Structure of finite groups with restrictions on the set of conjugacy classes sizes

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Abstract. Let \( N(G) \) be the set of conjugacy classes sizes of \( G \). We prove that if \( N(G) = \Omega \times \{1, n\} \) for specific set \( \Omega \) of integers, then \( G \cong A \times B \) where \( N(A) = \Omega \), \( N(B) = \{1, n\} \), and \( n \) is a power of prime.

Introduction

Let \( A,B \) be finite groups and \( G := A \times B \). It is easy to check that \( N(G) = N(A) \times N(B) \). We are interested in the converse of this assertion.

Question 0.1. Let \( G \) be a group such that \( N(G) = \Omega \times \Delta \). Which \( \Delta \) and \( \Omega \) guarantee that \( G \cong A \times B \), where \( A \) and \( B \) are subgroups such that \( N(A) = \Omega \) and \( N(B) = \Delta \)?

A. Camina proved in [4] that, if \( N(G) = \{1, p^m\} \times \{1, q^n\} \), where \( p \) and \( q \) are distinct primes, then \( G \) is nilpotent. In particular, \( G = P \times Q \) for a Sylow \( p \)-subgroup \( P \) and a Sylow \( q \)-subgroup \( Q \). Later A. Beltran and M. J. Felipe (see [1] and [2]) proved a more general result asserting that, if \( N(G) = \{1, m\} \times \{1, n\} \), where \( m \) and \( n \) are positive coprime integers, then \( G \) is nilpotent, \( n = p^a \) and \( m = q^b \) for some distinct primes \( p \) and \( q \).

In [13], C. Shao and Q. Jiang showed that if \( N(G) = \{1, m_1, m_2\} \times \{1, m_3\} \), where \( m_1, m_2, m_3 \) are positive integers such that \( m_1 \) and \( m_2 \) do not divide each other and \( m_1 m_2 \) is coprime to \( m_3 \), then \( G \cong A \times B \), where \( A \) and \( B \) are such that \( N(A) = \{1, m_1, m_2\} \) and \( N(B) = \{1, m_3\} \). In all these cases, the sets of prime divisors of the orders of \( A \) and \( B \) do not intersect. It was proved in [11] that if \( N(G) = N(Alt_5) \times N(Alt_5) \) and \( Z(G) = 1 \) then \( G \cong Alt_5 \times Alt_5 \).

In [7] a directed graph was introduced on the set \( N(G) \setminus \{1\} \). Given \( \Theta \subseteq \mathbb{N} \), with \( |\Theta| < \infty \), define the directed graph \( \Gamma(\Theta) \), with the vertex set \( \Theta \) and edges \( ab \) whenever \( a \) divides \( b \). Set \( \Gamma(G) = \Gamma(N(G) \setminus \{1\}) \).
In this article, the following theorem is proved.

**Theorem 0.2.** Let $\Omega$ be a set of integers and $\Gamma(\Omega \setminus \{1\})$ be disconnected, and $n$ be a positive integer such that $\gcd(n, \alpha) = 1$ for each $\alpha \in \Omega \setminus \{1\}$. Let $G$ be a finite group such that $N(G) = \Omega \times \{1, n\}$. Then $G \simeq A \times B$, where $N(A) = \Omega$, $N(B) = \{1, n\}$ and $n$ is a prime power.

## 1 Preliminaries

We fix the following notation: for an integer $k$, denote by $\pi(k)$ the set of prime divisors of $k$. If $\Omega$ is a set of integers, denote $\pi(\Omega) = \bigcup_{\alpha \in \Omega} \pi(\alpha)$. For a prime number $r$, denote by $k_r$ the highest power of $r$ dividing $k$. For integers $m_1, \ldots, m_s$, write $\gcd(m_1, m_2, \ldots, m_s)$ to denote their greatest common divisor, and write $\text{lcm}(m_1, m_2, \ldots, m_s)$ for their least common multiple.

Let $\Omega$ be a set of integers, and order it by the relation of divisibility. The subset of maximal elements is denoted by $\mu(\Omega)$ and the set of minimal elements is denoted by $\nu(\Omega)$.

**Definition 1.1.** We say that the set $\Omega$ is separated if, for each $\alpha \in \Omega$, there exists $\beta \in \mu(\Omega)$ such that $\alpha$ does not divide $\beta$.

