Noncompact Heisenberg spin magnets from high-energy QCD
II. Quantization conditions and energy spectrum

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Abstract:

We present a complete description of the spectrum of compound states of reggeized gluons in QCD in multi-colour limit. The analysis is based on the identification of these states as ground states of noncompact Heisenberg $SL(2,\mathbb{C})$ spin magnet. A unique feature of the magnet, leading to many unusual properties of its spectrum, is that the quantum space is infinite-dimensional and conventional methods, like the Algebraic Bethe Ansatz, are not applicable. Our solution relies on the method of the Baxter $Q$–operator. Solving the Baxter equations, we obtained the explicit expressions for the eigenvalues of the $Q$–operator. They allowed us to establish the quantization conditions for the integrals of motion and, finally, reconstruct the spectrum of the model. We found that intercept of the states built from even (odd) number of reggeized gluons, $N$, is bigger (smaller) than one and it decreases (increases) with $N$ approaching the same unit value for infinitely large $N$.

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1. Introduction

It has been recently realized that QCD possesses a hidden symmetry at high-energy \[1, 2\]. This symmetry is not seen at the level of classical QCD Lagrangian and it manifests itself through remarkable integrability properties of the Schrödinger equation for the partial waves of the scattering amplitudes in perturbative QCD in the so-called generalized leading logarithmic approximation (GLLA) \[3, 4\]. It turns out that in the multi-colour limit this equation coincides with the Schrödinger equation for two-dimensional quantum-mechanical completely integrable model, which was dubbed in \[2, 5\] as a noncompact Heisenberg spin magnet.

The asymptotics of the scattering amplitudes \(A(s, t)\) at high energy, \(s \gg -t\), is governed by the contribution of an infinite number of soft gluons exchanged between the scattered particles. In the GLLA approximation, the scattering amplitude is given by \[3, 4, 6\]

\[
A(s, t) \sim -is \sum_{N=2}^{\infty} (i\bar{\alpha}_s)^N \frac{s^{-\bar{\alpha}_s E_N/4}}{(\bar{\alpha}_s \sigma_N \ln s)^{1/2}} \xi_{a,N}(t) \xi_{b,N}(t),
\]

where \(\bar{\alpha}_s = \alpha_s N_c/\pi\) is the QCD coupling constant and the sum goes over an arbitrary number of reggeized gluons exchanged in the \(t-\)channel, \(N = 2, 3, \ldots\). In the GLLA approximation, the reggeized gluons interact with each other elastically and form colour-singlet compound states \[7\]. These states can be defined as solutions to the Bartels-Kwiecinski-Praszalowicz (BKP) equation \[3, 8\]

\[
H_N \Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N) = E_N \Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N).
\]

The effective QCD Hamiltonian \(H_N\) acts on the colour \(SU(N_c)\) charges of \(N\) reggeized gluons and their two-dimensional transverse coordinates, \(\vec{z}_k (k = 1, ..., N)\), which belong to the hyperplane orthogonal to the momenta of two scattered particles. The contribution to the scattering amplitude \(A(s, t)\) of the compound states built from \(N\) reggeized gluons, \(\Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)\), has the standard Regge form \(\sim s^{-\bar{\alpha}_s E_N/4}\). It is dominated at large \(s\) by the contribution of the ground state \(E_N\) for Eq. \(1.2\). Obviously, it increases (or decreases) with \(s\) if the energy of the ground state is negative (or positive). As we will show below, the spectrum of \(1.2\) is gapless (see Eq. \(5.6\) below) and, as a consequence, one has to retain in \(1.1\) the contribution of the excited states next to the ground state. This amounts to appearance of the additional factor \((\bar{\alpha}_s \sigma_N \ln s)^{-1/2}\) in the r.h.s. of \(1.1\). The residue factors \(\xi_{a(b),N}(t)\) measure the overlap of \(\Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)\) with the wave functions of the scattered particles and they depend, in general, on the momentum transferred, \(t\), and the colour factor, \(1/N_c^2\).

Calculation of the spectrum of the compound states \(1.2\) for arbitrary number of reggeized gluons \(N\) and eventual resummation of their contribution to the scattering amplitude \(1.1\) is a longstanding problem in high-energy QCD \[3, 11, 12\]. At \(N = 2\), the solution to \(1.2\) has been found a long time ago – the well-known Balitsky-Fadin-Kuraev-Lipatov (BFKL) Pomeron \[1\]. At \(N = 3\) the solution to \(1.2\) – the Odderon state in QCD \[13\], was formulated only a few years ago by Janik and Wosiek \[14\] by making use of the remarkable integrability properties of the effective QCD Hamiltonian \[1, 4\]. Their solution has been later verified in Refs. \[15, 16\]. The methods employed at \(N = 3\) in \[14\] can not be generalized, however, to higher \((N \geq 4)\) reggeized gluon compound states and very little is known about the solutions to \(1.2\) for \(N \geq 4\). Recently, a significant progress has been made in solving the Schrödinger equation \(1.2\) for higher \(N\) in the multi-colour limit, \(N_c \rightarrow \infty\) and \(\bar{\alpha}_s = \) fixed \[3, 16\]. The first results of the calculation of the ground state energy \(E_N\) for higher reggeized gluon compound states in multi-colour QCD...
were reported in a letter format \[7\]. In this paper, we shall provide a detailed account on the approach used in \[7\] and present new results for the spectrum of the Schrödinger equation (1.2). The reader interested in learning more about the latter could skip the first part of the paper and go directly to Section 5.

Our approach to solving the BKP equation (1.2) is based on the identification of the effective Hamiltonian in the multi-colour QCD, \(\mathcal{H}_N\), as the Hamiltonian of a completely integrable two-dimensional noncompact Heisenberg spin magnet \[1, 2\]. The latter model describes the nearest neighbour interaction between spins of \(N\) particles “living” on a two-dimensional plane of transverse coordinates \(\vec{z} = (x, y)\). The corresponding spin operators \(\vec{S}^0_k\) and \(\vec{S}^\pm_k\) (with \(k = 1, \ldots, N\)) are the generators of the unitary principal series representation of the \(SL(2, \mathbb{C})\) group specified by a pair of complex spins \((s, \bar{s})\). They act on the \(\vec{z}\)-plane as the differential operators

\[
S^0_k = z_k \partial_{z_k} + s, \quad S^-_k = -\partial_{z_k}, \quad S^+_k = z_k^2 \partial_{z_k} + 2sz_k, \quad (1.3)
\]

so that \(\vec{S}^2_k = (S^0_k)^2 + (S^+_k S^-_k + S^-_k S^+_k)/2 = s(s - 1)\). The operators \(\vec{S}^{\pm,0}_k\) are given by similar expressions with \(z_k\) and \(s\) replaced by \(\bar{z}_k\) and \(\bar{s}\), respectively. Here, the notation was introduced for the (anti)holomorphic coordinates on a two-dimensional \(\bar{z}\)-plane, \(z_k = x_k + i\bar{y}_k\) and \(\bar{z}_k = z_k^*\), so that \(d^2z_k = dz_k d\bar{z}_k/2\). By the definition, \([S^a_k, S^b_{\bar{k}}]\) = 0 for \(a, b = \pm, 0\). For the principal series of the \(SL(2, \mathbb{C})\), the possible values of the complex spins \((s, \bar{s})\) take the form \[18\]

\[
s = \frac{1 + n_s}{2} + i\nu_s, \quad \bar{s} = 1 - s^* = \frac{1 - n_s}{2} + i\nu_s \quad (1.4)
\]

with \(n_s\) integer and \(\nu_s\) real. For the reggeized gluon compound states, Eq. (1.2), the \(SL(2, \mathbb{C})\) spins take the values \(s = 0\) and \(\bar{s} = 1\), or equivalently \(n_s = -1\) and \(\nu_s = 0\).

The Hamiltonian of the noncompact \(SL(2, \mathbb{C})\) Heisenberg spin magnet is given by \[2, 5\]

\[
\mathcal{H}_N = \sum_{k=1}^{N} \left[ H(J_{k,k+1}) + H(\bar{J}_{k,k+1}) \right], \quad (1.5)
\]

where \(H(J) = \psi(1 - J) + \psi(J) - 2\psi(1)\) with \(\psi(x) = d\ln \Gamma(x)/dx\), \(J_{k,k+1}\) is the sum of two \(SL(2, \mathbb{C})\) spins, \(J_{k,k+1} + (J_{k,k+1} - 1) = (S^0_k + \bar{S}^{0}_{\bar{k}})^2\) with \(J_{N,N+1} = J_{N,1}\), and similar for \(\bar{J}_{k,k+1}\). The model possesses the set of mutually commuting conserved charges \(q_k\) and \(\bar{q}_k\) \((k = 2, \ldots, N)\). Their number is large enough for the Schrödinger equation (1.2) to be completely integrable. The charges \(q_k\) are given by the \(k\)-th order differential operators acting on the holomorphic coordinates of particles. They have particularly simple form for the \(SL(2, \mathbb{C})\) spins \(s = 0\) and \(\bar{s} = 1\) \[11, 12\]

\[
q_k \bigg|_{s=0,\bar{s}=1} = \imath^k \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq N} z_{j_1 j_2} \cdots z_{j_{k-1} j_k} \partial_{z_{j_1}} \cdots \partial_{z_{j_{k-1}}} \partial_{z_{j_k}} \quad (1.6)
\]

with \(z_{j_k} = z_j - z_k\). The charges \(\bar{q}_k\) are given by similar expressions in the \(\bar{z}\)-sector

\[
\bar{q}_k \bigg|_{s=0,\bar{s}=1} = \imath^k \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq N} \partial_{\bar{z}_{j_1}} \cdots \partial_{\bar{z}_{j_{k-1}}} \partial_{\bar{z}_{j_k}} \bar{z}_{j_1 j_2} \cdots \bar{z}_{j_{k-1} j_k} \bar{z}_{j_k j_1} \quad (1.7)
\]

so that \(\bar{q}_k = q_k^\dagger\) with respect to the \(SL(2, \mathbb{C})\) scalar product

\[
\|\Psi\|^2 = \int d^2z_1 \cdots d^2z_N \left| \Psi(z_1, \bar{z}_2, \ldots, \bar{z}_N) \right|^2. \quad (1.8)
\]

\(^1\)That is, the isotopic “spin” space coincides with the coordinate space of \(N\) particles.
The eigenstates $\Psi(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N)$ have to diagonalize these operators and be normalizable with respect to (1.8). The corresponding eigenvalues $q \equiv \{q_k, \bar{q}_k = q_k^*\}$, with $k = 2, \ldots, N$, form the complete set of quantum numbers parameterizing the spectrum of the Schrödinger equation (1.2), $E_N = E_N(q, \bar{q})$. The eigenproblem for the operators (1.6) and (1.7) leads to a complicated system of $(N-1)$-differential equations on $\Psi(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N)$, which was previously solved at $N = 2$ [7] and $N = 3$ [14]. For higher $N$, instead of dealing with this system, we apply the method developed in [5]. It represents an application of the Quantum Inverse Scattering Method [19] to noncompact Heisenberg spin magnet model.

The noncompact Heisenberg spin magnet, Eq. (1.5), can be considered as a generalization of the well-known spin–1/2 Heisenberg spin chain model (as well as its analogs for higher $SU(2)$ spins [20]) to arbitrary complex spins belonging to noncompact, unitary representations of the $SL(2, \mathbb{C})$ group. As we will demonstrate below, the noncompact and compact Heisenberg magnets have completely different properties, yet another manifestation of the fact that the quantum space of the model is infinite-dimensional in the former case. In particular, in distinction with the compact spins, the principal series of the $SL(2, \mathbb{C})$ group does not have the highest weight and, as a consequence, the conventional Algebraic Bethe Ansatz [21] is not applicable to diagonalization of the Hamiltonian (1.5). To solve the Schrödinger equation (1.2) for arbitrary number of particles $N$ we will rely instead on the method of the Baxter $Q$–operator [22].

In this method, the Hamiltonian (1.5), the integrals of motion (1.6) and, in general, all transfer matrices of the noncompact Heisenberg spin magnet are expressed in terms of a single operator $Q(u, \bar{u})$, which acts on the quantum space of the model and depends on a pair of spectral parameters, $u$ and $\bar{u}$. The explicit form of this operator was found in [8]. As a result, the Schrödinger equation (1.2) turns out to be equivalent to the eigenproblem for the Baxter $Q$–operator. It is this problem that we address in the present paper. Namely, we calculate the eigenvalues of the Baxter $Q$–operator, establish the quantization conditions for the integrals of motion (1.6) and, finally, obtain a complete description of the spectrum of the Schrödinger equation (1.2).

As we will show below, the system (1.2) has many features in common with two-dimensional conformal field theories (CFT) [23]. Since the Hamiltonian (1.5) is given by the sum of two mutually commuting operators acting in the $z$– and $\bar{z}$–sectors, the dynamics in the two sectors is independent on each other. As a consequence, the solutions to the Schrödinger equation (1.2) have the chiral structure similar to that of correlation functions in the CFT. Namely, the eigenstates $\Psi(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N)$ can be factorized into the product of “conformal blocks” depending on the (anti)holomorphic coordinates and the conserved charges $q$. Similar factorization holds for the eigenvalues of the Baxter operator $Q(u, \bar{u})$. For $\Psi(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N)$ to be a single-valued function on the two-dimensional $\vec{z}$–plane, the conserved charges $q$ have to satisfy the quantization conditions. The charges $q$ play the rôle analogous to that of the conformal weights of primary fields in the CFT. As we will show, the spectrum of their quantized values turns out to be very similar to the Kac spectrum of the conformal weights in the minimal CFT [23].

The paper is organized as follows. In Section 2, we summarize the main properties of the Baxter $Q$–operator for noncompact Heisenberg spin magnets. In Section 3, we show that the problem of finding the eigenvalues of the operator $Q(u, \bar{u})$ can be reduced to solving the $N$th order Fuchsian differential equation. Its solution leads to the set of consistency conditions which can be satisfied only if the integrals of motion $q$ take quantized values. In Section 4, we calculate the eigenvalues of the operator $Q(u, \bar{u})$ and demonstrate that they are factorized into a product of “conformal blocks”, which depend separately on the (anti)holomorphic spectral parameters, $u$
and \( \bar{u} \). Using these expressions, it becomes straightforward to determine the exact spectrum of the model for arbitrary number of particles, \( N \), and complex \( SL(2, \mathbb{C}) \) spins, \( s \) and \( \bar{s} \). In Section 5, we present the results of our calculations for the special values of the \( SL(2, \mathbb{C}) \) spins, \( s = 0 \) and \( \bar{s} = 1 \). The obtained expressions define the spectrum of the compound states of reggeized gluons in multi-colour QCD. Section 6 contains the concluding remarks. The details of the calculations are summarized in the Appendices.

2. Baxter \( \mathcal{Q} \)–operator

In this Section we shall describe, following [5], the general properties of the Baxter \( \mathcal{Q} \)–operator for the noncompact Heisenberg spin magnet. We assume that the number of particles \( N \) is arbitrary and the complex \( SL(2, \mathbb{C}) \) spins \( s \) and \( \bar{s} \) are given by (1.4). The operator \( \mathcal{Q}(u, \bar{u}) \) depends on two complex spectral parameters \( u \) and \( \bar{u} \). It acts on the quantum space of the model \( V_N = V \otimes ... \otimes V \), with \( V \equiv V^{(s, \bar{s})} \) being the representation space of the principal series of the \( SL(2, \mathbb{C}) \) group. For \( \mathcal{Q}(u, \bar{u}) \) to be a well-defined operator on \( V_N \), the spectral parameters have to satisfy the additional condition

\[
i(u - \bar{u}) = n \tag{2.1}
\]

with \( n \) being an arbitrary integer. The Baxter \( \mathcal{Q} \)–operator commutes with the Hamiltonian of the model (1.5) and shares the common set of the eigenstates

\[
\mathcal{Q}(u, \bar{u}) \Psi_{i\mathcal{P}(q, \bar{q})}(\bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = Q_{q,\bar{q}}(u, \bar{u}) \Psi_{i\mathcal{P}(q, \bar{q})}(\bar{z}_1, \bar{z}_2, ..., \bar{z}_N), \tag{2.2}
\]

with \( \mathcal{P} \) being the total two-dimensional momentum of the state and \( \{q, \bar{q}\} \) denoting the total set of the quantum numbers, \( q_k \) and \( \bar{q}_k \) with \( k = 2, ..., N \).

The Baxter operator plays the central role in our analysis as the energy spectrum of the model can be expressed in terms of its eigenvalues, \( Q_{q,\bar{q}}(u, \bar{u}) \). Indeed, there exist the following two (equivalent) relations between \( Q_{q,\bar{q}}(u, \bar{u}) \) and the energy \( E_N = E_N(q, \bar{q}) \)

\[
E_N(q, \bar{q}) = \varepsilon_N + i \frac{d}{du} \ln \left[ Q_{q,\bar{q}}(u + is, u + i\bar{s}) (Q_{q,\bar{q}}(u - is, u - i\bar{s}))^* \right] \bigg|_{u=0}, \tag{2.3}
\]

where \( \varepsilon_N = 2N \text{ Re} [\psi(2s) + \psi(2 - 2s) - 2\psi(1)] \) and

\[
E_N(q, \bar{q}) = - \text{Im} \frac{d}{du} \ln \left[ u^{2N} Q_{q,\bar{q}}(u + i(1 - s), u + i(1 - \bar{s})) \right. \\
\left. \times Q_{-q,-\bar{q}}(u + i(1 - s), u + i(1 - \bar{s})) \right] \bigg|_{u=0}. \tag{2.4}
\]

The Hamiltonian (1.3) is invariant under cyclic permutations of particles \( [\mathcal{P} \Psi](\bar{z}_1, ..., \bar{z}_{N-1}, \bar{z}_N) = \Psi(\bar{z}_2, ..., \bar{z}_N, \bar{z}_1) \) and, as a consequence, its eigenstates possess a definite value of the quasimomentum defined as

\[
[\mathcal{P} \Psi_{i\mathcal{P}(q, \bar{q})}](z_1, ..., z_N) = e^{i\theta_N(q, \bar{q})} \Psi_{i\mathcal{P}(q, \bar{q})}(z_1, ..., z_N), \quad \theta_N(q, \bar{q}) = 2\pi k/N, \tag{2.5}
\]

with \( k \) integer in virtue of \( \mathcal{P}^N = 1 \). The quasimomentum can be expressed in terms of the eigenvalues of the Baxter operator as

\[
\theta_N(q, \bar{q}) = i \ln \frac{Q_{q,\bar{q}}(is, i\bar{s})}{Q_{q,\bar{q}}(-is, -i\bar{s})}. \tag{2.6}
\]
Thus, the Schrödinger equation (2.2) turns out to be equivalent to the eigenproblem for the Baxter $Q$–operator, Eq. (2.2).

The spectral parameters satisfy (2.1) and their possible values can be parameterized as

\[
Q_{q,\bar{q}}(u, \bar{u}) = (u + is)^N Q_{q,\bar{q}}(u + i, \bar{u}) + (u - is)^N Q_{q,\bar{q}}(u - i, \bar{u}),
\]

where

\[
t_N(u) = 2u^N + q_2u^{N-2} + \ldots + q_N
\]

is the eigenvalue of the auxiliary transfer matrix with $q \equiv (q_2, \ldots, q_N)$ denoting the eigenvalues of the holomorphic integrals of motion. The “lowest” integral of motion, $q_2$, is related to the total $SL(2, \mathbb{C})$ spin, $h$, of the system of $N$ particles

\[
q_2 = -h(h - 1) + Ns(s - 1), \quad h = (1 + n_h)/2 + iv_h
\]

with $n_h$ integer and $v_h$ real. In addition, $Q_{q,\bar{q}}(u, \bar{u})$ obeys the equation similar to (2.7) in the $\bar{u}$–sector with $s$, $h$, and $q_k$ replaced, respectively, by

\[
s = 1 - s^*, \quad \bar{h} = 1 - h^*, \quad \bar{q}_k = q_k^*
\]

with $k = 2, \ldots, N$. The function $Q_{q,\bar{q}}(u, \bar{u})$ does not depend on the total momentum of the state, $p = (p, \bar{p})$ with $p = -i\sum_k S_k^s$ and $\bar{p} = -i\sum_k \bar{S}_k^s$, and it is invariant under $h \rightarrow 1 - h$ and $\bar{h} \rightarrow 1 - \bar{h}$. Indeed, the Baxter operator $Q(u, \bar{u})$ commutes with the generators of the $SL(2, \mathbb{C})$ group and, as a consequence, its eigenvalue $Q_{q,\bar{q}}(u, \bar{u})$ depends only on the $SL(2, \mathbb{C})$ Casimir operators, like $q_2$, which are symmetric under the above transformation of the total spin.

(ii) **Analytical properties:**

The spectral parameters satisfy (2.1) and their possible values can be parameterized as $u = \lambda - in/2$ and $\bar{u} = \lambda + in/2$, with $n$ arbitrary integer and $\lambda$ complex. Then, $Q_{q,\bar{q}}(u, \bar{u})$ should be a meromorphic function of $\lambda$ with an infinite set of poles of the order not higher than $N$ situated at the points

\[
\{u_m^\pm = \pm i(s - m), \quad \bar{u}_m^\pm = \pm i(s - \bar{m})\}, \quad m, \bar{m} = 1, 2, \ldots
\]

The behaviour of $Q_{q,\bar{q}}(u, \bar{u})$ in the vicinity of the pole at $m = \bar{m} = 1$ can be parameterized as

\[
Q_{q,\bar{q}}(u_1^\pm + \epsilon, \bar{u}_1^\pm + \epsilon) = R^\pm(q, \bar{q}) \left[ \frac{1}{\epsilon^N} + \frac{i E^\pm(q, \bar{q})}{\epsilon^{N-1}} + \ldots \right].
\]

The functions $R^\pm(q, \bar{q})$ fix an overall normalization of the Baxter operator, while the residue functions $E^\pm(q, \bar{q})$ define the energy of the system (see Eqs. (2.15) and (2.16) below).

(iii) **Asymptotic behaviour:**

In the above parameterization of the spectral parameters, $Q_{q,\bar{q}}(u, \bar{u})$ should have the following asymptotic behaviour for $|\text{Im} \lambda| < 1/2$ and $\text{Re} \lambda \rightarrow \infty$

\[
Q_{q,\bar{q}}(\lambda - in/2, \lambda + in/2) \sim e^{i\Theta_h(q, \bar{q})} \lambda^{h + \bar{h} - N(s + \bar{s})} + e^{-i\Theta_h(q, \bar{q})} \lambda^{1-h+1+\bar{h}-N(s+\bar{s})},
\]

\[
\Theta_h(q, \bar{q}) = \frac{\pi}{2} - \text{Im} \lambda (s - m) + \text{Im} \bar{\lambda} (s - \bar{m})
\]
with the phase \( \Theta_h(q, \bar{q}) \) depending on the quantum numbers of the state and the total \( SL(2, \mathbb{C}) \) spins \( h \) and \( \bar{h} \) defined in (2.9) and (2.10).

As we will show in Section 3, the Baxter equation (2.7) supplemented with the additional conditions on the pole structure of its solutions, Eq. (2.11), and asymptotic behaviour at infinity, Eq. (2.13), fixes uniquely the eigenvalues of the Baxter operator, \( Q_{q, \bar{q}}(u, \bar{u}) \), and, therefore, allows us to determine the spectrum of the model.

Additional properties of the function \( Q_{q, \bar{q}}(u, \bar{u}) \) can be deduced from the symmetry of the model under permutations of particles. Apart from the cycle symmetry, Eq. (2.5), the Hamiltonian into another one with the same energy but different set of the quantum numbers

\[
[\mathbb{M} \Psi_{q, \bar{q}}](\bar{z}_1, ..., \bar{z}_{N-1}, z_N) = \Psi_{-q, -\bar{q}}(\bar{z}_1, ..., \bar{z}_{N-1}, z_N)
\]

(2.14) with \(-q \equiv (q_2, -q_3, ..., (-1)^m q_m) \) and similar for \( \bar{q} \). For \( q_{2k+1} = 0 \), or equivalently \( q = -q \), Eq. (2.14) is replaced by \( \mathbb{M} \Psi_{q, \bar{q}} = (-1)^{N_k+i_{M_k}} \Psi_{q, \bar{q}} \). This property leads to the following parity relations for the residue functions \( R^+(q, \bar{q}) \) defined in (2.12)

\[
R^+(q, \bar{q})/R^+(-q, -\bar{q}) = e^{2i\theta_N(q, \bar{q})}
\]

(2.15) and for the eigenvalues of the Baxter operator

\[
Q_{q, \bar{q}}(-u, -\bar{u}) = e^{i\theta_N(q, \bar{q})} Q_{-q, -\bar{q}}(u, \bar{u})
\]

(2.16)

We recall that the spectral parameters \( u \) and \( \bar{u} \) have to satisfy (2.1). Examining the behaviour of the both sides of (2.16) around the pole at \( u = u^+_1 \) and \( \bar{u} = \bar{u}^+_1 \) and making use of Eq. (2.12) one gets

\[
R^\pm(q, \bar{q}) = (-1)^N e^{i\theta_N(q, \bar{q})} R^\mp(-q, -\bar{q}), \quad E^\pm(q, \bar{q}) = -E^\mp(-q, -\bar{q}).
\]

(2.17)

To obtain the expression for the energy \( E_N(q, \bar{q}) \), we apply (2.4). Calculating the logarithmic derivative of \( Q_{q, \bar{q}} \) in the r.h.s. of (2.4), we replace the function \( Q_{q, \bar{q}}(u \pm i(1-s), u \pm i(1-s)) \) by its pole expansion (2.12). Then, applying the second relation in (2.17), one finds

\[
E_N(q, \bar{q}) = E^+(-q, -\bar{q}) + (E^+(q, \bar{q}))^* = \text{Re} \left[ E^+(-q, -\bar{q}) + E^+(q, \bar{q}) \right],
\]

(2.18)

where the last relation follows from hermiticity of the Hamiltonian (1.5). We conclude from Eqs. (2.18) and (2.12), that in order to find the energy \( E_N(q, \bar{q}) \), one has to calculate the residue of \( Q_{q, \bar{q}}(u, \bar{u}) \) at the \((N-1)\)th order pole at \( u = i(s-1) \) and \( \bar{u} = i(\bar{s}-1) \).

