Abstract

We present a collection of matrix valued shape invariant potentials which give rise to new exactly solvable problems of SUSY quantum mechanics. It includes all irreducible matrix superpotentials of the generic form $W = kQ + \frac{1}{x}R + P$ where $k$ is a variable parameter, $Q$ is the unit matrix multiplied by a real valued function of independent variable $x$, and $P$, $R$ are hermitian matrices depending on $x$. In particular we recover the Pron’ko-Stroganov ”matrix Coulomb potential” and all known scalar shape invariant potentials of SUSY quantum mechanics.

In addition, five new shape invariant potentials are presented. Three of them admit a dual shape invariance, i.e., the related hamiltonians can be factorized using two non-equivalent superpotentials. We find discrete spectrum and eigenvectors for the corresponding Schrödinger equations and prove that these eigenvectors are normalizable.
1 Introduction

Invented by E. Witten [1] as a toy model supersymmetric quantum mechanics (SSQM) became a fundamental field including many interesting external and internal problems. In particular the SSQM presents powerful tools for explicit solution of quantum mechanical problems using the shape invariance approach [2]. Unfortunately, the number of problems satisfying the shape invariance condition is rather restricted. However, such problems include practically all cases when the related Schrödinger equation is exactly solvable and has an explicitly presentable potential. Well known exceptions are exactly solvable Schrödinger equations with Natanzon potentials [3] which are formulated in terms of implicit functions. The list of shape invariant potentials depending on one variable can be found in [4].

An interesting example of QM problem which admits a shape invariant supersymmetric formulation was discovered by Pron’ko and Stroganov [5] who studied a motion of a neutral non-relativistic fermion which interacts anomalously with the magnetic field generated by a thin current carrying wire.

The supersymmetric approach to the Pron’ko-Stroganov (PS) problem was first applied in paper [6] with using the momentum representation. In paper [7] this problem was solved using its shape invariance in the coordinate representation. Recently a relativistic generalization of the PS problem was proposed [8] which can also be integrated using its supersymmetry with shape invariance.

The specificity of the PS problem is that it is formulated using a matrix superpotential while in the standard SSQM the superpotential is simply a scalar function. Matrix superpotentials themselves were discussed in many papers, see, e.g., [9], [10], [11], [13] but this discussion was actually reduced to analysis of particular examples. In papers [14] such superpotentials were used for analysis of motion of a spin \( \frac{1}{2} \) particle in superposed magnetic and scalar fields. In paper [11] a certain class of such superpotentials was described which however was \( ad hoc \) restricted to \( 2 \times 2 \) matrices which depend linearly on the variable parameter. Thus, in contrast to the case of scalar superpotentials, the class of known matrix potentials includes only few examples which are important but rather particular, while the remaining part of this class is still “terra incognita”. It seems to be interesting to extend our knowledge of these potentials since this way it is possible to find new systems of Schrödinger equations which are exactly integrable.

In the present paper a certain class of matrix valued superpotentials is described which includes the superpotential for the PS problem as a particular case. Moreover, we do not make \( a \ priori \) suppositions about the dimension of the involved matrices but restrict ourselves to linear and inverse dependence of the superpotentials on variable parameters. Through our approach however the problem of classification of indecomposable matrix potentials which are shape invariant appears to be completely solvable. We present a classification of such potentials and discuss the corresponding exactly solvable problems for coupled systems of Schrödinger equations. In partic-
ular the discrete energy spectra and exact solutions for these models are found and normalizability of the ground and exited states is proven. Solutions corresponding to continuous spectra are not considered here.

Three out of five found hamiltonians admit alternative factorizations with using different superpotentials. The corresponding potentials are shape invariant w.r.t. shifts of two different parameters. Such dual shape invariance results in existence of two alternative spectra branches. Moreover, for some values of free parameters both these branches can be realized.

2 Superpotential for PS problem

The PS problem was discussed in numerous papers, see, e.g., [5]-[7]. Thus we will not present its physical motivations and calculation details but start with the corresponding equation for radial functions [7]

\[ \hat{H}_\kappa \psi = E_\kappa \psi \]  

where \( \hat{H}_\kappa \) is a Hamiltonian with a matrix potential, \( E_\kappa \) and \( \psi \) are its eigenvalue and eigenfunction correspondingly, moreover, \( \psi \) is a two-component spinor. Up to normalization of the radial variable \( x \) the Hamiltonian \( \hat{H}_\kappa \) can be represented as

\[ \hat{H}_\kappa = -\frac{\partial^2}{\partial x^2} + \kappa(\kappa - \sigma_3) \frac{1}{x^2} + \sigma_1 \frac{1}{x} \]  

where \( \sigma_1 \) and \( \sigma_3 \) are Pauli matrices and \( \kappa \) is a natural number. In addition, solutions of equation (1) must be normalizable and vanish at \( x = 0 \).

Hamiltonian \( \hat{H}_\kappa \) can be factorized as

\[ \hat{H}_\kappa = a^-_\kappa a^+_\kappa + c_\kappa \]  

where

\[ a^-_\kappa = \frac{\partial}{\partial x} + W_\kappa, \quad a^+_\kappa = -\frac{\partial}{\partial x} + W_\kappa, \quad c_\kappa = -\frac{1}{(2\kappa + 1)^2} \]

and \( W \) is a matrix superpotential

\[ W_\kappa = \frac{1}{2x} \sigma_3 - \frac{1}{2\kappa + 1} \sigma_1 - \frac{2\kappa + 1}{2x}. \]

Another nice property of Hamiltonian \( \hat{H}_\kappa \) is that its superpartner \( \hat{H}^+_\kappa \) is equal to \( \hat{H}_{\kappa+1} \), namely

\[ \hat{H}^+_\kappa = a^-_\kappa a^+_\kappa + c_\kappa = -\frac{\partial^2}{\partial x^2} + (\kappa + 1)(\kappa + 1 - \sigma_3) \frac{1}{x^2} + \sigma_1 \frac{1}{x} = \hat{H}_{\kappa+1} \]

Thus equation (1) admits supersymmetry with shape invariance and can be solved using the standard technique of SSQM [4].
3 Generic matrix shape invariant potentials

Following a natural desire to find other matrix potentials which are form invariant we consider equation (1) with

\[ H_k = -\frac{\partial^2}{\partial x^2} + V_k(x), \]  

where \( V_k(x) \) is an \( n \times n \) dimensional matrix potential depending on variable \( x \) and parameter \( k \).

Suppose that the Hamiltonian accepts factorization (3) where \( W_k(x) \) is a superpotential. Our goal is to find such superpotentials which generate form invariant potentials \( V_k(x) \).

Assume \( W_k(x) \) is Hermitian. Then the corresponding potential \( V_k(x) \) and its superpartner \( V_k^+(x) \), i.e.,

\[ V_k(x) = -\frac{\partial W_k}{\partial x} + W_k^2 \quad \text{and} \quad V_k^+(x) = \frac{\partial W_k}{\partial x} + W_k^2 \]  

are Hermitian too.

Suppose also that the Hamiltonian be shape invariant, i.e.,

\[ H_k^+ = H_{k+\alpha} + C_k, \]  

thus \( V_k^+ = V_{k+\alpha} + C_k \) or

\[ W_k^2 + W_k' = W_{k+\alpha}^2 - W_{k+\alpha}' + C_k \]  

were \( C_k \) and \( \alpha \) are constants.

Let us state the problem of classification of shape invariant superpotentials, i.e., \( n \times n \) matrices whose elements are functions of \( x, k \) satisfying conditions (8). In the following section we present such classification for a special class of superpotentials whose dependence on \( k \) is defined by terms proportional to \( k \) and \( \frac{1}{k} \) only.

4 Irreducible matrix superpotentials

To generalize (4) we consider superpotentials of the following special form

\[ W_k = kQ + \frac{1}{k}R + P, \]  

where \( P, R \) and \( Q \) are \( n \times n \) Hermitian matrices depending on \( x \). Moreover, we suppose that \( Q = Q(x) \) is proportional to the unit matrix. This supposition can be motivated by two reasons:
• our goal is to generalize superpotential (4) in which the term linear in \( k \) is proportional to the unit matrix;

• restricting ourselves to such \( Q \) it is possible to make a complete classification of the corresponding superpotentials (9) satisfying shape invariance condition (8).

