Diagonalization of $pp$-waves

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Abstract

A coordinate transformation is found which diagonalizes the axisymmetric $pp$-waves. Its effect upon concrete solutions, including impulsive and shock waves, is discussed.

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1 Introduction

A well-known class of gravitational waves are the plane-fronted waves with parallel rays (pp-waves) which admit a covariantly constant null vector \[1\], \[2\], \[3\], \[4\]. The metric can be written using cylindrical coordinates:

\[
 ds^2 = 2dudW + 2Hdu^2 - dP^2 - P^2d\Phi^2 
\]  

where \( H = H(u, P, \Phi) \) and is of Petrov type \( N \) or conformally flat. Mainly axisymmetric pp-waves have been discussed in the literature. Such is the Schwarzschild solution boosted to the speed of light, which is interpreted as a massless null particle \[5\], \[6\] or an ultrarelativistic black hole \[7\], \[8\], \[9\]. A ring of massless particles is produced by boosting the Kerr metric \[10\]. Axisymmetric pp-waves describe also the gravitational field of light beams \[11\], \[12\]. Plane-fronted electromagnetic waves generate pp-gravitational waves as exact solutions of the Einstein-Maxwell equations \[13\], \[14\], \[15\].

While superposition of pp-waves running in the same direction is trivial, their collisions have been studied mainly for the subclass of plane waves \[16\], \[17\] in which the physical invariants are constant over the wave surfaces. The reason is that (1) is unsuitable to describe two approaching waves, but in the case of plane waves a transformation due to Rosen \[18\] converts (1) into the Szekeres line element \[4\], \[17\], \[19\] which can encompass two approaching waves and the region of their interaction. For a wave of constant polarization it reads

\[
 ds^2 = 2dudv - e^{-U} \left( e^V dr^2 + e^{-V} d\varphi^2 \right) 
\]  

where \( U \) and \( V \) are functions of the null coordinate \( u \). Having in mind that \( \sqrt{2}u = t - z \), \( \sqrt{2}v = t + z \) it is clear that (2) is a diagonal metric. Unfortunately, such waves have infinite extent and energy. The waves given by (1) with a suitably chosen \( H \) are finite in extent and energy but are not asymptotically flat and contain metric discontinuities for impulsive and shock waves.

In the present paper we generalize the Rosen transformation to axisymmetric pp-waves and find for them a diagonal and asymptotically flat form like (2). This may be considered both as an alternative description and as a preparatory step before the investigation of their head-on collisions.

2 Generalized Rosen transformation

Let us investigate upon what conditions on \( H(u, P, \Phi) \) the metric (1) can be diagonalized. Following Rosen, we do not change \( u \) at all. The requirement \( g_{uv} = 1 \), like in (2), is ensured by \( W = v + W_1(u, r, \varphi) \). Next, the vanishing of \( g_{vv} \) takes place when \( P_v = \Phi_v = 0 \). The non-diagonal term \( g_{r\varphi} \) disappears if

\[
 P_rP_\varphi + P^2\Phi_r\Phi_\varphi = 0 
\]
Obviously $P_r \neq 0$, $\Phi_\varphi \neq 0$ and the simplest solution of (3) is $P_\varphi = \Phi_r = 0$.

Then the remaining non-diagonal terms disappear when

$$W_{1r} = P_u P_r \quad W_{1\varphi} = P^2 \Phi_u \Phi_\varphi$$

(4)

with $P(u,r)$ and $\Phi(u,\varphi)$. Integrating the first equation in (4) and plugging it into the second we find a l.h.s. independent of $r$, unlike the r.h.s. The simplest solution is $\Phi_u = 0$ and $\Phi = \varphi$ which does not change the range of the angular coordinate. Hence $W_1 = W_1(u,r)$. The nullification of $g_{uu}$ is guaranteed by the relation

$$2H = -2W_{1u} + P_u^2$$

(5)

The r.h.s. depends on $u$ and $r$, therefore after the coordinate transformation $H = H(u,r)$, which means that in the beginning it was $H(u,P)$. Thus the sufficient condition for the diagonalization of (1) is the axial symmetry of $H$.

