Abstract In this paper, we identify many important properties and develop criteria for the existence of subquasigroups in finite quasigroups. Based on these results, we propose an effective method that concludes the nonexistence of subquasigroup of a finite quasigroup, otherwise finds its all possible proper subquasigroups. This has an important application in checking the cryptographic suitability of a quasigroup.

Further, we propose a binary operation using arithmetic of finite fields to construct quasigroups of order $p^r$. We develop the criteria under which these quasigroups have desirable cryptographic properties, viz. polynomially completeness and possessing no proper subquasigroups. Then a practical method is given to construct cryptographically suitable quasigroups. We also illustrate these methods by some academic examples and implement all proposed algorithms in the computer algebra system SINGULAR.

Keywords Quasigroups · Subquasigroups · Polynomial Completeness · Cryptography
1 Introduction

Identifying some algebraic properties in quasigroups, such as polynomially completeness, having no proper subquasigroups, is vital for analyzing and designing quasigroup-based crypto primitives and schemes. Testing these properties has a two-fold advantage; on the one hand, we can find cryptographically suitable quasigroup for the design. On the other hand, the robustness of already existing quasigroup based crypto schemes can be checked.

Quasigroup based cryptography is an emerging research area for efficient secure communication. It has an advantage in resource constraint environments such as smart cards, RFID systems etc. The nonassociative and noncommutative properties of the quasigroup structure make them useful in the crypto designs, because many well-known cryptanalysis techniques to attack these crypto designs would not be directly applicable. Quasigroup is a well-studied algebraic combinatoric structure, see in Belyavskaya (1989); Belyavskaya and Tabarov (1992); Keedwell and Dénès (2013); Dénès and Keedwell (1991); Kepka (1978); Kepka and Nemec (1971). However, its intensive use by the crypto community can be traced back only in the last two decades, see Belyavskaya (1994); Shcherbacov (2017); Smith (2007). The significant properties of quasigroups for the crypto designs are polynomial completeness (simplicity, non-affineness), degree of non-commutativity and non-associativity, non-existence of proper subquasigroup, see Artamonov (2012); Artamonov et al. (2013, 2017); Galatenko et al. (2018); Grošek and Horák (2013).

Quasigroup string transformations are one of the fundamental building blocks in the design of quasigroup based crypto algorithms and the security of these transformations mainly depends on the proper choice of quasigroups Artamonov et al. (2016); Artamonov (2020); Markovski et al. (1997, 1999). In particular, the choice of quasigroup affects the rate of growth of the period and randomness of the output of quasigroup string transformations Dimitrova and Markovski (2004). A significant period and randomness provide good security to the crypto algorithms. Furthermore, these properties ensure that the search space for the brute-force attack can not be reduced; and the problem of solving an equivalent algebraic system of such quasigroups based crypto algorithm is NP-Complete Horváth et al. (2008).

Developing efficient methods to check and construct cryptographically suitable quasigroups of finite order is ongoing research. In this paper, we work on two important problems related to the quasigroup-based cryptography. At first, we develop methods to conclude the non-existence of proper subquasigroup and then propose a class of cryptographically suitable quasigroups of finite order $p^r$. In Section 2 we revise the main definitions and properties of quasigroups used in this paper. Then, in Section 3 we prove some important properties of subquasigroups. Based on these results, we propose an effective algorithm to determine all the possible proper subquasigroups of any given finite quasigroup, otherwise deduce the non-existence of any proper subquasigroup (the desirable property for crypto schemes). In section 4 we propose a method for the construction of cryptographically suitable finite quasigroup of order $p^r$. At first, we introduce a quasigroup binary operation based on the arithmetic of finite fields and then develop the criteria under which these quasigroups are polynomially complete and possess no subquasigroups. We then finalize all these
findings in an effective algorithm to construct quasigroups of given finite orders for any prime \( p \) and integer \( r \). In particular, the algorithm constructs the cryptographically desirable quasigroups of order \( 2^r \). Finally, the given methods are elaborated by their implementation in Singular [Decker et al. 2019] on some examples.

2 Preliminaries

In this section, we introduce some of the terminology used in this paper. A quasigroup is a set \( Q \) with a binary operation of multiplication \( x \cdot y \) such that for all \( a, b \in Q \) the equations \( ax = b \), \( ya = b \) have unique solutions \( x = a \backslash b \), \( y = b/a \). Then the class of quasigroups form a variety of algebras with three operations \( x \cdot y \), \( x \backslash y \), \( x / y \) which is defined by identities

\[
(xy)/y = x = (x/y)y, \quad x\backslash(xy) = y = x(x\backslash y).
\]  

(1)

Each quasigroup \( Q \) can be given by a Latin square

\[
\begin{array}{cccc}
  & x_1 & \ldots & x_n \\
 x_1 & a_{11} & \ldots & a_{1n} \\
 \vdots & \vdots & \ddots & \vdots \\
 x_n & a_{n1} & \ldots & a_{nn}
\end{array}
\]  

(2)

of size \( n \). The elements of \( Q \) are \( \{x_1, \ldots, x_n\} \), each entry \( a_{ij} \) stands for the product \( x_i y_j \) in the quasigroup \( Q \).

Let \( x \cdot y \), \( x * y \) be two quasigroup multiplications on a set \( Q \). We say that multiplication \( x * y \) is an isotope of multiplication \( x \cdot y \) if there exist permutations \( \pi_1, \pi_2, \pi_3 \) on \( Q \) such that

\[
x * y = \pi_1^{-1}(x) \pi_2^{-1}(y)
\]  

(3)

for all \( x, y \in Q \).

In terms of the Latin square (5) it means that we replace it by the square

\[
\begin{array}{cccc}
  * & x_1 & \ldots & x_n \\
 x_1 & b_{11} & \ldots & b_{1n} \\
 \vdots & \vdots & \ddots & \vdots \\
 x_n & b_{n1} & \ldots & b_{11}
\end{array}
\]

where

\[
b_{ij} = \pi_1^{-1}(x_i) \pi_2^{-1}(x_j) = \pi_1^{-1}(a_{\pi_1^{-1}(x_i), \pi_1^{-1}(x_j)}).
\]

In other words, we rearrange columns and rows of \( (Q, \cdot) \) using a permutations \( \pi_2 \) and \( \pi_1 \), respectively, and afterwards permute elements of the obtained Latin square using \( \pi_1^{-1} \).

The left and right multiplications by an element, namely

\[
L_a(y) = x * y \quad \text{and} \quad R_y(x) = x * y
\]
induce permutations on \( Q \). Note that a left multiplication \( L_i \) is the permutation of \( Q \) represented by the \( i^{th} \) row of Latin square given in (5) and denoted by \( \sigma_i \). Similarly, \( \tau_j \) represents the column permutation of \( Q \) induced by right multiplication \( R_j \).

The multiplication group \( \text{Mult}(Q) \) on a quasigroup \( (Q, \ast) \) of order \( n \) is a permutation group generated by the all row and column permutations of the Latin square of \( Q \). That is,

\[
\text{Mult}(Q) = \langle \sigma_i, \tau_j \mid 1 \leq i, j \leq n \rangle
\]

We denote by \( G(Q) \), the following subgroup of \( \text{Mult}(Q) \),

\[
G(Q) = \langle \sigma_i, \sigma_j^{-1}, \tau_j^{-1} \mid 1 \leq i, j \leq n \rangle
\]

**Definition 21** A finite quasigroup \( (Q, \ast) \) is affine if \( Q \) is equipped with a structure of additive Abelian group \( (Q, +) \) such that, for each \( x, y \in Q \),

\[
x \ast y = \alpha x + \beta y + c, \text{ where } c \in Q
\]

and \( \alpha, \beta \) are automorphisms of \( (Q, +) \).

