THE EXISTENCE OF QUASIMEROMORPHIC MAPPINGS IN
DIMENSION 3

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Abstract. We prove that a Kleinian group $G$ acting upon $H^3$ admits a non-constant $G$-automorphic function, even if it has torsion elements, provided that the orders of the elliptic (i.e. torsion) elements are uniformly bounded. This is accomplished by developing a technique for meshing distinct fat triangulations while preserving fatness. We further show how to adapt the proof to higher dimensions.

1. Introduction

The object of this article is the study of the existence of $G$-automorphic quasimeromorphic mappings (in the sense of Martio and Srebro – see [MS1]) $f : H^n \to \tilde{R}^n$, $\tilde{R}^n = \mathbb{R}^n \cup \{\infty\}$; i.e. such that

$$f(g(x)) = f(x) \quad \forall x \in H^n; \forall g \in G;$$

were $G$ is Kleinian group acting upon $H^n$.

Our principal goal is to prove the following:

**Theorem 1.1.** Let $G$ be a Kleinian group with torsion acting upon $\mathbb{H}^n$, $n \geq 3$. If the elliptic elements (i.e. torsion elements) of $G$ have uniformly bounded orders, then there exists a non constant $G$-automorphic quasimeromorphic mapping $f : H^n \to \tilde{R}^n$.

In this paper we restrict ourselves to the proof of the theorem in the classical case (i.e. $n = 3$) only. This restriction is motivated by two reasons: (a) the proof in the 3-dimensional case employs mainly elementary tools and (b) it develops and uses a technique for for meshing distinct fat triangulations while preserving fatness, technique that is relevant in Computational Geometry and Mathematical Biology. The proof of the general case is presented in [S1] and it is based upon a more general result concerning the existence of fat triangulations for manifolds with boundary – see [S2].

The question whether quasimeromorphic mappings exist was originally posed by Martio and Srebro in [MS1]; subsequently in [MS2] they proved the existence of fore-mentioned mappings in the case of co-finite groups i.e. groups such that $Vol_{hyp}(H^n/G) < \infty$ (the important case of geometrically finite groups being thus included). Also, it was later proved by Tukia ([Tu]) that the existence of non-constant quasimeromorphic mappings (or qm-maps, in short) is assured in the case when $G$ acts torsionless upon $\mathbb{H}^n$. Moreover, since for torsionless Kleinian groups...
$G, \mathbb{H}^n/G$ is a (analytic) manifold, the next natural question to ask is whether there exist \( \text{qm}\)-maps \( f : M^n \to \mathbb{R}^n \); where \( M^n \) is an orientable \( n \)-manifold. The affirmative answer to this question is due to K. Peltonen (see [Pe]); to be more precise she proved the existence of \( \text{qm}\)-maps in the case when \( M^n \) is a connected, orientable \( C^\infty \)-Riemannian manifold. 

In contrast with the above results it was proved by Srebro ([Sr]) that, for any \( n \geq 3 \), there exists a Kleinian group \( G \rangle \mathbb{H}^n \) s.t. there exists no non-constant, \( G \)-automorphic function \( f : \mathbb{H}^n \to \mathbb{R}^n \). More precisely, if \( G \) (as above) contains elliptics of unbounded orders (with non-degenerate fixed set), then \( G \) admits no non-constant \( G \)-automorphic \( \text{qm}\)-mappings.

Since all the existence results were obtained in constructive manner by using the classical ”Alexander trick” (See [Al]), it is only natural that we try to attack the problem using the same method. For this reason we present here in succinct manner Alexander’s method: One starts by constructing a suitable triangulation (Euclidian or hyperbolic) of \( \mathbb{H}^n \) or of \( M^n = \mathbb{H}^n/G \). Since \( \mathbb{H}^n \) and \( M^n \) are orientable, an orientation consistent with the given triangulation (i.e. such that two given \( n \)-simplices having a \((n-1)\)-dimensional face in common will have opposite orientations) can be chosen. Then one quasiconformally maps the simplices of the triangulation into \( \mathbb{R}^n \) in a chess-table manner: the positively oriented ones onto the interior of the respective simplex (see [Tu], [MS2]), and since the dilatation is to be uniformly bounded, we are naturally directing our efforts in the construction of a \textit{fat} triangulation, where:

**Definition 1.2.** A \( k \)-simplex \( \tau \subset \mathbb{R}^n \) (or \( \mathbb{H}^n \)); \( 2 \leq k \leq (n-1) \) is \textit{f-fat} if there exists \( f \geq 0 \) s.t. the ratio \( \frac{r}{R} \geq f \); where \( r \) denotes the radius of the inscribed sphere of \( \tau \) (inradius) and \( R \) denotes the radius of the circumscribed sphere of \( \tau \) (circumradius).

A triangulation (of a submanifold of \( \mathbb{R}^n \) or \( \mathbb{H}^n \)) \( \mathcal{T} = \{ \sigma_i \}_{i \in I} \) is \textit{f-fat} if all its simplices are f-fat.

A triangulation \( \mathcal{T} = \{ \sigma_i \}_{i \in I} \) is \textit{fat} if there exists \( f \geq 0 \) s.t. all its simplices are \( f \)-fat; \( \forall i \in I \).

The idea of the proof of Theorem 1.1. is first to build two fat triangulations: \( \mathcal{T}_c \) of a certain closed neighbourhood \( \overline{N}_c \) of the fixed set of \( G \) in \( \mathbb{H}^n \); and \( \mathcal{T}_g \) of \( \mathbb{H}^n \setminus \overline{N}_c \); and then to ”mash” the two triangulations into a new triangulation, while retaining their fatness\(^1\).

The first triangulation is constructive and is based upon the geometry of the elliptic transformations. The existence of the second triangulation is assured by Peltonen’s result. Unfortunately, these two triangulations are not \( G \)-invariant, so they are unsuited for our purpose of building a \( G \)-automorphic function. However, they induce fat triangulations: \( \mathcal{T}_c^* \) on \( (\overline{N}_c \cap \mathbb{H}^n)/G \), and \( \mathcal{T}_g^* \) on \( M_c = (\mathbb{H}^n \setminus \overline{N}_c)/G \), where \( M_c \) is a differential manifold with boundary \( \partial M_c = (\partial \overline{N}_c \cap \mathbb{H}^n)/G \).

Fortunately, we are provided with a ready made method of mashing triangulations, so we can direct our efforts towards the task of ”fattening” the simplices of the ”intermediate zone”; task which will be carried out in Section 4.

\(^1\) But see also [Ca1], [Ca2].
The “mashing” method mentioned above is based on a result and, even more, on the technique used in its proof, due to Munkres:

**Theorem 1.3.** Let $M^n$ be a $C^r$-manifold with boundary. Then any $C^r$-triangulation of $\partial M^n$ can be extended to a $C^r$-triangulation of $M^n$, $1 \leq r \leq \infty$.

Because of its importance to our own construction, we shall present the basic idea of the proof of Theorem 1.3. in the next Section.

This paper is organized as follows: in Section 2 we present the necessarily background on elliptic transformations and present in a nutshell the main techniques we employ: the Alexander trick, Peltonen’s method and the Proof of Munkres’ Theorem. In Section 3 we show how to choose and triangulate the closed neighbourhood of the $N$ of the fixed set of $G$, and how to select the ”intermediate zone” where the two different triangulations overlap. Section 4 is dedicated to the main task of fattening the common triangulation. Finally, in Section 5 we indicate the way of adapting our construction to higher dimensions.

2. Preliminaries

2.1. Elliptic Transformations. We shall restrict ourselves mainly to the 3-dimensional case, for, as we have already stated, this will be the direction in which our main efforts will be directed.

Let us first recall the basic definitions and notations: A transformation $f \in Isom(H^n)$, $f \neq Id$ is called elliptic if ($\exists$) $m \geq 2$ s.t. $f^m = Id$, and $m$ is called the order of $f$.