We can’t say that it is also necessary because the simplest solutions of (3,4) have been used.

Integrating eq(4), inserting the result into (5) and taking the $r$-derivative we obtain the main equation which governs the generalized Rosen transformation:

$$P(u,r)_{uu} = -H(u,P)_u$$

(6)

The line element (1) becomes

$$ds^2 = 2dudv - P_r^2 dr^2 - P^2 d\varphi^2$$

(7)

which is diagonal and the coordinate transformation reads

$$u = u \quad \Phi = \varphi \quad P = P(u,r) \quad W = v + W_1(u,r)$$

(8)

where $P$ is determined by (6) and $W_1$ by (4). $W$ does not appear in (6,7).

Usually (1) is written in cartesian coordinates:

$$ds^2 = 2dudW + 2H(u,X,Y)du^2 - dX^2 - dY^2$$

(9)

For comparison, we apply the same diagonalization process to (9) with the following results. $H$ must be separable

$$H = H_1(u,X) + H_2(u,Y)$$

(10)

there are two main equations

$$X(u,x)_{uu} = -H_1(u,X)_X \quad Y(u,y)_{uu} = -H_2(u,Y)_Y$$

(11)

the line element becomes

$$ds^2 = 2dudv - X_x^2 dx^2 - Y_y^2 dy^2$$

(12)

and the coordinate transformation reads

$$u = u \quad X = X(u,x) \quad Y = Y(u,y)$$
\[ W = v + \int^X X_u X_{x'} dx' + \int^Y Y_u Y_{y'} dy' \quad (13) \]

In vacuum the only surviving Einstein equation gives
\[ H_{1XX} + H_{2YY} = 0 \quad (14) \]

After the usual removal of linear terms in \( H_i \) (11) becomes linear:
\[ X_{uu} = -f(u) X \quad Y = f(u) Y \quad (15) \]

where \( f(u) \) is the second derivative of \( H_1 \). This permits the choice \( X = xF(u) \), \( Y = yG(u) \) which represents exactly the Rosen transformation for plane waves of constant polarization \([4],[18],[20]\). Thus in vacuum the diagonalization of \( pp \)-waves in cartesian coordinates requires separability (10) and leads directly to plane waves.

3 \( pp \)-waves in Brinkmann and diagonal form

We have shown that an axisymmetric \( pp \)-wave can be described by (7). At first sight (7) is a special case of the Szekeres line element (2) where now \( U \) and \( V \) depend on \( u \) and \( r \). In fact, there is no loss of generality and any metric (2) may be written as (7) provided the Einstein equations hold. Let us compare these equations for (1,7) and (2).

A \( pp \)-wave allows energy-momentum tensors of few types: vacuum, null electromagnetic field or pure radiation (null dust), which may be combined \([14],[15]\). All of them have only one non-trivial component:
\[ T_{uu} = 2\rho(u,P) = 2\rho_R + 2\rho_E \quad (16) \]

where \( \rho_E \) is the energy-density of pure radiation with no matter equations and \( \rho_E \) is the electromagnetic energy-density
\[ 2\rho_E = \nabla \psi \nabla \psi \quad \psi = A_u \quad (17) \]

We have used relativistic units with \( 8\pi G/c^4 = 1 \). \( A_u \) is the only component of the vector potential and satisfies the Maxwell equation \( \Delta \psi = 0 \). We suppose that no charges and currents are present. The gradient and Laplacian are with respect to \( P, \Phi \) (or \( X, Y \)). We have chosen this formalism instead of the Newman-Penrose one since \( T_{uu}, A_u \) and the Maxwell equation do not change under the generalized Rosen transformation (8).

