Group actions on matrices over local rings.
Annihilators of $T^1$-modules for the groups $G_{lr}, G_{congr}$.

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Abstract. We consider matrices with entries in a local ring, $A \in \text{Mat}_{m \times n}(R)$. Fix a group action, $G \circlearrowleft \text{Mat}_{m \times n}(R)$, and a subset of allowed deformations, $Σ \subseteq \text{Mat}_{m \times n}(R)$. The traditional objects of study in Singularity Theory and Algebraic Geometry are the tangent spaces $T_{Σ,A}$, $T_{(G,A),A}$, and their quotient, the tangent module to the universal deformation, $T^1_{(Σ,G,A)} = T_{(Σ,A)}/T_{(G,A,A)}$.

This module plays the key role in various deformation problems, e.g., deformations of maps, deformations of modules, of (skew)-symmetric forms. In particular, the first question is to determine the support/annihilator of this tangent module. In [BK.18] we have studied this tangent module for various $R$-linear group actions $G \circlearrowleft \text{Mat}_{m \times n}(R)$.

In the current work we study the support of the module $T^1_{(Σ,G,A)}$ for group actions that involve automorphisms of the ring. (Geometrically, these are group actions that involve the local coordinate changes.)

We obtain various bounds on localizations of $T^1_{(Σ,G,A)}$ and compute the radical of the annihilator of $T^1_{(Σ,G,A)}$, i.e., the set-theoretic support. This brings the definition of an (apparently new) type of singular locus, the “essential singular locus” of a map/sub-scheme. It reflects the “unexpected” singularities of a subscheme, ignoring those imposed by the singularities of the ambient space. Unlike the classical singular locus (defined by a fitting ideal of the module of differentials) the essential is defined by the annihilator ideal of the module of derivations.

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1. Introduction

1.1. Setup. Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring over a field $\kappa$ of zero characteristic. (The typical cases are $R = \kappa[[x]]/J$ or $R = \kappa(x)/J$ where $\kappa$ is a complete normed field. Here $x = (x_1, \ldots, x_p)$.) Denote by $\text{Mat}_{m \times n}(R)$ the $R$-module of $m \times n$ matrices. We always assume $m \leq n$. Various groups act on $\text{Mat}_{m \times n}(R)$ and show up in various areas.

Example 1.1. i. If the action $G \circlearrowleft \text{Mat}_{m \times n}(R)$ is $R$-linear and preserves the subset of degenerate matrices then $G$ is contained in the group of left-right multiplications, $G_{lr} := \text{GL}(m, R) \times \text{GL}(n, R)$. (See [BK.18] §3.6 for the precise statement.) Matrices considered up to $\text{GL}(n,R)$-transformations correspond to embedded modules, $\text{Im}(A) \subset \mathbb{R}^m_{\leq m}$. Matrices considered up to $\text{GL}(m,R) \times \text{GL}(n,R)$-transformations correspond to non-embedded modules, $\text{Coker}(A) = \mathbb{R}^m_{\leq m}/\text{Im}(A)$.

ii. The group of $k$-linear ring automorphisms, $\text{Aut}_k(R)$, acts on matrices entry-wise. Geometrically they are the local coordinate changes on $\text{Spec}(R)$. In Singularity Theory this group is known as the right equivalence, $R$.

iii. Accordingly one considers the semi-direct products, $G_r := \text{GL}(n,R) \rtimes \text{Aut}_k(R)$, $G_{lr} := \text{GL}(m,R) \rtimes \text{GL}(n,R) \rtimes \text{Aut}_k(R)$. The action on the modules $\text{Im}(A)$, $\text{Coker}(A)$ is by the base change, $\text{Coker}(A) \rightarrow \phi(\mathbb{R}) \otimes \text{Coker}(A)$. For one-row matrices, $m = 1$, the orbits of $G_{lr}, G_r$ coincide. In Singularity Theory this group is known as the contact equivalence of maps, $K$.

iv. The congruence, $G_{congr} = \text{GL}(m,R) \circlearrowleft \text{Mat}_{m \times m}(R)$, acts by $A \rightarrow UAU^t$. Matrices considered up to the congruence correspond to bilinear/symmetric/skew-symmetric forms. Accordingly one considers $G_{congr} := G_{congr} \rtimes \text{Aut}_k(R)$. 

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The traditional approach of deformation theory is to study the tangent space to the miniversal deformation. In our case this is the Tjurina module, \( T^1(\Sigma,G,A) := T(\Sigma,A)/T(G,A,A) \). Here \( \Sigma \) is one of \( \text{Mat}_{m \times n}(R) \), \( \text{Mat}_{m \times n}^\text{sym}(R) \), \( \text{Mat}_{m,n}^\text{sym}(R) \), while \( T(G,A,A) := T(G,1)A \) is the image tangent space to the orbit. (For our particular cases these are defined in \([3.2]\).)

In the simplest case of “functions”, \( m = 1 = n \), we get the Milnor algebra, \( T^1(R,\text{Aut}_R(R),A) \), and the Tjurina algebra, \( T^1(R,G_{\text{irr}},A) \).

### 1.2. The module \( T^1(\Sigma,G,A) \) is complicated. It is usually a torsion over \( R \), while over \( R/\text{Ann}(T^1(\Sigma,G,A)) \) it is far from being free and usually of high rank. The complexity and importance can be appreciated by the following particular cases.

**Example 1.2.**

i. For \( \Sigma = \text{Mat}_{m \times n}(R) \), \( G = GL(m,R) \times GL(n,R) \) and \( A = \text{Mat}_{m \times n}(m) \) there holds:

\[
T^1_{(\Sigma,G,A)} = \text{Ext}^1_R(\text{Coker}(A),\text{Coker}(A)).
\]

ii. Let \( m = 1 < n \), and identify \( \text{Mat}_{1 \times n}(R) \approx R^n \), then \( T^1_{(R^n,\text{Gr}_{irr},A)} \) is the classical Tjurina module of the map \( \text{Spec}(R) \to \text{A} \) \((k^n,0)\). This module is among the cornerstones of the Singularity Theory, see chapter 4 of [Looijenga]. Its structure is rich and is not completely understood. For \( R \) regular \( T^1 \) defines the singular locus of the map \( A \).

In both cases one cannot present \( T^1_{(\Sigma,G,A)} \) (or its annihilator) in any simple/more explicit form.

In this paper we study the annihilator/support of \( T^1_{(\Sigma,G,A)} \) for the actions \( G_{\text{irr}}, G_{\text{conorg}} \) of example 1.1. This information is needed e.g., for the studies of determinantal singularities, see [Bruce-Goryunov-Zakalyukin, Bruce-Tan], [Bruce, Frühbis-Krüger.99], [Frühbis-Krüger-Zaeh], [Frühbis-Krüger.18], [Damon-Pike], [Ahmed-Ruas, N-B-O-O-T.13], [N-B-O-O-T.18]. In particular, if the annihilator \( \text{Ann}(T^1_{(\Sigma,G,A)}) \) contains a power of the maximal ideal (and the ring \( R \) is henselian), then the studied object is “finitely determined” [BK.16, BGK]. Equivalently, the object is “algebraizable in families” [BK.18 §3.12]. This is the first step in establishing the finite dimensionality of the miniversal deformation, and then possible classification of simple/unimodal...singularity types.

The related questions of finite determinacy were studied in some particular cases, over the rings \( k[\{z\}], C\{z\}, \) [Greuel-Pham.17.a], [Greuel-Pham.17.b], [Greuel-Pham.18], but the approach was mostly algorithmic, translating the question into the case-by-case tasks for computer packages. Unlike the previous studies, we work in the generality of local Noetherian rings (without any regularity assumptions) and give explicit criteria, applicable without computer help.

The bounds on \( \text{Ann}(T^1_{(\Sigma,G,A)}) \) in theorems 2.1, 2.2 are somewhat involved. This is not a surprise, noticing the “complexity of the problem”. Recall that \( T^1_{(\Sigma,G,A)} \) encodes many of the singularity properties of the object, and the bounds clearly show the singularity invariants of the matrices. On the other hand, the bounds admit transparent geometric interpretation, in terms of certain degeneracy loci. We emphasize also that the formulae are “computationally simple”, and admit direct computer implementation.

In [BK.1,2] we apply these bounds to establish strong results on finite determinacy/algebraizability/properties of the miniversal deformation of matrices under various group actions. This gives criteria of algebraizability of embedded modules, (skew-)symmetric forms, complexes of modules, etc. (The statements in [BK.1,2] are of the following type. Let \( M_R = \text{coker}(A) \) be a finitely generated module over a complete Noetherian local ring \( \hat{R} \), over \( k \). If \( \sqrt{\text{Ann}(T^1_{(\Sigma,G_{\text{irr}},A)})} = m \) then there exists a finitely generated \( k \) subalgebra \( R \subset \hat{R} \), whose completion is \( \hat{R} \), and a finitely generated module \( M_R \) such that \( M_R = \hat{R} \otimes M_R \)).

Some of our statements hold in the generality of commutative unital rings, for some others it is enough to assume that \( R \) is local Noetherian (but not necessarily over a field of zero characteristic). Yet, we restrict to “\( k \) is a field of zero characteristic”, to avoid various pathologies of the modules of derivations \( \text{Der}_R(R) \), and differentials, \( \Omega^1_{R/k} \). In fact, our work has originated from the classical Singularity Theory, over the classical rings like \( k[[z]]/j, C\{z\} \). Already for these rings our results are new.

### 1.3. The content and the structure of the paper.

i. The bounds on \( \text{Ann}(T^1_{(\Sigma,G,A)}) \) are stated in 2.2. They are expressed in terms of the ideal Sing\(_r(J) \subset R \), the “essential singular locus” of the subscheme \( V(J) \subset \text{Spec}(R) \), see below. This essential singular locus measures the singularities of the determinantal strata, Sing\(_r(I_j(A)) \), here \( r \) is the expected grade of \( I_j(A) \). Recall that the scheme \( V(I_j(A)) \subset \text{Spec}(R) \) is always singular along \( V(I_{j-1}(A)) \). The module \( T^1_{(\Sigma,G,A)} \) is supported exactly at the “unexpected” singular points, \( V(\text{Sing}_r(I_j(A))) \setminus V(I_{j-1}(A)) \).

While the bounds are algebraic, we give transparent geometric interpretations. These bounds are extensively used in [BK.1,2].
ii. In §3 we prepare the tools. Trying to be understandable by non-experts in Commutative Algebra we emphasize the geometric meaning of various notions and recall some standard results.
(a) In §3.2 we describe the tangent spaces to the group orbits, $T_{(G,T)}A$.
(b) In §3.3 we establish the needed properties of localization.
(c) Sections 3.4.5 3.6 3.7 are about the properties of the determinantal ideals, $\{I_f(A)\}$, Pfaffian ideals, $\{ Pf_f(A)\}$, the annihilator-of-cokernel, $Ann.Coker(A)$, and its generalizations, $\{Ann.Coker_r(A)\}$.
(d) In §3.8 we study the essential singular locus, $Sing_r(J) \subset R$. (Here $r$ is the expected grade of $J \subset R$.)

Recall that the classical singular locus is defined by the Fitting ideal of the module of differentials, $Fit_{dim(J)}(R)\Omega^1_{R/J} \subset R$. The essential singular locus is defined using the module of derivations, $Der_r(R)$, and the annihilator scheme structure. Unlike the classical singularity locus, $Sing_r(J)$ measures only the “unexpected” singularities of $V(J) \subset Spec(R)$ and often ignores the singularities of the ambient space, $Spec(R)$.

If $R$ is a regular local ring and the ideal $J$ is pure, of grade $r$, we get the classical singular locus (but with annihilator rather than fitting ideal scheme formulation): $Sing_r(J) = Ann_r\Omega^1_{R/J}$. But for non-regular rings or non-pure ideals, one has usually:

$Sing_r(J) \supset Ann_r\Omega^1_{R/J} \supset \text{Fitt}_{dim(J)}(R)\Omega^1_{R/J}$.

