Rheological model of the fine particles viscoplastic suspension

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Abstract. The analysis of the mechanical behaviour of some types of fine particles suspensions is carried out. A rheological model is proposed to describe the shear stress and shear rate dependence, taking into account the appropriate conditions imposed on the model parameters. It is shown that such model can take into account the “solidification” effect manifestation. A variation of generalization of the rheological model for two- and three-dimensional hydrodynamics problems of fine particles suspensions is proposed, and some methodological features for solving such problems are considered.

1. Introduction

As a rule, traditional fluids exhibit a monotonic viscosity change over a fairly wide range of shear rate changes.

In particular, the classical Newtonian fluid is characterized by the constant viscosity. Well known pseudoplastic fluids correspond to the monotonic viscosity decrease with increasing shear rate. Another group of so-called dilatant fluids, on the contrary, manifests itself in the monotonic viscosity increase as the shear rate increases.

In contrast to continua of such kind, some types of suspensions based on polymer fluids and fine particles demonstrate a more complex mechanical behavior [1-4].

Such behavior manifests itself, first of all, in the specific dependence of their viscosity on the shear rate. At different shear rate ranges, the suspension can exhibit a different viscosity change pattern. For example, a monotonic decrease in viscosity in a certain shear rate range can change to a monotonic increase, but in a different range. And then, in the next shear rate variation range, its character again changes to a monotonic decrease. In this case, inflection points appear on the flow curve - the dependence of the shear stress on the shear rate - corresponding to the boundaries of such ranges.

Such mechanical behavior anomalies are based on the material internal structure changes that occur in various deformation modes and are associated with the formation of the fine particles into associations or clusters of the “solid” structures type.

Suspensions of this kind, with the predetermined combination of the corresponding particles characteristics, taking into account their size and concentration, as well as the rheological parameters of the suspension fluid component, are the basis for a number of materials with rather specific mechanical behavior.
This specific behavior is caused by viscosity anomalies. As the shear rate in the corresponding flow region zones approaches a certain critical value, such suspensions viscosity begins to increase sharply. Moreover, for some types of suspensions, this increase can be so significant that the viscosity value increases substantially. Then the suspension behavior in such flow region zones resembles the behavior of a solid.

Some examples of the use of suspensions of this kind in technical applications are given in [5-7].

Obviously, in order to manufacture products of “fixed” shape from such materials, they must demonstrate some plasticity. This can be achieved by using viscoplastic fluids with a sufficiently high shear stress as a liquid component.

Some examples of rheological models of similar fluids with viscous behavior anomalies were considered in [8-9].

Since these models do not fully describe all the features of the suspensions mechanical behavior, this article proposes a more complete rheological model of a nonlinear viscoplastic fluid, the flow curve of which has two inflection points corresponding to the changes in viscosity increase and decrease modes with increasing shear rate.

2. Rheological model

The review of the known experimental data, presented, for example, in [1-4], shows that fine particles suspensions in predetermined combination of their parameters (size, shape, concentration) and rheological characteristics of a viscous liquid base demonstrate a rather complex behavior.

As a rule, the dependence of the shear stress $\tau$ and viscosity $\mu$ on the shear rate $\dot{\gamma}$ is qualitatively described as follows.

At relatively low shear rate values, the continuum behaves like a pseudoplastic fluid. Moreover, its viscosity decreases monotonically as the shear rate increases, starting from the level $\mu_0$ at $\dot{\gamma} = 0$.

However, having reached the minimum level $\mu = \mu_{\text{min}}$ at the certain shear rate value $|\dot{\gamma}| = \dot{\gamma}_{\text{min}}$, the viscosity begins to increase and the suspension mechanical behavior begins to correspond to the dilatant fluid behavior. In these two ranges of the shear rate change, the viscosity obviously changes, but in large it takes on values of approximately one order from the certain range $\mu_{\text{min}} < \mu < \mu_{\text{s}}$.

Subsequently, when the shear rate exceeds a certain threshold level $|\dot{\gamma}| = \dot{\gamma}_{s}$, the viscosity continues to increase, but this increase rate rises sharply. As shown by the known experimental data, for example, from [2], in this range, the viscosity can increase by several orders of magnitude.

In some cases, the viscosity increase becomes so significant that the suspension begins to behave like a solid. In such a situation, we can talk about the fluid "solidification" or the "solidification" effect manifestation.

Having finally reached the certain maximum level $\mu = \mu_{\text{max}}$ at the corresponding shear rate value $|\dot{\gamma}| = \dot{\gamma}_{\text{max}}$, its further increase leads to a decrease in viscosity.

