Jucys–Murphy elements of partition algebra for the rook monoid

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Abstract

Kudryavtseva and Mazorchuk exhibited Schur–Weyl duality between the rook monoid algebra $\mathbb{C} R_n$ and a subalgebra, denoted by $\mathbb{C} I_k$, of the partition algebra $\mathbb{C} A_k(n)$ acting on $(\mathbb{C}^n)^{\otimes k}$. In this paper, we consider a subalgebra $\mathbb{C} I_{k+\frac{1}{2}}$ of $\mathbb{C} I_{k+1}$ such that there is Schur–Weyl duality between the actions of $\mathbb{C} R_{n-1}$ and $\mathbb{C} I_{k+\frac{1}{2}}$ on $(\mathbb{C}^n)^{\otimes k}$. We call $\mathbb{C} I_k$ and $\mathbb{C} I_{k+\frac{1}{2}}$ totally propagating partition algebras. This paper studies the representation theory of $\mathbb{C} I_k$ and $\mathbb{C} I_{k+\frac{1}{2}}$ inductively by considering the tower ($\mathbb{C} I_1 \subset \mathbb{C} I_2 \subset \mathbb{C} I_2 \subset \cdots$) whose Bratteli diagram turns out to be a simple graph. Furthermore, this inductive approach is established as a spectral approach by describing the Jucys–Murphy elements and their actions on the canonical Gelfand–Tsetlin bases of irreducible representations of $\mathbb{C} I_k$ and $\mathbb{C} I_{k+\frac{1}{2}}$. Also, we describe the Jucys–Murphy elements of $\mathbb{C} R_n$ which play a central role in the demonstration of the actions of Jucys–Murphy elements of $\mathbb{C} I_k$ and $\mathbb{C} I_{k+\frac{1}{2}}$. In addition, we compute the Kronecker product of $\mathbb{C}^n$ and an irreducible representation of $\mathbb{C} R_n$ which is further used to specify the decomposition of $(\mathbb{C}^n)^{\otimes k}$ inductively.

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1 Introduction

Jones [Jon94] and Martin [Mar94], independently, discovered partition algebras to study the Potts model in statistical mechanics and also as a generalization of Temperley–Lieb algebras. For \( k \in \mathbb{Z}_{\geq 0} \) and \( \xi \in \mathbb{C} \), the partition algebra \( \mathbb{C}A_k(\xi) \) has a basis \( A_k \) consisting of partition diagrams corresponding to the set partitions of \( \{1, 2, \ldots, k, 1', 2', \ldots, k'\} \). The symmetric group \( S_n \) on \( n \) letters acts on \( V = \mathbb{C}^n \) in a natural way. For \( \xi = n \), Jones [Jon94] defined an action of \( \mathbb{C}A_k(n) \) on the tensor space \( V \otimes k \) and showed Schur–Weyl duality between the actions of the group algebra \( \mathbb{C}S_n \) and the partition algebra \( \mathbb{C}A_k(n) \) on \( V \otimes k \). Furthermore, Martin and Saleur [MS94] showed \( \mathbb{C}A_k(\xi) \) is semisimple unless \( \xi \) is a nonnegative integer less than \( (2k - 1) \).

Martin and Rollet [MR98] introduced a subalgebra, denoted by \( \mathbb{C}A_k+k^2(\xi) \), of \( \mathbb{C}A_{k+1}(\xi) \). Also, they demonstrated Schur–Weyl duality between the actions of \( \mathbb{C}A_{k+\frac{1}{2}}(n) \) and \( \mathbb{C}S_{n-1} \) on \( V \otimes k \), where \( S_{n-1} \) consists of those permutations in \( S_n \) which fix \( n \). The algebra \( \mathbb{C}A_{k+\frac{1}{2}}(\xi) \) is also called partition algebra. Moreover, Martin described the irreducible representations of partition algebras in [Mar94]. Whenever partition algebras are semisimple, the branching rule for the inclusion

\[
\mathbb{C}A_k(\xi) \subset \mathbb{C}A_{k+\frac{1}{2}}(\xi) \subset \mathbb{C}A_{k+1}(\xi),
\]

was determined in [MR98, Proposition 1] and the Bratteli diagram for the tower of algebras

\[
\mathbb{C}A_0(\xi) \subset \mathbb{C}A_{\frac{1}{2}}(\xi) \subset \mathbb{C}A_1(\xi) \subset \mathbb{C}A_{1+\frac{1}{2}}(\xi) \subset \mathbb{C}A_2(\xi) \subset \cdots, \tag{1}
\]

was constructed. The Bratteli diagram of the tower (1) is a simple graph, i.e., for \( l \in \frac{1}{2}\mathbb{Z}_{\geq 0} \), the restriction of an irreducible representation of \( \mathbb{C}A_l(\xi) \) to \( \mathbb{C}A_{l-\frac{1}{2}}(\xi) \) is multiplicity free.

An important tower of groups which has a simple Bratteli diagram is the family of symmetric groups

\[
S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots. \tag{2}
\]

Over the years, the representation theory of symmetric groups has generated considerable interest because of its rich interaction with combinatorial techniques and methods. One of the recent approaches to the representation theory of symmetric groups was given by Okounkov and Vershik [OV96, VO04]. This is a spectral approach which explains the
appearance of Young diagrams and standard Young tableaux in the representation theory of symmetric groups. Using the simplicity of the Bratteli diagram (2), Okounkov and Vershik obtained the Gelfand–Tsetlin decomposition and the canonical Gelfand–Tsetlin basis of an irreducible representation of $S_n$. The actions of some special elements in the group algebra $\mathbb{C}S_n$, called Jucys–Murphy elements [VO04, Section 2], on the canonical Gelfand–Tsetlin basis constitute the spectral approach to the representation theory of $S_n$.

The Jucys–Murphy elements play an important role in the representation theory when we have a multiplicity free tower of algebras or groups, to mention some specific instances: for partition algebras by Halverson and Ram [HR05, p. 898], for generalized symmetric groups by Pushkarev [Pus97, Definition 2(a)], Mishra and Srinivasan [MS16, Section 4], for $q$-rook monoid algebras (when $q$ is not a root of unity) by Halverson [Hal04, Section 3.1], and for partition algebras for complex reflection groups by Mishra and Srivastava [MS, Section 7]. In all these cases, the Jucys–Murphy elements have two fundamental properties: (i) these elements commute with each other, and (ii) the eigenvalues of the actions of these elements distinguish the non-isomorphic irreducible representations of the algebras or groups to which these elements correspond. Thus, the Jucys–Murphy elements give a spectral approach to the representation theory.

Let $R_n$ be the set consisting of $n \times n$ matrices whose entries are either 0 or 1 such that each row and each column has at most one nonzero entry. The set $R_n$ is a monoid with respect to matrix multiplication, and Solomon [Sol02] called $R_n$ a rook monoid. Treating $R_n$ as a semigroup, Munn [Mun57] determined its representation theory. The rook monoid algebra $\mathbb{C}R_n$ is semisimple over $\mathbb{C}$. The irreducible representations of $\mathbb{C}R_n$ are indexed by the partitions of $r$, where $0 \leq r \leq n$. Grood [Gro02] constructed an analog of the Specht modules for $\mathbb{C}R_n$. Paget [Pag06] studied a characteristic free approach to the representation theory of $R_n$. Solomon [Sol02, Theorem 5.10] gave the Schur–Weyl duality between the actions of $\mathbb{C}R_n$ and the general linear group $GL(n)$ on $U^\otimes k$, where $U = \mathbb{C}^n \oplus \mathbb{C}$.

Solomon [Sol04] defined $q$-rook monoid algebra $\mathcal{I}_n(q)$ over $\mathbb{C}(q)$, and when $q \to 1$, it is the rook monoid algebra $\mathbb{C}R_n$. Halverson [Hal04] constructed the irreducible representations of $\mathcal{I}_n(q)$ and showed that the branching rule for the inclusion $\mathcal{I}_{n-1}(q) \subset \mathcal{I}_n(q)$ is multiplicity free. In particular, this implies that the Bratteli diagram of the following tower

$$\mathcal{I}_1(q) \subset \mathcal{I}_2(q) \subset \cdots$$

(3)

is a simple graph. Halverson and Ram [HR04, p. 240] showed that the Bratteli diagram of the tower (3) is a subgraph of the Bratteli diagram of the Iwahori–Hecke algebras of type $B$. When $q$ is a root of unity, the actions of the Jucys–Murphy elements defined in [Hal04, Section 3.1] are not sufficient to distinguish the non-isomorphic irreducible representations of $\mathcal{I}_n(q)$. In this paper, for $q = 1$, we define more elements which along with the elements in [Hal04, Section 3.1] perform the role of the Jucys–Murphy elements in $\mathbb{C}R_n$.

The space $V = \mathbb{C}^n$ is an irreducible representation of $\mathbb{C}R_n$. Kudryavtseva and Ma-
zorchuk [KM08] gave the Schur–Weyl duality between the actions of $\mathbb{C}R_n$ and a subalgebra of $\mathbb{C}A_k$ on $V^\otimes k$. This subalgebra, denoted by $CI_k$, is the monoid algebra of the submonoid $I_k$ of $A_k$, where $I_k$ consists of those partition diagrams each of whose block is propagating, i.e., a block which intersects non-trivially with both $\{1, 2, \ldots, k\}$ and $\{1', 2', \ldots, k'\}$. The monoid $I_k$ appeared first time in a work of Fitzgerald and Leech [FL98] as a dual symmetric inverse monoid and as a categorical dual of $R_n$. Maltcev [Mal07] described a generating set of $I_k$ and also determined the automorphisms of $I_k$. Easdown, East, and Fitzgerald [EEF08] gave a monoid presentation of $I_k$. In this paper, we study the representation theory of the monoid algebras $CI_k$ and $CI_{k+\frac{1}{2}} := CI_{k+1} \cap CA_{k+\frac{1}{2}}(\xi)$. These algebras are independent of the parameter $\xi$ and are always semisimple over $\mathbb{C}$. We call $CI_k$ and $CI_{k+\frac{1}{2}}$ totally propagating partition algebras.

The main results in this paper are as follows.

(a) For the rook monoid algebras:

(i) We give Jucys–Murphy elements (14) of $\mathbb{C}R_n$ and describe their actions on the Gelfand–Tsetlin bases of the irreducible representations of $\mathbb{C}R_n$ in Theorem 2.7.

(ii) In Theorem 2.10, we compute the Kronecker product of $\mathbb{C}^n$ with any irreducible representation of $\mathbb{C}R_n$.

(iii) As a consequence of Theorem 2.10, the multiplicity of an irreducible representation of $\mathbb{C}R_n$ in the decomposition of $(\mathbb{C}^n)^\otimes k$ can be encoded in terms of paths corresponding to a graph associated with $(\mathbb{C}^n)^\otimes k$. Using an idea from Benkart, Halverson, and Harman [BHH16, Section 5.3] involving Robinson–Schensted–Knuth row-insertion, we give a bijection in Corollary 2.12 between the set of these paths and the set of standard set-partition tableaux. In particular, we recover a result of Solomon [Sol02, Example 3.18] which gives the multiplicity of an irreducible representation of $\mathbb{C}R_n$ in $(\mathbb{C}^n)^\otimes k$ involving a Stirling number of the second kind and the number of standard Young tableaux.

(b) For the totally propagating partition algebras:

(i) We construct a tower (27) of totally propagating partition algebras and note that it is not induced from the tower (1) because the embedding (26) of $CI_k$ in $CI_{k+\frac{1}{2}}$ is different from the embedding [HR05, Equation (2.2)] of $CA_k(\xi)$ in $CA_{k+\frac{1}{2}}(\xi)$. In Theorem 3.17, we prove that the Bratteli diagram of the tower (27) is a simple graph.

(ii) We give Jucys–Murphy elements (41) of $CI_k$ and $CI_{k+\frac{1}{2}}$, and describe their actions on the canonical Gelfand–Tsetlin bases of irreducible representations of $CI_k$ and $CI_{k+\frac{1}{2}}$ in Theorem 4.6. Moreover, Corollary 4.7 observes how the Jucys–Murphy elements give a spectral approach to the representation theory of totally propagating partition algebras.
The outline of this paper is as follows. We start Section 2 with a brief overview of the irreducible representations of $CR_n$. In the rest of this section, we give new results about the representation theory of $CR_n$, in particular, Jucys–Murphy elements of $CR_n$, and the Kronecker product of $C^n$ with an irreducible representation of $CR_n$. This section ends with a proof of Frobenius reciprocity between modified induction and restriction rules for the inclusion $CR_{n-1} \subset CR_n$.

Section 3 includes the preliminaries on partition algebras and Schur–Weyl dualities. We define totally propagating partition algebras and give a parametrization of their irreducible representations. This section concludes with the construction of a tower of totally propagating partition algebras and a description of the Bratteli diagram of this tower. Section 4 details about Jucys–Murphy elements of totally propagating partition algebras.

2 The rook monoid algebra

Let $R_n$ be the set of all $n \times n$ matrices whose entries are either 0 or 1 such that there is at most one non-zero entry in each row and each column. Under the matrix multiplication, $R_n$ is a monoid, called the rook monoid. The algebra $CR_n$ is called the rook monoid algebra.

2.1 The irreducible representations of $CR_n$

Munn [Mun57] characterized the irreducible representations of $CR_n$ in terms of the irreducible representations of the group algebra $CS_r$ of symmetric group $S_r$, for $0 \leq r \leq n$. Solomon [Sol02] revisited the representation theory of $CR_n$, and also established Schur–Weyl duality between $CR_n$ and the general linear group $GL_n(\mathbb{C})$. A quantum deformation of $CR_n$, known as $q$-rook monoid algebra where $q$ is a parameter, was given by Solomon [Sol04]. Halverson [Hal04] studied the representation theory of $q$-rook monoid algebra. Upon substituting $q = 1$ in the $q$-rook monoid algebra, we recover $CR_n$. In this section, we recall a set of generators and relations of $CR_n$, as well as, the construction of the irreducible representations of $CR_n$ from [Hal04].

**Generators and relations.** The transposition $(l_1, l_2) \in S_n$ and its corresponding permutation matrix in $R_n$ both are denoted by $(l_1, l_2)$. For $1 \leq i \leq (n - 1)$, let $s_i$ denote $(i, i+1).$ For $1 \leq j \leq n$, let $P_j \in R_n$ be the diagonal matrix whose first $j$ diagonal entries are 0 and the remaining diagonal entries are 1. From [Hal04, Section 2], we have the set

$$\{ s_i, P_j \mid 1 \leq i \leq (n - 1) \text{ and } 1 \leq j \leq n \}$$

(4)

as a generating set of the algebra $CR_n$. Recursively using the relation

$$P_j = P_{j-1}s_{j-1}P_{j-1} \text{ for } 1 \leq j \leq n$$

(5)

from [Hal04, Lemma 1.4], we see that the set

$$\{ s_i, P_1 \mid 1 \leq i \leq (n - 1) \}$$

(6)
can also be considered as a generating set of $\mathbb{C}R_n$. These generators satisfy the following relations:

(a) $s_i^2 = id$, for $1 \leq i \leq n - 1$,

(b) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, for $1 \leq i \leq n - 1$,

(c) $s_is_j = s/js_i$, when $|i - j| \geq 2$,

(d) $s_iP_1 = P_1s_i$, for $2 \leq i \leq n - 1$, and

(e) $P_2^2 = P_1$.

