Quantum tunneling from three-dimensional black holes

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Abstract: We study Hawking radiation from three-dimensional black holes. For this purpose the emission of charged scalar and charged fermionic particles is investigated from charged BTZ black holes, with and without rotation. We use the quantum tunneling approach incorporating WKB approximation and spacetime symmetries. Another class of black hole which is asymptotic to a Sol three-manifold has also been investigated. This procedure gives us the tunneling probability of outgoing particles, and we compute the temperature of the radiation for these black holes. We also consider the quantum tunneling of particles from black hole asymptotic to Sol geometry.
1. Introduction

Quantum mechanical effects when studied in the background of classical general relativity give rise to many interesting phenomenon. These phenomena play an important role in understanding the theories of quantum gravity. One such process of significance is the evaporation of black holes as a result of Hawking radiations [1, 2]. These radiations have also been viewed as quantum tunneling of particles from the horizons of black holes [3, 4, 5]. Many well known black holes have been researched for these radiations [6-12]. In one of the procedures the wave equation governing the emission of particles is solved in the background of the black hole spacetimes by using complex path integration techniques and WKB approximation. This gives the tunneling probability of particles crossing the event horizon, which in turn gives the temperature and surface gravity of the black hole.

The study of (2+1)-dimensional black holes provides a valuable insight in understanding low-dimensional gravity theories and their quantum counterparts. The BTZ black hole [13] is such an example in (2+1)-dimensional gravity. This black hole is remarkably similar to (3+1)-dimensional black hole. Like the Kerr black hole it contains an inner and an outer horizon. It can be fully characterized by mass, angular momentum and charge. It also possesses thermodynamical properties analogous to the (3+1)-dimensional black hole. Its entropy is given by a law analogous to the Bekenstein bound in (3+1)-dimensions, where we replace the surface area by the circumference of the BTZ black holes. This black hole does arise from collapsing matter and can represent a gravitational collapse. The BTZ solution is also discussed in the realm of (2+1)-dimensional quantum gravity.

In this paper, first we consider the charged version of the BTZ black hole [13]. The charged BTZ black hole is characterized by a power-law curvature singularity generated by the electric charge of the hole. The curvature singularity gives rise to ln r terms when the gravitational field is expanded asymptotically and it has nontrivial boundary terms. This black hole solution exists in the presence of a negative or zero cosmological constant. In this paper we consider tunneling of charged scalar and charged fermionic particles from these black holes, and work out the Hawking temperature. We also study a class of topological three-dimensional black holes constructed from Sol geometry [14]. In our approach we solve charged Klein-Gordon and Dirac equations and calculate the tunneling probabilities of particles crossing their horizons and work out the temperatures.

The organization of this paper is as follows. After an introduction of BTZ black holes in the next section, we discuss tunneling of charged scalar particles from charged
BTZ black holes in Section 3, and from charged rotating BTZ black holes in Sections 4. After this, Section 5 and Section 6 are devoted to the emission of charged fermions from charged black holes, and from charged rotating black holes, respectively. In Section 7, we investigate the topological black hole from the Sol geometry and work out the tunneling probability and Hawking temperature. We conclude our paper with a discussion and brief summary of results.

2. Charged BTZ black holes

The BTZ black hole solutions in (2+1) spacetime dimensions are derived from a three-dimensional theory of gravity. The BTZ black holes [13, 15] are solutions of the Einstein field equations with cosmological constant in three dimensions. These solutions represent one of the main recent advances for low energy in three dimensional gravity theories. The total Einstein action in three dimensional gravity in the presence of cosmological constant is given by

\[ I = \frac{1}{16\pi G} \int d^3 x \sqrt{-g} (R - 2\Lambda), \]  

where \( G \) is the gravitational constant, \( \Lambda = -1/l^2 \), is the Cosmological constant, \( R \), the Ricci scalar and \( g \) is determinant of the metric tensor \( g_{\mu\nu} \). We use units for both \( G \) and \( l \) as \( \text{(length)}^3 \) and we work in the units that \( 8G = 1 \) in this paper.

The corresponding line element in Schwarzschild coordinates is given by

\[ ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 \left( d\phi - \frac{J}{2r^2} dt \right)^2, \]  

where

\[ f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}. \]  

Here \( M \) is the mass, \( J \) is the angular spin of the BTZ black hole and

\[-\infty < t < \infty, \quad 0 \leq r < \infty, \quad 0 \leq \phi < 2\pi. \]  

The horizons, \( r_+ \) (henceforth simply the black hole horizon) and \( r_- \) respectively, concerning the positive mass black hole spectrum with spin \( J \neq 0 \) of the line element (2.2) are given by putting \( f(r) = 0 \),

\[ r_{\pm}^2 = \frac{l^2}{2} \left( M \pm \sqrt{M^2 - \frac{J^2}{l^2}} \right), \]  

\[-3\]
where $'+'$ and $'−'$ denote the outer and inner horizons. The BTZ black hole without electric charge can be obtained as the quotient of AdS space.

