ERROR ESTIMATES FOR DISCRETE APPROXIMATIONS OF GAME OPTIONS WITH MULTIVARIATE DIFFUSION ASSET PRICES

YURI KIFER

INSTITUTE OF MATHEMATICS
HEBREW UNIVERSITY
JERUSALEM, ISRAEL

To the memory of Hiroshi Kunita

Abstract. We obtain error estimates for strong approximations of a diffusion with a diffusion matrix $\sigma$ and a drift $b$ by the discrete time process defined recursively

$$X_N\left(\frac{(n+1)}{N}\right) = X_N\left(\frac{n}{N}\right) + N^{-1/2}\sigma(X_N\left(\frac{n}{N}\right))\xi(n+1) + N^{-1}b(X_N\left(\frac{n}{N}\right)),$$

where $\xi(n), n \geq 1$ are i.i.d. random vectors, and apply this in order to approximate the fair price of a game option with a diffusion asset price evolution by values of Dynkin’s games with payoffs based on the above discrete time processes. This provides an effective tool for computations of fair prices of game options with path dependent payoffs in a multi asset market with diffusion evolution.

1. Introduction

In the present paper we continue the line of research in [12] and [13] approximating game options whose stocks evolutions are described by multidimensional diffusion processes. This is done constructing first strong approximations of the diffusion by a sequence of discrete time processes estimating $L^2$ errors of these approximations. These processes are discrete time processes obtained recursively for $n = 0, 1, ..., N - 1$ by

$$X_N\left(\frac{(n+1)}{N}\right) = X_N\left(\frac{n}{N}\right) + N^{-1/2}\sigma(X_N\left(\frac{n}{N}\right))\xi(n+1) + N^{-1}b(X_N\left(\frac{n}{N}\right)),$$

where $X_N(0) = x_0$, $\sigma$ and $b$ is a matrix and a vector functions, respectively and $\xi(n), n \geq 1$ is a sequence of i.i.d. random vectors with $E\xi(1) = 0$. The strong approximation method enables us to redefine both the sequence $\xi(n), n \geq 1$ and the limiting diffusion $d\Xi(t) = \sigma(\Xi(t))dW(t) + b(\Xi(t))dt$ preserving their distributions on a same sufficiently rich probability space so that the $L^2$-distance between them have the order $N^{-\delta}$ for some $\delta > 0$. In the second step we compare fair prices of options with payoffs based on these discrete approximations with the fair price of the option with the diffusion asset evolution. This is not straightforward since this

Date: December 28, 2021.

2000 Mathematics Subject Classification. Primary: 91G20 Secondary: 60F15, 60G40, 91A05.

Key words and phrases. game options, strong diffusion approximation, dynamical programming Dynkin game.
prices are given by values of the corresponding Dynkin games which depend on sets of stopping times involved and the latter are different for the approximations and for the limiting diffusion. The payoffs of the above game options are supposed to be path dependent, and so free boundary partial differential equations methods cannot help here and discrete time approximations is the only possible approach in this situation to fair price computations taking into account that in the discrete time case we can employ the dynamical programming (backward recursion) algorithm.

The setup of this paper is the special case of a more general discrete time setup in our recent paper [15] but the latter paper provides a detailed proof mostly in the continuous time averaging setup while here we deal with the more specific discrete time setup which can be described in the more transparent way. The motivation both for [15] and for the present paper comes, in particular, from the series of papers [11], [1], [2] and [7] on the weak diffusion limit in averaging and from the series of papers [4], [16], [18] and [9] on strong approximations (see [15] for more details). The former papers yielded only weak convergence results while the latter dealt only with approximations of the Brownian motion. We observe that in the one dimensional case it is still possible to use an extended version of the Skorokhod embedding into martingales which also yields error estimates for approximations (see [3]). Approximation similar to ours appeared previously in [10] but only the weak convergence to a diffusion was established there which, in principle, could not provide any error estimates. The reader may compare our approach with the well known Euler–Maruyama approximation of solutions of stochastic differential equations (see, for instance, [17]) where $\xi(n)$’s are increments of the Brownian motion. In our setup $\xi(n)$’s are quite general and, in particular, we can take i.i.d. random vectors taking on only few values which can be useful in applications since they are easier to simulate and compute than Gaussian random vectors.

2. Preliminaries and main results

We start with a complete probability space $(\Omega, \mathcal{F}, P)$, a sequence of independent identically distributed (i.i.d.) random vectors $\xi(n)$, $n \geq 1$ and a diffusion process $\Xi$ solving the stochastic differential equation

$$d\Xi(t) = \sigma(\Xi(t))dW(t) + b(\Xi(t))dt$$

where $W$ is the $d$-dimensional continuous Brownian motion while $\sigma$ and $b$ are bounded Lipschitz continuous $d \times d$ matrix and $d$-dimensional vector functions, respectively. Namely, we assume that for some constant $L \geq 1$ and all $x, y \in \mathbb{R}^d$,

$$|\sigma(x)| \leq L, \quad |b(x)| \leq L, \quad |\sigma(x) - \sigma(y)| \leq L|x - y|, \quad |b(x) - b(y)| \leq L|x - y|$$

where $| \cdot |$ denotes the Euclidean norm of a vector or of a matrix. We assume also that

$$E\xi(1) = 0, \quad E(\xi(1)\xi_j(1)) = \delta_{ij} \text{ and } |\xi(1)| \leq L \text{ almost surely (a.s.)}$$

where $\xi(n) = (\xi_1(n), ..., \xi_d(n))$ and $\delta_{ij}$ is the Kronecker delta. Next, we consider the sequence of discrete time processes $X_N$, $N \geq 1$ on $\mathbb{R}^d$ defined recursively for $n = 0, 1, ..., N - 1$ by

$$X_N((n+1)/N) = X_N(n/N) + N^{-1/2}\sigma(X_N(n/N))\xi(n+1) + N^{-1}b(X_N(n/N))$$

where $X_N(0) = \Xi(0) = x_0$ is fixed. We extend $X_N$ to the continuous time setting

$$X_N(t) = X_N(n/N) \quad \text{if} \quad n/N \leq t < (n + 1)/N$$
and without loss of generality we will assume that all our processes evolve on the
time interval $[0, 1]$ so that $n$ runs in (2.5) from 0 to $N$. The following result will be
proved in Sections 3 and 4.

2.1. **Theorem.** Suppose that the conditions (2.2) and (2.3) hold true and that
the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough so that there exist a sequence of
i.i.d. uniformly distributed random variables defined on it. Then for each integer
$N \geq N_0 = (10^s d^{34} + 1)^4$ there exists a $d$-dimensional Brownian motion
$W = W_N$ such that the strong solution $\Xi = \Xi_N$ of the stochastic differential equation (2.1)
with such $W$ and the initial condition $X_N(0) = \Xi(0) = x_0$ satisfies

$$(2.6) \quad E \sup_{0 \leq t \leq 1} |X_N(t) - \Xi(t)|^2 \leq C_0 [N^{4\frac{2}{3}}]^{-\frac{2}{3}}$$

where $C_0 = C_3 e^{C_1} + 2L^2 (L^2 + 1) + 40L^2$ with $C_3$ and $C_4$ defined at the end of
Section 3. In particular, the Prokhorov distance between the path distributions of
$X_N$ and of $\Xi$ is bounded by $C_0 [N^{4\frac{2}{3}}]^{-\frac{2}{3}}$.

We observe that though we may have to redefine the Brownian motion $W$ for
each $N$ separately the path distribution of the diffusion $\Xi$ remains the same since
it is continuous and the coefficients of the stochastic differential equation (2.1)
do not change, and so we have all the time the same Kolmogorov equation and the
same martingale problem (see [20]). Clearly, the estimate (2.6) is meaningful only
for large $N$ and we provide it for all $N \geq N_0$ though, of course, an explicit estimate in (2.6)
can be obtained also when $1 \leq N \leq N_0$ taking into account that then $|X_N(t)| \leq L(\sqrt{N_0}L + 1)$ and $E \sup_{0 \leq t \leq 1} |\Xi(t)|^2 \leq 5L^2$. Theorem 2.1 can also
be derived under some moment boundedness conditions rather than the uniform
bounds in (2.2) which are assumed to reduce technicalities in our exposition. Observe
also that if we consider time dependent coefficients $\sigma(t, x)$ and $b(t, x)$ Lipschitz
continuous in both variables and take $\sigma(n/N, X_N(n/N))$ and $b(n/N, X_N(n/N))$ in
(2.4) in place of $\sigma(X_N(n/N))$ and $b(X_N(n/N))$, then we will obtain $X_N(t)$ which
approximates a time inhomogeneous diffusion with coefficients $\sigma(t, x)$ and $b(t, x)$
having essentially the same error estimates as in (2.6).

Next, we will describe an application of our results to computations of values of
Dynkin’s optimal stopping games and fair prices of game options with the payoff
function having the form

$$(2.7) \quad R_{\Xi}(s, t) = G_s(\Xi) 1_{s < t} + F_t(\Xi) 1_{t \leq s}$$

where $\Xi$ is a diffusion solving the stochastic differential equation (2.1). Here, $G_t \geq F_t$ and both are functionals on paths for the time interval $[0, t]$ satisfying certain
regularity conditions specified below. Thus, if the first player stops at the time $s$
and the second one at the time $t$ then the former pays to the latter the amount $R_{\Xi}(s, t)$.
The game runs until the termination time 1 when the game stops automatically,
if it was not stopped before, and then the first player pays to the second one the
amount $G_1(\Xi) = F_1(\Xi)$. Clearly, the first player tries to minimize the payment
while the second one tries to maximize it. Under the conditions below this game
has the value (see, for instance, Section 6.2.2 in [14]),

$$(2.8) \quad V_{\Xi} = \inf_{\sigma \in \mathcal{T}_{\Xi}^F} \sup_{\tau \in \mathcal{T}_{\Xi}^F} E R_{\Xi}(\sigma, \tau)$$
where $T_{01}^t$ is the set of all stopping times $0 \leq \tau \leq 1$ with respect to the filtration $\mathcal{F}_t^\Xi$, $t \geq 0$ generated by the diffusion $\Xi$ or, which is the same, generated by the Brownian motion $W$.

When we are talking about asset prices then usually it is assumed that they are nonnegative, and so a diffusion with bounded coefficients maybe not a good model for a description of evolution of these prices. It maybe more appropriate to assume that the asset prices evolve according to the vector process described by exponents $(\exp(\Xi^i(t)), i = 1, \ldots, d)$ where $\Xi^{(1)}, \ldots, \Xi^{(d)}$ are components of the vector $\Xi$. Nevertheless, it will be more convenient for us to speak about the diffusion $\Xi$ itself and to impose conditions on the payoff functionals $F$ and $G$ such that exponential functionals will be allowed which will amount to the same effect as exponents describing the evolution of asset prices. Another important point that the fair price of a game option equals the value of the corresponding Dynkin optimal stopping game considered with respect to the equivalent martingale measure, i.e. with respect to the probability for which the assets evolution is described by a martingale (see [14]) provided that the interest rate is supposed to be zero. When the asset prices are considered with respect to the equivalent martingale measure, i.e. with respect to the probability for which the assets evolution is described by a martingale (see [14]),

$$\frac{dQ}{dP} = \Lambda \text{ where } \Lambda = \exp(-\int_0^1 \langle \zeta(s), dW(s) \rangle - \frac{1}{2} \int_0^1 |\zeta(s)|^2 ds)$$

where $\langle \cdot, \cdot \rangle$ is the inner product and the vector process $\zeta(s)$ satisfies

$$\sigma(\Xi(s))\zeta(s) = b(\Xi(s)) + \frac{1}{2} \eta(\Xi(s)) \text{ and } \eta = (\eta_1, \ldots, \eta_d), \eta_i(x) = \sum_{i=1}^d \sigma_{ij}^2(x).$$

Since our estimates do not depend explicitly on the probability measure once the setup above is preserved and since there is no one preferable stock evolution model here, we will not discuss this point further, and so, strictly speaking, we will deal with the approximation of the Dynkin game value $V^\Xi$ and not of the fair price of the corresponding game option, i.e. we will make estimates with respect to the probability $P$ and not with respect to an equivalent martingale measure which depends on a choice of the stock evolution model.

