On a 3D model of non-isothermal flows in a pipeline network

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Abstract. In this work, we propose a new mathematical model to describe non-isothermal steady flows of a fluid with temperature-dependent viscosity in a 3D pipeline network. Our approach is based on the renouncement of the averaging procedure for the velocity and temperature fields and the use of the conjugation conditions that express the mass and energy balance for interior joints of the network. We give weak formulation of the problem. The main result is an existence theorem (in the class of weak solutions) for small data.

1. Introduction and problem formulation

In contemporary technological processes to transfer liquids and gases are widely used pipeline networks of complex geometry, including main pipeline networks. Many works are devoted to problems of mathematical modeling of the transportation of liquids and gases (see, e.g., [1–5] and the literature cited there). Nevertheless, the mathematical theory of heat and mass transfer in pipeline networks is in the formative stage and is represented only by fragmentary results.

The current problems in this direction include the development and analysis of multi-dimensional mathematical models describing non-isothermal flows of a viscous fluid in pipeline networks, taking into account the conjugation conditions in pipe joints. One of such models is proposed in this note. Namely, we consider a model that describes the heat and mass transfer in a netlike domain $\tilde{\Omega} = \Omega \cup \omega$:

$$\Omega = \bigcup_{i=1}^{N} \Omega_i, \quad \omega = \bigcup_{j=1}^{M} \omega_j,$$

where $\Omega_i$ and $\omega_j$ are locally Lipschitz bounded domains in three-dimensional space $\mathbb{R}^3$ such that

$$\Omega_i \cap \Omega_k = \emptyset, \quad \omega_j \cap \omega_l = \emptyset, \quad \Omega_i \cap \omega_j = \emptyset,$$

for any $i, k \in \{1, 2, \ldots, N\}, i \neq k$, and $j, l \in \{1, 2, \ldots, M\}, j \neq l$.

Note that the domain $\tilde{\Omega}$ can be considered as a network of pipes: here $\Omega_1, \ldots, \Omega_N$ model pipes, while $\omega_1, \ldots, \omega_M$ represent joints in which pipes are connected. The shape of pipes and joints can be very diverse.

Let us assume that for each joint $\omega_j$ there exist exactly $m_j$ domains $\Omega_{j_1}, \Omega_{j_2}, \ldots, \Omega_{j_{m_j}}$, $1 \leq j_1 < j_2 < \ldots < j_{m_j} \leq N$, $1 \leq m_j \leq N$, such that

$$\omega_j \cap \Omega_{j_k} \neq \emptyset \quad \forall k \in \{1, 2, \ldots, m_j\}.$$
If \( m_j \geq 2 \), then we shall say that the joint \( \omega_j \) is interior; in the case \( m_j = 1 \), the joint \( \omega_j \) is called external.

By \( S_{jn} \) we denote the intersection \( \bar{\omega}_j \cap \bar{\omega}_{jn} \). Of course, \( S_{jn} \subset \partial \Omega_{jn} \). Let suppose that

\[
\text{meas}_2(S_{jn}) > 0 \quad \forall j \in \{1, 2, \ldots, M\}, \; n \in \{1, 2, \ldots, m_j\},
\]

where \( \text{meas}_2(\cdot) \) indicates the Lebesgue 2-dimensional measure of a set.

In addition, we assume that for each domain \( \Omega_i \) there exist exactly two joints \( \omega_{i1} \) and \( \omega_{i2} \) such that

\[
\bar{\Omega}_i \cap \bar{\omega}_{i1} \neq \emptyset, \quad \bar{\Omega}_i \cap \bar{\omega}_{i2} \neq \emptyset.
\]

It is clear that for each \( i \in \{1, 2, \ldots, N\} \) there exists a uniquely determined pair \( (i'_1, i'_2) \) such that

\[
\bar{\Omega}_i \cap \bar{\omega}_{i1} = S_{i1i'_1}, \quad \bar{\Omega}_i \cap \bar{\omega}_{i2} = S_{i2i'_2}.
\]

Set

\[
\Gamma_i := \partial \Omega_i \setminus (S_{i1i'_1} \cup S_{i2i'_2})
\]

and assume that \( \text{meas}_2(\Gamma_i) > 0 \) for each \( i \in \{1, 2, \ldots, N\} \).

