Vortex Line Nucleation of First-Order Transition $U(1)$-Symmetric Field Systems

H. Kleinert
Institut für Theoretische Physik
Arnimallee 14, D14195 Berlin, Germany

We show that in field systems with $U(1)$-symmetry, first-order transitions are nucleated by vortex lines, not bubbles, thus calling for a reinvestigation of the Kibble mechanism for the phase transition of the early universe.

1. Since Langer’s historic paper on bubble nucleation [12] of first-order transition in a real scalar field system, field theorists have assumed this mechanism to cause transitions in a large variety of physical systems. This belief was enhanced by a rigorous proof of Coleman at al., that the dominant classical solutions of rotationally-invariant field equations in instanton calculations are bubble-like [3]. Most importantly for our very existence, the first-order transitions in the early universe are supposed to be nucleated by bubbles via the celebrated Kibble mechanism [4,5].

To an unbiased observer, this assumption comes as a surprise, since the evolution of the early universe is described by a field theory which is a nonabelian generalization of the Ginzburg-Landau theory of superconductivity. For a superconductor, however, bubbles play no role in the phase transition. A superconductor can have a second- or a first-order transition, depending on the ratio of the two length scales $\kappa =$ magnetic penetration depth/coherence length [6,7]. In the second-order regime, the transition can be understood completely as a proliferation of magnetic vortex lines. This can shown convincingly in a lattice field theory of the system [10].

Moreover, there is a dual description of this theory which is a simple XY-model. The high-temperature expansion of the partition function of this model can be rewritten as a sum of closed loops which are direct pictures of the magnetic vortex lines in the superconductor at low temperatures. In this grand-canonical line ensemble one can easily calculate the temperature of proliferation [10].

In the continuous limit, this XY-model can be transformed via functional techniques into a $|\psi|^2$-field theory with a complex disorder field [4]. In this formulation, the Feynman diagrams in the perturbation expansion of the vacuum energy are the direct pictures of the magnetic vortex lines, which proliferation as the mass term becomes negative.

When lowering the parameter $\kappa$ into the regime of weak first-order transitions, there still exists a generalization of the XY-model describing this system, which has the same type of high-temperature expansion in terms of closed loops, thus showing again that only vortex lines can be relevant for understanding the transition $\bar{\mathbf{0}}$. Thus we must conclude that in a superconductor and related field theories which possess vortex lines as topological excitations, these excitations are also the relevant driving mechanism of the phase transitions.

It is the purpose of this note to make these qualitative arguments convincing, demonstrating the superior efficiency of vortex line over bubble nucleation, thus casting doubts on all studies of phase transition based on the Kibble mechanism.

2. The generalized XY-model which provides us with a disorder description of a superconductor on a lattice has the partition function

$$Z = \int \frac{D\theta}{2\pi} e^{-\beta \sum_{n} [\nabla_{i} \theta + \delta \cos 2\theta]} \tag{1}$$

where $\nabla_{i}$ are lattice gradients, and $\beta, \delta$ model parameters. The phase structure of this model has been studied in detail in the literature [8–10]. For $\delta = 0$ the model is known to describe the critical behavior of superfluid helium near the $\lambda$-transition. The same thing is true for a small interval around zero $\delta \in (0, 0, 2)$. In addition, there exists a regime of $\delta$ where the transition is of first order. In the disordered phase, the partition function [1] can be rewritten as a sum over non-self-backtracking loops of superflow. Under a duality transformation, these go over into the magnetic vortex lines of the superconductor. The parameter $\beta$ which is the inverse temperature in the XY-model grows with the temperature in the superconductor. The loops of superflow can have strengths $1, 2, 3, \ldots$ on the lattice. They are dual representation of the quantized flux strengths of the magnetic vortex lines in the superconductor. In the second-order regime, the critical properties of the model have been shown to be the same as for a simplified model which can contain only loops of unit strength [12,10].

For a single loop, the partition function of this simplified model can easily be written down. If $n$ is the length of the loop in lattice units, we have

$$Z = \sum_{n} N_{n} e^{\beta_{V} \varepsilon_{n}} \tag{2}$$

where $\beta_{V}$ is a function $\beta, \delta$ which plays the role of an inverse temperature for this one-loop model, $\varepsilon_{n}$ is the loop energy, and $N_{n}$ is the number of different loops of length $n$. For large $n$, the energy $\varepsilon_{n}$ is proportional to $n$, say $\varepsilon_{n} \approx \varepsilon n$. The notation $\varepsilon$ is really an approximation, since it neglects a slight dependence on the loop.
shape. This, however, is very weak for lines which are much longer than the length scale \( n^\kappa \) over which the lines show stiffness. This stiffness is a result of the non-self-backtracking property and the fact that if two (or more) portions of a loop merge into a line of strength two (or larger), the energy of this portion is much larger than the sum of the energies of the constituent lines, causing a strong Boltzmann suppression. Writing the number \( N_n \) as \( e^{-\beta n} \), we define the configurational entropy \( s_n \) of loops of length \( n \). Also \( s_n \) grows linearly for large \( n \), say like \( s n \). As \( \beta \nu \) becomes smaller than a critical value \( \beta^c \), the free energy of the loops

\[
f_n = \varepsilon_n - \beta^{-1} s_n
\]

(3)
goes to negative infinity for large \( n \), so that the sum over \( n \) in (3) diverges. The loop length diverges and the loop fills the entire system with superflow, a characteristic feature of the phase transition into the superfluid state. A large energy of a loop will always be canceled by the configurational entropy if the temperature is sufficiently large.

