Coding for the Lee and Manhattan Metrics with Weighing Matrices
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Abstract

This paper has two goals. The first one is to discuss good codes for packing problems in the Lee and Manhattan metrics. The second one is to consider weighing matrices for some of these coding problems. Weighing matrices were considered as building blocks for codes in the Hamming metric in various constructions. In this paper we will consider mainly two types of weighing matrices, namely conference matrices and Hadamard matrices, to construct codes in the Lee (and Manhattan) metric. We will show that these matrices have some desirable properties when considered as generator matrices for codes in these metrics. Two related packing problems will be considered. The first one is to find good codes for error-correction (i.e. dense packings of Lee spheres). The second one is to transform the space in a way that volumes are preserved and each Lee sphere (or conscribed cross-polytope), in the space, will be transformed into a shape inscribed in a small cube.

Index Terms

Conference matrices, cross-polytopes, Hadamard matrices, Lee metric, Lee spheres, Manhattan metric, space transformation, weighing matrices.

I. INTRODUCTION

The Lee metric was introduced in [22], [33] for transmission of signals taken from GF(p) over certain noisy channels. It was generalized for \( \mathbb{Z}_m \) in [16]. The Lee distance \( d_L(X, Y) \) between two words \( X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n) \in \mathbb{Z}_m^n \) is given by 
\[
d_L(X, Y) \overset{\text{def}}{=} \sum_{i=1}^{n} \min\{x_i - y_i \pmod{m}, y_i - x_i \pmod{m}\}.
\]
A related metric, the Manhattan metric, is defined for alphabet letters taken from the integers. For two words \( X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n) \in \mathbb{Z}^n \) the Manhattan distance between \( X \) and \( Y \), \( d_M(X, Y) \), is defined as 
\[
d_M(X, Y) \overset{\text{def}}{=} \sum_{i=1}^{n} |x_i - y_i|.
\]

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The material in this paper was presented in part in the 2010 IEEE Information Theory Workshop, Dublin, Ireland, August-September 2010. This work was supported in part by the United States — Israel Binational Science Foundation (BSF), Jerusalem, Israel, under Grant 2006097.
A code $C$ in either metric has minimum distance $d$ if for each two distinct codewords $c_1, c_2 \in C$ we have $d(c_1, c_2) \geq d$, where $d(\cdot, \cdot)$ stands for either the Lee distance or the Manhattan distance.

The main goal of this paper is to explore the properties of some interesting dense codes in the Lee and Manhattan metrics. Two related packing problems will be considered. The first one is to find good codes for error-correction (i.e. dense packings of Lee spheres) in the Lee and Manhattan metrics. The second one is to transform the space in such a way that volumes of shapes are preserved and each Lee sphere (or inscribed cross-polytope), in the space, will be transformed to a shape inscribed in a small cube. Some interesting connections between these two problems will be revealed in this paper.

An $n$-dimensional Lee sphere $S_{n,R}$, with radius $R$, is the shape centered at $(0, \ldots, 0)$ consisting of all the points $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ which satisfy

$$\sum_{i=1}^{n} |x_i| \leq R.$$  

Similarly, an $n$-dimensional cross-polytope is the set consisting of all the points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ which satisfy the equation

$$\sum_{i=1}^{n} |x_i| \leq 1.$$  

A Lee sphere, $S_{n,R}$, centered at a point $(y_1, \ldots, y_n) \in \mathbb{Z}^n$, contains all the points of $\mathbb{Z}^n$ whose Manhattan distance from $(y_1, \ldots, y_n)$ is at most $R$. The size of $S_{n,R}$ is well known [16]:

$$|S_{n,R}| = \sum_{i=0}^{\min\{n,R\}} 2^i \binom{n}{i} \binom{R}{i}$$

A code with minimum distance $d = 2R + 1$ (or $d = 2R + 2$) is a packing of Lee spheres with radius $R$. Asymptotically, the size of an $n$-dimensional Lee sphere with radius $R$ is $\frac{(2R)^n}{n!} + O(R^{n-1})$, when $n$ is fixed and $R \to \infty$.

The research on codes in the Manhattan metric is not extensive. It is mostly concerned with the existence and nonexistence of perfect codes [4], [16], [19], [27]. Nevertheless, all codes defined in the Lee metric over some finite alphabet, (subsets of $\mathbb{Z}_m^n$) can be extended to codes in the Manhattan metric over the integers (subsets of $\mathbb{Z}^n$). The literature on codes in the Lee metric is very extensive, e.g. [5], [10], [16], [25], [26], [29], [30]. Most of the interest at the beginning was in the existence of perfect codes in these metrics. The interest in Lee codes increased in the last decade due to many new applications of these codes. Some examples are constrained and partial-response channels [29], interleaving schemes [6], orthogonal frequency-division multiplexing [31], multidimensional burst-error-correction [14], and error-correction in the rank modulation scheme for flash memories [20]. The increased interest is also due to new attempts to settle the existence question of perfect codes in these metrics [19].

Linear codes are usually the codes which can be handled more effectively and hence we will consider only linear codes throughout this paper.
A linear code in $\mathbb{Z}^n$ is an integer lattice. A lattice $\Lambda$ is a discrete, additive subgroup of the real $n$-space $\mathbb{R}^n$,

$$\Lambda = \{ u_1 v_1 + u_2 v_2 + \cdots + u_n v_n : u_1, u_2, \ldots, u_n \in \mathbb{Z} \},$$  

(1)

where $\{v_1, v_2, \ldots, v_n\}$ is a set of linearly independent vectors in $\mathbb{R}^n$. A lattice $\Lambda$ defined by (1) is a sublattice of $\mathbb{Z}^n$ if and only if $\{v_1, v_2, \ldots, v_n\} \subset \mathbb{Z}^n$. We will be interested solely in sublattices of $\mathbb{Z}^n$. The vectors $v_1, v_2, \ldots, v_n$ are called basis for $\Lambda \subseteq \mathbb{Z}^n$, and the $n \times n$ matrix

$$G = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}$$

having these vectors as its rows is said to be a generator matrix for $\Lambda$. The lattice with generator matrix $G$ is denoted by $\Lambda(G)$.

The volume of a lattice $\Lambda$, denoted $V(\Lambda)$, is inversely proportional to the number of lattice points per unit volume. More precisely, $V(\Lambda)$ may be defined as the volume of the fundamental parallelogram $\Pi(\Lambda)$, which is given by

$$\Pi(\Lambda) = \{ \xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_n v_n : 0 \leq \xi_i < 1, \ 1 \leq i \leq n \}$$

There is a simple expression for the volume of $\Lambda$, namely, $V(\Lambda) = |\det G|$. An excellent reference, for more material on lattices and some comparison with our results, is [12].

Sublattices of $\mathbb{Z}^n$ are periodic. We say that the lattice $\Lambda$ has period $m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$ if for each $i, 1 \leq i \leq n$, the point $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ is a lattice point in $\Lambda$ if and only if $(x_1, \ldots, x_{i-1}, x_i + m_i, x_{i+1}, \ldots, x_n) \in \Lambda$. Let $m$ be the least common multiple of the integers $m_1, m_2, \ldots, m_n$. The lattice $\Lambda$ has also period $(m, m, \ldots, m)$ and it can be reduced to a code $C$ in the Lee metric over the alphabet $\mathbb{Z}_m$. It is easy to verify that the size of the code $C$ can be the same as the minimum distance of $\Lambda$, but it can be larger (for example, most binary codes of length $n$ can be reduced from a sublattice of $\mathbb{Z}^n$, where their Manhattan distance is at most $2$. This is the inverse of Construction A [12, p. 137]).

One should note that if the lattice $\Lambda$ in the Manhattan metric is reduced to a code over $\mathbb{Z}_p$, $p$ prime, in the Lee metric, then the code is over a finite field. But, usually the code in the Lee metric is over a ring which is not a field. It makes its behavior slightly different from a code over a finite field. Codes over rings were extensively studied in the last twenty years, see e.g. [3], [8], [9], [18], and references therein. In our discussion, a few concepts are important and for codes over $\mathbb{Z}_m$ these are essentially the same as the ones in traditional codes over a finite field. For example, the minimum distance of the code is the smallest distance between two codewords. The minimum distance is equal to the weight of the word with minimum Manhattan (Lee) weight.

