Shells without drilling rotations: A representation theorem in the framework of the geometrically nonlinear 6-parameter resultant shell theory

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Abstract

In the framework of the geometrically nonlinear 6–parameter resultant shell theory we give a characterization of the shells without drilling rotations. These are shells for which the strain energy function $W$ is invariant under the superposition of drilling rotations, i.e. $W$ is insensible to the arbitrary local rotations about the third director $d_3$. For this type of shells we show that the strain energy density $W$ can be represented as a function of certain combinations of the shell deformation gradient $F$ and the surface gradient of $d_3$, namely $W(F^TF, F^T d_3, F^T \text{Grad}_s d_3)$. For the case of isotropic shells we present explicit forms of the strain energy function $W$ having this property.

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1. Introduction

There exist many variants of the shell theory in the literature and various different models for shells and plates (Antman, 1995; Ciarlet, 2000). Some of them are classical, such as the Kirchhoff–Love shell theory or the Reissner–Mindlin theory (Love, 1944; Naghdi, 1972; Sokolnikoff, 1956; Timoshenko, 1951). These approaches do not take into account the drilling rotations in shells, i.e. the rotations of points about the shell filament. Although the classical theories can handle the majority of practical shell problems, there exist however situations when the drilling rotations cannot be neglected. For instance, in the case of shells with a certain microstructure or for branching and self-intersecting shells one should account also for the drilling rotations (Chróścielewski, Makowski, & Pietraszkiewicz, 2004; Konopińska & Pietraszkiewicz, 2007; Libai & Simmonds, 1998; Steigmann, 1999; Steigmann, 2013). The question whether one has to consider a drilling degree of freedom or not is intensively discussed in the literature. Since in the classical Kirchhoff–Love and Reissner–Mindlin there is no drilling degree of freedom present, there is also no energy storage due to this variable.

Other non-classical approaches are more general, such as the Cosserat–type theories of shells or micropolar shells (Cosserat & Cosserat, 1909; Altenbach, Altenbach, & Eremeyev, 2010). A general theory of micropolar shells has been developed by Eremeyev and Zubov (Eremeyev & Zubov, 2008). It is conceivable, that a Cosserat–type shell model has the
possibility of drilling degrees of freedom. In this case it is a matter of assumption of what is the contribution to the stored elastic energy. For these generalized models each material point has six degrees of freedom (parameters): 3 for the translations and 3 rotational degrees of freedom. In order to characterize the independent rotations of material points, one considers a triad of vectors \( \mathbf{d}_1 \) (called directors) attached to every point. The drilling rotations are then described as rotations about the third director \( \mathbf{d}_3 \), it is difficult to physically justify the drilling degree of freedom. Therefore we wish to characterize the shells without drilling rotations, i.e. the case when the strain energy of the shell is insensitive to rotations about the director \( \mathbf{d}_1 \). In the framework of the 6-parameter resultant shell theory we prove a representation theorem for shells without drilling rotations, which asserts that the strain energy density \( W \) of such shells can be represented as a function of the following arguments

\[
W = W\left( F^T \mathbf{f}, F^T \mathbf{d}_3, \mathbf{F} \mathbf{G} \mathbf{r} \right),
\]

where \( \mathbf{F} \) is the shell deformation gradient tensor and \( \mathbf{G} \mathbf{r} \) denotes the surface gradient operator.

It is now firmly established that the 6-parameter resultant shell theory (Libai & Simmonds, 1998; Chróścielewski et al., 2004; Eremeyev & Pietraszkiewicz, 2004) has the same kinematical structure as the theory of Cosserat shells. The shells of Cosserat-type without drilling rotations have been studied extensively by Zhilin in Zhilin (1976), Zhilin (2006). We compare next our results with the model of Zhilin and find a close relation, especially in the linearized theory.

In the last section we consider the explicit form of the strain energy function for isotropic shells made of a physically linear material. The existence of minimizers for 6–parameter elastic shells has been proved in Bîrsan and Neff (2013), Bîrsan and Neff (2014) under certain conditions on the constitutive coefficients which insure the positive definiteness of \( W \). However, the existence theorem presented in Bîrsan and Neff (2014) does not apply to shells insensitive to drilling rotations, since we obtain here a limit case when the strain energy function is only positive semi-definite. The similar problem in the case of Cosserat plates has been treated in Neff (2004a), Neff (2007), where the existence of minimizers has been proved.

This discussion on the drilling degree of freedom is intimately related to the value of the Cosserat couple modulus \( \mu_c \), in the shell models derived by dimensional reduction from three-dimensional Cosserat elasticity, where the parent model is a hyperelastic micropolar model (Neff, 2006a; Neff, 2006b; Neff, 2006c). Thus, the case of shells without drilling degree of freedom corresponds to \( \mu_c = 0 \). This is a degenerate case, which was studied extensively by Neff in Neff (2007), see also (Neff & Jeong, 2009, Jeong, Ramezani, Münch, & Neff, 2009, 2009, 2010). In general, it can be said, that the nonlinear theories without drill are not well-posed as minimization problems, since a certain control of the third director in combination with the coupling of the director to the shell surface is missing. Only in the linearized models this problem can be solved (Neff, 2004b; Neff, 2006c; Neff & Chelmiński, 2007; Neff, Hong, & Jeong, 2010). Thus, from a mathematical point of view, the inclusion of the drilling degree of freedom stabilizes somehow the model at the expense of physical clarity.

2. Field equations for 6–parameter elastic shells

In this section we repeat the basic equations of the 6–parameter resultant shell theory and present some useful relations.

Let us denote with \( S^0 \) the base surface of a general configuration and with \( S \) the base surface in the deformed configuration. The shell is referred to a fixed Cartesian frame with origin \( O \) and unit vectors \( \{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \} \) along the coordinate axes. The reference configuration of the shell will be described by the position vector \( y^0 \) and the structure tensor \( \mathbf{Q}^0 \), which is a proper orthogonal tensor. In order to represent the structure tensor \( \mathbf{Q}^0 \) we consider an orthonormal triad of vectors \( \{ \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3 \} \) (called directors) attached to every point of \( S^0 \) such that \( \mathbf{Q}^0 = \mathbf{d}_i \otimes \mathbf{e}_i \) (Chróścielewski et al., 2004; Eremeyev & Pietraszkiewicz, 2006). Here we employ the summation convention over repeated indexes. The Latin indexes \( i,j \)…take the values \( \{ 1, 2, 3 \} \), while Greek indexes \( \alpha, \beta \)…the values \( \{ 1, 2 \} \).

