Clifford Algebra of the Vector Space of Conics for decision boundary Hyperplanes in $m$-Euclidean Space.

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Abstract

In this paper we embed $m$-dimensional Euclidean space in the geometric algebra $\mathcal{C}_m$ to extend the operators of incidence in $\mathbb{R}^m$ to operators of incidence in the geometric algebra to generalize the notion of separator to a decision boundary hyperconic in the Clifford algebra of hyperconic sections denoted as $\mathcal{C}(\mathcal{V}_2)$. This allows us to extend the concept of a linear perceptron or the spherical perceptron in conformal geometry and introduce the more general conic perceptron, namely the elliptical perceptron. Using Clifford duality a vector orthogonal to the decision boundary hyperplane is determined. Experimental results are shown in 2-dimensional Euclidean space where we separate data that are naturally separated by some typical plane conic separators by this procedure. This procedure is more general in the sense that it is independent of the dimension of the input data and hence we can speak of the hyperconic elliptic perceptron.

Keywords: Computational Geometry, Geometric Algebra, Neural Networks, Projective Geometry of Hyperconics, Elliptical Perceptrons.

1 Introduction

In this paper we extend the operators of incidence in $m$-dimensional Euclidean space to operators of incidence in the geometric algebra to take the advantage of simple representation of geometric entities on the one hand and its low computational complexity on the other. More concretly, in the case of linear subspaces of $m$-dimensional Euclidean space, the perceptrons such as the hyperplanes, hyperspheres and hyperconic find a representation as hyperplanes or linear subspaces in the geometric algebra where the notions of incidence are exactly expressed as in the case of $m$-dimensional Euclidean space. In the

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simplest case, which is the linear perceptron the input data is divided in two classes of points by the hyperplane, namely the points of one side and the points of the other side of it. Another example is that of the spherical perceptron where we define two classes of input data which are the interior points and exterior points to the circle. In the latter, the circle is in fact represented as a hyperplane in the conformal geometric algebra $\mathbb{P}K^m$.

In the same way that the lines, planes or hyperplanes are the simplest and natural separators in the Euclidean space $\mathbb{R}^m$ of two classes where we define the perceptron, the circle, the sphere and hypersphere are the natural separators of two classes in the conformal space $\mathbb{P}K^m$. In this space we define spherical perceptron and also the spherical neural networks in which we can separate points from structures that have an interior and exterior.

Similarly as the perceptron is defined to separate linearly two classes and the spherical perceptron to separate spherically interior from exterior we define the elliptic perceptron to separate points of one side of the conic and points in the other side which have a conic as a natural boundary decision hypersurface; it includes hyperplanes, hyperspheres, hyperellipses and hyperbolic surfaces; with this separator we generalize any other separators. We also use this conic separator to extend the concept of spherical neural network to define the elliptic neural network which is a generalization to all others. The paper is organized as follows. In section 2 the basic notations and conventions of Clifford algebras are introduced used throughout this paper. In section 2.1 the real vector space of hyperconics is introduced and identified with a real vector space by means of the mapping $\tau$. This allows us to identify the space of hyperconic sections $\mathcal{V}_2$ with the set of symmetric matrices. The Clifford algebra $Cl(\mathcal{V}_2)$ is then defined. The decision hypersphere is briefly recalled in section 3.1 as a concept naturally introduced in conformal space which is used in defining the spherical perceptron. This leads us to define the concept of elliptical perceptron used throughout the paper as a special case of the spherical perceptron. In section 3.2 we state as lemma 5 the embedding $\iota : \mathbb{R}^m \hookrightarrow M^*$ and we introduce the embedding $\mathbb{R}^m \hookrightarrow Cl(\mathcal{V}_2)$. This allows us to characterize in lemma 8 the elementary but basic incidence property of a point lying on a hyperconic using only the Clifford product. We state and recall briefly the one to one correspondance between the space of conics in $\mathbb{P}^2$, the set of hyperplanes in $\mathbb{P}^5$ and the dual projective space $\mathbb{P}^*_5$. The definition of the $d$-uple embedding $\rho_d$ is briefly recalled and used for the special case of $d = 2$ to conclude that the relation of a point $x \in \mathbb{P}^2$ being incident to a plane conic is equivalent to find a hyperplane $\mathbb{P}^5$ containing $\rho_2(x)$. We solve the problem of determining the boundary decision hyperplane by means of duality in Clifford algebra in proposition 11. Duality in projective geometry and in $Cl(\mathcal{V}_2)$ are equivalent in the sense of remark 13 using corollary 12 to proposition 11. We relate the mappings $\tau, \rho_2$ and $\iota$ previously introduced by means of proposition 15. The relationship between proposition 15 and the definitions given by 5 is stated in remark 17. In section 3.3 the experimental results are given by producing input data for $m = 2$ training the elliptical perceptron as a neural network by means of the backpropagation algorithm. The results are stated in table 1. The final conclusions are stated in section 4.