The definition of a coset for a lattice $\Lambda$ is very simple. Let $\Lambda$ be a sublattice of $\mathbb{Z}^n$ and $x \in \mathbb{Z}^n$. The coset of $x$ is $x + \Lambda = \{ x + c \mid c \in \Lambda \}$. The set of cosets is clearly unique. For each coset we choose a coset leader, which is a point in the coset with minimum Manhattan weight. If there are a few points with the same minimum Manhattan weight we choose one of them (arbitrarily) as the coset leader. Once a set of coset leaders is chosen then each
point \( x \in \mathbb{Z}^n \) has a unique representation as \( x = c + s \), where \( c \) is a lattice point of \( \Lambda \) and \( s \) is a coset leader. The number of different cosets is equal to the volume of the lattice \( \Lambda \). In this context, the covering radius of a lattice \( \Lambda \) (respectively a code \( C \)) is the distance of the word \( x \) whose distance from the lattice (respectively code) is the highest among all words. It equals to the weight of the coset leader with the largest weight. The covering radius of a lattice \( \Lambda \) is the same as the covering radius of the code \( C \) reduced from \( \Lambda \) to \( \mathbb{Z}_m \), where \( m \) is the period of \( \Lambda \).

A **weighing matrix** \( W_{n,w} \) of order \( n \) and weight \( w \) is an \( n \times n \) matrix over the alphabet \( \{0, 1, -1\} \) such that each row and column has exactly \( w \) nonzero entries; and \( W \cdot W^T = wI_n \), where \( I_n \) is the identity matrix of order \( n \).

The most important families of weighing matrices are the Hadamard matrices in which \( w = n \), and the conference matrices in which \( w = n - 1 \). In most of the results in this paper these families are considered. Our construction in Section IV will use weighing matrices with some symmetry. A weighing matrix \( W \) is **symmetric** if \( W^T = W \) and **skew symmetric** if \( W^T = -W \).

In this paper we examine lattices and codes related to weighing matrices. We prove that the minimum Manhattan (respectively Lee) distance of the lattice (respectively code) derived from a generator matrix taken as a weighing matrix of weight \( w \), is \( w \). We discuss properties of Reed-Muller like codes, i.e. based on Sylvester Hadamard matrices, in the Lee and the Manhattan metrics. These codes were used before for power control in orthogonal frequency-division multiplexing transmission. We prove bounds on their covering radius and extend their range of parameters. We define transformations which transform \( \mathbb{R}^n \) to \( \mathbb{R}^n \) (respectively \( \mathbb{Z}^n \) to \( \mathbb{Z}^n \)), in which each conscribed cross-polytope (respectively Lee sphere) in \( \mathbb{R}^n \) (respectively \( \mathbb{Z}^n \)), is transformed into a shape which can be inscribed in a relatively small cube. The transformations will preserve the volume of the shape and we believe that they are optimal in the sense that there are no such transformations which preserve volume and transform conscribed cross-polytopes (respectively Lee spheres) into smaller cubes. Generalization of the transformations yield some interesting lattices and codes which are related to the codes based on Sylvester Hadamard matrices.

The rest of the paper is organized as follows. In Section II we discuss the use of weighing matrices as generator matrices for codes (respectively lattices) in the Lee (respectively Manhattan) metric. We will prove some properties of the constructed codes (respectively lattices), their size, minimum distance, and on which alphabet size they should be considered for the Lee metric. In Section III we will construct codes related to the doubling construction of Hadamard matrices. We will discuss their properties and also their covering radius. In Section IV we present the volume preserving transformations which transform each conscribed cross-polytope (respectively Lee sphere) in \( \mathbb{R}^n \) (respectively \( \mathbb{Z}^n \)), into a shape which can be inscribed in a relatively small cube. These transformations are part of a large family of transformations based on weighing matrices and they will also yield some interesting codes. Some connections to the codes obtained in Section III will be discussed. In Section V the existence of weighing matrices needed for our constructions will be discussed. Conclusions and problems for future research are given in Section VI.

\(^1\)there is a generalization for this definition for skew symmetric Hadamard matrices (see [15, p. 89]), but this generalization is not considered in our paper.
II. WEIGHING MATRICES CODES

This section is devoted to codes whose generator matrices are weighing matrices. We will discuss some basic properties of such codes.

Each weighing matrix can be written in a normal form such that its first row consists only of zeroes and +1’s, where all the zeroes precede the +1’s. For this we only have to negate and permute columns. We will now consider weighing matrices written in normal form, unless we will apply some operations on the original matrix in normal form and obtain one which is not in normal form. We note that a weighing matrix $W_1$ is equivalent to a weighing matrix $W_2$ if $W_1$ is obtained from $W_2$ by permuting rows and columns, and/or negating rows and columns.

In the sequel, let $e_i$ denote the unit vector with an one in the $i$-th coordinate, let $0$ denote the all-zero vector, and let $1$ denote the all-one vector.

**Theorem 1.** Let $W$ be a weighing matrix of order $n$ and weight $w$ and let $\Lambda(W)$ be the corresponding lattice.

- The minimum Manhattan distance of $\Lambda(W)$ is $w$.
- The volume of $\Lambda(W)$ is $w^n$.
- $\Lambda(W)$ can be reduced to a code $C$ of length $n$, in the Lee metric, over the alphabet $\mathbb{Z}_w$. The minimum Lee distance of $C$ is $w$.

**Proof:**

- The minimum distance of $\Lambda(W)$ is the weight of the nonzero lattice point of minimum Manhattan weight.

Let $x \in \Lambda(W)$ be a nonzero lattice point of minimum Manhattan weight. The point $x$ is obtained by a linear combination of a few rows from $W$.

Let $y$ be a row which is included in this linear combination with a coefficient $\rho$, $\rho \geq 1$, in this sum. Let $W'$ be the matrix obtained from $W$ by negating the columns in which $y$ has −1’s.

Let $x'$ be a lattice point in $\Lambda(W')$ formed from the linear combination of the same related rows of $W'$, as those from which $x$ was formed from $W$.

The point $x'$ has minimum Manhattan weight in $\Lambda(W')$ (the same Manhattan weight as $x$). The related row $y'$ has only zeroes and ones and without loss of generality we can assume that the ones are in the last $w$ coordinates. $y'$ is included $\rho$ times in the linear combination from which $x'$ is obtained. Hence, the total sum of the elements in these last $w$ entries of $x'$ is $\rho w$ (since in a row which is not $y'$ the sum of the entries in the last $w$ coordinates is zero). Thus, the Manhattan weight of $x'$ is at least $\rho w$.

All the rows in $W$ have Manhattan weight $w$ and hence the minimum Manhattan distance of $\Lambda(W)$ is exactly $w$.

- The volume of $\Lambda(W)$ is the determinant of $W$ known to be $w^n$ and this is easily inferred from the definition of a weighing matrix.

- Let $\omega_i$ be row $i$ of $W$ and let $\omega_j^{(i)}$ be the $j$-th entry in this row. Clearly, $\sum_{i=1}^{n} \omega_j^{(i)} \omega^{(i)} = w \cdot e_j$ (since the $j$-th column is orthogonal to all the columns of $W$ except itself). Thus, $\Lambda(W)$ can be reduced to a code $C$ of length $n$, in the Lee metric over, the alphabet $\mathbb{Z}_w$. The lattice points of minimum Manhattan weight $w$ are also codewords in $C$ (where $-1$ is replaced by $w - 1$) and hence the minimum Lee distance of $C$ is also $w$. 


A code is called self-dual if it equals its dual. Since the inner product of two rows from a weighing matrix \( W \) is either 0 or \( w \), it follows that the code \( C \) reduced from \( \Lambda(W) \) is contained in its dual. Since the size of the code is \( w^{n/2} \) and the size of the space is \( w^n \), it follows that the dual code has also size \( w^{n/2} \). Thus, we have

**Theorem 2.** Let \( W \) be a weighing matrix of order \( n \) and weight \( w \). If \( C \) is the code over \( \mathbb{Z}_w \) reduced from \( \Lambda(W) \) then \( C \) is a self-dual code.

Self-dual codes were considered extensively in coding theory, e.g. [28]. The lattice \( \Lambda' \) is the dual of the lattice \( \Lambda \) if \( \Lambda' \) contains all the points in \( \mathbb{R}^n \) whose inner product with the lattice points of \( \Lambda \) is an integer. We are not interested in dual lattices as the related lattice points have usually some non-integer entries.

Let \( A \) be an \( n \times n \) matrix over \( \mathbb{Z}_k \). The rank of \( A \) over \( \mathbb{Z}_k \) is defined to be the maximum number of linearly independent rows of \( A \) over \( \mathbb{Z}_k \).

