Asymptotically Free $\hat{U}(1)$ Kac-Moody Gauge Fields in 3 + 1 dimensions.

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Abstract

$\hat{U}(1)$ Kac-Moody gauge fields have the infinite dimensional $\hat{U}(1)$ Kac-Moody group as their gauge group. The pure gauge sector, unlike the usual $U(1)$ Maxwell lagrangian, is nonlinear and nonlocal; the Euclidean theory is defined on a $d + 1$-dimensional manifold $\mathcal{R}_d \times S^1$ and hence is also asymmetric. We quantize this theory using the background field method and examine its renormalizability at one-loop by analyzing all the relevant diagrams. We find that, for a suitable choice of the gauge field propagators, this theory is one-loop renormalizable in 3+1 dimensions. This pure abelian Kac-Moody gauge theory in 3 + 1 dimensions has only one running coupling constant and the theory is asymptotically free. When fermions are added the number of independent couplings increases and a richer structure is obtained. Finally, we note some features of the theory which suggest its possible relevance to the study of anisotropic condensed matter systems, in particular that of high-temperature superconductors.

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1 Introduction

Kac-Moody groups embody the infinite dimensional symmetries that underlie string theory and conformal field theory \[1\]. It is natural to realize the Kac-Moody group $\hat{G}$ as a gauge symmetry, that is as a group of gauge transformations of gauge fields, and to obtain a lagrangian invariant under these gauge transformations.

A lagrangian of vector gauge fields coupled to matter fields that is invariant under Kac-Moody gauge transformations for an arbitrary group $\hat{G}$ was obtained in \[2\]. A Kac-Moody group $\hat{G}$ consists of all mappings of the circle $S^1$ into the compact Lie group $G$. Let $\mathcal{R}_d$ be an Euclidean space consisting of one time dimension and $(d - 1)$ space dimensions. The gauge transformation at a point $x \in \mathcal{R}_d$, namely $\Phi(x)$, is an element of $\hat{G}$ and has the form ($f \equiv f_0^R d\sigma$, that is, $S^1$ has radius $R/2\pi$)

$$\Phi(x) = e^{i\Lambda(x)} e^{i \int \phi^\alpha(x,\sigma) Q^\alpha(\sigma)}.$$ 

(1.1)

$Q^\alpha(\sigma)$ are the generators of $\hat{G}$, $\phi^\alpha(x,\sigma)$ and $\Lambda(x)$ are dimensionless gauge functions, and $\alpha$ is the Lie group index. We can consider $\sigma \in S^1$ to be a new space-like direction in the theory, which is then effectively defined on $\mathcal{R}_d \times S^1$. The $S^1$ direction has a special significance since each point in $S^1$ carries its own non-commuting charge $Q(\sigma)$. Hence non-abelian gauge fields $A^\mu_\alpha(x,\sigma)$ are defined on the manifold $\mathcal{R}_d \times S^1$. Since $\Phi(x)$ has a central extension, there is also a $U(1)$ gauge field $B_\mu(x)$ defined on only $\mathcal{R}_d$.

Just as for the non-abelian case, the lagrangian for $\hat{U}(1)$ Kac-Moody gauge fields is nonlinear, nonlocal and asymmetric. Quantizing the abelian theory is of particular importance because: (i) this is the starting point for the analysis of the more complex non-abelian Kac-Moody gauge fields \[2\] and, (ii) the new nonlinearities arising from the Kac-Moody nature of the gauge group can be studied in their simplest and most essential form for the $\hat{U}(1)$ case.

In Section II we review the $\hat{U}(1)$ Kac-Moody gauge symmetry and derive the lagrangian; in Section III we quantize the theory via the Feynman path integral using the background field method. In Section IV we compute the one-loop $\beta$-function, and in Sect.V we analyze all the one loop Feynman diagrams, identify the divergent diagrams and show that the theory is one-loop renormalizable in 3+1 dimensions. In Section V we draw some conclusions and highlight a possible application of the theory as an effective low-energy description of anisotropic superconductors.

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\[^3\]The “Kac-Moody group” we refer to here is sometimes called an “affine Lie Group”, and the corresponding algebra is known to physicists as current algebra.
2 The Kac-Moody Gauge-Field Lagrangian

The $\hat{U}(1)$ Kac-Moody generators satisfy the commutation relation

\[ [Q(\sigma), Q(\sigma')] = i\kappa \delta'(\sigma - \sigma') \]  (2.1)

where the prime symbol ($'$) on functions means $\partial/\partial \sigma$, and $\kappa$ is a real number. For gauge transformations given by (1.1), we have

\[ \Phi(x) Q(\sigma) \Phi^\dagger(x) = Q(\sigma) + \kappa \phi'(x, \sigma). \]  (2.2)

To determine the gauge transformation of the gauge-fields, consider the link variable ($a = $ lattice spacing)

\[ U_\mu(x) = e^{iagB_\mu(x) + iae \int A_\mu(x, \sigma)Q(\sigma)} \]  (2.3)

where $g$ and $e$ are dimensionful coupling constants. We define Kac-Moody gauge transformations by

\[ U_\mu(x) \to \Phi(x) U_\mu(x) \Phi^\dagger(x + a \hat{\mu}) \]  (2.4)

and in the $a \to 0$ limit we obtain

\[ \delta A_\mu(x, \sigma) = -\frac{1}{e} \partial_\mu \phi(x, \sigma), \]

\[ \delta B_\mu(x) = -\frac{1}{g} \partial_\mu \Lambda(x) + \frac{\kappa e}{g} \int \phi'(x, \sigma)A_\mu(x, \sigma) - \frac{\kappa}{2g} \int \phi'(x, \sigma) \partial_\mu \phi(x, \sigma). \]  (2.5)

Consider next the open plaquette starting and ending at $x$, that is,

\[ W_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x + a \hat{\mu})U^\dagger_\mu(x + a \hat{\nu})U^\dagger_\nu(x). \]  (2.6)