By definition, put

\[
S := \bigcup_{j=1}^{M} \bigcup_{n=1}^{m_j} S_{jn}, \quad \Gamma := \bigcup_{i=1}^{N} \Gamma_i.
\]

Finally, by \( \mathbf{n} = \mathbf{n}(\mathbf{x}) \) we denote the outer (with respect to \( \Omega_i \)) unit normal to \( \partial \Omega_i \) at a point \( \mathbf{x} \in \partial \Omega_i \).

Consider a stationary mathematical model of non-isothermal incompressible flows in the domain \( \Omega \):

\[
\sum_{k=1}^{3} u_k \frac{\partial \mathbf{u}}{\partial x_k} - \text{div} \, \mathbf{T} = \mathbf{f}(\mathbf{x}, \theta) \quad \text{in} \; \Omega,
\]

\[
\mathbf{T} = \mu(\theta)\mathbf{D}(\mathbf{u}) - p \mathbf{I} \quad \text{in} \; \Omega,
\]

\[
\text{div} \, \mathbf{u} = 0 \quad \text{in} \; \Omega,
\]

\[
\sum_{k=1}^{3} u_k \frac{\partial \theta}{\partial x_k} - \kappa \Delta \theta = \varphi(\mathbf{x}, \theta) \quad \text{in} \; \Omega,
\]

\[
\mathbf{u} = \mathbf{0} \quad \text{on} \; \Gamma,
\]

\[
\kappa \frac{\partial \theta}{\partial \mathbf{n}} = -\beta \theta \quad \text{on} \; \Gamma_i \quad \forall \; i \in \{1, 2, \ldots, N\},
\]

\[
\mathbf{u}_r = \mathbf{0} \quad \text{on} \; S,
\]

\[
|\mathbf{u}|^2/2 - (\mathbf{T} \mathbf{n}) \cdot \mathbf{n} = h_i \quad \text{on} \; S_{i1i'_1} \cup S_{i2i'_2} \quad \forall \; i \in \{1, 2, \ldots, N\},
\]

\[
\kappa \frac{\partial \theta}{\partial \mathbf{n}} = -\psi_i \quad \text{on} \; S_{i1i'_1} \cup S_{i2i'_2} \quad \forall \; i \in \{1, 2, \ldots, N\},
\]

\[
\sum_{k=1}^{m_j} \int_{S_{jk}} \mathbf{u} \cdot \mathbf{n} \, d\sigma = 0 \quad \forall \; j \in \{1, 2, \ldots, M\} \quad \text{such that} \; m_j \geq 2,
\]

\[
\sum_{k=1}^{m_j} \int_{S_{jk}} \kappa \frac{\partial \theta}{\partial \mathbf{n}} \, d\sigma = 0 \quad \forall \; j \in \{1, 2, \ldots, M\} \quad \text{such that} \; m_j \geq 2.
\]

