Fractional Multidimensional System

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1 Abstract

The multidimensional (n-D) systems described by Roesser model are presented in this paper. These n-D systems consist of discrete systems and continuous fractional order systems with fractional order $\nu$, $0 < \nu < 1$. The stability and Robust stability of such n-D systems are investigated.

Keywords: n-D; fractional; stability; Robust

2 Introduction

The multidimensional (n-D) systems have been studied for almost four decades [1-5]. It has been applied in fields such as image process [1], n-D coding and decoding [6] and n-D filtering [7]. The n-D systems can represent dynamic processes that information propagates in many independent directions. However, the information of one dimensional systems only propagates in one direction.

As for a multidimensional system which consists of fractional order differential equations, Galkowski et al. first presented such a system in 2005 [8]. But until now, researches on fractional n-D systems are either discrete system [8] or continuous system with different fractional order [8,9]. To the best of our knowledge, fractional n-D systems which consist of discrete system and fractional order system are not studied.

This paper focus on a hybrid n-D system which consists of discrete system and continuous fractional order system.

Notation 1. For a matrix $X$, $X^*$, $X^T$ denote the transpose conjugate and transpose of matrix $X$, respectively. $\text{Sym}(X)$ denotes $X + X^*$. $I$ is the identity matrix with appropriate dimensions. For a matrix $X$, $X > 0$ ($X \geq 0$) means positive definite (semi-definite) and $X < 0$ ($X \leq 0$) means negative definite (semi-definite). The notation $\mathcal{H}_n$ stands for the set of Hermitian matrices of dimension $n$. And $\mathcal{H}_n^+ \subset \mathcal{H}_n$ stands for the subset of positive definite matrices while $\mathcal{H}_n^- \subset \mathcal{H}_n$ is the subset of negative definite matrices. Let the following notations be defined

\[ \bigoplus_{i=1}^{k} M_i = \text{diag} M_i \]

Let $\mathbb{I}(k)$ be

\[ \mathbb{I}(k) \triangleq 1, ..., k, \ k \in \mathbb{N} \]

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3 Preliminaries

Based on Bochniak’s model [10], Bachelier [4] presented a hybrid version of Roesser model [1], which combined integer order continuous system and discrete system. Here, we apply the Roesser model to a continuous-discrete fractional order system.

\[
\begin{bmatrix}
D^\nu x^1(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k) \\
\vdots \\
x^{r+1}(t_1, \ldots, t_r, j_{r+1}+1, \ldots, j_k) \\
x^k(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k+1)
\end{bmatrix}
= \begin{bmatrix}
A_c & A_{cd} \\
A_{dc} & A_d
\end{bmatrix}
\begin{bmatrix}
x^1(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k) \\
\vdots \\
x^{r+1}(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k) \\
x^k(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k)
\end{bmatrix}
+ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k)
\]

\[
y(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k)
+ Du(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k)
\]

where \(\sum_{i=1}^{k} n_i = n\) and \(0 < \nu \leq 1\).

The vectors \(x^i(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k), u(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k)\) and \(y(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k)\) are the local state subvectors, the input vector and the output vector, respectively. The matrices

\[
A = \begin{bmatrix} A_c & A_{cd} \\ A_{dc} & A_d \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}^{n \times p},
\]

\[
C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \in \mathbb{R}^{l \times n}, \quad D \in \mathbb{R}^{l \times p}
\]

are the state, control, observation and transfer matrices respectively.

By applying the Laplace transform and the Z-transform of system (1), the following can be obtained

\[
Y(\rho) = T(\rho)U(\rho)
\]

where

\[
T(\rho) = C(H(\rho) - A)^{-1} B + D, \quad H(\rho) \triangleq \bigoplus_{i=1}^{k} \rho_i I_{n_i}
\]

Then, \(\Delta(\rho, A)\) is defined by

\[
\Delta(\rho, A) \triangleq \det(H(\rho) - A)
\]

In this paper, we use the Caputo’s fractional derivative, of which the Laplace transform allows utilization of initial values. The Caputo’s fractional derivative is defined as [11]
\[ aD^\nu f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t f^{(n)}(\tau) d\tau \]

\[ (t - \tau)^{\alpha+1-n} \]

where \( n \) is an integer satisfying \( 0 \leq n - 1 < \alpha < n \); \( \Gamma(\cdot) \) is the Gamma function which is defined as

\[ \Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \]

The following lemmas are useful for presenting our results.

