On Whitney type inequalities for local anisotropic polynomial approximation

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Abstract

We prove a multivariate Whitney type theorem for the local anisotropic polynomial approximation in $L_p(Q)$ with $1 \leq p \leq \infty$. Here $Q$ is a $d$-parallelepiped in $\mathbb{R}^d$ with sides parallel to the coordinate axes. We consider the error of best approximation of a function $f$ by algebraic polynomials of fixed degree at most $r_i - 1$ in variable $x_i$, $i = 1, \ldots, d$, and relate it to a so-called total mixed modulus of smoothness appropriate to characterizing the convergence rate of the approximation error. This theorem is derived from a Johnen type theorem on equivalence between a certain $K$-functional and the total mixed modulus of smoothness which is proved in the present paper.

Keywords Whitney type inequality; Anisotropic approximation by polynomials; Total mixed modulus of smoothness; Mixed $K$-functional; Sobolev space of mixed smoothness;

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1 Introduction and main results

The classical Whitney theorem establishes the equivalence between the modulus of smoothness $\omega_r(f, |I|)_{p,I}$ and the error of best approximation $E_r(f)_{p,I}$ of a function $f : I \rightarrow \mathbb{R}$ by algebraic polynomials of degree at most $r - 1$ measured in $L_p$, $1 \leq p \leq \infty$. Namely, the following inequalities

$$2^{-r} \omega_r(f, |I|)_{p,I} \leq E_r(f)_{p,Q} \leq C \omega_r(f, |I|)_{p,I}$$

(1.1)

hold true with a constant $C$ depending only on $r$. This result was first proved by Whitney [20] for $p = \infty$ and extended by Brudnyi [1] to $1 \leq p < \infty$. The inequalities (1.1) provide, in particular, a convergence characterization for a local polynomial approximation when the degree $r - 1$ of polynomials is fixed and the interval $I$ is small.

Several authors have dealt with this topic in order to extend and generalize the result in various directions. Let us briefly mention them. A multivariate (isotropic) generalization for functions on a coordinate $d$-cube $Q$ in $\mathbb{R}^d$ was given by Brudnyi [2] and [3]. It turned out that the result keeps valid if one replaces the

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$d$-cube by a more general domain $\Omega$. The case of a convex domain $\Omega \subset \mathbb{R}^d$ is already treated in Brudnyi [2]. Let us also refer to the recent contributions by Dekel, Leviatan [3] and Dekel [4] with focus on convex and Lipschitz domains and the improvement of the constants involved.

A reasonable question is also to ask for the case $0 < p < 1$. We refer to the works of Storozhenko [13], Storozhenko, Oswald [16], and in addition to the appendix of the substantial paper by Hedberg and Netrusov [9] for a brief history and further references.

A natural question arises: Is there a Whitney type theorem for the anisotropic approximation of multivariate functions on a coordinate $d$-parallelepiped $Q$? Some work has been done in this direction, see for instance Garrigós, Tabacco [10]. However, the present paper deals with a rather different setting which is somehow related to the theory of function spaces with mixed smoothness properties [3, 17, 18, 19]. We intend to approximate a function $f$ by polynomials of fixed degree at most $r_i - 1$ in variable $x_i$, $i = 1, ..., d$, on a small $d$-parallelepiped $Q$. A total mixed modulus of smoothness is defined which turns out to be a suitable convergence characterization to this approximation. The classical Whitney inequality can be derived as a corollary of Johnen’s theorem [11] on the equivalence of the $r$-th Peetre $K$-functional $K_r(f, t^*)$ and the modulus of smoothness $\omega_r(f, t)$. A proof was given by Johnen and Scherer in [12]. Following this approach to Whitney type theorems we will introduce the notion of a mixed $K$-functional and prove its equivalence to the total mixed modulus of smoothness by generalizing the technique of Johnen and Scherer to the multivariate mixed situation.

1.1 Notation

In order to give an exact setting of the problem and formulate the main results let us preliminarily introduce some necessary notations. As usual, $\mathbb{N}$ is reserved for the natural numbers, by $\mathbb{Z}$ we denote the set of all integers, and by $\mathbb{R}$ the real numbers. Furthermore, $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of non-negative integers and real numbers, respectively. Elements $x$ of $\mathbb{R}^d$ will be denoted by $x = (x_1, ..., x_d)$. For a vector $r \in \mathbb{Z}_+^d$ and $r \in \mathbb{R}^d$, we will further write

$$x^r := (x_1^r, ..., x_d^r).$$

Moreover, if $x, y \in \mathbb{R}^d$, the inequality $x \leq y$ ($x < y$) means that $x_i \leq y_i$ ($x_i < y_i$), $i = 1, ..., d$. As usual, the notation $A \ll B$ indicates that there is a constant $c > 0$ (independent of the parameters which are relevant in the context) such that $A \leq cB$, whereas $A \asymp B$ is used if $A \ll B$ and $B \ll A$, respectively.

