Quantification and Control of Non-Markovian Evolution in Finite Quantum Systems via Feedback

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We consider the unitary time evolution of continuous quantum mechanical systems confined to a cavity in contact with a finite bath of variable size. We define a new measure relating to (non-)Markovianity which parallels the standard one for the case of integrable Lindbladian dynamics but has the advantage of being numerically tractable also for large many particles systems. The relevant time scales are identified, which characterize non-Markovian transient behavior, boundary scattering induced non-Markovian oscillations at intermediate times, and non-Markovian rephasing events at long time scales. It is shown how these time scales can be controlled by tunable parameters such as the bath size and the strength of the system-bath coupling.

I. INTRODUCTION

It is well known that the time evolution of quantum closed systems at finite size is intrinsically non-Markovian. The reason is that the reduced density matrix is necessarily an oscillating function which admits no infinite-time limit. In the quantum case, the time scale for partial rephasing events (revivals) is proportional to the system size, whereas full rephasing (Poincare recurrences) can occur at an astronomically large time scales [1, 2]. Nonetheless, previous studies of the transient dynamics in lattice models have revealed extended time domains within which the time evolution is pseudo-Markovian in the sense that memory of the initial state appears to be lost [3]. Strictly speaking, this conclusion would of course be erroneous, as this information is only temporarily dispersed among the system’s accessible modes before it re-emerges during rephasing events. However, this observation raises the interesting question if and how the time scales which determine such pseudo-Markovian time evolution (i.e. Markovian when restricted to a definite time domain) as well as non-Markovian features can be controlled.

Here we consider the illustrative example of a continuous one-dimensional array with periodic boundary conditions. We divide this array into an active “system” in region $A$ connected to a region $B$ which acts as a bath. A pair of thin barriers are than placed between the system and the bath. The concept of a finite bath is different from the conventional notion of an infinite reservoir, but this setup allows us to explore the size of the bath as a non-trivial tuning parameter, along with the system-bath coupling. In previous work, the non-Markovian dynamics in such arrangements have been explored using lattice Hamiltonians [3]. It was found that by breaking certain symmetries, features signaling non-Markovian time evolution disappear within a finite time scale which is governed by the system-bath coupling as well as by the amount of random symmetry breaking introduced in the bath Hamiltonian. Here, randomness raises the number of accessible states of the bath.

In this paper, we investigate control over the time evolution of continuum Hamiltonians, tuning their bath degrees of freedom by adjusting the bath size relative to the system. While this study is kept at a relatively abstract level, primarily focusing on the emerging time scales, experimental realizations of such configurations can easily be imagined in the context of coupled laser cavities, connected by semi-transparent mirrors, or in the context of electron wave packets tunneling through barriers in layered (Ga,Al)As heterostructures. In order to avoid the pitfalls of a more conventional treatment of open quantum systems based on the Lindblad formalism [4], including infinitesimal system-bath coupling and forced Markovianity, we study the time evolution of a complete, untruncated system-bath Hamiltonian. The price we have to pay for this is the restriction to non-interacting systems described by effective single-particle Hamiltonians, in order to keep the problem numerically manageable.

We will first introduce the concept of free Markovianity (FM) which parallels the definition of Markovianity a la Breuer-Laine-Piilo (BLP) [5], for the case in which the Lindblad Markovian dynamics, is integrable in the sense of [6]. Free Markovianity differs from BLP Markovianity in the general case, however the two definitions become equivalent when restricted to integrable dynamics. The resulting measure of free non-Markovianity (FNM), $D(t)$, detects a free non-Markovian event whenever $D(t)$ is an increasing function of time, in analogy with the BLP measure [5]. It should

![Figure 1](image_url)
be reminded at this point that properly quantifying the amount of non-Markovianity of a given dynamics has been subject of intense research in recent times and many different, non-necessarily equivalent, measures of non-Markovianity have been proposed \cite{Breuer}. The measure $D(t)$ that we propose, although identifies yet another class of non-Markovian systems, has a series of advantages with respect to the measures that have appeared so far. The main point being that $D(t)$ can easily be computed even for many-body systems composed of many particles. Moreover the quantity $D(t)$ is expressed in terms of single particles quantities thus allowing for a simple physical interpretation. While in general, the maximum of $D(t)$ over all possible initial states has to be calculated, even a single pair of linearly independent states can provide valuable insight into the overall dynamics of the system. We will refer to the distance between such a pair as $D_{\Gamma_1,r_2}(t)$, where $\Gamma$ are the covariance matrix of the two states.