One can obtain more general metrics by considering coupling of the pure Einstein gravity with other matter fields. For example, one can consider three-dimensional Einstein gravity with topological matter [16]. One can also discuss the Einstein-Maxwell theory. If we include the Maxwell tensor also, the action is given by

$$I = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left( R - 2\Lambda - 4\pi G F_{\mu\nu} F^{\mu\nu} \right),$$

(2.6)

with

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu},$$

(2.7)

where $A_\mu$ is the electrical potential. In addition to the black hole solutions (2.2) described above, charged black hole solutions similar to (2.2) exist [13, 17]. These are solutions following from the action (2.6).

The electrically charged black hole solutions of Einstein-Maxwell theory take the form

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 d\phi^2,$$

(2.8)

but with

$$f(r) = -M + \frac{r^2}{l^2} - \frac{1}{2} Q^2 \ln \left( \frac{r}{l} \right),$$

(2.9)

where $Q$ is the electric charge of the black hole. For this charged black hole, there is a limit $l \to \infty, \Lambda \to 0$, in which it goes over to charged black hole asymptotic to flat space.

The electrically charged and rotating black holes take the form (2.2), but with

$$f(r) = -M + \frac{r^2}{l^2} - \frac{1}{2} Q^2 \ln \left( \frac{r}{l} \right) + \frac{J^2}{4r^2},$$

(2.10)

where $Q$ is the electric charge, and $J$ is the angular momentum of the black hole.

Using the fact that for BTZ black hole electric potential, $A_\mu$, is given by $(A_t, 0, 0)$, we have from (2.6) and (2.9)

$$\frac{Q}{r} = -A_{t,r}.$$  

(2.11)

After integration the above expression comes out to be

$$A_t = -Q \ln \left( \frac{r}{l} \right).$$

(2.12)
Since we are considering the case of charged BTZ black hole, we consider the line
element (2.8) with \( f(r) \) given by (2.9). The Maxwell field is given by
\[
F_{tr} = \frac{Q}{r}.
\] (2.13)

We see that unlike the uncharged case where horizons can be found easily, the
function \( f(r) \) is more complicated for the charged BTZ black hole.

3. Quantum tunneling of scalar particles from charged BTZ
black hole

Here we treat the emission of scalar particles from charged BTZ black holes as tun-
neling phenomenon across the event horizon. This is done by solving charged Klein-
Gordon equation for scalar field, \( \Psi \), which is given by
\[
\frac{1}{\sqrt{-g}} \left( \partial_{\mu} - \frac{iq}{\hbar} A_{\mu} \right) \left( \sqrt{-g} g^{\mu\nu} (\partial_{\nu} - \frac{iq}{\hbar} A_{\nu}) \Psi \right) - \frac{m^2}{\hbar^2} \Psi = 0,
\] (3.1)
where \( \nu \) and \( \mu \) have values 0, 1, 2 for the coordinates \( t, r, \phi \). The \( m \) and \( q \) are the
mass and charge of the particle. We use WKB approximation and choose an ansatz
of the form
\[
\Psi(t, r, \phi) = e^{\left( \frac{i}{\hbar}(I(t,r,\phi)+I_1(t,r,\phi)+O(\hbar)) \right)}.
\] (3.2)
Using this in (3.1) in leading powers of \( \hbar \), dividing by the exponential term and
multiplying by \( \hbar^2 \), we get
\[
0 = g^{tt}(\partial_t I - qA_t)^2 + g^{rr}(\partial_r I)^2 + g^{\phi\phi}(\partial_\phi I)^2 + m^2,
\] (3.3)
or
\[
0 = - \left( -M + \frac{r^2}{l^2} - \frac{1}{2}Q^2 \ln r \right)^{-1} (\partial_t I - qA_t)^2 + \\
\left( -M + \frac{r^2}{l^2} - \frac{1}{2}Q^2 \ln r \right) (\partial_r I)^2 + r^{-2}(\partial_\phi I)^2 + m^2.
\] (3.4)

If we look at the symmetries of charged BTZ black hole then \( \partial_t \) and \( \partial_\phi \) are the
Killing fields. So there exists a solution for this differential equation which in terms
of the classical action \( I \) can be written as
\[
I = -\omega t + W(r) + j\phi + K,
\] (3.5)
where $\omega$ and $j$ are the energy and angular momentum of the particle, and $K$ is a constant which can be complex. Using this function in the above expression we get

$$W_\pm(r) = \pm \int \sqrt{-\frac{g^{tt}}{g^{rr}}} \left( (\omega + qA_t)^2 + \frac{g^{\phi\phi}}{g^{tt}} j^2 + \frac{1}{g^{tt}} m^2 \right) dr. \quad (3.6)$$

Putting the values of $g^{tt}$ and $g^{rr}$, we can write $W_\pm(r)$ as

$$W_\pm(r) = \pm \int \frac{\sqrt{(\omega + qA_t)^2 - f(r) \left( \frac{j^2}{r^2} + m^2 \right)}}{f(r)} dr. \quad (3.7)$$

