Energy Estimates for the Tracefree Curvature of Willmore Surfaces and Applications

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Abstract

We prove an $\varepsilon$-regularity result for the tracefree curvature of a Willmore surface with bounded second fundamental form. For such a surface, we obtain a pointwise control of the tracefree second fundamental form from a small control of its $L^2$-norm. Several applications are investigated. Notably, we derive a gap statement for surfaces of the aforementioned type. We further apply our results to deduce regularity results for conformal minimal spacelike immersions into the de Sitter space $S^{4,1}$.

1. Introduction

1.1. The Willmore energy

We consider an immersion $\Phi$ from a closed Riemann surface $\Sigma$ into $\mathbb{R}^3$. We denote by $g := \Phi^*\xi$ the induced metric on $\Sigma$, with $\xi$ the standard Euclidean metric on $\mathbb{R}^3$, and $d\operatorname{vol}_g$ the volume form associated with $g$. We denote by $\vec{n}$ the Gauss map of $\Phi$, that is, the normal to the surface. In local coordinates $(x, y)$, we have

$$\vec{n} := \frac{\Phi_x \times \Phi_y}{|\Phi_x \times \Phi_y|},$$

with $\Phi_x = \partial_x \Phi$, $\Phi_y = \partial_y \Phi$, and $\times$ is the usual cross product in $\mathbb{R}^3$. The second fundamental form of $\Phi$ is then defined as

$$\vec{A}(X, Y) := A(X, Y)\vec{n} := \langle d^2\Phi (X, Y) , \vec{n}\rangle\vec{n}.$$  

The two key objects of this paper are the mean curvature $H$ and the trace-free second fundamental form $\tilde{A}$ defined as follows:

$$\tilde{H}(p) = H(p)\vec{n} = \frac{1}{2}\operatorname{Tr}_g(A)\vec{n},$$

$$\tilde{A}(X, Y) = \tilde{A}(X, Y)\vec{n} = (A(X, Y) - H(p)g(X, Y))\vec{n}.$$
From these, the Willmore energy is defined as
\[
W(\Phi) := \int_{\Sigma} H^2 \, d\text{vol}_g.
\]
The Willmore energy was introduced in the early XIX\textsuperscript{th} century to study elastic plates. It was identified as a conformal invariant by Blaschke (see \cite{5}) and further studied by Willmore (\cite{33}). It is also linked to the conformal volume of Li-Yau, see \cite{19}. It must be pointed out that the conformal invariance of \( W \) is \textit{contextual}: an inversion centered on a round sphere sends it to a plane with a loss of Willmore energy of \( 4\pi \). The true \textit{pointwise} conformal invariant (see Willmore’s \cite{33}) is rather \( |\hat{A}|_g^2 \, d\text{vol}_g \). The tracefree total curvature is then a conformal invariant, defined as
\[
\hat{E}(\Phi) := \int_{\Sigma} |\hat{\hat{A}}|_g^2 \, d\text{vol}_g.
\]
As a consequence of the Gauss–Bonnet formula, with \( \chi(\Sigma) \) denoting the Euler characteristic of \( \Sigma \), one has that
\[
\hat{E}(\Phi) = 2W(\Phi) - 4\pi \chi(\Sigma).
\] (1)
The contextual conformal invariance of \( W \) is thus to be understood as follows: \( W \) is invariant under the action of conformal transformations that do not change the topology of the surface.

In the present article we will study Willmore immersions, that is critical points of \( W \) (or equivalently \( \hat{E} \)). Willmore immersions form a conformally invariant family satisfying the Willmore equation:
\[
\Delta_g H + |\hat{\hat{A}}|_g^2 H = 0.
\] (2)
Given the lackluster analytic properties of this equation (which is supercritical in the weak framework we will make explicit below), two pivotal results are the small energy estimates (termed \( \varepsilon \)-\textit{regularities}). The first of those is an \textit{extrinsic} result by Kuwert and Schätzle (see Theorem 2.10 in \cite{15}), followed by an \textit{intrinsic} version by Riviére (Theorem I.5 in \cite{30}). We refer the readers to the discussion in the last paragraphs of the introduction of Bernard et al. \cite{4} for an explanation of why these two results do not overlap and fundamentally differ in philosophy, while a concrete example can be found in \cite{23} (see the Proof of Theorem 1.3 in part 2), with further explanations in Remark 3.3.3 of \cite{21}.

The goal of the present paper is to prove an \( \varepsilon \)-regularity result for the tracefree second fundamental form, using T. Riviére’s formalism of weak Willmore immersions. Such a search is motivated first by a similar \textit{extrinsic} result obtained with E. Kuwert and R. Schätzle’s approach (see Theorem 2.9 in \cite{15}), but also by a recent intrinsic \( \varepsilon \)-regularity result for the mean curvature (Theorem 1.4 in \cite{24}). Our \( \hat{\hat{A}} \) \( \varepsilon \)-regularity result would thus complete the extent of possibilities offered by the weak Willmore immersions formalism. Further, the application of the \( H \) \( \varepsilon \)-regularity to minimal bubbling (by eliminating several minimal bubbling configurations, see \cite{23}) unlocks the possibility of applying the \( \hat{\hat{A}} \) version to control the appearance of round spheres as Willmore bubbles (the main results of \cite{2} and \cite{17} can also provide insight on Willmore bubbling).
1.2. Conformal weak Willmore immersions

In this subsection we establish the notation we will adopt and the notions we will use throughout this paper.

Let $\Sigma$ be an arbitrary closed compact two-dimensional manifold and $g_0$ be a smooth “reference” metric on $\Sigma$. The Sobolev space $W^{k,p}(\Sigma, \mathbb{R}^3)$ of measurable maps from $\Sigma$ into $\mathbb{R}^3$ is defined as

$$W^{k,p}(\Sigma, \mathbb{R}^3) := \left\{ f \text{ measurable} : \Sigma \to \mathbb{R}^3 \text{ s.t.} \sum_{l=0}^{k} \int_{\Sigma} \left| \nabla^l g_0 f \right|^p g_0 d\text{vol}_{g_0} < \infty \right\}.$$ 

Since $\Sigma$ is assumed to be compact, this definition does not depend on $g_0$. Now let us recall the notion of weak immersions (see Section 1.2 in Laurain and Riviére’s [17]).

**Definition 1.1.** Let $\Phi : \Sigma \to \mathbb{R}^3$. Let $g_\Phi = \Phi^* \xi$ be the first fundamental form of $\Phi$ and $\vec{n}$ its Gauss map. Then $\Phi$ is called a weak immersion with locally $L^2$-bounded second fundamental form if $\Phi \in W^{1,\infty}(\Sigma)$, if there exists a constant $C_{\Phi}$ such that

$$\frac{1}{C_{\Phi}} g_0 \leq g_\Phi \leq C_{\Phi} g_0,$$

and if

$$\int_{\Sigma} |d\vec{n}|^2 g_\Phi d\text{vol}_\Phi < \infty.$$

The set of weak immersions with $L^2$-bounded second fundamental form on $\Sigma$ will be denoted $\mathcal{E}(\Sigma)$.

Weak immersions are regular enough for us to work with conformal charts thanks to Müller and Šverák work [27] (see also as Section 5 of [12,17]).

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More recently, P. Laurain and T. Rivière have proven the existence of a specific atlas with a control of the conformal factor independent of the conformal class (see Theorem 3.1 of [18]).

**Theorem 1.2.** Let $(\Sigma, g)$ be a closed Riemann surface of fixed genus greater than one. Let $h$ denote the metric with constant curvature (and volume equal to one in the torus case) in the conformal class of $g$ and $\Phi \in \mathcal{E}(\Sigma)$ conformal, that is,

$$\Phi^* \xi = e^{2u} h.$$
Then, there exists a finite conformal atlas $(U_i, \Psi_i)$ and a positive constant $C$ depending only on the genus of $\Sigma$, such that

$$\|\nabla \lambda_i\|_{L^2,\infty(V_i)} \leq C \|\nabla \Phi^* \bar{n}\|_{L^2(\Sigma)}^2,$$

with $\lambda_i = \frac{1}{2} \log \frac{|\nabla \Phi|}{2}$ the conformal factor of $\Phi \circ \Psi_i^{-1}$ in $V_i = \Psi_i(U_i)$.\(^1\)

One can then automatically study any $\Phi \in E(\Sigma)$ in such local conformal charts defined on the unit disk $\mathbb{D}$, as a conformal bilipschitz map $\Phi \in E(\mathbb{D})$. Nevertheless, if the conformal class degenerates when studying a sequence, the chart of the collar will be conformally equivalent to degenerating annuli.

