Chaos in Beddington–DeAngelis food chain model with fear

Hiba Abdullah Ibrahim and Raid Kamel Naji
Department of the Mathematics, College of Science, University of Baghdad, Baghdad, IRAQ
haibrahim82@yahoo.com; rknaji@gmail.com

Abstract. In the current paper, the effect of fear in three species Beddington–DeAngelis food chain model is investigated. A three species food chain model incorporating Beddington-DeAngelis functional response is proposed, where the growth rate in the first and second level decreases due to existence of predator in the upper level. The existence, uniqueness and boundedness of the solution of the model are studied. All the possible equilibrium points are determined. The local as well as global stability of the system are investigated. The persistence conditions of the system are established. The local bifurcation analysis of the system is carried out. Finally, numerical simulations are used to investigate the existence of chaos and understand the effect of varying the system parameters. It is observed that the existence of fear up to a critical value has a stabilizing effect on the system; otherwise it works as an extinction factor in the system.

1. Introduction
It is well known that the study of the prey-predator systems is an important subject in ecology and biology, due to the wide existence of such type of interaction in the environment [1-2]. Such prey-predator models have been extensively studied in literatures through previous years [3-5]. Most of these studies in literatures mainly concentrated on the local stability as well as persistence [6-7], while recent studies display a direction in exploring dynamical behaviors, for example, local bifurcation and chaos [8-11]. Food chain system is an ecological system that depends completely on the prey-predator interaction in which the energy transfers directly from one level to the higher level.

The effect of predator on the prey population within ecological systems may be direct or indirect or both. In the state of direct effect, the predator preys upon prey through killing them directly [12]. While, in the state of indirect effect, predator motivate fear in prey and change prey's behavior due to decreasing of the prey growth rate [13]. The fear effect is appearance of stress on prey. Recent works presented that the fear is strong enough to affect into the dynamics of ecological systems [14-15]. Many researchers studied the effect of fear in the ecological models. For example, Wang et al [16] have suggested a prey-predator model, where the effect of fear plays important role in the growth of prey. They spotted that the fear can stabilize the system. Zhang et al [17] have investigated the effect of anti-predator behavior that resulting from the fear of predators. They adopted a Holling type-II prey-predator, which incorporating a prey refuge. Pal et al [18] have studied a two species prey-predator model with a functional response of Beddington–DeAngelis type in case of existence of fear. Panday et al [19] investigated the role of fear in a food chain model consisting of three levels with a functional response of Holling type-II, they observed that fear effect can stabilize the system from chaos to stable.
In the present study, we are particularly interested to the dynamics of a food chain model with Beddington–DeAngelis (BD) type of functional response that proposed in [20] in case of existence of fear. It is assumed that the growth rates of prey and middle predator are decreasing as a cost of fear of upper level predator.

In Section (2) the mathematical model is formulated and then all the mathematical properties of the solution of the model are studied. Section (3) studied the stability analysis and determined the conditions of persistent of the model. Local bifurcation near each equilibrium point is discussed in section (4). However, numerical simulation is investigated in section (5). Eventually, in section (6) the discussion and conclusions are carried out from our obtained analytical.

2. Mathematical Model

In this section, a BD food chain model with fear is suggested. The mathematical model is formulated according to the following hypotheses:

- Let the densities of prey, middle predator and top predator at time $T$ are given by $X(T)$, $Y(T)$ and $Z(T)$ respectively.
- In the absence of middle predator $Y(T)$, the prey grows according to logistic function with intrinsic growth rate $r > 0$ and carrying capacity $k > 0$. While, the growth rate of prey decreases due to fear from the predation by middle predator with fear rate constant $a > 0$.
- The middle predator $Y(T)$ consumes the prey according to BD functional response with maximum attack rate $a_1 > 0$, the half saturation level $b_1 > 0$ and middle predator’s encounters rate $C_1 > 0$. However, The food converted to middle predator $Y(T)$ with conversion rate $0 < e_1 < 1$. It is assumed that, in the absence of the prey, the middle predator decays exponentially with natural death rate $D_1 > 0$. On the other hand, since the middle predator facing predation by top predator $Z(T)$ too, the growth rate of middle predator decreases with fear constant $\beta > 0$.
- The top predator $Z(T)$ consumes the middle predator according to BD functional response with maximum attack rate $a_2 > 0$, the half saturation level $b_2 > 0$, top predator’s encounters rate $C_2 > 0$ and then the food consumed by top predator is converted with conversion rate $0 < e_2 < 1$. However, in the absence of middle predator, it is decay exponentially with natural death rate $D_2 > 0$.

According to the above mentioned hypotheses, the dynamics of BD food chain model with fear represented by the following set of differential equations.

$$
\frac{dx}{dt} = \left( \frac{r x}{1 + a y} \right) \left( 1 - \frac{x}{k} \right) - \frac{a_1 x y}{b_1 y + x + c_1} - D_1 X
$$

$$
\frac{dy}{dt} = \left( \frac{e_1 a_1 x y}{b_1 y + x + c_1} \right) \left( \frac{1 + \beta x}{1 + \beta z} \right) - \frac{a_2 y z}{b_2 z + y + c_2} - D_1 Y
$$

$$
\frac{dz}{dt} = \frac{e_2 a_2 y z}{b_2 z + y + c_2} - D_2 Z
$$

Now, to simplify the model, the following dimensionless variables and parameters are used:

$$
t = \frac{r T}{r}, x = \frac{x}{k}, y = \frac{a_1 y}{b_1}, z = \frac{a_2 a_2 Z}{r^2 k}, a_1 = \frac{r k}{a_1}, \beta = \frac{r b_1}{a_1}, \gamma_1 = \frac{c_1}{k},
$$

$$
\theta_1 = \frac{e_1 a_1}{r}, \theta_2 = \frac{r^2 \beta k}{a_1 a_2}, \beta_2 = \frac{r b_2}{a_2}, \gamma_2 = \frac{c_2 a_1}{r k}, d_1 = \frac{a_1}{r}, \theta_2 = \frac{e_2 a_2}{r}, d_2 = \frac{a_2}{r}
$$

Therefore, system (1) reduced to:

$$
\frac{dx}{dt} = \left( \frac{1-x}{(1+\theta_1 y)} \right) - \frac{y}{b_1 y + x + \gamma_1} = xf_1(x, y, z)
$$

$$
\frac{dy}{dt} = \left( \frac{\theta_1 x}{b_1 y + x + \gamma_1} \right) - \frac{z}{b_2 z + y + \gamma_2} = yf_2(x, y, z)
$$

$$
\frac{dz}{dt} = \left( \frac{\theta_2 y}{b_2 z + y + \gamma_2} \right) - d_2 = zf_3(x, y, z)
$$

Theorem 1: System (3) has a uniformly bounded (UB) solutions.
Proof: From the first equation, we get

$$\frac{dx}{dt} \leq x[1 - x]$$

By the usual comparison theorem the following is obtained:

$$x(t) \leq \frac{x_0}{x_0 + e^{-t}(1 - x_0)}$$

where $x_0 = x(0)$ and then for $t \to \infty$, we get $x(t) \leq 1$.