Here, we have simple pole at $r = r_+$, so by using the residue theory for semi circles, we get

$$W_\pm = \pm \frac{\pi i (\omega + qA_t)}{f'(r_+)} , \quad (3.8)$$

since $f(r_+) = 0$. This implies that

$$\text{Im} W_+ = \frac{\pi (\omega + qA_t)}{f'(r_+)} . \quad (3.9)$$

Hawking radiation from black holes can be studied as a process of quantum tunneling of particles from the black hole horizon. Using this approach we calculate the imaginary part of the classical action for this classically forbidden process of emission across the horizon. In this semi-classical method the probabilities of crossing the horizon from inside to outside, and from outside to inside, are given by [7, 8]

$$P_{\text{emission}} \propto \exp \left( -\frac{2}{\hbar} \text{Im} I \right) = \exp \left( -\frac{2}{\hbar} (\text{Im} W_+ + \text{Im} K) \right) , \quad (3.10)$$

$$P_{\text{absorption}} \propto \exp \left( -\frac{2}{\hbar} \text{Im} I \right) = \exp \left( -\frac{2}{\hbar} (\text{Im} W_- + \text{Im} K) \right) . \quad (3.11)$$

We know that any outside particle will certainly fall into the black hole. Thus we must take $\text{Im} K = -\text{Im} W_-$. From (3.8), we have $W_+ = -W_-$, and this means that the probability of a particle tunneling from inside to outside the horizon is

$$\Gamma = \exp \left( -\frac{4}{\hbar} \text{Im} W_+ \right) . \quad (3.12)$$

Putting the value of $\text{Im} W_+$ from equation (3.9) into (3.12), we get

$$\Gamma = \exp \left( -\frac{4\pi (\omega - qQ \ln \left( \frac{r_+}{r} \right))}{hf'(r_+)} \right) . \quad (3.13)$$
This is the tunneling probability of scalar particles from inside to outside the event horizon of the charged BTZ black hole.

If we compare this equation (3.13) with $\Gamma = \exp (-\beta \omega)$, which is Boltzmann factor for particle of energy $\omega$, and $\beta$ is the inverse temperature of the horizon [7, 8], we can derive the Hawking temperature for black holes. Comparing equation (3.13) with the Boltzmann factor of energy, we can find the Hawking temperature (taking $\hbar = 1$) as

$$T_H = \frac{f'(r_+)}{4\pi},$$

(3.14)

where $f'(r_+)$ is the derivative of $f$ with respect to $r$ at $r = r_+$. So from equation (3.14), the temperature becomes

$$T_H = \frac{1}{4\pi} \left( \frac{2r_+}{l^2} - \frac{Q^2}{2r_+} \right).$$

(3.15)

This situation is similar to the Reissner-Nordström black hole in $(3+1)$-dimensions. It has the interesting Boltzmann factor (3.13), with chemical potential conjugate to the charge.

Now we look at thermodynamic relations in this situation. From $f(r_+) = 0$, the mass of the black hole is given by

$$M_{bh}(Q) = \frac{r_+^2}{l^2} - \frac{1}{2} Q^2 \ln \left( \frac{r_+}{l} \right).$$

(3.16)

The entropy of the black hole is

$$S = 4\pi r^+, \quad (3.17)$$

which is the circumference, in the units $8G = 1$. For the electric potential $V$

$$\frac{\partial M}{\partial Q}|_S = V = -Q \ln \left( \frac{r_+}{l} \right) = A_t.$$

(3.18)

So that we have

$$\frac{\partial M}{\partial S}|_Q = T = \frac{1}{4\pi} \left( \frac{2r_+}{l^2} - \frac{Q^2}{2r_+} \right),$$

(3.19)

which is from the thermodynamic relation, and is the same as (3.15) by the above independent method of quantum tunneling.
4. Quantum tunneling of scalar particles from rotating charged BTZ black hole

In this section, we consider emission of scalar particles from rotating charged BTZ black hole. The line element is given by

\[ ds^2 = -f(r) \, dt^2 + f^{-1}(r) \, dr^2 + r^2 \left( d\phi + N^\phi(r) dt \right)^2 , \]  

(4.1)

where

\[ f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{1}{2} Q^2 \ln \left( \frac{r}{l} \right) , \]  

(4.2)

\[ N^\phi(r) = \frac{-J}{2r^2} . \]  

(4.3)

To deal with the quantum tunneling of scalar particles from this black hole, we will use the charged Klein-Gordon equation given by (3.1). Again assuming the function of the form in (3.2) for the solution and following the earlier procedure, we get the differential equation

\[ g^{\mu\nu}(\partial_\mu I - qA_\mu)^2 + g^{\nu\nu}(\partial_\nu I - qA_\nu)^2 + g^{\phi\phi}(\partial_\phi I)^2 + m^2 = 0 . \]  

(4.4)

As before we assume the function \( I \) of the form in (3.6) and obtain

\[ W'(r) = \pm \sqrt{\frac{-g^{tt}}{g^{rr}} \left( (\omega + qA_t)^2 + \frac{g^{\phi\phi}}{g^{tt}} j^2 - \frac{g^{t\phi} j}{g^{tt}} (\omega + qA_t) + \frac{1}{g^{tt} m^2} \right)} . \]  