Let us briefly describe the various terms in this problem statement: \( \mathbf{u} = (u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x})) \) is the velocity of a particle of the fluid at a point \( \mathbf{x} \in \bar{\Omega} \), \( \mathbf{T} = (T_{ij}(\mathbf{x})) \) is the Cauchy stress tensor,
\[ p = p(x) \] stand for the pressure function, \( \theta = \theta(x) \) is the deviation from the average temperature value, \( f(x, \theta(x)) \) is the external force, \( D(u) \) designates the strain velocity tensor,
\[
D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T),
\]
\( \mu(\theta) > 0 \) represents the viscosity of the fluid, \( \kappa > 0 \) is the thermal conductivity, \( \varphi(x, \theta(x)) \) denotes the power of the heat source, \( \beta_i > 0 \) is the Robin coefficient characterizing heat transfer on \( \Gamma_i \). Functions
\[
h_i: S_{i1} \cup S_{i2} \rightarrow \mathcal{R}, \quad \psi_i: S_{i1} \cup S_{i2} \rightarrow \mathcal{R}
\]
describe, respectively, the head (the dynamic pressure) and the heat fluxes on the surface \( S_{i1} \cup S_{i2} \). Here and in the succeeding discussion, the symbol \( \mathcal{R} \) denotes the set of real numbers. By \( (\cdot)_\tau \) we denote the tangential component of a vector, i.e., \( v_\tau := v - (v \cdot n)n \). The term \( d\sigma \) indicates an element of surface area on \( S \).

Note that equation (1.1) is the motion equation in the Cauchy form, (1.2) represents Newton’s constitutive relation with temperature-dependent viscosity, (1.3) is the incompressibility condition, (1.4) is the heat conduction equation with a heat source \( \varphi \), equality (1.5) is the no-slip boundary condition on walls of pipes, (1.6) is Newton’s law of cooling. Boundary conditions (1.7) and (1.8) give a partial description of the flowing through components of \( S \), whereas by (1.9) heat fluxes are specified on these surfaces. Finally, relations (1.10) and (1.11) are the conjugation conditions, which represent the mass and energy balance for interior joints of the network.

In system (1.1)–(1.11), the unknowns are \( u, \theta, T \) and \( p \), while all other quantities are assumed to be given.

This paper is mainly concerned with the question of the solvability of boundary value problem (1.1)–(1.11) within the framework of weak solutions.

**Remark 1.** Another approach to modeling flows in netlike domains is proposed in [6, 7], where the authors develop an idea, advanced in their works for the case of \( n = 1 \) (for problems with distributed parameters on a graph), in the direction of the dimension increase \( n \).

**Remark 2.** The Stokes and Navier–Stokes equations with boundary conditions involving the pressure (cf. condition (1.8)) are carefully examined in the seminal work [8]; see also the recent papers [9, 10].

### 2. Notations, main assumptions and weak formulation of the problem

For vectors \( a, b \in \mathcal{R}^3 \) and matrices \( A, B \in \mathcal{R}^{3 \times 3} \), by \( a \cdot b \) and \( A : B \) we denote the scalar products, respectively,
\[
a \cdot b := \sum_{i=1}^{3} a_i b_i, \quad A : B := \sum_{i,j=1}^{3} A_{ij} B_{ij}.
\]

We use Lebesgue spaces \( L^q(U) \), where \( U \) is a bounded domain in \( \mathcal{R}^3 \) and \( q \geq 1 \), with the norm
\[
\|v\|_{L^q(U)} := \left( \int_U |v|^q \, dx \right)^{1/q}
\]
and the Sobolev space
\[
H^1(U) := W^{1,2}(U)
\]
with the norm
\[
\|v\|_{H^1(U)} := \left( \int_U |v|^2 \, dx + \sum_{i=1}^{3} \int_U |\partial_x v|^2 \, dx \right)^{1/2}:
\]
see [11] for definitions of these spaces and systematic descriptions of their properties. To shorten notations, we shall denote the corresponding spaces of vector-valued functions by bold face letters, e.g.,

\[ L^2(U) := L^2(U)^3, \quad H^1(U) := H^1(U)^3. \]

By definition, put

\[ \mathcal{X} := \{ \mathbf{u} : \overline{\Omega} \rightarrow \mathbb{R}^3 : \mathbf{u}|_{\Omega_i} \in C^\infty(\Omega_i) \text{ for each } i = 1, 2, \ldots, N, \mathbf{u} \text{ satisfies (1.3), (1.5), (1.7), (1.10)} \}. \]

