BREAK-DOWN CRITERION FOR THE WATER-WAVE EQUATION

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Abstract. We study the break-down mechanism of smooth solution for the gravity water-wave equation of infinite depth. It is proved that if the mean curvature $\kappa$ of the free surface $\Sigma_t$, the trace $(V, B)$ of the velocity at the free surface, and the outer normal derivative $\frac{\partial P}{\partial n}$ of the pressure $P$ satisfy

$$\sup_{t \in [0, T]} \|\kappa(t)\|_{L^p \cap L^2} + \int_0^T \| \langle \nabla V, \nabla B \rangle(t) \|_{L^\infty} \, dt < +\infty,$$

$$\inf_{(t, x, y) \in [0, T] \times \Sigma_t} \frac{\partial P}{\partial n}(t, x, y) \geq c_0,$$

for some $p > 2d$ and $c_0 > 0$, then the solution can be extended after $t = T$.

1. Introduction

1.1. Presentation of the problem. In this paper, we are concerned with the motion of an ideal, incompressible, irrotational gravity fluid in a domain with free boundary of infinite depth:

$$\{(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} : (x, y) \in \Omega_t\},$$

where $\Omega_t$ is the fluid domain at time $t$ located by the free surface

$$\Sigma_t = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y = \eta(t, x)\}.$$

where $t, x, y$ denote the time variable, the horizontal and vertical spatial variables respectively. Throughout this paper, we will use the notations:

$$\nabla = (\partial_{x_i})_{1 \leq i \leq d}, \quad \nabla_{x,y} = (\nabla, \partial_y), \quad \Delta = \sum_{1 \leq i \leq d} \partial_{x_i}^2, \quad \Delta_{x,y} = \Delta + \partial_y^2.$$

The motion of the fluid is described by the incompressible Euler equation

$$\partial_t v + v \cdot \nabla_{x,y} v = -\nabla_{x,y} P \quad \text{in} \quad \Omega_t, \quad t \geq 0, \quad (1.1)$$

where $-\nabla_{x,y} P = \begin{pmatrix} 0, \cdots, 0, -g \end{pmatrix}$ denotes the acceleration of gravity and $v = (v_1, \cdots, v_d, v_{d+1})$ denotes the velocity field. The incompressibility of the fluid is expressed by

$$\text{div} \, v = 0 \quad \text{in} \quad \Omega_t, \quad t \geq 0, \quad (1.2)$$

and the irrotationality means that

$$\text{curl} \, v = 0 \quad \text{in} \quad \Omega_t, \quad t \geq 0. \quad (1.3)$$

At the free surface, the boundary conditions are given by

$$\partial_t \eta - \sqrt{1 + |\nabla \eta|^2} v_n|_{y = \eta(t, x)} = 0, \quad P(t, x, y)|_{y = \eta(t, x)} = 0, \quad \text{for} \quad t \geq 0, \, x \in \mathbb{R}^d, \quad (1.4)$$
where \( v_n = \mathbf{n}_+ \cdot v|_{y=\eta(t,x)} \), with \( \mathbf{n}_+ := \frac{1}{\sqrt{1+|\nabla \eta|^2}} (-\nabla \eta, 1)^T \) denoting the outward normal vector to the free surface \( \Sigma_t \). The first equation of (1.4) means that the free surface moves with the fluid. In general, the pressure at the free surface is given by

\[
P|_{y=\eta(t,x)} = -\kappa \nabla \cdot \left( \frac{\nabla \eta}{\sqrt{1+|\nabla \eta|^2}} \right) \quad \text{for} \quad t \geq 0, x \in \mathbb{R}^d,
\]

where \( \kappa \geq 0 \) is the surface tension coefficient. In this paper, we will consider the case without surface tension. In such case, the pressure at the free surface can be set to zero.

As in [13, 17], we use an alternative formulation of the water wave system (1.1)-(1.4). From (1.2) and (1.3), there exists a potential flow function \( \phi \) such that \( v = \nabla_{x,y} \phi \) and

\[
\Delta_{x,y} \phi = 0 \quad \text{in} \quad \Omega_t, \ t \geq 0. \tag{1.5}
\]

The boundary condition (1.4) can be expressed in terms of \( \phi \)

\[
\partial_t \eta - \nabla \cdot (\mathbf{n}_+ \cdot \nabla_{x,y} \phi)|_{y=\eta(t,x)} = 0, \quad \text{for} \quad t > 0, \ x \in \mathbb{R}^d, \tag{1.6}
\]

where we denote \( \partial_{n+} \equiv \mathbf{n}_+ \cdot \nabla_{x,y} \). The Euler’s equation (1.1) can be put into Bernoulli’s form

\[
\partial_t \phi + \frac{1}{2} |\nabla_{x,y} \phi|^2 + gy = -P \quad \text{in} \quad \Omega_t, \ t \geq 0. \tag{1.7}
\]

We next reduce the system (1.5)-(1.7) to a system where all the functions are evaluated at the free surface only. For this purpose, we introduce the trace of the velocity potential \( \phi \) at the free surface

\[
\psi(t,x) \equiv \phi(t, x, \eta(t, x)),
\]

and the (rescaled) Dirichlet-Neumann operator \( G(\eta) \)

\[
G(\eta) \psi \equiv \sqrt{1 + |\nabla \eta|^2} \partial_{n+} \phi|_{y=\eta(t,x)}.
\]

Taking the trace of (1.7) on the free surface, the system (1.5)-(1.7) is equivalent to the system

\[
\begin{cases}
\partial_t \eta - G(\eta) \psi = 0, \\
\partial_t \psi + g \eta + \frac{1}{2} |\nabla \psi|^2 - \frac{(G(\eta) \psi + \nabla \eta \nabla \psi)^2}{2(1 + |\nabla \eta|^2)} = 0,
\end{cases} \tag{1.8}
\]

which is an evolution equation for the height of the free surface \( \eta(t, x) \) and the trace of the velocity potential on the free surface \( \psi(t, x) \).

1.2. Main result. Let us first recall some known results on the well-posedness of the water-wave problem. Nalimov [21], Yoshihara [28] and Craig [13] proved the local well-posedness of the 2-D water-wave equation in the case when the motion of free surface is a small perturbation of still water. In general, the local well-posedness of the water wave equation without surface tension was solved by Wu [23, 24] in the case of infinite depth. See also Ambrose and Masmoudi [4, 5], where they studied the well-posedness of the water-wave equation with surface tension and zero surface tension limit. Based on the formulation (1.8), Lannes [17] proved the local well-posedness of the water-wave equation without surface tension in the case of finite depth; while Ming and Zhang [20] dealt with the case with surface tension. Recently, Alazard, Burq and Zuily [1, 2] proved the local well-posedness of the water-wave equation with surface tension for the low regularity initial data by using the paradifferential...
operator tools and Strichartz type estimates. We should mention some recent results [10, 19, 11, 22, 29] concerning the local well-posedness of the rotational water-wave equation.

For small initial data, Wu [25] proved the almost global well-posedness of 2-D water-wave equation, and Wu [26] and Germain, Masmoudi and Shatah [16] proved the global well-posedness of 3-D water-wave equation. On the other hand, Castro, Cordoba, Ferferman, Gancedo and Lopez-Fernandez [7] showed that there exists smooth initial data for the water-waves equation such that the solution overturns in finite time. See [8, 12] for the splash singularity. Wu [27] also construct a class of self-similar solution for the 2-D water-wave equation without the gravity.

In this paper, we will study the possible break-down mechanism of the local solution of the system (1.8). For the incompressible Euler equation in the whole space, Beale, Kato and Majda [6] showed that as long as
\[ \int_0^T \| \text{curl} v(t) \|_{L^\infty} dt < +\infty, \]
then the solution \( v \) can be extended after \( t = T \). For the water-wave equation, Craig and Wayne in a survey paper [15] propose the similar problem “How do solutions break down?” and state:

There are several versions of this question, including “What is the lowest exponent of a Sobolev space \( H^s \) in which one can produce an existence theorem local in time?” Or one could ask “For which \( \alpha \) is it true that, if one knows a priori that \( \sup_{[-T,T]} \| (\eta, \psi) \|_{C^\alpha} < +\infty \) and that \( (\eta_0, \psi_0) \in C^\infty \), then the solution is fact \( C^\infty \) over the time interval \([−T,T]\)?” · · · · · · It would be more satisfying to say that the solution fails to exist because the curvature of the surface has diverged at some point, or a related geometrical and(or) physical statement.

For the first version of Craig-Wayne’s problem, Alazard, Burq and Zuily make the important progress in a recent work [3]. To state their result, we denote by \((V, B)\) the horizontal and vertical traces of the velocity on \( \Sigma_t \), which is defined by
\[ V \text{ def } \nabla \phi|_{y = \eta}, \quad B \text{ def } \partial_y \phi|_{y = \eta}. \quad (1.9) \]

**Theorem 1.1.** [3] Let \( d \geq 1, s > 1 + \frac{d}{2} \). Assume that the initial data \((\eta_0, \psi_0)\) satisfy
\[ \eta_0 \in H^{s+1/2}(\mathbb{R}^d), \quad \psi_0 \in H^{s+\frac{7}{2}}(\mathbb{R}^d), \quad V_0 \in H^s(\mathbb{R}^d), \quad B_0 \in H^s(\mathbb{R}^d). \]
Then there exists \( T > 0 \) such that the system (1.8) with the initial data \((\eta_0, \psi_0)\) has a unique solution \((\eta, \psi)\) satisfying
\[ \eta \in C([0, T]; H^{s+\frac{7}{2}}(\mathbb{R}^d)), \quad \psi \in C([0, T]; H^{s+\frac{7}{2}}(\mathbb{R}^d)), \quad V \in C([0, T]; H^s(\mathbb{R}^d)), \quad B \in C([0, T]; H^s(\mathbb{R}^d)). \]

**Remark 1.2.** In fact, the authors in [3] consider the case of finite depth with rough bottom, in which case one need to impose an extra condition on the initial data:
\[ a(0, x) \geq c > 0 \quad \text{for} \quad x \in \mathbb{R}^d, \quad \text{where} \quad a(t, x) = - (\partial_y P)(t, x, \eta(t, x)), \]
which is the so-called Taylor sign condition. In the case of infinite depth or finite depth with the flat bottom, this condition automatically holds, see [17, 23, 24] for example.

The goal of this paper is to answer the second version of Craig-Wayne’s problem. Our main result is stated as follows.
Theorem 1.3. Let \((\eta, \psi)\) be the solution of the system (1.8) on \([0, T]\) stated in Theorem 1.1. If the solution \((\eta, \psi)\) satisfies

\[
M(T) \overset{\text{def}}{=} \sup_{t \in [0, T]} \|\kappa(t)\|_{L^p \cap L^2} + \int_0^T \|\nabla V, \nabla B\)(t)\|_{L^\infty} dt < +\infty,
\]

\[
TS \overset{\text{def}}{=} \inf_{(t,x,y) \in [0,T] \times \Sigma_t} \partial P(t,x,y) \geq c_0,
\]

for some \(p > 2d\) and \(c_0 > 0\), then we have

\[
\sup_{t \in [0,T]} E_s(t) \leq C(E_s(0), M(T), T, TS^{-1}).
\]

Especially, the solution \((\eta, \psi)\) can be extended after \(t = T\). Here \(\kappa(t, x)\) is the mean curvature of the free surface \(\Sigma_t\) defined by

\[
\kappa(t, x) \overset{\text{def}}{=} \nabla \cdot \left( \frac{\nabla \eta(t, x)}{\sqrt{1 + |\nabla \eta(t, x)|^2}} \right),
\]

and \(E_s(t) \overset{\text{def}}{=} \|\eta, \psi(t)\|_{H^{s+\frac{1}{4}}} + \|V, B\)(t)\|_{H^{s}}, \) and \(C(\cdot, \cdot, \cdot)\) is an increasing function.

Remark 1.4. Theorem 1.3 implies that there are three possible blow-up mechanisms for the solution of the gravity water-wave equation: rolling-over of the surface, blow-up of the curvature or the formation of shock.

Remark 1.5. The same result should be true for the case of finite depth. In a future work, we will extend a similar result to the rotational water-wave equation.

2. Paradifferential calculus

In this section, we recall some results about the paradifferential calculus from [18], see also [1, 3].

2.1. Paradifferential operators. Let us first introduce the definition of the symbol with limited spatial smoothness. We denote \(W^{k,\infty}(\mathbb{R}^d)\) the usual Sobolev spaces for \(k \in \mathbb{N}\), and the Hölder space with exponent \(k\) for \(k \in (0, 1)\).

Definition 2.1. Given \(\rho \in [0, 1]\) and \(m \in \mathbb{R}\), we denote by \(\Gamma_{\rho}^m(\mathbb{R}^d)\) the space of locally bounded functions \(a(x, \xi)\) on \(\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}\), which are \(C^\infty\) with respect to \(\xi\) for \(\xi \neq 0\) and such that, for all \(\alpha \in \mathbb{N}^d\) and all \(\xi \neq 0\), the function \(x \to \partial^\alpha_x a(x, \xi)\) belongs to \(W^{\rho, \infty}\) and there exists a constant \(C_\alpha\) such that

\[
\|\partial^\alpha_x a(\cdot, \xi)\|_{W^{\rho, \infty}} \leq C_\alpha (1 + |\xi|)^{m-|\alpha|} \text{ for any } |\xi| \geq \frac{1}{2}.
\]

The semi-norm of the symbol is defined by

\[
M^m_{\rho}(a) \overset{\text{def}}{=} \sup_{|\alpha| \leq 3d/2 + 1 + \rho} \sup_{|\xi| \geq 1/2} \|(1 + |\xi|)^{|\alpha|} \partial^\alpha_x a(\cdot, \xi)\|_{W^{\rho, \infty}}.
\]

Given a symbol \(a\), the paradifferential operator \(T_a\) is defined by

\[
\widehat{T_a u}(\xi) \overset{\text{def}}{=} (2\pi)^{-d} \int \chi(\xi - \eta, \eta) \hat{u}(\xi - \eta, \eta) \psi(\eta) \hat{u}(\eta) d\eta,
\]

\(\hat{a}(\xi) \overset{\text{def}}{=} \int a(x, \xi) dx.\)
where \(\hat{a}(\theta, \xi)\) is the Fourier transform of \(a\) with respect to the first variable; the \(\chi(\theta, \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)\) is an admissible cut-off function: there exists \(\varepsilon_1, \varepsilon_2\) such that \(0 < \varepsilon_1 < \varepsilon_2\) and

\[
\chi(\theta, \eta) = 1 \quad \text{for} \quad |\theta| \leq \varepsilon_1|\eta|, \quad \chi(\theta, \eta) = 0 \quad \text{for} \quad |\theta| \geq \varepsilon_2|\eta|,
\]

and such that for any \((\theta, \eta) \in \mathbb{R}^d \times \mathbb{R}^d,\)

\[
|\partial_\theta^\alpha \partial_\eta^\beta \chi(\theta, \eta)| \leq C_{\alpha, \beta}(1 + |\eta|)^{-|\alpha| - |\beta|}.
\]

The cut-off function \(\psi(\eta) \in C^\infty(\mathbb{R}^d)\) satisfies

\[
\psi(\eta) = 0 \quad \text{for} \quad |\eta| \leq 1, \quad \psi(\eta) = 1 \quad \text{for} \quad |\eta| \geq 2.
\]

Throughout this paper, we will take the admissible cut-off function \(\chi(\theta, \eta)\) as

\[
\chi(\theta, \eta) = \sum_{k=0}^{\infty} \zeta_{k-3}(\theta)\varphi_k(\eta),
\]

where \(\zeta(\theta) = 1\) for \(|\theta| \leq 1.1\) and \(\zeta(\theta) = 0\) for \(|\theta| \geq 1.9\); and

\[
\zeta_k(\theta) = \zeta(2^{-k}\theta) \quad \text{for} \quad k \in \mathbb{Z},
\]

\[
\varphi_0 = \zeta, \quad \varphi_k = \zeta_k - \zeta_{k-1} \quad \text{for} \quad k \geq 1.
\]

We also introduce the Littlewood-Paley operators \(\Delta_k, S_k\) defined by

\[
\Delta_k u = \mathcal{F}^{-1}(\varphi_k(\xi)\hat{u}(\xi)) \quad \text{for} \quad k \geq 0, \quad \Delta_k u = 0 \quad \text{for} \quad k < 0,
\]

\[
S_k u = \mathcal{F}^{-1}(\zeta_k(\xi)\hat{u}(\xi)) \quad \text{for} \quad k \in \mathbb{Z}.
\]

In the case when the function \(a\) depends only on the first variable \(x\) in \(T_a u\), we take \(\psi = 1\). Then \(T_a u\) is just the usual Bony’s paraproducts and

\[
T_a u = \sum_{k=0}^{\infty} S_{k-3} a \Delta_k u. \tag{2.2}
\]

Furthermore, we have Bony’s decomposition:

\[
a u = T_a u + T_a a + R(u, a), \tag{2.3}
\]

where the remainder term \(R(u, a)\) is defined by

\[
R(u, a) = \sum_{|k-\ell| \leq 2; \ell \geq -2} \Delta_k a \Delta_\ell u.
\]

Now we introduce the Besov space.

**Definition 2.2.** Let \(s \in \mathbb{R}, p, q \in [1, \infty]\). The inhomogeneous Besov space \(B^s_{p,q}(\mathbb{R}^d)\) consists of the temperate distribution \(f\) satisfying

\[
\|f\|_{B^s_{p,q}} \overset{\text{def}}{=} \left( \sum_k 2^{ksq} \|\Delta_k f\|_{L^p}^q \right)^{\frac{1}{q}} < \infty.
\]

When \(p = q = 2\), \(B^s_{p,q}(\mathbb{R}^d)\) is just the usual Sobolev space \(H^s(\mathbb{R}^d)\); When \(p = q = \infty\) and \(s \not\in \mathbb{N}\), it is the Zygmund space \(C^s(\mathbb{R}^d)\). From the definition, it is easy to see that if \(s_1 \leq s_2\) and \(q_2 \leq q_1\), then

\[
\|f\|_{B^{s_1}_{p,q}} \leq \|f\|_{B^{s_2}_{p,q}}; \tag{2.4}
\]

if \(s_2 > s_1\), then

\[
\|f\|_{B^{s_1}_{p,1}} \leq \|f\|_{B^{s_2}_{p,\infty}}. \tag{2.5}
\]
The following Berstein’s inequality will be repeatedly used.

**Lemma 2.3.** Let \(1 \leq p \leq q \leq \infty, \alpha \in \mathbb{N}^d\). Then it holds that
\[
\|\partial^\alpha S_k u\|_{L^q} \leq C 2^{kd(k \cdot \frac{1}{p} - \frac{1}{q} + |\alpha|)} \|S_k u\|_{L^p} \quad \text{for } k \in \mathbb{N},
\]
\[
\|\Delta_k u\|_{L^q} \leq C 2^{kd(k \cdot \frac{1}{p} - \frac{1}{q} + |\alpha|)} \sup_{|\beta| = |\alpha|} \|\partial^\beta \Delta_k u\|_{L^p} \quad \text{for } k \geq 1.
\]

**2.2. Symbolic calculus.** We recall the symbolic calculus for the paradifferential operators.

**Proposition 2.4.** Let \(m, m' \in \mathbb{R}\) and \(\rho \in [0, 1]\).

1. If \(a \in \Gamma_0^m(\mathbb{R}^d)\), then for any \(\mu \in \mathbb{R}\),
   \[
   \|T_a\|_{H^\mu \to H^{\mu - m}} \leq CM_0^m(a);
   \]
2. If \(a \in \Gamma_\rho^m(\mathbb{R}^d), b \in \Gamma_\rho^{m'}(\mathbb{R}^d)\), then for any \(\mu \in \mathbb{R}\),
   \[
   \|T_a T_b - T_{ab}\|_{H^\mu \to H^{\mu - m - m' + \rho}} \leq CM_\rho^m(a)M_\rho^{m'}(b) + K M_0^m(a)M_\rho^{m'}(b);
   \]
3. If \(a \in \Gamma_\rho^m(\mathbb{R}^d)\), then for any \(\mu \in \mathbb{R}\),
   \[
   \|T_a - (T_a)^*\|_{H^\mu \to H^{\mu - m + \rho}} \leq CM_\rho^m(a).
   \]

Here \((T_a)^*\) is the adjoint operator of \(T_a\), and \(C\) is a constant independent of \(a, b\).