We assume that $F_t$ and $G_t$, $t \in [0,1]$ are continuous functionals on the space $M_d[0,t]$ of bounded Borel measurable maps from $[0,t]$ to $\mathbb{R}^d$ considered with the uniform metric $d_{0\theta}(v,\tilde{v}) = \sup_{0 \leq s \leq t} |v_s - \tilde{v}_s|$ and there exists a constant $K > 0$ such that

$$|F_t(v) - F_t(\tilde{v})| + |G_t(v) - G_t(\tilde{v})|$$

$$\leq K(d_{0\theta}(v,\tilde{v}) + \|\sup_{0 \leq s \leq t} |v_s - \tilde{v}_s| > 1\|) \exp(K \sup_{0 \leq u \leq t} (|v_u| + |\tilde{v}_u|))$$

and

$$|F_t(v) - F_s(v)| + |G_t(v) - G_s(v)| \leq K(|t-s| + \sup_{0 \leq u \leq t} |v_u - \tilde{v}_u|) \exp(K \sup_{0 \leq u \leq t} |v_u|).$$
Next, we will consider Dynkin’s games with payoffs based on the process $X_N$, (2.11) $R_N(s, t) = G_s(X_N)1_{s < t} + F_t(X_N)1_{t \leq s}$.

Denote by $\mathcal{F}^\xi_{mn}$, $m \leq n$ the $\sigma$-algebra generated by $\xi(m), ..., \xi(n)$ and let $\mathcal{T}^\xi_{mn}$ be the set of all stopping times with respect to the filtration $\mathcal{F}^\xi_{0k}$, $k \geq 0$ taking on values $m, m + 1, ..., n$. We allow also any stopping time to take on the value $\infty$, i.e. we allow players not to stop the game at all, but anyway the game is stopped automatically at the termination time $1$ and then the first player pays to the second one the amount $G_1(X_N) = F_1(X_N)$. Now the game value of the Dynkin game in this setup is given by

$$V_N = \inf_{\xi \in \mathcal{T}^\xi_{0N}} \sup_{\eta \in \mathcal{T}^\xi_{0N}} E R_N(\xi/N, \eta/N).$$

2.2. **Theorem.** Suppose that the conditions (2.9) and (2.10) as well as the conditions of Theorem 2.1 hold true. Then for each $\delta > 0$ there exists $C_\delta > 0$ such that for any integer $N \geq N_0$,

$$|V^\Xi - V_N| \leq C_\delta [N^{\delta \over 2 - 1 \over 10 \delta}],$$

where $C_\delta$ does not depend on $N$ and for each $\delta$ it can be estimated explicitly from the proof in Section 3.

Since we use in Theorem 2.2 a specific construction of the diffusion $\Xi$ from Theorem 2.1 it is important to note that the game value $V^\Xi$ depends only on the path distribution of $\Xi$, i.e. only on the diffusion coefficients $\sigma$ and $b$, and not on a choice of the Brownian motion in the stochastic differential equation (2.1) (see [8]). We observe also that the main advantage in computation $V^\Xi$ in comparison to $V_N$ is the possibility to use the dynamical programming (backward recursion) algorithm. Namely, set $V_{N0} = F_1(X_N)$ and recursively for $n = N - 1, ..., 1, 0$,

$$V_{Nn} = \min \{G_{n/N}(X_N), \max(F_{n/N}(X_N), E(V_{N,n+1}|\mathcal{F}^\xi_{0,n}))\}.$$

Then $V_{N0} = V_N$ (see, for instance, Section 6.2.2 in [13]). Of course, the computation of conditional expectations above becomes complicated if the $\sigma$-algebras $\mathcal{F}^\xi_{0n}$ are big but if we choose independent random vectors $\xi(n)$ in (2.4) taking on only few values then these $\sigma$-algebras contain not so many sets and the conditional expectations can be computed easily. Observe also that in the particular case when the diffusion $\Xi$ is just a multidimensional Brownian motion, a result similar to Theorem 2.2 was obtained in [13] where it was sufficient to consider the standard normalized sums of random vectors $\xi(n)$ rather than the more subtle case of difference equations (2.4).

3. **Auxiliary estimates**

Set $n_k = k[N^{\delta \over 2}], k = 0, 1, ..., k_N$ where $k_N = [N^{1 \over [N^{\delta \over 2}]}]$ where $[\cdot]$ denotes the integral part. Define

$$\hat{X}_N(t) = x_0 + N^{-1/2}\sum_{0 \leq k \leq k_N(t)} (\sigma(X_N(\frac{n_k}{N})) \sum_{n_k < t \leq n_{k+1} \wedge [Nt]} \xi(l)) + N^{-1/2}b(X_N(\frac{n_k}{N}))(n_{k+1} \wedge [Nt] - n_k)$$

where $k_N(t) = \max\{k : n_k \leq Nt\}$.

3.1. **Lemma.** For any $N \geq 1$,

$$E \sup_{0 \leq \xi \leq 1} |X_N(t) - \hat{X}_N(t)|^2 \leq 136L^8N^{-1/2}.$$
Proof. First, we write

\begin{equation}
|X_N(t) - \bar{X}_N(t)|^2 \leq 2|M(t)|^2 + 2|J(t)|^2
\end{equation}

where

\[ M(t) = N^{-1/2} \sum_{0 \leq k < k_N(t), n_k < t \leq n_{k+1} \wedge [Nt]} (\sigma(X_N(\frac{l}{N})) - \sigma(X_N(\frac{n_k}{N})))\xi(l + 1) \]

and

\[ J(t) = N^{-1} \sum_{0 \leq k < k_N(t), n_k < t \leq n_{k+1} \wedge [Nt]} (b(X_N(\frac{l}{N})) - b(X_N(\frac{n_k}{N}))). \]

Recall that if \( h = h(x,y) \) is a bounded Borel function, \( G \subset F \) is a \( \sigma \)-algebra and \( Y,Z \) are random variables such that \( Y \) is \( G \)-measurable and \( Z \) is independent of \( G \), then \( E(h(Y,Z)|G) = g(Y) \) where \( g(x) = Eh(x,Z) \). It follows from here and from (3.3) that \( M(t), 0 \leq t \leq 1 \) is a martingale. Hence, by (3.3) and the Doob martingale inequality (see, for instance, Section 6.1.2 in [14]),

\begin{equation}
E\sup_{0 \leq t \leq 1} |M(t)|^2 \leq 4E|M(1)|^2
\end{equation}

\[ = 4N^{-1} \sum_{0 \leq k < k_N(t)} \sum_{n_k < t \leq n_{k+1} \wedge [Nt]} E[|\sigma(X_N(\frac{k}{N})) - \sigma(X_N(\frac{n_k}{N}))|]\xi(l + 1)|^2. \]

By (2.2), (2.3) for \( n_k < l \leq n_{k+1} \),

\begin{equation}
E[|\sigma(X_N(\frac{k}{N})) - \sigma(X_N(\frac{n_k}{N}))|]\xi(l + 1)|^2 \leq L^4 E|X_N(\frac{k}{N}) - X_N(\frac{n_k}{N})|^2
\end{equation}

\[ \leq 2L^4(N^{-1}E|\sum_{n_k \leq m < l} \sigma(X_N(m/N))\xi(m+1)|^2
\]

\[ + N^{-2}E|\sum_{n_k \leq m < l} b(X_N(m/N))\xi(m+1)|^2) \leq 16L^8N^{-1/2}. \]

Now, by (2.4) and (2.3),

\[ E\sup_{0 \leq t \leq 1} |J(t)|^2 \leq \sum_{0 \leq k < k_N(t)} \sum_{n_k < t \leq n_{k+1} \wedge N} E|X_N(\frac{l}{N}) - X_N(\frac{n_k}{N})|^2 \]

and for \( n_k < l \leq n_{k+1} \),

\[ |X_N(\frac{l}{N}) - X_N(\frac{n_k}{N})| \leq N^{-1/2}(L^2 + N^{-1/2}L)[N^\frac{1}{2}] \leq 2L^2N^{-\frac{1}{2}}. \]

These together with (3.3), (3.5) yield (3.2). \hfill \Box

Next, we estimate the characteristic function of a sum of independent random vectors which is well known but for completeness and in order to provide explicit constants we provide the details.

3.2. Lemma. For any integer \( n \geq 1 \) and \( x \in \mathbb{R}^d \),

\begin{equation}
|f_n(x,w) - \exp(-\frac{1}{2}\langle A(x)w,w \rangle)| \leq C_1n^{-\varphi}
\end{equation}

for all \( w \in \mathbb{R}^d \) with \( |w| \leq n^{\varphi/2} \) where \( A(x) = \sigma(x)\sigma^*(x) \),

\[ f_n(x,w) = E\exp(i\langle w, n^{-1/2}\sigma(x)\sum_{0 \leq i \leq n} \xi(i) \rangle), \]

\( \varphi = 1/6 \) and \( C_1 = \frac{\varphi}{2}L^6. \)
Proof. Set \( m_j = j[\sqrt{n}], j = 0, 1, ..., m(n), m(n) = \max\{j : j[\sqrt{n}] \leq n\}, y_j = \sigma(x) \sum_{m_j < l \leq m_{j+1} \land n} \xi(l) \) and \( \eta_j = \langle w, n^{-1/2}y_j \rangle \). Now we have

\[
|f_n(x, w) - \exp(-\frac{1}{2}(A(x)w, w))| \leq I_1 + I_2
\]

where

\[
I_1 = |E \exp(i \sum_{0 \leq j \leq m(n)} \eta_j) - \prod_{0 \leq j \leq m(n)} \exp(i \eta_j)| = 0,
\]

since \( \eta_j, j = 1, ..., m(n) + 1 \) are independent random variables, and

\[
I_2 = |\prod_{0 \leq j \leq m(n)} \exp(-\frac{1}{2}(A(x)w, w))| \leq \sum_{0 \leq j \leq m(n)} |\exp(-\frac{m_{j+1} \land n - m_j}{2n}(A(x)w, w))|
\]

where we use that

\[
|\prod_{1 \leq j \leq l} a_j - \prod_{1 \leq j \leq l} b_j| \leq \sum_{1 \leq j \leq l} |a_j - b_j|
\]

whenever \( 0 \leq |a_j|, |b_j| \leq 1, j = 1, ..., l \).

