A Construction of String 2-Group Models using a Transgression-Regression Technique

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Dedicated to Steven Rosenberg on the occasion of his 60th birthday

Abstract

In this note we present a new construction of the string group that ends optionally in two different contexts: strict diffeological 2-groups or finite-dimensional Lie 2-groups. It is canonical in the sense that no choices are involved; all the data is written down and can be looked up (at least somewhere). The basis of our construction is the basic gerbe of Gawędzki-Reis and Meinrenken. The main new insight is that under a transgression-regression procedure, the basic gerbe picks up a multiplicative structure coming from the Mickelsson product over the loop group. The conclusion of the construction is a relation between multiplicative gerbes and 2-group extensions for which we use recent work of Schommer-Pries.

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1 Introduction

The string group String(\(n\)) is a topological group defined up to homotopy equivalence as the 3-connected cover of Spin(\(n\)), for \(n = 3\) or \(n > 4\). Concrete models for String(\(n\)) have been provided by Stolz [Sto96] and Stolz-Teichner [ST04]. In order to understand, e.g. the differential geometry of String(\(n\)), the so-called “string geometry”, it is necessary to
have models in better categories than topological groups. Its 3-connectedness implies that String(n) is a $K(\mathbb{Z}, 2)$-fibration over Spin(n), so that it cannot be a (finite-dimensional) Lie group. Instead, it allows models in the following contexts (in the order of appearance):

(i) Strict Fréchet Lie 2-groups [BCSS07].
(ii) Banach Lie 2-groups [Hen08].
(iii) Finite-dimensional Lie 2-groups [SP11].
(iv) Strict diffeological 2-groups [Sch11].
(v) Fréchet Lie groups [NSW].

We recall that a strict Lie 2-group is a Lie groupoid equipped with a certain kind of monoidal structure. In the non-strict case the monoidal structure is generalized to a “stacky” product. A 2-group model for String(n) is a Lie 2-group $\Gamma$, possibly strict, Banach, Fréchet or diffeological, equipped with a Lie 2-group homomorphism

$$\Gamma \rightarrow \text{Spin}(n)$$

such that the geometric realization of $\text{(1.1)}$ is a 3-connected cover.

The purpose of this note is to construct a new 2-group model for String(n), which can – in the very last step – either be chosen to live in the context (iii) of finite-dimensional Lie 2-groups, or in the context (iv) of strict, diffeological 2-groups. The strategy we pursue is to reduce the problem of constructing 2-group models for String(n) to the construction of certain gerbes over Spin(n). For the context (iii), this reduction is possible due to an equivalence of bicategories

$$\left\{ \text{Multiplicative, smooth bundle gerbes over } G \right\} \rightarrow \left\{ \text{Central Lie 2-group extensions of } G \text{ by } BS^1 \right\},$$

which exists for any compact Lie group $G$ and reflects the fact that both bicategories are classified by $H^4(BG, \mathbb{Z})$ [Bry, SP11]. The equivalence (1.2) is designed such that any multiplicative bundle gerbe over Spin(n) whose class is a generator of $H^4(B\text{Spin}(n), \mathbb{Z}) \cong \mathbb{Z}$ automatically goes to a Lie 2-group model for String(n) in the context (iii) [SP11]. A version of the equivalence (1.2) exists in the strict, diffeological context (iv).

Sections 2 and 3 review the notions of bundle gerbes and multiplicative structures, and discuss the equivalence (1.2). In Section 4 we upgrade to the diffeological version.
The following two sections are concerned with the construction of the input data, certain multiplicative bundle gerbes. In short, the construction goes as follows: Gawędzki-Reis [GR02, GR03] and Meinrenken [Mei02] have described a canonical construction of a bundle gerbe $G_{bas}$ over a compact, simple, simply-connected Lie group $G$, whose Dixmier-Douady class generates $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$. A direct construction of a multiplicative structure on $G_{bas}$ is not known – this is the main problem we solve in this note.

We use a transgression-regression technique developed in a series of papers [Wala, Walb, Walc]. The transgression of $G_{bas}$ is a principal $S^1$-bundle $L_{G_{bas}}$ over the loop group $LG$. Our main insight is to combine two additional structures one naturally finds on $L_{G_{bas}}$: the Mickelsson product [Mic87] and the fusion product [Wala]. The fusion product allows one to regress $L_{G_{bas}}$ to a new, diffeological bundle gerbe $\mathcal{R}(L_{G_{bas}})$ over $G$. The Mickelsson product regresses alongside to a strictly multiplicative structure on $\mathcal{R}(L_{G_{bas}})$. Regression is inverse to transgression in the sense of a natural isomorphism

$$G_{bas} \cong \mathcal{R}(L_{G_{bas}})$$

of bundle gerbes over $G$. Since $H^4(BG, \mathbb{Z}) \cong H^3(G, \mathbb{Z})$ for the class of Lie groups we are looking at here, this implies that the class of $\mathcal{R}(L_{G_{bas}})$ generates $H^4(BG, \mathbb{Z})$.

We conclude our construction in Section 7 by either feeding the strictly multiplicative, diffeological bundle gerbe $\mathcal{R}(L_{G_{bas}})$ into the strict, diffeological version of the equivalence (1.2), or we conclude by using the isomorphism (1.3) to induce a finite-dimensional, non-strict multiplicative structure on $G_{bas}$ and feeding that into the equivalence (1.2). For $G = \text{Spin}(n)$, this yields the two new 2-group models for String($n$) in the contexts (iv) and (iii), respectively.

The construction in the context (iii) is probably the most interesting result of this note. It can be seen as a small addendum to the work of Schommer-Pries [SP11]. Indeed, the model of [SP11] is only defined up to a “contractible choice of isomorphisms”, while our model is canonical “on the nose”.

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2 Multiplicative Bundle Gerbes

In this section we review the notion of a multiplicative bundle gerbe, which is central for this note. Let $M$ be a smooth manifold.

**Definition 2.1.** Let $\pi : Y \rightarrow M$ be a surjective submersion, and let $P$ be a principal $S^1$-bundle over the two-fold fibre product $Y^{[2]} := Y \times_M Y$. A gerbe product on $P$ is an isomorphism

$$\mu : \text{pr}_{12}^*P \otimes \text{pr}_{23}^*P \rightarrow \text{pr}_{13}^*P$$

of bundles over $Y^{[3]}$ that is associative over $Y^{[4]}$.

In this definition, we have denoted by $\text{pr}_{ij} : Y^{[3]} \rightarrow Y^{[2]}$ the projection to the indexed factors, and we have denoted by $\otimes$ the tensor product of $S^1$-bundles. Thus, a gerbe product is for every point $(y_1, y_2, y_3) \in Y^{[3]}$ a smooth, equivariant map

$$\mu : P(y_1, y_2) \otimes P(y_2, y_3) \rightarrow P(y_1, y_3)$$

between fibres of $P$. The associativity condition is that

$$\mu(\mu(q_{12} \otimes q_{23}) \otimes q_{34}) = \mu(q_{12} \otimes \mu(q_{23} \otimes q_{34}))$$

for all $q_{ij} \in P(y_i, y_j)$ and all $(y_1, y_2, y_3, y_4) \in Y^{[4]}$.

**Definition 2.2 (Mur96).** A bundle gerbe over $M$ is a surjective submersion $\pi : Y \rightarrow M$, a principal $S^1$-bundle $P$ over $Y^{[2]}$ and a gerbe product $\mu$ on $P$.

