SPHERICALLY AVERAGED ENDPOINT STRICHARTZ ESTIMATES FOR THE TWO-DIMENSIONAL SCHRÖDINGER EQUATION

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Abstract. The endpoint Strichartz estimates for the Schrödinger equation is known to be false in two dimensions[7]. However, if one averages the solution in $L^2$ in the angular variable, we show that the homogeneous endpoint and the retarded half-endpoint estimates hold, but the full retarded endpoint fails. In particular, the original versions of these estimates hold for radial data.

1. Introduction

Let $\Delta$ be the Laplacian on $\mathbb{R}^n$ for $n \geq 1$, so that $e^{it\Delta}$ is the evolution operator corresponding to the free Schrödinger equation.

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We consider the problem of obtaining bounds for this operator in the mixed spacetime Lebesgue norms

\[ \|F\|_{L^q_t L^r_x} = \int \|F(t, \cdot)\|_{L^r_x}^q \, dt \]^{1/q}.\]

Such estimates are commonly known as Strichartz estimates and have application to the study of non-linear Schrödinger equations (see e.g. [1]). The following Strichartz estimates are known (see [4],[5]):

**Definition 1.1.** If \( n \) is given, we say that the exponent pair \((q, r)\) is *admissible* if \( q, r \geq 2, (q, r, n) \neq (2, \infty, 2) \) and

\[
\frac{1}{q} + \frac{n}{2r} = \frac{n}{4}.
\]

**Theorem 1.2.** [5] If \( n \) is given and \((q, r), (\tilde{q}, \tilde{r})\) are admissible, then we have the estimates

1. \(
\|e^{it\Delta} f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2_x},
\)
2. \(
\| \int e^{-is\Delta} F(s) \, ds \|_{L^q_t L^r_x} \lesssim \|F\|_{L^{q'}_t L^{r'}_x},
\)
3. \(
\| \int_{s<t} e^{i(t-s)\Delta} F(s) \, ds \|_{L^q_t L^r_x} \lesssim \|F\|_{L^{q'}_t L^{r'}_x}
\)

for all test functions \( f, F \) on \( \mathbb{R}^n, \mathbb{R}^{n+1} \) respectively.

The above conditions on \((q, r)\) are known to be necessary for the homogeneous estimates (1) and (2), but it is not known what the necessary and sufficient
conditions are for the inhomogeneous retarded estimate (3). For further discussion we refer the reader to [4], [7], [5]. In the radial case for \( n > 2 \) there is a further smoothing effect, see [11].

In this paper we investigate the “forbidden endpoint” \((q, r, n) = (2, \infty, 2)\). Accordingly we shall restrict ourselves to the two-dimensional case \( n = 2 \) for the remainder of this paper. With no further assumptions on \( f, F \), the estimates (1), (2), (3) are known to be false even if \((\tilde{q}, \tilde{r})\) are admissible, and even if the \( L^\infty_x \) norm is replaced with the BMO norm; see [7]. The counterexamples are non-radial and involve Brownian motion.

However, one can recover the endpoint estimate by averaging in \( L^2 \) over angular directions. More precisely, let \( L_r^\infty L_\theta^2 \) denote the norm

\[
\|f\|_{L_r^\infty L_\theta^2} = \sup_{r > 0} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right)^{1/2},
\]

with the dual norm \( L_r^1 L_\theta^2 \) defined similarly. Then we have

**Theorem 1.3.** Let \((q, r, n) = (2, L_r^\infty L_\theta^2, 2)\). Then (1) and (2) hold, and the estimate (3) holds if \((\tilde{q}, \tilde{r})\) is admissible.

For radial functions the \( L_r^\infty L_\theta^2 \) norm is just the \( L^\infty \) norm, and so we have

**Corollary 1.4.** Let \((q, r, n) = (2, \infty, 2)\), and let \( f \) and \( F \) be radial. Then (1) and (2) hold. The estimate (3) holds if \((\tilde{q}, \tilde{r})\) is admissible.

Finally, we present a very simple

**Proposition 1.5.** Let \((q, r, n) = (\tilde{q}, \tilde{r}, n) = (2, \infty, 2)\). Then (3) can fail even if \( F \) is radial.

