Mesoscopic fluctuations of eigenfunctions and “level velocity” distribution in disordered metals

Yan V. Fyodorov† and Alexander D. Mirlin‡

1 Fachbereich Physik, Universität-GH Essen, 45117 Essen, Germany
2 Institut für Theorie der Kondensierten Materie, Universität Karlsruhe, 76128 Karlsruhe, Germany
† Petersburg Nuclear Physics Institute, 188350 Gatchina, St. Petersburg, Russia

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We calculate the distribution of eigenfunction amplitudes and the variance of the “inverse participation ratio” (IPR) in disordered metallic samples. A relation is established between the distribution function of IPR and that of “level velocities” – parametric derivatives of energy levels with respect to a random perturbation.

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I. INTRODUCTION.

Statistical properties of disordered metals have attracted considerable research interest last years. It was understood that the old problem of a quantum particle moving in a quenched random potential considered earlier in the context of Anderson localization and mesoscopic phenomena [1] exemplified a particular class of chaotic quantum systems and had much in common with such paradigmatic problems in the domain of Quantum Chaos as quantum billiards [2]. The Wigner-Dyson energy level statistics first found in the framework of random matrix theory (RMT) [3] and considered to be a “fingerprint” of quantum chaotic systems [4] was shown to be relevant for disordered systems as well [5]. This fact gave a boost to a broad application of RMT results for qualitative and quantitative description of mesoscopic conductors and stimulated a common interest to statistical characteristics of spectra of disordered systems [6].

At the same time less attention was paid to statistical properties of eigenfunctions in disordered and chaotic quantum systems. Recently, however, the distribution of eigenfunction amplitudes was shown to be relevant for description of fluctuations of tunnelling conductance across the “quantum dots” [7] as well as for understanding some properties of atomic spectra [8]. Besides, a so called “microwave cavity” technique has emerged [9] as a laboratory tool to simulate a disordered quantum system. This technique allows to observe directly eigenfunctions spatial fluctuations and was used [10] to study experimentally the eigenfunction statistics in weak localization regime. All these facts make the issue of eigenfunction statistics to be of special interest and are calling for a detailed theoretical consideration.

In order to characterize eigenfunction statistics quantitatively, it is convenient to introduce a set of moments

\[ I_d = \int |\psi(r)|^{2d} d^3r \] of eigenfunction local intensity \(|\psi(r)|^2\) [13]. The second moment \(I_2\) is known as the inverse participation ratio (IPR). This quantity is a useful measure of eigenfunction localization: it is inversely proportional to a volume of a part of a system which contributes effectively to eigenstate normalization. For completely “ergodic” eigenfunctions covering randomly, but uniformly the whole sample \(I_2 \propto 1/V\), with \(V\) being a system volume. If, in contrast, the eigenfunctions are localized, i.e. concentrated in a region of linear size \(\xi\), the mean IPR scales as \(I_2 \propto \xi^{-D_2}\) where \(D_2\) is an effective dimension which can be different from a spatial dimensionality \(d\) because of a multifractal structure of eigenfunctions [13]. Correspondingly, IPR fluctuations reflect level-to-level variations of eigenfunction spatial structure.

The most complete analytical study of statistical characteristics of eigenfunctions was performed for the cases of 0d systems [14, 20], as well as for strictly 1d [15] and quasi 1d [17] geometry. Some analytical results were obtained also for a system in the vicinity of localization transition in the dimensionality \(d = 2 + \epsilon, \epsilon \ll 1\) [13] as well as for \(d \rightarrow \infty\) [13]. Let us note that in Ref. [14, 17, 13, 23] the supersymmetry method was used which is a very powerful tool to study distribution functions of various quantities characterizing eigenfunctions statistics.

