ERGODIC AUTOMORPHISMS WITH SIMPLE SPECTRUM 
CHARACTERIZED BY FAST CORRELATION DECAY

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Abstract. The existence of measure preserving invertible transformations $T$ on a Borel probability space $(X, \mathcal{B}, \mu)$ with simple spectrum is established possessing the following rate of correlation decay for a dense family of functions $f \in L^2(X, \mu)$:

$$\forall \varepsilon > 0 \quad \langle f(T^k x), f(x) \rangle = O(|k|^{-1/2 + \varepsilon}).$$

According to identity $\langle f(T^k x), f(x) \rangle = \hat{\sigma}(k)$, where $\sigma_f$ denotes the spectral measure associated with $f$, the rate of decay of the Fourier coefficients $\hat{\sigma}_f(k)$, observed for the class of transformations introduced in the paper, is the maximal possible for singular Borel measures on $[0,1]$.

This note summarizes the results of the paper arXiv:1008.4301.

Let us consider an invertible measure preserving transformation $T$ on a Borel probability space $(X, \mathcal{B}, \mu)$ and recall a question which is well-known in the spectral theory of ergodic dynamical systems and goes back to Banach: Does there exist a transformation with invariant probability measure having Lebesgue spectrum of multiplicity one?

Ulam [1] (ch. VI, § 6) states this problem in the following way.

Does there exist a function $f \in L^2(X, \mu)$ and a measure preserving invertible transformation $T: X \to X$ (an automorphism), such that the sequence of functions $\{f(T^k x): k \in \mathbb{Z}\}$ is a complete orthogonal system in the Hilbert space $L^2(X, \mathcal{B}, \mu)$?

One can easily construct an example of dynamical system of such kind on the space with infinite measure. Let us consider the set of integers $\mathbb{Z}$ as a phase space $X$ with the standard counting measure $\nu(\{j\}) \equiv 1$, and let us define $T: j \mapsto j + 1$. Then the functions $T^k \delta_0$ constitute a basis in $L^2(\mathbb{Z}, \nu)$, where $\delta_0(j) = 1$ if $j = 0$ and $\delta_0(j) = 0$ if $j \neq 0$.

In the class of finite measure preserving transformations the hypothesis of Banach is still open. Kirillov [2] states a generalized hypothesis for general Abelian group actions on a space with a finite probability measure.

We concentrate on the case $G = \mathbb{Z}$. For a survey of constructions and results in the spectral theory of ergodic dynamical systems the reader can refer to [3] and [4]. The dual group $\hat{\mathbb{Z}}$ is isomorphic to the unit circle $S^1$ in the complex plane and the Fourier coefficients of any Borel probability measure $\sigma$ on $S^1$ are recovered from the identity

$$\hat{\sigma}(k) = \int_{S^1} z^k d\sigma, \quad k \in \mathbb{Z}.$$
Definition 1. Let $\kappa(\sigma)$ denote the liminf of the values $\alpha \in \mathbb{R}$ satisfying the estimate 
$\hat{\sigma}(k) = O(|k|^{\alpha+\varepsilon})$ for any $\varepsilon > 0$.

Since the group $S^1$ is compact then any $\sigma$ satisfying $\hat{\sigma}(k) = O(|k|^{-1/2-c})$ for some $c > 0$ is absolutely continuous with respect to the normalized Lebesgue measure $\lambda$ on $S^1$. In particular, $\sigma = p(x)\lambda$, where $p(x) \in L^1(S^1, \lambda)$. Thus, any measure on $[0,1]$ with $\kappa(\sigma) < -1/2$ is absolutely continuous.

Given a function $f \in L^2(X, \mathcal{B}, \mu)$ consider a sequence of auto-correlations

$$R_f(k) = (f(T^k x), f(x)),$$

and the spectral measure $\sigma_f$ associated with $f$ and given by $\hat{\sigma}_f(k) = R_f(k)$. Whenever $\kappa(\sigma_{f_j}) < -1/2$ holds for a dense family $\{f_j\} \subseteq L^2(X) = L^2(X, \mathcal{B}, \mu)$, the spectrum of $T$ is absolutely continuous. We will prove that an extreme value $\kappa(\sigma_f) = -1/2$ (on a dense set of functions) is achieved in the class of ergodic transformations with simple spectrum.

