Gauge invariant action for superstring in Ramond-Ramond plane-wave background

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Abstract

We present a gauge invariant action for a superstring in the plane wave background with Ramond-Ramond (RR) five-form flux. The Wess-Zumino term is given explicitly in a bilinear form of the left invariant currents by introducing a fermionic center to define the nondegenerate group metric. The reparametrization invariance generators, whose combinations are conformal generators, and fermionic constraints, half of which generate $\kappa$-symmetry, are obtained. Equations of motion are obtained in conformal invariant and background covariant manners.

PACS: 11.30.Pb;11.17.+y;11.25.-w

Keywords: Wess-Zumino term; Superalgebra; plane wave background
1 Introduction

Recently the plane wave solution with the Ramond-Ramond (RR) 5-form flux was found as a maximally supersymmetric type IIB supergravity solution [1] in addition to the Minkowski flat and the AdS$_5 \times$S$^5$ spaces, based on studies of the plane wave solutions with the 4-form flux in the 11-dimensional supergravity [2]. The Penrose’s limiting procedure [3] was applied to the AdS spaces to obtain these plane wave solutions with fluxes (pp-wave) [4, 5, 6]. It is recognized as an approximation of AdS spaces and leads to interesting approaches to the AdS/CFT correspondence [7].

The superstring action in the RR plane wave background was presented [8] and it was also shown that the action in the light-cone gauge becomes simply an action for 8 bosons and 8 fermions which are free and massive in 2-dimensions. Brane actions in the RR pp-wave background have been widely studied [9] mostly in the light-cone gauge. In the light-cone gauge the conformal symmetry is broken by a 2-dimensional mass term at the gauge fixed level though it should be recovered in whole string theory even in the RR pp-wave background [10]. The light-cone Hamiltonian does not commute with other global space-time charges thus states in a supergravity multiplet have different light-cone energy values [11] and the light-cone energy is not minimized for BPS states. From a point of view of the symmetry, the light-cone approach is not suitable to understand systems. Manifest conformal invariant approaches have been providing us elegant formulations of string theories and practical computation methods. However the conformal invariant treatment has not been explored except in an alternative hybrid approach [12]. In this paper we will study the Green-Schwarz type superstring action in the RR pp-wave background in a covariant and manifest conformal invariant way.

The RR backgrounds are usually described by the Green-Schwarz type actions which contain Wess-Zumino (WZ) terms. In reference [8] the WZ term of the superstring in pp-wave background is given in a one parameter integral of a closed three form. The formal integral representation of the WZ term is often useful in discussing the invariance of the action but is not always convenient for the practical analysis. The integral is hardly performed especially for curved background cases except in the light-cone gauge, so local symmetry constraints and covariant equations of motion which will be needed in the covariant string field theories are hardly discussed. On the other hand, it was shown that the WZ term for the covariant superstring in AdS spaces can be constructed in a bilinear form of the left-invariant (LI) currents [13, 14, 15, 16], and local symmetry constraints [17] and global charges [16] were obtained explicitly. For the super-AdS group the WZ term which is bilinear in the LI currents can be constructed using a nondegenerate group metric depending on the scale parameter. In this paper we obtain the superstring action in the super-pp wave background from the one in the super-AdS background [13] using the Penrose limit [3]. The limiting procedure must be taken carefully, since the bilinear form WZ term contains a divergent term when the Penrose limit parameter, $\Omega$, is brought to zero. In a flat limit where the AdS radius goes to infinity, the bilinear form WZ term also contains a divergent term which is a total derivative term causing no difficulty. On the other hand the divergent term in the Penrose limit, which is a bilinear product of leading
terms of $\Omega$ power series of LI currents, is not a total derivative. The correct form of the bilinear WZ term is obtained by subtracting the divergent term and preserving a next to leading term of LI currents that gives a finite contribution. The next to leading term corresponds to the fermionic center introduced to make the group metric nondegenerate [18].

The organization of this paper is as followings. The Penrose limit of the bosonic string is explained as a simpler case in the section 2. In the section 3, the correct limiting procedure of the bilinear form WZ term is explained. The resulting WZ term is shown to produce the closed three form $H^{[3]}$. The $\kappa$-invariance of the action is also confirmed. Then the gauge invariant action for the RR pp-wave background is presented. It is also shown that this action reduces to the light-cone action obtained by Metsaev [8] and the Green-Schwarz action [19] in the flat limit. In the section 4, we examine a particle, a superparticle, a bosonic string and a superstring cases. For each systems generators of reparametrizations and fermionic constraints are calculated. We obtain Hamiltonians and equations of motion in the conformal gauge.

2 Penrose limit of bosonic string action

The superstring in the $\text{AdS}_5 \times S^5$ space is given as a sigma model action on $\frac{SU(2,2|4)}{SO(4,1) \times SO(5)}$, and we begin with a bosonic part of it. We use the same notation of the superalgebra as [20, 21]. Bosonic part of the LI Cartan 1-forms are given by

$$e^a_{\text{AdS}} = dy^a + \left( \frac{\sinh y}{y} - 1 \right) dy^b \Upsilon^b_{a} , \quad e^{a'}_{\text{AdS}} = dy^{a'} + \left( \frac{\sin y'}{y'} - 1 \right) dy^{b'} \Upsilon^{b'}_{a'} \quad (2.1)$$

with

$$y = \sqrt{y^2 + y^a y_a}, \quad y' = \sqrt{y'^2 + y^{a'} y_{a'}},$$

$$\Upsilon^b_{a} = \delta^b_a - \frac{y a y^b}{y^2}, \quad \Upsilon^{b'}_{a'} = \delta^{b'}_{a'} - \frac{y' a y^{b'}}{y'^2} . \quad (2.2)$$

A scale parameter $R$, which is the radius of $\text{AdS}_5$ and $S^5$, is introduced in the normalization of $F_5 = (1/2R)(\text{dvol}(\text{AdS}_5) + \text{dvol}(S^5))$, by rescaling $P_{\hat{a}} \rightarrow RP_{\hat{a}}$. The bosonic part of the sigma model action is written as

$$\mathcal{L}_{0,\text{AdS}} = e^a_{\text{AdS}} e_{\text{AdS},\hat{a}} = (dy)^2 + (\sinh \frac{y}{R})^2 d\Omega^2_4 + (dy')^2 + (\sin \frac{y'}{R})^2 d'\Omega^2_4 \quad (2.3)$$

where $d\Omega^2_4$ and $d'\Omega^2_4$ are 4-sphere metrics.

The Penrose limit is obtained as $\Omega \rightarrow 0$ after rescaling

$$y^+ \rightarrow \Omega^2 y^+, \quad y^- \rightarrow y^-, \quad y^i \rightarrow \Omega y^i \quad (2.4)$$
where \( y^\pm = (y^0 \pm y^9)/\sqrt{2} \), corresponding to \( P_+ \to \Omega^{-2} P_+ \), \( P_- \to P_- \), \( P_i \to \Omega^{-1} P_i \). Rewriting (2.1) in terms of (2.4) and taking \( \Omega \) infinitesimally small, (2.1) turns out to be a power series of \( \Omega \). Cartan 1-forms should be also rescaled as \( e^+ \to \Omega^2 e^+ \), \( e^- \to e^- \), \( e^i \to \Omega e^i \) for consistency. Taking \( \Omega \to 0 \) limit, leading terms of the expansion in \( \Omega \) are identified to the ones in the plane wave background

\[
\begin{align*}
\Omega^2 e^+_{AdS} &= \Omega^2 \left[ dy^+ + \left( \frac{\sin \frac{y^+}{\sqrt{2}R}}{\sqrt{2}R} - 1 \right) \frac{y^i}{y^-} \left( y^i dy^- - dy^i \right) \right] + o(\Omega^4) \equiv \Omega^2 e^+_{pp} + o(\Omega^4) \\
e^-_{AdS} &= dy^- + o(\Omega^4) \equiv e^-_{pp} + o(\Omega^4) \\
\Omega e^i_{AdS} &= \Omega \left[ dy^i - \left( \frac{\sin \frac{y^i}{\sqrt{2}R}}{\sqrt{2}R} - 1 \right) \left( y^i dy^- - dy^i \right) \right] + o(\Omega^4) \equiv \Omega e^i_{pp} + o(\Omega^4).
\end{align*}
\]

Bosonic part of the sigma model action in the pp-wave background is written as

\[
\mathcal{L}_{0,pp} = e^i_{pp} e^{pp, \dot{a}}
\]

\[
= 2 dy^+ dy^- + \left\{ \left( \frac{\sin \frac{y^+}{\sqrt{2}R}}{\sqrt{2}R} \right)^2 - 1 \right\} \left\{ \left( \frac{y^i}{y^-} \right)^2 (dy^-)^2 + 2 \left( \frac{y^i}{y^-} \right) dy^i dy^- \right\} + \left( \frac{\sin \frac{y^i}{\sqrt{2}R}}{\sqrt{2}R} \right)^2 dy^i dy^i
\]

\[
= 2 dx^+ dx^- - \sum_{i=1}^{8} \left( 2 \mu y^i \right)^2 (dx^-)^2 + \sum_{i=1}^{8} dx^i dx^i,
\]

where the last expression is obtained by the following field redefinition

\[
\begin{align*}
x^i &= \frac{\sin 2\mu y^- y^i}{2\mu y^-} \\
x^- &= y^- \\
x^+ &= y^+ + \sum_{i=1}^{8} \frac{(y^i)^2}{2y^-} \left( 1 - \frac{\sin 4\mu y^-}{4\mu y^-} \right).
\end{align*}
\]

A scale parameter \( \mu \) is introduced in the normalization \( F_5 = \mu dx^- \wedge (dx^1 dx^2 dx^3 dx^4 + dx^5 dx^6 dx^7 dx^8) \). This coordinate system is obtained if one begins with the coset parameterization \( G_{pp}(x) = e^{x^+ P_+} e^{x^- P_-} e^{x^i P_i} \).

