CONSERVATION LAWS OF DAMPED NONLINEAR WAVE EQUATIONS

STEPHEN C. ANCO\textsuperscript{1}, ALMUDENA P. MÁRQUEZ\textsuperscript{2}, TAMARA M. GARRIDO\textsuperscript{2}, MARÍA L. GANDARIAS\textsuperscript{2}

\textsuperscript{1}Department of Mathematics and Statistics
Brock University
St. Catharines, ON L2S3A1, Canada

\textsuperscript{2}Department of Mathematics
University of Cadiz
11510 Puerto Real, Cadiz, Spain

Abstract. All energy-momentum type conservation laws are found for a general class of damped nonlinear wave equations in one dimension. The classification shows that conservation laws exist only for linear damping if the wave equation is non-singular. In the simplest cases, the conservation laws generalize ordinary momentum and energy, and also null-energies. An explanation of the existence of these conservation laws is provided by a point transformation that maps the damped nonlinear wave equation to an undamped wave equation with a time-dependent nonlinearity. Specializations to damped nonlinear wave equations that are invariant under time and space translations are summarized. In the general case, the conservation laws describe a further generalization of momentum, energy, and null-energies, as well as a generalized energy-momentum. The latter two types of conservation laws are shown to arise from a mapping to a wave equation that has a null-form damping (namely, a combination of spatial and temporal damping tied to the null lines of the linear wave operator). This represents a certain type of null structure which is closely related to the null-form nonlinearities studied in analysis of nonlinear wave equations.

1. Introduction

The wave equation \( u_{tt} = c^2 u_{xx} \) is a commonplace simple model for wave propagation and vibrations in one dimension, where \( c \) is the wave speed and \( u(x,t) \) is the wave amplitude. In real world applications, it is of interest to consider more general models \([1]\) that include the effects of frictional damping and nonlinear self-interaction:

\[
 u_{tt} + au_t + g(u) = c^2 u_{xx} \tag{1}
\]

where \( a \) is the damping coefficient and \( g(u) \) is the self-interaction term. A main quantity for understanding the behaviour of solutions \( u(x,t) \) is the energy

\[
 E = \int_{\mathbb{R}} \left( \frac{1}{2} (u_t^2 + c^2 u_x^2) + G(u) \right) dx \tag{2}
\]

where \( G'(u) = g(u) \).

When damping is absent, \( a = 0 \), the energy is conserved, namely \( \frac{d}{dt} E = 0 \) for all solutions \( u(x,t) \) with either sufficient decay for \( |x| \to \infty \) or compact support in \( x \). As is well known,
energy conservation implies long-time existence of solutions having smooth initial data and finite energy if the self-interaction energy $G(u)$ is non-negative. However, if $G(u)$ is negative or has indefinite sign, then some finite-energy solutions with smooth initial data can exhibit blow-up. (See e.g. Ref. [2, 3].)

The effect of damping, $a > 0$, is to cause the energy to decrease, since $\frac{d}{dt}E = -a \int_U u_t^2 \, dx < 0$ whenever $u_t \neq 0$. This does not necessarily prevent solutions from blowing up when $G(u)$ is negative (or has indefinite sign), since $E$ in that case is unbounded from below and hence $E \to -\infty$ is possible as $t$ increases. An interesting mathematical question therefore is to study conditions under which blow-up occurs in the presence of damping and self-interaction such that $G(u)$ is negative. This has been a topic of much interest in the analysis literature [4, 5, 6, 7], particularly for the case of power nonlinearities $u_{tt} + au_t = c^2 u_{xx} + ku|u|^p$ with $p > 0$ and $k > 0$ [8, 9, 10]. Related questions of stability of solutions have also been addressed [5].

As a consequence, from the viewpoint of conserved quantities, wave equations (1) with frictional damping and negative self-interaction energy could be expected to lack any local conservation laws. Some very recent work [11], surprisingly, found that conservation laws do exist for certain forms of self-interaction, despite the fact that the energy is decreasing for non-equilibrium solutions.

This result motivates undertaking a study of conservation laws of nonlinear wave equations with a more general form

$$u_{tt} + g(t, x, u, u_t) = c^2 u_{xx},$$

including frictional damping and self-interaction, without any a priori constraints other than $g$ being non-singular. The main goals will be to obtain a classification of energy-type and momentum-type conserved quantities and to understand their physical meaning and their properties. A useful fact is that all conserved quantities arising from local conservation laws for wave equations (3) correspond to multipliers, as shown by general results in Ref. [12, 13, 14].

Firstly, the starting point is the example of a momentum-type conservation law given in Ref. [11] for a wave equation with linear damping in the class (1). In Section 2 this conservation law is shown to arise by applying a linear change of dependent variable to an undamped wave equation with a time-dependent self-interaction term for which ordinary momentum is conserved. The derivation is reformulated equivalently in terms of a multiplier which directly yields the conservation law from the original linearly damped wave equation. A physical explanation of the conservation law in terms of the properties of the damping for solutions $u(x, t)$ is also stated.

Secondly, as the first main result, a class of energy and momentum conservation laws are derived for general nonlinear wave equations (3) in Section 3. The method builds on the previous momentum example by seeking multipliers whose form is tailored to produce the type of conservation laws being sought. This can be viewed as a particular instance of solving a general inverse problem [15] that consists of finding PDEs (within a given class) possessing a conserved integral of a specified form.

The conservation laws describe conserved energy and momentum quantities as well as conserved energies associated to the light-cone structure, which arise when the damping is linear with a constant coefficient. Existence of these conserved quantities is shown to arise
from a point transformation (in particular a linear change of dependent variable) under which the damped wave equation is mapped into an undamped wave equation.

Thirdly, in Section 4, a general classification of all multipliers that are linear homogeneous in $u_t$, $u_x$, and $u$ is carried out. This class of multipliers corresponds to seeking generalized energy-type and momentum-type conservation laws. The results yield the forms of nonlinearity $g(t, x, u, u_t)$ such that a damped wave equation (3) possesses conserved quantities of these types, which are found to consist of generalized momentum, generalized energy, and a generalized form of light-cone energies. These quantities arise for linear damping, but unlike the previous ones, some exist when no point transformation to an undamped wave equation is possible. It will be shown that existence of these latter conservation laws is due to a point transformation to a wave equation with damping that has a certain null-form related to the light-cone. This is a close analog of null-form nonlinearities, which have been active topic of study in the analysis of wave equations for many years (see e.g. [16, 17] and more recently [18]).

The new results herein are relevant for analysis, since the existence of conserved quantities allows for refinement of theorems on decay, blow up, and stability.

Finally, some concluding remarks are made in Section 5. An Appendix contains some remarks on the computations.

A summary of the general theory of multipliers and local conservation laws for nonlinear wave equations can be found in Ref. [13, 14]. (See also Ref. [2, 12, 19, 20].)

2. Motivating example

Nonlinear wave equations (1) with damping and self-interaction turn out to possess a conserved quantity [11]

$$\tilde{P} = \int_{\mathbb{R}} e^{at} u_t u_x \, dx.$$  

It is easy to verify that $\frac{d}{dt} \tilde{P} = 0$ for all solutions $u(x, t)$ having either sufficient decay for $|x| \to \infty$ or compact support in $x$. This quantity is a generalization of the ordinary momentum $P = \int_{\mathbb{R}} u_t u_x \, dx$ which is a well-known conserved quantity for nonlinear wave equations (1) without damping, $a = 0$.

