Optimal Antipodal Configuration of 2d Points
on a Sphere in $\mathbb{R}^d$ for Covering

Sergiy Borodachov

Department of Mathematics, Towson University, 8000 York Rd., Towson, MD, 21252

Abstract

We show that among antipodal 2d-point configurations on the sphere $S^{d-1}$ in $\mathbb{R}^d$, the set of vertices of a regular cross-polytope inscribed in $S^{d-1}$ uniquely solves the best-covering problem (this is new for $d \geq 5$) and the maximal polarization problem for potentials given by a function of the distance squared with a positive and convex second derivative ($d \geq 3$).

Keywords: Antipodal spherical code, regular cross-polytope, best-covering problem, polarization problem, convex polytope, radial projection, area argument.

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1 Introduction

Let $S^{d-1} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1^2 + \ldots + x_d^2 = 1\}$, $d \geq 2$, be the unit sphere in the Euclidean space $\mathbb{R}^d$. For a given point configuration $\omega_N := \{x_1, \ldots, x_N\} \subset S^{d-1}$, let

$$\rho(\omega_N, S^{d-1}) := \max_{x \in S^{d-1}} \min_{i=1,N} |x - x_i|$$

denote its mesh norm (relative to the sphere). The classical optimal covering problem on the sphere is to minimize the mesh-norm.

Problem 1.1. For a given $N \in \mathbb{N}$, find the quantity

$$\rho_N(S^{d-1}) := \inf_{\omega_N \subset S^{d-1}} \rho(\omega_N, S^{d-1})$$

and optimal-covering N-point configurations on $S^{d-1}$; i.e., configurations $\omega_N^* \subset S^{d-1}$ attaining the infimum on the right-hand side of (1).

Solution to Problem 1.1 is known on $S^3$ for every $N$ (equally spaced points), on $S^2$ for $N = 1 - 8, 10, 12$, and 14, see the works by Fejes-Tóth [16, 17, 18], Schütte [25], and Wimmer [29], on $S^3$ for $N = 1 - 6$ and 8, see the works by Galiev [19], Böröczky and Wintsche [7], and Dalla, Larman, Mani-Levitska, and Zong [14], and on $S^{d-1}$, $d > 4$, for $1 \leq N \leq d + 2$, see [19, 7]. For more information on optimal covering, see, among others, books by Fejes-Tóth [16, 17], Rogers [24], and Böröczky [6].
The main goal of this paper is a further study of the optimal-covering property of the set of vertices of a regular cross-polytope inscribed in $S^{d-1}$, which is any $2d$-point configuration of the form $\omega_{2d} := \{\pm a_1, \ldots, \pm a_d\}$, where $\{a_1, \ldots, a_d\}$ is an orthonormal basis in $\mathbb{R}^d$. The optimal covering property of $\omega_{2d}$ is known on $S^{d-1}$ for $d = 2$ (a basic result), $d = 3$ [16, 17], and $d = 4$ [14]. It is a well-known open question for $d \geq 5$. We resolve it here for antipodal configurations.

The proof in [14] for $d = 4$ uses the area argument together with the Euler-Poincaré formula and the fact that (cf. [5]) the largest $(d-1)$-dimensional area of a spherical simplex inscribed in a given spherical cap of angular radius less than $\pi/2$ is that of a regular simplex. If the number $\tau$ of $(d-1)$-dimensional simplices obtained after triangulating the facets of the convex hull of a given $2d$-point configuration is less than or equal to $2^d$, the area argument combined with the above mentioned fact from [5] complete the proof. If $\tau > 2^d$ then the Euler-Poincaré formula is used to provide an efficient estimate for the number of one-dimensional edges of the convex hull. Since the number of terms in the Euler-Poincaré formula grows with $d$, such an estimate is not possible when $d \geq 5$ and the argument fails. However, if one makes an additional assumption that the configurations are antipodal, then it can be shown that $\tau = 2^d$, and the area argument can be used together with the above mentioned fact from [5] to complete the proof, see Theorem 2.2.

The paper is structured as follows. We state our main result, Theorem 2.2, in Section 2 and its consequences in Section 3. Section 4 contains the proof of Theorem 2.2. In Section 5 we state and prove Theorem 5.1 on the minimum value of the potential of a regular cross-polytope, which we use in Section 6 to prove consequences of Theorem 2.2. The known proof of Lemma 4.1 is given in Appendix (Section 7) for completeness.

2 Main result

Recall that a point configuration on $S^{d-1}$ is called antipodal if together with a point $x$ it contains $-x$. We say that a given point set in $\mathbb{R}^d$ is in general position if it is not contained in any hyperplane (or, equivalently, in any $(d-1)$-dimensional affine subspace). It will be convenient for us to recast the optimal-covering problem in terms of dot products.

