Split cuts in the plane

Amitabh Basu∗ Michele Conforti† Marco Di Summa‡ Hongyi Jiang†

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Abstract

We provide a polynomial time cutting plane algorithm based on split cuts to solve integer programs in the plane. We also prove that the split closure of a polyhedron in the plane has polynomial size.

1 Introduction

In this paper, we work in \( \mathbb{R}^2 \) and we always implicitly assume that all polyhedra, cones, half-planes, lines are rational. Given a polyhedron \( P \subseteq \mathbb{R}^2 \), we let \( P := \text{conv}(P \cap \mathbb{Z}^2) \), where "conv" denotes the convex hull operator.

Given \( \pi \in \mathbb{Z}^2 \setminus \{0\} \) and \( \pi_0 \in \mathbb{Z} \), let \( H_0 \) and \( H_1 \) be the half-planes defined by \( \pi x \leq \pi_0 \) and \( \pi x \geq \pi_0 + 1 \), respectively. Given a polyhedron \( P \subseteq \mathbb{R}^2 \), we let \( P_0 := P \cap H_0 \), \( P_1 := P \cap H_1 \) and \( P_{\pi,\pi_0} := \text{conv}(P_0 \cup P_1) \). \( P_{\pi,\pi_0} \) is a polyhedron that contains \( P \cap \mathbb{Z}^2 \). An inequality \( cx \leq d \) is a split inequality (or split cut) for \( P \) if there exist \( \pi \in \mathbb{Z}^2 \setminus \{0\} \) and \( \pi_0 \in \mathbb{Z} \) such that the inequality \( cx \leq d \) is valid for \( P_{\pi,\pi_0} \). The vector \((\pi,\pi_0)\), or the set \( H_0 \cup H_1 \), is a split disjunction, and we say that \( cx \leq d \) is a split inequality for \( P \) with respect to \((\pi,\pi_0)\). The closed complement of a split disjunction, i.e., the set defined by \( \pi_0 \leq \pi x \leq \pi_0 + 1 \), is called a split set. If one of \( P_0 \), \( P_1 \) is empty, say \( P_1 = \emptyset \), the split inequality \( \pi x \leq \pi_0 \) is called a Chvátal inequality.

A Chvátal inequality \( \pi x \leq \pi_0 \) where \( \pi \) is a primitive vector (i.e., its coefficients are relatively prime) has the following geometric interpretation: Let \( z := \max_{x \in P} cx \) and let \( H \) be the half-plane defined by \( \pi x \leq z \). Then \( P \subseteq H \) and \( \pi_0 = \lfloor z \rfloor \), because \( \pi_0 \in \mathbb{Z} \) and \( P_1 = \emptyset \). Since \( \pi \) is a primitive vector, \( H_I \) is defined by the inequality \( \pi x \leq \pi_0 \). We will say that the inequality \( \pi x \leq \pi_0 \) defines the Chvátal strengthening of the half-plane \( H \).

An inequality description of a polyhedron \( P \subseteq \mathbb{R}^2 \) is a system \( Ax \leq b \) such that \( P = \{ x \in \mathbb{R}^2 : Ax \leq b \} \), where \( A \in \mathbb{Z}^{m \times 2} \) and \( b \in \mathbb{Z}^m \) for some positive integer \( m \). The size of the description of \( P \), i.e., the number of bits needed to encode the linear system, is \( O(m \log \| A \|_\infty + m \log \| b \|_\infty) \). (Notation \( \| \cdot \|_\infty \) indicates the infinity-norm of a vector or a matrix, i.e., the maximum absolute value of its entries.) It follows from the above argument that when the coefficients in each row of

∗Department of Applied Mathematics and Statistics, Johns Hopkins University, Baltimore, MD, USA (basu.amitabh@jhu.edu, hjiang32@jhu.edu). Supported by the NSF Grant CMMI1452820 and the ONR Grant N000141812096.
†Dipartimento di Matematica "Tullio Levi-Civita", Università degli Studi Padova, Italy (conforti@math.unipd.it, disumma@math.unipd.it). Supported by the PRIN grant 2015B5F27W and by a SID grant of the University of Padova.
A are relatively prime integers, the inequalities defining \( P \) are Chvátal strengthened, i.e., they are the Chvátal strengthening of themselves.

Given a polyhedron \( P \), a cutting plane or cut is an inequality that defines a half-plane \( H \) such that \( P \not\subseteq H \) but \( P_I \subseteq H \). A cutting plane algorithm is a procedure that, given a polyhedron \( P \subseteq \mathbb{R}^2 \) and a vector \( c \in \mathbb{Z}^2 \), solves the integer program \( \max \{ cx : x \in P \cap \mathbb{Z}^2 \} \) by adding at each iteration a cut that eliminates an optimal vertex of the current continuous relaxation until integrality is achieved or infeasibility is proven.

Integer programming in the plane is the problem \( \max \{ c^T x : Ax \leq b, \ x \in \mathbb{Z}^2 \} \) where \( c \in \mathbb{Z}^2, A \in \mathbb{Z}^{m \times 2} \) and \( b \in \mathbb{Z}^m \). In Section 2 we provide a cutting plane algorithm for this problem that uses split inequalities as cutting planes and such that the number of iterations (i.e., cutting planes computed) is \( O(m(\log \| A \|_\infty)^2) \). (The derivation of every cutting plane can be carried out in polynomial time but involves a constant number of gcd computations.)

We note that integer programming in the plane is well-studied and understood. In particular, given a polyhedron \( P \subseteq \mathbb{R}^2 \), Harvey [10] gave an efficient procedure to produce an inequality description of \( P_I \). Eisenbrand and Laue [9] gave an algorithm to solve the problem that makes \( O(m + \max\{\log \| A \|_\infty, \log \| b \|_\infty, \log \| c \|_\infty\}) \) arithmetic operations.

As split cuts are widely used in integer programming solvers, the scope of the present research is to prove that this class of integer programs can also be solved in polynomial time with a cutting plane algorithm based on split cuts (albeit not as efficiently as in [9]).

The second part of this paper deals with the complexity of the split closure of a polyhedron in the plane. Given a polyhedron \( P \subseteq \mathbb{R}^2 \), the split closure \( P^{\text{split}} \) of \( P \) is defined as follows:

\[
P^{\text{split}} := \bigcap_{\pi \in \mathbb{Z}^2, \pi_0 \in \mathbb{Z}^P} P^{\pi, \pi_0}.
\]

Cook, Kannan and Schrijver [6] proved that \( P^{\text{split}} \) is a polyhedron. Polyhedrality results for cutting plane closures, such as the above split closure result, have a long history in discrete optimization starting from the classical result that the Chvátal closure of a rational polytope (the intersection of all Chvátal inequalities) is polyhedral (see, e.g., Theorem 23.1 in [12]), with several more recent results [2, 7, 8, 3, 13, 11], to sample a few. The complexity of cutting plane closures, i.e., the number of facets and the bit complexity of the facets, is relatively less understood. One of the most well-known results in this direction is due to Eisenbrand and Bockmayr [4], who show that the complexity of the Chvátal closure of a rational polytope is polynomial in the description size of the polytope, if the dimension is a fixed constant (see Theorem 21 in Section 3 below). It has long remained an open question whether the split closure is of polynomial complexity as well, even in the case of two dimensions. We settle this question in the affirmative in this paper; see Theorem 20 in Section 3.

Finally, as again shown in [6], if one defines \( P_0 := P \) and recursively \( P_i := (P_{i-1})^{\text{split}} \), then \( P^t = P_I \) for some \( t \). The split rank of \( P \) is the smallest \( t \) for which this occurs. It is well-known that if \( P \subseteq \mathbb{R}^2 \) is a polyhedron, its split rank is at most 2; we will observe in Remark 34 that this also follows from the arguments used in this paper.
2 Tilt cuts and the clockwise algorithm

To simplify the presentation, throughout the paper the notions of facet and facet defining inequality of a polyhedron will be interchangeably used.