For the sake of brevity, we set once and for all the notation pertaining to $\Phi$ that we will adopt, namely

**Definition 1.2.** Let $\Phi \in E(\mathbb{D})$ be a weak conformal immersion. We will denote the following:

- $\lambda$ its conformal factor, i.e. $e^{2\lambda} dx dy = \Phi^* \xi$,
- $\bar{n}$ its Gauss map,
- $A := \langle \nabla^2 \Phi, \bar{n} \rangle$ its second fundamental form,
- $H := e^{-\lambda} \text{Tr}(A)$ its mean curvature with $\bar{H} := H \bar{n}$,
- $\hat{A} := A - He^{2\lambda} \text{Id}$ its tracefree second fundamental form.

That all these quantities are well defined while requiring as little regularity as possible on $\Phi$ is a key reason to adopt the weak formalism to study Willmore immersions.

**Remark 1.1.** It must be pointed out that $|\hat{A}|^2 d\text{vol}_g = |\hat{A} e^{-\lambda}|^2 dx dy$. The conformal invariant is then written in a local conformal chart: $|\hat{A} e^{-\lambda}|$. It is this quantity which will appear in our estimates.

We now recall the notion of weak Willmore immersions (Definition I.2 in [30]):

**Definition 1.3.** Let $\Phi \in E(\Sigma)$. $\Phi$ is a weak Willmore immersion if

$$\text{div} \left( \nabla \bar{H} - 3\pi \bar{n} \left( \nabla \bar{H} \right) + \nabla^\perp \bar{n} \times \bar{H} \right) = 0 \quad (3)$$

holds in a distributional sense in every conformal parametrization $\Psi : \mathbb{D} \to D$ on every neighborhood $D$ of $x$, for every $x \in \Sigma$. Here the operators div, $\nabla$ and $\nabla^\perp = \left(-\frac{\partial_y}{\partial_x}\right)$ are to be understood with respect to the flat metric on $\mathbb{D}$.

Equation (3) is simply the Willmore Equation (2) written in divergence form well-defined for weak immersions. Thanks to the $\varepsilon$-regularity (Theorem I.5 in [30]) Willmore immersions are known to be smooth and thus to satisfy the Willmore Equation (2) in the classical sense.

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\(^1\) The weak-$L^2$ Marcinkiewicz space $L^{2,\infty}(B_1(0))$ is defined as those functions $f$ which satisfy $\sup_{\alpha > 0} \alpha^2 \left| \left\{ x \in B_1(0) : |f(x)| \geq \alpha \right\} \right| < \infty$. In dimension two, the prototype element of $L^{2,\infty}$ is $|x|^{-1}$. The space $L^{2,\infty}$ is also a Lorentz space, and in particular is a space of interpolation between Lebesgue spaces. See [12] for details.
1.3. Conformal Gauss map

As first shown by Bryant in his seminal work [7], the conformal Gauss map has proven a precious tool in the study of Willmore immersions, and will be pivotal in the present work (see also the nice presentation of Eschenburg’s [11]). We thus briefly review its main properties.

Definition 1.4. Let $\Phi : \Sigma \to \mathbb{R}^3$ be an immersion. The conformal Gauss map of $\Phi$ is a spacelike application $Y : \Sigma \to S^{4,1} \subset \mathbb{R}^{4,1}$ defined as follows:

$$Y := H \left( \frac{|\Phi|^2 - 1}{2|\Phi|^2 + 1} \right) + \left( \frac{\vec{n}}{\langle \vec{n}, \Phi \rangle} \right). \quad (4)$$

Throughout this paper, $S^{4,1}$ denotes the de Sitter space in the Lorentz space $\mathbb{R}^{4,1}$, that is, the set of vectors $v$ satisfying $\langle v, v \rangle_{4,1} = 1$ with $\langle ., . \rangle_{4,1}$ denoting the Lorentz product in $\mathbb{R}^{4,1}$, not to be confused with the Euclidean product $\langle ., . \rangle$.

Differentiating (4), one finds that

$$\nabla Y = \nabla H \left( \frac{|\Phi|^2 - 1}{2|\Phi|^2 + 1} \right) - e^{-2\lambda} \hat{\lambda} \left( \left. \frac{\nabla \Phi}{\langle \nabla \Phi, \Phi \rangle} \right) \right). \quad (5)$$

The conformal Gauss map is deeply linked with conformal geometry, as seen in the following proposition (which is a merger of Theorem 2.4 and Proposition 3.3 in [22]):

Proposition 1.1. Let $\varphi \in \text{Conf}(\mathbb{R}^3)$ and $\Phi : \Sigma \to \mathbb{R}^3$ be a smooth immersion with conformal Gauss map $Y$. We assume the set of umbilic points of $\Phi$ to be nowhere dense. Let $Y_\varphi$ be the conformal Gauss map of $\varphi \circ Y$. Then there exists $M \in SO(4,1)$ such that

$$Y_\varphi = MY.$$ 

Remark 1.2. The connection between $\varphi$ and $M$ is explicitly known, and we refer the reader to equalities (5)–(8) in [22] for details.

The following proposition (see Theorem A of [7] or Theorem 4.2 in [22]) will clarify our interest in the conformal Gauss map in the present context:

Proposition 1.2. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a conformal immersion. Its conformal Gauss map $Y : \mathbb{D} \to S^{4,1}$ is a conformal map with conformal factor $|\hat{\lambda}e^{-\lambda}|^2$. Further, $Y$ is minimal if and only if $\Phi$ is Willmore.
1.4. Summary of main results

The main result of our paper is

**Theorem 1.3.** Let \( \Phi \in \mathcal{E}(\mathbb{D}) \) be a conformal weak Willmore immersion with normal vector \( \vec{n} \) and conformal parameter \( \lambda \). Assume

\[
\int_{\mathbb{D}} |\nabla \vec{n}|^2 \leq \frac{4\pi}{3},
\]

and

\[
\|\nabla \lambda\|_{L^2,\infty(\mathbb{D})} \leq C_0,
\]

for some constant \( C_0 > 0 \).

There exist constants \( \varepsilon_0(C_0) > 0 \) and \( C(C_0) > 0 \) such that, if

\[
\left\| \hat{A}e^{-\lambda} \right\|_{L^2(\mathbb{D})} \leq \varepsilon_0,
\]

then

\[
\left\| \hat{A}e^{-\lambda} \right\|_{L^\infty\left(\mathbb{D}^{1/2}\right)} \leq C \left\| \hat{A}e^{-\lambda} \right\|_{L^2(\mathbb{D})}.
\]

The small energy hypothesis (6) also appears in Theorem I.5 of [30], the difference here is that the resulting inequality only involves the tracefree curvature without relying on the whole second fundamental form. We will in fact establish more than Theorem 1.3 and we show how to recover Theorem I.5 of [30] up to a conformal transformation whose conformal factor is controlled. This is the main idea of the proof: if \( \|\hat{A}\Phi\|_2 \) is small enough we can find a conformal transformation \( T \) such that \( \|\hat{A}\phi\|_2 \) is small, where \( \Phi = T \circ \Phi \). We have to point here the previous results of C.De Lellis and S. Müller, on the control of the second fundamental form on compact surfaces (not necessarily Willmore) when the its total trace free part is small enough, see [8,9].

**Corollary 1.1.** Let \( \Phi \in \mathcal{E}(\mathbb{D}) \) be a conformal weak Willmore immersion.

Assume

\[
\int_{\mathbb{D}} |\nabla \vec{n}|^2 \leq \frac{4\pi}{3},
\]

and

\[
\|\nabla \lambda\|_{L^2,\infty(\mathbb{D})} \leq C_0,
\]

for some constant \( C_0 > 0 \).

There exist constants \( \varepsilon_0(C_0) > 0 \), \( C(C_0) > 0 \) such that if

\[
\left\| \hat{A}e^{-\lambda} \right\|_{L^2(\mathbb{D})} \leq \varepsilon_0,
\]

then one can find a conformal transformation \( \Theta \) such that, setting \( \hat{\Phi} = \Theta \circ \Phi \) and denoting \( \vec{n}_{\hat{\Phi}} \) its Gauss map, one has

\[
\frac{\|\nabla \Phi\|_\infty}{C} \leq \|\nabla \hat{\Phi}\|_\infty \leq C \|\nabla \Phi\|_\infty
\]
and

\[ \| \nabla \hat{n}_\Phi \|_{L^\infty(D_{1/2})} \leq C \| \hat{A} e^{-\lambda} \|_{L^2(D)}. \]

The next achievement of the present paper consists in removing the small energy hypothesis (6) in Theorem 1.3, as indicated in the next statement.

**Theorem 1.4.** Let \( \Phi \in \mathcal{E}(D) \) be a conformal weak Willmore immersion. Assume

\[ \int_D |\nabla \hat{n}|^2 \leq C_0, \]

and

\[ \| \nabla \lambda \|_{L^{2,\infty}(D)} \leq C_0, \]

for some constant \( C_0 > 0 \).

