Quantum Teichmüller theory assigns invariants to three-manifolds via projective representations of mapping class groups derived from the representation of a noncommutative torus. Here, we focus on a representation of the simplest non-commutative torus which remains fixed by all elements of the mapping class group of the torus, $SL_2(\mathbb{Z})$. Also known as the modular group. We use this representation to associate a matrix to each element of $SL_2(\mathbb{Z})$; we then compute the trace and determinant of the associated matrix.

1. Introduction

Fock and Chekhov [2] defined a noncommutative algebra related to the Teichmüller space of a punctured surface. The algebra is a noncommutative torus whose defining relations come from a triangulation of the underlying surface. There is an action of the mapping class group on this algebra, and if a representation is fixed by this action, then this gives rise to a projective representation of the mapping class group.

In this paper, we study a toy model of quantum Teichmüller space, the noncommutative torus in two variables, $\mathcal{W}_q$, which can be thought of as being associated to the two torus $T^2 = S^1 \times S^1$. The “trivial” representation of $\mathcal{W}_q$ is fixed by all elements of the mapping class group, and so gives rise to a projective representation of the mapping class group of the torus, $SL_2(\mathbb{Z})$. In this paper we compute the trace, and the determinant of a matrix associated to each element of $SL_2(\mathbb{Z})$ by this projective representation in the case of $q$ being a root of unity of prime order.

We begin with a section of preliminaries, starting with the definition of $\mathcal{W}_q$ and a description of the action of $SL_2(\mathbb{Z})$ on $\mathcal{W}_q$ as automorphisms. After reviewing properties of matrix algebras in subsection 2.2 we will review representations of algebras in subsection 2.3. We finish the preliminaries with a review of Gauss sums.

In section 3 we give models of the irreducible representations of $\mathcal{W}_q$ and prove they are indeed irreducible.

Next, in section 4 we take an arbitrary element $B \in SL_2(\mathbb{Z})$ and find a matrix whose action by conjugation is the same as $B$’s action as an automorphism on the representation of $\mathcal{W}_q$. Finally in sections 5 and 6 we calculate the trace and determinant (respectively) of the conjugating matrix.

2. Preliminaries

2.1. Noncommutative Tori. Let $\mathcal{W}_q = \mathbb{C}[l, l^{-1}, m, m^{-1}]_q$ be the non-commuting Laurent polynomials in variables $l$ and $m$ with $lm = q^2 ml$ for some $q \in \mathbb{C}\{0\}$. We will study $\mathcal{W}_q$ using the following basis. Let $e_{r,s} = q^{-rs} l^r m^s$. The set $\{e_{r,s} \mid r, s \in \mathbb{Z}\}$ forms a basis for $\mathcal{W}_q$.
over $\mathbb{C}$ where we can take products of elements in this basis as follows,

$$e_{p,t} \ast e_{r,s} = q^{-pt-rs}p^i t^r m^s$$

$$= q^{-pt-rs}p(q^{-2t}r^t)m^s$$

$$= q^{-pt-rs-2tr}q^{(p+r)(t+s)}q^{-(p+r)(t+s)}p^p t^t m^s$$

$$= q^{ps-rt}e_{p+r,t+s}$$

$$= q\begin{pmatrix}p & t \\ r & s\end{pmatrix}e_{p+r,t+s}$$

Lastly, note that if $n$ is odd and $q$ is a primitive $n$th root of unity, then the center of $W_q$ (see Definition 1) is spanned by $\{e_{np,nt} \mid p, t \in \mathbb{Z}\}$. We can now view $W_q$ as a module over its center (see Definition 2).

**Proposition 1.** The algebra $W_q$ is free module over its center with rank $n^2$ and basis $\{e_{i,j} \mid 0 \leq i, j < n\}$.