Remark 2.1. A point configuration $\omega_N \subset S^{d-1}$ is a solution to Problem 1.1 if
and only if $\omega_N$ maximizes the quantity

$$\eta(\omega_N, S^{d-1}) := \min_{x \in S^{d-1}} \max_{i=1,N} x \cdot x_i$$

over all $N$-point configurations $\omega_N = \{x_1, \ldots, x_N\}$ on $S^{d-1}$.

We allow configurations, where points may coincide. However, if an antipodal configuration of $2d$ points on $S^{d-1}$ is in general position, its points are pairwise distinct.

The main result of this paper is the following.

**Theorem 2.2.** Let $d \geq 2$ and $\omega_{2d}$ be an antipodal configuration of $2d$ points on $S^{d-1}$. Then

$$\eta(\omega_{2d}, S^{d-1}) \leq \frac{1}{\sqrt{d}}. \quad (2)$$

Equality in (2) holds if and only if $\omega_{2d}$ is the set of vertices of a regular cross-polytope inscribed in $S^{d-1}$.

For $d = 3$ and 4, Theorem 2.2 is a special case of theorems proved in [16, 17] and in [14], respectively. For $d = 2$, it is a basic result.

### 3 A consequence of Theorem 2.2 for polarization

The best-covering problem is a limiting case of the polarization problem stated next. Let $f : [0, 4] \to (-\infty, \infty]$ be a function finite and continuous on $(0.4]$ such that $f(0) = \lim_{t \to 0^+} f(t)$. We will call $f$ a potential function. For a given point configuration $\omega_N = \{x_1, \ldots, x_N\} \subset S^{d-1}$, denote

$$P_f(\omega_N, S^{d-1}) := \min_{x \in S^{d-1}} \sum_{i=1}^N f \left( |x - x_i|^2 \right)$$

and let

$$P_f(S^{d-1}, N) := \sup_{\omega_N \subset S^{d-1}} P_f(\omega_N, S^{d-1}). \quad (3)$$

The (max-min) $N$-point polarization problem on the sphere is stated in the following way.

**Problem 3.1.** Find quantity (3) and $N$-point configurations $\omega_N^*$ on $S^{d-1}$ that attain the supremum on the right-hand side of (3).
In the case of a sphere, solution to Problem 3.1 is known on the unit circle $S^1$ for every $N \geq 1$ and $f(t) = t^{-s/2}$, $s > 0$, and $f(t) = -t^{-s/2}$, $-1 \leq s < 0$, which correspond to the Riesz potential, as well as for $f(t) = \frac{1}{2} \ln \frac{1}{t}$, which corresponds to the logarithmic potential, see the works by Stolarsky, Ambrus, Nikolov, Rafailov, Ball, Erdélyi, Saff, Hardin, and Kendall [26, 1, 22, 15, 20]. The solution is the set of vertices of a regular $N$-gon inscribed in $S^1$. In fact, paper [20] established this for arbitrary interactions given by a decreasing and convex function of the geodesic distance on $S^1$. Optimal $N$-point configurations for polarization on $S^1$ were also characterized by Bosuwan and Ruengrot [12] for the Riesz potential with $s = -2, -4, \ldots, 2 - 2N, N \geq 2$.

On the sphere $S^{d-1}, d \geq 3$, the solution to Problem 3.1 is known for $1 \leq N \leq d$ (basic result) and $N = d + 1$, see the works by Su [28] and the author [8] showing the optimality of a regular simplex. Also, for $N = 2d$, the optimality of a regular cross-polytope $\omega_{2d}$ was shown among centered configurations by Boyvalenkov, Dragnev, Hardin, Saff, and Stoyanova [13]. A configuration $\omega_{2d} \subset S^{d-1}$ is called centered if there is a point $y \in S^{d-1}$ such that $-\frac{1}{\sqrt{d}} \leq y \cdot x_i \leq \frac{1}{\sqrt{d}}$, $i = 1, \ldots, 2d$.

Other settings of polarization problem were studied in [26, 21, 15, 12, 3, 4, 9, 13]. More extensive reviews on polarization (including the continuous version and asymptotics) can be found, for example, in book [11].

Polarization problem can also be recast in terms of dot products. Let $g(t) := f(2 - 2t), t \in [-1, 1]$. Then $f$ is a potential function if and only if $g : [-1, 1] \to (-\infty, \infty]$ is a function finite and continuous on $[-1, 1]$ such that $g(1) = \lim_{t \to 1^-} g(t)$.

For every configuration $\omega_N = \{x_1, \ldots, x_N\} \subset S^{d-1}$, we have

\[ P^g(\omega_N, S^{d-1}) := \min_{x \in S^{d-1}} \sum_{i=1}^{N} g(x \cdot x_i) = \min_{x \in S^{d-1}} \sum_{i=1}^{N} f(|x - x_i|^2) = P_f(\omega_N, S^{d-1}). \]

Thus, a point configuration $\omega_N$ maximizes the quantity $P_f(\omega_N, S^{d-1})$ if and only if it maximizes the quantity $P^g(\omega_N, S^{d-1})$.