There exists constants \( \varepsilon_0(C_0) > 0 \) and \( C(C_0) > 0 \) such that, if

\[ \| \hat{A} e^{-\lambda} \|_{L^2(D)} \leq \varepsilon_0, \]

then

\[ \| \hat{A} e^{-\lambda} \|_{L^\infty(D_{1/2})} \leq C \| \hat{A} e^{-\lambda} \|_{L^2(D)}. \]

It should be noted that the hypothesis (11) is needed for analytical reasons as a minimal starting control on the metric. We have chosen (11) for convenience, but much weaker hypotheses are possible. Moreover, in the case of application of compact surfaces, it is automatically satisfies thanks to Theorem 1.2.

We will also bring forth two applications of these theorems. First, we will develop a translation of the \( \varepsilon \)-regularity into the conformal Gauss map framework to obtain a result for minimal surfaces into the de Sitter space \( S^{4,1} \).

**Theorem 1.5.** Let \( Y \) be the conformal Gauss map of a conformal weak Willmore immersion satisfying (10) and (11). There exists \( \varepsilon_0 > 0 \) and \( C > 0 \) such that if

\[ \int_D \langle \nabla Y, \nabla Y \rangle_{4,1} \leq \varepsilon_0, \]

then:

\[ \| \langle \nabla Y, \nabla Y \rangle_{4,1} \|_{L^\infty(D_{1/2})} \leq C \int_D \langle \nabla Y, \nabla Y \rangle_{4,1}. \]

It is known that any minimal spacelike immersion in \( S^{4,1} \) is the conformal Gauss map of a Willmore immersion (see for instance Theorem 3.9 of [22]). But it is also well-known that minimal surfaces, and more generally harmonic maps, play a key role in Physics, see [10, 14] for instance. For example, they are at the core of the AdS/CFT correspondence proposed by Maldacena [20], and more recently in his work with Alday [1]. In General Relativity, space-time is represented by a Lorentzian manifold [29] whose de Sitter space \( S^{4,1} \) is one important cosmological model. In this context, the following reformulation of Theorem 1.5 should find important applications, at least since \( \varepsilon \)-regularity is the first step of any asymptotic analysis of sequences of harmonic maps.
Corollary 1.2. Let \( Y : \Sigma \to S^{4,1} \) be a conformal minimal spacelike immersion such that
\[
\int_{\Sigma} \langle \nabla Y, \nabla Y \rangle_{4,1} < \infty.
\]
Then:

- It is the conformal Gauss map of a Willmore immersion \( \Phi \) of finite total curvature.
- Around any point \( p \), one can find a local conformal chart on a disk \( \mathbb{D} \) such that
  \[
  \| \langle \nabla Y, \nabla Y \rangle_{4,1} \|_{L^\infty(\mathbb{D}^{1/2})} \leq C \int_{\mathbb{D}} \langle \nabla Y, \nabla Y \rangle_{4,1}.
  \]

Proof. Since \( Y \) has finite energy, the Willmore immersion \( \Phi \) has finite tracefree total curvature, and thus, owing to the Gauss Bonnet formula, has finite total curvature. Hypotheses (10) and (11) (thanks to Theorem 1.2) are then satisfied on any local disk, and one can apply Theorem 1.5. \( \square \)

This result is to be compared with previous \( \varepsilon \)-regularity results for harmonic surfaces into \( S^{4,1} \), in particular Theorem 1.7 in [34] by M. Zhu:

Theorem 1.6. Any weakly harmonic map \( u \in W^{1,2}(\mathbb{D}, S^{4,1}) \) is Hölder continuous. In particular there exists \( \varepsilon_0 > 0 \) such that if \( \| \nabla u \|_{L^2(\mathbb{D})} \leq \varepsilon_0 \) then
\[
\| \nabla u \|_{L^\infty(\mathbb{D}^{1/2})} \leq C \| \nabla u \|_{L^2(\mathbb{D})}. \tag{14}
\]

The key difference between (13) and (14) lies in the norms they involve. In the case of (13), we make use of the more geometrically meaningful Lorentz norm, while Zhu’s result involves the Euclidean norm, more convenient analytically but with less geometrical meaning. This observation therefore gives Corollary 1.2 a particular advantage, and in that sense improves Theorem 1.7 of [34]. Naturally, our version requires more than a mere harmonicity hypothesis, and one has to further impose a minimality hypothesis, along with (10) and (11) (indeed, Corollary 1.2 is more geometric in nature than analytic). On the other hand, a significant gain is made on the regularity assumption of the underlying Willmore surface: the Euclidean \( L^2 \) norm required by Zhu is tantamount to asking a \( L^2 \) control on \( \nabla H \), which can be understood as a \( W^{3,2} \) control on the immersion. The estimate on \( \langle \nabla Y, \nabla Y \rangle_{4,1} \), however, does require as much. It is furthermore weak enough to be useful in the framework of weak immersions with \( L^2 \) second fundamental form. Despite this weaker regularity input, the output is a remarkable inequality involving Lorentzian quantities.

Our second application is a new gap result, which can be seen as a companion to the gap results of Wheeler and McCoy (see Theorem 1 in [25,31]). While in the latter, it is assumed that the immersion is proper, we instead assume our immersion has bounded energy.
Theorem 1.7. There exists $\varepsilon_0 > 0$ such any complete Willmore surface $\Sigma$ satisfying
\[ \int_{\Sigma} |A|^2 \, d\text{vol}_g < +\infty, \]
and
\[ \int_{\Sigma} |\hat{A}|^2 \, d\text{vol}_g < \varepsilon_0, \]
is totally umbilic, i.e. is either a plane or a round sphere.

Altogether, Theorems 1.3–1.4 offer a new, intrinsic and parametrized $\varepsilon$-regularity result on $\hat{A}$ for Willmore surfaces. It sharply differs from the extrinsic work of Kuwert and Schätzle (Theorems 2.9 and 2.10 in [15]): not only are both statements and proofs different, but in addition the ranges of the respective applications are distinct as well (see Remark 3.3.3 of [21] where it is explained why the configuration found in part 1 of [23] and speculated upon in the introduction of [4] can only be analyzed by the intrinsic formalism). Moreover, we offer new applications to gap results (Theorem 1.7), Semi-Riemannian estimates (Theorem 1.5) and Willmore bubbling (Section 2.6).

Finally, we point out that Theorems 1.3–1.4 complete the set of intrinsic tools for the study of Willmore surfaces: the $\varepsilon$-regularity on $A$ (Theorem 1.5 of [30]) and the $\varepsilon$-regularity on $H$ (Theorem 1.4 [24]) are now completed with the corresponding analogue for $\hat{A}$. As such, our results offer new prospects in understanding more completely sequences of Willmore surfaces, and perhaps even the Willmore flow.

2. Proof of the main theorems

The proof will proceed in four broad steps:

- First, we will use the Gauss–Codazzi formula to deduce a control on how the mean curvature differs from its average.
- We will then apply the same procedure to obtain an analogous control of the conformal Gauss map.
- Using suitable conformal transformations, we will show that the average of the mean curvature can be set to zero.
- Finally, we will show that under a small energy condition, the average of the mean curvature can be cancelled, which ultimately yields the desired estimates.

2.1. Controlling $H$

Proposition 2.1. Let $\Phi \in \mathcal{E}(\mathbb{D})$ be a conformal weak Willmore immersion. Assume
\[ \int_{\mathbb{D}} |\nabla \vec{n}|^2 \leq \frac{4\pi}{3}, \quad (15) \]
and
\[ \|\nabla \lambda\|_{L^2,\infty(\mathbb{D})} \leq C_0, \quad (16) \]
for some constant $C_0 > 0$.

There exists $C > 0$ depending only on $C_0$ such that for any $U \in W^{1,2}_0 \cap L^\infty\left(\mathbb{D}_1/2, \mathbb{R}^2\right)$, we have

$$
\int_{\mathbb{D}_1/2} \langle \nabla H, U \rangle e^{2\lambda} \, dxdy \leq C \left\| \hat{A} e^{-\lambda} \right\|_{L^2\left(\mathbb{D}_1/2\right)} \left\| \nabla U e^{\lambda} \right\|_{L^2\left(\mathbb{D}_1/2\right)}. \tag{17}
$$

**Proof.** Using Lemma II.1 in [3], we know that

$$
\left\| \nabla \lambda \right\|_{L^2\left(\mathbb{D}_1/2\right)} + \left| \lambda - \frac{1}{|\mathbb{D}_1/2|} \int_{\mathbb{D}_1/2} \lambda(x, y) \, dxdy \right| \leq C \left( \left\| \nabla \lambda \right\|_{L^{\infty}(\mathbb{D})}, \left\| \nabla \vec{n} \right\|_{L^2(\mathbb{D})} \right). \tag{18}
$$

In particular, there exists another constant $C > 0$ depending only on $C_0$, such that on $\mathbb{D}_1/2$ one has

$$
\frac{e^{\bar{\lambda}}}{C} \leq e^{\lambda} \leq C e^{\bar{\lambda}} \text{ on } \mathbb{D}_1/2, \tag{19}
$$

where $\bar{\lambda} = \frac{1}{|\mathbb{D}_1/2|} \int_{\mathbb{D}_1/2} \lambda \, dxdy$.