This paper is organized as follows. In the next section we define our measure of free non-Markovianity for non-interacting systems. We then describe the general framework. We proceed here to define a measure of free non-Markovianity $\mathcal{N}$ and canonical Bose operators $u_j$ and $v_j$ with a discussion of how feedback via tunable bath parameters can be utilized to control the time evolution in physically relevant systems.

II. MEASURE OF FREE NON-MARKOVIANITY FOR NON-INTERACTING SYSTEMS

We proceed here to define a measure of free non-Markovianity for a system of non-interacting particles. The results apply both to Fermi-Dirac and Bose-Einstein statistics although with some technical caveats in the latter case. The proper definition of a measure of non-Markovianity is an issue of current debate \cite{Breuer, Cubrovic, BLP, Serafini}. For our purposes it seems appropriate to take the point of view of Breuer-Laine-Piilo (BLP) \cite{BLP} which provides a physical interpretation in terms of information flow. According to Eq. (1) a violation of Markovianity is detected whenever the quantity $|\parallel \rho_1(t) - \rho_2(t)\parallel_1$ increases in time. Here $\parallel \cdot \parallel_1$ denotes the trace norm and $\rho_i(t)$ are the density matrices of the system of interest at time $t$, corresponding to different initial conditions $\rho_1(0)$, $\rho_2(0)$. As it turns out the trace norm $\parallel \rho_1(t) - \rho_2(t)\parallel_1$ is difficult to compute even for the simplest case of systems composed of a single qubit \cite{Serafini}. Therefore we take a different approach.

Consider a many-body system whose dynamical evolution is not necessarily unitary, but it is assumed to be non-interacting in a sense that we are going to specify shortly. Suppose that the quantum process is described by a Markovian master equation,

$$\frac{d\rho}{dt} = \mathcal{L}\rho$$

with generator $\mathcal{L}$ in Lindblad form

$$\mathcal{L}\rho = -i[H, \rho] + \sum_i \gamma_i \left[ A_i \rho A_i^\dagger - \frac{1}{2} \left\{ A_i^\dagger A_i, \rho \right\} \right].$$

In the spirit of \cite{Breuer} we call the dynamics non-interacting if the Hamiltonian $H$ is quadratic in the canonical creation and annihilation operators whereas the Lindblad terms $A_i$ are linear combinations thereof. The solution of such a free Markovian master equations is conveniently encoded in terms of the covariance matrix or two-point correlation function $\Gamma_{j_0,k_\beta}$ \cite{Breuer, Cubrovic}. For Fermions the covariance matrix has the form $\Gamma_{j_0,k_\beta} = -\text{Im}[\text{tr}(\rho u_{j_0}^\dagger \omega_{k_\beta}^2)]$ where the the Majorana operators $\omega_{k_\beta}^\alpha$ are given by $\omega_1^\alpha = f_j + f_j^\dagger$, $\omega_2^\alpha = i(f_j - f_j^\dagger)$ in terms of Fermi operators $f_i$. For bosons instead one has $\Gamma_{j_0,k_\beta} = \text{Re}[\text{tr}(\rho u_{j_0}^\dagger u_{k_\beta}^2)]$ for quadrature operators $u_j^\dagger = b_j + b_j^\dagger$ and $u_j^2 = i(b_j - b_j^\dagger)$ and canonical Bose operators $b_j$ (see Ref. \cite{Breuer} for more details).

From Eq. (1) one finds that the covariance matrix $\Gamma$ satisfies the equation of motion \cite{Breuer}

$$\frac{d\Gamma}{dt} = X^T \Gamma + \Gamma X - Y,$$

with matrices $X, Y$ which depend on $H$ and $A_i$. In the Fermionic case one can show that the spectrum of $X$ lies in the sector $\text{Re}(z) \geq 0$ of the complex plane \cite{Breuer, Cubrovic}. This in turns implies that the dynamics given by Eq. (1) gives rise to a contractive flow. In other words, using following basis-dependent identification between matrices and vectors $|n\rangle \langle m| \equiv |n, m\rangle$ (see e.g. \cite{Cubrovic} Sec. 2.4) given two different initial conditions $\Gamma_1(0),\Gamma_2(0)$, one has \cite{Serafini}

$$\parallel \Gamma_1(t) - \Gamma_2(t)\parallel_2 \leq \parallel \Gamma_1(s) - \Gamma_2(s)\parallel_2 , \quad s \leq t.$$

The same result does not apply directly in the case of bosons essentially because the mapping $\rho \rightarrow \Gamma[\rho]$ is not continuous. Physically this corresponds to the possibility of pumping-in infinite energy in the bosonic fields \cite{Breuer, Cubrovic}. A standard procedure to avoid such infinities is to introduce a (very large) cut-off in the number of particle thus avoiding infinite energy states. With this prescription Eq. (4) holds also for bosons.