We denote the material curvilinear coordinates on the surface \( S^0 \) with \( (x_1, x_2) \in \omega \). The set \( \omega \) is assumed to be a bounded open domain with Lipschitz boundary \( \partial \omega \) in the \( O x_1 x_2 \) plane. Then the reference configuration of the shell is characterized by the functions

\[
\begin{align*}
y^0 : \omega \subset \mathbb{R}^2 & \rightarrow \mathbb{R}^3, \\
\mathbf{Q}^0 : \omega \subset \mathbb{R}^2 & \rightarrow \text{SO}(3), \\
\mathbf{d}_i^0 : \omega \subset \mathbb{R}^2 & \rightarrow \mathbb{R}^3.
\end{align*}
\]

Let us designate by \( y(x_1, x_2) \) the position vector of the points of \( S \) and by \( \mathbf{R}(x_1, x_2) = \mathbf{d}_i(x_1, x_2) \otimes \mathbf{e}_i \in \text{SO}(3) \), the structure tensor in the deformed configuration. Here \( \{ \mathbf{d}_i(x_1, x_2) \} \) is the orthonormal triad of directors attached to the point of \( S \) with the initial curvilinear coordinates \( (x_1, x_2) \). Then, one can characterize the deformation of the elastic shell by means of the functions

\[
\begin{align*}
y = \chi(y^0), \\
\mathbf{Q} = \mathbf{d}_i \otimes \mathbf{d}_i^0 \in \text{SO}(3),
\end{align*}
\]

where \( \chi : S^0 \rightarrow \mathbb{R}^3 \) represents the deformation of the base surface of the shell and \( \mathbf{Q}^r = \mathbf{R} \mathbf{Q}^0 \mathbf{R}^{-1} \) describes the (effective) elastic rotation.

For the reference base surface \( S^0 \) we denote with \( \{ \mathbf{a}_1, \mathbf{a}_2 \} \) the (covariant) base vectors given by \( \mathbf{a}_2 = \partial y^0 / \partial x_2 \), and with \( \mathbf{n}^0 = \mathbf{a}_1 \times \mathbf{a}_2 / \| \mathbf{a}_1 \times \mathbf{a}_2 \| \), the unit normal to \( S^0 \). We introduce also the reciprocal (contravariant) base vectors \( \{ \mathbf{a}^1, \mathbf{a}^2 \} \) and the notations \( \mathbf{a}_3^0 = \mathbf{a}^1 \times \mathbf{a}^2 = \mathbf{n}^0 \) such that the relations \( \mathbf{a}_i \cdot \mathbf{a}^j = \delta^j_i \) (the Kronecker symbol) holds. In the reference configuration \( S^0 \) we choose the initial director \( \mathbf{d}_3^0 \) such that \( \mathbf{d}_3^0 = \mathbf{n}^0 \).
In order to present the elastic shell strain measures and bending–curvature measures we employ the notations
\[ a = a_x \otimes a^x = a_{xy} a_x \otimes a^y = a^{xy} a_x \otimes a^y = d_x \otimes d_y, \quad a = \sqrt{\det(a_{xy})} > 0, \]
\[ b = -\operatorname{Grad}_n n^0 = -\partial_n n^0 \otimes a^x, \]
\[ c = -n^0 \times a = -a \times n^0 = \frac{1}{\sqrt{a}} \epsilon_{xy} a_x \otimes a_y = \epsilon_{xy} d_x \otimes d_y, \]
where \( \epsilon_{xy} \) is the two-dimensional alternator (\( \epsilon_{12} = -\epsilon_{21} = 1, \quad \epsilon_{11} = \epsilon_{22} = 0 \)). The tensors \( a \) and \( b \) are the first and second fundamental tensors of the surface \( S^0 \), while \( c \) is called the alternator tensor of \( S^0 \) (Zhilin, 2006). We have \( a^T = a, b^T = b, c^T = -c \) and \( c c = -a \).

Then, the elastic shell strain tensor \( E^e \) in the material representation is (Eremeyev & Pietraszkiewicz, 2006; Chróścielewski et al., 2004)
\[ E^e = Q^e T \operatorname{Grad}_n y = \operatorname{Grad}_n y^0 = \left( Q^e T \partial_y y - \partial_y y^0 \right) \otimes a^x, \]
(2)

The bending–curvature tensor \( K^e \) in material representation is given by (Eremeyev & Pietraszkiewicz, 2006; Chróścielewski et al., 2004; Birsan & Neff, 2013)
\[ K^e = Q^e T \operatorname{axl} \left( \partial_y Q^e T \right) \otimes a^x = \operatorname{axl} \left( Q^e T \partial_y Q^e T \right) \otimes a^x, \]
(3)

We can represent \( K^e \) in terms of \( R \) and \( Q^0 \) in the form
\[ K^e = Q^0 \left[ \operatorname{axl} \left( R^T \partial_y R \right) - \operatorname{axl} \left( Q^{0T} \partial_y Q^0 \right) \right] \otimes a^x, \]
(4)

or equivalently
\[ K^e = K - K^0, \quad \text{with} \quad K = Q^0 \operatorname{axl} \left( R^T \partial_y R \right) \otimes a^x, \]
\[ K^0 = Q^0 \operatorname{axl} \left( Q^{0T} \partial_y Q^0 \right) \otimes a^x = \operatorname{axl} \left( \partial_y Q^0 Q^{0T} \right) \otimes a^x. \]
(5)

It is useful to express the tensors \( E^e \) and \( K^e \) decomposed in the tensor basis \( \{ d_x \otimes a^x \} \). Thus, we obtain
\[ E^e = \left[ \left( \partial_y y \cdot d_x \right) d_x^0 - a_x \right] \otimes a^x, \]
\[ K^e = \left[ \left( \partial_x d_2 \cdot d_2 + \left( \partial_x d_3 \cdot d_3 + \left( \partial_x d_1 \cdot d_1 \right) d_1^0 \right) \right] \right] \otimes a^x, \]
\[ K^0 = \left[ \left( \partial_x d_2 \cdot d_x^0 \right) d_2^0 + \left( \partial_x d_3 \cdot d_x^0 \right) d_3^0 + \left( \partial_x d_1 \cdot d_x^0 \right) d_1^0 \right] \otimes a^x. \]
(6)

Thus, the tensor components are
\[ E^e = E^e x d_x^0 \otimes a^x = \left( \partial_y y \cdot d_x - a_x \right) d_x^0 \otimes a^x, \]
\[ K^e = K^e x d_x^0 \otimes a^x = \frac{1}{2} \epsilon_{ijk} \left( \partial_y d_j \cdot d_k - \partial_y d_k \cdot d_j \right) d_i^0 \otimes a^x = \left( \partial_y d_2 \cdot d_1 - \partial_y d_1 \cdot d_2 \right) d_x^0 \otimes a^x + \left( \partial_y d_3 \cdot d_1 - \partial_y d_1 \cdot d_3 \right) d_x^0 \otimes a^x, \]
(7)

where \( \epsilon_{ijk} \) is the three-dimensional alternator (i.e. the signature of the permutation \( (1, 2, 3) \rightarrow (i, j, k) \)).

