Large-$N_c$ Higher Order Weak Chiral Lagrangians for Nonleptonic and Radiative Kaon Decays

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ABSTRACT

In pure chiral perturbation theory (ChPT) the couplings of higher order Lagrangian terms are running parameters and hence can be determined only empirically from various low-energy hadronic processes. While this scenario works well for strong interactions, it is unsatisfactory for nonleptonic and radiative nonleptonic weak interactions: It is impossible, from the outset, to determine all unknown higher-order chiral couplings by experiment; theory is tested only by certain chiral constraints rather than by its quantitative predictions. Based on a QCD-motivated model for $p^4$ strong chiral Lagrangian valid in the limit of large $N_c$, one can derive large-$N_c$ fourth-order effective chiral Lagrangians for $\Delta S = 1$ nonleptonic weak interactions and radiative weak transitions. Applications to $K \to \pi\pi\pi$, $K \to \pi\gamma\gamma$, and $K \to \pi\pi\gamma$ are discussed.

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1. Introduction

There are at least two reasons that lead us to study seriously the structure of higher-order effective chiral Lagrangians for weak interactions. First, the \( K(k) \to \pi(p_1)\pi(p_2)\pi(p_3) \) decay amplitude in the Dalitz plot is conventionally parametrized as

\[
A(K \to 3\pi) = a + bY + c(Y^2 + X^2/3) + d(Y^2 - X^2/3),
\]

where \( Y = (s_3 - s_0)/m_{\pi}^2, \ X = (s_2 - s_1)/m_{\pi}^2, \ s_i = (k - p_i)^2, \ s_0 = (s_1 + s_2 + s_3)/3. \)

The experimental signal for the quadratic terms (i.e., the parameters \( c \) and \( d \)) requires the inclusion of higher order weak Lagrangian terms containing four or more derivatives. Also, it is well known that current-algebra predictions for \( a \) and \( b \) are too small by 18% and 35% respectively. Second, the radiative kaon transition cannot be generated by the lowest-order chiral Lagrangian since Lorentz and gauge invariance requires at least two powers of momenta in the radiative decay amplitude. Therefore, it is necessary of higher order in chiral perturbation theory.

The couplings of higher-order chiral Lagrangians depend on the choice of the renormalization scale \( \mu \) as divergences of chiral loops are absorbed by the counterterms which have the same structure as higher derivative chiral terms. Consequently, the unknown running parameters are not really fundamental coupling constants; also they cannot be fixed by the requirement of chiral symmetry alone. For strong interactions, Gasser and Leutwyler\(^1\) have empirically determined the coupling parameters at the mass scale \( \mu = m_\eta \) from various low-energy hadronic processes in conjunction with the Zweig-rule argument.

Despite the fact that pure ChPT is phenomenologically successful when applied to low-energy strong-interaction physical processes, this approach is unsatisfactory for nonleptonic and radiative weak interactions. For example, in the chiral limit, there are seven independent \( p^4 \) Lagrangian \( \mathcal{L}_{W}^{(4)} \) responsible for \( \Delta S = 1, \Delta I = \frac{1}{2} \) weak transitions. Unfortunately, there is only one process, namely \( K \to 3\pi \) decay, relevant for the determination of \( \mathcal{L}_{W}^{(4)} \). Unlike the strong interaction, it is impossible from the outset to completely determine the structure of \( \mathcal{L}_{W}^{(4)} \); only certain chiral constraint relations can be tested to check the validity of ChPT at the four-derivative level.\(^2\) That is, although pure ChPT is a rigorous theory, its ability of making predictions for phenomenological \( p^4 \) weak transitions and for radiative nonleptonic decays is rather limited. While the unknown coupling constants in ChPT in principle must
be determined by experiment so that its prediction is truly model independent, it is also
important to appeal a dynamic model to help us understand the underlying physics if theory
by itself does not lead to any significant quantitative predictions owing to present limitation
from both experiment and theory.

A QCD-inspired model for nonanomalous four-derivative effective action for strong in-
teractions does exist in the large $N_c$ limit.\cite{3-6} It is obtained by coupling QCD to external
meson and gauge fields. In the limit of $N_c$, chiral loops are suppressed and the effects of
interest are due to the quark loops and the gluonic corrections arising from all planar dia-
grams without the internal quark loop. In the leading $1/N_c$ approach, coupling constants
become renormalization scale independent and are (almost) uniquely determined.