We do not make any \textit{a priori} supposition about possible values of the continuous independent variable \( x \). However we suppose that relations (8) are valid also for \( k \to k' \) where \( k' = k + \alpha, k + 2\alpha, \ldots, k + n\alpha \) and \( n \) is a natural number which is sufficiently large to make the following speculations.

It is reasonable to restrict ourselves to the case when the matrices \( P \) and \( R \) cannot be simultaneously transformed to a block diagonal form since if such (unitary) transformation is admissible, the related superpotentials are completely reducible. Thus we suppose that the pair of matrices \( <P, R> \) is irreducible. Let us show that in this case it is sufficient to consider \( 1 \times 1 \) and \( 2 \times 2 \) matrices only.

Considering the special case \( \alpha = 0 \) we conclude that the corresponding \( W_k \) should be linear in \( x \) provided relation (8) is satisfied:

\[
W_k = \frac{1}{2}C_k x + M_k
\]  

where \( C_k \) is a constant multiplied by the unit matrix and \( M_k \) is a constant hermitian matrix which can be diagonalized. In this way we obtain a direct sum of shifted one dimensional oscillators whose irreducible components can be represented in form (10) where \( C_k \) and \( M_k \) are constants.

Let \( \alpha \neq 0 \). Substituting (9) into (8) and multiplying the obtained expression by \( k^2(k+\alpha)^2 \) we obtain

\[
AB^2(Q' - \alpha Q^2) + 2B^2(P' - \alpha QP) + \alpha B\{R, P\} + ABR' + \alpha AR^2 = B^2C_k
\]  

where \( \{R, P\} = RP + PR \) is anticommutator of matrices \( P \) and \( Q \), \( A = 2k + \alpha, B = k(k+\alpha) \) and the prime denotes derivative w.r.t. \( x \).

All terms in the l.h.s. of equations (11) are polynomials in discrete variable \( k \). In order for this equation be consistent, its r.h.s. (which includes an arbitrary element \( C_k \)) should also be a polynomial of the same order whose general form is

\[
B^2C_k = \nu \alpha AB^2 - 2\mu B^2 - \alpha \lambda B + \rho AB + \alpha \omega^2 A
\]  

where the Greek letters denote arbitrary parameters. Substituting (12) into (11) and equating coefficients for linearly independent terms we obtain the following system:

\[
Q^2\alpha - Q' + \nu \alpha = 0, \tag{13}
\]

\[
P' - \alpha QP + \mu = 0, \tag{14}
\]

\[
\{R, P\} + \lambda = 0 \tag{15}
\]

\[
R' = \rho, \quad R^2 = \omega^2. \tag{16}
\]
It follows from (16) that $\rho = 0$ and $R$ is a constant matrix whose square is proportional to the unit one.

If $R$ is proportional to the unit matrix $I$ or is the zero matrix (in the last case $\omega = 0$) the corresponding superpotential (13) is reducible. Let $\omega \neq 0$ and $R \neq \pm \omega I$ then the general form of $P$ satisfying (15) is

$$P = \frac{\lambda}{2\omega} R + \tilde{P}$$

(17)

where $\tilde{P}$ is a matrix which anticommutes with $R$.

A straightforward analysis of equation (14) shows that it is easily integrable, but to obtain non-trivial $Q$ it is necessary to set $\mu = \lambda = 0$. Indeed, without loss of generality hermitian matrix $R$ whose square is proportional to the unit matrix can be chosen in the diagonal form:

$$R = \omega \left( \begin{array}{cc} I_{m \times m} & 0_{m \times s} \\ 0_{s \times m} & -I_{s \times s} \end{array} \right), \quad m + s = n$$

(18)

where $\omega \neq 0$ is a constant, $I_\cdot \cdot$ and $0_\cdot \cdot$ are the unit and zero matrices whose dimension is indicated in subindices, and without loss of generality we suppose that $s \geq m$.

The corresponding matrix $\tilde{P}$ satisfying (25) has the following generic form:

$$\tilde{P} = \left( \begin{array}{cc} 0_{m \times m} & M_{m \times s} \\ M_{s \times m} & 0_{s \times s} \end{array} \right)$$

(19)

where $M_{m \times s}$ is an arbitrary matrix of dimension $m \times s$. Substituting (17)–(19) into (14) we obtain the following equations:

$$\tilde{P}' = 2\alpha Q \tilde{P},$$

(20)

$$\left( \frac{\lambda}{\omega} R + \mu I_{n \times n} \right) Q = 0.$$  

(21)

Analyzing equation (21) we conclude that for $\lambda^2 + \mu^2 \neq 0$ the matrix in brackets is invertible and so we have to set $Q = 0$. If $Q$ is nontrivial we have to set $\lambda = \mu = 0$. As a result the system (14)–(16) is reduced to the following form

$$Q^2 \alpha - Q' + \nu \alpha = 0,$$

(22)

$$P = \tilde{P} \exp \left( \alpha \int Q dx \right),$$

(23)

$$\{ \tilde{P}, R \} = 0, \quad R^2 = \omega^2,$$

(24)

$$C_k = \frac{\alpha \omega^2 (2k + \alpha)}{k^2 (k + \alpha)^2} + \nu \alpha (2k + \alpha)$$

(25)

where both $R$ and $\tilde{P}$ are constant matrices.
Thus the problem of classification of matrix valued shape invariant potentials \( (9) \) is reduced to solving the first order differential equation (22) for function \( Q \) and the algebraic problem (24) for hermitian matrices \( R \) and \( \tilde{P} \).

Let us show that hermitian \( n \times n \) matrices \( \tilde{P} \) and \( R \) which satisfy conditions (24) can be simultaneously transformed to a block diagonal form. Moreover, irreducible matrices satisfying (24) are nothing but the \( 2 \times 2 \) Pauli matrices multiplied by constants, and ”\( 1 \times 1 \) matrices” (scalars) satisfying \( R\tilde{P} = 0 \). Starting with (18) and (19) and applying a unitary transformation

\[
R \rightarrow R' = URU^\dagger, \quad \tilde{P} \rightarrow \tilde{P}' = U\tilde{P}U^\dagger,
\]

where \( u_{m \times m} \) and \( u_{s \times s} \) are unitary submatrices, we obtain

\[
\tilde{P}' = \begin{pmatrix} 0_{m \times m} & M'_{m \times s} \\ M'_{s \times m} & 0_{s \times s} \end{pmatrix}, \quad R' = R \quad \text{(26)}
\]

with

\[
M'_{m \times s} = u_{m \times m} M_{m \times s} u_{s \times s}^\dagger.
\quad \text{(27)}
\]

Transformation (27) can be used to simplify submatrix \( M_{m \times s} \). In particular this submatrix can be reduced to the following form

\[
M'_{m \times s} = \begin{pmatrix} \tilde{M}_{m \times m} & 0_{m \times (s-m)} \end{pmatrix}
\quad \text{(28)}
\]

where \( \tilde{M}_{m \times m} \) is a diagonal matrix:

\[
\tilde{M}_{m \times m} = \text{diag}(\mu_1, \mu_2, \ldots, \mu_m)
\quad \text{(29)}
\]

where \( \mu_1, \mu_2, \ldots \) are real parameters. Without loss of generality we suppose that there are \( r \) nonzero parameters \( \mu_1, \mu_2, \ldots, \mu_r \) with \( 0 \leq r \leq m \) being the rank of matrix \( M \).

Notice that transformation (27)–(29) for rectangular matrices \( M \) is called singular value decomposition. Such transformations are widely used in linear algebra, see, e.g., [17].

But the set of matrices \( \{R, \tilde{P}_A\} \) with \( R \) and \( \tilde{P}_A \) given in (18) and (26), (28), (29) is completely reducible since by an accordant permutation of rows and columns they can be transformed to direct sums of \( 2 \times 2 \) matrices \( \{R_{2 \times 2}, \tilde{P}_{2 \times 2}\} \) where

\[
R_{2 \times 2} = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \equiv \omega \sigma_3, \quad \tilde{P}_{2 \times 2} = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix} \equiv \mu \sigma_1, \quad \mu = \mu_1, \mu_2, \ldots, \mu_r \quad \text{(30)}
\]

and of \( 1 \times 1 \) matrices

\[
R_{1 \times 1} = \pm \omega, \quad \tilde{P}_{1 \times 1} = \mu, \quad \mu \omega = 0 \quad \text{(31)}
\]
were $\omega$ and $\mu$ are arbitrary real numbers. The transformation of matrices (18), (26), (28), (29) to the direct sum of matrices (30) and (31) can be given explicitly as

$$R \rightarrow URU^\dagger, \quad \tilde{P} \rightarrow URU^\dagger$$

where $U$ is a unitary matrix whose nonzero entries are:

$$U_{a\,a} = U_{b\,m+b-1} = U_{m+s\,m+s+1} = 1,$$

for $a = 1, m + s, m + s + 1, m + s + 2, \ldots, n$, and $b = 2, 3, \ldots, s + 1$.

