Duality of Super D-brane Actions
in
Type II Supergravity Background

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Abstract
We show that supersymmetric and $\kappa$-symmetric Dp-brane actions in type II supergravity background have the same duality transformation properties as those in a flat Minkowskian background. Specially, it is shown that the super D-string transforms in a covariant way while the super D3-brane is self-dual under the $SL(2, Z)$ duality. Also, the D2-brane and the D4-brane transform in ways expected from the relation between type IIA superstring theory and M-theory. The present study proves that various duality symmetries, which were originally found in the flat background field, are precisely valid even in the curved background geometry.

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1 Introduction

One of the most important insights about nonperturbative behavior of superstring theory \[1\] and M-theory \[2, 3\] is the existence of the intricate network of duality symmetries. Much nonperturbative information has been gleaned by exploring the duality transformations connecting various string vacua and excitations. For instance, it was remarkably shown that the five known superstring theories and the still rather mysterious M-theory are indeed nonperturbatively equivalent by means of the web of string dualities \[4\]. This is a typical example that a chain of duality transformations reduces the degree of non-uniqueness of string theory. Another interesting aspect of the duality transformations is that it allows us to study a strongly coupled string theory by mapping that theory to a weakly coupled dual theory whenever such a dual theory exists.

Among three types of dualities appearing in superstring theory, named S-, T- and U-dualities, S-duality possesses a curious historical position. The appearance of a non-compact global symmetry group $G$ was already known in the 1970’s to be a characteristic feature of supergravity theory, which originally arose from the desire to include supersymmetry into the framework of general relativity with the hope that supersymmetry might tame the notorious ultraviolet divergences but is now considered to be the low energy effective theory of superstring theory. Such a global group is realized nonlinearly by the scalar fields parametrizing the coset space $G/H$ where $H$ is the maximal compact subgroup of $G$ \[5, 6, 7\]. However, in those days, the non-compact global symmetry was regarded as an artifact of the low energy approximation to underlying renormalizable theory and not taken seriously. It was recognized only recently that the $SL(2, Z)$ subgroup of the $SL(2, R)$ symmetry of Cremmer et al. \[5\] precisely corresponds to one example of S-duality in the toroidally compactified heterotic superstring.

Another line of development is the discovery of super D-branes carrying Ramond-Ramond charge \[8\]. These extended objects have been shown to play a fundamental role in nonperturbative aspects of string theory and quantum black holes. It is interesting to notice that historically the D-branes were first studied using T-duality symmetry on the string worldsheet \[8\], and their role in the web of string dualities is at present understood to some extent.

Despite these impressive developments, our understanding about nonperturbative regime in superstring theory is still far from complete. Perhaps, under such a situation, a sound direction of study is to achieve as thorough an understanding of symmetry and dynamics of super D-branes as possible in order to clarify nonpertubative aspects of superstring theory. Actually motivated by this reason and more, in recent years, the world-volume actions for supersymmetric and $\kappa$-symmetric D-branes have been constructed in a flat background \[9\] as well as a general type II supergravity background \[10, 11, 12\], and some of duality symmetries have been investigated.

One of the main distinctions between super Dp-branes and super p-branes is that in the former there is an abelian gauge field $A_i$ in addition to the superspace coordinates $(X^i, \theta)$. Then by carrying out a duality transformation of this abelian gauge potential, we can arrive at dual super D-brane actions. In the case of a flat background, it has been already observed
that the resulting world-volume actions after a duality transformation give rise to the expected properties \[13\]. Specifically, it was shown that the super D-string transforms in a covariant manner while the super D3-brane is self-dual under the SL(2, \(Z\)) S-duality. Also, the D2-brane and the D4-brane transform in ways expected from the duality relation between type IIA superstring theory and M-theory.

Until recently, however, we have had no idea whether or not these dualities symmetries existing in the (super-) D-brane actions in the flat background also exist in a curved background. A first step towards the proof of the SL(2, \(Z\)) S-duality of the super D-string and D3-brane actions on AdS\(_5 \times S^5\) was taken in \[14, 15\] where it was shown that this duality indeed exists in the specific background, and later one of the present authors has also verified this fact without fixing \(\kappa\)-symmetry \[16\]. The main motivation of the present study is to prove that various duality symmetries found in the flat background field, are valid even in a more general curved background geometry.

A preliminary work of a super D-string in type IIB supergravity background was reported by one of authors \[17\] where it was shown that the super D-string in this curved background is transformed to the type IIB Green-Schwarz superstring action \[18\], thereby proving the SL(2, \(Z\)) covariance of the super D-string. In this paper, as promised in \[17\], we will extend the ideas to broader situations, those are, the self-duality of the super D3-brane in type IIB supergravity background under an SL(2, \(Z\)) S-duality transformation, and the relations between D2 and M2-branes and the one between D4 and M5-branes in type IIA supergravity background which are expected from the IIA/M-duality.

This article is organized as follows. Section 2 reviews super D-brane actions in a general IIA and IIB supergravity background \[11, 12\]. In Section 3 it is then proved in a quantum-mechanically exact manner that the super D-string action in type IIB on-shell supergravity background is transformed to the type IIB Green-Schwarz superstring action with the SL(2, \(Z\)) covariant tension through an S-duality transformation. Section 4 deals with the super D2-brane in type IIA on-shell supergravity background and presents that the super D2-brane action can be transformed to the super M2-brane action with a circular eleventh dimension by a duality transformation. In Section 5 we show that the super D3-brane action in type IIB on-shell supergravity background is mapped into itself by an S-duality transformation, thereby verifying the SL(2, \(Z\)) self-duality of the action. We shall present both classical and quantum-mechanical proofs here. In Section 6 it is shown that the super D4-brane action becomes identical to the supersymmetric action which is obtained in terms of double-dimensional reduction of the super M5-brane action in the eleven dimensional space-time through a duality transformation. The final section will be devoted to discussions.

To close this section, we would like to stress that the analysis considered in this paper has two improvements over that in a paper \[13\] even if there exists an exact correspondence of the obtained results. One big improvement, of course, Aganagic et al. \[13\] have taken account of only a flat background while we have considered a general curved background. Another important improvement is that their analysis is purely classical, on the other hand, we have performed the quantum analysis at least for the super D-string and D3-brane because of their
importance.

2 Super D-brane actions in a general type II background

We start by reviewing super Dp-brane actions in a general type II supergravity background [11, 12]. It is well known nowadays that super Dp-brane actions consist of two terms, those are, the Dirac-Born-Infeld action and the Wess-Zumino action. The former includes the NS-NS two-form, dilaton and world-volume metric in addition to Abelian gauge field while the latter action contains the coupling of the D-brane to the R-R fields. The two terms are separately invariant under type II superspace reparametrizations as well as \((p + 1)\)-dimensional general coordinate transformations. However, local \(\kappa\) symmetry is achieved by a suitable conspiracy between the two terms.