**Lemma 2.** [12] For given matrices \( U \in \mathbb{C}^{n \times m}, \Phi = \Phi^* \in \mathbb{C}^{n \times n} \), the following two statements are equivalent

1. \( U^* \Phi U < 0 \)
2. There exists a matrix \( X \), such that \( \Phi + N_u X N_u^* < 0 \)

where \( N_u \) is the orthogonal complement of \( U \).

**Lemma 3.** [12] For given matrices \( U \in \mathbb{C}^{n \times m}, V \in \mathbb{C}^{k \times n}, \Phi = \Phi^* \in \mathbb{C}^{n \times n} \), then the following two statements are equivalent

1. There exists matrix \( X \in \mathbb{C}^{m \times k} \) such that \( \text{Sym}\{UXV\} + \Phi < 0 \) holds
2. \( N_u \Phi N_u^* < 0 \) and \( N_v^* \Phi N_v < 0 \) hold

where \( N_u, N_v \) are the orthogonal complement of \( U, V^* \), respectively.

**Lemma 4.** [13] Let \( F(s) \) be the Laplace transform of the function \( f(t) \), then for any \( \nu > 0 \),

\[ \lim_{t \to \infty} D^\nu f(t) = \lim_{s \to 0} s^{\nu+1}F(s) \quad \Re(s) > 0 \]

**Lemma 5.** [3] A multidimensional discrete system

\[
\begin{bmatrix}
  x^1(j_1 + 1, \ldots, j_k) \\
  \vdots \\
  x^k(j_1, \ldots, j_k + 1)
\end{bmatrix} = A 
\begin{bmatrix}
  x^1(j_1, \ldots, j_k) \\
  \vdots \\
  x^k(j_1, \ldots, j_k)
\end{bmatrix}
\]

is asymptotically stable if and only if

\[ \det(H(z) - A) \neq 0, \forall z \in \mathcal{U}_2 \]

where \( H(z) \) is defined as in [4] and \( \mathcal{U}_2 = \{ z = [z_1, \ldots, z_k]^T : |z_i| \geq 1, i = 1, \ldots, k \} \).

4 Main results

4.1 Stability

**Lemma 6.** A multidimensional continuous fractional order system with order \( 0 < \nu \leq 1 \) and \( \det(A) \neq 0 \)

\[
\begin{bmatrix}
  D^\nu x^1(t_1, \ldots, t_r) \\
  \vdots \\
  D^\nu x^r(t_1, \ldots, t_r)
\end{bmatrix} = A 
\begin{bmatrix}
  x^1(t_1, \ldots, t_r) \\
  \vdots \\
  x^r(t_1, \ldots, t_r)
\end{bmatrix}
\]

4
is asymptotically stable if
\[ \det(H(\lambda) - A) \neq 0, \forall \lambda \in \mathcal{U}_\lambda \] (7)
where \( H(\lambda) \) is defined as in (6) and \( \mathcal{U}_\lambda = \{ \lambda : \lambda = [\lambda_1, \ldots, \lambda_r]^T, \ |\arg(\lambda_i)| \leq \frac{\pi}{2\nu}, 0 < \nu \leq 1, i = 1, \ldots, r \} \).

