PARTIAL DETERMINANTS OF KRONECKER PRODUCTS

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Abstract. Let \( \det_2(A) \) be the block-wise determinant (partial determinant). We consider the condition for completing the determinant
\[
\det(\det_2(A)) = \det(A),
\]
and characterize the case when for an arbitrary Kronecker product \( A \) of matrices over an arbitrary field. Further insisting that \( \det_2(AB) = \det_2(A)\det_2(B) \), for Kronecker products \( A \) and \( B \), yields a multiplicative monoid of matrices. This leads to a determinant-root operation \( \text{Det} \) which satisfies \( \text{Det}(\det_2(A)) = \text{Det}(A) \) when \( A \) is a Kronecker product of matrices for which \( \text{Det} \) is defined.

1. Introduction

Let \( \{ E_{ij}^{[n]} : i, j \in \{1, \ldots, n\} \} \) denote the standard basis in \( M_n(\mathbb{F}) \), the algebra of \( n \times n \) matrices over a field \( \mathbb{F} \). In other words, \( E_{ij}^{[n]} \) is an \( n \times n \) matrix with a 1 in row \( i \) and column \( j \) and all other entries are zero. The standard basis is given entry-wise by \( (E_{ij}^{[n]})_{kl} = \delta_{ik}\delta_{jl} \), where \( \delta_{ik} \) is the Kronecker delta
\[
\delta_{ik} = \begin{cases} 1 & i = k, \\
0 & i \neq k. \end{cases}
\]

Let \( \otimes \) denote the Kronecker product of matrices, i.e. for \( A \in M_m(\mathbb{F}) \) and \( B \in M_n(\mathbb{F}) \) the matrix \( A \otimes B \) is given block-wise by
\[
A \otimes B := \begin{pmatrix}
(A)_{11}B & (A)_{12}B & \cdots & (A)_{1m}B \\
(A)_{21}B & (A)_{22}B & \cdots & (A)_{2m}B \\
\vdots & \vdots & \ddots & \vdots \\
(A)_{m1}B & (A)_{m2}B & \cdots & (A)_{mm}B
\end{pmatrix}.
\]

Choi gives the definition of the partial determinants \( \det_1(A) \) and \( \det_2(B) \) as follows [2].

Definition 1. Let \( A, B \in M_m(\mathbb{F}) = M_m(M_n(\mathbb{F})) = M_m(\mathbb{F}) \otimes M_n(\mathbb{F}) \) with
\[
A = \sum_{i,j=1}^{n} A_{ij} \otimes E_{ij}^{[n]} , \quad B = \sum_{i,j=1}^{m} E_{ij}^{[m]} \otimes B_{ij}.
\]
The partial determinants \( \det_1(A) \) and \( \det_2(B) \) are given by
\[
\det_1(A) := \sum_{i,j=1}^{n} \det(A_{ij})E_{ij}^{[n]} , \quad \det_2(B) := \sum_{i,j=1}^{m} \det(B_{ij})E_{ij}^{[m]}.
\]

This definition is analogous to that of the partial trace. The partial trace features prominently in quantum information theory (see for example [1]).
**Definition 2.** Let $A, B \in M_{mn}(\mathbb{F})$ with
\[ A = \sum_{i,j=1}^{n} A_{ij} \otimes E_{ij}^{[m]}, \quad B = \sum_{i,j=1}^{m} E_{ij}^{[m]} \otimes B_{ij}. \]
We define the partial traces
\[ \text{tr}_1(A) := \sum_{i,j=1}^{n} \text{tr}(A_{ij})E_{ij}^{[m]}, \quad \text{tr}_2(B) := \sum_{i,j=1}^{m} \text{tr}(B_{ij})E_{ij}^{[m]}. \]

The partial trace is linear while the partial determinant is not linear. Additionally, the partial trace is “partial” in the sense that the trace can be completed
\[ \text{tr}(\text{tr}_2(A)) = \text{tr}(\text{tr}_1(A)) = \text{tr}(A). \]
In general, $\det(\text{det}_2(A)) \neq \det(A)$. Thompson showed that if $B \in M_{mn}(\mathbb{C})$ is positive definite, then $\det(\text{det}_2(B)) \geq \det(B)$ and that $\det(\text{det}_2(B)) = \det(B)$ if and only if $B_{ij} = \delta_{ij}B_{ij}$, where $\delta_{ij}$ denotes the Kronecker delta $[7]$. This article provides initial results on when equality holds in a more general setting.

Another important property to consider is whether the identity $\det(AB) = \det(A)\det(B)$ carries over to the partial determinant. In other words, what are the conditions on $A$ and $B$ such that $\det_1(AB) = \det_1(A)\det_1(B)$?

Since $B \otimes C$ is permutation similar to $C \otimes B$ via the vec-permutation matrix $P$ (perfect shuffle matrix $[3][8]$), so that $\det_2(A) = \det_1(PAP^T)$, we confine our attention to $\det_1(A)$.

In the following, we only consider partial determinants of Kronecker products. Let $\circ$ denote the Hadamard product and $A^{(m)}$ denote the $m$-th Hadamard power of the matrix $A$ (i.e. the entry-wise power). For every $(0,1)$-matrix $A$ we have $A^{(m)} = A$, and most results are straightforward for $(0,1)$-matrices and over the field $\mathbb{F} = GF(2)$. Our main results are characterizations in terms of the distributivity of the Hadamard power over matrix products. In the case of $\det_1(AB) = \det_1(A)\det_1(B)$ for Kronecker products, we have the following theorem.

**Theorem.** Let $A, B \in M_m(\mathbb{F})$ and $C, D \in M_n(\mathbb{F})$. Then
\[ \det_1((AB) \otimes (CD)) = \det_1((A \otimes C)\det_1(B \otimes D)) \]
if and only if $\det(AB) = 0$ or $(CD)^{(m)} = C^{(m)}D^{(m)}$.

The partial determinant may be completed when the $m$-th Hadamard power and determinant commute, $\det(B^{(m)}) = (\det(B))^{(m)} = \det(B)^m$.