### 3. Quantization conditions

Let us construct the solution to the Baxter equation (2.7) satisfying the additional conditions (2.12) and (2.13). It proves convenient to use the following integral representation for \( Q_{q, \bar{q}}(u, \bar{u}) \)

\[
Q_{q, \bar{q}}(u, \bar{u}) = \int \frac{d^2z}{z \bar{z}} z^{-iu} \bar{z}^{-i\bar{u}} Q(z, \bar{z}),
\]

(3.1)

where integration goes over the two-dimensional \( z \)-plane with \( z = z^* \). This ansatz is advantageous in many respects. Firstly, the condition (2.1) is automatically satisfied since it is only for...
these values of the spectral parameters that the \( z \)-integral in the r.h.s. of (3.1) is well-defined. Secondly, the functional Baxter equation on \( Q_{q,q}(u, \bar{u}) \) is translated into the \( N \)th order differential equation for the function \( Q(z, \bar{z}) \). Its derivation is based on the identity

\[
P(u) Q_{q,q}(u, \bar{u}) = \int \frac{d^2z}{z\bar{z}} z^{-iu} \bar{z}^{-i\bar{u}} P(-iz\partial_z) Q(z, \bar{z}),
\]

(3.2)

with \( P(u) \) being a polynomial in \( u \). Substituting (3.1) into the Baxter equation (2.7) and applying (3.2), one arrives at

\[
\left[ z^s(z\partial_z)^N z^{1-s} + z^{-s} (z\partial_z)^N z^{s-1} - 2(z\partial_z)^N - \sum_{k=2}^N i^k q_k (z\partial_z)^{N-k} \right] Q(z, \bar{z}) = 0.
\]

(3.3)

The \( \bar{z} \)-dependence of \( Q(z, \bar{z}) \) is constrained by a similar equation in the antiholomorphic sector with \( s \) and \( q_k \) replaced by \( \bar{s} = 1 - s^* \) and \( \bar{q}_k = q_k^* \), respectively. Finally, as we will show below, the remaining conditions on the analytical properties and asymptotic behaviour of \( Q_{q,q}(u, \bar{u}) \), Eqs. (2.12) and (2.13), become equivalent to a requirement for \( Q(z, \bar{z} = z^*) \) to be a single-valued function on the complex \( z \)-plane.

Going through a standard analysis [24], one finds that the differential equation (3.3) is of Fuchsian type with three regular singular points located at \( z = 0, z = 1 \) and \( z = \infty \). Defining \( N \) linear independent solutions to Eq. (3.3), \( Q_a(z) \), and their antiholomorphic counterparts, \( \bar{Q}_b(\bar{z}) \), we construct the general expression for the function \( Q(z, \bar{z}) \) as

\[
Q(z, \bar{z}) = \sum_{a,b=1}^N Q_a(z) C_{ab} \bar{Q}_b(\bar{z}),
\]

(3.4)

with \( C_{ab} \) being an arbitrary mixing matrix. The functions \( Q_a(z) \) and \( \bar{Q}_b(\bar{z}) \) acquire a nontrivial monodromy around three singular points, \( z, \bar{z} = 0, 1 \) and \( \infty \). For \( Q(z, \bar{z} = z^*) \) to be well-defined on the whole plane, the monodromy should cancel in the r.h.s. of (3.4). This requirement leads to the set of nontrivial conditions on the matrix \( C_{ab} \). Solving them, we will be able not only to obtain the values of the mixing coefficients, \( C_{ab} \), but also determine the quantized values of the integrals of motion \( q_k \).

We would like to point out that our approach to defining the function \( Q(z, \bar{z}) \) in Eq. (3.4) is similar in many respects to the well-known approach to constructing the correlation functions in the minimal CFT [25]. There, \( Q(z, \bar{z}) \) plays the rôle of four-point correlation functions depending on the anharmonic ratios of the coordinates, \( z \) and \( \bar{z} \). The latter satisfy the differential equations (“the null vector condition”) similar to (3.3), in which the integrals of the motion \( q_k \) are replaced by some combinations of the conformal weights of the primary fields. In the minimal CFT, the conformal blocks \( Q_a(z) \) and \( \bar{Q}_b(\bar{z}) \) are given by multiple contour integrals and their monodromy around the singular points \( z = 0, 1 \) and \( \infty \) can be found in a closed form. Going over to Eq. (3.3), one finds (see Section 3.4 below) that similar representation exists only at \( N = 2 \) and it remains unclear whether it can be generalized for arbitrary number of particles \( N \). Our subsequent analysis does not rely on such representation.

To determine the function \( Q(z, \bar{z}) \), we shall construct the r.h.s. of (3.4) in the vicinity of three singular points, \( z = 0, z = 1 \) and \( z = \infty \), and analytically continue the obtained expressions onto the whole \( z \)-plane. Additional simplification occurs due to the symmetry of the differential
equation (3.3) under the transformation $z \to 1/z$ and $q_k \to (-1)^k q_k$. This symmetry is a manifestation of a general property of the eigenvalues of the Baxter operator, Eq. (2.16), which leads to

$$Q_{q,q}(z, \bar{z}) = e^{i\theta_N(q, \bar{q})} Q_{-q,-q}(1/z, 1/\bar{z}).$$

(3.5)

Here, we indicated explicitly the dependence of the function $Q(z, \bar{z})$ on the integrals of motion. Applying (3.3), one can define $Q(z, \bar{z})$ around $z = \infty$ from the solution at $z = 0$.

### 3.1. Solution around $z = 0$

Looking for the solution to (3.3) around $z = 0$ in the form $Q(z) \sim z^a$, we find that the exponent $a$ satisfies the indicial equation

$$(a - 1 + s)^N = 0.$$  

(3.6)

Since its solution, $a = 1 - s$, is $N$–times degenerate, the small–$z$ asymptotics of $Q(z)$ contains terms $\sim (\ln z)^k$ with $k \leq N - 1$. Let us define the fundamental set of linear independent solutions to (3.3) around $z = 0$ as

$$Q_1^{(0)}(z) = z^{1-s} u_1(z)$$

$$Q_m^{(0)}(z) = z^{1-s} \left[ u_1(z)(\ln z)^{m-1} + \sum_{k=1}^{m-1} c_{m-1}^k u_{k+1}(z)(\ln z)^{m-k-1} \right],$$

(3.7)

with $2 \leq m \leq N$ and the binomial coefficients $c_{m-1}^k = (m-1)!/(k!(m-k-1)!)$ inserted for later convenience. The functions $u_m(z)$ are given by power series

$$u_m(z) = 1 + \sum_{n=1}^{\infty} z^n u_n^{(m)}(q),$$

(3.8)

which converge uniformly inside the region $|z| < 1$. Inserting (3.7) and (3.8) into (3.3), one finds that the expansion coefficients $u_n^{(m)}(q)$ satisfy three-term (nonhomogeneous) recurrence relations with respect to $n$. To save space, we do not present here their explicit form.

The fundamental set of solutions to the antiholomorphic differential equation, $\overline{Q}_m^{(0)}(\bar{z})$, can be obtained from (3.7) by replacing $s$ and $q_k$ by $\bar{s} = 1 - s^*$ and $\overline{q}_k = q_k^*$, respectively. Then, the general solution for $Q(z, \bar{z})$ around $z = 0$ is given by

$$Q(z, \bar{z}) \mid_{z \to 0} = \sum_{m,n=1}^{N} Q_m^{(0)}(z) C_{mn}^{(0)} \overline{Q}_n^{(0)}(\bar{z}).$$

(3.9)

The mixing matrix $C_{mn}^{(0)}$ has to be chosen in such a way that $Q(z, \bar{z})$ should be single-valued at $z = 0$, or equivalently, the monodromy of $Q_m^{(0)}(z)$ and $\overline{Q}_m^{(0)}(\bar{z} = z^*)$ around $z = 0$ should cancel each other in the r.h.s. of (3.9). According to (3.7), the monodromy of $Q_m^{(0)}(z)$ is due to $z^{1-s}$–factor and $\ln z$–terms. Taking into account that $s - \bar{s} = n_s$ is an integer, we find that

\footnote{The monodromy matrix, defined as $Q_n^{(0)}(ze^{2\pi i}) = M_{nk}Q_n^{(0)}(z)$, has a Jordan structure and, therefore, it can not be brought to a diagonal form upon redefinition of the fundamental basis. It is interesting to note that similar situation occurs in the operator algebra of primary fields in the so-called Logarithmic CFT [24].}
the factor $z^{1-s}z^{1-s}$ does not affect the monodromy of the r.h.s. of (3.9). Then, for $Q(z, \bar{z})$ to be single valued, it should depend only on $\ln(z\bar{z})$ rather than on $\ln z$ and $\ln \bar{z}$ separately. It is straightforward to verify that this condition is satisfied provided that the matrix elements $C_{nm}^{(0)}$ vanish below the main anti-diagonal, that is for $n + m > N + 1$, and have the following form for $n + m \leq N + 1$

$$C_{nm}^{(0)} = \frac{\sigma}{(n-1)!(m-1)!} \sum_{k=0}^{N-n-m+1} (-2)^k k! \alpha_{k+n+m-1}$$

with $\sigma, \alpha_1, ..., \alpha_{N-1}$ being arbitrary complex parameters and $\alpha_N = 1$.

The mixing matrix $C_{nm}^{(0)}$ depends on $N$ arbitrary complex parameters $\sigma$ and $\alpha_k$. We can fix the normalization of the function $Q(z, \bar{z})$ by choosing the value of $\sigma$. Since the resulting expression for the eigenvalue of the Baxter operator has to satisfy simultaneously two parity relations, Eqs. (2.15) and (3.5), its value can not be arbitrary. As we will see in a moment, both relations are satisfied for $\sigma = \exp(i\theta_N(q, \bar{q}))$, with $\theta_N(q, \bar{q})$ being the quasimomentum. Later, we will use (3.5) to calculate the eigenvalues of $\theta_N(q, \bar{q})$ (see Eq. (3.24)).

Substituting (3.10) into (3.9) and taking into account (3.7), one gets small--$z$ expansion of the function $Q(z, \bar{z})$. The leading asymptotic behaviour for $z \to 0$ can be obtained by neglecting $O(z)$ corrections in (3.8). In this way, one finds

$$Q_{q,\bar{q}}(z, \bar{z}) = z^{1-s}z^{1-s}e^{i\theta_N(q, \bar{q})} \left[ \frac{\ln^{N-1}(z\bar{z})}{(N-1)!} + \frac{\ln^{N-2}(z\bar{z})}{(N-2)!} \alpha_{N-1} + \ldots + \frac{\ln(z\bar{z})}{1!} \alpha_2 + \alpha_1 \right] \left( 1 + O(z, \bar{z}) \right).$$

Using this expression, we can calculate the contribution of the small--$z$ region to the eigenvalue of the Baxter operator, Eq. (3.1). Introducing a cut-off, $\rho \ll 1$, and replacing $Q(z, \bar{z})$ in Eq. (3.1) by its expansion (3.11), one integrates term-by-term over the region $|z| < \rho$ by making use of the identity

$$\int_{|z|<\rho} \frac{d^2z}{z\bar{z}} z^{-iu}z^{-\bar{u}} \ln^n(z\bar{z}) z^{-s}z^{-\bar{s}} = \pi \delta_{m-s-iu, \bar{m}-\bar{s}-i\bar{u}} \left[ \frac{(-1)^n n!}{(m-s-iu)^{n+1}} + O((m-s-iu)^0) \right],$$

with $m$ and $\bar{m}$ positive integer. We find that, in agreement with the general properties of the Baxter operator, Eqs. (2.11) and (2.12), the function $Q_{q,\bar{q}}(u, \bar{u})$ has poles of the order $N$ at the points $u = i(s - m)$ and $\bar{u} = i(\bar{s} - \bar{m})$. At $m = \bar{m} = 1$ one finds from (3.11)

$$Q_{q,\bar{q}}(u_1^+, \epsilon, \bar{u}_1^+ + \epsilon) = -\frac{\pi e^{i\theta_N(q, \bar{q})}}{(i\epsilon)^N} \left[ 1 + i\epsilon \alpha_{N-1} + \ldots + (i\epsilon)^N \alpha_2 + (i\epsilon)^{N-1} \alpha_1 + O(\epsilon^N) \right]$$

with $u_1^+$ and $\bar{u}_1^+$ defined in (2.11). It is easy to see, using (3.5) and (3.11), that the remaining poles of $Q_{q,\bar{q}}(u, \bar{u})$ are located at $u = -i(s - m)$ and $\bar{u} = -i(\bar{s} - \bar{m})$ and they originate from integration in (3.1) over the region of large $|z| > 1/\rho$.

Matching (3.12) into (2.12) one gets

$$R^+(q, \bar{q}) = -\frac{\pi}{(i\epsilon)^N} e^{i\theta_N(q, \bar{q})}, \quad E^+(q, \bar{q}) = \alpha_{N-1}(q, \bar{q}).$$

We verify, using $\theta_N(q, \bar{q}) = -\theta_N(-q, -\bar{q})$, that the obtained expression for $R^+(q, \bar{q})$ satisfies the parity relation (2.15). According to their definition, Eq. (3.10), the $\alpha$--parameters are arbitrary.

---

3To cancel the monodromy, it is enough to require that the sum over $k$ should depend only on the sum $n + m$. We have chosen the sum in this particular form for later convenience (see Eq. (3.13)).
complex. We indicated in (3.13) the dependence of the \( \alpha \)-parameters on the integrals of motion, since we anticipate that their values will be fixed by the quantization conditions to be discussed below. Insertion of the second relation in (3.13) into (2.18) leads to the following remarkable expression for the energy

\[
E_N(q, \bar{q}) = \text{Re} \left[ \alpha_{N-1}(-q, -\bar{q}) + \alpha_{N-1}(q, \bar{q}) \right],
\]

(3.14)

where \( q = \{q_k\}, -q = \{(-1)^k q_k\} \) and similar for \( \bar{q} \).

We conclude that the small \( z \) asymptotics (3.11) leads to the correct analytical properties of the eigenvalues of the Baxter operator, Eq. (2.11). Moreover, the energy of the system, \( E_N(q, \bar{q}) \), is related to the matrix elements of the mixing matrix (3.10) in the fundamental basis (3.7).

### 3.2. Solution around \( z = 1 \)

Substituting \( Q(z) \sim (z - 1)^b \) into (3.3) one obtains after some calculation the following indicial equation

\[
(b + 1 + h - Ns)(b + 2 - h - Ns) \prod_{k=0}^{N-3} (b - k) = 0,
\]

(3.15)

with the total \( SL(2, \mathbb{C}) \) spin \( h \) defined in (2.9). Since the solutions \( b = k \) with \( k = 0, ..., N - 3 \) differ from each other by an integer, one expects to encounter logarithmically enhanced terms \( \sim \ln(1 - z) \). However, a close examination of (3.3) reveals that the solutions to (3.3) do not contain such terms provided that \( h \neq (1 + n_h)/2 \), or equivalently \( \text{Im} h \neq 0 \) (see Eq. (2.9)). At \( h = (1 + n_h)/2 \) and \( \text{Im} s \neq 0 \), the additional degeneracy occurs between the solutions to (3.15), \( b = Ns - h - 1 \) and \( b = Ns + h - 2 \). It leads to the appearance of the terms \( \sim \ln(1 - z) \) in the asymptotics of \( Q(z) \) for \( z \to 1 \). Obviously, similar relations hold in the \( z \)-sector.

The fundamental set of solutions to Eq. (3.3) around \( z = 1 \) is defined similarly to (3.7). For \( \text{Im} h \neq 0 \) one gets

\[
Q_1^{(1)}(z) = z^{1-s}(1 - z)^{Ns-h-1}v_1(z),
\]

\[
Q_2^{(1)}(z) = z^{1-s}(1 - z)^{Ns+h-2}v_2(z),
\]

\[
Q_m^{(1)}(z) = z^{1-s}(1 - z)^{m-3}v_m(z),
\]

(3.16)

with \( m = 3, ..., N \). The functions \( v_i(z) (i = 1, 2) \) and \( v_m(z) \) are given by the power series

\[
v_i(z) = 1 + \sum_{n=1}^{\infty} (1 - z)^n v_n^{(i)}(q), \quad v_m(z) = 1 + \sum_{n=N-m+1}^{\infty} (1 - z)^n v_n^{(m)}(q),
\]

(3.17)

which converge uniformly inside the region \( |1 - z| < 1 \). We notice that \( Q_2^{(1)}(z) \) can be obtained from the function \( Q_1^{(1)}(z) \) by replacing \( h \to 1 - h \). Substituting (3.16) into (3.3), one finds that the expansion coefficients \( v_n^{(i)} \) and \( v_n^{(m)} \) satisfy the \( N \)-term homogenous recurrence relations with respect to the index \( n \).

\(^4\)As already mentioned, at \( h = (1+n_h)/2 \) the solutions \( Q_{1,2}^{(1)}(z) \) become

Without this factor, the recursion will involve \( N + 1 \) terms.
degenerate and one of them, $Q_1^{(1)}(z)$ for $n_h \geq 0$, has to be redefined to include the additional
\[ \ln(1 - z) - \gamma \] -term
\[ Q_1^{(1)}(z) \bigg|_{h=(1+n_h)/2} = z^{1-s}(1 - z)^{N_s - (n_h + 3)/2} \left[(1 - z)^{n_h} \ln(1 - z) \psi_2(z) + \bar{\psi}_1(z)\right]. \tag{3.18} \]

Here the function $\psi_2(z)$ is the same as before, while $\bar{\psi}_1(z) = \sum_{k=0}^{\infty} \bar{v}_k z^k$ and the coefficients $\bar{v}_k$ satisfy the $N$-term recurrence relations with the boundary condition $\bar{v}_{n_h} = 1$.

The fundamental set in the antiholomorphic sector, $\mathcal{Q}_n^{(1)}(\bar{z})$, is obtained from the functions $Q_n^{(1)}(z)$ by replacing $s$ and $h$ by $\bar{s} = 1 - s^*$ and $\bar{h} = 1 - h^*$, respectively. Among all functions in the fundamental set (3.16) only two, $Q_1^{(1)}(z)$ and $Q_2^{(1)}(z)$, are not analytical at $z = 1$. As a consequence, a general solution for $Q(z, \bar{z})$ possessing a trivial monodromy around $z = 1$ can be constructed as
\[ Q(z, \bar{z}) \overset{z \to 1}{=} \beta_h Q_1^{(1)}(z) \bar{Q}_1^{(1)}(\bar{z}) + \beta_{1-h} Q_2^{(1)}(z) \bar{Q}_2^{(1)}(\bar{z}) + \sum_{m, \bar{m} = 3}^{N} Q_m^{(1)}(z) \gamma_{m\bar{m}} \bar{Q}_m^{(1)}(\bar{z}). \tag{3.19} \]

The $\beta-$coefficients depend, in general, on the total spin $h$ (and $\bar{h} = 1 - h^*$). They are chosen in (3.19) in such a way that the symmetry of the eigenvalues of the Baxter operator under $h \to 1 - h$ is manifest. It is convenient to rewrite (3.19) in a matrix form as
\[ Q(z, \bar{z}) = \mathcal{Q}^{(1)} \cdot C^{(1)} \cdot \bar{Q}^{(1)}, \quad C^{(1)} = \begin{pmatrix} \beta_h & 0 & 0 \\ 0 & \beta_{1-h} & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \tag{3.20} \]

with $\gamma \equiv \gamma_{m\bar{m}}$. The expansion (3.19) is valid only for $\text{Im} h \neq 0$. For $h = (1 + n_h)/2$ the first two terms in the r.h.s. of (3.19) look differently in virtue of (3.18)
\[ Q(z, \bar{z}) \bigg|_{h=(1+n_h)/2} = \beta_1 \left[ Q_1^{(1)}(z) \bar{Q}_2^{(1)}(\bar{z}) + Q_2^{(1)}(z) \bar{Q}_1^{(1)}(\bar{z}) \right] + \beta_2 Q_2^{(1)}(z) \bar{Q}_2^{(1)}(\bar{z}) + \ldots, \tag{3.21} \]

where ellipses denote the remaining terms. Substituting (3.19) into (3.3) and performing integration over the region of $|1 - z| \ll 1$, one can find the asymptotic behaviour of $Q(u, \bar{u})$ at large $u$ and $\bar{u}$. As we will show in Section 4, it turns out to be in agreement with the general properties of the Baxter operator, Eq. (2.13).

The mixing matrix $C^{(1)}$ defined in (3.20) has a block-diagonal structure. It depends on $2 + (N - 2)^2$ complex parameters $\beta_h$, $\beta_{1-h}$ and $\gamma_{m\bar{m}}$ which, in general, are some functions of the integrals of motion $(q, \bar{q})$ to be fixed by the quantization conditions. Let us take into account that the function $Q(z, \bar{z})$ has to satisfy the duality relation (3.3). For $|1 - z| \to 0$, one can apply (3.19) to evaluate the both sides of (3.3) in terms of the mixing matrices $C^{(1)}(q, \bar{q})$ and $C^{(1)}(-q, -\bar{q})$. This leads to the set of relations on the functions $\beta_i(q, \bar{q})$ and $\gamma_{m\bar{m}}(q, \bar{q})$.

To obtain these relations one uses the following property of the fundamental basis (3.16)
\[ Q_a^{(1)}(1/z; -q) = \sum_{b=1}^{N} S_{ab} Q_b^{(1)}(z; q), \tag{3.22} \]

where $\text{Im}(1/z) > 0$. Here, we indicated explicitly the dependence on the integrals of motion. Since the $Q-$functions in the both sides of (3.22) satisfy the same differential equation (3.3), the
Examining the leading asymptotic behaviour of the both sides of (3.22) for \( z \to 1 \). In this way, applying (3.16) and (3.17), one finds that the only nonvanishing matrix elements are given by

\[
S_{11} = e^{-i\pi(Ns-h-1)}, \quad S_{22} = e^{-i\pi(Ns+h-2)}, \quad S_{k,k+m} = (-1)^{k-3}(k-2s-1)m/m!
\]

(3.23)

with \( (x)_m \equiv \Gamma(x+m)/\Gamma(x), \ 3 \leq k \leq N \) and \( 0 \leq m \leq N-k \). Similar relations hold in the anti-holomorphic sector, \( \overline{S}_{11} = e^{i\pi(N\bar{s}-h-1)}, \overline{S}_{22} = e^{i\pi(N\bar{s}+h-2)} \) and \( \overline{S}_{k,k+m} = (-1)^{k-3}(k-2\bar{s}-1)m/m! \).

Finally, we substitute (3.19) and (3.22) into (3.5) and find

\[
\beta_h(q, \bar{q}) = e^{i\theta_N(q, \bar{q})}(-1)^{Nn_s+n_h}\beta_h(-q, -\bar{q}),
\]

\[
\gamma_{nm\bar{n}}(q, \bar{q}) = e^{i\theta_N(q, \bar{q})}\sum_{n, \bar{n} \geq 3}^N S_{nm}\gamma_{n\bar{n}}(-q, -\bar{q})\overline{S}_{n\bar{n}}.
\]

(3.24)

These relations imply that, similar to the energy, Eq. (3.14), the eigenvalues of the quasimomentum, \( \theta_N(q, \bar{q}) \), can be obtained from the mixing matrix at \( z = 1 \). In particular, it follows from the first relation in (3.24) that the quasimomentum of the eigenstates with \( q_{2k+1} = \bar{q}_{2k+1} = 0 \) \((k = 1, 2...,)\), or equivalently \( q = -\bar{q} \), is equal to

\[
e^{i\theta_N(q, \bar{q})} = (-1)^{Nn_s+n_h},
\]

(3.25)

since \( \beta_h(q, \bar{q}) = \beta_h(-q, -\bar{q}) \). At \( N = 2 \) one finds from (3.24) that \( e^{i\theta_2} = (-1)^{n_s+n_h} \).

### 3.3. Transition matrices

Eqs. (3.9) and (3.13) define the solution for \( Q(z, \bar{z}) \) in the vicinity of \( z = 0 \) and \( z = 1 \), respectively. To obtain the eigenvalues of the Baxter \( Q \)-operator, Eq. (3.14), one has to sew (3.9) and (3.19) inside the region \( |1 - z| < 1, |z| < 1 \) and, then, analytically continue the resulting expression for \( Q(z, \bar{z}) \) into the whole complex \( z- \)plane by making use of the duality relation (3.5). As we will see in a moment, this can be done only for the special values of integrals of motion \( (q, \bar{q}) \) satisfying the quantization conditions (see Eq. (3.28) below).

The sewing procedure is based on the relation between two fundamental sets of solutions, Eqs. (3.9) and (3.19). Choosing \( z \) to be inside the region of convergence of the both series, Eqs. (3.8) and (3.17), we define the transition matrices \( \Omega(q) \) and \( \overline{\Omega}(\bar{q}) \)

\[
Q_n^{(0)}(z) = \sum_{m=1}^N \Omega_{nm}(q)Q_{m}^{(1)}(z), \quad \overline{Q}_n^{(0)}(\bar{z}) = \sum_{m=1}^N \overline{\Omega}_{nm}(\bar{q})\overline{Q}_m^{(1)}(\bar{z}).
\]

(3.26)

Since the functions \( Q_n^{(0)}(z) \) and \( Q_m^{(1)}(z) \) satisfy the same differential equation (3.3), the transition matrices are \( z \)-independent. For the fundamental set of solutions, Eqs. (3.9) and (3.16), these matrices are uniquely fixed and they can be calculated as

\[
\Omega(q) = W^{(0)}[W^{(1)}]^{-1}, \quad W^{(j)}_{nk} = \partial_{z_0}^k Q_n^{(j)}(z_0)
\]

(3.27)

with \( j = 0, 1 \) and \( z_0 \) being some reference point, say \( z_0 = 1/2 \), and similar for \( \overline{\Omega}(\bar{q}) \). The resulting expressions for the matrices \( \Omega(q) \) and \( \overline{\Omega}(\bar{q}) \) take the form of infinite series in \( q \) and \( \bar{q} \), respectively.
respectively. The transition matrices in two sectors are related to each other as \( \Omega(q) = \Omega_s(h, q) \) and \( \Omega(q) = \Omega_q(h, \bar{q}) \).

