PUSH-FORWARDS ON PROJECTIVE TOWERS

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Abstract. In this paper we derive a simple and useful combinatorial formula for the push-forwards of cohomology classes down projective towers, in terms of the push-forwards down the individual steps in the tower.

1. Introduction

1.1. Consider a proper map of algebraic varieties:

\[ \pi : X' \to X. \]

Pick any class \( c \in A^*(X') \) and call it the tautological class of \( \pi \). Relative to this choice, we can define the Segre series of \( \pi \):

\[ s(\pi, u) = \pi^* \left( \frac{1}{u - c} \right). \quad (1.1) \]

This series in \( u^{-1} \) has coefficients in \( A^*(X) \), and it encodes the push-forwards of all powers of the tautological class \( c \).

The terminology is motivated by the case when \( X' = \mathbb{P}_X V \) is the projectivization of a cone on \( X \). In this case, we let \( c = c_1(O(1)) \) and the above notion coincides (up to normalization) with the Segre class introduced by Fulton in [1]. In the particular case when \( V \) is a vector bundle, the Segre series equals the inverse of the (properly renormalized) Chern polynomial of \( V \).

1.2. The main subject of this paper are projective towers, namely compositions of proper maps of algebraic varieties:

\[ \pi : X_k \xrightarrow{\pi^k} X_{k-1} \xrightarrow{\pi^{k-1}} \ldots \xrightarrow{\pi^2} X_1 \xrightarrow{\pi^1} X_0. \quad (1.2) \]

As before, pick \( c_i \in A^*(X_i) \) and call them the tautological classes of the tower. We want to encode the push-forwards of these tautological classes under \( \pi \), and the reasonable way to do this is to define the Segre series of the tower as:

\[ s(\pi, u_1, \ldots, u_k) = \pi^* \left( \frac{1}{u_1 - c_1} \cdot \ldots \cdot \frac{1}{u_k - c_k} \right). \quad (1.3) \]

We suppress the obvious pull-back maps to \( X_k \), and hope that this will cause no confusion.
1.3. One of the main technical results of [2] involves studying a particular projective tower (1.2). One needs to derive a closed formula for the Segre series of the whole tower $\pi$ from the Segre series of the individual maps $\pi^i$. The assumption we make on these individual Segre series is that:

$$s(\pi^i, u) = \prod_{m_i} Q_{m_i}(u + m_i^{i-1}c_{i-1} + ... + m_i^1c_1),$$

(1.4)

for some series $Q_{m_i}$ with coefficients in $A^*(X_0)$, where the product goes over finitely many vectors $m_i = (m_i^1, ..., m_i^{i-1})$ of integers. This assumption will be motivated in section 2.1, based on the particular example of a tower of projective bundles. Then our main Theorem 2.3 below implies that:

$$s(\pi, u_1, ..., u_k) = \left[ \prod_{i=1}^k \prod_{m_i} Q_{m_i}(u_i + m_i^{i-1}u_{i-1} + ... + m_i^1u_1) \right]^{-}$$

(1.5)

The notation $[...]^{-}$ means that we expand each $Q_{m_i}$ in non-negative powers of $u_{i-1}, ..., u_1$, and then we only keep the monomials with all negative exponents in the resulting formula.

1.4. The basic idea, naturally, is to successively push forward the tautological classes from $X_k$ to $X_{k-1}$ to $...$ to $X_1$ to $X_0$, and assumption (1.4) provides the means for this recursion. However, if one carried out this procedure straightforwardly, one would not obtain a closed formula. The reason why formula (1.5) looks so nice is that we are adding terms with non-negative exponents, only to get rid of them when we apply $[...]^{-}$ at the very end.

This closed formula is very useful in the papers [2] and [3]. In the present note, we will present a baby case of the main technical computation of these papers: we will rederive a closed formula for integrals on the complete flag variety of vector subspaces of a fixed vector space.

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2. Tautological Classes

2.1. Consider the special case of (1.2) where $X_i = \mathbb{P}_{X_{i-1}} \mathcal{V}_i$ for some vector bundle $\mathcal{V}_i$ of rank $r_i$ on $X_{i-1}$, and $c_i = c_1(\mathcal{O}_i(1))$ is the first Chern class of the tautological line bundle. It is well-known ([1], Section 3.2) that the individual Segre classes are equal to the inverse Chern classes:

$$s(\pi^i, u) = c^{-1}(\mathcal{V}_i, u) \quad \text{where} \quad c(\mathcal{V}_i, u) = u^{r_i} \cdot \sum_{k=0}^{r_i} u^{-k}c_k(\mathcal{V}_i).$$

(2.1)