In the present article we address systematically the issue of the eigenfunction statistics for arbitrary spatial dimensionality \(d\) in the weak localization domain. In the leading approximation (which neglects spatial structure of the system and treats it as a zero-dimensional one) these statistics are described by the RMT which predicts a Gaussian distribution of eigenfunctions amplitudes \(\psi(r)\) [3, 15] for systems with unbroken or completely broken time reversal symmetry [24]. It is known since the paper by Altshuler and Shklovskii [25] that the diffusion motion of a particle in a metallic sample produces deviations of spectral statistics from that predicted by RMT. To our best knowledge, the analogous problem for the eigenfunctions statistics for 2D or 3D systems has never been studied. It is just considered in the present paper. We use a recently developed method [21] which is based on the supersymmetry technique [3, 24] and combines a perturbative elimination of fast diffuse modes (in spirit of renormalization group ideas) and consequent non-perturbative evaluation of a resulting 0d integral. In this way, we calculate the deviations from the Gaussian
distribution of $\psi(r)$ in mesoscopic metallic samples. We calculate also the variance of the IPR which turns out to be of order of $1/g^2$, where $g$ is the dimensionless (measured in units of $e^2/h$) conductance of the sample. In the final part of the paper we establish a relation between the IPR distribution $P_I(I)$ and the distribution $P_r(r)$ of the so-called “level velocities” $v$, which characterize a motion of energy levels as a response to a small random perturbation. It turns out that the finite value of IPR variance gives rise to deviations of the form of $P_r(v)$ from the Gaussian. The latter form is known to be typical for completely “ergodic” systems with unbroken or completely broken time reversal invariance [22-24].

II. DISTRIBUTION OF EIGENFUNCTION AMPLITUDES AND IPR FLUCTUATIONS.

In order to calculate the distribution of eigenfunction amplitude and to find the IPR variance we use the fact that relevant quantities can be expressed in terms of correlation functions of certain supermatrix $\sigma$–model [17,18,25]. A quite general exposition of the method can be found in [18] and is not repeated here. Depending on whether the time reversal and spin rotation symmetries are broken or not, one of three different $\sigma$–models is relevant, with orthogonal, unitary or symplectic symmetry group. We consider first the case of the unitary symmetry (broken orthogonal, unitary or symplectic symmetry group. We have a form of a regular expansion in small parameter $\Delta/E_c = g^{-1}$. The systematic way to construct such an expansion can be briefly outlined as follows [17]. The matrix $Q(r)$ is decomposed as

$$Q(r) = T_0^{-1} \tilde{Q}(r) T_0 ,$$

where $T_0$ is a spatially uniform matrix and $\tilde{Q}$ describes all modes with non-zero momenta. When $\Delta \ll E_c$, the matrix $\tilde{Q}$ fluctuates only weakly around the value $\tilde{Q} = \Lambda$. It can be parametrized as

$$\tilde{Q} = \Lambda (1 + W/2) (1 - W/2)^{-1} = \Lambda \left(1 + \frac{W^2}{2} + \frac{W^3}{4} + \frac{W^4}{8} + \ldots \right) ,$$

where $W$ is a block off-diagonal supermatrix representing independent fluctuating degrees of freedom. Substituting this expansion into eqs. (1), (2) and integrating out the “fast” modes one obtains an expression for renormalized functional $\mathcal{F}_{ef}(u,Q_0)$, where $Q_0 = T_0^{-1} \Lambda T_0$ is an $r$–independent matrix (zero mode):

$$e^{-\mathcal{F}_{ef}(u,Q_0)} = (e^{-\mathcal{F}(u,Q_0,W)} + \ln J(W))_W .$$

Here $J(W)$ is the Jacobian of the transformation (1), (2) from the variable $Q$ to $(Q_0, W)$; and $(\cdot)_W$ denotes integration over $W$. Now one expands the exponential in (6) and calculates integrals over the fast modes $W$ using the Wick theorem and the contraction rules (3):
where the system is thought to be of the size $L_1 \times L_2 \times \ldots \times L_d$. Finally, the integrals over $Q_0$ are performed exactly, using the technique developed in [33].

