On a linear system of differential equations

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The linear systems of partial differential equations of the first order with the identical main parts is considered. Applying the well-known relation between a normal system of ordinary differential equations and a linear system of partial differential equations of the first order with the same main parts, the existence of integral basis of a linear inhomogeneous system of partial differential equations of the first order adjoining to some solution of the same linear inhomogeneous system of differential equations with partial derivatives of the first order is proved. A sign at which the nonlinear system of ordinary differential equations has a neighborhood such that any solution with initial values from it tends to zero is found. Using the equivalence of a linear system of partial differential equations of the first order with identical main parts to a linear differential equation with partial derivatives of the first order, the existence of integral basis of the adjoining to zero linear homogeneous system of partial differential equations of first order with nonlinear coefficients is shown.

Key words: equation, first order partial derivatives.

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Key words: equation, first order partial derivatives.
Рассматриваются линейные системы дифференциальных уравнений с частными производными первого порядка с одинаковыми главными частями. Применяя известную связь между нормальной системой обыкновенных дифференциальных уравнений и линейной системой дифференциальных уравнений с частными производными первого порядка с одинаковыми главными частями, доказано, что существует интегральный базис линейной неоднородной системы дифференциальных уравнений с частными производными первого порядка, примыкающий к некоторому решению этой же линейной неоднородной системы дифференциальных уравнений с частными производными первого порядка. Найден признак, по которому нелинейная система обыкновенных дифференциальных уравнений имеет такую окрестность, что любое решение с начальными значениями из этой окрестности стремится к нулю. Используя эквивалентность линейной системы дифференциальных уравнений с частными производными первого порядка с одинаковыми главными частями и линейного дифференциального уравнения с частными производными первого порядка, доказано, что существует интегральный базис, примыкающей к нулю линейной однородной системы дифференциальных уравнений с частными производными первого порядка с нелинейными коэффициентами.

Ключевые слова: уравнение, частные производные первого порядка.

1 Introduction

The article deals with linear systems of partial differential equations of the first order with the identical main parts. Applying the well-known relation between a normal system of ordinary differential equations and a linear system of partial differential equations of the first order with the same main parts, the existence of integral basis of a linear inhomogeneous system of partial differential equations of the first order adjacent to some solution of the same linear inhomogeneous system of differential equations with partial derivatives of the first order is proved. A sign at which the nonlinear system of ordinary differential equations has a neighborhood such that any solution with initial values from it tends to zero is found. Using the equivalence of a linear system of partial differential equations of the first order with identical main parts to a linear differential equation with partial derivatives of the first order, the existence of integral basis of the adjacent to zero linear homogeneous system of partial differential equations of first order with nonlinear coefficients is shown. In (see [3]), the equivalence of a linear system of partial differential equations with identical main parts to a certain system of ordinary differential equations is shown. In the article, the approach differs from said above and the well-known simple relations between solutions of a linear homogeneous and linear inhomogeneous system of partial differential equations of the first order is used.

2 Literature review

The general theory is presented in the books [1-10]. The domain of existence of solutions was investigated by Kamke and data is contained in the reference books [11-12]. The domain of existence of solutions was investigated in the work [13-15]. Non-analytic equations are considered in the papers [16-18]. In work of Kruzhkov, generalized solutions was considered [19]. Kovalevskaya’s theorem was published in [20]. The problems of Lyapunov's second method were shown in [21, 22]. An example of nonexistence of a solution was constructed in [23-27]. The first-order partial differential equations were considered in [28]. Questions related to the nonlinear system of ordinary differential equations are in [29, 30].
3 Materials and research methods

Let us consider a linear inhomogeneous system of partial differential equations of the first order

\[
\begin{align*}
\frac{\partial u_1}{\partial x} + & \sum_{k=1}^{n} p_{1k}(x) y_k \frac{\partial u_1}{\partial y_k} + \ldots + \sum_{k=1}^{n} p_{nk}(x) y_k \frac{\partial u_1}{\partial y_n} = b_1(x) u_1 \\
\frac{\partial u_2}{\partial x} + & \sum_{k=1}^{n} p_{2k}(x) y_k \frac{\partial u_2}{\partial y_k} + \ldots + \sum_{k=1}^{n} p_{nk}(x) y_k \frac{\partial u_2}{\partial y_n} = b_2(x) u_2 \\
\frac{\partial u_n}{\partial x} + & \sum_{k=1}^{n} p_{nk}(x) y_k \frac{\partial u_n}{\partial y_k} + \ldots + \sum_{k=1}^{n} p_{nk}(x) y_k \frac{\partial u_n}{\partial y_n} = b_n(x) u_n;
\end{align*}
\]

where \( u_1(x, y_1, \ldots, y_n), \ldots, u_n(x, y_1, \ldots, y_n) \) are unknown functions, \( p_{ik}(x), \ i = 1, \ldots, n; \ k = 1, \ldots, n; \) and \( b_k(x), \ s = 1, \ldots, n; \) have continuous first-order partial derivatives on the set \( x_0 \leq x < +\infty, \ -\infty < y_i < +\infty, \ i = 1, \ldots, n; \)

