Bipartite non-locality beyond Bell inequalities by means of multivariable correlations

Bruno Leggio,1 Bruno Bellomo,2 Romain Azaïs,1 Przemysław Prusinkiewicz,3 and Christophe Godin1

1Laboratoire Reproduction et Développement des Plantes, Univ Lyon, ENS de Lyon, UCB Lyon 1, CNRS, INRA, Irissa, F-69342, Lyon, France
2Institut UTINAM, CNRS UMR 6213, Université Bourgogne Franche-Comté, Observatoire des Sciences de l’Univers THETA, 41 bis avenue de l’Observatoire, F-25010 Besançon, France
3Department of Computer Science, University of Calgary, Calgary, AB T2N 1N4, Canada

We give a set of necessary conditions for locality in bipartite systems, which include and generalize known Bell’s inequalities. Each condition corresponds to a specific order of the expansion of random variables defined on graphs, in terms of genuinely $n$-variable correlation functions. The first non-trivial order leads to known Bell inequalities, while higher orders produce additional, non-equivalent conditions. In particular, in CHSH settings, we obtain at least two additional tight conditions which do not reduce to the CHSH inequality. This shows that tight Bell inequalities are sufficient but not necessary conditions for the non-locality of bipartite quantum correlations.

Ever since the revolutionary work of Bell [1], quantum systems have been known to present non-local correlations and to be intrinsically indeterministic [2, 3]. Numerous extensions of the original Bell’s inequality have been produced [2–9], and experimental verifications of their violations [10–13] have confirmed that quantum mechanics is non-local: no deterministic theory based on hidden variables can account for the correlations observed in the quantum realm [14]. This groundbreaking result has far-reaching theoretical, philosophical, and practical consequences [3].

Bell inequalities test the local determinism of correlations of measurement outcomes. They are characterized by the number of subsystems $N_S$ a physical system is partitioned into, the number of variables per subsystem $N_V$, and the number of possible outcomes per variable $N_m$. Each inequality sets a condition which classical systems obey. An important class of Bell inequalities are the tight ones, for which equality may be realized in classical systems: tight inequalities are regarded as necessary and sufficient conditions for non-locality since they set a sharp boundary between local and non-local behavior of correlations [2].

For $N_S = N_V = N_m = 2$, the only known tight condition [15] is the CHSH inequality [4]

$$|\langle x_1 y_1 \rangle + \langle x_1 y_2 \rangle + \langle x_2 y_1 \rangle - \langle x_2 y_2 \rangle| \leq 2,$$  \hfill (1)

where $x_1$, $x_2$ and $y_1$, $y_2$ are, respectively, pairs of random variables of the first and the second subsystem, and $\langle \cdot \rangle$ stands for the average value. For appropriately chosen $x_1$, $x_2$ and $y_1$, $y_2$, bipartite quantum states can violate inequality (1) [2].

In this Letter we show that tightness of Eq. (1) does not imply that its satisfaction is a sufficient condition for locality. By formalizing local determinism for any number of measurements in terms of conditions on oriented paths on a graph, we prove that Eq. (1) is obtained as the first non-trivial order of the cumulant expansion of random variables defined on these paths. Higher orders, given in terms of genuinely $n$-variable correlations, are shown to produce additional tight inequalities, which can detect non-locality in settings where Eq. (1) cannot.

Classicality.—Consider several copies of the same bipartite system $S_a \otimes S_b$, each subsystem being equipped with a set of binary random variables with values $\pm 1$, respectively $\{x_i\}$ and $\{y_j\}$, as shown in Fig. 1(a). On each copy of the system, pairs $(x_i, y_j)$ are measured. Classicality of these measures implies two conditions: local determinism applies and all measurements commute (i.e., any variable can be measured without affecting the outcomes of any other measurements). Local determinism postulates the existence of hidden variables $\lambda$, not known to the observer, which deterministically impose the value of each $z \in \{x_i\} \cup \{y_j\}$. When measuring the same random variable $x_i$ on several copies of $S_a$ (with potentially different values of $\lambda$), the observed distribution of outcomes is determined $a$
\[ \kappa_n (V_P) = \kappa_n (V_{P'}) \quad \forall n, \tag{2} \]

where \(\kappa_n\) denotes the \(n\)-th order cumulant \[17\] \[18\] (see also Appendix A). Since classicality implies the set of conditions in Eq. (2), the existence of at least one \(n_0\) such that \(\kappa_{n_0}(V_P) \neq \kappa_{n_0}(V_{P'})\) implies that the system behaves non-classically. We show in Appendix B that non-commutativity between local variables may lead to \(V_P \neq V_{P'}\) even in the absence of non-locality. On the other hand, as we now proceed to show, we can extract from Eq. (2) a class of conditions to test locality independently of commutativity.

**Second-order conditions.**—Let us start by examining Eq. (2) for \(n = 2\) in the case of the example of Fig. 1(b), where \(\mathcal{P}_1 = \{z_1 = x_2, z_2 = y_1, z_3 = x_1, z_4 = y_2\} \) and \(\mathcal{P}_{II} = \{z'_1 = x_2, z'_2 = y_2\} \). Using the identities \(\kappa_2(X) = \text{Var}(X)\) and \(\text{Var}(A + B) = \text{Var}(A) + \text{Var}(B) + \text{cov}(A, B) + \text{cov}(B, A)\), we can rewrite Eq. (2) for \(n = 2\) and for \(\mathcal{P} = \mathcal{P}_1 \) and \(\mathcal{P}' = \mathcal{P}_{II}\) as

\[ \tilde{\mathcal{C}}_2 = \text{Var}(\Lambda_{z_1}^{z_2}) + \text{Var}(\Lambda_{z_2}^{z_3}) + \text{Var}(\Lambda_{z_3}^{z_4}) \]

\[ - \text{Var}(\Lambda_{z_1}^{z'_2}) \tag{3} \]

\[ \mathcal{C}_2 = \frac{1}{2} \left[ \text{cov}(\Lambda_{z_2}^{z_2}, \Lambda_{z_2}^{z_3} + \Lambda_{z_2}^{z_4}) + \text{cov}(\Lambda_{z_2}^{z_2}, \Lambda_{z_2}^{z_3} + \Lambda_{z_2}^{z_4}) \right. \]

\[ + \text{cov}(\Lambda_{z_2}^{z_2} + \Lambda_{z_2}^{z_3} + \Lambda_{z_2}^{z_4} + \Lambda_{z_2}^{z_2} + \Lambda_{z_2}^{z_3} + \Lambda_{z_2}^{z_4}) \] \tag{4} \]

According to our general definition of paths, \(z_i\) and \(z_{i+1}\) are variables of different physical systems and then commute, while in general \(z_i\) does not commute with \(z_{i+h}\) if \(h\) is even. Hence, all
terms in $\tilde{C}_2$ involve products of commuting variables, whereas the term $\overline{C}_2$ contains all potentially non-commuting terms $\text{cov}(\Lambda_{z_{i+1}}^{z_i}, \Lambda_{z_{i+h+1}}^{z_{i+h}})$.

Expressing $\overline{C}_2$ in terms of vertex variables $x, y$ yields a linear combination of second-order cumulants $\kappa_2(z_i) \equiv \text{Var}(z_i)$ and two-variable joint cumulants $\kappa(z_i, z_{i+1}) \equiv \text{cov}(z_i, z_{i+1})$. Separating terms $\kappa_2(z_i)$ and $\kappa(z_i, z_{i+1})$, we obtain $\overline{C}_2 = \overline{\mathbb{A}}_2 + \overline{\mathbb{F}}_2$, where

\[
\overline{\mathbb{A}}_2 = \text{Var}(x_1) + \text{Var}(y_1), \quad (5)
\]

\[
\overline{\mathbb{F}}_2 = \text{cov}(x_2, y_2) - \text{cov}(x_1, y_1)
\quad \text{and} \quad -\text{cov}(x_1, y_2) - \text{cov}(x_2, y_1). \quad (6)
\]

Hence, for $n = 2$, Eq. (2) is equivalent to $\overline{\mathbb{A}}_2 + \overline{\mathbb{F}}_2 + \overline{\mathbb{C}}_2 = 0$. For $\overline{T}_2 = \overline{\mathbb{A}}_2 + \overline{\mathbb{F}}_2, \overline{\mathbb{C}}_2, \overline{\tilde{C}}_2$, we define $T_2 = T_2(t_i = 0 \forall i)$ and we observe that the modulus of $\overline{\mathbb{B}}_2$ coincides with the l.h.s. of Eq. (1). The r.h.s. of Eq. (1) is a consequence of the fact that, as we show in Appendix C for classical variables, $|\mathbb{A}_2 + \mathbb{C}_2| \leq 2$.

