Abstract

We introduce the notion of weight for the lattice dimension and the notion of topological dimension – hole dimension. The condensate in Bose–holes exists in the case when temperature is not low.

In our previous papers we applied the quantum statistics approach to linguistic statistics [1]. To each word a frequency dictionary assigns its number of occurrences in the corresponding corpus of texts. There may be several words with the same number of occurrences.

The rank of a word (i.e., the order number of a word in a frequency dictionary) is measured from the word $A_1$ with the highest number of occurrences $s$ in a given corpus of texts. The number of all words with the same number of occurrences $i$ we designate as $N_i$. Then, if $A_m$ is the last word in a list of words with number of occurrences $m$ (the words of the same number of occurrences can be arranged in an arbitrary way), then the rank $r_m$ of $A_m$ is obviously

$$r_m = \sum_{i=m}^{s} N_i.$$

There is an analogy between the Bose particles at the energy level of an oscillator $\lambda_i = i$ and the words with the occurrence number $i$, namely, the words with the same occurrence number can be ordered in an arbitrary way, say, alphabetically, inversely alphabetically, or in any other order. The indexing of the ranks (the indices) of words within the family of a given occurrence number is arbitrary. In this sense, the words are indistinguishable and are distributed according to the Bose statistic.

However, there is a difference between the approaches under consideration. In the frequency dictionary one evaluates the number of occurrences of every word and then orders the words, beginning with the most frequently occurring words.

When there were no computers, it was difficult for a person to evaluate the number of words with equal occurrence number. By looking at a page as if it were a picture, a person can determine a desired word on this page by its graphical form at every place of occurrence of the word. In this case, the person looks at a page of the text as if it were a photo, without going into the meaning. Similarly, if a person looks for a definite name in a long list of intrants who had entered a college, this person finds (or does not find) the desired name by eyes rather than reads all the names one after another.

An eye gets into the way of recognizing the desired image, and this ability intensifies as the viewed material increases: the more pages the eye scans, the less is the difficulty...
in finding the desired graphical form. Therefore, under a manual counting, it was simpler to recognize the desired word on a page without reading the text and to cross it out by a pencil, simultaneously counting the number of occurrences of the word. This procedure is repeated for any subsequent word, already using the text with the words crossed out (“holes”), which facilitates the search. In other words, the procedure is in the recognition of the image of the given word, similar to the recognition, say, of a desired mushroom in a forest without sorting out all the plants on the soil one after another. An ordinary computer solves problems of this kind by exhaustion, whereas a quantum computer (see [2, 3]) makes this by recognizing the image.

However, for an ordinary computer, the number of operations needed to find the occurrence frequency of a word is less than the number of operations needed to find the number of words in the text with a given occurrence frequency.

One can say that the number of mushrooms we gathered (took away from the forest) is the number of holes we left in the forest. Similarly, the words we had “got out” from the text in the above way is an analog of holes rather than particles. Therefore, the linguists count the rank of words starting from the opposite end as compared to the starting end which would be used by physicists. The physicists would count the particles starting from the lowest level, whereas the holes, the absent electrons, would be counted from the highest level.

For this reason, the words in a frequency dictionary are associated with holes rather than particles. Correspondingly, the dimension in the distribution of frequency dictionaries is to be chosen as a “hole” dimension (“Dirac’s hole”), which is negative.

We will prove a cumulative formula in which the densities coincide in shape with the Bose–Einstein distribution. The difference consists only in that, instead of the set \( \lambda_n \) of random variables or eigenvalues of the Hamiltonian operator, the Bose–Einstein formula contains some of their averages over the cells [4]. In view of the theorem given below, one can proof that the \( \varepsilon_i \), which are averages of the energy \( \lambda_k \) at the \( i \)th cell, are nonlinear averages in the sense of Kolmogorov [5].