The transition matrices allow us to analytically continue the solutions (3.9) valid for \(|z| < 1\) to the region \(|1 - z| < 1\). Substituting (3.26) into (3.9) and matching the result into (3.19), we find that the two expressions for the function \( Q(z, \bar{z}) \), Eqs. (3.3) and (3.19), can be sewed together provided that the mixing matrices \( C^{(0)} \) and \( C^{(1)} \) satisfy the following relation

\[
C^{(1)}(q, \bar{q}) = \left[ \Omega(q) \right]^T C^{(0)}(q, \bar{q}) \Omega(q). \tag{3.28}
\]

This matrix equation provides the quantization conditions for the integrals of motion of the model, \( q_k \) and \( \bar{q}_k \) with \( k = 3, ..., N \). In addition, it allows us to determine the matrices \( C^{(0)} \) and \( C^{(1)} \) and, as a consequence, evaluate the eigenvalues of the Baxter \( Q \)-operator, Eq. (3.1). Indeed, replacing in (3.28) the mixing matrices by their expressions, Eqs. (3.9) and (3.19), we obtain the system of \( N^2 \) equations involving \((N - 1)\) \( \alpha \)-parameters inside the matrix \( C^{(0)} \), \( \beta_{1,2} \) and \( \gamma_{mn} \) inside the matrix \( C^{(1)} \), as well as \((N - 2)\) integrals of motion \( q_3, ..., q_N \) (we recall that \( \bar{q}_k = q_k \)). Thus, the system (3.28) is overdetermined. It allows us to determine all parameters including the quantized \( q \) and, in addition, it provides \((2N - 3)\) nontrivial consistency conditions on the obtained solutions. Additional consistency conditions follow from (3.24).

The solutions to the quantization conditions (3.28) for different number of particles \( N \) and the emerging properties of the spectrum of the model will be described in details in Section 5.

3.4. Special case: \( N = 2 \)

As was already mentioned, the solution to the differential equation (3.3) for \( N = 2 \) admit representation in the form of contour integrals and, as a result, the quantization conditions (3.28) can be solved exactly. At \( N = 2 \), after the change of variables \( z = (x - 1)/x \) and \( Q(z) = [x(1 - x)]^{1-s}y(x) \), Eq. (3.3) takes the form of the Legendre’s differential equation [24]

\[
\left[ \frac{d}{dx} x(1 - x) \frac{d}{dx} + h(h - 1) \right] y(x) = 0. \tag{3.29}
\]

Its general solution is well-known as \( y(x) = \int_{C_w} dw w^{h-1}(w-1)^{h-1}(w-x)^{-h} \), where the integration contour \( C_w \) has to be chosen in such a way that the integrand resumes its original value after encircling \( C_w \). Then, two linear independent solutions to (3.29) are given by the Legendre’s functions of the first and second kind, \( P_{-h}(2x - 1) \) and \( Q_{-h}(2x - 1) \), respectively. Using the relation between these functions

\[
-\pi \cot(\pi h) P_{-h}(2x - 1) = Q_{-h}(2x - 1) - Q_{h-1}(2x - 1) \tag{3.30}
\]

and going back to the \( z \)-representation, we choose the fundamental set of solutions to Eq. (3.3) as \( Q_s(z; h) \) and \( Q_s(z; 1 - h) \), where the notation was introduced for\(^5\)

\[
Q_s(z; h) \equiv \left[ \frac{z}{(1 - z)^2} \right]^{1-s} Q_{-h} \left( \frac{1 + z}{1 - z} \right). \tag{3.31}
\]

The properties of the function \( Q_s(z; h) \), including its relation to the fundamental set (3.7) and (3.10), can be found in Appendix [A].

\(^5\)Obviously, this definition is ambiguous. Instead of using the \( Q \)-functions, one may define the fundamental set entirely in terms of the \( P \)-functions (see Eq. (A.3) below).
Following (3.24), we construct \( Q(z, \bar{z}) \) as a bilinear combination of the functions \( Q_s(z; h) \) and \( Q_s(z; 1 - h) \) and their antiholomorphic counterparts, \( Q_s(\bar{z}; h) \) and \( Q_s(\bar{z}; 1 - h) \). Requiring \( Q(z, \bar{z}) \) to have a trivial monodromy around \( z = 1 \) and taking into account that \( Q_s(z; h) \sim (1 - z)^{2s - h - 1} \) for \( z \to 1 \) (see Eq. (A.2)), one gets

\[
Q(z, \bar{z}) = c_h Q_s(z; h) Q_s(\bar{z}; \bar{h}) + c_{1-h} Q_s(z; 1-h) Q_s(\bar{z}; 1-\bar{h}).
\] (3.32)

To fix the coefficients \( c_h \) and \( c_{1-h} \), one examines the small-\( z \) asymptotics of (3.32) with a help of Eq. (A.1) and requires that the terms \( \sim \ln z \ln \bar{z} \) should cancel and the coefficients in front of \( \ln z \) and \( \ln \bar{z} \) should be the same. Applying the identity \( \psi(1 - h) - \psi(h) = \pi \cot(\pi h) \) one finds that the both conditions are fulfilled provided that \( c_h = -c_{1-h} \) and \( \cot(\pi h) = \cot(\pi \bar{h}) \). The second relation is automatically satisfied thanks to the property of the series, \( h - \bar{h} = n_h \) with \( n_h \) integer. Choosing

\[
c_h = \frac{2}{\pi} \tan(\pi h)(-1)^{n_h},
\] (3.33)

we obtain from (3.32) and (A.1) the small-\( z \) behaviour of the function \( Q(z, \bar{z}) \) as

\[
Q(z, \bar{z}) \sim z^{1-s} \bar{z}^{1-\bar{s}} (-1)^{n_h} \left\{ \ln(z\bar{z}) + 2 \text{Re} \left[ \psi(h) + \psi(1 - h) - 2\psi(1) \right] + O(z, \bar{z}) \right\}.
\] (3.34)

Taking into account that the quasimomentum of the \( N = 2 \) states is equal to \( e^{i\theta_2} = (-1)^{n_h} \), Eq. (3.25), we find that this relation is in agreement with (3.11). One determines the \( \alpha \)-parameter by matching (3.34) into (3.11)

\[
\alpha_1(h) = 2 \text{Re} \left[ \psi(h) + \psi(1 - h) - 2\psi(1) \right].
\] (3.35)

Finally, applying (3.14) we calculate the energy at \( N = 2 \) as

\[
E_2(h, \bar{h}) = 2\alpha_1(h) = 8 \text{Re} \left[ \psi \left( \frac{1 + |n_h|}{2} + i\nu_h \right) - \psi(1) \right].
\] (3.36)

The ground state corresponds to \( h = \bar{h} = 1/2 \), or equivalently \( n_h = \nu_h = 0 \),

\[
\min E_2(h, \bar{h}) = -16 \ln 2.
\] (3.37)

and it defines the intercept of the BFKL Pomeron [7].

The exact solution at \( N = 2 \) is based on the properties of the Legendre functions, Eq. (3.34). Going over to the systems with the number of particles \( N \geq 3 \), one encounters the following difficulties. Firstly, representation for the solution to (3.3) in the form of contour integrals does not exist or, at least, it is not warranted. Secondly, for \( N \geq 3 \) the function \( Q(z, \bar{z}) \) depends on the integrals of motion, \( q \) and \( \bar{q} \), whose values should be determined from the quantization conditions (3.28). As we will show in Section 5, both problems can be solved by using the power series representation for the fundamental set of solutions, Eqs. (3.7) and (3.16).

In this Section, we have demonstrated that in order for the eigenvalues of the Baxter \( Q \)-operator to possess the prescribed properties, Eqs. (2.7) – (2.13), the integrals of motion, \( q \), have to satisfy the quantization conditions (3.28). In this case, one can construct the function \( Q(z, \bar{z}) \) in the vicinity of the singular points, \( z = 0 \), \( z = 1 \) and \( z = \infty \), and analytically continue it onto the whole \( z \)-plane with a help of the transition matrices (3.26). The spectrum of the model – the energy and the quasimomentum, can be obtained from the mixing matrices \( C^{(0)} \) and \( C^{(1)} \), which define the asymptotic behaviour of \( Q(z, \bar{z}) \) around \( z = 0 \) and \( z = 1 \), respectively.
4. Eigenvalues of the Baxter $\mathbb{Q}$–operator

In the previous Section, we established the quantization conditions for the integrals of motion $q$ and obtained the expression for the energy $E_N$. Let us now construct the corresponding eigenstates $\Psi_{\vec{p},\{q,\bar{q}\}}(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)$, Eqs. (1.2) and (2.2).

The analysis is based on the method of Separated Variables (SoV) developed by Sklyanin [27]. It allows us to find the integral representation for the eigenstates of the model by going over to the representation of the separated coordinates $\vec{x} = (\vec{x}_1, ..., \vec{x}_{N-1})$ [3].

$$\Psi_{\vec{p},\{q,\bar{q}\}}(\vec{z}) = \int d^{N-1}\vec{x} \, \mu(\vec{x}) \, U_{\vec{p},\vec{x}}(\vec{z}) \, (\Phi_{\{q,\bar{q}\}}(\vec{x}))^*, \tag{4.1}$$

with $\vec{z} = (\vec{z}_1, ..., \vec{z}_N)$. Here, $U_{\vec{p},\vec{x}}(\vec{z})$ is a kernel of the unitary operator corresponding to this transformation and $\Phi_{\{q,\bar{q}\}}(\vec{x})$ is the wave function in the separated coordinates. The explicit expression for $U_{\vec{p},\vec{x}}(\vec{z})$ for arbitrary $N$ was found in [8]. Independently, similar expressions at $N = 2$ and $N = 3$ were obtained in [10].

Remarkable property of the SoV representation is that $\Phi_{\{q,\bar{q}\}}(\vec{x})$ is factorized into the product of the eigenvalues of the Baxter $\mathbb{Q}$–operator depending on different separated coordinates

$$\left(\Phi_{\{q,\bar{q}\}}(\vec{x})\right)^* = e^{i\theta_{N(q,\bar{q})/2}} \prod_{k=1}^{N-1} \left[ \frac{\Gamma(s + i\nu_k)\Gamma(\bar{s} - i\bar{x}_k)}{\Gamma(1 - s + i\bar{x}_k)\Gamma(1 - \bar{s} - i\bar{x}_k)} \right]^{N} Q_{q,\bar{q}}(x_k, \bar{x}_k), \tag{4.2}$$

where the additional factor $e^{i\theta_{N(q,\bar{q})/2}}$ is needed to ensure (2.14). Contrary to the $\vec{z} = (z, \bar{z})$–coordinates, the possible values of the separated coordinates $\vec{x}_k = (x_k, \bar{x}_k)$ are quantized as follows [3, 10]

$$x_k = \nu_k - \frac{n_k}{2}, \quad \bar{x}_k = \nu_k + \frac{n_k}{2}, \tag{4.3}$$

with $\nu_k$ real and $n_k$ integer. Integration on the space of separated variables in Eq. (4.1) implies summation over integer $n_k$ and integration over continuous $\nu_k$

$$\int d^{N-1}\vec{x} = \prod_{k=1}^{N-1} \left( \sum_{n_k = -\infty}^{\infty} \int_{-\infty}^{\infty} d\nu_k \right), \quad \mu(\vec{x}) = \frac{2\pi^{-N^2}}{(N-1)!} \prod_{j,k=1,\ j \neq k}^{N-1} |\vec{x}_k - \vec{x}_j|^2, \tag{4.4}$$

where $|\vec{x}_k - \vec{x}_j|^2 = (\nu_k - \nu_j)^2 + (n_k - n_j)^2 / 4$.

Eqs. (4.1) and (4.2) allow us to calculate the eigenfunctions of the model in terms of the eigenvalues of the Baxter $\mathbb{Q}$–operator. By the construction, the latter have poles specified in (2.14). One verifies, however, that they lie outside the integration contour in (4.1) and the integral in (4.1) is well-defined. Still, one can make use of the pole structure of $Q_{q,\bar{q}}(u, \bar{u})$ by closing the integration contour over $\nu_k$ into the upper (or lower) half-plane and calculating the asymptotics of the wave function $Ψ_{\vec{p},\{q,\bar{q}\}}(\vec{z})$ in the different regions of the $\vec{z}$–space.

According to (3.1) and (3.4), the eigenvalue of the Baxter $\mathbb{Q}$–operator is given by the following two-dimensional integral

$$Q_{q,\bar{q}}(u, \bar{u}) = \int \frac{d^2z}{z \bar{z}} z^{-iu} \bar{z}^{-i\bar{u}} \sum_{a,\bar{b}=1}^{N} Q_{a}(z) C_{ab} \bar{Q}_{\bar{b}}(\bar{z}). \tag{4.5}$$

Similar integrals have already appeared in the calculation of correlation functions in two-dimensional CFT. Applying the results of [28], one can convert $Q_{q,\bar{q}}(u, \bar{u})$ into a sum of products of...
holomorphic and antiholomorphic contour integrals, \( \int_C dz \, z^{-1-iu} Q_a(z) \) and \( \int_{\bar{C}} d\bar{z} \, \bar{z}^{-1-i\bar{u}} \bar{Q}_b(\bar{z}) \), respectively, with the integration contours \( C \) and \( \bar{C} \) starting and ending at one of the singular points \( z, \bar{z} = 0, 1 \) and \( \infty \). These contour integrals define the set of \( 2N \) functions of the (anti)holomorphic spectral parameters \( u \) and \( \bar{u} \). In analogy with the CFT, we shall refer to them as the \( Q \)-blocks. The resulting expression for \( Q_{q,\bar{q}}(u, \bar{u}) \), Eq. (4.5), is given by a bilinear combination of \( N \) blocks belonging to two sectors.

We would like to stress that, contrary to \( Q_{q,\bar{q}}(u, \bar{u}) \), the definition of the \( Q \)-blocks is ambiguous. The eigenvalues of the Baxter operator, \( Q_{q,\bar{q}}(u, \bar{u}) \), stay invariant if one replaces the \( Q \)-blocks by their linear combinations and redefines appropriately the expansion coefficients \( C_{ab} \) in the r.h.s. of (4.3). Making use of this ambiguity, one may look for the definition of the blocks, for which the expression for \( Q_{q,\bar{q}}(u, \bar{u}) \) looks particularly simple. In this Section, we shall present such a definition. We will demonstrate that the eigenvalues of the Baxter operator can be expressed in terms of only two \( Q \)-blocks – one in each sector, defined below in Eq. (4.12). We will also show that the quantization conditions for the integrals of motion and the expressions for the energy and the quasimomentum, established in Section 3, can be reformulated in terms of the \( Q \)-blocks.

4.1. Decomposition over the Baxter blocks

To proceed with calculation of (4.5), one has to specify the mixing matrix \( C_{ab} \), as well as the functions \( Q_a(z) \) and \( \bar{Q}_b(\bar{z}) \). These functions have to be defined uniformly on the whole complex \( z \)-plane with the cuts and their bilinear combination, Eq. (3.4), should match (3.9) and (3.19) for \( z \to 0 \) and \( z \to 1 \), respectively.

Let us choose \( C_{ab} \) to be the mixing matrix introduced in (3.19) and define the functions \( Q_a(z) \) in the vicinity of \( z = 1 \) as

\[
C_{ab} = C_{ab}^{(1)}(q, \bar{q}) , \qquad Q_a(z; q) \xrightarrow{z \to 1} Q_a^{(1)}(z; q) .
\]  

(4.6)

Analytical continuation of \( Q_a(z; q) \) to the region \( z \to 0 \) and \( z \to \infty \) can be obtained from (3.20) and (3.22) as

\[
Q_a(z; q) \xrightarrow{z \to 0} \sum_{b=1}^{N} [\Omega^{-1}(q)]_{ab} Q_b^{(0)}(z; q) ,
\]

\[
Q_a(z; q) \xrightarrow{z \to \infty} \sum_{b=1}^{N} [\Omega(-q) S]^{-1}_{ab} Q_b^{(0)}(1/z; -q) ,
\]

(4.7)

with the matrix \( S \) given by (3.23). The functions \( \bar{Q}_b(\bar{z}; \bar{q}) \) are defined similarly.

The functions \( Q_a(z) \) defined in this way possess a nontrivial monodromy at \( z = 1 \). Encircling the point \( z = 1 \) on the complex \( z \)-plane in anticlockwise direction, one calculates from (3.16) the corresponding monodromy matrix as

\[
Q_a(z) \mapsto M_{ab} Q_b(z) , \qquad M = \text{diag} \left( e^{2\pi i (N_s-h)}, e^{2\pi i (N_s+h)}, 1, ..., 1 \right) .
\]  

(4.8)

Unity entries in this matrix correspond to \((N-2)\) functions in the fundamental set (3.16) analytical at \( z = 1 \). Following [28], the two-dimensional integral in Eq. (4.3) can be evaluated as
a nonvanishing contribution to the calculation of (4.5) and making use of the monodromy of the functions

\[ M = \begin{pmatrix} e^{-2\pi i (N\bar{s} - \bar{h})} & e^{-2\pi i (N\bar{s} + \bar{h})} \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix} \]

with \( M^T = M \) according to (4.8). Here we tacitly assumed that the bilinear combination Eq. (3.4) is a single-valued function on the complex plane. This implies, in particular, that the integrals of motion \( q \) have to satisfy the quantization conditions (3.28).

As follows from their definition, Eq. (4.7), the functions \( Q_a(z) \) satisfy the relation (3.22)

\[ Q_a(1/z; -q) = \sum_{b=1}^{N} S_{ab} Q_b(z; q), \]

which holds for arbitrary \( z \) such that \( \text{Im}(1/z) > 0 \). Changing the integration variable in (4.9), \( z \to 1/z \), and applying (4.10) one gets

\[ Q_{\bar{q}, q}(u, \bar{u}) = \frac{1}{2i} \sum_{a, b=1}^{N} [S^T(1 - M) C^{(1)}]_{ab} \int_0^1 \frac{dz}{z} z^{-iu} Q_a(z; -q) \int_0^1 \frac{d\bar{z}}{\bar{z}} \bar{z}^{-i\bar{u}} \overline{Q}_b(\bar{z}; \bar{q}). \]

We recall that the matrices \( S, M \) and \( C^{(1)} \) were defined before in Eqs. (3.23), (4.8) and (3.19), respectively. Notice also that \( M^{-1} = S^2 \). Substituting the monodromy matrix (4.8) into (4.11) we find that among \( N^2 \) terms in the r.h.s. of (4.11) only two (with \( a = b = 1 \) and \( a = b = 2 \)) provide a nonvanishing contribution to \( Q_{\bar{q}, q}(u, \bar{u}) \). These two terms correspond to the \( Q \)-functions nonanalytical at \( z = 1 \), \( Q_1(z) \sim (1 - z)^{Ns-h-1} \) and \( Q_2(z) \sim (1 - z)^{Ns+h-2} \).

Let us introduce notation for the holomorphic and antiholomorphic Baxter blocks

\[ Q(u; h, q) = \frac{1}{\Gamma(Ns - h)} \int_0^1 \frac{dz}{z} z^{iu} Q_1(z; -q), \]

\[ \overline{Q}(\bar{u}; \bar{h}, \bar{q}) = \frac{1}{\Gamma(N\bar{s} - \bar{h})} \int_0^1 \frac{d\bar{z}}{\bar{z}} \bar{z}^{-i\bar{u}} \overline{Q}_1(\bar{z}; \bar{q}), \]

with the normalization factors chosen for later convenience (see Eq. (4.17) below). Taking into account Eqs. (3.23) and (3.19), one gets from (4.11)

\[ Q_{\bar{q}, q}(u, \bar{u}) = \pi \left[ \frac{\Gamma(N\bar{s} - \bar{h})}{\Gamma(1 - Ns + h)} \beta_h(q, \bar{q}) Q(u; h, q) \overline{Q}(\bar{u}; \bar{h}, \bar{q}) \right. \]

\[ + \left. \frac{\Gamma(N\bar{s} - 1 + \bar{h})}{\Gamma(2 - Ns - h)} \beta_{1-h}(q, \bar{q}) Q(u; 1 - h, q) \overline{Q}(\bar{u}; 1 - \bar{h}, \bar{q}) \right]. \]
one arrives at another (through equivalent) expression for the eigenvalues of the Baxter operator

\[ Q_{q,\bar{q}}(u, \bar{u}) = \pi \left[ \frac{\Gamma(Ns - h)}{\Gamma(1 - N\bar{s} + h)} \beta_h(q, \bar{q}) Q(-u; h, -q) \overline{Q}(-\bar{u}; h, -\bar{q}) \right. \\
+ \left. \frac{\Gamma(Ns - 1 + h)}{\Gamma(2 - N\bar{s} - h)} \beta_{1-h}(q, \bar{q}) Q(-u; 1 - h, -q) \overline{Q}(-\bar{u}; 1 - \bar{h}, -\bar{q}) \right]. \] (4.15)

One can verify the equivalence of Eqs. (4.15) and (4.13), by substituting (4.15) into the l.h.s. of (4.12) and taking into account the second relation in (3.24).

Thus, for a given set of the integrals of motion, \((q, \bar{q})\), satisfying the quantization conditions (3.28), the eigenvalue of the Baxter \(Q\)-operator, \(Q_{q,\bar{q}}(u, \bar{u})\), is unique. It is expressed in terms of two chiral blocks introduced in (4.12) and is given by Eq. (4.13).

### 4.2. Properties of the blocks

Let us show that the blocks \(Q(u; h, q)\) and \(\overline{Q}(\bar{u}; \bar{h}, \bar{q})\), defined in (4.12), have the following properties:

(i) \(Q(u; h, q)\) satisfies the chiral Baxter equation (2.7). \(\overline{Q}(\bar{u}; \bar{h}, \bar{q})\) obeys similar equation in the antiholomorphic sector.

(ii) \(Q(u; h, q)\) and \(\overline{Q}(\bar{u}; \bar{h}, \bar{q})\) are meromorphic functions [29, 3, 16] on the complex \(u-\) and \(\bar{u}-\)planes, respectively, with the only poles (which are all of order not higher than \(N\)) located at the points \(u_m = -i(s - m)\) and \(\bar{u}_m = i(\bar{s} - \bar{m})\), with \(m\) and \(\bar{m}\) positive integer. The same property can be expressed in a concise form as

\[ Q(u; h, q) = \Gamma^N(1 - s + iu) f(u), \quad \overline{Q}(\bar{u}; \bar{h}, \bar{q}) = \Gamma^N(1 - \bar{s} - i\bar{u}) \bar{f}(\bar{u}) \] (4.16)

with \(f(u)\) and \(\bar{f}(\bar{u})\) being entire functions.

(iii) At large \(u\) and \(\bar{u}\), away from the poles (1.16), that is for \(\text{Re}(1 - s + iu) > 0\) and \(\text{Re}(1 - \bar{s} - i\bar{u}) > 0\), the functions \(Q(u; h, q)\) and \(\overline{Q}(\bar{u}; \bar{h}, \bar{q})\) have the asymptotic behaviour

\[ Q(u; h, q) \sim (iu)^{-Ns+h} [1 + \mathcal{O}(1/u)] , \quad \overline{Q}(\bar{u}; \bar{h}, \bar{q}) \sim (-i\bar{u})^{-Ns+\bar{h}} [1 + \mathcal{O}(1/\bar{u})] . \] (4.17)

By the definition, Eq. (4.12), the blocks \(Q(u; h, q)\) and \(\overline{Q}(\bar{u}; \bar{h}, \bar{q})\) are related to the same universal function calculated for different values of the parameters

\[ Q(u; h, q) = Q_s(u; h, q) , \quad \overline{Q}(\bar{u}; \bar{h}, \bar{q}) = Q_s(-\bar{u}; \bar{h}, -\bar{q}) . \] (4.18)

To verify these properties one uses the integral representation for the blocks, Eq. (4.12). Then, the first property follows from the fact that the functions \(Q_1(z)\) and \(\overline{Q}_1(\bar{z})\) satisfy the differential equation (3.3). As to the second property, the poles of \(Q(u; h, q)\) and \(\overline{Q}(\bar{u}; \bar{h}, \bar{q})\) come from integration in (4.12) over the region of small \(z\) and \(\bar{z}\). The leading asymptotic behaviour of the functions \(Q_1(z)\) and \(\overline{Q}_1(\bar{z})\) in this region can be obtained from the first relation in (4.7) as \(Q_a(z) \sim z^1-s \ln^{\bar{N}-1} z\) and \(Q_b(\bar{z}) \sim z^{1-\bar{s}} \ln^{N-1} \bar{z}\). Finally, to obtain the asymptotics at infinity, Eq. (4.11), one integrates in (4.12) over the region \(z \to 1\) and \(\bar{z} \to 1\) and makes use of Eqs. (4.7) and (3.10) to replace \(Q_1(z) \sim (1 - z)^{Ns-h-1}\) and \(\overline{Q}_1(\bar{z}) \sim (1 - \bar{z})^{N\bar{s}-\bar{h}-1}\.\)
The above three properties uniquely specify the blocks for \( \text{Im } h \neq 0 \). As we have seen in Section 3.2 (see Eq. (B.18)), at \( \text{Im } h = 0 \) one of the fundamental solutions to the differential equation (1.3) has to be redefined in order to avoid a degeneracy. One encounters the same problem trying to define the block \( Q(u; h, q) \) at \( h = (1+nh)/2 \). In this case, the linear combination \( Q(u; h, q) + cQ(u; 1 - h, q) \) satisfies the three conditions on the \( Q \)-block for arbitrary \( c \) and, as a consequence, the blocks \( Q(u; h, q) \) and \( Q(u; 1 - h, q) \) become degenerate. The expression for the blocks at \( h = (1+nh)/2 \) can be found in the Appendix [4].