2 Clifford Algebras and the Clifford Algebra for the vector space of conics.

In this section we recall the basic notation, facts and well known properties of Clifford algebras, for a more comprehensive treatment we refer the reader to e.g., chapter 15 of [7] or chapters 3 and 4 of [4] and we will restrict ourselves to introduce the main notions and notational conventions used throughout this paper. We denote an $m$-euclidean vector space as $\mathbb{R}^m$ with its usual quadratic form. In $\mathbb{R}^m$ we fix as basis $e_1, e_2, \ldots, e_m$ and denote by $Cl(\mathbb{R}^m)$ or simply $Cl_m$ if it is clear that we are forming the Clifford algebra.
over the real field, the Clifford algebra associated to the \( m \)-dimensional euclidean quadratic space. A hypersurface of degree 2 in \( \mathbb{R}^m \) will be called a hyperconic section or hyperconic. If the Clifford algebra is to be emphasized associated to the quadratic space we enclose it within parenthesis as for example \( Cl(\mathbb{R}^m) \). The Clifford algebra \( Cl_m \) as a real vector space has dimension \( 2^m \). Considering the usual embedding of \( \mathbb{R}^m \) in \( Cl_m \) and denoting by the same symbols the vectors \( e_1, e_2, \ldots, e_m \) under this embedding these are called the basis blades. For mathematical applications, it is equally valid and useful to introduce the geometric algebra \( G_{p,q,r} \) as the geometric algebra of dimension \( 2^m \) where \( m = p + q + r \) which is defined from its underlying vector space \( \mathbb{R}^{p,q,r} \) endowed with a signature \((p,q,r)\) by application of a geometric product. In the sequel, we will only consider non-degenerate geometric algebras \( G_{p,q} \) where \( r = 0 \). Besides, we will write \( G_m \) if \( q = 0 \). In particular, note that \( Cl_m = G_m \) with this notation. Another example is projective space which is \( G_{3,1} \). Points in this space are represented by 1-blades. The geometric product of two multivectors \( a \) and \( b \) is simply denoted by \( ab \). The geometric product consists of an outer product (\( \wedge \)) and an inner product (\( \cdot \)). More precisely, as \( G_m \) is generated as an \( \mathbb{R} \)-algebra by its basis blades, the geometric product of two basis vectors is given by:

\[
e_{ij} \overset{\text{def}}{=} \begin{cases} 
 1 & \text{for } i = j \in \{1, \ldots, p\}, \\
 -1 & \text{for } i = j \in \{p + 1, \ldots, p + q\}, \\
 0 & \text{for } i = j \in \{p + q + 1, \ldots, m\}, \\
 e_{ij} = e_i \wedge e_j = -e_{ji} & \text{for } i \neq j.
\end{cases}
\]

The outer product is a special operation defined within Clifford algebra and is equivalent to the exterior product of the Grassmann algebra. It is associative and distributive. For vectors \( x, y \in \mathbb{R}^m \) it is also anticommutative, i.e. \( x \wedge y = -y \wedge x \). Another important property is that for a set \( \{x_1, \ldots, x_k\} \subset \mathbb{R}^m \) of \( k \leq m \) mutually linearly independent vectors, \( x_1 \wedge x_2 \cdots \wedge x_k \wedge y = 0 \) if and only if \( y \) is linearly dependent with respect to \( \{x_1, \ldots, x_k\} \). The outer product of \( k \) vectors is called a \( k \)-blade and is denoted by

\[
A_{<k>} = a_1 \wedge a_2 \cdots \wedge a_k \overset{\text{def}}{=} \bigwedge_{i=1}^{k} a_i
\]

The grade of a blade is simply the number of vectors that “wedged” together give the blade. Hence, the outer product of \( k \) linearly independent vectors gives a blade of grade \( k \), i.e. a \( k \)-blade. The unit pseudoscalar of \( Cl_m \) is a blade of grade \( m \) with magnitude 1 and denoted by \( I \). In geometric algebra, blades, as defined above, are given a geometric interpretation. As for example the 1-blades are the vectors, the 2-blades or bivectors are the oriented planes and so on. This is also based on their interpretation as linear subspaces. For example, given a vector \( a \in \mathbb{R}^m \), we can define a function \( O_a \) as

\[
O_a : \mathbb{R}^m \to Cl_m \\
x \mapsto x \wedge a
\]

The kernel of this function is called the outer product null space (OPNS) of \( a \) and denoted by \( NO(a) \). We can explicitly describe it as:

\[
NO = \{ x \in \mathbb{R}^m : x \wedge A_{<k>} = 0 \}.
\]

Therefore the OPNS of the vector \( a \) is a line through the origin with the direction given by \( a \). In general, the OPNS of some \( k \)-blade \( A_{<k>} \in Cl_m \) is a \( k \)-dimensional linear subspace of \( \mathbb{R}^m \). Another useful concept we will use is the null space of blades with respect to the inner product denoted as the inner product null space (IPNS) of a blade \( A_{<k>} \), denoted by \( \mathbb{N}(A_{<k>}) \) which is defined as the kernel of the function
It is well known from linear algebra that for a symmetric sense (see [1] for a recent survey of the methods to investigate this problem).