Now, define the function $\omega(t) = x(t) + y(t) + z(t)$; then the time derivative of $\omega(t)$ is determined by:

$$\frac{d\omega}{dt} = \frac{x(1-x)}{(1+\alpha_1 y)} - \frac{xy}{\beta_1 y + x y_1} \left(1 - \frac{\theta_1}{1 + \alpha_2 z}\right) - \frac{y z (1 - \theta_2)}{\beta_2 y + y y_2} - d_1 y - d_2 z.$$

Therefore, due to the biological meaning of the system’s parameters and the bound of $x(t)$, it is obtained that

$$\frac{d\omega}{dt} + \mu \omega \leq 2$$

where $\mu = \min\{1, d_1, d_2\}$. Hence, due to the Gronwall lemma [21], we obtain $\omega(t) \leq \omega_0 e^{-\mu t} + \frac{b}{\mu} (1 - e^{-\mu t})$. Thus, for $t \to \infty$, we have that $0 \leq \omega(t) \leq \frac{2}{\mu}$. Hence all solutions of system (3) are UB and the proof is done.

3. The stability analysis

In this section, the existence and stability of the equilibrium points (EPs) are discussed. It’s observed that, system (3) has at most four EPs, which can be stated as follows:

1- The trivial equilibrium point $q_0 = (0,0,0)$ always exists.
2- The axial equilibrium point (AEP) that given by $q_1 = (1,0,0)$ always exists.
3- The top predator free equilibrium point (TPFEP), which is given by $q_2 = (\bar{x}, \bar{y}, 0)$, where

$$\bar{x} = \frac{d_1 (\beta_1 y + y_1)}{(\theta_1 - d_1)} \quad (4a)$$

While, $\bar{y}$ is a unique positive root of the equation:

$$H_1 y^2 + H_2 y + H_3 = 0 \quad (4b)$$

where

$$H_1 = -(\beta_1^2 y_1) d_1 + \alpha_1 (\theta_1 - d_1) < 0$$
$$H_2 = \beta_1 y_1 d_1 - \beta_1 \theta_1 d_1 - 2 \beta_1 y_1 d_1 - \theta_1^2 + 2 \theta_1 d_1 - d_1^2$$
$$H_3 = \theta_1^2 y_1 - \theta_1 d_1 - \theta_1 y_1^2 d_1.$$

So by Descartes’ rule of sign [22], equation (4b) has a unique positive root provided that:

$$d_1 (1 + y_1) < \theta_1 \quad (5)$$

Therefore, $q_2$ exists uniquely under the above condition.

4- The positive equilibrium point (PEP), that given by $q_3 = (x^*, y^*, z^*)$, where

$$x^* = \frac{-G_2 + \sqrt{G_2^2 - 4G_3}}{2}; \quad z^* = \frac{\theta_2 y^* - d_2 (y^* + y_2)}{\beta_2 d_2} \quad (6a)$$

with

$$G_2 = \beta_1 y^* + y_1 - 1$$
$$G_3 = y^*(1 - \beta_1 + \alpha_1 y^*) - y_1.$$

However, $y^*$ is a positive root of the following equation:

$$K_1 y^2 + K_2 y + K_3 = 0 \quad (6b)$$

here

$$K_1 = -\beta_1 (1 + \alpha_2 z^*) < 0 ,$$
$$K_2 = \theta_2 x^* - (1 + \alpha_2 z^*) (\beta_1 z^*(1 + \beta_2 d_1) + d_1 (\beta_1 y_1 + x^* + y_1)) ,$$
$$K_3 = (\theta_1 - d_1) (\beta_2 x^* z^* + y_2 x^*) - (1 + \alpha_2 z^*) [z^*(x^* + y_1) + d_1 y_1 (\beta_2 z^* + y_2)] - \alpha_2 d_1 x^* z^* (\beta_2 z^* + y_2)$$

So by Descartes’ rule of sign [22], equation (6b) has a unique positive root provided that:

$$K_3 > 0 \quad (7a)$$
Therefore, the PEP exists uniquely in the $\text{Int. } \mathbb{R}_+^3$ provided that in addition to condition (7a) the following conditions hold.

$$y^*(1 + \alpha_1 y^*) < \beta_1 y^* + \gamma_1$$ \hspace{1cm} (7b) \\
$$d_2 (y^* + \gamma_2) < \theta_2 y^*$$ \hspace{1cm} (7c)

Now the dynamical behavior of system (3) can be studied locally using linearization technique. Observed that it is simple to verify that, the Jacobian matrix (JM) of system (3) at $q_0 = (0,0,0)$ can be written in the form:

$$f(q_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}$$ \hspace{1cm} (8a)

Thus, the eigenvalues of $f(q_0)$ are given by

$$\lambda_{01} = 1 > 0, \lambda_{02} = -d_1 < 0, \lambda_{03} = -d_2 < 0.$$ \hspace{1cm} (8b)

Therefore, the trivial equilibrium point is a saddle point.

The JM at the (AEP), that is given by $q_1 = (1,0,0)$, can be written as:

$$f(q_1) = \begin{bmatrix} -1 & -\left(\frac{1}{1+\gamma_1}\right) & 0 \\ 0 & \frac{\theta_1}{1+\gamma_1} - d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}$$ \hspace{1cm} (9a)

Hence, the eigenvalues of $f(q_1)$ are given by

$$\lambda_{11} = -1 < 0, \lambda_{12} = \frac{\theta_1}{1+\gamma_1} - d_1 \text{ and } \lambda_{13} = -d_2 < 0.$$ \hspace{1cm} (9b)

Clearly, the AEP is locally asymptotically stable (LAS) if the following condition holds:

$$\theta_1 < d_1 (1 + \gamma_1)$$ \hspace{1cm} (10)

Moreover, it is easy to verify that, the point $q_0$ is a saddle point if the condition (5) holds.

The JM at the (TPFEP), $q_2 = (\bar{x}, \bar{y}, 0)$, can be written in the form:

$$f(q_2) = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$$ \hspace{1cm} (11a)

where

$$b_{11} = \bar{x} \left(\frac{-1}{1+\alpha_1 \bar{y}} + \frac{\bar{y}}{(\beta_1 \bar{y} + x + \gamma_1)}\right), b_{12} = -\left(\frac{\bar{x} (1-\bar{x}) \alpha_1}{(1+\alpha_1 \bar{y})^2} + \frac{\bar{x} (\bar{x} + \gamma_1)}{(\beta_1 \bar{y} + x + \gamma_1)^2}\right)$$

$$b_{21} = -\frac{\theta_1 \beta_1 \bar{x} \bar{y}}{(\beta_1 \bar{y} + x + \gamma_1)^2}, b_{22} = -\frac{\theta_1 \beta_1 \bar{x} \bar{y}}{(\beta_1 \bar{y} + x + \gamma_1)^2}, b_{23} = -\frac{\theta_2 \bar{y}}{(\bar{x} + \gamma_1)} + \frac{\bar{y}}{\bar{y} + \gamma_2}$$

$$b_{33} = \frac{\theta_2 \bar{y}}{\bar{y} + \gamma_2} - d_2.$$

Then the characteristic equation of $f(q_2)$ can be determined as follows:

$$\left(\bar{x}^2 - T_2 \lambda + D_2\right) (b_{33} - \lambda) = 0$$ \hspace{1cm} (11b)

where

$$T_2 = b_{11} + b_{22} \hspace{1cm} D_2 = b_{11} b_{22} - b_{12} b_{21}$$

Consequently, the eigenvalues are written as:

$$\lambda_{21} = \frac{T_2}{2} - \sqrt{\frac{T_2^2 - 4 D_2}{2}}, \lambda_{22} = \frac{T_2}{2} + \sqrt{\frac{T_2^2 - 4 D_2}{2}}, \lambda_{23} = \frac{\theta_2 \bar{y}}{\bar{y} + \gamma_2} - d_2$$ \hspace{1cm} (11c)