To leading order in $a$ we have

\[ W_{\mu\nu}(x) = e^{ia^2 g \chi_{\mu\nu}} e^{ia^2 e \int F_{\mu\nu}(x, \sigma)Q(\sigma)}, \]  (2.7)

where the pure phase is

\[ \chi_{\mu\nu} = f_{\mu\nu} + \frac{\kappa e^2}{g} \int A'_\mu A_\nu \]  (2.8)

with

\[ f_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \]
and

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \]

Note that the term \( \kappa \int A'_\mu A_\nu \) in the phase \( \chi_{\mu\nu} \) arises due to the central extension in the Kac-Moody algebra and is absent for gauge theories based on compact Lie groups. Under gauge-transformations (2.4) we have, upon using (2.2),

\[
W_{\mu\nu}(x) \rightarrow \Phi(x)W_{\mu\nu}(x)\Phi^\dagger(x) \quad (2.9)
\]

which yields

\[
\delta F_{\mu\nu} = 0 \quad (2.11)
\]

and

\[
\delta \chi_{\mu\nu} = \frac{\kappa e}{g} \int F_{\mu\nu} \phi' . \quad (2.12)
\]

Note that unlike the case of compact Lie groups, the Kac-Moody gauge transformation (2.9) induces an inhomogeneous transformation of \( \int F_{\mu\nu} \phi' \) on \( F_{\mu\nu} \) and hence induces a change in the phase \( \chi_{\mu\nu} \). To obtain a gauge-invariant lagrangian we need to introduce a new field \( A_5(x,\sigma) \) to cancel the variation \( \int F_{\mu\nu} \phi' \), and which transforms as

\[
\delta A_5(x,\sigma) = -\frac{1}{e} \phi'(x,\sigma) . \quad (2.13)
\]

Defining the antisymmetric tensor

\[
\Gamma_{\mu\nu}(x) = \chi_{\mu\nu}(x) + \frac{\kappa e^2}{g} \int F_{\mu\nu} A_5(x,\sigma) , \quad (2.14)
\]

we have from (2.11, 2.12), and (2.13)

\[
\delta \Gamma_{\mu\nu} = 0 . \quad (2.15)
\]

It has been shown in [2] that \( \Gamma_{\mu\nu} \) can be generalised to the nonabelian case. Define

\[
F_{\mu5}(x,\sigma) = \partial_\mu A_5 - A'_\mu \quad (2.16)
\]

which from (2.5) and (2.13) is gauge invariant,

\[
\delta F_{\mu5} = 0 . \quad (2.17)
\]
Hence we have the following classical lagrangian\(^3\) in \( \mathcal{R}_d \times S^1 \) which is invariant under the Kac-Moody gauge transformations (2.5) and (2.13)

\[
\mathcal{L}(x) = \frac{1}{4} \Gamma_{\mu\nu}^2(x) + \frac{1}{4} \int d\sigma d\sigma' F_{\mu\nu}(x, \sigma) h(\sigma - \sigma') F_{\mu\nu}(x, \sigma') \\
+ \frac{1}{2} \int d\sigma d\sigma' F_{\mu5}(x, \sigma) f(\sigma - \sigma') F_{\mu5}(x, \sigma'),
\]

(2.18)

and where, as defined above \((\partial_5 = \partial/\partial\sigma)\),

\[
\Gamma_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \frac{\kappa e^2}{g} \int A'_\mu A_\nu + \frac{\kappa e^2}{g} \int F_{\mu\nu} A_5, \\
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \\
F_{\mu5} = \partial_\mu A_5 - \partial_5 A_\mu.
\]

(2.19)

Recall that \( A_\mu = A_\mu(x, \sigma) \), \( A_5 = A_5(x, \sigma) \) and \( B_\mu = B_\mu(x) \). Since \( \Gamma_{\mu\nu}(x) \) has an asymmetric coupling in the \( \sigma \)-direction, we anticipate a similar asymmetry in the pure \( A_\mu, A_5 \) sector and this is reflected in the functions \( h \) and \( f \) which have been introduced in (2.18). These functions are required to be positive but are otherwise undetermined.

From the kinetic term for the \( B_\mu \) field, its canonical mass dimension is deduced to be \((d - 2)/2\). We choose the \( A_\mu \) and \( A_5 \) fields to have the mass dimension \((d - 1)/2\), which is the appropriate canonical dimension for gauge fields in \( d + 1 \) dimensions. Then the coupling constant \( e \) has mass dimension \((3 - d)/2\) and

\[
\lambda \equiv \frac{\kappa e^2}{g}
\]

(2.20)

has mass dimension \((2 - d)/2\), while the mass dimension of the functions \( h(\sigma) \) and \( f(\sigma) \) in (2.18) is 1. The Fourier transform of \( h(\sigma) \) (and similarly for \( f(\sigma) \)) can therefore be written as \((\sum_n \equiv \sum_{n=-\infty}^{\infty})\)

\[
h(\sigma) = \frac{1}{R} \sum_n h_n e^{ip_n\sigma},
\]

(2.21)

where \( p_n = 2\pi n/R, \ n \in \mathbb{Z}, \ R \) is the length of the compact \( \sigma \)-dimension, and the Fourier coefficients \( h_n \) are dimensionless. We will eventually be interested in the case \( d = 3 \) and since \( R \) is the natural scale in the theory, we can define a dimensionless coupling constant

\[
\bar{\lambda} = \lambda/\sqrt{R}.
\]

(2.22)