Inequality (2) in Theorem 2.2 is equivalent to the following statement.

**Corollary 3.2.** Any antipodal $2d$-point configuration on $S^{d-1}, d \geq 2$, is centered.

Paper [13] proved universal bounds for polarization and used them to show that regular cross-polytope $\omega_{2d}$ is optimal for polarization among all centered
configurations on $S^{d-1}$. This result together with Corollary 3.2 imply the following (the equality in (4) below follows from Theorem 5.1).

**Corollary 3.3.** Suppose $g : [-1, 1] \to (-\infty, \infty]$ is a function continuous on $[-1, 1)$ with $g(1) = \lim_{t \to 1^-} g(t)$ and differentiable on $(-1, 1)$ such that $g''$ is non-negative and convex on $(-1, 1)$. Suppose also that $\omega_{2d} \subset S^{d-1}, d \geq 2,$ is any antipodal $2d$-point configuration. Then

$$P^g(\omega_{2d}, S^{d-1}) \leq P^g(\overline{\omega_{2d}}, S^{d-1}) = d \left( g \left( \frac{1}{\sqrt{d}} \right) + g \left( -\frac{1}{\sqrt{d}} \right) \right). \quad (4)$$

Here we provide an elementary proof of Corollary 3.3 that avoids the use of universal bounds for polarization and is valid for any centered configuration. Concerning uniqueness of the optimal configuration, we have the following.

**Corollary 3.4.** Under assumptions of Corollary 3.3, if $g''$ is strictly positive on $(-1, 1)$, then equality holds throughout (4) if and only if $\omega_{2d}$ is the set of vertices of a regular cross-polytope inscribed in $S^{d-1}$.

We remark that a similar uniqueness question for centered configurations remains open. Observe that we do not require the potential function $g$ to be monotone in Corollaries 3.3 and 3.4.

### 4 Proof of Theorem 2.2

We need the following auxiliary statement proved by Böröczky in [5] (see also [6, Lemma 6.7.2]). Let $r : \mathbb{R}^d \setminus \{0\} \to S^{d-1}, r(x) = x/|x|,$ denote the radial projection onto $S^{d-1}$. We will call a $(d - 1)$-simplex a simplex in $\mathbb{R}^d$ with $d$ vertices having dimension $(d - 1)$.

**Lemma 4.1.** Let $H$ be a hyperplane in $\mathbb{R}^d, d \geq 2,$ with equation $x_d = a$, where $0 < a < 1$. Let $Y$ be a non-degenerate $(d - 1)$-simplex inscribed in $H \cap S^{d-1}$. Then the $(d - 1)$-dimensional volume of the radial projection $r(Y)$ of $Y$ onto $S^{d-1}$ is the largest if and only if $Y$ is a regular $(d - 1)$-simplex inscribed in $H \cap S^{d-1}$.

For completeness, we provide the proof of Lemma 4.1 in the Appendix.

Let $D(\omega_{2d})$ denote the convex hull of a point configuration $\omega_{2d} \subset S^{d-1}$. Points from $\omega_{2d}$ will also be called vertices. Recall that a hyperplane $\mathbf{x} \cdot \mathbf{a} = \alpha$ in $\mathbb{R}^d$ is called a supporting hyperplane for a convex body $B$ if for every $\mathbf{z} \in B$, we have $\mathbf{z} \cdot \mathbf{a} \leq \alpha$, while for some $\mathbf{y} \in B$, we have $\mathbf{y} \cdot \mathbf{a} = \alpha$. 

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Lemma 4.2. Let \( \omega_{2d} \subseteq S^{d-1}, d \geq 2 \), be an antipodal configuration of \( 2d \) points in general position. Then the interior of \( D(\omega_{2d}) \) contains the origin, and the boundary of \( D(\omega_{2d}) \) is the union of \( 2^d \) \((d-1)\)-simplices whose vertices are in \( \omega_{2d} \), pairwise intersections have \((d-1)\)-dimensional measure 0, and the hyperplane containing each simplex does not pass through the origin.

Proof. Pick an arbitrary subset of \( d \) linearly independent vectors from \( \omega_{2d} \) and denote it by \( \{y_1, \ldots, y_d\} \). Such a subset exists, since \( \omega_{2d} \) is in general position. Let \( \Pi := \{-1, 1\}^d \). Then any of the \( 2^d \) sets \( T_\sigma := \{\sigma_1 y_1, \ldots, \sigma_d y_d\} \), where \( \sigma = (\sigma_1, \ldots, \sigma_d) \in \Pi \), is a linearly independent subset of \( \omega_{2d} \) and, hence, is a set of vertices of a \((d-1)\)-simplex, which we will denote by \( F_\sigma \). For every \( \sigma \in \Pi \), the set \( T_\sigma \) is contained in a unique hyperplane \( H_\sigma \). Then the set \( T_\sigma = -T_\sigma \) is contained in the hyperplane \(-H_\sigma\) with \(-H_\sigma \neq H_\sigma\), since \( \omega_{2d} = T_\sigma \cup T_{-\sigma} \) is in general position. Then \( D(\omega_{2d}) \) is contained in the closed subset of \( \mathbb{R}^d \) bounded by \( H_\sigma \) and \(-H_\sigma\). Then the simplex \( F_\sigma \) is contained in the boundary \( \partial D(\omega_{2d}) \) of \( D(\omega_{2d}) \) for every \( \sigma \in \Pi \).