\[ \mathcal{I}_{A_{<k>}} : \mathbb{R}^m \rightarrow Cl_m \]
\[ x \mapsto x \cdot A_{<k>} \]

which is given explicitly as \( \mathbb{N}(A_{<k>}) = \{ x \in \mathbb{R}^m : \mathcal{I}_{A_{<k>}}(x) = 0 \} \). An important notion is the dual operation in the Clifford algebra. The dual of a multivector \( A \in Cl \), denoted as \( A^* \) is defined as \( A \cdot I^{-1} = AI^{-1} \) where \( I^{-1} \) is the inverse unit pseudoscalar, which is also an \( m \)-blade. A property useful relating both \( \mathbb{N}, \mathbb{N}^0 \) which will be used in proposition [1] is the following:

**Lemma 1** For a \( k \)-blade \( A_{<k>} \):

\[ \mathbb{N}^0(A_{<k>}) = \mathbb{N}(A^*_{<k>}). \]

**Proof**: According to equation (3.34) of [2] if \( C, B_{<l>} \) are a \( 1 \)-blade (resp. an \( l \)-blade): \( (C \wedge B_{<l>})^* = C \cdot (B_{<l>})^* \) for \( l \leq m - 1 \) which gives directly the " \( \subset \) " contention. As for the other set theoretical contention, the last equation gives in fact \( (C \wedge A_{<k>})^* = 0 \) for \( C \in \mathbb{N}(A^*_{<k>}) \) hence \( C \wedge A_{<k>}I^{-1} = (C \wedge A_{<k>})I^{-1} = 0 \) multiplying by \( I \) gives in the \( \mathbb{R} \)-algebra: \( 1(C \wedge A_{<k>}) = 0. \)

**Example 2** The OPNS of a bivector in \( \mathbb{R}^3 \) is the IPNS of the cross product of its vectors, a nice property only valid for three euclidean vector space.

The projective space \( \mathbb{P}\mathbb{R}^m \) is the \( m + 1 \) dimensional vector space \( \mathbb{R}^{m+1} \) without the origin. In conformal geometric algebra \( G_{4,1} \) the spheres are the basis entities from which the other entities are involved, see e.g. \& 3 of [5]. Even though we will work in the sequel with the conformal space of 3-dimensional Euclidean space, all formulae extend directly to \( m \)-dimensions. In order to obtain a conformal space, the euclidean \( m \)-space \( \mathbb{R}^m \) is embedded in conformal space denoted by \( \mathbb{K}^m \) via the stereographic projection and this space will be denoted by \( \mathbb{PK}^m \). To obtain a basis, we extend the orthonormal basis \( \{ e_1, \ldots, e_m \} \) of \( \mathbb{R}^m \) by two orthogonal basis vectors \( \{ e_+, e_- \} \) with \( e_+^2 = -e_-^2 = 1 \).

A set of geometric entities of interest in computer vision are conic sections. It is therefore useful to construct the Clifford algebra over a real vector space such that the conics and their incidence properties such as the union, intersection etc; can be represented in terms of the INPS and the ONPS as represented above. The idea of using the Clifford algebra for the vector space of conics has already been introduced by e.g. \& 4 of [5]. The authors use this idea to express the classical problem of fitting a set of given points in \( \mathbb{R}^2 \) to a real conic and also to fit a set of conics as given input data to a cluster of points in a least square sense (see [1] for a recent survey of the methods to investigate this problem).

### 2.1 The Clifford Algebra for the real vector space of hyperconic sections

It is well known from linear algebra that for a symmetric \( 3 \times 3 \) matrix \( A \) the set of vectors \( x = (x_1, x_2, 1) \) that satisfy \( x^tAx = 0 \) where \( t \) denotes the transpose of a vector, lie on a conic containing the point \( (x_1, x_2) \). One then says that \( A \) represents the conic defined by the equation above. More generally, we introduce the following vector spaces precisely as: \( M \overset{\text{def}}{=} M_{m,m}(\mathbb{R}) \), the space of real \( m \) by \( m \) matrices and \( M^+ \overset{\text{def}}{=} \{ A \in M | A = A^t \} \) the subvector space of symmetric \( m \) by \( m \) matrices of \( M \). We can identify the first with the \( m^2 \) dimensional vector space \( \mathbb{R}^{m^2} \) by means of the isomorphism \( \tau : M \rightarrow \mathbb{R}^{m^2} \) given explicitly by:

\[
(x_{i,j})_{i,j \in \{1, \ldots, m\}} \mapsto (x_{1,m}, x_{2,m}, \ldots, x_{m-1,m}, \frac{x_{m,m}}{\sqrt{2}}, \frac{x_{1,1}}{\sqrt{2}}, \frac{x_{2,2}}{\sqrt{2}}, \frac{x_{1,2}}{\sqrt{2}}, \frac{x_{3,3}}{\sqrt{2}}, x_{2,3}, x_{1,3}, \frac{x_{m-1,m-1}}{\sqrt{2}}, x_{m-2,m-1}, \ldots, x_{1,m-1}).
\]
Note that to describe such an isomorphism we are choosing a special permutation of the orthonormal basis of $\mathbb{R}^{m^2}$ followed by a homothety. One reason for choosing such a special permutation is because of remark 17 in subsection 3.2. In particular for an element $A \in M^s$ :