**Theorem.** Let $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$. Then
\[ \det_1((A \otimes B)) = \det_1(A \otimes B) \]
if and only if $\det(A) = 0$ or $\det(B^{(m)}) = \det(B)^m$.

These two conditions on the matrices in $M_n(\mathbb{F})$ which satisfy the above two theorems allow us to characterize the matrices in terms of underlying monoids in $M_n(\mathbb{F})$. As we will see below, the $m$-th Hadamard power distributes over matrix products and distributes over column/row-wise Grassman products (wedge products) in these monoids. Finally, we consider a monoid $D_m(\mathbb{F})$ and a determinant-root operation $\text{Det}$, such that the partial determinant-root obeys

**Theorem.**

1. Let $A \in D_m(\mathbb{F})$ and $C \in D_n(\mathbb{F})$. Then
\[ \text{Det}(A \otimes C) = \text{Det}(\text{Det}_1(A \otimes C)). \]

2. Let $A, B \in D_m(\mathbb{F})$ and $C, D \in M_n(\mathbb{F})$. Then
\[ \text{Det}_1((A \otimes C)(B \otimes D)) = \text{Det}_1(A \otimes C)\text{Det}_1(B \otimes D). \]
2. Kronecker products

Let $I_n$ denote the $n \times n$ identity matrix, and $0_n$ the $n \times n$ zero matrix. First we note that $\det(I_{mn}) = \det(I_m \otimes I_n) = I_n$ analogous to $\det(I) = 1$.

**Lemma 1.** Let $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$. Then

$$\det(A \otimes B) = \det(A) B^{(m)}.$$  

**Proof.** The proof follows immediately from

$$A \otimes B = \sum_{i,j=1}^{n} ((B)_{ij} A) \otimes E_{ij}^{[n]},$$

where $B = (B)_{ij}$. □

This is a direct consequence of the remark by Choi that $\det_1(A)$ may be computed as a determinant where the blocks are treated as scalars and instead of the usual scalar product we use the Hadamard product [2].

**Theorem 1.** Let $A, C \in M_m(\mathbb{F})$ and $B, D \in M_n(\mathbb{F})$ with $B \circ D = 0_n$. Then

$$\det(A \otimes B + C \otimes D) = \det(A \otimes B) + \det(C \otimes D).$$

**Proof.** We have

$$A \otimes B + C \otimes D = \sum_{i,j=1}^{n} ((B)_{ij} A) \otimes E_{ij}^{[n]} + \sum_{i,j=1}^{n} ((D)_{ij} C) \otimes E_{ij}^{[n]},$$

and since $B \circ D = 0_n$ if follows that $(B)_{ij} \neq 0 \implies (D)_{ij} = 0$ and vice-versa. The two sums in (1) are over disjoint subsets of $\{1, \ldots, n\}^2$ and

$$\det(A \otimes B + C \otimes D) = \sum_{B \circ j=1}^{n} \det(B)_{ij}^{[m]} E_{ij}^{[n]} + \sum_{D \circ j=1}^{n} \det(C)_{ij}^{[m]} E_{ij}^{[n]} = \det(A \otimes B) + \det(C \otimes D).$$

□

Lemma 1 provides an immediate characterization of partial determinants that can be completed.

**Theorem 2.** Let $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$. Then

$$\det(\det(A \otimes B)) = \det(A \otimes B)$$

if and only if $\det(A) = 0$ or $\det(B^{(m)}) = \det(B)^m$.

Drnovšek considered a much stronger condition in [3] for matrices $B$ (which also satisfy Theorem 2), namely that $B^{(r)} = B^r$ for all $r \in \mathbb{N}$. Under the determinant, we have a weaker condition which leads to somewhat trivial cases for triangular matrices and $(0,1)$-matrices. For triangular matrices $B$ we have $\det(B^{(m)}) = \det(B^m) = \det(B)^m$.

**Corollary 1.** Let $A \in M_m(\mathbb{F})$ be arbitrary and let $B \in M_n(\mathbb{F})$ be triangular. Then

$$\det(\det(A \otimes B)) = \det(A \otimes B).$$

For $(0,1)$-matrices $B$ we have $B^{(m)} = B$, and hence the following corollary to Theorem 2.
Corollary 2. Let $A \in M_m(F)$ be arbitrary and let $B \in M_n(F)$ be a $(0,1)$-matrix. Then
\[
\det(\det_1(A \otimes B)) = \det(A \otimes B)
\]
if and only if $\det(A) \det(B) = 0$ or $\det(B)$ is an $(m - 1)$-th root of unity.

When $n = 2$, the characterization is straightforward since
\[
\det\begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2
\]
if and only if $b^2c^2 = abcd$, so that $bc = 0$ (triangular matrix) or $ad = bc \neq 0$ (rank-1 matrix) and both cases satisfy $\det(B^{(m)}) = \det(B)^m$. Here we used the fact that a rank-1 matrix $xy^T$ remains a rank-1 matrix $x^{(m)}(y^{(m)})^T$ under the Hadamard power, where $x$ and $y$ are column vectors. Now we consider transformations which preserve the partial determinant. The following theorem characterizes the multiplicative property of the partial determinant.

Theorem 3. Let $A, B \in M_m(F)$ and $C, D \in M_n(F)$. Then
\[
\det_1((A \otimes C)(B \otimes D)) = \det_1(A \otimes C) \det_1(B \otimes D)
\]
if and only if $\det(AB) = 0$ or $(CD)^{(m)} = C^{(m)}D^{(m)}$.

Proof. Applying Lemma 1 on both sides of the equation provides
\[
\det(AB)(CD)^{(m)} = \det(AB)C^{(m)}D^{(m)}
\]
and the result follows. $\square$

When $F = GF(2)$ we have the following corollary.

Corollary 3. Let $A, B \in M_m(GF(2))$ and $C, D \in M_n(GF(2))$. Then
\[
\det_1((A \otimes C)(B \otimes D)) = \det_1(A \otimes C) \det_1(B \otimes D).
\]

Since a row or column permutation of a matrix commutes with the Hadamard power, as does multiplying each row by a constant, we have the following corollary to Theorem 3.