A polyhedron in \( \mathbb{R}^2 \) which is the intersection of two non parallel half-planes is a full-dimensional translated pointed cone. However, to simplify terminology we will often refer to such a polyhedron as a translated cone. Its unique vertex is the apex of the cone. Given a half-plane \( H \), we let \( H^\perp \) denote its boundary.

**Definition 1.** Let \( C = H_1 \cap H_2 \) be a translated cone with apex not in \( \mathbb{Z}^2 \), and assume that \( H_1 \) is Chvátal strengthened.

Let \( \hat{H} \) be the line in \( H_1 \) parallel to \( H_1^\perp \) and closest to \( H_1^\perp \) such that \( \hat{H} \cap \mathbb{Z}^2 \neq \emptyset \). Let \( p \in H_1^\perp \cap C \cap \mathbb{Z}^2 \) and \( q \in (H_1^\perp \setminus C) \cap \mathbb{Z}^2 \) be the unique points such that the open line segment \((p,q)\) contains no integer point. Let \( x \in \hat{H} \cap C \cap \mathbb{Z}^2 \) and \( y \in (\hat{H} \cap \mathbb{Z}^2) \setminus C \) be the unique points such that the open line segment \((x,y)\) contains no integer point (possibly \( x \in H_2^\perp \)).

Two parallel sides of the parallelogram \( P := \text{conv}(p,q,y,x) \) are contained in \( H_1^\perp \cup \hat{H} \). The other two sides of \( P \) define a split disjunction in the following way. Let \( W_0, W_1 \) be the half-planes such that \( W_0^\perp \) is the line containing \( p \) and \( x \), \( W_1^\perp \) is the line containing \( q \) and \( y \), and \( W_0 \cap W_1 = \emptyset \). As \( P \) has integer vertices but contains no other integer point, \( P \) has area 1 and \( W_0 \), \( W_1 \) define a split disjunction \((\pi,\pi_0)\).

Let \( F_1 \) be the facet of \( C \) induced by \( H_1 \). We now define the tilt \( T \) of \( F_1 \) with pivot \( p \). If \( C \cap W_1 = \emptyset \), then \((\pi,\pi_0)\) defines a Chvátal cut for \( C \) (as in Fig. 1(i)), and we let \( T \) be this Chvátal cut. Otherwise let \( x' \in W_1^\perp \cap C \cap \mathbb{Z}^2 \) and \( y' \in (W_1^\perp \setminus C) \cap \mathbb{Z}^2 \) be the unique points such that the open line segment \((x',y')\) contains no integer point and let \( q' = [x',y'] \cap H_2^\perp \) (possibly \( q' = x' \)), see Fig. 1(ii). We define \( T \) as the split cut for \( C \) with respect to \((\pi,\pi_0)\) such that \( T^\perp \) contains \( p \) and \( y' \). (Note that in this case \( T \) is not the “best” split cut for \( C \) with respect to \((\pi,\pi_0)\), as it does not define a facet of \( \text{conv}((C \cap W_0) \cup (C \cap W_1)) \).)

In the next two lemmas we refer to the notation introduced in Definition 1.

**Lemma 2.** Let \( ax \leq \beta \) and \( dx \leq \delta \) be inequality descriptions of \( H_1 \) and \( H_2 \) respectively, where the coefficients of \( a \) are relatively prime. Then \( dq - \delta \leq |a_1d_2 - a_2d_1| \).

**Proof.** Let \( p \) be as in Definition 1. As \( a_1, a_2 \) are relatively prime, we may assume that \( q = p + \left( \begin{array}{c} -a_2 \\ a_1 \end{array} \right) \).

Hence, \( dq = dp + (-a_2d_1 + a_1d_2) \). Since \( p \in H_2 \), \( dp \leq \delta \). The result follows. \( \Box \)

**Lemma 3.** Let \( dx \leq \delta \) be an inequality description of \( H_2 \).

(i) \( T \) is always Chvátal strengthened.

(ii) \( T \) defines a facet of \( C_I \) if and only if \( T \) is a Chvátal cut for \( C \).

(iii) When \( C \cap W_1 \neq \emptyset \), we have \( 0 < dy' - \delta \leq \frac{dq - \delta}{2} \).

**Proof.** (i) As \( p \in T^\perp \cap \mathbb{Z}^2 \) and \( T \) is a rational half-plane, \( T \) is always Chvátal strengthened.

(ii) Recall that \( T \) is a Chvátal cut for \( C \) if and only if \( C \cap W_1 = \emptyset \). When \( C \cap W_1 = \emptyset \) we have \( x \in T^\perp \cap C \cap \mathbb{Z}^2 \), whereas when \( C \cap W_1 \neq \emptyset \) we have \( y' \in (T^\perp \setminus C) \cap \mathbb{Z}^2 \) and there is no integer
point in the open segment \((p, y')\). Since \(T\) is a facet of \(C\) if and only if \(T\)\(\cap C\) contains an integer point different from \(p\), this happens if and only if \(T\) is a Chvátal cut.

(iii) When \(C \cap W_1 \neq \emptyset\), we have that \(q, y, y' \in (W_1^= \cap \mathbb{Z}^2) \setminus C, x' \in (W_1^= \cap \mathbb{Z}^2) \cap C\), while \(q' \in W_1^= \cap H_2^=\). Therefore the length of \([y', q]\) is at most half the length of \([q, q']\). This implies that \(0 < dy' - \delta \leq dq - \delta/2\).

Let \(\text{dim}(P)\) denote the dimension of \(P\).

**Remark 4.** The algorithm below uses the following fact: If \(P \subseteq \mathbb{R}^2\) is not full-dimensional, the integer program \(\max\{cx : x \in P \cap \mathbb{Z}^2\}\) can be solved by applying at most two Chvátal cuts. Specifically, if \(\text{dim}(P) \leq 0\), the problem is trivial, and if \(\text{dim}(P) = 1\), with one cut we can determine if \(\text{aff}(P) \cap \mathbb{Z}^n = \emptyset\) (where \(\text{aff}(P)\) denotes the affine hull of \(P\)). In this case the problem is infeasible. Otherwise, if \(\text{aff}(P) \cap \mathbb{Z}^n \neq \emptyset\), the problem is unbounded if and only if \(\max\{cx : x \in P\} = \infty\). Finally, if \(\text{aff}(P) \cap \mathbb{Z}^n \neq \emptyset\) and \(\max\{cx : x \in P\}\) is finite, the integer program is either infeasible or admits a finite optimum: this can be determined by applying a second Chvátal cut.

**Definition 5.** Given an irredundant description \(Ax \leq b\) of a full-dimensional pointed polyhedron \(P\) with \(m\) facets, we denote by \(F_i\) the facet of \(P\) defined by the the \(i\)th inequality \(a_i^t x \leq b_i\) of the system \(Ax \leq b\). Given a vector \(c \in \mathbb{Z}^2 \setminus \{0\}\) such that the linear program \(\max\{cx : x \in P\}\) has finite optimum, and a specified optimal vertex \(v\), we assume that \(a_1, \ldots, a_m\) are ordered clockwise so that \(v \in F_m \cap F_1\) and \(c\) belongs to the cone generated by \(a_m\) and \(a_1\). We call \(F_m\) the late facet and \(F_1\) the early facet of \(P\) with respect to \(v\). This ordered pair defines a translated cone with apex \(v\) that we denote with \((F_m, F_1)\).

The “clockwise” cutting plane algorithm
**INPUT:** A pointed polyhedron $P \subseteq \mathbb{R}^2$ and a vector $c \in \mathbb{Z}^2 \setminus \{0\}$ such that $\max \{ cx : x \in P \}$ is finite.