As in [6], the values of the random variable \( \lambda_1, \ldots, \lambda_s \) are ordered in absolute value. Some of the numbers \( \lambda_1, \ldots, \lambda_s \) may coincide. Then these numbers are combined adding the corresponding ”probabilities”, i.e., the ratio of the number of “hits” at \( \lambda_i \) to the general number of trials. The number of equal \( \lambda_i : \lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+k} \) will be called the multiplicity \( q_i \) of the value \( \lambda_i \). In our consideration, both the number of trials \( N \) and \( s \) tend to infinity.

Let \( N_i \) be the number of ”appearances” of the value \( \lambda_i : \lambda_i < \lambda_{i+1} \), then

\[
\sum_{i=1}^{s} \frac{N_i}{N} \lambda_i = M, \tag{1}
\]

where \( M \) is the mathematical expectation.

The cumulative probability \( P_k \) is the sum of the first \( k \) probabilities in the sequence \( \lambda_i \): \( P_k = \frac{1}{N} \sum_{i=1}^{k} N_i \), where \( k < s \). We denote \( NP_k = B_k \).

If all the variants for which

\[
\sum_{i=1}^{s} N_i = N \tag{2}
\]

and

\[
\sum_{i=1}^{s} N_i \lambda_i \leq E, \quad E = MN \leq N \overline{X}, \tag{3}
\]
where $\lambda = \sum_{i=1}^{s} q_i \lambda_i$, $Q = \sum_{i=1}^{s} q_i$, are equivalent (equiprobable), then [5] [7] [8] the majority of the variants will accumulate near the following dependence of the "cumulative probability" $B_i\{N_i\} = \sum_{i=1}^{l} N_i$,

$$\sum_{i=1}^{l} N_i = \sum_{i=1}^{l} \frac{q_i}{e^{\beta \lambda_i - \nu - 1}},$$

where $\beta'$ and $\nu'$ are determined by the conditions

$$B_s = N,$$

$$\sum_{i=1}^{s} q_i \lambda_i e^{\beta \lambda_i - \nu - 1} = E,$$

as $N \to \infty$ and $s \to \infty$.

We introduce the notation: $\mathcal{M}$ is the set of all sets $\{N_i\}$ satisfying conditions [2] and [3]; $N \{\mathcal{M}\}$ is the number of elements of the set $\mathcal{M}$.

**Theorem 1** Suppose that all the variants of sets $\{N_i\}$ satisfying the conditions [2] and [3] are equiprobable. Then the number of variants $N$ of sets $\{N_i\}$ satisfying conditions [2] and [3] and the additional relation

$$| \sum_{i=1}^{l} N_i - \sum_{i=1}^{l} \frac{q_i}{e^{\beta \lambda_i - \nu - 1}} | \geq N^{(3/4 + \varepsilon)}$$

is less than $c_1 N \{\mathcal{M}\} / N^{m}$ (where $c_1$ and $m$ are any arbitrary numbers, $\sum_{i=1}^{l} q_i \geq \varepsilon Q$, and $\varepsilon$ is arbitrarily small).

**Proof of Theorem 1.**

Let $\mathcal{A}$ be a subset of $\mathcal{M}$ satisfying the condition

$$| \sum_{i=1}^{s} N_i - \sum_{i=1}^{s} \frac{q_i}{e^{\beta \lambda_i - \nu - 1}} | \leq \Delta;$$

$$| \sum_{i=1}^{l} N_i - \sum_{i=1}^{l} \frac{q_i}{e^{\beta \lambda_i - \nu - 1}} | \leq \Delta,$$

where $\Delta$, $\beta$, $\nu$ are some real numbers independent of $l$.

We denote

$$| \sum_{i=1}^{s} N_i - \sum_{i=1}^{s} \frac{q_i}{e^{\beta \lambda_i - \nu - 1}} | = S_{s-l};$$

$$| \sum_{i=1}^{l} N_i - \sum_{i=1}^{l} \frac{q_i}{e^{\beta \lambda_i - \nu - 1}} | = S_l.$$

Obviously, if $\{N_i\}$ is the set of all sets of integers on the whole, then

$$N \{\mathcal{M} \setminus \mathcal{A}\} = \sum_{\{N_i\}} \left( \Theta(E - \sum_{i=1}^{s} N_i \lambda_i) \delta_{(\sum_{i=1}^{s} N_i)\nu} \Theta(S_l - \Delta) \Theta(S_{s-l} - \Delta) \right),$$

(8)
where \( N_i \) are arbitrary integers.