(e) In §3.9 we prove that the ideal $Ann(T^j_{(G,A)})$ is invariant under the action of some elements of $G_{lt}$.

This fact is used repeatedly in §4.

iii. In §4 we prove the statements of §2. The proofs go by checking the support of $T^j_{(G,A)}$ “pointwise”, i.e., by localizations at prime ideals.

1.4. Notations and conventions.

i. The ideal quotient is $I : J = \{ f \in R \mid f \cdot J \subseteq I \}$.

ii. The saturation of $I \subset R$ by $J \subset R$ is the ideal $Sat_I(J) := \bigcap_{k \geq 1} I^k J^k$. We note that $Sat_I(J) := \bigcap_{k \geq 1} I^k J^k$. Geometrically one erases the subscheme $V(\sqrt{J}) \subset Spec(R)$ and then takes the Zariski closure, $V(Sat_I(J)) = \overline{V(I)} \setminus V(\sqrt{J})$.

We often use the relation $\sqrt{Sat_I(J)} = \sqrt{J}$, see lemma 3.3.

iii. Localization at a prime ideal, $R \to R_p$, induces the natural map $Mat_{m \times n}(R) \to Mat_{m \times n}(R_p)$. We denote the image of $A \in Mat_{m \times n}(R)$ by $A_p \in Mat_{m \times n}(R_p)$.

iv. Suppose an $R$-module $M$ is minimally generated by $m$ elements. The annihilator ideal $Ann(M)$ is a refined version of the fitting ideal, $Fitt(I,M)$. Similarly, the $j$th annihilator, $Ann_j(M)$, is the refinement of the ideal $Fitt_{m-j}(M)$.

Choosing a particular presentation matrix of a module, $M = Coker(A)$, we get the determinantal ideals, $I_f(A) = Fitt_{m-j}(Coker(A))$, and their refined versions, $\{Ann.Coker_r(A) := Ann_r(Coker(A))\}$. See §3.6 §3.7 for the definitions and properties.

v. Let $Der_r(R)$ be the module of (k-linear) derivations of $R$. The derivations act on matrices entrywise, for any $D \in Der_r(R)$ one has $D(A) \in Mat_{m \times n}(R)$. By applying the whole module $Der_r(R)$ we get the submodule $Der_r(R)(A) \subseteq Mat_{m \times n}(R)$. Similarly, for an ideal $J \subset R$ one gets the ideal $Der_r(R)(J) \subset R$.

Sometimes we need only the subodule $Der_r(R, m)$, the derivations sending $R$ to $m$. Accordingly we have $Der_r(R, m)(J) \subset R$ and $Der_r(R, m)(A) \subseteq Mat_{m \times n}(m)$.

vi. Fix an ideal, $J \subseteq R$, and a number $r \in \mathbb{N}$, which is often the expected height/grade of $r$. Assume $J$ is finitely generated, choose any system of generators, $J = \langle f_1, \ldots, f_N \rangle$, write them as a column, $J$. Applying the derivations of $R$ to this column we get the submodule $Der_r(R)(J) \subseteq R^\mathbb{N}$.

The essential singular locus of $J$ is defined as

$Sing_r(J) := \begin{cases} Ann_r R^\mathbb{N}/\langle J \cdot R^\mathbb{N} + Der_r(R)(J) \rangle & \text{for } r \leq N; \\
J & \text{for } r > N. \end{cases}$

(Here $Ann_r$ is the $r$th annihilator ideal, a refinement of the $r$th determinantal ideal, see §5.7.)

The typical context for $Sing_r(J)$ is the determinantal ideals, $J = I_{j+1}(A)$. Then $r$ is taken as the expected height/grade:

(a) for $A \in Mat_{m \times n}(m)$ one takes $r = (m - j)(n - j)$;

(b) for $A \in Mat_{m \times n}^\text{skew-sym}(m)$ one takes $r = \binom{m+1-j}{2}$

(c) for $A \in Mat_{m \times n}^\text{skew-sym}(m)$ and $j$-even one takes $r = \binom{m-j}{2}$;

(d) for $A \in Mat_{m \times n}^\text{skew-sym}(m)$ and $j$-odd one takes $r = \binom{m-j+1}{2}$.

Note that we do not take here the usual $\min((m-j)(n-j), \dim(R))$ or $\min((m-j)(n-j), \text{depth}(R))$.

Sometimes we use the $m$-singular locus, $Sing_m(J)$, with $Der_m(R, m)$ instead of $Der_r(R)$.

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2. The main results

2.1. The action $Aut_{\Sigma}(R) \cap \Sigma := Mat_{m \times n}(R)$ does not involve any matrix structure and the presentation of the tangent space $T_{(Aut_{\Sigma}(R),A)}$ (32) gives the obvious

(3) \[ Ann(T_{1(\Sigma,Aut_{\Sigma}(R),A)}) = Ann(\Sigma/\text{Der}_{R}(m,A)) \]

For the regular rings like $k[[x]], \mathbb{C}[x]$, this recovers the classically studied cases in Singularity Theory, e.g. the “local ring version” of the Milnor algebra for $m = n = 1$. For the non-regular rings $k[[x]], \mathbb{C}[x]$ see e.g., \cite{Bruce-Roberts}.

We remark that for $(m,n) \neq (1,1)$ and $A \in Mat_{m \times n}(m^2)$ the ideal $Ann(\Sigma/\text{Der}_{R}(m,A))$ does not contain any power of $m$. (See e.g., proposition 5.7 of \cite{BGK}.)

2.2. The action $G_{irr} \subset Mat_{m \times n}(R)$.

Theorem 2.1. Let $R$ be a Noetherian local ring over a field $k$ of characteristic $0$, $A \in \Sigma := Mat_{m \times n}(R)$.

1. If $m = 1$ then $Ann(T_{1(\Sigma,G_{irr},A)}(I_{1}(A)) = Sing_{n-1}(I_{1}(A)) + Ann(\Sigma/\text{Der}_{R}(m,A))$.

2. For any prime ideal ideal $\mathfrak{p}$ satisfying $\mathfrak{p} \supseteq I_{1}(A)$, the localizations of ideals at $\mathfrak{p}$ satisfy:

\[ Ann(T_{1(\Sigma,G_{irr},A)}(I_{1}(A))_{\mathfrak{p}} = Sing_{n-1+1}(I_{m}(A))_{\mathfrak{p}} \]

3. $Ann(Coker(A) + Ann(\Sigma/\text{Der}_{R}(m,A)) \subseteq Ann(T_{1(\Sigma,G_{irr},A)}) \subseteq \bigcap_{j=0}^{m-1} Sat_{I_{1}(A)}(Sing_{n-m-j}(I_{j+1}(A)))$.

4. Suppose for some $j$ the prime decomposition is $I_{j+1}(A) = (\cap R_{p_{j}}) \cap (\cap q_{j})$, where $\text{grade}(R_{p_{j}}) < (m-j)(n-j)$ and $\text{grade}(q_{j}) = (m-j)(n-j)$. Take the corresponding localizations, $R \rightarrow R_{p_{j}}$, and suppose the rings $\{R_{p_{j}}\}$ are regular. Then $Ann(T_{1(\Sigma,G_{irr},A)}) \subseteq Sat_{I_{1}(A)}(\cap R_{p_{j}}^{-1}(I_{j+1}(A)))$.

5. $\sqrt{Ann(T_{1(\Sigma,G_{irr},A)})} = \bigcap_{j=0}^{m-1} \sqrt{Sing_{n-m-j}(I_{j+1}(A))}$.

Below are some remarks and geometric interpretations (when the base field is algebraically closed, $k = \mathbb{C}$).

Part 1. Here the matrix structure plays no role and the $G_{irr}$-action induces the classical contact equivalence of maps, $K$. For $m = 1 = n$ denote $A$ by $f$, then $Ann(T_{1(\Sigma,G_{irr},f)}(f) + \text{Der}_{R}(m,f))$. This is the “local ring version” of the classical Tjurina ideal of a function.

More generally, for $m = 1 \leq n$, the ideal $Sing_{n}(I_{1}(A))$ defines the essential singular locus of the subscheme $A^{-1}(0) \subseteq Spec(R)$ or of the map $Spec(R) \rightarrow (k^{n},0)$. Recall that we use derivations instead of differentials and the annihilator scheme structure, rather than the fitting scheme structure.

Part 2. $T_{1(\Sigma,G_{irr},A)}$ is supported on the essential singular locus of $V(I_{m}(A))$, with the locus $V(I_{m-1}(A))$ erased:

\[ Supp(T_{1(\Sigma,G_{irr},A)}) \setminus (Supp(T_{1(\Sigma,G_{irr},A)}) \cap V(I_{m-1}(A))) = Sing_{n-1+1}(V(I_{m}(A))) \setminus V(I_{m-1}(A)) \subset Spec(R) \]

Recall that $A$ is “infinitesimally $G_{irr}$-stable”, i.e. $T_{(G_{irr},A)} = Mat_{m \times n}(R)$, at the points of $Spec(R) \setminus V(I_{m}(A))$, where $A$ is of full rank. Part 2 gives: $A$ is “infinitesimally $G_{irr}$-stable” at the points of $Spec(R) \setminus V(Sing_{n-1+1}(I_{m}(A)))$.

Part 3. The embedding $Ann(T_{1(\Sigma,G_{irr},A)}) \subseteq \ldots$ is the embedding of (germs of) schemes:

(4) \[ Supp(T_{1(\Sigma,G_{irr},A)}) \supseteq \bigcup_{j=0}^{m-1} Sing(V(I_{j+1}(A))) \setminus V(I_{1}(A)) \]

(The closure is taken here in Zariski topology.)

Recall that the singular locus of $V(I_{j+1}(A))$ always contains the locus $V(I_{1}(A))$. The upper bound in part 3 says that $Supp(T_{1(\Sigma,G_{irr},A)})$ contains the “unexpected” singular loci of all $V(I_{j+1}(A))$.

The lower bound in part 3 implies in particular that the support of $T_{1(\Sigma,G_{irr},A)}$ lies inside the locus $V(I_{m}(A))$. If $A$ is non-degenerate at some point of $Spec(R)$ then $T_{1(\Sigma,G_{irr},A)}$ is not supported at that point.

Part 4. This implies: the subscheme $Supp(T_{1(\Sigma,G_{irr},A)}) \subset Spec(R)$ contains (as subschemes) all the components of $V(I_{j+1}(A)) \subseteq Spec(R)$ that are not of expected co-dimension. In the classical case, $R = \mathbb{C}[x]$, the set theoretic version of this result is well known, see e.g., \cite{Bruce-Tar}, \cite{Gizatullin-}. (And the classical proofs use the Thom stratification/transversality theorems, over $\mathbb{R}, \mathbb{C}$.)

Part 5. Set-theoretically this is equality of the reduced subschemes:

(5) \[ Supp(T_{1(\Sigma,G_{irr},A)})_{red} = \bigcup_{j=0}^{m-1} Sing(V(I_{j+1}(A))) \setminus V(I_{1}(A))_{red} \subset Spec(R) \]

Thus $T_{1(\Sigma,G_{irr},A)}$ is (set-theoretically) supported exactly on the “unexpected” singular loci of the determinantal strata. This fact is of vital importance and brings numerous corollaries for the determinacy and deformations. Some versions of this are well known in the classical case, $R = \mathbb{C}[x]$, \cite{Bruce-Tar}, \cite{Gizatullin-}.
2.3. Theorem 2.2. Let $R$ be a local Noetherian ring over a field $k$ of characteristic zero.

1. Let $A \in \Sigma := \operatorname{Mat}_{m \times m}(R)$, $m > 1$, and assume $\dim(R) < \frac{2m}{2}$. Then $\operatorname{Ann}(T^1_{\Sigma, G_{\text{congr}}, A}) \subseteq \nil(R)$, the nilradical of $R$.