**Irreducible representations.** Let $\Lambda_{\leq n}$ denote the set of all partitions of $r$, where $0 \leq r \leq n$. Since a partition can be written equivalently as a Young diagram, therefore we use the same notation $\Lambda_{\leq n}$ to denote the set of all Young diagrams with the total number of boxes being $r$, where $0 \leq r \leq n$. Also, a partition and its Young diagram both are denoted by the symbol $\lambda$, and it will be clear from the context whether we are taking a partition or its Young diagram. For $\lambda \in \Lambda_{\leq n}$, an $n$-standard tableau of shape $\lambda$ is a filling of the Young diagram of shape $\lambda$ with entries from $\{1, \ldots, n\}$ such that entries along each row from left to right and along each column from top to bottom strictly increase. Let $\tau^\lambda_n$ be the set of all $n$-standard tableaux of shape $\lambda$. For $L \in \tau^\lambda_n$, let $v_L$ denote the symbol indexed by $L$. Define $V^\lambda_n := \mathbb{C}\text{-span}\{v_L \mid L \in \tau^\lambda_n\}$.

For $L \in \tau^\lambda_n$, if $i$ is an entry of a box in $L$ then we write $i \in L$. For $L \in \tau^\lambda_n$, define $s_iL$ as follows: if $i \in L$ then replace $i$ by $i + 1$, and if $i + 1 \in L$ then replace $i + 1$ by $i$, and the remaining entries in $L$ are fixed.

The content of a box $b$ at position $(x, y)$ in a Young diagram is $ct(b) = (y - x)$. For $\alpha \in L$, $ct(L(\alpha))$ denotes the content of the box in $L$ containing $\alpha$. If $\alpha, \alpha+1 \in L$, define

$$a_{L(\alpha)} = \begin{cases} 
\frac{1}{ct(L(\alpha + 1)) - ct(L(\alpha))} & 
\end{cases}.$$

(7)

For $1 \leq i \leq n - 1$, define an action of $s_i$ on $V^\lambda_n$ as follows:

$$s_i v_L = \begin{cases} 
v_{s_iL} & \text{if } i \in L, i + 1 \notin L, 
v_{s_iL} & \text{if } i \notin L, i + 1 \in L, 
v_L & \text{if } i \notin L, i + 1 \notin L, 
\alpha_{L(i)}v_L + (1 + \alpha_{L(i)})v_L' & \text{if } i \in L, i + 1 \in L,
\end{cases}$$

(8)

where

$$v_{L'} = \begin{cases} 
v_{s_iL} & \text{if } s_iL \text{ is } n\text{-standard tableau}, 
0 & \text{otherwise}.
\end{cases}$$
Define an action of $P_1$ on $V_n^\lambda$ as follows:

$$P_1 v_L = \begin{cases} v_L & \text{if } 1 \notin L, \\ 0 & \text{otherwise}. \end{cases}$$ (9)

For $2 \leq j \leq n$, using (5), (8) and (9), we get the action $P_j$ on $V_n^\lambda$ as

$$P_j v_L = \begin{cases} v_L & \text{if } 1, \ldots, j \notin L, \\ 0 & \text{otherwise}. \end{cases}$$ (10)

**Example 2.1.** (i) For $\lambda = \emptyset$, $V_n^\lambda$ is isomorphic to the trivial representation $\mathbb{C}$ of $\mathbb{C}R_n$, i.e., every element of $\mathbb{C}R_n$ fixes every element of $V_n^\lambda$ pointwise.

(ii) For $\lambda = (1)$, $V_n^\lambda$ is isomorphic to the defining representation $\mathbb{C}^n$ of $\mathbb{C}R_n$, i.e, the elements of $R_n$ act on $\mathbb{C}^n$ by the matrix multiplication.

By specializing $q = 1$ in [Hal04, Theorem 3.2], the following theorem gives a classification of the irreducible representations of $\mathbb{C}R_n$.

**Theorem 2.2.** The set $\{V_n^\lambda \mid \lambda \in \Lambda \leq_n \}$ is a complete set of pairwise non-isomorphic irreducible representations of $\mathbb{C}R_n$.

The monoid $R_{n-1}$ is isomorphic to the submonoid of $R_n$ consisting of elements whose $(n, n)$-th entry is 1, so $\mathbb{C}R_{n-1}$ embeds inside $\mathbb{C}R_n$. The following proposition is [Hal04, Corollary 3.3] which describes the branching rule for the embedding $\mathbb{C}R_{n-1} \subset \mathbb{C}R_n$.

For $\lambda \in \Lambda \leq_n$, let $\lambda^{-:, =}$ denote the set consisting of $\nu \in \Lambda \leq_{n-1}$ such that either $\nu$ is obtained by removing a box from an inner corner of $\lambda$ or $\nu = \lambda$. Similarly, for $\mu \in \Lambda \leq_{n-1}$, let $\mu^{+, =}$ denote the set consisting of $\nu' \in \Lambda \leq n$ such that either $\nu'$ is obtained by adding a box to an outer corner of $\mu$ or $\nu' = \mu$.

**Proposition 2.3** (Branching rule). For $\lambda \in \Lambda \leq_n$ and $\mu \in \Lambda \leq_{n-1}$, we have

$$\text{Res}_{\mathbb{C}R_{n-1}}^{\mathbb{C}R_n}(V_n^\lambda) \cong \bigoplus_{\nu \in \lambda^{-:, =}} V_{n-1}^\nu,$$ (11)

$$\text{Ind}_{\mathbb{C}R_{n-1}}^{\mathbb{C}R_n}(V_{n-1}^\mu) \cong \bigoplus_{\nu' \in \mu^{+, =}} V_n^{\nu'}.$$ (12)

From Proposition 2.3, we conclude that the Brattelli diagram for the tower of algebras

$$\mathbb{C} \subset \mathbb{C}R_1 \subset \mathbb{C}R_2 \subset \cdots \mathbb{C}R_n \subset \cdots$$ (13)

is a simple graph. Below is the Brattelli diagram for the tower (13) upto $n = 3$. 
For Young diagrams $\lambda$ and $\mu$, if $\mu$ is contained in $\lambda$ then we write $\mu \subset \lambda$, and moreover if $\lambda$ and $\mu$ differ by only one box then we denote this box as $\lambda/\mu = \square$. The following lemma is analogous to the correspondence between paths in Young’s graph for symmetric groups and standard tableaux.

**Lemma 2.4.** For $\lambda \in \Lambda_{\leq m}$, a path from the vertex $\emptyset$ at the level $n = 0$ to the vertex $\lambda$ at the level $n = m$ in the Bratteli diagram for the tower (13) corresponds to a unique $m$-standard tableau in $\tau_m^\lambda$. Moreover, this correspondence is a bijection.

**Proof.** Let $\lambda \in \Lambda_{\leq m}$. We associate an element of $L \in \tau_m^\lambda$ to a path $(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(m)})$, in the Bratteli diagram of the tower (13), from $\gamma^{(1)} = \emptyset$ at level $n = 0$ to $\gamma^{(m)} = \lambda$ at level $n = m$, as follows. In the given path, if a box is added first time at level $n = r$ (i.e., $\gamma^{(1)} = \gamma^2 = \cdots = \gamma^{(r-1)} = \emptyset$ and $\gamma^{(r)} = \square$), then fill $(1,1)$-th entry of $\lambda$ by $r$. Now, starting at level $n = r$, if a new box is added at level $n = r + s$, then fill the box $\gamma^{(r+s)}/\gamma^{(r)} = \square$ of $\lambda$ by $r + s$. By continuing this way along the given path until we reach at $\lambda$ at level $n = m$, we get an element of $L \in \tau_m^\lambda$. Since this process can be easily reversed, it is a bijection.

**Example 2.5.** For $m = 3$ and $\lambda = \square\square\square\square$, from Lemma 2.4, we have the following.

1. The path $(\emptyset, \square\square\square\square, \square\square\square\square)$ corresponds to the element $\square\square\square\square$.
2. The path $(\emptyset, \square\square\square\square, \square\square\square\square)$ corresponds to the element $\square\square\square\square$.
3. The path $(\emptyset, \emptyset, \square\square\square\square)$ corresponds to the element $\square\square\square\square$.

### 2.2 Jucys–Murphy elements of rook monoid algebra

Halverson [Hal04, Section 3.1] defined Jucys–Murphy elements of $q$-rook monoid algebra and described their actions on the basis vector parametrized by an $n$-standard tableau.
In this section, we define new elements, denoted by $\tilde{X}_i$ for $1 \leq i \leq n$, in $CR_n$, which along with the elements, denoted by $X_i$ for $1 \leq i \leq n$, defined by Halverson play the role of Jucys–Murphy elements in rook monoid algebra.

For $2 \leq i \leq n$, define

$$Q_i = (2, i - 1)(1, i)P_2(1, i)(2, i - 1).$$

Lemma 2.6. Let $V_n^\lambda$ be the irreducible representation of $CR_n$ corresponding to $\lambda \in \Lambda_\leq$. For the basis element $v_L \in V_n^\lambda$ corresponding to $L \in \tau_n^\lambda$ and for $2 \leq i \leq n$, we have

$$Q_i v_L = \begin{cases} v_L & \text{if } i - 1, i \notin L, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. We use induction on $i$ to prove the result. For $i = 2$, the result follows from (10). Suppose that the result is true for $i - 1$.

Observe that $Q_i \in R_n$ is the diagonal matrix whose $(i - 1)$-th and $i$-th diagonal entries are 0 and the remaining diagonal entries are 1. It can be seen that for $i > 2$, we have

$$Q_i = s_i(s_i s_{i - 1} Q_i s_{i - 1} s_i - 2).$$

Case(i): $i - 1 \notin L$ and $i \notin L$.

Subcase(a): $i - 2 \in L$. Then $s_i s_{i - 2} L$ is an $n$-standard tableau which does not contain both $i - 2$ and $i - 1$. Using (8) and the induction hypothesis,

$$Q_i s_i s_{i - 2} v_L = v_{s_i s_{i - 2} L}.$$ 

Thus $s_i(s_i s_{i - 1} Q_i s_{i - 1} s_i - 2)v_L = v_L$.

Subcase(b): $i - 2 \notin L$. Then $s_i s_{i - 2} L = L$, and using (8) $s_i s_{i - 2} v_L = v_L$. Since $i - 2 \notin L$ and $i - 1 \notin L$, by the induction hypothesis we get $Q_i s_i s_{i - 2} v_L = v_L$, and so $Q_i v_L = v_L$.

Case(ii): $i - 1 \in L$ and $i \notin L$.

Subcase(a): $i - 2 \in L$. Then using (8) $s_i s_{i - 2} v_L = a v_L + b v_{L'}$, for some $a, b \in \mathbb{C}$, and both $L, L'$ contain $i - 2$. So by the induction hypothesis $Q_i s_i s_{i - 2} v_L = 0$, and this implies $Q_i v_L = 0$.

Subcase(b): $i - 2 \notin L$. Then $s_i s_{i - 2} L$ is an $n$-standard tableau which contains $i - 2$. Therefore using (8) and by the induction hypothesis $Q_i s_i s_{i - 2} v_L = 0$, and so we get $Q_i v_L = 0$. 

[Hal04, Proposition 3.5]. It can be seen that when the parameter $q$ is not a root of unity, the actions of Jucys–Murphy elements distinguish the non-isomorphic irreducible representations of $q$-rook monoid algebra. This is similar to the property of Jucys–Murphy elements in the case of symmetric groups and partition algebras, as discussed in Section 1.

Therefore using (8) and by the induction hypothesis $Q_i s_i s_{i - 2} v_L = 0$, and so we get $Q_i v_L = 0$. 

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Case (iii): \( i - 1 \notin L \) and \( i \in L \).

Subcase (a): \( i - 2 \in L \). Then using (8) \( s_{i-1}s_{i-2}v_L = av_{L'} + bv_{L''} \), for some \( a, b \in \mathbb{C} \), and both \( L', L'' \) contain \( i - 1 \). So by the induction hypothesis \( Q_{i-1}s_{i-1}s_{i-2}v_L = 0 \), and this implies \( Q_i v_L = 0 \).

Subcase (b): \( i - 2 \notin L \). Then using (8) \( s_{i-1}s_{i-2}v_L = v_{L'} \), where \( L' \) is obtained from \( L \) by putting \( i - 1 \) at the place of \( i \). By the induction hypothesis \( Q_{i-1}v_{L'} = 0 \), and this gives \( Q_i v_L = 0 \).

Case (iv): \( i - 1 \in L \) and \( i \in L \).

Subcase (a): \( i - 2 \in L \). Then using (8) \( s_{i-1}s_{i-2}v_L \) is a linear combination of \( v_{L'} \) such that each \( L' \) contains \( i - 1 \). So by the induction hypothesis \( Q_{i-1}s_{i-1}s_{i-2}v_L = 0 \), and this implies \( Q_i v_L = 0 \).

Subcase (b): \( i - 2 \notin L \). Then using (8) \( s_{i-1}s_{i-2}v_L = v_{L'} \), where \( L' \) is obtained from \( L \) by first replacing \( i - 1 \) by \( i - 2 \) and then by replacing \( i \) by \( i - 1 \). Now using the the induction hypothesis we get \( Q_{i-1}s_{i-1}s_{i-2}v_L = 0 \), and hence \( Q_i v_L = 0 \). \( \square \)

From [Hal04, Section 3.1], we have the following elements of \( \mathbb{C}R_n \):

\[
X_1 = (1 - P_1), \quad \text{and} \\
X_i = s_{i-1}X_{i-1} - s_{i-1}v_L, \quad \text{for } 2 \leq i \leq n.
\]

For \( 1 \leq i \leq n \), define \( \gamma_i := (1 - X_i) \).

Now, we define the new elements in \( \mathbb{C}R_n \) as follows:

\[
\tilde{X}_1 = 0, \quad \text{and} \\
\tilde{X}_i = s_{i-1}\tilde{X}_{i-1} + s_{i-1} - s_{i-1}\gamma_{i-1} - \gamma_{i-1}s_{i-1} + Q_i, \quad \text{for } 2 \leq i \leq n.
\] (14)

In Lemma 2.4, we have observed that the Gelfand–Tsetlin basis, with respect to the tower (3), of an irreducible representation associated to a partition \( \lambda \) of \( r \), where \( 0 \leq r \leq n \), is in bijection with the set of \( n \)-standard tableaux of shape \( \lambda \). In the following theorem, we describe the actions of elements (14) on basis vectors corresponding to \( n \)-standard tableaux.