Let the tensors \( N \) and \( M \) be the internal surface stress resultant and stress couple tensors of the 1st Piola–Kirchhoff type for the shell. The equilibrium equations for 6–parameter shells are (Chróścielewski et al., 2004; Eremeyev & Pietraszkiewicz, 2006; Birsan & Neff, 2014)
\[ \operatorname{Div}_n N + f = 0, \]
\[ \operatorname{Div}_n M + \operatorname{axl} \left( NF^T - FN^T \right) + l = 0, \]
(8)

where \( f \) and \( l \) are the external surface resultant force and couple vectors applied to points of \( S \), but measured per unit area of \( S^0 \). Here \( \operatorname{Div}_n \) denotes the surface divergence and \( \operatorname{axl}(\cdot) \) represents the axial vector of a skew–symmetric tensor. To formulate the boundary–value problem, we consider boundary conditions of the type (Pietraszkiewicz, 2011)
\[ N v = n^*, \quad M v = m^* \quad \text{along} \quad \partial S^0, \]
\[ y = y^*, \quad R = R^* \quad \text{along} \quad \partial S^0. \]
(9)
where \( \mathbf{v} \) is the external unit normal vector to the boundary curve \( \partial S^0 \) (lying in the tangent plane) and \( \{ \partial S^0_1, \partial S^0_2 \} \) is a disjoint partition of \( \partial S^0 \). Let \( W = W(\mathbf{E}^e, \mathbf{K}^e) \) be the strain energy density of the elastic shell, measured per unit area of the base surface \( S^0 \). The principle of energy can be written in the form \( \text{Zhilin (2006)} \)

\[
W = \left( \mathbf{Q}^T \mathbf{N} \right) 
\mathbf{E}^e + \left( \mathbf{Q}^T \mathbf{M} \right) \mathbf{K}^e,
\]

where a superposed dot designates the material time derivative and \( \cdot \) means the scalar product of tensors, i.e. \( \mathbf{S} \cdot \mathbf{V} = \text{tr}(\mathbf{S}^T \mathbf{V}) \). Similar relations expressing the internal virtual power and the principle of virtual work have been presented in Eremeyev and Pietraszkiewicz (2004), Eremeyev and Pietraszkiewicz (2006). Under the hyperelasticity assumption, \( \mathbf{N} \) and \( \mathbf{M} \) are expressed by the constitutive equations

\[
\mathbf{N} = \mathbf{Q}^e \frac{\partial W}{\partial \mathbf{E}^e}, \quad \mathbf{M} = \mathbf{Q}^e \frac{\partial W}{\partial \mathbf{K}^e}.
\]

To resume, the boundary–value problem for non-linear elastic 6–parameter shells consists of the Eqs. (2), (3), (8), (9), and (11). The minimization problem associated to the deformation of elastic shells can be put in the following form: find the pair \( (\mathbf{y}, \mathbf{R}) \) in the admissible set \( \mathcal{A} \) which realizes the minimum of the functional

\[
l(\mathbf{y}, \mathbf{R}) = \int_{S^0} W(\mathbf{E}^e, \mathbf{K}^e) \, d\mathbf{S} - \Lambda(\mathbf{y}, \mathbf{R}) \quad \text{for} \quad (\mathbf{y}, \mathbf{R}) \in \mathcal{A},
\]

where \( d\mathbf{S} \) is the area element of the surface \( S^0 \) and \( \Lambda(\mathbf{y}, \mathbf{R}) \) is a function representing the potential of external surface loads \( \mathbf{f}, \mathbf{l} \), and boundary loads \( \mathbf{n}^t, \mathbf{m}^t \) (Eremeyev & Pietraszkiewicz, 2004; Bîrsan & Neff, 2014). The admissible set is

\[
\mathcal{A} = \left\{ (\mathbf{y}, \mathbf{R}) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \text{SO}(3)) \mid \mathbf{y}_{|\partial S^0} = \mathbf{y}^*, \quad \mathbf{R}_{|\partial S^0} = \mathbf{R}^* \right\},
\]

where \( H^1 \) designates the well-known Sobolev space and the boundary conditions are to be understood in the sense of traces.

We remark that the 6–parameter model of shells takes into account also the deformations caused by the so-called drilling rotations. The drilling rotation in a point of \( S \) is interpreted as the rotation about the director \( \mathbf{d}_i \).

In what follows, we describe the shells which are insensible to drilling rotations, in the framework of the 6–parameter theory. This class of shells is important since it is widely used in applications.

3. Characterization of shells without drilling rotations

For shells completely without drilling rotations the strain energy \( W \) must be insensible to the rotations about \( \mathbf{d}_i \). Let \( \mathbf{R}_0 \) denote the rotation of angle \( \theta(\mathbf{x}_1, \mathbf{x}_2) \) about \( \mathbf{d}_i \). The general form of \( \mathbf{R}_0 \) may be given by

\[
\mathbf{R}_0 = \mathbf{d}_i \otimes \mathbf{d}_j + \cos \theta(\mathbf{1} - \mathbf{d}_i \otimes \mathbf{d}_j) + \sin \theta(\mathbf{d}_i \times \mathbf{1}),
\]

or equivalently, written in the \( \{ \mathbf{d}_i \otimes \mathbf{d}_j \} \) tensor basis,

\[
\mathbf{R}_0 = \cos \theta(\mathbf{d}_i \otimes \mathbf{d}_i + \mathbf{d}_j \otimes \mathbf{d}_2 + \sin \theta(\mathbf{d}_2 \otimes \mathbf{d}_i - \mathbf{d}_1 \otimes \mathbf{d}_2) + \mathbf{d}_j \otimes \mathbf{d}_3.
(12)
\]

In other words, the shells without drilling rotations are characterized by the property that \( W(\mathbf{E}^e, \mathbf{K}^e) \) remains invariant under the superposition of arbitrary rotations angle \( \theta(\mathbf{x}_1, \mathbf{x}_2) \) about \( \mathbf{d}_i \), i.e. invariant under the transformation

\[
\mathbf{Q}^e \rightarrow \mathbf{R}_0 \mathbf{Q}^e.
(13)
\]

Taking into account the relations (2), (3), and (13), we see that the strain and bending-curvature measures \( \mathbf{E}^e \) and \( \mathbf{K}^e \) transform as

\[
\mathbf{E}^e \rightarrow \mathbf{Q}^e \mathbf{R}_0^T \mathbf{E}^e - \mathbf{a},
\]

\[
\mathbf{K}^e \rightarrow \text{axl} \left[ \mathbf{Q}^e \mathbf{R}_0^T \partial_2 (\mathbf{R}_0 \mathbf{Q}^e) \right] \otimes \mathbf{a}^2
\]

In view of (2), (3), and (14), the invariance condition on the strain function \( W \) can be written as

\[
W(\mathbf{Q}^e \mathbf{R}_0^T \mathbf{E}^e - \mathbf{a}, \text{axl} \left[ \mathbf{Q}^e \mathbf{R}_0^T \partial_2 (\mathbf{R}_0 \mathbf{Q}^e) \right] \otimes \mathbf{a}^2) = W(\mathbf{Q}^e \mathbf{R}_0^T \mathbf{E}^e - \mathbf{a}, \text{axl} \left[ \mathbf{Q}^e \mathbf{R}_0^T \partial_2 (\mathbf{R}_0 \mathbf{Q}^e) \right] \otimes \mathbf{a}^2)
\]

(15)

for all rotation angles \( \theta(\mathbf{x}_1, \mathbf{x}_2) \). If we employ the tensors components \( \mathbf{E}^e \) and \( \mathbf{K}^e \) introduced in (7), then the invariance condition (15) can be put into another form: we denote by \( \mathbf{d}_i \) the directors rotated with angle \( \theta \) about \( \mathbf{d}_i \) in the configuration, i.e.