$$\tau(A) = (a_{1,m}, a_{2,m}, \ldots, a_{m-1,m}, \frac{a_{m,m}}{\sqrt{2}}, \frac{a_{1,1}}{\sqrt{2}}, \frac{a_{2,2}}{\sqrt{2}}, a_{1,2}, \frac{a_{3,3}}{\sqrt{2}}, a_{2,3}, \frac{a_{4,4}}{\sqrt{2}}, a_{3,4}, a_{4,4}, a_{1,4}, \ldots, \frac{a_{m-1,m-1}}{\sqrt{2}}, a_{m-2,m-1}, \ldots, a_{1,m-1})$$

which implies that $\tau|_{M^s} = \tau \circ j$ where $j : M^s \to M$ is the inclusion. In the sequel, we will adapt the shorter notation $\tau|$ for the restriction instead of writing the full formula. We will use the proof of the following

**Lemma 3** There is an isomorphism $\tau| : M^s \cong \mathbb{R}^{\frac{1}{2}m(m+1)}$ of $\mathbb{R}$-vector spaces.

**Proof**: The dimension of both vector spaces are equal, so it is enough to show injectivity or surjectivity. We show the latter. If $r = (r_1, \ldots, r_N) \in \mathbb{R}^N$ where $N = \frac{1}{2}m(m+1)$ we will define the following matrix:

$$R = \begin{pmatrix}
\sqrt{2}r_{m+1} & r_{m+3} & r_{m+6} & r_{m+10} & r_{m+15} & \cdots & r_{N-1} & r_1 \\
r_{m+3} & \sqrt{2}r_{m+2} & r_{m+5} & r_{m+9} & r_{m+14} & \cdots & r_{N-2} & r_2 \\
r_{m+6} & r_{m+5} & \sqrt{2}r_{m+4} & r_{m+8} & r_{m+13} & \cdots & r_{N-3} & r_3 \\
r_{m+10} & r_{m+9} & r_{m+8} & \sqrt{2}r_{m+7} & r_{m+12} & \cdots & r_{N-4} & r_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
r_1 & r_2 & r_3 & r_4 & \cdots & \sqrt{2}r_{m^2-m+4} & r_{m-1} & \sqrt{2}r_m
\end{pmatrix}$$

It is clear that $\tau|(R) = r$ as required.

As a consequence of Lemma 3 the space of **hyperconic sections** denoted as $\mathcal{V}_2$ is represented by the $\frac{1}{2}m(m+1)$-dimensional euclidean vector space and its Clifford algebra is denoted by $\text{Cl}(\mathcal{V}_2)$; we will study some of its properties of incidence in subsection 3.2. In the sequel if $x \in \mathbb{R}^m$ we will denote by $x' = (x, 1) \in \mathbb{R}^n$, where $m = n - 1$. It will be useful to introduce the following:

**Definition 4** Let $\iota : \mathbb{R}^m \to M^s$ be defined as $x \mapsto x'x'$ where the product in the right is the usual matrix product.

## 3 A Decision boundary hyperplane in $\text{Cl}(\mathcal{V}_2)$ for $\mathbb{R}^{\frac{1}{2}m(m+1)}$.

### 3.1 The decision hypersphere for $\mathbb{P}K$, the spherical perceptron and the elliptical perceptron.

It is well known that Clifford algebra is used to represent geometric entities like lines and planes through the origin in $\text{Cl}_q$. Conformal space extends this idea by embedding the $m$-dimensional Euclidean space as a regular map (in the projective-geometric sense) in an $m + 2$-dimensional space. Conformal space derives its name from the fact that certain types of reflections in conformal space represent inversion in Euclidean space and conformal transformations can be represented as compositions of inversions in the sense of affine geometry. We have already introduced the conformal space $\mathbb{P}K$. The embedding of a Euclidean vector $x$ in conformal space is given by

$$X = x + \frac{1}{2}x^2e_\infty + e_o$$
where $e_{\infty} \overset{\text{def}}{=} e_- + e_+$ and $e_o \overset{\text{def}}{=} \frac{1}{2}(e_- - e_+)$. Using the null basis $\{e_{\infty}, e_o\}$ instead of $\{e_+, e_-\}$ leads to the representation of $e_{\infty}$ (resp. $e_o$) as the point at infinity (resp. the origin). A vector of the form $S = X - \frac{1}{2}\rho^2 e_{\infty}$ represents a sphere centered on $x$ with radius $\rho$ and in higher dimensions represents a hypersphere. A decision hypersphere has the property that it separates points of the input data into points outside and inside the sphere; such a decision hypersphere has been determined and is given as:

$$\frac{S \cdot X}{(S \cdot e_{\infty})(X \cdot e_{\infty})} \begin{cases} > 0 : x \text{ inside sphere,} \\ = 0 : x \text{ on sphere,} \\ < 0 : x \text{ outside sphere} \end{cases}$$

whenever $X \in \mathbb{H}_a^3$ where $\mathbb{H}_a^3$ is the affine null cone (see e.g. equation 2.26 of [6]). The significance of the affine null cone is that it represents the vectors in $\mathbb{P}K_m$ whose $e_o$ component is unity. The decision hypersphere allows us to define the spherical perceptron represented in figure 1 which shows that it has $m + 2$ weights $w_{ij}$ and $m + 2$ inputs $x_i$ and one output function $y$. As a special case we define the elliptical perceptron as the spherical perceptron with 6 weights, 6 inputs and one output function.

![Figure 1: Spherical Perceptron.](image)

### 3.2 Boundary Decision hyperplanes using duality in Projective Geometry and in $Cl(V_2)$.

Given a set of input data, to classify the set of two classes of points a decision hyperplane is determined whenever a linear one is possible. In order to determine the boundary decision hyperplane we are posing the problem of determining the boundary of the decision hyperplane. In this section we will determine one, using the concept of duality in projective geometry and relate it to its dual in $Cl(V_2)$. In the sequel, recall that $m = n - 1$.

**Lemma 5** $i$ embeds $\mathbb{R}^m$ into $M^a$. 


Proof: Explicitly

\[ \iota(x) = \begin{pmatrix} x_1^2 & x_1 x_2 & \cdots & x_1 x_m & x_1 \\ x_2 x_1 & x_2^2 & \cdots & x_2 x_m & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_m x_1 & x_m x_2 & \cdots & x_m^2 & x_m \\ x_1 & x_2 & \cdots & x_m & 1 \end{pmatrix} \]

and observe that the \( m + 1 \)-th column completely determines \( \iota \).

\[ \odot \]

**Example 6.** Note that \( \iota \) can be extended to \( \mathbb{R}^{m+1} \) but it is no longer an embedding. For example \( \iota(x_1, x_2, x_3) = \iota(-x_1, -x_2, -x_3) \).

A direct calculation shows that for \( x \in \mathbb{R}^m \):

\[ \tau(\iota(x)) = (x_1, x_2, \ldots, x_m, 1, \frac{1}{\sqrt{2}} x_1^2, 1, \frac{1}{\sqrt{2}} x_2^2, x_1 x_2, 1, \frac{1}{\sqrt{2}} x_3^2, x_2 x_3, x_1 x_3, 1, \frac{1}{\sqrt{2}} x_4^2, x_3 x_4, x_2 x_4, x_1 x_4, \ldots, 1, \frac{1}{\sqrt{2}} x_m^2, x_m x_{m-1} x_m, \ldots, x_1 x_m). \]

In particular for \( m = 2 \) the above formula reduces for \( x = (x_1, x_2, 1) \) to:

\[ \tau(\iota(x)) = (x_1, x_2, 1, \frac{1}{\sqrt{2}} x_1^2, 1, \frac{1}{\sqrt{2}} x_2^2, x_1 x_2). \]

**Definition 7** Let \( \overline{\mathfrak{x}} = \tau(\iota(x)) \), \( x \mapsto \overline{\mathfrak{x}} \) defines an embedding of \( \mathbb{R}^m \) into \( \mathcal{V}_2 \hookrightarrow \text{Cl}(\mathcal{V}_2) \).

**Lemma 8** Denote by \( \cdot \) the dot product in \( \text{Cl}(\mathcal{V}_2) \) and for \( A \in M^s \) let \( a = \tau(A) \) then:

\[ \overline{\mathfrak{x}} \cdot a = 0 \iff x_1^2 a_{11} + x_2^2 a_{22} + \cdots + x_m^2 a_{m,m} + 2 x_1 x_2 a_{12} + 2 x_1 x_3 a_{13} + \cdots + 2 x_1 x_m a_{1m} + \cdots + a_{m+1,m+1} = 0 \iff x^n A x' = 0. \]

**Proof**: This is a direct calculation and follows from the definitions.

Note that by lemma[8] in order to test if a conic defined by \( a = \tau(A) \) contains a point \( x \) it is enough to test whether its Clifford product \( \overline{\mathfrak{x}} \cdot a \) is zero or not.

**Example 9** For \( m = 2 \), \( \overline{\mathfrak{x}} = (x_1, x_2, 1) \):

\[ \overline{\mathfrak{x}} \cdot a = 0 \iff x_1^2 a_{11} + x_2^2 a_{22} + 2 x_1 x_2 a_{12} + 2 x_1 a_{13} + 2 x_2 a_{23} + a_{33} = 0. \]

where again \( \cdot \) is the dot product in \( \text{Cl}(\mathcal{V}_2) \).