**Remark 2.5.** If \(\mu, \mu + m \notin \mathbb{N}\), then we have
\[
\|T_a\|_{W^{\mu, m, \infty} \to W^{\mu, \infty}} \leq CM_\rho^m(a),
\]
and if \(\mu, \mu - m - m' + \rho \notin \mathbb{N}\), then we have
\[
\|T_a T_b - T_{ab}\|_{W^{\mu, \infty} \to W^{\mu - m - m' + \rho, \infty}} \leq CM_\rho^m(a)M_\rho^{m'}(b) + CM_0^m(a)M_\rho^{m'}(b).
\]

**Lemma 2.6.** Let \(m, m', \mu \in \mathbb{R}, q \in [1, \infty]\) and \(\rho \in [0, 1]\).

1. If \(a \in \Gamma_0^m(\mathbb{R}^d)\), then
   \[
   \|T_a\|_{B^m_{\infty, q} \to B^{m - \rho}_{\infty, q}} \leq CM_0^m(a);
   \]
2. If \(a \in \Gamma_\rho^m(\mathbb{R}^d), b \in \Gamma_\rho^{m'}(\mathbb{R}^d)\), then
   \[
   \|T_a T_b - T_{ab}\|_{B^m_{\infty, q} \to B^{m - m' - \rho}_{\infty, q}} \leq CM_\rho^m(a)M_\rho^{m'}(b) + CM_0^m(a)M_\rho^{m'}(b).
   \]

Here \(C\) is a constant independent of \(a, b\).

**Proof.** Take \(\alpha\) such that \(\alpha, \alpha + m \notin \mathbb{N}\). From the definition of \(T_a\), we know that there exists a constant \(N_0 \in \mathbb{N}\) such that
\[
\Delta_j(T_a u) = \sum_{|j - k| \leq N_0} \Delta_j(T_a \Delta_k u).
\]

Then it follows from Remark 2.5 that
\[
\|\Delta_j T_a u\|_{L^\infty} \leq \sum_{|j - k| \leq N_0} \|\Delta_j T_a \Delta_k u\|_{L^\infty} \leq \sum_{|j - k| \leq N_0} 2^{-j \alpha} \|T_a \Delta_k u\|_{W^{\alpha, \infty}} \leq C \sum_{|j - k| \leq N_0} 2^{-j \alpha} \|\Delta_k u\|_{W^{\alpha + m, \infty}} \leq C \sum_{|j - k| \leq N_0} 2^{mk} \|\Delta_k u\|_{L^\infty},
\]
which implies (1). The proof of (2) is similar. \(\square\)
Remark 2.7. If the symbol $a(x, \xi)$ satisfies
\[
||\partial_\xi^m a(\cdot, \xi)||_{C^{-\rho}} \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}
\] for any $|\xi| \geq \frac{1}{2}$, then $T_\alpha$ is bounded from $H^\mu(\mathbb{R}^d)$ to $H^{\mu-m-\rho}(\mathbb{R}^d)$ and $B^\mu_{\infty, q}$ to $B^\mu_{\infty, \infty q-\rho}(\mathbb{R}^d)$.

Lemma 2.8. Let $F$ be a smooth function with $F(0) = 0$ and $\mu > 0, q \in [1, \infty]$. Then it holds that
\[
\|F(u)\|_{H^\mu} \leq C\|u\|_{L^\infty} \|u\|_{H^\mu},
\]
\[
\|F(u)\|_{B^\mu_{\infty, q}} \leq C\|u\|_{L^\infty} \|u\|_{B^\mu_{\infty, q}}.
\]

2.3. Commutator estimates.

Proposition 2.9. Let $m, \mu \in \mathbb{R}, s > 0$ and $a \in \Gamma^m_\rho(\mathbb{R}^d)$ with $\rho \in (0, 1]$. Then
\[
\|(D)^s, T_\alpha\|_{H^\mu} \leq CM^m_\rho(a) \|u\|_{H^{s+m-\mu}}.
\]

Proof. We write
\[
\|(D)^s, T_\alpha\|_{H^\mu} = T(\xi)^s T_\alpha u - T(\xi)^s a + (D)^s - T(\xi)^s) T_\alpha u,
\]
then the proposition follows from Proposition 2.4 and
\[
\|(D)^s - T(\xi)^s) T_\alpha u\|_{H^\mu} \leq C\|(D)^s (1 - \psi(D)) T_\alpha u\|_{H^\mu} \leq CM^m_\rho(a) \|u\|_{H^{s-\mu}}
\]
for any $\mu' > 0$.

Proposition 2.10. Let $V \in C([0, T]; B^1_{1, 1}(\mathbb{R}^d))$ and $p = p(t, x, \xi)$ be homogenous in $\xi$ of order $m$. Then it holds that
\[
\|T_\xi + V \cdot \nabla)u(t)\|_{L^2} \leq C(M^m_0(p)\|V(t)\|_{B^{1}_{1, 1}} + M^m_0(\partial_t p + V \cdot \nabla p))\|u(t)\|_{H^m}.
\]

Proof. We follow the proof of Lemma 2.16 in [3]. As in [3], it suffices to consider the case when $p = p(t, x)$ by decomposing $p$ into a sum of spherical harmonic. By a direct calculation, we have
\[
[\partial_t + V \cdot \nabla, T_\xi)u = T_{\xi, p} u + V \cdot T_{\xi, p} u + V \cdot T_\xi u - T_\xi (V \cdot \nabla u)
\]
\[
= T_{\xi, p} u + V \cdot T_{\xi, p} u + (V \cdot T_{\xi, p} - T_\xi (V \cdot \nabla p))u + (V \cdot T_\xi u - V \cdot T_{\xi, p} (V \cdot \nabla u))
\]

First of all, we get by Proposition 2.4 that
\[
\|T_{\xi, p} u\|_{L^2} \leq C\|\partial_t p + V \cdot \nabla p\|_{L^\infty} \|u\|_{L^2}.
\]
Set $S^{j-3}(V) = \sum_{k \geq j-3} \Delta_k^p V$. Then $V = S^{j-3}(V) + S^{j-3}(V)$. Hence,
\[
V \cdot T_{\xi, p} u = \sum_j S^{j-3}(V) \cdot S^{j-3}(\nabla p) \Delta_j u + \sum_j S^{j-3}(V) \cdot S^{j-3}(\nabla p) \Delta_j u
\]
\[
= \sum_j (S^{j-3}(S^{j-3}(V) \cdot \nabla p) \Delta_j u + (S^{j-3}(V), S^{j-3}) \cdot \nabla p) \Delta_j u
\]
\[
= T_{V \cdot \nabla p} u - \sum_j (S^{j-3}(S^{j-3}(V) \cdot \nabla p) + (S^{j-3}(V), S^{j-3}) \cdot \nabla p) \Delta_j u + \sum_j S^{j-3}(V) \cdot S^{j-3}(\nabla p) \Delta_j u.
\]
By Lemma 2.3, we get
\[ \| \sum_j S_j^{-3}(V) \cdot S_j^{-3}(V) \Delta_j u \|_{L^2} \leq C \| p \|_{L^\infty} \sum_j 2^j \| S_j^{-3}(V) \|_{L^\infty} \| \Delta_j u \|_{L^2} \]
\[ \leq C \| p \|_{L^\infty} \| u \|_{L^2} \sum_j 2^j \sum_{k \geq j-3} \| \Delta_k V \|_{L^\infty} \]
\[ \leq C \| p \|_{L^\infty} \| V \|_{B_{\infty,1}^1} \| u \|_{L^2}. \]
Noting that \( S_j^{-3}(V) \cdot \nabla p \Delta_j u \) is spectrally supported in an annulus \( \{c_1 2^j \leq |\xi| \leq c_2 2^j\} \), we infer from Lemma 2.3 that
\[ \| \sum_j S_j^{-3}(V) \cdot \nabla p \Delta_j u \|_{L^2}^2 \]
\[ \leq C \| p \|_{L^\infty} \| V \|_{B_{\infty,1}^1} \| \Delta_j u \|_{L^2} \]
\[ \leq C \| p \|_{L^\infty} \| V \|_{B_{\infty,1}^1} \| u \|_{L^2}^2. \]

Similarly, \( \{[S_j^{-3}(V), S_j^{-3}(\nabla p)] \Delta_j u \) is also spectrally supported in an annulus \( \{c_1 2^j \leq |\xi| \leq c_2 2^j\} \), thus,
\[ \| \sum_j ([S_j^{-3}(V), S_j^{-3}(\nabla p)] \Delta_j u \|_{L^2}^2 \]
\[ \leq C \| p \|_{L^\infty} \| V \|_{B_{\infty,1}^1} \| u \|_{L^2}^2. \]

Here we used the commutator estimate
\[ \| [S_j, g] \nabla f \|_{L^\infty} \leq C \| \nabla g \|_{L^\infty} \| f \|_{L^\infty}, \]
which follows from the identity
\[ [S_j, g] \nabla f(x) = \int_{\mathbb{R}^d} \hat{\zeta}_k(x-x')(g(x') - g(x)) \nabla f(x') dx' \]
\[ = \int_{\mathbb{R}^d} \nabla \hat{\zeta}_k(x-x')(g(x') - g(x)) f(x') dx' \]
\[ - \int_{\mathbb{R}^d} \hat{\zeta}_k(x-x') \nabla g(x') f(x') dx', \]
and \( \| \hat{\zeta}_k \|_{L^1} + \| x \nabla \hat{\zeta}_k \|_{L^1} \leq C. \) This proves that
\[ \| (V \cdot T_{\nabla p} - T_{V \cdot \nabla p}) u \|_{L^2} \leq C \| p \|_{L^\infty} \| V \|_{B_{\infty,1}^1} \| u \|_{L^2}. \]
Next we write
\[ V \cdot (T_{\nabla p} u) = \sum_j S_j^{-3}(V) S_j^{-3}(p) \cdot \Delta_j \nabla u + \sum_j S_j^{-3}(V) S_j^{-3}(p) \cdot \Delta_j \nabla u. \]
It follows from Lemma 2.3 that
\[ \| \sum_j S_j^{-3}(V) S_j^{-3}(p) \Delta_j \nabla u \|_{L^2} \leq C \| p \|_{L^\infty} \| V \|_{B_{\infty,1}^1} \| u \|_{L^2}. \]
On the other hand, we have
\[ T_p(V \cdot \nabla u) = T_p T_V \cdot \nabla u + T_p(V - T_V) \cdot \nabla u = \sum_j S_{j-3}(p) \Delta_j \{S_{j-3}(V) \cdot \nabla u\} \]
\[ + \sum_{j,k} S_{j-3}(p) \Delta_j \{(S_{k-3} - S_{j-3})(V) \cdot \nabla \Delta_k u\} + T_p(V - T_V) \cdot \nabla u \]
\[ = \sum_j S_{j-3}(V) S_{j-3}(p) \Delta_j \nabla u + \sum_j S_{j-3}(p) \{\Delta_j, S_{j-3}(V)\} \cdot \nabla u \]
\[ + \sum_{j,k} S_{j-3}(p) \cdot \Delta_j \{(S_{k-3} - S_{j-3})(V) \cdot \nabla \Delta_k u\} + T_p(V - T_V) \cdot \nabla u \]
\[ \equiv \sum_j S_{j-3}(V) S_{j-3}(p) \cdot \nabla \Delta_j u + I_1 + I_2 + I_3. \]

We get by Proposition 2.4 and (2.3) that
\[ \|I_3\|_{L^2} \leq C\|p\|_{L^\infty} \|\nabla u\|_{L^2} \leq C\|p\|_{L^\infty} \|V\|_{B^1_{\infty,1}} \|u\|_{L^2}. \]

Note that the summation index \((j, k)\) in \(I_2\) should satisfy \(|k - j| \leq N_0\) for some \(N_0 \in \mathbb{N}\), hence,
\[ \|I_2\|_{L^2} \leq C\|p\|_{L^\infty} \|V\|_{B^1_{\infty,1}} \|u\|_{L^2}. \]

We rewrite \(I_1\) as
\[ I_1 = \sum_j S_{j-3}(p) \{\Delta_j, S_{j-N_0}(V)\} \cdot \nabla u + \sum_j S_{j-3}(p) \{\Delta_j, (S_{j-3} - S_{j-N_0})(V)\} \cdot \nabla u \]
for some \(N_0 \in \mathbb{N}\) so that \(S_{j-3}(p) \{\Delta_j, S_{j-N_0}(V)\} \cdot \nabla u\) is spectrally supported in an annulus \(\{c_1 2^j \leq |\xi| \leq c_2 2^j\}\). Then as in the above, it is easy to get
\[ \|I_1\|_{L^2} \leq C\|p\|_{L^\infty} \|V\|_{B^1_{\infty,1}} \|u\|_{L^2}. \]

Hence, we conclude
\[ \|V \cdot T_p \nabla u - T_p(V \cdot \nabla u)\|_{L^2} \leq C\|p\|_{L^\infty} \|V\|_{B^1_{\infty,1}} \|u\|_{L^2}. \]
This finishes the proof. \(\square\)

3. PARABOLIC EVOLUTION EQUATION

Let \(I = [z_0, z_1]\). We denote by \(\Gamma^m_\rho(I \times \mathbb{R}^d)\) the space of symbols \(a(z; x, \xi)\) satisfying
\[ \overline{M}_\rho^m(a) \overset{\text{def}}{=} \sup_{z \in I} \sup_{|\alpha| \leq 3d/2 + 1 + \rho} \sup_{|\xi| \geq 1/2} \|((1 + |\xi|)^{-m} \partial^\alpha_x a(z; \cdot, \xi))\|_{W^{\rho, \infty}} < +\infty. \]

In this section, we study the parabolic evolution equation
\[
\begin{align*}
\partial_t w + T_p w &= f, \\
|w|_{z = z_0} &= w_0,
\end{align*}
\] (3.1)

where the symbol \(p \in \Gamma^1_\rho(I \times \mathbb{R}^d)\) is elliptic in the sense that there exists \(c_1 > 0\) such that for any \(z \in I, (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d\), it holds that
\[ \text{Re } p(z; x, \xi) \geq c_1 |\xi|. \] (3.2)
In order to obtain the maximal parabolic regularity of the solution, we introduce Chemin-Lerner type space \( \tilde{L}^q_s(I; B^p_{\infty, \ell}(\mathbb{R}^d)) \), whose norm is defined by

\[
\|f\|_{\tilde{L}^q_s(I; B^p_{\infty, \ell}(\mathbb{R}^d))} := \left( \sum_k 2^{kr\ell} \|\Delta_k f\|_{L^q_s(I; L^p)}^\ell \right)^{\frac{1}{\ell}},
\]

which was firstly introduced by Chemin and Lerner [9] to study the incompressible Navier-Stokes equations. When \( p = \ell = \infty \), we denote it by \( \tilde{L}^q_s(I; C^s(\mathbb{R}^d)) \). When \( p = q = \ell = 2 \), we have \( \tilde{L}^q_2(I; B^p_{\infty, \ell}(\mathbb{R}^d)) = L^2(I; H^\ell(\mathbb{R}^d)); \) When \( q = \infty, p = \ell = 2 \), we denote it by \( \tilde{L}^\infty_2(I; H^\ell(\mathbb{R}^d)) \). In this case, we have

\[
\|f\|_{L^\infty_s(I; H^\ell)} \leq \|f\|_{\tilde{L}^\infty_s(I; H^\ell)}.
\]

**Proposition 3.1.** Let \( r \in \mathbb{R}, \ell \in [1, \infty] \) \( 1 \leq q \leq p \leq \infty \). Assume that \( p \in \Gamma^1_\rho(I \times \mathbb{R}^d) \) for \( \rho > 0 \) and \( w \) is a solution of (3.1). Then for any \( \delta > 0 \), we have

\[
\|w\|_{\tilde{L}^q_s(I; B^{\frac{p-1}{q}}_{\infty, \ell})} \leq C(M^1(p), c_1) \left( \|w_0\|_{B^{\frac{p-1}{q}}_{\infty, \ell}} + \|w\|_{\tilde{L}^q_s(I; B^{\frac{p-1}{q}+\frac{1}{\ell}}_{\infty, \ell})} + \|w\|_{\tilde{L}^q_s(I; C^{\delta})} \right),
\]

where \( C(\cdot) \) is a nondecreasing function independent of \( p \).

The proof is based on the following classical parabolic smoothing effect.

**Lemma 3.1.** Let \( \kappa > 0 \) and \( p \in [1, \infty] \). Then there exists some \( c > 0 \) such that for any \( t > 0, k \geq 1 \), we have

\[
\|e^{-t\kappa |D|} \Delta_k u\|_{L^p} \leq Ce^{-ct2^k} \|\Delta_k u\|_{L^p}.
\]

**Proof.** Take a function \( \chi_1 \in C^\infty_0(\mathbb{R}^d \setminus \{0\}) \) such that \( \chi_1(\xi/2^k) = 1 \) for \( \xi \in \text{supp} \varphi_k \). Then we have

\[
e^{-t\kappa |D|} \Delta_k u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi - \kappa t|\xi|} \chi_1(\xi/2^k) \tilde{\Delta_k u}(\xi) d\xi = G_{k,t} * \Delta_k u(x),
\]

where

\[
G_{k,t}(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi - \kappa t|\xi|} \chi_1(\xi/2^k) d\xi.
\]

Then the lemma will follows from Young’s inequality and the estimate

\[
|G_{k,t}(x)| \leq Ce^{-ct2^k} 2^{dk} (1 + |2^k x|)^{-N}\tag{3.4}
\]

for some \( N > d \).

Now we prove (3.4). Noting that

\[
G_{k,t}(x) = (2\pi)^{-d} 2^{dk} \int_{\mathbb{R}^d} e^{i2^k x \cdot \xi - \kappa t2^k |\xi|} \chi_1(\xi) d\xi \equiv 2^{dk} \tilde{G}_{k,t}(2^k x),
\]

thus it suffices to show that

\[
\tilde{G}_{k,t}(x) \leq Ce^{-ct2^k} (1 + |x|)^{-N}.
\]

(3.5)

It is easy to see that

\[
|\tilde{G}_{k,t}(x)| \leq C \int_{|\xi| \sim 1} e^{-\kappa t2^k} d\xi \leq Ce^{-ct2^k}.
\]

(3.6)
To obtain the behavior of $\tilde{G}_{k,t}(x)$ for large $x$, we need to integrate by parts. For this end, we introduce the operator $L(x, D) = \frac{x\cdot\nabla}{|x|^2}$. Since $L(x, D)e^{ix\cdot\xi} = e^{ix\cdot\xi}$, then for any $N \in \mathbb{N}$, we have

$$
\tilde{G}_{k,t}(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} L^N e^{ix\cdot\xi} e^{-\kappa t 2^k |\xi|} \chi_1(\xi) d\xi
$$

$$
= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} (L^* e^{-\kappa t 2^k |\xi|} \chi_1(\xi)) d\xi,
$$

where the integrand can be majorized by

$$
|x|^{-N} \max \{1, (t2^k)^N, t2^k |\xi|^{1-N} \} e^{-\kappa t 2^k |\xi|} |\xi|^{-1}.
$$

Hence, we infer that

$$
|\tilde{G}_{k,t}(x)| \leq C_N |x|^{-N} e^{-\kappa t 2^k} \int_{|\xi|^{1-N}} (1 + t2^k |\xi|)^N e^{-\kappa t 2^k |\xi|} d\xi
$$

$$
\leq C_N |x|^{-N} e^{-\kappa t 2^k},
$$

which along with (3.6) implies (3.5).

