Sparse universal graphs for planarity

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Abstract
We show that for every integer $n \geq 1$, there exists a graph $G_n$ with $(1 + o(1))n$ vertices and $n^{1+o(1)}$ edges such that every $n$-vertex planar graph is isomorphic to a subgraph of $G_n$. The best previous bound on the number of edges was $O(n^{3/2})$, proved by Babai, Chung, Erdős, Graham, and Spencer in 1982. We then show that for every integer $n \geq 1$, there is a graph $U_n$ with $n^{1+o(1)}$ vertices and edges that contains induced copies of every $n$-vertex planar graph. This significantly reduces the number of edges in a recent construction of the authors with Dujmović, Gavoille, and Micek.

MSC 2020
05C07, 05C70, 05D40 (primary)

1 | INTRODUCTION

Given a family $F$ of graphs, a graph $G$ is universal for $F$ if every graph in $F$ is isomorphic to a (not necessarily induced) subgraph of $G$. The topic of this paper is the following question: What is the minimum number of edges in a universal graph for the family of $n$-vertex planar graphs? Besides being a natural question, we note that finding sparse universal graphs is also motivated by applications in VLSI design [5, 19] and simulation of parallel computer architecture [3, 4].

A moment’s thought shows that $\Omega(n \log n)$ edges are needed: for $t = 1, \ldots, n$, consider the forest consisting of $t$ copies of the star $K_{1, \lfloor n/t \rfloor - 1}$. A universal graph for the class of $n$-vertex planar graphs must contain all these forests as subgraphs, and so, it must have a degree sequence which, once sorted in non-increasing order, dominates the sequence

$$(n - 1, \lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{3} \rfloor - 1, \lfloor \frac{n}{4} \rfloor - 1, \ldots),$$

hence the lower bound. As far as we are aware, no better lower bound is known for $n$-vertex planar graphs.
For $n$-vertex trees, a matching upper bound of $O(n \log n)$ on the number of edges in the universal graph is known [9]. For $n$-vertex planar graphs of bounded maximum degree, Capalbo constructed a universal graph with $O(n)$ edges [7]. However, for general $n$-vertex planar graphs, only a $O(n^{3/2})$ bound is known, proved by Babai, Chung, Erdős, Graham, and Spencer [2] in 1982 using the existence of separators of size $O(\sqrt{n})$.

In this paper, we show that universal graphs with a near-linear number of edges can be constructed.

**Theorem 1.** The family of $n$-vertex planar graphs has a universal graph with $(1 + o(1))n$ vertices and at most $n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}$ edges.

As the original construction of Babai et al. [2] only uses the existence of separators of size $O(\sqrt{n})$, it was later shown to apply to more general classes than planar graphs, for instance, to any proper minor-closed class [8]. Our result also holds in greater generality, but not quite as general as the construction of Babai et al. [2], as we now explain.

The **strong product** $A \Box B$ of two graphs $A$ and $B$ is the graph whose vertex set is the Cartesian product $V(A \Box B) := V(A) \times V(B)$ and in which two distinct vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if:

1. $x_1x_2 \in E(A)$ and $y_1y_2 \in E(B)$; or
2. $x_1 = x_2$ and $y_1y_2 \in E(B)$; or
3. $x_1x_2 \in E(A)$ and $y_1 = y_2$.

We may now state the main result of this paper.

**Theorem 2.** Fix a positive integer $t$ and let $Q_t$ denote the family of all graphs of the form $H \Box P$ where $H$ is a graph of treewidth $t$ and $P$ is a path, together with all their subgraphs. Then the family of $n$-vertex graphs in $Q_t$ has a universal graph with $(1 + o(1))n$ vertices and at most $t^2 \cdot n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}$ edges.

It was proved by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [13] that every planar graph is a subgraph of the strong product of a graph of treewidth at most 8 and a path (see also the recent improvement by Ueckerdt, Wood, and Yi [18]).

**Theorem 3** [13]. The class of planar graphs is a subset of $Q_8$.

Moreover, Bose, Morin, and Odak [6] gave a linear-time algorithm that, given a planar graph $G$, finds a graph $H$ of treewidth at most 8 and an embedding of $G$ in the strong product of $H$ with a path.

Note that Theorems 3 and 2 directly imply Theorem 1. It was proved that Theorem 3 can be generalized (replacing 8 with a larger constant) to bounded genus graphs, and more generally to apex-minor free graphs [13], as well as to $k$-planar graphs and related classes of graphs [10]. Thus, it follows that families of $n$-vertex graphs in these more general classes also admit universal graphs with $n^{1+o(1)}$ edges.

**Induced-universal graphs.** A related problem is to find an *induced-universal graph* for a family $F$, which is a graph that contains all the graphs of $F$ as *induced* subgraphs. In this context, the problem is usually to minimize the number of vertices of the induced-universal graph [15].
Recently, Dujmović, Esperet, Joret, Gavoille, Micek, and Morin used Theorem 3 to construct an induced-universal graph with \( n^{1+o(1)} \) vertices for the class of \( n \)-vertex planar graphs \([11, 12]\). Since an induced-universal graph for a class \( F \) is also universal for \( F \), their graph is universal for the class of \( n \)-vertex planar graphs. However, while that graph has a near-linear number of vertices, it is quite dense and it has order of \( n^2 \) edges. Thus, it is not directly useful in the context of minimizing the number of edges.

Nevertheless, in this paper, we reuse key ideas and techniques introduced in \([12]\). Very informally, a central idea in \([12]\) is the notion of *bulk tree sequences*, which is used to efficiently “encode” the rows from the product structure using almost perfectly balanced binary search trees, in such a way that the trees undergo minimal changes when moving from one row to the next one. (These tree sequences are described in the next section.)

Given that, for \( n \)-vertex planar graphs, there exist (1) a universal graph with a near-linear number of edges, and (2) an induced-universal graph with a near-linear number of vertices, it is natural to wonder if these two properties can be achieved simultaneously. In the second part of this paper, we show that this can be done.

**Theorem 4.** The family of \( n \)-vertex planar graphs has an induced-universal graph with at most \( n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})} \) edges and vertices.

In the same way that Theorem 1 is a special case of Theorem 2, Theorem 4 is obtained as a special case of Theorem 5:

**Theorem 5.** Fix a positive integer \( t \). Then the family of \( n \)-vertex graphs in \( Q_t \) has an induced-universal graph with at most \( n \cdot 2^{O(\sqrt{\log n \cdot \log \log n}) \cdot (\log n)^{O(t^2)}} \) edges and vertices.

The construction in Theorem 5 is based on a nontrivial modification of the construction of induced-universal graphs in \([12]\) and reuses ideas from the construction of universal graphs in the first part of the current paper. It is significantly more complicated than the construction used for Theorem 1, is more tightly coupled with the labeling scheme in \([12]\), and the end result has a greater dependence on \( t \) (a \( t^2 \) factor in Theorem 2 is replaced by a \((\log n)^{O(t^2)} \) factor in Theorem 5). Moreover, the classical techniques that allow us to reduce the number of vertices from \( n^{1+o(1)} \) to \((1 + o(1))n \) in Theorem 2 do not apply to induced-universal graphs, so decreasing further the number of vertices in Theorem 5 seems to require completely new ideas.

**Paper organization.** The first part of the paper consists of Sections 2 and 3, and is devoted to proving Theorem 2. In the second part of the paper, Section 3, we start by recalling the construction of induced-universal graphs from \([12]\). Then, we explain why these graphs are too dense, and describe how to modify the construction to achieve a near-linear number of edges.

## 2 | PRELIMINARIES

### 2.1 | Graph products

Given two graphs \( G_1, G_2 \), and \( v_1 \in V(G_1) \), the set \( \{(v_1, v_2) \mid v_2 \in V(G_2)\} \) is called a **column** of \( G_1 \boxtimes G_2 \). Similarly, for \( v_2 \in V(G_2) \), the set \( \{(v_1, v_2) \mid v_1 \in V(G_1)\} \) is called a **row** of \( G_1 \boxtimes G_2 \).
Lemma 6. Let $G_1$ and $G_2$ be two graphs, and let $H$ be an $n$-vertex subgraph of $G_1 \Box G_2$. Then $G_1$ and $G_2$ contain induced subgraphs $G_1'$ and $G_2'$ with at most $n$ vertices such that $H$ is a subgraph of $G_1' \Box G_2'$, and each row and column of $G_1' \Box G_2'$ contains at least one vertex of $H$.

Proof. If there is a vertex $x \in V(G_1)$ such that no vertex of $G_1 \Box G_2$ of the form $(x, y)$ is included in $H$, then $H$ is a subgraph of $(G_1 - x) \Box G_2$. So, by considering induced subgraphs $G_1'$ and $G_2'$ of $G_1$ and $G_2$ if necessary, we may assume that each column (and by symmetry each row) of $G_1' \Box G_2'$ contains a vertex of $H$. It follows that $G_1'$ and $G_2'$ contain at most $n$ vertices. □

We deduce the following result, which will be useful in the proof of our main result.

Lemma 7. Let $n$ be an integer, let $H_1$ be a graph with at least $n$ vertices, and let $G_1$ be a graph that is universal for the family of $n$-vertex subgraphs of $H_1$. Then for any graph $H_2$, the graph $G_1 \Box H_2$ is universal for the family of $n$-vertex subgraphs of $H_1 \Box H_2$.

Proof. Let $G$ be an $n$-vertex subgraph of $H_1 \Box H_2$. By Lemma 6, we can assume that there is a subgraph $H_1'$ of $H_1$ with at most $n$ vertices, such that $G$ is a subgraph of $H_1' \Box H_2$. By adding vertices of $H_1$ to $H_1'$ if necessary, we can assume that $H_1'$ contains precisely $n$ vertices, and is thus a subgraph of $G_1$. It follows that $G$ is a subgraph of $G_1 \Box H_2$, as desired. □

2.2 Binary search trees

A binary tree $T$ is a rooted tree in which each node except the root is either the left or right child of its parent and each node has at most one left and at most one right child. For any node $x$ in $T$, $P_T(x)$ denotes the path from the root of $T$ to $x$. The length of a path $P$ is the number of edges in $P$, that is, $|P| - 1$. The depth, $d_T(x)$, of $x$ is the length of $P_T(x)$. The height of $T$ is $\text{height}(T) := \max_{x \in V(T)} d_T(x)$. A node $x$ in $T$ is a $T$-ancestor of a node $y$ in $T$ if $x \in V(P_T(y))$. If $x$ is a $T$-ancestor of $y$, then $y$ is a $T$-descendant of $x$. A $T$-ancestor $x$ of $y$ is a strict $T$-ancestor if $x \neq y$. We use $\prec_T$ to denote the strict $T$-ancestor relation and $\preceq_T$ to denote the $T$-ancestor relation. Let $P_T(x_r) = x_0, \ldots, x_r$ be a path from the root $x_0$ of $T$ to some node $x_r$ (possibly $r = 0$). Then the signature of $x_r$ in $T$, denoted as $\sigma_T(x_r)$, is a binary string $b_1, \ldots, b_r$ where $b_i = 1$ if and only if $x_i$ is the left child of $x_{i-1}$. Note that the signature of the root of $T$ is the empty string.

A binary search tree $T$ is a binary tree whose node set $V(T)$ consists of distinct real numbers and that has the binary search tree property: For each node $x$ in $T$, $z < x$ for each node $z$ in $x$’s left subtree and $z > x$ for each node $z$ in $x$’s right subtree.

Let $\log x := \log_2 x$ denote the binary logarithm of $x$. We will use the following standard facts about binary search trees, which were also used in [12].

Lemma 8 Lemma 5 in [12]. For any finite $S \subset \mathbb{R}$ and any function $w : S \rightarrow \mathbb{R}^+$, there exists a binary search tree $T$ with $V(T) = S$ such that, for each $y \in S$, $d_T(y) \leq \log(W/w(y))$, where $W := \sum_{y \in S} w(y)$.

Observation 9 Observation 6 in [12]. Let $T$ be a binary search tree and let $x, y$ be nodes in $T$ such that $x < y$ and there is no node $z$ in $T$ such that $x < z < y$, that is, $x$ and $y$ are consecutive in the sorted order of $V(T)$. Then
(1) (if \( x \) has no right child) \( \sigma_T(y) \) is obtained from \( \sigma_T(x) \) by removing all trailing 1’s and the last 0; or
(2) (if \( x \) has a right child) \( \sigma_T(y) \) is obtained from \( \sigma_T(x) \) by appending a 1 followed by \( d_T(y) - d_T(x) - 1 \) 0’s.