The set of conics in the two dimensional projective space \( \mathbb{P}_2 \) in the homogeneous coordinates \( (x : y : z) \) is given by:

\[ C = \{ x^2 a_{11} + y^2 a_{22} + z^2 a_{33} + 2 x y a_{12} + 2 x z a_{13} + 2 y z a_{23} = 0 \} \]

If we introduce coordinates \( (a_{11} : a_{22} : a_{33} : 2 a_{12} : 2 a_{13} : 2 a_{23}) \) for \( \mathbb{P}_5^* \) then we can define a one-to-one correspondence: \( C \longleftrightarrow \mathbb{P}_5^* \) given as \( c \mapsto (a_{11} : a_{22} : a_{33} : 2 a_{12} : 2 a_{13} : 2 a_{23}) \).

It is well known, see e.g. exercise 2.12, chapter 1 of [3], from the projective geometric properties of regular maps of \( \mathbb{P}_2 \) that there is a regular mapping, which is an embedding, the so-called \( d \)-uple embedding \( \rho_d \) which considers all monomials of degree \( d \) in the variables \( x_0, \ldots, x_m \), which are \( \binom{m+d}{m} \) and substituting each homogeneous coordinate of the point \( P = (a_0 : \ldots : a_m) \) in the monomials thus giving a map \( \rho_d : \mathbb{P}_m \rightarrow \mathbb{P}_M \) where \( M = \binom{m+d}{m} - 1 \).
Example 10  If $m = 1, d = 2$ the double embedding of $\mathbb{P}_1$ in $\mathbb{P}_2$ has as image a conic curve.

Another typical example is given by $m = 2, d = 2$ the image $\rho_2(\mathbb{P}_2)$ is a surface called the Veronese surface. Let $N = \frac{1}{2}(m + 1)(m + 2)$ and for the application of $\rho_d$ to the case of hyperconics, $d = 2$ and $M = \left(\frac{m+2}{2}\right) - 1 = N - 1$.

As a consequence of Lemma 8 is that for $x = (x_1: x_2 : 1)$, $\rho_2(x) \in \mathbb{P}_5$ and a hyperplane $H_x$ in $\mathbb{P}_5$ containing this point is given by Eq. (1). Duality in projective geometry is an isomorphism of projective spaces which defines for each hyperplane $H$ in $\mathbb{P}_5$ as above, a point $a(a_1 : a_2 : 2a_13 : 2a_23 : a_{33}) \in \mathbb{P}_5^*$ and conversely, for each $a \in \mathbb{P}_5^*$ corresponds a unique hyperplane in $\mathbb{P}_5$, namely the hyperplane $H_a$ defined by the equation:

$$z_1a_{11} + z_2a_{22} + z_3a_{12} + z_4a_{13} + z_5a_{23} + z_6a_{33} = 0.$$ 

Note that the point $a$ defined in lemma 8 is up to an automorphism the point in the Veronese surface. We conclude from the previously stated isomorphisms $\mathcal{C} \longleftrightarrow \mathbb{P}_5^* \longleftrightarrow \{\text{hyperplanes in } \mathbb{P}_5\}$ that to find a conic in $\mathbb{P}_2$ containing $x$ it is sufficient to find a point in $\mathbb{P}_5^*$ or equivalently a hyperplane $H_x$ in $\mathbb{P}_5$ containing $\rho_2(x)$.

We solve the problem of determining the boundary decision hyperplane by means of duality in Clifford algebra. More precisely,

Proposition 11  Let $x^{(1)}, \ldots, x^{(N)}$ define a set of mutually linearly independent vectors in $\mathbb{R}^m$ and let $u = \underline{x}_1 \wedge \ldots \wedge \underline{x}_N$ where $\underline{x}_i = \tau(i(x^{(i)}))$ for $i = 1, \ldots, N$ then $\mathbb{N}(\underline{u}) = \mathbb{N}(u^*)$ where $u^*$ is unique up to a constant.

Proof: Each vector $\underline{x}_i$ is linearly dependent with $u$ hence $u \wedge \underline{x}_i = 0$ for all $i$. Hence by lemma 11 $u^* \cdot \underline{x}_i = 0$ for all $i$. Hence $u^* \in W$, where $W = \langle \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N \rangle^\perp$ which is one-dimensional. ♦

Corollary 12  With the same hypothesis as lemma 11 $u^*$ is the unique conic which incides through the points $x^{(1)}, \ldots, x^{(N)}$.

Proof: This follows immediately from lemma 11. ♦

Duality in projective geometry and duality in Clifford algebra are equivalent in the following sense:

Remark 13  Let $m, N$ as before and $x \in \mathbb{R}^m$. To find a hyperplane $H$ in $\mathbb{P}_N$ containing $\rho_2(x)$ it is sufficient to find $N$ points: $x^{(1)}, \ldots, x^{(N)}$ mutually linearly independent in $\mathbb{R}^m$ which determine the $N$-blade $x = \underline{x}_1 \wedge \ldots \wedge \underline{x}_N$ and its Clifford dual $x^*$ which is a vector in $\mathcal{C}(\mathcal{V}_2)$. In homogeneous coordinates it is the projective dual to the hyperplane $H$.