**Theorem 2.7.** Let \( V_n^\lambda \) be the irreducible representation of \( \mathbb{C}R_n \) corresponding to \( \lambda \in \Lambda_{\leq n} \). For the basis element \( v_L \in V_n^\lambda \) corresponding to \( L \in \pi_n^\lambda \) and for \( 1 \leq i \leq n \), we have

(i)

\[
X_i v_L = \begin{cases} 
    v_L & \text{if } i \in L, \\
    0 & \text{otherwise.}
\end{cases}
\] (15)

(ii)

\[
\tilde{X}_i v_L = \begin{cases} 
    \text{ct}(L(i))v_L & \text{if } i \in L, \\
    0 & \text{otherwise.}
\end{cases}
\] (16)
Proof. The first part of the theorem is proved in [Hal04, Proposition 3.5]. Recall
\[ \tilde{X}_1 = 0, \]
\[ \tilde{X}_i = s_{i-1}\tilde{X}_{i-1} s_{i-1} + s_{i-1} - s_{i-1} \gamma_{i-1} - \gamma_{i-1} s_{i-1} + Q_i, \text{ for } 2 \leq i \leq n. \]
We prove the second part using induction on \( i \). For \( i = 1 \), the result is true as \( \tilde{X}_1 = 0 \), and if \( 1 \in L \), then \( \text{ct}(L(1)) = 0 \). Assume that (16) holds for \( i - 1 \) and we prove it for \( i \).

Case(i): \( i-1 \notin L \) and \( i \notin L \). Then using (8) \( s_{i-1} v_L = v_L \) and by the induction hypothesis, \( s_{i-1} \tilde{X}_{i-1} s_{i-1} v_L = 0 \). By (15) \( \gamma_{i-1} s_{i-1} v_L = v_L \) and also \( s_{i-1} \gamma_{i-1} v_L = v_L \), by Lemma 2.6 \( Q_i v_L = v_L \). Therefore \( \tilde{X}_i v_L = 0 \).

Case(ii): \( i - 1 \in L \) and \( i \notin L \). Since \( s_{i-1} L \) does not contain \( i - 1 \) and \( s_{i-1} v_L = v_L \), by the induction hypothesis, we have \( s_{i-1} \tilde{X}_{i-1} s_{i-1} v_L = 0 \). Since \( L \) contains \( i - 1 \), by (15), \( s_{i-1} \gamma_{i-1} v_L = 0 \), and by Lemma 2.6 \( Q_i v_L = 0 \). By (15), \( \gamma_{i-1} s_{i-1} v_L = s_{i-1} v_L \). Combining all of these, we have \( \tilde{X}_i v_L = s_{i-1} v_L - s_{i-1} v_L = 0 \).

Case(iii): \( i - 1 \notin L \) and \( i \in L \). Since \( s_{i-1} L \) contains \( i - 1 \) such that \( (s_{i-1} L)(i - 1) = L(i) \) and also \( s_{i-1} v_L = v_{s_{i-1} L} \), by the induction hypothesis we have \( s_{i-1} \tilde{X}_{i-1} s_{i-1} v_L = \text{ct}(L(i)) v_L \).
By (15) \( \gamma_{i-1} s_{i-1} v_L = 0 \) and \( s_{i-1} \gamma_{i-1} v_L = s_{i-1} v_L \), and by Lemma 2.6 \( Q_i v_L = 0 \). Combining all of these, we have, \( \tilde{X}_i v_L = \text{ct}(L(i)) v_L + s_{i-1} v_L - s_{i-1} v_L = \text{ct}(L(i)) v_L \).

Case(iv): \( i - 1 \in L \) and \( i \in L \). Using (8), \( s_{i-1} v_L = a v_L + b v_L \) where \( a, b \in \mathbb{C} \), and if \( v_L \neq 0 \) then \( L' \) contains both \( i - 1 \) and \( i \). By (15), \( \gamma_{i-1} s_{i-1} v_L = 0 \) and \( s_{i-1} \gamma_{i-1} v_L = 0 \), and by Lemma 2.6 \( Q_i v_L = 0 \). Then we have \( \tilde{X}_i v_L = s_{i-1} \tilde{X}_{i-1} s_{i-1} v_L + s_{i-1} v_L \). Recall from (7) that \( a_{L(i-1)} = \frac{1}{ct(L(i)) - ct(L(i-1))} \).

Subcase(a): \( s_{i-1} L \) is not an \( n \)-standard tableau. Then, \( i - 1 \) and \( i \) occur consecutively either in the same row or in the same column of \( L \). So \( a_{L(i-1)} = \pm 1 \). Then,
\[ s_{i-1} \tilde{X}_{i-1} s_{i-1} v_L + s_{i-1} v_L = s_{i-1} \tilde{X}_{i-1} (a_{L(i-1)} v_L) + a_{L(i-1)} v_L \]
\[ = s_{i-1} (\text{ct}(L(i-1)) a_{L(i-1)} v_L) + a_{L(i-1)} v_L \]
\[ = (\text{ct}(L(i-1)) (a_{L(i-1)})^2 + a_{L(i-1)}) v_L, \]
where the second equality follows from the induction hypothesis. Now,
\[ (\text{ct}(L(i-1)) (a_{L(i-1)})^2 + a_{L(i-1)}) = a_{L(i-1)} (\text{ct}(L(i-1)) a_{L(i-1)} + 1) \]
\[ = \text{ct}(L(i)) (a_{L(i-1)})^2 \]
\[ = \text{ct}(L(i)), \text{ as } a_{L(i-1)} = \pm 1. \]
The second equality follows by substituting \( a_{L(i-1)} = \frac{1}{ct(L(i)) - ct(L(i-1))} \).

Subcase(b): \( s_{i-1} L \) is an \( n \)-standard tableau. It is easy to observe that
\[ \text{ct}(s_{i-1} L(i-1)) = \text{ct}(L(i)) \text{ and } a_{s_{i-1} L(i-1)} = -a_{L(i-1)}. \]
Then,

\[ s_{i-1} \tilde{X}_{i-1} v_L + s_{i-1} v_L = s_{i-1} \tilde{X}_{i-1} (a_{L(i-1)} v_L + (1 + a_{L(i-1)}) v_{s_{i-1}L}) + a_{L(i-1)} v_L + (1 + a_{L(i-1)}) v_{s_{i-1}L} \]

\[ = s_{i-1} (ct(L(i-1)) a_{L(i-1)} v_L + (1 + a_{L(i-1)}) ct(L(i)) v_{s_{i-1}L}) + a_{L(i-1)} v_L + (1 + a_{L(i-1)}) v_{s_{i-1}L} \]

\[ = ct(L(i-1)) a_{L(i-1)} v_L + (1 + a_{L(i-1)}) v_{s_{i-1}L} + (1 + a_{L(i-1)}) v_{s_{i-1}L} \]

\[ = Av_L + Bv_{s_{i-1}L} , \]

where

\[ A = ct(L(i-1))(a_{L(i-1)})^2 + (1 + a_{L(i-1)})(1 - a_{L(i-1)})ct(L(i)) + a_{L(i-1)} \]

\[ B = ct(L(i-1))a_{L(i-1)}(1 + a_{L(i-1)}) - ct(L(i))a_{L(i-1)}(1 + a_{L(i-1)}) + (1 + a_{L(i-1)}) \]

Now we simplify the expressions of \( A \) and \( B \). We have

\[ A = ct(L(i-1))(a_{L(i-1)})^2 + (1 + a_{L(i-1)})(1 - a_{L(i-1)})ct(L(i)) + a_{L(i-1)} \]

\[ = ct(L(i)) + a_{L(i-1)}(ct(L(i-1))a_{L(i-1)} - ct(L(i))a_{L(i-1)} + 1) \]

\[ = ct(L(i)) \]

and

\[ B = ct(L(i-1))a_{L(i-1)}(1 + a_{L(i-1)}) - ct(L(i))a_{L(i-1)}(1 + a_{L(i-1)}) + (1 + a_{L(i-1)}) \]

\[ = (1 + a_{L(i-1)}) \left( \frac{ct(L(i-1))}{ct(L(i)) - ct(L(i-1))} - \frac{ct(L(i))}{ct(L(i)) - ct(L(i-1))} + 1 \right) \]

\[ = 0 \]

Therefore we have \( \tilde{X}_i v_L = ct(L(i)) v_L \). \( \square \)

**Corollary 2.8.** For \( 1 \leq i, j \leq n \), we have

\( i \) \hspace{1em} \( X_i X_j = X_j X_i, \)

\( ii \) \hspace{1em} \( \tilde{X}_i \tilde{X}_j = \tilde{X}_j \tilde{X}_i, \)

\( iii \) \hspace{1em} \( X_i \tilde{X}_j = \tilde{X}_j X_i. \)

**Proof.** Given \( \lambda \in \Lambda_{\leq n} \), Theorem 2.7 says that there exists a basis \( \{ v_L \mid L \in \tau_\lambda^n \} \) with respect to which each \( X_i \) and each \( \tilde{X}_j \), for \( 1 \leq i, j \leq n \), act diagonally on the irreducible representation \( V_\lambda^n \) of \( \mathbb{C} R_n \). By considering the left regular representation of \( \mathbb{C} R_n \), which is a faithful representation, we obtain all three commutation results as stated in the corollary. \( \square \)
2.3 Kronecker product

In this section, we compute the Kronecker product of the character of $C_n$ with the character of an irreducible representation of $C R_n$ in Theorem 2.10. We also give an inductive rule to decompose $(C^n)^{\otimes k}$ into the irreducible representations of $C R_n$. As a corollary of this inductive rule, we obtain a result of Solomon [Sol02, Example 3.18] regarding the multiplicity formula of an irreducible representation in the decomposition of $(C^n)^{\otimes k}$ as a representation of $C R_n$.

We first define some notation. For $\sigma \in R_n$, let $I(\sigma)$ be the set of indices of non-zero rows of $\sigma$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $C_n$. For a subset $K$ of $\{1, 2, \ldots, n\}$, the meaning of $\sigma K = K$ is that for every $i \in K$, there exists $j \in K$ such that $\sigma e_i = e_j$. Define

$$C_{\sigma, r} = \{ K \subseteq I(\sigma) \mid |K| = r, \sigma K = K \}.$$ 

Given $K \subseteq \{1, 2, \ldots, n\}$ such that $|K| = r$, let $\theta_K$ be the element of $R_n$ whose non-zero rows and columns are indexed by the elements of $\{1, 2, \ldots, r\}$ and $K$, respectively. Let $\theta_K^{tr}$ denote the transpose of $\theta_K$. Following [Sol02], we identify $S_r$ with the submonoid of $R_n$ consisting of matrices $\sigma = (\sigma_{i,j})_{1 \leq i,j \leq n}$ such that the submatrix $(\sigma_{p,q})_{1 \leq p,q \leq r}$ is a permutation matrix of order $r$ and the entries $\sigma_{s,t} = 0$ for all $r+1 \leq s, t \leq n$.

Remark 2.9. One can also identify the symmetric group on $r$ symbols with the submonoid of $R_n$ consisting of matrices $\sigma = (\sigma_{i,j})_{1 \leq i,j \leq n}$ such that the submatrix $(\sigma_{p,q})_{1 \leq p,q \leq r}$ is a permutation matrix of order $r$ and the entries $\sigma_{s,t} = 0$ for all $r+1 \leq s, t \leq n$ except the entry $\sigma_{r+1,r+1}$ which is 1. We denote this copy of the symmetric group on $r$ symbols by $\tilde{S}_r$. Given $\tilde{\tau} \in \tilde{S}_r$, let $\tau$ denote the element in $S_r$ obtained from $\tilde{\tau}$ by making the $(r+1, r+1)$-th entry zero. Then the map

$$\tilde{S}_r \to S_r \quad (17)$$

$$\tilde{\tau} \mapsto \tau$$

is a group isomorphism. For a partition $\lambda$ of $r$, suppose that $\chi_\lambda$ is the character of the irreducible representation of $S_r$ corresponding to $\lambda$. Then the pullback of $\chi_\lambda$ is the character $\tilde{\chi}_\lambda$ of the irreducible representation of $\tilde{S}_r$ corresponding to $\lambda$. Also for $\tilde{\tau} \in \tilde{S}_r$, we have

$$\tilde{\chi}_\lambda(\tilde{\tau}) = \chi_\lambda(\tau).$$

For the character $\chi$ of a representation of $C S_r$ and $\sigma \in R_n$, define

$$\chi^*(\sigma) := \sum_{K \in C_{\sigma, r}} \chi(\theta_K \sigma \theta_K^{tr}).$$

(18)

Then from [Sol02, Theorem 2.30] $\chi^*$ is the character of a representation of $C R_n$. For $\lambda \in \Lambda_{\leq n}$ and the character $\chi_\lambda$ of the irreducible representation of symmetric group corresponding to $\lambda$ then $\chi^*_\lambda$, obtained using (18), is the character of the irreducible representation of $C R_n$ corresponding to $\lambda$. By combining [Sol02, Theorem 2.24] and [Sol02, Theorem...
2.30], we note that the characters of the irreducible representations of $\mathbb{C}R_n$ arise as defined in (18).

Given $\lambda \in \Lambda_{\leq n}$, we define $\lambda^- +$ to be the set of all Young diagrams $\mu$ obtained from $\lambda$ in the following way:

(i) First remove a box from an inner corner of $\lambda$ to obtain a Young diagram $\omega$;

(ii) Then add a box to an outer corner of $\omega$ to obtain $\mu$.

Also, we define $\lambda^+ + n$ to be the set of all Young diagrams in $\Lambda_{\leq n}$ obtained from $\lambda$ by adding a box to an outer corner.

