On the Vanishing Topology of Isolated Cohen-Macaulay Codimension 2 Singularities

Anne Frühbis-Krüger, Matthias Zach
Institut f. Alg. Geometrie, Leibniz Universität Hannover, Germany

January 9, 2015

Abstract

Isolated Cohen-Macaulay codimension 2 singularities share many common features with isolated complete intersection singularities, but they also exhibit some striking new behaviour. One such instance was recently observed by Damon and Pike [9] in their study of the vanishing topology and Euler characteristic, where they took this class of singularities as examples. In this article, we explore their findings further by determining the Betti numbers explicitly and explain the new phenomena. An important tool here is the Tjurina modification relating a Cohen-Macaulay codimension 2 singularity to a finite number of complete intersection singularities.

1 Introduction

Isolated hypersurfaces singularities and a bit more generally isolated complete intersection singularities have been a central focus of singularity theory ever since the famous A-D-E list of Arnold [1] classifying the simple hypersurface singularities. Classification questions as well as topological and analytic properties of these singularities have been studied intensively over the last decades (e.g. [14], [16]); many of the properties can be expressed in terms of invariants of the singularities and relations among these. The most famous ones are the Milnor number $\mu$ on the topological side and the Tjurina number $\tau$, i.e. the dimension of the $T^1$, describing important deformation properties. Following Damon and Pike we will view $\mu$ as the “defect of the Euler characteristic”. For an isolated singularity with unique Milnor fiber this is the difference between the topological Euler characteristic of any smooth and the central fiber in a deformation. For quasihomogeneous ICIS, equality of the two numbers holds (see [15]); for any general ICIS we still have $\mu \geq \tau$ (see [19]).

As soon as we pass beyond ICIS, however, only a few results are known. The easiest non-ICIS case is the case of isolated Cohen-Macaulay codimension 2 singularities where the Hilbert-Burch theorem allows a description by means
of the presentation matrix of the vanishing ideal. This case has recently come into focus of ongoing research, starting with the classification simple singularities in this case, first for space curves in [12] and later for arbitrary dimension in [11]. This, in turn led to further study of the properties of these singularities as in [25], [22], and most recently in [13] and [6]. But up to now the properties of the Milnor fibre of such singularities are far from being explored in all details.

Damon and Pike studied the vanishing topology of a generalizations of the Milnor fibre, the so-called singular Milnor fibre in [9], in a much broader context. They used the simple isolated Cohen-Macaulay codimension 2 singularities from the list by Frühbis-Kräger and Neumer [11] as examples to illustrate their methods. For the surface case, their approach (as well as independently by a different method introduced by da Silva Pereira and Ruas in [25]) provided direct computations for the Milnor number, which is the second (and only non-vanishing) Betti number of the Milnor fiber in this dimension.

Moving one dimension higher to isolated Cohen-Macaulay codimension 2 singularities of dimension 3, the situation becomes more delicate. Some facts are still known: the existence of a smoothing and the vanishing of the first Betti number have been shown by Greuel and Steenbrink in [16], whereas the vanishing of homology in degrees bigger than the complex dimension of the underlying variety is a well-known consequence of the Lefschetz hyperplane theorem (see [21]). The methods of Damon and Pike then allowed the computation of the difference \( b_3 - b_2 \) of the two remaining Betti-numbers, but not of each of the two separately. This is the defect of the Euler characteristic which we understand as Milnor number \( \mu \) as mentioned earlier. Nevertheless their results provided striking evidence for \( b_2 \) to be nonzero in some of the families of simple threefold singularities from [11], as the computed value of the difference was negative for these.

In this article, we extend a technique which was previously only used for surfaces, the Tjurina modification (see e.g. [27] or [26]), applying it to Cohen-Macaulay codimension 2 singularities in general. It allows us to relate the given singularity or family of singularities to a local complete intersection scheme or a family of local complete intersections factoring through a given deformation. Using this tool, we are then able to explain the observation of Damon and Pike, explicitly compute that Betti numbers \( b_2 \) and \( b_3 \) for all simple isolated Cohen-Macaulay codimension 2 threefold singularities and even state a large class of such singularities, including the simple ones, for which \( b_2 \) has to have the value 1.

It turns out that our results also allow some geometric insights into the interplay of Tjurina and Milnor number in the case of simple isolated Cohen-Macaulay codimension 2 3-fold singularities. Viewing the Tjurina modification from a different point of view, as a small birational map replacing the singular point by a set of codimension 2, we can also see new behaviour in comparison
to the results of Laufer, Pinkham and Friedman (see e.g. [10]) in the case of isolated Gorenstein 3-fold singularities.

In section 2 we briefly recall those of the known results about isolated Cohen-Macaulay codimension 2 singularities which will be needed later on. In the section 3, we consider the notion of a Tjurina modification in detail and extend it suitably to higher dimensions, larger matrices and families of singularities. We then recall important facts about the Milnor fibre and prove the main results in section 4. The last section then contains the application of the results to explicit examples.

We would like to thank Terence Gaffney, Wolfgang Ebeling, Wim Veys, Slawomir Rams, Victor Gonzalez Alonso, Miguel Marco and Jesse Kass for fruitful exchange of ideas on the topics related to in this article. This work is partially supported by funds of the research project 'Experimental methods in Computer Algebra' of the NTH.

2 Basic Facts on isolated Cohen-Macaulay codimension 2 singularities

Isolated Cohen-Macaulay codimension 2 singularities (abbreviated by ICMCd2 in the following) provide the most accessible setting for non-complete-intersection singularities. Contact equivalence, semi-universal deformation and simple objects are known in this case, see [12] and [11]. For the list of simple objects in the dimension 3 case, see table 1. In this section, we will briefly recall some of the basic facts for reader’s convenience.

Using the Hilbert-Burch theorem, all Cohen-Macaulay germs of codimension 2 can be expressed as the maximal minors of \((t+1) \times t\)-matrices \(M\) and vice versa. In the same way, flat deformations can be represented by perturbations of the matrix \(M\) and any perturbation gives rise to a flat deformation (cf. Burch [5], Schaps [24]). The minimal matrix size \(t\) is called the Cohen-Macaulay type of the singularity.

Classification up to contact-equivalence means that two singularities are considered equivalent, if their germs are isomorphic. The action of the contact-group translates directly to the application of coordinate changes and row and column operations on \(M\). A singularity is called simple, if it can only deform into finitely many different equivalence classes (types) of singularities.

For a more consistent notation, we prefer to describe the Cohen-Macaulay codimension 2 singularities by their presentation matrix instead of the vanishing ideal. This requires a reformulation of \(T_{1,0}^{1,0}\) in terms of the presentation matrix:
Lemma 2.1 ([12]). $T^1_{X,0}$ is given by

\[ T^1_{X,0} \cong \text{Mat}(t + 1, t; \mathbb{C}\{x_1, \ldots, x_n\})/(J_M + \text{Im}(g)) \]

where $J_M$ is the submodule generated by the matrices of the form

\[
\begin{pmatrix}
\frac{\partial M_{11}}{\partial x_j} & \cdots & \frac{\partial M_{11}}{\partial x_j} \\
\vdots & & \vdots \\
\frac{\partial M_{t+11}}{\partial x_j} & \cdots & \frac{\partial M_{t+11}}{\partial x_j}
\end{pmatrix}
\]

\[ \forall 1 \leq j \leq m \]

and $g$ is the map

\[ \text{Mat}(t + 1, t + 1; \mathbb{C}\{x_1, \ldots, x_m\}) \oplus \text{Mat}(t, t; \mathbb{C}\{x_1, \ldots, x_m\}) \mapsto \text{Mat}(t + 1, t; \mathbb{C}\{x_1, \ldots, x_m\}) \]

mapping $(A, B) \mapsto AM + MB$.

It is a well-known fact that $T^2_{X,0} = 0$ for Cohen-Macaulay codimension 2 singularities, i.e., that there are no obstructions to lifting first order deformations. As the Cohen-Macaulay codimension 2 singularities, we are considering, are isolated, $T^1_{X,0}$ is of finite dimension $\dim \mathbb{C}T^1_{X,0} = \tau$. Hence the base of the semiuniversal deformation of $(X, 0)$ is $\mathbb{C}^\tau$ and the total space is given by the minors of the matrix

\[ M_{su} = M + \sum_{i=1}^{\tau} s_im_i \in \text{Mat}(t + 1, t; \mathbb{C}[s_1, \ldots, s_\tau]\{x\}) \]

where the $s_i$ are the coordinates of $\mathbb{C}^\tau$ and $\{m_1, \ldots, m_\tau\}$ is a $\mathbb{C}$-basis of $T^1(X, 0)$ in the matrix notation of Lemma 1.

The above description of $T^1_{X,0}$ in terms of the presentation matrix, the nonexistence of obstructions and the explicit description of the semiuniversal deformation are the main reasons why this class of singularities is a natural choice for a first step beyond isolated complete intersection singularities: To study their deformations we can follow the main ideas used in the complete intersection case. This led to the complete classification of simple isolated Cohen-Macaulay codimension 2 singularities found in [11] which lists nearly 30 series and more than 20 exceptional cases in dimensions $0 \leq \dim(X, 0) \leq 4$.

On the other hand, there are certain structural properties of our singularities which do not coincide with the complete intersection case and are based on the fact that the ring of $(X, 0)$ is a determinantal ring (see e.g. [3] for a textbook on determinantal rings). A first occurrence of this situation is linked to the following well-known fact:

**Lemma 2.2.** For $k \leq l$, let $G \in \text{Mat}(l, k; \mathbb{C}\{y_{1,1}, \ldots, y_{l,k}\})$ be the matrix with entries $g_{i,j} = y_{i,j}$. Denote by $V_r \subset \mathbb{C}^{l,k} = \text{Mat}(l, k; \mathbb{C})$ the variety of matrices with rank $\leq k - r$. These form a chain

\[ \mathbb{C}^{l,k} = V_0 \supset V_1 \supset \cdots V_k = \{0\} , \]
where the variety $V_r$ is defined by the ideal generated by all $k - r + 1$-minors of $G$. For $k > r > 0$ the singular locus of $V_r$ is precisely $V_{r+1}$.