Corollary 4. Let $A, B \in M_m(F)$ and $C, P \in M_n(F)$, where $P$ is a permutation matrix or a diagonal matrix. Then
\[
\begin{align*}
\det_1((AB) \otimes (PC)) &= \det_1(A \otimes P) \det_1(B \otimes C), \\
\det_1((AB) \otimes (CP)) &= \det_1(A \otimes C) \det_1(B \otimes P).
\end{align*}
\]

Thus if $B \in M_n(F)$ has no more than $n(n + 1)/2$ non-zero entries, and there exist permutations $P, Q \in M_n(F)$ such that $PBQ^T$ is triangular, then $\det_1(A \otimes B) = \det(A \otimes B)$. Of course, this is not true in general. For example
\[
B = \begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
has $7 \leq 10$ non-zero entries, but no such $P$ and $Q$ exist. We note that multiplying by $P$ preserves the number of non-zero entries in each column and multiplying by $Q^T$ preserves the number of non-zero entries in each row.
3. Monoids for completable partial determinants

3.1. Characterizations in terms of monoids. Theorems 2 and 3 provide a closure result for matrix multiplication, if $C$ and $D$ are matrices satisfying both theorems then $CD$ satisfies Theorem 2. A set of such matrices, with the identity, form a monoid under matrix multiplication. This follows since $(CD)^{(m)} = C^{(m)}D^{(m)}$ implies
\[
\det ((CD)^{(m)}) = \det (C^{(m)}) \det (D^{(m)}) = \det (C)^m \det (D)^m = \det (CD)^m
\]
when $\det (C^{(m)}) = \det (C)^m$ and $\det (D^{(m)}) = \det (D)^m$. The identity $C = I_n$ is clearly in this set. The diagonal matrices in $M_n(\mathbb{F})$ provide a non-trivial example of such a monoid. Another example is the set of all matrices with at most one non-zero row together with the identity matrix.

In the sequel, we use the term monoid to mean a monoid under the usual matrix multiplication. Theorems 2 and 3 trivially provide the next corollary.

**Corollary 5.** A monoid $M$ of matrices $C, D \in M_n(\mathbb{F})$ satisfying
\[
\det(C^{(m)}) = \det(C)^m, \quad \det(D^{(m)}) = \det(D)^m \quad \text{and} \quad (CD)^{(m)} = C^{(m)}D^{(m)}
\]
satisfies
\[
\det (\det (A \otimes B)) = \det (A \otimes B),
\]
\[
\det ((A \otimes C) \otimes (B \otimes D)) = \det (A \otimes C) \det (B \otimes D)
\]
for all $A, B \in M_m(\mathbb{F})$ and $C, D \in M$.

A necessary condition for a matrix $C \in M$ to be in such a monoid, is that $(C^k)^{(m)} = (C^{(m)})^k$ for all $k \in \mathbb{N}$. In the case of $M_n(GF(2))$, the largest such monoid is $M = M_n(GF(2))$. Let $C_k$ denote the $k$-th column of $C$ and $D_{(k)}$ denote the $k$-th row of $D$. Consider the matrix
\[
\Phi(C, D) := (I_n \otimes C) \left( \sum_{i,j=1}^n E_{i,j}^{[n]} \otimes E_{i,j}^{[m]} \right) (I_n \otimes D) = \sum_{i,j=1}^n E_{i,j}^{[n]} \otimes (C_iD_{(j)}).
\]
Then $CD = \text{tr}_1(\Phi(C, D))$ is the sum of the matrices $C_kD_{(k)}$ on the block diagonal of $\Phi(C, D)$. Similarly, $DC = \text{tr}_2(\Phi(C, D))$. We also have $\Phi(C, D)^{(m)} = \Phi (C^{(m)}, D^{(m)}).$ Thus $(CD)^{(m)} = C^{(m)}D^{(m)}$ if and only if
\[
\text{tr}_1 \left( \Phi(C, D)^{(m)} \right) = (\text{tr}_1(\Phi(C, D)))^{(m)},
\]
or equivalently, since $(C_kD_{(k)})^{(m)} = C_k^{(m)}D_{(k)}^{(m)},$
\[
(2) \quad \sum_{k=1}^n C_k^{(m)}D_{(k)}^{(m)} = \sum_{(k_1, \ldots, k_m) \in \{1, \ldots, n\}^m} (C_{k_1} \circ \cdots \circ C_{k_m})D_{(k_1)} \circ \cdots \circ D_{(k_m)}.
\]
The set $\{ (k, \ldots, k) \} \subseteq \{1, \ldots, n\}^m$ selects the block diagonal matrices of the matrix $\Phi(C, D)^{(m)}$ in the sum, and the non-diagonal block entries sum to zero. When $F = \mathbb{R}$ and $C$ and $D$ are matrices with non-negative entries, then $(C_{k_1} \circ \cdots \circ C_{k_m}) = (D_{(k_1)} \circ \cdots \circ D_{(k_m)})^T = 0$ for all non-diagonal block entries, i.e. $C$ and $D$ are in the monoid $\{ PDQ^T \}$ of matrices with at most one non-zero entry in each row and column ($P$ and $Q$ are permutation matrices and $D$ is diagonal).

Let us now consider a stronger condition, namely we consider monoids satisfying
\[
\det(C^{(m)}) = \det(C)^m, \quad \det(D^{(m)}) = \det(D)^m \quad \text{and} \quad (CD)^{(m)} = C^{(m)}D^{(m)}
\]
for all $m \in \mathbb{N}$. On an ordered field we need only consider $m = 2$, with an additional condition on matrix multiplication of elements of the monoid.
The symmetric polynomials play a central role in this characterization. Let 
\[ p_m(x_1, \ldots, x_n) = x_1^m + \cdots + x_n^m \]
denote the \( m \)-th power sum, and
\[ e_m(x_1, \ldots, x_n) = \sum_{j_1 < j_2 < \cdots < j_m} x_{j_1} \cdots x_{j_m}. \]
denote the \( m \)-th elementary symmetric sum \([5, p. 28]\).

