Propagation of atomic matter waves inside an atom wave guide

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During the last few years, with the development of matter-wave interferometry and atom-manipulation technologies (e.g., guiding, cooling and trapping atoms), many rapid progresses have been made in the atom optics (atomic matter-wave optics) [1–5]. The techniques (including the atom optical devices) utilized for manipulating atoms are as follows: magnetic confinement inside both hollow glass tubes [6] and current carrying wires [7], permanent micromagnets [8], light-induced force trapping [9] and microfabricated-structure method [10]. The controllable manipulation of cold neutral atoms in field potentials created by substrate mounted microstructures has been advancing at an increasing pace recently. Many matter-wave optical devices such as tightly confining traps and guides [11,12], beam splitters [13] and mirrors [14] have been realized on such atom chips. More recently, Krüger et al. suggested a scheme of three-dimensional trap formed by modulating a magnetic guide using electrostatic fields, and observed atoms trapped in a string of up to six individual such traps [15]; Luo et al. presented an omnidirectional matter wave guide (atom fibre) on an atom chip, and demonstrated the guiding of thermal atoms around more than two complete turns along a spiral shaped 25 mm long curved path (curve radii down to 200 µm) at various atom-surface distances (35-450 µm) [16]. As for the theoretical study of atom trapping and manipulating with microfabricated structure (atom chips), Jääskeläinen et al. considered many related topics, including the localization in splitting of matter waves, adiabatic propagation and reflection of atoms in potential structures, self-imaging in atom wave guides, multimode interferometer for guided matter waves as well as quantum-state measurement through ballistic expansion of matter waves [17–22]. Since most of these investigations concentrated their attention primarily on the subjects of the matter-wave propagation in the straight atomic-wave guides, but did not consider the topics inside the curved (say, spiral shaped) guides [16], we proposed a formulation to deal with the wave propagation problem in the curved guides in Ref. [23], which is a phenomenological description of time evolution of atomic matter waves inside a noncoplanar atom fibre (atomic-wave guide). In this paper, we first establish a theoretical base for such a phenomenological description, and then treat the evolutionary behavior of neutral atomic matter wave in both classical and quantized potential fields. We discuss the evolutions of guided atomic matter wave in both spatial and temporal domains. As one of the most remarkable results of this treatment, the atomic matter-wave bandgap structure in a cavity field and a spatially periodic guiding field is considered. Here the concept of atomic bandgap medium (optical lattice bandgap medium) is analogous to the photonic bandgap material (photonic crystal) [24,25] and phononic crystal. Finally, we suggest and briefly discuss a scheme of the controllable manipulation of three-level atomic matter waves in a wave guide by an external controlling light.

II. PHENOMENOLOGICAL DESCRIPTION OF PROPAGATION OF ATOMIC MATTER WAVES INSIDE A SPIRAL SHAPED ATOM WAVE GUIDE

In this section, based on the effective action principle, we will suggest a phenomenological description of propagation of atomic matter waves inside a spiral shaped atom wave guide (atom fibre). It was verified in Ref. [23]
that the effective interaction that causes the curved-path motion of matter wave in the atom fibre is a “magnetic”-type (topological, global) interaction, which has a vanishing action [26]. So, we should construct such a vanishing action (or Lagrangian). In general, the propagation of matter wave inside the guide contains the transverse and longitudinal motions. The atomic motional state in the transverse direction is confined (trapped) by a side-guide potential (e.g., magnetic-wire guide potential that is the result of the homogeneous external bias field). Thus, atoms can only be guided along the third unconfined (longitudinal) direction. First we assume that the guiding potential allows us to separate the atomic matter wavefunction, i.e., in the wave guide it can be rewritten as the product of the transverse state and the longitudinal state. The transverse modes can be chosen to be the standing wave functions or harmonic oscillator eigenfunctions [18], which we will not consider further here. Instead, we are concerned with the atomic motion (wave propagation) in the one-dimensional longitudinal direction. Thus, the vanishing Lagrangian that characterizes the phenomenological (effective) interaction between the guided matter wave and the guiding potential fields can be written as

$$L = \eta \hbar (k \cdot v - k \cdot v) = \eta \hbar k \cdot \mathbf{r},$$

where $\mathbf{r}$ denotes the time derivative, and $v$ and $k$ the atomic velocity and the propagation constant (wave vector), respectively, along the guide path (one-dimensional longitudinal propagation). $v$ and $v$ are defined as $v = \mathbf{dr}/dt$ and $v = ds/dt$, where $s$ is the guide path length. Since the direction of $k$ is the same as that of the one-dimensional longitudinal guide path, then one gets the relation $ks = k \cdot \mathbf{r}$. For a matter wave, the relationship between the wave vector and the particle velocity is $k = \mu v/\hbar$, where $\mu$ is the mass of the atom. Thus one has $\mu v \cdot k/\hbar k^2 = 1$. So, the phenomenological Lagrangian can be rewritten as

$$L = \eta \hbar (\mu/k) (\mathbf{v} \cdot k) \cdot \mathbf{r}.$$  

It is assumed that the modulus of the wave vector $k$ does not alter much in the evolution process inside the curved atom waveguide, i.e., $k^2$ can be considered a constant number. Thus, we can have $\mathbf{v} \cdot k = 0$. If we add a vanishing term $-\eta \hbar (\mu/k^2) (\mathbf{v} \cdot k) \cdot \mathbf{k}$ to $L$, then the new Lagrangian is

$$L = \eta \mu v \cdot \left[ \mathbf{r} \times \left( \frac{\mathbf{k} \times \dot{\mathbf{k}}}{k^2} \right) \right]. \quad (1)$$