\(^3\)Following the convention of [3], the index \( \mu \) covers the \( d \) coordinates of the Euclidean manifold \( \mathcal{R}_d \) while ’5′ labels the \((d+1)\)-th coordinate in \( S^1 \).
Though we will mostly discuss the pure gauge theory in this paper, we summarize here the coupling of fermions to the gauge fields [2]. Their gauge-invariant coupling to the fields \( A_\mu, A_5 \) is given by the usual Dirac lagrangian 

\[
S_F^1 = \int d^d x \, d\sigma \, \bar{\psi}(x, \sigma) \left[ (\partial_\mu - ieA_\mu)\gamma_\mu + (\partial_5 - ieA_5)\gamma_5 + m \right] \psi(x, \sigma) \tag{2.23}
\]

where \( \psi(x, \sigma), \bar{\psi}(x, \sigma) \) are the Dirac fermions defined on the \((d+1)\) dimensional manifold \( \mathbb{R}^d \times S^1 \). The coupling of fermions to \( B_\mu(x) \) is more subtle [2]. Consider the \( \hat{U}(1) \) super Kac-Moody algebra in which Majorana fermions \( H(\sigma) \), satisfying 

\[
\{ H(\sigma), H(\sigma') \} = \delta(\sigma - \sigma'),
\]

together with \( Q(\sigma) \) form the generators. The super Virasoro algebra formed from \( Q(\sigma) \) and \( H(\sigma) \) has \( c = 1 + 1/2 \) with a (suitably regularized) super-charge operator \( G(\sigma) = \frac{1}{\kappa}Q(\sigma)H(\sigma) \). Kac-Moody fermions \( \Psi(x) \) are defined on \( \mathbb{R}_d \), and form an arbitrary irreducible representation of the super Virasoro algebra. They have Kac-Moody invariant coupling to the gauge fields as follows:

\[
S_F^2 = \int d^d x \, \bar{\Psi}(x) \left[ (\partial_\mu - ie \int A_\mu(x, \sigma)Q(\sigma) - igB_\mu(x))\gamma_\mu + M \right] \Psi(x) + \tau \int d^d x \, \bar{\Psi}(x) \int [G(\sigma) - A_5(x, \sigma)H(\sigma)] \Psi(x) .
\tag{2.24}
\]

The \( \tau \) term in \( S_F^2 \) has been introduced to attenuate the high mass states. The full fermion coupling is given by \( S_F^1 + S_F^2 \). The thermodynamical partition function of the free lagrangian for \( \Psi(x) \) given in (2.24) has been studied in [3] and exhibits a maximum limiting temperature. In Section 4.4 we will discuss the contribution of the ordinary Dirac fermions (2.23) to the beta-function.

### 3 Path-Integral Quantization.

We define the quantum theory through the Feynman path-integral and the Fadeev-Popov ansatz for the gauge-fixing. It is convenient to use the background field formalism [4] whereby the fields are split into background fields \( (A_\mu, A_5, B_\mu) \) and quantum fields \( (a_\mu, a_5, b_\mu) \). The classical action (2.18) is then invariant under the transformation

\[
\delta(A_5 + a_5) = -\frac{1}{e} \phi',
\]

\[
\delta(A_\mu + a_\mu) = -\frac{1}{e} \partial_\mu \phi,
\]

\[
\delta(B_\mu + b_\mu) = -\frac{1}{g} \partial_\mu \Lambda + \frac{\kappa e}{g} \int \phi'(A_\mu + a_\mu) - \frac{\kappa}{2g} \int \phi'(\partial_\mu \phi) .
\]

\footnote{Note again that the index ‘5’ refers to the compact coordinate, and so \( \gamma_5 \) has nothing to do with the usual chirality matrix.}
The action with given background fields is still invariant under a gauge-transformation of
the quantum fields alone. These quantum gauge transformations are deduced by deleting
the classical fields on the left-hand-side of (3.1-3.3),
\[ \delta^Q a_5 = -\frac{1}{e} \phi', \]  
(3.4)
\[ \delta^Q a_\mu = -\frac{1}{e} \partial_\mu \phi, \]  
(3.5)
\[ \delta^Q b_\mu = -\frac{1}{g} \partial_\mu \Lambda + \frac{\kappa e}{g} \int \phi' (A_\mu + a_\mu) - \frac{\kappa}{2g} \int \phi' (\partial_\mu \phi). \]  
(3.6)
Hence to define the background field path-integral \( Z[A_\mu, A_5, B_\mu] \), the quantum gauge fields
must be gauge-fixed and an infinite group volume factored out. However the gauge-fixed
action is required to be invariant under background field transformations which are of the
same form as the classical gauge-invariance. That is,
\[ \delta^B a_5 = -\frac{1}{e} \phi', \]  
(3.7)
\[ \delta^B A_\mu = -\frac{1}{e} \partial_\mu \phi, \]  
(3.8)
\[ \delta^B B_\mu = -\frac{1}{g} \partial_\mu \Lambda + \frac{\kappa e}{g} \int \phi' (A_\mu) - \frac{\kappa}{2g} \int \phi' (\partial_\mu \phi), \]  
(3.9)
\[ \delta^B b_\mu = \frac{\kappa e}{g} \int a_\mu \phi', \]  
(3.10)
\[ \delta^B a_\mu = \delta^B a_5 = 0. \]  
(3.11)
A gauge-fixing term for the \( a_\mu \) and \( b_\mu \) fields which is invariant under (3.7-3.11) is
\[ \delta (\partial_\mu a_\mu) \delta \left( \partial_\mu b_\mu + \frac{\kappa e^2}{g} \int A_\mu' a_\mu - c_\mu \right) \]  
(3.12)
with \( c_\mu(x) \) an arbitrary function. To see the invariance of the Dirac-delta-function con-
straints in (3.12), consider the variation of the argument of the second term,
\[ \delta^B \left( \partial_\mu b_\mu + \frac{\kappa e^2}{g} \int A_\mu' a_\mu - c_\mu \right) = \partial_\mu \delta^B b_\mu + \frac{\kappa e^2}{g} \int (a_\mu \delta^B A_\mu' + A_\mu' \delta^B a_\mu) \]
\[ = \partial_\mu \left( \frac{\kappa e}{g} \int a_\mu \phi' \right) + \frac{\kappa e^2}{g} \int \frac{(-\partial_\mu \phi')}{e} a_\mu \]
\[ = \frac{\kappa e}{g} \int \phi' \partial_\mu a_\mu \]
\[ = 0, \]
where in obtaining the last line we used that, from (3.12), \( \partial_\mu a_\mu = 0 \). Also, the constraint
on the \( a_\mu \) field in (3.12) is trivially invariant under (3.11). When the gauge-fixing (3.12)
is implemented in the path-integral, the constraint on the $b_\mu$ field may be exponentiated as usual by averaging over the $c_\mu$ variable with a Gaussian weight. However, as noted above, it is crucial for the invariance of (3.12) that the $a_\mu$ field be in the strict Landau gauge. The Fadeev-Popov ghosts due to the above gauge-fixing are non-interacting and so may be ignored. Thus the gauge-fixed action, including external sources $J_\mu, K_\mu$ and $L$, is

