma 2** implies one technical statement, needed in sequel

**Lemma 2.** Let \( r = u^nv^m \), \( n > k, m > l \). Then \( r \) has not common subwords with \( u^\infty \) of length \( \geq n|u| + |u| + |v| - 1 = (n + 1)|u| + |v| - 1 \) and has not common subwords with \( v^\infty \) of length \( \geq (m + 1)|v| + |u| - 1 \).

**2.2. Periods switching.**

**Proposition 4.** Let \( u, v \) are not powers of the same word, \( l|v| > 2|u| \) and \( s|u| > 2|v| \). Then \( v^lu^s \) is not a subword of \( u^\infty \) or \( v^\infty \).

This proposition follows from the following assertion which is follows at once from Lemma 1 (see [4]): if two periodical superwords of periods \( m \) and \( n \) have the common part of length \( > m + n - 2 \), then these words are identical. In this case, \( u^sv^l \) is a subword \( v^\infty \) and \( u^\infty \).

**Proposition 5.** Let \( u, v \) are not powers of the same word, \( l|v| > 2|u| \) and \( s|u| > 2|v| \). Then \( v^lu^s \) is not a proper power.

**Proof.** Without loss of generality we may suppose that both \( u \) and \( v \) are non-cyclic. Suppose that \( k > 1 \) and \( z^k = v^lu^s \) for some non-cyclic word \( z \). Without loss of generality we can assume that \( |v^l| \geq |u^s| \).

If \( k \geq 4 \), then \( v^l \) contains \( z^2 \) and from the other hand, \( v^l \) is subword of \( z^\infty \). It follows from overlapping lemma [1] that \( v \) is power of \( z \) and hence \( v = z \) (both \( z, v \) are non cyclic). Then \( u \) is also power of \( z \) and we are done.

If \( k = 2 \) then \( u^s \) is a subword of \( z \) and hence of \( v^l \). That contradicts overlapping lemma [1]?

If \( k = 3 \) then because \( |v^l| \geq |u^s| \) and \( s|u| > 2|v| \) we have \( l \geq 3 \). In this case \( |v| < |z|/2 \) and \( |v^l| \geq |v| + |z| \). By overlapping lemma [1] we have that \( v^\infty = z^\infty \) and hence \( v = z \) because both are non cyclic. Then \( z^3 = v^lu^s = z^lu^s \) and \( u^s = z^{3-l} \). Because \( u \) is noncyclic \( u = z \). Hence \( u = v \) that contradicts conditions of the proposition [5]. \( \Box \)

**Lemma 3.** Let \( r = u^nv^m \), \( n > k, m > l \). Then \( r \) is not a subword of \( W' = v^\infty/2 u^\infty/2 \) and hence of \( v^pu^q \) for all \( p, q \).
Proof. If \( r \) is a subword of \( W' \) then either \( u^n \) (i.e. left part of \( r \)) is a subword of \( v^\infty \) or \( v^m \) (i.e. right part of \( r \)) is a subword of \( u^\infty \). Both cases are excluded by proposition 4. \(\square\)

**Proposition 6.** Consider superword \( W = u^{k/2}v^{\infty/2} \), where \( u \neq v \) are different noncyclic words. Let \( S = u^{k_1}v^l \) and suppose \( |u^{k-1}| > 2|v|, |v^{k-1}| > 2|u|, k, l \geq 2. \)

Then \( S \) has just one occurrence in \( W \), which is the obvious one (which we call the “standard occurrence”).

Proof. Otherwise the extra occurrence of \( S \) is either to the left of the standard occurrence, or to the right. Without loss of generality it is enough to consider the left case.

In this case, by Proposition 2 \( W \) is shifted respect to the standard occurrence by a distance divisible by \( |u| \).

Hence we have: \( u^nW = WR, \) i.e. \( u^nW \) starts with \( W \). We can apply Proposition 3 and so we get that \( u^{\infty/2} \) starts with \( W \). Then from combining Lemma 1 and Proposition 2 we get that \( v \) is cyclically conjugate to \( u \), and \( |u| = |v| \).