**OUTPUT:** A solution of the integer program $\max \{ cx : x \in P \cap \mathbb{Z}^2 \}$ or a certificate of infeasibility.

1. Initialize $Q = P$.
2. If $\dim(Q) \leq 1$, apply at most two Chvátal cuts to output INFEASIBLE or an optimal solution.
3. Else solve the linear program $\max \{ cv : v \in Q \}$ and let $v^*$ be the optimal vertex. If $v^* \in \mathbb{Z}^2$, STOP and output $v^*$.
4. If $v^* \notin \mathbb{Z}^2$, number the facets of $Q$ in clockwise order $F_1, \ldots, F_m$ so that $v^* \in (F_m, F_1)$. If $F_m$ is not Chvátal strengthened, let $T$ be its Chvátal strengthening. Otherwise let $T$ be the tilt of $F_m$ with respect to $(F_m, F_1)$. Update $Q = Q \cap T$ and go to Step 2.

In the above algorithm, if $P$ is full-dimensional and has two vertices that maximize $cx$, then $\arg \max_{x \in P} cx$ is a bounded facet of $P$ (“optimal” facet). We assume that:

**Assumption 6.** The optimal vertex $v^*$ is the first vertex encountered when traversing the optimal facet in clockwise order.

Therefore if two vertices maximize $cx$, the optimal facet is $F_1$ in our numbering. With this convention, inequality $T$ computed in Step 4 of the algorithm at a given iteration will be tight for the vertex at the successive iteration. (Note that if $\arg \max_{x \in P} cx$ is an unbounded facet of $P$ and the unique vertex is the first point encountered on this facet when traversing it clockwise, this facet is $F_1$ in our numbering, and if the unique vertex is the last point encountered on this facet when traversing it clockwise, this facet is $F_m$ in our numbering.)

**Remark 7.** A cutting plane algorithm typically works with a vertex solution, so it is natural to assume that $P$ is pointed. (The integer program can be solved with at most one Chvátal cut when $P$ is a polyhedron in the plane which is not pointed.)

If $P$ is a pointed polyhedron but $\max \{ cx : x \in P \}$ is unbounded, the integer program is either infeasible or unbounded. There are ways to overcome this, however it seems difficult to efficiently distinguish these two cases by only using cutting planes, even when $\dim(P) = 2$.

We will need the following theorem about integer hulls of translated cones in the plane.

**Theorem 8** ([10]). Given a description $Ax \leq b$ of a translated cone $C \subseteq \mathbb{R}^2$, $C_I$ has $O(\log \|A\|_\infty)$ facets. Furthermore, each facet of $C_I$ has a description $ax \leq \beta$ where $\|a\|_\infty \leq \|A\|_\infty$.

We also need the following lemma.

**Lemma 9.** Let $P$ be a pointed polyhedron such that $P_I \neq \emptyset$. Let $u$ be the largest infinity norm of a vertex of $P$ or $P_I$. Let $ax \leq \beta$ be an inequality which is valid for $P_I$ but is not valid for $P$. Then $|\beta| \leq 2u\|a\|_\infty$.

**Proof.** As $P_I \neq \emptyset$, by Meyer’s theorem (see e.g. Theorem 4.30 in [5]) $P$ and $P_I$ have the same recession cone. Therefore $\max_{x \in P} ax$ is finite and is larger than $\beta$, because $ax \leq \beta$ is not valid for $P$. Since finite maxima are attained at vertices, we have that $-2u\|a\|_\infty \leq \max_{x \in P_I} ax \leq \beta < \max_{x \in P} ax \leq 2u\|a\|_\infty$, which proves the lemma.
The above lemma obviously extends to pointed polyhedra in $\mathbb{R}^n$: in this case the bound is $|\beta| \leq nux\|a\|_\infty$.

**Theorem 10.** Let $Ax \leq b$ be a description of a translated cone $C \subseteq \mathbb{R}^2$ and let $c \in \mathbb{Z}^2 \setminus \{0\}$ be such that $\max\{cx : x \in P\}$ is finite. Then the clockwise algorithm solves the integer program $\max\{cx : x \in C \cap \mathbb{Z}^2\}$ in $O((\log \|A\|_\infty)^2) \|b\|_\infty$ iterations. Furthermore, there is a polynomial function $f(\cdot, \cdot)$ (independent of the data) such that every cut computed by the algorithm admits a description $ax \leq \beta$ where $\|a\|_\infty \leq \|A\|_\infty$ and $\beta \leq f(\|A\|_\infty, \|b\|_\infty)$.

**Proof.** We use the same notation as in Definition 1 and the fact that since $C$ is a translated cone, if $p \in \mathbb{Z}^2$ is a pivot element of a cut computed by the algorithm, then $p \in C$.

Let $T_i$ be the cutting plane produced by the clockwise algorithm at iteration $i$, where we assume that $T_0$ is the Chvátal strengthening of $H_1$.

**Claim 1.** If $T_i$ defines the early facet of $C \cap T_0 \cap \cdots \cap T_i$, the clockwise algorithm computes an optimal solution in iteration $i + 1$.

**Proof of claim.** In this case $(T_{i-1}, T_i)$ is the new translated cone whose apex is the pivot element $p_i$ of iteration $i$. As $p_i \in C \cap \mathbb{Z}^2$, at iteration $i + 1$ the algorithm determines that $p_i$ is an optimal solution.

By the above claim, $T_i$ is the tilt (with pivot element $p_i$) with respect to the translated cone $(T_{i-1}, H_2)$. Also recall that by Lemma 3 (i), $T_i$ is Chvátal strengthened. This fact will be important because we will work with the translated cone $(T_i, H_2)$ below and use notions from Definition 1 and results based on these notions, which assume that the facet $H_1 = T_i$ of the translated cone is Chvátal strengthened.

**Claim 2.** There is a polynomial function $f(\cdot, \cdot)$ such that $T_i$ admits a description $ax \leq \beta$ where $\|a\|_\infty \leq \|A\|_\infty$ and $\beta \leq f(\|A\|_\infty, \|b\|_\infty)$.

**Proof of claim.** By induction on $i$. The base case $i = 0$ is trivial.

We first show that $T_i$ admits a description $ax \leq \beta$ where $\|a\|_\infty \leq \|A\|_\infty$. If $T_i$ defines a Chvátal cut with respect to the translated cone $(T_{i-1}, H_2)$, as by induction $T_{i-1}$ satisfies the claim, we are done by Lemma 3 (ii) and Theorem 8.

So we assume that $(T_{i-1}, H_2) \cap W_i \neq \emptyset$ (where $W_i$ is as in Definition 1 with respect to $(T_{i-1}, H_2)$). Consider the translated cone $(T_{i-1}, H'_2)$, where $H'_2$ is the translation of $H_2$ through $y'$ (see Figure 1(ii)). As $T_i$ is the tilt with respect to $(T_{i-1}, H_2)$, we have that $p_i, y' \in T_{i-1} \cap \mathbb{Z}^2$. Furthermore, $T_{i-1}$ satisfies the claim by induction. It follows that $T_i$ is a facet of $(T_{i-1}, H'_2)$ and, by Theorem 8, $T_i$ admits a description $ax \leq \beta$ where $\|a\|_\infty \leq \|A\|_\infty$.

Let $u$ be the largest infinity norm of a vertex of $C$ or $C_I$. Then $u$ is bounded by a polynomial function of $\|A\|_\infty$ and $\|b\|_\infty$ (see, e.g., [12, Theorems 10.2 and 17.1]). Therefore, by Lemma 9, there is a polynomial function $f(\cdot, \cdot)$ such that $\beta \leq f(\|A\|_\infty, \|b\|_\infty)$. This completes the proof of the claim.

