KÄHLER-EINSTEIN METRICS, CANONICAL RANDOM POINT PROCESSES AND BIRATIONAL GEOMETRY

ROBERT J. BERMAN

Abstract. A new probabilistic/statistical-mechanical approach to the study of canonical metrics and measures on a complex algebraic variety $X$ is introduced. On any such variety with positive Kodaira dimension a canonical (birationally invariant) random point processes is defined and shown to converge in probability towards a canonical deterministic measure $\mu_X$ on $X$, coinciding with the canonical measure of Song-Tian and Tsuji, previously introduced in a different setting. More generally, the convergence is shown to hold in the setting of log canonical pairs. In the case of variety $X$ of general type we obtain as a corollary that the (possibly singular) Kähler-Einstein metric on $X$ with negative Ricci curvature is the limit of a canonical sequence of quasi-explicit Bergman type metrics. When $X$ is defined over the integers the partition functions of the point processes define height type arithmetic invariants which, in the case of certain Shimura varieties, are shown to converge to a logarithmic derivative of the corresponding Dedekind zeta function. Finally, in the opposite setting of a Fano variety $X$ we relate the canonical point processes to a new notion of stability, that we call Gibbs stability, which admits a natural algebro-geometric formulation and which we conjecture is equivalent to the existence of a Kähler-Einstein metric on $X$ and hence to K-stability as in the Yau-Tian-Donaldson conjecture.

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1. Introduction

Kähler-Einstein metrics, i.e. Kähler metrics with constant Ricci curvature, play a key role in the study of complex algebraic varieties. When such a variety $X$ admits a (say negatively curved) Kähler-Einstein metric it is unique, i.e. canonically attached to $X$ and can thus be used to probe the space $X$ using differential-geometric
tools (e.g. as in Yau’s proof of the Miyaoka-Yau Chern number inequalities). Singular versions of Kähler-Einstein are also naturally linked to Mori’s Minimal Model Program (MMP) for complex algebraic varieties, notably through the Kähler-Ricci flow \[40\] \[45\] \[50\]. In the case when the variety \(X\) is defined over the integers, i.e. \(X\) is cut out by polynomials with integer coefficients, it is also expected that Kähler-Einstein metrics carry arithmetic information (even if there are very few direct results in this direction). For example, the role of Kähler-Einstein metrics in arithmetic (Arakelov) geometry was speculated on by Manin \[41\], as playing the role of minimal models over the prime/place at infinity (compare Remark \[6.5\] in the present paper). These metrics also make their appearance in Kudla’s program and the Maillot-Rössler’s conjectures concerning the relation between special values of derivatives of L-functions and arithmetic intersection numbers (see \[21\] \[15\] and references therein).

In this paper a new probabilistic (statistical-mechanical) approach to the study of Kähler-Einstein metrics is introduced, where the Kähler-Einstein metrics appear in the large \(N\)−limit of of certain canonical random point processes on \(X\) with \(N\) particles. The canonical point processes in question are directly defined in terms of algebro-geometric data and the thrust of this approach is thus that it gives a new direct link between algebraic geometry on one hand and complex differential (Kähler) geometry on the other. While our main results are centered around the case of negatively curved Kähler-Einstein metrics (and various singular and twisted generalizations of such metrics), we also formulate a conjectural picture relating the existence problem for Kähler-Einstein metrics with positive Ricci curvature on a Fano manifold \(X\) to a statistical mechanical notion of stability that we call Gibbs stability. The latter notion of stability thus replaces the notion of K-stability which appears in the seminal Yau-Tian-Donaldson conjecture for Kähler-Einstein metrics on Fano manifolds \[24\] \[49\]. Interestingly, the notion of Gibbs stability also admits a natural purely algebro-geometric formulation in terms of the standard singularity notions in the MMP. Moreover, in the arithmetic setting of a variety defined over the integers the asymptotics of the corresponding canonical partition functions turns out to be naturally linked to Kudla’s program and the Maillot-Rössler’s conjectures.

The connections to physics (in particular emergent gravity and fermion-boson correspondences) were emphasized in \[10\], where a heuristic argument for the convergence of the point processes was outlined. Here we will provide rigorous proofs - in a more general setting - building on \[10\] \[11\] \[12\] \[7\]. The key new technical feature is a mean value inequality for quasi-subharmonic functions on Riemannian quotients in large dimensions with a distortion coefficient which is sub-exponential in the dimension and which will be obtained as a refinement of a submean inequality of Li-Schoen \[38\].

Before turning to a precise statement of the main results it may also be worth pointing out that there are also numerical motivations for the current approach. Indeed, there are very few examples of Kähler-Einstein metrics that can be written down explicitly and one virtue of the present framework is that it offers an, essentially, explicit way of numerically sampling random points in order to approximate the Kähler-Einstein metric. It would thus be interesting to compare it with other recently proposed numerical schemes \[28\] \[27\].

1.1. Kähler-Einstein geometry and the canonical point processes. Let \(X\) be an \(n\)−dimensional compact complex manifold. A Kähler metric \(\omega\) on \(X\) is said
to be \textit{Kähler-Einstein} if its Ricci curvature is constant:

\begin{equation}
\text{Ric}_{\omega_{KE}} = -\beta \omega_{KE}
\end{equation}

where, after normalization one may assume that \( \beta = \pm 1 \) or \( \beta = 0 \). Since the Ricci form of a Kähler metric represents minus the first Chern class \( c_1(K_X) \) of the canonical line bundle \( K_X := \Lambda^n(TX) \) of \( X \) the Kähler-Einstein equations imposes cohomological conditions saying that \( c_1(K_X) \) vanishes when \( \beta = 0 \) and has a definite sign when \( \beta = \pm 1 \). In the latter case, which is the one we shall mainly focus on, this means in algebro-geometric terms that \( \pm K_X \) is ample (using additive notation for tensor products, so that \( -K_X \) denotes the dual of \( K_X \) and in particular \( X \) is a projective algebraic variety. As is well-known there are also singular versions of Kähler-Einstein metrics obtained by either relaxing the positivity (or negativity) condition on \( K_X \) or introducing a log structure on \( X \), i.e. a suitable divisor \( D \) on \( X \) (see below). We recall that these notions appear naturally in the Minimal Model Program, which aims at attaching a minimal model to a given algebraic variety (say with positive Kodaira dimension), i.e. a birational model whose canonical line bundle is nef (which is the numerical version of semi-positivity) \cite{17}. Recently, there has also been a rapid development of the theory of Kähler-Einstein metrics attached to a log pair \((X, D)\) (see for example \cite{29, 20, 23, 42, 24, 49, 13, 14}).

1.1.1. Varieties of general type (\( \beta = 1 \)). Let us start with the case when \( \beta = 1 \), the case when \( K_X \) is positive, i.e. ample (or more generally big, i.e. \( X \) is a variety of general type). We will show how to recover the unique Kähler-Einstein metric on such a manifold \( X \) from the large \( N \) limit of certain \textit{canonical} random point processes on \( X \) with \( N \) particles, defined as follows. First define the following sequence of positive integers:

\[ N_k := \dim H^0(X, kK_X), \]

where \( H^0(X, kK_X) \) is the space of all pluricanonical (holomorphic) \( n \)-forms of \( X \), at level \( k \) (recall that we are using additive notation for tensor products). In other words, \( N_k \) is the \( k \)th plurigenus of \( X \) and we recall that \( X \) said to be of \textit{general type} if \( N_k \) is of the order \( k^n \) for \( k \) large. In particular, this is the case if \( K_X \) is ample. The starting point of the present probabilistic approach is the observation that there is a canonical probability measure \( \mu^{(N_k)} \) on the \( N_k \)-fold product \( X^{N_k} \) which may be defined as follows, in terms of local holomorphic coordinates \( z \) on \( X \):

\begin{equation}
\mu^{(N_k)} := \frac{1}{Z_{N_k}} \left| (\text{det} S^{(k)})(z_1, \ldots, z_{N_k}) \right|^{2/k} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k} \wedge d\bar{z}_{N_k}
\end{equation}

where \( dz \wedge d\bar{z} \) is a short hand for the local Euclidean volume form on \( X \) determined by the local coordinates \( z \) and \( \text{det} S^{(k)} \) is a holomorphic section of the line bundle \( (kK_X)^{\otimes N_k} \to X^{N_k} \), defined by a generator of the determinantal line \( \Lambda^{N_k}(H^0(X, kK_X)) \) (and thus defined up to a multiplicative complex number). Concretely, we may take \( S^{(k)} \) as the following Vandermonde type determinant

\[ (\text{det} S^{(k)})(z_1, z_2, \ldots, z_N) := \text{det}(s_i^{(k)}(z_j)), \]

in terms of a given basis \( s_i^{(k)} \) in \( H^0(X, kK_X) \), which locally may be identified with a holomorphic function on \( X^{N_k} \). The point is that, by the very definition of the canonical line bundle \( K_X \), the local function \( (\text{det} S^{(k)})(z_1, \ldots, z_{N_k}) \) transforms as a density on \( X^{N_k} \) and after normalization by its total integral \( Z_{N_k} \) one obtains
a sequence of globally well-defined probability measures \( \mu^{(N_k)} \) on \( X^{N_k} \) which are independent of the choice of basis \( s_i^{(k)} \) in \( H^0(X, kK_X) \) and thus canonically attached to \( X \). From a statistical mechanical point of view the normalizing constant

\[
Z_{N_k} := \int_{X^{N_k}} \left| (\det S^{(k)}(z_1, \ldots, z_{N_k})) \right|^{2/k} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k} \wedge d\bar{z}_{N_k}
\]

is the partition function and it depends on the choice of generator (but the point is that \( \mu^{(N_k)} \) does not).

By construction the probability measure \( \mu^{(N_k)} \) is symmetric, i.e. invariant under the natural action of the permutation group \( \Sigma_{N_k} \) and hence defines a random point process on \( X \) with \( N_k \) particles (i.e. a probability measure on the space of all configurations of \( N_k \) points on \( X \)). To simplify the notation we will often omit the subscript \( k \) on \( N_k \). This should cause no confusion, since \( k \) tends to infinity precisely when \( N_k \) does.

We recall that the empirical measure of a random point process with \( N \) particles on a space \( X \) is the random measure

\[
\delta_N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}
\]

on the ensemble \((X^N, \mu^{(N)})\) which defines a map from \( X^N \) to the space \( \mathcal{M}_1(X) \) of all probability measures \( \mu \) on \( X \). By definition the law of \( \delta_N \) is the push-forward of \( \mu^{(N)} \) to \( \mathcal{M}_1(X) \) under the map \( \delta_N \), which thus defines a probability measure on \( \mathcal{M}_1(X) \).

**Theorem 1.1.** Let \( X \) be a variety of general type. Then the empirical measures of the canonical random point processes on \( X \) converge in probability towards the normalized volume form \( dV_{KE} \) of the Kähler-Einstein metric on \( X \). More precisely, the laws of the empirical measures satisfy a large deviation principle with speed \( N \) and rate functional \( F(\mu) \), where \( F(\omega^n/V) \) is Mabuchi’s K-energy of the Kähler form \( \omega \) (normalized so that \( F \) vanishes on \( dV_{KE} \)).

We recall that, loosely speaking, the large deviation principle referred to in the previous theorem says that the convergence in probability is exponential in the following sense:

\[
\text{Prob} \left( \frac{1}{N_k} \sum_i \delta_{x_i} \in B_\epsilon(\mu) \right) \sim e^{-NF(\mu)}
\]

as first \( N \to \infty \) and then \( \epsilon \to 0 \), where \( B_\epsilon(\mu) \) denotes a ball of radius \( \epsilon \) centered at a given probability measure \( \mu \) in the space \( \mathcal{P}(X) \) (equipped with a metric defining the weak topology). In words this means that the probability of finding a cloud of \( N \) points \( x_1, \ldots, x_N \) on \( X \) such that the corresponding measure \( \frac{1}{N_k} \sum_i \delta_{x_i} \) approximates a volume form \( \mu \) is exponentially small unless \( \mu \) is a minimizer of \( F \).

**Corollary 1.2.** Let \( X \) be a variety of general type. Then the one-point correlation measures \( \nu_k := \int_{X^{N-1}} \mu^{(N_k)} \) of the canonical point processes define a sequence of canonical measures on \( X \) converging weakly to \( dV_{KE} \). Moreover, the curvature forms of the corresponding metrics on \( K_X \) defined by the sequence \( \nu_k \) converge weakly to the unique Kähler-Einstein metric \( \omega_{KE} \) on \( X \).

The last statement in the previous corollary concretely says that the unique Kähler-Einstein metric \( \omega_{KE} \) on \( X \) may be recovered as the weak limit of the following sequence of quasi-explicit canonical positive currents in \( c_1(K_X) \):

\[
\]
\( \omega_k := \frac{i}{2\pi} \partial \bar{\partial} \log \int_{X^{N_k-1}} \left| (\det S^{(k)})(z_1, \ldots, z_{N_k-1}) \right|^{2/k} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k-1} \wedge d\bar{z}_{N_k-1} \)

which are smooth away from the base locus of \( kK_X \).

1.1.2. Birational invariance and varieties of positive Kodaira dimension. The results for varieties of general type above generalize to the setting of a projective variety \( X \) of positive Kodaira dimension \( \kappa \). The starting point is the observation that the canonical random point processes introduced above are well-defined as long as the plurigenera \( N_k \) and that they are invariant under birational equivalence of varieties. In order to do asymptotics we also need \( N_k \) to tend to infinity, which means that the natural setting for the canonical point processes is a projective variety \( X \) of positive Kodaira dimension \( \kappa \) (recall that \( \kappa \) is the natural number defined by the growth property \( N_k \sim k^\kappa \)).

Theorem 1.3. Let \( X \) be a projective variety of positive Kodaira dimension. Then the empirical measures of the canonical random point processes on \( X \) converge in probability towards a unique probability measure \( \mu_X \), which coincides with the canonical measure of Song-Tian and Tsuji.

The canonical measure \( \mu_X \) was introduced by Song-Tian [46] in their study of the Kähler-Ricci flow and, independently, by Tsuji [50] in his study of dynamical systems defined by Bergman kernels. The measure \( \mu_X \) is defined in terms of the Ithaka fibration \( F : X \rightarrow Y \) attached to \( X \) on the Zariski open subset \( Y_0 \) where \( F \) defines a smooth morphism the measure \( \mu_X \) may be written as

\[ \mu_X = (F^* \omega_Y)^\kappa \wedge (\omega_{CY})^{n-\kappa}, \]

where, fiberwise, \( \omega_{CY} \) is a Ricci flat metric with normalized volume and \( \omega_Y \) is a canonical metric defined on the base \( Y \) in terms of a twisted Kähler-Einstein equation: it satisfies (in a weak sense) the equation

\[ \text{Ric} \omega_Y = -\omega_Y + \omega_{WP} \]

away from the branch locus of the fibration, where \( \omega_{WP} \) is the generalized Weil-Petersson metric on \( Y \), measuring the infinitesimal variation of the complex moduli of the Calabi-Yau fibers (see section 5 for precise definitions). The proof of the previous theorem relies on Fujino-Mori’s canonical bundle formula [31] which allows one to reduce the problem to the base \( Y \) of the Ithaka fibration. A similar argument was used by Tsuji [50]. It was shown by Song-Tian [46] that in the case when \( K_X \) is semi-ample the Kähler-Ricci flow on \( X \) converges weakly towards a canonical positive current which coincides with \( F^* \omega_Y \). In our setting we obtain as an immediate corollary of the previous theorem that, for any variety of positive Kodaira dimension, the canonical sequence of currents \( \omega_k \in c_1(K_X) \) defined by formula (1.5) converges weakly to a canonical positive current in \( c_1(K_X) \) with minimal singularities, coinciding with \( F^* \omega_Y \) on a Zariski open subset of \( X \). This is hence more general then assuming that \( K_X \) be semi-ample. On the other hand, assuming the validity of the fundamental conjectures of the MMP, i.e. the existence of a minimal model and the abundance conjecture (which is not necessary for our approach) the convergence of \( \omega_k \) reduces to the case when \( K_X \) is semi-ample.
1.1.3. Logarithmic generalizations and twisted Kähler-Einstein metrics. The results stated above admit natural generalization to the logarithmic setting in the sense of MMP, which from the differential geometric point of view is related to Kähler-Einstein metrics with conical and cusp type singularities. To explain this first recall that (in the smooth setting) a log canonical pair \((X, D)\) consists of a smooth complex algebraic variety \(X\) and a \(\mathbb{Q}\)-divisor \(D\) on \(X\) with simple normal crossings and coefficients in \([-\infty, 1]\). In this setting the role of the canonical line bundle \(K_X\) is placed by the log canonical line bundle \(K_X + D\) and the role of the Ricci curvature \(\text{Ric}\ \omega\) of a metric \(\omega\) is played by twisted Ricci curvature \(\text{Ric}\ \omega - [D]\), where \([D]\) denotes the current of integration defined by \(D\). The corresponding (twisted) Kähler-Einstein equation thus reads

\[
\text{Ric}\ \omega = \beta \omega + [D]
\]

which should be interpreted in a weak sense (see [14] for a very general setting). In case the coefficients of \(D\) are in \([0, 1]\) the pair \((X, D)\) is said be Kawamata log terminal (klt, for short) and if \(K_X + D\) is ample the solution of the corresponding (twisted) Kähler-Einstein equation then has conical singularities along \(D\) \([12, 28]\). In the opposite case when \((X, D)\) is purely log canonical, i.e. \(D\) is a reduced divisor, the corresponding twisted Kähler-Einstein metric defines a complete Kähler-Einstein metric on the quasi-projective variety \(X - D\) with cusp/Poincare type singularities along \(D\) as first shown by Kobayashi and Tian-Yau; see \([33]\) for generalizations to the mixed case and \([14]\) for the even more general setting of semi-log canonical pairs.

To any log pair \((X, D)\) we may attach a sequence of canonical probability measures \(\mu^{(N_k)}\) on \(X^{N_k}\) defined as follows. First, in the case when \((X, D)\) is klt we take \(\det S^{(k)}\) to be a generator in the determinant line of \(H^0(X, k(K_X + D))\) and multiply the local volume forms \(dz \wedge d\bar{z}\) with \(1/|s_D|^2\), where \(s_D\) is a global holomorphic (multivalued) section cutting out \(D\). In the general log canonical case we replace \(H^0(X, k(K_X + D))\) with the subspace of all “cusp forms”, i.e. the subspace of all sections vanishing along the purely log canonical part \(D_{lc}\) of \(D\) (assuming that \(k\) is sufficiently divisible). This is needed to make sure that the corresponding partition function \(Z_{N_k}\) is finite (since \(1/|s_D|^2\) is not locally integrable close to \(D_{lc}\) of \(D\). In the case when \((X, D)\) is klt the proof of the corresponding convergence results can then be deduced from the general Theorem \([14, 15]\) below, using that the pair \((X, D)\) determines a finite measure \(\mu_0\) which is absolutely continuous with respect to the Lebesgue measure (with an \(L^p\)—density, for some \(p > 1\)). However, the general logarithmic case is considerably more subtle, which is related to the fact one has to work with metrics on \(K_X + D\) which are not locally bounded (as they are singular along \(D\)). Anyway, using the recent results in \([15]\) we show how to bypass this problem.

The terminology of “cups forms” used above is borrowed from the arithmetic situation where \((X, D)\) is the Baily-Borel compactification (or one of its log resolutions) of an arithmetic quotient \(X_0\) and where the cusp forms appear as subspaces of automorphic forms (of weight \(2k\)). This leads to some intriguing relations to arithmetic geometry to which we next turn.

1.2. Cusp forms and arithmetical aspects. As emphasized above the partition functions \(Z_{N_k}\) are not intrinsically defined by the complex variety \(X\) (or a log pair
(X, D)) as they depend on the choice of a generator \( \det S^{(k)} \) of the corresponding determinant line. However, when \( X \) is defined over \( \mathbb{Z} \), in the sense of arithmetic geometry, there is indeed a canonical sequence of partition functions \( Z_{N_k} \) and its asymptotics turns out to be related to Kudla’s program and Maillot-Rössler’s conjectures in Arakelov geometry concerning the relation between arithmetic intersection numbers and derivatives of \( L \)-functions at negative integers. We will recall the arithmetic setup in section 6.1. For the moment we just recall that the point is that if \((X_\mathbb{Q}, L_\mathbb{Q})\) is a polarized variety defined over \( \mathbb{Q} \), then fixing a model \((X, L)\) for a model for \((X_\mathbb{Q}, L_\mathbb{Q})\) over \( \mathbb{Z} \) allows one to consider the corresponding determinant lines defined over \( \mathbb{Z} \) whose generators are uniquely determined up to a sign. The main case of arithmetic interest appears when \( L = K_X + D \) for a log canonical divisor \( D \) defined over \( \mathbb{Q} \) with simple normal crossings or even more specifically: when \( X \) is the Baily-Borel compactification of an arithmetic quotient \( X_0 = B/G \) of a bounded symmetric domain (for example a Shimura variety) and \( D \) is the boundary divisor \( X - X_0 \). Passing to a log resolution of \((X, D)\) we may then assume that \( X \) is smooth if we replace the relative ampleness assumption by relative semi-ampleness. Combining the convergence results above with the singular generalization of Gillet-Soulé’s arithmetic Hilbert-Samuel formula in [15] we arrive at the following

**Theorem 1.4.** Let \((X, D)\) be a log canonical pair, which is log smooth and such that the log canonical line bundle \( K_X + D \) is ample and \((X, L)\) and \((\mathcal{X}, \mathcal{L})\) a model for \((X, K_X + D)\) over \( \mathbb{Z} \). Then the corresponding canonical partition functions \( Z_{N_k} \) satisfy

\[
- \lim_{k \to \infty} \frac{1}{N_k} \log Z_{N_k} = h_{KE}(\mathcal{X}, \mathcal{L})
\]

where \( h_{KE}(\mathcal{X}, \mathcal{L}) \) denotes the height of \((\mathcal{X}, \mathcal{L})\) (i.e. the arithmetic top intersection number) with respect to the Kähler-Einstein metric. More generally, the result holds for \( K_X + D \) semi-ample and big as long as the Kähler-Einstein metric has log-log type singularities.