Next, given $U = (U_1, U_2) \in \mathbb{R}^2 \otimes W^{1,2}_0\left(\mathbb{D}_1/2\right)$, we have

$$
\int_{\mathbb{D}_1/2} \langle \nabla H, U \rangle e^{2\lambda} \, dxdy = \int_{\mathbb{D}_1/2} \left( H_x U_1 + H_y U_2 \right) e^{2\lambda} \, dxdy. \tag{20}
$$

Let us now recall the Gauss–Codazzi formula (see (54)) below

$$
\begin{cases}
  e^{2\lambda} H_x = \left( \frac{l-n}{2} \right)_x U_1 + m_y U_1 - \left( \frac{l-n}{2} \right)_y U_2 + m_x U_2 \\
  e^{2\lambda} H_y = - \left( \frac{l-n}{2} \right)_y U_1 + \left( \frac{l-n}{2} \right)_x U_2 + m_x
\end{cases} \tag{21}
$$

where $A = \left( \begin{smallmatrix} l & m \\ m & n \end{smallmatrix} \right)$.

Injecting (21) into (20), one finds

$$
\int_{\mathbb{D}_1/2} \langle \nabla H, U \rangle e^{2\lambda} \, dxdy = \int_{\mathbb{D}_1/2} \left( \frac{l-n}{2} \right)_x U_1 + m_y U_1 - \left( \frac{l-n}{2} \right)_y U_2 + m_x U_2 \\
= \int_{\mathbb{D}_1/2} \nabla \left( \frac{l-n}{2} \right) \cdot \left( \begin{array}{c} U_1 \\ -U_2 \end{array} \right) + \nabla m \cdot \left( \begin{array}{c} U_2 \\ U_1 \end{array} \right) \\
= - \int_{\mathbb{D}_1/2} \left( \frac{l-n}{2} \right) \text{div} \left( \begin{array}{c} U_1 \\ -U_2 \end{array} \right) + m \text{div} \left( \begin{array}{c} U_2 \\ U_1 \end{array} \right). \tag{22}
$$
since \( U \) vanishes on \( \partial D_{1/2} \). We can then deduce that

\[
\left| \int_{D_{1/2}} \langle \nabla H, U \rangle e^{2\lambda} dxdy \right| \leq C \left( \frac{l-n}{2} \left\| \frac{l-n}{2} \right\|_{L^2(D_{1/2})} + \| m \|_{L^2(D_{1/2})} \right) \left\| \nabla U \right\|_{L^2(D_{1/2})}
\]

\[
\leq C \left\| \bar{A} \right\|_{L^2(D_{1/2})} \left\| \nabla U \right\|_{L^2(D_{1/2})}.
\]

(23)

The conclusion follows from (19). \( \square \)

We will use in a decisive way the following result from J. Bourgain and H. Brezis (Theorem 3’ in [6]):

**Theorem.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded set with Lipschitz boundary. If \( f \in L^2(\Omega) \) has null average on \( \Omega \), then there exists \( V \in \mathbb{R}^2 \otimes (W^{1,2}_0 \cap L^\infty)(\Omega) \) such that

\[
\text{div} \ V = f.
\]

Moreover

\[
\| V \|_{L^\infty(\Omega)} + \| \nabla V \|_{L^2(\Omega)} \leq C(\Omega) \| f \|_{L^2(\Omega)}.
\]

(24)

Let

\[
\bar{H} := \frac{1}{|D_{1/2}|} \int_{D_{1/2}} H dxdy.
\]

Applying the above Theorem and (19), we know that there exists \( V \in \mathbb{R}^2 \otimes (W^{1,2}_0 \cap L^\infty)(D_{1/2}) \) such that

\[
\text{div} \ V = H - \bar{H} \quad \text{on} \quad D_{1/2}
\]

and

\[
\| V \|_{L^\infty(D_{1/2})} + \| \nabla V \|_{L^2(D_{1/2})} \leq C e^{-\lambda} \| \bar{H} \|_{L^2(D_{1/2})}.
\]

(25)

Now replacing \( H \) by \( H - \bar{H} \) in (23), setting \( U = e^{-2\lambda} V \), and integrating by parts yields

\[
\int_{D_{1/2}} \left| H - \bar{H} \right|^2 dxdy = \left| \int_{D_{1/2}} \langle H - \bar{H}, \text{div}(V) \rangle dxdy \right| = \left| \int_{D_{1/2}} \langle \nabla H, U \rangle e^{2\lambda} dxdy \right| \leq C \left\| \bar{A} e^{-\lambda} \right\|_{L^2(D_{1/2})} \left\| e^\lambda \nabla U \right\|_{L^2(D_{1/2})}.
\]

(26)

But by (18), (19) and (25), we have

\[
\left\| e^\lambda \nabla U \right\|_{L^2(D_{1/2})} \leq e^{-\lambda} \left( \left\| \nabla V \right\|_{L^2(D_{1/2})} + \left\| \nabla \lambda \right\|_{L^2(D_{1/2})} \left\| V \right\|_{L^\infty(D_{1/2})} \right) \]

\[
\leq C e^{-2\lambda} \| \bar{H} \|_{L^2(D_{1/2})}.
\]

Introducing this into (26) and applying once more (19) yields
Corollary 2.1. Let $\Phi \in \mathcal{E}(\mathbb{D})$ be a conformal weak Willmore immersion. Assume
\[
\int_{\mathbb{D}} |\nabla \tilde{n}|^2 \leq \frac{4\pi}{3},
\]
and
\[
\|\nabla \lambda\|_{L^2,\infty(\mathbb{D})} \leq C_0,
\]
for some constant $C_0 > 0$.
There exists $C > 0$ depending only on $C_0$ such that
\[
\| (H - \bar{H}) e^\lambda \|_{L^2(\mathbb{D}_1^2)} \leq C \| \hat{\Delta} e^{-\lambda} \|_{L^2(\mathbb{D}_1^2)}.
\]

2.2. Controlling $Y$

In this subsection we will abundantly use the conformal Gauss map $Y$, and assume (28) and (19) hold on the whole of $\mathbb{D}$, for notational convenience. In addition, up to a translation, we may assume that $\Phi(0) = 0$ and up to a dilation that $\bar{\lambda} = 0$. Hence,
\[
\exists C_0(\mathcal{C}_0) \text{ s.t. } \frac{1}{C} \leq e^\lambda \leq C,
\]
and
\[
\| \Phi \|_{W^{1,\infty}(\mathbb{D})} \leq C.
\]

Proposition 2.2. Let $\Phi \in \mathcal{E}(\mathbb{D})$ be a conformal weak Willmore immersion. Assume (30) and (31) hold.
Then, there exists $C > 0$ depending only on $C_0$ such that:
\[
\| (Y - \bar{Y}) e^\lambda \|_{L^2(\mathbb{D})} \leq C \| \hat{\Delta} e^{-\lambda} \|_{L^2(\mathbb{D})}.
\]

Proof. Let $U \in W^{1,2}_0 \cap L^{\infty}(\mathbb{D}, \mathbb{R}^5 \otimes \mathbb{R}^2)$. Then, owing to (5), we have
\[
\left| \int_{\mathbb{D}} (\nabla Y, U) e^{2\lambda} \, dxdy \right| \leq C \left\| \hat{\Delta} e^{-\lambda} \right\|_{L^2(\mathbb{D})} \left\| U \right\|_{L^2(\mathbb{D})}.
\]
where we have used (30) and estimate (31). The proof then proceeds as in Corollary 2.1, by setting $\text{div} \, V = Y - \overline{Y}$ and $U = e^{-2\lambda} V$, so as to obtain the announced estimate:

$$\int_D |Y - \overline{Y}|^2 e^{2\lambda} \, dxdy \leq C \| \hat{A} e^{-\lambda} \|^2_{L^2(D)}.$$ 

\[ \square \]

Note that

$$\int_D \langle Y - \overline{Y}, Y - \overline{Y} \rangle_{4,1} \, dxdy = \int_D \left( \langle Y, Y \rangle_{4,1} - 2 \langle Y, \overline{Y} \rangle_{4,1} + \langle \overline{Y}, \overline{Y} \rangle_{4,1} \right) \, dxdy$$

$$= |D| - 2 |D| \langle \overline{Y}, \overline{Y} \rangle_{4,1} + |D| \langle \overline{Y}, \overline{Y} \rangle_{4,1}$$

$$= |D| \left( 1 - \langle \overline{Y}, \overline{Y} \rangle_{4,1} \right),$$

where we have used that $\langle Y, Y \rangle_{4,1} = 1$. It then follows that

$$\langle \overline{Y}, \overline{Y} \rangle_{4,1} = 1 - \frac{1}{|D|} \int_D \langle Y - \overline{Y}, Y - \overline{Y} \rangle_{4,1} \, dxdy. \quad (33)$$