In fact, one implication of the last statement is obvious: the entries of the jacobian matrix of the ideal of $r$-minors of $G$ are $\mathbb{C}\{y\}$-linear combinations of the $(r - 1)$-minors as we can easily check by direct computation.

Now suppose an ICMCd2 singularity $(X, 0) \subset (\mathbb{C}^n, 0)$ is given by a matrix $M \in \text{Mat}(t+1, t; \mathbb{C}\{x\})$. We can regard this matrix as a map

$$M : (\mathbb{C}^n, 0) \rightarrow \text{Mat}(t+1, t; \mathbb{C}) \cong \mathbb{C}^{t(t+1)},$$

which by abuse of notation we also denote by $M$. Then the singularity $(X, 0)$ appears as the preimage $M^{-1}(V_1)$.

**Corollary 2.3.** Let $M \in \text{Mat}(t+1, t; \mathbb{C}\{x_1, \ldots, x_n\})$ a presentation matrix of an isolated Cohen-Macaulay codimension 2 singularity $(X, 0)$ of Cohen-Macaulay type $t$. Then the locus defined by the $(t - 1)$-minors of $M$ is either the origin or empty.

**Proof.** As $(X, 0)$ is a germ of an isolated singularity, the singular locus is the origin. Regarding $M$ as a map to $\mathbb{C}^{t(t+1)}$ gives a ring homomorphism

$$M^* : \mathbb{C}\{y_{i,j} \mid 1 \leq i \leq t+1, 1 \leq j \leq t\} \rightarrow \mathbb{C}\{x_1, \ldots, x_n\} \quad y_{i,j} \mapsto m_{i,j}.$$ 

Here $m_{i,j} \in \mathbb{C}\{x_1, \ldots, x_n\}$ denotes the entry of the matrix $M$ in the $i$-th row and $j$-th column. Let $(\delta_1, \ldots, \delta_{t+1})$ be the ideal generated by the $t+1$ maximal minors $\delta_i \in \mathbb{C}\{y\}$ of the matrix $G$ from the previous lemma, i.e. the vanishing ideal of $V_1 \subset \mathbb{C}^{t(t+1)}$. Then the ideal $I \subset \mathbb{C}\{x\}$ defining $X$ is given by

$$f_i(x) = \delta_i(M(x)).$$

Hence the jacobian matrix factors

$$J_f(x) = J_g(M(x)) \cdot J_M(x).$$

Now by the preceeding lemma the matrix $J_g(M(x))$ has entries contained in the ideal of $(t - 1)$-minors of $M$ and hence does $J_f(x)$ and all ideals of minors thereof. The statement follows immediately from this inclusion of ideals.

**Remark 2.4.** Of course, Corollary 2.3 can also be proved more directly. However, that does not illustrate the point in question. This is the more elegant argument: The presentation matrix $M$ of the isolated singularity $(X, 0)$ describes the relations of the generators of the conormal module $I/I^2$. Thus the ideal of $t-1$-minors is the second fitting ideal $\text{Fitt}_2(I/I^2)$ of the conormal module. Its vanishing locus is the set of primes where $I/I^2$ cannot be generated by 2 elements. But $X$ is of codimension 2 and $I/I^2$ is locally free on the smooth part. Hence

$$V(\text{Fitt}_2(I/I^2)) \subset \text{Sing}(X) = \{0\}.$$
Remark 2.5. Regarding the matrix $M$ of an ICMCd2 singularity as a map to the space of matrices as used in Corollary (2.3) provides a different perspective to those of the properties of our singularities which originate from the determinantal structure: Cohen-Macaulay codimension 2 singularities are among the classes of singularities for which deformations of the space germs coincide with deformations of $M$ as a map (see [4], chapter 4 and 5). The appropriate notion of equivalence in this context is $K_V$-equivalence (see [7], [8]), but we will not need this notion in our considerations.

In this article, the interplay of both aspects, i.e. of the similarities to the ICIS case and of the structural properties of determinantal singularities, will be essential to studying the topology of the singularities in question. More precisely, we shall even see contributions of both kinds in the topology.

3 Tjurina modifications revisited

Central to our considerations will be a not so widely known tool that was developed by G. Tjurina in [27] for her study of rational triple point singularities. After finding that such surface singularities can be realized by a system of 3 equations, which we can easily recognize as 2-minors of a $2 \times 3$ matrix, she considers the map to $\mathbb{P}^1$ which maps each point of the rank-1-locus of the matrix to the corresponding (non-zero) column vector of the matrix. Resolving the locus of indeterminacy of this map (i.e. the rank-0-locus) then provides her with a local complete intersection which only possesses rational double point singularities.

This construction has later also been used in the thesis of D. van Straten [26], where its name was coined, and in a few other articles. However, it has – to our knowledge – never been applied beyond the case of surface singularities of Cohen-Macaulay-type $t = 2$. As we shall apply it to the 3-dimensional case and as we do not want to restrict our methods to the case of Cohen-Macaulay-type $t = 2$, we will generalize Tjurina’s construction here.

Construction 3.1. (Tjurina modification for generic determinantal varieties)

Consider the varieties $V_r \subset \mathbb{C}^{l \times k}$, $k \leq l$, as in Lemma 2.2. For a general point, i.e. a general $l \times k$-matrix $A \in V_r$, the row vectors of $A$ span a $k - r$ dimensional hyperplane $P_A \subset \mathbb{C}^k$. This determines a rational map to the Grassmannian of $(k - r)$-planes in $k$-space.

\[
P : V_r \to \text{Grass}(k - r, k)
A \mapsto P_A.
\]

Clearly $P$ is defined on the open set $V_r \setminus V_{r+1}$. Regarding $\text{Grass}(k - r, k) \subset \mathbb{P}(\wedge^{k-r} \mathbb{C}^k)$ as a subvariety of projective space it becomes clear that $P$ can
always be expressed in terms of $k - r$-minors of $A$. As a projective variety the Grassmannian is complete and we can blow up the rational map $P$ to obtain

$$W_r := \Gamma_P(V_r \setminus V_{r+1}) \subset \mathbb{C}^{l,k} \times \text{Grass}(k - r, k)$$

as the closure of the graph of $P$ restricted to $V_r \setminus V_{r+1}$ together with the canonical projection $\pi$ and the prolongation $\hat{P}$

$$\begin{array}{c}
W_r \\
\downarrow \pi \\
V_r \quad P \\
\dashrightarrow \dashrightarrow \\
\text{Grass}(k - r, k)
\end{array}$$

In particular $\pi$ is a resolution of the singularities of $V_r$.

**Remark 3.2.** For calculating $W_r$ explicitly, we cover the projective variety $\text{Grass}(k - r, k) \subset \mathbb{P}(\wedge^{k-r} \mathbb{C}^k)$ by the standard affine charts. Similarly to writing a point $p \in \mathbb{P}^n$ in projective $n$-space as $p = (s_0 : \cdots : s_n)$ in projective coordinates and thus also indicating the line $L(p) = \text{span}((s_0, \ldots, s_n)^T) \subset \mathbb{C}^{n+1}$ sitting over $p$ in the tautological bundle, we write a point $z \in \text{Grass}(k - r, k)$ as a $(k - r) \times k$-matrix $B$. The standard cover is indexed by subsets $\alpha \subset \{1, \ldots, k\}$ of cardinality $\# \alpha = k - r$. Analogous to normalizing the projective coordinates of a point $p = (s_0 : \cdots : s_i : \cdots : s_n)$ in the $i$-th chart of projective space to $p = \left(\frac{s_0}{s_i} : \cdots : \frac{s_n}{s_i}\right) = \left(s_0^{(i)} : \cdots : 1 : \cdots : s_n^{(i)}\right)$, we require the maximal square submatrix of $B$ indexed by $\alpha$ to be the unit matrix. Thus we obtain affine coordinates $(z_{i,j}^{(\alpha)})_{i,j}$. For example if $\alpha = \{1, \ldots, k - r\}$ we write a point $z \in U_\alpha \subset \text{Grass}(k - r, k)$ as

$$B_\alpha(z) = \begin{pmatrix}
1 & 0 & \cdots & 0 & z_{1,k-r+1}^{(\alpha)} & \cdots & z_{1,k}^{(\alpha)} \\
0 & 1 & \ddots & \vdots & z_{2,k-r+1}^{(\alpha)} & \cdots & z_{2,k}^{(\alpha)} \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & z_{k-r,k-r+1}^{(\alpha)} & \cdots & z_{k-r,k}^{(\alpha)}
\end{pmatrix}$$

The subspace $L(z) \subset \mathbb{C}^k$ sitting over $z$ is now given by the span of the rows of the above matrix. Given a generic $l \times k$ matrix $G \in V_r \subset \mathbb{C}^{l,k}$, requiring the span of the rows of $G$ to be contained in $L(z)$ therefore amounts to asking for the $k - r + 1$-minors of the matrix

$$\begin{pmatrix}
B_\alpha \\
G
\end{pmatrix}$$

to vanish. In fact the variety $W_r \subset \mathbb{C}^{l,k} \times \text{Grass}(k - r, k)$ which is locally defined by these minors is already the strict transform of $V_r$ under the blowup of $P$ in our construction.
In the setting of ICMCd2 singularities we will only be concerned with \((t + 1) \times t\)-matrices of rank \(t - 1\). A \(t - 1\)-dimensional subspace \(L\) in \(\mathbb{C}^t\) is uniquely determined by the class of a normal vector \([\vec{n}_L] \in \mathbb{P}^{t-1} \cong \text{Grass}(t-1,t)\). The identification of \(\text{Grass}(t-1,t)\) with \(\mathbb{P}^{t-1}\) is given on the standard cover by identifying a point \([\vec{n}_L]\) in the chart \(\{s_i \neq 0\}\) in \(\mathbb{P}^{t-1}\) with the matrix \(B_i(s) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -s_i^{(i)} & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -s_i^{(i)} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -s_i^{(i)} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -s_i^{(i)} & 0 & \cdots & 0 & 1 \end{pmatrix}\)

The equations of the Tjurina transform \(W_1\) of \(V_1 \subset \mathbb{C}^{t(1+1)}\) therefore take a particular simple form: For a point \(s = (s_1 : \cdots : s_t) \in \mathbb{P}^{t-1}\) we just require the vector \(\vec{s} = (s_1, \ldots, s_t)^T\) to be perpendicular to the columns of a matrix \(G \in V_1\).