If $r \in \mathbb{Z}_+^d$, let $\mathcal{P}_r$ be the set of algebraic polynomials of degree at most $r_i - 1$ at variable $x_i$, $i \in [d]$, where $[d]$ denotes the set of all natural numbers from 1 to $d$. We intend to approximate a function $f$ defined on a $d$-parallelepiped

$$Q := [a_1, b_1] \times ... \times [a_d, b_d]$$

by polynomials from the class $\mathcal{P}_r$. If $D \subset \mathbb{R}^d$ is a domain in $\mathbb{R}^d$ and $L_p(D)$ denote by $L_p(D)$, $0 < p \leq \infty$, the quasi-normed space of Lebesgue measurable functions on $D$ with the usual $p$-th integral quasi-norm $\| \cdot \|_{p,D}$ to be finite, whereas we use the ess sup norm if $p = \infty$. The error of the best approximation of $f \in L_p(D)$ by polynomials from $\mathcal{P}_r$ is measured by

$$E_r(f)_p := \inf_{\varphi \in \mathcal{P}_r} \| f - \varphi \|_{p,Q}.$$ 

For $r \in \mathbb{Z}_+$, $h \in \mathbb{R}$, and a univariate functions $f$, the $r$th difference operator $\Delta^r_h$ is defined by

$$\Delta^r_h(f, x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x + jh), \quad \Delta^0_h f(x) := f(x),$$

where $\binom{r}{j}$ denotes the set of natural numbers from 1 to $d$.
whereas for \( r \in \mathbb{Z}_+^d \), \( h \in \mathbb{R}^d \) and a \( d \)-variate function \( f : \mathbb{R}^d \to \mathbb{R} \), the mixed \( r \)th difference operator \( \Delta^r_h \) is defined by

\[
\Delta^r_h := \prod_{i=1}^d \Delta^r_{h,i}.
\]

Here, the univariate operator \( \Delta^r_{h,i} \) is applied to the univariate function \( f \) by considering \( f \) as a function of variable \( x_i \) with the other variables fixed. Let

\[
\omega_r(f, t)_{p,Q} := \sup_{|h_i| \leq t} \| \Delta^r_h(f) \|_{p,Q,t}, \quad t \in \mathbb{R}_+^d,
\]

be the mixed \( r \)th modulus of smoothness of \( f \), where for \( y, h \in \mathbb{R}^d \), we write \( yh := (y_1h_1, \ldots, y_dh_d) \) and \( Q_y := \{ x \in Q : x_i, x_i + y_i \in [a_i, b_i], i \in [d] \} \). For \( r \in \mathbb{Z}_+^d \) and \( e \subset [d] \), denote by \( r(e) \in \mathbb{Z}_+^d \) the vector with \( r(e)_i = r_i, i \in e \) and \( r(e)_i = 0, i \notin e \) (\( r() = 0 \)). We define the total mixed modulus of smoothness of order \( r \) by

\[
\Omega_r(f, t)_{p,Q} := \sum_{e \subset [d], e \neq \emptyset} \omega_r(e)(f, t)_{p,Q}, \quad t \in \mathbb{R}_+^d.
\]

The total mixed modulus of smoothness is a modification of mixed moduli of smoothness which is an important tool for characterizing function spaces with mixed smoothness properties, see [8, 17] and the recent contributions [18, 19].

### 1.2 Main results

In the present paper, we generalize the Whitney inequality [11] to the error of the best local anisotropic approximation \( E_r(f)_{p,Q} \) by polynomials from \( P_r \) and the total mixed modulus of smoothness \( \Omega_r(f, t)_{p,Q} \). More precisely, we prove the following Whitney type inequalities.

**Theorem 1.1** Let \( 1 \leq p \leq \infty \), \( r \in \mathbb{N}^d \). Then there is a constant \( C \) depending only on \( r, d \) such that for every \( f \in L_p(Q) \)

\[
\left( \sum_{e \subset [d]} \prod_{i \in e} 2^{r_i} \right)^{-1} \Omega_r(f, \delta)_{p,Q} \leq E_r(f)_{p,Q} \leq C \Omega_r(f, \delta)_{p,Q},
\]

where \( \delta = \delta(Q) := (b_1 - a_1, \ldots, b_d - a_d) \) is the size of \( Q \).

Theorem 1.1 shows that the total mixed modulus of smoothness \( \Omega_r(f, t)_{p,Q} \) gives a sharp convergence characterization of the best anisotropic polynomial approximation when \( r \) is fixed and the size \( \delta(Q) \) of the \( d \)-parallelepiped \( Q \) is small. This may have applications in the approximation of functions with mixed smoothness by piecewise polynomials or splines.

So far we focus on the case \( 1 \leq p \leq \infty \). This makes it possible to apply a technique developed by Johnen and Scherer [12]. As mentioned above they showed the equivalence of Peetre’s \( K \)-functional of order \( r \) with respect to a classical Sobolev space \( W^r_p \) and the modulus of smoothness of order \( r \) for the univariate case. The question of a \( K \)-functional suitable for mixed Sobolev spaces has been often considered in the past. We refer for instance to Sparr [14] and DeVore et al. [7]. By introducing a mixed \( K \)-functional \( K_r(f, t)_{p,Q}, t \in \mathbb{R}_+^d, \) (see the definition in Section 3), such an equivalence between \( K_r(f, t')_{p,Q} \) and the total mixed modulus of smoothness \( \Omega_r(f, t)_{p,Q} \) can be established as well. Namely, we prove the following

**Theorem 1.2** Let \( 1 \leq p \leq \infty \) and \( r \in \mathbb{N}^d \). Then for any \( f \in L_p(Q) \), the following inequalities

\[
\left( \sum_{e \subset [d]} \prod_{i \in e} 2^{r_i} \right)^{-1} \Omega_r(f, t)_{p,Q} \leq K_r(f, t')_{p,Q} \leq C \Omega_r(f, t)_{p,Q}, \quad t \in \mathbb{R}_+^d,
\]
hold true with a constants C depending on r, p, d only.