**Theorem 1.** If for some \( \mu > 0 \) and for some positive continuous function \( \varphi(x), \int_{x_0}^{x} \varphi(s)ds \uparrow +\infty, \) on \( x \geq x_0 \) next conditions are true:

1) inequalities are fulfilled: \( p_{ik}(x) - p_{k+1,k+1}(x) \geq \mu \varphi(x), \ \mu > 0, \ k = 1, \ldots, n - 1; \) and \( x_0 \leq x < +\infty; \)

2) the following limits exist: \( \lim_{x \to +\infty} \frac{|p_{ik}(x)|}{\varphi(x)} = 0, \ i \neq k, \ i = 1, 2, \ldots, n, \ k = 1, 2, \ldots, n; \)

3) \( \lim_{x \to +\infty} \frac{1}{q(x)} \int_{x_0}^{x} p_{kk}(s)ds = \beta_k, \ k = 1, 2, \ldots, n; \) and \( \beta_1 < 0; \ q(x) = \int_{x_0}^{x} \varphi(s)ds. \) then a linear inhomogeneous system of first-order partial differential equations (1) has an integral basis, which is adjacent to the solution of a linear inhomogeneous system. Proof. System (1) corresponds to a linear homogeneous system of partial differential equations of the first order

\[
\begin{align*}
\frac{\partial u_1}{\partial x} & + \sum_{k=1}^{n} p_{1k}(x) y_k \frac{\partial u_1}{\partial y_k} + \ldots + \sum_{k=1}^{n} p_{nk}(x) y_k \frac{\partial u_1}{\partial y_n} = 0 \\
\frac{\partial u_2}{\partial x} & + \sum_{k=1}^{n} p_{2k}(x) y_k \frac{\partial u_2}{\partial y_k} + \ldots + \sum_{k=1}^{n} p_{nk}(x) y_k \frac{\partial u_2}{\partial y_n} = 0 \\
\frac{\partial u_n}{\partial x} & + \sum_{k=1}^{n} p_{nk}(x) y_k \frac{\partial u_n}{\partial y_k} + \ldots + \sum_{k=1}^{n} p_{nk}(x) y_k \frac{\partial u_n}{\partial y_n} = 0;
\end{align*}
\]

Obviously, the linear homogeneous system (2) is equivalent to a linear partial differential equation of the first order

\[
\frac{\partial u}{\partial x} + \sum_{k=1}^{n} p_{1k}(x) y_k \frac{\partial u}{\partial y_1} + \ldots + \sum_{k=1}^{n} p_{nk}(x) y_k \frac{\partial u}{\partial y_n} = 0
\]