**Corollary 3.3.** Let \(G = (y_{i,j})_{i,j} \subset \mathbb{C} \{y\}\) be the generic \((t + 1) \times t\) matrix and \((s_1 : \cdots : s_t)\) the projective coordinates of \(\mathbb{P}^{t-1} = \text{Grass}(t-1,t)\). The Tjurina transform \(W_1 \subset \mathbb{C}^{(t+1)t} \times \mathbb{P}^{t-1}\) of \(V_1\) is the zero locus of the equations

\[
\begin{pmatrix} y_{1,1} & \cdots & y_{1,t} \\ \vdots & \vdots & \vdots \\ y_{t+1,1} & \cdots & y_{t+1,t} \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_t \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

**Proof.** Computing each chart with the matrix \(B_i\) as in [1] gives the local equations. But these can be easily recognized as the dehomogenization of the equations given by [2]. \qed

From now on we will denote a representative of an isolated singularity by \(X_0\), i.e. with an additional index 0. This is due to consideration of deformations in the sequel, where the singularity \((X_0, 0)\) is embedded as the special fiber in a total space \((X, 0) \xrightarrow{\varepsilon} (B, 0)\) fibered over some base \((B, 0)\) by deformation parameters \(\varepsilon\). Consequently for a choice of representatives the fiber over any nonzero \(\varepsilon \in B\) is denoted by \(X_\varepsilon\).

\(^1\) The non-standard numbering was chosen for consistency with the numbering of matrix entries.
**Construction 3.4.** (Tjurina modification for ICMCd2 singularities) As pointed out in remark 2.5 an ICMCd2 singularity \((X_0,0) \subseteq (\mathbb{C}^n,0)\) can be studied by means of a polynomial map \(M : U \longrightarrow \mathbb{C}^{(t+1)t}\) for a chosen representative \(X_0 \subset U\) of \((X_0,0)\) in a neighborhood \(U\) of the origin.\footnote{By abuse of notation we will also refer to such a representative as ICMCd2.} Concatenation with \(P : V_1 \longrightarrow \mathbb{P}^{t-1}\) gives a rational map
\[
P \circ M : X_0 \longrightarrow \mathbb{P}^{t-1}.
\]
Because \(P \circ M\) is expressed in the projective coordinates of \(\mathbb{P}^{t-1}\) in terms of \((t-1)\) minors of \(M\), it is well defined outside the singular locus of \(X\) by Corollary 2.3.

Now we define the Tjurina modification \(Y_0\) to be the fiber product \(X_0 \times_{V_1} W_1\) in the following diagram:
\[
\begin{array}{ccc}
X_0 & \xrightarrow{M} & W_1 \\
\downarrow \pi & & \downarrow \rho \\
X_0 & \xrightarrow{M} & V_1 \\
& & \downarrow P \searrow \mathbb{P}^r
\end{array}
\]

On the level of equations this means nothing but regarding \(M\) as a matrix with polynomial entries and requiring the equations of the system
\[
M \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_t \end{pmatrix} = 0
\]
to hold in \(U \times \mathbb{P}^{t-1}\). Clearly outside the singular locus \(\{0\}\) the map \(\pi : Y_0 \rightarrow X_0\) is an isomorphism, while the origin itself is substituted with the whole Grassmannian \(\mathbb{P}^{t-1}\).

**Example 3.5.** Let us consider the (non-simple) ICMCd2 singularity \((X_0,0) \subset (\mathbb{C}^5,0)\) given by the 3-minors of the matrix
\[
M = \begin{pmatrix} x & y - v & y + z \\ y & z - v & x + u \\ z & 0 & x - u \\ 0 & u & v \end{pmatrix}.
\]

Let \((s_1 : s_2 : s_3)\) be the projective coordinates of \(\mathbb{P}^2 = \text{Grass}(2,3)\). Then we obtain \(Y_0 \subset \mathbb{C}^5 \times \mathbb{P}^2\) as the zero locus of the equations
\[
\begin{pmatrix} x & y - v & y + z \\ y & z - v & x + u \\ z & 0 & x - u \\ 0 & u & v \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = 0.
\]

\[\text{(4)}\]
The Tjurina transform $Y_0$ is still singular at 10 distinct points in $\{0\} \times \mathbb{P}^2 \subset \mathbb{C}^5 \times \mathbb{P}^2$. But there we only find 3-dimensional $A_1$ singularities embedded in higher dimensional space. Thus the situation became much simpler. Consider e.g. the singularity at the point $p = (0, (1 : 0 : 0))$ in the chart $s_1 \neq 0$: The first three lines of the system define a smooth variety $H$ of dimension 4 around $p$. Inside $H$ the equation

$$s_2^{(1)} \cdot u + s_3^{(1)} \cdot v = 0$$

in the last line provides the $A_1$ singularity.

Any deformation of an ICMCd2 singularity $X_0$ is described by a perturbation of the entries of the matrix $M$ defining $X_0$. As the process of Tjurina modification is based on the determinantal structure, it can be applied to all fibers of such a family simultaneously. Therefore it makes sense to ask whether (or in which situations) a Tjurina modification is well-behaved within the family.

**Construction 3.6.** (Tjurina modification in family) Let $X_0 \hookrightarrow X \xrightarrow{\varepsilon} \mathbb{C}$ be a deformation of an ICMCd2 singularity $X_0 \subset \mathbb{C}^n$ of Cohen-Macaulay type $t$ described by a matrix $M(\bar{x}, \varepsilon) \in \text{Mat}(t+1, t; \mathbb{C}[\bar{x}, \varepsilon])$. The Tjurina modification in family for this deformation is the result of applying the Tjurina modification to the total space $X \xrightarrow{\varepsilon} \mathbb{C}$ which leads to a diagram extending diagram (3) above:

$$\begin{array}{cccc}
X_0 \times_{V_1} W_1 & \xrightarrow{\pi_0} & X \times_{V_1} W_1 & \xrightarrow{M} & W_1 \\
\downarrow \pi & & \downarrow \rho & & \downarrow \rho \\
X_0 & \xrightarrow{\varepsilon} & X & \xrightarrow{M} & V_1 \\
\downarrow \varepsilon & & \downarrow \rho & & \downarrow \rho \\
\{0\} & \xrightarrow{\varepsilon} & \mathbb{C} & & \mathbb{C} \\
\end{array}$$

The equations defining $Y = X \times_{V_1} W_1$ in $\mathbb{C}^n \times \mathbb{C} \times \mathbb{P}^{t-1}$ are again

$$M(\bar{x}, \varepsilon) \cdot \bar{s} = 0$$

with $\bar{s} = (s_1, \ldots, s_t)^T$ the vector whose entries are the homogeneous coordinates of $\mathbb{P}^{t-1}$. For the special fiber $Y_0 = X_0 \times_{V_1} W_1$ one always obtains the same result as in the case of applying the Tjurina modification to the singularity alone by Construction.

**Example 3.7.** Consider the deformation with a parameter $\varepsilon$ given by the matrix

$$M(\bar{x}, \varepsilon) = \begin{pmatrix}
x & y - v & y + z + 2\varepsilon \\
y & z - v & x + u + 2\varepsilon \\
z & 3\varepsilon & x - u \\
3\varepsilon & u & v
\end{pmatrix}.$$
Let $X \subset \mathbb{C}^5 \times \mathbb{C}$ be the total space of the deformation

\[
\begin{array}{ccc}
X_0 & \longrightarrow & X \\
\downarrow & & \downarrow \\
\{0\} & \longrightarrow & \mathbb{C} \\
\end{array}
\]

The Tjurina modification in family in $\mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^2$ is now described by the equations

\[
M(x, \varepsilon) \cdot \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = 0
\]

As a direct computation shows, all fibers of this family except the one over $\varepsilon = 0$ are smooth. This has an important consequence in the setting of Tjurina modification:

**Proposition 3.8.** If in the setting of Construction 3.6 above the deformation of $X_0$ over $\mathbb{C}$ is a smoothing, the restriction of $\pi$ to a smooth fiber

\[
\pi'_\varepsilon : Y_\varepsilon \rightarrow X_\varepsilon
\]

in diagram (5) is an isomorphism.

**Proof.** For fixed $\varepsilon$ the rational map

\[
P \circ M(-, \varepsilon) : X_\varepsilon \dashrightarrow \mathbb{P}^{d-1}
\]

is not well defined in the vanishing locus of $(t - 1)$-minors of $M$. By Corollary 2.3 this is contained in the singular locus of $X_\varepsilon$, which is empty for smooth fibers. Hence $P \circ M_\varepsilon$ is regular. 

Unfortunately it is not at all clear that the families $Y \xrightarrow{\varepsilon = 0} \mathbb{C}$ obtained by Tjurina modifications in family as in diagram (5) are flat. Whether or not this is the case, will in general depend on the deformation in question, the dimension of the singularity and the Cohen-Macaulay-type. Consider e.g. a space curve $X_0 \subset \mathbb{C}^3$ of Cohen-Macaulay-type 3 as the special fiber in a smoothing by a parameter $\varepsilon$, then the fiber $Y_0$ of the Tjurina transform over $\varepsilon = 0$ contains a $\mathbb{P}^2$, while the other fibers stay 1-dimensional. This clearly contradicts flatness.

