ON CERTAIN CLASSES OF TOTALLY DISCONNECTED LOCALLY
COMPACT GROUPS THAT HAVE A RIGID GROUP TOPOLOGY

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Abstract. In [2] Kramer proves for a large class of semisimple Lie groups that they admit just one locally compact \(\sigma\)-compact Hausdorff topology compatible with the group operations. We present two different ways of generalising this to the group of rational points of an absolutely quasi-simple algebraic group over a non-archimedean local field (in the second case, it is necessary to assume that the group is centreless and non-compact). The first method of argument involves demonstrating that, given any topological group \(G\) which is totally disconnected, locally compact, \(\sigma\)-compact, locally topologically finitely generated, and has the property that no compact open subgroup has an infinite abelian continuous quotient, then group \(G\) is topologically rigid in the previously described sense. Then the desired conclusion for the group of rational points of an absolutely quasi-simple algebraic group over a non-archimedean local field may be inferred as a special case. The other method of argument involves proving that any group of automorphisms of a regular locally finite building, which is closed in the compact-open topology and acts Weyl transitively on the building, has the topological rigidity property in question. This again yields the desired result in the case that the group is centreless and non-compact.

In [2] Kramer explores the question of when a semisimple Lie group \(G\) admits just one locally compact \(\sigma\)-compact Hausdorff group topology, or, to put it another way, when it is the case that any abstract isomorphism \(\varphi : \Gamma \rightarrow G\) whose domain \(\Gamma\) is a locally compact \(\sigma\)-compact Hausdorff topological group is necessarily a homeomorphism. He proves that this is the case for a connected centreless real semisimple Lie group for which all the simple ideals in the Lie algebra are absolutely simple (in the sense that the result of taking the tensor product of the real lie algebra with \(\mathbb{C}\) yields a simple complex Lie algebra). In this paper we wish to discuss two different ways by which this result can be generalised to the group of rational points of an absolutely quasi-simple algebraic group over a non-archimedean local field.

The following theorems play a key role in Linus Kramer’s argument regarding the Lie group case.

**Theorem 0.1** (Open Mapping Theorem). Let \(\psi : G \rightarrow H\) be a surjective continuous homomorphism between locally compact Hausdorff topological groups. If \(G\) is \(\sigma\)-compact, then \(\psi\) is an open map.

*Proof.* See Hewitt and Ross [1], II 5. 29 or Stroppel [?], 6.19. \(\square\)

**Remark 0.2.** In the statement of the theorem in [2] the hypothesis is given as “\(H\) is \(\sigma\)-compact”. This would appear to be a typographical error.

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Theorem 0.3 (Automatic Continuity). Suppose that $G$ is a locally compact Hausdorff topological group and that $H$ is a $\sigma$-compact Hausdorff topological group. Assume that $\psi : G \rightarrow H$ is a group homomorphism which is also a Borel map, i.e. that the preimage of every open set $U \subseteq H$ is a Borel set. Then $\psi$ is continuous.

Proof. This is a special case of Hewitt and Ross [1], V.22.18; see also Kleppner [?], Thm. 1. □

These two results yield an important technique for verifying the topological rigidity result for a specific locally compact $\sigma$-compact Hausdorff topological group $G$. Suppose that we have some base of open neighbourhoods of the identity in $G$, and we are able to prove for each member of the base that it is unconditionally Borel, that is, it is a Borel set with respect to any locally compact $\sigma$-compact Hausdorff topology compatible with the group operations on $G$. Then it follows that we may conclude that $G$ is topologically rigid in the previously described sense. For suppose that we are given some abstract isomorphism $\varphi : \Gamma \rightarrow G$ where the domain $\Gamma$ is a locally compact $\sigma$-compact Hausdorff topological group. Then for each member $K$ of the base of open neighbourhoods of the identity in question, we have that $\varphi^{-1}(K)$ is Borel. From this it follows that $\varphi$ is Borel measurable. We can now conclude from the two results given above that it is open and continuous and therefore a homeomorphism. The conclusion that an abstract isomorphism $\varphi$ of the kind described is always a homeomorphism is another way to state the topological rigidity result, so this completes the argument.

We now come to the proof of our first main theorem.

Theorem 0.4. Let $G$ be a totally disconnected locally compact $\sigma$-compact topological group that is locally topologically finitely generated and has the property that no compact open subgroup has an infinite abelian continuous quotient. Then $G$ admits just one locally compact $\sigma$-compact Hausdorff topology compatible with the group operations.

