On trapping surfaces in spheroidal space-times

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I. INTRODUCTION AND MOTIVATION

According to General Relativity, black holes are portions of Lorentzian manifolds characterised by the existence of an event horizon, from within which no signals can ever escape. In more general gravitating systems, the local counterpart of the event horizon is given by a trapping surface, which can be naively understood as the location where the escape velocity equals the speed of light at a given instant. If the system approaches an asymptotically static regime, the outermost trapping surface should then become the future event horizon, like it happens in the very simple Oppenheimer-Snyder model [1].

More formally, a trapping surface occurs where the expansion scalars associated with outgoing and ingoing geodesics vanishes [2–5]. In general, the expansion scalars associated with outgoing and ingoing geodesics are respectively given by

\[ \Theta_\ell = q^{\mu\nu} \nabla_\mu l_\nu , \quad \Theta_\mathbf{n} = q^{\mu\nu} \nabla_\mu n_\nu , \]  

where \( \mu, \nu = 0, \ldots, 3 \) and

\[ q_{\mu\nu} = g_{\mu\nu} + l_\mu n_\nu + n_\mu l_\nu \]  

(2)

is the metric induced by the space-time metric \( g_{\mu\nu} \) on the 2-dimensional space-like surface formed by spatial foliations of the null hypersurface generated by the outgoing tangent vector \( \ell \) and the ingoing tangent vector \( \mathbf{n} \). This 2-dimensional metric is purely spatial and has the following properties

\[ q_{\mu\nu} \ell^\mu = q_{\nu\mu} \ell^\mu = 0 , \quad q^{\mu}_{\mu} = 2 , \quad q^{\mu}_{\lambda} q^{\lambda}_{\nu} = q^{\mu}_{\nu} , \]  

where \( q^\mu_{\nu} \) represents the projection operator onto the 2-space orthogonal to \( \ell \).

Given these definitions, it is clear that the study of trapping surfaces in any realistic system is a very complex topic, and determining their existence and location is in general possible only by means of numerical methods. However, for the particular case of a spherically symmetric self-gravitating source, one can employ the gravitational radius, and the equivalent Misner-Sharp mass function. We recall that we can always write a spherically symmetric line element as

\[ ds^2 = g_{ij}(x^k) dx^i dx^j + r^2(x^k) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) , \]  

(4)

where \( x^i = (x^1, x^2) \) parametrise surfaces of constant angular coordinates \( \theta \) and \( \phi \). For the metric (1), the gradient \( \nabla_i r \) is orthogonal to surfaces of constant area \( A = 4\pi r^2 \), and one finds that the product

\[ \Theta_\ell \Theta_\mathbf{n} \propto g^{ij} \nabla_i r \nabla_j r \]  

(5)

precisely vanishes on trapped surfaces. Moreover, if we set \( x^1 = t \) and \( x^2 = r \), and denote the matter density as \( \rho = \rho(t, r) \), Einstein’s field equations yield the solution

\[ g^{rr} = 1 - \frac{r_H(t, r)}{r} , \]  

(6)

where

\[ r_H(t, r) = 2 m(t, r) \]  

(7)

is the gravitational radius determined by the Misner-Sharp mass function

\[ m(t, r) = 4\pi \int_0^r \rho(t, \tilde{r}) \tilde{r}^2 d\tilde{r} . \]  

(8)

According to Eq. (5), a trapping surface then exists where \( g^{rr} = 0 \), or where the gravitational radius satisfies

\[ r_H(t, r) = r , \]  

(9)

for \( r > 0 \). If the source is surrounded by the vacuum, the Misner-Sharp mass asymptotically approaches the Arnowitt-Deser-Misner (ADM) mass of the source, \( m(t, r \to \infty) = M \), and the gravitational radius likewise becomes the Schwarzschild radius \( R_H = 2M \). To summarise, the relevant properties of the Misner-Sharp

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1 We shall mostly use units with \( G_N = c = 1 \).
mass \([i]\) it only depends on the source energy density and \(ii\) it allows one to locate the (time-dependent) trapping surfaces via Eq. \([7]\).

In quantum physics, the energy density that defines the Misner-Sharp mass \(m\) (and ADM mass \(M\)) becomes a quantum observable and one expects the gravitational radius to admit a similar description. The horizon quantum mechanics (HQM) was in fact proposed \([7]\) in order to describe the “fuzzy” Schwarzschild (or gravitational) radius of a localised quantum source, by essentially lifting Eq. \([7]\) to a quantum constraint acting on the state vectors of matter and the gravitational radius. In this respect, the HQM differs from most other attempts in which the gravitational degrees of freedom of the horizon, or of the black hole metric, are instead quantised independently of the state of the source. It however follows that, in order to extend the HQM to non-spherical systems, we need to identify a mass function from which the location of trapping surfaces of constant \(r = r_0\) can be uniquely determined and which depends only on the state of the matter source, like the Misner-Sharp mass \([8]\) for isotropic sources. The latter property is crucial in a perspective in which one would eventually like to recover the geometric properties of space-times from the quantum state of the whole matter-gravity system.

Since we are interested in generalising the above quantum description to non-spherical sources, in this work we shall first try and generalise the classical analysis of trapping surfaces to systems with a slightly spheroidal symmetry. Moreover, since it is hardly possible to describe analytically such systems if they evolve in time, we shall consider static configurations as simple case studies. In particular, we shall deform a static and spherically symmetric space-time, and study the location of trapping surfaces perturbatively in the deformation parameter. In this respect, it is worth stressing that the assumption of staticity will ultimately lead to matter distributions which break some of the energy conditions. The cases presented here are therefore only intended to serve as toy models, whose purpose is to shed some light on the possible relation between these small perturbations and a mass function. Consequently, the development of a more precise analysis for dynamical horizons is left for future studies.

Explicit expressions will be given for the deformed de Sitter space-time. We shall also consider the case of a spheroidal space-time which contains a source whose energy and pressure depart from such a symmetry. In both cases, we will see that the location of trapping surfaces is given by surfaces of symmetry, and can therefore be determined by computing the Misner-Sharp mass on the reference unperturbed (spherically symmetric) space-time. The results of this analysis will serve in order to establish the adapted quantization rules for the HQM of such systems, but the whole quantum extension will be described in other publications (for some preliminary results, see Ref. \([8]\)).

