Properties of stress field and differential equation of motion

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Abstract. In continuum mechanics, the stress and the divergence of stress tensor are two vectors introduced to describe the internal force distribution and the resultant force of stress on material element, respectively. This study discusses the properties of the vectors under the basic assumption of classical continuum mechanics. To analyze the properties of the two vector fields, a vector whose gradient is the stress tensor is introduced. Then the stress and the divergence of the stress tensor are expressed with the introduced vector. The results show that the stress field is a curl free field when the stress tensor is symmetric and the traditional understanding on the divergence of stress tensor exceeds the limitation that stress field is a curl free field. Based on the conclusions, this paper further analyzes the motion of material element described with differential equation of motion, the shear wave in elastomer and viscous flow in fluid. It is concluded that the traditional understanding on the differential equation of motion is beyond Newton's second law of motion and the motion of material elements exceeds the assumption of particle motion. Then, the study proposes a new differential equation of motion to derive the traditional defined shear wave in elastomer and viscous flow in fluid. The results show that shear stress reciprocity is unnecessary in classical continuum mechanics, and the viscous force defined by Newton is essentially different from that based on deformation theory.

Keywords: Classical continuum mechanics, Stress field, Shear stress reciprocity, Elastic wave, Viscous flow
1. Introduction

Continuum mechanics studies the motion, deformation and failure of deformed media such as fluids and solids under the continuum hypothesis, where real fluids and solids are considered to be perfectly continuous and are paid no attention to their molecular structure [1, 2, 3]. The subject is the basis and framework of engineering science. With the continuous development of engineering and technology, continuum mechanics has been fully applied in aerospace [4, 5], information technology [6, 7], biomedical engineering [8, 9], micro/nano technology [10, 11, 12] and other fields. At the same time, the application of continuum mechanics in these fields promotes its development.

Continuum hypothesis enables the stress to be defined to describe the internal force distribution and the equilibrium for free body in continua via the powerful methods of calculus [1, 3]. In order to conveniently describe the stress on bounding surface of free body and the equilibrium for the free body whose volume tends to zero under the resultant force, the stress tensor is introduced into continuum mechanics [1, 2, 3, 13]. The introduce of stress tensor also brings convenience in describing the relationship between stress and deformation of continua. Since continuum mechanics is believed to be the branch of classical mechanics, the motion of nonzero-volume elements constituting continua is treated as the motion of particles and their dynamics is believed to be described with Newton’s second law of motion in the initial configuration [1, 2, 3, 14]. Under the assumption, the stress tensor is proved to be symmetric in classical continuum mechanics [13, 14, 15]. As mentioned above, stress tensor is a mathematical quantity rather than a physical quantity, which is introduced to conveniently describe the stress in continua and the resultant surface force acting on material elements whose volume tends to zero. The
stress and the divergence of stress tensor are vector fields that can be expressed with stress tensor. Therefore, the symmetry of stress tensor should imply the properties of these two vector fields like that displacement is curl free when rotation tensor, the antisymmetric part of the gradient of displacement, is vanished. According to what I have learnt, the properties of these two vector fields have not been studied yet.

Due to the lack of understanding of the properties of stress and divergence of stress tensor, there are lots of questions cannot be answered clearly in continuum mechanics. For example, the rotation of material element is ignored when its motion is described, but the rotation of a material element is admitted when the deformation of continua is described and the rotation tensor is described as local rigid body rotation of material element [2, 13, 14, 15]. In the theories of elasticity and fluid dynamics, the local rigid body rotation related to rotation tensor in the theory of elasticity and rotation rate tensor in the theory of fluid dynamics does not contribute stress. Then paradoxes are manifested in theory of elasticity and the theory of fluid dynamics that the divergence of stress tensor seems proportional to a spatial derivative of rotation vector in elastic wave equation and a spatial derivative of the vorticity in Navier-Stokes equation [2, 13, 14]. In particular, the both solenoidal and irrotational part of velocity field, which can be expressed by scalar potential without divergence, contributes to shear stress, but it does not show up in the Navier-Stokes equation. It looks like that shear stress (viscous force) does not always course energy dissipation in viscous flow. In my opinion, only the properties of the stress field are clarified, these questions can be clearly answered.