Note that this generalized momentum $\tilde{P}$ is conserved despite the fact that the energy $E$ is strictly decreasing due to the damping, $a > 0$. The form of $\tilde{P}$ suggests considering a change of dependent variable given by

$$v = e^{\frac{2a}{3}} u,$$  

under which

$$\tilde{P} = \int_{\mathbb{R}} v_t v_x \, dx$$

becomes the ordinary momentum in terms of $v$.

When this change of variable is applied to the damped wave equation (1), it yields an undamped wave equation

$$v_{tt} + \tilde{g}(t, v) = c^2 v_{xx}$$

in which the interaction term involves $t$ explicitly,

$$\tilde{g}(t, v) = e^{\frac{2a}{3}} g(e^{-\frac{2a}{3}} v) - \left(\frac{a}{2}\right)^2 v.$$  

(8)
Ordinary momentum (6) is readily verified to be conserved for any wave equation of the form (7), independently of the form of $\tilde{g}(t,v)$. In particular,

$$\frac{d}{dt}P = \int_{\mathbb{R}} (v_x(c^2v_{xx} - \tilde{g}(t,v)) + v_tv_{tx})
= \left(\frac{1}{2}(c^2v_x^2 + v_t^2) - \tilde{G}(t,v)\right)|^\infty_{-\infty}$$

(9)

will vanish for solutions $v(x,t)$ with suitable spatial decay or compact support, where

$$\tilde{G}(t,v) = \int \tilde{g}(t,v)
$$

This conservation property (9) implies that

$$P = \int_{\mathbb{R}} u_t u_x
dx = \int_{\mathbb{R}} e^{-at}v_t v_x
dx = e^{-at}\int_{\mathbb{R}} v_t v_x
dx$$

(10)

satisfies

$$\frac{d}{dt}P = -aP + e^{-at}\frac{d}{dt}\tilde{P} = -aP$$

modulo terms at $x = \pm\infty$. Hence, for the original damped wave equation (11),

$$\frac{d}{dt}P = -aP$$

(11)

holds for solutions $u(x,t)$ having suitable spatial decay or compact support, whereby the ordinary momentum exhibits exponential decay in $t$: $P = e^{-at}P_0 \to 0$ as $t \to \infty$, with $P_0 = P|_{t=0}$ denoting the initial momentum (namely, the momentum of the initial data).

Thus, a physical explanation of the conservation of the generalized momentum (14) is that the damping acts uniformly in $x$ such that the momentum density $u_t u_x$ decreases exponentially in $t$. Mathematically, the existence of this conserved momentum quantity is explained by the linear change of variable (5) combined with the conservation of ordinary momentum for a corresponding undamped wave equation (7)–(8).

A further useful remark is that the generalized momentum can be derived as a local conservation law through the multiplier $Q = e^{at}u_x$ for the damped nonlinear wave equation (11).

Specifically,

$$e^{at}u_x(u_{tt} + au_t + g(u) - c^2u_{xx}) = D_t(e^{at}u_t u_x) + D_x\left(e^{at}G(u) - \frac{1}{2}e^{at}(u_t^2 + c^2u_{xx}^2)\right)$$

(12)

holds as an identity, from which the conservation (balance) equation

$$\frac{d}{dt}\tilde{P} = \left(\frac{1}{2}e^{at}(u_t^2 + c^2u_{xx}^2) - e^{at}G(u)\right)|^\infty_{-\infty}$$

(13)

follows directly. Note that this equation is equivalent to equation (9) under the linear change of variable (5).

3. ENERGY-MOMENTUM CLASSIFICATION RESULTS

For nonlinear wave equations (11) in the case without damping, $a = 0$, the conserved energy

$$E = \int_{\mathbb{R}} \left(\frac{1}{2}(u_t^2 + c^2u_{xx}^2) + G(u)\right)
dx$$

and the conserved momentum $P = \int_{\mathbb{R}} u_t u_x
dx$ correspond to multipliers $Q = u_t = \delta E/\delta u_t$ and $Q = u_x = \delta P/\delta u_t$, respectively. Here $\delta/\delta u_t$ denotes the variational derivative with respect to $u_t$. A linear combination of these two multipliers $Q = c_1u_t + c_2u_x$ yields a corresponding conserved quantity $c_1E + c_2P$ which can be expressed equivalently as $c_+E_+ + c_-E_-$ with $c_{\pm} = \frac{1}{2}(c_1 \pm c_2/c)$ where

$$E_\pm = E \pm cP = \int_{\mathbb{R}} \left(\frac{1}{2}(u_t \pm cu_x)^2 + G(u)\right)
dx.$$  

(14)

This observation is not widely known in the literature on conservation laws of semilinear undamped wave equations (see e.g. [1, 2, 14, 21, 22]). These conserved quantities (14) will
be referred to as null energy-momenta (or simply null energies) because the kinetic term involves \( u_t \pm c u_x \), whose form is given by the derivative \( \partial_t \pm c \partial_x \) associated to the null lines \( x \pm ct \). Note that \( u_t \pm c u_x = \delta E_\pm / \delta u_t = Q_\pm \) is also the corresponding multiplier.

The relationship between conserved quantities and multipliers applies also to damped nonlinear wave equations \(^1\) and more generally to nonlinear wave equations of the form \(^3\). For such wave equations, an expression \( Q \) depending on \( t, x, u, u_t, \) and their \( x \)-derivatives will be a conservation law multiplier iff it satisfies a divergence identity \(^{12, 13, 14}\),

\[
(u_{tt} - c^2 u_{xx} + g(t, x, u, u_t))Q = D_t T + D_x \Psi, \tag{15}
\]

where \( D_t \) and \( D_x \) denote total derivatives. Here \( T \) is the conserved density, and \( \Psi \) is the spatial flux, which are some functions of \( t, x, u, \) and \( u_t, \) and their \( x \)-derivatives. On the space of solutions \( E \) of a given wave equation \(^3\), the integral

\[
I = \int_\mathbb{R} T \, dx \tag{16}
\]

thereby satisfies a balance equation

\[
\frac{d}{dt} I = -\Psi \big|_{-\infty}^{\infty}, \tag{17}
\]

so \( I \) is a conserved quantity for solutions whose net spatial flux vanishes at \( x = \pm \infty \). A conservation law is locally trivial if \( T = D_x \Theta \) and \( \Psi = -D_t \Theta \) hold for all solutions \( u(x, t) \), since then the balance equation for \( I \) holds identically (namely, it contains no useful information about solutions). Two conservation laws that differ by a trivial conservation law are said to be locally equivalent.

For general wave equations, multipliers and conserved quantities are related by \(^{13}\)

\[
Q = \delta C/\delta u_t. \tag{18}
\]

This relationship can be shown to imply that there is (up to equivalence) a one-to-one correspondence between non-trivial conserved quantities and multipliers.