The characteristic system of equation (3) is the following linear system

\[
\frac{dy_i}{dx} = \sum_{k=1}^{n} p_{ik}(t) y_k, \ i = 1, \ldots, n.
\]
due to the condition, the linear system (4) has \( n \) linearly independent solutions
\[
y_{1k}, y_{2k}, \ldots, y_{nk}, \ k = 1, \ldots, n;
\]
which satisfy equalities
\[
a) \lim_{x \to +\infty} \frac{y_{ik}}{y_{jk}} = 0, \ i \neq k; \ b) \lim_{x \to +\infty} \frac{1}{\varphi(x)} \frac{y'_{ik}}{y_{ik}} - \frac{p_{kk}(x)}{\varphi(x)} = 0;
\]
This implies that for any \( \varepsilon > 0 \) there exists a \( T \in I \) such that for any \( x > T, k = 1, \ldots, n; \) there are inequalities
\[
|y_{kk}(x_0)| e^{\int_{x_0}^x p_{kk}(t)dt + \varepsilon q(x)} \leq \|\bar{y}_k(x)\| \leq n|y_{kk}(x_0)| e^{\int_{x_0}^x p_{kk}(t)dt + \varepsilon q(x)}
\]
it implies
\[
\frac{1}{q(x)} \int_{x_0}^x p_{kk}(t)dt - \varepsilon \leq \frac{1}{q(x)} \ln \|\bar{y}_k(x)\| \leq \frac{1}{q(x)} \int_{x_0}^x p_{kk}(t)dt + \varepsilon;
\]
Passing to the limit as \( x \to +\infty \), by virtue of condition 3) we have
\[
\lim_{x \to +\infty} \frac{1}{q(x)} \ln \|\bar{y}_k(x)\| = \beta_k, \ k = 1, 2, \ldots, n;
\]
and besides
\[
\beta_1 > \beta_2 > \ldots > \beta_n.
\]
Therefore, \( \bar{y}_k, k = 1, 2, \ldots, n; \) forms the normal basis of the linear system (4) and \( \beta_k, k = 1, 2, \ldots, n; \) they are generalized exponents of the linear system (4) with respect to \( q(x); \) moreover, they are exact generalized exponents of the linear system (4). Due to the condition \( \beta_1 < 0 \), we find that the generalized exponents of the linear system (4) are negative. Consequently, any solution of the linear system (4) tends to zero as \( x \to +\infty \). Take the general solution of the linear system (4) of the Cauchy form
\[
\begin{align*}
y_1 &= \varphi_1(x, x_0, y_1^0, \ldots, y_n^0) \\
\cdots \\
y_n &= \varphi_n(x, x_0, y_1^0, \ldots, y_n^0)
\end{align*}
\tag{5}
\]
where the initial values \( y_1^0, \ldots, y_n^0 \) are arbitrary real numbers. There is equality
\[
\lim_{x \to +\infty} \varphi_k(x, x_0, y_1^0, \ldots, y_n^0) = 0, \ k = 1, \ldots, n
\tag{6}
\]
solving (5) with respect to \( y_1^0, \ldots, y_n^0 \), we obtain an integral basis
\[
\begin{align*}
\psi_1(x, x_0, y_1, \ldots, y_n) \\
\cdots \\
\psi_n(x, x_0, y_1, \ldots, y_n)
\end{align*}
\]
of linear equation (3). Consequently

\[
\begin{pmatrix}
\psi_1(x, x_0, y_1, \ldots, y_n) \\
0 \\
\vdots \\
0
\end{pmatrix}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
domain \|y\| = \left(\sum_{k=1}^{n} y_k^2\right)^{\frac{1}{2}} < h, \ y = colon [y_1, \ldots, y_n], \ g_i(x, 0, \ldots, 0) = 0, \ i = 1, \ldots, n;

**Theorem 2.** If for some \(\mu > 0\) and for some positive continuous function \(\varphi(x)\), \(\int_{x_0}^{x} \varphi(s)ds \uparrow +\infty\), on \(x \geq x_0\) next conditions are true:

1) inequalities are fulfilled: \(p_{k,k}(x) - p_{k+1,k+1}(x) \geq \mu \varphi(x), \ \mu > 0, \ k = 1, \ldots, n - 1;\) put \(x \geq x_0;\)

2) the following limits exist: \(\lim_{x \rightarrow +\infty} \frac{p_{kk}(x)}{\varphi(x)} = 0, \ i \neq k, \ i = 1, 2, \ldots, n, \ k = 1, 2, \ldots, n;\)

3) \(\lim_{x \rightarrow +\infty} \frac{1}{q(x)} \int_{x_0}^{x} p_{kk}(s)ds = \beta_k, \ k = 1, 2, \ldots, n;\) and \(\beta_1 < 0; q(x) = \int_{x_0}^{x} \varphi(s)ds.\)

4) \(0 < \mu < |\beta_1|, \ m > 1 \text{ and } 0 < \varepsilon < (m - 1)\mu \int_{x_0}^{x} e^{(\varepsilon + \mu (1-m))|q(s)-q(x_0)|}ds < \infty \) are fulfilled;

5) for vector function \(g(x, y) = colon[g_1(x, y_1, \ldots, y_n), \ldots, g_n(x, y_1, \ldots, y_n)]\) the inequality is true

\[
\|g(x, y)\| \leq K\|y\|^m, \ K > 0, \ m > 1;
\]

where

\[
\|g(x, y)\| = \left(\sum_{i=1}^{n} g_i^2(x, y_1, \ldots, y_n)\right)^{\frac{1}{2}};
\]

then a linear homogeneous system (8) has an integral basis, which is adjacent to zero.