(4.5)

Substituting the value of the metric tensor we get the integral

\[ W_{\pm}(r) = \pm \int \frac{\sqrt{(\omega + qA_t)^2 - \left( \frac{f(r)}{r^2} - (N^\phi)^2 \right) j^2 + 2N^\phi(r)(\omega + qA_t)j - f(r)m^2}}{f(r)} \, dr . \]  

(4.6)

Here, we have simple pole at \( r = r_+ \), and therefore, from the residue theory for semi circles, we get on integration

\[ W_{\pm} = \pm \pi i \sqrt{(\omega + qA_t)^2 + (N^\phi(r_+))^2 j^2 + 2N^\phi(r_+)(\omega + qA_t)j} . \]  

(4.7)

The above equation implies that

\[ \text{Im} W_{\pm} = \pi \frac{(\omega + qA_t(r_+) + j N^\phi(r_+))}{f'(r_+)} . \]  

(4.8)
As the probabilities of crossing the horizon from inside to outside and outside to inside is given by

\[ P_{\text{emission}} \propto \exp \left( \frac{-2}{\hbar} \text{Im} I \right) = \exp \left( \frac{-2}{\hbar} (\text{Im} W_+ + \text{Im} K) \right), \quad (4.9) \]

\[ P_{\text{absorption}} \propto \exp \left( \frac{-2}{\hbar} \text{Im} I \right) = \exp \left( \frac{-2}{\hbar} (\text{Im} W_- + \text{Im} K) \right). \quad (4.10) \]

The probability of a particle tunneling from inside to outside the horizon is given by \( \Gamma = \exp \left( \frac{-4}{\hbar} \text{Im} W_+ \right) \), which on substituting the value of \( \text{Im} W_+ \) from equation (4.8) yields

\[ \Gamma = \exp \left( \frac{-4\pi}{\hbar} \left( \frac{\omega - qQ \ln \left( \frac{r_+}{l} \right) - 2J^2}{4r_+^2} \right) \right). \quad (4.11) \]

This is the tunneling probability of scalars across the event horizon of the charged rotating BTZ black hole. We note that this does not depend on the mass of the tunneling particle but only on its charge. By comparing this with the Boltzmann factor of energy of particle, we can find the Hawking temperature of this black hole \( T_H = f'(r_+)/4\pi \) as

\[ T_H = \frac{1}{4\pi} \left( \frac{2r_+}{l^2} - \frac{Q^2}{2r_+} - \frac{J^2}{2r_+^2} \right). \quad (4.12) \]

Putting charge \( Q = 0 \) will correspond to the temperature for uncharged BTZ black hole [18].

Now we discuss the thermodynamic relations. The mass of the black hole is

\[ M_{bh}(Q, J) = \frac{r_+^2}{l^2} - \frac{1}{2} Q^2 \ln \left( \frac{r_+}{l} \right) + \frac{J^2}{4r_+^2}. \quad (4.13) \]

The entropy is

\[ S = 4\pi r_+, \quad (4.14) \]

which is the circumference, in the unit \( 8G = 1 \). We see that

\[ \frac{\partial M}{\partial J}|_{Q,S} = \Omega = \frac{J}{2r_+^2}, \quad J = 2\Omega r_+^2. \quad (4.15) \]

The mass of the black hole can also be expressed as

\[ M = \left( \frac{1}{l^2} + \Omega^2 \right) r_+^2 - \frac{1}{2} Q^2 \ln \left( \frac{r_+}{l} \right). \quad (4.16) \]

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For the electric potential $V$ we have
\[
\frac{\partial M}{\partial Q}|_{J,S} = V = -Q \ln \left( \frac{r_+}{l} \right) = A_t. \tag{4.17}
\]
So that we have
\[
\frac{\partial M}{\partial S}|_{Q,J} = T = \frac{1}{4\pi} \left( \frac{2r_+}{l^2} - \frac{Q^2}{2r_+} - \frac{J^2}{2r_+^3} \right), \tag{4.18}
\]
which is from the thermodynamic relation, and is the same as obtained by the quantum tunneling method above. The $J = 0$ limit of the temperature reduces to the result given by the quantum tunneling method in Section 3, and by another method [19].