We introduce the space

\[ \mathbf{X} := \text{the closure of } \mathcal{X} \text{ in the Sobolev space } H^1(\Omega) \]

with the scalar product

\[ (\mathbf{u}, \mathbf{v})_X := \sum_{i=1}^N \int_{\Omega_i} D(\mathbf{u}) : D(\mathbf{v}) \, dx. \]

Note that the norm \( \| \cdot \|_X = (\cdot, \cdot)_X^{1/2} \) is equivalent to the standard \( H^1 \)-norm. This follows from Korn’s inequality (see the Appendix, Proposition 3).

Moreover, let us introduce the Hilbert space

\[ \mathbf{Y} := \{ \theta : \overline{\Omega} \rightarrow \mathcal{R} : \theta|_{\Omega_i} \in H^1(\Omega_i) \text{ for each } i = 1, 2, \ldots, N \} \]

with the scalar product

\[ (\theta, \eta)_Y := \sum_{i=1}^N \int_{\Omega_i} \nabla \theta \cdot \nabla \eta \, dx + \sum_{i=1}^N \int_{\Gamma_i} \theta \eta \, d\sigma. \]

Since \( \text{meas}_2(\Gamma_i) > 0 \) for each \( i \in \{1, 2, \ldots, N\} \), it can be proved that the norm \( \| \cdot \|_Y = (\cdot, \cdot)_Y^{1/2} \) is equivalent to the norm \( \| \cdot \|_{H^1} \).

We shall suppose that

(i) the function \( \mu : \mathcal{R} \rightarrow \mathcal{R} \) is continuous and there exist constants \( \mathcal{M}_0, \mathcal{M}_1 \) such that

\[ 0 < \mathcal{M}_0 \leq \mu(y) \leq \mathcal{M}_1 \quad \forall y \in \mathcal{R}; \]

(ii) the functions \( f(\cdot, y) : \Omega \rightarrow \mathcal{R} \) and \( \varphi(\cdot, y) : \Omega \rightarrow \mathcal{R} \) are measurable for every \( y \in \mathcal{R} \);

(iii) the functions \( f(x, \cdot) : \mathcal{R} \rightarrow \mathcal{R} \) and \( \varphi(x, \cdot) : \mathcal{R} \rightarrow \mathcal{R} \) are continuous for almost every \( x \in \Omega \);

(iv) there exist constants \( F_0 \) and \( \Phi_0 \) such that

\[ |f(x, y)| \leq F_0, \quad |\varphi(x, y)| \leq \Phi_0. \]

for every \( y \in \mathcal{R} \) and for almost every \( x \in \Omega \);

(v) for each \( i \in \{1, 2, \ldots, N\} \) the functions \( h_i \) and \( \psi_i \) belong to the space \( L^2(S_{i1} \cup S_{i2}) \);

(vi) for each \( j \in \{1, 2, \ldots, M\} \) such that \( m_j \geq 2 \) the following equality is valid

\[ \sum_{k=1}^{m_j} \int_{S_{jk}} \psi_{jkj} \, d\sigma = 0. \]
**Definition.** We shall say that a pair \((u, \theta) \in X \times Y\) is a **weak solution** of boundary value problem (1.1)–(1.11) if the following equalities hold for any test functions \(v \in L^2\) hold for any test functions \(v \in \mathbb{X}\) and \(\eta \in Y\).