For simple ICMCd2 singularities of dimension $\dim(X_0, 0) > 0$ this does not pose a problem\footnote{We deliberately exclude the simple fat points here as their behaviour obviously differs from the higher dimensions, because any $\mathbb{P}^k$ in the $Y_0$ would violate flatness.}. All families in the classification of Neumer and Frühbis-Krüger \footnote{\cite{11}} have Cohen-Macaulay type $t = 2$. This will turn out to be sufficient to assure flatness in all of our cases of interest.
Proposition 3.9. Let \((X_0, 0) \subset (\mathbb{C}^{n+2}, 0)\) be an ICMCd2 singularity of dimension \(n > 0\) and Cohen-Macaulay type \(t \leq n + 1\). The Tjurina modification in family for a deformation \(X_0 \hookrightarrow X \xrightarrow{\varepsilon} \mathbb{C}\)

\[
\begin{array}{ccc}
Y_0 & \rightarrow & Y \\
\pi_0 & \downarrow & \pi \\
X_0 & \rightarrow & X \\
\downarrow & \downarrow & \varepsilon \\
\{0\} & \rightarrow & \mathbb{C}
\end{array}
\]

is flat over \(\mathbb{C}\).

Proof. The Grassmannian in question is a \(\mathbb{P}^{t-1}\). As usual let \((s_1 : \ldots : s_t)\) be its projective coordinates and \(M(\underline{x}, \varepsilon)\) the matrix describing the family \(X\). The variety \(Y_0 \subset \mathbb{C}^{n+2} \times \mathbb{P}^{t-1}\) is given by the \(t + 1\) equations

\[
M_0(\underline{x}) \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_t \end{pmatrix} = 0
\]

Now

\[
\dim Y_0 = \max \{\dim X_0, \dim \mathbb{P}^{t-1}\} = \max \{n, t - 1\} = \dim X_0,
\]

because \(\pi_0 : Y_0 \rightarrow X_0\) is an isomorphism on \(Y_0 \setminus \pi_0^{-1}(\{0\})\) and the exceptional set \(\pi_0^{-1}(\{0\})\) is a \(\mathbb{P}^{t-1}\). Since \(X_0\) had codimension 2 in \(\mathbb{C}^{n+2}\) we find \(Y_0\) to have codimension \(t + 1\) in \(\mathbb{C}^{n+2} \times \mathbb{P}^{t-1}\). But locally in all charts, there are exactly \(t + 1\) equations describing \(Y_0\). This means \(Y_0\) is a locally complete intersection, so the induced deformation by \(M(\underline{x}, \varepsilon)\) in the Tjurina modification in family is flat.

Remark 3.10. The above result was independently formulated by Jesse Kass for simple space curve singularities in his up-to-now unpublished work on Coxeter-Dynkin diagrams of space curve singularities.

An alternative way to check flatness of a family is checking the relation lifting property for the relations of generators of the defining ideal (cf. [2]).

Let \(X_0 \subset \mathbb{C}^n\) be an ICMCd2 singularity of Cohen-Macaulay type \(t\) and \(M(\underline{x}, \varepsilon)\) be a matrix defining a deformation of \(X_0\) over \((\mathbb{C}, 0)\). The ideal \(J \subset \mathbb{C}[\underline{x}, \varepsilon][s_1, \ldots, s_t]\) defining the Tjurina transform \(Y \subset \mathbb{C}^n \times \mathbb{C} \times \mathbb{P}^{t-1}\) is generated by the \(t\) equations \(H_i(\underline{x}, \varepsilon, \underline{s}) = 0\) originating from the lines of the system

\[
M(\underline{x}, \varepsilon) \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_t \end{pmatrix} = 0.
\]
The relation lifting property for flatness requires that any relation
\[ \sum_j r_j \cdot h_j = 0 \]
in \( \mathbb{C}(\mathbb{x})[\mathbf{s}] \) among the \( h_j = H_j(\varepsilon = 0) \) can be lifted to a relation \( \sum_j R_j \cdot H_j = 0 \) in \( \mathbb{C}(\mathbb{x},\varepsilon)[\mathbf{s}] \) with \( r_j = R_j(\varepsilon = 0) \). Now there is one relation among the \( H_j \) which comes naturally with a lifting:

Because the matrix \( M \) describes the syzygies of the generators of
\[ I = \langle F_1(\mathbb{x},\varepsilon), \ldots, F_{t+1}(\mathbb{x},\varepsilon) \rangle, \]
i.e. the ideal defining \( X \subset \mathbb{C}^n \times \mathbb{C} \), we can write
\[ 0 = (F_1, \ldots, F_{t+1}) \cdot M \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_t \end{pmatrix} = (F_1, \ldots, F_{t+1}) \cdot \begin{pmatrix} H_1 \\ \vdots \\ H_t \end{pmatrix}. \]
We call this the “relation by the maximal minors”. This leads to the following criterion for flatness of the Tjurina modification.

**Lemma 3.11.** If in the above setting the relations among the generators \( h_i \) of the ideal defining the Tjurina modification \( Y_0 \) of an ICM Cd2 singularity \( X_0 \) are generated by the Koszul relations and the relation by the maximal minors, then any Tjurina modification in family of a deformation of \( X_0 \) is again flat.

We finish the discussion of flatness by looking at one last example which does not satisfy the condition in the preceding criteria:

**Example 3.12.** The family of ICM Cd2 fat points defined by
\[ \begin{pmatrix} 0 & x \\ x & y \\ y & \varepsilon \end{pmatrix} \]
does not give rise to a flat family by Tjurina modification:

Tjurina modification provides
\[ \hat{X}_\varepsilon = V(s_2 x, s_1 x + s_2 y, s_1 y + s_2 \varepsilon) \]
with \( f_i = f_1 = f_2 = f_3 \).

For \( \varepsilon = 0 \) we have the additional relation
\[ s_1^2 f_1 - s_1 s_2 f_2 + s_2^2 f_3 = 0 \]
among the generators of the ideal of \( Y_0 \). The relation cannot be lifted to a relation of the whole family.

Considering this example from a geometric perspective, all fibers except the fiber at \( \varepsilon = 0 \) are zero-dimensional, but the special fiber additionally contains the \( \mathbb{P}^1 \) introduced by the Tjurina modification. As before such a jump in dimension clearly contradicts flatness.
To end this section, we want to study the effects of a Tjurina modification to versal families of IC Mc2 singularities of Cohen-Macaulay type \( t = 2 \). These observations originate from direct computations, but will be useful for explicit examples:

**Remark 3.13.** Let \((X_0, 0) \subset (\mathbb{C}^n, 0)\) be an IC Mc2 singularity at the origin with Cohen-Macaulay type \( t = 2 \) and \( \text{dim} \ X_0 > 0 \). Let \( M \) be the corresponding presentation matrix. Expanding the matrix entries up to degree \( r \) and taking equivalence classes modulo \( \langle x_1, \ldots, x_n \rangle^{r+1} \), we can represent each such class by a matrix with polynomial entries of degree at most \( r \). We shall refer to this representative as the \( r \)-jet of the presentation matrix, \( j_r M \). More precisely, we need to prepare the subsequent discussion of the relationship of the deformations of \( X_0 \) and \( Y_0 \) and thus determine a very coarse classification of occurring 1-jets.\(^4\)

As we are considering a germ around the origin, all entries of \( j_1 M \) are homogeneous linear polynomials. We know that row and column operations on \( M \) leave the germ \((X_0, 0)\) unchanged, and we can safely pass to sufficiently general \( \mathbb{C} \)-linear combinations of the two original columns. The second column of \( j_1 M \) thus holds up to 3 \( \mathbb{C} \)-linearly independent linear forms. By suitable row operations on \( M \), we can then cancel linearly dependent entries of this column of \( j_1 M \) and achieve that the zero entries are positioned below the non-zero entries. (Note that the sufficiently general linear combination of the columns now ensures that a row with a zero in the second entry also holds a zero in the first entry.) By an analytic change of coordinates, we can now choose the non-zero entries of the second column as new coordinates, starting with \( x_1 \), and obtain the following four cases:

\[
\begin{pmatrix}
* & x_1 \\
* & x_2 \\
* & x_3
\end{pmatrix},
\begin{pmatrix}
* & x_1 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
* & x_1 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

Here \( * \) denotes an arbitrary entry.

**Lemma 3.14.** Let \((X_0, 0) \subset (\mathbb{C}^5, 0)\) be an IC Mc2 threefold singularity of Cohen-Macaulay type \( t = 2 \). Then the Tjurina transform \( Y_0 \) has at most isolated singularities, iff \( X_0 \) is contact equivalent to a IC Mc2 with presentation matrix

\[
\begin{pmatrix}
a & x_1 \\
b & x_2 \\
c & x_3
\end{pmatrix},
\]

where \( a, b, c \subset \langle x_1, \ldots, x_3 \rangle \).