The paper is organized as follows. In Section 2 we establish an error formula for the anisotropic approximation by Taylor polynomials for functions from mixed Sobolev spaces. Section 3 is devoted to the equivalence of the total mixed modulus of smoothness and the mixed K-functional (Theorem 1.2) which is applied in Section 4 to derive the Whitney type inequality for the local anisotropic polynomial approximation (Theorem 1.1).

2 Anisotropic approximation by Taylor polynomials

By \( f^{(k)}, k \in \mathbb{Z}^d_+ \), we denote the \( k \)th order mixed weak derivative of \( f \), i.e., \( f^{(k)} := \partial^{k_1 + \cdots + k_d} f / \partial x_1^{k_1} \cdots \partial x_d^{k_d} (f^{(0)} = f) \). For \( r \in \mathbb{N}^d \) and \( 1 \leq p \leq \infty \), the Sobolev type space \( W^r_p(Q) \) of mixed smoothness \( r \) is defined as the set of functions \( f \in L_p(Q) \), for which the following norm is finite

\[
\| f \|_{W^r_p(Q)} := \sum_{e \subseteq [d]} \| f^{(r(e))} \|_{p,Q}.
\]

If \( f \in W^r_p(Q), k \leq r, k, r \in \mathbb{Z}^d_+ \), and \( x^0 \in Q \), then we can define the Taylor polynomial \( T_k(f) \) of order \( k \) by

\[
T_k(f, x) := T_k(f, x^0, x) = \sum_{0 \leq s < k} f^{(s)}(x^0)e_s(x - x^0)
\]

where \( e_s(x) := \prod_{i=1}^d e_{s_i}(x_i) \) and \( e_m(t) := t^m/m! \). We want to give an upper bound of the error of approximation of \( f \in W^r_p(Q) \) by the Taylor polynomial \( T_r \). For this purpose we need the following auxiliary lemma.

Lemma 2.1 Let \( 1 \leq p \leq \infty \), \( r \geq 1 \) and \( Q = [a,b] \). Then there exist constants \( C_1, C_2 \) depending only on \( r \) such that for \( k = 0, \ldots, r - 1 \) and \( 0 \leq t \leq b - a \) the inequalities

\[
\begin{align*}
&k||f^{(k)}||_{p,Q} \leq C_1(\| f \|_{p,Q} + t^r \| f^{(r)} \|_{p,Q}), \\
&k^{1/r}||f^{(k)}||_{\infty,Q} \leq C_2(\| f \|_{p,Q} + t^r \| f^{(r)} \|_{p,Q})
\end{align*}
\]

hold true for any \( f \in W^r_p(Q) \).

Proof. Step 1. A proof of the first inequality can be found in [3, page 38]. Let us prove the second one by using the first one. We start with \( r = 1 \) and \( k = 0 \). We fix \( y \in Q \) and consider a subinterval \( Q_t \) of length \( t \) containing \( y \). We define the function

\[
g_t(x) = \frac{\chi_{Q_t}(x)}{t} \int_{Q_t} f(s) \, ds.
\]

Since \( f \) is continuous the function \( f - g_t \) has a zero \( \xi \in Q_t \). By Taylor’s formula we have

\[
(f - g_t)(y) = \int_{\xi}^y (f' - g'_t)(s) \, ds = \int_{\xi}^y f'(s) \, ds.
\]

This gives \( |f(y)| - |g_t(y)| \leq |f(y) - g_t(y)| \leq t^{1/r'} \| f' \|_p \) and hence

\[
|f(y)| \leq |g_t(y)| + t^{1/r'} \| f' \|_p \leq t^{1/r - 1} \| f \|_p + t^{1/r} \| f' \|_p = t^{-1/p}(\| f \|_p + t \| f' \|_p).
\]

(2.2)
Since we chose \( y \in Q \) arbitrarily at the beginning we obtain (2.2) for every \( y \in Q \). This proves the second inequality in (2.1) for \( r = 1 \) and \( k = 0 \).

**Step 2.** Let now \( r > 1 \) and \( k < r \). Taylor’s formula gives

\[
f^{(k)}(x) = \sum_{\ell=0}^{r-k-1} f^{(k+\ell)}(\xi) \frac{(x-\xi)^\ell}{\ell!} + \int_{\xi}^{x} f^{(r)}(s) e_{r-k-1}(x-s) \, ds.
\]

Therefore, we have

\[
|f^{(k)}(x)| \ll \sum_{\ell=0}^{r-k-1} \|f^{(k+\ell)}\|_{\infty} t^\ell + \|f^{(r)}\|_p t^{r-k-1+1/p'}.
\]

Applying the formula from Step 1 leads to

\[
|f^{(k)}(x)| \ll \sum_{\ell=0}^{r-k-1} \|f^{(k+\ell)}\|_{\infty} t^\ell + \|f^{(r)}\|_p t^{r-k-1+1/p'}
\]

\[
\ll t^{-1/p} \sum_{\ell=0}^{r-k-1} (\|f^{(k+\ell)}\|_{\infty} t^\ell + \|f^{(k+\ell+1)}\|_{\infty} t^\ell + t^{r-k-1/p} \|f^{(r)}\|_p)
\]

\[
= t^{-(k+1/p)} \left( \|f^{(r)}\|_p t^r + \sum_{\ell=0}^{r-k-1} (\|f^{(k+\ell)}\|_{\infty} t^k t^\ell + \|f^{(k+\ell+1)}\|_p t^{k+\ell+1}) \right).
\]

