The new integrable deformations of a short pulse equation and sine-Gordon equation, and their solutions

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Abstract

We first derive an integrable deformed hierarchy of a short pulse equation and its Lax representation. Then we concentrate on the solution of the integrable deformed short pulse equation (IDSPE). By proposing a generalized reciprocal transformation, we find a new integrable deformed sine-Gordon equation (IDSGE) and its Lax representation. The multisoliton solutions, negaton solutions and positon solutions for the IDSGE and the N-loop soliton solutions, N-negaton and N-positon solutions for the IDSPE are presented. In the reduced case the new N-positon solutions and N-negaton solutions for the short pulse equation are obtained.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

It is known that pulse propagation in optical fibers is usually modeled by the cubic nonlinear Schrödinger equation [1]. However, it is not valid for the ultrashort pulses. In 2004, starting from the Maxwell equation of electric field in the fiber, Schäfer and Wayne derived the short pulse equation (SPE) [2]

\[ u_{xt} = u + \frac{1}{6} (u^3)_{xx} \]  

(1)

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as an alternative of the cubic nonlinear Schrödinger equation to describing the propagation of ultrashort optical pulses in nonlinear media, where \( u(x, t) \) represents the magnitude of the electric field. In recent years, the short pulse model has attracted considerable attention. In [3, 4], it was also deduced from the partial differential equation which describes pseudospherical surfaces. In [5], Sakovich and Sakovich proved that the SPE is integrable by discovering a Lax pair for the SPE. Brunelli proved the integrability of the SPE from the Hamiltonian point of view and studied the short pulse hierarchy in [6, 7]. In [8], Feng et al proposed the integrable semi-discrete and full-discrete analogs of the SPE. As far as the solutions of the SPE are considered, in [9] not only solitary wave solutions are obtained by making use of transformation between the SPE and the sine-Gordon equation but also the pulse solutions of the SPE were derived from the breather solutions of the SG equation. Some periodic and traveling wave solutions of SPE are given in [10]. Kuetche et al constructed the two-loop soliton solutions with use of a bilinear method and hodograph transformation [11]. In [12, 13], Matsuno developed a systematic procedure to construct the periodic solutions and multiloop solitons.

The integrable deformation of integrable system attract a lot of interests from both physical and mathematical points. One kind of the integrable deformation is the so-called soliton equation with self-consistent sources [14–21], which consists of the soliton equation with additional terms by coupling the corresponding eigenvalue problems, and has an important application. For example, the nonlinear Schrödinger equation with self-consistent sources is relevant to some problems of plasma physical and solid-state physics.

In this paper, we consider the integrable deformed short pulse equation (IDSPE) which has not been studied yet. First we derive the integrable deformed hierarchy of the short pulse equation (IDSPH) and its Lax representation, which includes IDSPE and its Lax representation. This implies that the IDSPE is Lax integrable. A generalized reciprocal transformation for the IDSPE is proposed. This transformation converts the IDSPE and its Lax representation into a new integrable deformed sine-Gordon equation (IDSGE) and its Lax representation. The IDSGE can be written in bilinear form by introducing an independent variable transformation. The \( N \)-soliton solutions of the sine-Gordon equation were obtained in [22, 23], and the positon solutions of the sine-Gordon equation and its properties were studied in [23]. Here, we find the \( N \)-soliton solutions, \( N \)-negaton solutions and \( N \)-positon solutions for the IDSPE. In the reduced case we obtain new \( N \)-negaton solutions and new \( N \)-positon solutions for the SPE.

This paper is organized as follows. In section 2, we establish the IDSPH and its Lax representation. In section 3, a generalized reciprocal transformation for the IDSPE is proposed and the IDSGE is worked out. In section 4, the solutions of the IDSGE are obtained. Section 5 gives the solutions of IDSPE. The conclusion is given in section 6.

2. The new IDSPH and its Lax pair

2.1. The IDSPH

Consider the eigenvalue problem [5]

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_x = U \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}, \quad U = \begin{pmatrix}
\lambda & \lambda u_x \\
\lambda u_x & -\lambda
\end{pmatrix}.
\] (2)

The adjoint representation reads

\[ V_x = [U, V]. \] (3)
Set
\[ V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} = \sum_{m=0}^{\infty} \begin{pmatrix} \lambda a_m & \lambda u_x a_m + b_m \\ \lambda u_x a_m + c_m & -\lambda a_m \end{pmatrix} \lambda^m. \] (4)

Equation (3) yields
\[ \begin{cases} 
    a_{m,x} = u_x (c_m - b_m), \\
    b_{m+1,x} = 2b_m - (u_x a_m)_x, \\
    c_{m+1,x} = -2c_m - (u_x a_m)_x.
\end{cases} \] (5)

Taking \( b_0 = c_0 = 0, a_0 = \frac{1}{4} \), we have
\[ \begin{cases} 
    b_1 = c_1 = -\frac{1}{2} u, \\
    b_3 = c_3 = -\frac{1}{2} u^2, \\
    \ldots
\end{cases} \] (6)

and in general,
\[ \begin{cases} 
    b_{2n} = -c_{2n} = 2\partial^{-1} b_{2n-1}, \\
    a_{2n} = -2\partial^{-1}(u_x b_{2n}), \\
    a_{2n+1} = 0, \\
    b_{2n+1} = c_{2n+1} = 2\partial^{-1} b_{2n} - u_x a_{2n} = L b_{2n-1}, \\
    L = 4(\partial^{-1} + u_x \partial^{-1} u_x) \partial^{-1},
\end{cases} \] (7)

where \( \partial = \frac{\partial}{\partial x} \). Then the compatibility condition of equations (2) and (9) gives rise to the short pulse hierarchy (SPH)
\[ u_{xt} = -\partial b_{2n+1}, \quad n = 0, 1, \ldots \] (10)