In the 3-dimensional case the fixed point set of $f$, i.e. $Fix(f) = \{x \in H^n | f(x) = x\}$, is a hyperbolic line and will be denoted by $A(f)$ - the axis of $f$. If $A$ is an axes of an elliptic of order $m$, then $A$ is called an $m$-axes.

If the discrete group $G$ is acting upon $H^3$, then by the discreteness of $G$, there exists no accumulation point of the elliptic axes in $H^3$. Moreover, if $G$ contains no elliptics with intersecting axes, then the distances between the axes are, in general, bounded from below. To be more precise, the following holds:

**Theorem 2.1 (GM1).** Let $G$ be a discrete group $G$ acting upon $H^3$, and let $f, g \in G$ be s.t. ord$(f) \geq 3$ or ord$(g) \geq 3$; and s.t. $A(f) \cap A(g) = \emptyset$.

Then $\exists \delta > 0$, that is independent of $G, f, g$ s.t.

$$\text{dist}_{hyp}(A(f), A(g)) \geq \delta;$$

where $\text{dist}_{hyp}$ denotes the hyperbolic distance in $H^3$.

It is extremely important to notice that the results above do not include the case when all the elliptic transformations of $G$ are of order 2, since they depend intrinsically upon:

**Theorem 2.2 (J, AH).** $G < H^n$ is discrete iff $< f, g >$ is discrete; $\forall f, g \in G$.

The theorem above is of little avail in the case of two half-turns (that is elliptics of order 2); since any two half-turns (in $H^3$) generate a discrete group. (See S3 for a proof of this "folkloric" result from an unpublished paper by Jørgensen.) Indeed, examples of discrete groups of isometries of hyperbolic 3-space can be constructed, such that the distances between the axes of the order 2 elliptics are not bound from below (see S3).

In the presence of node points (i.e. intersections of axes) the situation is more complicated. Fortunately, there are only a few types of such possible intersections
- for the orders of the elliptic axes meeting at a node point must satisfy certain conditions (determined by the Euler number for the orbifold\(^2\)); namely, the possible local situations in orbifold are: either (a) Dihedral, i.e. of the type \((2,2,n)\), \(n \geq 2\); or one of the following exceptional types: (b) Tetrahedral, i.e. of type \((2,2,3)\); (c) Octahedral, i.e. of type \((2,3,4)\); (d) Icosahedral, i.e. of type \((2,3,5)\); (see Fig. 1).

**Figure 1.**

Remark 2.3. The reduced number of possibilities is rather fortunate, for the computation of the distances between node points is more difficult than that of distances between disjoint axes. (For more specific information in this direction of study see \([DM]\) and \([Med]\); and more recently \([GMI]\), \([GM1]\) and \([GMMR]\)).

Since our main interest lies in Kleinian groups acting upon \(\mathbb{H}^3\) whose elliptic elements have orders bounded from above, the following theorem is highly relevant:

**Theorem 2.4** (\([FM]\)). *Let \(G\) be finitely generated Kleinian group acting on \(\mathbb{H}^3\). Then the number of conjugacy classes of elliptic elements is finite.*

For a sketch of an alternative proof of this Theorem see Appendix.

Remark 2.5. The Theorem above is not true for groups of isometries of \(\mathbb{H}^n\), \(n \geq 4\); indeed there exist counterexamples, one due to Mess and Feighn (\([FM]\)) and another due to Kapovitch and Potyagailo (\([KP]\)). It should be remarked that both of the examples cited above produce (albeit different) conjugacy classes of elliptics of the same order. Considering this and the goal of our investigation, the following recent result is highly relevant:

**Theorem 2.6** (\([H]\)). *There exists a discontinuous group \(\Gamma \leq \text{Isom}(\mathbb{H}^n)\), \(n \geq 4\); such that:
(i) \(\Gamma\) contains elliptics of arbitrary large orders and

\(^2\) See \([Th1]\), \([Th2]\) for the definition and the necessary proofs.

\(^3\) and also \([S3]\).
(ii) $\mathrm{Vol}(N_{\epsilon}(M_{\Gamma})) < \infty^4$, where $M_{\Gamma} = H_{\Gamma}/\Gamma$, and $H_{\Gamma}$ denotes the convex core (in $H^n$) of the limit set $\Lambda(\Gamma) \subset H^n$, and $N_{\epsilon}$ represents the $\epsilon$-neighbourhood of $M_{\Gamma}$.

2.2. Alexander's Trick. The technical ingredient in Alexander's trick is the following Lemma (which we formulate for $\mathbb{R}^3$ only, but which readily generalizes to higher dimensions):

**Lemma 2.7.** ([MS1], [Pe]) Let $T$ be a fat triangulation of $M \subset \mathbb{R}^3$, and let $\tau, \sigma \in T$, $\tau = (p_1, p_2, p_3, p_4)$, $\sigma = (q_1, q_2, q_3, q_4)$; and denote $|\tau| = \tau \cup \text{int}\tau$.

Then there exists a sense-preserving homeomorphism $h = h_\tau : |\tau| \to \hat{\mathbb{R}}^3$ s.t.

1. $h(|\tau|) = |\sigma|$, if $\det(p_1, p_2, p_3, p_4) > 0$
2. $h(|\tau|) = \hat{\mathbb{R}}^3 \setminus |\sigma|$, if $\det(p_1, p_2, p_3, p_4) < 0$.
3. $h|_{\partial|\sigma|}$ is a PL-homeomorphism.
4. $h|_{\text{int}|\sigma|}$ is quasiconformal.

**Remark 2.8.** The branching set of $h$ is the 1-skeleton of the triangulation.

2.3. Peltonen's Technique. Peltonen's method is an extension of one due to Cairns, developed in order to triangulate $C^2$-compact manifolds ([Ca3]). It is based on the subdivision of the given manifold into a closed cell complex generated by a Dirichlet (Voronoi) type partition whose vertices are the points of a maximal set that satisfy a certain density condition. We give below a sketch of the Peltonen's method, refereing the interested reader to the authoritative [Pe] for the full details.\(^5\)

The construction devised by Peltonen consists of two parts:

**Part 1** This part of the proof proceeds in two steps:

A We build an exhaustion $\{E_i\}$ of $M^n$, generated by the pair $(U_i, \eta_i)$, where:

1. $U_i$ is a relatively compact set $E_i \setminus E_{i-1}$ and
2. $\eta_i$ is a number that controls the "fatness" of the simplices of the triangulation of $E_i$, that will be constructed in Part 2, such that they don't differ too much on adjacent simplices, i.e.:
   (i) The sequence $(\eta_i)_{i \geq 1}$ descends to 0;
   (ii) $2\eta_i \geq \eta_{i-1}$.

B Produce a maximal set $A$, $|A| \leq N_0$, s.t. $A \cap U_i$ satisfies:

(i) a density condition, and
(ii) a "gluing" condition (for $U_i, U_{i=1}$).

A maximal set $A$ is a cell complex and every cell has a finite number of faces (so it can be triangulated in a standard manner).

**Part 2** Consider first the dual complex $\Gamma$ and prove that it is a Euclidian simplicial complex with a "good" density, then project $\Gamma$ on $M^n$ (using the normal map). Finally, prove that the resulting complex can be triangulated by fat simplices.

\(^4\)That $\Gamma$ is almost geometrically finite.