Bundle gerbes over $M$ form a bicategory $\mathcal{Grb}(M)$ [Ste00, Wal07]. In fact, they form a **double category with companions** in the sense of [GP04, Shu08]. This means that there are two types of 1-morphisms, “general” ones and “simple” ones, together with a certain map that assigns to each simple 1-morphism a general one, its “companion”. In the case of bundle gerbes, we call the general 1-morphisms **1-isomorphisms** (they are all invertible) and the simple ones **refinements**. For the definition of a 1-isomorphism we refer to [NWa, Definition 5.1.2]. A refinement $f : G \rightarrow G'$ between two bundle gerbes is a smooth map $f_1 : Y \rightarrow Y'$ that commutes with the two submersions to $M$, together with a bundle isomorphism $f_2 : P \rightarrow P'$ over the induced map $Y^{[2]} \rightarrow Y'^{[2]}$, such that $f_2$ is

\footnote{Sometimes the simpler ones are called “morphisms”, and the general ones “stable isomorphisms”}

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a homomorphism for the gerbe products $\mu$ and $\mu'$. The assignment of a 1-morphism to a refinement can be found in [NWA] Lemma 5.2.3.

The bicategory $\mathcal{G}rb(M)$ is equipped with many additional features. For instance, it is monoidal, and the assignment $M \mapsto \mathcal{G}rb(M)$ is a sheaf of monoidal bicategories over the site of smooth manifolds (with surjective submersions) [Ste00, Wal07, NWA]. This means in particular that one can consistently pull back and tensor bundle gerbes, refinements, 1-isomorphisms, and 2-morphisms. Denoting by $h_0$ the operation of taking the set of isomorphism classes we have:

**Theorem 2.3** ([MS00]). $h_0\mathcal{G}rb(M) \cong H^3(M, \mathbb{Z})$.

In the following we consider bundle gerbes over a Lie group $G$. For preparation, let us suppose that $\pi : Y \to G$ is a surjective submersion, such that $Y$ is another Lie group and $\pi$ is a group homomorphism. Then, the fibre products $Y^{[k]}$ are again Lie groups, and the projections $\text{pr}_{ij} : Y^{[3]} \to Y^{[2]}$ are Lie group homomorphisms. Suppose further that we have a central extension

$$1 \to S^1 \to P \to Y^{[2]} \to 1$$

of Lie groups, i.e. a central extension of groups such that $P$ is a principal $S^1$-bundle over $Y^{[2]}$. In this situation, a gerbe product $\mu$ on $P$ is called multiplicative if it is a group homomorphism, i.e. if

$$\mu(p_{12}p'_{12} \otimes p_{23}p'_{23}) = \mu(p_{12} \otimes p_{23}) \cdot \mu(p'_{12} \otimes p'_{23})$$

for all $p_{ij} \in P(y_i, y_j), p'_{ij} \in P(y'_i, y'_j)$ and all $(y_1, y_2, y_3), (y'_1, y'_2, y'_3) \in Y^{[3]}$.

**Definition 2.4.** Let $\mathcal{G} = (Y, \pi, P, \mu)$ be a bundle gerbe over $G$. A strictly multiplicative structure on $\mathcal{G}$ is a Lie group structure on $Y$ such that $\pi$ is a group homomorphism, together with a Lie group structure on $P$, such that $P$ is a central extension of $Y^{[2]}$ by $S^1$ and $\mu$ is multiplicative.

A bundle gerbe $\mathcal{G}$ together with a strictly multiplicative structure is called a strictly multiplicative bundle gerbe. The problem is that strictly multiplicative structures on bundle gerbes rarely exist. The following definition is a suitable generalization.

**Definition 2.5** ([Bry] CJM+05 [Wal10]). A multiplicative structure on a bundle gerbe $\mathcal{G}$ over $G$ is a 1-isomorphism

$$\mathcal{M} : \text{pr}_1^*\mathcal{G} \otimes \text{pr}_2^*\mathcal{G} \to m^*\mathcal{G}$$
of bundle gerbes over $G \times G$, and a 2-isomorphism

$$
\begin{array}{ccc}
G_1 \otimes G_2 \otimes G_3 & \xrightarrow{M_{1,2} \otimes \text{id}} & G_{12} \otimes G_3 \\
\downarrow \text{id} \otimes M_{2,3} & & \downarrow M_{12,3} \\
G_1 \otimes G_{23} & \xrightarrow{M_{1,23}} & G_{123}
\end{array}
$$

between 1-isomorphisms over $G \times G \times G$ that satisfies the obvious pentagon axiom.

In this definition, $m : G \times G \longrightarrow G$ denotes the multiplication of $G$, and the index convention is such that e.g. the index $(..)_{i,j,k}$ stands for the pullback along the map $(g_i, g_j, g_k) \longrightarrow (g_ig_j, g_k)$. For instance, $G_i = \text{pr}^*_i G$ and $G_{12} = m^* G$. A multiplicative bundle gerbe over $G$ is a bundle gerbe together with a multiplicative structure. Multiplicative bundle gerbes over $G$ form a bicategory that we denote by $\mathcal{M}ult\mathcal{G}rb(G)$. We have for compact Lie groups $G$:

**Theorem 2.6 ([Bry] Propositions 1.5 and 1.7).** $h_0 \mathcal{M}ult\mathcal{G}rb(G) \cong \mathbb{H}^4(BG, \mathbb{Z})$.

A strictly multiplicative structure on a bundle gerbe $G = (Y, \pi, P, \mu)$ induces a multiplicative structure in the following way. Over $G \times G$, we consider the bundle gerbes $G_{1,2} = \text{pr}^*_1 G \otimes \text{pr}^*_2 G$ and $G_{12} = m^* G$. Employing the definitions of pullbacks and tensor products [Wal07], the bundle gerbe $G_{1,2}$ consists of $Y_{1,2} := Y \times Y$ with $\pi_{1,2} := \pi \times \pi$, and the bundle $P^{1,2}$ over $Y_{1,2}$ with fibres

$$P^{1,2}_{(y_1, y_2), (y'_1, y'_2)} = P_{y_1, y'_1} \otimes P_{y_2, y'_2}.$$ 

The bundle gerbe product $\mu^{1,2}$ on $P^{1,2}$ is just the tensor product of $\mu$ with itself. On the other side, the bundle gerbe $G_{12}$ is $Y_{12} := G \times Y$ with $\pi_{12}(g, y) := (g, g^{-1}\pi(y))$, while the principal bundle $P^{12}$ and the bundle gerbe product $\mu^{12}$ are just pullbacks along the projection $Y_{12} \longrightarrow Y$. Now, the multiplication of the Lie group $Y$ defines a smooth map

$$f_1 : Y_{1,2} \longrightarrow Y_{12} : (y_1, y_2) \longrightarrow (\pi(y_1), y_1y_2)$$

that commutes with $\pi_{1,2}$ and $\pi_{12}$. Further, the multiplication on $P$ defines a bundle isomorphism

$$f_2 : P^{1,2} \longrightarrow P^{12} : (p, p') \longrightarrow pp'$$

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and the multiplicativity of $\mu$ assures that $f_2$ is a homomorphism for the bundle gerbe products $\mu^{1,2}$ and $\mu^{12}$. Thus, the pair $(f_1, f_2)$ is a refinement $f : \mathcal{G}_{1,2} \rightarrow \mathcal{G}_{12}$ which in turn defines the required 1-isomorphism $\mathcal{M}$. Next we look at the diagram over $G \times G \times G$ of Definition 2.5. It turns out that the associativity of the Lie groups $Y$ and $P$ imply the strict commutativity of the refinements representing the four 1-isomorphisms in the diagram. In this case, the coherence of companions in double categories provides the required 2-isomorphism $\alpha$, and a general coherence result implies the pentagon axiom. This concludes the construction of a multiplicative bundle gerbe $(\mathcal{G}, \mathcal{M}, \alpha)$ from a strictly multiplicative one.

The 2-functor $\text{MultGrb}(G) \rightarrow \mathcal{Grb}(G)$ that forgets the multiplicative structure corresponds [Wal10, Lemma 2.3.9] under the bijections of Theorems 2.3 and 2.6 to the usual “transgression” map

$$H^4(BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z}).$$

(2.1)

If $G$ is compact, simple, and simply connected, this map is a bijection, so that every bundle gerbe over $G$ has a (up to isomorphism) unique multiplicative structure. If $G$ is only compact and simple, the map (2.1) is still injective, but the existence of multiplicative structures is obstructed.\(^2\)

3 Lie 2-Group Extensions

We relate multiplicative bundle gerbes to central Lie 2-group extensions. The material presented here is well-known; the whole section can be seen as an expansion of [SP11, Remark 101].