We finally show that in $O((\log \|A\|_\infty)^2)$ iterations the clockwise algorithm finds an optimal solution to the program $\max\{cx : x \in C \cap \mathbb{Z}^2\}$. By Claim 1 if the cut $T_i$ becomes the early facet in iteration $i$, then the algorithm finds an optimal solution in iteration $i + 1$. By Claim 2 all cuts $T_1, \ldots, T_i$ admit a description $ax \leq \beta$ where $\|a\|_\infty \leq \|A\|_\infty$ and $\beta \leq f(\|A\|_\infty, \|b\|_\infty)$. Therefore, by Theorem 8, it suffices to show that within at most $O(\log \|A\|_\infty)$ iterations beyond any particular
iteration $i$, the algorithm either finds an optimal solution or computes the facet adjacent to $T_i$ of the integer hull of the translated cone $(T_i, H_2)$. Note that, as all pivot elements are in $C$, this is also a facet of $C_I$.

By Lemma 3 (ii), the facet adjacent to $T_i$ of the integer hull of the translated cone $(T_i, H_2)$ is obtained when $W_i \cap (T_i, H_2) = \emptyset$, i.e., when $T_{i+1}$ is a Chvátal cut (this $W_i$ is as in Definition 1 with respect to the translated cone $(T_i, H_2)$). If $W_i \cap (T_i, H_2) \neq \emptyset$, by Lemma 3 (iii), we have that $0 < dy' - \delta \leq \frac{dx - \delta}{2}$, where $dx \leq \delta$ is the inequality defining $H_2$ in the description of $(T_i, H_2)$. Since $dy' - \delta \in \mathbb{Z}$ at every iteration, by Lemma 2, after at most $O(\log \|A\|_\infty)$ iterations the algorithm will produce a Chvátal cut.

Corollary 11. Any cut derived during the execution of the clockwise cutting plane algorithm on any pointed polyhedron $P$ admits a description $ax \leq \beta$ where $\|a\|_\infty \leq \|A\|_\infty$ and $\beta \leq f(\|A\|_\infty, \|b\|_\infty)$, where $f(\cdot, \cdot)$ is the function from Theorem 10.

Proof. Any cut in the algorithm is derived as a cut from some relaxation that is a translated cone whose half-planes are either original inequalities for $P$ or cuts derived by the algorithm. Now apply Theorem 10.

We now turn to the case of a general pointed polyhedron $P \subseteq \mathbb{R}^2$.

Definition 12. We let $Q_i$ be the polyhedron computed at the beginning of iteration $i$ of the clockwise algorithm and $T_i$ be the cutting plane computed at iteration $i$. We start our iterations at $i = 0, 1, \ldots$, so $Q_0 = P$ and $Q_i = Q_{i-1} \cap T_{i-1}$. When $Q_i$ is full-dimensional, we let $F_{i,1}, \ldots, F_{i,m_i}$ be the facets of $Q_i$ so that the optimal vertex of $Q_i$ is the apex of $(F_{i,m_i}, F_{i,1})$ and $T_i$ is either the Chvátal strengthening or the tilt of $F_{i,m_i}$ with respect to $(F_{i,m_i}, F_{i,1})$.

Note that, when $Q_i$ is full-dimensional, $T_i$ is either the early or the late facet of $Q_{i+1}$, as $T_i$ defines the optimal vertex chosen by the algorithm.

Definition 13. Given vectors $a, b$, we define $\angle(a, b)$ as the clockwise angle between $a$ and $b$, starting from $a$. When $Q_i$ is full-dimensional, let $a_{i,1}, \ldots, a_{i,m_i}$ be the normals of $F_{i,1}, \ldots, F_{i,m_i}$ (as defined in Definition 12). Then $\angle(a_{i,m_i}, c) < 180^\circ$ and $\angle(c, a_{i,1}) < 180^\circ$. We define a facet $F$ of $Q_i$ with normal $a$ potentially late if either $F = F_{i,m_i}$ or $0 < \angle(a, c) < 180^\circ$ and potentially early if either $F = F_{i,1}$ or $0 < \angle(c, a) \leq 180^\circ$. Note that if a facet of $Q_i$ satisfies $\angle(c, a) = 180^\circ$, then it cannot define the optimal vertex of $Q_j$, $j > i$.

Lemma 14. Given two full-dimensional relaxations $Q_i$ and $Q_j$ computed at iterations $i$ and $j > i$ of the algorithm, if $F$ is a potentially early facet of $Q_i$, then it cannot become a potentially late facet of $Q_j$ and if $F$ is a potentially late facet of $Q_i$, then it cannot become a potentially early facet of $Q_j$.

Proof. Let $a$ be the normal of $F$. The result is obvious when $\angle(a, c) > 0$ and $\angle(c, a) > 0$. If $\angle(a, c) = 0$, i.e., $F = \arg \max_{x \in Q_i} c x$, then $F$ remains potentially late or potentially early, by the choice of the optimal vertex; see Assumption 6.

Definition 15. Given a full-dimensional pointed polyhedron $P = Q_0 \subseteq \mathbb{R}^2$ and an objective vector $c \in \mathbb{Z}^2 \setminus \{0\}$ such that $\max \{c x : x \in P\}$ is finite, let $F_{0,1}, \ldots, F_{0,k}$ be the potentially early facets of $Q_0$ and $F_{0,k+1}, \ldots, F_{0,m_0}$ be the potentially late facets of $Q_0$. We say that facet $F_{0,\ell}$ of $P$ belongs to family $\ell$ and we recursively define the family of a cut $T_i$ produced at iteration $i$ of the algorithm as
the family of the late inequality that is used to produce $T_i$. We finally say that family $\ell$ is extinct at iteration $k$ if no facet of $Q_k$ belongs to family $\ell$.

Remark 16. By Lemma 14, no facet that is potentially early can become potentially late and vice versa; therefore, all cuts produced by the clockwise algorithm belong to the $m-k$ families associated with the potentially late facets of $Q_0$ (assuming the input to the algorithm is full-dimensional; otherwise, the algorithm terminates in at most two iterations —see Step 2 of the algorithm).

Theorem 17. Let $Ax \leq b$ be a description of a pointed polyhedron $P \subseteq \mathbb{R}^2$ with $m$ facets, and $c \in \mathbb{Z}^2 \setminus \{0\}$ be such that $\max\{cx : x \in P\}$ is finite. Then the clockwise algorithm solves the integer program $\max\{cx : x \in P \cap \mathbb{Z}^2\}$ in $O(m(\log \|A\|_\infty)^2)$ iterations.

Proof. We refer to the definitions of $Q_i$ and $T_i$, and when $Q_i$ is full-dimensional, to the definitions of $F_{i,1},\ldots,F_{i,m_i}$ with corresponding normals $a_{i,1},\ldots,a_{i,m_i}$ (see Definition 12 and Definition 13). In this case, we assume $F_{i,1},\ldots,F_{i,k_i}$ are potentially early and $F_{i,k_i+1},\ldots,F_{i,m_i}$ are potentially late. Moreover, let $E_i$ be the number of facets of $Q_i$ that are potentially early (i.e., $E_i = k_i$) and let $L_i$ be the number of families that are not extinct at iteration $i$ and such that the last inequality added to the family is potentially late.

By Theorem 10, there exists a function $z \mapsto g(z)$, where $g \in O((\log z)^2)$, such that the clockwise algorithm applied to any translated cone with description $A'x \leq b'$ terminates in $g(\|A'\|_\infty)$ iterations. Define $t := g(\|A\|_\infty)$.