Apparently, such a Lagrangian has a form $\mu v \cdot \mathbf{A}$. Here the three-dimensional effective vector potential is $\mathbf{A} = \eta \mathbf{r} \times (\mathbf{k} \times \mathbf{k})/k^2$, and in consequence, the phenomenological (effective) field strength is $\mathbf{B}_{\text{eff}} = \nabla \times \mathbf{A} = -2\eta (\mathbf{k} \times \dot{\mathbf{k}})/k^2$. Further calculation shows that the phenomenological Hamiltonian is $H = (\mathbf{P} - \mu \mathbf{A})^2/2\mu$ with the canonical momentum being $\mathbf{P} = \mu \mathbf{v} + \mu \mathbf{A}$. So, the equation of motion of the guided atoms takes the form $\mathbf{P} = \mu \mathbf{v} \times (\nabla \times \mathbf{A}) = -2\eta \mu \mathbf{v} \times (\mathbf{k} \times \dot{\mathbf{k}})/k^2$, namely,

$$\dot{\mathbf{k}} + 2\eta \mathbf{k} \times \left( \frac{\mathbf{k} \times \dot{\mathbf{k}}}{k^2} \right) = \mathbf{0}. \quad (2)$$

Now we can determine the parameter $\eta$ in both the Lagrangian and the above equation. It is apparently seen that when $\eta$ is taken to be $1/2$, the obtained $\mathbf{k} + \mathbf{k} \times (\mathbf{k} \times \dot{\mathbf{k}})/k^2 = \mathbf{0}$ is an identity. In what follows, we take $\eta = 1/2$.

Because of $\nabla \cdot \mathbf{A} = 0$, the above phenomenological Hamiltonian can be rewritten as $H = \mathbf{P}^2/2\mu - \mathbf{A} \cdot \mathbf{P} + (\mu/2)\mathbf{A}^2$, where the term $-\mathbf{A} \cdot \mathbf{P}$ is

$$-\mathbf{A} \cdot \mathbf{P} = \frac{1}{2} \left( \frac{\mathbf{k} \times \dot{\mathbf{k}}}{k^2} \right) \cdot \mathbf{J}, \quad (3)$$

where $\mathbf{J} = \mathbf{r} \times \mathbf{P}$, which is the total angular momentum of both guided atoms and guiding potential fields. It should be noted that the above analysis is treated merely inside the classical Newtonian framework. If one utilizes the equivalence principle, the effective Hamiltonian that describes such a phenomenological interaction is two times that of the obtained result (3) [27–29]. Thus, the total effective Hamiltonian is

$$H_{\text{eff}}(t) = \left[ \frac{\mathbf{k}(t) \times \dot{\mathbf{k}}(t)}{k^2} \right] \cdot \mathbf{J}, \quad (4)$$

and consequently the equation governing the time evolution of the atomic matter wave inside the noncoplanarly curved atom fibre is of the form

$$H_{\text{eff}}(t)|m, \mathbf{k}(t)\rangle = i\hbar \frac{\partial}{\partial t} |m, \mathbf{k}(t)\rangle. \quad (5)$$

The solution to the above equation has been presented in Ref. [23]. In the following, we simply mention the principal result of Ref. [23]. In order to solve Eq. (5), the atom wave vector can be rewritten in the spherical polar coordinate system, i.e., $\mathbf{k}(t) = k(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, where both $\theta$ and $\varphi$ are the time-dependent functions. It is assumed that the initial wave vector is $\mathbf{k}(0) = (0, 0, k)$, i.e., the initial polar angle $\theta(0) = 0$. According to the Lewis-Riesenfeld invariant formulation [30,31], the wavefunction in Eq. (5) can be written in the form $|m, \mathbf{k}(t)\rangle = \exp \left[ \frac{i}{\hbar} \phi_m(t) \mathbf{J}(t) \right] |\mathbf{k}(t)\rangle$, where $|m\rangle$ is the initial state satisfying the eigenvalue equation $J_3|m\rangle = m\hbar|m\rangle$, and the expression for the phase is $\phi_m(t) = m \int_{t_0}^{t} \mathbf{\varphi}(t') [1 - \cos \theta(t')] \, dt'$. The time-dependent operator $\mathbf{V}(t)$ takes the form $\mathbf{V}(t) = \exp [\beta(t) \mathbf{J}_+ - \beta^*(t) \mathbf{J}_-]$ with $\beta = - (\theta/2) \exp(-i\varphi)$ and $\beta^* = - (\theta/2) \exp(i\varphi)$. Here the operators $\mathbf{J}_\pm = (\mathbf{J}_1 \pm i\mathbf{J}_2)/\hbar$.