Applying this method to eqs. (1), (3) one obtains:

$$
\overline{l_{I(q)}^2} = \frac{q!}{V^{q-1}} \left\{ 1 + \frac{a_1}{g} q(q-1) + O \left( \frac{1}{g^2} \right) \right\} \quad (9)
$$

where $q = 2 \pi v DL^d / 2$ is the conductance of the sample. The value of the coefficient $a_1$,

$$
a_1 = g \sum_q \Pi(q) = \frac{1}{\pi^2} \sum_{n_i \geq 0; \; n_2 > 0} \frac{1}{n_1^2 + \ldots + n_2^2}; \quad (10)
$$

depends on the spatial dimension and is equal to $a_1 = 1/6$ in quasi 1d systems. For $d \geq 2$ the corresponding sum over momenta $q$ diverges at large $|q|$ and is to be cut off at $|q| \sim l^{-1}$. This gives $a_1 = 1 / (2 \pi) \ln L / l$ for $d = 2$ and $a_1 \propto L / l$ for $d = 3$.

Knowing all the moments $\overline{l_{I(q)}^2}$ it is easy task to restore the whole probability distribution $P(y)$ of the eigenfunction local intensity $y = V |\langle \psi(r) \rangle|^2$:

$$
P^{(\psi)}(y) = e^{-y} \left\{ 1 + \frac{a_1}{g} (2 - 4y + y^2) + O \left( \frac{1}{g^2} \right) \right\} \quad (11)
$$

The leading terms here reproduce the well-known RMT result [8]; the rest is the weak localization correction.

Now we turn to the consideration of IPR fluctuations. It turns out that IPR variance is of order of $1/g^2$. Thus the expression (9) is insufficient for our needs and should be extended to the next order. By the same method we find:

$$
\overline{l_{I_{y_{y_2}}}^2} = \frac{2}{V} \left\{ 1 + \frac{2a_1}{g} + \frac{q}{g^2} (2a_1^2 - 5a_2) + O \left( \frac{1}{g^3} \right) \right\} \quad (12)
$$

Here the coefficient $a_2$ is defined as $a_2 = g^2 \sum_q \Pi^2(q)$ and is equal to

$$
a_2 = \frac{1}{\pi^2} \sum_{n_i \geq 0; \; n_2 > 0} \frac{1}{n_1^2 + \ldots + n_2^2}; \quad (14)
$$

the sum being convergent for $d < 4$. In particular, for $d = 1, 2, 3$ we have $a_1 = 1 / 90 \approx 0.0111, a_2 \approx 0.0266$ and $a_3 \approx 0.0527$ respectively.

Thus, we find the following expression for the relative variance of the IPR distribution:

$$
\delta^{(\psi)}(I_2) = \overline{l_{I_{y_{y_2}}}^2} \frac{\overline{\delta(l_{I_{y_{y_2}}}^2)^2}}{\overline{l_{I_{y_{y_2}}}^2}^2} = \frac{8a_2}{g^2} + O \left( \frac{1}{g^3} \right) \quad (15)
$$

B. Orthogonal and symplectic symmetry.

For the systems with unbroken time reversal symmetry (orthogonal and symplectic σ-models) the calculation goes along the same lines. The main difference is in the form of the contraction rules which in this case include an additional (Cooperon) contribution [5, 6] and can be presented in the following form:

$$
\langle \text{Str}(W(r) P W(r')) R \rangle = 2 \Pi(r, r') [\text{Str} P \text{Str} R - \text{Str} \text{Str} \text{Str} \text{Str} R] + \alpha \text{Str}(P \text{Str} R - \text{Str} P \text{Str} R) \right) \quad (16)
$$

where $\alpha = +1 (-1)$ for the orthogonal (resp. symplectic) symmetry, and $R = M^T R M$, with $M$ being a supermatrix representing the time reversal transformation and satisfying $M^T M = 1, M^2 = n k$ (see [27] for more details).