**Remark 3.** Equalities (2.1) and (2.2) are natural in the definition of weak solutions. Here the line of reasoning is as follows. Assume that \((u, \theta) \in X \times Y\) satisfies the following energy equalities:

\[
\begin{align*}
&-\sum_{i=1}^{N} \sum_{k=1}^{3} \int_{\Omega} u_{i} \cdot \frac{\partial v}{\partial x_{k}} \, dx + \sum_{i=1}^{N} \int_{\Omega} \mu(\theta)D(u) : D(v) \, dx \\
&+ \sum_{i=1}^{N} \int_{\Omega} \sum_{k=1}^{3} \left( h_{i} + |u|^{2}/2 \right) (v \cdot n) \, d\sigma = \sum_{i=1}^{N} \int_{\Omega} f(x, \theta) \cdot v \, dx, \tag{2.1}
\end{align*}
\]

\[
\begin{align*}
&\sum_{i=1}^{N} \sum_{k=1}^{3} \left( h_{i} + |u|^{2}/2 \right) \eta \, d\sigma + \sum_{i=1}^{N} \int_{\Omega} k\nabla \theta \cdot \nabla \eta \, dx \\
&+ \sum_{i=1}^{N} \int_{\Gamma_{i}} \beta_{i} \theta \, d\sigma + \sum_{i=1}^{N} \int_{\Omega} \psi_{i} \eta \, d\sigma = \sum_{i=1}^{N} \int_{\Omega} \varphi(x, \theta) \eta \, dx \tag{2.2}
\end{align*}
\]

hold for any test functions \(v \in \mathbb{X}\) and \(\eta \in Y\).

**3. Main results of the work**

Our main results read as follows.

**Theorem.** Assume that conditions (i)–(vi) hold. Then

(a) there exists a positive constant \(K(\Omega)\) such that if

\[
\frac{\sum_{i=1}^{N} \|h_{i}\|_{L^{2}(S_{1i_{1}' \cup S_{2i}'})} + F_{0} \sqrt{\text{meas}_{3}(\Omega)}}{M_{0} \min\{\beta_{1}, \ldots, \beta_{N}, \kappa\}} < K(\Omega), \tag{3.1}
\]

then problem (1.1)–(1.11) has at least one weak solution;

(b) any weak solution \((u, \theta) \in X \times Y\) of problem (1.1)–(1.11) satisfies the following energy equalities:

\[
\begin{align*}
&\sum_{i=1}^{N} \int_{\Omega_{i}} \mu(\theta)|D(u)|^{2} \, dx + \sum_{i=1}^{N} \sum_{\Omega_{1i_{1}' \cup S_{2i}'}} h_{i} u \cdot n \, d\sigma = \sum_{i=1}^{N} \int_{\Omega_{i}} f(x, \theta) \cdot u \, dx, \tag{3.2}
\end{align*}
\]

\[
\begin{align*}
&\sum_{i=1}^{N} \int_{\Omega_{i}} k|\nabla \theta|^{2} \, dx + \sum_{i=1}^{N} \sum_{\Omega_{1i_{1}' \cup S_{2i}'}} \left( \frac{|\theta|^{2}}{2} u \cdot n + \psi_{i} \theta \right) \, d\sigma \\
&+ \sum_{i=1}^{N} \int_{\Gamma_{i}} \beta_{i} |\theta|^{2} \, d\sigma = \sum_{i=1}^{N} \int_{\Omega_{i}} \varphi(x, \theta) \theta \, dx \tag{3.3}
\end{align*}
\]

(c) the set of weak solutions to problem (1.1)–(1.11) is sequentially weakly closed in the space \(X \times Y\).
To prove this theorem we use the Galerkin procedure and topological degree arguments. The proof is based on the energy estimates of approximate solutions in various function spaces, the Krasnoselskii theorem on the continuity of the superposition operator as well as compactness theorems for the imbedding and trace operators, allowing to extract from the sequence of approximate solutions a subsequence converging to a weak solution of problem (1.1)–(1.11).

4. Sketch of the proof of the Theorem

The proof of the existence result (a) is performed in three steps.