**II. STATIC SPHEROIDAL SOURCES**

In this section, we will investigate how the particular description for static spherically symmetric systems extends to the case in which the symmetry is associated with (slightly) spheroidal surfaces. We start from the spherically symmetric metric \([4]\), with \(r\) the areal radius constant on the 2-spheres of symmetry, and assume the time-dependence is negligible. Einstein equations then yield the solution \([6]\), in which the now time-independent Misner-Sharp mass \(m = m(r)\) is determined by a static density \(\rho = \rho(r)\) according to Eq. \([8]\). We will always assume that the matter source also contains a (isotropic) pressure term, such that the Tolman-Oppenheimer-Volkov equation of hydrostatic equilibrium is satisfied \([9]\). We then change to (prolate or oblate) spheroidal coordinates and consider a localised source of spheroidal radius \(r = r_0\), say with mass \(M_0\), surrounded by a fluid with the energy density \(\rho = \rho(r)\).

The central source only serves the purpose to avoid discussing coordinate singularities at \(r = 0\). In the interesting portion of space \(r > r_0\), we assume the metric \(g_{\mu\nu}\) is of the form

\[
\text{ds}^2 = -h(r, \theta; a) \, dr^2 + \frac{1}{h(r, \theta; a)} \left( \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \right) \, dr^2 + (r^2 + a^2 \sin^2 \theta) \, d\theta^2 + (r^2 + a^2) \sin^2 \theta \, d\phi^2 ,
\]

where \(h = h(r, \theta; a)\) is a function to be determined. Surfaces of constant \(r\) now represent ellipsoids of revolution, or spheroids, on which the density is constant. For

FIG. 1. Spheroids: prolate spheroid with \(a^2 > 0\) (in yellow) compared to oblate spheroid with \(a^2 < 0\) (in red) and to the reference sphere \(a^2 = 0\) (in green).
\( a^2 > 0 \), the above metric can describe the space-time outside a prolate spheroidal source, which extends more along the axis of symmetry than on the equatorial plane (see yellow surface in Fig. 1). In order to describe an oblate source, which is flatter along the axis (see red surface in Fig. 1), we can simply consider the mapping \( a \to i a \) (so that \( a^2 \to -a^2 \)). It is also important to remark that a space-time equipped with the metric \( G_{\mu\nu} \) admits two trivial Killing vectors, namely \( \partial_r \) and \( \partial_\theta \). Furthermore, it is also easy to see that the vanishing of \( g_{00} = -h(r, \theta; a) \) determines the location of the Killing horizons for spacetimes belonging to this class.²

For consistency, the energy-momentum tensor \( T_{\mu\nu} \) of the source can be inferred from the Einstein equations,

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8 \pi T_{\mu\nu} ,
\]

where \( G_{\mu\nu} \) is the Einstein tensor, \( R_{\mu\nu} \) the Ricci tensor and \( R \) the Ricci scalar. However, we are only interested in ensuring that the energy density is spheroidally symmetric, that is

\[
G^0_0 = -8 \pi \rho (r) ,
\]

and we will therefore assume the necessary pressure terms are present in order to maintain equilibrium. In order to solve Eq. (12), we change variable from the azimuthal angle \( \theta \) to \( x = \cos \theta \), after which the line element reads

\[
ds^2 = -h(r, x; a) dt^2 + \frac{1}{h(r, x; a)} \left( \frac{r^2 + a^2 x^2}{r^2 + a^2} \right) dr^2 + \frac{r^2 + a^2 x^2}{1 - x^2} dx^2 + (r^2 + a^2) (1 - x^2) d\theta^2 .
\]

Given the symmetry of the system, we can restrict the analysis to the upper half spatial volume \( 1 \geq x \geq 0 \) forgoing to \( 0 \leq \theta \leq \pi/2 \). We then find

\[
G_0^0 = \frac{1}{4 (r^2 + a^2 x^2)^3} h^2 \left\{ 4 \left[ r^4 + a^2 r^2 (4 x^2 - 1) + a^4 x^2 \right] h^3 + 3 (1 - x^2) (r^2 + a^2 x^2)^2 (\partial_r h)^2 
+ 2 (r^2 + a^2 x^2) h \left[ x (2 r^2 + a^2 \{ 3 x^2 - 1 \}) \partial_r h + (x^2 - 1) (r^2 + a^2 x^2) \partial_r^2 h \right] 
+ 2 h^2 \left[ 2 \left( a^2 r^2 \{ 1 - 4 x^2 \} - a^4 x^2 \{ 1 + x^2 \} - r^4 \right) + (r^2 + a^2 x^2) (2 r^2 + a^2 \{ 1 + x^2 \}) \partial_r h \right] \right\} ,
\]

so that Eq. (12) appears to be a rather convoluted differential equation for the unknown \( h = h(r, x; a) \).

We proceed by considering small departures from spherical symmetry, parameterised by \( a^2 \ll r_0^2 \), and expand all expressions up to order \( a^2 \). In particular, the energy density must have the form

\[
\rho \simeq \rho_{(0)}(r) + a^2 \rho_{(2)}(r) ,
\]

whereas the unknown metric function

\[
h \simeq h_{(00)}(r) + a^2 \left[ h_{(20)}(r) + x^2 h_{(22)}(r) \right] ,
\]

\[
\simeq 1 - \frac{2 m_{(00)}(r)}{r} - 2 a^2 x m_{(20)}(r) + x^2 m_{(22)}(r) ,
\]

where we introduced a Misner-Sharp mass function \( m_{(00)} \), like in Eq. (8), for the zero order term and correlative terms \( m_{(2i)} \) at order \( a^2 \), with \( i = 0,2 \). In fact, at zero order, Eq. (12) reads

\[
G_{(0)}^0 = -\frac{2 m_{(00)}(r)}{r^2} = -8 \pi \rho_{(0)}(r) ,
\]

with primes denoting derivatives with respect to \( r \). The solution \( m_{(00)} \) is correctly given by the relation (8).