**Proof.** It is clear that $\{e_{i,j} \mid 0 \leq i, j < n\}$ spans $W_q$ over $Z(W_q)$ since for any $r, s \in \mathbb{Z}$ we can find $p$ and $t$ so that $r = pn + i$ and $s = tn + j$ with $0 \leq i, j < n$ so that $e_{r,s} = q^{-i \begin{pmatrix}p & t \\ r & s\end{pmatrix}} e_{pn,tn} \ast e_{i,j}$. To show that this set is linearly independent, first note that $\{e_{np,nt} \mid p, t \in \mathbb{Z}\}$ is linearly independent over $\mathbb{C}$ as it is a subset of a basis for $W_q$ over $\mathbb{C}$. Suppose that $\sum_{i,j=0}^{n-1} \alpha_{i,j} e_{i,j} = 0$ with $\alpha_{i,j} \in Z(W_q) = \langle e_{np,nt} \mid p, t \in \mathbb{Z}\rangle$. Write $\alpha_{i,j} = \sum_k \beta_{i,j}^k e_{n^k p_i, n^k t_j}$ where $\beta_{i,j}^k \in \mathbb{C}$ and the choice of the $\beta_{i,j}^k$ is unique. Then

$$0 = \sum_{i,j=0}^{n-1} \alpha_{i,j} e_{i,j}$$

$$= \sum_{i,j=0}^{n-1} \left( \sum_k \beta_{i,j}^k e_{n^k p_i, n^k t_j} \right) e_{i,j}$$

$$= \sum_{i,j=0}^{n-1} \sum_k \beta_{i,j}^k q^{np_i^k \begin{pmatrix}p & t \\ i & j\end{pmatrix}} e_{n^k p_i, n^k t_j + j}$$

$$= \sum_{i,j=0}^{n-1} \sum_k \beta_{i,j}^k e_{n^k p_i, n^k t_j + j}$$

But this is a linear combination of unique elements in $\{e_{i,j} \mid i, j \in \mathbb{Z}\}$, our basis for $W_q$ over $\mathbb{C}$, and thus $\forall i, j, k$ it must be that $\beta_{i,j}^k = 0$ by the linear independence of our basis. Therefore, $\forall i, j \in \{0, \ldots, n-1\}$, $\alpha_{i,j} = 0$. \hfill $\square$

**Proposition 2.** There is a left action of $SL_2(\mathbb{Z})$ as automorphisms on $W_q$ defined by

$$\begin{pmatrix}a & b \\ c & d\end{pmatrix} e_{p,t} = e_{ap+bt,cp+dt}.$$
Proof. We need only show that \((a \ b) \ e_{p,t} * e_{r,s} = (a \ b) \ e_{p,t} * (a \ b) \ e_{r,s}\). This can be accomplished through direct computation:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} (e_{p,t} * e_{r,s}) = q \begin{pmatrix} p & t \\ r & s \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} e_{p+r,t+s} \\

= q \begin{pmatrix} p & t \\ r & s \end{pmatrix} e_{a(p+r)+b(t+s),c(p+r)+d(t+s)} \\

= q \begin{pmatrix} p & r \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} e_{ap+ar+bt+bs,cp+cr+dt+ds} \\

= q \begin{pmatrix} ar + bs \\ cp + dt \end{pmatrix} \begin{pmatrix} ap + bt \\ cr + ds \end{pmatrix} e_{ap+bt+ar+bs,cp+dt+cr+ds} \\

= e_{ap+bt,cp+dt} * e_{ar+bs,cr+ds} \\

= \begin{pmatrix} a & b \\ c & d \end{pmatrix} (e_{p,t} * \begin{pmatrix} a & b \\ c & d \end{pmatrix}) e_{r,s} \\

\]

\[ \square \]

2.2. Matrix Algebras.

Definition 1. An algebra \(A\) over a field \(F\) is a vector space with an additional bilinear binary operation \(\cdot : A \times A \to A\) usually called multiplication. We assume that the multiplication is associative, and there is a unit element. The center of \(A\), \(Z(A)\) are those elements of \(A\) that commute with all other elements, i.e. \(Z(A) = \{x \in A \mid x \cdot a = a \cdot x \ \forall a \in A\}\).

Definition 2. Given an algebra, \(A\), a left \(A\)-module or representation of \(A\) is a vector space \(V\) over \(\mathbb{C}\) along with a homomorphism \(\rho : A \to \text{End}(V)\). We say the representation is irreducible if \(\rho\) is onto.

