Restoring Reality for the Self-Dual N=2 String

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Abstract

It is known that the critical N=(2,2) string describes 2+2 dimensional self-dual gravity in a non-covariant form, since it requires the choice of a complex structure in the target, which leaves only U(1,1) Lorentz symmetry. We briefly review picture-changing and spectral flow and show that the world-sheet Maxwell instantons individually break the Lorentz group further to SU(1,1). However, their contributions conspire to restore full SO(2,2) global symmetry if dilaton and axion fields are assembled in a null anti-self-dual two-form, denying them the status of Lorentz scalars. We present the fully SO(2,2) invariant tree-level three-point amplitude and the corresponding extension of the Plebanski action for self-dual gravity.

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1 Introduction and Results

Closed strings with (2, 2) world-sheet supersymmetry are perhaps the only exactly solvable four-dimensional closed string theories, providing a consistent quantum theory of self-dual 4d gravity \[1\]. This hope derives from the observation that the spacetime background of signature 2+2 leaves no room for perturbative transverse string excitations and only a massless spectrum remains. Consistency of the absence of massive poles with duality then requires the perturbative vanishing of all string scattering amplitudes beyond the three-point function. In particular, the crucial vanishing of the four-point function is tied to the peculiar kinematics in 2+2 dimensions. This is as simple as a string theory can get. Still, one has to deal with the perturbative string expansion which, due to the presence of the graviphoton of the \(N=2\) world-sheet supergravity, sums over world-sheet genera as well as Maxwell instanton numbers.

Five years ago, Ooguri and Vafa \[2\] showed that the single massless physical field degree of freedom \(\phi\) present in the closed \(N=2\) string parametrizes the Kähler potential of self-dual gravity in 2+2 dimensions. After computing tree-level amplitudes with up to four legs they indeed identified the Plebanski action \[3\] of self-dual gravity as the (tree-level) effective spacetime action of the closed \(N=2\) string. Since self-dual structures in four dimensions are believed to unify all integrable systems in two and three dimensions, \(N=2\) strings should also teach us about the quantization of integrable models.

Two years ago, Berkovits, Ooguri and Vafa \[4, 5\] reformulated type II \(N=2\) strings in terms of \(N=4\) topological strings. They showed that a rotation of the spacetime complex structure (i.e. the choice of \(R^{2,2} \rightarrow C^{1,1}\)) mixes the amplitudes from the various Maxwell instanton numbers occurring at a given genus. Since such a (so-called flavor) rotation is nothing but a transformation under the \(SU(1, 1)_{f}\) factor of the would-be Lorentz group \(SO(2, 2) = [SU(1, 1)_{c} \otimes SU(1, 1)_{f}]/Z_{2}\), fixed-instanton-number amplitudes are in general only invariant under (so-called color) \(SU(1, 1)_{c}\). Only for zero instanton number do the amplitudes share the full global \([SU(1, 1)_{c} \otimes U(1)_{f}]/Z_{2} \otimes Z_{2}'\) symmetry of the Brink-Schwarz action \[4\]. This fact has been supported by explicit computations \[4, 8, 9\] and has led to some controversy. In particular, it raised the question whether it is possible to restore the global \(SO(2, 2)\) invariance, which had to be broken by choosing a complex structure in order to write down the Brink-Schwarz action in the first place.

In this letter we point out a way to restore \(SO(2, 2)\) invariance. We make use of the freedom in the definition of any fermionic string theory which arises when putting
together left- and right-moving monodromies for world-sheet fermions. It will also be necessary to invoke the dynamical nature of the string couplings $\kappa$ and $\lambda$ which weigh the different world-sheet topologies.

The genus $g$ of the world-sheet $\Sigma$ and the instanton number $c$ of the principal $U(1)$ bundle over $\Sigma$ are given by

$$\frac{1}{2\pi} \int_{\Sigma} R = 2 - 2g , \quad \frac{1}{2\pi} \int_{\Sigma} F = c ,$$

(1)

where $R$ and $F$ are the curvature two-forms of the spin and Maxwell connections of $N=2$ world-sheet supergravity, respectively. For each topology, labeled by $(g,c)$, there are metric, fermionic, and Maxwell moduli spaces to be integrated over \cite{8}. The integral over the Maxwell moduli of flat $U(1)$ connections combines with the spin structure sum to a continuous sum over world-sheet fermionic monodromies. It turns out that nothing depends on the NS or R (or interpolating) assignments for external states, a manifestation of the \textit{spectral flow} endomorphism of the $N=2$ superconformal constraint algebra.