$$S_{\text{eff}} = \lim_{\alpha \to 0} \int d^4x \left\{ \mathcal{L}(A_5 + a_5, A_\mu + a_\mu, B_\mu + b_\mu) + \frac{1}{2\alpha} \int (\partial_\mu a_\mu)^2 \right\}$$

$$+ \int d^4x \left\{ \frac{1}{2\eta} (\partial_\mu b_\mu + \frac{\kappa e^2}{g} \int A'_\mu a_\mu)^2 + \int J_\mu a_\mu + \int L a_5 + K_\mu b_\mu \right\},$$

with the lagrangian $\mathcal{L}$ given by (2.18-2.19). This action is invariant under the background transformation (3.7-3.11) plus the following transformation for the sources

$$\delta^B J_\mu = -\frac{\kappa e}{g} K_\mu \phi',$$

$$\delta^B K_\mu = \delta^B L = 0,$$

and therefore the quantum field theory defined by the background path-integral

$$Z[A_\mu, A_5, B_\mu, J_\mu, K, L] = \int [Da_\mu Da_5 Db_\mu] e^{-S_{\text{eff}}}$$

is invariant under the transformations (3.7-3.11) and (3.14). The one-particle-irreducible (1PI) effective action of the "vacuum" theory (that is without background fields) is obtained by computing, using (3.15), 1PI graphs with no external quantum fields [4] and by identifying the background fields in (3.14) with the fields appearing in the 1PI action. This 1PI action is invariant under the Kac-Moody gauge transformations (3.7-3.9).

For a one-loop calculation we only need to expand (3.14) up to second order in the quantum fields. Terms linear in the quantum fields may be ignored as they do not contribute to the 1PI effective action. The free propagators for the gauge-fields are given in momentum space by (for simplicity we choose the Feynman gauge $\eta = 1$ for the $b_\mu$ field)

$$<b_\mu b_\nu> = \frac{\delta_{\mu\nu}}{p^2},$$

$$<a_\mu a_\nu> = \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{1}{f_n p_n^2 + h_n b^2},$$

$$<a_5 a_5> = \frac{1}{f_n p_n^2},$$

$$<a_\mu a_5> = 0.$$
Note that because of the strict Landau gauge condition on the $a_\mu$ field, there is no mixed $<a_\mu a_5>$ propagator at tree-level.

Finally, the fermionic action (2.23), being conventional, can be incorporated in the usual way [4].

4 Beta function

4.1 General

The tree-level action (3.14) has been written in terms of "bare" quantum and background fields, and bare coupling constants $\lambda$ and $e$. When the quantum fields are integrated out in the path-integral (3.15) to generate the full 1PI effective action, UV divergences will be present in the new terms which have been added to the classical action. As the full 1PI action is guaranteed to be gauge-invariant in the background gauge, the divergences will take a gauge-invariant form. If the theory is renormalizable, which we show below to be the case at one-loop, the divergences can be removed by rescaling the fields and couplings in the bare classical lagrangian so that the new lagrangian, and hence the physical partition function, is finite when written in terms of finite "renormalized" quantities. Consider the one-loop UV divergent correction $\Delta \Gamma^{2}_{\mu\nu}$ to $\Gamma^{2}_{\mu\nu}$. Then we want to write

$$\Gamma^{2}_{\mu\nu} + \Delta \Gamma^{2}_{\mu\nu} \equiv \Gamma^{(r)2}_{\mu\nu},$$

by renormalizing the bare tensor (in $d = 3 - 2\epsilon$ dimensions)

$$\Gamma_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} + \mu_{0}^{\epsilon} \sqrt{R} \bar{\lambda} \int d\sigma ( A_{\mu} A_{\nu} + F_{\mu\nu} A_{5} ),$$

where $\mu_{0}$ is some fixed mass scale (such as $R$), into the renormalized tensor

$$\Gamma^{r}_{\mu\nu} = \partial_{\mu} B^{r}_{\nu} - \partial_{\nu} B^{r}_{\mu} + \mu^{\epsilon} \sqrt{R} \bar{\lambda} \bar{r} \int d\sigma ( A^{r}_{\mu} A^{r}_{\nu} + F^{r}_{\mu\nu} A^{r}_{5} )$$

defined at the new mass scale $\mu$. The bare quantities in (4.2) are related to the renormalized ones in (4.3) by the $Z$ factors,

$$B_{\mu} = \sqrt{Z_{B}} B^{r}_{\mu},$$

$$A_{\mu} = \sqrt{Z_{A}} A^{r}_{\mu},$$

$$A_{5} = \sqrt{Z_{5}} A^{r}_{5},$$

and

$$\bar{\lambda} = Z_{\lambda} \bar{\lambda} \bar{r}.$$
(We are implicitly assuming the theory is multiplicatively renormalizable, a fact which we will verify to one-loop by explicit calculations.) As the Kac-Moody gauge-invariance (3.7,3.9) of the 1PI action is maintained in the background gauge, the $Z$ factors must be related as follows

$$Z_5 = Z_A$$  \hspace{1cm} (4.8)

and

$$\sqrt{Z_B} = Z_\lambda \mu^\epsilon \sqrt{Z_A Z_5}$$  \hspace{1cm} (4.9)

$$= Z_\lambda Z_5 \left( \frac{\mu_0}{\mu} \right)^\epsilon.$$  \hspace{1cm} (4.10)

Hence

$$\bar{\lambda} = \frac{\sqrt{Z_B}}{Z_5} \left( \frac{\mu}{\mu_0} \right)^\epsilon \bar{\lambda}_r,$$  \hspace{1cm} (4.11)

from which the $\beta$-function $\beta_\lambda = \mu \frac{\partial \bar{\lambda}}{\partial \mu}$ be determined by differentiating both sides of (4.11) with respect to $\mu$ and taking $\epsilon \to 0$. In subsections 4.2 and 4.3, we determine the wavefunction renormalization factors required in (4.11) to one-loop order in the pure gauge theory. Note that though $Z_5$ and $Z_A$ are equal from (4.8), we have used the former in (4.11) as it is easier to determine directly (only two diagrams need be analysed). In Section 4.4 we will discuss how including the Dirac fermions (2.23) changes the number of independent couplings, and also compute the new one-loop beta-functions.