In Brinkmann coordinates the only non-trivial Ricci tensor component is \( R_{uu} \) which is a Laplacian. The Einstein equation is
\[ H_{PP} + \frac{1}{P} H_P = 2\rho \quad (18) \]

and its solution is well-known from classical potential theory:
\[ H_P = \frac{2}{P} \int_0^P \rho(u,P') \, P' dP' + \frac{2}{P} \rho_e(u) \quad (19) \]
\[ H = 2 \int_0^P \frac{dP'}{P'} \int_0^{P'} \rho(u, P'') P'' dP'' + 2 \rho_e(u) \ln \frac{P}{a} \]  

(20)

where an ignorable term has been omitted in (20). The second term in (19,20) is the exterior solution with arbitrary \( \rho_e \) and some constant length \( a \). The Maxwell equation is

\[ (P \psi_P)_P + \frac{1}{P} \psi_{\varphi \varphi} = 0 \]  

(21)

We have retained some \( \varphi \)-dependence in \( \psi \) but it must disappear in \( \rho_E \). The only non-zero Weyl scalar is given by

\[ \Psi_4 = \rho - \frac{H_P}{P} \]  

(22)

Now, let us concentrate on the metric (2) with \( U(u, r) \), \( V(u, r) \). The Einstein equations yield:

\[ 2U_{uu} = U_u^2 + V_u^2 + 4\rho \]  

(23)

\[ (U + V)_{ru} = (U + V)_r V_u \]  

(24)

\[ (U + V)_{rr} = (U + V)_r V_r \]  

(25)

Eqs(24,25) are easily integrated

\[ (U + V)_r = -2e^V \]  

(26)

We have chosen the integration constant in order to restore Minkowski space-time for \( V = U \), \( r = e^{-V} \). This allows smooth transition to it in front of the wave and is in accord with our demand for asymptotically flat solutions. The Maxwell equation is

\[ \left( e^{-U} \psi_r \right)_r + e^V \psi_{\varphi \varphi} = 0 \]  

(27)

where like in (21) we allow for some \( \varphi \)-dependence. Using the natural NP null tetrad \[ \text{[4]} \] we find three non-trivial Weyl scalars:

\[ \Psi_2 = -\frac{1}{12} e^{U-V} \left[ (U + V)_{rr} - (U + V)_r V_r \right] \quad \Psi_3 = -\frac{\sqrt{2}}{4} e^{-\frac{U}{2}} R_{ur} \]  

(28)

\[ \Psi_4 = \frac{1}{2} (V_u U_u - V_{uu}) \]  

(29)

A look at the Einstein equations (23-25) shows that \( \Psi_2 = \Psi_3 = 0 \) and the field is of type \( N \) not of type \( \Pi \).

One can try to solve the relevant eqs(23,26), when \( \rho \) is given, in two ways. First, we may take an arbitrary \( U \), solve for \( V \) from (23) and insert the result into (26). This leads to a condition on \( U \):

\[ (U + A)_r = a_1(r) e^A + a_2(r) \quad A = \int \sqrt{2U_{uu} - U_u^2 - 4\rho} du \]  

(30)

with arbitrary \( a_1 \neq 0 \) and \( a_2 \). Eq(30) is too complicated to be examined. Second, we take an arbitrary \( V \) and notice that (23) is a linear second-order equation for \( e^{-U/2} \). Then we integrate (26):

\[ U = -\int \left( 2e^V + V_r \right) dr + f_1(u) \]  

(31)
where $f_1(u)$ is some yet undetermined function. Substituting (31) into (23) we get an equation for $f_1$ with additional conditions on $V$ to yield a $r$-independent $f_1$ which again are very complicated.

