QUASI-PERIODIC SOLUTIONS FOR
MATRIX NONLINEAR SCHRÖDINGER EQUATIONS†

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Abstract

The Adler-Kostant-Symes theorem yields isospectral hamiltonian flows on the dual \( \tilde{g}^{++} \) of a Lie subalgebra \( \tilde{g}^+ \) of a loop algebra \( \tilde{g} \). A general approach relating the method of integration of Krichever, Novikov and Dubrovin to such flows is used to obtain finite-gap solutions of matrix Nonlinear Schrödinger Equations in terms of quotients of \( \theta \)-functions.

1. Introduction

In a recent paper [AHH1] the method of integrating nonlinear partial differential equations due to Krichever, Novikov and Dubrovin (see, for example, [KN, D]) was adapted to computing finite-gap solutions for PDE arising as integrability conditions for Lax equations arising in the framework of the Adler-Kostant-Symes (AKS) theorem. Consider a loop algebra \( \tilde{g} = \tilde{g}^+ \oplus \tilde{g}^- \), split into the direct sum of the subalgebra \( \tilde{g}^+ \) of loops \( X(\lambda) \), which, viewed as maps \( X : S^1 \to g \) into a Lie algebra \( g \) extend holomorphically inside the unit circle in the complex \( \lambda \)-plane and \( \tilde{g}^- \), the subalgebra of loops extending holomorphically outside the unit circle, normalized by the condition \( X(\infty) = 0 \). The Lax equations considered in [AHH1], such as (1.5a,b), determine completely integrable Hamiltonian systems. They follow from the AKS theorem and are are defined on the dual space \( \tilde{g}^{++} \) of the subalgebra \( \tilde{g}^+ \).

In an earlier paper [AHP], flows in a finite dimensional subspace consisting of rational elements in \( \tilde{g}^{++} \) were related to isospectral rank-\( r \) deformations of a fixed \( N \times N \) matrix. In [HW] the coupled nonlinear Schrödinger equations corresponding

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to the classical Lie algebras (cf. [FK]), which will henceforth be called “matrix NLS equations”, were related to the [AHP] framework. The general form of these equations is

\[
\sqrt{-1} q_t = q_{xx} + qpq \\
-\sqrt{-1} p_t = p_{xx} + pqp,
\]

where \(q, p \in \mathbb{C}^{a \times b}\).

Let \(\tilde{\mathfrak{sl}}(r, \mathbb{C}), r = a + b\) be the Lie algebra of loops in \(\mathfrak{sl}(r, \mathbb{C})\). Let \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^+\) be the subalgebra of loops extending holomorphically inside the unit circle. Using the ad-invariant inner product

\[
\langle X, Y \rangle = \oint_{S^1} \frac{\text{tr}(X(\lambda)Y(\lambda))}{\lambda} d\lambda, \quad X, Y \in \tilde{\mathfrak{sl}}(r, \mathbb{C})
\]

we may henceforth identify the regular part of \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^*\) with \(\tilde{\mathfrak{sl}}(r, \mathbb{C})\). In the same way \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^{++}\) is identified with \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^-\), the subalgebra of loops extending holomorphically outside the unit circle. No notational distinction will be made between elements of \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^*\) (resp. \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^{++}\)) and elements of \(\tilde{\mathfrak{sl}}(r, \mathbb{C})\) (resp. \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^-\)). Let \(\mathcal{F}_+ = \text{I}(\tilde{\mathfrak{sl}}(r, \mathbb{C})^*)\tilde{\mathfrak{sl}}(r, \mathbb{C})^{++}\) be the ring of Ad* -invariant functions on \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^*\) restricted to \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^{++}\). The AKS-theorem tells us that elements of \(\mathcal{F}_+\) commute in the Lie-Poisson structure of \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^{++}\). Furthermore, the Hamiltonian equations for functions in \(\mathcal{F}_+\) are given by Lax pairs. The Hamiltonians for the matrix nonlinear Schrödinger equation are given by

