ON A CONJECTURE OF A BOUND FOR THE EXPONENT OF THE SCHUR MULTIPLIER OF A FINITE $p$-GROUP

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Communicated by

ABSTRACT. Let $G$ be a $p$-group of nilpotency class $k$ with finite exponent $\text{exp}(G)$ and let $m = \lfloor \log_p k \rfloor$. We show that $\text{exp}(\mathcal{M}^c(G))$ divides $\text{exp}(G)p^{m(k-1)}$, for all $c \geq 1$, where $\mathcal{M}^c(G)$ denotes the $c$-nilpotent multiplier of $G$. This implies that $\text{exp}(\mathcal{M}(G))$ divides $\text{exp}(G)$ for all finite $p$-groups of class at most $p-1$. Moreover, we show that our result is an improvement of some previous bounds for the exponent of $\mathcal{M}^c(G)$ given by M. R. Jones, G. Ellis and P. Moravec in some cases.

1. Introduction and Motivation

Let a group $G$ be presented as a quotient of a free group $F$ by a normal subgroup $R$. Then the $c$-nilpotent multiplier of $G$ (the Baer invariant of $G$ with respect to the variety of nilpotent group of class at most $c$) is defined to be

$$\mathcal{M}^c(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, eF]}.$$ 

This research was supported by a grant from Ferdowsi University of Mashhad; (No. MP87150MSH).

MSC(2000): Primary: 20C25, 20D15; Secondary: 20E10, 20F12.

Keywords: Schur multiplier, Nilpotent multiplier, Exponent, Finite $p$-groups.

Received: April 17, 2009, Accepted: August 1, 2010.

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where \([R, c\times F]\) denotes the commutator subgroup \(\underbrace{[R, F, ..., F]}_{c\text{-times}}\) and \(c \geq 1\).

The case \(c = 1\) which has been much studied is the Schur multiplier of \(G\), denoted by \(M(G)\). When \(G\) is finite, \(M(G)\) is isomorphic to the second cohomology group \(H^2(G, \mathbb{C}^*)\) (see G. Karpilovsky [6] and C. R. Leedham-Green and S. McKay [8] for further details).

It has been interested to finding a relation between the exponent of \(M^{(c)}(G)\) and the exponent of \(G\). Let \(G\) be a finite \(p\)-group of nilpotency class \(k \geq 2\) with exponent \(\exp(G)\). M. R. Jones [5] proved that \(\exp(M(G))\) divides \(\exp(G)^{k-1}\). This has been improved by G. Ellis [3] who showed that \(\exp(M^{(c)}(G))\) divides \(\exp(G)^{\lceil k/2 \rceil}\), where \(\lceil k/2 \rceil\) denotes the smallest integer \(n\) such that \(n \geq k/2\). For \(c = 1\), P. Moravec [11] showed that \(\lceil k/2 \rceil\) can be replaced by \(2\lceil \log_2 k \rceil\) which is an improvement if \(k \geq 11\).

In this paper we will show that if \(G\) is a finite exponent \(p\)-group of class \(k \geq 1\), then \(\exp(M(G))\) divides \(\exp(G)^{p^m(k-1)}\), for all \(c \geq 1\), where \(m = \lfloor \log_p k \rfloor\). Note that this result is an improvement of the results of Jones, Ellis and Moravec if \(\lfloor \log_p k \rfloor(k-1)/k < e\), \(\lfloor \log_p k \rfloor(k-1)/\lceil k/2 \rceil - 1 < e\), \(\lfloor \log_p k \rfloor(k-1)/2\lfloor \log_2 k \rfloor - 1 < e\), respectively, where \(\exp(G) = p^e\).