It should however be stressed that finding a tractable model over \( \mathbb{Z} \) is a deep arithmetic problem (compare the discussion in the introduction in [15]). In the classical case when \( X_0 \) is the modular curve with its standard model over \( \mathbb{Z} \) it was shown by Kuhn and Bost, independently, that the height \( h_{KE}(\mathcal{X}, \mathcal{L}) \) may be explicitly expressed in terms of the logarithmic derivatives of the Dedekind zeta function. Subsequently, this result was generalized by Bruinier-Burgos-Kuhn [21] to some Hilbert modular surfaces determined by a a real quadratic field \( F \), where the \( L \)-function in question is the one defined by the Dedekind zeta function of the field \( F \) (see also [15]).

1.3. General \( \beta \)-deformations of Vandermonde type determinants. The results stated above will appear as special case of the following general setting: \( L \to X \) is a big line bundle over a compact complex manifold, \( || \cdot || \) is a continuous Hermitian metric on \( L \) (whose curvature current will be denoted by \( \theta \)) and \( \mu_0 \) a finite measure on \( X \). To this data we may, after fixing a positive number \( \beta \), associate the following sequence of probability measures on \( X^{N_k} \):

\[
\mu^{(N_k, \beta)} := \frac{\|(\det S^{(k)})(x_1, x_2, \ldots, x_{N_k})\|^{2\beta/k}}{Z_{N_k, \beta}} \mu_0^N
\]

(1.6)
where now $N_k$ is the dimension of $H^0(X, L^\otimes k)$ (recall that the condition that $L$ be big simply means that $N_k$ has maximal growth, i.e. $N_k \sim k^n$) and $\det S^{(k)}$ it a generator of the corresponding determinant line, i.e. the top exterior power of $H^0(X, kL)$. We will then show that, if the back-ground measure $\mu_0$ is a singular volume form whose zeroes and poles are defined by a klt divisor $D$, then the large $N-$limit of the corresponding random point processes may be described by the solutions $u_\beta$ of the following complex Monge-Ampère equations:

\begin{equation}
\tag{1.7}
MA(u_\beta) = e^{\beta u_\beta} \mu_0
\end{equation}

where $MA(u_\beta) := (\theta + dd^c u_\beta)^n$ is the complex Monge-Ampère measure of the $\theta-$psh function $u_\beta$. In general, such an equation is globally well-defined in the weak sense of pluripotential theory and admits a unique solution $u_\beta$ (which is globally continuous and smooth on $X - D$ if $L$ is semi-ample).

**Theorem 1.5.** Let $L \to X$ be a big line equipped with a continuous Hermitian metric $\|\|\|$ and $\mu_0$ a singular volume form on $X$ (in the sense above). Fix a positive number $\beta$. Then the empirical measures of the corresponding random point processes on $X$ converge in probability towards the measure $\mu_\beta$, where $\mu_\beta = MA(u_\beta)$ for the unique solution $u_\beta$ of the complex Monge-Ampère equation (1.7). More precisely, the laws of the empirical measures of the processes satisfy a large deviation principle with speed $\beta N$ and rate functional $F_\beta(\mu)$, where $F_\beta$ is the corresponding free energy functional at inverse temperature $\beta$.

Here the free energy functional is the following functional introduced in [7]

$$F_\beta(\mu) = E(\mu) + \frac{1}{\beta} D(\mu) - C_\beta$$

where $E(= E_\theta)$ is the pluricomplex energy of $\mu$ wrt the background form $\theta$, $D(\mu)$ is the entropy of $\mu$ wrt the back-ground measure $\mu_0$ and $C_\beta$ is the normalizing constant ensuring that the infimum of $F_\beta$ (which is attained precisely at $\mu_\beta$) is equal to zero.

More generally, we point out that the convergence in the previous theorem will be shown to hold for any finite background measure $\mu_0$ of the form $e^{-v} \mu_D$, where $\mu$ is a singular volume form with zeroes/poles defined by a klt divisor and $v$ is an upper-semicontinuous function with values in $[\infty, \infty]$. The extra flexibility offered by the presence of $v$ will be needed in the setting of a variety with positive Kodaira dimension. In the case when $L$ is semi-ample we will also be able to treat the case where the divisor defining the singularity class of the volume form $\mu_0$ is a general log canonical divisor. This is needed in the canonical setting of log canonical pairs of general type (see Section 6).

The convergence result in the previous theorem can be seen as a generalization of the large deviation principle obtained in [4] for the determinantal point processes on $X$ determined by the pair $(\|\|, \mu_0)$. These are defined by a probability measure as in formula (1.6), but with $\beta/k$ replaced with 1. More generally, the results in [4] were shown to hold in the setting when $\beta/k$ is replaced by a sequence $\beta_k$ such that $1/C \leq \beta_k k$ and $\beta_k k \to \infty$. In fact, the proof of Theorem 1.5 will apply in the general setting of a sequence $\beta_k$ such that

$$\beta_k k \to \beta \in [0, \infty]$$

The results then split into three “phases” of increasing “temperature”:
the “zero-temperature phase” when $\beta = \infty$: the speed of the LDP is then $\beta_k N$ and the limit measure $\mu_\infty$ is a minimizer of the energy $E$

- the “intermediate phase” when $\beta \in [0, \infty[$: the speed of the LDP is then $\beta_k N$ and the limit measure $\mu_\infty$ is a minimizer of the free energy $F_\beta = E + D/\beta$

- The “infinite temperature phase” when $\beta = 0$: the speed is then $N$ and $\mu_\infty$ is a minimizer of the entropy $D_{\mu_0}$

Concretely, Theorem 1.5 turns out to be equivalent to the following asymptotics for the $L^{2\beta_k/k}$-norm of the generator $\det S^{(k)}$ of the determinant line of $H^0(X, kL)$ which is orthonormal wrt the $L^2$-product determined by $(\|\cdot\|, \mu_0)$:

$$\frac{1}{N_k} \log \left\| \det S^{(k)} \right\|_{L^{2\beta_k/k}(X^{N_k}, \mu_0^{\otimes N_k})} \to - \inf_{\mu \in M_1(X)} F_\beta(\mu)$$

It should be pointed out that the last phase above ($\beta = 0$) already appears in the classical probabilistic setting of Sanov’s theorem (concerning the extreme case $\beta_k = 0$). In fact, at least heuristically, the idea of the proof of Theorem 1.5 is to treat the intermediate phase $\beta \in [0, \infty[$ “interpolating” between the two extreme phases $\beta = \infty$ and $\beta = 0$. As will be explained in the next section this fits into a rather general statistical mechanical framework. But let us first point out that the results above indicate that the unique solution $u_\beta$ of the complex Monge-Ampère equation $\partial^* \partial \phi = \det S^{(k)}$ converges in the “zero-temperature limit”, i.e. as $\beta \to \infty$ to a $\theta$-psh function $u_\infty$ with the property that $MA(u_\infty)$ is the minimizer of the pluricomplex energy $E_\theta$. As shown in a separate publication this is indeed the case. In fact, in the case of an ample line bundle $L$ the convergence of $u_\beta$ towards $u$ holds in the strongest possible sense, i.e. in the $C^{1, \alpha}(X)$-topology for any $\alpha \in ]0, 1]$. The point is that, unless the back-ground curvature $\theta$ is semi-positive the limit $u_\infty$ is not in $C^2(X)$, which is reflected by the fact that $MA(u_\infty) = 1_S \theta^n$ for a proper subset $S$ of $X$. This is in sharp contrast with the picture in the case $\beta \in [0, \infty[$ where the support of $\mu_\beta := MA(u_\beta)$ is all of $X$ for any $\beta < \infty$. The appearance of an ordered structure $S$ in the limit $\beta \to \infty$ can, from a statistical mechanical point of view, be interpreted as a second order phase transition. In the case of the Riemann sphere the subset $S$ coincides with the “droplet” which appears in the setting of random normal matrices (which corresponds to $\beta_k = k$; see [4] [34] [51] and references therein).

Let us finally point out that Theorem 1.5 extends to the “weighted setting” in $\mathbb{C}^n$ where $\det S^{(k)}$ may be taken as the classical Vandermonde determinant for multivariate polynomials of degree at most $k$, where the role of the weight $\phi$ of a metric is played by a continuous function in $\mathbb{C}^n$ with super logarithmic growth (compare [4] and $\mu_0$ is the ordinary Lebesgue measure. This may be shown by embedding $\mathbb{C}^n$ in $\mathbb{P}^n$ in the usual way; the details will appear elsewhere.

### 1.4. A large deviation result for singular Hamiltonians

A key ingredient in the proof of the results above (or more precisely Theorem 1.5) is a general result, of independent interest, concerning interacting particle systems in thermal equilibrium, or more precisely a large deviation result for Gibbs measures associated to singular Hamiltonians. Let us first recall the general setting where one is given a topological space $X$ equipped with a finite measure $\mu_0$. For a fixed positive integer $N$ (representing the number of particles) we assume given an $N$-particle Hamiltonian $H^{(N)}$, i.e. a function on the $N$-fold product $X^N$ which is symmetric, i.e. invariant
there exists a sequence \( \beta - 1 \). The corresponding Gibbs measure is the probability measure on \( X^N \) defined by
\[
\mu_{\beta N}^{(N)} := e^{-\beta_N H^{(N)}} \mu_0 \otimes N / Z_{\beta N},
\]
where we have also fixed a positive number \( \beta_N \) (the inverse temperature) and where \( Z_N \) is the normalizing constant (partition function), i.e.
\[
Z_{\beta N} := \int_{X^N} e^{-\beta_N H^{(N)}} \mu_0 \otimes N
\]
Given a continuous function \( u \) on \( X \) we denote by \( Z_{\beta N}[u] \) the “tilted” partition function obtained by replacing \( H^{(N)} \) with \( H^{(N)} + u \), where \( u(x_1, ..., x_N) := \sum u(x_i) \).
In other words \( Z_{\beta N}[u] \) is the (scaled) Laplace transform of the law of the empirical measure \( \delta_N \) defined by the Gibbs measure associated to \( H_N \). We will be particular, interested in the case when the Hamiltonian \( H^{(N)} \) is singular. Indeed, in the application to the complex geometric setting above
\[
H^{(N)}(x_1, x_2, ..., x_N) := -\frac{1}{k} \log \left\| (\det S^{(k)}) (x_1, x_2, ..., x_N) \right\|^2,
\]
which clearly blows-up when two points merge; more precisely, \( H^{(N_k)} = \infty \) along the hyper surface in \( X^N \) cut out by \( \det S^{(k)} \). The key property that we will use is that Hamiltonian \( [18] \) is uniformly quasi-superharmonic in the following sense: there exists a Riemannian metric \( g \) on the compact manifold \( X \) such that
\[
\Delta_{x_1} H^{(N_k)}(x_1, x_2, ..., x_N) \leq C
\]
on \( X^N \) for some uniform constant \( C \) (independent of \( N \)) where \( \Delta_{x_1} \) denotes the Laplacian, determined by \( g \), acting on the first variable (equivalently, \( H^{(N_k)}/N \) is uniformly quasi-super harmonic as a function on the Riemannian product \( X^N \)).

**Theorem 1.6.** Let \( H^{(N)} \) be an \( N \)-particle Hamiltonian on a compact Riemannian manifold \( (X,g) \) and denote by \( \mu_0 \) the corresponding volume form. Assume that there exists a sequence \( \beta_N \) of positive real numbers tending to infinity such that
\[
-\frac{1}{\beta_N} \log Z_{\beta N}[u] \text{ converges, when } N \to \infty,
\]
to a functional \( F(u) \) which is Gateaux differentiable on \( C^0(X) \) and that \( H^{(N)} \) is quasi-superharmonic. Then, for any fixed \( \beta > 0 \), the measure \( (\delta_N)_* e^{-\beta_N H^{(N)}} \) satisfies an LDP with speed \( \beta N \) and good rate functional
\[
F_\beta(\mu) = E(\mu) + \frac{1}{\beta} D_{\mu_0}(\mu)
\]
where the functional \( E(\mu) \) is the Legendre transform of \( F(u) \) (formula [5.3]) and \( D_{\mu_0}(\mu) \) is the entropy of \( \mu \) relative to \( \mu_0 \). In particular, the empirical measures \( \delta_N \) of the corresponding random point processes converge in probability to the deterministic measure given by the unique minimizer \( \mu_\beta \) of \( F_\beta \).

In fact, under a mild regularity assumption on the functional \( F_\beta \) the result extends to more general back-ground measures \( \mu_0 \) and non-compact spaces \( X \) (compare the proofs of Theorem [18] and Theorem [03]). In the complex geometric setting we can take \( \beta_N \) to be equal to \( k \) and deduce the Gateaux differentiability from the results in [10] (compare [14]); the point is that the corresponding point processes is then determinantal, which allows one to use various \( L^2 \)-tools such as Bergman reproducing kernels.
The previous theorem in particular generalizes (in the case $\beta > 0$) the mean field type results in [22, 36] concerning mathematical models for turbulence in two real dimensions (which in turn extend to a singular setting previous results in [39]). In the latter setting the Hamiltonian $H^{(N)}$ is of the explicit form

$$H^{(N)}(x_1, \ldots, x_N) = -\frac{1}{N(N-1)} \sum_{1 \leq i < j \leq N} G(x_i, x_j)$$

for a symmetric function $g$ (independent of $N$) - in fact, $G$ is the Green function of the corresponding Laplacian, which is thus singular along the diagonal. However, while the analysis in [22, 36] reduce, thanks to the explicit formula above, to properties of the function $G$, the main point of the present approach is that it applies in situations where $H^{(N)}(x_1, \ldots, x_N)$ does not admit any tractable explicit formula, as in the complex geometric setting above. In fact, a tractable formula does exist in one complex dimension, namely the so called bosonization formula on a Riemann surface which involves explicit theta functions and regularized determinants of Laplacians (the formula was used in a related large deviation setting in [52]). The bosonization formula is particularly useful in the case $\beta < 0$ on a Riemann surface as will be explained in a separate publication.

Acknowledgement. It is a pleasure to thank Sebastien Boucksom, David Witt-Nyström, Vincent Guedj and Ahmed Zeriahi for the stimulating collaborations [10, 11, 12], which paved the way for the present work. I am also grateful to Bo Berndtsson for infinitely many fruitful discussions on complex analysis and Kähler geometry over the years. This work was partly supported by an ERC grant.

Organization. In section 2 we prove the submean inequality in large dimensions, which will play a key role in the subsequent section 3 where a Large Deviation Principle is established for Gibbs measures with singular Hamiltonians. In Section 4 we introduce the complex geometry setting, first proving the LDP in the general setting of a big line bundle and then in the canonical setting of a klt pair of log general type. In the following Section 5 we show how to deal with any variety of positive Kodaria dimension, using Fujino-Mori’s canonical bundle formula. In Section 6 we confront the new difficulties that appear in the setting of a log canonical pair and then in the final Section 7 we show how to define canonical point processes on Fano manifolds and discuss the relations to a new notion of stability, stating a number of conjectures. The paper is concluded with an appendix where the dimensional dependence on the constant in the Cheng-Yau gradient estimate is obtained, by tracing through the usual proof.

2. Submean inequalities in large dimension

2.1. Setup. Let $(X, g)$ be compact $n$-dimensional Riemannian manifold (or more generally a complete one) and assume that

$$\text{Ric } g \geq -\kappa^2(n-1)$$

for some positive constant $\kappa$ (sometimes referred to as the normalized lower bound on the Ricci curvature). Let $G$ a finite group acting by isometries on $X$ and denote by $M := X/G$ the corresponding quotient equipped with the distance function induced by the metric $g$, i.e.

$$d_M(x, y) := \inf_{\gamma \in G} d_X(x, \gamma y),$$

11
where $d_X$ is the Riemannian distance function on $(X, g)$. Even though the quotient $M$ is not a manifold in general (since $G$ will in general have fixed points) it still comes with a smooth structure in the following sense. Denote by $p$ the natural projection map from $X^N$ to $M$. Using the projection $p$ we can identify a function $f$ on $M$ with $G$–invariant function $p^* f$ on $X$ and accordingly we say that $f$ is smooth if $p^* f$ is. Similarly, there is a natural notion of Laplacian $\Delta_0$ on the quotient $M$ : the Laplacian $\Delta u$ of a locally integrable function $u$ on $M$ is the signed Radon measure defined by

$$\int_M (\Delta u)f := \frac{1}{|G|} \int_X p^* u \Delta (p^* f)$$

for any smooth function $f$ on $M$. More generally, by localization, this setup naturally extends to the setting of Riemannian orbifolds (see [19]), but the present setting of global quotients will be adequate for our purposes.

### 2.2. Statement of the submean inequality.

**Theorem 2.1.** Let $(X, g)$ be a (complete) Riemannian manifold of dimension $n$ such that $\text{Ric}g \geq -\kappa^2(n-1)$ and $G$ a finite group acting by isometries on $X$. Denote by $M := X/G$ the corresponding quotient equipped with the distance function induced by the metric $g$ and let $v$ be a non-negative function on $M$ such that $\Delta_g v \geq -\lambda^2 v$ for some non-negative constant $\lambda$. Then, for any $\delta > 0$ and $\epsilon \in [0, 1]$ there exist constants $A$ and $C$ such that

$$\sup_{B_\delta(x_0)} v^2 \leq A e^{2\lambda \delta} e^{Cn(\delta + \epsilon)} \int_{B_\delta(x_0)} v^2 dV,$$

where $C$ only depends on an upper bound on $\kappa$ and $A$ only depends on $\delta$ and $\epsilon$.

The previous theorem will be applied in the complex geometric setting though the following

**Corollary 2.2.** Let $L \rightarrow X$ be a holomorphic line bundle over a compact complex manifold $X$ and $\|\cdot\|$ a Hermitian metric on $L$ and $g$ a Kähler metric on $X$. Let $s^{(N)}$ be a sequence of holomorphic sections of $(kL)^{\otimes N} \rightarrow X^N$ such that $s^{(N)}$ is symmetric, i.e. invariant under the permutation group on $N$ letters. Fix a positive number $\beta$. Then, for any $\delta > 0$ there exist constants $C_\delta$ and $C$ (where $C$ is independent of $\delta$) such that

$$\left\|s^{(N)}(x_1, x_2, \ldots, x_N)\right\|^{\beta/k} \leq C_\delta e^{Cn\delta} \int_{B_\delta} \left\|s^{(N)}(x_1, x_2, \ldots, x_N)\right\|^{\beta/k} dV^{\otimes N},$$

where $B_r$ denote the pull-back to $X^N$ of a ball of radius $r$ centered at the image of $(x_1, x_2, \ldots, x_N)$ in the quotient $X^{(N)}$ equipped with the scaled distance function $d^{(N)}$ induced by $g$. Here $dV$ is any fixed volume form on $X$.

**Proof.** First observe that we may as well assume that $dV$ is the volume form $dV_g$ of the metric $g$. Indeed, $dV = e^{-u\beta} dV_g$ for some smooth function $u$ and hence changing $dV$ corresponds to changing the metric $\|\cdot\|$ to $\|\cdot\| e^{-u}$. Next, since $\log \|s\|^2$ is $k\omega$–psh for any holomorphic section $s$ of $kL \rightarrow X$ (where $\omega$ is the curvature form of $\|\cdot\|$) we get $\Delta_g \log \|s\|^{2/k} \geq -\lambda$ for some positive constant $\lambda$ and applying this inequality $N$ times on $X^N$ gives $\Delta_g \log \|s^{(N)}\|^{2/k} \geq -N\lambda$ on $X^N$. As a consequence $\Delta_g \left\|s^{(N)}\right\|^{2\beta} \geq -\beta N\lambda \left\|s^{(N)}\right\|^{2\beta}$ on $X^N$. After rescaling the metric $g$ (i.e. replacing
2.3. Proof of the submean inequality in Theorem 2.1. We will follow closely the elegant proof of Li-Schoen [38] of a similar submean inequality. But there are two new features here that we have to deal with:

- We have to make explicit the dependence on the dimension $n$ of all constants and make sure that the final contribution is sub-exponential in $n$
- We have to adapt the results to the singular setting of a Riemannian quotient

Before turning to the proof we point out that it is well-known that submean inequalities with a multiplicative constant $C(n)$ do hold in the more general singular setting of Alexandrov spaces (with a strict lower bound $-\kappa$ on the sectional curvature). But it seems that the current proofs (see for example [35]), which combine local Poincaré and Sobolev inequalities with the Moser iteration technique, do not give the subexponential dependence on $C(n)$ that we need.