\(^5\)A rather detailed exposition of the main steps of the proof one can be find in [S3].
2.4. Munkres’ Theorem. The basics steps in the proof\(^6\) of Theorem 1.3. are as follows:

a) Prove that you can triangulate a smooth manifold without boundary in the following way: approximate \(M^n\) locally by a locally finite Euclidean triangulation, by means of the secant map (see [Mun], p. 90). Modify these local triangulations coordinate chart by chart, so they will be PL-compatible wherever they overlap. To extend the triangulation globally, we work in \(\mathbb{R}^n\), by using the coordinate charts and maps. Here again we have to approximate the given triangulation by a PL-map, s.t. the given triangulation and the one we produce will be compatible.

b) Triangulate a product neighbourhood \(P(\partial M^n)\) of \(\partial M^n\), \(P(\partial M^n) \subset M^n\), in a standard way, and mash it together with the triangulation of the non-bounded manifold \(int.M\), by using the same method as above.

It is important to emphasize that for most of the process the technique sketched above not only preserves the fatness of the simplices, but actually takes care that the said fatness will occur.

3. Constructing and Intersecting Triangulations

3.1. Geometric Neighbourhoods. If there are no elliptics with intersecting axes, a standard choice for a regular neighbourhood of an \(m\)-axes will be – for obvious geometric reasons – a doubly-infinite regular hyperbolic \(m\)-prism (henceforth called a geometric neighbourhood), and a fundamental domain will be a prismatic "slice", i.e. a fundamental region for the action of \(C_m\) on \(\{m\} \times A(f)\), where \(\{m\}\) denotes the regular hyperbolic polygon with \(m\) sides and \(C_m\) denotes the cyclic group of order \(m\), i.e. the rotation group of \(\{m\}\). In order to triangulate the geometric neighbourhood, we divide it in a finite number of radial strata of equal width \( \delta/\kappa_0 \), and further partition it into "slabs" of equal height \(h\). Each prismatic fundamental region thus obtained naturally decomposes into three congruent tetrahedra, generating a \(C_m\)-invariant triangulation of the geometric neighbourhood. (See Fig. 2 for a representation in the ball model of \(H^3\) in the case \(m = 4\).) The fatness of the triangulation of the geometric neighbourhood thus depends upon \(\delta, \kappa_0\) and \(h\), enabling one to control the initial fatness of the geometric triangulation by means of the parameters \(\delta\) and \(h\).

Remark 3.1. One can easily modify the construction above and produce instead of a \(C_m\)-invariant triangulation, a \(D_m\)-invariant one, where \(D_m\) denotes the dihedral group of order \(m\), i.e. the full-symmetry group of \(\{m\}\), thus allowing one to consider groups that contain orientation-reversing isometries of \(H^3\).

For the choice of geometric neighbourhoods for the node points, the natural choice is that of an Archimedean solid which is a natural carrier of the symmetry group of the desired type: \(D_n, T, O\) or \(I\) (or rather for its spherical counterpart – see [Cox]).

3.2. Mashing Triangulations. We start by showing first how to construct the desired fat triangulation and how to produced the quasimeromorphic mapping ensuing from it in the basic case of groups who’s elliptic elements axes do not intersect. Moreover, let us presume here that there exist a least an elliptic element of order \(\geq 3\). Since \(G\) is a discrete group, \(G\) is countable so we can write \(G = \{g_j\}_{j \geq 1}\) and

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\(^6\) A much more detailed exposition of the proof is given in [S3]. For the full proof one should consult, of course, the original study of Munkres [Mun].
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\[ N(A) = N(f) \]
\[ B^3 = \{4\} \]
\[ A(f) = \int \{4\} \]

Figure 2.

let \( \{f_i\}_{i \geq 1} \subset G \) denote the set of elliptic elements of \( G \). The steps in building the fat desired fat triangulation are as follows:

1. Let \( N_i = \{ x \in \mathbb{H}^3 \mid \text{dist}_{hyp}(A_i, x) < \delta/4 \} \), where \( \delta/4 \) is the constant provided by Theorem 2.1, and where \( A_i = A(g_i) \). We also put: \( N_e = \bigcup N_i \).

2. Consider the following quotients: \( N^*_i = (N_i \cap \mathbb{H}^3)/G \), and \( N^*_e = (N_e \cap \mathbb{H}^3)/G \).

3. Let \( T_e \) denote a \( G \)-invariant fat triangulation of \( N_e \) that induces a \( G \)-invariant fat triangulation \( T_i \) of \( N_i \), and let \( T^*_e \), \( T^*_i \) denote the fat triangulations induced upon \( N^*_e \), \( N^*_i \) by \( T_e \) and \( T_i \), respectively.

4. Denote by \( T_p \) the fat triangulation of \( M_c = (\mathbb{H}^3/G) \setminus N^*_e = (\mathbb{H}^3 \setminus N_e)/G \) assured by Peltonen’s Theorem. The lift of \( T_p \) to \( \mathbb{H}^3 \) is a \( G \)-invariant fat triangulation \( T_c \) of \( \mathbb{H}^3 \setminus N_e \).

5. The desired triangulation \( T^* \) of \( M = \mathbb{H}^3/G \) will consist of the simplices of \( T^*_c \), the simplices of \( T^*_p \) away from \( \partial M_c = (\partial N_e \cap \mathbb{H}^3)/G \), and other new simplices \( T_b \subset \partial M_c \times [0, 1] \subset M_c \), that are constructed by applying the mashing technique of Munkres.

6. Apply Alexander’s Trick to \( T^* \) and get a quasimeromorphic mapping \( f^* : \mathbb{H}^3/G \to \mathbb{R}^3 \).

7. The lifting of \( f^* \) to a mapping \( f : \mathbb{H}^3 \to \mathbb{R}^3 \) produces the desired \( G \)-invariant quasimeromorphic mapping.

Remark 3.2. The choice of "\( \delta/4 \)" instead of "\( \delta \)" in the definition of the geometric neighbourhoods \( N_i \) is dictated by the following Lemma:

Lemma 3.3. (Rat) Let \( X \) be a metric space, and let \( \Gamma \subset \text{Isom}(X) \) be a discontinuous group.

Then, for any \( x \in X \) and any \( r \in (0, \delta/4) \):

\[ \pi : B(x, r)/\Gamma_x \simeq B(\pi(x), \delta/4) \]

where: \( \Gamma_x \) is the stabilizer of \( x \), \( \pi \) denotes the natural projection, \( \delta := d(x, \Gamma(x) \setminus \{x\}) \), and where the metric on \( X/\Gamma \) is given by:

\[ d_{\Gamma}([\pi(x)], [\pi(y)]) = d(\Gamma(x), \Gamma(y)); \forall x, y \in X/\Gamma. \]
Note Instead of the triangulation scheme presented above, scheme that follows closely the Proof of Theorem 1.3, we could have used in this case the natural triangulation of geometric neighbourhods in \(H^3\), to devise a simpler method for mashing triangulations, as follows:

1. Consider again the geometric neighbourhod
   \[N_i = N_{i,1/4} = \{x \in \mathbb{H}^3 \mid \text{dist}_{hyp}(A_i, x) < \delta/4\}\]
   with its natural fat triangulation.

2. Replace the neighbourhods \(N_i = N_{i,1/4}\) by the neighbourhods
   \[N'_i = N_{i,3/16} = \{x \in \mathbb{H}^3 \mid \text{dist}_{hyp}(A_i, x) < 3\delta/16\},\]
   and triangle the \(N'_i\) in such a manner that the simplices of the triangulation of \(\partial N'_i\) are also simplices of the triangulation of \(\text{int} N_i\).

3. Consider instead of \(N_e\) the following manifold: \(N'_e = \bigcup N'_i\).

4. Replace \(M_e\) by \(M'_e = (\mathbb{H}^3 / G) \setminus N'_e = (\mathbb{H}^3 \setminus \mathbb{N}) / G\).

5. The role of the triangulation \(T_p\) is played by \(T'_p\), which consists of the simplices produced by Peltonen’s method and those simplices resulting from those of the geometric triangulation of \(T_i = N_{i,1/4} \setminus N_{i,3/16}\).