### 3 Penrose limit of superstring action
3.1 Wess-Zumino term and nondegenerate super-pp-wave algebra

For an Inönü-Wigner group contraction [21] the Cartan 1-forms are expanded with respect to a parameter \( s \) which is brought to zero as

\[ T_A \to s^{-N_A} T_A \Rightarrow L^A(z) \to s^{N_A} L^A(s^{N_B} z^B) = \sum_{N=N_A}^{\infty} s^N L^A_{(N)}(z), \tag{3.1} \]

as was seen in (2.5) for bosonic part. The Maurer-Cartan equations are also expanded as

\[ \sum_{N=N_A}^{\infty} s^N dL^A_{(N)} + \frac{1}{2} f^A_{BC} \sum_{M=N_B}^{\infty} \sum_{K=N_C}^{\infty} s^{M+K} L^B_{(M)} L^C_{(K)} = 0, \tag{3.2} \]

where \( f^A_{BC} \) is the structure constant of the original Lie algebra. Usually only leading terms of Cartan 1-forms are preserved in the limiting procedure as was seen in the previous section. The Maurer-Cartan equations for the leading terms

\[ dL^A_{(N_A)} + \frac{1}{2} f^A_{(N_A)(N_B)} c_{(N_C)} L^B_{(N_B)} L^C_{(N_C)} = 0, \]

\[ \begin{cases} f^A_{(N_A)(N_B)} c_{(N_C)} = f^A_{BC} & \text{for } N_A = N_B + N_C \\ f^A_{(N_A)(N_B)} c_{(N_C)} = 0 & \text{for } N_A \neq N_B + N_C \end{cases} \tag{3.3} \]

describe the resultant group structure.

The Penrose limit from the super-AdS group to the super-pp wave group makes the super-pp group metric to be degenerate. This is the similar situation as in the flat limit from the super-AdS group to the super-translation group where the metric is degenerate. However if the next to leading term in the fermionic Cartan 1-forms (3.1) is maintained in the limiting procedure, the nondegenerate group metric of the central extended super-pp group can be constructed as we will show below.

The nondegenerate group metric can be defined in the super-AdS space and the bilinear form WZ term can be constructed as follows. In terms of light-cone indices \( \hat{a} = (+, -, i, i') \)

\[ \hat{i} = (i, i') \]

and the light cone projection operators for spinors \( \varpi \)

\[ \theta^\pm = \varpi \theta, \quad \zeta^\pm = \zeta \varpi, \quad Q^\pm = Q \varpi, \quad \varpi = \frac{1}{\sqrt{2}} (\Gamma_0 \pm \Gamma_7), \quad \Gamma_7 = \frac{1}{\sqrt{2}} (\Gamma_9 \pm \Gamma_0), \tag{3.4} \]

the Cartan 1-forms for the super-AdS5\times S5 space are written as

\[ G^{-1}_{AdS} dG_{AdS} = L^+_{AdS} P_+ + L^-_{AdS} P_- + L^i_{AdS} P^i + L^*_{AdS} Q_+ + L^-_{AdS} Q_- + L^i_{AdS} P^i + \frac{1}{2} L^j_{AdS} M_{ij}, \tag{3.5} \]

where

\[ P^*_i = M_{0i}, \quad P^*_i = M_{0'i}. \tag{3.6} \]
The MC equation for $Q_-$ is given by

$$
\begin{align*}
\frac{dL_{AdS}}{\Lambda^{AdS}} + \frac{1}{4} L^{ij}_{AdS} \Gamma^{ij}_{AdS} L^{\dot{\alpha}}_{AdS} - \frac{\sqrt{2}}{4} L^{i}_{AdS} \Pi_{i} \Pi_{j} L_{AdS}^{+} + \left( -2 L^{\dot{\alpha}}_{AdS} \Pi L^{\dot{\beta}}_{AdS} - L^{\dot{\alpha}}_{AdS} \Pi \Gamma_{\dot{\beta}}^{+} L_{AdS}^{+} \right) = 0
\end{align*}
$$

with $\Pi = \Gamma_{1234}$ and the MC equation for $Q_+$ has a similar form to this. From these MC equations it can be seen that structure constants $f^{Q_+}_{P_+ Q_+}$, $f^{Q_+}_{P_+ Q_+}$, $f^{Q_+}_{P_+ Q_+}$, and $f^{Q_+}_{P_+ Q_+}$ are present and the nondegenerate group metric can be defined consistently. The bilinear form WZ term can be constructed for the super-AdS space [16]

$$
\tilde{B}_{[2],AdS} = R \left( L^{\alpha' \beta'} C_{\alpha' \beta'} \tau_{1,1} L^{\beta' \gamma} - d\theta^{\alpha' \beta}' C_{\alpha' \beta'} \tau_{1,1} d\theta^{\beta' \gamma} \right) .
$$

Here the nondegenerate metric for spinor index is $\rho \theta_{\alpha' \beta'} = C_{\alpha' \beta'} (\tau_{1})_{1,1} \equiv \rho_{\alpha' \beta'}$, and this is used for the WZ term as $L^{\alpha} L^{\beta} \rho_{\alpha' \beta'}$. It gives $d(L^{\alpha} L^{\beta} \rho_{\alpha' \beta'}) = L^{\alpha} L^{\beta} f_{\alpha' \beta'} \sim H_{[3]}$ with totally antisymmetric structure constants defined as $f^{\alpha \beta}_{\gamma \delta \epsilon}$. In [16] it was shown that (3.8) gives correct exterior derivative $\tilde{B}_{[2]} = H_{[3]}$, $\kappa$-invariance of the total action and the correct flat limit. It was also shown that the second term in (3.8) is required for the pseudo-supersymmetry invariance giving the correct string charge in the superalgebra.

Let us discuss the Penrose limit of this bilinear form WZ term, (3.8). The Penrose limit is taken as the following rescaling [4]

$$
\begin{align*}
\theta^{+} \rightarrow \Omega \theta^{+}, \quad \theta^{-} \rightarrow \theta^{-}
\end{align*}
$$

in addition to (2.4). This corresponds to $s = \Omega$ in (3.1) and the scaling dimensions $N_{\alpha}$ in (3.2) to be

$$
N_{P_{+}} = 2, \quad N_{P_{-}} = 1, \quad N_{P_{\alpha}} = 0, \quad N_{P_{\alpha}^{*}} = 1, \quad N_{M_{ij}} = 0, \quad N_{Q_{+}} = 1, \quad N_{Q_{-}} = 0 .
$$

Under this rescaling the AdS-WZ term (3.8) becomes

$$
\tilde{B}_{[2],AdS} = \frac{i}{\sqrt{2}} \left\{ \Omega^{2} L_{(0)}^{+} \Gamma^{+} \tau_{1} \Pi L_{(1)}^{+} - \bar{L}_{(0)}^{-} \Gamma^{-} \tau_{1} \Pi \bar{L}_{(2)}^{-} - \Omega^{2} 2 L_{(0)}^{-} \Gamma^{-} \tau_{1} \Pi \bar{L}_{(2)}^{-} \right. \\
\left. - \Omega^{2} d\bar{\theta}^{+} \Gamma^{+} \tau_{1} \Pi d\theta^{+} + d\bar{\theta}^{-} \Gamma^{-} \tau_{1} \Pi d\theta^{-} + o(O^{3}) \right\}
$$

where $\bar{L} = L \mathcal{C}$ and $\mathcal{C}$ is the charge conjugation matrix in 10 dimensions. The scaling factor $\Omega^{2}$ of $\tilde{B}_{[2]}$ is determined by requiring the same scaling weight as the string kinetic term, the Nambu-Goto action.

Under the Penrose limit the 0-th order equation of the MC equation for $Q_{-}$ (3.3) becomes a trivial one according to (3.10)

$$
\frac{dL_{(0)}^{-} + f^{Q_{-}}_{M_{ij}} Q_{-} L_{(0)}^{-}}{\Lambda^{AdS}} = 0 .
$$

This implies that the group metric is degenerate in the “–” spinor direction. In order to make it nondegenerate $f^{Q_{-}}_{P_{+} Q_{-}}$ and $f^{Q_{+}}_{P_{-} Q_{+}}$ must be included, so additional term is required.
whose scaling dimension is \( N_{P_+} + N_{Q_-} = N_{P_1} + N_{Q_+} = 2 \). It is \( L_{(2)}^- \equiv \tilde{L}_{pp} \) and must be maintained in the limiting procedure, and explicit computation confirms that \( \tilde{L}_{pp} = L_{(2)}^- \) is the next to leading term. Under the Penrose limit the Cartan 1-forms of the super-AdS group reduce to those of the super-pp background as

\[
\begin{align*}
\Omega^{N_A} L_A^{\text{AdS}} &= \Omega^{N_A} L_{pp}^A + o(\Omega^{N_A+2}) \quad \cdots \text{for } A \neq Q_-
L_{\text{AdS}}^- &= L_{pp}^- + \Omega^2 \tilde{L}_{pp}^- + o(\Omega^4)
\end{align*}
\] (3.13)

and they satisfy the following MC equations for bosonic and fermionic Cartan 1-forms respectively