Then super Dp-brane actions in a general type II on-shell supergravity background which we consider are given by

\[
S = S_{DBI} + S_{WZ},
\]

with

\[
S_{DBI} = -\int_{M^{p+1}} d^{p+1}\sigma \sqrt{-\det(G_{ij} + F_{ij})},
\]

\[
S_{WZ} = \int_{M^{p+1}} e^\mathcal{F} \wedge C = \int_{M^{p+1}} \Omega_{p+1} = \int_{M^{p+2}} I_{p+2},
\]

where \(\sigma^i (i = 0, 1, \ldots, p)\) are the world-volume coordinates, and \(G_{ij}\) is the metric of the world-volume. We have defined various quantities as follows:

\[
\mathcal{F} = F - b_2,
\]

\[
F = dA,
\]

\[
C = \bigoplus_{n=0}^9 C_{(n)},
\]

\[
I_{p+2} = d\Omega_{p+1} = d(e^\mathcal{F} \wedge C),
\]

\[
M_{p+1} = \partial M_{p+2},
\]

where \(F\) is the Maxwell field strength 2-form, and the 2-form \(b_2\) is introduced such that \(\mathcal{F}\) is invariant under supersymmetry. And the RR \(n\)-form fields \(C_{(n)}\) are collected in \(C\) with \(n\) taking odd integers for type IIA and even integers for type IIB.

In addition, in order to describe the curved target superspace geometry we have to introduce the superspace vielbein 1-form \(E^A\) defined by

\[
E^A = dZ^M E^A_M,
\]
with \( dZ^M \) denoting the superspace differential \( (dX^m, d\theta^\mu) \), and the torsion 2-form \( T^A = DE^A \)
as well as the curvature 2-form defined in terms of the spin connection \( \omega^B_A \) as
\[
R^B_A = d\omega^B_A + \omega^C_A \wedge \omega^B_C. \tag{5}
\]
Note that we have also defined as \( M = (m, \mu) \) in curved superspace while \( A = (a, \alpha) \) in flat superspace as usual. Then the world-volume metric \( G_{ij} \) is represented by
\[
G_{ij} = E^a_i E^b_j \eta_{ab}, \tag{6}
\]
where \( E^A_i = \partial_i Z^M E^A_M \) and \( \eta_{ab} = \text{diag}(-, +, \ldots, +) \).

Throughout this paper we use following conventions for superspace forms. Firstly, a general \( n \)-form superfield \( \Omega^{(n)} \) is expanded as
\[
\Omega^{(n)} = \frac{1}{n!} dZ^M_n \wedge \ldots \wedge dZ^M_1 \Omega^{M_1 \ldots M_n},
\]
\[
= \frac{1}{n!} E^A_n \wedge \ldots \wedge E^A_1 \Omega^{A_1 \ldots A_n}. \tag{7}
\]
Secondly, we define the exterior derivative as an operator acting from the right
\[
d(\Omega^{(m)} \wedge \Omega^{(n)}) = \Omega^{(m)} \wedge d\Omega^{(n)} + (-)^n d\Omega^{(m)} \wedge \Omega^{(n)}. \tag{8}
\]
Now, following the paper [11], let us define the NS-NS 3-form superfield \( H_3 \) and the R-R \( n \)-form superfield \( R \) as
\[
H_3 = db_2,
\]
\[
R = e^{b_2} \wedge d(e^{-b_2} \wedge C) = \bigoplus_{n=1}^{10} R_{(n)}. \tag{9}
\]
It is obvious that from these definitions the field strengths obey the following Bianchi identities
\[
dH_3 = 0,
\]
\[
e^{b_2} \wedge d(e^{-b_2} \wedge R) = dR - R \wedge H_3 = 0. \tag{10}
\]

In order to reduce the enormous unconstrained field content included in the superfields to the field content of the on-shell type II supergravity theory, one has to impose the constraints on the torsion and the field strengths by hand, which make various Bianchi identities to coincide with the equations of motion of supergravity. Under the assumption of vanishing (or constant) dilaton, the nontrivial constraints imposed on the torsion and field strength components [11] take the following forms for type IIA:
\[
T^c_{\alpha\beta} = 2i \gamma^c_{\alpha\beta},
\]
\[
H_{\alpha \alpha \beta} = -2i (\gamma_{11} \gamma_{\alpha})_{\alpha \beta},
\]
\[
R_{(n)a_1 \ldots a_{n-2}\alpha \beta} = 2i (\gamma_{a_1 \ldots a_{n-2}} (\gamma_{11})^2)_{\alpha \beta}, \tag{11}
\]
\[^3\text{See the ref. [12] for type IIA massive supergravity, i.e., } R_0 = m.\]
and for IIB:

\[ T_{\alpha\beta} = 2i\gamma_c^{\alpha\beta}, \]
\[ H_{a\alpha\beta} = -2i(K\gamma_a)_{\alpha\beta}, \]
\[ R_{(n)a_1...a_{n-2}\alpha\beta} = 2i(\gamma_{a_1...a_{n-2}}K^{n-1}_{\alpha\beta})E_{\alpha\beta}, \]  \( (12) \)

where \( \mathcal{E}, \mathcal{I}, \) and \( K \) describing the \( SO(2) \) matrices are defined in terms of the conventional Pauli matrices \( \sigma_i \) as follows:

\[ \mathcal{E} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{I} = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  \( (13) \)

Based on this formulation of the super Dp-brane actions in type II on-shell supergravity background, we shall explore various duality symmetries in subsequent sections. Of course, our formulation is not so general in that we have confined ourselves to the vanishing (or constant) dilaton and antisymmetric tensor fields et c. It is quite valuable to remove these restrictions and construct a more general formalism in future.

3 The super D-string

In this section we would like to consider the super D-string (i.e. the super D1-brane) first. The super D2, D3 and D4-branes will be treated in order in subsequent sections. In these sections we shall prove various duality symmetries of the super D-brane actions in type II on-shell supergravity background. The corresponding proofs have been already done in ref.\[13\] in the case of a flat Minkowskian background. Actually, we will see that the duality relations found in ref.\[13\] precisely hold even in the curved background. This fact is quite important for future development in string theory and M-theory since the global discrete symmetries such as the \( SL(2,\mathbb{Z}) \) S-duality are nowadays believed to be exact symmetries in still mysterious underlying theory \[19, 3\] so that these symmetries should be valid in arbitrary curved background geometries.

