Numerical iteration for nonlinear oscillators by Elzaki transform

Naveed Anjum, Muhammad Suleman, Dianchen Lu, Ji-Huan He and Muhammad Ramzan

Abstract
Iteration methods are widely used in numerical simulation. This paper suggests the Elzaki transform in the variational iteration method for simple identification of the Lagrange multiplier. The Elzaki transform is a modification of the Laplace transform, and it is extremely useful for treating with nonlinear oscillators as illustrated in this paper; a single iteration leads to a high accuracy of the solution.

Keywords
Variational iteration method, Lagrange multiplier, Laplace transform, Elzaki transform, nonlinear oscillator

Introduction
Iteration methods are widely used to deal with nonlinear problems, the most used one is the well-known Newton’s iteration method for weak nonlinear problems, while for strong nonlinear problems the variational iteration method has been widely applied, which has been proved to be accurate and efficient and it is also efficient for fractal differential equations. In this paper, we will introduce the Elzaki transform to simplify the identification of the Lagrange multiplier involved in the variational iteration algorithm. The Elzaki transform which was introduced by Tarig Elzaki in 2011 is a modification of the Laplace transform. This transformation can be used to solve ordinary, partial, and integral equations in the time domain. Many authors combined this transformation with the homotopy perturbation method, the Adomian decomposition method, and the variational iteration method to solve nonlinear problems in a convenient way.

In this manuscript, there are two basic objectives of coupling of the variational iteration method with the Elzaki transform. Initially is to identify the Lagrange multiplier and then to find amplitude–frequency relationship of a nonlinear oscillatory system. To achieve these objectives, we will use an oscillator with coordinate-dependent mass in the form

\[(1 + ax^2)x'' + axx' - x(1 - x^2) = 0\]  (1)

with initial conditions
\[x(0) = A, \quad x'(0) = 0\]  (2)

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Equation (1) can define phase transition in physics and takes significant part in field theory to describe the new phase formation, cosmos logical model, quark confinement, and spinodal decomposition.\textsuperscript{32}

Equation (1) can be expressed as

\[ x'' - x + 2xx'' + 2xx^2 + x^3 = 0 \]  

(3)

This nonlinear oscillator is difficult to solve as it involves a linear term with negative coefficient. Wu and He\textsuperscript{32} applied the homotopy perturbation method with an expanding parameter to overcome the difficulty. Anjum and He\textsuperscript{33} hybridized the variational iteration method and Laplace transform to propose a fairly accurate solution of equation (1). This paper shows the Elzaki transform works for the negative coefficient of the linear term for a nonlinear oscillator.

**Identification of variation iteration method’s Lagrange multiplier by the Elzaki transform**

Consider a general nonlinear oscillator in the form

\[ x''(t) + f(x) = 0 \]  

(4)

\[ x(0) = A, \quad x'(0) = 0 \]  

(5)

We can rewrite equation (4) as

\[ x'' + \Omega^2 x + g(x) = 0 \]  

(6)

where \( g(x) = f(x) - \Omega^2 x \).

According to the variation iteration method, the correction functional for equation (6) is given as

\[ x_{m+1}(t) = x_m(t) + \int_0^t \lambda(\xi) \left[ x''_m(\xi) + \Omega^2 x_m(\xi) + \bar{g}_m(x) \right] d\xi, \quad m = 0, 1, 2, \ldots \]  

(7)

where \( \lambda \) is the general Lagrange multiplier and can be calculated optimally using the variational theory. The subscript \( m \) represents the \( m^{th} \) approximation and \( \bar{g}_m \) is the restricted variation, i.e. \( \bar{g}_m = 0 \). There are many researchers discussing how to find the multiplier effectively but we will propose an alternative way for the identification of the multiplier. According to He\textsuperscript{1–4} and He and Wu,\textsuperscript{5} the general form of multiplier is

\[ \lambda = \tilde{\lambda}(t - \xi) \]

The integration in equation (7) is basically the convolution; hence, we can use convolution theorem of Elzaki transform easily. Applying Elzaki transform on both sides of equation (7) the correction functional will be transformed in the following manner

\[ E[x_{m+1}(t)] = E[x_m(t)] + E \left[ \int_0^t \lambda(\xi) \left[ x''_m(\xi) + \Omega^2 x_m(\xi) + \bar{g}_m(x) \right] d\xi \right], \quad m = 0, 1, 2, \ldots \]  

(8)

Thus

\[ E[x_{m+1}(t)] = E[x_m(t)] + E[\tilde{\lambda}(t) * (x''_m(t) + \Omega^2 x_m(t) + \bar{g}_m(x))] \]

\[ E[x_{m+1}(t)] = E[x_m(t)] + \frac{1}{w} E[\tilde{\lambda}(t)] E[x''_m(t) + \Omega^2 x_m(t) + \bar{g}_m(x)] \]
\[ E[x_{m+1}(t)] = E[x_m(t)] + \frac{1}{w} E[\lambda(t)] \left[ \left( \frac{1}{w^2} + \Omega^2 \right) E[x_m(t)] - x_m(0) - vx_m'(0) + E[g_m(x)] \right] \]