Therefore, for each \( \sigma \in \{0,1\}^* \) and integer \( h \) such that \( |\sigma| \leq h \), there exists a set \( L(\sigma, h) \) of bit-strings in \( \{0,1\}^* \), each of length at most \( h \), with \( |L(\sigma, h)| \leq h + 1 \) such that for every binary search tree \( T \) of height at most \( h \) and for every two consecutive nodes \( x, y \) in the sorted order of \( V(T) \), we have \( \sigma_T(y) \in L(\sigma_T(x), h) \).

The following lemma from [12] is a key tool in our proof.

**Lemma 10** Lemmas 8, 25, and 27 in [12]. Let \( n \) be a positive integer and define \( k = \max(5, \lceil \sqrt{\log n / \log \log n} \rceil) \). Then there exists a function \( B : (\{0,1\}^*)^2 \to \{0,1\}^* \) such that for any finite sets \( S_1, \ldots, S_h \subset \mathbb{R} \) with \( \sum_{y=1}^{h} |S_y| = n \), there exist binary search trees \( T_1, \ldots, T_h \) such that

1. for each \( y \in \{1, \ldots, h-1\} \), \( V(T_y) \supseteq S_y \cup S_{y+1} \), and \( V(T_h) \supseteq S_h \);
2. \( \sum_{y=1}^{h} \left| V(T_y) \right| \leq 4 \sum_{y=1}^{h} |S_y| = 4n \); [12, Lemma 8]
3. for each \( y \in \{1, \ldots, h\} \), \( \text{height}(T_y) \leq \log \left| V(T_y) \right| + O(k + k^{-1} \log |V(T_y)|) \). [12, Lemma 25]
4. for each \( y \in \{1, \ldots, h-1\} \), and each \( z \in V(T_y) \cap V(T_{y+1}) \), there exists \( \nu_y(z) \in \{0,1\}^* \) with \( \left| \nu_y(z) \right| = O(k \log(\text{height}(T_y))) \) such that \( B(\sigma_{T_y}(z), \nu_y(z)) = \sigma_{T_{y+1}}(z) \). [12, Lemma 27]

The sequence \( T_1, \ldots, T_h \) obtained in the lemma is called a **bulk tree sequence** in [12] and plays a fundamental role in [12] and the present paper.

**Observation 11.** There exists a function \( \lambda : \mathbb{N} \to \mathbb{N} \) with \( \lambda(n) \in O(\sqrt{\log n \log \log n}) \) such that

- \( \text{height}(T_y) \leq \log \left| V(T_y) \right| + \lambda(n) \) always holds in property (3) of Lemma 10, and
- \( \left| \nu_y(z) \right| \leq \lambda(n) \) always holds in property (4) of Lemma 10.

**Proof.** This follows from the bounds \( \text{height}(T_y) \leq \log \left| V(T_y) \right| + O(k + k^{-1} \log |V(T_y)|) \) in property (3) of Lemma 10 and \( \left| \nu_y(z) \right| = O(k \log(\text{height}(T_y))) \) in property (4) of Lemma 10, combined with properties (2) and (3) of that lemma.

It is important to note that the function \( L \) of Observation 9 and the function \( B \) of Lemma 10 are **explicit**, in the sense that [12] provides simple deterministic algorithms for producing the output of the functions (note that this is clear for Observation 9 by considering (1) and (2) in the statement of the observation).

## 2.3 Universal graphs for interval graphs

An **interval graph** is a graph \( G \) that admits an **interval representation**, defined as a collection \( (I_v)_{v \in V(G)} \) of closed intervals of the real line such that, for distinct vertices \( u, v \in V(G), uv \in E(G) \) if and only if \( I_u \cap I_v \neq \emptyset \).
Lemma 12. Let $G$ be an $n$-vertex interval graph with clique number at most $\omega$. Then $V(G)$ can be partitioned into three sets $X_1, X_2, Z$ such that $|Z| \leq \omega$, $|X_i| \leq \frac{1}{2} n$ for $i \in \{1, 2\}$, and there are no edges between $X_1$ and $X_2$.

Proof. Consider an interval representation $(I_v)_{v \in V(G)}$ of $G$, where $I_v = [a_v, b_v]$ for any $v \in V(G)$, and such that at most one interval $I_v$ starts at each point (it is well known that such a representation exists). Order the vertices of $G$ as $v_1, \ldots, v_n$ such that for any $1 \leq i \leq j \leq n$, $a_{v_i} \leq a_{v_j}$. For each $1 \leq i \leq n$, let $Z_i$ be the set of vertices $v$ of $G$ such that $I_v$ contains $a_{v_i}$. Since $G$ has clique size at most $\omega$, each set $Z_i$ contains at most $\omega$ vertices (and at least one vertex, namely $v_i$). Moreover, the vertex set of each $G - Z_i$ can be partitioned into two sets $A_i$ (the vertices $v$ such that $b_v < a_{v_i}$) and $B_i$ (the vertices $v$ such that $a_v > a_{v_i}$) with no edges between them. Recall that at most one interval starts at each $a_{v_i}$, so $|B_i| = n - i$ for any $1 \leq i \leq n$. So, there is $1 \leq i \leq n$ such that $n/2 - 1 \leq |B_i| \leq n/2$. It follows that $|A_i| \leq n/2 + 1 - |Z_i| \leq n/2$, as desired. □

For any integer $d \geq 0$, let $B_d$ be the unique binary search tree with $V(B_d) = \{1, \ldots, 2^{d+1} - 1\}$ and having height $d$. The closure $C_d$ of $B_d$ is the graph with vertex set $V(C_d) := V(B_d)$ and edge set $E(C_d) := \{vw : v <_{B_d} w\}$ (see Figure 1). The universal graph for the family of $n$-vertex planar graphs of Babai et al. [2], with $O(n^{3/2})$ edges, is precisely $C_{\lceil \log n \rceil} \boxtimes K_t$, with $t = O(\sqrt{n})$. Using the same idea, we now describe a universal graph for the family of $n$-vertex interval graphs of bounded clique number.

Lemma 13. For every positive integers $n \geq 1$ and $\omega \geq 1$, the graph $C_{\lceil \log n \rceil} \boxtimes K_\omega$ is universal for the class of $n$-vertex interval graphs with clique number at most $\omega$.

Proof. We prove the result by induction on $n$. If $n = 1$, then the result clearly holds, so we can assume that $n \geq 2$. Consider an $n$-vertex interval graph $G$ with clique number at most $\omega$. By Lemma 12, the vertex set of $G$ has a partition into three sets $X_1, X_2, Z$ such that $|Z| \leq \omega$, $|X_i| \leq \frac{1}{2} n$ for $i \in \{1, 2\}$, and there are no edges between $X_1$ and $X_2$. By the induction hypothesis, $G[X_1]$ is a subgraph of $C_{\lceil \log (n/2) \rceil} \boxtimes K_\omega = C_{\lceil \log n \rceil - 1} \boxtimes K_\omega$, and similarly, $G[X_2]$ is a subgraph of $C_{\lceil \log n \rceil - 1} \boxtimes K_\omega$. Note that for $n \geq 2$, $C_{\lceil \log n \rceil} \boxtimes K_\omega$ can be obtained from two disjoint copies of $C_{\lceil \log n \rceil - 1} \boxtimes K_\omega$ by adding $\omega$ universal vertices. Using that $|Z| \leq \omega$, this implies that $G$ is a subgraph of $C_{\lceil \log n \rceil} \boxtimes K_\omega$, as desired. □

Note that the proof of Lemma 13 is constructive: given any interval representation of an $n$-vertex interval graph $G$ with clique number at most $\omega$, it gives an efficient deterministic algorithm to find a copy of $G$ in $C_{\lceil \log n \rceil} \boxtimes K_\omega$. 
For a node \( v \in C_d \), define the interval \( I_{C_d}(v) := \{ w \in V(B_d) : v \preceq_{B_d} w \} \). We observe that any two intervals are either nested or disjoint.

**Observation 14.** For any two nodes \( v, w \) of \( C_d \),
\[
I_{C_d}(v) \supseteq I_{C_d}(w), \quad I_{C_d}(v) \subseteq I_{C_d}(w) \quad \text{or} \quad I_{C_d}(v) \cap I_{C_d}(w) = \emptyset \text{ and, in the first two cases, } vw \in E(C_d).
\]

Let \( G \) be an induced subgraph of \( C_d \) and let \( T \) be a binary search tree with \( V(G) \subseteq V(T) \subseteq V(C_d) \). (Let us remark that, while the node set of \( T \) is a subset of that of \( B_d \), the structure of \( T \) could potentially be very different from that of \( B_d \).) For each \( v \in V(G) \), let \( x_T(v) \) denote the node \( x \in V(T) \) of minimum \( T \)-depth such that \( x \in I_{C_d}(v) \).

For two strings \( x \) and \( y \), we use \( x \preceq y \) to denote that \( x \) is a prefix of \( y \), and \( x < y \) to denote that \( x \leq y \) and \( |x| < |y| \). We use \( x \triangleright y \) to denote that \( x \leq y \) or \( y \preceq x \) (note that the relation \( \triangleright \) is reflexive and symmetric but not transitive).

**Lemma 15.** Let \( G \) be an induced subgraph of \( C_d \) and let \( T \) be a binary search tree with \( V(G) \subseteq V(T) \subseteq V(C_d) \). Let \( vw \in E(G) \). Then \( x_T(v) \preceq_T x_T(w) \) or \( x_T(w) \preceq_T x_T(v) \), and hence, \( \sigma_T(x_T(v)) \triangleright \sigma_T(x_T(w)) \).

**Proof.** Note that \( vw \in E(G) \) implies that \( I_{C_d}(v) \subseteq I_{C_d}(w) \) or \( I_{C_d}(v) \supseteq I_{C_d}(w) \), say without loss of generality \( I_{C_d}(v) \subseteq I_{C_d}(w) \). Suppose that neither \( x_T(v) \preceq_T x_T(w) \) nor \( x_T(w) \preceq_T x_T(v) \) holds. Then there exists a common \( T \)-ancestor \( z \in V(T) \) of \( x_T(v) \) and \( x_T(w) \) with \( z \neq x_T(v), x_T(w) \). Since \( T \) is a binary search tree, it follows that \( x_T(v) < z < x_T(w) \) or \( x_T(w) < z < x_T(v) \). Since \( x_T(v) \in I_{C_d}(v) \subseteq I_{C_d}(w) \) and \( x_T(w) \in I_{C_d}(w) \), we also have \( z \in I_{C_d}(w) \), by definition of \( I_{C_d}(w) \). Hence, \( z \in I_{C_d}(w) \) and \( z \) is a strict \( T \)-ancestor of \( x_T(w) \), which contradicts the choice of \( x_T(w) \). \( \square \)

### 2.4 Treewidth and pathwidth

A tree-decomposition of a graph \( G \) is a tree \( T \) along with a collection of subsets \((X_t)_{t \in V(T)}\) of vertices of \( G \) (called the bags of the decomposition) such that for every edge \( uv \in E(G) \), there is a node \( t \in V(T) \) such that \( u, v \in X_t \), and for every vertex \( u \in V(G) \), the nodes \( t \in T \) such that \( u \in X_t \) form a (non-empty) subtree of \( T \). The tree decomposition is called a path decomposition if the tree \( T \) is a path. The width of a tree decomposition is the maximum size of a bag, minus 1. The treewidth of a graph \( G \) is the minimum width of a tree decomposition of \( G \), and the pathwidth of a graph \( G \) is the minimum width of a path decomposition of \( G \). Note that the treewidth of a graph \( G \) is at most the pathwidth of \( G \). We will use the following partial converse.

**Lemma 16** [17]. Every \( n \)-vertex graph of treewidth at most \( t \) has pathwidth at most \( (t + 1) \lfloor \log_3(2n + 1) \rfloor - 1 \).

Observe that an equivalent definition of pathwidth, which will be used in the proofs, is the following: A graph \( G \) has pathwidth at most \( k \) if and only if \( G \) is a spanning\(^\dagger\) subgraph of an interval graph with clique number at most \( k + 1 \).

\(^\dagger\) A subgraph \( G \) of a graph \( H \) is spanning if \( V(G) = V(H) \).
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In this section, we establish the following technical theorem.

**Theorem 17.** For every positive integer $n$, the family of $n$-vertex induced subgraphs of $C_{\lceil \log n \rceil} \boxtimes P_n$ has a universal graph $G_n$ with

$$|V(G_n)| \leq n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})} \quad \text{and} \quad |E(G_n)| \leq n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}.$$

Before proving Theorem 17, let us explain why it implies our main theorem, Theorem 2. The proof proceeds in two steps: we first show that for $\omega \approx t \log n$, $G_n \boxtimes K_\omega$ is a universal graph for the $n$-vertex graphs of $\mathcal{Q}_t$. This graph has the desired number of edges, but a fairly large number of vertices. The second step of the proof consists in reducing the number of vertices to $(1 + o(1))n$.