\[
\mathbf{d}_i' = \mathbf{R}_0 \mathbf{d}_i = \cos \theta \mathbf{d}_i + \sin \theta \mathbf{d}_2, \quad \mathbf{d}_j' = \mathbf{R}_0 \mathbf{d}_j = - \sin \theta \mathbf{d}_i + \cos \theta \mathbf{d}_2,
\]

\[
\mathbf{d}_3' = \mathbf{R}_0 \mathbf{d}_3 = \mathbf{d}_3, \quad \mathbf{R}_0 \mathbf{Q}^e = \mathbf{d}_i' \otimes \mathbf{d}_i'.
\]

Then, the transformation (14) can be written with the help of the components (7) as follows.
\[ E^r_{12} \mathbf{d}_1^r \otimes \mathbf{a}_2^r \rightarrow \left( \partial_r \mathbf{y} \cdot \mathbf{d}_1^r - \partial_r \mathbf{y}_0 \cdot \mathbf{d}_1^r \right) \mathbf{d}_2^r \otimes \mathbf{a}_2^r, \]
\[ K^r_{12} \mathbf{d}_1^r \otimes \mathbf{a}_2^r \rightarrow \frac{1}{2} e_{ijk} \left( \partial_r \mathbf{d}_1^r \cdot \mathbf{d}_2^r - \partial_r \mathbf{d}_1^r \cdot \mathbf{d}_3^r \right) \mathbf{d}_1^r \otimes \mathbf{a}_1^r \]

and the invariance condition (15) on the strain energy function \( W \) takes the form
\[ \bar{W}(E^r_{12}, K^r_{12}) = \bar{W}\left( \partial_r \mathbf{y} \cdot \mathbf{d}_1^r - \partial_r \mathbf{y}_0 \cdot \mathbf{d}_1^r, \frac{1}{2} e_{ijk} \left( \partial_r \mathbf{d}_1^r \cdot \mathbf{d}_2^r - \partial_r \mathbf{d}_1^r \cdot \mathbf{d}_3^r \right) \right). \]

where the components \( E^r_{12} \) and \( K^r_{12} \) are given in (7) and the function \( \bar{W} \) is defined by \( \bar{W}(E^r_{12}, K^r_{12}) = W(E^r_{12} \mathbf{d}_1^r \otimes \mathbf{a}_2^r, K^r_{12} \mathbf{d}_1^r \otimes \mathbf{a}_1^r) \).

We present now the main result, which gives a representation theorem for the strain energy function \( W \) that satisfy the invariance condition (15), i.e. for shells without drilling rotations.

Theorem 1. The strain energy function \( W \) of the shell is invariant under the transformations (13) (i.e. it is insensitive to drilling rotations) if and only if it can be represented as a function of the following arguments
\[ W = \bar{W}(F^r \mathbf{F}, F^r \mathbf{d}_3, F^r \text{Grad} \mathbf{d}_3). \]

Proof. We observe first that any function \( W \) which admits a representation of the type (19) is invariant under the transformations (13). Indeed, the vectors \( \mathbf{y} \) and \( \mathbf{d}_3 \) are invariant under rotations about \( \mathbf{d}_3 \). Therefore, the tensors \( F = \text{Grad} \mathbf{y} \) and \( \text{Grad} \mathbf{d}_3 \) are also invariant under rotations (12) and consequently, any function \( W \) of the combinations \((F^r \mathbf{F}, F^r \mathbf{d}_3, F^r \text{Grad} \mathbf{d}_3)\) is invariant under the transformation (13).

Conversely, let us prove that any strain energy function \( W \) satisfying this invariance condition admits necessarily a representation of the form (19). We assume that \( W \) fulfills the invariance relation (15), which is equivalent to (18). In order to write (18) in a more convenient form, we calculate the scalar products
\[ \partial_r \mathbf{d}_1^r \cdot \mathbf{d}_2^r = d_3 \cdot \partial_r \mathbf{d}_2^r, \]
\[ \partial_r \mathbf{d}_1^r \cdot \mathbf{d}_1^r = -d_3^2 \cdot \partial_r \mathbf{d}_1^r = -d_3 \cdot \partial_r \mathbf{d}_1^r, \]
\[ \partial_r \mathbf{d}_1^r \cdot \mathbf{d}_2^r = [\cos \theta \partial_r d_1 + \sin \theta \partial_r d_2] + [-\cos \theta d_1 + \sin \theta d_2] \cdot [\cos \theta d_1 + \sin \theta d_2] = d_3 \mathbf{d}_1 \cdot \mathbf{d}_2 + \partial_r \theta. \]

Using relations of this type we can put the condition (18) in the following form
\[ \bar{W}(E^r_{12}, K^r_{12}) = \bar{W}\left( \partial_r \mathbf{y} \cdot \mathbf{d}_1^r - \mathbf{a}_2 \cdot \mathbf{d}_1^r, \partial_r \mathbf{d}_1^r \cdot \mathbf{d}_3 - \partial_r \mathbf{d}_2^r \cdot \mathbf{n}^0 - \partial_r \mathbf{d}_3^r \cdot \mathbf{d}_1 + \partial_r \mathbf{d}_3^r \cdot \mathbf{n}^0 - \partial_r \theta, \partial_r \mathbf{d}_1 \cdot \mathbf{d}_2 - \partial_r \mathbf{d}_1 \cdot \mathbf{d}_1 \right), \]

where \( \mathbf{d}_1^r \) are expressed by (16). Relation (20) must hold for all angles of rotation \( \theta = \theta(x_1, x_2) \). Since the left-hand side is independent of \( \theta \) and \( \partial_r \theta \), it follows that the derivatives of the right-hand side with respect to \( \theta \) and \( \partial_r \theta \) are zero. Thus, taking the derivative of (20) with respect to \( \partial_r \theta \) we obtain by the chain rule
\[ \frac{\partial \bar{W}}{\partial \theta} = 0, \quad \text{or equivalently} \quad \frac{\partial \bar{W}}{\partial (\mathbf{n}^0 \mathbf{K}^r)} = 0. \]

Since \( \mathbf{n}^0 \mathbf{K}^r = \mathbf{d}_3^r (\mathbf{K}^r_{12} \mathbf{d}_3^r \otimes \mathbf{a}_3^r) = \mathbf{K}_{32} \mathbf{a}_3^r \). If we differentiate the relation (20) with respect to \( \theta \) and use the relations
\[ \frac{d}{d \theta} \left( \mathbf{d}_1^r \right) = \mathbf{d}_2^r, \quad \frac{d}{d \theta} \left( \mathbf{d}_2^r \right) = -\mathbf{d}_1^r, \]
\[ \frac{d}{d \theta} \left( \partial_r \mathbf{d}_1^r \cdot \mathbf{d}_1^r \right) = -\partial_r \mathbf{d}_1^r \cdot \mathbf{d}_1^r, \quad \frac{d}{d \theta} \left( \partial_r \mathbf{d}_1^r \cdot \mathbf{d}_2^r \right) = \partial_r \mathbf{d}_2^r \cdot \mathbf{d}_3, \]

then we get
\[ \frac{\partial \bar{W}}{\partial \mathbf{E}^{12}} \cdot \left( \partial_r \mathbf{y} \cdot \mathbf{d}_2^r \right) + \frac{\partial \bar{W}}{\partial \mathbf{E}^{22}} \cdot \left( -\partial_r \mathbf{y} \cdot \mathbf{d}_1^r \right) + \frac{\partial \bar{W}}{\partial \mathbf{K}^{12}} \cdot \left( -\partial_r \mathbf{d}_1^r \cdot \mathbf{d}_1^r \right) + \frac{\partial \bar{W}}{\partial \mathbf{K}^{22}} \cdot \left( -\partial_r \mathbf{d}_2^r \cdot \mathbf{d}_3 \right) = 0. \]