This paper is organized as follows. We first prove (1) for radial \( f \) in Section 2: the estimate (2) follows immediately by duality. It turns out that the estimate

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2After this paper was completed, we learnt that this Corollary was independently proved by Atanas Stefanov.
reduces easily to a maximal oscillatory integral estimate of the type discussed in [9], with a minor complication arising from the behaviour of the Bessel function

\[ J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix\cos\theta} e^{in\theta} d\theta \quad (4) \]

for \( x \) close to \( n \).

We then turn to the positive results for (3) in Section 3. Fortunately we shall be able to obtain these results as an automatic consequence of the homogeneous estimate, by a very general argument of Christ and Kiselev [2].

Finally, we discuss the negative results in Section 4.

We remark that analogous results hold for the wave equation (with \( n = 3 \) playing the role of \( n = 2 \)) but are proved differently. See [5], [6], [7].

2. The homogeneous estimate

In this paper \( C \) denotes an absolute positive constant which may vary from line to line, and we use the notation \( A \lesssim B \) as shorthand for \( A \leq CB \).

In this section we prove (1) for \( (q, r, n) = (L^2, L^\infty_r L^2_\theta, 2) \), which implies (2) by duality.

We will always make the a priori assumption that all functions are in the Schwarz class, and any singular integrals will be evaluated in the principal value sense.

Since \( e^{i\mu A} \) commutes with rotations, and our norms are \( L^2 \) in the angular variable, we may use standard orthogonality arguments to reduce to the case when \( f \) is given by a single spherical harmonic, i.e.
for some \( n \in \mathbb{Z} \) and some function \( f_n(r) \). Our task is then to prove (1) with a bound independent of \( n \).

Fix \( n \); we may assume that \( n \geq 0 \). From the explicit formula for the fundamental solution

\[
e^{it\Delta}f(x) = \frac{C}{t} \int e^{i|x-y|^2/4t} f(y) \, dy
\]

and a change to polar co-ordinates, we obtain

\[
e^{it\Delta}f(re^{i\theta}) = \frac{C}{t} \int e^{i|re^{i\theta} - Re^{i\phi}|^2/4t} f(Re^{i\phi}) \, d\phi \, RdR,
\]

which simplifies to

\[
e^{it\Delta}f(re^{i\theta}) = \frac{C}{t} e^{ir^2/4t} \int_0^\infty \int_0^{2\pi} e^{iR\cos(\theta - \phi)/2t} e^{iR^2/4t} f_n(R)e^{i\alpha} \, d\phi \, RdR.
\]

Making a change of variables \( \alpha = \theta - \phi \) and taking absolute values, we obtain

\[
|e^{it\Delta}f(re^{i\theta})| = \frac{C}{|t|} \left| \int_0^\infty \left( \int_0^{2\pi} e^{iR\cos(\alpha)/2t} e^{i\alpha} d\alpha \right) e^{iR^2/4t} f_n(R) \, RdR \right|.
\]

By (4). the inner integral is \( 2\pi J_n(\frac{rR}{2t}) \). Thus (1) can be rewritten as

\[
\left( \int_{r \geq 0} \sup_{t \geq 0} \int_0^\infty J_n(\frac{rR}{2t}) e^{iR^2/4t} f_n(R) \, RdR \right)^{1/2} \leq \left( \int_0^\infty |f_n(R)|^2 \, RdR \right)^{1/2} \leq \left( \int_0^\infty |f_n(R)|^2 \, RdR \right)^{1/2}
\]

From our a priori assumptions we may replace \( r \geq 0 \) by \( r > 0 \) in the supremum.
Write $\xi = R^2$, and $g(\xi) = f_n(R)$. Also write $x = 1/(8\pi t)$ and $\lambda = r/(2|t|)$. After a change of variables, the above estimate becomes

$$(\int \sup_{\lambda>0} |\int_0^\infty J_n(\lambda|\xi|^{1/2})e^{2\pi ix\xi} g(\xi) \, d\xi|^2 \, dx)^{1/2} \lesssim (\int_0^\infty |g(\xi)|^2 \, d\xi)^{1/2}.$$  

Clearly this estimate will be implied by

$$(\int \sup_{\lambda>0} |\int J_n(\lambda|\xi|^{1/2})e^{2\pi ix\xi} g(\xi) \, d\xi| \, dx)^{1/2} \lesssim (\int |g(\xi)|^2 \, d\xi)^{1/2}$$

where the integrations now range over all of $\mathbb{R}$.