In the above, we obtained the atomic matter-wave motional state of one-dimensional longitudinal propagation in the curved atom fibre.
III. ATOMIC MATTER WAVES IN THE PRESENCE OF A GUIDING POTENTIAL FIELD

A number of ways has been suggested for building atomic matter-wave guides in experiments. These include the methods by using electric (electrostatic) forces [32–34], magnetic forces [35–37], and light-induced forces [38–40]. Here, we will briefly consider the scheme of electric (electrostatic) and magnetic forces, and then investigate in details the time evolution of the interacting system (including the atomic internal levels, light fields and atomic centre of mass motion) in the scheme of light-induced forces.

A. Electrostatic field as a guiding field

The techniques of electric-magnetic traps and guides can be utilized to realize the monomode atom waveguides and are integrated into a mesoscopic “atom chip” technology. The interaction Hamiltonian of the electrostatic fields and neutral but polarizable atoms (polarizability α) is given \( H_{e-a} = -\alpha E^2/2 \) [15], which leads to the propagation of atomic matter wave along the one-dimensional longitudinal direction inside the atom waveguide. Such a motion results from the mechanism that the atomic centre of mass will be driven towards the larger electric fields since the interaction between the induced electric dipole and the electric field is always attractive. To realize a stable trapping configuration on atom chips, one can use a combination of the electric and magnetic interactions. The full Hamilton of such a three-dimensional confinement is \( H = \mu_B g_e m_F B - \alpha E^2/2 \) [15], where \( \mu_B, g_e, m_F, B \) denote the Bohr magneton, the Landé factor, the magnetic quantum number, and the magnetic field modulus, respectively. Here, the role of the magnetic interaction is to realize a magnetic wire guide (side guide), in which atoms are trapped in a potential tube along a line parallel to a straight current carrying wire [16]. Recently, Luo et al. reported an experiments of trapping and manipulating neutral atoms with electrostatic fields [15]. More recently, this group reported on the implementation and experimental test of a key element for the controlled manipulation of matter waves on the atom chip: an omnidirectional atom fibre [16]. As to the theoretical treatment for the scheme of electrostatic and magnetic forces, readers may be referred to Refs. [17–22].

B. Light-induced potential as a guiding potential

The total Hamiltonian of the atomic matter wave is \( H (r,t) = p^2/2\mu + H_a (r,t) \). Here \( H_a (r,t) \) denotes the Hamiltonian of the atomic internal levels coupled to a guiding potential field, and has the following form

\[
H_a (r,t) = \frac{1}{2} \hbar \Delta \left( |e\rangle \langle e| - |g\rangle \langle g| \right) - p_{eg} \cdot E (r,t) |g\rangle \langle e| - p_{ge} \cdot E (r,t) |e\rangle \langle g|,
\]

where \(|e\rangle\) and \(|g\rangle\) denote the ground and excited states of the two-level atom, respectively, and the frequency detuning \( \Delta = \omega_g - \omega \). Here \( \omega_g \) and \( \omega \) are the atomic transition frequency and the light-field mode frequency, respectively. In the representation of the two-state base vectors \(|e\rangle, |g\rangle\), the matrix form of \( H_a (r,t) \) reads

\[
H_a (r,t) = \left( \begin{array}{cc} \frac{1}{2} \hbar \Delta & V (r,t) \\ V^* (r,t) & -\frac{1}{2} \hbar \Delta \end{array} \right),
\]

where \( V (r,t) = -p_{ge} \cdot E (r,t) \) and \( V^* (r,t) = -p_{eg} \cdot E (r,t) \).

Here we only consider the ultracold atoms, whose centre of mass motion cannot easily drive the transitions between the atomic internal levels. If the external field varies spatially rather adiabatically, then the Born-Oppenheimer approximation is applicable to this problem, and the wavefunction, \( \Psi_\eta (r,t) \), of the atomic system can be separable: it can be rewritten as the product of the centre-of-mass wavefunction \( \Phi_\eta (r,t) \) and the atomic internal-level eigenstates \( |\psi (r,t; \eta)\rangle \), i.e., \( \Psi_\eta (r,t) = \Phi_\eta (r,t) |\psi (r,t; \eta)\rangle \). Because of the validity of the Born-Oppenheimer approximation, the derivatives of \( |\psi (r,t; \eta)\rangle \) with respect to the spatial coordinates can be ignored in the following calculation. Thus, the motional state of the atomic centre of mass and the wavefunction of atomic internal levels agree with the following time-dependent Schrödinger equations

\[
\left[ \frac{p^2}{2\mu} + U (r,t) \right] \Phi_\eta (r,t) = i\hbar \frac{\partial}{\partial t} \Phi_\eta (r,t)
\]

and

\[
[H_a (r,t) - U (r,t) |\psi (r,t; \eta)\rangle] = i\hbar \frac{\partial}{\partial t} |\psi (r,t; \eta)\rangle,
\]

respectively. If one substitutes the relation \( |\psi (r,t; \eta)\rangle = \exp \left[ (i/\hbar) \int_0^t U (r',t') dt' \right] |\psi (r,t; \eta)\rangle \) into Eq. (9), then one can obtain

\[
H_a (r,t) |\psi (r,t; \eta)\rangle = i\hbar \frac{\partial}{\partial t} |\psi (r,t; \eta)\rangle.
\]

In the following, we will solve the time-dependent Schrödinger equation (10).

