REAL ALGEBRAIC THREEFOLDS I.
TERMINAL SINGULARITIES

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1. INTRODUCTION

In real algebraic geometry, considerable attention has been paid to the study of real algebraic curves (in connection with Hilbert’s 16th problem) and also to real algebraic surfaces. See [Viro90], [Riesler93] and the references there.

In higher dimensions one of the main avenues of investigation was initiated by [Nash52], and later developed by many others (see [AK92] for some recent directions). One of these results says that every compact differentiable manifold can be realized as the set of real points of an algebraic variety. [Nash52] posed the problem of obtaining similar results using a restricted class of varieties, for instance rational varieties.

The aim of this series of papers is to develop the theory of minimal models for real algebraic threefolds. This approach gives very strong information about the topology of real algebraic threefolds, and it also answers the above mentioned question of [Nash52].

For algebraic threefolds over $\mathbb{C}$, the minimal model program provides a very powerful tool. The method of the program is the following. (See [Kollár87] or [CKM88] for introductions)

Starting with a smooth projective variety $X$, we perform a series of “elementary” birational transformations

$$X = X_0 \dasharrow X_1 \dasharrow \cdots \dasharrow X_n$$
until we reach a variety $X_n$ whose global structure is “simple”. In essence the minimal model program allows us to investigate many questions in two steps: first study the effect of the “elementary” transformations and then consider the “simple” global situation.

In practice both of these steps are frequently rather difficult. For instance, we still do not have a complete list of all possible “elementary” steps, despite repeated attempts to obtain it.

A somewhat unpleasant feature of the theory is that the varieties $X_i$ are not smooth, but have so-called terminal singularities. In developing the theory of minimal models for real algebraic threefolds, we again have to understand the occurring terminal singularities.

The aim of this paper is to give a classification of terminal 3-fold singularities over $\mathbb{R}$. Minimal models serve only as a background, the proofs depend entirely on well-established methods of singularity theory. I do not even use the definition of terminal singularities!

Terminal 3-fold singularities over $\mathbb{C}$ are completely classified. [Reid85] is a very readable introduction and survey. I will take the result of this classification as my definition, since the theory over $\mathbb{R}$ can be most naturally developed in this setting.

The classification is, in some sense, not complete. In a few cases I obtain unique normal forms (4.3), but in most cases this seems nearly impossible (see [Markush85] for a special case over $\mathbb{C}$). My aim is to write the singularities in a form that allows one to determine their topology over $\mathbb{R}$. The resulting lists and algorithms are given in sections 4–5.

It turns out that the normal forms of 3-fold terminal singularities are essentially the same over any field of characteristic zero. Thus in sections 2–3 I work with any subfield of $\mathbb{C}$.

As a consequence of the classification over $\mathbb{C}$, we know that 3-fold terminal singularities come in two types. Some are hypersurface singularities, and the others are quotients of these hypersurface singularities by a finite cyclic group. Accordingly, the classification over any field is done in two steps. Section 2 deals with terminal hypersurface singularities. These results are mostly routine generalizations of the theory over $\mathbb{C}$.

Quotient singularities frequently have “twisted” forms over a subfield of $\mathbb{C}$. “Twisted” forms do not appear for 3-fold terminal singularities, and so the classification ends up very similar to the one over $\mathbb{C}$.

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2. Terminal hypersurface singularities

**Notation 2.1.** For a field $K$ let $K[[x_1, \ldots, x_n]]$ denote the ring of formal power series in $n$ variables over $K$. For $K = \mathbb{R}$ or $K = \mathbb{C}$, let $K\{x_1, \ldots, x_n\}$ denote the ring of those formal power series which converge in some neighborhood of the origin.

For any $F \in K\{x_1, \ldots, x_n\}$ the set $(F = 0)$ is a germ of a real or complex analytic set. I will refer to it as a singularity. If $F \in K[[x_1, \ldots, x_n]]$ then by the singularity $(F = 0)$ I mean the scheme $\text{Spec}_K K[[x_1, \ldots, x_n]]/(F)$.

For a power series $F$, $F_d$ denotes the degree $d$ homogeneous part. The multiplicity, denoted by $\text{mult}_0 F$, is the smallest $d$ such that $F_d \neq 0$. If we write a power series as $F \geq d$ then it is assumed that its multiplicity is at least $d$.

Two power series $F, G \in K[[x_1, \ldots, x_n]]$ are called equivalent over $K$ if there is an automorphism of $K[[x_1, \ldots, x_n]]$ given by $x_i \mapsto \phi_i(x_1, \ldots, x_n) \in K[[x_1, \ldots, x_n]]$ and an invertible $u(x_1, \ldots, x_n) \in K[[x_1, \ldots, x_n]]$ such that $u(x_1, \ldots, x_n)G(x_1, \ldots, x_n) = F(\phi_1, \ldots, \phi_n)$.

Thus $F$ and $G$ are equivalent iff the corresponding singularities $(F = 0)$ and $(G = 0)$ are isomorphic (over $K$).

We have to pay special attention to cases when $F$ and $G$ are not equivalent over $K$ but are equivalent over some larger field. For instance, $F = x_1^2 + x_2^2$ and $G = x_1^2 - x_2^2$ are not equivalent over $\mathbb{R}$ but are equivalent over $\mathbb{C}$.

If $K = \mathbb{R}, \mathbb{C}$ and $F, G \in K\{x_1, \ldots, x_n\}$ then I am mainly interested in equivalences where $u, \phi_i \in K\{x_1, \ldots, x_n\}$.

If $F, G \in K\{x_1, \ldots, x_n\}$ have isolated critical points at the origin, then $F$ and $G$ are equivalent in $K\{x_1, \ldots, x_n\}$ iff they are equivalent in $K[[x_1, \ldots, x_n]]$ (cf. [AGV85, p.121]), thus we do not have to be careful about this distinction.

**Definition 2.2.** Let $K$ be a field of characteristic zero with algebraic closure $\bar{K}$. $(F(x, y, z) = 0)$ is called a Du Val singularity (or a rational double point) iff over $\bar{K}$ it is equivalent to one of the standard forms

- $A_n \quad x^2 + y^2 + z^{n+1} = 0$;
- $D_n \quad x^2 + y^2z + z^{n-1} = 0$;
- $E_6 \quad x^2 + y^3 + z^4 = 0$;
- $E_7 \quad x^2 + y^3 + yz^3 = 0$;
Du Val singularities have many interesting intrinsic characterizations, (cf. [Durfee79, Reid85]) but I will not use this.

The following definition introduces our basic objects of study.

**Definition 2.3.** Let $K$ be a field of characteristic zero with algebraic closure $\bar{K}$. $(F(x, y, z, t) = 0)$ is called a compound Du Val singularity (or $cDV$ for short) iff over $\bar{K}$ it is equivalent to

$$h(x, y, z) + tf(x, y, z, t) = 0$$

where $(h = 0)$ is a Du Val singularity.

$(F(x, y, z, t) = 0)$ is called a $cA_n$ (resp. $cD_n$ or $cE_n$) singularity if its equation can be written as above with $h$ having type $A_n$ (resp. $D_n$ or $E_n$), but it does not admit such representation with a smaller value of $n$. It is called a $cA$ (resp. $cD$ or $cE$) singularity if the value of $n$ is not specified.

The reason we are interested in $cDV$ singularities is the following:

**Theorem 2.4.** [Reid80] A 3-dimensional hypersurface singularity over $\mathbb{C}$ is terminal iff it is an isolated $cDV$ singularity.

The aim of this section is to develop “normal forms” for $cDV$ singularities over any field $K$. This will then give “normal forms” for 3-dimensional terminal hypersurface singularities over $K$.

The proof is a rather standard application of the methods of [AGVS].

**2.5.** We repeatedly use 3 methods:

1. The Weierstrass preparation theorem. This is frequently stated only over $\mathbb{C}$, but it works over any field since the Weierstrass normal form is unique.
2. The elimination of the $y^{n-1}$-term from the polynomial $a_n y^n + a_{n-1} y^{n-1} + \ldots$ by a coordinate change $y \mapsto y - a_{n-1}/na_n$ when $a_n$ is invertible.
3. Let $M_1, \ldots, M_k$ be monomials in the variables $x_1, \ldots, x_m$. Assume that $x_0 M_1, \ldots, x_0 M_k$ are multiplicatively independent. Then any power series of the form $\sum M_i \cdot u_i(x_1, \ldots, x_m)$ where $u_i(0) \neq 0$ for all $i$ is equivalent to $\sum M_i \cdot u_i(0)$ by a suitable coordinate change $x_i \mapsto x_i \cdot (\text{unit})$.

These elementary operations are sufficient to deal with the $cA$ and $cE$ cases. In the $cD$ case the following generalization of (2.3.2) is needed.

**Construction 2.6.** In $K[[x_1, \ldots, x_m]]$, assign positive integral weights to the variables $w(x_i) = w_i$. For a monomial set $w(\prod x_i^{c_i}) = \sum c_i w_i$. 