In the case at hand, the action \( (1), (2) \) and the constraints \( (12) \) reduce to

\[ S = S_{DBI} + S_{WZ}, \]
\[ S_{DBI} = -\int_{M_2} d^{2}\sigma \sqrt{-\det(G_{ij} + F_{ij})}, \]
\[ S_{WZ} = \int_{M_2=\partial M_3} C_2 = \int_{M_3} I_3, \]
\[ H_3 = db_2 = i\mathcal{E} \wedge \mathcal{\hat{E}} \wedge K\mathcal{E}, \]
\[ I_3 = dC_2 = -i\mathcal{E} \wedge \mathcal{\hat{E}} \wedge \mathcal{T}\mathcal{E}, \]  \( (14) \)

where we have used not only the superspace convention \( (7) \) but also the fact that the R-R 3-form field strength superfield \( R_{(3)} \) coincides with the Wess-Zumino form \( I_3 \). Moreover, for
a while we have neglected the axion $C(0)$ which will be considered later. In Eq.(14), $E$, $\hat{E}$ and $\bar{E}$ represent the Dirac conjugate of $E^{I\alpha}$, $E^a\gamma_a$ and $E^{I\alpha}$ with $I$ being the $N = 2$ index ($I = 1, 2$), respectively.

Now we are ready to present a quantum-mechanical exact proof of $SL(2, \mathbb{Z})$ S-duality covariance of the super D-string action in a general ten dimensional IIB supergravity background [17]. To this end, the crucial observations concern the fact that the action in Eq.(14) is of the form similar to that on $AdS_5 \times S^5$ [14]. Once this point is understood, the analysis is a fairly straightforward generalization of that presented in [14], though some points are a little more involved. Following the techniques developed in [20, 21], let us utilize the path integral of the first-order Hamiltonian form.

As the first step of the Hamiltonian formalism, let us introduce the canonical conjugate momenta $\pi^i$ corresponding to the gauge field $A_i$ defined as

$$\pi^i = \frac{\partial S_{DBI}}{\partial \dot{A}_i},$$

(15)

where we used the fact that the Wess-Zumino term is independent of the gauge potential, which holds only in the case of string theory. Then the canonical conjugate momenta $\pi^i$ are calculated to be

$$\pi^0 = 0, \quad \pi^1 = \frac{\mathcal{F}_{01}}{\sqrt{-\det(G_{ij} + \mathcal{F}_{ij})}},$$

(16)

where the former equation just shows the existence of the $U(1)$ gauge invariance. From these equations we will see that the Hamiltonian density takes the form

$$\mathcal{H} = \sqrt{1 + (\pi^1)^2} - \det G_{ij} - A_0 \partial_1 \pi^1 + \partial_1 (A_0 \pi^1) + \pi^1 b_{01} - C_{01},$$

(17)

Now the partition function is defined by the first-order Hamiltonian form with respect to only the gauge field as follows:

$$Z = \frac{1}{\mathcal{Z}} \int \mathcal{D}\pi^0 \mathcal{D}\pi^1 \mathcal{D}A_0 \mathcal{D}A_1 \exp \left[ \int d^2\sigma (\pi^1 \partial_0 A_1 - \mathcal{H}) \right],$$

(18)

where the integrations over $A_i$ explicitly, which gives rise to $\delta$ functions

$$Z = \int \mathcal{D}\pi^1 \delta(\partial_0 \pi^1) \delta(\partial_1 \pi^1) \exp \left[ \int d^2\sigma \left[ -\sqrt{1 + (\pi^1)^2} \sqrt{-\det G_{ij}} - C_{01} - \pi^1 b_{01} \right] \right].$$

(19)
The existence of the $\delta$ functions reduces the integral over $\pi^1$ to the one over only its zero-modes. If we require that one space component is compactified on a circle, these zero-modes are quantized to be integers \[22\]. As a consequence, the partition function becomes

$$Z = \sum_{m \in \mathbb{Z}} \exp i \int d^2\sigma \left[ -\sqrt{1 + m^2} \sqrt{-\det G_{ij} + C_{01} - mb_{01}} \right],$$

(20)

from which we can read off the effective action

$$S = \int d^2\sigma \left( -\sqrt{1 + m^2} \sqrt{-\det G_{ij} + C_{01} - mb_{01}} \right).$$

(21)

Moreover, recalling the relations in (14)

$$\int_{M_2 = \partial M_3} d^2\sigma (C_{01} - mb_{01}) = \int_{M_3} (I_3 - m H_3) = -i \int_{M_3} \bar{E} \wedge \hat{E} \wedge (mK + \mathcal{I})E,$$

(22)

and then carrying out an orthogonal transformation

$$U^T (mK + \mathcal{I}) U = -\sqrt{1 + m^2} \mathcal{K},$$

(23)

with an orthogonal matrix $U = \frac{1}{\sqrt{1 + (m - \sqrt{1 + m^2})^2}}[(m - \sqrt{1 + m^2})1 - \mathcal{E}]$, one finally obtains the action

$$S = -\sqrt{1 + m^2} \left( \int_{M_2} d^2\sigma \sqrt{-\det G_{ij}} - i \int_{M_3} \bar{E} \wedge \hat{E} \wedge K E \right).$$

(24)

This is nothing but type IIB Green-Schwarz superstring action with the modified tension $\sqrt{1 + m^2}$ in a type IIB supergravity background \[23\].

It is worthwhile to notice that the result obtained above agrees with the tension formula for the $SL(2, Z)$ S-duality spectrum of strings in the flat background \[24\] provided that we identify the integer value $\pi^1 = m$ with the NS-NS charge corresponding to the $(m, 1)$ string. This identification means that the D-string action is actually the action for an arbitrary number of 'fundamental' IIB strings bound to a single D-string. To show more clearly that the tension at hand is the $SL(2, Z)$ covariant tension, it would be more convenient to start with the following classical action

$$S = -n \int_{M_2} \left[ e^{-\phi} \sqrt{-\det (G_{ij} + F_{ij}) - C_2} + \frac{1}{2} \epsilon^{ij} \chi F_{ij} \right],$$

(25)

where $n$ is an integer, and we have introduced the constant dilaton $\phi$ and the constant axion $C_{(0)} \equiv \chi$. Then following the same path of derivation as above, we can obtain the manifestly $SL(2, Z)$ covariant tension

$$T = \sqrt{(m + n\chi)^2 + n^2 e^{-2\phi}}.$$

(26)

Here we would like to comment two important points. One point is that we have shown that there exists $SL(2, Z)$ S-duality in type IIB superstring theory even in a general type
IIB supergravity background without reference to any approximation. Thus this relation is quantum-mechanically exact.