But in that case \( W = u^{k}v^l \) is subword of \( u^\infty \), implying that the relative shifts of \( u \) and \( v \) are divisible by \( |u| = |v| \), and hence \( u = v \). The proposition is proved. \(\square\)

This proposition together with Lemmas 2 and 3 implies

**Corollary 1.** Let \( R = r^\infty = (u^nv^m)^\infty, n > k, m > 1. \) Then all the occurrences of \( S \) in \( R \) are separated by distances divisible by \( |r| = n|u| + m|v| \).

Proof. First of all, as in the proposition 4 one can define notion of standard occurrence of \( S \) in \( R \). Consider an occurrence of \( S \) in \( R \). Then only following cases are logically possible:

1. It naturally corresponds to occurrence of \( S \) in \( W \) (i.e. power of \( v \) in \( S \) starts in on the power of \( v \) in \( W \) and similarly power of \( u \) in \( S \) ends in on the power of \( u \) in \( W \)).
2. It contains completely either \( u^n \) or \( v^m \).
3. It lies on the position of period switching from \( v^m \) to \( v^n \).

Second possibility is excluded due to due to Lemma 2, third – to due to Lemma 3. First possibility due to proposition 4 corresponds only to standard occurrences and they are separated by distances divisible by \( |r| = |u^nv^m| = n|u| + m|v| \). \(\square\)

Note that \( r \) and \( t \) are cyclically conjugate, iff \( r^\infty = t^\infty \). Using this observation and the previous corollary we get a proposition needed in the sequel:

**Proposition 7.** Let \( u, v \) be different non-cyclic words, with \( |u^n| > 2|v| \) and \( |v^n| > 2|u| \). For all \( k_1, l_1 \geq n, \) if \( (k_1, l_1) \neq (k_2, l_2) \), then \( u^{k_1}v^{l_1} \) and \( u^{k_2}v^{l_2} \) are not cyclically conjugate.

Proof. Suppose that \( r = u^{k_1}v^{l_1} \) and \( t = u^{k_2}v^{l_2} \) are cyclically conjugate. Then \( r^\infty = t^\infty \) and because \( r, t \) are not cyclic, \( |r| = |t| = \mu \). Let us denote \( R = (u^{k_2}v^{l_2})^\infty \). Let \( S = u^{k}v^l \) and suppose \( |u^{k-1}| > 2|v|, |v^{k-1}| > 2|u|, k, l \geq 2 \) and also \( k \leq \min(k_1, k_2), l \geq \min(l_1, l_2) \). It is clear that such \( S \) exist and is a subword of both \( r \) and \( t \).

Then due to corollary 1 all occurrences of \( S \) in \( W \) are shifted by distance divisible by \( \mu \) – period of \( R \). It means that any occurrence of \( S \) can be extended to occurrence of \( r \) as well as to occurrence of \( t \). Hence there exists an occurrences of \( r = u^{n_1}Sv^{m_1} \) and \( t = u^{n_2}Sv^{m_2} \) in \( W \) with common part \( S \).

If \( r \neq t \), then \( n_1 \neq n_2 \) because \( |s| = |t| \). Without loss of generality we can suppose that \( n_1 < n_2 \). In this case \( m_1 > m_2 \). Word \( t \) is shifted to the left from the word \( r \) on the distance \( d = |u^{n_1} - u^{n_2}| = |v^{m_2} - v^{m_1}|. \)

Consider a union \( \omega \) of \( r \) and \( t \). Then \( \omega = fr = ft, |e| = |f| = d \). Note that \( r^2 \) is a subword of \( t^\infty = W \), occurrence of \( v^{m_1} \) (which is end of \( r \)) precedes an occurrence of \( r \). Because \( |v^{m_2}| > d, e = v^{m_2 - m_1} \). Similarly \( f = u^{n_1 - n_2} \).
From other hand \( \omega \) can be also obtained by extending the subword \( S \) of \( W \) to the left on the distance \( |u^{\max(n_1,n_2)}| \) and to the right on the distance \( |u^{\max(n_1,n_2)}| \) and \( \omega = u^{\max(n_1,n_2)}v^{\max(m_1,m_2)} = u^{n_1-n_2}t = \nu^{m_2-m_1}.