Proof. Call the given topology on $G$ the “standard topology”. Suppose that there were an exotic topology on $G$, different to the standard topology, that was locally compact, $\sigma$-compact, Hausdorff, and compatible with the group operations. This exotic topology would have to be also totally disconnected. For otherwise some compact open subgroup $K$ in the standard topology would have to have a subgroup which was a countable-index subgroup of a pro-Lie group $H$. But then by considering the image under the exponential map of a one-dimensional subalgebra of the pro-Lie algebra $\mathfrak{h}$ of $H$, we get that $H$ has a subgroup abstractly isomorphic to a one-dimensional Lie group, and so $K$ has a subgroup abstractly isomorphic to a countable-index subgroup of a one-dimensional Lie group. In particular it has a subgroup isomorphic to $\mathbb{Q}$, but $\mathbb{Q}$ has no proper finite-index subgroups so this is a contradiction. So by means of this argument we may assume that the exotic topology is totally disconnected, and therefore the compact open subgroups form a base of open neighbourhoods of the identity, by von Dantzig’s theorem.

Let $K$ be a compact open subgroup of $G$ in the exotic topology and let $H$ be a compact open subgroup of $G$ in the standard topology. $H$ is topologically finitely generated, locally topologically finitely generated, and has no infinite virtually abelian continuous quotients, and so, in particular, by a result of Nikolov and Segal [4], it has no countably infinite abstract
Next we examine the consequences for the group of rational points $G(k)$ of an absolutely quasi-simple algebraic group $G$ over a non-archimedean local field $k$.

It follows from the classification of the semisimple algebraic groups over local fields [6] that we may choose a global field $K$ contained in $k$ such that $G$ is defined over $K$ and $k$ is the completion of $K$ at a valuation $v$. Let $S$ be a finite set of places of $K$ containing all the Archimedean ones, and containing a non-archimedean place different to $v$, but not containing $v$, and such that $G(O_S)$ has rank at least two, where by the rank of $G(O_S)$ we mean the sum of the ranks of $G(K_{v'})$ where $v'$ ranges over a set of valuations representing each place in $S$. By strong approximation, $G(O_S)$ is dense in the compact open subgroup $G(O_k)$. If $U \subseteq G(k)$ is any compact open subgroup of $G(k)$, then the intersection $\Gamma := G(O_S) \cap U$ has finite index in $G(O_S)$. Furthermore, since $G(O_S)$ has rank at least two, it is finitely generated. Thus $\Gamma$ is finitely generated and dense in $U$, so that $U$ is topologically finitely generated. Thus we have shown that the groups $G(k)$ of the kind described are locally topologically finitely generated. Also, by the Margulis normal subgroup theorem, $G(O_S)$ does not have an infinite abelian quotient, so therefore the group $U$ does not have an infinite abelian continuous quotient. I am grateful to Tyakal Nanjundiah Venkataramana for suggesting this argument. Thus we obtain as a corollary to the previous theorem

**Theorem 0.5.** Let $G$ be an absolutely quasi-simple algebraic group over a non-archimedean local field $k$. Then the group of rational points $G(k)$ admits exactly one locally compact $\sigma$-compact Hausdorff topology compatible with the group operations.

Next we present an argument showing topological rigidity for closed Weyl transitive groups of automorphisms of a regular locally finite building.

**Definition 0.6.** A pair $(W, S)$ such that $W$ is an abstract group and $S$ is a set of generators of $W$ of order two is said to be a Coxeter system if $W$ admits the presentation $(S; (st)^m(s, t) = 1)$ where $m(s, t)$ is the order of $st$ and there is one relation for each pair $s, t$ with $m(s, t) < \infty$. 
Definition 0.7. Suppose that \((W, S)\) is a Coxeter system. A building of type \((W, S)\) is a pair \((C, \delta)\) consisting of a nonempty set \(C\), whose elements are called chambers, together with a map \(\delta : C \times C \to W\) called the Weyl distance function, such that for all \(C, D \in C\), the following three conditions hold:

1. \(\delta(C, D) = 1\) if and only if \(C = D\).
2. If \(\delta(C, D) = w\) and \(C' \in C\) satisfies \(\delta(C', C) = s \in S\), then \(\delta(C', D) = sw\) or \(w\). If, in addition, \(l(sw) = l(w) + 1\), then \(\delta(C', D) = sw\).
3. If \(\delta(C, D) = w\), then for any \(s \in S\) there is a chamber \(C' \in C\) such that \(\delta(C', C) = s\) and \(\delta(C', D) = sw\).

Definition 0.8. Suppose that a group \(G\) acts by isometries (that is, bijections preserving the Weyl distance function) on a building \(\Delta\) of type \((W, S)\). The action is said to be Weyl transitive if it is transitive on the set of ordered pairs of chambers \((C, D)\) such that \(\delta(C, D) = w\) for each fixed \(w \in W\).