We say that a subset $X$ of vertices of a graph $G$ is saturated by a matching $M$ of $G$ if every vertex of $X$ is contained in some edge of $M$. We will need the following result proved (in a slightly different form) in [1]. As the proof there is only alluded to, we give the complete details in the Appendix.

**Lemma 18.** For any sufficiently large integer $n$, any integer $k \geq 1$, any real number $0 < \epsilon \leq 1$, and any integer $N_0 \geq k(1 + \epsilon)n$, there is a bipartite graph $H$ with bipartition $(V, U)$ such that the following holds.

- $|V| = N$, and $|U| = N/k$, where $N$ is divisible by $k$ and $N_0 \leq N \leq N_0 + k$,
- each vertex of $V$ has degree $O(1/\epsilon \log k)$, and
- each $n$-vertex subset of $V$ is saturated by a matching of $H$.

We are now ready to prove Theorem 2.

**Proof of Theorem 2 assuming Theorem 17.** Let $G_n$ be the universal graph for the family of $n$-vertex subgraphs of $C_{\lceil \log n \rceil} \boxtimes P_n$ given by Theorem 17. Let $t$ be an integer and let $\omega = (t + 1)\lceil \log_3(2n + 1) + 1 \rceil$. By Lemma 7, $G'_n = G_n \boxtimes K_\omega$ is universal for the class of $n$-vertex subgraphs of $C_{\lceil \log n \rceil} \boxtimes P_n \boxtimes K_\omega$. Note that $G'_n$ has precisely $\omega |V(G_n)| = \omega \cdot n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}$ vertices and

$$|E(G_n)| \cdot \omega^2 + |V(G_n)| \cdot \left(\frac{\omega}{2}\right) \leq \omega^2 \cdot n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})} \leq t^2 \cdot n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}$$

edges. We will see shortly how to reduce the number of vertices from $\omega n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}$ to $(1 + o(1))n$, but for now we prove that $G'_n$ is universal for the $n$-vertex graphs of $Q_t$. For this, it suffices to show that any $n$-vertex graph $G \in Q_t$ is a subgraph of $C_{\lceil \log n \rceil} \boxtimes P_n \boxtimes K_\omega$.

We consider an $n$-vertex graph $G \in Q_t$. By the definition of $Q_t$ and Lemma 6, there exists a graph $H$ with treewidth at most $t$ and at most $n$ vertices such that $G$ is a subgraph of $H \boxtimes P_n$. By Lemma 16, $H$ has pathwidth at most $(t + 1)\lceil \log_3(2n + 1) + 1 \rceil - 1 = \omega - 1$. Hence, there exists an interval graph $I$ with clique number at most $\omega$ containing $H$ as a spanning subgraph. In particular, $I$ has at most $n$ vertices. By Lemma 13, $I$ (and thus $H$) is a subgraph of $C_{\lceil \log n \rceil} \boxtimes K_\omega$. It follows that $G$ is a subgraph of $C_{\lceil \log n \rceil} \boxtimes P_n \boxtimes K_\omega$. This proves that $G'_n$ is indeed universal for the class of $n$-vertex graphs of $Q_t$, as desired.
The final step consists in reducing the number of vertices in our universal graph from \(n^{1+o(1)}\) to \((1 + o(1))n\). We consider our universal graph \(G_n' = G_n \boxtimes K_\omega\) for the family of \(n\)-vertex planar graphs, with \(N_0 = n^{1+o(1)}\) vertices and \(n^{1+o(1)}\) edges. Take \(\epsilon = \log^{-1} n\), and let \(k = N_0/(1 + \epsilon)n = n^{o(1)}\). By Lemma 18, there exist \(d = O(\frac{1}{\log n}) = O(\log^2 n)\) and a bipartite graph \(H\) with partite sets \(V \supseteq V(G_n')\) and \(U\), with \(|V| = N \leq N_0 + k\) and \(|U| = N/k \leq (1 + \epsilon)n + 1\), such that the vertices of \(V\) have degree at most \(d\) in \(H\) and every \(n\)-vertex subset of \(V\) is saturated by a matching in \(H\).

We define a graph \(H_n\) from \(G_n'\) and \(H\) as follows: the vertex set of \(H_n\) is \(U \subseteq V(H)\), and two vertices \(u, u'\) are adjacent in \(H_n\) if there are \(v, v' \in V = V(G_n')\) such that \(vv' \in E(G_n')\), \(vu \in E(H)\), and \(v'u' \in E(H)\). Note that \(H_n\) has at most \(d^2 |E(G_n')| = O((\log n)^4)\cdot n^{1+o(1)} = n^{1+o(1)}\) edges and \(|U| \leq (1 + \epsilon)n + 1 = n + O(n/\log n) = (1 + o(1))n\) vertices.

It remains to prove that \(H_n\) contains all \(n\)-vertex graphs of \(\mathsf{\mathcal{G}}_t\) as subgraphs. Take an \(n\)-vertex graph \(F \in \mathsf{\mathcal{G}}_t\). Then \(F\) is a subgraph of \(G_n'\), so there is a set \(X\) of \(n\) vertices of \(G_n'\) such that \(F\) is a subgraph of \(G_n'[X]\). By Lemma 18, there is a matching between \(X\) and \(N_H(X)\) in \(H\) that saturates \(X\). The intersection of this matching with \(U\) consists of a set \(Y\) of \(n\) vertices, and it follows from the definition of \(H_n\) that \(F\) is a subgraph of \(H_n[Y]\), as desired. \(\Box\)

In the remainder of Section 3, we prove Theorem 17.

### 3.1 Definition of the universal graphs

Let \(n\) be a positive integer. We define a graph \(G_n\) that will be universal for \(n\)-vertex subgraphs of \(C_{\lceil \log n \rceil} \boxtimes P_n\). For convenience, let \(d := \lceil \log n \rceil\). Let \(k := \max(5, \lceil \sqrt{\log n/\log \log n} \rceil)\), as in Lemma 10. With a slight abuse of notation, let \(\lambda := \lambda(n)\), where \(\lambda(n)\) is the function from Observation 11.

The vertices of the graph \(G_n\) are all the triples \((x, y, z)\) where \(x, y \in \{0, 1\}^*\) are bitstrings such that \(|x| + |y| \leq d + \lambda + 2\) and \(z\) is an integer with \(z \in \{0, \ldots, d\}\). When defining the edge set of \(G_n\), it will be convenient to orient the edges to simplify the discussions later on, the graph \(G_n\) itself is, of course, undirected. Given two distinct vertices \((x_1, y_1, z_1), (x_2, y_2, z_2)\), we put a directed edge from \((x_1, y_1, z_1)\) to \((x_2, y_2, z_2)\) if one of the following conditions is satisfied:

1. \(y_1 = y_2\) and \(x_2 \leq x_1\),
2. \(y_1 \neq y_2\), and
   a. \(y_2 \in L(y_1, d + 2)\), where \(L\) is defined in Observation 9,
   b. there exists \(x'_2 \in \{0, 1\}^*\) with \(|x'_2| \leq d + \lambda + 2 - |y_1|\) such that \(x_1 \text{ubit} x'_2\), and
   c. there exists \(\nu \in \{0, 1\}^*\) with \(|\nu| \leq \lambda\) such that \(B(x'_2, \nu) = x_2\), where \(B\) is the function from Lemma 10.

Observe that the third coordinate of the triples is not used when defining adjacencies in \(G_n\). It will be used when proving the universality of \(G_n\). Note also that the definition of our universal graph is explicit, as the functions \(L\) and \(B\) are explicit themselves (see the discussion at the end of Section 2.2).

We start by bounding the number of vertices and edges in \(G_n\).

**Lemma 19.** The following bounds hold:

- \(|V(G_n)| \leq 2^{d+\lambda+3} \cdot (d + \lambda + 3)^2 \leq n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}\), and
- \(|E(G_n)| \leq 2^{d+2\lambda+4} \cdot (d + \lambda + 3)^6 \leq n \cdot 2^{O(\sqrt{\log n \cdot \log \log n})}\).
Proof. For each $0 \leq r \leq d + \lambda + 2$, there are $(r + 1)2^r$ pairs $(x, y)$ with $x, y \in \{0, 1\}^*$ such that $|x| + |y| = r$. It follows that for each $z \in \{0, \ldots, d\}$, $G_n$ contains at most

$$\sum_{r=0}^{d+\lambda+2} (r + 1) \cdot 2^r \leq (d + \lambda + 3)2^{d+\lambda+3}$$

vertices of the form $(x, y, z)$. It follows that $|V(G_n)| \leq 2^{d+\lambda+3} \cdot (d + \lambda + 3)(d + 1) \leq 2^{d+\lambda+3} \cdot (d + \lambda + 3)^2$.

In order to bound $|E(G_n)|$, we will bound the number of outgoing edges from a given vertex $(x_1, y_1, z_1)$ of $G_n$.

The number of choices for $(x_2, z_2)$ that result in an edge of Type (1) is at most $(|x_1| + 1) \cdot (d + 1) \leq (d + \lambda + 3) \cdot (d + 1)$. It follows that the number of edges of Type (1) is at most

$$|V(G_n)| \cdot (d + \lambda + 3)(d + 1) \leq 2^{d+\lambda+3} \cdot (d + \lambda + 3)^4.$$ 

To count outgoing edges of Type (2), we again fix $(x_1, y_1, z_1)$ with $|x_1| + |y_1| = r$. By Observation 9, the number of choices for $y_2 \in L(y_1, d + 2)$ is at most $d + 3$. The number of choices for $x'_2$ is at most

$$|x_1| + 1 + 2^{d+\lambda+2} - |y_1| - |x_1| = |x_1| + 1 + 2^{d+\lambda+2} - r \leq d + \lambda + 3 + 2^{d+\lambda+2} - r \leq (d + \lambda + 3)2^{d+\lambda+2} - r.$$

The number of choices of $\nu$ is at most $2^{d+\lambda+1}$. The choices of $x'_2$ and $\nu$ determine $x_2$. The number of choices for $z_2$ is $d + 1$. As before, the number of vertices $(x_1, y_1, z_1)$ with $|x_1| + |y_1| = r$ is $(r + 1) \cdot 2^r \cdot (d + 1)$. Therefore, the total number of edges of Type (2) is at most

$$\sum_{r=0}^{d+\lambda+2} (r + 1) \cdot 2^r \cdot (d + 1) \cdot 2^{d+\lambda+2} - r \cdot (d + 3)(d + \lambda + 3) \cdot 2^{d+\lambda+2} - r \cdot (d + 1) \leq 2^{d+2\lambda+3} \cdot (d + \lambda + 3)^6.$$ 

We obtain that the total number of edges in $G_n$ is at most $2^{d+2\lambda+4} \cdot (d + \lambda + 3)^6$. \hfill \Box

### 3.2 Proof of universality

**Lemma 20.** The graph $G_n$ is universal for the class of $n$-vertex subgraphs of $C_{\lfloor \log n \rfloor} \boxtimes P_n$.

**Proof.** Let $d := \lfloor \log n \rfloor$, $k := \max(5, \lfloor \sqrt{\log n / \log \log n} \rfloor)$, and $\lambda := \lambda(n)$. Let $G$ be an $n$-vertex subgraph of $C_d \boxtimes P_h$. By Lemma 6, we may assume that $G$ is a subgraph of $C_d \boxtimes P_h$ for some integer $1 \leq h \leq n$ and that for each $i \in \{1, \ldots, h\}$, there exists at least one vertex $v$ in $C_d$ such that $(v, i) \in V(G)$. Clearly, it suffices to prove the result when $G$ is an induced subgraph of $C_d \boxtimes P_h$.

