Critical index of Anderson transition in 3D systems

Arisato Kawabata
Department of Physics, Gakushuin University,
1-5-1 Mejiro, Toshima-ku, Tokyo 171
March 22, 2022

Abstract
Anderson transition in three-dimensional systems is investigated using renormalization group theory. $\beta$-function of a very simple form is derived from a self-consistent consideration, and it gives a value $1 + 1/\sqrt{3} = 1.58$ for the critical index of the transition, which is very close to those obtained by numerical studies.

keyword: Anderson localization, metal-insulator transition, critical index, renormalization group, self-consistent theory

1 Introduction
Metal-insulator transition of electronic systems due to randomness was predicted by Anderson in 1958. Let $x$ be a quantity which controls the conductivity, such as impurity density or electron density. Then the system undergoes a transition from a conductor to an insulator at $x = x_c$, the critical value of $x$ (in the following we assume that the system is metallic for $x > x_c$). One of the main interests on this subject is the critical behavior of the transition, namely, how the conductivity $\sigma(x)$ depends on $x$ near the critical point. In spite of an amount of theoretical studies, little progress has been made about this problem, until the appearance of the renormalization group (or scaling) theory developed by Wegner [2] and by Abrahams et al. [3].

The scaling hypothesis by Abrahams et al. leads to the renormalization group equation [4]

$$\frac{d \log g(L)}{d \log L} = \beta(g(L)),$$

1 E-mail: arisato.kawabata@gakushuin.ac.jp
where \( g(L) \) is the conductance of a system of size \( L \). The renormalization group theory predicts that the transition is continuous in three-dimension: \( \sigma(x) \) vanishes continuously at \( x = x_c \). The critical index (or exponent) \( \nu \) of the transition is defined by the equation

\[
\sigma(x) = C(x - x_c)^\nu.
\]

Here \( C \) is a constant which is dependent on the microscopic structure of the system, while \( \nu \) is believed to be universal and dependent only on the universality class, i.e., the fundamental symmetry of the Hamiltonian. Below we will treat the orthogonal class, in which the Hamiltonian is composed of the kinetic energy and the impurity potential, without spin-orbit coupling nor magnetic field.

The critical index \( \nu \) is given in terms of the \( \beta \)-function \( \beta(g) \):

\[
\nu = \frac{1}{g_c \beta'(g_c)},
\]

where the prime on \( \beta(g) \) indicates the derivative, and \( g_c \) is the fixed point value of \( g \), satisfying

\[
\beta(g_c) = 0.
\]

Elaborate calculations of the \( \beta \)-function have been done by Wegner [5] and Hikami, [6] and their results lead to \( \nu \leq 1 \). This problem was investigated also by numerical methods. Kramer and macKinnon developed a method to estimate \( \nu \) from numerical data, and they obtained a value considerably larger than unity. [7, 8] Calculations with high accuracy have been performed recently by Slevin and Ohtsuki, and they found that [9, 10]

\[
\nu = 1.57 \sim 1.58.
\]

Thus the discrepancy between the analytic and the numerical methods is significant. At present, as long as the accuracy of the results are concerned, the analytic methods seem less reliable than the numerical methods. However, it is still important to pursue better analytic methods, because there are some aspects of which the better understandings are obtained only by analytic methods.

In this paper we propose an analytic theory based on an idea different from those of existing theories. We will find it well reproduces the results of numerical studies.

## 2 Perturbational Calculation of \( \beta \)-function

In this section we briefly review the method applied so far to calculate \( \beta \)-function, for it is the starting point of the present theory, too. The details
of the calculations are reviewed in Ref. [4] To be specific, we consider an electron gas system with randomly distributed impurities. The hamiltonian is of the form

\[ H = \frac{p^2}{2m} + \sum_i u \delta(r - R_i), \]

where \( R_i \)'s are the positions of the impurities, which are assume to be distributed uniformly with the average density \( c_i \), independently of each other.

So far the \( \beta \)-function was calculated in the form of the power series in \( 1/g \). The term of \( 1/g \) arises from the lowest order interference effect of electron wave scattered by impurities. Without the interference correction, the conductivity is given by Boltzmann (classical) conductivity

\[ \sigma = \sigma_B \equiv \frac{n_e e^2 \tau}{m}, \]

where \( n_e \) is the electron density, and \( \tau \) is the electron scattering time by impurities given by

\[ \frac{1}{\tau} = \frac{2\pi}{\hbar} c_i u^2 N_0 \]

\( N_0 \) being the density of states at the Fermi level.