It was a longstanding open problem as to wether \(\exp(M(G))\) divides \(\exp(G)\) for every finite group \(G\). In fact it was conjectured that the exponent of the Schur multiplier of a finite \(p\)-group is a divisor of the exponent of the group itself. I. D. Macdonald and J. W. Wamsley [1] constructed an example of a group of order \(2^{21}\) which has exponent 4, whereas its Schur multiplier has exponent 8, hence the conjecture is not true in general. Also Moravec [12] gave an example of a group of order 2048 and nilpotency class 6 which has exponent 4 and multiplier of exponent 8. He also proved that if \(G\) is a group of exponent 4, then \(\exp(M(G))\) divides 8. Nevertheless, Jones [5] has shown that the conjecture is true for \(p\)-groups of class 2 and emphasized that it is true for some \(p\)-groups of class 3. S. Kayvanfar and M. A. Sanati [7] have proved the conjecture for \(p\)-groups of class 4 and 5, with some conditions. A. Lubotzky and A. Mann [9] showed that the conjecture is true for powerful \(p\)-groups. The first and the third authors [10] showed that the conjecture is true for nilpotent multipliers of powerful \(p\)-groups. Finally, Moravec [11, 12] showed that the conjecture is true for metabelian groups of exponent \(p\), \(p\)-groups with potent filtration and \(p\)-groups of maximal
class. Note that a consequence of our result shows that the conjecture is true for all finite $p$-groups of class at most $p - 1$.

2. Preliminaries

In this section, we are going to recall some notions we will use in the next section.

**Definition 2.1.** (M. Hall [4]). Let $X$ be an independent subset of a free group, and select an arbitrary total order for $X$. We define the basic commutators on $X$, their weight $\text{wt}$, and the ordering among them as follows:

1. The elements of $X$ are basic commutators of weight one, ordered according to the total order previously chosen.
2. Having defined the basic commutators of weight less than $n$, the basic commutators of weight $n$ are the $c_k = [c_i, c_j]$, where:
   - (a) $c_i$ and $c_j$ are basic commutators and $\text{wt}(c_i) + \text{wt}(c_j) = n$, and
   - (b) $c_i > c_j$, and if $c_i = [c_s, c_t]$, then $c_j \geq c_t$.
3. The basic commutators of weight $n$ follow those of weight less than $n$. The basic commutators of weight $n$ are ordered among themselves lexicographically; that is, if $[b_1, a_1]$ and $[b_2, a_2]$ are basic commutators of weight $n$, then $[b_1, a_1] \leq [b_2, a_2]$ if and only if $b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$.

**Lemma 2.2.** (R. R. Struik [13]). Let $x_1, x_2, ..., x_r$ be any elements of a group and let $v_1, v_2, ...$ be the sequence of basic commutators of weight at least two in the $x_i$'s, in ascending order. Then

$$(x_1 x_2 ... x_r)^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_i^{\alpha_i} v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} ... v_i^{f_i(\alpha)} ...,$$

where $\{i_1, i_2, ..., i_r\} = \{1, 2, ..., r\}$, $\alpha$ is a nonnegative integer and

$$f_i(\alpha) = a_1 \left(\frac{\alpha}{1}\right) + a_2 \left(\frac{\alpha}{2}\right) + ... + a_{w_i} \left(\frac{\alpha}{w_i}\right), \quad (I)$$

with $a_1, ..., a_{w_i} \in \mathbb{Z}$ and $w_i$ is the weight of $v_i$ in the $x_i$'s.

**Lemma 2.3.** (Struik [13]). Let $\alpha$ be a fixed integer and $G$ be a nilpotent group of class at most $k$. If $b_1, ..., b_r \in G$ and $r < k$, then

$$[b_1, ..., b_{i-1}, b_i^\alpha, b_{i+1}, ..., b_r] = [b_1, ..., b_r]^\alpha v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} ...,$$
where $\nu_i$’s are commutators in $b_1, \ldots, b_r$ of weight strictly greater than $r$, and every $b_j$, $1 \leq j \leq r$, appears in each commutator $\nu_i$, the $\nu_i$’s listed in ascending order. The $f_i(\alpha)$’s are of the form (I), with $a_1, \ldots, a_{w_i} \in \mathbb{Z}$ and $w_i$ is the weight of $\nu_i$ (in the $b_j$’s) minus $(r-1)$.

Remark 2.4. Outer commutators on the letters $x_1, x_2, \ldots, x_n, \ldots$ are defined inductively as follows:

The letter $x_i$ is an outer commutator word of weight one. If $u = u(x_1, \ldots, x_s)$ and $v = v(x_{s+1}, \ldots, x_{s+t})$ are outer commutator words of weights $s$ and $t$, then $w(x_1, \ldots, x_{s+t}) = [u(x_1, \ldots, x_s), v(x_{s+1}, \ldots, x_{s+t})]$ is an outer commutator word of weight $s+t$ and will be written $w = [u, v]$.