**Corollary 2.2.** Under the hypotheses of Proposition 2.2, there exists $C > 0$ depending only on $C_0$ such that:

$$\langle \overline{Y}, \overline{Y} \rangle_{4,1} \geq 1 - C \| \hat{A} e^{-\lambda} \|^2_{L^2(D)}.$$

Hence, there exists $\varepsilon_0 > 0$ depending only on $C_0$ such that, if

$$\| \hat{A} e^{-\lambda} \|_{L^2(D)} \leq \varepsilon_0,$$

then

$$\langle \overline{Y}, \overline{Y} \rangle_{4,1} \geq \frac{1}{2}.$$

**Proof.** The result is readily obtained from combining Proposition 2.2 with (33). In order to reintroduce the conformal factor, one calls upon (30). Moreover, one uses that for any $v \in \mathbb{R}^{4,1}$, it holds

$$\langle v, v \rangle_{4,1} \leq \langle v, v \rangle.$$ 

\[ \square \]
2.3. Cancelling $\overline{H}$

**Proposition 2.3.** Let $\Phi \in \mathcal{E}(\mathbb{D})$ be a conformal weak Willmore immersion satisfying (27) and (28), while (19) holds on the whole disk $\mathbb{D}$. Then, there exists $\varepsilon_0 > 0$ depending only on $C_0$ such that if

$$\| \bar{A} e^{-\lambda} \|_{L^2(\mathbb{D})} \leq \varepsilon_0,$$

then there exists $\Theta \in \text{Conf}(\mathbb{R}^3)$ such that $\Psi := \Theta \circ \Phi$ still satisfies (28) and (19), as well as

$$\overline{H}_\Psi = 0.$$

**Proof.** Without loss of generality, as previously seen, we can arrange for (30) and (31) to hold (instead of merely (27)). To do so, it suffices to apply suitable translation and dilation. For notational simplicity, the resulting immersion will continue to be denoted by $\Phi$.

Let

$$Y = H \left( \frac{\Phi}{|\Phi|^2 - 1} \right) + \frac{\tilde{n}}{|\tilde{n}, \Phi|} =: (\tilde{Y}_{123}, Y_4, Y_5)$$

be the conformal Gauss map of $\Phi$. We adopt the same notation as in [22], with $\tilde{Y}_{123}(x, y) \in \mathbb{R}^3$, and $Y_4(x, y), Y_5(x, y) \in \mathbb{R}$.

We remark that

$$H = Y_5 - Y_4.$$  \hfill (34)

Theorem 2.4 of [22] ensures that a conformal transformation acting on $\Phi$ induces on $Y$ a change by a matrix $M \in SO(4, 1)$. More precisely, for an inversion $\Psi = \frac{\Phi - \tilde{a}}{|\Phi - \tilde{a}|^2}$ about a point $\tilde{a} \in \mathbb{R}^3$, we find $Y_\Psi = MY$ with

$$M = \begin{pmatrix} -\text{Id} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \text{Id} & \tilde{a} & -\tilde{a} \\ -\tilde{a}^T & 1 - |\tilde{a}|^2/2 & |\tilde{a}|^2/2 \\ -\tilde{a}^T & 1 + |\tilde{a}|^2/2 & -1 - |\tilde{a}|^2/2 \end{pmatrix} = \begin{pmatrix} -\text{Id} & -\tilde{a} & \tilde{a} \\ -\tilde{a}^T & 1 - |\tilde{a}|^2/2 & |\tilde{a}|^2/2 \\ \tilde{a}^T & |\tilde{a}|^2/2 & -1 - |\tilde{a}|^2/2 \end{pmatrix}.$$  \hfill (35)

Hence

$$Y_\Psi = \begin{pmatrix} -\tilde{Y}_{123} + \tilde{a}(Y_5 - Y_4) \\ -\langle \tilde{a}, \tilde{Y}_{123} \rangle + Y_4 + \frac{|\tilde{a}|^2}{2} (Y_5 - Y_4) \\ \langle \tilde{a}, \tilde{Y}_{123} \rangle - Y_5 - \frac{|\tilde{a}|^2}{2} (Y_5 - Y_4) \end{pmatrix}.$$  

This yields that $H_\Psi = 2\langle \tilde{a}, \tilde{Y}_{123} \rangle - Y_5 - Y_4 - |\tilde{a}|^2 (Y_5 - Y_4)$, or in other symbols,

$$\overline{H}_\Psi = \langle \tilde{a}, \tilde{Y}_{123} \rangle - \overline{Y}_5 - \overline{Y}_4 - |\tilde{a}|^2 (\overline{Y}_5 - \overline{Y}_4).$$  \hfill (36)

Note that if $\overline{H} = \overline{Y}_5 - \overline{Y}_4 = 0$, the result is immediate by choosing $\Theta = \text{Id}$. We will thus assume without loss of generality that $\overline{H} \neq 0$. 


We decompose \( \tilde{a} = x \frac{\bar{y}_{123}}{|\bar{y}_{123}|} + y \bar{v}_1 + z \bar{v}_2 \) where \( \left\{ \frac{\bar{y}_{123}}{|\bar{y}_{123}|}, \bar{v}_1, \bar{v}_2 \right\} \) is an orthonormal basis of \( \mathbb{R}^3 \). From (36) we then deduce that

\[
\overline{H}_\psi = 2x \frac{\bar{y}_{123}}{|\bar{y}_{123}|} - \bar{y}_5 - \bar{y}_4 - (x^2 + y^2 + z^2) (\bar{y}_5 - \bar{y}_4)
\]

\[
= - (\bar{y}_5 - \bar{y}_4) \left( x - \frac{\bar{y}_{123}}{\bar{y}_5 - \bar{y}_4} \right)^2 + y^2 + z^2 - \frac{|\bar{y}_{123}|^2}{(\bar{y}_5 - \bar{y}_4)^2} + \frac{\bar{y}_4 + \bar{y}_5}{\bar{y}_5 - \bar{y}_4} \right)
\]

\[
= - (\bar{y}_5 - \bar{y}_4) \left( x - \frac{\bar{y}_{123}}{\bar{y}_5 - \bar{y}_4} \right)^2 + y^2 + z^2 - \frac{|\bar{y}_{123}|^2 + \bar{y}_4^2 - \bar{y}_5^2}{(\bar{y}_5 - \bar{y}_4)^2} \right)
\]

Thus there exists \( \tilde{a} \) such that \( \overline{H}_\psi = 0 \) if and only if \( |\bar{y}, \bar{y}|_{4,1} \geq 0 \), and in that case any \( \tilde{a} \) belonging to the sphere \( S_Y \) of center \( \frac{\bar{y}_{123}}{|\bar{y}_{123}|} \in \mathbb{R}^3 \) and radius \( \sqrt{\frac{|\bar{y}, \bar{y}|_{4,1}}{(\bar{y}_5 - \bar{y}_4)^2}} \)

satisfies \( \overline{H}_\psi = 0 \). Owing to Corollary 2.2, we know there exists \( \varepsilon_0 > 0 \) depending only on \( C_0 \) such that if \( \| \tilde{A}e^{-\lambda} \|_{L^2(D)} \leq \varepsilon_0 \) then \( |\bar{y}, \bar{y}|_{4,1} \geq \frac{1}{2} \). Thus indeed there exists \( \tilde{a} \in \mathbb{R}^3 \) cancelling \( \overline{H}_\psi \). A straightforward computation shows that, if we denote by \( \lambda_\psi \) the conformal factor of \( \Psi \), then we have

\[
\lambda_\psi = \lambda + \log |\Phi - \tilde{a}|. \tag{37}
\]

Since \( \log \) has its gradient in the weak Marcinkiewicz space \( L^{2,\infty} \), from (31), it follows that \( \lambda_\psi \) inherits the bound (28) on \( \lambda \) so soon as there exists a constant \( C > 0 \) depending only on \( C_0 \) such that \( \frac{1}{C} \leq |\Phi - \tilde{a}| \leq C \). This ensures that (28) holds for \( \lambda_\psi \) and enables us to conclude the proof. We now establish the existence of such a constant \( C \).

**Claim.** We can choose \( \tilde{a} \in S_Y \) such that

\[
\frac{1}{C} \leq |\Phi - \tilde{a}| \leq C, \tag{38}
\]

with \( C \) depending only on \( C_0 \).

**Proof.** By hypothesis, \( \Phi(D) \subset B(0, R_0) \) for some \( R_0 \) depending only on \( C_0 \). We prove there exists \( R'_0 \geq 2R_0 \) depending only on \( C_0 \) such that

\[
B(0, R'_0) \cap S_Y \neq \emptyset.
\]

In doing so, we first establish that there exists \( C > 0 \) depending only on \( C_0 \), such that

\[
\| \bar{y} \| \leq C, \tag{39}
\]
where $\| \cdot \|$ is the Euclidean metric. Combining (27) and (31), one finds that

$$\| H \|_{L^2(D)} + \left\| \left( \frac{\Phi}{|\Phi|^2 + 1} \right) \right\|_{L^2(D)} + \left\| \left( \frac{\tilde{n}}{|\tilde{n}|}, \Phi \right) \right\|_{L^2(D)} \leq C. \quad (40)$$

Given Definition (4) for the conformal Gauss map, one has that

$$\| Y \|_{L^2(D)} \leq C, \quad (41)$$

and consequently we deduce (39).

