The minority game: effects of strategy correlations and timing of adaptation

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Abstract

A brief review is given of the minority game, an idealized model of a market of speculative agents, and its complex many-body behaviour. Particular consideration is given to the consequences and implications of correlations between strategies and different frequencies and timings of adaptation.

There is currently much interest in the statistical physics community in the emergence of complex co-operative behaviour as a consequence of frustration and disorder in systems of simple microscopic constituents and rules of interaction. Examples are found in spin and structural glasses, neural networks and hard optimization problems [1]. Minimal models, designed to capture the essence of real world problems without the latter’s peripheral complications or to go beyond them to provide solubility and conceptual insight, have played crucial roles in the understanding of such systems, via their computer simulation, mean field analyses and extrapolation. The minority game is a corresponding minimalist econophysics model introduced to mimic a simple market of speculators trying to profit by buying low and selling high. In this paper we review some of its features and consider it in the context of complexity.

The model comprises an ensemble of a large number $N$ of agents each of whom at each step of a discrete dynamics makes a choice which has a scalar value that can be either positive or negative (buy or sell). The objective of each agent is to make a choice of opposite sign from that of the sum of all the actions. No agent has any direct knowledge of the actions or propensities of the others but is aware of the cumulative action (sum of the choices). Each agent’s actual choice is determined by the application of a personal strategy operator to some common information available identically to all. In the simplest versions of the model, to which we restrict here, the strategy operators are chosen randomly and independently for each agent before play commences and are not modified during play. Each agent does however have a finite set of strategies, which
for simplicity we restrict to two each. At each step one strategy is chosen. The choice is determined by points allocated to each strategy and augmented regularly according to a comparison between the behaviour its play would have yielded and the actual outcome; the points are increased for minority prediction. This is the only mechanism for co-operation but is sufficient to yield complex macroscopic behaviour. As we shall see this complex behaviour has both similarities and differences as compared to condensed matter systems studied earlier.

In the original version of the model [2] the information on which decisions were made was the history of the actual play over a finite window (e.g. the last $m$ steps). However, simulations demonstrated that utilising instead a random fictitious ‘history’ at each time-step produced essentially identical behaviour, suggesting that its relevance is just to provide a mechanism for interaction [3].

Effective interaction via the common information is demonstrated by a deviation of the macroscopic behaviour from that of agents making independent random choices at each time-step. Phase transitions occur as a function of scaled information dimension $d = D/N$, where $D$ is the unscaled information dimension [4], and also as a function of ‘temperature’ $T$ in the case of stochastic indeterminacy in strategy choice [5,6]. These manifest in singularities in macroscopic measures, such as the fraction of frozen agents [7] and appropriate response functions [8]. Complexity arises in that a critical phase line in $(d, T)$ space separates a region in which the starting point allocations are irrelevant, after an ‘equilibration’ period, from one in which different starting extremes yield different behaviours; we refer to these situations respectively as ergodic and non-ergodic. The non-ergodic behaviour manifests in the volatility being sensitive to initial conditions, reminiscent of the differences between field-cooled and zero-field-cooled linear susceptibilities in a spin glass [1]. While the volatility remains finite for all $(d, T)$, a divergence does occur in an appropriate response function [8], reminiscent of that in the spin-glass or third-order non-linear susceptibilities in a spin glass.

Since the information on which the agents act is the same for all, this problem is manifestly mean-field. It therefore offers the potential for exact solution for its macro-behaviour in the sense of the elimination of the microscopic variables in favour of self-consistently determined macro-parameters in the thermodynamic limit. The actual solution of these self-consistency equations is, however, non-trivial in general, and not all scenarios have yet been fully explored.

In this paper we restrict discussion to systems with the ‘information’ (on which the strategies operate to yield choices) chosen randomly at each time-step. However we consider several variations on further convention, in particular with respect to independence of strategies and frequency or timing of the up-
dating of strategy points. These variations have demonstrated several novelties
some of whose investigation has thrown new light on the underlying behaviour
and others which require further study.

The physics seems robust to variations of detail concerning the use of continuous
or discrete ‘bid’ or strategy spaces, nor on the number of strategies each
agent holds provided it is finite. However, for completeness we indicate the ver-
sions used in the figures. At the basic level each agent \( i, i = 1, \ldots, N \), is taken
to have two \( D = dN \)-dimensional strategies \( R_{ia} = (R_{1ia}, \ldots, R_{dNia}) \), \( a = \pm 1 \),
with each component \( R_{\mu ia} \) chosen independently randomly \( \pm 1 \) at the outset
and thereafter fixed. \( \mu(t) \) is chosen stochastically randomly at each time-step
\( t \) from the set \( \mu(t) \in \{1, \ldots, D\} \) and each agent plays one of his or her two
strategies \( R_{\mu ia} \), \( a = \pm 1 \). The actual choices of \( a \) used, \( b_i(t) \), are determined
by the current values of point differences \( p_i(t) \). Here we restrict to determin-
istic choices, \( b_i(t) = \text{sgn}(p_i(t)) \), but a simple extension to stochastic choices is
possible \([5,6,9]\). The \( p_i(t) \) are updated every \( M \) time-steps according to

\[
p_i(t + M) = p_i(t) - M^{-1} \sum_{\ell=t}^{t+M-1} \xi_i^{\mu(\ell)} \left\{ N^{-1/2} \sum_j (\omega_{\mu j}^{(\ell)} + \xi_j^{(\ell)} \text{sgn}(p_j(t))) \right\},
\]

where \( \omega_i = (R_{i1} + R_{i2})/2, \xi_i = (R_{i1} - R_{i2})/2 \). The case where \( M = 1 \)
is referred to as the online model. For \( M \geq O(N) \) one expects (and simulations
confirm) the behaviour to be the same as that for the so-called batch version
in which the sum on the actual \( \mu(\ell) \) in (1) is replaced by an average:

\[
p_i(t + 1) = p_i(t) - \lambda N^{-1} \sum_{\mu=1}^D \xi_i^\mu \left\{ \sum_j (\omega_j^\mu + \xi_j^\mu \text{sgn}(p_j(t))) \right\}, \lambda = O(1).
\]