Hence \( u^{n_1-n_2} = v^{m_2-m_1}; n_1 \neq n_2; m_1 \neq m_2 \). It follows that \( u, v \) are powers of the same word \( s \). Because \( u \neq v \) one of these powers is greater than 1 and booth \( u \) and \( v \) can not be non cyclic words. But this contradicts to their initial choice. \( \square \)

3. Regular Words and Lie brackets

We shall extend the relation \( < \) by defining the following \( \triangleright \)-relation (“Ufnarovsky order”): \( f \triangleright g \), if, for any two right superwords \( W_1, W_2 \), such that \( W_2(a,b) \triangleright W_1(a,b) \), when ever \( b \triangleright a \), the inequality \( W_2(g,f) \triangleright W_1(g,f) \) holds. This condition is well defined and equivalent to following: \( f \triangleright g \) iff \( f^{\infty/2} \triangleright g^{\infty/2} \) (i.e., \( f^m \triangleright g^n \), for some \( m \) and \( n \)). It is clear that if \( f \triangleright g \), then \( f \triangleright g \).

The relation \( \triangleright \) is a linear ordering on the following set of equivalence classes: \( f \sim g \), if for some \( s, f = s^k, g = s^l \).

Let us note that each finite word \( u \) uniquely corresponds to the right superword \( u^\infty \). To equivalent words correspond the same superwords. The relation \( \triangleright \) corresponds to the relation \( > \) on the set of superwords.

It is known \((4), (3)\) that:

A word \( u \) is called regular, if one of the following equivalent conditions holds:

a) \( u \) word is greater all its cyclic conjugates: \( u_1u_2 = u \), then \( u \triangleright u_2u_1 \).

b) \( u_1u_2 = u \), then \( u \triangleright u_2 \).

c) \( u_1u_2 = u \), then \( u_1 \triangleright u \).

A word \( u \) is called semi-regular in the following case: If \( u = u_1u_2 \), then, either \( u \triangleright u_2 \), or \( u_2 \) is a beginning of \( u \). (An equivalent definition can be obtained if the relation \( < \) is replaced by the relation \( \leq \) in the definition of a regular word.) Every semi-regular word is a power of a regular one.

It is well-known that every regular word \( u \) defines the unique bracket arrangement \([u]\) such that after opening all Lie brackets \( u \) will be a highest term in this expression. Moreover, monomials of such type form a basis in the free Lie algebra (so called Hall–Shirshov basis) (see \( [2], [6] \)).

We shall need some technical statements:

Lemma 4 \((4)\). Suppose \( |u^k| \triangleright |v^l| \) and \( u^k \) is a subword of \( v^\infty \). Then there exists \( S' \) cyclically conjugate to \( S \), such that \( u = (S')^m \) and \( v = (S^2)^n \). If, moreover, the initial symbols of \( u \) and \( v \) are at a distance divisible by \( |S| \) in \( v^\infty \), then \( S = S' \).

Corollary 2. Let \( u \triangleright v \) be semi-regular words. Then, for sufficiently large \( k \) and \( l \), the words \( u^k v^l \) are regular and

\( u^{k_1}v^{l_1} \triangleright u^{k_2}v^{l_2} \)

for \( k_1 > k_2 \).

Proof. Let \( \delta \) be a cyclic conjugate of \( u^k v^l \). It is clear that \( \delta \triangleright u^k v^l \), we only need to prove inequality \( \delta \neq u^k v^l \). In order to do this, we need only to show that \( u^k v^l \) is not cyclic word, but it follows from the proposition \( 4 \).

The next lemma follows from Lemma 4 and Corollary 2:

Lemma 5. Let \( k_i > |d|, l_i > |u| \), for \( i = 1, 2 \). Then \( u^{k_1}d^{l_1} \) and \( u^{k_2}d^{l_2} \) are not cyclically conjugate, provided that \( u \triangleright d \) and \( u, d \) are not conjugate to proper powers of the same word.
4. Words in automata algebras

By $\Phi\langle x_1, \ldots, x_s \rangle$ will be denoted the free associative $\Phi$-algebra with generators $x_1, \ldots, x_s$. By $A\langle a_1, \ldots, a_s \rangle$ will be denoted an arbitrary $\Phi$-algebra with a fixed set of generators $a_1, \ldots, a_s$. A word or a monomial from the set of generators $\mathcal{M}$ is an arbitrary product of elements in $\mathcal{M}$. The set of all words constitutes a semigroup, which will be denoted by $Wd\langle \mathcal{M} \rangle$. The order $a_1 \prec \cdots \prec a_s$ generates the lexicographic order on the set of words: The greater of two words is the one whose first letter is greater; if the first symbols coincide, then the second letter are compared, then the third letters and so on. Two words are incomparable, only if one of them is initial in the other.