Our definition of the $N=2$ string declares left and right monodromies as independent and, consequently, sums over left and right spin structures separately. The same prescription turns the $N=1$ closed fermionic string into the type II superstring by way of the GSO projection. For the $N=2$ string, it amounts to having independent right and left spectral flow symmetries and is compatible with modular invariance. External state vertex operators can be twisted by a pair of spectral-flow operators

$$V(k) \longrightarrow V^{(\theta_L, \theta_R)}(k) := SFO_L(\theta_L) SFO_R(\theta_R) V(k) ,$$

(2)

and it is easy to see that correlators of vertex operators are unchanged as long as their total twists (left and right) vanish \cite{8}. Nonzero integral total twists $(\theta_L, \theta_R) = (c_L, c_R)$, however, shift the instanton number, i.e. they amount to a topology change \cite{9}! Even though geometrically not obvious, we are led to sum over a \textit{pair} of instanton numbers in the expression for the full $n$-point scattering amplitude,

$$A(k_1, \ldots, k_n) = \sum_{g \in \mathbb{Z}_+} \kappa^{2g-2+n} \sum_{c_L \in \mathbb{Z}} \sum_{c_R \in \mathbb{Z}} \ell_{c_L,c_R} \lambda^{-c_L-c_R} A^g_{c_L,c_R}(k_1, \ldots, k_n) ,$$

(3)

with a priori unknown integral multiplicities $\ell_{c_L,c_R}$, to be fixed later. \footnote{We might absorb those into $A^g_{c_L,c_R}$ but the latter are naturally related by spectral flow.} Because $(A^g_{c_L,c_R})^* = A^{g-c_L-c_R}$, reality demands that $\kappa$ is real and $\lambda$ is a phase. It should be noted that fermion zero modes make $A^g_{c_L,c_R}$ vanish for any $|c| > 2g-2+n$, restricting the instanton sum in eq. (3) to $|c_L| \leq 2g-2+n \geq |c_R|$.
Since flavor rotations of $A$ mingle the $A_{\mu L}^{a R}$ for $g$ fixed, $SO(2,2)$ invariance can be achieved only if the string couplings $\kappa$ and $\lambda$ are allowed to vary in compensation! This seems strange at first since, in a background more general than $C^{1,1}$, we should identify

$$\kappa = e^{i\langle d \rangle} \quad , \quad \lambda = e^{i\langle a \rangle} \quad ,$$

where $d$ and $a$ stand for the spacetime dilaton and axion fields, which couple to $R$ and $F$, respectively. One is used to view these fields as real spacetime Lorentz scalars. Here, we propose unusual Lorentz properties for the dilaton and axion fields! Using the isomorphism $SU(1,1)/Z_2 = SO^+(2,1)$ for the flavor subgroup, we find that the triple $w$ given by

$$\begin{pmatrix}
    w^1 \\
    w^2 \\
    w^3
\end{pmatrix} := \sqrt{\kappa} \begin{pmatrix}
    (\lambda+\lambda^{-1})/2 \\
    (\lambda-\lambda^{-1})/2i \\
    1
\end{pmatrix} = \exp\{\langle d \rangle/2\} \begin{pmatrix}
    \cos \langle a \rangle \\
    \sin \langle a \rangle \\
    1
\end{pmatrix} \quad (5)$$

must transform as a (massless) vector in 2+1 Minkowski space. Other options encode the dilaton and axion degrees of freedom in a Majorana-Weyl spinor $v$ of $SO(2,2)$ or a null anti-self-dual two-form $\Omega^-$ in the full 2+2 dimensional background. We suggest that the lack of a covariant action for self-dual gravity may be overcome when such a two-form is employed with the metric. Alternatively, the presence of an $SO(2,2)$ spinor hints at the possibility of spacetime supersymmetry (see, however, refs. [10, 11]).

In the following, we shall show how the Lorentz transformations of $\kappa$ and $\lambda$ arise naturally already on the level of the vertex operators, i.e. from BRST cohomology. After a discussion of physical states in the different pictures of the $N=2$ string and their relation by picture-changing and spectral flow, we detail the behavior of the theory under $SU(1,1)_I$ rotations, which change the complex structure and complete the would-be Lorentz group. We demonstrate how the instanton sum can restore $SO(2,2)$ invariance of the scattering amplitudes. Finally, the tree-level three-point function is worked out completely, and the corresponding spacetime action is constructed, providing a stringy extension of self-dual gravity.

2 Physical States, Pictures and Spectral Flow

The physical states or vertex operators for the type II $N=2$ string are obtained as elements of the relative BRST cohomology, meaning that one imposes as subsidiary conditions the vanishing of the reparametrization and Maxwell anti-ghost zero modes
on physical states. This cohomology factorizes into the relative BRST cohomologies for the two chiral halves of the string. Let us consider the left-moving cohomology, dropping the $L$ subscript. It is graded by