It is noted by Struik [13] that Lemma 2.3 can be proved by a similar method if $[b_1, \ldots, b_{i-1}, b_\alpha i, b_i+1, \ldots, b_r]$ and $[b_1, \ldots, b_r]$ are replaced with outer commutators.

By a routine calculation we have the following useful fact.

Lemma 2.5. Let $p$ be a prime number and $k$ be a nonnegative integer. If $m = \lfloor \log_p k \rfloor$, then $p^t$ divides $\binom{p^m+t}{k}$, for all integers $t \geq 1$.

3. Main Results

In order to prove the main result we need the following lemma.

Lemma 3.1. Let $G$ be a $p$-group of class $k$ and exponent $p^e$ with a free presentation $F/R$. Then for any $c \geq 1$, every outer commutator of weight $w > c$ in $F/[R, cF]$ has an order dividing $p^{e+m(c+k-w)}$, where $m = \lfloor \log_p k \rfloor$.

Proof. Since $\gamma_{k+1}(F) \subseteq R$, we have $\gamma_{c+k+1}(F) \subseteq [R, cF]$. Also, for all $x$ in $F$ and $t \geq 0$ we have $x^{p^{c+t}} \in R$ and hence every outer commutator of weight $w > c$ in $F$, in which $x^{p^{e+t}}$ appears, belongs to $[R, cF]$. Now we use inverse induction on $w$ to prove the lemma. For the first step, $w = c+k$, the result follows by the above argument and Lemma 2.3. Now assume that the result is true for all $l > w$. Put $\alpha = p^{e+m(c+k-w)}$ and let $u = [x_1, \ldots, x_w]$ be an outer commutator of weight $w$. Then by Lemma 2.3 and Remark 2.4 we have

$$[x_1^\alpha, \ldots, x_w] = [x_1, \ldots, x_w]^\alpha v_1 f_1(\alpha) v_2 f_2(\alpha) \ldots,$$
where the $v_i^{f_i(\alpha)}$ are as in Lemma 2.3. Note that $w < w_i = wt(v_i) \leq c+k$ modulo $[R, cF]$ and hence $f_i(\alpha) = a_1(\alpha) + a_2(\alpha) + \ldots + a_{w_i}(\alpha)$, where $k_i = w_i - w + 1 \leq c + k - w + 1 \leq k$, for all $i \geq 1$. Thus Lemma 2.5 implies that $p^{e+m(c+k-w-1)}$ divides the $f_i(\alpha)$’s. Now by induction hypothesis $v_i^{f_i(\alpha)} \in [R, cF]$, for all $i \geq 1$. On the other hand, since $x_i^\alpha \in R$ and $w > c$, $[x_1^\alpha, \ldots, x_w] \in [R, cF]$. Therefore $u^\alpha \in [R, cF]$ and this completes the proof.

Theorem 3.2. Let $G$ be a $p$-group of class $k$ and exponent $p^e$. Let $G = F/R$ be any free presentation of $G$. Then the exponent of $\gamma_{c+1}(F)/[R, cF]$ divides $p^{e+m(k-1)}$, where $m = [\log_p k]$, for all $c \geq 1$.

Proof. It is easy to see that every element $g$ of $\gamma_{c+1}(F)$ can be expressed as $g = y_1 y_2 \ldots y_n$, where $y_i$’s are commutators of weight at least $c+1$. Put $\alpha = p^{e+m(k-1)}$. Now Lemma 2.2 implies the identity

$$g^\alpha = y_1^\alpha y_2^\alpha \ldots y_n^\alpha v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} \ldots,$$

where $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$ and $v_i^{f_i(\alpha)}$’s are as in Lemma 2.2. Then the $v_i$’s are basic commutators of weight at least two and at most $k$ in the $y_i$’s modulo $[R, cF]$ (note that $\gamma_{c+k+1}(F) \subseteq [R, cF]$). Thus Lemma 2.5 yields that $p^{e+m(k-2)}$ divides the $f_i(\alpha)$’s. Hence $v_i^{f_i(\alpha)} \in [R, cF]$, for all $i \geq 1$ and $y_j^\alpha \in [R, cF]$, for all $1 \leq j \leq n$, by Lemma 3.1. Therefore we have $g^\alpha \in [R, cF]$ and the desired result now follows.