The optimal value of \( \lambda \) can be obtained to take the variation with respect to \( x_m(t) \). So

\[
\frac{\delta}{\delta x_m} E[x_{m+1}(t)] = \frac{\delta}{\delta x_m} E[x_m(t)] + \frac{\delta}{\delta x_m} \frac{1}{w} E[\lambda(t)] \\
\times \left( \left( \frac{1}{w^2} + \Omega^2 \right) E[x_m(t)] - x_m(0) - vx_m'(0) + E[g_m(x)] \right)
\]

and hence equation (9) can be simplified upon applying the variation to

\[
E[\delta x_{m+1}] = E[\delta x_m] + \frac{1}{w} E[\lambda] \left( \frac{1}{w^2} + \Omega^2 \right) E[\delta x_m]
\]

By applying the extremum condition, we have the stationary condition as

\[
E[\delta x_m] + \frac{1}{w} E[\lambda] \left( \frac{1}{w^2} + \Omega^2 \right) E[\delta x_m] = 0
\]

\[
E[\lambda] = - \frac{w^3}{(1 + \Omega^2 w^2)}
\]

By applying the Elzaki inverse on the last equation yields the optimal Lagrange multiplier \( \lambda \)

\[
\lambda(t) = - \frac{1}{\Omega} \sin \Omega t
\]

which is same as obtained in Anjum and He. 33

Using equation (8), the iterative formula has the form

\[
E[x_{m+1}(t)] = E[x_m(t)] - \frac{1}{\Omega} E \left[ \int_0^t \sin \Omega(t - \xi) \left( x''_m(\xi) + \Omega^2 x_m(\xi) + g_m(\xi) \right) d\xi \right], \quad m = 0, 1, 2, \ldots
\]

**Example**

To apply the variational iteration method with Elzaki transform on equation (1), we can write it in the form

\[
x'' + \Omega^2 x + g(x) = 0
\]

where \( g(x) = -(1 + \Omega^2)x + 2xx'' + xx' + x = 0 \).

Using equation (13), the iterative formula is developed as

\[
E[x_{m+1}(t)] = E[x_m(t)] - E \left[ \int_0^t \frac{1}{\Omega} \sin \Omega(t - \xi) \left( x''_m(\xi) + \Omega^2 x_m(\xi) + g(x_m) \right) d\xi \right]
\]

\[
E[x_{m+1}(t)] = E[x_m(t)] - \frac{1}{\Omega} E[\sin \Omega t] E \left[ x''_m(t) + \Omega^2 x_m(t) + g(x_m) \right]
\]
\[ E[x_{m+1}] = E[x_m] - \frac{1}{\Omega} E[\sin\Omega t]E[x''_m - x_m + \alpha x_m^2 x'_m + \alpha x_m x'_m + x'_m] \]

Assuming

\[ x_0(t) = A\cos\Omega t \] (15)

\[ E[x_1(t)] = E[A\cos\Omega t] - \frac{1}{\Omega} \left( -A\Omega^2 - A + \frac{3}{4} A^3 - \frac{1}{2} \alpha A^3 \Omega^2 \right) E[\sin\Omega t] E[\cos\Omega t] \]

\[ - \frac{1}{\Omega} \left( \frac{1}{4} A^3 - \frac{1}{2} \alpha A^3 \Omega^2 \right) E[\sin\Omega t] E[\cos\Omega t] \] (16)

After applying Elzaki and inverse Elzaki transform in equation (16), we have

\[ x_1(t) = A\cos\Omega t - \frac{1}{\Omega} \left( -A\Omega^2 - A + \frac{3}{4} A^3 - \frac{1}{2} \alpha A^3 \Omega^2 \right) \left( \frac{1}{2\Omega^2} \sin\Omega t - \frac{1}{2\Omega} \cos\Omega t \right) \]

\[ - \frac{1}{\Omega} \left( \frac{1}{4} A^3 - \frac{1}{2} \alpha A^3 \Omega^2 \right) \left( \frac{1}{8\Omega^2} \sin\Omega t - \frac{1}{24\Omega^2} \sin\Omega t \right) \]

\[ = A\cos\Omega t - \frac{1}{2\Omega^2} \left( -A\Omega^2 - A + \frac{13}{16} A^3 - \frac{5}{8} \alpha A^3 \Omega^2 \right) \sin\Omega t \]

\[ + \frac{1}{24\Omega^2} \left( \frac{1}{4} A^3 - \frac{1}{2} \alpha A^3 \Omega^2 \right) \sin\Omega t + \frac{1}{2\Omega^2} \left( -A\Omega^2 - A + \frac{3}{4} A^3 - \frac{1}{2} \alpha A^3 \Omega^2 \right) \cos\Omega t \] (17)

No secular-term in \( x_1 \) requires that

\[ \frac{1}{2\Omega^2} \left( -A\Omega^2 - A + \frac{3}{4} A^3 - \frac{1}{2} \alpha A^3 \Omega^2 \right) = 0 \] (18)

\[ \Omega = \sqrt{\frac{3}{4} A^2 - \frac{1}{1 + \frac{1}{2} \alpha A^2}} \] (19)

Equation (19) is valid when

\[ A > \frac{2}{\sqrt{3}} \] (20)

Equations (19) and (20) are the same as obtained in Wu and He\(^{32}\) and Anjum and He,\(^{33}\) showing the correctness of the solution.

**Conclusion**

This paper, for the first time ever, applies the Elzaki transform to the variational iteration algorithm with great success, the identification of Lagrange multiplier, which was identified by the variational theory, becomes simpler.\(^{34,35}\) An optimal variational iteration algorithm is obtained by the Elzaki transform, and the iteration algorithm converges fast and only one iteration results in a high accurate solution.

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