When \( n = 1 \) (10) gives the SPE
\[ u_{xt} = u + \frac{1}{4}(u^3)_{xx}, \] (11)

and \( V^{(2)} \) in (9) is given by
\[ V^{(2)} = \begin{pmatrix} \frac{1}{2}\lambda u^2 + \frac{1}{4}\lambda & \frac{1}{2}\lambda u^2 u_x - \frac{u}{2} \\
    \frac{1}{2}\lambda u^2 u_x + \frac{u}{2} & -\frac{1}{2}\lambda u^2 - \frac{1}{4}\lambda \end{pmatrix}. \] (12)

For \( n \) distinct real \( \lambda_j \), consider the following spectral problem:
\[ \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}_x = \begin{pmatrix} \lambda_j & \lambda_j u_x \\ \lambda_j u_x & -\lambda_j \end{pmatrix} \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}. \]

It is easy to find that
\[ \frac{\delta \lambda_j}{\delta u} = -2\lambda_j (\varphi_{1j}^2 + \varphi_{2j}^2), \quad L (\varphi_{1j}^2 + \varphi_{2j}^2)_x = \frac{1}{\lambda_j^2} (\varphi_{1j}^2 + \varphi_{2j}^2)_x. \] (13)
According to the approach proposed in [17–21], the short pulse hierarchy with self-consistent sources (SPHSCS) or the integrable deformed short pulse hierarchy (IDSPH) is defined by

\[ u_{xtn} = -\partial \left[ b_{2n+1} - \sum_{j=1}^{N} \frac{1}{2\lambda_j} (\psi_{1j}^2 + \psi_{2j}^2) x \right], \tag{14a} \]

\[ \phi_{1jx} = \lambda_j \phi_{1j} + \lambda_j x \phi_{2j}, \quad \phi_{2jx} = \lambda_j x \phi_{1j} - \lambda_j \phi_{2j}, \quad j = 1, 2, \ldots, N. \tag{14b} \]

When \( n = 1 \) (14) gives the short pulse equation with self-consistent sources (SPESCS) or IDSPE:

\[ u_{xt} = u + \frac{1}{6} (u^3)_{xx} + \sum_{j=1}^{N} \frac{1}{2\lambda_j} (\psi_{1j}^2 + \psi_{2j}^2)_{xx}, \tag{15a} \]

\[ \phi_{1jx} = \lambda_j \phi_{1j} + \lambda_j x \phi_{2j}, \quad \phi_{2jx} = \lambda_j x \phi_{1j} - \lambda_j \phi_{2j}, \quad j = 1, 2, \ldots, N. \tag{15b} \]

2.2. Lax pair of the IDSPH

In order to find the Lax pair for the IDSPE (15), we first consider the following stationary equation of (15):

\[ b_3 = \sum_{j=1}^{N} \frac{1}{2\lambda_j} (\psi_{1j}^2 + \psi_{2j}^2) x = 0, \tag{16a} \]

\[ \phi_{1jx} = \lambda_j \phi_{1j} + \lambda_j x \phi_{2j}, \quad \phi_{2jx} = \lambda_j x \phi_{1j} - \lambda_j \phi_{2j}, \quad j = 1, 2, \ldots, N. \tag{16b} \]

According to equations (6), (7), (13) and (16), we may define

\[ \bar{a}_0 = \frac{1}{4}, \quad \bar{b}_0 = \bar{c}_0 = 0, \quad \bar{b}_1 = \bar{c}_1 = -\frac{1}{4} u_x, \]

\[ \bar{a}_1 = 0, \quad \bar{b}_2 = -\bar{c}_2 = -\frac{1}{2} u, \quad \bar{a}_2 = \frac{1}{2} u^2, \]

\[ \bar{b}_{2n+1} = \bar{c}_{2n+1} = L^{n-1} \bar{b}_3 = L^{n-1} \sum_{j=1}^{N} \frac{1}{2\lambda_j} (\psi_{1j}^2 + \psi_{2j}^2) x \]

\[ = \sum_{j=1}^{N} \frac{1}{2\lambda_j} (\psi_{1j}^2 + \psi_{2j}^2) x, \quad n = 1, 2, \ldots, \]

\[ \bar{b}_n = -\bar{c}_n = 2\partial^{-1} \bar{b}_{2n-1} = \sum_{j=1}^{N} \frac{1}{\lambda_j} (\psi_{1j}^2 + \psi_{2j}^2), \quad n = 2, 3, \ldots, \]

\[ \bar{a}_{2n} = -2\partial^{-1} (u_x \bar{b}_n) = -2\partial^{-1} \sum_{j=1}^{N} \frac{1}{\lambda_{2n-2}} u_x (\psi_{1j}^2 + \psi_{2j}^2) \]

\[ = -2 \sum_{j=1}^{N} \frac{1}{\lambda_j} \psi_{1j} \psi_{2j}, \quad n = 2, 3, \ldots, \]

\[ \bar{a}_{2n+1} = 0, \quad n = 1, 2, 3, \ldots. \tag{17} \]
Then we have
\[
\bar{\Lambda} = \lambda^{-2} \sum_{n=0}^{\infty} \bar{a}_n \lambda^{n+1} = \frac{1}{4\lambda} + \frac{1}{2} \bar{u}^2 + \bar{\Lambda}_0,
\]
\[
\bar{\Lambda}_0 = \sum_{n=2}^{\infty} \bar{a}_{2n} \lambda^{2n-1} = -2 \sum_{j=1}^{N} \frac{\lambda}{\lambda_j} \sum_{n=2}^{\infty} \left( \frac{\lambda}{\lambda_j} \right)^{2n-1} \psi_{1j} \psi_{2j}
\]
\[
= 2\lambda \sum_{j=1}^{N} \frac{1}{\lambda_j} \psi_{1j} \psi_{2j} + 2\lambda \sum_{j=1}^{N} \frac{\lambda_j}{\lambda^2 - \lambda_j^2} \psi_{1j} \psi_{2j}.
\]