The above Chern classes only depend on the class of $\mathcal{V}_i$ in the $K-$theory of $X_{i-1}$. We will make the following assumption on this class:
\[ [V_i] = \sum_{m_i} [V_{m_i}] \otimes [\mathcal{O}_1(m_i^1)] \otimes \ldots \otimes [\mathcal{O}_{i-1}(m_i^{i-1})], \]

in \( K \)-theory, where the sum goes over finitely many vectors \( m_i = (m_i^1, \ldots, m_i^{i-1}) \) of integers and \([V_{m_i}] \in K(X_0)\) are arbitrary classes (we are suppressing the obvious pull-back maps, and hope that this will cause no confusion). In other words, we assume that in \( K \)-theory each \( V_i \) is constructed by twisting bundles on the lower steps in the tower by various tautological line bundles. Then the Whitney sum formula tells us that:

\[ s(V_i, u) = \prod_{m_i} s(V_{m_i}, u + m_i^{i-1}c_{i-1} + \ldots + m_1^1c_1), \]

where the Segre series \( s(V_{m_i}, u) \) now have coefficients in \( A^*(X_0) \). This setup justifies our assumption (1.4).

### 2.2

Let us now go to a general projective tower (1.2) that satisfies assumption (1.4). Along with the variable \( u_i \), for each \( i \in \{1, \ldots, k\} \) pick an extra set of variables \( A_i \). Then our main result is the following theorem:

**Theorem 2.3.** We have the following relation:

\[ \pi_* \prod_{i=1}^k \left( \frac{1}{u_i - c_i} \prod_{u \in A_i} \frac{1}{u - c_i} \right) = \]

\[ = \prod_{i=1}^k \prod_{m_i} Q_{m_i}(u_i + m_i^{i-1}u_{i-1} + \ldots + m_1^1u_1) \prod_{u \in A_i} \frac{1}{u - u_i} \]

(2.2)

where we expand each \( Q_{m_i} \) in non-negative powers of \( u_{i-1}, \ldots, u_1 \), and each \( (u - u_i)^{-1} \) in non-negative powers of \( u_i \). The notation \( \ldots \ldots \ldots \ldots \) means that we only keep the terms for which all the \( u_i \)'s and \( u \)'s have negative exponents.

Relation (1.5) is simply the case when all the \( A_i \) are empty. Though the difference between (1.5) and (2.2) is a purely formal manipulation of series, we are working with this more general format for the purposes of [2].

**Proof** For each \( i \) between 0 and \( k \), let us define the quantity:

\[ Z_j = \pi_1^1 \ldots \pi_j^j \left[ \prod_{i=1}^j \left( \frac{1}{u_i - c_i} \prod_{u \in A_i} \frac{1}{u - c_i} \right) \right]. \]

\[ \cdot \prod_{i=j+1}^k \prod_{m_i} Q_{m_i}(u_i + m_i^{i-1}u_{i-1} + \ldots + m_i^{j+1}u_{j+1} + m_i^jc_j + \ldots + m_1^1c_1) \prod_{u \in A_i} \frac{1}{u - u_i} \]

It is easy to see that \( Z_k \) is the LHS and \( Z_0 \) is the RHS of (2.2). Therefore, to complete the proof of our theorem, we need to show that \( Z_j = Z_{j-1} \), or in other words that:

\[ \prod_{i=j+1}^k \prod_{m_i} Q_{m_i}(u_i + m_i^{i-1}u_{i-1} + \ldots + m_i^{j+1}u_{j+1} + m_i^jc_j + \ldots + m_1^1c_1) \prod_{u \in A_i} \frac{1}{u - u_i} \]

\[ = \prod_{i=j+1}^k \prod_{m_i} Q_{m_i}(u_i + m_i^{i-1}u_{i-1} + \ldots + m_i^{j}u_{j+1} + m_i^jc_j + \ldots + m_1^1c_1) \prod_{u \in A_i} \frac{1}{u - u_i} \]

...
\[
\pi^j_* \left[ \left( \frac{1}{u_j - c_j} \prod_{u \in A_j} \frac{1}{u - c_j} \right) \cdot \frac{1}{u_j - c_j} \prod_{i = j+1}^k \prod_{m_i} Q_{m_i}(u_i + \ldots + m_j^i c_j + \ldots m_1^i c_1) \right] = \\
= \left[ \prod_{u \in A_j} \frac{1}{u - u_j} \prod_{i = j}^k \prod_{m_i} Q_{m_i}(u_i + \ldots + m_j^i u_j + \ldots + m_1^i c_1) \right]
\]  \hspace{1cm} (2.3)