Here the sum is taken over all integers \( N_i \), \( \Theta(\lambda) \) is the Heaviside function, and \( \delta_{k_1,k_2} \) is the Kronecker symbol.

We use the integral representations

\[
\delta_{NN'} = \frac{e^{-\nu N}}{2\pi} \int_{-\pi}^{\pi} d\varphi e^{-iN\varphi} e^{\nu N'} e^{iN'\varphi},
\]

(9)

\[
\Theta(y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\lambda \frac{1}{\lambda - i} e^{\beta y(1+i\lambda)}.
\]

(10)

Now we perform the standard regularization. We replace the first Heaviside function \( \Theta \) in (8) by the continuous function

\[
\Theta_\alpha(y) = \begin{cases} 0 & \text{for } \alpha > 1, \ y < 0 \\ 1 - e^{\beta y(1-\alpha)} & \text{for } \alpha > 1, \ y \geq 0 \\ e^{\beta y(1-\alpha)} & \text{for } \alpha < 0, \ y < 0 \\ 1 & \text{for } \alpha < 0, \ y \geq 0 \end{cases}
\]

where \( \alpha \in (-\infty, 0) \cup (1, \infty) \) is a parameter, and obtain

\[
\Theta_\alpha(y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\beta y(1+ix)} \left( \frac{1}{x+i} - \frac{1}{x-\alpha i} \right) dx.
\]

(11)

If \( \alpha > 1 \), then \( \Theta(y) \leq \Theta_\alpha(y) \).

Let \( \nu < 0 \). We substitute (9) and (10) into (8), interchange the integration and summation, then pass to the limit as \( \alpha \to \infty \) and obtain the estimate

\[
N \{ M \setminus A \} \leq \left| \frac{e^{-\nu N + \beta E}}{i(2\pi)^2} \int_{-\pi}^{\pi} \exp(-iN\varphi) \sum_{\{N_j\}} \exp\left\{ -\beta \sum_{j=1}^{s} N_j\lambda_j + (i\varphi + \nu) \sum_{j=1}^{s} N_j \right\} d\varphi \times \right.
\]

\[
\times \Theta(S_{l} - \Delta) \Theta(S_{s-l} - \Delta) \right|,
\]

(12)

where \( \beta \) and \( \nu \) are real parameters such that the series converges for them.

To estimate the expression in the right-hand side, we bring the absolute value sign inside the integral sign and then inside the sum sign, integrate over \( \varphi \), and obtain

\[
N \{ M \setminus A \} \leq \frac{e^{-\nu N + \beta E}}{2\pi} \sum_{\{N_i\}} \exp\left\{ -\beta \sum_{i=1}^{s} N_i\lambda_i + \nu \sum_{i=1}^{s} N_i \right\} \times \Theta(S_{l} - \Delta) \Theta(S_{s-l} - \Delta).
\]

(13)

We denote

\[
Z(\beta, N) = \sum_{\{N_i\}} e^{-\beta \sum_{i=1}^{s} N_i\lambda_i},
\]

(14)

where the sum is taken over all \( N_i \) such that \( \sum_{i=1}^{s} N_i = N \),

\[
\zeta_1(\nu, \beta) = \prod_{i=1}^{l} \zeta_i(\nu, \beta); \zeta_{s-l}(\nu, \beta) = \prod_{i=l+1}^{s} \zeta_i(\nu, \beta);
\]
\[ \xi_i(\nu, \beta) = \frac{1}{(1 - e^{\nu - \beta \lambda_i}) q_i}, \quad i = 1, \ldots, l. \]