**Theorem 2.10.** For $\lambda \in \Lambda_{\leq n}$ and for $\nu = (1)$, the Kronecker product $\chi_\nu^* \chi_\lambda^*$ of $\chi_\nu^*$ and $\chi_\lambda^*$ is given by

$$\chi_\nu^* \chi_\lambda^* = \bigoplus_{\mu \in \lambda^- + \cup \lambda^+ + n} \chi_\mu^*.$$

**Proof.** Let $\sigma \in R_n$. Then,

$$(\chi_\nu^* \chi_\lambda^*)(\sigma) = \chi_\nu^*(\sigma) \chi_\lambda^*(\sigma)$$

$$= \left( \sum_{L \in C_{\sigma,1}} \chi_\nu(\theta_L \sigma \theta_L^{tr}) \right) \left( \sum_{K \in C_{\sigma,r}} \chi_\lambda(\theta_K \sigma \theta_K^{tr}) \right)$$

$$= \sum_{K \in C_{\sigma,r}} \left( \sum_{L \in C_{\sigma,1}} \chi_\nu(\theta_L \sigma \theta_L^{tr}) \right) \chi_\lambda(\theta_K \sigma \theta_K^{tr})$$

$$= \sum_{K \in C_{\sigma,r}} \left( \sum_{L \in C_{\sigma,1}, L \subseteq K} \chi_\nu(\theta_L \sigma \theta_L^{tr}) \right) \chi_\lambda(\theta_K \sigma \theta_K^{tr}) + \sum_{K \in C_{\sigma,r}} \left( \sum_{L \in C_{\sigma,1}, L \not\subseteq K} \chi_\nu(\theta_L \sigma \theta_L^{tr}) \right) \chi_\lambda(\theta_K \sigma \theta_K^{tr}).$$

Since $\nu = (1)$, $\sum_{L \in C_{\sigma,1}, L \subseteq K} \chi_\nu(\theta_L \sigma \theta_L^{tr})$ is equal to the number of fixed points of $\theta_K \sigma \theta_K^{tr}$, which is also the character value at $\theta_K \sigma \theta_K^{tr}$ of the representation $\mathbb{C}^r$ of $S_r$. By the tensor identity [HR05, Equation(3.18)] followed by the induction and restriction rules for the symmetric groups, we have

$$(\text{the number of fixed points of } \theta_K \sigma \theta_K^{tr}) \chi_\lambda(\theta_K \sigma \theta_K^{tr}) = \bigoplus_{\mu \in \lambda^- +} \chi_\mu(\theta_K \sigma \theta_K^{tr}).$$

Since $\chi_\nu(\theta_L \sigma \theta_L^{tr}) = 1$, therefore the second sum in the above expansion of $(\chi_\nu^* \chi_\lambda^*)(\sigma)$ becomes $\sum_{K \in C_{\sigma,r}, L \in C_{\sigma,1}, L \not\subseteq K} \chi_\lambda(\theta_K \sigma \theta_K^{tr})$, which we prove in the following that it is equal to

$(\text{Ind}_{S_r}^{S_{r+1}}(\bar{\chi}_\lambda))^{(\sigma)}$. Using the formula for an induced character (for example, see [FH91,
Equation(3.18)), we have
\[
(\text{Ind}^{S_{r+1}}(\bar{\chi}_\lambda))^*(\sigma) = \sum_{M \in C_{\sigma,r+1}} \text{Ind}^{S_{r+1}}(\bar{\chi}_\lambda)(\theta_M \sigma \theta_{M}^{tr})
\]
\[
= \sum_{M \in C_{\sigma,r+1}} \left( \sum_{i \in [r+1]} \chi(\theta_M \sigma \theta_{M}^{tr}(i, r + 1)) \right).
\]  
(19)

Note that, for \(i \in [r+1]\), \(\theta_M \sigma \theta_{M}^{tr}(i) = i\) if only if \(\sigma(\theta_{M}^{tr}(i)) = \theta_{M}^{tr}(i)\). So the inner sum contributes to zero for \(M \in C_{\sigma,r+1}\) which does not contain a fixed point of \(\sigma\). Also there is one-one correspondence between the sets \(\{(M,j) \mid M \in C_{\sigma,r+1} \text{ and } j \in M \text{ such that } \sigma(j) = j\}\) and \(\{(K,L) \in C_{\sigma,r} \times C_{\sigma,1} \mid L \not\subseteq K\}\). Therefore, the sum (19) can be rewritten as
\[
\sum_{K \in C_{\sigma,r}} \left( \sum_{L \in C_{\sigma,1} \setminus L \not\subseteq K} \bar{\chi}_\lambda((\theta_{K\cup L}(L), r + 1)\theta_{K\cup L}\sigma\theta_{K\cup L}^{tr}(\theta_{K\cup L}(L), r + 1)) \right).
\]

The matrix \((\theta_{K\cup L}(L), r + 1)\theta_{K\cup L}\sigma\theta_{K\cup L}^{tr}(\theta_{K\cup L}(L), r + 1))\) is of rank \(r + 1\) whose first \(r \times r\) submatrix is \(\theta_K \sigma \theta_K^{tr}\) and \((r + 1, r + 1)\)-th entry is 1. So by Remark 2.9, we have
\[
\bar{\chi}_\lambda((\theta_{K\cup L}(L), r + 1)\theta_{K\cup L}\sigma\theta_{K\cup L}^{tr}(\theta_{K\cup L}(L), r + 1)) = \chi_\lambda(\theta_K \sigma \theta_K^{tr}).
\]

This completes the proof. \(\square\)

**Corollary 2.11.** Let \(V^\lambda_n\) be the irreducible representation of \(\mathbb{C}R_n\) corresponding to \(\lambda \in \Lambda_{\leq n}\). Then,
\[
\mathbb{C}^n \otimes V^\lambda_n \cong \bigoplus_{\mu \in \lambda^- + \lambda^+, n} V^\mu_n.
\]

**Proof.** The character of the representation \(\mathbb{C}^n\) of \(\mathbb{C}R_n\) is given by \(\chi_\nu^\ast\) where \(\nu = (1)\). And the character of \(V^\lambda_n\) is \(\chi_\lambda^\ast\). Since the character of the tensor product of two representations is the product of their characters, the result follows from Proposition 2.10. \(\square\)

Using Corollary 2.11, we build a graph \(\hat{R}(n)\) which encodes the multiplicity of an irreducible representation of \(\mathbb{C}R_n\) in the decomposition of \((\mathbb{C}^n)^\otimes k\) as \(k\) varies over positive integers. The set \(\hat{R}_k(n)\) of vertices at the level \(k\) of \(\hat{R}(n)\) is \(\{\lambda \vdash r \mid 1 \leq r \leq \min\{k, n\}\}\). For \(\lambda \in \hat{R}_k(n)\) and \(\mu \in \hat{R}_{k+1}(n)\), there is an edge between \(\lambda\) and \(\mu\) if and only if \(\mu \in \lambda^- \cup \lambda^+, n\) (see Page 14 for the definitions of \(\lambda^-\) and \(\lambda^+, n\)).

The following graph depicts \(\hat{R}(3)\) up to level 3.
Solomon [Sol02, Example 3.18] gave a formula for the multiplicity of an irreducible representation of $\mathbb{CR}_n$ in the decomposition of $(\mathbb{C}^n)^{\otimes k}$. In Corollary 2.12, we give a bijective proof of his formula, and for this, we need the following definitions and notation.

1. (Maximum-entry order) For a set partition $\pi$ of $\{1, 2, \ldots, k\}$, we underline maximum-entry of each block of $\pi$. This defines a total order on $\pi$, and in [BH19a, Definition 3.14], it is called the maximum-entry order. For the definition of a set partition and its block, see Section 3.1.

2. (Set-partition tableau) For $1 \leq r \leq n$, let $\lambda$ be a partition of $r$. From [BH19a, Definition 3.14], a set-partition tableau $T_k$ of shape $\lambda$ is a filling of boxes of $\lambda$ with a set partition $\pi$ of $\{1, 2, \ldots, k\}$ into $r$-parts. The set partition $\pi$ is called the content of $T_k$. We say $T_k$ is a standard set-partition tableau if the entries of $T_k$ strictly increase along the rows from left to right and along the columns from top to bottom with respect to the maximum-entry order on the blocks of $\pi$.

3. (Robinson–Schensted–Knuth row-insertion) Given a set-partition tableau $T_k$ and a subset $b$ of $\{1, 2, \ldots, k\}$, let $(T_k \leftarrow b)$ denote the Robinson–Schensted–Knuth (RSK) row-insertion of $b$ in the tableau $T_k$. For an explicit algorithm of RSK row-insertion, we refer to [Sta99, Section 7.11].

**Corollary 2.12.** Let $\lambda \in \Lambda_{\leq n}$ be a partition of $r$. Then the multiplicity of the irreducible representation corresponding to $\lambda$ in $(\mathbb{C}^n)^{\otimes k}$ is $S(k, r)f^\lambda$, where $S(k, r)$ is a Stirling number of the second kind and $f^\lambda$ is the number of standard Young tableaux of shape $\lambda$.

**Proof.** From Corollary 2.11, the multiplicity of the irreducible representation, corresponding to $\lambda \in \Lambda_{\leq n}$, of $\mathbb{CR}_n$ in the decomposition of $(\mathbb{C}^n)^{\otimes k}$ is the number of paths, in the graph $\mathcal{R}(n)$, starting from the Young diagram corresponding to (1) at level 1 and ending at the Young diagram $\lambda$ at level $k$. Let $\mathcal{R}_{k, \lambda}(n)$ denote the set of such paths.

The cardinality of the set of standard set-partition tableaux with content a set-partition $\pi$ into $r$-blocks is $S(k, r)f^\lambda$. In order to prove the corollary, we give a bijection
between $\hat{R}_{k,\lambda}(n)$ to the set of standard set-partition tableau with content a set-partition $\pi$ into $r$-blocks.

Given a standard set-partition tableau $T_k$ of shape $\lambda$, following [BHH16, Section 5.3], below we define a path $(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(k)})$ such that $\gamma^{(1)} = (1)$ and $\gamma^{(k)} = \lambda$.

1. Remove the inner corner filled with the set $b$ containing $k$ from $T_k$, and let $T_{k-\frac{1}{2}}$ denote the resulting tableau with total number of boxes $(r - 1)$.

2. Let $b'$ be obtained from $b$ by deleting the element $k$ from $b$. If $b'$ is empty then $\gamma^{(k-1)}$ is the shape of tableau $T_{k-\frac{1}{2}}$. If $b'$ is nonempty then perform RSK row-insertion ($T_{k-\frac{1}{2}} \leftarrow b'$) to obtain a tableau $T_{k-1}$ filled with a set-partition of $\{1, 2, \ldots, k-1\}$ into $r$-blocks. Define $\gamma^{(k-1)}$ to be the shape of $T_{k-1}$.

3. Repeat the steps above until we reach $\gamma^{(1)}$ at the level 1.

Conversely, given a path $(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(k)})$ with $\gamma^{(1)} = (1)$ and $\gamma^{(k)} = \lambda$ we recursively construct a sequence $T_1, T_2, \ldots, T_k$ such that, for each $i$, $T_i$ is a standard set-partition tableau of shape $\gamma^{(i)}$ on $\{1, 2, \ldots, i\}$. Let $T_1$ be the set-partition tableau of shape $(1)$ filled with $\{1\}$. Then for $i = 2, \ldots, k$, construct $T_i$ from $T_{i-1}$ by performing the following steps:

1. If $\gamma^{(i)}/\gamma^{(i-1)} = \square$, then $T_i$ is same as $T_{i-1}$ with the additional box of $T_i$ filled with $\{i\}$.

2. Suppose $\gamma^{(i)}$ is obtained from $\gamma^{(i-1)}$ by removing an inner corner resulting into a Young diagram $\gamma^{(i-\frac{1}{2})}$ and then adding an outer corner to $\gamma^{(i-\frac{1}{2})}$. Then using RSK row-insertion algorithm, there exist unique tableau $T_{i-\frac{1}{2}}$ of shape $\gamma^{(i-\frac{1}{2})}$ and unique set partition $b$ of $\{1, 2, \ldots, i-1\}$ such that $T_{i-1} = (T_{i-\frac{1}{2}} \leftarrow b)$. Note that $\gamma^{(i-\frac{1}{2})} \subset \gamma^{(i)}$ such that $\gamma^{(i)}/\gamma^{(i-\frac{1}{2})} = \square$. The tableau $T_i$ is same as $T_{(i-\frac{1}{2})}$ with the additional box $\gamma^{(i)}/\gamma^{(i-\frac{1}{2})} = \square$ filled with $b \cup \{i\}$.

Example 2.13. We illustrate that the path $((\square, \square, \square))$ in $\hat{R}(3)$ corresponds to the standard set-partition tableau $\begin{array}{c} \{1\} \\ \{2,3\} \end{array}$. For the given path, we have $\gamma^{(1)} = \square$, $\gamma^{(2)} = \square$, and $\gamma^{(3)} = \square$. So $T_1 = \begin{array}{c} \{1\} \\ \{2,3\} \end{array}$ and since $\gamma^{(2)}$ is obtained from $\gamma^{(1)}$ by adding a box to it, we have $T_2 = \begin{array}{c} \{1\} \\ \{2\} \end{array}$. Also, $\gamma^{(3)}$ is obtained from $\gamma^{(2)}$ by first removing a box which results in $\gamma^{(\frac{3}{2})} = \square$ and then adding a box to $\gamma^{(\frac{3}{2})}$. By RSK row-insertion, we have $T_{\frac{3}{2}} = \begin{array}{c} \{1\} \\ \{2\} \end{array}$ and $b = \{2\}$ such that $T_2 = T_{\frac{3}{2}} \leftarrow b$. 

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By the algorithm discussed in the proof of Corollary 2.12, \( T_3 \) is same as \( T_{\frac{3}{2}} \) with the additional box \( \gamma^{(3)}/\gamma^{(\frac{3}{2})} = \square \) filled with \( \{2, 3\} \), i.e, \( T_3 = \begin{array}{c} 1 \\ 2, 3 \end{array} \)

Similarly, the paths \( (\gamma^{(1)} = \square, \gamma^{(2)} = \begin{array}{c} 2 \\ \end{array}, \gamma^{(3)} = \end{array}) \) and \( (\gamma^{(1)} = \square, \gamma^{(2)} = \square, \gamma^{(3)} = \square) \) correspond to the standard set-partition tableaux \( \begin{array}{c} 2 \\ 1, 3 \end{array} \) and \( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \) respectively.

### 2.4 Modified induction and restriction rules

The restriction of the trivial representation of \( S_n \) to \( S_{n-1} \) followed by the induction from \( S_{n-1} \) to \( S_n \) gives the representation \( C_n \) of \( S_n \). In order to obtain a similar result, in this section, we define modified restriction and modified induction rules from \( CR_{n-1} \) to \( CR_n \), which also satisfy the Frobenius reciprocity. As a crucial application of this, we prove that the branching rule for inclusion \( CI_{k+\frac{1}{2}} \subset CI_{k+1} \) of totally propagating partition algebras is multiplicity free, see Theorem 3.17.

For \( \lambda \in \Lambda_{\leq n-1} \), define \( \lambda^+ \) to be the set of all Young diagrams in \( \Lambda_{\leq n} \) obtained from \( \lambda \) by adding a box to an outer corner. Define the modified induction rule as follows

\[
\widetilde{\text{Ind}}_{CR_n}^{CR_{n-1}} \left( \bigoplus_{\lambda \in \Lambda_{\leq n-1}} (V_{n-1}^{\lambda})^{\oplus m_{\lambda}} \right) = \bigoplus_{\lambda \in \Lambda_{\leq n-1}} \left( \bigoplus_{\nu \in \lambda^+} V_{n}^{\nu} \right)^{\oplus m_{\lambda}},
\]

where \( m_{\lambda} \) is a nonnegative integer.

**Proposition 2.14** (Tensor identity). For a representation \( M \) of \( CR_n \), we have

\[
\widetilde{\text{Ind}}_{CR_{n-1}}^{CR_n} (\text{Res}^{CR_n}_{CR_{n-1}} M) \cong (M \otimes \mathbb{C}^n).
\]  

**Proof.** When \( M \) is an irreducible representation of \( CR_n \), the isomorphism (20) holds because of Corollary 2.11 and the definition of \( \widetilde{\text{Ind}}_{CR_{n-1}}^{CR_n} (-) \). For arbitrary representation of \( CR_n \), the results follows by using the complete reducibility and by observing that \( \widetilde{\text{Ind}}_{CR_{n-1}}^{CR_n} (-) \) preserves the direct sums. \( \square \)

**Proposition 2.15.** For a nonnegative integer \( k \), we have

\[
(\widetilde{\text{Ind}}_{CR_{n-1}}^{CR_n} (\text{Res}^{CR_n}_{CR_{n-1}} V_n^{\emptyset}))^k \cong (\mathbb{C}^n)^{\otimes k}.
\]  

**Proof.** For \( k = 0 \), both sides of (21) are isomorphic to the trivial representation \( \mathbb{C} \) of \( CR_n \).