Theorem 1. There exists an automorphism $T$ on a Borel space $(X, \mathcal{B}, \mu)$ with simple spectrum such that $\kappa(\sigma_f) \leq -1/2$ for a dense set of functions $f \in L^2(X)$.

Let us define

$$\kappa(T) = \inf_{\mathcal{F} \text{ dense in } L^2(X)} \sup f \in \mathcal{F} \kappa(\sigma_f).$$

Thus, theorem 1 states the existence of $T$ with simple spectrums and $\kappa(T) \leq -1/2$. Observe that $\kappa(T) = -\infty$ for any $T$ with Lebesgue spectrum.

Theorem 2. Let $T$ be an automorphism satisfying the equality $\kappa(T) = -1/2$, and let $\sigma$ be the maximal spectral type of $T$. Then $\sigma \ast \sigma \ll \lambda$, and, furthermore, the spectrum of $T$ either contains an absolutely continuous component or is purely singular; and for any spectral measure $\sigma_f$ we have $\kappa(\sigma_f) = -1/2$.

Throughout this paper we call singular Borel measures satisfying $\kappa(\sigma_f) = -1/2$ Salem–Schaeffer measures. This class of probability distributions were studied in the works of Schaeffer, Salem, Sigmund, Ivashev-Musatov et al. (see [5, 6, 7]).

The main idea of this work is to show that Salem–Schaeffer measures are found among spectral measures of ergodic dynamical systems. We propose a construction of a class of automorphisms that will serve an example of such kind.

Definition 2. (Symbolic construction) Let $\mathcal{A}$ be a finite alphabet and let $w_0$ be a finite word in $\mathcal{A}$ containing at least two different letters. Denote by $\rho_{\alpha}(w)$ the cyclic shift of $w$ to the left:

$$\rho_1(au) = ua, \quad \rho_\alpha(u) = \rho_{\alpha+1}(u), \quad a \in \mathcal{A} \text{ — a letter, } u \text{ — a word.}$$

Let us construct the sequence of words $w_n$ applying the next rule:

$$w_{n+1} = \rho_{\alpha_n,0}(w_n) \rho_{\alpha_{n-1},1}(w_n) \ldots \rho_{\alpha_{n-2},0}(w_n).$$

Here the sequences $q_n \in \mathbb{N}$ and $\alpha_{n,j} \in \mathbb{Z}/h_n \mathbb{Z}$, $h_n = |w_n|$ serve as parameters of the construction, and $|w|$ denotes the length of the word $w$. Without loss of generality, one can assume the first entry of $w_0$ inside the bigger word $w_{n+1}$ is not touched, $\alpha_{n,0} \equiv 0$. Then every $w_n$ is a prefix of the successor word $w_{n+1}$, and we can define a unique infinite word $w_\infty$ expanding every word $w_n$ to the right.

Further, applying a standard procedure let us define the minimal compact subset $K \subset \mathbb{R}^2$ containing all the shifts of the word $w_\infty$. The left shift transformation $T: (x_j) \mapsto (x_{j+1})$ provides a topological dynamical system acting on the set $K$. 

Futher, let us endow the set $K$ with a natural Borel measure $\mu$ invariant under $T$. We define the probability $\mu(u)$ of the word $u$ to be the asymptotic frequency of observing $u$ as subword in $w_\infty$.

The construction and ergodic properties of the ergodic system $(T, K, \mathcal{B}, \mu)$ are discussed in details in [8]. The complexity characteristics of the topological system $(T, K, \mathcal{B}, \mu)$ are studied in [9]. An infinite sequence of concatenated cyclic shifts of a fixed word was previously studied in the theory of recursive functions [10]. Setting $h_1 = 2$, $q_n = 2$, $\rho_{n,0} = 0$, $\rho_{n,1} = h_n/2$, we see that the classical *Morse automorphism* (see [11]) is included in the class defined above. It can be also shown that the constructed systems possess adic representation.

**Definition 3. (Algebraic construction)** Let $h_n$ be a sequence of positive integers such that $h_{n+1} = q_n h_n$, $q_n \in \mathbb{N}$, $q_n \neq 1$. Consider then a sequence of embedded lattices $\Gamma_n = h_n \mathbb{Z}$, where $\Gamma_{n+1} \subset \Gamma_n$, and the corresponding homogeneous spaces $M_n = \mathbb{Z}/\Gamma_n = \mathbb{Z}_{h_n}$.