2. Let $A \in \Sigma^{\text{sym}} := \operatorname{Mat}_{m \times m}^{\text{sym}}(R)$.
   i. Suppose a prime ideal $\mathfrak{p} \subseteq \mathfrak{m}$ satisfies $I_{m-1}(A) \subseteq \mathfrak{p} \supseteq (\det(A))$. Then the localizations at $\mathfrak{p}$ satisfy:
      \[ \operatorname{Ann}(\operatorname{Coker}(A) + \operatorname{Ann} \big/ \operatorname{Der}_R(\mathfrak{m})(A)) \subseteq \operatorname{Ann}(T^1_{\Sigma^{\text{sym}}, G_{\text{congr}}, A}) \subseteq \bigcap_{j=0}^{m-1} \operatorname{Sat}_{j+1}(A) \left( \operatorname{Sing}^{(m-1)/2}(I_{j+1}(A)) \right) \cdot \]
   ii. Suppose for some $j$ the prime decomposition is $\sqrt{I_{j+1}(A)} = (\cap_{\mathfrak{p}_a} \cap (\cap_{\mathfrak{p}_b}))$, where $\operatorname{grade}(\mathfrak{p}_a) < (m-1)/2$ and $\operatorname{grade}(\mathfrak{p}_b) = (m-1)/2$. Take the corresponding localizations, $R \cong \mathfrak{p}_a$, and suppose $\{R_{\mathfrak{p}_a}\}$ are regular. Then $\operatorname{Ann}(T^1_{\Sigma^{\text{sym}}, G_{\text{congr}}, A, \mathfrak{p}_a}) \subseteq \operatorname{Sat}_{j+1}(A) \left( \cap_{\mathfrak{p}_a} \operatorname{Sing}^{(m-1)/2}(I_{j+1}(A)) \right)$.

3. Let $A \in \Sigma := \operatorname{Mat}_{m \times m}^{\text{skew-sym}}(R)$, $m \geq 2$. Below $Pf_{m-1}(A)$ is the generalized Pfaffian ideal, defined in [7,2]
   i. Suppose $m$ is even and a prime ideal $\mathfrak{p} \subseteq \mathfrak{m}$ satisfies $I_{m-2}(A) \subseteq \mathfrak{p} \supseteq (\det(A))$. Then the localizations satisfy $\operatorname{Ann}(T^1_{\Sigma, G_{\text{congr}}, A}) = \operatorname{Ann}(I_{m-1}(A))$.
   ii. Suppose $m$ is odd and a prime ideal $\mathfrak{p} \subseteq \mathfrak{m}$ satisfies $I_{m-1}(A) \subseteq \mathfrak{p} \supseteq I_{m-1}(A)$. Then the localizations satisfy $\operatorname{Ann}(T^1_{\Sigma, G_{\text{congr}}, A}) = \operatorname{Ann}(Pf_{m-1}(A))$.
   iii. Suppose $m$ is odd and a prime ideal $\mathfrak{p} \subseteq \mathfrak{m}$ satisfies $I_{m-1}(A) \subseteq \mathfrak{p} \supseteq I_{m-1}(A)$. Then the localizations satisfy $\operatorname{Ann}(T^1_{\Sigma, G_{\text{congr}}, A}) = \operatorname{Ann}(Pf_{m-1}(A))$.
   iv. Suppose for some $j$ the prime decomposition is: $\sqrt{I_{j+1}(A)} = (\cap_{\mathfrak{p}_a} \cap (\cap_{\mathfrak{p}_b}))$, where $\operatorname{grade}(\mathfrak{p}_a) < (m-1)/2$ and $\operatorname{grade}(\mathfrak{p}_b) = (m-1)/2$. Take the corresponding localizations, $R \cong \mathfrak{p}_a$, and assume $\{R_{\mathfrak{p}_a}\}$ are regular. Then $\operatorname{Ann}(T^1_{\Sigma, G_{\text{congr}}, A}) = \operatorname{Sat}_{j+1}(A) \left( \cap_{\mathfrak{p}_a} \operatorname{Sing}^{(m-1)/2}(I_{j+1}(A)) \right)$.
   v. For any $m$ (even or odd) holds:
      \[ \operatorname{Ann}(T^1_{\Sigma, G_{\text{congr}}, A}) = \bigcap_{j=0}^{m-1} \left( \operatorname{Sing}^{(m-1)/2}(I_{j+1}(A)) : I_j(A) \right) \]

As in the case of $G_t$, the statements have direct geometric interpretations.

Part 1. implies: if $\dim(R) < \frac{2m}{2}$ then ideal $\operatorname{Ann}(T^1)$ contains no power of the maximal ideal. This implies (e.g., [K,16], [K,2], [GK]) that no matrix $A \in \operatorname{Mat}_{m \times m}(R)$ is $G_{\text{congr}}$-finitely determined.

Part 2. Here i. says that $T^1_{\Sigma, G_{\text{congr}}, A}$ is supported on the essential singular locus of $V(\det(A)) \subseteq \operatorname{Spec}(R)$, with the sublocus $V(I_{m-1}(A))$ erased. As in the $G_t$-case: $A$ is infinitesimally $G_{\text{congr}}$-stable at the points of $\operatorname{Spec}(R) \setminus V((\det(A) + \operatorname{Der}_R(\det(A)))$.

The interpretation of part iii. is as in the same as for $G_t$-case.

Part iv. means: the full (set-theoretic) support of $T^1_{\Sigma, G_{\text{congr}}, A}$ consists of the “unexpected” singularities of the determinantal strata, cf. equation [13].

Part 3. Part i. can be written also as $\operatorname{Ann}(T^1_{\Sigma, G_{\text{congr}}, A}) = \operatorname{Sing}^m P(A)$, using the properties of Pfaffian ideals. [3.3]

The geometric interpretations of the statements are as in Part 2 and for $G_t$-case.

The even-odd differences and the conditions $j \in \mathbb{Z}$ are due to peculiarities of $I_j(A)$ for skew-symmetric matrices, e.g., $\sqrt{I_2} = \sqrt{I_0} = 1$ and $I_m(A) = 0$ for $m$-odd.

3. Preparations

Unless stated otherwise, $R$ is a commutative ring over a field $k$ of zero characteristic.
3.1. The module of $k$-linear derivations. For the (regular) rings $k[[x]]$, $k\{x\}$, $k(x)$ one has $\text{Der}_k(R) = R\langle \partial_i \rangle$, generated by the first order partial derivatives.

The module of those derivations of $R$ that preserve $m$ satisfies: $\text{Der}_k(R,m) \supseteq m \cdot \text{Der}_k(R)$. The equality holds here for many regular rings.

The module $\text{Der}_k(R)$ localizes nicely, for a prime $p \subseteq R$ holds: $\text{Der}_k(R)_p = \text{Der}_k(R_p)$, see proposition 16.9 of [Eisenbud]

3.2. Tangent spaces to the orbits. We recall their presentation e.g., from [BK.16 §3.7]. The tangent space to the orbit of a matrix is obtained by applying the tangent space of a group, $T_{(G,A,A)} = T_{(G,1)}A$.

i. $G_{ir} = GL(m,R) \times GL(n,R)$ acts by $A \to UAVA^{-1}$. Here

$$T_{(G_{ir},A)} := T_{(G_{ir},1)}A = \text{Mat}_{m \times n}(R) \cdot A \cdot \text{Mat}_{m \times n}(R) \subseteq \text{Mat}_{m \times n}(R).$$

Similarly for $G_1$ and $G_2$.

ii. $\text{Aut}_R(R)$. Let $(R,m)$ be a local ring, then $T_{(\text{Aut}_R(R,1))} = \text{Der}_R(R,m)$. Here we have only the submodule $\text{Der}_R(R,m) \subseteq \text{Der}_R(R)$ because the automorphisms of the local ring correspond to the local coordinate changes, i.e. preserve the origin of $\text{Spec}(R)$. Therefore

$$T_{(\text{Aut}_R(R,1))}A = \text{Der}_R(R,m)(A) = \text{Span}_R\{D(A) \}_{D \in \text{Der}_R(R,m)} \subseteq \text{Mat}_{m \times n}(R).$$

iii. $G_{sr} : A \to U\phi(A)V^{-1}$. Here $T_{(G_{sr},1)}A = \text{Mat}_{m \times n}(R) \cdot A + A \cdot \text{Mat}_{m \times n}(R) + \text{Der}_R(R,m)(A) \subseteq \text{Mat}_{m \times n}(R)$.

iv. $G_{s\text{cong}} : A \to U\phi(A)U^t$. Here $T_{(G_{s\text{cong}},1)}A = \text{Span}_R\{uA + Au^t\} \subseteq \text{Mat}_{m \times n}(R) + \text{Der}_R(R,m)(A) \subseteq \text{Mat}_{m \times m}(R)$.

3.3. Basic results on localizations.

**Lemma 3.1.** Let $p \subseteq R$ be a prime ideal. Then the saturations/localizations satisfy:

1. $p \nsubseteq J$ iff $J = p$.
2. If $p \nsubseteq I$ then $Sat_I(J) = J_p$.
3. Suppose $R$ is Noetherian and fix some ideals $I, J_1, J_2 \subseteq R$. Then $Sat_I(J_1) \subseteq Sat_I(J_2)$ iff for any prime $p \nsubseteq I$ holds: $(J_1)_p \subseteq (J_2)_p$.

**Proof.**

1. See the remark on page 71 of [Bourbakiz].

2. $\nsubseteq$ is obvious as $Sat_I(J) \nsubseteq J$. For the part $\subseteq$ it is enough to prove: $Sat_I(J) \subseteq J_p$. Suppose $f \in Sat_I(J)$ then $INf \nsubseteq J$ for some $N$. As $p \nsubseteq I$ we have, by part 1, $I_p = R_p$. Hence $(f) = (INf)_p \subseteq J_p$.

3. $\nsubseteq$ Consider the quotient module $Sat_I(J_1) + Sat_I(J_2)/Sat_I(J_2)$. The localization of this quotient at any prime $p \nsubseteq I$ vanishes:

$$\left(\frac{Sat_I(J_1) + Sat_I(J_2)}{Sat_I(J_2)}\right)_p = \frac{Sat_I(J_1)_p + Sat_I(J_2)_p}{Sat_I(J_2)_p} = \frac{(J_1)_p + (J_2)_p}{(J_2)_p} = \{0\}.$$

Therefore this quotient is not supported on $Spec(R) \setminus V(I)$. As $R$ is Noetherian, the ideals are finitely generated and there exists $N$ satisfying:

$$Ann\left(\frac{Sat_I(J_1) + Sat_I(J_2)}{Sat_I(J_2)}\right) \supseteq I^N \Rightarrow I^N \cdot Sat_I(J_1) \subseteq Sat_I(J_2) \Rightarrow Sat_I(J_1) \subseteq Sat_I(J_2).$$

**The geometric interpretation.** ($k = \overline{k}$ is a field.) Take a point $pt \in Spec(R)$ then:

1. $pt \notin V(J)$ iff $(V(J),pt)$ is empty.
2. Suppose $pt \notin V(I)$. Then $pt \in V(J) \setminus V(I)$ iff $pt \in V(J)$.
3. $V(J_1) \setminus V(I) \supseteq V(J_2) \setminus V(I)$ iff for any point $pt \notin V(I)$ holds: $(V(J_1),pt) \supseteq (V(J_2),pt)$.

**Lemma 3.2.** Let $R$ be a local Noetherian ring and $J_1, J_2 \subseteq R$ some proper ideals. The following conditions are equivalent:

1. $\sqrt{J_1} = \sqrt{J_2}$.
2. For any non-maximal prime ideal, $p \nsubseteq m$, holds: $(J_1)_p = R_p$ if $(J_2)_p = R_p$.
3. For any non-maximal prime ideal, $p \nsubseteq m$, holds: $(J_1)_p \neq R_p$ if $(J_2)_p \neq R_p$.