\[
\Phi_x(X) = \frac{1}{2} \text{tr} \left( \left( \frac{a(\lambda)}{\lambda^{n-1}} X(\lambda) \right)_0 \right) \quad (1.3a)
\]

\[
\Phi_t(X) = \frac{1}{2} \text{tr} \left( \left( \frac{a(\lambda)}{\lambda^{n-2}} X(\lambda) \right)_0 \right) \quad (1.3b)
\]

for the \(x\) and \(t\) flows respectively, where \(X \in \tilde{\mathfrak{sl}}(r, \mathbb{C})^*\) and \(a(\lambda) = \prod_{i=1}^n (\lambda - \alpha_i)\). The \(\alpha_i\) are constants chosen inside the unit circle. (N.B. This restriction is inessential; we can always choose our circle \(S^1\) to have a larger radius, so as to enclose arbitrarily chosen \(\alpha_i\)'s). Let

\[
\mathcal{N}(\lambda) = \lambda \sum_{i=1}^n \frac{\mathcal{N}_i}{\alpha_i - \lambda} \in \tilde{\mathfrak{sl}}(r, \mathbb{C})^-,
\]

where the \(\mathcal{N}_i \in \mathfrak{sl}(r, \mathbb{C})\) are of fixed rank \(k_i \leq r - 1\). Hamilton’s equations for (1.3a,b) for elements of the form (1.4) are given by

\[
\frac{d}{dx} \mathcal{N}(\lambda) = [d\Phi_x(\mathcal{N})_+, \mathcal{N}] \quad (1.5a)
\]

\[
\frac{d}{dt} \mathcal{N}(\lambda) = [d\Phi_t(\mathcal{N})_+, \mathcal{N}] \quad (1.5b)
\]
where the “+” subscript means projection to \( \widetilde{\mathfrak{sl}}(r, \mathbb{C})^+ \). The co-adjoint orbit \( \mathcal{O}_N \) through \( N \) preserves the pole structure of \( N(\lambda) \) and the rank of \( N_i \) and hence these are preserved under any Hamiltonian flow. This orbit is therefore finite dimensional.

The Lax equations for

\[
\mathcal{L}(\lambda) \equiv \frac{a(\lambda)}{\lambda} N(\lambda) = \lambda^{n-1} \mathcal{L}_0 + \lambda^{n-2} \mathcal{L}_1 + \cdots + \mathcal{L}_{n-1}
\]  

are given by

\[
\frac{d}{dx} \mathcal{L}(\lambda) = [\lambda \mathcal{L}_0 + \mathcal{L}_1, \mathcal{L}]
\]

\[
\frac{d}{dt} \mathcal{L}(\lambda) = [\lambda^2 \mathcal{L}_0 + \lambda \mathcal{L}_1 + \mathcal{L}_2, \mathcal{L}]
\]  

These equations may also be viewed as due to the AKS theorem on the orbit \( \mathcal{O}_L \) through \( L \), with Hamiltonians

\[
\widetilde{\Phi}_x(X) = \frac{1}{2} \text{tr}(\lambda^{2-n} X(\lambda))_0)
\]

\[
\widetilde{\Phi}_t(X) = \frac{1}{2} \text{tr}(\lambda^{3-n} X(\lambda))_0)
\]  

Equations (1.1) are the integrability conditions for (1.7a,b) if the leading terms of \( \mathcal{L}(\lambda) \) are

\[
\mathcal{L}_0 = \frac{\sqrt{-1}}{a + b} \begin{pmatrix} bI_a & 0 \\ 0 & -aI_b \end{pmatrix}
\]

\[
\mathcal{L}_1 = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}
\]

\[
\mathcal{L}_2 = \frac{\sqrt{-1}}{q} \begin{pmatrix} qp & -q x \\ px & -pq \end{pmatrix}
\]  

The underlying constraints on \( \mathcal{L}_0 \), \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are invariant under the flows of all Hamiltonians in \( \mathcal{F}_+ \).