Using (2.3) and the inequalities

\[
|e^{ia} - 1 - ia + \frac{a^2}{2}| \leq |a|^3 \quad \text{and} \quad |e^{-a} - 1 + a| \leq a^2 \quad \text{if} \quad a \geq 0,
\]

we obtain that

\[
|\exp(i \eta_j) - \exp(-\frac{m_{j+1} \land n - m_j}{2n}(A(x)w, w))| \leq \frac{1}{2}|\exp(-\frac{m_{j+1} \land n - m_j}{n}(A(x)w, w)) + \exp(i \eta_j)|^3 + \frac{1}{4n}|\langle A(x)w, w \rangle|^2.
\]

Now, by (2.3) and the independency of \( \xi(l) \)’s,

\[
E \eta_j^2 = n^{-1} \sum_{m_j < l \leq m_{j+1} \land n} E\langle w, \sigma(x)\xi(l) \rangle^2 = n^{-1}(m_{j+1} \land n - m_j)(A(x)w, w).
\]

Hence,

\[
I_2 \leq (\sqrt{n} + 1)(n^{-3/2}L^6 |w|^6 + \frac{1}{4}n^{-1}L^4 |w|^4)
\]

and (3.6) follows. \( \square \)

Set \( Y_{N,k}(x) = \sigma(x) \sum m_k < l \leq m_{k+1} \land n \xi(l) \) for \( k = 0, 1, ..., k_N - 1 \) and \( Y_{N,k_N}(x) = \sigma(x) \sum m_{k_N} < l \leq N \xi(l) \). As a corollary of Lemma 3.2 we obtain

3.3. Lemma. For any integer \( N \geq 1 \) and \( k = 0, 1, ..., k_N - 1, \)

\[
|E \exp(i \langle w, (n_{k+1} - n_k)^{-1/2}Y_{N,k}(X_N(n_k/N)) \rangle) | \mathcal{F}_{0n_k}^\xi \rangle - g_{x_k}(\frac{w}{n_k})(w) \rangle \leq C_1(n_{k+1} - n_k)^{-\psi}
\]

for all \( w \in \mathbb{R}^d \) with \( |w| \leq (n_{k+1} - n_k)^{\psi/2} \), where \( g_{x_k}(w) = \exp(-\frac{1}{2}(A(x)w, w)) \) and, recall, \( \mathcal{F}_{0n_k}^\xi = \sigma\{\xi(1), ..., \xi(n)\} \).

Proof. Since \( X_N(\frac{n_k}{N}) \) is \( \mathcal{F}_{0n_k}^\xi \)-measurable and \( \sum m_k < l \leq m_{k+1} \land n \xi(l) \) is independent of \( \mathcal{F}_{0n_k}^\xi \), it follows that

\[
E \exp(i \langle w, (n_{k+1} - n_k)^{-1/2}Y_{N,k}(X_N(n_k/N)) \rangle) | \mathcal{F}_{0n_k}^\xi \rangle = f_{n_{k+1} - n_k}(X_N(n_k/N), w),
\]

where \( f_n(x, w) \) was defined in Lemma 3.2 and so (3.12) follows from (3.6). \( \square \)
4. Strong approximation

The strong approximations here will be based on the following result which is a slight variation of Theorem 3 and Remark 2.6 from [13] with the additional feature from Theorem 4.6 of [9] that we enrich the probability space by a sequence of i.i.d. uniformly distributed random variables and not just by one such random variable and this result follows by essentially the same proofs as in the cited above papers.

4.1. Theorem. Let \( \{V_m, m \geq 1\} \) be a sequence of random vectors with values in \( \mathbb{R}^d \) defined on some probability space \((\Omega, F, P)\) and such that \( V_m \) is measurable with respect to \( F_m, m = 1, 2, \ldots \), where \( F_m, m \geq 1 \) is a filtration of sub-\( \sigma \)-algebras of \( F \). Let \( G_m \) and \( H_m, m = 0, 1, \ldots \) be two increasing sequences of countably generated sub-\( \sigma \)-algebras of \( F \) such that \( H_m \subset G_m \subset F_m \) for each \( m \geq 1 \). Assume that the probability space is rich enough so that there exists on it a sequence of uniformly distributed on \([0, 1]\) independent random variables \( U_m, m \geq 1 \) independent of \( \forall \nu \geq 0 \). For each \( m \geq 1 \), let \( G_m(\cdot |H_m-1) \) be a regular conditional distribution on \( \mathbb{R}^d \), measurable with respect to \( H_m-1 \) and with the conditional characteristic function

\[
g_m(w | H_m-1) = \int_{\mathbb{R}^d} \exp(i\langle w, x \rangle) G_m(dx | H_m-1), \quad w \in \mathbb{R}^d.
\]

Suppose that for some non-negative numbers \( \nu_m, \delta_m \) and \( K_m \geq 10^8 d \),

\[
\int_{|w| \leq K_m} E|E[\exp(i\langle w, V_m \rangle)] G_m \langle\cdot | H_m-1\rangle - g_m(w | H_m-1)] dw \leq \nu_m (2K_m)^d
\]

and that

\[
E(G_m(\{x : |x| \geq 1/2K_m\} | H_m-1)) < \delta_m.
\]

Then there exists a sequence \( \{W_m, m \geq 1\} \) of \( \mathbb{R}^d \)-valued random vectors defined on \((\Omega, F, P)\) with the properties

(i) \( W_m \) is \( G_m \cup \sigma\{U_m\} \)-measurable for each \( m \geq 1 \);

(ii) \( G_m(\cdot |H_m-1) \) is conditional distribution of \( W_m \) given \( \sigma\{U_1, \ldots, U_{m-1}\} \cup G_{m-1} \), in particular, \( W_m \) is conditionally independent of \( \sigma\{U_1, \ldots, U_{m-1}\} \cup G_{m-1} \) (and so also of \( U_1, \ldots, U_{m-1} \)) given \( H_m-1 \), \( m \geq 1 \);

(iii) Let \( \varrho_m = 16K_m^{-1} \log K_m + 2\nu_m^{1/2}K_m + 2\delta_m^{1/2} \). Then

\[
P(\{V_m - W_m \geq \varrho_m\} \subseteq \varrho_m)
\]

and, in particular, the Prokhorov distance between the distributions \( L(V_m) \) and \( L(W_m) \) of \( V_m \) and \( W_m \), respectively, does not exceed \( \varrho_m \).

Now, in the notations of Theorem 4.1 we set \( V_k = (n_k - n_{k-1})^{-1/2} Y_{N,k-1}(X_N(\frac{n_k}{N})) \), \( F_k = G_k = \mathcal{F}_{n_{k-1}}, H_k = \sigma\{X_N(\frac{n_k}{N})\} \) and \( g_k(w | H_k-1) = g_{X_N(\frac{n_k}{N})}(w) \) where \( g_x \) was defined in Lemma 3.3. Thus, \( G_k(\cdot | H_k-1) = G_{X_N(\frac{n_k}{N})}(\cdot) \) where \( G_x \) is the mean zero \( d \)-dimensional Gaussian distribution with the covariance matrix \( A(x) \) and the characteristic function \( g_x \).

By Lemma 3.3

\[
\int_{|w| \leq K_1} E[\exp(i\langle w, V_k \rangle)] - g_k(w | H_k-1)] dw \leq C_1(n_k - n_{k-1})^{-\nu}(2K_k)^d \leq 2^d C_1[N^{1/4}]^{-1/8}
\]
where we take $K_k = [N^{\frac{1}{2}}]^{\frac{1}{d^{\frac{1}{2}}}} < (n_k - n_{k-1})^{1/2}$. Next, for each $x \in \mathbb{R}^d$ let $\Theta_x$ be a mean zero Gaussian random variable with the covariance matrix $A(x)$. Then by (2.2) and the Chebyshev inequality,

$$E(G_k(\{y \in \mathbb{R}^d : |y| \geq \frac{1}{K}K_k\} | H_{k-1}))$$

$$\leq \sup_{y \in \mathbb{R}^d} P(|\Theta_y| \geq \frac{1}{K}K_k) \geq \frac{1}{K}K_k$$

In order to use Theorem 4.1 we need that $N \geq N_0 = ((10^d24d + 1)^4$ which is the assumption of Theorem 2.1. Now, Theorem 4.1 provides us with random vectors $\{W_k, k \geq 1\}$ satisfying the properties (i)–(iii), in particular, given $X_N(n_k - n_{k-1})$, the random vector $W_k$ has the mean zero Gaussian distribution with the covariance matrix $A(X_N((n_k - n_{k-1}))$ and it is conditionally independent of $G_{k-1}$ and of $W_1, ..., W_{k-1}$ while in view of (4.4) and (4.5) the property (iii) holds true with

$$\varrho_k = \frac{2}{2d}[N^{\frac{1}{2}}]^{\frac{1}{d^{\frac{1}{2}}}} \log([N^\frac{1}{2}]) + 2\sqrt{C_1}[N^{\frac{1}{2}}]^{\frac{1}{d^{\frac{1}{2}}}} + 4L^2d([N^\frac{1}{2}]) = \frac{1}{2}([N^\frac{1}{2}]) + \sqrt{C_1} + 2L\sqrt{d}.$$  

Next, we obtain the uniform $L^2$-bound for the difference between the sums of $(n_k - n_{k-1})^{1/2}V_k$’s and of $(n_k - n_{k-1})^{1/2}W_k$’s. Set

$$I(t) = \sum_{0 \leq k \leq k_N(t)} (n_k - n_{k-1})^{1/2}(V_k - W_k).$$

4.2. Lemma. For any integer $N \geq N_0$,

$$E \max_{0 \leq t \leq 1} |I(t)|^2 \leq C_2N[N^{\frac{1}{2}}]^{\frac{1}{d^{\frac{1}{2}}}}$$

where

$$C_2 = \sup_{N \geq 1} ([N^{\frac{1}{2}}]^{\frac{1}{d^{\frac{1}{2}}}} \sqrt{\log N})(1 + 4L^2(L^2 + d) + 2L^2d)(1 + \sqrt{2\sqrt{C_1} + 2L\sqrt{d}}).$$

Proof. Set

$$M_k = \sum_{0 \leq l \leq k} (n_l - n_{l-1})^{1/2}(V_l - W_l).$$

Then

$$\max_{0 \leq t \leq 1} |I(t)|^2 = \max_{1 \leq k \leq k_N} |M_k|^2$$

and by the properties (i) and (ii) of Theorem 4.1 together with the conditional independence of each $V_l - W_l$ of $F_{t-1} \vee \sigma\{U_1, ..., U_{l-1}\}$ given $X_N(n_{l-1})$, it is easy to see that $M_k, k = 1, 2, ..., k_N$ is a martingale with respect to the filtration $F_k \vee \sigma\{U_1, ..., U_k\}, k = 1, ..., k_N$. Hence, by the Doob martingale inequality

$$E \max_{1 \leq k \leq k_N} |M_k|^2 \leq 4E|M_{k_N}|^2 = 4[N^{\frac{1}{2}}] \sum_{1 \leq k \leq k_N} E|V_k - W_k|^2$$

where we use also that $(V_k - W_k), k = 1, ..., k_N$ are uncorrelated for different $k$’s.