One can then proceed to derive a large $N_c$ effective Lagrangian for nonleptonic $\Delta S = 1$
weak interactions at order $p^4$ based on the following three ingredients:\cite{7}: a rather simple
structure of the effective weak Hamiltonian in the leading $1/N_c$ expansion, bosonization up
to the subleading order, and factorization valid in the limit of large $N_c$. Confrontation with
experiment for $K \to 3\pi$ decays reveals a good agreement for two of the measured parameters
in the Dalitz expansion of $K \to 3\pi$ amplitudes.\cite{8} Based on the same approach, a derivation
of non-anomalous and anomalous fourth-order chiral Lagrangians responsible for $\Delta S = 1$
radiative weak transitions is also straightforward.\cite{9}

2. Higher-order Chiral Lagrangians for Nonleptonic Weak Interactions

The lowest order chiral Lagrangian including explicit chiral-symmetry breaking for low-
energy QCD is given by\cite{10}

$$\mathcal{L}_S^{(2)} = \frac{f_\pi^2}{8} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{f_\pi^2}{8} \text{Tr}(MU^\dagger + UM^\dagger),$$

where $U = \exp(2i\phi f_\pi)$, $\phi = \frac{1}{\sqrt{2}} \phi^a \lambda^a$, $\text{Tr}(\lambda^a \lambda^b) = 2\delta^{ab}$, $f_\pi = 132$ MeV, and $M$ is a meson
mass matrix with the non-vanishing matrix elements $M_{11} = M_{22} = m^2_\pi$, $M_{33} = 2m^2_K - m^2_\pi$.
It was established by Gasser and Leutwyler\cite{1} that in the chiral limit the most general
expressions for the $p^4$ effective chiral Lagrangians including external vector $V_\mu$ and axial-
vector $A_\mu$ gauge fields are

$$\mathcal{L}_S^{(4)} = L_1[\text{Tr}(D^\mu U^\dagger D_\mu U)]^2 + L_2[\text{Tr}(D_\mu U^\dagger D_\nu U)]^2 + L_3\text{Tr}(D^\mu U^\dagger D_\mu U)^2$$
$$- iL_9\text{Tr}(F^R_{\mu\nu} D^\mu U^\dagger D^\nu U + F^L_{\mu\nu} D^\mu U D^\nu U^\dagger) + L_{10}\text{Tr}(U^\dagger F^R_{\mu\nu} U F^{\mu\nu L})$$

(2.2)
with

\[ D_\mu U = \partial_\mu U + A^L_\mu U - U A^R_\mu, \]
\[ F_{\mu\nu}^{LR} = \partial_\mu A^{LR}_\nu - \partial_\nu A^{LR}_\mu + [A^{LR}_\mu, A^{LR}_\nu], \]
\[ A^{L,R}_\mu = V_\mu \pm A_\mu. \] (2.3)

Gasser and Leutwyler have empirically determined the parameters \( L_1, \ldots L_{10} \) at the mass scale \( \mu = m_\eta \) from various low energy hadronic processes in conjunction with the Zweig-rule argument.

In the limit of large \( N_c \) (\( N_c \) being the number of colors), the aforementioned chiral couplings are theoretically manageable at least to the zeroth order in \( \alpha_s^2 \). There exist several approaches for the computation of \( L_i \).\(^{[3-6]} \) Here, we will only mention a formal one.\(^{[3,5]} \) First of all, the chiral-loop contribution is suppressed by at least a factor of \( 1/N_c \) relative to the quark loop at the same order of \( p^n \) in the leading \( 1/N_c \) expansion. Subsequently, the higher order couplings in the large \( N_c \) chiral perturbation theory are renormalization scale independent. Second, consider QCD coupled to external gauge fields. The integration of both quark and gluonic degrees of freedom yields two different categories of global chiral anomalies: proper (Bardeen) anomalies which contain the totally antisymmetric tensor \( \epsilon_{\mu\nu\alpha\beta} \) and spurious anomalies which do not. Now the variation of the generating function under a local chiral transformation is governed by chiral anomalies. It is well known that the integration of topological anomalies gives the Wess-Zumino-Witten effective action. Likewise, the integration of nontopological anomalies yields an action for \( p^4 \) nonanomalous chiral Lagrangians. A consistent leading \( 1/N_c \) expansion requires one to include not only the contributions from the quark loops but also the gluonic effects arising from all planar diagrams without the internal quark loops. The gluonic corrections which have been neglected in previous publications were dicussed in ref.[5]. In the chiral limit, the large-\( N_c \) chiral couplings valid to the leading order in gluonic corrections read\(^{[3-6]} \)

\[ 8L_1 = 4L_2 = L_9 = \frac{N_c}{48\pi^2}, \quad L_3 = L_{10} = \frac{N_c}{96\pi^2}(1 + \xi). \] (2.4)