### 4.2 $Z_B$

Two diagrams (see Fig.2) contribute to the one-loop correction $\Delta \Gamma^2_{\mu\nu}$ and these determine $Z_B$. We obtain

$$\Delta \Gamma^2_{\mu\nu}(x) = -4 T \Gamma^2_{\mu\nu}(x)$$  \hspace{1cm} (4.12)

where

$$T = \frac{-\bar{\lambda}^2 R}{8(4-d)} S(d) \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2(q^2+1)} + \text{convergent terms as } d \to 3,$$

and

$$S(d) = \mu^2 \left( \frac{2\pi}{R} \right)^{d-2} \sum_n \frac{|n|^{d-2}}{\mu_n^{d/2}} f_n^{d-4}.$$  \hspace{1cm} (4.13)
Note that although the functions \( f_n \) and \( h_n \) appear in very different ways in the two diagrams, they finally contribute in the same manner through the sum above. Since the \( d \)-dimensional loop momentum integral is ultraviolet (UV) finite for \( d \leq 3 \), the only possible ultraviolet (UV) divergence as \( d \to 3 \) comes from the sum over \( n \). To evaluate (4.13) we have to specify the functions \( f_n \) and \( h_n \). We choose (see below)

\[
\begin{align*}
h_n &= c_1|n| + c'_1, \\
f_n &= c_2|n| + c'_2.
\end{align*}
\] (4.14)

For the boundedness of the path-integral we require the constants \( c_i \) and \( c'_i \) to be non-negative. If either of \( c_1 \) or \( c_2 \) were zero then, since the Riemann zeta-function \( \zeta(s) = \sum_1^\infty \frac{1}{n^s} \) has a pole only for \( s = 1 \), (specifically \( \zeta(s \to 1) \to 1/(s - 1) \)), the sum (4.13) would be finite as \( d \to 3 \). For the choice (4.14 - 4.15) with \( c_1 \) and \( c_2 \) nonzero, and as \( \epsilon \to 0 \),

\[
S(d = 3 - 2\epsilon) \to \frac{2\pi}{R} \frac{1}{\sqrt{c_1^3 c_2}} \frac{1}{\epsilon} + O(\epsilon^0). \] (4.16)

Hence, using the minimal subtraction scheme,

\[
\Delta \Gamma^2_{\mu\nu}(x) = \frac{\lambda^2}{4\epsilon} \frac{1}{\sqrt{c_1^3 c_2}} \Gamma^2_{\mu\nu}(x),
\] (4.17)

and since \( \Gamma^2_{\mu\nu} + \Delta \Gamma^2_{\mu\nu} = \Gamma^{(r)}^2 \equiv Z_B^{-1} \Gamma^2_{\mu\nu} \), this implies

\[
Z_B^{-1} = 1 + \frac{\lambda^2}{4\epsilon} \frac{1}{\sqrt{c_1^3 c_2}}.
\] (4.18)

As \( \lambda = \lambda_r + O(\lambda_r^3) \), therefore

\[
Z_B = 1 - \frac{\lambda^2}{4\epsilon} \frac{1}{\sqrt{c_1^3 c_2}}.
\] (4.19)

To motivate the choice (4.14 - 4.15), let us assume that \( h_n \) and \( f_n \), for large \( n \), have the power-like asymptotic forms

\[
\begin{align*}
h_n &\to c_s|n|^s, \\
f_n &\to c_t|n|^t.
\end{align*}
\] (4.20)

For integral \( s, t > 0 \), the sum (4.13) for large \( n \) is then

\[
\sim \sum_n |n|^{d-2} |n|^{\left(\frac{d-4}{2}\right)} |n|^{-\frac{d}{2}}.
\] (4.22)
From the properties of the Riemann \( \zeta \)-function, as \( d \to 3 \), this has a divergence only if

\[
3s + t = 4 .
\] (4.23)

The only positive integral solution to (4.23) is \( s = t = 1 \). If the condition (4.23) is not satisfied then the \( n \) sum is finite as \( d \to 3 \) and we would obtain \( Z_B = 1 \). Furthermore as \( Z_5 = 1 \) (see next subsection) independent of \( f_n \) and \( h_n \), this implies that the one-loop \( \beta \) function for \( \lambda \) is zero for the \textit{ansatz} (4.20-4.21) unless \( s = t = 1 \). Thus our choice (4.14-4.15), and any others equivalent to it at large \( n \), would give a running coupling while other choices of \( h_n \) and \( f_n \) with asymptotic forms (4.20-4.21) would give a UV finite theory at one-loop.

### 4.3 \( Z_5 \)

This is determined from the two diagrams contributing to the self-energy of the \( A_5 \) field: The tadpole diagram Fig.(3a) and the non-tadpole Fig.(3b) diagrams. Neither of these involves an internal \( n \) summation. Since the \( a_\mu \) propagator is massive, the tadpole digram has a potential UV divergence but this divergence is linear and vanishes in dimensional regularization: Recall that in dimensional regularization (DR) the scaleless integral \((d > 2)\)

\[
\int \frac{d^d q}{q^2} = 0 ,
\] (4.24)

so that

\[
\int \frac{d^d q}{q^2 + m^2} = \int d^d q \left( \frac{1}{q^2 + m^2} - \frac{1}{q^2} \right) = \int d^d q \left( \frac{-m^2}{q^2 (q^2 + m^2)} \right)
\] (4.25)

is finite as \( d \to 3 \). Similarly a potential linear UV divergence in the other diagram is removed in DR. (We mention in passing that as \( p \to 0 \), the UV finite pieces of the two diagrams cancel so that the \( A_5 \) field self-energy vanishes at zero momentum, as required by gauge invariance.) Hence \( Z_5 = 1 \) at one-loop order. As a cross-check, we have verified, by looking at the \( < A_\mu A_\nu > \) and \( < A_\mu A_5 > \) self-energies that they are also finite near \( d = 3 \) so that \( Z_5 = Z_{A_\mu} = 1 \) as required by gauge-invariance in the background gauge. Furthermore, the fact that the divergence in the \( B_\mu \) field two-point function could be renormalised by a wavefunction renormalisation and that the \( A_\mu \) and \( A_5 \) two-point functions are finite, justifies \textit{a posteriori} our assumption in the last subsection.
of multiplicative renormalizibility of the two-point functions at one-loop. Thus it follows from (4.11) and (4.19) that

$$\beta_\lambda = -\frac{\lambda^3}{4\sqrt{c_1c_2}}.$$  \hspace{1cm} (4.26)