It follows from the inequality for the hyperbolic cosine \( \cosh(x) = (e^x + e^{-x})/2 \) for \(|x_1| \geq \delta; |x_2| \geq \delta:\)

\[ \cosh(x_1) \cosh(x_2) = \cosh(x_1 + x_2) + \cosh(x_1 - x_2) > \frac{e\delta}{2} \tag{15} \]
that the inequality

\[ \Theta(S_{s-I} - \Delta) \Theta(S_I - \Delta) \leq e^{-c\Delta} \cosh \left( c \sum_{i=1}^{l} N_i - c\phi_I \right) \cosh \left( c \sum_{i=l+1}^{s} N_i - c\phi_{s-l} \right), \tag{16} \]

where

\[ \phi_I = \sum_{i=1}^{l} \frac{q_i e^{\beta \lambda_i - \nu'}}{e^{\beta \lambda_i - \nu'} - 1}; \quad \phi_{s-l} = \sum_{i=l+1}^{s} \frac{q_i e^{\beta \lambda_i - \nu'}}{e^{\beta \lambda_i - \nu'} - 1}, \]

holds for all positive \( c \) and \( \Delta. \)

We obtain

\[ \mathcal{N} \{ \mathcal{M} \setminus \mathcal{A} \} \leq e^{-c\Delta} \exp(\beta E - \nu N) \times \]
\[ \times \sum_{\{N_i\}} \exp \left\{ -\beta \sum_{i=1}^{l} N_i \lambda_i + \nu \sum_{i=1}^{l} N_i \right\} \cosh \left( \sum_{i=1}^{l} cN_i - c\phi \right) \times \]
\[ \times \exp \left\{ -\beta \sum_{i=l+1}^{s} N_i \lambda_i + \nu \sum_{i=l+1}^{s} N_i \right\} \cosh \left( \sum_{i=l+1}^{s} cN_i - c\phi \right) = \]
\[ = e^{\beta E} e^{-c\Delta} \times \]
\[ \times (\zeta_I(\nu - c, \beta) \exp(-c\phi_I) + \zeta_I(\nu + c, \beta) \exp(c\phi_I)) \times \]
\[ \times (\zeta_{s-l}(\nu - c, \beta) \exp(-c\phi_{s-l}) + \zeta_{s-l}(\nu + c, \beta) \exp(c\phi_{s-l})). \tag{17} \]

Now we use the relations

\[ \frac{\partial}{\partial \nu} \ln \zeta_I|_{\beta=\beta',\nu=\nu'} \equiv \phi_I; \quad \frac{\partial}{\partial \nu} \ln \zeta_{s-l}|_{\beta=\beta',\nu=\nu'} \equiv \phi_{s-l} \tag{18} \]

and the expansion \( \zeta_I(\nu \pm c, \beta) \) by the Taylor formula. There exists a \( \gamma < 1 \) such that

\[ \ln(\zeta_I(\nu \pm c, \beta)) = \ln(\zeta_I(\nu, \beta) \pm c(\ln \zeta_I)'(\nu, \beta) + \frac{c^2}{2}(\ln \zeta_I)''(\nu \pm \gamma c, \beta). \]

We substitute this expansion, use formula (15), and see that \( \phi_{\nu, \beta} \) is cancelled.

Another representation of the Taylor formula implies

\[ \ln(\zeta_I(\nu + c, \beta)) = \ln(\zeta_I(\beta, \nu)) + \frac{c}{\beta} \frac{\partial}{\partial \nu} \ln(\zeta_I(\beta, \nu)) + \]
\[ + \int_{\nu}^{\nu+c/\beta} dv' (\nu + c/\beta - v') \frac{\partial^2}{\partial v'^2} \ln(\zeta_I(\nu', \beta')). \tag{19} \]

A similar expression holds for \( \zeta_{s-l}. \)

From the explicit form of the function \( \zeta_I(\beta, \nu) \), we obtain

\[ \frac{\partial^2}{\partial \nu^2} \ln(\zeta_I(\beta, \nu)) = \beta^2 \sum_{i=1}^{l} \frac{g_i \exp(-\beta(\lambda_i + \nu)}{\exp(-\beta(\lambda_i + \nu)) - 1)^2} \leq \beta^2 Qd, \tag{20} \]
where \(d\) is given by the formula
\[
d = \frac{\exp(-\beta(\lambda_1 + \nu))}{(\exp(-\beta(\lambda_1 + \nu)) - 1)^2}.
\]
The same estimate holds for \(\zeta_{s-l}\).