C. Time evolution in the presence of light fields

The study of topics on the time evolution in atom optics is of interest. For example, recently, several proposals for atomic matter-wave interferometers have been made in the context of splitting and combining a microtrap ground state with a time-dependent potential [20,41,42].
Here, we also consider the behavior of atomic matter wave in the presence of a time-dependent guiding potential field.

For convenience, in the following we will temporarily not mention the variable \( \mathbf{r} \) in \( V \). The Hamiltonian (7) can be written in the form \( H_a = v_1 \sigma_1 + v_2 \sigma_2 + (h \Delta/2) \sigma_3 \), where \( \sigma_i \)’s denote the Pauli matrices, and \( V = v_1 - iv_2 \), \( V^* = v_1 + iv_2 \). Thus, \( H_a(t) \) can be rewritten as

\[
H_a(t) = \hbar \Omega(t) \left( \frac{1}{2} \sin \theta(t) \exp[-i \phi(t)] S_+ + \frac{1}{2} \sin \theta(t) \exp[i \phi(t)] S_- + \cos \theta(t) S_3 \right),
\]

where the instantaneous energy eigenvalue \( \hbar \Omega(t) = 2 \sqrt{\hbar^2 \Delta^2 / 4 + V^*(t)V(t)} \), \( \theta = \arccos(\Delta / \Omega) \) and \( \phi = \arctan(v_2 / v_1) \). The operators \( S_i = (1/2) \sigma_i \) (\( i = 1, 2, 3 \)) and \( S_\pm = S_1 \pm i S_2 \), and agree with the commutation relations \( [S_i, S_\pm] = 2 S_3 \) and \( [S_3, S_\pm] = \pm S_\pm \). According to the Lewis-Riesenfeld invariant theory \([30]\), the Lewis-Riesenfeld invariant that has time-independent eigenvalues obeys the following Liouville-von Neumann equation \([30]\)

\[
\frac{\partial}{\partial t} I(t) + \frac{1}{i \hbar} [I(t), H(t)] = 0.
\]

The eigenvalue equation of the invariant \( I(t) \) is \( I(t)|t; \eta \rangle = \eta |t; \eta \rangle \), where the eigenvalue \( \eta \) is time-independent, i.e., \( \partial \eta / \partial t = 0 \). As the Hamiltonian is constructed in terms of the three operators \( S_3 \) and \( S_\pm \), the invariant \( I(t) \) in Eq. (12) should also take the form expressed by these operators. Thus, we have

\[
I(t) = \frac{1}{2} \sin \zeta(t) \exp[-i \zeta(t)] S_+ + \frac{1}{2} \sin \zeta(t) \exp[i \zeta(t)] S_- + \cos \zeta(t) S_3.
\]

The time-dependent parameters \( \zeta(t) \) and \( \zeta(t) \) can be determined by Eq. (12). Insertion of Eq. (13) into Eq. (12) yields the following set of auxiliary equations

\[
\begin{cases}
\exp(-i \zeta) \left( \zeta \cos \zeta - i \zeta \sin \zeta \right) - i \Omega \exp(-i \phi) \cos \zeta \sin \theta \\
\zeta + \Omega \sin \theta \sin(\zeta - \phi) = 0,
\end{cases}
\]

which can determine the parameters \( \zeta(t) \) and \( \zeta(t) \) of the Lewis-Riesenfeld invariant (13).

It follows from the Lewis-Riesenfeld invariant theory that the particular solution \( |\psi(\mathbf{r}, t; \eta)\rangle \) of the time-dependent Schrödinger equation (10) is different from the eigenstate \( |t; \eta \rangle \) of the invariant \( I(t) \) only by a time-dependent c-number factor \( \exp \left[ \frac{i}{\hbar} \varphi_\eta(t) \right] \), namely, the general solutions of Eq. (10) can be written in the form

\[
|\Psi(t)\rangle = \sum_\eta C_\eta \exp \left[ \frac{i}{\hbar} \varphi_\eta(t) \right] |t; \eta\rangle,
\]

where \( C_\eta \) stands for some certain time-independent coefficients, and the time-dependent phases in (15) are given by

\[
\varphi_\eta(t) = \int_0^t \langle \eta' | \left[ H(t') - i \hbar \frac{\partial}{\partial t'} \right] |t'; \eta\rangle \, dt'.
\]

In view of the above discussion, one can see that if the eigenstate \( |t; \eta \rangle \) is obtained, then the time-dependent Schrödinger equation (10) is solved by the Lewis-Riesenfeld invariant theory \([30]\). To solve the eigenvalue equation \( I(t)|t; \eta \rangle = \eta |t; \eta \rangle \) of the invariant, one can utilize a time-dependent unitary transformation \( V(t) \), which leads to \([43]\)

\[
[V^\dagger(t) I(t) V(t)] |t; \eta \rangle = \eta V^\dagger(t) |t; \eta \rangle.
\]