We apply the first inequality in (2.1) and obtain for every \( x \)

\[
t^{k+1/p} |f^{(k)}(x)| \ll \|f\|_p + t^r \|f^{(r)}\|_p
\]

which concludes the proof. \( \Box \)

**Theorem 2.2** Let \( 1 \leq p \leq \infty \), \( r \in \mathbb{N}^d \). Then there is a constant \( C \) depending only on \( r, d \) such that for every \( f \in W^r_p(Q) \)

\[
E_r(f)_{p,Q} \leq C \sum_{e \subseteq [d], e \neq \emptyset} \prod_{i \in e} \delta_i^{r_i} \|f^{(r_i)}\|_{p,Q},
\]

where \( \delta = \delta(Q) \).

**Proof.** For simplicity we prove the theorem for the case \( d = 2 \) and \( Q = [0,b_1] \times [0,b_2] \). If \( g \) is a univariate function on the interval \([0,b]\) and \( g \in W^r_p([0,b]) \), then \( g \) can be represented as

\[
g(x) = T_k(g,0,x) + \int_{0}^{x} g^{(k)}(\xi) e_{k-1}(x-\xi) \, d\xi.
\]

Hence, for a function \( f \in W^r_p(Q) \) we get

\[
f(x) = T_r(f, x)
\]

\[
\quad + \sum_{k_1=0}^{r_1-1} e_{k_1}(x_1) \int_{0}^{x_2} f^{((k_1,r_2))}(0,\xi_2) e_{r_2-1}(x_2-\xi_2) \, d\xi_2
\]

\[
\quad + \sum_{k_2=0}^{r_2-1} e_{k_2}(x_2) \int_{0}^{x_1} f^{((r_1,k_2))}(\xi_1,0) e_{r_1-1}(x_1-\xi_1) \, d\xi_1
\]

\[
\quad + \int_{0}^{x_2} \int_{0}^{x_1} f^{(r)}(\xi_1,\xi_2) e_{r-1}(x_1-\xi_1) e_{r-2}(x_2-\xi_2) \, d\xi_1 d\xi_2,
\]

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where

\[ T_r(f, x) = \sum_{k_2=0}^{r_2-1} \sum_{k_1=0}^{r_1-1} f^{((k_1,k_2))}(0,0)e_{k_2}(x_2)e_{k_1}(x_1). \]

By triangle and Hölder’s inequality we obtain

\[ \|f - T_r(f)\|_{p, Q} \leq \sum_{k_1=0}^{r_1-1} b_1^{k_1+1/p} b_2^{k_2} \|f((k_1,r_2)) (0, \cdot)\|_{p, [0,b_2]} \]

\[ + \sum_{k_2=0}^{r_2-1} b_1^r b_2^{k_2+1/p} \|f((r_1,k_2)) (\cdot, 0)\|_{p, [0,b_1]} + b_1^r b_2^r \|f(r)\|_{p, Q}. \]

(2.3)

The following inequalities are a direct consequence of Lemma 2.1

\[ b_1^{k_1+1/p} \|f((k_1,r_2)) (0, \cdot)\|_{p, [0,b_2]} \leq \|f((0,r_2))\|_{p, Q} + b_1^r \|f(r)\|_{p, Q}, \]

\[ b_2^{k_2+1/p} \|f((r_1,k_2)) (\cdot, 0)\|_{p, [0,b_1]} \leq \|f((r_1,0))\|_{p, Q} + b_2^r \|f(r)\|_{p, Q}. \]

(2.4)

Indeed, put \( I_i := [0, b_i], \ j = 1, 2). If \( f \) is in \( W_p^1(Q) \), then almost all (with respect to \( t \in I_1 \)) functions \( g_i := f((r_1,0))(t, \cdot) \) belong to \( W_p^2(I_2) \). Now we can use Lemma 2.1 to obtain for \( k_2 < r_2 \)

\[ b_2^{k_2+1/p} \|g_i \|_{p, I_2} \leq b_2^{k_2+1/p} \|g_i \|_{p, \infty} \leq \|g_i \|_{p, I_2} + b_2^r \|g_i \|_{p, I_2} \]

for almost all \( t \). The weak derivative \( g_i^{k_2} \) coincides with \( f((r_1,k_2))(t, \cdot) \) in \( L_p(I_2) \). If \( k_2 < r_2 \), then \( g_i^{k_2} \) and \( f((r_1,k_2))(t, \cdot) \) are continuous functions and the equation \( g_i^{k_2}(0) = f((r_1,k_2))(t,0) \) makes sense and holds true (we take the continuous representative) for almost all \( t \). Hence, using the estimate from above (the norm \( \| \cdot \|_{p, I_2, t} \) on the right-hand side is with respect to \( t \)) we have

\[ b_2^{k_2+1/p} \|f((r_1,k_2)) (\cdot, 0)\|_{p, I_1} = b_2^{k_2+1/p} \|g_i(0)\|_{p, I_1, t} \]

\[ \leq \|g_i(\cdot)\|_{p, I_2} \|g_i(\cdot)\|_{p, I_1, t} + b_2^r \|g_i^{k_2}(\cdot)\|_{p, I_2} \|g_i(\cdot)\|_{p, I_1, t} \]

\[ = \|f((r_1,0))\|_{p, Q} + b_2^r \|f((r_1,r_2))\|_{p, Q}. \]

This proves the second line in (2.4), the first one is obtained analogously.