The other point is the problem of whether one can interpret the orthogonal transformation (23) as the $SO(2)$ rotation of the $N = 2$ spinor coordinates. In our previous paper [14, 15] this problem was emphasized too much, but on reflection it turns out that this problem is rather trivial by the following reasons. Notice that the torsion constraint in (10) is obviously invariant under this rotation. Moreover, since we require that the original super D-string action and the fundamental Green-Schwarz action reduce to the well-known forms of the corresponding flat space actions in the flat space limit, $E^{I\alpha}$ with the $SO(2)$ index $I$ and $E^a$ must take the following forms at the lowest order expansion with respect to the spinor coordinates $\theta$,

\begin{align*}
E^I_i & = \partial_i \theta^I + \ldots , \\
E^a_i & = \partial_i X^a - i \bar{\theta} \gamma^a \partial_i \theta + \ldots , \\
\end{align*}

where the dots indicate the higher order terms reflecting the curved nature of the background metric. These facts mean that $E^I$ transforms as the adjoint representation of the $SO(2)$ group, on the other hand, $E^a$ must be invariant under an $SO(2)$ rotation. Accordingly, we can understand that the orthogonal transformation (23) is indeed performed by an $SO(2)$ rotation of the $N = 2$ spinor coordinates. In this way, we have succeeded in deriving the $SL(2, \mathbb{Z})$ S-duality of type IIB superstring theory in type IIB on-shell supergravity background at least within the present context.

4 The super D2-brane

Next we turn to the classical derivation of a duality transformation between the super D2-brane (i.e., the super D-membrane) in type IIA supergravity background and the super M2-brane in eleven dimensional supergravity. The authors in a paper [12] have already dealt with this problem from a different viewpoint. The method adopted there is start with the super M2-brane in eleven dimensions, achieve the dimensional reduction to ten dimensions a la KK ansatz, then perform a duality transformation for the purpose of getting the super D2-brane action and its $\kappa$-symmetry. Our method is similar to that of Aganagic et al. [13] where the above arguments were reversed, namely, the super M2-brane action was obtained from starting with the super D2-brane action through a duality transformation of the world-volume gauge field.

From Eqs.(1) and (3), the super D2-brane action in the string metric becomes

\begin{align*}
S & = S_{DBI} + S_{WZ} + S_{\tilde{H}}, \\
S_{DBI} & = - \int_{M_3} d^3 \sigma \sqrt{- \det (G_{ij} + F_{ij})}, \\
S_{WZ} & = \int_{M_3 = \partial M_4} (C_3 + C_1 \wedge F) = \int_{M_4} I_4, \\
\end{align*}

8
\[ S_{\tilde{H}} = \int_{M_3} d^3\sigma \frac{1}{2} \tilde{H}^{ij}(F_{ij} - 2\partial_i A_j), \]  

(28)

where we have added \( S_{\tilde{H}} \) to the original action to perform a duality transformation. Moreover, in this case the constraints (11) on the field strengths reduce to

\[
\begin{align*}
H_3 &= db_2 = i\bar{E} \wedge \gamma_{11} E, \\
R_{(4)} &= \frac{i}{2} \bar{E} \wedge \gamma_{ab} E \wedge E^b \wedge E^a, \\
R_{(2)} &= i\bar{E} \wedge \gamma_{11} E.
\end{align*}
\]  

(29)

From these equations and the definitions (3) and (9), we find that \( C_3 \) and \( C_1 \) are determined by the conditions

\[
\begin{align*}
R_{(4)} &= dC_3 + db_2 \wedge C_1 = \frac{i}{2} \bar{E} \wedge \gamma_{ab} E \wedge E^b \wedge E^a, \\
R_{(2)} &= dC_1 = i\bar{E} \wedge \gamma_{11} E.
\end{align*}
\]  

(30)

At this stage, we take the variation with respect to \( A_i \), which gives us the solution \( \tilde{H}^{ij} = \epsilon^{ijk} \partial_k B \) with \( B \) being a scalar superfield. Then after substituting this solution into the action and solving the equation of motion for \( F_{ij} \) in order to rewrite the action in terms of \( B \) instead of \( F_{ij} \), we arrive at the dual action \( S_D \) of (28)

\[ S_D = -\int_{M_3} d^3\sigma \sqrt{-\det G''_{ij}} + \int_{M_3} (C_3 + b_2 \wedge dB), \]  

(31)

where we have defined as

\[ G''_{ij} = G_{ij} + (\partial_i B + C_i)(\partial_j B + C_j). \]  

(32)

Incidentally, in order to derive the dual action we have used the mathematical formulas holding for \( 3 \times 3 \) matrices

\[
\begin{align*}
\det(G_{ij} + A_i A_j) &= (\det G_{ij}) \times (1 + G''_{ij} A_i A_j), \\
\det(G_{ij} + F_{ij}) &= (\det G_{ij}) \times (1 + \frac{1}{2} G''_{ij} G''_{kl} F_{ik} F_{jl}),
\end{align*}
\]  

(33)

where \( F_{ij} = -F_{ji} \).

Eq. (32) suggests the identification \( E^{11} = C_1 + dB \), in other words, identifying the world-volume scalar with the coordinate of a compact extra target-space dimension. Consequently, the Dirac-Born-Infeld action in Eq. (31) takes the standard form for the induced metric of the M2-brane. The remaining work is to show that the second term in the right hand side of Eq. (31) equals to the expression for the Wess-Zumino term of the super M2-brane. Indeed, taking the exterior derivative and using the relation Eq. (30) we can arrive at the following equation:

\[
\begin{align*}
d(C_3 + b_2 \wedge dB) &= \frac{i}{2} \bar{E} \wedge \gamma_{ab} E \wedge E^b \wedge E^a + i\bar{E} \wedge \gamma_{11} \gamma_a E \wedge E^a \wedge E^{11} \\
&= \frac{i}{2} \bar{E} \wedge \gamma_{ab} E \wedge E^b \wedge E^a \\
&\equiv d\Omega^{11},
\end{align*}
\]  

(34)
where $\hat{a} \equiv (a, 11)$ denotes 11 dimensional index. As implied in the above last equation, the left hand side in Eq. (34) exactly coincides with the Wess-Zumino term in the super M2-brane action [24] except antisymmetric field which we have neglected from the beginning in this paper. Accordingly, the dual action (31) of the super D2-brane can be written to

$$S_D = -\int_{M_3} d^3\sigma \sqrt{-\det G_{ij}^{11}} + \int_{M_3} \Omega^{11},$$

(35)

where $G_{ij}^{11} = E_i^\hat{a} E_j^\hat{b} \eta_{\hat{a}\hat{b}}$ and $d\Omega^{11} = \frac{i}{2} \tilde{E} \wedge \gamma_{\hat{a}\hat{b}} E \wedge E^\hat{b} \wedge E^\hat{a}$. Thus, we have proved that the super D2-brane action in type IIA supergravity background is transformed to the super M2-brane action with a circular compactified 11th dimension in eleven dimensional supergravity background through a duality transformation of the world-volume gauge field as expected from IIA/M-duality.