**Theorem 3.** The rank of a Hadamard matrix of order \( n \) over \( \mathbb{Z}_n \) is \( n - 1 \).

**Proof:** Let \( H \) be a Hadamard matrix of order \( n \). By Theorem 2 we have that the code \( C \) reduced from \( \Lambda(H) \), to the Lee metric over \( \mathbb{Z}_n \), is a self-dual code. Therefore, it is easy to verify that the words of length \( n \) with exactly two nonzero entries equal \( \frac{n}{2} \) are codewords of \( C \). Consider the generator matrix

\[
G = \begin{bmatrix}
\frac{n}{2} & 0 & 0 & \ldots & 0 & \frac{n}{2} \\
0 & \frac{n}{2} & 0 & \ldots & 0 & \frac{n}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \frac{n}{2} & \frac{n}{2} \\
0 & 0 & 0 & \ldots & 0 & n
\end{bmatrix}
\]

the lattice \( \Lambda(G) \) is a sub-lattice of \( \Lambda(H) \). Thus, the rank of the code \( C \) reduced from \( \Lambda(H) \) has rank at least \( n - 1 \). If the rank of \( C \) is \( n \) then it has at least one row with Hamming weight one in each generator matrix. But this row can only be \( n \cdot e_i \), for some \( i, 1 \leq i \leq n \). It implies that the rank of \( C \) is less than \( n \), i.e. \( n - 1 \).

**Theorem 4.** If \( W \) is a conference matrix of order \( n = p + 1 \), where \( p \) is a prime, then its rank over \( \mathbb{Z}_p \) (also \( \mathbb{F}_p \)) is \( \frac{p+1}{2} \).

**Proof:** Since the volume of \( \Lambda(W) \) is \( p^{\frac{p+1}{2}} \) and \( p \) is a prime, it follows that in a lower diagonal matrix representation, the generator matrix of \( \Lambda(W) \) has a diagonal with \( \frac{p+1}{2} \) \( p \)'s and \( \frac{p+1}{2} \) 1's. Thus, the rank of \( W \) over \( \mathbb{Z}_p \) is \( \frac{p+1}{2} \). Since \( p \) is a prime, it follows that the ring \( \mathbb{Z}_p \) is equal to the finite field \( \mathbb{F}_p \).

**Conjecture 5** If \( C \) is a code of length \( p + 1 \) constructed from a generator matrix which is a conference matrix then \( C \) is an MDS code of dimension \( \frac{p+1}{2} \) and minimum Hamming distance \( \frac{p+3}{2} \).

Conjecture 5 was verified to be true up to \( n = 23 \), where the conference matrices are based on the Paley’s construction from quadratic residues modulo \( p \). Codes with these parameters (self-dual MDS of length \( q + 1 \), \( q \) a prime power) were constructed in [17].
III. CODES FROM THE DOUBLING CONSTRUCTION

The most simple and celebrated method to construct Hadamard matrices of large orders from Hadamard matrices of small orders is the doubling construction. Given a Hadamard matrix $\mathcal{H}$ of order $n$, the matrix

$$ \begin{bmatrix} \mathcal{H} & \mathcal{H} \\ \mathcal{H} & -\mathcal{H} \end{bmatrix}, $$

is a Hadamard matrix of order $2n$.

A Sylvester Hadamard matrix of order $m$, $\mathcal{H}_m$, is a $2^m \times 2^m$ Hadamard matrix obtained by the doubling construction starting with the Hadamard matrix $\mathcal{H}_0 = [1]$ of order one. This matrix is also based on the first order Reed-Muller code \cite{23}. Let $H_0 = [1]$ and $H_{m+1} = \begin{bmatrix} H_m & H_m \\ 0 & H_m \end{bmatrix}$, $m \geq 0$. Let $G(m, j)$, $0 \leq j \leq m$, be the $2^m \times 2^m$ matrix constructed from $H_m$ as follows. Let $2^j$ be the Hamming weight of the $s$-th row of $H_m$. If $\ell \geq j$ then the $s$-th row of $G(m, j)$ will be the same as the $s$-th row of $H_m$. If $\ell < j$ then the $s$-th row of $G(m, j)$ will be the $s$-th row of $H_m$ multiplied by $2^{j-\ell}$.

It is easy to verify that $G(m, j)$ can be defined recursively as follows. For $1 \leq j < m$, $G(m, j)$ is given by

$$ G(m, j) = \begin{bmatrix} G(m-1, j-1) & G(m-1, j-1) \\ 0 & G(m-1, j) \end{bmatrix}, $$

where $G(m, m)$ is given by

$$ G(m, m) = \begin{bmatrix} G(m-1, m-1) & G(m-1, m-1) \\ 0 & 2G(m-1, m-1) \end{bmatrix}, $$

and $G(m, 0) = H_m$.

The following lemma can be proved by applying a simple induction.

**Lemma 6.**

$$ \Lambda(G(m, m)) = \Lambda(H_m), $$

for all $m \geq 0$.

**Example 1.** The Sylvester Hadamard matrix of order 2, is a Hadamard matrix of order 4, given by

$$ \mathcal{H}_2 = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{bmatrix}. $$
In $\mathbb{Z}^4$ it generates the same lattice as the generator matrix

$$G(2, 2) = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 2 & 0 & 2 \\
0 & 0 & 2 & 2 \\
0 & 0 & 0 & 4
\end{bmatrix}.$$ 

Reducing the entries of $G(2, 2)$ into zeroes and ones yields

$$H_2 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$ 

Clearly, the rows of $G(m, j)$ are linearly independent. Let $\Lambda(m, j)$ be the lattice whose generator matrix is $G(m, j)$, and $C(m, j)$ the code reduced from $\Lambda(m, j)$, over $\mathbb{Z}_2$, whose generator matrix is $G(m, j)$. $C(m, j)$ was constructed by a completely different approach for the control of the peak-to-mean envelope power ratio in orthogonal frequency-division multiplexing in [31], where its size and minimum distance were discussed. How this sequence of codes can be generalized for length which is not a power of two and to Hadamard matrices which are not based on Sylvester matrices? A possible answer to this question and our different approach for these codes will be demonstrated in Section IV.

The following lemma is an immediate result from the recursive construction of $H_m$.

**Lemma 7.** The number of rows with weight $2^i$, $0 \leq i \leq m$, in $H_m$ is $\binom{m}{i}$.

By Lemma 7 and by the definition of $G(m, j)$, for each $\ell, j \leq \ell \leq m$, there exist rows in $G(m, j)$ with Manhattan weight $2^\ell$. These are the only weights of rows in $G(m, j)$.

**Theorem 8.**

- The minimum Manhattan distance of $\Lambda(m, j)$ is $2^j$.
- The volume of the lattice $\Lambda(m, j)$ is $\prod_{i=0}^{j} 2^{(j-i)\binom{m}{i}}$.
- $\Lambda(m, j)$ is reduced to the code $C(m, j)$. $C(m, j)$ has minimum Lee distance $2^j$.

**Proof:**

- The minimum distance of $\Lambda(m, j)$ can be derived by a simple induction from the recursive definition of $G(m, j)$.
- The volume of $\Lambda(m, j)$ can be derived easily by induction from the recursive definition of $G(m, j)$ or by a very simple direct computation from Lemma 7.
• It is easily verified by using induction that for each \( i, 1 \leq i \leq n \), the point \( 2^i \cdot e_i \) is contained in \( \Lambda(m, j) \). Thus, \( \Lambda(m, j) \) can be reduced to a code \( C \) of length \( 2^m \), in the Lee metric, over the alphabet \( \mathbb{Z}_{2^j} \). The minimum Lee distance can be derived also by a simple induction.

In Section IV, we will consider codes related to the lattice \( \Lambda(m, j) \). The covering radius of these codes will be an important factor in our construction for a space transformation. Therefore, we will devote the rest of this section to find bounds on the covering radius of the lattice \( \Lambda(m, j) \), which is equal to the covering radius of the code \( C(m, j) \).

\( \Lambda(m, 0) \) is equal to \( \mathbb{Z}^{2m} \) and hence its covering radius is 0. \( \Lambda(m, 1) \) consists of all the points in \( \mathbb{Z}^{2m} \) which have an even sum of elements. The covering radius of this code is clearly 1. \( C(m, 2) \) is a diameter perfect code with minimum Lee distance 4 and covering radius 2 \([2, 13]\). In general we don’t know the exact covering radius of \( \Lambda(m, j) \) except for two lattices (codes) for which the covering radius was found with a computer aid. The covering radius of \( \Lambda(3, 3) \) equals 6 and the covering radius of \( \Lambda(4, 3) \) equals 8. We also found that the covering radius of \( \Lambda(4, 4) \) is at most 20. However, two bounds can be derived from the structure of \( G(m, j) \). Let \( r(m, j) \) be the covering radius of the lattice \( \Lambda(m, j) \) (and also the code \( C(m, j) \)).

**Theorem 9.** \( r(m, m) \leq 3r(m − 1, m − 1) + 2^{m−1}, m \geq 5 \), where \( r(2, 2) = 2, r(3, 3) = 6 \) and \( r(4, 4) \leq 20 \).