Next, let $S(c, r) := \overline{S}_Y$. We have

$$Vertc|| - r = \frac{|\overline{Y}_{123}|}{|\overline{Y}_5 - \overline{Y}_4|} - \frac{|\overline{Y}, \overline{Y}|_{4,1}}{(\overline{Y}_5 - \overline{Y}_4)^2} = \frac{|\overline{Y}_{123}| - \sqrt{|\overline{Y}, \overline{Y}|_{4,1}}}{|\overline{Y}_5 - \overline{Y}_4|} = \frac{|\overline{Y}_{123}|^2 - |\overline{Y}, \overline{Y}|_{4,1}}{|\overline{Y}_5 - \overline{Y}_4| (|\overline{Y}_{123}| + \sqrt{|\overline{Y}, \overline{Y}|_{4,1}})} = \pm \frac{|\overline{Y}_4 + \overline{Y}_5|}{|\overline{Y}_{123}| + \sqrt{|\overline{Y}, \overline{Y}|_{4,1}}},$$

Calling upon Corollary 2.2 and (39), we obtain the desired result with

$$R_0' = \max(2R_0, 2|\overline{Y}_4 + \overline{Y}_5|^{1/2}).$$

It is important to note that the radius of $\overline{S}_Y$ is bounded from below thanks to (39) and Corollary 2.2.

Clearly, if $\overline{S}_Y \not\subset B(0, R_0')$, then any $\tilde{a} \in S(0, R_0') \cap \overline{S}_Y$ satisfies the desired property (38). On the other hand, suppose that $\overline{S}_Y \subset B(0, R_0')$. The only problem would then arise with a sequence $\{\Phi_k\}$ satisfying (27) and for which $\Phi_k(D)$ gets arbitrarily close to $\overline{S}_Y$, in the sense that it ends up covering the whole sphere. In that case, we could not bound $|\Phi_k(D) - \tilde{a}|$ from below. This issue is avoided thanks to the smallness hypothesis on $\|\nabla \tilde{n}\|_{L^2}$. Indeed, taking $\|\nabla \tilde{n}\|_{L^2}$ smaller than the energy of a half-sphere (which does not depend on the radius), we can conclude that the $\Phi_k(D)$ can cover at most half a sphere. Taking $\tilde{a}$ the pole opposite to this half-sphere, we find that $|\Phi_k(D) - \tilde{a}|$ is bounded from below by the radius of $\overline{S}_Y$. Since, as remarked above this radius is bounded from below, this concludes the proof of the claim and thus of the Proposition.

$\square$

Remark 2.1. It might seem strange to favor working with the Euclidean average, rather than with the average computed against the metric. Equality (36) reveals why: the exchange law enables recovering the mean curvature after the conformal transformation. The metric change induced by the transformation modifies the exchange law into a less convenient form.
Remark 2.2. The fact that $\varepsilon_0$ depends only on $C_0$ stems from the dilation used to neutralize the constant $\tilde{\lambda}$ in (19). This dilation is not merely used to simplify the problem, but it is also part of the resulting conformal transformation.

2.4. Proof of Theorem 1.3

Proof. Owing to Corollary 2.7 of [24], we know that $\Phi$ satisfies (19) on $D_{3/4}$. By hypothesis, $\Phi$ satisfies (27) and (28) on $D$ and a fortiori on $D_{3/4}$. We can then apply Proposition 2.3 on $D_{3/4}$, assuming $\varepsilon_0$ small enough, so as to find $\Theta \in \text{Conf}(\mathbb{R}^3)$ such that $\Theta \circ \Phi$ still satisfies (19), and such that $\overline{H}_\Psi = 0$ (with the average taken over $D_{3/4}$).

Since $\Psi$ satisfies (19), we can apply Proposition 2.1 and its corollaries (specifically 2.1) on $D_{3/4}$ to obtain, from (29) and $\overline{H}_\Psi = 0$ that

$$\left\| (H_\Psi - \overline{H}_\Psi) e^{\lambda_\Psi} \right\|_{L^2(D_{3/4})} = \left\| H_\Psi e^{\lambda_\Psi} \right\|_{L^2(D_{3/4})} \leq C \left\| \hat{A}_\Psi e^{-\lambda_\Psi} \right\|_{L^2(D_{3/4})}. \quad (42)$$

Thus,

$$\left\| \nabla \tilde{n}_\Psi \right\|_{L^2(D_{3/4})} \leq \left\| (H_\Psi - \overline{H}_\Psi) e^{\lambda_\Psi} \right\|_{L^2(D_{3/4})} + \left\| \hat{A}_\Psi e^{-\lambda_\Psi} \right\|_{L^2(D_{3/4})} \leq C \left\| \hat{A}_\Psi e^{-\lambda_\Psi} \right\|_{L^2(D_{3/4})}. \quad (43)$$

Choosing $\varepsilon_0 = \min \left( \tilde{\varepsilon}_0, \frac{4\pi}{3C(C_0)} \right)$, with $C(C_0)$ the final constant in (43), we find that $\Psi$ satisfies

$$\left\| \nabla \tilde{n}_\Psi \right\|_{L^2(D_{3/4})} \leq \frac{4\pi}{3}. \quad (44)$$

The classical $\varepsilon$-regularity for Willmore immersions [30] states that

$$\left\| \nabla \tilde{n}_\Psi \right\|_{L^2(D_{3/4})} \leq C \left\| \nabla \tilde{n}_\Psi \right\|_{L^2(D_{3/4})}. \quad (45)$$

Combining (43) and (45), we deduce that

$$\left\| \hat{A}_\Psi e^{-\lambda_\Psi} \right\|_{L^\infty(D_{3/4})} \leq \left\| \nabla \tilde{n}_\Psi \right\|_{L^\infty(D_{3/4})} \leq C \left\| \nabla \tilde{n}_\Psi \right\|_{L^2(D_{3/4})} \leq C \left\| \hat{A}_\Psi e^{-\lambda_\Psi} \right\|_{L^2(D_{3/4})}. \quad (46)$$

Since (46) is conformally invariant, it holds with $\Phi = \Theta^{-1} \circ \Psi$ in place of $\Psi$, which concludes the proof. \qed
2.5. Proof of Theorem 1.4

We end this section by removing the small bound on the total curvature. As we have seen, this bound is decisive in controlling the conformal factor. When the conformal factor can no longer be controlled, there must be concentration of energy and a bubbling phenomenon ensues. However, owing to our hypotheses, we will see that all bubbles must be round spheres. In that case, the conformal factor still satisfies some Harnack estimate, which, as we will see, is sufficient to conclude.

The Proof of Theorem 1.4 goes in 4 steps:

- We first show that the statement fails only when bubbling develops.
- We prove all bubbles must be Euclidean spheres.
- We eliminate those bubbles with the help of an inversion.
- Finally, conformal invariance leads to a contradiction.

Proof. For the sake of contradiction, consider a sequence \( \Phi_k : \mathbb{D} \to \mathbb{R}^3 \) such that \( \Phi_k \in \mathcal{E}(\mathbb{D}) \) is a conformal Willmore immersion satisfying (10) and (11). We further assume that the induced conformal classes lie in a compact subset of Moduli space, and that there exists \( C(k) \to \infty \) such that:

\[
\| \hat{A}_k e^{-\lambda_k} \|_{L^\infty(D_{1/2})} \geq C(k) \| \hat{A}_k e^{-\lambda_k} \|_{L^2(D)} .
\] (47)

Up to a dilation, we can also assume that there exists \( A > 0 \) such that \( \Phi_k(\mathbb{D}_{1/2}) \cap B_{1/4}(0)^c \neq \emptyset \). We are then precisely in the situation of Theorem I.3 of [2], which states that there exist a finite number \( N \) of radii \( \rho_i^k \to 0 \) and points \( a_i^k \to a^i \in \mathbb{D} \), a Willmore immersion \( \Phi_\infty : \mathbb{D} \to \mathbb{R}^3 \), and some possibly branched Willmore immersions \( \omega^i : \mathbb{R}^2 \to \mathbb{R}^3 \), as well as conformal transformations \( \theta_k, \xi^i_k \), such that:

\[
\theta_k \circ \Phi_k \to \Phi_\infty \quad C^{\infty}_{\text{loc}}(\mathbb{D}\setminus\{a^1, \ldots, a^i\})
\]

\[
\xi^i_k \circ \Phi_k(\rho_i^k x + a_i^k) \to \omega^i \quad C^{\infty}_{\text{loc}}(\mathbb{R}^2\setminus\{\text{finite set}\})
\]

\[
\| \nabla \vec{n}_k \|_{L^2(D)}^2 \to \| \nabla \vec{n}_\infty \|_{L^2(D)}^2 + \sum_{i=1}^{N} \| \nabla \vec{n}_{\omega^i} \|_{L^2(\mathbb{R}^2)}^2
\]

\[
\| \hat{A}_k e^{-\lambda_k} \|_{L^2(D)}^2 \to \| \hat{A}_\infty e^{-\lambda_\infty} \|_{L^2(D)}^2 + \sum_{i=1}^{N} \| \hat{A}_{\omega^i} e^{-\lambda_{\omega^i}} \|_{L^2(\mathbb{R}^2)}^2 .
\] (48)

In [2], it is in fact \( \| H \|_{L^2}^2 \) which is quantized, but as remarked in Lemma 3.1 of [23], one has also quantization for the full second fundamental form and in particular for its traceless part as well.