\[
\begin{align*}
\left\{ \begin{array}{l}
dL_{pp}^+ - \frac{1}{\sqrt{2}} L_{pp}^i L_{pp}^i - i\tilde{L}_{pp}^i \Gamma^i L = 0 \\
-dL_{pp}^- - iL_{pp}^i \Gamma^- L_{pp}^i = 0 \\
-dL_{pp}^i = 4\sqrt{2} \mu^2 L_{pp}^i L_{pp}^i + L_{pp}^i \tilde{L}_{pp}^i - i\tilde{L}_{pp}^i \Gamma^i L_{pp}^i = 0 \\
-dL_{pp}^i = -4\sqrt{2} \mu^2 L_{pp}^i L_{pp}^i + L_{pp}^i \tilde{L}_{pp}^i - 2i\mu \tilde{L}_{pp}^i \Pi \Gamma^i \epsilon L_{pp}^i = 0 \\
-dL_{pp}^i = L_{pp}^i L_{pp}^i + 2i\mu \tilde{L}_{pp}^i \Pi \Gamma^i \epsilon L_{pp}^i = 0 \\
-dL_{pp}^i = L_{pp}^i L_{pp}^i + 2i\mu \tilde{L}_{pp}^i \Pi \Gamma^i \epsilon L_{pp}^i = 0 \\
\end{array} \right.
\] (3.14)

\[
\begin{align*}
\left\{ \begin{array}{l}
dL_{pp}^+ + \frac{1}{4} (L_{pp}^i \bar{\Gamma}^i L_{pp}^i + \sqrt{2} \tilde{L}_{pp}^i \Gamma_+=\tilde{L}_{pp}^i) + \mu \epsilon \left( 2L_{pp}^i \Pi L^+ - \Pi L_{pp}^i \Gamma^i L_{pp}^i \right) = 0 \\
-dL_{pp}^- + \frac{1}{4} (L_{pp}^i \bar{\Gamma}^i L_{pp}^i = 0 \\
-dL_{pp}^i + \frac{1}{4} (L_{pp}^i \bar{\Gamma}^i L_{pp}^i + \sqrt{2} \tilde{L}_{pp}^i \Pi \Gamma^i \Pi L_{pp}^i) + \mu \epsilon \left( -2L_{pp}^i \Pi L_{pp}^- - L_{pp}^i \Pi \Gamma^i L_{pp}^i \right) = 0
\end{array} \right.
\] (3.15)

where \( \Pi' = \Gamma_{5678} \). These MC equations coincide with the ones obtained by the direct computation [8] using

\[
G_{pp}^{-1} G_{pp} = L_{pp}^+ P_+ + L_{pp}^- P_- + L_{pp}^i P_i + L_{pp}^+ Q_+ + L_{pp}^- Q_- + L_{pp}^i P_i^* + \frac{1}{2} L_{pp}^i M_{ij}
\] (3.16)

except for the last equation for \( \tilde{L}_{pp}^- \). It would be obtained if one uses an extended super-pp algebra with a fermionic center. The last equation of (3.15) has the same form as the MC equation of the AdS [3.7] then it becomes nondegenerate.

Following to above limiting procedure from the super-AdS WZ term (3.8) we propose a superstring action in the super-pp-wave background as

\[
S = \int d^2 \sigma \mathcal{L} = \int d^2 \sigma (\mathcal{L}_0 + \mathcal{L}_{WZ})
\]

\[
\mathcal{L}_0 = -T \sqrt{-h} \epsilon^{uv} L_u^i \bar{L}^i_v \gamma_{ab} \, , \quad \mathcal{L}_{WZ} = T \tilde{B}_{[2],pp}
\]

where the WZ term is

\[
\tilde{B}_{[2],pp} = \frac{i}{4\mu} \left( \tilde{L}_{pp}^i \Gamma_-(\tau_1)_{IJ} \Pi L_{pp}^{J+} - 2\tilde{L}_{pp}^i \Gamma_+(\tau_1)_{IJ} \Pi \tilde{L}_{pp}^{J-} - d\bar{\theta}^+ \Gamma_-(\tau_1)_{IJ} \Pi d\theta^+ \right)
\] (3.17)

and \( u, v = \tau, \sigma = 0, 1 \) are the world volume indices and Cartan 1-forms are \( L^A = dz^M L^A_M = d\sigma^u L^A_u \) with \( z^M = (x^m, \theta^\mu) \).
It satisfies the following criteria:

(i) giving the correct three form, \( d\overline{B}_2 = H_3 \) for \( dH_3 = 0 \),

(ii) \( \kappa \)-invariance of the total action.

(i) **Three form** \( H_3 \)

From now on the currents \( L^A_{\mu
u} \) is denoted just as \( L^A \). The first condition (i) is checked by using the MC equations (3.14) and (3.15),

\[
d\overline{B}_2 = \frac{i}{4\mu} \left( 2L^+\Gamma_-(\tau_1)_{IJ}\Pi dL^{+J} - 2dL^{-I}\Gamma_+(\tau_1)_{IJ}\Pi L^{-J} - 2L^+L_+ - L^-L_- \right) \\
= -i(\overline{L}^+\Gamma_-\tau_3L^+ + \overline{L}^+\overline{\Gamma}_\tau\tau_3L^-) + i\frac{1}{2\sqrt{2\mu}}\overline{L}^+(-L_i^\dagger\Gamma_i + L_i^\dagger\overline{\Gamma}_\nu)\Pi\tau_1L^- \\
- i(\overline{L}^+\Gamma_-\tau_3L^- + \overline{L}^+\overline{\Gamma}_\tau\tau_3L^+) + i\frac{1}{2\sqrt{2\mu}}\overline{L}^+(L_s^\dagger\Gamma_i - L_s^\dagger\overline{\Gamma}_\nu)\Pi\tau_1L^- \\
= -i\overline{L}^a\overline{\Gamma}_a\tau_3L = H_3 \quad .
\]

(ii) **\( \kappa \)-invariance**

The second condition (ii) \( \kappa \)-invariance of the action is confirmed as follows. Let us denote an arbitrary variation of the coset element \( \delta G \) as the following combination

\[
G^{-1}\delta G = \Delta L^A T_A = \delta z^M L_M^A T_A \quad ,
\]
and the variation of the Cartan 1-form is given by

\[
\delta L^A T_A = \delta(G^{-1}dG) = \left\{ d(\Delta L^A) - \Delta L^B L^C f_{BC}^A \right\} T_A \quad .
\]

Then an arbitrary variation of a Cartan 1-form is obtained by referring to the corresponding MC equation.

The \( \kappa \)-variations (\( \delta \theta^\mu = \kappa^\mu \)) of Cartan 1-forms are given by

\[
\delta_\kappa L^a = -f_{BC}^a \Delta_\kappa L^B L^C \\
\delta_\kappa L^\alpha = d(\Delta_\kappa L^\alpha) - f_{BC}^\alpha \Delta_\kappa L^B L^C
\]

where \( \Delta_\kappa L^a = 0 \) is imposed for the \( \kappa \)-variations. Concrete expression is

\[
\begin{align*}
\delta_\kappa L^+ - \frac{1}{\sqrt{2}} (\Delta_\kappa L^+ L^+_s + L^+_s \Delta_\kappa L^+_s) - 2i\overline{L}^+ \Delta_\kappa L = 0 \\
\delta_\kappa L^- - 2iL^- L^- \Delta_\kappa L = 0 \\
\delta_\kappa L^i + \frac{1}{\sqrt{2}} (\Delta_\kappa L^j L^i_s + L^j \Delta_\kappa L^i_s) + (\Delta_\kappa L^j L^j^i_s + L^j \Delta_\kappa L^j^i_s) - 2i\overline{L}^i \Delta_\kappa L = 0 \\
\delta_\kappa L^+ + \mu \epsilon \left\{ 2(\Delta_\kappa L^+ L^i_+ + L^i_+ \Delta_\kappa L^+ + L^j \Gamma_i \Gamma_j \Delta_\kappa L^+) \right\} \\
+ \frac{1}{4}(\Delta_\kappa L^j L^j L^i_+ + L^j \Gamma_i \Gamma_j \Delta_\kappa L^+) - \frac{1}{2\sqrt{2}} (\Delta_\kappa L^i \Gamma_i \Gamma_i + L^i_+ \Gamma_i \Gamma_i \Delta_\kappa L^-) = 0 \\
\delta_\kappa L^- + \frac{1}{4}(\Delta_\kappa L^j L^j L^- + L^j \Gamma_i \Gamma_i \Delta_\kappa L^-) = 0 \\
\delta_\kappa L^i - \mu \epsilon \left\{ 2(\Delta_\kappa L^i L^i + L^i \Delta_\kappa L^i + L^j \Gamma_i \Gamma_j \Delta_\kappa L^i) \right\} \\
+ \frac{1}{4}(\Delta_\kappa L^j \Gamma_i \Gamma_i L^- + L^j \Gamma_i \Gamma_i \Delta_\kappa L^-) - \frac{1}{2\sqrt{2}} (\Delta_\kappa L^i \Gamma_i \Gamma_i L^- + L^i \Gamma_i \Gamma_i \Delta_\kappa L^+) = 0 
\end{align*}
\]

(3.22)
Using these relations, the $\kappa$-variation of the total action is calculated as

$$
\delta_\kappa (L_0 + b L_{WZ}) = -T \delta_\kappa (\sqrt{-gg}\, L_u \hat{a}_u \hat{L}_v) + b T \epsilon_{uv} \hat{B}_{[2]uv} = -2iT (\nabla_\kappa L_v) (\sqrt{-\det G_{uv}} G_{uv} \Psi_v 1 + b e^{uv} \Psi_u \tau_3 \, L_v) = -2iT (\nabla_\kappa L_v) (\Gamma_{(1)} + b) e^{uv} \Psi_v \tau_3 \, L_v
$$

with

$$
\Gamma_{(1)} = \frac{1}{2\sqrt{-\det G_{uv}}} \tau_u \bar{\Lambda}, \quad \Gamma_{(1)}^2 = 1, \quad \text{tr}\Gamma_{(1)} = 0, \quad -\sqrt{-\det G_{uv}} G_{uv} \Psi_v = \Gamma_{(1)} \tau_3 e^{uv} \Psi_v.
$$