3.1 Lie 2-Groups

We recall that a Lie groupoid is a groupoid $\Gamma$ whose objects $\Gamma_0$ and morphisms $\Gamma_1$ form smooth manifolds, whose source and target maps are surjective submersions, and whose composition and inversion are smooth maps.

Example 3.1.1.

\(^2\)This can e.g. be seen by looking at the descent theory for multiplicative gerbes [GW09].
(i) Every smooth manifold $X$ defines a “discrete” Lie groupoid $X_{dis}$ with objects $X$ and only identity morphisms.

(ii) Every Lie group $G$ defines a Lie groupoid $BG$ with one object and automorphism group $G$.

(iii) Let $G = (Y, π, P, µ)$ be a bundle gerbe over $M$. Then, we have a Lie groupoid $Γ_G$ with objects $Y$ and morphisms $P$. Source and target maps are defined by $s := π_1 ∘ χ$ and $t := π_2 ∘ χ$, where $χ : P → Y^{[2]}$ denotes the bundle projection, and the composition is the gerbe product $µ$. Identities and inversion are also induced by $µ$ [NWa Corollary 5.2.6 (iii)].

Lie groupoids – like bundle gerbes – form a double category with companions, denoted $\mathbf{LieGrpd}$. The simple 1-morphisms are smooth functors. The general ones are smooth anafunctors $P : Γ → Ω$, which are principal $Ω$-bundles $P$ over $Γ$, see [NWa Section 2] for a detailed discussion and references. The 2-morphisms are $Ω$-bundle isomorphisms over $Γ$, and will be called smooth transformations.

**Proposition 3.1.2.** The assignment $\mathcal{G} → Γ_G$ of a Lie groupoid to a bundle gerbe extends to a 2-functor $\mathcal{Grb}(M) → \mathbf{LieGrpd}$ that respects companions.

Proof. The 2-functor is constructed in [NWa Section 7.2]. The claim that this 2-functor respects companions means additionally that it sends a refinement $f : \mathcal{G} → \mathcal{G}'$ to a smooth functor $Γ_G → Γ_G'$; this can easily be checked using the given definitions. □

**Definition 3.1.3 ([BL04]).** A strict Lie 2-group is a Lie groupoid $Γ$ whose objects $Γ_0$ and morphisms $Γ_1$ form Lie groups, such that source, target, and composition are group homomorphisms.

Continuing Example 3.1.1, it is easy to check the following statements:

(i) If $G$ is a Lie group, the Lie groupoid $G_{dis}$ is a strict Lie 2-group.

(ii) If $A$ is an abelian Lie group, the Lie groupoid $BA$ is a strict Lie group.

(iii) If $G$ is a strictly multiplicative bundle gerbe over $G$, the Lie groupoid $Γ_G$ is a strict Lie 2-group.

\[^3\]Sometimes smooth anafunctors are called “Hilsum-Skandalis morphisms” or “bibundles”.

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In order to include non-strictly multiplicative bundle gerbes, we need the following generalization:

**Definition 3.1.4** ([BL04, SP11]). A Lie 2-group is a Lie groupoid \( \Gamma \) with smooth anafunctors

\[
m : \Gamma \times \Gamma \to \Gamma \quad \text{and} \quad e : 1 \to \Gamma,
\]

and smooth transformations \( \alpha, l, r \), where \( \alpha \) expresses that \( m \) is an associative product and \( l, r \) express that \( e \) is a left and right unit for this product, such that the smooth anafunctor

\[
(\text{pr}_1, m) : \Gamma \times \Gamma \to \Gamma \times \Gamma
\]

is invertible.

In this definition, 1 denotes the trivial Lie groupoid. The details about the smooth transformations can e.g. be found in [SP11, Definition 41]. We have the following examples of Lie 2-groups:

1.) If \( \Gamma \) is a strict Lie 2-group, the Lie group structures on \( \Gamma_0 \) and \( \Gamma_1 \) can be bundled into smooth functors \( m : \Gamma \times \Gamma \to \Gamma \) and \( e : 1 \to \Gamma \) satisfying strictly the axioms of an associative multiplication and of a unit. The coherence of companions in the double category \( \text{LieGrpd} \) provides associated smooth anafunctors and the required smooth transformations. Thus, strict Lie 2-groups are particular Lie 2-groups.

2.) A multiplicative structure \((\mathcal{M}, \alpha)\) on a bundle gerbe \( \mathcal{G} \) equips the Lie groupoid \( \Gamma_\mathcal{G} \) with a Lie 2-group structure. Indeed, one can check explicitly that \( \Gamma_{\text{pr}_1^*\mathcal{G} \otimes \text{pr}_2^*\mathcal{G}} = \Gamma_{\mathcal{G}} \times \Gamma_{\mathcal{G}} \) as Lie groupoids, and also produce an evident smooth functor \( \text{pr} : \Gamma_{\mathcal{M}^*\mathcal{G}} \to \Gamma_{\mathcal{G}} \). Using that \( \Gamma \) is functorial (Proposition 3.1.2), we obtain a smooth anafunctor

\[
\Gamma_{\mathcal{G}} \times \Gamma_{\mathcal{G}} = \Gamma_{\text{pr}_1^*\mathcal{G} \otimes \text{pr}_2^*\mathcal{G}} \xrightarrow{\Gamma_{\mathcal{M}}} \Gamma_{\mathcal{M}^*\mathcal{G}} \xrightarrow{\text{pr}} \Gamma_{\mathcal{G}}.
\]

Similarly, one can check that the 2-isomorphism \( \alpha \) provides the required associator for this multiplication. Let \( 1 : pt \to G \) denote the unit element of the group \( G \). Using duals of bundle gerbes one can show that the 1-isomorphism \( \mathcal{M} \) induces a distinguished 1-isomorphism \( \mathcal{E} : \mathcal{I} \to 1^*\mathcal{G} \), where \( \mathcal{I} \) is the trivial \( S^1 \)-bundle gerbe over the point. We have \( \Gamma_{\mathcal{I}} = BS^1 \), and obtain, again by functorality of \( \Gamma \), the required smooth anafunctor

\[
1 \xrightarrow{\Gamma_{\mathcal{I}}} BS^1 \xrightarrow{\Gamma_{1^*\mathcal{G}}} \Gamma_{\mathcal{G}}.
\]

The smooth transformations \( l \) and \( r \) can both be deduced from the 2-isomorphism \( \alpha \).
3.2 Central Extensions

We briefly review some aspects of principal 2-bundles [Bar04, SP11]. Let $\Gamma$ be a Lie 2-group. A principal $\Gamma$-2-bundle over a smooth manifold $M$ is a Lie groupoid $P$ "total space", a smooth functor $\pi : P \to M_{\text{dis}}$ "projection", a smooth anafunctor $\tau : P \times \Gamma \to P$ "right action" together with two smooth transformations satisfying several axioms. If $\Gamma$ is a strict Lie 2-group, a principal $\Gamma$-2-bundle is called strict if $\tau$ is a smooth functor, and both smooth transformations are identities. Strict principal $\Gamma$-2-bundles have been studied in detail in [NWa].

Example 3.2.1. We recall from Example 3.1.1 (iii) that there is a Lie groupoid $\Gamma_G$ associated to any bundle gerbe $G$ over $M$. Together with the smooth functor $\pi : \Gamma_G \to M_{\text{dis}}$ given by the surjective submersion of $G$, and the smooth functor $\tau : \Gamma_G \times B S^1 \to \Gamma_G$ induced by the action of $S^1$ on $P$, this yields a strict principal $B S^1$-2-bundle over $M$, see [SP11] Example 73, [NWa Section 7.2].