Let us now compare the two theories. The link is given by (7):

$$\begin{align*}
P &= e^{-U} + V^2 \\
P_r &= e^{-U}
\end{align*}$$

Eq(32) gives at first sight an additional constraint between $U$ and $V$ but this turns out to be exactly (26) which is necessarily satisfied. Going backwards, (26) shows that (32) holds for some $P$. It can be shown further that (6,18) are equivalent via (32) to (23). The same is true about (21) and (27) which is not so surprising for a Laplacian. At last, under (32) the Weyl scalar (22) coincides with (29). Consequently, axisymmetric $pp$-waves (1) are in a one-to-one correspondence with the solutions $U(u,r), V(u,r)$ for metric (2). Thus we can replace metric (2) with two functions by metric (7) with one function $P$ or by metric (1) with one function $H$. Each of these forms has its own merits. The Brinkmann metric (1) has simple Einstein equations (compare (18) with (30,31)) but is not asymptotically flat for exterior solutions, sometimes has a discontinuous $H$ and is unfit for studying collisions of $pp$-waves because of only one null coordinate. The metric given by (7) or (2,32) is worth as a starting point for the interaction problem and is asymptotically flat for realistic $\rho$ as will be shown in the following. However, $0 \leq P \leq \infty$ as a radial coordinate in (1) which makes $g_{\varphi\varphi}$ in (7) singular at some points. This coordinate singularity is innocuous when it is due to the cylindrical character of the coordinate system. If not, the experience with plane waves teaches that it becomes a fold singularity and is intimately related to the curvature singularity in the interaction region [21].

4 Solutions: general features

Eq(6) with $H$ satisfying (20) is a second-order nonlinear differential equation with respect to $P$. In the process of solving it arbitrary functions of $r$ arise which reflect the residual freedom in the coordinate transformation and may be selected to further simplify the solution and satisfy boundary conditions.

The trivial Minkowski solution is given by $H = 0$, $P = r$, $P_r = 1$. Having in mind the setting of the collision problem, $u = 0$ must be the boundary between the running wave ($u > 0$) and Minkowski spacetime ($u < 0$) where the wave has not yet arrived. This gives the universal boundary condition

$$P(0,r) = r$$

We also demand asymptotic flatness i.e. $P(u,\infty) = r$ for fixed $u$.

It is clear from (20) that the exterior solution is always separable, the interior is separable when $\rho = \rho_1(P)\rho_2(u)$. Almost all $pp$-waves discussed in the literature are of this kind with $H(u,P) = H_1(P)\rho_2(u)$ and we shall consider only them in the following. A natural question arises: when $H$ is static in (1) is it possible that $P$ is also static in (7)? This is not allowed by (4,5). A static
$P$ has $P_u = 0$, $W_1 = W_1(u)$ and $H = H(u)$ contrary to our assumption that $H = H(P(r))$. Hence $P$ depends on $u$ even when $\rho_2(u) = 1$.

For the exterior solution given in (19) eqs(6,22) become

$$P_{uu} = -\frac{2}{r^2} \rho_e(u) \quad \Psi_4 = -\frac{2}{r^2} \rho_e(u)$$

(34)

Unlike (15), eq(34) is non-linear and we can’t get rid of the $r$-dependence. For many simple choices of $\rho_e$ (34) falls in the class of Emden-Fowler equations [22]. They are quite difficult to solve and many of them remain non-integrable. For example, when $\rho_e(u) = u^n$ the integrable cases are just $n = 0; -1; -2$. For asymptotically flat solutions (34) shows that $\Psi_4 \to 0$ when $r \to \infty$.