Inserting (16) into (22) we get the equivalent form
\[ \sin \theta \left[ \frac{\partial \bar{W}}{\partial \mathbf{E}^{12}} \cdot \left( \partial_r \mathbf{y} \cdot \mathbf{d}_1^r \right) + \frac{\partial \bar{W}}{\partial \mathbf{E}^{22}} \cdot \left( \partial_r \mathbf{y} \cdot \mathbf{d}_2^r \right) + \frac{\partial \bar{W}}{\partial \mathbf{K}^{12}} \cdot \left( \partial_r \mathbf{d}_2^r \cdot \mathbf{d}_1^r \right) + \frac{\partial \bar{W}}{\partial \mathbf{K}^{22}} \cdot \left( -\partial_r \mathbf{d}_1^r \cdot \mathbf{d}_3 \right) \right] \]
\[ + \cos \theta \left[ \frac{\partial \bar{W}}{\partial \mathbf{E}^{12}} \cdot \left( -\partial_r \mathbf{y} \cdot \mathbf{d}_1^r \right) + \frac{\partial \bar{W}}{\partial \mathbf{E}^{22}} \cdot \left( \partial_r \mathbf{y} \cdot \mathbf{d}_2^r \right) + \frac{\partial \bar{W}}{\partial \mathbf{K}^{12}} \cdot \left( \partial_r \mathbf{d}_1^r \cdot \mathbf{d}_1^r \right) + \frac{\partial \bar{W}}{\partial \mathbf{K}^{22}} \cdot \left( \partial_r \mathbf{d}_2^r \cdot \mathbf{d}_3 \right) \right] = 0. \]

which must hold for every angle \( \theta = \theta(x_1, x_2) \). Thus, both square brackets in (23) must be zero. We show next that the relation (21) implies the equation
\[ \frac{\partial \bar{W}}{\partial \mathbf{E}^2} \cdot (\mathbf{c}^{T} \mathbf{F}) + \frac{\partial \bar{W}}{\partial \mathbf{K}^{22}} \cdot (\mathbf{c}) = 0. \]
where · denotes the scalar product of tensors and c is defined in (1). Indeed, we have

\[
\frac{\partial W}{\partial E} = \frac{\partial \tilde{W}}{\partial \tilde{E}} d_t \otimes a_s, \quad \frac{\partial W}{\partial K} = \frac{\partial \tilde{W}}{\partial \tilde{K}} d_0 \otimes a_s,
\]

\[
c_{Q^T}F = \left(\varepsilon_{xy} d_t^y \otimes d_t^x \right) \left( \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \right) c_{xy} = \varepsilon_{xy} \left( d_t^y \otimes d_t^x \right) \left( \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \right) c_{xy} = \varepsilon_{xy} \left( \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial x} \right) d_t^y \otimes a^x
\]

\[
\mathbf{c}_K = \left( \varepsilon_{xy} d_t^y \otimes d_t^x \right) \left[ (\partial_x d_1 - \partial_y d_2) d_1^y + (\partial_y d_1 - \partial_x d_2) d_2^y \right] \otimes a^x = - (\partial_x d_1 \cdot d_1) d_1^y \otimes a^x - (\partial_y d_2 \cdot d_2) d_2^y \otimes a^x
\]

and inserting this in the left-hand side of (24) we obtain the last square bracket in (23), which is equal to zero.

Thus, the relation (24) holds true. In view of (2) and (5) we can express the relation (24) in terms of E^c and K^c in the form

\[
\frac{\partial W}{\partial E^c} \cdot c(E^c + a) + \frac{\partial W}{\partial K^c} \cdot c(K^c + K^0) = 0.
\]

(25)

We regard the relation (25) as a first order linear partial differential equation for the unknown function W(E^c, K^c). The characteristic system attached to the differential Eq. (25) is (see e.g. (Vrabie, 2004), Section 6.3)

\[
\frac{dE^c}{ds} = c(E^c + a), \quad \frac{dK^c}{ds} = c(K^c + K^0).
\]

(26)

Since the unknown function W(E^c, K^c) depends in total on 12 independent scalar arguments (6 components of E^c and 6 components of K^c in the tensor basis \(\{d_t^0 \otimes a_s^0\}\)) it suffices to determine 11 independent first integrals of the system of ordinary differential Eqs. (26). Then, the general solution of the Eq. (24) can be represented as an arbitrary function of the 11 first integrals.

Let us introduce the functions \(U_1, U_2, U_3, \) and \(U_4\) given by

\[
U_1 = (E^c + a)^T (E^c + a), \quad U_2 = n^0 E^c, \quad U_3 = (E^c + a)^T c(K^c + K^0), \quad U_4 = n^0 K^c.
\]

(27)

We show that \(U_1, U_2, U_3, U_4\) represent first integrals of the system (26). Indeed, taking into account \(n^0 c = 0, a^T = a,\) and \(c^T = -c\) and (26) we find

\[
\frac{dU_1}{ds} = \left( \frac{dE^c}{ds} \right)^T (E^c + a) + (E^c + a) \frac{dE^c}{ds} = (E^c + a) \cdot (-c) (E^c + a) + (E^c + a) c (E^c + a) = 0.
\]

\[
\frac{dU_2}{ds} = n^0 \frac{dE^c}{ds} = n^0 [c(E^c + a)] = 0.
\]

\[
\frac{dU_3}{ds} = \left( \frac{dE^c}{ds} \right)^T c(K^c + K^0) + (E^c + a) c \frac{dK^c}{ds} = (E^c + a) \cdot (-c) c (K^c + K^0) + (E^c + a) c c (K^c + K^0) = 0.
\]

\[
\frac{dU_4}{ds} = n^0 \frac{dK^c}{ds} = n^0 [c(K^c + K^0)] = 0.
\]

The functions \(U_1, U_2, U_3, U_4\) give in total 11 independent first integrals: 3 components of \(U_1\) (symmetric) and 4 components of \(U_3\) in the tensor basis \(\{d_t^0 \otimes a_s^0\}\), 2 components of the vector \(U_2\) and 2 components of the vector \(U_4\) in the basis \(\{a^T, a^T\}\).