Proposition 3.2.2 ([NWa Theorem 7.1]). Example 3.2.1 establishes an equivalence between the bicategories of bundle gerbes over $M$ and strict principal $B S^1$-2-bundles over $M$.

Schommer-Pries has introduced a very general notion of Lie 2-group extensions [SP11 Definition 75]. For the purpose of this note we may reduce it to the case that a "discrete" Lie 2-group $G_{\text{dis}}$ is extended by the "codiscrete" Lie 2-group $B S^1$.

Definition 3.2.3. Let $G$ be a Lie group. A Lie 2-group extension of $G$ by $B S^1$ is a Lie 2-group $\Gamma$ with Lie 2-group homomorphisms

$$
\begin{array}{c}
B S^1 \overset{i}{\longrightarrow} \Gamma \overset{\pi}{\longrightarrow} G_{\text{dis}}
\end{array}
$$

such that:

(i) The composite $\pi \circ i$ is the constant functor $1 : B S^1 \to G_{\text{dis}}$.

(ii) $\pi : \Gamma \to G_{\text{dis}}$ is a principal $B S^1$-2-bundle over $G$.

The extension is called strict if $\Gamma$ is a strict Lie 2-group, and $\pi$, $i$ are strict 2-group homomorphisms.
The notion of central Lie 2-group extensions introduced in [SP11, Definition 8.3] requires a certain group homomorphism $\alpha : G \to \text{Aut}(S^1) \cong \mathbb{Z}/2\mathbb{Z}$ to be trivial. Central Lie 2-group extensions of $G$ by $BS^1$ form a bicategory $\text{Ext}(G, BS^1)$, and for $G$ compact we have:

**Theorem 3.2.4 ([SP11]).** $h_0 \text{Ext}(G, BS^1) \cong H^4(BG, \mathbb{Z})$.

As discussed in Section 3.1, the Lie groupoid $\Gamma_G$ associated to a strictly multiplicative bundle gerbe $G = (Y, \pi, P, \mu)$ over $G$ is a strict Lie 2-group. We have the functor $\pi : \Gamma_G \to G_{\text{dis}}$ from Example 3.2.1 and a functor $i : BS^1 \to \Gamma_G$ induced by the second arrow of the central extension

$$1 \to S^1 \to P \to Y^{[2]} \to 1. \quad (3.2.1)$$

Condition (i) is clear, and (ii) is proved by Example 3.2.1. Centrality follows from the one of (3.2.1). Thus, every strictly multiplicative bundle gerbe defines a central, strict Lie 2-group extension.

If $G$ is a multiplicative bundle gerbe over $G$, the Lie groupoid $\Gamma_G$ is a Lie 2-group. The functor $\pi : \Gamma_G \to G_{\text{dis}}$ is the same as before, and the smooth anafunctor $i : BS^1 \to \Gamma_G$ is defined by

$$BS^1 = pt \times BS^1 \xrightarrow{e \times \text{id}} \Gamma_G \times BS^1 \xrightarrow{\tau} \Gamma_G,$$

where $\tau$ is the action functor of Example 3.2.1. Conditions (i) and (ii) are still satisfied, and centrality can be concluded from the strict case, since it only affects the underlying “discrete” 2-groups and every Lie 2-group can be strictified upon discretization. Thus, every multiplicative bundle gerbe defines a central Lie 2-group extension. Summarizing, we obtain the following (commutative) diagram of bicategories and 2-functors:

$$\begin{array}{ccc}
\left\{ \text{Strictly multiplicative bundle gerbes over } G \right\} & \longrightarrow & \left\{ \text{Central strict Lie 2-group extensions of } G \text{ by } BS^1 \right\} \\
\downarrow & & \downarrow \\
\left\{ \text{Multiplicative bundle gerbes over } G \right\} & \longrightarrow & \left\{ \text{Central Lie 2-group extensions of } G \text{ by } BS^1 \right\}
\end{array} \quad (3.2.2)
$$

**Theorem 3.2.5.** The horizontal 2-functors in diagram (3.2.2) are equivalences of bicategories. If $G$ is compact, they induce the identity on $H^4(BG, \mathbb{Z})$ under the bijections of Theorems 2.6 and 3.2.4.
Proof. For the purposes of this note, it suffices to prove the second statement for \( G \) compact and simple. Then, since \( H^4(BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z}) \) is injective, it suffices to observe that the horizontal 2-functors induce the identity on \( H^3(G, \mathbb{Z}) \). The maps to \( H^3(G, \mathbb{Z}) \) induced by the bijections of Theorems 2.6 and 3.2.4 are, respectively, the projection to the underlying bundle gerbe, see (2.1), and the projection to the underlying principal \( BS^1 \)-2-bundle of a 2-group extension, see [SP11]. Under both horizontal 2-functors, these are related by the assignment of Example 3.2.1 which is an equivalence of bicategories (Proposition 3.2.2).
\[ \square \]

4 The Site of Diffeological Spaces

We recall that a site is a category together with a Grothendieck (pre-)topology: a class of morphisms called coverings, containing all identities, closed under composition, and stable under pullbacks along arbitrary morphisms. Above we have presented the definitions of bundle gerbes, groupoids, 2-groups, and 2-group extensions internal to the familiar site \( C^\infty \) of smooth (finite-dimensional) manifolds, with the coverings given by surjective submersions.

Schommer-Pries proved that the site \( C^\infty \) allows 2-group models for the string group [SP11, Theorem 2]. However, one can show that it does not allow strict 2-group models. As mentioned in Section 1, strictness can be achieved by passing to a bigger site, e.g. the site \( F^\infty \) of (possibly infinite-dimensional) Fréchet manifolds [BCSS07]. For the transgression-regression technique we want to use in the next section we have to pass to a yet bigger site, the site \( D^\infty \) of diffeological spaces.

We refer to [Wala, Appendix A.1] for an introduction to diffeological spaces and ref-

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4By Theorem 3.2.5 such a strict Lie 2-group extension of \( G \) by \( BS^1 \) would correspond to a strictly multiplicative bundle gerbe \( \mathcal{G} \) over \( G \) whose Dixmier-Douady class generates \( H^3(G, \mathbb{Z}) \). We may assume that \( G = SU(2) \), otherwise we consider the restriction of \( \mathcal{G} \) to an SU(2) subgroup (with still non-trivial Dixmier-Douady class). The strict Lie 2-group \( \Gamma_G \) induces an exact sequence

\[
1 \rightarrow S^1 \rightarrow \ker(s) \rightarrow Y \rightarrow SU(2) \rightarrow 1
\]

of Lie groups [NWb, Section 3], where \( s,t \) are the source and target maps of \( \Gamma_G \). Thus, the submersion \( \pi : Y \rightarrow SU(2) \) of \( \mathcal{G} \) is a principal bundle for the structure group \( H := \ker(s)/S^1 \). Such bundles are classified by \( \pi_2(H) = 0 \), which implies that it has a global section, in contradiction to the non-triviality of \( \mathcal{G} \), see [Walb, Lemma 3.2.3].

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ereferences. In short, a **diffeological space** is a set $X$ together with a collection of generalized charts called “plots”. A **plot** is a triple $(n, U, c)$ consisting of a number $n \in \mathbb{N}$, an open subset $U \subseteq \mathbb{R}^n$ and a map $c : U \to X$. A map $f : X \to X'$ between diffeological spaces is **smooth** if its composition $f \circ c$ with every plot $c$ of $X$ is a plot of $X'$. This defines the category $D^\infty$ of diffeological spaces. A Grothendieck topology on $D^\infty$ is provided by **subductions**: smooth maps $\pi : Y \to X$ such that every plot $c : U \to X$ lifts locally to $Y$.

A manifold $M$ (either smooth or Fréchet) can be regarded as a diffeological space with the underlying set $M$, and the plots given by all smooth maps $c : U \to M$, for all open subsets of $\mathbb{R}^n$ and all $n$. We obtain a sequence

$$C^\infty \to F^\infty \to D^\infty$$

(4.1)

of functors. These preserve the Grothendieck topologies in the sense that they send surjective submersions to subductions. Furthermore, they are full and faithful: this means that upon embedding two objects into a bigger site, the set of all maps between them is not getting bigger or smaller.