The condition $\langle t_i \rangle = 0$ is only used here to formally obtain $\mathbb{A}_2, \mathbb{B}_2$, and $\mathbb{C}_2$ from $\overline{\mathbb{A}}_2, \overline{\mathbb{F}}_2, \overline{\mathbb{C}}_2$. It does not mean, however, that the inequality $|\mathbb{B}_2| \leq 2$ only holds for zero-mean variables: the functions $\mathbb{A}_2, \mathbb{B}_2$, and $\mathbb{C}_2$ must still satisfy $\mathbb{A}_2 + \mathbb{B}_2 + \mathbb{C}_2 = 0$ for any classical variables, including those with non-zero mean, due to the fact that $\overline{\tilde{C}}_2 + \overline{\mathbb{C}}_2 = 0$ is a rewriting of the identity $0 = 0$. Hence, the condition $|\mathbb{B}_2| \leq 2$ is valid for any classical random binary variable, which is a known property of the CHSH inequality.

Since $\overline{\tilde{C}}_2$ is independent of commutativity, the same holds for $\mathbb{B}_2$. Hence, a violation of $|\mathbb{B}_2| \leq 2$ can only be due to non-locality and implies that $\text{Var}(V_{\mathcal{P}}) \neq \text{Var}(V_{\mathcal{P}'}).$

Following the same procedure, different choices of compatible paths lead to different Bell’s functions at second order. Panels (c), (d), and (e) of Fig. [panel (e)] show the choices of $\mathcal{P}$ and $\mathcal{P}'$ to obtain the CHSH function [panel (e)], the original function derived by Bell [1] [panel (d)], and an additional function $\mathbb{B}_2(\mathcal{P}_{III}, \mathcal{P}_{IV}) = \langle x_2 y_3 \rangle - \langle x_1 y_1 \rangle - \langle x_1 y_2 \rangle - \langle x_2 y_1 \rangle - \langle x_3 y_2 \rangle - \langle x_3 y_3 \rangle$ [panel (e)].

The exhaustive characterization of all possible bipartite Bell inequalities for any numbers of variables per subsystem $N_V$ is challenging and has until now only been achieved for $N_V = 2$ and $N_V = 3$ [2]. By enumerating all compatible paths for a given number of variables, our method paves the way for a more systematic exploration of such multi-variable scenarios.

**General result.** We now consider the general case of cumulants of order $n$ in Eq. (2). Note that [19]

\[
\kappa_n(a + b) = \sum_{j=0}^{n} \sum_{\Pi \in \mathcal{S}_{j}^{(n)}} \kappa(\Pi(a[b], b[n-j])) = \sum_{j=0}^{n} \sum_{\Pi \in \mathcal{S}_{j}^{(n)}} \kappa(\Pi(a[b], b[n-j])) = 0
\]

in terms of the joint (or multivariable) cumulants $\kappa(a, \ldots, b)$ [17]. Here, $\mathcal{S}_{j}^{(n)}$ is the set of all possible anagrams $\Pi$ of the word $a[b], b[n-j]$, with $x[i] = x, \ldots, x$. Joint cumulants, measuring genuine multivariable correlations [20–22], quantify here $n$-variable correlations in bipartite systems. As we show in Appendix D, the difference of $n$-th order cumulants of path variables can be decomposed into a sum of $n$-th order cumulants of each edge variable ($\mathcal{C}_n$) and joint cumulants connecting different edge variables ($\mathcal{C}_n$). In analogy to the case $n = 2$, only $\mathcal{C}_n$ is potentially affected by non-commutativity, since it connects variables $z_i$ with $z_{i+h}$ for $h > 1$. Expressing $\mathcal{C}_n$ in terms of vertex variables, grouping $n$-th order cumulants in an $\mathcal{A}_n$ term and joint cumulants in a $\mathcal{B}_n$ term and introducing, for $\mathcal{T}_n = \mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n$, the functions $T_n = T_n|_{(t_i) = 0 \forall i}$ we obtain

\[
\kappa_n(V_{\mathcal{P}}) - \kappa_n(V_{\mathcal{P}'}) = 0 \Rightarrow \mathcal{A}_n + \mathcal{B}_n + \mathcal{C}_n = 0 \quad (8)
\]

where we have defined the generalized $n$-th order Bell function $\mathbb{B}_n$ as

\[
\mathbb{B}_n(\mathcal{P}, \mathcal{P}'') = \frac{1}{2} \sum_{j=1}^{n-1} \binom{n}{j} \left[ \sum_{z_i, z_{i+1} \in \mathcal{P}} \kappa^*(z_j, z_{j+1}) - \sum_{z_i, z_{i+1} \in \mathcal{P}'} \kappa^*(z_i, z_{i+1}) \right].
\]
with $\kappa^*(a^{[j]}, b^{[n-j]}) := \kappa(a^{[j]}, b^{[n-j]})|_{(a) = (b) = 0}$ and $-z^{[i]} := (-z)^{[i]}$. Equation (9) is derived in detail in Appendix $\mathcal{D}$. Given $\mathcal{P}$ and $\mathcal{P}'$, $\mathcal{B}_n$ in Eq. (9) is classically bounded by the extrema of $A_n + C_n$. These bounds produce locality-testing inequalities at every order in the cumulant expansion, valid for variables with any mean: as for the case $n = 2$, the condition $\langle t_i \rangle = 0$ is only used to formally obtain the specific expression in Eq. (9), but $A_n + B_n + C_n = 0$ must be satisfied by all classical binary variables, independently of their mean. These inequalities, whose bounds are discussed in the following, are the main result of this Letter.

Before doing that, we remark that additional inequalities can be obtained using bounded functions of a finite number of $\mathcal{B}_n$, with different orders $n$. In analogy to the link between moments $\mu_n$ and cumulants $\kappa_n$ given by the recursive function $\mu_n = \sum_{i=0}^{n-1} \binom{n-1}{i} \kappa_{n-i} \mu_i \ $ [23], we define a moment-inspired $n$-th order Bell function $\mathcal{B}_n^M$ from the first $n$ $\mathcal{B}_i$ in Eq. (9) as

$$\mathcal{B}_n^M = \sum_{i=0}^{n-1} \binom{n-1}{i} \mathcal{B}_{n-i} \mathcal{B}_i^M,$$  

where $\mathcal{B}_0^M = 1$. This definition has the advantage of inducing a notion of expansion order also to the functions $\mathcal{B}_n^M$. We give the explicit expressions for the first few orders in Appendix $\mathcal{E}$. In the following we show that at fourth order the inequalities related, respectively, to Eq. (9) and (10) are not equivalent.