Hence the (TPFEP) is LAS provided the following conditions hold:

$$\frac{\bar{y}}{\bar{x} (\beta_1 \bar{y} + x + \gamma_1)^2} < \frac{1}{1+\alpha_1 \bar{y}}$$ \hspace{1cm} (12a)

$$\theta_2 \bar{y} < d_2 (\bar{x} + \gamma_2)$$ \hspace{1cm} (12b)

The JM at the PEP, that given by $q_3 = (x^*, y^*, z^*)$, can be written in the form

$$f(q_3) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$ \hspace{1cm} (13a)

where

$$a_{11} = x^* \left(\frac{-1}{1+\alpha_1 y^*} + \frac{y^*}{(\beta_1 y^* + x^* + \gamma_1)^2}\right), a_{12} = -\left(\frac{x^* (1-x^*) \alpha_1}{(1+\alpha_1 y^*)^2} + \frac{x^* (\gamma_1 + x^*)}{(\beta_1 y^* + x^* + \gamma_1)^2}\right), a_{13} = 0$$
Then the characteristic equation of $J(q_3)$ is
\[ \lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0 \]  
(13b)
where
\[
\begin{align*}
A_1 &= -(a_{11} + a_{22} + a_{33}) \\
A_2 &= a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32} - a_{12}a_{21} \\
A_3 &= a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33}
\end{align*}
\]
while
\[
\Delta = A_1 A_2 - A_3 = -(a_{11} + a_{22})[a_{11}a_{22} - a_{12}a_{21}] - 2a_{11}a_{22}a_{33} \\
&\quad - (a_{22} + a_{33})[a_{11}a_{23}a_{32} - a_{23}a_{32}] - a_{11}a_{33}[a_{11} + a_{33}]
\]

Now, according to the Routh-Hurwitz criterion [23], the roots of equation (13b) have negative real parts provided that $A_1 > 0$, $A_3 > 0$ and $\Delta > 0$. Direct calculation shows that these conditions hold provided that
\[
\frac{y^*}{(\beta_1y^*+x^*+y_1)^2} < \frac{1}{1+a_{11}y^*} \quad \text{(14a)}
\]
\[
\frac{x^*}{(\beta_2x^*+y^*+y_2)^2} < \frac{1}{(1+a_{22}y^*)} \quad \text{(14b)}
\]

Therefore the PEP is LAS under the conditions (14a)-(14b).

Obviously system (3) has only one possible subsystem lying in the first quadrant of $xy$–plane. This subsystem can be written as:
\[
\begin{align*}
\frac{dx}{dt} &= x\left[\frac{(1-x)}{(1+a_{11}y)} - \frac{y}{\beta_1y+x+y_1}\right] = g_1(x,y) \\
\frac{dy}{dt} &= y\left[\frac{-\theta_1 x}{\beta_1y+x+y_1} - d_1\right] = g_2(x,y)
\end{align*}
\]
(15)

Now, in order to investigate the existence of periodic dynamics in the interior of the first quadrant of $xy$–plane, define the Dulac function as $(x,y) = \frac{1}{xy}$. Clearly $B(x,y) > 0$ and $C^1$ function in the Int. $\mathbb{R}_+^2$ of the $xy$–plane. Further, we have
\[
\Delta(x,y) = \frac{\partial(B g_1)}{\partial x} + \frac{\partial(B g_2)}{\partial y} = -\frac{1}{y(1+a_{11}y)} + \frac{1-\theta_1 \beta_1}{(\beta_2y+x+y_1)^2}
\]

Then $\Delta(x,y)$ does not identically zero in the Int. $\mathbb{R}_+^2$ of the $xy$–plane and does not change sign under one of the following two conditions:
\[
\frac{1-\theta_1 \beta_1}{(\beta_2y+x+y_1)^2} < \frac{1}{y(1+a_{11}y)} \quad \text{(16a)}
\]
or
\[
\frac{1-\theta_1 \beta_1}{(\beta_2y+x+y_1)^2} > \frac{1}{y(1+a_{11}y)} \quad \text{(16b)}
\]

Therefore, by using Dulac-Bendixon criterion [24], there is no closed curve lying in the Int. $\mathbb{R}_+^2$ of the $xy$–plane for all the trajectories satisfying condition (16a) or condition (16b). Hence according to the Poincare-Bendixon theorem [24], the unique equilibrium point in the Int. $\mathbb{R}_+^2$ of the $xy$–plane that given by $d_2$ will be a globally asymptotically stable (GAS) whenever it is LAS.

Theorem 2: Assume that either conditions (16a) or (16b) holds and let the following conditions hold then system (3) is uniformly persistent.
\[
\begin{align*}
d_1(1+y_1) &< \theta_1 \\
d_2(y+y_2) &< \theta_2 y
\end{align*}
\]
(17a)
(17b)

Proof: Let us use the average Lyapunov method [25]. Consider the following function $(x,y,z) = x^{p_1} y^{p_2} z^{p_3}$, where $p_j$, $\forall j = 1,2,3$ are positive constants. Obviously $\varphi(x,y,z) > 0$ for all $(x,y,z) \in \text{Int. } \mathbb{R}_+^2$ and $\varphi(x,y,z) \to 0$ when $x \to 0$ or $y \to 0$ or $z \to 0$. Consequently, we obtain
Now, according to average Lyapunov method, the proof follows if $\Omega(E) > 0$ for any boundary equilibrium point $E$, with suitable choice of constants $p_1 > 0$, $p_2 > 0$, and $p_3 > 0$.

$$\Omega(q_1) = p_2 \left( \frac{\theta_1}{y_1} - d_1 \right) + p_3 (-d_2)$$

$$\Omega(q_2) = p_3 \left( \frac{\theta_2 y}{y + y_2} - d_2 \right)$$

Clearly, $\Omega(q_1) > 0$ under condition (17a) for appropriate choice of positive constants $p_2$ and $p_3$, so that $p_2$ is large enough with respect to the constant $p_3$. While, $\Omega(q_2) > 0$ under condition (17b). Hence the proof is complete.

**Theorem 3**: Assume that the AEP is **LASS**, then it is a **GASS** in the Int. $\mathbb{R}^3_+$ provided that the following condition holds.

$$\frac{1 + \theta_1}{y_1} < d_1$$

**Proof**: Define the function

$$u(x, y, z) = \int_{\frac{1}{m}}^{x} \ln dm + y + \frac{1}{\theta_2} z$$

Clearly the function $u$ is positive definite so that $u(1,0,0) = 0$ and $u(x, y, z) > 0$ for all $(x, y, z) \in \mathbb{R}^3_+$ with $(x, y, z) \neq (1,0,0)$ and $x > 0$.

Now, straightforward calculations give that

$$\frac{du}{dt} \leq - \left( 1 - \frac{\theta_1}{\theta_2} \right) \frac{\theta_1}{y_1} y - \frac{d_2}{\theta_2} z$$

Hence under condition (18), we obtain that $\frac{du}{dt}$ will be negative definite. Then $u$ is a Lyapunov function (L5). Therefore AEP is a **GASS**.

**Theorem 4**: Assume that the PFEP is **LASS**, then it is a **GASS** in the Int. $\mathbb{R}^3_+$ provided that the following conditions hold.

$$R_1 < (1 + \alpha_1 \bar{y}) R_2$$

$$\frac{\theta_2 \bar{y}}{y_2} < \frac{4 q_{11} q_{22}}{d_2}$$

where all the symbols are described clearly in the proof.