The paper studies the properties of stress field when stress tensor is a symmetric and the limitation of stress field properties on the divergence of stress tensor. It is shown from
this work that the stress field is curl free when stress tensor is symmetric and the divergence of stress tensor can be expressed by the Laplacian of a vector field whose gradient is the stress tensor. The definition of shear wave in elastomer and viscous flow in fluid means that the stress tensor can be asymmetric, the stress field can be a curl field and the motion of material element goes beyond the particle motion.

2. Property of stress field

Due to the complexity of the relationship between stress field and strain (strain rate) field in continuum, the stress tensor, strain (or strain rate) tensor and rotation (or rotation rate) tensor are introduced in continuum mechanics, and the relationship between stress field and strain field is shown via the relationship between stress tensor and strain tensor [1-3, 13-15]. In order to figure out the properties of the stress and divergence of stress tensor when stress tensor is symmetric, the study first analyzes the relationship between the properties of a vector field and its gradient.

Assuming that $A$ represents an arbitrary vector field, two vectors infinitesimal close in distance satisfy the following relationship under linear expansion:

$$A(R + \delta R) = A(R) + \delta R \cdot \nabla A,$$

where $\nabla$ is the vector operator del, $R$ is the radius vector. Separating the gradient of $A$ into three tensors: a spherical tensor $\alpha$, a deviatoric tensor $\alpha'$ and a rotation tensor $\chi$. They can be expressed with the gradient of $A$ as:

$$\alpha = \frac{1}{6} tr (\nabla A + A \nabla) I,$$

$$\alpha' = \frac{1}{2} (\nabla A + A \nabla) - \alpha,$$
\[ \chi = \frac{1}{2} (\nabla A - A \nabla), \tag{4} \]

where, \( I \) represents second order unit tensor. When vector \( A \) represents displacement field in elasticity, the spherical tensor \( a \), deviatoric tensor \( a' \) and rotation tensor \( \chi \) correspondingly describe the volume expansion, shear deformation and rigid body rotation of material element at a point. With Equations (2) to (4), the divergence and curl of \( A \) can be expressed as:

\[
\nabla \cdot A = \text{tr} \left( a + a' + \chi \right) = \text{tr} \left( a \right), \tag{5}
\]

\[
\nabla \times A = \varepsilon : \left( a + a' + \chi \right) = \varepsilon : \chi, \tag{6}
\]

here, \( \varepsilon \) is the permutation symbol. It is obtained from Equations (5) and (6) that the divergence and curl of a vector field are included in spherical tensor and rotation tensor, respectively. The vector field \( A \) is a curl free field when the gradient of \( A \) is a symmetric tensor, and the vector field \( A \) is a curl field when the gradient of \( A \) is an asymmetric tensor.

By taking the divergence of a gradient, the Laplace operator is obtained. The Laplacian of a vector field is another vector field and is expressed as:

\[
\nabla^2 A = \nabla \nabla \cdot A - \nabla \times \nabla \times A. \tag{7}
\]

By submitting Equations (5) and (6) into Equation (7), the Laplacian of the vector field \( A \) can be rewritten as:

\[
\nabla^2 A = \nabla \left( \text{tr} \left( a \right) \right) - \nabla \times \left( \varepsilon : \chi \right). \tag{8}
\]

It is seen from Equations (7) and (8) that a vector field can be divided into two parts: the divergence part and curl part. The two parts can both reduced to the field both solenoidal and irrotational.