The action has the $\kappa$ invariance whose parameter $\kappa$ is satisfying

$$
(-\Gamma_{(1)} + b)\kappa = 0, \quad b = \pm 1.
$$

### 3.2 Gauge invariant action for a super-pp-string

We present an explicit expression of the gauge invariant action for a superstring in the RR pp-wave background, with $b = 1$ of (3.27),

$$
S = \int d^2\sigma \, \mathcal{L} = \int d^2\sigma \, (L_0 + L_{WZ}) \quad L_0 = -T \sqrt{-h} h^{uv} L_u \hat{a}_u \hat{L}_v \eta_{ab}
$$

$$
L_{WZ} = T \epsilon_{uv} \frac{i}{4\mu} (-\bar{L}_u + \Gamma_{-} \Pi L_v^+ + 2 \bar{L}_u \Gamma_{+} \Pi \bar{L}_v^- + \partial_u \bar{\theta}^+ \Gamma_{-} \Pi \partial_v \bar{\theta}^+)
$$

(3.28)

where the Cartan 1-forms in the pp-wave background are obtained from the ones in the super-AdS$_5 \times$S$^5$ given in the appendix. The results are

$$
\begin{align*}
L^+ &= e^+ + i\bar{\theta}^+ \Gamma^+ \left( \frac{\sin \Psi_+}{\Psi_+} \right)^2 (D\theta)^+ + i\bar{\theta}^+ \Gamma^+ \left( \frac{\sin \Psi_-}{\Psi_-} \right)^2 (D\theta^-)_{|\Omega^0} \\
L^- &= e^- + i\bar{\theta}^- \Gamma^- \left( \frac{\sin \Psi_-}{\Psi_-} \right)^2 (D\theta^-)_{|\Omega^0} \\
L^i &= e^i + i\bar{\theta}^+ \Gamma^i \left( \frac{\sin \Psi_+}{\Psi_+} \right)^2 (D\theta^-)_{|\Omega^0} + i\bar{\theta}^- \Gamma^i \left( \frac{\sin \Psi_-}{\Psi_-} \right)^2 (D\theta^+)_{|\Omega^0} \\
L^+ &= \frac{\sin \Psi_+}{\Psi_+} (D\theta)^+ + \left( \frac{\bar{\omega}_+ \sin \Psi_-}{\Psi_-} \right)_{|\Omega} (D\theta^-)_{|\Omega^0} \\
L^- &= \frac{\sin \Psi_-}{\Psi_-} (D\theta^-)_{|\Omega^0} \\
\bar{L}^- &= \left( \frac{\sin \Psi_-}{\Psi_-} \right) (D\theta^-)_{|\Omega^2} + \left( \frac{\sin \Psi_-}{\Psi_-} \right)_{|\Omega^2} (D\theta^-)_{|\Omega^0} + \left( \bar{\omega}_- \sin \Psi_- \bar{\omega}_+ \right)_{|\Omega} (D\theta^+). \quad (3.29)
\end{align*}
$$
with the covariant derivatives

\[
\begin{align*}
(D\theta)^{-}|_{\Omega^0} & = d\theta^- + \frac{1}{2} \omega^{ij} \Gamma_{ij} \theta^- \\
(D\theta)^{-}|_{\Omega^2} & = -\mu \epsilon \left( 2e^+ \Pi \theta^- - \Pi e^+ \Gamma_{ij} \theta^+ \right) - \frac{\sqrt{2}}{4} \epsilon_i^j \Pi \Pi \theta^+ \\
(D\theta)^+ & = d\theta^+ + \frac{1}{2} \omega^{ij} \Gamma_{ij} \theta^+ + \mu \epsilon \left( 2e^- \Pi \theta^+ - \Pi e^- \Gamma_{ij} \theta^- \right) - \frac{\sqrt{2}}{4} \epsilon_i^j \Gamma_{ij} \theta^-
\end{align*}
\]

(3.30)

and arguments of the sin’s are

\[
\begin{align*}
\Psi^2 & = 2\mu \epsilon \left[ \Pi \left( 2\theta^+ \bar{\theta} - \Gamma_+ - 2\theta^- \bar{\theta} + \Gamma_+ \hat{\theta} \bar{\theta} \Gamma_i - \Gamma_+ \hat{\theta} \bar{\theta} \Gamma_i \right) \\
& \quad + \left( \Gamma_+ \left( \hat{\theta} \bar{\theta} - \Gamma_i \Pi - \hat{\theta} \bar{\theta} \Pi \Gamma_i \right) + \Gamma_- \left( \hat{\theta} \bar{\theta} - \Pi \Gamma_i \right) - \Gamma_+ \hat{\theta} \bar{\theta} \Gamma_i \right) \epsilon \\
& \quad - \frac{1}{2} \left( (\Gamma^{ij} \hat{\theta} \bar{\theta} - \Gamma_i \Gamma_j) \Gamma_+ - (\Gamma^{ij} \hat{\theta} \bar{\theta} - \Gamma_i \Gamma_j) \Gamma_- \right) \Pi \epsilon \right]
\end{align*}
\]

(3.31)

\[
\begin{align*}
\varpi_+ \Psi^2 \varpi_+ & \equiv \Psi_+^2 = 2\mu \epsilon \left[ \Pi \left( \epsilon \Gamma_i \hat{\theta} \bar{\theta} - \Gamma_i \Pi + \Gamma_+ \hat{\theta} \bar{\theta} \Gamma_i - \Gamma_+ \hat{\theta} \bar{\theta} \Gamma_i \Pi \epsilon \right) \\
& \quad - \frac{1}{2} (\Gamma^{ij} \hat{\theta} \bar{\theta} - \Gamma_i \Gamma_j) \Gamma_+ \Pi \epsilon \right]
\end{align*}
\]

(3.32)

The Cartan 1-forms are same as the one in [8] except for \( \tilde{L}^- \). From now on we use the “\( x \)-coordinates” in (2.7) for simplicity in which the bosonic Cartan 1-forms are given by

\[
\begin{align*}
e^+ & = dx^+ - 2\mu^2 (x^i)^2 dx^-

e^- & = dx^-
\epsilon^i & = dx^i
\epsilon^i_\lambda & = -4\sqrt{2} \mu^2 x^i dx^-
\omega^{ij} & = 0
\end{align*}
\]

(3.32)

We confirm that this action reduces to following known ones

(i) light-cone action in the light cone gauge,

(ii) flat action in the flat limit \( \mu \to 0 \).

(i) **Light-cone gauge action**

In the light-cone gauge with the conformal metric

\[
x_+ = p_+ \tau , \quad \theta^- = 0 , \quad h^{uv} = \eta^{uv}
\]

(3.33)
the Cartan 1-forms reduce into
\[
\begin{align*}
L^+ &= e^+ + i \bar{\theta}^+ \Gamma^+ (d\theta^+ + 2 \mu e_+ \Pi \theta^+) \\
L^- &= e_+ \\
L i &= e^i \\
L^+ &= d\theta^+ + 2 \mu e_+ \Pi \theta^+ \\
L^- &= 0 \\
\bar{L}^- &= \mu e \Pi \bar{e}^\dagger \Gamma_{i- \theta^+} - \sqrt{\frac{2}{3}} \bar{e}^\dagger \Pi \Gamma_{i- \theta^+} .
\end{align*}
\]
(3.34)

The light-cone action of a superstring in the super-pp-wave background is given by
\[
S = -T \int d^2 \sigma \left[ \frac{1}{2} \left( \partial^\mu x^i \partial_\mu x^i + m^2 (x^3)^2 \right) - i p_+ \left( \bar{\theta}^+ \Gamma^+ (\partial_0 + \tau_3 \partial_1) \theta^+ + m \bar{\theta}^+ \Gamma^+ \epsilon \Pi \theta^+ \right) \right] ,
\]
\[
m \equiv 2 \mu p_+ 
\]
(3.35)
which coincides with the one obtained by Metsaev [8].

(ii) Flat action

The flat limit is realized by taking \( \mu \to 0 \) limit. The kinetic term \( L_0 \) in the flat limit is straightforwardly obtained by taking \( \mu \to 0 \) in Cartan 1-forms \( L^a \) as
\[
L^a = dx^a + i \bar{\theta} \Gamma^a d \theta .
\]
(3.36)
The WZ term \( L_{WZ} \) in the flat limit must be evaluated with care. Term proportional to \( 1/\mu \) is absent in (3.17). Terms proportional to \( \mu^0 \) are obtained from terms proportional to \( \mu^1 \) in \( L^i \)
\[
\begin{align*}
L^+ &= d\theta^+ + \mu \left[ e \Pi (2e^- \theta^+ - e^i \Gamma_{i+ \theta^+}) + \frac{1}{3!} (\Psi^2 d\theta^+ + \varpi_+ \Psi^2 \varpi_- d\theta^-) \right] + o(\mu^2) \\
\bar{L}^- &= \mu \left[ -e \Pi (2e^+ \theta^- - e^i \Gamma_{i- \theta^+}) + \frac{1}{3!} (\Psi^2 d\theta^- + \varpi_- \Psi^2 \varpi_+ d\theta^+) \right] + o(\mu^2) \\
L^- &= d\theta^- + o(\mu^2)
\end{align*}
\]
(3.37)
The WZ term becomes
\[
L_{WZ} = T \left[ i \frac{4}{4 \mu} \left( 2 d\theta^+ \Gamma_{+ \tau_1} \Pi \left( L^+ \right) \bigg|_{\mu^1} - 2 d\bar{\theta}^- \Gamma_{+ \tau_1} \Pi \left( \bar{L}^- \right) \bigg|_{\mu^1} + o(\mu^3) \right) \right]_{\mu \to 0}
\]
\[
= T \left[ i d x^a \bar{\theta} \Gamma_{a} d \theta + \frac{1}{3} \bar{\theta} \Gamma_{a} \tau_3 d \theta \bar{\theta} \Gamma_{a} d \theta + \frac{1}{3} \bar{\theta} \Pi \bar{\theta} \Gamma_{i} \tau_1 d \theta \bar{\theta} \Pi \bar{\theta} \Gamma_{i} \epsilon d \theta - \frac{1}{12} \left( \bar{\theta} \Pi \bar{\theta} \Gamma_{a} \tau_1 d \theta \bar{\theta} \Pi \bar{\theta} \Gamma_{a} \Gamma_{i} d \theta - \bar{\theta} \Pi \bar{\theta} \Gamma_{i} d \theta \bar{\theta} \Pi \bar{\theta} \Gamma_{a} \epsilon d \theta \right) \right]
\]
\[
= T \left[ i d x^a \bar{\theta} \Gamma_{a} d \theta + \frac{1}{2} \bar{\theta} \Gamma_{a} \tau_3 d \theta \bar{\theta} \Gamma_{a} d \theta \right]
\]
(3.38)
where the last equality is derived by using the relation in the section 2.3 of [16]. This action is the Green-Schwarz superstring action in a flat space [17].
4 Hamiltonian and equations of motion

In this section we calculate the Hamiltonian, constraints and equations of motion of the system. Before examining the superstring in the pp-wave background, particle, superparticle and bosonic string cases are examined.