Combining the inequalities (2.3) and (2.4) proves the theorem. \( \square \)

### 3 Johnen type inequalities for mixed K-functionals

For \( r \in \mathbb{N}^d \), the mixed K-functional \( K_r(f, t)_{p, Q} \) is defined for functions \( f \in L_p(Q) \) and \( t \in \mathbb{R}^d \) by

\[ K_r(f, t)_{p, Q} := \inf_{g \in W_p^r(Q)} \{ \|f - g\|_{p, Q} + \sum_{e \subseteq [d], e \neq \emptyset} \left( \prod_{i \in e} t_i \right) \|g^{(r(e))}\|_{p, Q} \}. \]

The following technical lemma needs a further notation. Let us assume \( a_i \leq c_i < d_i \leq b_i \) for \( i \in [d] \). We put \( I^i = [a_i, b_i], \ I^i_1 = [a_i, d_i], \) and \( I^i_0 = [c_i, b_i] \) and further

\[ Q_e := \prod_{i=1}^d I^i_{\chi_e(i)}, \]

where \( \chi_e \) denotes the characteristic function of the set \( e \subseteq [d] \).
Lemma 3.1 Let $1 \leq p \leq \infty$ and $r \in \mathbb{N}^d$. Then for any $f \in L_p(Q)$ the inequality

$$K_r(f, t^r)_{p,Q} \leq C \sum_{c \subset [d]} K_r(f, t^r)_{p,Q_c}$$

holds true for all $t \in \mathbb{R}^d_+$ with $t_i \leq d_i - c_i$, $i \in [d]$. The constant $C$ only depends on $r$ and $d$.

Proof. The proof is based on an iterative argument. The first step is to observe

$$Q = Q_1 \cup Q_0 = (I_1^1 \times \prod_{i \in [d] \setminus \{1\}} I_i^1) \cup (I_0^1 \times \prod_{i \in [d] \setminus \{1\}} I_i)$$

and to show that

$$K_r(f, t^r)_{p,Q} \leq K_r(f, t^r)_{p,Q_1} + K_r(f, t^r)_{p,Q_0}. \tag{3.2}$$

We start with an increasing function $\varphi \in C^\infty(\mathbb{R})$ such that

$$\varphi(s) = \begin{cases} 0 & : s < 0 \\ 1 & : s > 1 \end{cases}.$$ 

Putting $h = d_1 - c_1$ and

$$\lambda(s) = \varphi\left(\frac{s - c_1}{h}\right), \quad s \in \mathbb{R},$$

we obtain a $C^\infty(\mathbb{R})$-function $\lambda$ that equals zero on $(a_1, c_1)$, equals one on $[d_1, b_1]$, and is increasing on $[c_1, d_1]$. As a direct consequence we get

$$\|\varphi^{(k)}\|_{\infty, \mathbb{R}} \leq h^{-k}\|\varphi^{(k)}\|_{\infty, \mathbb{R}}, \quad k \in \mathbb{N}.$$ 

Let now $f \in W^r_p(Q)$ and $t \in \mathbb{R}^d_+$ with $t_i \leq d_i - c_i$, $i \in [d]$. For arbitrary $g_1 \in W^r_p(Q_1)$ and $g_0 \in W^r_p(Q_0)$, put

$$g(x) = \lambda(x_1)g_0(x) + (1 - \lambda(x_1))g_1(x)$$

$$= g_1(x) + \lambda(x_1)(g_0(x) - g_1(x)).$$

First of all, the function $g$ is defined on $Q_0 \cap Q_1 \subseteq Q$. We extend $g$ by $g_0$ on $Q_0 \setminus Q_1$ and by $g_1$ on $Q_1 \setminus Q_0$ and denote the result also by $g$. By the construction of $\lambda$ this $g$ belongs to $W^r_p(Q)$ and we have

$$\|f - g\|_{p,Q} \leq \|\lambda(x_1)f(x) - \lambda(x_1)g_0(x) + (1 - \lambda(x_1))f(x) - (1 - \lambda(x_1))g_1(x)\|_{p,Q}$$

$$\leq \|f - g_0\|_{p,Q_0} + \|f - g_1\|_{p,Q_1}. \tag{3.3}$$

Furthermore, for any non-empty fixed subset $e \subset [d]$ we have

$$g^{(r(e))}(x) = g_1^{(r(e))}(x) + \sum_{k=0}^{r_1} \binom{r_1}{k} \lambda^{(r_1-k)}(x_1)(g_1^{(k, \tilde{r}(e))}(x) - g_0^{(k, \tilde{r}(e))}(x))$$

on $Q_0 \cap Q_1$, where $\tilde{r}(e)$ denotes the vector $r(e \setminus \{1\})$ without the leading 0. Note that $\tilde{r}(e)$ might be zero, which happens in case $e = \{1\}$. Consequently,