The general separable interior solution emerges from

$$P_{uu} = -H_1(P) \rho_2(u)$$

(35)

and again is reducible in many cases to the Emden-Fowler equations and their generalizations. The case $\rho_1(P) = 1$ is special. Then (35) is linear in $P$ and we can use an analog of the ansatz applied after (15), namely $P(u,r) = rp(u)$:

$$H = \frac{1}{2} (X^2 + Y^2) \rho_2(u) \quad \Psi_4 = 0$$

(36)

$$p_{uu} = -\rho_2(u) p$$

(37)

$$ds^2 = 2 dudv - p(u)^2 \left( dX^2 + dY^2 \right)$$

(38)

This is the case of pure radiation with density $\rho_R = \rho_2$ or an electromagnetic wave with potential, Maxwell scalar and energy-density given by

$$\psi = a_3(u) X + a_4(u) Y \quad \Phi_2(u) = -\frac{1}{\sqrt{2}} \left[ a_3(u) - ia_4(u) \right]$$

(39)

$$\rho_E(u) = \rho_2(u) = \frac{1}{2} \left( a_3^2 + a_4^2 \right)$$

(40)

where $a_3, a_4$ are arbitrary functions. Obviously $\psi$ depends on $\varphi$ while $\rho_E$ does not. The potential satisfies the Maxwell equation (21). In fact eqs(36-40) represent a plane electromagnetic wave [8] and an axisymmetric electromagnetic $pp$-wave at the same time. There is no pure gravitational wave in addition because $\Psi_4 = 0$. The Ricci scalar $\Phi_{22} = \Phi_2 \Phi_2$ is constant over the wave surface. On the contrary, pure plane gravitational waves can not be axisymmetric because their $H \sim X^2 - Y^2$ which is $\varphi$-dependent.

The case discussed above provides a link between plane and axisymmetric $pp$-waves. Even for it, eq(37) is the normal form of the general linear second-order equation and its general solution is given analytically only if a non-trivial concrete solution is known. Therefore we are going to discuss two cases of simple $u$-dependence when the solution of (6) may be found. These are the impulsive and shock waves.
5 Impulsive waves

These are waves with $\rho_2 = \delta(u)$. Eq(37) may be integrated with the help of (33):

\[ P = r \left( 1 - \frac{H_{1r}}{r} u \right) \quad P_r = 1 - H_{1r} u \quad (41) \]

Eq(22) transforms into

\[ \Psi_4 = \left( \rho_1 (r) - \frac{H_{1r}}{r} \right) \delta(u) \quad (42) \]

which clearly demonstrates the impulsive character of the wave. It is seen from (19) that $H_{1r} > 0$ and (41) shows that $P$ always possesses a coordinate singularity for some $u > 0$, different from the cylindrical singularity at $r = 0$. This is a consequence of the positive energy condition and the idealized impulsive character of the wave.

For a boosted Schwarzschild solution [5, 6, 9] $H = 2\mu \delta(u) \ln P^2$ where $\mu$ is the momentum of the null point-like particle and (41, 42) give

\[ ds^2 = 2dudv - \left[ 1 + \frac{4\mu}{r^2} u \theta(u) \right]^2 \left( 1 - \frac{4\mu}{r^2} u \theta(u) \right)^2 r^2 d\varphi^2 \quad (43) \]

\[ \Psi_4 = \left( \delta(r) - \frac{4}{r^2} \right) \mu \delta(u) \quad (44) \]

Eq(43) is exactly the line element found in [7, 8]. There is a curvature singularity at the point of the source $r = 0$. $H$ is also the function for an exterior impulsive solution given in (20). If $t$ is fixed, for any $z$ and $r \to \infty$ the solution is asymptotically flat. There is a coordinate singularity at $\sqrt{2}r^2 = 4\mu (t - z)$. For a fixed $z$, as time goes by, the singular circle centred at $z$ expands towards infinity.

For a boosted Kerr solution [10]:

\[ H = 2\mu \delta(u) \ln \left| P^2 - b^2 \right| \quad (45) \]

According to (41):

\[ P = r \left( 1 - \frac{4\mu}{r^2 - b^2} u \right) \quad P_r = 1 + \frac{4\mu}{r^2 - b^2} u \quad (46) \]

where $b$ is the radius of the ring of massless particles. The curvature singularity moves to $r = b$ and the region $r \leq b$ is free of coordinate singularities. The metric is asymptotically flat.