Definition 0.9. Two chambers \(C, D\) in a building of type \((W, S)\) are said to be adjacent if and only if \(\delta(C, D) = s\) for some \(s \in S\). A building is said to be locally finite if the number of chambers adjacent to any given chamber is finite. A building is said to be regular if the number of chambers \(D\) such that \(\delta(C, D) = s\) is the same for all chambers \(C\), for each \(s \in S\).

Definition 0.10. Suppose that \(\Delta\) is a regular locally finite building of type \((W, S)\). The compact-open topology on the full automorphism group \(G\) of \(\Delta\) is the topology such that the class of all pointwise stabilisers of finite sets of chambers is a base for the topology.

Theorem 0.11. Suppose that \(\Delta\) is a regular locally finite building of type \((W, S)\). Let \(H\) be a group of automorphisms of \(\Delta\) which is closed in the compact-open topology and which acts Weyl transitively on \(\Delta\). Equip \(H\) with the compact-open topology. Suppose that \(\Gamma\) is a locally compact \(\sigma\)-compact Hausdorff topological group and \(\varphi : \Gamma \to H\) is an abstract isomorphism. Then \(\varphi\) is a homeomorphism.

Proof. Suppose for a contradiction that the group \(H\) admitted an exotic topology other than the compact-open topology, which was locally compact, \(\sigma\)-compact, Hausdorff, and compatible with the group operations. First we shall prove that the pointwise stabiliser of any finite set of chambers must be dense in the exotic topology.

Let \(C\) be a fixed chamber and let \(H_C\) denote the stabiliser of \(C\) in \(H\). From the Weyl transitivity of the action of \(H\) we can infer that \(H_C\) is either closed or dense in the exotic topology on \(H\). We argue this point as follows. Suppose that the closure of \(H_C\) in the exotic topology (which we shall denote by \(\overline{H_C}\)) were strictly larger than \(H_C\). Then we would have an element \(h \in \overline{H_C}\) with the property that \(C \neq D := h(C)\). Since \(\overline{H_C}\) is a subgroup of \(H\), it would then follow that \(H_D \subseteq \overline{H_C}\). Hence \(\overline{H_C}\) would contain the orbit of \(h^{-1}\) under conjugation by \(H_D\). Thus it would contain elements mapping \(D\) to any chamber the same Weyl distance from \(D\) as \(C\). In particular it will contain an element mapping \(D\) to at least one chamber adjacent to \(C\). Thus it follows that \(\overline{H_D}\) would contain an element mapping \(C\) to some chamber adjacent to \(C\), and therefore would contain elements mapping \(C\) to every chamber adjacent to \(C\). Now it follows by induction that \(\overline{H_C}\) is equal to all of \(H\).
We have established that any chamber stabiliser is either closed or dense in the exotic topology on $H$. Now we wish to show that the pointwise stabiliser of any finite set of chambers is dense in the exotic topology on $H$. We can see this as follows. Suppose that $F$ is a finite set of chambers. The closure of the pointwise stabiliser of $F$ in the exotic topology on $H$ would have to have finite index in $H$. In particular if we let $C$ be a chamber such that $C \in F$ then the orbit of $C$ under the closure of the pointwise stabiliser of $F$ would have to be unbounded, and the closure of the pointwise stabiliser of $F$ would have to contain the pointwise stabiliser of each one of a family of finite sets of chambers containing respectively the elements of the unbounded orbit in question. This shows that the closure must be all of $H$. (What we have just done is equivalent to proving that $H$ cannot have any proper finite-index subgroups.)

It can be seen that the exotic topology on $H$ must be totally disconnected. For if not, then $H$ would have a subgroup isomorphic to $\mathbb{Q}$, but this is not possible for a group which acts faithfully on a regular locally finite building. So it follows that the compact open subgroups of $H$ in the exotic topology form a base of open neighbourhoods of the identity in the exotic topology, by van Dantzig’s theorem. Now let $K$ be a compact open subgroup of $H$ in the exotic topology, and let $H_C$ be the stabiliser in $H$ of a chamber $C$. $H_C \cap K$ has countable index in both $H_C$ and $K$. Since every pointwise stabiliser of a finite set of chambers is dense in the exotic topology, it follows that $H_C \cap K$ is a dense subgroup of $H_C$ in the compact-open topology. Hence $H_C \cap K$ acts transitively on the chambers a fixed Weyl distance from $C$. It follows that the action of $K$ on $\Delta$ is Weyl transitive, and, by similar reasoning to before, that $K$ has no proper finite-index subgroups, but this is a contradiction. We conclude that it is not possible for the exotic topology to be different to the compact-open topology. This completes the proof of the result. \qed
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