- conformal dimension $h$ as eigenvalue of $L_0^{\text{tot}}$; physical states must have $h = 0$.
- local Maxwell charge $e$ as eigenvalue of $J_0^{\text{tot}}$; physical states must have $e = 0$.
- picture numbers $(\pi^+,\pi^-)$, with $\pi^+ + \pi^- \in \mathbb{Z}$ and $\pi^+ - \pi^- \in \mathbb{R}$, labeling inequivalent superconformal ghost vacua.
- total ghost number $u \in \mathbb{Z}$ as eigenvalue of the total ghost charge $U$; physical states are presumed to occur only for $\tilde{u} \equiv u - \pi^+ - \pi^- = 1$.
- color quantum numbers $(j,m)$ labeling $SU(1,1)_c$ behavior; physical states are believed to be singlets.
- global $U(1)_f$ charge $q \in \mathbb{R}$; it will play a key role.
- complex center-of-mass momentum $k^\mu$ as eigenvalue of $P^\mu = \frac{1}{2\pi} \oint \partial X^\mu$; physical states are massless, i.e. $k^\mu k^\mu = 0$.

The relative BRST cohomology has been worked out for the “canonical” picture of $(\pi^+,\pi^-) = (-1, -1)$, finding indeed a single massless physical state $|\phi\rangle = V_{\text{can}}(k) |0\rangle$ with $u = -1$, $(j,m) = (0,0)$, and $q = 0$ [12, 13].

It is very helpful that there exist BRST-invariant operators which carry non-zero picture numbers and thus may relate the cohomologies in the various pictures. One has picture-changing [14, 13] and spectral-flow [4, 8] operators, with charges

| operator     | $\pi^+$ | $\pi^-$ | $u$ | $v$ | $(j,m)$ | $q$ |
|--------------|---------|---------|-----|-----|---------|-----|
| $PCO^+$      | 1       | 0       | 1   | 0   | (0,0)   | 0   |
| $PCO^-$      | 0       | 1       | 1   | 0   | (0,0)   | 0   |
| $SFO(\theta)$ | $+\theta$ | $-\theta$ | 0   | 0   | (0,0)   | $+\theta$ |

and $h = e = 0 = k^\mu k^\mu$ for everybody. It is important to note that $PCO^+$, $PCO^-$, and $SFO(\theta)$ commute with one another, up to BRST-exact terms.

Since $SFO(\theta, z) = \exp \left[ 2\theta \int_{0}^{z} J^{\text{tot}} \right]$, spectral flow is additive and therefore invertible via $\theta \rightarrow -\theta$. Hence, it provides a one-to-one map between cohomology classes for the same value of $\pi \equiv \pi^+ + \pi^-$. In particular, in all $\pi = -2$ sectors we find a single physical state, with $q = \frac{1}{2}(\pi^+ - \pi^-)$. In fact, there is a subtlety in defining a local
spectral-flow operator. Namely, \( SFO(\theta, z) \) must depend on an arbitrary reference point \( z_0 \), where all charges are reversed compared to those at \( z \). Fortunately, \( z_0 \) dependence drops out if the spectral flows of all vertex operators in the correlator sum to zero. Given that \( \partial SFO(\theta) \) is BRST-exact, the correlator is even invariant under such a flow, allowing one to freely change the relative picture numbers of its vertex operators \[8\]. Somewhat surprisingly, the spectral flow angle \( \theta \) is not a compact variable. Instead, \( ICO \equiv SFO(\theta=1) \) creates a Maxwell instanton at the location of the vertex operator. Thus, an overall spectral flow of \( \sum_{i=1}^{n} \theta_i = \Delta c \in \mathbb{Z} \) does change the amplitude and connects different instanton sectors. The above table then implies that the amplitude \( A_{cL,cR} \) carries a flavor charge of \( q = c_L + c_R \). Moreover, the worrisome \( z_0 \) dependence of an instanton-changing flow has a beautiful interpretation \[9\]: It is nothing but the inherent ambiguity in comparing different topologies in the first-quantized approach and must be absorbed in the string coupling \( \lambda \). Consequently, \( \lambda \) must also carry the compensating unit of \( q \) charge! The picture numbers, in contrast, are already balanced due to the selection rule

\[
\sum_{i=1}^{n} (\pi^+, \pi^-) = (2g-2-c, 2g-2+c)
\]

and any such picture-number assignment to the vertex operators leads to the same amplitude.

Finally, we employ the picture-changing operators \( PCO^\pm \) to map the known BRST cohomology for \( \pi = -2 \) to that in higher pictures. Regrettably, this map is not one-to-one, because picture-changing cannot be inverted for the \( N=2 \) string \[13, 7\]. Indeed, \( PCO^+ \) and \( PCO^- ICO \) map to the same picture but differ by one unit in the flavor charge \( q \)! Hence, starting from \( \pi = -2 \) and \( q=0 \) and successively applying \( r \) picture-changing operators, we have \( r+1 \) possibilities,

\[
(POC^+)^r, \ (POC^+)^{r-1}POC^-, \ (POC^+)^{r-2}(POC^-)^2, \ldots, \ (POC^-)^r
\]