We prove the result using induction on \( k \). For \( k = 1 \),

\[
\widetilde{\text{Ind}}_{CR_{n-1}}^{CR_n} (\text{Res}^{CR_n}_{CR_{n-1}} V_n^{\emptyset}) \cong V_n^{(1)} \cong \mathbb{C}^n.
\]
Suppose (21) holds for \( k - 1 \). Then,
\[
(\hat{\text{Ind}}_{\mathcal{C}R_{n-1}}^{\mathcal{C}R_n}(\text{Res}_{\mathcal{C}R_{n-1}}^{\mathcal{C}R_n}V_n^\theta))^k = \hat{\text{Ind}}_{\mathcal{C}R_{n-1}}^{\mathcal{C}R_n}(\text{Res}_{\mathcal{C}R_{n-1}}^{\mathcal{C}R_n}((\hat{\text{Ind}}(\text{Res}_{\mathcal{C}R_{n-1}}^{\mathcal{C}R_n}V_n^\theta)^{k-1})
\cong \hat{\text{Ind}}_{\mathcal{C}R_{n-1}}^{\mathcal{C}R_n}(\text{Res}_{\mathcal{C}R_{n-1}}^{\mathcal{C}R_n}((\mathbb{C}^n)^\otimes (k-1)), \text{ by the induction hypothesis}
\cong (\mathbb{C}^n)^\otimes (k-1) \otimes \mathbb{C}^n, \text{ from Proposition 2.14}
= (\mathbb{C}^n)^\otimes k.
\]
\[
\square
\]

For \( \lambda \in \Lambda_{\leq n} \), define \( \lambda^- \) to be the set of all Young diagrams in \( \Lambda_{\leq n-1} \) obtained from \( \lambda \) by removing a box from an inner corner. Define the modified restriction rule as follows
\[
\hat{\text{Res}}_{\mathcal{C}R_{n-1}}^{\mathcal{C}R_n} (\bigoplus_{\lambda \in \Lambda_{\leq n}} (V_n^\lambda)^{\otimes n_\lambda}) = \bigoplus_{\lambda \in \Lambda_{\leq n}} \bigoplus_{\mu \in \lambda^-} V_{n-1}^\mu, \quad \text{where } n_\lambda \text{ is a nonnegative integer.}
\]

**Proposition 2.16** (Frobenius reciprocity). Let \( V \) and \( W \) be representations of \( \mathcal{C}R_n \) and \( \mathcal{C}R_{n-1} \), respectively. Then
\[
\text{Hom}_{\mathcal{C}R_n}(\hat{\text{Ind}}_{\mathcal{C}R_{n-1}}^{\mathcal{C}R_n}(W), V) \cong \text{Hom}_{\mathcal{C}R_{n-1}}(W; \hat{\text{Res}}_{\mathcal{C}R_{n-1}}^{\mathcal{C}R_n}(V)).
\]

**Proof.** Since the rook monoid algebras over \( \mathbb{C} \) are semisimple, it is enough to prove the isomorphism (22) for irreducible representations \( V = V_n^\lambda \) and \( W = V_{n-1}^\mu \) of \( \mathcal{C}R_n \) and \( \mathcal{C}R_{n-1} \) for \( \lambda \in \Lambda_{\leq n} \) and \( \mu \in \Lambda_{\leq n-1} \), respectively. Then,
\[
\text{Hom}_{\mathcal{C}R_n}(\hat{\text{Ind}}_{\mathcal{C}R_{n-1}}^{\mathcal{C}R_n}(V_{n-1}^\mu), V_n^\lambda) = \text{Hom}_{\mathcal{C}R_n}(\bigoplus_{\nu \in \mu^+} V_n^\nu, V_n^\lambda)
= \begin{cases} 
\text{Hom}_{\mathcal{C}R_n}(V_n^\lambda, V_n^\lambda) & \text{if } \lambda \in \mu^+, \\
\{0\} & \text{otherwise.}
\end{cases}
\]
Similarly,
\[
\text{Hom}_{\mathcal{C}R_{n-1}}(V_{n-1}^\mu, \hat{\text{Res}}_{\mathcal{C}R_{n-1}}^{\mathcal{C}R_n}(V_n^\lambda)) = \text{Hom}_{\mathcal{C}R_{n-1}}(V_{n-1}^\mu, \bigoplus_{\nu' \in \lambda^-} V_{n-1}^{\nu'})
= \begin{cases} 
\text{Hom}_{\mathcal{C}R_{n-1}}(V_{n-1}^{\mu}, V_{n-1}^{\mu}) & \text{if } \mu \in \lambda^-, \\
\{0\} & \text{otherwise.}
\end{cases}
\]
Since \( \lambda \in \mu^+ \) if and only if \( \mu \in \lambda^- \) and also \( \text{Hom}_{\mathcal{C}R_n}(V_n^\lambda, V_n^\lambda) \cong \mathbb{C} \cong \text{Hom}_{\mathcal{C}R_{n-1}}(V_{n-1}^{\mu}, V_{n-1}^{\mu}) \), the isomorphism (22) holds.
\[
\square
\]
3 Totally propagating partition algebra and its representation theory

The totally propagating partition algebra $\mathbb{C}I_k$ (Definition 3.8) is a subalgebra of partition algebra $\mathbb{C}A_k(\xi)$ (Definition 3.2). The Schur–Weyl duality between the actions of $\mathbb{C}R_n$ and $\mathbb{C}I_k$ on $(\mathbb{C}^n)^{\otimes k}$ was given in [KM08]. In this section, we give an indexing set of the irreducible representations of $\mathbb{C}I_k$, and further by introducing $\mathbb{C}I_k^{+\frac{1}{2}}$ (also called totally propagating partition algebra, Definition 3.8), we build a tower consisting of $\mathbb{C}I_k^{+\frac{1}{2}}$ and $\mathbb{C}I_k$ whose Bratteli diagram is a simple graph. Let us first begin with a brief overview of partition algebra.

3.1 Partition algebra

Given a positive integer $k$, a set partition of $\{1, 2, \ldots, k, 1', 2', \ldots, k'\}$ is a collection $\{B_1, B_2, \ldots, B_s\}$ of nonempty sets such that

(i) $B_p \cap B_q = \emptyset$ for all $1 \leq p \neq q \leq s$, and

(ii) $\sqcup_{p=1}^s B_i = \{1, 2, \ldots, k, 1', 2', \ldots, k'\}$.

Given a set partition as above, draw an undirected graph whose vertices are arranged in two rows such that the top row consists of the vertices $1, 2, \ldots, k$, the bottom row consists of the vertices $1', 2', \ldots, k'$; and there is a path between two vertices if and only if both vertices lie in the same $B_p$ for some $1 \leq p \leq s$. This graph is called the partition diagram corresponding to the given set partition. The connected components (i.e., $B_p$ for $1 \leq p \leq s$) of a partition diagram $d$ are called the blocks of $d$.

Example 3.1. The set partition $\{\{1, 2, 1', 3'\}, \{4, 2'\}, \{3, 4'\}\}$ of $\{1, 2, 3, 4, 1', 2', 3', 4'\}$ corresponds to the following partition diagram:

Let $A_k$ denote the set of all partition diagrams on $\{1, 2, \ldots, k, 1', 2', \ldots, k'\}$. For $d_1, d_2 \in A_k$, the composition $d_1 \circ d_2$ is a partition diagram obtained by placing $d_1$ above $d_2$, identifying the bottom row of $d_1$ with the top row of $d_2$, and excluding the connected components that lie entirely in the middle row. With respect to the composition of partition diagrams, $A_k$ is a monoid.
Definition 3.2. For $\xi \in \mathbb{C}$, let $\mathbb{C}A_k(\xi) = \mathbb{C}\text{-span}\{d \mid d \in A_k\}$. Given basis elements $d_1, d_2 \in A_k$, define a multiplication in $\mathbb{C}A_k(\xi)$ as follows:

$$d_1d_2 := \xi^l d_1 \circ d_2,$$

(23)

where $l$ is the number of connected components that lie entirely in the middle row while computing $d_1 \circ d_2$. With respect to the multiplication (23), $\mathbb{C}A_k(\xi)$ is a unital associative algebra over $\mathbb{C}$, and it is called the partition algebra.

Example 3.3. Let $d_1$ and $d_2$ be the partition diagrams corresponding to the set partitions $\{\{1,3\}, \{2,1'\}, \{4\}, \{2',3'\}, \{4'\}\}$ and $\{\{1, 4'\}, \{2\}, \{3\}, \{4\}, \{1'\}, \{2', 3'\}\}$. The multiplication $d_1d_2$ in $\mathbb{C}A_4(\xi)$ is illustrated below.

![Figure 1: Multiplication](image)

Definition 3.4. Let $A_{k+\frac{1}{2}}$ be the submonoid of $A_{k+1}$ consisting of partition diagrams whose vertices $(k+1)$ and $(k+1)'$ are always in the same block. Let

$$\mathbb{C}A_{k+\frac{1}{2}}(\xi) = \mathbb{C}\text{-span}\{d \mid d \in A_{k+\frac{1}{2}}\}.$$

The subspace $\mathbb{C}A_{k+\frac{1}{2}}(\xi)$ is a subalgebra of $\mathbb{C}A_{k+1}(\xi)$. It is also called partition algebra.

Example 3.5. The following partition diagram is an element of $A_{3+\frac{1}{2}}$:

![Figure 2: Example 3.5](image)
Remark 3.6. It is well known that the partition algebra $\mathbb{C}A_k(\xi)$ contains (a deformation of) the rook monoid algebra; however, this copy is different from the rook monoid algebra that we study in this paper.

We recall from [Jon94, p. 5] the orbit basis of $\mathbb{C}A_k(\xi)$ which will be relevant in Section 4. For $d_1, d_2 \in A_k$, we say $d_1$ is coarser than $d_2$ if for $i$ and $j$ in the same block of $d_2$ then $i$ and $j$ are in the same block of $d_1$. The notation $d_1 \leq d_2$ means that $d_1$ is coarser than $d_2$ and this defines a partial order on $A_k$. For $d \in A_k$, define $x_d \in \mathbb{C}A_k(\xi)$ to be the element uniquely satisfying the following relation

$$d = \sum_{d' \leq d} x_{d'}.$$ 

By linearly extending the partial order, it can be seen that the transition matrix between \{d | d \in A_k\} and \{x_d | d \in A_k\} is unitriangular, so \{x_d | d \in A_k\} is also a basis of $\mathbb{C}A_k(\xi)$, and it is called the orbit basis of $\mathbb{C}A_k(\xi)$. The structure constants of $\mathbb{C}A_k(\xi)$ with respect to the orbit basis were first given in online notes [Ram10] and later these also appeared in [BH19b, Theorem 4.8] and [MS, Lemma 3.1].

For $d_1, d_2 \in A_k$, a block in $d_1 \circ d_2$ is called an internal block if it lies entirely in the middle row while computing $d_1 \circ d_2$. For $1 \leq i, j \leq k$, whenever $i'$ and $j'$ are in the same block in $d_1$ if and only if $i$ and $j$ are in the same block in $d_2$ then we say that the bottom row of $d_1$ matches with the top row of $d_2$.

Lemma 3.7. For $d_1, d_2 \in A_k$, the multiplication of $x_{d_1}$ and $x_{d_2}$ in $\mathbb{C}A_k(\xi)$ is given by

$$x_{d_1}x_{d_2} = \begin{cases} \sum_d c_d x_d & \text{if the bottom row of } d_1 \text{ matches with the top row of } d_2, \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is taken over all those $d$ in $A_k$ such that $d$ is coarser than $d_1 \circ d_2$ and the coarsening is done by connecting a block of $d_1$ which is contained entirely in the top row of $d_1$ with a block of $d_2$ which is contained entirely in the bottom row of $d_2$ and

$$c_d = (\xi - |d|)|d_1 \circ d_2|,$$

where $|d|$ is the number of blocks in $d$, $[d_1 \circ d_2]$ is the number of internal blocks in $d_1 \circ d_2$, and for any polynomial $f(\xi)$ in $\xi$, $b \in \mathbb{Z}_{\geq 0}$,

$$(f(\xi))_b := \begin{cases} f(\xi)(f(\xi) - 1) \cdots (f(\xi) - b + 1) & \text{if } b > 0, \\ 1 & \text{if } b = 0. \end{cases}$$

3.2 Totally propagating partition algebra

A block $B$ of a partition diagram in $A_k$ is called a propagating block if $B$ contains vertices of both top and bottom rows, i.e.,

$$B \cap \{1, 2, \ldots, k\} \neq \emptyset \quad \text{and} \quad B \cap \{1', 2', \ldots, k'\} \neq \emptyset.$$
Let \( I_k \) be the subset of \( A_k \) consisting of partition diagrams each of whose blocks are propagating blocks. The subset \( I_k \) is a submonoid of \( A_k \).

**Definition 3.8.** (i) Define \( CI_k = \mathbb{C}\text{-span}\{d \mid d \in I_k\} \). The subspace \( CI_k \) is a subalgebra of \( CA_k(\xi) \).

(ii) For \( I_{k+\frac{1}{2}} = A_{k+\frac{1}{2}} \cap I_{k+1} \), define \( CI_{k+\frac{1}{2}} = \mathbb{C}\text{-span}\{d \mid d \in I_{k+\frac{1}{2}}\} \). The subspace \( CI_{k+\frac{1}{2}} \) is a subalgebra of both \( CA_{k+\frac{1}{2}}(\xi) \) and \( CI_{k+1} \).

An important observation is that the multiplications in \( CI_k \) and \( CI_{k+\frac{1}{2}} \) do not depend on the multiplication factor \( \xi \). Specifically, for \( d_1, d_2 \in I_k \), the multiplication in \( CI_k \) is given by

\[
d_1 d_2 = d_1 \circ d_2
\]

since there are no connected components which lie entirely in the middle row while computing \( d_1 \circ d_2 \). We call the algebras \( CI_k \) and \( CI_{k+\frac{1}{2}} \) the totally propagating partition algebras.

For \( d \in I_k \) (respectively, \( d \in I_{k+\frac{1}{2}} \)), any partition diagram coarser than \( d \) is also an element of \( I_k \) (respectively, \( I_{k+\frac{1}{2}} \)) and therefore, \( x_d \in CI_k \) (respectively, \( x_d \in CI_{k+\frac{1}{2}} \)). Lemma 3.7 implies the following corollary which specializes \( x_{d_1} x_{d_2} \) for \( d_1, d_2 \in I_k \).

