On the quasi-ergodic distribution of absorbing Markov processes

Guoman He¹,², Hanjun Zhang³, Yixia Zhu⁴

Abstract

In this paper, we give a sufficient condition for the existence and uniqueness of a quasi-ergodic distribution for absorbing Markov processes. Using an orthogonal-polynomial approach, we prove that the previous main result is valid for the birth-death process on the nonnegative integers with 0 an absorbing boundary and ∞ an entrance boundary. We also show that the unique quasi-ergodic distribution is stochastically larger than the unique quasi-stationary distribution in the sense of monotone likelihood-ratio ordering for the birth-death process. Moreover, we give an alternative proof of eigentime identity for the birth-death process with exit boundary.

Keywords: Process with absorption; quasi-ergodicity; quasi-stationary distribution; eigentime identity; birth-death process

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1 Introduction

Let \((\Omega, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (P_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathcal{E} \cup \{\partial}\})\) be a time-homogeneous Markov process with state space \(\mathcal{E} \cup \{\partial\}\), where \((\mathcal{E}, \mathcal{E})\) is a measurable space and \(\partial \notin \mathcal{E}\) is a cemetery state. Let \(\mathbb{P}_x\) and \(\mathbb{E}_x\) stand for the probability and the expectation, respectively, associated with the process \(X\) when initiated from \(x\). We assume that the process \(X\) has a finite lifetime \(T\), i.e., for all \(x \in \mathcal{E}\),

\[\mathbb{P}_x(T < \infty) = 1,\]

where \(T = \inf\{t \geq 0 : X_t = \partial\}\). We also assume that for all \(t \geq 0\) and \(\forall x \in \mathcal{E}\),

\[\mathbb{P}_x(t < T) > 0.\]

¹School of Mathematics and Statistics, Hunan University of Commerce, Changsha, Hunan 410205, PR China. Email address: hgm0164@163.com

²Key Laboratory of Hunan Province for New Retail Virtual Reality Technology, Hunan University of Commerce, Changsha, Hunan 410205, PR China

³School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, PR China. Email address: hjz001@xtu.edu.cn

⁴School of Mathematics and Statistics, Hunan University of Finance and Economics, Changsha, Hunan 410205, PR China. Email address: zhuyixia62@163.com
The main purpose of this work is to study the existence and uniqueness of a quasi-ergodic distribution for an absorbing Markov process, and give a comparison between the quasi-ergodic distribution and the quasi-stationary distribution for a class of birth-death process.

A probability measure $\nu$ on $E$ is called a \textit{quasi-stationary distribution} if, for all $t \geq 0$ and any $A \in \mathcal{E}$,

$$P_{\nu}(X_t \in A | T > t) = \nu(A).$$

Quasi-stationary distribution for a killed Markov process has been studied by various authors since 1940s. On this topic, we refer the reader to survey papers [16, 19] and the book [7] for the background and more informations.

A probability measure $m$ on $E$ is called a \textit{quasi-ergodic distribution} if, for any $x \in E$ and any bounded measurable function $f$ on $E$, the following limit exists:

$$\lim_{t \to \infty} \mathbb{E}_x \left( \frac{1}{t} \int_0^t f(X_s) ds | T > t \right) = \int_E f(x) m(dx).$$

We remark that the above limiting law of the time-average, which we called quasi-ergodic distribution, comes from the paper [2], where the authors proved initially a conditioned version of the ergodic theorem for Markov processes. Under some mild conditions, Chen and Deng [4] showed that a quasi-ergodic distribution can be characterized by the Donsker-Varadhan rate functional which is typically used as the large deviation rate function for Markov processes. Recently, many authors have extensively studied the quasi-ergodic distribution; see [5, 11, 21] for example. In existing research works, it often needs to assume the process is $\lambda$-positive (see, e.g., [2, 5, 8]), except some specific cases. However, for general, almost surely absorbed Markov processes, checking whether it is $\lambda$-positive is not an easy thing to do. This leads us to look for some alternative conditions ensuring the existence and uniqueness of quasi-ergodic distributions of absorbing Markov processes.

Quasi-ergodic distribution, sometimes referred to as the limiting conditional mean ratio quasi-stationary distribution [8], is quite different from quasi-stationary distribution (see, e.g., [5, 11, 21]). A natural question is whether there is a relationship between them? In this paper, we plan to give a comparison between them for a class of birth-death process, who admits a unique quasi-ergodic distribution and a unique quasi-stationary distribution. We show that the unique quasi-ergodic distribution is stochastically larger than the unique quasi-stationary distribution in the sense of monotone likelihood-ratio ordering (see, e.g., [20]) for the process. In other words, the quasi-ergodic distribution provides an upper bound for the quasi-stationary distribution in the sense of monotone likelihood-ratio ordering. Although the quasi-stationary distributions of absorbing Markov processes are known to have considerable practical importance in, e.g., ecology, biological, and physical chemistry, computation of the quasi-stationary distributions is often nontrivial. Thus, the comparison result seems to be meaningful.
In this work, we first prove that, under suitable assumptions, there exists a unique quasi-ergodic distribution for the absorbing Markov process. In order to illustrate the previous main result, we then consider a birth-death process on the nonnegative integers with 0 as an absorbing state and \( \infty \) as an entrance boundary. Based on orthogonal polynomials techniques of \([13]\), we show that the previous main result is valid for the birth-death process. Moreover, using a duality method, we can give another new proof of eigentime identity for the birth-death process with exit boundary.