In our case, there is at least one concentration point inside \( \mathbb{D}_{3/4} \), else we would simply conclude by covering \( \mathbb{D}_{1/2} \) with a finite number of disks of radius bounded
from below and satisfying the hypothesis of Theorem 1.3. We then see that the energies of the bubble and of the limit are controlled, namely,

$$\left\| \hat{A}_\infty e^{-\lambda_\infty} \right\|_{L^2(\mathbb{D})} \leq \varepsilon_0$$

$$\forall i \left\| \hat{A}_{\omega_i} e^{-\lambda_{\omega_i}} \right\|_{L^2(\mathbb{R}^2)} \leq \varepsilon_0.$$ \hspace{1cm} (49)

Owing to (49), we see that each bubble is a round sphere. Indeed Theorem H in [26] (see also Lemma 3.1 in the Appendix) guarantees that the bubbles, even if branched, are conformal inversions of minimal surfaces. But a classical result states that the total curvature of a minimal surface is a multiple of $4\pi$. Hence, assuming that $\varepsilon_0$ is small enough ensures that the bubbles are round spheres.

From the Proof of Theorem 0.2 of [17], it is known that a round sphere cannot be glued onto a compact surface without a third surface appearing in between, and this surface is necessarily non-umbilic.\(^2\) Hence all the round bubbles are simple, and the concentration points must be isolated.\(^3\)

In Willmore bubbling, singular points (branched or non-compact) can only appear at concentration points. Since all round bubbles are simple, they may have at most one singular branched point. However, there exists no conformal parametrization of the Euclidean sphere with one single branch point. Accordingly, none of the bubbles $\omega_i$ may have branch points, and thus are all immersions.

Next, let $x_k \in \mathbb{D}_{\frac{1}{4}}$ be such that:

$$\left| \hat{A}_k e^{-\lambda_k} (x_k) \right| \geq C(k) \left\| \hat{A} e^{-\lambda} \right\|_{L^2(\mathbb{D})}.$$ \hspace{1cm} (50)

There exists $x_0 \in \mathbb{D}_{\frac{3}{4}}$ such that $x_k \rightarrow x_0$. Necessarily, $x_0$ is a concentration point (one of the aforementioned points $a^i$). We choose $\rho > 0$ such that $B(x_0, \rho)$ does not contain any other concentration point (since those are isolated), and moreover

$$\xi_k \circ \Phi_k (\rho_k x + a_k) \rightarrow \omega \quad C^\infty_{\text{loc}}(\mathbb{R}^2),$$

where $\omega$ parametrizes a round sphere. Consider $p := \omega(\infty)$ and $\iota_p$ the inversion of $\mathbb{R}^3$ centered on $p$. Put $\Psi_k := \iota_p \circ \xi_k \circ \Phi_k$. If the energy $\|\nabla \tilde{n}_k\|_2$ were to concentrate, we would be able to blow-up a round sphere, but since a plane ($\iota_p \circ \omega$) develops at scale $\rho_k$, using again the argument of the Proof of Theorem 0.2 of LAURAIN AND RIVIÈRE [17] and recalled above, it is then possible to generate a non-umbilic bubble between the sphere and the plane (in the same manner as one proves the simplicity of the bubbles), which yields a contradiction.

\(^2\) The argument is as follows: between the round sphere and the compact piece, there is a small geodesic circle. Blowing up the surface around this geodesic gives rise to a non-compact Willmore surface with at least two ends which cannot be umbilic. Hence all the involved concentration points develop only one simple bubble which is a round sphere and $\Phi_\infty$ must be constant.

\(^3\) In fact there is only one concentration point since the argument of Theorem 0.2 of [17] applies between two bubbles.
We may now apply Theorem 1.3 to $\Phi_k$ on a finite cover of $B(x_0, \rho)$, thereby obtaining by conformal invariance the estimate
\[
\|\hat{A}_k e^{-\lambda_k}\|_{L^\infty(B(x_0, \frac{\rho}{2}))} \leq C \|\hat{A} e^{-\lambda}\|_{L^2(\mathbb{D})}.
\]
This contradicts (47) and concludes the Proof of Theorem 1.4.

Remark 2.3. In the Proof of Theorem 1.4, we do not exclude Euclidean spheres as bubbles. We merely show they appear through a more regular concentration phenomenon, one not affecting the tracefree curvature. A parallel reasoning should be drawn with [24], where the $\epsilon$-regularity for $H$ yields an improved regularity for minimal bubbling, because of the high impact a control on the mean curvature has on the regularity of the immersion. In the present case, while control on the tracefree curvature does not immediately yield control on the immersions, it sufficiently restricts the appearance of bubbles.

By making the change of variables $\Phi_\rho = \Phi(\rho \cdot)$ (as in [24], for instance Corollary 2.1) one can easily extend these results to disks of arbitrary radius $\rho$:

**Corollary 2.3.** Let $\Phi \in \mathcal{E}(D_\rho)$ be a conformal weak Willmore immersion. Assume
\[
\int_{D_\rho} |\nabla \tilde{n}|^2 \leq C_0, \quad (51)
\]
and
\[
\|\nabla \lambda\|_{L^{2,\infty}(D_\rho)} \leq C_0, \quad (52)
\]
for some constant $C_0 > 0$.
Then there exists a constant $\epsilon_0(C_0) > 0$ such that, if
\[
\|\hat{A} e^{-\lambda}\|_{L^2(D_\rho)} \leq \epsilon_0,
\]
there exists a constant $C(C_0) > 0$ such that
\[
\|\hat{A} e^{-\lambda}\|_{L^\infty(D_{\frac{\rho}{2}})} \leq \frac{C}{\rho} \left\|\hat{A} e^{-\lambda}\right\|_{L^2(D_\rho)}.
\]

2.6. An Umbilical Willmore bubble

In order to clarify the Proof of Theorem 1.4, it is instructive to consider a concrete example of an umbilic Willmore bubble. In the first section of [23], an example of Willmore bubbling with a minimal bubble is given: there exists a sequence of Willmore immersions $\Phi_\mu$ of the sphere converging smoothly to an inverted López surface, away from a single concentration point. We remind the reader that a López surface is a minimal sphere with one branched end of multiplicity 3 and one immersed end. The inverted López surface is thus a Willmore branched surface (see Fig. 1), with a point of density 4, decomposed into a branch point of multiplicity 3 and a regular point.
The appearance of a branch point is symptomatic of bubbling phenomena, and a blow-up analysis shows that the sequence $\frac{\Phi_\mu(\mu^{-1})}{\mu^3}$ converges smoothly to an Enneper surface (a minimal sphere with one branched end of multiplicity 3, see Fig. 2).

Given the complexity of situations involving multiple points (branch ends and branch points) we adopt a schematic representation to illustrate the bubbling configurations. The López surface (and its inverse) will be represented according to Fig. 3 while the Enneper (and its inverse) will be represented by Fig. 4. The bubbling configuration of $\Phi_\mu$ is schematically depicted on Fig. 5.

In what follows, we exploit the topological symmetry of this bubbling configuration: *both the limit surface and the bubble are topological spheres*. We can then
revers the situation and consider the compactified bubble as the limit surface. Let us then consider $p \in \mathbb{R}^3$ and $\eta > 0$ such that $d(p, \Phi_{\mu}(\mu^3z)) \geq \eta > 0$, and define:

$$\Psi_{\mu}(z) := \iota_p \circ \left( \frac{\Phi_{\mu}(\mu^3z)}{\mu^9} \right).$$

Necessarily, $\Phi_{\mu}$ converges to an inverted Enneper surface whose branch point at 0 must be desingularized by a non-compact bubble. Let

$$\tilde{\Psi}_{\mu} := \frac{\Psi_{\mu}(\mu^3z)}{\mu^9} = \frac{\iota_p \circ \left( \Phi_{\mu}(\mu^3z) \right)}{\mu^9} = \mu^9 \iota_{\Psi_{\mu}(1)} \Phi_{\mu} \left( \frac{1}{z} \right).$$

Given the asymptotic behavior of $\Phi_{\mu}$, the map $\tilde{\Psi}_{\mu}$ may only converge to a López surface (whose branched end is this time at $\infty$ while its simple end is at 0). Since a López minimal surface has two ends, i.e. two singular points, $\tilde{\Psi}_{\mu}$ still has a concentration point desingularizing the simple end. However, given the conformal invariance of $|\tilde{A}e^{-\lambda}|$, all the tracefree total curvature is accounted for within the inverted Enneper and the López surfaces. The only remaining bubble must then be totally umbilic, that is, a Euclidean sphere. This bubbling configuration is schematically represented on Fig. 6.