It is known that stress tensor is introduced to describe the internal force in continua.
Therefore, the symmetric second-order stress tensor can be expressed by the gradient of a curl free vector field. Assuming that the curl free vector field is $\Sigma^S$ called the stress potential here, the symmetric second-order stress tensor symbolled with $\sigma^S$ can be expressed with $\Sigma^S$ as:

$$\sigma^S = \nabla \Sigma^S.$$  

(9)

With Equations (1) and (9), the stress symbolled with $F$ can be expressed as:

$$F = n \cdot \sigma^S = \frac{\Sigma^S (R + \delta R) - \Sigma^S (R)}{|\delta R|}.$$  

(10)

with $n$ the unit vector of outer normal of surface element. It is seen from Equation (10) that the stress $F$ is the directional derivatives of the stress potential $\Sigma^S$. Due to the stress potential $\Sigma^S$ is a curl free field, the stress field $F$ is also a curl free field. With Equation (9) and the properties of the stress potential $\Sigma^S$, the divergence of stress tensor can be rewritten as follows:

$$\nabla \cdot \sigma^S = \nabla^2 \Sigma^S = \nabla \nabla \cdot \Sigma^S = \nabla \left( tr\left(\sigma^S\right) \right).$$  

(11)

It is obtained from Equation (11) that the divergence of stress tensor $\sigma^S$ equals the gradient of the trace of $\sigma^S$. Since the trace of $\sigma^S$ is an invariant, the divergence of stress tensor $\sigma^S$ doesn’t change with the selection of coordinates.

Reviewing the divergence of stress tensor $\sigma^S$ without considering the properties of stress field, the properties of the traditional expression of the divergence of stress tensor $\sigma^S$ changes with the selection of coordinates and the divergence of stress tensor $\sigma^S$ degenerates into a curl free field in the coordinate system when the coordinate axes are consistent with the eigenvectors of the stress tensor. For example, when plane shear wave propagation in an elastomer, the eigenvectors of the stress tensors at different point are the same. Therefore,
no shear wave equation should have been derived because the stress tensor $\sigma^S$ in the coordinate system whose coordinate axes are consistent with the eigenvectors of the stress tensor does not have shear stress components. This means that the traditional understanding of the divergence of stress tensor $\sigma^S$ exceeds the limitation that the stress field is a curl free field.

In the differential equation of motion, the divergence of stress tensor $\sigma^S$ describes the resultant force of the stress acting on the material element. Since the traditional understanding of the divergence of stress tensor $\sigma^S$ exceeds the limitation that the stress field is a curl free field, the differential equation of motion breaks through the assumption that the motion of a material element forming continuum is particle motion.

3. **Elastic wave equation derivation from traditional motion description**

In classical continuum mechanics, a continuum is regarded as a set of particles. By treating a material element as particle and only considering its translation, the motion equation of material element in differential form is expressed in the following form [2, 3]:

$$\rho \frac{Dv}{Dt} - \nabla \cdot \sigma^S - f = 0,$$

where $D/Dt$ is the material derivative, $v$ is the velocity of material element translation, $\rho$ is the mass density, $\sigma^S$ is the symmetric second-order stress tensor, $f$ is the body force which is a curl free field.

With Equation (11), the motion equation of material element forming a continuum can be rewritten as:

$$\rho \frac{Dv}{Dt} - \nabla \nabla \cdot \Sigma^S - f = 0.$$
It is obtained from Equation (13) that the translation of material element is determined by the normal stress. The deviatoric stress does not contribute to the translation of material element, which is different from the traditional understanding.

For elastomer with small deformation, the element translation is expressed in differential form as [13, 14]:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla \cdot \mathbf{\sigma}^s = \mathbf{f}, \quad (14)$$

here, $\partial / \partial t$ is the time derivative, $\mathbf{u}$ is the displacement. The constitutive relation and strain-displacement relation of an isotropic elastomer in component form is expressed as:

$$\mathbf{\sigma}^s_{ij} = C_{ijkl} e^s_{kl}, \quad (15)$$

$$e^s_{ij} = \frac{1}{2} (u_{k,j} + u_{l,k}), \quad (16)$$

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right), \quad (17)$$

where, $e^s_{ij}$ is strain tensor, $C_{ijkl}$ is elastic tensor, $\delta_{ij}$ is the Kronecker delta. $\lambda$ and $\mu$ are the Lamé constants. When the displacement field is only caused by deformation, the displacement field is a curl free field.

