Analytical solution of a plane strain pure bending problem in second gradient electroelasticity

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Abstract

Semi-inverse analytical solution of a pure bending problem for piezoelectric layer is developed in the framework of linear electroelasticity theory with strain gradient and electric field gradient effects. Two-dimensional solution is derived assuming plane strain state of a layer. It is shown that obtained solution can be used for the validation of size-dependent beam and plate models in second gradient electroelasticity theory.

1 Introduction

Size-dependent behavior of piezoelectric structures attracts high attention in last years due to its possible importance for novel micro- and nanoelectromechanical systems such that sensors, energy harvesters, nanopositioners etc. Different non-classical theories of piezoelectric materials have been proposed to explain and to predict new experimental results taking into account flexoelectricity, polarization field and electric field gradients, stress and strain gradients, couple stresses, surface effects. Corresponding beam and plate theories were developed to explain non-classical dependence of apparent electromechanical properties of thin piezoelectric cantilevers and membranes on its size. However, for the best of the authors knowledge, despite some experimental tests and molecular dynamics simulations, theoretical validation of these simplified models has not been presented yet. Nevertheless, it is well known that comparison with exact three-dimensional or plane strain/plain stress solutions should be provided to demonstrate a correctness and strong theoretical background of established beam and plate models.

For example, in classical elasticity validation of an Euler-Bernoulli beam model can be obtained by using Saint-Venant’s problems solutions assuming...
high length-to-thickness ratio of a beam and neglecting of Poisson’s effect [20]. In classical piezoelectricity, Saint-Venant’s solutions for piezoelectric cylinders have been also obtained [27, 28, 29, 30, 31]. These solutions together with finite-element simulations can be used to verify a classical theory of piezoelectric beams with different order of through-thickness approximation of the displacement and electric fields (see, [32] and references therein). Plane strain and three-dimensional solutions for verification of classical piezoelectric plate models have been given in [33, 34].

Verification of non-classical beam and plate models in generalized continuum theories is of highly importance due to its complex nature and a lot of additional constitutive relations and boundary conditions. Such validation of an Euler-Bernoulli beam model in microdilatation elasticity [35] have been given in [36] by using semi-inverse solution of a beam pure bending problem. Validation of beam theories in micropolar and microstretch elasticity can be obtained by using known solutions of corresponding generalized Saint-Venant’s problems [37, 38, 39]. In the authors recent work [40, 41], validation of gradient beam bending theories in Mindlin Form I and Form II [42, 43] have been presented based on three-dimensional semi-inverse solution of a beam pure bending problem. It was shown a significant importance of correct formulation of gradient beam models taking into account additional boundary conditions on the top and bottom surfaces of the beam that were ignored in a number of works.

Subject of the present paper is a second gradient electroelasticity theory that takes into account an influence of strain and electric field gradients. This theory has been developed in [44, 45, 12]. Recently it was applied to the problems of fracture mechanics [46] and micromechanics [47, 48]. Beam-type models in this theory were developed in [23] and with strain gradient effects only in [21, 12]. Similar theory with distortion gradient and thermal effects were presented recently in [14]. In the present paper we derive a semi-inverse analytical solution of a pure bending problem for piezoelectric layer that can be used for verification of corresponding beam and plate models in considered theory. We obtained analytical solution following an approach that were used in [40] generalizing it for piezoelectric effects. Due to complex form of second gradient electroelasticity theory, we consider static plane strain statement of a problem, i.e. cylindrical bending of a plate. For the same reason, simplified variant of constitutive equations with single additional length scale parameter is used following [47, 23]. Thus, in the absence of electromechanical coupling, considered theory can be reduced to the widely known simplified strain gradient elasticity theory [49, 50].