6. The desired triangulation \(T'^*\) are composed of those of \(N_{i,1/4}\), those of the original \(M_e\) and those obtained by mashing the two triangulations of the tubes \(T_i\).

Remark 3.4. The second construction, besides being more simple and geometrically intuitive, reduces more rapidly the original problem to that of mashing and uniformly fattening two locally finite Euclidian triangulations.

In the case when there exist intersecting elliptic axes, the following modification of our construction is required: instead of \(\delta\) one has to consider \(\delta^* = \min(\delta, \delta_0)\), where \(\delta_0\) represents the minimal distance between node-points.\(^7\)

We still have to deal with the case when all the elliptic transformations are half-turns, since, as we have seen, no minimal distance between the axes can be computed in this case. However, by the discreetness of \(G\) it follows that there is no accumulation point of the axes in \(H^3\). Let \(D = \{d_{ij} \mid d_{ij} = \text{dist}_{hyp}(A_i, A_j)\}\) denote the set of mutual distances between the axes of the elliptic elements of \(G\). Then, since \(G\) is countable, so will be \(D\), thus \(D = \{d_k\}_{k \geq 1}\). Then the set of neighbourhods \(N_2 = \bigcup_{k \in \mathbb{N}} N_k = \bigcup_{k \in \mathbb{N}} \{x \in \mathbb{H}^3 \mid \text{dist}_{hyp}(A_k, x) < \delta/4k\}\) will constitute a proper geometric neighbourhod of \(A_G = \bigcup_{i \in \mathbb{N}} A_i\). The fatness of the simplices of the geometric triangulation of \(A_G\) can be controlled, as before, by a proper choice of \(h\) and \(q\).

Remark 3.5. The existence of \(N_2\) is easy to justify geometrically if one uses the upper-half space model of \(H^3\): up to conjugation one can choose \(x\) to be an accumulation point for \(A_G\), therefore the axes accumulated at this point are represented as parallel Euclidian half-lines, perpendicular to the plane \(\mathbb{R}^2\). Consider a family of disjoint Euclidian cylinders \(C_i = \{x \mid \text{dist}_{Eucl}(A_i, x) < r_i\}\), \(C_i \cap \mathbb{R}^2 = p_i\). Let \(l_i^\pm\) represent two parallel generators of \(C_i\), let \(h_i\) be the hyperbola of vertex \(p_i\) and asymptotes \(l_i^\pm\), and let \(H_i\) the hyperboloid of rotation with axes \(A_i\) and generatrix

\(^7\) See [GM1], [GM1], [GMMR].
Then \(\{\text{int } H_i \mid i \geq 1\}\) represents a proper geometric neighbourhood for the set of axes accumulating at \(\infty\).

## 4. Fattening Triangulations

### 4.1. Preliminaries

We have seen that we reduced the problem to that of "fattening" the intersection of two 3-dimensional finite, fat Euclidean triangulations. We do this piecemeal, first fattening the 2-simplices, then the 3-simplices. It is natural to do so, for the following holds:

**Lemma 4.1.** If an \(n\)-dimensional simplex is fat, then all its \(k\)-dimensional faces, \(2 \leq k \leq n - 1\), are fat.

In particular, in order that an \(n\)-dimensional simplex be fat, its 2-simplices have to be fat. Note that for triangles, with angles \(\alpha, \beta, \gamma\), and \(r\) and \(R\) as above, the conditions: \(r/R \geq f\) and \(\min \{\alpha, \beta, \gamma\} \geq \varphi\), where \(\varphi = \varphi(f)\) is an angle depending on \(f\), are equivalent. Thus we start "fattening" 2-simplices, by ensuring that

\[
\min_i \{\alpha_i, \beta_i, \gamma_i\} \geq \frac{\varphi_0}{10};
\]

where the minimum is taken over all the triangles of the resulting triangulation, and where \(\varphi_0\) is the minimal angle of the original triangulations – ensured by their uniform fatness.

From now on, let \(S = \{s_i\}_{i \in I}\) and \(\Sigma = \{\sigma_j\}_{j \in J}\) stand for the simplices of the triangulations of \(T_i\) and \(M'_c\), respectively.\(^8\)

Since the intersections of tetrahedra can be rather unruly, we simplify the situation by requiring that the simplices of one of the triangulations be much smaller than those of the other:

\[
diam s_i \leq \frac{1}{10^{k_0}} \text{diam } \sigma_i; \forall i \in I, \forall j \in J;
\]

where \(k_0\) is to be determined later.

By using general position arguments\(^9\) (see [Hu], [Mun]), the relative positions of the \(s_i\)'s and the \(\sigma_j\)'s are now reduced to the following relevant possibilities:

(a) \(s_i \subset \text{int } \sigma_j\), or (a') \(s_i \subset \text{ext } \sigma_j\);
(b) \(\exists \eta_j \text{ s.t. } \eta_j \cap \text{int } s_i \neq \emptyset\), where \(\eta_j\) is an edge of \(\sigma_j\), but \(\not\exists v_{im} \in \eta_j\), where \(v_{im}\) is a vertex of \(s_i\);
(c) \(\exists \nu_{jk} \text{ s.t. } \nu_{jk} \in \text{int } s_i\), where \(\nu_{jk}\) is a vertex of \(\sigma_j\);
(d) \(\text{int } s_i \cap \text{int } \sigma_j \neq \emptyset\), but we are not in one of the previous cases (see Fig. 3).

### 4.2. Fattening 2-dimensional Triangulations

We start our triangulation "fattening" process by dealing with the 2-dimensional case first:

Let \(\sigma_j \in \Sigma\) be such that there exists a regular neighbourhood \(N_{j_0}\) of \(\sigma_j\), triangulated by elements of \(S = \{\sigma_i\}\). Now \(S\) is partitioned by \(\sigma_j\) into three disjoint families \(S_{0,1}, S_{0,2}, S_{0,3}\), where:

\[
S_{0,1} = \{s_i \in \Sigma \mid s_i \subset \text{int}(\sigma_j_0) \text{ or } s_i \subset \text{int}(\sigma_{j_0})\}
\]

\[
S_{0,2} = \{s_i \in \Sigma \mid \exists v_{0,k} \text{ vertex of } \sigma_{j_0} \text{ s.t. } v_{0,k} \in \text{int}(s_i)\}
\]

\[
S_{0,3} = \{s_i \in \Sigma \mid \exists e_{0,l} \text{ edge of } \sigma_{j_0} \text{ s.t. } e_{0,l} \cap \text{int}s_i \neq \emptyset\}
\]

---

\(^8\) or \(\partial M_c \times [0,1)\) and \(M_c\).

\(^9\) completely analogous to the technique used in the proof of Theorem 1.3.
It is easy to assure – by eventual further subdivision and $\varepsilon$-moves\textsuperscript{10} – that $S_{0,3} \cap S_{1,3} = \emptyset$; where $S_{1,3}$ is the family corresponding to $S_{0,3}$, induced by $\sigma_{j_1}$, that is adjacent to $\sigma_{j_0}$.

The intersections belonging to the family $S_{0,2}$ are the principal generators of "un-fatness", for $\angle(e_0,l,e_i,m)$ may be arbitrarily small, where $e_{i,m}$; $m = 1, 2, 3$ are the edges of $s_i$ (see Fig. 3).

\textbf{Figure 3.}

Let $\nu_{0,p}$, $\nu_{0,m}$, $\nu_{0,n}$ be three consecutive intersection points of $e_{0,l}$ with edges of two adjacent simplices $s_i, s'_i$ and let $\phi_{0,p}, \phi_{0,m}, \phi_{0,n}$ denote the resulting angles for \textsuperscript{10}\textit{i.e.} we "move a bit" the triangulation $S = \{ s_i \}$ s.t. no intersection $s_i \cap \sigma_j$ is of one of the forbidden types. (See \textsuperscript{Mun} for definition and further details.)