The pure, abelian, Kac-Moody gauge theory is asymptotically free! The only other known asymptotically free gauge theory in four dimensions is Yang-Mills (YM) theory (and linearizations thereof [5]), which is based on a non-abelian group. It has been understood [5] that the crucial ingredient for asymptotic freedom in YM theory is the self-interaction of spin one (vector) fields which makes the vacuum of YM theory behave like a paramagnetic medium. As the abelian Kac-Moody gauge theory also has self-interacting vector fields, asymptotic freedom in this case has probably a similar physical interpretation. However we remind that unlike YM theory, the Kac-Moody gauge theory is nonlocal, or equivalently it may be interpreted as a local theory but with an infinite set of non-commuting charges.

### 4.4 Including Dirac Fermions.

The Dirac fermions $\psi(x, \sigma), \bar{\psi}(x, \sigma)$ couple only to the gauge-fields $A_\mu, A_5$ as indicated in (2.23), and the resulting theory with gauge-fields and Dirac fermions has two independent coupling constants: $\lambda$ and $e$. The wave-function renormalization constant $Z_A$ now obtains a quantum-electrodynamic-like contribution [4] from the fermion-loop diagram of Fig.(4),

$$Z_\psi = Z_A = 1 - \frac{e^2N}{12\pi^2\epsilon},$$ \hspace{1cm} (4.27)

where $N$ is the number of fermion flavours. The divergence which (4.27) absorbs (through a rescaling of the bare fields $A_\mu, A_5$) is that in the loop correction to the usual kinetic terms $\frac{1}{4}F_{\mu\nu}^2$ and $\frac{1}{2}F_{\mu5}^2$, that is, Eq.(2.18) with $h(\sigma - \sigma') = f(\sigma - \sigma') = \delta(\sigma - \sigma')$. Therefore with Dirac fermions present, the renormalizability of the theory seems to restrict the choice (4.14-4.15) further to $h_n = f_n = 1$. However, as we argue below, it is still possible to choose the more general form

$$h_n = f_n = 1 + c|n|.$$ \hspace{1cm} (4.28)

With (4.28), the kinetic terms for the $A_\mu$ and $A_5$ fields in (2.18) are now split into two pieces each: The ‘1’ in (4.28), gives rise to the conventional kinetic term which is renormalized by (4.27) while the $c|n|$ part gives rise to another kinetic term. Since
at one-loop there is no ultraviolet divergence corresponding to the latter kinetic term, consistency demands that $c$ may be nonzero only if it renormalizes inversely to (4.27), so that the unconventional kinetic term is unrenormalized. That is, we require

$$Z_c Z_A = 1,$$  \hspace{1cm} (4.29)

where the bare parameter $c$ is related to the renormalized parameter $c_r$ by

$$c = Z_c c_r.$$  \hspace{1cm} (4.30)

(Actually what is being renormalized is a mass parameter $c/R$).

Now the bare charge $e$ is related to the renormalized charge $e_r$ by

$$e = \left(\frac{\mu}{\mu_0}\right)^\epsilon Z_c e_r.$$  \hspace{1cm} (4.31)

Since from (2.23) and gauge-invariance in the background gauge

$$Z_c = \frac{1}{\sqrt{Z_A}} = \frac{1}{\sqrt{Z_5}},$$  \hspace{1cm} (4.32)

therefore

$$\beta_e = \mu \frac{\partial e_r}{\partial \mu} = -e_r \epsilon + \frac{e^3_r N}{12\pi^2},$$  \hspace{1cm} (4.33)

as in quantum electrodynamics (QED). (We have kept the term of order $\epsilon$ in $\beta_e$ as it is needed below). Then one obtains

$$\gamma_c = \mu \frac{\partial c_r}{\partial \mu} = \frac{c_r e^2_r N}{6\pi^2}.$$  \hspace{1cm} (4.34)

Note that there is no term of order $\epsilon$ on the right-hand-side of the last equation because, as mentioned before, $c/R$ is a mass parameter of fixed mass dimension 1, so that no compensating mass parameter $\mu'$ enters in its dimensional continuation.

Using the values of $\beta_e, \gamma_c, Z_5$ above, and $Z_b$ from (4.19) with $c_1 = c_2 = c$, we obtain from (4.11)

$$\beta_\lambda = -\frac{\lambda_r^3}{4e_r^2} + \frac{\lambda_r e_r^2 N}{6\pi^2}.$$  \hspace{1cm} (4.35)
to lowest order. The contribution of the fermions is of opposite sign to that of the gauge-fields just as in quantum-chromodynamics (QCD). However unlike QCD, we have two independent coupling constants appearing in (4.35) so that $\beta_\lambda$ can be arranged to have a nontrivial zero in weak coupling: For $\epsilon_r$ small, $\beta_\lambda$ vanishes at

$$\lambda^*_r = \epsilon_r \frac{2c_r}{\pi} \left( \frac{N}{6} \right)^{\frac{1}{2}}.$$  (4.36)

Since $\lambda^*_r \sim O(\epsilon_r)$ even for $c_r$ of order 1 and $N$ not too large, the result (4.36) is self-consistent as it is within the perturbative regime for both $\lambda_r$ and $\epsilon_r$. Thus starting from short distances (but $\epsilon_r$ still small), $\lambda_r$ tends to increase at longer distances, but before it can become too large the fermion coupling drives it to $\lambda^*_r$. This point is infrared stable since $\beta'_\lambda(\lambda^*_r) > 0$. However $\lambda^*_r$ is not a fixed point of the whole system since $\epsilon_r$ continues to run.