Section 2 contains a brief summary of the main results of [AHH1]. The first part is devoted to the properties of the invariant spectral curve arising from Lax equations of the form

\[ \mathcal{L}_\tau = [P(\mathcal{L}(\lambda), \lambda^{-1})_+, \mathcal{L}(\lambda)] \]  

where \( \mathcal{L}(\lambda) \) is a matrix polynomial of the form (1.6), \( \tau \) is the flow parameter and \( P(z, \lambda^{-1}) \) is a polynomial in \( z \) and \( \lambda^{-1} \). The second part of section 2 summarizes the integration method for equations of type (1.10) yielding solutions in terms of \( \theta \)-functions. Section 3.1 discusses the singularities of the spectral curve underlying equations (1.7a,b). Solutions to (1.1) in terms of quotients of \( \theta \)-functions are obtained in section 3.2.
Although only the generic case of $a \times b$ dimensional rectangular matrices $q, p^T$ is treated here, we emphasize that finite-gap solutions for matrix NLS equations corresponding to the various reductions by involutive automorphisms of the hermitian symmetric Lie algebra $(\mathfrak{sl}(r, \mathbb{C}), \mathfrak{sl}(a, \mathbb{C}) \oplus \mathfrak{sl}(b, \mathbb{C}) \oplus \mathbb{C})$ (see [FK, HW]) can be obtained by imposing appropriate initial conditions on the matrix polynomial (1.6). (See also [AHH1] for a general approach to reductions to sublagebras of $\mathfrak{sl}(r, \mathbb{C})$).

2. Preliminaries

In this section we summarize the basic results of [AHH1].

2.1 The spectral curve for flows in a finite dimensional rational co-adjoint orbit

This subsection is devoted to the specific properties of the spectral curve $S_0$ given by

$$P(z, \lambda) = \det(L(\lambda) - zI) = 0. \quad (2.1)$$

which is invariant under flows of the Lax equation (1.10).

Let $\mathcal{O}(i)$ denote the $i$-th power of the hyperplane bundle $\mathcal{O}(1)$ over $\mathbb{P}^1(\mathbb{C})$. The technique of [AHH1] consists in compactifying $S_0$ by embedding it into the surface $T = \mathcal{O}(n-1)$. On $\mathbb{P}^1(\mathbb{C})$ consider standard coordinate charts $(V_0, \lambda)$ and $(V_1, \tilde{\lambda})$ over $V_0 = \mathbb{P}^1(\mathbb{C})\backslash\{\infty\}$, $V_1 = \mathbb{P}^1(\mathbb{C})\backslash\{0\}$, with $\tilde{\lambda} = \lambda^{-1}$ on $V_0 \cap V_1$. The line bundle $\pi : \mathcal{O}(n-1) \to \mathbb{P}^1(\mathbb{C})$ is covered by the coordinate charts $U_i = \pi^{-1}(V_i)$, $i = 1, 2$ with coordinates $(\lambda, z)$ on $U_0$ and $(\tilde{\lambda}, \tilde{z})$ on $U_1$, where $\tilde{z} = z\lambda^{-(n-1)}$ on $U_0 \cap U_1$. Expanding (2.1) in $z$ and $\lambda$ shows how $S_0$ may be embedded into $U_0$. Changing to $(\tilde{\lambda}, \tilde{z})$ over $U_0 \cap U_1$ extends the embedding of $S_0$ into $U_0$ and determines a compact curve $\tilde{S}$ on the surface $T$ that coincides with $S_0$ over $U_0$. Using the adjunction formula [GH, p. 471] the virtual genus $g$ of $S$ is easily computed to be

$$g = \frac{1}{2}(r - 1)(r(n - 1) - 2). \quad (2.2)$$

Since $L$ is given by (1.6), the curve has singularities at the points $(\lambda, z) = (\alpha_i, 0)$, $i = 1, \ldots, n$ if $k_i = \text{rank}(N_i) \leq r - 1$. Generically we have an ordinary $(r - k_i)$-fold intersection over the points $\lambda = \alpha_i$. This is the case when the matrices $N_i$, $i = 1, \ldots, n$ are diagonalizable. Blowing up $S$ at these points we obtain the (possibly singular) curve $\tilde{S}$, with virtual genus