We first define the embedding of $G$ onto $G_n$. For each $i \in \{1, \ldots, h\}$, let $S_i := \{v \in V(C_d) : (v, i) \in V(G)\}$. Recall that $V(C_d) = \{1, \ldots, 2^{d+1} - 1\}$, thus $S_i \subset \mathbb{R}$. Let $T_1, \ldots, T_h$ be the sequence of binary search trees obtained by applying Lemma 10 to the sequence $S_1, \ldots, S_h$. Let $T$ be a binary search tree with $V(T) = \{1, \ldots, h\}$ obtained by applying Lemma 8 with the weight function $w(i) = |V(T_i)|$, for each $i \in \{1, \ldots, h\}$. Let $\varphi : V(C_d) \to \{0, \ldots, d\}$ be a proper
coloring of $C_d$. (For instance, one could set $\varphi(v) := d_{B_d}(v)$. ) Each vertex $(v, i)$ of $G$ maps to the vertex

$$\zeta(v, i) := (\sigma_{T_i}(x_{T_i}(v)), \sigma_T(i), \varphi(v)).$$

First, we verify that $\zeta$ does indeed take vertices of $G$ onto vertices of $G_n$. Let $(v, i)$ be a vertex of $G$ and let $\zeta(v, i) = (x := \sigma_{T_i}(x_{T_i}(v)), y := \sigma_T(i), z := \varphi(v))$. Clearly, $z \in \{0, \ldots, d\}$. Note that $\sum_{j=1}^h w(j) \leq 4n$, by Lemma 10. Thus, by Lemma 8, we have $|y| \leq \log(4n) - \log |V(T_i)| \leq d + 2 - \log |V(T_i)|$. By Lemma 10 (complemented by Observation 11), $\text{height}(T_i) \leq \log |V(T_i)| + \lambda$ and since $|x| = |\sigma_{T_i}(x_{T_i}(v))| \leq \text{height}(T_i)$, we have $|x| + |y| \leq d + \lambda + 2$. Thus, $(x, y, z)$ is indeed a vertex of $G_n$.

Next, we verify that $\zeta : V(G) \to V(G_n)$ is injective. Let $(v, i)$ and $(w, j)$ be two distinct vertices of $G$. If $i \neq j$, then $\sigma_T(i) \neq \sigma_T(j)$. We may thus assume that $i = j$, so $v \neq w$ and both $v$ and $w$ are nodes of $T_i$. If $x_{T_i}(v) \neq x_{T_i}(w)$, then $\sigma_{T_i}(x_{T_i}(v)) \neq \sigma_{T_i}(x_{T_i}(w))$. We may therefore assume that $x := x_{T_i}(v) = x_{T_i}(w)$. This implies that $x \in I_{C_d}(v) \cap I_{C_d}(w)$, so by Observation 14, $vw \in E(C_d)$. Since $v \neq w$, this implies that $z_1 = \varphi(v) \neq \varphi(w) = z_2$. Thus, $\zeta(v, i) \neq \zeta(w, j)$ for $(v, i) \neq (w, j)$, so $\zeta$ is injective.

Finally, we need to verify that, for each edge $(v, i)(w, j) \in E(G)$, $G_n$ contains the edge $\zeta(v, i)\zeta(w, j)$. Let $(x_1, y_1, z_1) := \zeta(v, i)$ and let $(x_2, y_2, z_2) := \zeta(w, j)$. There are two cases to consider:

**Case 1:** $j = i$. In this case, $y_1 = y_2 = \sigma_T(i)$, $v \neq w$ and $vw \in E(C_d)$, and $v, w \in V(T_i)$. By Lemma 15, $x_1 = \sigma_{T_i}(x_{T_i}(v)) \diamond \sigma_{T_i}(x_{T_i}(w)) = x_2$. Therefore, $\zeta(v, i)\zeta(w, j) \in E(G_n)$ since it is included in $G_n$ as an edge of Type (1).

**Case 2:** $j = i + 1$. In this case, $y_2 \in L(y_1, \text{height}(T))$ by Observation 9. Lemma 8 ensures that $\text{height}(T) \leq \max\{d_T(i) : i \in \{1, \ldots, h\}\} = \max\{\log(4n/w(i)) : i \in \{1, \ldots, h\}\} \leq \log(4n) \leq d + 2$. Thus, $y_2 \in L(y_1, d + 2)$, and so condition (2a) for edges of Type (2) is satisfied.

Next, let $x_2' := \sigma_{T_i}(x_{T_i}(w))$. Observe that $w \in V(T_i)$ by property (1) of Lemma 10, so $x_2'$ is well defined. Since $(v, i)(w, j) \in E(G)$, either $v = w$ or $vw \in E(C_d)$. In the former case, we immediately have $x_2' = x_1$, and thus, $x_2' \diamond x_1$ holds. In the latter case, Lemma 15 implies that $x_2' = \sigma_{T_i}(x_{T_i}(w)) \diamond \sigma_{T_i}(x_{T_i}(v)) = x_1$. Therefore, $x_2'$ satisfies condition (2b) for edges of Type (2).

Next, by the definition of $\lambda$ and by property (4) of Lemma 10 (complemented by Observation 11), there exists a bitstring $\nu$ of length at most $\lambda$ such that $B(x_2', \nu) = \sigma_{T_i}(x_{T_i}(w)) \diamond \sigma_{T_i}(x_{T_i}(v)) = x_1$. Hence, $x_2'$ and $\nu$ satisfy condition (2c) for edges of Type (2). Therefore, $\zeta(v, i)\zeta(w, j) \in E(G_n)$ since it is included in $G_n$ as an edge of Type (2). □

Theorem 17 follows from Lemmas 19 and 20.

## 4 | INDUCED-UNIVERSAL GRAPHS

In this section, we prove Theorem 5. We describe a graph $U_n$ that is induced-universal for $n$-vertex members of $Q_t$ and has $n \cdot 2^{O(\sqrt{\log n \log \log n} \cdot (\log n)^{O(t)}}$ edges and vertices. The construction of $U_n$ relies on a relationship between induced-universal graphs and adjacency labeling schemes, which we now describe. Throughout this section, for the sake of brevity, we use $n^{O(1)}$ factors in place of more precise quantities like $2^{O(\sqrt{\log n \log \log n})}$ and $(\log n)^{O(t)}$. 
At the end of this section, we give a brief discussion of how the precise result in Theorem 5 appears.

Dujmović et al. [12] describe a \((1 + o(1)) \log n\)-bit adjacency labeling scheme for graphs in \(Q_t\). This means that there is a single function \(A : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}\) such that, for any \(n\)-vertex graph \(G \in Q_t\), there is an injective labeling \(\ell_G : V(G) \to \{0,1\}(1+o(1))\log n\) for which \(A(\ell_G(v), \ell_G(w)) = 1\) if and only if \(vw \in E(G)\). The existence of such a labeling scheme has the following immediate consequence: For every positive integer \(n\), there exists a graph \(I_n\) having \(n^{1+o(1)}\) vertices such that, for every \(n\)-vertex graph \(G \in Q_t\), \(I_n\) contains an induced subgraph isomorphic to \(G\). To see this, let \(I_n\) be the graph with vertex set \(V(I_n) = \{0,1\}(1+o(1))\log n\) and for which \(xy \in E(I_n)\) if and only \(A(x, y) = 1\). Then, for any \(n\)-vertex graph \(G \in Q_t\), the induced subgraph \(G' := I_n[\{\ell_G(v) : v \in V(G)\}]\) is isomorphic to \(V(G)\) (and \(\ell_G\) gives the isomorphism from \(G\) into \(G'\)).

In Section 4.1, we begin by reviewing the adjacency labeling scheme of Dujmović et al. [12]. In Section 4.2, we show that the induced-universal graph \(I_n\) defined in the previous paragraph has \(\Omega(n^2)\) edges. In Section 4.3, we show how the adjacency labeling scheme can be modified so that the resulting induced-universal graph \(U_n\) has \(n^{1+o(1)}\) edges.

### 4.1 Review of adjacency labeling

In this section, we review the adjacency labeling scheme in [12]. This review closely follows the presentation of [12] with a few exceptions that we discuss in footnotes when they occur. The main purpose of this review is to focus on a list of properties (P1)–(P6) that allow the adjacency labeling scheme to work correctly. Later, we will modify this labeling scheme and show that the modified scheme also has (suitably modified versions of) properties (P1)–(P6).

A \(t\)-tree \(H\) is a graph that is either a clique on \(t + 1\) vertices or contains a vertex \(v\) of degree \(t\) that is part of a \((t + 1)\)-clique and such that \(H - \{v\}\) is a \(t\)-tree. This definition implies that every \(t\)-tree \(H\) has a construction order \(v_1, ..., v_n\) of its vertices such that \(v_1, ..., v_i\) form a clique and, for each \(i \in \{1, ..., n\}\), \(v_i\) is adjacent to exactly \(\min\{i - 1, t\}\) vertices among \(v_1, ..., v_{i-1}\) and these vertices form a clique.

Fix a construction order \(v_1, ..., v_n\) of \(H\) and define

\[
C_{v_i} := \{v_i\} \cup \{v_j : v_i v_j \in E(H), j \in \{1, ..., \max(t + 1, i)\}\}
\]

for each \(i \in \{1, ..., n\}\). Then the vertices in \(C_{v_i}\) form a clique of order \(t + 1\) in \(H\) that we call the family clique of \(v_i\). For each \(v \in V(H)\), each vertex \(w \in C_v\) is called an \(H\)-parent of \(v\). A vertex \(a\) of \(H\) is an \(H\)-ancestor of \(v\) if \(a = v\) or \(a\) is an \(H\)-ancestor of some \(H\)-parent of \(v\). Note that \(v\) is an \(H\)-parent and an \(H\)-ancestor of itself.

The construction order \(v_1, ..., v_n\) implies that every \(t\)-tree \(H\) has a proper coloring using \(t + 1\) colors. Fix such a coloring \(\varphi : V(H) \to \{1, ..., t + 1\}\). For any vertex \(v\) of \(H\), the \(i\)-parent of \(v\), denoted by \(p_i(v)\), is the unique node \(w \in C_v\) with \(\varphi(w) = i\). Note that \(v\) is the \(\varphi(v)\)-parent of itself, that is, \(p_{\varphi(v)}(v) = v\).

It is well known that every graph of treewidth at most \(t\) is a subgraph of some \(t\)-tree. Thus, it is sufficient to describe how the adjacency labeling scheme in [12] works for any \(n\)-vertex subgraph \(G\) of \(H \boxtimes P\) where \(H\) is a \(t\)-tree and \(P\) is a path. Without loss of generality, we may assume that the vertices of \(P\) are the integers \(1, ..., h\) in the order they occur on the path \(P\) and that, for each
y \in \{1, \ldots, h\}, there exists at least one v \in V(H) such that (v, y) \in E(G), so h \leq n. Similarly, we may assume that |V(H)| \leq n.†

The adjacency labeling scheme in [12] makes use of an interval supergraph of H. Each vertex v of H is mapped to a real interval [a_v, b_v] in such a way that uv \in E(H) implies that [a_u, b_u] \cap [a_v, b_v] \neq \emptyset. Lemma 16 essentially says that this mapping is thin, in the following sense:

(P1) for any x \in \mathbb{R}, |\{v \in V(H) : x \in [a_v, b_v]\}| \in O(t \log n).

For each y \in \{0, \ldots, h+1\}, let L_y := \{v \in V(H) : (v, y) \in E(G)\} and let S_y := \bigcup_{v \in L_y} C_v. The labeling scheme first finds sets S_1^+, \ldots, S_h^+ of total size O(n) such that S_y^+ \supseteq S_{y-1}^+ \cup S_y^+ \cup S_{y+1}^+.‡

The adjacency labeling scheme uses a sequence of binary search trees T_1, \ldots, T_h such that, for each y \in \{1, \ldots, h\} and each v \in S_y^+, T_y contains at least one value x \in [a_v, b_v]. (T_1, \ldots, T_h form a bulk tree sequence as defined in Lemma 10, which also plays a central role in the proof of Theorem 2.) This leads to the following very important definition: For each v \in S_y^+, x_y(v) is the minimum-depth node x of T_y such that x \in [a_v, b_v]. Note that x_y(v) is well defined since T_y contains at least one node x \in [a_v, b_v]. The following property follows from these definitions and Helly’s Theorem:\§

(P2) For any v \in L_{y-1} \cup L_y \cup L_{y+1}, there exists a path P_y(v) that begins at the root of T_y and contains every node in X_y(v) := \{x_y(w) : w \in C_v\}.

For each y \in \{1, \ldots, h\} and each v \in L_y, we define P_y(v) to be the minimum length path in T_y that satisfies (P2), so that P_y(v) begins at the root of T and ends at the node in X_y(v) of maximum T_y-depth. For each y \in \{1, \ldots, h\} and each v \in V(T_y), d_y(x) denotes the depth of x in the tree T_y. It is helpful to think of x_y as a function x_y : S_y^+ \rightarrow V(T_y). For each y \in \{1, \ldots, h\} and each node x of T_y, let B_{y,x} := \{v \in S_y^+ : x_y(v) = x\} = x_y^{-1}(x). Since x \in [a_v, b_v] for each v \in B_{y,x}, (P1) implies the following property:

(P3) For each y \in \{1, \ldots, h\} and each x \in V(T_y), |B_{y,x}| \in O(t \log n).