\textbf{Figure 4.}
THE EXISTENCE OF QUASIMEROMORPHIC MAPPINGS IN DIMENSION 3

(we always choose the acute angle) (see Fig. 3). Now it is not possible that two consecutive of the angles $\phi_{0, 1}, \phi_{0, 2}, \phi_{0, 3}$ are smaller than $\phi_0$: indeed, let us suppose that both $\phi_{0, 1} < \phi_0$ and $\phi_{0, 2} < \phi_0$. Then $\phi_{0, 1} > \pi - 2\phi_0$, so $\phi_{0, 1} + \phi_{0, 2} < 2\phi_0$, and thus either $\phi_{0, 2} < \phi_0$ or $\phi_{0, 3} < \phi_0$, in contradiction to the fatness of $s_i$.

The fact above implies that we obtain two quadrilaterals which contain the "bad" points $\nu_{0, p}$ and $\nu_{0, n}$ in their (respective) interiors, let them be: $Q_1 = \square \nu_{0, 1} \nu_{0, 4} \nu_{0, 5} \nu_{0, 2}$ and $Q_1 = \square \nu_{0, 2} \nu_{0, 3} \nu_{0, 6} \nu_{0, 1}$.

We erase the segments $\nu_{0, q} \nu_{0, p}, \nu_{0, 2} \nu_{0, p}, \ldots, \nu_{0, n} \nu_{0, r}$ and we replace them with segments that will "fattily" triangulate the quadrilaterals in question. These triangles have "big" angles (since their angles are belonging to fat triangles or are the sum of two such angles).

We distinguish between two cases:

(a) Let $Q$ be a convex quadrangle such that all its angles are $\geq \phi_0$ and let $\overrightarrow{l_{i,k}}$ be the ray interior to $\angle A_{i-1} A_i A_{i+1}$ s.t. $\overrightarrow{l_{i,k}} A_i A_k = \frac{1}{4} \alpha_i$; $i = 0, \ldots, 3$; $k \in \{i - 1, i + 1\}$ are considered mod 4, of course. (See Fig. 4, where $Q^\ast = \chi_1 \ldots \chi_9$.) Then the rays $\{\overrightarrow{l_{i,k}}\}_{i,k}$ generate a convex polygon $Q^\ast \subset \text{int} Q$. (See [53].)

By its very definition this quadrangle has the property that, for any $\nu_{0, p} \in \text{int} Q$ we have that:

\[ \beta_{ik} = \angle \nu_{0, p} A_i A_k > \frac{\alpha_i}{4} \geq \frac{\phi_0}{4} ; \quad k \in \{(i - 1) \text{ mod } 4, (i + 1) \text{ mod } 4\}. \]

Also:

\[ \gamma_{ik} = \angle A_i \nu_{0, p} A_k > \min \{\angle A_{i-1} A_i A_{i+1}, \angle A_i A_{i+1} A_{i-1}\}. \]

(See Fig. 5) But one of the angles $\angle A_{i-1} A_i A_{i+1}$ and $\angle A_i A_{i+2} A_{i-1}$ belongs to one of the original fat triangles, so:

\[ \phi_{ik} > \phi_0. \]
So, from (4.1) and (4.3) it follows that the triangles \( \triangle A_i \nu_0^* A_{i+1} : i = 0, \ldots, 3 \ (mod\ 4) \) are fat.

**(b)** In this case, rather than tracking back our steps through the same argument as in the previous case; we prefer to dissect \( Q \) into two triangles and one convex quadrilateral, in the following way: if \( \square A_0 A_0 A_0 A_3 \) is such that \( \angle A_3 A_0 A_1 > \pi \) and such that \( \nu_0, p \in A_2 \), then consider the bisectors \( A_0 B_{12} \) of and \( \angle A_1 A_0 A_2 \) and \( A_0 B_{23} \) of \( \angle A_2 A_0 A_3 \). (See Fig. 6.)

![Figure 6](image)

**(Figure 6.**

Now, instead of returning once over to the argument used in case (a), it is easy to proceed directly and show that the triangles \( \triangle A_1 A_0 B_{12}, \ldots, \triangle A_0 B_{23} \) are fat. (See [S3] for details.)

We are now faced with a new fat triangulation. However, new points of intersection with \( \varepsilon_{0,l} \) are introduced: let them be \( \nu_{0,p-1} \) and \( \nu_{0,p+1} \). If

\[
\varepsilon_{0,l} \cap \square A_0 B_{12} A_2 B_{23} \cap \{B_{12} B_{23}\} = \emptyset,
\]
then the same argument we used for the original triangulation shows that \( \nu_{0,p-1} \) and \( \nu_{0,p+1} \) are \( \geq \phi_0 \). If

\[ e_{i,l} \cap \square A_0 B_{12} A_2 B_{23} \cap \{ B_{12} B_{23} \} \neq \emptyset, \]

we can employ either one of the following two methods to remedy the situation:

(a) use the general position technique again and bring the new triangulation to the required position; or (b) consider, instead of the bisector \( A_0 B_{23} \) the meridian \( A_0 M_{23} (M_{23} \in A_2 A_3) \) – see \([S3]\) for details.

We still have to face the problem of "mashing" the triangulations "over the wave front" of the "s_i"-s. Away from the vertices of the complex \( S \), we are faced with two possibilities: \( S \) contains: (a) one or (b) two of the vertices of \( \sigma \in \Sigma \). We shall deal first with:

**Case (b)** By further dividing the triangles of:

\[ (4.6) \quad \text{Front } S = \{ s \in S \mid \exists e \text{ edge of } s \text{ s.t. } e \in \partial S \}, \]

we are able to "erase" the families \( S_1 \) and \( S_2 \), where \( S_1 \cup S_2 = \text{Front } S \cap \{ e_i \}, i = 1, 2 \) (see Fig. 7 (a), (c)) our only problem being that some of the simplices \( s \in S \) may intersect the edges \( e_i \) of \( \sigma \) at an angle \( \phi_{s,1,i} < \phi_0, i = 1, 2 \). However, the angles \( \phi_{s,1,i}^2 \) (see Fig. 7 (b)) are, by the previous argument \( \geq \phi_0 \).

\[ \text{Figure 8.} \]

Let \( S' = S \setminus \text{Front } S \). Then \( \partial S' \) will still be convex, so we can consider the joins (cones) \( J(v_{12}, \varepsilon_k) \), where \( \varepsilon_k \) are the edges of \( \partial S' \) that are included in \( \sigma \), and \( v_{12} = e_1 \cap e_2 \) (see Fig. 8).

Now, since the number \( n_0 \) of conjugacy classes of elliptics is finite, we can consider

\[ (4.7) \quad \delta_0 = \min \{ \delta_1, \delta_2, \ldots, \delta_{n_0} \}; \]

where the bounds \( \delta_i, 1 \leq i \leq n_0 \) are those given by Theorem 2.1. Then in \( \partial M_c \times [3/4, 1] \) (or alternatively in the tubes \( T_{0,i} = N_i(\delta_0/4) \setminus N_i(3\delta_0/16) \)), the radii \( r_0, R_0 \) of the simplices \( \sigma_0, \sigma_0 \bigcup_{i \in N} J_0 \neq \emptyset \), where \( J_0 = \bigcup T_{0,i} \), will be uniformly

\[ \text{since we are in the planar case, the division may be done by lines parallel to the edges of } s. \]
bounded.\textsuperscript{12}
Thus, there exist numbers $m_0$ and $m_1$ such that:

\begin{equation}
    m_0 \leq \text{diam}(\sigma) \leq m_1; \quad \forall \sigma \text{ s.t. } \sigma \cap T_0 \neq \emptyset.
\end{equation}