Let \(M_n(\mathbb{C})\) be the algebra of \(n \times n\)-matrices with complex entries. There is a standard basis denoted \(E_{i,j}\) of matrices that are zero except in the \((i, j)\)-entry. In this paper the indices \(i\) and \(j\) run from 0 to \(n - 1\). It is well known that

\[
E_{i,j}E_{k,l} = \delta_{k,i}E_{i,l}
\]

where \(\delta_{k,i}\) is the Kronecker delta. Note that the center of \(M_n(\mathbb{C})\) is exactly all scalar multiples of the identity. We will also use the Skolem-Noether \([3]\) theorem, which ensures that every automorphism of \(M_n(\mathbb{C})\) is inner, i.e. if \(\theta : M_n(\mathbb{C}) \to M_n(\mathbb{C})\) is an automorphism, then there exists \(C \in M_n(\mathbb{C})\), unique up to scalar multiples, so that for all \(A \in M_n(\mathbb{C})\),

\[
\theta(A) = C^{-1}AC.
\]

Remark 3. After choosing a basis for the vector space \(V\), \(\text{End}(V)\) can be identified with \(M_n(\mathbb{C})\). In this paper we treat representations as homomorphisms into \(M_n(\mathbb{C})\).

2.3. Irreducible Representations. We begin this section, by showing that irreducible representations of an algebra, \(A\), are determined, up to equivalence, by their kernels. Then we show that under certain circumstances (that will appear in Section \([3]\)) those kernels are fully determined by their intersections with \(Z(A)\).
Proposition 3. If $A$ is an associative algebra, and $\rho : A \to M_n(\mathbb{C})$ is irreducible with $I = \ker \rho$, then $\rho$ is completely determined by $I$.

Proof. Suppose $\rho_1, \rho_2 : A \to M_n(\mathbb{C})$ both onto, such that $\ker \rho_1 = \ker \rho_2$. Then there exists a well defined endomorphism on $M_n(\mathbb{C})$, $\rho_2 \circ \rho_1^{-1}$. Therefore, by the Skolem-Noether theorem, this map must be inner so that there is some $C \in GL_n(\mathbb{C})$ such that $C \rho_1 C^{-1} = \rho_2$. □

Proposition 4. Let $A$ be an associative algebra that is a free module of rank $n^2$ over its center. Let $\rho : A \to M_n(\mathbb{C})$ be an irreducible representation, and let $I = \ker(\rho)$, then $(I \cap Z(A)) \cdot A = I$. In particular, this means the kernel of $\rho$ is determined by its intersection with the center of $A$.

Proof. Because $I$ is an ideal of $A$, it is clear that $(I \cap Z(A)) \cdot A \subseteq I$, so it is sufficient show the other inclusion. Notice that $\rho|_{Z(A)} : Z(A) \to Z(M_n(\mathbb{C})) = \mathbb{C}I_n$ is a homomorphism from $Z(A)$ onto a field; consequently, $\ker(\rho|_{Z(A)}) = I \cap Z(A)$ is a maximal ideal of $Z(A)$. If $B = \{e_i\}_{i=1}^{n^2}$ is a basis for $A$ over $Z(A)$ (of order $n^2$), then $\rho(B)$ must be a spanning set of $M_n(\mathbb{C})$ since $\rho$ is onto. Therefore, the elements of $\rho(B)$ are linearly independent as they span an $n^2$-dimensional vector space. Let

$$J_i := \{z \in Z(A) | \exists z_{e_i} + \sum_{j \neq i} z_{e_j} \in I\}.$$ 

This is an ideal of $Z(A)$ because the $e_i$ are a basis so that the expression $ze_i + \sum_{j \neq i} z_{e_j}$ is unique. Notice the fact that for all $i$, $I \cap Z(A)$ is contained in $J_i$ which implies $I \cap Z(A) = J_i$, since $I \cap Z(A)$ is maximal. As this is true for each $i$, we get that for any $i$ (since $J_{i_1} = J_{i_2}$), $I \subseteq J_i \cdot A$. Therefore, $(I \cap Z(A)) \cdot A = I$. □