According to the theory of differential equations (see e.g. (Vrabie, 2004), Section 6.1), the general solution of the partial differential Eq. (25) has the form

\[
W(E^c, K^c) = \tilde{W}(U_1, U_2, U_3, U_4).
\]

(28)

On the other hand, by virtue of (21) we have \(\frac{\partial W}{\partial E^c} = 0\) and from (28) we deduce that the energy function \(W\) can be represented as

\[
W(E^c, K^c) = \tilde{W}(U_1, U_2, U_3).
\]

(29)

Finally, by a straightforward calculation we obtain from (27) and (2–6) the relations

\[
U_1 = (E^c + a)^T (E^c + a) = (Q^T F)^T (Q^T F) = F^T F,
\]

\[
U_2 = n^0 E^c = d_t^3 (Q^T \partial_x y - a_s) \otimes a^s = (Q^T d_t^3 \cdot \partial_x y) a^s = (d_3 \cdot \partial_x y) a^s = d_3 (\partial_x y \otimes a^s) = F^T d_3,
\]

\[
U_3 = (E^c + a)^T c(K^c + K^0) = (Q^T F)^T c K = F^T Q (d_t^0 \otimes d_t^3 - d_t^2 \otimes d_t^1) c K
\]

\[
= F^T \left[ (\partial_x d_1 \cdot d_1) d_1^3 + (\partial_x d_2 \cdot d_2) d_2^3 \right] \otimes a^x
\]

\[
= F^T \left[ (\partial_x d_1 \cdot d_1) d_1 + (\partial_x d_2 \cdot d_2) d_2 \right] \otimes a^x = F^T \left[ (d_1 \otimes d_1 + d_2 \otimes d_2) \partial_x d_3 \right] \otimes a^x = F^T (d_1 \otimes d_1) (\partial_x d_3 \otimes a^s) = F^T \text{Grad}_x d_3.
\]

Inserting these expressions for \(U_1\) in (29) we obtain that the representation (19) holds true. The proof is complete. \(\square\)
Remark 1. The values of the arguments of the function \( W \) in (19), calculated in the reference configuration \( S^0 \), are

\[
F^T F_{S^0} = a^T a = a, \quad F^T d_{3_{S^0}} = a^T d_0^0 = 0.
\]

Then, one can introduce the measures of deformation \( \mathcal{E}, \gamma, \Psi \) defined by

\[
\mathcal{E} = \frac{1}{2} (F^T F - a), \quad \gamma = F^T d_3, \quad \Psi = (F^T \text{Grad} d_3 + b) + \mathcal{E} b.
\]  

(30)

and the strain energy function (19) can be represented as

\[
W = \mathcal{W}(\mathcal{E}, \gamma, \Psi).
\]  

(31)

The tensor \( \mathcal{E} \) is a second order symmetric tensor accounting for extensional and in-plane shear strains, \( \gamma \) is the vector of transverse shear deformation, and \( \Psi \) is a second order tensor for the bending and twist strains. We designate by \( E_{ij}^e = a E_{ij}^e = E_{ij}^e d_0^0 \otimes a^x \) the “planar part” of \( E^e \) (in the tangent plane) and by \( E_{ij}^n = n^i E_{ij}^e = E_{ij}^e a^x \) the “normal part” of \( E^e \), and analogously \( K_{ij}^e = a K_{ij}^e = K_{ij}^e d_0^0 \otimes a^x \). Then, in view of (2), (3), the tensors (30) can be written in the alternative forms

\[
\mathcal{E} = \frac{1}{2} E^T E^e + \text{sym}(E_{ij}^e), \quad \gamma = n^i E_{ij}^e = E_{ij}^e^2,
\]

\[
\Psi = (E^{eT} + a) c K^e + \left[ \frac{1}{2} E^T E^e + \text{skew}(E_{ij}^e) \right] b.
\]  

(32)

Remark 2. Instead of the first integral \( U_3 \) introduced in the proof of the Theorem 1, one can consider alternatively the following first integral:

\[
U_3 = (E_{ij}^T + a)(K_{ij}^e + K_0^0).
\]  

(33)

Indeed \( U_3 \) is a first integral of the system (26) since we have

\[
\frac{dU_3}{ds} = \frac{d}{ds} \left[ (E_{ij}^e + a) a(b K^e + K_0^0) \right] = \left( \frac{dE_{ij}^e}{ds} \right)^T a (K_{ij}^e + K_0^0) + (E_{ij}^e + a) \frac{dK_{ij}^e}{ds}
\]

\[
= (E_{ij}^e + a)(-c) a (K_{ij}^e + K_0^0) + (E_{ij}^e + a) a (c K_{ij}^e + K_0^0) = 0.
\]

On the other hand, the function \( U_3 \) given by (33) can be expressed in terms of \( F \) and \( \text{Grad} d_3 \) as follows

\[
U_3 = (E_{ij}^e + a) \frac{T}{2} a (K_{ij}^e + K_0^0) = F^T Q^e a K
\]

\[
= F^T Q^e a \left[ \partial_{ij} d_3 \cdot d_3^0 + \left( \partial_{ij} d_3 \cdot d_1 d_2 \right) d_2^0 + \left( \partial_{ij} d_1 d_2 \cdot d_2^0 \right) d_2^0 \right] \otimes a^x
\]

\[
= F^T [\partial_{ij} d_3 d_3^0 + \left( \partial_{ij} d_1 d_2 \right) d_2^0 + \left( \partial_{ij} d_1 d_2 \right) d_2^0] \otimes a^x
\]

\[
= F^T [\left( \partial_{ij} d_3 d_3^0 + \left( \partial_{ij} d_1 d_2 \right) d_2^0 + \left( \partial_{ij} d_1 d_2 \right) d_2^0 \right) \otimes a^x = F^T (d_3 \otimes d_3 + d_1 \otimes d_2 + d_2 \otimes d_1) \otimes a^x
\]

\[
= F^T [d_3 \times d_3 \otimes d_3 \text{Grad} d_3 + F^T (d_3 \otimes d_3) \text{Grad} d_3] = F^T (d_3 \otimes d_3)\text{Grad} d_3 = F^T (d_3 \times \text{Grad} d_3).
\]

Thus, if we employ the first integrals \( U_1, U_2 \), and \( U_3 \) we obtain the following alternative representation of the strain energy function

\[
W = \mathcal{W}(F^T F, F^T d_3, F^T (d_3 \times \text{Grad} d_3)).
\]  

(34)

Like in Remark 1, one can introduce the measures of deformation \( \mathcal{E}, \gamma \) and \( \Phi \) defined by (30)1,2 and respectively

\[
\Phi = [F^T (d_3 \times \text{Grad} d_3) + n^0 \times b] + \mathcal{E} (n^0 \times b).
\]  

(35)

The tensor \( \Phi \) accounts for bending and twist strains and was previously introduced by Zhilin (2006). It can be rewritten in the form

\[
\Phi = (E^{eT} + a) K^e - \left[ \frac{1}{2} E^T E^e + \text{skew}(E_{ij}^e) \right] c^h.
\]  

(36)

With the help of \( \mathcal{E}, \gamma \) and \( \Phi \) one can give the following representation for the strain energy function

\[
W = \mathcal{W}^*(\mathcal{E}, \gamma, \Phi).
\]  

(37)

The form (37) was established previously in Zhilin (2006). The only difference to the representation (31) is the definition of the bending–twist tensor \( \Phi \) in (35), as compared to \( \Psi \) in (30).
By comparison of the relations (30) and (35) one can see that the definition (30) for the bending–twist tensor is more appropriate since the vector product in (35) introduces an additional (unnecessary) rotation of Grad\(d_j\) in the plane of \(\{d_1, d_2\}\).