The remainder of this paper is organized as follows. The main result and its proof are presented in Section 2. In Section 3 we study the case of birth-death processes. We conclude in Section 4 with an example.

## 2 Main result

In this paper, our main goal is to prove that Assumption (A) below is a sufficient criterion for the existence of a unique quasi-ergodic distribution for an absorbing Markov process. We point out that, for an absorbing Markov process, Assumption (A) is a necessary and sufficient condition for exponential convergence to a unique quasi-stationary distribution in the total variation norm (see \([3, \text{Theorem 2.1}]\)).

**Assumption (A)** There exists a probability measure \( \nu_1 \) on \( E \) such that

(A1) there exist \( t_0, c_1 > 0 \) such that, for all \( x \in E \),
\[
\mathbb{P}_x(X_{t_0} \in \cdot | T > t_0) \geq c_1 \nu_1(\cdot);
\]

(A2) there exists \( c_2 > 0 \) such that, for all \( x \in E \) and \( t \geq 0 \),
\[
\mathbb{P}_{\nu_1}(T > t) \geq c_2 \mathbb{P}_x(T > t).
\]

In order to let the reader have a better understanding of Assumption (A), its specific meaning (see \([3]\)) is restated here: If \( E \) is a Polish space, then Assumption (A1) implies that the process \( X \) comes back fast in compact sets from any initial conditions. If \( E = \mathbb{N} \) or \( \mathbb{R}_+ := (0, +\infty) \) and \( \partial = 0 \), then Assumption (A1) implies that the process \( X \) comes down from infinity (see \([7]\)); Assumption (A2) means that the highest non-absorption probability among all initial points in \( E \) has the same order of magnitude as the non-absorption probability starting from the probability distribution \( \nu_1 \).

According to \([3, \text{Proposition 2.3}]\), we know that Assumption (A) implies that there exists a non-negative function \( \eta \) on \( E \cup \{\partial\} \), which is positive on \( E \) and vanishes on \( \partial \), such that
\[
\eta(x) = \lim_{t \to \infty} e^{\lambda t} \mathbb{P}_x(T > t), \quad (2.1)
\]
where the convergence holds for the uniform norm on \( E \cup \{ \partial \} \) and \( \lambda > 0 \). Moreover, \( \eta \) belongs to the domain of the infinitesimal generator \( L \) of the semigroup \( (P_t)_{t \geq 0} \) on the set of bounded Borel functions on \( E \cup \{ \partial \} \) equipped with uniform norm and

\[
L \eta = -\lambda \eta.
\]

According to [3, Theorem 3.1], we know that Assumption (A) implies that the \( h \)-process (the process conditioned to never be absorbed) exists. More precisely, if Assumption (A) holds, then the family \( (Q_x)_{x \in E} \) of probability measures on \( \Omega \), defined by

\[
Q_x(A) = \lim_{t \to \infty} P_x(x|T > t), \quad \forall A \in \mathcal{F}_s, \quad \forall s \geq 0,
\]

is well-defined, and the process \( (\Omega, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (Q_x)_{x \in E}) \) is an \( E \)-valued homogeneous Markov process.

The following theorem is our main result.

**Theorem 2.1.** Assume that Assumption (A) holds. Then, there exists a unique quasi-ergodic distribution

\[
m(dx) = \eta(x) \nu(dx)
\]

for the process \( X \), where \( \nu \) is the unique quasi-stationary distribution of the process \( X \). In particular, \( m \) is just the unique stationary distribution of the \( h \)-process.

**Proof.** From [3, Proposition 2.3], we know that \( \int_E \eta(x)\nu(dx) = 1 \). Then, \( m \) is a probability distribution on \( E \). Next, we first assume that \( f \) is positive and bounded. For fixed \( u \), we set

\[
h_u(x) = \inf \{e^{\lambda t}P_x(T > r)/\eta(x) : r \geq u\}.
\]

From (2.1), one can easily see that \( h_u(x) \uparrow 1 \), as \( u \to \infty \). Let \( 0 < q < 1 \). When \((1-q)t \geq u \), by the Markov property, we obtain

\[
\mathbb{E}_x(f(X_{qt})|T > t) = \frac{\mathbb{E}_x(f(X_{qt})|T > t)}{\mathbb{P}_x(T > t)} = \frac{\mathbb{E}_x[f(X_{qt})1(T > qt)]\mathbb{P}_x(T > (1-q)t]}{\mathbb{P}_x(T > t)} \geq \frac{e^{\lambda qt}\mathbb{E}_x[f(X_{qt})h_u(X_{qt})\eta(X_{qt})1(T > qt)]}{e^{\lambda t}\mathbb{P}_x(T > t)},
\]

where \( 1_A \) denotes the indicator function of \( A \).