$$\tilde{g} = g - \sum_{i=1}^{n} \frac{(r - k_i)(r - k_i - 1)}{2}. \quad (2.3)$$

2.2 Periodic Solutions in terms of $\theta$-functions for AKS-flows

In what follows, it is assumed that the partially desingularized curve $\tilde{S}$ does not have other singularities. In the case of equation (1.1), however, there are generically two
additional singular points at $\lambda = \infty$. Section 3 shows how solutions can be computed considering these singular points.

Let $O_T(i) = \pi^* O(i)$ denote the pullback of $O(i)$ to $T$. The flow of a matrix polynomial (1.6) gives rise to a flow of sheaves $E$ over $T$, defined by the exact sequence

$$0 \rightarrow O_T(-n + 1)^{\oplus r} \xrightarrow{L(\lambda)^{-zI}} O_T^{\oplus r} \rightarrow E \rightarrow 0. \quad (2.4)$$

where $O_T$ is the sheaf of holomorphic functions on $T$, identified with the trivial line bundle. By a standard abuse of notation we make no distinction between a line bundle and its sheaf of sections.

If $S$ is nonsingular and irreducible with genus $g$, $E$ is a line bundle of degree $g + r - 1$ over $S$. If $S$ is singular and irreducible, $E$ is a pushdown by some (possibly partial) desingularization map $\Psi : \tilde{S} \rightarrow S$ of a line bundle of degree $\tilde{g} + r - 1$ on $\tilde{S}$.

Conversely, a linear flow of line bundles $E_\tau$ over $S$ gives rise to a flow of matricial polynomials (1.6), governed by a Lax equation (1.10).

Let $L(\lambda; 0)$ be an initial value polynomial defining a spectral curve $S$. Let $\tilde{S}$ be the desingularization of $S$ with genus $\tilde{g}$. Let $E_0$ be the initial value line bundle over $\tilde{S}$ defined by (2.4). The degree of $E_0$ is generically $\tilde{g} + r - 1$. If $E_\tau$ is a line bundle of degree $\tilde{g} + r - 1$ undergoing linear flow and having $E_0$ as initial value, the tensor product $F_\tau = E_0^* \otimes E_\tau$ is a line bundle of degree zero and hence has a transition function $g_{01}$ from $U_0$ to $U_1$ given by an exponential,

$$g_{01}(z, \lambda) = \exp(\tau \mu(z, \lambda^{-1})), \quad (2.5)$$

where $\mu$ is a polynomial in $z$ and $\lambda^{-1}$. Fix a basis $\{\psi^1_\tau, \ldots, \psi^r_\tau\}$ of sections $H^0(\tilde{S}, E_\tau)$ (viewed as functions over a neighborhood of $\lambda = \infty$ by fixing a local trivialization) normalized by the condition

$$\psi^i_\tau(\infty_j) = \delta^i_j, \quad \forall \tau, \forall i, j = 1, \ldots, r. \quad (2.6)$$

where $\{\infty_1, \ldots, \infty_r\} \in \tilde{S}$ are the $r$ points over $\lambda = \infty$. For $p_j$ in a neighborhood of $\infty_j$ and $\lambda = \pi(p_j)$ its projection to $\mathbb{P}^1(C)$, define the matrix $\psi_\tau(\lambda)$ by

$$(\psi_\tau(\lambda))^i_j = \psi^i_j(p_j). \quad (2.7)$$

Let $z(p_j), j = 1, \ldots, r$ be the (generically distinct) eigenvalues of $L(\lambda; 0)$. The matrix polynomial $L(\lambda)$ given by

$$L(\lambda; \tau) = \psi^{-1}_\tau \begin{pmatrix} z(p_1) & \cdots & \cdots & \cdots & z(p_r) \end{pmatrix} \psi_\tau. \quad (2.8)$$
Define \( E \) such a function is also a section of \( \theta \) of \( r \) points over \( (a, \ldots, b) \) an element of \( H \). \( E \) is in general position there is exactly one meromorphic function \( \Delta \) such that \( (\mathcal{L}(\lambda_j) - z_j I)_{adj} v = 0 \), but \( (\mathcal{L}(\lambda_j) - z_j I)_{adj} \neq 0 \). Here the subscript denotes the classical adjoint matrix and the vector \( v \) exists.