$E_8 \quad x^2 + y^3 + z^5 = 0.$
Write a power series in terms of its weighted homogeneous pieces $F = F_d + F_{d+1} + \ldots$. Choose $g_i \in K[[x_1, \ldots, x_m]]$ such that $w(g_i) = w(x_i) + e$ for some $e > 0$. Then

$$F(x_i + g_i) = F(x_i) + \sum g_i \frac{\partial F_d}{\partial x_i} + R_{> (d+e)}(x_i).$$

Repeatedly using this for higher and higher degrees, we see that, for every $N > 0$, $F$ is equivalent to a power series $F^N + R^N$ where $F^N$ is a polynomial of degree $N$ and no linear combination of the monomials in $F^N$ can be written in the form $\sum g_i(\partial F_d/\partial x_i)$ as above.

In the ring of formal power series this can be continued indefinitely, thus at the end we can kill all the degree $> d$ elements of the Jacobian ideal

$$\Delta(F_d) := \left( \frac{\partial F_d}{\partial x_1}, \ldots, \frac{\partial F_d}{\partial x_m} \right).$$

If $F \in K \{x_1, \ldots, x_m\}$ defines an isolated singularity, then by Tougeron’s lemma (cf. [AGV85, p.121]), $F^N + R^N$ is equivalent to $F^N$ by an analytic coordinate change for $N \gg 1$. Thus the final conclusion is the same.

**Proposition 2.7.** Any power series $F_{\geq 2}(x_1, \ldots, x_n)$ is equivalent to a power series

$$a_1 x_1^2 + \cdots + a_k x_k^2 + G_{\geq 3}(x_{k+1}, \ldots, x_n).$$

**Proof.** By a linear change of coordinates we can diagonalize $F_2$, thus we can assume that $F_2 = a_1 x_1^2 + \cdots + a_k x_k^2$. Repeatedly applying (2.5.1) to the variables $x_1, \ldots, x_k$ we reach a situation when $F$ is a quadratic polynomial in the variables $x_1, \ldots, x_k$. (2.5.2) can then be used to eliminate the linear terms in $x_1, \ldots, x_k$. □

**Theorem 2.8.** Assume that $F_{> 1}(x, y, z, t) \in K[[x, y, z, t]]$ defines a terminal singularity of type $cA$. Then $F$ is equivalent to one of the following:

- $cA_0$ $x = 0.$
- $cA_1$ $ax^2 + by^2 + cz^2 + dt^m = 0,$ where $abcd \neq 0.$
- $cA_{> 1}$ $ax^2 + by^2 + f_{> 3}(z, t) = 0,$ where $ab \neq 0.$ This has type $cA_n$ for $n = \text{mult}_0 f - 1.$

**Proof.** If $F_1 \neq 0$ then (2.5.1) gives $cA_0$. Thus assume that $F_1 = 0$. $F$ has type $cA$, hence $F_2$ is a quadric of rank at least 2. If the rank is 2 then (2.7) gives the $cA_{> 1}$ cases.

Assume finally that $F_2$ has rank 3 or 4. By (2.7) we can write $F$ as $ax^2 + by^2 + cz^2 + g(t) = 0$. Using (2.5.3) we obtain $ax^2 + by^2 + cz^2 + dt^m = 0.$
In all these cases we can multiply through by $a^{-1}$ to get a somewhat simpler form when the coefficient of $x^2$ is 1.

**Theorem 2.9.** Assume that $F_{\geq 2}(x, y, z, t) \in K[[x, y, z, t]]$ defines a terminal singularity of type $cD$. Then $F$ is equivalent to one of the following:

$cD_4 \quad x^2 + f_{\geq 3}(y, z, t)$, where $f_3$ is not divisible by the square of a linear form.

$cD_{\geq 4} \quad x^2 + y^2z + ayt^r + h_{\geq s}(z, t)$, where $a \in K$, $r \geq 3$, $s \geq 4$ and $h_s \neq 0$. This has type $cD_n$ where $n = \min\{2r, s + 1\}$ if $a \neq 0$ and $n = s + 1$ if $a = 0$.

Proof. $F_2$ is a rank one quadric, thus in suitable coordinates the equation becomes $ax^2 + f_{\geq 3}(y, z, t)$. Here $f_3 \neq 0$ is not the cube of a linear form since otherwise we would have a type $cE$ singularity. If $f_3$ is not divisible by the square of a linear form then we have case $cD_4$.

If $f_3$ is divisible by the square of a linear form, then $f_3 = l_1^2l_2$ for two linear forms $l_i$, and both of them are defined over $K$. We can change coordinates $l_1 \mapsto y$ and $l_2 \mapsto z$.

At this point our power series is $x^2 + y^2z + (\text{higher order terms})$. Assign weights $w(x) = 3, w(y) = w(z) = 2, w(t) = 6$. The leading term is $x^2 + y^2z$. Using (2.9) we can eliminate all monomials which contain $y^2$ or $yz$.

To see the last part, take the hyperplane section $t = \lambda z$. The term $ay\lambda^rz^r$ can be eliminated by a substitution $y \mapsto y + (a/2)\lambda z^{-1}$. This creates a term $-(a/2)^2\lambda z^{2r-1}$. The only problem could be that $h(z, \lambda z)$ has multiplicity $2r - 1$ and there is cancellation. However, $h_{2r-1}(z, \lambda z) = z^{2r-1}h_{2r-1}(1, \lambda)$ is a polynomial of degree $2r - 1$ in $\lambda$, thus it does not equal $-(a/2)^2\lambda z^{2r-1}$.

**Theorem 2.10.** Assume that $F_{\geq 2}(x, y, z, t) \in K[[x, y, z, t]]$ defines a terminal singularity of type $cE$. Then $F$ is equivalent to one of the following:

$cE_6 \quad x^2 + y^3 + yg_{\geq 3}(z, t) + h_{\geq 4}(z, t)$, where $h_4 \neq 0$.

$cE_7 \quad x^2 + y^3 + yg_{\geq 3}(z, t) + h_{\geq 5}(z, t)$, where $g_3 \neq 0$.

$cE_8 \quad x^2 + y^3 + yg_{\geq 4}(z, t) + h_{\geq 5}(z, t)$, where $h_5 \neq 0$.

Proof. $F_2$ is a rank one quadric by (2.3), thus in suitable coordinates the equation becomes $ax^2 + f_{\geq 3}(y, z, t)$. Here $f_3 \neq 0$ and it is the cube of a linear form since otherwise we would have a type $cD$ singularity. (2.7–1, 2) gives an equation

$$ax^2 + by^3 + yg_{\geq 3}(z, t) + h_{\geq 4}(z, t).$$
Multiply the equation by $a^3b^2$ and then make the substitutions $x \mapsto xa^{-2}b^{-1}$ and $y \mapsto ya^{-1}b^{-1}$ to get the required normal forms.

### 3. Higher Index Terminal Singularities

The classification of non-hypersurface terminal 3-fold singularities over $\mathbb{C}$ relies on the following construction:

Let $\mathbb{Z}_n$ denote the cyclic group of order $n$ and $\epsilon$ a primitive $n^{th}$ root of unity. Assume that $\mathbb{Z}_n$ acts on $\mathbb{C}^4$ by

$$\sigma : (x, y, z, t) \mapsto (\epsilon^{az}x, \epsilon^{ay}y, \epsilon^{az}z, \epsilon^{at}t).$$

I will use the shorter notation $\frac{1}{n}(a_x, a_y, a_z, a_t)$ to denote such an action.

If $F(x, y, z, t)$ is equivariant with respect to this action, then $\mathbb{Z}_n$ acts on the hypersurface $(F = 0)$ and we can take the quotient, denoted by $(F = 0)/\frac{1}{n}(a_x, a_y, a_z, a_t)$.

By [Reid80], every terminal 3-fold singularity $X$ over $\mathbb{C}$ is of the form $(F = 0)/\frac{1}{n}(a_x, a_y, a_z, a_t)$, where $F$ defines a terminal hypersurface singularity. The value of $n$ is uniquely determined by $X$, it is called the index of $X$.

It is not easy to come up with a complete list of terminal 3-fold singularities, but by now the list is well understood; see [Reid85] for a good survey. It turns out that most actions do not produce terminal quotients and we have only a few cases:

**Theorem 3.1.** [Mori85] Let $0 \in X$ be a 3-fold terminal nonhypersurface singularity over $\mathbb{C}$. Then $0 \in X$ is isomorphic to a singularity described by the following list:

| name   | equation       | index | action | condition        |
|--------|----------------|-------|--------|------------------|
| $cA/n$ | $xy + f(z, t)$ | $n$   | $(r, -r, 1, 0)$ | $(n, r) = 1$   |
| $cAx/2$ | $x^2 + y^2 + f_{\geq 4}(z, t)$ | 2 | $(0, 1, 1, 1)$ |         |
| $cAx/4$ | $x^4 + y^4 + f_{\geq 2}(z, t)$ | 4 | $(1, 3, 1, 2)$ | $f_2(0, 1) = 0$ |
| $cD/2$  | $x^2 + f_{\geq 3}(y, z, t)$ | 2 | $(1, 0, 1, 1)$ |         |
| $cD/3$  | $x^2 + f_{\geq 3}(y, z, t)$ | 3 | $(0, 2, 1, 1)$ | $f_3(1, 0, 0) \neq 0$ |
| $cE/2$  | $x^2 + y^2 + f_{\geq 4}(y, z, t)$ | 2 | $(1, 0, 1, 1)$ |         |

The equations have to satisfy 2 obvious conditions:

1. The equations define a terminal hypersurface singularity.
2. The equations are $\mathbb{Z}_n$-equivariant. (In fact $\mathbb{Z}_n$-invariant, except for $cAx/4$.)