Since \( \omega_{2d} \) is in general position, the origin is in the interior of \( D(\omega_{2d}) \) as a point on the line segment joining two points in relative interiors of opposite facets.

Let now \( x \) be any point in \( \partial D(\omega_{2d}) \). Then \( x \neq 0 \) and there is a unique set of numbers \( \alpha_1, \ldots, \alpha_d \) such that \( x = \alpha_1 y_1 + \ldots + \alpha_d y_d \). For some \( \sigma = (\sigma_1, \ldots, \sigma_d) \in \Pi \), we have \( x = \beta_1 \sigma_1 y_1 + \ldots + \beta_d \sigma_d y_d \), where \( \beta_i := |\alpha_i| \geq 0 \), \( i = 1, \ldots, d \). Let \( \beta := \sum_{i=1}^d \beta_i \). Then \( \beta > 0 \) and \( (1/\beta) x \in F_\sigma \subseteq \partial D(\omega_{2d}) \). If it were that \( \beta > 1 \), then, since \( (1/\beta) x \in H_\sigma \), the origin and \( x \) would lie in different half-spaces relative to \( H_\sigma \), which is a supporting hyperplane for \( D(\omega_{2d}) \). Since the origin is in \( D(\omega_{2d}) \), we have \( x \notin D(\omega_{2d}) \); that is, \( x \notin \partial D(\omega_{2d}) \). If it were that \( \beta < 1 \), then \( x \) would be in the relative interior of the line segment joining \((1/\beta)x\) and \( 0 \). Since \((1/\beta)x\) is in \( D(\omega_{2d}) \) and \( 0 \) is in its interior, then \( x \) is also in the interior of \( D(\omega_{2d}) \); that is, \( x \notin \partial D(\omega_{2d}) \). This contradiction shows that \( \beta = 1 \). Then \( x \in F_\sigma \).

Thus, \( \partial D(\omega_{2d}) = \bigcup_{\sigma \in \Pi} F_\sigma \). Assume that the intersection of two simplices \( F_\sigma \) and \( F_{\sigma'} \), with \( \sigma = (\sigma_1, \ldots, \sigma_d) \neq \sigma' = (\sigma'_1, \ldots, \sigma'_d) \) is non-empty. For every point \( z \) in both simplices, there are numbers \( \beta_1, \ldots, \beta_d, \beta'_1, \ldots, \beta'_d \geq 0 \) such that

\[
\sum_{i=1}^d \beta_i = \sum_{i=1}^d \beta'_i = 1 \quad \text{and} \quad \sum_{i=1}^d \beta_i \sigma_i y_i = \sum_{i=1}^d \beta'_i \sigma'_i y_i.
\]

Since vectors \( y_1, \ldots, y_d \) are linearly independent, we have \( \beta_i \sigma_i = \beta'_i \sigma'_i, \quad i = 1, \ldots, d \).
1, . . . , d. Since \( \sigma_j \neq \sigma'_j \) for some \( j \), we have \( \beta_j = -\beta'_j \). In view of the non-negativity, we have \( \beta_j = \beta'_j = 0 \). Therefore, \( z \) belongs to a lower-dimensional face of \( F_\sigma \). Thus, \( \text{Vol}_{d-1}(F_\sigma \cap F_{\sigma'}) = 0 \). Furthermore, the hyperplane \( H_\sigma \) containing \( F_\sigma \) does not contain the origin, since if it did, vectors \( y_1, \ldots, y_d \) would be linearly dependent.

Recall that an intersection \( U \) of a convex polytope \( P \) with its supporting hyperplane, such that \( \text{dim } U = d - 1 \), is called a facet of \( P \). The polytope \( D(\omega_{2d}) \) in Lemma 4.2 has \( 2^d \) facets.

**Proof of Theorem 2.2.** We first verify that \( \eta(\overline{\omega}_{2d}, S^{d-1}) = \frac{1}{\sqrt{d}} \). Without loss of generality, we can assume that \( \overline{\omega}_{2d} = \tilde{\omega}_{2d} := \{ \pm e_1, \ldots, \pm e_d \} \), where \( e_1, \ldots, e_d \) are standard basis vectors in \( \mathbb{R}^d \). Then for every vector \( x = (x_1, \ldots, x_d) \in S^{d-1} \), we have \( \max_{i=1,d} x_i = \max_{i=1,d} |x_i| \geq \frac{1}{\sqrt{d}} \), since \( \sum_{i=1}^d |x_i|^2 = 1 \), with equality occurring whenever each coordinate of \( x \) is \( \pm \frac{1}{\sqrt{d}} \).