It turns out however that the formulae (1), (13) can be generalized to the orthogonal and symplectic cases without additional calculations. The leading weak–localization correction to $I_q$ originates from the one-diffuson-loop diagram which is proportional to $q(q-1) a_1/g$. The numerical coefficient can be restored by using the results we obtained earlier for the case of quasi-1D systems [8, 27].

$$
\overline{l_{q_{1D}}^2} = I_q^{(GE)} \left\{ 1 + \frac{1}{90} (q - 1) x + \frac{1}{180} q(q-1)(q-2)(3q-1)x^2 + O(x^3) \right\}, \quad (17)
$$

where $x = 2 / \beta g; \; \beta = 1(2, 4)$ for the orthogonal (resp. unitary, symplectic) symmetry, and $I_q^{(GE)}$ denotes the value of $I_q$ for the corresponding Gaussian ensemble. We see that the quasi-1D result is in agreement with our general formulae (1), (13). We restore therefore the coefficients of the leading corrections to $I_q$ in the orthogonal and symplectic cases:

$$
\overline{l_{q_{0}}^2} = \frac{q - 1)!}{V^{q-1}} \left\{ 1 + \frac{2a_1}{g} q(q-1) + O \left( \frac{1}{g^2} \right) \right\} \quad (18)
$$

The corresponding corrections to distributions of eigenfunctions intensities are as follows [27]:

$$
P^{(\psi)}(y) = e^{-y/2} \left\{ 1 + \frac{a_1}{g} \left( \frac{3}{2} - 3y + \frac{y^2}{2} \right) + O \left( \frac{1}{g^2} \right) \right\} \quad (20)
$$

$$
P^{(\psi)}(y) = 4y e^{-2y} \left\{ 1 + \frac{a_1}{g} \left( 3 - 6y + 2y^2 \right) + O \left( \frac{1}{g^2} \right) \right\} \quad (21)
$$

As to the IPR variance, $\delta(I_2)$, the leading contribution to it is given by the diagrams with a two-diffuson-loop
and is therefore proportional to $a_2/g^2$. The coefficient can be again restored by comparison with the quasi–1D result [18,23]:

$$\delta^{(1D)}(I_2) \simeq \frac{16}{45\beta^2g^2}$$

(22)

In the quasi–1D case $a_2 = 1/90$, so that [22] agrees with [15]. For orthogonal and symplectic systems of a general dimensionality, we find

$$\delta^{(o)}(I_2) \simeq 32a_2/g^2;$$

(23)

$$\delta^{(sp)}(I_2) \simeq 2a_2/g^2$$

(24)

We have checked the above results by direct calculation in the orthogonal case. It should be noted that in order to get the correct results for $I_2$ and $I_2^2$ to $1/g^2$ order in this case, one should take into account the Jacobian $J(W)$ in eq.(10) (whereas in the unitary case this Jacobian contributes only starting from the $1/g^3$ order). For the orthogonal symmetry, we find

$$\ln J(W) = -\frac{1}{4} \int \text{Str}W^2 + O(W^4)$$

(25)

Carrying out the calculations we obtain

$$\overline{I_2^{(o)}} = \frac{3}{V} \left[ 1 + \frac{4a_1}{g} \right] + \frac{1}{g^2}(12a_1^2 - 30a_2) + O\left(\frac{1}{g^3}\right)$$

(26)

$$\overline{I_2^{(sp)}} = \left( \frac{3}{V} \right)^2 \left[ 1 + \frac{8a_1}{g} \right] + \frac{1}{g^2}(40a_1^2 - 28a_2) + O\left(\frac{1}{g^3}\right);$$

(27)

and consequently, $\delta^{(o)}(I_2) = 32a_2/g^2$, in agreement with [15], [23].

C. Discussion.

Let us now give some comments on the results obtained in this section. Eqs.(11), (21), (21) present leading (one–loop) corrections to the distribution of eigenfunction amplitudes which are due to the weak–localization effects. It is easy to see that the corrections lead to an increase in probability to find a value of the amplitude considerably smaller or considerably larger than the average value. It is exactly what one would intuitively expect assuming a tendency of eigenfunctions to localization. It is interesting to note that the correction is of the same type for all three ensemble, in contrast to the well known difference in weak–localization corrections to the conductivity.

Deviation of the eigenfunctions amplitude distribution from its RMT form was observed very recently in an experiment with a disordered 2D microwave cavity [21]. The form of the correction reported in [21] is in agreement with our formula (21).