**Remark 3.** Let us write the above representations in the linearized theory. For the linear theory we introduce the small (infinitesimal) displacement \(u = y - y^0\) and the vector of small rotations \(\psi\) such that (Chróścielewski et al., 2004; Zhilin, 2006)

\[
Q^f = \eta_3 + \psi \times \eta_3, \quad Q^{fT} = \eta_3 - \psi \times \eta_3.
\]

Then, we have

\[
F = \text{Grad}(u + a) = (\partial_x u + a_x) \otimes a^x,
\]

\[
\text{axl}(\partial_x Q^f Q^{fT}) = \partial_x \psi, \quad \text{axl}(Q^f Q^{fT}) = \psi
\]

and from (2), (3) we find, in the approximation of the linear theory,

\[
E^f = \text{Grad}(u - (\psi \times a) = (\partial_x u - \psi \times a_x) \otimes a^x,
\]

\[
K^f = \text{Grad}(\psi) = \partial_x \psi \otimes a^x.
\]

Using (32), (36) and (38) we find the expressions of \(E^f\), \(\gamma\) and \(\Psi\) in the linear theory

\[
E^f = \text{sym}([a \text{Grad}(u)], \quad \gamma = n^0 \text{Grad}(u + c \psi),
\]

\[
\Psi = c \Phi = c \text{Grad}(a \psi) + [\text{skew}(a \text{Grad}(u))] b.
\]

We note that the relation between the tensors \(\Psi\) and \(\Phi\) in the linear theory is very simple: \(\Psi = c \Phi\). If we decompose the vector of small rotations as \(\psi = \psi^a a^a\), then the drilling rotations are described by the component \(n^0 \cdot \psi = \psi_3\). We remark that

\[
c \psi = \frac{1}{\sqrt{a}} (a_1 \otimes a_2 - a_2 \otimes a_1) \psi = \frac{1}{\sqrt{a}} (\psi_2 a_1 - \psi_1 a_2),
\]

\[
\text{Grad}(a \psi) = \text{Grad}(\psi^a a^a) = \partial_x (\psi^a a^a) \otimes a^x
\]

and from (39) we see that the tensors \(E^f\), \(\gamma\) and \(\Psi\) are indeed independent of the drilling rotations \(\psi_3\).

In this case one gets the Reissner-type kinematics of shells (Wiśniewski, 2010; Neff et al., 2010) with 5 degrees of freedom.

### 4. The case of isotropic shells

The local symmetry group for 6-parameter elastic shells has been studied in Eremeyev and Pietraszkiewicz (2006). The expression of the strain energy for a physically linear model has the general form

\[
2W(E^f, K^f) = \alpha_1 \left( \text{tr}(E^f)^2 + \alpha_2 \text{tr}(E^f)^2 \right) + \alpha_3 \text{tr}(E^f)^2 + \alpha_4 (n^0 E^f)^2 + \beta_1 \text{tr}(K^f)^2 + \beta_2 \text{tr}(K^f)^2 + \beta_3 \text{tr}(K^f)^2 + \beta_4 (n^0 K^f)^2.
\]

The constitutive coefficients \(\alpha_1, \ldots, \alpha_4, \beta_1, \ldots, \beta_4\) can depend in general on the initial structure curvature tensor \(K^0\), but we assume for simplicity that they are constant. Provided that the coefficients \(\alpha_k\) and \(\beta_k\) satisfy the following inequalities

\[
2 \alpha_4 + 2 \alpha_2 + \alpha_3 > 0, \quad 2 \alpha_2 + \alpha_3 > 0, \quad \alpha_3 - \alpha_2 > 0, \quad \alpha_4 > 0,
\]

\[
2 \beta_4 + 2 \beta_2 + \beta_3 > 0, \quad 2 \beta_2 + \beta_3 > 0, \quad \beta_3 - \beta_2 > 0, \quad \beta_4 > 0,
\]

the energy function (40) is coercive in the sense that there exists a constant \(C > 0\) with

\[
W(E^f, K^f) \geq C \left( ||E^f||^2 + ||K^f||^2 \right).
\]

Under the conditions (41) we can prove the existence of minimizers for isotropic elastic shells. To this aim we apply the recent existence result in the theory of 6-parameter shells given by Theorem 1 in Birsan and Neff (2014), see also Neff (2004a).

**Remark 4.** One can find in the literature some simplified versions of the strain energy (40) for 6-parameter isotropic shells. For instance, in Chróścielewski et al. (2004), Chróścielewski, Pietraszkiewicz, and Witkowsi (2010) the following special form is employed

\[
2W(E^f, K^f) = C \left[ v \left( \text{tr}(E^f)^2 + (1 - v) \text{tr}(E^f)^2 \right) + \alpha_1 C(1 - v) n^0 E^f E^{fT} n^0 + D \left[ v \left( \text{tr}(K^f)^2 + (1 - v) \text{tr}(K^f)^2 \right) \right] \right]
\]

\[
+ \alpha_1 D(1 - v) n^0 K^f K^{fT} n^0,
\]

(42)
where \( C = \frac{E h}{1 - \nu^2} \) is the stretching (in-plane) stiffness of the shell, \( D = \frac{E h^3}{12(1 - \nu^2)} \) is the bending stiffness, \( h \) is the thickness of the shell, and \( \alpha_s, \alpha_t \) are two shear correction factors. Also, \( E \) and \( \nu \) denote the Young modulus and Poisson ratio of the isotropic and homogeneous material. By the numerical treatment of non-linear shell problems, the values of the shear correction factors have been set to \( \alpha_s = 5/6, \alpha_t = 7/10 \) in \( \text{Chrościelewski et al. (2010)} \). In this simpler case, the coefficients \( \alpha_t \) and \( \beta_t \) from (40) have the expressions

\[
\begin{align*}
\alpha_1 &= C v, & \alpha_2 &= 0, & \alpha_3 &= C(1 - v), & \alpha_4 &= \alpha_t C(1 - v), \\
\beta_1 &= D v, & \beta_2 &= 0, & \beta_3 &= D(1 - v), & \beta_4 &= \alpha_t D(1 - v).
\end{align*}
\]

We remark that the conditions (41) are satisfied for the values (43), by virtue of the well-known inequalities \( E > 0 \) and \( -1 < \nu < \frac{1}{2} \) (or equivalently, \( \mu > 0 \) and \( 2\mu + 3\lambda > 0 \), in terms of the Lamé moduli \( \lambda, \mu \)).