It is believed that by choosing the appropriate parameters in \( V(t) \), one can obtain \( V^\dagger(t) I(t) V(t) = S_3 \) and \( |\eta \rangle = V^\dagger(t)|t; \eta \rangle \), and thus Eq. (17) will be rewritten as a very simple form \( S_3|\eta \rangle = \eta |\eta \rangle \), which is easily solved. Such a time-dependent unitary transformation operator takes the form \([31]\)

\[
V(t) = \exp \left\{ \left[ \frac{- \zeta(t)}{2} e^{-i \zeta(t)} \right] S_+ - \left[ \frac{- \zeta(t)}{2} e^{i \zeta(t)} \right] S_- \right\}.
\]

By using such a time-dependent unitary transformation, the time-dependent phase (16) can be rewritten as

\[
\varphi_\eta(t) = \int_0^t \langle \eta | \left[ V^\dagger(t') H(t') V(t') - i \hbar V^\dagger(t') \frac{\partial V(t')}{\partial t'} \right] |\eta\rangle \, dt'.
\]

Further calculation shows that \( \varphi_\eta(t) = \varphi_\eta^{(d)}(t) + \varphi_\eta^{(g)}(t) \), where the dynamical phase is

\[
\varphi_\eta^{(d)}(t) = \eta \int_0^t \hbar \Omega(t') \left[ \cos \zeta(t') \cos \theta(t') + \sin \zeta(t') \sin \theta(t') \cos [\zeta(t') - \phi(t')] \right] \, dt',
\]

and the geometric phase is

\[
\varphi_\eta^{(g)}(t) = \eta \int_0^t \hbar \zeta(t') [1 - \cos \zeta(t')] \, dt'.
\]

As the eigenvalues of \( S_3 \) are \( \pm 1/2 \) corresponding to the eigenstates \( |\eta = \pm 1/2 \rangle \), the eigenvalues of the invariant \( I(t) \) are \( \eta = \pm 1/2 \). Thus, the eigenvectors of the invariant are as follows \( |t; \eta = \pm 1/2 \rangle = V(t)|\eta = \pm 1/2 \rangle \), which can be rewritten as
\[
\begin{align*}
|t; \eta = +\frac{1}{2} \rangle &= \cos \frac{\zeta(t)}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{i\phi(t)} \sin \frac{\zeta(t)}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
|t; \eta = -\frac{1}{2} \rangle &= \cos \frac{\zeta(t)}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - e^{-i\phi(t)} \sin \frac{\zeta(t)}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\end{align*}
\]

(22)

where \(|\eta = \pm \frac{1}{2}\rangle\) have been expressed by the two-dimensional column matrices, \(i.e.,\)
\[
|\eta = +\frac{1}{2} \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\eta = -\frac{1}{2} \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(23)

D. Discussions

In what follows, we will discuss the adiabatic case. It is apparent that when the Hamiltonian (11) is time-independent, \(i.e.,\) the parameters \(\theta = \text{const.} \) and \(\phi = \text{const.}\), according to the auxiliary equation (14), the solution of the parameters in the invariant (13) is \(\zeta = \theta, \phi\). For the adiabatic case, \(\zeta(t) \approx \theta(t)\) and \(\phi(t)\) can be viewed as the approximate solution of the auxiliary equation (14). So, in the adiabatic approximation, the eigenbasis (22) of the invariant \(I(t)\) will be reduced to the following form
\[
\begin{align*}
|t; \eta = +\frac{1}{2} \rangle &= \cos \frac{\zeta(t)}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{i\phi(t)} \sin \frac{\zeta(t)}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
|t; \eta = -\frac{1}{2} \rangle &= \cos \frac{\zeta(t)}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - e^{-i\phi(t)} \sin \frac{\zeta(t)}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\end{align*}
\]

(24)

The general solution of the time-dependent Schrodinger equation (10) is
\[
|\Psi(t)\rangle = c_+(t) \begin{pmatrix} t; \eta = +\frac{1}{2} \rangle + c_-(t) \begin{pmatrix} t; \eta = -\frac{1}{2} \rangle,
\end{pmatrix}
\]

(25)

where the time-dependent coefficients are defined as \(c_{\pm}(t) \propto \exp \left[-i\varphi_\eta = \pm 1/2(t)/\hbar\right].\)

According to the two expressions (20) and (21), the phase in the time-dependent coefficients \(c_{\pm}(t)\) is expressed by \(\varphi_\eta(t) \approx \eta \int_0^t \hbar \Omega(t')dt'.\) In the meantime, the invariant \(I(t)\) is also reduced to the Hamiltonian, \(i.e., I \rightarrow \hbar/\Omega.\) This, therefore, means that in the adiabatic approximation, the eigenvalue equation of the invariant acts as the instantaneous eigenvalue equation of the Hamiltonian. In the nonadiabatic case, however, the instantaneous eigenvalue equation of the Hamiltonian is no longer valid, and it should be replaced with the eigenvalue equation of the Lewis-Riesenfeld invariant.

Now we consider the atomic centre of mass motion in the presence of external fields. For the stationary (or adiabatic) case, it follows from Eqs. (8), (9) and (10) that the eigenvalue equation of atomic centre of mass motion is
\[
\begin{align*}
\frac{p^2}{2\mu} \pm \sqrt{\frac{\hbar^2 \Delta^2}{4} + V^*(r)V(r)} \Phi_{\pm}(r) = E\Phi_{\pm}(r).
\end{align*}
\]

(26)

In principle, we presented a theoretical treatment for the time evolution of the atom-light interacting system.