To prove this relation, it is enough to assume \( Q_{m_i}(u) = u^{\alpha_{m_i}} \) and then the LHS becomes

\[
\pi^j_* \left[ \sum_{\beta_j, \beta_u, \beta_{m_i}} \frac{\beta_j u_j - \sum \beta_u + \sum \beta_{m_i}}{u_j - \sum \beta_u \prod_{i \geq j} (m_i^j)^{\beta_{m_i}} u_i + \ldots + m_j^i u_j + \ldots + m_1^i c_1)} \left( \prod_{u \in A_j} \frac{1}{u - u_j} \prod_{i = j}^k \prod_{m_i} Q_{m_i}(u_i + \ldots + m_j^i u_j + \ldots + m_1^i c_1) \right) \right] - 
\]

where all the \( \beta \)'s range over the non-negative integers, \( u \) ranges over \( A_j \) and \( i \) ranges over \( \{ j + 1, \ldots, k \} \). Now if we denote by \( \gamma \) the exponent of \( c_j \) and solve for \( \beta_j \), the above becomes:

\[
\pi^j_* \left[ \sum_{\gamma, \beta_u, \beta_{m_i}} \frac{\beta_j u_j - \gamma - 1}{u_j - \gamma - 1} \prod_{u \in A_j} (\frac{\beta_u - \gamma - 1}{u_j - \gamma - 1}) \prod_{i > j} (m_i^j)^{\beta_{m_i}} u_i + \ldots + m_j^i u_j + \ldots + m_1^i c_1)} \left( \prod_{u \in A_j} \frac{1}{u - u_j} \prod_{i = j}^k \prod_{m_i} Q_{m_i}(u_i + \ldots + m_j^i u_j + \ldots + m_1^i c_1) \right) \right] - 
\]

The condition that \( \beta_j \geq 0 \), which was lost when we replaced it by the variable \( \gamma \), is recovered by the condition \([ \ldots ]\). Since \( \gamma \) and the \( \beta \)'s sum independently, the above equals:

\[
\left[ \left( \frac{1}{u_j - c_j} \prod_{u \in A_j} \frac{1}{u - u_j} \prod_{i = j+1}^k \prod_{m_i} Q_{m_i}(u_i + \ldots + m_j^i u_j + \ldots + m_1^i c_1) \right) \right] - 
\]

Then if we replace \( \pi^j_*(u_j - c_j)^{-1} \) by (1.4), the above yields the RHS of (2.3), thus completing the proof.

\[\square\]

3. A Basic Example

Theorem (2.3) works equally well if we replace Chow rings by cohomology rings. For a simple example, let us consider the variety \( F \) of complete flags in \( \mathbb{C}^{k+1} \):

\[
V_1 \subset \ldots \subset V_k \subset \mathbb{C}^{k+1},
\]

where \( V_i \) is an \( i \)-dimensional subspace. On \( F \), we will consider the universal vector bundle \( \mathcal{V}_i \) whose fiber over (3.1) is \( V_i \), and also the tautological line bundle:

\[
\mathcal{O}_i(1) = \mathcal{V}_{k+1-i}/\mathcal{V}_{k+1-i}.
\]

(3.2)
It is well known that \( c_i = c_1(\mathcal{O}_i(1)) \) as \( i \in \{1,\ldots,k\} \) generate the cohomology of \( F \). Then we have the following result:

**Proposition 3.1.** The following identity tells us how to integrate any cohomology class on \( F \):

\[
\int_F \frac{1}{(u_1 - c_1) \cdots (u_k - c_k)} = (u_1 \cdots u_k)^{-k-1} \prod_{1 \leq i < j \leq k} (u_j - u_i).
\]

**Proof** If we let \( F_i \) parametrize flags:

\[
V_{k+1-i} \subset \ldots \subset V_k \subset \mathbb{C}^{k+1},
\]

where each \( V_j \) still has dimension \( j \), then \( F_0 = \text{pt} \) and \( F_k = F \). All these spaces fit into a projective tower:

\[
\pi : F_k \xrightarrow{\pi^k} F_{k-1} \xrightarrow{\pi^{k-1}} \ldots \xrightarrow{\pi^2} F_1 \xrightarrow{\pi^1} F_0 = \text{pt}.
\]

It is easy to see that \( F_i = \mathbb{P}_{F_i} (V_{k+2-i}^{\vee}) \), so this realizes the flag variety as a tower of projective bundles over the point. It’s easy to see that \( \mathcal{O}_i(1) \) of this tower are precisely the line bundles (3.2), and therefore we have the following equality in the Grothendieck group of \( F_i \):

\[
[V_{k+2-i}^{\vee}] = [\mathcal{O}_i^{k+1}] - [\mathcal{O}_1(-1)] - \ldots - [\mathcal{O}_{i-1}(-1)],
\]

By the Whitney sum formula and (2.1), one therefore has:

\[
s(\pi^i, u) = \frac{(u - c_1) \cdots (u - c_{i-1})}{u^{k+1}},
\]

Then (1.5) implies the desired result. We do not need the \([\ldots]\) anymore, because all terms only consist of negative monomials already.

\[\square\]

**References**

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