Proof. The equation
\[ \sum_{i=1}^n x_i^m = 2 \]
using
\[ (1) = \sum_{i=1}^n x_i^m \]
we find
\[ \sum_{i=1}^n x_i = m. \]
and that
\[ \sum_{i=1}^n x_i = m. \]
and
\[ \sum_{i=1}^n x_i = m. \]

Lemma 2. Let \( \mathbb{F} \) be a field with characteristic 0, and \( x_1, \ldots, x_n \in \mathbb{F} \). The equations
\[ p_m(x_1, \ldots, x_n) = (e_1(x_1, \ldots, x_n))^m \quad \text{for all } m \in \mathbb{N} \]
hold if and only if \( x_ix_j = 0 \) for \( j \neq k \) (\( j,k \in \{1, \ldots, n\} \)).

Proof. The equation \( p_2(x_1, \ldots, x_n) = (e_1(x_1, \ldots, x_n))^2 \) yields \( e_2(x_1, \ldots, x_n) = 0 \)
in \([4]\), when \( m = 2 \). Using \( e_2(x_1, \ldots, x_n) = 0 \) in \([4]\), when \( m = 3 \) yields that \( e_3(x_1, \ldots, x_n) = 0 \). Continuing in the same manner for \( p_4, p_5, \ldots, p_n \), we find that \( e_n = x_1 \cdots x_n = 0 \). It follows that there exists an \( x_j \) satisfying \( x_j = 0 \). Now \( e_{n-1}(x_1, \ldots, x_n) = x_1 \cdots x_{j-1}x_{j+1} \cdots x_n = 0 \) which yields another \( x_k \) with \( x_k = 0 \), where \( k \neq j \). Proceeding in this way we arrive at \( e_2(x_1, \ldots, x_n) = x'_jx'_k = 0 \) for some \( j' \neq k' \), where \( x_j = 0 \) for \( j \notin \{j', k'\} \).

Theorem 4. Let \( \mathbb{F} \) be an ordered field and \( M \subseteq M_n(\mathbb{F}) \). Then the following statements are equivalent.

1. \( M \) is a monoid of matrices satisfying \((CD)^{(m)} = C^{(m)}D^{(m)}\) for all \( C,D \in M \) and \( m \in \mathbb{N} \);
2. \( M \) is a monoid of matrices satisfying
   (a) \((CD)^{(2)} = C^{(2)}D^{(2)}\) for all \( C,D \in M \),
   (b) for all \( C,D \in M \) and all \( i,j \in \{1, \ldots, n\} \) there exists \( k \in \{1, \ldots, n\} \) such that \((CD)_{ij} = (C)_{ik}(D)_{kj}\).

Proof. Each monoid includes the identity matrix, and the properties are all identically satisfied for \( C = I_n \) and \( D \in M \). Now we show that monoids of type 1 are also monoids of type 2. Comparing the entries of the matrices in equation \([2]\) in row \( i \) and column \( j \) we find
\[ \sum_{k=1}^n (C)_{ik}^{(m)}(D)_{kj}^{(m)} = \left( \sum_{k=1}^n (C)_{ik}(D)_{kj} \right)^m. \]

Applying Lemma \([2]\) yields for all \( i,j,k,k' \in \{1, \ldots, n\} \) with \( k \neq k' \),
\[ (C)_{ik}(D)_{kj}(C)_{ik'}(D)_{kj} = 0. \]
Next we show that monoids of type 2 are also monoids of type 1. Assume \((CD)^{(2)} = C^{(2)}D^{(2)}\) and that \((CD)_{ij} = (C)_{ik}(D)_{kj}\) for some \( k \in \{1, \ldots, n\} \). Then
\[ ((C)_{ik}(D)_{kj})^2 = ((CD)^{(2)})_{ij} = (C^{(2)}D^{(2)})_{ij} = \sum_{k'=1}^n (C)_{ik'}^{(2)}(D)_{kj}^{(2)}. \]
so that \( \sum_{k' = 1}^{n} (C)_{k'^{2}}^{j} (D)_{k'j} = 0 \) and, since \( \mathbb{F} \) is an ordered field, equation (4) follows.

\[
\square
\]

**Corollary 6.** Let \( \mathbb{F} \) be an ordered field and let \( M \subseteq M_n(\mathbb{F}) \). Then the following statements are equivalent.

1. \( M \) is a monoid of matrices closed under the matrix transpose and satisfying \((CD)^{(m)} = C^{(m)}D^{(m)}\) for all \( C, D \in M \) and \( m \in \mathbb{N} \);
2. \( M \) is a submonoid of the monoid of matrices with at most one non-zero entry in each row and column.

If we insist that Theorem 1 holds for all \( m \in \mathbb{N} \), applying Lemma 2 yields the following theorem, which we state without proof.

**Theorem 5.** Let \( B, C \in M_n(\mathbb{F}) \). Then

\[
\det(A \otimes (B + C)) = \det(A \otimes B) + \det(A \otimes C)
\]

holds for all \( m \in \mathbb{N} \) and \( A \in M_m(\mathbb{F}) \) if and only if \( B \circ C = 0_n \).

For the \( 2 \times 2 \) matrices, we find that the property \( \det(C^{(m)}) = \det(C)^m \) for all \( m \in \mathbb{N} \) is satisfied only when \( C \) is triangular, which is consistent with the requirement that \((C^2)^{(2)} = (C^{(2)})^2\). Let \( D_n(\mathbb{F}) \) denote the set of diagonal \( n \times n \) matrices over \( \mathbb{F} \), \( R_n(\mathbb{F}) \) the set of matrices with at most one non-zero row, and \( C_n(\mathbb{F}) \) the set of matrices with at most one non-zero column.

**Theorem 6.** Let \( \mathbb{F} \) be an ordered field and \( M \subseteq M_2(\mathbb{F}) \). Then the following statements are equivalent.