3.1. **Multipliers and energy-momentum conservation laws.** The problem of finding all energy-momentum type conservation laws for damped nonlinear wave equations \(^3\) reduces to the corresponding problem of finding all multipliers that yield such conservation laws. Motivated by the examples of null energies \(^{14}\) and generalized momentum \(^4\), it is natural to seek multipliers having the linear form

\[
Q = e^{at+bx} (c_1 u_t + c_2 u_x + c_0 u) \tag{19}
\]

where \( a, b \) and \( c_0, c_1, c_2 \) (not all zero) are constants. From the general relationship \(^{18}\), the corresponding conserved integrals will have the form

\[
I = \int_\mathbb{R} (e^{at+bx} (\frac{1}{2} c_1 u_t^2 + c_2 u_x u_t + c_0 uu_t) + C(t, x, u, u_x)) \, dx, \tag{20}
\]

for some function \( C \). More discussion of the physical interpretation of this form will be given later.

The determining equation \(^{15}\) for multipliers is equivalent to the condition

\[
E_u \left( (u_{tt} - c^2 u_{xx} + g(t, x, u, u_t)) e^{at+bx} (c_1 u_t + c_2 u_x + c_0 u) \right) = 0 \tag{21}
\]

where \( E_u \) is the Euler operator whose kernel consists of divergence expressions. Note that this condition is required to hold as an identity (namely, off of \( E \)). Consequently, it splits with
respect to $x$-derivatives of $u$ and $u_t$, and thereby yields a system of equations in which the unknowns consist of $g(t, x, u, u_t)$, $a$, $b$, $c_0$, $c_1$, $c_2$. This system is nonlinear in these unknowns and turns out to contain one algebraic equation and five partial differential equations. Each solution of the system determines a multiplier, from which a corresponding conservation law is obtained through equation (15).

In the present problem, $g$ needs to be nonlinear and contain damping, namely $g_u^2 + g_{u_t}^2 \neq 0$ and $g_{u_t} \neq 0$. Additionally, for the resulting nonlinear wave equation to be of physical interest, $g$ needs to be non-singular as a function of $t$, $x$, $u$, $u_t$.

The full system is straightforward to solve (as discussed in the Appendix) and leads to the following result.

**Proposition 1.** A damped nonlinear wave equation (3), with $g$ being non-singular, admits a multiplier of the form (19) only in the following cases:

(i) \[ g = au_t + f(t, u), \]
\[ Q = e^{at} u_x; \] (22)

(ii) \[ g = au_t + f(x - ct, e^{\frac{at}{2}} u) e^{-\frac{at}{2}}, \]
\[ Q = e^{at} (u_t + cu_x + \frac{a}{2} u), \] \[ \tilde{c} = \text{const.}; \] (23)

(iii) \[ g = (a \pm cb)u_t + \frac{1}{4} (a \pm cb)^2 u + f(x \pm ct, e^{\frac{at}{2}} u) e^{-\frac{at}{2}}, \]
\[ Q = e^{at+bx} (u_t \mp cu_x + \frac{a \pm cb}{2} u). \] (24)

In each case, $f$ is an arbitrary function of its arguments.

In case (i), the multiplier is the same as the multiplier for the generalized momentum (1). In cases (ii) and (iii), the multipliers are similar to those for energy and null energies.

The corresponding conservation laws and conserved integrals are given as follows, where $F = \int f \, du$ in terms of the function $f$ in $g(t, x, u, u_t)$.

**Theorem 1.** For damped nonlinear wave equations (3), with $g$ being non-singular, the only energy-momentum type conservation laws consist of:

(i) \[ T = e^{at} u_t u_x, \]
\[ \Psi = -e^{at} \left( \frac{1}{2} (u_t^2 + c^2 u_x^2) + e^{at} F(t, u), \right) \] (25a, 25b)

for $g$ of the form (22);

(ii) \[ T = e^{at} \left( \frac{1}{2} (u_t^2 + c^2 u_x^2) + (cu_x + \frac{1}{2} au) u_t \right) + e^{\frac{at}{2}} F(x - ct, e^{\frac{at}{2}} u), \]
\[ \Psi = -e^{at} \left( \frac{1}{2} (u_t^2 + c^2 u_x^2) + c^2 (u_t + \frac{1}{2} au) u_x \right) + \tilde{c} e^{\frac{at}{2}} F(x - ct, e^{\frac{at}{2}} u), \] (26a, 26b)

for $g$ of the form (23);

(iii) \[ T = e^{at+bx} \left( \frac{1}{2} (u_t^2 \mp cu_x)^2 + \frac{1}{2} (a \pm cb) (u_t + \frac{1}{2} (a \pm 3cb) u) u \right) + e^{\frac{a \pm 3cb}{2} t + bx} F(x \pm ct, e^{\frac{a \pm cb}{2} t} u), \] (27a)
\[
\Psi = \pm ce^{at+bx}(\frac{1}{2}(u_t^2 \pm cu_x)^2 - \frac{1}{2}(a \mp cb)(\pm cu_x + \frac{1}{4}(a \mp cb)u)u) + ce^{\frac{a+3cb}{2}t+bx}F(x \pm ct, e^{\frac{a+cb}{2}t}u),
\]

for \( g \) of the form (24). The conserved quantities (16) given by these conservation laws are, respectively, generalized momentum (4), generalized energy (24), and generalized null energies. Corollary 1. Damped nonlinear wave equations of the form

\[
\tilde{E} = \int_{\mathbb{R}} \left( e^{at+bx} \frac{1}{2}(u_t^2 + c^2u_x^2 + (2\tilde{c}u_x + au)u_t) + e^{\frac{c}{2}t}F(x - \tilde{c}t, e^{\frac{c}{2}t}u) \right) dx,
\]

and

\[
\tilde{E}_x = \int_{\mathbb{R}} \left( e^{at+bx} \frac{1}{2}((u_t + cu_x)^2 + (a \pm cb)(u_t + \frac{1}{4}(a \pm 3cb)u)) + e^{\frac{a+3cb}{2}t+bx}F(x \pm ct, e^{\frac{a+cb}{2}t}u) \right) dx,
\]

which describes generalized null energies.

A few remarks are worthwhile.

Observe that cases (i) and (ii) overlap when \( f = h(e^{\frac{c}{2}t}u)e^{-\frac{c}{2}t} \). Hence, both generalized momentum and generalized energy with \( \tilde{c} = 0 \) are conserved quantities

\[
\tilde{P} = \int_{\mathbb{R}} e^{at}u_t u_x dx,
\]

\[
\tilde{E} = \int_{\mathbb{R}} \left( e^{at} \frac{1}{2}(u_t^2 + c^2u_x^2 + auu_t) + e^{\frac{c}{2}t}H(e^{\frac{c}{2}t}u) \right) dx
\]

for damped nonlinear wave equations of the form

\[
\tilde{u}_tt + au_t + h(e^{\frac{c}{2}t}u)e^{-\frac{c}{2}t} = c^2u_{xx}
\]

where \( H = \int h(e^{\frac{c}{2}t}u) du \).