**Proof**: Consider the following function

$$V(x, y, z) = \int_{\frac{x}{u}}^{x} \frac{u}{u} du + \frac{1}{\theta_2} \int_{\frac{y}{v}}^{y} \frac{v}{v} dv + z.$$ 

Obviously the function $V(x, y, z) > 0$ is a continuously differentiable real valued function for all $(x, y, z) \in \mathbb{R}^3_+$ and $(x, y, z) \neq (\bar{x}, \bar{y}, 0)$ with $x > 0$, $y > 0$, while $V(\bar{x}, \bar{y}, 0) = 0$.

Now, straightforward calculations give that

$$\frac{dV}{dt} \leq - q_{11} (x - \bar{x})^2 - q_{12} (x - \bar{x})(y - \bar{y}) - q_{22} (y - \bar{y})^2 - z R_1 R_2 \left[ \frac{d_2}{\theta_2 \bar{y}} \right]$$

where

$$q_{11} = (1 + \alpha_1 \bar{y}) R_2 - R_1,$$

$$q_{12} = (1 + \alpha_1 \bar{x}) R_2 + \gamma_1 \left( 1 + \frac{\theta_1 \theta_2}{1 + \alpha_2 z} \right) R_1 + \left( \bar{x} - \frac{\theta_1 \theta_2 \beta_1 \bar{y}}{1 + \alpha_2 z} \right) R_1,$$

$$q_{22} = \frac{\theta_1 \theta_2 \beta_1 \bar{y}}{(1 + \alpha_2 z)} R_1.$$ 

with $R_1 = (1 + \alpha_1 \bar{y})(1 + \alpha_1 \bar{y})$ and $R_2 = (\beta_1 y + x + \gamma_1)(\beta_1 \bar{y} + \bar{x} + \gamma_1)$. Accordingly, by using the given conditions (19a)–(19c), we obtain
\[
\frac{dv}{dt} \leq -\left[\sqrt{q_{11}(x - \bar{x})} + \sqrt{q_{22}(y - \bar{y})}\right]^2 - z R_1 R_2 \left[d_2 - \frac{\theta_2 y}{y_2}\right].
\]

Therefore, the derivative \(\frac{dv}{dt}\) is negative definite and then \(V\) is a \(\mathcal{L}\). Thus the PFEP is a \(\mathcal{GAS}\).

**Theorem 5:** Assume that the PEP is \(\mathcal{LAS}\) in the Int. \(\mathbb{R}_+^3\), then it is a \(\mathcal{GAS}\) provided that the following conditions hold:

\[
\begin{align*}
q_{122} &< 2 q_{11} q_{22} \\
q_{233} &< 2 q_{22} q_{33} \\
y' &< \left(\frac{1 + \alpha_1 \gamma'}{1 + \alpha_2 \gamma'}\right) \\
\frac{z'}{R_4} &< \frac{\theta_1 \beta_1 x' (1 + \alpha_1 z')}{R_2 R_3}
\end{align*}
\]  

(20a) (20b) (20c) (20d)

where all the symbols are described clearly in the proof.

**Proof:** Consider the positive definite function

\[
l(x, y, z) = \int_x^{x} \int_y^{y} \int_z^{z} du + \int_y^{y} \int_z^{z} dv + \frac{1}{\theta_2} \int_z^{z} dw
\]

Clearly, the function \(l(x, y, z) > 0\) is a continuously differentiable real valued function for all \((x, y, z) \in \mathbb{R}_+^3\) with \((x, y, z) \neq (x', y', z')\) and \(x > 0, y > 0, z > 0\), while \(l(x', y', z') = 0\).

Now, the derivative of this function with respect to time can be written as

\[
\frac{dl}{dt} = -q_{11}(x - \bar{x})^2 - q_{12}(x - \bar{x})(y - \bar{y}) - q_{22}(y - \bar{y})^2
\]

\[
\begin{align*}
qu_{11} &= R_2(1 + \alpha_1 \gamma')(1 - y')R_1 y' \\
qu_{12} &= \frac{\alpha_1 (1 - \bar{x})}{R_1} + \frac{y' + \bar{x}}{R_2} - \frac{\theta_1 (1 + \alpha_1 \gamma')(1 + \alpha_2 z')}{R_2 R_3} \\
qu_{22} &= \frac{\theta_1 \beta_1 x' (1 + \alpha_1 z')}{R_2 R_3} - \frac{\bar{x}}{R_4} \\
qu_{33} &= \frac{\theta_2 (1 + \alpha_2 z')}{R_4}
\end{align*}
\]

and

\[
\begin{align*}
q_{122} &< 2 q_{11} q_{22} \\
q_{233} &< 2 q_{22} q_{33} \\
y' &< \left(\frac{1 + \alpha_1 \gamma'}{1 + \alpha_2 \gamma'}\right) \\
\frac{z'}{R_4} &< \frac{\theta_1 \beta_1 x' (1 + \alpha_1 z')}{R_2 R_3}
\end{align*}
\]

(20a) (20b) (20c) (20d)

Accordingly, by using the given conditions (20a)–(20d) we obtain

\[
\frac{dl}{dt} \leq - \left[\sqrt{q_{11}(x - \bar{x})} + \sqrt{q_{22}(y - \bar{y})}\right]^2 - \left[\sqrt{\frac{q_{122}}{2}}(y - \bar{y})\right] - \left[\sqrt{\frac{q_{233}}{2}}(z - \bar{z})\right]
\]

Therefore, the derivative \(\frac{dl}{dt}\) is negative definite and hence \(l\) is a \(\mathcal{L}\). Thus, the PEP is a \(\mathcal{GAS}\).

### 4. Local Bifurcation

In this section, the local bifurcation near the possible EPs of system (3) is discussed with the help of Sotomayor’s theorem [21]. It is well known that the existence of non-hyperbolic equilibrium point represents a necessary but not sufficient condition for occurrence of bifurcation. Therefore the candidate bifurcation parameter that makes the equilibrium point non-hyperbolic at a specific value of that parameter is selected. Now rewrite system (3) in the form:

\[
\frac{dX}{dt} = F(X)
\]

where \(X = (x, y, z)^T\) and \(F = (xf_1, yf_2, zf_3)^T\) with \(f_i; i = 1, 2, 3\) represent the interaction functions in the right hand side of system (3). Then straightforward computation on the JM of system (3) with any non-zero vector \(V = (v_1, v_2, v_3)^T\), gives the following second directional derivative

\[
D^2F(x, y, z)(V, V) = (e_{ij})_{3 \times 1}
\]

(22)

where
Theorem 6: System (3) at AEP undergoes a transcritical bifurcation (TB) but neither saddle node bifurcation (SNB) nor pitchfork bifurcation (PB) can occur when the parameter passes through the value \( d_1 = d_1 (1 + \gamma_1) \).

Proof: According to the JM that given in equation (9a), system (3) at AEP with \( \theta_1 = \theta_1^* \) has the following JM, say \( J(q_1; \theta_1^*) = J_1 \), where

\[
J_1 = \begin{bmatrix}
-1 & -\frac{1}{1 + \gamma_1} & 0 \\
0 & 0 & 0 \\
0 & 0 & -d_1
\end{bmatrix}
\]

Clearly, \( J_1 \) has a zero eigenvalue given by \( \lambda_{12}^* = 0 \) and hence AEP is a nonhyperbolic point.

Now, let \( U^{[1]} = (u_1^{[1]}, u_2^{[1]}, u_3^{[1]})^T \) be the eigenvector corresponding to the eigenvalue \( \lambda_{12}^* = 0 \).

Thus \( J_1 U^{[1]} = 0 \) gives that \( U^{[1]} = (n u_2^{[1]}, u_2^{[1]}, 0)^T \), where \( n = -\frac{1}{1 + \gamma_1} < 0 \) and \( u_2^{[1]} \) represents any nonzero real number. Also, let \( \psi^{[1]} = (\psi_1^{[1]}, \psi_2^{[1]}, \psi_3^{[1]})^T \), represents the eigenvector corresponding to the eigenvalue \( \lambda_{12}^* = 0 \) of \( J_1^T \).