Here and elsewhere, viscosity is understood as the following quantity

$$\mu(\dot{\gamma}) = \frac{d[\tau(\dot{\gamma})]}{d[\dot{\gamma}]}.$$  \hfill (2.1)

Note that in case of the shear stress and the shear rate linear dependence, which corresponds to the Newtonian fluid, the latter expression immediately leads to the traditional dynamic viscosity.

Let us assume that the suspension fluid component is a viscoplastic fluid. Then, taking into account the above-mentioned suspensions features based on a liquid component, showing only viscous behavior,
we arrive at the following representation for the dependences of the shear stress and viscosity on the shear rate, which are schematically shown in figure 1.

![Figure 1](image)

**Figure 1.** Dependences of the shear stress (a) and viscosity (b) on the shear rate.

It is quite acceptable to describe the flow curve (the shear stress and the shear rate dependence) with one function, but it doesn’t seem entirely rational. This is due to the fact that, as can be expected, such function will have a rather complex form. In turn, this will lead to difficulties in obtaining explicit analytical solutions, even for relatively simple hydrodynamics problems.

In this regard, taking into account the occurrence of characteristic sections of different mechanical behavior on the flow curve, it is proposed to take a rheological model of suspensions in the following form

\[
|\tau| = \begin{cases} 
\tau_1 + k_1 \cdot (|\dot{\gamma}| + \dot{\gamma}_j)^{n_1}; & 0 \leq |\dot{\gamma}| \leq \dot{\gamma}_\text{min}; \\
\tau_2 + k_2 \cdot (|\dot{\gamma}| - \dot{\gamma}_j)^{n_2}; & \dot{\gamma}_\text{min} \leq |\dot{\gamma}| \leq \dot{\gamma}_s; \\
\tau_3 - k_3 \cdot (\dot{\gamma}_3 - |\dot{\gamma}|)^{n_3}; & \dot{\gamma}_s \leq |\dot{\gamma}| \leq \dot{\gamma}_\text{max}; \\
\tau_4 + k_4 \cdot (|\dot{\gamma}| - \dot{\gamma}_4)^{n_4}; & |\dot{\gamma}| \geq \dot{\gamma}_\text{max};
\end{cases}
\]  

(2.2)

Where

\[
\tau_j, \quad k_j > 0, \quad \dot{\gamma}_j, \quad n_j, \quad j = 1, 2, 3, 4;
\]

(2.3)

are the rheological model parameters, some of which are restricted by

\[
\tau_1 < \tau_p; \quad \tau_4 < \tau_\text{max} < \tau_3; \quad \dot{\gamma}_1 > 0; \quad \dot{\gamma}_2 < \dot{\gamma}_\text{min}; \quad \dot{\gamma}_4 < \dot{\gamma}_\text{max} < \dot{\gamma}_3; \\
0 < n_1 < 1; \quad n_2 > 1; \quad 0 < n_3 < 1; \quad 0 < n_4 < 1.
\]

(2.4)
In addition to these restrictions, the model relation (2.2) assumes that the parameters (2.3) must satisfy the following conditions

\[ \tau_p = \tau_1 + k_1 \cdot (\dot{\gamma})^{n_1}; \]
\[ \tau_{\text{min}} = \tau_1 + k_1 \cdot (\dot{\gamma}_{\text{min}} + \dot{\gamma})^{n_1} = \tau_2 + k_2 \cdot (\dot{\gamma}_{\text{min}} - \dot{\gamma})^{n_2}; \]
\[ k_1 \cdot n_1 \cdot (\dot{\gamma}_{\text{min}} + \dot{\gamma})^{n_1-1} = k_2 \cdot n_2 \cdot (\dot{\gamma}_{\text{min}} - \dot{\gamma})^{n_2-1}; \]
\[ \tau_s = \tau_2 + k_2 \cdot (\dot{\gamma}_s - \dot{\gamma})^{n_2} = \tau_3 - k_3 \cdot (\dot{\gamma}_3 - \dot{\gamma}_s)^{n_3}; \]
\[ k_2 \cdot n_2 \cdot (\dot{\gamma}_s - \dot{\gamma})^{n_2-1} = k_3 \cdot n_3 \cdot (\dot{\gamma}_3 - \dot{\gamma}_s)^{n_3-1}; \]
\[ \tau_{\text{max}} = \tau_3 - k_3 \cdot (\dot{\gamma}_3 - \dot{\gamma}_{\text{max}})^{n_3} = \tau_4 + k_4 \cdot (\dot{\gamma}_{\text{max}} - \dot{\gamma}_4)^{n_4}; \]
\[ k_3 \cdot n_3 \cdot (\dot{\gamma}_3 - \dot{\gamma}_{\text{max}})^{n_3-1} = k_4 \cdot n_4 \cdot (\dot{\gamma}_{\text{max}} - \dot{\gamma}_4)^{n_4-1}. \]

In relations (2.4), (2.5), the parameters \( \tau_p, \tau_{\text{min}}, \tau_s, \tau_{\text{max}} \) are the shear stress modulus values characteristic of the given suspension.