In thermal Green’s function formalism, [11] the interference correction to the Boltzmann conductivity comes from the term corresponding to the Feynmann graph shown in Fig. 1. Here the solid lines and the dotted lines indicate the one electron Green’s functions and the impurity potential, respectively; the wavy line indicates the cooperon propagator. Up to this order, the conductivity of an infinitely large system is given by

\[ \sigma = \sigma_B \left[ 1 - \frac{1}{\pi \hbar N_0} \int \frac{dq}{(2\pi)^3} \frac{1}{D_0 q^2} \right], \]

Figure 1: The Feynmann graph for the lowest order interference correction

\[ \text{\includegraphics[width=0.5\textwidth]{feynmann_graph.png}} \]
where $D_0$ is the diffusion coefficient defined by

$$D_0 \equiv \frac{\sigma_B}{2e^2 N_0} = \frac{v_F^2 \tau}{3},$$

$v_F$ being the Fermi velocity, and the integral is to be done in the region $q < q_c \approx 1/v_F \tau$.

The conductance $g(L)$ is defined in terms of the conductivity $\sigma(L)$ of the system of size $L$ as

$$g(L) = r L \sigma(L),$$

where $r$ is a constant of dimension of resistance to make $g(L)$ dimensionless, to be determined later. We postulate that $\sigma(L)$ is given by the right hand side of eq. (9) in which the integral is cut off at $q = \alpha/L$, $\alpha$ being a numerical constant of order 1 (we consider the region $\alpha/L < q_c$). Then we obtain

$$g(L) = r L \left[ \sigma_B - \frac{e^2}{\pi^3 \hbar} \left( q_c - \frac{\alpha}{L} \right) \right],$$

and $\beta$-function is obtained from eq. (11):

$$\beta(g) = 1 - \frac{2}{g},$$

where we have taken $r$ so that

$$r \alpha e^2 \frac{\pi^3 \hbar}{\alpha} = 2.$$  (14)

As is seen from eqs. (3) and (4), the ambiguity of $r$ within a numerical factor does not affect the value of $\nu$, and we find that

$$\nu = 1.$$  (15)

Calculations of $\beta$-function up to the order $1/g^4$ were done by Wegner [5] and Hikami [6]. Their $\beta$-functions lead to $\nu = 0.56$, and the deviation from the numerical value becomes even larger. The inclusion of the higher order seems very difficult. Moreover, it is not certain if the $1/g$ expansion is a proper expansion. Thus we have to investigate the problem from a different point of view.

3 Self-Consistent Treatment

Self-consistent treatments of Anderson transition were developed by Vollhardt and Woelfle [12] and by the present author [13]. Although those theories contributed to the deeper understanding of the problem, they give $\nu = 1$ like the renormalization group theory with the $\beta$-function (13). Those theories are based on the requirement of the consistency between $\sigma$ and $D_0$ in the integrand.
(cooperon propagator) in eq. (9) for \( L \to \infty \). In this paper we require the consistency including the \( L \) dependence of these quantities.

In this sense eq. (9) is not self-consistent, for \( \sigma(L) \) depends on \( L \) while \( D_0 \) is a constant. The diffusion coefficient must be independent of \( q \) for small \( q \), but it can depend on \( q \) for large \( q \). Therefore, we assume that the diffusion coefficient behave like \( \propto q^7 \) for large \( q \). We consider the case when \( \alpha/L \) is large in some sense. Then, assuming that the cooperon propagator is proportional to \( 1/q^{2+\gamma} \), and putting it into eq. (9) we find that

\[
\sigma(L) \propto \left( \frac{\alpha}{L} \right)^{1-\gamma},
\]

if we neglect the terms independent of \( L \). We identify \( \alpha/L \) with \( q \), and from the self-consistency requirement we find that \( \gamma = 1/2 \) and that the diffusion coefficient should behave like \( \sqrt{q} \) for large \( q \). Scaling arguments predict a different \( q \) dependence for large \( q \), and this problem will be discussed in the last section.

Thus, among various possibilities, we assume a simple form for the cooperon propagator:

\[
\frac{1}{D_0 q^2 (1 + c \sqrt{\lambda_0 q})},
\]

where \( c \) is a numerical constant to be determined self-consistently, and

\[
\lambda_0 = \frac{1}{2 \pi h N_0 D_0}.
\]

The combination of \( q \) with \( \lambda_0 \) is based on the scaling assumption that a length should be scaled by \( \lambda_0 \).