The norm appearing in Eq. (4) is in principle any norm for the vectors $|\Gamma_j(t)\rangle$. The $l^2$ norm seems to be the most natural one which induces a basis-independent norm on the matrices $\Gamma_j(t)$. The induced norm in this case is the Hilbert-Schmidt (HS) norm for the matrices $\Gamma_j$, and one has

$$\parallel X\parallel_E = \parallel X\parallel_{HS} = \sqrt{\text{tr}(X^\dagger X)},$$

To summarize, the BLP measure of Markovianity \cite{Serafini} is defined as the trace distance between two density matrices,

$$D_{\rho_1,\rho_2}^{BLP}(t) = \parallel \rho_1(t) - \rho_2(t)\parallel_1$$

while our measure of free Markovianity is defined as the Hilbert-Schmidt distance between two covariance matrices

$$D_{\Gamma_1,\Gamma_2}(t) = \parallel \Gamma_1(t) - \Gamma_2(t)\parallel_{HS}.$$
(d/dt)(∥Γ(1) − Γ(2)∥_{HS}/2), we say that the dynamics is
free Markovian (FM) if for some time interval and initial
states with covariance matrix Γ_{1,2}(0), we have ∥σ(t, Γ_1, Γ_2)∥ ≤
0. In principle one could even define a measure of free non-
Markovianity paralleling the definition of \[ N = \sup_{\Gamma_1, \Gamma_2} \int_0^\infty \sigma(t, \Gamma_1, \Gamma_2) dt, \] for quasi-free
quantities. The quantity:

encodes the amount of free non-Markovianity in the process
dynamics from \( t = 0 \) to \( t = \infty \). As we will see, for one-
particle states \( \|Γ(1) − Γ(2)\|_{HS} \) has a particularly simple
form so that one can hope to be able to perform the maximization
in Eq. (7). This project is left for future investigations.

In general free Markovianity is different from the notion
of Markovianity a-la BLP. However, when restricted to the
class of free, Lindblad dynamics the two definitions become

equivalent.

Our procedure will be the following. We will consider a
system of identical particles described by a free Hamiltonian
dynamics. The setting is relevant to experts, so that particles
will be fermions, but, as shown above, the statistic is
essentially unimportant modulo a technical caveat. Restrict-
ing to free Hamiltonians will allow us to easily integrate the
equation of motions going to the one particle sector. We then
consider the sub-dynamics of the system, obtained by trac-
ing over the bath degrees of freedom. The central question
we ask, is if such sub-dynamics can be described by a free-
Markovian master equation. To this end we compute the quan-
tity in Eq. (4) for different initial states and Hamiltonian
parameters. Note that the result, Eq. (4), is expressed in terms of
single-particle quantities. This fact will make the phys-
ical interpretation simpler. In the following, for simplicity of
language, we will simply refer to free (non) Markovianity as (non) Markovianity. The definition given here is intended
throughout unless otherwise specified.

### III. METHODS

We now explain the general setting. We place a single
particle in a one-dimensional array of length \( L = L_A + L_B \),
with periodic boundary conditions (PBC), i.e. a ring. We will later
trace out the segment \( B \) of length \( L_B \) which plays the role of
external bath, whereas \( A \) is the system of interest. The one-
particle Hamiltonian is of the form \( H = -\partial^2/\partial x^2 + V(x) \)
where the external potential \( V(x) \) will be specified later.
In a particle-number conserving system the covariance matrix
\( \Gamma \) is a function of \( R_{x,y} := \text{tr}(\rho c_j^\dagger c_y) \) only and one has
\( \|\Gamma\|_{HS} = \sqrt{\|R\|_{HS}} \). The system is initialized in the one-particle state \( |\psi(0)\rangle \), which is then evolved according to \( |\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \). In the Fock space this

corresponds to the state \( |\psi(t)\rangle = c_j^\dagger \psi(t)|0\rangle \), where \( c_j^\dagger \) create a
Fermion or Boson in state \( j \). Now, in the position “basis”
|\psi(t)\rangle = \sum\psi(x, t)|x\rangle \) and \( c_j^\dagger \psi(t) = \sum\psi(x, t)c_j^\dagger \)
and one obtains the following integral kernel

\[
R_{x,y} = \psi(x, t)|\psi(y, t). \tag{8}
\]