5. Quantum tunneling of fermionic particles from charged BTZ black holes

We will now calculate Dirac particle’s Hawking radiation from the charged BTZ black hole. In this case the function $f(r)$ will be, as in (2.8) and (2.9),
\[
f(r) = -M + \frac{r^2}{l^2} - \frac{1}{2} Q^2 \ln \left( \frac{r}{l} \right). \tag{5.1}
\]

We consider the two-component massive spinor field $\psi$, with mass $\mu$ and charge $q$, which obeys the covariant Dirac equation
\[
\frac{i}{\hbar} \gamma^a e^a_{\mu} \left( \nabla_{\mu} - \frac{i}{\hbar} qA_{\mu} \right) \psi - \mu \psi = 0, \tag{5.2}
\]
where $\nabla_{\mu}$ is the spinor covariant derivative given by $\nabla_{\mu} = \partial_{\mu} + \Omega_{\mu}$, where
\[
\Omega_{\mu} = \frac{i}{2} \Gamma^{\alpha\beta}_{\mu} \Sigma_{\alpha\beta}, \tag{5.3}
\]
\[
\Sigma_{\alpha\beta} = i \frac{1}{4} [\gamma^\alpha, \gamma^\beta], \quad \Omega_{\mu} = -\frac{1}{8} \Gamma^{\alpha\beta}_{\mu} [\gamma^\alpha, \gamma^\beta]. \tag{5.4}
\]

The $\gamma$ matrices in three spacetime dimensions are selected to be
\[
\gamma^a = (-i\sigma^1, \sigma^0, \sigma^2), \tag{5.5}
\]
where $\sigma^i$ are the Pauli sigma matrices. For the line element (2.8) the vielbein field $e^a_{\mu}$ can be selected as
\[
e^0_{\mu} = \left( f^{-\frac{1}{2}} 0 0 \right), \quad e^1_{\mu} = \left( 0 f^{\frac{1}{2}} 0 \right), \quad e^2_{\mu} = \left( 0 0 \frac{1}{r} \right). \tag{5.6}
\]
We use the ansatz for the two-component spinor $\psi$ as

$$\psi = \begin{pmatrix} A(t, r, \phi) \\ B(t, r, \phi) \end{pmatrix} e^{i \frac{I(t, r, \phi)}{\hbar}}.$$  \hfill{(5.7)}

In order to apply WKB approximation, we insert ansatz for spinor field $\psi$ into the Dirac equation. Dividing by the exponential term with $\hbar$, we have the following two equations

$$A \left( \mu + \frac{1}{r} \partial_\phi I(t, r, \phi) \right) + B \left[ \sqrt{f} \partial_t I(t, r, \phi) + \left( \frac{1}{\sqrt{f}} \partial_t I(t, r, \phi) + \frac{1}{\sqrt{f}} Q q \ln \left( \frac{r}{l} \right) \right) \right] = 0,$$

$$A \left[ \sqrt{f} \partial_t I(t, r, \phi) - \left( \frac{1}{\sqrt{f}} \partial_t I(t, r, \phi) + \frac{1}{\sqrt{f}} Q q \ln \left( \frac{r}{l} \right) \right) \right] + B \left( \mu - \frac{1}{r} \partial_\phi I(t, r, \phi) \right) = 0.$$

Note that although $A$ and $B$ are not constant, their derivatives and the component $\Omega_\mu$ are all of order $\hbar$, so they can be neglected to the lowest order in WKB approximation. These two equations have a non-trivial solution for $A$ and $B$ if and only if the determinant of coefficient matrix vanishes. Thus we get

$$\frac{1}{r^2} (\partial_\phi I(t, r, \phi))^2 + \mu^2 + \left( \sqrt{f} \partial_t I(t, r, \phi) \right)^2 - \left( \frac{1}{\sqrt{f}} \partial_t I(t, r, \phi) + \frac{1}{\sqrt{f}} Q q \ln \left( \frac{r}{l} \right) \right)^2 = 0.$$  \hfill{(5.10)}

Because there are two Killing vectors $\left( \frac{\partial}{\partial t} \right)^\mu$ and $\left( \frac{\partial}{\partial \phi} \right)^\mu$ in the charged BTZ spacetime, so we can make the separation of variables for $I(t, r, \phi)$ as

$$I = -\omega t + j \phi + W(r) + K,$$

where $\omega$ and $j$ are Dirac particle’s energy and angular momentum respectively, and $K$ is a complex constant. Now putting

$$\partial_t I = \partial_r W(r), \quad \partial_\phi I = j, \quad \partial_\phi I = -\omega,$$

in (5.10) we get

$$\partial_r W(r) = \pm \frac{1}{f} \sqrt{\left( \omega - q Q \ln \left( \frac{r}{l} \right) \right)^2 - f \left( \frac{\mu^2}{r^2} + \frac{j^2}{r^2} \right)}.$$  \hfill{(5.13)}

In view of (3.10) and (3.11), we have that $W_- = -W_+$. Integration gives

$$W_+(r) = \int \frac{dr}{f} \sqrt{\left( \omega - q Q \ln \left( \frac{r}{l} \right) \right)^2 - f \left( \frac{\mu^2}{r^2} + \frac{j^2}{r^2} \right)}.$$  \hfill{(5.14)}
Substituting the imaginary part of $W_+$ in tunneling probability we obtain

$$W_+ = \frac{\pi i}{F(r_+)} \left( \omega - qQ \ln \left( \frac{r_+}{l} \right) \right),$$

or in simplified form we have

$$\text{Im}W_+ = \frac{\pi}{2\kappa} \left( \omega - qQ \ln \left( \frac{r_+}{l} \right) \right),$$

where $\kappa = \left( \frac{r_+}{l} - \frac{2GQ^2}{r_+} \right)$ is the surface gravity of outer event horizon. This leads to the tunneling probability

$$\Gamma = \exp \left[ -\frac{2\pi}{\hbar \kappa} \left( \omega - qQ \ln \left( \frac{r_+}{l} \right) \right) \right].$$

Thus the Hawking temperature $T_H = f'(r_+)/4\pi$ is

$$T_H = \frac{r_+}{2\pi l^2} - \frac{Q^2}{8\pi r_+}.$$  

This is the same as calculated in the case of scalar particles in (3.15) in Section 3, and agrees with the thermodynamic relation.