IV. ATOMIC MATTER WAVE IN A WEAKLY GUIDING FIELD

A. Atomic matter wave in a quantized light field

The full Hamiltonian of the two-level atomic ensemble interacting with a photon field is \(H = p^2/2\mu + H_{A-F},\) where \(H_{A-F}\) denotes the total Hamiltonian that includes the free Hamiltonians of the atomic internal levels, the external photon field as well as their interaction Hamiltonian. \(H_{A-F}\) is written in the form
\[
H_{A-F} = \frac{1}{2} \hbar \omega_0 \langle |e\rangle - |g\rangle\rangle + \hbar g(r,t) a\langle e\rangle|g\rangle + g^*(r,t) a^\dagger \langle g\rangle \langle e\rangle + \hbar \omega a^\dagger a,
\]

(27)

where \(a\) and \(a^\dagger\) stand for the annihilation and creation operators of the photons, respectively. The coupling coefficient in the atom-field Hamiltonian (27) is \(g(r,t) = -E_0 d \cdot u^*(r,t)/\hbar,\) where \(E_0 = \sqrt{\hbar\omega/2\epsilon_0 V}\) stands for the electric field strength of monophoton, and \(d\) and \(u\) denote the electric-dipole transition matrix element and the electric-field normal mode, respectively. In the expression for \(E_0,\) the quantization volume \(V = \int u^* \cdot d\). According to the Born-Oppenheimer approximation, the wavefunction, \(\Psi_{\eta,m}(r,t),\) of the system under consideration can be separable, \(i.e.,\)
\[
\Psi_{\eta,m}(r,t) = \Phi_{\eta,m}(r,t)V_\omega(t)|\psi_m(r,t;\eta)\rangle,
\]

(28)

where \(\Phi_{\eta,m}(r,t)\) and \(|\psi_m(r,t;\eta)\rangle\) denote the motional state of the atomic centre of mass and the wavefunction of atomic internal levels, respectively. The time-dependent unitary transformation \(V_\omega(t)\) in (28) takes the form
\[
V_\omega(t) = \exp \left[\frac{1}{i\omega} \left(\langle e| - |g\rangle\rangle - \frac{\hbar}{2} a^\dagger a\right) t\right].
\]

(29)

It follows that the motional state \(\Phi_{\eta,m}(r,t)\) of the atomic centre of mass and the wavefunction \(|\psi_m(r,t;\eta)\rangle\) of atomic internal levels satisfy the following two time-dependent Schrodinger equations
\[
\begin{align*}
\frac{p^2}{2\mu} + U_{\eta,m}(r,t) \Phi_{\eta,m}(r,t) = i\hbar \frac{\partial}{\partial t} \Phi_{\eta,m}(r,t),
\end{align*}
\]

(30)

and
\[
\begin{align*}
[H_A(r,t) - U_{\eta,m}(r,t)] |\psi_m(r,t;\eta)\rangle = i\hbar \frac{\partial}{\partial t} |\psi_m(r,t;\eta)\rangle,
\end{align*}
\]

(31)
respectively. Here the Hamiltonian $H_A(r,t)$ is

$$H_A(r,t) = \frac{1}{2} \hbar \Delta (|e\rangle\langle e| - |g\rangle\langle g|) + \hbar [g(r,t) a|e\rangle\langle g| + g^*(r,t) a^\dagger|g\rangle\langle e|]. \quad (32)$$

We should simplify Eq. (31). By inserting the relation $\psi_m(r,t;\eta) = \exp \left(\frac{i}{\hbar} \int_0^t U_{m,r}(r,t')dt'\right)\psi_m(r,t;\eta)_{\eta}$ into Eq. (31), one can yield

$$H_A(r,t)\psi_m(r,t;\eta)_{\eta} = i\hbar \frac{\partial}{\partial \eta} \psi_m(r,t;\eta)_{\eta}. \quad (33)$$

In the representation of the two-state base vectors ($|e\rangle, |g\rangle$), the operator $|e\rangle\langle e| - |g\rangle\langle g|$ can be rewritten as a matrix form, i.e., $|e\rangle - |g\rangle = \sigma_3$. Thus, one can obtain the commutation relation $[a|e\rangle\langle g|, a^\dagger|g\rangle\langle e|] = N'\sigma_3$, where the operator $N'$ is

$$N' = \begin{pmatrix} a^\dagger a & 0 \\ 0 & a^\dagger a \end{pmatrix}. \quad (34)$$

The eigenvalue equation of $N'$ is as follows

$$N' \begin{pmatrix} |m\rangle \\ |m+1\rangle \end{pmatrix} = (m+1) \begin{pmatrix} |m\rangle \\ |m+1\rangle \end{pmatrix}. \quad (35)$$

Note that the operator $N'$ commutes with all the generators in the Hamiltonian (32). For this reason, the eigenstates of $N'$ have an essential significance in treating Eq. (31) or (33), namely, the solutions of Eq. (33) can be obtained in the sub-Hilbert-space corresponding to the particular eigenvalue of the operator $N'$.