Recall that, for any node x in a binary search tree T, \sigma_T(x) is the binary string b_1, \ldots, b_k obtained from the root-to-x path x_0, \ldots, x_k in T by setting b_i = 0 or b_i = 1 depending on whether x_i is the left or right child of x_{i-1}, respectively. Note that the function \sigma_T : V(T) \rightarrow \{0, 1\}^* is injective. We extend this notation to paths in T so that if P is a path from the root of T to some node x, then \sigma_T(P) := \sigma_T(x). We will use \sigma_y as a shorthand for \sigma_{T_y}.

Let \psi_y : S_y^+ \rightarrow \{1, \ldots, O(t \log n)\} be a coloring of S_y^+ such that, for each x \in V(T_y) and each distinct pair v, w \in B_{y,x}, \psi_y(v) \neq \psi_y(w). Such a coloring exists by (P3) and because x_y is a function, so each v \in S_y^+ appears in B_{y,x} for exactly one x \in V(T_y). Note that, for any v \in S_y^+, the pair (x_y(v), \psi_y(v)) uniquely identifies v. Since the signature function \sigma_y := \sigma_{T_y} is injective, this means that the pair (\sigma_y(x_y(v)), \psi_y(v)) also uniquely identifies v:

† This assumption requires that n \geq t + 1. We ignore the graphs in \mathcal{Q}_t having fewer than t + 1 vertices since there are only O(2^{(t)}) such graphs.

‡ The original labeling scheme only uses S_y^+ \supseteq S_{y-1} \cup S_y but it is convenient for us to include S_{y+1} as well and this change does not invalidate anything in the original scheme.

§ Helly’s theorem (in one dimension): Any finite set of pairwise intersecting intervals has a nonempty common intersection.
For any $y \in \{1, \ldots, h\}$ and any $v, w \in S_y^+$, $v = w$ if and only if $\sigma_y(x_y(v)) = \sigma_y(x_y(w))$ and $\psi_y(v) = \psi_y(w)$.

The binary search tree sequence $T_1, \ldots, T_h$ has two additional properties that are crucial:

(P5) For each $y \in \{1, \ldots, h\}$, $T_y$ has height $\text{height}(T_y) \leq \log |S_y^+| + o(\log n)$.

(P6) There exists a universal function $J : \{0, 1\}^* \times \{0, 1\}^* \to \{0, 1\}^*$ such that for each $y \in \{1, \ldots, h-1\}$ and each $v \in S_y^+ \cap S_{y+1}^+$, there exists a bitstring $\mu_y(v)$ of length $o(\log n)$ such that $J(\sigma_y(x_y(v)), \mu_y(v)) = \sigma_{y+1}(x_{y+1}(v))$.

The bitstring $\mu_y(v)$ is called a transition code.†

4.1.1 The labels

For each vertex $(v, y)$ of $G \subseteq H \boxtimes P$, the label $\ell_G(v, y)$ has these parts:

(L1) $\alpha(y)$: a bitstring of length of $\log n - \log |S_y^+| + o(\log n)$. Given $\alpha(y_1)$ and $\alpha(y_2)$ for any $y_1, y_2 \in \{1, \ldots, h\}$, it is possible to distinguish between the following cases: (a) $y_1 = y_2$; (b) $y_1 = y_2 + 1$; (c) $y_1 = y_2 - 1$; and (d) $|y_1 - y_2| \geq 2$.

(L2) $\sigma_y(p_y(v))$: this is a bitstring of length at most $\text{height}(T_y) \leq \log |S_y^+| + o(\log n)$.

(L3) $\eta_y(v)$: a bitstring of length $O(\log n)$. This bitstring is designed so that, for any vertex $v \in S_y^+ \cap S_{y+1}^+$, it is possible to recover $\sigma_{y+1}(p_{y+1}(v))$ given only $\sigma_y(p_y(v))$ and $\eta_y(v)$. The existence of $\eta_y(v)$ follows easily from the existence of $\mu_y(v)$ in (P6) and from the knowledge of the content of (L5) below.

(L4) $\varphi(v)$: the color of $v$ in the proper coloring of $H$ (a bitstring of length $\lceil \log(t + 1) \rceil$).

(L5) $d_y(x_y(p_i(v)))$ for each $i \in \{1, \ldots, t + 1\}$ (a bitstring of length $O(t \log \log n)$).

(L6) $\psi_{y+b}(p_i(v))$ for each $i \in \{1, \ldots, t + 1\}$ and each $b \in \{-1, 0, 1\}$ (a bitstring of length $O(t \log \log n + t \log t)$).‡

(L7) $a_y(v)$: a bitstring of length $3(t + 1)$ that indicates, for each $i \in \{1, \ldots, t + 1\}$ and each $b \in \{-1, 0, 1\}$ whether or not $G$ contains the edge with endpoints $(v, y)$ and $(p_i(v), y + b)$.

The label (L1) comes from Observation 9 but requires some further explanation. First, we remark that, like all parts of $\ell_G(v, y)$, the string $\alpha(y) := \alpha_G(y)$ depends on both $G$ and $y$. The string $\alpha(y)$ consists of two parts: $\alpha_1(y)$ is a bitstring of length at most $\log n - \log |S_y^+|$ and $\alpha_2(y)$ is a bitstring of length at most $\log \log n + O(1)$. These strings are designed so that there is a universal function $N$, which does not depend on $G$, such that $N(\alpha(y_1)) = \alpha_1(y_2)$ if and only if $y_2 = y_1 + 1$. Clearly, this makes it possible to distinguish between cases (a)–(d). It also has the following implication: For any fixed binary string $\tilde{y}_1$ that we interpret as $\alpha_1(y_1)$, there are at most $2^{\log \log n + O(1)} = O(\log n)$ binary strings that result in case (b). Indeed, these are strings $\tilde{y}_2 := \alpha_1 \circ \alpha_2$ (where $\circ$ denotes concatenation of strings) such that $N(\alpha(y_1)) = \alpha_1$ and $|\alpha_2| \leq \log \log n + O(1)$. The set of such strings turns out to be useful, so we denote it with $L(\alpha(y_1)) := \{N(\alpha(y_1)) \circ s : s \in \{0, 1\}^{\log \log n + O(1)}\}$.

† Our presentation here differs slightly from that in [12]. In [12], the transition code is used to take $\sigma(p_y(v))$ onto $\sigma(p_{y+1}(v))$. However, the proof that this is possible [12, Section 5.3] uses the existence of the transition code described in (P6) for each $w \in C_v$ and the fact that $\sigma(p_{y+1}(v)) = \sigma(x_{y+1}(w))$ for some $w \in C_v$.

‡ This is another place where our presentation differs slightly from that in [12]. In [12], the information contained in (L4), (L5), and (L6) is spread across several different parts of the label.
4.1.2 Adjacency testing

Given inputs $\ell_G(v_1, y_1)$ and $\ell_G(v_2, y_2)$, the adjacency testing function $A$ uses $\alpha(y_1)$ and $\alpha(y_2)$ to determine which of the following cases applies:

(a) $y := y_1 = y_2$. For each $i \in \{1, ..., t + 1\}$, determine if $v_1 = p_i(v_2)$ (or vice versa) and, if so, use $a_y(v_1)$ (or $a_y(v_2)$, respectively) to determine if $(v_1, y)$ and $(v_2, y)$ are adjacent in $G$. Specifically, if $v_1 = p_i(v_2)$, then one of the bits in $a_y(v_2)$ indicates whether or not $(v_1, y)$ and $(v_2, y)$ are adjacent in $G$. If $v_1 \neq p_i(v_2)$ and $v_2 \neq p_i(v_1)$ for every $i \in \{1, ..., h\}$, then $v_1v_2 \notin E(H)$ and hence $(v_1, y)$ and $(v_2, y)$ are not adjacent in $G \subseteq H \boxtimes P$.

By (P4), testing if $v_1 = p_i(v_2)$ is equivalent to testing if $\sigma_y(x_y(v_1)) = \sigma_y(x_y(p_i(v_2)))$ and $\psi_y(v_1) = \psi_y(p_i(v_2))$. We now show that $\ell_G(v_1, y_1)$ and $\ell_G(v_2, y_2)$ contain enough information to perform this test.

- We can recover $d_y(x_y(v_1)) = d_y(x_y(p_i(v_1)))$ and using this, recover $\sigma_y(x_y(v_1))$ from $\sigma_y(x_y(p_i(v_1)))$ and $d_y(x_y(v_1))$. Next, we can recover $\sigma_y(x_y(p_i(v_2)))$ from $\sigma_y(P_y(v_2))$ and $d_y(x_y(p_i(v_2)))$. This makes it possible to test if $\sigma_y(x_y(v_1)) = \sigma_y(x_y(p_i(v_2)))$.

- The color $\psi_y(v_1)$ can be recovered from $\ell_G(v_1, y_1)$ since $\psi_y(v_1) = \psi_y(P_y(v_1))$. The color $\psi_y(p_i(v_2))$ is stored explicitly in part (L6) of $\ell_G(v_2, y_2)$. This makes it possible to test if $\psi_y(v_1) = \psi_y(p_i(v_2))$.

(b) $y := y_2 = y_1 + 1$. In this case, recover $\sigma_y(P_y(v_1))$ from $\sigma_y(P_y(v_1))$ and $\eta_y(v_1)$. At this point, the algorithm proceeds exactly as in the previous case except for two small changes: (i) the value of $\psi_y(v_1) = \psi_{y+1}(v_1)$ is obtained from (L6); and (ii) in the final step, one bit of $a_y(v_2)$ (L7) is used to check if $(v_2, y_2)$ is adjacent to $(v_1, y_1) = (p_i(v_2), y_2 - 1)$ in $G$.

(c) $y := y_1 = y_2 + 1$. This case is symmetric to the previous case with the roles of $(v_1, y_1)$ and $(v_2, y_2)$ reversed.

(d) $|y_1 - y_2| \geq 2$. In this case, $y_1 \neq y_2$ and $y_1y_2 \notin E(P)$ and therefore $(v_1, y_1)$ and $(v_2, y_2)$ are not adjacent in $G \subseteq H \boxtimes P$.

4.2 Edge density of the induced-universal graph $I_n$

We now explain why the induced-universal graph $I_n$ defined by the labeling scheme in [12] is not sparse. It produces a universal graph $I_n$ having $\Omega(n^2)$ edges. The main issue is the definition of $P_y(v)$ as a path in $T_y$ that contains every node in $X_y(v) := \{x_y(w) : w \in C_v\}$. The problem comes from the fact that there can be nodes in $X_y(v)$ that have much greater $T_y$-depth than $x_y(v)$. As we will show below, this ultimately leads to a large complete bipartite graph in $I_n$ with sides $L$ and $R$ in which the elements of $L$ all correspond to a single vertex $(v, y)$ of $H \boxtimes P$. This problem even occurs when $P$ consists of a single vertex and $H$ is a tree.

Consider the tree $H$ illustrated in Figure 2 that consists of a five-vertex path $\beta, u, v, w, \alpha$ and a set of $n - 5$ leaves. Exactly half of these leaves are adjacent to $\beta$ and exactly half are adjacent to $\alpha$. If we root $H$ at $w$ and perform a preorder traversal, we obtain a construction order $v_1, ..., v_n$ of $H$ in which $C_d = \{v, w\}$ and $C_a$ contains $a$ and the parent of $a$ for each $a \in V(H) \setminus \{w\}$.

Observe that $H - \{v\}$ has two components each of size exactly $(n - 1)/2$. Therefore, when the vertices of $H$ are mapped onto intervals, it is natural to map $v$ onto the dominating interval $[a_v, b_v] := [1, n]$. Since $H - \{v\}$ consists of two stars centered at $\alpha$ and $\beta$, it is then natural to have $[a_\alpha, b_\alpha] := [1, (n - 1)/2]$ and $[a_\beta, b_\beta] := [n/2 + 1, n - 1]$. Now, $H - \{v, \alpha, \beta\}$ has no edges, so the remaining vertices can be mapped to appropriate zero-length intervals. All nodes adjacent
to $\alpha$ (including $w$) are mapped to $[i, i]$ for distinct $i \in \{1, \ldots, (n - 5)/2\}$. All nodes adjacent to $\beta$ (including $u$) are mapped to $[n/2 + j, n/2 + j]$ for distinct $j \in \{1, \ldots, (n - 5)/2\}$.