In particular, \( \tau_p \) is the shear stress, which is determined by the plastic properties of the suspension fluid component. The parameters \( \tau_{\text{min}} \) and \( \tau_{\text{max}} \) correspond to the shear stresses at which the viscosity reaches, respectively, the minimum and maximum values.

Another parameter \( \tau_s \) defines the shear stress at a certain shear rate \( |\dot{\gamma}| = \dot{\gamma}_s \). This value belongs to the range \( \dot{\gamma}_{\text{min}} \leq |\dot{\gamma}| \leq \dot{\gamma}_{\text{max}} \) of monotonic increase in suspension viscosity from the minimum value \( \mu_{\text{min}} \) to the maximum level \( \mu_{\text{max}} \). In its meaning, such shear rate value \( \dot{\gamma}_s \) determines the level above which a significant increase in the suspension viscosity is observed.

Note that, in a sense, the choice of the value \( \dot{\gamma}_s \), and its corresponding value \( \tau_s \), is not univalent and is not entirely clear-cut. In this regard, let us take such shear rate value \( \dot{\gamma} \in [\dot{\gamma}_{\text{min}}, \dot{\gamma}_{\text{max}}] \) in place of \( \dot{\gamma}_s \) at which the suspension viscosity exceeds its initial value \( \mu_{\theta} \) (or, for example, the minimum value \( \mu_{\text{min}} \)) by a predetermined value.

Another possible choice of the value \( \dot{\gamma}_s \) can be associated with the choice of the inflection point on the flow curve in the shear rate variation range \( \dot{\gamma} \in [\dot{\gamma}_{\text{min}}, \dot{\gamma}_{\text{max}}] \).

Note that the fulfillment of the conditions (2.5) ensures continuous differentiability for the shear stress and the shear rate dependence (2.2). As this takes place, only the continuity condition is satisfied for the viscosity and the shear rate dependence.

One of the features of the model (2.2) is that its corresponding flow curve has two inflection points at \( |\dot{\gamma}| = \dot{\gamma}_{\text{min}} \) and \( |\dot{\gamma}| = \dot{\gamma}_{\text{max}} \).

3. A rheological model with the "solidification" effect

Within the framework of the rheological model proposed above, a special case can be considered, which involves taking into account the "solidification" effect manifestation.

This effect should correspond to a significant increase in viscosity (by several orders of magnitude) as the shear rate approaches a certain critical value. Let us take \( \dot{\gamma}_{\text{max}} \) corresponding to the maximum viscosity value in place of such shear rate critical value.

Then, taking into account the viscosity representation by the relation (2.1), at least at the model level, the condition for the "solidification" effect manifestation can be represented in the form
\[
\lim_{\dot{\gamma} \to \dot{\gamma}_{\text{max}}} \left( \frac{d[\tau(\dot{\gamma})]}{d[\dot{\gamma}]} \right) = \infty. \tag{3.1}
\]

This result is immediate from the model (2.2) considered above, if we assume that for the corresponding model parameters, the following relations are satisfied:
\[
\tau_3 = \tau_4 = \tau_{\text{max}}; \quad \dot{\gamma}_3 = \dot{\gamma}_4 = \dot{\gamma}_{\text{max}}. \tag{3.2}
\]

Note that relations (3.2) act as the last two conditions in (2.5). In this case, at the model level, the viscosity and the shear rate dependence, taking into account (3.1), has a discontinuity of the second kind at the point \( |\dot{\gamma}| = \dot{\gamma}_{\text{max}} \).

4. The rheological model generalization

The shear stress and the shear rate rheological dependences, in particular, of the type (2.2), can be used to solve only one-dimensional problems of hydrodynamics. In this regard, it is of interest to generalize the rheological model (2.2) to the higher dimensional flows case, which in such situation can be represented in the form

\[
\tau_{ij} = -P \cdot \delta_{ij} + 2 \cdot \varphi(I_2) \cdot \varepsilon_{ij}; \quad i, j = 1, 2, 3; \tag{4.1}
\]