If some definition is given in terms of the ingredients of a certain site, the same definition can obviously be repeated in any other site. For example, a **smooth principal $S^1$-bundle** over a smooth manifold $X$ can be defined as a surjective submersion $\pi : P \to X$ together with a smooth map $\tau : P \times S^1 \to P$ that defines a fibrewise action, such that $(pr_1, \tau) : P \times S \to P \times_X P$ is a diffeomorphism. Accordingly, a **diffeological principal $S^1$-bundle** over a diffeological space $X$ is a subduction $\pi : P \to X$ and a smooth map $\tau$ satisfying the same conditions; see [Wala] for a thorough discussion. Similarly, one repeats the definition of a bundle gerbe, of a Lie groupoid, of a Lie 2-group, and of a Lie 2-group extension in the site of diffeological spaces.

The classification Theorems 2.3, 2.6 and 3.2.4 remain true for (multiplicative) diffeological bundle gerbes and diffeological 2-group extensions, as long as the base spaces $M$ and $G$ are finite-dimensional smooth manifolds; see e.g. [Walb, Theorem 3.1.3]. Similarly, the relation between multiplicative bundle gerbes and 2-group extensions of Theorem 3.2.5 remains true in the diffeological context. In particular, there is a 2-functor

$$\left\{ \begin{array}{c} \text{Strictly multiplicative} \\ \text{diffeological bundle gerbes over } G \end{array} \right\} \to \left\{ \begin{array}{c} \text{Central, strict diffeological} \\ \text{2-group extensions of } G \text{ by } BS^1 \end{array} \right\}$$

(4.2)

that induces the identity on $H^4(BG, \mathbb{Z})$ for $G$ compact and simple. This will be used in
Section 7

5 The Transgression-Regression Machine

Brylinski and McLaughlin have defined a procedure to transform a bundle gerbe over a smooth manifold $M$ into a Fréchet principal $S^1$-bundle over the Fréchet manifold $LM := C^\infty(S^1, M)$, setting up an important relation between geometry on a manifold and geometry on its loop space \cite{Bry93, BM94}. Their procedure uses, as an auxiliary datum, a connection on the bundle gerbe. If $Grb^\nabla(M)$ denotes the bicategory of bundle gerbes with connection over $M$, and $h_1$ denotes the operation of producing a category (by identifying 2-isomorphic morphisms), then Brylinski’s and McLaughlin’s construction furnishes a functor

$$L : h_1 Grb^\nabla(M) \to \text{Bun}_{S^1}(LM).$$

We shall describe some details of the construction following \cite{Wal10} \cite{Walsb}. If $G$ is a bundle gerbe with connection over $M$, the fibre of $LG$ over a loop $\tau \in LM$ is

$$LG|_{\tau} := h_0 \text{Triv}^\nabla(\tau^* G),$$

i.e. it consists of isomorphism classes of (connection-preserving) trivializations of $\tau^* G$. In general, trivializations of a bundle gerbe $K$ over a smooth manifold $X$ form a category that is a torsor for the monoidal category $\text{Bun}_{S^1}^{\nabla_0}(X)$ of flat principal $S^1$-bundles over $X$, under a certain action functor

$$\text{Triv}^\nabla(K) \times \text{Bun}_{S^1}^{\nabla_0}(X) \to \text{Triv}^\nabla(K) : (T, P) \mapsto T \otimes P.$$ (5.2)

The fibres (5.1) are thus torsors over the group $h_0 \text{Bun}_{S^1}^{\nabla_0}(S^1) \cong S^1$. There exists a unique Fréchet manifold structure on $LG$ turning it into a Fréchet principal $S^1$-bundle \cite{Wal10} Proposition 3.1.2].

It is easier to pass to the site of diffeological spaces. The plots of $LM$ are maps $c : U \to LM$ whose adjoint map $U \times S^1 \to M : (u, z) \mapsto c(u)(z)$ is smooth (in the ordinary sense) \cite{Wala} Lemma A.1.7]. The plots of the total space $LG$ are maps $c : U \to LG$ for which every point $w \in U$ has an open neighborhood $w \in W \subseteq U$ such
that

(i) the map \( d : W \times S^1 \longrightarrow M \) defined by
\[
W \times S^1 \xrightarrow{\circ \mid W \times \text{id}} LG \times S^1 \xrightarrow{\text{pr} \times \text{id}} LM \times S^1 \xrightarrow{\text{ev}} M
\]
is a smooth map, and

(ii) there exists a trivialization \( \mathcal{T} \) of \( d^*G \) with \( c(x) \cong \iota_x^* \mathcal{T} \) for all \( x \in W \), where \( \iota_x : S^1 \longrightarrow W \times S^1 \) is \( \iota_x(z) := (x, z) \).

Diffeological principal \( S^1 \)-bundles over \( LM \) in the image of the transgression functor \( L \) are equipped with more structure. Relevant for this note is a fusion product [Walb]. We denote by \( PM \) the set of smooth paths \( \gamma : [0, 1] \longrightarrow M \) with sitting instants, i.e. \( \gamma \) is constant near the endpoints. This ensures that two paths \( \gamma_1, \gamma_2 \) with a common end can be composed to another smooth path \( \gamma_2 \ast \gamma_1 \). The set \( PM \) is not a Fréchet manifold, but a nice diffeological space whose plots are again those maps \( c : U \longrightarrow PM \) whose adjoint map \( (u, t) \longmapsto c(u)(t) \) is smooth. The evaluation map
\[
ev : PM \longrightarrow M \times M : \gamma \longmapsto (\gamma(0), \gamma(1))
\]
is obviously smooth, and a subduction if \( M \) is connected. We denote by \( PM^{[k]} \) the fibre product of \( PM \) over \( M \times M \); it consists of \( k \)-tuples of paths with a common initial point and a common end point. If we denote by \( \overline{\gamma} \) the inverse of a path \( \gamma \), we obtain a smooth map [Walb, Section 2.2]
\[
\ell : PM^{[2]} \longrightarrow LM : (\gamma_1, \gamma_2) \longmapsto \overline{\gamma_2} \ast \gamma_1.
\]

**Definition 5.1** ([Walb Definition 2.1.3]). Let \( P \) be a diffeological principal \( S^1 \)-bundle over \( LM \). A fusion product on \( P \) is a gerbe product \( \lambda \) on \( \ell^*P \) in the sense of Definition 2.7.

Explicitly, a fusion product \( \lambda \) provides, for each triple \( (\gamma_1, \gamma_2, \gamma_3) \in PM^{[3]} \) a smooth map
\[
\lambda : P_{\gamma_2 \ast \gamma_1} \cong P_{\gamma_3 \ast \gamma_2} \longrightarrow P_{\gamma_3 \ast \gamma_1},
\]
and these maps are associative over quadruples of paths. A pair \( (P, \lambda) \) is called a fusion bundle. We denote by \( \text{FusBun}(LM) \) the category of fusion bundles over \( LM \). The important point established in [Walb, Section 4.2] is that the functor \( L \) lifts to a functor
\[
\text{Grb}^\nabla(M) \longrightarrow \text{FusBun}(LM),
\]

- 15 -
i.e. a principal $S^1$-bundle in the image of transgression is equipped with a *canonical fusion product* $\lambda_G$. Let us briefly recall how $\lambda_G$ is characterized. We denote by $\iota_1, \iota_2 : [0, 1] \to S^1$ the inclusion of the interval into the left and the right half of the circle. Let $(\gamma_1, \gamma_2, \gamma_3)$ be a triple of paths with a common initial point $x$ and a common end point $y$, and let $T_{ij}$ be trivializations of the pullback of $G$ to the loops $\ell(\gamma_i, \gamma_j)$, for $(ij) = (12), (23), (13)$. Then, the relation

$$\lambda_G(T_{12} \otimes T_{23}) = T_{13}$$

holds if and only if there exist 2-isomorphisms

$$\phi_1 : \iota_1^*T_{12} \Longrightarrow \iota_1^*T_{13}, \quad \phi_2 : \iota_2^*T_{12} \Longrightarrow \iota_1^*T_{23} \quad \text{and} \quad \phi_3 : \iota_2^*T_{23} \Longrightarrow \iota_2^*T_{13}$$

between trivializations of the pullbacks of $G$ to the paths $\gamma_1$, $\gamma_2$, and $\gamma_3$, respectively, such that their restrictions to the two common points $x$ and $y$ satisfy the cocycle condition $\phi_1 = \phi_3 \circ \phi_2$.