The full density matrix is a many body state \( \rho(t) =
|\Psi(t)\rangle\langle\Psi(t)| \), and tracing out the bath \( B \) corresponds to dis-
carding from \( R \) in Eq. (8) all the labels \( (x, y) \) belonging to \( B \), i.e. projecting \( R \) onto \( A \). The one-particle Hilbert space is
decomposed into a direct sum \( \mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B \), where \( \mathcal{H}_A/B \)
describes wave functions with support only in \( A/B \). Calling \( P_A \) the operator which projects onto \( \mathcal{H}_A \), the restriction of \( R = \langle\psi|\langle\psi| \) on \( \mathcal{H}_A \) is \( R^A = P_A|\psi\rangle\langle\psi|P_A = |\psi_A\rangle\langle\psi_A| \),
for a non-normalized state \( |\psi_A\rangle \). With the notation \( \rho_R \) for the
Gaussian state given by covariance \( R \), the tracing out the bath is
simply achieved via \( tr_{B|PR} = \rho_{RA} \). In the general case in
which the initial states has \( N \) particles, \( R \) and \( R^A \) are rank \( N \)
operators. Considering the evolution of two different initial
states \( |\psi_0(t)\rangle \), the difference of covariance matrices restricted
to region \( A \) is

\[
P_A(|\psi_1\rangle\langle\psi_1| - |\psi_2\rangle\langle\psi_2|)P_A = |\psi_{1, A}\rangle\langle\psi_{1, A}| - |\psi_{2, A}\rangle\langle\psi_{2, A}| \tag{9}
\]

with unnormalized states \( |\psi_{j, A}\rangle \) supported in \( \mathcal{H}_A \). In the, non-
orthonormal, basis \( |\psi_{j, A}\rangle \) the above operator has the form

\[
\left( \begin{array}{cc}
\frac{p_{1, 1}}{p_{1, 2}} & \frac{p_{1, 2}}{p_{2, 2}} \\
\frac{p_{1, 1}}{p_{1, 2}} & \frac{p_{1, 2}}{p_{2, 2}} \\
\end{array} \right)
\tag{10}
\]

where

\[
p_{i, j} = \langle\psi_{i, A}|\psi_{j, A}\rangle
\tag{11}
\]

\[
= \langle\psi_{i}(t)|P_A|\psi_{j}(t)\rangle
\tag{12}
\]

\[
= \sum_{k, q} e^{iHt-E_k} \langle\psi_{i}|\phi_{k}\rangle\langle\phi_{q}|\psi_{j}\rangle \Delta^A_{L}(k, q)
\tag{13}
\]

with \( i, j = 1, 2 \) and having defined

\[
\Delta^A_{L}(k, q) = \langle\phi_{k}|P_A|\phi_{q}\rangle = \int_{A} \phi_{k}(x)\phi_{q}(x)dx.
\tag{14}
\]

The term \( p_{i, j} \) gives the probability that the particle initial-
ized in \( |\psi_{j}\rangle \) is in region \( A \) at time \( t \). Our indicators of non-
Markovianity are given in terms of the eigenvalues of the matrix
(10) which read

\[
\lambda_{1, 2} = \frac{(p_{1, 1} - p_{2, 2}) \pm \sqrt{(p_{1, 1} + p_{2, 2})^2 - 4|p_{1, 2}|^2}}{2}.
\tag{15}
\]

Finally, according to the discussion in Sec. [II] the distance that we consider to characterize non-Markovianity is given by

\[
D_{\psi_{1, 2}} = \frac{1}{\sqrt{2}} \left\| R^A_{1}(t) - R^A_{2}(t) \right\| = \frac{1}{\sqrt{2}} \sqrt{\lambda^2_1 + \lambda^2_2}
\tag{16}
\]

\[
= \frac{1}{\sqrt{2}} \sqrt{p_{1, 1}^2 + p_{2, 2}^2 - 2|p_{1, 2}|^2},
\tag{17}
\]