One might in principle obtain local bounds on $\mathcal{B}_n$ and $\mathcal{B}_n^M$ analytically, as we have done for $\mathcal{B}_2$ (Appendix $\mathcal{C}$), but the task becomes rapidly intractable. Instead, as detailed in Appendix $\mathcal{E}$ we have numerically maximized each $\mathcal{B}_n$ as a function of the classical probabilities of all possible measurement outcomes. This means automatically that our inequalities are tight. For the paths of Fig. 1(c) (\(\mathcal{P}_i, \mathcal{P}_I\)) and Fig. 1(e) (\(\mathcal{P}_{II}, \mathcal{P}_{IV}\)), we obtain the values reported in Table I. A violation of these bounds implies that the two random variables $V_P$ and $V_{P'}$ differ in their $n$-th order cumulants because of non-locality. As we now proceed to show, $n$-th order ($n > 2$) inequalities can detect non-locality when known

| Function | \((\mathcal{P}, \mathcal{P}')\) | min | Max |
|----------|------------------|-----|-----|
| $\mathcal{B}_1$ | $\mathcal{P}_I, \mathcal{P}_I$ | 20 | 20 |
| $\mathcal{B}_2$ | $\mathcal{P}_I, \mathcal{P}_I$ | 32 | 04 |
| $\mathcal{B}_2^M$ | $\mathcal{P}_{II}, \mathcal{P}_{IV}$ | 27.2 | 04 |
| $\mathcal{B}_3$ | $\mathcal{P}_{II}, \mathcal{P}_{IV}$ | 75.2 | 04 |

* For binary random variables, $\mathcal{B}_1 = \mathcal{B}_3 = 0$ (see Appendix $\mathcal{F}$).

**TABLE I.** Maximal and minimal classically-allowed values for the first few orders in Bell functions for binary random variables. These values are obtained for classical bounded random variables, with no assumption on their averages. Higher order values are given in Appendix $\mathcal{F}$.

**FIG. 2.** Rescaled Bell functions $\mathcal{B}_n$ and $\mathcal{B}_n^M$, obtained by scaling $\mathcal{B}_n$ and $\mathcal{B}_n^M$ such that their classical bounds are $\pm 2$. All functions are shown in absolute value as functions of $\theta_{y_1}$. (a) and (b) $|\mathcal{B}_n|$ and $|\mathcal{B}_n^M|$ for paths $\mathcal{P}_1, \mathcal{P}_I$ of Fig 1(c). The state is $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. In (a): $\theta_{x_1} = 0, \theta_{x_2} = 2.55$ rad, and $\theta_{y_2} = 3.7$ rad; in (b): $\theta_{x_1} = 0, \theta_{x_2} = 3.6$ rad, and $\theta_{y_2} = 3.1$ rad; the inset is a zoom around the second maximum of $\mathcal{B}_4$. (c) $|\mathcal{B}_n|$ and $|\mathcal{B}_n^M|$ for paths $\mathcal{P}_{II}, \mathcal{P}_{IV}$ of Fig 1(e). The state is $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $\theta_{x_1} = \theta_{x_2} = 0, \theta_{y_2} = \theta_{y_3} = 5.5$ rad, and $\theta_{x_3} = 0.46$ rad. For comparison, $|\mathcal{B}_2(\mathcal{P}_I, \mathcal{P}_I)| = |\mathcal{B}_2(\mathcal{P}_I, \mathcal{P}_I)|$ [i.e., the l.h.s. of Eq. (1)] is also shown.

Bell inequalities cannot.

**Examples.**—Consider a two spin-1/2 system in a pure entangled state $|\psi\rangle$ and the observables $\hat{S}^{(j)}(\theta) = \sin \theta \hat{\sigma}_j^z + \cos \theta \hat{\sigma}_j^y$, where $\hat{\sigma}_j^z$ are Pauli operators for the two particles ($j = 1, 2$). The measurement angles $\theta_{x_1}$ and $\theta_{y_1}$ define the random variables $x_i$ and $y_i$. Not every choice of these four angles leads to a violation of Eq. (1), despite the state being entangled. One could wonder whether, for these choices, non-locality of correlations hides in higher orders.

Panels (a) and (b) of Fig. 2 show that this is indeed the case: for paths $\mathcal{P}_I$ and $\mathcal{P}_I$, and for the state and angles given in the caption, functions
$B_4$ and $B_4^{M} = B_4 + 3B_2^2$ violate their classical limits even if $B_2$ does not. Correlations between the measurement outcomes on particles 1 and 2 at certain angles are non-local as captured by 4-th order inequalities, but this non-locality is not detected by $B_2$. In this case, correlations are compatible with $\text{Var} \left( V_{P} \right) = \text{Var} \left( V_{P'} \right)$. This proves that a violation of Eq. (1) is not a necessary condition for non-locality. Figure 2 (c) shows that similar results occur using the paths $P_{III}$ and $P_{IV}$, as detailed in the caption.

Conclusions.—In this Letter we have developed a novel approach to study quantum non-locality of arbitrary bipartite systems based on a graph representation of pairs of random variables defined on paths. The locality of the correlations between the outcomes of the measurements of these variables has been explored through an expansion in terms of genuinely $n$-variable correlation functions. We have obtained Bell-like inequalities from each order in this expansion, recovering, at the second order and by appropriate choices of paths, known Bell inequalities. We have explicitly shown for a two-qubit system that higher order inequalities may reveal non-locality even when the tight CHSH condition cannot. Our results point out that, even for pure states, tight Bell inequalities do not fully characterize quantum non-locality. We stress that the new inequalities introduced here can be measured with exactly the same experimental setups used to test well-known biparite Bell functions, and have hence a direct experimental relevance.

Our graph-based approach paves the way for a more comprehensive characterization of quantum non-locality and could possibly be extended to other scenarios, such as multipartite systems.

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Appendix A: Definition of cumulants

Here, we provide the textbook definition of cumulants and joint cumulants [17, 18]. Consider a random variable $X$ and its probability function $p_X$. Let us start by introducing the moment-generating function $M(h, X)$ for a random variable $X$, defined as

$$ M(h, X) = \langle e^{hX} \rangle, \quad (A1) $$

with $h \in \mathbb{R}$ and where $\langle f(X) \rangle$ stands for the mean value $\int dx f(x)p_X(x)$. The function $M(h, X)$ allows one to obtain all moments $\mu_n(X)$ of $X$ as

$$ \mu_n(X) = \left. \frac{d^n M(h, X)}{dh^n} \right|_{h=0}. \quad (A2) $$

The cumulant-generating function $K(h, X)$ is defined from $M(h, X)$ as

$$ K(h, X) = \ln M(h, X), \quad (A3) $$

and the cumulants $\kappa_n(X)$ of the random variable $X$ are defined as

$$ \kappa_n(X) = \left. \frac{d^n K(h, X)}{dh^n} \right|_{h=0}. \quad (A4) $$

Note that, while Eq. (A2) is not the textbook definition of moments but rather a property of $M(h, X)$, cumulants are usually defined through Eq. (A4). The set of cumulants $\{\kappa_n\}$ contains the same information on the random variable $X$ as the set of moments $\{\mu_n\}$. In particular, if the support of the probability function $p_X$ is a bounded interval, the set $\{\mu_n\}$ (and hence the set $\{\kappa_n\}$) fully and uniquely characterizes the variable $X$ [16].

Joint cumulants of $m$ variables $X_1, \ldots, X_m$ are defined equivalently through the joint cumulant generating function

$$ K_f(h_1, \ldots, h_m, X_1, \ldots, X_m) = \ln \langle e^{\sum_i h_i X_i} \rangle, \quad (A5) $$

with $h_1, \ldots, h_m \in \mathbb{R}$. Using the notation $X_{[i]} = \underbrace{X, \ldots, X}_i$, joint cumulants of the random varia-
ables $X_1, \ldots, X_m$ are then defined as
$$\kappa(X_1^{[i_1]}, \ldots, X_m^{[i_m]}) = \frac{\partial^{i_1 + \cdots + i_m}}{\partial h_1^{i_1} \cdots \partial h_m^{i_m}}$$
$$K_f(h_1, \ldots, h_m, X_1, \ldots, X_m) \Big|_{h_1=\ldots=h_m=0} \tag{A6}$$

Incidentally, note that joint cumulants can be expressed in terms of mixed moments by means of Möbius inversion of the partition lattice $[18]$. Consider the set $\Gamma$ of all possible partitions of a set $\{X_1, \ldots, X_m\}$. Each element $p \in \Gamma$ contains $n_p$ subsets $s_1, \ldots, s_{n_p}$ representing a partition of the set $\{X_1, \ldots, X_m\}$. Be $\mu_s(s_i) = \langle X_{i_1} \cdots X_{i_k} \rangle$ the average product of the $k$ elements $X_{i_1}, \ldots, X_{i_k}$ of $s_i$. Then
$$\kappa(X_1, \ldots, X_m) = \sum_{p \in \Gamma} (-1)^{n_p-1} (n_p-1)! \prod_{s_i \in p} \mu_s(s_i). \tag{A7}$$

Appendix B: Non-commutativity contribution to non-classicality of $V$ and $V'$

We have shown in the main text that non-locality can generate violations of the classical identity $V = V'$, independently of local commutators. Here, we show that non-commutativity of local variables can also lead to $V \neq V'$, even in the absence of non-locality. The variables $V = (x_2 - y_1) + (y_1 - x_1) + (x_1 - y_2)$ and $V' = (x_2 - y_2)$ are defined in the main text.