**Proof:** Let \( (x, y) \in \mathbb{Z}_{2^m}^2 \), where \( x, y \in \mathbb{Z}_{2^m}^2 \). We have to show that there exists a codeword \( c \in C(m, m) \) such that \( d_L(c, (x, y)) \leq 3r(m − 1, m − 1) + 2^{m−1} \).

Let \( x' \in \mathbb{Z}_{2^m}^{2m−1} \) be the word obtained from \( x \) by reducing each entry of \( x \) modulo \( 2^{m−1} \). Let \( z' \) be a codeword in \( C(m − 1, m − 1) \) such that \( d_L(x', z') \leq r(m − 1, m − 1) \). By using the same linear combination of rows from \( G(m − 1, m − 1) \), which was used to obtain \( z' \) in \( \mathbb{Z}_{2^m−1}^2 \), with computation modulo \( 2^m \) instead of modulo \( 2^{m−1} \) which was used for \( z' \), we obtain a word \( z \in \mathbb{Z}_{2^m−1}^2 \) not necessarily equals to \( z' \) (for each coordinate, the values of \( z \) and \( z' \) are equal modulo \( 2^{m−1} \)).

For each \( i, 1 \leq i \leq 2^{m−1} \) we form a word \( p^{(i)} \in \mathbb{Z}_{2^m}^2 \) as follows. Let \( \ell_i = \min\{z_i − x_i \mod 2^m, x_i − z_i \mod 2^m\} \), such that \( 0 \leq \ell_i \leq 2^{m−2} \). If \( \ell_i < 2^{m−2} \) then \( p^{(i)} = (0, 0) \). If \( \ell_i \geq 2^{m−2} \) then \( p^{(i)} = 2^{m−1}(e_i, e_i) \). It is easy to verify by the definition of \( G(m, m) \) that \( (z, z) \in C(m, m) \) and \( p^{(i)} \in C(m, m) \) for each \( 1 \leq i \leq 2^{m−1} \). Therefore, \( (v, v) = (z, z) + \sum_{i=1}^{2^{m−1}} p^{(i)} \) is a codeword in \( C(m, m) \) and \( d_L(x, v) = d_L(x', z') \leq r(m − 1, m − 1) \).

Let \( y' \in \mathbb{Z}_{2^m−1}^2 \) be the word defined as follows. If \( v_i − y_i \) is an even integer then \( y'_i = y_i \) and if \( v_i − y_i \) is an odd integer then \( y'_i = y_i + 1 \mod 2^m \). Since the covering radius of \( C(m − 1, m − 1) \) is \( r(m − 1, m − 1) \), it follows that there exists a codeword \( z'' \in C(m − 1, m − 1) \) such that \( d_L(y' − v, 2z'') \leq 2r(m − 1, m − 1) \). Clearly, \( (v, v+2z'') \in C(m, m) \) and \( d_L((x, y'), (v, v+2z'')) \leq 3r(m − 1, m − 1) \). It implies that \( d_L((x, y), (v, v+2z'')) \leq 3r(m − 1, m − 1) + 2^{m−1} \). Thus, \( r(m, m) \leq 3r(m − 1, m − 1) + 2^{m−1} \).

One can analyze the bound of Theorem 9 and obtain that when \( m \) is large \( r(m, m) \) is less than approximately \( 4 \cdot 3^{m−2} \), or \( n^{1.585} \). But, we believe that the covering radius of \( C(m, m) \) is considerably smaller.
Theorem 10. \( r(m, j) \leq r(m - 1, j - 1) + r(m - 1, j) \), \( 2 < j < m \), where \( r(m, 2) = 2 \) for \( m \geq 2 \) and upper bound on \( r(m, m) \) is given in Theorem[3]

Proof: Let \((x, y) \in \mathbb{Z}_2^m \), where \( x, y \in \mathbb{Z}_2^{m-1} \). We have to show that there exists a codeword \( c \in \mathbb{C}(m, j) \) such that \( d_L(e, (x, y)) \leq r(m - 1, j - 1) + r(m - 1, j) \).

Let \( x' \in \mathbb{Z}_2^{m-1} \) be the word obtained from \( x \) by reducing each entry of \( x \) modulo \( 2^j-1 \). Let \( z' \in \mathbb{Z}_2^{m-1} \) be a codeword in \( \mathbb{C}(m - 1, j - 1) \) such that \( d_L(x', z') \leq r(m - 1, j - 1) \). By using the same linear combination of rows from \( G(m - 1, j - 1) \), which was used to obtain \( z' \) in \( \mathbb{Z}_2^{m-1} \), but with computation modulo \( 2^j \) instead of modulo \( 2^j-1 \) which was used for \( z' \), we obtain a word \( z \in \mathbb{Z}_2^{m-1} \) not necessarily equal to \( z' \) (for each coordinate, the values of \( z \) and \( z' \) are equal modulo \( 2^j-1 \)).

For each \( i, 1 \leq i \leq 2^{m-1} \) we form a word \( p^{(i)} \in \mathbb{Z}_2^m \) as follows. Let \( \ell_i = \min\{z_i - x_i \ (\text{mod} \ 2^j), x_i - z_i \ (\text{mod} \ 2^j)\} \), such that \( 0 \leq \ell_i \leq 2^{j-1} \). If \( \ell_i < 2^{j-2} \) then \( p^{(i)} = (0, 0) \). If \( \ell_i \geq 2^{j-2} \) then \( p^{(i)} = 2^{j-1}(e_i, e_i) \). It is easy to verify by the definition of \( G(m, j) \) that \((z, z) \in \mathbb{C}(m, j)\) and \( p^{(i)} \in \mathbb{C}(m, j) \) for each \( 1 \leq i \leq 2^{m-1} \). Therefore, \((v, v) = (z, z) + \sum_{i=1}^{2^{m-1}} p^{(i)} \) is a codeword in \( \mathbb{C}(m, j) \) and \( d_L(x, v) = d_L(x', z') \leq r(m - 1, j - 1) \).

Since the covering radius of \( \mathbb{C}(m - 1, j) \) is \( r(m - 1, j) \), it follows that there exists a codeword \( z'' \in \mathbb{C}(m - 1, j) \) such that \( d_L(y - v, z'') \leq r(m - 1, j) \). Clearly, \((v, v + z'') \in \mathbb{C}(m, j)\) and \( d_L((x, y), (v, v + z'')) \leq r(m - 1, j - 1) + r(m - 1, j) \). Thus, \( r(m, j) \leq r(m - 1, j - 1) + r(m - 1, j) \). \( \blacksquare \)

The covering radius of a code can be computed also from the parity-check matrix of the code. Hence, it would be interesting to examine the parity-check matrix of the code \( \mathbb{C}(m, j) \). We construct the parity check matrix \( F(m, j) \) of the code \( \mathbb{C}(m, j) \) from the matrix \( H_m \) as follows. Let \( 2^\ell \) be the Hamming weight in the \( s \)-th row of \( H_m \). If \( m - \ell < j \) then \( F(m, j) \) will contain the \( s \)-th row of \( H_m \) multiplied by \( 2^{m-\ell} \). There are no other rows in \( F(m, j) \).

One can verify that \( F(m, j) \), \( 0 \leq j \leq m \), is defined recursively as follows (we leave the formal proof to the reader). For \( 1 \leq j \leq m \), \( F'(m, j) \) is given by

\[
F'(m, j) = \begin{bmatrix}
F'(m - 1, j) & F'(m - 1, j) \\
0 & 2F'(m - 1, j - 1)
\end{bmatrix},
\]

where \( F'(m, m) \) is defined by

\[
F'(m, m) = \begin{bmatrix}
F'(m - 1, m - 1) & F'(m - 1, m - 1) \\
0 & 2F'(m - 1, m - 1)
\end{bmatrix},
\]

and \( F'(m, 0) = [1 \ 1 \ \cdots \ 1] \), for \( m \geq 0 \).

\( F(m, j) \) is constructed from \( F'(m, j) \) by omitting the last row.