2 Second gradient electroelasticity theory

Let us consider an electro-elastic body occupying the region Ω with boundary ∂Ω. The electric Gibbs energy density in considered theory can be expressed in
the following form \[45, 23, 47\]:

\[
g(\varepsilon_{ij}, \varepsilon_{ij,k}, E_i, E_{i,j}) = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \varepsilon_{kij}\varepsilon_{ij}E_k - \frac{1}{2}\kappa_{ij}E_iE_j + \\
\frac{1}{2}A_{ijklmn}\varepsilon_{ij,k}\varepsilon_{lm,n} - \frac{1}{2}\alpha_{ijkl}E_{i,j}E_{k,l} \tag{1}
\]

where \(C_{ijkl}\) and \(A_{ijklmn}\) are the fourth- and sixth-order tensors of the elastic moduli; \(\varepsilon_{kij}\) is the third-order tensor of piezoelectric moduli; \(\kappa_{ij}\) and \(\alpha_{ijkl}\) are the second- and fourth-order tensors of dielectric permittivity constants; \(\varepsilon_{ij}\) is an infinitesimal strain tensor and \(E_i\) is electric field vector that are defined by the relations:

\[
\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}
\]

\[
E_i = -\phi_{,i} \tag{3}
\]

where \(u(x)\) and \(\phi(x)\) are the vector of mechanical displacements and electric potential function, respectively, at a point \(x = \{x_1, x_2, x_3\}\); the comma denotes the differentiation with respect to spatial variables and repeated indices imply summation.

Note, that in (1) we neglect the flexoelectric effects and high-order coupling effects, including coupling between strain and strain gradient tensors that can persist in the general case of non-centrosymmetric piezoelectric materials. From one side we make these assumptions to consider a most simple gradient electroelasticity theory because our goal is to derive a closed form analytical solution that could be used as the test solution for validation of beam and plate theories. From the other side, experience has shown that even simplified and reduced gradient theories can be useful in practical application \[51\].

Following \[47, 23\], we assume the relations between high-order and classical moduli: \(A_{ijklmn} = C_{ijlm}\ell_{kn}\) and \(\alpha_{ijkl} = \kappa_{ij}\rho_{jl}\), where \(\ell_{kn}\) and \(\rho_{jl}\) are the tensors of the length scale parameters that define the gradient effects in elastic and electrical fields, respectively. For simplification, we additionally assume that the length scale parameters are the same in elasticity and electrostatics problems and do not depend on the direction, i.e. \(\ell_{ij} = \rho_{ij} = \ell^2\delta_{ij}\). Thus, the model consists single additional length scale parameter \(\ell\). In the case of \(\ell = 0\), electric Gibbs energy density (1) reduces to the classical form of electroelasticity theory \[52\].

Constitutive equations of linear theory follow from (1):

\[
\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - \varepsilon_{kij}E_k \tag{4}
\]

\[
\tau_{ijk} = A_{ijklmn}\varepsilon_{lm,n} \tag{5}
\]

\[
D_i = \varepsilon_{ijk}\varepsilon_{jk} + \kappa_{ij}E_j \tag{6}
\]

\[
Q_{ij} = \alpha_{ijkl}E_{k,l} \tag{7}
\]

where \(\sigma_{ij}\) is Cauchy stress tensor, \(\tau_{ijk}\) is the third-order double stress tensor \[43\], \(D_i\) is the electric displacement vector, \(Q_{ij}\) is the electric quadrupole tensor \[10, 53\].
Field equations and boundary conditions of the model can be obtained based on variational approach (see [45, 47, 14]). Assuming the absence of body forces and free charges one can obtain the equations of elastic equilibrium and generalized Gauss’ law of electrostatics:

\[
\sigma_{ij,j} - \tau_{ijk,jk} = 0 \quad x \in \Omega
\]

\[
D_{i,i} - Q_{ij,ij} = 0 \quad x \in \Omega
\]

Boundary conditions are prescribed on the body surface \(\partial \Omega\) assuming absence of high-order loading (zero right parts in the equations (12)-(13)) and have the following form:

\[
(\sigma_{ij} - \tau_{ijk,k})n_j - (\tau_{ijk}n_k)_{,j} + (\tau_{ijk}n_kn_l)_{,l}n_j = t_i \quad \text{or} \quad u_i = n_i
\]

\[
(D_j - Q_{jk,k})n_j - (Q_{jk,k})_{,j} + (Q_{jk}n_kn_l)_{,l}n_j = q \quad \text{or} \quad \phi = \bar{\phi}
\]

\[
\tau_{ijk}n_jn_k = 0 \quad \text{or} \quad u_{i,j}n_j = 0
\]

\[
Q_{ij}n_in_j = 0 \quad \text{or} \quad E_i n_i = 0
\]

where \(n_i\) is the vector of the external unit normal to \(\partial \Omega\); \(t_i, u_i, q, \phi\) are the prescribed mechanical traction vector, displacement vector, surface electric charge density and electric potential, respectively.