Proof. Linear homogeneous system (8) is equivalent to a linear first-order partial differential equation

\[
\frac{\partial u}{\partial x} + \left(\sum_{k=1}^{n} p_{1k}(x)y_k + g_1(x, y_1, \ldots, y_n)\right)\frac{\partial u}{\partial y_1} + \ldots + \left(\sum_{k=1}^{n} p_{nk}(x)y_k + g_n(x, y_1, \ldots, y_n)\right)\frac{\partial u}{\partial y_n} = 0 \quad (9)
\]

The characteristic system of equation (9) is the following nonlinear system of differential equations.

\[
\frac{dy_k}{dx} = \sum_{k=1}^{n} p_{1k}(x)y_k + g_i(x, y_1, \ldots, y_n), \ i = 1, \ldots, n. \quad (10)
\]

The corresponding linear homogeneous system of differential equations is

\[
\frac{dy_k}{dx} = \sum_{k=1}^{n} p_{ik}(x)y_k, \ i = 1, \ldots, n. \quad (11)
\]

In the proof of Theorem 1 it was established that linear system (11) has exact generalized exponents \(\beta_k, \ k = 1, 2, \ldots, n;\) with respect to \(q(x)\). There are equality

\[
\sum_{k=1}^{n} \beta_k = \sum_{k=1}^{n} \lim_{x \rightarrow +\infty} \frac{1}{q(x)} \int_{x_0}^{x} p_{kk}(s)ds = \lim_{x \rightarrow +\infty} \frac{1}{q(x)} \int_{x_0}^{x} \sum_{k=1}^{n} p_{kk}(s)ds
\]

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Consequently, linear system (11) is a generalized regular linear system with respect to \( q(x) \). Let
\[
\frac{dy}{dx} = P(x)y
\]
(12)
is a vector matrix view of the linear system (11) and
\[
\frac{dy}{dx} = P(x)y + g(x, y)
\]
(13)
of nonlinear system (11). Further next lemma will be used: **Lemma 1.** If the conditions of Theorem 2 are satisfied, then there exists a neighborhood of the point \( y \) and any solution of the nonlinear system of differential equations (13) with initial values from this neighborhood tends to zero as \( x \to +\infty \).

Proof. Let \( \alpha > 0 \) from 4th condition, i.e. \( 0 < \alpha < \beta_1 \), where \( \beta_1 \) is senior generalized exponent with respect to \( q(x) \) of the system (12) and in the system (13) irreplace
\[
y = ue^{-\alpha[q(x)-q(x_0)]}
\]
(14)
where \( u(x) \) is new unknown function. Furthermore
\[
\frac{du}{dx} = B(x)u + \nu(x, u)
\]
(15)
where \( B(x) = P(x) + \alpha \frac{dq}{dx}E, \ \nu(x, u) = e^{\alpha[q(x)-q(x_0)]}g(x, ue^{-\alpha[q(x)-q(x_0)]}) \). Vector function \( \nu(x, u) \) is continuous on \( x \in I \) and has continuous partials on \( u \) in the domain
\[
\|u\| < \beta e^{\alpha[q(x)-q(x_0)]}
\]
Replacing in (14), due to choosing \( \alpha \), preserves the negativity of the senior generalized exponent and generalized correctness, therefore the linear system of differential equations
\[
\frac{du}{dx} = B(x)u
\]
(16)
is generalized regular and has negative generalized exponents with respect to \( q(x) \). The nonlinear system of differential equations (15) with the initial condition \( u(x_0) = y(x_0) \) is equivalent to an integral equation
\[
u(x) = H(x)u(x_0) + \int_{x_0}^{x} K(x, s)\nu(s, u)ds
\]
(17)
where \( K(x, s) = H(x)H^{-1}(s) \) is a Cauchy matrix, \( H(x) \) is a normalized fundamental matrix of the linear system (16).

Due to the negativeness of the generalized exponents of the linear system (16), there is an estimate
\[
\|H(x)\| < C_1, \ x \in I, \ C_1 > 1
\]
(18)
and for \( \varepsilon \in (0, (m-1)\alpha) \) (from condition 4), due to the negativity of the generalized exponents and regarding the generalized correctness of the linear system (16), the next estimate is true:

\[
\|K(x, s)\| < C_2e^{\alpha[q(s)-q(x_0)]}, \quad x \geq s \geq x_0, \quad C_2 > 1.
\]  

(19)

For \( \nu(x, u) \) in the nonlinear system (15) the estimate holds:

\[
\|\nu(x, u)\| \leq K_1\|u(x)\|^m e^{\alpha[1-m(q(s)-q(x_0))]}. \quad (20)
\]