Thus up to unitary equivalence we have only two versions of irreducible matrices $R$ and $P$ which are given by equations (30) and (31).

The remaining equation (22) is easily integrable, thus we can find all inequivalent irreducible superpotentials (9) in explicit form.

## 5 Superpotentials and shape invariant potentials

There are six different types of solutions of equation (22), namely

$$Q = 0, \quad \nu = 0,$$

and

$$Q = -\frac{1}{\alpha x}, \quad \nu = 0,$$

$$Q = -\frac{\lambda}{\alpha}, \quad \nu = -\frac{\lambda^2}{\alpha^2} < 0,$$

$$Q = -\frac{\lambda}{\alpha} \tanh \lambda x, \quad \nu = -\frac{\lambda^2}{\alpha^2} < 0,$$

$$Q = -\frac{\lambda}{\alpha} \coth \lambda x, \quad \nu = -\frac{\lambda^2}{\alpha^2} < 0,$$

$$Q = \frac{\lambda}{\alpha} \tan \lambda x, \quad \nu = \frac{\lambda^2}{\alpha^2} > 0$$

that are defined up to translations $x \rightarrow x + c$, $c$ is an integration constant, and $\alpha$ is supposed to be nonzero.

The corresponding matrices $\tilde{P}$ are easily calculated using equations (23) and (30) or (31).

Let us note that using solutions (32) or solutions (33) for scalar $P$ and $R$ given by relations (31), we simply recover the known list of shape invariant potentials which is presented, e.g., in [4], see the table on pages 291-292 (this list includes also the harmonic oscillator (10)). We will not present this list here but note that our approach gives a simple and straightforward way to find it.

Consider the case when $P$ and $R$ are $2 \times 2$ matrices (30). Now solutions (32) for $Q$ and solutions with trivial matrices $P$ are not available since they lead to reducible
superpotentials. However, solutions (33) are consistent. Substituting (23), (30), (33) into (9) we obtain the following list of matrix superpotentials

\[ W_{\kappa,\mu} = ((2\mu + 1)\sigma_3 - 2\kappa - 1) \frac{1}{2x} + \frac{\omega}{2\kappa + 1}\sigma_1, \quad \mu > -\frac{1}{2}, \]  

(34)

\[ W_{\kappa,\mu} = \lambda \left(-\kappa + \mu \exp(-\lambda x)\sigma_1 - \frac{\omega}{\kappa}\sigma_3\right), \]  

(35)

\[ W_{\kappa,\mu} = \lambda \left(\kappa \tan \lambda x + \mu \sec \lambda x\sigma_3 + \frac{\omega}{\kappa}\sigma_1\right), \]  

(36)

\[ W_{\kappa,\mu} = \lambda \left(-\kappa \coth \lambda x + \mu \csch \lambda x\sigma_3 - \frac{\omega}{\kappa}\sigma_1\right), \quad \mu < 0, \ \omega > 0, \]  

(37)

\[ W_{\kappa,\mu} = \lambda \left(-\kappa \tanh \lambda x + \mu \sech \lambda x\sigma_1 - \frac{\omega}{\kappa}\sigma_3\right), \]  

(38)

where we introduce the normalized parameter \( \kappa = \frac{k}{\alpha} \). These superpotentials are defined up to translations \( x \to x + c, \kappa \to \kappa + \gamma \), and up to unitary transformations \( W_{\kappa,\mu} \to U_k W_{\kappa,\mu} U_k^\dagger \) where \( U_1 = \sigma_1, U_2 = \frac{1}{\sqrt{2}}(1 \pm i\sigma_2) \) and \( U_3 = \sigma_3 \). In particular these transformations change signs of parameters \( \mu \) and \( \omega \) in (35)–(38) and of \( \mu + \frac{1}{2} \) in (34), thus without loss of generality we can set

\[ \omega > 0, \quad \mu > 0 \]  

(39)

in all superpotentials (35)–(38). Zero values of these parameters are excluded if superpotentials (34)–(37) are irreducible.

Conditions (39) can be imposed also for superpotential (37). To unify some following calculations we prefer to fix the signs of \( \mu \) and \( \kappa \) in the way indicated in (37).

Notice that the transformations \( k \to k' = k + \alpha \) correspond to the following transformations for \( \kappa \):

\[ \kappa \to \kappa' = \kappa + 1. \]  

(40)

If \( \mu = 0 \) and \( \omega = 1 \) then operator (34) coincides with the well known superpotential for PS problem (1), but for \( \mu \neq 0 \) superpotential (34) is not equivalent to (4). The other found superpotentials are new also and make it possible to formulate consistent, exactly solvable problems for Schrödinger equation with matrix potential. The corresponding potentials \( V_\kappa \) can be found starting with (34)–(37) and using definition (6). Let us rewrite equation (6) as follows:

\[ W_{2,\kappa,\mu}' - W_{2,\kappa,\mu} = V_\kappa = \hat{V}_\kappa + c_\kappa \]  

(41)

where \( c_\kappa \) is a constant and \( \hat{V}_\kappa \) does not include constant terms proportional to the unit matrix. As a result we obtain

\[ \hat{V}_\kappa = \left(\mu(\mu + 1) + \kappa^2 - \kappa(2\mu + 1)\sigma_3\right) \frac{1}{x^2} - \frac{\omega}{x}\sigma_1, \]  

(42)
\[ \hat{V}_\kappa = \lambda^2 \left( \mu^2 \exp(-2\lambda x) - (2\kappa - 1)\mu \exp(-\lambda x)\sigma_1 + 2\omega\sigma_3 \right), \tag{43} \]
\[ \hat{V}_\kappa = \lambda^2 \left( (\kappa(\kappa - 1) + \mu^2) \sec^2 \lambda x + 2\omega \tan \lambda x \sigma_1 ight. 
  + \left. \mu(2\kappa - 1) \sec \lambda x \tan \lambda x \sigma_3 \right), \tag{44} \]
\[ \hat{V}_\kappa = \lambda^2 \left( (\kappa(\kappa - 1) + \mu^2) \csch^2(\lambda x) + 2\omega \coth \lambda x \sigma_1 ight. 
  + \left. \mu(1 - 2\kappa) \coth \lambda x \csch \lambda x \sigma_3 \right), \tag{45} \]
\[ \hat{V}_\kappa = \lambda^2 \left( (\mu^2 - \kappa(\kappa - 1)) \sech^2 \lambda x + 2\omega \tanh \lambda x \sigma_3 ight. 
  - \left. \mu(2\kappa - 1) \sech \lambda x \tanh \lambda x \sigma_1 \right). \tag{46} \]

Potentials (42), (43), (44) and (45) and (46) are generated by superpotentials (34), (35), (36), (37) and (38) respectively. The corresponding constants \( c_\kappa \) in (41) are

\[ c_\kappa = \frac{\omega^2}{(2\kappa + 1)^2} \tag{47} \]

for potential (42),

\[ c_\kappa = \lambda^2 \left( \kappa^2 + \frac{\omega^2}{\kappa^2} \right) \tag{48} \]

for potentials (43), (45), (46) and

\[ c_\kappa = \lambda^2 \left( \frac{\omega^2}{\kappa^2} - \kappa^2 \right) \tag{49} \]

for potential (44).

All the above potentials are shape invariant and give rise to exactly solvable problems for systems of two coupled Schrödinger equations, i.e., for systems of Schrödinger-Pauli type.

### 6 Dual shape invariance

To find potentials (42)–(45) we ask for their shape invariance w.r.t. shifts of parameter \( \kappa \). The shape invariance condition together with the supposition concerning the generic form (9) of the corresponding superpotential make it possible to define these potentials up to arbitrary parameters \( \lambda, \omega, \kappa \) and \( \mu \).