From [3, Proposition 2.3], we know that \( \eta \) is bounded. Moreover, because the convergence in (2.1) is uniform in \( x \in E \), there exists a constant \( C > 0 \) such that, for all \( r \geq u \) and \( x \in E \),

\[
|f(x)h_u(x)\eta(x)| \leq |f(x)e^{\lambda t}\mathbb{P}_x(T > r)| \leq C\|f\|_{\infty}\|\eta\|_{\infty}.
\]
Therefore, the function $f h_u \eta$ is bounded and measurable. According to [3, Theorem 2.1], we know that for any $x \in E$ and any bounded measurable function $g$ on $E$,

$$
\lim_{t \to \infty} \mathbb{E}_x(g(X_t)|T > t) = \int_E g(x)\nu(dx).
$$

(2.2)

So, by (2.1) and (2.2), we obtain

$$
\liminf_{t \to \infty} \mathbb{E}_x(f(X_{qt})|T > t) \geq \liminf_{t \to \infty} \frac{e^{\lambda qt} \mathbb{E}_x[f(X_{qt})h_u(X_{qt})\eta(X_{qt})1_{\{T > qt\}}]}{e^{\lambda t} \mathbb{P}_x(T > t)}
= \int_E f(x)h_u(x)\eta(x)\nu(dx).
$$

Based on the monotone convergence theorem, by letting $u \to \infty$ in the above formula, we have

$$
\liminf_{t \to \infty} \mathbb{E}_x(f(X_{qt})|T > t) \geq \int_E f(x)m(dx). 
$$

(2.3)

On the other hand, since $f$ is bounded, we can repeat the argument, replacing $f$ by $\|f\|_\infty - f$, which gives

$$
\limsup_{t \to \infty} \mathbb{E}_x(f(X_{qt})|T > t) \leq \int_E f(x)m(dx). 
$$

(2.4)

Combining (2.3) and (2.4), for positive and bounded function $f$, we have

$$
\lim_{t \to \infty} \mathbb{E}_x(f(X_{qt})|T > t) = \int_E f(x)m(dx). 
$$

(2.5)

For (2.5), we can extend it to arbitrary bounded $f$ by subtraction.

So, by change of variable in the Lebesgue integral and the dominated convergence theorem, we get

$$
\lim_{t \to \infty} \mathbb{E}_x \left( \frac{1}{t} \int_0^t f(X_s)ds | T > t \right) = \lim_{t \to \infty} \mathbb{E}_x \left( \int_0^1 f(X_{qt})dq | T > t \right)
= \lim_{t \to \infty} \int_0^1 \mathbb{E}_x(f(X_{qt})|T > t)dq
= \int_E f(x)m(dx).
$$

The uniqueness of $m$ is due to the uniqueness of $\nu$. Thus, we have proved that there exists a unique quasi-ergodic distribution for the process $X$.

If Assumption (A) holds, then we know from [3, Theorem 3.1] that the $h$-process admits the unique invariant probability measure

$$
\beta(dx) = \eta(x)\nu(dx).
$$

Thus, $m$ coincides with the unique stationary distribution $\beta$ of the $h$-process. This ends the proof of the theorem.
According to [3, Theorem 2.1], we know that Assumption (A) implies that for all probability measure \( \mu \) on \( E \) and all \( A \in \mathcal{E} \),
\[
\lim_{t \to \infty} \mathbb{P}_\mu(X_t \in A|T > t) = \nu(A).
\]
From [3, Proposition 1.2], we also know that Assumption (A) implies that for all probability measure \( \mu \) on \( E \),
\[
\lim_{t \to \infty} e^{\lambda t} \mathbb{P}_\mu(T > t) = \int_E \eta(x) \mu(dx).
\]
Thus, by using a similar argument as in the proof of Theorem 2.1, we have the following result.

**Corollary 2.2.** Assume that Assumption (A) is satisfied. Then, for any initial distribution \( \mu \) on \( E \) and any bounded measurable function \( f \) on \( E \), we have
\[
\lim_{t \to \infty} \mathbb{E}_\mu \left( \frac{1}{t} \int_0^t f(X_s)ds|T > t \right) = \int_E f(x) m(dx),
\]
where \( m \) is as in Theorem 2.1.

## 3 Birth-death processes

In this section, we study the quasi-ergodic distribution and the eigentime identity of birth-death processes. Section 3.1 is presented to illustrate that our main result is valid by using a new proof method which is different from the one used in the main result. Moreover, we also give a comparison between the quasi-ergodic distribution and the quasi-stationary distribution for the birth-death process. Section 3.2 is a little related to the main theme of this paper, but is of independent interest.