Hence \( J_1^T \psi^{[1]} = 0 \) gives that \( \psi^{[1]} = (0, \psi_2^{[1]}, 0)^T \), where \( \psi_2^{[1]} \) stands for any nonzero real number.

Now because

\[
\frac{\partial F}{\partial \theta_1} = F_{\theta_1}(X, \theta_1) = \begin{bmatrix} 0 & x & y \\ \frac{1}{1 + \alpha_2 + (\beta_1 y + \lambda_1 y_1)} \end{bmatrix}
\]

Thus, \( F_{\theta_1}(q_1, \theta_1^*) = (0, 0, 0)^T \), which gives \( (\psi_1^{[1]})^T F_{\theta_1}(q_1, \theta_1^*) = 0 \). So according to Sotomayor’s theorem for local bifurcation, system (3) has no SNB at \( \theta_1 = \theta_1^* \). Furthermore because we have

\[
DF_{\theta_1}(q_1, \theta_1^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 + \gamma_1 \gamma_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Then we obtain,

\[
(\psi_1^{[1]})^T (DF_{\theta_1}(q_1, \theta_1^*) U^{[1]}) = \psi_2^{[1]} \frac{u_2^{[1]}}{1 + \gamma_1} \neq 0
\]

Moreover using equation (22) with \( q_1, \theta_1^* \) and \( U^{[1]} \) gives

\[
D^2F(q_1, \theta_1^*) (U^{[1]}, U^{[1]}) = 2 \left( u_2^{[1]} \right)^2 \begin{bmatrix} -n^2 \gamma_1 \left(-\alpha_1 + \frac{\gamma_1}{(1 + \gamma_1)^2} + \frac{\beta_1}{(1 + \gamma_1)^2} \right) \\ n \gamma_1 \frac{1}{1 + \gamma_1} \theta_1^* - \frac{\gamma_1 \gamma_1}{(1 + \gamma_1)^2} \theta_1^* \end{bmatrix}
\]

Hence, it is obtained that
Thus, based on Sotomayor’s theorem, system (3) at AEP has a TB as the parameter $\theta_1$ passes through the bifurcation value $\theta_1^*$, while $S\mathcal{N}B$ cannot occurs and that complete the proof.

**Theorem 7:** Assume that condition (12a) holds, then system (3) at TPFEP undergoes a $\mathcal{N}B$ but neither $S\mathcal{N}B$ nor $S\mathcal{B}$ can occurs when the parameter $d_2$ passes through the value $d_2^* = \frac{\theta_2 y}{(y+y_2)^2}$.

**Proof:** From the JM that given in equation (11a), system (3) at TPFEP with $d_2 = d_2^*$ has the following JM, say $f(q_2, d_2^*) = J_2$, which has zero eigenvalue, say $\lambda_{23}^* = 0$.

$$
J_2 = \begin{bmatrix}
    b_{11} & b_{12} & 0 \\
    b_{21} & b_{22} & b_{23} \\
    0 & 0 & 0
\end{bmatrix}
$$

where $b_{ij}; \forall i,j = 1,2,3$ are given in equation (11a).

Now, let $U^{[2]} = \left( u_1^{[2]}, u_2^{[2]}, u_3^{[2]} \right)^T$ represents the eigenvector corresponding to the eigenvalue $\lambda_{23}^* = 0$.

Therefore, $J_2 U^{[2]} = 0$ gives that $U^{[2]} = \left( m_1 u_1^{[2]}, m_2 u_2^{[2]}, u_3^{[2]} \right)^T$, where $m_1 = \frac{b_{12} b_{23}}{b_{11}b_{22} - b_{12} b_{21}} > 0$,

$m_2 = -\frac{b_{11} b_{23}}{b_{11}b_{22} - b_{12} b_{21}} < 0$ and $u_3^{[2]}$ represents any nonzero real number. Also, let $\psi^{[2]} = \left( \psi_1^{[2]}, \psi_2^{[2]}, \psi_3^{[2]} \right)^T$ represents the eigenvector corresponding to the eigenvalue $\lambda_{23}^* = 0$ of $J_2^T$.

Hence $J_2^T \psi^{[2]} = 0$ gives that $\psi^{[2]} = \left( 0,0,\psi_3^{[2]} \right)^T$, where $\psi_3^{[2]}$ stands for any nonzero real number.

Now because we have

$$
\frac{\partial F}{\partial d_2} = F_{d_2}(X, d_2) = \left( 0,0,-z \right)^T
$$

Thus $F_{d_2}(q_2, d_2^*) = \left( 0,0,0 \right)^T$, which gives $\left( \psi^{[2]} \right)^T F_{d_2}(q_2, d_2^*) = 0$. So according to Sotomayor’s theorem for local bifurcation, system (3) has no $S\mathcal{N}B$ at $d_2 = d_2^*$. Furthermore because we have

$$
DF_{d_2}(q_2, d_2^*) = \begin{bmatrix}
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & -1
\end{bmatrix}
$$

We can show that

$$
\left( \psi^{[2]} \right)^T \left( DF_{d_2}(q_2, d_2^*) U^{[2]} \right) = \left( 0,0,\psi_3^{[2]} \right) \left( 0,0,-\psi_3^{[2]} \right)^T = -\psi_3^{[2]} \psi_3^{[2]} \neq 0
$$

Moreover, using equation (22) with $q_2, d_2^*$ and $U^{[2]}$ gives

$$
D^2 F(q_2, d_2^*) \left( U^{[2]}, U^{[2]} \right) = 2 \left( u_3^{[2]} \right)^2 \left( c_{ij}^{[2]} \right)_{3 \times 1}
$$

Where

$$
c_{11}^{[2]} = \left( -\frac{1}{1+\alpha_1 y} + \frac{y(\beta_1 y + \gamma_1)}{(\beta_1 y + \gamma_1)^2} \right) m_1^2 - \left( \frac{a_3 (1-2x)}{(1+\alpha_3 y)^3} + \frac{2 \beta_1 y + \gamma_1 (\beta_1 y + \gamma_1 y + \gamma_1)}{(\beta_1 y + \gamma_1)^3} \right) m_1 m_2
$$

$$
c_{21}^{[2]} = -\left( \frac{\theta_2 \beta_1 y + \gamma_1 y}{(\beta_1 y + \gamma_1)^2} \right) m_1 m_2 + \left( \frac{2 \beta_1 y + \gamma_1 + \theta_1 \gamma_1 (\beta_1 y + \gamma_1 y + \gamma_1)}{(\beta_1 y + \gamma_1)^3} \right) m_1 m_2
$$

$$
c_{31}^{[2]} = \left( \frac{\theta_2 \beta_1 y + \gamma_1 y}{(\beta_1 y + \gamma_1)^2} \right) m_2^2 - \left( \frac{2 \beta_1 y + \gamma_1}{(\beta_1 y + \gamma_1)^2} \right)
$$

Hence, it is obtained that

$$
\left( \psi^{[1]} \right)^T D^2 F(q_1, \theta_1^*) \left( U^{[1]}, U^{[1]} \right) = \frac{2 d_1}{(1+\gamma)} (n \gamma_1 - \beta_1) \psi_2^{[1]} \left( u_2^{[1]} \right)^2 \neq 0
$$
Therefore, by Sotomayor’s theorem, system (3) at TPFEP has a $\mathcal{B}$ as the parameter $d_2$ passes through the bifurcation value $d_2^*$, while $\mathcal{B}$ cannot occurs and hence the proof is complete.