Starting with superpotentials (34)–(37) we can find the related potentials (42)–(45) in a unique fashion. But let us consider the inverse problem: to find possible superpotentials corresponding to given potentials which in our case are given by equations (42)–(45). The problems of this kind are very interesting since their solutions can be used to generate families of isospectral hamiltonians. It happens that in the case of matrix superpotentials everything is much more interesting since there exist additional superpotentials compatible with the shape invariance condition.
To find the mentioned additional superpotentials we use the following observation: potentials (42), (44) and (45) are invariant with respect to the simultaneous change

\[ \mu \to \kappa - \frac{1}{2}, \quad \kappa \to \mu + \frac{1}{2}, \]  

(50)

In addition, there exist another transformations of \( \mu \) and \( \kappa \) but they lead to the same results.

Thus in addition to the shape invariance w.r.t. shifts of \( \kappa \) potentials (42), (44) and (45) should be shape invariant w.r.t. shifts of parameter \( \mu \) also. In other words, superpotentials in Section 5, should be considered together with superpotentials which can be obtained from (34), (36) and (37) using the change (50).

Thus, we also can represent potentials (34), (36) and (37) in the following form

\[ \tilde{W}_{\mu,\kappa}^2 - \tilde{W}_{\mu,\kappa}' = \hat{V}_{\mu} + c_{\mu} \]  

(51)

where \( \hat{V}_{\mu} = \hat{V}_{\kappa} \), and

\[ \tilde{W}_{\mu,\kappa} = \frac{\kappa \sigma_3 - \mu - 1}{x} + \frac{\omega}{2(\mu + 1)} \sigma_1, \quad c_{\mu} = \frac{\omega^2}{4(\mu + 1)^2} \]  

(52)

for \( \hat{V}_{k} \) given by equation (42),

\[ \tilde{W}_{\mu,\kappa} = \frac{\lambda}{2} \left( (2\mu + 1) \tan \lambda x + (2\kappa - 1) \sec \lambda x \sigma_3 + \frac{4\omega}{2\mu + 1} \sigma_1 \right) \]  

(53)

for potential (44), and

\[ \tilde{W}_{\mu,\kappa} = \frac{\lambda}{2} \left( -(2\mu + 1) \coth \lambda x + (2\kappa - 1) \csc \lambda x \sigma_3 - \frac{4\omega}{2\mu + 1} \sigma_1 \right) \]  

(54)

for potential (45). The related constant constant \( c_{\mu} \) is:

\[ c_{\mu} = \lambda^2 \left( \pm \frac{1}{4} (2\mu + 1)^2 + \frac{4\omega^2}{(2\mu + 1)^2} \right) \]  

(55)

where the sign “+” and “−” corresponds to the cases (53) and (54) respectively.

We stress that superpartners of potentials (51) constructed using superpotentials \( \tilde{W}_{\mu,\kappa} \), i.e.,

\[ V_{\mu}^+ = \tilde{W}_{\mu,\kappa}^2 + \tilde{W}_{\mu,\kappa}' \]  

(56)

satisfy the shape invariance condition since

\[ V_{\mu}^+ = V_{\mu+1} + C_{\mu} \]
with \( C_\mu = c_{\mu+1} - c_\mu \).

Thus potentials (33), (36) and (37) admit a dual supersymmetry, i.e., they are shape invariant w.r.t. shifts of two parameters, namely, \( \kappa \) and \( \mu \). More exactly, superpartners for potentials (42), (44) and (45) can be obtained either by shifts of \( \kappa \) or by shifts of \( \mu \) while simultaneous shifts are forbidden. We call this phenomena dual shape invariance.

Notice that the remaining potentials (43) and (46) do not possess the dual shape invariance in the sense formulated above. In potential (43) parameter \( \mu \) is not essential. It is supposed to be non-vanishing (since for \( \mu = 0 \) the corresponding superpotential is reducible) and can be normalized to the unity by shifting independent variable \( x \).

The hamiltonian with potential (46) is not invariant w.r.t. change (50). However if we suppose that parameter \( \mu \) be purely imaginary, i.e., set \( \mu = i\tilde{\mu} \) with \( \tilde{\mu} \) real, the corresponding potential admits discrete symmetry (50) for parameters \( \kappa \) and \( \tilde{\mu} \) and thus possesses dual supersymmetry with shape invariance. In this way we obtain a consistent model of "PT-symmetric quantum mechanics [15]" with the dual shape invariance. Discussion of this model lies out the scope of present paper. We only note that for \( \omega = 0 \) the corresponding potential is decoupled to a direct sum of potentials discussed in [16].

7 Exactly solvable problems
for systems of Schrödinger equations

Consider the Schrödinger equations

\[
\hat{H}_\kappa \psi \equiv \left( -\frac{\partial^2}{\partial x^2} + \hat{V}_\kappa \right) \psi = E_\kappa \psi \tag{57}
\]

where \( \hat{H}_\kappa = a_{\kappa,\mu}^+ a_{\kappa,\mu}^- + c_\kappa \) and \( \hat{V}_\kappa \) are matrix potentials represented in (43)–(48). Since all these potentials are shape invariant, equations (57) can be integrated using the standard technique of SSQM. An algorithm for construction of exact solutions of supersymmetric a shape invariant Schrödinger equations includes the following steps (see, e.g., [4]):

- To find the ground state solutions \( \psi_0(\kappa, \mu, x) \) which are proportional to square integrable solutions of the first order equation

\[
a_{\kappa,\mu} \psi_0(\kappa, \mu, x) \equiv \left( \frac{\partial}{\partial x} + \hat{W}_{\kappa,\mu} \right) \psi_0(\kappa, \mu, x) = 0. \tag{58}
\]

In view of (51) function \( \psi_0(\kappa, \mu, x) \) solves equation (57) with

\[
E_\kappa = E_{\kappa,0} = -c_\kappa. \tag{59}
\]
To find a solution $\psi_{1}(\kappa, \mu, x)$ for the first excited state which is defined by the following relation:

$$\psi_{1}(\kappa, \mu, x) = a_{\kappa, \mu}^{+} \psi_{0}(\kappa + 1, \mu, x) \equiv \left( -\frac{\partial}{\partial x} + W_{\kappa, \mu} \right) \psi_{0}(\kappa + 1, \mu, x). \quad (60)$$

Since $a_{\kappa}^{+}$ and $\hat{H}_\kappa$ satisfy the interwining relations

$$\hat{H}_\kappa a_{\kappa, \mu}^{+} = a_{\kappa, \mu}^{+} \hat{H}_{\kappa + 1}$$

function (60) solves equation (57) with $E_{\kappa} = E_{\kappa, 1} = -c_{\kappa+1}$.

Solutions for the second excited state can be found as $\psi_{2}(\kappa, \mu, x) = a_{\kappa, \mu}^{+} \psi_{1}(\kappa + 1, \mu, x)$, etc. Finally, solutions which correspond to $n^{th}$ exited state for any admissible natural number $n > 0$ can be represented as

$$\psi_{n}(\kappa, \mu, x) = a_{\kappa, \mu}^{+} a_{\kappa+1, \mu}^{+} \cdots a_{\kappa+n-1, \mu}^{+} \psi_{0}(\kappa + n, \mu, x). \quad (61)$$

The corresponding eigenvalue $E_{\kappa, n}$ is equal to $-c_{\kappa+n}$.

For systems admitting the eigenvalue $E_{\kappa, n}$ it is necessary to repeat the steps enumerated above using alternative (or additional) superpotentials.

All potentials presented in the previous section generate integrable models with Hamiltonian (57). However, it is desirable to analyze their consistency. In particular, it is necessary to verify that there exist square integrable solutions of equation (58) for the ground state.

In the following sections we prove that such solutions exist for all superpotentials given by equations (34)–(37) and (52)–(54). We will see that to obtain normalizable ground state solutions it is necessary to impose certain conditions on parameters of these superpotentials.

To finish this section we present energy spectra for models (57) with potentials (42)–(45):

$$E = -\frac{\omega^2}{(2N+1)^2} \quad (62)$$

for potential (42),

$$E = -\lambda^2 \left( N^2 + \frac{\omega^2}{N^2} \right) \quad (63)$$

for potentials (43), (46), (45), and

$$E = \lambda^2 \left( N^2 - \frac{\omega^2}{N^2} \right) \quad (64)$$
for potentials (44).

In equations (62)–(62) we omit subindices labeling the energy levels. The spectral parameter $N$ can take the following values

$$N = n + \kappa,$$

and (or)

$$N = n + \mu + \frac{1}{2}$$

(66)

where $n = 0, 1, 2, \ldots$ are natural numbers which can take any values for potentials (42)–(44). For potentials (43), (46) and (45) with a fixed $k < 0$ the admissible values of $n$ are bound by the condition $(k + n)^2 > |\omega|$, see section 9.