Let us consider the simple case of two 1/2-spin particles in the pure product state $|\phi\rangle = |0\rangle \otimes |1\rangle$, $|0\rangle$ and $|1\rangle$ being the eigenstates of the Pauli operators $\sigma_z^{(j)}$, $j = 1, 2$, with, respectively, eigenvalues $-1$ and $+1$. Be also $x_1 = \sigma_x^{(1)}$, $y_1 = \sigma_x^{(2)}$, $x_2 = \sigma_z^{(1)}$, and $y_2 = \sigma_z^{(2)}$ the outcome of a measurement of the observable represented by the Pauli operator $\sigma_z^{(j)}$. Of course, since there is no entanglement initially between the two particles, and since all we perform on them are local operations, no non-locality is at play during the measurements leading to $V$ and $V'$. However, the local operators involved in the measurement protocol do not commute between themselves.

Let us start looking at what happens when performing the measurements involved in the definition of $V$. After measuring the first pair of operators to obtain $(x_2, y_1) = (\sigma_x^{(1)}, \sigma_x^{(2)})$, one gets $x_2 = -1$ with certainty and, depending on the result of the measurement $y_1 = \sigma_z^{(2)}$, the state collapses to
$$|\phi\rangle \to |x_\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes (|0\rangle \pm |1\rangle)). \tag{B1}$$

The second measurement, leading to the pair of values $(x_1, y_1) = (\sigma_x^{(1)}, \sigma_x^{(2)})$, produces the collapse
$$|x_\pm\rangle \to |\xi_{\pm}\rangle = \frac{1}{2} (|0\rangle \pm |1\rangle) \otimes (|0\rangle \pm |1\rangle). \tag{B2}$$

Hence, the third pair of operators is measured on one of the four states $|\xi_{\pm}\rangle$ in order to obtain the couple of values $(x_1, y_2) = (\sigma_x^{(1)}, \sigma_z^{(2)})$. For all these states one has $1/2$ probability of obtaining $y_2 = \sigma_z = 1$ and $1/2$ of obtaining $y_2 = \sigma_z = -1$. This means that $V$ is a random variable with two possible outcomes,
$$V = \begin{cases} 0, & p = \frac{1}{2} \\ -2, & p = \frac{1}{2} \end{cases}. \tag{B3}$$

Concerning $V'$, it is straightforward to see that differently from $V$, this variable is not random and deterministically takes the value $-2$.

This proves that, even in the absence of non-locality, local non-commutativity alone may lead to $V \neq V'$.

Appendix C: CHSH inequality from path equivalence

Here, we derive the known CHSH bound $|B_2(P_1, P_{11})| \leq 2$. We recall that $A_2 = \overline{A_2}_{|t_i = 0 \forall i}$, $B_2 = \overline{B_2}_{|t_i = 0 \forall i}$, $C_2 = \overline{C_2}_{|t_i = 0 \forall i}$, with $\overline{A_2}, \overline{B_2}$ and $\overline{C_2}$ defined in Eqs. (5), (6), and (4) of the main manuscript, and that $A_2 + B_2 + C_2 = 0$. We will exploit the properties of $A_2$ and $C_2$ for classical binary variables with values $\pm 1$. Note that, by their very definition, the edge variables satisfy the following properties
$$\Lambda^z_{zi} + \Lambda^z_{zj} = \Lambda^z_{zi}, \tag{C1}$$
$$\Lambda^z_{zi} = -\Lambda^z_{zi}, \tag{C2}$$
$$z_i, z_j \in [-1, 1] \Rightarrow \Lambda^z_{zi} \in [-2, 2]. \tag{C3}$$
\[
\langle i \rangle = \langle j \rangle = 0 \Rightarrow \langle A^2 \rangle_i = 0. \tag{C4}
\]

Concerning \( A_2 \), it is straightforward to see from its definition that for binary variables it holds \( A_2 = 2 \).

Now, we need to study the limits within which \( C_2 \) is classically allowed to vary. Let us begin by recasting \( C_2 \) into a more appropriate form. Using property (C1), the fact that the covariance is bilinear with respect to its arguments, and the relation, valid for classical variables, \( \text{Var}(A + B) = \text{Var}(A) + \text{Var}(B) + 2 \text{cov}(A, B) \), \( C_2 \) can be rewritten as

\[
C_2 = \text{cov} \left( \Lambda_2^{y_1}, \Lambda_2^{y_2} \right) + \text{cov} \left( \Lambda_2^{x_1}, \Lambda_2^{x_2} \right)
= \left[ \frac{1}{2} \left( \text{Var}(\Lambda_2^{x_2}) - \text{Var}(\Lambda_2^{y_1}) - \text{Var}(\Lambda_2^{a_{x_1}}) \right) \right]. \tag{C5}
\]

Due to Eq. (C3), \( \text{Var}(z_i) \leq 1 \) and \( \text{Var}(\Lambda_2^{z_i}) \leq 4 \). For the sake of notational simplicity, let us introduce \( \alpha = \Lambda_2^{y_1}, \beta = \Lambda_2^{y_2}, \gamma = \Lambda_2^{x_1} \). Hence, \( \Lambda_2^{y_2} = \alpha + \gamma - \beta \) and

\[
C_2 = -V_\beta - V_\gamma - c_{\alpha\gamma} + c_{\alpha\beta} + c_{\beta\gamma}, \tag{C6}
\]

having set \( \text{Var}(t) = V_t \) and \( \text{cov}(t, h) = c_{th} \). Using Eq. (C4) in Eq. (C6), we then obtain

\[
C_2 = -\langle \beta^2 \rangle - \langle \gamma^2 \rangle - \langle \alpha \gamma \rangle + \langle \alpha \beta \rangle + \langle \beta \gamma \rangle, \tag{C7}
\]

with \( \alpha, \beta, \gamma \in \{ -2, 0, 2 \} \). Let us now introduce the probabilities \( p_i \) for each of the 27 possible values of the vector \( \{ \alpha, \beta, \gamma \} \) as given in Table II.

One can now express average values of functions of the variables \( \{ \alpha, \beta, \gamma \} \) in terms of the 27 \( p_i \)’s.

For instance,

\[
\langle \alpha \rangle = -2 \left( p_1 - p_{19} + p_2 - p_{20} - p_{21} - p_{22} - p_{23} - p_{24} - p_{25} - p_{26} - p_{27} + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 \right). \tag{C8}
\]

It is then just a matter of lengthy but straightforward calculations to express Eq. (C7) as a function of the \( p_i \)’s. One gets

\[
C_2 = -4 \left( p_1 + p_{10} + p_{11} + 3p_{12} + p_{13} + p_{15} + 3p_{16} + p_{17} + p_{18} + p_{19} + 2p_{20} + 5p_{21} + 2p_{22} + p_{23} + p_{24} + p_{25} + p_{26} + p_{27} + p_3 + p_4 + 5p_7 + 2p_8 + p_9 \right). \tag{C9}
\]