**Definition 22** A finite quasigroup is simple if it has only trivial congruences.

**Theorem 23** Artamonov et al. (2013); Hagemann and Herrmann (1982) A finite quasigroup is polynomially complete if and only if it is simple and non-affine.

### 3 Subquasigroups

**Proposition 31** Let \( Q \) be a finite quasigroup and \( H \) be a subset of \( Q \) closed under multiplication. Then \( H \) is a subquasigroup in \( Q \).

**Proof** Let \( a, b \in H \). We need to prove that the equations \( ax = b, ya = b \) have solutions in \( H \). Note that the left multiplication \( L_a \) maps \( H \) into \( H \). The map \( L_a \) has a finite order \( d \) in \( Q \). Hence \( L_a^{d-1} = L_a^{-1} \). Therefore \( H \) contains the element

\[
L_a^{d-1} b = L_a^{d-1}(ax) = L_a^{d} x = x.
\]

Similarly \( y \in H \).

Let a quasigroup \( Q \) with elements \( x_1, \ldots, x_n \) contains a subquasigroup \( W \) with elements \( x_1, \ldots, x_k \). Then the Latin square (5) of \( Q \) has the form

\[
\begin{array}{cccc}
* & x_1 & \ldots & x_k \\
\hline
x_1 & a_{11} & \ldots & a_{1k} \\
\vdots & \ldots & \ldots & \ldots \\
x_k & a_{k1} & \ldots & a_{kk} \\
x_{k+1} & a_{k+1,1} & \ldots & a_{k+1,k} \\
\vdots & \ldots & \ldots & \ldots \\
x_n & a_{n1} & \ldots & a_{nk} \\
\end{array}
\]

(4)
where 
\[ a_{ij} \in \begin{cases} \{W, & \text{if } 1 \leq i, j \leq k; \\ Q \setminus W, & \text{if either } i > k, j \leq k, \text{ or } j > k, i \leq k. \end{cases} \]

It follows immediately that \( W \) is contained in each column and each row 
\[
\begin{pmatrix}
  a_{k+1,j} \\
  \vdots \\
  a_{n,j}
\end{pmatrix}, \quad (a_{j,k+1 \ldots a_{jn}}) \text{ for any } j = k+1, \ldots, n.
\]

In particular \( n - k \geq |W| \).

**Proposition 32 (Wall and Drury (1957))** Let \( W \) be a subquasigroup in \( Q \). Then \( |W| \leq \frac{|Q|}{2} \).

**Proposition 33** Let \( Q \) be a quasigroup of order 4. Then the following are equivalent:
1. \( Q \) has a subquasigroup \( W \) such that \( 1 < |W| < 4 \);
2. \( W \) is a congruence class of order 2 and therefore \( Q \) is not simple.

**Proof** Let \( 1 < |W| < 4 \). Then \( |W| = 2 \) by Proposition 32. Hence, \( W = \{x_i, x_j\} \) is a cyclic group of order 2.

Let \( W = \{x_i, x_j\} \subseteq Q = \{x_i, x_j, x_u, x_v\} \). The Latin square of \( Q \) is

\[
\begin{array}{cccc}
  x_i & x_j & x_u & x_v \\
  x_i & a_{ii} & a_{ij} & a_{iu} & a_{iv} \\
  x_j & a_{ji} & a_{jj} & a_{ju} & a_{jv} \\
  x_u & a_{ui} & a_{uj} & a_{uu} & a_{uv} \\
  x_v & a_{vi} & a_{vj} & a_{vv} & a_{vv} \\
\end{array}
\]

Clearly, up to rearrangement of \( x_i \) and \( x_j \), we have
\[
x_i^2 = x_i, \quad x_ix_j = x_jx_i = x_j, \quad x_j^2 = x_i, \quad \text{i.e. } a_{ii}, a_{jj} \in W \text{ for } t = i, j.
\]

Therefore,
\[
\{x_i, x_j\} \cdot \{x_u, x_v\} \subseteq \{x_i, x_j\} \cdot \{x_u, x_v\} \subseteq \{x_i, x_j\} \quad \text{and} \quad \{x_u, x_v\} \cdot \{x_u, x_v\} \subseteq \{x_i, x_j\}.
\]

Let us consider a partition of \( Q \), \( P = \{\{x_i, x_j\}, \{x_u, x_v\}\} \). Now, we have
\[
\begin{align*}
\{x_i, x_j\} \cdot \{x_i, x_j\} &= \{x_i, x_j\}, \\
\{x_u, x_v\} \cdot \{x_u, x_v\} &= \{x_i, x_j\}, \\
\{x_i, x_j\} \cdot \{x_u, x_v\} &= \{x_u, x_v\}, \\
\{x_u, x_v\} \cdot \{x_i, x_j\} &= \{x_i, x_j\}.
\end{align*}
\]

That is, \( Q/P \) is a group of order 2 with the identity element \( \{x_i, x_j\} \). Hence, the partition \( P \) is a congruence relation.

Converse statement follows from Artamonov et al. (2013) Theorem 7).

**Theorem 34 (Recognition of subquasigroups)** Let \( (Q = \{x_1, \ldots, x_n\}, \ast) \) be a quasigroup of order \( n \). Take a subset \( W = \{x_{i_1}, \ldots, x_{i_k}\}, k \leq \frac{n}{2} \), in \( Q \) which is a union of orbits of any permutation \( \sigma_{i_j} \) for \( j = 1, \ldots, k \). \( W \) is a subquasigroup in \( Q \) if and only if \( x_{i_j} \in W \) implies \( \sigma_{i_t}(x_{i_j}) = x_{i_t} \ast x_{i_j} \in W \) for every \( t = 1, \ldots, k \).
Proof Let \( x_{ij}, x'_{ij} \in W \). Then \( x_{ij} \ast x'_{ij} = \sigma_{x_{ij}}(x_{ij}) \in W \) because \( \sigma_{x_{ij}}(x_{ij}) \) belongs to an orbit of \( x_{ij} \), which is contained in \( W \). Thus \( W \) is a subquasigroup.

Conversely if \( W \) is a subquasigroup in the quasigroups \( Q \) with the Latin square take \( C_i \) as the set of entries of \( i^{th} \) row \( (a_{i1} \ldots a_{ik}) \) which is equal to \( W \). Then all properties are satisfied.

Note that the orbit of each \( x_{ij} \) with respect to \( \sigma_{x_{ij}} \) for all \( t = 1, \ldots, k \) is contained in \( W \). Moreover \( W \) is a union of all orbits \( \bigcup_{1 \leq i, j \leq k, s \geq 0} (\sigma^s_{x_{ij}}(x_{ij})) \).

Now, we illustrate an efficient procedure to obtain all proper subquasigroups of a quasigroup \( (Q, \ast) \) based on Theorem 34 by means of some concrete examples.