For potential (43) the spectral parameter is defined by equation (65). For potentials (42), (44), (46) the form of $N$ depends on relations between parameters $\kappa$ and $\mu$, see section 9.

8 Some special values of parameters and isospectrality

Let us show that for some values of parameters $\mu$ and $\kappa$ potentials (42)–(46) are isospectral with direct sums of known scalar potentials.

Considering potential (42) and using its dual shape invariance it is possible to discover that for half integer $\mu$ $V_\kappa$ can be transformed to a direct sum of scalar Coulomb potentials. Indeed, its superpartner obtained with using superpotential (51) with opposite sign, i.e., $\hat{W}_{\mu,\kappa} = -\tilde{W}_{\mu,\kappa}$ looks as:

$$\hat{V}_{\kappa,\mu}^+ = \hat{W}_{\mu,\kappa}^2 + \hat{W}_{\mu,\kappa}^\prime + c_\mu = (\mu(\mu - 1) + \kappa^2 - \kappa(2\mu - 1)\sigma_3) \frac{1}{x^2} - \frac{\omega}{x}\sigma_1.$$ (67)

Considering $\hat{V}_{\kappa,\mu}^+ = \hat{V}_{\kappa,\mu+1}^+$ as the main potential and calculating its superpartner with using superpotential $\hat{W}_{\mu+1,\kappa}$ we come to equation (67) with $\mu \to \mu - 2$, etc. It is easy to see that continuing this procedure we obtain on some step the following result:

$$\hat{V}_{\kappa,\hat{\mu}} = \frac{l(l + 1)}{x^2} - \frac{\omega}{x}\sigma_1, \quad l = \kappa - \frac{1}{2}$$ (68)

where $\hat{\mu} = \mu + n$, $n = -\mu - \frac{1}{2}$. Diagonalizing matrix $\sigma_1 \to \sigma_3$ we reduce (68) to a direct sum of attractive and repulsive Coulomb potentials written in radial variables. It means that for negative and half integer $\mu$ our potential (42) is isospectral with the Coulomb one.
In analogous way we can show that potentials (44) with half integer \( \kappa \) or integer \( \mu \) is isospectral with the potential

\[
\hat{V}_\kappa = \lambda^2 \left( r(r - 1) \sec^2 \lambda x + 2 \omega \tan \lambda x \sigma_1 \right), \quad r = \frac{1}{2} \pm \mu \text{ or } r = \kappa,
\]

which is equivalent to the direct sum of two trigonometric Rosen-Morse potentials. Under the same conditions for parameters \( \mu \) and \( \kappa \) potential (46) is isospectral with the following potential:

\[
\hat{V}_\kappa = \lambda^2 \left( r(r - 1) \csch^2(\lambda x) + 2 \omega \coth \lambda x \sigma_1 \right)
\]

which is equivalent to the direct sum of two Eckart potentials. Finally, potential (46) is isospectral with

\[
\hat{V}_\kappa = \lambda^2 \left( r(r - 1) \sech^2 \lambda x + 2 \omega \tanh \lambda x \sigma_3 \right), \quad r = \frac{1}{2} \pm \sqrt{\mu^2 + \frac{1}{2}}
\]

provided \( \kappa \) is negative half integer. Potential (71) is equivalent to the direct sum of two hyperbolic Rosen-Morse potentials.

Thus for some special values of parameters \( \mu \) and \( \kappa \) we can establish the isospectrality relations of matrix potentials (42)–(46) with well known scalar potentials. This observation is supported by the direct comparison of spectra (62)–(62) with the spectra of Schrödinger equation with Coulomb, Rosen-Morse and Eckart potentials which can be found, e.g., in [4].

Let us note that setting in (43)–(46) \( \omega = 0 \) we also come to the direct sums of shape invariant potentials, namely, Morse, Sraft and generalized Pöschl-Teller ones. However for nonzero \( \omega = 0 \) and \( \mu, \kappa \) which do not satisfy conditions imposed to obtain (68)–(71) the found potentials cannot be transformed to the mentioned direct sums using the consequent Darboux transformations.

9 Ground state solutions

Let us find the ground state solutions for equations (57) with shape invariant potentials (42)–(45). To do this it is necessary to solve equations (58) where \( W_{\kappa,\mu} \) are superpotentials given in (34)–(37), and analogous equation with superpotentials (52)–(54). The corresponding solutions are square integrable two component functions which we denote as:

\[
\psi_0(\kappa, \mu, x) = \left( \begin{array}{c} \varphi \\ \xi \end{array} \right).
\]

In this section we find the ground state solutions considering consequently all the mentioned potentials.
9.1 Ground states for systems with potentials (42) and (43)

Let us start with the superpotential defined by equation (34). Substituting (34) and (72) into (58) we obtain the following system:

\[
\begin{align*}
\frac{\partial \varphi}{\partial x} + (\mu - \kappa) \frac{\varphi}{x} + \frac{\omega}{2\kappa + 1} \xi &= 0, \\
\frac{\partial \xi}{\partial x} - (\mu + \kappa + 1) \frac{\xi}{x} + \frac{\omega}{2\kappa + 1} \varphi &= 0.
\end{align*}
\]

(73) (74)

Solving (74) for \(\varphi\), substituting the solution into (73) and making the change

\[\xi = y^{\kappa+1} \hat{\xi}(y), \quad y = \frac{\omega x}{2\kappa + 1}\]

we obtain the equation

\[y^2 \frac{\partial^2 \hat{\xi}}{\partial y^2} + y \frac{\partial \hat{\xi}}{\partial y} - (y^2 + \mu^2) \hat{\xi} = 0.
\]

(76)

Its solution is a linear combination of modified Bessel functions:

\[\hat{\xi} = C_1 K_{\mu}(y) + C_2 I_{\mu}(y).
\]

(77)

To obtain a square integrable solution we have to set in (77) \(C_2 = 0\) since \(I_{\mu}(y)\) turns to infinity with \(x \to \infty\). Then substituting (77) into (75) and using (74) we obtain solutions for system (73), (74) in the following form:

\[\varphi = y^{\kappa+1} K_{\mu+1}(y), \quad \xi = y^{\kappa+1} K_{\mu\kappa}(y)
\]

(78)

where \(y\) is the variable defined in (75), \(\omega x/(2\kappa + 1) \geq 0\).

Functions (78) are square integrable provided parameter \(\kappa\) is positive and satisfies the following relation:

\[\kappa - \mu > 0.
\]

(79)

If this condition is violated, i.e.,

\[\kappa - \mu \leq 0
\]

(80)

we cannot find ground state vector using equation (58) with superpotential (34) since its solutions (78) are not square integrable. But since potential (42) admits the dual shape invariance, it is possible to make an alternative factorization of equation (57) using superpotential (52) and search for normalizable solutions of the following equation:

\[\tilde{a}_{\mu,\kappa} \tilde{\psi}_0(\mu, \kappa, x) \tilde{\psi}_0(\mu, \kappa, x) = 0.
\]

(81)
where (and in the following)

\[ \tilde{a}_{\mu,\kappa}^- = \frac{\partial}{\partial x} + \tilde{W}_{\mu,\kappa}, \quad \tilde{a}_{\mu,\kappa}^+ = -\frac{\partial}{\partial x} + \tilde{W}_{\mu,\kappa}. \]  

(82)

Indeed, solving (81) we obtain a perfect ground state vector:

\[ \tilde{\psi}_0(\mu, \kappa, x) = \left( \tilde{\phi} \tilde{\xi} \right), \quad \tilde{\varphi} = y^{\mu + \frac{3}{2}} K_{[\nu]}(y), \quad \tilde{\xi} = y^{\mu + \frac{3}{2}} K_{[\nu-1]}(y) \]  

(83)

where \( y = \frac{\omega x}{2(\mu+1)} \) and \( \nu = \kappa + 1/2 \). The normalizability conditions for solution (83) are:

\[ \kappa - \mu < 1, \quad \text{if} \quad \kappa \geq 0 \]  

(84)

and

\[ \kappa + \mu > 1, \quad \text{if} \quad \kappa < 0. \]  

(85)

It is important to note that conditions (79) and (84) are compatible provided

\[ \kappa > 0, \quad 0 < \kappa - \mu < 1. \]  

(86)

Conditions (79) and (85) are incompatible.