We recall that the two main ingredients in the proof of the result of Li-Schoen referred to above is the gradient estimate of Cheng-Yau [25] and a Poincaré-Dirichlet inequality on balls. Let us start with the gradient estimate that we will need:

**Proposition 2.3.** Let $u$ be a harmonic function on $B_a(x_0)$ in $M$. Set $\rho_{x_0}(x) := d(x, x_0)$ (the distance between $x$ and $x_0$). Then

$$\sup_{B_a(x_0)} (|\nabla \log |u| \rho_{x_0} - a)) \leq Cn(1 + \kappa a) \quad (C_n \leq Cn)$$

for some absolute constant $C$ (in particular, independent of $n$, \(\kappa\) and $a$).

**Proof.** In the smooth case this is the celebrated Cheng-Yau gradient estimate [25]. The result is usually stated without an explicit estimate of the multiplicative constant $C_n$ in terms of $n$, but tracing through the proof in [25] gives $C_n \leq Cn$ (see the appendix in the present paper and also [1] for a probabilistic proof providing an explicit constant). We claim that the same estimate holds in the singular setting using a lifting argument. To see this recall that the usual proof of the gradient estimate proceeds as follows (see the appendix). Set $\phi(x) := |\nabla \log |u| |(= |\nabla u|/u)$ and $F(x) := \phi(x)(\rho_{x_0} - a)^2$. Then $F$ attains its maximum in a point $x_1$ in the interior of $B_a(x_0)$ (otherwise $|\nabla u|$ vanishes identically and then we are trivially done). Hence, $F(x) \leq F(x_1)$ on some neighborhood $U$ of $x_1$. Now, in case $F$ (or equivalently $\rho_{x_0}$) is smooth on $U$ we get $\Delta F \leq 0$ and $\nabla F = 0$ at $x_1$. Calculating $\Delta F$ and using Bochner formula and Laplacian comparison then gives

$$\phi(x_1)(\rho_{x_0}(x_1) - a) \leq Cn(1 + \kappa(x_0)a) \tag{2.1}$$

which is the desired estimate. In the case when $\rho_{x_0}$ is not smooth on $U$, i.e. $x_1$ is contained in the cut locus of $x_0$ one first replaces $\rho_{x_0}$ with a smooth approximation $\rho_{x_0}^{(c)}$ of $\rho_{x_0}$ (which is a local barrier for $\rho_{x_0}$) and then lets $\epsilon \to 0$ to get the same conclusion as before. In the singular case $M = X/G$ we proceed as follows. First we identify $F$ with a $G$–invariant function on the inverse image of $B_R(x_0)$ in $X$ (and $x_0$ and $x_1$ with fixed points in the corresponding $G$–orbits) and set $\tilde{F} := (x)(\tilde{\rho}_{x_0} - a)^2$, where $\tilde{\rho}_{x_0} := d_X(x_0, x_1)$. By definition $\tilde{\rho}_{x_0} \geq \rho_{x_0}$ on $X$ and, after possibly changing the lift of the point $x_1$ we may assume that $\tilde{\rho}_{x_0} = \rho_{x_0}$ at $x = x_1$. In particular, $\tilde{F} \leq F$ on $U$ and $\tilde{F} = F$ at $x_1$ and hence $\tilde{F}$ also has a local maximum at $x_1$. 

\[\square\]
But then the previous argument in the smooth case gives that $2.1$ holds with $\rho_{x_0}$ replaced by $\tilde{\rho}_{x_0}$. But since the two functions agree at $x_1$ this concludes the proof in the general case. □

**Corollary 2.4.** Let $h$ be a positive harmonic function on $B_\delta(x_0)$. Then there exists a constant $C$ only depending on an upper bound on $\kappa$ such that

$$\sup_{B_{\delta}(x_0)} h^2 \leq e^{Cn} \frac{\int_{B_{\delta}(x_0)} h^2 dV}{\int_{B_{\delta}(x_0)} dV}$$

for $0 < \epsilon < 1$.

**Proof.** Set $v := \log h$ and fix $x \in B_{\delta}(x_0)$. Integrating along a minimizing geodesic connecting $x_0$ and $x$ and using the gradient estimate in the previous proposition gives

$$|v(x) - v(x_0)| \leq C n \int_0^\delta \frac{1}{\delta - t} dt = C n (\log(\delta - 0) - \log(\delta - \epsilon \delta)) = -C n \log(1 - \epsilon).$$

In particular, for any two points $x, y \in B_{\delta}(x_0)$ we get $|v(x) - v(y)| \leq |v(x) - v(x_0)| + |v(y) - v(x)| \leq -2 C n \log(1 - \epsilon)$, i.e. $h(x) \leq (1 - \epsilon)^{-2 C n} h(y)$. In particular, $\sup_{B_{\delta}(x_0)} h^2 \leq (1 - \epsilon)^{-4 C n} \inf_{B_{\delta}(x_0)} h^2$, which implies the proposition after renaming the constant $C$. □

The second key ingredient in the proof of Theorem 2.1 is the following Poincaré-Dirichlet inequality:

**Proposition.** Let $f$ be a smooth function on $B_\delta(x_0)$ vanishing on the boundary. Then

$$\int_{B_\delta(x_0)} |f|^2 dV_g \leq 4 e^{C n \delta} \int_{B_\delta(x_0)} |f|^2 dV_g$$

where the constant $C$ only depends on an upper bound on $\kappa$.

**Proof.** We follow the proof in [38] with one crucial modification (compare the remark below). To fix ideas we first consider the case of a Riemannian manifold. Fix a point $p$ in the boundary of the ball $B_1(x_0)$ and denote by $r_1(x)$ the distance between $x \in M$ and $p$. From the standard comparison estimate for the Laplacian we get

$$\Delta r_1 \leq (n - 1) \left( \frac{1}{r_1} + \kappa \right)$$

(in the weak sense and point-wise away from the cut locus of $p$). In particular, for any positive number $a$ we deduce the following inequality on $B_\delta(x_0)$ (using that $g(\nabla r_1, \nabla r_1) = 1$ a.e.)

$$\Delta_g(e^{-a r_1}) = a e^{-a r_1} (a - \Delta r_1) \geq a e^{-a (1 + \delta)} \left( a - (n - 1) \left( \frac{1}{(1 - \delta)} + \kappa \right) \right)$$

Hence, setting $a := n \left( \frac{1}{(1 - \delta)} + \kappa \right)$ gives

$$\Delta_g(e^{-a r_1}) \geq a e^{-a (1 + \delta)} \left( \frac{1}{(1 - \delta)} + \kappa \right) > 0$$
Multiplying by $|f|$ and integrating once by parts (and using that $\|\nabla r_1\| \leq 1$) we deduce that

$$a \int_{B_\delta(x_0)} |\nabla f| e^{-a r_1} dV \geq a \left( \frac{1}{(1-\delta)} + \kappa \right) \int_{B_\delta(x_0)} |f| e^{-a(1+\delta)} dV.$$  

Estimating $e^{-a r_1} \leq e^{-a(1-\delta)}$ in the lhs above and rearranging gives

$$\int_{B_\delta(x_0)} |\nabla f| dV e^{2a\delta} \left( \frac{1}{(1-\delta)} + \kappa \right)^{-1} \geq \int_{B_{\delta}(x_0)} |f| dV,$$  

(using that $g(\nabla r_1, \nabla r_1) \leq 1$ in the sense of upper gradients). This shows that the $L^1$--version of the Poincaré inequality in question holds with the constant $\left( \frac{1}{(1-\delta)} + \kappa \right)^{-1} e^{2\delta n/(1-\delta)}$, which for $\delta$ sufficiently small is bounded from above by $e^{n(1+2\kappa)\delta}$. The general Riemannian $L^2$--Poincare inequality now follows from replacing $|f|$ with $|f|^2$ and using Hölder’s inequality. Finally, in the case of the a Riemannian quotient $M$ we can proceed exactly as above using that the Laplacian comparison estimate in formula $[2.12]$ is still valid. Indeed, the pull-back $p^*r_1$ of $r_1$ to $X$ is an infimum of functions for which the corresponding estimate holds (by the usual Laplacian comparison estimate and the assumption that $G$ acts by isometries). But then the estimate also holds for the function $p^*r_1$, by basic properties of Laplacians. More generally, the required Laplacian comparison estimate was shown in $[19]$ for general Riemannian orbifolds.

\[\square\]

**Remark 2.5.** The only difference from the argument used in $[38]$ is that we have taken the point $p$ to be of distance 1 from $x_0$ rather than distance $2\delta$, as used in $[38]$. For $\delta$ small this change has the effect of improving the exponential factor from $e^{n(1+2\kappa)}$ to $e^{n(1+\delta\kappa)}$, which is crucial as we need a constant in the Poincare inequality which has subexponential growth in $n$ as $\delta \to 0$.

### 2.3.1. End of proof of Theorem 2.1

Let us first consider the case when $\lambda = 0$. Denote by $h$ the harmonic function on $B_\delta$ coinciding with $v$ on $\partial B_\delta$. By Cor 2.4 and the subharmonicity of $v$

$$\sup_{B_{\delta}(x_0)} v^2 \leq e^{C_n} \frac{\int_{B_{\delta}(x_0)} |h|^2 dV_g}{\int_{B_{\delta}(x_0)} dV_g}.$$  

Next, by the triangle inequality

$$\int_{B_{\delta}(x_0)} |h|^2 dV_g / 2 \leq \int_{B_{\delta}(x_0)} |h-v|^2 dV + \int_{B_{\delta}(x_0)} |v|^2 dV$$

Since $h - v$ vanishes on the boundary of $B_{\delta}(x_0)$ applying the Poincaré inequality in Prop 2.3 then gives

$$\int_{B_{\delta}(x_0)} |h-v|^2 dV \leq A e^{B_{n\delta}} \int_{B_{\delta}(x_0)} |\nabla h-v|^2 dV \leq 2 A e^{B_{n\delta}} \int_{B_{\delta}(x_0)} |\nabla h|^2 + |\nabla v|^2 dV$$

But $h$ is the solution to a Dirichlet problem and as such minimizes the Dirichlet norm $\int_{B_{\delta}(x_0)} |\nabla h|^2$ over all subharmonic functions with the same boundary values as $h$. Accordingly,

$$\int_{B_{\delta}(x_0)} |h-v|^2 dV \leq A e^{B_{n\delta}} \int_{B_{\delta}(x_0)} |\nabla v|^2 dV$$
Finally, using that \( v \) is subharmonic we get
\[
\int_{B_{\delta}(x_0)} |\nabla v|^2 dV \leq C_5 \int_{B_{2\delta}(x_0)} |v|^2 dV
\]
(as is seen by multiplying with a suitable smooth function \( \chi \) supported on \( B_{2\delta} \) such that \( \chi = 1 \) on \( B_\delta \)). All in all this concludes the proof of Theorem 2.1 in the case \( \lambda = 0 \).

Finally, to handle the general case (i.e. \( \lambda \neq 0 \)) we set \( N := M \times [-1,1] \) equipped with the standard product metric and apply the previous case to the function \( \nu e^{\lambda t} \) to get
\[
\sup_{B_\epsilon(x_0) \subset N} v^2 e^{2\lambda t} \leq A_\delta e^{B_n(\delta+\epsilon)} \frac{\int_{B_{2\delta}(x_0) \subset N} v^2 e^{2\lambda t} dV}{\int_{B_{\epsilon}(x_0) \subset N} dV}.
\]
But restricting the sup in the lhs to \( B_\delta(x_0) \times \{0\} \) and using that \( B_{\epsilon\delta/2}(x_0,0) \times [-\epsilon\delta/2,\epsilon\delta/2] \subset B_{\delta}(x_0,0) \) and \( B_{2\delta}(x_0,0) \subset B_{2\delta}(x_0,0) \times [2\delta,2\delta] \) gives
\[
\sup_{B_{\epsilon}(x_0) \subset M} v^2 \leq A_{\delta,\epsilon} e^{2\lambda\delta + \epsilon} \frac{\int_{B_{2\delta}(x_0) \subset M} v^2 dV}{\int_{B_{\epsilon}(x_0) \subset M} dV},
\]
which concludes the proof of the general case (after rescaling suitably).

3. A LARGE DEVIATION PRINCIPLE FOR GIBBS MEASURES WITH SINGULAR HAMILTONIAN

3.1. Setup: the Gibbs measure \( \mu^{(N)} \) associated to the Hamiltonian \( H^{(N)} \).

Let \( X \) be a compact topological space. A random point process with \( N \) particles is by definition a probability measure \( \mu^{(N)} \) on the \( N \)-particle space \( X^N \) which is symmetric, i.e. invariant under permutations of the factors of \( X^N \). Its \( j \)-point correlation measure \( \mu_j^{(N)} \) is the probability measure on \( X^j \) defined as the push forward of \( \mu^{(N)} \) to \( X \) under the map \( X^N \to X^j \) given by projection onto the first \( j \) factors (or any \( j \) factors, by symmetry):
\[
\mu_j^{(N)} := \int_{X^{N-j}} \mu^{(N)}.
\]

In the following we will denote by \( \mathcal{M}_1(Y) \) the space of all probability measures on a space \( Y \) and we will be particularly concerned with the case when \( Y = X^N \). In the latter case we will usually use the notation \( \mu_N \) for (not necessarily symmetric) elements of \( \mathcal{M}_1(\mu_N) \) and reserve the notation \( \mu^{(N)} \) for specific Gibbs measures defined below. The empirical measure of a given random point process is the following random measure
\[
(3.1)\quad \delta_N : X^N \to \mathcal{M}_1(X), \ (x_1, \ldots, x_N) \mapsto \delta_N(x_1, \ldots, x_N) := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}
\]
on the ensemble \( (X^N, \mu^{(N)}) \). By definition the law of \( \delta_N \) is the push-forward of \( \mu^{(N)} \) to \( \mathcal{P}(X) \) under the map \( \delta_N \), which thus defines a probability measure on \( \mathcal{M}_1(X) \). Now fix a background measure \( \mu_0 \) on \( X \) and let \( H^{(N)} \) be a given \( N \)-particle Hamiltonian, i.e. a symmetric function on \( X^N \), which we will assume is lower semi-continuous (and in particular bounded from below, since \( X \) is assumed compact). Also fixing a positive number \( \beta \) the corresponding Gibbs measure (at inverse
temperature \( \beta \) is the symmetric probability measure on \( X^N \) defined as

\[
\mu^{(N,\beta)} := \frac{e^{-\beta H^{(N)}}}{Z_N} \mu_0^\otimes N,
\]

where the normalizing constant

\[
Z_{N,\beta} := \int_{X^N} e^{-\beta H^{(N)}} \mu_0^\otimes N
\]

is called the \((N\text{–particle})\) partition function.

### 3.2. Preliminaries (mean entropy, energy and free energy)

First we recall the general definition of the relative entropy (or the Kullback–Leibler divergence) of two measures \( \nu_1 \) and \( \nu_2 \) on a space \( Y \): if \( \nu_1 \) is absolutely continuous with respect to \( \nu_2 \), i.e. \( \nu_1 = f \nu_2 \), one defines

\[
D(\nu_1, \nu_2) := \int_Y \log(\nu_1/\nu_2) \nu_1
\]

and otherwise one declares that \( D(\mu) := \infty \). Note the sign convention used: \( D \) is minus the physical entropy. Next, we define the mean entropy (relatively \( \mu_0^\otimes N \)) of a probability measure \( \mu_N \) on \( X^N \) (i.e. \( \mu_N \in M_1(X^N) \)) as

\[
D^{(N)}(\mu_N) := \frac{1}{N} D(\mu_N, \mu_0^\otimes N).
\]

When \( N = 1 \) we will simply write \( D(\mu) := D^{(1)}(\mu) = D(\mu, \mu_0) \). On the other hand the mean energy of \( \mu_N \) is defined as

\[
E^{(N)}(\mu_N) := \frac{1}{N} \int_{X^N} H^{(N)} \mu_N
\]

Finally, the mean (Gibbs) free energy functional on \( M_1(X^N) \) is now defined as

\[
F^{(N)} := E^{(N)} + \frac{1}{\beta} D^{(N)}
\]

Next, we will collect some basic general lemmas. First we have the following simple special case of the well-known sub-additivity of the entropy.

**Lemma 3.1.** The following properties of the entropy hold:

- \( D(\nu_1, \nu_2) \geq 0 \) with equality iff \( \nu_1 = \nu_2 \)
- For a product measure on \( X^N \)

\[
D^{(N)}(\mu^\otimes N) = D(\mu)
\]

- More generally,

\[
D^{(N)}(\mu_N) \geq D(\mu_{N,1}),
\]

where \( \mu_{N,1} \) is the corresponding first marginal (one point correlation measure) on \( X \).

The proof of the previous lemma uses only the (strict) concavity of the function \( t \mapsto \log t \) on \( \mathbb{R} \) (see for example [36]). The latter (strict) concavity also immediately gives the following
Lemma 3.2. (Gibbs variational principle). Fix $\beta > 0$. Given a function $H^{(N)}$ on $X^N$ and a measure $\mu_0$ on $X$, the corresponding free energy functional $F^{(N)}$ on $\mathcal{M}_1(X^N)$ attains its minimum value on the corresponding Gibbs measure $\mu^{(N)}_\beta$ and only there. More precisely,

$$\inf_{\mathcal{M}_1(X^N)} \beta F^{(N)} = \beta F^{(N)}(\mu^{(N)}_\beta) = -\frac{1}{N} \log Z_N$$

Proof. We recall the simple proof: since $\log(ab) = \log a + \log b$, we have

$$F^{(N)}(\mu_N) = \frac{1}{\beta N} \int_{X^N} \log(\mu_N/e^{-\beta H^{(N)}(\mu_N)} \mu_0) \mu_N = \frac{1}{\beta N} \int_{X^N} \log(\mu_N/\mu^{(N)}_\beta \mu_0) - \frac{1}{N} \beta \log Z_N$$

which proves the lemma using Jensen’s inequality (i.e. the first point in the previous lemma). \qed

Note that the same argument applies if $\beta < 0$, since we can simply replace $H^N$ with $-H^N$ in the previous argument (as long as the corresponding partition function $Z_N$ is finite).

3.3. Definition of a general Large Deviation Principle (LDP). The statement that the empirical measure $\delta_N$ of random point process converges in probability towards a deterministic measure $\mu$ equivalently means that the law of $\delta_N$, which defines a probability measure $\Gamma_N$ on the space $\mathcal{M}_1(X)$, converges weakly towards $\delta_\mu$, the Dirac mass at the point $\mu \in \mathcal{M}_1(X)$. The notion of a Large Deviation Principle (LDP), introduced by Varadhan, allows one to gives a stronger notion of exponential convergence, which can be seen as an infinite dimensional version of the Laplace principle [26].

Let us first recall the general definition of a Large Deviation Principle (LDP) for a general sequence of measures.

Definition 3.3. Let $\mathcal{P}$ be a Polish space, i.e. a complete separable metric space.

(i) A function $I : \mathcal{P} \to ]-\infty, \infty]$ is a rate function if it is lower semi-continuous. It is a good rate function if it is also proper.

(ii) A sequence $\Gamma_k$ of measures on $\mathcal{P}$ satisfies a large deviation principle with speed $r_k$ and rate function $I$ if

$$\limsup_{k \to \infty} \frac{1}{r_k} \log \Gamma_k(F) \leq -\inf_{\mu \in F} I$$

for any closed subset $F$ of $\mathcal{P}$ and

$$\liminf_{k \to \infty} \frac{1}{r_k} \log \Gamma_k(G) \geq -\inf_{\mu \in G} I(\mu)$$

for any open subset $G$ of $\mathcal{P}$.

We will be mainly interested in the case when $\Gamma_k$ is a probability measure (which implies that $I \geq 0$ with infimum equal to 0). Then it will be convenient to use the following alternative formulation of a LDP (see Theorems 4.1.11 and 4.1.18 in [26]):

Proposition 3.4. Let $\mathcal{P}$ be a compact metric space and denote by $B_\epsilon(\nu)$ the ball of radius $\epsilon$ centered at $\nu \in \mathcal{P}$. Then a sequence $\Gamma_k$ of probability measures on $\mathcal{P}$ satisfies a LDP with speed $r_k$ and a rate functional $I$ iff

$$-\lim_{\epsilon \to 0} \liminf_{N \to \infty} \frac{1}{r_k} \log \Gamma_k(B_\epsilon(\nu)) = I(\nu) = -\lim_{\epsilon \to 0} \limsup_{N \to \infty} \frac{1}{r_k} \log \Gamma_k(B_\epsilon(\nu))$$

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We will also use the following classical result of Sanov, which is the standard example of a LDP for point processes.