**Proof of Proposition 3.1.** For $y \in I$ and $z \in [z_0, y]$, we set

$$
e(y, z; x, \xi) = \exp \left( - \int_z^y p(s; x, \xi) ds \right).
$$

Noting that $\partial_z e = ep$, we get by (3.1) that

$$
\partial_z (Tew) = T_{\partial_z e} w + T_e \partial_z w = (T_{ep} - T_{ep} T_p) w + T_e f.
$$

Integrating it on $[z_0, y]$, we get

$$
T_1 w(y) = T_{\mid_{z=0}} w_0 + \int_{z_0}^y T_e f(z) dz + \int_{z_0}^y (T_{ep} - T_{ep} T_p) w(z) dz
$$

$$
\triangleq G_1 + G_2 + Rw
$$

so that for any $N \in \mathbb{N}$, there holds

$$
w = (I + R + \cdots + R^N)(G_1 + G_2 + T_1 w + w) + R^{N+1} w,
$$

where for any $\delta > 0$, we have

$$
||w - T_1 w||_{L^p(I; B_{\infty,C^{-\delta}}^r)} = ||(1 - \psi(D))w||_{L^p(I; B_{\infty,C^{-\delta}}^r)} \leq C ||w||_{L^p(I; C^{-\delta})}.
$$

It is easy to verify that $e(y, z; x, \xi) \exp(c_1 (y - z)|\xi|/2) \in \Gamma^0_p(\mathbb{R}^d)$ for $y, z \in I, z \leq y$ with the bound

$$
M^0_p (e(y, z; x, \xi) \exp(c_1 (y - z)|\xi|/2)) \leq C M^1_p(p).
$$
Thus by (2.6), Remark 2.5 and Lemma 3.1, we have
\[
\| \Delta_j T_e u \|_{L^\infty} \leq \sum_{|j-k| \leq N_0} \| \Delta_j T_e \Delta_k u \|_{L^\infty}
\]
\[
\leq \sum_{|j-k| \leq N_0} 2^{-\frac{j}{r}} \| \Delta_j T_e \exp(c_1 D |(y-z)/2) \exp(-c_1 D |(y-z)/2) \Delta_k u \|_{C^2}
\]
\[
\leq C M_p^1(p) \sum_{|j-k| \leq N_0} 2^{-\frac{j}{r}} \| \exp(-c_1 D |(y-z)/2) \Delta_k u \|_{C^2}
\]
\[
\leq C M_p^1(p) \sum_{|j-k| \leq N_0} \exp(-c(y-z)2^k) \| \Delta_k u \|_{L^\infty}
\] (3.8)

for some $N_0 \in \mathbb{N}$ and $c > 0$ (Important note: the summation index $k \geq 1$ due to the definition of $T_e$).

Now let us turn to the estimates of $G_i$. We get by (3.8) that
\[
\| G_i \|_{L_0^p(I; B^r_{\infty, \ell})} = \left( \sum_{j} 2^{j(r+\frac{1}{p})} \| \Delta_j G_i \|_{L_0^p(I; L^\infty)} \right)^{\frac{1}{r}}
\]
\[
\leq C M_p^1(p) \left( \sum_{j} \sum_{|j-k| \leq N_0} 2^{j(r+\frac{1}{p})} \| \exp(-c(y-z)2^k) \| \Delta_k w_0 \|_{L^\infty} \| L_0^p(I) \right)^{\frac{1}{r}}
\]
\[
\leq C M_p^1(p) \left( \sum_{j} \sum_{|j-k| \leq N_0} 2^{j(r+\frac{1}{p})} 2^{-\frac{k}{p}} \| \Delta_k w_0 \|_{L^\infty} \right)^{\frac{1}{r}}
\]
\[
\leq C M_p^1(p) \| w_0 \|_{B^r_{\infty, \ell}}. \quad (3.9)
\]

For $G_2$, we have by (3.8) that
\[
\int_{z_0}^{y} \| \Delta_j T_e f(z) \|_{L^\infty} dz \leq C M_p^1(p) \sum_{|j-k| \leq N_0} \int_{z_0}^{y} \exp(-c(y-z)2^k) \| \Delta_k f(z) \|_{L^\infty} dz,
\]
from which and Young's inequality, we infer that
\[
\left\| \int_{z_0}^{y} \| \Delta_j T_e f(z) \|_{L^\infty} dz \right\|_{L^p(I)} \leq C M_p^1(p) \sum_{|j-k| \leq N_0} 2^{-k(1+\frac{1}{p}-\frac{1}{r})} \| \Delta_k f \|_{L_0^p(I; L^\infty)}.
\]

This implies that
\[
\| G_2 \|_{L_0^p(I; B^r_{\infty, \ell})} \leq C M_p^1(p) \| f \|_{L_0^p(I; B^r_{\infty, \ell})}. \quad (3.10)
\]

Similar to the proof of (3.8), we can get
\[
\| \Delta_j (T_e T_p - T_{ep}) w \|_{L^\infty} \leq C (M_p^1(p)) \sum_{|j-k| \leq N_0} 2^{k(1-\rho)} \exp(-c(y-z)2^k) \| \Delta_k w \|_{L^\infty},
\]
which implies that
\[
\| R w \|_{L_0^p(I; B^r_{\infty, \ell})} \leq C (M_p^1(p)) \| w \|_{L_0^p(I; B^r_{\infty, \ell})}. \quad (3.11)
\]

Take $N$ big enough so that $r + \frac{1}{p} - (N+1)\rho \leq -\delta$. Then the proposition follows from (3.7) and (3.9)-(3.11). \qed
Given $r \in \mathbb{R}$, let us introduce the spaces
\[
X^r(I) \overset{\text{def}}{=} \tilde{L}_2^\infty(I; H^r(\mathbb{R}^d)) \cap L_2^2(I; H^{r+1/2}(\mathbb{R}^d)),
\]
\[
Y^r(I) \overset{\text{def}}{=} \tilde{L}_2^1(I; H^r(\mathbb{R}^d)) + L_2^2(I; H^{r-1/2}(\mathbb{R}^d)).
\]

In a similar way as in Proposition 3.1, one can show that

**Proposition 3.2.** Let $r \in \mathbb{R}$. Assume that $p \in \Gamma^1_\rho(I \times \mathbb{R}^d)$ for $\rho > 0$ and $w$ is a solution of (3.1). Then it holds that
\[
\|w\|_{X^r(I)} \leq C(\tilde{M}^1_\rho(p), c_1) \left( \|w_0\|_{H^r} + \|f\|_{Y^r(I)} + \|w\|_{L^2(I; H^r)} \right),
\]
where $C(\cdot)$ is a nondecreasing function independent of $p$.

Let us conclude this section by presenting some product estimates in the Chemin-Lerner type space.

**Lemma 3.3.** Let $r \in \mathbb{R}$ and $q, q_1, q_2 \in [1, \infty]$ with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then for any $r_1, r_2 > 0$, we have
\[
\|T_g f\|_{L^q(I; L^{q_1})} \leq C \|g\|_{L^q(I; L^{q_2})} \|f\|_{L^q(I; L^{q_1})},
\]
\[
\|T_g f\|_{L^q(I; L^{q_1})} \leq C \|g\|_{L^q(I; L^{q_2})} \|f\|_{L^q(I; L^{q_1})},
\]
\[
\|T_g f\|_{L^q(I; L^{q_1})} \leq C \|g\|_{L^q(I; L^{q_2})} \|f\|_{L^q(I; L^{q_1})}.
\]

**Proof.** By the definition of paraproduct, we have
\[
\Delta_j T_g f = \sum_{|j-k| \leq N_0} \Delta_j (S_{k-3g} \Delta_k f)
\]
for some $N_0 \in \mathbb{N}$. Hence, we get by Lemma 2.3 that
\[
\|\Delta_j T_g f\|_{L^q(I; L^{q_1})} \leq C \sum_{|j-k| \leq N_0} \|S_{k-3g}\|_{L^q(I; L^{q_1})} \|\Delta_k f\|_{L^q(I; L^{q_1})}
\]
\[
\leq C \sum_{|j-k| \leq N_0} \|g\|_{L^q(I; L^{q_1})} \|\Delta_k f\|_{L^q(I; L^{q_1})},
\]
which implies the first inequality of the lemma. On the other hand, by the definition of $S_k$, we have
\[
\|S_{k-3g}\|_{L^q(I; L^{q_1})} \leq \sum_{\ell \leq k-2} \|\Delta_{j\ell} g\|_{L^q(I; L^{q_1})} \leq 2^{\ell r_1} \|g\|_{L^q(I; L^{q_1})} \quad \text{or}
\]
\[
\|S_{k-3g}\|_{L^q(I; L^{q_1})} \leq C k \|g\|_{L^q(I; L^{q_1})} \leq C 2^{k r_2} \|g\|_{L^q(I; L^{q_1})},
\]
which imply the last two inequalities. \(\square\)

In a similar way, one can show that

**Lemma 3.4.** Let $r \in \mathbb{R}$ and $q, q_1, q_2, \ell \in [1, \infty]$ with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then for any $r_1, r_2 > 0$, we have
\[
\|T_g f\|_{L^q(I; B^r_{\infty,\ell})} \leq C \|g\|_{L^q(I; L^{q_1})} \|f\|_{L^q(I; B^r_{\infty,\ell})},
\]
\[
\|T_g f\|_{L^q(I; B^r_{\infty,\ell})} \leq C \|g\|_{L^q(I; L^{q_1})} \|f\|_{L^q(I; B^r_{\infty,\ell})},
\]
\[
\|T_g f\|_{L^q(I; B^r_{\infty,\ell})} \leq C \|g\|_{L^q(I; L^{q_1})} \|f\|_{L^q(I; B^r_{\infty,\ell})}.
\]
Lemma 3.5. Let \( q, q_1, q_2, \ell \in [1, \infty] \) with \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \). Then for any \( r > 0 \) and \( r_1 \in R \), we have
\[
\| R(f, g) \|_{L^q(I; H^r)} \leq C\| g \|_{L^{q_1}(I; C^{r_1})} \| f \|_{L^{q_2}(I; H^{r-r_1})},
\]
\[
\| R(f, g) \|_{L^q(I; B^{r_2}_{\infty, \ell})} \leq C\| g \|_{L^{q_1}(I; C^{r_1})} \| f \|_{L^{q_2}(I; B^{r_1}_{\infty, \ell})}.
\]
If \( r \leq 0 \) and \( r_1 + r_2 > 0 \), then we have
\[
\| R(f, g) \|_{L^q(I; H^r)} \leq C\| g \|_{L^{q_1}(I; C^{r_1})} \| f \|_{L^{q_2}(I; H^{r_2})},
\]
\[
\| R(f, g) \|_{L^q(I; B^{r_2}_{\infty, \ell})} \leq C\| g \|_{L^{q_1}(I; C^{r_1})} \| f \|_{L^{q_2}(I; B^{r_1}_{\infty, \ell})}.
\]

Proof. Due to the definition of \( R(f, g) \), we have
\[
\Delta_j R(f, g) = \sum_{|k-\ell| \leq 2k, \ell \geq j-N_0} \Delta_j (\Delta_k f \Delta_j g) \quad \text{for some } N_0 \in \mathbb{N},
\]
from which and Lemma 2.3, we infer that
\[
\| \Delta_j R(f, g) \|_{L^2(I; L^2)} \leq C \sum_{|k-\ell| \leq 2k, \ell \geq j-N_0} \| \Delta_k f \|_{L^2(I; L^2)} \| \Delta_j g \|_{L^2(I; L^2)}
\]
\[
\leq C\| g \|_{L^{q_1}(I; C^{r_1})} \sum_{k \geq j-N_0} 2^{-kr_1} \| \Delta_k f \|_{L^{q_2}(I; L^2)},
\]
which implies the first inequality of the lemma. The proof of the other three inequalities are similar.

4. Elliptic estimates in a strip of infinite depth

In this section, we consider the elliptic equation in a strip of infinite depth \( S = \{(x, y) : x \in \mathbb{R}^d, y < \eta(x)\} \):
\[
\begin{cases}
\Delta_{x,y} \phi = g & \text{in } S, \\
\phi|_{y=\eta} = f.
\end{cases}
\] (4.1)
Throughout this section, we assume that \( \eta \in H^{s+\frac{1}{2}}(\mathbb{R}^d) \cap C^{\frac{1}{2}+\varepsilon}(\mathbb{R}^d) \) for \( s > 1 + \frac{d}{2} \) and some \( \varepsilon > 0 \). We denote by \( K_\eta = K_\eta \bigl( \| \eta \|_{C^{\frac{1}{2}+\varepsilon}}, \| \eta \|_{L^2} \bigr) \) a nondecreasing function, which may be different from line to line, \( I = (-\infty, 0) \).

First of all, we flatten the boundary of \( S \) by the following regularized mapping:
\[
(x, z) \in \mathbb{R}^d \times (-\infty, 0] \mapsto (x, \rho_\delta(x, z)) \in \mathcal{S},
\]
where \( \rho_\delta \) with \( \delta > 0 \) is given by
\[
\rho_\delta(x, z) = z + (e^{\delta |D|} \eta)(x).
\] (4.2)

Remark 4.1. For any \( z < 0 \), we have
\[
\| \partial_z \rho_\delta - 1 \|_{L^\infty} \leq \delta \| e^{\delta |D|} |D| \|_{L^\infty} = \delta \| P_{-\delta z} \ast |D| \eta \|_{L^\infty}
\]
\[
\leq C\delta \| P_{-\delta z} \|_{L^1} \| |D| \eta \|_{L^\infty} \leq C\delta \| \eta \|_{C^{\frac{1}{2}+\varepsilon}}.
\]
Here \( P_z(x) \) is the poisson kernel. Throughout this paper, we will fix \( \delta \) small enough depending only on \( \| \eta \|_{C^{\frac{1}{2}+\varepsilon}} \) such that
\[
\| \partial_z \rho_\delta - 1 \|_{L^\infty} \leq \frac{1}{2}, \text{ hence } \partial_z \rho_\delta \geq \frac{1}{2}.
\]
We set \( v(x, z) = \phi(x, \rho_8(x, z)) \). It is easy to find that \( v \) satisfies
\[
\begin{aligned}
&\partial_z^2 v + \alpha \Delta v + \beta \cdot \nabla \partial_z v - \gamma \partial_z v = F_0, \\
&v|_{z=0} = f,
\end{aligned}
\]
(4.3)
where \( F_0 = \alpha g \) and the coefficients \( \alpha, \beta, \gamma \) are defined by
\[
\alpha = \frac{(\partial_z \rho_8)^2}{1 + |\nabla \rho_8|^2}, \quad \beta = -2 \frac{\partial_z \rho_8 \nabla \rho_8}{1 + |\nabla \rho_8|^2}, \quad \gamma = \frac{1}{\partial_z \rho_8} (\partial^2 \rho_8 + \alpha \Delta \rho_8 + \beta \cdot \nabla \partial_z \rho_8). \quad (4.4)
\]
By the definition of \( \rho_8 \), we find
\[
\begin{aligned}
&\partial_z (\rho_8 - z) - \delta |D| (\rho_8 - z) = 0, \\
&\rho_8 - z|_{z=0} = \eta.
\end{aligned}
\]
Then we infer from Proposition 3.2 and Proposition 3.1 that
\[
\begin{aligned}
&\|\nabla \rho_8\|_{X^{\frac{1}{4} + \frac{1}{2}}(I)} + \|\partial_z \rho_8 - 1\|_{X^{\frac{1}{4} + \frac{1}{2}}(I)} \leq C (\|\eta\|_{C^{\frac{3}{2} + \varepsilon}}, \|\eta\|_{H^{\frac{1}{2} + \varepsilon}}), \\
&\|\nabla_{x,z} \rho_8\|_{L^2(I; C^{\frac{1}{2} + \varepsilon})} + \|\partial^2_x \rho_8\|_{L^2(I; C^{-\frac{1}{2} + \varepsilon})} \leq C (\|\eta\|_{C^{\frac{3}{2} + \varepsilon}}).
\end{aligned}
\]
(4.5) (4.6)
In order to obtain the tame elliptic estimates, we paralinearize the elliptic equation (4.3) as
\[
\partial_z^2 v + T_\alpha \Delta v + T_\beta \cdot \nabla \partial_z v = F_0 + F_1 + F_2,
\]
(4.7)
with \( F_1, F_2 \) given by
\[
F_1 = \gamma \partial_z v, \quad F_2 = (T_\alpha - \alpha) \Delta v + (T_\beta - \beta) \cdot \nabla \partial_z v.
\]
As in [3], the equation (4.7) can be decoupled into a forward and a backward parabolic evolution equations:
\[
(\partial_z - T_\alpha)(\partial_z - T_A)v = F_0 + F_1 + F_2 + F_3 \triangleq F,
\]
(4.8)
where
\[
\begin{aligned}
a &= \frac{1}{2} \left(-i \beta \cdot \xi - \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2}\right), \\
A &= \frac{1}{2} \left(-i \beta \cdot \xi + \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2}\right), \\
F_3 &= (T_\alpha T_A - T_A \Delta)v - (T_\alpha + T_A + T_\beta \cdot \nabla) \partial_z v - T_{\partial_z A} v.
\end{aligned}
\]
Remark 4.2. The symbols \( a, A \) satisfy
\[
a(z; x, \xi) \cdot A(z; x, \xi) = -\alpha(x, z)|\xi|^2, \quad a(z; x, \xi) + A(z; x, \xi) = -i \beta(x, z) \cdot \xi.
\]
Noticing that
\[
4\alpha|\xi|^2 - (\beta \cdot \xi)^2 \geq c_2 |\xi|^2
\]
for some \( c_2 > 0 \) depending only on \( \|\eta\|_{C^{\frac{3}{2} + \varepsilon}} \), it follows from (4.6) that
\[
\bar{M}_{\frac{3}{2} + \varepsilon}^1(a) \leq C (\|\eta\|_{C^{\frac{3}{2} + \varepsilon}}), \quad \bar{M}_{\frac{3}{2} + \varepsilon}^1(A) \leq C (\|\eta\|_{C^{\frac{3}{2} + \varepsilon}}).
\]
4.1. Elliptic estimates in Sobolev space.

**Proposition 4.3.** Let $v$ be a solution of (4.3) on $I \times \mathbb{R}^d$. Then for all $\sigma \in [-\frac{1}{2}, s-\frac{1}{2}]$, it holds that

$$\|\nabla_{x,z} v\|_{X^{\sigma}(I)} \leq K_\eta \left( \|\nabla_{x,z} v\|_{L^2(I \times \mathbb{R}^d)} + \|f\|_{H^{\sigma+1}} + \|F_0\|_{Y^\sigma(I)} + \|\eta\|_{H^{\sigma+\frac{1}{2}}} \|\nabla_{x,z} v\|_{L^\infty(I \times \mathbb{R}^d)} \right).$$

Moreover, for $\sigma = -\frac{1}{2}$, we have

$$\|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}(I)} \leq K_\eta \left( \|F_0\|_{Y^{-\frac{1}{2}}(I)} + \|\nabla_{x,z} v\|_{L^2(I \times \mathbb{R}^d)} \right).$$

Before proving the proposition, we make the estimates for the coefficients $\alpha, \beta, \gamma$ and $F_i (i = 1, 2, 3)$.