IV. Lee Sphere Transformations

In multidimensional coding, many techniques are applied on multidimensional cubes of \( \mathbb{Z}^n \) and cannot be applied on other shapes in \( \mathbb{Z}^n \), e.g. [1], [7], [14]. Assume we want to apply a technique which is applied on any \( n \)-dimensional cube of \( \mathbb{Z}^n \) to a different \( n \)-dimensional shape \( S \) of \( \mathbb{Z}^n \). This problem can be solved by a transformation from \( \mathbb{Z}^n \) to \( \mathbb{Z}^n \), which preserves volumes, in which each \( n \)-dimensional shape \( S \) of \( \mathbb{Z}^n \) is transformed into a shape \( S' \) which
can be inscribed in a relatively small \( n \)-dimensional cube of \( \mathbb{Z}^n \). The technique is now applied on the image of the transformation and then transformed back into the domain. One of the most important shapes in this context is the \( n \)-dimensional Lee sphere with radius \( R \), \( S_{n,R} \). Clearly, an \( n \)-dimensional Lee sphere with radius \( R \) can be inscribed in an \( (2R + 1) \times \cdots \times (2R + 1) \) \( n \)-dimensional cube. In [14] a transformation of \( \mathbb{Z}^n \) is given for which \( S_{n,R} \) is transformed into a shape inscribed in a cube of size \( (R + 1) \times (R + 1) \times \cdots \times (R + 1) \) \( n \times n \) times. The gap from the theoretical size of the cube is still large since the size of the \( n \)-dimensional Lee sphere with radius \( R \) is \( \frac{(2R)^n}{n!} + O(R^{n-1}) \), when \( n \) is fixed and \( R \to \infty \). The goal of this section is to close on this gap. In the process, some interesting codes and coding problems will arise. The transformation we have to define is clearly a discrete transformation, but for completeness, and since it has an interest of its own, we will consider also the more simple case of a continuous transformation \( T : \mathbb{R}^n \to \mathbb{R}^n \). This can be viewed also as a transformation on conscribed cross-polytopes, which were defined in [16], rather than on Lee spheres. For every Lee sphere, \( S_{n,R} \), the conscribed cross-polytope, \( CP_{n,R} \), is defined [16] to be the convex hull of the \( 2n \) centers points of the \( (n-1) \)-dimensional extremal hyperfaces of \( S_{n,R} \). What makes this figure more attractive to us than similar figures is that the volume of \( CP_{n,R} \) is exactly \( \frac{(2R+1)^n}{n!} \).

The continuous transformation will also be interesting from error-correcting codes point of view as it will be understood in the sequel. The transformation which will be described will make use of weighing matrices with some symmetry (symmetric or skew-symmetric). In Section V we will discuss the existence of such matrices and their relevance in our construction.

A. The Continuous Transformation

In this subsection we are going to define a sequence of transformations based on symmetric or skew symmetric weighing matrices, some of which will transform Lee spheres (or conscribed cross-polytopes) in the space, into shapes inscribed in a relatively small cubes. These transformations also form some interesting codes in the Lee and Manhattan metrics which are related to the codes defined in Section III. Of these transformations there is one which will preserve volumes and will serve as our main transformation.

Let \( \mathcal{W} \) be a symmetric or skew symmetric weighing matrix of order \( n \) and weight \( w \). Given a real number \( s > 0 \), we define a transformation \( T^\mathcal{W}_s : \mathbb{R}^n \to \mathbb{R}^n \), as follows. For each \( x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n \),

\[
T^\mathcal{W}_s(x) \stackrel{\text{def}}{=} \frac{\mathcal{W}x}{s}.
\]  

Lemma 11. Let \( \mathcal{W} \) be a weighing matrix of order \( n \) and weight \( w \) and let \( s > 0 \) be a positive real number.

1. If \( \mathcal{W} \) is symmetric then for all \( x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n \),

\[
T^\mathcal{W}_s(T^\mathcal{W}_s(x)) = x.
\]

2. If \( \mathcal{W} \) is skew symmetric then for all \( x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n \),

\[
T^\mathcal{W}_s(T^\mathcal{W}_s(x)) = -x.
\]
Proof: We will prove the case where $W$ is a symmetric matrix. For each $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$,

$$T_s^W(T_s^W(x)) = \frac{W \cdot Wx}{w} = \frac{W \cdot W^T x}{w} = \frac{wI_n \cdot x}{w} = x.$$ 

The case in which $W$ is a skew symmetric matrix has an identical proof.

Let $W$ be a symmetric or a skew symmetric weighing matrix of order $n$ and weight $w$ and let $s$ be a positive integer which divides $w$. Let $\Lambda_s^W$ be the set of points in $\mathbb{Z}^n$ which are mapped to points of $\mathbb{Z}^n$ by the transformation $T_s^W$ given by (2), i.e.

$$\Lambda_s^W \overset{\text{def}}{=} \{x \in \mathbb{Z}^n : T_s^W(x) \in \mathbb{Z}^n\}.$$

The proof of the next theorem can be deduced from the theory of dual lattices [12]. But, to avoid a new sequence of definitions and known results we will provide another direct proof.

**Theorem 12.** Let $W$ be a symmetric or a skew symmetric weighing matrix of order $n$ and weight $w$, and let $s$ be a positive integer which divides $w$. Then $\Lambda_s^W$ is a lattice with minimum Manhattan distance $s$; moreover $\Lambda_s^W = T_s^W(\Lambda_s^W)$. Finally, $\Lambda_s^W$ can be reduced to a code $C_s^W$ of length $n$, in the Lee metric, over the alphabet $\mathbb{Z}_s$.

**Proof:** We break the proof into three parts. First, we will prove that $\Lambda_s^W$ is a lattice. We will proceed to prove that the minimum Manhattan distance of $\Lambda_s^W$ is $s$; and that $\Lambda_s^W$ can be reduced to a code $C_s^W$ of length $n$, in the Lee metric, over the alphabet $\mathbb{Z}_s$. Finally, we will prove that $\Lambda_s^W = T_s^W(\Lambda_s^W)$.

1) If $x_1, x_2 \in \Lambda_s^W$ then $x_1, x_2 \in \mathbb{Z}^n$ and $T_s^W(x_1), T_s^W(x_2) \in \mathbb{Z}^n$. Hence,

$$T_s^W(x_1 + x_2) = \frac{W(x_1 + x_2)}{s} = \frac{Wx_1}{s} + \frac{Wx_2}{s} = T_s^W(x_1) + T_s^W(x_2) \in \mathbb{Z}^n,$$

and therefore $x_1 + x_2 \in \Lambda_s^W$, i.e. $\Lambda_s^W$ is a lattice.

2) Since $\Lambda_s^W$ is a lattice it follows that its minimum Manhattan distance is the Manhattan weight of a nonzero lattice point with minimum Manhattan weight. Let $x = (x_1, x_2, \ldots, x_n)^t \in \Lambda_s^W$ be a nonzero lattice point, i.e. $(y_1, y_2, \ldots, y_n)^t = \frac{Wx}{s} = \left(\frac{s \cdot y_1}{s}, \frac{s \cdot y_2}{s}, \ldots, \frac{s \cdot y_n}{s}\right) \in \mathbb{Z}^n$. There exists at least one $i$ for which $y_i \neq 0$. For this $i$, we have $s \cdot y_i = \sum_{j=1}^n \omega_j^{(i)} x_j$. Since $|w_j^{(i)}| \leq 1$ for every $j, 1 \leq j \leq n$, it follows that $\sum_{j=1}^n |x_j| \geq s$. Thus, the minimum Manhattan weight of $x$ is at least $s$ and the same it true for the minimum Manhattan distance of $\Lambda_s^W$. It is easy to verify that $(0, \ldots, 0, s, 0, \ldots, 0)^t$ is a point in $\Lambda_s^W$ and hence the minimum Manhattan distance of $\Lambda_s^W$ is exactly $s$; and $\Lambda_s^W$ can be reduced to a code $C_s^W$ of length $n$, in the Lee metric, over the alphabet $\mathbb{Z}_s$.

3) Let $x \in \Lambda_s^W$, i.e. $x \in \mathbb{Z}^n$, $y = T_s(x) \in \mathbb{Z}^n$. By Lemma [11] we have $T_s^W(y) = T_s^W(T_s^W(x))$ equals either $x$ or $-x$, i.e. $y \in \Lambda_s^W$, which implies that $x \in T_s^W(\Lambda_s^W)$. Therefore, $\Lambda_s^W \subseteq T_s^W(\Lambda_s^W)$.

Let $x \in T_s^W(\Lambda_s^W)$, i.e. $x \in \mathbb{Z}^n$, $x = T_s^W(y)$, where $y \in \Lambda_s^W \subseteq \mathbb{Z}^n$. By Lemma [11] we have that $T_s^W(x) = T_s^W(T_s^W(y))$ equals either $y$ or $-y$, and hence $x \in \Lambda_s^W$. Therefore, $T_s^W(\Lambda_s^W) \subseteq \Lambda_s^W$.

Thus, $\Lambda_s^W = T_s^W(\Lambda_s^W)$. 
Theorem 13. Let \( \mathcal{H} \) be a Hadamard matrix of order \( n > 4 \) and let \( s \) be a positive integer which divides \( n \). If \( s \) is even then the minimum Lee distance of \( \mathbb{C}_s^\mathcal{H} \) is \( s \). If \( s > 1 \) is odd then the minimum Lee distance of \( \mathbb{C}_s^\mathcal{H} \) is greater than \( s \) and is at most \( \frac{n}{2} \).