Step 1. The Galerkin approximation. Let \( \{v_j\}^\infty_{j=1} \subset X \) be an orthonormal basis of the space \( X \) and let \( \{\eta_j\}^\infty_{j=1} \subset C^\infty(\overline{\Omega}) \) be an orthonormal basis of the space \( Y \). For arbitrary fixed integer \( m \) we consider an auxiliary problem:

Find a pair of functions \((u^m, \theta^m)\) such that

\[
-\xi \sum_{i=1}^{N} \sum_{k=1}^{3} \int_{\Omega} u^m_i \partial v^j_i \partial x_k \, dx + \sum_{i=1}^{N} \int_{\Omega} \mu(\xi \theta^m) D(u^m) : D(v^j) \, dx
\]

\[+ \xi \sum_{i=1}^{N} \int_{S_{1i1'} \cup S_{2i2'}} (h_i + |u^m|^2/2)(\psi_j \cdot n) \, d\sigma = \xi \sum_{i=1}^{N} \int_{\Omega_i} f(x, \theta^m) \cdot v^j \, dx, \quad j = 1, \ldots, m, \quad (4.1)\]

\[
\sum_{i=1}^{N} \int_{\Gamma_i} \beta_i \theta^m \eta_j \, d\sigma + \xi \sum_{i=1}^{N} \int_{S_{1i1'} \cup S_{2i2'}} \psi_i \eta_j \, d\sigma \quad \text{and} \quad \sum_{i=1}^{N} \int_{\Gamma_i} \beta_i \theta^m \eta_j \, d\sigma + \xi \sum_{i=1}^{N} \int_{S_{1i1'} \cup S_{2i2'}} \psi_i \eta_j \, d\sigma, \quad j = 1, \ldots, m, \quad (4.2)\]

\[
u^m = \sum_{i=1}^{m} a_{mi} v^i, \quad \theta^m = \sum_{i=1}^{m} b_{mi} \theta^i, \quad (4.3)\]

where \( a_{m1}, \ldots, a_{mm} \) and \( b_{m1}, \ldots, b_{mm} \) are unknown numbers, \( \xi \) is a parameter, and \( \xi \in [0, 1] \).

Step 2. A priori estimates for the Galerkin solutions. Assume that a pair \((u^m, \theta^m)\) satisfies (4.1)–(4.3) for some \( \xi \in [0, 1] \). Then the following estimates can be derived from (4.1)–(4.3):

\[
M_0 \sum_{i=1}^{N} \int_{\Omega_i} |D(u^m)|^2 \, dx \leq F_0 \sum_{i=1}^{N} \int_{\Omega_i} |u^m| \, dx + \sum_{i=1}^{N} \int_{S_{1i1'} \cup S_{2i2'}} |h_i| |u^m| \, d\sigma,
\]

\[
\kappa \sum_{i=1}^{N} \int_{\Omega_i} |\nabla \theta^m|^2 \, dx + \sum_{i=1}^{N} \int_{\Gamma_i} |\theta^m|^2 \, d\sigma \leq \sum_{i=1}^{N} \int_{S_{1i1'} \cup S_{2i2'}} |u^m| |\theta^m|^2 \, d\sigma + \sum_{i=1}^{N} \int_{S_{1i1'} \cup S_{2i2'}} |\psi_i| |\theta^m| \, d\sigma + \Phi_0 \sum_{i=1}^{N} \int_{\Omega_i} |\theta^m| \, dx.
\]

Using these relations and the Hölder inequality, it can be shown that there exists a positive constant \( K(\overline{\Omega}) \) such that if (3.1) holds, then

\[
\| (u^m, \theta^m) \|_{X \times Y} \leq C \quad (4.4)
\]

with a constant \( C \) that is independent of \( m \) and \( \xi \).