$$\left(\prod_{i \in e} t_i^{r_i}\right)\|g^{(r(e))}\|_{p, Q_0 \cap Q_1}$$

$$\ll \left(\prod_{i \in e} t_i^{r_i}\right)\left(\|g_1^{(r(e))}\|_{p, Q_0 \cap Q_1} + \max_{0 \leq k \leq r_1} h^{-k}\|g_1^{(k, \tilde{r}(e))} - g_0^{(k, \tilde{r}(e))}\|_{p, Q_0 \cap Q_1}\right) \tag{3.4}$$

$$\ll \left(\prod_{i \in e \setminus \{1\}} t_i^{r_i}\right)\left(t_1^{r_1+1}\|g_1^{(r(e))}\|_{p, Q_0 \cap Q_1} + \max_{0 \leq k \leq r_1} \left(\frac{t_1}{h}\right)^{-k}\|g_1^{(k, \tilde{r}(e))} - g_0^{(k, \tilde{r}(e))}\|_{p, Q_0 \cap Q_1}\right).$$

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We apply the first relation in Lemma 2.1 in the first component of \( g^{(k, \tilde{r}(e))}_1 - g^{(k, \tilde{r}(e))}_0 \) using a similar argument as after (3.4) and get
\[
\mathbb{E}_i \mathbb{R} \mathbb{Z} \mathbb{C} \mathbb{D} \mathbb{E} \mathbb{F} \mathbb{G} \mathbb{H} \mathbb{I} \mathbb{J} \mathbb{K} \mathbb{L} \mathbb{M} \mathbb{N} \mathbb{O} \mathbb{P} \mathbb{Q} \mathbb{R} \mathbb{S} \mathbb{T} \mathbb{U} \mathbb{V} \mathbb{W} \mathbb{X} \mathbb{Y} \mathbb{Z}
\]
Plugging this into (3.4) and taking \( t_1 \leq h \) into account gives in case \( \tilde{r}(e) \neq 0 \)
\[
\left( \prod_{i \in \mathbb{E}_i} t_i \right) ||g^{(r(e))}||_{p, Q_0 \cap Q_1} \leq \left( \prod_{i \in \mathbb{E}_i} t_i \right) ||g^{(0, \tilde{r}(e))}||_{p, Q_0} + \left( \prod_{i \in \mathbb{E}_i} t_i \right) ||g^{(0, \tilde{r}(e))}||_{p, Q_0} + \left( \prod_{i \in \mathbb{E}_i} t_i \right) ||g^{(0, \tilde{r}(e))}||_{p, Q_1},
\]
and in case \( \tilde{r}(e) = 0 \), i.e., if \( e = \{1\} \),
\[
\left( \prod_{i \in \mathbb{E}_i} t_i \right) ||g^{(r(e))}||_{p, Q_0 \cap Q_1} \leq \left( \prod_{i \in \mathbb{E}_i} t_i \right) ||g^{(r(e))}||_{p, Q_0} + \left( \prod_{i \in \mathbb{E}_i} t_i \right) ||g^{(r(e))}||_{p, Q_1} + \left( \prod_{i \in \mathbb{E}_i} t_i \right) ||g^{(r(e))}||_{p, Q_1}.
\]
Using that
\[
||g^{(r(e))}||_{p, Q} \leq ||g^{(r(e))}||_{p, Q_0 \cap Q_1} + ||g^{(r(e))}||_{p, Q_0} + ||g^{(r(e))}||_{p, Q_1},
\]
we obtain together with (3.3), (3.5), and (3.6) the relation
\[
K_r(f, t')_{p, Q} \leq K_r(f, t')_{p, Q_0} + K_r(f, t')_{p, Q_0}
\]
which is (3.2). We continue with the same procedure, this time with \( Q_1 \) and \( Q_0 \) instead of \( Q \), proving that (analogously for \( Q_1 \))
\[
K_r(f, t')_{p, Q_0} \leq K_r(f, t')_{p, Q_0} + K_r(f, t')_{p, Q_0},
\]
where
\[
Q_{00} = \left( I_0^1 \times I_0^2 \times \prod_{i \in \mathbb{E}_i \setminus \{1,2\}} I_i \right) \quad \text{and} \quad Q_{01} = \left( I_0^1 \times I_0^2 \times \prod_{i \in \mathbb{E}_i \setminus \{1,2\}} I_i \right),
\]
and so forth. An iteration of this argument finishes the proof. \( \square \)

3.1 Proof of Theorem 1.2

Proof. The first inequality in (1.3) follows from the definition. Namely, if \( f \in L_p(Q) \), for any non-empty \( e \subset [d] \) and any \( g \in W^r_p(Q) \), we have
\[
\omega_{r(e)}(f, t)_{p, Q} \leq \omega_{r(e)}(f - g, t)_{p, Q} + \omega_{r(e)}(g, t)_{p, Q}
\]
\[
\leq \left( \prod_{i \in \mathbb{E}_i} 2^{r_i} \right) \left\{ \| f - g \|_{p, Q} + \left( \prod_{i \in \mathbb{E}_i} t_i \right) \| g^{(r(e))} \|_{p, Q} \right\}.
\]
Hence, we obtain the first inequality in (1.3). Let us prove the second one. For simplicity we prove it for \( d = 2 \) and \( t \in \mathbb{R}^+ \), \( t > 0 \). If \( k \) is a natural number, then we define for univariate functions \( \varphi \) on the interval \( [a, b] \) the operator \( P^k_t \), \( t \geq 0 \), by
\[
P^k_t(\varphi, x) := \varphi(x) + (-1)^{k+1} \int_{-\infty}^{\infty} \Delta^k_{\varphi, x}(h) M_k(h) dh,
\]
where $M_k$ is the B-spline of order $k$ with knots at the integer points $0, \ldots, k$, and support $[0, k]$. The function $P_t^k(\varphi)$ is defined on $[a, b - h/4]$ for $t \leq \bar{t} := h/4k^2$, where $h := b - a$. We have, see \[ page 177],