Next, by the Cauchy-Schwarz inequality

$$E|V_k - W_k|^2 = E((V_k - W_k)^2|V_k - W_k| \leq \varrho_k)$$

$$+ E((V_k - W_k)^2|V_k - W_k| > \varrho_k)$$

$$\leq \varrho_k^2 + (E|V_k - W_k|^{1/2}(P(|V_k - W_k| > \varrho_k))^{1/2}$$

$$\leq \varrho_k^2 + 4\varrho_k^{1/2}(E|V_k|^{1/2} + (E|W_k|^{1/2})^2).$$
Now, by (2.2) and (2.3),
\[(4.10)\]
\[E[V^4 \leq [N^\frac{1}{2}]^{-2}L^4E|\sum_{n_k-1 \leq t \leq n_k} \xi(t)|^4 \leq [N^\frac{1}{2}]^{-2}L^4([N^\frac{1}{2}]E|\xi(1)|^4 + [N^\frac{1}{2}]^2E|\xi(1)|^2) \leq L^4(L^2 + d^2).\]
Since \(W_k\) is distributed as \(\sigma(X_N(\frac{n_k}{N}))\), where \(N\) is the \(d\)-dimensional Gaussian random vector with the identity covariance matrix, we obtain that
\[(4.11)\]
\[E|W_k|^4 \leq 3L^4d^2.\]
Finally, (4.7) follows from (4.8)–(4.11). \(\square\)

Next, let \(W(t), t \geq 0\) be a \(d\)-dimensional Brownian motion such that the increments \(W(n_k) - W(n_k-1)\) are independent of \(X_N(\frac{n_k}{N})\) for any \(k = 1, ..., k_N\). Then, given \(X_N(\frac{n_k}{N})\), the sequences of random vectors \(\hat{W}_k = \sigma(X_N(\frac{n_k}{N}))(W(n_k) - W(n_k-1))\) and \((n_k - n_{k-1})^{1/2}W_k, k = 1, ..., k_N\) have the same distributions. Moreover, we can redefine the process \(\xi(n)\), \(1 \leq n < \infty\) and choose a Brownian motion \(W(s), s \geq 0\) preserving their distributions so that the joint distribution of the sequences of pairs \((\hat{V}_k, W_k)\) and of \((V_k, W_k)\) will be the same and, in particular, that (4.7) will hold true with \(\hat{W}_k\) in place of \(W_k\). Indeed, by the Kolmogorov extension theorem (see, for instance, [20]) such pair of processes exists if we impose consistent restrictions on their joint finite dimensional distributions. But since the pair of processes \(\xi\) and \(W_k\), \(1 \leq k \leq k_N\) satisfying our conditions exist by Theorem 4.1 and Lemma 4.2, these restrictions are consistent and the required pair of processes exists. From now on we will drop tilde and denote \(\sigma(X_N(\frac{n_k}{N}))(W(n_k) - W(n_k-1))\) by \(W_k\) which is supposed to satisfy (4.7).

Now, using the Brownian motion \(W(t), t \geq 0\) constructed above we consider the new Brownian motion \(W_N(t) = N^{-1/2}W(tN), 0 \leq t \leq 1\) and introduce the diffusion process \(\Xi_N(t), t \geq 0\) solving the stochastic differential equation (2.1) which we write now with \(W_N\),
\[d\Xi_N(t) = \sigma(\Xi_N(t))dW_N(t) + b(\Xi_N(t))dt, \quad \Xi_N(0) = x_0.\]
Now, we introduce the auxiliary process \(\hat{\Xi}_N\) with coefficients frozen at times \(n_k\),
\[\hat{\Xi}_N(t) = x_0 + \sum_{1 \leq k \leq k_N(tN)} (\sigma(\Xi_N(\frac{n_k}{N}))(W_N(\frac{n_k}{N}) - W_N(\frac{n_k}{N} - 1))) + N^{-1}b(\Xi_N(\frac{n_k}{N}))(n_k - n_{k-1}).\]

4.3. Lemma. For any integer \(N \geq 1\),
\[(4.12)\]
\[E_{\max_{0 \leq k \leq k_N}|\Xi_N(n_k/N) - \hat{\Xi}_N(n_k/N)|^2 \leq 32\Delta(N)\]
where \(\Delta(N) = N^{-1}[N^\frac{1}{2}].\)

Proof. First, we write
\[(4.13)\]
\[E_{\max_{0 \leq k \leq k_N}}|\Xi_N(n_k/N) - \hat{\Xi}_N(n_k/N)|^2 \leq 2(E_{\max_{0 \leq k \leq k_N}}|J_1(n_k/N)|^2 + E_{\max_{0 \leq k \leq k_N}}|J_2(n_k/N)|^2)\]
where
\[J_1(t) = \int_0^t \left(\sigma(\Xi_N(s)) - \sigma(\Xi_N([s/\Delta(N)]\Delta(N)))\right)dW_N(s)\]
and
\[J_2(t) = \int_0^t \left(b(\Xi_N(s)) - b(\Xi_N([s/\Delta(N)]\Delta(N)))\right)ds.\]
By the Doob martingale inequality and the Itô isometry for stochastic integrals (see, for instance, [14], Sections 6.1.2 and 7.2.1),

\begin{align}
(4.14) & \quad E \max_{0 \leq k \leq k_N} |J_1(n_k/N)|^2 \\
& \leq 4 \int_0^{T/(\Delta(N))} \max_{0 \leq k \leq k_N} \left| E[\sigma(\hat{\Xi}(s)) - \sigma(\hat{\Xi}_N([s/\Delta(N)]\Delta(N)))^2 ds \\
& \leq 4L^2 \sum_{1 \leq k \leq k_N} E|\hat{\Xi}(s) - \hat{\Xi}_N(n_k-1/N)|^2 ds.
\end{align}

By (2.2) and the Cauchy–Schwarz inequality,

\begin{align}
(4.15) & \quad E \max_{0 \leq k \leq k_N} |J_2(n_k/N)|^2 \leq L^2 \int_0^1 \left| \Xi_N(s) - \hat{\Xi}_N([s/\Delta(N)]\Delta(N)) \right|^2 ds.
\end{align}

Again, by (2.2) and the moment inequalities for stochastic integrals

\begin{align}
(4.16) & \quad E|\Xi_N(s) - \hat{\Xi}_N(n_k-1/N)|^2 \leq 2(\max_{0 \leq k < k_N(T_N)} |\hat{\Xi}_N(n_k/N)|^2 \\
& + L^2(s - n_{k-1}/N)^2) \leq 2L^2(1 + \Delta(N)) \leq 4L^2(\Delta(N))
\end{align}

since \( s \in [n_{k-1}/N, n_k/N] \) here, and so \( s - n_{k-1}/N \leq \Delta(N) \). Now, (4.12) follows from (4.13)–(4.16).

Next, we introduce another auxiliary process \( \hat{\Xi}_N \) defined by,

\[ \hat{\Xi}_N(t) = x_0 + \sum_{1 \leq k \leq k_N(t)} \left( \sigma(X_N(\frac{n_k}{N})) (W_N(\frac{n_k}{N}) - W_N(\frac{n_{k-1}}{N})) \\
+ N^{-1}b(X_N(\frac{n_k}{N})) (n_k - n_{k-1}) \right). \]

Then we can write

\begin{align}
(4.17) & \quad E\sup_{0 \leq s \leq 1} |\hat{\Xi}_N(s) - \hat{\Xi}_N(s)|^2 = E \max_{0 \leq k < k_N(T_N)} |\hat{\Xi}_N(n_k/N) \\
& - \hat{\Xi}_N(n_k/N)|^2 \leq 2(\max_{0 \leq k < k_N(T_N)} |\hat{\Xi}_N(n_k/N) - \Xi_N(n_k/N)|^2 \\
& + E \max_{0 \leq k < k_N(T_N)} |\Xi_N(n_k/N) - \Xi_N(n_k/N)|^2).
\end{align}

By Lemma 4.2

\begin{align}
(4.18) & \quad E \max_{0 \leq k \leq n} |\hat{\Xi}_N(n_k/N) - \Xi_N(n_k/N)|^2 = E \max_{0 \leq k \leq n} |\Xi_N(n_k/N) - \Xi_N(n_k/N)|^2 \\
& \leq N^{-1} E\sup_{0 \leq t \leq 1} |I(t)|^2 \leq C_2[N^2]^{-1/2},
\end{align}

In order to estimate the second term in the right hand side of (4.17) introduce the \( \sigma \)-algebras \( \mathcal{Q}_n = \mathcal{F}_0 \vee \sigma\{W(u), 0 \leq u \leq n\} \) and observe that by our construction for each \( k \) the increment \( W(n_{k+1}) - W(n_k) \) is independent of \( \mathcal{Q}_{n_k} \). On the other hand, for any \( k \geq n \) both \( X_N(n_k/N) \) and \( \Xi_N(n_k/N) \) are \( \mathcal{Q}_{n_k} \)-measurable. Hence,

\[ \mathcal{I}_1(n_k) = \sum_{0 \leq t < k} (\sigma(X_N(\frac{n_t}{N})) - \sigma(\Xi_N(\frac{n_t}{N}))) (W_N(\frac{n_t}{N}) - W_N(\frac{n_{t+1}}{N})) \]

is a martingale in \( k \) with respect to the filtration \( \mathcal{Q}_k, k = 1, 2, \ldots, k_N - 1 \). Thus, by (2.2) and the Doob martingale inequality,