**Theorem 8:** Assume that condition (14a) along with the following sufficient conditions hold

\begin{align}
\frac{\theta_1 \beta_1 x^*}{\rho_3^2 \rho_3} &< \frac{x^*}{\rho_4^2} \\
\rho_3 &< \theta_1 \beta_1 \\
1 &< \theta_2 \beta_2 \\
\frac{x^* + x_0^*}{\rho_1 \rho_2^2 \rho_3} + \frac{\theta_1 \beta_1 x^* y^*}{\rho_1 \rho_2^2 \rho_3} + \frac{\theta_1 a_1 x^*(1-x^*)(\beta_1 y^* + \gamma_1)}{\rho_1^2 \rho_2^2 \rho_3} &< \frac{\theta_3 y_1 x^* y^*}{\rho_2^2 \rho_3} \\
\frac{x^*}{\rho_1} + \frac{x_0^*}{\rho_2^2 \rho_3} (\theta_1 \beta_1 - \rho_3) &< \frac{x^*}{\rho_4} \\
[A_1(\theta_2^*) A_2(\theta_2^*)] &< A_3^* (\theta_2^*)
\end{align}

where $\rho_1 = 1 + \alpha_1 y^*$; $\rho_2 = \beta_1 y^* + x^* + \gamma_1$; $\rho_3 = 1 + \alpha_2 z^*$ and $\rho_4 = \beta_2 z^* + y^* + \gamma_2$; while $A_i; i = 1, 2, 3$ are given in equation (13b). Then system (3) undergoes a Hopf bifurcation ($\mathcal{HB}$) around the equilibrium point $q_3$ as the parameter $\theta_2$ passes through the positive value $\theta_2^*$, that given in the proof.

**Proof:** Recall that, according to the $\mathcal{HB}$ theorem [26] for the three dimensional autonomous system, such as system (3), undergoes a $\mathcal{HB}$ as the parameter $\theta_2$ passes through the positive value $\theta_2^*$, provided that:

The JM of system (3) at the equilibrium point $q_3$ has a simple pair of complex eigenvalues, say $\sigma_3(\theta_2) \pm i \sigma_2(\theta_2)$, such that they become purely imaginary at $\theta_2 = \theta_2^*$, while the third eigenvalue remain real and negative. Moreover, the transversality condition $\frac{d \sigma_1(\theta_2)}{d \theta_2} |_{\theta_2 = \theta_2^*} \neq 0$ should be hold.

Note that the above first condition will be satisfied if the coefficients of the characteristic equation given by (13b) satisfy that $\Delta = A_1 A_2 - A_3 = 0$. So straightforward computation gives that this is equivalent to

$$r_1 \theta_2^2 + r_2 \theta_2 + r_3 = 0$$

(24a)

Where

$$r_1 = -[\beta_2 y^*(a_{11} + a_{22}) + (\beta_2 z^* + \gamma_2)a_{23}] \frac{\beta_2 y^*}{\rho_4^2},$$

$$r_2 = [\beta_2 y^*(a_{11} + a_{22}) + (\beta_2 z^* + \gamma_2)a_{23}] \frac{x^*}{\rho_4^2},$$

$$r_3 = -(a_{11} + a_{22})(a_{11} a_{22} - a_{12} a_{21})$$

Clearly, according to the signs of JM elements that given by equation (13a) in addition to the sufficient conditions (14a), (23a), (23b), (23d) and (23e) it is easy to verify that $a_{11} < 0, a_{22} > 0, r_1 > 0$ and $r_3 < 0$, and then equation (24a) has a unique positive root denoted by $\theta_2^*$ that satisfies $A_1(\theta_2^*) A_2(\theta_2^*) = A_3(\theta_2^*)$. Consequently, as $\theta_2 = \theta_2^*$ the characteristic equation given by (13b) will be

$$(\lambda + A_1)(\lambda^2 + A_2) = 0$$

(24b)

Thus, equation (24b) has the following roots

$$\lambda_1 = -A_1(\theta_2^*)$$

And

$$\lambda_{2,3} = \pm i \sqrt{A_2(\theta_2^*)} = \pm i \sigma_2(\theta_2^*).$$

Again, the given conditions (23b)-(23d) with the signs of JM elements guarantee that $A_i > 0$ for all $i = 1, 2, 3$. Therefore the first condition of the $\mathcal{HB}$ follows.

Now in order to check the occurrence of the transversality condition, substitute $a_1(\theta_2) + i a_2(\theta_2)$, where $\theta_2$ in the neighborhood of $\theta_2^*$, in the equation (24b) and then take the derivative with respect to the bifurcation parameter $\theta_2$. Then comparing the two sides of this equation and then equating their real and imaginary parts, we get
Solving the linear system (25a) by using Cramer’s rule for the unknowns $\sigma_1'(\theta_2)$ and $\sigma_2'(\theta_2)$ gives that

\[
\begin{align*}
\sigma_1'(\theta_2) &= -\frac{\theta_2^2 + \theta_2}{\Psi(\theta_2)\Phi(\theta_2)} + \frac{\theta_2}{\Phi(\theta_2)}, \\
\sigma_2'(\theta_2) &= -\frac{\theta_2^2 + \theta_2}{\Psi(\theta_2)\Phi(\theta_2)} + \frac{\theta_2}{\Phi(\theta_2)}.
\end{align*}
\]

Therefore the transversality condition is satisfied provided that

\[
\theta(\theta_2^*)\Psi(\theta_2^*) + \Gamma(\theta_2^*)\Phi(\theta_2^*) = 0.
\]

Obviously, we have that $\sigma_1(\theta_2^*) = 0$ and $\sigma_2(\theta_2^*) = \sqrt{A_2(\theta_2^*)}$, so we obtain that

\[
\begin{align*}
\sigma_1'(\theta_2^*) &= -A_1'(\theta_2^*)A_2(\theta_2^*) + \alpha_2'(\theta_2^*) \\
\sigma_2'(\theta_2^*) &= 2A_1(\theta_2^*)A_2(\theta_2^*)
\end{align*}
\]

Accordingly, we get that

\[
\begin{align*}
\sigma_1'(\theta_2^*) &= 2A_2(\theta_2^*) \left[ A_2(\theta_2^*) - \alpha_2'(\theta_2^*) \alpha_2(\theta_2^*) + \alpha_2(\theta_2^*) \alpha_2(\theta_2^*) \right] \\
&= \frac{[\Psi(\theta_2^*)]^2 + [\Phi(\theta_2^*)]^2}{\Phi(\theta_2^*)}.
\end{align*}
\]

Consequently, $\sigma_1'(\theta_2^*) > 0$ under condition (23f), and then the transversality condition hold. Hence $\mathcal{HB}$ occurs at $\theta_2^*$. Not that, according the above theorem, we have that for $\theta_2 > \theta_2^*$ the equilibrium point $q_3$ of system (3) is stable; when $\theta_2 = \theta_2^*$ loses its stability and the $\mathcal{HB}$ occurs at $q_3$, and when $\theta_2 < \theta_2^*$, the equilibrium point $q_3$ becomes unstable and a family of periodic solutions bifurcates from $q_3$.

5. Numerical Simulation

In this section, the global dynamics of system (3) is investigated numerically. The main objectives understand the effect of fear on the dynamics of system (3), specify the set of parameters that control the dynamical behavior of the system (3) and confirm our obtained results. Different tools are used through this investigation such as bifurcation diagram (BD), chaotic attractor, 3D phase plot and time series. Predictor-Corrector method with six-order Runge Kutta methods are used for solving the system, while MATLAB version 6 is used to present these numerical results.

