A quantum field algebra

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Abstract. The Laplace Hopf algebra created by Rota and coll. is generalized to provide an algebraic tool for combinatorial problems of quantum field theory. This framework encompasses commutation relations, normal products, time-ordered products and renormalisation. It considers the operator product and the time-ordered product as deformations of the normal product. In particular, it gives an algebraic meaning to Wick’s theorem and it extends the concept of Laplace pairing to prove that the renormalised time-ordered product is an associative deformation of the normal product involving an infinite number of parameters. The parameters themselves form a group: the renormalisation group, which acts on the product instead of on the algebra.

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1. Introduction

Quantum field theory is one of the most powerful methods of contemporary physics. It exhibits exciting analytical and combinatorial problems that are tightly intermingled. The purpose of this paper is to develop an algebraic framework which simplifies the solution of its combinatorial problems. This framework encompasses the main concepts used in the practical calculation of observable quantities: normal products, time-ordered products and renormalisation. With these three tools, one can calculate the S-matrix and the Green functions, i.e. about everything that can be compared with experiments. The analytical difficulties will be avoided by working in a finite-dimensional vector space instead of with field operators (i.e. operator-valued distributions [1]). The algebraic framework is an extension of the “Laplace Hopf algebras” created and elaborated by Rota and coll. [2, 3]. A Laplace Hopf algebra is a Hopf algebra equipped with a bilinear form called the “Laplace pairing”. With this algebra, Rota and coll. were able to transform intricate combinatorial problems into clear algebraic manipulations [4, 5, 6, 7, 8].

In quantum field theory, the commutator of two field operators is not an operator but a scalar function. Therefore, all products of field operators can be written as a linear combination of symmetric (i.e. normal) products of operators. We denote by $S(V)$ the vector space spanned by all normal products of operators. Similarly, the difference between the time-ordered product of two operators and the normal product of these operators is a scalar function. Thus, all time-ordered products of operators belong also to $S(V)$. In other words, quantum field theory can be entirely discussed in the space $S(V)$ of normal products. Working in $S(V)$ has several advantages, already noted by Houriet and Kind [9], and later stressed by Wick [10] and by Gupta [11]: $S(V)$ is equipped with a commutative product (the symmetric product) and the expectation value of an element $u$ of $S(V)$ over the vacuum is zero if $u$ has no scalar component.

Fauser discovered recently [12] that the time-ordered product is an instance of the “circle product”, which is a deformation of the symmetric product of $S(V)$ constructed from the Laplace pairing. In particular, Wick’s theorem becomes a very simple algebraic expression. Moreover, we prove that the operator product is also an instance of the circle product derived from a different Laplace pairing. Thus, the bare quantities of quantum field theory can be obtained from circle products. However, renormalisation does not fit into Rota and Stein’s original framework. Renormalisation was created in 1949 by Dyson [13], and its algebraic meaning remained mysterious until Kreimer discovered that it is ruled by a Hopf algebra [14]. Renormalisation was then refined by Connes and Kreimer [15, 16]. There are several ways to see renormalisation. In the standard approach, an analytic expression is renormalised by adding counterterms to the Lagrangian. Epstein and Glaser proposed an alternative point of view, where renormalisation is seen as a modification of the time-ordered product [17]. This approach was elaborated into its most powerful form by Pinter [18, 19]. It turns out that her work can be used to define a deformation of the circle product through a modified Laplace pairing, which satisfies axioms weaker than those of Rota’s Laplace pairing, but enables us to define an associative product, the renormalised circle product. This deformation depends on an infinite number of free parameters. For a certain choice of the bilinear form, the renormalised circle product is identical with the renormalised t-product.

We call quantum field algebra the symmetric Hopf algebra $S(V)$ equipped with
Quantum field algebra

a Laplace pairing and the group of renormalisation parameters. This name seems appropriate because such a structure enables us to define all the quantities that can be calculated from quantum field theory and compared with experiment.

The purpose of this paper is threefold: (i) to define an algebraic framework which simplifies the combinatorics of quantum field theory, (ii) to extend Rota’s Laplace pairing so that more general combinatorial problems can be solved, (iii) to extract as much algebra as possible from the quantum field formalism, so that algebraists can contribute to quantum field theory without having to read physics textbooks.

In the next section, we define the Laplace Hopf algebra and the circle product, we prove that the circle product is associative and that, for a symmetric Laplace pairing, the circle product reproduces the time-ordered product and satisfies Wick’s theorem. Then, we introduce the renormalised circle product, we prove that it is associative and we show that it reproduces the renormalised time-ordered product. Then we give some algebraic properties of the bare and renormalised time-ordered products. Finally, we describe further connections between our algebraic approach and quantum field theory. An appendix gathers the basic concepts of Hopf algebras.

In this article we concentrate on the presentation of the formalism and we give several proofs in full detail, to show how straightforward they become in the new algebraic framework. To make things as simple as possible we treat here scalar bosons fields, but the extension to fermions is possible.

2. Laplace Hopf algebra

As pointed out by Fauser [12], Wick’s theorem is related to a mathematical structure discovered by Rota and Stein and called a Laplace Hopf algebra [8]. Thus, our first step will be to give a definition and a few properties of Laplace Hopf algebras. The readers who are not familiar with Hopf algebras should first read the appendix.

2.1. The symmetric Hopf algebra

Since we are working with bosons, our starting point will be the symmetric algebra. Later, the symmetric product will designate the normal product of bosonic creation and annihilation operators (see section 7.1).

Let $V$ be a finite dimensional vector space, with basis $\{e_i\}$. We define the symmetric algebra $S(V)$ as the direct sum

$$S(V) = \bigoplus_{n=0}^{\infty} S^n(V),$$

where $S^0(V) = \mathbb{C}$, $S^1(V) = V$ and $S^n(V)$ is spanned by elements of the form $e_{i_1} \vee \cdots \vee e_{i_n}$, with $i_1 \leq i_2 \leq \cdots \leq i_n$. The symbol $\vee$ denotes the associative and commutative product $\vee : S^m(V) \otimes S^n(V) \to S^{m+n}(V)$ defined on elements of the basis of $V$ by $(e_{i_1} \vee \cdots \vee e_{i_m}) \vee (e_{i_{m+1}} \vee \cdots \vee e_{i_{m+n}}) = e_{i_{\sigma(1)}} \vee \cdots \vee e_{i_{\sigma(m+n)}}$, where $\sigma$ is the permutation on $m + n$ elements such that $i_{\sigma(1)} \leq \cdots \leq i_{\sigma(m+n)}$, and then extended by linearity and associativity to all elements of $S(V)$. The unit of $S(V)$ is the scalar unit $1 \in \mathbb{C}$ (i.e. for any $u \in S(V)$: $1 \vee u = u \vee 1 = u$). Moreover, this algebra is graded: if $u \in S^n(V)$ then we say that its grading is $|u| = n$. The symmetric product is a graded map: if $|u| = m$ and $|v| = n$ then $|u \vee v| = m + n$. In fact, $S(V)$ can be seen as the algebra of polynomials in the variables $\{e_i\}$, the elements of $S^n(V)$ being homogeneous polynomials of degree $n$. 
In this paper, $u$, $v$, $w$ designate elements of $S(V)$, $a$, $a_i$, $b$, $c$ designate elements of $V$ and $e_i$ are elements of a basis of $V$.

We define a coproduct over $S(V)$ as follows

\[
\begin{align*}
\Delta 1 &= 1 \otimes 1, \\
\Delta a &= a \otimes 1 + 1 \otimes a \quad \text{for} \quad a \in V, \\
\Delta (u \vee v) &= \sum (u_{(1)} \vee v_{(1)}) \otimes (u_{(2)} \vee v_{(2)}),
\end{align*}
\]  

(1)

(2)

where Sweedler’s notation was used for the coproduct of $u$ and $v$: $\Delta u = \sum u_{(1)} \otimes u_{(2)}$ and $\Delta v = \sum v_{(1)} \otimes v_{(2)}$. This coproduct is coassociative and cocommutative. It is equivalent to the coproduct defined by Pinter (equation (5) of reference [19]). As an example, we calculate the coproduct $\Delta (a \vee b)$, where $a$ and $b$ are in $V$. From equation (2)

\[
\Delta (a \vee b) = \sum (a_{(1)} \vee b_{(1)}) \otimes (a_{(2)} \vee b_{(2)}),
\]

From the definition (1) of the coproduct acting on elements of $V$ we know that $\Delta a = a \otimes 1 + 1 \otimes a$ and $\Delta b = b \otimes 1 + 1 \otimes b$. Thus,

\[
\begin{align*}
\Delta (a \vee b) &= \sum (a_{(1)} \vee b_{(1)}) \otimes (a_{(2)} \vee b_{(2)}) \\
&= (a \vee b) \otimes (1 \vee 1) + (a \vee 1) \otimes (1 \vee b) + (1 \vee b) \otimes (a \vee 1) \\
&\quad + (1 \vee 1) \otimes (a \vee b) \\
&= (a \vee b) \otimes 1 + a \otimes b + b \otimes a + 1 \otimes (a \vee b).
\end{align*}
\]

At the next order, we have

\[
\begin{align*}
\Delta (a \vee b \vee c) &= 1 \otimes a \vee b \vee c + a \otimes b \vee c + b \otimes a \vee c + c \otimes a \vee b \\
&\quad + a \vee b \otimes c + a \vee c \otimes b + b \vee c \otimes a + a \vee b \vee c \otimes 1.
\end{align*}
\]

Generally, if $u = a_1 \vee a_2 \vee \ldots \vee a_n$, the coproduct of $u$ can be written explicitly as (20 p.450)

\[
\begin{align*}
\Delta u &= u \otimes 1 + 1 \otimes u \\
&\quad + \sum_{p=1}^{n-1} \sum_{\sigma} a_{\sigma(1)} \vee \ldots \vee a_{\sigma(p)} \otimes a_{\sigma(p+1)} \vee \ldots \vee a_{\sigma(n)},
\end{align*}
\]

(3)

where $\sigma$ runs over the $(p, n-p)$-shuffles. A $(p, n-p)$-shuffle is a permutation $\sigma$ of $(1, \ldots, n)$ such that $\sigma(1) < \sigma(2) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(n)$.