At first order in \( a^2 \), the component of the Einstein tensor in Eq. (14) contains two terms,

\[
G_{(2)}^0 = F(r) + x^2 L(r) ,
\]

where \( F(r) \) and \( L(r) \) do not depend on \( x \). Since \( \rho \) does not depend on \( x \) by construction, we must have \( L(r) = 0 \), which yields

\[
m_{(22)} + \left( 1 - \frac{2 m_{(00)}}{r} \right)^{-1} \frac{3 m_{(22)}}{r} 
- \frac{3}{2 r^2} \left( m_{(00)}' - \frac{5 m_{(00)}}{3 r} \right) = 0 .
\]

Finally, we are left with

\[
F(r) = -\frac{2 m_{(20)}}{r^2} - \frac{m_{(00)}}{r^4} + \frac{3 m_{(00)}}{r^2}
+ \frac{2 m_{(22)}}{r^3} \left( 1 - \frac{2 m_{(00)}}{r} \right)^{-1}
= -8 \pi \rho_{(2)} ,
\]

in which \( m_{(00)} \) is determined by Eq. (17) and \( m_{(22)} \) by Eq. (19), respectively. Eq. (20) can then be used to determine \( m_{(20)} \).

Once the metric function \( h = h(r, x; a) \) is obtained, one can determine the locations of trapping surfaces from the expansions of null geodesics defined in Eq. (1). It will then be interesting to compare the result with the solutions of the generalised Eq. (7), namely

\[
2 m(r_H, x; a) = r_{H}(x) ,
\]

² In a more general, time-dependent space-time, no such Killing structure would of course exist.
where
\[ m(r, x; a) \simeq m_{(00)}(r) + a^2 \left[ m_{(20)}(r) + x^2 m_{(22)}(r) \right], \quad (22) \]
is now the extended Misner-Sharp mass. We also note that Eq. \((21)\) is equivalent to
\[ h(r_H, x; a) = 0, \quad (23) \]
which will be checked below with a specific example. We can just anticipate that we expect the location of the trapping surface respects the spheroidal symmetry of the system and is thus given by the spheroidal deformation of the isotropic horizon obtained for \(a \to 0\).

### III. SLIGHTLY SPHEROIDAL DE SITTER

In order to proceed and find more explicit results, we shall now apply the above general construction to the specific example of the spheroidally deformed de Sitter metric.

Like in the previous general treatment, we start by assuming the presence of an inner core of radius \(r = r_0\) and mass \(M_0\), which is here surrounded by a fluid with energy density
\[ \rho(r) = \rho_0(r) = \frac{\alpha^2}{4\pi r}, \quad (24) \]
where \(r > r_0\), and \(\alpha\) is a positive constant independent of \(a\) (so that \(\rho_2 = 0\)). From Eq. \((17)\), we obtain
\[ m_{(00)} = M_0 + \frac{\alpha^2 (r^2 - r_0^2)}{2}, \quad (25) \]
which of course holds for \(r > r_0\). We further set \(\alpha^2 r_0^2 \simeq M_0\), so that
\[ m_{(00)}(r) \simeq \frac{\alpha^2 r^2}{2}. \quad (26) \]

This case admits a trapping surface when \(2 m_{(00)}(r) = r\), that is
\[ r_H = \alpha^{-2}, \quad (27) \]
which is just the usual horizon for the isotropic de Sitter space.

Next, Eq. \((19)\) reads
\[ m_1^{(22)} + \frac{3 m_{(22)}}{1 - \alpha^2 r} - \frac{\alpha^2}{4 r} = 0, \quad (28) \]
and admits the general solution
\[ m_2^{(22)} = \frac{3 m_{(22)}}{1 - \alpha^2 r} + \frac{\alpha^2}{4 r}, \quad (29) \]
for \(r \simeq \alpha^{-2}\). After substituting \(m_{(00)}\), \(m_{(20)}\) and \(m_{(22)}\) into Eq. \((16)\), we have
\[ h(r, x; a) \simeq \left( 1 - \alpha^2 r \right) - \frac{\alpha^2}{4 r} \left[ 3 \log(\alpha^2 r) + x^2 (1 - \alpha^2 r) \right]. \quad (34) \]
The condition \((23)\) then admits two separate solutions, namely
\[ r_H^{(1)} \simeq \alpha^{-2}, \quad (35a) \]
\[ r_H^{(2)}(x) \simeq \frac{\alpha^2}{3 \alpha^2 \alpha^2 - 8} \simeq \frac{\alpha^2}{2} \frac{2 x^2 - 9}{8}, \quad (35b) \]
the latter of which is clearly negative for \(\alpha^2 a \ll 1\) (since \(0 \leq x^2 \leq 1\)). Therefore, we expect there exists a horizon, whose location \(r_h^{(1)} \simeq r_H\) is given exactly by the original (spherically symmetric) solution \((27)\) for the unperturbed space-time. This expectation will have to be confirmed from the study of null geodesics, but we should also add that this calculation does not imply uniqueness and more trapping surfaces could in principle develop.
A. Trapping surfaces

Since the $g_{xx}$ component of the metric (13) is not well defined at $x = 1$ ($\theta = 0$), it will be more convenient to work with the metric in the form given originally in Eq. (10). In particular, the function $h(r, \theta)$ is obtained from Eq. (34), and reads

$$h(r, \theta; a) \simeq (1 - \alpha^2 r) + \frac{\alpha^2 + \alpha^2}{4 r} \left[ 3 \ln(\alpha^2 r) + \cos^2 \theta (1 - \alpha^2 r) \right].$$

(36)

The Lagrangian $2 \mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ for a point particle moving on this space-time can be written as

$$2 \mathcal{L} = -h(r, \theta; a) \dot{t}^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \dot{\rho}^2 + \left( r^2 + a^2 \cos^2 \theta \right) \dot{\phi}^2 + \left( r^2 + a^2 \sin^2 \theta \right) \dot{\phi}^2 .$$