Similarly, cases (i) and (iii) overlap when \( a = \alpha, f = (\alpha^2)u + h(e^{\frac{c}{2}t}u)e^{-(\frac{a+2cb}{2})t} \) in case (i), and \( a \pm cb = \alpha, f = h(e^{\frac{c}{2}t}u)e^{-(\frac{a+2cb}{2})t} \) in case (iii). Hence, for damped nonlinear wave equations of the form

\[
\tilde{u}_tt + \alpha u_t + (\frac{\alpha}{2})^2u + h(e^{\frac{c}{2}t}u)e^{-(\frac{a+2cb}{2})t} = c^2u_{xx},
\]

both generalized momentum and generalized null energies

\[
\tilde{P} = \int_{\mathbb{R}} e^{at}u_t u_x dx,
\]

\[
\tilde{E}_x = \int_{\mathbb{R}} \left( e^{at}e^{b(x \mp ct)} \left( \frac{1}{2}(u_t + cu_x)^2 + \frac{\alpha}{2}(u_t + \frac{a+2cb}{4}u)u \right) + e^{-\frac{c}{2}t}e^{b(x \mp ct)}H(e^{\frac{c}{2}t}u) \right) dx
\]

are conserved quantities, where \( H = \int h(e^{\frac{c}{2}t}u) du \).

As a result, from comparing the forms of the damped wave equations (31) and (32), there is a non-trivial overlap which will have all three conserved quantities.

**Corollary 1.** Damped nonlinear wave equations

\[
\tilde{u}_tt + \alpha u_t + (\frac{\alpha}{2})^2u + h(e^{\frac{c}{2}t}u)e^{-\frac{c}{2}t} = c^2u_{xx}
\]

possess the conserved quantities

\[
\tilde{P} = \int_{\mathbb{R}} e^{at}u_t u_x dx,
\]

\[
\tilde{E} = \int_{\mathbb{R}} \left( e^{at} \left( \frac{1}{2}(u_t^2 + c^2u_x^2) + \frac{\alpha}{2}uu_t + \frac{a^2}{8}u^2 \right) + e^{\frac{c}{2}t}H(e^{\frac{c}{2}t}u) \right) dx,
\]
\[
\tilde{E}_\mp = \tilde{E} + c\tilde{P} = \int_\mathbb{R} \left( e^{c\alpha t} \left( \frac{1}{2} (u_t + cu_x)^2 + \frac{\alpha}{2} u_t u + \frac{\alpha^2}{8} u^2 \right) + e^{\mp \frac{a}{2} t} H(e^{\pm t} u) \right) \, dx, \tag{37}
\]

where \( H = \int h(e^{\frac{a}{2} t} u) \, du. \)

It is of physical interest to specialize the preceding results to the situation when the wave equations \((3)\) are time-translation invariant. This restricts the nonlinear term \(f\) in \(g\) as follows in Proposition \([1]\). In case (i), the condition is simply \(\partial_t f = 0\), which gives \(f = h(u)\) where \(h\) is an arbitrary function. In case (ii), time-translation invariance imposes the condition \(-\tilde{c}\partial_y f + \frac{1}{2} \tilde{a} \omega \partial_w f - \frac{1}{2} \partial f = 0\), where \(y = x - \tilde{c} t\), and \(w = u e^{\tilde{a} t}\). This gives \(f = \tilde{h}(we^{\tilde{a} ty})\), where \(\tilde{h}\) is an arbitrary function. Similarly, in case (iii), the invariance condition is \(\pm \tilde{c}\partial_y f + \frac{1}{2} \tilde{a} \omega \partial_w f - (\frac{1}{2} \alpha \mp 2\tilde{c} b) f = 0\), where \(y = x \pm ct\) and \(w = u e^{\tilde{a} t}\), with \(\alpha = a \pm cb\). This yields \(f = w^{1 \mp 4\tilde{c} b / \alpha} \tilde{h}(we^{\tilde{a} ty})\), where \(\tilde{h}\) is an arbitrary function.

The resulting wave equations and conserved energy-momentum quantities are shown in Table \([1]\). Note that the second and third cases coincide for \(\tilde{c} = \pm c\), \(b = 0\), with \(h\) in the second case replaced by \(h + \frac{\alpha^2}{4}\). Also note that the first case overlaps the second case only if \(f = ku\) which is linear.

| nonlinearity \(f(x, u)\) | conserved integral | quantity |
|--------------------------|-------------------|----------|
| \(h(u)\) | \(\tilde{P} = e^{\alpha t} \int_\mathbb{R} u_t u_x \, dx\) | gen. momentum |
| \(uh(ue^{\mp x})\) | \(\tilde{E} = e^{\alpha t} \int_\mathbb{R} \left( \frac{1}{2} (u_t^2 + c^2 u_x^2) + (\tilde{c} u_x + \frac{\alpha}{2} u) u_t + H(x, u) \right) \, dx\) | gen. energy |
| & \(H_u = uh(e^{\mp x} u)\) |
| \(\frac{\alpha^2}{3} u + u^{1 \mp 4cb / \alpha} \tilde{h}(ue^{\mp x})\) | \(\tilde{E}_\mp = e^{\alpha t} \int_\mathbb{R} \left( \frac{1}{2} (u_t + cu_x)^2 + \frac{\alpha}{2} (u_t + \frac{\alpha + 2cb}{4}) u + H(x, u) \right) e^{b(x \mp ct)} \, dx\) | gen. energies |
| & \(H_u = u^{1 \mp 4cb / \alpha} \tilde{h}(e^{\mp x} u)\) |

Table 1. Energy-momentum quantities for damped nonlinear wave equations \(u_{tt} + \alpha u_t + f(x, u) = c^2 u_{xx}\).

Finally, it is useful further to specialize Table \([1]\) to wave equations \((3)\) that are invariant with respect to both time-translation and space-translation. This situation arises by respectively taking \(h = ku^p\) in the first case and \(h = k\), \(b = \mp (p - 1)\frac{\alpha}{4c}\) in the third case, with \(k = \text{const.}, \, p > 0\). The resulting wave equation contains a power nonlinearity plus a mass term:

\[
u_{tt} + \alpha u_t + (\frac{1}{2} \alpha)^2 u + ku^p = c^2 u_{xx}. \tag{38}\]

Any such wave equation possesses both generalized momentum and generalized null energies

\[
\tilde{P} = e^{\alpha t} \int_\mathbb{R} u_t u_x \, dx, \tag{39}
\]

\[
\tilde{E}_\mp = e^{\alpha t} \int_\mathbb{R} \left( \frac{1}{2} (u_t + cu_x)^2 + \frac{\alpha}{2} u_t u - \frac{(p-3)\alpha^2}{16} u^2 + \frac{k}{p+1} u^{p+1} \right) e^{\mp \frac{1}{4} \alpha (x \mp ct)} \, dx. \tag{40}
\]

The existence of these conserved quantities has not been widely recognized in the analysis literature.
3.2. **Equivalence transformations.** The conservation of generalized momentum \( \mathbf{I} \) for damped wave equations \( \mathbf{II} \) was explained by existence of a point transformation that maps this equation into an undamped wave equation \( \mathbf{I} \)–\( \mathbf{II} \).