$$\{P_t^k(\varphi)\}_r^{(k)}(x) = t^{-k} \sum_{j=1}^{k} (-1)^{j+1} j^{k-j} \Delta_j^{k}(\varphi, x).$$

(3.7)

Put $h_i := b_i - a_i$ and $c_i := a_i + h_i/4$, $d_i := b_i - h_i/4$, $i \in [2]$. It holds $a_i < c_i < d_i < b_i$, and we will use the notation $Q_e$ given in \[ page 3.1] for any $e \in [d]$. In particular, we have $Q_{[2]} = [a_1, d_1] \times [a_2, d_2]$. For functions $f$ on the parallelepiped $Q = [a_1, b_1] \times [a_2, b_2]$ the operator $P_t^r, t \in \mathbb{R}_+^2$, is defined by

$$P_t^r(f) := \prod_{i=1}^{2} P_{t,1}^r(f),$$

where the univariate operator $P_{t,1}^r$ is applied to the univariate function $f$ by considering $f$ as a function of variable $x_i$ with the remaining variables fixed. The function $P_t^k(\varphi)$ is defined on $Q_{[2]}$ for $t \leq \bar{t}$, where $\bar{t} := h_i/4r_i^2$. We have

$$P_t^r(f, x) = f(x) + \int_{-\infty}^{\infty} \Delta_{t,1}(f, x) M_{r_1}(h_1)dh_1 + \int_{-\infty}^{\infty} \Delta_{t,2}(f, x) M_{r_2}(h_2)dh_2$$

and

$$P_t^r(f, x) = P_t^r(f, x) + \int_{-\infty}^{\infty} \Delta_{t,1}(f, x) M_{r_1}(h_1)dh_1 + \int_{-\infty}^{\infty} \Delta_{t,2}(f, x) M_{r_2}(h_2)dh_2,$$

where $r_1 := (r_1, 0)$ and $r_2 := (0, r_2)$. Let us define the function $g_t = P_t^r(f)$. If $f \in L_p(Q)$, by Minkowski’s inequality and properties of the B-spline $M_{r_1}$, we get

$$\|f - g_t\|_{p, Q_{[2]}} \ll \omega_{r_1}(f, t)_{p, Q_{[2]}}, \quad \omega_{r_2}(f, t)_{p, Q_{[2]}}, \quad \omega_{r}(f, t)_{p, Q_{[2]}} = \Omega_{r}(f, t)_{p, Q_{[2]}}.$$ 

(3.8)

Further, by (3.7) we obtain

$$g_t^{(r_1)} = P_{t,1}^{r_1}(P_{t,1}^{r_1}(f))^{(r_1)} = P_{t,2}^{r_2}(t_1^{-r_1} \sum_{j=1}^{r_1} (-1)^{j_1+1} j_1^{-r_1} \Delta_{j_1 t_1,1}^{r_1}(f)).$$

Since $P_{t,2}^{r_2}$ is a linear bounded operator from $L_p(Q_{[2]})$ into $L_p(Q_{[2]})$ and further $\|\Delta_{j_1 t_1,1}^{r_1}(f)\|_{p, Q_{[2]}} \ll \omega_{r_1}(f, t)_{p, Q_{[2]}},$ we have

$$t_1^{r_1}\|g_t^{(r_1)}\|_{p, Q_{[2]}} \ll \omega_{r_1}(f, t)_{p, Q_{[2]}},$$

(3.9)

Similarly, we can prove that

$$t_2^{r_2}\|g_t^{(r_2)}\|_{p, Q_{[2]}} \ll \omega_{r_2}(f, t)_{p, Q_{[2]}},$$

Again, by (3.7) we get

$$g_t^{(r)} = \sum_{j_1=1}^{r_1} j_1^{-r_1} \sum_{j_2=1}^{r_2} j_2^{-r_2} \omega_{r_1}(f, t)_{p, Q_{[2]}}.$$ 

From the inequality $\|\Delta_{j_1 t_1,1}^{r_1}(f)\|_{p, Q} \ll \omega_{r_1}(f, t)_{p, Q_{[2]}}$ it follows that

$$t_1^{r_1}t_2^{r_2}\|g_t^{(r)}\|_{p, Q_{[2]}} \ll \omega_{r}(f, t)_{p, Q_{[2]}}.$$ 

(3.10)

Combining (3.8), (3.9) – (3.10) gives

$$\|f - g_t\|_{p, Q_{[2]}} + t_1^{r_1}\|g_t^{(r_1)}\|_{p, Q_{[2]}} + t_2^{r_2}\|g_t^{(r_2)}\|_{p, Q_{[2]}} + t_1^{r_1}t_2^{r_2}\|g_t^{(r)}\|_{p, Q_{[2]}} \ll \Omega_{r}(f, t)_{p, Q}.$$ 

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Therefore, we get
\[ K_r(f, t')_{p, Q} \ll \Omega_r(f, t)_{p, Q}, \]
and in a similar way
\[ K_r(f, t')_{p, Q} \ll \Omega_r(f, t)_{p, Q} \]
for any subset \( e \subset [2] \), where \( Q_e \) is given by (3.11). The last inequality and Lemma 3.1 prove (1.3) for \( t \leq \bar{t} \).