### 3.1 Quasi-ergodic distributions for the birth-death process

Let \( X = (X_t, t \geq 0) \) be a continuous-time birth-death process taking values in \( \{0\} \cup \mathbb{N} \), where 0 is an absorbing state and \( \mathbb{N} = \{1, 2, \cdots \} \) is an irreducible transient class. Its jump rate matrix \( Q := (q_{ij}, i, j \in \mathbb{N}) \) satisfies
\[
q_{ij} = \begin{cases} 
  b_i & \text{if } j = i+1, i \geq 0, \\
  d_i & \text{if } j = i-1, i \geq 1, \\
  -(b_i + d_i) & \text{if } j = i, i \geq 0, \\
  0 & \text{otherwise},
\end{cases}
\]
where the birth rates \( (b_i, i \in \mathbb{N}) \) and death rates \( (d_i, i \in \mathbb{N}) \) are strictly positive, and \( d_0 = b_0 = 0 \).

Define the potential coefficients \( \pi = (\pi_i, i \in \mathbb{N}) \) by
\[
\pi_1 = 1 \quad \text{and} \quad \pi_i = \frac{b_1 b_2 \cdots b_{i-1}}{d_2 d_3 \cdots d_i}, \quad i \geq 2. \quad (3.1)
\]
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Then, we have \( b_i \pi_i = d_{i+1} \pi_{i+1} \), for \( i \in \mathbb{N} \).

Put

\[
A = \sum_{i=1}^{\infty} \frac{1}{b_i \pi_i}, \quad B = \sum_{i=1}^{\infty} \pi_i, \quad R = \sum_{i=1}^{\infty} \frac{1}{b_i \pi_i} \sum_{j=1}^{i} \pi_j, \quad S = \sum_{i=1}^{\infty} \frac{1}{b_i \pi_i} \sum_{j=i+1}^{\infty} \pi_j. \tag{3.2}
\]

We observe that

\[
R + S = AB, \quad A = \infty \Rightarrow R = \infty, \quad S < \infty \Rightarrow B < \infty.
\]

If absorption at 0 is certain which means that \( \mathbb{P}_i(T < \infty) = 1 \), for \( i \in \mathbb{N} \), where \( T = \inf\{t \geq 0 : X_t = 0\} \) is the absorption time of \( X \), then it is equivalent to \( A = \infty \) (see [12]). Therefore, \( A = \infty \) implies the process \( X \) is non-explosive. In this subsection, we assume that \( A = \infty \).

We write \( P_{ij}(t) = \mathbb{P}_i(X_t = j) \). It is well known (see, e.g., [1, Theorem 5.1.9]) that under our assumptions, there exists a parameter \( \lambda \geq 0 \), called the decay parameter of the process \( X \), such that

\[
\lambda = -\lim_{t \to \infty} \frac{1}{t} \log P_{ij}(t), \quad i, j \in \mathbb{N}. \tag{3.3}
\]

In [17, Theorem 3.2], van Doorn proved that: (i) if \( \infty \) is an entrance boundary (i.e., \( R = \infty, S < \infty \)), then \( \lambda > 0 \) and there is a unique quasi-stationary distribution for the process \( X \); (ii) if \( \infty \) is a natural boundary (i.e., \( R = \infty, S = \infty \)), then either \( \lambda > 0 \) and there is an infinite continuum of quasi-stationary distributions, or \( \lambda = 0 \) and there is no quasi-stationary distribution.

Let \( (Q_i(x), i \geq 0) \) be the birth-death polynomials, given as

\[
Q_0(x) = 0, \\
Q_1(x) = 1, \\
b_i Q_{i+1}(x) - (b_i + d_i)Q_i(x) + d_i Q_{i-1}(x) = -x Q_i(x), \quad i \in \mathbb{N}. \tag{3.4}
\]

It is well known (see, e.g., [6]) that \( Q_i(x) \) has \( i - 1 \) positive, simple zeros, \( x_{ij} (j = 1, 2, \ldots, i-1) \), which verify the interlacing property

\[
0 < x_{i+1,j} < x_{i,j} < x_{i+1,j+1}, \quad j = 1, 2, \ldots, i-1, \quad i \geq 2. \tag{3.5}
\]

Therefore, the following limits

\[
\xi_j \equiv \lim_{i \to \infty} x_{ij}, \quad j \geq 1, \tag{3.6}
\]

exist and satisfy \( 0 \leq \xi_j \leq \xi_{j+1} < \infty \). It is easy to see from (3.4) that, as a result,

\[
x \leq \xi_1 \iff Q_i(x) > 0 \quad \text{for all} \quad i \in \mathbb{N}. \tag{3.7}
\]

When \( A = \infty \) and \( S < \infty \), we have \( 0 < \lambda = \xi_1 < \xi_2 < \cdots \) with \( \lim_{j \to \infty} \xi_j = \infty \).