Claim. Assume $\dim(Q_i) = 2$. Let $q$ be the largest natural number such that at iteration $i + q$, $\dim(Q_{i+q}) = 2$, $F_{i+q+m_{i+q}}$ is in the same family as $F_{i,m_i}$, and $F_{i+q,1}$ and $F_{i,1}$ are both defined by the same normal. Then

1. $q \leq t$, and

2. either $\dim(Q_{i+q+1}) \leq 1$, or the algorithm terminates at iteration $i + q + 1$, or $E_{i+q+1} + 2L_{i+q+1} \leq E_i + 2L_i - 1$.

Proof of claim. 1. follows from Theorem 10 and Corollary 11, after observing that during iterations $i,\ldots,i+q$ the algorithm computes the same cuts as those that it would compute if the polyhedron at iteration $i$ was the translated cone $(F_{i,m_i},F_{i,1})$.

We now prove 2. Suppose $\dim(Q_{i+q+1}) = 2$. We will establish that either the algorithm terminates at iteration $i + q + 1$ or $E_{i+q+1} + 2L_{i+q+1} \leq E_i + 2L_i - 1$. Let $(F_{i+q+1,m_i+q+1},F_{i+q+1,1})$ be the translated cone at iteration $i + q + 1$. Recall that, by the choice of the optimal vertex, $T_{i+q}$ must be either $F_{i+q+1,m_{i+q+1}}$ or $F_{i+q+1,1}$. We distinguish two cases.

Case 1 Assume $T_{i+q} = F_{i+q+1,m_{i+q+1}}$. Since $T_{i+q}$ is in the same family as $F_{i+q,m_{i+q}}$, which is in the same family as $F_{i,m_i}$, by definition of $q$ we must have that $F_{i,1}$ is distinct from $F_{i+q+1,1}$. Then $F_{i,1}$ is a redundant inequality for $Q_{i+q+1}$. Therefore, by Lemma 14, $E_{i+q+1} \leq E_i - 1$ and $L_{i+q+1} = L_i$, thus $E_{i+q+1} + 2L_{i+q+1} \leq E_i + 2L_i - 1$.

Case 2 Assume $T_{i+q} = F_{i+q+1,1}$. Let $p$ be the pivot element of the translated cone $(F_{i+q+1,m_{i+q+1}},F_{i+q,1})$. If $p$ is feasible, i.e., $p \in P \cap \mathbb{Z}^2$, then $p$ will be the optimal integral vertex in the iteration $i + q + 1$ and the algorithm terminates. Else $p$ is infeasible. This means that $p$ must violate the inequality defining $F_{i+q+1,m_{i+q+1}}$. Consider the facet $F$ of $P$ that is the original facet of $p$ from the same family as $T_{i+q}$. We will now show that $F$ and all the inequalities in this family except for $T_{i+q}$ are redundant for $(F_{i+q+1,m_{i+q+1}},T_{i+q})$ and therefore for $Q_{i+q+1} \subseteq (F_{i+q+1,m_{i+q+1}},T_{i+q})$. Since we
have processed this family during the algorithm, there must have been an optimal vertex defined by 
\((F,F')\) for some inequality \(F'\) that is facet defining at some point during the algorithm. Let \(Q'\) be 
the relaxation at the iteration of the algorithm when the vertex \(v\) defined by \((F,F')\) was optimal.

Since \(F_{i+q+1,m_{i+q+1}}\) is not redundant for \(Q'\) (as it is not redundant for \(Q_{i+q+1}\) which is a subset 
of \(Q'\)), and the inequality defining \(F_{i+q+1,m_{i+q+1}}\) is valid for the optimal vertex \(v\) of \(Q'\), its normal 
vector cannot be contained in the cone generated by the normals of \(F\) and \(F'\). Since the vertex 
defined by \((F,F')\) was optimal for the relaxation \(Q'\), \(c\) is contained in the cone generated by the 
normals of \(F\) and \(F'\). Thus, the normal of \(F_{i+q+1,m_{i+q+1}}\) cannot be contained in the cone generated 
by the normal of \(F\) and \(c\). This means that the normal of \(F\) is contained in the cone between the 
normal of \(F_{i+q+1,m_{i+q+1}}\) and \(c\), since both \(F\) and \(F_{i+q+1,m_{i+q+1}}\) are late facets at some time 
during the algorithm. Moreover, the normals of the inequalities in the family of \(F\) are contained in the 
cone generated by the normal of \(F\) and \(T_{i+q}\), and therefore, in the cone generated by the normals 
of \(F_{i+q+1,m_{i+q+1}}\) and \(T_{i+q}\). Since our current optimal vertex is defined by \(F_{i+q+1,m_{i+q+1}}\) and \(T_{i+q}\), 
all these inequalities from the family must be redundant for \((F_{i+q+1,m_{i+q+1}}, T_{i+q})\).

Thus, we have established that \(F\) and all the inequalities in its family except for \(T_{i+q}\) are 
redundant for \(Q_{i+q+1}\). Since \(T_{i+q}\) is from the same family and is early at iteration \(i + q + 1\), we 
must have \(L_{i+q+1} \leq L_{i} - 1\) by Lemma 14. Moreover, \(T_{i+q}\) is the only new early facet, and therefore, 
\(E_{i+q} \leq E_{i} + 1\) by Lemma 14. Thus, \(E_{i+q+1} + 2L_{i+q+1} \leq E_{i} + 2L_{i} - 1\). This completes the proof 
of the claim.

By the above claim, in at most \(O(\log \|A\|_{\infty}^2)\) iterations after iteration \(i\), either the algorithm 
terminates or the number \(E_{i} + 2L_{i}\) must decrease by at least 1. Since the maximum value of \(E_{i} + 2L_{i}\) 
is at most \(2m\), we have the result.

\[\text{Remark 18.} \quad \text{The upper bound on the number of iterations given in the above theorem does not depend on } c.\]

\[\text{Remark 19.} \quad \text{By Lemma 3, when } C \cap W_1 \neq \emptyset, \text{ the tilt } T \text{ produced by the algorithm may not be a facet of the split closure of the translated cone } C. \text{ However, this property is crucial for the convergence of the algorithm. Indeed, if the “best cut” is used, the algorithm may not converge as shown by the following example.} \]

Define \(p_i := 3\) and \(p_{i+1} := 2p_i - 2\) for all integers \(i \geq 1\). Given \(i \geq 0\), consider the following integer program, which we denote by \(P_i:\)

\[
\begin{align*}
\max \ & x_2 \\
\text{s.t.} \ & x_1 \leq 4 \\
& (2p_i - 1)x_1 - (4p_i - 4)x_2 \geq 0 \\
& 5x_1 - 8x_2 \geq 0 \\
& x_1, \ x_2 \in \mathbb{Z} 
\end{align*}
\]

(Note that for \(i = 0\) the inequalities Eq. (3) and Eq. (4) coincide.) The optimal solution of the 
continuous relaxation is \(\left(4, \frac{2p_{i+1} - 1}{p_i - 1}\right) \notin \mathbb{Z}^2\), which is the unique point satisfying both constraints 
Eq. (2) and Eq. (3) at equality.

We will use the same notation as in Definition 1, where \(H_1\) and \(H_2\) are the two half-planes 
defined by Eq. (2) and Eq. (3), respectively. Since \((2p_i - 1)\) and \((4p_i - 4)\) are coprime numbers, \(H\)
is defined by the equation $(2p_i - 1)x_1 - (4p_i - 4)x_2 = 1$. Combined with the fact that the pivot is $p = (0, 0)$, this implies that $q = (4p_i - 4, 2p_i - 1)$.

We claim that $x = (-2p_i + 3, -p_i + 1)$ and $y = (2p_i - 1, p_i)$. This follows from the following three observations: (i) these two points are integer and belong to $H$; (ii) $-2p_i + 3 < 4 < 2p_i - 1$ (because this is true for $i = 0$ and $p_i$ increases as $i$ increases); (iii) $y - x = q - p$.