A fusion product permits one to define a functor inverse to transgression [Walb, Section 5.1]. Suppose $(P, \lambda)$ is a fusion bundle over $LM$, and $x \in M$. We denote by $P_xM \subseteq PM$ the subspace of those paths that start at $x$. Then, there is a diffeological bundle gerbe $\mathcal{R}_x(P, \lambda)$ over $M$ consisting of

(i) the subduction $\text{ev}_1 : P_xM \longrightarrow M : \gamma \longmapsto \gamma(1)$.

(ii) the diffeological principal $S^1$-bundle $\ell^*P$ over $P_xM$.

(iii) the gerbe product $\lambda$ on $\ell^*P$.

This defines a *regression functor*

$$\mathcal{R}_x : \mathcal{FusBun}(LM) \longrightarrow \mathcal{h}_1\mathcal{Grb}(M).$$

The main theorem of the transgression-regression machine is that regression is inverse to transgression, in the following sense:

**Theorem 5.2.** Let $M$ be a connected smooth manifold. Then, the diagram

$$\begin{array}{ccc}
\mathcal{FusBun}(LM) & \xrightarrow{L} & \mathcal{h}_1\mathcal{Grb}(M) \\
\mathcal{R}_x & \lor & \mathcal{R}_x \\
\mathcal{h}_1\mathcal{Grb}(M) & \xrightarrow{\mathcal{R}_x} & \mathcal{h}_1\mathcal{Grb}(M) \\
\end{array}$$
of functors, which has on the bottom the functor that forgets connections and embeds bundle gerbes into diffeological bundle gerbes, is commutative up to a canonical natural equivalence.

Theorem 5.2 is proved in [Walb, Section 6.1] by constructing for each bundle gerbe $\mathcal{G}$ with connection over $M$ a 1-isomorphism

$$A_{\mathcal{G},y} : \mathcal{G} \rightarrow \mathcal{R}_x(L\mathcal{G}, \lambda_\mathcal{G}).$$

This 1-isomorphism depends on the additional choice of a lift $y \in Y$ of the base point $x \in M$ along the surjective submersion of the bundle gerbe $\mathcal{G}$. Different choices of $y$ lead to 2-isomorphic 1-isomorphisms, $A_{\mathcal{G},y} \cong A_{\mathcal{G},y'}$. Under the operation $h_1$, these 2-isomorphisms become equalities; the resulting morphism $h_1 A_{\mathcal{G},y}$ is thus independent of the choice of $y$.

**Remark 5.3.** Transgression and regression can be made an equivalence of categories by either incorporating the connections on the side of the fusion bundles, or dropping the connections on the side of the gerbes; see the main theorems of [Walb, Walc].

Transgression and regression can be promoted to a multiplicative setting, i.e. with multiplicative bundle gerbes (with connection) on the left hand side. On the loop space side we need:

**Definition 5.4.** A *fusion extension* of $LG$ is a central extension

$$1 \rightarrow S^1 \rightarrow P \rightarrow LG \rightarrow 1$$

of diffeological groups together with a multiplicative fusion product $\lambda$ on $P$.

Here it is important that the evaluation map $ev : PG[2] \rightarrow G$, as well as path composition and inversion are group homomorphisms. In particular, the map $\ell : PG[2] \rightarrow LG$ is a group homomorphism. The multiplicativity condition for the fusion product is that

$$\lambda(q_{12} \otimes q_{23}) \cdot \lambda(q'_ {12} \otimes q'_{23}) = \lambda(q_{12}q'_ {12} \otimes q_{23}q'_{23})$$

for all elements $q_{ij} \in P_{\ell(\gamma_1, \gamma_2)}$ and $q'_{ij} \in P_{\ell(\gamma'_1, \gamma'_2)}$ and all $(\gamma_1, \gamma_2, \gamma_3), (\gamma'_1, \gamma'_2, \gamma'_3) \in PG[3]$.

One can show that transgression sends a multiplicative bundle gerbe with connection to a fusion extension [Walb, Section 1.3]. Here, it will be more important to look at regression. With the base point $1 \in G$ understood, a fusion bundle $(P, \lambda)$ over $LG$ regresses to a diffeological bundle gerbe $\mathcal{R}(P, \lambda)$. It is easy to check that the additional structure
of a fusion extension (the group structure on $P$) makes $\mathcal{R}(P,\lambda)$ a strictly multiplicative, diffeological bundle gerbe.

**Remark 5.5.** Transgression and regression can be seen as a functorial strictification

\[
\begin{align*}
\{ \text{Multiplicative} & \} \\
\{ \text{bundle gerbes with} & \} \rightarrow \{ \text{Fusion} & \} \\
\{ \text{connection over} & \} \rightarrow \{ \text{extensions} & \} \rightarrow \{ \text{Strictly multiplicative,} & \} \\
\{ \text{of} & \} \rightarrow \{ \text{diffeological bundle} & \} \\
\{ \text{over} & \} \rightarrow \{ \text{gerbes over} & \} \\
\{ \text{G} & \}.
\end{align*}
\]

If $G$ is compact and simple, so that $H^4(BG,\mathbb{Z}) \longrightarrow H^3(G,\mathbb{Z})$ is injective, it follows from Theorem 5.2 that this strictification preserves the characteristic class in $H^4(BG,\mathbb{Z})$.

### 6 The Mickelsson Product

In this section we suppose that $G$ is compact, connected and simply-connected, for example $G = \text{Spin}(n)$ for $n > 2$. We consider the differential forms

\[
H := \frac{1}{6} \langle \theta \wedge [\theta \wedge \theta] \rangle \in \Omega^3(G) \quad \text{and} \quad \rho := \frac{1}{2} \langle \text{pr}_1^*\theta \wedge \text{pr}_2^*\bar{\theta} \rangle \in \Omega^2(G \times G),
\]
where $\theta$ and $\bar{\theta}$ are the left and right invariant Maurer-Cartan forms on $G$, respectively, and $\langle -, - \rangle$ is an invariant bilinear form on the Lie algebra $\mathfrak{g}$ of $G$. The forms $H$ and $\rho$ satisfy the identities

\[
dH = 0 \quad \text{and} \quad \Delta H = d\rho \quad \text{and} \quad \Delta \rho = 0,
\]
where $\Delta : \Omega^q(G^k) \longrightarrow \Omega^q(G^{k+1})$ is the alternating sum over the pullbacks along the face maps of the nerve of $BG$. Hence, the second and third equation become (in the notation of Section 2)

\[
\text{pr}_1^*H - m^*H + \text{pr}_2^*H = d\rho \quad \text{and} \quad \rho_{1,2} - \rho_{2,3} + \rho_{12,3} - \rho_{1,23} = 0.
\]

Suppose $\mathcal{G}$ is a bundle gerbe over $G$ with connection of curvature $H$. The *Mickelsson product*

\[
* : \text{LG} \times \text{LG} \longrightarrow \text{LG}
\]
on the transgression of $\mathcal{G}$ is defined as follows [Mic87; see Wal10, Section 3.1] and Bry93, Theorem 6.4.1]. First of all, we recall that the connection on the bundle gerbe $\mathcal{G}$ determines a *surface holonomy* $\text{Hol}_{\mathcal{G}}(\varphi) \in S^1$ for every closed oriented surface $\Sigma$ and a smooth map
In its application to two-dimensional field theories, the surface holonomy provides the Feynman amplitude of the so-called Wess-Zumino term \cite{Gaw88}. If the surface \( \Sigma \) has a boundary one has to impose a boundary condition in order to keep the holonomy well-defined. The boundary condition may be provided by a trivialization \( \mathcal{T} \) of \( \varphi^* \mathcal{G}|_{\partial \Sigma} \) \cite{CJM02}; the surface holonomy in this case is denoted by \( \mathcal{A}_G(\varphi, \mathcal{T}) \). We refer to \cite[Section 3.3]{Walb} for a detailed treatment with more references.