One already sees that \( C_2 \leq 0 \). However, since \( \alpha = x_2 - y_2, \beta = x_2 - y_1 \) and \( \gamma = x_1 - y_1 \), with \( x_1, x_2, y_1, y_2 \in \{ -1, 1 \} \), strong constraints exist between \( \alpha, \beta, \) and \( \gamma \). For example, consider a configuration in which \( \alpha = 2 \), as the 21st in Table II. Since \( \alpha = 2 \), one must necessarily have \( x_2 = 1 \), which in turn implies that \( \beta \) cannot be negative (it can only assume the values 2 or 0 depending on the value of \( y_1 \)). This argument holds in general: every time the same vertex variable enters (with the same sign) the expression of two

| \{\alpha, \beta, \gamma\} | p_1 | \alpha, \beta, \gamma, (\beta - \alpha)\gamma, (\beta - \gamma)\alpha | Attainable |
|-------------------------|------|------------------------------------------------|------------|
| \{-2, -2, -2\}          | \( p_1 \) | 4, 4, 0, 0 | Yes |
| \{-2, -2, 0\}           | \( p_2 \) | 4, 4, 0, 0 | Yes |
| \{-2, -2, 2\}           | \( p_3 \) | 4, -4, 0, 0 | No |
| \{-2, 0, -2\}           | \( p_4 \) | 0, 0, -4, 0 | No |
| \{-2, 0, 0\}            | \( p_5 \) | 0, 0, 0, 0 | Yes |
| \{-2, 0, 2\}            | \( p_6 \) | 0, 0, 4, 0 | Yes |
| \{-2, 2, -2\}           | \( p_7 \) | -4, -4, 0, 0 | No |
| \{-2, 2, 0\}            | \( p_8 \) | -4, 0, 0, 0 | No |
| \{-2, 2, 2\}            | \( p_9 \) | -4, 0, 4, 0 | No |
| \{0, -2, -2\}           | \( p_{10} \) | 0, 4, 0, 0 | Yes |
| \{0, -2, 0\}            | \( p_{11} \) | 0, 0, 0, 0 | Yes |
| \{0, -2, 2\}            | \( p_{12} \) | 0, -4, 0, 0 | No |
| \{0, 0, -2\}            | \( p_{13} \) | 0, 0, 0, 0 | Yes |
| \{0, 0, 0\}             | \( p_{14} \) | 0, 0, 0, 0 | Yes |
| \{0, 0, 2\}             | \( p_{15} \) | 0, 0, 0, 0 | Yes |
| \{0, 2, -2\}            | \( p_{16} \) | 0, -4, 0, 0 | No |
| \{0, 2, 0\}             | \( p_{17} \) | 0, 0, 0, 0 | Yes |
| \{0, 2, 2\}             | \( p_{18} \) | 0, 4, 0, 0 | Yes |
| \{2, -2, -2\}           | \( p_{19} \) | -4, 4, 0, 0 | No |
| \{2, -2, 0\}            | \( p_{20} \) | -4, 0, 0, 0 | No |
| \{2, -2, 2\}            | \( p_{21} \) | -4, 0, -4, 0 | No |
| \{2, 0, -2\}            | \( p_{22} \) | 0, 0, 4, 0 | Yes |
| \{2, 0, 0\}             | \( p_{23} \) | 0, 0, 0, 0 | Yes |
| \{2, 0, 2\}             | \( p_{24} \) | 0, 0, -4, 0 | No |
| \{2, 2, -2\}            | \( p_{25} \) | 4, -4, 0, 0 | No |
| \{2, 2, 0\}             | \( p_{26} \) | 4, 0, 0, 0 | Yes |
| \{2, 2, 2\}             | \( p_{27} \) | 4, 4, 0, 0 | Yes |
different edge variables, the product of the latter cannot be negative. This imposes the constraint $\alpha\beta \geq 0$. Analogous reasoning lead to the set of conditions

$$
\alpha\beta \geq 0, \quad (C10)
\beta\gamma \geq 0, \quad (C11)
(\beta - \alpha)\gamma \geq 0, \quad (C12)
(\beta - \gamma)\alpha \geq 0. \quad (C13)
$$

This means that, due to the constraints between $\alpha, \beta,$ and $\gamma$, those among the possibilities listed in Table I which do not respect constraints $(C10)$-$(C13)$ cannot exist and must hence be associated with vanishing probabilities. For instance, $p_{21} = 0$ because $\alpha\beta < 0$ for the corresponding configuration. Using the last two columns of Table I it is straightforward to see that Eqs. $(C10)$-$(C13)$ translate to

$$
p_3 = p_4 = p_7 = p_8 = p_9 = p_{12} = p_{16} = p_{19} = p_{20} = p_{21} = p_{24} = p_{25} = 0. \quad (C14)
$$

Using now Eq. $(C14)$ into Eq. $(C9)$ one obtains

$$
\mathbb{C}_2 = -4(p_1 + p_{10} + p_{11} + p_{13} + p_{15} + p_{17} + p_{18} + p_{27}). \quad (C15)
$$

Since one always has $0 \leq p_1 + p_{10} + p_{11} + p_{13} + p_{15} + p_{17} + p_{18} + p_{27} \leq 1$ thanks to the properties of probabilities, we obtain

$$
-4 \leq \mathbb{C}_2 \leq 0. \quad (C16)
$$

Since $A_2 + B_2 + C_2 = 0$, one has

$$
\mathbb{B}_2 = -\mathbb{C}_2 - A_2 \Rightarrow -A_2 \leq \mathbb{B}_2 \leq 4 - A_2. \quad (C17)
$$

Using the fact that $A_2 = 2$, we finally obtain

$$
-2 \leq \mathbb{B}_2 \leq 2. \quad (C18)
$$

### Appendix D: Higher-order Bell functions

Here, we prove Eq. (9) of the main manuscript. Given two compatible paths $P$ and $P'$, and their associated path variables $V_P = \sum_i \Lambda_{zi}^{z_{i+1}}$ and $V_{P'} = \sum_i \Lambda_{zi}^{z_{i+1}}'$, the condition on the equality of their cumulants reads

$$
\kappa_n \left( \sum_i \Lambda_{zi}^{z_{i+1}} \right) - \kappa_n \left( \sum_i \Lambda_{zi}^{z_{i+1}}' \right) = 0. \quad (D1)
$$

The main step is to extract from Eq. $(D1)$ a condition on the $n$-th order cumulants of single edge variables. To this end, we rewrite Eq. $(D1)$ as

$$
\sum_i \kappa_n \left( \Lambda_{zi}^{z_{i+1}} \right) - \sum_i \kappa_n \left( \Lambda_{zi}^{z_{i+1}}' \right) + 2 \mathcal{T}_n = 0, \quad (D2)
$$

where formally $2 \mathcal{T}_n = \kappa_n(\sum_i \Lambda_{zi}^{z_{i+1}}) - \kappa_n(\sum_i \Lambda_{zi}^{z_{i+1}}') - \sum_i \kappa_n(\Lambda_{zi}^{z_{i+1}}) + \sum_i \kappa_n(\Lambda_{zi}^{z_{i+1}}')$. Hence, the condition on the $n$-th order cumulants of single edge variables is

$$
\sum_i \kappa_n \left( \Lambda_{zi}^{z_{i+1}} \right) - \sum_i \kappa_n \left( \Lambda_{zi}^{z_{i+1}}' \right) = -2 \mathcal{T}_n. \quad (D3)
$$

We define the term $2 \mathcal{T}_n$ as the left-hand-side of Eq. $(D3)$. We now proceed to obtain an explicit expression for $\mathcal{T}_n$. This is achieved by repeated use of the expression for the cumulant of the sum of two variables [19], given by

$$
\kappa_n (A + B) = \sum_{j=0}^{n} \sum_{\Pi \in S_j^{(n)}} \kappa \left( \Pi (A^{[j]}, B^{[n-j]}) \right), \quad (D4)
$$

in terms of the joint cumulants $\kappa(A, \ldots, B)$ [18], where $S_j^{(n)}$ is the set of all possible anagrams $\Pi$ of $n$-sized words composed of two letters $A, B$, the first one appearing $j$ times (we recall that $x^{[i]} = x^i_1, \ldots, x^i_i$). Note that one needs to keep the explicit sum over all anagrams in Eq. $(D4)$ due to the potential non-commutativity of some of the terms involved, in the same way as one must write $\text{Var}(a + b) = \text{Var}(a) + \text{Var}(b) + \text{cov}(a, b) + \text{cov}(b, a)$ in the case of non-commuting variables.