Example 1 Let us consider a quasigroup \( (Q = \{1,2,\ldots,8\}, \ast) \) of order 8 with the following Latin square, \( L \):

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 1 & 4 & 3 & 5 & 6 & 7 & 8 \\
3 & 4 & 2 & 1 & 7 & 8 & 5 & 6 \\
4 & 3 & 1 & 2 & 8 & 5 & 6 & 7 \\
5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
6 & 7 & 8 & 5 & 2 & 3 & 4 & 1 \\
7 & 8 & 5 & 6 & 3 & 1 & 2 & 4 \\
8 & 5 & 6 & 7 & 4 & 1 & 2 & 3 \\
\end{array}
\]

Let’s consider the first row permutation, say \( \sigma_1 \), of the Latin square \( L \). We use two distinct sets, denoted by \( \Sigma \) and \( \Sigma' \), to store the indices of the row permutations. The set \( \Sigma \) is composed of all indices \( j \), if \( \sigma_j \) are being considered in the method. Whereas, the set \( \Sigma' \) is composed of some particular indices (not by all) which will be illustrated later. Initially, \( \Sigma' = \emptyset \) and since we are considering the first permutation, \( \Sigma = \{1\} \).

Now, we start by decomposing \( \sigma_1 \) into disjoint cycles. If \( O_{ij} \) denotes the \( j^{th} \) cycle of \( \sigma_1 \) row permutation, then the cycles of \( \sigma_1 \) are

\[
O_{11} = (1, 2), \quad O_{12} = (3, 4), \quad O_{13} = (5), \quad O_{14} = (6), \quad O_{15} = (7), \quad O_{16} = (8)
\]

By Theorem 34 if a subset \( W = \{i_1, i_2, \ldots, i_k\} \subset Q \) is a subquasigroup, then \( W \) is a union of cycles of any permutation \( \sigma_{ij} \) such that \( i_j \in O_{ik} \subset W \) and \( |W| \leq \frac{|Q|}{2} \). So, first we locate the cycle, say \( O_{ik} \), which has \( j^{th} \) element of \( Q \) and consider all possible subsets formed by the unions of cycles with \( O_{ik} \) such that the cardinality of each subset is not greater than \( \frac{|Q|}{2} \) and their elements do not belong to \( \Sigma' \). Mathematically, we can express the set \( \mathcal{P} \) of all possible candidates in the following way

\[
\mathcal{P} = \left\{ W \in \mathcal{P}' \mid |W| + |O_{ik}| \leq n/2 \right\}, \text{ where}
\]

\[
\mathcal{P}' = \text{PowerSet}\{O_{ij} \mid O_{ij} \subset Q \setminus \Sigma' \land |O_{ij}| + |O_{ik}| \leq n/2\}.
\]
So in the above example, the set denoted by \( \mathcal{P} \), of all possible candidates for being subquasigroup of \( Q \) corresponding to \( \sigma_1 \), is given by

\[
\mathcal{P} = \left\{ \{O_{11}\}, \{O_{11} \cup O_{12}\}, \{O_{11} \cup O_{13}\}, \{O_{11} \cup O_{15}\}, \{O_{11} \cup O_{16}\}, \{O_{11} \cup O_{13} \cup O_{14}\}, \{O_{11} \cup O_{13} \cup O_{15}\}, \{O_{11} \cup O_{13} \cup O_{16}\} \right\}
\]

i.e.

\[
\mathcal{P} = \left\{ \{1, 2\}, \{1, 2, 3, 4\}, \{1, 2, 5\}, \{1, 2, 6\}, \{1, 2, 7\}, \{1, 2, 8\}, \{1, 2, 5, 6\}, \{1, 2, 5, 7\}, \{1, 2, 5, 8\}, \{1, 2, 6, 7\}, \{1, 2, 6, 8\}, \{1, 2, 7, 8\} \right\}
\]

If a subset \( W \in \mathcal{P} \) is closed under the operation \( * \) then \( W \) is a subquasigroup of \( Q \). For instance in the ongoing example, only the subset \( \{1, 2, 3, 4\} \) of \( \mathcal{P} \) is closed. So the list \( \mathcal{L} \) of subquasigroup obtained by considering the permutation \( \sigma_1 \) is

\[
\mathcal{L} = \left\{ \{1, 2, 3, 4\} \right\}
\]

Now, we find the subquasigroups consisting of only elements from the set \( S = \bigcup_{W \in \mathcal{L}} W \setminus \Sigma = \{2, 3, 4\} \). That is, we need to consider only those orbits of permutations \( \sigma_j, j \in S \) for the possible subquasigroups which are subsets of \( S \). We repeat the process until \( S \) is empty.

Here, on considering \( 2 \in S \),

\[
O_{21} = (1), \quad O_{22} = (2, 3), \quad O_{23} = (4), \quad O_{24} = (5, 6, 7, 8)
\]

Since \( O_{22}, O_{23} \subset S \),

\[
\mathcal{P} = \left\{ \{2, 3\}, \{2, 3, 4\} \right\}
\]

The subset \( \{2, 3\} \) is closed w.r.t. \( * \). So, the subquasigroup list is

\[
\mathcal{L} = \left\{ \{1, 2, 3, 4\}, \{2, 3\} \right\}
\]

The updated sets \( S = S \setminus \{2\} = \{3, 4\} \) and \( \Sigma = \Sigma \cup \{2\} = \{1, 2\} \). Next, consider \( 3 \in S \),

\[
O_{31} = (1, 4), \quad O_{32} = (2), \quad O_{33} = (3), \quad O_{34} = (5, 7), \quad O_{35} = (6, 8)
\]

So, \( \mathcal{P} = \left\{ \{3\} \right\} \) and \( \{3\} \) is closed. Hence the subquasigroups are

\[
\mathcal{L} = \left\{ \{1, 2, 3, 4\}, \{2, 3\}, \{3\} \right\}
\]

Now, \( S = S \setminus \{3\} = \{4\} \) and \( \Sigma = \Sigma \cup \{3\} = \{1, 2, 3\} \). Decomposition of \( \sigma_1 \) provides

\[
O_{41} = (1, 3), \quad O_{42} = (2, 4), \quad O_{43} = (5, 6, 7, 8)
\]

Since \( O_{42} \not\subset S \), \( \mathcal{P} = \emptyset \). The sets \( S = S \setminus \{4\} = \emptyset \) and \( \Sigma = \Sigma \cup \{4\} = \{1, 2, 3, 4\} \).
At this step, we recall that we started with first permutation $\sigma_1$ and then based on obtained subquasigroups we consider other permutations for finding the subquasigroups consisting only elements form the set $S$. The set $\Sigma'$ contains indices of such considered permutations $\sigma_i$, $\Sigma' = \{1\}$.

Now, one has to consider the next permutation, in the process, is $\sigma_i$, where

$$i = \min(Q \setminus \Sigma).$$

So for the example, we are illustrating here, the index for the next permutation

$$i = \min\{\{1,2,\ldots,8\} \setminus \{1,2,3,4\}\} = \min\{5,6,7,8\} = 5.$$

Now we repeat the above described complete process starting with the permutation $\sigma_5$. First get the cycle decomposition of $\sigma_5$, which is

$$O_{51} = (1,5), \ O_{52} = (2,6), \ O_{53} = (3,7), \ O_{54} = (4,8)$$

Since, $5 \in O_{51}$ and $O_{51} \not\subseteq Q \setminus \Sigma'$, the set of possible candidates $\mathcal{P} = \emptyset$. The sets $\Sigma = \Sigma \cup \{5\} = \{1,2,3,4,5\}$ and $S = \bigcup_{W \in \mathcal{L}} W \setminus \Sigma = \emptyset$. $\Sigma' = \{1,5\}$.