It is straightforward to check that we can also get the relation between the string metric in ten dimensions and the eleven dimensional metric [3] by introducing a constant dilaton in the original action in an appropriate way. To this end, let us begin by the following action with the dependence of the constant dilaton background:

$$S' = -\int d^3\sigma e^{-\phi} \sqrt{-\det(G_{ij} + \mathcal{F}_{ij})} + \int e^{-\phi}(C_3 + C_1 \wedge \mathcal{F}).$$

(36)

The same procedure as before leads to the dual action

$$S'_D = -\int d^3\sigma e^{-\phi} \sqrt{-\det G'_{ij}} + \int (e^{-\phi}C_3 + b_2 \wedge dB),$$

(37)

where

$$G'_{ij} = G_{ij} + (e^{\phi}\partial_i B + C_i)(e^{\phi}\partial_j B + C_j),$$

(38)

which exactly reduces to (31) and (32) in the absence of the dilaton background. Then we can rewrite $S'_D$ in the previous form $S_D$ of the standard M2-brane action with the obvious rescalings

$$E^\hat{a} \rightarrow e^{\frac{3}{2}\phi} E^\hat{a}, \ E^\alpha \rightarrow e^{\frac{1}{2}\phi} E^\alpha.$$  

(39)

Comparing $G'_{ij}$ and $G_{ij}^{11}$, one finds the relation

$$G_{ij}^{11} = e^{-\frac{3}{2}\phi} G'_{ij} = e^{-\frac{3}{2}\phi} G_{ij} + e^{\frac{1}{2}\phi}(\partial_i B + e^{-\phi} C_i)(\partial_j B + e^{-\phi} C_j).$$

(40)

This equation correctly reproduces the relationship between the string metric in ten dimensions and the eleven dimensional metric, in particular, the well-known relation, $R_{11} = e^{\frac{3}{2}\phi} [3]$ read from the coefficient in front of $(\partial B)^2$ where $R_{11}$ is the radius of the compactified 11th dimension on a circle.
5 The super D3-brane

In this section let us show the self-duality of the super D3-brane action in the general type IIB supergravity background in two ways. The first is the semi-classical way and the second is the exact one without resort to any semi-classical approximation.

From Eqs. (1) and (2), the super D3-brane action in the Einstein metric becomes

\[ S = S_{DBI} + S_{WZ}, \]

\[ S_{DBI} = - \int_{M_4} d^4 \sigma \sqrt{- \det(G_{ij} + e^{-\phi/2} F_{ij} - b_{2ij})}, \]

\[ S_{WZ} = \int_{M_5=\partial M_4} (C_4 + C_2 \wedge (e^{-\phi/2} F - b_2) + \frac{1}{2} C_0 F \wedge F) \]

\[ = \int_{M_5} I_5, \]  

(41)

where we have explicitly written down the dependence of the dilaton field. And the constraints (12) on the field strengths are given by

\[ H_{(3)} = db_2 = i \hat{E} \wedge \hat{E} \wedge K E, \]

\[ R_{(5)} = \frac{i}{6} \hat{E} \wedge \gamma_{abc} E E \wedge E^c \wedge E^b \wedge E^a, \]

\[ R_{(3)} = -i \hat{E} \wedge \hat{E} \wedge I E. \]  

(42)

From these equations and the definitions (3) and (9), \( C_4 \) and \( C_2 \) are determined by the conditions

\[ R_{(5)} = dC_4 - db_2 \wedge C_2 = i \frac{1}{6} \hat{E} \wedge \gamma_{abc} E E \wedge E^c \wedge E^b \wedge E^a, \]

\[ R_{(3)} = dC_2 = -i \hat{E} \wedge \hat{E} \wedge I E. \]  

(43)

5.1 The semi-classical self-duality

In this subsection we show that the super D3-brane action in the general type IIB supergravity background is semiclassically self-dual. We first consider the case of vanishing dilaton and axion. Adding a Lagrangian multiplier term

\[ S_{\tilde{H}} = \int_{M_4} d^4 \sigma \frac{1}{2} \tilde{H}^{ij}(F_{ij} - 2 \partial_i A_j), \]  

(44)

to the above action (41), the equation of motion for \( A_i \) can be solved by \( \tilde{H}^{ij} = \epsilon^{ijkl} \partial_k B_l \) with a dual vector potential \( B_i \). Then after substituting this solution into the action and solving the equation of motion for \( F_{ij} \), we arrive at the dual action \( S_D \) of (41)

\[ S_D = - \int_{M_4} \sqrt{- \det(G_{ij} + \tilde{F}_{ij} + C_{2ij})} + \int_{M_4} \Omega_D, \]

\[ \Omega_D = C_4 - b_2 \wedge C_2 + b_2 \wedge (\tilde{F} + C_2), \]  

(45)
where $\tilde{F} = dB$.

Next let us perform the following $SO(2)$ rotation of the spinor coordinate $\theta$

$$\theta' = \frac{1}{\sqrt{2}}(1 + \mathcal{E})\theta, \quad \bar{\theta}' = \bar{\theta} \frac{1}{\sqrt{2}}(1 - \mathcal{E}),$$

then the spinor components of the vielbeins rotate in the same way as explained at the end of Section 3. Under these $SO(2)$ rotations of vielbeins it is shown that from (42) and (43) the $b_2, C_2$ and $C_4$ transform as follows;

$$b_2' = -C_2, \quad C_2' = b_2, \quad C_4' = C_4 - b_2 \wedge C_2.$$  \hspace{1cm} (47)

Then the dual action can be written in terms of transformed fields as

$$S_D = -\int_{M_4} \sqrt{-\det(G_{ij} + \tilde{F}_{ij} - b_2'_{ij})} + \int_{M_4} (C_4' + C_2' \wedge (\tilde{F} - b_2')).$$  \hspace{1cm} (48)

The resulting action is completely the same form as the original action (41) from which we have started. Thus we have established the semi-classical self-duality of the super D3-brane action in the generic type IIB supergravity background.