**Lemma 4.4.** It holds that

$$\|\alpha - 1\|_{X^{\sigma+\frac{1}{2}}(I)} + \|\beta\|_{X^{\sigma+\frac{1}{2}}(I)} + \|\gamma\|_{X^{\sigma+\frac{1}{2}}(I)} \leq K_\eta \|\eta\|_{H^{\sigma+\frac{1}{2}}},$$

$$\|\alpha\|_{\bar{L}_z^{\infty}(I; C^{\frac{1}{2}+\epsilon})} + \|\beta\|_{\bar{L}_z^{\infty}(I; C^{\frac{1}{2}+\epsilon})} + \|\gamma\|_{\bar{L}_z^{\infty}(I; C^{\frac{1}{2}+\epsilon})} \leq K_\eta,$$

$$\|\alpha - 1\|_{\bar{L}_z^{1}(I; H^{\sigma+\frac{1}{2}})} + \|\beta\|_{\bar{L}_z^{1}(I; H^{\sigma+\frac{1}{2}})} + \|\gamma\|_{\bar{L}_z^{1}(I; H^{\sigma+\frac{1}{2}})} \leq K_\eta \|\eta\|_{H^{\sigma+\frac{1}{2}}},$$

$$\|\alpha\|_{\bar{L}_z^{2}(I; C^{1+\epsilon})} + \|\beta\|_{\bar{L}_z^{2}(I; C^{1+\epsilon})} + \|\gamma\|_{\bar{L}_z^{2}(I; C^{1})} \leq K_\eta.$$

**Proof.** Noting $s - \frac{3}{2} > 0$, the first two inequalities of the lemma follows from Lemma 2.8 and (4.5)-(4.6) except that $\|\gamma\|_{\bar{L}_z^{\infty}(I; C^{-\frac{1}{2}+\epsilon})}$. Thanks to (4.6) and Lemma 2.8, $\gamma$ can be written as

$$\gamma = \gamma_1 \nabla^2 \gamma_2, \quad \text{where} \quad \gamma_1 \in \tilde{L}_z^{\infty}(I; C^{\frac{1}{2}+\epsilon}), \; \gamma_2 \in \tilde{L}_z^{\infty}(I; C^{\frac{3}{2}+\epsilon})$$

with the following bounds

$$\|\gamma_1\|_{\tilde{L}_z^{\infty}(I; C^{\frac{1}{2}+\epsilon})} + \|\gamma_2\|_{\tilde{L}_z^{\infty}(I; C^{\frac{3}{2}+\epsilon})} \leq K_\eta.$$

We use Bony’s decomposition (2.3) to write $\gamma_1 \nabla^2 \gamma_2$ as

$$\gamma = T_{\gamma_1} \nabla^2 \gamma_2 + T_{\nabla \gamma_2} \gamma_1 + R(\gamma_1, \nabla^2 \gamma_2).$$

We infer from Lemma 3.4 and Lemma 3.5 that

$$\|T_{\gamma_1} \nabla^2 \gamma_2\|_{\tilde{L}_z^{\infty}(I; C^{-\frac{1}{2}+\epsilon})} \leq C \|\gamma_1\|_{L^\infty} \|\gamma_2\|_{\tilde{L}_z^{\infty}(I; C^{\frac{3}{2}+\epsilon})} \leq K_\eta,$$

$$\|T_{\nabla \gamma_2} \gamma_1\|_{\tilde{L}_z^{\infty}(I; C^{2\epsilon})} \leq \|\gamma_1\|_{\tilde{L}_z^{\infty}(I; C^{\frac{1}{2}+\epsilon})} \|\gamma_2\|_{\tilde{L}_z^{\infty}(I; C^{\frac{3}{2}+\epsilon})} \leq K_\eta,$$

$$\|R(\gamma_1, \nabla^2 \gamma_2)\|_{\tilde{L}_z^{\infty}(I; C^{2\epsilon})} \leq \|\gamma_1\|_{\tilde{L}_z^{\infty}(I; C^{\frac{1}{2}+\epsilon})} \|\gamma_2\|_{\tilde{L}_z^{\infty}(I; C^{\frac{3}{2}+\epsilon})} \leq K_\eta.$$
Indeed, by the definition of $\rho_\delta$, we have
\[
\|\nabla_{x,z}^2 \rho_\delta\|^2_{L^2(I; H^{s-\frac{1}{2}})} \leq \sum_j 2^{2j(s-\frac{1}{2})} \|\Delta_j \nabla_{x,z}^2 \rho_\delta\|^2_{L^2(I; L^2)} \\
\leq C \sum_j 2^{2j(s-\frac{1}{2})} \|\Delta_j |D| \epsilon^\delta z |D| \eta\|^2_{L^2(I; L^2)} \\
\leq C \sum_j 2^{2j(s+\frac{1}{2})} \|\Delta_j \eta\|^2_{L^2} \leq C \|\eta\|_{H^{s+\frac{1}{2}}},
\]
and by Lemma 3.1, we get
\[
\|\nabla_{x,z} (\rho_\delta - z)\|_{L^2(I; C^{1+\varepsilon})} \leq \sup_j 2^{j(1+\varepsilon)} \|\Delta_j \nabla_{x,z} \rho_\delta\|_{L^2(I; L^\infty)} \\
\leq \sup_j 2^{j(1+\varepsilon)} \|\Delta_j |D| \epsilon^\delta z |D| \eta\|_{L^2(I; L^\infty)} \\
\leq \sup_{j>0} 2^{j(2+\varepsilon)} \|\epsilon^\delta z \Delta_j \eta\|_{L^\infty} + \|\Delta_0 |D| \epsilon^\delta z \eta\|_{L^2(I; L^\infty)} \\
\leq C \|\eta\|_{C^{\frac{1}{2}+\varepsilon}} + C \|\eta\|_{L^2},
\]
where we use the estimate in the last inequality:
\[
\|\Delta_0 |D| \epsilon^\delta z \eta\|_{L^2(I; L^\infty)} \leq C \|\epsilon^\delta z |D| |\Delta_0 \eta\|_{L^2(I \times \mathbb{R}^d)} \leq C \|\eta\|_{L^2}.
\]
The proof is finished. \(\square\)

**Lemma 4.5.** For any \(0 < \delta_1 \leq \frac{1}{2}\) and \(\sigma + \delta_1 \leq s - \frac{1}{2}\), it holds that
\[
\|F_1\|_{Y^{s+\delta_1}(I)} \leq K_\eta (\|\partial_z v\|_{L^2(I; H^{s+\delta_1-\varepsilon})} + \|\partial_z v\|_{L^\infty(I \times \mathbb{R}^d)} \|\eta\|_{H^{s+\frac{1}{2}}}).
\]

**Proof.** Using Bony’s decomposition (2.3), we write \(F_1\) as
\[
F_1 = \gamma \partial_z v = T_\gamma \partial_z v + T_\partial_z v \gamma + R(\gamma, \partial_z v).
\]
We infer from Lemma 3.3 that
\[
\|T_\gamma \partial_z v\|_{L^2(I; H^{s+\delta_1-\frac{1}{2}})} \leq C \|\gamma\|_{L^\infty(I; C^{-\frac{1}{2}+\varepsilon})} \|\partial_z v\|_{L^2(I; H^{s+\delta_1-\varepsilon})}, \\
\|T_\partial_z v \gamma\|_{L^2(I; H^{s+\delta_1-\frac{1}{2}})} \leq C \|\partial_z v\|_{L^\infty(I \times \mathbb{R}^d)} \|\gamma\|_{L^2(I; H^{s+\delta_1-\frac{1}{2}})},
\]
and by noting \(\sigma + \delta_1 \leq s - \frac{1}{2}\),
\[
\|R(\gamma, \partial_z v)\|_{L^1(I; H^{s+\delta_1})} \leq C \|\partial_z v\|_{L^\infty(I; C^\alpha)} \|\gamma\|_{L^1(I; H^{s+\frac{1}{2}})}.
\]
This together with Lemma 4.4 gives the lemma. \(\square\)

**Lemma 4.6.** For any \(0 < \delta_1 \leq \frac{1}{2}\) and \(\sigma + \delta_1 \leq s - \frac{1}{2}\), it holds that
\[
\|F_2\|_{Y^{s+\delta_1}(I)} \leq K_\eta \|\nabla_{x,z} v\|_{L^\infty(I \times \mathbb{R}^d)} \|\eta\|_{H^{s+\frac{1}{2}}},
\]

**Proof.** Recalling \(F_2 = (T_\alpha - \alpha) \Delta v + (T_\beta - \beta) \cdot \nabla \partial_z v\), it suffices to consider \((T_\alpha - \alpha) \Delta v\).
We get by (2.3) that
\[
(T_\alpha - \alpha) \Delta v = -T_\Delta v (\alpha - 1) - R(\Delta v, \alpha - 1).
\]
Due to $\sigma + \delta_1 \leq s - \frac{1}{2}$, we infer from Lemma 3.4 and Lemma 3.5 that
\[
\|T_{\Delta v}(\alpha - 1)\|_{L^2_2(I;H^{\sigma + \delta_1 - \frac{1}{2}})} \leq C\|\nabla v\|_{L^\infty_2(I \times \mathbb{R}^d)}\|\alpha - 1\|_{L^2_2(I;H^{\sigma + \delta_1 + \frac{1}{2}})}
\leq C\|\nabla v\|_{L^\infty_2(I \times \mathbb{R}^d)}\|\alpha - 1\|_{L^2_2(I;H^s)},
\]
\[
\|R(\Delta v, \alpha - 1)\|_{L^2_2(I;H^{\sigma + \delta_1})} \leq \|R(\Delta v, \alpha - 1)\|_{L^2_2(I;H^{s - \frac{1}{2}})}
\leq C\|\nabla v\|_{L^\infty_2(I \times C^0)}\|\alpha - 1\|_{L^2_2(I;H^{s + \frac{1}{2}})},
\]
which along with Lemma 4.4 give the lemma.

\begin{proof}
\end{proof}

Lemma 4.7. For any $\sigma, \delta_1 \in \mathbb{R}$, it holds that
\[
\|F_3\|_{Y^{\sigma + \delta_1}(I)} \leq K_\eta\|\nabla v\|_{L^2_2(I;H^{\sigma + \delta_1 - \epsilon})}.
\]

\begin{proof}
\end{proof}

Lemma 4.8. It holds that
\[
\|\partial_\xi^2 \partial_z A(z; \cdot, \xi)\|_{C^{-\frac{1}{2}, \epsilon}_z} \leq K_\eta(1 + |\xi|)^{1-|\alpha|} \quad \text{for any } |\xi| \geq \frac{1}{2},
\]
from which and Remark 2.7, it follows that
\[
\|T_\partial_z A\|_{L^2_2(I;H^{\sigma + \delta_1 - \frac{1}{2}})} \leq K_\eta\|\nabla v\|_{L^2_2(I;H^{\sigma + \delta_1 - \epsilon})}.
\]

The proof is completed.

\begin{proof}
\end{proof}

Lemma 4.9. It holds that
\[
\|\nabla_\alpha \nabla v\|_{L^1_2(I;H^{-\frac{1}{2}})} + \|\nabla_\beta \partial_z v\|_{L^1_1(I;H^{-\frac{1}{2}})} + \|\gamma \partial_z v\|_{L^1_1(I;H^{-\frac{1}{2}})}
\leq K_\eta\|\nabla_{x,z} v\|_{L^2_2(I \times \mathbb{R}^d)}.
\]

\begin{proof}
\end{proof}

and by the proof of Lemma 3.5, we see that
\[
\|R(g, \partial_z v)\|_{L^2_2(I;H^{s - \epsilon})} \leq \int \left( \sum_j 2^{-j} \left( \sum_{|k-\ell| \leq 2j, \ell \geq j} \|\Delta_{k,\ell} g\|_{L^\infty} \|\Delta_{k,\ell} \partial_z v\|_{L^2} \right)^2 dz \right)^{\frac{1}{2}} dz
\leq C \int \sum_j 2^{-j} \left( \sum_{|k-\ell| \leq 2j, \ell \geq j} \|\Delta_{k,\ell} g\|_{L^\infty} \|\Delta_{k,\ell} \partial_z v\|_{L^2} dz \right)^{\frac{1}{2}} dz
\leq C \|\gamma\|_{L^2_2(I;C^0)}\|\partial_z v\|_{L^2_2(I \times \mathbb{R}^d)},
\]
which along with Lemma 4.4 and (2.3) give
\[
\|\gamma \partial_z v\|_{L^1_2(I;H^{-\frac{1}{2}})} \leq K_\eta\|\partial_z v\|_{L^2_2(I \times \mathbb{R}^d)}.
\]
The estimates for the other two terms are similar.
Now let us turn to the proof of Proposition 4.3.

**Proof of Proposition 4.3.** First of all, we consider $\sigma = -\frac{1}{2}$. We have

$$
\left(\nabla v(z), \nabla v(z)\right)_{H^{-\frac{1}{2}}} \leq 2 \int_{-\infty}^{z} \left(\partial_{z}^2 \nabla v(z'), \nabla v(z')\right)_{H^{-\frac{1}{2}}} dz' \leq 2 \left\| \nabla_{x,z} v \right\|^2_{L^2(I \times \mathbb{R}^d)},
$$

and by the equation (4.3) and Lemma 4.8, we get

$$
\left(\partial_{z} v(z), \partial_{z} v(z)\right)_{H^{-\frac{1}{2}}} = 2 \int_{-\infty}^{z} \left(\partial_{z}^2 v(z'), \partial_{z} v(z')\right)_{H^{-\frac{1}{2}}} dz' = 2 \int_{-\infty}^{z} \left(F_0 - \alpha \Delta v + \beta \partial_{z} v - \gamma \partial_{z} v, \partial_{z} v\right)_{H^{-\frac{1}{2}}} dz' \leq \left(\left\| F_0 \right\|_{Y^{'-\frac{1}{2}}(I)} + \left\| \text{div} (\alpha \nabla v + \beta \partial_{z} v) \right\|_{L^2(I;H^{-1})} + \left\| \nabla \alpha \nabla v + \nabla \beta \partial_{z} v + \gamma \partial_{z} v \right\|_{L^2(I;H^{-\frac{1}{2}})} \right) \left\| \partial_{z} v \right\|_{X^{-\frac{1}{2}}(I)} \leq K_{\eta} \left(\left\| F_0 \right\|_{Y^{'-\frac{1}{2}}(I)} + \left\| \nabla_{x,z} v \right\|_{L^2(I \times \mathbb{R}^d)} \right) \left\| \partial_{z} v \right\|_{X^{-\frac{1}{2}}(I)}.
$$

This implies the case of $\sigma = -\frac{1}{2}$.

For general $\sigma$, we use the bootstrap argument. To this end, let us first assume

$$
\left\| \nabla_{x,z} v \right\|_{X^{\sigma}(I)} \leq K_{\eta} \left(\left\| \nabla_{x,z} v \right\|_{L^2(I \times \mathbb{R}^d)} + \left\| f \right\|_{H^{\sigma+1}} + \left\| F_0 \right\|_{Y^{'\sigma}(I)} + \left\| \eta \right\|_{H^{\sigma+\frac{1}{2}}(I)} \right) \left\| \nabla_{x,z} v \right\|_{L^\infty(I \times \mathbb{R}^d)}.
$$

Then we show that the inequality remains true for $r + \delta_1 \leq s - \frac{1}{2}$ with $\delta_1 \leq \frac{1}{2}$, thus the proposition follows since it is true for $r = -\frac{1}{2}$.

Set $w = (\partial_{z} - T_A)v$. Then $(v, w)$ satisfies the forward and backward parabolic equation respectively:

$$
(\partial_{z} - T_A)w = F \quad \text{on } I \times \mathbb{R}^d, \quad w|_{z=\infty} = 0,
$$

$$
(\partial_{z} - T_A)v = w \quad \text{on } I \times \mathbb{R}^d, \quad v|_{z=0} = f.
$$

By Proposition 3.2 and Lemma 4.5-Lemma 4.7, we infer that

$$
\left\| w \right\|_{X^{r+\delta_1}(I)} \leq K_{\eta} \left(\left\| w \right\|_{L^2(I;H^{r+\delta_1})} \right) \leq K_{\eta} \left(\left\| F \right\|_{Y^{r+\delta_1}(I)} + \left\| \nabla_{x,z} v \right\|_{L^2(I;H^{r+\delta_1-\epsilon})} \right) \left\| \nabla_{x,z} v \right\|_{L^\infty(I \times \mathbb{R}^d)} \left\| \eta \right\|_{H^{r+\frac{1}{2}}(I)}. \quad (4.11)
$$

Here we use the estimate

$$
\left\| w \right\|_{L^2(I;H^{r+\delta_1})} \leq K_{\eta} \left\| \nabla_{x,z} v \right\|_{L^2(I;H^{r+\delta_1})} \quad (\text{by Proposition 2.4}).
$$

Take $\nabla$ to the equation of $v$ to get

$$
(\partial_{z} - T_A)\nabla v = \nabla w + T_{\nabla A} v \quad \text{on } I \times \mathbb{R}^d, \quad \nabla v|_{z=0} = \nabla f.
$$

By Remark 2.7 and Remark 4.2, we have

$$
\left\| T_{\nabla A} v \right\|_{L^2(I;H^{r+\delta_1-\frac{1}{2}})} \leq K_{\eta} \left\| \nabla v \right\|_{L^2(I;H^{r+\delta_1-\epsilon})}.
$$

Then by Proposition 3.2 and (4.11), we get by using $\partial_{z}v = T_A v + w$ that

$$
\left\| \nabla_{x,z} v \right\|_{X^{r+\delta_1}(I)} \leq K_{\eta} \left(\left\| w \right\|_{X^{r+\delta_1}(I)} + \left\| \nabla v \right\|_{L^2(I;H^{r+\delta_1})} + \left\| f \right\|_{H^{r+1+\delta_1}} \right)
$$

$$
\leq K_{\eta} \left(\left\| f \right\|_{H^{r+1+\delta_1}} + \left\| F_0 \right\|_{Y^{r+\delta_1}(I)} + \left\| \nabla_{x,z} v \right\|_{L^2(I;H^{r+\delta_1-\epsilon})} \right) \left\| \nabla_{x,z} v \right\|_{L^\infty(I \times \mathbb{R}^d)} \left\| \eta \right\|_{H^{r+\frac{1}{2}}(I)} \leq K_{\eta} \left(\left\| \nabla_{x,z} v \right\|_{L^2(I \times \mathbb{R}^d)} + \left\| f \right\|_{H^{r+\delta_1+1}} + \left\| F_0 \right\|_{Y^{r+\delta_1}(I)} + \left\| \eta \right\|_{H^{r+\frac{1}{2}}(I)} \right) \left\| \nabla_{x,z} v \right\|_{L^\infty(I \times \mathbb{R}^d)}.
$$
This completes the proof of the proposition. 

4.2. Elliptic estimates in Besov space.

**Proposition 4.9.** Let \( q \in [1, \infty] \) and \( v \) be a solution of (4.3) on \( I \times \mathbb{R}^d \). Then for any \( r \in [0, \frac{1}{2}] \), \( \delta_2 > 0 \), it holds that

\[
\| \nabla x,v \|_{L^2_q(I;B^r_{\infty,q})} + \| \nabla x,v \|_{L^2_q(I;B^{r+\frac{1}{2}}_{\infty,q})} \\
\leq K_\eta (\| \nabla f \|_{B^r_{\infty,q}} + \| \partial \bar{\gamma}_q \|_{L^2_q(I;C^{-\delta_2})})
\]

where \( \bar{\gamma}_q(I) \overset{\text{def}}{=} \bar{L}_z(I;B^r_{\infty,q}) + \bar{L}_z(I;B^{r-\frac{1}{2}}_{\infty,q}) \).

Let us first present the Hölder estimates of \( F \).

**Lemma 4.10.** For any \( r \leq \frac{1}{2} \) and \( q \in [1, \infty] \), we have

\[
\| F_1 \|_{L^2_q(I;B^{r-\frac{1}{2}}_{\infty,q})} \leq K_\eta (\| \nabla x,v \|_{L^2_q(I;C^{-\delta_2})} + \| \partial z,v \|_{L^2_q(I;B^{r-\frac{1}{2}}_{\infty,q})})
\]

**Proof.** By Lemma 3.4 and Lemma 3.5, we have

\[
\| T_\gamma \partial_z v \|_{L^2_q(I;B^{r-\frac{1}{2}}_{\infty,q})} \leq C(\| \nabla x,v \|_{L^2_q(I;C^{-\delta_2})} + \| \partial z,v \|_{L^2_q(I;B^{r-\frac{1}{2}}_{\infty,q})})
\]

which along with Lemma 4.4 and (2.3) gives the lemma.

**Lemma 4.11.** For any \( r \leq \frac{1}{2} \) and \( q \in [1, \infty] \), we have

\[
\| F_2 \|_{L^2_q(I;B^{r-\frac{1}{2}}_{\infty,q})} \leq K_\eta \| \nabla x,v \|_{L^2_q(I;C^{-\delta_2})}
\]

**Proof.** Using (2.3), we infer from Lemma 3.4 and Lemma 3.5 that

\[
\| F_2 \|_{L^2_q(I;B^{r-\frac{1}{2}}_{\infty,q})} \leq \| (T_\alpha - \alpha) \Delta v \|_{L^2_q(I;B^{r-\frac{1}{2}}_{\infty,q})} + \| (T_\beta - \beta) \nabla v \|_{L^2_q(I;B^{r-\frac{1}{2}}_{\infty,q})}
\]

The proof is finished.

**Lemma 4.12.** For any \( r \leq \frac{1}{2} \) and \( q \in [1, \infty] \), we have

\[
\| F_3 \|_{L^2_q(I;B^{r-\frac{1}{2}}_{\infty,q})} \leq K_\eta \| \nabla x,v \|_{L^2_q(I;C^{-\delta_2})}
\]

**Proof.** From the proof of Lemma 2.6, we see that

\[
\| (T_\alpha T_\lambda - T_\lambda \Delta) v \|_{L^2_q(I;B^{r-\frac{1}{2}}_{\infty,q})} \leq C(\| M_0^1(a) \|_{L^2_q(I;M^{1+\epsilon}_{1+\epsilon})} + \| a \|_{L^2_q(I;M^{1+\epsilon}_{1+\epsilon})} \| \nabla v \|_{L^2_q(I;C^{-\delta_2})})
\]

where we denote

\[
\| a \|_{L^2_q(I;M^{1+\epsilon}_{1+\epsilon})} \overset{\text{def}}{=} \sup_{|\alpha|\leq d/2+1+\rho} \sup_{|\xi|\geq 1/2} \| (1 + |\xi|)^{|\alpha|-m} \partial^\alpha v \|_{L^2_q(I;C^{1+\epsilon})},
\]

and
and by Lemma 4.4, we have
\[ \|a\| \tilde{L}_2(I; M_{1+\varepsilon}) + \|A\| \tilde{L}_2(I; M_{1+\varepsilon}) \leq K_\eta. \]

The estimate for the other parts of \( F_3 \) is similar. \( \square \)

Now let us turn to the proof of Proposition 4.9.