Of importance to us in the proof of our main results will be the process \( X \) conditioned to never be absorbed, usually referred to as the \( h \)-process. Let \( \overline{P}_{ij}(t) = \mathbb{P}_i(Y_t = j) \) be transition
kernel of the \( h \)-process \( Y = (Y_t, t \geq 0) \). From \([10]\), we know that \( \mathbb{T}_{ii}(t) = e^{\lambda t}P_{ii}(t) \), for all \( i \in \mathbb{N} \). Thus, the \( \lambda \)-classification of the killed process \( X^T \) can be presented in the following form. If \( Y \) is positive recurrent (resp. recurrent, null recurrent, transient), then the killed process \( X^T \) is said to be \( \lambda \)-positive (resp. \( \lambda \)-recurrent, \( \lambda \)-null, \( \lambda \)-transient).

For two probability vectors \( \rho = (\rho(i), i \in \mathbb{N}) \) and \( \rho' = (\rho'(i), i \in \mathbb{N}) \), we put \( \rho' \lesssim \rho \) and said that \( \rho' \) is stochastically smaller than \( \rho \) in the sense of monotone likelihood-ratio ordering if and only if \( (\rho(i)/\rho'(i), i \in \mathbb{N}) \) is increasing.

For the birth-death process, we have the following result. Note that if absorption at 0 is certain, then \( S < \infty \) is equivalent to Assumption (A) (see \([3\), Theorem 4.1\]). Thus, the main result is illustrated on the birth-death process.

**Theorem 3.1.** Let \( X \) be a birth-death process for which 0 is an absorbing state and \( \infty \) is an entrance boundary. Then, there exists a unique quasi-ergodic distribution \( m = (m_i, i \in \mathbb{N}) \) for the process \( X \), where

\[
m_i = \frac{\pi_i Q_i^2(\lambda)}{\sum_{j \in \mathbb{N}} \pi_j Q_j^2(\lambda)}.
\]

In particular, \( m \) is just the unique stationary distribution of the \( h \)-process. Moreover, \( \nu \lesssim m \), where \( \nu \) is the unique quasi-stationary distribution of the process \( X \).

**Proof.** (i) We first prove that the \( h \)-process \( Y \) is strongly ergodic, which implies the killed process \( X^T \) is \( \lambda \)-positive. Although for general, almost surely absorbed Markov processes, Champagnat and Villemonais have proved that the process \( Y \) is exponentially ergodic (see \([3\), Theorem 3.1\]), we will prove that the process \( Y \) is strongly ergodic by using a different proof method here. According to \([7\), Proposition 5.9\], the process \( Y \), whose law starting from \( i \in \mathbb{N} \) is given by

\[
P_i(Y_{s_1} = i_1, \cdots, Y_{s_k} = i_k) := \lim_{t \to \infty} P_i(X_{s_1} = i_1, \cdots, X_{s_k} = i_k | T > t),
\]

is a Markov chain with transition kernel

\[
\forall i, j \in \mathbb{N} : \quad P_i(Y_s = j) = e^{\lambda s} \frac{Q_j(\lambda)}{Q_i(\lambda)} P_i(X_s = j).
\]  (3.8)

From (3.8), we get that the process \( Y \) is still a birth-death process taking values in \( \mathbb{N} \), and its birth and death parameters are given respectively by

\[
\forall i \in \mathbb{N} : \quad \bar{b}_i = \frac{Q_{i+1}(\lambda)}{Q_i(\lambda)} b_i,
\]

\[
\forall i \in \mathbb{N} : \quad \bar{d}_i = \frac{Q_{i-1}(\lambda)}{Q_i(\lambda)} d_i.
\]

So, we can compute the potential coefficients \( \pi = (\pi_i, i \in \mathbb{N}) \) analogous to (3.1): \( \pi_1 = 1 \) and

\[
\pi_i = \frac{\bar{b}_1 \bar{b}_2 \cdots \bar{b}_{i-1}}{d_2 d_3 \cdots d_i} = Q_i^2(\lambda) \pi_i, \quad i \geq 2.
\]
Similarly, we can compute the constants $\mathcal{A}, \mathcal{B}, \mathcal{R}, \mathcal{S}$ analogous to (3.2):

$$\mathcal{A} = \sum_{i=1}^{\infty} \frac{1}{Q_{i+1}(\lambda)Q_i(\lambda)b_i\pi_i}, \quad \mathcal{B} = \sum_{i=1}^{\infty} Q_i^2(\lambda)\pi_i,$$

$$\mathcal{R} = \sum_{i=1}^{\infty} \frac{1}{Q_{i+1}(\lambda)Q_i(\lambda)b_i\pi_i} \sum_{j=1}^{i} Q_j^2(\lambda)\pi_j, \quad \mathcal{S} = \sum_{i=1}^{\infty} \frac{1}{Q_{i+1}(\lambda)Q_i(\lambda)b_i\pi_i} \sum_{j=i+1}^{\infty} Q_j^2(\lambda)\pi_j.$$