By continuing, the index of next row permutation to consider,

$$i = \min(Q \setminus \Sigma) = 6$$

and the cycle decomposition of $\sigma_6$ is

$$O_{61} = (1,3,6,8), \ O_{62} = (2,4,5,7)$$

Again $O_{61} \not\subseteq Q \setminus \Sigma'$, so $\mathcal{P} = \emptyset$. The sets $\Sigma = \{1,2,3,4,5,6\}$, $S$ remains $\emptyset$, and $\Sigma' = \Sigma' \cup \{6\} = \{1,5,6\}$.

The next row index, $i = \min(Q \setminus \Sigma) = 7$. The cycle decomposition of $\sigma_7$ is

$$O_{71} = (1,7), \ O_{72} = (2,8), \ O_{73} = (3,5), \ O_{74} = (4,6)$$

So, once again $O_{71} \not\subseteq Q \setminus \Sigma'$, $\mathcal{P} = \emptyset$. Then, $\Sigma = \{1,2,3,4,5,6,7\}$ and $\Sigma' = \Sigma' \cup \{7\} = \{1,5,6,7\}$.

Now, $t = \min(Q \setminus \Sigma) = 8$. The decomposition of $\sigma_8$

$$O_{81} = (1,3,6,8), \ O_{82} = (2,4,5,7)$$

$O_{81} \not\subseteq Q \setminus \Sigma'$. All the row permutations have been considered, hence the procedure terminates and the list of all proper subquasigroups of $(Q, \ast)$.

$$\mathcal{L} = \left\{\{1,2,3,4\}, \{2,3\}, \{3\}\right\}$$
Below we display the pseudo code of the above described method to find out all proper subquasigroups of a given quasigroup.

**Algorithm 1** Finding all Proper Subquasigroups

Require: A Latin square \( L \) of Quasigroup \((Q = \{1, 2, \ldots, n\}, \ast)\)

Ensure: List of all subquasigroups

1: list \( L = 0 \), \( i = 1 \), \( \Sigma = \emptyset \), \( \Sigma' = \emptyset \)
2: while \( i \leq n \) do
3: \( \Sigma = \Sigma \cup \{i\} \)
4: decompose \( \sigma_i \) of \( L \) into disjoint cycles \( O_{ij} \)
5: fix \( k \), such that \( O_{ik} \) contains \( i \)
6: \( P' = \text{PowerSet}\{O_{ij} \mid O_{ij} \subset Q \land |O_{ij}| + |O_{ik}| \leq n/2\} \)
7: \( P = \{W \in P' \mid |W| + |O_{ik}| \leq n/2\} \)
8: while \( P \neq \emptyset \) do
9: \( W \in P, \quad P = P \setminus \{W\} \)
10: \( W = O_{ik} \cup W \)
11: if \( W \) is closed under \( \ast \) then
12: \( L = L \cup \{W\} \)
13: \( S = \bigcup_{W \in L} W \setminus \Sigma \)
14: while \( S \neq \emptyset \) do
15: decompose \( \sigma_j, \ j \in S \)
16: \( \Sigma = \Sigma \cup \{j\} \)
17: \( P' = \text{PowerSet}\{O_{ij} \mid O_{ij} \subset S \text{ and } |O_{ij}| + |O_{ik}| \leq n/2\} \)
18: \( P = \{W \in P' \mid |W| + |O_{ik}| \leq n/2\} \)
19: while \( P \neq \emptyset \) do
20: \( W \in P, \quad P = P \setminus \{W\} \)
21: \( W = O_{ik} \cup W \)
22: if \( W \) is closed under \( \ast \) then
23: \( L = L \cup \{W\} \)
24: \( S = S \setminus \{W\} \)
25: \( \Sigma' = \Sigma' \cup \{i\} \)
26: \( i = \min\{Q \setminus \Sigma\} \)
27: return \( L \)

Note that if the above algorithm returns empty list \( L \), then there is no any proper subquasigroup in \( Q \).

**Proposition 35** Let \( D_1 \) be the set of all diagonal entries of the Latin square of \( Q \). Define by induction \( D_{k+1} \) as a set of all squares of elements from \( D_k \). Since \( Q \) is finite there exists a positive integer \( k \) such that \( D = D_k = D_{k+1} \). The quasigroup \( Q \) has no proper subquasigroups if and only if \( Q \) is generated by every element from \( D \).

**Proof** Suppose that \( Q \) has no proper subquasigroup and \( a \in D \). Then \( a \) generates \( Q \).

Conversely, let \( A \) be a subquasigroup in \( Q \). If \( a \in A \) then \( a^2 \in A \cap D_1 \). By induction for any \( k \) we can find an element \( a_k \in A \cap D_k \). Hence by assumption \( A = Q \).

**Example 2** Consider a quasigroup \((Q = \{1, 2, 3, 4, 5\}, \ast)\) as follows
The elements of diagonal set \( D_1 = \{2, 5, 4, 3, 1\} \) are placed with in the box. The set of square of the diagonal elements \( D_2 = D_1 \), so \( D = D_1 \). Now,

\[
2 \ast 2 = 5, \quad (2 \ast 2) \ast 2 = 5 \ast 2 = 4, \quad 2 \ast (2 \ast 2) = 2 \ast 5 = 1,
\]

\[
(2 \ast (2 \ast 2)) \ast 2 = (2 \ast 5) \ast 2 = 1 \ast 2 = 3
\]

Hence, 2 generates \( Q \). Similary one can check that all other diagonal elements generates \( Q \). Note that here the order of parenthesis has to be taken into consideration.

**Example 3** Consider the following quasigroup of order 8.

\[
\begin{array}{cccccccc}
* & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 5 & 8 & 3 & 7 & 6 & 4 & 1 \\
2 & 3 & 1 & 6 & 2 & 4 & 8 & 7 & 5 \\
3 & 4 & 6 & 1 & 7 & 3 & 5 & 2 & 8 \\
4 & 8 & 7 & 2 & 6 & 5 & 3 & 1 & 4 \\
5 & 6 & 4 & 3 & 8 & 1 & 2 & 5 & 7 \\
6 & 5 & 2 & 7 & 1 & 8 & 4 & 6 & 3 \\
7 & 7 & 8 & 5 & 4 & 2 & 1 & 3 & 6 \\
8 & 1 & 3 & 4 & 5 & 6 & 7 & 8 & 2 \\
\end{array}
\]

In this case, the set of diagonal elements \( D_1 = \{2, 1, 6, 4, 3\} \) is a proper subset of \( Q \), \( D_2 = \{1, 2, 4, 6\} \) and \( D_3 = D_2 \), so \( D = D_2 \). We can check as above that any element of \( D \) generates \( Q \), hence \( (Q, *) \) has no proper subquasigroup. The same has also been shown by Algorithm[1].