Likewise, for each of the cases (i), (ii), (iii) in Proposition \( \mathbf{I} \) there exists a point transformation

\[
u = e^{-\frac{a}{2}t}v
\]

under which the damped nonlinear wave equation is mapped into the undamped wave equation

\[
\nu_{tt} + \tilde{g}(t, x, \nu) = c^2\nu_{xx}.
\]

The constant \( \alpha \) is equal to the damping coefficient in the original wave equation, and the nonlinear term \( \tilde{g} \) is given as follows:

(i) \[
\tilde{g} = f(t, e^{-\frac{a}{2}t}v)e^{\frac{a}{2}t} - (\frac{a}{2})^2\nu, \quad \alpha = a,
\]

for \( g \) of the form \( \mathbf{II} \);

(ii) \[
\tilde{g} = f(x - \tilde{c}x, \nu)e^{\frac{a}{2}t} - (\frac{a}{2})^2\nu, \quad \alpha = a,
\]

for \( g \) of the form \( \mathbf{II} \);

(iii) \[
\tilde{g} = f(x \pm ct, \nu)e^{\pm 2\alpha t}, \quad \alpha = a \pm cb,
\]

for \( g \) of the form \( \mathbf{II} \).

Under these mappings, the energy-momentum conservation laws in Theorem \( \mathbf{I} \) are equivalent to the following conservation laws of undamped wave equations \( \mathbf{I} \), where \( F_v = f \).

(i) 
\[
T = \nu_t\nu_x, \quad \Psi = -\frac{1}{2}(\nu_t^2 + c^2\nu_x^2) - \frac{1}{8}a^2\nu^2 + e^{\frac{a}{2}t}F(t, e^{-\frac{a}{2}t}v),
\]

for \( g \) of the form \( \mathbf{II} \);

(ii) 
\[
T = \frac{1}{2}(\nu_t^2 + c^2\nu_x^2) + \tilde{c}\nu_t\nu_x - \frac{1}{8}a^2\nu^2 + F(x - \tilde{c}t, \nu),
\]

\[
\Psi = -\tilde{c}(\frac{1}{2}(\nu_t^2 + c^2\nu_x^2) + \frac{1}{8}a^2\nu^2) - c^2\nu_t\nu_x + \tilde{c}F(x - \tilde{c}t, \nu),
\]

for \( g \) of the form \( \mathbf{II} \);

(iii) 
\[
T = \frac{1}{2}e^{b(x \mp ct)}(\nu_t \mp cv_x)^2 + e^{-b(x \pm ct)}F(x \pm ct, \nu),
\]

\[
\Psi = \pm\frac{1}{2}e^{b(x \mp ct)}(\nu_t \mp cv_x)^2 \mp ce^{-b(x \pm ct)}F(x \pm ct, \nu),
\]

for \( g \) of the form \( \mathbf{II} \).

Similarly, the conserved quantities expressed in terms of \( v \) are, respectively, ordinary momentum \( \mathbf{I} \), energy

\[
\tilde{E} = \int_\mathbb{R} \left( \frac{1}{2}(v_t^2 + c^2v_x^2) + \tilde{c}v_t v_x - \frac{1}{8}a^2\nu^2 + F(x - \tilde{c}t, \nu) \right) dx
\]

\( \mathbf{II} \).
and null energies
\[ \tilde{E}_x = \int_{\mathbb{R}} (\frac{1}{2} e^{b(x \mp ct)} (v_t \mp cv_x)^2 + e^{-b(x \mp ct)} F(x \pm ct, v)) \, dx. \] (50)

The multipliers for these conserved quantities are given by
\[ \delta \tilde{P} / \delta v_t = v_x, \quad \delta \tilde{E} / \delta v_t = v_t + cv_x, \quad \delta \tilde{E}_x / \delta v_t = e^{b(x \mp ct)} (v_t \mp cv_x). \] (51)

A summary of the preceding results will now be stated.

**Theorem 2.** A damped nonlinear wave equation (3), with \( g \) being non-singular, possesses an energy-momentum type conservation law (20) if and only if the equation has linear damping and is mapped into an undamped wave equation (12) by a point transformation (11) where \( \alpha = \text{const.} \) is the damping coefficient.

3.3. Physical meaning. The physical meaning of generalized momentum (14), generalized energy (28), and generalized null energies (29) can be understood most readily by considering damped nonlinear wave equations (3) that are \( x \)-translation invariant. Then one has
\[ g = g(t, u, u_t) = \alpha u_t + f(t, u) \] (52)
where the damping coefficient is given by \( \alpha = a \) in cases (i) and (ii), and \( \alpha = a \pm cb \) in case (iii), and where the self-interaction term is independent of \( t \) in cases (ii) and (iii), \( f = f(u) \).

Firstly, when \( \alpha = 0 \), so that the equation is undamped, the conserved quantities (14), (28) and (29) reduce to the ordinary momentum \( P = \int_{\mathbb{R}} u_t u_x \, dx, \) and the ordinary energy \( E = \int_{\mathbb{R}} (\frac{1}{2} (u_t^2 + c^2 u_x^2) + \frac{\alpha}{2} u_t u + e^{-\frac{\alpha}{2} t} F(e^{\frac{\alpha}{2} t} u)) \, dx \) and null energies \( E_x = \int_{\mathbb{R}} (\frac{1}{2} e^{b(x \mp ct)} (u_t \mp cu_x)^2 + e^{b(x \mp ct)} F(u)) \, dx, \) with \( F'(u) = f(u) \). Note that the latter quantities further reduce to the well-known energies (14) when the free parameter \( b \) is put equal to 0. The existence of this parameter is related to the fact that disturbances propagate along the null lines \( x = \pm ct \) (namely the light-cone) for undamped wave equations \( u_{tt} + f(u) = c^2 u_{xx} \) [2, 3].

Secondly, when damping is present, \( \alpha \neq 0 \), these three conserved quantities can be expressed in a factorized form
\[ \tilde{P} = e^{\alpha t} \int_{\mathbb{R}} u_t u_x \, dx, \] (53)
\[ \tilde{E} = e^{\alpha t} \int_{\mathbb{R}} \left( \frac{1}{2} (u_t^2 + c^2 u_x^2) + \frac{\alpha}{2} u_t u + e^{-\frac{\alpha}{2} t} F(e^{\frac{\alpha}{2} t} u) \right) \, dx, \] (54)
\[ \tilde{E}_x = e^{\alpha t} \int_{\mathbb{R}} \left( e^{b(x \mp ct)} \left( \frac{1}{2} (u_t \mp cu_x)^2 + \frac{\alpha}{2} (u_t + \frac{\alpha + 2b}{4} u) u \right) + e^{b(x \mp ct)} e^{-\frac{\alpha}{2} t} F(e^{\frac{\alpha}{2} t} u) \right) \, dx, \] (55)

similarly to the expressions in Table 1. This implies that the integral expressions in these quantities decay like \( e^{-\alpha t} \) in time:
\[ P = \int_{\mathbb{R}} u_t u_x \, dx = e^{-\alpha t} P_0, \] (56)
\[ E = \int_{\mathbb{R}} \left( \frac{1}{2} (u_t^2 + c^2 u_x^2) + \frac{\alpha}{2} u_t u + e^{-\frac{\alpha}{2} t} F(e^{\frac{\alpha}{2} t} u) \right) \, dx = e^{-\alpha t} E_0, \] (57)
\[ E_x = \int_{\mathbb{R}} \left( e^{b(x \mp ct)} \left( \frac{1}{2} (u_t \mp cu_x)^2 + \frac{\alpha}{2} (u_t + \frac{\alpha + 2b}{4} u) u \right) + e^{b(x \mp ct)} e^{-\frac{\alpha}{2} t} F(e^{\frac{\alpha}{2} t} u) \right) \, dx = e^{-\alpha t} E_{x0}, \] (58)
where

\[ P_0 = P|_{t=0} = \tilde{P}|_{t=0}, \quad E_0 = E|_{t=0} = \tilde{E}|_{t=0}, \quad E_\mp 0 = E_\mp|_{t=0} = \tilde{E}_\mp|_{t=0} \]

are the initial values of the quantities. The decay behaviour shows that the damping in the wave equation acts uniformly in \( x \) as measured by the ordinary momentum \( P \), ordinary energy \( E \), and ordinary null energies \( E_\mp \).