(37)

where a dot represents the derivative with respect to the parameter $\lambda$ along the trajectories. Since $t$ and $\phi$ are cyclic variables, one has the conserved conjugate momenta

$$p_t = -h(r, \theta; a) \dot{t} = -E,$$

(38a)

$$p_\phi = (r^2 + a^2) \sin^2 \theta \dot{\phi} = J,$$

(38b)

where $E$ and $J$ are constants of motion. In particular, one can always set $\dot{\phi} = J = 0$, but for purely radial geodesics to exist, the equation of motion for $\theta = \theta(\lambda)$

$$2 h^2 \left[ (r^2 + a^2 \cos^2 \theta) \ddot{\theta} + 2 r \dot{r} \dot{\theta} - a^2 \cos \theta \sin \theta \ddot{\theta}^2 \right]
= \left[ 2 a^2 h \cos \theta \sin \theta + (r^2 + a^2 \cos^2 \theta) \partial_\theta h \right]
\times \left( h \ddot{\theta}^2 - \frac{E^2}{r^2 + a^2 \cos^2 \theta} \right) + E^2 \partial_\theta h ,$$

(39)

must admit solutions with $\theta(\lambda) = \theta_0$ and constant. It is clear that $\theta = 0$ and $\theta = \pi$ (corresponding to a motion along the polar lines) and $\theta = \pi/2$ (motion on the equatorial plane) are allowed. We shall then see in the next two subsections that trapping surfaces again occur where $h = 0$ in these particular cases. We shall afterwards check that the condition $h = 0$ in Eq. (23) holds on the trapping surface and determines the radius (27)

for all angles $0 \leq \theta \leq \pi$.

1. Trapping surface on the equatorial plane

The radial null geodesics on the equatorial plane can be determined directly by setting

$$\theta = \pi/2 , \quad \dot{\theta} = J = 0 ,$$

(40)

so that $2 \mathcal{L} = 0$ reads

$$2 \mathcal{L} = \frac{1}{h(r, \pi/2; a)} \left( \frac{r^2 \dot{r}^2}{r^2 + a^2} - E^2 \right) = 0 ,$$

(41)

or

$$\frac{\dot{r}}{E} = \pm \sqrt{\frac{r^2 + a^2}{r}} ,$$

(42)

in which

$$h(r, \pi/2; a) \simeq (1 - a^2 r) - 3 a^2 \alpha^2 \ln(\alpha^2 r) .$$

(43)

From Eqs. (38a) and (42), we can write the 4-vectors respectively tangent to outgoing and ingoing null trajectories as

$$\ell = \frac{1}{2} \partial_t + h(r, \pi/2; a) \frac{\sqrt{r^2 + a^2}}{2 r} \partial_r ,$$

(44a)

$$n = \frac{1}{h(r, \pi/2; a)} \partial_t - \frac{\sqrt{r^2 + a^2}}{r} \partial_r ,$$

(44b)

where we multiplied the outgoing null vector by $h(r, \pi/2; a)/2$ in order to satisfy the normalisation condition $\ell^\mu n_\mu = -1$. Since $\ell^\mu$ and $n^\mu$ are zero for $\mu = 2$ and 3, the induced metric (2) has $q_{22} = g_{22}$ and $q_{33} = g_{33}$. Using Eqs. (44), the outgoing and ingoing expansion scalar are given by

$$\Theta_\ell = \frac{r^2 + a^2/2}{r^2 \sqrt{r^2 + a^2}} h(r, \pi/2; a) = -\frac{\Theta_n}{2} h(r, \pi/2; a) ,$$

(45a)

$$\Theta_n = -\frac{2 r^2 + a^2}{r^2 \sqrt{r^2 + a^2}} = -\frac{2 \Theta_\ell}{h(r, \pi/2; a)} ,$$

(45b)

with $h(r, \pi/2; a)$ given in Eq. (43). The location of the trapping surfaces is now given by $\Theta_\ell = 0$, which, from Eq. (45a) gives $h(r, \pi/2; a) = 0$. This condition is precisely the same as Eq. (23) for $x = 0$, to wit

$$4 r (1 - \alpha^2 r) = 3 a^2 \alpha^2 \ln(\alpha^2 r) ,$$

(46)

which can be simplified by expanding the logarithmic term for $r \simeq 1/\alpha^2 = r_H$, and yields the two solutions (27a) and (27b). Finally, we should note that, according to our construction, there cannot be other points along these geodesics where $\Theta_\ell = 0$, and the above trapping surface is therefore unique on the equatorial plane.

2. Trapping surface at the pole

In order to determine the location of the trapping surface at $\theta = 0$, we need to study the radial null geodesics that fall along the pole line $\theta = 0$. However, setting $\theta = 0$ from the start would lead to spurious singularities in the relevant expressions. We must instead expand all relevant quantities for small $\theta$ and take the limit $\theta \to 0$ at the end of the calculation. The components of the metric tensor (10) with $h(r, \theta; a)$ given in Eq. (36) are

$$g_{00} = -h(r, \theta; a) \simeq -h_0(r) - h_1(r) \theta^2 ,$$

(47a)

$$g_{11} \simeq \frac{1}{h_0(r)} + h_2(r) \theta^2 ,$$

(47b)

$$g_{22} \simeq r^2 + a^2 (1 - \theta^2) ,$$

(47c)

$$g_{33} \simeq (r^2 + a^2) \theta^2 ,$$

(47d)
where

\[ h_0(r) = (1 - a^2 r) \left( 1 - \frac{a^2 \alpha^2}{4 r} \right) - \frac{3 a^2 \alpha^2 \ln(\alpha^2 r)}{4 r} \]  \tag{48a}

\[ h_1(r) = \frac{a^2 \alpha^2}{4 r} (1 - a^2 r) , \]  \tag{48b}

\[ h_2(r) = \frac{a^2 (4 + a^2 r)}{4 r^2 (1 - a^2 r)} . \]  \tag{48c}

Again, following the same procedure of the previous subsection, we note that the radial geodesic \( \dot{\theta} = 0 \) is allowed for \( \theta = 0 \), and we can write the outgoing null 4-vector

\[ \ell = \frac{1}{2} \partial_t + \frac{1}{2} \sqrt{\frac{h(r, \theta; a)}{g_{11}(r, \theta)}} \partial_r , \]  \tag{49}

and the ingoing null 4-vector

\[ n = \frac{1}{h(r, \theta; a)} \partial_t - \frac{1}{\sqrt{h(r, \theta; a) g_{11}(r, \theta)}} \partial_r , \]  \tag{50}