If we work over a field $K$ which does not contain the $n^{th}$ roots of unity, then the action $\frac{1}{n}(a_1, \ldots, a_m)$ is not defined over $K$. There is, however, another way of looking at the quotient which does make sense over any field.
Any action of the cyclic group $\mathbb{Z}_n$ on $\mathbb{C}^m$ defines a $\mathbb{Z}_n$-grading $w$ of $\mathbb{C}[[x_1, \ldots, x_m]]$ by

$$w(\prod x_i^{c_i}) = a \text{ iff } \sigma(\prod x_i^{c_i}) = e^a \cdot \prod x_i^{c_i}.$$ 

If $F$ is $\mathbb{Z}_n$-equivariant then $(F) \subset \mathbb{C}[[x_1, \ldots, x_m]]$ is a homogeneous ideal, hence the grading descends to a grading of $\mathbb{C}[[x_1, \ldots, x_m]]/(F)$. The ring of functions on the quotient $(F = 0)/\mathbb{C}^m(a_1, \ldots, a_m)$ can be identified with the ring of grade zero elements of $\mathbb{C}[[x_1, \ldots, x_m]]/(F)$.

If $K$ is any field, $n \in \mathbb{N}$ and $a_i \in \mathbb{Z}$, then we obtain a $\mathbb{Z}_n$-grading $w = w(a_1, \ldots, a_m)$ of $K[[x_1, \ldots, x_m]]$ (or of $\mathbb{R}\{x_1, \ldots, x_m\}$) by

$$w(\prod x_i^{c_i}) = \sum c_i a_i \in \mathbb{Z}_n.$$ 

Let $R \subset K[[x_1, \ldots, x_m]]$ denote the subring of grade zero elements. Then $\text{Spec}_K R$ gives a singularity over $K$ which is denoted by

$$\mathbb{A}^m_j(\mathbb{R})^n(a_1, \ldots, a_m).$$

(Especially when $K = \mathbb{R}$, one might be tempted to write $\mathbb{R}^m/\mathbb{Z}_n(a_1, \ldots, a_m)$ instead. However, the set of real points of $\mathbb{A}^m_j\mathbb{R}/\mathbb{Z}_n(a_1, \ldots, a_m)$ is not in any sense a quotient of the set $\mathbb{R}^n$ (cf. (5.3)), so this may lead to confusion.)

If $F \in K[[x_1, \ldots, x_m]]$ is graded homogeneous, then $w$ gives a grading of $K[[x_1, \ldots, x_m]]/(F)$. Let $R/(R \cap (F)) \subset K[[x_1, \ldots, x_m]]/(F)$ be the subring of grade zero elements. $\text{Spec}_K R/(R \cap (F))$ defines a singularity over $K$. By construction,

$$\text{Spec}_K R/(R \cap (F)) \times_{\text{Spec}_K \mathbb{K}} \text{Spec} \mathbb{K} \cong (F = 0)/\mathbb{Z}_n(a_1, \ldots, a_m).$$

Thus $\text{Spec}_K R$ is a terminal singularity over $K$ iff $(F = 0)/\mathbb{Z}_n(a_1, \ldots, a_m)$ is a terminal singularity over $\mathbb{K}$.

Under certain conditions, every $K$-form of a quotient is obtained this way:

**Theorem 3.2.** $K$ be a field of characteristic zero with algebraic closure $\mathbb{K}$. Let $\mathbb{Z}_n$ denote the cyclic group of order $n$ and $e$ a primitive $n$th root of unity. Assume that $\mathbb{Z}_n$ acts on $\mathbb{K}^m$ by $\sigma : (x_i) \mapsto (e^{w_i}x_i)$. Let $F \in \mathbb{K}[[x_1, \ldots, x_m]]$ be equivariant with respect to this action, and assume that the fixed point set of $\sigma$ has codimension at least 2 in $(F = 0)$. Assume in addition that

$$w(F) - \sum w_i \text{ is relatively prime to } n.$$ 

Let $0 \in X$ be a singularity over $K$ such that

$$X \times_{\text{Spec}_K \mathbb{K}} \text{Spec} \mathbb{K} \cong (F = 0)/\mathbb{Z}_n(w_1, \ldots, w_m).$$
Then there is an $F^K \in K[[x_1, \ldots, x_m]]$ such that $F$ and $F^K$ are equivalent over $K$ and

$$X \cong (F^K = 0)/\frac{1}{n}(w_1, \ldots, w_m).$$

It is worthwhile to note that the condition about $n$ and $w(F) - \sum w_i$ being relatively prime is essential:

**Example 3.3.** Consider the quotient singularity $\mathbb{C}[u, v]/\frac{1}{n}(1, -1)$. It is isomorphic to $(xy - z^n = 0)$ via the substitutions $x = u^n, y = v^n, z = uv$. Over $\mathbb{C}$ we have a Du Val singularity $A_{n-1} = (x^2 + y^2 + z^n = 0)$.

Over $\mathbb{R}$ we see that $(x^2 - y^2 - z^n = 0) \cong \mathbb{A}^2/\frac{1}{n}(1, -1)$. Another $\mathbb{R}$-form of $A_{n-1}$ is $x^2 + y^2 - z^n$. This can also be obtained as a quotient, but this time we act on $\mathbb{A}^2$ by rotation with angle $2\pi/n$.

Finally, if $n$ is even, then there is another $\mathbb{R}$-form of $A_{n-1}$ given by $(x^2 + y^2 + z^n = 0)$. The only $\mathbb{R}$-point is the origin, so we do not even have a nonzero map $\mathbb{R}^2 \to (x^2 + y^2 + z^n = 0)$.

As another example, take the 4-dimensional terminal singularity $\mathbb{C}^4/\frac{1}{n}(a, -a, b, -b)$ for any $(ab, n) = 1$. It has another $\mathbb{R}$-form given as $\mathbb{A}^4/\mathbb{Z}_n$ where we act on the first two coordinates by rotation with angle $2a\pi/n$ and on the last two coordinates by rotation with angle $2b\pi/n$.

In some special cases there are further $\mathbb{R}$-forms. Take for instance $\mathbb{C}^4/\frac{1}{2}(1, 1, 1, 1)$. This can be realized as the cone over $\mathbb{CP}^3$ embedded by the quadrics to $\mathbb{CP}^9$.

Let $C \subset \mathbb{RP}^2$ be a smooth conic. Taking symmetric powers we have $S^3C \subset S^3\mathbb{RP}^2$ and $S^3H^0(\mathbb{RP}^2, \mathcal{O}(1))$ embeds it to $\mathbb{RP}^9$. If $C$ has a real point, then $S^3C \cong \mathbb{RP}^3$ and we get the Veronese embedding. If $C$ has no real points then the image is a variety over $\mathbb{R}$ without real points. The cone over it is a real form of $\mathbb{A}^4/\frac{1}{2}(1, 1, 1, 1)$ with an isolated real point at the origin.

Proof of (3.2). Set $S = \tilde{K}[[x_1, \ldots, x_m]]/(F)$, the ring of functions on $\tilde{X}_K := (F = 0)$. The $\mathbb{Z}_n$-action defines a $\mathbb{Z}_n$-grading $S = \sum_{i=0}^{n-1} S_i$. $S_0$, the ring of grade 0 elements, is exactly the ring of functions on $X_{\tilde{K}}$. Our aim is to find an algebraic way of reconstructing $S$ from $S_0$, which then hopefully generalizes to nonclosed fields.

There is another summand which can be easily seen algebraically. Set $d = w(F) - \sum w_i$. Note that

$$\frac{1}{\partial F/\partial x_m}dx_1 \wedge \cdots \wedge dx_{m-1}$$
is a local generator of $\omega_S$ and it has weight $-d$. Thus

$$\omega_{S_0} \cong S_d \frac{1}{\partial F/\partial x} dx_1 \wedge \cdots \wedge dx_{m-1}.$$ 

Once $S_d$ is determined, we obtain $S_{jd}$ as follows. The multiplication map

$$S_a \otimes_{S_0} S_b \to S_{a+b} \quad \text{(subscripts modulo $n$)}$$

are isomorphisms over the open set where the $\mathbb{Z}_n$-action is free. We assumed that the complement has codimension at least 2, thus $S_{jd} \cong S_{d}^{[j]}$, where $S_{d}^{[j]}$ denotes the double dual of $S_{d}^{\otimes j}$. If $d$ and $n$ are relatively prime, then we obtain every summand $S_i$ this way. In particular,

$$S = \sum_{i=0}^{n-1} S_i \cong \sum_{j=0}^{n-1} \omega_{S_0}^{[j]}.$$ 

Over an arbitrary field, we can thus proceed as follows. Let $\omega_X$ be the dualizing sheaf of $X$. This is also the reflexive sheaf $\mathcal{O}_X(K_X)$ where $K_X$ is the canonical class.

Then $\omega_X^{[n]}$ is isomorphic to $\mathcal{O}_X$, where $n$ is the index. (We know this over $\bar{K}$. Isomorphism of two sheaves $F, G$ is a question about $\text{Hom}(F, G)$ and this commutes with base field extensions.) Fix such an isomorphism $s : \omega_X^{[n]} \to \mathcal{O}_X$.