### 6. Quantum tunneling of fermionic particles from rotating charged BTZ black holes

In this section we work out the tunneling probability of fermions from rotating charged BTZ black hole. We consider the Dirac equation for electromagnetic field

$$i\gamma^\mu \left( \partial_\mu + \Omega_\mu - \frac{i}{\hbar} qA_\mu \right) \psi - \frac{\mu}{\hbar} \psi = 0,$$  

where

$$\Omega_\mu = \frac{i}{2} \Gamma_\mu^{\alpha\beta} \Sigma_{\alpha\beta},$$

$$\Sigma_{\alpha\beta} = \frac{i}{4} \left[ \gamma^\alpha, \gamma^\beta \right], \quad \Omega_\mu = \frac{-1}{8} \Gamma_\mu^{\alpha\beta} \left[ \gamma^\alpha, \gamma^\beta \right].$$

With the Pauli sigma matrices given by

$$\sigma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

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we choose the curved space $\gamma^\mu$ matrices as
\[
\gamma^t = \begin{pmatrix} 0 & -\frac{1}{\sqrt{f}} \\ \frac{1}{\sqrt{f}} & 0 \end{pmatrix}, \quad \gamma^r = \begin{pmatrix} 0 & \sqrt{f} \\ \sqrt{f} & 0 \end{pmatrix}, \\
\gamma^\phi = \begin{pmatrix} \frac{1}{r} & -\frac{4JG}{r^2\sqrt{f}} \\ \frac{4JG}{r^2\sqrt{f}} & -\frac{1}{r} \end{pmatrix},
\]
which also satisfy the condition $\{\gamma^\mu, \gamma^{\nu}\} = 2g^{\mu\nu} \times I$ where $I$ is the identity matrix.

Now inserting the value of electric potential for charged BTZ black hole the equation of motion becomes
\[
i\gamma^t \left( \partial_t - \frac{i}{\hbar} q A_t \right) \psi + i\gamma^r (\partial_r) \psi + i\gamma^\phi (\partial_\phi) \psi - \mu \psi = 0. \tag{6.6}
\]
For a fermion with spin 1/2 the wave function has two states namely spin-up ($\uparrow$) and spin-down ($\downarrow$), and therefore, we can take the following ansatz for the solution
\[
\psi_{\uparrow} = \begin{pmatrix} A(t, r, \phi) \\ 0 \end{pmatrix} e^{\frac{i}{\hbar} I_{\uparrow}(t, r, \phi)}, \tag{6.7}
\]
\[
\psi_{\downarrow} = \begin{pmatrix} 0 \\ B(t, r, \phi) \end{pmatrix} e^{\frac{i}{\hbar} I_{\downarrow}(t, r, \phi)}, \tag{6.8}
\]
where $\psi_{\uparrow}$ denotes the wave function of the spin-up particle and $\psi_{\downarrow}$ is for the spin-down case. Inserting equation (6.7) for the spin-up particle into the Dirac equation (6.6) and dividing by exponential term and multiplying by $\hbar$, we get the following equation
\[
-\frac{A}{\sqrt{f}} \partial_t I_{\uparrow}(t, r, \phi) + \frac{q A_t}{\sqrt{f}} A - \sqrt{f} A \partial_r I_{\uparrow}(t, r, \phi) - \frac{4JG A}{r^2\sqrt{f}} \partial_\phi I_{\uparrow}(t, r, \phi) = 0. \tag{6.9}
\]
Now considering the method of separation of variables for the spin-up case we have
\[
I_{\uparrow} = -\omega t + W(r) + \Theta(\phi) + K = -\omega t + W(r) + j\phi + K. \tag{6.10}
\]
Here $\omega$ and $j$ are the energy and angular momentum of the emitted particle, and $K$ is a complex constant. Using this expression in the above equation we get
\[
\frac{A}{\sqrt{f}} \omega + \frac{q A_t}{\sqrt{f}} A - \sqrt{f} A \partial_r W - \frac{4JG A}{r^2\sqrt{f}} \partial_\phi \Theta = 0. \tag{6.11}
\]
For simplification we put equation (6.10) in (6.11) to get
\[
\frac{\omega}{\sqrt{f}} + \frac{q A_t}{\sqrt{f}} - \sqrt{f} \partial_r W - j \frac{4JG}{r^2\sqrt{f}} = 0, \tag{6.12}
\]
or
\[ \partial_r W = \frac{1}{f} \left( \omega + qA_t - j \frac{4JG}{r^2} \right). \] (6.13)