Additional boundary conditions of a gradient theory should be prescribed at the sharp edges of the body \(\Gamma(\partial \Omega)\) as follows:

\[
[\tau_{ijk}m_jn_k] = 0, \quad [Q_{jk,m}n_k] = 0
\]

where \(m_j = \epsilon_{jml}s_mn_l\) is the outer co-normal vector; \(s_m\) is the unit vector tangent to the given edge; the brackets \([...]\) indicate that the enclosed quantity is the difference between the values on the body faces to which the given edge belongs; \(\epsilon_{jml}\) is the permutation symbol.

### 3 Pure bending of piezoelectric layer

Consider a piezoelectric layer of length \(L\) and thickness \(2h\), i.e. \(\Omega = \{x : x_1 \in [0, L], x_3 \in [-h, h]\\}\) (Fig. 1). Layer has infinite dimension in out-of-plane direction \(x_2\), thus, a plane strain state hypothesis is valid: we assume that there are no displacements in transversal direction \((u_2 = 0)\), all fields quantities do not depend on \(x_2\) coordinate and boundary conditions on the lateral sides \((n_i = \{0, 1, 0\}\}) can be neglected. The layer is poled along the thickness direction, so that it exhibits the material symmetry of a hexagonal crystal in class 6mm – transversely isotropic about \(x_3\) axis.

Boundary conditions for the considered problem are the following. At the end faces of the layer \(x_1 = 0\) and \(x_1 = L\) the distributed loading \(\bar{T}_i = \{\bar{T}_i(x_3), 0, 0\}\) is acted with resultant bending moments \(M\) (distributed moments per unit width) and with zero force resultants, such that pure bending is realized in the
$x_1x_3$ plane about the $x_2$ axis. There are no stresses acting on the top and on the bottom faces of the layer. The surface charge density $q$ is zero over the whole boundary, thus, the electric permittivity of the surrounding medium (for example air) assumed to be much less than that of the piezoelectric layer. The layer is fixed at the origin of coordinate system, where the zero value of electric potential is also prescribed.

For a given statement of the problem, the component representations of the boundary conditions (10)-(13) become:

$$x_1 = 0, L :$$

$$\sigma_{11} - \tau_{111,1} - \tau_{113,3} - \tau_{131,3} = \tilde{t}_1(x_3)$$ \hspace{1cm} (15)

$$\sigma_{21} - \tau_{211,1} - \tau_{213,3} - \tau_{231,3} = 0$$ \hspace{1cm} (16)

$$\sigma_{31} - \tau_{311,1} - \tau_{313,3} - \tau_{331,3} = 0$$ \hspace{1cm} (17)

$$\tau_{111} = 0, \tau_{211} = 0, \tau_{311} = 0$$ \hspace{1cm} (18)

$$D_1 - Q_{11,1} - Q_{13,3} - Q_{31,3} = 0$$ \hspace{1cm} (19)

$$Q_{11} = 0$$ \hspace{1cm} (20)

$$x_3 = \pm h :$$

$$\sigma_{13} - \tau_{113,1} - \tau_{131,1} - \tau_{133,3} = 0$$ \hspace{1cm} (21)

$$\sigma_{23} - \tau_{213,1} - \tau_{231,1} - \tau_{233,3} = 0$$ \hspace{1cm} (22)

$$\sigma_{33} - \tau_{313,1} - \tau_{331,1} - \tau_{333,3} = 0$$ \hspace{1cm} (23)

$$\tau_{133} = 0, \tau_{233} = 0, \tau_{333} = 0$$ \hspace{1cm} (24)

$$D_3 - Q_{13,1} - Q_{31,1} - Q_{33,3} = 0$$ \hspace{1cm} (25)

$$Q_{33} = 0$$ \hspace{1cm} (26)