Using the definition of the blocks \( Q(u; h, q) \) and \( \overline{Q}(\bar{u}; \bar{h}, \bar{q}) \), one can establish different useful relations between them. In particular, as shown in the Appendix [3] the blocks in each sector satisfy nontrivial Wronskian relations (Eqs. (B.4) and (B.5)). Moreover, the blocks in two sectors are related to each other as (see Eq. (B.3))

\[
Q(u; h, q) = \left[ \frac{\Gamma(1-s+iu)}{\Gamma(s+iu)} \right]^N (\overline{Q}(u^*; 1 - \bar{h}, \bar{q}))^*. 
\] (4.19)

At \( N = 2 \) the eigenvalues of the Baxter \( Q \)-operator and the \( Q \)-blocks can be expressed in terms of the \( _3F_2 \)-hypergeometric series of a unit argument. As was shown in Section 3.4, at \( N = 2 \) the function \( Q_1(z) \) entering (4.12) is equal (up to an overall normalization) to the Legendre function of the second kind. Substituting (B.31) into (4.12) and using integral representation of the Legendre functions [24], one obtains after some algebra

\[
Q_s(u; h) = \frac{1}{2} \Gamma \left[ \begin{array}{c} 1 - s + iu, 1 - s + iu, 1 - h \\ 1 + s - h + iu, 2 - s + iu - h \end{array} \right] _3F_2 \left( \begin{array}{c} s + iu, 1 - s + iu, 1 - h \\ 1 + s - h + iu, 2 - s + iu - h \end{array} \right), 
\] (4.20)

where \( \Gamma[...] \) denotes the ratio of the products of the \( \Gamma \)-functions with the arguments listed in the upper and lower rows, respectively. Together with (4.13) and (A.7) this leads to the following expression for the eigenvalues of the Baxter operator at \( N = 2 \)

\[
Q(u, \bar{u}) = 2(-1)^{nh} \tan(\pi h) 
\times \left[ \frac{\Gamma(2\bar{s} - \bar{h})}{\Gamma(2\bar{s} - 1 - \bar{h})} Q(u; h) \overline{Q}(\bar{u}; \bar{h}) - \frac{\Gamma(2s - 1 + h)}{\Gamma(2s - h)} Q(u; 1 - h) \overline{Q}(\bar{u}; 1 - \bar{h}) \right], 
\] (4.21)

where \( Q(u; h) = Q_s(u; h) \) and \( Q(\bar{u}; \bar{h}) = Q_s(-\bar{u}; \bar{h}) \) are (non-normalized) \( Q \)-blocks at \( N = 2 \).

For \( N \geq 3 \) one can calculate the blocks by replacing the functions \( Q_1(z; q) \) in (4.12) by their expressions in terms of the fundamental solutions around \( z = 0 \) and \( z = 1 \), Eqs. (1.7) and (1.8), respectively. As shown in the Appendix [3], this leads to two different series representations for the block \( Q(u; h, q) \), Eqs. (B.8) and (B.10), which are valid in the different regions on the complex \( u \)-plane.

We are now in position to demonstrate that the eigenvalues of the Baxter operator constructed in this Section have correct asymptotic behaviour at infinity, Eq. (2.13). Substituting (4.17) into (4.13), we verify that \( Q_{q,q}(u, \bar{u}) \) satisfies Eq. (2.13) with the phase \( \Theta_h(q, \bar{q}) \) given by

\[
e^{2i\Theta_h(q, \bar{q})} = (-1)^{nh} \frac{\beta_h(q, \bar{q})}{\beta_{1-h}(q, \bar{q})} \frac{\Gamma(2 -Ns - h)\Gamma(Ns - \bar{h})}{\Gamma(2 -Ns + h) \Gamma(Ns - 1 + \bar{h})}. 
\] (4.22)

In particular, for \( h = \bar{h} = 1/2 + ivh \) and \( \nu_h \to 0 \) one has \( \beta_h(q, \bar{q}) \sim 1/\nu_h \) leading to \( e^{2i\Theta_{1/2}(q, \bar{q})} = -1 \).

\[\text{Due to the additional factor } \Gamma^2(1-h)/(2\Gamma(2-2h)) \text{ in the r.h.s. of } (A.4), \text{ this expression has asymptotics at infinity that differs from } (4.17) \text{ by the same factor.}\]
4.3. Quantization conditions from the $Q$–blocks

As we have seen in the previous Section, the eigenvalues of the Baxter $Q$–operator can be expressed in terms of the $Q$–blocks satisfying the conditions $(i)$–$(iii)$. In distinction with $Q_{q,q}(u, \bar{u})$, the $Q$–blocks can be constructed for arbitrary values of the integrals of motion $q, \bar{q}$. In this Section, we will show that the requirement for $Q_{q,q}(u, \bar{u})$ to have correct analytical properties leads to the quantization conditions for the integrals of motion which are equivalent to (3.28).

The general expression for the eigenvalue of the Baxter operator $Q_{q,q}(u, \bar{u})$ in terms of the blocks $Q(u; h, q)$ and $Q(u; 1 - h, q)$ and their antiholomorphic counterparts looks like

$$Q_{q,q}(u, \bar{u}) = c_h Q(u; h, q) \overline{Q(\bar{u}; \bar{h}, \bar{q})} + c_{1-h} Q(u; 1 - h, q) \overline{Q(\bar{u}; 1 - \bar{h}, \bar{q})}$$

(4.23)

with $c_h$ (and $c_{1-h}$) being arbitrary function of $h$ and the integrals of motion. $Q_{q,q}(u, \bar{u})$ defined in this way is symmetric under $h \rightarrow 1 - h$ and $\bar{h} \rightarrow 1 - \bar{h}$, it satisfies the Baxter equations in the holomorphic and antiholomorphic sectors, and its asymptotic behaviour at infinity is in agreement with Eqs. (2.13). Therefore, it remains to show that $Q_{q,q}(u, \bar{u})$ has the correct structure of the poles, Eq. (2.11). To this end, one applies (4.19) (see also (B.3)) and rewrites (4.23) in two equivalent forms

$$Q_{q,q}(u, \bar{u}) = \left[ \frac{\Gamma(1 - \bar{s} - i\bar{q}u)}{\Gamma(s - i\bar{u})} \right]^N \left\{ c_h Q(u; h, q)(Q(\bar{u}^*; 1 - h, q))^* + c_{1-h} Q(u; 1 - h, q)(Q(\bar{u}^*; h, q))^* \right\}$$

$$= \left[ \frac{\Gamma(1 - s + i\bar{u}u)}{\Gamma(s + i\bar{u})} \right]^N \left\{ c_h \overline{Q(\bar{u}; \bar{h}, \bar{q})}(\overline{Q(u^*; 1 - \bar{h}, \bar{q}))^* + c_{1-h} \overline{Q(\bar{u}; 1 - \bar{h}, \bar{q})}(\overline{Q(u^*; \bar{h}, \bar{q})})^* \right\} (4.24)$$

The analytical properties of $Q_{q,q}(u, \bar{u})$ are now manifest – the poles of $Q_{q,q}(u, \bar{u})$ in $\bar{u}$ and $u$ are generated by the $\Gamma$–functions in the first and the second relation, respectively. One deduces from (4.24) that for arbitrary $c_h$ and $c_{1-h}$ the function $Q_{q,q}(u, \bar{u})$ has the $N$–th order poles at $\bar{u} = i(\bar{s} - \bar{m})$ and, separately, at $u = -i(s - m)$ with $m, \bar{m} = \mathbb{Z}_+.$

Let us now require that the analytical properties of (4.24) should match similar properties of the Baxter $Q$–operator, Eq. (2.11). We remind that the operator $Q(u, \bar{u})$ is well-defined only if the spectral parameters $u$ and $\bar{u}$ satisfy (2.1). Imposing this condition, we find that (4.24) has the prescribed poles, (2.11), plus additional “spurious” $N$–th order poles located at

$$\{u = i(s + m - 1), \ \bar{u} = i(\bar{s} - \bar{m})\},$$

$$\{u = -i(s - m), \ \bar{u} = -i(\bar{s} + \bar{m} - 1)\},$$

(4.25)

with $m, \bar{m} = 1, 2, \ldots$. Thus, the coefficients $c_h$ (and $c_{1-h}$) and the integrals of motions $(q, \bar{q})$ have to be chosen in such a way that (4.24) has to have vanishing residues at the poles (4.25). Introducing the functions

$$\Phi(\epsilon) = \frac{Q(i(s + \epsilon); h, q)}{Q(i(s + \epsilon); 1 - h, q)}, \quad \overline{\Phi(\epsilon)} = \frac{\overline{Q(-i(s + \epsilon); \bar{h}, \bar{q})}}{\overline{Q(-i(s + \epsilon); 1 - h, \bar{q})}}$$

(4.26)

and examining the first and the second relation in (4.24) for $\{u = i(s + m - 1 + \epsilon), \ \bar{u} = i(\bar{s} - \bar{m} + \epsilon)\}$ and $\{u = -i(s - m + \epsilon), \ \bar{u} = i(\bar{s} + \bar{m} - 1 + \epsilon)\},$ respectively, as $\epsilon \rightarrow 0$, one finds that this requirement leads to

$$c_h \Phi(m - 1 + \epsilon) + c_{1-h} (\Phi(\bar{m} - 1 + \epsilon))^* = \mathcal{O}(\epsilon^N)$$

$$c_h \overline{\Phi}(\bar{m} - 1 + \epsilon) + c_{1-h} (\overline{\Phi}(m - 1 + \epsilon))^* = \mathcal{O}(\epsilon^N),$$

(4.27)
with $m$ and $\bar{m}$ being positive integer. It follows from the Wronskian relations (see Appendix B, Eq. (3.6)), that the infinite system of equations (1.27) becomes equivalent to a single condition at $m = \bar{m} = 1$

$$\frac{c_{1-h}}{c_h} \left( \Phi(\epsilon) \right)^* = -1 + \mathcal{O}(\epsilon^N), \quad \frac{c_{1-h}}{c_h} \left( \overline{\Phi}(\epsilon) \right)^* = -1 + \mathcal{O}(\epsilon^N),$$

(4.28)

with the functions $\Phi(\epsilon)$ and $\overline{\Phi}(\epsilon)$ defined in (4.26).

One finds from (4.28) that $c_h/c_{1-h}$ is a pure phase. Its value can be obtained by matching (4.28) into (2.13) at large $u$ and $\bar{u}$ with a help of (4.17),

$$\frac{c_{1-h}}{c_h} = (-1)^{n_h} e^{-2i\Theta_h(q, \bar{q})},$$

(4.29)

Then, recalling the definition of the function $\Phi(\epsilon)$, Eq. (1.26), we obtain from (4.28)

$$\arg \left[ \frac{Q(i(s + \epsilon); h, q)}{Q(i(s + \epsilon); 1 - h, q)} \right] = \pi \left( \frac{n_h + 1}{2} + \ell \right) - \Theta_h(q, \bar{q}) + \mathcal{O}(\epsilon^N)$$

(4.30)

and similar relation for the antiholomorphic block

$$\arg \left[ \frac{\overline{Q}(-i(s + \epsilon); h, q)}{\overline{Q}(-i(s + \epsilon); 1 - h, q)} \right] = \pi \left( \frac{n_h + 1}{2} + \bar{\ell} \right) - \Theta_h(q, \bar{q}) + \mathcal{O}(\epsilon^N),$$

(4.31)

with $\ell$ and $\bar{\ell}$ integer, such that, in general, $\bar{\ell} \neq \ell$. Applying (4.19), one can express (4.31) entirely in terms of the holomorphic blocks. The blocks entering the relations (4.30) and (4.31) are given by (3.8).

Expanding the both sides of (1.30) and (1.31) in powers of $\epsilon$, one obtains the (overdetermined) system of $2N$ real quantization conditions on $N - 2$ complex charges $q_3, \ldots, q_N$ and real phase $\Theta_h(q, \bar{q})$. We will verify in Section 5 that their solutions are consistent with the quantization conditions (3.28).

### 4.4. Energy spectrum from the $Q$–blocks

Let us show that the energy, $E_N(q, \bar{q})$, and quasimomentum, $\theta_N(q, \bar{q})$, admit a simple representation in terms of the $Q$–blocks. To this end, we introduce new blocks

$$Q_0(u; q) = a_h Q(u; h, q) + a_{1-h} Q(u; 1 - h, q),$$

$$\overline{Q}_0(\bar{u}; \bar{q}) = \bar{a}_h \overline{Q}(\bar{u}; \bar{h}, \bar{q}) + \bar{a}_{1-h} \overline{Q}(\bar{u}; 1 - \bar{h}, \bar{q}),$$

(4.32)

which are symmetric under $h \rightarrow 1 - h$ and $\bar{h} \rightarrow 1 - \bar{h}$.

We require that $Q_0(u; q)$ and $\overline{Q}_0(\bar{u}; \bar{q})$ should have the same poles as the $Q$–blocks, Eq. (1.19), but of the order not be higher than $N - 1$. Applying (1.19), it is straightforward to verify that the linear combination of the $Q$–blocks in the r.h.s. of (4.32) has a vanishing residues at the $N$th pole at $u^-_1 = -i(s - 1)$ and $\bar{u}^+_1 = i(\bar{s} - 1)$ provided that (up to an overall normalization)

$$a_h = \tan(\pi h)(\overline{Q}(-is; \bar{h}, \bar{q}))^*, \quad \bar{a}_h = \tan(\pi \bar{h})(Q(is; h, q))^*. $$

(4.33)

Then, the residues at the remaining $N$th order poles, $\{u_m, \bar{u}_{m}^+\}$, vanish automatically, since otherwise $Q_0(u; q)$ and $\overline{Q}_0(\bar{u}; \bar{q})$ will not satisfy the Baxter equations (2.4).
\[ Q(\alpha + i\gamma)Q(\alpha + i\gamma) - Q(\alpha + i\gamma)Q(\alpha + i\gamma) = \text{const} \times \left[ \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \right] \]

(4.40)

\[ \Delta Q = \frac{B_w Q(\alpha + i\gamma)}{\Delta \alpha Q(\alpha + i\gamma); Q(\alpha + i\gamma)} = \text{const} \times \left[ \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \right] \]

(4.39)

\[ Q(\alpha + i\gamma) = A_0 Q(\alpha + i\gamma) + B_0 Q(\alpha + i\gamma) \]

(4.38)

\[ \theta(\delta, s, \nu) = \frac{\Gamma(s + \nu)}{\Gamma(s + \nu + 1)} \]

(4.37)

\[ Q(\alpha + i\gamma) = \text{const} \times \left[ \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \right] \]

(4.36)

\[ \Omega(\alpha + i\gamma) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.35)

\[ \Omega(u, \nu) = Q(\alpha + i\gamma) + aQ(\alpha + i\gamma) \]

(4.34)

\[ \Omega(u, \nu) = \frac{\Gamma(1 - s - \nu)}{\Gamma(s + 1 - \nu)} \]

(4.33)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.32)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.31)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.30)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.29)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.28)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.27)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.26)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.25)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.24)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.23)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.22)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.21)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.20)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.19)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.18)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.17)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.16)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.15)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.14)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.13)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.12)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.11)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.10)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.09)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.08)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.07)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.06)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.05)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.04)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.03)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.02)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.01)

\[ \Omega(u, \nu) = \frac{\Gamma(u - i\gamma)}{\Gamma(u + i\gamma)} \]

(4.00)
which follows from (3.4) and the definition of the $Q_0$– and $Q_1$–blocks, Eqs. (1.32) and (1.34), respectively. The normalization constant in the r.h.s. of (4.40) can be obtained for $u = -i(s + \varepsilon)$ from the comparison of the residues of the both sides of (4.40) at the $N$th order pole in $\varepsilon$. In this way, one finds from (4.38)

$$\frac{d}{du} \ln Q_{q,\bar{q}}(u - i s, u - i \bar{s}) \bigg|_{u=0} = \frac{i N}{1 - 2s} + (\ln \overline{Q}_0(-i \bar{s}; \bar{q}))' - (\ln [Q_0(-i s; q)])'. \quad (4.41)$$

Repeating similar calculation of the logarithmic derivative of (4.36) at $u = i s$ and $\bar{u} = i \bar{s}$ we find

$$\frac{d}{du} \ln Q_{q,\bar{q}}(u + i s, u + i \bar{s}) \bigg|_{u=0} = -\frac{i N}{1 - 2s} + (\ln Q_0(i s; q))' - (\ln [Q_0(i s; q)]'). \quad (4.42)$$

Finally, we substitute the last two relations into (2.3) and obtain the following expression for the energy

$$E_N(q, \bar{q}) = -2 \text{Im} (\ln Q_0(i s; q))' + 2 \text{Im} (\ln \overline{Q}_0(-i \bar{s}; \bar{q}))' + \varepsilon_N, \quad (4.43)$$

where $\varepsilon_N = 2N \text{Re}[\psi(2s) + \psi(2 - 2s) - 2\psi(1)]$. One can further simplify this expression by using the relation $(\ln Q_{q,\bar{q}}(-i s, -i \bar{s}))' = -(\ln Q_{q,\bar{q}}(i s, i \bar{s}))'$ that follows from (2.16). Together with (4.41) and (4.42), it leads to

$$E_N(q, \bar{q}) = -2 \text{Im} (\ln Q_0(i s; q))' - 2 \text{Im} (\ln Q_0(i s; -q))' + E_N^{(0)}, \quad (4.44)$$

where $E_N^{(0)} = 2N \text{Re}[\psi(2s) + \psi(1 - 2s) - 2\psi(1)]$. The symmetry of the energy $E_N(q, \bar{q}) = E_N(-q, -\bar{q})$ becomes manifest in this form.

The results obtained in Sections 3 and 4 provide the basis for calculating the energy spectrum of the model. We have demonstrated that

- The eigenvalues of the Baxter operator, $Q_{q,\bar{q}}(u, \bar{u})$, possessing the correct analytical properties and asymptotic behaviour at infinity (see Section 2) can be constructed only for the special values of the integrals of motion $q$ and $\bar{q}$ satisfying the quantization conditions, Eqs. (3.28), (4.30) and (4.31).

- The functions $Q_{q,\bar{q}}(u, \bar{u})$ can be decomposed over the chiral $Q$–blocks, which depend on the spectral parameters and the integrals of motion in the (anti)holomorphic sector, Eq. (4.13) and (4.30). In distinction with $Q_{q,\bar{q}}(u, \bar{u})$, the definition of the $Q$–blocks is ambiguous and, therefore, they do not have any physical meaning per se.

- Once the quantization conditions for the integrals of motion are fulfilled, the corresponding eigenvalues of the Baxter operator are uniquely fixed (up to an overall normalization). They allow us to calculate the energy spectrum of the model by using two different expressions Eqs. (3.14) and (4.44), which lead to the same value of $E_N$.

We would like to mention that we disagree on these points with the approach of Ref. [16], in which the Baxter equation for noncompact magnet of spin $s = 1$ has been investigated.

5. Energy spectrum

In this Section we solve the quantization conditions (3.28) and describe the spectrum of the Schrödinger equation (1.2) for different number of particles $N$. 

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For arbitrary $N$, the spectrum of the model – the energy, $E_N(q, \bar{q})$, and the corresponding eigenstates, $\Psi_{\tilde{p}^I(q, \bar{q})}(\tilde{z})$, are uniquely specified by the total set of quantum numbers in the holomorphic and antiholomorphic sectors, $q = (q_2, q_3, \ldots, q_N)$ and $\bar{q} = (\bar{q}_2, \bar{q}_3, \ldots, \bar{q}_N)$, respectively. Since $q_2 = \bar{q}_2$, the total number of independent complex valued quantum numbers is equal to $(N - 1)$. One of them, $q_2$, fixes the total $SL(2, \mathbb{C})$ spin of the state $(h, \bar{h})$ defined in (2.9). According to Eq. (2.9), the quantized values of $h$, or equivalently $q_2$, are parameterized by integer $n_h$ and real $\nu_h$

$$q_2(n_h, \nu_h) = \frac{1}{4} - \left(\frac{n_h}{2} + i\nu_h\right)^2 + Ns(s - 1).$$

At $N = 2$ this becomes the only quantum number parameterizing the spectrum of the model (3.36). For $N \geq 3$ one has to deal with a bigger set of the quantum numbers.

To find the spectrum of the integrals of motion $q_3, \ldots, q_N$ for $N \geq 3$, one has to solve the overdetermined system of the quantization conditions (3.28). As was explained in Section 3, their solutions give us the expressions for the $\alpha-$, $\beta-$ and $\gamma-$parameters entering the mixing matrices $C^{(0)}$ and $C^{(1)}$, Eqs. (3.9) and (3.19). Then, the corresponding energy, $E_N(q, \bar{q})$, and the quasimomentum, $\theta_N(q, \bar{q})$, are calculated by inserting these expressions into (3.14) and (3.24), respectively. In distinction with the $N = 2$ case, Eqs. (3.36) and (3.35), the resulting expressions for the spectrum of the model can not be expressed for arbitrary $N$ in a closed analytical form. Nevertheless, the results that we are going to present in this Section exhibit a remarkable regularity and they suggest that there should exist a WKB-like description of the spectrum.

To analyze the quantization conditions (3.28), one has to specify the values of the $SL(2, \mathbb{C})$ spins $(s, \bar{s})$ defined in (1.4). Since our main motivation for studying the noncompact Heisenberg spin magnets came from high-energy QCD, we shall fix them to be the same as for the $N-$reggeized gluons compound state, Eqs. (1.1) and (1.2),

$$s = 0, \quad \bar{s} = 1.$$  

Another specifics of QCD is that, solving the Schrödinger equation (1.2), one is mainly interested in finding the ground state. It is this state that provides the dominant contribution to the asymptotic behaviour of the scattering amplitudes (1.1) at high-energy. Additional constraint on the solutions to (1.2) is imposed in QCD by the Bose symmetry of the $N-$reggeized gluon compound states. As we will argue below, this condition selects among all eigenstates only those with the quasimomentum $\exp(i\theta_N(q, \bar{q})) = 1$. It is automatically fulfilled for the ground state for which one expects $\theta^\text{ground}_N(q, \bar{q}) = 0$.

To solve the quantization conditions (3.28), one needs the expressions for the transition matrices $\Omega(q)$ and $\Omega(\bar{q})$, defined in (3.26). For $N = 2$ they can be determined exactly from the properties of the Legendre functions (see Appendix A). For $N \geq 3$ the calculation of these matrices is based on the relation (3.27), which in turn relies on the power series representation of the fundamental solutions, Eqs. (3.7) and (3.16). In general, the resulting expressions for the transition matrices take the form of an infinite series in $q$. Solving the quantization conditions, we truncated infinite series in Eqs. (3.8) and (1.14) and retained a sufficiently large number of terms ($n_{\text{max}} \sim 10^3$). This allowed us to calculate the energy spectrum numerically with a rather high accuracy (see Table 3 below).

As was already mentioned, the system of $N^2$ quantization conditions (3.28) is overdetermined. Using the subset of $(N^2 - 3N + 5)$ conditions, one can calculate the $\alpha-$, $\beta-$ and $\gamma-$parameters in terms of the charges $q$ and define the following test function

$$f(q_2, q_3, \ldots, q_N) = \text{Tr}(TT^\dagger), \quad T = C^{(1)}(q, \bar{q}) - [\Omega(q)]^T C^{(0)}(q, \bar{q}) \Omega(\bar{q}).$$
Obviously, \( f(q) \) is a positive definite function on the \((N-1)\)-dimensional complex moduli space
\[ q = (q_2, q_3, ..., q_N). \]
The solutions to the quantization conditions \((3.28)\) correspond to points on this space, in which the test function vanishes, \( f(q) = 0 \). Since the dimension of the moduli space increases with \( N \), the problem of finding zeros of \( f(q) \) becomes very nontrivial at higher \( N \). To solve it, we applied the algorithm developed in \[30\]. It allowed us to identify the zeros of the test function \((3.3)\) by the “steepest descent” method starting from some reference point on the moduli space. As yet another test of our approach, we verified that the obtained expressions for the integrals of motion satisfy the conditions \((4.30)\) and \((4.31)\).

### 5.1. Fine structure of the spectrum

Before summarizing the results of our calculations for \( N \geq 3 \), let us describe a general structure of the spectrum. We find that for given total \( SL(2, \mathbb{C}) \) spin of the system, \( h = (1 + n_h)/2 + i \nu_h \), the quantization conditions \((3.28)\) provide us with an infinite number of discrete values of the integrals of motion. They can be parameterized by the set of integers as

\[
q_k = q_k(\nu_h; n_h, \{\ell\}), \quad E_N = E_N(\nu_h; n_h, \{\ell\}), \quad \{\ell\} = (\ell_1, \ell_2, ..., \ell_{2(N-2)}) \quad (5.4)
\]

with \( k = 3, ..., N \). The explicit form of this dependence and interpretation of integers \( \{\ell\} \) will be given below. Eqs. \((5.4)\) imply that, as a function of the total spin \( h \), the quantized values of \( q_k \) form the family of one-dimensional continuous nonintersecting trajectories in the \((N-1)\)-dimensional space of \( q = (q_2, q_3, ..., q_N) \). The “proper time” along each trajectory, \( \nu_h \), is defined by the imaginary part of the total spin, \( \Im h = \nu_h \), whereas the integers \( n_h, \ell_1, ..., \ell_{2(N-2)} \) specify different members of the family. Each trajectory in the \( q \)-space induces the corresponding trajectory for the energy \( E_N = E_N(q, \tilde{q}) \). We recall that, in contrast with the integrals of motion, \( q \) and \( \tilde{q} \), the energy takes only real values.

In what follows, we shall separate all eigenstates of the model into two groups according to the value of the “highest” charge \( q_N \): \( q_N \neq 0 \) and \( q_N = 0 \). The reason for this is that, as we will show in Section 5.5, there exists an intrinsic relation between the eigenstates with \( q_N = 0 \) and the eigenstates of the system with the number of particles equal to \( N-1 \) and \( q_{N-1} \neq 0 \). Namely, they share the same spectrum of the energy and the integrals of motion \( q_k \) (with \( k = 2, ..., N-1 \)) and the corresponding wave functions are related to each other in a simple way.