Taking into account the fact that \(\zeta_l \zeta_{s-l} = \zeta_s\), we obtain the following estimate for \(\beta = \beta'\) and \(\nu = \nu'\):
\[
N\{M \setminus A\} \leq \zeta_s(\beta', \nu') \exp(-c\Delta + \frac{\nu'^2}{2} \beta'^2 Qd) \exp(E\beta' - \nu'N). \tag{21}
\]

Now we express \(\zeta_s(\nu', \beta')\) in terms \(Z(\beta, N)\). To do this, we prove the following lemma.

**Lemma 1** Under the above assumptions, the asymptotics of the integral
\[
Z(\beta, N) = \frac{e^{-\nu N}}{2\pi} \int_{-\pi}^{\pi} d\alpha e^{-iN\alpha} \zeta_s(\beta, \nu + i\alpha) \tag{22}
\]
has the form
\[
Z(\beta, N) = C e^{-\nu N} \frac{\zeta_s(\beta, \nu)}{|(\partial^2 \ln \zeta_s(\beta, \nu))/((\partial^2 \nu))|} (1 + O\left(\frac{1}{N}\right)), \tag{23}
\]
where \(C\) is a constant.

The proof of the lemma readily follows from the saddle-point method and the inequalities
\[
|\xi_i(\nu + i\alpha, \beta)| < \xi_i(\nu, \beta), \quad |\zeta_s(\nu + i\alpha, \beta)| < \zeta_s(\nu, \beta), \tag{24}
\]
which hold, because \(e^{\nu - \beta \lambda_i} < 1\) for all \(\alpha \neq 2\pi n\), where \(n\) is an integer. It follows from these inequalities that \(\alpha = 0\) is a saddle point of integral (22) [9, 10].

**Lemma 2** The quantity
\[
\frac{1}{N(M)} \sum_{\{N_i\}} e^{-\beta \sum_{i=1}^{s} N_i \lambda_i}, \tag{25}
\]
where \(\sum N_i = N\) and \(\lambda_i N_i \leq E - N^{1/2+\varepsilon}\), tends to zero faster than \(N^{-k}\) for any \(k, \varepsilon > 0\).

We consider the point of minimum in \(\beta\) of the right-hand side of (17) with \(\nu(\beta, N)\) satisfying the condition
\[
\sum e^{\beta \lambda_i - \nu(\beta, N)} - 1 = N.
\]
It is easy to see that it satisfies condition (5). Now we assume that the assumption of the lemma is not satisfied.

Then for \(\sum N_i = N\), \(\sum \lambda_i N_i \geq E - N^{1/2+\varepsilon}\), we have
\[
e^{-\beta E} \sum_{\{N_i\}} e^{-\beta \sum_{i=1}^{s} N_i \lambda_i} \geq e^{(N^{1/2+\varepsilon})\beta}.
\]
Obviously, \(\beta < \frac{1}{\sqrt{N}}\) provides a minimum of (17) if the assumptions of Lemma 1 are satisfied, which contradicts the assumption that the minimum in \(\beta\) of the right-hand side of (17) is equal to \(\beta'\).
We set \( c = \frac{\Delta}{N^{1+\alpha}} \) in formula (21) after the substitution (23); then it is easy to see that the ratio
\[
\frac{\mathcal{N}(\mathcal{M} \setminus \mathcal{A})}{\mathcal{N}(\mathcal{M})} \approx \frac{1}{N^m},
\]
where \( m \) is an arbitrary integer, holds for \( \Delta = N^{3/4+\varepsilon} \). The proof of the theorem is complete.

Now we consider the notion of the lattice dimension.