The counit is defined by $\epsilon(1) = 1$ and $\epsilon(u) = 0$ if $u \in S^n(V)$ and $n > 0$. The antipode is defined by $s(u) = (-1)^n u$ if $u \in S^n(V)$. In particular, $s(1) = 1$.

Since the symmetric product is commutative, the antipode is an algebra morphism: $s(u \vee v) = s(u) \vee s(v)$.

The fact that this coproduct is coassociative and that $\Delta$, $\epsilon$ and $S$ give $S(V)$ the structure of a cocommutative Hopf algebra is a classical result (20 p.450 and references therein).

This algebra is called the symmetric Hopf algebra.
2.2. The Laplace pairing

We define a Laplace pairing on $S(V)$ as a bilinear form $V \times V \rightarrow \mathbb{C}$, that we denote by $(a|b)$, and which is extended to $S(V)$ by the following recursions

$$(u \lor v|w) = \sum (u|w(1))(v|w(2)),$$  

(4)

$$(u|v \lor w) = \sum (u(1)|v)(u(2)|w).$$  

(5)

It is important to stress that we do not assume any special symmetry for the bilinear form (i.e. it is a priori neither symmetric nor antisymmetric). The pairing defined by Rota and coll. in [3] is more general, but for quantum field theory our restricted definition is sufficient.

The name “Laplace pairing” comes from the fact that equations (4) and (5) are an elegant way of writing the Laplace identities for determinants (in the case of fermion operators). These identities express the determinant in terms of minors (see Ref.[21] p.26 and Ref.[22] p.93). They were derived by Laplace in 1772 [23].

In reference [3], it is proved that these recursions have the following unique solution: If $u = a_1 \lor a_2 \lor \ldots \lor a_k$ and $v = b_1 \lor b_2 \lor \ldots \lor b_n$, then $(u|v) = 0$ if $n \neq k$ and $(u|v) = \text{perm}(a_i|b_j)$ if $n = k$. We recall that the permanent of the matrix $(a_i|b_j)$ is

$$\text{perm}(a_i|b_j) = \sum_\sigma (a_1|b_{\sigma(1)}) \cdots (a_k|b_{\sigma(k)}),$$  

(6)

where the sum is over all permutations $\sigma$ of $(1,\ldots,k)$. The permanent is a kind of determinant where all signs are positive. For instance

$$(a \lor b|c \lor d) = (a|c)(b|d) + (a|d)(b|c).$$

In this paper, a Laplace Hopf algebra is the symmetric Hopf algebra equipped with a Laplace pairing.

3. The circle product

This section introduces the circle product. Its importance stems from the fact that the circle product treats in one single stroke the operator product and the time-ordered product, according to the definition of the bilinear form $(a|b)$. Following [8], the circle product is the operation on $S(V)$ defined by

$$u \circ v = \sum u(1) \lor v(1) (u(2)|v(2)).$$  

(7)

By cocommutativity of the coproduct on $S(V)$, this definition is equivalent to

$$u \circ v = \sum u(1) \lor v(2) (u(2)|v(1)) = \sum u(2) \lor v(1) (u(1)|v(2))$$

$$= \sum (u(1)|v(1)) u(2) \lor v(2).$$

In reference [8], Rota and Stein define the circle product from a Laplace pairing that is not necessarily scalar. This more general Laplace pairing must satisfy two additional identities (called (c) and (d)) which are not necessary when the pairing is scalar. In fact, their additional conditions are not even true in our case (take for example $w = a$, $w' = b$ and $w'' = 1$ in their equation (c)). Thus, the present circle product is not strictly a particular case of Rota and Stein’s. However, the main properties of their circle product remain true with a scalar pairing.
A few examples might be useful
\[
\begin{align*}
    a \circ b &= a \lor b + (a|b), \\
    (a \lor b) \circ c &= a \lor b \lor c + (a|c)b + (b|c)a, \\
    a \circ (b \lor c) &= a \lor b \lor c + (a|c)b + (a|b)c, \\
    a \circ b \circ c &= a \lor b \lor c + (a|b)c + (a|c)b + (b|c)a, \\
    a \circ b \circ c \circ d &= a \lor b \lor c \lor d + (a|b)c \lor d + (a|c)b \lor d + (b|a)c \lor d \\
    &\quad + (b|c)a \lor d + (a|d)b \lor c + (b|d)a \lor c + (c|d)a \lor b \\
    &\quad + (a|b)(c|d) + (a|c)(b|d) + (b|c)(a|d), \\
    (a \lor b) \circ (c \lor d) &= a \lor b \lor c \lor d + (a|c)b \lor d + (b|c)a \lor d \\
    &\quad + (a|d)b \lor c + (b|d)a \lor c + (a|c)(b|d) + (b|c)(a|d).
\end{align*}
\]

We apply the counit \(\epsilon\) to definition (2) of the coproduct of \(u\) and \(v\):
\[
\epsilon(u \circ v) = \sum \epsilon(u^{(1)} \lor v^{(1)})(u^{(2)}|v^{(2)}).
\]
The counit is an algebra morphism, i.e. \(\epsilon(u^{(1)} \lor v^{(1)}) = \epsilon(u^{(1)})\epsilon(v^{(1)})\). Thus, by linearity of the Laplace pairing,
\[
\epsilon(u \circ v) = \sum (\epsilon(u^{(1)})u^{(2)})\epsilon(v^{(1)})v^{(2)}.
\]
But the counit property is precisely the identity \(\sum \epsilon(u^{(1)})u^{(2)} = u\). This proves equation (2).

In the next section, we prove that the circle product is associative: \(u \circ (v \circ w) = (u \circ v) \circ w\).

### 3.1. Proof of associativity

We give a detailed proof of the associativity of the circle product, because it is an important result (and because neither Fauser nor Rota and Stein provided it). We first need a useful lemma:
\[
\Delta(u \circ v) = \sum (u^{(1)} \lor v^{(1)}) \otimes (u^{(2)}|v^{(2)})
\]
\[
= \sum (u^{(1)} \circ v^{(1)}) \otimes (u^{(2)} \lor v^{(2)}).
\]
The proof is easy, we start from the definition of the circle product (2) to write
\[
\Delta(u \circ v) = \sum \Delta(u^{(1)} \lor v^{(1)})(u^{(2)}|v^{(2)}).
\]
Now we use the definition (2) of the coproduct of \(u^{(1)} \lor v^{(1)}\):
\[
\Delta(u \circ v) = \sum (u^{(11)} \lor v^{(11)}) \otimes (u^{(12)} \lor v^{(12)})(u^{(2)}|v^{(2)}).
\]
Quantum field algebra

The coassociativity of the coproduct of \( u \) means that any triple \( (u_{(1)}, u_{(2)}, u_{(3)}) \) can be replaced by \( (u_{(1)}, u_{(21)}, u_{(22)}) \). Therefore,

\[
\Delta(u \circ v) = \sum (u_{(1)} \lor v_{(1)}) \otimes (u_{(21)} \lor v_{(2)}) (u_{(22)} \mid v_{(2)}).
\]

We use now the coassociativity of the coproduct of \( v \)

\[
\Delta(u \circ v) = \sum (u_{(1)} \lor v_{(1)}) \otimes (u_{(2)} \lor v_{(2)}) (u_{(22)} \mid v_{(2)}),
\]

and the definition of the circle product brings

\[
\Delta(u \circ v) = \sum (u_{(1)} \lor v_{(1)}) \otimes (u_{(2)} \circ v_{(2)}).
\]

To obtain the other identity of the lemma, we start from

\[
\Delta(u \circ v) = \sum (u_{(1)} \lor v_{(1)}) \otimes (u_{(2)} \circ v_{(2)}).
\]

The second lemma is

\[
(u \mid v \circ w) = (u \circ v \mid w).
\]

The proof is straightforward

\[
(u \mid v \circ w) = \sum (u_{(1)} \lor v_{(1)}) (v_{(2)} \mid w_{(2)})
\]

\[
= \sum (u_{(1)} \mid v_{(1)}) (u_{(2)} \mid v_{(2)}) (v_{(2)} \mid w_{(2)})
\]

\[
= \sum (u_{(1)} \mid v_{(1)}) (u_{(2)} \lor v_{(2)} \mid w) = (u \circ v \mid w).
\]

The first line is the definition of the circle product (7), the second line is the expansion of the Laplace pairing (5), the third one is the Laplace identity (4) and the last equality is again equation (7).

The associativity of the circle product follows immediately from these two lemmas.

From definition (7) and the first lemma we obtain

\[
u \circ (v \circ w) = \sum u_{(1)} \lor (v \circ w)_{(1)} (u_{(2)} \mid (v \circ w)_{(2)})
\]

\[
= \sum u_{(1)} \lor (v_{(1)} \lor w_{(1)}) (u_{(2)} \mid v_{(2)} \circ w_{(2)}).
\]

From the associativity of the symmetric product and the second lemma we obtain

\[
u \circ (v \circ w) = \sum u_{(1)} \lor v_{(1)} \lor w_{(1)} (u_{(2)} \circ v_{(2)} \mid w_{(2)}).
\]

Now the first lemma enables us to rewrite this as

\[
u \circ (v \circ w) = \sum (u \circ v)_{(1)} \lor w_{(1)} ((u \circ v)_{(2)} \mid w_{(2)}) = (u \circ v) \circ w.
\]

In comparison with the standard proofs of quantum field theory, the combinatorial derivations have been replaced by purely algebraic ones. This is the great advantage of the Hopf algebraic approach to combinatorics advocated by Rota.