**Proof.** Tjurina modification is an isomorphism outside the singular locus, which implies that the singular locus of \( Y_0 \) is contained in \( E = \pi_0^{-1}(\{0\}) \cong \mathbb{P}^1 \). Because \( E \) is irreducible, the singular locus of \( Y_0 \) is either a finite number of points or

\(^4\)This list is, of course, loosely related to the lists of 1-jets in \([11]\), but it contains significantly fewer classes, because here we are only hunting for a criterion for isolatedness of the singularities of the Tjurina transform.
If the matrix has the desired structure, we focus on one of the two standard affine charts of the exceptional \( \mathbb{P}^1 \) to show that there can be at most isolated singularities. As usual let \( (s_1 : s_2) \) be the homogeneous coordinates of the exceptional curve \( \mathbb{P}^1 \). The equations for \( Y_0 \subset \mathbb{C}^5 \times \mathbb{P}^1 \) in the chart \( s_1 \neq 0 \) are given by the ideal \( I = \langle a + s_2 x_1, b + s_2 x_2, c + s_2 x_3 \rangle \). The jacobian of this complete intersection reads

\[
\begin{pmatrix}
\frac{\partial a}{\partial x_1} + s_2 & \frac{\partial a}{\partial x_2} & \frac{\partial a}{\partial x_3} & \frac{\partial a}{\partial x_4} & \frac{\partial a}{\partial x_5} \\
\frac{\partial b}{\partial x_1} & \frac{\partial b}{\partial x_2} + s_2 & \frac{\partial b}{\partial x_3} & \frac{\partial b}{\partial x_4} & \frac{\partial b}{\partial x_5} \\
\frac{\partial c}{\partial x_1} & \frac{\partial c}{\partial x_2} & \frac{\partial c}{\partial x_3} + s_2 & \frac{\partial c}{\partial x_4} & \frac{\partial c}{\partial x_5} 
\end{pmatrix} \cdot x_1.
\]

One of its 3-minors (first 3 columns) and hence of the ideal of the singular locus contains an element of the form \( s_2^3 + \phi \) where the \( s_2 \)-degree of the remaining part \( \phi \) is at most 2. This excludes the case of the singular locus being the whole exceptional curve \( E \).

If, on the other hand, the matrix is not of the desired form, at least one row and hence at least one generator of \( I \) is contained in \( \langle x_1, \ldots, x_5 \rangle^2 \), whence at least one row of the jacobian matrix – and thus the ideal of its 3-minors – is contained in \( \langle x_1, \ldots, x_5 \rangle \). Hence the singular locus would be 1-dimensional in this case.

**Lemma 3.15.** Let \( (X_0, 0) \subset (\mathbb{C}^5, 0) \) be an ICMCd2 threefold singularity of Cohen-Macaulay type \( t = 2 \) such that the Tjurina transform \( Y_0 \) has at most isolated singularities. Furthermore let \( X_0 \hookrightarrow X \twoheadrightarrow \mathbb{C}^t \) be a semi-universal deformation of \( X_0 \). Then the induced family \( Y_0 \hookrightarrow Y \twoheadrightarrow \mathbb{C}^t \) is again versal for each of the arising singularities.

Note that the induced local deformations for the isolated singularities of \( Y_0 \) do not need to be semi-universal, i.e. \( \tau \) might not be minimal.

**Proof.** By means of Lemma 3.14 we can safely assume, that the matrix defining \( (X_0, 0) \) is of the form

\[
M = \begin{pmatrix}
a & x_1 \\
b & x_2 \\
c & x_3 
\end{pmatrix} \in \text{Mat}(3, 2; \mathbb{C}\{x_1, \ldots, x_5\}).
\]

Keeping the above notation let

\[
H_1 = s_1 \cdot a + s_2 \cdot x_1, \quad H_2 = s_1 \cdot b + s_2 \cdot x_2, \quad H_3 = s_1 \cdot c + s_2 \cdot x_3 \in \mathbb{C}\{x\}[s_1, s_2]
\]

be the three equations defining the Tjurina transform \( Y_0 \) in \( \mathbb{C}^5 \times \mathbb{P}^1 \), which are homogeneous in \( s \).

Assume \( p \in Y_0 \) to be an isolated singular point. Then \( p \) cannot be the point \( (0, (0 : 1)) \in \mathbb{C}^5 \times \mathbb{P}^1 \) as a direct computation of the rank of the Jacobian matrix at this point shows. Hence we can pass to the affine chart \( \{s_1 \neq 0\} \); after a change of coordinate in \( s_2 \) we can w.l.o.g. assume \( p \) to be the origin of this
Let $h_i$ be the respective dehomogenizations of the $H_i$. We now relate the $T^1_{Y_0,p}$ of the singular point $p \in Y_0$ to the $T^1_{X_0,0}$ by explicit computation:

First note that every deformation of $(X_0,0)$ induces a deformation of $(Y_0,p)$. More fundamental there is a canonical map

$$\{\text{Perturbations of } M\} \xrightarrow{\Lambda} \{\text{Perturbations of } h\}$$

$$\text{Mat}(3, 2; \mathbb{C}\{s\}) \xrightarrow{1:1} (\mathbb{C}\{s\}[s_2])^3$$

According to Lemma 2.1 we get a matrix presentation of $T^1_{X_0,0}$ as

$$T^1_{X_0,0} \cong \text{Mat}(3, 2; \mathbb{C}\{x\})/(J_M + \text{Im}(g))$$

On the other hand the space $T^1_{Y_0,p}$ is the quotient

$$T^1_{Y_0,p} \cong (\mathbb{C}\{s\}[s_2])^3 / J_h$$

where $J_h$ is the jacobian matrix, see Equation (6). We now show that $\Lambda$ induces a well defined and surjective map $T^1_{X_0,0} \to T^1_{Y_0,p}$ on the respective quotients.

To this end observe, that perturbations of the matrix $M$ in the second column can be shifted to perturbations of the first column through reduction by a suitable linear combination of

$$\frac{\partial M}{\partial x_1} = \begin{pmatrix} \frac{\partial a}{\partial x_1} & 1 \\ \frac{\partial b}{\partial x_1} & 0 \\ \frac{\partial c}{\partial x_1} & 0 \end{pmatrix}, \quad \frac{\partial M}{\partial x_2} = \begin{pmatrix} \frac{\partial a}{\partial x_2} & 0 \\ \frac{\partial b}{\partial x_2} & 1 \\ \frac{\partial c}{\partial x_2} & 0 \end{pmatrix}, \quad \frac{\partial M}{\partial x_3} = \begin{pmatrix} \frac{\partial a}{\partial x_3} & 0 \\ \frac{\partial b}{\partial x_3} & 0 \\ \frac{\partial c}{\partial x_3} & 1 \end{pmatrix}$$

Correspondingly for $(Y_0,p)$ perturbations involving the variable $s_2$ in any of the three equations can be replaced by perturbations with elements of $\mathbb{C}\{s\}$ through suitable reduction by the partial derivatives

$$\begin{pmatrix} \frac{\partial h_1}{\partial x_1} \\ \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_1}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial a}{\partial x_1} + s_2 \\ \frac{\partial b}{\partial x_2} \\ \frac{\partial c}{\partial x_3} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial h_2}{\partial x_1} \\ \frac{\partial h_2}{\partial x_2} \\ \frac{\partial h_2}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial a}{\partial x_2} + s_2 \\ \frac{\partial b}{\partial x_3} \\ \frac{\partial c}{\partial x_1} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial h_3}{\partial x_1} \\ \frac{\partial h_3}{\partial x_2} \\ \frac{\partial h_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial a}{\partial x_3} + s_2 \\ \frac{\partial b}{\partial x_1} \\ \frac{\partial c}{\partial x_2} \end{pmatrix}$$

Thus a perturbation of the first column of $M$ by a vector $\eta = (\eta_1, \eta_2, \eta_3)^T \in \mathbb{C}\{s\}^3$ translates 1:1 to a perturbation of the $h_i$ by $\eta$. This already assures surjectivity on the intermediate quotients.

We now explicitly check the compatibility with the remaining relations in $T^1_{X_0,0}$. The ones given by $\frac{\partial M}{\partial x_4}$ and $\frac{\partial M}{\partial x_5}$ immediately translate to perturbations by $\frac{\partial h_1}{\partial x_4}$ and $\frac{\partial h_1}{\partial x_5}$. Also the row operations on $M$ result in canonical operations.

---

5This coordinate change is of the form $s_{2,\text{new}} = s_2 + c$ for some constant $c$. We can compensate for this change of setting by adding the $c$-times the second column to the first in our matrix $M$. 

16
on the generators $h_i$. More subtle are the column operations: As generators we choose to consider

$$M \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

according to 2.1. These translate to the respective entries of the following list:

$$h - \frac{\partial h}{\partial s}, \quad \frac{\partial h}{\partial s}, \quad s_2(h - \frac{\partial h}{\partial s}), \quad s_2 \frac{\partial h}{\partial s}.$$

Remark 3.16. For all $(X_0, 0)$ in the table of simple ICMCd2 singularities of dimension 3, the Tjurina modification $Y_0$ has at most simple ICIS as can be read off from table 1.

4 Vanishing cycles

From now on we’ll often be concerned with the homology groups of a given topological space. By this we mean Simplicial homology with integer coefficients and we’ll just write $H_\bullet(\cdot)$ for $H_\bullet(\cdot, \mathbb{Z})$ for short. To fix notation, we briefly recall the definition of Milnor fiber and vanishing cycles for isolated singularities (see e.g. [20] for a reference on these topics).

Let $(X_0, 0) \subset (\mathbb{C}^n, 0)$ be an isolated singularity at the origin and $X_0$ a representative thereof. Then there is a real $\eta_0 > 0$ such that the intersection of $X_0$ with the sphere $S_\eta$ of radius $\eta$ is transversal for all $\eta \geq \eta > 0$. For any $\eta > 0$ chosen in this way, we will refer to a closed ball $B_\eta$ of radius $\eta$ around $0 \in \mathbb{C}^n$ as a Milnor ball for the singularity $X_0$. Furthermore we denote by $\overline{X}_0$ the topological space

$$\overline{X}_0 := X_0 \cap B_\eta.$$

We explicitly cite the following well-known theorem, which ensures that these definitions are independent of the chosen (sufficiently small) $\eta$.

Theorem 4.1 (conical structure, [20]). For an isolated singularity $(X_0, 0) \subset (\mathbb{C}^n, 0)$ and a Milnor ball $B_\eta$ for $X_0$ the pair of spaces $(B_\eta, \overline{X}_0)$ is homeomorphic to the pair $(C(S_\eta), C(\partial \overline{X}_0))$, where $C(L)$ denotes the cone over $L \subset S_\eta$, i.e. the set of real line segments to the origin. This can be chosen to be a diffeomorphism on the open set $(B_\eta \setminus \{0\}, \overline{X}_0 \setminus \{0\})$.