Therefore, exists $k_1 \in N_+$ such that:

\begin{equation}
    \frac{1}{10^{k_1}} \text{diam}(\sigma) \leq \text{diam}(\bar{s}); \quad \forall s, \sigma \text{ s.t. } \bar{s} \cap T_0 \neq \emptyset.
\end{equation}

This, in conjunction with (4.2), assures us that the number $\rho_1$ of triangles $s_i$ that intersect the edge $e_{0,l}$ (see Fig. 9. (a)) will be bounded by two natural numbers, i.e.

\begin{equation}
    n_1 \leq \rho_1 \leq n_2.
\end{equation}

Moreover, our last subdivision ensures us that the number $\rho_2$ of triangles intersecting $\partial S'$ is:

\begin{equation}
    \rho_2 \leq 3(\rho_1 - 1) + 2,
\end{equation}

so there exists $\lambda_0 > 0$ s.t. the angles $\psi_k = \angle v_1, \psi^1, \psi^2$ (see Fig. 8) satisfy

\begin{equation}
    \psi_k \geq \lambda_0 \varphi_0, \quad \psi^i \geq \lambda_0 \varphi_0, \quad i = 1, 2;
\end{equation}

\textsuperscript{12}See [P3].
as desired. We still have to deal with Case (a) We repeat once more the procedure used for the subdivision of ∂S and Front S that we employed in Case (b). We divide the edge e2 into ρ2 − 1 equal segments and consider the joins (cones) J(v′ j, εj+1) and J(v′ k, v′ k+1) (see Fig. 9(a)). Using the same arguments as before one easily checks that the resulting triangulation of s \ S will be fat. Moreover, although this procedure dramatically reduces the "fatness" of the next stratum of simplices of the family Σ, it leaves the other strata unchanged. This procedure takes care of the intersections of Σ with "the front wave" S ∩ {x | dbyp(Ai, x) = δ0/4}. To deal with the case S ∩ {x | dbyp(Ai, x) = 3δ0/16} one proceeds along the same lines and then fits the new triangulation S’ to S in a properly chosen tubular region J1, using, instead of σ, triangles s0 ∈ S ∩ T1. We do have yet to contend with the problem posed by "corners" i.e. by triangles s ∈ S such that (i) s ∈ Front S and (ii) ∂s ∩ ∂S = v (were v is a vertex). The two cases that ensue may be treated with the methods developed before. (See [S3].)

Remark 4.2. We can dispense with these last considerations altogether by considering the following: we are concerned – in fact – only with patching together triangulations already contained in a δ0-neighbourhood of an axis. But the geometric triangulations employed partitioned these neighbourhoods into "levels" or "heights", so one can fit the triangulation of two consecutive levels – say "m" and "m + 1" – by considering the intermediary adjusting patch delimited by the levels "m − 1/2" and "m + 1/2". (A finite number of further barycentric subdivisions may still be required.)

This concludes the "fattening" process for the two dimensional triangulations.

4.3. Fattening 3-dimensional Triangulations. We shall divide the "fattening" process of two intersecting 3-dimensional tetrahedra into two parts: A) The "fattening" of the 2-dimensional intersection between the triangles belonging to ∂S and a face f123 = ∆v1v2v3 of a tetrahedron σ ∈ Σ, and B) The extension of the new triangulation to a fat 3-dimensional triangulation.

4.3.1. Fattening 2-dimensional intersections. Let us start by noticing that, since the fatness of the tetrahedra s ∈ S is bounded from below, so will be their dihedral angles. It follows that even after the partition of f123 into triangles {τk} and quadrilaterals {ηj}, to a triangulation {τk}, the number of triangles around each vertex will be bounded by a natural number m1. We shall exploit this fact to our advantage, so that we will be able to replace {τk} by a fat triangulation {τk} that is "close" to {τk}.

Indeed, if we denote by å0 j, j = 1, ..., m0, m0 ≤ m; the angles of the triangles {τk0} that are respectively adjacent to the vertex ρ0 = \tag{10} then there are two sources of "thiness": either å0 j = å0 j is smaller than ϕ0, for some j0 ∈ {1, ..., m} or one of the angles å0 jk å0 jk+1 produced by the division of the quadrilateral ηj into two triangles: å0 jk and å0 jk+1.

If we bisect the angles å0 j, j = 1, ..., m0; we will receive angles å0 jk, k = 1, 2. Let us consider the angles å0 jk, j = 1, ..., m0; k = 1, 2; 

\textsuperscript{13} by the "trihedral sinus formula" (see [BAC]).
where: \( \tilde{\beta}_{j_2} = \frac{\tilde{\alpha}_{j_1} + \tilde{\alpha}_{j_2}}{2} \), \( \tilde{\beta}_{j_2} = \frac{\tilde{\alpha}_{j_1} + \tilde{\alpha}_{j_2}}{2} \).\(^{14}\) If each end every of the angles \( \tilde{\beta}_{j_k} \) defined above is greater than \( \varphi_1 = \varphi_0/10 \), then we desist and proceed towards part. But it may be that, for instance, both \( \tilde{\alpha}_{j_1} \) and \( \tilde{\alpha}_{j_2} \) are smaller than \( \varphi_0/5 \). If this happens we continue the process of "mixing the angles". To be more precise: let us delate – for commodity reasons – the upper index "0" in the enumeration of the angles \( \beta^0 \), and denote them by \( \tilde{\beta}_j \), \( j = 1, \ldots, m_0 \); and let us form the sequence of angles \( \tilde{\beta}_j', \tilde{\beta}_j'' \), \( j = 1, \ldots, m_0 \); and so on. But this process will halt – inasmuch as we are concerned – in a finite number of steps, for the following inequality holds:

\[(4.13) \quad \tilde{\beta}_{j_0} > \frac{\alpha_1 + \cdots + \alpha_{m_0}}{2^{m_0+1}} \quad j = 1, \ldots, m_0 ;
\]

that is:

\[(4.14) \quad \tilde{\beta}_{j_0}^{m_0+1} > \frac{2\pi}{2^{m_0+1}} > \frac{\varphi_0}{10^k} \quad j = 1, \ldots, m_0 ;
\]

and so:

\[(4.15) \quad \beta_j^{m_0-1} > \varphi_1 = \frac{\varphi_0}{10^k} \quad 1, \ldots, m_0 ;
\]

where \( k \) is the least natural power that satisfies the right-handed inequality in (4.12).

We shall use the bound above in order to produce a fat triangulation. However, some care is needed in doing this, for in general, both the number of iterations used for each vertex and the number of bisectors \( b_{j_k}^k \), \( j = 1, \ldots, m_0 \); \( k = 1, 2, 3 \); that intersect the triangle \( \tilde{\tau} \) will be different, thus affecting the sizes of the angles \( \alpha_j^k \).

Let \( b \) denote the shortest of the segments \( b_{j_k}^k \cap \tilde{\tau} \), \( k = 1, 2, 3 \). (By elementary geometry, \( b \) should be the segment \( b_{j_0}^{m_0} \) that is the nearest to the shortest side of

\(^{14}\)The indices are to be taken \( \mod(m_0) \), of course.
\[ \nu \] permutation of indices – see Fig. 11 (b). (Here \( \tilde{\nu} \) (see Fig. 11 (a)). It may be that we will have to consider \( \nu \) closest to the edge \[ \tau \] (\( \nu \) illustrated the conning of the segments \( \nu \).)