The simplest case of remark 13 is the following:

Example 14  For $m = 2, N = 5$ and $x \in \mathbb{R}^2$, to determine a hyperplane $H$ in $\mathbb{P}_5$ containing $\rho_2(x)$ it is enough to find five points no three of which are collinear. Denoting by $x^{(1)}, \ldots, x^{(5)}$ these points and by $x = \underline{x}_1 \wedge \ldots \wedge \underline{x}_5$, $x^*$ is a vector in $\mathcal{C}(\mathcal{V}_2)$ which is an element of $\mathbb{P}_5^*$.

Let $\mathbb{A}_M \overset{\text{def}}{=} \{x \in \mathbb{P}_M | x_{m+1} = 1\}$, $\mathbb{A}_m \overset{\text{def}}{=} \{x \in \mathbb{P}_m | x_{m+1} = 1\}$ and

$$\rho_2 : \mathbb{A}_M \rightarrow \mathbb{A}_M$$

$$x \mapsto [x_1 : x_2 : \ldots : x_m : 1 : x_1^2 : x_2^2 : x_1x_2 : \ldots : x_1x_m].$$
which is the restriction of the double embedding. Let

\[ s(l) = \begin{cases}
  m + 1 & \text{if } l = 0, \\
  m + 2 & \text{if } l = 1, \\
  s(l - 1) + l - 1 & \text{if } 2 \leq l \leq m.
\]

Note that in particular \( s(m) = \frac{(m+1)^2 - (m+1) + 4}{2} \). The integers \( \{s(i)\}_{i=0}^{m} \) define a set with \( m + 1 \) elements \( S \).

Define the following mappings:

\[
T : A_M \rightarrow A_M, \quad \{x_i\}_{i=1}^{N} \mapsto \left\{ \frac{\sqrt{2}x_i}{x_i} \right\}_{i \in S}, \quad \text{otherwise.}
\]

\[
p : \mathbb{R}^N - \{0\} \rightarrow A_M, \quad (x_1, \ldots, x_N) \mapsto (x_1 : \ldots : x_N),
\]

\[
q : A_m \rightarrow \mathbb{R}^m, \quad (z_1 : \ldots : z_m : 1) \mapsto (z_1, \ldots, z_m).
\]

The relation between all the maps above is given by the following:

**Proposition 15** The following diagramme:

\[
\begin{array}{cccc}
\mathbb{P}_m & \supseteq & A_m & \quad \xymatrix{
\{x_i\}_{i=1}^{N} & \ar[r]_{T} & \mathbb{R}^{N} - \{0\} & \ar[l]^{p} \ar[l]_{\rho_2} \ar[r] & \mathbb{P}_M
}
\end{array}
\]

is commutative, where the open ended arrows are isomorphisms, \( \rho_2 | \) is only an embedding and \( p \) is only surjective. More precisely, \( T \circ p \circ (\tau|) \circ q = \rho_2 | \).

**Proof**: This is a direct consequence of the definitions of the mappings given above.

**Example 16** For the space of plane conic sections \( d = m = 2 \), \( M = 5 \). By fixing an ordering on the monomials, \( \rho_2 : \mathbb{P}_2 \rightarrow \mathbb{P}_5 \) is the mapping \( (x_1 : x_2 : x_3) \mapsto (x_1^2 : x_2^2 : x_3^2 : x_1x_2 : x_1x_3 : x_2x_3) \). For this case, the double embedding is defined at the corresponding affine charts \( A_2 = \{ x \in \mathbb{P}_2 | x_3 = 1 \} \), \( A_5 = \{ x \in \mathbb{P}_5 | x_3 = 1 \} \) given by its restriction: \( \rho_2 | : A_2 \rightarrow A_5 ; \) in this case \( S = \{ 3, 4, 5 \} \) and \( T : A_5 \rightarrow A_5 \) is the automorphism given by:

\[ (\xi_1 : \xi_2 : \xi_3 : \xi_4 : \xi_5 : \xi_6) \mapsto (\xi_1 : \xi_2 : \sqrt{2}\xi_3 : \sqrt{2}\xi_4 : \sqrt{2}\xi_5 : \xi_6). \]

Note that the inverse image of a hyperplane of \( \mathbb{P}_5 \) under \( \rho_2 \) is a conic in \( \mathbb{P}_2 \).

**Remarks 17** The authors in [5] introduce \( Cl(V_2) \) only for the case of plane conic sections and define the mappings \( T \) and \( D \) stating no apparent relation amongst these mappings. In our case \( T = \tau \) and \( D(x) = \frac{1}{2}x \) in our notation hence stating their close relationship. We complete the relation amongst these mappings by introducing the mappings \( p, T \) and \( \rho_2 | \) which is summarized by prop. [15].