Thus if parameters \( \mu \) and \( \kappa \) satisfy (80) and (84), equation (57) admits ground state solutions (83). If (79) is satisfied but (84) is not true, the ground state solutions are given by relations (78). If condition (86) is satisfied both solutions (78) and (83) are available. In the special case \( \kappa = \mu + 1/2 \) solutions (78) and (83) coincide.

Notice that our convention that parameter \( \mu \) is positive excludes the case \( \mu = -1/2 \) when potential (42) is reduced to a direct sum of Coulomb potentials, see section 8.

Analogously, considering equation (58) with superpotential (35) and representing its solution in the form (72) with

\[ \xi = y^{\frac{1}{2} - \kappa} \hat{\xi}(y), \quad \varphi = y^{\frac{1}{2} - \kappa} \hat{\varphi}(y), \quad y = \mu \exp(-\lambda x) \]

we find the following solutions:

\[ \varphi = y^{\frac{1}{2} - \kappa} K_{[\nu]}(y), \quad \xi = -y^{\frac{1}{2} - \kappa} K_{[\nu-1]}(y) \]  

(87)

where \( \nu = \omega/\kappa + 1/2 \) and parameters \( \omega \) and \( \kappa \) should satisfy the conditions

\[ \kappa < 0, \quad \kappa^2 > \omega. \]  

(88)

Since potential (13) does not admit the dual shape invariance, there are no other ground state solutions.
9.2 Ground states for systems with potentials (44)–(46)

In analogous manner we find solutions of equations (58) and (81) for the remaining superpotentials (35)–(37). Let us present them without calculational details.

Solving equation (58) for superpotential (36) we obtain two normalizable solutions, the first of which is:

\[ \varphi_1 = y^{\frac{\omega - \mu}{2}} (1 - y)^{\frac{\omega + \mu}{2}} {}_2F_1(a, b, c; y), \]

\[ \xi_1 = \frac{2\omega}{\kappa(2\mu - 1)} y^{\frac{\omega + \mu}{2}} (1 - y)^{\frac{\omega + \mu}{2}} {}_2F_1(a + 1, b + 1, c + 1; y). \]

Here \(_2F_1(a, b, c; y)\) is the hypergeometric function,

\[ a = -i\frac{\omega}{\kappa}, \quad b = i\frac{\omega}{\kappa}, \quad c = \frac{1}{2} - \mu \neq 0, \]

\[ y = \frac{1}{2} \sin \lambda x + 1, \quad -\frac{\pi}{2} \leq \lambda x \leq \frac{\pi}{2}, \]

and parameters \(\mu\) and \(\kappa\) are constrained by the conditions (79) and (91).

The second solution is

\[ \varphi_2 = y^{\frac{1 + \omega + \mu}{2}} (1 - y)^{\frac{\omega + \mu}{2}} {}_2F_1(a, b, c; y), \]

\[ \xi_2 = \frac{(2\mu + 1)\kappa}{2\omega} \left( \frac{1}{2} - \frac{1}{y} \right)^{\frac{1}{2}} \varphi_2 \]

\[ - \frac{\kappa^2(2\mu + 1)^2 + 4\omega^2}{2\omega\kappa(2\mu + 3)} y^{\frac{\omega + \mu}{2}} (1 - y)^{\frac{\omega + \mu}{2}} {}_2F_1(a + 1, b + 1, c + 1; y) \]

where variable \(y\) is the same as in (89),

\[ a = \mu + \frac{1}{2} - i\frac{\omega}{\kappa}, \quad b = \mu + \frac{1}{2} + i\frac{\omega}{\kappa}, \quad c = \mu + \frac{3}{2}, \]

and parameters \(\kappa, \mu\) again should satisfy conditions (79) and (91).

Using the dual shape invariance of potential (44) we can find additional (or alternative) ground state solutions using equation (81) with superpotential (53). In this way we obtain

\[ \tilde{\varphi}_1 = y^{\frac{\mu - \omega + 1}{2}} (1 - y)^{\frac{\omega + \mu}{2}} {}_2F_1(a, b, c; y), \]

\[ \tilde{\xi}_1 = \frac{4\omega}{(2\kappa - 3)(2\mu + 1)} y^{\frac{\omega + \mu}{2}} (1 - y)^{\frac{\omega + \mu}{2}} {}_2F_1(a + 1, b + 1, c + 1; y). \]

Here variable \(y\) and its domain are the same as given in (90),

\[ -a = b = \frac{2i\omega}{2\mu + 1}, \quad c = 1 - \kappa, \]

(95)
and parameters \( \mu \) and \( \kappa \) are constrained by the conditions (91) and
\[
\kappa - \mu < 1.
\] (96)
The second solution is
\[
\tilde{\varphi}_2 = \varphi_2, \quad \tilde{\xi}_2 = \xi_2
\] (97)
where \( \varphi_2 \) and \( \xi_2 \) are functions defined by equation (92) where arguments \( a, b \) and \( c \) of the hypergeometric function differs from (93) and have the following form:
\[
a = \kappa - \frac{2i\omega}{2\mu + 1}, \quad b = \kappa - \frac{2i\omega}{2\mu + 1}, \quad c = \kappa + \frac{1}{2},
\]
and parameters \( \kappa, \mu \) should satisfy conditions (91) and (96).

Thus for potential (44) we have three versions of constraints for parameters \( \mu \) and \( \kappa \):
\[
\kappa - \mu \geq 1, \quad \kappa - \mu \leq 0 \tag{98}
\]
or
\[
0 < \kappa - \mu < 1. \tag{100}
\]
In addition, condition (91) should be imposed.

For the cases (98) and (99) the ground state solutions are given by equations (89), (92) and (94), (97) correspondingly while in the case (100) all solutions (89), (92), (94) and (97) are available.

For superpotential (37) we obtain the following solution of equation (58):
\[
\varphi_1 = (1 - y^2)\kappa^{-\kappa}y^{\kappa-\mu} \quad _2F_1 \left( a, b, c, y^2 \right),
\]
\[
\xi_1 = -y\varphi_1 + \frac{a + c}{c}y^{\kappa-\mu+1}(1 - y^2)^{1-\kappa+\frac{2\mu}{\omega}} \quad _2F_1 \left( a + 1, b + 1, c + 1; y^2 \right)
\] (101)
where
\[
a = \frac{\omega}{\kappa}, \quad b = a + c, \quad c = \frac{1}{2} - \mu, \quad y = \tanh \frac{\lambda x}{2}
\] (102)
and parameters \( \kappa, \mu, \omega \) satisfy conditions (79) and (88). These conditions are compatible iff \( \mu < 0 \).

One more solution for superpotential (37) has components given below:
\[
\xi_2 = (1 - y^2)\kappa^{-\kappa}y^{\kappa-\mu+1} \quad _2F_1 \left( a, b, c, y^2 \right),
\]
\[
\varphi_2 = \left( \frac{\kappa(2\mu - 1)(y^2 - 1)}{2\omega y} - y \right) \varphi_2
\]
\[
+ \frac{\kappa ab}{\omega c} \quad _2F_1 \left( a + 1, b + 1, c + 1; y^2 \right)
\] (103)
where \( y \) is the variable given in (102),

\[
a = 1 + \frac{\omega}{\kappa}, \quad b = c - a, \quad c = \frac{3}{2} - \mu.
\] (104)

Solution (103) is normalizable provided conditions (79) and (88) are satisfied.

Potential (45) possesses the dual shape invariance thus we also should find solutions of equation (81) with superpotential (54). The explicit expression of these solutions can be obtained from (101) and (103) using the change (50). To obtain consistent solutions the additional conditions

\[
\mu < 0, \quad (2\mu + 1)^2 > 4\omega > 0
\] (105)

should be imposed instead of (88).