As a final example we present the diagonalization of an impulsive beam of light with transverse radius $a$ [11, 12]. This is a global solution the interior being given by (36) and the exterior by (20). The junction conditions require that

\[ H = \frac{4m P^2}{a^2} \theta(a - P) \delta(u) + 4m \left( 1 + 2 \ln \frac{P}{a} \right) \theta(P - a) \delta(u) \quad (47) \]

where $m$ is the constant energy density. With the help of (18) eq(41) may be rewritten as

\[ P = r \left( 1 - \frac{H_{1r}}{r} u \right) \quad P_r = 1 + \left( \frac{H_{1r}}{r} - 2\rho_1 \right) u \quad (48) \]
Inserting (47) into (48) we obtain for the interior and exterior solutions:

\[ P_i = r \left(1 - \frac{8m}{\sigma^2} u\right) \quad \quad P_{ir} = 1 - \frac{8m}{\sigma^2} u \quad (49) \]

\[ P_e = r \left(1 - \frac{8m}{\sigma^2} u\right) \quad \quad P_{er} = 1 + \frac{8m}{\sigma^2} u \quad (50) \]

It is seen that \( P \) is continuous at \( r = a \) but \( P_r \) makes a finite jump. According to (48) the reason is the jump in \( \rho_1 \) from zero to a finite constant, since the junction conditions require that \( H_1 \) and \( H_{1r} \) should be continuous. Consequently, solutions which are perfectly well joined in Brinkmann coordinates acquire discontinuous metric upon diagonalization due to unrealistic densities with \( \theta (r) \) terms. The problem disappears when the density smoothly falls to zero. Take for example \( \rho_1 (P) = e^{-P^2} \). Then

\[ \delta (u) \]

\[ H = \left[ \ln P - \frac{1}{2} \text{Ei} \left(-P^2\right) \right] \delta (u) \quad (51) \]

\[ P = r \left(1 - \frac{1-e^{-r^2}}{r^2} u\right) \quad \quad P_{r} = 1 + \frac{1-e^{-r^2}-2r^2 e^{-r^2}}{r^2} u \quad (52) \]

When \( P \to 0, \infty \) \( H \) in (51) approaches the first or the second term in (47). Correspondingly, when \( r \to 0 \) (52) approaches (49) and when \( r \to \infty \) it approaches (50) with \( 8m = a = 1 \). The metric (52) is asymptotically flat but the coordinate singularities still exist.

### 6 Shock waves

These waves have \( H = H_1 (P) \theta (u) \) and (6) has a first integral:

\[ P_u^2 = c (r) - 2H_1 (P) \quad (53) \]

It is clear from (20) that \( H_1 \) is a positive and increasing function. This is the reason to keep the arbitrary function \( c (r) \) in (53) so that the r.h.s. is positive. Eq(53) is easily integrated. Imposing (33) we obtain

\[ \pm u = \int_r^P \frac{dP'}{\sqrt{c(r) - 2H_1 (P')}} = K (P, r) - K (r) \quad (54) \]

which gives \( P (u, r) \) indirectly. For future convenience we have introduced also the indefinite integral \( K \).

In order to understand the meaning of \( c (r) \) let us discuss the interior solution (36-38) with \( \rho_2 (u) = \theta (u) = 1 \) in the region occupied by the wave. The integral in (54) can be evaluated:

\[ \arcsin \frac{P}{\sqrt{c}} = -u + \arcsin \frac{r}{\sqrt{c}} \quad (55) \]

and \( P \) is found by inverting (55). Let us choose

\[ c (r) = 2H_1 (r) \quad (56) \]
Then we obtain

\[ P = r \cos u \]  (57)

\[ ds^2 = 2dudv - \cos^2 u \left( dr^2 + r^2 d\varphi^2 \right) \]  (58)

This, however, is the line element of an electromagnetic shock wave with Ricci scalar \( \Phi_{22} = \theta(u) \) and this is really the case here because we can choose in (39) \( a_3 = \sqrt{2}\theta(u), a_4 = 0, \Phi_2 = -\theta(u) \). We conclude that plane waves recommend the receipt (56). Eq(57) shows that it is equivalent to the method used in (35-37) for the linear case.