2.4. Legendre Symbols and Gauss Sums. Lastly, we introduce Gauss Sums which will be instrumental in Section 5.

Definition 4. Given two integers $a$ and $p$ such that $a \not\equiv 0 \mod p$, then the associated quadratic symbol, or Legendre symbol, is defined to be

$$\left(\frac{a}{p}\right) = \begin{cases} 
1 & \text{if } a \equiv x^2 \mod p \\
-1 & \text{if } a \not\equiv x^2 \mod p
\end{cases}$$

for any integer $x$. This value depends only on the residue class of $a \mod p$.

Definition 5. A Gauss quadratic sum is a sum of the form,

$$\sum_{x \mod b} e^{\frac{2\pi i}{b} ax^2} =: G(a, b)$$

where $a$ and $b$ are relatively prime, non-zero integers with $b > 0$.

Lang gives an exposition of Gauss’ calculation showing that if $b \geq 1$ is odd, then $G(a, b) = \left(\frac{a}{b}\right) G(1, b)$, and that $G(1, b) = \frac{1 + i^{b-1}}{1 + i^{b-2}} \sqrt{b}$ so that for $b \geq 1$ odd,

$$G(a, b) = \left(\frac{a}{b}\right) \frac{1 + i^{-b}}{1 + i^{-1}} \sqrt{b}.$$
3. Irreducible Representations of the Noncommutative Torus

We now describe representatives of every irreducible representation of $W_q$. Choose an $n$th root $b^{1/n}$ of $b$. Define $\rho_{a,b} : W_q \to M_n(\mathbb{C})$ to be the representation of $W_q$ determined by $\rho_{a,b}(l) = L_a$ and $\rho_{a,b}(m) = M_b$ where $a, b \in \mathbb{C}$ and

$$L_a = \begin{pmatrix} 0 & 0 & \cdots & 0 & a \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad M_b = \begin{pmatrix} b^{\frac{1}{n}} & 0 & \cdots & 0 \\ 0 & b^{\frac{1}{n}q^{-2}} & \cdots & 0 \\ 0 & 0 & b^{\frac{1}{n}q^{-4}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b^{\frac{1}{n}q^{-2(n-1)}} \end{pmatrix}$$

Remark 6. Even though the matrix $M_b$ depends on the choice of $b^{1/n}$, the equivalence class of the representation does not. As this is a representation of an associative algebra that is free of rank $n^2$ over its center, by Proposition 4 the representation is determined by the intersection of its kernel with $Z(W_q)$. In this case that is the ideal $(l^n - a, m^n - b)$, which does not depend on the choice of $n$th root of $b$.

Proposition 5. The representation, $\rho_{a,b} : W_q \to M_n(\mathbb{C})$, is irreducible (surjective).

Proof. Recall that $1 + q + q^2 + \cdots + q^{n-1} = \frac{q^n - 1}{q-1} = 0$. We have that $\sum_{i=0}^{n-1} M_i^i = nE_{0,0}$, the matrix where every entry but the top left corner is zero. Hence, $\frac{1}{n} \sum_{i=0}^{n-1} (b^{\frac{1}{n}i}M_b)^i = \frac{1}{n} \sum_{i=0}^{n-1} M_i^i = E_{0,0}$. Subsequently, $L_a^i E_{0,0} = E_{i,0}$ for $0 \leq i \leq n - 1$. Finally, $E_{i,0}(\frac{1}{n}L_a)L_a^{k-1} = E_{i,n-k}$ for $1 \leq k \leq n - 1$ so that $L_a, M_b$ together span every $E_{i,j}$ which spans all of $M_n(\mathbb{C})$. Therefore, $\rho_{a,b}$ is irreducible.

Proposition 6. If $B \in SL_2(\mathbb{Z})$, and 1 is not an eigenvalue of $B$ the only representations of $W_q$ that can be fixed by $B$ are $\rho_{a,b} : W_q \to M_n(\mathbb{C})$ were $a$ and $b$ are roots of unity whose order divides the determinant of $B - I_2$.