**Proof.** Obviously 2 $\iff$ 3, thus we prove 1 $\iff$ 2.

1 $\Rightarrow$ 2 If $(J_1)_p = R_p$ then there exists $f \in J_1$ whose image in $(J_1)_p$ is invertible. Thus $f \notin p$. But $f^N \in J_2$ for some $N < \infty$. And the image of $f^N$ in $(J_2)_p$ is still invertible, hence $(J_2)_p = R_p$.

2 $\Rightarrow$ 1 Take the prime decomposition, $\sqrt{J_1} = \cap p_i$. (As $\sqrt{J_1}$ is a radical ideal, its primary decomposition consists of prime ideals.) Suppose for some $i$ happens $p_i \nsubseteq \sqrt{J_2}$ then $p_i \nsubseteq J_2$, thus $(J_2)_p = R_p$, $(J_1)_p$. Thus, if $p$ is a minimal prime for $J_1$ then $p \nsubseteq J_2$. Suppose $p_i$ is not a minimal prime for $J_2$, then exists
a smaller prime ideal \( q \subseteq p_i \), which is a minimal prime for \( J_2 \). But then, by the same argument as above, \( q \supseteq J_1 \), thus \( p \) could not be a minimal prime.

Therefore; \( p \) is a minimal prime ideal for \( \sqrt{J_1} \) iff it is the one for \( \sqrt{J_2} \). In other words, \( \sqrt{J_1}, \sqrt{J_2} \) have the same minimal primes. Thus, as both are radical, their primary decompositions coincide. Hence \( \sqrt{J_1} = \sqrt{J_2} \).

**The geometric interpretation.** (\( k = \bar{k} \) is a field.) Denote by \( 0 \in \text{Spec}(R) \) the base point of the germ. The following are equivalent:
1. Two proper ideals define (set-theoretically) the same locus, \( V(J_1)_{\text{red}} = V(J_2)_{\text{red}} \).
2. For any closed point, \( 0 \neq pt \in \text{Spec}(R) \), there holds: \( pt \in V(J_1) \) iff \( pt \in V(J_2) \).

### 3.4. Saturation vs radicals.

**Lemma 3.3.** Given two ideals \( I, J \subset R \), with \( I \) finitely generated, there holds: \( \sqrt{\text{Sat}_I(J)} = \sqrt{J : I} \).

**Proof.** If \( f \in \sqrt{\text{Sat}_I(J)} \) then \( f^n \in J : I^n \), for some \( n \in \mathbb{N} \). Thus \( f^n \cdot I^n \subseteq J \), hence \( f \cdot I \subseteq \sqrt{J} \).

Let \( I = (g_1, \ldots, g_n) \) and suppose \( f \cdot I \subseteq \sqrt{J} \). Then \( f^n \cdot I^n = f^n(g_1, \ldots, g_n)^n \subseteq J \) for some \( N \gg 1 \). Thus \( f^n \in \text{Sat}_I(J) \).

The finiteness assumption on \( I \) is important, due to the following standard example.

(9) \( R = \bar{k}[x_1, x_2, \ldots] \supseteq m = \{x_i\} \supseteq J = \langle x_1, x_2^2, x_3^3, \ldots \rangle \). Then \( \sqrt{J} : m = R \) but \( \sqrt{\text{Sat}_m(J)} = \sqrt{J} = m \).

### 3.5. Determinantal and Pfaffian ideals.

For \( 1 \leq j \leq m \) and \( A \in \text{Mat}_{m \times n}(R) \) denote by \( I_j(A) \subset R \) the determinant ideal generated by all the \( j \times j \) minors of \( A \). By definition \( I_0(A) = R \) and \( I_{m-1}(A) = \{0\} \).

Determinantal ideals of skew-symmetric matrices, \( A \in \text{Mat}_{m \times m}^{\text{skew-sym}}(R) \), have special properties, see e.g., theorem 3.8 in [Ko,La,Sw]. Recall the Pfaffian ideal, \( Pf(A) \), and its generalizations \( Pf_i(A) = \) the ideal generalized by Pfaffians of the principal \( i \times i \) submatrices of \( A \). We use the following:

- For \( m \)-even: \( I_m(A) = Pf(A)^{2} \) and \( I_{m-1}(A) = Pf(A) \cdot Pf_{m-2}(A) \).
- For \( m \)-odd: \( I_m(A) = 0 \) and \( I_{m-1}(A) = Pf_{m-1}(A)^2 \).
- For any \( j \) holds: \( \sqrt{I_j(A)} = \sqrt{I_{2j-1}(A)} \).

### 3.6. Annihilator of cokernel.

**[Eisenbud] [20]** Consider \( A \in \text{Mat}_{m \times n}(m) \) as a presentation matrix of its cokernel, \( R^m \xrightarrow{A} R^n \to \text{Coker}(A) \to 0 \).

The support of the module \( \text{Coker}(A) \) is the annihilator-of-cokernel ideal:

\[
\text{Ann.}\text{Coker}(A) = \text{Ann}\left( \frac{R^m}{\text{Im}(A)} \right) = \{f \in R \mid f \cdot R^m \subseteq \text{Im}(A)\} \subset R.
\]

This ideal is \( G_r = GL(m, R) \times GL(n, R) \)-invariant and refines the ideal \( I_m(A) \).

The annihilator-of-cokernel is a rather delicate invariant but it is controlled by the ideals \( \{I_j(A)\} \), see [Eisenbud] proposition 20.7] and [Eisenbud] exercise 20.6]:

**Properties 3.4.**

1. \( \forall j < m: \text{Ann.}\text{Coker}(A) \cdot I_j(A) \subseteq I_{j+1}(A) \), and \( \text{Ann.}\text{Coker}(A)^n \subseteq I_m(A) \subseteq \text{Ann.}\text{Coker}(A) \subseteq \sqrt{I_m(A)} \).

2. If \( m = n \) and \( \det(A) \in R \) is not a zero divisor, then \( \text{Ann.}\text{Coker}(A) = I_n(A) : I_{m-1}(A) \).

3. If \( m < n \), and \( \text{grade}(I_m) = (n - m + 1) \), then \( \text{Ann.}\text{Coker}(A) = I_m(A) \).

In particular, for one-row matrices, \( m = 1 \), or when \( I_m(A) \) is a radical ideal, \( \text{Ann.}\text{Coker}(A) = I_m(A) \).

We use also the following properties of the ideal \( \text{Ann.}\text{Coker} \):

**Properties 3.5.**

1. (Block-diagonal case) \( \text{Ann.}\text{Coker}(A \oplus B) = \text{Ann.}\text{Coker}(A) \cap \text{Ann.}\text{Coker}(B) \).

2. If \( A \) is a square matrix and \( \det(A) \) is not a zero divisor then \( \text{Ann.}\text{Coker}(A) = \text{Ann.}\text{Coker}(A^T) \).

3. If \( R \) is a unique factorization domain (UFD) and \( A \) is square then \( \text{Ann.}\text{Coker}(A) \) is a principal ideal.

**Proof.**

**Part 1** is immediate.

**Part 2** Let \( f \in \text{Ann.}\text{Coker}(A) \) then \( AB = f \cdot \mathbb{I} \) for some \( B \in \text{Mat}_{m \times m}(R) \). Thus \( A^T \cdot A = f \cdot A^T \), implying: \( \det(A)B \cdot A = f \cdot \det(A) \). As \( \det(A) \) is not a zero divisor we get \( BA = f \cdot \mathbb{I} \), hence \( A^T B^T = f \cdot \mathbb{I} \). Thus \( f \in \text{Ann.}\text{Coker}(A^T) \).

**Part 3** Suppose \( 0 \neq f, g \in \text{Ann.}\text{Coker}(A) \), then for some \( B_1, B_g \in \text{Mat}_{m \times m}(R) \) holds: \( AB_1 = f \cdot \mathbb{I}, AB_g = g \cdot \mathbb{I} \). Thus (as \( g \) is not a zero divisor), by part two we have: \( B_g A = g \cdot \mathbb{I} \). Together we get: \( g \cdot B_1B_g = f \cdot B_g \). Present \( g = \tilde{g} \cdot c \), where \( (c) \supseteq f \), while \( \text{gcd}(\tilde{g}, f) = 1 \), i.e., \( (\tilde{g}) \cap (f) = (\tilde{g} \cdot f) \). Then, as \( R \) is UFD, we get: the entries of \( B_g \) are divisible by \( \tilde{g} \). But then \( A \cdot \frac{1}{f} = \tilde{g} \cdot c \), i.e., \( c \in \text{Ann.}\text{Coker}(A) \), and \( (c) \supseteq f \).

Finally, as \( R \) is UFD there exists a finite decomposition of \( c \) into irreducibles. Thus, after a finite such steps we get a generator of \( \text{Ann.}\text{Coker}(A) \).
Remark 3.6. Part 2 does not hold when $\text{det}(A)$ is nilpotent. For example:

$$R = \mathbb{k}[x,y,z]/(y^2,z^2), \quad A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \quad \text{Ann.Coker}(A) = (yx, xz), \quad \text{Ann.Coker}(A^t) = (xz).$$

Part 3 does not hold for domains with no unique factorization. For example:

$$R = \mathbb{k}[x,y,z,w]/(xy - zw), \quad A = \begin{bmatrix} x & 0 \\ 0 & z \end{bmatrix}. \quad \text{Ann.Coker}(A) = (x \cap (z) = (xz, xy) \text{ is non-principal}.$$  

3.7. The generalization of the annihilator of cokernel. The ideal $\text{Ann.Coker}(A)$ is a ‘partially reduced’ version of the ideal of maximal minors $\text{Im}_m(A)$. Equivalently, the annihilator of a module, $\text{Ann}(M)$, is a refinement of the minimal Fitting ideal of that module, $\text{Fitt}_0(M)$. More generally, the counterparts of the ideals $\{f_i(A)\}$ (or the Fitting ideals $\{\text{Fitt}_{m-j}(M)\}$) are described in [Buchsbaum-Eisenbud], see also [Eisenbud] exercise 20.9.

We recall briefly the definition and the main properties.

Fix a morphism of free $R$-modules, $E \overset{\phi}{\to} F$, here $\text{rank}(F) = m < \infty$. For each $1 \leq j \leq m$ define the associated morphism $E \otimes R^{1 \to j} F \overset{\phi}{\to} F$ by $a \otimes w \mapsto \phi(a) \wedge w$.

**Definition 3.37.** $\text{Ann.Coker}_j(\phi) := \text{Ann.Coker}(\phi_{m+1-j})$, for $1 \leq j \leq m$.

**Properties 3.38.**
1. $\text{Ann.Coker}_j(\phi) = \text{Ann}\left(\frac{m+1-j}{m} \text{ Coker}(\phi)\right)$. In particular, this ideal is fully determined by the module $\text{Coker}(\phi) = \frac{E}{\phi(E)}$.
2. The ideals $\text{Ann.Coker}_j(\phi)$ refine the determinantal ideals, in the following sense:
   i. $\text{Ann.Coker}_j(\phi) = \text{Ann.Coker}_m(\phi) \subseteq \cdots \subseteq \text{Ann.Coker}_j(\phi) \subseteq \cdots \subseteq \text{Ann.Coker}_1(\phi) = I_1(\phi)$.
   ii. For any $i > j \geq 0$ holds: $\text{Ann.Coker}_i(\phi) \subseteq \text{Ann.Coker}_j(\phi)$.
   iii. $\text{Ann.Coker}_j(\phi) \supseteq I_j(\phi) \supseteq \text{Ann.Coker}_j(\phi)$.
   iv. $I_j(\phi) \subseteq \text{Ann.Coker}_j(\phi) \subseteq I_j(\phi) : I_{j-1}(\phi)$.
3. Suppose the map $\phi$ splits block-diagonally, i.e., $E_1 \oplus E_2 \overset{\phi_1 \oplus \phi_2}{\to} F_1 \oplus F_2$. Suppose moreover $\phi_1$ is invertible (thus in particular $\text{rank}(E_1) = \text{rank}(F_1)$). Then $\text{Ann.Coker}_j(\phi) = \text{Ann.Coker}_{\text{rank}(E_1)}(\phi_2)$.
   ii. If $A = \text{diag}(\lambda_1, \ldots, \lambda_m) \in \text{Mat}_{m \times m}(R)$ and $(\lambda_1) \supseteq (\lambda_2) \supseteq \cdots \supseteq (\lambda_m)$ then $\text{Ann.Coker}_j(\phi) = (\lambda_j)$.
4. The ideals $\text{Ann.Coker}_j(\phi)$ are functorial under localizations, i.e., $\text{Ann.Coker}_j(\phi_p) = \text{Ann.Coker}_j(\phi_p)$ for any prime $p \subset R$.
5. Suppose $\text{rank}(\text{Im}(\phi)) < r$, then $\text{Ann.Coker}_j(\phi) = \{0\}$ for $j \geq r$.