Substituting the strain-displacement relations (Equation (16)) into the constitutive relations (Equation (15)) and subsequently substituting the constitutive relations expressed with displacement into the equation of motion (Equation (14)), the displacement equation of motion without considering the limitation of stress field on the divergence of stress tensor $\mathbf{\sigma}^s$ can be expressed in vector notation as [14]:

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \quad (18)$$

which is also called the Navier equation. In the traditional derivation of elastic wave
equation, the following formula is considered to be true:
\[ \nabla^2 u = \nabla (\nabla \cdot u) - \nabla \times \nabla \times u. \] (19)

Then the Navier equation is rewritten as:
\[ (\lambda + 2\mu) \nabla \nabla \cdot u - \mu \nabla \times \nabla \times u + f = \rho \frac{\partial^2 u}{\partial t^2}. \] (20)

From the analysis in above section, we know that the stress field and the displacement caused by deformation are curl free fields. For elastomer with small deformation, the stress potential \( \Sigma^s \) and the displacement \( u \) caused by deformation (the strain potential) should meet the following relationship:
\[ \nabla \nabla \cdot \Sigma^s = (\lambda + 2\mu) \nabla \nabla \cdot u. \] (21)

Therefore, under the traditional motion description, the displacement field should be a curl free field and only longitudinal wave can be derived. Equation (20) breaks the precondition that local rigid body rotation does not contribute to stress and stress field is a curl free field. Since shear waves exist objectively in elastic media, the motion of material element forming a continuum should be beyond the particle motion description and cannot be described fully by Newton's second law of motion.

4. Elastic wave equation derivation from new motion description

Though the existence of shear wave is objective in real solid media, it is hard to say whether the shear wave is consistent with our definition. The study does not delve into this issue and only derives the elastic wave equation including the traditional defined shear wave. To theoretically obtain the elastic wave equation, a new motion equation needs to be proposed which can generally describe the motion of material elements in classical
continuum mechanics domain. Since the constitutive relation and strain-displacement relationship of elastomer is established based on the motion of material element, the constitutive relation and strain-displacement relationship of elastomer should be modified correspondingly when the motion equation is changed.

According to the properties of displacement field during shear wave propagation, the study believes that the motion equation of material element forming a classical continuum can be expressed as:

$$\rho \frac{D\mathbf{v}}{Dt} - \nabla \Sigma - \mathbf{f} = 0,$$

(22)

here, $\Sigma$ is an arbitrary vector field which can be a curl field. In this case, the stress tensor is an asymmetric tensor. The symmetric part and antisymmetric part of stress tensor can be separated respectively as:

$$\sigma^s = \frac{1}{2} \left( \nabla \Sigma + \Sigma \nabla \right),$$

(23)

$$\sigma^A = \frac{1}{2} \left( \nabla \Sigma - \Sigma \nabla \right).$$

(24)

Replacing the stress potential $\Sigma$ with stress tensor, the motion equation of material element is rewritten as:

$$\nabla \left( tr (\sigma^s) \right) - \nabla \times \left( \mathbf{e} : \sigma^A \right) + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}.$$

(25)

It is seen that under the new motion description of material element shear stress reciprocity is unnecessary in classical continuum mechanics. When stress field is a curl field, the stress tensor is asymmetric.

For elastomer with small deformation, the convective acceleration is ignored and the motion equation can be written as:
\[ \nabla \left( tr(\sigma^s) \right) - \nabla \times (\varepsilon : \sigma^A) + f = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \] (26)

Under the new motion description, the study believes that the constitutive relation and strain-displacement relationship of isotropic elastomer should be expressed as follows:

\[ \sigma = C : e, \] (27)

\[ e = \nabla \mathbf{u}, \] (28)

here, \( e \) describes the strain including traditional defined one and local rigid body rotation.