From (3.4), we have

$$b_i(Q_{i+1}(\lambda) - Q_i(\lambda)) + d_i(Q_{i-1}(\lambda) - Q_i(\lambda)) = -\lambda Q_i(\lambda). \quad (3.9)$$

Multiplying both sides of (3.9) by $\pi_i$, and then sum of $i$ from 1 to $k$, we get

$$b_k\pi_k(Q_{k+1}(\lambda) - Q_k(\lambda)) - d_1\pi_1 Q_1(\lambda) = -\lambda \sum_{i=1}^{k} \pi_i Q_i(\lambda).$$

Note that $Q_1(\lambda) = 1$, $\pi_1 = 1$ and $\lambda \sum_{i \in \mathbb{N}} \pi_i Q_i(\lambda) = d_1$ by (3.10) below. Then

$$Q_{k+1}(\lambda) - Q_k(\lambda) = \frac{\lambda}{b_k\pi_k} \sum_{i=k+1}^{\infty} \pi_i Q_i(\lambda) > 0, \quad k \in \mathbb{N}.$$ 

Therefore, $Q_i(\lambda)$ is strictly increasing with $i \in \mathbb{N}$ and has the minimum 1. Also, we know from [9, Lemma 3.4] that $Q_i(\lambda)$ is bounded, denoted by $W$ an upper bound. Thus, we have

$$\mathcal{R} \geq \sum_{i=1}^{\infty} \frac{1}{W^2b_i\pi_i} \sum_{j=1}^{i} \pi_j$$

$$\geq \sum_{i=1}^{\infty} \frac{1}{W^2} \sum_{j=1}^{i} \pi_j$$

$$= \frac{1}{W^2} \mathcal{R}$$

$$= \infty$$

and

$$\mathcal{S} \leq \sum_{i=1}^{\infty} \frac{1}{W^2b_i\pi_i} \sum_{j=i+1}^{\infty} W^2\pi_j$$

$$\leq W^2 \sum_{i=1}^{\infty} \frac{1}{b_i\pi_i} \sum_{j=i+1}^{\infty} \pi_j$$

$$= W^2 \mathcal{S}$$

$$< \infty.$$ 

Note that 1 is a reflecting boundary for the process $Y$. Hence, we know from [14, Theorem 3.1] that the process $Y$ is strongly ergodic. Therefore, the killed process $X^T$ is $\lambda$-positive. And,
it is well known that there exists a unique stationary distribution \(\left(\sum_{j \in \mathbb{N}} \pi_j Q_1^{2(\lambda)}(\cdot), i \in \mathbb{N}\right)\) for the process \(Y\). From \cite{17}, we know that \(Q(\lambda) := (Q_i(\lambda), i \in \mathbb{N})\) is a \(\lambda\)-invariant function for \(Q\), that is, \(QQ(\lambda) = -\lambda Q(\lambda)\), and the process \(X\) admits \(\nu = (\nu_i, i \in \mathbb{N})\) as the unique quasi-stationary distribution, where
\[
\nu_i = \frac{\pi_i Q_i(\lambda)}{\sum_{j \in \mathbb{N}} \pi_j Q_j(\lambda)} = \frac{\lambda}{d_1} \pi_i Q_i(\lambda). \tag{3.10}
\]
This implies that the series \(\sum_{i \in \mathbb{N}} \pi_i Q_i(\lambda)\) is summable, and \(\theta = (\theta_i, i \in \mathbb{N})\) is the unique \(\lambda\)-invariant measure for \(Q\), where \(\theta_i = \pi_i Q_i(\lambda)\), that is, \(\theta Q = -\lambda \theta\).

Because the killed process \(X_T\) is \(\lambda\)-positive, we know from \cite[Theorem 5.2.8]{1} that
\[
\lim_{t \to \infty} e^{\lambda t} P_{ij}(t) = \frac{Q_i(\lambda) \pi_j Q_j(\lambda)}{\sum_{k \in \mathbb{N}} \pi_k Q_k^2(\lambda)}. \tag{3.11}
\]
Also, we know from the proof of \cite[Proposition 5.9]{7} that
\[
\lim_{t \to \infty} \frac{\mathbb{P}_j(T > t)}{\mathbb{P}_i(T > t)} = \frac{Q_j(\lambda)}{Q_i(\lambda)}. \tag{3.12}
\]
Thus, we have
\[
\lim_{t \to \infty} \mathbb{E}_i \left(\frac{1}{t} \int_0^t 1_{\{X_s = j\}} ds | T > t\right) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{P_{ij}(s) \sum_{k \neq 0} P_{jk}(t - s)}{\sum_{k \neq 0} P_{jk}(t)} ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{\lambda x} P_{ij}(x) \sum_{k \neq 0} e^{\lambda (t - s)} P_{jk}(t - s) ds \sum_{k \neq 0} e^{\lambda t} P_{ik}(t) = \lim_{x \to \infty} e^{\lambda x} P_{ij}(x) \lim_{y \to \infty} \frac{P_j(T > y)}{P_i(T > y)} = \frac{\pi_j Q_j^2(\lambda)}{\sum_{k \in \mathbb{N}} \pi_k Q_k^2(\lambda)}.
\]
Hence, we know that there exists a unique quasi-ergodic distribution \(m\) for the process \(X\).