Proof. Suppose that $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\rho_{e,f} : W_q \to M_n(\mathbb{C})$ is fixed by $B$. Since irreducible representations of $W_q$ are determined by the intersection of their kernel with the center we are looking for $\lambda, \mu \in \mathbb{C} - \{0\}$ that solve the system of equations

$$\begin{align*}
\lambda^n &= e, \quad \mu^n = f \\
\lambda^{na} \mu^{nb} &= e, \quad \lambda^{nc} \mu^{nd} = f.
\end{align*}$$

Performing a multiplicative row operation we get,

$$\lambda^n(a-1) \mu^{nb} = 1, \quad \lambda^{nc} \mu^{nd(a-1)} = 1.$$

Using elementary operations of determinant 1, done multiplicatively, we can make this system,

$$\lambda^{ne_1} = 1, \quad \mu^{ne_2} = 1,$$

where $e_1e_2$ is equal to the determinant of $\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}$. From this we see that $\lambda^n$ and $\mu^n$ are roots of unity whose order divides the determinant of $B - I_2$.

□

Remark 7. For a particular $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ there can be some fixed representations, that hold out promise of invariants of the mapping cylinders of those matrices. In order to get a
problem that we can solve uniformly we now restrict our attention to the representation \( \rho_{1,1} : \mathcal{W}_q \to M_n(\mathbb{C}) \), which is fixed by all elements of \( SL_2(\mathbb{Z}) \). We are in fact studying a projective representation of the mapping class group, of the torus that should be related to the Witten-Reshetikhin-Turaev representations \([1]\).

Remark 8. Proposition 6 shows us that the only representation \( \rho_{a,b} \) fixed by all of \( SL_2(\mathbb{Z}) \) is \( \rho_{1,1} \). From now on, we will refer to \( \rho_{1,1} \) as \( \rho \), \( L_1 \) as \( L \), and \( M_1 \) as \( M \).

4. Finding the Conjugating Matrix using the Skolem-Noether Theorem

We now know that every \( B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \) acts as automorphisms of \( \mathcal{W}_q \), and consequently on \( Z(\mathcal{W}_q) \). Thus, if any ideal \( I \subseteq Z(\mathcal{W}_q) \) is fixed by \( B \), then \( B \) induces an automorphism on \( M_n(\mathbb{C}) \) as inner automorphisms. Our goal in this section is to determine the conjugating matrix associated with the automorphism induced by \( B \). We begin with the following crucial observation,

Lemma 1. The first column of \( CE_{k,0}C^{-1} \) is a constant multiple of the \( k \)th column of \( C \).

Proof. Note that

\[
ME_{k,0}N = (m_{i,j})E_{k,0}(n_{i,j}) = (m_{i,j})(\delta^k_jn_{i,j}) = (m_{i,k}n_{i,j})
\]

where \( \delta^k_j = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases} \). Specifically, the first column of \( (m_{i,j})E_{k,0}(n_{i,j}) = (m_{i,k}n_{0,j}) \) is

\[
(\begin{array}{c} m_{1,k} \\ \vdots \\ m_{n,k} \end{array})^T,
\]

a constant multiple of the \( k \)th column of \( M \). □

Thus, if we know what our automorphism does to the matrices \( E_{k,0} \), then we can determine our conjugating matrix (which is only defined up to a scalar multiple).

The last thing we need in order to compute the conjugating matrix is the following lemma, which will be helpful in all future computations.

Lemma 2. If \( A = (a_{i,j})_{i,j=0}^{n-1} \), then \( L^rM^sA = (q^{-2s(i-r)}a_{i-r,j}) \) (where the index \( i - r \) is taken modulo \( n \)).