Consider the $\mathcal{O}_X$-algebra

$$R(X, s) := \sum_{j=0}^{n-1} \omega_X^{[j]},$$

where multiplication for $j + k \geq n$ is given by

$$\omega_X^{[j]} \otimes \omega_X^{[k]} \mapsto \omega_X^{[j+k]} \cong \omega_X^{[n]} \otimes \omega_X^{[j+k-n]} \otimes 1 \omega_X^{[j+k-n]}.$$ 

This has a $\mathbb{Z}_n$ grading by declaring $\omega_X^{[j]}$ to have grade $j$.

(Note. Two isomorphisms $s_1, s_2 : \omega_X^{[n]} \to \mathcal{O}_X$ differ by an invertible function $h \in \mathcal{O}_X^*$. If $h$ is an $n^{th}$-power, then the resulting algebras $R(X, s_i)$ are isomorphic, but they need not be isomorphic otherwise. This is connected with the topological aspects observed in [3.3].)

Over $\bar{K}$, $R(X, s)$ is isomorphic to $\mathcal{O}_X$. Thus $R(X, s)$ is a $K$-form of $\mathcal{O}_X$. In particular, $R(X, s)$ is an algebra of the form $\bar{K}[[x_1, \ldots, x_m]]/(F^K)$, where $F$ and $F^K$ are equivalent over $\bar{K}$.

The grading lifts to a grading of $\bar{K}[[x_1, \ldots, x_m]]$ such that $F^K$ is graded homogeneous. We can choose $x_i$ to be homogeneous. \qed

As a corollary, we obtain the following classification of terminal 3-fold nonhypersurface singularities over nonclosed fields:
Theorem 3.4. Let $K$ be a field of characteristic zero and $0 \in X$ a 3-fold terminal nonhypersurface singularity over $K$. Then $0 \in X$ is isomorphic over $K$ to a singularity described by the following list:

| name   | equation                  | index | weights | condition |
|--------|---------------------------|-------|---------|-----------|
| $cA/2$ | $ax^2 + by^2 + f(z, t)$   | 2     | $(1, 1, 0)$ |
| $cA/n$ | $xy + f(z, t)$            | $n \geq 3$ | $(r, -r, 1, 0)$ | $(n, r) = 1$ |
| $cAx/2$ | $ax^2 + by^2 + f_{>4}(z, t)$ | 2     | $(0, 1, 1, 1)$ |
| $cAx/4$ | $ax^2 + by^2 + f_{>2}(z, t)$ | 4     | $(1, 3, 1, 2)$ | $f_2(0, 1) = 0$ |
| $cD/2$ | $x^2 + f_{>3}(y, z, t)$   | 2     | $(1, 0, 1, 1)$ |
| $cD/3$ | $x^2 + f_{>3}(y, z, t)$   | 3     | $(0, 2, 1, 1)$ | $f_3(1, 0, 0) \neq 0$ |
| $cE/2$ | $x^2 + y^2 + f_{>4}(y, z, t)$ | 2     | $(1, 0, 1, 1)$ |

Complement 3.5. The corresponding quotient singularity is terminal iff the equations satisfy 2 obvious conditions:

1. The equations define a terminal hypersurface singularity.
2. The equations are graded homogeneous.

With these assumptions, a terminal singularity corresponds to exactly one case on the above list.

Proof. By looking at the list of (3.1), we see that the assumptions of (3.2) are satisfied. Hence we know that $X$ is of the form $(F^K = 0)/\frac{1}{n}(a_x, a_y, a_z, a_t)$ where $\frac{1}{n}(a_x, a_y, a_z, a_t)$ is on the list of (3.1).

Once we know a $\mathbb{Z}_n$-grading on $K[[x, y, z, t]]$ and a graded homogeneous power series $F^K$, we can try to bring it to some normal form using the methods (2.5) and (2.6). They are set up in such a way that if $F^K$ is homogeneous in a $\mathbb{Z}_n$-grading the all coordinate changes respect the grading.

The proofs of (2.3, 2.9, 2.10) remain unchanged. The only difference is in (2.7). It is not true that a quadratic form can be diagonalized using a linear transformation which respects the $\mathbb{Z}_n$-grading. The best one can achieve is a sum of forms in disjoint sets of variables $\sum q_i$ where each $q_i$ is either $au^2_i$ or $u_iv_i$. The latter case is necessary iff the two variables have different $\mathbb{Z}_n$-grading.

In the $cD$ and $cE$ cases the quadric has rank 1, so it can be diagonalized.

In the $cA/2$ and $cAx/2$ cases every grade 0 quadratic can be diagonalized.

In the $cAx/4$ case $x^2, xz, y^2, z^2$ are the only grade 2 quadratic monomials. A quadratic form like this can again be diagonalized.

Finally let us look at the $cA/n$-case for $n \geq 3$. The only grade 0 degree 2 monomials are $xy, t^2$ and $xz$ if $r = -1$ or $yz$ if $r = 1$. We need to get a rank $\geq 2$ quadratic, so $xy$ (or $xz$ if $r = -1$, $yz$ if $r = 1$) must
appear. In the $r = \pm 1$ case we may need to perform a linear change of variables to get the normal form $xy + f(z, t)$. □

4. The topology of terminal hypersurface singularities

Let $0 \in X$ be a real singularity. It’s real points $X(\mathbb{R})$ form a topological space, which can be triangulated (cf. [BCR87, 9.2]). We may assume that $0$ is a vertex of the triangulation. Then locally near $0$, $X(\mathbb{R})$ is PL-homeomorphic to the cone over a simplicial complex $L = L(X(\mathbb{R}))$, which is called the link of $0$ in $X(\mathbb{R})$. The local topology of $X(\mathbb{R})$ at $0$ is thus determined by $L$.

In general one needs to contemplate the dependence of $L$ on various choices made. I am mainly interested in the case when $X$ is a 3-dimensional isolated singularity. In this case $L$ is a compact surface (without boundary) and so $L$ and $X(\mathbb{R})$ determine each other up to homeomorphism.

The aim of this section is to classify terminal singularities over $\mathbb{R}$ according to their local topology. To be precise, we give a classification in the $cA$ cases and provide a procedure in the $cD$ and $cE$ cases which reduces the 3-dimensional problem to some questions about plane curve singularities.

Notation 4.1. $M \sim N$ denotes that $M$ and $N$ are homeomorphic.

$\sqcup$ denotes disjoint union. $M \sqcup rN$ denotes the disjoint union of $M$ and of $r$ copies of $N$.

$M_g$ denotes the unique compact, closed and orientable surface of genus $g$.

We start with a general lemma.

Lemma 4.2. Let $X$ be a smooth real hypersurface. Then $X(\mathbb{R})$ is orientable.

Proof. Let $X = (f = 0)$ be a real equation where $f \in \mathbb{R}[x_1, \ldots, x_n]$ or $f \in \mathbb{R}\{x_1, \ldots, x_n\}$. At each point $p \in X$, $X$ divides a neighborhood of $p$ into two halves. $f$ is positive on one half and negative on the other half. Choosing a sign thus determines an orientation. □

Theorem 4.3. The following table gives a complete list of 3-dimensional terminal singularities of type $cA_1$ over $\mathbb{R}$.

In the table $n \geq 1$. Case 4, $n = 1$ and case 5, $n = 1$ are isomorphic. Aside from this, two singularities are isomorphic iff they correspond to
the same case and the same value of $n$.

| case   | equation                       | $L$          |
|--------|--------------------------------|--------------|
| $cA_1(1)$ | $x^2 + y^2 + z^2 \pm t^{2n+1}$ | $S^2$        |
| $cA_1(2)$ | $x^2 + y^2 - z^2 \pm t^{2n+1}$ | $S^2$        |
| $cA_1(3)$ | $x^2 + y^2 + z'^2 + t^{2n}$   | $\emptyset$ |
| $cA_1(4)$ | $x^2 + y^2 + z^2 - t^{2n}$    | $S^2 \cup S^2$ |
| $cA_1(5)$ | $x^2 + y^2 - z^2 + t^{2n}$    | $S^2 \cup S^2$ |
| $cA_1(6)$ | $x^2 + y^2 - z^2 - t^{2n}$    | $S^1 \times S^1$ |

Proof. The equations follow from (2.8), once we note that after multiplying by $\pm 1$ we may assume that the quadratic part has at least 2 positive eigenvalues.

The topology is easy to figure out. Since all the claims are special cases of the next result, I discuss them in more detail there.