An application of Proposition 1 (see the Appendix) yields that problem (4.1)–(4.3) is solvable for each \( m \in \{1, 2, \ldots\} \) and any \( \xi \in [0, 1] \).
Step 3. Passing to the limit. Let \((\tilde{u}^m, \tilde{\theta}^m)\) be a solution to \((4.1)-(4.3)\) with \(\xi = 1\), i.e.,

\[
- \sum_{i=1}^{N} \sum_{k=1}^{3} \int_{\Omega_i} \tilde{u}^m \frac{\partial v^i}{\partial x_k} \, dx + \sum_{i=1}^{N} \int_{\Omega_i} \mu(\tilde{\theta}^m) D(\tilde{u}^m) : D(\nabla v^i) \, dx \\
+ \sum_{i=1}^{N} \int_{S_{1i'} \cup S_{2i'}^2} (h_i + |\tilde{u}^m|^2/2)(\nabla v^i \cdot \mathbf{n}) \, d\sigma = \sum_{i=1}^{N} \int_{\Omega_i} f(\mathbf{x}, \tilde{\theta}^m) \cdot v^i \, dx, \quad j = 1, \ldots, m,
\]

\[(4.5)\]

In view of \((4.4)\), of course, we have

\[
\|(\tilde{u}^m, \tilde{\theta}^m)\|_{X \times Y} \leq C.
\]

Therefore, without loss of generality, we can assume that

\[
\tilde{u}^m \to u^0 \text{ weakly in } X \text{ as } m \to \infty,
\]

\[(4.7)\]

\[
\tilde{\theta}^m \to \theta^0 \text{ weakly in } Y \text{ as } m \to \infty,
\]

\[(4.8)\]

for some pair \((u^0, \theta^0) \in X \times Y\). Moreover, by theorems on the compactness of the imbedding and trace operators (see, e.g., [12, Chap. 2, Sect. 2.6, Theorems 6.1 and 6.2]), we obtain

\[
\tilde{u}^m \to u^0 \text{ strongly in } L^4(\Omega) \text{ as } m \to \infty,
\]

\[(4.9)\]

\[
\tilde{\theta}^m \to \theta^0 \text{ strongly in } L^4(\Omega) \text{ as } m \to \infty,
\]

\[(4.10)\]

\[
\tilde{u}^m|_{S_{1i'} \cup S_{2i'}^2} \to u^0|_{S_{1i'} \cup S_{2i'}^2} \text{ strongly in } L^2(S_{1i'} \cup S_{2i'}^2) \text{ as } m \to \infty,
\]

\[(4.11)\]

\[
\tilde{\theta}^m|_{\Gamma_i} \to \theta^0|_{\Gamma_i} \text{ strongly in } L^2(\Gamma_i) \text{ as } m \to \infty,
\]

\[(4.12)\]

for each \(i \in \{1, 2, \ldots, N\}\).

Using the Krasnoselskii theorem on the continuity of Nemytskii’s operator (see the Appendix, Proposition 2) and the convergences \((4.7)-(4.12)\), we pass to the limit \(m \to \infty\) in equalities \((4.5), (4.6)\) and obtain:

\[
- \sum_{i=1}^{N} \sum_{k=1}^{3} \int_{\Omega_i} u^0 \frac{\partial v^i}{\partial x_k} \, dx + \sum_{i=1}^{N} \int_{\Omega_i} \mu(\theta^0) D(u^0) : D(v^i) \, dx \\
+ \sum_{i=1}^{N} \int_{S_{1i'} \cup S_{2i'}^2} (h_i + |u^0|^2/2)(v^i \cdot \mathbf{n}) \, d\sigma = \sum_{i=1}^{N} \int_{\Omega_i} f(\mathbf{x}, \theta^0) \cdot v^i \, dx, \quad j = 1, \ldots, m,
\]

\[(4.13)\]