5 Renormalizability

We want to show that the theory as defined by (3.14) is at least one-loop renormalizable. That is, for all one-loop diagrams with an arbitrary number of external legs, the UV divergences can be absorbed by rescaling the terms in the lagrangian (2.18). Note that since the tensors $\Gamma_{\mu\nu}, F_{\mu\nu}, F_{\mu 5}$ are individually gauge-invariant, radiative corrections could in principle generate more gauge-invariant terms than are in the lagrangian (2.18), such as

$$\Delta L = \int \left\{ \alpha_1(\partial_{\mu} F_{\mu\nu}) F_{\nu 5} + \alpha_2 \Gamma_{\mu\nu} F_{\mu\nu} + \alpha_3 F_{\mu\nu} \partial^2 F_{\mu\nu} + \ldots \right\}$$  (5.1)

In this section we show that, remarkably, all the one-loop UV divergences have the same form as the minimal lagrangian (2.18) so that no extra terms such as (5.1) are needed for renormalization. This is also fortunate because some of the terms in (5.1) are not positive definite and so unsuitable for inclusion in the tree-level lagrangian. More specifically, we will sketch renormalizability of the theory for the case

$$h_n = c_1 |n| + c'_1$$  (5.2)
$$f_n = c_2 |n| + c'_2$$  (5.3)

discussed in the last section (for (4.20-4.21) the theory is even more convergent). For the purpose of determining the UV degree of divergence of the one loop diagrams, it is
sufficient to consider the large $n$ limit in which case we can treat the discrete sum as a continuous integral by making the replacements

$$\frac{n}{R} \rightarrow qa \quad \text{and} \quad \frac{1}{R} \sum_n \int d^3q \rightarrow \int dqz d^3q.$$ 

(5.4)

In sections 5.1 and 5.2 we consider various subsets of one-loop diagrams contributing to the 1PI action to the pure gauge theory and in section 5.3 we consider the fermions (2.23).

## 5.1 Case A: diagrams containing at least one external $\Gamma$ vertex

(i) Exactly two $\Gamma$’s:

We have already shown (in Sect 4.2) that the one-loop diagrams with two $\Gamma(\equiv \Gamma_{\mu\nu})$ vertices has a logarithmic UV divergence which is removed by wavefunction renormalization, and that no other divergences remain in those diagrams.

(ii) More than two $\Gamma$’s:

Since the two-$\Gamma$ diagrams were logarithmically divergent, adding any more external vertices makes the diagrams UV convergent.

(iii) One $\Gamma$ and some other external vertices:

From the Feynman rules, one sees that at one-loop no gauge-invariant term in the 1PI action can be formed from exactly one $\Gamma$ and one $F_{\mu\nu}$ (or $F_{\mu5}$). So consider possible terms of the form $\Gamma(F)^n, n \geq 2,$ (we have used a symbolic notation: indices and possible derivatives have been suppressed, and $F$ here denotes $F_{\mu\nu}$ or $F_{\mu5}$). Consider first $n = 2$; Some of these diagrams (see Fig.(5)) actually contribute to $\Delta\Gamma_{\mu\nu}^2$ but by gauge-invariance their UV divergence is related to the wavefunction renormalization of the $B_\mu$ field already discussed in case A(i) above. There are also diagrams which are not part of corrections to $\Gamma_{\mu\nu}^2$ and so must be part of corrections to $\Gamma(F)^2$ (see e.g. Fig.(6)). We wish to show that the net contribution of diagrams such as in Fig.(6) to $\Gamma(F)^2$ is only a finite renormalization. Note that superficially Fig.(6) is logarithmically UV divergent but gauge-invariance demands that the external $A_5$ vertex must eventually (either after doing the detailed calculation or after adding other similar diagrams) become a $\partial_\mu A_5$. This then makes the diagram UV convergent.

With the above effective power counting (that is, (5.4) plus gauge invariance), one can show that all diagrams contributing to $\Gamma(F)^2$ are convergent. Then it follows that $\Gamma(F)^n (n \geq 2)$, radiative contributions to the 1PI action at one-loop order are UV finite.
5.2 Case B: diagrams not containing an external $\Gamma$ vertex.

(i) Terms of the form $(F)^{2n+1}, n \geq 1$.
It is impossible to form such terms at one-loop order as it requires an odd number of
$A - b - a$ vertices which cannot be contracted since the $b$ fields must occur in pairs.

(ii) $(F)^4$
Some of the diagrams containing four external $A$ lines (see eg. Fig.(7)) contribute to $\Delta \Gamma^2_{\mu\nu}$
and therefore their UV divergence is as before related to the wavefunction renormalization
of the $B_\mu$ field. Here we need to consider those four-$A$ diagrams which contribute to $F^4$.
Consider for example the diagram of Fig.(8). There is no internal $n$-sum because of the
$b$ propagator, so superficially the UV degree of divergence is that of the integral $\int \frac{d^dq}{q^6}$
which is logarithmic. More explicitly integrals of the form
\[
\int \frac{d^dq}{q^6(m^2 + (p - q)^2)}
\] (5.5)
appear but these are finite as $d \to 3$ because in an expansion around $p = 0$, the first term
is zero by symmetry and the rest are finite. Thus the superficially divergent diagram is
actually UV finite.

(iii) $F^{2n}, n \geq 3$.
Since the $n = 2$ diagrams were only (superficially) logarithmically divergent, the $n \geq 3$
diagrams are therefore convergent by the effective power counting rules discussed above.