**Theorem 4.4.** A 3-dimensional terminal singularity of type $cA > 1$ over $\mathbb{R}$ is equivalent to a form

$$x^2 \pm y^2 \pm h(z, t) \prod_{i=1}^{m} f_i(z, t) = 0,$$

where the $f_i$ are irreducible power series (over $\mathbb{R}$) such that $(f_i(z, t) = 0)$ changes sign on $\mathbb{R}^2 \setminus \{0\}$ and $h(z, t)$ is positive on $\mathbb{R}^2 \setminus \{0\}$. The following table gives a complete list of the possibilities for the topology of $X(\mathbb{R})$.

| case   | equation                       | $L(X(\mathbb{R}))$ |
|--------|--------------------------------|--------------------|
| $cA^+_{>1}(0, +)$ | $x^2 + y^2 + h$    | $\emptyset$        |
| $cA^+_{>1}(0, -)$ | $x^2 + y^2 - h$    | $S^1 \times S^1$  |
| $cA^+_{>1}(m)$    | $x^2 + y^2 \pm hf_1 \cdots f_m$ | $\sqcup mS^2$ |
| $cA^-_{>1}(0)$    | $x^2 - y^2 \pm h$   | $S^2 \cup S^2$    |
| $cA^-_{>1}(m)$    | $x^2 - y^2 \pm hf_1 \cdots f_m$ | $M_{m-1}$ |

Proof. We already have the form $x^2 \pm y^2 + f(z, t)$ by (2.8). Write $f$ as a product of irreducible power series over $\mathbb{R}$. Those factors which do not vanish on $\mathbb{R}^2 \setminus \{0\}$ are multiplied together to get $h$. By writing $\pm h$ we may assume that $h$ is positive on $\mathbb{R}^2 \setminus \{0\}$. (Since the signs of the other factors are not fixed, the sign of $h$ matters only if there are no other factors.) Let $f_i$ be the remaining factors of $f$.

Assume now that we are in the $cA^+$-case: $x^2 + y^2 \pm h \prod f_i$. Projection to the $(z, t)$-plane is a proper map whose fibers are as follows:

1. $S^1$ if $\pm h(z, t) \prod f_i(z, t) < 0$,
2. a point if $\pm h(z, t) \prod f_i(z, t) = 0$,
3. empty if $\pm h(z, t) \prod f_i(z, t) > 0$. 


If \( m = 0 \) then \( X(\mathbb{R}) \setminus \{0\} \) is a circle bundle over either \( \mathbb{R}^2 \setminus \{0\} \) or over the empty set. The first case gives \( L \sim S^1 \times S^1 \) by (1.2).

If \( m > 0 \), we have to describe the semi-analytic set \( U := (\prod f_i(z, t) \leq 0) \subset \mathbb{R}^2 \). Semi-analytic sets can be triangulated (cf. [BCR87, 9.2]), thus in a neighborhood of the origin, \( U \) is the cone over \( U \cap (z^2 + t^2 = \epsilon) \).

Each \( (f_i = 0) \) is an irreducible curve germ over \( \mathbb{R} \), thus homeomorphic to \( \mathbb{R}^1 \). So each \( f_i \) has 2 roots on the circle \( (z^2 + t^2 = \epsilon) \). Hence \( U \cap (z^2 + t^2 = \epsilon) \) is the disjoint union of \( m \) closed arcs. Therefore \( L \) has \( m \) connected components, each homeomorphic to \( S^2 \).

The second possibility is the \( cA^- \)-case: \( x^2 - y^2 - h \prod f_i \). (The two choices of \( \pm h \) are equivalent by interchanging \( x \) and \( y \).) Here we project to the \((y, z, t)\)-hyperplane. The fiber over a point \((y, z, t)\) is

1. 2 points if \( y^2 + h(z, t) \prod f_i(z, t) > 0 \),
2. 1 point if \( y^2 + h(z, t) \prod f_i(z, t) = 0 \),
3. empty if \( y^2 + h(z, t) \prod f_i(z, t) < 0 \).

Thus we have to determine the region

\[
U := \{ y^2 + h(z, t) \prod f_i(z, t) \geq 0 \} \subset (y^2 + z^2 + t^2 = \epsilon) \sim S^2,
\]

and then take its double cover to get \( L \).

If \( m = 0 \) then \( U = S^2 \) and so \( L = S^2 \cup S^2 \). If \( m > 0 \) then \( h \prod f_i \) is negative on \( m \) disjoint arcs in the circle \( (z^2 + t^2 = \epsilon) \), and \( y^2 + h(z, t) \prod f_i(z, t) \) is negative in contractible neighborhoods of these intervals. Thus \( U = S^2 \setminus (m\text{-discs}) \) and so \( L \) is a surface of genus \( m - 1 \), orientable by \((1.2)\).

**Example 4.5.** It is quite instructive to consider the following incorrect approach to the topology of \( cD \) and \( cE \)-type singularities. I illustrate it in the \( cE_6 \)-case.

Over \( \mathbb{R} \), a surface singularity of type \( E_6 \) is \( x^2 + y^2 \pm z^4 \). In both cases, projection to the \((x, z)\)-plane is a homeomorphism. Consider a \( cE_6 \)-type point \( X \). If \( L(X(\mathbb{R})) \) has several connected components, then a suitable hyperplane intersects at least two of them. By a small perturbation we obtain an \( E_6 \)-singularity as the intersection, thus we conclude that \( L(X(\mathbb{R})) \) is connected. This is especially suggestive if we note that instead of a plane we could use a small perturbation of any smooth hypersurface.

Unfortunately the conclusion is false, as we see in (4.16). \( L(X(\mathbb{R})) \) can have several components, and some of them are not seen by general hypersurface sections. These look like very “thin” cones, as opposed to the main component which is “thick”. It would be interesting to give precise meaning to this observation and to see its significance in the study of singularities.
The following approach to the topology of $cD$ and $cE$-type singularities is taken from [AGV85, Sec.12].

4.6 (Deformation to the weighted tangent cone).

Let $X := (f(x_1, \ldots, x_n) = 0)$ be a hypersurface singularity. For simplicity of notation I assume that $f$ converges for $|x_i| < 1 + \delta$. Assign integral weights to the variables $w(x_i) = w_i$ and write $f$ as the sum of weighted homogeneous pieces

$$f = f_d + f_{d+1} + f_{d+2} + \ldots,$$

where $f_s$ is weighted homogeneous of degree $s$. For a parameter $\lambda \neq 0$
set

$$f^\lambda(x_1, \ldots, x_n) := \lambda^{-d} f(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n)$$

$$= f_d + \lambda f_{d+1} + \lambda^2 f_{d+2} + \ldots$$

This suggests that if we define $f^0 := f_d$ then

$$X^\lambda := \{f^\lambda = 0\} \quad \text{for} \quad \lambda \in \mathbb{R}$$

is a “nice” family of hypersurface singularities. For $\lambda \neq 0$ they are all isomorphic to $(f = 0)$ and for $\lambda = 0$ we obtain the weighted tangent cone $(f_d = 0)$.

This can be used to determine the topology of $X(\mathbb{R})$ in 2 steps. First describe $X^0$ and then try to relate $X^\lambda$ and $X^0$ for small values of $\lambda$.

Let $w$ be a common multiple of the $w_i$ and set $u_i = w/w_i$.

**Proposition 4.7.** Notation as above. Assume that $X = (f = 0)$ is an isolated hypersurface singularity.

Then there is a $0 < \lambda_0$ such that for every $0 < \lambda \leq \lambda_0$

1. $L^\lambda := X^\lambda \cap (\sum x_i^{2u_i} = 1)$ is smooth, and

2. $X^\lambda \cap (\sum x_i^{2u_i} \leq 1)$ is homeomorphic to the cone over $L^\lambda$.

Proof. The map $\mathbb{R}^n \to \mathbb{R}^+$ given by $(x_1, \ldots, x_n) \mapsto \sum x_i^{2u_i}$ is proper. Thus its restriction to $X^\lambda$ is also proper. The proposition follows once we establish that the resulting map

$$t : X^\lambda \to \mathbb{R}$$

has no critical points with critical value in $(0, 1]$ for $0 < \lambda \leq \lambda_0$.

The critical values of a real algebraic morphism form a semi-algebraic set (cf. [BCR87, 9.5]), thus there is a $0 < \mu_0$ such that $t : X^1 \to \mathbb{R}$ has no critical values in $(0, \mu_0)$. The following diagram is commutative

$$\begin{array}{ccc}
X^\lambda & \xrightarrow{x_i \mapsto \lambda^{-w_i} x_i} & X^1 \\
t & \downarrow & \downarrow t \\
\mathbb{R} & \xrightarrow{s \mapsto \lambda^{-w_s}} & \mathbb{R}
\end{array}$$
which shows that \( (4.7) \) holds with \( \lambda_0 = \mu_0^{1/w} \).

So far we have not done much, but the advantage of this approach is that we can view \( L^\lambda \) as a deformation of the compact real algebraic variety \( L^0 \). If \( L^0 \) is smooth then this deformation is locally trivial differentiably. Thus we obtain:

**Corollary 4.8.** Assume that \((f_3 = 0)\) defines an isolated singularity. Then \( L^\lambda \) is diffeomorphic to \( L^0 \).