**Proof of Proposition 4.9.** Recall that if we set \( w = (\partial_z - T_A)v \), then \((v, w)\) satisfies
\[ (\partial_z - T_A)w = F \quad \text{on} \quad I \times \mathbb{R}^d, \quad w|_{z = -\infty} = 0, \]
\[ (\partial_z - T_A)v = w \quad \text{on} \quad I \times \mathbb{R}^d, \quad v|_{z = 0} = f. \]

By Proposition 3.1 and Lemma 4.10-Lemma 4.12, we deduce that
\[ \|w\|_{\tilde{L}_2(I; B_{\infty,q}^\alpha)} + \|w\|_{E_2(I; B_{\infty,q}^{\alpha + \frac{1}{2}})} \leq K_\eta \left( \|F\|_{Y_q(I)} + \|w\|_{\tilde{L}_2(I; C^{-\frac{1}{2}})} + \|\nabla_x z v\|_{L_2^\infty(I; C^{-\frac{1}{2}})} \right), \]
and noting that \( (\partial_z - T_A)\nabla v = \nabla w + T_{\nabla A}v \), we get by Proposition 3.1 that
\[ \|\nabla v\|_{\tilde{L}_2(I; B_{\infty,q}^\alpha)} + \|\nabla v\|_{E_2(I; B_{\infty,q}^{\alpha + \frac{1}{2}})} \leq K_\eta \left( \|\nabla f\|_{B_{\infty,q}^\alpha} + \|w\|_{L_2(I; B_{\infty,q}^{\alpha + \frac{1}{2}})} + \|\nabla x, z v\|_{L_2^\infty(I; B_{\infty,q}^{\alpha + \frac{1}{2}})} + \|\nabla x, z v\|_{E_2(I; C^{-\frac{1}{2}})} \right). \]

Thus, we obtain
\[ \|\nabla x, z v\|_{L_2^\infty(I; B_{\infty,q}^\alpha)} + \|\nabla x, z v\|_{E_2(I; B_{\infty,q}^{\alpha + \frac{1}{2}})} \leq K_\eta \left( \|\nabla f\|_{B_{\infty,q}^\alpha} + \|F_0\|_{\tilde{L}_2(I; B_{\infty,q}^{\alpha + \frac{1}{2}})} + \|\nabla x, z v\|_{L_2^\infty(I; B_{\infty,q}^{\alpha + \frac{1}{2}})} + \|\nabla x, z v\|_{E_2(I; C^{-\frac{1}{2}})} \right), \]

from which and the interpolation, we conclude the proof of the proposition. \( \square \)

5. **Dirichlet-Neumann operator**

5.1. **Definition and basic properties.** We consider the boundary value problem

\[
\begin{align*}
\Delta_{x,y} \phi &= 0 \quad \text{in} \quad S, \\
\phi|_{y = \eta} &= f,
\end{align*}
\]  

where \( S = \{(x, y) : x \in \mathbb{R}^d, y < \eta(x)\} \). Given \( f \in H^{\frac{1}{2}}(\mathbb{R}^d) \), the existence of the variation solution \( \phi \) with \( \nabla_{x,y} \phi \in L^2(S) \) can be deduced by using Riesz theorem, see [3] for example. Moreover, it holds that
\[ \|\nabla_{x,y} \phi\|_{L^2(S)} \leq C(\|\nabla \eta\|_{L^\infty} \|f\|_{H^{\frac{1}{2}}}). \]  

**Definition 5.1.** Given \( \eta, f, \phi \) as above, the Dirichlet-Neumann operator \( G(\eta) \) is defined by
\[ G(\eta)f \overset{\text{def}}{=} \sqrt{1 + |\nabla \eta|^2} \partial_n \phi|_{y = \eta}. \]

We have the following basic properties for \( G(\eta) \), see [17].

**Proposition 5.2.** It holds that
1. the operator \(G(\eta)\) is self-adjoint:
   \[
   (G(\eta)f, g) = (f, G(\eta)g), \quad \forall f, g \in H^{\frac{1}{2}}(\mathbb{R}^d);
   \]

2. the operator \(G(\eta)\) is positive:
   \[
   (G(\eta)f, f) = \|\nabla_x, g\|_{L^2(S)} \geq 0, \quad \forall f \in H^{\frac{1}{2}}(\mathbb{R}^d);
   \]

3. for any \(f, g \in H^{\frac{1}{2}}(\mathbb{R}^d)\), we have
   \[
   \|(G(\eta)f, g)\| \leq C(\|\nabla \eta\|_{L^\infty}) \|f\|_{H^{\frac{1}{2}}} \|g\|_{H^{\frac{1}{2}}};
   \]

4. the shape derivative \(d_\eta G(\eta)\) of \(G(\eta)\) is
   \[
   d_\eta G(\eta) \psi \cdot \partial_\eta = -G(\eta) (\partial_\eta B) - \text{div}(\partial_\eta V),
   \]
   where \(V = \nabla \phi |_{y = \eta}, B = \partial_\eta \phi |_{y = \eta}\).

**Remark 5.3.** By the definition of Dirichlet-Neumann operator \(G(\eta)\), it is easy to see that

\[
B = \frac{\nabla \eta \cdot \nabla \psi + G(\eta)\psi}{1 + |\nabla \eta|^2}, \quad V = \nabla \psi - B\nabla \eta.
\]

With the notations in Section 4, we denote \(v(x, z) = \phi(x, \rho_\delta(x, z))\). In terms of \(v\), the Dirichlet-Neumann operator \(G(\eta)\) can be written as

\[
G(\eta)f = \left(\frac{1 + |\nabla \rho_\delta|^2}{\partial_z \rho_\delta} \partial_z v - \nabla \rho_\delta \cdot \nabla v \right)_{|z=0}.
\] (5.3)

**5.2. Tame estimates of the Dirichlet-Neumann operator.** In this subsection, we assume that \(\eta \in H^{s+\frac{1}{2}}(\mathbb{R}^d) \cap C^{2,\epsilon} \cdot (\mathbb{R}^d)\) for \(s > 1 + \frac{d}{2}\) and \(\epsilon > 0\). We denote \(I = (-\infty, 0)\) and by \(K_\eta = K_\eta(\|\eta\|_{C^{2,\epsilon}}(\mathbb{R}^d), \|\eta\|_{L^2})\) an increasing function.

Following [3], we first paralinearize \(G(\eta)\). We set

\[
\zeta_1 = \frac{1 + |\nabla \rho_\delta|^2}{\partial_z \rho_\delta}, \quad \zeta_2 = \nabla \rho_\delta.
\]

By Lemma 2.8 and (4.6), we have

\[
\|\zeta_1 - 1\|_{L^\infty(I; H^{s+\frac{3}{2}})} + \|\zeta_2\|_{L^\infty(I; H^{s+\frac{1}{2}})} \leq K_\eta \|\eta\|_{H^{s+\frac{1}{2}}},
\] (5.4)

\[
\|\zeta_1\|_{L^\infty(I; C^{2,\epsilon})} + \|\zeta_2\|_{L^\infty(I; C^{2,\epsilon})} \leq K_\eta.
\] (5.5)

Using Bony's decomposition (2.3), we decompose \(G(\eta)\) as

\[
G(\eta)f = T_{\zeta_1} \partial_z v + T_{\partial_z v} \zeta_1 - R(\zeta_1, \partial_z v) - T_{\partial_z v} \zeta_2 - R(\zeta_2, \nabla v)_{|z=0}.
\]

Replacing \(\partial_z v\) by \(T_A v\), we get

\[
G(\eta)f = T_A f + R(\eta)f, \quad \text{where } \lambda = \zeta_1 A - i \zeta_2 \cdot \xi \text{ with } A = \frac{1}{2}(-i\vec{\beta} \cdot \xi + \sqrt{4\alpha |\xi|^2 - (\vec{\beta} \cdot \xi)^2}) \text{ and } R(\eta)f = \left[ (T_{\zeta_1} T_A - T_{\zeta_1 A}) v - T_{\zeta_1} (\partial_z - T_A) v \
+ (T_{\partial_z v} \zeta_1 + R(\zeta_1, \partial_z v) - T_{\partial_v v} \zeta_2 - R(\nabla v, \zeta_2)) \right]_{|z=0} = R_1(\eta)f + R_2(\eta)f + R_3(\eta)f.
\] (5.6)
Proposition 5.4. It holds that
\[ \|R(\eta)f\|_{H^{s-\frac{1}{2}}} \leq K_\eta \left( \|f\|_{H^s} + \|\nabla x,z v\|_{L^\infty(I \times \mathbb{R}^d)} \|\eta\|_{H^{s+\frac{1}{2}}} \right), \]
\[ \|R(\eta)f\|_{H^{s-1}} \leq K_\eta \left( \|f\|_{H^{s-\frac{1}{2}}} + \|\nabla x,z v\|_{L^\infty(I \times \mathbb{R}^d)} \|\eta\|_{H^{s+\frac{1}{2}}} \right). \]

Proof. Recalling that \( A \in \Gamma_{\frac{1}{2} + \varepsilon}^1(I \times \mathbb{R}^d) \), we get by Proposition 2.4, (5.5), Proposition 4.3 and (5.2) that
\[ \|R_1(\eta)f\|_{H^{s-\frac{1}{2}}} \leq K_\eta \|\nabla v\|_{L^\infty(I ; H^{s-1})}, \]
\[ \leq K_\eta \left( \|\nabla x,z v\|_{L^2(I \times \mathbb{R}^d)} + \|f\|_{H^s} + \|\nabla x,z v\|_{L^\infty(I ; L^\infty)} \|\eta\|_{H^{s+\frac{1}{2}}} \right), \]
\[ \leq K_\eta \left( \|f\|_{H^s} + \|\nabla x,z v\|_{L^\infty(I \times \mathbb{R}^d)} \|\eta\|_{H^{s+\frac{1}{2}}} \right), \]
and by Proposition 2.4, we have
\[ \|R_2(\eta)f\|_{H^{s-\frac{1}{2}}} \leq K_\eta \|\partial_z - T_A v\|_{L^\infty(I ; H^{s-1})} \leq K_\eta \|\nabla x,z v\|_{X^{-1}(I)} \leq K_\eta \left( \|f\|_{H^s} + \|\nabla x,z v\|_{L^\infty(I \times \mathbb{R}^d)} \|\eta\|_{H^{s+\frac{1}{2}}} \right). \]

For \( R_3(\eta) \), we infer from Lemma 3.3, Lemma 3.5 and (5.4) that
\[ \|R_3(\eta)f\|_{H^{s-\frac{1}{2}}} \leq K_\eta \|\nabla x,z v\|_{L^\infty(I \times \mathbb{R}^d)} \|\eta\|_{H^{s+\frac{1}{2}}}. \]
This gives the first inequality. The proof of the second inequality is similar. \( \square \)

Proposition 5.5. For any \( \delta_3 > 0 \), it holds that
\[ \|R(\eta)f\|_{C_{\frac{1}{2}}} \leq K_\eta \left( \|f\|_{C^1} + \|\nabla x,z v\|_{\tilde{L}^2_\infty(I ; C^{-\delta_3})} + \|\nabla x,z v\|_{L^2_\infty(I ; C^{-\delta_3})} \right). \]

Proof. By Lemma 2.6, Proposition 4.9 and (5.5), we get
\[ \|R_1(\eta)f\|_{C_{\frac{1}{2}}} \leq K_\eta \|\nabla v\|_{L^\infty(I ; C^0)} \leq K_\eta \left( \|f\|_{C^1} + \|\nabla x,z v\|_{\tilde{L}^2_\infty(I ; C^{-\delta_3})} + \|\nabla x,z v\|_{L^2_\infty(I ; C^{-\delta_3})} \right). \]

Set \( w = (\partial_z - T_A) v \). From the proof of Proposition 4.9(with \( r = \frac{1}{2} \)), we see that
\[ \|w\|_{L^\infty(I ; C^\frac{1}{2})} \leq K_\eta \|\nabla x,z v\|_{\tilde{L}^2_\infty(I ; C^0)}, \]

hence,
\[ \|R_2(\eta)f\|_{C_{\frac{1}{2}}} \leq K_\eta \|\partial_z - T_A v\|_{L^\infty(I ; C^\frac{1}{2})} \leq K_\eta \left( \|f\|_{C^1} + \|\nabla x,z v\|_{\tilde{L}^2_\infty(I ; C^{-\delta_3})} + \|\nabla x,z v\|_{L^2_\infty(I ; C^{-\delta_3})} \right). \]

It is easy to show by Lemma 3.4 and Lemma 3.5 that
\[ \|R_3(\eta)f\|_{C_{\frac{1}{2}}} \leq K_\eta \|\nabla x,z v\|_{L^\infty(I ; C^0)} \leq K_\eta \left( \|f\|_{C^1} + \|\nabla x,z v\|_{\tilde{L}^2_\infty(I ; C^{-\delta_3})} + \|\nabla x,z v\|_{L^2_\infty(I ; C^{-\delta_3})} \right). \]

This finishes the proof. \( \square \)

Remark 5.6. The estimates of \( R(\eta) \) may not be optimal, but it is suitable for our application. We refer to [3] for more sharper estimates.

Remark 5.7. By (5.5) and Remark 4.2, we know that \( \lambda \in \Gamma_{\frac{1}{2} + \varepsilon}^1(I \times \mathbb{R}^d) \) with the bound
\[ \tilde{M}_{\frac{1}{2}+\varepsilon}^1(\lambda) \leq K_\eta. \]
6. The estimate of the pressure

Throughout this section, we denote \( \Omega_t = \{(x, y) : x \in \mathbb{R}^d, y < \eta(t,x)\}, I = (-\infty, 0), \bar{f}(t, x, z) = f(t, x, \rho_\delta(x, z)), \) and by \( K_\eta = K(||\eta||_{C^{2+\varepsilon}}, ||\eta||_{L^2}) \) a nondecreasing function.

6.1. The estimates of the velocity potential. Recall that the velocity potential \( \phi \) satisfies

\[
\Delta_{x,y} \phi = 0 \quad \text{in} \quad \Omega_t, \quad \phi|_{y=\eta} = \psi. \tag{6.1}
\]

We infer from Proposition 5.2 that

\[
||\nabla_{x,y} \phi||_{L^2(\Omega_t)} = (G(\eta)\psi, \psi)^\frac{1}{2} \equiv E(\psi). \tag{6.2}
\]

By the definition of \( (V, B) \), we have

\[
\Delta_{x,y}(\nabla \phi) = 0 \quad \text{in} \quad \Omega_t, \quad \nabla \phi|_{y=\eta} = V, \\
\Delta_{x,y}(\partial_y \phi) = 0 \quad \text{in} \quad \Omega_t, \quad \partial_y \phi|_{y=\eta} = B.
\]

Then it follows from (5.2) and the maximum principle that

\[
||\nabla_{x,y} \phi||_{L^2(\Omega_t)} \leq K_\eta ||(V, B)||_{H^\frac{1}{2}}, \tag{6.3}
\]

\[
||\nabla_{x,y} \phi||_{L^\infty(\Omega_t)} \leq ||(V, B)||_{L^\infty}. \tag{6.4}
\]

Further more, we find that

\[
\left\{\begin{array}{l}
\Delta_{x,y} \partial_{x_1}(\nabla \phi) = 0 \quad \text{in} \quad \Omega_t, \\
\partial_{x_1} \nabla \phi|_{y=\eta} = \partial_{x_1} V - \partial_{x_1} \eta \left( \nabla B + \frac{\text{div} V - \nabla B \cdot \nabla \eta}{1 + ||\nabla \eta||^2} \right),
\end{array}\right.
\]

and

\[
\left\{\begin{array}{l}
\Delta_{x,y} \partial_y(\nabla \phi) = 0 \quad \text{in} \quad \Omega_t, \\
\partial_y \nabla \phi|_{y=\eta} = \nabla \eta \left( \frac{\text{div} V - \nabla B \cdot \nabla \eta}{1 + ||\nabla \eta||^2} \right),
\end{array}\right.
\]

and

\[
\left\{\begin{array}{l}
\Delta_{x,y}(\partial_y^2 \phi) = 0 \quad \text{in} \quad \Omega_t, \\
\partial_y^2 \phi|_{y=\eta} = - \frac{\text{div} V - \nabla B \cdot \nabla \eta}{1 + ||\nabla \eta||^2}.
\end{array}\right.
\]

Using the maximum principle, we deduce that

\[
||\nabla_{x,y} \phi||_{L^\infty(\Omega_t)} \leq K_\eta ||(V, \nabla \phi)||_{L^\infty}. \tag{6.5}
\]

Then by Proposition 4.3 and (6.3), we infer that

\[
||\nabla_{x,z}(\vec{\nabla} \phi, \partial_y \phi)||_{X^{s-1}(I)} \leq K_\eta \left( ||(V, B)||_{H^s} + ||\eta||_{H^{s+\frac{1}{2}}} ||\nabla_{x,y} \phi||_{L^\infty(\Omega_t)} \right) \\
\leq K_\eta \left( ||(V, B)||_{H^s} + ||\eta||_{H^{s+\frac{1}{2}}} ||(\nabla V, \nabla B)||_{L^\infty} \right) \tag{6.6}
\]

And by Proposition 4.9 with \( \delta_2 > \frac{d}{2} \), (6.4) and (6.2), we get

\[
||\nabla_{x,z} \phi||_{L^\infty(I; C^\frac{3}{2})} + ||\nabla_{x,z} \phi||_{L^2(I; C^1)} \\
\leq K_\eta \left( ||\nabla \psi||_{C^\frac{3}{2}} + ||\nabla_{x,z} \phi||_{L^\infty(I; C^{s-\delta_2})} + ||\nabla_{x,z} \phi||_{L^2(I; C^{s-\delta_2})} \right) \\
\leq K_\eta \left( ||\nabla \psi||_{C^\frac{3}{2}} + ||\nabla_{x,z} \phi||_{L^\infty(I \times \mathbb{R}^d)} + ||\nabla_{x,z} \phi||_{L^2(I \times \mathbb{R}^d)} \right) \\
\leq K_\eta \left( ||(V, B)||_{W^{1,\infty}} + E_0(\psi) \right). \tag{6.7}
\]

Here we use the fact that \( \nabla \psi = V + B \nabla \eta \) so that \( ||\nabla \psi||_{C^\frac{3}{2}} \leq K_\eta ||(V, B)||_{W^{1,\infty}}. \)
Using (2.3), we get by Lemma 3.4 and Lemma 3.5 that
\[ \| \alpha \Delta \tilde{\phi} + \beta \cdot \nabla \partial_\xi \tilde{\phi} \|_{L^2(I;C^0)} \leq C \| \nabla_{x,z} \tilde{\phi} \|_{L^2(I;C^1)} \|(\alpha, \beta)\|_{L^\infty(I;C^1)}, \]
\[ \| \gamma \partial_\xi \tilde{\phi} \|_{L^2(I;C^0)} \leq C \| \partial_\xi \tilde{\phi} \|_{L^\infty(I;C^1)} \| \gamma \|_{L^2(I;C^0)}. \]
Hence, using the equation
\[ \partial_\xi^2 \tilde{\phi} = -\alpha \Delta \tilde{\phi} - \beta \cdot \nabla \partial_\xi \tilde{\phi} + \gamma \partial_\xi \tilde{\phi}, \]
we infer from Lemma 4.4 that
\[ \| \partial_\xi^2 \tilde{\phi} \|_{L^2(I;C^0)} \leq K_\eta \left( \| (V, B) \|_{W^{1,\infty}} + E_0(\psi) \right). \tag{6.8} \]
Noticing that
\[ \partial_\xi \tilde{\phi} = \frac{\partial_\xi \tilde{\phi}}{\partial_\xi \phi} = \frac{\partial_\xi \tilde{\phi} \partial_\xi \phi}{(\partial_\xi \phi)^2}, \]
thus by (2.3), Lemma 3.4, Lemma 3.5, (6.7), (6.8) and (4.10), we have
\[ \| \partial_\xi \tilde{\phi} \|_{L^2(I;C^0)} \leq K_\eta \left( \| \partial_\xi \tilde{\phi} \|_{L^2(I;C^0)} + \| \partial_\xi \tilde{\phi} \|_{L^\infty(I;C^0)} \| \partial_\xi \phi \|_{L^2(I;C^0)} \right) \]
\[ \leq K_\eta \left( \| (V, B) \|_{W^{1,\infty}} + E_0(\psi) \right). \]
Similarly, we can prove
\[ \| \nabla_{x,z} \nabla_{x,y} \tilde{\phi} \|_{L^2(I;C^0)} \leq K_\eta \left( \| (V, B) \|_{W^{1,\infty}} + E_0(\psi) \right). \]
Then by Proposition 4.9 with \( \delta_2 > \frac{d}{2} \) again, (6.5) and (6.2), we get
\[ \| \nabla_{x,z} (\nabla \tilde{\phi}, \nabla_\xi \phi) \|_{L^2(I;C^0)} + \| \nabla_{x,z} (\nabla \tilde{\phi}, \nabla_\xi \phi) \|_{L^2(I;C^0)} \]
\[ \leq K_\eta \left( \| (V, B) \|_{C^1} + \| \nabla_{x,z} (\nabla \tilde{\phi}, \nabla_\xi \phi) \|_{L^\infty(I \times \mathbb{R}^d)} + \| \nabla_{x,z} (\nabla \tilde{\phi}, \nabla_\xi \phi) \|_{L^2(I;C^{-\delta_2})} \right) \]
\[ \leq K_\eta \left( \| (V, B) \|_{W^{1,\infty}} + E_0(\psi) \right). \tag{6.9} \]
As an application of (6.9), we infer from Proposition 5.4 and Proposition 5.5 that