**Corollary 3.9.** For \( d_1, d_2 \in I_k \), the multiplication of \( x_{d_1} \) and \( x_{d_2} \) in \( CI_k \) is given by

\[
x_{d_1} x_{d_2} = \begin{cases} x_{d_1 \circ d_2} & \text{if the bottom row of } d_1 \text{ matches with the top row of } d_2, \\ 0 & \text{otherwise.} \end{cases}
\]

### 3.3 Schur–Weyl dualities

For \( V = \mathbb{C}^n \), recall that \( \{e_1, e_2, \ldots, e_n\} \) is the standard basis of \( V \). The symmetric group \( S_n \) acts on \( V \) as follows:

\[
\beta e_i = e_{\beta(i)},
\]

where \( \beta \in S_n \) and \( 1 \leq i \leq n \). The \( k \)-fold tensor space \( V \otimes V \) with a basis

\[
\{e_i \otimes e_i \otimes \cdots \otimes e_i \mid 1 \leq i_1, i_2, \ldots, i_k \leq n\}
\]

is a representation of \( S_n \) with respect to the diagonal action. We identify the symmetric group \( S_{n-1} \) with the subgroup consisting of permutation matrices, whose \((n, n)\)-th entry is 1, in \( S_n \). The space \( V \otimes (k+\frac{1}{2}) := V \otimes k \otimes e_n \) is a representation of \( S_{n-1} \).

The partition algebra \( CA_k(n) \) acts on \( V \otimes k \) on the right by the following map:

\[
\phi_k : CA_k(n) \to \text{End}_\mathbb{C}(V \otimes k) \\
d \mapsto \phi_k(d),
\]
where the map $\phi_k(d)$ on the basis element $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ is given by

$$(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k})\phi_k(d) = \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n} (\phi_k(d))^{i_1, i_2, \ldots, i_k}_{i_1, i_2, \ldots, i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$$

with the coefficients

$$(\phi_k(d))^{i_1, i_2, \ldots, i_k}_{i_1, i_2, \ldots, i_k} = \begin{cases} 1 & \text{if } i_r = i_s \text{ when } r \text{ and } s \text{ are in the same block of } d, \\ 0 & \text{otherwise.} \end{cases}$$

By definition of $x_d$, we see that the coefficient of $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ in the linear expansion of $(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k})\phi(x_d)$ with respect to the basis (24) is

$$(\phi_k(x_d))^{i_1, i_2, \ldots, i_k}_{i_1, i_2, \ldots, i_k} = \begin{cases} 1 & \text{if } i_r = i_s \text{ if and only if } r \text{ and } s \text{ are in the same block of } d, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi_{k+\frac{1}{2}} : \mathbb{C}A_{k+\frac{1}{2}}(n) \to \text{End}_{\mathbb{C}}(V^{\otimes(k+\frac{1}{2})})$ denote the restriction $\phi_{(k+1)|A_{k+\frac{1}{2}}}(n)$ of the map $\phi_{k+1}$ to $\mathbb{C}A_{k+\frac{1}{2}}(n)$. The map $\phi_{k+\frac{1}{2}}$ gives us a right action of $\mathbb{C}A_{k+\frac{1}{2}}(n)$ on $V^{\otimes(k+\frac{1}{2})}$. We recall Schur–Weyl dualities between the partition algebras and the symmetric groups from [HR05, Theorem 3.6].

**Theorem 3.10.**  
(i) The image of the map $\phi_k : \mathbb{C}A_k(n) \to \text{End}_{\mathbb{C}}(V^{\otimes k})$ is $\text{End}_{S_n}(V^{\otimes k})$. The kernel of $\phi_k$ is $\mathbb{C}$-span $\{x_d \mid d \text{ has more than } n \text{ blocks}\}$. In particular, for $n \geq 2k$, $\mathbb{C}A_k(n) \cong \text{End}_{S_n}(V^{\otimes k})$.

(ii) The image of the map $\phi_k : \mathbb{C}A_{k+\frac{1}{2}}(n) \to \text{End}_{\mathbb{C}}(V^{\otimes(k+\frac{1}{2})})$ is $\text{End}_{S_{n-1}}(V^{\otimes(k+\frac{1}{2})})$. The kernel of $\phi_k$ is $\mathbb{C}$-span $\{x_d \mid d \text{ has more than } n \text{ blocks}\}$. In particular, for $n \geq 2k+1$, $\mathbb{C}A_{k+\frac{1}{2}}(n) \cong \text{End}_{S_n}(V^{\otimes(k+\frac{1}{2})})$.

Let $\tilde{\psi}_k : \mathbb{C}R_n \to \text{End}_{\mathbb{C}}(V^{\otimes k})$ be the algebra homomorphism arising from the action of $\mathbb{C}R_n$ on $V^{\otimes k}$ and let $\tilde{\phi}_k$ be the restriction map $\phi_k|_{\mathbb{C}I_k}$ of $\phi_k$ to $\mathbb{C}I_k$. For $d \in I_k$, since each block of $d$ is propagating, therefore we have:

$$(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k})\tilde{\phi}_k(d) = \begin{cases} e_{i_r} \otimes e_{i_s} \otimes \cdots \otimes e_{i_{k'}} & \text{if } i_r = i_s \text{ when } r \text{ and } s \text{ are in the same block of } d, \\ 0 & \text{otherwise.} \end{cases}$$

The rook monoid algebra $\mathbb{C}R_n$ contains the group algebra $\mathbb{C}S_n$, therefore, the centralizer algebra $\text{End}_{\mathbb{C}R_n}(V^{\otimes k})$ is a subalgebra of the centralizer algebra $\text{End}_{\mathbb{C}S_n}(V^{\otimes k})$. The following theorem is the Schur–Weyl duality [KM08, Theorem 1] between the actions of $\mathbb{C}I_k$ and $\mathbb{C}R_n$ on $V^{\otimes k}$.
Theorem 3.11. (i) The image of the map $\tilde{\phi}_k : CI_k \to \text{End}_C(V^{\otimes k})$ is $\text{End}_{CR_n}(V^{\otimes k})$.

The kernel of $\tilde{\phi}_k$ is $C$-span $\{x_d \mid d \in I_k$ and $d$ has more than $n$ blocks$\}$. In particular, when $n \geq k$, $CI_k \cong \text{End}_{CR_n}(V^{\otimes k})$.

(ii) The image of the map $\tilde{\psi}_k : \mathbb{C}R_n \to \text{End}_C(V^{\otimes k})$ is $\text{End}_{I_k}(V^{\otimes k})$.

Let $\tilde{\psi}_{k+\frac{1}{2}} : \mathbb{C}R_{n-1} \to \text{End}_C(V^{\otimes (k+\frac{1}{2})})$ be the algebra homomorphism arising from the action of $\mathbb{C}R_{n-1}$ on $V^{\otimes (k+\frac{1}{2})}$ and let $\tilde{\phi}_{k+\frac{1}{2}}$ be the restriction map $\phi_{k+\frac{1}{2}|I_{k+\frac{1}{2}}}$ of $\phi_{k+\frac{1}{2}}$ to $CI_{k+\frac{1}{2}}$. Then as a corollary of Theorem 3.10 and Theorem 3.11 we obtain:

Corollary 3.12. (i) The image of the map $\tilde{\phi}_{k+\frac{1}{2}} : CI_{k+\frac{1}{2}} \to \text{End}_C(V^{\otimes (k+\frac{1}{2})})$ is given by $\text{End}_{CR_{n-1}}(V^{\otimes (k+\frac{1}{2})})$. The kernel of $\tilde{\phi}_{(k+\frac{1}{2})}$ is given by $C$-span $\{x_d \mid d \in I_{k+\frac{1}{2}}$ and $d$ has more than $n$ blocks$\}$.

In particular, when $n \geq k+1$, $CI_{k+\frac{1}{2}} \cong \text{End}_{CR_{n-1}}(V^{\otimes (k+\frac{1}{2})})$.

(ii) The image of the map $\tilde{\psi}_{k+\frac{1}{2}} : \mathbb{C}R_{n-1} \to \text{End}_C(V^{\otimes (k+\frac{1}{2})})$ is $\text{End}_{I_{k+\frac{1}{2}}}(V^{\otimes (k+\frac{1}{2})})$.

3.4 Irreducible representations of totally propagating partition algebras

We describe an indexing set of the irreducible representations of $CI_k$ and $CI_{k+\frac{1}{2}}$, and define an embedding of $CI_k$ inside $CI_{k+\frac{1}{2}}$ whose branching rule is also described.

Theorem 3.13. For $k \geq 1$, the irreducible representations of $CI_k$ and $CI_{k+\frac{1}{2}}$ are indexed by the elements of $\widehat{I}_k := \{\lambda \vdash r \mid 1 \leq r \leq k\}$ and $\widehat{I}_{k+\frac{1}{2}} := \{\mu \vdash r \mid 0 \leq r \leq k\}$, respectively.

Proof. Choose an integer $n$ such that $n \geq k$. By applying Corollary 2.11, we see that in the decomposition of $V^{\otimes k}$ only those irreducible representations of $CR_n$ appear which correspond to the partitions of $r$ for $1 \leq r \leq k$. Since $n \geq k$, by Theorem 3.11 we have $CI_k \cong \text{End}_{CR_n}(V^{\otimes k})$.

So, by the centralizer theorem [HR05, Theorem 5.4], the irreducible representations of $CI_k$ are indexed by the elements of $\widehat{I}_k$.

Similarly, by choosing an integer $n$ such that $n \geq k+1$ and then applying Corollary 3.12 and Proposition 2.3, we get that the irreducible representations of $CI_{k+\frac{1}{2}}$ are indexed by the elements of $\widehat{I}_{k+\frac{1}{2}}$. $\square$

Suppose $I^\lambda_k$ and $I^\mu_{k+\frac{1}{2}}$ denote the irreducible representations corresponding to $\lambda \in \widehat{I}_k$ and $\mu \in \widehat{I}_{k+\frac{1}{2}}$ of $CI_k$ and $CI_{k+\frac{1}{2}}$, respectively.
Theorem 3.14.  
(i) For \( n \geq k \), as \((\mathbb{C}R_n, \mathbb{C}I_k)\)-bimodule we have  
\[
V^{\otimes k} \cong \bigoplus_{\lambda \in \hat{I}_k} V^n_{\lambda} \otimes I_k^\lambda.
\]

(ii) For \( n \geq k + 1 \), as \((\mathbb{C}R_{n-1}, \mathbb{C}I_{k+\frac{1}{2}})\)-bimodule we have  
\[
V^{\otimes k} \cong \bigoplus_{\mu \in \hat{I}_{k+\frac{1}{2}}} V_{n-1}^\mu \otimes I_{k+\frac{1}{2}}^\mu.
\]

Proof. The proof of the first part (respectively, the second part) is an application of Theorem 3.11 (respectively, Corollary 3.12), Theorem 3.13, and the centralizer theorem [HR05, Theorem 5.4].

\[\square\]

A tower, branching rule, and the Bratteli diagram. We first give the following embedding. For \( k \geq 1 \), define  
\[
\eta_k : \mathbb{C}I_k \to \mathbb{C}I_{k+\frac{1}{2}} \quad \tag{26}
\]
\[
x_d \mapsto x_{d'},
\]
where \( d \in I_k \) and \( d' \) is obtained from \( d \) by adding the block \( \{(k + 1), (k + 1)\}' \) to \( d \). Then \( \eta_k \) is an injective map and Corollary 3.9 implies that it is also an algebra homomorphism. Using the embedding (26), we have the following tower of totally propagating partition algebras:  
\[
\mathbb{C}I_{\frac{1}{2}} \subseteq \mathbb{C}I_1 \subset \mathbb{C}I_{\frac{3}{2}} \subset \mathbb{C}I_2 \subset \cdots. \quad \tag{27}
\]
Note that \( \mathbb{C}I_{\frac{1}{2}} \cong \mathbb{C}I_1 \).

Remark 3.15. The embedding (26) is not induced from the embedding [HR05, Equation (2.2)] of partition algebras \( \mathbb{C}A_k(\xi) \to \mathbb{C}A_{k+\frac{1}{2}}(\xi) \).

We state the following theorem in order to describe the branching rule for the embedding \( \mathbb{C}I_k \subset \mathbb{C}I_{k+\frac{1}{2}} \).

Theorem 3.16.  [Ram, Theorem 5.9] Let \( A \) be a subalgebra of an algebra \( B \) and let \( M \) be semisimple as both \( B \)-module and \( A \)-module. Then, the multiplicity of a representation \( W \) of \( \text{End}_B(M) \) in the restriction of a representation \( \overline{V} \) of \( \text{End}_A(M) \) to \( \text{End}_B(M) \) is equal to the multiplicity of the corresponding representation \( V \) of \( A \) in the restriction of the corresponding representation \( W \) of \( B \) to \( A \).

Corollary 3.17.  
(i) \( \text{Res}_{\mathbb{C}I_{k+\frac{1}{2}}}^{\mathbb{C}I_{k+\frac{1}{2}}} I^\emptyset_{k+\frac{1}{2}} \cong I_k^\nu \), where \( \nu = (1) \).

(ii) For \( \mu \in \hat{I}_{k+\frac{1}{2}} \setminus \{\emptyset\} \), we have  
\[
\text{Res}_{\mathbb{C}I_k}^{\mathbb{C}I_{k+\frac{1}{2}}} I_{k+\frac{1}{2}}^\mu \cong \bigoplus_{\nu \in \hat{I}_{k+\frac{1}{2}}, \nu \neq \emptyset} I_k^\nu.
\]

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Proof. The rule for decomposing $\text{Res}^{C_{I_{k+\frac{1}{2}}}^\mu}_{C_{I_{k}^+}} I_{k+\frac{1}{2}}$, for $\mu \in \tilde{I}_{k+\frac{1}{2}}^+$, follows from Theroem 3.14 and Theorem 3.16.

In the following theorem, we give the branching rule for $C_{I_{k+\frac{1}{2}}} \subset C_{I_{k+1}}$.