This is sufficient to describe the the topology of “general” members of several families of terminal singularities:

**Corollary 4.9.** Let \( X \) be a terminal singularity given by one of the of the following equations:

\[
\begin{align*}
cD_4 & \quad x^2 + f_{\geq 3}(y, z, t), \text{ where } f_3 = 0 \text{ has no real singular point.} \\
cE_6 & \quad x^2 + y^3 + yg_{\geq 3}(z, t) + h_{\geq 4}(z, t), \text{ where } h_4 \text{ has no multiple real linear factor.} \\
cE_8 & \quad x^2 + y^3 + yg_{\geq 4}(z, t) + h_{\geq 5}(z, t), \text{ where } h_5 \text{ has no multiple real linear factor.}
\end{align*}
\]

Then:

\[
\begin{align*}
cD_4 & \quad L(X(\mathbb{R})) \sim S^2 \text{ if } (f_3 = 0) \subset \mathbb{RP}^2 \text{ has one connected component and } L(X(\mathbb{R})) \sim S^2 \uplus (S^1 \times S^1) \text{ if } (f_3 = 0) \text{ has two connected components.} \\
cE_6 & \quad L(X(\mathbb{R})) \sim S^2. \\
cE_8 & \quad L(X(\mathbb{R})) \sim S^2.
\end{align*}
\]

Proof. We use deformation to the weighted tangent cone with (suitable integral multiples of the) weights \((1/2, 1/3, 1/3, 1/3)\) in the \(cD_4\)-case, \((1/2, 1/3, 1/4, 1/4)\) in the \(cE_6\)-case, and \((1/2, 1/3, 1/5, 1/5)\) in the \(cE_8\)-case. The equations for \( X^0 \) are \( x^2 + f_3(y, z, t) = 0, x^2 + y^3 + h_4(z, t) = 0 \) and \( x^2 + y^3 + h_5(z, t) = 0 \). Our conditions guarantee that \( X^0(\mathbb{R}) \) has isolated singularities, thus it is sufficient to determine \( L(X^0(\mathbb{R})) \).

In the \(cE\)-cases, projection to the \((x, z, t)\) hyperplane is a homeomorphism from \( X^0(\mathbb{R}) \) to \( \mathbb{R}^3 \), thus \( L(X^0(\mathbb{R})) \sim S^2 \)

In the \(cD_4\)-cases we project to the \((y, z, t)\)-hyperplane. As in the proof of \((4.4)\), we can get \( L(X^0(\mathbb{R})) \) once we know the set \( U \subset (y^2 + z^2 + t^2 = 1) \) where \( f_3 \) is nonnegative. The boundary \( \partial U \) doubly covers the projective curve \( (f_3 = 0) \subset \mathbb{RP}^3 \). If \( (f_3 = 0) \subset \mathbb{RP}^2 \) has one connected component then it is a pseudo-line and \( \partial U \) is a connected double cover, hence \( U \) is a disc. If \( (f_3 = 0) \) has two connected components, then one is a pseudo-line the other an oval. \( \partial U \) has 3 connected components, and \( U \) is a disc plus an annulus. Thus \( L(X^0(\mathbb{R})) \sim S^2 \uplus (S^1 \times S^1) \). \( \Box \)
Remark 4.10. In the $cE$ cases of the above example, projection to the $(x, z, t)$ plane is a homeomorphism from $X^0(\mathbb{R})$ to $\mathbb{R}^3$ even if $h_4$ or $h_5$ have multiple factors. In these cases, however, we cannot conclude that $X(\mathbb{R})$ is also homeomorphic to $\mathbb{R}^3$. In fact we see in (1.16) that this is not always true.

Similar arguments work in some of the $cD_{>4}$-cases:

Corollary 4.11. Let $X$ be a terminal singularity given by equation $x^2 + y^2z + h_{>s}(z, t)$, where $z \not| h_s$ and $h_s$ has no multiple real linear factors. Let $s$ be the number of real linear factors of $h_s$. There are three cases:

1. $s = 2r + 1$ and $L(X(\mathbb{R})) \sim M_r \cup rS^2$;
2. $s = 2r$, $h(0, 1) < 0$ and $L(X(\mathbb{R})) \sim M_r \cup (r-1)S^2$;
3. $s = 2r$, $h(0, 1) > 0$ and $L(X(\mathbb{R})) \sim M_{r-1} \cup rS^2$.

Proof. We use deformation to the weighted tangent cone with weights $(1/2, (s-1)/2s, 1/s, 1/s)$. Thus we need to figure out the topology of $x^2 + y^2z + h_{>s}(z, t) = 0$. As before, this reduces to understanding the set where $y^2z + h_{>s}(z, t) \leq 0$. This can be done by projecting to the $(z, t)$ plane. Details are left to the reader.

4.12. For many terminal singularities one can not choose weights so that the weighted tangent cone has an isolated singularity at the origin, but in all cases it is possible to choose weights so that the weighted tangent cone has 1-dimensional singular locus:

| name   | equation is $x^2+$ | $w(y)$ | $w(z)$ | $w(t)$ |
|--------|--------------------|--------|--------|--------|
| $cD_4$ | $f_{>2}(y, z, t)$  | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $cD_{>4}(1)$ | $y^2z + ayt^r + h_{>s}(z, t)$ | $\frac{s-1}{2s}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $cD_{>4}(2)$ | $y^2z \pm y^2z + h_{>s}(z, t)$ | $\frac{1}{2r-1} - \epsilon$ | $\frac{1}{2r-1} + 2\epsilon$ | $\frac{1}{2r-1} + \frac{1}{r}$ |
| $cE_6$  | $y^3 + yg_{>3}(z, t) + h_{>4}(z, t)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $cE_7$  | $y^3 + yg_{>3}(z, t) + h_{>5}(z, t)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $cE_8$  | $y^3 + yg_{>4}(z, t) + h_{>5}(z, t)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

In the $cD_{>4}$ case we assume that $h_s \neq 0$ and use the first weight sequence if $a = 0$ or $2r > s + 1$ and the second weight sequence if $2r \leq s + 1$, where $\epsilon$ is a small positive number. (We could use $\epsilon = 0$ except when $2r = s - 1$.) Let $(w_1, w_2, w_3, w_4)$ be integral multiples of these weights.
The weighted tangent cone, and its singularities are the following:

| name     | weighted tangent cone | singularities                  |
|----------|-----------------------|--------------------------------|
| $cD_4$   | $x^2 + f_3(y, z, t)$  | at singular pts of $(f_3 = 0)$ |
| $cD_{>4}(1)$ | $x^2 + y^2 z + h_8(z, t)$ | at multiple real factors of $zh_8$ |
| $cD_{>4}(2)$ | $x^2 + y^2 z \pm yt^r$ | at the $z$-axis                |
| $cE_6$   | $x^2 + y^3 + h_4(z, t)$ | at multiple real factors of $h_4$ |
| $cE_7$   | $x^2 + y^3 + yg_3(z, t)$ | at real factors of $g_3$       |
| $cE_8$   | $x^2 + y^4 + h_5(z, t)$ | at multiple real factors of $h_5$ |

These equations have the form $x^2 + F(y, z, t)$ and the deformation to the weighted tangent cone leaves this form invariant:

$$x^2 + F^\lambda(y, z, t) = x^2 + \lambda^{-d}F(\lambda^{u_2}y, \lambda^{u_3}z, \lambda^{u_4}t).$$

Set

$$U^\lambda := \{(y, z, t)|F^\lambda(y, z, t) \leq 0 \subset (y^{2u_2} + z^{2u_3} + t^{2u_4} = 1)\}.$$  

$U^\lambda$ is a semi-algebraic set and its boundary is the real algebraic curve

$$C^\lambda := \{(y, z, t)|F^\lambda(y, z, t) = 0 \subset (y^{2u_2} + z^{2u_3} + t^{2u_4} = 1)\}.$$  

We have established the following:

**Proposition 4.13.** $C^\lambda$ is a deformation of the real algebraic curve $C^0$ inside the smooth real algebraic surface $(y^{2u_2} + z^{2u_3} + t^{2u_4} = 1)$.

4.14. The deformations of real algebraic curves can be understood in two steps (cf. [Viro90]). Put small discs around the singularities. Outside the discs all small deformations are topologically trivial and inside the discs we have a local problem involving real curve singularities. Here we have the advantage that $(y^{2u_2} + z^{2u_3} + t^{2u_4} = 1)$ is an affine algebraic surface, thus we can choose the local deformations independently and they can always be patched together.

Thus we can describe the possible cases for $C^\lambda$, and thereby the topological types of the corresponding 3-dimensional terminal singularities, if we can describe the deformations of the occurring real plane curve singularities. By looking at the equations we see that the only singularities that we have to deal with are the 2-variable versions of the Du Val singularities:

- $A_n$ $y^2 \pm z^{n+1} = 0$;
- $D_n$ $y^2z \pm z^{n-1} = 0$;
- $E_6$ $y^3 + z^4 = 0$;
- $E_7$ $y^3 \pm y^2z^3 = 0$;
- $E_8$ $y^3 + z^5 = 0$. 

For all of these cases, a complete list of the topological types of real deformations is known [Chislenko88]. The list can also be found in [Viro90], Figs. 16–28, which contains many further examples.

Example 4.15. Consider for instance the $cD_4$ cases. The various possibilities for $f_3$ are easy to enumerate. The most interesting is $f_3 = yzt$. Here $C^0$ is the union of the 3 coordinate hyperplanes intersecting $(y^6 + z^6 + t^6 = 1)$. We have 6 singular points of type $u^2 - v^2 = 0$. At each of them we can choose a deformation $u^2 - v^2 \pm \epsilon = 0$. This gives $2^6$ possibilities. The symmetries of the octahedron act on the configurations so it is easy to get a complete list.

At the end we get 7 possible topological types for $L(X(\mathbb{R})$ where $X = (x^2 + yzt + f_{\geq 4}(y, z, t) = 0)$:

$$M_2, M_1 \sqcup S^2, M_1, S^2, 2S^2, 3S^2, 4S^2.$$  

It turns out that these exhaust all the cases given by $cD_4$.