**Proof:** If \( s \) is even and divides \( n \) then \((0, \ldots, 0, \frac{n}{2}, \frac{n}{2}\) is a codeword in \( \mathbb{C}_s^\mathcal{H} \) and the minimum Lee distance \( s \) follows from Theorem [12]

If \( s \) is odd and divides \( n \) then \( s \) also divides \( \frac{n}{2} \). W.l.o.g. we assume that the first row of \( \mathcal{H} \) consists of \( n + 1 \)'s and the second row consists of \( \frac{n}{2} + 1 \)'s followed by \( \frac{n}{2} - 1 \)'s. Therefore, in all the other rows of \( \mathcal{H} \) we have exactly \( \frac{n}{2} + 1 \)'s and \( \frac{n}{2} - 1 \)'s in the first \( \frac{n}{2} \) entries. It implies that the word of length \( n \) with \( \frac{n}{2} \) ones followed by \( \frac{n}{2} \) zeroes is a codeword in \( \mathbb{C}_s^\mathcal{H} \). Hence, the minimum Lee distance of \( \mathbb{C}_s^\mathcal{H} \) is at most \( \frac{n}{2} \).

Assume that the minimum Lee distance of \( \mathbb{C}_s^\mathcal{H} \) is \( s \). Hence, there is a lattice point \( \mathbf{x} \) in \( \Lambda_s^\mathcal{H} \) with Manhattan weight \( s \) and at least two nonzero entries. Since \( \mathcal{H} \) contains a row with all entries +1's it follows that the sum of elements in \( \mathbf{x} \) is \(-s, 0, \text{ or } s \). Since \( s \) is odd, it follows that this sum cannot be 0. Hence, \( \mathbf{x} \) cannot contain both positive and negative entries. Since the projection, of the entries with nonzero elements in \( \mathbf{x} \), on \( \mathcal{H} \), has rows with both +1 and -1, it follows that the inner product of these rows with \( \mathbf{x} \) is not divisible by \( s \). Hence \( \mathbf{x} \) is not a lattice point in \( \Lambda_s^\mathcal{H} \), a contradiction. By Theorem [12] the minimum Lee distance of \( \mathbb{C}_s^\mathcal{H} \) is at least \( s \) and thus the minimum Lee distance of \( \mathbb{C}_s^\mathcal{H} \) is greater than \( s \) and is at most \( \frac{n}{2} \).

Theorem 13 provides some information on the minimum Lee distance of the code \( \mathbb{C}_s^\mathcal{H} \), where \( \mathcal{H} \) is a Hadamard matrix. In general, for a weighing matrix \( \mathcal{W} \), what is the Lee distance of the code \( \mathbb{C}_s^\mathcal{W} \)? It appears that it is not always reduced to \( s \) as the Manhattan distance of \( \Lambda_s^\mathcal{W} \). In fact, if \( \frac{n}{2} < w < n \) we conjecture that it is always \( w \), in contrast to the result in Theorem [14] for \( w = n \).

**Theorem 14.** If \( \mathcal{W} \) is a weighing matrix of order \( n \) and weight \( w \) then \( \Lambda_w^\mathcal{W} = \Lambda(\mathcal{W}) \).

**Proof:** Clearly, \( \mathbb{C}_1^\mathcal{W} \) is equal \( \mathbb{Z}^n \). By Theorem [12] we have that \( \Lambda_w^\mathcal{W} = T_1^\mathcal{W}(\Lambda_1^\mathcal{W}) \). Since \( \Lambda_1^\mathcal{W} = \mathbb{Z}^n \) it follows that \( \Lambda_w^\mathcal{W} \) contains exactly all the linear combinations of the rows from \( \mathcal{W} \). Thus, \( \Lambda_w^\mathcal{W} = \Lambda(\mathcal{W}) \).

**Lemma 15.** If \( s_1 \) divides \( s_2 \) and \( s_1 < s_2 \), then \( \Lambda_{s_2}^\mathcal{W} \subset \Lambda_{s_1}^\mathcal{W} \).

**Proof:** If \( \mathbf{x} \in \mathbb{Z}^n \) and the entries of \( \mathcal{W}\mathbf{x} \) are divisible by \( s_2 \) then by definition we have \( \mathbf{x} \in \Lambda_{s_2}^\mathcal{W} \). Since \( s_1 \) divides \( s_2 \), it follows that the entries of \( \mathcal{W}\mathbf{x} \) are divisible also by \( s_1 \). Hence, \( \mathbf{x} \in \Lambda_{s_1}^\mathcal{W} \) and \( \Lambda_{s_2}^\mathcal{W} \subset \Lambda_{s_1}^\mathcal{W} \). By Theorem [12] the minimum distance of \( \Lambda_{s_1}^\mathcal{W} \) is \( s_1 \) and the minimum distance of \( \Lambda_{s_2}^\mathcal{W} \) is \( s_2 \). Thus, \( \Lambda_{s_2}^\mathcal{W} \subset \Lambda_{s_1}^\mathcal{W} \).

**Corollary 16.** If \( s \) divides \( w \) then \( \Lambda_s^\mathcal{W} \) contains \( \Lambda(\mathcal{W}) \).
We now turn to a volume preserving transformation from the set of all transformations which were defined. This transformation is $T^W$ and redefined as $T^W : \mathbb{R}^n \to \mathbb{R}^n$ to be

$$T^W(x) = Wx \sqrt{w}$$  \hspace{1cm} (3)

Theorem 12 is applied also with the transformation $T^W$. In this case $w = D^2$, where $D$ is a positive integer, $\Lambda^W \triangleq \Lambda^W_D$ is a lattice with minimum Manhattan distance $D$, and $\Lambda^W = T^W(\Lambda^W)$. Finally, $\Lambda^W$ can be reduced to a code $C^W$ of length $n$, in the Lee metric, over the alphabet $Z_D$.

**Lemma 17.** A conscribed cross-polytope, centered at $c = (c_1, \ldots, c_n)^t \in \mathbb{R}^n$, $CP_{n,R}(c)$, is inscribed after the transformation $T^W$ inside an $n$-dimensional cube of size

$$\left(\frac{2R + 1}{\sqrt{w}}\right) \times \cdots \times \left(\frac{2R + 1}{\sqrt{w}}\right).$$

**Proof:** $CP_{n,R}(c)$ is contained in the following set of points

$$CP_{n,R}(c) = \left\{ c + (x_1, \ldots, x_n)^t \left| \sum_{i=1}^n |x_i| \leq R + \frac{1}{2} \right. \right\},$$

where $(x_1, \ldots, x_n)^t \in \mathbb{R}^n$. This set of points is transformed by the transformation $T^W$ into the following set of points,

$$T^W(CP_{n,R}(c))$$

$$= \left\{ T^W(c + (x_1, \ldots, x_n)^t) \left| \sum_{i=1}^n |x_i| \leq R + \frac{1}{2} \right. \right\}$$

$$= \left\{ Wc \sqrt{w} + W(x_1, \ldots, x_n)^t \left| \sum_{i=1}^n |x_i| \leq R + \frac{1}{2} \right. \right\}. \hspace{1cm}$$

If $W(x_1, \ldots, x_n)^t = (y_1, \ldots, y_n)^t$ then, for $1 \leq i \leq n$,

$$|y_i| = \left| \sum_{j=1}^n \omega_j^{(i)} x_j \right| \leq \sum_{j=1}^n |x_j| \leq R + \frac{1}{2}.$$  \hspace{1cm}

Therefore, the set $T^W(CP_{n,R}(c))$ is located inside the following $\left(\frac{2R + 1}{\sqrt{w}}\right) \times \cdots \times \left(\frac{2R + 1}{\sqrt{w}}\right)$ $n$-dimensional cube

$$\left\{ Wc \sqrt{w} + (\ell_1, \ldots, \ell_n)^t \left| |\ell_i| \leq R + \frac{1}{2} \sqrt{w}, \hspace{0.5cm} 1 \leq i \leq n \right. \right\}.$$  \hspace{1cm}

Note that since $\det(W/\sqrt{w}) = 1$, the transformation $T^W$ also preserves volumes. The volume of the conscribed cross-polytope is $\frac{(2R+1)^n}{n!}$ while the volume of the inscribing $n$-dimensional cube is $\frac{(2R+1)^n}{\sqrt{w}}$. If we choose $w = n$, i.e. a Hadamard matrix of order $n$, then we get that the ratio between the volumes of the $n$-dimensional cube and the conscribed cross-polytope is $\frac{n!}{n^{n/2}}$. The shape of the Lee sphere is very similar to the one of the conscribed cross-polytope and hence a similar result can be obtained for a Lee sphere. But, a continuous shape like the conscribed cross-polytope is more appropriate when we consider a continuous transformation.
B. On the connection between \( C(m, j) \) and \( C_{2^j}^H \)

In this subsection, we consider some interesting connections between the code \( C(m, j) \) defined in Section \[III\] and the code \( C_{2^j}^H \) defined in Theorem \[12\] where the weighing matrix \( W \) is the Hadamard matrix \( H_m \).