Proof. Note that \( L \) is a permutation matrix so that \( LA = (a_{i-1,j}) \) where the index is taken modulo \( n \). It is also clear that \( MA = (q^{-2i}a_{i,j}) \) since \( M \) is diagonal. More generally, this means \( L^rA = (a_{i-r,j}) \) and \( M^sA = (q^{-2is}a_{i,j}) \) so that \( L^rM^sA = (q^{-2s(i-r)}a_{i-r,j}) \). □

Theorem 9. The conjugating matrix associated with \( B \) is

\[
C = (c_{i,j})_{i,j=0}^{n-1} = (q^{-b^{-1}d(i-a)j^2 - 2c(i-a)j - acj^2}).
\]

Proof. Because the sum \( 1 + q + q^2 + \cdots + q^{n-1} = \frac{q^n-1}{q-1} = 0 \), and since \( n \) is prime, we have that \( \sum_{i=0}^{n-1} M^i = \sum_{i=0}^{n-1} e_{0,i} = nE_{0,0} \), and from here, we can use \( L = e_{1,0} \) to get our desired matrices (up to a scalar multiple):

\[
L^j \sum_{i=0}^{n-1} M^i = e_{j,0} \sum_{i=0}^{n-1} e_{0,i} = nE_{j,0}.
\]
Now we can see where our automorphism sends these matrices to determine our conjugating matrix as in Lemma 1. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ sends $e_{j,0} \sum_{i=0}^{n-1} e_{0,i}$ to $e_{a,j,cj} \sum_{i=0}^{n-1} e_{bi,di}$ whose first column (indexed by $j = 0$) is the first column of $\sum_{i=0}^{n-1} e_{bi,di}$, call this vector $v = \begin{pmatrix} v_0 \\ \vdots \\ v_{n-1} \end{pmatrix}$.

Since $e_{bi,di} = q^{-bd_i^2} L^b_i M^d_i$, and the first column of $M$ is $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, we have that $v_b = q^{-bd_i^2}$ where the index is taken modulo $n$. Hence, $v_i = q^{-bd_i^2} = q^{-b^{-1}d_i^2}$ where $b^{-1}$ is the multiplicative inverse of $b$ in $\mathbb{Z}/n\mathbb{Z}$.

In general, the first column of $e_{aj,cj} \sum_{i=0}^{n-1} e_{bi,di}$ is $e_{aj,cj}v = q^{-acj^2} L^aj M^cj (q^{-b^{-1}d_i^2})^{n-1}_{i=0}$ from which we can finally write down our desired matrix by using Lemma 2.

\[ q^{-acj^2} \left[ L^aj M^cj (q^{-b^{-1}d_i^2})^{n-1}_{i=0} \right] = q^{-acj^2} (q^{-b^{-1}d(i-a)^2} q^{-2cj(i-a)}) = (q^{-b^{-1}d(i-a)^2 - 2cj(i-a) - acj^2}). \]

5. Trace Calculation

We now compute the trace of the matrix found in Section 4.

**Theorem 10.** Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The matrix $C$ we computed in the last section, associated to $B$ has

\[ \text{Tr}(C) = \left( \frac{K_B}{n} \right) \frac{1 + i^{-n}}{1 + i^{-1} \sqrt{n}} \]

where $K_B := -(b^{-1}d(1-a)^2 + c(2-a))$.

**Proof.** Above we see that the elements along the diagonal of our conjugating matrix are of the form $c_{i,i} = q^{-b^{-1}d(i-a)^2} q^{-i^{-1}d(1-a)^2 + 2c(i-a) + ac} = q^{-i^{-2}d(1-a)^2 + 2c(i-a) + ac}$.

Define $K_B := -(b^{-1}d(1-a)^2 + c(2-a))$. Then we have that \( \text{Tr}(C) = \sum_{i=0}^{n-1} q^{i^2K_B} \) and this is just the Gauss sum, $G(K_B, n) = \left( \frac{K_B}{n} \right) G(1, n) = \left( \frac{K_B}{n} \right) \frac{1 + i^{-n}}{1 + i^{-1} \sqrt{n}}$ where $\left( \frac{K_B}{n} \right)$ is the Legendre symbol of $K_B$ with respect to $n$. \( \Box \)

6. Determinant Calculation

We now wish to calculate the determinant of $C$, which would be impractical to find through direct calculation and so I will use the following.