where \( R^A_{1}(t) \) are the covariance matrices of system \( A \) at time \( t \). A factor \( 1/\sqrt{2} \) has been inserted to scale the measure, so
that its maximum value is 1 (attained when \( p_{1, 1} = p_{2, 2} = 1 \)
and \( p_{1, 2} = 0 \)). Note that by Schwartz inequality one has
\( |p_{1, 2}|^2 \leq p_{1, 1}p_{2, 2} \) implying that \( D_{\psi_{1, 2}} \) is indeed real.
The final result Eq. (17) is extremely simple and physically quite compelling. Assume for simplicity that the states are orthogonal on $A$ (so that $p_{1,2} = 0$). Eq. (17) then is simply the geometric mean of the probabilities of the particles being in region $A$. As such, as a function of $t$, it is quite clear that $D_{\psi_1,\psi_2}(t)$ increases when particles move into region $A$ signaling a violation of Markovianity. Since $D_{\psi_1,\psi_2}(t)$ is the (Hilbert-Schmidt) distance of two covariance matrices it trivially characterizes the distinguishability of the $R_j^A(t)$. The “information flow” of the BLP measure becomes in this setting a flow of probability. With slight abuse of language we will speak of Markovian behavior when $D_{\psi_1,\psi_2}(t)$ decreases in time although this is only consistent with Markovian dynamics. This is in accordance with the general intuition of Markovian evolution occurring due to information leakage from the system.

IV. RESULTS

We consider two specific initial Gaussian packets localized around $x_j$: $\psi_j(x) = C_j e^{-(x-x_j)^2/(4\sigma_j^2)}$ ($j = 1, 2$). The normalization factor is given by the equation $C_j^2 = \frac{\sqrt{\pi/2}\sigma_j}{\sqrt{\sqrt{\pi\sigma_j} \{ [L - 2x_j]/(\sqrt{\pi\sigma_j}) \} + \text{Erf} \{ [L - 2x_j]/(\sqrt{\pi\sigma_j}) \}}} = 1$, [24] which boils down to $C_j \simeq (2\pi\sigma_j^2)^{-1/4}$ for $\sigma_j \ll L$. We choose initial states symmetrically displaced with respect to the origin, i.e. $x_1 = -x_2$ as shown in Fig. 1. In this case, for symmetric potentials $V(x)$ one has $p_{1,1}(t) = p_{2,2}(t)$ [25]. For such symmetric configurations $\lambda_1 = \lambda_2$, and one has the more simplified $D_{\psi_1,\psi_2} = |\lambda_1| = |\lambda_2|$. As a first example we consider in some detail the purely kinetic evolution corresponding to $V(x) = 0$.

A. Purely kinetic evolution

Let us first consider the case when there are no delta barriers separating the system from the bath, as shown in the insets (a) and (c) of Fig. 3. The spectrum acquires the familiar form $E_k = k^2$ where the quasi-momenta satisfy $k = 2\pi n/L$, $n \in \mathbb{Z}$. The “geometric” factor $\Delta_A^L$ becomes

$$\Delta_A^L(k, q) = \frac{1}{L} \int_{-L}^{L} e^{i k x - q y} dx = \frac{\sin [L_A(k - q)/2]}{[L(k - q)/2]}.$$  \hspace{1cm} (18)

In this case we have simple dispersing wave packets. However, since the system-bath configuration has finite spatial extent, there are rephasing events (or revivals) that occur at time scales of the order of the total system size, i.e. $\tau_L \propto L$ [18] [19]. This timescale measures the time it takes for a packet to go around the periodic boundaries and come back. As Fig. 2 shows, recurrences are not present if the bath is infinite (discussed in more detail later). However, for a finite bath recurrences are observed in Fig. 3(a) and (c), where the time evolution of the non-Markovian indicator between two initial wave packets is shown. In addition to these rephasing events one also observes smaller amplitude, more rapid oscillations in the non-Markovian indicator. A measure of these oscillations is roughly given by the effective numbers of Hamiltonian eigenstates needed to express the initial wave-packet. One can obtain this number by imposing that the fidelity of the initial state be $F = 1 - \epsilon$. One obtains that roughly $N^* = - (L/\sigma) \sqrt{-\ln(\epsilon)}$ eigenstates are required to obtain the desired fidelity. The number $N^*$ also gives the number of effective energy eigenstates involved in the dynamics. It is evident from the figure that broad initial wave-packets (Fig. 3 (a) and (b)) contain more frequencies than narrow initial wave-packets (Fig. 3(a) and (b)). In fact smaller $\sigma$, means larger $N^*$ so that the measure $D_{\psi_1,\psi_2}(t)$ contains more frequencies and consequently shows faster oscillations.

We conclude that for this most simple example non-Markovian time evolution, i.e. deviations from a monotonically decaying HS distance, occurs at two time scales, $\tau_L$ controlled by the total system size, and $\tau_\sigma$ controlled by the width of the initial wave packets and their subsequent dispersion.