It is a straightforward task to introduce the dilaton and axion fields and establish the semi-classical $SL(2, R)$ self-duality of the action. In this case, the $SO(2)$ transformation rules of the spinor coordinates $\theta$, 2- and 4-form potentials $b_2, C_2$ and $C_4$, and the dilaton and axion fields $\tau = C_0 + ie^{-\phi}$ are given by

$$\theta' = \frac{1}{\sqrt{2(1 + e^{2\phi}C_0^2 - e^{\phi}C_0)}} \left[\sqrt{1 + e^{2\phi}C_0^2 - e^{\phi}C_0 + \mathcal{E}} \right] \theta,$$

$$b_2' = \frac{1}{\sqrt{1 + e^{2\phi}C_0^2}}(-C_2 - e^{\phi}C_0b_2),$$

$$C_2' = \frac{1}{\sqrt{1 + e^{2\phi}C_0^2}}(-e^{\phi}C_0C_2 + b_2),$$

$$C_4' = C_4 - \frac{1}{1 + e^{2\phi}C_0^2}C_2 \wedge b_2 + \frac{e^{\phi}C_0}{2(1 + e^{2\phi}C_0^2)}(C_2 \wedge C_2 - b_2 \wedge b_2),$$

and

$$\tau' = -\frac{1}{\tau}.$$  \hspace{1cm} (51)

Combining this transformation with the symmetry under a constant shift of $C_0$ at the classical level, one deduce the $SL(2, R)$ self-duality of the super D3-brane action.
5.2 The exact self-duality

In this subsection we show that the super D3-brane action in the type IIB supergravity background satisfies the Gaillard and Zumino (GZ) self-duality condition, thereby establishing its exact self-duality without resort to any semiclassical approximation.

First let us review the GZ duality condition and its some properties. Given a generic Lagrangian density \( L(F_{\mu \nu}, g_{\mu \nu}, \phi^A) = \sqrt{-g}L(F_{\mu \nu}, g_{\mu \nu}, \phi^A) \) in four dimensional spacetime which contains a U(1) gauge field strength \( F_{\mu \nu} \), gravitational field \( g_{\mu \nu} \) and generic matter fields \( \Phi^A \), the constructive relation is given by

\[
\tilde{K}_{\mu \nu} \equiv \frac{\partial L}{\partial F_{\mu \nu}}, \quad \frac{\partial F_{\alpha \beta}}{\partial F_{\mu \nu}} \equiv \delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta,
\]

where the Hodge dual components for the anti-symmetric tensor \( K_{\mu \nu} \) are defined by

\[
\tilde{K}_{\mu \nu} \equiv \frac{1}{2} \eta^{\rho \sigma}_{\mu \nu} K_{\rho \sigma}, \quad \tilde{K}_{\mu \nu} = -K_{\mu \nu},
\]

where \( \eta^{\mu \nu \rho \sigma} = \sqrt{-g} \epsilon^{\mu \nu \rho \sigma}, \epsilon^{0123} = 1 \) and the signature of \( g_{\mu \nu} \) is \((-+-+,-+,+-,+,-)\).

If one defines the infinitesimal SO(2) duality transformation by

\[
\delta F_{\mu \nu} = \lambda K_{\mu \nu}, \quad \delta K_{\mu \nu} = -\lambda F_{\mu \nu}, \\
\delta \Phi^A = \xi^A(\Phi),
\]

then the consistency of the constructive relation and the invariance of the field equations under this SO(2) duality transformation require the following condition:

\[
\frac{\lambda}{4}(F_{\mu \nu} \tilde{F}^{\mu \nu} + K_{\mu \nu} \tilde{K}^{\mu \nu}) + \delta \Phi L = 0,
\]

and the invariance of the energy-momentum tensor requires \( \delta g_{\mu \nu} = 0 \). We call the condition the GZ self-duality condition \([26, 27]\).

It has been known that the SO(2) duality is lifted to the SL(2, R) duality by introducing a dilaton \( \phi \) and an axion \( \chi \) \([28]\). Moreover in \([29]\) it has also been shown explicitly that the GZ condition \((55)\) is actually the necessary and sufficient condition in order that one can define the off-shell (non-local) duality transformation for the U(1) gauge potential itself under which the action is invariant. Therefore if one can show for an action to satisfy the condition under the transformation \((57)\) with suitable transformation rule for matter fields, then one establishes the exact self-duality of the theory described by this action without resort to any semiclassical approximation. In \([30, 16]\) it has been shown that the super D3-brane actions on the flat and \( AdS_5 \times S^5 \) background indeed satisfy the GZ self-duality condition.

Now let us show that the super D3-brane action \((41)\) in the general type IIB supergravity background satisfies the GZ duality condition under the following SO(2) duality transformation:

\[
\delta F_{ij} = \lambda K_{ij}, \quad \delta K_{ij} = -\lambda F_{ij}, \\
\delta \theta = -\frac{\lambda}{2} \xi \theta, \quad \delta \bar{\theta} = \frac{\lambda}{2} \bar{\xi}, \quad \delta X = 0.
\]

13
The $N = 2$ spinor coordinates transform as an $SO(2)$ doublet. As we already noted that the spinor components of the vielbeins transform as the same way as $\theta$ under the $SO(2)$ transformation;

$$\delta E = -\frac{\lambda}{2} \xi E, \quad \delta \bar{E} = \frac{\lambda}{2} \bar{\xi} \bar{E}.$$  \hspace{1cm} (57)

Therefore using Eqs. (12) and (13) we obtain the $SO(2)$ transformation rule of the 2- and 4-form potentials as follows;

$$\delta b_2 = \lambda C_2, \quad \delta C_2 = -\lambda b_2,$$

$$\delta C_4 = \frac{\lambda}{2} (C_2 \wedge C_2 - b_2 \wedge b_2).$$  \hspace{1cm} (58)

Now let us first prove the self-duality of the super D3-brane action (11) with vanishing dilaton and axion fields. First let us calculate $\frac{\lambda}{4} (F_{ij} \tilde{F}^{ij} + K_{ij} \tilde{K}^{ij})$. From the constructive relation (52) and the action (11) with vanishing $\phi$ and $C_0$ we obtain

$$\tilde{K}^{ij} = \frac{\partial L}{\partial F_{ij}} = \sqrt{-\det G_{ij}} \left( -\tilde{F}^{ij} + T \tilde{F}^{ij} + \tilde{C}_2^{ij} \right),$$  \hspace{1cm} (59)

where we have used the determinant formula for the four-by-four matrix:

$$\det(G_{ij} + F_{ij}) = \det G_{ij} (1 + \frac{1}{2} F_{ij} F^{ij} - T^2), \quad T \equiv \frac{1}{4} F_{ij} \tilde{F}^{ij}.$$  \hspace{1cm} (60)

Taking the Hodge dual of (59), we find

$$K_{ij} = -\frac{1}{2} \eta_{ijkl} \tilde{K}^{kl} = \frac{\sqrt{-\det G_{ij}}}{\sqrt{-\det(G_{ij} + F_{ij})}} (\tilde{F}^{ij} + T F_{ij}) + C_{2ij}.$$  \hspace{1cm} (61)