Let $G$ be a group and take $a \in G$. We denote by $a^G$ the conjugacy class of $G$ containing $a$. If $N$ is a subgroup of $G$, then $\text{Ind}(N, a) = |N|/|C_N(a)|$. Note that $\text{Ind}(G, a) = |a^G|$. Denote by $|G||_p$ the highest power $p^n$ of $p$ such that $N(G)$ contains multiples of $p^n$ while avoiding multiples of $p^{n+1}$. For $\pi \subseteq \pi(G)$ put $|G||_\pi = \prod_{p \in \pi} |G||_p$. For brevity, write $|G|$ to mean $|G||_\pi(G)$. Observe that $|G||_p$ divides $|G||_p$ for each $p \in \pi(G)$. In general, $|G||_p$ is less than $|G||_p$.

**Definition 1.2.** We say that a group $G$ satisfies the condition $R(p)$, or that $G$ is an $R(p)$-group, if there exists an integer $\alpha > 0$ such that $a_p \in \{1, p^\alpha\}$ for each $a \in N(G)$. In that case, we write $G \in R(p)$.

The set of $R(p)$-groups can be seen as the disjoints of the two subsets $R(p)^*$ and $R(p)^{**}$:

- $a)$ $G \in R(p)^*$ if $G \in R(p)$ and contains a $p$-element $h$ such that $\text{Ind}(G, h)_p > 1$;
- $b)$ $G \in R(p)^{**}$ if $G \in R(p)$ and $\text{Ind}(G, h)_p = 1$ for each $p$-element $h \in G$.

**Lemma 1.3 ([9, Main theorem]).** If $G \in R(p)^*$, then $G$ has a normal $p$-complement.

**Lemma 1.4 ([9, Corollary]).** If $G \in R(p)^*$ and $P \in \text{Syl}_p(G)$, then $Z(P) \leq Z(G)$.

**Lemma 1.5 ([8, Lemma 1.4]).** For a finite group $G$, take $K \trianglelefteq G$ and put $\overline{G} = G/K$. Take $x \in G$ and $\overline{x} = xK \in G/K$. The following claims hold:

(i) $|x^K|$ and $|\overline{x}|$ divide $|x|$. 


Lemma 1.8 with either where Lemma 1.6 (\([10, \text{Lemma 4}])\).

Lemma 1.9.

Proof. Lemma 1.10.

If \(|x|, |K|\) = 1, then \(C_G(xy) = C_G(x) \cap C_G(y)\).

\(\text{Lemma 1.10.}\) Let \(g \in G\). If each conjugacy class of \(G\) contains an element \(h\) such that \(g \in C_G(h)\) then \(g \in Z(G)\).

Lemma 1.7 ([3, Theorem A]). Let \(G\) be a finite group, and let \(p\) and \(q\) be distinct primes. Then some Sylow \(p\)-subgroup of \(G\) commutes with some Sylow \(q\)-subgroup of \(G\) if and only if the class sizes of the \(q\)-elements of \(G\) are not divisible by \(p\) and the class sizes of the \(p\)-elements of \(G\) are not divisible by \(q\).

We call a \(p\)-element \(x\) of \(G\) \(p\)-central if \(x \in Z(P)\) for some Sylow \(p\)-subgroup \(P\) of \(G\).

Lemma 1.8 ([12, Theorem B]). Let \(G\) be a finite group and \(p\) a prime. Suppose that every \(p\)-element of \(G\) is \(p\)-central. Then

\[O^p(G/O_p(G)) = S_1 \times \cdots \times S_r \times H,\]

where \(H\) has an abelian Sylow \(p\)-subgroup, \(r \geq 0\), and \(S_i\) is a non-abelian simple group with either

(i) \(p = 3\) and: \(S_i \simeq Ru\), or \(J_4\), or \(S_i \simeq 2F_4(q_i)'\), \(9 \not| (q_i + 1)\); or

(ii) \(p = 5\) and \(S_i \simeq Th\) for all \(i\).

Lemma 1.9. If \(G \in R^{**}(p)\), then the Sylow \(p\)-subgroups of \(G\) are abelian.