**Proposition 3.5.** Let $X$ be a topological space and $\mu_0$ a finite measure on $X$. Then the law $\Gamma_N$ of the empirical measures of the corresponding Gibbs measure $\mu_0^{\otimes N}$ (i.e. $H^{(N)} = 0$) satisfies an LDP with speed $N$ and rate functional the relative entropy $D_{\mu_0}$.

### 3.4. A LDP for quasi-superharmonic Hamiltonians.

It will be convenient to identify a continuous function $u$ on $X$ with a function on the products $X^N$:

$$u(x_1, ..., x_N) := \sum_{i=1}^N u(x_i)$$

In the following we will assume that $(X,g)$ is an $m-$dimensional compact Riemannian manifold and $H^{(N)}$ is a Hamiltonian satisfying the following assumptions:

- $H^{(N)}$ is uniformly quasi super-harmonic, i.e. $\Delta_x H^{(N)}(x_1, x_2, ... x_N) \leq C$ on $X^N$

- The following limit exists for any $u \in C^0(X)$:

$$\mathcal{F}(u) := \lim_{N \to \infty} \inf_{X^N} \frac{1}{N}(H^{(N)} + u),$$

and $\mathcal{F}$ is Gateaux differentiable on $C^0(X)$

- Regularization assumption: for any measure $\nu$ such that $E(\nu) < \infty$ there exists a sequence $\nu_j$ in the image $(d\mathcal{F})(C^0(X))$ such that $\nu_j \to \nu$ weakly and $E(\nu_j) \to E(\nu)$

Given an element $\mu$ in the space $M_1(X)$ of probability measures on $X$ we define its thermodynamic energy $E(\mu)$ as the Legendre transform of $\mathcal{F}$ at $\mu$ (with a somewhat non-standard sign convention):

$$E(\mu) := \sup_{u \in C^0(X)} (\mathcal{F}(u) - \int_X u \nu) := \sup_{u \in C^0(X)} \mathcal{F}_\mu$$

The corresponding free energy functional at inverse temperature $\beta$ (relative to $\mu_0$) is defined by

$$F(\mu) := E(\mu) + \frac{1}{\beta} D_{\mu_0}(\mu)$$

We will rely on the following simple but very useful lemma (which was used in the similar context of Fekete points in [11]).

**Lemma 3.6.** Fix $u_* \in C^0(X)$ and assume that $x_*^{(N)} \in X^N$ is a minimizer of the function $(H^{(N)} + u_*)/N$ on $X^N$. If the corresponding large $N-$ limit $\mathcal{F}(u)$ exists for all $u \in C^0(X)$ and $\mathcal{F}$ is Gateaux differentiable at $u_*$, then $\delta_N(x_*^{(N)})$ converges weakly towards $\mu_* := d\mathcal{F}_{|u_*}$.

**Proof.** Fix $v \in C^0(X)$ and a real number $t$. Let $f_N(t) := \frac{1}{N}(H^{(N)} + u_*)(x_*^{(N)})$ and $f(t) := \mathcal{F}(u + tv)$. By assumption $\lim_{N \to \infty} f_N(0) = f(0)$ and $\lim_{N \to \infty} f_N(t) \geq f(t)$. Note that $f$ is a concave function in $t$ (since it is defined as an inf of affine functions) and $f_N(t)$ is affine in $t$. Then it follows from the differentiability of $f$ at $t = 0$ that $\lim_{N \to \infty} df_N(t)/dt|_{t=0} = df(t)/dt|_{t=0}$, i.e. that

$$\lim_{N \to \infty} \left\langle \delta_N(x_*^{(N)}), v \right\rangle = \left\langle d\mathcal{F}_|u_*, v \right\rangle.$$
which thus concludes the proof of the lemma.

We can now prove the LDP for Hamiltonians satisfying the assumptions above. In fact, we will provide two different proofs, which mainly differ in the (easiest) part, which concerns the “upper bound”. The main reason for including two proofs is that they have different merits when considering more singular situations, or the case when $X$ is non-compact. Another reason for including the second proof, which is based on Gibbs variational principle, is that, in the complex geometric setting, it gives a direct link to the variational principles which play a prominent role in complex geometry. For example, the latter proof directly shows that the limit of the corresponding one-point correlation measures converge to a minimizer of the free energy functional.

**Theorem 3.7.** Let $(X,g)$ be a compact Riemannian manifold and $H^{(N)}$ a sequence of Hamiltonians satisfying the assumptions in the beginning of the section. Let $\mu_0$ be a volume form on $X$ and denote by $\mu^{(N,\beta)}$ the corresponding Gibbs measures. Then $(\delta_N)_* e^{-\beta H^{(N)}} \mu_0^{\otimes N}$ and $\mu^{(N,\beta)}$ satisfy LDPs with speed $\beta N$ and rate functionals $F_\beta$ and $F_\beta + C_\beta$, respectively, where $F_\beta$ is the corresponding free energy functional and $C_\beta$ is a constant. In particular,

$$-\frac{1}{\beta N} \log Z_{N,\beta} \to \inf_{M_1(X)} F_\beta$$

**Proof 1 (using asymptotics on small balls).** Fix a metric on $M_1(X)$ defining the weak topology and denote by $B_\epsilon(\mu)$ the corresponding ball of radius $\epsilon$ centered at $\mu \in M_1(X)$. Abusing notation slightly we will also denote by $B_\epsilon(\mu)$ the inverse image of $B_\epsilon(\mu)$ in $X^N$ under the map $\delta_N$ (defined in formula (3.1)). In the case when $\mu = \delta_N(x(N))$ we will sometimes write $B_\epsilon(x(N))$ for the corresponding set in $X^N$.

Letting $\Gamma_N := (\delta_N)_* e^{-\beta H^{(N)}} \mu_0^{\otimes N}$ we claim that it is enough to prove that the asymptotics in Prop. 3.4 hold for $\Gamma_N$ with $I = F_\beta$ to deduce that $\Gamma_N$ satisfies a LDP. The point is that $1/N$ times $\log Z_N(\Gamma_N)$ is then uniformly bounded. Indeed, the lower bound follows immediately from the lower bound on $\frac{1}{\beta} \log \Gamma_N(B(\nu))$ for $\nu$ fixed and the upper bound follows from the uniform lower bound on $H^{(N)}/N$ guaranteed by the second assumption on $H^{(N)}/N$. After passing to a subsequence we may thus assume that $-\frac{1}{\beta} \log Z_N \to C$ and it then follows from Prop. 3.4 that the probability measures $\mu^{(N)} := \Gamma_N/\Gamma_N(X)$ satisfy an LDP with rate functional $F_\beta - C$. But then it must be that $C$ is equal to the infimum of $F_\beta$ over $M_1(X)$ which is finite and independent of the subsequence. Hence, the LDP holds for the whole sequence $\Gamma_N/\Gamma_N(X)$, which in turn is equivalent to an LDP for $\Gamma_N$. In conclusion, it will be enough to prove that the upper and lower bounds appearing in the formulation of Prop. 3.4.

**The upper bound on $\frac{1}{N\beta} \log \Gamma_N(B_\epsilon(\nu))$**

We first rewrite

$$\Gamma_N(B_\epsilon(\nu)) = \int_{B_\epsilon(\nu)} e^{-\beta H^{(N)}} \mu_0^{\otimes N} = \int_{B_\epsilon(\nu)} e^{-\beta (H^{(N)} + u)} \mu_0^{\otimes N}$$

to get

$$\frac{1}{N\beta} \log \Gamma_N(B_\epsilon(\nu)) \leq -\inf_{X^N} \frac{H^{(N)} + u}{N} + \frac{1}{N\beta} \log \int_{B_\epsilon(\nu)} \mu_0^{\otimes N}$$
for $\epsilon B \delta$

exists a positive constant $C$ for a fixed $j$

all $u D$

Finally, by definition $D e^{-\beta \mu_0}(\nu) = D \mu_0(\nu) + \int_X u \nu$ and hence taking the sup over all $u \in C^0(X)$ shows that the rhs is bounded from above by $-F_\beta(\nu)$, as desired.

The lower bound on $\frac{1}{N^2} \log \Gamma_N(B_\epsilon(\nu))$

Let us first prove that it will be enough to show that for any given $\epsilon > 0$ there exists a positive constant $C_\epsilon$ such that the following submean inequality holds on $X^N$, for any $N$,

$$e^{-\beta H(N)}(x^{(N)}) \leq C_\epsilon e^{\epsilon N\int_{B_\epsilon(x^{(N)})} e^{-\beta H(N)} dV \otimes N} \int_{B_\epsilon(x^{(N)})} e^{-\beta (H(N) + u)} \mu_0 \otimes N$$

To this end we fix a probability measure $\nu$ on $X$ and $\epsilon > 0$. To fix ideas first assume that $\nu = dF_u$ for some $u \in C^0(X)$. Denote by $x^{(N)} \in X^N$ a sequence of minimizers of $H(N) + u$. By Lemma 3.6 we have that $\delta(x^{(N)}) \rightarrow \nu$ weakly and hence for $N$ sufficiently large $\Gamma_N(B_{2\epsilon}(\nu)) = \int_{B_{2\epsilon}(\nu)} e^{-\beta H(N)} \mu_0 \otimes N \geq \int_{B_{2\epsilon}(x^{(N)})} e^{-\beta (H(N) + u)} \mu_0 \otimes N$.

where $\delta(\epsilon)$ is a modulus of continuity for $u$ on $X$ tending to zero with $\epsilon$ (by the compactness of $X$). Next, using the submean inequality above gives

$$\frac{1}{N} \log \Gamma_N(B_\epsilon(\nu)) \geq \beta \langle u, \nu \rangle - \delta(\epsilon) + \frac{1}{N} \log \int_{B_{2\epsilon}(x^{(N)})} \mu_0 \otimes N + \beta (H(N) + u)(x^{(N)}) / N - \epsilon - C_\epsilon \frac{N}{N}$$

Note that since $\delta(x^{(N)}) \rightarrow \nu$ we may, for $N$ sufficiently large, assume that $B_{\epsilon 2/\epsilon}(\nu) \subset B_{\epsilon 2}(x^{(N)})$ and hence letting $N \rightarrow \infty$ and using Sanov’s theorem (i.e. Prop 3.5) for $\epsilon$ fixed and the assumed convergence of $(H(N) + u)(x^{(N)}) / N$

does give

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \Gamma_N(B_\epsilon(\nu)) \geq \beta \langle u, \nu \rangle - \delta(\epsilon) - \inf_{\mu \in B_{\epsilon 2/\epsilon}(\nu)} D \mu_0(\mu) + \beta \mathcal{F}(u) - \epsilon$$

Since $\nu$ is a candidate for the inf in the rhs above the inf in question may be estimated from above by $D \mu_0(\nu)$ and hence letting $\epsilon \rightarrow 0$ concludes the proof under the assumption that $\nu := dF_u$ for some $u \in C^0(X)$. To prove the general case we use the regularization assumption to write $\nu$ as a weak limit of $\nu_j := dF_{u_j}$ for $u_j \in C^0(X)$. We may then replace $u$ in the previous argument with $u_j$ for a fixed $j$ and replace $\nu$ with $\nu_j$ in the previous argument to get, for $j \geq j_0$,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \Gamma_N(B_\epsilon(\nu_j)) \geq \beta \langle u_j, \nu_j \rangle - \delta_j(\epsilon) - \inf_{\mu \in B_{\epsilon 2/\epsilon}(\nu_j)} D \mu_0(\mu) + \beta \mathcal{F}(u_j) - \epsilon$$

But for $j$ sufficiently large $\nu_j$ is in the ball $B_{\epsilon 2/\epsilon}(\nu)$ and hence the inf above is bounded from above by $D \mu_0(\nu)$ and hence

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \Gamma_N(B_\epsilon(\nu)) \geq \beta \langle u_j, \nu_j \rangle - \delta_j(\epsilon) - D \mu_0(\nu) + \beta \mathcal{F}(u_j) - \epsilon$$
Letting first $\epsilon \to 0$ and then $j \to \infty$ gives
\[
\liminf_{N \to \infty} \frac{1}{N} \log \Gamma_N(B_{3\epsilon}(\nu)) \geq -\beta(\lim_{j \to \infty} (E(\nu_j) + \frac{1}{\beta} D_{\mu_0}(\nu))
\]
By the regularization assumption we may assume that $E(\nu_j) \to E(\nu)$ which concludes the proof under the assumption of the validity of the submean property in formula [3.5] to whose proof we next turn.

Will show how to deduce the submean property in question from Theorem 2.4. First we recall that, since $X$ is compact, the weak topology on $M_1(X)$ is metrized by the Wasserstein 2-metric $d$ induced by a given Riemannian metric $g$ on $X$, where
\[
d(\mu, \nu)^2 := \inf_{\Gamma \in \Gamma(\mu, \nu)} \int d_\mu(x, y)^2 d\Gamma,
\]
where $\Gamma(\mu, \nu)$ is the space of all couplings between $\mu$ and $\nu$, i.e. all probability measures $\Gamma$ on $X \times X$ such that the push forward of $\Gamma$ to the first and second factor is equal to $\mu$ and $\nu$ respectively. We next observe that the pull-back of $d$ on $M_1(X)$ to the quotient space $X^{(N)} := X^N/\Sigma_N$ under the map $\delta_N$ defined by the empirical measure (formula [3.5]) coincides with $1/N^{1/2}$ times the quotient distance function on $X^{(N)}$, induced by the product Riemannian metric on $X^N$:
\[
(3.6) \quad \delta_N^* d = \frac{1}{N^{1/2}} d_{X^{(N)}} := d^{(N)}
\]
Indeed, this follows from the Birkhoff-Von Neumann theorem which gives that for any symmetric function $c(x, y)$ on $X \times X$ we have that if $\mu = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$ and $\nu = \frac{1}{N} \sum_{i=1}^N \delta_{y_i}$, for given $(x_1, ..., x_N), (y_1, ..., y_N) \in X^N$, then
\[
\inf_{\Gamma(\mu, \nu)} \int c(x, y) d\Gamma = \inf_{\Gamma_N(\mu, \nu)} \int c(x, y) d\Gamma
\]
where $\Gamma_N(\mu, \nu) \subset \Gamma(\mu, \nu)$ consists of couplings of the form $\Gamma_\sigma := \frac{1}{N} \sum \delta_{x_i} \otimes \delta_{y_{\sigma(i)}}$, for $\sigma \in \Sigma_N$, where $\Sigma_N$ is the symmetric group on $N$ letters.

Now consider the metric space $(X^{(N)}, d^{(N)})$ which is the quotient space defined with respect to the finite group $\Sigma_N$ acting isometrically on the Riemannian manifold $(X^N, g_N)$, where $g_N$ denotes $1/N$ times the product Riemannian metric. By assumption $H^{(N)}$ is $\Sigma_N$-invariant and $\Delta_{g_N} H^{(N)} \leq C$ on $X^N$ (using the obvious scaling of the Laplacian). Moreover, since $X$ is compact there exists a non-negative number $k$ such that $\text{Ric} \geq -kg$ on $X$ and hence rescaling gives $\text{Ric} \ g_N \geq -kN g_N$ on $(X^N, g_N)$. But the dimension of $X^N$ is equal to $nN$ and hence setting $\kappa^2 := k/n + 1$ shows that, for $N$ large, the assumptions in Theorem 2.4 are satisfied for $u := e^{-\beta H^{(N)}}$ and $(X, g)$ replaced by $(X^N, g_N)$. Applying the latter theorem with $\delta = \epsilon$ and using the pull-back property in formula [3.6] then shows that the submean property [3.5] indeed holds, which thus concludes the first proof of the theorem.

Proof 2 (using the Gibbs variational principle). By general principles (the Gärtner-Ellis theorem) it will be enough to prove the convergence of $-\frac{1}{N\beta} \log Z_{N, \beta}$ [see [9] and references therein].

The upper bound on $\frac{1}{N\beta} \log Z_{N, \beta}$

As will be clear the proof of the lower bound (just as the upper bound above) does not use the assumption about super-harmonicity of $H^{(N)}$, nor the differentiability
of $F$. First we apply the second point in Lemma [3.1] to get
\[
\frac{1}{N} \int H^{(N)} \mu^{(N)} + \frac{1}{\beta} D(\mu_1^{(N)}) \leq F_\beta^{(N)}(\mu^{(N)}) = -\frac{1}{\beta N} \log Z_{N,\beta}
\]
Next, we fix $u \in C^0(X)$ and rewrite the first term in the lhs above as follows:
\[
\frac{1}{N} \int H^{(N)} \mu^{(N)} = \int \frac{1}{N} (H^{(N)} + u) \mu^{(N)} - \int_X u \mu_1^{(N)}
\]
and hence replacing the first integral in the rhs with its infimum gives
\[
\inf_{X^N} \frac{1}{N} (H^{(N)} + u) - \int_X u \mu_1^{(N)} \leq \frac{1}{N} \int H^{(N)} \mu^{(N)},
\]
which by definition means that, if $\mu_*$ denotes a weak limit point of the sequence $(\mu^{(N)})_1$, then
\[
F(u) - \int_X u \mu_* \leq \lim \inf_{N \to \infty} \frac{1}{N} \int H^{(N)} \mu^{(N)},
\]
Taking the sup over all $u \in C^0(X)$ thus gives
\[
E(\mu_*) \leq \lim \inf_{N \to \infty} \frac{1}{N} \int H^{(N)} \mu^{(N)}.
\]
Next, since $D$ is lower-semi continuous we have
\[
D(\mu_*) \leq \lim \inf_{N \to \infty} D(\mu_1^{(N)}),
\]
which concludes the proof of the lower bound on $-\frac{1}{\beta N} \log Z_{N,\beta}$.

The lower bound on $-\frac{1}{\beta N} \log Z_{N,\beta}$

The upper bound is proved just as above. Indeed, we may fix an arbitrary $\nu$ and directly restrict the integration to $B_\beta(\nu)$ and finally optimize over all $\nu$.

**Corollary 3.8.** Under the assumptions in the previous theorem the corresponding j-point correlation measures $(\mu^{(N,\beta)})_1$ converge weakly towards $\mu_\beta^{\otimes j}$, where $\mu_\beta$ is the unique minimizer of $F_\beta$.

**Proof.** This is a standard consequence of the LDP in the previous theorem (alternatively the convergence of $(\mu^{(N)})_1$ appeared in second proof of the previous theorem). For completeness for the recall the argument: by the LDP $(\delta_N)_* \mu^{(N)} \to \delta_{\mu_\beta}$ weakly on $M_1(X)$, where $\delta_{\mu_\beta}$ denote the Dirac measure supported at the point $\mu_\beta$ in $M_1(X)$ defined as the unique minimizer of $F_\beta$ (the uniqueness follows directly from the fact that $E(\mu)$ and $D(\mu)$ are both convex and $D(\mu)$ is even strictly convex on the set where it is finite). But, by definition, \[
\int_{M_1(X)} (\delta_N)_* \mu^{(N)} u_1 \cdots u_j = \int_{M_1(X)} \delta_{\mu_\beta} u_1 \cdots u_j = \int_{M_1(X)} \mu_\beta u_1 \cdots u_j,
\]
 desired (using the notation $(u_1 \otimes \cdots \otimes u_j)(\mu) := \int_X u_1 \mu_1 \cdots \int_X u_j \mu_\beta$).

4. **The Complex Geometric Setting ($\beta = 1$) and Point Processes on KLT Pairs of General Type**

4.1. **Setup.** Let $L \to X$ be a holomorphic line bundle over an $n$-dimensional compact complex manifold $X$. 

4.1.1. **Metrics, weights and \(\theta-\text{psh} \)** functions. We will denote by \(|\cdot||\cdot|\) a Hermitian metric on \(L\). Occasionally we will use additive notation for metrics, where the metric is represented by a *weight* denoted by \(\phi\), which is a short hand for a collection of local functions: if \(s\) is a trivializing local holomorphic section of \(L\), i.e. \(s\) is non-vanishing on a given open set \(U\) in \(X\), then \(\phi|_U := \log \|s\|^2\). In this notation the normalized curvature current of \(L\) may be (locally) represented as \(\theta := dd^c \phi\). Following standard practice it will also be convenient to identify positively curved \(\theta\) with\(\phi\). Following the standard notation in Minimal Model Program (MMP) a pair \((X, D)\) is said to be *log canonical* (lc) if \(c_i \leq 1\) and *Kawamata Log Terminal* (klt) if \(c_i < 1\). Denoting by \(s_i\) a holomorphic section of the line bundle \(\mathcal{O}(D_i)\) cutting out \(D_i\) we will use the symbolic notation \(s_D := s_1^{c_1} \cdots s_m^{c_m}\) (which can viewed as a multi-section when \(c_i \in \mathbb{Q}\)). The point is that \(\phi_D := \log |s_D|^2\) is then a well-defined weight on

For a continuous function \(\varphi\) on \(X\), i.e. \(\varphi \in C^0(X)\). In particular, the curvature current of the metric \(|\cdot||\cdot|\) may then be written as

\[
\theta_{\varphi} := \theta + dd^c \varphi
\]

More generally, this procedure gives a correspondence between the space of all (singular) metrics on \(L\) with positive curvature current and the space \(\mathcal{PSh}(X, \theta)\) defined as the space of all upper-semi continuous functions \(\varphi\) on \(X\) such that \(\theta_{\varphi} \geq 0\) holds in the sense of currents. We will use the same notation \(|\cdot||\cdot|\) for the induced metrics on tensor powers of \(L\) etc. In the weight notation the induced weight on the \(k\)th tensor power \(kL\) is thus given by \(k\phi\).