Using Eq. $(D4)$ let us write

$$
\kappa_n \left( \sum_{i=1}^{k} A_i \right) = \sum_{j=0}^{n} \sum_{\Pi^{(1)} \in S_j^{(n)}} \kappa \left( \Pi^{(1)} \left( A_1^{[j_1]}, \left( \sum_{i=2}^{k} A_i \right)^{[n-j_1]} \right) \right). \quad (D5)
$$

Note now that joint cumulants are multilinear with respect to their arguments, so that for instance $\kappa(A, B + C, B + C) = \kappa(A, B, B) + \kappa(A, B, C) + \kappa(A, C, B) + \kappa(A, C, C) + \kappa(B, A, B) + \kappa(B, A, C) + \kappa(B, C, A) + \kappa(B, C, C) + \kappa(C, A, B) + \kappa(C, A, C) + \kappa(C, B, A) + \kappa(C, B, C).$
κ(A, B, C) + κ(A, C, B) + κ(A, C, C). Hence, a term of the kind κ(A^{[j_1]}_1, (\sum_{i=2}^{k} A_i)^{[n-j_1]}) gives rise to a sum over all possible (n − j_1)-sized anagrams composed of the k − 1 letters A_2, . . . , A_k, appearing respectively j_2, . . . , j_k−1, n − j_1 − \sum_{i=2}^{k-1} j_i times. Be \mathcal{S}_{j_2, . . . , j_k−1} the set of these anagrams. Thanks to the multilinearity of joint cumulants we can thus write Eq. (D5) as

\[
\kappa_n \left( \sum_{i=1}^{k} A_i \right) = \sum_{j_1=1}^{n} \sum_{\Pi(1) \in \mathcal{S}_{j_1}^{(n)}} \sum_{\Pi(2) \in \mathcal{S}_{j_2, . . . , j_k−1}^{(n)}} \kappa\left( \Pi^{(1)}(A^{[j_1]}_1, A^{[j_2]}_2, . . . , A^{[j_k]}_k) \right).
\]

Finally, since the composition of two anagrams is itself an anagram and since \sum_{j_1=0}^{n} \sum_{j_2+\ldots+j_k=n−j_1} = \sum_{j_1+\ldots+j_k=n}, Eq. (D6) can be cast under the form

\[
\kappa_n \left( \sum_{i=1}^{k} A_i \right) = \sum_{j_1+\ldots+j_k=n} \kappa \left( \sum_{\Pi(1) \in \mathcal{S}_{j_1}^{(n)}} \Pi^{(1)}(A^{[j_1]}_1, . . . , A^{[j_k]}_k) \right),
\]

(D7)

where \mathcal{S}_{j_1, . . . , j_k−1}^{(n)} is the set of n-sized anagrams composed of the k letters A_1, . . . , A_k, appearing respectively j_1, . . . , j_k−1, n − \sum_{i=1}^{k−1} j_i times, so that \sum_{i=1}^{k} j_i = n.

Note that in Eq. (D7) terms of the form κ(A^{[n]}_h) appear \forall h \in \{1, . . . , k\}, in correspondence to \sum_{j_1=0}^{\text{any}} \sum_{\sum_{j\neq h} n−j_1} = 0. For each of these terms, the sum \sum_{\Pi(1) \in \mathcal{S}_{j_1}^{(n)}} = \sum_{\Pi(1) \in \mathcal{S}_{j_2}^{(n)}} contains one term only, since only one anagram exists of a word composed of the same letter repeated n times. Hence, Eq. (D7) contains, among other terms, the sum \sum_{h=1}^{k} κ(A^{[n]}_h). Since finally κ(A^{[n]}_h) = κ_n(A), by using Eq. (D7) in Eq. (D1) one can isolate all n-th order cumulants of single edge variables, i.e., the l.h.s. of Eq. (D3). What is left, i.e., joint cumulants of edge variables, constitutes the term 2 \mathcal{C}_n.

Expressing now the l.h.s. of Eq. (D3) in terms of vertex variables by using again Eq. (D4) and collecting all n-th order cumulants of single vertex variables in a 2 \mathcal{A}_n term, one is left with an expression for 2 \mathcal{B}_n in terms of joint cumulants of vertex variables only. Directly from Eq. (D4) and with the notation \( -x^{[i]} := (−x)^{[i]} \) one obtains

\[
2 \mathcal{B}_n = \sum_{j=1}^{n−1} \sum_{z_i, z_{i+1} \in P} \kappa \left( \sum_{\Pi(1) \in \mathcal{S}_{j}^{(n)}} \Pi^{(1)}(z_i^{[j]}, -z_{i+1}^{[n-j]}) \right)
\]

(D8)

Since one always has that \( z_i \) commutes with \( z_{i+1} \) and \( z'_i \) commutes with \( z'_{i+1} \) because edge variables only connect vertices belonging to different subsystems, and since joint cumulants are invariant after permutations of their arguments if the latter commute, one can replace the sums \( \sum_{\Pi(1) \in \mathcal{S}_{j}^{(n)}} \) by \( \binom{n}{j} \), thus obtaining

\[
\mathcal{B}_n = \frac{1}{2} \sum_{j=1}^{n−1} \binom{n}{j} \left[ \sum_{z_i, z_{i+1} \in P} \kappa \left( z_i^{[j]}, -z_{i+1}^{[n-j]} \right) \right. \\
- \left. \sum_{z'_i, z'_{i+1} \in P'} \kappa \left( z'_i^{[j]}, -z'_{i+1}^{[n-j]} \right) \right].
\]

(D9)

Finally note that, due to the fact that \( \mathcal{A}_n + \mathcal{B}_n + \mathcal{C}_n = 0 \) is not a condition on the t variables but stems from rewriting the identity 0 = 0,

\[
\mathcal{A}_n + \mathcal{B}_n + \mathcal{C}_n = 0 \Rightarrow \mathcal{A}_n + \mathcal{B}_n + \mathcal{C}_n = 0,
\]

(D10)

where \( \mathcal{A}_n = \mathcal{A}_n|_{(t_i)=0 \forall i} \), \( \mathcal{B}_n = \mathcal{B}_n|_{(t_i)=0 \forall i} \) and \( \mathcal{C}_n = \mathcal{C}_n|_{(t_i)=0 \forall i} \). One can thus obtain locality-testing inequalities by bounding \( \mathcal{B}_n \) through \( \mathcal{A}_n + \mathcal{C}_n \). Hence, introducing the modified joint cumulant \( \kappa^* (a^{[j]}, b^{[n-j]}) := \kappa(a^{[j]}, b^{[n-j]})|_{(a)=(b)=0} \), the generalized n-th order Bell function \( \mathcal{B}_n = \)
\[ \mathbb{B}_n |_{t_i = 0} \] reads as

\[ \mathbb{B}_n = \frac{1}{2} \sum_{j=1}^{n-1} \binom{n}{j} \left[ \sum_{z_i, z_{i+1} \in P} \kappa^* \left( z[j], -z'[n-j] \right) - \sum_{z'_i, z'_{i+1} \in P'} \kappa^* \left( z'[j], -z[n-j] \right) \right], \tag{D11} \]

which is Eq. (9) of the main manuscript.