It should be noted that Eqs.(11), (21), (21) are valid for not too large values of $y$: $y \lesssim \sqrt{g/a_1}$, when the correction term is small as compared to the leading one. For larger $y$ (i.e. in the far “tail”), the distribution $\mathcal{P}(\gamma)$ differs strongly from its RMT form and can not be found by the method used in this paper. At present we are able to present the result for the “tail” of $\mathcal{P}(\gamma)$ for the quasi–1D geometry only. This can be done using the exact analytical expressions for $\mathcal{P}(\gamma)$ we have found earlier for quasi–1D systems (see Eqs.(16), (17) of Ref. [18] or Eqs.(79), (89) of Ref. [22]). We find three different regions of the variable $y$:

$$\mathcal{P}^{(u)}(y) \simeq e^{-y} \left[ 1 + \frac{x}{6}(2 - 4y + y^2) + \ldots \right]; \quad y \lesssim x^{-1/2}$$

(28)

$$\mathcal{P}^{(u)}(y) \sim \exp \left[ -y + \frac{1}{15}y^2x + \ldots \right] ; \quad x^{-1/2} \lesssim y \lesssim x^{-1}$$

(29)

$$\mathcal{P}^{(a)}(y) \sim \exp \left[ -4\sqrt{y/x} \right] ; \quad y \gtrsim x^{-1}$$

(30)

where $x = 1/g$. For completeness, we repeated again the result of preceding subsections, Eq.(28), which presents a relatively small correction to the RMT formula. In the intermediate region (29), the correction in the exponent is small compared to the leading term but much larger than unity, so that $\mathcal{P}(\gamma) \gg \mathcal{P}_{\text{RMT}}(\gamma)$ though $\ln \mathcal{P}(\gamma) \simeq \ln \mathcal{P}_{\text{RMT}}(\gamma)$. Finally, in the region (30), the distribution function $\mathcal{P}(\gamma)$ has nothing to do with its RMT form. To understand the physical origin of the asymptotical behavior (30) we note that it has exactly the same form as in the case of a system of an infinite length, when all states are localized (see Eq.(21) of Ref. [18] or Eq.(114) of Ref. [23]). In the quasi–1D case $x = 1/g = L/L_c$, where $L$ is the sample length and $L_c$ is the localization length in an infinite system. Thus, the ratio $y/x$ in the exponent in Eq.(30) is equal to $y/x = V_c|\psi(r)|^2$, where $V_c$ is the localization volume. Therefore, the asymptotics (30) is controlled by the correlation length $L_c$ and describes a probability to have a quasi–localized “bump” with an extent much less than $L_c$ (28). We stress again that the formulae (28), (30) have been derived for quasi–1D systems; the problem of their generalization to higher $d$ is still open.

The obtained value of the IPR fluctuations $\delta(I_2) \sim 1/g^2 \propto L^{4-2d}$ is much larger than the RMT result $\delta(I_2) \propto 1/V \sim L^{-d}$. These fluctuations have much in common with the universal conductance fluctuations which have the same relative magnitude. It is interesting to note that extrapolating this result to the Anderson transition point, where $g \sim 1$, we get $\delta(I_2) \sim 1$, so that the magnitude of fluctuations is of order of the mean value. Thus, despite the fact that the wave function is extended through the whole sample, there is no self–averaging of the IPR at the mobility edge.
III. DISTRIBUTION OF LEVEL VELOCITIES.

Let us now demonstrate that the IPR fluctuations are intimately related to level response characteristics of disordered system subject to a random perturbation.

The issue of the energy level motion under an external perturbation was under intensive study recently [24,32]. In the paper [24] the statistical properties of the “level velocities” (LV) defined as \( v_n = \partial E_n / \partial \alpha \), \( E_n \) standing for a given energy level and \( \alpha \) being some control parameter, were studied for systems with completely “ergodic” eigenstates. In particular, the LV distribution \( P(v) \) was shown to be Gaussian within the zero-mode approximation for supersymmetric \( \sigma \)-model. On the other hand, in the recent paper [32] the LV distribution was calculated analytically for a long quasi 1d system subject to a random perturbation. It was found that in the limit \( g \to 0 \) which corresponds to complete localization of eigenstates, \( P(v) \) is essentially non-Gaussian.