Then, by replacing the integrand in eq. (9) with the expression (17), we obtain

\[
D(L) = \frac{\sigma(L)}{2 e^2 N_0} = D_0 \left[ 1 + \frac{2}{\pi^2 c} \sqrt{\frac{\alpha \lambda_0}{L}} - \frac{2}{\pi^2 c} \sqrt{\lambda_0 q_c} + \frac{2}{\pi^2 c^2} \log \frac{c \sqrt{\lambda_0 q_c} + 1}{c \sqrt{\alpha \lambda_0/L + 1}} \right].
\]

In comparing it with the denominator of the expression (17), we neglect the last two terms in the square bracket. In fact, we will see that the terms independent of \( L \) does not affect the renormalization group equation. Moreover, we have to compare them in the regions \( \sqrt{\alpha \lambda_0/L} > 1 \) and the \( L \) dependence of the logarithmic term is weaker than that of \( \sqrt{\alpha \lambda_0/L} \). Thus, identifying \( \alpha/L \) with \( q \), the self-consistency requirement gives

\[
c = \frac{2}{\pi^2 c}, \quad \text{i.e.,} \quad c = \frac{\sqrt{2}}{\pi}.
\]
4 Renormalization Group Equation

We will derive a renormalization group equation from eq. (19). Here we define

\[ g(L) \equiv b \frac{L}{\lambda(L)}, \]

(21)

\[ \lambda(L) \equiv \frac{1}{2\pi \hbar N_0 D(L)}, \]

(22)

where \( b \) is a numerical constant, and we easily find that this is equivalent to eq. (13). Then we obtain

\[ g(L) = b \frac{L}{\lambda_0} \left( 1 - \frac{2}{\pi} \sqrt{\lambda_0 q_c} \right) \]

\[ + b \frac{\sqrt{2}}{\pi} \sqrt{\frac{L}{\alpha \lambda_0}} + \frac{L}{\lambda_0} \log \frac{\sqrt{\lambda_0 q_c} + \pi/\sqrt{2}}{\alpha \lambda_0 / L + \pi/\sqrt{2}}, \]

(23)

and

\[ \frac{d \log g(L)}{d \log L} = \frac{1}{g(L)} \left[ g(L) - \eta \sqrt{\frac{b L}{\lambda_0} + \frac{b L}{\lambda_0} \frac{1}{2 + \eta^{-1} \sqrt{b L/\lambda_0}}} \right], \]

(24)

with \( \eta \equiv \sqrt{b \alpha / (\sqrt{2}\pi)} \).

We assume that the Anderson transition is described by renormalization group equation. Then, the right hand side of this equation have to be a universal function only of \( g(L) \). From eqs. (18), (21) and (22), we find that \( b L / \lambda_0 \) is \( g(L) \) without the interference correction. Therefore, it is reasonable to identify \( b L / \lambda_0 \) with exact \( g(L) \). With this replacement, the \( \beta \)-function is obtained:

\[ \beta(g) = 1 - \eta \sqrt{g} + \frac{1}{2 + \eta^{-1} \sqrt{g}}, \]

(25)

where we have chosen \( b \) so that \( \eta = 1 \).

5 Critical Index

The critical index \( \nu \) is obtained from eqs. (3) and (4). Using eq. (25) we easily find that

\[ g_c = 4 - 2\sqrt{3}, \]

(26)

\[ \nu = 1 + \frac{1}{\sqrt{3}} = 1.58. \]

(27)

This value of \( \nu \) agrees with that of numerical studies, i.e., \( \nu = 1.57 \sim 1.58 \).
6 Summaries and Discussions

On the basis of a self-consistent consideration, we have derived a renormalization group equation for Anderson transition. It gives a critical index very close to those by numerical studies. We have not intended to derive an exact result, and such a good agreement is not the most important result of the present theory. It should be noted, however, that so far the analytic methods have never given a critical index comparable with those by numerical methods, within the author’s knowledge, except for the rather phenomenological theory by Shapiro. [13, 14]

As regards the one parameter scaling, the recent numerical study by Ohtsuki and Slevin revealed that the critical index is little dependent on the definition of $g(L)$, i.e., what kind of averaged conductance we take for it. [17] This result is very important because it indicates that a simple one parameter renormalization group theory is reliable.