(ii) Notice that
\[
\frac{m_i}{\nu_i} = \frac{\sum_{j \in \mathbb{N}} \pi_j Q_j^2(\lambda)}{\lambda_1 \sum_{j \in \mathbb{N}} \pi_j Q_j(\lambda)} = \frac{Q_i(\lambda)}{\lambda_1 B}.
\]
Furthermore, from the proof of the above result, we know that \(Q_i(\lambda)\) is increasing with \(i \in \mathbb{N}\). Therefore, \((m_i/\nu_i, i \in \mathbb{N})\) is increasing. Thus, the result follows.

\(\square\)

3.2 A note on the eigentime identity for the birth-death process with exit boundary

Let \(\tilde{X} = (\tilde{X}_t, t \geq 0)\) be a continuous-time birth-death process taking values on \(\mathbb{Z}_+ = \{0, 1, 2, \cdots\}\) with birth rates \(\tilde{b}_i > 0 (i \in \mathbb{Z}_+)\) and death rates \(\tilde{d}_i > 0 (i \in \mathbb{N})\). Define the
potential coefficients to be \( \tilde{\pi}_0 = 1, \tilde{\pi}_i = \tilde{b}_0 \tilde{b}_1 \cdots \tilde{b}_{i-1} / \tilde{d}_1 \tilde{d}_2 \cdots \tilde{d}_i \) \((i \in \mathbb{N})\). Set

\[
\tilde{A} = \sum_{i=0}^{\infty} \frac{1}{b_i \tilde{\pi}_i}, \quad \tilde{B} = \sum_{i=0}^{\infty} \tilde{\pi}_i, \quad \tilde{R} = \sum_{i=0}^{\infty} \frac{1}{b_i \tilde{\pi}_i} \sum_{j=0}^{i} \tilde{\pi}_j, \quad \tilde{S} = \sum_{i=0}^{\infty} \frac{1}{b_i \tilde{\pi}_i} \sum_{j=i+1}^{\infty} \tilde{\pi}_j. \tag{3.13}
\]

According to [1, Chapter 8], when \( \tilde{R} < \infty \), the corresponding \( \tilde{Q} \)-process is not unique. Due to this reason, we consider here only the minimal birth-death process. Suppose that \( \tilde{X} = (\tilde{X}_t, t \geq 0) \) is the minimal birth-death process. From [1, Chapter 8], we also know that \( \tilde{R} \) is the expectation of first passage time of the process \( \tilde{X} \) from 0 to \( \infty \).

The following result is obtained essentially by Mao [15]. Here, we give an alternative proof for it.

**Theorem 3.2.** Let \( \tilde{X} \) be the minimal birth-death process for which 0 is a reflecting boundary \((i.e., \tilde{d}_0 = 0)\) and \( \infty \) is an exit boundary \((i.e., \tilde{R} < \infty, \tilde{S} = \infty)\). Then, the eigentime identity reads as

\[
\tilde{R} = \sum_{i=1}^{\infty} \frac{1}{\tilde{\xi}_i},
\]

where \((\tilde{\xi}_i, i \in \mathbb{N})\) is the spectrum of infinitesimal generator of the process \( \tilde{X} \).

**Remark 3.3.** We point out that, the study of eigentime identity for the minimal birth-death process \( \tilde{X} \) with 0 an absorbing boundary and \( \infty \) an exit boundary can be transformed into the present case via a Doob’s \( h \)-transform; see [9] for details.

**Proof of Theorem 3.2.** Using a duality method, we can transfer the process \( \tilde{X} \) to the process \( X \), where the process \( X \) is defined in Subsection 3.1. In fact, it will be seen from the following definition:

\[
d_0 = 0; \quad b_i = \tilde{d}_i, \quad d_{i+1} = \tilde{b}_i, \quad i \in \mathbb{Z}_+.
\]

Straightforward calculations show that

\[
\pi_i = \tilde{b}_0 \frac{1}{d_i \tilde{\pi}_i}, \quad \frac{1}{b_i \pi_i} = \frac{1}{b_0 \tilde{\pi}_i}, \quad i \in \mathbb{N}; \tag{3.14}
\]

\[
A = \frac{1}{b_0} (\tilde{B} - 1), \quad B = \tilde{b}_0 \tilde{A}, \quad R = \tilde{S}, \quad S = \tilde{R} - \tilde{A}. \tag{3.15}
\]