Proof. Applying Laplace transform to the multidimensional fractional system (3), the following holds
\[ (H(s^\nu) - A) \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_r(s) \end{bmatrix} = \begin{bmatrix} s^\nu x_1(0) \\ s^\nu x_2(0) \\ \vdots \\ s^\nu x_r(0) \end{bmatrix} \] (8)

It leads to
\[ (H(s^\nu) - A) \begin{bmatrix} s^\nu+1 X_1(s) \\ s^\nu X_2(s) \\ \vdots \\ s^\nu X_r(s) \end{bmatrix} = \begin{bmatrix} s^\nu x_1(0) \\ s^\nu x_2(0) \\ \vdots \\ s^\nu x_r(0) \end{bmatrix} \] (9)

\( \forall \lambda \in \mathcal{U}_\lambda, \det(H(\lambda) - A) \neq 0 \), thus \( \det(H(s^\nu) - A) \neq 0 \) when \( \text{Re}(s) > 0 \). It means that (9) has the only solution
\[ \begin{bmatrix} s^\nu+1 X_1(s) \\ s^\nu X_2(s) \\ \vdots \\ s^\nu X_r(s) \end{bmatrix} = (H(s^\nu) - A)^{-1} \begin{bmatrix} s^\nu x_1(0) \\ s^\nu x_2(0) \\ \vdots \\ s^\nu x_r(0) \end{bmatrix} \] (10)

Therefore, the following holds
\[ \lim_{s \to 0, \text{Re}(s) > 0} s^\nu+1 X_i(s) = 0, \ i = 1, 2, \ldots, r \]

According to Lemma 4
\[ \lim_{t \to +\infty} D^{\nu} x^i(t) = \lim_{s \to 0, \text{Re}(s) > 0} s^\nu+1 X_i(s) = 0, \ i = 1, \ldots, r \]

Therefore,
\[ \lim_{t \to +\infty} x(t) = A^{-1} \lim_{t \to +\infty} D^{\nu} x(t) = 0 \]

It implies that the system is asymptotically stable.

This completes the proof. \( \square \)

Theorem 7. Consider a multidimensional system represented by (7). Then, it’s asymptotically stable if
\[ \Delta(p, A) \neq 0, \ \forall p \in \mathcal{U}_{\mathcal{E}_{\lambda}} \] (11)
where \( \Delta(\cdot) \) is defined by (3) and \( \mathcal{U}_{\mathcal{E}_{\lambda}} \) is defined by
\[ \mathcal{U}_{\mathcal{E}_{\lambda}} = \left\{ \rho = \begin{bmatrix} \rho_1 \\ \vdots \\ \rho_k \end{bmatrix} \in \mathbb{C}^k : |\arg(\rho_i)| \leq \frac{\pi}{2\nu}, 0 < \nu \leq 1, i = 1, \ldots, r; \right\} \] (12)

\[ |\rho_i| \geq 1, i = r + 1, \ldots, k \]
Proof. It can be proved by straightforward combinations of Lemma 5 and 6. 

4.2 Point-clustering

To proceed, consider the following matrices

\[ P_i = \begin{bmatrix} p_{i1} & p_{i2} \\ p_{i2}^* & p_{i2} \end{bmatrix} \in \mathbb{C}^{2 \times 2}, \quad Q_i = \begin{bmatrix} q_{i1} & q_{i2} \\ q_{i2}^* & q_{i2} \end{bmatrix} \in \mathbb{C}^{2 \times 2} \] (13)

Define the sets \( D_i \) as

\[ D_i \triangleq \{ s \in \mathbb{C} : F_{P_i}(s) \geq 0, F_{Q_i}(s) \geq 0, \forall i \in I(k) \} \] (14)

where the functions \( F_{X_i}(s) \) are defined by

\[ F_{X_i}(s) \triangleq \begin{bmatrix} sI \\ I \end{bmatrix} X_i \begin{bmatrix} sI \\ I \end{bmatrix}, \quad \forall i \in I(k) \] (15)

We limit our consideration to sets described by \( D_i \). Define the "k-region" \( D \) as

\[ D \triangleq D_1 \times D_2 \times ... \times D_k \] (16)

Let \( D \) represent \( U_{\lambda,\Lambda} \), then the matrices \( P_i \) and \( Q_i \) are

\[ P_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q_i = \begin{bmatrix} 0 & \sin(\frac{\pi}{2} \nu) - j \cos(\frac{\pi}{2} \nu) \\ \sin(\frac{\pi}{2} \nu) + j \cos(\frac{\pi}{2} \nu) & 0 \end{bmatrix} \] (17)

where \( 0 < \nu \leq 1 \) and \( \forall i \in I(r) \).