If we set the operators $S_+ = (m + 1)^{1/2} a|e\rangle\langle g|$, $S_- = (m + 1)^{-1/2} a^\dagger|g\rangle\langle e|$ and $S_3 = (1/2)\sigma_3$, we can obtain the following commutation relations $[S_+, S_-] = 2S_3$ and $[S_3, S_\pm] = \pm S_\pm$. Thus, the Hamiltonian (32) of the atomic internal levels (coupled to a radiation field) can be rewritten as

$$H_A(r,t) = \hbar \Delta S_3 + \hbar \sqrt{m+1} [g(r,t) S_+ + g^*(r,t) S_-]. \quad (36)$$

**B. Adiabatic eigenstates**

It is apparent that the Hamiltonian (36) has a same form as (11). Therefore, one can easily obtain the time-evolution solution of Eq. (33) by using the method presented in the preceding section. So, here we will not further discuss this problem. Instead, we will consider the stationary (or adiabatic) case for this quantum mechanical system. Calculation shows that the eigenstates of the atom-field Hamiltonian (27) are as follows

$$\begin{align*}
| r; m, + \rangle & = \cos \frac{\Delta}{2} | e, m \rangle - \sin \frac{\Delta}{2} | g, m + 1 \rangle, \\
| r; m, - \rangle & = \cos \frac{\Delta}{2} | g, m + 1 \rangle + \sin \frac{\Delta}{2} | e, m \rangle.
\end{align*} \quad (37)$$

corresponding to the energy eigenvalues

$$U_{\pm,m}(r) = \left(m + \frac{1}{2}\right) \hbar \omega \pm \sqrt{\frac{\hbar^2 \Delta^2}{4} + (m + 1)|g(r)|^2}, \quad (38)$$

respectively. The parameter angle $\theta_m$ in the eigenstates (37) is defined as $\theta_m(r) = \arctan \left(-2\sqrt{m+1}g(r)/\Delta\right)$, where $g(r)$ is assumed to be a real function.

In the meanwhile, the motional state of the atomic centre of mass satisfies

$$[p^2/2\mu + U_{\pm,m}(r)] \Phi_{\pm,m}(r) = E \Phi_{\pm,m}(r). \quad (39)$$

This means that the atom in the presence of a light field will undergo a light-induced force, which can be used to guide atomic matter waves.

**C. Atomic matter-wave bandgap structure in a weakly guiding field**

External fields (light, electrostatic field and magnetic field) can cool, guide and trap atoms. Apart from these roles which external fields play, there may be a new effect of atomic matter wave arising from the interaction between atomic internal levels and external optical fields, i.e., the existence of the atomic matter-wave bandgap structure in a spatially-periodic guiding field or a cavity field. As the spatially-periodic potential (or force) acting upon the atoms is provided by the optical fields, such a medium can also be referred to as “optical lattice bandgap medium”. One of the properties of this optical lattice bandgap medium (atomic matter-wave bandgap medium) is such that some atoms with certain momentum can propagate through the potential region, and other atoms which have momentum within the atomic band gap will, however, be recoiled and therefore cannot propagate in the atomic matter-wave bandgap medium.

As the “lattice constant” of the spatially periodic potential (optical lattice) is the wavelength of the optical field, in order to form such an optical lattice bandgap structure, the atomic momentum should be of about $10^{-27}$ Kg-m/s. Thus, such a kind of atomic media (optical lattice bandgap media) can be realized below $T = 10^{-5}$ K.

During the last decades, a kind of materials referred to as photonic crystals, which is patterned with a periodicity in dielectric constant and can therefore create a range of forbidden frequencies called a photonic band gap, focus considerable attention of many researchers [24,25]. Likewise, we believe that it is also of physical interest to investigate its atomic counterpart, particularly in the field of Bose-Einstein condensation (BEC). The photonic crystal is, however, a classically electromagnetic material, since the behavior of light can be governed by the
classical Maxwellian equations. Such an atomic (optical lattice) bandgap structure medium is a purely quantum optical material. Since it will offer the possibility of molding the flow of atoms in the guiding potential fields, this optical lattice bandgap structure may enable the atomic waveguide technique to be achieved more effectively in the technology of atom chips.

Additionally, it has been suggested that the atoms and molecules can be trapped inside a single-wall carbon nanotube. Recently, a new molecular conformation of carbon, i.e., the supramolecular assembly comprising C60 molecules inside single-wall carbon nanotubes was synthesized by using transmission electron microscopy [44] and vapor phase methods [45]. As the encapsulated C60 molecules with a neighbor distance of 1 nm are in the carbon nanotubes, such a molecular assembly is referred to as the nanoscopic carbon peapod [44]. The potential application of the carbon peapod is such that it can be utilized as a memory device capable of storing a bit of information by the voltage-driven shuttling of interior C60 between the two endcaps of the single-wall carbon nanotubes [45]. We believe that in addition to the method of controlling the C60 molecules trapped in the nanoscopic carbon peapod by using the moderate voltage, it can be manipulated by the optical fields in the carbon nanotube. Such optical fields itself has also been trapped in the carbon peapod because the electric permittivity inside the carbon nanotube is of spatial periodicity. Thus, the carbon peapod can be considered a one-dimensional photonic crystal. In the next section, we will propose a theoretical mechanism for the coherent control of C60 molecules trapped in the nanoscopic carbon peapod by both the external controlling light and the interior optical fields trapped in the carbon nanotube.