For superpotential (38) we also have two ground state vectors which solve equation (58). The first of them has the following components:

\[
\varphi_1 = y^{-\frac{\omega}{2} + \frac{\kappa}{2}}(1 - y)^{-\frac{\omega}{2} - \frac{\kappa}{2}}F_1(a, b, c; y),
\]

\[
\xi_1 = -\frac{2\mu \kappa}{2\omega + \kappa} y^{\frac{1}{2} - \frac{\omega}{2} + \frac{\kappa}{2}}(1 - y)^{\frac{1}{2} - \frac{\omega}{2} - \frac{\kappa}{2}}F_1(a + 1, b + 1, c + 1; y),
\] (106)

where the parameters and variables are

\[
a = -i\mu, \quad b = i\mu, \quad c = \frac{1}{2} + \frac{\omega}{\kappa} \neq 0, \quad y = \frac{1}{2}(\tanh \lambda x + 1)
\] (107)

with \( \kappa \) and \( \mu \) satisfying (88). The second solution looks as:

\[
\varphi_2 = y^{\frac{1}{2} - \frac{\omega}{2}}(1 - y)^{-\frac{\omega}{2} - \frac{\kappa}{2}}F_1(a, b, c; y),
\]

\[
\xi_2 = \frac{2\omega - \kappa}{2\kappa \mu} \left( \frac{1 - y}{y} \right)^{\frac{1}{2}} \varphi_2
\]

\[
-\frac{(\kappa - 2\omega)^2 + 4\mu^2 \kappa^2}{2\mu \kappa(3\kappa - 2\omega)} y^{1 - \frac{\omega}{2} - \frac{\kappa}{2}}(1 - y)^{\frac{1}{2} - \frac{\omega}{2} - \frac{\kappa}{2}}F_1(a + 1, b + 1, c + 1; y)
\] (108)

where \( y \) is the variable defined in (107),

\[
a = \frac{1}{2} - \frac{\omega}{\kappa} - i\mu, \quad b = \frac{1}{2} - \frac{\omega}{\kappa} + i\mu, \quad c = \frac{3}{2} - \frac{\omega}{\kappa},
\] (109)

and parameters \( \kappa, \omega \) should satisfy condition (88).

Due to the absence of dual shape invariance of potential (46) there are no alternative ground state solutions.

Thus we find ground state solutions for equation (57) with all potentials (42)–(45). These solutions are square integrable and correspond to the eigenvalues \( E_\kappa = -c_\kappa \) or \( E_\mu = -c_\mu \) where \( c_\kappa \) and \( c_\mu \) are given by equations (47)–(49) and (55).
10 Exited states

We already know ground state solutions for Schrödinger equations with potentials given by relations (42)–(45). Solutions which correspond to $n^{\text{th}}$ energy level can be obtained starting, e.g., with the ground state solutions (78), (87), (89), (92), (101), (103), (106), (108) and applying equation (61). However it is necessary to make sure that such defined ground and exited states are square integrable.

Let us first consider potential (42) which admits the dual shape invariance. This invariance enables to make factorization of the corresponding Schrödinger equation and find ground state solutions using either superpotential (34) or (52), or even both of them, depending on given initial values of parameters $\kappa$ and $\mu$. Namely if (79) is satisfied but (84) is not true, the ground state solutions are given by equations (72) and (78) and exited states are given by equation (61).

The corresponding energy levels are given by formulae (62) and (66).

If parameters $\mu$ and $\kappa$ satisfy (80) and one of conditions (84) or (84), equation (57) admits ground state solutions (83) and exited states are defined by the following equation:

$$\tilde{\psi}_n(\kappa, \mu, x) = \tilde{a}_0 + \tilde{a}_{\kappa, \mu} \tilde{a}_{\kappa+1, \mu} \cdots \tilde{a}_{\kappa+n-1, \mu} \tilde{\psi}_0(\kappa + n, \mu, x)$$

(110)

where $\tilde{a}_{\kappa, \mu} = -\frac{\partial}{\partial y} + \tilde{W}_{\mu, \kappa}$ and $\tilde{W}_{\mu, \kappa}$ is the alternative superpotential given by (52). The corresponding energy levels are given by formulae (62) and (66).

If condition (86) is satisfied both versions of solutions and energy levels given above are available. In the special case $\kappa = \mu + 1/2$ solutions (78) and (83) coincide.

Let us start with the ground state solution (78). Its normalizability is almost evident since the modified Bessel function has the only singular point, namely, $y = 0$, and decreases exponentially at infinity. Moreover, at $y = 0$ there is the inverse power singularity, i.e., $K_{\nu} \sim \frac{1}{y^\nu}$ with $\nu = \mu$ or $\nu = \mu + 1$ which is perfectly compensated by the multiplier $y^{\kappa+1}$ provided $\kappa$ satisfies (79).

Calculating the first exited state (60) we can use relation (58) where $\kappa \rightarrow \kappa + 1$, thus

$$\psi_1(\kappa, \mu, y) = (W_{\kappa+1, \mu} + W_{\kappa, \mu}) \psi_0(\kappa + 1, \mu, y).$$

(111)

Again we recognize a good behavior at the singularity point $y = 0$ since $(W_{\kappa+1, \mu} + W_{\kappa, \mu}) \sim \frac{1}{y} + \cdots$ and $\psi_0(\kappa + 1, \mu, y) \sim y \psi_0(\kappa, \mu, y)$.

Let us suppose that the wave function corresponding to $n^{\text{th}}$ exited state is regular at $y = 0$ and has the following form

$$\psi_n(\kappa, \mu, y) = W \psi_0(\kappa + n, \mu, y)$$

(112)

where $\psi_0(\kappa + n, \mu, y)$ is the ground state solution given by relations (72), (78) and $W$.
is a matrix depending on $y$ and $k$. Then

$$
\psi_{n+1}(\kappa, \mu, y) = \left(-\frac{\partial}{\partial y} + W_\kappa\right) \psi_n(\kappa + n + 1, \mu, y)
= \left(-\frac{\partial W}{\partial y} + W_\kappa W + WW_{\kappa+n+1}\right) \psi_0(\kappa + n + 1, \mu, y)
$$

(113)

where we use the fact that $\psi_0(\kappa + n + 1, \mu, y)$ solves the equation (58) where $\kappa \to \kappa + n + 1$.

Using (113) it is not difficult to show that if function $\psi_n(\kappa + 1, \mu, y)$ be square integrable then $\psi_{n+1}(\kappa, \mu, y)$ is square integrable too. Indeed, at the neighborhood of the singularity point $y = 0$ the ground state functions are related as $\psi_0(\kappa + n + 1, \mu, y) \sim y \psi_0(\kappa + n, \mu, y)$, and $\left(-\frac{\partial W}{\partial y} + W_\kappa W + WW_{\kappa+n+1}\right) \sim \frac{1}{y} W$. Thus $\psi_{n+1}(\kappa, \mu, y)$ is regular at $y = 0$ provided $\psi_n(\kappa, \mu, y)$ be regular. Since for $n = 0, 1$ our supposition is fulfilled we conclude by induction that wave functions $\psi_n(\kappa, \mu, y)$ (61) are normalizable for any $n$.

By direct repeating the above speculations we can prove the square integrability of ground state vector (83) and excited state vectors given by relation (110). In fact the only thing we need is to change $\psi_n(\kappa, \mu, y)$ and $W_{\kappa,\mu}$ by their counterparts $\tilde{\psi}_n(\kappa, \mu, y)$ and $\tilde{W}_{\mu,\kappa}$.

In complete analogy with the above one can prove the normalizability of the exited states for the case when the superpotential is given by equation (35). However there is an essentially new point which is generated by condition (88). The think is that solutions (87) being well defined for $\kappa$ and $\mu$ satisfying conditions (79) and (88), can loose their square integrability after the change $\kappa \to \kappa + n$ for a sufficiently large $n$. Namely, in order to obtain a normalizable solution $\psi_0(\kappa + n, \mu, x)$ we have to ask for $(\kappa + n)^2 \geq \omega > 0$. Since $\kappa$ is negative we have the following restriction for $n$:

$$
n < |\kappa| - \sqrt{\omega}.
$$

(114)

It is possible to show that if $\psi_0(\kappa + n, \mu, x)$ is not normalizable the same is true for exited states (61).

Let us consider the ground state solutions (89). This solution like all the remaining solutions (92), (94), (97), (101), (103), (106), (108) is expressed via linear combinations of the following elements:

$$
y^A(1 - y)^B 2F_1(a, b, c; y)
$$

(115)

where parameters $a, b$ and $c$ are given by equation (91), $A = \frac{\kappa - \mu}{2}, B = \frac{\kappa + \mu}{2}$ for component $\varphi_1$, etc.

In accordance with its definition, variable $y$ belongs to the interval $[0, 1]$ and there are two points which are "suspicious w.r.t. singularity", namely, $y = 0$ and $y = 1$. In order the solution to be regular (and equal to zero) in these points it is necessary
and sufficient to ask for $A > 0, B > 0$ and $\Re (B + c - a - b) > 0$. Exactly these
conditions generate restrictions (88) for parameters $\kappa$ which guarantee the solution
normalizability. The same is true for solutions (92), (106), (108), (101) and (103).