**Lemma 6.1.** It holds that
\[ \| R(\eta)(V, B) \|_{H^{s+\frac{1}{2}}} \leq K_\eta \left( \| (V, B) \|_{H^s} + \| (V, B) \|_{W^{1,\infty}} + E_0(\psi) \right) \eta_{H^{s+\frac{1}{2}}}, \]
\[ \| R(\eta)(V, B) \|_{H^{s+1}} \leq K_\eta \left( \| (V, B) \|_{H^{s+\frac{1}{2}}} + \| (V, B) \|_{W^{1,\infty}} + E_0(\psi) \right) \eta_{H^{s+\frac{1}{2}}}, \]
\[ \| R(\eta)(V, B) \|_{C^{\frac{1}{2}}} \leq K_\eta \left( \| (V, B) \|_{W^{1,\infty}} + E_0(\psi) \right). \]

**6.2. The estimates of the pressure.** Recall that the pressure \( P \) satisfies
\[ -P = \partial_t \phi + \frac{1}{2} |\nabla_{x,y} \phi|^2 + gy. \]
Take \( \Delta_{x,y} \) on both sides and use the fact \( \Delta_{x,y} \phi = 0 \) to get
\[ \Delta_{x,y} P = -|\nabla_{x,y} \phi|^2 = -\nabla_{x,y} \cdot \left( \nabla_{x,y} \phi \cdot \nabla_{x,y} \phi \right) \quad \text{in} \quad \Omega_t, \quad P|_{y=\eta} = 0. \tag{6.10} \]
By the \( L^2 \) energy estimate and (6.5), we get
\[ \| \nabla_{x,y}(P - y) \|_{L^2(\Omega_t)} \leq \| \nabla_{x,y} \phi \nabla_{x,y} \phi \|_{L^2(\Omega_t)} \]
\[ \leq K_\eta \left( \| \nabla \phi - \nabla_\xi \phi \|_{L^2(I \times \mathbb{R}^d)} \right) \]
\[ \leq K_\eta \left( \| \nabla \phi - \nabla_\xi \phi \|_{L^2(I \times \mathbb{R}^d)} \| \nabla_{x,z} (\nabla \tilde{\phi}, \nabla_\xi \phi) \|_{L^\infty(I \times \mathbb{R}^d)} \right) \]
\[ \leq K_\eta E_0(\psi) \left( \| (V, B) \|_{W^{1,\infty}} \right). \]
which implies that
\[ \| \nabla_x \bar{P}_1 \|_{L^\infty(I; H^{-\frac{1}{2}})} + \| \nabla_{x,z} \bar{P}_1 \|_{L^2(I \times \mathbb{R}^d)} \leq K_\eta \| E_0(\psi) \| (V, B) \|_{W^{1,\infty}}. \] (6.11)
Here and in what follows, we denote \( P_1 = P - y \). Indeed, if \( y < \eta(x) - 2 \), following the proof of Proposition 9.2, we can get
\[ |\nabla_{x,y} P_1(x, y)| \leq C(E_0(\psi) + \|(V, B)\|_{W^{1,\infty}})^2. \]
For \( z \in [-2, 0] \), we have
\[
\partial_z \bar{P}_1(x, z) - \partial_z \bar{P}_1(x, -2) = \int_{-2}^z \partial_z^2 \bar{P}_1(z')dz' \\
= \int_{-2}^0 (F_0 - \alpha \Delta \bar{P}_1 + \beta \nabla \partial_z \bar{P}_1 - \gamma \partial_z \bar{P}_1)dz'
\]
with \( F_0 = -\alpha |\nabla_{x,y} \phi|^2 \). Using (2.3), we get by Lemma 3.4, Lemma 3.5 and Lemma 4.4 that
\[ \| \alpha \Delta \bar{P}_1 - \beta \nabla \partial_z \bar{P}_1 + \gamma \partial_z \bar{P}_1 \|_{L^1(-2, 0; H^{-1})} \leq K_\eta \| \nabla_{x,z} \bar{P}_1 \|_{L^2} \]
So, we conclude that
\[ \| \partial_z \bar{P}_1 \|_{L^\infty(I; L^\infty+H^{-1})} \leq K_\eta (E_0(\psi) + \|(V, B)\|_{W^{1,\infty}})^2. \] (6.12)

**Lemma 6.2.** It holds that
\[ \| \alpha |\nabla_{x,y} \phi|^2 \|_{L^2(I; H^0_{-\infty,1})} \leq K_\eta \| E_0(\psi) \| (V, B) \|_{W^{1,\infty}})^2, \]
\[ \| \alpha |\nabla_{x,y} \phi|^2 \|_{L^2(I; H^0_{-\infty,1})} \leq K_\eta (\| (V, B) \|_{W^{1,\infty}} \| (V, B) \|_{H^s} + \| (V, B) \|_{W^{1,\infty}} \| \eta \|_{H^s+\frac{1}{2}}). \]

**Proof.** Using (2.3), we get by Lemma 3.4 and Lemma 3.5 along with Lemma 4.4 and (4.6) that
\[ \| \alpha |\nabla_{x,y} \phi|^2 \|_{L^2(I; H^0_{-\infty,1})} \leq K_\eta \| |\nabla_{x,y} \phi|^2 \|_{L^2(I; H^0_{-\infty,1})} \]
\[ \leq K_\eta \| |\nabla_{x,y} \phi| \|_{L^\infty(I; H^0_{-\infty,1})} \| |\nabla_{x,y} \phi| \|_{L^2(I; \mathbb{C})} \]
\[ \leq K_\eta \| \nabla_{x,z} |\nabla_{x,y} \phi| \|_{L^\infty(I; \mathbb{C})} \| \nabla_{x,z} |\nabla_{x,y} \phi| \|_{L^2(I; \mathbb{C})}, \]
where we use the chain rule so that
\[ \partial_{x_i} |\nabla_{x,y} \phi| = \partial_{x_i} \nabla_{x,y} \phi - \partial_{x_i} \rho \frac{\partial_{x_i} \nabla_{x,y} \phi}{\partial \rho_i}, \quad \partial_{y} |\nabla_{x,y} \phi| = \frac{\partial_{y} \nabla_{x,y} \phi}{\partial \rho_i}. \]
Then the first inequality of the lemma follows from (6.5) and (6.9).

Using (2.3), we get by Lemma 3.3- Lemma 3.5 along with Lemma 4.4 and (4.5)-(4.6) that
\[ \| \alpha |\nabla_{x,y} \phi|^2 \|_{L^2(I; H^{s-1})} \]
\[ \leq K_\eta (\| |\nabla_{x,y} \phi|^2 \|_{L^\infty(\Omega_t)} \| \eta \|_{H^{s+\frac{1}{2}}} + \| |\nabla_{x,y} \phi| \|_{L^\infty(\Omega_t)} \| \nabla_{x,y} \phi \|_{L^2(I; H^{s-1})}) \]
\[ \leq K_\eta (\| |\nabla_{x,y} \phi|^2 \|_{L^\infty(\Omega_t)} \| \eta \|_{H^{s+\frac{1}{2}}} + \| |\nabla_{x,y} \phi| \|_{L^\infty(\Omega_t)} \| \nabla_{x,z} \nabla_{x,y} \phi \|_{L^2(I; H^{s-1})}), \]
from which, (6.6) and (6.5), we deduce the second inequality. \( \square \)
Now we infer from Proposition 4.9 with $\delta_2 > \frac{d}{2} + 1$, Lemma 6.2, (6.11) and (6.12) that
\[
\|\nabla x,z \tilde{P}_1\|_{L^\infty_x(I;L^2_{x,z},1)} + \|\nabla x,z \tilde{P}_1\|_{L^2_x(I;L^\infty_{x,z},1)} \\
\leq K\eta(\|\alpha|\nabla x,y \phi|^2\|_{L^2_x(I;B^0_{x,y})} + \|\nabla x,z \tilde{P}_1\|_{L^\infty_x(I;C^{-\delta_2})} + \|\nabla x,z \tilde{P}_1\|_{L^2_x(I;C^{-\delta_2})}) \\
\leq K\eta(\|\alpha|\nabla x,y \phi|^2\|_{L^2_x(I;B^0_{x,y})} + \|\nabla x,z \tilde{P}_1\|_{L^\infty_x(I;L^\infty_{x,z} + H^{-1})} + \|\nabla x,z \tilde{P}_1\|_{L^2_x(I;L^2_{x,z},1)}) \\
\leq K\eta(E_0(\psi) + \|(V, B)\|_{W^{1,\infty}})^2.
\] (6.13)

On the other hand, using the equation
\[
\partial_t^2 \tilde{P}_1 = -\alpha \Delta \tilde{P}_1 - \beta \cdot \nabla \tilde{P}_1 + \gamma \partial_z \tilde{P}_1 - \alpha \overline{|\nabla x,y \phi|^2},
\] (6.14)
we infer from Lemma 3.4, Lemma 3.5, Lemma 4.4 and Lemma 6.2 that
\[
\|\partial_t^2 \tilde{P}_1\|_{L^2_x(I;B^0_{x,z},1)} \leq C\|\alpha, \beta\|_{L^\infty_x(I;C^0)} \|\nabla x,z \tilde{P}_1\|_{L^2_x(I;B^1_{x,z},1)} \\
+ C\|\gamma\|_{L^\infty_x(I;C^0)} \|\partial_z \tilde{P}_1\|_{L^\infty_x(I;B^1_{x,z},1)} + \|\alpha|\nabla x,y \phi|^2\|_{L^2_x(I;B^0_{x,y})} \\
\leq K\eta(E_0(\psi) + \|(V, B)\|_{W^{1,\infty}})^2.
\] (6.15)

While, it follows from Proposition 4.3, Lemma 6.2, (6.11) and (6.14) that
\[
\|\nabla x,z \tilde{P}_1\|_{X^{s+\frac{1}{2}}(I)} + \|\partial_t^2 \tilde{P}_1\|_{L^2_x(I;H^{s-1})} \\
\leq K\eta(\|\nabla x,z \tilde{P}_1\|_{L^2_x(I;L^\infty_{x,z},1)} + \|\alpha|\nabla x,y \phi|^2\|_{L^2_x(I;H^{s-1})} + \|\tilde{\eta}\|_{H^{s+\frac{1}{2}}} \|\nabla x,z \tilde{P}_1\|_{L^\infty_x(I;L^\infty_{x,z},1)}) \\
\leq K\eta(1 + E_0(\psi) + \|(V, B)\|_{W^{1,\infty}})^2(\|\tilde{\eta}\|_{H^{s+\frac{1}{2}} + \|(V, B)\|_{H^s}).
\] (6.16)

Here we use the fact $\|\nabla x,z \tilde{P}_1\|_{L^\infty_x(I;L^\infty_{x,z},1)} \leq C\|\nabla x,z \tilde{P}_1\|_{L^\infty_x(I;L^2_{x,z},1)}$ and
\[
\|\partial_t^2 \tilde{P}_1\|_{L^2_x(I;H^{s-1})} \leq K\eta(\|\nabla x,z \tilde{P}_1\|_{L^\infty_x(I;L^\infty_{x,z},1)} \|\tilde{\eta}\|_{H^{s+\frac{1}{2}}} + \|\nabla x,z \tilde{P}_1\|_{X^{s+\frac{1}{2}}(I)} + \|\alpha|\nabla x,y \phi|^2\|_{L^2_x(I;H^{s-1})}),
\]
which follows from Lemma 3.3, Lemma 3.5 and Lemma 4.4.

To estimate $(\partial_t + V \cdot \nabla)a$, we derive the equation of $\hat{P} \overset{\text{def}}{=} (\partial_t + \nabla x,y \phi \cdot \nabla x,y)P$.

**Lemma 6.3.** Assume that $(\phi, \eta, P)$ is a smooth solution of the water-wave system (1.5)-(1.7). Then we have
\[
\left\{
\begin{array}{l}
\Delta x,y \hat{P} = 4 \nabla x,y \phi \cdot \nabla x,y P + 2 \sum_{i,j,k} (\partial_i \partial_j \phi)(\partial_i \partial_k \phi)(\partial_j \partial_k \phi) \equiv F \quad \text{in} \quad \Omega_t, \\
\hat{P}|_{y=\eta} = 0.
\end{array}
\right.
\] (6.17)

**Proof.** By (1.5) and (6.10), we get
\[
\Delta x,y (\nabla x,y \phi \cdot \nabla x,y P) = 2 \nabla x,y \phi \cdot \nabla x,y P + \Delta x,y \nabla x,y P \\
= 2 \nabla x,y \phi \cdot \nabla x,y P - 2 \partial_k \phi(\partial_i \partial_j \phi)(\partial_i \partial_j \phi).
\] Hence,
\[
\Delta x,y \hat{P} = 2 \nabla x,y \phi \cdot \nabla x,y P - 2 (\partial_i \partial_j \phi)(\partial_i \partial_j \phi) + \partial_k \phi(\partial_i \partial_j \phi)(\partial_i \partial_j \phi).
\]
Taking $\partial_i \partial_j$ on both sides of (1.7), we get
\[
\partial_i (\partial_i \partial_j \phi) + \partial_k \phi(\partial_i \partial_j \phi) = -\partial_i \partial_j P - (\partial_i \partial_k \phi)(\partial_j \partial_k \phi).
\]
This gives the first equation.

Due to $P(t, x, y) = 0$, we infer that

$$P_t + \eta_t \partial_y P|_{y=\eta} = 0, \quad \nabla P + \nabla \eta \partial_y P|_{y=\eta} = 0,$$

which implies that

$$P_t + \nabla_{x,y} \phi \cdot \nabla_{x,y} P|_{y=\eta} = -\partial_y P(\eta_t + \nabla \phi \cdot \nabla \eta - \partial_y \phi)|_{y=\eta},$$

from which and (1.6), it follows that $\dot{P}|_{y=\eta} = 0$.

**Remark 6.4.** Using $\Delta_{x,y} \phi = 0$, the second term on the right hand side of (6.17) can be written as the divergence form:

$$\sum_{i,j,k} (\partial_i \partial_j \phi)(\partial_i \partial_k \phi)(\partial_j \partial_k \phi)$$

$$= \sum_{i,j,k} \partial_i ((\partial_j \phi)(\partial_i \partial_k \phi)(\partial_j \partial_k \phi)) - \sum_{i,j,k} (\partial_j \phi)(\partial_i \partial_k \phi)(\partial_i \partial_j \partial_k \phi)$$

$$= \sum_{i,j,k} \partial_i ((\partial_j \phi)(\partial_i \partial_k \phi)(\partial_j \partial_k \phi)) - \frac{1}{2} \sum_{i,j,k} \partial_j ((\partial_i \partial_k \phi)(\partial_i \partial_k \phi))$$

$$= \sum_{i,j,k} \partial_i ((\partial_j \phi)(\partial_i \partial_k \phi)(\partial_j \partial_k \phi)) - \frac{1}{2} \nabla_{x,y} \cdot (|\nabla_{x,y} \phi|^2 \nabla \phi).$$

Now we infer from (6.17) and Remark 6.4 that

$$\| \nabla_{x,y} \dot{P} \|_{L^2(\Omega_t)} \leq 4 \| \nabla_{x,y} \phi \|_{L^\infty(\Omega_t)} \| \nabla_{x,y} P \|_{L^2(\Omega_t)} + 2 \| \nabla_{x,y} \dot{P} \|_{L^2(\Omega_t)}.$$

Then following the proof of (6.12), we get by (5.2), (6.2), (6.5), (6.11) and (6.13) that

$$\| \nabla_{x,y} \dot{P} \|_{L^\infty(I; L^\infty + H^{-1})} + \| \nabla_{x,y} \dot{P} \|_{L^2(I \times \mathbb{R}^d)} \leq K_\eta (E_0(\psi) + \|(V, B)\|_{W^{1,\infty}})^3. \quad \text{(6.18)}$$

**Lemma 6.5.** It holds that

$$\| \alpha \ddot{F} \|_{L^1(I; B_{\infty,1}^0)} \leq K_\eta (\| (V, B) \|_{W^{1,\infty}} + E_0(\psi))^3.$$

**Proof.** Using (2.3), we get by Lemma 3.4 and Lemma 3.5 that

$$\| \alpha \ddot{F} \|_{L^1(I; B_{\infty,1}^0)} \leq K_\eta (\| \nabla_{x,y} \phi \|_{L^\infty(\Omega_t)} 2 \| \nabla_{x,y} \dot{P} \|_{L^2(I; C^0)}\| \nabla_{x,y} \phi \|_{L^2(\Omega_t)} + \| \nabla_{x,y} \phi \|_{L^2(I; C^0)}\| \nabla_{x,y} \phi \|_{L^\infty(\Omega_t)})$$

By the chain rule, we have that for $i = 1, \ldots, d$,

$$\partial_i \ddot{P} = \partial_i \dot{P} + \partial_i \dot{P} \cdot \partial_i \rho_\delta, \quad \partial_z \ddot{P} = \partial_z \dot{P} \cdot \partial_z \rho_\delta, \quad (\partial_i = \partial_{x_i})$$

$$\partial_{i}^2 \ddot{P} = \partial_{i}^2 \dot{P} + \partial_{i}^2 \dot{P} \cdot \partial_{i}^2 \rho_\delta,$$

$$\partial_{i,j}^2 \ddot{P} = \partial_{i,j}^2 \dot{P} + \partial_{i,j}^2 \dot{P} \cdot \partial_{i,j} \rho_\delta + \partial_{i,j}^2 \dot{P} \cdot \partial_{i,j} \rho_\delta + \partial_{i,j} \dot{P} \cdot \partial_{i,j} \rho_\delta \cdot \partial_{i,j} \dot{P} + \partial_{i,j} \dot{P} \cdot \partial_{i,j} \rho_\delta \cdot \partial_{i,j} \dot{P},$$

which along with Lemma 3.4, Lemma 3.5, (4.6) and (4.10) implies that

$$\| \nabla_{x,y} \dot{P} \|_{L^2(I; B_{\infty,1}^0)} \leq K_\eta (\| \nabla_{x,y} \ddot{P} \|_{L^2(I; B_{\infty,1}^0)} + \| \partial_{i}^2 \ddot{P} \|_{L^2(I; B_{\infty,1}^0)}).$$

Then the lemma follows from (6.4), (6.9), (6.13) and (6.15).
Then we infer from Proposition 4.9 with $\delta_2 > \frac{d}{2} + 1$, Lemma 6.5 and (6.18) that
\[
\|\nabla_{x,z}\tilde{P}\|_{L^\infty(I;B^0_{\infty,1})} \leq K_0\left(\|\alpha\tilde{F}\|_{L^1(I;B^0_{\infty,1})} + \|\nabla_{x,z}\tilde{P}\|_{L^\infty(I;L^\infty + H^{-1})} + \|\nabla_{x,z}\tilde{P}\|_{L^2(I \times \mathbb{R}^d)}\right)
\leq K_0\left(\|(V,B)\|_{W^{1,\infty} + E_0(\psi)}\right)^3.
\] (6.19)