### 3.3 Experimental results to determine the boundary decision hyperplane.

In order to obtain experimental results we produced data of points for \( m = 2 \), that is to say plane conics in \( \mathbb{R}^2 \). The decision conic is to be determined by using the **elliptical perceptron** defined at the end of subsection[3.1] which has weights \( \{\omega_i\}_{i=1}^{6} \) and with 6 inputs and one output function. Each of the examples considered for the elliptical perceptron is tabulated in table[11] given below. We give in each case as data for
the MLP a set of points divided in two classes to be separated by a decision boundary hyperplane. For the data, to train the neural network 6 nodes for the input and one node for the output with no hidden layers in both cases were chosen. The learning rule is the backpropagation algorithm where the input function was chosen to be the dot product with typical transfer functions as the sigmoid bipolar and the sine bipolar to properly bound the output in the interval $[-1, +1]$. In order to obtain the equation of the conic we obtained the set of weights $\omega_1, \ldots, \omega_6$. If we let $\omega = (\omega_1, \ldots, \omega_6)$ using $\tau$:

$$
\tau^{-1}(\omega) = \begin{pmatrix}
\sqrt{2}\omega_4 & \omega_6 & \omega_1 \\
\omega_6 & \sqrt{2}\omega_5 & \omega_2 \\
\omega_1 & \omega_2 & \sqrt{2}\omega_3
\end{pmatrix}
$$

and the equation of the conic in this case is:

$$\sqrt{2}x^2\omega_4 + \sqrt{2}y^2\omega_5 + 2xy\omega_6 + 2x\omega_1 + 2y\omega_2 + \sqrt{2}\omega_3 = 0.$$

This equation was then transformed into the standard form to obtain the equation of the estimated conic described in the last column of table (1) for each vector $\omega$ of weights. In figure 2 we graph in the first column the two classes of points to be separated for each of the examples of table 1. The first class of points is denoted by a cross and the second by a diamond. A decision boundary hyperplane is to be determined in $\mathbb{R}^2$. In the second column the decision conic is drawn, showing the separation between both classes of points.

| Conic  | Weights $(\omega_1, \ldots, \omega_6)$ | Estimated Conic Equation |
|--------|-------------------------------------|--------------------------|
| Ellipse | $(0.00, 0.00, -3.30, 5.00, 6.36, 0.00)$ | $\frac{x^2}{0.00} + \frac{y^2}{0.71} = 1$ |
| Ellipse | $(8.48, 0.00, -2.84, -1.50, -14.43, 0.00)$ | $\frac{(x-10.05)^2}{14.075} + \frac{y^2}{1.49} = 1$ |
| Hyperbola | $(-2.23, 0.00, -8.26, -19.05, 20.2, 0.00)$ | $\frac{(x+0.07)^2}{1.29} - \frac{y^2}{1.79} = 1$ |

Table 1: Results for the experimental points.

4 Conclusion

The elliptical perceptron introduced in this paper generalizes the spherical perceptron used in conformal geometry to determine the boundary decision hypersurface in euclidean $m$-dimensional space. We have shown that, by means of Clifford algebra the usual space of hyperconic sections embeds into the Clifford algebra of hyperconic sections; this allows us to use all the properties of the geometric product enjoyed by this Clifford algebra and as we have shown, also the Clifford Dual is essential to determine the vector orthogonal to the boundary of the decision hyperplane. A projective property of the space of hyperconics is that it is equivalent to the set of hyperplanes in the projective dual and then it is proved that for each such hyperplane its orthogonal vector is in fact the Clifford dual since to find a decision boundary hyperplane in the euclidean $m$-dimensional space, it is enough in terms of the space $\text{Cl}(\mathbb{V}_2)$ to determine an $m-1$-blade generated by $m-1$ pairwise independent vectors and evaluate its Clifford dual which is a fortiori the orthogonal vector to the original hyperplane. In the experiments to test the theory introduced in subsection 2.1 to determine a boundary decision hyperconic we linearize the problem of finding the hyperconic section by embedding the input data by means of the double-embedding $\rho_2$. The MLP of the elliptical perceptron is introduced to determine a vector orthogonal to the hyperplane in this feature space.
Figure 2: Points to separate (a), (c) and (e) and decision conic (b), (d) and (f)
and then the inverse mapping $\tau^{-1}$ is applied to the vector. We then use equation 1 for this special case to evaluate the equation of the estimated conic. Note that the procedure we have outlined is completely general and does not depend on the dimension of the ambient input space. The experimental results in subsection 3.3 are only done for typical examples which is for plane conics, where it is shown that there exists one decision boundary conic for each of the input data given in table 1. By training the elliptical perceptron the estimated vector orthogonal to the boundary of the decision hyperplane is evaluated. Using $\tau^{-1}$ the estimated equation of the conic is computed. This procedure might at first hand seem very special but the theory developed so far can be done is developed for the higher dimensional case as the maps $\rho_2$ and $\tau$ are completely independent of the dimension of the ambient space and the typical examples in such cases will then be the more general hyperconics sections, where again a vector orthogonal to the boundary decision hyperplane needs to be determined by exactly the same procedure and $\tau^{-1}$ is used to determine the equation of the estimated general hyperconic and only the values for $n, m, N$ have to be once again determined.

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