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right); \quad I_2 = \varepsilon_{11} \cdot \varepsilon_{22} + \varepsilon_{33} \cdot \varepsilon_{11} - \varepsilon_{12}^2 - \varepsilon_{23}^2 - \varepsilon_{31}^2; \tag{4.2}
\]

\[
\varphi(I_2) = \begin{cases}
0.5 \cdot |I_2|^{-0.5} \cdot \tau_1 + k_1 \cdot \left[ 2 \cdot \left( \sqrt{|I_2|} + \sqrt{|I_2, 1|} \right) \right]^{q_1}, & 0 \leq |I_2| \leq I_{2, \text{min}}; \\
0.5 \cdot |I_2|^{-0.5} \cdot \tau_2 + k_2 \cdot \left[ 2 \cdot \left( \sqrt{|I_2|} - \sqrt{|I_2, 2|} \right) \right]^{q_2}, & I_{2, \text{min}} \leq |I_2| \leq I_{2, s}; \\
0.5 \cdot |I_2|^{-0.5} \cdot \tau_3 - k_3 \cdot \left[ 2 \cdot \left( \sqrt{|I_2, 3|} - \sqrt{|I_2|} \right) \right]^{q_3}, & I_{2, s} \leq |I_2| \leq I_{2, \text{max}}; \\
0.5 \cdot |I_2|^{-0.5} \cdot \tau_4 + k_4 \cdot \left[ 2 \cdot \left( \sqrt{|I_2|} - \sqrt{|I_2, 4|} \right) \right]^{q_4}, & |I_2| \geq I_{2, \text{max}}.
\end{cases}
\]

Where \( \tau_{ij}, \varepsilon_{ij} \) are the stress tensors and strain rates components; \( P \) is the hydrostatic pressure; \( u_i \) are the velocity vector components; \( \delta_{ij} \) is the Kronecker symbol; \( I_2 \) is the strain rate tensor second invariant; \( \varphi(I_2) \) is a function of the second strain rate tensor invariant, that characterizes the suspension transverse viscosity; \( I_{2,1}, I_{2,2}, I_{2,3}, I_{2,4}, I_{2,\text{min}}, I_{2,s}, I_{2,\text{max}} \) are the rheological model parameters, which in their value are similar, respectively, to the parameters \( \dot{\gamma}_1, \dot{\gamma}_2, \dot{\gamma}_3, \dot{\gamma}_4, \dot{\gamma}_{\text{min}}, \dot{\gamma}_s, \dot{\gamma}_{\text{max}} \) within the model framework (2.2).

In the particular case of a one-dimensional flow, when
\[
I_2 = -0.25 \cdot \dot{\gamma}^2,
\]
relations (4.1) with allowance for (4.2) are immediately reduced to the expression (2.2) for the shear stress.
The rheological model type (4.2) predetermines the methodology for solving hydrodynamic problems for a viscoplastic suspension of fine particles.

The main feature of this method is the obligatory division of the flow region, in the most general case, into five separate zones. One of these zones must correspond to the plastic flow. In each of the remaining four zones, a shear flow corresponding to one of the ranges (4.2) of the strain rate tensor second invariant modulus variation must be realized.

In this case, the shear flow zones number is not fixed. It can vary from one to four zones.

In each specific case, the number of realized zones is determined by the flow mode corresponding parameters.

For example, for a flow in a channel, the pressure drop along its length can act as a flow mode parameter. In this case, in addition to the plastic flow zone, a shear flow zone will correspond to relatively small pressure drop values. If the pressure drop exceeds a certain threshold level, it will lead to the formation of the shear flow second zone. Further, as the pressure drop increases, the number of the realized shear flow zones will also increase. An example of this approach implementation, but for the case of a simpler rheological model, is given in [10].

Another feature of this method is that the separate spatial zones boundaries are not known in advance and must be determined in the course of solving the hydrodynamic problems under consideration.

Speaking about the formulation of the fine particles viscoplastic suspension hydrodynamic problems that satisfies the model of type (4.2), let us note the following. The possibility of the existence of separate zones with different mechanical behavior in the flow region presupposes, along with the traditional boundary conditions (at the flow region outer boundaries), the setting of additional boundary conditions at the common boundaries of these zones. Such boundary conditions should ensure the velocity and stress fields “stitching”.

5. Conclusions

The proposed rheological model allows one to describe the mechanical behavior of the fine particles suspensions, the fluid component of which is a viscoplastic medium, over a wide range of the shear rate variation.

Such model takes into account the sequential alternation of the three sections as the shear rate increases: a decrease, an increase, and then again a decrease in viscosity.

The model configuration allows obtaining solutions of one-dimensional hydrodynamics problems in an analytical form. In this case, the model admits a special case of the “solidification” effect manifestation, when the viscosity increases so significantly (by several orders of magnitude) that at the model level it is permissible to assume $\mu_{\text{max}} \rightarrow \infty$.

The proposed generalization of the considered rheological model to the flows can be used when considering hydrodynamics problems in a higher dimension.

The main features of solving hydrodynamic problems for viscoplastic suspensions of fine particles were shown.

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