Proof. Note that \(R^{**}(p)\)-groups satisfy the condition of Lemma 1.8. Hence, if a Sylow \(p\)-subgroup is non-abelian, then \(p \in \{3, 5\}\) and \(O^p(G/O_p(G)) = S_1 \times \cdots \times S_r \times H\), where \(S_i\) is isomorphic to one of the groups \(Ru, J_4, 2F_4(q_i)', Th\). Note that if \(r > 1\), then the group \(G\) is not an \(R^{**}(p)\) group. It follows from the description of conjugacy class sizes in [17] and [15] that \(S\) contains a \(p'\)-element \(g_1\) such that \(1 < \text{Ind}(S, g_1)_p < |S|_p\) and a \(p'\)-element \(g_2\) such that \(\text{Ind}(S, g_2)_p = |S|_p\). Since \(p\) and \(|O_p(G)|\) are relatively prime, there exists \(g_2' \in G\) such that \(g_2'O_p(G) = g_1\) and \(\text{Ind}(G, g_1')_p = \text{Ind}(G/O_p(G), g_1)_p\). Let \(g_2' \in G\) be such that \(g_2'O_p(G) = g_2\). We have \(C_G(g_2')O_p(G) / O_p(G) \leq C_G/O_p(G)(g_2)\). In particular \(\text{Ind}(G, g_2)_p \geq \text{Ind}(G/O_p(G), g_2)_p > \text{Ind}(G, g_1')_p\), contradicting the definition of \(R^{**}(p)\)-groups.

Lemma 1.10. Any \(R^{**}(p)\)-group contains at most one non-abelian composition factor whose order is divisible by \(p\).
Proof. Let $G$ be an $R^*(p)$-group. Lemma 1.9 implies that the Sylow $p$-subgroup of $G$ is abelian. Let $1 < G_1 < \cdots < G_k = G$ be the chief series. Assume that $G_i/G_{i-1} = H$ is a non-solvable group and the order of $H$ is divisible by $p$. Lemma 1.5 implies that the conjugacy class sizes of the group $H$ divide the corresponding conjugacy class sizes of $G$. We have $H = S_1 \times S_2 \times \cdots \times S_t$, where the $S_i$ are isomorphic non-abelian finite simple groups, for $1 \leq i \leq t$.

Assume that $|G_{i-1}|$ is divisible by $p$. Let $P \leq G_{i-1}$ be a Sylow $p$-subgroup of $G_{i-1}$. From Frattini’s argument, it follows that $N_{G_i}(P)/N_{G_{i-1}}(P) \cong G_i/G_{i-1}$. Let $\hat{H} \leq N_{G_i}(P)$ be a subgroup generated by all Sylow $p$-subgroups of $N_{G_i}(P)$. Since any Sylow $p$-subgroup of $G$ is abelian and $H$ is generated by $p$-elements, we infer that $\hat{H}G_{i-1}/G_{i-1} = H$ and $\hat{H}$ centralizes some Sylow $p$-subgroup of the group $G_{i-1}$.

Assume that $g \in G/G_{i-1}$ is a $p$-element acting on $H$ as an outer automorphism. The fact that the Sylow $p$-subgroups of $G$ are abelian implies that $S_j^g = S_j$ for any $1 \leq j \leq t$. Assume that $g$ acts non-trivially on $S_j$. Since the Sylow 2-subgroup of a simple alternating group of degree greater than 5 is non-abelian and the outer automorphism group of an alternating group is a 2-group, we obtain that $S_j$ cannot be isomorphic to any of the alternating groups. It follows from [17] and Lemma 1.8 that $S_j$ cannot be isomorphic to any of the sporadic groups, and therefore $S_j$ is a group of Lie type. In [14, Theorem 1] and in [16] it is described when a Sylow $p$-subgroup of a simple group of Lie type is abelian. We can show that $g$ acts on $S_j$ as a field automorphism. It follows from the description of the centralizers of field automorphisms (see [6, Theorem 4.9.1]) that the Sylow $p$-subgroup of $S_j \langle g \rangle$ is non-abelian, and hence the Sylow $p$-subgroup of $G$ is non-abelian, which is a contradiction. Therefore, it can be considered that $H$ contains a Sylow $p$-subgroup of $G/G_{i-1}$.

Assume that $t > 1$. For each $j \in \{1, \ldots, t\}$, there is an element $h_j \in S_j$ such that $\text{Ind}(S_j, h_j)_p = |S_j|_p$. Let $g = h_1 \cdots h_t$ and $\hat{g} \in \hat{H}$ be some pre-image of the element $g$. Since $\hat{H}$ centralizes a Sylow $p$-subgroup of $G_{i-1}$ and $\text{Ind}(H, g)$ divides $\text{Ind}(G, \hat{g})$, we infer that $\text{Ind}(G, \hat{g})_p = (\text{Ind}(H, g))_p = |H|_p$. If $t > 1$, then $\hat{H}$ contains an element $\hat{h}_1$, which is the pre-image of the element $h_1$ such that $1 < \text{Ind}(G, \hat{h}_1)_p < |H|_p$. This contradicts the definition of an $R(p)$-group.