which satisfy the normalization condition \( \ell^\mu n_\mu = -1 \). The non-vanishing components of the induced 2-dimensional metric are \( g_{22} = g_{22} = r^2 + a^2 (1 - \theta^2) \) and \( g_{33} = g_{33} = (r^2 + a^2) \theta^2 \), and the expansion scalars then become

\[ \Theta_\ell = \frac{r}{2} \sqrt{\frac{h(r, \theta; a)}{g_{11}(r, \theta)}} \left[ \frac{1}{r^2 + a^2 (1 - \theta^2)} + \frac{1}{r^2 + a^2} \right] \]

\[ = -\frac{\Theta_n}{2} h(r, \theta; a). \]  \tag{51}

It is already clear that, for \( \theta \approx 0 \), the location of the trapping surface where \( \Theta_\ell = 0 \) will be given by Eq. \( 23 \). In fact, after substituting the expressions of \( h(r, \theta; a) \) and \( g_{11}(r, \theta) \) in the above, the limit \( \theta \to 0 \) yields the expansion scalars

\[ \Theta_\ell = \frac{r}{r^2 + a^2} h_0(r) = \frac{r}{r^2 + a^2} h(r, 0) , \]  \tag{52a}

\[ \Theta_n = -\frac{2 r}{r^2 + a^2} . \]  \tag{52b}

The location of the trapping surface is therefore the same \( r = r_H^{(1)} \approx r_H \) of Eq. \( 35a \).

3. General trapping surface

In order to locate the trapping surface for a generic angle \( \theta \), let us denote with \( \theta_0 \) the azimuthal angle of the point where the null geodesic crosses the trapping surface and assume \( \dot{\theta}_0 = 0 = \dot{\phi} \). At this point, Eq. \( 39 \) takes the simple form

\[ 2 \left( r^2 + a^2 \cos^2 \theta \right) \partial_\theta h_a + 2 h_a a^2 \sin \theta \cos \theta = 0 , \]  \tag{53}

where, from Eq. \( 36 \), we have

\[ \partial_\theta h_a \simeq a^2 \alpha^2 \frac{1}{2 r} (1 - a^2 r) \sin \theta \cos \theta . \]  \tag{54}

Moreover, the Lagrangian \( 37 \) takes the form

\[ 2 \mathcal{L} = \frac{1}{h(r, \theta_0; a)} \left( \frac{r^2 + a^2 \cos^2 \theta_0}{r^2 + a^2} r^2 - E^2 \right) = 0 , \]  \tag{55}

where we used the equation of motion \( 38a \). From Eq. \( 55 \), we then obtain

\[ \frac{\dot{r}}{E} = \pm \frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta_0} , \]  \tag{56}

By using the Eqs. \( 38a \) and \( 56 \), we can write the two normalised null tangent vectors as

\[ \ell = \frac{1}{2} \partial_t + \frac{h(r, \theta_0; a)}{\sqrt{r^2 + a^2 \cos^2 \theta_0}} \partial_r , \]  \tag{57a}

\[ n = \frac{1}{h(r, \theta_0; a)} \partial_t - \sqrt{r^2 + a^2 \cos^2 \theta_0} \partial_r , \]  \tag{57b}

and their expansions are given by

\[ \Theta_\ell = \frac{r (2 r^2 + a^2 + a^2 \cos^2 \theta_0)}{2 \sqrt{(r^2 + a^2)(r^2 + a^2 \cos^2 \theta_0)}} h(r, \theta_0; a) \]

\[ = -\frac{\Theta_n}{2} h(r, \theta_0; a) . \]  \tag{58}

We again find that \( \Theta_\ell = 0 \) where Eq. \( 23 \) holds and that, in turn, Eq. \( 53 \) is satisfied on the trapping surface, as we expected. Again, no other point exists where \( \Theta_\ell = 0 \), and this trapping surface is unique along the poles line as well.

B. Misner-Sharp mass

In the example considered in this section, we have found two results: fist of all, the location of the trapping surface is given by the same value of the radial coordinate as for the isotropic case. In particular, we have seen that

\[ \Theta_\ell = -\frac{\Theta_n}{2} h(r, \theta; a) , \]  \tag{59}

for all angles \( \theta \); the second result is that \( h(r, \theta; a) = 0 \) where the spherically symmetric \( h(r, \theta; a = 0) \) is 0.

Putting the two results together, we then find that

\[ 2 m(r_H, \theta; a) = 2 m(r_H) = r_H , \]  \tag{60}

where \( m(r) = m(r, \theta; a = 0) \). We can therefore conjecture that the relevant mass function for determining the location of trapping surfaces in (slightly) spheroidal systems is given by the Misner-Sharp mass computed according to Eq. \( 8 \) on the reference isotropic space-time. This conjecture is somewhat reminiscent of the property of the original Misner-Sharp mass that it is given by the volume integral over the flat reference space.
IV. A NON-SPHEROIDAL SOURCE

In this section, we want to consider the more complex case of a localised source of spheroidal radius $r = r_0$, with mass $M_0$ and charge $Q$, surrounded by its static electric field, with energy-momentum tensor $T_{\mu\nu}^{(Q)}$, and a suitable (electrically neutral) fluid. We are again not interested in the inner structure of the central source, but only in the portion of space for $r > r_0$, where we assume the metric is of the form given in Ref. [10], that is Eq. (10) with

$$h(r,\theta) = h(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2},$$

(61)

where $M$ is the total ADM mass.

It is clear that the deformation parameter $a$ now measures deviations from the spherically symmetric Reissner-Nordström metric. In this respect, it is worth stressing that such a deformation should be regarded as a simple example of a (most likely) unstable intermediate configuration within the framework of a dynamical gravitational collapse.