4.1.2. **Holomorphic sections and big line bundles.** We will denote by \(H^0(X, kL)\) the space of all global holomorphic sections with values in the \(k\)th tensor power of \(L\). We will usually assume that \(L\) is *big*, i.e.

\[
N_k := \dim H^0(X, kL) = Vk^n + o(k^n),
\]

for a positive number \(V\), called the *volume of \(L\).* In the case when \(L = K_X\) and \(s_k \in H^0(X, kK_X)\) we can attach a canonical measure \(\mu_{s_k}\) on \(X\) to \(s_k\) which we will, abusing notation somewhat, sometimes write as

\[
\mu_{s_k} := \frac{n^2}{s_k} \left(s_k \wedge \frac{1}{s_k}\right)^{1/k} = \left|s_k\right|^{2/k} idz \wedge d\bar{z},
\]

where in the rhs we have locally identified \(s_k\) with a holomorphic function, defined wrt the local trivialization \(dz^\otimes k\) of \(K_X\) (\(dz := dz_1 \wedge \cdots \wedge dz_n\)) induced by a choice of local coordinates \(z\).

4.1.3. **Divisors, log pairs and singular volume forms.** We recall that an \((\mathbb{R}-)\) divisor \(D\) on a complex manifold \(X\) is a formal finite sum of one-dimensional irreducible subvarieties:

\[
D = \sum_{i=1}^m c_i D_i, \quad c_i \in \mathbb{R}
\]

Following the standard notation in Minimal Model Program (MMP) a pair \((X, D)\) is called a *log pair* and \((X, D)\) it is said to be *log smooth* if \(D\) has simple normal crossings (snc), i.e. locally we can always choose holomorphic coordinates so that \(D_i = \{z_i = 0\}\). Henceforth \((X, D)\) will always refer to a log smooth pair (anyway this can always be arranged by passing to a log resolution). The pair \((X, D)\) is said to be *log canonical* (lc) if \(c_i \leq 1\) and *Kawamata Log Terminal* (klt) if \(c_i < 1\).
the corresponding $\mathbb{R}$–line bundle $\mathcal{O}(D)$ and its curvature current coincides with the integration current $[D]$ defined by the divisor $D$. Given a volume form $dV$ on $X$ and a continuous metric on the $\mathbb{R}$–line bundle $\mathcal{O}(D)$ we obtain a measure

$$\mu = \|s_D\|^{-2} dV$$

on $X$, which thus has zeroes and poles along the irreducible divisors $D_i$ with negative and positive coefficients, respectively. The klt assumption is equivalent to demanding that $\mu$ be a finite measure. We will say that $\mu$ defined as above is a singular volume form in the singularity class defined by $D$.

4.2. Preliminaries. In this section we will recall some definitions and results of global pluripotential theory in [18, 12, 10, 11, 7] that will be used in the proofs of the main results.

4.2.1. The complex Monge-Ampère operator and the pluricomplex energy. Let $L \to X$ be a big line bundle and fix, as above, a continuous Hermitian metric $\|\cdot\|$ on $L$ with curvature current $\theta$. For a smooth curvature form $\theta(\phi)(:=\theta + dd^c \phi)$ one defines the Monge-Ampère measure of the function $\phi$ (wrt $\theta$) as

$$\text{MA}(\phi) := \theta^n \phi / V,$$

i.e. the top exterior power of the corresponding curvature form divided by the volume $V$ of the class $[\theta]$ (see below). More generally, according to the classical local pluripotential theory of Bedford-Taylor the expression in formula (4.1) makes locally sense for any bounded $\theta$–psh function (and the corresponding Monge-Ampère measure does not charge pluripolar subsets, i.e. sets of the form $\{\phi = -\infty\}$ for a $\theta$–psh function $\phi$. In general, following [18], for any $\phi \in \text{PSH}(X, \theta)$ we will denote by $\text{MA}(\phi)$ the non-pluripolar Monge-Ampère measure $\text{MA}(\phi)$, which is a globally well-defined measure on $X$ (defined by replacing the ordinary wedge products with the so called non-pluripolar products introduced in [18]). In particular, we recall that for $\phi$ an element in the subspace $\text{PSH}(X, \theta)_{\text{min}}$ of functions with minimal singularities the the total mass of $\theta^n \phi$ is independent of $\phi$ and may be taken as the definition of the volume $V$ of the class $[\theta]$. With the normalization in (4.1) this thus means that for any $\phi$ in $\text{PSH}(X, \theta)_{\text{min}}$ the measure $\text{MA}(\phi)$ is a probability measure on $X$. On the space $\text{PSH}(X, \theta)_{\text{min}}$ there is an energy type functional, denoted by $\mathcal{E}$, which may be defined as a primitive for the one-form defined by $\text{MA}(\phi)/V$, i.e.

$$d\mathcal{E}(\phi) = \text{MA}(\phi)$$

(in the sense that $d\mathcal{E}(\phi + tu)/dt = \int_X \text{MA}(\phi)u/V$ at $t = 0$). The functional $\mathcal{E}$ is only defined up to an additive constant which may be fixed by the normalization condition $\mathcal{E}(v_\theta) = 0$ for some reference element $v_\theta$ in $\text{PSH}(X, \theta)_{\text{min}}$. Occasionally, we will write $\mathcal{E}_\theta$ to indicate the dependence of $\mathcal{E}$ on the fixed normalization. We will also denote by $\mathcal{E}_\theta$ the unique upper semi-continuous extension of $\mathcal{E}_\theta$ to all of $\text{PSH}(X, \theta)$ and write

$$\mathcal{E}^1(X, \theta) := \{ \phi \in \text{PSH}(X, \theta) : \mathcal{E}_\theta(\phi) > -\infty \},$$

which is called the space of all functions on $X$ with finite energy. Now following [12] the pluricomplex energy $E_\theta(\mu)$ of a probability measure $\mu$ is defined by

$$E_\theta(\mu) := \sup_{\phi \in \text{PSH}(X, \theta)} \mathcal{E}_\theta(\phi) - \langle \phi, \mu \rangle,$$

(4.2)
As recalled in the following theorem the sup defining $E_\theta$ is in fact attained:

\[
E_\theta(\mu) := \mathcal{E}_\theta(\varphi_\mu) - \langle \varphi_\mu, \mu \rangle
\]

for a unique function $\varphi_\mu \in \mathcal{E}^1(X, \theta)$ if $E_\theta(\mu) < \infty$ where

\[
MA(\varphi_\mu) = \mu
\]

**Theorem 4.1.** [12] The following is equivalent for a probability measure $\mu$ on $X$:

- $E_\theta(\mu) < \infty$
- $\langle \varphi, \mu \rangle < \infty$ for all $\varphi \in \mathcal{E}^1(X, \theta)$
- $\mu$ has a potential $\varphi_\mu \in \mathcal{E}^1(X, \theta)$, i.e. equation (4.4) holds

Moreover, $\varphi_\mu$ is uniquely determined mod $\mathbb{R}$, i.e. up to an additive constant and can be characterized as the function maximizing the functional whose sup defines $E_\theta(\mu)$ (formula (4.2)). Even more generally: if $\varphi_j$ is an asymptotically maximizing sequence sequence (normalized so that $\sup_{x \in X} \varphi_j = 0$), i.e.

\[
\liminf_{j \to \infty} \mathcal{E}(\varphi_j) - \langle \varphi_j, \mu \rangle = E_\omega(\mu)
\]

then $\varphi_j \to \varphi_\mu$ in $L^1(X, \mu)$ and $\mathcal{E}(\varphi_j) \to \mathcal{E}(\varphi_j)$.

**Example 4.2.** When $n = 1$, i.e. $X$ is a Riemann surface and $\omega$ is smooth and strictly positive the functional $E_\omega(\mu)$ is the classical Dirichlet energy of $\mu$ in the “neutralizing back-ground charge $\omega$”. In other words, $E_\omega(\mu) = \frac{1}{2} \int_{X \times X} G_\omega(x, y) d\mu \otimes \mu$, where $G_\omega$ is the Green function of the Laplacian $\Delta$ defined wrt the metric $\omega$, i.e. $\Delta G_\omega(x, y) = \delta_x - 1$ and $\int_{x \in X} G_\omega(x, y) d\mu = 1$

In the case when the class $[\theta]$ is Kähler and $\mu$ is a volume form the existence of a smooth solution to equation 4.4 was first shown by Yau [54] in his celebrated solution of the Calabi conjecture (the uniqueness of such solutions is due to Calabi).  

4.2.2. The psh-projection $P_\theta$, the functional $F_\theta$ and asymptotics. The “psh-projection” is the operator $P_\theta$ from $C_0^0(X)$ to $PSH(X, \theta)_{\min}$ defined by the following envelope:

\[
(P_\theta u)(x) := \sup\{\varphi(x) : \varphi \leq u, \}
\]

Using the latter projection operator it will be convenient to take the reference element $v_\theta$ in $PSH(X, \theta)_{\min}$, referred to above, to be defined by

\[
v_\theta := P_\theta 0
\]

We may then define the following functional on $C_0^0(X)$:

\[
F_\theta(u) := (\mathcal{E}_\theta \circ P_\theta)(u)
\]

Using the latter functional the pluricomplex energy $E_\theta$, defined above, may be realized as a Legendre transform (as shown in [18] in the ample case and in [4] in the general big case):

**Proposition 4.3.** The pluricomplex energy $E_\theta$ is the Legendre-Fenchel transform of the functional $u \mapsto - (\mathcal{E}_\theta \circ P_\theta)(-u)$, i.e.

\[
E_\theta(\mu) := \sup_{u \in C_0^0(X)} \mathcal{E}_\theta(P_\theta u) - \langle u, \mu \rangle,
\]

Moreover

\[
E_\theta(\mu) \geq 0
\]

with equality precisely for $\mu := MA(v_\theta)$, where $v_\theta := P_\theta 0$.  

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Finally, we will have great use for the following result concerning the existence and differentiability of the transfinite diameter associated to a big Hermitian line bundle (which is equivalent to Theorem A and B in [10]).

**Theorem 4.4.** ([10]) Let $L \to X$ be a big line bundle equipped with a continuous Hermitian metric $\| \cdot \|$ on $L$ with curvature current $\theta$. Then

- If $\det S^{(k)}$ denotes the element in the determinant line of $H^0(X, kL)$ induced by a basis in $H^0(X, kL)$ which is orthonormal with respect to the $L^2$-norm determined by $\| \|$, and a volume form $dV$ on $X$ (or more generally a measure $\mu_0$ which has the Bernstein-Markov property) then
  \[
  \lim_{k \to \infty} -\frac{1}{kN} \inf_{X^N} \left( \log \left( \det S^{(k)} \right)^2 (x_1, \ldots, x_N) + ku(x_1) + \cdots + ku(x_N) \right) = \mathcal{F}_\theta(u),
  \]

- The functional $\mathcal{F}_\theta$ is Gateaux differentiable on $C^0(X)$ with differential $(d\mathcal{F}_\theta)|_u = MA(P_\theta u)$.

### 4.3. $\beta$-deformation of Vandermonde type determinants attached to a big line bundle $L$

As above we let $L \to X$ be a given big line bundle over a compact complex manifold, $\| \cdot \|$ is a continuous Hermitian metric on $L$ (whose curvature current will be denoted by $\theta$). We also fix a finite measure $\mu_0$ on $X$ a positive number $\beta$. To this data we may associate the following sequence of probability measures on $X^{N_k}$:

\[
\mu_{(N_k, \beta)} := \frac{\left( \left( \det S^{(k)} \right)(x_1, x_2, \ldots, x_{N_k}) \right)^{2\beta/k}}{Z_{N_k, \beta}} \mu_0^{\otimes N_k}
\]

where we recall that $N_k$ is the dimension of $H^0(X, kL)$ and $\det S^{(k)}$ it a generator of the corresponding determinant line $\Lambda^{N_k} H^0(X, kL)$ viewed as a one-dimensional subspace of $H^0(X^{N_k}, (kL)^{\otimes N_k})$ (the totally anti-symmetric part). As usual, $Z_{N_k, \beta}$ is the normalizing constant (partition function):

\[
Z_{N_k, \beta} := \int_{X^{N_k}} \left( \det S^{(k)} \right)^{2\beta/k} \mu_0^{\otimes N_k}
\]

and it will often be convenient to take $\det S^{(k)}$ to be the generator determined by a basis in $H^0(X, kL)$ which is orthonormal with respect to the $L^2$-product determined by $\(\| \|, \mu_0\)$. In this section we will assume that the background measure $\mu_0$ is a singular volume form in the singularity class defined by a klt divisor $D$ on $X$ (see section 4.1) and accordingly we will sometimes write $\mu_0 = \mu_D$ to indicate the dependence on $D$. The probability measure above is thus the Gibbs measure defined by the following Hamiltonian on $X^{N_k}$:

\[
H^{(N_k)}(x_1, x_2, \ldots, x_{N_k}) := -\frac{1}{k} \log \left( \det S^{(k)}(x_1, x_2, \ldots, x_{N_k}) \right)^2
\]

We will occasionally denote by $H^{(N_k)}_u$ the perturbed Hamiltonian $H^{(N_k)}_u = H^{(N_k)} + u$, determined by $u \in C^0(X)$ (identified with a function on $X^N$). In the present setting perturbing by $u$ corresponds to replacing the metric $\|\cdot\|^2$ by the twisted metric $\|\cdot\|^2 e^{-u}$. Since the metric $\|\cdot\|$ is determined up to a multiplicative constant by its curvature current $\theta$, the probability measures above thus only depends on the metric through its curvature and is hence determined by the triple $(\mu_0, \theta, \beta)$. 

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To the latter triple we may also attach the following free energy type functional on the space $\mathcal{M}_1(X)$ of all probability measures on $X$:

$$(4.9) \quad F_\beta = E_\theta + \frac{1}{\beta} D_{\mu_0}$$

Note that the energy functional $E_\theta$ is minimized on the measure $MA(\nu_0)$ (see Prop 4.3), while the entropy functional $D_{\mu_0}$ is minimized on $\mu_0$. As explained in [7] the functional $F_\beta$ may be identified with a twisted version of Mabuchi’s $K$-energy functional and as shown in [7,13] its minimizers may be identified with solutions to a complex Monge-Ampère equation (the existence of smooth solutions for an ample line bundle was first shown in the seminal works of Aubin [2] and Yau [54]):

**Proposition 4.5.** Let $[\theta]$ be a big class (for example, $[\theta] = c_1(L)$ for $L$ a big line bundle) and consider the free energy functional $F_\beta$ attached to the triple $(\mu_0, \theta, \beta)$ (for $\beta > 0$). Then any minimizer $\mu_\beta$ of $F_\beta$ on $\mathcal{M}_1(X)$ can be written as $\mu_\beta = MA(u_\beta)$ where $u_\beta$ is the unique finite energy solution of the equation

$$(4.10) \quad MA(u_\beta) = e^{\beta u_\beta} \mu_0$$

In fact, the results in [7,13] hold as long as $\mu_0$ has finite energy and also yield the existence of a finite energy minimizer. In the present setting the existence of a minimizer will also be a consequence of the proof of the following theorem, as the minimizer will be obtained as the limit of the one-point correlation measures of the random point processes above.

**Theorem 4.6.** Let $L \to X$ be a big line equipped with a continuous Hermitian metric $\|\cdot\|$ and $\mu_0$ a volume form on $X$ with singularities described by a klt divisor $\Delta$. Fix a positive number $\beta$. Then the empirical measures of the corresponding random point processes on $X$ converge in probability towards the measure $\mu_\beta$, where $\mu_\beta = MA(u_\beta)$ for the unique solution $u_\beta$ of the complex Monge-Ampère equation

$$(4.11) \quad C_{\beta}(\mu_\beta) = \inf_{\mathcal{M}_1(X)} F_\beta = C$$

More precisely:

- The laws of the empirical measures of the processes satisfy a large deviation principle (LDP) with speed $\beta N$ and rate functional $F_\beta(\mu) - C$ where $F_\beta$ is the corresponding free energy functional and $C$ is the normalizing constant.

- Similarly, the non-normalized measures $(\delta_N)_* \left( \|\det S(k)\|^{2\beta/k} \mu_0^{\otimes N_k} \right)$, where $\det S(k)$ an element in the determinant line of $H^0(X,kL)$ induced by a basis in $H^0(X,kL)$ which is orthonormal with respect to the $L^2$-norm determined by $(\|\cdot\|, \mu_0)$ satisfy a LDP with rate functional $F_\beta$ and the corresponding partition functions satisfy

$$- \lim_{N_k \to \infty} \frac{1}{N_k} \log Z_{N_k} = \inf_{\mathcal{M}_1(X)} F_\beta = C$$

**Proof.** Let us first verify that the Hamiltonian $C_{\beta}$ satisfies the assumptions in Theorem 3.7. First, the superharmonicity follows as in the proof of Cor 2.2. Next, the second point follows from Theorem 4.4 above. Finally, the required regularization property is proved as follows. If $E(\mu) < \infty$, then by Theorem 4.1 we can write $\mu = MA(\varphi)$ for a function $\varphi$ with finite energy. Since the function $\varphi$ is usc it is a decreasing limit of continuous functions $u_j$ on $X$. It then follows that the projections $Pu_j$ also decrease to $\varphi$ and hence, by the continuity of mixed Monge-Ampere expression under monotone limits [13] it follows that $\mu_j := MA(Pu_j) \to \mu$ and $E(\mu_j) \to E(\mu)$ as desired. Using the Legendre transform representation of the
pluricomplex energy $E(\mu)$ (Prop 4.3) this proves the theorem in the case that $\mu_0$ is a bona fide volume form.

To prove the case when $\mu_0$ has singularities encoded by a klt divisor $D$ we first decompose $D = D_- + D_+$ where $D_-$ are the components with negative coefficients (and hence $D_+$ consists of components with coefficients in $[0, 1]$). Let $s_\pm$ be holomorphic (multi-) section cutting out $\pm D_\pm$ and fix appropriate metrics so that

$$\mu_0 = \frac{\|s_-\|^2}{\|s_+\|^2}dV$$

for some volume form $dV$. We will next show how to adapt the proof of Theorem 3.7 to this singular setting. We have two options: two generalize either the first or the second proof of the theorem in question and we choose the second one concerning bound on $Z_N$ (but the first one would also be possible).

The upper bound on $\log Z_N$ in the proof of the theorem proceeds exactly as before, as it works for any back-ground measure $\mu_0$. To prove the lower bound we will modify the previous proof by first assuming that $\nu$ (and its approximations $\nu_j$) have supports which are compactly included in $X - D_-$. To simplify the notation we also assume that $\beta = 1$ (but the general argument is the same). We factorize $\mu_0 =: \|s_-\|^2 \mu_+$ and rewrite

$$\|\det S^{(k)}\|^{2/k} \mu_0^\otimes N = \|\det S^{(k)}\|^{2/k} \mu_+^\otimes N, \quad \det S_{-}^{(k)} := \det S^{(k)} \cdot s_{-}^{\otimes k N}$$

Next, we fix a sequence $\mu_i$ of volume forms on $X$ increasing to $\mu_+$ (for example, obtained by replacing $\|s_+\|^2$ with $(\|s_+\|^2 + 1/i)$) to get

$$-\frac{1}{\beta N} \log Z_N \leq -\log \frac{1}{\beta N} \int \|\det S_{-}^{(k)}\|^{2/k} \mu_i^\otimes N,$$

We can then repeat the previous arguments with $\det S^{(k)}$ replaced by $\det S_{-}^{(k)}$ and $\mu_0$ replaced by $\mu_i$ to get

$$\lim_{N \to \infty} \frac{1}{\beta N} \log Z_N \leq \int \log \|s_-\|^2 \nu + D_{\mu_+}(\nu)$$

where we have applied the submean inequality in Cor 2.2 to $\det S_{-}^{(k)}$ and then used that

$$-\frac{1}{kN} \log \det S_{-}^{(k)}(x(N)) = -\frac{1}{kN} \log \det S^{(k)}(x(N)) - \frac{1}{N} \log \|s_-\|^2 (x(N)) \to \mathcal{F}_\theta(u) - \int \log \|s_-\|^2 \nu$$

(as first $N$ and then $j$ tends to infinity), using that $\nu$ and $\nu_j$ are supported in compact sets where $\log \|s_-\|^2$ is continuous. All in all, letting $i$ tend to infinity in the rhs of equation 4.11 and using the Lebesgue monotone convergence theorem gives

$$\lim_{N \to \infty} -\frac{1}{\beta N} \log Z_N \leq E(\nu) - \int \log \|s_-\|^2 \nu + D_{\mu_+}(\nu)$$

Since by definition $D_{\mu_0}(\nu) = -\int \log \|s_-\|^2 \nu + D_{\mu_+}(\nu)$ this shows that

$$\lim_{N \to \infty} -\frac{1}{\beta N} \log Z_N \leq \inf_{\nu} (E(\nu) + D_{\mu_0}(\nu)),$$

where the infimum ranges over all probability measure $\nu$ supported in the open subset $X - D_-$ of $X$. But by an approximation argument (see below) the latter
inf coincides with the inf over all of \( \mathcal{M}_1(X) \), which concludes the proof. As for the approximation argument it goes as follows: writing \( \nu \in \mathcal{M}_1(X) \) as the limit of \( \chi_i \nu \) where \( \chi_i := 1_{K_i} \nu / \int K_i \nu \) for an exhaustion of \( X - D_- \) by compact sets \( K_i \), it is enough to show that \( E(\mu_i) \to E(\nu) \) (since the converge of the entropies follows from the Lebesgue monotone convergence theorem). But this is shown exactly as in Lemma 5.5 in [4].