3.2. Using the antipode

In reference [8], Rota and Stein use the antipode of the symmetric Hopf algebra to write the symmetric product in terms of the circle product. The same holds in our case:

\[
u \lor v = \sum (s(u_{(1)}) \mid v_{(1)}) u_{(2)} \circ v_{(2)} = \sum (u_{(1)} \mid s(v_{(1)})) u_{(2)} \circ v_{(2)}.
\]
The proof is simple:
\[ \sum (s(u_{(1)})v_{(1)})u_{(2)} \vartriangleright v_{(2)} = \sum (s(u_{(1)})v_{(1)})u_{(21)}v_{(2)} \vee v_{(22)} = \sum (s(u_{(1)})v_{(1)})u_{(12)}v_{(21)}u_{(22)} \vee v_{(2)} \]

\[ = \sum (s(u_{(11)})v_{(1)})u_{(12)}v_{(21)}u_{(22)} \vee v_{(2)} = \sum (u_{(1)})v_{(1)}u_{(2)} \vee v_{(2)} = u \vee v. \]

The last line was obtained because, from equation (8), \( (1 \circ v_{(1)}) = \epsilon(1 \circ v_{(1)}) = \epsilon(v_{(1)}). \) As for (8), it is also possible to recover the Laplace pairing from the circle product:
\[ (u|v) = \sum s(u_{(1)})v_{(1)} \vee (u_{(2)} \circ v_{(2)}). \]  

(12)

Again, the proof is straightforward:
\[ \sum s(u_{(1)} \vee v_{(1)}) \vee (u_{(2)} \circ v_{(2)}) = \sum s(u_{(1)} \vee v_{(1)}) \vee s(u_{(2)} \circ v_{(2)}) = \sum s(u_{(1)}) \vee s(v_{(1)}) \vee u_{(21)} \vee v_{(21)}u_{(22)}v_{(22)} = \sum s(u_{(1)}) \vee s(u_{(2)}) \vee v_{(22)} = \sum \epsilon(u_{(1)})\epsilon(v_{(1)})u_{(2)} \vee v_{(2)} = (u|v). \]

Rota and Stein (8) also provide a sort of distributivity formula for circle and symmetric products, which still holds:
\[ u \circ (v \vee w) = \sum (u_{(11)} \circ v) \vee (u_{(12)} \circ w) \vee s(u_{(2)}). \]  

(13)

For this result we need the following lemma
\[ \sum (u_{(1)} \circ v) \vee s(u_{(2)}) = \sum (u|v_{(1)})v_{(2)}, \]

which is proved easily
\[ \sum (u_{(1)} \circ v) \vee s(u_{(2)}) = \sum (u_{(11)}|v_{(1)})u_{(12)} \vee v_{(2)} \vee s(u_{(2)}) = \sum (u_{(11)}|v_{(1)})u_{(21)} \vee v_{(2)} \vee s(u_{(2)}) = \sum (u_{(1)|v_{(1)})\epsilon(u_{(2)})v_{(2)} = \sum (u|v_{(1)})v_{(2)}. \]

Now, we can prove (13).
\[ X = \sum (u_{(1)} \circ v) \vee (u_{(12)} \circ w) \vee s(u_{(2)}) = \sum (u_{(1)} \circ v) \vee (u_{(21)} \circ w) \vee s(u_{(22)}) = \sum (u_{(1)} \circ v) \vee w_{(2)}(u_{(2)}|w_{(1)}), \]

where the last line was obtained using the previous lemma. Now we rewrite the last line as
\[ X = \sum (u_{(1)}|w_{(1)})u_{(2)} \vartriangleright w_{(2)} = \sum (u_{(1)}|w_{(1)})u_{(21)}v_{(2)} \vee w_{(2)} = \sum (u_{(1)}|w_{(1)})u_{(12)}v_{(1)}u_{(2)} \vee w_{(2)} = \sum (u_{(1)}|w_{(1)})u_{(2)} \vee v_{(2)} \vee w_{(2)} = u \circ (v \vee w). \]
3.3. Wick’s theorem

In this section, we show that the circle product satisfies a generalized version of Wick’s theorem. This result is important because it will be used to show that, according to the choice of the bilinear form \((a|b)\), the circle product is identical with the time-ordered product or with the operator product.

There are two Wick’s theorems, one for operator products and one for time-ordered products \([24]\), but they have an identical structure. Wick’s theorem is very well known, so we recall it briefly. It states that the time-ordered (resp. operator) product of a given number of elements of \(V\) is equal to the sum over all possible pairs of contractions (resp. pairings). The reader who is not familiar with Wick’s theorem will be referred to standard references (e.g. \([24]\) p.209, \([26]\) p.261, \([27]\) p.85). A contraction \(a^*b^*\) is the difference between the time-ordered product and the normal product. In Wick’s notation \([10]\) \(a^*b^* = T(ab) - :ab:\). A pairing \(a^\diamond b^\diamond\) is the difference between the operator product and the normal product: \(a^\diamond b^\diamond = ab - :ab:\). This pairing was used by Houriet and Kind even before Wick’s article \([9]\). Both the contraction and the pairing are scalars. To express these in our notation, we identify the normal product \(ab\) with the symmetric product \(a \lor b\). This is valid because the normal product has all the properties required for a symmetric product (see section 7.1).

On the one hand, the time-ordered product is symmetric, and \(a^*b^*\) is a symmetric bilinear form that we can identify with our \((a|b)\). Thus, the time-ordered product of two operators is equal to the circle product obtained from this symmetric bilinear form.

On the other hand, the pairing defined from the operator product is an antisymmetric bilinear form: \(a^\diamond b^\diamond = (ab - ba)/2\), that we can also identify with our \((a|b)\) (not the same as in the previous paragraph, of course). The operator product of two elements of \(V\) is now equal to the circle product obtained from this antisymmetric bilinear form.

Therefore, we have identified the symmetric products with the normal products and we have shown that the circle product can reproduce the time-ordered product and the operator product of two elements of \(V\). To prove that the equality remains true for the product of more than two elements of \(V\), we shall show that the circle product satisfies Wick’s theorem.

We recall that the main ingredient used by Wick \([10]\) to go from a product of \(n\) elements of \(V\) to a product of \(n + 1\) elements of \(V\) is the following recursive identity:

\[
:a_1 \ldots a_n:b = :a_1 \ldots a_n b: + \sum_{j=1}^{n} a_j^* b^*:a_1 \ldots a_{j-1}a_{j+1} \ldots a_n:\ 
\]

To show this, we use the definition \([7]\) of the circle product and equation \([1]\) to find

\[
u \circ b = \sum u \lor b + \sum (u(1)|b)u(2).
\]
The Laplace pairing \((u_{(1)}|b)\) is zero if the grading of \(u_{(1)}\) is different from 1. In other words, \(u_{(1)}\) must be an element of \(V\). According to the general equation (3) for \(\Delta u\), this happens only for the \((1,n-1)\)-shuffles. By definition, a \((1,n-1)\)-shuffle is a permutation \(\sigma\) of \((1,\ldots,n)\) such that \(\sigma(2) < \ldots < \sigma(n)\), and the corresponding terms in the coproduct of \(\Delta u\) are
\[
\sum_{j=1}^{n} a_j \otimes a_1 \lor \ldots \lor a_{j-1} \lor a_{j+1} \lor \ldots \lor a_n.
\]
This proves the required identity, and we have shown that the circle product satisfies Wick’s theorem.

Our proof of Wick’s theorem for the circle product did not use any symmetry of the bilinear form. Thus it is more general than the usual Wick’s theorem. If the bilinear form is half of the commutator, as in Houri and Kind’s article [9], then we obtain Wick’s theorem for operator products ([24] p.212) and the circle product is equal to the operator product. If the bilinear form is the Feynman propagator, as in Wick’s article [10], then we obtain Wick’s theorem for time-ordered products ([24] p.215) and the circle product is equal to the time-ordered product. The fact that the commutator and the Feynman propagator are sufficient to determine the operator and time-ordered products was noticed by Fauser for the case of fermion operators [28].

4. Renormalisation

In this section, we introduce the renormalised circle product that we write \(\tilde{\otimes}\). In a finite dimensional vector space, the purpose of renormalisation is no longer to remove infinities, since everything is finite, but to provide a deformation of the circle product involving an infinite number of parameters. The physical meaning of these renormalisation parameters is the following: time ordering of operators is clear when two operators are defined at different times. However, the meaning of the time-ordering of two operators taken at the same time is ambiguous. Renormalisation is here to parametrise this ambiguity. The renormalised circle product will be defined in several steps.