For loops \( \tau, \tau' \in LG \), let \( \mathcal{T}, \mathcal{T}' \) be trivializations of \( \tau^* \mathcal{G} \) and \( \tau'^* \mathcal{G} \); these represent elements in \( LG \) over \( \tau \) and \( \tau' \), respectively. We choose extensions \( \varphi, \varphi' : D^2 \longrightarrow G \) of \( \tau \) and \( \tau' \) from the circle \( S^1 \) to its bounding disc \( D^2 \); these exist because \( G \) is simply connected. The pointwise product \( \tilde{\varphi} := \varphi \varphi' \) is a similar extension of \( \tilde{\tau} := \tau \tau' \). We choose any trivialization \( \tilde{\mathcal{T}} \) of \( \tilde{\tau}^* \mathcal{G} \). Finally, we consider the combined map \( \Phi := (\varphi, \varphi') : D^2 \longrightarrow G \times G \). Then, we define the Mickelsson product by

\[
\mathcal{T} \star \mathcal{T}' := \tilde{\mathcal{T}} \cdot \mathcal{A}_G(\varphi, \mathcal{T})^{-1} \cdot \mathcal{A}_G(\varphi', \mathcal{T}')^{-1} \cdot \mathcal{A}_G(\tilde{\varphi}, \tilde{\mathcal{T}}) \cdot \exp \left( \int_{D^2} \Phi^* \rho \right),
\]

(6.3)

where \( \cdot \) denotes the action of \( S^1 \) on the element \( \tilde{\mathcal{T}} \in LG \).

**Lemma 6.6.** Definition (6.3) is independent of all choices and turns \( LG \) into a central extension

\[
1 \longrightarrow S^1 \longrightarrow LG \longrightarrow LG \longrightarrow 1
\]

of diffeological groups.

**Proof.** Suppose \( \tilde{\mathcal{T}}_1 \) and \( \tilde{\mathcal{T}}_2 \) are two choices of trivializations. By (5.2) there exists a principal \( S^1 \)-bundle \( P \) with flat connection over \( S^1 \) such that \( \tilde{\mathcal{T}}_2 \cong \tilde{\mathcal{T}}_1 \otimes P \). The associated surface holonomies satisfy \( \mathcal{A}_G(\tilde{\varphi}, \tilde{\mathcal{T}}_2) = \mathcal{A}_G(\tilde{\varphi}, \tilde{\mathcal{T}}_1) \otimes \text{Hol}_P(S^1)^{-1} \) \cite[Lemma 3.3.2 (a)]{Walb}; this shows that (6.3) is independent of the choice of \( \tilde{\mathcal{T}} \). Suppose further that \((\varphi_1,\varphi'_1)\) and \((\varphi_2,\varphi'_2)\) are two choices of extensions of \( \tau, \tau' \). We consider the 2-sphere \( S^2 = D^2 \# D^2 \) as glued together from two discs, equipped with piecewise defined maps \( \alpha := \varphi_1 \# \varphi_2 \), \( \alpha' := \varphi'_1 \# \varphi'_2 \) and \( \tilde{\alpha} := \tilde{\varphi}_1 \# \tilde{\varphi}_2 \), where \( \tilde{\varphi}_k := \varphi_k \varphi'_k \). The gluing law for surface holonomies \cite[Lemma 3.3.2 (c)]{Walb} implies

\[
\mathcal{A}_G(\varphi_2, \mathcal{T}) = \mathcal{A}_G(\varphi_1, \mathcal{T}) \cdot \text{Hol}_G(\alpha),
\]

(6.4)

and analogous formulae with primes and tildes. Further, we consider the map \( \Phi := \Phi_1 \# \Phi_2 \) with \( \Phi_i := (\varphi_i, \varphi'_i) \). The identity \( \Delta H = d\rho \) implies \cite{Gaw00, GW09} the Polyakov-
Wiegmann formula

$$\text{Hol}_G(\alpha \alpha') = \text{Hol}_G(\alpha) \cdot \text{Hol}_G(\alpha') \cdot \exp \left( \int_{S^2} \Phi^* \rho \right). \quad (6.5)$$

Formulas (6.4) and (6.5) prove that (6.3) is independent of the choice of the extensions $\varphi$ and $\varphi'$.

Associativity of $*$ follows from $\Delta \rho = 0$; smoothness from the smoothness of the surface holonomy $\mathcal{A}_G$ [Walb, Lemma 4.2.2]. The construction of a unit and of inverses is straightforward. Thus, $L_G$ is a diffeological group and also a principal $S^1$-bundle over $L_G$, i.e. a central extension.

Next we recall from Section 5 that $L_G$ carries a fusion product $\lambda_G$.

**Lemma 6.7.** The fusion product $\lambda_G$ is multiplicative with respect to the Mickelsson product.

**Proof.** First we mention the following general fact, for a bundle gerbe $\mathcal{G}$ with connection over a compact, simply-connected manifold $M$. Suppose $(\gamma_1, \gamma_2, \gamma_3) \in PM^3$. Since $M$ is simply-connected, there exists a smooth path $\Gamma : [0, 1] \rightarrow PM^3$ such that $\Gamma(0) = (\gamma_1, \gamma_2, \gamma_3)$, and $\Gamma(1)$ is a triple of identity paths at some point in $M$. The paths $\varphi_{ij} := \ell \circ \text{pr}_{ij} \circ \Gamma$ in $LM$ can be regarded as extensions of the loops $\tau_{ij} := \ell(\gamma_i, \gamma_j)$ to the disc. Then, [Walb, Proposition 4.3.4] implies that

$$\mathcal{A}_G(\varphi_{12}, T_{12}) \cdot \mathcal{A}_G(\varphi_{23}, T_{23}) = \mathcal{A}_G(\varphi_{13}, T_{13}) \quad (6.6)$$

for any triple of trivializations $T_{ij}$ of $\tau_{ij}^* \mathcal{G}$ satisfying $\lambda_G(T_{12} \otimes T_{23}) = T_{13}$.

Now suppose $(\gamma_1, \gamma_2, \gamma_3), (\gamma'_1, \gamma'_2, \gamma'_3) \in PM^3$ and $T_{ij}, T'_{ij}$ are trivializations over $\tau_{ij}, \tau'_{ij}$ such that $\lambda_G(T_{12} \otimes T_{23}) = T_{13}$ and $\lambda_G(T'_{12} \otimes T'_{23}) = T'_{13}$. We choose paths $\Gamma, \Gamma'$ as above, and extract the extensions $\varphi_{ij}, \varphi'_{ij}$ each satisfying (6.6). The product $\Gamma := \Gamma \cdot \Gamma'$ produces the extensions $\tilde{\varphi}_{ij} = \varphi_{ij} \varphi'_{ij}$ also satisfying (6.6). For the combined maps $\Phi_{ij} = (\varphi_{ij}, \varphi'_{ij})$ we have by construction

$$\int_{D^2} \Phi^*_{13} \rho = \int_{D^2} \Phi^*_{12} \rho + \Phi^*_{23} \rho. \quad (6.7)$$

Define $\tilde{T}_{12} := T_{12} * T'_{12}$ and $\tilde{T}_{23} := T_{23} * T'_{23}$, i.e. these are trivializations that satisfy via (6.3)

$$\mathcal{A}_G(\varphi_{ij}, T_{ij}) \cdot \mathcal{A}_G(\varphi'_{ij}, T'_{ij}) = \mathcal{A}_G(\tilde{\varphi}_{ij}, \tilde{T}_{ij}) \cdot \exp \left( \int_{D^2} \Phi^*_{ij} \rho \right). \quad (6.8)$$
The multiplicativity we have to show is now equivalent to the identity

\[ T_{13} \ast T'_{13} = \lambda_G(\widetilde{T}_{12} \otimes \widetilde{T}_{23}). \]

It follows from (6.6), (6.7) and (6.8) upon computing the left hand side with \( \widetilde{T}_{13} := \lambda_G(\widetilde{T}_{12} \otimes \widetilde{T}_{23}) \).