V. COHERENT CONTROL OF THREE-LEVEL ATOMIC MATTER WAVE BY AN EXTERNAL CONTROLLING FIELD

In this section, a scheme to manipulate atomic matter waves inside the wave guide by using an external controlling field is suggested. If, for example, such guided atoms have a system of three-level configuration, the full Hamiltonian of which is

\[ H = \frac{p^2}{2\mu} + H_{A-F} (r), \]

where the atom-field Hamiltonian \( H_{A-F} (r) \) reads

\[
H_{A-F} (r) = \hbar \omega^e |e\rangle \langle e| + \hbar \omega^{s1} |g_1\rangle \langle g_1| + \hbar \omega^{s2} |g_2\rangle \langle g_2| \\
+ \hbar \omega_1 a_1^\dagger a_1 + \hbar \omega_2 a_2^\dagger a_2 \\
+ \hbar g_1 (r) \left( a_1^\dagger |e\rangle + a_1 |g_1\rangle \right) \\
+ \hbar g_2 (r) \left( a_2^\dagger |e\rangle + a_2 |g_2\rangle \right). \tag{40}
\]

Here we assume that the three-level atom has one upper (excited) and two lower (ground) states, which are denoted by |e⟩, |g1⟩ and |g2⟩, respectively. The corresponding energy eigenvalues of these three states are \( \hbar \omega^e, \hbar \omega^{s1}, \) and \( \hbar \omega^{s2}, \) respectively. \( \omega_1 \) and \( \omega_2 \) in (40) denote the mode frequencies of the two photon fields coupled to the \( |g_1\rangle - |e\rangle \) and \( |g_2\rangle - |e\rangle \) transitions, the coupling coefficients of which are \( g_1 \) and \( g_2, \) respectively. By using the Born-Oppenheimer approximation, the separable wavefunction \( \Psi_{\eta,n_1,n_2} (r) \) can be written as \( \Psi_{\eta,n_1,n_2} (r) = \Phi_{\eta,n_1,n_2} (r) |r; n_1, n_2, \eta\rangle, \) where \( \Phi_{\eta,n_1,n_2} (r) \) and \( |r; n_1, n_2, \eta\rangle \) stand for the motional state of atomic centre of mass and the eigenstate of the interaction between atomic internal levels and two radiation fields, the latter of which agrees with the eigenvalue equation

\[
H_{A-F} (r) |r; n_1, n_2, \eta\rangle = E_{\eta,n_1,n_2} (r) |r; n_1, n_2, \eta\rangle \tag{41}
\]

where the parameter

\[
k_\pm = \frac{1}{2} \left[ \frac{N_0^2 E_{0,n_1,n_2}}{\hbar} \pm \sqrt{\left( \frac{N_0^2 E_{0,n_1,n_2}}{\hbar} \right)^2 + 4N_0^2} \right]. \tag{42}
\]

The normalized coefficients in (41) are \( N_0 = 1/\sqrt{g_1^2 (n_1 + 1) + g_2^2 (n_2 + 1)}. \) \( N_\pm = 1/\sqrt{1 + k_\pm^2/N_0^2}. \) The corresponding energy eigenvalues of the three eigenstates (41) are given by

\[
E_{0,n_1,n_2} = \hbar (\omega^e + n_1 \omega_1 + n_2 \omega_2),
\]

\[
E_{\pm,n_1,n_2} = \frac{1}{2} \left( E_{0,n_1,n_2} \pm \sqrt{E_{0,n_1,n_2}^2 + \frac{4N_0^2}{N_0^2}} \right). \tag{43}
\]

Thus the state \( \Phi_{\eta,n_1,n_2} (r) \) of the atomic centre of mass motion satisfies

\[
\left[ \frac{p^2}{2\mu} + E_{\eta,n_1,n_2} (r) \right] \Phi_{\eta,n_1,n_2} (r) = E \Phi_{\eta,n_1,n_2} (r). \tag{44}
\]

It should be noted that the presented formulation can be applied to the investigation of the controllable manipulation of three-level atomic matter-wave bandgap structure by the external controlling light. It follows from Eq. (44) that the atomic matter-wave bandgap structure caused by one of the optical fields, say, the field coupled to the \( |g_1\rangle - |e\rangle \) level pair, can be controllably manipulated by another optical fields (i.e., the optical field coupled to the \( |g_2\rangle - |e\rangle \) pair).
VI. CONCLUDING REMARKS

The motion of atoms inside the guides in both temporal and spatial domains were discussed in the present paper. We considered four topics: (i) a phenomenological model to describe the wave propagation of atoms in the curved atom guides; (ii) the motion of atomic matter wave in the presence of both classical and quantized guiding fields; (iii) the existence of the atomic matter-wave bandgap (optical lattice bandgap) structure in a cavity field or a spatially periodic guiding field; (iv) the controllable manipulation of three-level atomic matter wave in a wave guide by an external controlling light. The paper provided a theoretical formulation for treating the time evolutional process of the atomic matter wave by the time-dependent guiding potential. One of the most remarkable concepts in this paper is the atomic matter-wave bandgap structure. We pointed out that such an atomic bandgap medium can be coherently controlled by external fields, which may have some potential applications in information technology, e.g., the technique of information storage.

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