1. First three orders

Let us now explore the explicit structure of the first three orders of Eq. (D11) for the CHSH compatible paths \( P_I \) and \( P_{II} \) introduced in the main text. Calculating directly Eq. (D1) for \( n = 1 \) one has

\[ \langle \Lambda_{y_1}^{x_2} + \Lambda_{y_2}^{x_1} + \Lambda_{y_2}^{x_2} \rangle - \langle \Lambda_{y_1}^{x_2} \rangle = 0. \tag{D12} \]

The l.h.s. of Eq. (D12) is of course equal to \( \langle \Lambda_{y_1}^{x_2} \rangle + \langle \Lambda_{y_2}^{x_1} \rangle + \langle \Lambda_{y_2}^{x_2} \rangle - \langle \Lambda_{y_1}^{x_2} \rangle \), which is the l.h.s. of Eq. (D3). This means that \( \mathbb{C}_1 = 0 \). It is obvious to see, writing this expression in terms of vertex variables \( x_i \) and \( y_j \), that all terms are first order moments of single vertex variables, hence only contributing to \( \mathbb{A}_1 \). This means that \( \mathbb{B}_1 = 0 \). This is the result we were after for the first order Bell function. In addition, in order to test the self-consistency of our formalism, we check whether the condition \( \mathbb{A}_1 + \mathbb{B}_1 + \mathbb{C}_1 = 0 \) implied by Eq. (D3) is satisfied here. This means that, since \( \mathbb{B}_1 = \mathbb{C}_1 = 0 \), \( \mathbb{A}_1 \) must be zero as well. This is indeed the case, since Eq. (D12) identically vanishes because it reduces to

\[ \mathbb{A}_1 = \langle x_2 \rangle - \langle y_1 \rangle + \langle y_1 \rangle - \langle x_1 \rangle + \langle x_1 \rangle - \langle y_2 \rangle - \langle x_2 \rangle + \langle y_2 \rangle = 0. \tag{D13} \]

As shown in the main text, at the second order one obtains the known CHSH inequality.

At the third order, one gets

\[ \mathbb{B}_3 = \frac{3}{2} \left( \kappa^* (x_2, -y_1, -y_1) + \kappa^* (x_2, x_2, -y_1) + \kappa^* (y_1, x_1, -x_1) + \kappa^* (y_1, y_1, -x_1) + \kappa^* (x_1, x_1, -y_2) + \kappa^* (x_1, x_1, -y_2) - \kappa^* (x_2, -y_2, -y_2) - \kappa^* (x_2, x_2, -y_2) \right), \tag{D14} \]

which is a sum of terms of the form \( \kappa^*(a, a, b) \).

Specifying Eq. (A7) for the case \( n = 3 \), one has to enumerate all possible partitions of a set \( \{a, b, c\} \). These are

\[ \Gamma \{ \{a, b, c\} \} = \{ \{a\}, \{b, c\} \}, \{ \{b\}, \{a, c\} \}, \{ \{c\}, \{b, a\} \}, \{ \{a\}, \{b\}, \{c\} \}, \{ \{a, b, c\} \} \}. \tag{D15} \]

Among the five elements of \( \Gamma \{ \{a, b, c\} \} \), the first four have at least one subset containing one element only. For each of them, the factor \( \prod_{i \in S} \mu_S (s_i) \) entering the expression of \( \kappa \) in Eq. (A7) vanishes for \( \kappa^* \) due to the condition \( \langle t_i \rangle = 0 \). Hence, only the last element of \( \Gamma \{ \{a, b, c\} \} \) contributes to \( \kappa^* \) and one obtains \( \kappa^*(a, b, c) = \langle abc \rangle \). Specialising this result to \( \kappa^*(a, a, b) \) and for binary variables with values \( \pm = 1 \) (e.g., variables of spin-1/2 particles) one has \( \kappa^*(a, a, b) = \langle a^2 b \rangle = \langle b \rangle \). Hence,

\[ \mathbb{B}_3 = \frac{3}{2} \left( \langle x_2 \rangle - \langle y_1 \rangle + \langle y_1 \rangle - \langle x_1 \rangle + \langle x_1 \rangle \right) - \langle y_2 \rangle - \langle x_2 \rangle + \langle y_2 \rangle = 0, \tag{D16} \]

which proves that for binary random variables one always has \( \mathbb{B}_3 = 0 \) and the first non-trivial order after \( n = 2 \) is the fourth.

2. Fourth-order cumulant and moment Bell functions

Using Eqs. (D1), (D3), (D4), (D7), and (D11), one obtains at fourth order
\[ A_4 = \kappa_4^* (x_1) + \kappa_4^* (y_1), \]
\[ B_4 = 3\kappa^* (x_1^2, y_1^2) - 2\kappa^* (x_1, y_1^2) - 2\kappa^* (x_1^2, y_2^2) + 3\kappa^* (x_1, y_2^2) - 2\kappa^* (x_1, y_1^3) - 3\kappa^* (x_1^2, y_2^3) + 2\kappa^* (x_2, y_1^3) + 3\kappa^* (x_2^2, y_2^3) \]
\[ + 2\kappa^* (x_2, y_1^3) \]
\[ C_4 = \frac{1}{2} \left[ \kappa^* (x_1^2, x_2^2) + \kappa^* (x_1^2, y_1^2) + \kappa^* (y_1^2, x_2^2) + \kappa^* (x_1^2, y_1^2) + \kappa^* (y_1^2, x_2^2) \right] \]
\[ = 0 \]

with \( \kappa_n^*(a) := \kappa_n(a)|a\rangle \langle a | = 0 \) and where we have used the fact that, thanks to the multilinearity of joint cumulants, \( \kappa(A^{[i]}, B^{[j-i]}) = (-1)^i \kappa(A^{[i]}, B^{[j-i]}) \). Equation (D18) is the 4th order Bell function in the CHSH setup.

Note that, from Eq. (A7), the 4th order modified joint cumulant for variables of zero mean is \( \kappa^*(A, B, C, D) = \langle ABCD \rangle - \langle AB \rangle \langle CD \rangle - \langle AC \rangle \langle BD \rangle - \langle AD \rangle \langle BC \rangle \).

Specializing Eq. (D18) to the case of binary
variables with values ±1 we obtain
\[ B_4 = 8B_2 \]
\[ -6 \left( \langle x_1y_2 \rangle^2 + \langle x_1y_1 \rangle^2 + \langle x_2y_1 \rangle^2 - \langle x_2y_2 \rangle^2 \right), \]  
(D20)

where \( B_2 \) is the function given in Eq. (6) of the main text under the condition \( \langle t_i \rangle = 0 \) ∀\( i \).

**Appendix E: Moments from cumulants**

One elegant and general way of obtaining moments in terms of cumulants, needed to evaluate Eq. (10) of the main text, is given in Ref. [23] in terms of the recursive function

\[ \mu_n = \sum_{i=0}^{n-1} \left( \begin{array}{c} n-1 \\ i \end{array} \right) \kappa_{n-i} \mu_i, \]  
(E1)

with the initial condition \( \mu_0 = 1 \). Solving Eq. (E1) for the first few orders one obtains

\[ \mu_1 = \kappa_1 \]
\[ \mu_2 = \kappa_1^2 + \kappa_2 \]
\[ \mu_3 = \kappa_1^3 + 3\kappa_1\kappa_2 + \kappa_3 \]
\[ \mu_4 = \kappa_1^4 + 6\kappa_1^2\kappa_2 + 4\kappa_1\kappa_3 + 3\kappa_2^2 + \kappa_4 \]
\[ \mu_5 = \kappa_1^5 + 10\kappa_1^3\kappa_2 + 15\kappa_1^2\kappa_3 + 10\kappa_1\kappa_2^2 + 5\kappa_1\kappa_4 + 10\kappa_2\kappa_3 + \kappa_5 \]
\[ \mu_6 = \kappa_1^6 + 15\kappa_1^4\kappa_2 + 45\kappa_1^2\kappa_3 + 20\kappa_1^3\kappa_3 + 60\kappa_1\kappa_2\kappa_3 + 15\kappa_2^2\kappa_4 + 6\kappa_1\kappa_5 + 15\kappa_2^3 + 10\kappa_3 + 15\kappa_2\kappa_4 + \kappa_6 \]
\[ \mu_7 = \kappa_1^7 + 21\kappa_1^5\kappa_2 + 105\kappa_1^3\kappa_3 + 105\kappa_1^2\kappa_4 + 35\kappa_1^3\kappa_4 + 35\kappa_1\kappa_2\kappa_5 + 70\kappa_1\kappa_6 + 35\kappa_1\kappa_7 + 105\kappa_2\kappa_4 + 21\kappa_2^2\kappa_5 + 7\kappa_1\kappa_6 + 105\kappa_2^3\kappa_3 + 35\kappa_3 + 21\kappa_2\kappa_5 + \kappa_7 \]
\[ \mu_8 = \kappa_1^8 + 28\kappa_1^6\kappa_2 + 210\kappa_1^4\kappa_2 + 420\kappa_1^2\kappa_3 + 56\kappa_1^5\kappa_3 + 56\kappa_1^3\kappa_2\kappa_3 + 840\kappa_1^2\kappa_2^2\kappa_3 + 280\kappa_1^2\kappa_3 + 280\kappa_2^2\kappa_3 + 70\kappa_1^4\kappa_4 + 420\kappa_1^2\kappa_2\kappa_4 + 210\kappa_2^3\kappa_3 + 280\kappa_1\kappa_3\kappa_4 + 35\kappa_1^2 + 56\kappa_1^3\kappa_5 + 168\kappa_1\kappa_2\kappa_5 + 56\kappa_3 + 56\kappa_2\kappa_6 + 28\kappa_1\kappa_7 + 8\kappa_1\kappa_7 + \kappa_8, \]  
(E2)