4.1 Bosonic particle

The action for a bosonic particle system in a curved background is

\[ S_{PA} = \int dt \frac{1}{2e} \dot{x}^m g_{mn} \dot{x}^n. \]

The canonical momentum of \( x^m \) is

\[ p_m = \frac{\delta S_{PA}}{\delta \dot{x}^m} = (1/e) g_{mn} \dot{x}^n \]

and the gauge invariant Hamiltonian is

\[ \mathcal{H}_{PA} = cH_{PA}, \quad H_{PA} = \frac{1}{2} p_m g^{mn} p_n = 0. \] (4.1)

It leads to the following equations of motion in \( e = 1 \) gauge

\[ \begin{cases} \dot{x}^m = g^{mn} p_n, \\ \dot{p}_m = -\frac{1}{2} (\partial_m g^{nl}) p_n p_l \end{cases} \] (4.2)

then

\[ \ddot{x}^m = -\Gamma^m_{nl} \dot{x}^n \dot{x}^l \] (4.3)

with the Affine connection coefficients defined by

\[ \Gamma^l_{nk} = \frac{1}{2} g^{lm} \left( -\partial_m g_{nk} + \partial_n g_{km} \right). \]

For the pp-wave background the metric is given by

\[ g_{mn} = \begin{pmatrix} g_{++} & g_{+-} & g_{+j} \\ g_{-+} & g_{--} & g_{-j} \\ g_{i+} & g_{i-} & g_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4\mu^2 x^2 & 0 \\ 0 & 0 & \delta_{ij} \end{pmatrix}, \quad g^{mn} = \begin{pmatrix} 4\mu^2 x^2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \delta_{ij} \end{pmatrix}, \] (4.4)

the Hamiltonian in the \( e = 1 \) gauge is given by

\[ \mathcal{H}_{PA} = \frac{1}{2} \left[ 2p_+ p_- + \dot{p}^2 + (2\mu p_+)^2 x^2 \right] \] (4.5)

and equations of motion are

\[ \begin{cases} \dot{x}^+ = p_- + (2\mu)^2 x^2 p_+ \\ \dot{x}^- = p_+ \\ \dot{x} = p \end{cases}, \quad \begin{cases} \dot{p}_- = 0 \\ \dot{p}_+ = 0 \\ \dot{p} = -(2\mu p_+)^2 x \end{cases} \] (4.6)
with \( \mathbf{x} = x^i \) and \( \mathbf{p} = p_i \). The second order equations of motion can be written as

\[
\begin{cases}
\dot{x}^+ = (2\mu)^2 p_+ (\dot{x}^2) \\
\dot{x}^- = 0 \\
\ddot{x} = -\omega^2 x, \quad \omega = 2\mu p_+
\end{cases}
\]  \hspace{1cm} (4.7)

and (4.6) and (4.7) are solved as

\[
\begin{cases}
x^+ = \left(p_+ + 2\mu^2 p_+ (\alpha^2 + \dot{\alpha}^2)\right) \tau - \frac{1}{2\mu} (\alpha^2 - \dot{\alpha}^2) \sin(4\mu x^-) - \mu \alpha \cdot \dot{\alpha} (\cos(4\mu x^-) - 1) + x_0^+ \\
x^- = p_+ \tau + x_0^- \\
x = \alpha \sin(2\mu x^-) + \dot{\alpha} \cos(2\mu x^-)
\end{cases}
\]  \hspace{1cm} (4.8)

where constants \( \alpha \) and \( \dot{\alpha} \) are amplitudes of harmonic motions in transverse directions. The Hamiltonian is also written as

\[ H_{PA} = \frac{1}{2} \left[ 2p_+ - \omega^2 (\alpha^2 + \dot{\alpha}^2) \right] \]  \hspace{1cm} (4.9)

The canonical momenta \( p_i \) are not constants of motion but the constants of motion are the global translation and boost charges \( P_a \) and \( P_i^* \). Their forms in terms of the canonical variables are calculated from

\[
\lambda^a P_a = p_m \delta_{\lambda} x^m = p_m \Delta_{\lambda} L^A (L^{-1})_A^m, \quad C^{-1}_{ab} \delta_{ab} G_{pp} = C^{-1}_{pp} \lambda a P_a G_{pp} = \Delta_{\lambda} L^A T_A, \]  \hspace{1cm} (4.10)

etc. and are

\[
P_a = \begin{cases}
P_+ = p_+ \\
P_- = -p_- \\
\hat{P}_i = p_i \cos(2\mu x^-) + 2\mu p_i x_i \sin(2\mu x^-) = \omega \alpha_i
\end{cases}
\]  \hspace{1cm} (4.11)

\[
P_i^* = (\frac{1}{2\sqrt{2}\mu})(p_i \sin(2\mu x^-) + 2\mu p_i x_i \cos(2\mu x^-)) = (\frac{1}{2\sqrt{2}\mu}) \omega \alpha^*_i.
\]  \hspace{1cm} (4.12)

In terms of global charges the Hamiltonian can be written as the quadratic Casimir operator of the pp-algebra

\[ H_{PA} = \frac{1}{2}(P_a \eta^{ab} P_b + P_i^* \eta^i_{\ j} P^*_j) \quad \eta^i_{\ j} \equiv \delta^i_{\ j}(2\sqrt{2}\mu)^2. \]  \hspace{1cm} (4.13)

This Hamiltonian is justified algebraically as it commutes with all global space-time symmetry charges. It is also obtained by the Penrose limit of the Hamiltonian of the AdS particle which is also the quadratic Casimir operator of the AdS algebra:

\[
\begin{cases}
c_{\text{AdS}} = \sum_{a=0,1,\ldots,4} P_a P_b \eta^{ab} + \sum_{ab=0,\ldots,4} (M_{ab})^2 \\
c_{\text{S}} = \sum_{a'=5,\ldots,9} P_{a'} P_{b'} \eta^{a'b'} + \sum_{a'\ b'=5,\ldots,9} (M_{a'b'})^2 \\
\text{Penrose limit} \quad \begin{cases}
c_{pp(-2)} = \sum_{a=0,1,\ldots,9} P_a P_b \eta^{ab} + \sum_{i=1,\ldots,8} P_i^* P_j^* \eta^i_{\ j} \Rightarrow 2H_{\text{PA}} \\
c_{pp(-4)} = P^2_+ .
\end{cases}
\end{cases}
\]  \hspace{1cm} (4.13)
There are two quadratic Casimir operators each for the $AdS_5$ and $S^5$. They turn to become two quadratic Casimir operators in the pp algebra. The one, which contains the Lorentz invariant mass term in a flat limit $\mu \rightarrow 0$, is the Hamiltonian of this system.

### 4.2 Superparticle

The action for a superparticle system in the pp-wave background is obtained by eliminating $\sigma$ dependence in (3.17), (3.29), (3.30), (3.31),

$$S_{\text{SUPA}} = \int d\tau \frac{1}{2e} L_0 \dot{a} L_0 \dot{b} \eta_{\dot{a}\dot{b}}$$

(4.14)

where $L_0 \dot{a}$ is the coefficient of $d\tau$ in the pullback of the MC form $L \dot{a}$,

$$L_0 \dot{a} = \dot{x}^m L_m \dot{a} + \dot{\theta}^\mu L_\mu \dot{a}, \quad L_m \dot{a} = e_m \dot{a} + \Theta_m ^\mu L_\mu \dot{a}.$$  

(4.15)

Canonical momenta for $x^m$ and $\theta^\mu$ are

$$p_m \equiv \frac{\delta S_{\text{SUPA}}}{\delta \dot{x}^m} = T e \dot{L}_m \dot{a} L_0 \dot{b} \eta_{\dot{a}\dot{b}}$$

(4.16)

$$\zeta_\mu \equiv \frac{\delta S_{\text{SUPA}}}{\delta \dot{\theta}^\mu} = T e \dot{L}_\mu \dot{a} L_0 \dot{b} \eta_{\dot{a}\dot{b}}.$$  

(4.17)

The gauge invariant Hamiltonian of the pp-superstring is obtained as

$$H_{\text{SUPA}} = e H_{\text{SUPA}} + F_{\text{SUPA},\mu} \Lambda^\mu$$

(4.18)

where primary constraints are

$$H_{\text{SUPA}} = \frac{1}{2} \pi_\dot{a} \pi_\dot{b} \eta_{\dot{a}\dot{b}} = 0$$

(4.19)

$$F_{\text{SUPA},\mu} = \zeta_\mu - \pi_\dot{a} L_\mu \dot{a} = 0$$

with super-invariant (up to local Lorentz) combination

$$\pi_\dot{a} = (e^{-1})^m_\dot{a} (p_m + \zeta_\mu \Theta_\mu ^\mu) = T e \dot{L}_0 \dot{b} \eta_{\dot{a}\dot{b}} \equiv (e^{-1})^m_\dot{a} \pi_m.$$  

(4.20)

Half of the fermionic constraints generates the $\kappa$-symmetry transformations the existence of which is given in the subsection 3.2. Other half is the set of second class constraints and the multipliers $\Lambda^\mu$’s in (1.19), associating with the second class constraints vanish by the consistency condition.