Assume to the contrary that there is an antipodal configuration \( \omega_{2d} \subset S^{d-1} \) such that \( \eta(\omega_{2d}, S^{d-1}) > 1/\sqrt{d} \). Then \( D(\omega_{2d}) \) contains \( 0 \) in its interior (if it did not, we would have \( \eta(\omega_{2d}, S^{d-1}) \leq 0 < 1/\sqrt{d} \)). Furthermore, \( \omega_{2d} \) is in general position and, hence, its points are pairwise distinct. Polytope \( D(\omega_{2d}) \) contains the sphere \( S \) of radius \( 1/\sqrt{d} \) centered at the origin. If it didn’t, then any point \( a \in S \setminus D(\omega_{2d}) \) would be strictly separated from \( D(\omega_{2d}) \) by some hyperplane \( L = \{ x : x \cdot v = c \} \), where we can take \( |v| = 1 \) and \( v \cdot a > c \). Then for every \( x_i \in \omega_{2d} \), we would have \( x_i \cdot v < c < a \cdot v \leq |a||v| = 1/\sqrt{d} \) which would contradict the contrary assumption.

By Lemma 4.2, boundary of \( D(\omega_{2d}) \) is the union of \((d - 1)\)-simplices (which we denote by \( F_1, \ldots, F_{2d} \)) with vertices in \( \omega_{2d} \). Their pairwise intersections have \((d - 1)\)-dimensional volume 0, and there are \( a_i > 0 \) and \( z_i \in S^{d-1}, i = 1, \ldots, 2^d \), such that the hyperplane \( H_i := \{ x : x \cdot z_i = a_i \} \) contains \( F_i \). Denote by \( k \) an index such that the radial projection \( r(F_k) \) onto \( S^{d-1} \) of the simplex \( F_k \) has \((d - 1)\)-dimensional volume at least \( 1/2^d \) of the \((d - 1)\)-dimensional volume of \( S^{d-1} \). If \( V \) denotes a regular \((d - 1)\)-simplex inscribed in \( H_k \cap S^{d-1} \), then in view of Lemma 4.1, we have

\[
2^{-d} \text{Vol}_{d-1}(S^{d-1}) \leq \text{Vol}_{d-1}(r(F_k)) \leq \text{Vol}_{d-1}(r(V)).
\]

Since \( H_k \) does not contain the origin (by Lemma 4.2), there are no antipodal pairs among the vertices of \( F_k \). Then the remaining \( d \) points from \( \omega_{2d} \)
are contained in the hyperplane \( x \cdot z_k = -a_k \). We have \( a_k = \max_{i=1,2d} z_k \cdot x_i \geq \eta(\omega_{2d}, S^{d-1}) > 1/\sqrt{d} \). The \((d - 1)\)-dimensional volume (denoted by \( \nu \)) of the radial projection \( r(W) \) of a regular simplex \( W \) inscribed in \( M \cap S^{d-1} \), where \( M = \{ x : x \cdot z_k = 1/\sqrt{d} \} \), will be strictly larger than \( \text{Vol}_{d-1}(r(V)) \). This is because \( a_k > 1/\sqrt{d} \) and the radius of the intersection of a hyperplane perpendicular to \( z_k \) with \( S^{d-1} \) decreases as the hyperplane moves further away from the origin. Since \( r(W) \) is the same as the radial projection of any facet of \( D(\omega_{2d}) \) and there are exactly \( 2^d \) facets, we have

\[
2^{-d} \text{Vol}_{d-1}(S^{d-1}) \leq \text{Vol}_{d-1}(r(V)) < \text{Vol}_{d-1}(r(W)) = 2^{-d} \text{Vol}_{d-1}(S^{d-1}).
\]

This contradiction proves (1).

To complete the proof of Theorem 2.2, assume that equality holds in (2). Then the origin is in the interior of \( D(\omega_{2d}) \). Assume to the contrary that there is a facet \( Q \) of \( D(\omega_{2d}) \) such that the hyperplane containing it is at a distance strictly greater than \( 1/\sqrt{d} \) from the origin. Then by Lemma 4.1, the radial projection of \( Q \) onto \( S^{d-1} \) has \((d - 1)\)-dimensional volume strictly less than \( \nu \). Equality in (2) implies that every other facet \( Y \) of \( D(\omega_{2d}) \) is contained in a hyperplane whose distance to the origin is at least \( 1/\sqrt{d} \). By Lemma 4.1, \( \text{Vol}_{d-1}(r(Y)) \leq \nu \) and, by Lemma 4.2, \( D(\omega_{2d}) \) has \( 2^d \) facets. Then the radial projection onto \( S^{d-1} \) of the whole boundary of \( D(\omega_{2d}) \) has \((d - 1)\)-dimensional volume strictly less than \( 2^d \nu = \text{Vol}_{d-1}(S^{d-1}) \). This contradiction shows that the hyperplane containing each facet of \( D(\omega_{2d}) \) is at a distance exactly \( 1/\sqrt{d} \) from the origin. Each facet of \( D(\omega_{2d}) \) must be a regular simplex. If some facet \( U \) were not, by Lemma 4.1, we would have \( \text{Vol}_{d-1}(r(U)) < \nu \) while for any other facet \( J \), we would have \( \text{Vol}_{d-1}(r(J)) \leq \nu \) leading to a similar contradiction. Let \( a_1, \ldots, a_d \) be the vertices of one of the facets of \( D(\omega_{2d}) \). Then \( \omega_{2d} = \{ \pm a_1, \ldots, \pm a_d \} \). It is not difficult to verify that \( a_i \cdot a_j = 0, i \neq j \). Then \( \{ a_1, \ldots, a_d \} \) is an orthonormal basis in \( \mathbb{R}^d \) and \( \omega_{2d} \) is the set of vertices of a regular cross-polytope inscribed in \( S^{d-1} \).