Inspecting $\frac{\psi_{\mu} |_{D}}{\text{diam}(\psi_{\mu}(D))}$, we observe the situation described in Theorem 1.4: a Willmore disk becoming a non compact surface up to a proper rescaling, as a Euclidean bubble concentrates at the origin. However, this bubble arises after an
inversion at a regular point of another bubbling configuration (the blue regular sheet of Fig. 5 becomes the blue end and the blue bubble of 6). This configuration is thus regular and does satisfy (12). One key consequence of the Proof of Theorem 1.4 is then that all umbilic bubbles are “artificial” ones arising from the inversion of a regular point, and are thus smoother than expected.

3. Applications

3.1. Proof of Theorem 1.5: Lorentzian $\varepsilon$-Regularity

Proof. The statement is merely a reformulation of 1.4 with $(\nabla Y, \nabla Y)_{4,1} = |\hat{A}e^{-\lambda}|^2$ (see Proposition 1.2).

$\square$

3.2. Proof of Theorem 1.7: Intrinsic gap

Proof. We first show that $\Sigma$ is topologically either a plane or a sphere. By a classical result of Huber [13], see also [32], we know that $\Sigma$ is conformally equivalent to a compact Riemann surface $\hat{\Sigma}$ with possibly a finite set of points $\{p_1, \ldots, p_N\}$ removed. So there exists a conformal parametrization $\Phi : \hat{\Sigma} \setminus \{p_i\}_i \to \Sigma$. Since $\Sigma$ is complete, each $p_i$ corresponds to some end of $\Sigma$. Hence, thanks to the generalized Gauss–Bonnet formula (see Theorem 10 of [28]), we have

$$\int_{\Sigma} K d\sigma = 4\pi (1 - g(\hat{\Sigma})) - 2\pi \sum_{i=1}^{n} (m_i + 1),$$

where $g(\hat{\Sigma})$ is the genus of $\hat{\Sigma}$ and $m_i$ is the multiplicity of the end $p_i$. Moreover we have on $\Sigma$ the identity

$$|A|_g^2 = 2|\hat{A}|_g^2 + 2K,$$
\[
\frac{1}{2} \int_{\Sigma} |A|_g^2 \, d\text{vol}_g \leq \int_{\Sigma} |\hat{A}|_g^2 \, d\text{vol}_g + 4\pi (1 - g(\hat{\Sigma})) - 4\pi N. \tag{53}
\]

Then for \(\varepsilon_0\) small enough, \(\hat{\Sigma}\) is a topological sphere with at most one end or a torus with no end. But the latter case is excluded since the left-hand side of (53) is bigger than \(4\pi\) owing to a classical estimate by Willmore (Theorem 7.2.2 [33]).

In conclusion, there exists a conformal Willmore immersion \(\Phi : \mathbb{R}^2 \to \mathbb{R}^2\) whose image is \(\Sigma\) (up to the removal of a point in the compact case). Applying Corollary 2.3 around any point \(p \in \mathbb{R}^2\) and letting \(\rho \to +\infty\) yields \(|\hat{A}|(p) = 0\), which implies the announced statement.

\[\Box\]

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Appendix

Variational Bubbles are Conformally Minimal

In Theorem H of [26], the authors obtain as a byproduct that any branched sphere that appears as a bubble must be conformally minimal. The Proof of Theorem H in [26] is quite involved due to the general assumptions used by the authors. For the sake of completeness of the present paper, we give an elementary argument to obtain the same result.

Lemma 3.1. Let \(\Phi_k : \mathbb{D} \to \mathbb{R}^3\) be a sequence of conformal Willmore immersions, \(a_k \in \mathbb{D}\) a sequence of points converging to some \(a_\infty\), \(\mu_k\) converging to 0, and \(T_k\) a sequence of conformal transformations of \(\mathbb{R}^3\). Suppose that

\[
\Phi_k := T_k \circ \Phi_k(a_k + \mu_k \cdot) \to \omega \text{ on } \mathbb{R}^2 \setminus S,
\]

where \(S\) is the finite set and \(\omega\) a branched Willmore sphere. Then \(\omega\) is conformally minimal.

Proof. Following Bryant’s work, in order to prove that a sphere is conformally minimal, it suffices to prove that its associated quartic form \(Q_\omega\) vanishes (see Theorem E in [7]). Let \(Q_{\Phi_k}\) be the quartic form associated with \(\Phi_k\). Per Theorem B in [7], \(Q_{\Phi_k}\) is holomorphic. Let \(p \in \mathbb{R}^2\) be a branch point. Our strong convergence
hypothesis away from branch points guarantees that the quartic form $Q_\omega$ is holomorphic around $p$, because $\tilde{Q}_{\Phi_1}$ is bounded, owing to the maximum principle. Hence $Q_\omega$ may have at most one pole at infinity. Letting $\tilde{Q} = Q_\omega \left( \frac{1}{z} \right)$, by Theorem 3.1 of [16], the order of the pole of $\tilde{Q}$ at 0 is at most 2. One easily checks that $\tilde{Q}(z) = O \left( \frac{1}{|z|^2} \right)$, so that $Q_\omega \equiv 0$ by Liouville’s Theorem, thereby concluding the proof.

**Convenient Reformulation of the Gauss–Codazzi Equation**

Recall that $\nabla\tilde{n} = -e^{-2\lambda} A \nabla \Phi$, that is,

$$-\tilde{n}_x = \left( H + \left( \frac{l-n}{2} \right) e^{-2\lambda} \right) \Phi_x + me^{-2\lambda} \Phi_y$$

and

$$-\tilde{n}_y = me^{-2\lambda} \Phi_x + \left( H - \left( \frac{l-n}{2} \right) e^{-2\lambda} \right) \Phi_y.$$ 

Differentiating yields

$$-\tilde{n}_{xy} = \left( H_y + \left( \frac{l-n}{2} \right) e^{-2\lambda} x + \lambda_y H - \lambda_y \left( \frac{l-n}{2} \right) e^{-2\lambda} - \lambda_x me^{-2\lambda} \right) \Phi_x$$

and

$$-\tilde{n}_{yx} = \left( me^{-2\lambda} x - \lambda_x me^{-2\lambda} + \lambda_y \left( H + \left( \frac{l-n}{2} \right) e^{-2\lambda} \right) \right) \Phi_x$$

Identifying $\tilde{n}_{xy}$ and $\tilde{n}_{yx}$, one finds that

$$e^{2\lambda} H_x = \left( \frac{l-n}{2} \right) + m_y$$

and

$$e^{2\lambda} H_y = -\left( \frac{l-n}{2} \right) + m_x. \tag{54}$$

**Brief Proof of Theorem 1.6**

In order to contrast 1.5 from 1.6, we sketch a Proof of Theorem 1.6.

Proof. A weakly harmonic application $u$ satisfies

$$\Delta u + \langle \nabla u, \nabla u \rangle_{4,1} u = 0.$$
Equivalently, the latter may be recast in the conservative form
\[ \text{div}(\nabla uu^T - u \nabla u^T) = 0. \]

Accordingly, on \( \mathbb{D} \), there exists a matrix \( B \) such that \( \nabla \perp B = \nabla uu^T - u \nabla u^T \). As a consequence, \( B \) satisfies \( \nabla B = u \nabla \perp u^T - \nabla \perp uu^T \) which implies
\[ \Delta B = 2 \nabla u \nabla \perp u^T. \] (55)

On the other hand, if we denote by \( \varepsilon \) the signature matrix of \( \mathbb{R}^{4,1} \), then for any two vectors \( a, b \), we have \( \langle a, b \rangle_{4,1} = a^T \varepsilon b \). Then
\[ \nabla \perp B \varepsilon \nabla u = \nabla uu^T \varepsilon \nabla u - u \nabla u^T \varepsilon \nabla u = \langle u, \nabla u \rangle_{4,1} \nabla u - \langle \nabla u, \nabla u \rangle_{4,1} u = -\langle \nabla u, \nabla u \rangle_{4,1} u = \Delta u, \]

since, given that \( u \in S^{4,1} \), we have \( \langle u, \nabla u \rangle_{4,1} = 0 \). Combining this to (55) yields the system:
\[ \Delta u = \nabla \perp B \nabla (\varepsilon u) \]
\[ \Delta B = 2 \nabla u \nabla \perp u^T. \]

Provided that \( \| \nabla u \|_{L^2} \leq \varepsilon_0 \), the statement (14) now follows from classical integration by compensation techniques, such as those presented in [12]. \( \square \)

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