And

\[ P_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q_i = \begin{bmatrix} 0 & \sin(\frac{\pi}{2} \nu) - j \cos(\frac{\pi}{2} \nu) \\ \sin(\frac{\pi}{2} \nu) + j \cos(\frac{\pi}{2} \nu) & 0 \end{bmatrix} \] (18)

where \( \forall i \in \{r+1, ..., k\} \).

The following gives a sufficient condition for the stability of system (1). 

**Theorem 8.** The system (1) is asymptotically stable if there exist matrices \( U_{n_i}, V_{n_i} \in \mathcal{H}_n^{+} \) and a matrix \( J = J^* \) such that

\[ Z = G + [I - A]^* J [I - A] < 0 \] (19)

where

\[ G \triangleq \begin{bmatrix} k \oplus (U_i p_{i11} + V_i q_{i11}) & k \oplus (U_i p_{i12} + V_i q_{i12}) \\ k \oplus (U_i p_{i21} + V_i q_{i21}) & k \oplus (U_i p_{i22} + V_i q_{i22}) \end{bmatrix} \] (20)

and \( p_{i11}, p_{i12}, p_{i21}, q_{i11}, q_{i12}, q_{i22} \) are defined in (13) with (17) and (18).
Proof. We’ll prove that for \( \forall \rho \) that satisfies \( \det(\bigoplus_{i=1}^k \rho_i I_{n_i} - A) = 0 \), then \( \rho \notin \mathcal{D} \).

Let \( A(\rho) = \bigoplus_{i=1}^k \rho_i I_{n_i} - A \). If \( \det(A(\rho)) = 0 \), then there exists a nonzero vector \( y \in \mathbb{C}^n \) such that

\[
A(\rho)y = 0
\]

(21)

Let

\[
y = \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix}
\]

where \( y_i \in \mathbb{C}^{n_i} \).

And let

\[
x = \begin{bmatrix} (\bigoplus_{i=1}^k \rho_i I_{n_i})y \\ y \end{bmatrix}
\]

From (19), we get

\[
x^*Zx < 0
\]

which leads to

\[
\sum_{i=1}^k (F_{p_i}(\rho_i)y_i^*U_i y_i + F_{q_i}(\rho_i)y_i^*V_i y_i) + (A(\rho)y)^*J(A(\rho)y) < 0
\]

(22)

According to (21), the second term of (22) is zero. If \( \rho \in \mathcal{D} \), then according to (14), the first term of (22) is positive or zero. Therefore, when \( \rho \in \mathcal{D} \), \( \det(A(\rho)) \neq 0 \). Due to Theorem 7, the system (1) is stable.

Corollary 9. The system (1) is asymptotically stable if there exist matrices \( U_{n_i}, V_{n_i} \in \mathcal{H}_{n_i}^+ \) and a matrix \( J = J^* \) such that

\[
G + \begin{bmatrix} I & 0 \\ -A & -A \end{bmatrix} J \begin{bmatrix} I & 0 \\ -A & -A \end{bmatrix}^* < 0
\]

(23)

where \( G \) is defined as in (20).

Proof. Similar to the proof of Theorem 8 if inequality (23) holds, then

\[
\det(\bigoplus_{i=1}^k \rho_i I_{n_i} - A^T) \neq 0, \rho \in \mathcal{D}
\]

which is equivalent to

\[
\det(\bigoplus_{i=1}^k \rho_i I_{n_i} - A) \neq 0, \rho \in \mathcal{D}
\]

Therefore, the system is asymptotically stable due to Theorem 7.