For the moment, we shall exclude the eigenstates with \( q_N = 0 \) from our consideration and return to them in Section 5.5. We find that among all eigenstates of the \( N \)-particle system with \( q_N \neq 0 \), the minimal energy occurs at \( \nu_h = n_h = 0 \), or equivalently \( \hbar = 1/2 \),

\[
E_N^{\text{ground}} = \min E_N(\nu_h; n_h, \{\ell\}) = E_N(0; 0, \{\ell_{\text{ground}}\}) \quad (5.5)
\]

It belongs to the special trajectory \( \{\ell_{\text{ground}}\} \), to which we shall refer as the ground state trajectory. The energy along this trajectory is a continuous function of \( \nu_h \) and it approaches its minimal value, \( E_N^{\text{ground}} \), at \( \nu_h = 0 \). In the vicinity of \( \nu_h = 0 \), one finds an accumulation of the energy levels

\[
E_N(\nu_h; 0, \{\ell_{\text{ground}}\}) - E_N^{\text{ground}} = \sigma_N \nu_h^2 + \mathcal{O}(\nu_h^4), \quad (5.6)
\]

with \( \sigma_N \) being the diffusion coefficients.

\[7\text{This method was implemented in the form of the Fortran-90 code, which is available from the authors upon request.} \]
The spectrum of quantized \( q_2, ..., q_N \) possesses the following symmetry

\[
q_k \rightarrow (-1)^k q_k \rightarrow q^*_k
\]

with \( k = 2, ..., N \). Here, the first relation follows from invariance of the Hamiltonian under mirror permutations of particles, Eq. (2.14). The second relation is a consequence of the symmetry of the model at \( s = 0 \) and \( \bar{s} = 1 \) under interchange of the \( z- \) and \( \bar{z}- \) sectors, or equivalently \( q_k \leftrightarrow \bar{q}_k \) and \( s(s - 1) \leftrightarrow \bar{s}(\bar{s} - 1) \). The relation (5.7) implies that if the quantization conditions (3.28) are satisfied at some point \( \{q_k\} \) on the moduli space, then the same holds true at the points \( \{(-1)^k q_k\}, \{q^*_k\} \) and \( \{(-1)^k q^*_k\} \).

As we will see in the next Section, the spectrum of quantized charges \( q \) has a hidden structure, which can be revealed by examining the distribution of the quantized values of the “highest” charge \( q_N^{1/N} \). Since \( q_N^{1/N} \) is a multi-valued function of complex \( q_N \), each eigenstate of the model will be represented on the complex \( q_N^{1/N} \)-plane by \( N \) different points. Together with the symmetry property of the spectrum, Eq. (5.7), this leads to the following transformation on the moduli space

\[
q_N^{1/N} \rightarrow \exp (\pi i k / N) q_N^{1/N}, \quad [0 < k < 2N]
\]

with \( k \) integer for odd \( N \) and even for even \( N \). It maps one of the eigenstates into itself or into another one with the same energy.

In the rest of this Section we shall present the results of our calculations of the spectrum of the model for the number of particles \( 3 \leq N \leq 8 \).

### 5.2. Quantum numbers of the \( N = 3 \) states

At \( N = 3 \) the eigenstates depend on the quantum number \( q_3 \), which is an eigenvalue of the operator \( q_3 \) defined in (1.6). Some of the eigenvalues of this operator have been already found in (14, 15, 16) using different methods. The eigenstates found in (14) have pure imaginary values of quantized \( q_3 \) and their quasimomentum is equal to \( \theta_3 = 0 \). We will demonstrate in this Section, that contrary to the statements made in (14) the spectrum of the operator \( q_3 \) is not exhausted by pure imaginary values, even in the sector with \( \theta_3 = 0 \).

Solving the quantization conditions (3.28) at \( N = 3 \), we reconstructed the full spectrum of quantized \( q_3 \). We found that apart from pure imaginary \( q_3 \) calculated in (14, 15, 16), the spectrum also contains (an infinite number of) complex values of \( q_3 \), including pure real ones. Enumerating the quantized \( q_3 \) according to their absolute value starting from the smallest one, we notice that \( |q_3| \) grows cubically with its number. This suggests to describe the spectrum in terms of \( q_3^{1/3} \) rather than \( q_3 \). To illustrate this point, we present in Figure 4 the results of our calculations of quantized \( q_3^{1/3} \) for the total \( SL(2, \mathbb{C}) \) spin of the system \( h = 1/2 \), or equivalently \( n_h = \nu_h = 0 \). Similar picture emerges at \( h = (1 + n_h)/2 + i\nu_h \) for \( n_h = 1, 2, ... \) and \( \nu_h \) real.

The spectrum of quantized \( q_3^{1/3} \), shown in Figure 4, is in agreement with the symmetry properties (5.7) and (5.8). Defining the fundamental domain as \( 0 \leq \arg(q_3^{1/3}) < \pi/3 \), we find that the whole spectrum of quantized \( q_3^{1/3} \) can be obtained by applying the transformations (5.7) and (5.8) to the points belonging to this domain.

It is difficult do not notice a remarkable regularity in the distribution of quantized charges in Figure 4. Apart from a few points close to the origin, the quantized values of \( q_3^{1/3} \) are located at the vertices of the lattice built from equilateral triangles. As a consequence, they can be
Figure 1: The spectrum of quantized $q_3^{1/3}$ for the system of $N = 3$ particles. The total $SL(2, \mathbb{C})$ spin of the system is equal to $h = 1/2$.

described parameterized as

$$[q_3^\text{WKB}(\ell_1, \ell_2)]^{1/3} = \Delta_{N=3} \cdot \left( \frac{1}{2} \ell_1 + \frac{i}{2} \sqrt{3} \ell_2 \right),$$

where $\ell_1$ and $\ell_2$ are integers, such that their sum $\ell_1 + \ell_2$ is even. Here, $\Delta_3$ denotes the lattice spacing. Its value can be calculated from the leading-order WKB solution of the Baxter equation \cite{31}

$$\Delta_3 = \left[ \frac{3}{4^{1/3} \pi} \int_{-\infty}^{1} \frac{dx}{\sqrt{1-x^3}} \right]^{-1} = \frac{\Gamma^3(2/3)}{2\pi} = .395175...$$

Quantized $q_3^{1/3}$ occupy the whole complex plane except the interior of the disk of the radius $\Delta_3$

$$|q_3^{1/3}| > \Delta_3.$$  

The comparison of (5.9) with the exact expressions for $q_3$ at $h = 1/2$ is shown in Figure 2 and Table 1. We find that the expression (5.9) describes the excited eigenstates with a high accuracy. The agreement becomes less impressive for the eigenstates with smaller $q_3$. For instance, for the ground state with $iq_3 = 0.205258...$ and the first excited state with $q_3 = 0.368293...$ the accuracy of (5.9) is $\sim 16\%$ and $\sim 10\%$, respectively. Eq. (5.9) can be systematically improved by including subleading WKB corrections. Notice that the same WKB formula (5.9) is valid not only at $h = 1/2$ but also for arbitrary spin $h$. In the latter case, it describes correctly the excited states with $|q_3^{1/3}| \gg |q_2^{1/2}|$.

According to (5.3), the quantized values of $q_3$ are parameterized by a pair of integers $\ell_1$ and $\ell_2$ which define the coordinates on the lattice shown in Figures 1 and 2. Eq. (5.3) provides...

\footnote{To save space, we truncated in the Table the last few digits in the obtained numerical values of the charges and the energy.}
Table 1: Comparison of the exact spectrum of \( q_3^{1/3} \) at \( \hbar = 1/2 \) with the approximate WKB expression (5.9). The last line defines the corresponding energy \( E_3(\nu_\hbar,0;\ell_1,\ell_2) \).

| \((\ell_1,\ell_2)\) | \((0,2)\) | \((2,2)\) | \((4,2)\) | \((6,2)\) | \((8,2)\) | \((10,2)\) |
|---------------------|--------|--------|--------|--------|--------|--------|
| \((q_3^{\text{exact}})^{1/3}\) | .590i | .358 + .621i | .749 + .649i | 1.150 + .664i | 1.551 + .672i | 1.951 + .676i |
| \((q_3^{\text{WKB}})^{1/3}\) | .684i | .395 + .684i | .790 + .684i | 1.186 + .684i | 1.581 + .684i | 1.976 + .684i |
| \(-E_3/4\) | -.2472 | -.6910 | -1.7080 | -2.5847 | -3.3073 | -3.9071 |

Figure 2: Comparison of the exact spectrum of quantized \( q_3^{1/3} \) at \( h = 1/2 \) (crosses) with the WKB expression (5.9) (circles).

In particular, the quasimomentum vanishes for the eigenstates with \( \ell_1 = 0 \pmod{3} \). It is easy to see from (5.9) that the corresponding \( q_3 \) take, in general, complex values. There are, however, special cases, like \( \ell_1 = 0 \) or \( \ell_2 = 0 \), when \( q_3 \) becomes, respectively, pure imaginary or real. The former values have been previously found in [14, 15].

We recall that the spectrum of \( q_3 \), shown in Figure 4, corresponds to the total spin \( h = 1/2 \), or equivalently \( n_\hbar = \nu_\hbar = 0 \) in Eq. (5.4). In general, quantized \( q_3 \) depend on integer \( n_\hbar \) and continuous \( \nu_\hbar \). For simplicity, we present here our results only at \( n_\hbar = 0 \), or equivalently \( h = 1/2 + i\nu_\hbar \). For \( n_\hbar = 1, 2, 3, \ldots \) the spectrum of \( q_3 \) exhibits a similar structure.

We find that quantized \( q_3 = q_3(\nu_\hbar,0;\ell_1,\ell_2) \) are continuous functions of \( \nu_\hbar \). For different integers \( \ell_1 \) and \( \ell_2 \), the functions \( q_3(\nu_\hbar,0;\ell_1,\ell_2) \) define an infinite set of trajectories in the three-dimensional space.

\(^9\text{For pure imaginary } q_3, \text{ the } n_\hbar - \text{dependence has been studied in [30].}\)
dimensional \((\nu_h, \text{Re}(q_3^{1/3}), \text{Im}(q_3^{1/3}))\) moduli space. The trajectories cross the hyperplane \(\nu_h = 0\) at the points shown in Figure 3 and go to infinity for \(\nu_h \to \pm \infty\). To illustrate the properties of these trajectories, three representatives, corresponding to \((\ell_1, \ell_2) = (0, 2), (2, 2)\) and \((4, 2)\), are shown in Figure 3. The quasimomentum \(\theta_3\) takes a constant value along each trajectory, \(\theta_3(\ell_1, \ell_2) = 0, 4\pi/3\) and \(2\pi/3\), respectively.

Let us now consider the energy spectrum at \(N = 3\). Since the energy is a function of the total spin \(h\) and the charge \(q_3\), each trajectory \(q_3 = q_3(\nu_h; \ell_1, \ell_2)\) shown in Figure 3 induces the corresponding trajectory for the energy, \(E_3 = E_3(\nu_h; \ell_1, \ell_2)\). Solving the quantization conditions (3.28) and applying (3.14) we obtain the energy spectrum shown in Figure 4.

We find that the energy is a continuous function of \(\nu_h\) along each \((\ell_1, \ell_2)\)-trajectory and it
approaches its minimal value, \( \min_{\nu_h} E_3(\nu_h; \ell_1, \ell_2) \), at \( \nu_h = 0 \), or equivalently \( h = 1/2 \). Examining the value of \( E_3(0; \ell_1, \ell_2) \) for different sets of integers \((\ell_1, \ell_2)\), we find that \( E_3(0; \ell_1, \ell_2) \) increases as one goes towards larger \( |q_3^{1/3}(0; \ell_1, \ell_2)| \) (see Table I). This implies that the ground state corresponds to the point(s) on the plane of quantized \( q_3^{1/3} \) (see Figure 4) closest to the origin. It is easy to see from Figure 4 that, in total, there are six such points, \((\ell_1, \ell_2) = (0, \pm 2), (\pm 3, \pm 1)\) and \((\mp 3, \pm 1)\). According to (5.12), their quasimomentum is equal to zero. Going over from \( q_3^{1/3} \) to \( q_3 \), we find that these six points define two nontrivial eigenstates, which have opposite values of \( q_3 \) and the same energy

\[
\begin{align*}
\imath q_3^{\text{ground}} = \pm 0.205258... , \quad &E_3^{\text{ground}} = .988678... \quad (5.13)
\end{align*}
\]

This implies that at \( N = 3 \) and \( q_3 \neq 0 \) the ground state is double degenerate. As we will demonstrate below, this property is rather general – the ground state is double degenerate for the systems with odd number of particles \( N \) and \( q_N \neq 0 \), but it is unique for even \( N \). Eq. (5.13) is in agreement with the results of the previous calculations [14, 15]. To verify that for \( q_3 \neq 0 \) the ground state occurs at \( n_h = 0 \), one has to compare (5.13) with the minimal energy in the sectors with higher \( n_h \). At \( n_h = 1, 2, 3 \) our results can be found in the first three columns of the Table 4 (see below). They indicate that for \( q_3 \neq 0 \) the minimal energy grows as the Lorentz spin of the system, \( n_h \), increases.

At \( N = 3 \) the ground state is located on the \((0,2)\)–trajectory at \( \nu_h = n_h = 0 \) as shown in Figure 4. We notice that close to \( \nu_h = 0 \) there is an accumulation of the energy levels. At small \( \nu_h \) the energy of excited states is described by a general expression (5.4) with the dispersion parameter \( \sigma_3 \) given below in the Table 4.

### 5.3. Quantum numbers of the \( N = 4 \) states

At \( N = 4 \) the spectrum of the model depends on two complex quantum numbers \( q_3 \) and \( q_4 \). Similar to the \( N = 3 \) case, we shall determine their spectrum by solving the quantization conditions (3.28) under the additional condition \( q_4 \neq 0 \). The only difference with the \( N = 3 \) case is that one has to increase the dimension of the mixing matrices, \( C^{(0)} \) and \( C^{(1)} \), and recalculate the transition matrices, \( \Omega \) and \( \overline{\Omega} \), as functions of \( q_3 \) and \( q_4 \). As before, to understand the structure of the spectrum at \( h = 1/2 \), it becomes convenient to deal with multi-valued complex variables \( q_4^{1/4} \). Then, each eigenstate will be represented on the complex \( q_4^{1/4} \)–plane by four different points.

At \( h = 1/2 \) the quantized values of \((q_3, q_4)\) are parameterized by four real numbers and, as a consequence, they do not admit a simple pictorial representation. The results of our calculations are presented in Figure 5. There, each point on the complex \( q_4^{1/4} \)–plane has additional complex coordinate defined by the quantized values of the charge \( q_3 \). The latter are not displayed for simplicity. It is convenient to separate the eigenstates into two sets according to their quasimomentum \( \theta_4 = (2\pi k)/4, k = 0, 2 \) and \( k = 1, 3 \) as shown in Figure 5. We notice that some points have very close values of \( q_4 \). Nevertheless, the charge \( q_3 \) and the energy \( E_4 \) corresponding to these points are different. Here is an example of a pair of such states at \( h = 1/2 \) and \( \theta_4 = 0: \ (q_3 = 0, q_4 = -2.185790, E_4 = 12.563898) \) and \((q_3 = 1.524585 i, q_4 = -2.195368, E_4 = 12.383790)\). Aside from this spurious degeneracy, the

\[\text{As we will show below, this property is rather general and it holds for the energy } E_N \text{ as a function on the } (N-2)\text{–dimensional space of the integrals of motion } (q_3, ..., q_N).\]
Figure 5: The spectrum of the integrals of motion \( q_4 \) at \( N = 4 \) and the total spin \( h = 1/2 \). The left and the right panels correspond to the eigenstates with the quasimomentum \( e^{i\theta_4} = \pm 1 \) and \( \pm i \), respectively.

The spectrum of quantized \( q_4 \) has many features in common with the spectrum of \( q_3 \) at \( N = 3 \) shown in Figure 1.

We notice that quantized \( q_4 \) are located close to the vertices of a square-like lattice. To verify this property we selected among all eigenstates of the \( N = 4 \) system only those with \( h = 1/2 \), \( q_3 = 0 \) and nonzero values of \( q_4 \) (see Figure 6). These states have the quasimomentum \( \theta_4 = 0 \) and they play an important role in our discussion as the ground state of the system has the same quantum numbers. The WKB analysis of the Baxter equation at \( N = 4 \) leads to the following expression for quantized \( q_4 \) at \( q_3 = 0 \) (see Eq. (5.19) below)

\[
[q_4^{WKB}(\ell_1, \ell_2)]^{1/4} = \Delta_{N=4} \cdot \left( \frac{\ell_1}{\sqrt{2}} + i \frac{\ell_2}{\sqrt{2}} \right),
\]

(5.14)

where the integers \( \ell_1 \) and \( \ell_2 \) are such that their sum \( \ell_1 + \ell_2 \) is even and the lattice spacing is

\[
\Delta_4 = \left[ \frac{4^{3/4}}{\pi} \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^4}} \right]^{-1} = \frac{\Gamma^2(3/4)}{2\sqrt{\pi}} = 0.423606...
\]

(5.15)

As before, the leading-order WKB formula (5.14) is valid only for \( |q_4^{1/4}| \gg |q_2^{1/2}| \). The comparison of (5.14) with the exact results for \( q_4 \) at \( h = 1/2 \) is shown in Figure 6 and Table 2 (see footnote 8). We find that quantized \( q_4^{1/4} \) occupy the whole complex plane except the interior of the disk of radius \( \Delta_4 \)

\[
|q_4^{1/4}| > \Delta_4,
\]

(5.16)

and the WKB formula (5.14) describes their spectrum with a good accuracy.

We recall that the points shown in Figure 6 describe the \( N = 4 \) eigenstates with \( h = 1/2 \), \( q_3 = 0 \) and the quasimomentum \( \theta_4 = 0 \). Similar to the \( N = 3 \) case, the spectrum of \( q_4 \) is parameterized by the pair of integers \( (\ell_1, \ell_2) \), Eq. (5.14). Still, as one can see from Figure 5, there exist the eigenstates which have the same quasimomentum, close value of \( q_4 \) but \( q_3 \neq 0 \). Similar phenomenon also occurs for other values of the quasimomentum. In order to distinguish these additional eigenstates, one has to introduce the second pair of integers \( (\ell_3, \ell_4) \). The dependence
Figure 6: The spectrum of quantized $q_4$ for $h = 1/2$ and $q_3 = 0$. The exact results are shown by crosses and the WKB predictions based on Eq. (5.14) are denoted by circles.

Table 2: Comparison of the exact spectrum of $q_4^{1/4}$ at $q_3 = 0$ and $h = 1/2$ with the approximate WKB expression (5.14). The last line defines the exact energy $E_4$.

| $(\ell_1, \ell_2)$ | $(2, 0)$ | $(2, 2)$ | $(3, 1)$ | $(4, 0)$ | $(3, 3)$ | $(4, 2)$ |
|-------------------|----------|----------|----------|----------|----------|----------|
| $(q_4^{\text{exact}})^{1/4}$ | .626     | .520 + .520 $i$ | .847 + .268 $i$ | 1.158     | .860 + .860 $i$ | 1.159 + .574 $i$ |
| $(q_4^{\text{WKB}})^{1/4}$ | .599     | .599 + .599 $i$ | .899 + .299 $i$ | 1.198     | .899 + .899 $i$ | 1.198 + .599 $i$ |
| $-E_4/4$ | 0.6742   | -1.3783  | -1.7919  | -2.8356  | -3.1410  | -3.3487  |

It has the quasimomentum $\theta_4 = 0$ and, in contrast with the $N = 3$ case, it is unique.

Going over from $\nu_h = 0$ to nonzero values of $\nu_h$, one finds that, similar to the $N = 3$ case (see Figure 3), the eigenvalues of the integrals of motion flow along the trajectories on the moduli space $(\nu_h, q_3, q_4)$. Namely, each point shown in Figure 5 creates its own trajectory labelled by the set of integers $\{\ell\}$. Among all trajectories the one with $(\ell_1, \ell_2) = (2, 0)$ plays the special role as it contains the ground state of the model. We find that $q_3 = \text{Im} q_4 = 0$ for arbitrary $\nu_h$ on the ground state trajectory, whereas $\text{Re}(q_4)$ and $E_4$ vary with $\nu_h$ as shown in Figure 7. Accumulation of the energy levels next to the ground state at $\nu_h = 0$ is described by Eq. (5.6) with the dispersion parameter $\sigma_4$ given below in the Table 3.
The dependence of the energy, $-E_4/4$, and the quantum number, $q_4/q_2$, with $q_2 = 1/4 + \nu h$, on the total spin $h = 1/2 + i\nu h$ along the ground state trajectory at $N = 4$.

Figure 7: The dependence of the energy, $-E_4/4$, and the quantum number, $q_4/q_2$, with $q_2 = 1/4 + \nu h$, on the total spin $h = 1/2 + i\nu h$ along the ground state trajectory at $N = 4$.

The ground state (5.17) has the Lorentz spin $n_h = 0$. In analogy with the $N = 3$ case, we calculated the minimal energy in the sectors with $n_h = 1, 2, 3$ (see the first four columns in the Table 3 below) and verified that it increases with $n_h$.

As we have seen before, the spectrum of quantized charges at $N = 3$ and $N = 4$ is described by the simple formulae, Eqs. (5.9) and (5.14). Their derivation is based on the WKB approach to the eigenproblem for the Baxter operator [31]. In this approach, $Q_{q,q}(x, \bar{x})$ is constructed as a quasiclassical wave function in the separated coordinates $\bar{x} = (x, \bar{x})$. The underlying classical dynamics is described by the spectral curve ("equal energy level" equation)

$$\Gamma_N : \quad y^2 = t_N^2(x) - 4x^{2N} = (4x^N + q_2 x^{N-2} + \ldots + q_N)(q_2 x^{N-2} + \ldots + q_N),$$

(5.18)

where $t_N(x)$ was defined in (2.8) and $y(x) = 2x^N \sinh p_x$ is related to the holomorphic part of the momentum of a particle in the separated coordinates, $p_x$. For arbitrary complex $x$, the equation (5.18) has two solutions for $y(x)$. As a consequence, $y(x)$ becomes a single-valued function on the Riemann surface corresponding to the complex curve $\Gamma_N$. This surface is constructed by gluing together two copies of the complex $x$–plane along the cuts $[\sigma_1, \sigma_2], \ldots, [\sigma_{2N-3}, \sigma_{2N-2}]$ running between the branching points $\sigma_j$ of the curve (5.18). The latter are defined as simple roots of the equation $t_N^2(\sigma_j) = 4\sigma_j^{2N}$. Their positions on the complex plane depend on the values of the integrals of motion $q_2, q_3, \ldots, q_N$. In general, the Riemann surface defined in this way has a genus $g = N - 2$, which depends on the number of reggeons, $N$. It is a sphere at $N = 2$, a torus at $N = 3$ and so on.

Let us define on $\Gamma_N$ the set of oriented closed $\alpha$– and $\beta$–cycles. The cycles $\alpha_k$ encircle the cuts $[\sigma_{2k-1}, \sigma_{2k}]$ with $k = 1, \ldots, N - 2$ and belong to the both sheets of $\Gamma_N$. The cycles $\beta_k$ run from the cut $[\sigma_{2N-3}, \sigma_{2N-2}]$ to $[\sigma_{2k-1}, \sigma_{2k}]$ on the upper sheet and, then, back on the lower sheet. Then, classical trajectories of a particle in the separated coordinates correspond to wrapping around $\alpha$– and $\beta$–cycles on the Riemann surface (5.18). Requiring $Q_{q,q}^{\text{WKB}}(x, \bar{x} = x^*)$ to be single-valued on the complex $x$–plane, one obtains the following WKB quantization conditions\footnote{We are most grateful to A. Gorsky for collaboration on this point.}

$$\text{Re} \oint_{\alpha_k} dS_0 = \pi \ell_{2k-1}, \quad \text{Re} \oint_{\beta_k} dS_0 = \pi \ell_{2k}, \quad (k = 1, \ldots, N - 2)$$

(5.19)
where \( \ell_k \) are integer and \( dS_0 \) is the “action” differential on the curve \([5.18]\) \([31]\)

\[
dS_0 = dx p_x \cong \frac{N t_N(x) - x t_N'(x)}{\sqrt{t_N^2(x) - 4x^{2N}}} \, dx. \quad (5.20)
\]

Solving \((5.19)\), one can find the explicit expressions for the integrals of motion, Eq. \((5.4)\). At \( N = 3 \) and \( N = 4 \), for \( |q_N^{1/N}| \gg q_2^{1/2} \), one arrives at \((5.9)\) and \((5.14)\). The general analysis of \((5.19)\) turns out to be rather involved and it will be presented in the forthcoming publication.

### 5.4. Quantum numbers of the states with higher \( N \)

In this Section we will describe the results of our calculations of the spectrum of the model for higher \( N \) and \( q_N \neq 0 \). As we have seen in the previous Sections, the structure of the spectrum gets more complicated as one increases \( N \). Therefore, instead of presenting a detailed description of the whole spectrum, as was done before for \( N = 3 \) and \( N = 4 \), we will restrict our analysis to the properties of the ground state trajectory only.

Solving the quantization conditions \((3.28)\) for \( N \geq 5 \), we found that the ground state trajectories have different properties for even and odd number of particles. Namely, for even \( N \) the integrals of motion \( q_k \) with odd indices \( k \) vanish and for even \( k \) they take pure real values

\[
q_3 = q_5 = ... = 0, \quad \text{Im } q_4 = \text{Im } q_6 = ... = 0, \quad [N = \text{even}] \quad (5.21)
\]

For odd \( N \), the integrals of motion \( q_k \) take nonvanishing values. They are pure imaginary for odd \( k \) and real otherwise

\[
\text{Re } q_3 = \text{Re } q_5 = ... = 0, \quad \text{Im } q_4 = \text{Im } q_6 = ... = 0, \quad [N = \text{odd}] \quad (5.22)
\]

We recall that the total spin of the system on the ground state trajectory is equal to \( h = 1/2 + i \nu_h \), so that the quantized charges \( q_k \) and the energy \( E_N \) are functions of \( \nu_h \). At \( N = 3 \) and \( N = 4 \) the \( \nu_h \)-dependence of \( q_N \) and \( E_N \) is shown in Figures \([3] [4] [7]\), respectively.

It follows from Eqs. \((5.21)\) and \((5.22)\) that for even \( N \) the ground state trajectory is invariant under the symmetry transformations \((5.7)\), whereas for odd \( N \) it is mapped into another trajectory with the different quantum numbers \((-1)^k q_k \) but the same energy \( E_N \). This implies, that the ground state of the system is double degenerate for odd number of particles \( N \) and it is unique for even \( N \). The degeneracy is related to the properties of the system under mirror permutations of particles \([3] [4] [43]\).