By a word in an algebra we understand a nonzero word from its generators $\{a_i\}$. We cannot speak about the value of a superword in an algebra, but can speak about its equality or nonequality to zero (and, in some cases, about linear dependence). A superword $W$ is called zero superword, if it has a finite zero subword, and it is called a nonzero superword, if it has no finite zero subwords.

An algebra $A$ is called monomial, if it has a base of defining relations of the type $c = 0$, where $c$ is a word from $a_1, \ldots, a_s$. Obviously, a monomial algebra is a semigroup algebra (it coincides with the semigroup algebra over the semigroup of its words).

4.1. Automata algebras. First we recall some well known definitions from [4]. Suppose we are given an alphabet (i.e., a finite set) $X$. By finite automaton (FA) with the alphabet $X$ of input symbols we shall understand an oriented graph $G$, whose edges are marked with the letters from $X$. One of the vertices of this graph is marked as initial, and some vertices are marked as final. A word $w$ in the alphabet $X$ is called accepted by a finite automaton, if there exists a path in the graph, which begins at the initial vertex and finishes in some final vertex, such that marks on the path edges in the order of passage constitute the word $w$.

By a language in the alphabet $X$ we understand some subset in the set of all words (chains) in $X$. A language $L$ is called regular or automata, if there exists a finite automaton which accepts all words from $L$ and only them.

An automaton is called deterministic, if all edges, which start from one vertex are marked by different letters (and there are no edges, marked by the empty chain). If we reject such restriction and also allow edges, marked by the empty chain, then we shall come to the notion of a non-deterministic finite automaton. Also we can allow an automaton to have several initial vertices. The following result from the theory of finite automata is well known:

For each non-deterministic FA there exists a deterministic FA, which accepts the same set of words (i.e. the same language).

It will be convenient for us to consider the class of FA, such that all vertices are initial and final simultaneously. The reason of this is that the language of nonzero words in a monomial algebra has the following property: each subword of a word belonging to the language, also belongs to it.

Suppose throughout that $G$ is the graph of a deterministic FA, $v$ is a vertex of $G$, and $w$ is a word. If the corresponding path $C$ starting from $v$ exist in $G$, then one can define the vertex $vw$ terminal vertex for $C$.

Let $A$ be a monomial algebra (not necessary finitely defined). $A$ is called an automata algebra, if the set of all of its nonzero words from $A$ generators is a regular language. Obviously, a monomial algebra is an automata algebra, only if the set of its nonzero words is the set of all subwords of words of some regular language.

It is known that every automata algebra can be given by a certain deterministic graph, and that every finitely defined monomial algebra is automata ([4], [6]).

The Hilbert series for an automata algebra is rational (Proposition 5.9 [4]). An automata algebra has exponential growth if and only if $G$ has two cycles $C_1$ and $C_2$ with common vertex.
v, such that the corresponding words \( w_1, w_2 \) (we read them starting from v) are not powers of the same word. In this case the words \( w_1 \) and \( w_2 \) generate a free 2-generated associative algebra. If there are no such cycles, \( A \) has a polynomial growth. No intermediate growth is possible.

The following theorem is the aim of this section:

**Theorem 3.** Let \( A = \langle a_1, \ldots, a_n \rangle \) be an automata algebra of exponential growth. Then the Lie algebra \( A^\ast \) contains a free 2-generator subalgebra.

We continue to assume that \( G \) is the graph of a deterministic FA. Call a semi-regular word \( u \) well–based if it written on a certain cycle \( C \) with an initial vertex \( v \); i.e. \( vu = v \). Two semi–regular words \( u_1 \) and \( u_2 \) are pair–wisely well–based if \( u_1 \) and \( u_2 \) are written on cycles \( C_1 \) and \( C_2 \) with a common initial vertex \( v \) and \( vC_1 = vC_2 = v \). In this case, for any word \( W(a, b) \) the word \( W(u_1, u_2) \neq 0 \) in \( A \); in particular \( u^k_1u^k_2 \neq 0 \).