Corollary 10. The system (1) is asymptotically stable if there exist matrices \( U_{n_i}, V_{n_i} \in \mathcal{H}_{n_i}^+ \) and a matrix \( R \) such that

\[
G + \text{Sym}\left\{ \begin{bmatrix} I \\ -A \end{bmatrix} R \begin{bmatrix} I \\ I \end{bmatrix} \right\} < 0
\]

(24)

where \( G \) is defined as in (20).
Proof. Let

\[ G_{11} \triangleq \bigoplus_{i=1}^{k} (U_i p_{i1} + V_i q_{i1}) \]
\[ G_{12} \triangleq \bigoplus_{i=1}^{k} (U_i p_{i2} + V_i q_{i2}) \]
\[ G_{22} \triangleq \bigoplus_{i=1}^{k} (U_i p_{i2} + V_i q_{i2}) \] (25)

Then,

\[
[I - I] G [I - I]^* = G_{11} - G_{12}^* - G_{12} + G_{22}
\]
\[
= \begin{bmatrix}
-2 \left( \bigoplus_{i=1}^{r} (U_i \sin(\frac{\pi}{2} \nu) + V_i \sin(\frac{\pi}{2} \nu)) \right) & 0 \\
0 & k \oplus_{i=r+1}^{n} 0
\end{bmatrix}
\]
\[
< 0
\] (26)

It’s obvious that

\[
[I - I] [I] = 0
\]

Therefore, according to Lemma 3 and inequality (24) and (26), the following inequality holds

\[
[A \ I] G \begin{bmatrix} A^T \end{bmatrix} < 0
\] (27)

According to Lemma 2, inequality (27) is then equivalent to

\[
G + \begin{bmatrix} I \ -A \end{bmatrix} J [I - A^T] < 0
\] (28)

Thus, according to Corollary 9, the system is asymptotically stable. This completes the proof.

\[
\square
\]

5 Numerical Examples

5.1 Example 1

The following example presents a (1+1)D system of (1), i.e. a system with one continuous independent variable and one discrete independent variable. The system is considered:

\[
\nu = 0.5
\]
\[
A_c = \begin{bmatrix} -0.8 & 0 \\ 0 & -1.2 \end{bmatrix}, \quad A_{cd} = \begin{bmatrix} 0.5 & 0.3 \\ 0.7 & 0.2 \end{bmatrix}
\]
\[
A_{dc} = \begin{bmatrix} 0.4 & 0.3 \\ 0.8 & 0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.3 & 0 \\ 0 & -0.6 \end{bmatrix}
\]
\[
A = \begin{bmatrix} A_c & A_{cd} \\ A_{dc} & A_d \end{bmatrix}
\] (29)
Let a system be system (1) with (29). Applying Theorem 8, the variables can be calculated by the Matlab LMI toolbox. The solution is

\[
U_1 = \begin{bmatrix} 146.84 & 0 \\ 0 & 146.84 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 24.3 & 5.99 \\ 5.99 & 1.9 \end{bmatrix}, \\
V_1 = \begin{bmatrix} 4.24 & 14.68 \\ 14.68 & 194.82 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 146.84 & 0 \\ 0 & 146.84 \end{bmatrix}, \\
J = \begin{bmatrix} -164.9 & -57.19 & -210.42 & -75.7 \\ -57.19 & -108.77 & -154.38 & -62.71 \\ -210.42 & -154.38 & -643.18 & -137.73 \\ -75.7 & -62.71 & -137.73 & -116.65 \end{bmatrix} \tag{30}
\]

It means that the continuous-discrete (1+1)D system is stable.