If we look at the spin-down particle, its phase \( I_\downarrow \) and its \( r \)-dependence have the similar expressions as equations (6.10) and (6.13), respectively. Integration of equation (6.13) gives
\[ W = \int \frac{dr}{f} \left( \omega + qA_t - j \frac{4JG}{r^2} \right). \] (6.14)

We integrate along a semi circle around the pole at \( r_+ = 0 \). Now, at the horizon the radial function can be given as
\[ W = \frac{\pi i(\omega + qA_t - j \frac{4JG}{r_+^2})}{\left( \frac{2r_+}{r^2} - \frac{4GQ^2}{r_+} - \frac{32G^2J^2}{r_+^2} \right)}. \] (6.15)

Tunneling probability for this is given by equation (3.12) that is
\[ \Gamma = \exp \left( -\pi \left( \omega - qQ \ln \left( \frac{r_+}{r^2} \right) - j \frac{4JG}{r_+^2} \right) \frac{r^2}{\kappa} \right), \] (6.16)

where \( \kappa = \left( \frac{r_+}{r^2} - \frac{2GQ^2}{r_+} - \frac{16G^2J^2}{r_+^2} \right) \) is the surface gravity. We work in the units \( 8G = 1 \).

We see that this is the same as obtained by solving the Klein-Gordon equation in Section 4.

7. Quantum tunneling from three-dimensional topological black holes

Some classes of three-manifolds have been studied for their interesting physical properties. One such example is the Sol geometry [14] defined by
\[ ds^2 = e^{2u} dx^2 + du^2 + e^{-2u} dy^2. \] (7.1)

It is the \( \mathbb{R}^2 \) bundle over \( \mathbb{R} \), in which \((x, y)\) is the \( \mathbb{R}^2 \), and \( u \) is the \( \mathbb{R} \). From this class we consider the following metric representing a three-dimensional topological black hole
\[ ds^2 = -\frac{f}{M^2} dt^2 + \frac{1}{l^2 f} dr^2 + \frac{M}{f} d\phi^2, \] (7.2)
where
\[ f(r) = \frac{r^2}{l^2} - M. \] (7.3)

In the asymptotic region, when \( f(r) \to \frac{r^2}{l^2} \), the above metric becomes
\[
ds^2 = \frac{-r^2}{Ml^4} dt^2 + \frac{dr^2}{r^2} + \frac{Ml^2}{r^2} d\phi^2.\] (7.4)

We redefine
\[
\tilde{t} = \frac{t}{\sqrt{M}} = l^2 x,
\quad r = e^u,
\quad u = \ln r,
\quad y = \sqrt{Ml} \phi.
\] (7.5)

In this case (7.2) is asymptotic to
\[
ds^2 = \frac{-f}{l^2} dt^2 + \frac{1}{l^2 f} dr^2 + \frac{M}{f} d\phi^2
\rightarrow -r^2 dx^2 + \frac{dr^2}{r^2} + \frac{1}{r^2} dy^2
= -e^{2u} dx^2 + du^2 + e^{-2u} dy^2.\] (7.6)

We see that (7.8) is exactly the Lorentzian version of the Sol geometry (7.1), by analytical continuation \( x \to ix \). So the black hole in (7.2) is asymptotic to the Sol geometry. The metric (7.2) on three-manifolds can be interpreted as black holes or black hole like objects. They can arise from three dimensional gravity with matter fields.

Here we study the emission of particles from these Sol black holes described above. We use the Dirac equation
\[
i\hbar \gamma^a e^a_\mu \tilde{\nabla}_\mu \psi - \mu \psi = 0,\] (7.9)
where \( \tilde{\nabla}_\mu \) is the spinor covariant derivative. It is worth mentioning here that using Klein-Gordon equation will also give the same results. We select the \( \gamma \) matrices as before and write the vielbein field \( e^\mu_a \) as
\[
e^\mu_0 = \left( \frac{M^{1/2}}{l^{1/2}} 0 0 \right),
\quad e^\mu_1 = \left( 0 1 f^{1/2} 0 \right),
\quad e^\mu_2 = \left( 0 0 \frac{f^{1/2}}{M^{1/2}} \right).\] (7.10)

We use the ansatz for the two-component spinor \( \psi \) as
\[
\psi = \left( A(t, r, \phi) \right) e^{\frac{i}{\hbar} t(t, \gamma, \phi)}.\] (7.11)
In order to apply WKB approximation, we insert this ansatz for spinor field $\psi$ into the Dirac equation. Dividing by the exponential term with $\hbar$, one can get the following two equations

$$A \left( \mu + \sqrt{\frac{f}{M}} \partial_\phi I(t, r, \phi) \right) + B \left( \sqrt{fl} \partial_t I(t, r, \phi) + \sqrt{\frac{M}{f}} l \partial_r I(t, r, \phi) \right) = 0,$$