Lemma 1.11 ([5, Theorem 5.2.3]). Let $A$ be a $\pi(G)'$-group of automorphisms of an abelian group $G$. Then $G = C_G(A) \times [G, A]$.

Lemma 1.12. Let $P \leq G$ be a Sylow $p$-subgroup of $G$. If $P = A \times B$ with $A, B$ normal subgroups of $G$, then $C_G(ab) = C_G(a) \cap C_G(b)$ for any $a \in A$ and $b \in B$.

Proof. The assertion of the lemma follows from the fact that any $p$-element $x$ is uniquely represented as $x = x_ax_b$ where $x_a \in A$ and $x_b \in B$. 

\hfill \Box
2 Proof of the Main Theorem

Let $G$ be as in the hypothesis of the theorem. We divide the proof of the theorem into 3 propositions. In the preliminary lemma and in Propositions 2.2 and 2.4, we only use the separation property of the set $\Omega$. The disconnection of the graph $\Gamma(\Omega \setminus \{1\})$ is used only in the proof of Proposition 2.

Note that $G$ has the property $R(p)$ for any $p \in \pi(n)$. In Propositions 2.2 and 2.3 we prove that $G \not\in R^{**}(p)$. In Proposition 2.4 we analyze the case $G \in R^*(p)$ and thus complete the proof of the Main Theorem.

Assume that $G \in R^{**}(p)$ for any $p \in \pi(n)$. In this case, Lemma 1.9 implies that a Sylow $p$-subgroup of $G$ is abelian. It follows from Lemma 1.7 that a Hall $\pi(n)$-subgroup exists and is abelian. It follows from the well-known Wielandt theorem that all Hall $\pi(n)$-subgroups are conjugate.

Lemma 2.1. The order of any non-abelian composition factor of $G$ is not divisible by $p$.

Proof. Lemma 1.10 implies that $G$ contains at most one non-abelian composition factor $S$ whose order is divisible by $p$. Let $R \lhd G$ be such that $S \leq G/R$. Let $g \in G$ be a $p$-element such that its image $gR \in S$ is not trivial. Let $x \in G$ be an element of minimal order such that $\text{Ind}(G, x) = n$. Since $n$ is minimal with respect to divisibility in $N(G)$, we infer that $|x| = r^\alpha$ is a power of a prime $r$. We have that $x$ centralizes Sylow $t$-subgroups for any $t \in \pi(\Omega)$ and, in particular, $x$ centralizes Sylow $t$-subgroups for any $t \in \pi(\text{Ind}(G, g))$. Put $C = C_G(x)$. Since $S$ is the unique non-abelian composition factor whose order is divisible by $p$, we infer that $S$ is a normal subgroup of $G/R$. Note that $CR/R$ contains Sylow $t$-subgroups of $G/R$ for any $t \in \pi(\text{Ind}(S, \bar{g}))$. Let $T$ be a Sylow $t$-subgroup of $G/R$ for some prime $t \in \pi(G/R)$. Since $S$ is a normal subgroup of $G/R$ we infer that $T \cap S$ is a Sylow $t$-subgroup of $S$. From the fact that finite simple groups do not have Hall $p'$-subgroups for each prime divisor $p$ of its order, we get that group $S$ is generated by its Sylow $t$-subgroups, where $t \in \pi(\text{Ind}(S, \bar{g}))$. Hence $S \leq CR/R$. In particular, $C$ contains a pre-image of the group $S$. Therefore, $C$ contains an $r'$-element $y$ such that $\text{Ind}(C, y)_p > 1$. Thus, $\text{Ind}(G, xy)_p > \text{Ind}(G, x)_p$, which is a contradiction. \hfill \Box

Let $O = O_{\pi(n)'}(G)$. Lemma 2.1 implies that $G/O$ contains a normal $p$-subgroup $\mathcal{P}$, for some $p \in \pi(n)$. Let $T = O_{\pi(n)}(G/O)$. Assume that $T$ is not a Hall $\pi(n)$-subgroup of $G/O$. Since a Hall $\pi(n)$-subgroup of $G$ is abelian, we have $T$ is abelian. The centralizer of $R$ in $G/O$ is a normal subgroup of $G/O$ for each Sylow subgroup $R$ of $T$. For any $g \in G/O$ it follows from the inequality $\text{Ind}(G/O, g)_p > 1$ that $\text{Ind}(G/O, g)_{\pi(n)} = n$. Using these facts it is easy to obtain a contradiction. Therefore, $G/O$ contains a normal Hall $\pi(n)$-subgroup $\mathcal{P}$. In particular, we can assume that $\mathcal{P}$ is a Sylow $p$-subgroup of $G/O$. Let $x \in G$ be an element of minimal order such that $\text{Ind}(G, x) = n$. Since $n$ is minimal by divisibility number of $N(G)$, we infer that $x$ is an element of order $t^\alpha$, where $t$ is some prime and $t \not\in \pi(n)$.