In the same way, we find that
\[
\bar{V} = \left( \begin{array}{cc}
\bar{A} & \bar{B} \\
\bar{C} & -\bar{A}
\end{array} \right) = \lambda^{-2} \sum_{n=0}^{\infty} \begin{pmatrix}
\lambda \bar{a}_n & \lambda \bar{u}_n \bar{a}_n + \bar{b}_n \\
-\lambda \bar{a}_n & \bar{a}_n
\end{pmatrix} \lambda^n
\]
\[
= V^{(2)} + N_0, \quad N_0 = \left( \begin{array}{cc}
\bar{A}_0 & \bar{B}_0 \\
\bar{C}_0 & -\bar{A}_0
\end{array} \right),
\]
\[
B_0 = \sum_{j=1}^{N} \frac{2\lambda}{\lambda_j} \bar{u}_j \bar{a}_j \psi_{1j} \psi_{2j} - \left( \psi_{1j}^2 + \psi_{2j}^2 \right) - \frac{\lambda_j}{\lambda} \left( \psi_{1j}^2 - \psi_{2j}^2 \right)
\]
\[
+ \frac{\lambda_j^2}{\lambda^2 - \lambda_j^2} \left[ \frac{\lambda_j}{\lambda} \left( \psi_{2j}^2 - \psi_{1j}^2 \right) - \left( \psi_{1j}^2 + \psi_{2j}^2 \right) \right],
\]
\[
C_0 = \sum_{j=1}^{N} \left[ \frac{2\lambda}{\lambda_j} \bar{u}_j \bar{a}_j \psi_{1j} \psi_{2j} + \left( \psi_{1j}^2 + \psi_{2j}^2 \right) - \frac{\lambda_j}{\lambda} \left( \psi_{1j}^2 - \psi_{2j}^2 \right) \right]
\]
\[
+ \frac{\lambda_j^2}{\lambda^2 - \lambda_j^2} \left[ \frac{\lambda_j}{\lambda} \left( \psi_{2j}^2 - \psi_{1j}^2 \right) + \left( \psi_{1j}^2 + \psi_{2j}^2 \right) \right].
\]

Since \( \bar{a}_n, \bar{b}_n \) and \( \bar{c}_n \) satisfy the same recursion relations as (5). It is obvious that \( \bar{V} \) satisfies
\[
\bar{V}_t = [U, \bar{V}],
\]
(18)
In fact, it is easy to verify that (18) under (16b) leads to (16a). Since (16) is the stationary equation of (15), we immediately obtain the zero curvature representation for the IDSPE (15):
\[
U_t - \bar{V}_x + [U, \bar{V}] = 0,
\]
(19)
with the Lax pair for the IDSPE (15):
\[
\left( \begin{array}{c}
\phi_1 \\
\phi_2
\end{array} \right)_x = \left( \begin{array}{cc}
\lambda & \lambda \bar{u}_x \\
\lambda \bar{u}_x & -\lambda
\end{array} \right) \left( \begin{array}{c}
\phi_1 \\
\phi_2
\end{array} \right),
\]
(20a)
\[
\left( \begin{array}{c}
\phi_1 \\
\phi_2
\end{array} \right)_t = \bar{V} \left( \begin{array}{c}
\phi_1 \\
\phi_2
\end{array} \right).
\]
(20b)

Furthermore, the zero curvature representation and Lax pair for the IDSPH (14) are given by (19) and (20) with
\[
\bar{V} = V^{(n)} + N_0.
\]
(21)

**Remark 1.** The zero curvature representation and Lax pair for the IDSPH (14) are given by (19), (20) and (21). This implies that the new IDSPH is integrable in the Lax sense.
3. The new integrable deformed sine-Gordon equation

By introducing the new dependent variable [12]
\[ r^2 = 1 + \frac{u^2}{2}. \]  
(22)

Equation (15a) is transformed into the form
\[ r_t = \frac{1}{2}u^2r_x + \frac{u_x}{2r} \sum_{j=1}^{N} \lambda_j^{-2}(\psi_{1j}^2 + \psi_{2j}^2)_{xx} = \left( \frac{1}{2}u^2r + 2r \sum_{j=1}^{N} \lambda_j^{-1} \psi_{1j} \psi_{2j} \right)_x. \]  
(23)

So we can define a reciprocal transformation \((x, t) \rightarrow (y, s)\) by the relation
\[ dy = r \, dx + \left( \frac{1}{2}u^2r + 2r \sum_{j=1}^{N} \lambda_j^{-1} \psi_{1j} \psi_{2j} \right) \, dx, \quad ds = dt. \]  
(24)

and we have
\[ \frac{\partial}{\partial x} = r \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial s} + \left( \frac{1}{2}u^2r + 2r \sum_{j=1}^{N} \lambda_j^{-1} \psi_{1j} \psi_{2j} \right) \frac{\partial}{\partial y}. \]  
(25)