Remark. In the AKS framework, \( \mu(z, \lambda^{-1}) \) is given by \( d\phi_\tau(z) \), where \( \phi_\tau \) is the Hamiltonian defining the \( \tau \)-flow.

To obtain a representation of the sections \( \psi^i_j, \; j = 1, \ldots, r \) in terms of \( \theta \) functions, we first describe explicitly the sections of \( E_0 \). Let \( \Delta \) be an effective divisor representing \( E_0 \). For instance, picking a fixed vector \( v \in \mathbb{C}^r \) we can take \( \Delta \) to be the sum of points \( \Delta^j = (\lambda_j, z_j)_{j=1,\ldots,\tilde{g}+r-1} \) in \( \tilde{S} \) such that \( (\mathcal{L}(\lambda_j) - z_j I)_{adj} v = 0 \), but \( (\mathcal{L}(\lambda_j) - z_j I)_{adj} \neq 0 \). Here the subscript denotes the classical adjoint matrix and the vector \( v \in \mathbb{C}^r \) represents an element of \( H^0(\tilde{S}, E_0) \). Let \( \Delta_\infty = \infty_1 + \cdots + \infty_r \) be the divisor consisting of the \( r \) points over \( \lambda = \infty \). Due to the Riemann-Roch theorem, assuming \( \Delta \) is in general position there is exactly one meromorphic function \( \psi^i_0 \) on \( \tilde{S} \) such that \( (\psi^i_0) \geq -\Delta + D_\infty - \infty_i \) and

\[
\psi^i_0(\infty_j) = \delta^i_j. \tag{2.10}
\]

Such a function is also a section of \( E_0 \). The \( r \) sections \( \{\psi^1_0, \ldots, \psi^r_0\} \) form a basis of \( H^0(\tilde{S}, E_0) \). Following [AHH1] such sections may be expressed in terms of quotients of \( \theta \)-functions defined on the Jacobi variety \( J \) of \( \tilde{S} \).

Choose a basis of cycles \( (a_1, \ldots, a_{\tilde{g}}, b_1, \ldots, b_{\tilde{g}}) \) on \( \tilde{S} \) with the intersection property \( a_i \cdot a_j = 0, \; b_i \cdot b_j = 0 \) and \( a_i \cdot b_j = \delta_{ij}, \; i, j = 1, \ldots, \tilde{g} \). Let \( \{\omega_1, \ldots, \omega_{\tilde{g}}\} \) be a basis of holomorphic differentials on \( \tilde{S} \), normalized by the condition that \( \hat{\delta} \omega_i = \delta^i_j, \; i, j = 1, \ldots, \tilde{g} \). Define the Abel map \( A : \tilde{S} \times \tilde{S} \to J \) by

\[
A(x, y) = \left( \int_x^y \omega_1, \ldots, \int_x^y \omega_{\tilde{g}} \right). \tag{2.11}
\]