To analyze solutions for exited states we rewrite superpotential (36) in terms of
variable $y$:

$$W_{\kappa, \mu} = \lambda \left( \kappa (2y - 1) + \frac{\mu}{2} \sqrt{y(1-y)}\sigma_1 + \frac{\omega}{\kappa} \sigma_3 \right).$$

(116)

We see that $W_{\kappa}$ is nonsingular at $y = 0$ and $y = 1$, the same is true for $W_{\kappa+n, \mu}$ for
any natural number $n$. Functions (89) are still regular at these points if we change
$\kappa \to \kappa + n$, thus we can again apply relations (111)–(113) to prove the normalizability
of the corresponding solutions (61) for arbitrary $n$.

In a similar way we can prove the square integrability for solutions (61) corresponding to ground state solutions (92), (94), (97), (101), (103), (106) and (108). However in these cases we again have constraint (88) which generates restriction (114)
for the number $n$ enumerating the exited states (61). Analogously, starting with condition (105) we come to conclusion that solutions (110) for the Schrödinger equation
with potential (45) are square integrable provided quantum number $n$ satisfies the condition

$$n < |\mu| - \sqrt{\omega} - \frac{1}{2}.$$  

(117)

Thus for any fixed $\kappa, \mu$ and $\omega$ equation (57) with potentials (43) (45) and (46)
describes a system which has a finite number of states with discrete spectrum. These
states are enumerated by non-negative natural numbers $n$ satisfying condition (114)
or (117).

The systems with potentials (42)–(46) can also have states with continuous spectrum.
In particular, such states should change the bound states when conditions (114) and (117) are violated. Analysis of the states with continuous spectra lies out
of frames of the present paper.

11 Discussion

Generalizing the supersymmetric PS problem we find a family of matrix potentials
for Schrödinger equation satisfying the shape invariance condition. In this way we
find five exactly solvable problems for systems of coupled Schrödinger equations. The
related matrix potentials are given by equations (42)–(46).

Let us stress that we present the completed classification of shape invariant superpotentials of the generic form (9) where $P$ and $R$ are hermitian matrices of arbitrary
finite dimension and $Q$ is proportional to the unit matrix. Namely, we show that such
objects can be reduced to direct sums of known scalar superpotentials and superpotentials presented in section 5.
The found potentials include parameters $\lambda, \kappa, \mu$ and $\omega$ whose possible values are restricted but quite arbitrary. Moreover, parameters $\omega$ in (42) and $\mu$ in (43) can be reduced to unity by scaling and shifting the independent variable $x$ correspondingly.

Taking into account all possibilities enumerated in (79), (84), (85) and (98)–(100) we conclude that in the case of potentials (42), (44) and (45) there are discrete spectrum states for all real values of arbitrary parameters $\lambda, \kappa$ and $\mu$ except the case $\kappa = \mu$. Parameter $\omega$ can be constrained by equations (88) or (105).

Potential (42) is a slightly generalized effective potential for the PS problem. Moreover, these potentials coincide for a particular value $\mu = 0$ of arbitrary parameter $\mu$. However, if $\mu \neq 0$ potential (42) is not equivalent to the potential appearing in the PS problem and corresponds to a more general interaction in the initial three-dimensional problem.

At the best of our knowledge the remaining potentials (43)–(46) are new. The related Schrödinger equations can be integrated using tools of the SUSY quantum mechanics. The corresponding spectrum and eigenvectors are given by equations (62)–(66) and (61) or (110) while the ground state solutions are discussed in section 9. Notice that the “matrix supersymmetry” has a new feature in comparison with the standard (i.e., scalar) one. Namely, matrix models with shape invariance can have degenerated ground states in spite of that there exists a normalizable solution for equation (58). Example of models with such specific spontaneously broken SUSY is given by the Schrödinger equation (57) with potentials (44)–(46).

Mathematically, there are natural reasons for appearance of a zero energy doublet of the ground states in systems with the matrix supersymmetry. The thing is that equation (58) is a system of two the first order equations whose solutions are linear combinations of two functions while in the ordinary SUSY quantum mechanics we have a one first order equation for ground states. For potentials (42) and (43) only one of these functions is normalizable but for potentials (44)–(46) there are two ground state solutions.

Let us note that existence of zero energy doublets of the ground states was already registered in periodic quantum systems, see [18] and [19] for discussion of this phenomenon. In this connection it seems to be interesting to extend our approach to the case of periodic systems. Formally speaking, the only new constructive elements of potentials (42)–(46) in comparison with the standard scalar shape invariant potentials are matrices $\sigma_1$ and $\sigma_3$ which are involutions anticommuting between themselves. In fact the nature of these involutions is not essential for deducing the shape invariant potentials, and many of the results discussed in present paper can be generalized to the case of another involutions. For example, it is possible to change the mentioned matrices by reflection and shift operators which also can be anticommuting involutions being applied to functions with an appropriate parity and periodicity. In this way it seems to be possible to extend the list of potentials which admit supersymmetry including shifts of arguments [20]. A classification of anticommuting discrete symmetries and the corresponding supersymmetric versions of the Schrödinger and
Pauli equations can be found in [22].

An interesting phenomena which appears to be typical for systems with matrix SUSY is the dual shape invariance discussed in Section 6. It enables to impose much less restrictive constrains on parameters of potentials then the ordinary shape invariance. In addition, it can be used to explain the insensitivity of the spectra (62)–(62), (65) on parameter $\mu$. Namely, hamiltonians with shifted $\mu$ should be almost isospectral thanks to the dual shape invariance, which is incompatible with $\mu$-dependence of energy values (excluding the exotic case when these values are periodic functions of $\mu$).

For some values of parameters $\mu$ and $\kappa$ the additional branch of spectrum caused by the dual shape invariance can appear. In particular it is true for potential (12) with $\mu = 0$ and $0 < \kappa < 1/2$. Enhanced analysis of such potentials was made in paper [12]. In the present paper we slightly refine results of [12].

Let us note that the dual shape invariance can be recognized for two potentials of the ordinary SUSY quantum mechanics, namely, for the trigonometric Scraft 1 and generalized Pöschl-Teller potentials:

\[
V_1 = (\kappa(\kappa - 1) + \mu^2) \sec^2 x + \mu(1 - 2\kappa) \sec \lambda x \tan \lambda x,
\]

\[
V_2 = (\kappa(\kappa - 1) + \mu^2) \csc h^2(x) + \mu(1 - 2\kappa) \coth \lambda x \csc h x
\]

both of which admit symmetries (50). The corresponding energy spectra is $\mu$-independent also, and the dual shape invariance can be used to explain this phenomena.

It is shown in section 8 that for some values of parameters $\mu$ and $\kappa$ the matrix potentials (12)–(16) are isospectral with direct sums of one dimensional shape invariant potentials. Unfortunately, in this way we cannot establish isospectrality with the reflectionless hyperbolic Poschl-Teller (HPT) system which has a lot of interesting applications and admits a hidden (bosonized) nonlinear supersymmetry [21]. A natural question arises if there are other matrix superpotentials which can add the list given by equations (34)–(37) and may include a matrix counterpart of the reflectionless HPT potential?

In sections 3–5 we show that irreducible matrix potentials of generic form (9) are exhausted by known scalar ones and $2 \times 2$ matrix operators given by equations (34)–(37). Of course we restrict ourselves to the shape invariance of type 1 when variable parameter $\kappa$ is changing by shifts.

Nevertheless it is possible to search for matrix superpotentials in a more general approach, when matrix $Q$ in (9) is not restricted to be proportional to the unit one. Let us present an example of such superpotential:

\[
W_{\kappa,\mu} = \left( \kappa + \frac{1}{2} \right) \frac{x - c\sigma_3}{c^2 - x^2} + \frac{\omega}{(2\kappa + 1)} \sigma_1
\]

where $c$ is a a constant. The corresponding potential $V_{\kappa}$ (3) is shape invariant and has the following form:

\[
V_{\kappa} = (4\kappa^2 - 1) \frac{x^2 + c^2 - 2cx\sigma_3}{4(x^2 - c^2)^2} + \frac{\omega x}{c^2 - x^2} \sigma_1
\]

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while the energy spectrum is given by equation (62). The other examples (which, however, are restricted to a linear dependence of $W_{\kappa,\mu}$ on variable parameter $\kappa$) can be found in paper [11].

The problem of classification of matrix potentials (9) with generic hermitian matrices $Q, P$ and $R$ is a subject of our contemporary research.

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