leading to a spectrum of \( q=0 \) states with \( |\pi^+-\pi^-| \leq r \) at \( \pi = r-2 \). Using spectral flow we may move all these states to the same picture \( (\pi^+, \pi^-) \) and arrive at \( r+1 \) distinctive states

\[
\left| q = -\pi^- - 1 \right>, \quad \left| q = -\pi^- \right>, \quad \ldots, \quad \left| q = \pi^+ \right>, \quad \left| q = \pi^+ + 1 \right>
\]

with a range of \( U(1)_f \) charges, as sketched in fig. 1. We will argue that all those states are proportional to one another, with \( q \)-charged functions of momentum \( k^\mu \) as proportionality factors. For generic momenta then, the states of eq. (9) either are all BRST-exact or represent different cohomology classes. Since in the first case the
three-point function vanishes for \( g \geq (r-2)/4 \), we assume the second variant to hold. This scheme suggests that the number of physical states depend on the picture and no physical states exist in subcanonical pictures \( \pi < -2 \).

It is instructive to look at the simplest example, \( r = 1 \). We use the following notation. Complex momenta \( k^\mu \), complex string coordinates \( X^\mu \), and complex NSR fermions \( \psi^\mu \), with Lorentz index \( \mu = 0, 1 \), are expressed in real bispinor notation with respect to the \( SU(1,1)_f \) and \( SU(1,1)_c \) factors of the would-be Lorentz group. For instance,

\[
k^{q,m} \in \{k^{++}, k^{+-}, k^{-+}, k^{--}\} \quad \text{with} \quad q = \pm \frac{1}{2} \quad \text{and} \quad m = \pm \frac{1}{2}
\]

transforms as a \((2, 2)\) of \((SU(1,1)_f, SU(1,1)_c)\). The masslessness condition reads

\[
k^{++}k^{--} + k^{+-}k^{-+} = 0.
\]

Sometimes we hide the color index \( m \) and simply write \( k^q \). Denoting the superconformal ghost vacuum at momentum \( k \) by \( |\pi^+, \pi^-; k\rangle \), one finds

\[
\begin{align*}
\text{PCO}^- & \quad k^+\psi^- | -1, 0; k \rangle \\
| -1, -1; k \rangle \\
\text{PCO}^+ & \quad k^-\psi^+ | 0, -1; k \rangle \\
\end{align*}
\]
where we introduced the color contractions
\[ a \cdot b := a^+ b^- + a^- b^+ , \quad a \wedge b := a^+ b^- - a^- b^+ \] (13)
among spinors. Although the two physical states in the lower row of eq. (12) differ by one unit in flavor charge \( q \), they are proportional because
\[ k^+ \wedge \psi^+ = k^{++} \psi^{+-} - k^{+-} \psi^{++} \]
\[ = k^{++} \left( k^{-+} \psi^{+-} + k^{--} \psi^{++} \right) \]
\[ = k^{++} k^- \cdot \psi^+ , \] (14)
with the help of eq. (11). This proportionality has been exploited in refs. [4, 15] to derive vanishing theorems for scattering amplitudes.

BRST analysis in the \((0, -1)\) picture also reproduces the lower row of eq. (12). The requirement of BRST invariance for physical states simply restricts the ansatz \( \epsilon \cdot \psi^+ |0, -1; k\) by \( k^+ \cdot k^- = 0 \) and \( \epsilon \cdot k^+ = 0 \). The general solution to the second condition is \( \epsilon = v^+ k^- - v^- k^+ \wedge \) with free coefficients \( v^\pm \), or
\[ |0, -1; \text{phys} \rangle = (v^+ k^- \cdot \psi^+ - v^- k^+ \wedge \psi^+) |0, -1; k\rangle \]
\[ = (v^+ - \frac{k^{++}}{k^-} v^-) k^- \cdot \psi^+ |0, -1; k\rangle . \] (15)

More generally, in any picture one finds a \((\pi + 3)\)-plet of physical states which are all proportional to one another but carry different flavor charges.

3 Rotating the Complex Structure

We have made explicit use of a complex structure in spacetime, e.g. when formulating the BRST cohomology problem. As already mentioned, such a choice corresponds to a breaking of \( SU(1, 1)_f \) to \( U(1)_f \). Thus, the two-dimensional moduli space of complex structures is given by the pseudo-sphere \( SU(1, 1)_f / U(1)_f \). In order to reconstruct full \( SU(1, 1)_f \) symmetry, we must understand how objects with different \( U(1)_f \) charge \( q \) combine to multiplets under flavor rotations.