The corresponding \( \theta \) function is defined by the matrix of \( \beta \)-periods of \( \{\omega_1, \ldots, \omega_{\tilde{g}}\} \). Define \( Q^i_1 + \cdots + Q^i_{\tilde{g}} = (\psi^i_0) - D_\infty + \infty_i + \Delta \), with \( \Delta = \Delta^1 + \cdots + \Delta^{\tilde{g}+r-1} \). There exists \( e \in \mathbb{C}^g \) such that

\[
\begin{align*}
\theta(e) &= 0 \\
\theta(e + A(Q^i_j, y)) &\neq 0, \; \forall j = 1, \ldots, \tilde{g} \\
\theta(e + A(\infty_j, y)) &\neq 0, \; \forall j = 1, \ldots, r \\
\theta(e + A(\Delta^j, y)) &\neq 0, \; \forall j = 1, \ldots, \tilde{g} + r - 1.
\end{align*} \tag{2.12}
\]
Fix a base point $p \in \tilde{S}$ and define constants

$$
\alpha' = \sum_{j=1}^{\tilde{g}} A(p, \Delta^j) \quad \alpha'' = \sum_{j=1}^{r-1} A(p, \Delta^{\tilde{g}+j})
$$

$$
\beta = \sum_{j=1}^{r} A(p, \infty_j) \quad \beta_i = A(p, \infty_i) \quad (2.13)
$$

$$
\gamma_i = \sum_{j=1}^{\tilde{g}} A(p, Q^j_i).
$$

Let $\delta$ be the Riemann constant. With these definitions the functions

$$
\tilde{F}^i(y) = \frac{\theta(A(p,y) + \delta - \alpha' - \alpha'' + \beta - \beta_i) \prod_{j \neq i} \theta(A(\infty_j, y) + e)}{\theta(A(p,y) + \delta - \alpha') \prod_{j=1}^{r-1} \theta(A(\Delta^{\tilde{g}+j}, y) + e)}
$$

(2.14)

define sections of $E_0$ with the correct zeros and poles. The normalisation condition (2.10) is obtained by setting

$$
\psi_0^i(y) = \frac{\tilde{F}^i(y)}{\tilde{F}^i(\infty_i)}.
$$

(2.15)

To compute sections of the time dependant line bundle $E_\tau = E_0 \otimes F_\tau$ we recall that $F_\tau$ is given by the exponential transition function (2.5). Thus sections of $E_\tau$ can be represented as functions with zeros at $D_\infty - \infty_i$, poles at $\Delta$ and exponential singularities at $D_0 = 0_1 + \cdots + 0_r$, the $r$ points over $\lambda = 0$. Such functions are called $r$-point Baker-Akhiezer functions.

For an explicit representaion of $\psi^i_\tau$ let $\tilde{\mu}$ be a differential of the second kind on $\tilde{S}$ with zero $\alpha$-cycle integrals and the same polar part as $d\mu$. Let $U \in C^{\tilde{g}}$ be given by

$$
U = \frac{1}{2\pi \sqrt{-1}} \left( \oint_{b_1} \tilde{\mu}, \ldots, \oint_{b_{\tilde{g}}} \tilde{\mu} \right).
$$

(2.16)

Define

$$
h^i_\tau(y) = \exp \left( \tau \int_p^y \tilde{\mu} \right) \frac{\theta(A(p,y) + \tau U + \delta - \gamma_i)}{\theta(A(p,y) + \delta - \gamma_i)}
$$

(2.17)

then the functions

$$
\psi^i_\tau(y) = \frac{h^i_\tau(y) \tilde{F}^i(y)}{h^i_\tau(\infty_i) \tilde{F}^i(\infty_i)}
$$

(2.18)

define a basis of $H^0(\tilde{S}, E_\tau)$ satisfying properties (2.6).
3. The Matrix Nonlinear Schrödinger Equation

The techniques of section 2.2 also apply to the matrix NLS equation once the corresponding spectral curve is desingularized at $\lambda = \infty$.