7. New formulation and symmetrization

Recall the water-wave system
\[
\begin{aligned}
\partial_t \eta - G(\eta)\psi &= 0, \\
\partial_t \psi + g\psi + \frac{1}{2} |\nabla \psi|^2 - \frac{(G(\eta)\psi + \nabla \eta \cdot \nabla \psi)^2}{2(1 + |\nabla \eta|^2)} &= 0.
\end{aligned}
\] (7.1)

Following the framework of [3], we introduce the new unknowns
\[
\zeta = \nabla \eta, \quad B = \partial_y \phi|_{y=\eta}, \quad V = \nabla \phi|_{y=\eta}, \quad a = -\partial_y P|_{y=\eta}.
\] (7.2)

Recall that the pressure $P$ satisfies
\[
-P = \partial_t \phi + \frac{1}{2} |\nabla_{x,y} \phi|^2 + gy,
\] (7.3)

where $\phi$ is the solution of the elliptic equation
\[
\Delta_{x,y} \phi = 0 \quad \text{in} \quad \Omega_t, \quad \phi|_{y=\eta} = \psi.
\]

Then the system (7.1) can be reformulated as (see [3]):

**Lemma 7.1.** The new unknowns $(V,B,\zeta)$ satisfy
\[
\begin{aligned}
(\partial_t + V \cdot \nabla)B &= a - g, \quad (7.4) \\
(\partial_t + V \cdot \nabla)V + a\zeta &= 0, \quad (7.5) \\
(\partial_t + V \cdot \nabla)\zeta &= G(\eta)V + \zeta G(\eta)B. \quad (7.6)
\end{aligned}
\]

**Proof.** For the reader’s convenience, we present a proof. By the chain rule, for any function $f = f(t,x,y)$, we have
\[
(\partial_t + V \cdot \nabla)(f|_{y=\eta}) = (\partial_t + V \cdot \nabla)(f(t,x,\eta))
= \left[[\partial_t f + \nabla \phi \cdot \nabla f] + \partial_y f(\partial_t \eta + V \cdot \nabla \eta)\right]|_{y=\eta}
= (\partial_t f + \nabla_{x,y} \phi \cdot \nabla_{x,y} f)|_{y=\eta}.
\] (7.7)

Here in the last equality we use the fact that
\[
\partial_t \eta + V \cdot \nabla \eta = B. \quad (7.8)
\]

Taking $\nabla_{x,y}$ to (7.3), we deduce the equalities (7.4)-(7.5) from (7.7) and $P(t,x,\eta) = 0$. Taking $\partial_{x_1}$ to (7.8), we get
\[
(\partial_t + V \cdot \nabla)\partial_{x_1} \eta = \partial_{x_1} B - \sum_j \partial_{x_1} V_j \partial_{x_j} \eta. \quad (7.9)
\]

By the definitions of $(V,B)$ and $G(\eta)$, we find
\[
\partial_{x_1} B - \sum_j \partial_{x_1} V_j \partial_{x_j} \eta = \partial_y \partial_{x_1} \phi - \sum_j \partial_{x_1} \eta \partial_{x_j} \partial_{x_1} \phi|_{y=\eta} + \partial_{x_1} \eta(\partial_y(\partial_y \phi) - \nabla \eta \cdot \nabla \partial_y \phi)|_{y=\eta}
= G(\eta)V_i + \partial_{x_1} \eta G(\eta)B,
\]
which along with (7.9) gives (7.6). □
Now we introduce the so called good unknown \((U_s, \zeta_s)\) defined by
\[
U_s \overset{\text{def}}{=} \langle D \rangle^8 V + T_\zeta \langle D \rangle^8 B, \quad \zeta_s \overset{\text{def}}{=} \langle D \rangle^8 \zeta.
\]

**Lemma 7.2.** The unknown \((U_s, \zeta_s)\) satisfies
\[
\begin{align*}
\left\{ \begin{array}{l}
(\partial_t + TV \cdot \nabla)U_s + T_\zeta \zeta_s = f_1, \\
(\partial_t + TV \cdot \nabla)\zeta_s = T_\lambda U_s + f_2,
\end{array} \right.
\end{align*}
\]
where \((f_1, f_2)\) is given by
\[
\begin{align*}
f_1 &= \langle D \rangle^8 h_1 - \langle D \rangle^8 T_V \cdot \nabla V - \langle D \rangle^8 T_a \zeta - \langle D \rangle^8 T_\zeta (\partial_t + TV \cdot \nabla)B \\
&\quad - T_\zeta \langle [D \rangle^8, TV \cdot \nabla \rangle B - [T_\zeta, \partial_t + TV \cdot \nabla \rangle \langle D \rangle^8 B, \\
f_2 &= \langle D \rangle^8 h_2 - \langle D \rangle^8 TV \cdot \nabla \zeta + \langle [T_\lambda, D \rangle^8 \rangle U,
\end{align*}
\]
with \(U = V + T_\zeta B\) and \((h_1, h_2)\) given by
\[
\begin{align*}
h_1 &= (TV - V) \cdot \nabla V - R(a, \zeta) + T_\zeta (TV - V) : \nabla B, \\
h_2 &= (TV - V) \cdot \nabla \zeta + [T_\zeta, T_\lambda] B + (\zeta - T_\zeta) T_\lambda B + R(\eta) V + \zeta R(\eta) B.
\end{align*}
\]

**Proof.** Applying Bony’s decomposition (2.3) to (7.4)-(7.6), we get
\[
\begin{align*}
(\partial_t + TV \cdot \nabla)V + T_\zeta \zeta + T_\zeta (\partial_t + TV \cdot \nabla)B &= h_1, \\
(\partial_t + TV \cdot \nabla)\zeta = T_\lambda U + h_2.
\end{align*}
\]
Then the system (7.10) follows by applying \(\langle D \rangle^8\) to the above equations. \(\square\)

We denote
\[
\gamma = \sqrt{a\lambda}, \quad q = \sqrt{\frac{a}{\lambda}}, \quad \theta_s = T_\eta \zeta_s.
\]
Taking \(T_\eta\) on the both sides of the second equation of (7.10), we obtain the following symmetrized system:
\[
\begin{align*}
\left\{ \begin{array}{l}
(\partial_t + TV \cdot \nabla)U_s + T_\zeta \zeta_s = F_1, \\
(\partial_t + TV \cdot \nabla)\zeta_s = T_\lambda U_s + F_2,
\end{array} \right.
\end{align*}
\]
with \((F_1, F_2)\) given by
\[
\begin{align*}
F_1 &= f_1 + (T_\eta T_\xi - T_a) \zeta_s, \\
F_2 &= T_\eta f_2 + (T_\eta T_\lambda - T_\xi) U_s - [T_\eta, \partial_t + TV \cdot \nabla] \zeta_s.
\end{align*}
\]

8. Energy estimates

Assume that \((U_s, \zeta_s)\) is a solution of (7.11) on \([0, T]\), and \(a(t, x)\) satisfies
\[
\inf_{(t,x)\in[0,T] \times \mathbb{R}^d} a(t, x) \geq c_0.
\]

We denote by \(K^1_\eta = K^1_\eta \left( \sup_{t \in [0, T]} \left( \|a(t)\|_{C^{\frac{3}{2} + \epsilon}} + \|a(t)\|_{L^2} \right), c_0 \right) \) by an increasing function, which may be different from line to line. By the definition of \((\gamma, q)\), it is easy to show that
\[
M^{\frac{1}{2}}_0 (\gamma) + M^{-\frac{1}{2}}_0 (q) \leq K^1_\eta \left( \|a\|_{C^{\frac{3}{2}}} \right), \quad M^{\frac{1}{2}}_2 (\gamma) + M^{-\frac{1}{2}}_2 (q) \leq K^1_\eta \left( \|a\|_{C^{\frac{3}{2}}} \right).
\]
(8.2)

And by Lemma 3.3, we have
\[
\| (U_s, \zeta_s) \|_{L^2} \leq K^1_\eta \left( \| (V, B) \|_{H^{s}} + \|\eta\|_{H^{s+\frac{1}{2}}} \right).
\]
The goal of this section is to prove that
Proposition 8.1. \[
\frac{d}{dt} \|(U_s, \theta_s)\|_{L^2} \leq K_1^1 \left( \left| \|G_1(t)(V, B)\|_{H^s} + G_2(t)\|\eta\|_{H^{s+\frac{1}{2}}} + \|a - g\|_{H^{s-\frac{1}{2}}} \right) \right), \tag{8.3}
\]
where \(G_i(i = 1, 2)\) is defined by
\[
G_1(t) = 1 + \|a\|_{C^\frac{1}{2}} + \|(V, B)\|_{B_{s,1}^1},
\]
\[
G_2(t) = 1 + \|a\|_{C^\frac{1}{2}}^2 + \|\partial_t a + V \cdot \nabla a\|_{L^\infty} + \|a\|_{C^\frac{1}{2}} \left( \|(V, B)\|_{B_{s,1}^1} + E_0(\psi) \right).
\]

**Proof.** We multiply \((U_s, \theta_s)\) by both sides of (7.11) and integrate on \(\mathbb{R}^d\) to obtain
\[
\frac{1}{2} \frac{d}{dt} \|(U_s, \theta_s)\|_{L^2}^2 = I_1 + I_2 + I_3, \tag{8.4}
\]
with \(I_i\) given by
\[
I_1 = -\langle TV \cdot \nabla U_s, U_s \rangle - \langle TV \cdot \nabla \theta_s, \theta_s \rangle,
\]
\[
I_2 = -\langle T\gamma_{s_1}, U_s \rangle + \langle T\gamma_{s_1}, \theta_s \rangle,
\]
\[
I_3 = \langle F_1, U_s \rangle + \langle F_s, \theta_s \rangle.
\]
By Proposition 2.4, we know that
\[
\|(TV \cdot \nabla)^s + TV \cdot \nabla\|_{L^2 \to L^2} \leq C\|V\|_{W^{1,\infty}},
\]
\[
\|(T\gamma - (T\gamma)^s\|_{L^2 \to L^2} \leq CM_{\frac{3}{2}}(\gamma) \leq K_1^1 \|a\|_{C_{\frac{1}{2}}^\frac{1}{2}},
\]
from which and (8.4), we infer that
\[
\frac{d}{dt} \|(U_s, \theta_s)\|_{L^2} \leq K_1^1 \left( \|(V, B)\|_{W^{1,\infty}} + \|a\|_{C^\frac{1}{2}} \right) \|(U_s, \theta_s)\|_{L^2} + \|(F_1, F_2)\|_{L^2}. \tag{8.5}
\]
It remains to estimate \(\|(F_1, F_2)\|_{L^2}\). By Proposition 2.4 and (8.2), we get
\[
\|(T\gamma - T\gamma)^s\|_{L^2} \leq K_1^1 \|a\|_{C^\frac{1}{2}} \|\gamma\|_{H^{s+\frac{1}{2}}} \leq K_1^1 \|a\|_{C^\frac{1}{2}} \|\gamma\|_{H^{s+\frac{1}{2}}},
\]
\[
\|(T\gamma - T\gamma)\|_{L^2} \leq K_1^1 \|a\|_{C^\frac{1}{2}} \|U_s\|_{L^2} \leq K_1^1 \|a\|_{C^\frac{1}{2}} \|(V, B)\|_{H^s},
\]
By Proposition 2.4 and Proposition 2.9, we get
\[
\|(D)^s(TV \cdot \nabla)\|_{L^2} \leq C\|V\|_{W^{1,\infty}}\|V\|_{H^s},
\]
\[
\|(T\gamma - T\gamma)^s\|_{L^2} \leq K_1^1 \|a\|_{C^\frac{1}{2}} \|\gamma\|_{H^{s+\frac{1}{2}}} \leq K_1^1 \|a\|_{C^\frac{1}{2}} \|\gamma\|_{H^{s+\frac{1}{2}}},
\]
and by Remark 5.7,
\[
\|(D)^s(T\gamma)^s\|_{L^2} \leq C\|a\|_{C^\frac{1}{2}} \|\nabla \gamma\|_{H^{s+\frac{1}{2}}} \leq C\|a\|_{C^\frac{1}{2}} \|\nabla \gamma\|_{H^{s+\frac{1}{2}}},
\]
\[
\|(D)^s(T\gamma)\|_{L^2} \leq K_1^1 \|a\|_{C^\frac{1}{2}} \|U\|_{H^s} \leq K_1^1 \|a\|_{C^\frac{1}{2}} \|(V, B)\|_{H^s}.
\]
Using the equation \(\partial_t B + V \cdot \nabla B = a - g\), we get by Proposition 2.9 that
\[
\|(D)^s(T\gamma)^s(\partial_t B + TV \cdot \nabla B)\|_{L^2} \leq K_1^1 \|\partial_t B + TV \cdot \nabla B\|_{H^{s+\frac{1}{2}}}
\leq K_1^1 (\|a - g\|_{H^{s-\frac{1}{2}}} + \|B\|_{W^{1,\infty}}\|V\|_{H^s}).
By Proposition 2.10, we get
\[
\| [T_\zeta, \partial_t + T_V \cdot \nabla] (D)^s B \|_{L^2} \\
\leq K_\eta^1 (\| V \|_{B_{\infty,1}^1} + \| \partial_t \zeta + V \cdot \nabla \zeta \|_{L^\infty}) \| B \|_{H^s} \\
\leq K_\eta^1 (\| V \|_{B_{\infty,1}^1} + \| (V, B) \|_{W^{1,\infty}}) \| B \|_{H^s}.
\]

Here we use the fact that
\[
(\partial_t + V \cdot \nabla) \partial_t \eta = \partial_t B - \sum_{j=1}^d \partial_i V_j \partial_j \eta.
\]
Similarly, we have
\[
\| [T_q, \partial_t + T_V \cdot \nabla] \zeta \|_{L^2} \\
\leq C \left( \mathcal{M}_0^{1/2} (q) \| V \|_{B_{\infty,1}^1} + (\partial_t q + V \cdot \nabla q) \right) \| \zeta \|_{H^{-1/2}} \\
\leq K_\eta^1 (\| \partial_t a + V \cdot \nabla a \|_{L^\infty} + \| a \|_{C^{1/2}_B} \| \partial_t \eta + V \cdot \nabla \eta \|_{L^\infty}) \| \eta \|_{H^{s+1/2}} \\
\leq K_\eta^1 (\| \partial_t a + V \cdot \nabla a \|_{L^\infty} + \| a \|_{C^{1/2}_B} \| (V, B) \|_{W^{1,\infty}}) \| \eta \|_{H^{s+1/2}}.
\]

Using (2.3), we infer from Lemma 3.3 and Lemma 3.5 that
\[
\| (D)^s h_1 \|_{L^2} \leq K_\eta^1 (\| (V, B) \|_{W^{1,\infty}} \| (V, B) \|_{H^s} + \| a \|_{C^{1/2}_B} \| \eta \|_{H^{s+1/2}}).
\]

Next we present the estimate of \( h_2 \). First of all, we have by Lemma 3.3 and Lemma 3.5 that
\[
\| (T_V - V) \cdot \nabla \zeta \|_{H^{s-1/2}} \leq K_\eta^1 \| V \|_{H^s}, \\
\| (\zeta - T_\zeta) T_\lambda B \|_{H^{s-1/2}} \leq C \| T_\lambda B \|_{L^\infty} \| \eta \|_{H^{s+1/2}}.
\]

It follows from Proposition 2.4 that
\[
\| [T_\zeta, T_\lambda] B \|_{H^{s-1/2}} \leq K_\eta^1 \| B \|_{H^s}.
\]

And by Lemma 6.1, we get
\[
\| R(\eta) V \|_{H^{s-1/2}} \leq K_\eta^1 (\| V \|_{H^s} + (\| (V, B) \|_{W^{1,\infty}} + E_0(\psi)) \| \eta \|_{H^{s+1/2}}), \\
\| \zeta R(\eta) B \|_{H^{s-1/2}} \leq K_\eta^1 (\| (R(\eta) B) \|_{H^{s-1/2}} + \| (R(\eta) B) \|_{L^\infty} \| \zeta \|_{H^{s-1/2}}) \\
\leq K_\eta^1 (\| B \|_{H^s} + (\| (V, B) \|_{W^{1,\infty}} + E_0(\psi)) \| \eta \|_{H^{s+1/2}}).
\]

Hence, we deduce that
\[
\| (D)^s T_\eta h_2 \|_{L^2} \\
\leq K_\eta^1 \| a \|_{C^{1/2}_B} (\| (V, B) \|_{H^s} + (\| (V, B) \|_{W^{1,\infty}} + E_0(\psi) + \| T_\lambda B \|_{L^\infty}) \| \eta \|_{H^{s+1/2}}).
\]

Noting that
\[
\| T_\lambda B \|_{L^\infty} \leq \| T_\lambda B \|_{B_{\infty,1}^1} \leq K_\eta^1 \| B \|_{B_{\infty,1}^1}^2,
\]
then by summing up the above estimates, we conclude that
\[
\| (F_1, F_2) \|_{L^2} \leq K_\eta^1 (G_1(t) \| (V, B) \|_{H^s} + G_2(t) \| \eta \|_{H^{s+1/2}} + \| a - g \|_{H^{s-1/2}}),
\]
from which and (8.5), we deduce (8.3). \(\square\)
Next we recover the estimate of \((V, B, \eta)\) from that of \((U_s, \theta_s)\).