The mean squared LV \( \langle v_n^2 \rangle \) was found to be of order of inverse localization length \( \xi^{-1} \), whereas the RMT predicts \( \langle v_n^2 \rangle \propto 1/V \).

In order to get understanding of these facts in broader context let us consider a disordered system as described by a one-particle random Hamiltonian \( \mathcal{H} \). Let us impose on it some random perturbation \( \mathcal{W} \) so that perturbed Hamiltonian is \( \mathcal{H} + \alpha \mathcal{W} \), with the parameter \( \alpha \) controlling the strength of perturbation. We assume that both the Hamiltonian \( \mathcal{H} \) and the perturbation \( \mathcal{W} \) belong to the same pure symmetry class [24]. In the sequel, we denote the averaging over the realizations of \( \mathcal{W} \) by \( \langle \ldots \rangle \), which should be distinguished from the averaging over unperturbed Hamiltonian \( \mathcal{H} \) denoted by bar. The level velocity \( v_n \) corresponding to an energy level \( E_n \) can be found within the conventional perturbation theory as

\[
v_n = \int d^d r \int d^d r' W_{rr'} \psi_n^*(r) \psi_n(r') , \tag{31}
\]

where \( W_{rr'} \) are matrix elements of the perturbation. To be specific, these matrix elements are supposed to be independent Gaussian distributed random variables with the mean value equal to zero and the variance \( \langle W_{rr'} W_{rr'} \rangle = \mathcal{W}_0 |r - r'| \). We also make a natural assumption that the perturbation \( \mathcal{W} \) is of finite range \( \zeta \), i.e. \( \mathcal{W}_0 |r| \gg \zeta \) is assumed to vanish.

By using eq.(31) one easily obtains:

\[
\langle v_n^2 \rangle = \int d^d r \int d^d r' \mathcal{W}_0 |r - r'| |\psi_n(r)\psi_n(r')|^2 \tag{32}
\]

To simplify eq.(32) we need to know the short-scale behaviour of the correlator \( \langle \psi_n^2(r) \psi_n^*(r') \rangle \). We have [33]:

\[
\langle \psi_n^2(r) \psi_n^*(r') \rangle = \frac{1}{2} [1 + k_d (r - r')] |\psi_n^2(r)| , \tag{33}
\]

where \( k_d(r) = \operatorname{Im} G(r^2) / (\pi \nu)^2 \) and \( G(r - r') = \langle r | (E - \mathcal{H})^{-1} | r' \rangle \) is the Green function of the Hamiltonian \( \mathcal{H} \).

The explicit form of \( k_d(r) \) for \( d = 2, 3 \) can be found in [33]; for our purposes it is enough to mention its asymptotic behaviour:

\[
k_d(r) \simeq 1 \quad ; \quad k_f r \ll 1
\]

\[
k_d(r) \ll 1 \quad ; \quad k_f r \gg 1 \tag{34}
\]

where \( k_f \) is the Fermi momentum.

Substituting (33) in (32) and using (34) we find

\[
\langle v_n^2 \rangle = w_0 I_2 ; \quad w_0 = c \int d^d r \mathcal{W}_0(r) , \tag{35}
\]

where \( I_2 \) is the mean IPR introduced above and \( c = 1 (1/2) \) if \( k_f \zeta \ll 1 \) (resp. \( k_f \zeta \gg 1 \)).

Therefore, the mean squared LV \( \langle v_n^2 \rangle \) is proportional to the mean IPR \( I_2 \). In the infinite quasi 1d system \( I_2 \) is inversely proportional to the localization length \( \xi \) in agreement with the behaviour \( \langle v_n^2 \rangle \propto \xi^{-1} \) found in [32].