Let $\alpha_i$ (respectively, $\beta_j$) denote the node adjacent to $\alpha$ (respectively, $\beta$) that maps to the interval $[i, i]$ (respectively, $[n/2 + j, n/2 + j]$). It is entirely possible that $w = \alpha_p$ and $u = \beta_q$ for some $n/12 < p, q \leq 2n/12$. Suppose this is the case. For each $i, j \in \{1, \ldots, n/12\}$, consider the induced subgraph $H_{i,j}$ of $H$ having vertex set $V(H_{i,j})$ that contains

1. $\beta, u, v, w, \alpha$;
2. $\alpha_1, \ldots, \alpha_i$ and $\alpha_{2n/12+1}, \ldots, \alpha_{2n/12+n/12-i}$;
3. $\alpha_{n/4+1}, \ldots, \alpha_{n/4+n/12}$;
4. $\beta_1, \ldots, \beta_j$ and $\beta_{2n/12+1}, \ldots, \beta_{2n/12+n/12-j}$;
5. $\beta_{n/4+1}, \ldots, \beta_{n/4+n/12}$.

Let $P_1$ be a path consisting of a single vertex. If we apply the labeling scheme of Dujmović et al. [12] to $H_{i,j} \boxtimes P_1$, to obtain a labeling $\ell_{i,j} : V(H_{i,j}) \rightarrow \{0, 1\}^*$, then the binary search tree $T_1$ used in defining $\ell_{i,j}$ could be any balanced binary search tree containing

1. a root $r := n/2$ so that $x_y(v) = r$,
2. depth-1 nodes $a = n/4$ and $b = 3n/4$ so that $x_y(\alpha) = a$ and $x_y(\beta) = b$,
3. $\{k : \alpha_k \in V(H_{i,j})\}$,
4. $\{n/2 + k : \beta_k \in V(H_{i,j})\}$.

The first two levels of $T_1$ are fixed, independent of $i, j$ and each of the four depth-2 nodes is the root of a subtree of size exactly $n/12$. In particular, the “shape” of $T_1$ can be the same for any $i, j \in \{1, \ldots, n/12\}$. For example, if $n/12 = 2^k - 1$ for some integer $k$, then $T_1$ could be a complete binary tree of height $k + 2$. Suppose that this is the case. Then $\sigma_1(P_1(u)) = \sigma_1(x_1(u))$ depends only on the choice of $j$. Similarly, $\sigma_1(P_1(v)) = \sigma_1(x_1(w))$ depends only on the choice of $i$. 

**Figure 2** A tree $H$ that leads to $\Omega(n^2)$ edges in $I_n$. 

\[ \begin{array}{cc}
& \alpha \\
\hline
w & u \\
\hline
& \beta \\
\end{array} \]
This means that the label $\ell_i(v) := \ell_{i,j}(v,1)$ depends only on $i$. Furthermore, for any $i \neq i_2$, $\ell_{i_1}(v) \neq \ell_{i_2}(v)$. Similarly, the label $\ell_j(u) := \ell_{i,j}(u,1)$ depends only on $j$ and is distinct for each $j \in \{1, \ldots, n/12\}$. Furthermore, $uv$ is an edge of $T_{i,j}$ for each $i, j \in \{1, \ldots, n/12\}$, so $A(\ell_i(v), \ell_j(u)) = 1$ for each $i, j \in \{1, \ldots, n/12\}$. Therefore, the universal graph $I_n$ contains a complete bipartite subgraph with parts $L := \{\ell_i(v) : i \in \{1, \ldots, n/12\}\}$ and $R := \{\ell_j(u) : j \in \{1, \ldots, n/12\}\}$. Therefore, $|E(I_n)| \geq n^2/144$.

### 4.3 A sparse induced-universal graph

We now describe how to modify the adjacency labeling scheme of Dujmović et al. [12] so that the resulting induced-universal graph is sparse. As discussed above, the main difficulty comes from the fact that, for some vertex $(v, y) \in V(G)$, $v$ can have an $H$-parent $w$ such that $x_y(w)$ has $T_y$-depth much greater than $x_y(v)$. In order to avoid this, we modify the function $x_y : S^+ \to V(T_y)$ to create a new function $x'_y$ such that if $w$ is an $H$-parent of $v$ then $d_y(x'_y(w)) \leq d_y(x'_y(v)) + 1$.

This has to be done carefully in order to preserve (P2) and (P3). Initially, $x'_y(v) = x_y(v)$ for each $v \in S^+_y$, but then modifications are performed by calling the following recursive procedure with the root of $T_y$ as its argument:

**Fixup($x$):**

1. **for each** $v \in S^+_y$ **such that** $x'_y(v) = x$ **do**
2. **for each** $u \in C_v \cap S^+_y$ **do**
3. **if** $d_y(x'_y(u)) > d_y(x) + 1$ **then**
4.  **[this implies that]** $x'_y(w) = x_y(w)$
5.  $x'_y(w) \leftarrow$ the depth $-(d_y(x) + 1)T_y$ — ancestor of $x'_y(w)$
6.  **[so]** $x'_y(w)$ becomes a child of $x = x'_y(v)$
7.  **Fixup** left child of $x$ (if any)
8.  **Fixup** right child of $x$ (if any)

Observe that the only modifications to $x'_y$ occur in Line 5 and they involve setting $x'_y(w)$ to a $T_y$-ancestor of $x'_y(w)$. For each $v \in S^+_y$, $x'_y(v) = x_y(v)$ before the algorithm runs. Therefore, after the algorithm runs to completion, $x'_y(v)$ is a $T_y$-ancestor of $x_y(v)$. This ensures that (P2) holds for $x'_y$. Furthermore, Lines 3–6 of the algorithm ensure that for any $H$-parent $w$ of $v$, $d_y(x'_y(w)) \leq d_y(x'_y(v)) + 1$. Therefore, after running Fixup($r$), the following strengthening of (P2) holds:

**(P2′)** For any $v \in L_{y-1} \cup L_y \cup L_{y+1}$, there exists a path $P'_y(v)$ of length at most $d_y(x'_y(v)) + 1$ that begins at the root of $T_y$ and contains every node in $X'_y(v) := \{x'_y(w) : w \in C_v\}$.

Property (P2) is one of two critical properties needed by the function $x_y$. The other, (P3), bounds the size of $B_{y,x} := \{v \in S^+_y : x_y(v) = x\}$ by $O(t \log n)$. However, it is not the case that $x'_y$ satisfies (P3). Indeed, $B'_{y,x} := \{v \in S^+_y : x'_y(v) = x\}$ can be much larger than $B_{y,x}$, and even larger than $O(t \log n)$. Nevertheless, the next lemma shows that, for fixed $t$, the size of $B'_{y,x}$ remains polylogarithmic in $n$.

**Lemma 21.** For each $y \in \{1, \ldots, h\}$ and each node $x$ of $T_y$, $|B'_{y,x}| \in O(t(\log n)^{t+2})$. 

Proof. Let \( x \) be some node of \( T_y \) and suppose that \( x'(w) = x \) for some \( w \in S^+_y \). We now define a path \( w_0, w_1, w_2, \ldots, w_d \) in \( H \) by the following procedure. We start with \( w_0 = w \). At each step \( i \geq 0 \), we first check whether \( x'(w_i) = x'(w_{i+1}) \), and, if so, we set \( d := i \) and stop the process. Otherwise, it means that the definition of \( x'(w_i) \) was modified at some point by Fixup, and thus, \( w_i \) has a neighbor \( w_{i+1} \) in \( H \) with \( w_{i+1} \in C_{w_{i+1}} \), such that \( x'(w_i) \) was set to be a child of \( x'(w_{i+1}) \) in \( T_y \) by Fixup. In this way, we obtain a path \( w_0, w_1, w_2, \ldots, w_d \), \( d \geq 0 \), in \( H \) such that

(a) \( w_0 = w \);
(b) \( w_{i-1} \) is an \( H \)-parent of \( w_i \) for each \( i \in \{1, \ldots, d\} \);
(c) \( x'(w_i) \) is the \( T_y \)-parent of \( x'(w_{i+1}) \) for each \( i \in \{1, \ldots, d\} \); and
(d) \( x(y(w_d)) = x'(w_d) \).

In particular, \( w \) is an \( H \)-ancestor of \( w_d \) and there is a path \( w_0, w_1, w_2, \ldots, w_d \) in \( H \) of length at most \( d \) with endpoints \( w \) and \( w_d \). In the language of Pilipczuk and Siebertz [16] \( w_0 \) is \( d \)-reachable from \( w_d \).

Pilipczuk and Siebertz [16, Lemma 13] show that the number of \( d \)-reachable \( H \)-ancestors of any node \( v \) in a \( t \)-tree \( H \) is at most \((d + t)^t \). Now, let \( x = x_0, x_1, \ldots, x_k \) be the path from \( x = x_0 \) to the root \( x_k \) of \( T_y \). By the preceding argument, for each \( w \in B'_{y,x} \) there exists some \( d \in \{0, \ldots, k\} \) such that \( w \) is a \( d \)-reachable \( H \)-ancestor of some node \( v \in B_{y,x} \). Recall that by (P5), \( k \leq \text{height}(T_y) = (1 + o(1)) \log n \), so it follows that

\[
|B_{y,x}| \leq \sum_{d=0}^{k} |B_{y,x}| \left( \frac{d + t}{t} \right) \in O(t \log n \cdot k^{t+1}) \subseteq O(t \log n)^{t+2}.
\]

Therefore, by Lemma 21, \( x' \) satisfies the following weakening of (P3):

(P3') For each \( y \in \{1, \ldots, h\} \) and each \( x \in V(T_y) \), \( |B'_{y,x}| \in O(t \log n)^{t+2} \).

Let \( \psi'_y : S^+_y \to \{1, \ldots, O(t \log n)^{t+2}\} \) be a coloring of \( S^+_y \) such that, for each \( x \in V(T) \) and each distinct pair \( v, w \in B'_{y,x} \), \( \psi'_y(v) \neq \psi'_y(w) \). Such a coloring exists because, by (P3'), \( |B'_{y,x}| \in O(t \log n)^{t+2} \) and \( x' \) is a function, so each \( v \in S^+_y \) appears in \( B'_{y,x} \) for exactly one \( x \in V(T_y) \). Since \( x' : S^+_y \to V(T_y) \) is a function and \( \sigma_y \) is injective, we have the following variant of (P4):

(P4') For any \( y \in \{1, \ldots, h\} \) and any \( v, w \in S^+_y \), \( v = w \) if and only if \( \sigma_y(x'_y(v)) = \sigma_y(x'_y(w)) \) and \( \psi'_y(v) = \psi'_y(w) \).

4.4 The new labels

For each vertex \((v, y)\) of \( G \), the label \( \ell_G'(v, y) \) has these parts:

(NL1) \( \alpha(y) \): this is unmodified from the original scheme.
(NL2) \( \sigma_y(x'_y(v)) \): note that this is not \( \sigma_y(P'_y(v)) \), but \( \sigma_y(x'_y(v)) \) can be recovered from \( \sigma_y(x_y(v)) \) and \( d_y(x'_y(P_y(v))) \). This makes it possible to recover \( \sigma_y(P'_y(v)) = \sigma_y(x'_y(v)) \odot \tau_y(v) \) where \( \tau_y(v) \) is defined in (NL8), below (recall that \( \odot \) is used to denote string concatenation).
(NL3) \( \mu_y(v) \): a bitstring of length \( o(\log n) \). This bitstring, defined in (P6), is designed so that for any vertex \( v \in S^+_y \cap S^+_{y+1} \), it is possible to recover \( \sigma_{y+1}(x_{y+1}(v)) \) given only \( \sigma_y(x_y(v)) \) and \( \mu_y(v) \).
(NL4) \( \varphi(v) \): the color of \( v \) in the proper coloring of \( H \) (a bitstring of length \( O(\log t) \)).
(NL5) $d_y(x'_y(p_i(v)))$ for each $i \in \{1, \ldots, t+1\}$ (a bitstring of length $O(t \log \log n)$).

(NL6) $\psi'_y(p_i(v))$ for each $i \in \{1, \ldots, t+1\}$ and each $b \in \{-1, 0, 1\}$ (by (P3'), this is a bitstring of length $O(t^2 \log \log n)$).

(NL7) $a_y(v)$: this is unmodified from the original scheme.

(NL8) $r_{y+b}(v)$ for each $b \in \{-1, 0, 1\}$: Three binary strings, each of length at most 1 such that $\sigma_y(v) = \psi'_y(p_i(v)) \circ r_{y+b}(v)$ for each $b \in \{-1, 0, 1\}$.