**Theorem 3.18.** For $\lambda \in \tilde{I}_{k+1}$, we have

$$\text{Res}^{C_{I_{k+1}}}_{C_{I_{k+\frac{1}{2}}}} I_{k+1}^\lambda \cong \bigoplus_{\mu \in \lambda^-} I_{k+\frac{1}{2}}^\mu.$$

Proof. Let $\lambda \in \tilde{I}_{k+1}$. Choose a positive integer $n$ such that $n \geq k + 1$. Then,

$$\text{Res}^{C_{I_{k+1}}}_{C_{I_{k+\frac{1}{2}}}} I_{k+1}^\lambda$$

$$\cong \text{Res}^{C_{I_{k+1}}}_{C_{I_{k+\frac{1}{2}}}} \text{Hom}_{CR_{n}}((C^n)^{\otimes k}, V_n^\lambda), \text{ by Theorem 3.14}$$

$$\cong \text{Res}^{C_{I_{k+1}}}_{C_{I_{k+\frac{1}{2}}}} \text{Hom}_{CR_{n}}((\hat{\text{Ind}}_{CR_{n-1}}^{CR_{n}} \text{Res}_{CR_{n-1}}^{V^0} V_n^k, V_n^\lambda), \text{ by Proposition 2.15}$$

$$\cong \text{Res}^{C_{I_{k+1}}}_{C_{I_{k+\frac{1}{2}}}} \text{Hom}_{CR_{n-1}}((\hat{\text{Ind}}_{CR_{n-1}}^{CR_{n-1}} \text{Res}_{CR_{n-1}}^{V^0} V_n^k, \hat{\text{Res}}_{CR_{n-1}}^{CR_{n}} V_n^\lambda), \text{ by Proposition 2.16}$$

$$\cong \text{Hom}_{CR_{n-1}}((C^n)^{\otimes k}, \bigoplus_{\mu \in \lambda^-} V_{n-1}^\mu), \text{ by definition of } \hat{\text{Res}}_{CR_{n-1}}(-)$$

$$\cong \bigoplus_{\mu \in \lambda^-} \text{Hom}_{CR_{n-1}}((C^n)^{\otimes k}, V_{n-1}^\mu)$$

$$\cong \bigoplus_{\mu \in \lambda^-} I_{k}^\mu, \text{ by Theorem 3.14}. \hfill \Box$$

Using Corollary 3.17 and Theorem 3.18, we get the Bratteli diagram $\hat{I}$ of the tower (27) in which the sets of vertices at level $k$ and at level $k + \frac{1}{2}$ are $\hat{I}_k = \{ \mu \vdash r \mid 1 \leq r \leq k \}$ and $\hat{I}_{k+\frac{1}{2}} = \{ \mu \vdash r \mid 0 \leq r \leq k \}$, respectively. For $\mu \in \hat{I}_k$ and $\nu \in \hat{I}_{k+\frac{1}{2}}$, there is an edge between $\mu$ and $\nu$ if and only if $\nu = \mu$ or $\nu \in \mu^-$. For $\nu \in \hat{I}_{k+\frac{1}{2}}$ and $\lambda \in \hat{I}_{k+1}$, there is an edge between $\nu$ and $\lambda$ if and only if $\lambda \in \nu^+$.

**Example 3.19.** The Bratteli diagram of the tower (27) up to level 3 is illustrated below.
Corollary 3.20. For \( t \in \frac{1}{2} \mathbb{Z}_{>0} \), the dimension of the irreducible representation \( I_t^\lambda \) of \( C_t \) is the number of paths from \( \emptyset \in \hat{I}_\frac{1}{2} \) to \( \lambda \in \hat{I}_t \) in the Bratteli diagram \( \hat{I} \).

### 4 Jucys–Murphy elements of totally propagating partition algebras

We begin this section with some definitions and notation which are needed to define Jucys–Murphy elements of the algebras \( C_k \) and \( C_{k+\frac{1}{2}} \).

For \( S \subseteq \{1, 2, \ldots, k\} \), let \( S' \) be the set \( \{i' \in \{1', 2', \ldots, k'\} \mid i \in S\} \). Let \( \mathcal{U}_k \) be the collection of ordered set partitions of \( \{1, 2, \ldots, k\} \). For \( B_t = (B_1, \ldots, B_l) \in \mathcal{U}_k \), let \( d_{B_t} \in I_k \) be the partition diagram corresponding to the set partition \( \{B_1 \cup B'_1, \ldots, B_l \cup B'_l\} \).

Let \( \mathcal{S}_k \) be the collection of ordered set partitions of \( \{1, 2, \ldots, k\} \) with at least two blocks. For \( B_t = (B_1, \ldots, B_l) \in \mathcal{S}_k \) and \( 1 \leq i \neq j \leq l \leq k \), define an element in \( I_k \) by

\[
d_{B_t,\{i,j\}} = \{B_1 \cup B'_1, \ldots, B_i \cup B'_i, \ldots, B_j \cup B'_j, \ldots, B_l \cup B'_l\}.
\]

**Example 4.1.** For \( k = 5 \), let \( B_3 = (\{1, 3\}, \{4\}, \{2, 5\}) \) be a set partition of \( \{1, 2, 3, 4, 5\} \). Here \( l = 3 \). Let \( i = 1 \) and \( j = 3 \). Then the partition diagram \( d_{B_3,\{1,3\}} \in I_5 \), depicted below, corresponds to the set partition \( \{\{1,3,2',5'\},\{4,4'\},\{2,5,1',3'\}\} \).
Define the following elements in $\mathbb{C}I_k$

$$Z_k := \sum_{B_l \in U_k} |B_l| x_{d_{B_l}},$$

(28)

$$\tilde{Z}_k := \sum_{B_l \in S_k} \sum_{1 \leq i < j \leq l} x_{d_{B_l(i,j)}}.$$  

(29)

Let $U_{k+\frac{1}{2}}$ be the collection of ordered set partitions of $\{1, 2, \ldots, k + 1, 1', 2', \ldots, (k + 1)\}'$ with at least two blocks. For $B_s = (B_1, \ldots, B_s, \ldots, B_l) \in U_{k+\frac{1}{2}}$ where $B_s$ contains $k + 1$, let $\tilde{d}_B \in I_{k+\frac{1}{2}}$ be the partition diagram corresponding to the set partition $\{B_1 \cup B_1', \ldots, B_l \cup B_l'\}$.

**Example 4.2.** The set partition $B_3$, as given in Example 4.1, is in $U_{4+\frac{1}{2}}$. The partition diagram $\tilde{d}_{B_3} \in I_{4+\frac{1}{2}}$ corresponding to $B_3$ is as follows:

![Diagram](https://via.placeholder.com/150)

Let $S_{k+\frac{1}{2}}$ be the collection of ordered set partitions of $\{1, 2, \ldots, k + 1\}$ with at least three blocks. For $B_{l,s} = (B_1, \ldots, B_s, \ldots, B_l) \in S_{k+\frac{1}{2}}$ with block $B_s$ containing $k + 1$ and $1 \leq i \neq j \leq l \leq (k + 1)$, $i \neq s$ and $j \neq s$, define an element in $I_{k+\frac{1}{2}}$ as follows:

$$\tilde{d}_{B_{l,s}(i,j)} := \{(B_1 \cup B_1', \ldots, B_s \cup B_s', \ldots, B_l \cup B_l')\}.$$  

**Example 4.3.** For $k = 4$, let $B_{3,3} = (\{1, 3\}, \{4\}, \{2, 5\})$ be a set partition of $\{1, 2, 3, 4, 5\}$. Here $l = 3$ and $s = 3$. Let $i = 1$ and $j = 2$. Then the partition diagram $d_{B_{3,3}(1,3)} \in I_{4+\frac{1}{2}}$, depicted below, corresponds to the set partition $\{(1, 3, 4'), \{4', 3', 2, 5, 2', 5'\}\}$.

![Diagram](https://via.placeholder.com/150)

Let $Z_{\frac{1}{2}} = 1$ and $\tilde{Z}_{\frac{1}{2}} = 1$. For $k \geq 1$, we define the following elements in $\mathbb{C}I_{k+\frac{1}{2}}$:

$$Z_{k+\frac{1}{2}} := \sum_{B_s \in U_{k+\frac{1}{2}}} (\{|B_s| - 1\}) x_{\tilde{d}_{B_s}} = Z_k + \sum_{B_s \in U_{k+\frac{1}{2}} |B_s| > 1} (\{|B_s| - 1\}) x_{\tilde{d}_{B_s}},$$

(30)

$$\tilde{Z}_{k+\frac{1}{2}} := \sum_{B_{l,s} \in S_{k+\frac{1}{2}} 1 \leq i \neq j \leq l, i \neq s, j \neq s} x_{\tilde{d}_{B_{l,s}(i,j)}}$$

(31)

$$= \tilde{Z}_k + \sum_{B_{l,s} \in S_{k+\frac{1}{2}} 1 \leq i \neq j \leq l, i \neq s, j \neq s, |B_s| > 1} x_{\tilde{d}_{B_{l,s}(i,j)}}.$$
In the above, we view $Z_k, \tilde{Z}_k \in \mathbb{C}I_{k+\frac{1}{2}}$ using the embedding \((26)\) $\eta_k : \mathbb{C}I_k \to \mathbb{C}I_{k+\frac{1}{2}}$.

**Theorem 4.4.** (a) The elements $\kappa_n := X_1 + \cdots + X_n$ and $\tilde{\kappa}_n := \tilde{X}_1 + \cdots + \tilde{X}_n$ are in the center of $\mathbb{C}R_n$.

(b) The elements $Z_k$ and $Z_{k+\frac{1}{2}}$ are in the centers of $\mathbb{C}I_k$ and $\mathbb{C}I_{k+\frac{1}{2}}$, respectively.

**Proof.** (a) Let $s_0 = \text{id}$ and recall from Page 10 that $\gamma_i = s_{i-1}s_{i-2}\cdots s_1P_1s_1\cdots s_{i-2}s_{i-1}$. Then by definition,

$$\kappa_n = n.1 - \left(\sum_{i=1}^{n} \gamma_i\right).$$

To show $\kappa_n$ is central, it is enough to show that $\left(\sum_{i=1}^{n} \gamma_i\right)$ commutes with the generators $s_1, s_2, \ldots, s_{n-1}, P_1$ of $\mathbb{C}R_n$.

For $1 \leq i \leq n$, $\gamma_i \in R_n$ is a diagonal matrix with $(i, i)$-th entry zero and the remaining entries one. So $P_1\gamma_i = \gamma_i P_1$ which implies $P_1\left(\sum_{i=1}^{n} \gamma_i\right) = \left(\sum_{i=1}^{n} \gamma_i\right) P_1$. For $1 \leq j \leq n$, we have

- $s_j\gamma_i s_j = \gamma_i$ for $i \neq j, j+1$,
- $s_j\gamma_j s_j = \gamma_{j+1}$ and $s_j\gamma_{j+1}s_j = \gamma_j$,

so $s_j\left(\sum_{i=1}^{n} \gamma_i\right) = \left(\sum_{i=1}^{n} \gamma_i\right) s_j$. Thus $\kappa_n$ is a central element of $\mathbb{C}R_n$.

Now we proceed towards proving that $\tilde{\kappa}_n$ is a central element of $\mathbb{C}R_n$. Recall that

$$\tilde{X}_1 = 0,$$

$$\tilde{X}_i = s_{i-1}\tilde{X}_{i-1}s_{i-1} + s_{i-1} - s_{i-1}\gamma_{i-1} - \gamma_{i-1}s_{i-1} + Q_i, \quad \text{for } 2 \leq i \leq n.$$

For $i = 1, 2, 3$, it is easy to verify that $P_1\tilde{X}_i = \tilde{X}_i P_1$. By taking base step $i = 3$, we use induction on $i$ to prove

$$P_1\tilde{X}_i = \tilde{X}_i P_1 \text{ for } 3 \leq i \leq n. \quad (32)$$

Note that

$$P_1s_j = s_j P_1 \text{ for } 2 \leq j \leq n - 1 \text{ and } \quad (33)$$

$$P_1D = DP_1 \text{ for any diagonal matrix } D \in R_n. \quad (34)$$

Suppose (32) holds for $3 \leq i < n$. Then using (33), induction hypothesis and that $Q_i$ (see Page 9) and $\gamma_i$ are the diagonal matrices, we get

$$P_1X_{i+1} = X_{i+1}P_1.$$
Hence, we have $P_1 \tilde{\kappa}_n = \tilde{\kappa}_n P_1$.

Since the zero element $\tilde{X}_1 = 0 \in R_n$ is a central element of $\mathbb{C}R_n$ therefore it is enough to show that $\sum_{i=2}^{n} \tilde{X}_i$ is a central element of $\mathbb{C}R_n$. We have

$$
\sum_{i=2}^{n} \tilde{X}_i = \sum_{i=2}^{n} \sum_{r=1}^{i-1} (r, i) + \sum_{i=2}^{n} \sum_{r=1}^{i-1} (2, r)(1, i)P_2(1, i)(2, r)
- \sum_{i=2}^{n} \sum_{r=1}^{i-1} (r, i) \gamma_r - \sum_{i=2}^{n} \sum_{r=1}^{i-1} \gamma_r (r, i).
$$

Since $\sum_{i=2}^{n} \sum_{r=1}^{i-1} (r, i) \in \mathbb{C}S_n$ is the sum of all transpositions in $S_n$, we have

$$
s_j \sum_{i=2}^{n} \sum_{r=1}^{i-1} (r, i) = \sum_{i=2}^{n} \sum_{r=1}^{i-1} (r, i)s_j \text{ for all } 1 \leq j \leq n.
$$

Let $\mathcal{L} \subset R_n$ consists of diagonal matrices with exactly two diagonal entries zero and the remaining diagonal entries one. Then

$$
\sum_{i=2}^{n} \sum_{r=1}^{i-1} (2, r)(1, i)P_2(1, i)(2, r) = \sum_{M \in \mathcal{L}} M.
$$

Since $\{s_j Ms_j \mid M \in \mathcal{L}\} = \mathcal{L}$, we have

$$
s_j \left( \sum_{i=2}^{n} \sum_{r=1}^{i-1} (2, r)(1, i)P_2(1, i)(2, r) \right) s_j = s_j \left( \sum_{M \in \mathcal{L}} M \right) s_j
= \sum_{M \in \mathcal{L}} s_j Ms_j = \sum_{M' \in \mathcal{L}} M'
= \sum_{i=2}^{n} \sum_{r=1}^{i-1} (2, r)(1, i)P_2(1, i)(2, r).
$$

Now in order to conclude that $\tilde{\kappa}_n$ is a central element of $\mathbb{C}R_n$, it remains to show the following for $1 \leq j \leq n - 1$:

$$
s_j \left( \sum_{i=2}^{n} \sum_{r=1}^{i-1} (r, i) \gamma_r + \sum_{i=2}^{n} \sum_{r=1}^{i-1} \gamma_r (r, i) \right) = \left( \sum_{i=2}^{n} \sum_{r=1}^{i-1} (r, i) \gamma_r + \sum_{i=2}^{n} \sum_{r=1}^{i-1} \gamma_r (r, i) \right) s_j. \quad (35)
$$

We have

$$
s_j(j, i) \gamma_j = (j + 1, i) \gamma_{j+1}s_1, \quad s_j \gamma_j(j, i) = \gamma_{j+1}(j + 1, i)s_j, \quad (36)
$$

$$
s_j(j + 1, i) \gamma_{j+1} = (j, i) \gamma js_j, \quad s_j \gamma_{j+1}(j + 1, i) = \gamma_j(j, i)s_j, \quad (37)
$$

for $j + 2 \leq r \leq i - 1$, $s_j(r, i) \gamma_r = (r, i) \gamma.rs_j$ and $s_j \gamma_r(r, i) = \gamma_r(r, i)s_j. \quad (38)$
For \( j = 1 \), (35) holds using (36), (37), and (38). For \( 1 < j \leq n \), we have
\[
\sum_{i=2}^{n} \sum_{r=1}^{i-1} (r, i) \gamma_r + \sum_{i=2}^{n} \sum_{r=1}^{i-1} \gamma_r (r, i) \\
= \sum_{i=2, i \neq j, i \neq j+1, i \neq j+2}^{n} \sum_{r=1}^{i-1} (r, i) \gamma_r + \sum_{i=2, i \neq j, i \neq j+1, i \neq j+2}^{n} \sum_{r=1}^{i-1} \gamma_r (r, i) \\
+ \sum_{r=1}^{j-1} (r, j) \gamma_r + \sum_{r=1}^{j-1} \gamma_r (r, j) + \sum_{r=1}^{j} (r, j + 1) \gamma_r + \sum_{r=1}^{j} \gamma_r (r, j + 1) \\
+ \sum_{r=1}^{j} (r, j + 2) \gamma_r + \sum_{r=1}^{j+2} \gamma_r (r, j + 2).
\]

So, for \( 1 < j \leq n \), (35) holds by further observing:

- \( s_j (r, j) \gamma_r = (r, j + 1) \gamma_r s_j \), \( s_j (r, j + 1) \gamma_r = (r, j) \gamma_r s_j \),
- \( s_j \gamma_r (r, j) = \gamma_r (r, j + 1) s_j \), \( s_j \gamma_r (r, j + 1) = \gamma_r (r, j) s_j \),
- \( s_j (j, j + 2) \gamma_j = (j + 1, j + 2) \gamma_{j+1} s_j \) and \( s_j (j + 1, j + 2) \gamma_{j+1} = (j, j + 2) \gamma_j s_j \).