\begin{align}
(4.19) & \quad E \max_{1 \leq k \leq m} |\mathcal{I}_1(n_k)|^2 \leq 4E|\mathcal{I}_1(n_m)|^2 \\
& \leq 4 \sum_{0 \leq t \leq m-1} E|\sigma(X_N(\frac{n_t}{N})) - \sigma(\Xi_N(\frac{n_t}{N}))) (W_N(\frac{n_t}{N}) - W_N(\frac{n_{t+1}}{N}))|^2 \\
& \leq 4dL^2(\Delta(N) \sum_{0 \leq t < m} E|X_N(\frac{n_t}{N}) - \Xi_N(\frac{n_t}{N})|^2).\]
Next, observe that
\[
\max_{0 \leq k \leq k_N} |\Xi_N(n_k/N) - \hat{\Xi}_N(n_k/n)|^2 \\
\leq 2(E \max_{0 \leq k \leq k_N} |J_1(n_k)|^2 + E \max_{0 \leq k \leq k_N} |J_2(n_k)|^2)
\]
where
\[
J_2(n_k) = N^{-1} \sum_{0 \leq l \leq k-1} (b(X_N(n_l)) - b(\Xi_N(n_l)))(n_{l+1} - n_l).
\]
By (2.2) we have
\[
|J_2(n_k)|^2 \leq N(\Delta(N))^2(\sum_{0 \leq l \leq k} |X_N(n_l) - \Xi_N(n_l/1)|^2)
\]
\[
\leq 2(\Delta(N))^2(N^{-1} \sum_{0 \leq l \leq k} |X_N(n_l) - \Xi_N(n_l/1)|^2)
\]
for
\[
Q_k = E \max_{0 \leq k \leq k_N} |X_N(n_k/N) - \Xi_N(n_k/n)|^2.
\]
Then we obtain from (4.23), (4.24) and (4.27) that for
\[
Q_n \leq C_3|N^{1/2}|^{-1/2} + C_4\Delta(N) \sum_{0 \leq k \leq n-1} Q_k
\]
where $C_3 = 408L^8 + 6C_2 + 96$ and $C_4 = L^2(16d + 4)$. By the discrete (time)
Gronwall inequality (see, for instance, [6]),
\[
Q_{k_N} \leq C_3|N^{1/2}|^{-1/2} \exp(C_4).
\]
It remains to estimate deviations of our continuous time processes within in-
tervals of time $(n_k/N, n_{k+1}/N)$ which where not taken into account in previous
estimates, i.e. we have to deal now with
\[
J_1 = E \sup_{0 \leq t \leq 1} |X_N(t) - X_N(n_{k_N}(tN))|^2
\]
and
\[
J_2 = E \sup_{0 \leq t \leq 1} |\Xi_N(t) - \Xi_N(n_{k_N}(tN))|^2.
\]
By the straightforward estimates using (2.4) and (2.5) we obtain
\[
J_1 \leq 2(\Delta(N)L^2(L^2 + 1)
\]
and
\[
J_2 \leq 4(J_3 + (2L)^2(\Delta(N))^2)
\]
where
\[
J_3 = E \max_{0 \leq k \leq k_N} \sup_{0 \leq u \leq 1} \int_{n_k/N}^{N^{-1}n_k+s} \sigma(\Xi_N(u))dW_N(u)^2.
\]
By the Jensen (or Cauchy-Schwarz) inequality and the uniform moment esti-
mates for stochastic integrals
\[
J_3 \leq \left(E \max_{0 \leq k \leq k_N} \sup_{0 \leq u \leq 1} \int_{n_k/N}^{N^{-1}n_k+s} \sigma(\Xi_N(u))dW_N(u)^4\right)^{1/2}
\]
\[
\leq \left(\sum_{0 \leq k \leq k_N} E \sup_{0 \leq u \leq 1} \int_{n_k/N}^{N^{-1}n_k+s} \sigma(\Xi_N(u))dW_N(u)^4\right)^{1/2}
\]
\[
\leq \left(\frac{4}{3} \right)^{2} \left(\sum_{0 \leq k \leq k_N} E \int_{n_k/N}^{N^{-1}n_k+s} \sigma(\Xi_N(u))dW_N(u)^4\right)^{1/2}
\]
\[
\leq 6L^2(\Delta(N))^{1/2}.
\]
Combining (4.23)–(4.26) we complete the proof of Theorem 2.1.
5. Dynkin Games

In view of the form of our regularity conditions (2.9) and (2.10) on the payoff functionals $F$ and $G$ we will need the following exponential estimates.

5.1. Lemma. (i) For any $M > 0$ and an integer $N \geq 1$,

$$\max_{0 \leq n \leq N} E \exp(M|X_N(n/N)|) \leq D_M^X e^{M|x|}$$

and

$$\max_{0 \leq n \leq N} E \exp(M|\tilde{X}_N(n/N)|) \leq D_M^X e^{M|x|}$$

where $x = X_N(0) = \tilde{X}_N(0)$ and $D_M^X = 2d \exp(\frac{1}{2}d^4L^4 + L + \frac{1}{6}M^3d^6L^6e^{Md^2L^2})$ does not depend on $N$;

(ii) For any $\delta, M > 0$ and an integer $N \geq 1$,

$$E \exp(M \max_{0 \leq n \leq N} |X_N(n/N)|) \leq D_M^X e^{M|x|} N^\delta$$

and

$$E \exp(M \max_{0 \leq n \leq N} |\tilde{X}_N(n/N)|) \leq D_M^X e^{M|x|} N^\delta$$

where $D_M^X, \delta = 1 + (D_{2M/\delta}^X D_M^X)^{1/2}$ also does not depend on $N$;

(iii) For any $M > 0$,

$$E \exp(M \sup_{0 \leq t \leq 1} |\Xi(t)|) \leq D_M^\Xi e^{M|x|}$$

and $E \exp(M \sup_{0 \leq t \leq 1} |\tilde{\Xi}_N(t)|) \leq D_M^\Xi e^{M|x|}$

where $x = \Xi(0)$ and $D_M^\Xi = 2 \exp(L + \frac{1}{2}ML^2d^2)$.

Proof. (i) Writing

$$X_N(n/N) = x + \sum_{k=0}^{n-1} (N^{-1/2} \sigma(X_N(k/N)) \xi(k+1) + N^{-1/2}b(X_N(k/N)))$$

we obtain

$$E \exp(M|X_N(n/N)|)$$

$$\leq e^{M(|x|+L)} E \exp(MN^{-1/2} \sum_{k=0}^{n-1} \sigma(X_N(k/N)) \xi(k+1))$$

$$\leq E \max_{1 \leq i \leq d} \exp(MdN^{-1/2} \sum_{k=0}^{n-1} \sum_{j=1}^d \sigma_{ij}(X_N(k/N)) \xi_j(k+1))$$

$$\leq \sum_{1 \leq i \leq d} (E \exp(MdN^{-1/2} \sum_{k=0}^{n-1} \sum_{j=1}^d \sigma_{ij}(X_N(k/N)) \xi_j(k+1))$$

$$+ E \exp(-MdN^{-1/2} \sum_{k=0}^{n-1} \sum_{j=1}^d \sigma_{ij}(X_N(k/N)) \xi_j(k+1)))$$

To shorten a bit notations we set for this proof $g(x) = (g_1(x), ..., g_d(x))$ where $g_j(x) = \pm Md\sigma_{ij}(x)$. Then we have to estimate

$$E \exp \left( N^{-1/2} \sum_{k=0}^{n-1} g(X_N(k/N)), \xi(k+1)) \right)$$

$$= E \left( \exp(N^{-1/2} \sum_{k=0}^{n-2} g(X_N(k/N)), \xi(k+1)) \right)$$

$$\times E \left( \exp(N^{-1/2} \langle g(X_N(n-1/N)), \xi(n) \rangle |F_{0,n-1}) \right).$$

Since $|g_j(x)| \leq MdL$, $j = 1, ..., d$, it follows that

$$|\exp(N^{-1/2} \langle g(X_N(n-1/N)), \xi(n) \rangle) - 1 - N^{-1/2} \langle g(X_N(n-1/N)), \xi(n) \rangle - \frac{1}{2}N^{-1} \langle g(X_N(n-1/N)), \xi(n) \rangle^2| \leq \sum_{i=3}^{\infty} \frac{(MdL^3)^i}{N^{i/2}} \leq \tilde{D}N^{-3/2}$$

where $\tilde{D}$ is a constant.
where 

\[ \hat{D} = \frac{1}{6} M^3 d^6 L^6 e^{M^2 L^2}. \]

Hence,

\begin{align}
(5.9) \quad & E \left( \exp(N^{-1/2} \langle g(X_N(n - 1/N)), \xi(n) \rangle \right| \mathcal{F}_{0,n-1} \\
& \leq 1 + \frac{1}{2} N^{-1} \langle g(X_N(n - 1/N)), \xi(n) \rangle^2 \leq 1 + \frac{1}{2} N^{-1} d^4 L^4 + \hat{D} N^{-3/2}
\end{align}

where we used that \( E(\xi(n))\mathcal{F}_{0,n-1} = E\xi(n) = 0 \). Continuing in the same way with the sums in the exponent till \( n - 2, n - 3, \ldots, 1 \) we obtain that

\begin{align}
(5.10) \quad & E \exp \left( N^{-1/2} \sum_{k=0}^{n-1} \langle g(X_N(k/N)), \xi(k + 1) \rangle \right) \\
& \leq (1 + \frac{1}{2} N^{-1} d^4 L^4 + N^{-3/2} \hat{D})^N \\
& \leq (1 + \frac{1}{2} N^{-1} d^4 L^4)^N (1 + \hat{D} N^{-3/2})^N \leq \exp(\hat{D} + \frac{1}{2} d^4 L^4)
\end{align}

proving \( 5.3 \) while \( 5.2 \) follows in the same way.

(ii) Set \( \Gamma(y) = \{|X_N(n/N) - x| \geq y\} \). By (i) and the exponential Chebyshev inequality for any \( n \leq N, y \geq 0 \) and \( \delta > 0 \),

\[ P\{\Gamma(\frac{\delta y}{2M})\} \leq D \frac{\exp(-y)}{2M}. \]

Then, taking \( y = 2 \log N \) we have

\begin{align}
(5.11) \quad & E \exp(M \max_{0 \leq n \leq N} |X_N(n/N)|) \\
& \leq e^{M|x|} E \exp(M \max_{1 \leq n \leq N} |X_N(n/N) - x|) \\
& \leq e^{M|x|} (N^\delta + \sum_{n=1}^{N} E(1_{\Gamma_n(\frac{\delta}{M} \log N)} \exp(M|X_N(n/N) - x|))) \\
& \leq e^{M|x|} (N^\delta + \sum_{n=1}^{N} (P(\Gamma_n(\frac{\delta}{M} \log N)))^{1/2} (E(\exp(2M|X_N(n/N) - x|)))^{1/2} \\
& \leq e^{M|x|} (N^\delta + (D_{2M}M^{1/2})^{1/2})
\end{align}

proving \( 5.3 \) while \( 5.4 \) follows in the same way.

For (iii) we have

\begin{align}
(5.12) \quad & E \exp(M \sup_{0 \leq t \leq 1} |\Xi(t)|) \\
& \leq e^{M(|x|+L)} E(\sup_{0 \leq t \leq 1} |\int_{0}^{t} \sigma(\Xi(s))dW(s)|) \\
& \leq e^{M(|x|+L)} \sum_{i=1}^{d} \left( E(\sup_{0 \leq t \leq 1} |\int_{0}^{t} \sigma_{ij}(\Xi(s))dW_{j}(s)|) + E(\sup_{0 \leq t \leq 1} |\int_{0}^{t} \sigma_{ij}(\Xi(s))dW_{j}(s)|) \right)
\end{align}

Since

\[ \exp \left( \pm Md \sum_{j=1}^{d} \int_{0}^{t} \sigma_{ij}(\Xi(s))dW_{j}(s) - \frac{M^2 d^2}{2} \sum_{j=1}^{d} \sigma_{ij}^2(\Xi(s))ds \right) \]

is a martingale with the expectation equal one, it follows from \( 2.22 \) and the Doob martingale inequality that

\[ E(\sup_{0 \leq t \leq 1} |\Xi(t)|) \leq e^{\frac{1}{2} M^2 L^2 d^2}, \]

and so the first inequality in \( 5.5 \) follows while we obtain the second one in the same way. \( \square \)

Let \( T^\Delta \) be the set of all stopping times with respect to the filtration \( \mathcal{F}^\xi_{0,n_k}, k \geq 0 \)

taking on values \( n_k, k = 0, 1, \ldots, k_{\text{max}} \) where \( k_{\text{max}} = k_N \) if \( k_N = N/[N^\frac{1}{2}] \) and \( k_{\text{max}} = k_N + 1 \) and \( n_{k_{\text{max}}} = N \) if \( n_{k_N} < N \). Denote by \( Q_{n_k} \) the \( \sigma \)-algebra \( \mathcal{F}^\xi_{0,n_k} \vee \sigma\{U_i, 1 \leq \)
\(i \leq k\) where, recall, \(U_1, U_2, \ldots\) is a sequence of i.i.d. uniformly distributed random variables appearing in Theorem 4.1. Let \(\mathcal{T}^Q\) be the set of all stopping times with respect to the filtration \(Q_{n_k}, \ k \geq 0\) taking on values \(n_k, \ k = 0, 1, \ldots, k_{\text{max}}\). Next, introduce the payoffs based on \(\hat{X}_N\) (the same as in Lemma 3.1),

\[
\hat{R}_N(s, t) = G_s(\hat{X}_N)1_{s < t} + F_t(\hat{X}_N)1_{t \leq s}
\]

and the Dynkin game values corresponding to the sets of stopping times \(\mathcal{T}^\Delta\) and \(\mathcal{T}^Q\),

\[
V_N^\Delta = \inf_{\sigma \in \mathcal{T}^\Delta} \sup_{\tau \in \mathcal{T}^\Delta} ER_N(\sigma/N, \tau/N),
\]

\[
\hat{V}_N^\Delta = \inf_{\sigma \in \mathcal{T}^\Delta} \sup_{\tau \in \mathcal{T}^\Delta} E\hat{R}_N(\sigma/N, \tau/N),
\]

and \(\hat{V}_N^Q = \inf_{\sigma \in \mathcal{T}^Q} \sup_{\tau \in \mathcal{T}^Q} E\hat{R}_N(\sigma/N, \tau/N)\).