We consider now the situation where $L \rightarrow \infty$. In this case many time scales which depend on $L$, such as $\tau_L$, are sent to infinity. We then keep $L_A$ constant and send $L \rightarrow \infty$ in Eq. (13). In this limit $L^{-1} \sum_k \int_{-L}^{L} dk$, and we obtain

$$p_{i,j} = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dq \frac{e^{it(k-q)^2}}{2\pi} \frac{\psi_i(k)\psi_j(q)}{(k-q/2)^2 \pi^2} \sin \left[ \frac{L_A(k-q)/2}{(k-q)/2} \right],$$  \hspace{1cm} (19)

where the initial wave functions in Fourier space are given by

$$\tilde{\psi}_j(q) = \int_{-\infty}^{\infty} dx e^{iqx} \psi_j(x) = \sqrt{4\pi\sigma_j^2} C_j e^{iqx} e^{-q^2\sigma_j^2}. \hspace{1cm} (20)$$

Changing variables to $r = (k-q)$, $R = (k+q)/2$, the integral over $R$ is Gaussian, and we obtain

$$p_{i,j} = \int_{-\infty}^{\infty} dr \sin (rL_A/2) \exp \left\{ -r^2 \sigma_t^2 / 2 \right\}$$
$$+ \frac{r}{2} \left[ \frac{t}{\sigma_t^2} (x_i - x_j) - i (x_i + x_j) - \frac{(x_i - x_j)^2}{8\sigma_t^2} \right], \hspace{1cm} (21)$$

where $\sigma_t = \sqrt{(t^2/\sigma_t^2 + \sigma^2)}$. Writing $\sin (L_Ar)/r = \int_0^{L_A} \cos (yr) dy$, the integral over $r$ is Gaussian, and one is left with an incomplete Gaussian integral over $y$. The final result is

$$p_{i,j} = \frac{e^{-(x_i-x_j)^2/(8\sigma_t^2) \sigma_t^2}}{2} \times$$
$$\left[ \text{Erf} \left( \frac{L_A^+\sigma_t^2 + it(x_i - x_j)}{\sqrt{8\sigma_t^2 \sigma_t^2}} \right) \right.$$$$+ \text{Erf} \left( \frac{L_A^+\sigma_t^2 - it(x_i - x_j)}{\sqrt{8\sigma_t^2 \sigma_t^2}} \right) \left\{ \frac{L_A^+\sigma_t^2}{\sqrt{8\sigma_t^2 \sigma_t^2}} \right\} \right] \hspace{1cm} (23)$$

having defined $L_A^+ = L_A \pm (x_i + x_j)$. Plugging Eq. (23) into
Eq. (17) one obtains
\[
(D_{\psi_1, \psi_2})^2 = \frac{1}{8} \sum_{j=1}^{2} \left[ \text{Erf} \left( \frac{L_A - 2x_j}{\sqrt{8\sigma t}} \right) + \text{Erf} \left( \frac{L_A + 2x_j}{\sqrt{8\sigma t}} \right) \right]^2 - \frac{1}{4} e^{\frac{(x_1 - x_2)^2}{2\sigma^2}} \text{Erf} \left( \frac{L_A^+ + it(x_1 - x_2)}{\sqrt{8\sigma t}(2\sigma^2)} \right) + \text{Erf} \left( \frac{L_A^- - it(x_1 - x_2)}{\sqrt{8\sigma t}(2\sigma^2)} \right)^2.
\]

For simplicity, as stated already, we initiate the evolution in a symmetric configuration where \(x_2 = -x_1\). To discuss the Markovian character of the evolution, encoded in the above equation we have to distinguish two cases according to whether the initial wave packets are centered inside or outside region \(A\) (i.e. \(|x_i| \leq L_A/2\) or \(|x_i| > L_A/2\)). Intuitively, in the first case particles can only escape from region \(A\), and so we always expect Markovianity for any parameter value. Indeed this intuition can be confirmed after a lengthy calculation taking the time derivative of Eq. (24).

Let us now consider the other situation where the particles are initialized outside \(A\), i.e. \(|x_i| > L_A/2\). In this case, the wave front first enters region \(A\) after having traveled a distance \(x_2 - L_A/2\), then travels inside \(A\) (length \(L_A\)), and finally escapes region \(A\) after having traveled a length \(x_1 + L_A/2\). At the beginning, particles enter region \(A\), and we expect an increase in \(D_{\psi_1, \psi_2}(t)\). Assuming that the front moves at constant speed \(v \sim 2/\sigma\) (which can be read off from \(\sigma_t\) at large times) the time scales after which we expect to see Markovian behavior is given roughly by \(\tau_M \sim \sigma (x_1 + L_A/2)/2\). In general, the appearance of a region of non-Markovianity can be observed as long as the initial wave functions are sufficiently localized, so that the initial front does not surpass region \(A\), i.e. roughly \(\sigma < (x_1 + L_A/2)/2\).