**Lemma 8.2.** It holds that

\[
\| \eta \|_{H^{s+\frac{1}{2}}} \leq K^1_\eta(\| \eta \|_{L^2} + \| a \|_{C^\frac{3}{2}}) \| \zeta_s \|_{H^{-1}},
\]

\[
\| (V, B) \|_{H^s} \leq K^1_\eta(\| U_s \|_{L^2} + \| (V, B) \|_{L^2} + \| (V, B) \|_{W^{1, \infty}} + E_0(\psi)) \| \eta \|_{H^{s+\frac{1}{2}}}.
\]

**Proof.** First of all, we have

\[
\zeta_s = (1 - T_1/q T_q) \zeta_s + T_1/q T_q \zeta_s = (1 - T_1/q T_q) \zeta_s + T_1/q \theta_s,
\]

which along with Proposition 2.4 and (8.2) implies

\[
\| \zeta_s \|_{H^{s+\frac{1}{2}}} \leq K^1_\eta(\| a \|_{C^\frac{3}{2}}) \| \zeta_s \|_{H^{-1}} + \| \theta_s \|_{L^2}).
\]

Hence, we get

\[
\| \eta \|_{H^{s+\frac{1}{2}}} \leq \| \eta \|_{L^2} + \| \zeta_s \|_{H^{s+\frac{1}{2}}} \leq K^1_\eta(\| \eta \|_{L^2} + \| a \|_{C^\frac{3}{2}}) \| \zeta_s \|_{H^{-1}} + \| \theta_s \|_{L^2}).
\] (8.6)

Recall that

\[
U = V + T_\zeta B, \quad \nabla B = G(\eta)V.
\]

Thus, we get

\[
\nabla U = \nabla V + T_\zeta \nabla B + T_{\nabla \zeta} B \equiv \nabla V + T_\zeta G(\eta)V + T_{\nabla \zeta} B.
\]

Let \(T_p V = T_{i\xi + \zeta} V\), then

\[
T_p V = \nabla U - (T_\zeta T_\lambda - T_{i\xi} \lambda) V - T_\zeta (R(\eta) V) + (T_\xi \nabla V - \nabla V) - T_{\nabla \zeta} B
\equiv \nabla U + R'(\eta)V - T_{\nabla \zeta} B,
\]

which implies

\[
V = T_1/p(\nabla U + R'(\eta)V - T_{\nabla \zeta} B) + (1 - T_1/p T_p) V. \quad (8.7)
\]

Then by Proposition 2.4 and Lemma 6.1, we get

\[
\| V \|_{H^s} \leq K^1_\eta(\| U \|_{H^s} + \| R(\eta) V \|_{H^{s-1}} + \| B \|_{H^{s+\frac{1}{2}}} + \| V \|_{H^{s+\frac{1}{2}}})
\]

\[
\leq K^1_\eta(\| U \|_{H^s} + \| (V, B) \|_{L^2} + \frac{1}{2} \| (V, B) \|_{H^s} + K^1_\eta(\| (V, B) \|_{W^{1, \infty}} + E_0(\psi)) \| \eta \|_{H^{s+\frac{1}{2}}}.
\]

on the other hand, we have

\[
\| B \|_{H^s} \leq \| B \|_{L^2} + \| G(\eta)V \|_{H^{s-1}}
\]

\[
\leq \| B \|_{L^2} + K^1_\eta(\| V \|_{H^s} + \| (V, B) \|_{W^{1, \infty}} + E_0(\psi)) \| \eta \|_{H^{s+\frac{1}{2}}},
\]

from which, it follows that

\[
\| (V, B) \|_{H^s} \leq K^1_\eta(\| U_s \|_{L^2} + \| (V, B) \|_{L^2} + \| (V, B) \|_{W^{1, \infty}} + E_0(\psi)) \| \eta \|_{H^{s+\frac{1}{2}}}). \quad (8.8)
\]

Then the lemma follows from (8.6) and (8.8). \(\square\)
Lemma 8.3. It holds that
\[ \|(V, B)\|_{B^{1,\infty}_{\infty,1}} \leq K_\eta^1 (1 + \|(V, B)\|_{W^{1,\infty}} + E_0(\psi)) \ln(e + \|U_s\|_{L^2} + \|B\|_{H^s}). \]

**Proof.** It follows from (8.7), Lemma 2.6 and Lemma 6.1 that
\[ \|V\|_{B^{1,\infty}_{\infty,1}} \leq K_\eta^1 \left( \|U\|_{B^{1,\infty}_{\infty,1}} + \|R(\eta)V\|_{B^{0}_{\infty,1}} + \|(V, B)\|_{B^{1/2}_{\infty,1}} \right) \]
\[ \leq K_\eta^1 \left( \|U\|_{B^{1,\infty}_{\infty,1}} + \|(V, B)\|_{W^{1,\infty}} + E_0(\psi) \right), \]
hence by \( \nabla B = G(\eta)V \), we get
\[ \|\nabla B\|_{B^{0}_{\infty,1}} \leq \|G(\eta)V\|_{B^{0}_{\infty,1}} \leq \|T_s V\|_{B^{0}_{\infty,1}} + \|R(\eta)V\|_{C^{1/2}} \]
\[ \leq K_1^1 \left( \|V\|_{B^{1,\infty}_{\infty,1}} + \|(V, B)\|_{W^{1,\infty}} + E_0(\psi) \right) \]
\[ \leq K_1^1 \left( \|U\|_{B^{1,\infty}_{\infty,1}} + \|(V, B)\|_{W^{1,\infty}} + E_0(\psi) \right). \]
This proves that
\[ \|(V, B)\|_{B^{1,\infty}_{\infty,1}} \leq K_\eta^1 (\|U\|_{B^{1,\infty}_{\infty,1}} + \|(V, B)\|_{W^{1,\infty}} + E_0(\psi)). \]

Give any \( N \in \mathbb{N} \), we have
\[ \|U\|_{B^{1,\infty}_{1,\infty}} \leq \sum_{j \leq N} 2^j \|\Delta_j U\|_{L^\infty} + \sum_{j > N} 2^j \|\Delta_j U\|_{L^\infty} \]
\[ \leq CN \|(V, B)\|_{W^{1,\infty}} + \sum_{j > N} 2^{(1-s)j} \|\Delta_j \langle D \rangle^s U\|_{L^\infty} \]
\[ \leq CN \|(V, B)\|_{W^{1,\infty}} + \sum_{j > N} 2^{(1-s)j} \|\Delta_j \langle D \rangle^s U\|_{L^\infty} \]
\[ \leq CN \|(V, B)\|_{W^{1,\infty}} + 2^{-N(s-1+\frac{s}{2})} \|\langle D \rangle^s U\|_{L^2}, \]
taking \( N \) such that \( 2^{-N(s-1+\frac{s}{2})} \|\langle D \rangle^s U\|_{L^2} \sim 1 \), we get
\[ \|U\|_{B^{1,\infty}_{1,\infty}} \leq C (1 + \|(V, B)\|_{W^{1,\infty}}) \ln(e + \|\langle D \rangle^s U\|_{L^2}) \]
\[ \leq K_1^1 (1 + \|(V, B)\|_{W^{1,\infty}}) \ln(e + \|U_s\|_{L^2} + \|B\|_{H^s}). \]
The proof is finished. \( \square \)

9. **Proof of Theorem 1.3**

9.1. **The basic energy law.** We introduce the total energy functional \( H(\eta, \psi) \) as
\[ H(\eta, \psi) \overset{\text{def}}{=} \int_{\mathbb{R}^d} (g|\eta|^2 + \psi G(\eta)\psi) \, dx. \] (9.1)

**Proposition 9.1.** Assume that \((\eta, \psi)\) is a smooth solution of (1.8) with the initial data \((\eta_0, \psi_0)\) on \([0, T]\). Then it holds that
\[ H(\eta(t), \psi(t)) = H(\eta_0, \psi_0) \quad \text{for any } t \in [0, T]. \]

**Proof.** Multiplying \( g\eta \) and \( G(\eta)\psi \) on the both side of the first and second equation of water wave respectively, and integrating on \( \mathbb{R}^d \), then we add the resulting equations to obtain
\[ (\partial_t \eta, g\eta) + (\partial_t \psi, G(\eta)\psi) = -\frac{1}{2} \left( |\nabla \psi|^2 - \frac{(G(\eta)\psi + \nabla \eta \cdot \nabla \psi)^2}{(1 + |\nabla \eta|^2)} \right), G(\eta)\psi. \]
First of all, we have
\[(\partial_t \psi, G(\eta)\psi) = \partial_t (\psi, G(\eta)\psi) - (d_\eta G(\eta) \psi \cdot \partial_t \eta, \psi) - (\psi, G(\eta) \partial_t \psi).\]
Then by Proposition 5.2 and (1.8), we get
\[
2(\partial_t \psi, G(\eta)\psi) = \partial_t (\psi, G(\eta)\psi) - (d_\eta G(\eta) \psi \cdot \partial_t \eta, \psi) \\
= \partial_t (\psi, G(\eta)\psi) + (G(\eta) (\partial_t \eta B) + \text{div}(\partial_t \eta V), \psi) \\
= \partial_t (\psi, G(\eta)\psi) + (\partial_t \eta B, G(\eta)\psi) - (\partial_t \eta V, \nabla \psi) \\
= \partial_t (\psi, G(\eta)\psi) + (G(\eta)\psi B, G(\eta)\psi) - (G(\eta)\psi V, \nabla \psi) \\
= \partial_t (\psi, G(\eta)\psi) + (G(\eta)\psi B) - (G(\eta)\psi, V \cdot \nabla \psi) \\
= \partial_t (\psi, G(\eta)\psi) + (G(\eta)\psi B - V \cdot \nabla \psi). \tag{9.2}
\]
It follows from Remark 5.3 that
\[
\|\nabla \psi\|^2 - \frac{(G(\eta)\psi + \nabla \eta \cdot \nabla \psi)^2}{(1 + |\nabla \eta|^2)} = (V + B\nabla \eta) \cdot \nabla \psi - B(G(\eta)\psi + \nabla \eta \cdot \nabla \psi) \\
= V \cdot \nabla \psi - B G(\eta)\psi,
\]
which along with (9.2) implies that
\[
\frac{d}{dt} \int_{\mathbb{R}^d} (g|\eta|^2 + \psi G(\eta)\psi) dx = 0.
\]
This implies the proposition. \qed

9.2. **Hölder estimate of the free surface from the mean curvature.** Recall that the equation of the mean curvature
\[
\nabla \cdot \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) = \kappa, \tag{9.3}
\]
where \(\kappa\) is the mean curvature of the free surface \(y = \eta(t, x)\).

**Proposition 9.2.** Assume that \(\eta \in W^{1,\infty}\) and \(\kappa \in L^2 \cap L^p\) for some \(p > d\). Then \(\eta \in C^{2-\frac{d}{p}}\) with the bound
\[
\|\eta\|_{C^{2-\frac{d}{p}}} \leq C(\|\nabla \eta\|_{L^\infty}, \|\kappa\|_{L^2 \cap L^p}).
\]

**Proof.** Taking \(\partial_t = \partial_{x_\ell}\) on both sides of (9.3), we get
\[
\partial_{x_\ell} \kappa = \partial_\ell \left( (1 + |\nabla \eta|^2)^{-\frac{1}{2}} \partial_\ell \partial_t \eta - (1 + |\nabla \eta|^2)^{-\frac{1}{2}} \partial_j \eta \partial_{x_\ell} \partial_t \eta \right).
\]
We set \(\eta_\ell = \partial_{x_\ell} \eta\) and \(a_{ij} = (1 + |\nabla \eta|^2)^{-\frac{3}{2}} ((1 + |\nabla \eta|^2) \delta_{ij} - \partial_i \eta \partial_j \eta)\). Then we find that
\[
\partial_j (a_{ij} \partial_{x_\ell} \eta_\ell) = \partial_{x_\ell} \kappa. \tag{9.4}
\]
It is easy to verify that the matrix \((a_{ij})\) is uniformly elliptic with the elliptic constants depending on \(\|\nabla \eta\|_{L^\infty}\). Using the De Giorgi method, it can be proved that \(\eta_\ell \in C^\epsilon\) for some \(\epsilon > 0\) and
\[
\|\eta_\ell\|_{C^\epsilon} \leq C(\|\nabla \eta\|_{L^\infty}, \|\kappa\|_{L^p}).
\]
This means that \(\eta \in C^{1+\epsilon}\), hence \(a_{ij} \in C^\epsilon\).
Next we prove Hölder regularity of $\eta$ by freezing the leading coefficients method. For any ball $B_r(x_0) \subset \mathbb{R}^d$ with radius $r$ and center $x_0$, let $w$ be a unique solution of the Dirichlet problem

$$\int_{B_r(x_0)} a_{ij}(x_0) \partial_i w \partial_j \varphi dx = 0 \quad \text{for any } \varphi \in H^1_0(B_r(x_0))$$

with $w - \eta \in H^1_0(B_r(x_0))$. Then $v = \eta - w$ satisfies

$$\int_{B_r(x_0)} a_{ij}(x_0) \partial_i v \partial_j \varphi dx = \int_{B_r(x_0)} \left( - \kappa \partial_k \varphi + (a_{ij}(x_0) - a_{ij}(x)) \partial_i \eta \partial_j \varphi \right) dx$$

for any $\varphi \in H^1_0(B_r(x_0))$. Take $\varphi = v$ to get

$$\int_{B_r(x_0)} |\nabla v|^2 dx \leq C \left( \int_{B_r(x_0)} |\nabla \eta|^2 dx + \int_{B_r(x_0)} |k|^2 dx \right),$$

which along with Lemma 9.3 gives for any $0 < \rho \leq r$

$$\int_{B_{\rho}(x_0)} |\nabla \eta|^2 dx \leq C \left( \left( r^{2^*} + \left( \frac{\rho}{r} \right)^d \right) \int_{B_r(x_0)} |\nabla \eta|^2 dx + r^{d(1 - \frac{2}{p})} \| \kappa \|_{L^p}^2 \right),$$

from which and a standard iteration, we infer that there exists $R_0 > 0$ such that for any $0 < \rho < r < R_0$,

$$\int_{B_{\rho}(x_0)} |\nabla \eta|^2 dx \leq C \left( \left( \frac{\rho}{r} \right)^d \int_{B_r(x_0)} |\nabla \eta|^2 dx + \rho^{d(1 - \frac{2}{p})} \| \kappa \|_{L^p}^2 \right).$$

In particular, taking $r = R_0$ yields that for any $0 < \rho < R_0$,

$$\int_{B_{\rho}(x_0)} |\nabla \eta|^2 dx \leq C \rho^{d(1 - \frac{2}{p})} \left( \| \nabla \eta \|_{L^2}^2 + \| \kappa \|_{L^p}^2 \right) \leq C \rho^{d(1 - \frac{2}{p})} \left( \| \kappa \|_{L^2}^2 + \| \kappa \|_{L^p}^2 \right).$$

This implies that $\eta \in C^{1 - \frac{d}{p}}$ and

$$\|\eta\|_{C^{1 - \frac{d}{p}}} \leq C \left( \| \nabla \eta \|_{L^\infty}, \| \kappa \|_{L^2 \cap L^p} \right).$$

Hence, $\eta \in C^{2 - \frac{d}{p}}$ and the proposition follows. \hfill \Box

**Lemma 9.3.** Let $w$ be as in the proof of Proposition 9.2. Then for any $u \in H^1(B_r(x_0))$ and $0 < \rho \leq r$, it holds that

$$\int_{B_\rho(x_0)} |\nabla u|^2 dx \leq C \left( \left( \frac{\rho}{r} \right)^d \int_{B_r(x_0)} |\nabla u|^2 dx + \int_{B_r(x_0)} |\nabla (u - w)|^2 dx \right),$$

where $C$ is a constant depending only on the elliptic constants of $(a_{ij}(x_0))$.

**Proof.** Set $v = u - w$, we have for any $0 < \rho \leq r$,

$$\int_{B_\rho(x_0)} |\nabla u|^2 dx \leq 2 \int_{B_\rho(x_0)} |\nabla w|^2 dx + |\nabla v|^2 dx \leq C \left( \left( \frac{\rho}{r} \right)^d \int_{B_r(x_0)} |\nabla w|^2 dx + 2 \int_{B_r(x_0)} |\nabla v|^2 dx \right) \leq C \left( \left( \frac{\rho}{r} \right)^d \int_{B_r(x_0)} |\nabla w|^2 dx + C \int_{B_r(x_0)} |\nabla v|^2 dx \right).$$

Here we used the property of Harmonic function for $w$, since it satisfies an elliptic equation with constant coefficients. \hfill \Box
9.3. Proof of Theorem 1.3. Recall the assumption of the theorem:

\[ M(T) \equiv \sup_{t \in [0, T]} \| \kappa(t) \|_{L^p \cap L^2} + \int_0^T \| (\nabla V, \nabla B)(t) \|_{L^\infty}^6 dt < +\infty, \]

\[ \inf_{(t, x, y) \in [0, T] \times \Sigma_t} -\frac{\partial P}{\partial n}(t, x, y) \geq c_0. \]

In order to prove Theorem 1.3, it suffices to show that

\[ \sup_{t \in [0, T]} E_s(t) \leq C(E_s(0), M(T), T, TS(a)^{-1}), \]

where \( C(\cdots) \) is an increasing function, and

\[ E_s(t) \equiv \| (\eta, \psi)(t) \|_{H^s} + \| (V, B)(t) \|_{H^s}, \quad TS(a) \equiv \inf_{(t, x) \in [0, T] \times \Omega_t} a(t, x). \]

In what follows, we denote by \( K^1_\eta = K^1_\eta(\sup_{t \in [0, T]} \| \eta(t) \|_{C^{\frac{1}{p} + \epsilon}}, \| \eta(t) \|_{L^2}) \), \( TS(a)^{-1} \) an increasing function.

Thanks to the equation of \( \eta \), we find that

\[ (\partial_t + V \cdot \nabla) \partial_t \eta = \partial_t B - \sum_{j=1}^d \partial_j V \partial_j \eta, \]

which implies that

\[ \| \nabla \eta \|_{L^\infty((0, T) \times \mathbb{R}^d)} \leq C(E_s(0), M(T), T). \]

Hence by Proposition 9.1 and Proposition 9.2, we obtain

\[ \sup_{t \in [0, T]} \left( \| \eta(t) \|_{C^{\frac{1}{p} + \epsilon}} + \| \eta(t) \|_{L^2} \right) \leq C(E_s(0), M(T), T), \]

with \( \epsilon = \frac{1}{2} - \frac{d}{p} > 0. \) Hence, \( K_\eta \leq C(E_s(0), M(T), T). \) Note that

\[ a(t, x) = -\partial_y P|_{y=\eta} = -\frac{1}{\sqrt{1 + |\nabla \eta|^2}} \left( \frac{\partial P}{\partial n} \right)|_{y=\eta}. \]

Hence, \( TS(a) \geq c_1 \) for some \( c_1 > 0. \)

By the definition of \( a \), we infer from (6.13) and (6.16) that

\[ \| a \|_{C^{\frac{1}{p}}} \leq K^1_\eta \left( \| (V, B) \|_{W^{1, \infty}} + E_0(\psi) \right)^2, \quad \| a - g \|_{H^{s-\frac{1}{2}}} \leq K^1_\eta \left( 1 + \| (V, B) \|_{W^{1, \infty}} + E_0(\psi) \right)^2 \left( \| \eta \|_{H^{s+\frac{1}{2}}} + \| (V, B) \|_{H^s} \right)^2 \]

By (7.7), we find that

\[ \partial_t a + V \cdot \nabla a = \partial_t \bar{P} - \partial_y \nabla_{x,y} \partial_y \nabla_{x,y} \bar{P}|_{y=\eta}, \]

which along with (6.19), (6.13) and (6.5) implies

\[ \| \partial_t a + V \cdot \nabla a \|_{L^\infty} \leq K^1_\eta \left( \| (V, B) \|_{W^{1, \infty}} + E_0(\psi) \right)^3. \]

Recall that \((V, B, \zeta)\) satisfies

\[ (\partial_t + V \cdot \nabla) B = a - g, \]
\[ (\partial_t + V \cdot \nabla) V + a \zeta = 0, \]
\[ (\partial_t + V \cdot \nabla) \zeta = G(\eta) V + \zeta G(\eta) B. \]

Making \( L^2 \) energy estimate for \((V, B)\), we get

\[ \frac{d}{dt} \| (V, B) \|_{L^2} \leq \| (\nabla V, \nabla B) \|_{L^\infty} \| (V, B) \|_{L^2} + \| a - g \|_{L^2} + \| a \|_{L^\infty} \| \zeta \|_{L^2}. \]
While, making $H^{s-1}$ energy estimates for $\zeta$, it is easy to obtain
\[
\frac{d}{dt}\|\zeta\|_{H^{s-1}} \leq C\|\nabla V\|_{L^\infty}\|\zeta\|_{H^{-1}} + \|\zeta\|_{L^\infty}\|V\|_{H^s} + \|G(\eta)V\|_{H^{s-1}} + \|\zeta G(\eta)B\|_{H^{s-1}}
\]
\[
\leq K_1^1\left(\|\nabla V\|_{L^\infty}\|\zeta\|_{H^{s-1}} + \|(V,B)\|_{H^s} + (\|V,B\|_{W^{1,\infty}} + E_0(\psi))\|\zeta\|_{H^{s+\frac{1}{2}}}\right).
\]

Then by \(8.3\), \(9.5\)-(9.7), Lemma 8.2 and Lemma 8.3, we obtain
\[
\frac{d}{dt}\left(\|(U_s, \theta_s)\|_{L^2} + \|(V,B)\|_{L^2} + \|\zeta_s\|_{H^{-1}}\right)
\]
\[
\leq K_1^3 G(t)\left(\|(U_s, \theta_s)\|_{L^2} + \|\zeta_s\|_{H^{-1}} + \|(V,B, \eta)\|_{L^2}\right) \ln \left(e + \|(U_s, \theta_s)\|_{L^2} + \|(V,B, \eta)\|_{L^2}\right).
\]
with $G(t) = (1 + E_0(\psi) + \|(V,B)\|_{W^{1,\infty}})^6$. Note that
\[
\|(V,B)\|_{L^\infty} \leq C\left(\|(V,B)\|_{H^{-\frac{1}{2}}} + \|\nabla V, \nabla B\|_{L^\infty}\right) \leq K_1(\|\psi_0\|_{H^{\frac{1}{2}}} + \|\eta_0\|_{L^2})
\]
\[
E_0(\psi) + \|\eta\|_{L^2} \leq C\left(\|\psi_0\|_{H^{\frac{1}{2}}} + \|\eta_0\|_{L^2}\right).
\]

Then we apply Gronwall’s inequality to obtain
\[
\|(U_s, \theta_s)\|_{L^2} + \|(V,B)\|_{L^2} \leq C(E_s(0), M(T), T, TS(a)^{-1}).
\]
Noting that for any $\epsilon > 0$,
\[
\|(V,B)\|_{W^{1,\infty}} \leq C\|\psi_0\|_{H^{\frac{1}{2}}} + \|\eta_0\|_{L^2}.
\]
Then by Lemma 8.2 again, we deduce that
\[
\|\eta\|_{H^{s+\frac{1}{2}}} + \|(V,B)\|_{H^s} \leq C(E_s(0), M(T), T, TS(a)^{-1}).
\]
This completes the proof of the theorem. \(\square\)

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