It was shown in ref.[5] that to the first order in \( \alpha_s \) only the couplings \( L_3 \) and \( L_{10} \) receive
Several remarks are in order. (i) The strong effective Lagrangian given by Eq.(2.4) should be viewed as a QCD-motivated model rather than a formal chiral Lagrangian derived from large-\(N_c\) QCD: It is obtained by coupling QCD to external gauge fields and considering its anomalous variation. (ii) Since the chiral parameters \(L_i\) in the leading \(1/N_c\) expansion are scale-independent constants, one should in principle not compare Eq.(2.4) with the running couplings \(L_i(\mu)\) determined from experiment. Nevertheless, empirically they are quite similar at the mass scale between 0.5 and 1 GeV.[13]

It is known that the lowest-order chiral Lagrangian responsible for \(\Delta S = 1\) and \(\Delta I = \frac{1}{2}\) nonleptonic weak interactions reads

\[
\mathcal{L}^{(2)}_W = -g_s \text{Tr}(\lambda_6 L_\mu L_\mu),
\]

where \(L_\mu \equiv (\overline{\partial}_\mu U)U^\dagger\) is an \(SU(3)_R\) singlet and \(L_\mu^\dagger = -L_\mu\). The parameter \(g_s\) of the octet weak interaction is determined from the measured \(K \to \pi\pi\) rates. In the chiral limit and in the absence of external gauge fields there are seven independent CP-even quartic-derivative weak Lagrangian terms[14] which transform as \((8L, 1_R)\) under chiral rotations:

\[
\mathcal{L}^{(4)}_W = \frac{g_8}{f_\pi^2} \left\{ h_1 \text{Tr}(\lambda_6 L_\mu L_\mu L_\nu L_\nu) + h_2 \text{Tr}(\lambda_6 L_\mu L_\nu L_\mu L_\nu) \\
+ h_3 \text{Tr}(\lambda_6 L_\mu L_\nu L_\nu L_\mu) + h_4 \text{Tr}(\lambda_6 L_\mu L_\nu) \text{Tr}(L_\mu L_\nu) \\
+ h_5 \text{Tr}(\lambda_6 \tilde{Y} \tilde{Y}) + h_6 \text{Tr}\left( [\lambda_6, \tilde{Y}] L_\mu L_\mu \right) + h_7 \text{Tr}\left( [\lambda_6, \tilde{Y}_\mu] L_\mu L_\nu \right) \right\},
\]

where \(Y_{\mu\nu} = (\overline{\partial}_\mu \partial_\nu U)^\dagger, \tilde{Y}_{\mu\nu} = Y_{\mu\nu} - Y_{\nu\mu}^\dagger,\) and \(\tilde{Y} = g^{\mu\nu} Y_{\mu\nu}.\) Under the CP transformation, \(\tilde{Y}_{\mu\nu} \to -\tilde{Y}_{\nu\mu}.\)

\* There exists an inconsistency for the chiral coupling \(L_{10}^\prime.\) The value of \(L_{10}^\prime(\mu = m_\rho) = -(5.2 \pm 0.3) \times 10^{-3}\) often quoted in the literature is obtained from the experimental measurement of the axial-to-vector form factor ratio \(f_A/f_V = 0.52 \pm 0.06\) (The updated value is 0.45 \pm 0.06[11]) in the radiative pion decay. On the other hand, it is extracted to be \(-(2.9 \pm 0.8) \times 10^{-3}\) from the pion polarizability measured in the pion Compton scattering.[12] While the former value is based on more precise experiments, the latter is favored by theory (though some experts may disagree on this point) for the reason that both \(L_3\) and \(L_{10}\) receive the same amount of gluonic corrections[9] and that the predicted \(L_3 = 3.2 \times 10^{-3}\) (note that \(L_3\) is \(\mu\)-independent) to the zeroth order in \(\alpha_s\) is already in good agreement with experiment. It is thus not clear to us what is the origin for the discrepancy. Fortunately, the coupling \(L_{10}\) is not relevant for our ensuing discussion. Therefore, we can safely put \(\xi = 0\) for our later purposes.