1. \( M \) is a monoid of matrices satisfying
   a. \( \det(C^{(m)}) = \det(C)^m \) for all \( C \in M \) and \( m \in \mathbb{N} \);
   b. \( (CD)^{(m)} = C^{(m)}D^{(m)} \) for all \( C, D \in M \) and \( m \in \mathbb{N} \);
2. \( M \) is a monoid of matrices with
   a. \( M \subseteq D_2(\mathbb{F}) \cup C_2(\mathbb{F}) \), or
   b. \( M \subseteq D_2(\mathbb{F}) \cup R_2(\mathbb{F}) \).

**Proof.** First we prove that monoids of type 1 are monoids of type 2. Let

\[
C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in M.
\]

As in the proof of Theorem 4 above, the condition \((C^2)^{(2)} = (C^{(2)})C^{(2)}\) yields

\[
(c_{11}c_{11})(c_{12}c_{21}) = (c_{22}c_{22})(c_{21}c_{12}) = (c_{11}c_{12})(c_{12}c_{22}) = (c_{21}c_{11})(c_{22}c_{21}) = 0.
\]

The zero matrix satisfies all the conditions, so assume \( C \neq 0 \). If either of \( c_{11} \) or \( c_{22} \) are non-zero, then \( c_{12}c_{21} = 0 \) and \( C \) is a triangular matrix with \( \det(C^{(m)}) = \det(C)^m \). Then \( C \) has one of the forms

\[
\begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \end{pmatrix}, \quad \begin{pmatrix} c_{11} & c_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{21} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & c_{12} & 0 \\ 0 & 0 & c_{22} \end{pmatrix}.
\]

If \( c_{11} = c_{22} = 0 \), then \( \det(C^{(m)}) = -c_{12}c_{21}^m = (-1)^m c_{12}c_{21}^m = \det(C)^m \). When \( m = 2 \) we find again that \( C \) must be of the form

\[
\begin{pmatrix} 0 & c_{12} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ c_{21} & 0 \end{pmatrix}.
\]
This means that every monoid of type 1 satisfies $M \subseteq D_2(\mathbb{F}) \cup C_2(\mathbb{F}) \cup R_2(\mathbb{F})$. Let $\{e_1, e_2\}$ be the standard basis in $\mathbb{F}^2$. Suppose that $x, y \in \mathbb{F}^2$ and $j, k \in \{1, 2\}$. Let $A = xe_j^T \in M \cap (C_2(\mathbb{F}) \setminus R_2(\mathbb{F}))$ and $B = e_ky^T \in M \cap (R_2(\mathbb{F}) \setminus C_2(\mathbb{F}))$. Then

$$AB = xe_j^Te_ky^T = \delta_{jk}xy^T \in M.$$ 

Thus $x$ or $y$ has at most one non-zero entry, which contradicts $A \notin R_2(\mathbb{F})$ or $B \notin C_2(\mathbb{F})$. On the other hand,

$$BA = e_ky^Txe_j^T = (y^Tx)e_ke_j^T \in M$$

if and only if $(y^Tx)^m = (y^Tx)^{m'}$ for all $m \in \mathbb{N}$, and again $e_ky^T \in C_2(\mathbb{F})$ or $xe_j^T \in R_2(\mathbb{F})$ which yields a contradiction. Thus $M \cap (C_2(\mathbb{F}) \setminus R_2(\mathbb{F})) = \emptyset$ or $M \cap (R_2(\mathbb{F}) \setminus C_2(\mathbb{F})) = \emptyset$. It follows that each monoid of type 1 is a monoid of type 2.

It is straightforward to verify that every monoid of type 2 is also of type 1. Each of the three types of matrices (diagonal, at most one non-zero row, at most one non-zero column) immediately satisfy $\det(C^{(m)}) = \det(C)^m$ for all $m \in \mathbb{N}$. If $D$ is diagonal then $(CD)^{(m)} = C^{(m)}D^{(m)}$ and $(DC)^{(m)} = D^{(m)}C^{(m)}$ for all $C \in M$. In $C_2(\mathbb{F})$ and $R_2(\mathbb{F})$, we have

$$(xe_j^T)(ye_j^T)^{(m)} = (e_ky^T)(xe_j^T)^{(m)} = (xe_j^T)(ye_j^T)^{(m)} = (e_ky^T)(xe_j^T)^{(m)}.$$ 

for any column vectors $x, y \in \mathbb{F}^2$ and $j, k \in \{1, 2\}$. □

We note that some of the observations in the previous proof hold in general. In a monoid $M \subseteq M_n(\mathbb{F})$, over an ordered field $\mathbb{F}$ and satisfying Theorem 4, we have for each $C \in M$ and $i, j \in \{1, \ldots, n\}$

$$(C)_{ii}(C)_{ij}(C)_{jj} = (C)_{ii}(C)_{ij}(C)_{jj} = 0,$$

$$(C)_{ij}(C)_{jj}(C)_{ij} = (C)_{ij}(C)_{jj}(C)_{ij} = 0.$$ 

This means that every $2 \times 2$ submatrix of $C \in M$ with entries in columns and rows $i$ and $j$ has one of the forms

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \quad \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix}.$$ 

The set of equations given by (5) is invariant under any permutation similarity transformation, i.e. interchanging $i \leftrightarrow k$ and $j \leftrightarrow l$ leaves (5) invariant (since we consider all $i$ and $j$). Consider $M_3(\mathbb{F})$ over an ordered field $\mathbb{F}$. The set of equations in (5) concern three $2 \times 2$ submatrices, two of which lie on the diagonal (indicated by squares) and the outermost $2 \times 2$ submatrix, in other words, we consider the corners of each of the squares:

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

Any reordering $PCPT^T$ of the rows and columns of the matrix $C$ preserves the equations (5) (here $P$ is a permutation matrix). We also have that

$$\det(PCPT)^m = \det((PCPT)^{(m)})$$

if and only if $\det(C)^m = \det(C^{(m)})$, and that

$$((PCPT)(PDP^T))^{(m)} = (PCPT)^{(m)}(PDP^T)^{(m)}$$
In this way, we can proceed to characterize monoids of Corollary 5 in $M_{\text{product}}$ (see for example [6, p. 173]), considering the 4 cases if and only if (C) and for each matrix $\det(C)^m = \det(C^{(m)})$ for all $m \in \mathbb{N}$. In the second case ($c_{11}c_{22} \neq 0$ and $c_{33} = 0$) we have one of the matrices
\[
\begin{pmatrix}
  c_{11} & 0 & c_{13} \\
  0 & c_{22} & c_{23} \\
  0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
  c_{11} & 0 & 0 \\
  0 & c_{22} & c_{23} \\
  0 & c_{31} & 0
\end{pmatrix},
\begin{pmatrix}
  c_{11} & 0 & c_{13} \\
  0 & c_{22} & 0 \\
  0 & c_{32} & 0
\end{pmatrix},
\begin{pmatrix}
  c_{11} & 0 & 0 \\
  0 & c_{22} & 0 \\
  c_{31} & c_{32} & 0
\end{pmatrix},
\]
and for each matrix $\det(C)^m = \det(C^{(m)})$. In the third case ($c_{11} \neq 0$ and $c_{22} = c_{33} = 0$) we have one of the matrices
\[
\begin{pmatrix}
  c_{11} & 0 & c_{13} \\
  0 & 0 & c_{24} \\
  0 & c_{32} & 0
\end{pmatrix},
\begin{pmatrix}
  c_{11} & 0 & 0 \\
  0 & 0 & c_{23} \\
  c_{31} & c_{32} & 0
\end{pmatrix},
\]
and for each matrix $\det(C) = -c_{11}c_{23}c_{32}$ so that $\det(C)^2 = \det(C^{(2)})$ if and only if $c_{23}c_{32} = 0$. In the fourth case ($c_{11} = c_{22} = c_{33} = 0$) we have the matrix
\[
\begin{pmatrix}
  0 & c_{12} & c_{13} \\
  c_{21} & 0 & c_{23} \\
  c_{31} & c_{32} & 0
\end{pmatrix}
\]
and within the monoid $M \subseteq M_3(\mathbb{F})$ satisfying Theorem \[ we have
\[
\det C^m = (c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32})^m = c_{12}^m c_{23}^m c_{31}^m + c_{13}^m c_{21}^m c_{32}^m = \det C^{(m)}
\]
since
\[
(c_{12}c_{23}c_{31})(c_{13}c_{21}c_{32}) = (c_{12}c_{21}c_{13}c_{31})c_{23}c_{32} = 0.
\]
The fourth case then yields one of the matrices (up to permutation similarity)
\[
\begin{pmatrix}
  0 & c_{12} & c_{13} \\
  c_{21} & 0 & c_{23} \\
  0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
  0 & c_{12} & 0 \\
  c_{21} & 0 & c_{23} \\
  0 & c_{31} & 0
\end{pmatrix},
\begin{pmatrix}
  0 & c_{12} & c_{13} \\
  c_{21} & 0 & 0 \\
  0 & c_{32} & 0
\end{pmatrix},
\begin{pmatrix}
  0 & c_{12} & 0 \\
  c_{21} & 0 & 0 \\
  c_{13} & c_{32} & 0
\end{pmatrix}.
\]
These observations in $M_3(\mathbb{F})$ and $M_3(\mathbb{F})$ may be applied in $M_4(\mathbb{F})$, so we consider the $2 \times 2$ and $3 \times 3$ submatrices in $M_4(\mathbb{F})$:
\[
C = \begin{pmatrix}
  c_{11} & c_{12} & c_{13} & c_{14} \\
  c_{21} & c_{22} & c_{23} & c_{24} \\
  c_{31} & c_{32} & c_{33} & c_{34} \\
  c_{41} & c_{42} & c_{43} & c_{44}
\end{pmatrix} = \begin{pmatrix}
  c_{11} & c_{12} & c_{13} & c_{14} \\
  c_{21} & c_{22} & c_{23} & c_{24} \\
  c_{31} & c_{32} & c_{33} & c_{34} \\
  c_{41} & c_{42} & c_{43} & c_{44}
\end{pmatrix}.
\]

In this way, we can proceed to characterize monoids of Corollary \[ in $M_{n+1}(\mathbb{F})$ based on observations in monoids in $M_n(\mathbb{F})$, when $\mathbb{F}$ is an ordered field.

The fact that $\det(C^{(m)}) = \det(C)^m$ can be expressed in terms of the Grassmann product (see for example [6, p. 173]),
\[
C^{(m)} \land \cdots \land C^{(m)} = (C_1 \land \cdots \land C_n)^{m}
\]
where
\[
C_1 \land \cdots \land C_n = \det(C)e_1 \land \cdots \land e_n
\]
and \{ e_1, \ldots, e_n \} is the standard basis in \mathbb{F}^n. Similarly, \det(D^{(m)}) = \det(D)^m is equivalent to

\[ D^{(m)}_{(1)} \land \cdots \land D^{(m)}_{(n)} = (D_{(1)} \land \cdots \land D_{(n)})^{(m)}. \]

To summarize, the Hadamard power distributes over matrix products and over column/row-wise Grassman products in the monoid of Corollary 5.

3.2. **Determinant-roots.** The preceding section suggests that a monoid with appropriate properties may yield a partial determinant that completes in a straightforward way. In this section we define such monoids and a “determinant-root” operation \( \text{Det} \) which is completable on Kronecker products of matrices in these monoids. First, let us define the \( m \)-th roots on the multiplicative group \( \mathbb{F}^\times \) of \( \mathbb{F} \).