We start by noticing that “fatness” of a quadrilateral means that:

1. the angles \( \nu \) and \( \tilde{\nu} \) where \( \nu \) and \( \tilde{\nu} \) denote the vertices closest to the edge \( \nu \) and \( \tilde{\nu} \), \( \tilde{\nu} \) play the same role in the adjacent triangle \( \tilde{\tau} \) (see Fig. 11 (a)). It may be that we will have to consider \( \nu \) (or any permutation of indices) – see Fig. 11 (b). (Here \( \tilde{\nu} \) plays in the triangle \( \tilde{\tau} \) the role \( \nu \) plays in the triangle \( \tilde{\tau} \).) In this case consider a point \( \tilde{\nu} \in \text{int} \ \tilde{\tau} \), such that:

1. \( \text{length}(\tilde{\nu} \tilde{\nu}) = \hat{b} \);
2. \( \angle \tilde{\nu} \tilde{\nu} \tilde{\nu} \geq \frac{\hat{\omega}}{2} = \varphi_2 \);
3. \( \angle \tilde{\nu} \tilde{\nu} \tilde{\nu} \geq \frac{\hat{\omega}}{2} = \varphi_2 \).

**Remark 4.3.** The existence of the positive integer \( k_2 \) with the desired properties is guaranteed by the fact that the angles \( \tilde{\alpha} \), \( \tilde{\alpha}' \), \( \tilde{\alpha}_1 \), \( \tilde{\alpha}_1 \), \( \tilde{\alpha}_m \) and \( \tilde{\alpha}_m \) are \( \geq \varphi_1 \), thus

\[
\begin{align*}
c_1 b & \leq \text{length}(\tilde{\nu} \tilde{\nu}) \leq c_2 b; \\
c_1 b' & \leq \text{length}(\tilde{\nu} \tilde{\nu}) \leq c_2 b'.
\end{align*}
\]

In consequence we are facing the situation depicted in Fig. 12, where we also illustrated the conning of the segments \( \tilde{\nu} \tilde{\nu} \) and \( \tilde{\nu} \tilde{\nu} \) from the barycenter \( \hat{b} \) of \( \tau \); \( k = 1, 2, 3; l = 1, 2, 3 \).

Now we have to show the fatness of three types of polygons:
1) triangles \( T_{ij} = \triangle \tilde{\nu} \tilde{\nu} \hat{b} \);
2) quadrilaterals \( Q_{kl} = \square \tilde{\nu} \tilde{\nu} \hat{b} \); \( \hat{b} \);
3) triangles \( T_{kl} = \triangle \tilde{\nu} \tilde{\nu} \hat{b} \), \( k, l = 1, 2, 3 \).

We start by noticing that ”fatness” of a quadrilateral means that:

a) its angles are bounded from below, and:

b) the ratios \( l \lambda \), \( \lambda, \lambda = 1, \ldots, 4m \) between the lengths of its sides are also bounded from below.

---

\( \nu \) is the smallest angle of \( \tilde{\tau} \). The best possibility occurs, of course, when we have to use the bisection only once, so we will have only (ordinary) bisectors \( \hat{b}, \hat{b}', \hat{b} \) that will meet, of course, at the barycenter \( \hat{b} \) of \( \tilde{\tau} \).
An easy computation (see [S3]) shows that

\[ \text{length}(\tilde{y}_1^l \tilde{\nu}_1^l) \geq 2\hat{b} \sin \frac{\pi}{m_0}; \]

\[ \text{length}(\tilde{y}_1^l \tilde{\nu}_1^k) \geq \text{length}(\tilde{\nu}_1^k \tilde{\nu}_1^l) - 2\hat{b}; \]

\[ \angle \tilde{\nu}_1^l \geq \arctan \frac{\hat{b} \cos 2\pi \hat{m}_0}{\text{length}(\tilde{\nu}_1^k \tilde{\nu}_1^l) - 2\hat{b} \sin \frac{2\pi}{m_0}}; \]

where \( \hat{b} = \text{length}(\tilde{\nu}_1^k \tilde{y}_1^l) \).\(^{16}\)

Therefore, the arguments employed before show that \( \Box Q_{kl} \) is decomposable into fat triangles, so case 2) is dealt with.

In a manner similar to that used in case 2) show that the triangles of types \( T_{ij} \) and \( T_{kl} \) are also fat; indeed:

\[ \angle \tilde{\nu}_1^i \tilde{\nu}_1^j \tilde{\nu}_1^{i+1} < 2 \arctan \frac{3\pi}{2\hat{m}_0 \hat{b}}; \]

where

\[ \text{length}(\tilde{y}_1^l \tilde{\nu}_1^k) \geq \hat{b} \geq \text{length}(\tilde{y}_1^l \tilde{\nu}_1^m); \]

and also:

\[ \angle \tilde{\nu}_1^i \tilde{\nu}_1^j \tilde{\nu}_1^k > \angle \tilde{\nu}_1^j \tilde{\nu}_1^k \tilde{\nu}_1^m \geq \varphi_1; \]

and so we dispose with cases 1) and 3) too, thus concluding the proof of part A).

\(^{16}\) and similar formulas hold for the other pairs of vertices.
4.3.2. The extension to a fat 3-dimensional triangulation. We start by observing that, if $u$ is a vertex of the simplex $s_i$ s.t. $f_{123} \cap s_i = \tilde{\tau}$ – where $f_{123}$ is a face of the tetrahedron $\sigma$ – then the triangles $T_{kl}$, $T_{ij}$ and those produced by the subdivision of $Q_{kl}$ are also may also be deduced from $u$.

We want to show that the simplices thus generated – denoted by $V_{ij}^\delta = J(u_\delta, T_{ij})$, $\delta = 1, 2, 3, 4; \ldots$ – have big angles. We shall justify this affirmation for tetrahedra of type $V_{ij}^\delta$, the other cases being completely analogous. Indeed:

a) The angles $\tilde{\nu}_i$, $\tilde{\nu}_j$, and $\angle \tilde{\nu}_i \tilde{\nu}_j$ are "big" by the very construction of $\triangle \tilde{\nu}_i \tilde{\nu}_j$;

b) The plane angles around the vertex $u_\delta$ are "big", since $\text{length}(\tilde{\nu}_i \tilde{\nu}_j) \geq c_1 b$, and because $\tilde{\nu}_i \tilde{\nu}_j$ is included in the plane of $\tilde{\tau}$;

c) The dihedral angles around $u_\delta$ are also "big" (by the argument above and by the "tetrahedral sinus formula"). To sum up, the angles denoted by "L" in Fig. 13 are larger than some constant $\phi^2$. However, the basic type of small angles may still occur (see Fig. 14, where they are denoted by "s").

In fact, the case of Fig. 3 (a) is associated with a low ratio "height/base side"; whereas, in this instance:

\begin{equation}
(4.22) \quad \frac{u_\delta \tilde{O}_{ij}}{\tilde{\nu}_i \tilde{\nu}_j} > \frac{u_\delta \tilde{O}_{ij}}{\tilde{\nu}_i \tilde{\nu}_m} > c^0 = \text{const. ;}
\end{equation}

(by the fatness of the tetrahedron $u_1 u_2 u_3 u_4$), so this case is excluded.

Part of the angles covered by case (b) of Fig 14 are "big", for the following inequality (and its analogues hold)

\begin{equation}
(4.23) \quad \frac{u_\delta \tilde{\nu}_k}{\tilde{\nu}_i \tilde{\nu}_j} > e^* \frac{u_\delta \tilde{\nu}_k}{\tilde{\nu}_i \tilde{\nu}_k} ; \quad e^* = \text{const. ;}
\end{equation}

\[\text{Figure 14.}\]

For a full proof of the "fatness" of $u_\delta \tilde{\nu}_i \tilde{\nu}_j \tilde{\nu}_k$ (to wit) we still have to check the size of each of the following angles: $\angle \tilde{\nu}_i \tilde{\nu}_j \tilde{\nu}_k$\(^{17}\), $\angle \tilde{\nu}_i u_\delta \tilde{\nu}_j$ and $\angle \tilde{\nu}_k u_\delta \tilde{\nu}_j$.