It follows that $W^i$ is defined by the equation $(p_i - 1)x_1 - (2p_i - 3)x_2 = 1$. This line intersects the edge $\{(x_1, x_2) : x_1 = 4, x_2 \leq \frac{2p_i - 1}{p_i - 1}\}$ of the continuous relaxation at $q' = \left(4, \frac{4p_i - 5}{2p_i - 3}\right)$ (note that $\frac{4p_i - 5}{2p_i - 3} < \frac{2p_i - 1}{p_i - 1}$). Thus the strongest cut would be $(4p_i - 5)x_1 - (8p_i - 12)x_2 \geq 0$. Since $p_i + 1 = 2p_i - 2$, the cut can be written as $(2p_i + 1 - 1)x_1 - (4p_i + 1 - 4)x_2 \geq 0$. When we add this cut to the continuous relaxation, Eq. (3) becomes redundant and we obtain problem $P_{i+1}$. Then this procedure never terminates.

### 3 Polynomiality of the split closure

In this section we prove the following result:

**Theorem 20.** Let $Ax \leq b$ be a description of a polyhedron $P \subseteq \mathbb{R}^2$ consisting of $m$ inequalities. Then the split closure of $P$ admits a description whose size is polynomial in $m$, $\log \|A\|_{\infty}$ and $\log \|b\|_{\infty}$.

We will make use of the following result, which holds in any fixed dimension.

**Theorem 21.** [4] Let $d \geq 1$ be a fixed integer and let $Ax \leq b$ be a description of a polyhedron $P \subseteq \mathbb{R}^d$ consisting of $m$ inequalities. Then the Chvátal closure of $P$ admits a description whose size is polynomial in $m$, $\log \|A\|_{\infty}$ and $\log \|b\|_{\infty}$.

Because of Theorem 21, in order to prove Theorem 20 it is sufficient to show that the intersection of all the split cuts for $P$ that are not Chvátal cuts is a polyhedron that admits a description of polynomial size.

We now start the proof of Theorem 20. We can assume that $P \subseteq \mathbb{R}^2$ is pointed, as otherwise it is immediate to see that the split closure of $P$ is $P_I$ and is defined by at most two inequalities. The following result holds in any dimension.

**Lemma 22** ([1]; see also [5, Corollary 5.7]). The split closure of $P$ is the intersection of the split closures of all the corner relaxations of $P$ (i.e., relaxations obtained by selecting a feasible or infeasible basis of the system $Ax \leq b$).

Since there are at most $\binom{m}{2}$ corner relaxations of $P$ (i.e., bases of $Ax \leq b$), because of Lemma 22 in the following we will work with a corner relaxation of $P$, which we denote by $C$. Thus $C$ is a full-dimensional translated pointed cone. We denote its apex by $v$.

**Definition 23.** We say that a split set is effective for $C$ if $v$ lies in its interior; note that this happens if and only if there is a split cut for $C$ derived from $S$ that cuts off $v$. Such a split cut will also be called effective.

Since $C$ is a translated cone, for every effective split disjunction $(\pi, \pi_0)$ we have $C^{\pi, \pi_0} = C \cap H$ for a unique split cut $H$ derived from this disjunction. In the following, whenever we say “the
split cut derived from a given disjunction” we refer to this specific split cut. Note that when the boundary of an effective split set $S$ intersects the facets of $C$ in precisely two points, the split cut derived from $S$ is delimited by the line containing these two points, while when the boundary of $S$ intersects the facets of $C$ in a single point, the line delimiting the split cut derived from $S$ contains this point and is parallel to the lineality space of $S$. (In the latter case, the split cut is necessarily a Chvátal cut.)

In the following, we let $\text{intr}(X)$ denote the interior of a set $X \subseteq \mathbb{R}^2$.

**Lemma 24.** Every effective split cut for $C$ is of one of the following types:

1. a Chvátal cut;

2. a cut derived from a split set $S$ such that $S \cap \text{intr}(C_I) \neq \emptyset$; in this case, both lines delimiting $S$ intersect the same facet of $C_I$.

**Proof.** Consider any effective split cut that is not a Chvátal cut, given by a split set $S$. Then the interior of the recession cone of $C$ contains a recession direction of $S$. Since $C$ and $C_I$ have the same recession cone, $S$ intersects the interior of $C_I$. As no vertex of $C_I$ can be in the interior of $S$, the bounding lines of $S$ must intersect the same facet of $C_I$.

**Definition 25.** Let $F_1^I, \ldots, F_n^I$ be the facets of $C_I$. For every $i \in \{1, \ldots, n\}$ we denote by $\ell_i^I$ the line containing $F_i^I$. Furthermore, we define $\hat{\ell}_i^I$ as the unique line with the following properties:

1. $\hat{\ell}_i^I$ is parallel to $\ell_i^I$;

2. $\hat{\ell}_i^I$ contains integer points;

3. there is no integer point strictly between $\ell_i^I$ and $\hat{\ell}_i^I$;

4. $\hat{\ell}_i^I \cap C_I = \emptyset$.

Given two split cuts $H, H'$ for $C$, we say that $H$ dominates $H'$ if $C \cap H \subseteq C \cap H'$.

**Lemma 26.** Fix $i \in \{1, \ldots, n\}$ and define the split set $S := \text{conv}(\ell_i^I, \hat{\ell}_i^I)$. If $v \in \text{intr}(S)$, then the split cut for $C$ derived from $S$ is a Chvátal cut that dominates every split cut derived from a split set intersecting $F_i^I$.

**Proof.** Since $v \in \text{intr}(S)$, the facets of $C$ do not intersect $\hat{\ell}_i^I$. This implies that the split cut for $C$ derived from $S$ is a Chvátal cut.

Let $S'$ be any split set intersecting $F_i^I$, and denote by $h_1, h_2$ the two lines delimiting $S'$. Since there is no integer point in $\text{intr}(S')$, both $h_1$ and $h_2$ intersect $F_i^I$. We denote by $x^1$ (resp., $x^2$) the intersection point of $h_1$ (resp., $h_2$) and $F_i^I$. Also we define $y^1$ (resp., $y^2$) as the intersection point of $h_1$ (resp., $h_2$) and $\hat{\ell}_i^I$. Since $x^1, x^2 \in C$ and $y^1, y^2 \notin C$ (as $\hat{\ell}_i^I$ does not intersect $C$), $h_1$ and $h_2$ intersect the facets of $C$ in two points contained in $\text{intr}(S)$. This implies that any split cut derived from $S'$ is dominated by the Chvátal cut derived from $S$.

**Definition 27.** Given a line containing integer points, we call each closed segment whose endpoints are two consecutive integer points of $\ell$ as a unit interval of $\ell$. 

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Observation 28. Any split set can intersect at most one unit interval of a given line.

Definition 29. Fix $i \in \{1, \ldots, n\}$. Given a unit interval $J$ of $\ell_I$ and a unit interval $\hat{J}$ of $\ell_{I'}$, there exists a unique parallelogram of area 1 having $J$ and $\hat{J}$ as two of its sides. We denote by $S(J, \hat{J})$ the split set delimited by the lines containing the other two sides of this parallelogram. If $S(J, \hat{J})$ is effective, we denote by $H(J, \hat{J})$ the split cut for $C$ derived from $S(J, \hat{J})$.

Lemma 30. Fix $i \in \{1, \ldots, n\}$ such that $v \notin \text{intr}(\text{conv}(\ell_I, \ell_{I'}))$. Then there exists a unique unit interval $\hat{J}$ of $\ell_I$ such that $\hat{J} \cap C \neq \emptyset$. Furthermore, for each unit interval $J$ of $\ell_I$ contained in $F_I$, $S(J, \hat{J})$ is an effective split set.