\[\text{For the construction of the most general expressions of the counterterm Lagrangians relevant for the nonleptonic weak interactions, see ref.[15].}\]
To determine the weak chiral parameters $h_i$ in the $1/N_c$ expansion requires three ingredients: the $\Delta S = 1$ effective weak Hamiltonian at the quark level, bosonization and factorization, as we are going to elaborate on. The $\Delta S = 1$ effective nonleptonic Hamiltonian in the limit of large $N_c$ has a rather simple structure:

$$H_{\text{eff}}^\Delta S=1 \propto \frac{G_F}{\sqrt{2}} \sin \theta_c \cos \theta_c \left\{ c_8 (Q_2 - Q_1) + c_27 (Q_2 + 2Q_1) \right\},$$

(2.7)

where $(\bar{q}_i q_j) \equiv \bar{q}_i \gamma_\mu (1 - \gamma_5) q_j$. The combination $(Q_2 - Q_1)$ is a $\Delta I = \frac{1}{2}$ four-quark operator which transforms as $(8_L, 1_R)$ under chiral rotation, while $(Q_2 + 2Q_1)$ is equivalent to a 27-plet $\Delta I = \frac{3}{2}$ operator in the large-$N_c$ approximation.

Using Eqs.(2.2) and (2.4) one can determine the bosonization of the quark current $J_{ij}^\mu \equiv (\bar{q}^i q^j)$ to the next-to-leading order in chiral expansion. Writing $(J_{ij})_{ji} = (ij^2/2)(\hat{L}_\mu)_{ij}$, the result is

$$\hat{L}_\mu = L_\mu + \frac{N_c}{24\pi^2 f_\pi^2} \left\{ (L_\nu L_\mu L_\nu + L_\mu L_\nu L_\nu) + [\hat{Y}, L_\mu] + [L_\nu', \hat{Y}_\nu]\right\}.$$

(2.8)

Since factorization is valid in the large-$N_c$ approximation, one may substitute (2.8) into (2.7) to obtain the quartic-derivative weak chiral couplings $h_i$:

$$h_1 = -\frac{1}{3} h_2 = \frac{1}{3} h_4 = h_6 = -h_7 = \frac{N_c}{24\pi^2}, \quad h_3 = h_5 = 0.$$  

(2.9)

The effective weak chiral Lagrangians $L_W^{(2)} + L_W^{(4)} (1/N_c)$ have been tested successfully in the study of the nonleptonic $K \to \pi\pi\pi$ decay.

3. Electromagnetically Induced Anomalous and Non-anomalous Weak Chiral Lagrangians

The most general $p^4$ electromagnetically induced $\Delta S = 1$ non-anomalous weak Lagrangians with at most two external photon fields which satisfy the constraints of chiral and
CPS symmetry have the form

\[
\mathcal{L}_{\text{non-anom}}^{S=1} = i \left( \frac{2}{f_\pi} \right) g_8 e^{F_{\mu\nu}} \left[ \omega_1 \text{Tr}(\lambda_6 L_\mu L_\nu Q) + \omega_2 \text{Tr}(\lambda_6 L_\nu Q L_\mu) \right] + \omega_3 \left( \frac{2}{f_\pi} \right) g_8 e^{2F_{\mu\nu}} F_{\mu\nu} \text{Tr}(\lambda_6 Q U Q U^\dagger),
\]

while the anomalous terms are (\( \tilde{F}_{\mu\nu} \equiv \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \)) \(^\dagger\)

\[
\mathcal{L}_{\text{anom}}^{S=1} = i \omega_4 \left( \frac{2}{f_\pi} \right) g_8 e \tilde{F}_{\mu\nu} \text{Tr}(Q L_\mu) \text{Tr}(\lambda_6 L_\nu)
+ i \omega_5 \left( \frac{2}{f_\pi} \right) g_8 e \tilde{F}_{\mu\nu} \text{Tr}(Q U L_\mu U) \text{Tr}(\lambda_6 L_\nu)
+ i \omega_6 \left( \frac{2}{f_\pi} \right) g_8 e \tilde{F}_{\mu\nu} \text{Tr}(\lambda_6 [U Q U^\dagger, L_\mu L_\nu])
+ \omega_7 \left( \frac{2}{f_\pi} \right) g_8 e^{2F_{\mu\nu}} F_{\mu\nu} \epsilon_{\alpha\beta\rho\sigma} \text{Tr}(\lambda_6 L_\alpha) \text{Tr}(L_\beta L_\rho L_\sigma),
\]