For isotropic elastomers, the elastic tensor should be expressed as:

\[ C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl}. \] (29)

It is seen from Equations (27) to (29) that the stress tensor is asymmetric when displacement field is rotational and is symmetric when displacement field is curl free. This means that Equations (27) to (29) contain the case in the classical theory of elasticity. In fact, the traditional expression of the elastic tensor is redundant because the local rigid body rotation does not contribute to the stress has been declared in the classical theory of elasticity. Though the traditional elastic tensor is replaced with Equation (29), the stress tensor is symmetric only if strain tensor is symmetric.

Submitting Equation (28) into Equation (27), the motion equation of elastomer is expressed as:

\[ (\lambda + \mu) \nabla \left( tr(\varepsilon^s) \right) - \mu \nabla \times (\varepsilon : e^A) + f = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \] (30)

with

\[ \varepsilon^s = \frac{1}{2} (\nabla \mathbf{u} + \mathbf{u} \nabla), \] (31)

\[ e^A = \frac{1}{2} (\nabla \mathbf{u} - \mathbf{u} \nabla), \] (32)
where, $e^S$ and $e^t$ are the strain tensor and rotation tensor. Via the relationship between vector field and its gradient (Equations (5) and (6)), Equation (30) is rewritten as:

$$\left(\lambda + \mu\right)\nabla \nabla \cdot u - \mu \nabla \times \nabla \times u + f = \rho \frac{\partial^2 u}{\partial t^2}.$$  \hspace{1cm} (33)

It is seen from Equation (33) that the new wave equation derived from new differential equation of motion can both predict the existence of longitudinal wave and shear wave in elastomer.

In the classical theory of elasticity, though the local rigid body rotation is admitted, the traditionally defined deformation is only considered for the deformation coordination, which means that the material elements composing an elastomer can rotate freely. This is inconsistent with the facts. In admitting the rotation of material element, the study considers that the deformation coordination of an elastomer should be described as follows [16]:

$$\nabla \times e = 0.$$  \hspace{1cm} (34)

5. Derivation of Navier-Stokes equation from new motion equation

The derivation of elastic wave equation can only illustrate that the new differential equation of motion is appropriate to explain the existence of traditional defined shear wave. Whether the differential equation of motion is universal in describing the motion of material element forming continua is questionable. Here I verify the generality of the new differential equation of motion in describing the motion of continua by deriving the Navier-Stokes equation.

For viscous fluids, the stress is related to the volume deformation and shear flow. When the shear flow occurs in viscous fluid, the velocity field is a curl field. Therefore, the
stress field in the viscous fluid is a curl field, and the stress tensor is an asymmetric tensor. Separating the stress caused by volume deformation from the stress caused by shear flow, the motion equation of material element is expressed as:

\[ \nabla \left( tr(\mathbf{\sigma}) \right) - \nabla \times (\mathbf{\varepsilon} : \mathbf{d}) + f = \rho \frac{D\mathbf{v}}{Dt}, \]  

(35)

with \[ \mathbf{d} = \mathbf{\sigma} - \frac{1}{3} tr(\mathbf{\sigma}) \mathbf{I}, \]  

(36)

here, \( \mathbf{d} \) is a deviatoric stress. For a Newtonian fluid, the relation between deviatoric stress and rate of deviatoric strain \( \xi \) in component form is expressed as:

\[ d_{ij} = \eta \delta_{ik} \delta_{lj} \xi_{kl}, \]  

(37)

where, \( \eta \) is the viscosity of fluid. The relation between deviatoric strain rate \( \xi \) and velocity \( \mathbf{v} \) is

\[ \xi = \nabla \mathbf{v} - \frac{1}{3} tr(\nabla \mathbf{v}) \mathbf{I}. \]  

(38)

Submitting Equation (38) into Equation (37) and consequently submitting the relation between deviatoric stress and deviatoric strain rate expressed with velocity into Equation (35), the motion equation can be expressed with velocity as:

\[ -\nabla p - \eta \nabla \times (\mathbf{\varepsilon} : \nabla \mathbf{v}) + f = \rho \frac{D\mathbf{v}}{Dt}. \]  