5.2. Lemma. For any \(\delta > 0\) and an integer \(N \geq 1\),

\[
|V_N - V_N^\Delta| \leq D_X^K e^{K|x|} N^{\delta - \frac{1}{2}} (1 + L + L^2),
\]

where \(x = X_N(0)\), and

\[
|V_N^\Delta - \hat{V}_N^\Delta| \leq 24 \sqrt{D_X^K e^{K|x|} L^4 N^{\frac{\delta}{2}}}. \tag{5.14}
\]

Proof. For any \(\zeta \in T_0^\xi\) set \(\zeta^\Delta = \min\{n_k : n_k \geq \zeta\}\) which defines a stopping time from \(\mathcal{T}^\Delta\) satisfying

\[
N^{-1} \zeta + \Delta(N) \geq N^{-1} \zeta^\Delta \geq N^{-1} \zeta. \tag{5.15}
\]

Since \(T_0^\xi \supset \mathcal{T}^\Delta\) we see that

\[
V_N \geq \inf_{\zeta \in T_0^\xi} \sup_{\eta \in \mathcal{T}^\Delta} ER(\zeta/N, \eta/N).
\]

Then for any \(\vartheta > 0\) there exists \(\zeta_\vartheta \in T_0^\xi\) such that

\[
V_N \geq \sup_{\eta \in \mathcal{T}^\Delta} ER_N(\zeta_\vartheta/N, \eta/N) - \vartheta,
\]

and so

\[
V_N \geq \sup_{\eta \in \mathcal{T}^\Delta} ER_N(\zeta^\Delta_\vartheta/N, \eta/N) - \vartheta - \sup_{\eta \in \mathcal{T}^\Delta} E(R_N(\zeta^\Delta_\vartheta/N, \eta/N) - R_N(\zeta_\vartheta/N, \eta/N)) \geq V_N^\Delta - \vartheta - \sup_{\eta \in \mathcal{T}^\Delta} J_1(\zeta_\vartheta/N, \eta/N)
\]

where for any \(\zeta \in T_0^\xi\) and \(\eta \in \mathcal{T}^\Delta\),

\[
J_1(\zeta/N, \eta/N) = E(R_N(\zeta^\Delta/N, \eta/N) - R_N(\zeta/N, \eta/N)).
\]

Since \(\zeta^\Delta \geq \zeta\),

\[
R_N(\zeta/N, \eta/N) = G_{\zeta/N}(X_N) \text{ whenever } R_N(\zeta^\Delta/N, \eta/N) = G_{\zeta^\Delta/N}(X_N).
\]
Hence, by (2.10) and (5.15),

\[(5.17) R_N(\zeta^\Delta/N, \eta/N) - R_N(\zeta/N, \eta/N) \leq \max \{|G_{\zeta^\Delta/N}(X_N) - G_{\zeta/N}(X_N)|, |F_{\zeta^\Delta/N}(X_N) - F_{\zeta/N}(X_N)|\} \]

\[\leq K(\Delta(N)(1 + L) + N^{-1/2}\max_{0 \leq k \leq \max \max_{1 \leq i \leq N^\Delta} |\sum_{n_k + l \leq j \leq n_{k+1}} \sigma(X_N(j/N)\xi(j))| \exp(K\max_{0 \leq n \leq N} |X_N(n/N)|)\]

\[\leq K(\Delta(N)(1 + L) + L^2N^{-1/4})\exp(K\max_{0 \leq n \leq N} |X_N(n/N)|)\]

Taking here \(\zeta_0\) in place of \(\zeta\) we obtain from (5.16), (5.17) and Lemma 5.1(ii) that

\[V_N \geq V_N^\Delta - \theta - D_{K,\delta}^X K e^{K|x|} N^{\delta - 1/4}(1 + L + L^2)\]

and since \(\theta > 0\) is arbitrary we have that

\[(5.18) V_N \geq V_N^\Delta - D_{K,\delta}^X K e^{K|x|} N^{\delta - 1/4}(1 + L + L^2)\]

On the other hand, since the Dynkin game here has a value (see, for instance, [14], Section 6.2.2) we can write also that

\[(5.19) V_N = \sup_{\eta \in T^\Delta_{0N}} \inf_{\zeta \in T^\Delta_{0N}} E R_N(\zeta/N, \eta/N) \leq \inf_{\zeta \in T^\Delta_{0N}} E R(\zeta/N, \eta_0/N) + \theta\]

for each \(\theta > 0\) and some \(\eta_0 \in T^\xi_{0N}\). Introducing \(\eta_0^\Delta\) and arguing as above we obtain that

\[V_N \leq V_N^\Delta + D_{K,\delta}^X K e^{K|x|} N^{\delta - 1/4}(1 + L + L^2)\]

which together with (5.18) completes the proof of (5.13).

In order to prove (5.14) we observe that by (2.9), Lemma 3.1, Lemma 5.1(ii), the Chebyshev and the Cauchy-Schwarz inequalities

\[(5.20) |V_N^\Delta - \hat{V}_N^\Delta| \leq \sup_{\zeta \in T^\Delta} \sup_{\eta \in T^\Delta} E|R_N(\zeta/N, \eta/N) - \hat{R}_N(\zeta/N, \eta/N)| \]

\[\leq \max \left(E \sup_{0 \leq t \leq 1} |F_t(X_N) - F_t(\hat{X}_N)|, |G_t(X_N) - G_t(\hat{X}_N)|\right) \]

\[\leq K E \left(\sum_{0 \leq t \leq 1} |X_N(t) - \hat{X}_N(t)|\right) \]

\[\leq 2K \left(E \sup_{0 \leq t \leq 1} |X_N(t) - \hat{X}_N(t)|^2\right)^{1/2} \]

\[\times \left(E \exp(4K \sup_{0 \leq t \leq 1} |X_N(t)|)^{1/2} \right) \left(E \exp(4K \sup_{0 \leq t \leq 1} |\hat{X}_N(t)|)^{1/4}\right) \]

\[\leq 24 \sqrt{D_{4K,\delta}^X K e^{K|x|} L^4 N^{\delta - 1/4}}\]

yielding (5.14). \(\square\)

5.3. **Lemma.** For any integer \(N \geq 1\),

\[(5.21) \hat{V}_N^\Delta = \hat{V}_N^\Delta\]

**Proof.** We prove (5.21) obtaining both \(V_N^\Delta\) and \(\hat{V}_N^\Delta\) by the standard dynamical programming (backward recursion) procedure (see, for instance, Section 1.3.2 in [14]). Namely, we have \(\hat{V}_N^\Delta = \hat{V}_N^\Delta_{N,0}\) and \(\hat{V}_N^\Delta = \hat{V}_N^\Delta_{N,0}\) where

\[(5.22) \hat{V}_N^\Delta_{N,k} = F_T(\hat{X}) = \hat{V}_N^\Delta_{N,k_{\max}}\]

proceeding recursively

\[\hat{V}_N^\Delta_{N,k} = \min \{G_{nk/N}(\hat{X}_N), \max(F_{nk/N}(\hat{X}_N), E(\hat{V}_N^\Delta_{N,k+1}|\mathcal{F}_{0,n_k}))\}\]
where, recall, $T$ we have to use moment estimates for diffusions. Set $T$ and so starting from (5.22) we proceed recursively to $\hat{X}$ of the process $\xi$ given by (5.22) and observe that by the construction

$$\hat{\Xi} = \hat{\Xi}(0)$$

and so it is independent of all stopping times with respect to the filtration $\mathcal{G}_{n_k}$. The proof is similar to Lemma 5.2 but here in place of estimates for $\xi$, we have $\xi = \xi(0)$.

Next, we turn our attention to the diffusion $\xi$ constructed in Theorem 2.1 and consider the corresponding Dynkin game value $V^\xi$ given by (5.23). Set $\sigma_{n_k}^\xi = \sigma\{W_N(u/N) : u \leq n_k\}$ and observe that by the construction

$$(5.23) \quad \mathcal{G}_{n_k}^\xi \subset Q_{n_k} = \mathcal{F}_{0,n_k} \ \forall \{U_i, 1 \leq i \leq k\}$$

where $W_N$ is the Brownian motion constructed in Section 4. Let $\mathcal{T}_{\mathcal{F}}$ be the set of all stopping times with respect to the filtration $\mathcal{G}_{n_k}^\xi$, $k \geq 0$ and $\mathcal{T}_{\mathcal{F}}$ be the set of all stopping times with respect to the filtration $Q_{n_k}$, $k \geq 0$, both taking values $n_k$ when $k$ runs from 0 to $k_{\max}$. Set

$$\hat{R}^\xi(s, t) = G_s(\hat{\Xi})I_{s < t} + F_t(\hat{\Xi})I_{t \leq s}$$

where the process $\hat{\Xi}$ is the same as in Lemma 4.3. Set

$$V^\xi = \inf_{\zeta \in \mathcal{T}_\Delta} \sup_{\eta \in \mathcal{T}_\Delta} E R^\xi(\zeta/N, \eta/N),$$

$$\hat{V}^\xi = \inf_{\zeta \in \mathcal{T}_\Delta} \sup_{\eta \in \mathcal{T}_\Delta} E \hat{R}^\xi(\zeta/N, \eta/N)$$

and

$$\hat{V}^\xi = \inf_{\zeta \in \mathcal{T}_\Delta} \sup_{\eta \in \mathcal{T}_\Delta} E \hat{R}^\xi(\zeta/N, \eta/N).$$

5.4. Lemma. For any integer $N \geq 1$,

$$|V^\xi - \hat{V}^\xi| \leq K(\Delta(N)D^\xi + 18L)\sqrt{2D^\xi_{2K}}\sqrt{\Delta(N)}, \tag{5.24}$$

where $x = \Xi(0)$, and

$$|V^\xi - \hat{V}^\xi| \leq (4\sqrt{2} + 2L)Ke^{K|x|}\sqrt{D^\xi_{4K}}\sqrt{\Delta(N)}. \tag{5.25}$$