Estimating now by the norm in the interval \( x_0 \leq x < x_0 + l \) of the existence of solutions of the integral equation (17), by virtue of (18), (19), (20), we will have

\[
\|u(x)\| \leq C_1\|u(x_0)\| \int_{x_0}^{x} C_2Ke^{\alpha[1-m(q(s)-q(x_0))]}\|u(x)\|^m ds. \quad (21)
\]

From inequality (21), using the Bihari Lemma ([29], p. 112), we find

\[
\|u(x)\| \leq \frac{C_1\|u(x_0)\|}{[1 - (m-1)C_1^{-1}\|u(x_0)\|^{-1}C_2Ke^{\alpha[1-m(q(s)-q(x_0))]}]^{m-1}} \quad (22)
\]

only if

\[
(m-1)C_1^{-1}\|u(x_0)\|^{-1}C_2K \int_{x_0}^{x} e^{\alpha[1-m(q(s)-q(x_0))]} ds < 1 \quad (23)
\]

Since, due to condition 4), the inequality holds:

\[
\int_{x_0}^{x} e^{\alpha[1-m(q(s)-q(x_0))]} ds < \infty;
\]

then inequality (23) can always be considered fulfilled by choosing a neighborhood of the initial values \( u(x_0) = y(x_0) \). From formula (22) it follows that if \( \|u(x_0)\| \) is sufficiently small, then for any \( x \in [x_0, x_0 + l] \) the point \( u(x) \) is the interior point of the domain

\[
Z = \left\{ x_0 \leq x < \infty, \quad \|u\| \leq \frac{h}{2} < h \right\}
\]

and, therefore, the solution \( u(x) \) is infinitely continued to the right, i.e. we can set \( l = \infty \). Thus, in the infinite interval \( x_0 \leq x < \infty \), the inequality

\[
\|u\| \leq D\|u(x_0)\| < \frac{h}{2} \quad (24)
\]

where \( D \) is some constant depending on the initial moment \( x_0 \). Returning to the variable \( y \), by virtue of formula (14), when \( x_0 \leq x < \infty \) and \( \|y(x_0)\| < \delta < h \), we will have

\[
\|y(x)\| \leq D\|y(x_0)\|e^{\alpha[q(x)-q(x_0)]} \quad (25)
\]
where the constant $\delta$ is small enough. It follows that any solution of the nonlinear system of differential equations (13) with initial values from $\|y(x_0)\| < \delta$ neighborhoods tends to zero as $x \to +\infty$.

Lemma 1 is proved. Now, by Lemma 1, we take the general solution of the characteristic system (10) of the Cauchy form.

$$
\begin{align*}
    y_1 &= y_1(x, x_0, y^0_1, \ldots, y^0_n) \\
    \vdots \\
    y_n &= y_n(x, x_0, y^0_1, \ldots, y^0_n)
\end{align*}
$$

where $y_i(x_0) = y^0_i, i = 1, \ldots, n; \sqrt{(y^0_1)^2 + \ldots + (y^0_n)^2} < \delta$. Solving (26) with respect to $y_1^0, \ldots, y_n^0$, we obtain an integral basis

$$
\begin{align*}
    u_1(x, x_0, y_1, \ldots, y_n) \\
    \vdots \\
    u_n(x, x_0, y_1, \ldots, y_n)
\end{align*}
$$

of the equation (9). Consequently

$$
\begin{pmatrix}
    u_1(x, x_0, y_1, \ldots, y_n) \\
    0 \\
    \ldots \\
    0
\end{pmatrix}
= 
\begin{pmatrix}
    0 \\
    \ldots \\
    0
\end{pmatrix}
\begin{pmatrix}
    u_1(x, x_0, y_1, \ldots, y_n) \\
    \vdots \\
    u_n(x, x_0, y_1, \ldots, y_n)
\end{pmatrix}
$$

integral basis of the linear homogeneous system (8), which is adjacent to zero in the parameter $x_0$, as $x_0 \to +\infty$. Theorem 2 is proved.

4 Results and discussion

The paper considers linear systems of first-order partial differential equations. Existing of integral basis of a linear system of partial differential equations of the first order adjoining to some solution was proved. In the proof established a sighn, which became interesting, because a nonlinear system of ordinary differential equations has a solution tending to zero. The existence of an integral basis of an adjacent to zero linear homogeneous system of first-order partial differential equations with nonlinear coefficients is proved.

5 Conclusion

Exists of the integral basis of a linear system of partial differential equations of the first order adjoining to some solution was proved. A sign at which a nonlinear system of ordinary differential equations has a solution tending to zero was established. It is proved that there exists an integral basis of an adjacent to zero linear homogeneous system of first-order partial differential equations with nonlinear coefficients.
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