Remark 4.7. Note that when \( \beta = 1 \) the probability measure \( \mu^{(N, \beta)} \) is invariant under the transformation \( (\| \cdot \|^2, \mu_0) \mapsto (\| \cdot \|^2 e^{-v}, e^v \mu_0) \) of the defining data. Hence, by the previous theorem the corresponding free energy functional \( F_\beta \) is invariant up to an additive constant. In fact, this is easy to see directly, since it follows from the definitions that \( D e^v \mu_0(\mu) = D e^v \mu_0(\mu) - \int v \mu \) and \( E_{\theta + dv} e^v(\mu) = E_\theta(\mu) + \int v \mu - C_v \) (where the constant \( C_v \) ensures that the infimum of \( E_{\theta + dv} e^v \) vanishes).

4.4. Canonical point processes attached to a klt pair \((X, D)\) of log general type. Assume given a smooth log pair \((X, D)\), where \( D \) is klt \( \mathbb{Q} \)-divisor and assume that \((X, D)\) is of log general type, i.e. \( L := K_X + D \) is big. To the pair \((X, D)\) we may attach a canonical sequence of random point processes on \( X \) as follows. Fix a continuous metric on \( L \) represented by a weight \( \phi_0 \). It determines a singular volume form \( \mu_{(\phi_0, D)} \), locally represented as

\[
\mu_{(\phi_0, D)} = e^{\phi_0 - \phi_D} idz \wedge d\bar{z},
\]

where \( idz \wedge d\bar{z} \) is a short hand for the local Euclidean volume form determined by the local holomorphic coordinates \( z \) and \( \phi_0 \) is the corresponding local representation of the weight. Then it if follows immediately from the definitions that the probability measure in formula [4.4] determined by the triple \( (\mu, \phi_0, \beta) = (\mu_{(\phi_0, D)}, \phi_0, 1) \) is independent of \( \phi_0 \) and thus canonically attached to \((X, D)\). In fact, it coincides with the probability measure of the canonical random point defined in the introduction of the paper. The point is that if \( s_k \) is a holomorphic section of \( k(K_X + D) \) then the measure

\[
\| s_k \|_{k(\phi_0)}^{2/k} \mu_{(\phi_0, D)} = \| s_k \|_{k(\phi_0)}^{2/k} \mu_{(\phi_0, D)} = \| s_k \|_{k(\phi_0)}^{2/k} e^{-\phi_D} idz \wedge d\bar{z},
\]

is clearly independent of \( \phi_0 \).

If \( u_{KE} \) is the solution of the corresponding Monge-Ampère equation [4.10] (with \( \theta = dd^c \phi_0 \) and \( \mu_0 = \mu_{(\phi_0, D)} \)) then it will sometimes be convenient to rewrite the equation in terms of the corresponding weight \( \phi_{KE} := \phi_0 + u_{KE} \):

\[
(dd^c \phi_{KE})^n = e^{\phi_{KE} - \phi_D} idz \wedge d\bar{z}
\]

Its curvature current \( \omega_{KE} := dd^c \phi_{KE} = \theta + dd^c u_{KE} \) satisfies the following log Kähler-Einstein equation associated to \((X, D)\):

\[
Ric \omega_{KE} = -\omega_{KE} + [D]
\]

where \( Ric \omega \) denotes the Ricci curvature of \( \omega \) viewed as a current on \( X \). See [4] for the precise meaning of the previous equation in the general setting when \( K_X + D \) is merely assumed big. Anyway, for \( K_X + D \) semi-ample and big (or nef and big) it was shown in [13] that the solution \( \omega \) is smooth on the log regular locus (i.e. on \( X - D \)) and defines a bona fide Kähler-Einstein metric there and its potential \( u_{KE} \) is globally continuous on \( X \). Moreover, in the case when \( K_X + D \) is ample the current \( \omega \) globally defines a singular Kähler-Einstein metric with edge-cone
The free energy functional

\[ F = E_{dd^c\phi_0} + D_{\mu_{(\phi_0, D)}} \]

determined by the background data \((\mu, \phi_0, \beta) = (\mu_{(\phi_0, D)}, \phi_0, 1)\) is also independent
of \(\phi_0\) (as follows from the transformation property pointed out in Remark 4.7). In
fact, the functional \(F\) corresponds to the log version \(\kappa_{(X, D)}\) of Mabuchi’s K-energy
functional in the sense that

\[ F(\frac{\omega^n}{n!}) = \kappa_{(X, D)}(\omega) \]

(where \(\kappa_{(X, D)}\) has been normalized so that it vanishes on \(\omega_{KE}\)). This can be seen
as a generalization of a formula of Tian and Chen for the K-energy (see 7 13 and
references therein).

**Theorem 4.8.** Let \((X, D)\) be a smooth klt pair of general type. Then the empirical
measures of the corresponding canonical random point processes on \(X\) converge
in probability towards the normalized volume form \(dV_{KE}\) of the Kähler-Einstein
metric on \((X, D)\). More precisely, the laws of the empirical measures satisfy a large
deviation principle with speed \(N\) and rate functional \(F(\mu)\), where \(F(\omega^n/V)\) is the
Mabuchi’s (log) K-energy of \(\omega\).

**Proof.** Setting \((\mu, \phi_0, \beta) = (\mu_{(\phi_0, D)}, \phi_0, 1)\) this is a direct consequence of Theorem
4.6 once we have verified that the infimum of \(F_\beta\) vanishes in this setting. To see
this note that, by the invariance discussed above, we may as well take \(\phi_0 = \phi_{KE}\)
so that \(\mu_{(\phi_0, D)} = MA(\phi_{KE})\). But \(MA(\phi_{KE})\) minimizes \(F_\beta\) (by Prop 4.5) and
\(E_{dd^c\phi_{KE}}(MA(\phi_{KE})) = 0\) (by Prop 4.3) and \(D_{MA(\phi_{KE})/V}(MA(\phi_{KE})/V) = 0\) (by
definition) which shows that \(F(MA(\phi_{KE})) = 0 + 0 = 0\), as desired.

As a rather direct consequence of the previous theorem we get the following

**Corollary 4.9.** Let \((X, D)\) be a smooth klt pair of general type. Then the first
correlation measures \(\nu_k := \int_{X_{N-1}} \mu_{(N_k)}\) of the canonical point processes define a
sequence of canonical measures on \(X\) converging weakly to \(dV_{KE}\). Moreover, the
curvature forms of the corresponding metrics on \(K_X\) defined by the sequence \(\nu_k\)
converge weakly to the unique Kähler-Einstein metric \(\omega_{KE}\) on \(X\).

**Proof.** First observe that, by definition, the one point-correlation measure we may
be written as

\[ \mu_1^{(N_k)} = e^{\phi_k} idz \wedge d\bar{z} = e^{u_k} dV \]

where \(\phi_0\) is a fixed smooth weight on \(K_X\), \(dV = \mu_{\phi_0}\) and \(u_k \in PSH(X, \theta)\), for \(\theta =
dd^c \phi_0\). In particular, \(\int_X e^{u_k} dV = 1\) and hence by Jensen’s inequality, \(\sup_X u_k \leq C_0\).

But, by standard compactness results for \(\theta-\text{psh functions}\) 32, it follows that either
\(u_k\) converges in \(L^1(X)\) towards some \(u \in PSH(X, \theta)\), or there is a subsequence
\(u_{k_j}\) such that \(u_{k_j} \to -\infty\) uniformly. But the latter alternative is not compatible
with the condition \(\int_X e^{u_k} dV = 1\) and hence \(u_k \to u\) in \(L^1(X)\). Equivalently, this
means that \(\phi_k \to \phi\) in \(L^1_{loc}\) where \(\phi\) is a weight on \(K_X\) with positive curvature
current. On the other hand, by the previous theorem \(\mu_1^{(N_k)} \to e^{\phi_{KE}} idz \wedge d\bar{z}\),
where \(dd^c \phi_{KE} = \omega_{KE}\) (compare Cor 3.3). But, since a subsequence of \(\phi_k\) converges a.e.
on $X$ it then follows that $\phi_{KE} = \phi$. In particular, $dd^c\phi_k \to dd^c\phi_{KE} = \omega_{KE}$ weakly and that concludes the proof. □

5. VARIETIES OF POSITIVE KODAIRA DIMENSION

5.1. BIRATIONAL SETUP. Let us start by recalling the standard setup in birational geometry. Let $X$ and $X'$ be (normal) projective varieties. A rational mapping $F$ from $X$ to $X'$, denoted by a dashed arrow $X \dasharrow X'$, is defined by a morphism $F : U \to X'$ from a Zariski open subset $U$ of $X$. It is called birational if it has an inverse. Then there is a maximal Zariski open subset $U \subset X$, where $F$ defines a well-defined isomorphism onto its image (the complement of $U$ is called the exceptional locus of $F$). Given a rational mapping $F$ from $X$ to $X'$ and a probability measure $X$ which is is absolutely continuous wrt Lebesgue measure, we can define $F_*\mu$ by pushing forward the restriction of $\mu$ to any Zariski open subset $U$ where $F$ is well-defined and pull-backs of such measures can be similarly defined. If $F : X \dasharrow X'$ is birational then there exists a non-singular variety $Z$ and birational morphisms $f : Z \to X$ and $f' : Z \to X'$ such that $f' = F \circ f$ (in fact, $f$ and $f'$ can even be obtained as a sequence of blow-ups and blow-downs respectively). The Kodaira dimension $\kappa(X)$ of an $n$-dimensional (say non-singular) variety $X$ is the birational invariant defined as the smallest number $\kappa \in \{-\infty, 0, 1, \ldots, n\}$ such that $N_k = O(k^{\kappa})$, where $N_k$ denotes the $k$th plurigenus of $X$, i.e. the dimension of $H^0(X, kK_X)$. In the strictly positive case $\kappa(X)$ may be equivalently defined as the the dimension of the image of $X$ under the $k$th canonical rational mappings

$$F_k : X \dasharrow \mathbb{P}^* H^0(X, kK_X), \ Y_k := F_k(X), \ x \mapsto \{s_k \in H^0(X, kK_X) : s_k(x) = 0\}$$

where here and subsequently $k$ stands for a sufficiently large, or sufficiently divisible, positive integer. By construction $kK_X$ is trivial along the fibers of $F_k$. Next, we recall that by classical results of Ithaka there exist non-singular varieties $X'$ and $Y'$ and a surjective morphism $F$ with connected fibers:

$$F : X' \to Y'$$

such that $X'$ and $Y'$ are birational to $X$ and $Y$, respectively and such that $F$ is conjugate to $F_k$. The fibration defined by $F$ is uniquely determined up to birational equivalence and usually referred to as the Ithaka fibration. A very general fiber of the fibration has vanishing Kodaira dimension.

Finally, it should be pointed out that by the deep results in [17, 44], proved in the context of the MMP, the canonical ring $R(X) := \bigoplus_{k \in \mathbb{N}} H^0(X, kK_X)$ of any non-singular projective variety $X$ is finitely generated. In particular, $Y_k$ (as defined above) is, for $k$ large, independent of $k$ (up to isomorphism) and coincides with the canonical model of $X$ (i.e. the Proj of $R(X)$). But this information will not be needed for our arguments.

5.2. CANONICAL POINT PROCESSES ON VARIETIES OF POSITIVE KODAIRA DIMENSION. Let us now consider a non-singular variety $X$ of positive Kodaira dimension (there is also a logarithmic version of this setup concerning klt pairs $(X, D)$, but for simplicity we will assume that $D = 0$). On such a variety $X$ we can define the canonical random point processes just as in section [6.3] (since $N_k > 0$ for $k$ large).

\[1\] More generally, the results will apply to $X$ a possibly singular normal variety, by defining the corresponding probability measures on the regular locus of $X$ and using the birational invariance below to replace $X$ with any resolution.
Proposition 5.1. The canonical random point processes attached to a variety \( X \) of positive Kodaira dimension are birationally invariant in the sense that if \( F : X \rightarrow X' \) is a birational mapping, then the canonical probability measures on \( X^{N_k} \) and \( X'^{N_k} \) are invariant under \( F_* \).

Proof. This follows from the usual proof of the birational invariance of the spaces \( H^0(X, kK_X) \). Indeed, \( F \) defines an isomorphism from \( U \) in \( X \) to \( U' \) in \( X' \), where \( U \) has codimension at least two. Hence, by the usual unique extension properties of holomorphic sections \( F^{*}_V \) induces an isomorphism between \( H^0(X, kK_X) \) and \( H^0(X', kK_{X'}) \), which (by the change of variables formula) respects the measure \( (S_k \wedge \overline{S_k})^{1/k} \) defined by an element \( S_k \in H^0(X, kK_X) \). Applying this argument on the products \( X^{N_k} \) and \( X'^{N_k} \) then concludes the proof. \( \square \)

When studying the random point processes on \( X \) we may without loss of generality, by the previous proposition, assume that there is morphisms \( F \) of \( X \) to the base \( Y \) of the Ithaka fibration. Next, we note that there is a canonical family of relative measures \( \mu_{X/Y} \) defined over an open dense subset \( Y_0 \) of \( Y \), such that \( Y - Y_0 \) is a null set, defined as follows. First, by the construction of the Ithaka fibration, we may assume that \( F \) is a submersion over some open dense subset \( Y_0 \) of \( Y \) and that \( kK_X \) is trivial along the fibers of \( F \) over \( Y_0 \). Let \( \Omega_y^{(k)} \) be a generator of \( H^0(X_y, kK_{X_y}) \) for \( y \in Y_0 \). Then

\[
(\mu_{X/Y})_y := \left( \Omega_y^{(k)} \wedge \overline{\Omega_y^{(k)}} \right)^{1/k} / \int_{X_y} \left( \Omega_y^{(k)} \wedge \overline{\Omega_y^{(k)}} \right)^{1/k}
\]

is a probability measure on \( X_y \) which is independent of the generator and of \( k \) (since \( (\Omega_y^{(k)})^{\otimes m} \) generates \( H^0(X_y, kmK_{X_y}) \)) and it defines a smooth family of relative \((n - \kappa, n - \kappa)\)-forms over \( Y_0 \). Let us also introduce some further notation: if \( \nu_Y \) is a measure on the base \( Y \) which is absolutely continuous wrt\ Lebesgue measure, then we will write \( F^* \nu_Y \wedge \mu_{X/Y} \) for the measure on \( X \) defined as a fiber-product, i.e. if \( u \) is a smooth function on \( X \) then

\[
\int_X F^* \nu_Y \wedge \mu_{X/Y} u := \int_{Y_0} \left( \int_{X_y} u \mu_{X/Y} \right) \nu_Y
\]

(which is independent of the choice of \( Y_0 \) since the complement is a null set).

Lemma 5.2. Let \( X \) be a variety of positive Kodaira dimension and assume that the Ithaka fibration \( F : X \rightarrow Y \) is a morphism and that the branch locus \( Y - Y_0 \) is equal to the support of a divisor \( D \) in \( Y \) with normal crossings. Then there exists a line bundle \( L_{X/Y} \) over \( Y \) equipped with a (singular) metric whose weight will be denoted by \( \phi_H \), with the property that \( K_Y + L_{X/Y} \) is big and for any \( S_k \in H^0(X, kK_X) \) there exists a unique \( s_k \in H^0(Y, k(K_Y + L_{X/Y})) \) such that

\[
(S_k \wedge \overline{S_k})^{1/k} = F^* \left( (s_k \wedge \overline{s_k})^{1/k} e^{-\phi_H} \right) \wedge \mu_{X/Y}
\]

over \( Y_0 \). The weight \( \phi_H \) is smooth on \( Y_0 \) and locally around any given point in \( Y - Y_0 \)

\[
\phi_H = q \log \log(-|s_D|^2) + \log |s_{D_{X/Y}}|^2 + O(1)
\]
for some positive number $q$, where $D_{X/Y}$ is a klt divisor whose support coincides with $D$. The line bundle $L_{X/Y}$ will be referred to as the Hodge line bundle and $\phi_H$ as the weight of the Hodge metric.

**Proof.** By assumption the morphism $F$ restricts to define a submersion $\pi : X_0 \to Y_0$ between Zariski open subsets. The (tautological) decomposition $K_X = F^* K_Y + K_{X/Y}$ restricted over $Y^0$ gives

\[ kK_{X_0} = F^* k( K_{Y_0} + F^* L_0 ), \]

where $L_0 := \pi_s(K_{X_0}/Y_0)$. The latter direct image sheaf is defined as a $\mathbb{Q}$–line bundle over $Y_0 : F_*(K_{X_0}/Y_0) = \frac{1}{k} F_*(kK_{X_0}/Y_0)$ for any fixed $k$ which is sufficiently large. Concretely, $kL_0$ is locally trivialized by $\Omega_y^{(k)}$, where $\Omega_y^{(k)}$ is as in formula 5.1.

We equip the direct image line bundle $L_0$ with the canonical $L^2$–metric, usually referred to as the Hodge metric. Concretely, the $k$ tensor power of the letter metric is defined by $\left\| \left( \Omega_y^{(k)} \right) \right\|^2 := \left( \int_{X_0} \left( \Omega_y^{(k)} \wedge \Omega_y^{(k)} \right)^{1/k} \right)^k$, i.e the local weight $\phi_H$ for $L_0$ determined by the trivialization (multi-) section $\left( \Omega_y^{(k)} \right)^{1/k}$ is given by

\[ \phi_H(y) := - \log \left\| \left( \Omega_y^{(k)} \right)^{1/k} \right\|^2 = - \log \int_{X_0} \left( \Omega_y^{(k)} \wedge \Omega_y^{(k)} \right)^{1/k} \]

Next, we recall the basic fact that any point $x$ in $X^0$ has a neighborhood $U$ with local coordinates $(w, z) \in \mathbb{C}^n \times \mathbb{C}^{n-k}$, such that the the morphism $F$ corresponds to projection on the first factor. Moreover, by a standard argument we may assume that $\left( \Omega_y^{(k)} \right)^{1/k} = dz$ on $U$, where $dz := dz_1 \cdot \cdots \cdot dz_{n-k}$. Now, by the relation 5.4 and the fact that we may assume that $kK_X$ is trivial along the fibers of $F$ over $Y_0$, the restriction of $S_k \in H^0(X, kK_X)$ to $U$ may be written as $S_k = f_k(w)dz_1 \otimes \cdots \otimes dw_{n-k}$ for a local holomorphic function $f_k(w)$ on $V := F(U)$ and hence

\[ \left( S_k \wedge \overline{S_k} \right)^{1/k} = |f_k(w)|^{2/k} dw \wedge d\bar{w} \wedge dz \wedge d\bar{z} = |f_k(w)|^{2/k} e^{-\phi_H(w)} dw \wedge d\bar{w} \wedge \mu_{X/Y}, \]

where we recall that $\mu_{X/Y}$ is the relative probability measure defined in formula 5.1. Since $x$ was an arbitrary point in $X_0$ this proves the relation 5.2 over $Y_0$ if $s_k$ is taken in $H^0(Y_0, k(K_Y + L_0))$.

Next, we will give the construction of the line bundle $L_{X/Y}$ extending $L_0$ and show that $s_k$ above can be taken as the restriction to $Y_0$ of an element in $H^0(Y, k(K_Y + L_{X/Y}))$. First, following Fujino-Mori 311, we may assume that the double dual of the torsion free sheaf $\pi_s(kK_{X/Y})/k$ is a well-defined $\mathbb{Q}$–line bundle and set $L_{X/Y} := L_0$.

The canonical bundle formula of Fujino-Mori (see Prop 2.2. in 311) says that

\[ K_X + B_- = \pi^*(K_Y + L_{X/Y}) + B_+ \]

where $B_\pm$ are effective $\mathbb{Q}$–divisors (supported in $Y - Y_0$) such that $\text{codim} F(\text{supp}(B_-)) \geq 2$ and $F_*(\mathcal{O}(B_\pm)) = \mathcal{O}_Y$. This formula implies that

- If $S_k \in H^0(X, kK_X)$, then the restriction of $S_k$ to $X_0$ may be written as $S_{k|X_0} = F^* s_k|Y_0$ for a unique section $s_k \in H^0(Y, k(K_Y + L_{X/Y}))$.