**Main Lemma.** The graph \( G \) contains two regular pairwise well–based words \( u \neq v \).

**Deduction of Theorem 3 from the Main Lemma.** We may always assume that \( u \triangleright v \). Let \( a > b \) and \( w \) a regular word. Then (see [4]) \( w(u, v) \) also is a regular word.

For every regular word \( u \) we can choose a unique presentation \( u = u_1u_2 \) with regular \( u_1 \) and regular \( u_2 \) of maximal length. In this case \( \{u\} = \{[u_1], [u_2]\} \) (see [6]). Therefore, \( w(u, v) \) can be obtained by setting \( [u] \mapsto a, [v] \mapsto b \) to the word with brackets \([w]\).

Since \( u \) and \( v \) are well-based, for every word \( R(a, b) \), one has \( R(u, v) \neq 0 \). Let \( [u], [v] \) be the results of the regular arrangement of the brackets for \( u \) and \( v \) respectively. Then \( [u] \neq 0, [v] \neq 0 \).

Thus we have constructed a one-to-one correspondence between the Hall basis of a Lie algebra, generated by \( [u], [v] \) and the Hall basis of a free 2-generated Lie algebra with generators \( a, b \). The theorem follows. □

**4.2. Proof of the Main Lemma.** Corollary 2 implies the following:

**Proposition 8.** Suppose the graph \( G \) contains two ordered (in the sense of the operation \( \triangleright \)) pairwise well-based words. Then \( G \) contains also two regular pairwise well-based words.

It is sufficient to find two ordered semi-regular pairwise well-based words, i.e. with common final and initial vertices. For that it is enough to prove the existence of a sufficiently large number of well-based ordered semi-regular words. In this case, infinitely many of them will have a common initial vertex, hence pairwise well–based, and the main lemma follows.

**Lemma 6.** Let \( u_1 \) be a well-based word and \( u_2 \) a cyclically conjugate word. Then \( u_2 \) is also well-based.

**Proof.** Suppose \( u_1 = w_1w_2, u_2 = w_2w_1 \) and \( v \) is a base vertex of \( u_1 \). Then \( v' = vw_1 \) is a base vertex of \( u_2 \). Indeed, \( v'u_2 = vw_1(w_2w_1) = v(w_1w_2)w_1 = vw_1 = v' \). □

**Corollary 3.** If \( u \) is well-based, then semi-regular word conjugate to \( u \) is also well-based.

Let \( u \) and \( d \) be ordered pairwise well-based words (necessarily not semi-regular). Then, for every \( w(a, b) \), the word \( w(u, d) \) is non-zero. In particular, \( u^k d^l \) are non-zero for all \( k, l \).

Now Lemma 5 together with the fact that every non–cyclic word uniquely corresponds to a cyclically conjugated regular word, implies that there are infinitely many well–based words. Infinitely many of them will have the same initial vertex and so will be pairwise well based; hence the Main Lemma follows.

5. **Group theoretical applications**

\( \text{Id}(S) \) denotes the ideal, generated by the set \( S \).
Lemma 7. Suppose $a$ and $b$ are homogeneous elements of a graded associative algebra $A$, such that the subalgebra generated by $a, b$ is free associative algebra with free generators $a, b$. Let $a'$ (resp. $b'$) be a linear combination of elements in $A$ with degrees strictly greater than the degree of $a$ (resp. $b$). Let $\bar{a} = a + a'$, $\bar{b} = b + b'$. Then the algebra generated by $\bar{a}, \bar{b}$ is a free associative with free generators $\bar{a}, \bar{b}$.

This lemma follows from the fact that for every polynomial $h(u, v)$ with non-zero minimal component $h'(u, v)$, the minimal component of $h(\bar{a}, \bar{b})$ is $h'(a, b) \neq 0$.

We call an algebra homogenous if all its defining relations are homogenous respect to the set of generators. Let $A$ be a homogenous algebra, and $J$ be an ideal of $A$, generated by elements of degree $\geq 1$. We call such algebra good. If $A/J \equiv k$ and $\bigcap J^n = 0$. Every monomial algebra is good. For any $x \in A$, the image of $x$ in $A/J$ is not zero for some $n$, so $A$ can be embedded into the projective limit $\varprojlim A/J^n$.