Now, we present an algorithm for the construction of a subquasigroup generated by an element \( a \in Q \).
Algorithm 2 Generation of a subquasigroup by an element

**Require:** An element \( a \in Q \)

**Ensure:** Subquasigroup of \( Q \) generated by \( a \)

1: \( A_0 = \{ a \} \)
2: \( k = 0 \)
3: \( A_{k+1} = A_k A_k \cup A_k = \{ a \times a, a \} \)
4: while \( A_{k+1} \neq A_k \) do
5: \( k = k + 1 \)
6: \( A_{k+1} = A_k A_k \cup A_k \)

\[ = \left( (A_k \setminus A_{k-1}) (A_k \setminus A_{k-1}) \right) \cup \left( (A_k \setminus A_{k-1}) A_{k-1} \right) \cup \left( A_{k-1} (A_k \setminus A_{k-1}) \right) \cup A_k \]

7: if \( |A_{k+1}| > |Q|/2 \) then
8: \( A_{k+1} = Q \)
9: break
10: return \((A_{k+1},\ast)\)

The above algorithm terminates since \( Q \) is finite so there exists a positive integer \( k \) such that \( A_{k+1} = A_k \). Note that all the possible order of parenthesis are considered in the above algorithm.

**Proposition 36** \( A_k \) is a subquasigroup in \( Q \) generated by an element \( a \in Q \). Then, \( a \) generates \( Q \) if and only if \( Q = A_k \). The quasigroup \( Q \) has no proper subquasigroup if and only if \( Q = A_k \) for every element \( a \in Q \).

**Proof** If \( A_{k+1} = A_k \) then by definition \( A_k \) is closed under multiplication. Then, the result follows from Proposition 31.

**Proposition 37** Let \( A_k \) be as above and \( |A_k| > \frac{|Q|}{2} \). Then \( A_k = Q \).

**Remark 1** It has to be noted that Algorithm 1 and Algorithm 2 both can be used to find whether a given quasigroup does not have any proper subquasigroup. But using Algorithm 2 one has to check the possibility of subquasigroups corresponding to each diagonal element of the quasigroup. So, in the case of no proper subquasigroup, which is of our interest, this algorithm will be less efficient, therefore confine its interest to the theoretical aspects.

## 4 Finite Field Based Polynomially Complete Quasigroup of Order \( p^r \) Without Proper Subquasigroups

In this section, we propose a binary operation to construct quasigroups based on finite fields which gives polynomially complete quasigroups of order \( p^r \), having no subquasigroups. We develop a methodology and an algorithm to construct such quasigroups. It provides cryptographically suitable quasigroups of order \( 2^r \) \((r > 1)\). Furthermore, we illustrate some important properties of these quasigroups.
Let us consider a finite field \( \mathbb{F}_q (q=p') \) with \( q > 2 \) elements, the set \( Q = \{ x \mid x \in \mathbb{F}_q \} \), and \( \alpha, \beta, c \in \mathbb{F}_q \). Suppose that \( m, d \) are positive integers coprime with \( q - 1 \). Define a multiplication in \( Q \) as follows

\[
x * y = \alpha x^m + \beta y^d + c
\]  

(5)

**Theorem 41** \((Q, *)\) is a quasigroup isotopic to the quasigroup \((Q, \cdot)\), where \( \cdot \) is the multiplication defined in (5) for \( m = d = 1 \). The quasigroup \((Q, \cdot)\) is affine.

**Proof** Since \( m, d \) are coprime with \( q - 1 \) the maps \( \pi_1(x) = x^m \) and \( \pi_2(y) = y^d \) are permutations on \( Q \). Hence,

\[
x * y = \alpha \pi_1(x) + \beta \pi_2(y) + c.
\]

That is,

\[
x * y = \pi^{-1}(\pi_1(x) \cdot \pi_2(y))
\]

It follows that \((Q, *)\) is a quasigroup with necessary isotopy with \( \pi \) as the identity map.

**Theorem 42** The quasigroup \((Q, *)\) from (5) is affine if and only if \( md^{-1} \in \{ 1, p, \ldots, p^{r-1} \} \mod (q - 1) \) where \( p \) is the characteristic of \( \mathbb{F}_q \), \( q = p' \).

**Proof** Let \( L_x \) and \( R_y \) be the maps of left and right multiplications in the quasigroup \((Q, *)\) from (5). Then

\[
L_x(y) = \beta y^d + \alpha x^m + c, \quad R_y(x) = \alpha x^m + \beta y^d + c.
\]

Hence

\[
L_x^{-1}(y) = \frac{\beta^{-1} y - \beta^{-1}(\alpha x^m + c)}{\alpha}, \quad R_y^{-1}(x) = \frac{\alpha^{-1} x - \alpha^{-1}(\beta y^d + c)}{\beta}.
\]

Hence

\[
L_x L_x^{-1}(y) = y + \alpha(x^m - z^m), \quad R_y R_y^{-1}(x) = x + \beta(y^d - t^d).
\]  

(6)

There exists a group isomorphism \( \pi : G(Q, *) \rightarrow (\mathbb{F}_q, +) \) such that

\[
\pi (L_x L_x^{-1}) = \alpha (x^m - z^m), \quad \pi (R_y R_y^{-1}) = \beta (y^d - t^d).
\]

Suppose that \((Q, *)\) is affine. Then there exists a structure of an additive abelian group \((Q, \oplus)\) such that

\[
x * y = \xi x \oplus \zeta y \oplus d, \quad \xi, \zeta \in \text{Aut}(Q, \oplus).
\]

Hence as in (5)

\[
L_x L_x^{-1}(y) = y \oplus \xi x \oplus \zeta z, \quad R_y R_y^{-1}(x) = x \oplus \zeta y \oplus \xi z, \quad \xi, \zeta \in \text{Aut}(Q, \oplus).
\]

Again we get a group isomorphism \( \omega : G(Q, *) \rightarrow (Q, \oplus) \) where

\[
\omega (L_x L_x^{-1}) = \xi x \oplus \zeta z, \quad \omega (R_y R_y^{-1}) = \zeta y \oplus \xi z.
\]
Consequently,
\[ \omega \pi^{-1} \alpha(x^m - z^m) = \xi(x \circ z), \quad \omega \pi^{-1} \beta(y^d - z^d) = \zeta(y \circ z), \]
or
\[ \zeta^{-1} \omega \pi^{-1} \alpha(x^m - z^m) = x \circ z, \quad \zeta^{-1} \omega \pi^{-1} \beta(y^d - z^d) = y \circ z, \]
and therefore
\[ \zeta^{-1} \omega \pi^{-1} \alpha(x^m - z^m) = \zeta^{-1} \omega \pi^{-1} \beta(x^d - z^d) = x \circ z. \]

It \( z = 0 \) then \( \zeta^{-1} \omega \pi^{-1} \alpha(x^m) = x \circ 0 \) that is \( \zeta^{-1} \omega \pi^{-1} \alpha(x) = x^{1/m} \circ 0 \) and therefore
\[ \zeta^{-1} \omega \pi^{-1} \alpha(x^m - z^m) = (x^m - z^m)^{1/m} \circ 0 = x \circ z. \]

Hence \( (x^m - z^m)^{1/m} = x \circ z \oplus 0. \)

Similarly
\[ \zeta^{-1} \omega \pi^{-1} \beta(x^d - z^d) = y \circ z = (x^d - z^d)^{1/d} \circ 0, \]

Hence
\[ \alpha^{-1} \pi \omega^{-1} \xi \zeta^{-1} \omega \pi^{-1} \beta(x^d - z^d) = x^m - z^m. \]

Recall that \( \xi \zeta^{-1} \in \text{Aut}(Q, \circ) \). Thus
\[ \omega \pi^{-1} \xi \zeta^{-1} \omega \pi^{-1} \text{ and } \theta = \alpha^{-1} \pi \omega^{-1} \xi \zeta^{-1} \omega \pi^{-1} \beta \in \text{Aut}(\mathbb{F}_q, +). \]

Thus in the group \( (\mathbb{F}_q, +) \) we have \( \theta(x^d - z^d) = x^m - z^m \) for some \( \theta \in \text{Aut}(\mathbb{F}_q, +) \). In particular \( \theta(x^d) = x^m \) or \( \theta(x) = x^{md^{-1}} \). Since the map \( \theta \) is additive, we can conclude that \( md^{-1} \) is a power of \( p \).