Some remarks/explanations are needed here.

1. Fix some bases of $E, F$, so that $\phi$ is presented by a matrix $A \in \text{Mat}_{m \times n}(R)$. Then $\text{Ann.Coker}_j(\phi)$ is invariant under $\text{GL}(m, R) \times \text{GL}(n, R)$-action on $A$. Similarly, fix $A \in \text{Mat}_{m \times n}(R)$ and $B \in \text{Mat}_{m \times n_2}(R)$.
   If $\text{Span}_R(\text{Columns}(A)) = \text{Span}_R(\text{Columns}(B))$ then $\text{Ann.Coker}_j(A) = \text{Ann.Coker}_j(B)$. If $n_1 = n_2$ and $\text{Span}_R(\text{Rows}(A)) = \text{Span}_R(\text{Rows}(B))$ then $\text{Ann.Coker}_j(A) = \text{Ann.Coker}_j(B)$.
2. $\text{i}$. This sequence of inclusions and the equalities are immediate.
   $\text{ii}$. and $\text{iii}$ see [Eisenbud] exercise 20.9 and [Eisenbud] exercise 20.10.
   For $\text{iv}$ see corollary 1.4. of [Buchsbaum-Eisenbud].
3. $\text{i}$. In this case $\text{Coker}(\phi) \approx \text{Coker}(\phi_2)$, now use part 1.
   $\text{ii}$. Follows by explicit check.
4. Follows straight from $\text{Ann}(M_p) = \text{Ann}(M)_{\mathfrak{p}}$.
5. Here $\text{rank}(\text{Im}(\phi)) = \max\{j : I_j(\phi) \neq 0\}$. If $\text{rank}(\text{Im}(\phi)) < r$ then $\text{rank}(\text{Im}(\phi_{m+1-r})) < \text{rank}^{m+1-r}(F)$.
   But then $\text{Ann.Coker}(\phi_{m+1-r}) = \{0\}$.

3.8. The properties of essential singular locus $\text{Sing}_r(J)$. (defined in [1.4]) First we give an explicit presentation. Let $J = (\bigcap), \text{Der}_2(R) = \{\mathcal{D}_\alpha\}$ be any (not necessarily minimal) choices of generators. Then equation (2) gives:

$$\text{Sing}_r(J) = \text{Ann.Coker}_{r} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} \mathcal{D}_\alpha f_1 \\ \mathcal{D}_\alpha f_2 \\ \vdots \end{bmatrix}. \begin{bmatrix} \{D_\alpha f_1\} \\ \{D_\alpha f_2\} \\ \vdots \end{bmatrix}$$

(Here the last column represents the block of columns.)

For the ideal $\text{Sing}_r^{(m)}(J)$ one takes $\mathcal{D}_\alpha \in \text{Der}_2(R, m)$.

**Example 3.9.** For a principal ideal, $J = (f)$, we get the traditional Tjurina ideal of a function,

$$\text{Sing}_r(f) = (f) + \text{Der}_2(R)(f) \subseteq R.$$  

More generally, for $J = (f_1, \ldots, f_N)$ and $R$ regular, equation (11) gives the traditional presentation of the singular locus of $V(J) \subset \text{Spec}(R)$, but with the annihilator scheme structure instead of the Fitting ideal.
3.8.1. Basic properties of $\text{Sing}_r(J)$. Though the definition involved various choices of generators, $\text{Sing}_r(J)$ depends on the ideal $J$ only. Moreover, $\text{Sing}_r(J)$ localizes nicely and has other good properties.

**Lemma 3.10.** Let $R$ be a commutative unital ring. Fix an ideal $J \subseteq R$ and some $r \in \mathbb{N}$.

1. The ideal $\text{Sing}_r(J)$ does not depend on the choice of the generators of $J$, $\text{Der}_k(R)$.

2. If $J_1 \subseteq J_2$ then $\text{Sing}_r(J_1) \subseteq \text{Sing}_r(J_2)$.

3. $\text{Sing}_r(J) \supseteq J + \text{Ann}_r R^{\otimes N} / \text{Der}_k(R)(J)$, and the inclusion can be proper.

4. For any prime ideal $p \subseteq m$ the localization satisfy: $\text{Sing}_r(J)_p = \text{Sing}_r(J_p)$. If $p \subsetneq m$ then $\text{Sing}_r^{(m)}(J)_p = \text{Sing}_r(J_p)$.

5. For any $A \in \text{Mat}_{m \times n}(R)$ and any $1 \leq r, j \leq m$ holds: $I_j(A) \subseteq \text{Sing}_r(I_j(A)) \subseteq I_j(A) + \text{Der}_k(R)(I_j(A))$.

**Proof.**

1. Suppose $(f_j, \tilde{f}_j)$ are two (finite) tuples of generators of $J$. We can assume (extending by zeros) that they are of the same length. The two tuples are related by $f = U \tilde{f}, \tilde{f} = V f$, for some square $R$-matrices $U, V$.

Then the matrices in equation (11), for $f, \tilde{f}$, are related by the left-right multiplication by some $R$-matrices. Hence we get: $\text{Sing}_r(J) \subseteq \text{Sing}_r^{(J)}(J)$, and thus $\text{Sing}_r(J) = \text{Sing}_r^{(J)}(J)$.

2. Immediate, just notice that for $\{f_i\} \subseteq J_1$ holds: $D_{\alpha}(\sum a_i f_i) \equiv \sum a_i D_{\alpha}(f_i) \text{ mod } (J_1)$.

3. The inclusion is obvious from the presentation in equation (11), and the following example shows the possible inequality. Let $k$ a field of zero characteristic and take $J = (x^7 + y^8, x^8 + y^9) \subset k[x, y]$. The height of this ideal is two. We claim that $\text{Sing}_2(J) \supseteq (x^8, y^9)$. Indeed:

$$\text{Sing}_2(J) = \text{Ann.Coker} \begin{bmatrix} x^7 + y^8 & x^8 + y^9 & x^7 + y^8 & 0 & 0 & 0 & 7x^6 & 8y^7 \\ 0 & 0 & x^7 + y^8 & x^8 + y^9 & 8x^7 & 9y^8 & 0 & 0 \end{bmatrix}.$$  

Denote the columns of this matrix by $\{c_i\}$, then $7c_1 + 8c_2 - x \cdot c_3 = (7y^8, 8y^9)^t$. Together with $c_6$ this gives:

$$\text{Sing}_2(J) = \text{Ann.Coker} \begin{bmatrix} x^7 & y^8 & 0 & 0 & 0 & 0 & 7x^6 & 8y^7 \\ 0 & 0 & x^8 & y^9 & x^7 + y^8 & 8x^7 & 9y^8 & 0 \end{bmatrix}.$$  

Thus $\text{Sing}_2(J) \supseteq (x^8, y^9)$. But

$$J + \text{Ann}_2 R^{\otimes 2} / \text{Der}_k(R)(J) = (x^7 + y^8, x^8 + y^9, 63x^6 y^8 - 64x^7 y^7) = (x^7 + y^8, xy^8 + y^9, m^6 \cdot y^8).$$

(The last transition uses the Gröbner basis.) From here one sees that e.g. $\text{Sing}_2(J) \supseteq x^8 \notin J + \text{Ann}_2 R^{\otimes 2} / \text{Der}_k(R)(J).$

4. The equality $\text{Sing}_r(J)_p = \text{Sing}_r(J_p)$ holds because the annihilator is functorial on localizations, $\text{Ann}_r(M_p) = \text{Ann}_r(M), $ and the module of derivations as well (see [3.3]).

If $p \subseteq m$ then $\text{Der}_k(R, M)_p \supseteq (m \cdot \text{Der}_k(R))_p = \text{Der}_k(R)_p = \text{Der}_k(R_p)$. Therefore $\text{Sing}_r^{(m)}(J)_p = \text{Sing}_r(J_p)$.

5. The inclusion $I_j(A) \subseteq \text{Sing}_r(I_j(A))$ holds by part 3. For the inclusion $\text{Sing}_r(I_j(A)) \subseteq I_j(A) + \text{Der}_k(R)(I_j(A))$ we have:

$$\text{Ann}_r R^{\otimes N} / I_j(A) \cdot R^{\otimes N} + \text{Der}_k(R) \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} \subseteq \text{Ann}_r R^{\otimes N} / I_j(A) \cdot R^{\otimes N} + \text{Der}_k(R)(I_j(A)) \cdot R^{\otimes N} = I_j(A) + \text{Der}_k(R)(I_j(A)).$$

**Remark 3.11.** Note that $\text{Der}_k(R)(I_{j+1}(A)) \subseteq I_j(A) \cdot \text{Der}_k(R)(I_{j+1}(A))$. (Expand the $(j+1) \times (j+1)$ minors in terms of $j \times j$ minors.) Therefore the upper bound of part 5 of this lemma implies: the singular locus of $V(I_{j+1}(A))$ contains $V(I_j(A))$. The inclusion $V(\text{Sing}_r(I_{j+1}(A))) \supseteq V(I_j(A))$ is often proper. However, if $\text{Spec}(R)$ is smooth and $A$ is generic then the two sets coincide.

**Lemma 3.12.** Suppose $R$ is a local, Noetherian ring.

1. When working with radicals one can replace the annihilator of cokernel by determinantal ideal, $\sqrt{\text{Sing}_r(J)} = \sqrt{J + I_r(\text{Der}_k(R)(J))}$.

2. Suppose $p \subset R$ is a minimal associated prime of $J$ and $\text{grade}(p) < r$ and $R_p$ is a regular ring. Then $\text{Sing}_r(J)_p = J_p$. 

3. Suppose the prime decomposition is \( \sqrt{J} = (p_{\alpha}) \cap (q_{\beta}) \), where \( \text{grade}(p_{\alpha}) < r \) and \( \text{grade}(q_{\beta}) = r \).

Take the corresponding localizations \( R_{p_{\alpha}} \) and suppose the rings \( \{R_{p_{\alpha}}\} \) are regular. Then
\[
\text{Sing}_r(J) \subseteq \cap_{\alpha} i_{\alpha}^{-1}(J_{p_{\alpha}}).
\]

**Proof.**

1. By lemma 3.2 it is enough to verify that for any prime ideal \( p \subseteq m \) holds:
\[
(17) \quad \text{Sing}_r(J)_p = R_p \iff \left( J + I_r(Der_k(R)(J)) \right)_p = R_p.
\]

If \( p \nsubseteq J \) then both sides are \( R_p \), as both sides contain \( J \), and \( J_p = R_p \) by lemma 3.11. If \( p \supseteq J \) then
\[
(18) \quad \text{Sing}_r(J)_p = R_p \iff \left( \text{Ann}_r R^{\oplus N}/Der_k(R)(J) \right)_p = R_p \iff I_r(Der_k(R)(J))_p = R_p.
\]

2. By part 4 of lemma 3.10 we can localize at \( p \). Thus we can assume: \((R, m)\) is a regular local ring and \( J \subseteq m \). Denote by \( x = (x_1, \ldots, x_n) \) a minimal set of generators of the ideal \( m \subset R \). By the regularity, \( n = \dim(R) < r \).