Summarizing, we obtain:

**Theorem 6.8.** Let \( G \) be a compact, connected, simply-connected Lie group, and let \( \mathcal{G} \) be a bundle gerbe over \( G \) with connection of curvature \( H \). Then, the Mickelsson product equips the transgression \( L_G \) with the structure of a fusion extension of \( LG \).

### 7 The Construction of String 2-Group Models

In this section we consider a compact, simple, simply-connected Lie group \( G \) such as \( \text{Spin}(n) \) for \( n = 3 \) or \( n > 4 \). We briefly review the “basic” bundle gerbe \( G_{\text{bas}} \) over \( G \) whose Dixmier-Douady class generates \( H^3(G, \mathbb{Z}) \cong \mathbb{Z} \), following Gawędzki-Reis [GR02, GR03], Meinrenken [Mei02], and Nikolaus [Nik09].

We choose a Weyl alcove \( \mathfrak{A} \) in the dual \( t^* \) of the Lie algebra of a maximal torus of \( G \). For these exist canonical choices [GR03, Section 4]. The alcove \( \mathfrak{A} \) parameterizes conjugacy classes of \( G \) in terms of a continuous map \( q : G \rightarrow \mathfrak{A} \). We denote by \( \mathfrak{A}_\mu := \mathfrak{A} \setminus f_\mu \) the complement of the closed face \( f_\mu \) opposite to a vertex \( \mu \) of \( \mathfrak{A} \). The preimages \( U_\mu \) of \( \mathfrak{A}_\mu \) under \( q \) form a cover of \( G \) by open sets. We denote by \( G_\mu \) the centralizer of \( \mu \) in \( G \) under the coadjoint action. These centralizer groups come with central \( S^1 \)-extensions \( \hat{G}_\mu \) which are trivial if and only if \( G_\mu \) is simply-connected. Each open set \( U_\mu \) supports a smooth map \( \rho_\mu : U_\mu \rightarrow G/G_\mu \), and thus the principal \( G_\mu \)-bundle \( P_\mu := \rho_\mu^* G \). The problem of lifting the structure group of \( P_\mu \) from \( G_\mu \) to \( \hat{G}_\mu \) defines a lifting bundle gerbe \( L_\mu \) over \( U_\mu \). These local lifting bundle gerbes glue together and yield the basic gerbe \( G_{\text{bas}} \). Further, each \( L_\mu \) can be equipped with a connection, and the glued connection on \( G_{\text{bas}} \) has curvature \( H \), for a certain normalization of the bilinear form \( \langle -, - \rangle \) in (6.1).

The transgression \( L_{G_{\text{bas}}} \) is a fusion extension of \( LG \) (Theorem 6.8), so that the multiplicative regression functor of Section 5 produces a strictly multiplicative, diffeological bundle gerbe

\[ \mathcal{R} := \mathcal{R}(L_{G_{\text{bas}}}, \lambda_{G_{\text{bas}}}) \]
over $G$. We may now optionally proceed in the following two ways:

1.) Theorem 5.2 shows that $\mathcal{R} \cong \mathcal{G}_{bas}$; whence the class of $\mathcal{R}$ generates $H^3(G, \mathbb{Z}) \cong H^4(BG, \mathbb{Z})$. Thus, the 2-functor $\mathcal{R}$ produces a central, strict, diffeological 2-group extension

$$\mathcal{B}S^1 \longrightarrow \Gamma_\mathcal{R} \longrightarrow \mathcal{G}_{dis}$$

with the same class, so that, for $G = \text{Spin}(n)$, $\Gamma_\mathcal{R}$ is a 2-group model for $\text{String}(n)$. Let us summarize the structure of $\Gamma_\mathcal{R}$ by assembling the various constructions: its space of objects is $P_1 G$ and its space of morphisms is $\ell^* L_{\mathcal{G}_{bas}} = P_1 G[2] \times \text{pr} L_{\mathcal{G}_{bas}}$, composition is the fusion product $\lambda_{\mathcal{G}_{bas}}$, and multiplication is the Mickelsson product. We remark that $\Gamma_\mathcal{R}$ has (essentially) the same objects and morphisms as the model of $\text{BCSS07}$, but the composition is defined in $\text{BCSS07}$ using the multiplication (the Mickelsson product) and here using the fusion product.

2.) Theorem 5.2 not only shows that $\mathcal{R} \cong \mathcal{G}_{bas}$, it also provides a distinguished 1-isomorphism

$$\mathcal{A}_{\mathcal{G}_{bas}, y} : \mathcal{G}_{bas} \longrightarrow \mathcal{R},$$

where $y \in Y$ is an element in the surjective submersion of $\mathcal{G}_{bas}$ that projects to $1 \in G$. In the construction of $\mathcal{G}_{bas}$ outlined above there is a such an element: the identity element $1 \in G$ lies in the open set $U_0$ associated to the origin $0 \in \mathfrak{g}^*$. Accordingly, its stabilizer is $G_0 = G$, and $P_0$ is the trivial principal $G$-bundle over $U_0$. As such, it has a canonical element $p = (1, 1) \in P_0 = U_0 \times G$. In the gluing construction of the local lifting gerbes $\mathcal{L}_\mu$ the surjective submersion $\pi : Y \longrightarrow G$ of $\mathcal{G}_{bas}$ is the disjoint union of total spaces $P_\mu$ of the submersions of $\mathcal{L}_\mu$; thus, $p \in Y$. Now, the multiplicative structure on $\mathcal{R}$ can be “pulled back” to $\mathcal{G}_{bas}$ along $\mathcal{A}_{\mathcal{G}, p}$.

The result is a diffeological multiplicative structure on the finite-dimensional bundle gerbe $\mathcal{G}_{bas}$. Its 1-isomorphism $\mathcal{M}$ involves a certain subduction $\chi : Z \longrightarrow Y' := Y_{1,2} \times_G Y_{12}$, where $Y_{1,2}$ and $Y_{12}$ are the smooth manifolds we have encountered in Section 2. It further involves a diffeological principal $S^1$-bundle $Q$ over $Z$. General bundle gerbe theory $\text{Wal07}$ Theorem 1] shows that $Q$ descends along $\chi : Z \longrightarrow Y'$. But a diffeological principal $S^1$-bundle over a smooth manifold is automatically smooth $\text{Wal07}$ Theorem 3.1.7]. This defines a new, smooth 1-isomorphism $\mathcal{M}'$. Both steps are functorial so that the associator $\alpha$ for $\mathcal{M}$ descends to an associator $\alpha'$ for $\mathcal{M}'$. Since smooth manifolds embed fully and faithfully into diffeological spaces, it follows that $\alpha'$ is smooth. Thus, $(\mathcal{G}_{bas}, \mathcal{M}', \alpha')$ is a smooth, multiplicative bundle gerbe over $G$ whose class generates $H^4(BG, \mathbb{Z})$. Under the
2-functor (3.2.2) it hence yields a smooth, finite-dimensional Lie 2-group extension $\Gamma_{G_{bas}}$ of $G$ by $BS^1$ of the same class. In particular, for $G = \text{Spin}(n)$, it is a 2-group model for $\text{String}(n)$.

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