3.1 The Spectral Curve for the Matrix NLS Equation

For the matrix NLS equation the leading terms of $L(\lambda)$ are given by (1.9a-c). In order to study the behaviour near $\lambda = \infty$ of the corresponding spectral curve $S$, given by (2.1) embedded into $T$, we switch to coordinates $(\tilde{\lambda}, \tilde{z})$ which leads to the representation

$$\tilde{P}(\tilde{\lambda}, \tilde{z}) = \det(\tilde{L}(\tilde{\lambda}) - \tilde{z}I) = 0. \tag{3.1}$$

with $\tilde{L}(\tilde{\lambda})$ given by

$$\tilde{L}(\tilde{\lambda}) = L_0 + \tilde{\lambda}L_1 + \tilde{\lambda}^2L_2 + \cdots + \tilde{\lambda}^{n-1}L_{n-1}. \tag{3.2}$$

Expanding (3.1) in coordinates $(\tilde{\lambda}, z') = -(\tilde{z} + \sqrt{\frac{a}{a+b}})$ around $(\tilde{\lambda}, z') = (0, 0)$ gives, under invariant genericity conditions on $L_3$,

$$\prod_{j=1}^{b}(z' + \tilde{\lambda}^3d_j + O(\tilde{\lambda}^4)) \prod_{i=1}^{a}(z' + e_i + O(\tilde{\lambda}^2)) = 0 \tag{3.3}$$

where $d_j, e_i$ are constants. This shows that $S$ has a $b$-fold intersection of order 3 at $(\tilde{\lambda} = 0, \tilde{z} = -\sqrt{\frac{a}{a+b}})$. Blowing up three times at this point reduces the virtual genus of $S$ by $\frac{3b(b-1)}{2}$. Similar things happen at the (generically) $a$-fold point of order 3 in $(\tilde{\lambda} = 0, \tilde{z} = \sqrt{\frac{b}{a+b}})$. Blowing up three times at this point reduces the (virtual) genus by $\frac{3a(a-1)}{2}$.

Call $\tilde{S}$ the curve obtained after desingularization at the $n+2$ points $(\alpha_i, 0)_{i=1,\ldots,n}$, $(\infty, -\sqrt{\frac{a}{a+b}})$, $(\infty, \sqrt{\frac{b}{a+b}})$, in coordinates $(\lambda, z)$. Invariant genericity conditions on $L$ may be chosen such that $\tilde{S}$ does not have other singularities. The arithmetic genus $\tilde{g}$ of $\tilde{S}$ is therefore

$$\tilde{g} = \frac{1}{2} \left( (r-1)(r(n-1) - 2) - \sum_{i=1}^{n}(r - k_i)(r - k_i - 1) \right) - 3b(b-1) - 3a(a-1). \tag{3.4}$$

3.2 Solutions for the Matrix NLS Equation in terms of $\theta$-Functions

In order to use formulas (2.16-18) we need an explicit representation of the polynomials $\mu(z, \lambda^{-1})$ and $\nu(z, \lambda^{-1})$ defining transition functions $g_{01}$ from $U_0$ to $U_1$ for the degree zero line bundle $F_{x,t}$ by

$$g_{01}(\lambda, z) = \exp(x\mu + t\nu). \tag{3.5}$$
As mentioned before, in the AKS framework \( \mu \) and \( \nu \) are given by the differentials of the AKS hamiltonians (1.8a,b). An easy computation gives

\[
\mu(z, \lambda^{-1}) = \frac{z}{\lambda^{n-2}} \quad (3.6a)
\]

\[
\nu(z, \lambda^{-1}) = \frac{z}{\lambda^{n-3}}. \quad (3.6b)
\]

The \((x,t)\)-dependance for the matrix polynomial (3.2) is computed as in (2.8). The components of \( L_1 \) carry the solution for the matrix nonlinear Schrödinger equation (1.1). It is recovered by differentiating (3.2) with respect to \( \tilde{\lambda} \) at \( \tilde{\lambda} = 0 \). The solutions \( q, p \) to (1.1) are given by

\[
q^j_i(x,t) = \left. \frac{d}{d\lambda} \right|_{\tilde{\lambda}=0} \tilde{\mathcal{L}}(\tilde{\lambda})_{j+a}^i \quad (3.7a)
\]

\[
p^j_i(x,t) = \left. \frac{d}{d\lambda} \right|_{\tilde{\lambda}=0} \tilde{\mathcal{L}}(\tilde{\lambda})^i_{j+a} \quad (3.7b)
\]