A similar discussion holds for the conserved quantities (4), (28), (29) for general damped wave equations (3).

4. Extension of results and null-form damping

All of the conserved quantities found so far in the investigation exist only for linear constant-coefficient damping, as summarized in Theorem 2. Generalizations of these quantities will now be derived for nonlinear wave equations having linear damping with a coefficient that depends on \( t, x \).

Multipliers are determined by the condition

\[ E_u ((u_{tt} - c^2 u_{xx} + g(t, x, u, u_t))(Q_1 u_t + Q_2 u_x + Q_0 u)) = 0, \]

which splits with respect to \( x \)-derivatives of \( u \) and \( u_t \), yielding a system of six partial differential equations for the unknowns \( g(t, x, u, u_t), Q_1(t, x), Q_2(t, x), Q_0(t, x) \). Each solution of the system produces a corresponding conservation law through equation (15).

Under the same requirements on \( g \) considered earlier, it is straightforward to find (as discussed in the Appendix) that the determining system has a solution in four cases. For two of these cases, the complete system can be solved explicitly, while in the remaining two cases the system can be reduced to a single first-order partial differential equation which gives the form of \( g \) in terms of a function of \( t, x \) appearing in the expressions for \( Q_1 \) and \( Q_2 \).

**Proposition 2.** A damped nonlinear wave equation (3), with \( g \) being non-singular, admits a linear homogeneous multiplier (60) only in the following cases:

(i)

\[ g = a'(t)u_t + f(t, u), \quad Q = e^{a(t)}u_x; \]

(ii)

\[ g = a'(t)u_t + \frac{1}{4}(a'(t))^2 + 2a''(t))u + f(x, e^{a(t)}u)e^{-a(t)}, \quad Q = e^{a(t)}(u_t + \frac{a'(t)}{2}u); \]
(iii)

\[ g = (a_{tt} - c^2 a_{xx}) u_t + ku + \frac{e^{\pm 3ca_x - at}}{2} f(x \pm ct, ue^{\pm ca_x + at}) , \]
\[ Q = e^{at + ca_x} (u_t \mp cu_x + \frac{1}{2}(a_{tt} - c^2 a_{xx}) u) , \]

where \( k(t, x) \) satisfies a first-order linear equation in terms of \( a(t, x) \);

(iv)

\[ g = (a_{tt} - c^2 a_{xx}) u_t + ku + \frac{e^{3b - 2at}}{2} f(b_0, ue^\frac{b}{2}) , \]
\[ Q = e^{at} (\cosh(ca_x) (u_t + \frac{1}{2}(a_{tt} - c^2 a_{xx}) u) + c \sinh(ca_x) u_x) , \]

where \( b_0(t, x) \) satisfies \( b_0 + c \tanh(ca_x) b_{0} = 0 \), \( b(t, x) \) satisfies \( b_t + c \tanh(ca_x) b_x = a_{tt} - c^2 a_{xx} \), and \( k(t, x) \) satisfies a first-order linear equation in terms of \( a(t, x) \).

In all four cases, \( f \) is an arbitrary function of its arguments.

The first-order linear equations for \( k(t, x) \) in cases (iii) and (iv) are shown in the Appendix.

The corresponding conservation laws and conserved integrals are given as follows, where \( F = \int f \, du \) in terms of the function \( f \) in \( g \).

**Theorem 3.** For damped nonlinear wave equations (33), with \( g \) being non-singular, the conservation laws from the multipliers (63)–(66) consist of:

(i)

\[ T = e^{a(t)} u_t u_x , \]
\[ \Psi = -e^{a(t)} \left( \frac{1}{2}(u_t^2 + c^2 u_x^2) + e^{a(t)} F(t, u) \right) , \]

for \( g \) of the form (63);

(ii)

\[ T = e^{a(t)} \left( \frac{1}{2}(u_t^2 + c^2 u_x^2) + \frac{1}{2}a'(t) uu_t + \frac{1}{2}a'(t) u^2 \right) + e^{a(t)} F(x, e^{\frac{a(t)}{2}} u) , \]
\[ \Psi = -c^2 e^{a(t)} (u_t u_x + \frac{1}{2}a'(t) uu_x) , \]

for \( g \) of the form (61);

(iii)

\[ T = e^{at + ca_x} \left( \frac{1}{2}(u_t \mp cu_x)^2 + \frac{1}{2}(a_{tt} - c^2 a_{xx}) u_t u - \frac{1}{4}(a_{ttt} - c^2 a_{txx}) + c(a_{tt} - c^2 a_{xx})(ca_{xx} \mp a_{tx}) - 2c^2 k u^2 \right) + e^{c \frac{a_t - ca_x}{2}} F(x \pm ct, e^{\frac{c(a_{tt} - c^2 a_{xx})}{2}} u) , \]
\[ \Psi = \pm ce^{at + ca_x} \left( \frac{1}{2}(u_t \mp cu_x)^2 + \frac{1}{2}c(a_{tt} - c^2 a_{xx}) u_x u \pm \frac{1}{4}c^2 (a_{tt} - c^2 a_{xx}) u_x \right) \mp ce^{c \frac{a_t - ca_x}{2}} F(x \pm ct, e^{\frac{c(a_{tt} - c^2 a_{xx})}{2}} u) , \]

for \( g \) of the form (65);

(iv)

\[ T = e^{at} \left( \cosh(ca_x) \left( \frac{1}{2}(u_t^2 + c^2 u_x^2) + \frac{1}{2}(a_{tt} - c^2 a_{xx}) u_t u - \frac{1}{4}(a_{ttt} - c^2 a_{txx}) + \frac{1}{4}c^2 (a_{tt} - c^2 a_{xx}) a_{xx} - 2k u^2 \right) + c \sinh(ca_x) \left( u_t u_x \mp \frac{1}{4}(a_{tt} - c^2 a_{xx}) u_x \right) \right) + \cosh(ca_x) e^{-a(t) + \frac{2}{b}b} F(b_0, e^\frac{b}{2} u) , \]

12
\[ \Psi = -ce^{at}(\cosh(\text{ca}_x)e^{2}(u_tu_x + \frac{1}{4}(a_{tt} - c^2a_{xx})u_xu - \frac{1}{4}(a_{ttt} - c^2a_{txx} + (a_{tt} - c^2a_{xx})a_{tx})) \\
+ c\sinh(\text{ca}_x)(\frac{1}{2}(a_t^2 + c^2u_x^2) - \frac{1}{4}(c^2(a_{tt} - c^2a_{xx})a_{xx} - 2k)u^2)) \\
+ c\sinh(\text{ca}_x)e^{-a_1 + \frac{3}{2}b}F(b_0, e^{\frac{a}{2}u}), \]  