Let us parametrize \( SU(1, 1)_f \) by complex \( 2 \times 2 \) matrices
\[ U = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \] with \( \alpha^* \alpha - \beta^* \beta = 1 \).
\[ (16) \]

\[ ^2 \text{The entries } \alpha \text{ and } \beta \text{ correspond to the harmonic variables } u_1 \text{ and } u_2 \text{ of ref. [4, 5, 15]. As they parametrize Lorentz transformations, we identify left- and right-moving flavor rotations.} \]
The compact $U(1)_f$ subgroup is given by $\beta = 0$. Decomposing $k^{q,m}$ of eq. (10) into two flavor spinors $k^{(m)}$,

\[ k^{(+)} := \begin{pmatrix} k^+ & k^- \\ k^- & -k^- \end{pmatrix} \quad k^{(-)} := \begin{pmatrix} k^+ & -k^- \\ k^- & k^- \end{pmatrix}, \tag{17} \]

one sees that $k^{(m)} \rightarrow U k^{(m)}$ under a flavor rotation. The matter fields $X$ and $\psi$ behave in the same way. Picture-changing and spectral-flow operators may also be combined to the flavor doublet

\[ PCO := \begin{pmatrix} +P CO^- ICO+1/2 \\ -P CO^+ ICO-1/2 \end{pmatrix} \tag{18} \]

carrying $(\pi^+, \pi^-) = (\frac{1}{2}, \frac{1}{2})$. In the process of computing an amplitude, all world-sheet fields inside the correlators have to be Wick-contracted, leaving a function of the external momenta only. Since those Wick contractions are fully $SO(2,2)$ invariant and momentum factors always arise from $\partial^s X e^{ikX}$ contractions, the behavior of the NSR field $\psi$ is irrelevant. Therefore, the flavor properties of an amplitude may be invoked by transforming only the string coordinates $X$ and external momenta $k_i$ but leaving $\psi$ inert. In the following we shall consider such restricted flavor rotations. In particular,

\[ \begin{pmatrix} k^+ \land \psi^+ \\ k^- \cdot \psi^+ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k^+ \cdot \psi^- \\ k^- \land \psi^- \end{pmatrix} \tag{19} \]

form restricted flavor doublets, which appear in eq. (12). In higher pictures, the $\pi+3$ physical states found above simply manufacture a spin $(\pi+2)/2$ flavor representation.

We shall now reinstall $SO(2,2)$ Lorentz symmetry separately for left and right movers. Ignoring possible left-right coupling due to $\partial^s X \bar{\partial}^f X$ contractions, the amplitudes factorize as $A^g_{c_L c_R} = \tilde{A}^g_{c_L} \tilde{\tilde{A}}^g_{c_R}$. The picture-number selection rule (7) and the flavor-charge selection rule $q_{tot}=c$ imply that the chiral $n$-point correlator $\tilde{A}^g_c$ is built from $n$ canonical flavor-singlet vertex operators with $\pi=-2$ together with $r=2s\equiv 4g-4+2n$ picture-changing operators and $c$ instanton-creation operators. Since $c$ ranges from $-s$ to $+s$, one has $2s+1$ chiral correlators for a given $g$. Eq. (18) then strongly suggests that they all combine in a spin $s$ flavor representation, which obtains from the symmetrized tensor product of $2s$ picture-changing doublets $[4]$.

The only way to produce flavor singlets is to tensor the $(2s+1)$-plet $(\tilde{A}^g_{-s}, \ldots, \tilde{A}^g_{+s})$ with its dual representation, which must sit in the coefficient factors weighing the different instanton contributions. In other words, the string coupling constants $\kappa$ and $\lambda$ should flavor-transform non-trivially. This goal is achieved most easily if we
form flavor singlets already at the level of vertex operators and/or picture-changing operators. Since the antisymmetric product of two flavor doublets yields the singlet,

\[ [a, b] := a^+ b^- - a^- b^+ , \]

(20)
it suffices to declare that the coefficients \((v^+, v^-)\) in eq. (15) form a flavor doublet \(v\), i.e. \(v \rightarrow Uv\) under \(SU(1,1)_f\). Eq. (15) can then be rearranged to

\[ |0, -1; \text{phys} \rangle = \left( [v, k^{(+)}] \psi^+- [v, k^{(-)}] \psi^{++} \right) |0, -1; k \rangle , \]

(21)
making restricted \(SU(1,1)_f\) invariance manifest. More generally, the chiral \(n\)-point correlator

\[ \tilde{A}^g := \langle V_{\text{can}}(k_1) \ldots V_{\text{can}}(k_n) [v, PCO]^{4g-4+2n} \rangle \]

(22)
(suppressing anti-ghost zero mode insertions and modular integrations) produces the complete bunch of \(\tilde{A}^g\) and is manifestly \(SO(2,2)\) invariant!