Conversely, let \( m = dp^l \), \( 0 \leq l \leq r - 1 \). Then (5) has the form
\[ x \ast y = \alpha x^{d^lp^l} + \beta y^{d^l} + c. \]

The map \( \pi(x) = x^d \) defines an isomorphism of \( (Q, \ast) \) and the quasigroup with multiplication \( x \ast y = \alpha \pi(x)^p + \beta \pi(y) + c \). The last one is affine since the maps \( x \mapsto \alpha \pi(x)^p \) and \( y \mapsto \beta \pi(y) \) are automorphisms of \( (\mathbb{F}_q, +) \).

**Theorem 43.** Suppose that \( Q \) is from (5) and \( \beta \) is a generator of the cyclic group \( \mathbb{F}_q^* \). Then \( Q \) is simple.

**Proof.** By (6) the group \( G(Q) \) contains all translations \( x \mapsto x + w \) for all \( w \in \mathbb{F}_q \). As it is shown in the proof of Theorem 42, \( L_\nu(y) = \beta y^d + u \), \( u = \alpha x^m + c \in \mathbb{F}_q \). Hence if we take \( x \) such that \( \alpha x^m + c = 0 \) then we see that the map \( f(y) = \beta y^d \) belongs to \( \text{Mult}(Q) \).

Consider the stabilizer subgroup \( H \) of zero element in \( \text{Mult}(Q) \). It contains the map \( f \) which acts transitively in \( \mathbb{F}_q^* \) because \( \beta \) is the generator of \( \mathbb{F}_q^* \). Hence \( \text{Mult}(Q) \) is a doubly transitive permutation group and therefore \( Q \) is simple. Phillips and Smith [1999] [Proposition 1].

**Corollary 44.** Suppose that \( md^{-1} \notin \{1, p, \ldots, p^{r-1}\} \mod (q - 1) \) and \( \beta \) is a generator of the cyclic group \( \mathbb{F}_q^* \). Then \( Q \) with multiplication (5) is polynomially complete.
**Theorem 45** Let $Q$ be the quasigroup defined in $\mathbb{F}_q$, $q > 2$, with multiplication

$$x \ast y = \alpha x^m + \beta y^d + c, \quad c \in \mathbb{F}_q, \quad \alpha, \beta \in \mathbb{F}_q^*,$$

where $\beta$ is a generator of the cyclic group $\mathbb{F}_q^*$. Here $0 < m, d < q - 1$ are coprime with $q - 1$. Then, any two distinct elements in $\mathbb{F}_q$ generate the quasigroup $Q$.

**Proof** Consider in $\mathbb{F}_q$ new multiplication

$$x \circ y = \alpha x^{md^{-1}} + \beta y + c. \quad (7)$$

Denote by $L^\circ_\alpha, L^\circ_\beta$ the operators of left multiplication by $x$ with respect to multiplications $\circ, \ast$. Since $x \ast y = x^d \circ y^d$ we have $L^\circ_\alpha(y) = L^\circ_\alpha(y^d)$.

Let’s fix elements $x, y \in \mathbb{F}_q$.

**Lemma 46** If $k \geq 1$ then $(L^\circ_\alpha)^k(y) = \alpha^{1 - \beta^k} x^{md^{-1}} + \frac{1 - \beta^k}{1 - \beta} c + \beta^k y$.

**Proof** The case $k = 1$ follows from (7). Suppose that the formula is true for some $k$. Then by induction

$$(L^\circ_\alpha)^{k+1}(y) = L^\circ_\alpha \left[ \alpha^{1 - \beta^k} x^{md^{-1}} + \frac{1 - \beta^k}{1 - \beta} c + \beta^k y \right]$$

$$= \alpha x^{md^{-1}} + \beta \left[ \alpha^{1 - \beta^k} x^{md^{-1}} + \frac{1 - \beta^k}{1 - \beta} c + \beta^k y \right] + c$$

$$= \alpha \left[ 1 + \beta \frac{1 - \beta^k}{1 - \beta} \right] x^{md^{-1}} + \beta^{k+1} y + \left[ 1 + \beta \frac{1 - \beta^k}{1 - \beta} \right] c.$$

It suffices to notice that $1 + \beta \frac{1 - \beta^k}{1 - \beta} = \frac{1 - \beta^{k+1}}{1 - \beta}$.

**Lemma 47** Let $W$ be a subquasigroup of $(Q, \ast)$ containing elements $x, y$. If $Q \neq W$, then

$$\alpha x^m + (\beta - 1) y^d + c = 0. \quad (8)$$

**Proof** As we have already noticed $L^\ast_\alpha(y) = L^\circ_\alpha(y^d)$. Suppose that $(L^\ast_\alpha)^k(y) = y$ for some minimal $1 \leq k \leq q - 2$. Then $(L^\circ_\alpha)^{k+1}(y^d) = L^\circ_\alpha(y^d)$. By Lemma 46,

$$\alpha \left[ 1 - \beta^{k+1} \right] x^m + \beta^{k+1} y^d + \frac{1 - \beta^{k+1}}{1 - \beta} c = \alpha x^m + \beta y^d + c.$$

Hence

$$\alpha \left[ 1 - \beta^{k+1} \right] \left[ x^m + \beta^{k+1} y^d + \frac{1 - \beta^{k+1}}{1 - \beta} c \right] = 0.$$

Note that $\frac{1 - \beta^{k+1}}{1 - \beta} = \frac{\beta(1 - \beta^k)}{1 - \beta}$. Multiplying by $\frac{1 - \beta}{\beta(1 - \beta^k)}$, we obtain (8). Note that the order of $\beta$ is equal to $q - 1$. So if $k \leq q - 2$ then $1 \neq \beta^k$.

Suppose that $k = q - 1$, Then $W$ contains distinct elements

$$y, (L^\ast_\alpha(y), (L^\ast_\alpha)^2(y), \ldots, (L^\ast_\alpha)^{q-2}(y))$$

Thus, $W = Q$ since $q - 1 > \frac{q}{2}$ for $q > 2$. 

Suppose that $W$ is a proper subquasigroup of $(Q, \ast)$ containing elements $x, y$. Then (8) holds for $x, y \in W$. Let $W'$ be a subquasigroup generated by $x$. Then $W' \subseteq W$ and therefore $W' \neq Q$. In particular, on taking $y = x$ in (8) we obtain
\[ \alpha x^m + (\beta - 1)x^d + c = 0. \] (9)
That is for an element $x \in W$ the equations (9) holds. By combining (8) for $x \in W$ with (9) for the same $x$ and another element $y \in W$, we obtain $(\beta - 1)x^d = (\beta - 1)y^d$. Since $1 - \beta \neq 0$, we can conclude that $y = x$ because $d$ is coprime with $q - 1$.

Thus, if $W$ is a proper subquasigroup of $(Q, \ast)$, it is an idempotent. That is, any two distinct elements generate the $(Q, \ast)$.