Equations of motion in $e = 1$ and $\Lambda = 0$ gauge are obtained in a background covariant way as

$$\begin{aligned}
\dot{x}^m &= \pi^m \\
\dot{p}_m &= -\frac{1}{2} \partial_m g^{nl} \pi_n \pi_l - \zeta (\partial_m \Xi_n ) \theta \pi^n \\
\dot{\theta}^\mu &= \pi_\dot{a} (\Xi_\dot{a} \theta)^\mu \\
\dot{\zeta}_\mu &= -\pi_\dot{a} (\Xi_\dot{a} )_\mu
\end{aligned}$$

(4.21)
with $\Theta_m^\mu = (\Xi_m)^\mu\nu\theta^\nu$ and second order equation is given by
\[
\ddot{x}^m = -\Gamma^m_{nl} \dot{x}^n \dot{x}^l - g^{mn} \dot{x}^l \zeta(\partial_m \Xi_l - [\Xi_m, \Xi_l]) \theta.
\] (4.23)

where indices $\hat{a}$ and $m$ are raised and lowered by $\eta^{\hat{a}\hat{b}}$ and $g^{mn}$ respectively.

The world line element is calculated as $ds_{\text{SUPA}}^2 = eH_{\text{SUPA}} = 0$ where (4.20) is used. The superparticle, which is the zero mode of the superstring, in the pp-wave background moves along the null geodesics and it satisfies the “massless” constraint $H_{\text{SUPA}} = 0$.

In order to solve equations of motion the concrete expression of the RR pp-wave background metric (4.4) and
\[
\Xi_m = \begin{cases} 
\Xi_+ = 0 \\
\Xi_- = \omega_+(2\mu \epsilon \Pi - 2\mu^2 x^3 \Gamma_+ \Gamma_-) \\
\Xi_i = -\mu \epsilon \Pi \Gamma_+ \Gamma_-
\end{cases}
\] (4.24)
are inserted. The Hamiltonian is given by
\[
H_{\text{SUPA}} = \frac{1}{2} \left[ 2p_+(p_- + \xi_\Xi \Xi_- \theta) + 4\mu^2 p_+ x^2 + (p_i + \zeta_\Xi \Xi_\theta)^2 \right] \\
= \frac{1}{2} \left[ 2p_+ \pi_- + \omega^2 x^2 + \pi^2 \right],
\] (4.25)
with $p_m = (p_+, p_-, \mathbf{p})$, $\pi_m = (\pi_+, \pi_-, \pi)$ and $x^m = (x^+, x^-, \mathbf{x})$. The equations of motion become
\[
\begin{align*}
\dot{x}^+ &= \pi_- + (2\mu)^2 x^2 p_+ \\
\dot{x}^- &= p_+ \\
\dot{x}^i &= \pi_i \\
\dot{p}_+ &= 0 \\
\dot{p}_- &= -\omega^2 x_1 + 2\mu^2 p_+ \zeta_\Xi \Gamma_+ \\
\dot{p}_i &= \pi_+ (\xi_\Xi \Xi_- \theta)^\mu + \pi_i (\xi_\Xi \Xi_\theta)^\mu \\
\dot{\xi}_\mu &= -p_+ (\xi_\Xi \Xi_- \theta)^\mu - p_i (\xi_\Xi \Xi_i)^\mu.
\end{align*}
\] (4.26)

Using facts $\dot{\pi}_- = 0$, $\dot{\pi} = -\omega^2 \mathbf{x}$, the second order equations for $x^m$ take same form as the bosonic particle (4.7). Furthermore $\dot{\theta}_\mu$ and $\dot{\zeta}_\mu$ satisfy second order harmonic equation with frequency $\omega$. It is shown by using $\partial_\xi \Xi_- = -\Xi_- \Xi_i$, $(\Xi_-)^2 = -4\mu^2 + 4\mu^2 x^3 \Xi_i$ and $\Xi_i \Xi_j = 0$. Solutions are found in the same form as (4.18) and (4.12) with replacing $p_-, p_i$ by $\pi_-, \pi_i$. The Hamiltonian is expressed as
\[
H_{\text{SUPA}} = E_B + E_F
\]
\[
E_B = \frac{1}{2} \left[ 2p_+ \pi_- + \omega^2 (\alpha^2 + \bar{\alpha}^2) \right] = \text{const}.
\]
\[
E_F = 2p_+^2 \mu \bar{\theta}_+ \Gamma^+ \left( \frac{\sin \frac{\psi}{2}}{\Psi_+/2} \right)^2 \epsilon \Pi \theta^+ - 2p_+^2 \mu \bar{\theta}_- \bar{\pi} \left( \frac{\sin \frac{\psi}{2}}{\Psi_+/2} \right)^2 \bar{\pi} \Gamma^- \theta^- = -\text{const}.
\]

where “+” projected part of fermionic constraints (4.19) are used.
In terms of global charges the Hamiltonian can be written as the quadratic Casimir operator of the super-pp-algebra which is obtained by the Penrose limit from the quadratic Casimir operator of the super-AdS algebra;

\[ c_{\text{AdS}} = \sum_{a=0,1,\ldots,9} P_a^\dagger P_a - \frac{1}{4} Q_{\alpha\alpha'} (\epsilon^{AB} C^{-1} C^{\gamma\delta} C^{\alpha\alpha'} C^{\beta\beta'}) Q_{\beta\beta'} B \]

\[ \text{Penrose limit} \quad c_{\text{supp}(-2)} = \sum_{a=0,1,\ldots,9} P_a^\dagger P_a + \sum_{i=1,\ldots,9} M_i^2 + \frac{1}{4} \eta_{ij} Q_+ (C^{-1} \Gamma + \Pi) Q_+ \Rightarrow 2H_{\text{SUPA}} \]

(4.28)

with \( \eta_{ij} = (2\sqrt{2}\mu)^2 \delta_{ij} \) and \( \eta_\nu = 2\sqrt{2}\mu \). The supercharge is obtained by the Penrose limit from the one of the super-AdS result [16]

\[ Q_+ = \left\{ \zeta_+ (1 - \frac{i}{2\sqrt{2}} \Gamma_+ i \theta - \Gamma_+ \Gamma^{ij} \Pi) (p_+ i \theta' + p_+ i \bar{\theta}' - \gamma_i) \left( \frac{\sin \psi_+}{2} \right)^2 \left( \frac{\psi_+}{\sin \psi_+} \right) \right\} \left( \cos(2\mu x^-) - \epsilon \Pi \sin(2\mu x^-) \right) \]

(4.29)

which is consistent with that the Hamiltonian (4.25) and the quadratic Casimir operator in (4.28).

### 4.3 Bosonic string

The action for a bosonic string system is obtained by eliminating \( \theta \) dependence in (3.28),

\[ S_{\text{ST}} = -T \int d^2 \sigma \sqrt{-h} h^{uv} \partial_u x^m \partial_v x^n . \]

Canonical momentum of \( x^m(\sigma) \) is \( p_m(\sigma) = \delta S_{\text{ST}} / \delta \dot{x}^m(\sigma) = -T \sqrt{-h} h^{uv} \partial_u x^m g_{mn} \partial_v x^n \). The gauge invariant Hamiltonian of the pp-string is obtained as

\[ H_{\text{ST}} = \int d\sigma \left( \frac{\sqrt{-h}}{h_{11}} H_\perp + \frac{h_{01}}{h_{11}} H_\parallel \right) , \]

(4.30)

where primary constraints are

\[ H_\perp(\sigma) = \frac{1}{2T} (p_m g^{mn} p_n + T^2 x^m g_{mn} x^n) = 0 \]

(4.32)

\[ H_\parallel(\sigma) = p_m x^m = 0 . \]

(4.33)

The equation of motion in the conformal gauge, \( h_{uv} = \eta_{uv} \), are obtained as

\[ \left\{ \begin{array}{l} \dot{x}^m(\sigma) = \frac{1}{T} g^{mn} p_n \\ \dot{p}_m(\sigma) = T (g_{mn} x^n)' + \frac{T}{2} (\partial_m g_{nl}) \left( \frac{1}{T^2} p^np^n - x^m x^n \right) \end{array} \right. \]

(4.34)
and
\[ \Box x^m(\sigma) = -\Gamma^m_{nl}(\dot{x}^n \dot{x}^l - x^m x^l) \]  
(4.35)

with \( \Box = \partial^2_{\tau} - \partial^2_{\sigma} = \partial^2_0 - \partial^2_1 \).