4.5 | Adjacency testing

Given inputs $\ell'_G(v_1, y_1)$ and $\ell'_G(v_2, y_2)$, the adjacency testing function $A$ for the new labeling scheme uses $\alpha(y_1)$ and $\alpha(y_2)$ to determine which of the following cases applies:

(a) $y := y_1 = y_2$. For each $i \in \{1, \ldots, t+1\}$, determine if $v_1 = p_i(v_2)$ (or vice versa) and, if so, use $a_y(v_2)$ (or $a_y(v_1)$, respectively) to determine if $(v_1, y)$ and $(v_2, y)$ are adjacent in $G$. Specifically, if $v_1 = p_i(v_2)$, then one of the bits in $a_y(v_2)$ indicates whether or $(v_1, y_1)$ and $(v_2, y_1)$ are adjacent in $G$. If $v_1 \neq p_i(v_2)$ and $v_2 \neq p_i(v_1)$ for every $i \in \{1, \ldots, h\}$, then $v_1, v_2 \not\in E(H)$ and $(v_1, y)$ and $(v_2, y)$ are not adjacent in $G \subseteq H \boxtimes P$.

By (P4'), testing if $v_1 = p_i(v_2)$, is equivalent to testing if $\sigma_y(x'_y(p_i(v_1))) = \sigma_y(x'_y(v_1))$ and $\psi'_y(v_1) = \psi'_y(v_2)$. We now show that $\ell'_G(v_1, y_1)$ and $\ell'_G(v_2, y_2)$ contain enough information to perform this test.

- We can recover $d_y(x'_y(v_1)) = d_y(x'_y(p\varphi(v_1)(v_1)))$ and using this, recover $\sigma_y(x'_y(v_1))$ from $\sigma_y(x'_y(v_1))$ and $d_y(x'_y(p_i(v_1)))$. Next, we can recover $\sigma_y(x'_y(p_i(v_2)))$ from $\sigma_y(x'_y(v_2))$ and $d_y(x'_y(p_i(v_2)))$. This makes it possible to test if $\sigma_y(x'_y(v_1)) = \sigma_y(x'_y(p_i(v_2)))$.
- The color $\psi'_y(v_1)$ can be recovered from $\ell'_G(v_1, y_1)$ since $\psi'_y(v_1) = \psi'_y(p\varphi(v_1)(v_1))$. The color $\psi'_y(p_i(v_2))$ is stored explicitly in $\ell'_G(v_2, y_2)$. This makes it possible to test if $\psi'_y(v_1) = \psi'_y(p_i(v_2))$.

(b) $y := y_1 = y_2 + 1$. In this case, recover $\sigma_y(x'_y(v_1))$ from $\sigma_y(x'_y(v_1))$ and $\mu_y(v_1)$. Next, recover $\sigma_y(P_y(v_1)) = \sigma_y(x'_y(v_1)) \circ r_{y_1+1}(v)$. At this point, the algorithm proceeds exactly as in the previous case except for two small changes: (i) the value of $\psi'_y(p_i(v_1)) = \psi'_{y+1}(v_1)$ is obtained from (NL6); and (ii) in the final step, one bit of $a_{y_2}(v_2)$ (NL7) is used to check whether $(v_2, y_2)$ is adjacent to $(v_1, y_1) = (p_i(v_2), y_2 - 1)$ in $G$.

(c) $y := y_1 = y_2 + 1$. This case is symmetric to the previous case with the roles of $(v_1, y_1)$ and $(v_2, y_2)$ reversed.

(d) $|y_1 - y_2| \geq 2$. In this case $y_1 \neq y_2$ and $y_1y_2 \not\in E(P)$, and therefore, $(v_1, y_1)$ and $(v_2, y_2)$ are not adjacent in $G \subseteq H \boxtimes P$.

4.6 | Bounding the number of edges

In the preceding sections, we have described an adjacency testing function $A$ such that, for any $n$-vertex graph $G \subseteq \mathcal{Q}_t$, there exists an injective labeling $\ell'_G : V(G) \to \{0, 1\}^{(1+o(1)) \log n}$ such that, for any $v, w \in V(G)$, $A(\ell'_G(v), \ell'_G(w)) = 1$ if and only if $v w \in E(G)$. We define the induced-universal graph $U_n$ as follows: $V(U_n)$ consists of $\ell'_G(v, y)$ for each $n$-vertex graph $G \subseteq \mathcal{Q}_t$ and each $(v, y) \in V(G)$. Similarly, an edge $\ell'_1 \ell'_2$ is in $U_n$ if and only if there exists an $n$-vertex graph $G \subseteq \mathcal{Q}_t$ that contains an edge $v w$ such that $\ell'_G(v) = \ell'_1$ and $\ell'_G(w) = \ell'_2$. As already discussed, it follows from the correctness of the labeling scheme that $U_n$ is induced-universal for $n$-vertex graphs in $\mathcal{Q}_t$. 
We will now show that \( U_n \) has \( n^{1+o(1)} \) vertices and edges. This analysis mostly follows along the same lines as the analysis of Section 3 but is, by necessity, a little less modular.\(^1\) In this analysis, it will be helpful to think of each label \( \ell_G'(v, y) \) in the labeling of a graph \( G \) as a triple \((x, \bar{y}, z)\) where \( x = \sigma_y(x_y(v)), \bar{y} = \alpha(y), \) and \( z \) is the concatenation of the bitstrings (NL3)-(NL8). Of course, since each vertex of \( U_n \) is \( \ell_G'(v, y) \) for some \( n \)-vertex \( G \in \mathcal{Q}_t \) and some \((v, y) \in V(G)\), we can also treat the vertices of \( U_n \) as triples. Thus, each vertex of \( U_n \) is a triple \((x, \bar{y}, z)\) where \( x, \bar{y}, \) and \( z \) are bitstrings with \( |x| + |\bar{y}| \leq \log n + \lambda, |z| \leq \lambda, \) and \( \lambda \in o(\log n) \).

In the proofs below, whenever we use Property (P2\(^{'}\)) explicitly, what we really use is only the weaker Property (P2). So, let us first explain where Property (P2\(^{'}\)) is really being used and makes a crucial difference with the previous labeling scheme with parts (L1)-(L7). Part (NL8) of \( \ell_G'(v, y) \), which is part of \( z \), has constant length and makes it possible to recover \( \sigma_y(P'_y(v)) \) from (NL2), which has length \( d_y(x_y(v)) \). With the original Property (P2), this would not be possible: recovering \( \sigma_y(P_y(v)) \) from \( \sigma_y(x_y(v)) \) requires a string of length \(|\sigma_y(P_y(v))| - d_y(x_y(v))\). In this case, the length of (NL8), and hence, the length of \( z \) could only be bounded by \( h(T_y) - d_y(x_y(v)) \) that, as shown in Section 4.2, maybe \( \Omega(\log n) \).

**Lemma 22.** The graph \( U_n \) has \( n^{1+o(1)} \) vertices.

**Proof.** Consider a vertex \((x, \bar{y}, z)\) of \( U_n \). The pair \((x, \bar{y})\) consists of two bitstrings of total length \( r := |x| + |\bar{y}| \leq \log n + \lambda \). For a fixed \( r \), the number of such \((x, \bar{y})\) is \((r + 1)2^r \). Therefore, the number of such \((x, \bar{y})\) over all choices of \( r \) is

\[
\sum_{r=0}^{\log n + \lambda} (r + 1)2^r \leq 2^{\log n + \lambda + 1}(\log n + \lambda + 1) = n^{1+o(1)}.
\]

The third coordinate, \( z \), is a bitstring of length at most \( \lambda \). The number of such bitstrings is \( 2^{\lambda + 1} - 1 = n^{o(1)} \). Therefore, the number of choices for \((x, \bar{y}, z)\) is \( n^{1+o(1)}\cdot n^{o(1)} = n^{1+o(1)} \). \(\Box\)

As in Section 3, we distinguish between two kinds of edges in \( U_n \). An edge with endpoints \((x_1, \bar{y}_1, z_1)\) and \((x_2, \bar{y}_2, z_2)\) is a **Type 1** edge if \( \bar{y}_1 = \bar{y}_2 \) and is a **Type 2** edge otherwise. We count Type 1 and Type 2 edges separately.

**Lemma 23.** The graph \( U_n \) contains \( n^{1+o(1)} \) **Type 1** edges.

**Proof.** Let \((x_1, \bar{y}_1, z_1)(x_2, \bar{y}_2, z_2)\) be a Type 1 edge of \( U_n \) and, for each \( i \in \{1, 2\} \), let \( \ell_i := (x_i, \bar{y}_i, z_i) \).
Since \( \ell_1, \ell_2 \) lies in \( E(U_n) \), there exists some \( t \)-tree \( H \), some path \( P \), some \( n \)-vertex subgraph \( G \) of \( H \boxtimes P \), and some edge \((v_1, y_1)(v_2, y_2)\) of \( G \) such that \( \ell_1 = \ell'_G(v_1, y_1) \) and \( \ell_2 = \ell'_G(v_2, y_2) \). For this graph \( G \), \( \alpha(y_1) = \bar{y} = \alpha(y_2) \), which implies that \( y := y_1 = y_2 \) for some integer \( y \).

\(^1\)The modular approach used in Section 3 to describe a universal graph can be ruled out by a simple counting argument. Section 3 describes a universal graph for the class \( \mathcal{C} \) of \( n \)-vertex subgraphs of \( C_d \boxtimes K_\omega \boxtimes P_n \) for \( d, \omega \in \Theta(\log n) \). However, the graph \( G := C_{\log n - \log \log n} \boxtimes K_{\log n} \) has \( n \) vertices and \( \Theta(n \log^2 n) \) edges, and lies in \( \mathcal{F} \). The graph \( G \) has at least \( 2^{\Omega(n \log^2 n)} \) nonisomorphic \( n \)-vertex subgraphs, and thus, \( \mathcal{C} \) contains at least \( 2^{\Omega(n \log^2 n)} \) nonisomorphic graphs. On the other hand, any graph with \( n^{1+o(1)} \) vertices has at most \( n^{1+o(1)} \) \( n \)-vertex induced subgraphs and since \( \binom{n^{1+o(1)}}{n} < n^{(1+o(1))n} = 2^{(1+o(1))n \log n} \ll 2^{\Omega(n \log^2 n)} \), it follows that a graph on at most \( n^{1+o(1)} \) vertices cannot be induced-universal for \( \mathcal{C} \).
The existence of the edge \((v_1, y)(v_2, y)\) in \(G\) implies the existence of the edge \(v_1v_2\) in \(H\). Therefore, \(v_1\) is an \(H\)-parent of \(v_2\), or vice versa. Property (P2') implies that one of \(x_1 = \sigma_y(x_y(v_1))\) or \(x_2 = \sigma_y(x_y(v_2))\) is a prefix of the other. Assume, without loss of generality, that \(x_2\) is a prefix of \(x_1\) and direct the edge \(\ell_1\ell_2\) away from \(\ell_1\). For a fixed \((x_1, \hat{y}, z_1)\), the number of \(x_2\) that are a prefix of \(x_1\) is most \(|x_1| + 1 \leq \log n + \lambda + 1 = n^{o(1)}\). For a fixed \((x_1, \hat{y}, z_1)\), the number of \((x_2, \hat{y}, z_2)\) in which \(x_2\) is a prefix of \(x_1\) is at most \(n^{o(1)} \cdot 2^{\lambda + 1} = n^{o(1)}\).

Therefore, each vertex \((x_1, \hat{y}, z_1)\) of \(U_n\) has at most \(n^{o(1)}\) Type 1 edges directed away from it. Therefore, the number of Type 1 edges in \(U_n\) is at most \(|V(U_n)| \cdot n^{o(1)} = n^{1+o(1)}\), where the upper bound on \(|V(U_n)|\) comes from Lemma 22.

Lemma 24. The graph \(U_n\) contains at most \(n^{1+o(1)}\) Type 2 edges.

Proof. Let \((x_1, \hat{y}_1, z_1)(x_2, \hat{y}_2, z_2)\) be a Type 2 edge of \(U_n\) and, for each \(i \in \{1, 2\}\), let \(\ell'_i := (x_i, \hat{y}_i, z_i)\).

Since \(\ell'_1\ell'_2\) lies in \(E(U_n)\), there exists some \(t\)-tree \(H\), some path \(P\), some \(n\)-vertex subgraph \(G\) of \(H \boxtimes P\), and some edge \((v_1, y_1)(v_2, y_2)\) of \(G\) such that \(\ell'_1 = \ell'_G(v_1, y_1)\) and \(\ell'_2 = \ell'_G(v_2, y_2)\).

Since \(\alpha(y_1) = \hat{y}_1 \neq \hat{y}_2 = \alpha(y_2)\), we have \(y_1 \neq y_2\). The existence of the edge \((v_1, y_1)(v_2, y_2)\) in \(G\) therefore implies that \(y_2\) is an edge of \(P\) so that (without loss of generality) \(y_1 = y\) and \(y_2 = y + 1\) for some \(y \in \{1, \ldots, h - 1\}\). Now, \(\hat{y}_1 = \alpha(y)\) and \(\hat{y}_2 = \alpha(y + 1)\). Specifically \(\hat{y}_2 \in L(\hat{y}_1)\) (see Section 4.1.1) and \(|L(\hat{y}_1)| \in O(\log n)|. Therefore, for a fixed \(\hat{y}_1\), the number of possible choices for \(\hat{y}_2\) is \(O(\log n)\).