5 Minimum of the potential of a regular cross-polytope

In order to show the optimality of \( \omega_{2d} \) for Problem 3.1, one will need to know the quantity \( P_f(\omega_{2d}, S^{d-1}) \) by locating the absolute minima of the potential of \( \omega_{2d} \) on the sphere \( S^{d-1} \). This was done earlier for the vertices of a regular \( N \)-gon inscribed in \( S^1 \) and for the vertices of a regular simplex, cross-polytope, and
cube inscribed in $S^{d-1}$ for Riesz potential functions ($s \neq 0$) and their horizontal translations, see the works by Stolarsky, Nikolov, and Rafa ilov [26, 27, 22, 23]. For general potentials, this has been recently done in [8] for vertices of a regular simplex. Below, we extend one of the results of [27, 23] by finding the absolute minima of the potential of $\tilde{\omega}_{2d}$ for potential functions $g$ that have a convex second derivative. Our proof is different from the one in [27, 23] and uses polynomial interpolation and convexity of $g''$. Without loss of generality, we can assume in this section that $\tilde{\omega}_{2d} = \tilde{\omega}_{2d} = \{\pm e_1, \ldots, \pm e_d\}$, where $e_1, \ldots, e_d$ is the standard basis in $\mathbb{R}^d$.

**Theorem 5.1.** Let $d \geq 2$ and $g : [-1, 1] \to (-\infty, \infty]$ be a function continuous on $[-1, 1)$ and differentiable on $(-1, 1)$ such that $g(1) = \lim_{t \to 1^-} g(t)$ and $g''$ is convex on $(-1, 1)$. Then the potential

$$p^g(\tilde{\omega}_{2d}, x) := \sum_{y \in \tilde{\omega}_{2d}} g(x \cdot y)$$

achieves its absolute minimum over $S^{d-1}$ at any point of $S^{d-1}$ whose every co-
ordinate is $1/\sqrt{d}$ or $-1/\sqrt{d}$ (these points are vertices of the cube dual to $\tilde{\omega}_{2d}$). Furthermore,

$$P^g(\tilde{\omega}_{2d}, S^{d-1}) = d \left( g \left( \frac{1}{\sqrt{d}} \right) + g \left( -\frac{1}{\sqrt{d}} \right) \right).$$

We also remark that in the upcoming paper [10], we obtain the locations of absolute minima of the potential of any configuration on $S^{d-1}$ which is a tight spherical design of an even strength or a $(2m-1)$-design contained in the union of $m$ parallel hyperplanes. Such is, for example, $\omega_{2d}$ for $m = 2$.

**Proof of Theorem 5.1.** Let $h(t) := g(t) + g(-t)$. Then $h$ is even and $h''$ is convex on $(-1, 1)$. Let $p$ be the Hermite interpolating polynomial for $h$ at points $t_1 = -1/\sqrt{d}$ and $t_2 = 1/\sqrt{d}$. Then the even polynomial $(p(t) + p(-t))/2$ is also Hermite for $h$ at $t_1$ and $t_2$. By uniqueness, $p$ must be even. Since $p$ has degree at most 3, it has the form $p(t) = at^2 + b$. Furthermore, $\frac{a}{d} + b = p(1/\sqrt{d}) = h\left(\frac{1}{\sqrt{d}}\right)$.

We also have, $h(t) \geq p(t), t \in [-1, 1]$. Indeed, assume to the contrary that $v(t) := h(t) - p(t)$ is negative for some $t = t_0 \in (-1, 1)$. Note that $v(t_1) = v(t_2) = v'(t_1) = v'(t_2) = 0$. Then the Mean value theorem and the Rolle’s theorem imply that there are points $-1 < \tau_1 < \tau_2 < \tau_3 < 1$ such that
$v''(\tau_1) < 0$ and $v''(\tau_2) = v''(\tau_3) = 0$ (if $t_0 < t_1$) or $v''(\tau_1) < 0$, $v''(\tau_2) > 0$, and $v''(\tau_3) < 0$ (if $t_1 < t_0 < t_2$) or $v''(\tau_1) = v''(\tau_2) = 0$ and $v''(\tau_3) < 0$ (if $t_0 > t_2$). None of these cases is possible, since $h''$ (and, hence, $v''$) is convex on $(-1,1)$. Thus, $v$ is non-negative on $(-1,1)$; that is, $h(t) \geq p(t)$, $t \in (-1,1)$. We extend this inequality to the endpoints by passing to the limit.