Let \( G \) be a multiplicative abelian group, and \( R_m \) be the subgroup

\[ R_m = \{ g \in G : g^m = 1 \} \]

of \( G \) (which generates the equivalence classes \( a \cdot R_m \) where \( a \cdot R_m = b \cdot R_m \) if and only if \( a^m = b^m \) for \( a, b \in G \)). We will write \( a \cdot R_m = \sqrt[m]{b} \) when \( a^m = b \). If no such \( a \) exists for a given \( b \in G \), then \( \sqrt[m]{b} \) is undefined. This definition of the \( m \)-th root may be extended with \( \sqrt[m]{0} := 0 \cdot R_m \) when \( G = \mathbb{F}^\times \) is the multiplicative group of \( \mathbb{F} \) and 0 is the additive identity in \( \mathbb{F} \). We note that \( \sqrt[m]{ab} = \sqrt[m]{a} \cdot \sqrt[m]{b} \) when the relevant \( m \)-th roots exist. The non-zero \( m \)-th roots of \( G \) form a multiplicative group, and the matrices \( M \in M_n(\mathbb{F}) \) where \( \sqrt[n]{\det(M)} \) exists form a monoid \( D_n(\mathbb{F}) \).

**Definition 3.** We define the determinant-root \( \text{Det} \) by

\[ \text{Det}(M) := \sqrt[n]{\det(M)} \]

for all \( M \in D_n(\mathbb{F}) \).

This definition provides the usual multiplicative property

\[ \text{Det}(AB) = \text{Det}(A) \text{Det}(B) \]

for \( A, B \in D_n(\mathbb{F}) \). There is a natural embedding \( h_{m,n} : a \cdot R_m \mapsto a^n \cdot R_{mn} \) and we may define the product \( * \) as follows

\[ (a \cdot R_m) * (b \cdot R_n) := h_{m,n}(a \cdot R_m)h_{n,m}(b \cdot R_n) = (a^n b^m)R_{mn}. \]

Consequently, \( \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a^n b^m} \) when the given roots exist. It follows that

\[ \text{Det}(A \otimes C) = \text{Det}(A) * \text{Det}(C) \]

for \( A \in D_m(\mathbb{F}) \) and \( C \in D_n(\mathbb{F}) \), from \( \text{det}(A \otimes C) = (\text{det}(A))^m (\text{det}(C))^n \). Let \( \sqrt[n]{a} \) be the \( n \)-th root of \( a \in \mathbb{F} \). Since \( \sqrt[n]{a} \) is in the (multiplicative) quotient \( \mathbb{Q} / R_m \), we consider the \( n \)-th root \( \sqrt[n]{\sqrt[n]{a}} \), when it exists. In this case, we may write \( \sqrt[n]{a} \equiv \sqrt[n]{\sqrt[n]{a}} \), since

\[ G/R_{mn}(G) \cong (G/R_m(G)) / R_n(G/R_m(G)) \]

where \( R_m(G) \) is the subgroup of an abelian group \( G \) as above. The isomorphism is simply

\[ aR_{mn}(G) \mapsto (a \cdot R_m(G)) \cdot R_n(G/R_m(G)). \]

Now we are ready to consider partial determinant-roots and their properties. The partial determinant-root \( \text{Det}_2 \) is again the block-wise determinant-root.

**Definition 4.** Let \( A, B \in M_{mn}(\mathbb{F}) = M_m(\mathbb{F}^n) = M_m(\mathbb{F}) \otimes M_n(\mathbb{F}) \) with

\[ A = \sum_{i,j=1}^n A_{ij} \otimes E_{ij}^{[n]}, \quad B = \sum_{i,j=1}^m E_{ij}^{[m]} \otimes B_{ij} \]
and assume that $A_{ij} \in D_m(F)$ and $B_{ij} \in D_n(F)$ for all $i, j$. The partial determinant-roots $\text{Det}_1(A)$ and $\text{Det}_2(B)$ are given by

$$\text{Det}_1(A) := \sum_{i,j=1}^n \text{Det}(A_{ij})E_{ij}[n], \quad \text{Det}_2(B) := \sum_{i,j=1}^m \text{Det}(B_{ij})E_{ij}[m].$$

In general, we work in the group ring $\mathbb{Z}[F \times R_m(F \times)]$. However, since we are concerned only with Kronecker products of matrices, we will express our results in $F \times R_m(F \times)$ where possible.

**Lemma 3.** The partial determinant-root obeys $\text{Det}_1(A \otimes B) = \text{Det}(A)B$ where $\text{Det}(A)B$ is the entry-wise product of the entries in $B$ with $\text{Det}(A)$.

**Proof.** We have $A \otimes B = \sum_{i,j=1}^n ((B)_{ij}A) \otimes E_{ij}[n]$, where $B = (B)_{ij}$. It follows that

$$\text{Det}_1(A \otimes B) = \sum_{i,j=1}^n \text{Det}(B_{ij}A_{ij})E_{ij}[n] = \sum_{i,j=1}^n B_{ij} \text{Det}(A_{ij})E_{ij}[n]$$

since $\sqrt[n]{B_{ij}^m} = B_{ij} \cdot R_m(F \times)$. \qed

**Theorem 7.**

1. Let $A \in D_m(F)$ and $C \in D_n(F)$. Then $\text{Det}(A \otimes C) = \text{Det}(\text{Det}_1(A \otimes C))$.

2. Let $A, B \in D_m(F)$ and $C, D \in M_n(F)$. Then $\text{Det}_1((A \otimes C)(B \otimes D)) = \text{Det}_1(A \otimes C) \text{Det}_1(B \otimes D)$.

**Proof.** Since $\text{Det}(A \otimes C) = \text{Det}(A) \cdot \text{Det}(C)$ and

$$\text{Det}(\text{Det}_1(A \otimes C)) = \sqrt[n]{\text{Det}(\text{Det}(A))C}$$

$$= \sqrt[n]{(\text{Det}(A))^n \cdot \text{Det}(C)}$$

$$\equiv \sqrt[n]{\text{Det}(A)^n \cdot \text{Det}(C)^m} \quad \text{(by the isomorphism $[6]$)}$$

$$= \text{Det}(A) \cdot \text{Det}(C)$$

statement (1) follows. For statement (2), Lemma [3] provides $\text{Det}_1((A \otimes C)(B \otimes D)) = \text{Det}(AB)(CD)$.