(b) This part follows from the previous part, Theorem 3.11, and Corollary 3.12.

For \( \lambda \in \Lambda_{\leq n} \), we write \( b \in \lambda \) to mean that \( b \) is a box in \( \lambda \).

**Theorem 4.5.**

(i) If \( \lambda \in \Lambda_{\leq n} \), then as operators on \( V_n^\lambda \)

\[
\kappa_n = |\lambda| \id_{V_n^\lambda} \quad \text{and} \quad \tilde{\kappa}_n = \sum_{b \in \lambda} \ct(b) \id_{V_n^\lambda}.
\]

(ii) We have

\[
Z_k = \kappa_n, \quad Z_{k+\frac{1}{2}} = \kappa_{n-1}, \quad \text{as operators on } V^\otimes k, \quad \text{and}
\]

\[
\tilde{Z}_k = \tilde{\kappa}_n, \quad \tilde{Z}_{k+\frac{1}{2}} = \tilde{\kappa}_{n-1}, \quad \text{as operators on } V^\otimes k + \frac{1}{2}.
\]

(iii) (a) For \( \lambda \in \hat{I}_k \),

\[
Z_k = |\lambda| \id_{I_k^\lambda} \quad \text{and} \quad \tilde{Z}_k = \sum_{b \in \lambda} \ct(b) \id_{I_k^\lambda}, \quad \text{as operators on } I_k^\lambda.
\]

(b) For \( \mu \in \hat{I}_{k+\frac{1}{2}} \),

\[
Z_{k+\frac{1}{2}} = |\lambda| \id_{I_{k+\frac{1}{2}}^\lambda} \quad \text{and} \quad \tilde{Z}_{k+\frac{1}{2}} = \sum_{b \in \lambda} \ct(b) \id_{I_{k+\frac{1}{2}}^\lambda}, \quad \text{as operators on } I_{k+\frac{1}{2}}^\lambda.
\]
Proof. (i) This follows from the definitions of $\kappa_n$, $\tilde{\kappa}_n$ and Theorem 2.7.

(ii) Let $e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k} \in (C^n)^{\otimes k}$. Then,

$$X_i(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}) = \begin{cases} 
   e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k} & \text{if } i \notin \tilde{j}, \\
   0 & \text{otherwise},
\end{cases}$$

where $\tilde{j}$ is the multiset $\{j_1, \ldots, j_k\}$.

For $1 \leq i \leq n$, let $B_i = \{s \in \{1, 2, \ldots, k\} | j_s = i\}$. The ordered set $(B_1, \ldots, B_n)$ is a set partition of $\{1, 2, \ldots, k\}$, and let $l$ be the number of nonempty blocks of it, and call $B_l = (B_1, \ldots, B_n)$. Then,

$$\kappa_n(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = l(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}).$$

Every term, except $|B_l| x_{\tilde{d}B_l}$ (where $|B_l|$ is the number of nonempty blocks of $B_l$), in the expression of $Z_k$ acts on $(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k})$ as zero. Since,

$$|B_l| x_{\tilde{d}B_l}(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = l(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}),$$

therefore, $Z_k = \kappa_n$ as operators on $(C^n)^{\otimes k}$.

For $2 \leq i \leq n$, using induction on $i$, we have

$$\tilde{X}_i(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}) = \begin{cases} 
   0 & \text{if } i \notin \tilde{j}, \\
   0 & \text{if } i \in \tilde{j} \text{ and } 1, 2, \ldots, i - 1 \notin \tilde{j}, \\
   ((l_1, i) + \cdots + (l_s, i))e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k} & \text{if } i \in \tilde{j} \text{ and } 1 \leq l_1, \ldots, l_s \in \tilde{j} \text{ such that } l_1, l_2, \ldots, l_s \leq i - 1.
\end{cases}$$

Let $m_1 < \cdots < m_l$ be such that the set $\{m_1, m_2, \ldots, m_l\}$ is the underlying set of the multiset $\tilde{j}$. Then

$$(\tilde{X}_1 + \cdots + \tilde{X}_n)(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}) = \begin{cases} 
   \sigma_l(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}) & \text{if } 2 \leq l, \\
   0 & \text{otherwise},
\end{cases}$$

where $\sigma_s = (m_1, m_2) + (m_1, m_3) + (m_2, m_3) + \cdots + (m_1, m_l) + \cdots + (m_{l-1}, m_l)$. 

Note that $B_{m_1}, \ldots, B_{m_l}$ are the only nonempty blocks of $B_t$. Thus
\[
\tilde{Z}_k(e_{j_1} \otimes \cdots \otimes e_{j_k}) = \sum_{B_{l'} \in \mathcal{S}_k} \sum_{1 \leq i \neq j \leq l'} x_{dB_{l'}\{i,j\}}(e_{j_1} \otimes \cdots \otimes e_{j_k})
= \sum_{1 \leq p \neq q \leq l} x_{dB_{l'}\{mp,mq\}}(e_{j_1} \otimes \cdots \otimes e_{j_k}).
\]
(39)
From the definition $d_{B_{l'}\{mp,mq\}}$ and (25), we have
\[
x_{dB_{l'}\{mp,mq\}}(e_{j_1} \otimes \cdots \otimes e_{j_k}) = (mp, mq)(e_{j_1} \otimes \cdots \otimes e_{j_k}).
\]
(40)
Substituting from (40) in (39), we see that
\[
\tilde{Z}_k(e_{j_1} \otimes \cdots \otimes e_{j_k}) = \tilde{\kappa}_n(e_{j_1} \otimes \cdots \otimes e_{j_k}).
\]

(iii) This part follows from parts (i), (ii), Theorem 3.11 and Corollary 3.12.

For $t \in \frac{1}{2} \mathbb{Z}_{>0}$, define the Jucys–Murphy elements of $\mathbb{C}I_t$ as follows:
\[
M_{\frac{1}{2}} = 1, \quad \tilde{M}_{\frac{1}{2}} = 1,
M_y = Z_y - Z_{y-\frac{1}{2}} \quad \text{and} \quad \tilde{M}_y = \tilde{Z}_y - \tilde{Z}_{y-\frac{1}{2}}, \quad \text{for} \quad y \in \frac{1}{2} \mathbb{Z}_{>0} \quad \text{and} \quad \frac{1}{2} < y \leq t.
\]
(41)
In the following theorem, we show that the elements (41) satisfy the fundamental properties of Jucys–Murphy elements as discussed in Section 1.

**Theorem 4.6.** Let $t \in \frac{1}{2} \mathbb{Z}_{>0}$.

(a) The elements $M_{\frac{1}{2}}, M_1, \ldots, M_{t-\frac{1}{2}}, \tilde{M}_t, \tilde{M}_{\frac{1}{2}}, \tilde{M}_1, \ldots, \tilde{M}_{t-\frac{1}{2}}, \tilde{M}_t$, all commute with each other in $\mathbb{C}I_t$.

(b) Let $\tilde{I}_t^\mu$ denote the set of paths from $\emptyset \in \tilde{I}_2$ to $\mu \in \tilde{I}_t$ in the Bratteli diagram $\tilde{I}_t$. Then there is a unique, up to multiplication by scalars, basis $\{v_\gamma \mid \gamma \in \tilde{I}_t^\mu\}$ of the irreducible representation $I_t^\mu$ of $\mathbb{C}I_t$. Furthermore, for $\gamma = (\gamma(\frac{1}{2}), \gamma(1), \gamma(\frac{3}{2}), \ldots, \gamma(t-\frac{1}{2}), \gamma(t)) \in \tilde{I}_t^\mu$, and $l \in \mathbb{Z}_{>0}$ such that $l \leq t$, we have
\[
\tilde{M}_l v_\gamma = ct(\gamma(l)/\gamma(l-\frac{1}{2})) v_\gamma,
M_l v_\gamma = v_\gamma,
\]
\[
\tilde{M}_{l-\frac{1}{2}} v_\gamma = \begin{cases} 
-ct(\gamma(l-1)/\gamma(l-\frac{1}{2})) v_\gamma & \text{if } \gamma(l-1)/\gamma(l-\frac{1}{2}) = \Box \\
0 & \text{if } \gamma(l-1) = \gamma(l-\frac{1}{2}),
\end{cases}
\]
and
\[
M_{l-\frac{1}{2}} v_\gamma = \begin{cases} 
-v_\gamma & \text{if } \gamma(l-1)/\gamma(l-\frac{1}{2}) = \Box \\
0 & \text{if } \gamma(l-1) = \gamma(l-\frac{1}{2}).
\end{cases}
\]
Proof. (a) Using the tower $CI_{\frac{1}{2}} \subseteq CI_1 \subseteq \cdots \subseteq CI_t$ we can view $Z_{\frac{1}{2}}, \ldots, Z_t, \tilde{Z}_{\frac{1}{2}}, \ldots, \tilde{Z}_t$ as elements of $CI_t$. By Theorem 4.6(c), both $Z_t$ and $\tilde{Z}_t$ are in the center of $CI_t$. Therefore, for $l \in \frac{1}{2}\mathbb{Z}_{>0}$ and $l \leq t$, we have

$$Z_l Z_t = Z_t Z_l, \quad Z_l \tilde{Z}_t = \tilde{Z}_t Z_l,$$

$$\tilde{Z}_l Z_t = Z_t \tilde{Z}_l, \quad \tilde{Z}_l \tilde{Z}_t = \tilde{Z}_t \tilde{Z}_l.$$

Using the above commutation relations, we conclude that $M_l$ and $\tilde{M}_l$ commute with each other inside $CI_t$.

(b) For $t = \frac{1}{2}, 1$, the algebras $CI_t$ are one dimensional and so there is only one irreducible representation of $CI_t$. In particular, $\dim I^{0}_{\frac{1}{2}} = 1 = \dim I^{(1)}_1$, thus there are unique choices of bases, up to scalars, of representations $I^{0}_{\frac{1}{2}}$ and $I^{(1)}_1$. For $\mu \in \widehat{I}_t$, from Theorem 3.18, an irreducible representation of $CI_{t-\frac{1}{2}}$ in the restriction $\text{Res}_{CI_{t-\frac{1}{2}}}^CI_t I^\mu$ can occur at most once. So by induction, we get a canonical basis for each irreducible representation of $CI_{t-\frac{1}{2}}$, and union of these bases form a basis for $I^\mu$.

Let $l \in \mathbb{Z}_{>0}$ such that $l \leq t$. For $\gamma = (\gamma^{(\frac{1}{2})}, \gamma^{(1)}, \gamma^{(\frac{3}{2})}, \ldots, \gamma^{(t-\frac{1}{2}), \gamma^{(t)}}) \in \widehat{I}_t$, let $\lambda = l$ and $\mu = l - 1$ such that $\lambda = \gamma(l)$ and $\mu = \gamma(l-\frac{1}{2})$ (this implies $\mu \subset \lambda$). By the above discussion, $v_\gamma \in I^{\lambda}_l$ and also $v_\gamma \in I^\mu_{l-\frac{1}{2}}$. Then by Theorem 4.5,

$$M_l v_\gamma = Z_l v_\gamma - Z_{l-\frac{1}{2}} v_\gamma = (|\lambda| - |\mu|) v_\gamma = v_\gamma \quad \text{and} \quad \tilde{M}_l v_\gamma = \tilde{Z}_l v_\gamma - \tilde{Z}_{l-\frac{1}{2}} v_\gamma = \left( \sum_{b \in \lambda} \text{ct}(b) - \sum_{b' \in \mu} \text{ct}(b') \right) v_\gamma = \text{ct}(\gamma^{(l-\gamma^{(l-\frac{1}{2})})}) v_\gamma.$$

Let $\nu = \gamma^{(l-1)}$ then $\mu \subset \nu$ and $v_\gamma \in I^\nu_{l-\frac{1}{2}}$. Again by Theorem 4.5, we have

$$M_{l-\frac{1}{2}} v_\gamma = Z_{l-\frac{1}{2}} v_\gamma - Z_{l-1} v_\gamma = (|\mu| - |\nu|) v_\gamma \quad \text{and} \quad M_{l-\frac{1}{2}} v_\gamma = Z_{l-\frac{1}{2}} v_\gamma - Z_{l-1} v_\gamma = \left( \sum_{b' \in \mu} \text{ct}(b') - \sum_{b'' \in \nu} \text{ct}(b'') \right) v_\gamma = -\text{ct}(\nu/\mu) v_\gamma.$$

Now the result follows by noticing that (a) if $\gamma^{(l-1)}/\gamma^{(l-\frac{1}{2})} = \emptyset$, then $|\mu| - |\nu| = -1$ and $-\text{ct}(\nu/\mu) = -\text{ct}(\gamma^{(l-1)}/\gamma^{(l-\frac{1}{2})})$; and (b) if $\gamma^{(l-1)} = \gamma^{(l-\frac{1}{2})}$, then $|\mu| - |\nu| = 0$ and $\text{ct}(\nu/\mu) = 0$. □
Now, the following corollary implies the spectral significance of Jucys–Murphy elements in the representation theory of totally propagating partition algebras.

**Corollary 4.7.** Let \( l \) be a positive integer. The actions of the elements \( \tilde{M}_y \), for \( y \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) and \( y \leq l \), are sufficient to distinguish the non-isomorphic irreducible representations of \( \mathbb{C}I_l \). While in order to distinguish the non-isomorphic irreducible representations of \( \mathbb{C}I_{l-\frac{1}{2}} \), the actions of both \( M_y \) and \( \tilde{M}_y \) for \( y \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) and \( y \leq l - \frac{1}{2} \) are needed to be considered.

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