4.1. Renormalisation parameters

We first need the renormalisation parameters. They are defined as a linear map \(\zeta\) from \(S(V)\) to \(\mathbb{C}\), such that \(\zeta(1) = 1\) and \(\zeta(a) = 0\) for \(a \in V\). These parameters form a group: the renormalisation group. The group product \(\star\) is defined by
\[
(\zeta \star \zeta')(u) = \sum \zeta(u_{(1)})\zeta'(u_{(2)}).
\]
This product is called the convolution of the Hopf algebra. The coassociativity and cocommutativity of the Hopf algebra implies that the product \(\star\) is associative and commutative. The unit of the group is the counit \(\epsilon\) of the Hopf algebra. The inverse of \(\zeta\) is \(\zeta^{-1}\), defined by
\[
(\zeta \star \zeta^{-1})(u) = \epsilon(u),
\]
or, recursively, by
\[
\zeta^{-1}(1) = 1,
\]
\[
\zeta^{-1}(u) = -\zeta(u) - \sum' \zeta(u_{(1)})\zeta^{-1}(u_{(2)}),
\]
To prove the coupling identity, we expand the Z-pairing in terms of the renormalisation then the coassociativity of $A$ few examples and properties might be useful

4.2. Z-pairing

We first use the definition of

$$\sum (\zeta^{-1}(u_{(1)} \otimes u_{(2)}) = \Delta u - 1 \otimes u - u \otimes 1 \text{ for } u \in S^n(V), n > 0. \text{ For instance,}
$$

$$\zeta^{-1}(a) = 0,$$

$$\zeta^{-1}(a \triangledown b) = -\zeta(a \triangledown b),$$

$$\zeta^{-1}(a \triangledown b \triangledown c) = -\zeta(a \triangledown b \triangledown c),$$

$$\zeta^{-1}(a \triangledown b \triangledown c \triangledown d) = -\zeta(a \triangledown b \triangledown c \triangledown d) + 2\zeta(a \triangledown b)\zeta(c \triangledown d) + 2\zeta(a \triangledown c)\zeta(b \triangledown d) + 2\zeta(a \triangledown d)\zeta(b \triangledown c).$$

4.2. Z-pairing

From the renormalisation parameters, we can define a pairing that we call a Z-pairing

$$Z(u, v) = \sum \zeta^{-1}(u_{(1)})\zeta^{-1}(v_{(1)})\zeta(u_{(2)} \triangledown v_{(2)}). \quad (15)$$

A few examples and properties might be useful

$$Z(u, v) = Z(v, u),$$

$$Z(1, u) = e(u),$$

$$Z(a, b) = \zeta(a \triangledown b),$$

$$Z(a, b \triangledown c) = \zeta(a \triangledown b \triangledown c),$$

$$Z(a \triangledown b, c \triangledown d) = \zeta(a \triangledown b \triangledown c \triangledown d) - \zeta(a \triangledown b)\zeta(c \triangledown d),$$

$$Z(a \triangledown b \triangledown c \triangledown d) = \zeta(a \triangledown b \triangledown c \triangledown d) - \zeta(a \triangledown b)\zeta(c \triangledown d) - \zeta(a \triangledown c)\zeta(b \triangledown d) - \zeta(b \triangledown c)\zeta(a \triangledown d).$$

The main property of the Z-pairing is the coupling identity

$$\sum Z(u_{(1)} \triangledown v_{(1)}, w)Z(u_{(2)}, v_{(2)}) = \sum Z(u, v_{(1)} \triangledown w_{(1)})Z(v_{(2)}, w_{(2)}). \quad (16)$$

For instance, the reader can check this identity for the case $u = a, v = b, w = c \triangledown d$:

$$Z(a \triangledown b, c \triangledown d) = Z(a, b \triangledown c \triangledown d) + Z(a, c)Z(b, d) + Z(b, c)Z(a, d).$$

To show equation (16), we first need an easy lemma

$$\sum \zeta^{-1}(u_{(1)} \triangledown v_{(1)})Z(u_{(2)}, v_{(2)}) = \zeta^{-1}(u)\zeta^{-1}(v).$$

We first use the definition of $Z(u_{(2)}, v_{(2)})$, then the cocommutativity of $u_{(2)}$ and $v_{(2)}$, then the coassociativity of $u$ and $v$, then the definition of $\zeta^{-1}(u_{(1)} \triangledown v_{(1)})$, and finally the definition of the counit:

$$\sum \zeta^{-1}(u_{(1)} \triangledown v_{(1)})Z(u_{(2)}, v_{(2)}) = \sum \zeta^{-1}(u_{(1)} \triangledown v_{(1)})\zeta^{-1}(u_{(21)})\zeta^{-1}(v_{(21)})\zeta^{-1}(u_{(22)})\zeta^{-1}(v_{(22)})$$

$$= \sum \zeta^{-1}(u_{(1)} \triangledown v_{(1)})\zeta^{-1}(u_{(21)} \triangledown v_{(21)})\zeta^{-1}(u_{(22)})\zeta^{-1}(v_{(22)})$$

$$= \sum \zeta^{-1}(u_{(11)} \triangledown v_{(11)})\zeta^{-1}(u_{(12)} \triangledown v_{(12)})\zeta^{-1}(u_{(21)})\zeta^{-1}(v_{(21)})$$

$$= \sum e(u_{(1)} \triangledown v_{(1)})\zeta^{-1}(u_{(21)})\zeta^{-1}(v_{(21)}) = \zeta^{-1}(u)\zeta^{-1}(v).$$

To prove the coupling identity, we expand the Z-pairing in terms of the renormalisation parameters

$$\sum Z(u_{(1)} \triangledown v_{(1)}, w)Z(u_{(2)}, v_{(2)}) = \sum \zeta^{-1}(u_{(1)} \triangledown v_{(1)})\zeta^{-1}(u_{(11)})\zeta^{-1}(w_{(11)})\zeta^{-1}(u_{(12)})\zeta^{-1}(v_{(12)})\zeta^{-1}(w_{(12)})Z(u_{(2)}, v_{(2)})Z(u_{(2)}).$$
Now we use the easy lemma to eliminate $Z(u(1), v(1))$ and the coassiciativity of the coproduct of $u$ and $v$:

$$\sum Z(u(1) \lor v(1), w)Z(u(2), v(2)) = \zeta^{-1}(u(12) \lor v(12))\zeta^{-1}(w(1)) \zeta(u(11) \lor v(11) \lor w(2))Z(u(2), v(2)) \zeta^{-1}(u(21))\zeta^{-1}(v(21))\zeta^{-1}(w(1)) \zeta(u(1) \lor v(1) \lor w(2))Z(u(22), v(22)).$$

Now we use the cocommutativity of the coproduct of permutations of $u$, $v$, and $w$. Therefore, all permutations of $u$, $v$, and $w$ in the left hand side give the same result. In particular, the permutation $(u,v,w) \rightarrow (v,w,u)$ and the symmetry of $Z$ transform the left hand side of equation (16) into its right hand side, which proves equation (14). The reader can check that the Laplace pairing also satisfies the coupling identity:

$$\sum (u(1) \lor v(1)|w)(u(2)|v(2)) = \sum (u|v(1) \lor w(1))(u(2)|v(2)). \quad (17)$$

For the renormalisation theory, it would be interesting to know the most general solution of the coupling identity. More precisely, if $Z$ is a bilinear map from $S(V) \times S(V)$ to $\mathbb{C}$ such that equation (16) is satisfied, $Z(u,v) = Z(v,u)$ and $Z(1,u) = \epsilon(u)$, what is the most general form of $Z$?

4.3. Modified Laplace pairing

From the $Z$-pairing, we can define the modified Laplace pairing as

$$\overline{(u|v)} = \sum Z(u(1), v(1))(u(2)|v(2)). \quad (18)$$

A few examples and properties might be appropriate

$$\overline{(u|1)} = (1|u) = \epsilon(u),$$

$$\overline{(a|b)} = \zeta(a \lor b) + (a|b),$$

$$\overline{(a\lor b|c)} = \zeta(a \lor b \lor c),$$

$$\overline{(a \lor b \lor c|d)} = Z(a \lor b \lor c, d),$$

$$\overline{(a \lor b|c \lor d)} = Z(a \lor b, c \lor d) + (a \lor b|c \lor d) + Z(a, c|b|d) + Z(a, d|b|c) + Z(b, c|a|d) + Z(b, d|a|c).$$

The modified Laplace pairing satisfies also the coupling identity

$$\sum (u(1) \lor v(1)|w)(u(2)|v(2)) = \sum (u|v(1) \lor w(1))(v(2)|w(2)). \quad (19)$$

We show this by expanding the definition of the modified Laplace pairing

$$Y = \sum (u(1) \lor v(1)|w)(u(2)|v(2))$$

$$= \sum Z(u(11) \lor v(11), w(1))(u(12) \lor v(12)|w(2))(u(2)|v(2))$$

$$= \sum Z(u(1) \lor v(1), w(1))(u(21) \lor v(21)|w(2))(u(22)|v(22))$$

$$= \sum Z(u(1) \lor v(1), w(1))(u(22) \lor v(22)|w(2))(u(21)|v(21))$$

$$= \sum Z(u(11) \lor v(11), w(1))(u(2) \lor v(2)|w(2))(u(12)|v(12)).$$
Now the same operations are done starting from the right. This leads us to
\[ u, v, w \]
only do the permutation \((u, v, w) \to (v, w, u)\) and put \(u\) on the left when \(w\) was on the right. This leads us to
\[
\sum (u|v(1) \lor w(1)) \rightarrow (v(2)|w(2)) = \sum Z(u(1), v(11) \lor w(11)) Z(v(12), w(12)) (u(2) \lor v(22) \lor w(22)) (v(21)|w(21)).
\]
The right hand side of this equation is related to the last form of \(Y\) by the coupling identities for the \(Z\)-pairing and the Laplace pairing. Thus they are identical and the proof of equation (10) is complete.