The idea is that the monomials of \(v^\pm\) in eq. (22) (and its right-moving counterpart) are simply provided by the powers of \(\kappa\) and \(\lambda\) in eq. (3), since those always match. We choose the multiplicities to factorize as \(\ell_{cL,cR} = \ell_{cL} \ell_{cR}\) and identify

\[ \tilde{A}^g = \sqrt{\kappa}^{2g-2+n} \sum_c \ell_c \lambda^{-c} A^g_c . \]

(23)
This prescription determines the multiplicities as \([4, 5]\)

\[ \ell_c = \frac{(4g-4+2n)!}{(2g-2+n-c)! (2g-2+n+c)!} . \]

(24)
More importantly, we read off

\[ v^\pm = \kappa^{1/4} \lambda^{\pm1/2} = e^{\pm(d)\pm i(a)} , \]

(25)
\[ \sqrt{\kappa} = v^+ v^- , \quad \lambda = v^+/v^- , \]

(26)
which demonstrates that dilaton and axion fields essentially encode the length and phase of an \(SO(2,2)\) Majorana-Weyl spinor. For later convenience we form the symmetric square \(w \equiv v \times v\), the \(SU(1,1)_f\) vector

\[ \begin{pmatrix} w^+ \\ w^0 \\ w^- \end{pmatrix} := \begin{pmatrix} +v^+ v^+ / \sqrt{2} \\ -v^- v^- / \sqrt{2} \\ +v^+ v^- / \sqrt{2} \end{pmatrix} = \sqrt{\kappa} \begin{pmatrix} +\lambda^{+1} / \sqrt{2} \\ -1 \\ -\lambda^{-1} / \sqrt{2} \end{pmatrix} , \]

(27)
which is automatically massless, i.e. \(w^0 w^0 + 2w^+ w^- = 0\). Under \(SU(1,1)_c \otimes SU(1,1)_f\) it transforms as a \(1, 3\), which may be interpreted as a anti-self-dual two-form in \(2,2\) spacetime. Converting to a real basis via \(w^\pm = (\pm w^1 + i w^2) / \sqrt{2}\) and \(w^0 = -w^3\), one arrives at eq. (5). Apparently, we may \(U(1)_f\) rotate away the vev of the axion, i.e. set \(\lambda = 1\) by a special coordinate choice. Then, one boost freedom remains to scale the dilaton vev to an arbitrary number. Yet, different “observers” in general will disagree on the size of both \(\langle d \rangle\) and \(\langle a \rangle\).
4 Invariant Amplitudes and Spacetime Actions

We can now study the change of scattering amplitudes under flavor transformations and assemble them into $SO(2, 2)$ invariants. The essential ingredients are the $SU(1, 1)_c$ invariant momentum bilinears

$$s_{ij} := k^+_i \cdot k^+_j + k^-_i \cdot k^-_j = k^{++}_i k^{--}_j + k^{+-}_i k^{-+}_j + k^{+-}_i k^{-+}_j$$
$$c^0_{ij} := k^+_i \cdot k^-_j - k^-_i \cdot k^+_j = k^{++}_i k^{--}_j - k^{+-}_i k^{-+}_j$$
$$c^+_{ij} := \sqrt{2} k^+_i \cdot k^+_j = \sqrt{2}(k^{++}_i k^{--}_j - k^{+-}_i k^{-+}_j)$$
$$c^-_{ij} := \sqrt{2} k^-_i \cdot k^-_j = \sqrt{2}(k^{++}_i k^{--}_j - k^{+-}_i k^{-+}_j)$$

(28)

Under $SU(1, 1)_f$, the Mandelstam variable $s_{ij}$ is a singlet, while $c^\epsilon_{ij}$ form a triplet, $\epsilon = +, 0, -$, as may be checked explicitly from eqs. (16) and (17). Complex conjugation exchanges $c^+_{ij}$ and $c^-_{ij}$, while $c^0_{ij}$ is purely imaginary. Any scattering amplitude is expressed in terms of $s_{ij}$ and $c^\epsilon_{ij}$, and various identities can be derived on-shell, for $s_{ii} = 0 = \sum_i k_i$ [4, 16].

Since only the three-point function is non-vanishing, let us be explicit for its chiral half, $\tilde{A}^\epsilon_c(k_1, k_2, k_3)$. Massless kinematics dictate that all $s_{ij} = 0$ and $c^\epsilon_{i,i+1} = -c^\epsilon_{i+1,i} =: c^\epsilon$. The first non-trivial identity is quadratic,

$$c^0 c^0 + 2 c^+ c^- = 0 \ ,$$

(29)

and expresses the lightlike nature of the flavor vector $c$. At tree-level, $c_L$ and $c_R$ range from $-1$ to $+1$. Straightforward computation of the chiral correlators $\tilde{A}^\epsilon_c$ yields [4, 3]

$$\tilde{A}^0_c = -\frac{1}{2} c^0 \ , \quad \tilde{A}^0_{\pm 1} = \pm \frac{1}{\sqrt{2}} c^\pm \ .$$

(30)