**Corollary 48** Let $p$ be a prime and $q = p^r$. Suppose that $m/\in \{1, p, \ldots, p^{r-1}\} \mod (q - 1)$ and $\beta$ is a generator of the cyclic group $F_q^*$. Suppose that $1 < m < q - 1$ is coprime with $q - 1$ and $d = 1$. Then there exists an element $c \in F_q^*$ such that $Q = \{x \mid x \in F_q\}$ with multiplication
\[ x \ast y = (1 - \beta)x^m + \beta y + c \] (10)
has no subquasigroups and it is polynomially complete. An element $x \in Q$ form a subquasigroup if and only if
\[ x^m - x + \frac{c}{1 - \beta} = 0. \] (11)

**Proof** The corollary follows immediately from the previous theorem except the existence of an element $c \in F_q^*$ such that $(Q, \ast)$ does not have any proper subquasi-
group.

Let’s consider a map
\[ f : F_q \to F_q, \quad x \mapsto f(x) = x^m - x \]
Since $f(0) = f(1) = 0$, the map $f$ is not bijective while the map
\[ g : F_q \to F_q, \quad c \mapsto g(c) = c/(\beta - 1) \]
is one-one. So, there exists a constant $c \in F_q^*$ such that $c/(\beta - 1) \in F_q^* \setminus \text{Range}(f)$, hence Equation (11) has no solution in $F_q$, consequently, $(Q, \ast)$ has no proper sub-
quasigroup.

Furthermore, the number of roots of the polynomial in (11) can be found by using König-Rados theorem, given in [Lidl and Niederreiter (1996), Chapter 6.1], which states that the equation (11) will have $q - 1 - r$ number of solutions in $F_q$, where $r$ is the rank of left circulant matrix corresponding to polynomial in (11) of order $(q - 1) \times (q - 1)$

\[
\begin{bmatrix}
\gamma & -1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 \\
-1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 & \gamma \\
\vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \gamma & -1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0
\end{bmatrix},
\]
where $\gamma = \frac{c}{1 - \beta}$.
Remark 2 There are only \(q - 1 - r\) idempotent subquasigroups in \((Q, \ast)\) for any \(c \in F_q^\ast\). Precisely, they are \((\{x\}, \ast)\), where \(x\) is a solution of Equation (11).

Based on Corollary [48] we propose the following algorithm for the construction of polynomially complete quasigroups of order \(p^r\) having no proper subquasigroup.

**Algorithm 3** Construction of Cryptographically Suitable Quasigroups of order \(q = p^r\)

**Require:** A prime \(p\) and a positive integer \(r\)

**Ensure:** A polynomially complete quasigroup of order \(p^r\) having no proper subquasigroups

1. select an integer \(m \in \{1, p, p^2, \ldots, p^r - 1\}\), \(1 < m < (q - 1)\), and \(\gcd(m; q - 1) = 1\)
2. fix a generator \(\beta\) of \(F_q^\ast\)
3. while True do
4. choose an element \(c \in F_q^\ast\)
5. compute rank \(r\) of the left circulant matrix of \(x^m - x + \frac{c}{1 - \beta}\)
6. if \(q - 1 = r\) then
7. break
8. \(Q = \{x \mid x \in F_q\}\)
9. define binary operation \(\ast\) on \(Q\):
   \[x \ast y = (1 - \beta)x^m + \beta y + c, \ \forall x, y \in Q\]
10. return \((Q, \ast)\)

**Remark 3** One can avoid computing the rank of left circulant matrix at Step 5, in the above algorithm by determining the range of the map \(f : F_q \to F_q, \ x \mapsto x^m - x\). Since the map \(f\) is not bijective, the nonempty set \(F_q \setminus \text{Range} (f)\) provides all those \(c\) for which there will not be any subquasigroup.

Now, we give some concrete examples of construction of quasigroups of order 8 over the field \(F_{2^3}\) to illustrate the above described method. For these constructions, possible choices of the integer \(m\), mentioned in the algorithm, are 3, 5, 6. In the following examples, we denote a primitive element of \(F_q\) by \(a\) and for convenience, tag the elements 0, 1, \(a, a + 1, a^2, a^2 + 1, a^2 + a, a^2 + a + 1\) of \(F_q\) by 0, 1, 2, 3, 4, 5, 6, 7, respectively.

**Example 4** Let \(c = 1 \in F_{2^3}\), \(m = 3\), and a generator \(\beta = a \in F_{2^3}^\ast\). For these choices of \(c, m\) and \(\beta\), the set \(Q = \{x \mid x \in F_{2^3}\}\), with respect to binary operation \(\ast\) as given in (10), yields the following polynomially complete quasigroup

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 3 & 5 & 7 & 2 & 0 & 6 \\
1 & 2 & 0 & 6 & 4 & 1 & 3 & 5 \\
2 & 4 & 6 & 0 & 2 & 7 & 5 & 3 \\
3 & 6 & 4 & 2 & 0 & 5 & 7 & 1 \\
4 & 5 & 7 & 1 & 3 & 6 & 4 & 2 \\
5 & 0 & 2 & 4 & 6 & 3 & 1 & 7 \\
6 & 3 & 1 & 7 & 5 & 0 & 2 & 4 \\
7 & 5 & 3 & 1 & 4 & 6 & 0 & 2 \\
\end{array}
\]
Now, the left circulant matrix corresponding to (11) is
\[
\begin{pmatrix}
a^2 + a & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & a^2 + a \\
0 & 1 & 0 & 0 & 0 & a^2 + a & 1 \\
1 & 0 & 0 & 0 & a^2 + a & 1 & 0 \\
0 & 0 & 0 & a^2 + a & 1 & 0 & 1 \\
0 & 0 & a^2 + a & 1 & 0 & 1 & 0 \\
0 & a^2 + a & 1 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]
whose rank is 7. So, Equation (11) has no solution in \( F_{23} \), hence the constructed quasigroup, \((Q, \ast)\), has no proper subquasigroup.

**Example 5** Now, let \( c = a^2 \in F_{23} \), \( m = 5 \), and a generator \( \beta = a \) of \( F_{23}^* \). For these choices of \( c, m \) and \( \beta \), the set \( Q = \{ x \mid x \in F_{23} \} \), with respect to binary operation \( \ast \) as given in (10), yields the following polynomially complete quasigroup

\[
\begin{align*}
\ast & | 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 4 & 6 & 0 & 2 & 7 & 5 & 3 & 1 \\
1 & 7 & 5 & 3 & 1 & 4 & 6 & 0 & 2 \\
2 & 6 & 4 & 2 & 0 & 5 & 7 & 1 & 3 \\
3 & 2 & 0 & 6 & 4 & 1 & 3 & 5 & 7 \\
4 & 1 & 3 & 5 & 7 & 2 & 0 & 6 & 4 \\
5 & 3 & 1 & 7 & 5 & 0 & 2 & 4 & 6 \\
6 & 0 & 2 & 4 & 6 & 3 & 1 & 7 & 5 \\
7 & 5 & 7 & 1 & 3 & 6 & 4 & 2 & 0 \\
\end{align*}
\]

Note that the above quasigroup is only permutation of some rows in the quasigroup constructed in Example 4.