This consideration can be extended to higher LV moments \( \langle v_n^2 \rangle \) as well. After some manipulations we find the general relation between the two distributions:

\[
P_v(v) = \frac{\delta(v - v_n)}{2\pi w_0 I_2} \exp \left[ -v^2 / 2w_0 I_2 \right] p_I(I_2) \tag{36}
\]

The following comment is appropriate here. For a given realization of the random perturbation \( \mathcal{W} \) one can define the mean level velocity \( \langle v \rangle \) (note that the averaging here goes only over the unperturbed Hamiltonian \( \mathcal{H} \), but not over \( \mathcal{W} \)). It follows from eq.(32) that \( \langle v \rangle \) defined in such a way shows Gaussian fluctuations around zero from one realization of the perturbation to another.

The mean square \( \langle v^2 \rangle \) can be easily found to be \( \langle v^2 \rangle = V^{-1} \int d^d r \mathcal{W}_0(r) |G(r)|^2 \) and is proportional to the inverse system volume \( V^{-1} \). At the same time the distribution function \( p_u(u) \) of the quantity \( u = v - \langle v \rangle \) turns out to be self-averaging, i.e. its form is the same for any realization of the perturbation \( \mathcal{W} \). The function \( p_u(u) \) can be shown to satisfy the relation similar to eq.(36):

\[
p_u(u) = \int_0^\infty dI_2 p_I(I_2) \frac{dI_2}{2\pi(w_0 I_2 - \langle v^2 \rangle)^{1/2}} \exp \left[ -u^2 / 2(w_0 I_2 - \langle v^2 \rangle) \right] \tag{37}
\]

The relation (37) is then recovered as a convolution of eq.(36) with the Gaussian distribution of \( \langle v \rangle \).

Eq.(30) shows that the Gaussian form of \( p_v(v) \) found earlier for generic chaotic systems [24,25] is a consequence of the IPR being a self-averaging quantity within zero-mode approximation to nonlinear \( \sigma \)-model [24]. For disordered systems with finite conductance \( g \) the function \( p_v(v) \) becomes nongaussian due to mesoscopic fluctuations of eigenfunctions. As a natural measure of the deviations from the Gaussian form one can use the cumulant:
\[
\left( \frac{\langle v^2 \rangle}{\langle v^3 \rangle^2} \right) - 3 \left( \frac{\langle v^2 \rangle^2}{\langle v^4 \rangle} \right) = 3 \frac{I_2^3 - I_2^2}{I_2^2} \approx 3 \delta(I_2) \propto \frac{1}{g^2}
\]  

(38)

With increasing disorder the conductance decreases and the system passes to the strong localization regime \( g \ll 1 \). For this regime the explicit expression for both distribution functions \( P_l(I_2) \) and \( P_v(v) \) were found in quasi 1d systems in the course of independent calculations [17, 23] (see also [14] and [32]). The results indeed satisfy the relation (38).

It could be interesting to study the functions \( P_v(v) \) and \( P_l(I_2) \) also in \( d > 2 \) and in the vicinity of the Anderson transition for \( d > 2 \) where their form is expected to be universal. Relation (38) may provide then a convenient background for investigations of the statistical properties of eigenfunctions. In particular, the behaviour of the mean square level velocity \( \langle v^2 \rangle \) in the critical region can be used for extracting a value of the anomalous effective dimension \( D_2 \).

IV. SUMMARY

In this paper we have studied deviations of the eigenfunction statistical characteristics in a disordered metallic sample from those predicted within the Random Matrix Theory. The found correction to the distribution of eigenfunction amplitudes reflect the influence of weak localization to an eigenfunction. The relative magnitude of fluctuations of the inverse participation ratio is of the order of \( 1/g^2 \), where \( g \) is the dimensionless conductance, in close analogy with the conductance fluctuations. We have also revealed the relationship between the mesoscopic fluctuations of the inverse participation ratio and the form of the “level velocity” distribution in a system.

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The conclusion about the existence of quasilocalized eigenstates in a diffusive sample was also drawn recently by B.A. Muzykantskii and D.E. Khmelnitskii (preprint, 1994) from consideration of long time relaxation phenomena.

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