The results of our calculations of the ground state for the system with the number of particles \( N \leq 8 \) are summarized in the Table \([3] [4] [7]\) and Figure \([8]\). We recall that for odd \( N \) there exists the second eigenstate with the same energy \( E_N \) and the charges \((-1)^k q_k \).

The energy of the ground state \( E_N \), as a function of the number of particles \( N \), has a number of interesting properties. \( E_N \) changes a sign as one increases the number of particles – it is negative for even \( N \) and positive for odd \( N \). Our results also indicate that the absolute value of the energy decreases with \( N \), \( |E_N| \sim 1/N \) for \( N \to \infty \) (see Figure \([8]\)). For large even and odd number of particles \( N \) it approaches the same asymptotic value \( E_{2\infty} = E_{2\infty+1} = 0 \).

These properties are unique (and quite unexpected) features of noncompact \( SL(2, \mathbb{C}) \) Heisenberg spin magnets. It is instructive to compare them with similar properties of the \( SL(2, \mathbb{R}) \) Heisenberg magnets studied earlier in \([32] [33] [34] [35]\) in the relation with the Evolution Equations for high-twist operators in high-energy QCD. There, the number of sites of the spin magnet

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Table 3: The exact quantum numbers, \( q \), and the energy, \( E_N \), of the ground state of \( N \) reggeized gluons in multi-colour QCD. The dispersion parameter, \( \sigma_N \), defines the energy of the lowest excited states, Eq. (5.6).

| \( N \) | \( q_3 \)   | \( q_4 \)  | \( q_5 \) | \( q_6 \) | \( q_7 \) | \( q_8 \) | \( -E_N/4 \) | \( \sigma_N/4 \) |
|-------|-------------|-------------|-------------|-------------|-------------|-------------|----------------|----------------|
| 2     |             |             |             |             |             |             | 2.772589       | 16.8288        |
| 3     | .205258     |             |             |             |             |             | -2.47170       | .9082          |
| 4     | 0           | .153589     |             |             |             |             | .674160        | 1.3176         |
| 5     | .267682     | .039452     | .060243     |             |             |             | -.127516       | .4928          |
| 6     | 0           | .281825     | 0           | .070488     |             |             | .394582        | .5644          |
| 7     | .313072     | .070993     | .128455     | .008494     | .019502     |             | -.081410       | .3194          |
| 8     | 0           | .391171     | 0           | .179077     | 0           | .030428      | .280987        | .3409          |

is equal to the twist of the operators and the ground state energy defines the minimal anomalous dimension of these operators. Roughly speaking, the Hamiltonian of the \( SL(2, \mathbb{R}) \) magnet is given by the holomorphic part of the reggeon Hamiltonian (1.5) with the only difference that the holomorphic \( z \)–coordinates take real values and the eigenstates \( \Psi(z_1, ..., z_N) \) are polynomials in \( z \). One finds \([33, 32, 34, 35]\), that for the \( SL(2, \mathbb{R}) \) spin magnet all charges \( q_3, ..., q_N \) take real quantized values, the ground state energy \( E_N \) is positive and it monotonically increases with \( N \). Thus, the properties of the \( SL(2, \mathbb{C}) \) and \( SL(2, \mathbb{R}) \) Heisenberg spin magnets turn out to be completely different.

Figure 8: The dependence of the ground state energy, \(-E_N/4\), on the number of particles \( N \). The exact values of the energy are denoted by crosses. The upper and the lower dashed curves stand for the functions \( 1.8402/(N - 1.3143) \) and \(-2.0594/(N - 1.0877)\), respectively.

The spectrum of the \( SL(2, \mathbb{C}) \) magnet is gapless for arbitrary \( N \). Accumulation of the energy levels next to the ground state is described by (5.6) with the dispersion parameter \( \sigma_N \) given in the Table 3. The states belonging to the ground state trajectory have the total \( SL(2, \mathbb{C}) \) spin \( h = 1/2 + i\nu_h \) and their quantum numbers \( q_3, ..., q_N \) satisfy (5.21) and (5.22) for even and odd
$N$, respectively. The explicit expressions for $q_k$ can be obtained in the WKB approach from Eq. (5.19).

Figure 9: The dependence of the energy $-E_N(\nu_h)/4$ and the “highest” integral of motion $|q_N|/q_2$ with $q_2 = (1/4 + \nu_h^2)$ on the total spin $h = 1/2 + i\nu_h$ along the ground state trajectory for different number of particles $2 \leq N \leq 8$. At large $\nu_h$, $-E_8 > \ldots > -E_3 > -E_2$ on the left panel and $|q_8/q_2| < \ldots < |q_2/q_2|$ on the right panel.

Another unusual feature of the $SL(2, \mathbb{C})$ magnet can be revealed by examining the dependence $E_N = E_N(\nu_h)$ on the total $SL(2, \mathbb{C})$ spin $h = 1/2 + i\nu_h$ along the ground state trajectory for different number of particles $N$. For $2 \leq N \leq 8$ this dependence is shown in Figure 8. We notice that the flow of the energy $E_N$ with $\nu_h$ is such that the hierarchy of the energy levels at $\nu_h = 0$ and large $\nu_h$ is completely different. At $\nu_h = 0$ the values of $E_N(0)$ coincide with those depicted by crosses in Figure 8. For large $\nu_h$ the system approaches a quasiclassical regime \[34, 31\] in which the energy $E_N(\nu_h)$ and the quantum numbers $q_N$ have a universal scaling behaviour

$$E_N(\nu_h) \sim 4 \ln |q_N|, \quad |q_N| \sim C^N \nu_h^2,$$

(5.23)

with $C \approx 0.52$. As can be seen from the right panel in Figures 7 and 8, this regime starts already at $\nu_h \approx 2$. These results suggest that the ground state of the $SL(2, \mathbb{C})$ magnet has the properties of quantum antiferromagnet, whereas the excited states are ferromagnetic.

5.5. Descendent states

Until now, we have excluded from consideration the eigenstates with $q_N = 0$. Let us now demonstrate that these states can be expressed in terms of the $(N - 1)$–particle eigenstates. That is why we shall refer to them as descendent states.

To see this, one examines the Baxter equation (2.7) for the $SL(2, \mathbb{C})$ spins $s = 0$ and $\bar{s} = 1$. One finds from (2.8) that for $q_N = \bar{q}_N = 0$ the transfer matrix factorizes as $t_N(u) = ut_{N-1}(u)$ and the holomorphic Baxter equation (2.7) at $s = 0$ takes the form

$$t_{N-1}(u) Q_N(u, \bar{u}) = u^{N-1} [Q_N(u + i, \bar{u}) + Q_N(u - i, \bar{u})].$$

(5.24)

Here, we introduced a subscript to indicate that $Q_N(u, \bar{u})$ corresponds to the $N$–particle system. Eq. (5.24) coincides with the Baxter equation for the eigenvalue of the Baxter operator for the
system of \((N-1)\)-particles, \(Q_{N-1}(u, \bar{u})\). The same happens in the antiholomorphic sector. One verifies that for \(s = 1\) the function \(\bar{u} Q_N(u, \bar{u})\) satisfies the same antiholomorphic Baxter equation as \(Q_{N-1}(u, \bar{u})\). This suggests that up to a unessential normalization factor

\[ Q_N^{(qN=0)}(u, \bar{u}) = Q_{N-1}(u, \bar{u})/\bar{u}. \tag{5.25} \]

However, in order to identify the r.h.s. of this relation as the eigenvalue of the Baxter operator for \(q_N = \tilde{q}_N = 0\), one has to verify that \(Q_N(u, \bar{u})\) has appropriate pole structure, Eqs. (2.12) and (2.13). By the construction, the function \(Q_{N-1}(u, \bar{u})\) possesses correct analytical properties. \(Q_N(u, \bar{u})\) inherits \((N-1)\)th order poles of the function \(Q_{N-1}(u, \bar{u})\), and, in addition, it acquires a spurious pole at \(u = \bar{u} = 0\). Therefore, Eq. (5.23) defines the eigenvalue of the Baxter operator only if the residue at this spurious pole vanishes, or equivalently \(Q_{N-1}(0, 0) = 0\). As we will show in a moment, this condition is satisfied provided that \(q_{N-1} \neq 0\) and the quasimomentum of the \((N-1)\)-particle state defined by the function \(Q_{N-1}(u, \bar{u})\) is equal to \(e^{i\theta_{N-1}} = (-1)^N\).

To evaluate \(Q_{N-1}(0, 0)\), we examine of the both sides of (5.24) at \(u = \bar{u} = \varepsilon\) as \(\varepsilon \to 0\). Taking into account that \(Q_{N-1}(u, \bar{u})\) is finite at \(u = \bar{u} = 0\) and it has \((N-1)\)th order poles at \((u = \pm i, \bar{u} = 0)\), Eq. (2.12),

\[ Q_{N-1}(\pm i, \varepsilon, \varepsilon) = \frac{R_{N-1}^\pm (q, \tilde{q})}{\varepsilon^{N-1}} \left[ 1 + O(\varepsilon) \right], \tag{5.26} \]

we find from (5.24) that

\[ q_{N-1} Q_{N-1}(0, 0) = R_{N-1}^+(q, \tilde{q}) + R_{N-1}^-(q, \tilde{q}). \tag{5.27} \]

The value of the residue functions \(R_{N-1}^\pm(q, \tilde{q})\) depends on the normalization of the Baxter operator, while their ratio is fixed by the quasimomentum of the state. Indeed, it follows from (2.13) and (2.17) that \(R_{N-1}^+(q, \tilde{q})/R_{N-1}^-(q, \tilde{q}) = (-1)^{N-1} e^{i\theta_{N-1}(q, \tilde{q})}\). Combining this relation together with (5.27), we obtain

\[ q_{N-1} Q_{N-1}(0, 0) = \left[ 1 - (-1)^N e^{-i\theta_{N-1}} \right] R_{N-1}^+(q, \tilde{q}). \tag{5.28} \]

This relation is rather general and it holds for all eigenstates of the \((N-1)\)-particle system with the single-particle \(SL(2, \mathbb{C})\) spin \(s = 0\) and \(\bar{s} = 1\). In particular, replacing \(N \to N + 1\) and putting \(q_N = 0\), we deduce from (5.28) that the descendent states have the quasimomentum

\[ e^{i\theta_N} \bigg|_{q_N=0} = (-1)^{N+1}. \tag{5.29} \]

Going back to (5.28), one finds that for \(q_{N-1} \neq 0\) the condition \(Q_{N-1}(0, 0) = 0\) is satisfied provided that the quasimomentum of the “ancestor” \((N-1)\)-particle state equals \(e^{i\theta_{N-1}} = (-1)^N\).

Thus, starting from arbitrary \((N-1)\)-particle eigenstate with the quasimomentum \(e^{i\theta_{N-1}} = (-1)^N\) and applying (5.23), one can obtain the \(N\)-particle solution to the Baxter equation with \(q_N = 0\). Remarkably enough, the \(N\)-particle state defined in this way has exactly the same energy as its ancestor \((N-1)\)-particle state

\[ E_N(q_2, \ldots, q_{N-1}, q_N = 0) = E_{N-1}(q_2, \ldots, q_{N-1}). \tag{5.30} \]
Indeed, as follows from (2.18) and (2.12), at \( s = 0 \) and \( \bar{s} = 1 \) the energy \( E_N \) is related to the behaviour of \( u^N Q_N(u + i, u) \) around \( u = 0 \). According to (5.25), at \( q_N = 0 \) this function coincides with \( u^{N-1} Q_{N-1}(u + i, u) \).

We conclude that the spectrum of the \((N-1)\)-particle system contains (an infinite) number of descendent \( q_N = 0 \) states, which have the same energy as \((N-1)\)-particle states. An example of such states at \( N = 3 \) and \( N = 4 \) can be found in the Tables 4 and 5. To establish the correspondence between two different sets of the states, it is convenient to define a new quantum number, \( \vartheta_N = (-1)^{N+1} e^{-i\vartheta N} \), so that \( \vartheta_N^N = 1 \). It follows from our analysis that there is the one-to-one correspondence between the \((N-1)\)-particle eigenstates with \( \vartheta_{N-1} = 1 \) and \( q_{N-1} \neq 0 \) and the \( N \)-particle eigenstates with \( \vartheta_N = 1 \) and \( q_N = 0 \).\footnote{We would like to stress that the charges \( q_k \) (with \( k = 2, \ldots, N \)) are continuous functions of the parameter \( \nu_h \) defining the total spin \( h = (1 + \nu_h)/2 + \nu_h \). The statement \( q_N = 0 \) means that \( q_N(\nu_h) \) vanishes for all \( \nu_h \), that is along the whole trajectory on the moduli space. If \( q_N \) vanishes only for some \( \nu_h \), the corresponding eigenstate is \textit{not} descendant. Similarly, \( q_{N-1} \neq 0 \) means that \( q_{N-1}(\nu_h) \) does not vanish except for some distinct \( \nu_h \).} This agrees with the results found previously in [37]. One can argue that all \( N \)-particle eigenfunctions with the highest charge \( q_N = 0 \) can be obtained in this way. We demonstrated that the \((N-1)\)-particle state with \( \vartheta_{N-1} = 1 \) and \( q_{N-1} \neq 0 \) can be transformed into the \( N \)-particle state with \( q_N = 0 \). Is it possible to continue this process and construct the \((N+1)\)-particle state with \( q_{N-1} \neq 0 \) and \( q_N = q_{N+1} = 0 \)? In this case, applying (5.23) one gets \( Q_{N+1}(u, \bar{u}) = Q_{N-1}(u, \bar{u})/u^2 \). To compensate the spurious pole at \( u = \bar{u} = 0 \) one has to require that \( Q_{N+1}(0, 0) = 0 \), or equivalently \( Q_{N-1}(\epsilon, \bar{\epsilon}) \sim \epsilon^2 \). Substituting this relation into the Baxter equation (5.24) for \( u = \bar{u} = \epsilon \) and comparing the small \( \epsilon \)-behaviour of the both sides, one obtains the relation between the residue functions \( E^\pm(q, \bar{q}) \), Eq. (2.12), \( E^+_N(q, \bar{q}) = E^-_{N-1}(q, \bar{q}) \). Together with (2.17) and (2.18), this leads to \( E_{N-1}(q_2, \ldots, q_{N-1}) = 0 \). Thus, a positive answer to the above question would imply that the energy of the \((N-1)\)-particle state should vanish for arbitrary values of \( \nu_h \), that is along the whole trajectory on the moduli space. We do not find such states in the spectrum of the model.

This implies that if one defines a linear operator \( \Delta \) that maps the subspace \( V^{q_{N-1}=1}_{\vartheta_{N-1}}(\vartheta_N=1) \) of the eigenstates of the \((N-1)\)-particle system with \( q_{N-1} \neq 0 \) and \( \vartheta_{N-1} = 1 \) into the \( N \)-particle states with \( q_N = 0 \) and \( \vartheta_N = 1 \),

\[
\Delta : \quad V^{q_{N-1}=1}_{\vartheta_{N-1}}(\vartheta_N=1) \to V^{q_{N-1}=1}_{\vartheta_{N-1}}(\vartheta_N=1),
\]

then this operator is nilpotent \footnote{We would like to stress that the charges \( q_k \) (with \( k = 2, \ldots, N \)) are continuous functions of the parameter \( \nu_h \) defining the total spin \( h = (1 + \nu_h)/2 + \nu_h \). The statement \( q_N = 0 \) means that \( q_N(\nu_h) \) vanishes for all \( \nu_h \), that is along the whole trajectory on the moduli space. If \( q_N \) vanishes only for some \( \nu_h \), the corresponding eigenstate is \textit{not} descendant. Similarly, \( q_{N-1} \neq 0 \) means that \( q_{N-1}(\nu_h) \) does not vanish except for some distinct \( \nu_h \).}. In addition, it has the following properties

\[
\Delta^2 = 0, \quad \Delta \cdot \mathcal{H}_{N-1} = \mathcal{H}_N \cdot \Delta, \quad \Delta \cdot \mathcal{P}_{N-1} = -\mathcal{P}_N \cdot \Delta = (-1)^N \Delta,
\]

where the last relation follows from \( \vartheta_N = \mathcal{P}_N (-1)^{N+1} \) with \( \mathcal{P}_N \) being the generator of cyclic permutations of \( N \) particles, Eq. (2.3). Then, one can show that the wave function of the \( N \)-particle state with \( q_N = 0 \) is related to its “ancestor” \((N-1)\)-particle state as

\[
\Psi^{(q_N=0)}_N(z_1, \ldots, z_N) = \Delta \cdot \Psi_{N-1}(\tilde{z}_1, \ldots, \tilde{z}_{N-1}) = N! \Pi_N \left[ \frac{1}{z_{N,1}} + \frac{1}{\bar{z}_{N-1,N}} \right] \Pi_{N-1} \Psi_{N-1}(\tilde{z}_1, \ldots, \tilde{z}_{N-1}),
\]

with \( \tilde{z}_h = \tilde{z}_h - \tilde{z}_h \) and \( \Pi_N = \sum_{n=0}^{N-1} (-1)^{N+1} \mathcal{P}_n \mathcal{P}_N \) being the projector onto eigenstates with the quasimomentum (5.29), \( \mathcal{P}_N \mathcal{P}_N = (-1)^{N+1} \mathcal{P}_N \). The explicit expressions at \( N = 3 \) and \( N = 4 \)
Generalization to arbitrary look as follows

\[
\Psi_3^{(q_3=0)} = \frac{\bar{z}_{12}}{z_{23} z_{31}} \Psi_2(z_1, z_2) + \frac{\bar{z}_{23}}{z_{31} z_{12}} \Psi_2(z_2, z_3) + \frac{\bar{z}_{31}}{z_{12} z_{23}} \Psi_2(z_3, z_1),
\]

\[
\Psi_4^{(q_4=0)} = \frac{\bar{z}_{31}}{z_{34} z_{41}} \Psi_3(z_1, z_2, z_3) - \frac{\bar{z}_{24}}{z_{12} z_{41}} \Psi_3(z_2, z_3, z_4) + \frac{\bar{z}_{13}}{z_{12} z_{23}} \Psi_3(z_3, z_4, z_1) - \frac{\bar{z}_{24}}{z_{23} z_{34}} \Psi_3(z_4, z_1, z_2),
\]

(5.34)

where the states \(\Psi_2\) and \(\Psi_3\) have the quasimomentum \(e^{i\theta_2} = -1\) and \(e^{i\theta_3} = 1\), respectively. According to (3.25), \(e^{i\theta_2} = (-1)^{n_h}\) and, therefore, the eigenstate \(\Psi_3^{(q_3=0)}\) can be defined only for odd Lorentz spins \(n_h\). For \(N \geq 4\) the spin \(n_h\) of the state \(\Psi_N^{(q_N=0)}\) can be arbitrary integer.

Let us transform the descendant states (5.33) into the momentum space using the relation

\[
k_1...k_N \cdot \hat{\Psi}_N(k_1, ..., k_N) = \int \prod_{k=1}^{N} d^2 z_k e^{i\bar{z}_k \cdot \bar{k}_k} \Psi_N(z_1, ..., z_N),
\]

(5.35)

where \(k = (k_x + i k_y)/2\) is holomorphic component of \(\bar{k} = (k_x, k_y)\) and the additional factor in the l.h.s. was introduced for later convenience. One finds from (5.34) after some algebra that up to a normalization factor

\[
\hat{\Psi}_3^{(q_3=0)} = \frac{(\bar{k}_3 + \bar{k}_1)^2}{k_3^2 k_1^2} \hat{\Psi}_2(k_3 + \bar{k}_1, k_2) + \frac{(\bar{k}_1 + \bar{k}_2)^2}{k_1^2 k_2^2} \hat{\Psi}_2(k_1 + \bar{k}_2, k_3) + \frac{(\bar{k}_2 + \bar{k}_3)^2}{k_2^2 k_3^2} \hat{\Psi}_2(k_2 + \bar{k}_3, k_1),
\]

\[
\hat{\Psi}_4^{(q_4=0)} = \frac{(\bar{k}_4 + \bar{k}_1)^2}{k_4^2 k_1^2} \hat{\Psi}_3(k_4 + \bar{k}_1, k_2, k_3) - \frac{(\bar{k}_3 + \bar{k}_4)^2}{k_3^2 k_4^2} \hat{\Psi}_3(k_3 + \bar{k}_4, \bar{k}_1, k_2) + \frac{(\bar{k}_2 + \bar{k}_3)^2}{k_2^2 k_3^2} \hat{\Psi}_3(k_2 + \bar{k}_3, k_4, \bar{k}_1) - \frac{(\bar{k}_1 + \bar{k}_2)^2}{k_1^2 k_2^2} \hat{\Psi}_3(k_1 + \bar{k}_2, k_3, k_4).
\]

(5.36)

Generalization to arbitrary \(N\) is straightforward. The expression for \(\hat{\Psi}_N^{(q_N=0)}\) coincides with the \(q_N = 0\) solution found at \(N = 3\) in Ref. [38] and for arbitrary \(N\) in Ref. [37] by different methods. One can verify Eq. (5.33) in a number of different ways. For instance, applying the operators (1.4) to the both sides of (5.33) one finds that the state \(\Psi_N^{(q_N=0)}\) is annihilated by the operator \(q_N\) and has the same spectrum \(q_k\) (with \(k = 2, ..., N - 1\)) as the state \(\Psi_{N-1}\) entering the r.h.s. of (5.33). It is more tedious to verify that (5.33) is in agreement with (5.25). To show this, one examines the action of the Baxter operator on the state (5.33), \(Q(u, \bar{u})\Psi_N^{(q_N=0)}(z_1, ..., z_N)\), replaces the \(Q\)-operator by its integral representation and uses the Feynman diagram technique to calculate the emerging two-dimensional integrals (see Ref. [3] for details). Finally, one can check by explicit calculation that \(\mathcal{P}_N \Psi_N^{(q_N=0)} = (-1)^{N+1} \Psi_N^{(q_N=0)}\) and \(\Delta \cdot \Psi_N^{(q_N=0)} = 0\), in agreement with Eqs. (5.23) and (5.32), respectively.

Using the results of the previous Sections, one can apply Eqs. (5.30) and (5.33) and reconstruct the spectrum of the descendant states for arbitrary \(N\). As before, we shall identify among these states the one with the minimal energy, \(E_N^{(\text{min})}(q_N = 0)\). In virtue of (5.30), this amounts to finding the minimal energy in the spectrum of the \((N - 1)\)-particle system with \(q_{N-1} \neq 0\) and the quasimomentum \(e^{i\theta_{N-1}} = (-1)^N\). Obviously, it can not be smaller than the energy of the ground state, \(E_N^{\text{ground}}\), whose quasimomentum is \(\theta_N^{\text{ground}} = 0\) (see Table 3). The results of our calculations at \(N = 3\) and \(N = 4\) are given in the Tables 3 and 4, respectively. As was already
Table 4: The dependence of the minimal energy on the Lorentz spin $n_h$ at $N = 3$. The total $SL(2, \mathbb{C})$ spin equals $h = (1 + n_h)/2$ for all states except $n_h = 2$ when $h = 3/2 + 17.6i$. Integer $\ell$ defines the quasimomentum, $\theta_3 = 2\pi \ell/3$. The subscript (d) refers to the descendant states (see Section 5.5). The states with $q_3 \neq 0$ are degenerate with respect to (5.7).

| $n_h$ | $q_3$   | $E_3$   | $\ell$ | $E_{3,d}$ | $\ell_d$ |
|-------|---------|---------|--------|-----------|----------|
| 0     | 0.2052i | 0.9884  | 0      | -         | -        |
| 1     | 0.3315  | 1.5368  | 1      | 0         | 0        |
| 2     | -0.4017 + 0.4201i | 4.6077 | 1      | -         | -        |
| 3     | 1.1766i | 6.9592  | 0      | 8         | 0        |

Table 5: The dependence of the minimal energy on the Lorentz spin $n_h$ at $N = 4$. The total $SL(2, \mathbb{C})$ spin equals $h = (1 + n_h)/2$. The last three columns correspond to the descendant states. The quasimomentum is defined as $\theta_4 = (2\pi \ell)/4$. The states are degenerate with respect to (5.7).

| $n_h$ | $q_3$   | $q_4$   | $E_4$   | $\ell$ | $q_{3,d}$ | $E_{4,d}$ | $\ell_d$ |
|-------|---------|---------|---------|--------|-----------|-----------|----------|
| 0     | 0       | 0.1535  | -2.6964 | 0      | 0.2052i   | 0.9884    | 2        |
| 1     | 0       | 0       | 0       | 2      | 1.3659    | 8.1080    | 2        |
| 2     | 0       | -0.2869 | 2.6268  | 0      | 1.0236    | 6.8888    | 2        |
| 3     | -0.6951 | -0.6337 | 5.8836  | 1      | 1.1766i   | 6.9592    | 2        |

mentioned, at $N = 3$ the descendant states can be constructed only for odd $n_h$. For $n_h = 1$ and $n_h = 3$ they are descendants of the $N = 2$ states with the energy (3.36). At $N = 4$ the $q_4 = 0$ states are descendants of the $N = 3$ states with the quasimomentum $e^{i\theta_4} = 1$. Notice that the $N = 4$ state with $n_h = 1$ and $q_3 = q_4 = 0$ is not descendant since $q_4 \neq 0$ for $\nu_h \neq 0$ (see footnote [12]).

In general, for even $N$ one has $e^{i\theta_{N-1}} = 1$ and, therefore, the eigenstate with the minimal energy, $\Psi^{(q_N=0)}_N$, is a descendant of the ground state, $\Psi^{\text{ground}}_{N-1}$. As follows from our results (see Table 3 and Figure 8), its energy is positive, $E^{\text{min}}_N(q_N = 0) = E^{\text{ground}}_{N-1} > 0$. For odd $N$ one has $e^{i\theta_{N-1}} = -1$ and, therefore, this state can not be a descendant of the ground state. In this case, the minimal value of the energy is given by $E^{\text{min}}_N(q_N = 0) = 0$ and the corresponding integrals of motion are equal to $q_2 = q_3 = \ldots = q_{N-1} = 0$. This eigenstate is located on the trajectory on the moduli space at $h = 1 + i\nu_h$ and $\nu_h = 0$. Going over to $\nu_h \neq 0$ we find that $q_2, q_4, \ldots, q_{N-1} \neq 0$ while $q_3 = q_5 = \ldots = q_{N-2} = 0$ for arbitrary $\nu_h$. The energy $E_N(q_N = 0)$ is a continuous function of $\nu_h$ along this trajectory and it takes its minimal value, $E^{\text{min}}_N(q_N = 0) = 0$, at $\nu_h = 0$. We notice that it is smaller than the energy of the ground state on the subspace with $q_N \neq 0$.