Indeed, by the canonical bundle formula and the property of $B_\pm$, the restriction of $S_k$ to $Y - \text{supp}(B_-)$ may be written as $S_k = F^* s_k \otimes s_{B_\pm}^n$ for a unique $s_k \in H^0(Y - F(\text{supp}(B_-)), kK_Y)$. But since $\text{codim} F(\text{supp}(B_-)) \geq 2$ the section $s_k$ extends to a unique element in $H^0(Y, k(K_Y + L_{X/Y}))$. Since $\text{supp}(B_-) \subset Y - Y_0$
this proves the point above. Note that this is essentially the same argument as the one used in [31] to prove that $H^0(X, kK_X) = H^0(X, k(K_Y + L_{X/Y})$ (see the proof of Theorem 4.5 in op.cit.) and it also shows that $K_Y + L_{X/Y}$ is big (since $N_k \sim k^e$). Finally, let us briefly recall the proof of the singularity structure of the Hodge metric on $L_{X/Y}$ in a neighborhood of a point contained in $Y - Y_0$, which follows from Tsuji’s argument in [50]. First, as shown in [31] may assume that $L_{X/Y} = M_{X/Y} + D_{X/Y}$, where $M_{X/Y}$ (“the semi-stable part”) is a nef line bundle on $Y$ and $D_{X/Y}$ (the “discriminant part”) is a klt divisor on $Y$. By the work of Kawamata and Schmidt on variations of Hodge structures (see [50] and references therein) the lines bundles $M_{X/Y}$ and $D_{X/Y}$ contribute over $Y - Y_0$ to the first and second term in the decomposition 5.3 of the weight $\phi_H$ defined wrt a given trivialization of $L_{X/Y}$ over a neighborhood of a point in $Y - Y_0$. □

Next, let us recall the definition of the (singular) canonical metric $\omega_Y$ on the base of the base $Y$ of the Ithaka fibration (which is a birational invariant). For our purposes it will be enough to define it in the case when $X$ fibers over $Y$ as in the previous lemma. Then we define $\omega_Y \in c_1(K_Y + L_{X/Y})$ as $\omega_Y = dd^c \phi_Y$, where $\phi_Y$ is the weight of a (possible singular) positively curved metric $\phi_Y$ on $K_X + L_{X/Y}$ defined as the unique finite energy weight $\phi$ on the big line bundle $K_X + L_{X/Y}$ solving

$$MA(\phi) = e^{\phi - \mu} dw \wedge d\bar{w}. \tag{5.5}$$

Note that by the previous lemma we have that locally $e^{-\phi} \in L^p_{loc}$ for some $p > 1$ and hence by the Kolodziej type estimates in [18] $\phi$ has minimal singularities. We may hence define $\omega_Y$ equivalently as the unique current in $c_1(K_Y + L_{X/Y})$ with minimal singularities such that

$$\text{Ric } \omega_Y = -\omega_Y + \omega_{WP} + [\Delta],$$

where $\Delta$ is the klt divisor of Fujino-Mori (the “discriminant divisor”) supported on the branch locus in $Y$ and $\omega_{WP}$ is equal to $1_Y, \omega_{WP}$ where $\omega_{WP}$ is the generalized Weil-Petersson type metric of the fibration over $Y_0$ (compare [46]). It may be defined as the curvature form of the Hodge metric on $L_{X/Y} \rightarrow Y_0$. Alternatively, we note that arguing as in the beginning of the proof of the previous lemma gives the following equivalent equation for $\phi_Y$, where the pull-back and push-forward is defined over $Y_0$ (and then extended by zero):

$$MA(\phi_Y) = F_*(e^{F^*\phi_Y} dz \wedge d\bar{z}) \tag{5.6}$$

Finally, we define the canonical probability measure $\mu_X$ on $X$ of Song-Tian-Tsuji as

$$\mu_X := F^* \mu_Y \wedge \mu_{X/Y}$$

Equivalently, $\mu_X = F^* \omega^n_X \wedge \omega^n_{CY}$, where $\omega^n_{CY}$ denotes a family of Ricci flat Kähler metrics defined over the very general Calabi-Yau fibers, where the metrics are normalized to have unit-volume.

**Theorem 5.3.** Let $X$ be projective variety of positive Kodaira dimension. Then the empirical measures of the canonical random point processes on $X$ converge in probability towards the canonical probability measure $\mu_X$ of Song-Tian-Tsuji.

**Proof.** By the previous lemma we can write $\mu^{(N_k)} = F^* \mu_Y^{(N_k)} \wedge \mu_{X/Y}^{(N_k)}$, where $\mu_Y^{(N_k)}$ is defined wrt the big line bundle $K_Y + L_{X/Y} \rightarrow Y$ equipped, where $L_{X/Y}$ is equipped
with the Hodge metric. In particular, the \( j \)-point correlation measures \( (\mu^{(N)})_j \) are given by
\[
(\mu^{(N)})_j := \int_{X_{N-j}} \mu^{(N)} = F^*(\mu_Y^{(N)})_j \wedge \mu_{X/Y}^j.
\]
Next we note that we may proceed as in the proof of Theorem 4.6 to see that the empirical measures of \( \mu_Y^{(N)} \) converge in probability towards \( \mu_Y \) (and even with a LDP), which implies that
\[
(\mu_Y^{(N)})_j \to \mu_Y^j.
\]
(remark 5.5. Indeed, fixing smooth Hermitian metrics (weights) \( \phi \) and \( \phi_0 \) on \( K_Y \) and \( L_{X/Y} \) the measure \( (s_k \wedge s_k)^{1/k} e^{-\phi \Delta} \) defined by an element \( s_k \in H^0(Y, k(K_Y + L_{X/Y})) \) may be written as \( \|s_k\|^2 \mu_\phi e^{-(\phi Y - \phi_0)} \) and by the previous lemma
\[
\mu_\phi e^{-(\phi Y - \phi_0)} = e^{-v} \mu_\Delta := \mu_0
\]
for a klt divisor \( \Delta \) where \( v \) is upper semi-continuous and such that \( e^{-v} \mu_\Delta \) is a finite measure (even with an \( L^p \)-density for some \( p > 1 \)). But then we may proceed precisely as in the proof of Theorem 4.6 the argument for the upper bound works the same for any finite measure \( \mu_0 \) and for the proof of the lower bound we can take a sequence \( v_j \) of continuous functions decreasing to \( v \) and replace \( \mu_0 \) with \( \mu_j := e^{-v_j} \mu_\Delta \). Then we let \( j \to \infty \) in the end of the argument, just as before and obtain the desired convergence in probability towards the deterministic measure \( \mu_Y \) satisfying \( \mu_Y = MA(\phi Y) \), where \( MA(\phi Y) = e^{\phi Y - \phi_0} idw \wedge d\bar{w} \), as desired. \( \square \)

Finally, combining the previous convergence with Sanov’s theorem gives
\[
(\mu^{(N)})_j \to F^*(\mu_Y^j) \wedge \mu_{X/Y}^j = (F^*(\mu_Y) \wedge \mu_{X/Y})^j,
\]
which equivalently means that the canonical empirical measures on \( X \) converge in probability towards \( (F^*(\mu_Y) \wedge \mu_{X/Y}) \), as desired.

**Corollary 5.4.** Let \( X \) be projective variety of positive Kodaira dimension. Then the currents
\[
\omega_k := \int \frac{i}{2\pi} \partial \bar{\partial} \log \left| \left( \det S^{(k)}(\cdot, z_1, \ldots, z_{N_k-1}) \right)^{2/k} dz_1 \wedge \bar{dz}_1 \wedge \cdots \wedge dz_{N_k-1} \wedge \bar{dz}_{N_k-1} \right|
\]
converge, as \( k \to \infty \), weakly towards a canonical positive current in \( c_1(K_X) \) which, on a Zariski open subset coincides with \( F^*\omega_Y \), i.e. the pull-back of the canonical metric on the base of the Ithaka fibration.

**Proof:** Arguing exactly as in the proof of Corollary 4.9 gives that \( \omega_k \to d\bar{d} \log(\mu_X) \). But specializing formula 4.6 to \( X_0 \) over \( Y_0 \) gives that \( d\bar{d} \log(\mu_X) = d\bar{d} F^*\omega_Y + 0 \) on \( X_0 \), using that \( d\bar{d} \log(\Omega^{(k)}_{2})^2 = 0 \) and that the terms involving \( d\bar{d} \phi_H \) cancel (as is seen by working on a local set \( U \) in \( X_0 \) as in the beginning of the proof of the previous lemma).

**Remark 5.5.** If \( K_X \) is semi-ample we can take \( F \) to be the morphism defined by the canonical map at some fixed level \( k \) so that \( Y := X_{can} \) is the canonical model of \( X \). Then we can define \( \omega_{can} \) as \( \omega_{can} := d\bar{d} \phi_{can} \) where \( \phi_{can} \) is the unique (locally bounded) positively curved metric on \( \mathcal{O}(1)|_{X_{can}} \) solving the equation 5.3 (using that \( \mathcal{O}(1) \) is naturally isomorphic to \( K_Y + L_{X/Y} \) over \( Y_0 \)). The metric \( \omega_{can} \) thus defined
yields a canonical Kähler metric in $c_1(\mathcal{O}(1)|_{X_{can}})$ which, by the uniqueness argument in [40] coincides with the canonical metric constructed in [40]. Accordingly, the limiting current obtained in the previous corollary coincides with $F^*\omega_{can}$ on $X_0$ and hence everywhere since the currents are elements in the same cohomology class $c_1(K_X)$. In this setting the previous corollary is thus analogous to the convergence result for the Kähler-Ricci flow for a variety with $K_X$ semi-ample obtained in [40].

6. LOG CANONICAL PAIRS, CUSP FORMS AND ARITHMETIC VARIETIES

In this section $(X, D)$ will denote a log smooth pair such that $D$ is a log canonical $\mathbb{Q}$–divisor and assume that $L := K_X + D$ is ample. We decompose $$D = D_{klt} + D_{lc},$$ where $D_{lc}$ is the purely log canonical part, i.e. containing the components with coefficient $c_i = 1$. If $D_{lc}$ in non-trivial then the corresponding singular volume forms $\mu_D$ have a density which is non-integrable close to the support of $D_{lc}$ and hence the previously defined canonical random point process is not well-defined (since the corresponding partition functions will be non-finite). The natural way around this problem is to replace the space $H^0(X, k(K_X + D))$ with the subspace of cusp forms, i.e. the subspace of all section vanishing along the divisor $D_{lc}$:

$$H^0(X, k(K_X + D))_{\text{cusp}} := \{ s_k \in H^0(X, k(K_X + D_{lc}) : s_k = 0 \text{ on } D_{lc} \}$$

(whose dimension will be denoted by $N_k$). We can then take the generator $\det S^{(k)}$ in the corresponding determinant line, denoting it by $\det S^{(k)}_{\text{cusp}}$. Then the corresponding partition function

$$(6.1) \quad Z_{N_k} := \int_{X_{N_k}} \left| (\det S^{(k)}_{\text{cusp}})(z_1, \ldots, z_{N_k}) \right|^{2/k} \frac{dz_1 \wedge d\bar{z}_1}{|s_D|^2} \wedge \cdots \wedge \frac{dz_{N_k} \wedge d\bar{z}_{N_k}}{|s_D|^2}$$

is finite. Indeed, freezing all but one variables of $\det S^{(k)}_{\text{cusp}}$, so that it can be identified with a holomorphic section of $L \to X$ vanishing along $D_{lc}$, we see that the density $|\det S^{(k)}_{\text{cusp}}|^{2/k}|s_D|^{-2}$ on $X$ may be locally estimated from above by a constant times $|s_{D_{lc}}|^{2(1-1/k)}|s_D|^{-2}$. But the latter function is integrable for $k > 0$ fixed, since $D$ is assumed log canonical. Iterating the argument $N$ times thus reveals that $Z_N$ is indeed finite.

To any log canonical pair $(X, D)$ as above one can attach a canonical Kähler-Einstein metric $\omega_{KE}$, which defines a current on $X$ and satisfies the equation (6.13) above in a weak sense (see [14]). However in this setting, the potential of $\omega_{KE}$, i.e. the corresponding solution $u_{KE}$ to the equation (6.10) will not be bounded even if $K_X + D$ is ample. Indeed, since the reference weight $\phi_0$ is continuous (and in particular locally bounded) the corresponding measure $\mu_0 := \mu(\phi_0, D)$ is not locally finite close to the support of $D_{lc}$ and hence $u_{KE}$ has to be singular along $D_{lc}$ in order for the rhs in equation (6.10) to have finite mass (which is a necessary condition since the lhs of the equation always has finite mass). In fact, when $K_X + D$ is ample it follows form the work of Kobayashi and Tian-Yau (see and references therein) that $u$ has log log singularities along $D_{lc}$, or more precisely:

$$(6.2) \quad u_{KE} := -2 \log \log(-\|s_{D_{lc}}\|^2) + O(1),$$

where $\|\cdot\|$ is a fixed metric on $\mathcal{O}(D_{lc})$. In particular, when $D = D_{lc}$ (i.e. $D$ is reduced) $\omega_{KE}$ defines a complete Kähler-Einstein metric on the quasi-projective variety $X - D$.  

As will be next shown Theorem 6.8 extends to this log canonical setting. However, there are some new technical difficulties that we have to deal with. These problems arise since we are forced to work with singular metrics on \( L := X + D \). Indeed, as explained above taking a locally bound reference weight \( \phi_0 \) gives a non-finite measure \( \mu_0 := \mu(\phi_0,D) \). Accordingly, we will instead take a singular weight \( \phi \) of the form

\[
\phi = \phi_0 + v, \quad v := \chi(\| s_{D,k} \|^2),
\]

where \( \chi(t) \) is a convex function such that \( |d\chi(t)/dt| \leq 1 \) for \( t \geq 0 \) with the property that \( v \) has finite energy and the measures \( \mu_0 := \mu(\phi_0,D) \) are finite for any sufficiently small positive number \( \epsilon \) (for example, \( \chi(t) = -2\log(-t) \) as in formula \( (6.2) \) would do, but another convenient choice is \( \chi(t) = -(\alpha t) \) for \( \alpha \) a sufficiently small positive number; compare Lemma 3.3 in [14] for the general setting when \( L \) is big).

Using the weight \( \phi \) thus corresponds to introducing a singular metric \( \| \cdot \|^2 = \| \cdot \|^2 e^{-v} \) on \( K_X + D \). The drawback of using the latter singular metric is that we ignore if the first point of Theorem 6.4 still holds in this setting. On the other hand it follows from the recent results in [15] that the \( L^2 \)-analog of the corresponding result does hold:

**Theorem 6.1.** Let \( (X,D) \) be a smooth log canonical pair. Assume that \( K_X + D \) is ample and let \( \| \cdot \| \) be a smooth metric on \( K_X + D \) with positive curvature form \( \theta_0 \). Assume that \( v \) has finite energy (wrt \( \theta_0 \)). Then, for any \( u \in C^0(X), \)

\[
\lim_{k \to \infty} -\frac{1}{kN_k} \log \int_X \left\| \det S_{\text{cusp}}^{(k)} \right\|^2 e^{-k(v+u)} \mu_0^{\otimes N} = \mathcal{E}_{\theta_0}(P_{\theta_0}(v+u)),
\]

where \( \mu_0 := \mu(\phi_0,D) \) and \( \det S_{\text{cusp}}^{(k)} \) is the generator of the determinant line of \( H^0(X,k(K_X + D))_{\text{cusp}} \) determined by bases in \( H^0(X,k(K_X + D))_{\text{cusp}} \) which is orthonormal wrt the \( L^2 \)-norm determined by \( \| \cdot \| \mu(\phi_0,D) \). In particular, the convergence holds for \( v \) with log log singularities along \( D_{lc} \) and \( \mu_0 \) with Poincaré growth along \( D_{lc} \).

**Proof.** Let us for simplicity assume that \( D = D_{lc} \) but the general argument is similar. First observe that the map \( s_k \mapsto s_k/s_D \) induces an isometry

\[
H^0(X,k(K_X + D))_{\text{cusp}} \cong H^0(X,(k-1)L + K_X)
\]

where the lhs is equipped with the \( L^2 \)-norm determined by \( \| \cdot \|_{v,\mu(\phi_0+v,D)} \) and the rhs is equipped with the natural adjoint \( L^2 \)-norm induced by the weight \( \phi \). Decomposing \( kL = (k-1)L + K_X + D \) the integral in the statement of the theorem may for \( u = 0 \) hence be rewritten as

\[
\mathcal{L}_k(\phi) := -\frac{1}{kN_k} \log \int_X \left| \det S^{(k-1)}(\phi) e^{-(k-1)\phi} i\omega \right| d\omega \wedge \cdots,
\]

where now \( \det S^{(k-1)}(\phi) \) is a generator for the determinant line of \( H^0(X,(k-1)L + K_X) \). By Theorem 3.5 in [15] the latter expression converges to \( \mathcal{E}_\theta(v) \) as \( k \to \infty \).

To handle the general case when \( u \neq 0 \) one proceeds as follows. Let \( \psi := P(\phi + u) \), where \( P \) denotes the analogue of the projection \( P_\theta \) in the language of weights (compare [10]). By definition \( \psi \leq \phi + u \) and hence \( \mathcal{L}_k(\psi) \leq \mathcal{L}_k(\phi + u) \). But \( \psi \) has finite energy and hence we can again apply Theorem 3.5 in [15] to deduce that \( \mathcal{L}_k(\psi) \to \mathcal{E}(P(\psi)) \). Finally, to prove the upper bound on \( \mathcal{L}_k(\phi + u) \) we use that
\(\phi\) is usc, i.e. \(\phi\) may be written as a decreasing limit \(\phi_j\) of continuous weights. In particular, for \(j\) fixed, \(L_k(\phi + u) \leq L_k(\phi_j + u)\). Applying Theorem A in [10] (which we recall is equivalent to the first point of Theorem 4.4) hence gives that \(L_k(\phi_j + u) \rightarrow \mathcal{E}(P(\phi_j + u), k \rightarrow \infty)\). Finally, letting \(j \rightarrow \infty\) and using the continuity of \(\mathcal{E}\) under decreasing limits concludes the proof. \(\square\)

In fact, the proof of Theorem 4.4 passes via the corresponding \(L^2\)-result, using the Bernstein-Markov property of a pair \((|||, dV)\) consisting of a smooth metric and a volume form, i.e the property that the distortion between the corresponding \(L^\infty\) and \(L^2\)-norms on \(H^0(X, kL)\) is subexponential in \(k\) (which follows from the local sub-mean property of holomorphic functions). It seems naturally to conjecture that this property also holds in the singular setting of the previous theorem, but for the moment this only seems to be known in the one-dimensional case (compare the discussion in the introduction of [15], where relations to some highly non-trivial number theory, such as Deligne’s proof of the Weil conjectures and the Ramanujan bounds, are pointed out). Still, as formulated in the following lemma, a weaker form of the Bernstein-Markov property does hold and it will be adequate for our purposes.

**Lemma 6.2.** Let \(L \rightarrow X\) be a compact manifold and \((X, D)\) a log smooth log canonical pair. Assume that \(L\) is equipped with a singular weight \(\phi\) as in formula (6.3) (for example a weight with log-log singularities). Then, for any \(\epsilon > 0\) there exists a constant \(C_\epsilon\) and a smooth weight \(\phi_0\) on \(L\) such that

\[
|s_k(x)|^2 e^{-k\phi(x)} \leq C_k e^\epsilon e^{(\phi_0 - \phi)(x)} \int_X |s_k(x)|^2 e^{-k\phi(x)} d\mu(\phi, D)
\]

for any holomorphic section \(s_k\) of \(kL\) vanishing along the purely log canonical component \(D_{lc}\) of \(D\).