Fix some generators \( (f_1) \) of \( J \), we have
\[
(19) \quad \text{Sing}_r(J) = \text{Ann}_r R^{\oplus N}/J \cdot R^{\oplus N} + Der_k(R) \left[ \frac{f_1}{f_N} \right] = \text{Ann}_r \left( \frac{R/J}{J \otimes Der_k(R)} \right) \left[ \frac{f_1}{f_N} \right] =: \text{Ann.Coker}_r \left[ \frac{Der_k(R)}{J} \right].
\]

Extend the \( N \)-tuple \( (f_1, \ldots, f_N) \) to the \( N+n \)-tuple \( (f_1, \ldots, f_N, 0, \ldots, 0) \), and compare it to the \( N+n \)-tuple \( (f_1, \ldots, f_N, x_1, \ldots, x_n) \). The latter is a (non-minimal) system of generators of \( m \subset R \). Therefore
\[
(20) \quad \text{Ann.Coker}_r \left[ \frac{Der_k(R)}{J} \right] = \text{Ann.Coker}_r \left[ \frac{Der_k(R)}{J} \right] \subseteq \text{Ann.Coker}_r \left[ \frac{Der_k(R)}{J} \right] = \text{Ann.Coker}_r \left[ \frac{Der_k(R)}{J} \right].
\]

Here the two equalities hold by part 1 of lemma 3.10 while the central inclusion holds because \( J \subseteq m \).

We have obviously \( \text{Ann.Coker}_r \left[ \frac{Der_k(R)}{J} \right] \supseteq J \) and it remains to prove the equality. As the ring \((R, m)\) is Noetherian, the completion is faithful. Therefore it is enough to check
\[
(21) \quad R \otimes \text{Ann.Coker}_r \left[ \frac{Der_k(R)}{J} \right] = R \cdot J \subset R.
\]

By Cohen structure theorem \( \bar{R} = \mathbb{K}[[x]], \) where \( \mathbb{K} \supseteq k \) is a field. Therefore \( Der_k(\mathbb{K}) = \bar{R}(\partial_1, \ldots, \partial_n) + \bar{R}.\)

Here \( \partial_j \) are the classical partial derivatives, while \( Der_k(\mathbb{K}) \) consists of derivations of \( \mathbb{K} \), thus \( Der_k(\mathbb{K})(x) = 0 \).

Therefore we have \( Der_k(\bar{R})(x) = 1_{n \times n} \). Finally, as \( n < r \), we get:
\[
(22) \quad \bar{R} \otimes \text{Ann.Coker}_r \left[ \frac{Der_k(R)}{J} \right] \subseteq \text{Ann.Coker}_r \left[ \frac{Der_k(\bar{R})}{J} \right] = \bar{R} \cdot J.
\]

Therefore, for the initial ring, \( \text{Ann.Coker}_r \left[ \frac{Der_k(R)}{J} \right] = J \).

3. Follows straight from the previous part, just notice \( \text{Sing}_r(J) \subseteq \cap_{\alpha} i_{\alpha}^{-1}(\text{Sing}_r(J_{p_{\alpha}})). \)

**Remark 3.13.**

i. Parts 2,3 read geometrically: the essential singular locus contains all the components of the subscheme \( V(J) \subset \text{Spec}(R) \) that are not of expected codimension. (Even if these components are smooth in the classical sense.)

ii. The inequality in part 3 does not hold, even when \( \sqrt{J} = p \), with \( \text{grade}(p) < r \). For example, let \( R = \mathbb{K}[x, y] \supset J = (x^p, xy^q) \), of grade 1. Then \( \text{Sing}_2(J) = (x^p, xy^q) \). But \( \sqrt{J} = (x) \) and \( J(x) = (x) \), thus \( i_{\alpha}^{-1}J(x) = (x) \subset R \).

3.8.2. **Relation of Sing_\alpha(J) to the classical singular locus of the subscheme V(J) C Spec(R).** The singular locus is classically defined using the module of Kähler differentials, \( \Omega^1_{R/J} \), with the Fitting ideal structure, \( \text{Fitt}_{\dim(R/J)} \Omega^1_{R/J} \subset R \).

For complete rings in zero characteristic the module of differentials is often pathological, e.g. uncountably generated, see e.g., §11 of [Knu]. Thus we work in this subsection with universally finite differentials/separated differentials.
For regular rings, the ideal $\text{Sing}(J) \subset R$ is the refinement of the classical ideal $\text{Sing}(V(J))$, with the annihilator instead of Fitting scheme structure:

**Lemma 3.14.** Suppose $R$ is a complete regular local Noetherian ring of dimension $n$ and $J \subset R$ is pure of height $r$. Then $\text{Sing}(J) = \text{Ann}_R \Omega^1_{R/J} \subset R$.

**Proof.** For a complete regular local ring $R$ of dimension $n$, and $J = (f) \subset R$, the conormal sequence gives, 

\[(26) \quad \Omega^1_{R/J} = R/J \otimes \frac{R[dx_1, \ldots, dx_n]}{\sum \phi(\partial f_i dx_i)_{j=1,\ldots,N}}.\]

As both $\text{Sing}(J)$ and $\text{Ann}_R \Omega^1_{R/J}$ contain $J$, we compare their images in $R/J$. We have the presentation \((R/J)^N \xrightarrow{\phi} (R/J)^n \xrightarrow{\partial} \Omega^1_{R/J} \to 0\), with the presentation matrix

\[(24) \quad \mathcal{A} = \begin{bmatrix} \partial f_1 & \cdots & \partial f_N \\ \partial f_1 & \cdots & \partial f_N \\ \vdots & \cdots & \vdots \\ \partial f_1 & \cdots & \partial f_N \end{bmatrix}.\]

Here $\mathcal{A}$ is the transpose of the block of derivatives in equation (26). Now we notice that $I_r(\mathcal{A})$ contains a non-zero divisor modulo $I_{r-1}(\mathcal{A})$, therefore, by part 2.iv of proposition 3.4,

\[(25) \quad \text{Fitt}_{r-1}(\Omega^1_{R/J}) = I_r(\mathcal{A}) = I_r(\mathcal{A}') = \text{Ann}_r(\mathcal{A}') = R/J \otimes \text{Sing}_r(J).\]

But in general the two ideals differ essentially, even their radicals differ.

**Example 3.15.** i. (The case of non-pure ideal) Let $R = k[[x,y,z]] / J = (xz, xy)$. Then

$$
\text{Fitt}_{\dim(R/J)}(\Omega^1_{R/J}) = \text{Fitt}_2 \left( \frac{R[dx, dy, dz]}{(dx)(dy), dz} \otimes R/J \right) = I_1 \begin{bmatrix} J & z & y \\ 0 & J & 0 \end{bmatrix} = (x, y, z) \subset k[[x,y,z]](xz,xy).
$$

On the other hand, the expected grade is 2 and:

$$
\text{Sing}_2(J) = \text{Ann}_2(\text{Coker}_2) \begin{bmatrix} J & z & 0 \\ 0 & J & x \end{bmatrix} = (x) \subset k[[x,y,z]].
$$

Thus $\sqrt{\text{Fitt}_2(\Omega^1_{R/J})} \subset \text{Sing}_2(J)$.

We observe also: $\text{Sing}_1(J) = \text{Ann}_1(\mathcal{A}_1) \begin{bmatrix} J & z & 0 \\ 0 & J & x \end{bmatrix} = (x, y, z) \subset k[[x,y,z]]$.

ii. (The case of non-regular rings) Let $R = k[[x,y,z]](xz) \supset J = (z)$. Here $\text{Der}_k(R) = R[\partial_x, \partial_y, \partial_z]$, thus

$\text{Sing}_1(J) = \text{Ann}_1(\mathcal{A}_1) \begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix} = (x, y, z) \subset k[[x,y,z]]$.

3.9. **Invariance of $\text{Ann}(T^1)$**. An element $h = (U, V, \phi) \in \mathcal{G}_r$ acts on $R$ by $f \to \phi(f) = J \to \phi(J)$.

Suppose $h \in \mathcal{G}_r$ acts on a submodular $\Sigma \subseteq \text{Mat}_{m \times n}(R)$, thus it sends the pair $(\Sigma, A)$ to the pair $(\Sigma, hA)$.

**Lemma 3.16.** Let $G$ be one of the groups $G_1, G_2, G_4, \text{Aut}_k(R), \mathcal{G}_r, \mathcal{G}_r, \mathcal{G}_r, \mathcal{G}_r, \mathcal{G}_r$. Suppose $h = (U, V, \phi) \in \mathcal{G}_r$ acts on $\Sigma$ and also acts on the $G$-orbits of $A$, i.e., $h(GA) = G(hA)$; then $\phi(\text{Ann}(T^1_{(\Sigma, G, hA)})) = \text{Ann}(T^1_{(\Sigma, G, hA)})$.

**Proof.** Consider $h$ as a $k$-linear automorphism of $\text{Mat}_{m \times n}(R)$. It induces the isomorphism of the tangent spaces, the first row of the diagram. Its restriction $(\Sigma, A) \to T_{(\Sigma, hA)}(\Sigma, hA)$ induces the second row. The restriction $(GA, A) \to T_{(GA, A)}(\Sigma, A)$ induces the third row.

If $h \in \mathcal{G}_r$, then the map $h_*$ is $R$-linear, i.e., $\phi = 1d$. If $h \in \mathcal{G}_r$, then the map is $R$-multiplicative:

\[(26) \quad h_*(f \cdot T_{(\Sigma, A)}) = \phi(f) \cdot h_*(T_{(\Sigma, A)}), \quad h_*(f \cdot T_{(GA, A)}) = \phi(f) \cdot h_*(T_{(GA, A)}).\]

Thus $h$ induces the isomorphism (of $k$-modules) $T^1_{(\Sigma, G, hA)} \xrightarrow{h_*} T^1_{(\Sigma, G, hA)}$ that satisfies: $h_*(f \cdot T^1_{(\Sigma, G, hA)}) = \phi(f) h_*(T^1_{(\Sigma, G, hA)})$. In particular, if $f \in \text{Ann}(T^1_{(\Sigma, G, hA)})$ then $\phi(f) \in \text{Ann}(T^1_{(\Sigma, G, hA)})$, i.e., $h^* \text{Ann}(T^1_{(\Sigma, G, hA)}) \subseteq \text{Ann}(T^1_{(\Sigma, G, hA)})$. As $h$ is invertible, we get the inverse inclusion as well. \hfill \blacksquare

**Example 3.17.** i. The assumptions of this lemma are obviously satisfied when $h \in G$. This is used to bring $A$ to a particular form in the proof of theorems 2.11 2.2.
ii. In many cases no choice of $h \in G$ helps, e.g., $A$ has no nice canonical form under the $G$-action. Then one takes $h$ in the normalizer of $G$, to ensure $hGA = GhA$. For example, we use the following normal extensions: $G_1, G_2 \triangleleft G$, $SGL(m, k) \times G_1, G_2 \triangleleft G \times GL(n, k)$.

iii. Note that $h \in G_{\mathbb{G}} \setminus \{GL(m, k) \times G_r\}$ does not in general normalize the $G_r$-action. Similarly for $h \in G_r \setminus \{G_r \times GL(n, k)\}$.

4. PROOFS OF THE MAIN RESULTS

4.1. The $G_{\mathbb{G}}$-action.

Proof. (of theorem 3.11)

1. Fix some $A = (a_1, \ldots, a_n) \in Mat_{1 \times n}(R)$. The tangent space $T_{(G_{\mathbb{G}}, A,A)}$ is written in $[3.2]$. We record the generating matrix of the submodule $T_{(G_{\mathbb{G}}, A,A)} \subseteq T_{(\Sigma, A)} = Mat_{1 \times n}(R):$

\[
\begin{pmatrix}
A & \emptyset & \ldots & \ldots & \emptyset & \{D_{a_1}\} \\
\emptyset & A & \emptyset & \ldots & \emptyset & \{D_{a_2}\} \\
\ldots & \ldots & \ldots & \ldots & \emptyset & A \{D_{a_n}\}
\end{pmatrix}
\]

(The right column here denotes the block of columns, as $D$ runs over the generators of $Der_k(R, m)$.)