At $q_2 = \ldots = q_N = 0$ the eigenvalue of the Baxter $\mathcal{Q}$–operator can be easily found from the integral representation (3.1). Solving the differential equations (3.3), one can construct a single-valued function $Q(z, \bar{z})$ as

$$Q^{(q=0)}(z, \bar{z}) = \frac{z}{(1-z)^2} \left[ c_0 \ln^{N-1}(z\bar{z}) + c_1 \ln^{N-2}(z\bar{z}) + \ldots + c_{N-1} \right] ,$$

with $c_i$ being arbitrary coefficients. To fix the coefficients, one substitutes (5.37) into (3.1) and compares the asymptotics of the resulting expression for $Q(u, \bar{u})$ at large $u$ and $\bar{u}$ with Eq. (2.13)
for $h = 1$, $s = 0$ and $\bar{s} = 1$. One finds that $c_1 = ... = c_{N-1} = 0$ and up to a normalization factor

$$Q_N^{(q=0)}(u, \bar{u}) = \frac{u - \bar{u}}{u^N} = -\frac{i n}{u^N},$$

(5.38)

with integer $n$ defined in (2.4). In this expression, the numerator compensates a spurious pole at $u = \bar{u} = 0$. Substituting (5.38) into (2.4) one calculates the corresponding energy, $E_N(q = 0) = 0$.

6. Summary

In this paper, we have continued the study of noncompact Heisenberg $SL(2, \mathbb{C})$ spin magnets initiated in [3]. Having solved this model, we obtained for the first time a complete description of the spectrum of the multi-reggeon compound states in QCD at large $N_c$.

From point of view of integrable models, the results presented in this paper provide an exact solution of the spectral problem for completely integrable quantum mechanical model of $N$ interacting spinning particles in two-dimensional space. A unique feature of this model, leading to many unusual properties of the energy spectrum, is that its quantum space is infinite-dimensional for finite $N$ and conventional methods, like the Algebraic Bethe Ansatz, are not applicable. To overcome this problem, we applied the method of the Baxter $Q-$operator developed in application to the $SL(2, \mathbb{C})$ spin magnets in [3]. Solving the Baxter equations, we were able to find the exact expressions for the eigenvalues of the $Q-$operator. They allowed us to establish the quantization conditions for the integrals of motion and, finally, reconstruct the spectrum of the model.

From point of view of high-energy QCD, we calculated the spectrum of the colour-singlet compound states built from $N$ reggeized gluons for $N \leq 8$ in the multi-colour limit, $N_c \to \infty$. The obtained expressions allowed us to reveal the general properties of the spectrum for arbitrary $N$. Our analysis was based on the identification of the $N-$reggeized gluon states in multi-colour QCD as the ground states for the noncompact Heisenberg magnet of the length $N$ and the $SL(2, \mathbb{C})$ spins ($s = 0, \bar{s} = 1$). The identification however is not straightforward. The contribution of the $N-$reggeon states to the scattering amplitude (1.1) takes the form $\mathcal{A} = is \sum_N (i\bar{a}_s)^N \mathcal{A}_N$ with

$$\mathcal{A}_N(s, t) = \int d^2 z_0 e^{i\bar{z}_0 \bar{\rho}} \langle \Phi_{a_0} (\bar{z}_0) | e^{-Y N^{(QCD)}_{\mathcal{H}}} \left( \frac{\partial}{\partial \bar{z}_1} ... \frac{\partial}{\partial \bar{z}_N} \right)^{-1} \Phi_{b(0)} \rangle,$$

(6.1)

where $\partial_k \equiv \partial/\partial \bar{z}_k$, the rapidity $Y = \ln s$ plays the role of the “evolution time”. The wave functions $|\Phi_{a(b)} (\bar{z})\rangle \equiv \Phi_{a(b)} (\bar{z}_1 - \bar{z}_0, ..., \bar{z}_N - \bar{z}_0)$ describe the coupling of $N-$gluons to the scattered particles. The $\bar{z}_0-$integration fixes the momentum transfer, $t = -\bar{p}^2$. The operators $1/\partial_k^2$ stand for two-dimensional transverse propagators of $N-$gluons and the scalar product is taken with respect to the $SL(2, \mathbb{C})$ scalar product (1.8). The Hamiltonian $\mathcal{H}_{N}^{(QCD)} = H_{12}^{\text{BFKL}} + ... + H_{N,1}^{\text{BFKL}}$ describes the interaction between $N$ reggeized gluons in multi-colour QCD and it is given by the sum of the BFKL kernels [3, 7]. Notice that it is different from the Hamiltonian of the magnet, Eq. (1.3).

The Hamiltonians $\mathcal{H}_{N}^{(QCD)}$ and $\mathcal{H}_{N}$ act on different Hilbert spaces. For the magnet, it coincides with the representation space of the principal series of the $SL(2, \mathbb{C})$ endowed with the scalar product (1.8). For the QCD Hamiltonian, the choice of the scalar product is dictated by physical requirements that the wave functions $\Phi_{a(b)}$ have to be normalizable and the Hamiltonian $\mathcal{H}_{N}^{(QCD)}$ has to be bounded from below. These conditions are satisfied if one normalizes the eigenstates
of \( \mathcal{H}_N^{(QCD)} \) as \[ \| \Psi_N^{(QCD)} \|^2 = \int \prod_{k=1}^N d^2 z_k \left| \partial_1 ... \partial_N \Psi_N^{(QCD)}(\vec{z}) \right|^2. \] (6.2)

Under this choice of the normalization condition, \( \mathcal{H}_N^{(QCD)} \) is related to the Hamiltonian of the \( SL(2, \mathbb{C}) \) magnet of the spin \( (s = 0, \bar{s} = 1) \) as \[ \mathcal{H}_N^{(QCD)} = \frac{\tilde{\alpha}_s}{4} (\partial_1 ... \partial_N)^{-1} \mathcal{H}_N (\partial_1 ... \partial_N), \] (6.3)

where \( \tilde{\partial}_k = \partial/\partial z_k \). One can verify using (1.5) and (1.3) that this transformation changes the spin in the antiholomorphic sector from \( \bar{s} = 1 \) to \( \bar{s} = 0 \). Thus, on the space of functions normalizable with respect to (6.2), two Hamiltonians have the same energy spectrum and their eigenstates are related as \( \Psi_{N,q}(\vec{z}) = \tilde{\partial}_1 ... \tilde{\partial}_N \Psi^{(QCD)}_N(\vec{z}) \). It remains unclear however whether physical solutions for \( \mathcal{H}_N^{(QCD)} \) can be constructed on a larger class of functions.

The transformation (6.3) is not well defined on the subspace of zero modes of the operator \( \tilde{\partial}_1 ... \tilde{\partial}_N \). However, these modes do not contribute to (6.1) due to gauge-invariance of the wave functions \( \Phi_{a(b)}(\vec{z}) \). However, the both factors have to possess the same parity under the cyclic and mirror permutations of reggeons. Since the wave function of the reggeon state \( \bar{N} \) is factorized, as \( N_e \rightarrow \infty \), into a product of the colour tensor and the function of reggeon coordinates, the both factors have to possess the same parity under the cyclic and mirror permutations simultaneously, \( P = 1 \) and \( M = \pm 1 \). Since the operators of the corresponding transformations, \( P \) and \( M \), do not commute with each other, this requirement leads to the selection rules on the eigenstates of the noncompact Heisenberg magnet.

By the construction, the eigenstates \( \Psi^{(\pm)}_{\bar{P}(q,\bar{q})}(\vec{z}) \) diagonalize the operator of cyclic permutations, Eq. (2.15). Then, making use of (2.14), the eigenstates of the operator of mirror permutations, \( M \Psi_{\bar{P}(\bar{q},q)}^{(+)} = \pm \Psi_{\bar{P}(q,\bar{q})}^{(\pm)} \), can be defined as
\[ \Psi_{\bar{P}(q,\bar{q})}^{(\pm)}(\vec{z}) = \frac{1 \pm M}{2} \Psi_{\bar{P}(q,\bar{q})}(\vec{z}) = \frac{1}{2} \left[ \Psi_{\bar{P}(q,\bar{q})}(\vec{z}) \pm \Psi_{\bar{P}(-q,-\bar{q})}(\vec{z}) \right], \] (6.5)

where \( q = \{ q_k \} \) and \( -q = \{ (-1)^k q_k \} \). Although these states do not diagonalize the integrals of motion, \( \{ q, \bar{q} \} \), they are the eigenstates of the Hamiltonian having the same energy \( E_N = \)

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13 This corresponds to switching from strength tensors to gauge potentials in the description of gluon correlations.
\( E_N(q, \bar{q}) = E_N(-q, -\bar{q}) \). Using \( \theta_N(-q, -\bar{q}) = -\theta_N(q, \bar{q}) \), we find from (2.3) that the operator of cyclic permutations acts on them as

\[
\mathbb{P} \Psi_{\vec{p}(q, \bar{q})}^{(\pm)}(\vec{z}) = \cos(\theta_N(q, \bar{q})) \cdot \Psi_{\vec{p}(q, \bar{q})}^{(\pm)}(\vec{z}) + i \sin(\theta_N(q, \bar{q})) \cdot \Psi_{\vec{p}(q, \bar{q})}^{(\mp)}(\vec{z}).
\]  

(6.6)

Thus, the eigenstates \( \Psi_{\vec{p}(q, \bar{q})}^{(\pm)}(\vec{z}) \) diagonalize the operators \( \mathbb{P} \) and \( \mathbb{M} \) simultaneously only at \( \sin(\theta_N(q, \bar{q})) = 0 \). Together with (2.3), this condition selects among all eigenstates of the magnet only those with the quasimomentum \( e^{\theta_N(q, \bar{q})} = \pm 1 \) for even \( N \) and \( e^{\theta_N(q, \bar{q})} = 1 \) for odd \( N \). Obviously, it is satisfied for the ground state, \( \Psi_{N=0}^\text{ground}(q, \bar{q}) = 0 \).

We conclude that the ground state of the magnet has a definite parity with respect to the cyclic and mirror permutations simultaneously and, therefore, its wave function \( \Psi_{\vec{p}(q, \bar{q})}^{(\pm)}(\vec{z}) \) can be identified as the \( \vec{z} \)-dependent part of the full wave function of the compound states of \( N \) reggeized gluons as \( N_c \to \infty \). The ground state of the magnet has different properties for even and odd number of particles. For even \( N \), its quantum numbers satisfy \( q_{2k+1} = 0 \), Eq (5.21), and, as a consequence, two sets of the quantum numbers, \( (q, \bar{q}) \) and \( (-q, -\bar{q}) \) coincide leading to \( \Psi_{\vec{p}(q, \bar{q})}^{(+)}(\vec{z}) = \Psi_{\vec{p}(q, \bar{q})}^{(-)}(\vec{z}) \) and \( \Psi_{\vec{p}(q, \bar{q})}^{(-)}(\vec{z}) = 0 \). For odd \( N \), the ground state in the sector with \( q_N \neq 0 \) is double degenerate. The degeneracy occurs due to the symmetry of the energy \( E_N \) under \( q_{2k+1} \to -q_{2k+1} \), Eq. (2.18). This allows us to construct two mutually orthogonal ground states, \( \Psi_{\vec{p}(q, \bar{q})}^{(\pm)}(\vec{z}) \), which are invariant under the cyclic permutations, \( \theta_N = 0 \), and possess a definite parity under the mirror permutations, \( \mathbb{M} = 1 \) and \( \mathbb{M} = -1 \) have the same \( C \)-parity as the Pomeron, \( C = 1 \). For odd \( N \), the sector with \( q_N \neq 0 \), the ground states with the parity \( \mathbb{M} = 1 \) and \( \mathbb{M} = -1 \) have the \( C \)-parity of the Odderon \{3\}, \( C = -1 \), and the Pomeron, \( C = 1 \), respectively. For the descendant states, \( q_N = 0 \), one deduces from (5.29) that the physical states with \( \theta_N = 0 \) can be constructed only for odd \( N \) \{37\}. Their minimal energy is \( E_{N,\text{min}}^{(q_N=0)} = 0 \) and the corresponding state with \( q_2 = \ldots = q_N = 0 \) has the quantum numbers of the Odderon.

Our results indicate that, in the multi-colour limit, in the Pomeron sector, only compound states built from even number of reggeized gluons \( N \) provide the contribution to the scattering amplitude \( \mathcal{A}(s, t)/s \), Eq. (1.1), rising with the energy \( s \). Their intercept \( \alpha_N \equiv 1 - \bar{\alpha}_N E_N/4 \) is bigger than one, but it decreases at large \( N \) as \( \alpha_N \sim 1 - 1/N \).

In the Odderon sector, the situation is different. Depending on the value of the “highest” charge, \( q_N \neq 0 \) and \( q_N = 0 \), one can construct two different solutions for the Odderon state. For the first solution, \( q_N \neq 0 \), the intercept of the compound states is smaller than one, but it increases with \( N \) as \( \alpha_N \sim 1 - 1/N \). As a consequence, the contribution to the scattering amplitude (1.1) from the \( N = 3 \) state (“bare Odderon”) is subdominant at high-energy with respect to the contribution of the \( N = 5 \) state and so on. The high-energy asymptotics of the scattering amplitude (1.1) in the Odderon sector with \( q_N \neq 0 \) is governed, as \( N_c \to \infty \), by the contribution of the states with an arbitrary large odd number of reggeized gluons. It increases the effective value of the Odderon intercept and leads to \( \alpha_{\text{Odderon}} = \alpha_{2N} = 1 \). For the second solution, \( q_N = 0 \), the intercept of the compound states equals 1 for arbitrary odd \( N \). At \( N = 3 \) such state was first constructed in \{38\}. To calculate the contribution of the \( q_N = 0 \) states to the scattering amplitude, one has to resum in (1.1) an arbitrary number of terms with \( N = 3, 5, \ldots \). They have the same energy behaviour \( \sim s^1/(\sigma_N \ln s)^{1/2} \), with the dispersion parameter \( \sigma_N \), which scales at large \( N \) as \( \sigma_N \sim 1/N^2 \) and, therefore, enhances the contribution of higher reggeon states.
Thus, the two solutions, \( q_N \neq 0 \) and \( q_N = 0 \), lead to the same value of the Odderon intercept, \( \alpha_{\text{Odderon}} = 1 \), but the properties of the underlying Odderon states are quite different. The Odderon state with \( q_N \neq 0 \) does not couple to a point-like hadronic impact factors of the form \(|\Phi_a\rangle \sim \delta(z_1 - z_2)\phi(z_2, ..., z_N) + \text{[cyclic permutations]}\), like the one for the \( \gamma^* \to \eta_c \) transition, whereas the Odderon state with \( q_N = 0 \) provides a nontrivial contribution \[38\]. Another difference comes from the analysis of the dependence of the scattering amplitude on the invariant mass of one of the scattered hadrons, \( Q^2 \). One can show that in the limit \( x_{\text{Bj}} = Q^2/s \ll 1 \) and \( Q^2 \to \infty \), the Odderon states with \( q_N \neq 0 \) and \( q_N = 0 \) provide a contribution to the scattering amplitude \( \sim 1/Q^p \) of the twist-4 \( (p = 4) \) and twist-3 \( (p = 3) \), respectively. Both properties have to do with the fact that the wave function of the \( N - \text{reggeon state}, \Psi_{\vec{p}, \{q, \bar{q}\}}(z_1, ..., z_N) \), vanishes as \( |z_k - z_{k+1}| \to 0 \) for \( q_N \neq 0 \) and it stays finite for \( q_N = 0 \). It remains unclear which of these solutions corresponds to a physical Odderon state in QCD.

We found that in the Pomeron and the Odderon sectors, the intercept of the \( N - \text{reggeon states} \) approaches the same value \( \alpha_{\infty} = 1 \) as \( N \to \infty \) and their contribution to the scattering amplitude, \( \mathcal{A}(s, t)/s \), ceases to depend on the energy \( s \) as \( N \to \infty \). It is interesting to notice that this result has been anticipated a long time ago within the bootstrap approach \[9\]. It is also in agreement with the upper bound on the energy of the compound reggeized gluon states established in \[12\]. We would like to remind, however, that the calculations were performed in the multi-colour limit and the important question remains: may the nonplanar corrections change the \( N - \)dependence of the energy \( E_N \) of the compound reggeized gluon states? One expects that the nonplanar \( 1/N^2 \) corrections to the reggeon Hamiltonian will break integrability of the Schrödinger equation \( (1.2) \) and calculations will be more involved. This problem deserves additional studies.

Analyzing the high-energy asymptotics of the scattering amplitudes, one is trying to identify the effective theory, which describes the QCD dynamics in the Regge limit. The main objects of this effective theory are the \( N = 2, 3, ... \) reggeon compound states constructed in this paper. In the generalized leading logarithmic approximation, these states propagate between the scattered hadrons and do not interact with each other. In the topological \( 1/N^2 \)-expansion \[9\], these states emerge from the summation of cylinder-like diagrams, whose walls are built from the reggeized gluons. These diagrams can be interpreted as describing the propagation a closed string between two scattered hadrons. This suggests that the effective dynamics of the multi-reggeon compound states in multi-colour QCD has to admit a stringy representation \[10\].

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Note added

After this paper has been submitted for publication, the preprint by H. de Vega and L. Lipatov (dVL) appeared, [arXiv: hep-ph/0204245], in which the spectrum of the \( N = 3 \) and \( N = 4 \) reggeon compound states was investigated. It contains a number of statements, which are in
contradiction with our results. We would like to comment on them below.

In the dVL paper, it is claimed that, contrary to our findings (see Figures [1, 5] and Tables [1, 2]), the quantized valued of the integrals of motion \( q_k \) for arbitrary \( N \) can take only pure imaginary values for odd \( k \) and real values for even \( k \). In particular, at \( N = 3 \) this would imply that \( \text{Re} \, q_3 = 0 \) for all eigenstates of the model. Moreover, for complex values of the charges \( q_k \) found in our paper, the dVL approach leads to complex values of the energy \( E_N \). The authors attributed a disagreement between their and our results to the fact that the quantization procedure proposed in our paper is erroneous. They did not offer, however, any further explanations and refer instead to [14]. In order to check the above assertions and, at the same time, to test the dVL approach, we decided to verify our results at \( N = 3 \) using the approach by R. Janik and J. Wosiek [14]. We found that all eigenstates at \( N = 3 \) constructed in our paper, including those with \( \text{Re} \, q_3 \neq 0 \), satisfy the quantization conditions formulated in [14]. As an example, we present two such states with \((\ell_1, \ell_2) = (3, 3) \) and \((3, 5) \), which have the total spin \( \hbar = 1/2 \) and the quasimomentum \( \theta_3 = 0 \) (see Eqs. (5.9) and (5.12))

\[
q_3(3, 3) = -1.475327, \quad E_3 = 8.469248, \\
q_3(3, 5) = -4.752678 - 3.048722i, \quad E_3 = 13.850368.
\]

In the notations of [14], the wave functions of these states are specified by the parameters, correspondingly,

\[
\alpha_{JW} = 0.4332, \quad \beta_{JW} = -0.6345 - 0.0361i, \quad \gamma_{JW} = 0.6391, \\
\alpha_{JW} = 0.0895 + 0.0007i, \quad \beta_{JW} = 0.8582, \quad \gamma_{JW} = 0.5055 + 0.0037i.
\]

Moreover, one can argue that the set of eigenstates with \( \text{Re} \, q_3 = 0 \) can not be complete. If it were complete, the operator \( q_3 + \bar{q}_3 \) defined in Eqs. (1.4) and (1.7) would be identically equal to zero on the \( SL(2, \mathbb{C}) \) representation space (1.8). Applying this operator to an arbitrary test function on this space \( \Psi(\bar{z}_1, \bar{z}_2, \bar{z}_3) \), one verifies that \( (q_3 + \bar{q}_3)\Psi \neq 0 \). In a similar manner, one can show that for higher \( N \) the eigenvalues of the integrals of motion \( q_3, ..., q_N \) can not take only real or pure imaginary values.

Another issue concerns the ground state at \( N = 4 \), Eq. (5.1). In the first version of the dVL paper, it was claimed that it is located at \( \hbar = 1 \) and its energy is smaller than the energy of the state defined in Eq. (5.17) above. Later, in the second version of the paper, the authors found yet another state at \( \hbar = 3/2 \), which has even smaller energy. We remind that for the principal series of the \( SL(2, \mathbb{C}) \) group the total spin \( \hbar \) has the form \((2, 3)\), so that \( \hbar = 1/2, 1 \) and \( 3/2 \) correspond to \( \nu_h = 0 \) and the Lorentz spin \( n_h = 0, 1 \) and \( 2 \), respectively. Thus, the results of the dVL paper imply that the minimal energy of the system of \( N = 4 \) particles decreases as the total angular momentum of their rotation on two-dimensional plane, \( n_h \), increases. Our results (see Table 3, 4th column) indicate that the dependence is just opposite. In addition, in our approach we do not find such states with \( \hbar = 1 \) and \( \hbar = 3/2 \) in the spectrum of the \( SL(2, \mathbb{C}) \) magnet and the ground state occurs at \( \hbar = 1/2 \), Eq. (5.17). Analyzing the quantization conditions (3.28), we were able to identify the physical meaning of the solutions found in the dVL paper. They fulfil the quantization conditions at \( N = 4 \), but have unusual quantum numbers, \( \hbar = \bar{h} = 1 \) and \( h = \bar{h} = 3/2 \), which do not match the principal series of the \( SL(2, \mathbb{C}) \) group, Eq. (2.9). One can show that these states are located on the trajectory with \( h = \bar{h} = 1/2 + i\nu_h \), which is obtained from the ground state trajectory with \( h = \bar{h} = 1/2 + i\nu_h \) by analytical continuation from real to

\footnote{We are grateful to J. Wosiek for making the Mathematica code used in [14] available to us.}
imaginary $\nu_h$. The same happens for the $q_3 = 0$ solution at $N = 3$ found there – its total spin is $h = \bar{h} = 1$. As a consequence, the states found in the dVL paper do not belong to the quantum space of the $SL(2, \mathbb{C})$ magnet and, therefore, are unphysical.

We conclude that the criticism of our results in the dVL paper is groundless. At $N = 3$ the quantization procedure proposed in that paper does not reproduce correctly the part of the spectrum corresponding to $\text{Re} \ q_3 \neq 0$, whereas at $N = 4$ it generates spurious states, which do not belong to the quantum space of the model.

A Appendix: Solution to the Baxter equation at $N = 2$

In this Appendix we summarize the properties of the eigenvalues of the Baxter operator $Q(u, \bar{u})$ at $N = 2$. As was shown in the Section 3, $Q(u, \bar{u})$ is equal to the integral (3.1) of the function $Q(z, \bar{z})$ defined in (3.32) over the two-dimensional plane.

The function $Q_s(z; h)$ entering (3.32) is expressed in terms of the Legendre function on the second-kind, Eq. (3.31). Using the properties of the Legendre functions [24], one finds the spectrum corresponding to $\text{Re} \ q_3$.

$$Q_s(z; h) \sim z^{1-s} \left[ -\frac{1}{2} \ln z - \psi(1 - h) + \psi(1) + \mathcal{O}(z) \right], \quad (A.1)$$

$$Q_s(z; h) \sim (1 - z)^{2s - h - 1} \left[ \frac{\Gamma^2(1 - h)}{2\Gamma(2 - 2h)} + \mathcal{O}(1 - z) \right], \quad (A.2)$$

Comparing (A.1) and (A.2) with the asymptotic behaviour of the functions $Q_n^{(0)}(z)$ and $Q_n^{(1)}(z)$ defined in (3.7) and (3.10), one finds that $Q_s(z; h)$ can be decomposed over the fundamental set of solutions around $z = 0$ as

$$Q_s(z; h) = - \left[ \psi(1 - h) - \psi(1) - \frac{1}{2} \right] Q_1^{(0)}(z) - \frac{1}{2} Q_2^{(0)}(z), \quad (A.3)$$

and over the fundamental set of solutions around $z = 1$ as

$$Q_s(z; h) = \frac{\Gamma^2(1 - h)}{2\Gamma(2 - 2h)} Q_1^{(1)}(z), \quad Q_s(z; 1 - h) = \frac{\Gamma^2(h)}{2\Gamma(2h)} Q_2^{(1)}(z). \quad (A.4)$$

The functions $Q_n^{(0)}(z)$ and $Q_n^{(1)}(z)$ are related to each other by the transition matrix $Q_n^{(0)}(z) = \sum_m \Omega_{nm} Q_m^{(1)}(z)$ defined in (3.26). Comparison of the r.h.s. of (A.3) and (A.4) yields

$$\Omega(h) = \left( \begin{array}{c} \Delta(h) \\ \delta(h) \end{array} \right), \quad \delta(h) = -2 \left[ \psi(h) - \psi(1) - \frac{1}{2} \right] \Delta(h).$$

where $\Delta(h) = \Gamma(2h - 1)/\Gamma^2(h)$. Similar matrix in the antiholomorphic sector equals $\overline{\Omega} = \Omega(\bar{h})$.

The general expression for the function $Q(z, \bar{z})$ is given by (3.32) with the expansion coefficients $c_h$ defined in (3.33). Substituting (A.3) and (A.4) as well as analogous relations in the
antiholomorphic sector into (3.32), one arrives at Eqs. (3.35) and (3.20) with the mixing matrices given by

\[
C^{(0)} = \begin{pmatrix}
\alpha_1(h) - 2 & 1 \\
1 & 0
\end{pmatrix}, \quad C^{(1)} = \begin{pmatrix}
\beta_h & 0 \\
0 & \beta_{1-h}
\end{pmatrix},
\]

(A.6)

where \(\alpha_1(h)\) was defined in (3.35) and

\[
\beta_h = (-1)^{n_h+1} \frac{\Gamma^2(1 - \bar{h}) \Gamma(2h - 1)}{\Gamma^2(h) \Gamma(2 - 2h)}.
\]

(A.7)

It is straightforward to verify that the matrices \(C^{(0)}\) and \(C^{(1)}\) satisfy the quantization conditions (3.28).