Using $\ell_{\pm 1} = 1$ and $\ell_0 = 2$, this fits perfectly with

$$\tilde{A}^0 = \langle V_{\text{can}}(k_1) V_{\text{can}}(k_2) V_{\text{can}}(k_3) \ [v, PCO]^2 \rangle$$
$$= \sqrt{\kappa} \left( \lambda^{-1} \tilde{A}^0_{+1} + 2\tilde{A}^0_0 + \lambda^{+1} \tilde{A}^0_{-1} \right)$$
$$= \sqrt{\kappa} \left( -\frac{1}{\sqrt{2}} \lambda^{-2} c^+ - c^0 + \frac{1}{\sqrt{2}} \lambda^{+1} c^- \right)$$
$$= w^- c^+ + w^0 c^0 + w^+ c^-$$

(31)

which is imaginary and manifestly flavor invariant as a scalar product of the two lightlike vectors $c$ and $w$ (see eq. (27)).

The full tree-level three-point function for the closed string is simply the square of the above sum,

$$A^0 = (\tilde{A}^0)^2 = w^2 c^2_+ + 2 w^- w_0 c_+ c_0 + \frac{3}{2} w^2_0 c^0_+ + 2 w_0 w^+ c_0 c^- + w^2_+ c^2_-$$
$$= \kappa \left( \frac{1}{2} \lambda^{-2} c^2_+ + \sqrt{2} \lambda^{-1} c_+ c_0 + \frac{3}{2} c^2_0 - \sqrt{2} \lambda^{+1} c_0 c^- + \frac{1}{2} \lambda^2 c^2_- \right) \ ,$$

(32)
where we have lowered the flavor superscripts for convenience. With our conjectured Lorentz behavior (5) of the string couplings understood, it is $SO(2, 2)$ invariant.

It is well-known that the zero-instanton tree-level amplitudes,

$$A_{0,0}^0(k_1, \ldots, k_n) = \delta_{n,2} + \frac{1}{4} c_0^2 \delta_{n,3} \quad ,$$

(33)

are reproduced (at least up to $n=6$) by the Plebanski action $S_{0,0} = \int d^4x L_{0,0}$, with

$$L_{0,0} = \frac{1}{2} \partial_+ \phi \cdot \partial_- \phi + \frac{2}{3} \kappa \phi \partial_+ \partial_- \phi \wedge \partial_+ \partial_- \phi$$

$$= \frac{1}{2} (\partial_+^+ \phi \partial_-^- \phi + \partial_+^- \phi \partial_-^+ \phi) + \frac{4}{3} \kappa \phi (\partial_+^+ \phi \partial_-^- \phi - \partial_+^- \phi \partial_-^+ \phi) \quad .$$

(34)

We have defined $\partial^m_q := \partial / \partial x^m$ and used the notation of eqs. (10) and (13). In the double contractions $\partial \partial \phi s_1 s_2 \partial \partial \phi$, the first symbol ($s_1$) refers to the pairing $\partial \partial \phi s_1 s_2 \partial \partial \phi$ while the second symbol ($s_2$) specifies $\partial \partial \phi s_1 s_2 \partial \partial \phi$. Note that this action yields $-\frac{1}{2} c^+ c^-$ for the three-point function, which agrees with eq. (33) due to eq. (29).

Under flavor rotation, the kinetic term in eq. (34) is invariant. The cubic interaction, however, transforms into a linear combination of $L_{cL,cR}^{int}$, with $-1 \leq c_L, c_R \leq +1$, as noted in ref. [16]. In particular, one generates

$$L_{1,1}^{int} = -\frac{2}{3} \kappa \phi \partial_+ \partial_+ \phi \wedge \partial_+ \partial_+ \phi$$

$$= -\frac{4}{3} \kappa \phi (\partial_+^+ \partial_+^+ \phi \partial_-^- \phi - \partial_+^- \partial_-^- \phi \partial_+^+ \partial_-^+ \phi)$$

$$L_{0,1}^{int} = -\frac{2}{3} \kappa \phi \partial_+ \partial_+ \phi \wedge \cdot \partial_+ \partial_- \phi$$

$$= -\frac{4}{3} \kappa \phi (\partial_+^+ \partial_+^+ \phi \partial_-^- \phi + \partial_+^- \partial_-^- \phi \partial_+^+ \partial_-^+ \phi - \partial_+^+ \partial_+^+ \phi \partial_-^- \phi - \partial_+^- \partial_-^- \phi \partial_+^+ \partial_-^+ \phi) \quad .$$

(35)