Let $x = (x_1, \ldots, x_d)$ be an arbitrary point on $S^{d-1}$. Then

$$p^g(\omega_{2d}, x) = \sum_{i=1}^{d} (g(x_i) + g(-x_i)) = \sum_{i=1}^{d} h(x_i) \geq \sum_{i=1}^{d} p(x_i) = \sum_{i=1}^{d} (ax_i^2 + b) = a + d \cdot b = d \cdot h \left(\frac{1}{\sqrt{d}}\right) = d \left( g \left( \frac{1}{\sqrt{d}} \right) + g \left( -\frac{1}{\sqrt{d}} \right) \right) = p^g(\omega_{2d}, x^*)$$

where $x^*$ is any point on $S^{d-1}$ whose every coordinate is $1/\sqrt{d}$ or $-1/\sqrt{d}$. □

6 Proof of Corollaries 3.3 and 3.4

**Proof of Corollary 3.3.** We choose an arbitrary centered configuration $\omega_{2d} = \{y_1, \ldots, y_{2d}\} \subset S^{d-1}$ and let $x^* \in S^{d-1}$ be a point such that

$$-\frac{1}{\sqrt{d}} \leq x^* \cdot y_i \leq \frac{1}{\sqrt{d}}, \quad i = 1, \ldots, 2d.$$ (5)

In fact, $-\frac{1}{\sqrt{d}} \leq \pm x^* \cdot y_i \leq \frac{1}{\sqrt{d}}$ for all $i$. Let $-t_1 \leq \ldots \leq -t_{2d} \leq t_{2d} \leq \ldots \leq t_1$ denote all the dot products that $x^*$ and $-x^*$ form with points $y_i$. Since $g''$ is non-negative on $(-1,1)$, $g'$ is non-decreasing on $(-1,1)$, and, hence, the function $h(t) = g(t) + g(-t)$ is non-decreasing on $[0,1)$. Since $t_1, \ldots, t_{2d} \in [0, 1/\sqrt{d}]$, we have

$$P^g(\omega_{2d}, S^{d-1}) \leq \frac{1}{2} \sum_{i=1}^{2d} g(x^* \cdot y_i) + \frac{1}{2} \sum_{i=1}^{2d} g(-x^* \cdot y_i) = \frac{1}{2} \sum_{i=1}^{2d} (g(t_i) + g(-t_i))
= \frac{1}{2} \sum_{i=1}^{2d} h(t_i) \leq d \cdot h \left(\frac{1}{\sqrt{d}}\right) = d \left( g \left( \frac{1}{\sqrt{d}} \right) + g \left( -\frac{1}{\sqrt{d}} \right) \right) = P^g(\omega_{2d}, S^{d-1}),$$ (6)

where the last equality in (6) holds in view of Theorem 5.1. This proves (4) for any centered configuration $\omega_{2d}$. In view of Corollary 3.2, we have Corollary 3.3. □
Proof of Corollary 3.4. Assume that equality holds throughout (4) for a given antipodal configuration \( \omega_{2d} \subset S^{d-1} \). Since \( \omega_{2d} \) is antipodal, \( x^* \) can be chosen in the beginning of the proof of Corollary 3.3 with the additional property that \( \max_{i=1,2d} x^* \cdot y_i = \eta(\omega_{2d}, S^{d-1}) \) (inequalities (5) will then hold in view of Theorem 2.2). Since equality now holds throughout (6), we have \( \sum_{i=1}^{2d} h(t_i) = 2d \cdot h(1/\sqrt{d}). \) Since \( 0 \leq t_i \leq 1/\sqrt{d}, i = 1, \ldots, 2d \) and \( h \) is strictly increasing on \([0, 1]\), we have \( t_i = 1/\sqrt{d}, i = 1, \ldots, 2d. \) Then \( x^* \) forms only dot products \( 1/\sqrt{d} \) and \(-1/\sqrt{d}\) with points of \( \omega_{2d} \). Thus, \( \eta(\omega_{2d}, S^{d-1}) = \max_{i=1,2d} x^* \cdot y_i = 1/\sqrt{d}. \) By the uniqueness part of Theorem 2.2, configuration \( \omega_{2d} \) is the set of vertices of a regular cross-polytope inscribed in \( S^{d-1} \). \( \square \)