in which the ordinary derivative in \( L_\mu \) is replaced by the covariant derivative in the presence of external gauge fields. Presently, there are only two information on the unknown parameters \( \omega_i \). First of all, Ecker, Pich and de Rafael (EPR) \(^{[17]}\) found the relation \( \omega_2 = 4L_9 \), which must hold at least for the divergent parts of the counterterm coupling constants because they must render the divergent loop amplitudes finite. Second, from the BNL measurement of the \( K^+ \to \pi^+ e^+ e^- \) decay rate, \(^{[18]}\) one finds a scale-independent relation

\[
\omega_1 + 2\omega_2 - 12L_9 \simeq -7.5 \times 10^{-3}.
\]

This together with the empirical value of \( L_9^r(\mu = m_\rho) = 6.7 \times 10^{-3} \) leads to

\[
\omega_1^r(\mu = m_\rho) + 2\omega_2^r(\mu = m_\rho) \simeq 0.074.
\]

In the presence of the external electromagnetic field, the gauge field in Eq.(2.3) is identified with

\[
A_\mu^L = -ieA_\mu Q, \quad A_\mu^R = -ieA_\mu Q,
\]

with \( Q = \text{diag}(2/3, -1/3, -1/3) \) and \( A_\mu \) being the photon field. Just as in Sec.II, one first finds out the bosonization of the quark current in the presence of external photon field

\(^\dagger\) Note that our \( \omega_3, \omega_4 \) are the couplings \( \omega_3, \omega_3 \) respectively in ref.[17]. The last three terms in (3.2) are missed in the same reference.
and then substitutes it into the four-quark operator \((Q_2 - Q_1)\) and gets the non-anomalous \(\Delta S = 1\) Lagrangian \(\mathcal{L}_{\text{non-anom}}^{\Delta S=1}\) [Eq.(3.1)] with

\[
\omega_1 = \omega_2 = \frac{N_c}{12\pi^2}, \quad \omega_3 = 0.
\] (3.6)

The previous observation of \(\omega_2 = 4L_9\) made by EPR is reproduced here. Evidently, the large-\(N_c\) prediction of \(\omega_1 + 2\omega_2 = 0.076\) is remarkably in agreement with (3.4).\(\dagger\)

The derivation of the large-\(N_c\) anomalous weak chiral Lagrangian coupled to external photon fields is more complicated but more interesting as it is governed by chiral anomalies. To do this, one first writes down the relevant Wess-Zumino-Witten terms

\[
\mathcal{L}_{\text{WZW}} = -\frac{N_c}{48\pi^2} \epsilon^{\mu
u\rho\sigma} \text{Tr}\{ - (A^R_\mu R_\nu R_\rho R_\sigma + A^L_\mu L_\nu L_\rho L_\sigma) \\
- \frac{1}{2} A^L_\mu L_\nu A^R_\rho L_\sigma - A^R_\mu U^\dagger_A^L_\nu U^R R_\rho R_\sigma + A^L_\mu U A^R_\nu U^\dagger L_\rho L_\sigma \\
+ \partial_\mu A^R_\nu U^\dagger_A^L_\rho U R_\sigma + \partial_\mu A^L_\nu U A^R_\rho U^\dagger L_\sigma \\
+ (A^L_\mu \partial_\nu A^L_\rho + \partial_\mu A^L_\nu A^L_\rho) L_\sigma \} + \ldots,
\] (3.7)

where \(R_\mu \equiv U^\dagger \partial_\mu U\). Once the bosonization in the anomalous case is found after a lengthy manipulation, it is straightforward to show that\(\ast\)

\[
\omega_4 = 2\omega_5 = 4\omega_6 = -8\omega_7 = \frac{N_c}{12\pi^2}.
\] (3.8)

This result first obtained in ref.[9] was recently confirmed by ref.[21]. It should be stressed that the anomalous chiral coupling constants are free of gluonic corrections.

4. **Application to \(K \rightarrow 3\pi\) Decay**

\(\dagger\) Recently, very different large-\(N_c\) predictions \(\omega_1 = \omega_2 = 8L_9\), \(\omega_3 = 12L_{10}\) were obtained by Bruno and Prades.\(\textsuperscript{19}\) This is attributed to the fact that the effect of the quark operator \(Q_1\) is not considered by them.

\(\dagger\) The large-\(N_c\) prediction \(\omega_1 + 2\omega_2 - 12L_9 = 0\) is also in good agreement with (3.3) in view of the fact that \(12L_9(\mu = m_\rho) \approx 0.08\).