The following hypothetical set of parameters is used.

\[
\begin{align*}
\alpha_1 = 0, & \quad \beta_1 = 0.2, \quad \gamma_1 = 0.2, \quad \theta_1 = 0.5, \quad \alpha_2 = 0, \quad \beta_2 = 0.2, \\
\gamma_2 = 0.2, & \quad d_1 = 0.2, \quad \theta_2 = 0.3, \quad d_2 = 0.1
\end{align*}
\]

Clearly, in the above set of data, there is no fear in the system (3). It is observed that system (3) undergoes a chaotic dynamics for the above set of data as shown in the Figure 1.
Obviously from Figure 1, system (3) without fear has a chaotic dynamics at the data (26), which indicates to existence of complex dynamics. Now, to investigate the impact of varying the parameters $\theta_1$, $\theta_2$ and $d_2$ on the dynamics of system (3), the $\mathcal{B}D$ for the trajectory of system (3) as a function of each parameter are drawn in Figure 2 – Figure 4 respectively. It is known that, the $\mathcal{B}D$ summarizes the dynamical behavior of the system as a function of a specific. These parameters are selected according to the analytical study in section (4).

**Figure 1.** The trajectory of system (3) for the data (26). (a) Chaotic attractor. (b) Time series of the attractor in (a).

**Figure 2.** $\mathcal{B}D$ of system (3) as a function of $\theta_1 \in (0,2,1)$ keeping other parameters as in the data (26). (a) Maximum of the trajectory of $y$ versus $\theta_1$. (b) Maximum of the trajectory of $z$ versus $\theta_1$. 
Figure 3. BDF of system (3) as a function of $\theta_2 \in (0,1)$ keeping other parameters as in the data (26). (a) Maximum of the trajectory of $y$ versus $\theta_2$. (b) Maximum of the trajectory of $z$ versus $\theta_2$.

Figure 4. BDF of system (3) as a function of $d_2 \in (0,0.25)$ keeping other parameters as in the data (26). (a) Maximum of the trajectory of $y$ versus $d_2$. (b) Maximum of the trajectory of $z$ versus $d_2$.

Clearly, as shown in the above BDF, system (3) is very sensitive for varying the parameters $\theta_1$, $\theta_2$ and $d_2$. Different types of bifurcations are obtained and system (3) enters to chaotic and periodic regions. Furthermore, it is obtained that system (3) approaches asymptotically to AEP for the range $\theta_1 \in (0,0.24)$, which is confirm stability condition (10). It is approaches asymptotically to TPFEP, where $q_2 = (0.89,0.11,0)$, for the range $\theta_1 \in (0.24,0.26)$. While it is approach asymptotically to PEP, with $q_3 = (0.91,0.1,0.005)$, for the range $\theta_1 \in (0.26,0.28)$. Finally, system (3) approaches asymptotically to a periodic dynamics in $Int. \mathbb{R}^2$, see Figure 5 for typical values of $\theta_1$ and Table 1 for varying other parameters.
Figure 5. The trajectory of system (3) for the data (26) with different values of $\theta_1$. (a) System (3) approach asymptotically to $q_1 = (1,0,0)$ for $\theta_1 = 0.2$. (b) Time series of the attractor in (a). (c) System (3) approach asymptotically to $q_2 = (0.89,0.11,0)$ for $\theta_1 = 0.25$. (d) Time series of the attractor in (c). (e) System (3) approach asymptotically to $q_3 = (0.91,0.1,0.005)$ for $\theta_1 = 0.27$. (g) System (3) approach asymptotically to period two attractor for $\theta_1 = 0.7$. (h) Time series of the attractor in (g).

Table 1. The dynamical behavior of system (3) using data (26) with varying one parameter each time

| The parameter | The range of varying | The dynamical behavior of system (3) |
|---------------|----------------------|--------------------------------------|
| $\beta_1$     | $0 < \beta_1 < 0.26$ | Complex dynamics involving periodic and chaos |
|               | $0.26 \leq \beta_1 < 1.5$ | Periodic dynamics in $Int. \mathbb{R}_+^3$ |
|               | $1.5 \leq \beta_1$ | Approaches to PEP in $Int. \mathbb{R}_+^3$ |
| $\gamma_1$   | $0 < \gamma_1 < 0.09$ | Periodic in the $xy$–plane |
|               | $0.09 < \gamma_1 < 0.23$ | Complex dynamics involving periodic and chaos |
|               | $0.23 \leq \gamma_1$ | Periodic dynamics in $Int. \mathbb{R}_+^3$ |
| $d_1$         | $0 < d_1 < 0.36$ | Complex dynamics involving periodic and chaos |
|               | $0.36 \leq d_1 < 0.4$ | Approaches to PEP in $Int. \mathbb{R}_+^3$ |
|               | $0.4 \leq d_1 < 0.42$ | Approaches to TPFEP in $xy$–plane |
|               | $0.42 \leq d_1 < 1$ | Approaches to AEP |
| $\beta_2$     | $0 < \beta_2 < 0.71$ | Complex dynamics involving periodic and chaos |
|               | $0.71 \leq \beta_2 < 1$ | Periodic dynamics in $Int. \mathbb{R}_+^3$ |
| $\gamma_2$   | $0 < \gamma_2 < 0.3$ | Complex dynamics involving periodic and chaos |
|               | $0.3 \leq \gamma_2 < 0.47$ | Periodic dynamics in $Int. \mathbb{R}_+^3$ |
|               | $0.47 \leq \gamma_2 < 0.69$ | Approaches to PEP in $Int. \mathbb{R}_+^3$ |
|               | $0.69 \leq \gamma_2 < 1$ | Periodic in the $xy$–plane |
| $\theta_2$    | $0 < \theta_2 < 0.18$ | Periodic in the $xy$–plane |
|               | $0.18 \leq \theta_2 < 0.7$ | Complex dynamics involving periodic and chaos |
|               | $0.7 \leq \theta_2 < 1$ | Periodic dynamics in $Int. \mathbb{R}_+^3$ |
Now, in order to understand the effects of varying the fear rates on the dynamics of system (3) using the data (26), the system is solved numerically with different values of prey’s fear rate $\alpha_1$ and different values of intermediate predator’s fear rate $\alpha_2$ as shown in Figure 6 and Figure 7 respectively.

Figure 6. The trajectory of system (3) for the data (26) with different values of $\alpha_1$. (a) System (3) approach asymptotically to chaotic attractor for $\alpha_1 = 0.5$. (b) Time series of the attractor in (a). (c) System (3) approach asymptotically to periodic attractor in $Int. \mathbb{R}_+^3$ for $\alpha_1 = 10$. (d) Time series of the attractor in (c). (e) System (3) approach asymptotically to $q_3 = (0.73, 0.1, 0.05)$ for $\alpha_4 = 13$. (f) Time series of the attractor in (e). (g) System (3) approach asymptotically periodic dynamics in the $xy-$plane for $\alpha_4 = 15$. (h) Time series of the attractor in (g).
Figure 7. The trajectory of system (3) for the data (26) with different values of $\alpha_2$. (a) System (3) approach asymptotically to chaotic attractor for $\alpha_2 = 1$. (b) Time series of the attractor in (a). (c) System (3) approach asymptotically to period two attractor in $\text{Int. } \mathbb{R}^3$ for $\alpha_2 = 10$. (d) Time series of the attractor in (c). (e) System (3) approach asymptotically to periodic attractor for $\alpha_2 = 15$. (f) Time series of the attractor in (e). (g) System (3) approach asymptotically to $q_3 = (0.91,0.1,0.006)$ for $\alpha_2 = 125$. (h) Time series of the attractor in (g).