One can easily compute the corresponding energy-momentum tensor $T_{\mu\nu}$ by means of the Einstein equations [11], from which one can see that it splits into two separate contributions, respectively proportional to the charge $Q^2$ and the mass $M$,

$$T_{\mu\nu} = M T_{\mu\nu}^{(M)} + Q^2 T_{\mu\nu}^{(Q)},$$

(62)

which will be analysed separately. For the part of the energy-momentum tensor associated to $M$, we consider an anisotropic fluid form,

$$M T_{\mu\nu}^{(M)} = \left(\rho + p\right) u_{\mu} u_{\nu} + p \delta_{\mu\nu} + \Pi_{\mu\nu},$$

(63)

where $\rho$ is the energy density, $p$ the radial pressure, $u_{\mu}$ the time-like 4-velocity of the fluid and $\Pi_{\mu\nu}$ the traceless pressure tensor orthogonal to $u_{\mu}$,

$$\Pi_{\mu\nu} = \Pi_{\mu\nu} u^\nu = 0.$$

(64)

Since the system is static, we can take

$$u = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1/2} \partial_\tau.$$

(65)

In particular, the (only) relevant component of the energy-momentum tensor, as far as our argument is concerned, reads

$$T_0^0 = -\frac{\alpha^2 M \left(\cos^2 \theta - 3\right) \left(r^2 + a^2 \cos^2 \theta\right) + 4 \left(r^2 + a^2\right) \cos^2 \theta}{8\pi r \left(r^2 + a^2 \cos^2 \theta\right)^3} - \frac{Q^2}{8\pi \left(r^2 + a^2 \cos^2 \theta\right)^3} \frac{2 \left(r^2 + a^2\right) - r^2 - a^2 \cos^2 \theta}{2 \left(r^2 + a^2\right) - r^2 - a^2 \cos^2 \theta},$$

(66)

from which one can easily see that the electrostatic contribution is constant on spheroids of constant $r$, whereas the contribution proportional to $M$ is not. The electrostatic contribution falls within the treatment of the previous sections, and we are here particularly interested in analysing the effects of the latter.

A. Trapping surfaces

In this section we will study the radial null geodesics for the metric defined by Eq. [10] with the metric function [61], which can be obtained from the Euler-Lagrange equations for the Lagrangian

$$2 \mathcal{L} = -h(r) \dot{r}^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \frac{\dot{\theta}^2}{h(r)} + \left(r^2 + a^2 \cos^2 \theta\right) \dot{\phi}^2 + \left(r^2 + a^2\right) \sin^2 \theta \phi^2,$$

(70)

where $h = h(r)$ is given in Eq. [61]. Since $t$ and $\phi$ are cyclic variables for the Lagrangian [70], we still have the
conserved momenta

\[
p_t = - \left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right) \dot{t} = -E, \quad (71a)
\]
\[
p_\phi = (r^2 + a^2) \sin^2 \theta \dot{\phi} = J, \quad (71b)
\]
where \( E \) and \( J \) are constants.

Whether the space-time at hand admits radial geodesics can then be determined from the dynamical equations for \( \theta \) with \( J = 0 \),

\[
(r^2 + a^2 \cos^2 \theta) \ddot{\theta} = -2r \dot{r} \dot{\theta} - \frac{a^2 \sin \theta \cos \theta}{r^2 + a^2} \frac{\dot{r}^2}{h(r)} + a^2 \sin \theta \cos \theta \dot{\theta}^2, \quad (72)
\]

From this equation, it straightforward to see that purely radial (null) geodesics exist on the equatorial plane \((\theta = \pi/2)\) and along the poles \((\theta = 0 \text{ and } \theta = \pi)\).

These are the only cases we shall consider explicitly for the metric \((61)\).

1. **Trapping surface on the equatorial plane**

We wish to determine the location of the trapping surface on the equatorial plane by means of the expansion scalars. For \( \theta = \pi/2 \) and \( J = 0 \), the Lagrangian given in Eq. \((70)\) can be rewritten as

\[
2\mathcal{L} = \frac{1}{h(r)} \left( \frac{r^2 \dot{\theta}^2}{r^2 + a^2} - E^2 \right) = 0, \quad (73)
\]

which yields

\[
\frac{\dot{r}}{E} = \pm \sqrt{\frac{r^2 + a^2}{r}}, \quad (74)
\]

and we note that, by rescaling the parameter \( \lambda \), we can always set \( E = 1 \).

We next consider the out-going null 4-vector

\[
\ell = \frac{1}{2} \partial_t + h(r) \frac{\sqrt{r^2 + a^2}}{2r} \partial_r, \quad (75a)
\]

and the in-going null 4-vector

\[
n = \frac{1}{h(r)} \partial_t - \frac{\sqrt{r^2 + a^2}}{r} \partial_r, \quad (75b)
\]

which satisfy the normalization condition \( \ell^\mu n_\mu = -1 \).

Using the expression of the expansion scalars given in Eqs. \(1\), we have

\[
\Theta_\ell = -\frac{2r^2 + a^2}{2r^2 \sqrt{r^2 + a^2}} h(r), \quad (76a)
\]
\[
\Theta_n = -\frac{2r^2 + a^2}{r^2 \sqrt{r^2 + a^2}}. \quad (76b)
\]

In particular, we find that \( \Theta_\ell = 0 \) where \( h(r) = 0 \), and we recover the Reissner-Nordström horizons \((67)\), although \( r \) is no more the areal radius for \( a^2 \neq 0 \).

2. **Trapping surface at the poles**

Using the approach discussed in Section \(\text{III A 2}\) for the spheroidal de Sitter case, we compute the metric components for small \( \theta \) and take the limit \( \theta \rightarrow 0 \) at the end. In particular, we now have

\[
g_{\theta \theta} = -h(r), \quad (77a)
\]
\[
g_{11} = \frac{1}{h(r)} \left( \frac{r^2 + a^2(1 - \theta^2)}{r^2 + a^2} \right), \quad (77b)
\]
\[
g_{22} = r^2 + a^2(1 - \theta^2), \quad (77c)
\]
\[
g_{33} = (r^2 + a^2) \theta^2. \quad (77d)
\]

Then, considering radial null geodesics along the pole lines, the Lagrangian simplifies to

\[
2\mathcal{L} = \frac{1}{h(r)} \left[ \frac{r^2 + a^2(1 - \theta^2)}{r^2 + a^2} \right] \dot{r}^2 - E^2 = 0, \quad (78)
\]

which yields

\[
\frac{\dot{r}}{E} = \pm \sqrt{\frac{r^2 + a^2}{r^2 + a^2(1 - \theta^2)}}, \quad (79)
\]

and again we can set \( E = 1 \).