Let $G$ be the Fourier transform of $g$. By Plancherel’s theorem, the above estimate is equivalent to

$$(6) \quad \| \sup_{\lambda>0} |T_\lambda G| \|_2 \lesssim \|G\|_2$$

where $T_\lambda$ is the multiplier defined by

$$\hat{T_\lambda G}(\xi) = J_n(\lambda|\xi|^{1/2})\hat{G}(\xi).$$

We partition the Bessel function $J_n$ smoothly as

$$J_n(r) = m_0(r) + m_1(r) + \sum_{2^j \gg n} m_j(r)$$

where $m_0$, $m_1$, and $m_j$ are supported on $|r| \ll n$, $|r| \sim n$, and $|r| \sim 2^j \gg n$ respectively. We similarly decompose $T_\lambda$ as

$$T_\lambda = T_\lambda^0 + T_\lambda^1 + \sum_{2^j \gg n} T_\lambda^j.$$

Finally, for $j = 0, 1$ or $2^j \gg n$ we let $K^j_\lambda$ be the convolution kernel of the operator $T^j_\lambda$; note that

$$K^j_\lambda(x) = \int e^{2\pi i x \xi} m_j(\lambda|\xi|^{1/2}) \, dx.$$  

When $|r| \ll n$, the phase in (4) is non-stationary. From this one can easily obtain the bounds

$$\|m_0\|_{C^k} \lesssim n^{-N}$$

for any $N, k > 0$. Since $m$ is compactly supported, we thus have

$$|K^0_\lambda(x)| \lesssim n^{-N} \lambda^{-2}(1 + \lambda^{-2}x)^{-k}$$

for any $N, k > 0$. This in turn implies that

$$|T^0_\lambda G(x)| \lesssim n^{-N} MG(x)$$

where $M$ is the Hardy-Littlewood maximal function. Thus the contribution of $T^0_\lambda$ to (6) is acceptable.

We now turn to the contribution of $T^1_\lambda$. We will not attempt to estimate this contribution efficiently, and rely instead on very crude tools. We begin with the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, which we write as

$$\sup_y |f(y)| \lesssim \left( \int |f(y)|^2 + |f'(y)|^2 \, dy \right)^{1/2}.$$  

We apply the change of variables $\lambda = e^{y/n}$ to obtain

$$\sup_\lambda |g(\lambda)| \lesssim \left( \int (n|g(\lambda)|^2 + \frac{1}{n}|\lambda g'(\lambda)|^2) \frac{d\lambda}{\lambda} \right)^{1/2}.$$
Applying this to $g(\lambda) = T^{1}_{\lambda}G(x)$ and taking $L^2$ norms of both sides, we obtain

$$\| \sup_{\lambda} |T^{1}_{\lambda}G|\|_2 \lesssim \left( \int (n\|T^{1}_{\lambda}G\|_2^2 + \frac{1}{n}\|\lambda \frac{\partial}{\partial \lambda} T^{1}_{\lambda}G\|_2^2) \frac{d\lambda}{\lambda} \right)^{1/2}. $$

Applying Plancherel’s theorem, we see that we will be done provided that

$$\int (n|m_1(\lambda)|\xi|^{1/2})^2 + \frac{1}{n}|\lambda \frac{\partial}{\partial \lambda} m_1(\lambda)|\xi|^{1/2})^2 \frac{d\lambda}{\lambda} \lesssim 1$$

uniformly in $\xi$. By rescaling $\lambda$ by $|\xi|^{1/2}$ we may assume $\xi = 1$; from the support of $m_1$ we can thus restrict the integration to the region $\lambda \sim n$. It thus suffices to show that

$$\int_{\lambda \sim n} |m_1(\lambda)|^2 + |m'_1(\lambda)|^2 \lesssim 1.$$ 

However, from (4), the definition of $m_1$, and Van der Corput’s lemma (See e.g. [8]) we have the estimates

$$|m_1(\lambda)| \lesssim n^{-1/3}(1 + n^{-1/3}|\lambda - n|)^{-1/4}$$

and

$$|m'_1(\lambda)| \lesssim n^{-1/2},$$

and the claim follows.