\[
\sum_{i=1}^{N} \sum_{k=1}^{3} \int_{\Omega_i} u^0 \frac{\partial \theta^0}{\partial x_k} \eta^j \, dx + \sum_{i=1}^{N} \int_{\Omega_i} \kappa \nabla \theta^0 \cdot \nabla \eta^j \, dx \\
+ \sum_{i=1}^{N} \int_{\Gamma_i} \beta \theta^0 \eta^j \, d\sigma + \sum_{i=1}^{N} \int_{S_{1i'} \cup S_{2i'}^2} \psi_i \eta^j \, d\sigma = \sum_{i=1}^{N} \int_{\Omega_i} \varphi(\mathbf{x}, \theta^0) \eta^j \, dx,
\]

\[(4.14)\]
for any \( j \in \{1, 2, \ldots\} \). Since \( \{v^j\}_{j=1}^{\infty} \) is a basis of \( X \) and \( \{\eta^j\}_{j=1}^{\infty} \) is a basis of \( Y \), we see that equalities (4.13) and (4.14) remain valid if we replace \( v^j \) and \( \eta^j \) with arbitrary functions \( v \in X \) and \( \eta \in Y \), respectively. This means that \( (u^0, \theta^0) \) is a weak solution of (1.1)–(1.11) and hence statement (a) is proved.

Setting \( v = u \) in (2.1) and \( \eta = \theta \) in (2.2), by integration by parts we arrive at (3.2) and (3.3).

Finally, by using the passage-to-limit procedure as above, it can be proved that the set of weak solutions to problem (1.1)–(1.11) is sequentially weakly closed in the space \( X \times Y \).

5. Appendix

For the convenience of readers, we provide three statements used in the previous sections of this paper.

**Proposition 1.** Let \( B_r = \{ x \in \mathbb{R}^n : |x| \leq r \} \) be a closed ball. Suppose that the continuous mapping \( F: B_r \times [0, 1] \to \mathbb{R}^n \) satisfies the following conditions:

- \( F(x, \xi) \neq 0 \) for any \( (x, \xi) \in \partial B_r \times [0, 1] \),
- \( F(x, 0) = A x \) for any \( x \in B_r \), where \( A: \mathbb{R}^n \to \mathbb{R}^n \) is an isomorphism;

then for any \( \xi \in [0, 1] \) the equation \( F(x, \xi) = 0 \) has at least one solution \( x_\xi \in B_r \).

This proposition can be proved by methods of the theory of topological degree (see [13, 14]).

**Proposition 2** (Krasnoselskii’s theorem [13, Chap. I, §2]). Let \( G \subset \mathbb{R}^n \) be a (Lebesgue) measurable set and let \( f: G \times \mathbb{R} \to \mathbb{R} \) be a function such that

- the function \( f(\cdot, y): G \to \mathbb{R} \) is measurable for every \( y \in \mathbb{R} \);
- the function \( f(x, \cdot): \mathbb{R} \to \mathbb{R} \) is continuous for almost every \( x \in G \);
- there exist constants \( p_1 \geq 1, p_2 \geq 1, \beta > 0 \) and a function \( \alpha \in L^{p_2}(G) \) such that the following inequality holds

\[
|f(x, y)| \leq \alpha(x) + \beta |y|^{p_1/p_2},
\]

for every \( y \in \mathbb{R} \) and for almost every \( x \in G \).

Then the Nemytskii operator

\[
N_f: L^{p_1}(G) \to L^{p_2}(G), \quad N_f[w](x) := f(x, w(x))
\]

is a bounded and continuous map.

**Proposition 3** (Korn’s inequality). Let us assume that \( U \) is a bounded domain in space \( \mathbb{R}^3 \) with boundary \( \partial U \in C^{0,1} \). If \( \Sigma \subset \partial U \) and \( \text{meas}_2(\Sigma) > 0 \), then there is a positive constant \( C = C(U) \) such that the inequality

\[
\|D(v)\|_{L^2(U)} \geq C(U)\|v\|_{H^1(U)}
\]

holds for all \( v \in H^1(U) \) satisfying the boundary condition \( v|\Sigma = 0 \).

This proposition is a consequence of Theorems 2.2 and 2.3 that is given in [15, Chap. I].
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