(70b)

for \( g \) of the form (66). The conserved quantities (16) given by these conservation laws are, respectively, generalized momentum

\[ \tilde{P} = \int e^{a(t)}u_tu_x \, dx, \]  

generalized energy

\[ \tilde{E} = \int e^{a(t)}(\frac{1}{2}(a_t^2 + c^2u_x^2) + \frac{1}{2}a'(t)u_{tt} + e^{a(t)}F(x, e^{a(t)}u)) \, dx, \]  

generalized null energies

\[ \tilde{E}_\pm = \int e^{a\pm \text{ca}_x}(\frac{1}{2}(u_t \mp cu_x)^2 + \frac{1}{2}(a_{tt} - c^2a_{xx})u_tu - \frac{1}{4}(a_{ttt} - c^2a_{txx} \\
+ c(a_{tt} - c^2a_{xx})(a_{xx} \mp a_{tx}) - 2k)u^2) + e^{a\pm \text{ca}_x}F(x \pm ct, e^{a\pm \text{ca}_x}u) \, dx, \]  

(73)

and generalized energy-momentum

\[ \tilde{W} = \int (e^{a\pm \text{ca}_x}(\frac{1}{2}(u_t \pm cu_x)^2 + \frac{1}{2}(a_{tt} - c^2a_{xx})u_tu - \frac{1}{4}(a_{ttt} - c^2a_{txx} \\
+ \frac{1}{4}(a_{tt} - c^2a_{xx})a_{xx} - 2k)u^2) + c\sinh(\text{ca}_x)(u_tu_x - \frac{1}{4}a_{tx}(a_{tt} - c^2a_{xx})u^2) \\
+ \cosh(\text{ca}_x)e^{-a_1 + \frac{3}{2}b}F(b_0, e^{\frac{a}{2}u}). \]  

(74)

It is worthwhile to observe that the generalized energy-momentum (74) can be expressed as a sum of null energies when the \( \cosh \) and \( \sinh \) are expanded in terms of exponentials:

\[ \tilde{W} = \int e^{a\pm \text{ca}_x}(\frac{1}{2}(u_t \pm cu_x)^2 + \frac{1}{4}(a_{tt} - c^2a_{xx})u_tu - \frac{1}{4}(a_{ttt} - c^2a_{txx}) \\
- \frac{1}{4}c(a_{tt} - c^2a_{xx})(a_{tx} \pm ca_{xx}) - 2k)u^2) + \frac{1}{2}e^{-a_1 + \text{ca}_x + \frac{3}{2}b}F(b_0, e^{\frac{a}{2}u}) \, dx \\
+ \int e^{a\pm \text{ca}_x}(\frac{1}{2}(u_t - cu_x)^2 + \frac{1}{4}(a_{tt} - c^2a_{xx})u_tu - \frac{1}{4}(a_{ttt} - c^2a_{txx}) \\
+ \frac{1}{4}c(a_{tt} - c^2a_{xx})(a_{tx} - ca_{xx}) + 2k)u^2) + \frac{1}{2}e^{-a_1 - \text{ca}_x + \frac{3}{2}b}F(b_0, e^{\frac{a}{2}u}) \, dx. \]  

(75)

Conservation of generalized momentum (71) and generalized energy (72) can be explained by existence of a point transformation

\[ v = e^{a(t)}u \]  

(76)

that maps the damped wave equation (3) in cases (i) and (ii) into an undamped wave equation (42).

In contrast, conservation of generalized null energies (73) and generalized energy-momentum (74)–(75) only arises similarly from an undamped wave equation if \( a(t, x) \) satisfies the condition \( a_{tx} = c^2a_{xxx} \). When this condition does not hold, the linear damping is intrinsic, namely, mapping to an undamped wave equation is not possible.
This raises a question of whether some other underlying structure can account for existence of the conserved quantities (73) and (74). Remarkably, they can be explained via a point transformation to a wave equation that has a space-time damping whose form is tied to the light-cone given by the null lines \( x = \pm ct \).

4.1. Null-form damping. Consider a wave equation

\[
\dot{v}_t - c^2\ddot{v}_{xx} + \alpha(t, x)(v_t \pm cv_x) + \kappa(t, x)v + h(t, x, v)
\]  

(77)

where \( h \) is the nonlinear term. In the space-time plane \((t, x)\), the light-cone associated to this equation consists of the null lines \( x = \pm ct \) modulo translations in \( t \) and \( x \). These lines correspond to the directional derivatives \( \partial_t \pm c\partial_x \). Hence, the damping term \( \alpha(t, x)(v_t \pm cv_x) \) acts on disturbances that propagate along the corresponding null line.

A multiplier turns out to be given by

\[
Q = v_t \mp cv_x + \alpha(t, x)v,
\]

provided that \( \kappa(t, x) \) and \( h(t, x, v) \) have a suitable form which will now be derived.

First look at the linear terms in the wave equation (77). By a direct computation using integration by parts:

\[
\begin{align*}
(v_t \mp cv_x + \alpha v)(\dot{v}_t - c^2\ddot{v}_{xx} + \alpha(v_t \pm cv_x) + \kappa v) \\
= & \quad D_t \left( \frac{1}{2}(v_t \mp cv_x)^2 + \alpha v_t v + \frac{1}{2}(\alpha^2 - \alpha_t + \kappa)v^2 \right) \\
& + D_x \left( \pm \frac{1}{4}c(v_t \mp cv_x)^2 - c^2\alpha v_x v + \frac{1}{2}c(\pm \alpha^2 + c\alpha_x \mp \kappa)v^2 \right) \\
& - \frac{1}{2}(\kappa_t \mp c\kappa_x - 2\kappa + (\alpha_t - \alpha^2)_t + c(c\alpha_x \pm \alpha^2)_x)v^2.
\end{align*}
\]

(79)

The key to this identity is that the term \( v_t \mp cv_x \) in the multiplier and the damping term \( \alpha(t, x)(v_t \pm cv_x) \) involve opposite null directions whereby their product is the null quadratic \( \alpha(t, x)(v_t^2 - c^2v_x^2) \). This quadratic is canceled by the terms arising from \( \alpha(t, x)v(v_t - c^2v_{xx}) \) after integration by parts.

Consequently, expression (79) will be a total space-time divergence if the coefficient of \( v^2 \) on the righthand side vanishes. This yields a first-order PDE to be satisfied by \( \kappa(t, x) \):

\[
\kappa_t \mp c\kappa_x - 2\kappa - (\alpha_t - \alpha^2)_t + c(c\alpha_x \pm \alpha^2)_x = 0.
\]

(80)

A similar computation applied to the nonlinear terms in the wave equation (77) gives

\[
(v_t \mp cv_x + \alpha v)h(t, x, v) = D_t H \mp cD_x H - \left( H_t \mp cH_x - \alpha vH_v \right)
\]

(81)

where \( H = \int h \, dv \). This identity will yield a total space-time divergence if \( H(t, x, v) \) satisfies the first-order PDE \( H_t \mp cH_x - \alpha vH_v = 0 \), which has the solution \( H = H(x \pm ct, ve^{-\int_0^s \alpha(s, x \pm c(t-s)) \, ds}} \). Thus, \( h = H \) has the form

\[
h = e^{-\int_0^t \alpha(s, x \pm c(t-s)) \, ds} \tilde{h}(x \pm ct, ve^{-\int_0^s \alpha(s, x \pm c(t-s)) \, ds}}
\]

(82)

where \( \tilde{h} \) is an arbitrary function of its arguments.