Let us consider next isotropic shells without drilling rotations. Zhilin (2006) proposed the following form of the strain energy \( W \) as a quadratic function of its arguments \( E, \gamma, \Phi \)

\[
2W = 2\tilde{W}(E, \gamma, \Phi) = C\Big[ (1 - v)\|E\|^2 + v(\text{tr} E)^2 \Big] + \frac{1}{2} C(1 - v)\gamma^2 + D \Big[ C(1 - v)(\text{tr} \Phi)^2 - v\text{tr}(\Phi^2) - \frac{1}{2} (1 - v)(\text{tr} \Phi)^2 \Big],
\]

where \( \gamma \) is a shear correction factor. The role of shear correction factors has been extensively discussed in the literature, see e.g. \( \text{Neff & Chełmiński, 2007} \). The expressions of the constitutive coefficients in (44) were suggested by Zhilin (2006) after the comparison between solutions of some shell problems in the linear theory and the corresponding three-dimensional solutions. Taking into account the relations \( \Phi = -c \Psi \) and

\[
\text{tr} \left[ (c \Psi)^2 \right] = \text{tr}(\Psi^T \Psi) - (\text{tr} \Psi)^2 = 2\|\text{dev}_2 \text{sym} \Psi\|^2 - \text{tr}(\Psi^2),
\]

then we deduce from (44) the expression of the strain energy \( W \) as function of \( E, \gamma, \Psi \):

\[
2W = 2\tilde{W}(E, \gamma, \Psi) = C\Big[ (1 - v)\|E\|^2 + v(\text{tr} E)^2 \Big] + \frac{1}{2} C(1 - v)\gamma^2 + D \Big[ C(1 - v)(\text{tr} \Psi)^2 + (1 - v)\|\text{dev}_2 \text{sym} \Psi\|^2 \Big],
\]

or equivalently,

\[
2W = 2\tilde{W}(E, \gamma, \Psi) = C\Big[ (1 - v)\|E\|^2 + v(\text{tr} E)^2 \Big] + \frac{1}{2} C(1 - v)\gamma^2 + D \Big[ (1 + v)(\text{tr} \Psi)^2 + (1 - v)\|\text{dev}_2 \text{sym} \Psi\|^2 \Big].
\]

In order to compare this energy with the energy (40) for 6-parameter isotropic shells, we insert the expressions (32) into (46) and we find

\[
2W(E^*, K^*) = C\Big[ \frac{1}{2} (1 + v)\left( \frac{1}{2} \|E^*\|^2 + \text{tr}(E^*^T E^*) \right)^2 + (1 - v)\left( \frac{1}{2} E^T E^* - \frac{1}{4} \|E^*\|^2 a + \|\text{dev}_2 \text{sym} (E^*)\|^2 \right) \Big] + \frac{1}{2} C(1 - v)\gamma^2 + D \Big[ \frac{1}{2} (1 + v)(\text{tr} \Psi)^2 + (1 - v)\|\text{dev}_2 \text{sym} \Psi\|^2 \Big].
\]

We observe that the energy (47) is super-quadratic as a function of the arguments \( (E^*, K^*) \). In the case of physically linear shells, when only the quadratic terms in \( (E^*, K^*) \) are taken into account, we obtain the simplified expression of the energy density (for the case when the constitutive coefficients are independent of \( K^0 \))

\[
2W(E^*, K^*) = C\Big[ v(\text{tr} E^*)^2 + \frac{1}{2} (1 - v)\|E^*\|^2 + \frac{1}{2} (1 - v) v\text{tr}(E^*^T E^*) \Big] + \frac{1}{2} C(1 - v)\|n^0 E^*\|^2 + D \Big[ \text{tr}(K^*^T K^*) - \frac{1}{2} (1 - v)(\text{tr} K^*)^2 \Big].
\]

By comparison of the relations (40) and (48) we see that these two expressions for the energy \( W \), coincide provided that the coefficients \( \alpha_t \) and \( \beta_t \) are

\[
\begin{align*}
\alpha_1 &= C v, & \alpha_2 &= \alpha_3 = \frac{1}{2} C(1 - v), & \alpha_4 &= \frac{1}{2} C(1 - v) \gamma, \\
\beta_1 &= -\frac{1}{2} D(1 - v), & \beta_2 &= -D v, & \beta_3 &= D, & \beta_4 &= 0.
\end{align*}
\]
or equivalently,
\[
x_1 = \frac{h \cdot 2\mu}{2\mu + \lambda}, \quad x_2 = x_3 = h \mu, \quad x_4 = h \mu \kappa, \\
\beta_1 = -\frac{h^3}{12} \mu, \quad \beta_2 = -\frac{h^3}{12} \cdot 2\mu + \lambda, \quad \beta_3 = \frac{h^3}{12} \cdot 4\mu(\mu + \lambda), \quad \beta_4 = 0.
\]

In view of the above arguments, we deduce that the constitutive coefficients \( x_k \) and \( \beta_k \) given by (49) (or equivalently (50)) are appropriate for isotropic 6-parameter physically linear shells insensitive to drilling rotations.

**Remark 5.** We see that the inequalities (41) are not satisfied for the set of constitutive coefficients (49). Indeed, we find
\[
2x_1 + x_2 + x_3 = h \frac{E}{1 - \nu} = h \frac{2\mu(2\mu + 3\lambda)}{2\mu + \lambda} > 0, \quad x_2 + x_3 = h \frac{E}{1 + \nu} = 2h \mu > 0,
\]
\[
\beta_2 + \beta_3 = \frac{h^3}{12} \frac{E}{1 + \nu} = \frac{h^3}{6} \mu > 0, \quad \beta_3 - \beta_2 = \frac{h^3}{12} \frac{E}{1 - \nu} = \frac{h^3}{6} \mu(2\mu + 3\lambda) > 0.
\]

but also
\[
x_3 - x_2 = 0, \quad 2\beta_1 + \beta_2 + \beta_3 = 0, \quad \beta_4 = 0.
\]

This means that the strain energy (48) for shells without drilling rotations is not positive definite, but only positive semi-definite. The existence theorem presented in Bîrsan and Neff (2014) does not apply here. The proof of the existence of minimizers is more difficult in this case, and this matter is left open. In the case when \( W \) is a super-quadratic function in the bending–curvature tensor \( K^r \), an existence theorem could be obtained using the same methods as in Neff (2007). Theorem 1 which asserts that
\[
W = W\left( F^r F^r, d_1, F^r \text{Grad} d_1 \right), \quad \text{or equivalently on } b, \quad a b K^r = c b.
\]

**Remark 6.** In this section we have considered for simplicity the case when the constitutive coefficients \( x_k \) and \( \beta_k \) are independent of the initial curvature tensor \( K^0 \) (or equivalently on \( b \), since \( a b K^0 = c b \)). However, a similar analysis can be performed also in the more complicated case when the constitutive coefficients of the strain energy function \( W \) depend on \( K^0 \).

### 5. Conclusions

In the framework of the resultant geometrically nonlinear 6-parameter theory of shells we have investigated shells for which the strain energy \( W \) is insensitive to drilling rotations. We have proved a representation theorem for this type of shells (Theorem 1) which asserts that \( W \) can be written as a function of these arguments: \( W(\{F^r F^r, d_1, F^r \text{Grad} d_1\}) \). Alternatively, in terms of the shell strain tensor \( E^r \) and bending–curvature tensor \( K^r \), the strain energy function \( W(E^r, K^r) \) depends on the following combinations (see (29))
\[
W = W(\{E^r + a^\top, E^r + a\}, n^\top E^r, (E^r + a)\top c (K^r + K^0)\}).
\]

We have compared these results with a related theory of directed surfaces (Zhiling, 2006) and we have indicated some advantages of our approach (see Remark 2).

In Section 4 we have considered isotropic shells and proposed explicit expressions for the constitutive coefficients of 6-parameter shells insensitive to drilling rotations (relations (49) and (50)). In this case, we observe that the strain energy function \( W(E^r, K^r) \) is only positive semi-definite (cf. Remark 5). This situation is similar to the Cosserat plate model established in Neff (2004a, 2007) in the case of zero Cosserat couple modulus \( (\mu_c = 0) \).

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