The other interactions obtain from $L_{cL,cR}^{int} = L_{cR,cL}^{int} = L_{-c_{L,-cR}}^{int}$ and $L_{1,-1}^{int} = L_{0,0}^{int}$. It is by now obvious that an $SO(2, 2)$ invariant action must combine all these terms with the same coefficients as in eq. (32). Expressed in terms of dilaton and axion fields, we finally arrive at the extended Plebanski action

$$S_{inv}[\phi, d, a] = \int d^4x \left[ \frac{1}{2} \partial_+ \phi \cdot \partial_- \phi + \frac{2}{3} e^d \phi \left( 6 \partial_+ \partial_- \phi \wedge \partial_+ \partial_- \phi \right. \right.$$  

$$- 4 e^{-ia} \partial_+ \partial_+ \phi \wedge \partial_+ \partial_- \phi \right.$$  

$$- e^{-2ia} \partial_+ \partial_+ \phi \wedge \partial_+ \partial_- \phi$$  

$$\left. + 4 e^{ia} \partial_- \partial_+ \phi \wedge \partial_- \partial_- \phi \right.$$  

$$\left. - e^{+2ia} \partial_- \partial_- \phi \wedge \partial_- \partial_- \phi \right) \right] .$$

(36)

Let us finally recast this into a more standard notation. Introducing dotted spinor indices via

$$\{k^{++}, k^{+-}, k^{-+}, k^{--}\} \equiv \{k^{\ast \ast}, k^{\ast -}, k^{- \ast}, k^{--}\} \supset k^{\alpha \dot{\alpha}} \quad (37)$$
(see also eq. (17)), we may use the fact that the only numerically invariant \( SU(1,1) \) tensors at our disposal are \( \delta^\alpha _\beta \) and \( \epsilon _{\alpha \beta} \) (and their dotted versions). The bilinears of eq. (28) read

\[
\begin{align*}
\epsilon _{\alpha \beta} & = -\epsilon _{\alpha \beta} \epsilon _{\dot{\alpha} \dot{\beta}} \kappa _{\dot{i}} \kappa _{\dot{j}}, \\
\omega _{\alpha \beta} & = -\omega _{\alpha \beta} \epsilon _{\dot{\alpha} \dot{\beta}} \epsilon _{\alpha \beta} \kappa _{\dot{i}} \kappa _{\dot{j}},
\end{align*}
\]

\( s_{ij} = -\epsilon _{\alpha \beta} \epsilon _{\dot{\alpha} \dot{\beta}} \kappa _{\dot{i}} \kappa _{\dot{j}}, \quad c^0 _{ij} = -\left( \delta^\alpha _{\alpha} \delta^\beta _{\beta} + \delta^\alpha _{\beta} \delta^\beta _{\alpha} \right) \epsilon _{\alpha \beta} \kappa _{\dot{i}} \kappa _{\dot{j}}, \quad c^+ _{ij} = \sqrt{2} \delta^\alpha _{\alpha} \delta^\beta _{\beta} \epsilon _{\alpha \beta} \kappa _{\dot{i}} \kappa _{\dot{j}}, \quad c^- _{ij} = -\sqrt{2} \delta^\alpha _{\alpha} \delta^\beta _{\beta} \epsilon _{\alpha \beta} \kappa _{\dot{i}} \kappa _{\dot{j}}
\]

(38)

so that eq. (31) becomes

\[ \tilde{A}^0 = \omega _{\alpha \beta} \epsilon _{\alpha \beta} \epsilon _{\dot{\alpha} \dot{\beta}} \kappa _{\dot{i}} \kappa _{\dot{j}} = \Omega^- _{mn} \epsilon _{\alpha \beta} \epsilon _{\dot{\alpha} \dot{\beta}} \kappa _{\dot{i}} \kappa _{\dot{j}} \]

(39)

with \( m = (\alpha \dot{\alpha}) \) and \( n = (\beta \dot{\beta}) \) denoting 2+2 dimensional Lorentz indices. Here, we defined the symmetric degenerate \( SU(1,1) \) tensor

\[ \omega (d,a) := \sqrt{\kappa} \begin{pmatrix} -\lambda^{-1} & 1 \\ 1 & -\lambda \end{pmatrix} = e^{d/2} \begin{pmatrix} -e^{-ia} & 1 \\ 1 & -e^{ia} \end{pmatrix} \]

(40)

and its antisymmetric anti-self-dual \( SO(2,2) \) cousin \( \Omega^- _{mn} = \Omega^- _{\alpha \dot{\alpha} \beta \dot{\beta}} := \omega _{\alpha \beta} \epsilon _{\alpha \beta} \epsilon _{\dot{\alpha} \dot{\beta}} \). The extended Plebanski action of eq. (36) then takes the form

\[ S_{inv}[\phi, d, a] = \frac{1}{2} \eta ^{mn} \partial _m \phi \partial _n \phi - \frac{2}{3} \Omega^- mn \Omega^- pq \partial _m \phi \partial _p \partial _q \phi \]

(41)

with the metric \( \eta ^{\alpha \beta} = -\epsilon ^{\dot{\alpha} \dot{\beta}} \). Of course, it would be very interesting to construct a manifestly \( SO(2,2) \) invariant action \( S[g_{mn}, d, a] \) for self-dual gravity with the full 4d metric \( g_{mn} \) from these data.

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