The relevant normalized null 4-vectors for \( \theta \approx 0 \) are then given by

\[
\ell = \frac{1}{2} \partial_t + \frac{h(r)}{2} \sqrt{\frac{r^2 + a^2}{r^2 + a^2(1 - \theta^2)}} \partial_r, \quad (80a)
\]
\[
n = \frac{1}{h(r)} \partial_t - \sqrt{\frac{r^2 + a^2}{r^2 + a^2(1 - \theta^2)}} \partial_r, \quad (80b)
\]

and the expansion scalars, in the limit \( \theta \rightarrow 0 \), read

\[
\Theta_\ell = \frac{r}{r^2 + a^2} h(r), \quad (81)
\]
\[
\Theta_n = -\frac{2r}{r^2 + a^2}. \quad (82)
\]

The expression of \( \Theta_\ell \) shows that the locations of the trapping surfaces are again given by the same condition that holds on the equatorial plane, and reduce to the isotropic solutions \((67)\), although \( r \) is not the areal radius.

B. **Misner-Sharp and ADM mass**

We should finally recall that the Misner-Sharp mass for the isotropic Reissner-Nordström space-time is given by (see Appendix A)

\[
m(r) \simeq M - \frac{Q^2}{2r}, \quad (83)
\]

and the condition \( h(r_{\pm}) = 0 \) that yields the horizons \((67)\) can indeed be written in the form of Eq. \((7)\), that is
2m(r±) = r±. This means that the results of the above analysis for the metric do not really differ from those for the de Sitter space-time in Section III and the isotropic Misner-Sharp mass remains a precious indicator of the location of horizons. In this perspective, it actually appears just like an accident that the asymptotic ADM mass computed for the isotropic reference space-time (obtained by setting \(a^2 = 0\)) also determines the location of the horizons.

The conjecture that the isotropic Misner-Sharp mass determines the location of slightly spheroidal horizons nonetheless remains somewhat surprising, if one considers that the above isotropic Misner-Sharp mass \(m = m(r)\) does not coincide with the Misner-Sharp mass adapted to the surfaces of symmetry of the spheroidal geometry. The latter is also computed in Appendix A where we show that it coincides with the Hawking quasi-local mass for the system.

V. CONCLUSIONS AND OUTLOOK

We have considered small spheroidal deformations of static isotropic systems and studied how the trapping surface is correspondingly deformed. Our main motivation for this investigation is to generalise the HQM beyond the spherical symmetry, for which we need a way to locate the horizon from quantities solely determined by the quantum state of the source. By analysing a purely spheroidal system in section III we conjectured that such a quantity is given by the isotropic Misner-Sharp mass, obtained by simply taking the deformation parameter to zero. More details about the quantum description are given in Ref. 8, where the formalism is described for the spheroidal de Sitter space of Section III. In this work, we have also considered a spheroidal deformation of the Reissner-Nordström metric in section IV for which a similar result is found for the location of the trapping surface, thus further supporting the main conjecture.

It is finally important to remark again that, despite the classical instability of the last example, it is still possible that such a configuration appears as an intermediate step during the collapse that leads to the formation of a black hole. In any case, one should not a priori exclude that it has a non-vanishing probability to be realised at the quantum level (described by the HQM). In fact, we recall that the quantum description is the main reason of our interest in this kind of (small) spheroidal deformations.

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Appendix A: Mass functions

We here compute the Hawking mass for the metric of Section V and show that it coincides with a Misner-Sharp mass adapted to the spheroidal symmetry.

1. Hawking(-Hayward) mass

The general Hawking-Hayward mass [12] is defined as the surface integral

\[
\hat{M} = \frac{A^{1/2}}{32 \pi^{3/2}} \int_{\Sigma} \mu \left[ R + \Theta_+ \Theta_- - \frac{\sigma^+_{\alpha\beta} \sigma^-_{\alpha\beta}}{2} - 2 \omega_{\alpha} \omega^{\alpha} \right]
\]

where \(R\) denotes the induced Ricci scalar on the 2-surface \(\Sigma\), \(\Theta_{(\pm)}\) denote the expansion scalars and shear tensors of a pair of outgoing and ingoing null geodesic congruences from the surface \(\Sigma\), respectively, \(\omega_{\alpha}\) is the projection onto \(\Sigma\) of the commutator of the null normal vectors to \(\Sigma\), \(\mu\) is the volume 2-form on \(\Sigma\) and \(A\) the area of \(\Sigma\). For the metric (61), one immediately finds that \(\omega_{\alpha}\) is of order \(a^2\) and, since we are considering all expressions only up to order \(a^2\), the last term can be dropped. The Hawking-Hayward mass then reduces to the Hawking mass, which we are now going to determine.

According to the general contracted Gauss equation [13]

\[
R + \Theta_+ \Theta_- - \frac{1}{2} \sigma^+_{\alpha\beta} \sigma^-_{\alpha\beta} + h^{\alpha\gamma} h^{\beta\delta} R_{\alpha\beta\gamma\delta},
\]

where \(h^{\alpha\gamma}\) is the induced metric on the 2-surface \(\Sigma\) and \(R_{\alpha\beta\gamma\delta}\) is the Riemann tensor. The tensor \(h^{\alpha\gamma}\) can be written as

\[
h^{\alpha\gamma} = g^{\alpha\gamma} + l^\alpha n^\gamma + l^\gamma n^\alpha,
\]

where \(g^{\alpha\gamma}\) is the inverse of metric tensor \(g_{\alpha\gamma}\), \(l^\alpha\) and \(n^\alpha\) are null vectors. On expanding in powers of \(a^2\),

\[
n^\alpha h^{\alpha\gamma} + a^2 n^\alpha h^{\alpha\gamma} \approx g^{\alpha\gamma} + a^2 g^{\alpha\gamma} + (l^\alpha + a^2 l^\alpha)(n^\gamma + a^2 n^\gamma)
\]

we obtain

\[
h^{\alpha\gamma} h^{\beta\delta} R_{\alpha\beta\gamma\delta} \approx h_0^{\alpha\gamma} h_0^{\beta\delta} R_{\alpha\beta\gamma\delta}(0) + a^2 h_0^{\alpha\gamma} h_0^{\beta\delta} R_{\alpha\beta\gamma\delta}(1) + a^2 \left[h_1^{\alpha\gamma} h_0^{\beta\delta} + h_0^{\alpha\gamma} h_1^{\beta\delta}\right] R_{\alpha\beta\gamma\delta}(0)
\]