Example 4.16. Consider the $cE_6$-type points

$$x^2 + y^3 + yg_{\geq 3}(z, t) \pm z^2t^2 + h_{\geq 5}(z, t).$$

Using the methods of (2.6) these can be brought to the form

$$x^2 + y^3 \pm z^2t^2 + ya(z) + yb(t) + c(z) + d(t).$$

The weighted tangent cone, $(x^2 + y^3 \pm z^2t^2 = 0)$ is singular along the $z$ and $t$-axes. In order to understand the singularity type of $C^\lambda$, say along the positive $z$-halfaxis, set $t = \epsilon$. We get an equation

$$x^2 + y^3 \pm z^2\epsilon^2 + ya(z) + yb(\epsilon) + c(z) + d(\epsilon).$$

$\text{mult}_0 a \geq 3$ and $\text{mult}_0 c \geq 5$, thus all the terms involving $z$ can be absorbed into $z^2$, and we obtain the equivalent form

$$x^2 \pm z^2 + y^3 + yb(\epsilon) + d(\epsilon).$$

The cubic $y^3 + yb(\epsilon) + d(\epsilon)$ has 3 real roots if $4b(\epsilon)^3 + 27d(\epsilon)^2 < 0$ and 1 real root if $4b(\epsilon)^3 + 27d(\epsilon)^2 > 0$.

$C^0$ is homeomorphic to $S^1$ and it has 4 singular points (along the $z$ and $t$ halfaxes). $C^\lambda$ is a smooth curve which has an oval near a singular point of $C^0$ if the corresponding cubic has 3 real roots and no ovals if only 1 real root. Thus $C^\lambda$ has at most 5 connected components.

We can also determine the location of the ovals relative to the “main component” of $C^\lambda$. In deforming $z^2 - y^3 = 0$, the oval can appear only toward the negative $y$-direction. Putting all this together, we get the following possibilities for $L(X(\mathbb{R}))$:

$$rS^2, 1 \leq r \leq 5 \text{ in the } +z^2t^2\text{-case, and}$$

$$M_r, 0 \leq r \leq 4 \text{ in the } -z^2t^2\text{-case.}$$
5. THE TOPOLOGY OF TERMINAL QUOTIENT SINGULARITIES

Let $0 \in X$ be 3-fold terminal singularity and $\pi : \tilde{X} \to X$ its index one cover. As we proved, $X = \tilde{X}/\mathbb{Z}_n(a_1, \ldots, a_m)$ where $n$ is the index of $X$ and the $a_i$ are integers. We use this representation to determine the topology of $X$ in terms of the already known topology of $\tilde{X}$.

The main question is to determine the real points of $\mathbb{A}^m/\mathbb{Z}_n(a_1, \ldots, a_m)$. Let $\sigma : \mathbb{C}^m \to \mathbb{C}^m$ be the corresponding action of $1 \in \mathbb{Z}_n$.

The answer depends on the parity of $n$. First we discuss the odd index cases which are easier.

**Proposition 5.1.** Assume that $n$ is odd and set $Y = \mathbb{A}^m/\mathbb{Z}_n(a_1, \ldots, a_m)$. Then the induced map $\mathbb{R}^n \to Y(\mathbb{R})$ is a homeomorphism.

**Proof.** Let $R \subset \mathbb{R}[[x_1, \ldots, x_m]]$ denote the ring of invariant functions. A point $P \in \mathbb{A}^m$ maps to a real point of $X$ iff $f(P) \in R$ for every $f \in R$. Let $\epsilon$ be a primitive $n^{th}$ root of unity. If $P = (p_1, \ldots, p_m)$ is real then $\sigma^b(P) = P$. This shows that the quotient map $\mathbb{R}^n \to X(\mathbb{R})$ is an injection.

Let $Q \in X(\mathbb{R})$ be a point. Then $\pi^{-1}(Q) \subset \mathbb{A}^m$ has an odd number of closed points over $\mathbb{C}$ (usually $n$ of them) and as a scheme it is defined over $\mathbb{R}$. Thus it has a real point, hence $\mathbb{R}^n \to X(\mathbb{R})$ is also surjective. \hfill $\square$

**Corollary 5.2.** Let $0 \in X$ be a 3-fold terminal singularity of odd index and $\pi : \tilde{X} \to X$ its index one cover. Then $\pi : \tilde{X}(\mathbb{R}) \to X(\mathbb{R})$ is a homeomorphism. \hfill $\square$

The even index case is more subtle. For purposes of induction we allow the case when $n$ is odd. Consider the action $\mathbb{Z}_n(a_1, \ldots, a_m)$ on $\mathbb{A}^m$. Write $n = 2^c n'$ where $n'$ is odd. Let $\eta$ be a primitive $2^{c+1}$-st root of unity and $j : \mathbb{R}^m \to \mathbb{C}^m$ the map

$$j(x_1, \ldots, x_m) = (\eta^{a_1}x_1, \ldots, \eta^{a_m}x_m).$$

(If $n$ is odd then $\eta = -1$, hence $j(\mathbb{R}^m) = \mathbb{R}^m$.) Write $a_i = 2^ca'_i$ such that $a'_i$ is odd for some $i$. If $s > c$, let $\tau : \mathbb{C}^m \to \mathbb{C}^m$ be the $\mathbb{Z}_2$-action

$$\tau(x_1, \ldots, x_m) = ((-1)^{a'_i}x_1, \ldots, (-1)^{a'_m}x_m).$$

For $s = c$ let $\tau$ be the identity. Note that both $\mathbb{R}^n$ and $j(\mathbb{R}^m)$ are $\tau$-invariant.

**Proposition 5.3.** Set $Y = \mathbb{A}^m/\mathbb{Z}_n(a_1, \ldots, a_m)$. Define $j$ and $\tau$ as above. Then $Y(\mathbb{R})$ is the quotient of $\mathbb{R}^m \cup j(\mathbb{R}^m)$ by $\tau$. 

Proof. The proof is by induction on \( m \) and \( n \). We can assume that the action is faithful, that is \( \sum b_i a_i = 1 \) is solvable in integers. Indeed, for non faithful actions we get the same quotient from a smaller group action. The definitions of \( j \) and \( \tau \) are set up such that they do not change if we change the group this way.

Set \( F := \prod_i x_i^{b_i} \). By induction on \( m \) we know that (5.3) holds on each coordinate hyperplane. Thus we have to deal with points \( P = (p_1, \ldots, p_m) \) such that each \( p_i \neq 0 \).

Assume that \( \pi(P) \) is real. Let \( \epsilon \) be a primitive \((2n)^{\text{th}}\) root of unity. \( p_i^n \) is real, hence \( p_i = \epsilon^{c_i} \cdot (\text{real number}) \) for some \( c_i \in \mathbb{Z} \). Thus \( F(P) = \epsilon^c \cdot (\text{real number}) \). \( F(\sigma(P)) = \epsilon^2 F(P) \), hence by replacing \( P \) by \( \sigma^r(P) \) for some \( r \) we may assume that \( F(P) \in \mathbb{R} \) or \( F(P) \in \eta \cdot \mathbb{R} \).

Assume first that \( F(P) \) is real. For each \( i \) the function \( F^{n-a_i} x_i \) is invariant, hence has a real value at \( P \). Thus \( P \in \mathbb{R}^n \). If \( F(P) \in \eta \cdot \mathbb{R} \) then the same argument shows that \( p_i \in \eta^{a_i} \cdot \mathbb{R} \), thus \( P \in j(\mathbb{R}^m) \).

This shows that \( \mathbb{R}^m \cup j(\mathbb{R}^m) \to Y(\mathbb{R}) \) is surjective. It is also \( \tau \)-invariant. Finally, if \( P = (p_1, \ldots, p_m) \in \mathbb{R}^m \cup j(\mathbb{R}^m) \) then

\[
\sigma^s(P) = (\epsilon^{ba_1} p_1, \ldots, \epsilon^{ba_m} p_m) \in \mathbb{R}^m \cup j(\mathbb{R}^m)
\]

iff \( \sigma^s(P) = P \) or \( \sigma^s(P) = \tau(P) \). Thus \( \mathbb{R}^m \cup j(\mathbb{R}^m) \to Y(\mathbb{R}) \) is 2 : 1 for \( s > c \) and 1 : 1 for \( s = c \).

**Definition 5.4.** Let \( F \in \mathbb{R}[[x_1, \ldots, x_m]] \) be a power series, homogeneous of grade \( d \) under the grading \( \frac{1}{n}(a_1, \ldots, a_m) \). Let \( \eta \) be as above. Define the *companion* \( F^c \) of \( F \) with respect to the action \( \frac{1}{n}(a_1, \ldots, a_m) \) by

\[
F^c(x_1, \ldots, x_m) := \eta^{-d} F(\eta^{a_1} x_1, \ldots, \eta^{a_m} x_m).
\]

Note that \( F^c \in \mathbb{R}[[x_1, \ldots, x_m]] \).