We consider a straight line, a plane, and a three-dimensional space. We separate points \( i = 0, 1, 2, \ldots \) on the line and points \( x = i = 0, 1, 2, \ldots \) on the coordinate axes \( x, y \) on the plane. We associate this set of points \((i, j)\) with the points on the straight line (with the positive integers \( l = 1, 2, \ldots \)) up to the quantum constant \( \chi \) of the lattice.

According to M. Gromov’s definition [11], the asymptotic (topological) dimension of this lattice is equal to two.

We associate each point with a pair of points \( i \) and \( j \) according to the rule \( i + j = l \). The number of such points \( n_l \) is equal to \( l + 1 \). In addition, we assume that \( z = k = 0, 1, 2, \ldots \) on the axis, i.e., we set \( i + j + k = l \). In this case, the number of points \( q_l \) is equal to
\[
q_l = \frac{(l + 1)(l + 2)}{2}.
\]

If we set \( \lambda_i = l \) in formula (11), then, in the three-dimensional case, each \( i \) is associated with \( \frac{(l+1)(l+2)}{2} \) of mutually equal \( x_i = l \) (these are the multiplicities or the \( q_l \)–hold degeneracies of the spectrum of the oscillator). Formula (11) in this special case becomes
\[
N_i = \text{const} \sum_{i=0}^{l} \frac{(i + 1)(i + 2)}{2(e^{\beta_i} - 1)}; \quad \Delta N_i = \text{const} \frac{(i + 1)(i + 2)}{2(e^{\beta_i} - 1)} \Delta_i, \quad \Delta_i = 1,
\]
\[
\Delta E_i = \text{const} \frac{i(i + 1)(i + 2)}{2(e^{\beta_i} - 1)} \Delta_i
\]
for large \( i \), \( \frac{\Delta_i}{i} \to 0 \),
\[
dE = \text{const} \frac{\omega^3 d\omega}{e^{\beta \omega} - 1}; \quad \beta = \frac{h}{T}
\]
(cf. formula (60.4) in [12]).

Thus, we obtain a somewhat sharper version of the famous Planck formula for the radiation of a black body.

For the \( D \)-dimensional case, it is easy to verify that the sequence of weights (multiplicities) of the number of versions \( i = \sum_{k=1}^{D} m_k \), where \( m_k \) are arbitrary positive integers, has the form of the binomial coefficient
\[
q_i(D) = \text{const} \frac{(i + D - 1)!}{i! D!},
\]
where the constant depends on \( D \).

Thus, for any \( D \), formula (11) has the form
\[
N_i = \text{const} \sum_{i=1}^{l} \frac{q_i(D)}{e^{\beta_i} - 1}.
\]
For the positive integers, we have a sequence of weights \( q_i \) (or, simply, a weight) of the form \((30)\).

Our weight series can easily be continued to an arbitrary case by replacing the factorials with the \( \Gamma \)-functions; in this case, we assume that \( D \) is negative.

This is the negative topological dimension (the hole dimension) of the quantized space (lattice).

If \( D > 1 \), then, as \( i \to \infty \), a condensation of a sufficiently small perturbation occurs in the spectrum of the oscillator and the multiplicities split, i.e., the spectrum becomes denser as \( i \) increases. The fact that \( D \) is negative means that there is strong rarefaction in the spectrum as \( i \to \infty \) (the constant in formula \((31)\) must be sufficiently large).

For non-positive integer \( D \), the terms \( i = 0, 1, 2, 3, \ldots, -D \) become infinite. This means that they are very large in the experiment, which permits determining the lattice negative dimension corresponding to a given problem. We note that a new condensate occurs, which is possible for small \( \beta \).

Now we return to frequency dictionaries. A frequency dictionary is composed using texts from a certain array. Each word in the dictionary is associated with the number of its encounters in the original array of texts. We denote this number by \( \omega_i \) and the number of words corresponding to this number of encounters by \( N_i \). The sum of all \( N_i \) is equal to the volume of the entire dictionary, and the sum of products \( \omega_i n_i \) is equal to the volume of the entire array of texts used to compose the dictionary.