Proof. The proof is similar to Lemma 5.2 but here in place of estimates for $\xi$, we have $\xi(0)$ and use moment estimates for diffusions. Set $\mathcal{T}_\mathcal{F}^{\xi,N} = \{\zeta : \zeta/N \in \mathcal{T}_\mathcal{F}^{\xi}\}$ where, recall, $\mathcal{T}_\mathcal{F}^{\xi,N}$ is the set of stopping times with respect to the filtration $\mathcal{F}_{\xi,N} = \sigma\{W_N(s), s \leq t\}$ having values in $[0, 1]$. For any $\xi \in \mathcal{T}_\mathcal{F}^{\xi,N}$ define $\zeta^\Delta = \min\{n_k : n_k \geq \zeta\}$ which yields a stopping time from $\mathcal{T}_\mathcal{F}^{\xi,N}$ satisfying (5.19). Since $\mathcal{T}_\mathcal{F}^{\xi,N} \subset \mathcal{T}_\mathcal{F}^{\xi,N}$, we have that

$$V^\xi \geq \inf_{\zeta \in \mathcal{T}_\mathcal{F}^{\xi,N}} \sup_{\eta \in \mathcal{T}_\mathcal{F}^{\xi,N}} E R^\xi(\zeta/N, \eta/N).$$

In the same way as in (5.16) we obtain that for some $\zeta_0 \in \mathcal{T}_\mathcal{F}^{\xi,N}$,

$$V^\xi \geq V^\xi_\Delta - \vartheta - \sup_{\eta \in \mathcal{T}_\mathcal{F}^{\xi,N}} J_\Delta(\zeta_0/N, \eta/N). \tag{5.26}$$
where for any $\zeta \in T_{\Delta}^{\Xi}$ and $\eta \in T_{\Delta}^{\Xi}$,
\[
J_2(\zeta/N, \eta/N) = E(R^\Xi(\zeta^N/N, \eta/N) - R^\Xi(\zeta/N, \eta/N)).
\]

As in (5.17) we obtain from (2.10) and (5.15) that
\[
(5.27) \quad R^\Xi(\zeta/N, \eta/N) - R^\Xi(\zeta/N, \eta/N) \leq K(1/N) \\
+ \max_{0 \leq k \leq k_{\max}} \sup_{n_k/N \leq s \leq n_{k+1}/N} |\Xi(n_{k+1}/N) - \Xi(s)| \\
\times \exp(K \max_{0 \leq t \leq 1} |\Xi(t)|).
\]

By the Cauchy-Schwarz inequality,
\[
(5.28) \quad |E(R^\Xi(\zeta/N, \eta/N) - R^\Xi(\zeta/N, \eta/N))| \\
\leq K(1/N)E\exp(K \max_{0 \leq t \leq 1} |\Xi(t)|) \\
+ K\left(E \max_{0 \leq k \leq k_{\max}} \sup_{n_k/N \leq s \leq n_{k+1}/N} |\Xi(n_{k+1}/N) - \Xi(s)|^2 \right)^{1/2} \\
\times (E \exp(2K \max_{0 \leq t \leq 1} |\Xi(t)|))^{1/2}.
\]

Next, we write
\[
(5.29) \quad \left(\sum_{1 \leq k \leq k_{\max}} E \sup_{n_k/N \leq s \leq n_{k+1}/N} |\Xi(n_{k+1}/N) - \Xi(s)|^2 \right)^{1/2} \\
\leq 8E|\Xi(n_{k+1}/N) - \Xi(n_k/N)|^4 + 8E \sup_{n_k/N \leq s \leq n_{k+1}/N} |\Xi(s) - \Xi(n_k/N)|^4.
\]

By the standard moment estimates for stochastic integrals
\[
(5.31) \quad E|\Xi(n_{k+1}/N) - \Xi(n_k/N)|^4 \leq 8E|\int_{n_k/N}^{n_{k+1}/N} \sigma(\Xi(u))dW_N(u)|^4 \\
+ 8E(\int_{n_k/N}^{n_{k+1}/N} b(\Xi(u))du)^4 \leq 288(1/N)E|\sigma(\Xi(u))|^3du \\
+ 8L^4(1/N)^4 \leq 8L^4(1/N)^2(36 + (1/N)^2)
\]

and
\[
(5.32) \quad E \sup_{n_k/N \leq s \leq n_{k+1}/N} |\Xi(s) - \Xi(n_k/N)|^4 \\
\leq 8(4/3)^4E|\int_{n_k/N}^{n_{k+1}/N} \sigma(\Xi(u))dW_N(u)|^4 \\
+ 8E(\int_{n_k/N}^{n_{k+1}/N} b(\Xi(u))du)^4 \leq 8L^4(1/N)^2(36(4/3)^4 + (1/N)^2).
\]

Combining (5.26)-(5.32) together with Lemma 5.1(ii) we obtain the required lower bound for $V^\Xi - V^\Xi_{\Delta}$ taking into account that $\vartheta > 0$ is arbitrary. On the other hand, since the Dynkin game has a value under our conditions (see, for instance, [14], Section 6.2.2) we can write that
\[
V^\Xi = \sup_{\eta \in T^{\Xi}_{\delta}} \inf_{\zeta \in T^{\Xi}_{\delta}} E R^\Xi(\zeta/N, \eta/N) \leq \inf_{\zeta \in T^{\Xi}_{\delta}} E R^\Xi(\zeta/N, \eta_0/N) + \vartheta
\]
for any $\vartheta > 0$ and some $\eta_0 \in T^{\Xi}_{\delta}$. Introducing $\eta^\Delta_0$ and relying on the same arguments as above we obtain the corresponding upper bound for $V^\Xi - V^\Xi_{\Delta}$ and complete the proof of (5.24).
Next, we obtain \((7.25)\) by \((2.9)\), Lemma 3.3, Lemma 5.1(iii), the Chebyshev and the Cauchy-Schwarz inequalities,

\[
(5.33) \quad |V_\delta^\Xi - \hat{V}_\delta^\Xi| \leq \sup_{\xi \in T_\delta^\Xi} \sup_{\eta \in T_\delta^\Xi} E|\hat{V}_\delta^\Xi(\zeta/N, \eta/N) - \hat{V}_\delta^\Xi(\zeta/N, \eta/N)|
\]

\[
\leq K(E\max_{0 \leq k \leq k_{\max}}|\hat{\Xi}(k/N) - \hat{\Xi}(n/N)|) + \|\max_{0 \leq k \leq k_{\max}}|\hat{\Xi}(k/N) - \hat{\Xi}(n/N)|\|_1^{1/2}
\]

\[
\times (E\exp(4K\max_{0 \leq k \leq k_{\max}}|\hat{\Xi}(k/N)|)^{1/4}(E\exp(4K\max_{0 \leq k \leq k_{\max}}|\hat{\Xi}(k/N)|)^{1/4}
\]

\[
\leq 2K\sqrt{D_4^\Xi}(E\max_{0 \leq k \leq k_{\max}}|\hat{\Xi}(k/N) - \hat{\Xi}(n/N)|^2 + E|\hat{\Xi}(n/N)|^2)^{1/2}
\]

\[
\leq (4\sqrt{\pi} + 2L)Ke^{K^2}\sqrt{D_4^\Xi}\sqrt{\Delta(N)}
\]

completing the proof of the lemma.

Next, we introduce the new process \(\Psi_N\), first recursively at the times \(N^{-1}n_k\) and then extending it for all \(t \in [0, T]\) in the piece-wise constant fashion. Namely, we set \(\Psi_N(0) = x\) and (with \(n_0 = 0\)),

\[
\Psi_N(N^{-1}n_{k+1}) = \Psi_N(N^{-1}n_k) + \sigma(\Psi_N(N^{-1}n_k))(W_N(N^{-1}n_{k+1}) - W_N(N^{-1}n_k))
\]

\[
+ N^{-1}b(\Psi_N(N^{-1}n_k))(n_{k+1} - n_k)
\]

for \(k = 0, 1, ..., k_{\max} - 1\). Set also \(\Psi_N(t) = \Psi_N(N^{-1}n_k)\) if \(N^{-1}n_k \leq t < N^{-1}n_{k+1}\).

5.5. \textbf{Lemma.} \textit{For any integer} \(N \geq 1\),

\[
E\max_{0 \leq k \leq k_{\max}}|\Xi(N^{-1}n_k) - \Psi_N(N^{-1}n_k)|^2 \leq 96\Delta(N)\exp(24L^2d).
\]

\textbf{Proof.} We have

\[
\max_{0 \leq k \leq n}|\Xi(N^{-1}n_k) - \Psi_N(N^{-1}n_k)|^2 \leq 3\max_{0 \leq k \leq n}|\Xi(N^{-1}n_k) - \hat{\Xi}(N^{-1}n_k)|^2
\]

\[
+ |\sum_{0 \leq l < k}\sigma(\Xi(N^{-1}n_l))\sigma(\Psi_N(N^{-1}n_l))(W_N(N^{-1}n_{l+1}) - W_N(N^{-1}n_l))|^2
\]

\[
+ (N^{-1}\sum_{0 \leq l < k}|b(\Xi(N^{-1}n_l)) - b(\Psi_N(N^{-1}n_l))|(n_{l+1} - n_l)|^2),
\]

and so

\[
\max_{0 \leq k \leq n}|\Xi(N^{-1}n_k) - \Psi_N(N^{-1}n_k)|^2
\]

\[
\leq 3\max_{0 \leq k \leq n}|\Xi(N^{-1}n_k) - \hat{\Xi}(N^{-1}n_k)|^2 + \max_{0 \leq k \leq n}|M_k|^2
\]

\[
+ 4k_{\max}(\Delta(N))^2\sum_{0 \leq l < n}|b(\Xi(N^{-1}n_l)) - b(\Psi_N(N^{-1}n_l))|^2
\]

where

\[
M_k = \sum_{0 \leq l < k}\sigma(\Xi(N^{-1}n_l))\sigma(\Psi_N(N^{-1}n_l))(W_N(N^{-1}n_{l+1}) - W_N(N^{-1}n_l))
\]

is a martingale with respect to the filtration \(\{G_{n_k}^\Xi, k \geq 0\}\) since \(\sigma(\Xi(N^{-1}n_l)) - \sigma(\Psi_N(N^{-1}n_l))\) is \(G_{n_l}\)-measurable while \(W_N(N^{-1}n_{l+1}) - W_N(N^{-1}n_l)\) is independent of \(G_{n_l}^\Xi\).