\( i = 1, \ldots, a, j = 1, \ldots, b \). A short computation yields

\[
q^j_i(x,t) = \sqrt{-1}(\psi^j_{i+a})'(\infty_i) \quad (3.8a)
\]

\[
p^j_i(x,t) = -\sqrt{-1}(\psi^j_i)'(\infty_{j+a}) \quad (3.8b)
\]

where the prime designates derivation with respect to \( \tilde{\lambda} \). Let \( \tilde{\mu}, \tilde{\nu} \) be normalized differentials of the second kind on \( \tilde{S} \) with same polar part as \( d\mu, d\nu \) respectively. Set

\[
U = \frac{1}{2\pi\sqrt{-1}} \left( \oint_{b_1} \tilde{\mu}, \ldots, \oint_{b_g} \tilde{\mu} \right) \quad (3.9a)
\]

\[
V = \frac{1}{2\pi\sqrt{-1}} \left( \oint_{b_1} \tilde{\nu}, \ldots, \oint_{b_g} \tilde{\nu} \right). \quad (3.9b)
\]

Fix a point \( p \in \tilde{S} \) and substitute formulas (2.14, 2.17) into (2.18). Use (3.8a,b) to get

\[
q^j_i(x,t) = K^j_i \exp(e^j_i x + f^j_i t) \frac{\theta(A(p, \infty_i) + xU + tV + \delta - \gamma_{j+a})}{\theta(A(p, \infty_{j+a}) + xU + tV + \delta - \gamma_{j+a})} \quad (3.10a)
\]

\[
p^j_i(x,t) = -K^i_{j+a} \exp(-e^j_i x - f^i_j t) \frac{\theta(A(p, \infty_{j+a}) + xU + tV + \delta - \gamma_i)}{\theta(A(p, \infty_i) + xU + tV + \delta - \gamma_i)} \quad (3.10b)
\]
where the constants $K^k_l, e^j_i$ and $f^j_i$ are given by

$$K^k_l = \sqrt{-1} \frac{\theta(A(p, \infty_k) + \delta - \gamma_k) \theta(A(p, \infty_l) + \delta - \alpha' - \alpha'' + \beta - \beta_l)}{F^k(\infty_k) \theta(A(p, \infty_l) + \delta - \alpha')}$$

$$\left. \frac{d}{dy} \right|_{y=\infty_l} \theta(A(\infty_l, y) + e) \sum_{m \neq k, l} \theta(A(\infty_m, \infty_l) + e) \prod_{m=1}^{r-1} \theta(A(\Delta \tilde{\gamma} m, \infty_l) + e)$$

$$e^j_i = \int_{\infty_j + a}^{\infty_i} \tilde{\mu}$$

$$f^j_i = \int_{\infty_j + a}^{\infty_i} \tilde{\nu}$$

where the integrals are evaluated in the same polygonization as that defining the Abel map $A$.

The functions (3.10a,b) are the desired periodic solutions to the matrix nonlinear Schrödinger equation (1.1).

Comments

The solutions for the matrix NLS equations obtained above could also have been obtained by applying the Hamiltonian integration method based upon computation of the Liouville-Arnold generating function (see [AHH2,3]) determining a canonical transformation to linearizing coordinates. That method explicitly shows how nonlinear Lax equations (1.7a,b) linearize on the Jacobi variety associated to the invariant spectral curve and makes the evaluation of many of the constants more explicit. However, use of the Krichever-Novikov-Dubrovin method avoids problems associated with “missing Darboux coordinates” due to the singular spectral data at $\lambda = \infty$ implied by the structure of the leading terms in $L$ given by (1.9a-c). In the Hamiltonian approach this requires the imposition of other symplectic constraints defining a reduced family of spectral data involving curves of the same genus as the desingularized ones. The main shortcoming of the present approach lies in the difficulty on explicitly computing the constants $\gamma_k$ and $K^k_l$ in view of the complicated implicit way they arise through formulas (2.13), (3.9a,b) and (3.11).

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