Then we obtain

$$\frac{\lambda}{4} (F_{ij} \tilde{F}^{ij} + K_{ij} \tilde{K}^{ij}) = \frac{\lambda}{4} (2b_{2ij} \tilde{F}^{ij} + 2C_{2ij} \tilde{K}^{ij} - C_{2ij} \tilde{C}_2^{ij} - b_{2ij} \tilde{b}_2^{ij})$$

$$= \frac{\lambda}{4} (4b_2 \wedge F + 4C_2 \wedge K - 2C_2 \wedge C_2 - 2b_2 \wedge b_2).$$  \hspace{1cm} (62)

Next let us calculate $\delta_\theta L$. In the language of differential forms,

$$\delta_\theta L = \frac{\partial L}{\partial F} \wedge \delta(-b_2) + \delta F \wedge \delta C_2 + \delta C_4$$

$$= \lambda [-K \wedge C_2 - (F - b_2) \wedge b_2 + \frac{1}{2} (C_2 \wedge C_2 - b_2 \wedge b_2)]$$

$$= \lambda [-K \wedge C_2 - F \wedge b_2 + \frac{1}{2} (C_2 \wedge C_2 + b_2 \wedge b_2)].$$  \hspace{1cm} (63)
It is clearly seen that the right-hand sides of (62) and (63) exactly cancel with each other and the GZ-duality condition is indeed satisfied. As we have proved the invariance of the action under the infinitesimal $SO(2)$ duality transformation, the action is also invariant under the finite $SO(2)$ duality transformation.

Now let us discuss the case with constant non-vanishing dilaton and axion background. The action is given by (41). In this case the $SO(2)$ self-duality is lifted to the $SL(2, R)$ self-duality [28]. Let us write the Lagrangian (41) as

$$
\hat{L}(G, F, \theta, \phi, C_0) = L(G, e^{-\phi/2}F, \theta) + \frac{1}{4}C_0F\tilde{F},
$$

(64)

where $L(G, F, \theta)$ is the Lagrangian density without dilaton and axion which satisfies the $SO(2)$ self-duality. Then if one define $\hat{F} = e^{-\phi/2}F$ and $\hat{K}$ by taking the dual of $\frac{\partial L(G, \hat{F}, \theta)}{\partial \hat{F}}$, the background dependence is absorbed in the rescaled variables $(\hat{K}, \hat{F})$. These are related with the background dependent $(K, F)$ by

$$
\begin{pmatrix}
K \\
F
\end{pmatrix}
= V \begin{pmatrix}
\hat{K} \\
\hat{F}
\end{pmatrix},
$$

$$
V = e^{\frac{\phi}{2}} \begin{pmatrix}
e^{-\phi} & C_0 \\
0 & 1
\end{pmatrix}.
$$

(65)

Here $V$ is a non-linear realization of $SL(2, R)/SO(2)$ transforming as

$$
V \rightarrow V' = \Lambda VO(\Lambda)^{-1}.
$$

(66)

Here $\Lambda$ is a global $SL(2, R)$ matrix

$$
\Lambda = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix},
$$

(67)

where $a, b, c, d$ are real numbers satisfying $ad - bc = 1$, and $O(\Lambda)$ is an $SO(2)$ transformation

$$
O(\Lambda)^{-1} = \begin{pmatrix}
\cos \lambda & \sin \lambda \\
-\sin \lambda & \cos \lambda
\end{pmatrix}.
$$

(68)

The condition that the form of $V$ (66) is unchanged under the transformation (67) determines the $SO(2)$ rotation angle $\lambda$ and the transformation rule of the background fields $\phi$ and $C_0$;

$$
\tan \lambda = \frac{ce^{-\phi}}{cC_0 + d},
$$

(69)

and

$$
\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d},
$$

(70)
where $\tau \equiv C_0 + ie^{-\phi}$.

These results show that if the original Lagrangian $L(G, F, \theta)$ is invariant under the $SO(2)$ duality transformation the extended Lagrangian $\hat{L}(G, F, \theta, \phi, C_0)$ with a dilaton and an axion fields is invariant under the $SL(2, R)$ duality transformation of $(K, F)$ and $\tau \equiv C_0 + ie^{-\phi}$ and $SO(2)$ rotation of $N = 2$ spinor with rotation angle $\lambda$ given by (69).

6 The super D4-brane

In this section let us start with the super D4-brane action and perform a duality transformation of the world-volume gauge field to reach the action obtained by the double-dimensional reduction of the super M5-brane [31, 32]. The method we consider is similar to that adopted in Section 4, so we shall follow a similar path of arguments as in the super D2-brane. Like the super D2-brane, the analysis in this section is purely classical.

This time, the super D4-brane action with a Lagrange multiplier term in the string metric becomes

$$S = S_{DBI} + S_{WZ} + S_{\tilde{H}},$$

$$S_{DBI} = -\int_{M_5} d^5\sigma \sqrt{-\det(G_{ij} + F_{ij})},$$

$$S_{WZ} = \int_{M_5=\partial M_6} (C_5 + C_3 \wedge F + \frac{1}{2} C_1 \wedge F \wedge F) = \int_{M_6} I_6,$$

$$S_{\tilde{H}} = \int_{M_5} d^5\sigma \frac{1}{2} \tilde{H}^{ij}(F_{ij} - 2\partial_i A_j).$$

(71)

And the constraints (11) on the field strengths are given by

$$H_3 = dB_2 = i\tilde{E} \wedge \gamma_{11}\tilde{E} \wedge E,$$

$$R_{(6)} = \frac{i}{24} \tilde{E} \wedge \gamma_{abcd}\gamma_{11}E \wedge E^d \wedge E^c \wedge E^b \wedge E^a,$$

$$R_{(4)} = \frac{i}{2} \tilde{E} \wedge \gamma_{ab}E \wedge E^b \wedge E^a,$$

$$R_{(2)} = i\tilde{E} \wedge \gamma_{11}E.$$  

(72)

In this case, $C_5$, $C_3$ and $C_1$ are determined by the conditions

$$R_{(6)} = dC_5 + dB_2 \wedge C_3 = \frac{i}{24} \tilde{E} \wedge \gamma_{abcd}\gamma_{11}E \wedge E^d \wedge E^c \wedge E^b \wedge E^a,$$

$$R_{(4)} = dC_3 + dB_2 \wedge C_1 = \frac{i}{2} \tilde{E} \wedge \gamma_{ab}E \wedge E^b \wedge E^a,$$

$$R_{(2)} = dC_1 = i\tilde{E} \wedge \gamma_{11}E.$$  

(73)