4.4. Renormalised circle product

After these preliminaries, we can now define the renormalised circle product
\[
u \bar{\circ} v = \sum (u(1)|v(1)) \times (u(2) \lor v(2)). \quad (20)
\]
A few examples and properties might again be useful
\[
\begin{align*}
1 \bar{\circ} u &= u, \\
(u|v) &= \epsilon(u \bar{\circ} v), \\
a \bar{\circ} b &= a \lor b + \zeta(a \lor b) + (a|b), \\
(a \lor b) \bar{\circ} c &= a \lor b \lor c + \zeta(a \lor b \lor c) + (a|c)b + (b|c)a + \zeta(b \lor c)a + \zeta(a \lor c)b.
\end{align*}
\]

The renormalised circle product is associative. To show that, we need the following identity
\[
\Delta(u \bar{\circ} v) = \sum (u(1) \lor v(1)) \otimes (u(2) \bar{\circ} v(2)) \\
= \sum (u(1) \bar{\circ} v(1)) \otimes (u(2) \lor v(2)), \quad (21)
\]
which is derived exactly as equations (11) and (10). Therefore, using equation (21) and the cocommutativity and coassociativity of the coproduct,
\[
\begin{align*}
(u \bar{\circ} v) \bar{\circ} w &= \sum ((u \bar{\circ} v(1)|w(1))(u \bar{\circ} v(2)) \lor w(2)) \\
&= \sum (u(1) \lor v(1)|w(1))(u(2) \bar{\circ} v(2)) \lor w(2) \\
&= \sum (u(1) \lor v(1)|w(1))(u(21)|v(21)) u(22) \lor v(22) \lor w(2) \\
&= \sum (u(11) \lor v(11)|w(11))(u(12)|v(12)) u(22) \lor v(22) \lor w(2).
\end{align*}
\]
From the coupling identity \((a|b) = (b|a)\) we obtain
\[
(u \circ v) \circ w = \sum (u(1) | v(1) \lor w(1)) (v(2) | w(2)) u(2) \lor v(2) \lor w(2)
\]
\[
= \sum (u(1) | v(1) \lor w(1)) (v(2) | w(2)) u(2) \lor v(2) \lor w(2)
\]
\[
= \sum (u(1) | (v \circ w)(1)) u(2) \lor (v(2) \circ w(2)) = u \circ (v \circ w).
\]

Thus, the renormalised circle product is associative. This is the first main result of the paper. Note that the action of the renormalisation group is not on the elements of the algebra but on the circle product. The Laplace pairing and the renormalisation parameters provide a deformation of the symmetric product with an infinite number of parameters. Now we shall define the t-product and the renormalised t-product, and we shall prove that the renormalised circle product reproduces the renormalised t-product of quantum field theory.

5. The time-ordered product

In quantum field theory, the boson operators commute inside a t-product. Thus, from now on, the circle product will be considered commutative. We first show that this is equivalent to \((a|b) = (b|a)\) for all \(a, b \in V\). The corresponding result for Clifford algebras was obtained by Fauser [24].

5.1. Commutative circle product

If the circle product is commutative, then in particular \(a \circ b = b \circ a\), so that \((a|b) = (b|a)\). Inversely, if \((a|b) = (b|a)\) for all \(a, b \in V\), then \(u \circ v = v \circ u\) for all \(u, v \in S(V)\).

The first step is to show that, if \((a|b) = (b|a)\) for all \(a, b \in V\), then \((u|v) = (v|u)\) for all \(u, v \in S(V)\). But this follows immediately from the definition \((\ref{eq:6})\) of the permanent. Because of the commutativity of the symmetric product we obtain
\[
u \circ w = \sum u(1) \lor v(1) (u(2) | v(2)) = \sum v(1) \lor u(1) (v(2) | u(2)) = v \circ u.
\]

5.2. The T-map

We define recursively a linear map \(T\) from \(S(V)\) to \(S(V)\) by \(T(1) = 1, T(a) = a\) for \(a \in V\) and \(T(u \lor v) = T(u) \circ T(v)\). More explicitly, \(T(a_1 \lor \ldots \lor a_n) = a_1 \circ \ldots \circ a_n\).

The circle product is associative and (in this section) commutative, therefore \(T\) is well defined. This T-map is the usual t-product of quantum field theory.

The main property we shall need in the following is that
\[
\Delta T(u) = \sum u(1) \otimes T(u(2)) = \sum T(u(1)) \otimes u(2).
\]

We use a recursive argument. Equation \((\ref{eq:23})\) is true for \(u = 1\) or \(u = a\) and \(v = 1\) or \(v = b\). Now
\[
\Delta T(u \lor v) = \Delta (T(u) \circ T(v)).
\]

By equation \((\ref{eq:23})\) we obtain
\[
\Delta T(u \lor v) = \sum (T(u(1)) \lor T(v(1))) \otimes (T(u(2)) \circ T(v(2))).
\]

The recursion hypothesis yields
\[
\Delta T(u \lor v) = \sum (u(1) \lor v(1)) \otimes (T(u(2)) \circ T(v(2)))
\]
\[
= \sum (u \lor v(1)) \otimes T(u(2) \lor v(2)).
\]
This gives us the expected result. The symmetric identity in equation (23) is obtained by cocommutativity of the coproduct.

5.3. Exponentiation

Now, we shall derive a result obtained by Anderson [29] and rediscovered several times [30]: the T-map can be written as the exponential of an operator Σ. As an application of the Laplace Hopf algebra, we give a purely algebraic proof of this.

First, we define a derivation \( \delta_k \) attached to a basis \( \{ e_k \} \) of vector space \( V \). This derivation is defined as the linear operator on \( S(V) \) satisfying the following properties

\[
\delta_k 1 = 0, \\
\delta_k e_j = \delta_{kj}, \\
\delta_k (u \lor v) = (\delta_k u) \lor v + u \lor (\delta_k v).
\]

In the second equation, \( \delta_{kj} \) is 1 for \( j = k \) and zero otherwise. The Leibniz relation (25) gives us \( \delta_1 (e_1 \lor e_2) = e_2 \) and \( \delta_2 (e_1 \lor e_2) = e_1 \), by example. From this definition, it can be shown recursively that the derivatives commute: \( \delta_i \delta_j = \delta_j \delta_i \). Note that the derivation does not act on the Laplace pairing: \( \delta_i (u|v) = 0 \).

Now we define the infinitesimal T-map as

\[
\Sigma = \frac{1}{2} \sum_{ij} (e_i|e_j) \delta_i \delta_j,
\]

where the sum is over all elements of the basis of \( V \).

We proceed in several steps to show that \( T = \exp \Sigma \). First, we show that

\[
[\Sigma, a] = \sum_i (a|e_i) \delta_i.
\]

To do that, we apply \( \Sigma \) to an element \( e_k \lor u \)

\[
\Sigma(e_k \lor u) = \frac{1}{2} \sum_{ij} (e_i|e_j) \delta_i \delta_j (e_k \lor u)
\]

\[
= \frac{1}{2} \sum_{ij} (e_i|e_j) \delta_i (\delta_{jk} u + e_k \lor \delta_j u)
\]

\[
= \frac{1}{2} \sum_i (e_i|e_k) \delta_i u + \frac{1}{2} \sum_j (e_k|e_j) \delta_j u + \frac{1}{2} \sum_{ij} (e_i|e_j) e_k \lor \delta_i \delta_j u
\]

\[
= \sum_j (e_k|e_j) \delta_j u + e_k \lor \Sigma u.
\]

If we extend this by linearity to \( V \) we obtain

\[
\Sigma(a \lor u) = a \lor \Sigma u + \sum (a|e_j) \delta_j u.
\]

And more generally

\[
\Sigma(u \lor v) = (\Sigma u) \lor v + u \lor (\Sigma v) + \sum_{ij} (a_i|e_j) (\delta_i u) \lor (\delta_j v).
\]

From the commutation of the derivations we obtain \( [\Sigma, \delta_k] = 0 \) and \( [\Sigma, [\Sigma, a]] = 0 \). Therefore, the classical formula (11) p.167) yields

\[
e^{\Sigma}ae^{-\Sigma} = a + [\Sigma, a],
\]
so that

\[ e^\Sigma, a] = \sum_i (a|e_i) \delta_i e^\Sigma = e^\Sigma \sum_i (a|e_i) \delta_i. \]

This can be written more precisely as

\[ e^\Sigma(a \vee u) = a \vee (e^\Sigma u) + [\Sigma, a](e^\Sigma u). \tag{28} \]

In the course of the proof of Wick’s theorem, we derived equation \([14]\) which can be rewritten

\[ a \circ u = a \vee u + [\Sigma, a] u. \tag{29} \]

Now we have all we need to prove inductively that \(T = e^\Sigma\). We have \(T(1) = e^\Sigma 1 = 1\) and \(T(a) = a = e^\Sigma a\) because \(\Sigma a = 0\). Now assume that the property is true up to grading \(k\), we take an element \(u\) of \(S^k(V)\) and calculate

\[ T(a \vee u) = a \circ T(u) = a \vee T(u) + [\Sigma, a] T(u) = a \vee e^\Sigma u + [\Sigma, a] e^\Sigma u = e^\Sigma (a \vee u), \]

where we used equation \((29)\), then equation \((28)\). Thus the property is true for \(a \vee u\) whose grading is \(k + 1\).