\(\ast\) It was wrongly conjectured in ref.[20] that the couplings \(\omega_4, \ldots, \omega_7\) have nothing to do with the chiral anomaly. This has been corrected in ref.[21] and is now consistent with ref.[9].
As stressed in Introduction, it is necessary to introduce a weak chiral Lagrangian with higher derivatives in order to account for non-vanishing quadratic coefficients and the discrepancy between theory and experiment for the constant and linear terms in the Dalitz expansion of \( K \to 3\pi \) amplitude [Eq.(1.1)]. For \( \Delta I = \frac{1}{2} \) amplitudes, we find a remarkable agreement between \( 1/N_c \) theory and experiment within 3% for the constant and linear terms. (The reader is referred to ref.[8] for more details.) This means that very little room is left for chiral-loop corrections. The predicted coefficient \( c \) is just marginally in accord with data within the experimental errors, while the other coefficient \( d \) is off by three standard deviations. Clearly more accurate \( K3\pi \) data are urgent to clarify this discrepancy. In the \( \Delta I = \frac{3}{2} \) sector, we see that the linear term is generally in agreement with data, whereas the constant term is four standard deviations off from experiment. Obviously, more high-statistics experiments are required to improve the determination of \( \Delta I = \frac{3}{2} \) coefficients \( a \) and \( b \), and to extract the quadratic terms \( c \) and \( d \) in order to test the chiral-Lagrangian approach.

Since the quadratic slope parameter in the \( K_L \to 3\pi^0 \) Dalitz plot was recently measured at Fermilab based on a sample of \( 5.1 \times 10^6 \) decays,\(^{[22]} \) it is very interesting to compare the \( 1/N_c \) prediction with experiment. The isospin structure of the \( K_L \to 3\pi^0 \) Dalitz amplitude is given by

\[
A(K_L \to 3\pi^0) = -3(a_1 - 2a_3) - 3(c_1 - 2c_3)(Y^2 + X^2/3),
\]

(4.1)

where the subscript 1 (3) refers to \( \Delta I = \frac{1}{2} (\frac{3}{2}) \) transition. The quadratic slope parameter \( h \) for the decay is \( 2(c_1 - 2c_3)/(a_1 - 2a_3) \). From Tables 1 and 2 of ref.[8] we find

\[
h = -4.7 \times 10^{-3},
\]

(4.2)
in accord with the result from the Fermilab E731 experiment

\[
h_{\text{expt}} = -(3.3 \pm 1.1 \pm 0.7) \times 10^{-3}.
\]

(4.3)

For comparison, a somewhat large value of \( -(1.2 \pm 0.4) \times 10^{-2} \) for \( h \) is predicted by ref.[2].

Finally, we would also like to mention the decay \( K_S \to \pi^+\pi^-\pi^0 \), whose Dalitz amplitu
is of the form $X(1 + \alpha Y)$. Explicitly,$^{[8]}$

$$A(K_S \to \pi^+\pi^-\pi^0) = -\frac{2}{3}b_3'X + \frac{4}{3}d_3'XY.$$ (4.4)

We predict that$^{[8]}$

$$Br(K_S \to \pi^+\pi^-\pi^0) = 3.9 \times 10^{-7}, \quad \alpha = 4.4 \times 10^{-2}. \quad (4.5)$$

The experimental feasibility for measuring this decay mode is not pessimistic.

5. Applications to Radiative Kaon Decay

5.1 The $K^+ \to \pi^+\gamma\gamma$ decay

As first pointed out by EPR,$^{[17]}$ the loop amplitudes of $K_{L,S} \to \pi\gamma\gamma$ and $K^+ \to \pi^+\gamma\gamma$ are finite. From the point of view of large $N_c$ chiral-Lagrangian approach, the mode $K^+ \to \pi^+\gamma\gamma$ is more interesting since it also receives contributions from the tree Lagrangians $\mathcal{L}_{\text{non-anom}}^{\Delta S=1}$ and $\mathcal{L}_S^{(4)}$ via pole diagrams (except for the $\omega_3$ term which contributes via the direct-emission diagram). The total decay rate of $K^+ \to \pi^+\gamma\gamma$ was calculated in ref.[17] to be

$$\Gamma(K^+ \to \pi^+\gamma\gamma) = \Gamma_{\text{loop}} + \Gamma_{\text{tree}} + \Gamma_{\text{int}} + \Gamma_{\text{WZW}}, \quad (5.1)$$