Finally, we consider the contribution of the $T^j_{\lambda}$ to (6). We will show

$$\| \sup_{\lambda > 0} |T^j_{\lambda}G|\|_2 \lesssim 2^{-j/4}\|G\|_2$$

uniformly for $j, n$ such that $2^j \gg n$; this clearly implies that the contribution of the $T^j_{\lambda}$ is acceptable.

Fix $j$. It suffices to show that

$$\|T^j_{\lambda(x)}G\|_{L^2_x} \lesssim 2^{-j/4}\|G\|_2$$
for an arbitrary positive function $\lambda(x)$ which we now fix. We write this as

$$\| \int G(y)K^j_{\lambda(x)}(x-y) \, dy \|_{L^2_x} \lesssim 2^{-j/4}\|G\|_2$$

By the $TT^*$ method, it suffices to show that

$$\| \left( \int K^j_{\lambda(x)}(x-y)K^j_{\lambda(x')} (x'-y) \, dy \right)F(x') \, dx' \|_2 \lesssim 2^{-j/2}\|F\|_{L^2_{x'}}$$

for all test functions $F$. This will follow from the estimate

**Lemma 2.1.** For any $a, b > 0$, $x, x' \in \mathbb{R}$ we have

$$| \int K^j_a(x-y)K^j_b(x'-y) \, dy | \lesssim \Phi_{j,a}(|x-x'|)$$

where $\Phi_{j,a}$ is a radial decreasing non-negative function with

$$\sup_a \| \Phi_{j,a} \|_1 \lesssim 2^{-j/2}.$$ 

Indeed, from this lemma we may bound the left-hand side of (7) pointwise by $C2^{-j/2}MF(x)$.

**Proof** By Parseval’s theorem and the definition of $K^j_{\lambda}$, the left-hand side is

$$C| \int m_j(a|\xi|^{1/2})m_j(b|\xi|^{1/2})e^{2\pi i(x-x')\xi} \, d\xi |.$$ 

On the other hand, from the standard asymptotics of $J_n$ (see e.g. [8]) we have

$$m_j(\xi) = \sum_{\pm} 2^{-j/2}e^{\pm i|\xi|\mu_j^\pm(2^{-j} \xi)}$$
where $\psi_j^\pm(\xi)$ is a bump function on $|\xi| \sim 1$ uniformly in $j, n$. We can therefore rewrite (8) as a finite number of expressions of the form

$$C2^{-j} \int e^{i(\pm a \pm b)|\xi|^{1/2}} e^{2\pi i (x-x') \xi} \psi_j^\pm (2^{-j} a|\xi|^{1/2}) \psi_j^\pm (2^{-j} b|\xi|^{1/2}) d\xi$$

where the $\pm$ signs need not agree.

It suffices to estimate the $\xi > 0$ portion of the integral. From the change of variables $\xi = 2^{2j} a^{-2} t^2$, this becomes

$$C2^j a^{-1} \int e^{2\pi i \left( \frac{2j(\pm a \pm b)}{2\pi a} t + \frac{2j(x-x')}{a^2} t^2 \right)} \psi_\pm(t) \psi_\pm \left( \frac{b}{a} t \right) t dt.$$

We will majorize this expression by $\Phi_{j,a}(|x-x'|)$, where

$$\Phi_{j,a}(r) = C \min(r^{-1/2}, 2^j a^{-1}, 2^j a^{-1} (2^j a^{-1} r)^{-10});$$

it is easy to verify that $\Phi$ satisfies the desired properties.