The number of words encountered only once in the array of texts is approximately equal to \( 1/3 \) of the entire frequency dictionary which the number of words equal to \( N \). So as \( N \to \infty \) this is the condensate. It follows from the above that \( D = -1 \) for the dictionary. Hence, for \( \beta \ll 1 \) and \( \nu \sim 1 \), we have

\[
N_i = \text{const} \sum_{i=2}^{l} \frac{1}{i(i-1)(e^{\beta i - \nu} - 1)} \sim \text{const} \int_\omega^\omega \frac{d\omega}{\alpha \omega(\alpha \omega - 1)(e^{\beta \alpha \omega - \nu} - 1)}, \tag{32}
\]

where \( \omega = l \) and \( \alpha \) is the scale constant. If \( \omega \) is finite and \( \beta \ll 1 \) the integral may be taken.

For the frequency of ”Japanese candles” of 30 stocks in the stock market (see [13]), the dimension is equal to 0.

Figures 1 and 2 show the rank-frequency curves for Leo Tolstoy’s War and Peace. In Fig. 1, the low-frequency part of the dictionary is approximated by formula \((32)\) as \( \beta = 0 \) (words with frequencies \( \leq 3 \); \( r_0 = 43408.8, \alpha = 0.891995, \) and \( c_1 = -189.321 \)). For the entire dictionary, Fig. 2 shows the deviation of the theoretical data from the dictionary data (the difference between the frequency given in the dictionary and the frequency given by the formula) against rank. The deviation is seen not to exceed 1.5 words if the rank is larger than 300.

**Examples of holes.**

In the Frenkel’ theory of crystals, a hole (Dirac’s hole) is an absent electron, and it behaves as a particle moving along the crystal. However, in contrast to an electron, this particle cannot exist outside the crystal; there is no hole in vacuum. In the same way, a hole, as we understand it, cannot exist without a sufficiently large scale of spaces in which it is “made.”

In fact, the fractal dimension of a hole, a crack in a rock, etc., depends on the scale at which “our eye” distinguishes points. For instance, the coastline is of fractal nature when observed from a plane. However, if we stand at the very coast, then this line is smooth, and its dimension is equal to that of a smooth curve, i.e., to one.
When considering cracks in metal, the dimension depends on the resolving capacity of the device used to observe the crack [14, 15, 16]. Denote the resolving capacity of the device by $\kappa$. The usual definition of the Hausdorff dimension is based on the condition that some balls (as a rule, three-dimensional in practice) vanish. This means that the volume $\Omega$ of the metal under consideration must be much greater than $\kappa^3$. Thus, reducing the problem to unit volume, we can see that the radius of the Hausdorff ball tends to zero as $\Omega \to \infty$.

The hole–wave propagation is especially visual when considering elastic waves in media in which the Young modulus is sufficiently large with respect to compression and vanishes with respect to extension. In particular, sand has this property. The waves-holes were studied in detail by the author together with P. P. Mosolov in the case of one spatial coordinate ([18], Appendix).

A worm corrodes a complicated serpiginous path-hole in the stem of a tree. This path is very small with respect to the stem, and we can assume that the thickness of the stem is infinitely large as compared with the hole.

The dimension of a wire of the same thickness and the same “sinuosity” can be defined independently of the surrounding medium, whereas one cannot consider the hole (made by the worm) beyond the stem, although both the dimensions have the same absolute value.

Let us now present an economical example.

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1That is, of the ball used in the definition of the Hausdorff dimension, see [17].
Assume that somebody has inherited a great legacy in diverse forms and frivols it away “to the left and to the right.” If it is complicated to evaluate the amount of legacy, then the amount of expenses increasing in time $t$ according to the power law $t^k$ (the appetite of expenses can increase, as well as the appetite of profits!), then $k$ is the negative dimension, or $k$ is the dimension of the “hole” thus occurring.