Thus $T_{(\Sigma, G_{\mathbb{G}}, A)}$ is the cokernel of this matrix, while the annihilator of $T_{(\Sigma, G_{\mathbb{G}}, A)}$, i.e., the $Ann.Coker$ of this matrix, is precisely $Sing^m(I_1(A))$. This proves the equality $Ann(T_{(\Sigma, G_{\mathbb{G}}, A)}) = Sing^m(I_1(A))$. The embedding $Sing(I_1(A), A, A) \supseteq I_1(A) + Ann.Coker(Der_k(R, m), (A), (A))$, follows by part 3 of lemma 3.10.

2. Fix a prime ideal $p$. As $I_m(A) \nsubseteq (p)$, we get $I_m(A) \subseteq (p)p$ thus $I_m(A)p = R_p$. But then at least one of the $(m-1) \times (m-1)$ minors of $A$ becomes invertible in $R_p$. Therefore the localized matrix is equivalent to a block-diagonal, $A_p \sim \begin{pmatrix} I_{(m-1)\times(m-1)} & 0 \\ 0 & A \end{pmatrix}$.

We assume $A_p$ in this form, by $[3.2]$ such a transition preserves $Ann(T_{(\Sigma, G_{\mathbb{G}}, A)})$. Then the tangent space to the orbit (see $[3.2]$) decomposes into the direct sum:

\[
(T_{(G_{\mathbb{G}}, A,A)})_p = Mat_{m \times m}(R_p) \cdot A + A_p \cdot Mat_{n \times n}(R_p) + Der_k(R_p, m, (A)) = Mat_{(m-1)\times n}(R_p) \oplus Mat_{1 \times (m-1)}(R_p) \oplus \left( I_{(m-1)\times (n-m+1)} \oplus (p)^{n-m+1} \right).
\]

We use this direct sum decomposition, together with the corresponding direct sum decomposition of $T_{(\Sigma, A)}$, to get:

\[
Ann(T_{(\Sigma, G_{\mathbb{G}}, A)})_p = Ann(T_{(\Sigma, G_{\mathbb{G}}, A)}) = Ann(T_{(\Sigma, A)})_p 
\]

\[
\approx Ann(Mat_{1 \times (n-m+1)}(R_p))_p \approx \left( I_{(m-1)\times (n-m+1)} \oplus (p)^{n-m+1} \right) = Sing_{n-m+1}(I_1(A)),
\]

Note that $I_1(A) = I_m(A) \subseteq (p)^{n-m+1},$ and the expected height for these ideals is $(n-m+1)$. Altogether:

\[
Sing_{n-m+1}(I_1(A)) = Sing_{n-m+1}(I_m(A)p) = Sing_{n-m+1}(I_m(A)),
\]

(For the last step we use $p \nsubseteq m$ and part 4 of lemma 3.11)

Thus $Ann(T_{(\Sigma, G_{\mathbb{G}}, A)})_p = Sing_{n-m+1}(I_m(A))_p$.

3. The embedding

\[
Ann(T_{(\Sigma, G_{\mathbb{G}}, A)}) \supseteq Ann.Coker(A) + Ann(Mat_{m \times n}(R)) / Der_k(R, m, (A))
\]

holds because $G_{\mathbb{G}} \supset G \times Aut_k(R)$ gives $T_{(G_{\mathbb{G}}, A,A)} \supseteq T_{(G, A,A)} + T_{(Aut_k(R), A,A)}$, and $Ann(T_{(\Sigma, G_{\mathbb{G}}, A)}) = Ann.Coker(A)$.

For any $0 \leq j < m$ we have to prove: $Ann(T_{(\Sigma, G_{\mathbb{G}}, A)}) \subseteq Sat_{I_j(A)}(Sing_{m-j}(I_{m-j}(I_{j+1}(A))))$.

Note that $Ann(T_{(\Sigma, G_{\mathbb{G}}, A)}) \subseteq Sat_{I_j(A)}(Ann(T_{(\Sigma, G_{\mathbb{G}}, A)}))$, thus it is enough to prove:

\[
Sat_{I_j(A)}(Ann(T_{(\Sigma, G_{\mathbb{G}}, A)})) \subseteq Sat_{I_j(A)}(Sing_{m-j}(I_{m-j}(I_{j+1}(A))))
\]

Now, by part 3 of lemma 3.11 it is enough to verify the embedding

\[
Ann(T_{(\Sigma, G_{\mathbb{G}}, A)})_p \subseteq Sing_{m-j}(I_{m-j}(I_{j+1}(A)))_p
\]

for any prime ideal $p$ with $p \nsubseteq I_j(A)$.

Note that $Sing_{m-j}(I_{m-j}(I_{j+1}(A)))_p \supseteq I_{j+1}(A)$. Thus if $p \nsubseteq I_{j+1}(A)$ then $Sing_{m-j}(I_{m-j}(I_{j+1}(A)))_p = R_p$. Therefore it is enough to check only the case when $I_j(A) \nsubseteq p \supseteq I_{j+1}(A)$. 

The proof below is similar to that of part two, just for $j < m - 1$ we obtain weaker statements. Take such a prime $p$, then $(I_j(A))_p = R_p$, by part 1 of lemma 3.1. Thus at least one $j \times j$ minor of $A$ is invertible in $R_p$. Therefore the localization of $A$ is block-diagonalizable, $A_p \cong (\mathbb{C}_p)^{13} \oplus \tilde{A}$, where $\tilde{A} \in \text{Mat}_{m-j \times (n-j)}(R_p)$. By the invariance of annihilator, we assume $A_p$ in this form.

Note that $I_1(\tilde{A}) = I_{j+1}(A_p) = I_{j+1}(A_p)$ and, as $I_j(A) \subseteq p$, we get $I_{j+1}(A) \subseteq (p)_p$, i.e., none of the entries of $\tilde{A}$ is invertible in $R_p$. As in part two we decompose the tangent space to the orbit into the direct sum.

$$
(13.1) \quad (T_{(\mathcal{G}_r,A)} \cdot I_p) = \text{Mat}_{m \times m}(R_p) \cdot (A_p + A_p \cdot \text{Mat}_{n \times n}(R_p) + \text{Der}_R(R_p, m_p)(A_p)) = \\
= \text{Mat}_{n \times n}(R_p) \oplus \text{Mat}_{(n-j) \times (n-j)}(R_p) \oplus \left( \text{Mat}_{(n-j) \times (n-j)}(R_p) \cdot \tilde{A} + \tilde{A} \cdot \text{Mat}_{(n-j) \times (n-j)}(R_p) + \text{Der}_R(R_p, m_p)(\tilde{A}) \right).
$$

We simplify the annihilator according to this decomposition:

$$
\text{Ann}(T_{(\mathcal{G}_r,A)})_p = \text{Ann}(T_{(\Sigma, A)})_p = \\
\approx \text{Ann}(T_{(\Sigma, A)})_p \setminus T_{(\mathcal{G}_r, A)}.
$$

Unlike part two, for $m-j > 1$ we cannot "pack" the last term in a nice form. Instead we enlarge the annihilator ideal by observing that $T_{(\mathcal{G}_r, A)} \subseteq \text{Mat}_{(n-j) \times (n-j)}(I_1(\tilde{A}))$. (This is equality for $m-j = 1$, but can be a proper embedding for $m-j > 1$.) Therefore:

$$
\text{Ann}(T_{(\Sigma, A)})_p \subseteq \text{Ann}(T_{(\Sigma, A)})_p \setminus \text{Mat}_{(n-j) \times (n-j)}(I_1(\tilde{A})) + \text{Der}_R(R_p, m_p)(\tilde{A}) = \\
= \text{Sing}_{m_p}(\text{Der}_R(R_p, m_p)(\tilde{A})).
$$

Now, as in part two, we observe: $I_1(\tilde{A}) = I_{j+1}(A_p) = I_{j+1}(A)_p$. Therefore:

$$
\text{Ann}(T_{(\Sigma, A)})_p \subseteq \text{Sing}_{m_p}(I_{j+1}(A)) = \left( \text{Sing}_{m_p}(I_{j+1}(A)) \right)_p.
$$

As this embedding holds for any localization at $p \nsubseteq I_j(A)$, we get (by part 3 of lemma 3.1):

$$
\text{Ann}(T_{(\Sigma, A)})_p \subseteq \text{Sat}_A \left( \text{Sing}_{m_p}(I_{j+1}(A)) \right).
$$

4. Follows right from the bound of part 3 and (part 3 of) lemma 3.2.

5. The embedding $\subseteq$ follows by applying lemma 3.3 to the embedding $\text{Ann}(T_{(\Sigma, A)})_p \subseteq \ldots$ of part three.

For the embedding $\supseteq$ we use lemma 3.2. Thus it is enough to verify that for any non-maximal prime ideal, $p \subseteq m$, the localizations of the ideals satisfy:

$$
\text{Ann}(T_{(\Sigma, A)})_p \neq R_p \quad \text{then} \quad \bigcap_{j=0}^{m-1} \text{Sat}_A \left( \text{Sing}_{m_p}(I_{j+1}(A)) \right)_p \neq R_p.
$$

Fix a prime ideal $p \subseteq m$ and fix the number $r$ satisfying $p \nsubseteq I_{r+1}(A)$, $p \nsubseteq I_r(A)$. Such $r$ exists, because $I_0(A) = R$, $I_{m+1}(A) = \{0\}$, and is unique, as the chain of ideals $\{I_j(A)\}$ is monotonic. Moreover, there holds: $0 \leq r < m$. If $r = m$ then $I_m(A) \nsubseteq p$ implies $I_m(A)_p = R_p$, hence $\text{Ann}(T_{(\Sigma, A)})_p = R_p$.

It is enough to prove: $\text{Sat}_A \left( \text{Sing}_{m_p}(I_{j+1}(A)) \right)_p \neq R_p$ at least for one value of $j$. Note that $I_{j+1}(A)_p = R_p$ for $j = 1, \ldots, r$, because $I_1(A) \nsubseteq p$, lemma 3.1 Therefore

$$
\text{Sat}_A \left( \text{Sing}_{m_p}(I_{j+1}(A)) \right)_p = R_p \quad \text{for} \quad j = 1, \ldots, r.
$$

Thus we take $j = r$ and prove:

$$
\text{Ann}(T_{(\Sigma, A)})_p \neq R_p \quad \text{then} \quad \text{Sat}_A \left( \text{Sing}_{m_p}(I_{r+1}(A)) \right)_p \neq R_p.
$$

As $p \nsubseteq I_r(A)$ we have $I_r(A)_p = R_p$, so the localization $A_p$ of $A$ has at least one invertible minor of size $r \times r$. Thus $A_p$ is $(G_{ir})_p$-equivalent to $\mathbb{I}_{r \times r} \oplus \tilde{A}$, where $\tilde{A} \in \text{Mat}_{m-r \times (n-r)}(p)$. Recall that $\text{Ann}(T_{(\Sigma, A)})_p$ is invariant under the $G_{ir}$-equivalence, therefore from now on we assume $A_p$ in this form.