**Proposition 7.** $CC^* = nI_n$ where $C^*$ is the conjugate transpose of $C$.

**Proof.** Let $v = (q^{-b^{-1}d_i^2})^{n-1}_{i=0}$ as above. I have shown that the $j$th column of $C$ is $e_{aj,cj}v = (e_{a,c})^j v$, that is, $C = (v Av A^2 v \cdots A^{n-1}v)$ where $A := e_{a,c}$. Hence,

\[ CC^* = (v Av A^2 v \cdots A^{n-1}v) \begin{pmatrix} v^* A^* \\ v^*(A^*)^2 \\ \vdots \\ v^*(A^*)^{n-1} \end{pmatrix} = \sum_{k=0}^{n-1} A^k v v^*(A^*)^k. \]
We know that \( v^* = (q^{k-1}d^{j^2}) \) and so \( vv^* \) is the matrix \( (q^{b-1}d^{j^2-\sqrt{i^2}}) \). By definition, \( A_k = q^{-ack^2}L^kM^k \) so that \( A_kvv^* = (q^{-ack^2}q^{-2ck(i-ak)}q^{b-1}d(j^2-(i-ak)^2)) \) by Lemma 2. Finally, consider \( (A^*)^k = q^{ack^2}(M^*)^k(L^*)^k \). To compute this, we will need an analogous statement to Lemma 2 but for \( L^* \) and \( M^* \).

**Lemma 3.** If \( A = (a_{i,j})_{i,j=0}^{n-1} \), then \( A(M^*)^s(L^*)^r = (q^{2s(j-r)}a_{i,j-r}) \) (where the index \( j-r \) is taken modulo \( n \)).

**Proof.** Note \( L^* \) is a (column) permutation matrix so that \( (a_{i,j})L^* = (a_{i,j-1}) \), and note \( (a_{i,j})M^* = (q^{2j}a_{i,j}) \) so that in general \( (a_{i,j})(L^*M^*)^s = (a_{i,j})(M^*)^s(L^*)^r = (q^{2s(j-r)}a_{i,j-r}) \). \( \square \)

We can now use this to calculate,

\[
A_kvv^*(A^*)^k = (q^{-ack^2}q^{2ck(j-ak)}q^{-ack^2}q^{-2ck(i-ak)}q^{b-1}d((j-ak)^2-(i-ak)^2))
\]

\[
= (q^{2ck((j-ak)-(i-ak))}+b^{-1}d((j-ak)^2-(i-ak)^2))
\]

\[
= (q^{2ck((-i-ak))}+b^{-1}d(j^2-i^2-2ak(j-i))}
\]

Therefore, \( CC^* = (\sum_{k=0}^{n-1}q^{2ck((j-ak)-(i-ak))}+b^{-1}d(j^2-i^2-2ak(j-i))) \) and it is clear that if \( i = j \), then the above becomes \( \sum_{k=0}^{n-1}q^0 = n \). If \( i \neq j \), then the exponent of \( q \) is linear in \( k \) and since \( n \) is prime and \( q \) is a primitive \( n \)th root of unity, we have that we will get a unique \( n \)th root of unity for each \( k \) so that we get the sum \( \sum_{k=0}^{n-1}q^k = 0 \). Therefore, \( CC^* = nI_n \). \( \square \)

Using this fact, we can conclude,

**Theorem 11.** \( \det(C) = \pm \sqrt{n^m} \).

**Proof.** \( \det(CC^*) = \det(C) \det(C^*) = \det(C)^2 = n^m \Rightarrow \det(C) = \pm \sqrt{n^m} \). \( \square \)

Therefore,

\[
\frac{\text{Tr}(C)}{\sqrt{\det(C)}} = \pm \left( \frac{K_B}{n} \right) \frac{1+\frac{n}{2}}{1+\frac{1}{2}}
\]

Although this should have to do with the Witten-Reshetikhin-Turaev invariant of the mapping cylinder of the corresponding element of the mapping class group \([1]\). We inspected these invariants as computed by Lisa Jeffrey \([1]\), but do not personally see the connection.

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