Proof. The existence of $\hat{J}$ follows from the assumption $v \notin \text{intr}(\text{conv}(\ell_I, \ell_{I'}))$. Furthermore, $\hat{J}$ is unique because, by definition of $\ell_I$, there are no integer points in $C \cap \ell_I$.

We now prove that for each unit interval $J$ of $\ell_I$ contained in $F_I$, $S(J, \hat{J})$ is an effective split cut. Up to a unimodular transformation, we can assume that $J = \{x \in \mathbb{R}^2 : x_2 = 0, 0 \leq x_1 \leq 1\}$ and $\hat{J} = \{x \in \mathbb{R}^2 : x_2 = 1, 0 \leq x_1 \leq 1\}$. Then the split set $S(J, \hat{J})$ is defined by the inequalities $0 \leq x_1 \leq 1$.

Since the second coordinate of $v$ is $v_2 \geq 1$ and $C_I$ is contained in the half-plane defined by $x_2 \leq 0$ (as this inequality induces facet $F_I$ of $C_I$), it follows that both facets of $C$ intersect the lines defined by $x_2 = 0$ and $x_2 = 1$. Thus one facet of $C$ contains points $(a_1, 0)$ and $(b_1, 1)$, and the other facet contains points $(a_2, 0)$ and $(b_2, 1)$, where $a_1 \leq 0$, $a_2 \geq 1$ and $0 < b_1 < b_2 < 1$. It is now straightforward to verify that $v$, which is the intersection point of the two facets, satisfies $0 < v_1 < 1$. This shows that $S(J, \hat{J})$ is an effective split set.

Lemma 31. Fix $i \in \{1, \ldots, n\}$. Let $S$ be a split set that gives an effective split cut of type 2 from Lemma 24, where $S$ intersects $F_I$. Suppose that this split cut is not dominated by a Chvátal cut. Let $J$ be the unit interval of $\ell_I$ that intersects both lines delimiting $S$ (see Observation 28), and let $\hat{J}$ be the unit interval of $\ell_{I'}$ such that $\hat{J} \cap C \neq \emptyset$ (see Lemma 30). Then:

(i) both lines delimiting $S$ intersect $\hat{J}$;

(ii) any cut produced by $S$ is dominated by $H(J, \hat{J})$.

Proof. Up to a unimodular transformation, we can assume that $J = \{x \in \mathbb{R}^2 : x_2 = 0, 0 \leq x_1 \leq 1\}$ and $\hat{J} = \{x \in \mathbb{R}^2 : x_2 = 1, 0 \leq x_1 \leq 1\}$. Since the split cut derived from $S$ is not dominated by a Chvátal cut, by Lemma 26 the apex $v$ does not lie strictly between $\ell_I$ and $\ell_{I'}$. In other words, $v_2 \geq 1$. Furthermore $0 < v_1 < 1$, as shown in the proof of Lemma 30.

Let $h_1$ and $h_2$ be the lines delimiting $S$, and define the segments $J'_1 := \{x \in \mathbb{R}^2 : x_1 = 0, 0 \leq x_2 \leq 1\}$ and $J'_2 := \{x \in \mathbb{R}^2 : x_1 = 1, 0 \leq x_2 \leq 1\}$. Since both $h_1$ and $h_2$ intersect $J$ and there is no integer point strictly between $h_1$ and $h_2$, we have that $h_1 \cup h_2$ can contain points from the relative interior of at most one of $J'_1$, $J'_2$ and $\hat{J}$.

Assume that $h_1$ and $h_2$ intersect the relative interior of $J'_1$. Since $h_1$ and $h_2$ also intersect $J$, we have

$$h_1 = \{x \in \mathbb{R}^2 : x_2 = ux_1 + r_1\}, \quad h_2 = \{x \in \mathbb{R}^2 : x_2 = ux_1 + r_2\},$$

for some $u < 0$. 

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Given any $\bar{x} \in h_1 \cap \{x \in \mathbb{R}^2 : x_2 \geq 1\}$, we have

$$\bar{x}_1 = \frac{\bar{x}_2 - r_1}{u} \leq \frac{1 - r_1}{u} < 0.$$ 

Thus $h_1 \cap \{x \in \mathbb{R}^2 : x_2 \geq 1\} \subseteq \{x \in \mathbb{R}^2 : x_1 < 0\}$. Similarly, $h_2 \cap \{x \in \mathbb{R}^2 : x_2 \geq 1\} \subseteq \{x \in \mathbb{R}^2 : x_1 < 0\}$. As $0 < v_1 < 1$ and $v_2 \geq 1$, it follows that $v$ does not lie strictly between $h_1$ and $h_2$, a contradiction.

A similar argument shows that $h_1$ and $h_2$ do not intersect the relative interior of $J'_2$. It follows that $h_1$ and $h_2$ intersect $\hat{J}$, and (i) is proven.

We now prove (ii). Since, by part (i), each of $h_1$ and $h_2$ intersects both $J$ and $\hat{J}$, each of $h^1$ and $h^2$ intersects the boundary of $C$. Moreover, because $S$ is an effective split set, $h_1 \cup h_2$ intersects the boundary of $C$ in at most two points. It follows that each of $h_1$ and $h_2$ intersects the boundary of $C$ in a single point, say $q^1$ and $q^2$, respectively. Note that $q^1_2 > 0$ and $q^2_2 > 0$, because $h_1$ and $h_2$ intersect $J$. Label $q^1$ and $q^2$ in such a way that $q^1$ (resp., $q^2$) belongs to the facet of $C$ contained in the half-plane $x_1 \leq v_1$ (resp., $x_1 \geq v_1$).

If $0 < q^2_2 < 1$ for some $j \in \{1, 2\}$, then $0 < q^1_j < 1$, because $h_1$ and $h_2$ intersect both $J$ and $\hat{J}$. If $q^j_2 \geq 1$, then again $0 < q^j_1 < 1$, as $\{x \in C : x_2 \geq 1\} \subseteq \{x \in \mathbb{R}^2 : 0 < x_2 < 1\}$.

The split set $S(J, \hat{J})$ is effective by Lemma 30, and its boundary intersects the facets of $C$ in two points $r^1, r^2$ that satisfy $r^1_1 = 0$ and $r^2_1 = 1$. Then $r^1$ (resp., $r^2$) is further from the apex than $q^1$ (resp., $q^2$) is, as $q^1, q^2$ and $v$ all satisfy $0 < x_1 < 1$. It follows that the cut $H(J, \hat{J})$ dominates any split cut derived from $S$.

Lemma 32. Fix $i \in \{1, \ldots, n\}$ such that $v \notin \text{intr}(\text{conv}(\ell_i', \ell_i))$, and let $\hat{J}$ be as in Lemma 30. Write $F'_j = J_0 \cup \cdots \cup J_t$, where $J_0, \ldots, J_t$ are the unit intervals contained in $F'_j$ ordered consecutively (i.e., $|J_{k-1} \cap J_k| = 1$ for every $k \in \{1, \ldots, t\}$). Then every point that violates $H(J_k, \hat{J})$ for some $k \in \{0, \ldots, t\}$ also violates $H(J_0, \hat{J})$ or $H(J_t, \hat{J})$.

Proof. Up to a unimodular transformation, we can assume that $\hat{J} = \{x \in \mathbb{R}^2 : x_2 = 1, 0 \leq x_1 \leq 1\}$ and $J_k = \{x \in \mathbb{R}^2 : x_2 = 0, k \leq x_1 \leq k + 1\}$ for every $k \in \{0, \ldots, t\}$. As argued in the proof of Lemma 30, one facet $G_1$ of $C$ contains points $(a_1, 0)$ and $(b_1, 1)$, and the other facet $G_2$ contains points $(a_2, 0)$ and $(b_2, 1)$, where $a_1 \leq 0, a_2 \geq 1$ (in fact, $a_2 - 1 \geq t \geq 0$) and $0 < b_1 < b_2 < 1$. Then the lines containing $G_1$ and $G_2$ are defined by the equations $x_1 + (a_1 - b_1)x_2 = a_1$ and $x_1 + (a_2 - b_2)x_2 = a_2$, respectively.