5.2 Example 2

The system is considered:

\[
\nu = 0.9, \quad A_c = \begin{bmatrix} -0.8 & 0.5 \\ 0.3 & -1.2 \end{bmatrix}, \quad A_{cd} = \begin{bmatrix} 0.5 & 0.6 \\ 0.7 & 0.8 \end{bmatrix}, \\
A_{dc} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.7 & 0 \\ 0 & -0.2 \end{bmatrix}, \\
A = \begin{bmatrix} A_c & A_{cd} \\ A_{dc} & A_d \end{bmatrix} \tag{31}
\]

Let a system be system (1) with (31). Applying Corollary 10, the variables can be calculated by the Matlab LMI toolbox. The solution is

\[
U_1 = \begin{bmatrix} 35333.44 & 0 \\ 0 & 35333.44 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 36674.54 & 4958.14 \\ 4958.14 & 7924.014 \end{bmatrix}, \\
V_1 = \begin{bmatrix} 32747.79 & 0 \\ 0 & 14331.51 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 51948.27 & 29479.11 \\ 29479.11 & 51948.27 \end{bmatrix}, \\
R = \begin{bmatrix} -9111.82 & -53.82 & -29013.36 & -830.36 \\ -1813.13 & -12214.89 & -13744.63 & -6499.66 \\ 26731.07 & 4233.10 & -30626.82 & -11869.47 \\ 2051.31 & 1540.31 & 2773.74 & -8884.96 \end{bmatrix} \tag{32}
\]

It means that the continuous-discrete (1+1)D system is stable.

6 Conclusion

In this paper, the fractional continuous-discrete systems are presented, where the fractional order is \(0 < \nu \leq 1\). The stability and Robust stability of fractional continuous-discrete systems have been investigated. Invoking fractional final value theorem, the sufficient condition of stability of such systems is proved. Then, we prove the sufficient condition of Robust multidimensional interval system. Finally, examples are given to verify the theorems.
References

[1] Robert P Roesser. A discrete state-space model for linear image processing. Automatic Control, IEEE Transactions on, 20(1):1–10, 1975.

[2] P Agathoklis. The lyapunov equation for n-dimensional discrete systems. Circuits and Systems, IEEE Transactions on, 35(4):448–451, 1988.

[3] Krzysztof Galkowski. LMI based stability analysis for 2D continuous systems. In Electronics, Circuits and Systems, 2002. 9th International Conference on, volume 3, pages 923–926. IEEE, 2002.

[4] Olivier Bachelier, Wojciech Paszke, and Driss Mehdi. On the kalman-yakubovich-popov lemma and the multidimensional models. Multidimensional Systems and Signal Processing, 19(3):425–447, 2008.

[5] Olivier Bachelier, Pawel Dabkowski, Krzysztof Galkowski, and Anton Kummert. Fractional and nd systems: a continuous case. Multidimensional Systems and Signal Processing, 23(3):329–347, 2012.

[6] Yun Q Shi and Xi Min Zhang. A new two-dimensional interleaving technique using successive packing. Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on, 49(6):779–789, 2002.

[7] Sankar Basu. Multidimensional causal, stable, perfect reconstruction filter banks. Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on, 49(6):832–842, 2002.

[8] Krzysztof Galkowski and Anton Kummert. Fractional polynomials and nd systems. In Circuits and Systems, 2005. ISCAS 2005. IEEE International Symposium on, pages 2040–2043. IEEE, 2005.

[9] Krzysztof Galkowski, Olivier Bachelier, and Anton Kummert. Fractional polynomials and nd systems: A continuous case. In PROCEEDINGS OF THE 45TH IEEE CONFERENCE ON DECISION AND CONTROL, VOLS, pages 1–14, 2006.

[10] Jacek Bochniak and Krzysztof Galkowski. LMI-based analysis for continuous-discrete linear shift-invariant nd systems. Journal of Circuits, Systems, and Computers, 14(02):307–332, 2005.

[11] Igor Podlubny. Fractional differential equations, volume 198 of MATHEMATICS IN SCIENCE AND ENGINEERING. Academic Press, San Diego, 1999.

[12] RE Skelton, T Iwasaki, and KM Grigoriadis. A unified algebraic approach to linear control design, 1998.

[13] M.D. Ortigueira. Introduction to fractional linear systems. part 1. continuous-time case. Vision, Image and Signal Processing, IEE Proceedings, 147(1):62 – 70, 2000.