A decrease of the parameter \( \delta \) in (3) brings the phase transition into the first-order regime. In the loop picture, this change the \( n \)-dependence of the energy \( \varepsilon_n \). In the partition function (3), The entropy \( s_n \) of the loops in the small-\( \beta \) expansion of (3) depends on \( n \) as shown in Fig. 1. After an initial rise it flattens out somewhat around \( n \approx 10 \), where it merges into the asymptotic linear behavior \( \sigma n \). The energy may depend on \( n \) in different characteristic ways, also indicated in Fig. 1. The region \( n \approx 10 \) where the linear behavior is reached is determined by the effective stiffness of the vortex lines.

The associated free energies \( f_n \) have the typical shapes displayed in Fig. 1. The left-hand plot shows the free energy for \( \varepsilon \) in an ordinary \( \chi \)-model. For sufficiently large temperatures, it possesses a minimum at a nonzero value of \( n \), say at \( n_m \). This value moves continuously from zero to infinity as \( \beta \nu \) is raised above the critical value \( \beta^c \). The transition is of second order. Even before the critical value is reached, there are loops of size \( n^\kappa \) in the system. Such precritical loops are found in Monte-Carlo simulations of the model (3). They are plotted as 3D-figures in Ref. 4.

The free energy in the right-hand plot of Fig. 2 corresponds to the energy \( \varepsilon \) in an ordinary \( \chi \)-model, and shows a completely different behavior. As the critical value \( \beta^c \) is reached, the free energy has a barrier at \( n_m \) which prevents the lines from growing infinitely long. Thermal fluctuations have to create a loop of length \( n_m \), which can then expand and fill the entire system with superflow, thereby converting the normal state of the \( \chi \)-model into a superfluid one, or the ordered state of a superconductor with magnetic vortex lines into the normal state. The size of \( n_m \) is of the order of the length scale of stiffness \( n^\kappa \).

Since the superconductor on a lattice can be represented exactly in terms of loops, there is no place for bubble nucleation in such a system. But there are also simple energy-entropy arguments to justify this conclusion. 3. Consider a possible nucleation of the transition by bubbles (3). Such bubbles can be calculated in a continuous approximation to the partition function (2) which can be derived by standard field-theoretic techniques (5). In this approximation, the partition function (2) becomes a functional integral over a complex disorder field \( \phi(x) \) with quartic and sextic self-interactions (5). When cooling the disordered phase slightly below the transition point, such a field theory possesses spherically-symmetric solutions whose inside contains the ordered phase whose energy is slightly lower than that of the disordered phase. Let \( \epsilon \) be the difference in energy density and \( \sigma \) the surface energy density. The total energy of the bubble is then

\[
E_{\text{bubble}} = S_D R^{D-1} \sigma - \frac{S_D}{D} R^D \epsilon,
\]

(4)
where \( S_D = 2\pi^{D/2}/\Gamma(D/2) \) is the surface of a unit sphere in \( D \) dimensions. This energy is maximal at \( R_c = (D-1)\sigma/\epsilon \), where it is equal to

\[
E_c = \frac{S_D}{D} (D-1)^{D-1} \frac{\sigma^D}{\epsilon^{D-1}}.
\]

(5)
The important point is now that for temperatures which lie only very little beyond the transition temperature, the energy difference \( \epsilon \) between the two phases is very small, corresponding a huge bubble radius and energy. The probability of nucleating such a bubble is therefore infinitesimally small. Only after considerable overheating (or overcooling) does the bubble energy become small enough to nucleate spontaneously (in the absence of other condensation nuclei such as dirt). In freezing transition of water, the radius \( r_c \) is about 50 Å.

In a superconductor, however, the phase transition proceeds without overheating, and the reason for this is the vortex nucleation mechanism discussed above. The energy of a critical vortex may be estimated by imagining a planar phase boundary rolled up to a thin line whose radius is the coherence length \( \xi_0 \) of the disorder theory. This, in turn, is bent into a doughnut of radius \( n^\kappa \xi_0 \). Neglecting the bending energy, we estimate the critical vortex energy to be of the order

\[
E_{\text{vort}} \approx 2\pi \xi_0 \times n^\kappa \xi_0 \sigma.
\]

(6)
This energy does not depend on the energy difference \( \epsilon \) between the two phases, so that the rate is practically independent of the degree of overheating (or undercooling), this being in contrast to the energy of the critical bubble which is extremely large slightly beyond the transition point.

Note that in contrast to vortex nucleation, bubble nucleation is not enhanced significantly by the configurational entropy of the bubble surface. The reason is that apart from translations, all surface fluctuations are massive (3). Configurational distortions of a long vortex line,
on the other hand, require practically no energy if tak-
ing place over length scales larger than the finite stiffness
length.

4. The above argument imply that bubble nucleation is
of no relevance to first-order phase transition in supercon-
ductors, and for that matter, to the first-order transitions
in the early universe, as long as the theory describing the
latter allows for line-like topological excitations. These
drive the transition with much greater efficiency than
bubbles due to their larger configurational entropy.

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culated by
P. Butera and M. Comi, (to be published).
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numbers to me prior to publication.
FIG. 1. Dependence of the entropy of lines of length \( n \) in the partition function (2) on the length \( n \) of the loops. The curves above and below are possible energies \( \varepsilon_n \) leading to second- or first-order phase transitions.

FIG. 2. Free energy as a function of loop length \( n \) for second- and first-order phase transitions at different inverse model temperatures \( \beta_V \).