Consequently $\overline{X}_0$ and $\partial \overline{X}_0$ are well defined topological spaces up to homeomorphism for a germ $(X_0, 0)$ of an isolated singularity, i.e. do not depend on the representative $X_0$. Now, consider a deformation of an isolated singularity
$(X_0, 0) \subset (\mathbb{C}^n, 0)$ by some parameter $\varepsilon$, i.e. a flat family

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X \\
\downarrow & & \downarrow \\
\{0\} & \longrightarrow & \mathbb{C}
\end{array}
$$

where $X_0 \subset \mathbb{C}^n$ and $X \subset \mathbb{C}^n \times \mathbb{C}$ are representatives of the respective germs.

Having chosen a Milnor ball $B_\eta$ for $X_0$ there exists an open neighborhood $0 \in D \subset \mathbb{C}$ in the deformation base $\mathbb{C}$, such that for all $\varepsilon \in D$ the intersection

$$
\partial X_\varepsilon = X_\varepsilon \cap \partial B_\varepsilon
$$

of the fiber $X_\varepsilon$ with the Milnor ball in the fiber over $\varepsilon$ is transversal. The cylinder $B_\eta \times D$ is called a Milnor tube for the deformation of $X_0$.

Theorem 4.1 ensures that all the homology groups of $X_0$ vanish except in degree 0. But in any deformed fiber $X_\varepsilon$ there may exist nontrivial cycles. If a fiber $X_\varepsilon$ is smooth, i.e. a smooth complex manifold with boundary, it is called a Milnor fiber of the singularity $X_0$. Any nontrivial cycles in the homology of $X_\varepsilon$ are called vanishing cycles of the singularity $X_0$. It is well known what these vanishing cycles look like for any ICIS of complex dimension $n$: they form a bouquet of spheres of real dimension $n$, see [17].

For the ICMCd2 singularities of dimension 3 which we are considering in this article, known results on the vanishing cycles of the Milnor fibers are scarce.

**Remark 4.2.** It is a priori not clear and in general wrong to expect exactly one Milnor fiber for a given isolated singularity $X_0$ (up to diffeomorphism). First of all there may not exist any deformation with smooth fibers at all, like for the rigid isolated 4-fold singularity appearing in the classification of Frühbis-Krüger and Neumer.

On the other hand, given two different smoothings $\pi : X \to \mathbb{C}$ and $\pi' : X' \to \mathbb{C}$ two smooth fibers $X_\varepsilon$ and $X'_\varepsilon$ are not necessarily diffeomorphic as the famous example of Pinkham [23] shows. They are, however, diffeomorphic if they belong to the same connected component of the deformation base. This is an immediate corollary of the Ehresmann fibration theorem.

If $X_0$ is a smoothable ICMCd2 singularity, the set of points $\varepsilon \in \mathbb{C}^r$ with smooth fibers is open and connected since its complement (the discriminant) has real codimension at least 2. Therefore in all our cases of interest in this article the singularities have a unique Milnor fiber up to diffeomorphism.

We need one more preliminary result which will be applied to determine the topology of the Tjurina modification $Y_0$.

**Proposition 4.3.** Let $(X_0, 0) \subset (\mathbb{C}^n, 0)$ be an isolated singularity and $\pi_0 : Y_0 \to X_0$ a morphism defined on suitably small representatives such that the restriction

$$
\pi_0 : Y_0 \setminus \pi^{-1}(\{0\}) \to X_0 \setminus \{0\}
$$
is an isomorphism and the exceptional set $E = \pi_0^{-1}(\{0\})$ is closed and projective. Then $E$ is a deformation retract of $Y_0$.

**Proof.** The variety $E$ is closed and projective, hence compact. It follows from [18], that $E$ is a Euclidean Neighborhood Retract of an open neighborhood $U$ of $E$ in $Y_0$. But outside $E$ the map $\pi_0$ is an isomorphism, so $\pi_0(U) \subset X_0$ is open. With the theorem about the conical structure 4.1, we can now shrink $\overline{Y}_0 \setminus E = \overline{X}_0 \setminus \{0\}$ to something homotopic to $\overline{Y}_0$ inside the open set $\pi_0(U)$ and subsequently to $E$. \(\square\)

Using Tjurina modification in family, we are now ready to explain the observations of [9] in the case of a simple ICMCd2 threefold $(X_0, 0) \subset (\mathbb{C}^5, 0)$. Applying the Tjurina modification we get a transform $Y_0$ with only A-D-E singularities according to Remark 3.16. Since we necessarily have Cohen-Macaulay type $t = 2$, the homotopy type of $Y_0$ is given by the exceptional set $\mathbb{P}^1 = E \subset Y_0$ as a consequence of Proposition 4.3. So we always find

$$b_0(Y_0) = 1, \quad b_1(Y_0) = 0, \quad b_2(Y_0) = 1, \quad b_3(Y_0) = 0. \quad (7)$$

Now Proposition 3.9 assures the Tjurina modification to be well behaved within families. Hence we can choose any smoothing $X_0 \hookrightarrow X \xrightarrow{\pi} \mathbb{C}$ and carefully observe the interplay of cycles present in $Y_0$ with upcoming vanishing cycles of the ICIS when passing from $Y_0$ to a deformed fiber $Y_\varepsilon$ in the induced deformation. This is covered in Theorem 4.4. Finally we can use the identification $Y_\varepsilon \cong X_\varepsilon$ from Proposition 3.8 to obtain the desired vanishing topology.

We slightly weaken the assumptions and also allow $(X_0, 0) \subset (\mathbb{C}^5, 0)$ to be a non-simple ICMCd2 threefold singularity. However we still require the Cohen-Macaulay type to be $t = 2$ and the jet type as in Lemma 3.14 i.e. only ICIS in $Y_0$. The results are gathered in the Tables 1 and 2 in the next section.

**Theorem 4.4.** In the above setting consider $X_0$ as the special fiber in a smoothing $X_0 \hookrightarrow X \xrightarrow{\pi} \mathbb{C}$ together with the Tjurina modification in family $Y_0 \hookrightarrow Y \xrightarrow{\pi_0} \mathbb{C}$ as in diagram 5. We denote the Milnor tube arising from $B$ by

$$T = B \times D \subset \mathbb{C}^5 \times \mathbb{C}$$

and the one originating from $\hat{B}$ by

$$\hat{T} = \pi^{-1}(T) \subset \mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1,$$

which also allows us to refer to the Milnor fiber $\overline{X}_\varepsilon$ and the fiber $\overline{Y}_\varepsilon = \hat{T} \cap Y_\varepsilon$ sitting over it. The Betti numbers of a smooth fiber $\overline{X}_\varepsilon$ are given by

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = 1, \quad b_3 = r$$

where $r \in \mathbb{N}$ is the sum of the Milnor numbers of the ICIS of $Y_0$. 

\(19\)
Proof. Throughout the proof many steps require shrinking the open set of admissible deformation parameters $\varepsilon$ in the deformation base. However, those steps are finitely many and no harm is done, since we only consider representatives of germs. For the reader’s convenience, we will suppress mentioning this obvious technical detail each time it occurs.

Let $n$ be the number of singularities of $Y_0$. Fix local analytic embeddings of these ICIS to some affine space and let $T = \bigcup_{i=1}^n T_i$ be a collection of Milnor tubes around the singularities of $Y_0$ for the induced deformation $Y_0 \hookrightarrow Y \xrightarrow{\pi} C$. For each $i = 1, \ldots, n$, let $S_i$ be the Milnor tubes swept out by Milnor balls of half of the radius of $T_i$. Decompose the total space $Y$ in the two open sets

$$U := Y \setminus \left( \bigcup_{i=1}^n S_i \cap Y \right) \quad \text{and} \quad V \text{ the interior of } \left( \bigcup_{i=1}^n T_i \cap Y \right).$$

Let $\overline{U}, \overline{V}$ and $\overline{W} = U \cap V$ be the closures of the respective open sets in the Euclidean topology. Each of them is compact, $\overline{U}$ and $\overline{W}$ even compact manifolds with boundary. An illustration of this setting can be found in Figure 1.

![Figure 1: $Y$ and the Milnor tubes $S$ and $T$](image)

By the Ehresmann fibration theorem we can choose a differentiable flow

$$\Phi : D \times U' \to U'$$

defined on an open set $0 \in D \subset \mathbb{C}$ in the deformation base and a neighborhood $U'$ of the closure $\overline{U}$ in $Y$, such that the restriction to $\overline{U}_0$

$$\Phi|_{\{\varepsilon\} \times \overline{U}_0} : \overline{U}_0 \to \overline{U}_\varepsilon$$

and the restriction to the overlap

$$\Phi|_{\{\varepsilon\} \times \overline{W}_0} : \overline{W}_0 \to \overline{W}_\varepsilon.$$
are diffeomorphisms of manifolds with boundary for sufficiently small $|\varepsilon|$. Now consider the long exact sequence in reduced homology for the pair of spaces $(Y_0, V_0)$:

$$
\ldots \to H_i(\overline{V}_0) \to H_i(Y_0) \to H_i(Y_0, \overline{V}_0) \to H_{i-1}(\overline{V}_0) \to \ldots \tag{8}
$$

Because $\overline{V}_0$ is contractible to a union of points, we find $H_i(\overline{Y}_0) \to H_i(Y_0, \overline{V}_0)$ to be an isomorphism for $i > 0$. Clearly,

$$H_i(Y_0, \overline{V}_0) = H_i(\overline{Y}_0/\overline{V}_0) = H_i(\overline{U}_0/\partial \overline{U}_0) = H_i(\overline{U}_\varepsilon/\partial \overline{U}_\varepsilon) = H_i(\overline{Y}_\varepsilon, \overline{V}_\varepsilon), \tag{9}
$$

because of excision. The identifications in equation (9) and equation (7) provide us with zeros for the terms $H_i(\overline{Y}_\varepsilon, \overline{V}_\varepsilon) = 0$ for $i \neq 0, 2$. Now consider the analogous long exact sequence for the pair of spaces $(\overline{V}_\varepsilon, \overline{V}_\varepsilon)$ and obtain in degree 3

$$0 \to H_3(\overline{V}_\varepsilon) \to H_3(\overline{Y}_\varepsilon) \to 0 \tag{10}
$$

which means that every vanishing cycle of the occurring singularities in $Y_0$ is preserved in the whole fiber $X_\varepsilon$. Since for ICIS singularities the Milnor fiber has the topological type of a bouquet of $\mu$ 3-spheres and we have a decomposition

$$H_3(\overline{V}_\varepsilon) = \bigoplus_{i=1}^n H_3(\overline{V}_{i,\varepsilon}),$$

the middle Betti number $b_3$ of $\overline{Y}_\varepsilon$ is the sum of Milnor numbers of the singularities of $Y_0$.