It is of interest to compare the cases of differently correlated \( R_{i1,2} \) \([10]\). In
Fig. 1 are shown the volatilities obtained in simulations with strategies chosen
randomly with correlation probability \( P(R_{i1}^\mu = R_{i2}^\mu) = \rho, \rho \in [0,0.5] \),
for both the on-line and the batch models. For the uncorrelated case the on-
line and batch models are almost identical, both exhibiting a minimum in
volatility lower than the random-bid value at a critical \( d \) \([4,11]\), but as the
level of (anti-)correlation is increased the volatilities of the online and batch
models become increasingly different from one another, as well as different
from the uncorrelated situation. Earlier studies on systems with uncorrelated
strategies showed slight differences between online and batch models both in
simulations \([12]\) and theoretical studies \([13]\) but anti-correlation as studied
here significantly amplifies and highlights the differences.

In the thermodynamic limit \((N \to \infty)\) the macrodynamics of the batch problem
for arbitrarily correlated local strategies \( R_{i1,2} \) can be expressed in terms
Fig. 1. Volatilities with varying degrees of anticorrelation between pairs of strategies, $P(R_{11}^\mu = R_{12}^\mu) \equiv \rho = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$, top to bottom, (a) online, (b) batch.

of non-linear coupled equations involving two-time correlation and response functions, via an extension of the generating functional method of Heimel and Coolen [8]. The full equations are complicated but can be conveniently expressed via a self-consistently determined coloured effective single-particle noise [8,14]. Here we just give some results. Let us concentrate first on the case of independent $R_{1,2}$. In this case the volatility exhibits a minimum at a critical $d_c$ above which the dynamics is ergodic and beneath which it is non-ergodic, the volatility yielded by a tabula rasa point initiation, $p_i(0) = 0$, being greater than that for a strongly biased start $|p_i(0)| \sim O(1)$. For $d \geq d_c$ a simple ansatz provides the stationary limit in accord with simulations [11]. For $d < d_c$ the behaviour is non-ergodic and a full closed analytic solution is still awaited, but iteration of the coupled macrodynamical equations is possible numerically using the self-consistent noise-sampling procedure of Eissfeller and Opper [15]. This shows good accord with corresponding simulations, as is illustrated in Fig. 2a. An analagous treatment of the anti-correlated case demonstrates that the non-ergodic transition persists, even though the volatility itself appears smooth; see Fig. 2b. The existence of a phase transition separating ergodic and non-ergodic regions is reminiscent of transitions between equilibrating and non-equilibrating regions in the ‘condensed matter’ examples mentioned earlier. The existence of a critical dimension $D_c$ scaling as $N$ is reminiscent of the critical scaling limit on the number of retrievable stored patterns in a neural network [16], but there are also crucial differences [17,10].
Fig. 2. Comparison of volatilities for tabula rasa and strongly biased-point starts in batch minority games after 40 time-steps ($p_i(0) = 0.0$ (circles), 0.5 (squares) and 1.0 (diamonds) for normalisation $\lambda = (1 - \rho)^{-1}$). (a) Uncorrelated strategies. (b) Fully anti-correlated case. Open symbols denote simulations for $N = 1000$ players. Solid symbols represent results from a numerical evaluation of the macroscopic self-consistent equations derived analytically in the limit $N \to \infty$. An average is performed over 50000 realisations of the single-particle noise.

For $M \geq O(N)$ the temporal correlation functions

$$C(\tau) = \lim_{t \to \infty} N^{-1} \sum_i \text{sgn}(p_i(t + \tau)) \text{sgn}(p_i(t))$$

(3)

exhibit oscillations with period $2M$. In the ergodic region these oscillations die away with a finite decay time for $\rho > 0$. In the non-ergodic region oscillatory behaviour persists. Randomizing the precise time of updating, independently for each agent, but maintaining the average updating frequency, reduces and eventually (if the time-spread is sufficient) removes the oscillations. This does not have an obvious well-known spin glass analogue and suggests a different complexity.

The minority game was already recognized as having interesting features from the perspective of complex emergent behaviour in many body physics, including an ergodic-nonergodic phase transition. By considering correlated strategies we have re-iterated that anti-correlation increases volatility and have demonstrated that it amplifies differences between online and batch behaviour, implying that updating frequency is relevant in generalizations of online learning models. We have also shown that analytic theory of the batch model can be extended to correlated strategies and is in good accord with simulations.
for both ergodic and non-ergodic regions. An ergodic-nonergodic transition is found in the fully anti-correlated case even though the *tabula rasa* volatility is smooth.

A more complete description of our results and analysis will be presented elsewhere [14].

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