However, for the data set given by (26) with $\alpha_1 = 10$ and $\alpha_2 = 15$, it is observed that the trajectory of system (3) approaches asymptotically to PEP represented by $q_3 = (0.79,0.1,0.02)$ as shown in Figure 8.

Figure 8. The trajectory of system (3) for the data (26) with $\alpha_1 = 10$ and $\alpha_2 = 15$. (a) System (3) approaches asymptotically to $q_3 = (0.79,0.1,0.02)$. (b) Time series of the attractor in (a).

Keeping the obtained results in view, the effect of varying the parameters of system (3) on the dynamical behavior of the system in case of having asymptotically stable PEP using the data given by equation (26) with $\alpha_1 = 10$ and $\alpha_2 = 15$ is also studied numerically and obtained results are summarized in Table 2.
Table 2. The dynamical behavior of system (3) using data (26) with $\alpha_1 = 10$ and $\alpha_2 = 15$ in case of varying one parameter each time

| The parameter | The range of varying | The dynamical behavior of system (3) |
|---------------|----------------------|------------------------------------|
| $\beta_1$     | $0 < \beta_1 < 0.09$ | Periodic in the $xy$ plane         |
|               | $0.09 \leq \beta_1$ | Approaches to PEP in $\mathbb{R}^3_+$ |
| $\gamma_1$   | $0 < \gamma_1 < 0.18$ | Periodic in the $xy$ plane         |
|               | $0.18 \leq \gamma_1 \leq 1$ | Approaches to PEP in $\mathbb{R}^3_+$ |
| $\theta_1$   | $0 < \theta_1 < 0.25$ | Approaches to AEP                  |
|               | $0.25 \leq \theta_1 < 0.28$ | Approaches to TPFEP in $xy$ plane |
|               | $0.28 \leq \theta_1 < 0.6$ | Approaches to PEP in $\mathbb{R}^3_+$ |
|               | $0.6 \leq \theta_1 < 0.8$ | Periodic dynamics in $\mathbb{R}^3_+$ |
|               | $0.8 \leq \theta_1 < 1$ | Periodic in the $xy$ plane         |
| $d_1$         | $0 < d_1 < 0.13$    | Periodic in the $xy$ plane         |
|               | $0.13 \leq d_1 < 0.18$ | Periodic dynamics in $\mathbb{R}^3_+$ |
|               | $0.18 \leq d_1 < 0.37$ | Approaches to PEP in $\mathbb{R}^3_+$ |
|               | $0.37 \leq d_1 < 0.41$ | Approaches to TPFEP in $xy$ plane |
|               | $0.41 \leq d_1 < 1$ | Approaches to AEP                  |
| $\beta_2$     | $0 < \beta_2 < 1$   | Approaches to PEP in $\mathbb{R}^3_+$ |
| $\gamma_2$   | $0 < \gamma_2 < 0.19$ | Periodic dynamics in $\mathbb{R}^3_+$ |
|               | $0.19 \leq \gamma_2 < 0.23$ | Approaches to PEP in $\mathbb{R}^3_+$ |
|               | $0.23 \leq \gamma_2 < 1$ | Periodic in the $xy$ plane         |
| $\theta_2$   | $0 < \theta_2 < 0.29$ | Periodic in the $xy$ plane         |
|               | $0.29 \leq \theta_2 < 0.32$ | Approaches to PEP in $\mathbb{R}^3_+$ |
|               | $0.32 \leq \theta_2 < 1$ | Periodic dynamics in $\mathbb{R}^3_+$ |
| $d_2$         | $0 < d_2 < 0.1$     | Periodic dynamics in $\mathbb{R}^3_+$ |
|               | $0.1 \leq d_2 < 0.11$ | Approaches to PEP in $\mathbb{R}^3_+$ |
|               | $0.11 \leq d_2 < 1$ | Approaches to PEP in $\mathbb{R}^3_+$ |

6. Conclusion and discussion

In this paper, a BD food chain model incorporating fear factors in the first and second traffic levels of the chain is proposed and studied. The objective is to investigating the role of fear on the dynamical behavior of the system. The boundedness of the solution is proved. All the EPs are determined and their stability analyses are investigated locally as well as globally. The persistence conditions of the system are established. The occurrence of local bifurcation around the equilibrium points is investigated too. Finally, the numerical simulation of the system in case of nonexistence and existence of fear is carried out. It is observed that using the hypothetical set of data given by equation (26) the food chain without fear has a complex dynamics involving chaos that is very sensitive for varying of most the parameters. Furthermore, it is clear that the existence of fear has a stabilizing effect, through removing the complex dynamics of the system. Now, according the numerical simulation results using the hypothetical set of data (26) the following observations are obtained.

1. System (3) without fear has complex dynamics including chaos and periodic.
2. Increasing the fear in the first level up to a specific value removes the chaotic dynamics and the trajectory of system (3) approaches asymptotically to stable PEP. However, further increasing the fear at the first level more than a critical value makes the system losing the persistence and then the trajectory approaches asymptotically to a periodic dynamics in the $xy$ plane.
3. Increasing the fear rate in the second level up to a specific value removes the chaos too and the trajectory of system (3) approaches asymptotically to periodic attractor in \( \mathbb{R}_+^4 \). Moreover, increasing the fear rate further above a critical value stabilizes the system and the trajectory approaches asymptotically to PEP.

4. The BDSs show that the system is very sensitive to varying in the conversion rates \( \theta_1, \theta_2 \) and the top predator death rate \( d_2 \). Different points of bifurcation have been obtained. In fact, decreasing the value of the conversion rates \( \theta_1, \theta_2 \) or increasing the value of predators death rates \( d_1, d_2 \) causes extinction in top predator and the system loses their persistence.

5. Similar observation has been obtained regarding increasing the values of top predator half saturation constant \( \gamma_2 \) as that obtained in case of increasing the predators death rate.

6. In case of existence of constant values of fear rates \( \alpha_1 = 10 \) and \( \alpha_2 = 15 \) with rest of parameters as given in equation (26), it is observed that the system persists at the PEP. While decreasing the value of encounters between the intermediate predator individuals or the intermediate predator half saturation constant causes extinction in top predator and system (3) approaches asymptotically top periodic dynamics in the \( xy \) –plane.

7. Decreasing (increasing) the conversion rate of the intermediate predator \( \theta_1 \) (death rate of intermediate predator \( d_1 \)) below (above) a specific value causes extinction in top predator and the solution of system (3) approaches asymptotically to TPFEP in the \( xy \) –plane. Further decreasing (increasing) in these parameters leads to extinction in intermediate predator too and the system approaches asymptotically to AEP. On the other hand, increasing \( \theta_1 \) (decreasing \( d_1 \)) above (below) a specific value leads to extinction in top predator and the solution approaches asymptotically to TPFEP in \( xy \) –plane.

8. Increasing the half saturation constant \( \gamma_2 \) or the death rate \( d_2 \) of top predator above a specific value causes losing the persistence of the system and the solution approaches asymptotically to periodic dynamics in \( xy \) –plane. However, decreasing these rates leads to losing the stability of the PEP and the system still persist at periodic attractor in \( \mathbb{R}_+^4 \).

9. Finally, decreasing the top predator conversion rate \( \theta_2 \) below a specific value causes losing the persistence of the system and the solution approaches asymptotically to periodic dynamics in \( xy \) –plane. However, increasing this rate leads to losing the stability of the PEP and the system still persist at periodic attractor in \( \mathbb{R}_+^4 \).

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