Denoting \(\phi_i(x, t) = \psi_i(y, s), \psi_{ij}(x, t) = \psi_{ij}(y, s) \) \((i = 1, 2)\), with the new variable \(y\) and \(s\) (22) and (23) are transformed into
\[ r^2 = 1 + r^2u_y^2, \]  
(26)

\[ r_s = r^2uu_y + 2r^2 \sum_{j=1}^{N} \lambda_j^{-1} (\psi_{1j} \psi_{2j})_y, \]  
(27)

respectively. Furthermore, we define
\[ u_y = \sin z, \quad z = z(y, s). \]  
(28)

Inserting (28) into (26) gives rise to
\[ r = \frac{1}{\cos z}. \]  
(29)

Using (25) and (29), (15b) is converted the following form:
\[ \psi_{1j} = \lambda_j \cos z \psi_{1j} + \lambda_j \sin z \psi_{2j}, \quad \psi_{2j} = \lambda_j \sin z \psi_{1j} - \lambda_j \cos z \psi_{2j}, \]  
\( j = 1, 2, \ldots, N. \)  
(30)

Under equations (28)–(30), (27) becomes
\[ z_s = u + 2 \sum_{j=1}^{N} (\psi_{1j}^2 + \psi_{2j}^2). \]  
(31)

So under the reciprocal transformation (25), the IDSPE (15) is transformed into the following IDSGE:
\[ z_{ys} = \sin z + 2 \sum_{j=1}^{N} (\psi_{1j}^2 + \psi_{2j}^2)_y, \]  
(32a)

\[ \psi_{1j} = \lambda_j \cos z \psi_{1j} + \lambda_j \sin z \psi_{2j}, \quad \psi_{2j} = \lambda_j \sin z \psi_{1j} - \lambda_j \cos z \psi_{2j}, \]  
\( j = 1, 2, \ldots, N. \)  
(32b)
Under the reciprocal transformation (25), (28), (29) and (31), the Lax pair (20) for the IDSPE (15) are transformed into the Lax pair for (32):

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_y = \begin{pmatrix}
\lambda \cos z & \lambda \sin z \\
\lambda \sin z & -\lambda \cos z
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix},
\]
\( \lambda, z \) (33a)

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_s = N \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix},
\]
(33b)

\[
N = \left( \frac{1}{4\lambda} - \frac{1}{2}z_0 \right) + \sum_{j=1}^N \frac{\lambda_j}{\lambda^2 - \lambda_j^2}
\times \begin{pmatrix}
2\lambda \psi_1 \psi_2 & -\lambda_j (\psi_1^2 + \psi_2^2) - \lambda (\psi_1^2 - \psi_2^2)
\end{pmatrix}.
\]

**Remark 2.** The system (32) is a new integrable deformation of the sine-Gordon equation with the Lax pair given by (33). The system (32) can be regarded as a new kind of sine-Gordon equation with a self-consistent source which is different from the sine-Gordon equation with the self-consistent source in [24].

4. Solutions to the integrable deformed sine-Gordon equation

4.1. Multisoliton solutions

Introducing the dependent variable transformation

\[
z = 2i \ln f,
\]
(34a)

\[
\psi_{1j} = i\left( \frac{\tilde{g}_j}{f} - \frac{\bar{g}_j}{\bar{f}} \right), \quad \psi_{2j} = -\left( \frac{\tilde{g}_j}{f} + \frac{g_j}{f} \right), \quad j = 1, \ldots, N
\]
(34b)

the IDSGE (32) can be transformed into the bilinear form:

\[
D_y D_s f \cdot f = \frac{1}{2} (f^2 - \bar{f}^2) - 8i \sum_{j=1}^N \lambda_j \bar{g}_j^2,
\]
(35a)

\[
D_y g_j \cdot f = -\lambda_j \bar{g}_j \bar{f}, \quad j = 1, 2, \ldots, N,
\]
(35b)

where \( \tilde{f} \) and \( \tilde{g}_j \) are the complex conjugates of \( f \) and \( g_j \), \( D \) is the well-known Hirota bilinear operator defined by [22]

\[
D^n_y D^n_s f \cdot g = (\partial_y - \partial_y')^n(\partial_s - \partial_s')^n f(y, s) g(y', s')|_{y'=y, s'=s}.
\]

The Wronskian determinant is defined as [25]

\[
W = |\Psi^{(0)}, \Psi^{(1)}, \ldots, \Psi^{(N-1)}| = 0, 1, \ldots, N - 1 = |N - 1|,
\]
(36)

where \( \Psi^{(0)} = \Psi = (\Psi_1(y, s), \Psi_2(y, s), \ldots, \Psi_N(y, s))^T \) and \( \Psi^{(j)} = \frac{\partial\Psi}{\partial y^j} \).

Since the bilinear form (35) is same as that in [24] except replacing \( g_j \) by \( \tilde{g}_j \), we may find the solution of the IDSGE (32) by directly using the formulae and notation in [24]. We have the following theorem.
Figure 1. (a), (b) Single-soliton solutions $z_1$ when $\lambda_1 = 0.5$, $\alpha_1(s) = s$, and $\lambda_1 = 0.5$, $\alpha_1(s) = s^2$, respectively. (c) The eigenfunction $\psi_{11}$ for $\lambda_1 = 0.5$, $\alpha_1(s) = s$.