The components of \(h^{\alpha\gamma}\) are determined from Eqs. (A3) and (A4a), which gives

\[
h_0^{\alpha\gamma} + a^2 h_1^{\alpha\gamma} \approx g^{\alpha\gamma} + a^2 g^{\alpha\gamma} + (l^\alpha + a^2 l^\alpha)(n^\gamma + a^2 n^\gamma) + (l_0^\alpha + a^2 l_1^\alpha)(n_0^\gamma + a^2 n_1^\gamma),
\]

where \(g_0^{\alpha\gamma}\) and \(g_1^{\alpha\gamma}\) are zeroth and first order terms of the metric tensor respectively, and similarly for \(l_0^\alpha, l_1^\alpha\) and \(n_0^\alpha, n_1^\alpha\). For the unperturbed Reissner-Nordström space-time
In particular, for the metric (61), we find
\[ h_{00}^2 + a^2 h_{11}^2 = g_{00}^2 + a^2 q_{12}^2 + 2 a^4 h_{11}^2 n_1 \approx g_{00}^2 + a^2 q_{12}^2, \]  
(7a)
\[ h_{03}^3 + a^2 h_{33}^3 = g_{03}^3 + a^2 q_{33}^3 + 2 a^4 h_{33}^3 n_3 \approx g_{03}^3 + a^2 q_{33}^3. \]  
(7b)
In particular, for the metric (61), we find
\[ h_{22} \approx \frac{1}{r^2} - a^2 \frac{\cos^2 \theta}{r^4}. \]  
(8)
Similarly,
\[ h_{33} \approx \frac{1}{r^2 \sin^2 \theta} - \frac{a^2}{r^4 \sin^2 \theta}. \]  
(9)
Eq. (A5) now reduces to
\[ h^{\alpha \gamma} h^{\beta \delta} R_{\alpha \beta \gamma \delta} \approx 2 \left[ h_{00}^2 h_{33} R_{2323(0)} + a^2 h_{00}^2 h_{33} R_{2323(1)} + a^2 \left( h_{01}^2 h_{33}^2 + h_{33}^2 h_{01}^2 \right) R_{2323(0)} \right] \]  
(A10)
where
\[ R_{2323} = \frac{\sin^2 \theta (r^2 + a^2)(2Mr - q^2)}{r^2 + a^2 \cos^2 \theta} \approx (2Mr - q^2) \sin^2 \theta + 2a^2 M \sin^4 \theta \frac{M}{r}. \]  
(A11)
By using equations (A8)(A11), the contracted Gauss equation (A2) takes the form
\[ h^{\alpha \gamma} h^{\beta \delta} R_{\alpha \beta \gamma \delta} = \mathcal{R} + \theta_+ \theta_- - \frac{1}{2} \sigma^+ \sigma^- = \frac{2 (2Mr - q^2)}{r^4} - \frac{8a^2 M \cos^2 \theta}{r^5}. \]  
(A12)
Since \( h_{22} = g_{22} = (r^2 + a^2 \cos^2 \theta) \) and \( h_{33} = g_{33} = (r^2 + a^2) \sin^2 \theta \), the volume 2-form
\[ \mu = \sqrt{\det(h_{\alpha \beta})} d\theta d\phi \approx \left( r^2 + a^2 \frac{3 + \cos 2\theta}{4} \right) \sin \theta d\theta d\phi. \]  
(13)
The area of \( \Sigma \) is then given by
\[ \mathcal{A} = \int_{\Sigma} \mu \approx 4 \pi \left( r^2 + \frac{2}{3} a^2 \right). \]  
(A14)
The Hawking mass is finally obtained from Eq. (A1) with Eqs. (12) and (A14), which yields
\[ M_H = M \left( 1 + \frac{a^2}{3r^2} \right) - \frac{q^2}{2r}. \]  
(15)
We are next going to recover this result in a different way.

2. Adapted Misner-Sharp mass

The Misner-Sharp mass (8) is properly defined only for spherically symmetric space-times. One could generalise it by integrating the matter density on the spatial volume inside surfaces of symmetry, which are given by spheroids in the present case. In other words, we replace Eq. (8) with
\[ m(r) = M_0 + 2\pi \int_0^r \int_0^\pi \sqrt{\gamma(\rho, \theta)} \rho(\rho, \theta) d\rho d\theta, \]  
(A16)
where we recall that \( r = r_0 \) is the coordinate of the inner core, and \( \gamma = (r^2 + a^2 \cos^2 \theta)^2 \sin^2 \theta \) is the determinant of the flat 3-metric in spheroidal coordinates,
\[ \gamma_{ij} dx^i dx^j = \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2. \]  
(A17)
Eq. (A16) then yields
\[ m(r) \approx M_0 - \frac{Q^2}{2} \left( \frac{1}{r} - \frac{1}{r_0} \right) + \frac{a^2 M}{3} \left( \frac{1}{r^2} - \frac{1}{r_0^2} \right), \]  
(A18)
For \( r \to \infty \), the above expression should equal the total ADM mass \( M \), that is
\[ M \approx M_0 - \frac{Ma^2}{3r_0^2} + \frac{Q^2}{2r_0}. \]  
(A19)
This allows us to express \( r_0 \) and \( M_0 \) so that Eq. (A18) becomes
\[ m(r) \approx M \left( 1 + \frac{a^2}{3r^2} \right) - \frac{Q^2}{2r} = M_H(r), \]  
(A20)
from which the isotropic Misner-Sharp mass (83) is obtained by taking \( a^2 \to 0 \). This calculations therefore shows that, at least for spheroidal space-times like (61), one can expect the Hawking mass function evaluated on surfaces of symmetry equals the adapted Misner-Sharp function evaluated inside volumes bounded by the same surfaces of symmetry.

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