Now, we are in a position to state and prove the main result of the paper.

Theorem 3.3. Let $G$ be a $p$-group of class $k$ and exponent $p^e$. Then $\exp(M^{(c)}(G))$ divides $\exp(G)p^{m(k-1)}$, where $m = [\log_p k]$, for all $c \geq 1$.

Proof. Let $G = F/R$ be any free presentation of $G$. Then $M^{(c)}(G) \leq \gamma_{c+1}(F)/[R, cF]$. Therefore $\exp(M^{(c)}(G))$ divides $\exp(\gamma_{c+1}(F)/[R, cF])$. Now the result follows by Theorem 2.3.

Note that the above result improves some previous bounds for the exponent of $M(G)$ and $M^{(c)}(G)$ as follows.

Let $G$ be a $p$-group of class $k$ and exponent $p^e$, then we have the following improvements.
(i) If $\lfloor \log_p k \rfloor (k - 1)/k < e$, then $\exp(G)p^{\lfloor \log_p k \rfloor (k - 1)} < \exp(G)^{k-1}$. Hence in this case our result is an improvement of Jones’s result [5]. In particular our result improves the Jones’s one for every $p$-group of exponent $p^e$ and of class at most $p^e - 1$.

(ii) If $\lfloor \log_p k \rfloor (k - 1)/k < e$, then $\exp(G)p^{\lfloor \log_p k \rfloor (k - 1)} < \exp(G)^{k-1}$ which shows that in this case our result is an improvement of Ellis’s result [3]. In particular our result improves the Ellis’s one for every $p$-group of exponent $p^e$ and of class $k < p^{e/3}$, for all $k \geq 3$, or of class $k < p^{e/4}$, for all $k \geq 4$.

(iii) If $\lfloor \log_p k \rfloor (k - 1)/2 \lfloor \log_2 k \rfloor - 1 < e$, then $\exp(G)p^{\lfloor \log_p k \rfloor (k - 1)} < \exp(G)^{2 \lfloor \log_2 k \rfloor}$ which shows that in this case our result is an improvement of Ellis’s result [3]. In particular our result improves the Ellis’s one for every $p$-group of exponent $p^e$ and of class $k < e$, for all $k \geq 2$.

**Corollary 3.4.** Let $G$ be a finite $p$-group of class at most $p - 1$, then $\exp(M^{(c)}(G))$ divides $\exp(G)$, for all $c \geq 1$. In particular $\exp(M(G))$ divides $\exp(G)$.

Note that the above corollary shows that the mentioned conjecture on the exponent of the Schur multiplier of a finite $p$-group holds for all finite $p$-group of class at most $p - 1$.

**Remark 3.5.** Let $G$ be a finite nilpotent group of class $k$. Then $G$ is the direct product of its Sylow subgroups, $G = S_{p_1} \times \cdots \times S_{p_n}$. Clearly

$$\exp(G) = \prod_{i=1}^{n} \exp(S_{p_i}).$$

By a result of G. Ellis [2, Theorem 5] we have

$$M^{(c)}(G) = M^{(c)}(S_{p_1}) \times \cdots \times M^{(c)}(S_{p_n}).$$

For all $1 \leq i \leq n$, put $m_i = \lfloor \log_{p_i} k \rfloor$. Then by Theorem 3.3 we have

$$\exp(M^{(c)}(G)) \mid \exp(G) \prod_{i=1}^{n} p_i^{m_i(k-1)}.$$

Hence the conjecture on the exponent of the Schur multiplier holds for all finite nilpotent group $G$ of class at most $\text{Max}\{p_1 - 1, \ldots, p_n - 1\}$, where $p_1, \ldots, p_n$ are all the distinct prime divisors of the order of $G$. 
Acknowledgments

The authors would like to thank the referee for useful comments.

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