These predictions are confirmed by our numerical experiments shown in Fig. 2. The purely Markovian behavior observed for \(|x_i| \leq L_A/2\) indicates, as it is reasonable to expect, that for infinite spatial extent, wave packets always leak out of region \(A\) and never come back. An important ingredient in reaching this conclusion is the fact that, for the case considered, the spectrum is purely continuous. The presence of bound states in the spectrum may lead to oscillations of information back and forth from region \(A\). This in turn may lead to a breaking of Markovianity in case such bound states are initially populated.

### B. Double delta barrier

In this section we modify the free dynamical evolution by introducing two delta barriers at the boundaries of region \(A\), at positions \(\pm L_A/2\), i.e. we consider the potential \(V(x) = V_1[\delta(x - L_A/2) + \delta(x + L_A/2)]\), where \(V_1\) measures the strengths of the barriers. Eigenstates and eigenvalues of this system can be found by integrating Schrödinger’s equation in the neighborhood of the barriers, and imposing continuity of the wave function. As a result one obtains a transcendental equation for the quantum number \(k\), which is solved using a numerical root finder based on the bisection method. Again, for finite size \(L\), the spectrum is purely (countably) discrete. The initial state is expressed in the Hamiltonian eigenbasis according to \(\psi_j(0, x) = \sum_k \phi_k(x)\langle \phi_k | \psi_j \rangle\), keeping as many terms in order to reach a fidelity of at least 0.99. The eigenfunctions themselves are just piece-wise con-
The pseudo-Markovian event observed in Fig. 4 clearly defines a characteristic time scale $t_{NM}$ which is the transi-
tive for which the system behaves in a Markovian fashion before non-Markovian oscillations occur [20]. We have operation-
ally defined $t_{NM}$ as the time when the HS distance $D_{\psi_1,\psi_2}(t)$ increases by 10% from its minimum value before $t_{NM}$. We have verified that this time constant $t_{NM}$ scales linearly with the strength of the barriers $V_0$, (for the case $V_1 = V_0$ and $V_2 = 2V_0$), and with the width of the initial Gaussian wave packets, $\sigma$ (see Fig. 5). Indeed, stronger barriers mean longer tunneling times, and therefore a longer time for the particles to return from the bath (Fig. 5(a)). On the other hand narrower wave packets lead to higher occupancy of the high-
energy modes which can more easily pass through the barriers (Fig. 5(b)).

It can be argued that the dips and peaks observed in Figs. 4(b) and (c) are artifacts of a very fine-tuned system-bath aspect ratio. Let us therefore examine more generic situations with variable bath size. This will also allow us to explore the level of control we can exert on the system by tuning feedback effects due to variable bath size, as well as examine the crossover to a more conventional notion of bath in the limit $L_B \to \infty$.

The fine tuning required to control the tunneling can be seen in Fig. 4(a)-(d). Fig. 4(a) shows the time averaged HS distance $\overline{D}_{\psi_1,\psi_2} := \lim_{T_{max} \to \infty} T_{max}^{-1} \int_0^{T_{max}} D_{\psi_1,\psi_2}(t)dt$ versus size of the bath. One observes that only at very specific values of $L_B$ a significant deviation from perfect distinguishability is achieved. Comparison to Fig. 4(d) which shows average probability for one of the particles to be found in the system, indicates that the underlying cause of this lack of distinguishability is that the wave function cannot significantly tunnel out of the system for most values of $L_B$. We notice...
for broad initial Gaussians: narrow dips in trace distance occur at lengths for which a significant portion of modes are able to tunnel into the bath. b) Same as (a), but with initially narrow Gaussians. c) A zoom on the peak seen in (b) at $L_B$. d) Time average of $p_{1,1}$ versus $L_B$ for narrow Gaussians, (red) stars, and broad Gaussians, (blue) squares. Note the similarity between these plots and plots (a) and (b). In all of these plots barriers are of strength $V_1 = 10^6$ and $V_2 = 2.0 \times 10^6$, “narrow” Gaussians have width $\sigma = 0.005$ whereas “broad” Gaussians have width $\sigma = 0.125$.

that in the case of narrow initial Gaussians (Fig. 6 (b)) the particle is able to tunnel out of the system for more values of $L_B$, as more eigenmodes of the system have significant amplitudes, but otherwise this result is analogous to the one shown in Fig. 6 (a).