As in the case of the super D2-brane, we take the variation with respect to $A_i$, which gives rise to the solution $\tilde{H}^{ij} = \frac{1}{6} e^{ijklm} H_{klm}$ with $H = dB$ with $B$ being a second-rank tensor.
superfield \( \Phi \). After substituting this solution into the action, we obtain the action \( S = S_1 + S_2 \) where \( S_1 \) and \( S_2 \) are defined as

\[
S_1 = - \int_{M_5} d^5 \sigma \sqrt{-\det(G_{ij} + F_{ij})} + \int_{M_5} (\mathcal{H} \wedge \mathcal{F} + \frac{1}{2} C_1 \wedge \mathcal{F} \wedge \mathcal{F}),
\]

\[
S_2 = \int_{M_5} (C_5 + H \wedge b_2),
\]

with \( \mathcal{H} = H + C_3 \). The duality transformation amounts to solving the equation of motion for \( F_{ij} \) in order to rewrite the action in terms of \( B \) (or its field strength \( H \)) instead of \( F_{ij} \). Since \( S_2 \) does not contain \( F \), this part of the action is invariant under the duality transformation. Therefore, one has only to concentrate on \( S_1 \). Following the formula in ref.[13], it is straightforward to derive the dual action \( S_D = S_{D1} + S_2 \) where \( S_{D1} \) is given by

\[
S_{D1} = - \int_{M_5} d^5 \sigma \left[ \sqrt{-G} \sqrt{1 + z_1 + \frac{z_2^2}{2} - z_2} - \frac{1}{8(1 + C_1^2)} \epsilon_{ijklm} C^i \tilde{\mathcal{H}}^j k \tilde{\mathcal{H}}^l \right],
\]

where

\[
\begin{align*}
    z_1 &= \frac{1}{2(-G)(1 + C_1^2)} \text{tr}(\tilde{\mathcal{G}} \tilde{\mathcal{H}} \tilde{\mathcal{G}} \tilde{\mathcal{H}}), \\
    z_2 &= \frac{1}{4(-G)^2(1 + C_1^2)^2} \text{tr}(\tilde{\mathcal{G}} \tilde{\mathcal{H}} \tilde{\mathcal{G}} \tilde{\mathcal{H}} \tilde{\mathcal{G}} \tilde{\mathcal{H}}), \\
    G &= \det G_{ij}, \\
    \tilde{G}_{ij} &= G_{ij} + C_i C_j, \\
    \tilde{\mathcal{H}}^{ij} &= \frac{1}{6} \epsilon^{ijklm} H_{klm}.
\end{align*}
\]

Now let us consider the Wess-Zumino action \( S_2 \). The conditions (73) yield the equation

\[
d(C_5 + H \wedge b_2) = \frac{i}{24} \tilde{E} \wedge \gamma_{abcd} \gamma_{11} E^d \wedge E^c \wedge E^b \wedge E^a - i \tilde{E} \wedge \gamma_{11} \hat{E} \wedge \mathcal{H}.
\]

As a result, we have the dual action of the super D4-brane in type IIA supergravity background

\[
S_D = - \int_{M_5} d^5 \sigma \left[ \sqrt{-G} \sqrt{1 + z_1 + \frac{z_2^2}{2} - z_2} - \frac{1}{8(1 + C_1^2)} \epsilon_{ijklm} C^i \tilde{\mathcal{H}}^j k \tilde{\mathcal{H}}^l \right] + \int_{M_5} \Omega_D,
\]

where \( d\Omega_D = \frac{i}{24} \tilde{E} \wedge \gamma_{abcd} \gamma_{11} E^d \wedge E^c \wedge E^b \wedge E^a - i \tilde{E} \wedge \gamma_{11} \hat{E} \wedge \mathcal{H} \). This dual action of the super D4-brane is identical to the action which is obtained by the double-dimensional reduction of the super M5-brane [31]. (In the above, we have neglected the dilaton field, but as in the other super D-branes it is easy to include a constant dilaton background \( p = 1, 2, 3 \) in the present formulation, from which a more manifest correspondence of the double dimensional reduction would be obtained.) Hence we have shown that the double-dimensional reduction of the super M5-brane action coincide with the dual super D4-brane action in type IIA supergravity background as suggested by the duality between M-theory and IIA superstring theory.

\[
\text{We apologize for using often the same alphabet } H \text{ to express different quantities. Here } H \text{ just means the field strength of the newly introduced tensor field } B.
\]
7 Discussions

In this paper, we have studied the properties of the duality transformation of super Dp-brane actions \((p = 1, 2, 3, 4)\) in type II on-shell supergravity background. In each case, the obtained results agreed with the corresponding results in a flat background. Thus we have succeeded in showing that various duality symmetries in the super D-brane actions are independent of the background geometry.

In the last section in a paper [13], it is stated that "...... For the most part, our analysis has been classical and limited to flat backgrounds. The results should not depend on these restrictions, however." In this paper, we have removed such restrictions completely for the super D1-brane and D3-brane. On the other hand, for the super D2-brane and D4-brane we have removed the restriction of 'flat background', but we have presented only the classical analysis. This restriction should be also removed in future. Concerning this problem, there may be a different opinion. Namely, since the Dp-brane actions with \(p > 1\) are in essence unrenormalizable, these actions might describe the low energy effective theory of underlying renormalizable theory so that the quantum-mechanical analysis is too much demanding. This problem still deserves further investigation.

The present study may also shed some light on symmetries of the underlying fundamental theory where it is widely believed that the \(SL(2,\mathbb{Z})\) duality found in the super D-string survives as an exact symmetry of the underlying theory [19, 3]. So far, symmetries have given us a useful guiding principle for establishing a theory in theoretical physics. Maybe, one of the most challenging studies in future would be to promote this global discrete symmetry to the local gauge symmetry, from which we could draw some very powerful general conclusions about the relation between the strong coupling phase and the weak coupling phase and compactified dimensions as well as the implications for physical four-dimensional spacetime.

Moreover, in the case of type IIB background we have spelled out the problem of the \(SO(2)\) rotation of the \(N = 2\) spinor coordinates. Our proof utilizes only an invariance of the constraints and the boundary condition in the flat background limit so that it can be applied to the other situations in a straightforward way.

Even if we have limited ourselves to the case of the constant (or vanishing) dilaton and the vanishing antisymmetric fields, it may be possible to generalize the analysis adopted in this paper to a more general situation. But we should be reminded that even in the case of a flat background the analysis of dualities is restricted to be only the constant dilaton field and the vanishing antisymmetric tensor fields [7]. Perhaps, particularly in the case of type IIB branes, provided that we would like to consider the non-constant dilaton, we may have to deal with the non-constant axion as well on an equal footing since they together magically parametrize the coset space \(SL(2,\mathbb{R})/SO(2)\).

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