For the RR pp-wave background case, by using (4.4), the Hamiltonian becomes
\[ \mathcal{H}_{ST} = \frac{1}{2T} \int d\sigma \left[ 2p_+ p_- + p^2 + (2\mu p_+)^2 x^2 + T^2 \left( x^+ x^-' + x^2 - (2\mu)^2 x^2 x^{-l^2} \right) \right]. \]
(4.36)

The equations of motion are
\[
\begin{align*}
\dot{x}^+ (\sigma) &= \frac{1}{T} (p_+ - (2\mu)^2 x^2 p_+) \\
\dot{x}^- (\sigma) &= \frac{1}{T} p_+ \\
\dot{x}(\sigma) &= \frac{1}{T} p
\end{align*}
\]
and ones of the second order form become
\[
\begin{align*}
\Box x^+ (\sigma) &= (2\mu)^2 x^2 \ddot{x}^- \\
\Box x^- (\sigma) &= 0 \\
\Box x(\sigma) &= (2\mu)^2 x \left( (x^-)^2 - (\dot{x}^-)^2 \right)
\end{align*}
\]
(4.38)

By solving equation of motion for \( x^- \) in (4.38) as
\[
x^- (\sigma) = \frac{p_{+,0}}{T} \tau + \frac{1}{\sqrt{2\pi}} \sum_{n \neq 0} \left( \alpha_n e^{2i n (\sigma - \tau)} + \bar{\alpha}_n e^{-2i n (\sigma + \tau)} \right), \]
(4.39)
the last term of the last equation in (4.38) becomes
\[
(x^-)^2 - (\dot{x}^-)^2 = - \left( \frac{p_{+,0}}{T} - 4i \sum_{n \neq 0} n \alpha_n e^{2i n (\sigma - \tau)} \right) \left( \frac{p_{+,0}}{T} - 4i \sum_{n \neq 0} n \bar{\alpha}_n e^{-2i n (\sigma + \tau)} \right) \]
(4.40)

It is difficult to solve the equation for \( x \) except in a case where only the zero-mode term \((p_{+,0}/T)^2\) is present. It is the same as the light-cone result [20]
\[
x = \alpha_0 \sin(\omega_0 \tau) + \bar{\alpha}_0 \cos(\omega_0 \tau) + \frac{1}{\sqrt{2\pi}} \sum_{n \neq 0} \left( \alpha_n e^{i(\omega_n \tau + 2n \sigma)} + \bar{\alpha}_n e^{i(\omega_n \tau - 2n \sigma)} \right) \]
(4.41)

with \( \alpha^\dagger_n = \alpha_{-n} \) and \( \bar{\alpha}^\dagger_n = \bar{\alpha}_{-n} \) and the Hamiltonian becomes
\[
\mathcal{H}_{ST} = \frac{1}{2T} \left[ \omega_0^2 (\alpha_0^2 + \bar{\alpha}_0^2) + \sum_{n \neq 0} \omega_n^2 \alpha_n \alpha_{-n} + \cdots \right]. \]
(4.42)
4.4 Superstring

Now let us examine a superstring in the RR pp-wave background which is described by the covariant action (3.28). The conjugate momenta are

\[
p_m(\sigma) \equiv \frac{\delta \mathcal{L}}{\delta \dot{x}^m} = \frac{\delta \mathcal{L}_0}{\delta \dot{x}^m} \mathbf{L}_m \dot{\hat{a}} + \frac{\delta \mathcal{L}_{WZ}}{\delta \dot{L}_0^\alpha} L_\mu^\alpha
\]

(4.43)

\[
\zeta_\mu(\sigma) \equiv \frac{\delta^\nu \mathcal{L}}{\delta \dot{\theta}_\mu} = \frac{\delta \mathcal{L}_0}{\delta \dot{\theta}_\mu^\alpha} L_\mu^\alpha + \frac{\delta^\nu \mathcal{L}_{WZ}}{\delta \dot{L}_0^\alpha} L_\mu^\alpha.
\]

(4.44)

The gauge invariant Hamiltonian of the pp-superstring is obtained as

\[
\mathcal{H} = \frac{\sqrt{-g}}{h_{11}} \mathcal{H}_\perp + \frac{h_{01}}{h_{11}} \mathcal{H}_\parallel
\]

\[
\mathcal{H}_\perp(\sigma) = H_\perp - F_\tau \theta'
\]

(4.45)

\[
\mathcal{H}_\parallel(\sigma) = H_\parallel + F \theta'
\]

where primary constraints are

\[
H_\perp(\sigma) = \frac{1}{2T}(\tilde{\pi}_a \tilde{\pi}_b \delta \hat{a} \dot{\hat{b}} + T^2 \mathbf{L}_1 \dot{\hat{a}} \mathbf{L}_1 \dot{\hat{b}} \eta_{\hat{a}\hat{b}}) = 0
\]

(4.46)

\[
H_\parallel(\sigma) = \tilde{\pi}_a \mathbf{L}_1 \dot{\hat{a}} = 0
\]

(4.47)

\[
F_\nu(\sigma) = \zeta_\nu - \tilde{\pi}_a \mathbf{L}_\nu \dot{\hat{a}} - \frac{\delta \mathcal{L}_{WZ}}{\delta \dot{L}_0^\beta} L_\nu^\beta = 0
\]

(4.48)

Half of the above fermionic constraints generate \(\kappa\)-symmetry as shown in the section 3.2 for the action (3.28).

Super-invariant (up to local Lorentz) combinations are \(\mathbf{L}_\mu^{\hat{a}}\) and

\[
\tilde{\pi}_a = (e^{-1})_{\hat{a} m} (\tilde{p}_m + \zeta_\mu \tilde{\pi}_m \theta^\mu) = -\frac{T}{\sqrt{-G}}(-\mathbf{L}_0 \dot{\hat{a}} G_{11} + \mathbf{L}_1 \dot{\hat{b}} G_{01}) \eta_{\hat{a}\hat{b}} \equiv (e^{-1})_{\hat{a} m} \tilde{p}_m
\]

\[
\tilde{p}_m = p_m - \frac{\delta \mathcal{L}_{WZ}}{\delta \dot{L}_0^\beta} L_\mu^\beta, \quad \zeta_\mu = \zeta_\mu - \frac{\delta \mathcal{L}_{WZ}}{\delta \dot{L}_0^\beta} L_\mu^\beta.
\]

The equations of motion for a superstring in the pp-wave background in the conformal gauge, \(h_{uv} = \eta_{uv}\), are obtained as

\[
\begin{align*}
\dot{x}^m &= \frac{1}{T} \tilde{\pi}^m + (e^{-1})_{\hat{a} m} \mathbf{L}_\mu^\alpha \dot{\hat{a}} \zeta_\mu \theta^\mu \\
\dot{p}_m &= T \left( \mathbf{L}_\mu^{\hat{a}} \mathbf{L}_{1\hat{a}} \right) + \left( \partial_m g_{\hat{a} l} \right) \left( \frac{1}{2T} \pi_{\hat{a} m} \pi^{l} - \frac{T}{2} \mathbf{L}_1 \dot{\hat{a}} (e^{-1})_{\hat{a} m} \mathbf{L}_1 \dot{\hat{b}} (e^{-1})_{\hat{b} l} - \tilde{\pi}_a \mathbf{L}_\mu^\alpha (e^{-1})_{\hat{a} m} l (\tau_3 \theta')^\mu \right) \\
&\quad - \frac{1}{T} \zeta (\partial_m \pi_n) \theta \tilde{\pi}^n - T \pi^{mn} ((\partial_m \pi_n) \theta)^\mu \mathbf{L}_\mu^\alpha \mathbf{L}_{1\hat{a}} \\
&\quad - \partial_m \left( \frac{\delta \mathcal{L}_{WZ}}{\delta \dot{L}_0^\alpha} \right) L_\mu^\alpha (\tau_3 \theta')^\mu
\end{align*}
\]

(4.49)
\begin{align}
\dot{\mu} & = - (\tau_3 \theta')^\mu (\Xi_i \theta (x^{-1}))_{,\alpha} \hat{\theta} \{ T (L_{\mu} \hat{\alpha} L_{1 \alpha})' - \tilde{\Xi} \tilde{\rho} L_{\mu} \hat{\alpha} \tau_3 \} \\
\dot{\zeta} & = - (\zeta \tau_3)' - T (L_{\mu} \hat{\alpha} L_{1 \alpha})' - \tilde{\Xi} \tilde{\rho} L_{\mu} \hat{\alpha} \tau_3 \\
& \quad - (\zeta \Xi_{\alpha}) \left( \hat{\pi} + L_{\mu} \hat{\alpha} (\tau_3 \theta')^\mu \right) \\
& \quad + T \left( x^m (\Xi m \tau_3 \theta')^\nu \right) L_{\nu} \hat{\alpha} - \left( x^m (\Xi m \tau_3 \theta')^\nu + \theta^\nu \right) \frac{\partial_{\theta}^\text{left}}{\partial_{\theta}^\text{left}} L_{\nu} \hat{\alpha} \\
& \quad + \frac{\partial_{\theta}^\text{left}}{\partial_{\theta}^\text{left}} \left( \delta L_{WZ} \alpha \right) \left( \tau_3 \theta')^\nu + \tilde{\Xi} \tilde{\rho} \left( \frac{\partial_{\theta}^\text{left}}{\partial_{\theta}^\text{left}} L_{\nu} \hat{\alpha} \right) \left( \tau_3 \theta')^\nu \right)
\end{align}

(4.50)

which are background covariant.

In the components by using (4.3) and (4.24) the Hamiltonian in the conformal gauge is rewritten as

\begin{align}
\mathcal{H} & = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 \\
\mathcal{H}_1 & = \int d\sigma \left[ \frac{1}{2 T} \left( 2 p_+ \hat{\pi} + 4 \mu^2 x^2 p_+^2 + \hat{\pi}^2 \right) \\
& \quad + p_+ L_{\mu}^\mu (\tau_3 \theta')^\mu + (\hat{\pi} - 2 \mu^2 x^2 p_+) L_{\nu}^\nu (\tau_3 \theta')^\nu - \tilde{\Xi} \tilde{\rho} L_{\mu}^\mu \tilde{\Xi} \tilde{\rho} \tau_3 \right]
\end{align}

(4.51)

\begin{align}
\mathcal{H}_2 & = \int d\sigma \left[ \frac{T}{2} \left( 2 \left( x^{+} - 2 \mu^2 x^2 x^{-} \right) \theta + x^{-} \Xi_\theta + x^{+} \Xi_\theta \right) L_{\mu}^\mu \left( x^{+} + (\theta')^\nu \right) L_{\nu} \right. \\
& \left. \quad + \left( x^{+} + (\theta')^\nu \right) L_{\nu} \right)
\end{align}

(4.52)

\begin{align}
\mathcal{H}_3 & = \int d\sigma \left[ - \zeta \tau_3 \theta' \right].
\end{align}

For a flat case \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) reduce into simply \( \int (1/2T)p^2 \) and \( \int (T/2)x^2 \) respectively.