5.3 Diagrams with Dirac fermion lines

Since the fermionic part of the action (2.23) looks like QED, all the one-loop diagrams
are the same as there. As in QED, there is only one UV divergent diagram with a closed
fermion loop, Fig(4). However unlike QED, the fermion self-energy diagram and the vertex
diagram in Fig.(9) are UV finite for $d = 3$ because the $a_\mu$ propagator is more damped (see
eq.(3.17)) than the photon in normal QED. Thus the vertex renormalization factor $Z_v$
and the fermion wave-function renormalization factor $Z_\psi$ corresponding to Fig.(9) equal
identity.

6 Conclusion

We have studied in this paper the quantization of Kac-Moody gauge-field theory intro-
duced in Ref.[2]. Although such a theory had been written in [2] for the non-abelian
case, we considered here the abelian limit which still retained the interesting features of
nonlinerality, nonlocality and spatial asymmetry. The last two features are a result of interpreting the internal Kac-Moody space as a compact spatial coordinate.

We quantized the pure $\hat{U}(1)$ Kac-Moody gauge theory (2.18) using the path-integral formulation in a background field and showed that it was one-loop renormalizable. Indeed, at one-loop, we found that because of the unusual propagators for the $a_\mu$ and $a_5$ fields, the theory was UV finite within dimensional regularisation for a wide range of the functions $h_n$ and $f_n$ appearing in the classical theory. However for $h_n$ and $f_n$ behaving like $|n|$ for large $n$, we found the dimensionless coupling $\bar{\lambda}_r$ was asymptotically free. When Dirac fermions (2.23) were added, the theory gained another coupling constant, $e$. At first sight, renormalizability and gauge-invariance now seemed to demand that the functions $h_n$ and $f_n$ took the form (4.28) with $c = 0$. However by keeping $c$ nonzero, we obtained a more interesting structure for the beta-functions: This could be done by promoting $c$ from a fixed quantity to a mass-like parameter which was allowed to be renormalized according to (4.29). In this way the constraints of gauge-invariance and renormalizability could be satisfied and the beta function $\beta_\lambda$ obtained two terms of opposing sign (4.35). The couplings $e_r$ and $\bar{\lambda}_r$ could be chosen in the perturbative regime to make $\beta_\lambda$ vanish.

If the theory is to be regarded as a fundamental theory then its renormalizability should in the future be studied to all orders. It would also be interesting to quantize the theory by including the Kac-Moody fermions (2.24). We only note here from the actions (2.18), (2.23) and (2.24) that the partition function for the full theory comprising gauge-fields and Kac-Moody fermions has the intriguing property of apparently being sensitive to the sign of the parameter $\kappa$ appearing in the Kac-Moody algebra (2.1).

One might wonder about the theory for other values of $d$. We found that already at one-loop the theory (3.14) was not renormalizable for $d = 4$ : For example a divergence of the form $p^4 R^2/(d - 4)$ appeared in the self-energy of the $A_5$ field and such a term could not be absorbed by wave-function renormalization. For $d = 1$ and $d = 2$ the pure gauge-theory is free.

Even if the $3 + 1$ dimensional theory is not ultimately renormalizable to all orders in the sense of QED, it might be profitable to treat it as a low energy effective theory. As the lagrangian (2.18) is naturally asymmetric, one is tempted then to associate it with a description of high-temperature superconductors (HTS), which are anisotropic systems.\footnote{At non-zero temperature the Kac-Moody gauge theory would be defined on $S^1 \times R_{d-1} \times S^1$.} It has been argued (see \cite{7} and references therein) that the non-Fermi liquid behaviour of the normal state of HTS requires for its explanation a planar dynamical gauge field in the effective low energy theory. This role in the lagrangian (2.18) could be played by $B_\mu(x)$,
while $A_\mu(x, \sigma), A_5(x, \sigma)$ and the fermions would describe some other fields (perhaps the effective electromagnetic fields and fermionic quasi-particles respectively). Of course detailed calculations are required to see if our proposal of (2.18) has indeed the required properties to describe the properties of HTS in the normal state, and if it has a phase transition at non-zero temperature to a superconducting state. In any case, regardless of a particular application, it would be interesting to study how the infrared dynamics of (2.18) depends on the anisotropy scale $R$, and what physical consequences issue from the Kac-Moody symmetry of the system.

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**Figure Captions**

Fig.1. The $a_\mu$ propagator (wavy line), the $a_5$ propagator (curly line), the $b_\mu$ propagator (dashed line) and the Dirac fermion propagator (unbroken line).

Fig.2. Diagrams contributing to $Z_B$.

Fig.3. Gauge field diagrams contributing to $Z_5$.

Fig.4. Fermion loop contributing to $Z_5$.

Fig.5. UV divergent diagrams contained in the one-loop correction to $\Gamma^2_{\mu\nu}$.

Fig.6. A diagram not contained in $\Gamma^2_{\mu\nu}$. The curly external line is $A_5$, the wavy external line is $A'_\mu$, while the dashed external line is $\partial_\mu B_\tau$.

Fig.7. A diagram contained in the one-loop correction to $\Gamma^2_{\mu\nu}$.

Fig.8. A superficially UV divergent diagram not contained in the one-loop correction to $\Gamma^2_{\mu\nu}$. It is UV finite.

Fig.9. Fermion self-energy and vertex-correction diagrams.
Fig. 1. The $a_\mu$ propagator (wavy line), the $a_5$ propagator (curly line), the $b_\mu$ propagator (dashed line) and the Dirac fermion propagator (unbroken line).

Fig. 2. Diagrams contributing to $Z_B$. 
Fig. 3. Gauge field diagrams contributing to $Z_5$.

Fig. 4. Fermion loop contributing to $Z_5$. 
Fig. 5. UV divergent diagrams contained in the one-loop correction to $\Gamma_{\mu
u}^2$.

Fig. 6. A diagram not contained in $\Gamma_{\mu
u}^2$. The curly external line is $A_5$, the wavy external line is $A'_\rho$, while the dashed external line is $\delta_\mu B_\tau$.

Fig. 7. A diagram contained in the one-loop correction to $\Gamma_{\mu
u}^2$. 

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Fig. 8 A superficially UV divergent diagram not contained in the one-loop correction to $\Gamma^2_{\mu\nu}$. It is UV finite.

Fig. 9. Fermion self-energy and vertex-correction diagrams.