Theorem 1. Let

$$\Psi_j = ie^{\xi_j} - (-1)^j e^{-\xi_j}, \quad j = 1, 2, \ldots, N,$$

where $\xi_j = -\lambda_j y - \frac{\alpha_j(s)}{\lambda_j} + \alpha_j(s)$, $\lambda_j$ are real number and we set $\lambda_1 < \lambda_2 < \cdots < \lambda_N$, then the IDSGE (32) has the Wronskian determinant solutions given by (34) with

$$f = |N - 1|, \quad \text{(38a)}$$

$$g_h = (-1)^{hN} \sqrt{\alpha'_h(s) \prod_{l=1}^{h-1} (\lambda_h^2 - \lambda_l^2) \prod_{l=h+1}^{N} (\lambda_h^2 - \lambda_l^2)} |N - 2, \tau_h|, \quad \tau_h = (\delta_{h,1}, \ldots, \delta_{h,N})^T, \quad h = 1, \ldots, N, \quad \text{(38b)}$$

where $|N - 2, \tau_h|$ is complex conjugates of $|N - 2, \tau_h|$.

Theorem 1 can be proved in the same way as in [24], we omit it.

When take $N = 1$ (34) and (38) give rise to one-soliton solution for the IDSGE (32):

$$z_1 = 4 \arctan e^{2\xi_1} = 2i \ln \frac{1 - ie^{2\xi_1}}{1 + ie^{2\xi_1}}, \quad \text{(39a)}$$

$$\psi_{11} = \frac{2\sqrt{\alpha'_1(s)e^{2\xi_1}}}{1 + e^{2\xi_1}}, \quad \psi_{21} = \frac{-2\sqrt{\alpha'_1(s)e^{2\xi_1}}}{1 + e^{2\xi_1}}. \quad \text{(39b)}$$

In figure 1, we plot the single-soliton solution of $z_1$ and $\psi_{11}$.

Similarly, when take $N = 2$ in (38), we have

$$f = (e^{\xi_1 + \xi_2} + e^{-\xi_1 + \xi_2})(\lambda_2 - \lambda_1) - i(e^{\xi_1 - \xi_2} - e^{\xi_2 - \xi_1})(\lambda_1 + \lambda_2), \quad \text{(40a)}$$

$$g_1 = -\sqrt{\alpha'_1(s)(\lambda_2^2 - \lambda_1^2)}(e^{-\xi_1} + ie^{\xi_1}), \quad g_2 = \sqrt{\alpha'_1(s)(\lambda_2^2 - \lambda_1^2)}(e^{-\xi_1} - ie^{\xi_1}). \quad \text{(40b)}$$

Equations (34) and (40) give to the two-soliton solutions for the IDSGE (32). Figure 2 describes the shapes and interactions of two-soliton solution for $z_2$ and $\psi_{11}$. Also figure 2 shows the interactions are elastic collisions and the influence on two-soliton solution for $z_2$ and $\psi_{11}$ for taking different $\alpha_j(s)$.

Note that solutions (38) contains arbitrary $s$ functions $\alpha_j(s)$. This implies that the insertion of non-homogeneous terms into the soliton equation may cause the variation of the speed and shape of soliton. So the dynamics of solutions of IDSGE (32) turns out to be much richer than that of solutions of the sine-Gordon equation.
Figure 2. (a), (b) The shapes and interactions for the two-soliton solutions \( z_2 \) when \( \lambda_1 = -0.1, \lambda_2 = 1, a_1(s) = s, a_2(s) = s \) and \( \lambda_1 = -0.1, \lambda_2 = 1, a_1(s) = \sin s, a_2(s) = \cos s \), respectively. (c), (d) The shapes and interactions for \( \psi_{11} \) when \( \lambda_1 = -0.1, \lambda_2 = 0.2, a_1(s) = 2s, a_2(s) = s \), and \( \lambda_1 = -0.1, \lambda_2 = 0.2, a_1(s) = 4s, a_2(s) = \cos s \), respectively.

4.2. Negaton solutions and positon solutions

For \( N = 2 \), taking \( a_1(s) = c_1, a_2(s) = (\lambda_2 - \lambda_1)e(s) + c_1 - \frac{1}{2}i\pi \), then (37) leads to

\[
\Psi_1 = e^{-\xi_1} + ie^{i\xi_1}, \quad \Psi_2 = -i(e^{2i\lambda} + ie^{i\xi_1}),
\]

where \( \xi_1 = -\frac{z_1}{4\lambda_1} + c_1, \xi_2 = -\frac{z_2}{4\lambda_2} + (\lambda_2 - \lambda_1)e(s) + c_1 - \frac{1}{2}i\pi, c_1 \) is a constant and \( e(s) \) is a function for \( s \). We have

\[
f = \begin{vmatrix} \Psi_1 & \Psi_{1y} \\ \Psi_2 & \Psi_{2y} \end{vmatrix} = \begin{vmatrix} \Psi_1 & \Psi_{1y} \\ \frac{\partial \Psi_2}{\partial \lambda_2} |_{\lambda_2 = \lambda_1} & \frac{\partial^2 \Psi_2}{\partial \lambda_2^2 \partial y} |_{\lambda_2 = \lambda_1} \end{vmatrix} (\lambda_2 - \lambda_1) + o(\lambda_2 - \lambda_1),
\]

\begin{align}
g_1 &= 0, \\
g_2 &= \sqrt{(\lambda_2 - \lambda_1)e(s)(\lambda_2^2 - \lambda_1^2)} \begin{vmatrix} \Psi_1 & 0 \\ \Psi_2 & 1 \end{vmatrix} = (\lambda_2 - \lambda_1)\sqrt{e(s)(\lambda_2 + \lambda_1)(e^{-\xi_1} - ie^{\xi_1})},
\end{align}

where \( \gamma = y - \frac{z}{4\lambda_1} - e(s) \). Then we obtain the one-negaton solution from (34) by taking \( \lambda_2 \rightarrow \lambda_1 \)

\[
z = 2i \ln \frac{\text{ch}2\xi_1 - 2i\lambda_1\gamma}{\text{ch}2\xi_1 + 2i\lambda_1\gamma},
\]