### 5.4. The scalar t-map

It is possible to write the T-map as a sum of scalars multiplied by elements of \(S(V)\). In fact we can show that

\[ T(u) = \sum t(u(1)) u(2), \tag{30} \]

where the map \(t\) is a linear map from \(S(V)\) to \(\mathbb{C}\) defined recursively by \(t(1) = 1, t(a) = 0\) for \(a \in V\) and

\[ t(u \vee v) = \sum t(u(1)) t(v(1))(u(2) | v(2)). \tag{31} \]

This scalar map is called the t-map. The proof is recursive. If the property is true up to grading \(k\), we take \(w = u \vee v\), where \(u\) and \(v\) have grading \(k\) or smaller and we calculate

\[ T(w) = T(u) \circ T(v) = \sum t(u(1)) t(v(1)) u(2) \circ v(2) \]

\[ = \sum t(u(1)) t(v(1)) (u(21) | v(21)) u(22) \vee v(22) \]

\[ = \sum t(u(1)) t(v(1)) (u(12) | v(12)) u(2) \vee v(2) \]

\[ = \sum t(u(1) \vee v(1)) u(2) \vee v(2) = \sum t(w(1)) u(2), \]

where the first line is the definition of \(T(u \vee v)\) and the recursion hypothesis, the second line is the definition of the circle product, the third line is the coassociativity of the coproduct and the last line is the definition of \(t\).

Moreover, the t-map is well defined because \(t(u) = \epsilon(T(u))\). Again, this can be proved by recursion. It is true for \(u = 1\) and \(u = a\). If it is true up to grading \(k\), we take the same \(w\) as for the previous proof and we have

\[ \epsilon(T(w)) = \sum t(w(1)) \epsilon(w(2)) = t(\sum w(1) \epsilon(w(2))) = t(w). \]

We close this section with a few examples: \(t(u) = 0\) if \(u \in S^n(V)\) with \(n\) odd.

\[ t(a \vee b) = (a|b), \]

\[ t(a \vee b \vee c \vee d) = (a|b)(c|d) + (a|c)(b|d) + (a|d)(b|c). \]
The general formula for \( t(a_1 \lor \ldots \lor a_{2n}) \) has \((2^n - 1)!!\) terms which can be written
\[
t(a_1 \lor \ldots \lor a_{2n}) = \sum_{\sigma} \prod_{j=1}^{n} (a_{\sigma(j)}|a_{\sigma(j+n)}),
\]
where the sum is over the permutations \( \sigma \) of \( \{1, \ldots, 2n\} \) such that \( \sigma(1) < \sigma(2) < \cdots < \sigma(n) \) and \( \sigma(j) < \sigma(n+j) \) for all \( j = 1, \ldots, n \). Or, as a sum over all the permutations \( \sigma \) of \( \{1, \ldots, 2n\} \).

6. The renormalised T-maps

In this section, we define the renormalised T-maps, and show that they coincide with the renormalised t-products of quantum field theory. A renormalised T-map \( \bar{T} \) is a linear map from \( S(V) \) to \( S(V) \) such that \( \bar{T}(1) = 1 \), \( \bar{T}(a) = a \) for \( a \) in \( V \) and \( \bar{T}(u \lor v) = \bar{T}(u) \circ \bar{T}(v) \). Since the circle product is assumed commutative, the renormalised circle product is also commutative, and the map \( \bar{T} \) is well defined on \( S(V) \).

Following the proof of equation (23), we can show that
\[
\Delta \bar{T}(u) = \sum Z(u(1), v(1))T(u(2)) \circ T(v(2)).
\]
In this section we derive two renormalisation identities. These identities show that our renormalised T-map coincides with the renormalised t-product of quantum field theory, as formalized by Pinter.

6.1. First identity

The first identity is:
\[
T(u) \circ T(v) = \sum Z(u(1), v(1))T(u(2)) \circ T(v(2)). \tag{32}
\]
This is proved as follows. By definition
\[
T(u) \circ T(v) = \sum (T(u(1))T(v(1))T(u(2)) \lor T(v(2))).
\]
Because of equation (23), this becomes
\[
T(u) \circ T(v) = \sum (u(1)|v(1))T(u(2)) \lor T(v(2)).
\]
By definition of the modified Laplace pairing, this can be rewritten
\[
T(u) \circ T(v) = \sum Z(u(11), v(11))(u(12)|v(12))T(u(2)) \lor T(v(2)),
\]
and the results follows from the coassociativity of the coproduct and the definition of the circle product.

6.2. Second identity

Now we show the following formula
\[
\bar{T}(u) = \sum \zeta(u(1))T(u(2)). \tag{33}
\]
This is the second main result of the paper. Equation (33) is Pinter’s identity for the renormalisation of a t-product (equation (17) in reference [19]). It is a very
general renormalisation formula. Its importance stems from the fact that it is valid also for field theories that are only renormalisable through an infinite number of renormalisation parameters. Its history starts with Dyson and Salam [13, 32, 33]. A crypted version of it can be recognized with hindsight in Bogoliubov’s works [14, 15, 16]. It finally appeared in full light in Pinter’s papers [18, 19]. Note that formula (33) does not encompass the complete Epstein-Glaser renormalisation [7, 77, 78] which completely avoids infinities. It only describes the transition from one renormalisation to another one. However, in the usual BPHZ renormalisation of quantum field theory, the removal of infinities is made by a formula which is equivalent to (33). Thus, equation (33) is the standard BPHZ renormalisation in the Epstein-Glaser guise.

We are going to prove equation (33) recursively. It is true for $u = 1$ or $u = a$ and $v = 1$ or $v = b$. Now, suppose it holds for $u$ and $v$, then by definition, and using the recursion hypothesis,

$$\bar{T}(u \lor v) = \bar{T}(u) \circ \bar{T}(v) = \sum \zeta(u^{(1)}) \zeta(v^{(1)}) T(u^{(2)}) \circ T(v^{(2)})$$

Equation (32) gives us

$$\bar{T}(u \lor v) = \sum \zeta(u^{(1)}) \zeta(v^{(1)}) Z(u^{(21)}, v^{(21)}) T(u^{(22)}) \circ T(v^{(22)}).$$

From the coassociativity of the coproduct and the definition (17) of the Z-pairing we obtain

$$\bar{T}(u \lor v) = \sum \zeta(u^{(1)} \lor v^{(1)}) T(u^{(2)}) \circ T(v^{(2)})$$

$$= \sum \zeta((u \lor v)^{(1)}) T((u \lor v)^{(2)}).$$

This is the required identity for $u \lor v$.

6.3. The scalar renormalised t-map

The reasoning that lead to the scalar t-map can be followed exactly to define a scalar renormalised t-map as a linear map $\bar{t}$ from $S(V)$ to $\mathbb{C}$ such that $\bar{t}(1) = 1$, $\bar{t}(a) = 0$ for $a \in V$ and

$$\bar{t}(u \lor v) = \sum \bar{t}(u^{(1)}) \bar{t}(v^{(1)}) \overline{(u^{(2)}) (v^{(2)})}. \tag{34}$$

Then $\bar{t}(u) = e(T(u))$ and

$$\bar{T}(u) = \sum \bar{t}(u^{(1)}) u^{(2)}, \tag{35}$$

Moreover, equation (34) enables us to show that

$$\bar{t}(u) = \sum \zeta(u^{(1)}) \bar{t}(u^{(2)}). \tag{36}$$

We give a few examples

$$\bar{t}(a \lor b) = (a|b) + \zeta(a \lor b),$$

$$\bar{t}(a \lor b \lor c) = \zeta(a \lor b \lor c),$$

$$\bar{t}(a \lor b \lor c \lor d) = \zeta(a \lor b \lor c \lor d) + \zeta(a \lor b)(c|d) + \zeta(a \lor c)(b|d)$$

$$+ \zeta(a \lor d)(b|c) + \zeta(b \lor c)(a|d) + \zeta(b \lor d)(a|c)$$

$$+ \zeta(c \lor d)(a|b) + (a|b)(c|d) + (a|c)(b|d) + (a|d)(b|c).$$

The scalar renormalised t-product corresponds to the numerical distributions of the causal approach 37, 38.
7. Relation to physics

In this section, we make a closer connection between the Hopf algebra approach and the usual quantum field formalism.

7.1. Normal products

To define a normal product, we start from creation and annihilation operators $a_k^+$ and $a_k^-$ that are taken as the basis vectors of two vector spaces $V^+$ and $V^-$. These two bases are in involution, i.e. there is an operator $\ast$ such that $a_k^{+\ast} = a_k^-$ and $a_k^{-\ast} = a_k^+$. The creation operators commute and there is no other relation between them. Thus, the Hopf algebra of the creation operators is the symmetric algebra $S(V^+)$. Similarly, the Hopf algebra of the annihilation operators is $S(V^-)$. From $V^+$ and $V^-$, we define a vector space $V = V^+ \oplus V^-$, so that each vector $a \in V$ can be written $a = a^+ + a^-$. We call $P$ the projector from $V$ to $V^+$ and $M$ the projector from $V$ to $V^-$. Thus $P(a) = a^+$ and $M(a) = a^-$. There is an isomorphism between $S(V)$, the symmetric algebra of $V$ and the tensor product of the symmetric algebras $S(V^+)$ and $S(V^-)$. $S(V)$ is the vector space of normal products. The isomorphism $\varphi : S(V) \to S(V^+) \otimes S(V^-)$ is defined by $\varphi(u) = \sum P(u_{(1)}) \otimes M(u_{(2)})$ and we recover the fact that a normal product puts all annihilation operators on the right of all creation operators. This isomorphism and the projectors $P : S(V) \to S(V^+)$ and $M : S(V) \to S(V^-)$ can be defined recursively by

\[
P(1) = 1, \quad M(1) = 1,
\]
\[
P(a) = a^+, \quad M(a) = a^-,
\]
\[
\varphi(1) = 1 \otimes 1,
\]
\[
\varphi(a) = P(a) \otimes 1 + 1 \otimes M(a),
\]
\[
\varphi(u \lor v) = \sum P(u_{(1)}) P(v_{(1)}) \otimes M(u_{(2)}) M(v_{(2)}),
\]
\[
P(u \lor v) = P(u) P(v), \quad M(u \lor v) = M(u) M(v).
\]

The algebra $S(V)$ is graded by the number of creation and annihilation operators.