Proposition 2.2. The image $\bar{x} \in G/O$ of $x$ is trivial.
Proof. Assume that \( \overline{x} \) is not trivial. Lemma 1.11 implies that \( \overline{P} = [\overline{x}, \overline{P}] \times C_{\overline{P}}(\overline{x}) \). Let \( \widetilde{x} \in G/OH \) denote the image of \( x \). Since \( \pi(G/OH) \) does not contain numbers from the set \( \pi(n) = \pi(\text{Ind}(G, x)) \), and \( \text{Ind}(G/OH, \overline{x}) \) divides \( \text{Ind}(G, x) \), we infer that \( \text{Ind}(G/OH, \overline{x}) \) is equal to 1. Hence \( \widetilde{x} \in Z(G/OH) \). Thus the subgroup \( C_{\overline{P}}(\overline{x}) \) is a normal subgroup of \( G/OH \). Since \( p \not\in \pi(G/OH) \), it follows from Maschke’s theorem that \( C_{\overline{P}}(\overline{x}) \) has compliment in \( P \). In particular \( [\overline{x}, \overline{P}] \) is a normal subgroup of \( G/OH \).

Let \( P \) be a Sylow \( p \)-subgroup of \( G \), and let \( P_1, P_2 \leq P \) be such that \( P_1.O/O = [\overline{x}, \overline{P}] \) and \( P_2/O/O = C_{\overline{P}}(\overline{x}) \). Since \( \text{lcm}(\text{Ind}(O, x), O) = 1 \), we have \( x \in C_G(O) \). The group \( C_G(O) \) is a normal subgroup of \( G \). We have \( C_G(O)/O/O \leq G \) and \( \overline{x} \in C_G(O)/O/O \). From the fact that \( \overline{x} \) acts without fixed points on \( [\overline{x}, \overline{P}] \) and \( [\overline{x}, \overline{P}] \leq C_{\overline{P}}(\overline{x}) \) it follows that \( [\overline{x}, \overline{P}] \) is the minimal normal subgroup of \( C_G(O) \) which includes \( \overline{x} \). In particular \( P_1 \leq C_G(O) \).

The fact that the number \( \text{Ind}(G, x)_p \) is maximal implies that centralizer of any \( t' \)-element of \( C_G(x) \) contains some Sylow \( p \)-subgroup of the group \( C_G(x) \). Since \( O.P_2 \trianglelefteq G \), we infer that the centralizer of any \( t' \)-element of \( O \) contains a subgroup conjugate to \( P_2 \) in \( O.P_2 \). Suppose there is a \( t \)-element \( y \in O \) such that \( \text{Ind}(O.P_2, y)_p > 1 \). Since \( \text{Ind}(G, x)_t = 1 \), we infer that \( C_G(x) \) contains some Sylow \( t \)-subgroup of \( G \). In particular, one can assume that \( y \in C_G(x) \). Consider \( C_G(xy) \). Let \( R \) be a Sylow \( p \)-subgroup of \( G \) such that \( \widetilde{R} = R \cap C_G(xy) \) is a Sylow \( p \)-subgroup of \( C_G(xy) \). Since \( P_1 \leq C_G(O) \), we have \( P_1 \leq R \) and \( \widetilde{R} \) is a Sylow \( p \)-subgroup of \( C_G(xy) \). It follows from the fact that \( \text{Ind}(G, x)_p = \text{Ind}(G, xy)_p = |P_1| \) and the fact that \( R \) is an abelian group that \( \widetilde{R} \) is an abelian group. Note that \( \widetilde{R} \leq C_G(x) \), and hence \( \widetilde{R} \) is conjugate to \( P_2 \) in \( C_G(x) \). In particular \( \widetilde{R} \) is conjugate to \( P_2 \) in \( O.P_2 \). Therefore \( y \) centralizes \( \widetilde{R} \) and \( \text{Ind}(O.P_2, y)_p = 1 \), which is a contradiction. Thus any element of \( O \) centralizes some Sylow \( p \)-subgroup. Lemma 1.6 implies that \( P_2 \leq C_G(O) \). Thus \( G \) contains a normal abelian Hall \( \pi(n) \)-subgroup \( N \).