The first bound of $Cr^{-1/2}$ follows from Van der Corput’s lemma (see e.g. [8]). The second bound of $C2^j a^{-1}$ simply follows from replacing everything by absolute values. To show the third bound, we may assume from the second bound that $|x-x'| \gg 2^{-j} a$. But then the phase $\frac{2j(\pm a \pm b)}{2\pi a} t + \frac{2j(x-x')}{a^2} t^2$ has a derivative of magnitude at least $2^j a^{-1} r$ on the support of $\psi_\pm$, and the bound follows from non-stationary phase.

One can improve the estimate on $T_1^\lambda$ by incorporating techniques from the treatment of $T_0^\lambda$ and $T_2^\lambda$. This will eventually yield a gain of $n^{-\epsilon}$ to (6) for some $n > 0$. This translates to a gain of angular regularity, so that we may replace $L_\theta^2$ by an angular Sobolev space $H_\theta^\epsilon$. By Sobolev embedding this implies that the $L_\theta^2$ can be improved to $L_p^\theta$ for some $p > 2$. A possibly related smoothing effect in higher dimensions was observed in [11].
It is not clear what the best value of \( p \) is. However the negative results in [7] show that this cannot be improved to \( p = \infty \) or to \( p = BMO \).

3. The inhomogeneous estimate

We now prove (3) when \((q, r, n) = (2, L_t^\infty L_{\theta}^2, 2)\) and \((\tilde{q}, \tilde{r})\) is admissible. We first observe that if the restriction \( s < t \) were somehow removed from the integral, the left-hand side of (3) would factor as

\[
\int e^{i(t-s)\Delta} F(s) \, ds = e^{it\Delta} \left( \int e^{-is\Delta} F(s) \, ds \right),
\]

and the claim would then follow by combining (1) and (2).

To finish the proof we need to reinstate the restriction \( s < t \). This can in fact be done very general circumstances, as observed by Christ and Kiselev [2], [3]. More precisely, we have

**Lemma 3.1.** [2] Let

\[
Tf(t) = \int_{\mathbb{R}} K(t, s)f(s) \, ds
\]

be a linear transformation which maps \( L^p(\mathbb{R}) \) to \( L^q(\mathbb{R}) \) for some \( 1 < p < q < \infty \). Then the map

\[
\tilde{T}f(t) = \int_{s<t} K(t, s)f(s) \, ds
\]

also maps \( L^p(\mathbb{R}) \) to \( L^q(\mathbb{R}) \).

For our purposes we need the trivial observation that the argument below extends to the case when \( K \) takes values in \( B(X,Y) \), the space of bounded mappings from one Banach space to another.
Proof We will prove the claim for smooth $f$ only, to avoid technical problems. We normalize so that $\|f\|_p = 1$. Define the function $F(t)$ by $F(t) = \int_{s<t} |f(s)|^p \, ds$. This map $F$ is an order-preserving bijection from $\mathbb{R}$ to $[0,1]$. Partition the interval $[0,1]$ into dyadic intervals in the usual manner. We define a relationship $I \sim J$ on dyadic intervals as follows: $I \sim J$ if and only if $I$ and $J$ are the same size, are adjacent, and the elements of $I$ are strictly less than the elements of $J$. It is easy to verify that for almost every $x < y$ there is a unique pair $I, J$ such that $x \in I$, $y \in J$, and $I \sim J$. Applying this with $x = F(s)$, $y = F(t)$, we obtain

$$\int_{s<t} ds = \int_{F(s)<F(t)} ds = \sum_{I,J:I \sim J} \chi_{F^{-1}(J)}(t) \int_{F^{-1}(I)} ds.$$ 

We thus have

$$\tilde{T}f = \sum_{I,J:I \sim J} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f).$$

We need to show that $\|\tilde{T}f\|_q \lesssim 1$. It suffices to prove that

$$\sum_{I,J:I \sim J, l(I) = 2^{-j}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f) \|_q \lesssim 2^{-\varepsilon j}$$

uniformly in $j \geq 0$ for some $\varepsilon > 0$, where $l(I)$ denotes the sidelength of $I$.