**Corollary 5.5.** Let \( 0 \in X \) be a 3-fold terminal singularity of even index and \( \pi : \tilde{X} \to X \) its index one cover. Then

\[
L(X(\mathbb{R})) \sim L(\tilde{X}(\mathbb{R}))/((\tau) \cup L(\tilde{X}^c(\mathbb{R}))/((\tau).
\]

Proof. We use the notation of (5.3). Let \( F = 0 \) be the equation of \( \tilde{X} \) and let \( W := (F = 0) \cap (\mathbb{R}^m \cup j(\mathbb{R}^m)) \). Then \( X(\mathbb{R}) \) is the quotient of \( W \) by \( \tau \).

\( (F = 0) \cap \mathbb{R}^m = \tilde{X}(\mathbb{R}) \). \( (F = 0) \cap j(\mathbb{R}^m) \) can be identified with the set of real zeros of \( F(\eta^{a_1} x_1, \ldots, \eta^{a_m} x_m) = 0 \). (The normalizing factor \( \eta^{-d} \) does not change the set of zeros.)

In the terminal case the group action is fixed point free on \( (F = 0) \setminus \{0\} \), thus \( (F = 0) \cap \mathbb{R}^m \) and \( (F = 0) \cap j(\mathbb{R}^m) \) intersect only at the origin. \( \square \)

As a byproduct we obtain the following:
Corollary 5.6. Let $0 \in X$ be a 3-fold terminal singularity of index $>1$. Then $0 \in X(\mathbb{R})$ is not an isolated point.

Proof. Let $\tilde{X}$ be the index 1 cover. We are done, unless $0 \in \tilde{X}(\mathbb{R})$ is an isolated point. This happens only in cases $cA/2$, $cAx/2$ (and maybe for $cAx/4$) where the equation of $\tilde{X}$ is $F = x^2 + y^2 + f(z, t)$ and $f(z, t)$ is positive on $\mathbb{R}^2 \setminus \{0\}$.

Let us compute $F^c$. In the $cA/2$ case we get $-x^2 - y^2 + f(iz, t)$ and this has nontrivial solutions in the $(x, y, t)$-hyperplane. In the $cAx/2$ case we get $x^2 - y^2 + f(iz, it)$ and this has nontrivial solutions in the $(x, y)$-plane.

In the $cAx/4$ case already $\tilde{X}$ has nontrivial $\mathbb{R}$-points. Indeed, here $f$ has grade 2, thus every $t$-power in it has an odd exponent. Thus $f(z, t)$ is not positive on the $t$-axis.

5.7 (Orientability of index 2 quotients). We have seen in (4.2) that every real algebraic hypersurface is orientable, and so are their quotients by odd order groups (5.1). With index 2 quotients, the question of orientability is interesting.

Consider a quotient $(F = 0)/\frac{1}{2}(w_1, \ldots, w_m)$. Let $\sigma$ be the corresponding $\mathbb{Z}_2$ action. We can orient $X := (F = 0)$ by choosing an orientation of $\mathbb{R}^m$ and at each smooth point of $X$ we choose the normal vector pointing in the direction where $F$ is positive. $\sigma$ preserves the orientation of $\mathbb{R}^m$ iff $\sum w_i$ is even. The parity of $w(F)$ determines the sign in $\sigma(F) = \pm F$. Thus $\sigma$ preserves the induced orientation of $X$ iff $w(F) + \sum w_i$ is even. If $w(F) + \sum w_i$ is odd, the induced orientation is not preserved. If $\sigma$ fixes a connected component of (the nonsingular part of) $X(\mathbb{R})$, then the corresponding quotient is not orientable. If, however, $\sigma$ only permutes the connected components of $X(\mathbb{R})$ then the quotient is still orientable. Thus we obtain:

Lemma 5.8. Let $0 \in X := (F = 0)/\frac{1}{2}(w_1, \ldots, w_m)$ be an isolated singular point, with index one cover $\tilde{X}$ and companion $\tilde{X}^c$. Then $L(X(\mathbb{R}))$ is nonorientable iff $w(F) + \sum w_i$ is odd and $\sigma$ fixes at least one of the connected components of $L(\tilde{X}(\mathbb{R}))$ or $L(\tilde{X}^c(\mathbb{R}))$. □

Example 5.9 (The topology of $cA/2$ points).

The simplest case is $\mathbb{A}^3/\frac{1}{2}(1, 1, 1)$. The link of $\mathbb{A}^3$ is the sphere $(x^2 + y^2 + z^2 = \epsilon^2)$. We act by the antipodal map, and the quotient is $\mathbb{R}\mathbb{P}^2$. The quotient of the purely imaginary subspace also gives real points, thus $L(X(\mathbb{R})) \sim 2\mathbb{R}\mathbb{P}^2$.

In the $cA_{>0}/2$ cases write

$$X := (x^2 \pm y^2 + f(z, t) = 0)/\frac{1}{2}(1, 1, 1, 0)$$
with cover \( \tilde{X} := (x^2 \pm y^2 + f(z, t) = 0) \). Only even powers of \( z \) occur in \( f(z, t) \), thus we can write \( f(z, t) = G(z^2, t) \). As in (4.4) we factor it as \( G(z^2, t) = \pm h(z, t) \prod_{i=1} f_i(z, t) \).

The companion cover is \( \tilde{X}^c = (x^2 \pm y^2 - G(-z^2, t) = 0) \). We have to be careful since the product decomposition of \( G \) is not preserved. A factor of \( h \) may become indefinite and it can also happen that two factors become conjugate over \( \mathbb{R} \). Thus we write \(-G(-z^2, t) = \pm h'(z, t) \prod_{j=1} f'_j(z, t) \).

By (5.3), \( L(X(\mathbb{R})) = L(\tilde{X}(r))/\tau \cup L(\tilde{X}^c(r))/\tau \). In all these cases (5.8) shows that if \( \tau \) fixes a connected component, the quotient is not orientable. Thus if \( L(\tilde{X}(r)) \) or \( L(\tilde{X}^c(r)) \) is connected, the quotient is not orientable. This holds in all the \( cA_{\geq 0}(r > 0) \) cases. In the \( cA_{\geq 0}(0) \) case the equation is \((x^2 = y^2 + h)\), and each halfspace of \((x \neq 0)\) contains a unique connected component. Thus \( \tau \) interchanges the two connected components and the quotient is orientable.

In the \( cA_{\leq 0}(r > 0) \) case \((\pm h(z, t) \prod_{i=1} f_i(z, t) = 0) \) is negative on \( r \) connected regions \( P_1, \ldots, P_r \subset \mathbb{R}^2 \), and \( L(X(\mathbb{R})) \) consist of \( r \) copies of \( S^2 \), one for each \( P_j \). The involution \( \tau \) fixes the \( t \)-axis pointwise, thus if one of the half \( t \)-axes is contained in some \( P_j \), then \( \tau \) fixes the \( S^2 \) over that region. The other copies of \( S^2 \) are interchanged. The same holds for \( \tilde{X}^c \).

Along the \( t \)-axis \( G(z^2, t) \) and \(-G(-z^2, t) \) have opposite signs. Thus among the 4 pairs

\[
\begin{align*}
&\text{(positive half \( t \)-axis, \( G(z^2, t) \))} & \text{(positive half \( t \)-axis, \(-G(-z^2, t) \))} \\
&\text{(negative half \( t \)-axis, \( G(z^2, t) \))} & \text{(negative half \( t \)-axis, \(-G(-z^2, t) \))}
\end{align*}
\]

there are two where the function is negative along the half axis.

We obtain the following list of possibilities. (We use the notation \( K_r := S^2 \# r \mathbb{R}P^2 \), thus \( K_2 \) is the Klein bottle.)

| \( X \) | \( X^c \) | \( L(X(\mathbb{R})) \) |
|---|---|---|
| \( cA_0 \) | \( cA_0 \) | \( \mathbb{R}P^2 \cup \mathbb{R}P^2 \) |
| \( cA_{\leq 0}(r > 0) \) | \( cA_{\leq 0}(r' > 0) \) | \( K_r \cup K_{r'} \) |
| \( cA_{\leq 0}(r > 0) \) | \( cA_{\leq 0}(0) \) | \( K_r \cup S^2 \) |
| \( cA_{\geq 0}(0) \) | \( cA_{\geq 0}(0) \) | \( S^2 \cup S^2 \) |
| \( cA_{\leq 0}(r > 0) \) | \( cA_{\leq 0}(r' > 0) \) | \( 2\mathbb{R}P^2 \cup \frac{-2}{t+z}S^2 \) |
| \( cA_{\leq 0}(r > 0) \) | \( cA_{\leq 0}(0, +) \) | \( 2\mathbb{R}P^2 \cup \frac{-2}{t+z}S^2 \) |
| \( cA_{\geq 0}(0, -) \) | \( cA_{\geq 0}(0, +) \) | \( K_2 \) |

Thus \( X(\mathbb{R}) \) is not orientable, except in the fourth case. This indeed occurs:
Let $\tilde{X} := (x^2 - y^2 + z^{4m} + t^{2n} = 0)$ and $X := \tilde{X}/\mathbb{Z}(1, 1, 1, 0)$ with companion $\tilde{X}^c = (-x^2 + y^2 + z^{4m} + t^{2n} = 0)$. $\tilde{X}^c \cong \tilde{X}$ and $L(\tilde{X}(\mathbb{R})) \sim S^2 \uplus S^2$. $\tau$ interchanges the two copies of $S^2$. Thus $X(\mathbb{R})$ is orientable and $L(X(\mathbb{R})) \sim S^2 \uplus S^2$.

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