Following Saint-Venant’s approach of classical elasticity, that was also used in electroelasticity (see, e.g. [28]) and in strain gradient elasticity [40] we will use a relaxed formulation of boundary conditions on the end faces of the layer.
Thus, we assume that stress resultants, rather than pointwise tractions, are prescribed there and we use the following conditions instead of (15):

\[ \int_{-h}^{h} (\sigma_{11} - \tau_{111,1} - \tau_{131,3} - \tau_{131,3}) dx_3 = 0 \]  

(27)

\[ \int_{-h}^{h} x_3 (\sigma_{11} - \tau_{111,1} - \tau_{131,3} - \tau_{131,3}) dx_3 = M \]  

(28)

Edge boundary conditions (14) reduce to the following relations at the layer corners:

\[ x_1 = 0, L, x_3 = \pm h : \]
\[ \tau_{113} = -\tau_{131}, \quad \tau_{213} = -\tau_{231}, \quad \tau_{313} = -\tau_{331}, \quad Q_{13} = -Q_{31} \]  

(29)

Next, let suggest the solution for the displacements and electric potential in the following form:

\[ u_1 = K x_1 x_3, \quad u_3 = -\frac{1}{2} K x_1^2 + w(x_3), \quad \phi = \varphi(x_3) \]  

(30)

where \( K, w(x_3) \) and \( \varphi(x_3) \) are the unknown constant and functions that define the deformations and electric field in the layer and that should be determined from the governing equations of the problem.

From (30) it follows that non-zero strains, strain gradients, electric field component and its gradient in the layer are:

\[ \varepsilon_{11} = K x_3, \quad \varepsilon_{33} = w', \]
\[ \varepsilon_{11,3} = K, \quad \varepsilon_{33,3} = w'', \]
\[ E_3 = -\varphi', \quad E_{3,3} = -\varphi'' \]  

(31)

where we introduce notation: \( f' = \partial f/\partial x_3 \).

We can mention here that the physical meaning of the constant \( K \) is a through-thickness gradient of axial strains, which has a constant value in given solution like in classical elasticity. Thus, like in classical elasticity, hypotheses (30) impose that there are no shear deformations under pure bending in the layer and that its cross sections remain plane after deformations. This is a common assumption that used in the analogous problems in generalized continuum theories and it is valid far from the end faces of the layer (see discussion in [40, 53, 54]). In the case of transverse isotropy this hypothesis also remain valid due absence of coupling between shear and normal strain for the chosen direction of poling.

Substituting (3) into constitutive equations (4)-(7) and taking into account transverse isotropy of the layer material in considered simplified gradient theory, one can find the following non-zero stresses, double-stresses, electric displacement and electric quadruple:

\[ \sigma_{11} = C_{11} K x_3 + C_{13} w' + e_{31} \varphi', \]
\[ \sigma_{22} = C_{12} K x_3 + C_{13} w' + e_{31} \varphi', \]
\[ \sigma_{33} = C_{13} K x_3 + C_{33} w' + e_{33} \varphi' \]  

(32)
\[ \tau_{113} = \ell^2(C_{11}K + C_{13}w'') \]
\[ \tau_{223} = \ell^2(C_{12}K + C_{13}w'') \]
\[ \tau_{333} = \ell^2(C_{13}K + C_{33}w'') \]
\[ D_3 = e_{31}Kx_3 + e_{33}w' - \kappa_{33}\varphi' \]
\[ Q_{33} = -\ell^2\kappa_{33}\varphi'' \]

where the Voigt notation is used for the material constants.

One can see that proposed solution (30)-(35) satisfy boundary conditions (16)–(22), (24.1), (24.2), (29.2), (29.3). Equilibrium equations (8) and generalized Gauss’ law (9) in the transversal \( x_2 \) and longitudinal \( x_1 \) directions are also satisfied identically. Boundary conditions at the edges (29.1) can not be satisfied with the considered solution because the double stress \( \tau_{113} \) does not equal to zero (see (33)), while \( \tau_{131} \) vanishes. However, this edge condition can be ignored in the Saint-Venant’s sense because its influence will be significant only near to the layer end faces. For example, these edge conditions should not affect the apparent bending stiffness of the layer. In strain gradient elasticity, it was shown by using numerical simulations in [40].