According to the proof of [7, Theorem 5.4], we know that

\[
\sum_{i=1}^{\infty} \frac{1}{\tilde{\xi}_i} = \frac{1}{d_1} B + S.
\]

Since \( \tilde{\xi}_i = \xi_i \), for \( i \in \mathbb{N} \) (see, e.g., [9, Proposition 3.1]), from (3.15) we get the conclusion.
The following result may be useful to study the existence of quasi-ergodic distributions for the birth-death process with exit boundary. We also point out that such an equality is a necessary and sufficient condition for some absorbing birth-death processes to be $\lambda$-positive (see [18, Theorem 5.2]).

**Proposition 3.4.** Let $\tilde{X}$ be the minimal birth-death process for which $0$ is a reflecting boundary and $\infty$ is an exit boundary. Then, we have

$$\sum_{i=0}^{\infty} \tilde{\pi}_i \tilde{Q}_i^2(\lambda) < \infty,$$

where $(\tilde{Q}_i(\lambda), i \in \mathbb{Z}_+)$ is analogous to (3.4).

**Proof.** According to [7, formula (5.26)], we know that

$$\tilde{Q}_i(x) - \tilde{Q}_{i+1}(x) = \frac{x}{d_i} \pi_i \tilde{Q}_i(x), \quad i \in \mathbb{N}. \quad (3.16)$$

By a direct calculation, we get

$$\tilde{Q}_i(x) = 1 - \frac{x}{d_i} \sum_{j=1}^{i-1} \pi_j Q_j(x), \quad i \in \mathbb{N}. \quad (3.17)$$

So, from (3.17), we can see that $\tilde{Q}_i(\lambda) \in [0, 1]$.

According to [7, formula (5.25)], we have

$$\tilde{\pi}_i \tilde{Q}_i(\lambda) = Q_i(\lambda) - Q_{i-1}(\lambda), \quad i \in \mathbb{N}. \quad (3.18)$$

So, we get

$$\tilde{\pi}_i \tilde{Q}_i^2(\lambda) = [Q_i(\lambda) - Q_{i-1}(\lambda)] \tilde{Q}_i(\lambda) \leq Q_i(\lambda) - Q_{i-1}(\lambda), \quad i \in \mathbb{N}.$$

Hence,

$$\sum_{i=0}^{\infty} \tilde{\pi}_i \tilde{Q}_i^2(\lambda) \leq \sum_{i=1}^{\infty} [Q_i(\lambda) - Q_{i-1}(\lambda)] = Q_\infty(\lambda) < \infty,$$

where with the convention that $\tilde{Q}_0(\lambda) \equiv 0$. This completes the proof. $\square$

4 An example

Set $d_0 = b_0 = 0; b_i = i + 3, d_i = (i + 1)^2, i \in \mathbb{N}$. Then $\pi_1 = 1,$

$$\pi_i = \frac{2(i + 2)}{3(i + 1)!},$$
On the quasi-ergodic distribution of absorbing Markov processes

\[ A = \sum_{i=1}^{\infty} \frac{1}{b_i \pi_i} = \frac{3}{2} \sum_{i=1}^{\infty} \frac{(i+1)!}{(i+3)(i+2)} = \infty, \]

\[ S = \sum_{i=1}^{\infty} \frac{1}{b_i \pi_i} \sum_{j=i+1}^{\infty} \pi_j = \sum_{i=1}^{\infty} \frac{(i+1)!}{(i+3)(i+2)} \sum_{j=i+1}^{\infty} \frac{(j+2)}{(j+1)!} < \infty. \]

Moreover, the decay parameter \( \lambda = 2 \) and the corresponding eigenfunction is \( Q_i(\lambda) = \frac{i}{i+2} \) (see [9, Example 5.1]). From [17, Theorem 3.2], we know that there exists a unique quasi-stationary distribution \( \nu = (\nu_i, i \in \mathbb{N}) \) for the birth-death process \( X \), where

\[ \nu_i = \frac{i}{(i+1)!}. \]

By Theorem 3.1, we know that \( m = (m_i, i \in \mathbb{N}) \) is the unique quasi-ergodic distribution for the process \( X \), where

\[ m_i = \frac{\pi_i Q_i^2(\lambda)}{\sum_{j \in \mathbb{N}} \pi_j Q_j^2(\lambda)} = \frac{j^2}{\sum_{j \in \mathbb{N}} (j^2)(j+1)!}. \]

Further, it is easy to check that

\[ \frac{m_i}{\nu_i} = \frac{i^{i+2}}{j^2(j+1)!} \]

is increasing with \( i \in \mathbb{N} \), which implies \( \nu \preceq m \). Thus, this example illustrates our results.

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