This establishes the following result.

**Proposition 3.** For a nonlinear wave equation (77) having a null-form damping with coefficient \( \alpha(t, x) \), if \( \kappa(t, x) \) and \( h(t, x, v) \) are given by equations (80) and (82), then the multiplier
yields a null-energy conservation law

\[ T = \frac{1}{2}(v_t \mp cv_x)^2 + \alpha v_t v + \frac{1}{2}(\alpha^2 - \alpha_t + \mathcal{K})v^2 + \int h \, dv, \]
\[ \Psi = \pm \frac{1}{2}c(v_t \mp cv_x)^2 - c^2 v_x v \pm \frac{1}{2}c(\alpha^2 \pm c\alpha_x - \kappa)v^2 \mp c \int h \, dv. \]  

(83a)

A comparison of this conservation law to the generalized null-energy conservation law

\[ v = e^{\mp \alpha (a_t \mp cax)} u, \]  

(84)

under which the null-form damped wave equation (77) is mapped to the damped nonlinear wave equation (3) given by case (65) in Theorem 3. Specifically, the coefficients in these respective wave equations are related by

\[ \alpha = \pm c(a_t \mp cax), \quad \mathcal{K} = k - \frac{1}{2}(a_t - c^2 a_{xx})^2 - \frac{1}{2}((a_t - c^2 a_{xx})_t \mp c(a_t - c^2 a_{xx}), \quad \tilde{h} = f. \]  

(85)

Therefore, Proposition 3 accounts for existence of the conserved null-energies (73).

In a similar way, existence of the conserved energy-momentum (74) can be explained by a mapping to a null-form damped wave equation (77) where the corresponding multiplier contains a combination of the two null-form terms \( v_t \mp cv_x \).

5. Concluding remarks

As main results, all energy-momentum type conservation laws have been found for a general class of damped nonlinear wave equations (3), where the nonlinearity is taken to be non-singular in terms of \( u \). These conservation laws are characterized by multipliers that are linear in \( u_t, u_x \), and \( u \).

The classification shows that conservation laws exist only for linear damping. Consequently, nonlinear damping prohibits existence of energy-type and momentum-type conservation laws if the wave equation is non-singular.

The explicit form of the conservation laws depends essentially on the form of the linear damping term and the form of the nonlinearity. In the simplest cases (cf Theorem 1), the conservation laws straightforwardly generalize ordinary momentum, energy, and null-energies. An explanation of the existence of these three conservation laws is provided by a point transformation (given by a linear change of dependent variable) that maps the damped nonlinear wave equation to an undamped wave equation with a time-dependent nonlinearity. (This accounts for the conservation laws found in Ref. [11] for a smaller class of nonlinear wave equations.)

Specializations to damped nonlinear wave equations that are invariant under time and space translations have been summarized (cf Corollary 1 and Table 1).

In the general case (cf Theorem 3), the conservation laws describe a further generalization of momentum, energy, and null-energies, as well as a generalized energy-momentum. The latter two types of conservation laws do not arise from mappings to an undamped wave equation with a time-dependent nonlinearity. Instead, they can be explained by a mapping to a wave equation that has a null-form damping; namely the damping is combination of spatial and temporal damping whose form is tied to the light-cone of the linear wave operator terms in the equation. In particular, this gives the wave equation a certain type of null structure which is closely related to the null-form nonlinearities studied in analysis.

For future work, the present results can be naturally extended to wave equations in higher dimensions, and applications in analysis can be expected to emerge.
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APPENDIX

The overdetermined systems leading to Propositions 1 and 2 have been solved by use of the software Maple. In particular, the Maple command ‘rifsimp’ is able to yield a complete classification of all cases, with the conditions $g_u^2 + g_t^2 \neq 0$ and $g_u \neq 0$ being appended to the overdetermined system in each computation. The additional condition that $g(t,x,u,u_t)$ is a non-singular function is imposed during the solution process of the system. At the end, the solutions are directly verified to satisfy the system and the conditions. The final case tree of solutions is organized by the generality of the function $g(t,x,u,u_t)$.

In the first computation, ‘rifsimp’ gives four cases, which are distinguished by the form of the solution for the function $g(t,x,u,u_t)$. In one of the cases, $g(t,x,u,u_t)$ contains a term which has a non-trivial denominator involving $u_t$, so the coefficient of this singular term is manually set to zero. The resulting solution case then turns to coincide with another case. After these cases have been merged, the remaining three cases yield Proposition 1.

In the second computation, ‘rifsimp’ is run with priority given to the function $g(t,x,u,u_t)$. This yields three cases. Two are straightforward to solve, and in one of them a singular term in $g(t,x,u,u_t)$ is eliminated during the solution process. The remaining case is more complicated. It involves two nonlinear first-order equations for the functions $Q_0(t,x)$, $Q_1(t,x)$, $Q_2(t,x)$. One of these equations is used to solve for $Q_0(t,x)$ in terms of the other two functions, while the other equation becomes a nonlinear PDE containing $Q_1(t,x)$ and $Q_2(t,x)$, which ‘pdsolve’ is unable to handle. This equation nevertheless has a nice structure which allows it to be solved via a change of dependent variables after splitting into two subcases distinguished by whether $Q_1(t,x)^2 - Q_2(t,x)^2$ is identically zero or not. In each of the subcases, the nonlinear term in $g(t,x,u,u_t)$ is obtained in an explicit form, while the coefficient of the linear $u$ term is given by a linear first-order PDE:

$$2c^2 \partial_{\pm} k + 4c^3(\partial_{\pm} a_x)k + c(\partial_{\pm} a_x)(4c\Box a_x \pm (\Box a)^2) - (\partial_{\pm} a \partial_{\pm}^2 a) - \Box^2 a = 0, \quad (86)$$

and

$$2 \cosh(ca_x)k_t + 2c \sinh(ca_x)k_x + 4(a_{tx} \sinh(ca_x) + ca_{xx} \cosh(ca_x))k$$

$$- (c^2a_{xx}(\Box a)^2 + 2\Box a_t) - 2c^2a_{tx}\Box a_x + \Box (\Box a_t) + \Box^2 a) \cosh(ca_x)$$

$$- c(a_{tx}(\Box a)^2 + 2\Box a_t) - 2c^2a_{xx}\Box a_x + \Box (\Box a_x)) \sinh(ca_x) = 0, \quad (87)$$

which arise respectively for cases (iii) and (iv) of Proposition 2. Here $\partial_{\pm} = \partial_t \pm c\partial_x$ is the derivative along the lightcone, and $\Box = \partial_t^2 - c^2\partial_x^2 = \partial_+ \partial_-$ is the wave operator.

Hence, a total of four distinct solution cases arise from the overdetermined system in the second computation.

For each multiplier arising in the two classifications, the corresponding conserved density $T$ and spatial flux $X$ are obtained through an integration by parts procedure applied to the right-hand side of the divergence identity (15). This yields the conservation laws stated in Theorems 1 and 3.
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