The Tjurina transform $Y_0$ has only ICIS, so we get zeros for $H_i(\overline{V}_\varepsilon)$ for $i \neq 0, 3$ which leads to

$$0 \to H_2(\overline{Y}_\varepsilon) \to H_2(\overline{Y}_\varepsilon, \overline{V}_\varepsilon) \to 0 \tag{11}
$$

in degree 2. As the fibers $\overline{V}_\varepsilon$ and $\overline{X}_\varepsilon$ are isomorphic according to Proposition 3.8, the claim is proved.

**Example 4.5.** The applied ideas also work for higher Cohen-Macaulay type, as we would like to illustrate by revisiting our previous Example 3.7 from Section 3: also in the case $t = 3$ Proposition 3.9 assures the Tjurina modification to work in family. Contrary to the case $t = 2$, the central fiber in the Tjurina modification $Y_0$ now has the homotopy type of $E = \pi_0^{-1}(\{0\}) = \mathbb{P}^2$. Thus the Betti numbers read

$$b_0(Y_0) = 1, \quad b_1(Y_0) = 0, \quad b_2(Y_0) = 1, \quad b_3(Y_0) = 0, \quad b_4(Y_0) = 1. \tag{12}
$$

Clearly the 4-cycle generated by the $\mathbb{P}^2$ itself can not be preserved in a smooth fiber $Y_\varepsilon \cong X_\varepsilon$, because $X_\varepsilon$ is affine and we would get a contradiction to the Lefschetz Hyperplane Theorem. In fact this cycle breaks at the 10 points of
the $A_1$ singularities of $Y_0$ when resolving them. We can again calculate along the lines of the proof of Theorem 4.4 using the long exact sequence of pairs of spaces. The degree 2 part stays isolated, so we again get

$$b_2(X_\varepsilon) = 1.$$ 

For degree 3 the calculations show, that the broken $\mathbb{P}^2$ leads to a relation among the vanishing cycles of the $A_1$'s. Hence we have

$$b_3(X_\varepsilon) = 9$$

and not 10 as one might have expected.

Remark 4.6. A similar phenomenon can be observed when projectivizing the column space of the matrix $M$ of an ICMCd2 threefold $(X_0,0)$ of Cohen-Macaulay type $t = 2$. As in the case of a Tjurina modification, projectivizing the column space is compatible with deformations, because we again get a locally complete intersection transform, say $Z_0$. Only this time we find a $\mathbb{P}^2$ as exceptional set. If we apply this to the first entry of the table $\Pi$ also called the $A_0^+$ singularity, we find

$$b_0(Z_0) = 1, \quad b_1(Z_0) = 0, \quad b_2(Z_0) = 1, \quad b_3(Z_0) = 0, \quad b_4(Z_0) = 1$$

and one $A_1$ singularity in $Z_0$. Again, the induced deformation destroys the 4-cycle of $Z_0$ leading to the vanishing cycle of the $A_1$ singularity being homologous to zero in $X_\varepsilon = Z_\varepsilon$.

5 The topological type of the simple ICMCd2 singularities

Using the results of the previous section, direct computation now provides explicit results for the structure of the Milnor fiber for the simple ICMCd2 singularities and for the bounding non-simple ones. We have summarized the results in the following two tables. Subsequently, we finish this article by pointing out and explaining some notable observations and stating some arising questions.
| Expression | Degree | Group $E_d$ | Index $n$ |
|------------|--------|-------------|-----------|
| $(x, y, z)$ | $x^3 + y^3$ | $E_7$ | 9 |
| $(x, y, z)$ | $x^3 + y^3$ | $E_8$ | 10 |
| $(w, y, x)$ | $(x, y) + v^k$ | $2k - 1$ | - |
| $(w, y, x)$ | $k + 2$ | $A_{k-1}$ | $k - 1$ |
| $(w, y, x)$ | $2k$ | $A_1$ | 1 |
| $(w, y, x)$ | $2k + 1$ | $A_1$ | 1 |
| $(w, y, x)$ | $k + 3$ | $A_{k-1}$ | $k - 1$ |
| $(w, y, x)$ | 7 | $A_2$ | 2 |
| $(v^2 + w^2, y, x)$ | $z^2 + y^2 + v^2$ | $k + l + 1$ | $A_{k-1}, A_{l-1}$ |
| $(v^2 + w^2, y, x)$ | $k + 4$ | $A_{k-1}, A_1$ | $k$ |
| $(v^2 + w^2, y, x)$ | $k + l + 2$ | $A_{k-1}, A_{l-1}$ | $k + l - 2$ |
| $(w, y, x)$ | $z^2 + y^3 + v^k$ | $2k - 1$ | $A_1, A_1$ |
| $(w, y, x)$ | 2 | $A_1, A_1$ | 2 |
| $(w, y, x)$ | $z^2 + y^3 + v^k$ | $2k + 2$ | $A_1, A_1$ |
| $(w, y, x)$ | 9 | $A_1, A_2$ | 3 |
| $(w, y, x)$ | 9 | $A_2, A_2$ | 4 |
| $(z, y, x)$ | $z^2 + y^{2} + z^k$ | $k + 4$ | $D_{k+1}$ |
| $(z, y, x)$ | $2k + 5$ | $A_{2k+2}$ | $2k + 2$ |
| $(z, y, x)$ | $2k + 4$ | $A_{2k+1}$ | $2k + 1$ |
| $(z, y, x)$ | 8 | $D_5$ | 5 |
| $(z, y, x)$ | 9 | $E_6$ | 6 |
| $(z, y, x)$ | 7 | $D_3$ | 3 |
\[
\begin{array}{|c|c|c|}
\hline
(z + y \ x + v^2) & 8 & A_4 \\
(x + v^2 \ vz + y^2) & 9 & D_5 \\
(z + y \ x + v^2) & & \\
(x + z^2 + y^2) & & \\
\hline
\end{array}
\]

Table 1: Homology of Milnor fibers computed by means of the Tjurina modification

\[
\begin{array}{|c|c|c|c|}
\hline
M^t & \tau & \text{sing. in } Y_0 & b_2 \\
(\begin{array}{cccc}
x & y & z & x^4 + y^4 \\
w & v & x^3 + y^6 \\
\end{array}) & 11 & X_9 & 1 \\
(\begin{array}{cccc}
w + v^2 & y & x & z \\
v & y^3 + v^3 & w & x \\
\end{array}) & 8 & D_4 & 1 \\
(\begin{array}{cccc}
w + v^4 & y & x & z \\
v & y^2 + v^4 & w & x \\
\end{array}) & 9 & A_3 & 1 \\
(\begin{array}{cccc}
z & y & x & x^2 + y^3 + z^3 \\
x & w & v & x^3 + y^2 + z^3 \\
\end{array}) & 11 & T_{3,3,3} & 1 \\
(\begin{array}{cccc}
z & y & x & x^3 + y^3 + z^2 \\
x & w & v & x^2 + y^3 + z^2 \\
\end{array}) & 13 & T_{3,3,3} & 1 \\
(\begin{array}{cccc}
z & y & x & x^3 + y^3 + z^2 \\
x & w & v & x^2 + y^3 + z^2 \\
\end{array}) & 17 & U_{12} & 1 \\
(\begin{array}{cccc}
z & y & x & x^3 + y^3 + z^2 \\
x & w & v & x^2 + y^3 + z^2 \\
\end{array}) & 12 & X_9 & 1 \\
(\begin{array}{cccc}
z & y & x + v^2 & x \\
x & w & vz + yz + vw & v \\
\end{array}) & 10 & D_6 & 1 \\
(\begin{array}{cccc}
z & y & x + v^3 & x \\
x & w & vy + z^2 & v \\
\end{array}) & 9 & A_3 & 1 \\
(\begin{array}{cccc}
z & y & x + v^3 & x \\
x & w & y^2 + yz + z^2 & v \\
\end{array}) & 15 & X_9 & 1 \\
(\begin{array}{cccc}
z & y & x + v^3 & x \\
x & w & y^2 + yz + z^2 & v \\
\end{array}) & 8 & D_4 & 1 \\
\hline
\end{array}
\]

Table 2: Homology of Milnor fibers for the bounding non-simple singularities

\footnote{There is a typesetting error in this matrix in \cite{11}. The right-hand lower entry here is the correct one.}
Remark 5.1. (direct observations from the table)

1. We only see simple singularities occurring in $Y_0$ in table 1. In table 2, where the listed singularities are non-simple, there are some simple and some non-simple singularities arising from Tjurina modification. In particular, the non-simple ones arise in the cases with 1-jet types $J^{(5,2)}$ and in some subcases of $J^{(4,4)}$ in the notation of [11], whereas the simple ones occur for $J^{(4,2)}$ and the remaining subcases for $J^{(4,4)}$. The 1-jet types $J^{(4,5)}$ and $J^{(4,6)}$ have not been included in the table, because they lead to non-isolated singularities in $Y_0$, the singular locus being the whole exceptional $\mathbb{P}^1$.