\(^{17}\)i.e. the dihedral angle between the faces $\triangle u_\delta \tilde{\nu}_i \tilde{\nu}_j$ and $\triangle \tilde{\nu}_k u_\delta \tilde{\nu}_j$. 

To ensure the proper size of the last two angles one has to notice that the size of – e.g. \( \angle \hat{\nu}_k u_\delta \hat{\nu}^k \) – inversely proportional to:

(a) the distance \( \hat{s} \) from \( u_\delta \) to \( \hat{\nu}^k \hat{\nu}^k \),

and

(b) the distance \( \delta^* \) from \( \hat{u}_\delta \) to \( \hat{\nu}^k \hat{\nu}^k \).

But, since the lengths \( \hat{\nu}^k \hat{\nu}^k \) are bounded from below, there exists a minimal universal distance \( \delta^*_0 \) s.t. if \( \delta^* < \delta^*_0 \) then \( \angle \hat{\nu}_k u_\delta \hat{\nu}^k > \varphi^*_2 \), for a suitable \( \varphi^*_2 \).

![Figure 15.](image)

But, since the lengths \( \hat{\nu}^k \hat{\nu}^k \) are bounded from below, there exists a minimal universal distance \( \delta^*_0 \) s.t. if \( \delta^* < \delta^*_0 \) then \( \angle \hat{\nu}_k u_\delta \hat{\nu}^k > \varphi^*_2 \), for a suitable \( \varphi^*_2 \).

By eventually decreasing \( \delta^*_0 \) to a new \( \delta^*_1 \), we can ensure that the angles of type \( \angle \hat{\nu}_k u_\delta \hat{\nu}^k \) are strictly greater than some \( \varphi^*_2 \), \( \varphi^*_2 \leq \varphi^*_2 \), for each \( \hat{\nu}_k \in St(u_\delta) \cap f_{123} \) and for each \( \hat{\nu}^k \), which concludes the proof of case (b). Indeed, we can vary the position of \( \hat{u}_\delta \) in the region \( \Lambda_{\hat{u}_\delta} \), such that all the required angles will be large – see Fig. 15. (Such a small movement won’t affect the fatness of the next stratum of tetrahedra of type ”s”.)

![Figure 16.](image)

After disposing with this case, we can turn our attention to and to case (a). Now, since \( |\tan \angle \hat{\nu}^k \hat{\nu}^k| = \frac{u_\delta \hat{u}_\delta}{u_\delta \hat{u}_\delta} \), where \( u_\delta \hat{u}_\delta \perp (\hat{\nu}^k \nu_k \hat{\nu}^k) \), \( u_\delta \hat{u}_\delta \perp \hat{\nu}^k \hat{\nu}^k \), and since \( u_\delta \hat{u}_\delta \) is known, we only have to ensure that \( u_\delta \hat{u}_\delta \) is bounded from below.

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18 Here \( u_\delta \hat{u}_\delta \perp f_{123} \)
Remark 4.4. The method we have just used is adaptable to the general context: consider instead of $u_δ$ vertices $u_δ^++$, such that $\text{dist}_{eucl}(u_δ, u_δ^+) = h_0$, where $h_0 = \frac{1}{2}h_{\text{min}}$, and $h_{\text{min}}$ = the minimal height of the simplices $s^+ \in St(u_δ)$. The points $u_δ^+$ are to be chosen on the normal through $u_δ$ to the plane $f_{123}$, so that the combinatorics of the triangulation will suffer no alteration. Clearly, uniform bounds will again be attained.

In order to conclude the proof of Theorem 1.1 we still have to provide for a fat triangulation around the node points. However, this missing case is easily dealt with by considering the intersections of the geometric neighbourhoods of the elliptic axes. Such an intersection will automatically inherit from the tubular neighbourhoods that generate it a natural, stratified, fat triangulation. (See Section 3.) So, the arguments involved in the proof of the restricted case of un-intersecting axes do apply here, too. Thus we conclude the

**Proof of Theorem 1.1**

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19 Of course – as before – we may have to subdivide the tetrahedra once or twice.
5. Higher Dimensions

Our approach to the extension of our results to dimension \( n \geq 4 \) – which we will expose only briefly – is based upon reduction to smaller dimensions. To do this, let us observe that if \( S(n+1) \) is a simplex in \( \mathbb{R}^n \), then any sectioning hyperplane separates the vertices of \( S(n+1) \) into two groups of \( p \) and \( n+1-p \) vertices, respectively. The notation will be as follows: \( (p, n+1-p) \equiv (n+1-p, p) \) for the polytopes of the section; and \( (p|n+1-p) \) and \( (n+1-p|p) \) for the resulting frusta.\(^{20}\)

The reduction to lower dimensions is permitted by:

\textbf{Lemma 5.1} (Som). The frustum of type \((p|2)\) of \( S(p+q) \) is isomorphic to the section of type \((p,q+1)\) of \( S(p+q+1) \).

The fattening algorithm is, in a nutshell, as follows: (i) Divide the polytopes obtained by sectioning into simplices: first the section polytope, then the faces that are part of the original \( S(n+1) \), while ensuring fatness by the methods exposed in Section 3. If needed apply an inductive process on the dimension of the simplices. (ii) ”Fatten” the frusta by proving the existence of a locus of points where from all the \((n-k)\)-faces are seen at big \((n-k)\)-dimensional angles.\(^{21}\)

Theorem 1.1 now follows. However, we have to understand much better the geometry of the elliptic locus of a Kleinian group with torsion, acting upon \( H^n \), \( n \geq 4 \).

We are, however, fortunate, for the fixed set of an elliptic transformation is a \( k \)-dimensional hyperbolic plane, \( 0 \leq k \leq n-2 \); thus providing the fixed point set with a geometric neighbourhood – together with its natural fat triangulation. Some complications may arise because different elliptics may well have fixed loci of different dimensions,\(^{22}\) so the respective simplices will have different dimensions and fatnesses. However this is easy to remedy by completing them to \( n \)-dimensional simplices, by ”expanding” the low dimensional neighbourhoods to maximal dimension in a product manner.

6. Appendix – Sketch of Proof of Theorem 2.4.

Since in dimension 3 parabolic elements may have only rank 1 or 2 (See [MS, Abi1]), suffice to analyze only the following two cases:

(A) If \( G \) contains no parabolic elements or if \( G \) contains only rank 1 parabolics (see [P]) then, by a theorem of Milnor-Gromov (see [Gro, GdlH]), \( \pi(G) \) is word-hyperbolic and it follows, by a corollary of Rips’ theorem (see [GdlH]), that it contains only a finite number of classes of elements of finite order, i.e. elliptics.

(B) If \( G \) contains parabolics of rank 2, then one can opt for one of the following paths:

(a) Use Scott’s Theorem (See [Abi1]) that states that 3-manifolds are compact core manifolds,\(^{23}\) then apply the Milnor-Gromov Theorem for \( M_0 \).

or

(b)”Double” the manifold with respect to its cuspidal ends (see [Mor, Abi1]). Since the number of cusps is finite (by a theorem of Sullivan’s – see [Abi1, KP])

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\(^{20}\)See Som for notations and precise definitions.

\(^{21}\)See Som for the relation between the \( p \)-dimensional angles of a \( n \)-simplex, \( p = 2, \ldots, n-1 \).

\(^{22}\)For a detailed discussion on the diversity of the elliptics (and their fixed loci) see AP.

\(^{23}\)i.e. if \( M \) is a 3-manifold s.t. \( \pi_1(M) \) is finitely generated, then there exists \( M_0 \subset M \), \( M_0 \) compact and s.t. \( \pi_1(M_0) \cong \pi_1(M_1) \).
and since the double $\widetilde{M}$ is compact it follows, exactly as above, that $\pi(\widetilde{M})$ has only a finite number of conjugacy classes of elliptics, hence so has $\pi_1(M)$.

\[\square\]

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