Let us apply this receipt to the exterior solution \( H_1(P) = b \ln \frac{P}{a}, b > 0 \). The integral in (54) yields the error function

\[ \text{erf} \sqrt{\ln \frac{r}{P}} = \sqrt{\frac{2b}{\pi r}} u \]  (59)

This formula may be inverted

\[ P = r \exp \left\{ - \left[ \text{erf}^{-1} \left( \sqrt{\frac{2b}{\pi r}} u \right) \right]^2 \right\} \]  (60)

The metric satisfies the necessary boundary condition (33) and is asymptotically flat for fixed \( u \). The problem is that \( \text{erf} z \leq 1 \) and (59) imposes the constraint

\[ u \leq \sqrt{\frac{\pi}{2b} r} \]  (61)

The solution (60) does not cover the whole region \( u \geq 0, 0 \leq r \leq \infty \).

This is a generic feature of the choice (56) into (54). Then \( P \leq r \) because \( H_1 \) is an increasing function. Hence, the minus sign must be chosen in (54). When \( u \) increases \( P \) necessarily decreases, becomes null and sometimes even negative as (57) demonstrates. However, for fixed \( r \) it remains bounded in order to keep the root in (54) real. The integral in (54) also remains finite so there should be some limit for the growth of \( u \) like (61). The same happens in (55) if we stick to the main branch of arcsin \( x \). Fortunately, \( u(P) \) may be a multivalued function while \( P(u) \) can not be. This explains why there are no problems in this case. Multivalued functions resulted from the inversion of periodic functions. In the general case periodic functions do not appear in \( K \) and that causes the limit problem. If \( P \) is not extended to negative values the limit of \( u \) is also a coordinate singularity and is given by

\[ u = K(r) - K(0) \]  (62)

This singularity is present generically in solutions with (56).

Another choice is

\[ c(r) = 2H_0 = 2H_1(P_0) \]  (63)

where \( H_0 \) is some very big constant. Then we may take the positive sign in (54) and \( P_0 \geq P \geq r \). The \( P, u \) and \( r \) dependencies separate:

\[ K(P) = u + K(r) \]  (64)
There is no coordinate singularity but the region \( P > P_0 \) is not described by this coordinate system because \( K \) is ill-defined there. In turn this means

\[
r < P_0 \quad \quad u < K(P_0) - K(0)
\]  

(65)

For the exterior solution these inequalities look like

\[
r < ae^{\frac{H_0}{b}} \quad \quad u < a \sqrt{\frac{2}{\pi}} e^{-\frac{H_0}{b}} \text{erf} \sqrt{\frac{H_0}{b} - \ln \frac{r}{a}}
\]  

(66)

If the first of them is made stronger and \( H_0 \) is taken big enough, \( u \) and \( r \) can cover a lot of their range. The choice (63) is perfect if \( H_1(P) \) were bounded from above and \( H_0 > H_{1\text{max}} \). Unfortunately, this does not happen due to the lower limit of the inside integral in (20). It cures the singular behaviour at small \( P \) but generates a logarithmic term in \( H_1 \) like the first term in (51).

A problem arises when we try to join the interior and exterior solutions discussed above. We start with (47) and \( \delta(u) \) replaced by \( \theta(u) \). Now we can’t replace e.g. \( \theta(a-P) \) by \( \theta(a-r) \) and that makes eq(54) intractable. Like in the case of impulsive waves it is preferable to have one smoothly falling out \( \rho \) for all \( r \) like the example given by (51). With such \( H_1 \) the integral in (54) can not be evaluated analytically and the limit problem still exists. This is the best we can do for realistic shock waves.

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