(39)

Since the following relations are hold:

\[ \nabla \times (\mathbf{\varepsilon} : \nabla \mathbf{v}) = \nabla \times \nabla \times \nabla \mathbf{v} = -\nabla^2 \mathbf{v} + \nabla \nabla \cdot \mathbf{v}, \]  

(40)

Equation (40) can be rewritten as:

\[ -\nabla p + \eta \left( \nabla^2 \mathbf{v} - \nabla \nabla \cdot \mathbf{v} \right) + f = \rho \frac{D\mathbf{v}}{Dt}. \]  

(41)
Equation (41) is identical to the Navier-Stokes equation of motion. For further special case of incompressible fluid, the mass conservation reduces to $\nabla \cdot \mathbf{v} = 0$, Equation (41) reduces to:

$$-\nabla p + \eta \nabla^2 \mathbf{v} + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}.$$  \hspace{1cm} (42)

Equation (42) is the same as the Navier-Stokes equation of motion for incompressible fluid. This means that the new motion equation proposed in this study is also suitable for describing the motion of material elements forming fluids.

It is seen from Equation (39) that the viscous force is generated only by the shear flow of fluid (or relative slide of fluid elements) rather than shear deformation of fluid. This means that the viscous force defined by Newton is essentially different from that based on deformation theory. By comparing the viscous force defined by Newton and deformation theory, there are two main differences between them. One is that the former shows that the fluid flow cannot produce viscous force when its velocity field is described by a scalar potential, while the latter shows that the flow may produce viscous force when its velocity field is described by scalar potential. The other is that the former believes that viscous force is related to local rigid body rotation (local vorticity), while the latter believes that viscous force is independent of local rigid body rotation. The difference seems to have been found by Batchelor. He pointed out that there is a paradox in the description of viscous force by deformation theory that viscous force should be independent of the local vorticity [2]. Since the viscous force term in the Navier-Stokes equation is only related to the rotational velocity field, it is proved that Newton's definition is right and the definition based on deformation theory is wrong. This result once again shows that the current deformation theory used in continuum mechanics has serious problems and a new theory is urgently
needed to describe the motion and deformation of continuum.

6. Discussion and conclusions

The study analyzes the properties of stress field when stress tensor is a symmetric and the limitation of stress field properties on differential equation of motion. It is concluded that the symmetry of stress tensor indicates that the stress field in continuum is a curl free field and the longitudinal wave can only be derived from the traditional motion equation. The traditional understanding on the divergence of stress tensor $\sigma^S$ exceeds the limitation that the stress field is a curl free field. Consequently, the differential equation of motion breaks through the assumption that the motion of a material element forming continuum is a particle motion.

In order to obtain the wave equation that can describe both longitudinal wave and shear wave in elastomer, the study proposed a new motion equation. The new motion equation shows that the reciprocity of stress tensor is unnecessary in continuum mechanics and the stress field is a curl field when stress tensor is asymmetric. In the derivation, the constitutive relations, strain-displacement relations and deformation coordination are correspondingly modified. The new constitutive relations indicate that the local rigid body rotation contribute to stress and a rotational stress field related to local rigid body rotation is produced to balance the rotational acceleration field when the shear wave propagates in elastomer. The study also verified the universality of the new motion equation in describing the motion of material element by deriving the Navier-Stokes equation of motion. The result shows that the Navier-Stokes equation of motion can be derived from the new motion equation when the relations between deviatoric stress and rate of deviatoric strain is correspondingly modified to fit the new stress tensor. It is obtained from the derivation of
the Navier-Stokes equation that the viscous force field related to shear flow is a curl field and can balance the curl free force field like pressure field. It is pointed out from the derivation of the Navier-Stokes equation of motion that the viscous force defined by Newton is essentially different from that based on deformation theory. The current deformation theory is unsuitable for the description of viscous force.