2. Looking at the preceding tables, one fact immediately attracts attention: For any of the explicitly computed ICMMCd2 threefold singularities the second Betti number is always 1. The mechanism behind this fact can be explained as follows: In the outlines preceding Theorem 4.4 we find Equation 7 as a consequence of the CM-type being $t = 2$. The low CM-type also assures the Tjurina modification to work in family an in the calculations of Theorem 4.4 the degree 2 part is preserved (Equation 11). This answers negatively to a question of J. Damon and B. Pike who expected both Betti numbers $b_2$ and $b_3$ to grow at the same rate in families of ICMMCd2 with $\mu = b_3 - b_2$ constant. But our computations show that there are infinite families in which neither of the two changes. The first occurrence of this phenomenon can be found in line 7 of the preceding table 1, given by the matrix

$$\begin{pmatrix} w & y & x \\ z & w & y + v^k \end{pmatrix}$$

Following the notation of [11], we call these singularities the $\Pi_k$ family. The topological type stays the same within this family, whereas the isomorphism classes of the space germs do not coincide for different $k$.

In the following we want to consider this unexpected behaviour from different perspectives to shed some more light on these effects: We have already observed at the end of section 3 that there are $\mathbb{C}$-basis elements of the $T^{1}_{X_{0,0}}$ which seem to have no corresponding element in the $T^{1}$ of any singularity of the Tjurina modification $Y_0$. The first occurrence of this phenomenon is already the very first singularity in the list, named $A^+_{\mathbb{C}}$. It is given by the matrix

$$\begin{pmatrix} x & y & z \\ v & w & x \end{pmatrix}$$

and has a smooth Tjurina transform $Y_0$ isomorphic to the total space of the bundle $\mathbb{O}_{\mathbb{P}^1}(-2) \oplus \mathbb{O}_{\mathbb{P}^1}(-1)$. The deformation of $Y_0$ induced by the semi-universal
deformation of \( X_0 \) is locally trivial in this case. It is natural to ask: How many degrees of freedom to deform \((X_0, 0)\) become locally trivial as deformations of the Tjurina transform? Can we relate this difference to some invariants of the singularities?

Let \( \{p_1, \ldots, p_N\} \subset Y_0 \) be the isolated singularities of the Tjurina transform \( Y_0 \). As a consequence of Lemma 3.15 they all have to be simple singularities for the simple ICICMd2 singularities in case we started with a simple ICICMd2 threefold. They can only be A-D-E singularities, as these are the only simple ICIS in dimension 3 (see [14]). Let \( \mu_i \) be the Milnor number of \((Y_0, p_i)\). It is well known, that for the A-D-E case Milnor, Tjurina and middle Betti number coincide. Hence the above question can be reformulated as asking for the difference

\[ \tau - \sum_{i=1}^{N} \mu_i \]

This is still a bit unfortunate, because it is a priori not clear that deforming locally at say \( p_1 \) does not affect any other of the \( p_i \). But Equation 10 in Theorem 4.4 assures all local vanishing cycles of the \((Y_0, p_i)\) to be preserved when passing to a representative of the global smooth fiber \( Y_0 \cong X_0 \). Thus we may write

\[ \eta := \tau - b_3 \]

with \( b_3 = b_3(X_0) \) the number of vanishing cycles in degree 3 of \((X_0, 0)\).

Returning to our previous example of the \( \Pi_k \) family we find \( b_3 = 0 \), so

\[ b_2 = 1 \neq 2k - 1 = \eta = \tau. \]

Thus there cannot be any relation between \( b_2 \) and \( \eta \). However \( \eta \) seems to be related to the maximal number of \( A_0^+ \) singularities in a deformed fiber of a \( \Pi_k \) singularity. Namely consider the family over \( \mathbb{C} \) given by

\[
\begin{pmatrix}
w & y & x \\
z & w & y + v^k \\
\end{pmatrix} - \begin{pmatrix}0 & 0 & 0 \\
0 & 0 & \varepsilon \end{pmatrix}.
\]

For \( \varepsilon \neq 0 \) we find \( k \) distinct singularities at the points \((x, y, z, v, w) = (0, 0, 0, \sqrt[k]{\varepsilon}, 0)\). Using the analytic coordinate change \( v' = y + v^k - \varepsilon \) locally at any of these points gives the standard form of the \( A_0^+ \).

Topologically the fiber over \( \varepsilon \neq 0 \) contains a bouquet of \( k \) real 3-spheres as indicated in Figure 2 which somehow reminds us of a chain of wrapped candy. Their total sum is homologous to zero, while either \( k - 1 \) of them generate \( H_3(Y_0) \). The \( A_0^+ \) singularities sit at the “wrapping points” between these spheres and are unraveled when passing to a smooth fiber. Consequently all local 2-cycles become pairwise homologous. A direct computation then shows that the appearing 3-chain is in fact homologous to 0. First experiments with other entries from the table of simple singularities hint at \( \eta \) growing roughly
Figure 2: The wrapped candy chain for $k = 5$ and $\varepsilon \neq 0$ before and after deforming the $A_0^+$ singularities linearly with the maximal number of $A_0^+$ singularities in a deformed fiber for each specific entry. It remains to be explored whether this observation is a coincidence or whether there is a theoretical reason behind it.

References

[1] V. Arnold. “Normal forms for functions near degenerate critical point”. In: FAA 6 (1972), pp. 254–272.
[2] M. Artin. Lectures on Deformations of Singularities. Tata Institute of Fundamental Research, Bombay, 1976.
[3] W. Bruns and U. Vetter. Determinantal Rings. Vol. 1327. Lecture Notes in Mathematics. Springer, 1988.
[4] R. Buchweitz. “Contributions à la théorie des singularités”. Thse. Paris, 1981.
[5] L. Burch. “On ideals of finite homological dimension in local rings”. In: Proc. Camb. Phil. Soc. 64 (1964), pp. 941–948.
[6] N. Chachapoyas Siesquen. “Invariants de variedades determinantais”. PhD-Thesis. Universidade de Sao Paulo, Sao Carlos, 2014.
[7] J. Damon. “Deformations of Sections of Singularities and Gorenstein Surface Singularities”. In: Amer. J. Math. 109 (1987), pp. 695–721.
[8] J. Damon. “Nonlinear sections of nonisolated complete intersections”. In: New Developments in Singularity Theory. Vol. 21. NATO Conf. Series. Kluwer, 2001, pp. 405–445.
[9] J. Damon and B. Pike. “Solvable groups, free divisors and nonisolated matrix singularities II:Vanishing topology”. In: Geom. Topol. 18 (2014), pp. 911–962.
10. R. Friedman. “Simultaneous Resolution of Threefold Double Points”. In: Math. Ann. 274 (1986), pp. 671–689.
11. A. Fruehbis-Krueger and A. Neumer. “Simple Cohen-Macaulay Codimension 2 Singularities”. In: Comm. in Alg. 38.2 (2010), pp. 454–495.
12. A. Frübbs-Krüger. “Classification of simple space curve singularities”. In: Comm. Algebra 27.8 (1999), pp. 3993–4013.
13. T. Gaffney and A. Rangachev. “Pairs of Modules and Determinantal Isolated Singularities”. In: (2015). eprint: arXiv:1501.00201.
14. M. Giusti. “Classification des singularités isolées simples d’intersections complètes”. In: Proc. Symp Pure Math. 40 (1983), pp. 457–494.
15. G.-M. Greuel. “On deformation of curves and a formula of Deligne”. In: Algebraic Geometry, La Rabida 1981. Vol. 961. Lecture Notes in Math. Springer, 1983, pp. 141–168.
16. G.-M. Greuel and J. Steenbrink. “On the topology of smoothable singularities”. In: Singularities, Part 1 (Arcata, Calif., 1981). Vol. 40. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, R.I., 1983, pp. 535–545.
17. H. Hamm. “Lokale topologische Eigenschaften komplexer Räume”. In: Math. Ann. 191 (1972), pp. 235–252.
18. S. Lojasiewicz. “Triangulizations of semi analytic sets”. In: Ann. Scoula Norm. Sup. Pisa (3) 18 (1964), pp. 449–474.
19. E. Looijenga and G.-M. Greuel. “Milnor number and Tjurina number of Complete Intersections”. In: Math. Ann. 271 (1985), pp. 121–124.
20. J. Milnor. Singular points of complex hypersurfaces. Princenton University Press, 1968.
21. J.W. Milnor. Morse Theory. Princeton, NJ: Princeton University Press, 1963.
22. J.J. Nuno-Ballesteros, B. Orféice-Okamoto, and J.N. Tomazella. “The vanishing Euler characteristic of an isolated determinantal singularity”. In: Israel J. Math. 197 (2013), pp. 475–495.
23. H. Pinkham. “Deformations of algebraic varieties with $G_m$ action”. In: vol. 20. Astérisque. Soc. Math. France, 1974.
24. M. Schaps. “Deformations of Cohen-Macaulay schemes of codimension 2 and nonsingular deformations of space curves”. In: Amer. J. Math. 99 (1977), pp. 669–684.
25. M. da Silva Pereira and M. Soares Ruas. “Codimension two Determinantal Varieties with Isolated Singularities”. In: Math.Scand. 115 (2014), pp. 161–172.
26. D. van Straten. “Weakly Normal Surface Singularities and Their Improvements”. PhD-Thesis. Universiteit Leiden, 1987.
[27] G. N. Tjurina. “Absolute isolatedness of rational singularities and triple rational points”. In: *Func. Anal. Appl.* 2 (1968), pp. 324–333.