Fix $j$. Since for each $I$ there are at most two $J$, and the functions $\chi_{F^{-1}(J)}$ have essentially disjoint support, we can estimate the left-hand side of (9) by

$$\left( \sum_{I, l(I) = 2^{-j}} \| T(\chi_{F^{-1}(I)} f) \|_q^q \right)^{1/q}.$$

By the assumption on $T$, this is bounded by
\[
\left( \sum_{I:|I| = 2^{-j}} \| \chi_{F^{-1}(I)} f \|_p^q \right)^{1/q}.
\]

But by construction \( \| \chi_{F^{-1}(I)} f \|_p = 2^{-j/p} \), hence this sum is just

\[2^{-j \left( \frac{1}{p} - \frac{1}{q} \right)},\]

and the claim follows from the hypothesis \( p < q \).

The requirement \( p < q \) is necessary, as can be seen by considering the Hilbert transform. The lemma also holds in the ranges \( 1 = p \leq q \leq \infty \) and \( 1 \leq p \leq q = \infty \), but for more trivial reasons. We remark that a stronger maximal version of this lemma appears in [2].

4. Negative results

We now show why (3) fails when \((q,r,n) = (2,\infty,2)\) and \((\tilde{q},\tilde{r})\) is not admissible, even when \( F \) is radial.

From dimensional analysis (recalling that time has twice the dimensionality of space for the purposes of the Schrödinger equation) we obtain the necessary condition for (3)

\[
\frac{2}{q} + \frac{2}{r} + 2 = \frac{2}{\tilde{q}} + \frac{2}{\tilde{r}}.
\]

Thus we must have

\[
\frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} = \frac{1}{2}.
\]

Therefore the only case left to consider is the double forbidden endpoint
\((q, r, n) = (\tilde{q}, \tilde{r}, n) = (2, \infty, 2)\).

By a limiting argument we may assume that \(F\) is a measure on the time axis \(x = 0\):

\[ F(x, s) = g(s)\delta(x). \]

Since \(G(0) \leq \|G\|_\infty\) for any \(G\), it suffices to disprove the estimate

\[ \left\| \int_{s<t} [e^{i(t-s)\Delta} F(s)](0)\, ds \right\|_{L_t^2} \lesssim \|g\|_{L_t^2}. \]

By (5), this is

\[ \left\| \int_{s<t} \frac{1}{s-t} g(s)\, ds \right\|_{L_t^2} \lesssim \|g\|_{L_t^2}, \]

which is clearly false (e.g. take \(g = \chi_{[0,1]}\)).

It is easy to modify this argument to show that the estimate continues to fail if the \(L^\infty\) or \(L^1\) norms are replaced by BMO or \(H^1\) norms, or if some frequency restriction or smoothness condition is placed on \(F\).

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REFERENCES

1. T. Cazenave, F.B. Weissler, *Critical nonlinear Schrödinger Equation*, Non. Anal. TMA, 14 (1990), 807–836.
2. M. Christ, A. Kiselev, *A maximal inequality*, preprint.
3. M. Christ, A. Kiselev, *Absolutely continuous spectrum for one-dimensional Schrödinger operators with slowly decaying potentials: some optimal results*, J. Amer. Math. Soc., to appear.
4. J. Ginibre, G. Velo, *Smoothing Properties and Retarded Estimates for Some Dispersive Evolution Equations*, Comm. Math. Phys., 123 (1989), 535–573.
5. M. Keel, T. Tao, *Endpoint Strichartz Estimates*, to appear, Amer. Math. J.
6. S. Klainerman, M. Machedon, *Space-time Estimates for Null Forms and the Local Existence Theorem*, Comm. Pure Appl. Math., 46 (1993), 1221–1268.
7. S. J. Montgomery-Smith, *Time Decay for the Bounded Mean Oscillation of Solutions of the Schrödinger and Wave Equation*, Duke Math J. 19 (1998), 393–408.
8. E. M. Stein, *Harmonic Analysis*, Princeton University Press, 1993.
9. E. M. Stein, *Oscillatory integrals related to Radon-like transforms*, J. Fourier Anal. and Appl., Kahane special Issue (1995): 535–557.
10. T. Tao, A. Vargas, L. Vega, *A bilinear approach to the restriction and Kakeya conjectures*, to appear, J. Amer. Math. Soc.
11. M. Vilela, *Regularity of solutions to the free Schrödinger equation with radial initial data*, preprint.