Thus, our task now is to solve the governing equations (8), (9) in the layer thickness direction with boundary conditions on the top/bottom surfaces (23), (24.3), (25), (26) and satisfy the last relaxed boundary conditions at the beam end faces (28), (27). In terms of introducing unknown functions we have the following statement of the boundary value problem:

\[ x_3 \in [-h,h] : \]
\[ \ell^2 C_{33}w''' - C_{33}w'' - e_{33}\varphi'' - KC_{13} = 0, \]
\[ \ell^2 \kappa_{33}\varphi''' - \kappa_{33}\varphi'' + e_{33}w'' + Ke_{31} = 0 \]
\[ x_3 = \pm h : \]
\[ \ell^2 C_{33}w''' - C_{33}w'' - e_{33}\varphi' - KC_{13}x_3 = 0, \]
\[ \ell^2 \kappa_{33}\varphi''' - \kappa_{33}\varphi' + e_{33}w' + Ke_{31}x_3 = 0, \]
\[ C_{33}w'' + KC_{13} = 0 \]
\[ \varphi'' = 0 \]

General solution of the system (36) can be represented in the following form accounting for the symmetry of boundary conditions (37):

\[ \varphi(x_3) = K \left( \frac{C_{33}e_{31} - C_{13}e_{33}}{2(C_{33}\kappa_{33} + e_{33}^2)} \right) x_3^2 + A_0 + \sum_{i=1}^{2} A_i \cosh \frac{\lambda_i x_3}{\ell}, \]
\[ w(x_3) = -K \left( \frac{e_{33}e_{31} + C_{13}e_{33}}{2(C_{33}\kappa_{33} + e_{33}^2)} \right) x_3^2 + A_0 + \frac{k_{33}}{e_{33}} \sum_{i=1}^{2} A_i (1 - \lambda_i^2) \cosh \frac{\lambda_i x_3}{\ell}, \]

where \( k_{33} = e_{33}/\sqrt{C_{33}\kappa_{33}} \) is the electromechanical coupling factor of the layer material and \( \lambda_{1,2} = \sqrt{1 \pm i k_{33}} \) are the eigenvalues of ODE system (36).
Constants $A_{01}$, $A_{02}$, $A_1$, $A_2$ in (38) should be found from the boundary conditions on the top or bottom surface of the beam (37) taking into account fixed condition for displacement and zero value of electric potential at the origin of coordinate system, i.e. $w(0) = 0$ and $\varphi(0) = 0$. Solutions for this constants are the following:

$$A_{01} = -S \left( \frac{(k_{33} - i)^2}{\cosh h_1} + \frac{(k_{33} + i)^2}{\cosh h_2} \right)$$

$$A_{02} = -S \sqrt{\frac{k_{33}}{c_{33}}} \left( \frac{i(k_{33} - i)^2}{\cosh h_1} - \frac{i(k_{33} + i)^2}{\cosh h_2} \right)$$

$$A_1 = S \frac{(k_{33} + i)^2}{\cosh h_2} , \quad A_2 = S \frac{(k_{33} - i)^2}{\cosh h_1}$$

where we use the notations:

$$S = K \ell^2 \frac{C_{33}e_{31} - C_{13}e_{33}}{(1 + k_{33}^2)^2 c_{33}k_{33}},$$

$$\bar{h}_1 = \frac{\lambda_2 h}{\ell}, \quad \bar{h}_2 = \frac{\lambda_1 h}{\ell}$$

It can be checked, that despite the fact that the eigenvalues $\lambda_i$ are imagine in solution (38), the values of functions $w(x_3)$ and $\varphi(x_3)$ will be always real for any physically admissible values of materials constants and geometrical parameters of the problem. These functions are found now up to the constant $K$ that should be found by using relaxed boundary conditions at the layer end faces. To do this, we should substitute (38), (39) into (2), (3) and then into (32), (33) to find the stresses and double stresses. After that one can evaluate the integrals in boundary conditions (27), (28). It can be shown that the first condition (27) for the force resultants will be satisfied identically. The use of a bending moments condition (28) provides us the following result:

$$K = \frac{M}{DJ}$$

where $D = E^* I$ is classical apparent bending stiffness of the piezoelectric layer under pure bending, which is realized in the absence of gradient effects, i.e. if $\ell = 0$; $I = 2h^3/3$ is the moment of inertia per unit width of the layer and $E^*$ is classical apparent Young’s modulus of piezoelectric layer that is defined by the relation:

$$E^* = \frac{C_{33}e_{31}^2 - 2C_{13}e_{31}e_{33} + C_{11}e_{33}^2 - C_{13}^2k_{33} + C_{11}C_{33}k_{33}}{e_{33}^2 + C_{33}k_{33}}$$