7 Appendix. Proof of Lemma 4.1

The assertion of Lemma 4.1 is trivial for \( d = 2 \). Therefore, we assume that \( d \geq 3 \). Let \( Y \) be a \((d-1)\)-simplex inscribed in \( H \cap S^{d-1} \) whose radial projection \( r(Y) \) onto \( S^{d-1} \) has the largest \((d-1)\)-dimensional volume. Let \( y_1, \ldots, y_d \) be the vertices of \( Y \). Let \( \Omega(Y) \) be the intersection of the closed unit ball \( B^d \) centered at the origin with the cone, denoted by \( \text{cone}\{y_1, \ldots, y_d\} \), constructed as the convex hull of the rays starting at \( 0 \) and passing through each vertex of \( Y \). Then \( \text{Vol}_{d-1}(r(Y)) = d \cdot \text{Vol}_d(\Omega(Y)). \)

Assume to the contrary that \( Y \) is not a regular simplex. Then there is a vertex of \( Y \) such that some two edges stemming out of it have non-equal lengths. Without loss of generality, we can assume that \( |y_3 - y_1| \neq |y_3 - y_2| \). Let \( L \) be the hyperplane that is the perpendicular bisector for the line segment with endpoints \( y_1 \) and \( y_2 \). Observe that \( 0 \in L \). Denote by \( y'_i \) the orthogonal projection of the point \( y_i \) onto \( L, i = 3, \ldots, d, \) and let \( Y' \) be the convex hull of \( \{y_1, y_2, y'_3, \ldots, y'_d\} \). Since \( y_1, y_2, y'_3, \ldots, y'_d \in H \cap B^d, \) we have \( Y' \subset H \cap B^d \) and \( r(Y') \subset C := \{(x_1, \ldots, x_d) \in S^{d-1} : x_d \geq a}\). Also, \( r(Y') = \text{cone}\{y_1, y_2, y'_3, \ldots, y'_d\} \cap S^{d-1} \).

Recall that a Steiner symmetrization of a set \( A \subset \mathbb{R}^d \) relative to a hyperplane \( L \) is the set

\[
\text{St}(A) := \bigcup_{\ell \in \mathcal{Q}} \left( \frac{1}{2} (A \cap \ell) + \frac{1}{2} (\tilde{A} \cap \ell) \right),
\]

where \( \mathcal{Q} \) is the set of all lines passing through points of \( A \) that are perpendicular to hyperplane \( L \) and \( \tilde{A} \) is the reflection of \( A \) with respect to \( L \). The Steiner sym-
metrization of cone\{y_1, \ldots, y_d\} is contained in cone\{y_1, y_2, y'_3, \ldots, y'_d\}. Since \(L\) passes through 0, whenever \(A, D \subset \mathbb{R}^d\) are such that \(\text{St}(A) \subset D\), we have \(\text{St}(A \cap B^d) \subset D \cap B^d\). Therefore, \(\text{St}(\Omega(Y)) \subset \Omega(Y')\), where \(\Omega(Y') := \text{cone}\{y_1, y_2, y'_3, \ldots, y'_d\} \cap B^d\). Since Steiner symmetrization preserves the volume, we have

\[
\text{Vol}_{d-1}(r(Y)) = d\text{Vol}_d(\Omega(Y)) = d\text{Vol}_d[\text{St}(\Omega(Y))] \\
\leq d\text{Vol}_d(\Omega(Y')) = \text{Vol}_{d-1}(r(Y')).
\]

Since the \((d - 1)\)-dimensional volume of \(r(Y)\) is positive, so is the one of \(r(Y')\) and, hence of \(Y'\). Then \(Y'\) is a \((d - 1)\)-simplex. Since \(y_3 \notin L\), its projection \(y'_3\) is an interior point of \(H \cap B^d\). Then there is a point \(y''_3 \in H \cap B^d\) into which we can move \(y'_3\) so that the new simplex \(Y''\) with vertices \(y_1, y_2, y'_3, y'_4, \ldots, y'_d\) contains \(Y'\) together with some open set that lies outside of \(Y'.\) Then we will have \(\text{Vol}_{d-1}(r(Y')) < \text{Vol}_{d-1}(r(Y''))\) with \(Y'' \subset H \cap B^d\). The simplex \(Y''\) is contained in some \((d - 1)\)-simplex \(Y'''\) inscribed in \(H \cap S^{d-1}\). Then \(\text{Vol}_{d-1}(r(Y)) < \text{Vol}_{d-1}(r(Y''''))\). This contradicts the assumption that \(r(Y)\) has the largest volume over all \((d - 1)\)-simplices \(Y\) inscribed in \(H \cap S^{d-1}\).

Thus, if \(\text{Vol}_{d-1}(r(Y))\) is maximal, then \(Y\) must be a regular simplex. At the same time, for all regular \((d - 1)\)-simplices \(V\) inscribed in \(H \cap S^{d-1}\), the volumes \(\text{Vol}_{d-1}(r(V))\) are the same. Then \(\text{Vol}_{d-1}(r(V))\) is maximal.

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