Let us fix projections $\phi_n : M_{n+1} \to M_n$ defined by

$$\phi_n : jh_n + k \mapsto k + \alpha_{n,j} \pmod{h_n}, \quad 0 \leq k < h_n, \quad j = 0, 1, \ldots, q_n.$$ 

Evidently, $\phi_n$ preserve normalized Haar measures $\mu_n$ on the Borel spaces $M_n$. Define the phase space $X$ as inverse limit of spaces $(M_n, \mathcal{B}_n, \mu_n)$, namely, set

$$X = \{ x = (x_1, x_2, \ldots, x_n, \ldots) : \phi_n(x_{n+1}) = x_n \}.$$ 

The measures $\mu_n$ become Borel measures $\mu$ on $X$. Let us define the transformation $T$ on the space $(X, \mathcal{B}, \mu)$ as follows. Any projection $\phi_n$ almost commutes with the shift transformation on $M_n$,

$$\mu \{ x : \phi_n(S_{n+1}(x_{n+1})) \neq S_n(\phi_n(x_{n+1})) \} \leq h_n^{-1},$$

hence, applying Borrel–Cantelli lemma, we see that $\mu$-almost surely the equality $\phi_n(S_{n+1}(x_{n+1})) = S_n(\phi_n(x_{n+1}))$ holds for $n \geq n^*(x)$, where $n^*(x)$ is a measurable function. Set

$$(Tx)_n = x_n + 1, \quad n \geq n^*(x), \quad \text{and} \quad (Tx)_n = \phi_n(Tx_{n+1}), \quad n < n^*(x).$$

**Lemma 1** (see [8]). *The map $T$ is a measure preserving invertible transformation of the probability space $(X, \mathcal{B}, \mu)$.***

The equivalence of the two constructions introduced above is verified via coding $T$-orbits by words $w_n$ induced by functions $M_{n_0} \to \mathcal{A}$ for some $n_0$.

In order to prove theorem 3, we consider a certain stochastic family of dynamical systems $(T, X, \mathcal{B}, \mu)$ constructed above, depending on random parameters. Then we show that $T$ has simple spectrum and the required rate of correlation decay almost surely with respect to the probability on the set of parameters.

**Theorem 3.** *There exists a sequence $q_n \in \mathbb{N}$ such that the transformation $T$ defined above with $\alpha_{n,j}$ independent and uniformly distributed on $M_n$ has simple spectrum and satisfy the inequality $\kappa(T) \leq -1/2$.***

**Proof.** The detailed proof of the simplicity of spectrum is given in [8]. It is based on the following lemma.
Lemma 2 (see [12]). Let $U$ be a unitary operator in a separable Hilbert space $H$, let $\sigma$ be the measure of maximal spectral type and let $M(z)$ denote the multiplicity function of the operator $U$. If $M(z) \geq m$ on a set of positive $\sigma$-measure then there exist $m$ orthogonal elements of unit length $f_1, \ldots, f_m$ such that for any cyclic space $Z \subset H$ (with respect to $U$) and for any $m$ elements $g_1, \ldots, g_m \in Z$ of the same length $\|g_i\| = a$ the inequality inequality holds

$$\sum_{i=1}^{m} \|f_i - g_i\|^2 \geq m(1 + a^2 - 2a/\sqrt{m}).$$

In order to prove the second statement of the theorem it is enough to estimate the decay of correlations for $\mathcal{B}_{n_0}$-measurable (cylinder) functions $f(x)$ which are dense in $L^2(X)$. Any such function $f(x)$ can be represented in the form $f(x) = f_{n_0}(x_{n_0})$, where $f_{n_0}: M_{n_0} \to \mathbb{C}$ and $x_n$ is the $n$-th coordinate of a point $x$. Then for any $n > n_0$

$$f(x) = f_n(x_n), \quad \text{where} \quad f_{n+1}(x_{n+1}) = f_n(\phi_n(x_{n+1})).$$