Let us now consider the simplest examples of measures in the general case for the $n$-dimensional space. Let $S_n$ be an $n$-dimensional ball of radius $r$. In the spherical coordinates, the volume $\mu(S_n)$ of the ball is equal to $\text{const} \int_0^1 r^{n-1} dr = \text{const} r^n$. Here $r^{n-1}$ stands for the density.

In the sense of the Fourier transform, the multiplication by a coordinate is dual to the corresponding derivation. Therefore, we can speak of dual $n$ times differentiable functions in the Sobolev space $W_2^n$. Dirac distinguished between the left and right components “bra” and ”ket” in the ”bracket” inner product. The “dual” space of this space according to Dirac is the space $W_2^{-n}$ of Sobolev distributions (generalized functions).

In the same way we can define the functions in $W_2^s$ by the “inner product,” where $s$ is a positive noninteger number, and the space $W_2^{-s}$ as the “inner product” conjugate to $W_2^s$.

One can similarly proceed with the density (or the weight) $r^s$ and $r^{-s}$, by using, for instance, the Riesz kernel to represent functions in $W_2^s$.

Let us present an example of a space (of noninteger positive dimension) equipped with the Haar measure $r^\sigma$, where $0 \leq \sigma \leq 1$.

On the closed interval $0 \leq x \leq 1$ there is a scale $0 \leq \sigma \leq 1$ of Cantor dust with the Haar measure equal to $x^\sigma$ for any interval $(0, x)$. The direct product of this scale by the Euclidean cube of dimension $k - 1$ gives the entire scale $k + \sigma$, where $k \in \mathbb{Z}$ and $\sigma \in (0, 1)$. We consider the space of negative
The Japanese candle order number vs. frequency for 30 stocks: $R$ is the candle order number, starting from the smallest candle; $\omega$ is the candle frequency.

dimension $-D = -k - \sigma$ with respect to this very space. “Quantizing” of the densities considered above leads to the density of the form $\Gamma(D + l)/(\Gamma(D + 1)\Gamma(l + 1))$.

The following problem arises further: How one can understand a scale of spaces in specific problems on a crack whose dimension $t$ increases as $t \to \infty$? The size (volume) of the metal or rock is not a dimension because all these objects are three-dimensional. Where a sufficiently large value of dimension arises?

For comparison, consider a gas in a three-dimensional vessel. Every molecule of gas moves. The number of molecules is $\approx 10^{23}$. If a motion of every molecule is considered, then, as a rule, one assumes that the entire dimension is equal to $10^{69}$.

On the other hand, one can consider all molecules as points of three-dimensional space. If we also take into account the displacement at time $t$, then we can consider the points in the four-dimensional space as well. This very duality enabled us (together with my co-author P. P. Mosolov) to pass from the Newton equations for every molecule to equations of field-like hydrodynamics of the entire gas [19, 20].

If we choose two moments of time for the displacement of the molecules of gas in the $10^{69}$-dimensional space, i.e., consider the displacement of all $10^{23}$ molecules as a single point at the first moment and as another point at the other moment, then we can introduce the natural distance between these points, i.e., a metric.

One can construct a scale of metrics between this metric and the metric in three-dimensional space by combining diverse groups of particles.

From the viewpoint of mechanics (elasticity theory) and the scales typical for this science, metals, as well as rocks, consist of “grains” of sufficiently small (rather than atomic) measure–volume. These grains can be of different size, and we can regard the set of these grains as a point in the space of sufficiently large dimension [21, 22]. This very interpretation enables us to pass to the general definition of a fractal crack as a hole and present the abstract mathematical definition.
General definition of spaces of negative negative dimension. Let \( M_{t_0} \) be a compactum, of Hausdorff dimension \( t_0 \), which is an element of a \( t \)-parameter scale of mutually embedded compacta, \( 0 < t < \infty \). Two scales of this kind are said to be equivalent with respect to the compactum \( M_{t_0} \) if all compacta in these scales coincide for any \( t \geq t_0 \). We say that the compactum \( M_{t_0} \) is a hole in this equivalent set of scales and the number \(-t_0\) is the negative dimension of this equivalence class.

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