(43a)
Figure 3. The shapes for one-negaton solution $z$ and $\psi_{12}$ when $\lambda_1 = 0.1$, $e(s) = 2s$, $s = 0.5$.

$$
\psi_{12} = \frac{2\sqrt{2\lambda_1 e'(s)}(-4\lambda_1 y e^{-\xi_1} + e^{-\xi_1} + e^{3\xi_1})}{(e^{2\xi_1} + e^{-2\xi_1})^2 + 16\lambda_1^2 y^2},
$$
\hspace{1cm}(43b)

$$
\psi_{22} = \frac{2\sqrt{2\lambda_1 e'(s)}(4\lambda_1 y e^{3\xi_1} + e^{\xi_1} + e^{-3\xi_1})}{(e^{2\xi_1} + e^{-2\xi_1})^2 + 16\lambda_1^2 y^2}.
$$
\hspace{1cm}(43c)

The shapes are given in figure 3. In general, as proposed in [26], the $N$-negaton solution can be obtained from (34), (36) and (38) by replacing $N$ by $2N$, taking

$$
\Psi_{1} = \Psi_{1}(\lambda_1 = \lambda_0, \lambda_2, \lambda_3, \ldots, \lambda_{2N-1}, \lambda_{2N} = \lambda_{2N-1}),
$$
\hspace{1cm}(44a)

$$
\xi_{2k-1} = -\lambda_{2k-1} y - \frac{s}{4\lambda_{2k-1}} + c_{2k-1},
$$
\hspace{1cm}(44b)

$$
\xi_{2k} = -\lambda_{2k} y - \frac{s}{4\lambda_{2k}} + (\lambda_{2k} - \lambda_{2k-1})e_{2k}(s) + c_{2k-1} - \frac{1}{2}i\pi,
$$
\hspace{1cm}(44c)

and taking $\lambda_{2k} \to \lambda_{2k-1}$.

In order to derive the positon solution as pointed out in [23], we have to take

$$
\lambda_1 = i\mu_1, \quad \lambda_2 = i\mu_2, \quad c_1 = -i\bar{c}_1.
$$
\hspace{1cm}(45)

By a similar calculation, we obtain the following one-positon solution:

$$
z = 2i \ln \frac{\cos 2\eta_1 + 2\mu_1 \tilde{\eta}}{\cos 2\eta_1 - 2\mu_1 \tilde{\eta}},
$$
\hspace{1cm}(46a)

$$
\psi_{12} = \frac{\sqrt{2\mu_1 e'(s)}(-4\mu_1 \tilde{\eta} e^{i\eta_1} + e^{i\eta_1} + e^{-3i\eta_1})}{2((\cos^2 2\eta_1 - 4\mu_1^2 \tilde{\eta}^2)^2),}
$$
\hspace{1cm}(46b)

$$
\psi_{22} = \frac{\sqrt{2\mu_1 e'(s)}(4\mu_1 \tilde{\eta} e^{-3i\eta_1} + e^{-i\eta_1} + e^{3i\eta_1})}{2((\cos^2 2\eta_1 - 4\mu_1^2 \tilde{\eta}^2)^2),}
$$
\hspace{1cm}(46c)

where $\tilde{\eta} = \gamma + \frac{e(s)}{4\eta_1} - e(s)$, $\eta_1 = \mu_1 y - \frac{e(s)}{4\eta_1} + \bar{c}_1$. Just as the positon solutions of the sine-Gordon equation are complex [23], one-positon solution (46) is also complex and reduces to the positon solution of the sine-Gordon equation by taking $e(s)$ to be a constant.

Figure 4 describes the shapes for the modulus of the one-positon solution. Similarly, we can find $N$-positon solutions by using (44) and $\lambda_j = i\mu_j, c_k = -i\bar{c}_k$ and $\mu_{2k} \to \mu_{2k-1}$.
5. Solutions for the integrable deformed short pulse equation

5.1. N-loop soliton solutions

Proposition 1. Suppose that \( z \) and \( \psi_{ij} (i = 1, 2, j = 1, 2, \ldots, N) \) are solutions of the IDSGE (32); then the solutions of the IDSPE (15) with a parametric representation of \((y, s)\) are given by

\[
\begin{align*}
  u &= z_s - 2 \sum_{j=1}^{N} (\psi_{1j}^2 + \psi_{2j}^2), \quad (47a) \\
  \varphi_{1j}(x, t) &= \psi_{1j}(y, s), \quad \varphi_{2j}(x, t) = \psi_{2j}(y, s), \quad j = 1, 2, \ldots, N, \quad (47b) \\
  x(y, s) &= \int \cos z \, dy = y - 2(\ln f_f |_{\alpha_j(s)} = \alpha_j(s)) |_{\alpha_j(s)}, \quad (47c)
\end{align*}
\]

where \( \alpha_j \) are arbitrary constants.