The existence of the edge \((v_1, y_1)(v_2, y_2)\) in \(G\) implies that \(v_1 = v_2\) or that \(v_1v_2 \in E(H)\).

1) If \(v_1 = v_2\), then \(x_2 = J(x_1, \mu_y(v_1))\). Since \(\mu_y(v_1)\) is included as part of \(z_1\) the condition \(v_1 = v_2\) implies that fixing \((x_1, \hat{y}_1, z_1) = \ell'_G(v_1, y_1)\) fixes the value of \(x_2\). We have already established that, for a fixed \(\hat{y}_1\), the number of options for \(\hat{y}_2\) is \(O(\log n)\). Finally, \(z_2\) is a bitstring of length at most \(\lambda\), so the number of options for \(z_2\) is at most \(2^{\lambda + 1} - 1 = n^{o(1)}\). Therefore, for a fixed \((x_1, \hat{y}_1, z_1)\), the number of options for \((x_2, \hat{y}_2, z_2)\) in this case is at most

\[1 \cdot O(\log n) \cdot n^{o(1)} = n^{o(1)}\]

By Lemma 22, the number of choices for \((x_1, \hat{y}_1, z_1)\) is at most \(n^{1+o(1)}\). Therefore, the number of Type 2 edges in \(U_n\) contributed by edges \((v_1, y_1)(v_2, y_2)\) in \(n\)-vertex graphs \(G \in \mathcal{Q}_t\), where \(v_1 = v_2\) is at most \(n^{1+o(1)} \cdot n^{o(1)} = n^{1+o(1)}\).

2) If \(v_1v_2 \in E(H)\), then recall the definition of \(S^+\), which implies that \(v_1, v_2 \in S^+_y \cap S^+_{y+1}\). Since \(v_1, v_2 \in E(H)\), at least one of \(v_1\) or \(v_2\) is an \(H\)-parent of the other. Since \((v_2, y + 1) \in V(G), v_2 \in S^+_y\) so \(x_y(v_2)\) is defined. By (P2'), one of \(x'_2 := \sigma_y(x_y(v_2))\) or \(x_1 = \sigma_y(x_y(v_1))\) is a prefix of the other. By (P5), \(|x'_2| \leq \text{height}(T_y) \leq \log |S^+_y| + \lambda \leq \log n + |\hat{y}_1|\), where the final inequality comes from the property of \(\alpha\) in (L1) and (NL1). Therefore, for a fixed \((x_1, \hat{y}_1, z_1)\), the number of choices for \(x'_2\) is at most \(|x_1| + 1 + 2^{\log n + |\hat{y}_1| - |x_1|} = n^{1+o(1)} \cdot 2^{-|x_1| - |\hat{y}_1|}\).

Since \(x'_2 = x_y(v_2)\), by (P6), there exists a bitstring \(\mu_y(v_2)\) of length \(o(\log n)\) such that \(J(x'_2, \mu_y(v_2)) = \sigma_{y+1}(x_{y+1}(v_2)) = x_2\). Therefore, for a fixed \(x'_2\), the number of choices for \(x_2\) is at most \(2^{o(\log n)} = n^{o(1)}\). Thus, for a fixed \((x_1, \hat{y}_1, z_1)\), the number of choices for \((x_2, \hat{y}_2, z_2)\) is at most

\[n^{1+o(1)} \cdot 2^{-|x_1| - |\hat{y}_1|} \cdot O(\log n) \cdot n^{o(1)} = n^{1+o(1)} \cdot 2^{-|x_1| - |\hat{y}_1|}\],

as required.
where the first factor counts the number of options for \( x_2 \), the second the number of options for \( \tilde{y}_2 \in L(\tilde{y}_1) \), and the third the number of options for \( z_2 \). For fixed \( r := |x_1| + |\tilde{y}_1| \), the number of choices for \((x_1, \tilde{y}_1)\) is \((r + 1) \cdot 2^r\). Therefore, for a fixed \( r \), the number of choices for \((x_1, \tilde{y}_1, z_1)\) is \((r + 1) \cdot 2^r \cdot (2^{r+1} - 1) = 2^r \cdot n^{o(1)}\). We can now sum over \( r \) to determine that the total number of Type 2 edges contributed by some edge \((v_1, y_1)(v_2, y_2)\) in some \( n \)-vertex graph \( G \in Q_t \) with \( v_1 \neq v_2 \) is at most

\[
\sum_{r=0}^{\log n + \lambda} 2^r \cdot n^{1+o(1)} \cdot 2^{-r} = n^{1+o(1)}(\log n + \lambda + 1) = n^{1+o(1)}.
\]

Each Type 2 edge \((x_1, \tilde{y}_1, z_1)(x_2, \tilde{y}_2, z_2)\) of \( U_n \) is contributed by some edge \((v_1, y_1)(v_2, y_2)\) in some \( n \)-vertex graph \( G \in Q_t \) such that either \( v_1 = v_2 \) or \( v_1 \neq v_2 \). Therefore, the two cases analyzed above establish that \( U_n \) has \( n^{1+o(1)} \) Type 2 edges.

A more careful handling of \( n^{o(1)} \) factors in the proofs of Lemmas 22–24 gives an upper bound of

\[
n \cdot 2^{O(\sqrt{\log n \log \log n})} \cdot (\log n)^{O(t^2)}
\]

on the number of edges and vertices in \( U_n \) and establishes Theorem 5. The bottleneck in the analysis is the value \( \lambda \) that represents the trade-off between the lengths of the transition codes \( \mu_y \) and the excess height of trees \( T_1, \ldots, T_h \) (this trade-off is captured by the parameter \( k \) in [12]). In particular, the optimal trade-off is obtained when \( |\mu_y(u)| \in O(\sqrt{\log n \log \log n}) \) and \( \text{height}(T_y) \leq \log |S^+_y| + O(\sqrt{\log n \log \log n}) \). The \((\log n)^{O(t^2)} \) factor comes from storing the colors \( \psi'_{y+b}(\rho_i(u)) \) in each for each \( i \in \{1, \ldots, t + 1\} \), since each color comes from a set of size \((\log n)^{O(t)}\).

We remark that our proof includes within it a labeling scheme for graphs of treewidth at most \( t \). Analyzing this labeling scheme separately shows that it gives rise to a graph \( H_n \) that has \( n(\log n)^{O(t^2)} \) edges and vertices, and contains each \( n \)-vertex subgraph of treewidth at most \( t \) as an induced subgraph.

5 | CONCLUSION

Our construction of universal graphs is based on the product structure theorem of [13], which does not apply to every proper minor-closed classes of graphs, only to apex-minor-free classes. A natural problem is thus to construct universal graphs with \( o(n^{3/2}) \) edges for \( n \)-vertex graphs from an arbitrary proper minor-closed class.

NOTE ADDED IN PROOF

Very recently, Gawrychowski and Janczewski [14] showed that the use of bulk tree sequences in [12] could be replaced with a simpler approach based on B-trees, while leaving the rest of the proof essentially unchanged. This simplifies the data-structure part of the proof in [12] and also gives a slightly improved bound of \( n \cdot 2^{O(\sqrt{\log n})} \) on the number of vertices in the resulting induced-universal graph for \( n \)-vertex planar graphs, compared to \( n \cdot 2^{O(\sqrt{\log n \log \log n})} \) in [12]. Given our
use of the bulk tree sequences from [12] as a “black box” in Sections 2 and 3, they can also be replaced with the approach based on B-trees from [14] in these proofs. This reduces the factor $n \cdot 2^{O(\log n \log \log n)}$ in Theorems 1 and 2 to $n \cdot 2^{O(\sqrt{\log n})}$. On the other hand, the proofs in Section 4 do depend on the inner workings of bulk tree sequences, and as such it is not immediately clear whether they could be replaced with B-trees. As in [14], we leave this as an open problem.

**ACKNOWLEDGEMENTS**

We thank Noga Alon for providing the details of the argument used in [1] to decrease the number of vertices in a universal graph. We are grateful to an anonymous referee for their helpful comments on an earlier version of the paper.

L. Esperet is partially supported by the French ANR Projects ANR-16-CE40-0009-01 (GATO) and ANR-18-CE40-0032 (GrR). G. Joret is supported by an ARC grant from the Wallonia-Brussels Federation of Belgium and a CDR grant from the National Fund for Scientific Research (FNRS). P. Morin is partially supported by NSERC.

**JOURNAL INFORMATION**

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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APPENDIX A: PROOF OF LEMMA 18

Let $N$ be the smallest integer divisible by $k$ such that $N \geq N_0$. Note that we have $N \leq N_0 + k$. Consider an integer $d = \Theta\left(\frac{1}{\varepsilon} \log k\right)$ (whose precise value will be determined later). Let $H$ be a bipartite graph with parts $V$ of size $N$, and $U$ of size $N/k$ in which each vertex of $V$ is connected to $d$ random vertices of $U$ (with replacement, so it might be the case that some vertices of $V$ have less than $d$ neighbors).

Claim A1. The following holds with positive probability: for each subset $X$ of $V$ with $|X| \leq n$, $|N_H(X)| \geq |X|$.

Proof. For subsets $S \subseteq V$ and $T \subseteq U$ of size $s \leq n$ and $t < s$, respectively, we denote by $\mathcal{E}_{S,T}$ the event that all neighbors of $S$ are in $T$. Note that $\mathcal{E}_{S,T}$ occurs with probability $(tk/N)^{sd}$. By the union bound, the probability that there are two subsets $S \subseteq V$ of size $s \leq n$ and $T \subseteq U$ of size $t < s$, such that all neighbors of $S$ are in $T$ is at most

$$
\sum_{s=1}^{n} \sum_{t=1}^{s-1} \binom{N}{s} \binom{N/k}{t} (tkN)^{sd} \leq \sum_{s=1}^{n} s \left( \frac{Ne}{s} \right)^s \left( \frac{Ne}{ks} \right)^s \left( \frac{sk}{N} \right)^{sd} \\
\leq \sum_{s=1}^{n} s \left[ \frac{Ne}{s} \cdot \frac{Ne}{sk} \cdot \left( \frac{sk}{N} \right)^d \right]^s \\
\leq \sum_{s=1}^{n} \left[ 2e^2 k \left( \frac{sk}{N} \right)^{d-2} \right]^s ,
$$
where we have used the inequalities \( s \leq 2^s \) and \((\frac{a}{b})^b \leq (ae/b)^b\). We have also used the fact that the function \( x \mapsto (c/x)^x \) is increasing on the interval \((0, c/e)\) for any fixed \( c > 0 \), and thus, \( \left(\frac{N}{k}\right) \leq \left(\frac{Ne}{sk}\right) \leq \left(\frac{Ne}{sk}\right)^s \) for any \( t \leq s \leq n \leq \frac{1}{e}(Ne/k) = N/k \).

Since \( s \leq n \leq \frac{N}{(1+\epsilon)k} \leq \frac{N}{(1+\epsilon)k} \), we have \((sk/N)^{d-2} \leq (\frac{1}{1+\epsilon})^{d-2} \leq \exp(-\frac{\epsilon}{2}(d-2))\) for any \( 0 < \epsilon < 1 \). It follows that by taking \( d \) to be a sufficiently large in \( \Omega(\frac{1}{\epsilon} \log k) \), we have \( 2e^2k(\frac{sk}{N})^{d-2} < 1/10 \) and thus the probability that there exist two subsets \( S \subseteq V \) of size \( s \leq n \) and \( T \subseteq U \) of size \( t < s \), such that all neighbors of \( S \) are in \( T \) is at most \( \sum_{s \geq 1} 10^{-s} < 1 \). This completes the proof of the claim. \( \square \)

The property that any \( n \)-vertex subset \( X \) of \( V \) is saturated by a matching of \( H \) is now a direct consequence of Hall’s theorem (applied to the subgraph of \( H \) induced by \( X \cup U \)). This concludes the proof of Lemma 18.

We note that the probabilistic construction of \( H \) in the proof above can be replaced by a purely deterministic (and explicit) construction using expander graphs, at the cost of a more tedious analysis and a worse bound on the degree \( d \) (as a function of \( k \) and \( \epsilon \)). The advantage of such an explicit construction is that (together with the other components of our proof), it provides an explicit description of the universal graph with \( (1 + o(1))n \) vertices, and an efficient deterministic algorithm giving an embedding of any \( n \)-vertex planar graph in the universal graph.