There is a subtlety here. It seems that we have forgotten the operator product of elements of $V^+$ with elements of $V^-$. We have replaced $a_k^+ a_l^-$ by $a_k^+ \otimes a_l^-$ and we lost all information concerning the commutation of creation and annihilation operators. However, this is also true in standard quantum field theory. Textbooks sometimes describe a kind of “normal product operator” which takes a product of operators and puts all creation operators on the left and all annihilation operators on the right. But such an operator would not be well-defined (p.28). For instance, $a_k^- a_k^+$ and $a_k^+ a_k^- + 1$ are equal as operators, but their normal products $a_k^+ a_k^-$ and $a_k^+ a_k^- + 1$ are different. Thus, it is not consistent to consider a normal product as obtained from the transformation of an operator product and we also lose all information concerning the commutation relations in standard quantum field theory.

However, we saw that an antisymmetric Laplace pairing enables us to build a circle product in $S(V)$ which is an operator product. In that sense normal products are a basic concept and operator product a derived concept of quantum field theory. The problem of quantum field theory in curved space times has also revealed that normal products (i.e. Wick polynomials) are fundamental elements of quantum field theory. Therefore, from our point of view, quantum field theory starts from the
Quantum field algebra

space $S(V)$ of normal products and the operator product and time-ordered products are obtained as deformations of the symmetric product.

In the quantum field theory of scalar particles, our “basis” $a_k$ is not indexed by a finite number of $k$ but by a continuous set of $x$. By an unfortunate convention, the creation operators are $\varphi^{(-)}(x)$ and the annihilation operators are $\varphi^{(+)}(x)$. Thus, for instance, the basis vectors of $V^+$ are the operators $\varphi^{(-)}(x)$ where $x$ plays the role of the index of the basis vectors. This uncountable basis of $V^+$ is one of the sources of the analytical complexity of quantum field theory. For instance, these basic vectors enable us to define a bilinear form for the definition of operator products as (p.117)

$$ (\varphi(x)|\varphi(y)) = iD(x-y), $$

and a bilinear form for the definition of time-ordered products as (p.124)

$$ (\varphi(x)|\varphi(y)) = iG_F(x-y), $$

where, for massless bosons (p.93)

$$ D(x) = -\frac{1}{2\pi} \text{sign}(x^0) \delta(x^2), $$

$$ G_F(x) = \frac{1}{4\pi^2} \frac{1}{x^2 + i0}, $$

are distributions. The manipulation of these distributions creates many analytical problems that we do not want to address here. This is why we consider only finite-dimensional vector spaces in the present paper.

7.2. The counit

The counit is a basic element of Hopf algebras, and it has a particularly simple expression for quantum fields: it is the expectation value over the vacuum. In other words, for any element $u$ of $S(V)$ (i.e. for any sum of normal products of quantum fields) the following striking identity holds: $\epsilon(u) = \langle 0|u|0 \rangle$, where $\epsilon$ is the counit of the symmetric Hopf algebra. This is very easy to show. Firstly, $\langle 0|u|0 \rangle$ is a linear map from $S(V)$ to $\mathbb{C}$. Secondly, for elements $u$ of $S^n(V)$ with $n > 0$ $\epsilon(u) = 0$ and $\langle 0|u|0 \rangle = 0$ because $u$ is a normal product. Finally, the symmetric Hopf algebra is connected, thus all elements of $S^0(V)$ are multiple of the unit: for any $u \in S^0(V)$ there is a complex number $\lambda(u)$ such that $u = \lambda(u) 1$. Moreover, $\epsilon(u) = \lambda(u) \epsilon(1) = \lambda(u)$. Thus $u = \epsilon(u) 1$. But the vacuum is assumed to be normalized, thus we have $\langle 0|u|0 \rangle = \epsilon(u) \langle 0|1|0 \rangle = \epsilon(u)$. Thus the counit and the expectation value over the vacuum are identical. This identity is a additional argument in favor of the fact that Hopf algebras are a natural framework for quantum field theory.

7.3. The $S$-matrix and the Green function

For any $u \in S(V)$ ($u$ plays the role of a Lagrangian) we can define the normal ordered S-matrix $S$ by

$$ S = \exp^{\vee}(u), $$

where

$$ \exp^{\vee}(u) = \sum_{n=0}^{\infty} \frac{1}{n!} u^{\vee n}, $$

are.
and the symmetric power is defined recursively by $u^\vee 0 = 1$, $u^\vee 1 = u$ and $u^\vee (n+1) = u \vee (u^\vee n)$. Now the bare S-matrix is defined as $T(S)$ and the renormalised S-matrix is $\bar{T}(S)$. Of course, for a physical definition of the renormalised S-matrix, we must inject various renormalisation and causality conditions which determine the renormalisation parameters. The mathematically-oriented reader has noticed that $\exp^\vee(u)$ is an infinite series that does not belong to $S(V)$. There are two ways out of this problem. The first solution is to put a topology on $S(V)$ and to extend $S(V)$ so that infinite series are admitted. If the circle product is still defined on such an extended states, our approach enables us to manipulate non perturbative quantites, such as the S-matrix and the Green function (see below). This is an advantage over the Feynman diagram approach. The second solution is more conservative: we can consider formal power series such as $\exp^\vee(\lambda u)$. Such a formal power series is a compact way to represent an infinite number of terms (i.e. one term for each $\lambda^n$) [40].

In section 7.2 we saw that the expectation value over the vacuum is equal to the counit. Thus, the Gell-Man and Low definition of the Green function ([31] p.264) can be translated into the Hopf algebraic approach as: the bare electron Green function $G_{ij}$ is defined by

$$G_{ij} = \frac{\epsilon(e_i \circ e_j \circ T(S))}{\epsilon(T(S))},$$

the renormalised electron Green function $\bar{G}_{ij}$ is defined by

$$\bar{G}_{ij} = \frac{\epsilon(e_i \circ e_j \circ \bar{T}(S))}{\epsilon(\bar{T}(S))}.$$

A Schwinger-Dyson equation can be derived for $G_{ij}$.

These examples were chosen to show that the circle product allows for a compact definition of quantum field quantities.

7.4. The simplest Lagrangians

For the simplest Lagrangian $u = a \in V$ we obtain the following identity

$$T(S) = \exp^\vee((a + (a|a)/2) = \exp((a|a)/2) \exp^\vee(a).$$

A slightly more complicated Lagrangian is

$$u = \sum_i e_i \vee e_i.$$ 

This Lagrangian is local in the sense that there is no cross term $e_i \vee e_j$. It corresponds to a mass term $\varphi(x)^2$ of quantum field theory. To write the S-matrix we define the matrix $M$ by its matrix elements $M_{ij} = (e_i|e_j)$. Then

$$T(S) = \frac{1}{\sqrt{\det(1 - 2M)}} \exp^\vee \left( \sum_{ij} e_i(1 - 2M)_{ij}^{-1} e_j \right).$$

Here, there is a small difference between the algebraic and the quantum field approaches. To make $T$ a map from $S(V)$ to $S(V)$, we must replace all symmetric products by circle products. In particular, $T(S)$ contains terms such as

$$T(u) = \sum_i e_i \circ e_i = \sum_i e_i \vee e_i + \sum_i (e_i|e_i).$$

These terms $(e_i|e_i)$ are usually avoided in quantum field theory because they are infinite. From the physical point of view, we can say that $u$ is a local Lagrangian and
all the terms of \( u \) are taken at the same time. Therefore, the time-ordered product inside \( u \) is not well defined and we must choose a prescription for it that agrees with experiment. According to the Epstein-Glaser approach, the correct time-ordered product taken at the same time is determined by renormalisation. In other words, the divergent term \((e_i|e_i)\) is removed by choosing a renormalisation parameter (for example \(\zeta(e_i \lor e_i) = -(e_i|e_i)\)). This corresponds to the removal of the density of occupied negative energy states in the Dirac sea picture \([41]\). In fact, in quantum electrodynamics in an external field, only a part of \((e_i|e_i)\) must be removed and the polarisation charge \((e_i|e_i) + \zeta(e_i \lor e_i)\) creates the Uehling potential which is observable \([42, 43]\).

Thus, for quantum electrodynamics, it is necessary to take the circle product inside the Lagrangian \( u \). This is natural from the algebraic point of view, and this agrees with experiment. However, for scalar fields this is not necessarily the case and we need a way to keep normal products inside \( T(u) \). This is again provided by Rota and coll. \([3]\), who define a divided power as the quantity \(a^{(n)} = a \otimes n! \) for \( a \in V \), with \( a^{(0)} = 1 \) and \( a^{(1)} = a \). The divided powers have the following properties, which are deduced from the properties of \( a \otimes n \):

\[
\Delta a^{(n)} = \sum_{k=0}^{n} a^{(k)} \otimes a^{(n-k)},
\]

\[
a^{(m)} \otimes a^{(n)} = \left( \begin{array}{c} m+n \\ m \end{array} \right) a^{(m+n)}, \quad s(a^{(n)}) = (-1)^n a^{(n)} \text{ and}
\]

\[
(a^{(n)}) a_1 \lor a_2 \lor \ldots \lor a_n = (a|a_1)(a|a_2) \cdots (a|a_n).
\]

Thus

\[
(a^{(n)}|b^{(n)}) = \frac{(a|b)^n}{n!} = (a|b)^{(n)}.
\]

The normal product is preserved inside \( T(u) \) by considering the divided powers as independent variables and by defining the \( T \)-map on them by \( T(a^{(n)}) = a^{(n)} \). The use of divided powers and their relations with the elements of \( V \) are treated in detail in reference \([3]\).