Proof. It is obvious that (47a) is given by (31). In the following, we prove (47c). From the reciprocal transformation, we have the following linear PDEs for \( x \):

\[
\begin{align*}
  \frac{\partial x}{\partial y} &= \frac{1}{r}, \quad \frac{\partial x}{\partial s} = -\frac{1}{2} u^2 - 2 \sum_{j=1}^{N} \lambda_j^{-1} \psi_{1j} \psi_{2j}. \quad (48)
\end{align*}
\]

By making use of the compatibility of the above two equations, we have

\[
\begin{align*}
  x(y, s) &= \int \frac{1}{r} \, dy = \int \cos z \, dy. \quad (49)
\end{align*}
\]

From (34) and (35), a direct calculation gives

\[
\cos z = 1 - 2(\ln f_f)_{ys} + 8i \sum_{j=1}^{N} \lambda_j \left( \frac{u_j^2}{f_j^2} - \frac{g_j^2}{f_j^2} \right). \quad (50)
\]

When \( f \) and \( g_j \) are given by (37) and (38), the terms with \( \alpha(s) \) in \( 2(\ln f_f)_{ys} \) and the terms \( 8i \sum_{j=1}^{N} \lambda_j \left( \frac{u_j^2}{f_j^2} - \frac{g_j^2}{f_j^2} \right) \) are canceled. So the above equation becomes

\[
\cos z = 1 - 2(\ln f_f |_{\alpha_j(s)} = \alpha_j(s))_{ys} |_{\alpha_j(s)}. \quad (51)
\]

namely, for calculating the derivatives with respect to \( s \), we regard \( \alpha_j(s) \) as independent of \( s \). Substituting this equation in (49) leads to (47c). \( \square \)
When we take $N = 1$, by making use of (39) and (47), we obtain the one-loop soliton solution for the IDSPE (15):

$$u_1 = \frac{2e^{2\xi_1}}{\lambda_1(1 + e^{4\xi_1})}, \quad \varphi_{11} = \frac{2\alpha_1'(s)e^{2\xi_1}}{(1 + e^{4\xi_1})}, \quad \varphi_{21} = -\frac{2\sqrt{\alpha_1'(s)}e^{2\xi_1}}{(1 + e^{4\xi_1})}, \quad x(y,s) = y + \frac{2}{\lambda_1(1 + e^{4\xi_1})}. \quad (52)$$

Figure 5 shows that the one-loop soliton solutions move to the left and keep the shapes. When take $N = 2$ in (38), (34) and (47) give to the two-loop solution of the IDSPE (15):

$$u_2 = \frac{2(\lambda_1^2 - \lambda_2^2)(\lambda_1(1 + e^{4\xi_1})e^{2\xi_1}(1 - 12\lambda_2\alpha_2'(s)) - \lambda_2(1 + e^{4\xi_1})e^{2\xi_1}(1 - 12\lambda_1\alpha_1'(s)))}{\lambda_1\lambda_2(1 + e^{4\xi_1})(1 + e^{4\xi_2})(\lambda_1^2 + \lambda_2^2) - 2\lambda_1\lambda_2(1 - e^{4\xi_1} - e^{4\xi_2} - 4e^{2\xi_1+\xi_2} + 4e^{4\xi_1+2\xi_2})}, \quad (53)$$

$$\varphi_{11} = \frac{4\sqrt{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 + \lambda_2^2)}(1 - e^{4\xi_1} + 2e^{2\xi_1+\xi_2})\lambda_1 - (1 + e^{4\xi_1})\lambda_1 e^{2\xi_1}}{1 + e^{4\xi_1})(1 + e^{4\xi_2})(\lambda_1^2 + \lambda_2^2) - 2\lambda_1\lambda_2(1 - e^{4\xi_1} - e^{4\xi_2} - 4e^{2\xi_1+\xi_2} + 4e^{4\xi_1+2\xi_2})}, \quad (53b)$$

$$\varphi_{21} = \frac{4\sqrt{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 + \lambda_2^2)}(e^{2\xi_1} - e^{2\xi_2} + e^{2\xi_1+\xi_2})\lambda_2 - e^{2\xi_1}(1 + e^{4\xi_1})\lambda_1 e^{2\xi_1}}{1 + e^{4\xi_1})(1 + e^{4\xi_2})(\lambda_1^2 + \lambda_2^2) - 2\lambda_1\lambda_2(1 - e^{4\xi_1} - e^{4\xi_2} + 4e^{2\xi_1+\xi_2} + e^{4\xi_1+2\xi_2})}, \quad (53c)$$

$$\varphi_{12} = \frac{4\sqrt{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 + \lambda_2^2)}(2e^{2\xi_1} - e^{2\xi_2} + e^{2\xi_1+\xi_2})\lambda_2 - e^{2\xi_1}(1 + e^{4\xi_1})\lambda_1 e^{2\xi_1}}{1 + e^{4\xi_1})(1 + e^{4\xi_2})(\lambda_1^2 + \lambda_2^2) - 2\lambda_1\lambda_2(1 - e^{4\xi_1} - e^{4\xi_2} + 4e^{2\xi_1+\xi_2} + e^{4\xi_1+2\xi_2})}, \quad (53d)$$