The left circulant matrix corresponding to (11) is
\[
\begin{pmatrix}
a^2 + 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & a^2 + 1 \\
0 & 1 & 0 & 0 & 0 & a^2 + 1 & 1 \\
1 & 0 & 0 & 0 & a^2 + 1 & 1 & 0 \\
0 & 0 & 0 & a^2 + 1 & 1 & 0 & 1 \\
0 & 0 & a^2 + 1 & 1 & 0 & 1 & 0 \\
0 & a^2 + 1 & 1 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]
whose rank is 6. So, Equation (11) has one solution in \( F_{23} \). The element 2, that is, \( a \) satisfy the Equation (11). Hence, the constructed quasigroup, \((Q, \ast)\), has one idempotent subquasigroup \((\{2\}, \ast)\).

**Remark 4** All the quasigroups constructed by varying the constant \( c \in F_q^* \) and integer \( m \) while keeping the generator \( \beta \) of \( F_q^* \) same in Equation (10), belong to the same isotopic class.

In connection with a consideration of subsquasigroups it is necessary to mention the following.
Theorem 49 (Kepka (1978)) A countable quasigroup with at least three members is necessarily isotopic to a quasigroup which has no proper subquasigroups.

Consider associative triples \(x, y, z\) in the quasigroup \(Q = \{ x | x \in \mathbb{F}_q \}\) with multiplication \([5]\) from Corollary \([48]\). It means that \(x * y * z = x * (y * z)\). Then we have

\[
(1 - \beta) ((1 - \beta) x^m + \beta y + c)^m + \beta z + c =
(1 - \beta) x^m + \beta ((1 - \beta) y^m + \beta z + c) + c.
\]

\[
\beta (1 - \beta) z = (1 - \beta) x^m + \beta (1 - \beta) y^m - (1 - \beta) ((1 - \beta) x^m + \beta y + c)^m + \beta c
\]

Since \(\beta (1 - \beta) \neq 0\) we obtain,

\[
z = \frac{x^m}{\beta} + y^m - \frac{(1 - \beta) x^m + \beta y + c}{\beta} + \frac{c}{1 - \beta}.
\]

So the pair \(x, y\) uniquely determines \(z\). Hence we prove the following result.

Theorem 410 The number of associative triples in the quasigroup \(Q\) constructed from Corollary \([48]\) is equal to \(q^2 = |Q|^2\).

5 Conclusion

Identifying the polynomially complete quasigroups with no proper subquasigroup is essential for the design and analysis of the quasigroup based crypto primitives and schemes. In this context, we have developed an effective algorithm that concludes if a given quasigroup has no proper subquasigroup, otherwise lists its all proper subquasigroups. We have constructed a class of quasigroups of order \(p^r\) by defining a binary operation based on arithmetic of finite fields. Further, we have given the criteria under which the quasigroups of this class are polynomially complete and have no proper subquasigroup. This work provides an effective technique to construct cryptographically suitable quasigroups. We have also demonstrated the effectiveness of our methods by their implementations in Singular on some examples.

In the future work, we aim to construct cryptographic S-boxes by using quasigroups of class presented herein, and then analyze their cryptographic properties with respect to the S-boxes based on arbitrarily chosen quasigroups. Moreover, we will also compare the cryptographic properties of these S-boxes with the benchmark S-boxes of AES \(\text{Daemen and Rijmen} (2002)\).

6 Acknowledgement

We are thankful to Ms. Anu Khosla, Director SAG, DRDO and Dr. Sudhir Kamath, DG, MED&CoS, DRDO for their supports and encouragements to carry out this collaborative research work. Authors are also thankful to all the team members of Indo-Russian joint project QGSEC for their technical supports and scientific discussions.
References

Artamonov VA (2012) Polynomially complete algebras. Series Natural Tech Med Sci 6(2):23–29
Artamonov VA (2020) Automorphisms of finite quasigroups with no proper no subquasigroups. Vestnik StPetersbourg university, Mathematics, Mechanics, Astronomy
Artamonov VA, Chakrabarti S, Gangopadhyay S, Pal SK (2013) On Latin squares of polynomially complete quasigroups and quasigroups generated by shifts. Quasigroups Relat Syst 21(2):117–130
Artamonov VA, Chakrabarti S, Pal SK (2016) Characterization of polynomially complete quasigroups based on Latin squares for cryptographic transformations. Discrete Appl Math 200:5–17
Artamonov VA, Chakrabarti S, Pal SK (2017) Characterizations of highly nonassociative quasigroups and associative triples. Quasigroups Relat Syst 25(1):1–19
Belyavskaya GB (1989) T-quasi-groups and the center of a quasi-group. Mat Issled 111:24–43
Belyavskaya GB (1994) Abelian quasigroups are T-quasigroups. Quasigroups Relat Syst 1(1):1–7
Belyavskaya GB, Tabarov AK (1992) Characteristic of linear and alinear quasigroups. Diskretn Mat 4(2):142–147
Daemen J, Rijmen V (2002) The Design of Rijndael: AES - The Advanced Encryption Standard (Information Security and Cryptography). 1st edn. Springer
Decker W, Greuel GM, Pfister G, Schönemann H (2019) SINGULAR 4-1-2 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de
Dénès J, Keedwell AD (eds) (1991) Latin squares. New developments in the theory and applications., vol 46. Amsterdam etc.: North-Holland
Dimitrova V, Markovski J (2004) On quasigroup pseudo random sequence generator. In: Proc. of the 1-st Balkan Conference in Informatics, Thessaloniki, pp 393–401
Galatentko AV, Pankrat’ev AE, Rodin SB (2018) Polynomially complete quasigroups of prime order. Algebra Logica 57(5):327–335
Grošek O, Horák P (2012) On quasigroups with few associative triples. Des Codes Cryptography 64(1-2):221–227
Hagemann J, Herrmann C (1982) Arithmetical locally equational classes and representation of partial functions. Universal algebra, Proc. Colloq., Esztergom/Hung. 1977, Colloq. Math. Soc. Janos Bolyai 29, 345-360 (1982).
Horváth G, Nehaniv CL, Szabó C (2008) An assertion concerning functionally complete algebras and NP-completeness. Theor Comput Sci 407(1-3):591–595
Keedwell AD, Dénes J (2015) Latin squares and their applications. 2nd ed., 2nd edn. Amsterdam: Elsevier
Kepka T (1978) A note on simple quasigroups. Acta Univ Carol, Math Phys 19(2):59–60
Kepka T, Nemec P (1971) Affine quasigroups. Acta Univ Carol, Math Phys 12(1):39–49
Lidl R, Niederreiter H (1996) Finite fields. 2nd ed., vol 20, 2nd edn. Cambridge: Cambridge Univ. Press

Markovski S, Gligoroski D, Andova S (1997) Using quasigroups for one-one secure encoding. In: Proceedings of the VIII international conference on logic and computer science: theoretical foundations of computer science, Lira ’97, Novi Sad, Yugoslavia, September 1–4, 1997, Novi Sad: Univ. of Novi Sad, Faculty of Science, Institute of Mathematics, pp 157–162

Markovski S, Gligoroski D, Bakeva V (1999) Quasigroup string processing-part 1. Contributions, Sec Math Tech Sci MANU XX:13–28

Phillips J, Smith J (1999) Quasiprimitivity and quasigroups. Bulletin of the Australian Mathematical Society 59(3):473–475, DOI 10.1017/S0004972700033165

Shcherbacov VA (2017) Elements of quasigroup theory and applications. Boca Raton, FL: CRC Press

Smith JDH (2007) An introduction to quasigroups and their representations. Boca Raton, FL: Chapman & Hall/CRC

Wall W, Drury W (1957) Subquasigroups of finite quasigroup. Pacific J of Mathematics 7(4):1711–1714