Lemma 18. The inner product of a lattice point from \( \Lambda(m, j) \) and a lattice point from \( \Lambda(m, m) \) is divisible by \( 2^j \).

Proof: The lattice \( \Lambda(m, m) \) is equal to the lattice \( \Lambda(H_m) \). By Theorem \[1\] this lattice is reduced to a code, in the Lee metric, over the alphabet \( \mathbb{Z}_{2^m} \). It follows that the inner product between lattice points of \( \Lambda(m, m) \) is divisible by \( 2^m \). The sum of the entries in a row of \( G(m, m) \) is exactly \( 2^m \). The sum of elements in a given row of \( G(m, j) \) is \( 2^j \), for some \( \ell \) such that \( j \leq \ell \leq m \). This row is obtained by dividing the entries of the related row in \( G(m, m) \) by \( 2^{m-\ell} \). Hence, the inner product of this row with any lattice point of \( \Lambda(m, m) \) is divisible by \( 2^j \). Therefore, the inner product of any row in \( G(m, j) \) and a lattice point from \( \Lambda(m, m) \) is divisible by \( 2^j \).

Thus, the inner product of a lattice point from \( \Lambda(m, j) \) and a lattice point from \( \Lambda(m, m) \) is divisible by \( 2^j \). \( \blacksquare \)

Corollary 19 The inner product of a codeword from \( C(m, j) \) and a codeword from \( C(m, m) \) is divisible by \( 2^j \).

Lemma 20. \( C(m, j) \subseteq C_{2^j}^H \).

Proof: By Corollary \[19\] the inner product between a codeword of \( C(m, j) \) and a codeword of \( C(m, m) \) is divisible by \( 2^j \). Since \( H_m \) is the generator matrix of \( C(m, m) \), it follows that if \( x \in C(m, j) \) then the entries of \( H_m x^\ell \) are divisible by \( 2^j \) and therefore \( x \in C_{2^j}^H \).

Thus, \( C(m, j) \subseteq C_{2^j}^H \). \( \blacksquare \)

Corollary 21 The covering radius of the code \( C_{2^j}^H \) is less than or equal to the covering radius of the code \( C(m, j) \).

Conjecture 22 \( C(m, j) = C_{2^j}^H \).

For the next result, we need one more definition and a few observations. For a given word \( x = (x_1, \ldots, x_n) \), the reverse of \( x \), \( x^R \), is the word obtained from \( x \) by reading its elements from the last to the first, i.e. \( x^R \eqdef (x_n, \ldots, x_1) \). It is readily verified that for each \( j \), \( 1 \leq j \leq m \), \( x \in C(m, j) \) if and only if \( x^R \in C(m, j) \). Moreover, if we take the matrix which consists of all the reverse rows of \( G(m, j) \) we will obtain another generator matrix for \( C(m, j) \).

Lemma 23. If \( x \), the \( i \)-th row of the matrix \( H_m \), has Manhattan weight \( 2^\ell \) then \( x^R \cdot H_m \) is a multiple by \( 2^\ell \) of the reverse for the \( (2^m + 1 - i) \)-th row of the matrix \( H_m \).

Proof: The proof is by induction on \( m \). The trivial basis is \( m = 2 \). Assume the claim is true when the matrices involved are \( H_m \) and \( H_m \). Let \( x = (x, x) \) be the \( i \)-th row of \( H_{m+1} \), and let \( v \) be the \( (2^m + 1 - i) \)-th row of \( H_m \), \( 1 \leq i \leq 2^m \). If the weight of \( x \) is \( 2^\ell \) then the weight of \( (x, x) \) is \( 2^{\ell+1} \). \( x \) is the \( i \)-th row of \( H_m \) and hence by the induction hypothesis \( x^R \cdot H_m \) is a multiple by \( 2^\ell \) of \( v^R \). It is also easy to verify that if \( z = y \cdot H_m \) then...
(2z, 0) = (y, y) \cdot \mathcal{H}_{m+1}. The (2^{m+1} + 1 - i)\text{-th row of } \mathcal{H}_{m+1} \text{ is } (0, v) \text{ and hence } (x, \tilde{x})^{R} \cdot \mathcal{H}_{m+1} \text{ is a multiple by } 2^{\ell+1} \text{ of } (v^{R}, 0).

A simple proof by induction on } m \text{ using the structure of } \mathcal{H}_{m} \text{ can be given for the following lemma.}

**Lemma 24.** If the } i \text{-th row of } \mathcal{H}_{m} \text{ has weight } 2^{\ell} \text{ then the } (2^{m} + 1 - i)\text{-th row of } \mathcal{H}_{m} \text{ has weight } 2^{m-\ell}.

**Lemma 25.** \(T^{H_{m}}_{2j}(\Lambda(m, j)) = \Lambda(m, m - j).\)

**Proof:** If } x \text{, the } i \text{-th row of } \mathcal{H}_{m} \text{, has weight } 2^{\ell}, \ell \geq j, \text{ then by definition the } i \text{-th row of } G(m, j) \text{ is equal } x. \text{ By } \text{Lemma 24 } \text{we have that } x^{R} \cdot \mathcal{H}_{m} \text{ is a multiple by } 2^{\ell} \text{ of the reverse of the } (2^{m} + 1 - i)\text{-th row of } \mathcal{H}_{m}. \text{ Hence } T^{H_{m}}_{2j}(x^{R}) \text{ is a multiple by } 2^{\ell-j} \text{ of the reverse of the } (2^{m} + 1 - i)\text{-th row of } \mathcal{H}_{m}. \text{ By } \text{Lemma 24 } \text{the } (2^{m} + 1 - i)\text{-th row of } \mathcal{H}_{m} \text{ has weight } 2^{m-\ell}. \text{ Therefore, the Manhattan weight of } T^{H_{m}}_{2j}(x^{R}) \text{ is } 2^{m-j} \text{ which implies by the definition of } G(m, m - j) \text{ that } T^{H_{m}}_{2j}(x^{R}) \text{ is a row in } G(m, m - j).

If } x \text{, the } i \text{-th row of } \mathcal{H}_{m} \text{, has weight } 2^{\ell}, \ell < j, \text{ then the } i \text{-th row of } G(m, j) \text{ is equal } 2^{i-\ell}x \text{ and its weight is } 2^{i}. \text{ By } \text{Lemma 24 } \text{we have that } x^{R} \cdot \mathcal{H}_{m} \text{ is a multiple by } 2^{i} \text{ of the reverse of the } (2^{m} + 1 - i)\text{-th row of } \mathcal{H}_{m}. \text{ Hence } T^{H_{m}}_{2j}(2^{i-\ell}x^{R}) \text{ is the reverse of the } (2^{m} + 1 - i)\text{-th row of } \mathcal{H}_{m}. \text{ By } \text{Lemma 24 } \text{the } (2^{m} + 1 - i)\text{-th row of } \mathcal{H}_{m} \text{ has weight } 2^{m-\ell} > 2^{m-j}. \text{ Therefore, by the definition of } G(m, m - j) \text{ we have that } T^{H_{m}}_{2j}(2^{i-\ell}x^{R}) \text{ is a row in } G(m, m - j).

We have shown that a basis of } \Lambda(m, j) \text{ is transformed by the transformation } T^{H_{m}}_{2j} \text{ to a basis of } \Lambda(m, m - j). \text{ Since } \Lambda(m, j) \text{ and } \Lambda(m, m - j) \text{ are lattices, and } T^{H_{m}}_{2j} \text{ is a linear transformation, it implies that } T^{H_{m}}_{2j}(\Lambda(m, j)) = \Lambda(m, m - j).

**Corollary 26** \(C^{H_{m}}_{2j} = \mathbb{C}(m, j) \text{ if and only if } C^{H_{m}}_{2m-j} = \mathbb{C}(m, m - j).\)

**C. The Discrete Transformation**

For the discrete case we want to modify the transformation } T^{W} \text{, used for the continuous case. Let } D \text{ be a positive integer and } W \text{ a symmetric weighing matrix of order } n \text{ and weight } w = D^{2}. \text{ Let } \mathcal{S} \text{ be the set of coset leaders of the lattice } \Lambda^{W} \text{ defined in Theorem 12 based on } \mathcal{S}. \text{ The discrete transformation } \tilde{T}^{W} : \mathbb{Z}^{n} \rightarrow \mathbb{Z}^{n} \text{ is defined as follows. For each } (x_{1}, \ldots, x_{n}) \in \mathbb{Z}^{n}, \text{ let } (x_{1}, \ldots, x_{n}) = (c_{1}, \ldots, c_{n}) + (s_{1}, \ldots, s_{n}) \text{, where } (c_{1}, \ldots, c_{n})^{t} \in \Lambda^{W} \text{ and } (s_{1}, \ldots, s_{n})^{t} \in \mathcal{S}. \text{ The choice of the pair } (c_{1}, \ldots, c_{n}) \text{ and } (s_{1}, \ldots, s_{n}) \text{ is unique once the set of coset leaders } \mathcal{S} \text{ is defined. Let } \tilde{T}^{W}((x_{1}, \ldots, x_{n})^{t}) = T^{W}((c_{1}, \ldots, c_{n})^{t}) + (s_{1}, \ldots, s_{n})^{t}, \text{ where } T^{W} \text{ is defined in } \mathcal{S}.