It should be pointed out that the new motion equation has made the motion of material elements forming classical continua beyond the description of particle motion. At present only two motion models, particle model and rigid body model, are proposed in classical mechanics and the latter is considered to be the collection of the motion of the former. Only from the derivation of the wave equation, the shear wave can be derived from the conservation of momentum moment by adding rotational degrees of freedom to the material element. In this case, the longitudinal and shear waves in elastomer correspond to the translation and rotation of material element, respectively. However, if we believe that the motion of classical continua can be described by the same equation of motion, it is not advisable to regard the translation and rotation of material elements as independent because the coupling between the curl field and curl free field in fluid cannot be reasonably explained. This implies that continuum mechanics needs to be re-established based on a new theory.

**Declaration of competing interest**

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
Acknowledgments

This work was supported by the scientific research and technology development project of China National Petroleum Corporation (2020B-3713).

Data availability

The data that support the findings of this study are available within the article.

References

[1] Lai, W.M., Rubin, D. H. Krempl, E.: Introduction to continuum mechanics. Butterworth-Heinemann, Oxford (2009).
[2] Batchelor, G. K.: An introduction to fluid dynamics. Cambridge Univ. press, Cambridge (2000).
[3] Malvern, L. E.: Introduction to the Mechanics of a Continuous Medium. Prentice Hall, New Jersey (1969).
[4] Drikakis, D., Kwak, D., Kiris, C. C.: Computational aerodynamics: advances and challenges. Aeronaut. J. (2016). https://doi.org/10.1017/aer.2015.2
[5] Jain, S., Bhatt, V. D., Mittal, S.: Shape optimization of corrugated airfoils. Comput. Mech. (2015). https://doi.org/10.1007/s00466-015-1210-x
[6] Basser, P. J., Mattiello, J., LeBihan, D.: MR diffusion tensor spectroscopy and imaging. Biophys. J. (1994). https://doi.org/10.1016/S0006-3495(94)80775-1
[7] Moura, L. M., Luccas, R., De Paiva, J. P. Q., Amaro Jr, E., Leemans, A., Leite, C. D. C., Otaduy, M. C. G., Conforto, A. B.: Diffusion tensor imaging biomarkers to predict motor outcomes in stroke: a narrative review. Front. Neurol. (2019).
[8] Mealy, J. E., Chung, J. J., Jeong, H. H., Issadore, D., Lee, D., Atluri, P., Burdick, J. A.: Injectable granular hydrogels with multifunctional properties for biomedical applications. Adv. Mater. (2018). https://doi.org/10.1002/adma.20175912

[9] Franze, K., Janmey, P. A., Guck, J., Mechanics in neuronal development and repair. Annu. Rev. Biomed. Eng. (2013). https://doi.org/10.1146/annurev-bioeng-071811-150045

[10] Nix, W. D., Gao H. Indentation size effects in crystalline materials: a law for strain gradient plasticity. J. Mech. Phys. Solids. (1998). https://doi.org/10.1016/S0022-5096(97)00086-0

[11] Yin, Y., Chen, C., Lü, C., Zheng, Q. S.: Shape gradient and classical gradient of curvatures: driving forces on micro/nano curved surfaces. Appl. Math. Mech. (2011). https://doi.org/10.1007/s10483-011-1436-6

[12] Chavoshi, S. Z., Xu, S.: A review on micro-and nanoscratching/tribology at high temperatures: instrumentation and experimentation. J. Mater. Eng. Perform. (2018). https://doi.org/10.1007/s11665-018-3493-5

[13] Graff, K. F.: Wave Motion in Elastic Solids, Dover publications, New York (1975).

[14] Achenbach, J. D.: Wave propagation in elastic solids. Elsevier, Amsterdam (1973).

[15] Landau, L. D., Lifšic, E. M., Lifshitz, E. M., Kosevich, A. M., Pitaevskii, L. P.: Theory of elasticity: volume 7. Elsevier, Oxford (1986).

[16] Qiu, Z.: A simple theory of asymmetric linear elasticity. World J. Mech. (2020). https://doi.org/10.4236/wjm.2020.1010012