Figure 5. The shapes and motions for the one-loop soliton solutions $u_1$ and $\varphi_{11}$ when $\lambda_1 = -0.5, \alpha_1(s) = s$. For example, when we take $N = 1$, by making use of (39) and (47), we obtain the one-loop soliton solution for the IDSPE (15):
Figure 6. The shapes and interactions for the two-loop soliton solutions $u_2$ and $\psi_{11}$ when $\lambda_1 = -1, \lambda_2 = 0.5, \alpha_1(s) = 2s, \alpha_2(s) = s$.

\[
\psi_{22} = \frac{4\sqrt{(\lambda_2^2 - \lambda_1^2)}|s'(s)|((1 - e^{4\xi_1} + 2 \cos(2\xi_1 + \xi_2))\lambda_2 - (1 + e^{4\xi_1})\lambda_1|e^{\xi_2}}{(1 + e^{4\xi_1})(1 + e^{4\xi_2})(\lambda_1^2 + \lambda_2^2) - 2\lambda_1\lambda_2(1 - e^{4\xi_1} - 4 \cos(\xi_1 + \xi_2) + 4 e^{2(\xi_1 + \xi_2)})}.
\]

Similarly, by making use of (37), (38) and (47), we can obtain the $N$-loop soliton solution for the IDSPE (15). Figure 6 describes the shapes and interactions of the two-loop soliton solutions and the interactions of two-loop soliton solution for $u_2$ and $\psi_{11}$ are shown to be elastic collisions.

5.2. Negaton solutions and positon solutions

By making use of (34), (43) and (47), the one-negaton solution for the IDSPE (15) is given:

\[
u = \frac{-2e^{\xi_1}[(s - 4\lambda_1^2(y - e(s))(e^{4\xi_1} - 1) + 2\lambda_1(1 + 8\lambda_1^2e(s))(e^{4\xi_1} + 1)]}{[s^2 + 16\lambda_1^2(y - e(s))^2 + 2\lambda_1^2(1 - 4ys) - 8\lambda_1^2se(s)]e^{4\xi_1} + \lambda_1^2(e^{8\xi_1} + 1)},
\]

\[
\psi_{12} = \sqrt{2\lambda_1e'(s)}e^{-\xi_1} + 4\lambda_1y e^{-\xi_1}, \quad \psi_{22} = \sqrt{2\lambda_1e'(s)}e^{\xi_1} + e^{3\xi_1} + 4\lambda_1y e^{\xi_1},
\]

\[
x(y, s) = \frac{-2[(2s + 32\lambda_1^2(y + e(s))(e'(s) - 8\lambda_1^2(y + e(s) + s')e'(s))e^{-4\xi_1} + \lambda_1(e^{-8\xi_1} - 1)]}{[s^2 + 16\lambda_1^2(y - e(s))^2 + 2\lambda_1^2(1 - 4ys) + 8se(s)]e^{-4\xi_1} + \lambda_1^2(e^{-8\xi_1} + 1)}.
\]

The shapes of the one-negaton solutions are given in figure 7. In the same way, by making use of (34), (46) and (47), the one-positon solution for the IDSPE (15) is given:

\[
u = \frac{4[i\xi_2(s + 4\mu_1^2(y - e(s)))sin 2\eta_1 - 2\mu_1(1 - 8\mu_1^2e'(s))cos 2\eta_1]}{2s^2 + 16\mu_1^2(y + e(s))^2 - 2\mu_1^2(1 - 4ys + cos 4\eta_1 - 4se(s))}.
\]
Figure 7. The shapes for the one-negaton solutions when \( \lambda_1 = 1, e(s) = s^3, s = -0.2 \).

Figure 8. The shapes for \( |u|^2 \), the real part and imaginary part of \( \phi_{12} \) with \( \lambda_1 = 0.1, e(s) = 2s, s = 2 \), respectively.

\[
\begin{align*}
\phi_{12} &= \frac{\sqrt{2}j_1 e'(s)(-4j_1 e^{i\eta_1} + e^{i\eta_1} + e^{-3i\eta_1})}{2(\cos^2 2\eta_1 - 4\mu_1^2 e^{2i\eta_1})}, \\
\phi_{22} &= \frac{\sqrt{2}j_1 e'(s)(4j_1 e^{-i\eta_1} + e^{-i\eta_1} + e^{3i\eta_1})}{2(\cos^3 2\eta_1 - 4\mu_1^2 e^{3i\eta_1})}, \\
x(y, s) &= y - \frac{4[s + 4\mu_1^2(y - e(s)) - \sin 4\eta_1]}{s^2 + 16\mu_1^4(y - e(s))^2 - 2\mu_1^2(1 - 4s(y - e(s)) + \cos 4\eta_1)}. 
\end{align*}
\]

The shapes for the one-position solutions are given in figure 8. By using the N-negaton solutions and N-positon solutions of the IDsGE and the inverse reciprocal transformation (47), we can find the N-negaton solutions and N-positon solutions for the IDSPE. In the reduced case we can find the new N-negaton solutions and N-positon solutions for the SPE by taking all \( e_j(s) \) to be constants.

6. Conclusion

We first derived the integrable deformed short pulse hierarchy and its zero curvature representation. Then we concentrated on the solution of the integrable deformed short pulse equation (IDSPE). By proposing a generalized reciprocal transformation, we found a new IDsGE and its zero curvature representation. The bilinear equation and the Wronskian determinant solutions for the IDSGE are given. Furthermore, based on the inverse reciprocal transformation and the solutions of the IDSGE, the N-loop soliton solutions, N-negaton and
$N$-positon solutions of the IDSPH are worked out. In the reduced case the new $N$-negaton solutions and new $N$-positon solutions for the SPE are obtained.

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