We have that \( P_2 \) is a Sylow \( p \)-subgroup of \( C_G(x) \) and \( P_2 \unlhd C_G(x) \). From the fact that \( \text{Ind}(G, x)_p \) is maximal it follows that any \( t' \)-element centralizes \( P_2 \). Therefore, we have that \( \pi(\text{Ind}(C_G(x), h)) \subseteq \{t\} \) for any \( h \in P_2 \). Since \( C_G(x) \) contains Sylow \( r \)-subgroups of \( G \) for any \( r \in \pi(\Omega) \) and a Hall \( \pi(n) \)-subgroup of \( G \) is abelian, we infer that \( \pi(\text{Ind}(G, h)) \subseteq \{t\} \) for any \( h \in P_2 \). Let \( g \in C_G(x) \) be some \( t' \)-element. Then \( g \) acts on \( P_1 \), and

\[
\text{Ind}(G, g)_p = \text{Ind}(P_1, g)_p.
\]

Since \( \text{Ind}(P_1, g)_p \in \{1, |P_1|\} \), we see that \( g \) acts on \( P_1 \) either trivially or without fixed points. Note that \( x^G = x^N \). Thus, \( \text{Ind}(G, a)_\nu = \text{Ind}(G, b)_\nu \) for any \( a, b \in P_1 \) and \( \pi(\text{Ind}(G, c)) \subseteq \{t\} \) for any \( c \in P_2 \). It follows from Lemma 1.12 that, for any \( p \)-element \( a \), there exists \( k \) such that \( \text{Ind}(G, a)_\nu \in \{1, k\} \). Thus, \( \Omega \) contains a number \( \alpha \) dividing the index of any \( p \)-element. Let \( h_1 \in P_1 \) be such that \( \text{Ind}(G, h_1) \) is minimal among \( \{\text{Ind}(G, g)|g \in P_1\} \), and let \( h_2 \in P_2 \) be such that \( \text{Ind}(G, h_2) \) is minimal among \( \{\text{Ind}(G, g)|g \in P_2\} \).

Assume that \( \text{Ind}(G, h_2)_t \leq \text{Ind}(G, h_1)_t \). Then \( \text{Ind}(G, h_2) \) divides \( \text{Ind}(G, g) \) for any \( p \)-element \( g \). Since \( \Omega \) is separated, we obtain that \( \mu(\Omega) \) contains an element \( \beta \) that is not divisible by \( \text{Ind}(G, h_2) \). Let \( l \in G \) be such that \( \text{Ind}(G, l) = \beta \). Since \( \text{Ind}(G, h_2) \) does not divide \( \beta \), we infer that \( C_G(l) \) does not contain \( p \)-elements. But \( \beta \) is not divisible by \( p \) and hence \( C_G(l) \) contains some Sylow \( p \)-subgroup, therefore we have a contradiction.
Thus, \( \text{Ind}(G, h_2)_t > \text{Ind}(G, h_1)_t \). Since \( \Omega \) is separated, we infer that \( \mu(\Omega) \) contains an element \( \beta \) that is not divisible by \( \text{Ind}(G, h_2) \). Let \( l \in G \) be such that \( \text{Ind}(G, l) = \beta \). Since \( \text{Ind}(G, h_2) \) does not divide \( \beta \), we see that \( |l| \) is divisible by \( p \). Further, we have \( l = ab \), where \( a \) is a \( p \)-element and \( b \) is a \( p' \)-element. We have that \( \text{Ind}(G, a) \) divides \( \beta \). From Lemma 1.12 and the fact that \( \beta \) is not divisible by numbers in \( \{\text{Ind}(G, g) | g \in P_2\} \), it follows that \( a \in P_1 \). It follows from Lemma 1.12 that \( \text{Ind}(G, abh_2) \) is divisible by \( \beta \) and \( \text{Ind}(G, h_2) \) contradicting the fact that \( \beta \) is maximal in \( \Omega \).

\( \square \)

**Proposition 2.3.** The element \( x \notin O \).

**Proof.** Assume that \( x \in O \). Since \( \text{Ind}(G, x) \) is relatively prime to \( |O| \), we have \( O \leq C_G(x) \).

Let \( X = \langle x^G \rangle \). The fact that \( O \) is a normal subgroup of \( G \) implies that \( O \leq C_G(X) \).