Non-classical effects in (40) are defined by the non-dimensional function $J$ that can be presented in the following form:

$$J = 1 + \frac{\ell^2}{h^2} \frac{3k_{33}(C_{33}e_{31} - C_{13}e_{33})^2}{2E^*(e_{33}^2 + C_{33}k_{33})^2} \sum_{i=1}^2 \left(1 + k_{33}^2 - 2\lambda_i^2 \right) \frac{\tanh \bar{h}_i}{\bar{h}_i}$$
Thus, we derived a closed-form semi-inverse analytical solution (38)-(41) for a pure bending problem of the piezoelectric layer taking into account electric field and strain gradient effects. This solution can be reduced to the classical one (see e.g. [56]) by assuming zero value of the length scale parameter $\ell = 0$. In absence of electromechanical coupling ($e_{33} = e_{31} = 0$) this solution becomes to the corresponding solution of the strain gradient elasticity theory [40]. In absence both of the gradient and electromechanical effects the solution coincides with famous classical elasticity result, namely, the constant, which define the linear variation of axial strain across the layer thickness, is defined by relation $K = M/(E^*I)$, where $E^* = E_1 = C_{11} - C_{13}^2 C_{33}$ is an elastic modulus of the layer material in $x_1$ direction.

4 Analytical solution for the thin piezoelectric layer

Now, we can use obtained semi-inverse solution to derive an asymptotic solution for a layer of very small thickness. In considered gradient model we should assume then that the ratio $h/\ell$ is very small and tend to zero. It can be shown, that in this limiting case the solution (38)-(39), reduces to the classical one and does not depend on the length scale parameter. Namely, the deflections of the layer neutral axis (at $x_3 = 0$) and electric potential function can be found in the following form:

$$
\begin{align*}
    u_3 &= -\frac{M}{2E_1I} x_1^2, \\
    \phi &= \frac{M}{2E_1I} \left( \frac{C_{33} e_{31} - C_{13} e_{33}}{e_{33}^2 + C_{33} e_{33}} \right) x_3^3
\end{align*}
$$

(42)

where we take into account that $w|_{x_3=0} = 0$ and that $\cosh \frac{h}{\ell} \approx 1$ if $z \in [-h,h]$ and $h/\ell \to 0$.

Obtained simple result can be used for verification of the Kirchoff plate and Euler-Bernoulli beam models (taking into account the relations between elastic constants in the plane strain and plane stress statements of the theory) in second gradient electroelasticity. One can see, that the electromechanical response of thin structures under pure bending should be classical according to (42).

Note, that similar solutions have been used recently to verify the beam theories in the strain gradient and distortion gradient elasticity theories (Mindlin Form I and II) in [40] and in the framework of micro-dilatation elasticity in [36]. It was shown that the apparent bending stiffness of the thin beams in these theories should have classical values. The same effects we obtain in the second gradient electroelasticity – apparent bending stiffness and voltage of gradient piezoelectric plates and beams are classical.

5 Conclusions

Semi-inverse analytical solution is developed for a pure bending problem in the framework of plane strain statement of the second gradient electroelasticity
theory. Obtained solution is valid far from the layer end faces where the shear stress and corresponding distortions arise. Numerical finite-element validation of this fact is given in the forthcoming article by the authors.

Presented solution can be used as the test solution for the plate and beam models in considered gradient theory. It is shown that the electromechanical response of thin layer under pure bending will correspond to the classical solution without arising non-classical size effects. Additionally, it is proved that the size-dependent behavior of thin piezoelectric structures that were found in experiments cannot be related to the strain and electric filed gradient effects. Another model should be involved to describe such phenomena, for example, taking into account surface or micropolar effects.

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