Now take a function \( \bar{g} \in W^r_p(Q) \) such that
\[ \| f - \bar{g} \|_{p, Q} \ll \bar{r}_1 \| g^{(r_1)} \|_{p, Q} + \bar{r}_2 \| g^{(r_2)} \|_{p, Q} + \bar{r}_3 \| g^{(r_3)} \|_{p, Q} \ll \Omega_r(f, \bar{t})_{p, Q}. \] (3.11)

By Theorem 2.2 we have
\[ \| \bar{g} - T_r(\bar{g}) \|_{p, Q} \ll \bar{r}_1 \| g^{(r_1)} \|_{p, Q} + \bar{r}_2 \| g^{(r_2)} \|_{p, Q} + \bar{r}_3 \| g^{(r_3)} \|_{p, Q} \ll \Omega_r(f, \bar{t})_{p, Q}. \] (3.12)

Since \( T_r(\bar{g}) \in W^r_p(Q) \) and \( (T_r(\bar{g}))^{(r(e))} = 0 \) for every non-empty subset \( e \subset [d] \), it holds for all \( t > \bar{t} \)
\[ K_r(f, t') \leq \| f - T_r(\bar{g}) \|_{p, Q} \ll \| f - \bar{g} \|_{p, Q} + \| \bar{g} - T_r(\bar{g}) \|_{p, Q} \ll \Omega_r(f, \bar{t})_{p, Q}. \]

where the third step combines (3.11) and (3.12). Therefore, (1.3) has been proved for arbitrary \( t > 0. \]

4 Whitney type inequalities

Using the results from Section 3 we are now able to prove Theorem 1.1.

Proof. The first inequality in (1.2) is trivial. Indeed, if \( f \in L_p(Q) \) then for any non-empty \( e \subset [d] \) and any \( \varphi \in \mathcal{P}_r \) we have
\[ \omega_{r(e)}(f, \delta)_{p, Q} = \omega_{r(e)}(f - \varphi, \delta)_{p, Q} \ll \left( \prod_{i \in e} 2^{r_e} \right) \| f - \varphi \|_{p, Q}. \]

Hence, we obtain the first inequality in (1.2). On the other hand, from Theorem 2.2 it follows that for any \( g \in W^r_p(Q) \)
\[ E_r(f)_{p, Q} \leq \| f - g \|_{p, Q} + E_r(g)_{p, Q} \ll \| f - g \|_{p, Q} + \| g - T_r(g) \|_{p, Q} \ll \| f - g \|_{p, Q} + \left( \prod_{i \in e} \delta_{r_i} \right) \| g^{(r(e))} \|_{p, Q}. \]

Hence, we get
\[ E_r(f)_{p, Q} \ll K_r(f, \delta^r)_{p, Q}. \]

By Theorem 1.2 we have proved the second inequality in (1.2). }

The result in Theorem 1.1 can be slightly modified. For \( r \in \mathbb{Z}_+^d, \ h \in \mathbb{R}^d, \ e \subset [d] \) and a \( d \)-variate function \( f : \mathbb{R}^d \to \mathbb{R} \) the mixed \( p \)-mean modulus of smoothness of order \( r(e) \) is given by
\[ w_{r(e)}(f, t)_p := \left( \prod_{i \in e} t_i^{-1} \right)^{1/p} \int_{U(t)} \int_{Q_{r(e)}h} |\Delta^{r(e)}_h(f, x)|^p \, dx \, dh \right)^{1/p}, \ t \in \mathbb{R}^d_+, \]
where $U(t) := \{ h \in \mathbb{R}^d : |h_i| \leq t_i, \ i \in [d]\}$, with the usual change of the outer mean integral to sup if $p = \infty$. This leads to the definition of the total mixed $p$-mean modulus of smoothness of order $r$ by

$$W_r(f, t)_{p,Q} := \sum_{e \subset [d], e \neq \emptyset} w_{r(e)}(f, t)_{p,Q}, \ t \in \mathbb{R}^d_+.$$ 

Note that $W_r(f, t)_{p,Q}$ coincides with $\Omega_r(f, t)_{p,Q}$ when $p = \infty$. In a way similar to the proof of Theorem 1.1 we can prove the following slightly stronger result.

**Theorem 4.1** Let $1 \leq p \leq \infty$, $r \in \mathbb{N}^d$. Then there are constants $C, C'$ depending only on $r, d$ such that for every $f \in L^p(Q)$

$$CW_r(f, \delta)_{p,Q} \leq E_r(f)_{p,Q} \leq C'W_r(f, \delta)_{p,Q}$$

where $\delta = \delta(Q)$.

**Remark 4.2** A corresponding inequality in the case $0 < p < 1$ is so far left open for subsequent contributions. It seems that the modulus $W_r(f, t)_{p,Q}$ is suitable to treat this case, cf. the appendix of [9].

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