Hence, \( X \) is an abelian \( t \)-subgroup of the group \( O \). Let \( P \) be a Sylow \( p \)-subgroup of \( G \) such that \( P_1 = P \cap C_G(x) = \text{a Sylow \( p \)-subgroup of} \ C_G(x) \). The fact that \( x^G = x^{O.H} \) implies that \( P_1 < C_G(X) \). Thus, any \( t' \)-element of \( O \) centralizes some subgroup conjugate to \( P_1 \).

Consider \( X \) as a \( \tilde{P} = P/P_1 \)-module. It follows from Lemma 1.11 that the group \( X \) can be represented as \( [X, \tilde{P}] \times C_X(\tilde{P}) \). Since \( \tilde{P} \) acts non-trivially on \( X \), we see that \( [X, \tilde{P}] \) is non-trivial. Since for any element \( y \in [X, \tilde{P}] \) we have \( \tilde{P} \cap C_{O.P}(y) = 1 \), we infer that \( \tilde{P} \) acts without fixed points on \( [X, \tilde{P}] \). Hence \( \tilde{P} \) is a cyclic group.

Assume that \( P_1 \) contains an element \( f \) such that \( \text{Ind}(G, f) > 1 \). We will use the fact that \( \text{graph } \Gamma(\Omega \setminus \{1\}) \) is disconnected. Let \( \Gamma_1 \) be a connected component of the graph \( \Gamma(\Omega \setminus \{1\}) \) such that \( \text{Ind}(G, f) \in \Gamma_1 \). Since any \( t' \)-element centralizes some element from \( f^G \), we infer that \( \text{Ind}(G, g)_{\pi(n)'} \in \Gamma_1 \cup \{1\} \) for any \( \{p, t\}' \)-element \( g \).

Denote by \( \Gamma_2 \) some connected component of the graph \( \Gamma(\Omega \setminus \{1\}) \) different from \( \Gamma_1 \). Let \( y \in G \) be such that \( \text{Ind}(G, y) \in \Gamma_2 \). We have that \( y \) is a \( \{p, t\} \)-element. Assume that \( y \) is a \( t \)-element. Since \( \text{Ind}(G, y)_p = 0 \), we infer that \( C_G(y) \) contains a subgroup conjugate to \( P_1 \), and hence \( \text{Ind}(G, y) \in \Gamma_1 \cup \{1\} \), deriving a contradiction.

Therefore, if \( \text{Ind}(G, g)_p = 1 \), then \( g \) is the product of a \( p \)-element and an element from the center of \( G \). In particular, if \( \text{Ind}(G, g) \in \Gamma_2 \), where \( g \) is an element of primary order, then \( \pi(g) = \{p\} \). It also follows from here that \( \pi(n) = \{p\} \).

Since \( \Gamma_2 \) is an arbitrary connected component of \( \Gamma(\Omega \setminus \{1\}) \) different from \( \Gamma_1 \), then we can assume that there exists \( z \in P \) such that \( \text{Ind}(G, z) \in \Gamma_2 \). Then \( \text{Ind}(G, y) \in \Gamma_2 \cup \{1\} \) for any \( y \in \langle z \rangle \). This means that \( \langle z \rangle \cap P_1 \leq Z(G) \). Let \( g \in P \) be such that \( z \in \langle g \rangle \). Since \( \text{Ind}(G, z) \) divides \( \text{Ind}(G, g) \) it follows that \( \text{Ind}(G, g) \in \Gamma_2 \). Since \( P/P_1 \) is a cyclic group and \( P \) is an abelian group, we can write \( P = \langle z, P_1 \rangle \).