A crucial assumption in this theory is the form of the cooperon propagator given by eq. (17). As was mentioned in Sec. 3, scaling arguments predict that the diffusion coefficient $D(q)$ should depend on $q$ like $D(q) \propto q^{1+\eta'}$ with $\eta' \approx 1.3$ at the critical point, or for such $q$ that $q\lambda \gg 1$, where $\lambda$ is the coherence length defined by eq. (22). On the other hand, from eq. (21) we find that $g \approx L/\lambda \approx 1/(q\lambda)$, and as regards the critical behavior of the transition, the relevant values of $g$ are $g_c \approx 1$. Therefore, the relevant region of $q$ is $q\lambda \approx 1$, and the assumption $D(q) \propto \sqrt{q}$ in this region does not contradict to the scaling arguments.

Since the $\beta$-function (25) can not be obtained by the expansion in $1/g$, probably it is impossible to derive the $\sqrt{q}$ term in the expression (17) by perturbational approaches. The $\beta$-function, however, reduces to eq. (13) for $g \gg 1$, and it might be the reason why the $1/g$ expansion seems to work for first order. For small $g$, on the other hand, the $\beta$-function (25) does not reduce to an expected form $\beta(g) \approx \log g$, but, it does not invalidate the present results, for, as was mentioned above, the important region of $g$ is $g \approx 1$.

In this context, the weak localization theory, which was very successful in explaining the magneto-resistance in fully metallic regions [4, 18, 19, 20], is not affected by the introduction of $\sqrt{q}$ term into the cooperon propagator, for the relevant scale of $q$ in that theory is $1/\ell_B = \sqrt{eB/\hbar}$, $B$ being the magnetic flux density, and is much smaller than $1/\lambda_0$.

As for the experiments, many of them suggest that $\nu \approx 1$. [21] However, recent careful analyses by Itoh et al. indicate a possibility of $\nu > 1$. [22] From a theoretical point of view, it is important to clarify the roles of electron-electron interaction.

Acknowledgment

This work is partly supported by "High Technology Research Center Project" of Ministry of Education, Culture, Sports, Sciences and Technology. The author is grateful to Prof. T. Ohtsuki for the information on the present subject.
References

[1] P.W. Anderson: Phys. Rev. 109 (1958) 1492.
[2] F.J. Wegner: Z. Phys. B 25 (1976) 327.
[3] E. Abrahams, P.W. Anderson, D.C. Licciardello and T.V. Ramakrishnan: Phys. Rev. Lett. 42 (1979) 673.
[4] For a review, the reader is referred to; A. Kawabata: Prog. Theor. Phys. Supple. No.84 (1985) 16.
[5] F.J. Wegner: Nucl Phys. B 316 (1989) 663.
[6] S. Hikami: Proceedings of Anderson Transition and Mesoscopic Fluctuations Workshop, Braunschweig, West Germany, 9-12 Jan. 1990, Physica A 167 (1990) 149.
[7] A. MacKinnon and B Kramer: Z. Phys. B 53 (1989) 1.
[8] B Kramer and A. MacKinnon: Rep. Prog. Phys. 56 (1993) 1469.
[9] K. Slevin and T. Ohtsuki: Phys. Rev. Lett. 82 (1999) 382.
[10] T. Ohtsuki, K. Slevin and T. Kawarabayashi: Proc. Intern. Conf. Localization 1999, (Ann. Phys. 8 (1999) 655).
[11] A.A. Abrikosov, L.P. Gor’kov and I.E. Dzyaloshinskii: Methods of Quantum Field Theory in Statistical Physics ( Pergamon Press, London, 1963, and Dover Publication Inc., New York, 1963).
[12] D. Vollhardt and P. Woelfle: Phys. Rev. Lett. 45 (1980) 842.
[13] A. Kawabata: Solid State Commun. 38 (1981) 823.
[14] J.T. Chalker: Physica A167 (1990) 253.
[15] B. Shapiro: Phil. Mag. B56 (1987) 1031.
[16] M. Jansen, Phys. Rep. 295 (1998) 1, Chapter 7.
[17] K. Slevin, P. Markos and T. Ohtsuki: Phys. Rev Lett. 86 (2001) 3594.
[18] S. Hikami, A.I. Larkin and Y. Nagaoka: Prog. Theor. Phys. 63 (1980) 707.
[19] A. Kawabata: Solid State Commun. 34 (1980) 432.
[20] A. Kawabata: J. Phys. Soc. Jpn. 49 (1980) 628.
[21] S. Katsumoto, F. Komori, S. Kobayashi and N. Sano: J.Phys. Soc. Jpn. 56 (1987) 2259.
[22] K.M. Itoh, M. Watanabe, Y. Ootuka, and E.E. Haller: Proc. Intern. Conf. Localization 1999, (Ann. Phys. 8 (1999) 631).