which, together with the conditions \( B_1 = B_3 = 0 \), valid for binary variables as shown in Sec. [D1], can be used to obtain the explicit expressions of \( B^M \) in Eq. (10) of the main manuscript. The first orders of \( B^M \) read

\[ B^M_1 = 0, \]
\[ B^M_2 = B_2, \]
\[ B^M_3 = 0, \]
\[ B^M_4 = 3B_2^2 + B_4, \]
\[ B^M_5 = 10B_2B_3 + B_5, \]
\[ B^M_6 = 15B_2^3 + 15B_2B_4 + B_6, \]
\[ B^M_7 = 105B_2^3B_3 + 35B_3B_4 + 21B_2B_5 + B_7, \]
\[ B^M_8 = 105B_4^2 + 210B_2^2B_4 + 35B_2^2 + 56B_3B_5 + 28B_2(10B_2^2 + B_6) + B_8. \]  
(E3)

**Appendix F: Search for classical bounds in the case of binary random variables**

Section [C] shows how classical bounds on \( B_2 \) can be obtained by finding the extrema of \( A_2 + C_2 \). However, such a procedure rapidly becomes too complex to be of any use for higher orders and/or for more complex paths. Instead, we have developed a mixed analytical/numerical strategy to find the classical bounds for \( B_n \).

In the case of \( N \) binary random variables \( z_i \in \{-1, 1\} \), one can easily enumerate all possible combinations \( c_i \) of their joint values as

\[ c_i = \{h_1, \ldots, h_N\}, \]  
(F1)

with \( h_j = \pm 1 \) and \( i = 0, 2^N - 1 \). The term \( c_i \) is nothing but the number \( i \) written in binary base with \( N \) digits and with 0 replaced by −1. For classical variables, one can associate each \( c_i \) with the probability \( p_i \) of its occurrence, independently of the order at which these variables are observed (local determinism). Note now that through Eq. (9) of the main manuscript and Eq. (A7), one can write Bell functions at any order as a linear combination of products of mixed moments of random variables. These mixed moments are in turn average values of products of variables, and they can be recast in terms of the probabilities \( p_i \) introduced above. In such a way, each Bell function becomes a polynomial function of the \( p_i \)’s. For instance, for \( P_I, P_{II}, P_{III}, \) and \( P_{IV} \), defined in the main text, one can obtain
\( \mathbb{B}_2 (\mathcal{P}_1, \mathcal{P}_{11}) = -2 + 4p_{11} + 4p_{13} + 4p_{14} + 4p_3 + 4p_4 + 4p_6 + 4p_8 + 4p_9, \)  
\( \mathbb{B}_4 (\mathcal{P}_1, \mathcal{P}_{11}) = 8 (1 - 2p_{11} - 2p_{12} - 2p_{13} - 2p_{14} - 2p_3 - 2p_4 - 2p_5 - 2p_6) \)
\[-6 (1 - 2p_{11} - 2p_{12} - 2p_{13} - 2p_{14} - 2p_3 - 2p_4 - 2p_5 - 2p_6)^2 \]
\[-8 (1 - 2p_{10} - 2p_{12} - 2p_{13} - 2p_{15} - 2p_2 - 2p_4 - 2p_5 - 2p_7) \]
\[+6 (1 - 2p_{10} - 2p_{12} - 2p_{13} - 2p_{15} - 2p_2 - 2p_4 - 2p_5 - 2p_7)^2 \]
\[+8 (1 - 2p_{11} - 2p_{13} - 2p_{15} - 2p_2 - 2p_4 - 2p_6 - 2p_8 - 2p_9) \]
\[-6 (1 - 2p_{11} - 2p_{13} - 2p_{15} - 2p_2 - 2p_4 - 2p_6 - 2p_8 - 2p_9)^2 \]
\[+8 (1 - 2p_{10} - 2p_{13} - 2p_{14} - 2p_3 - 2p_4 - 2p_7 - 2p_8 - 2p_9) \]
\[-6 (1 - 2p_{10} - 2p_{13} - 2p_{14} - 2p_3 - 2p_4 - 2p_7 - 2p_8 - 2p_9)^2 \],  
\( \mathbb{B}_2 (\mathcal{P}_{III}, \mathcal{P}_{IV}) = -2 + 4p_{13} + 4p_{14} + 4p_{15} + 4p_{16} + 4p_{19} + 4p_{20} + 4p_{23} + 4p_{24} + 4p_{27} + 4p_{28} + 4p_{31} \)
\[+ 4p_{32} + 4p_{33} + 4p_{34} + 4p_{37} + 4p_{38} + 4p_{41} + 4p_{42} + 4p_{45} + 4p_{46} + 4p_{5} + 4p_{50} \]
\[+ 4p_{51} + 4p_{52} + 4p_{57} + 4p_{58} + 4p_{59} + 4p_{6} + 4p_{60} + 4p_{7} + 4p_{8}. \]  

These polynomial functions are then numerically minimized and maximized, which gives the classical (local) limits for each Bell function. By doing so we obtain the bounds reported in Table III.

| Function | \((\mathcal{P}, \mathcal{P}')\) | Min | Max |
|----------|-----------------|-----|-----|
| \(\mathbb{B}_2\) | \(\mathcal{P}_1, \mathcal{P}_{11}\) | -2  | 2   |
| \(\mathbb{B}_4\) | \(\mathcal{P}_1, \mathcal{P}_{11}\) | -28 | 20  |
| \(\mathbb{B}_4^M\) | \(\mathcal{P}_1, \mathcal{P}_{11}\) | -106.7 | 106.7 |
| \(\mathbb{B}_5\) | \(\mathcal{P}_1, \mathcal{P}_{11}\) | -693.16 | 992 |
| \(\mathbb{B}_5^M\) | \(\mathcal{P}_1, \mathcal{P}_{11}\) | -1370.2 | 542.7 |
| \(\mathbb{B}_7\) | \(\mathcal{P}_1, \mathcal{P}_{11}\) | -9557.34 | 9557.34 |
| \(\mathbb{B}_7^M\) | \(\mathcal{P}_1, \mathcal{P}_{11}\) | -14037.4 | 14037.4 |
| \(\mathbb{B}_8\) | \(\mathcal{P}_1, \mathcal{P}_{11}\) | -69088 | 86310.4 |
| \(\mathbb{B}_8^M\) | \(\mathcal{P}_1, \mathcal{P}_{11}\) | -50519.6 | 110569 |
| \(\mathbb{B}_2\) | \(\mathcal{P}_{III}, \mathcal{P}_{IV}\) | -4 | 4 |
| \(\mathbb{B}_4\) | \(\mathcal{P}_{III}, \mathcal{P}_{IV}\) | -56 | 27.2 |
| \(\mathbb{B}_4^M\) | \(\mathcal{P}_{III}, \mathcal{P}_{IV}\) | -29.3 | 75.2 |

TABLE III. Maximal and minimal classically-allowed values for the first few orders of Bell functions. These values have been obtained for classical binary random variables with values ±1, with no assumption on their averages.

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