We have \( \text{Ind}(G, g) \in \Gamma_1 \) for any non-central \( \{p, t\}' \)-element \( g \). Therefore for any \( h \) such that \( \text{Ind}(G, h) \in \Gamma_2 \) it is true that \( C_G(h)/Z(G) \) is a \( \{p, t\} \)-group. In particular, \( \text{Ind}(G, h)_{\{p, t\}'} = |G|_{\{p, t\}'}. \) Assume that there exists \( z' \in P \setminus P_1 \) such that \( \text{Ind}(G, z') \in \Gamma_1 \). If \( C_G(z') \) does not contain non-central \( \{p, t\}' \)-elements, then \( \text{Ind}(G, z) \) is connected to \( \text{Ind}(G, z') \) in \( \Gamma(\Omega \setminus \{1\}) \) and hence \( \text{Ind}(G, z') \in \Gamma_2 \), contradicting \( \text{Ind}(G, z') \in \Gamma_1 \). Let \( s \in C_G(z') \) be a \( \{p, t\}' \)-element and \( E \in \text{Syl}_p(C_G(z')) \). Since \( C_G(z) \) contains some Sylow \( p \)-subgroup of \( C_G(s) \), we can assume that \( C_G(x^g) \) contains \( E \) for some \( g \). But \( C_G(x^g) \cap P = P_1 \), and we have a contradiction.
Let \( y \in C_G(z) \setminus (Z(G) \cup P) \). As noted above, \( y \) is a \( t \)-element. We can assume that \( C_G(y) \cap P \in Syl_p(C_G(y)) \). Hence \( C_G(xy) \cap P \in Syl_p(C_G(xy)) \). Obviously, \( z \) and \( P_1 \) do not lie in \( C_G(xy) \). Let \( zg \in C_G(xy) \), where \( g \in P_1 \setminus Z(G) \). Let \( \tilde{\gamma} : G \to G/O \) be a natural homomorphism. Note \( \tilde{\gamma}(zg) = \tilde{\gamma}(y) \). Hence \( zg \in C_G'(xy) \), and thus \( \tilde{\gamma} \in C_G'(\tilde{xy}) \). Since \( |O| \) is coprime to \( |g| \), then \( C_G(g) = C_G'(\tilde{g}) \). Hence \( C_G(g) \) contains the group \( O.(\tilde{y}) \), and therefore \( y \in C_G(g) \) contradicting the fact that \( C_G(y) \cap P < Z(G) \). Thus, it is proved that \( P_1 < Z(G) \).

Since \( z \) acts without fixed points on \( x^G \), \( \text{Ind}(G,z) > 1 \). Denote by \( \Gamma' \) the connected component of \( \Gamma(\Omega \setminus \{1\}) \) containing \( \text{Ind}(G,z) \). Note that \( \text{Ind}(G,g)_{\mu'} \in \Gamma' \cup \{1\} \) for any \( g \in C_G(z) \). Assume that there exists \( h \notin \langle z \rangle \) such that \( \text{Ind}(G,h) \in \Omega \setminus (\Gamma' \cup \{1\}) \). Then \( h \) centralizes some Sylow \( p \)-subgroup and, therefore, we can assume that \( h \in C_G(z) \). Thus \( \text{Ind}(G,h) \in \Gamma' \), contradicting the hypothesis on \( h \). We have \( |\Gamma'| = 1 \) and \( C_G(z)/Z(G) = \langle z \rangle \). Therefore \( \Omega = N(\langle z \rangle) \), and in particular \( \Gamma(\Omega) \) is connected, which is a contradiction. \( \square \)

It follows from the proposition 2.2 and 2.3 that \( G \in R(p)^* \).

**Proposition 2.4.** If \( G \in R(p)^* \) for some \( p \in \pi(n) \) then \( G = A \times B \), where \( N(A) = \Omega \) and \( N(B) = \{1,p^a\} \). In particular, \( n \) is a \( p \)-number.

**Proof.** Lemma 1.3 implies that \( G = N \times P \) where \( P \) is a Sylow \( p \)-subgroup of \( G \). Lemma 1.4 implies that \( Z(P) \leq Z(G) \).

Assume that there is \( z \in P \) such that \( \text{Ind}(G,z)_{\mu'} > 1 \). The separation of \( \Omega \) implies that there exists \( k \in \mu(\Omega) \) such that \( k \) is not divisible by \( \text{Ind}(G,z)_{\mu'} \). Let \( g \in G \) be such that \( \text{Ind}(G,g) = k \). We have \( g = g_1g_2 \), where \( g_1 \) is a \( p' \)-element and \( g_2 \) is a \( p \)-element. Since \( C_G(g) = C_G(g_1) \cap C_G(g_2) \) and \( \text{Ind}(G,g)_p = 1 \), it follows that \( \text{Ind}(G,g_2)_p = 1 \). Hence \( g_2 \in Z(G) \). Thus, \( \text{Ind}(G,g) = \text{Ind}(G,g_1) \). We have that \( C_G(g_1) \) contains some Sylow \( p \)-subgroup of \( G \), and therefore there is \( z' \in C_G(g_1) \cap z'^G \), deriving a contradiction.

Thus any \( p \)-element centralizes \( N \) and hence \( G \simeq N \times P \). Therefore for each \( g \in P \) we have \( \pi(\text{Ind}(G,g)) = \{p\} \). In particular \( n \) is a \( \{p\} \)-number. \( \square \)

The assertion of the theorem follows from Propositions 2.2, 2.3 and 2.4.

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