To obtain the \(Q\)-block at \(N = 2\) one inserts (3.31) into (4.12)

\[
Q(u; h) = \frac{1}{\Gamma(2s - h)} \int_0^1 dz \, z^{-s}(1 - z)^{2(s-1)} Q_{-h} \left( \frac{1 + z}{1 - z} \right).
\]

(A.8)

Integration can be performed by replacing the Legendre function by its integral representation. In this way, one arrives at (4.21). The function \(Q(u, \bar{u})\) with required analytical properties is given by the bilinear combination of the holomorphic and antiholomorphic \(Q\)-blocks, Eq. (4.21).

As was explained in Section 4.4, to calculate the energy and quasimomentum it is convenient to introduce the blocks, \(Q_0(u)\) and \(Q_1(u)\), Eqs. (4.32) and (4.34), respectively. At \(N = 2\) the block \(Q_0(u)\) is fixed (up to an overall normalization) by the requirement to have simple poles at the points \(u = -i(s - m)\) with \(m > 0\)

\[
Q_0(u) = \frac{1}{\Gamma(2s - 1 + h) \Gamma(2s - h)} \int_0^1 dz \, z^{-s}(1 - z)^{2(s-1)} P_{-h} \left( \frac{1 + z}{1 - z} \right)
\]

\[= \frac{1}{\Gamma(2s)} \, _3F_2 \left( s - iu, 2s - h, 2s - 1 + h \mid 2s, 2s \right).
\]

(A.9)

The antiholomorphic block \(Q_0(\bar{u})\) can be obtained from (A.9) by replacing \(u \to -\bar{u}\), \(s \to \bar{s}\) and \(h \to \bar{h}\). The block \(Q_1(u)\) is defined according to (4.34) as

\[
Q_1(u) = \left[ \frac{\Gamma(1 - s + iu)}{\Gamma(s + iu)} \right]^2 (Q_0(u^*))^* = \left[ \frac{\Gamma(1 - s + iu)}{\Gamma(s + iu)} \right]^2 Q_0(u; 1 - s),
\]

(A.10)

where in the last relation we indicated explicitly the dependence of the block \(Q_0\) on the spin \(s\).

The two sets of the blocks are linearly dependent

\[
\frac{1}{\Gamma(2s - 1 + h)} Q(u; h) - \frac{1}{\Gamma(2s - h)} Q(u; 1 - h) = -\pi \cot(\pi h) Q_0(u),
\]

\[
\frac{1}{\Gamma(1 - 2s + h)} Q(u; h) - \frac{1}{\Gamma(2 - 2s - h)} Q(u; 1 - h) = -\pi \cot(\pi h) Q_1(u).
\]

(A.11)

The inverse relation reads

\[
Q(u, h) = \rho(s, h) \left[ \frac{1}{\Gamma(2 - 2s - h)} Q_0(u) - \frac{1}{\Gamma(2s - h)} Q_1(u) \right],
\]

\[
Q(u, 1 - h) = \rho(s, h) \left[ \frac{1}{\Gamma(1 - 2s + h)} Q_0(u) - \frac{1}{\Gamma(2s - 1 + h)} Q_1(u) \right],
\]

(A.12)
where \( \rho(s, h) = \pi^2/(2 \sin(2\pi s) \sin(\pi h)) \). Substituting (A.12) into (A.21) one finds that the resulting expression matches (4.36).

Using the properties of the \( _3F_2 \)–series \([4]\), one can show that the block \( Q_0(u) \), Eq. (A.9), satisfies the following relation

\[
Q_0(-u; s) - \frac{\sin(\pi h)}{\sin(2\pi s)} Q_0(u; s) = -\Gamma \left[ \frac{2s, 1 - 2s, 1 - s - iu, 1 - s + iu}{2s - h, 2s - 1 + h, s - iu, s + iu} \right] Q_0(u; 1 - s),
\]

where we indicated explicitly the dependence on the spin \( s \). Applying (A.10), (A.12) and (A.13) one can verify that the eigenvalue of the Baxter operator at \( N = 2 \), Eq. (A.12), can be rewritten (up to an overall normalization) as

\[
Q(u, \bar{u}) \approx \Gamma \left[ \frac{1 - \bar{s} - i\bar{u}, 1 - \bar{s} + i\bar{u}}{\bar{s} - i\bar{u}, \bar{s} + i\bar{u}} \right] \left\{ Q_0(u) \left( Q_0(-\bar{u}^*)\right)^* + (-1)^n Q_0(-u) \left( Q_0(\bar{u}^*)\right)^* \right\}.
\]

Finally, two equivalent expressions for the eigenvalues of the Baxter \( Q \)–operator at \( N = 2 \), Eqs. (A.14) and (A.21), admit an elegant representation in terms of two-dimensional Feynman diagrams (see Figures 10a and b in Ref. [3]).

\section{Appendix: Properties of the \( Q \)–blocks}

In this Appendix we establish different useful relations between the blocks \( Q(u; h, q) \) and \( Q(\bar{u}; \bar{h}, q) \) defined in Eq. (1.12).

\subsection*{Intertwining relations}

It is well-known that the \( SL(2, \mathbb{C}) \) representations of the principal series of the spins \( (s, \bar{s}) \) and \( (1 - s, 1 - \bar{s}) \) are unitary equivalent. At the level of the eigenvalues of the Baxter \( Q \)–operator, this property leads to the following intertwining relation between the blocks

\[
Q_s(u; h, q) = \left[ \frac{\Gamma(1 - s + iu)}{\Gamma(s + iu)} \right]^N Q_{1-s}(u; h, q),
\]

where subscript indicates the corresponding value of the holomorphic spin. Indeed, one verifies that the both sides of this relation satisfy the Baxter equation (2.7), have the same analytical properties (4.16) and asymptotic behaviour (4.17) at infinity. In a similar manner, using the identities \( s^* = 1 - \bar{s}, h^* = 1 - \bar{h} \) and \( q_k = q^*_k \), one can show that

\[
(Q_{1-s}(\bar{u}^*; h, q))^* = Q_s(-\bar{u}; 1 - \bar{h}, -\bar{q}).
\]

Combining together Eqs. (B.1), (B.2) and (4.18), we obtain the relations between the holomorphic and antiholomorphic blocks

\[
Q(u; h, q) = \left[ \frac{\Gamma(1 - s + iu)}{\Gamma(s + iu)} \right]^N (Q(u^*; 1 - \bar{h}, \bar{q}))^*,
\]

\[
Q(\bar{u}; \bar{h}, \bar{q}) = \left[ \frac{\Gamma(1 - \bar{s} - i\bar{u})}{\Gamma(\bar{s} - i\bar{u})} \right]^N (Q(\bar{u}^*; 1 - h, q))^*.
\]

Here, the ratio of \( \Gamma \)–functions compensates the difference in the analytical properties of two blocks and their asymptotic behaviour at infinity.
Wronskian relations

The functions \( Q(u; h, q) \) and \( Q(u; 1 - h, q) \) satisfy the same Baxter equation (2.7). This suggests to define their Wronskian as

\[
W(u) = \left[ \frac{\Gamma(iu + s)}{\Gamma(iu - s)} \right]^N Q(u + i; h, q) Q(u; 1 - h, q) - Q(u; h, q) Q(u + i; 1 - h, q).
\]

(B.4)

It follows from (2.7) that \( W(u) \) is periodic, \( W(u + i) = W(u) \). In addition, taking into account the properties of the blocks, Eqs. (4.16) and (4.17), one finds that \( W(u) \) is analytical in the half-plane \( \text{Re}(iu - s + 1) > 0 \) and behaves there at large \( u \) as \( W(u) \sim u^0 \). This implies that \( W(u) \) takes constant values for arbitrary \( u \). Then, substituting (4.17) into (B.4), one finds \( W(u) = 1 - 2h \).

Similar consideration in the antiholomorphic sector leads to

\[
\overline{Q}(\bar{u} - i; \bar{h}, \bar{q}) \overline{Q}(\bar{u}; 1 - \bar{h}, \bar{q}) - \overline{Q}(\bar{u}; \bar{h}, \bar{q}) \overline{Q}(\bar{u} - i; 1 - \bar{h}, \bar{q}) = (1 - 2\bar{h}) \left[ \frac{\Gamma(-i\bar{u} - \bar{s})}{\Gamma(-i\bar{u} + \bar{s})} \right]^N.
\]

(B.5)

For \( u = i(s + \epsilon) \) and \( \bar{u} = -i(\bar{s} + \epsilon) \) we find from the Wronskians (B.4) and (B.5) that the functions \( \Phi(\epsilon) \) and \( \bar{\Phi}(\epsilon) \), defined in (4.26), satisfy the relations

\[
\Phi(\epsilon) - \Phi(\epsilon + k) = \mathcal{O}(\epsilon^N), \quad \bar{\Phi}(\epsilon) - \bar{\Phi}(\epsilon + k) = \mathcal{O}(\epsilon^N), \quad \text{with } k \text{ being positive integer.}
\]

(B.6)

Series representation

We can obtain a series representation for the block \( Q(u; h, q) \) in the different regions on the complex \( u \)-plane, by replacing the function \( Q_1(z) \) in (1.12) by its expressions in terms of the fundamental solutions, Eqs. (4.10) and (4.11), defined in (3.7) and (3.16). For \( z \to 1 \) the function \( Q_1(z) \) is given by

\[
Q_1(z) = z^{1-s} (1-z)^{Ns-h-1} \sum_{n=0}^{\infty} v_n (1-z)^n
\]

(B.7)

with \( v_0 = 1 \) and the expansion coefficients \( v_n \equiv v_n^{(1)}(-q) \) defined in (3.16) and (3.17). In this way, one gets from (4.12)

\[
Q(u; h, q) = \frac{\Gamma(1-s+iu)}{\Gamma(-h+Ns)} \sum_{n=0}^{\infty} \frac{\Gamma(n-h+Ns)}{\Gamma(n+1-h-s+iu+Ns)} v_n.
\]

(B.8)

For \( z \to 0 \) the function \( Q_1(z) \) is given by

\[
Q_1(z) = z^{1-s} \sum_{k=0}^{N-1} (\ln z)^k \sum_{n=0}^{\infty} w_n^{(k)} z^n,
\]

(B.9)

with \( w_n^{(k)} = \sum_{b=k+1}^{N} \Omega^{-1}(q)_{[b]} c_{b-1}^{k} u_{n-k}^{(b-k)}(-q) \) and the expansion coefficients \( u_n^{(m)} \) defined in (3.7) and (3.8). This leads to

\[
Q(u; h, q) = \frac{1}{\Gamma(Ns-h)} \sum_{n=0}^{\infty} \sum_{k=0}^{N-1} \frac{(-1)^k w_n^{(k)} k!}{(iu-s+n+1)^{k+1}}.
\]

(B.10)
where the sum over $n$ goes over the $k$–th order poles located at $u = i(n+1-s)$.

We notice that (B.8) reproduces correctly the asymptotic behaviour of $Q(u; h, q)$ at large $u$, Eq. (4.17), and the position of its poles on the $u$–plane, Eq. (4.16), but not their order. The reason for this is that the series (B.8) is convergent only for $\Re(1-s+iu) \geq 0$. Indeed, the large-order behaviour of the expansion coefficients $v_n$ in (B.7) is determined by the asymptotics of the function $Q_1(z)$ at $z = 0$, Eq. (B.9)

$$v_n = \frac{1}{2\pi i} \int_{|z|<\epsilon} dz \frac{z^{-s}Q_1(z)}{(1-z)^{n-h+N_l}} \sim \int_{-\infty}^{0} dz \frac{\ln^{-1} z}{(1-z)^{n+1}} \sim \frac{\ln^{-1} n}{n}.$$  

(B.11)

Therefore, for $\Re(1-s+iu) < 0$ the series (B.8) diverges at large $n$ as $\sum_n v_n/n^{1-s+iu} \sim \sum_n \ln^{N-1} n/n^{2-s+iu}$.

C  Appendix: Contour integral representation

Let us demonstrate that the two-dimensional integral (4.3) can be decomposed into the sum of products of simple contour integrals, Eq. (4.9). The derivation is based on the technique developed in [28] for calculation of the correlation functions in CFT. To simplify notations, we rewrite the integral over (4.5) as

$$Q = \int d^2 z \sum_{n,k=1}^{N} q_n(z) C_{nk} \tilde{q}_k(z) \equiv \int d^2 z q^T(z) \cdot C \cdot \tilde{q}(z) .$$  

(C.1)

Here, the functions $q_n(z) = z^{-iu} Q_n(z)$ and $\tilde{q}_k(z) = \tilde{z}^{-iu} \tilde{Q}_k(\tilde{z})$ have three isolated singular points located at $z_i = \tilde{z}_i = 0$, 1 and $\infty$. Around these points, they have a nontrivial monodromy

$$q_n(z) \xrightarrow{\tilde{z}\rightarrow_0} [M_i]_{nk} q_k(z) , \quad \tilde{q}_n(\tilde{z}) \xrightarrow{z\rightarrow_0} [\overline{M_i}]_{nk} \tilde{q}_k(\tilde{z}) ,$$  

(C.2)

with $M_i$ and $\overline{M_i}$ being the corresponding monodromy matrices such that $M_0 M_1 M_\infty = \mathbb{I}$ and $\overline{M_0} M_1 \overline{M_\infty} = \mathbb{I}$. Here, $z$ and $\tilde{z}$ encircle the singular points on the complex plane in the counterclockwise and clockwise directions, respectively. For the integrand in (C.1) to be a single-valued function on the two-dimensional plane, the mixing matrix in Eq. (C.1) has to satisfy the condition

$$M_i^T C \overline{M_i} = C , \quad (i = 0, 1, \infty).$$  

(C.3)

Since $q(z)$ and $\tilde{q}(\tilde{z})$ are analytical functions on the complex plane with two cuts running from the singular points 0 and 1 to infinity, as shown in Figure 10, one can apply the Stokes's theorem

$$Q = \int_{\Sigma} d^2 z \frac{\partial}{\partial \tilde{z}} \left[ q^T(z) \cdot C \cdot \int_{\tilde{z}_{aux}}^{\tilde{z}} d\tilde{z}' \tilde{q}(\tilde{z}') \right] = \frac{1}{2i} \int_{\partial \Sigma} dz q^T(z) \cdot C \cdot \int_{\tilde{z}_{aux}}^{\tilde{z}} d\tilde{z}' \tilde{q}(\tilde{z}') .$$  

(C.4)

Here, $\tilde{z}_{aux}$ is an arbitrary reference point and the $\tilde{z}'$–integration goes along the contour that does not cross the cut. It is convenient to choose $\tilde{z}_{aux} = 0$ and split the integral over (infinite) contour $\partial \Sigma$ into four integrals along the different edges of two cuts. Their contribution to (C.4) can be calculated as follows

$$Q_{[a' a]} = -\frac{1}{2i} \int_{a'}^{a} dz q^T(z) \cdot C \cdot \left[ \int_{1}^{\tilde{z}} d\tilde{z}' \tilde{q}(\tilde{z}') + \int_{0}^{1} d\tilde{z}' \tilde{q}(\tilde{z}') \right] ,$$

$$Q_{[a b]} = \frac{1}{2i} \int_{a}^{b} dz q^T(z) \cdot C \cdot \left[ \int_{1}^{\tilde{z}_{aux}} d\tilde{z}' \tilde{q}(\tilde{z}') + \int_{0}^{1} d\tilde{z}' \tilde{q}(\tilde{z}') \right] .$$  

(C.5)
In these relations the integration goes along two different edges of the same cut. In spite of the fact that \( q(z) \) and \( \bar{q}(\bar{z}) \) are discontinuous across the cut, their bilinear combination remains continuous due to (C.3):

\[
\int_{a'}^b dz \, q^T(z) \, C \left[ \int_1^\infty d\bar{z}' \, \bar{M}_1 \bar{q}(\bar{z}') \right] = \int_{a'}^b dz \, q^T(z) \, C \left[ \int_1^\infty d\bar{z}' \, \bar{q}(\bar{z}') \right].
\]

This leads to a partial cancellation of terms in the sum \( Q_{[b'a']} + Q_{[ab]} \). Then, one calculates the contribution of the second cut

\[
Q_{[cd]} = -\frac{1}{2i} \int_{d}^{c} dz \, q^T(z) \cdot C \cdot \int_0^{\bar{z}} d\bar{z}' \, \bar{q}(\bar{z}') ,
\]

\[
Q_{[d'c']} = \frac{1}{2i} \int_{d'}^{c'} dz \, q^T(z) \cdot C \cdot \int_0^{\bar{z}} d\bar{z}' \, \bar{q}(\bar{z}').
\] (C.6)

and finds that the same property leads to \( Q_{[cd]} + Q_{[d'c']} = 0 \). Combining together (C.5) and (C.6), we obtain the following expression for the two-dimensional integral (C.4)

\[
Q = \frac{1}{2i} \int_{1}^{\infty} dz \, q^T(z) \cdot C \cdot \int_0^{\infty} d\bar{z} \, \bar{q}(\bar{z}) \cdot \left[ 1 - M_1^T \right] C \cdot \int_0^{\infty} d\bar{z} \, \bar{q}(\bar{z}).
\] (C.7)

Finally, one replaces the functions \( q(z) \) and \( \bar{q}(\bar{z}) \) by their actual expressions, Eq. (C.1), and arrives at (L.9).

One can obtain another (though equivalent) representation for \( Q \) if one starts with

\[
Q = \int_{\Sigma} d^2z \frac{\partial}{\partial z} \left[ \int_0^{\bar{z}} dz' q^T(z') \cdot C \cdot \bar{q}(\bar{z}) \right] = -\frac{1}{2i} \int_{\partial \Sigma} d\bar{z} \int_0^{\bar{z}} dz' q^T(z') \cdot C \cdot \bar{q}(\bar{z})
\] (C.8)

instead of (C.4). Repeating the same analysis one gets

\[
Q = -\frac{1}{2i} \int_0^{1} dz' q^T(z') \cdot C \left( 1 - M_1 \right) \int_{1}^{\infty} d\bar{z} \, \bar{q}(\bar{z}).
\] (C.9)
Let us now take into account that \( q(z) \) is analytical inside \( \Sigma \) and, therefore, \( \oint_{\partial \Sigma} dz \; q(z) = 0 \). As before, splitting the integration contour into four pieces and taking into account that \( M_1 \int_a^b dz \; q(z) = \int_a^b dz \; q(z) \) and \( \int_c^d dz \; q(z) = M_0 \int_c^d dz \; q(z) \) one obtains

\[
-(1 - M_0^{-1}) \int_0^\infty dz \; q(z) + (1 - M_1) \int_1^\infty dz \; q(z) = 0 . \tag{C.10}
\]

Here, two terms in the l.h.s. correspond to the contribution of two cuts. We conclude from \( (C.10) \) that

\[
-(1 - M_0) \int_0^1 dz \; q(z) = (1 - M_0 M_1) \int_1^\infty dz \; q(z) . \tag{C.11}
\]

Obviously, similar property holds for the function \( \bar{q}(z) \) in the antiholomorphic sector. Recalling the definition of the function \( q(z) \), Eq. \( (C.1) \), one obtains

\[
\int_0^1 dz \; q_n(z) = \int_0^1 dz \; z^{-iu-1}Q_n(z) , \quad \int_1^\infty dz \; q_n(z) = \int_0^1 dz \; z^{iu-1}Q_n(1/z) . \tag{C.12}
\]

Eq. \( (C.11) \) allows us to rewrite \( (C.7) \) and \( (C.9) \) in the form

\[
Q = \frac{1}{2i} \int_0^1 dz \; q^T(z) \cdot \left( 1 - (1 - M_0^T)^{-1} - (1 - M_1^T)^{-1} \right)^{-1} C \cdot \int_0^1 d\bar{z} \; \bar{q}(\bar{z}) , \tag{C.13}
\]

in which the symmetry between \( z \)- and \( \bar{z} \)-sectors becomes manifest.

\section{Appendix: Degenerate \( Q \)-blocks}

In our analysis of the quantization conditions performed in Section 4.3, we have tacitly assumed that the blocks \( Q(u; h, q) \) are well-defined for arbitrary spins \( h = (1+n_h)/2 + i\nu_h \) and, in addition, \( Q(u; h, q) \) is finite at \( u = i(s + n - 1) \) with \( n \geq 1 \). As was already mentioned in Section 4.2, the first condition is not satisfied at \( \nu_h = 0 \). The second condition does not hold for (half)integer spins \( s \) since, by the definition, the block \( Q(u; h, q) \) has poles at \( u_m = -i(s - m) \) with \( m \geq 1 \) and for positive integer \( n \) and \( m \), such that \( 2s - 1 = m - n \), the point \( u = i(s + n - 1) \) coincides with the pole \( u_m \). In this Section, we will work out the quantization conditions for (half)integer spins \( h = (1+n_h)/2 \) and \( s \). It worth mentioning that one has to deal with these two cases calculating the ground state of the Schrödinger equation \( (1.2) \) at \( h = 1/2 \) and \( s = 0 \).

To start with, let us examine the series representation \( (B.7) \) for the block \( Q(u; h, q) \) at \( h = (1+n_h)/2 + i\nu_h \) in the limit \( \nu_h \to 0 \) and \( n_h > 0 \). The expansion coefficients \( v_n \) entering \( (B.7) \) satisfy the \( N \)-term recurrence relations, which lead to \( v_n \sim 1/\nu_h \) as \( \nu_h \to 0 \) for \( n \geq n_h \). The resulting expression for \( Q(u; h, q) \) can be written as

\[
Q(u; h, q) = \frac{A_{n_h}(q)}{\nu_h}Q(u; 1 - h, q) + \tilde{Q}(u; n_h, q) + \mathcal{O}(\nu_h) , \tag{D.1}
\]

where \( \tilde{Q}(u; h, q) \) and \( Q(u; 1 - h, q) \) are finite for \( \nu_h \to 0 \) and \( A_{n_h}(q) = \lim_{\nu_h \to 0}[v_{n_h}(q)\nu_h] \). Taking into account \( (B.17) \), we find that the function \( \tilde{Q}(u; n_h, q) \) defined in this way has the following asymptotic behaviour at large \( u \)

\[
\tilde{Q}(u; n_h, q) \sim (iu)^{-N_s+(1+n_h)/2} \left[ 1 + \mathcal{O}(1/u) \right] + A_{n_h}(q)(iu)^{-N_s+(1-n_h)/2} \ln u \left[ 1 + \mathcal{O}(1/u) \right] . \tag{D.2}
\]
According to the definition (D.1), the function $\tilde{Q}(u; n_h, q)$ is a linear combination of two degenerate blocks and, therefore, it satisfies the holomorphic Baxter equation (2.7). Eqs. (4.28) and (D.1) are valid only for $n_h > 0$. At $n_h = 0$, or equivalently $h = 1/2$, the function $\tilde{Q}(u; 0, q)$ is defined as

$$\tilde{Q}(u; 0, q) = \partial_{\nu_h} Q(u; 1/2 + i\nu_h, q) \bigg|_{\nu_h = 0} \sim \nu_{s+1/2} \ln u \left[ 1 + \mathcal{O}(1/u) \right]. \quad (D.3)$$

It is straightforward to verify that for $n_h \geq 0$ the function $\tilde{Q}(u; n_h, q)$ satisfies the chiral Baxter equation (2.7).

Let us now insert (D.1) into the quantization conditions (4.28) and examine the limit $\nu_h \to 0$. It follows from (4.26) that

$$\Phi(\epsilon) = \frac{A_{n_h}(q)}{\nu_h} + \tilde{Q}(i(s + \epsilon); n_h, q) \frac{\tilde{Q}(i(s + \epsilon); (1 - n_h)/2, q)}{Q(i(s + \epsilon); (1 - n_h)/2, q)} + \mathcal{O}(\nu_h). \quad (D.4)$$

Then, comparing the coefficients in front of different powers of $\nu_h$ we find from (4.28) the set of $N - 1$ quantization conditions on the charges $q_3, ..., q_N$

$$\frac{\partial^n}{\partial \epsilon^n} \Im \left[ (A_{n_h}(q))^* \frac{\tilde{Q}(i(s + \epsilon); n_h, q)}{Q(i(s + \epsilon); (1 - n_h)/2, q)} \right] \bigg|_{\epsilon = 0} = 0, \quad (D.5)$$

with $n = 1, ..., N - 1$ and $h = (1 + n_h)/2$. We recall that the additional set of $N$ quantization conditions follows from (4.31) in the antiholomorphic sector. At $h = 1/2$ the relation (D.3) leads to

$$\frac{\partial^n}{\partial \epsilon^n} \frac{\partial}{\partial \nu_h} \Im \ln Q(i(s + \epsilon); 1/2 + i\nu_h, q) \bigg|_{\epsilon = \nu_h = 0} = 0. \quad (D.6)$$

Let us now consider the quantization conditions for (half) integer spins $s$. In distinction with the previous case, the blocks $Q_s(u; h, q)$ as well as the eigenvalues of the Baxter operator $Q(u, \bar{u})$ remain finite in the limit $s = (1 + n_s)/2$ and possess a correct analytical properties as functions of $u$ and $\bar{u}$. Nevertheless, the quantization conditions have to be modified because the two sets of points (4.25) overlap. Repeating the analysis of Section 4.4, we find that the function $Q(u, \bar{u})$ has correct analytical properties for (half)integer $s$ provided that the following conditions are satisfied

$$\text{arg} \left[ \frac{Q(i(1 + |n_s|)/2 + \epsilon; h, q)}{Q(i(1 + |n_s|)/2 + \epsilon; 1 - h, q)} \right] = \pi \left( \frac{n_h}{2} + \ell \right) - \Theta_{q,q} + \mathcal{O}(\epsilon^{2N}). \quad (D.7)$$

Here, in distinction with (4.30), the $Q-$blocks are calculated in the vicinity of the point $u = i(1 + |n_s|)/2$ that belongs to the both sets (4.25) simultaneously. In addition, the small $\epsilon-$expansion in the r.h.s. starts with the terms $\sim \epsilon^{2N}$.

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