with

$$\Gamma_{\text{loop}} = 2.80 \times 10^{-23}\text{GeV}, \quad \Gamma_{\text{tree}} = 0.17\hat{c}^2 \times 10^{-23}\text{GeV},$$
$$\Gamma_{\text{int}} = 0.87\hat{c} \times 10^{-23}\text{GeV}, \quad \Gamma_{\text{WZW}} = 0.26 \times 10^{-23}\text{GeV}, \quad (5.2)$$

and

$$\hat{c} = 32\pi^2 \left[ 4(L_9 + L_{10}) - \frac{1}{3}(\omega_1 + 2\omega_2 + 2\omega_3) \right]. \quad (5.3)$$

Note that the combinations $\omega_1 + 2\omega_2 + 2\omega_3$ and $L_9 + L_{10}$ are separately scale independent. From Eqs.(2.4) and (3.6) we obtain $\hat{c} = -4$ and the branching ratio

$$Br(K^+ \to \pi^+\gamma\gamma) = 5.1 \times 10^{-7}. \quad (5.4)$$

Since this decay is dominated by chiral-loop effects, its decay rate is rather insensitive to the model of higher-derivative chiral Lagrangians. For example, $\hat{c}$ is predicted to be zero in
the so-called "weak deformation model"\textsuperscript{[23]} but the corresponding branching ratio $5.8 \times 10^{-7}$ is very close to that in the $1/N_c$ approach. In order to discriminate these two models, experimentally it is important to measure the two-photon spectrum around $z = m_{\gamma\gamma}^2/m_K^2 = 0.3$ where the spectrum behaves quite differently for $\hat{c} = -4$ and 0 (see Fig.2 of ref.[9]).

The present best upper limit\textsuperscript{[24]} $1.0 \times 10^{-6}$ for $K^+ \rightarrow \pi^+ \gamma\gamma$ was obtained by assuming a $\pi^+$ energy distribution given by phase space. If the theoretical spectrum for $\hat{c} = -4$ is used, then the upper limit will be pulled back to the level of $1.5 \times 10^{-4}$\textsuperscript{[24]}

5.2 Direct $K \rightarrow \pi\pi\gamma$ transitions

The structure-dependent photon-emission decay $K \rightarrow \pi\pi\gamma$ provides an excellent probe on the $p^4$ weak chiral Lagrangian coupled to external electromagnetic fields. Under Lorentz and gauge invariance, the general expression for the invariant direct emission (DE) amplitude of the decay $K(k) \rightarrow \pi(p_1)\pi(p_2)\gamma(q)$ reads

\[ A_{DE} = \tilde{\beta}M + \tilde{\gamma}E, \]
\[ M \equiv e\epsilon_{\mu\nu\rho\sigma}p_1^\mu p_2^\nu q^\rho \epsilon^\sigma, \]
\[ E \equiv e[(p_1 \cdot \epsilon)(p_2 \cdot q) - (p_2 \cdot \epsilon)(p_1 \cdot q)], \]

where $\epsilon_\mu$ is the polarization vector of the photon. The first term of $A_{DE}$ corresponds to magnetic transitions whereas the second term is caused by electric transitions. Evidently, the DE amplitude is of third power in momenta. Taking into account the experimental cutoff on the photon energy, we have the following branching ratios

\[ Br(K^+ \rightarrow \pi^+\pi^0\gamma)_{DE} = 1.32 \times 10^5 \text{GeV}^6 (|\tilde{\beta}|^2 + |\tilde{\gamma}|^2), \]
\[ Br(K_L \rightarrow \pi^+\pi^-\gamma)_{DE} = 1.33 \times 10^6 \text{GeV}^6 (|\tilde{\beta}|^2 + |\tilde{\gamma}|^2), \]
\[ Br(K_S \rightarrow \pi^+\pi^-\gamma)_{DE} = 2.28 \times 10^3 \text{GeV}^6 (|\tilde{\beta}|^2 + |\tilde{\gamma}|^2). \]