(7.12)

$$A \left( \sqrt{fl} \partial_\phi I(t, r, \phi) - \sqrt{\frac{M}{f}} l \partial_t I(t, r, \phi) \right) + B \left( \mu - \sqrt{\frac{f}{M}} \partial_\phi I(t, r, \phi) \right) = 0.$$

(7.13)

Now as we have discussed in the previous sections, for nontrivial solution we put the determinant of coefficient matrix equal to zero

$$\frac{M}{f} l^2 (\partial_t I)^2 - \frac{f}{M} (\partial_\phi I)^2 - l^2 f (\partial_r I)^2 - \mu^2 = 0.$$  

(7.14)

To calculate the classical action of trajectory we use the method of separation of variables and suppose

$$I = j \phi - \omega \tilde{t} + W(r) + K$$

(7.15)

$$= j \phi - \frac{1}{\sqrt{M}} \omega t + W(r) + K,$$

(7.16)

where $K$ is a complex constant and $j$ is the angular momentum of the particle. Now we put the relations

$$\partial_\phi I = j, \quad \partial_t I = -\frac{\omega}{\sqrt{M}}, \quad \partial_r I = \partial_r W,$$

(7.17)

in equation (7.14) and obtain

$$\partial_r W(r) = \frac{1}{f} \sqrt{\omega^2 - f \left( \frac{\mu}{l} \right)^2 - \frac{f^2}{M} \left( \frac{j}{l} \right)^2}.$$  

(7.18)

So we see that

$$W = \int dr \frac{1}{f} \sqrt{\omega^2 - f \left( \frac{\mu}{l} \right)^2 - \frac{f^2}{M} \left( \frac{j}{l} \right)^2}$$

(7.19)

$$= \frac{\pi i(\omega)}{F(r_+)}.$$  

(7.20)
where we have integrated along the semi-circle around the pole for $f = 0$ at $r_+$.

This implies that the probability of a particle tunneling from inside to outside the horizon becomes

$$
\Gamma = \exp \left( -\frac{4}{\hbar} \text{Im} W \right) = \exp \left( -\frac{4\pi \omega}{f'(r_+)\hbar} \right).
$$

Comparing with the Boltzmann factor $\exp (-\beta \omega)$ we obtain the temperature $T = f'(r_+)/4\pi$. Substituting the expression of $f(r)$, for $r_+ = \sqrt{Ml}$,

$$
T = \frac{f'(r_+)}{4\pi} = \frac{r_+}{2\pi l^2} = \frac{\sqrt{M}}{2\pi l}.
$$

From the thermodynamic relation

$$
\frac{\partial M}{\partial S} = T,
$$

the entropy $S$ is

$$
S = 4\pi r_+ = 4\pi \sqrt{Ml}.
$$

The temperature and thermodynamic behavior are similar to those of BTZ black hole.

8. Discussion

The theory of three-manifolds has been studied for its interesting geometric and topological properties [14], and classifications on one hand, and its physical applications on the other. Some of the metrics on three-manifolds have been interpreted as black holes or black hole like objects. They play a crucial role in understanding lower dimensional gravity theories.

Here we have analyzed the quantum tunneling approach of Hawking radiations for three-dimensional BTZ and other topological black holes. The charged BTZ black hole represents a solution of the Einstein-Maxwell equations. When we include the charge it becomes a three-dimensional analogue of the Reissner-Nordström black hole. We also computed the quantum tunneling for the topological black holes asymptotic to Sol geometry. Laws of black hole mechanics can be extended to these objects as well.

The BTZ spacetime also appears from the near horizon geometry of higher dimensional black holes. In the near horizon geometry of higher dimensional black holes, under appropriate limit, the time and radial direction of the black hole, and a
circle direction in the extra dimensions, combine into a BTZ geometry [20, 21]. It is interesting to see the connection of this to the discussion in this paper.

We have studied the emission of charged scalar particles and fermions from charged BTZ black holes, with and without rotation. For this purpose we have solved the charged Klein-Gordon and Dirac equations using WKB approximation and symmetries of the background spacetime. Using complex path integration we worked out the tunneling probability of particles from charged BTZ black holes. This also yields Hawking temperature from these three dimensional objects. The temperature is given by \( f'(r_+)/4\pi \), where the function \( f(r) \) for the charged black hole is given by equation (2.9) and that for the charged rotating black hole is given by (2.10). The charged BTZ black holes have a \(-\ln r\) potential term which is very interesting. We also obtained the Boltzmann factors with chemical potentials conjugate to the charge and to the angular momentum of the particles. It is worth emphasizing here that for no value of the energy, charge and angular momentum of the particles will the tunneling probabilities be greater than 1. Thus they will not violate unitarity. This is taken care of by the temporal contribution to the imaginary part of the action [22, 23].

A three-dimensional topological black hole which is asymptotic to a Sol three-manifold has also been investigated for its thermodynamical properties. We have studied the emission of particles and find that the Hawking temperature is again given by \( f'(r_+)/4\pi \), for \( f(r) \) given by (7.3), and it goes as the square root of \( M \). These objects have some other interesting mathematical and physical properties as well.

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