Similarly,

$$\text{Det}_1(A \otimes C) \text{Det}_1(B \otimes D) = (\text{Det}(A)C)(\text{Det}(B)D) \quad \text{(over $\mathbb{Z}[F \times R_m(F \times)$])}$$

$$= (\text{Det}(A) \text{Det}(B))(CD)$$

$$= \text{Det}(AB)(CD). \quad \square$$

4. **Conclusion**

We have provided some characterizations of matrices satisfying

$$\text{det}(\text{Det}_1(A \otimes B)) = \text{det}(A \otimes B) ,$$

$$\text{det}_1(A \otimes C \otimes (B \otimes D)) = \text{det}_1(A \otimes C) \text{det}_1(B \otimes D).$$

This falls under the problem of partial operations which can be completed, i.e. we have determined conditions under which the partial determinant of Kronecker products can be completed.
Let \( r : \mathbb{N} \to \mathbb{N} \) and let \( g_n : M_1(\mathbb{F}) \to M_1(\mathbb{F}) \) be some operation on matrices. For example, take \( r(n) = 1 \) and \( g_n(A) = \text{tr}(A) \) to be the trace operation, or \( r(n) = n \) and \( g_n(A) = A^T \) to be the matrix transpose. Let \( f_{m,n} : M_{mn}(\mathbb{F}) \to M_{mr(n)}(\mathbb{F}) \) be the partial operation of \( g \) defined by

\[
f_{m,n} \left( \sum_{i,j=1}^{n} E_{ij}^{[m]} \otimes A_{ij} \right) = \sum_{i,j=1}^{n} E_{ij}^{[m]} \otimes g_n(A_{ij}).
\]

The partial operation \( f_{m,n} \) on a matrix \( M \in M_{mn}(\mathbb{F}) \) can be completed when

\[
g_{mn}(M) = g_{mr(n)}(f_{m,n}(M)).
\]

This definition of completability yields that the partial trace is the uniquely completable linear partial operation of the trace, i.e. \( f_{m,n} = \text{tr}_2(M) \). To see this, let \( g_n \) be the trace operation, \( g_n : M_n(\mathbb{F}) \to \mathbb{F} \) be linear, and

\[
f'_{m,n} \left( \sum_{i,j=1}^{n} E_{ij}^{[m]} \otimes A_{ij} \right) = \sum_{i,j=1}^{n} E_{ij}^{[m]} \otimes g'_n(A_{ij})
\]

with

\[
g_{mn}(M) = g_{mr(n)}(f'_{m,n}(M)).
\]

We must have

\[
g_{mn}(E_{ij}^{[m]} \otimes E_{kl}^{[n]}) = \delta_{ij}\delta_{kl} = g_{mr(n)}(E_{ij}^{[m]} \otimes g'_n(E_{kl}^{[n]})) = \delta_{ij}g'_n(E_{kl}^{[n]})
\]

from which follows that \( g'_n(E_{kl}^{[n]}) = \delta_{kl} \), i.e. \( g'_n = \text{tr} \).

Consider the partial transpose, i.e. \( g_n(A) = A^T \). Let \( M \in M_{mn}(\mathbb{F}) \) be the matrix

\[
M = \sum_{i,j=1}^{n} E_{ij}^{[m]} \otimes M_{ij}.
\]

Then

\[
f_{m,n}(M) = \sum_{i,j=1}^{n} E_{ij}^{[m]} \otimes g_n(M_{ij}) = \sum_{i,j=1}^{n} E_{ij}^{[m]} \otimes M_{ij}^T
\]

and the partial transpose is not completable, since

\[
M^T = g_{mn}(M) \neq g_{mn}(f_{m,n}(M)) = \sum_{i,j=1}^{n} (E_{ij}^{[m]})^T \otimes M_{ij},
\]

unless \( M \) is block-wise symmetric (\( M_{ij}^T = M_{ij} \)).

Finally, we note the following regarding partial determinants and their relation to partial traces and the exponential map. When \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{C} \), we have

\[
\text{det}(\exp(A)) = \exp(\text{tr}(A))
\]

for all square matrices \( A \). W.-H. Steeb posed the following question in a private communication: for which matrices \( A \) is it true that \( \text{det}_1(\exp(A)) = \exp(\text{tr}_1(A)) \)?

In the case of Kronecker sums, we have a straightforward answer.

**Theorem 8.** Let \( B \in M_m(\mathbb{F}) \) and \( C \in M_n(\mathbb{C}) \) and let \( A \) be the Kronecker sum \( A = B \otimes I_n + I_m \otimes C \). Then \( \text{det}_1(\exp(A)) = \exp(\text{tr}_1(A)) \) if and only if the \( m \)-th Hadamard and matrix powers of \( \exp(C) \) coincide, i.e. \( (\exp(C))^{(m)} = (\exp(C))^m \).

The proof follows from \( \exp(A) = \exp(B) \otimes \exp(C) \), Lemma 4 and the fact that \( \text{det}(\exp(B)) = \exp(\text{tr}(B)) \). Assume that \( n \neq 0 \) in \( \mathbb{F} \). Define \( \text{Tr} : M_n(\mathbb{F}) \to \mathbb{F} \) by \( \text{Tr} : A \mapsto \frac{\text{tr}(A)}{n} \), and define the partial operation \( \text{Tr}_1 \) in the natural way. For determinant-roots we obtain

\[
\text{Det}(\exp(A)) = \exp(\text{Tr}(A)) \cdot R_n(\mathbb{F}_\infty)
\]
where $A \in D_n(F)$. Partial determinant-roots obey the exponential-determinant-trace relation as follows.

**Theorem 9.** Let $F$ be the field $F = \mathbb{R}$ of real numbers or the field $F = \mathbb{C}$ of complex numbers, and let $A$ be the Kronecker sum $A = B \otimes I_n + I_m \otimes C$ where $B \in D_m(F)$. Then

$$\text{Det}_1(\exp(A)) = (\exp(\text{Tr}_1(A)) \cdot R_m(F) \cdot \text{exp}(C)).$$

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