Given any $k \in \{0, \ldots, t\}$, the lines delimiting the split cut $H(J_k, \hat{J})$ are defined by the equations $x_1 + kx_2 = k$ and $x_1 + kx_2 = k + 1$. The intersection points of the former line with $G_1$ and of the latter line with $G_2$ are respectively the following:

$$q^k = \left(\frac{b_1k}{b_1-a_1+k}, \frac{k-a_1}{b_1-a_1+k}\right), \quad r^k = \left(\frac{b_2k + b_2 - a_2}{b_2-a_2+k}, \frac{k + 1 - a_2}{b_2-a_2+k}\right).$$

Consider any $k \in \{0, \ldots, t\}$. Since $H(J_k, \hat{J})$ is a half-plane that does not contain $v$, it is defined by an inequality of the form $c^k(x - v) \geq 1$, where $c^k \in \mathbb{R}^2$. Note that and $c^k(q^k - v) = c^k(r^k - v) = 1$, as $q^k$ and $r^k$ belong to the line delimiting $H(J_k, \hat{J})$.

Let $\bar{x}$ be any point in $G_1$ and $k \in \{0, \ldots, t\}$. Then $\bar{x} = v + \mu(q^k - v)$ for some $\mu \geq 0$ and therefore

$$c^k(\bar{x} - v) = \mu c^k(q^k - v) = \mu = \frac{v_1 - \bar{x}_1}{v_1 - q^k_1}.$$
Let \( \ell \) between \( F \) and thus there is a facet \( H \) which suffices to prove the theorem.

Proof. F and thus \( \bar{x} > 0 \) and \( v_1 - \bar{x} > 0 \).

We can calculate
\[
v_1 - q_1^k = v_1 - \frac{b_1 k}{b_1 - a_1 + k} = \frac{(v_1 - b_1)k + v_1 b_1 - v_1 a_1}{b_1 - a_1 + k},
\]
from which we obtain
\[
\frac{v_1 - b_1}{v_1 - q_1^k} = \frac{(v_1 - b_1)k + (v_1 - b_1)(b_1 - a_1)}{(v_1 - b_1)k + v_1 b_1 - v_1 a_1}
\]
\[
= 1 + \frac{(v_1 - b_1)(b_1 - a_1) - (v_1 b_1 - v_1 a_1)}{(v_1 - b_1)k + v_1 b_1 - v_1 a_1}
\]
\[
= 1 + \frac{b_1 a_1 - b_1}{(v_1 - b_1)k + v_1 b_1 - v_1 a_1}.
\]

Since \( b_1 > 0, a_1 - b_1 < 0 \) and \( v_1 - b_1 > 0 \), the last fraction above is of the form \( \frac{\alpha}{\beta k + \gamma} \), where \( \alpha < 0, \beta > 0 \) and \( \gamma \in \mathbb{R} \). It is immediate to verify that such a function of \( k \) is concave for \( k > -\frac{\gamma}{\beta} \). In our context, this condition reads \( k > -\frac{v_1(b_1-a_1)}{v_1-b_1} \), which is a negative number. Thus the function \( k \mapsto c^k(\bar{x} - v) \) is concave over the domain \([0,t]\). A similar argument shows that, for any fixed \( \bar{x} \in G_2 \), the function \( k \mapsto c^k(\bar{x} - v) \) is concave over \([0,t] \) (using the fact that \( t \leq a_2 - 1 \)).

If \( \bar{x} \) is any point in \( C \), then we can write \( \bar{x} = \lambda x_1 + (1 - \lambda) x_2 \), where \( x_1 \in G_1, x_2 \in G_2 \) and \( 0 \leq \lambda \leq 1 \). Then \( c^k(\bar{x} - v) = \lambda c^k(x_1 - v) + (1 - \lambda)c^k(x_2 - v) \) is a concave function of \( k \). This implies that the minimum of the set \( \{ c^k(\bar{x} - v) : k \in \{0, \ldots, t\} \} \) is achieved for \( k = 0 \) or \( k = t \). In particular, if \( \bar{x} \) violates \( H(J_k, \hat{J}) \) for some \( k \in \{0, \ldots, t\} \), then \( \min \{ c^0(\bar{x} - v), c^t(\bar{x} - v) \} \leq c^k(\bar{x} - v) < 1 \), and thus \( \bar{x} \) also violates \( H(J_0, \hat{J}) \) or \( H(J_t, \hat{J}) \).

\[ \square \]

**Theorem 33.** The number of facets of the split closure of a translated cone \( C \subseteq \mathbb{R}^2 \) is at most twice the number of facets of the Chvátal closure of \( C \).

**Proof.** Let \( E \subseteq \{1, \ldots, n\} \) be the index set of the facets of \( C_I \) such that \( v \) does not lie strictly between \( \ell^i \) and \( \ell^j \). By Lemma 30, for every \( i \in E \) there exists a unit interval \( \hat{J}^i \) of \( \hat{L}^i \) such that \( \hat{J}^i \cap C \neq \emptyset \). Moreover, let \( J_{0,0}, \ldots, J_{1,0} \) be the unit intervals of \( \ell^0 \) contained in \( F^i \), ordered consecutively. Let \( Q \) denote the Chvátal closure of \( C \). We show that the split closure of \( C \) is given by
\[
Q \cap \bigcap_{i \in E} \left( H(J_{0,0}^i, \hat{J}^i) \cap H(J_{1,0}^i, \hat{J}^i) \right),
\]
which suffices to prove the theorem.

Consider any split cut \( H \) derived from a split set \( S \). Since Eq. (6) is contained in \( Q \), we may assume that \( H \) is not dominated by a Chvátal cut. Therefore it must be of type 2 in Lemma 24, and thus there is a facet \( F^i \) of \( C_I \) that intersects the two lines delimiting \( S \). By Lemma 26, \( i \in E \). By Lemma 31 part (ii), \( H \) is dominated by \( H(J_k^i, \hat{J}^i) \) for some \( k \in \{0, \ldots, t_i\} \). By Lemma 32, \( H(J_k^i, \hat{J}^i) \) is in turn dominated by \( H(J_0^i, \hat{J}^i) \cap H(J_{t_i}^i, \hat{J}^i) \). \[ \square \]

To conclude the proof of Theorem 20, we note that by Lemma 22, Theorem 33 and Theorem 8, the number of inequalities needed to define the split closure of \( P \) is polynomial in \( m, \log \|A\|_{\infty} \) and
log\|b\|_\infty. Furthermore, the above arguments show that the size of every inequality is polynomially bounded. (However, it is known that also in variable dimension every facet of the split closure of a polyhedron \( P \) is polynomially bounded; see, e.g., [5, Theorem 5.5].)

**Remark 34.** Given a translated cone \( C \subseteq \mathbb{R}^2 \), the arguments used in this section show that, for every facet \( F_i \) of \( C_1 \), the split closure \( C' \) of \( C \) is contained in the half-plane delimited by \( \hat{F}_i \) and containing \( C \) (where we adopt the notation introduced in Definition 25). This implies that the Chvátal closure of \( C' \) is \( C_1 \). In particular, the split rank of \( C \) is at most 2.

Now let \( P \) be a polyhedron in \( \mathbb{R}^2 \). It is folklore that the integer hull of \( P \) is the intersection of the integer hulls of all the corner relaxations of \( P \). (This is not true in higher dimensions.) Then, by the previous argument, the split rank of \( P \) is at most 2.

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