Hence, by the Doob martingale moment inequality and by the Lipschitz continuity of \(\sigma\) (with the constant \(L\)),

\[
E\max_{0 \leq k \leq n}|M_k|^2 \leq 4E|M_n|^2 \leq 4L^2dN^{-1}\sum_{0 \leq k < n}Q_k(n_{k+1} - n_k)
\]

where

\[
Q_n = E\max_{0 \leq k \leq n}|\Xi(N^{-1}n_k) - \Psi_N(N^{-1}n_k)|^2.
\]
By (5.35), (5.36) and Lemma 4.3 we obtain that
\[ Q_n \leq 96\Delta(N) + 24L^2d\Delta(N) \sum_{0 \leq k < n} Q_k. \]

Thus, by the discrete (time) Gronwall inequality (see [6]),
\[ Q_n \leq 96\Delta(N) \exp(24L^2d\Delta(N)n) \]
and since \( n \leq k_{\max} \), (5.34) follows. \( \square \)

Next, we introduce the values of Dynkin games with payoffs based on the process \( \Psi_N \). Namely, we set
\[ V_\Delta^\Psi = \inf_{\zeta \in T_\Delta} \sup_{\eta \in T_\Psi} ER_N^\Psi(N^{-1}\zeta, N^{-1}\eta) \]
and
\[ V_\Psi^\Psi = \inf_{\zeta \in T^\Psi} \sup_{\eta \in T^\Psi} ER_N^\Psi(N^{-1}\zeta, N^{-1}\eta). \]

5.6. Lemma. For any \( \varepsilon > 0 \),
\[ (5.37) \]
\[ V_\Delta^\Psi = V_\Psi^\Psi. \]

Proof. As in Lemma 5.3 we will prove (5.37) obtaining both \( V_\Delta^\Psi \) and \( V_\Psi^\Psi \) by the dynamical programming procedure. Again, we have \( V_\Delta^\Psi = V_\Delta^\Psi,0 \) and \( V_\Psi^\Psi = V_{\Psi,0} \) where \( V_{\Delta, k_{\max}}^\Psi = F_T(\Psi_N) = V_{\Psi, k_{\max}}^\Psi \) and for \( k = k_{\max} - 1, k_{\max} - 2, \ldots, 0 \),
\[ V_{\Delta, k}^\Psi = \min\left( G_{N^{-1}n_k}(\Psi_N), \max(F_{N^{-1}n_k}(\Psi_N), E(V_{\Delta, k+1}^\Psi|G_{n_k})) \right) \]
and
\[ V_{\Psi, k}^\Psi = \min\left( G_{N^{-1}n_k}(\Psi_N), \max(F_{N^{-1}n_k}(\Psi_N), E(V_{\Psi, k+1}^\Psi|Q_{n_k})) \right). \]

For any vectors \( x_0, x_1, x_2, \ldots, x_{k_{\max}} \in \mathbb{R}^d \) set \( x(0) = x_0 \), \( x(t) = x_k \) if \( N^{-1}n_k \leq t < N^{-1}n_{k+1} \) and define the functions
\[ q_{k_N}(t)(x_1, \ldots, x_{k_N}(t)) = F_t(x) \quad \text{and} \quad r_{k_N}(t)(x_1, \ldots, x_{k_N}(t)) = G_t(x). \]

Introduce
\[ \Phi_t(x_1, \ldots, x_k) = \min\left( r_t(x_1, \ldots, x_k), \max(q_t(x_1, \ldots, x_k), h(x_1, \ldots, x_k)) \right) \]
where
\[ h(x_1, \ldots, x_k) = E\Phi_{t+1}(x_1, \ldots, x_k, x_k + \sigma(x_k)(W_N(N^{-1}n_{t+1}) - W_N(N^{-1}n_t))). \]

Since \( \Psi_N(N^{-1}n_k) \) is both \( G_{n_k} \) and \( Q_{n_k} \)-measurable while \( W_N(N^{-1}n_{t+1}) - W_N(N^{-1}n_t) \) is independent of both \( G_{n_k} \) and \( Q_{n_k} \) we see by induction that
\[ V_{\Phi, t}^\Psi = \Phi_t(\Psi_N(N^{-1}n_1), \Psi_N(N^{-1}n_2), \ldots, \Psi_N(N^{-1}n_k)) = V_{\Delta, t}^\Psi, \]
for all \( l = k_{\max}, k_{\max} - 1, \ldots, 0 \) where \( \Phi_0 = \min(F_0(x_0), \max(G_0(x_0), E\Phi_1(x_0 + \sigma(x_0)W_N(N^{-1}n_1)))) \), and (5.37) follows. \( \square \)

Now we can complete the proof of Theorem 2.2 writing first,
\[ (5.38) \]
\[ |V_\Psi^\Psi - V_N| \leq |V_N - V_\Delta^\Psi| + |V_\Delta^\Psi - \hat{V}_\Delta^\Psi| + |\hat{V}_\Delta^\Psi - \hat{V}_\Psi^\Psi| \]
\[ + |\hat{V}_\Psi^\Psi - V_\Psi^\Psi| + |\hat{V}_\Psi^\Psi - \hat{V}_\Delta^\Psi| + |\hat{V}_\Delta^\Psi - V_\Psi^\Psi|. \]

It remains to estimate \( |V_\Delta^\Psi - V_\Psi^\Psi| \) and \( |V_\Psi^\Psi - \hat{V}_\Delta^\Psi| = |V_\Psi^\Psi - \hat{V}_\Psi^\Psi| \) since all other terms in the right hand side of (5.38) are dealt with by Lemmas 5.2, 5.3. In both remaining estimates we use the fact that the game values there are defined with respect to
the same sets of stopping times which will allow us to rely on uniform bounds on
distances between the corresponding processes. By (2.9) and the Cauchy-Schwarz
inequality,
\begin{align}
(5.39) \quad |\hat{V}_N^\varphi - V_N^\varphi| & \leq \sup_{\zeta \in \mathcal{T}} \sup_{\eta \in \mathcal{T}} E|\hat{R}(N^{-1}\zeta, N^{-1}\eta) - R_N^\varphi(N^{-1}\zeta, N^{-1}\eta)| \\
& \leq \max(E \sup_{0 \leq t \leq 1} |F_t(\hat{X}_N) - F_t(\Psi_N)|, E \sup_{0 \leq t \leq 1} |G_t(\hat{X}_N) - F_t(\Psi_N)|) \\
& \leq \sqrt{K} (E \max_{0 \leq k \leq k_{\text{max}}} |\hat{X}_N(N^{-1}n_k) - \Psi_N(N^{-1}n_k)|^2 \\
& + P\left(\max_{0 \leq k \leq k_{\text{max}}} |\hat{X}_N(N^{-1}n_k) - \Psi_N(N^{-1}n_k)| > 1\right) \right)^{1/2} \\
& \times (E \exp(2K (\max_{0 \leq k \leq k_{\text{max}}} (|\hat{X}_N(N^{-1}n_k)| + |\Psi_N(N^{-1}n_k)|))) \right)^{1/2}.
\end{align}

Next, by Lemmas 3.1, 5.1 and Theorem 2.1
\begin{align}
(5.40) \quad E \max_{0 \leq k \leq k_{\text{max}}} |\hat{X}_N(N^{-1}n_k) - \Psi_N(N^{-1}n_k)|^2 \\
& \leq 3E \max_{0 \leq k \leq k_{\text{max}}} |\hat{X}_N(N^{-1}n_k) - X_N(N^{-1}n_k)|^2 \\
& + 3E \max_{0 \leq k \leq k_{\text{max}}} |X_N(N^{-1}n_k) - \Xi(N^{-1}n_k)|^2 \\
& + 3E \max_{0 \leq k \leq k_{\text{max}}} |\Xi(N^{-1}n_k) - \Psi_N(N^{-1}n_k)|^2 \\
& \leq 408L^8N^{-1/2} + 3C_0[N^d]^{-1/2} + 96 \exp(24L^2d)\Delta(N).
\end{align}

In view of the Chebyshev inequality the probability in (5.39) is also estimated by the
right hand side of (5.40).
Similarly, by (2.9) and by Lemmas 3.1, 5.1 and 5.5
\begin{align}
(5.41) \quad |V_N^\varphi - \hat{V}_N^\varphi| & \leq \sup_{\zeta \in \mathcal{T}} \sup_{\eta \in \mathcal{T}} E|\hat{R}_N^\varphi(N^{-1}\zeta, N^{-1}\eta) \\
& - \hat{R}_N^\varphi(N^{-1}\zeta, N^{-1}\eta)| \\
& \leq 2\Delta(1 + 3 \exp(24L^2d))^{1/2}(D_{4K}^\varphi)^{1/2}.
\end{align}

Combining (5.38) together with (5.39)–(5.41) and Lemmas 5.1, 5.4 we complete the
proof of Theorem 2.2.

REFERENCES

[1] A.N. Borodin, A limit theorem for solutions of differential equations with random right-
hand side, Theory Probab. Appl. 22 (1977), 482–497.
[2] A.N. Borodin and M.I. Freidlin, Fast oscillating random perturbations of dynamical sys-
tems with conservation laws, Annales de l'I.H.P., sec. B, 31 (1995), 485–525.
[3] E. Bayraktar, Ya. Dolinsky and J. Guo, Recombining tree approximations for optimal
stopping for diffusions, Annales de l'I.H.P., sec. B, 31 (1995), 485–525.
[4] I. Berkes and W. Philipp, Approximation theorems for independent and weakly dependent
random vectors, Annals Probab. 7 (1979), 29–54.
[5] K.-L. Chung, A Course in Probability, 3d edition, Acad. Press, San Diego, Ca., 2001.
[6] D. S. Clark, A short proof of a discrete Gronwall inequality, Discrete Appl. Math. 16
(1987), 279–281.
[7] R. Cogburn and J.A. Ellison, A stochastic theory of adiabatic invariance, Commun.
Math. Phys. 149 (1992), 97–126.
[8] Y. Dolinsky, Applications of weak convergence for hedging of game options, Ann. Appl.
Probab. 20 (2010), 1891–1906.
[9] H. Dehling and W. Philipp, Empirical process technique for dependent data, In: H.G. Dehling, T. Mikosch and MSorenson (Eds.), Empirical Process Technique for Dependent Data, p.p. 3–113, Birkhäuser, Boston, 2002.

[10] H. He, Convergence from discrete-to continuous-time contingent claims prices, Review Financial Studies 3 (1990), 523–546.

[11] R.Z. Khasminskii, A limit theorem for the solution of differential equations with random rand-hand sides, Theory Probab. Appl. 11 (1966), 390–406.

[12] Yu. Kifer, Error estimate for binomial approximation of game options, Annals of Appl. Probab. 16 (2006), 984-1033.

[13] Yu. Kifer, Optimal stopping and strong approximation theorems, Stochastics 79 (2007), 253–273.

[14] Yu. Kifer, Lectures on Mathematical Finance and Related Topics, World Scientific, Singapore, 2020.

[15] Yu. Kifer, Strong diffusion approximation in averaging and value computation in Dynkin’s games, arXiv: 2011.07907.

[16] J. Kuelbs and W. Philipp, Almost sure invariance principles for partial sums of mixing B-valued random variables, Annals Probab. 8 (1980), 1003–1036.

[17] X. Mao, Stochastic Differential Equations and Applications, 2nd. ed., Woodhead, Oxford, 2010.

[18] D. Monrad and W. Philipp, Nearby variables with nearby laws and a strong approximation theorem for Hilbert space valued martingales, Probab. Th. Rel. Fields 88 (1991), 381–404.

[19] D. Monrad and W. Philipp, The problem of embedding vector-valued martingales in a Gaussian process, Theory Probab. Appl. 35 (1991), 374–377.

[20] D.W. Stroock and S.R.S. Varadhan, Multidimensional Diffusion processes, Springer-Verlag, Berlin, 1997.

Institute of Mathematics, The Hebrew University, Jerusalem 91904, Israel
Email address: kifer@math.huji.ac.il