Lemma 8. Let $B$ be a good homogenous algebra, such that $1 + a$ and $1 + b$ are invertible, $a, b \in J$, and the elements $a$ and $b$ are free generators of a free associative subalgebra $C$ of $B$. Then the group generated by $1 + a$ and $1 + b$ is free.

Remark. Note that the pair of two different pairwise well–based words in a monomial algebra generates a free associative subalgebra.

Proof. Suppose $W(1 + a, 1 + b) = 1$ for some non-trivial word $W(x, y) \neq 1$ in the free group. Consider free algebra $k\langle x, y \rangle$ and its localization by $1 + x, 1 + y$. Then $W(1 + x, 1 + y) \neq 1$, and, for some $n_0 = n_0(W)$, $W(1 + \bar{x}, 1 + \bar{y}) \neq 1$ in $\pi(k\langle x, y \rangle) = k\langle x, y \rangle/\text{Id}(x, y)^{n_0}$ for all $n \geq n_0$. In each such image, the elements $1 + x$ and $1 + y$ are invertible, so there is no need for localization.

On the other hand, because $A$ is good homogenous, the image of $J^n \cap C$ under isomorphism $\phi$ generated by $a \rightarrow x, b \rightarrow y$ lies in $\text{Id}(x, y)^{n_0}$, and

$$1 = \pi(\phi(W(1 + a, 1 + b))) = W(1 + \bar{x}, 1 + \bar{y}) \neq 1.$$  

Contradiction.

Let $u$ and $v$ be two pairwise well based words. They have canonical Lie bracket arrangement; let $[u]$ and $[v]$ be corresponding Lie elements (obtained via opening the Lie brackets). Notice that

$$[u] = u + \text{lexicographically smaller terms, } [v] = v + \text{lexicographically smaller terms.}$$

Hence we have following

Lemma 9. Let $u$ and $v$ be two pairwise well based words, and $[u]$ and $[v]$ be the corresponding Lie elements (obtained via opening Lie brackets). Then $[u], [v]$ generate as a free generators 2-generated free associative algebra (and also free Lie algebra via commutator operation).

For $n \geq 1$, consider the monomial algebra $A_{n+2}$ with generators $x_1, \ldots, x_{n+2}$ (see the Introduction). Adjoin a unit $A'_{n+2} = A_{n+2} \cup \{1\}$. The elements $\bar{x}_i := 1 + x_i$ have inverses $\bar{x}_i^{-1} := 1 - x_i$. Consider the group $A^n_{n+2}$ generated by the elements $1 + x_i$. Consider the configuration of brackets in the generators of the free subalgebra in the Lie algebra $A^n_{n+2}$ and write the correspondent Lie elements in the group $A^n_{n+2}$. Lemmas 7 and 8 imply that the subgroup $A^n_{n+2}$ generated by these two elements will be free. It is clear from the construction that all left-normalized commutators of length $k$ in $A^n_{n+2}$ which consists of fewer that $k$ different symbols from $\{x_1, \ldots, x_{n+2}\}$, are trivial. Hence, Theorem 2 follows.
References

[1] S.A. Amitsur, L.W. Small: Affine algebras with polynomial identities, Suppl. ai Rendiconti del Circolo Mat. di Palermo, Serie 2, 31 (1992), 9-43.
[2] Bahturin Yu., A.: Identities in Lie algebras. M.: Nauka, 1985., pages 448 (Russian). Engl. transl. (by Bahturin): Identical relations in Lie algebras. VNU Science Press, b.v., Utrecht, 1987. x+309 pp.
[3] Belov A.: About height theorem. Comm. in Algebra, 1995, vol. 23, N 9, p. 3551–3553.
[4] A. Belov, V. Borisenko and V. Latyshev: Monomial algebras. Algebra 4, J, Math. Sci. (New York) 87 (1997), 3463-3575.
[5] M. Gromov: Entropy and Isoperimetry for Linear and non-Linear Group Actions, preprint.
[6] V. Ufnarovskij: Combinatorial and asymptotic methods in algebra. Algebra, VI, 1–196, Encyclopaedia Math. Sci., 57, Springer, Berlin, (1995)