Given a function $f(z)$ with zero mean define the cyclic correlations

$$R_n^o(t) = \int f_n(j + t) \overline{f_n(j)} \, d\mu_n(j),$$

where $j + t$ is the sum in the group $M_n$, i.e. $j + t \pmod h_n$. Taking into account the conditions on the distribution of the random parameters, $\alpha_{n,j}$, it can be easily shown that $\mu$-a.s. $R_n^o(t) \to R_f(t)$ for any $t$, hence, the distributions $\hat{R}^o_n$ converges weakly to the spectral measure $\sigma_f$ (but we need only the first convergence). Now let us consider (the most important) case, when $t = sh_n$, $s \neq 0$. For this special value of the argument $t$ the following recurrent identity holds

$$R_{n+1}^o(t) = \frac{1}{q_n} \sum_{k=0}^{q_n-1} R_n^o(\alpha_{n,k+s} - \alpha_{n,k}).$$

Applying expectation operator with respect to the probability on the parameters’ space, we obtain $E[R_{n+1}^o(t)] = E[R_n^o(\alpha_{n,k+s} - \alpha_{n,k}) = 0$ and

$$E|R_{n+1}^o(t)|^2 = \frac{1}{q_n^2} \sum_{k=0}^{q_n-1} R_n^o(\alpha_{n,k+s} - \alpha_{n,k}) R_n^o(\alpha_{n,k+s} - \alpha_{n,k}).$$

Observe that all the terms in the above sum are zero except the terms such that \{k, k+s\} = \{\ell, \ell+s\}. If $h_n$ is odd these are always the terms with $k = \ell$, and if $n$ is even we should also count the terms given by $k + s = \ell$, and $\ell + s = k$. Clearly, the latter contributes $O(q_n^{-1})$ to the sum over all $s$, so without loss of generality we can assume that all $h_n$ are odd. We have then

$$E|R_{n+1}^o(t)|^2 = \frac{1}{q_n} E|R_n^o(\alpha_{n,k+s} - \alpha_{n,k})|^2,$$

hence, $E|R_{n+1}^o(t)|^2 = h_n^{-1} E\|R_n^o\|^2$, where $\| \cdot \|$ is a standard form in $L^2(Z)$, and the function $R_n^o$ is restricted to $[0, h_n - 1]$. The same arguments slightly modified lead the equality $E|R_{n+1}^o(t)|^2 = h_n^{-1} E\|R_n^o\|^2$ for any $t \in (h_n, h_n + 1)$. Thus, one can see that

$$E\|R_{n+1}^o\|^2 \leq 2 \cdot E\|R_n^o\|^2.$$

It follows that $R_{n+1}^o(t) = O(|t|^{-1/2+\varepsilon})$ for any $\varepsilon > 0$ the second statement of the theorem is verified. □
Definition 4. We say that an automorphism $T$ of a Borel probability space $(X, \mathcal{B}, \mu)$ admits \textit{approximation of type I} if for any finite partition $\mathcal{P}$ and any $\varepsilon > 0$ there exists a subset $\Omega_\varepsilon \subset X$ of the measure $1 - \varepsilon$ and a word $W_\varepsilon$ such that for all $x \in \Omega_\varepsilon$ the infinite word generated by $\mathcal{P}$-coding of the $x$-orbit is $\varepsilon$-covered by a sequence of words $\tilde{W}_j$ which are $d$-$\varepsilon$-close to cyclic shifts $\rho_{\alpha_j}(W_\varepsilon)$ of the word $W_\varepsilon$.

This property is a metric invariant. In particular, the class of maps satisfying type I approximation includes rank one transformations. Clearly, the main construction of this paper generates transformations of type I.

Hypothesis 1. Let $T$ be an automorphism constructed according to definition \cite{2} (or definition \cite{3}). Consider an arbitrary $B_n$-measurable function $f$ with zero mean. Then $\kappa(\sigma_f) \geq -1/2$.

Hypothesis 2. Assume that an automorphism $T$ admits approximation of type I, and suppose that $\xi_n$ are finite partitions generating, for any fixed finite partition $\mathcal{P}$, approximating sequence of words $W_{\varepsilon_n}$ with $\varepsilon_n \to 0$. Then

$$\liminf_{k \to \infty} \inf_{f \text{ is } \xi_n\text{-measurable}} \kappa(\sigma_f) \geq -1/2$$

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