There are two contributions to direct photon emission of $K \rightarrow \pi\pi\gamma$: long-distance pole contributions and contact-term ones (i.e., direct weak transitions). The long-distance pole contribution is governed by the anomalous Wess-Zumino-Witten interaction. Note that in the limit of $CP$ symmetry, $K_L \rightarrow \pi^+\pi^-\gamma$ proceeds only via the magnetic transition, whereas $K_S \rightarrow \pi^+\pi^-\gamma$ is caused by electric transition. At first sight, one may tempt to conclude that the theoretical prediction for $K_L \rightarrow \pi^+\pi^-\gamma$ should be most reliable because it is entirely determined by chiral anomalies. We will see later that it is not the case.
Numerical values of the $1/N_c$ predictions are shown in Table I of ref.[9]. It is evident from Table I that the agreement between theory and experiment for the direct emission of $K^+ \to \pi^+\pi^0\gamma$ is striking, implying that very little room is left for chiral-loop corrections. For $K_S \to \pi^+\pi^-\gamma$, the branching ratio of the structure-dependent component is predicted to be $2 \times 10^{-7}$, which is beyond the present upper limit$^{[25]} 6 \times 10^{-5}$.

We cannot make a definite prediction for the direct emission of $K_L \to \pi^+\pi^-\gamma$ owing to a large theoretical uncertainty in the estimate of the long-distance effect, as we shall discuss shortly. In the absence of the $\eta'$ pole, it is easily seen that the pole contribution to $K_L \to \pi^+\pi^-\gamma$ vanishes due to the Gell-Mann-Okubo mass relation $m_{\eta}^2 = \frac{1}{3}(4m_K^2 - m_{\pi})$. However, the direct weak contribution alone will yield a branching ratio of $2 \times 10^{-4}$, which is too large by an order of magnitude when compared with experimental branching fraction$^{[26]}$ of $(2.89 \pm 0.28) \times 10^{-5}$. This means that a large destructive interference between pole and direct-transition amplitudes of $K_L \to \pi^+\pi^-\gamma$ is required in order to explain data. The $\eta'$ pole is thus called for.

The inclusion of the $\eta'$ intermediate state introduces two complications: First, the $SU(3)$ singlet $\eta_0$ is outside of the framework of $SU(3) \times SU(3)$ ChPT; that is, the matrix element $\langle \eta_0 | L_W | K_L \rangle$ is not related to the $K_L - \pi^0$ transition by $SU(3)$ symmetry. Second, there will be an $\eta - \eta'$ mixing effect. In the $U(3) \times U(3)$ version of $L_W$, the above-mentioned two matrix elements are connected via nonet symmetry, viz.

$$\langle \eta_0 | L_W | K_L \rangle = -2\sqrt{\frac{2}{3}}\rho \langle \pi^0 | L_W | K_L \rangle , \quad (5.7)$$

where the parameter $\rho$ is introduced so that the deviation of $\rho$ from unity implies the breakdown of nonet symmetry. The pole contribution is quite sensitive to $SU(3)$ symmetry and nonet symmetry breaking. For example, in the absence of symmetry breaking effects the branching ratio is calculated to be $6.4 \times 10^{-5}$, which is two times large. Neglecting $SU(3)$ breaking and fitting to the experimental central value, we find $\rho \approx 1.1$. This illustrates that presently no definite prediction on the pole effects can be made with certainty.*

Finally, two remarks are in order. (i) Pole and contact-term contributions are equally important for the direct radiative transition of $K^+$ and $K_L$, whereas only the latter one

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* Efforts of relating the pole contributions of $K_L \to \pi^+\pi^-\gamma$ and $K_L \to \gamma\gamma$ have been made before (see e.g., refs.[9, 20,27]). However, as emphasized by Shore and Veneziano,$^{[28]}$ the naive PCAC analysis does not apply to the decay $\eta' \to \gamma\gamma$. 

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contributes to $K_S \rightarrow \pi^+ \pi^- \gamma$ in the limit of CP symmetry. (ii) Unlike inner bremsstrahlung, the direct-emission amplitudes of $K^{\pm} \rightarrow \pi^{\pm} \pi^0 \gamma$ and $K_L \rightarrow \pi^+ \pi^- \gamma$ are no longer subject to the $\Delta I = \frac{1}{2}$ rule and CP violation, respectively. This explains why the branching ratio of $K^+$ and $K_L$ is larger than that of $K_S$ by two orders of magnitude and why structure-dependent effects can be seen in those two decay modes.

6. Conclusion

We have applied the large $N_c$ chiral Lagrangian approach to nonleptonic and radiative kaon decays. Decay rates and spectra are unambiguously predictable to the leading $1/N_c$ expansion and to the zeroth order in gluonic modifications. Future high-statistics experiments with great sensitivity will be able to test those predictions.

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