**Lemma 27.** For each } x = (x_{1}, \ldots, x_{n})^{t} \in \mathbb{Z}^{n}, \text{ \[ \tilde{T}^{W}(\tilde{T}^{W}(x)) = x. \]
**Proof:** Let \( x = (x_1, \ldots, x_n)^t \in \mathbb{Z}^n \) be a point such that \((x_1, \ldots, x_n) = (c_1, \ldots, c_n) + (s_1, \ldots, s_n)\), where \((c_1, \ldots, c_n)^t \in \Lambda^W\) and \((s_1, \ldots, s_n)^t \in \mathbb{S}\). By the definition of \( \tilde{T}^W \) we have that

\[
\tilde{T}^W(\tilde{T}^W(x)) = \tilde{T}^W(T^W((c_1, \ldots, c_n)^t) + (s_1, \ldots, s_n)^t).
\]

Since \((c_1, \ldots, c_n)^t \in \Lambda^W\) it follows by Theorem 12 that \(T^W((c_1, \ldots, c_n)^t) \in \Lambda^W\) and hence by the definition of \( \tilde{T}^W \) we have that

\[
\tilde{T}^W(T^W((c_1, \ldots, c_n)^t)) + (s_1, \ldots, s_n)^t.
\]

Finally, by Lemma 11 we have that

\[
T^W(T^W((c_1, \ldots, c_n)^t)) + (s_1, \ldots, s_n)^t = (c_1, \ldots, c_n)^t + (s_1, \ldots, s_n)^t = (x_1, \ldots, x_n)^t.
\]

Thus,

\[
\tilde{T}^W(\tilde{T}^W(x)) = x.
\]

**Theorem 28.** Let \( \rho \) be the covering radius of the lattice \( \Lambda^W \). A Lee sphere with radius \( R \) is inscribed after the transformation \( \tilde{T}^W \), inside an \( n \)-dimensional cube of size

\[
\left( 2 \left\lfloor \frac{R + \rho}{D} \right\rfloor + 2\rho + 1 \right) \times \cdots \times \left( 2 \left\lfloor \frac{R + \rho}{D} \right\rfloor + 2\rho + 1 \right).
\]

**Proof:** A Lee sphere with radius \( R \) and center \( c = (c_1, \ldots, c_n)^t \in \mathbb{Z}^n \), \( S_{n,R}(c) \), is the following set of points

\[
S_{n,R}(c) = \left\{ c + (x_1, \ldots, x_n)^t \mid \sum_{i=1}^n |x_i| \leq R \right\},
\]

where \( x = (x_1, \ldots, x_n)^t \in \mathbb{Z}^n \). For each \( c + x \in \mathbb{Z}^n \) let \( s(c + x) \in \mathbb{S} \), be the coset leader in the coset of \( c + x \) with respect to the lattice \( \Lambda^W \). The set \( S_{n,R}(c) \) is transformed after the transformation \( \tilde{T}^W \) into the following set

\[
\tilde{T}^W(S_{n,R}(c)) = \left\{ \tilde{T}^W(c + x) \mid \sum_{i=1}^n |x_i| \leq R \right\}
\]

\[
= \left\{ T^W(c + x - s(c + x)) + s(c + x) \mid \sum_{i=1}^n |x_i| \leq R \right\}
\]

\[
= \left\{ \frac{Wc}{D} + \frac{W(x - s(c + x))}{D} + s(c + x) \mid \sum_{i=1}^n |x_i| \leq R \right\}
\]

Let \( \frac{W(x-s(c+x))}{D} + s(c+x) = y = (y_1, \ldots, y_n) \). Since the covering radius of the lattice \( \Lambda^W \), defined in Theorem 12 is \( \rho \), it follows that \( |s(c + x)| \leq \rho \). Then, for \( 1 \leq i \leq n \),

\[
|y_i| = \frac{\sum_{j=1}^n h_{i,j}(x_j - s(c + x)_j)}{D} + s(c + x)_i
\]

\[
\leq \frac{\sum_{j=1}^n (|x_j| + |s(c + x)_j|)}{D} + s(c + x)_i \leq \left\lfloor \frac{R + \rho}{D} \right\rfloor + \rho.
\]
inscribing the discrete transformation we will need a symmetric weighing matrix whose order is a square. There are several constructions which yield symmetric Hadamard matrices. The matrix of order exists then

\[ m \equiv 36, 100, 196, \text{ and } 484. \]  

But, other values such as \( n = 144, 324, 400, \) and \( 576, \) are not covered by this construction. Another construction is given in [24] for all orders of the form \( 4m^4, \) where \( m \) is odd. It covers a large set of values, for example \( n = 324 \) is covered by this construction.

We now turn our attention to conference matrices. It is well known [11] that if a conference matrix of order \( n \) exists then \( n \equiv 0(\mod 4) \). If \( n \equiv 0(\mod 4) \) then the matrix can be made skew symmetric and if \( n \equiv 2(\mod 4) \)
it can be made symmetric. It is conjectured that a conference matrix of order \( n \) exists for each \( n \) divisible by 4. If \( n \equiv 2 \pmod{4} \) then a necessary condition for the existence of a conference matrix of order \( n \) is that \( n - 1 \) can be represented as a sum of two squares. It is conjectured that this condition is also sufficient. More information on orders of other conference matrices and weighing matrices in general can be found in [11], [15].

In general, a weighting matrix of odd order \( n \) and weight \( w \) implies that \( w \) is a square. An infinite family in this context are weighting matrices of order \( q^2 + q + 1 \) and weight \( q^2 \), for each \( q \) which is a power of a prime [32].

When the discrete transformation was discussed we have considered the code \( C^W_{\sqrt{w}} \) for a given weighing matrix \( W \) with weight \( w \). We note that the code \( C^W_s \), \( s < w \), defined in Theorem 12 seems to be not interesting when \( W \) is a weighing matrix of order \( n \) and weight \( w < n \). The reason for this is that except for the fact that the code is reduced to a code in the Lee metric over the alphabet \( \mathbb{Z}_s \), the code essentially equals to the code \( C^W_w \). All the codewords of \( C^W_s \) are contained in \( C^W_w \). Even so we have considered the code \( C^W_{\sqrt{w}} \), for the discrete transformation, since the covering radius of the code makes it still attractive for the discrete transformation.

VI. CONCLUSION AND PROBLEMS FOR FUTURE RESEARCH

We have considered the linear span of weighing matrices as codes in the Lee and the Manhattan metrics. We have proved that the minimum Lee distance of such a code is equal to the weight of a row in the matrix. A set of codes related to Sylvester Hadamard matrices were defined. Properties of these codes, such as their size, minimum distance, and covering radius were explored. We have defined a transformation which transforms any Lee sphere in the space (also a conscribed cross-polytope in the continuous space) into a shape with the same volume (in the continuous space) located in a relatively small cube. The transformation was defined as one of sequence of transformations which yield a sequence of error-correcting codes in the Lee metric. These codes are related to the codes obtained from Sylvester type Hadamard matrices. Many interesting questions arise from our discussion, some of which were already mentioned. The following questions summarize all of them.

1) What is the covering radius of the code obtained from a Hadamard matrix?
2) Is the code of length \( p + 1 \), \( p \) prime, obtained from a conference matrix of order \( p + 1 \) is an MDS code?
3) Are the code \( C(m, j) \) and the related code \( C^{H_m}_{2j} \), equal?
4) Determine the size of the code \( C^H_s \) obtained from a general Hadamard matrix \( H \) of order \( n \).
5) What is the covering radius of the code \( C^{H_m}_{2j} \)?
6) Is it possible to find a volume preserving transformation which transfers each Lee sphere into a shape inscribed in a cube whose size is smaller than the one given in our constructions? What about the same question for a conscribed cross-polytope?
7) What is the minimum Lee distance and the covering radius of \( C^{W}_{s} \), for a given weighing matrix \( W \)? The first interesting cases are when \( W \) is a Hadamard matrix or a conference matrix.

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