Equations of motion are written as

\begin{align}
\left\{ \begin{array}{l}
\dot{x}^+ = \frac{1}{2} \left( \hat{\pi} + 4 \mu^2 x^2 p_+ + L_{\mu}^\mu (\tau_3 \theta')^\mu + 2 \mu^2 x^2 L_{\nu}^\nu (\tau_3 \theta')^\nu \\
\dot{x}^- = \frac{1}{2} p_+ + L_{\mu}^\mu (\tau_3 \theta')^\mu \\
\dot{x}^i = \frac{1}{2} \hat{\pi} + L_{\mu}^\mu (\tau_3 \theta')^\mu \\
\dot{p}_- = T \left( -2 \mu^2 x^2 L_{1 \nu} + (\Xi_\theta)^\mu L_{\mu}^\mu L_{1 \nu} - L_{\mu}^\mu L_{1 \nu} \hat{\alpha} \right) \\
+ \int d\sigma' \frac{\partial}{\partial x^i} \left( \frac{\delta L_{WZ} \mu}{\delta L_{0 \alpha}} \right) \left\{ 2 L_{\mu}^\mu \left( \Xi_\theta (\tau_3 \theta')^\mu \right) - L_{\mu}^\mu (\tau_3 \theta')^\mu \right\}
\end{array} \right.
\end{align}

(4.53)
In this paper a form of the gauge invariant action for the superstring in the RR pp-wave background is proposed. It is explicit since the Wess-Zumino term is bilinear with respect to the LI currents of the super-pp-wave algebra. It is obtained by the Penrose limit from the superstring action in the AdS background with the bilinear WZ term [5]. The Penrose limit of the WZ term is given explicitly as follows:

\[
\begin{align*}
\square x^+ (\sigma) &= (2\mu)^2 \dot{x}^2 \dot{x}^- + \theta^\pm \text{ dependent terms} \\
\square x^- (\sigma) &= \partial_\sigma \left( L_{\nu}^- (\tau_3 \theta^-)^{\nu} \right) + \partial_\sigma \left( L_{\nu}^- (\theta^-)^{\nu} \right) + i \left( \frac{\sin \Psi_- c \Pi \theta^-}{\Psi_-} \right) C \Gamma - \tau_1 \Pi \frac{\sin \Psi_-}{\Psi_-} \tau_3 \theta^- \right) \\
\square x (\sigma) &= (2\mu)^2 x \left( (x^-)^2 - (\dot{x}^-)^2 \right) + \theta^- \text{ dependent terms}
\end{align*}
\]

The right hand sides of the first and third equations are complicated functions of $\theta^-$ or $\theta^-$ and $\theta^+$.

5 Conclusions and discussions

In this paper a form of the gauge invariant action for the superstring in the RR pp-wave background is proposed. It is explicit since the Wess-Zumino term is bilinear with respect to the LI currents of the super-pp-wave algebra. It is obtained by the Penrose limit from the superstring action in the AdS background with the bilinear WZ term [5]. The Penrose limit of the WZ term is given explicitly as follows:

1. Rescale the coordinates in the LI 1-forms with a parameter $\Omega$ with suitable weights, for example

\[
L_{\text{AdS}}^{-} (z^m \rightarrow \Omega^N z^m) = L_{(0)}^{-} (z^m) + \Omega^2 L_{(2)}^{-} (z^m) + o(\Omega^4) .
\]  

2. Rescale the WZ term with the same weight of the Nambu-Goto term as

\[
L_{\text{WZ,AdS}} \rightarrow \Omega^{-2} L_{\text{WZ,AdS}} .
\]
3. Subtract the divergent term in $\Omega \to 0$ limit which is proportional to $1/\Omega^2$ and closed.

4. Take the $\Omega \to 0$ limit in the bilinear WZ term

$$L_{WZ, AdS} \to L_{WZ, pp} = 2L_{(0)}^+ \Gamma^\tau \Pi L_{(2)}^- + L^+ \text{ dependent term}.$$ 

This procedure corresponds to make a nondegenerate super-pp-wave group by introducing the fermionic center associating to $L_{(2)}^- = \tilde{L}^-$. This is contrasted with the conventional WZ term case:

1. Rescale coordinates in the LI 1-forms with a parameter $\Omega$ with suitable weights and take the leading terms in the limit $\Omega \to 0$

$$L_{AdS}^-(z^m \to \Omega^N z^m) \to L_{(0)}^-(z^m), \quad L_{AdS}^+(z^m \to \Omega^N z^m) \to \Omega^2 L_{(2)}^+(z^m). \quad (5.3)$$

2. Construct the WZ term in the conventional form given by an integral of the three form $\mathcal{L}$

$$\int d^2\sigma L_{CWZ, AdS} \to \int d^2\sigma L_{CWZ, pp} = \int d^3\sigma [L_{(0)}^+ \mathcal{L}_{(2)}^+ \tau_3 L_{(0)}^- + L^\pm \text{ dependent terms}].$$

In this case the next to leading term $L_{(2)}^-$ is not necessary to be preserved in taking the Penrose limit of the WZ term. However the resultant WZ term does not allow us simple treatment of the Hamiltonian and equations of motion.

The Hamiltonians for the bosonic particle, superparticle, string and superstring in the RR pp-wave background are obtained in the conformal gauge. The particle and the superparticle Hamiltonians are identified with the quadratic Casimir operators of the pp-wave and the super-pp-wave algebras respectively. Once the superparticle Hamiltonian (4.18) is recognized as the super-pp-invariant “mass operator” of the superstring theory, all states in a supergravity multiplet are “massless” and the supersymmetry is manifest.

The world-sheet reparametrization generators and the local fermionic constraints are also obtained. The combinations of the reparametrization constraints (4.46) and (4.47) as $\mathcal{H}_\| \pm \mathcal{H}_\perp$ and the first class part of $F_\nu^I$, where $I = 1, 2$ correspond to right/left $(\pm)$ modes, will make a closed set of constraint algebra, namely $ABCD$ constraint system $[2]$. It was shown that the local constraints of the AdS superstring satisfy the $ABCD$ algebra [17] as well as of the flat superstring. Since the super-pp-wave algebra is obtained by the Penrose limit from the super-AdS algebra [3] as well as the flat algebra by the flat limit, the local symmetry algebra is also expected to be obtained by the same limiting procedure preserving the same structure of the $ABCD$ algebra. The background independence of the local symmetries is plausible.

Equations of motion in the conformal gauge are obtained and are background covariant. The equations of motion for $x^\pm$ are obtained and will be important for taking into account interactions. In order to quantize the superstring theory in the conformal gauge there may be required a suitable change of variables such as to the GL$(4\mid 4)$ matrix variable
as was done for the AdS$_5 \times $S$^5$ case [13]. The covariant approach will be useful to examine symmetry structures toward the covariant superstring field theory, S-T-U dualities which are deeply related to the background symmetry and non-perturbative properties such as BPS conditions.

A Cartan 1-forms for AdS$_5 \times $S$^5$

The Cartan 1-forms in the AdS$_5 \times $S$^5$ space are presented [20, 16]. The left-invariant Cartan one-forms of a coset SU(2|4)/[SO(4,1) × SO(5)] ≃ G = G(x, θ) = e$^{xP}e^{θQ}$ are defined by $G^{-1}\,dG = L^a P_a + L^a' P'_a + \frac{1}{2} L^{ab} J_{ab} + \frac{1}{2} L^{a'b'} J_{a'b'} + L^{αα'}Q_{αα'}$. They are given by

$$
\left\{
\begin{align*}
L^a &= e^a + iθCC'γ^a \left(\frac{sin(\frac{Ψ}{2})}{Ψ/2}\right)^2 Dθ, \\
L^{ab} &= ω^{ab} - θCC'γ^{αβ} \left(\frac{sin(\frac{Ψ}{2})}{Ψ/2}\right)^2 Dθ, \\
L^a &= \frac{sinΨ}{Ψ} Dθ, \\
ω^{ab} &= \frac{1}{2} \left(\frac{sin(\frac{Ψ}{2})}{Ψ/2}\right)^2 dx^{[a}x^{b]}, \\
ω^{a'b'} &= -\frac{1}{2} \left(\frac{sin(\frac{Ψ}{2})}{Ψ/2}\right)^2 dx^{[a'}x^{b']}
\end{align*}
\right.
$$

where $[ab] = ab - ba$ and charge conjugation matrix for AdS$_5$ space and S$^5$ space are $C$ and $C'$ respectively and

$$
Dθ = \left[ d - \frac{i}{2} (γ^a e_a + iγ^a'e_{a'}) + \frac{1}{4} (γ^{ab} ω_{ab} + γ^{a'b'} ω_{a'b'}) \right] θ
$$

\[
(Ψ^2)^{αα'}_{ββ'} = (εγ^aθ)^{αα'} (θCC'γ_a)_{ββ'} - (εγ^aθ)^{αα'} (θCC'γ_a')_{ββ'} + \frac{1}{2} (γ^{a'b'} θ)^{αα'} (θCC'γ_{a'b'})_{ββ'} - \frac{1}{2} (γ^{ab} θ)^{αα'} (θCC'γ_{ab})_{ββ'}
\]

(A.2)

After the Penrose limit using (2.4) and (3.9) they are written as (B.29). The relation between the AdS variables and the Penrose variables are described in [5].

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