For this form of $A_p$ we have the direct sum decomposition

$$
(T_{(\mathcal{G}_r, A)} \cdot I_p) \approx \text{Mat}_{r \times n}(R_p) \oplus \text{Mat}_{m-r \times n}(R_p) \oplus T_{(\mathcal{G}_r, A)}.
$$
as in equations (25) and (33). Thus $Ann(T^1_{(\Sigma,G_{\text{congr}})})_p \approx Ann(T^1_{(\Sigma,G_{\text{congr}})}),$ where $\Sigma = \text{Mat}(m-r) \times (n-r) \cdot R_p$ and $\tilde{G}_t$ is the corresponding group. Therefore we must prove:

$$\text{If } Ann(T^1_{(\Sigma,G_{\text{congr}})})_p \neq R_p \text{ then } \text{Sat}_{I_j(A)}\left(\text{Sing}_{(m-r)(n-r)}(I_j+1(A))\right)_p \neq R_p.$$  

Recall that $T(\tilde{G}_t,\tilde{A}) = \text{Mat}(m-r) \times (n-r) \cdot (R_p) \cdot (\tilde{A} + \tilde{A} \cdot \text{Mat}(m-r) \times (n-r) \cdot (R_p) + \text{Der}_{\Sigma}(R_p)(\tilde{A})$ and all the entries of $\tilde{A}$ belong to $p$. Therefore $T(\tilde{G}_t,\tilde{A}) \subseteq T(\tilde{G}_t,\tilde{A})$ iff $I_j(\tilde{A}) \cdot T(\tilde{G}_t,\tilde{A}) + \text{Der}_{\Sigma}(R_p)(\tilde{A}) \subseteq T(\tilde{G}_t,\tilde{A})$. Thus $Ann(T^1_{(\Sigma,G_{\text{congr}})})_p \neq R_p$ iff $\text{Sing}_{(m-r)(n-r)}(I_j(A)) \neq R_p$. Finally, we observe: $\text{Sing}_{(m-r)(n-r)}(I_j(A)) = \text{Sing}_{(m-r)(n-r)}(I_j(A)_p)$. This proves the implication (41).

Altogether we have:

$$\frac{m-1}{n-j} \cdot \text{Sat}_{I_j(A)}\left(\text{Sing}_{(m-j)(n-j)}(I_j+1(A))\right) = \text{Sat}_{I_0(A)}\left(\text{Sing}_{(m-j)}(I_1(A))\right) = \text{Sing}_{(m-j)}(I_1(A)) \geq I_1(A).$$

On the other hand, suppose $A$ is diagonal, then: $Ann(T^1_{(\Sigma,G_{\text{congr}})}) = Ann.Coker(A)$. Therefore we have trivially

$$\sqrt{Ann(T^1_{(\Sigma,G_{\text{congr}})})} = \sqrt{Ann.Coker(A)} = m = I_1(A) = \sqrt{\frac{m-1}{n-j} \cdot \text{Sat}_{I_j(A)}\left(\text{Sing}_{(m-j)(n-j)}(I_j+1(A))\right)}.$$

But the ideal $I_1(A)$ can be much larger than $Ann.Coker(A)$, and the radicals here cannot be lifted in any universal way (suitable also for $dim(R) > 1$).

4.2. The congruence action, $G_{\text{congr}}.$

Proof. (of theorem 2.2)

1. To verify $Ann(T^1_{(\Sigma,G_{\text{congr}})}) \subseteq \text{nil}(R)$ we prove: for any prime $p \subseteq R$ holds

$$\text{Frac}(R_p) \otimes Ann(T^1_{(\Sigma,G_{\text{congr}})}) = 0 \in \text{Frac}(R_p).$$

This implies: $Ann(T^1_{(\Sigma,G_{\text{congr}})})$ is inside the intersection of all the prime ideals of $R$, thus inside $\text{nil}(R)$. (See e.g., proposition 1.8 of [Atiyah-Macdonald].)

Geometrically we check the vanishing of $Ann(T^1_{(\Sigma,G_{\text{congr}})})$ at the points of $\text{Spec}(R) \setminus 0$.

Thus, for any prime $p \subseteq R$ we study the vector subspace

$$T(\tilde{G}_{\text{congr}},A) \otimes \text{Frac}(R_p) = \text{Span}_{\text{Frac}(R_p)}(UA + AU^T) + \text{Frac}(R_p) \otimes \text{Der}_{\Sigma}(R_p)(A) \subseteq \text{Mat}_{m \times m}(\text{Frac}(R_p)).$$

As $dim(R) < \lfloor \frac{m}{2} \rfloor$ we have $\text{rank}(\text{Der}_{\Sigma}(R_p)) < \lfloor \frac{m}{2} \rfloor$, and therefore $\dim \left(\text{Frac}(R_p) \otimes \text{Der}_{\Sigma}(R_p)(A)\right) < \lfloor \frac{m}{2} \rfloor.

To bound the dimension of $\text{Span}_{\text{Frac}(R_p)}(UA + AU^T)$ we study the following vector space of solutions:

$$\{U \in \text{Mat}_{m \times m}(\text{Frac}(R_p)) \mid UA + AU^T = 0\}.$$  

The later equation is well studied, the dimension of the space of solutions is precisely the codimension of the orbit of $A$ under the congruence. The minimal codimension equals $\lfloor \frac{m}{2} \rfloor$, see e.g., Theorem 3 in [De Terán-Dopico], and it is achieved for $A \in \text{Mat}_{m \times m}(\text{Frac}(R_p))$ generic. Therefore we get:

$$\dim(T(\tilde{G}_{\text{congr}},A) \otimes \text{Frac}(R_p)) \leq m^2 - \lfloor \frac{m}{2} \rfloor.$$}

Therefore $dim(T(\tilde{G}_{\text{congr}},A) \otimes \text{Frac}(R_p)) < m^2$. Hence the vanishing in equation (45).

Therefore $Ann(T^1_{(\Sigma,G_{\text{congr}})})$ lies inside the intersection of all the prime ideals $\subseteq \cap p$. The proofs of the remaining parts are essentially the same as in theorem (2.1), thus we just indicate the main steps.
2. i. Localize at \( p \), then bring \( A_p \) to the block-diagonal form \( A \mathbf{G}^{\cong} \mathbf{Diag} \oplus \mathbf{A} \), where \( \mathbf{Diag} \in \text{Mat}_{(m-1) \times (m-1)}(R_p) \) is invertible, while \( \mathbf{A} \in \text{Mat}_{1 \times 1}(p) \). (See e.g. [Birkhoff-MacLane] theorem 3, page 345.)

Now, as in equation (29), one has

\[
\left( T_{(1, 1)}^{G^{\cong}, \mathbf{A}} \right) \mathbf{R}_p \cong \frac{\mathbf{R}_p \left\langle A \right\rangle + \text{Der}_R(R_p, \mathbf{m}_p) (\mathbf{A})}{\left\langle (\text{det}(A)) + \text{Der}_R(R, \mathbf{m})(\text{det}(A)) \right\rangle}.
\]

Finally, as \( p \subseteq \mathbf{m} \) and \( \text{Der}_R(R, \mathbf{m}) \cong \mathbf{m} \cdot \text{Der}_R(R) \), one has \( \text{Der}_R(R, \mathbf{m}) \mathbf{p} = \text{Der}_R(R_p) \).

2.ii and 2.iv The proof is the same as in theorem 2.1 just replace \( G_t \) by \( G^{\cong} \). We have \( A_p \mathbf{G}^{\cong} \mathbf{Diag} \oplus \mathbf{A} \), where \( \mathbf{Diag} \in \text{Mat}_{1 \times 1}(R_p) \) is invertible. Then the analogue of equation (30) is

\[
\left( \text{Ann}(T_{(1, 1)}^{G^{\cong}, \mathbf{A}}) \right) \mathbf{p} \subseteq \text{Ann} R_p / I(\mathbf{A}) \cong \text{Mat}_{1 \times 1}^*(R_p) / \text{Der}_R(R_p, I(\mathbf{A}))/\left( \text{Sing}_m(\mathbf{m}_p) / I(\mathbf{A}) \right).
\]

2.iii Follows by (part 3 of) lemma 3.12 applied to the bound of part 2.ii.

3. i. (m even) In this case \( A_p \mathbf{G}^{\cong} \mathbf{E} \oplus \mathbf{A} \), where \( E \in \text{Mat}_{2 \times 2}^{\text{skew-sym}}(R_p) \) is invertible, while \( \mathbf{A} \in \text{Mat}_{2 \times 2}^{\text{skew-sym}}(p) \). (See e.g., [Birkhoff-MacLane] exercise 9, page 347.) Thus \( I_m(\mathbf{A}) \mathbf{p} = I_1(\mathbf{A}) \).

As before we get:

\[
\left( T_{(1, 1)}^{G^{\cong}, \mathbf{A}} \right) \mathbf{p} \cong \text{Mat}_{2 \times 2}^{\text{skew-sym}}(R_p) / \{ u \mathbf{A} + \mathbf{A} u \}_{u \in \text{Mat}_{2 \times 2}(R_p)} + \text{Der}_R(R, \mathbf{m}_p) (\mathbf{A}).
\]

Hence \( \text{Ann}(T_{(1, 1)}^{G^{\cong}, \mathbf{A}}) / \mathbf{p} \cong \text{Sing}_1(\mathbf{m}_p) / I(\mathbf{A}) = \text{Sing}_1(\mathbf{m}_p) / I(\mathbf{A}) = \text{Sing}_1(\mathbf{m}_p) / I(\mathbf{A}) \). (The last equality because of \( \mathbf{p} \subseteq \mathbf{m} \).

ii. (m even) The bound \( \cdots \subseteq \text{Ann}(T_{(1, 1)}^{G^{\cong}, \mathbf{A}}) \subseteq \cdots \) is proved as in part 3 of theorem 2.1. One just notes that \( \sqrt{I_2(\mathbf{A})} = \sqrt{I_2(\mathbf{A})} \).

iii. (m odd) In this case we have \( A_p \mathbf{G}^{\cong} \mathbf{E}_{(m-2) \times (m-2)} \oplus \mathbf{A} \), again by [Birkhoff-MacLane] exercise 9, page 347. Therefore \( I_m(\mathbf{A}) \mathbf{p} = I_2(\mathbf{A}) \).

As before we get:

\[
\left( T_{(1, 1)}^{G^{\cong}, \mathbf{A}} \right) \mathbf{p} \cong \text{Mat}_{3 \times 3}^{\text{skew-sym}}(R_p) / \{ u \mathbf{A} + \mathbf{A} u \}_{u \in \text{Mat}_{3 \times 3}(R_p)} + \text{Der}_R(R, \mathbf{m}_p) (\mathbf{A}).
\]

As before, \( \text{Der}_R(R, \mathbf{m}) \mathbf{p} = \text{Der}_R(R) \) as \( p \subseteq \mathbf{m} \).

Recall: \( \{ u \mathbf{A} + \mathbf{A} u \}_{u \in \text{Mat}_{3 \times 3}(R_p)} = \text{Mat}_{3 \times 3}^{\text{skew-sym}}(P f_2(\mathbf{A})) \), see e.g., lemma 3.5 in [BK.18]. Thus

\[
\text{Ann}(T_{(1, 1)}^{G^{\cong}, \mathbf{A}}) / \mathbf{p} \cong \text{Sing}_3(P f_2(\mathbf{A})) = \text{Sing}_3(P f_{m-1}(\mathbf{A}) \mathbf{p}).
\]

iv. For the bound \( \cdots \subseteq \text{Ann}(T_{(1, 1)}^{G^{\cong}, \mathbf{A}}) \) we note that \( T_{(G^{\cong}, 1)}^{1, 1} \mathbf{A} \cong \text{Mat}_{m \times m}^{\text{skew-sym}}(P f_{m-1}(\mathbf{A})) \), see e.g., lemma 3.5. of [BK.16]. The bound \( \text{Ann}(T_{(1, 1)}^{G^{\cong}, \mathbf{A}}) \subseteq \cdots \) is proved as before.

v. As before, this follows by (part 3 of) lemma 3.12 applied to the bound of part 3.iv.

vi. As in the proof of theorem 2.1 we fix a prime ideal \( I_j(\mathbf{A}) \not\subseteq \mathbf{p} \subseteq I_{j+1}(\mathbf{A}) \). For \( j \) odd one has \( \sqrt{I_j(\mathbf{A})} \not\subseteq \mathbf{p} \subseteq I_{j+1}(\mathbf{A}) \), (see 3.3), thus \( \mathbf{p} \) as above exists only for \( j \)-even. Otherwise the proof is the same.

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