8. Conclusion

We hope that we have convinced the reader that Hopf algebra is a powerful and natural tool for quantum field theory. It enables us to transform combinatorial reasoning into algebraic manipulations.

This paper was expository, and only the simple case of scalar fermions was treated. In a forthcoming publication we shall consider the cases of fermions and more complex problems such as the relation of the renormalization parameters with the usual counterterms of the Lagrangian. We shall also discuss the question of connected products \([44]\), which have a nice operadic interpretation \([45]\).

Fermions are a relatively straightforward extension of the present formalism. The main trick is to twist the tensor product so as to account for the anticommutation property of fermions. Moreover, bosons and fermions can be merged into a supersymmetric algebra \([3]\). All this is necessary to consider quantum electrodynamics. The circle product of fermions is a generalization of the Clifford algebra and of Hestenes’ Geometric Algebra \([46, 47]\). By considering a more complicated twist, it
might be possible to work with quantum groups [48] and braided quantum field theory [49].

Since our algebra is finite dimensional, it is well suited to treat problems in lattice field theory.

The circle and renormalised circle products are deformations of the normal product, thus they could be relevant to the deformation theory of quantum fields [50, 51, 44] or to quantize Fedorov [52] and symplectic [53] manifolds.

But the main reason why the present formalism was developed is the calculation of the Green function for a degenerate vacuum. Some steps in this direction were done by Kutzelnigg and Mukherjee [54], but the combinatorial formulas are very complex because they mix time-ordered and commutation functions. We hope that our quantum field algebra can be helpful for that problem.

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Appendix

In this appendix, we make a short presentation of Hopf algebras.

To define a Hopf algebra, we first need an algebra. An algebra is a vector space $\mathcal{A}$ over $\mathbb{C}$ equipped with an associative linear product over $\mathcal{A}$, denoted $\cdot$ and a unit, denoted $1$. A product is linear if for any $a, b, c \in \mathcal{A}$:

$$(a + b) \cdot c = a \cdot c + b \cdot c \quad \text{and} \quad a \cdot (\lambda b) = \lambda(a \cdot b) = (\lambda a) \cdot b.$$ 

It is associative if for any $a, b, c \in \mathcal{A}$:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

The unit is the element of $\mathcal{A}$ such that, for any $a \in \mathcal{A}$, we have $1 \cdot a = a \cdot 1 = a$.

As an example, we consider the algebra $\mathcal{X}$ generated by all sets of elements $\{x_1, \ldots, x_n\}$, where $x_i$ are vectors in $\mathbb{R}^3$ for example. The associative product is given by the union of sets

$$\{x_1, \ldots, x_m\} \cdot \{y_1, \ldots, y_n\} = \{x_1, \ldots, x_m, y_1, \ldots, y_n\}.$$ 

The unit $1$ is the empty set.

The most unusual concept in a Hopf algebra is the coproduct, denoted $\Delta$. In general, a coproduct is a linear application from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{A}$, denoted $\Delta$. The coproduct has various physical meanings. Quite often, a coproduct can be considered as giving all the ways to split an element of $\mathcal{A}$ into two “parts”. For the example of $\mathcal{X}$, the coproduct of $\{x_1, \ldots, x_n\}$ is defined as

$$\Delta\{x_1, \ldots, x_n\} = \sum_{I, I^c} I \otimes I^c,$$

where the sum is over all subsets $I$ of $\{x_1, \ldots, x_n\}$, and $I^c = \{x_1, \ldots, x_n\} \setminus I$. For instance,

$$\Delta 1 = 1 \otimes 1,$$
This property is called the coassociativity of the coproduct, and we can define the action of the angular momentum operators on many-particle systems. It can be checked that the coproduct of the counit \( \sum a \) (i.e., the general property is that \( a + b \)) depends on which side you apply the \( \Delta \) on \( a \). But since \( \Delta \) is a natural to go from a two-particle operator to a three-particle operator by repeating the action of the \( \Delta \). For products of operators \( a \cdot b \) is the operator product. Then the coproduct of these operators is

\[
\Delta(J_x \cdot J_y) = J_x \cdot (J_y \otimes 1 + J_y \otimes J_x + 1 \otimes J_x \cdot J_y).
\]

Now it is natural to go from a two-particle operator to a three-particle operator by repeating the action of the \( \Delta \). But since \( \Delta a = \sum a a \) it is not clear on which side the second \( \Delta \) should act: on \( a(1) \) or on \( a(2) \)? A very important property of Hopf algebras is that the result does not depend on which side you apply the coproduct. For each \( a \) we denote the action of the coproduct by \( \Delta a = a_{(1)} \otimes a_{(2)} \) and for each \( a \), \( \Delta a = a_{(1)} \otimes a_{(2)} \). So, the fact that the action of the \( \Delta \) on three particles does not depend on the order of the coproducts amounts to

\[
(\Delta \otimes \text{Id})\Delta a = \sum (\Delta a_{(1)} \otimes a_{(2)}) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(2)}
= \sum a_{(1)} \otimes a_{(2)} \otimes a_{(2)} = \sum a_{(1)} \otimes (\Delta a_{(2)})
= (\text{Id} \otimes \Delta)\Delta a.
\]

This property is called the coassociativity of the coproduct. By applying again the \( \Delta \), we can define the action of the angular momentum operators on many-particle states. The coassociativity ensures that the result does not depend on the order used. It can be checked that the coproduct of \( X \) and \( Y \) are coassociative.

We still need two ingredients to make a Hopf algebra: a counit and an antipode. A counit is a linear map from \( A \) to \( \mathbb{C} \), denoted by \( \epsilon \), such that \( \sum a_{(1)} \epsilon(a_{(2)}) = \sum \epsilon(a_{(1)}) a_{(2)} = a \). In a Hopf algebra, the counit \( \epsilon \) is an algebra homomorphism (i.e. \( \epsilon(ab) = \epsilon(a) \epsilon(b) \)). In most cases, the counit is very simple. For \( X \) we have \( \epsilon(1) = 1 \), \( \epsilon([x_1, \ldots, x_n]) = 0 \) for \( n > 0 \), for \( Y \) we have \( \epsilon(1) = 1 \), \( \epsilon([x_1, \ldots, x_n]) = 0 \) for \( n > 0 \). Finally the antipode is a linear map from \( A \) to \( A \) such that \( \sum a_{(1)} \cdot s(a_{(2)}) = \sum s(a_{(1)}) \cdot a_{(2)} = \epsilon(a) 1 \). The antipode is a kind of inverse. Its main general property is that \( s(ab) = s(b) \cdot s(a) \). In our examples: for \( X \) the antipode is...
defined by \( s(1) = 1 \) and \( s\{x\} = -\{x\} \), so that \( s(\{x_1, \ldots, x_n\}) = (-1)^n\{x_1, \ldots, x_n\} \) for \( n > 0 \), for \( \mathcal{J} \) the antipode is defined by \( s(1) = 1 \) and \( s(J_i) = -J_i \), so that \( s(J_{i_1} \cdot \cdots \cdot J_{i_n}) = (-1)^n J_{i_n} \cdot \cdots \cdot J_{i_1} \) for \( n > 0 \).

To summarize a Hopf algebra an algebra \( \mathcal{A} \) equipped with a coassociative coproduct \( \Delta \), a counit \( \varepsilon \) and an antipode \( s \), such that \( \Delta \) and \( \varepsilon \) are algebra homomorphisms.

A Hopf algebra is commutative if, for any \( a, b \in \mathcal{A} \): \( a \cdot b = b \cdot a \). A Hopf algebra is cocommutative if, for any \( a \in \mathcal{A} \): \( \sum a_{(1)} \otimes a_{(2)} = \sum a_{(2)} \otimes a_{(1)} \), in other words, if \( a \) can be split into \( a_{(1)} \) and \( a_{(2)} \), then there is also a splitting of \( a \) where \( a_{(2)} \) is the first part and \( a_{(1)} \) is the second part. The reader can check that \( \mathcal{X} \) and \( \mathcal{J} \) are cocommutative Hopf algebras.

The last concept we shall use is that of a graded Hopf algebra. An algebra is graded if it can be written as the vector sum of subsets \( \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots \). If \( a \in \mathcal{A}_m \), we denote the grading of \( a \) by \( |a| = m \). The grading must be compatible with the product: if \( a \in \mathcal{A}_m \) and \( b \in \mathcal{A}_n \), then \( (a \cdot b) \in \mathcal{A}_{m+n} \). Therefore, \( 1 \in \mathcal{A}_0 \). If all elements of \( \mathcal{A}_0 \) can be written \( \lambda 1 \), the algebra is said to be connected.

In our two examples, \( \mathcal{X} \) is a connected graded Hopf algebra, where the grading is given by \( |\{x_1, \ldots, x_n\}| = n \), but \( \mathcal{J} \) is not a graded algebra: If it were graded, we could consider that the grading of \( J_i \) is 1. In that case, the grading of \( J_x \cdot J_y \) and of \( J_y \cdot J_x \) would be 2. But the commutator identity \( J_x \cdot J_y - J_y \cdot J_x = iJ_z \) tells us that the grading of \( J_x \cdot J_y - J_y \cdot J_x \) would be one, which is incompatible with the fact that the grading of \( J_x \cdot J_y \) and of \( J_y \cdot J_x \) are 2.

For more information on Hopf algebras, see Refs.[53, 20].

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