The reason for the behavior seen in Fig. 6 relates to the fact that the energy scale of all eigenmodes of the system considered here is much smaller than the barrier strengths. For this reason, the only modes which are allowed to have appreciable amplitudes both in the system and the bath are those where both of the barriers are very close to a node.

Fig. 6 (c) shows that both the average HS distance and the average probability to be in the bath can be controlled by fine tuning $L_B$. As the ratio of $L_B$ vs. $L$ is moved further from away from 1/2, the modes become separated into those which are isolated in the system, and those which are isolated in the bath. As this happens, tunneling is reduced, and consequently the particles become more distinguishable on average.

We now turn our interest to the case where $L_A$ is fixed and $L_B \to \infty$. Here we set the two barrier strengths equal for simplicity. In this case the particles do not return after escaping the system, so the evolution is Markovian. Fig. 7 shows the resulting trace distance for the case of a pair of initially narrow Gaussians. As expected this plot indicates Markovian decay as the wave functions escape the system.


c) d) Time average of $p_{1,1}$ versus $L_B$ for narrow Gaussians, (red) stars, and broad Gaussians, (blue) squares. Note the similarity between these plots and plots (a) and (b). In all of these plots barriers are of strength $V_1 = 10^6$ and $V_2 = 2.0 \times 10^6$, “narrow” Gaussians have width $\sigma = 0.005$ whereas “broad” Gaussians have width $\sigma = 0.125$.

V. CONCLUSIONS

In summary, we have investigated non-Markovian effects arising from a tunable finite bath. These can be quantified by a measure that tracks particles flow out of and into a system that is connected to the bath. We have identified various time scales for rephasing events that depend non-trivially on the bath size, on the tunneling barrier potentials between the system and the bath, and on the shape of the wave functions with which the evolution is initiated. In particular, we found that substantial rephasing can be achieved by fine tuning the bath length and by choosing initial states with significant high frequency components, allowing the wave packets to tunnel efficiently between the system and the bath. One can envision physical realizations of such a setup in the context of nanoelectronics and nanophotonics. For example, a photonic microcavity can act as the system, connected via semitransparent mirrors to an external cavity that acts as the bath. Then the transparency of the mirrors corresponds to the barrier potential, and the length of the external cavity sets the time scale for major rephasing events. An implementation in the context of nanoelectronics may be even more interesting, because in this case the effects of electron interactions on non-Markovian system dynamics could be studied as well.

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[20] As will be clear below, the computation simply requires diagonalization of a matrix of size $O(N)$ for a system of $N$ particles.

[21] In vector notation Eq. (3) becomes $|\Gamma| \gg |\mathbb{I} \otimes X^T + X^T \otimes \mathbb{I}|$. Calling $\mathcal{M} = |\mathbb{I} \otimes X^T + X^T \otimes \mathbb{I}|$ the solution is $|\Gamma(t)| \gg e^{\mathcal{M} \Gamma(0)} \gg -|Y| \gg + |\Gamma(\infty)\rangle \gg |\Gamma(\infty)\rangle \gg \mathcal{M}^{-1} |Y\rangle \gg$ in case $\mathcal{M}$ is invertible. Under the above hypothesis the spectrum of $\mathcal{M}$ has non positive real part and the result (4) follows.

[22] We are aware that a particle-number conserving evolution is not consistent with Eqs. (1) and (3). In practice we are asking if it is possible to have a Markovian evolution [of the form of Eqs. (1) and (3)] for the subsystem $A$ giving rise to the observed matrix $R$ in this subsystem.

[23] The error function is defined as $\text{Erf}(z) := \pi^{-1/2} \int_0^z e^{-t^2} dt$.

[24] Indeed, consider $U_{LR}$ the unitary operator which implements the Left-Right inversion around the origin. Then, at all times, for symmetric potentials $V(x)$, $U_{LR} |\psi_1(t)\rangle = e^{i\phi} |\psi_2(t)\rangle$. Then $p_{11}(t) = \langle \psi_1(t) | U_{LR}^† U_{LR} P_A U_{LR}^† U_{LR} |\psi_1(t)\rangle = \langle \psi_2(t) | P_A |\psi_2(t)\rangle = p_{22}(t)$ since $[P_A, U_{LR}] = 0$.

[25] Operationally $t_{NM}$ is the first occurrence of time $t$ for which $\sigma(t, \Gamma_1, \Gamma_2) > 0$.