**Proof.** In the klt case this is a simple consequence of the local submean property of holomorphic functions and to simplify the exposition we will thus assume that \(D = D_{lc}\). First, it is easy to see that the \(L^\infty - L^2\)-distortion may be estimated by \(C_k e^{k\epsilon}\) on a compact subset of \(X - D\) (using the submean property of holomorphic functions) so the problem is to get a bound close to \(D\). Let us first consider the one-dimensional case and fix local coordinates centered around a point in the support of \(D\), so that locally around \(D\) we have \(s_D = z\). We fix a point \(z_0\) in the unit-disc, say. Let us first show that for any holomorphic function in the unit-disc

\[
|f|^2 e^{-(k-1)\phi}(z_0) \leq C e^{-\phi_D(z_0)} \int_{|z|<1} |f|^2 e^{-(k-1)\phi} d\phi d\bar{\phi} \wedge d\bar{z}
\]

(where now \(e^{-\phi_D(z)} = 1/|z|^2\)). To this end first note that by the submean value property

\[
|f|^2(z_0) \leq \frac{1}{r^2} \int_{|z-z_0|<r} |f|^2 dz \wedge d\bar{z}
\]

The idea is now to let \(r\) depend on \(z_0\) in a suitable way related to the oscillation of \(\phi\), which we recall is assumed to be of the form in formula (6.3). Since \(|\chi'| \leq 1\) we have \(|\partial \phi(z)/\partial z| \leq C_1/|z|\). Let now

\[
r(z_0) := \delta|z_0|
\]
for $0 < \delta < 1/2$ fixed. Then on the disc $\Delta_{r_0}(z_0)$ centered at $z_0$ of radius $r(z_0)$ we have, by the previous estimate on the derivative,

$$|\phi(z) - \phi(z_0)| \leq C\frac{1}{(1 - \delta)|z_0|} \delta |z_0| \leq 2C\delta$$

Hence,

$$|f|^2 e^{-(k-1)\phi}(z_0) \leq \delta^{-2}|z_0|^{-2} e^{2C\delta(k-1)} \int_{\Delta_{r_0}(z_0)} |f|^2 e^{-(k-1)\phi(z)} dz \wedge d\bar{z}$$

and since for $z_0$ close to 0 the disc $\Delta_{r_0}(z_0)$ is contained in the unit-disc this proves (6.3). Now we take a cusp section $s_k$ in $H^0(X, kL)$. Its restriction to the unit-disc above can be written as $s_k = f z$ for a holomorphic function $f$ in the unit-disc. Hence, applying (6.3) gives

$$|s_k(z_0)|^2 e^{-(k-1)\phi(z_0)} \leq \delta^{-2} e^{2C\delta(k-1)} \int_{|z|<1} |s_k|^2 e^{-k\phi} dz \wedge d\bar{z},$$

which gives the desired estimate (after rescaling $\delta$). To treat the case of a smooth reducible divisor $D$ in general dimensions we locally write $D = \{z_1 = 0\}$ and replace the disc above with a poly-disc

$$\Delta_{r_0} \times \Delta_{\delta} \times ... \times \Delta_{\delta}$$

centered at $z_0$ and apply the previous argument in the first variable and the usual mean-value property in the other variables. The general case of a log canonical divisor may be essentially reduced to the previous case, by passing to a log resolution. \hfill \Box

We are now in a position to prove the LDP for log canonical pairs. But let us first comment on the definition of the canonical free energy functional $F$ in this setting. It may be defined as in the klt case (formula 4.14) but replacing the smooth fixed weight $\phi_0$ with a weight $\phi$. Alternatively, in order to avoid using the singular background curvature form $\theta := d\bar{\partial} \phi$ it will be convenient to fix a smooth weight $\phi_0$, as before and write $\phi := \phi_0 + v$ as in formula (6.3). Then we can first simply define

$$E_\phi(\mu) := E_{\phi_0}(\mu) + \int_X v\mu - C_v,$$

where the constant $C_v$ is there to ensure that the infimum of $E_\phi$ vanishes (compare Remark 4.7) and then define $F$ as in the klt case, i.e. by $F = E_{ddc\phi} + D_{\mu(\phi, \Delta)}$.

**Theorem 6.3.** Let $(X, D)$ be a log canonical pair such that $X$ is smooth and $K_X + D$ is ample. Then the empirical measures of the corresponding random point process converge in probability towards the normalized volume form $dV_{KE}$ of the Kähler-Einstein metric attached to $(X, D)$. More precisely, the laws of the empirical measures of the corresponding canonical random point processes satisfy a LDP with rate functional $F$.

**Proof.** We equip $K_X + D$ with a weight $\phi = \phi_0 + v$ of the form in formula (6.3) and set $\mu_0 := \mu(\phi, D)$ for the corresponding measure with Poincaré growth. We will write $|||^2_v$ for the corresponding singular metric on $K_X + D$ (so that $|||^2_v (= |||^2)$ is the smooth metric corresponding to $\phi_0$). We observe that

$$\sup_{u \in C^0(X)} (E_{\phi_0}(P_{\phi_0}(v + u) - (u, \mu)) = E_{\phi_0}(\mu) + \int v\mu$$

(6.5)
as follows from Prop \ref{prop:1.3} and the fact that \(v\) is upper semi-continuous, i.e. a decreasing limit of continuous functions. We will adapt the first proof of Theorem \ref{thm:3.7} to the present setting. Accordingly we set \(\Gamma_N := (\delta_N)_* \|\det S^{(k)}\|_{k^0}^{2/k} \mu_0^{\otimes N}\), where \(\det S^{(k)}\) is a generator of the determinant line of \(H^0(X, k(K_X + D))_{\text{cusp}}\), defined by a basis in \(H^0(X, k(K_X + D))_{\text{cusp}}\) which is orthonormal wrt \((\|\cdot\|, \mu_0)\). We are going to show that \(\Gamma_N\) satisfies a LDP with rate functional \(E_{\theta_0}(\mu) + \int \nu \mu + D_{\mu_0}(\mu)\).

To reduce the problem to the asymptotics in Prop \ref{prop:4.4} we first need to verify that \(\frac{1}{N} \log Z_N\) is uniformly bounded (the new feature here is that we ignore if \(H^{(N)}/N\) is uniformly bounded from below). Combining Lemma \ref{lem:6.2} (applied \(N\) times, i.e. to each factor of \(X^N\)) with Theorem \ref{thm:6.1} gives

\[
\left(\|\det S^{(k)}\|_{k^0}^{2/k} e^{v/k}\right) \leq C e^{N/k} e^{N(-E_{\theta_0}(P_{\theta_0}(v+u)+o(1))}
\]

for any \(u \in C^0(X)\). In particular, applying this to \(u = 0\) and integrating over \(X^N\) gives \(\frac{1}{N} \log Z_N \leq C\) using that, by assumption, \(\int_X e^{-v/k}\mu_0 \leq C\), for \(k\) large. We next go on to establish the required upper bound on \(\Gamma_N(B_\epsilon(\nu))\). To this end we first rewrite \(\Gamma_N(B_\epsilon(\nu)) :=
\]

\[
= \int_{B_\epsilon(\nu)} \|\det S^{(k)}\|_{k^0}^{2/k} \mu_0^{\otimes N} = \int_{B_\epsilon(\nu)} \left(\|\det S^{(k)}\|_{k^0}^{2/k} e^{v/k} \right) (e^{u-v/k} \mu_0)^{\otimes N}
\]

Applying the inequality \ref{thm:6.1},

\[
\frac{1}{N} \log \int_{B_\epsilon(\nu)} \|\det S^{(k)}\|_{k^0}^{2/k} \mu_0^{\otimes N} \leq \frac{1}{k} \log C_{\epsilon} + E_{\theta_0}(P_{\theta_0}(v+u)+o(1)) + \frac{1}{N} \log \int_{B_\epsilon(\nu)} (e^{u-v/k} \mu_0)^{\otimes N}
\]

In particular, for \(k = k_0\) fixed (and sufficiently large) applying Sanov’s theorem to the measure \(e^{u-v/k_0} \mu_0\), by which the assumption on \(v\) is finite for \(k_0\) sufficiently large, gives

\[
\lim \limsup_{\epsilon \to 0} \frac{1}{N} \log \int_{B_\epsilon(\nu)} \|\det S^{(k)}\|_{k^0}^{2/k} \mu_0^{\otimes N} \leq -E_{\theta_0}(P_{\theta_0}(v+u)) - \int_X uv + D_{e^{-v/k_0} \mu_0}(\nu),
\]

which, when taking the sup over all \(u\), may be estimated from above by

\[
- \left( E_{\theta_0}(\mu) + \int \nu \mu + D_{\mu_0}(\mu) \right) - D_{e^{-v/k_0} \mu_0}(\nu) - \frac{1}{k_0} \int \nu \nu
\]

if \(\int (-\nu) \nu\) is finite. In particular, letting \(k_0 \to \infty\) gives, for any such \(\nu\),

\[
\lim \limsup_{\epsilon \to 0} \frac{1}{N} \log \int_{B_\epsilon(\nu)} \|\det S^{(k)}\|_{k^0}^{2/k} \mu_0^{\otimes N} \leq - \left( E_{\theta_0}(\mu) + \int \nu \mu + D_{\mu_0}(\mu) \right).
\]

To handle the general case we write a general probability measure \(\nu\) as a weak limit of probability measures \(\nu_j\) compactly supported in \(X - D\) (compare the proof of Theorem \ref{thm:4.4}). Then, for \(\epsilon\) fixed, we have that \(B_\epsilon(\nu) \subset B_{2\epsilon}(\nu_j) \subset B_{2\epsilon}(\nu_j) \subset B_{3\epsilon}(\nu)\) for \(j\) sufficiently large. Hence, proceeding as above, but using Sanov’s theorem for \(\epsilon\) fixed gives, for any \(j \geq j_0\)

\[
\limsup_{N \to \infty} \frac{1}{N} \log \int_{B_\epsilon(\nu)} \|\det S^{(k)}\|_{k^0}^{2/k} \mu_0^{\otimes N} \leq -E_{\theta}(P_{\theta}(v+u)) - \inf_{B_{2\epsilon}(\nu_j)} D_{e^{v} \mu_0} \leq -F_{\theta}(\nu) - \inf_{B_{3\epsilon}(\nu)} D_{e^{v} \mu_0}
\]
and (i.e. a scheme, projective, proper and flat over Spec \( \text{variety} \), fiber of the structure morphism \( H \)).

In the sense of arithmetic geometry, there is indeed a canonical sequence of partition functions that allows one to define arithmetic intersection numbers and derivatives of Maillot-Sleeper’s conjectures in Arakelov geometry concerning the relation between complexification may be identified with the \( N \) of the corresponding determinant line. However, when \( X \) is defined over \( \mathbb{Z} \), this is not intrinsically defined by the complex variety \( X \) (or a log pair \( (X, D) \)) as they depend on the choice of a generator \( \det S^{(k)} \) of the corresponding determinant line. However, when \( X \) is defined over \( \mathbb{Z} \), in the sense of arithmetic geometry, there is indeed a canonical sequence of partition functions \( Z_{N_0} \) and its asymptotics turns out to be related to Kudla’s program and Maillot-Sleeper’s conjectures in Arakelov geometry concerning the relation between arithmetic intersection numbers and derivatives of \( L \)-functions at negative integers.

Let us start by briefly recalling the usual setup in Arakelov geometry \([47]\). Let \((X_\mathbb{Q}, L_\mathbb{Q})\) be an \( n \)-dimensional polarised projective algebraic variety defined over \( \mathbb{Q} \) and \((\mathcal{X}, \mathcal{L})\) a model for \((X_\mathbb{Q}, L_\mathbb{Q})\) over \( \mathbb{Z} \). This means that \( \mathcal{X} \) is an arithmetic variety, i.e. a scheme, projective, proper and flat over \( \text{Spec}\mathbb{Z} \), where the generic fiber of the structure morphism \( \mathcal{X} \to \text{Spec}\mathbb{Z} \) is equal to \( X_\mathbb{Q} \) and \( \mathcal{L} \to \mathcal{X} \) is a relatively ample line bundle (i.e. an invertible sheaf) whose restriction to the generic fiber coincides with \( L_\mathbb{Q} \). By construction, \( H^0(\mathcal{X}, \mathcal{L}) \) is a \( \mathbb{Z} \)-module whose complexification may be identified with the \( N_0 \)-dimensional complex vector space \( H^0(X, L) \), where \((X, L)\) denotes the polarised complex manifold corresponding to \((X_\mathbb{Q}, L_\mathbb{Q})\). Equipping \( L \) with an Hermitian metric (invariant under conjugation) allows one to define arithmetic intersection numbers in the sense of Gillet-Soule; the corresponding arithmetic top intersection number is usually called the height.

The main case of arithmetic interest appears when \( L = K_X + D \) for a log canonical divisor \( D \) defined over \( \mathbb{Q} \) with simple normal crossings or even more specifically: when \( X \) is the Baily-Borel compactification of an arithmetic quotient \( X_0 := B/G \) of a bounded symmetric domain (for example a Shimura variety) and \( D \) is the boundary divisor \( X - X_0 \). Passing to a log resolution of \((X, D)\) we may then assume that \( X \) is smooth if we replace the relative ampleness assumption by relative semi-ampleness. In this setting we will, for simplicity also assume that the divisor

\[
-\mathcal{E}_\varnothing(P_0(v + u) - D_{\varnothing\mu_0}(v)) = -(\mathcal{E}_\varnothing(P_0(v + u) - \int_X uv) - D_{\mu_0}(v))
\]

Hence, taking the sup over all \( u \in C^0(X) \) and using \([6,5]\) concludes the proof in the general case.

Finally, the lower bound on \( \int_B \| \det S^{(k)} \|^{2/k}_{\text{lev}} \mu^N_{\text{lev}} \) is proved essentially as in the klt case by regularizing \( v \), i.e. writing \( v \) is a decreasing limit of continuous function \( v_j \). All in all this shows that \( \Gamma_N := (\delta_\mathcal{N})_* \| \det S^{(k)} \|^{2/k}_{\text{lev}} \mu^N_{\text{lev}} \) satisfies a LDP with rate functional \( E_{\varnothing_0}(\mu) + f v_\mu + D_{\mu_0}(\mu) \). But then it follows immediately that the corresponding normalized measures \( \mu^{(N)} \), defining the canonical random point process, satisfy and LDP with rate functional \( E_{\varnothing_0}(\mu) + f v_\mu + D_{\mu_0}(\mu) - C' \) for some constant \( C' \), which by the definition of the canonical free energy functional \( F \) attached to \((X, D)\) may be written as \( F(\mu) - C \), for another constant \( C \). To conclude the proof we just have to verify that \( C = 0 \). But this is shown just as in the klt case (compare the proof of Theorem 1.5).

6.1. **The canonical partition functions on arithmetic varieties.** As emphasized above the partition functions \( Z_{N_0} \) are not intrinsically defined by the complex variety \( X \) (or a log pair \((X, D)\)) as they depend on the choice of a generator \( \det S^{(k)} \) of the corresponding determinant line. However, when \( X \) is defined over \( \mathbb{Z} \), in the sense of arithmetic geometry, there is indeed a canonical sequence of partition functions \( Z_{N_0} \) and its asymptotics turns out to be related to Kudla’s program and Maillot-Sleeper’s conjectures in Arakelov geometry concerning the relation between arithmetic intersection numbers and derivatives of \( L \)-functions at negative integers.
$D$ is defined over $\mathbb{Z}$ as a Cartier divisor. It should be stressed that when working with automorphic line bundles in this setting one has to allow metrics which are (mildly) singular along the boundary divisor, or more precisely metrics which are good in the sense of Mumford (or more generally, the weights of the metrics have log log singularities) \cite{21}.

To see the relation to the present setup we note that the choice of a model over $\mathbb{Z}$ singles out a particular generator $\det S^{(k)}$ in the determinant line of $H^0(X, kL)$, defined up to a factor of $\pm 1$, by taking the bases elements $s_i^{(k)}$ as generators of the $\mathbb{Z}$-module $H^0(\mathcal{F}, kL)$ in $H^0(X, kL)$. In particular, the density $|\det S^{(k)}|^{2/k}$ is canonically defined and the model over $\mathbb{Z}$ thus allows one to define a canonical partition function $Z_{N_k}$. Combining the convergence results above with the generalization of Gillet-Soulé’s arithmetic Hilbert-Samuel formula in \cite{15} we arrive at the following

**Theorem 6.4.** Let $(X, D)$ be a log canonical pair defined over $\mathbb{Q}$, which is log smooth and such that the log canonical line bundle $K_X + D$ is ample and let $(\mathcal{F}, \mathcal{L})$ be a model for $(X, K_X + D)$ over $\mathbb{Z}$. Then the corresponding canonical partition functions $Z_{N_k}$ satisfy

$$- \lim_{k \to \infty} \frac{1}{N_k} \log Z_{N_k} = h_{\omega_{KE}}(\mathcal{F}, \mathcal{L})$$

where $h_{\omega_{KE}}(\mathcal{F}, \mathcal{L})$ denotes the height of $(\mathcal{F}, \mathcal{L})$ with respect to the metric on $K_X$ induced by the Kähler-Einstein metric $\omega_{KE}$. More generally, the result holds for $K_X + D$ semi-ample and big as long as $\mathcal{L}$ is nef on vertical fibers and the Kähler-Einstein metric has log-log type singularities.

**Proof.** Denote by $\det S^{(k)}_{KE}$ the generator defined by a basis in $H^0_{cusp}(X, K_X + D)$ which is orthonormal wrt the $L^2$-product induced by the Kähler-Einstein metric (attached to $(X, D)$), i.e. the $L^2$-product determined by $(\phi_{KE}, \mu_{(\phi_{KE}, D)})$. By basic linear algebra the canonical generator $\det S^{(k)}$, determined by the fixed arithmetic model, satisfies $|\det S^{(k)}|^2 = |\det S^{(k)}_{KE}|^2 \det H^{(k)}$ where the Hermitian matrix $H^{(k)}$ is defined by $H^{(k)}_{ij} = \langle s_i^{(k)} , s_j^{(k)} \rangle_{KE}$, where $s_i^{(k)}$ are fixed generators of the integral lattice in $H^0_{cusp}(X, k(K_X + D))$ determined by the arithmetic model. Hence, (6.7)

$$- \frac{1}{N_k} \log Z_{N_k} = - \frac{1}{N_k} \log \int_X |\det S^{(k)}_{cusp}|^{2/k} \frac{dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k} \wedge d\bar{z}_{N_k}}{|s_D|^2} - \frac{1}{N_k} \log \det H^{(k)}$$

By the arithmetic Hilbert-Samuel formula in \cite{15} the second term (which coincides with the arithmetic degree of the corresponding determinant line) converges to the top arithmetic intersection number $h_{KE}(\mathcal{F}, \mathcal{L})$ determined by the arithmetic model and the Kähler-Einstein metric and the proof will just be concluded once we have verified that the first term above tends to zero, which we will deduce from Theorem 6.3. First take $\phi = \phi_{KE}$ in the construction of the canonical random point process attached to the log canonical $(X, D)$. Writing $\phi_{KE} = \phi_0 + v$ gives $|\det S^{(k)}_0|^2 = |\det S^{(k-1)}_0|^2 \det H^{(k)}_0$, where now $\det S^{(k)}_0$ is

\footnote{In practice, this condition is non-trivial to check and may be replaced by a choice of a model $\mathcal{F}$ for $K_X$ and then defining the space of integral cusp forms at level $k$ to be $H^0(\mathcal{F}, (k-1)\mathcal{L} + \mathcal{F})$; compare the discussion in the introduction of \cite{15}.}
the generator in $H^0_{\text{cusp}}(X, k(K_X + D))$ which is orthonormal wrt the $L^2$-norm determined by $(\phi_0, \mu(\phi_0, D))$. Hence, the first term in the rhs of formula (6.7) may be written as
\[
\frac{1}{N_k} \log \int_{X^N_k} \left\| \det \zeta_{KE}^{(k)} \right\|^2 \mu(\phi_0, D) + \frac{1}{N_k} \log \det H_0^{(k)}
\]
As explained in the proof of Theorem 6.3 the first term in the rhs above tends to $\inf_\mu \left( E_{\phi_0}(\mu) + \int v \mu + D_{\mu_0}(\mu) \right)$. As for the second term it tends to $-\mathcal{E}_{\phi_0}(v)$ (by Theorem 6.1). But $\mathcal{E}_{\phi_0}(v) = \inf_\mu E_{\phi_0}(\mu) + \int v \mu$, as is seen by inverting the Legendre transform relation in Prop 4.3 (compare [12, 7]). Hence,
\[
-\frac{1}{N_k} \log \int_{X^N_k} \left\| \det \zeta_{KE}^{(k)} \right\|^2 \mu(\phi_0, D) \to \inf_\mu F,
\]
where $F$ is the canonical free energy functional defined in the discussion proceeding the statement of Theorem 6.3. Finally, as shown in the proof of the latter theorem the inf in the previous line vanishes, which concludes the proof in the ample case. Finally, we point out that the arguments extends to the case when $L \to X$ is merely semi-ample and big as long as $\mathcal{L}$ is nef on vertical fibers and $\phi_{KE}$ has log log singularities. The point is that, as shown in [15], the arithmetic Hilbert-Samuel formula still holds in this setting (and it can be used as a replacement of Theorem 6.3).

The extension to the semi-ample case is important for arithmetic applications. Indeed, starting with an arithmetic quotient $X_\mathbb{Q}$ its Baily-Borel compactification $X := X - D$ is, in general, singular. But pulling back the corresponding log canonical line bundle to a non-singular (toroidal) resolution of $X$ gives a semi-ample log canonical line bundle to which the previous theorem applies (the corresponding Kähler-Einstein metrics indeed have log log singularities as a consequence of Mumford’s theory for good metrics on automorphic vector bundles; see [34]).

Remark 6.5. Let $X_\mathbb{Q}$ be defined over $\mathbb{Q}$ with $K_{X_\mathbb{Q}}$ ample (more generally, one could consider log pairs as above). To any arithmetic model $(\mathcal{X}, \mathcal{L})$ for $(X_\mathbb{Q}, K_{X_\mathbb{Q}})$ we can attach the height type invariant $h_k(\mathcal{X}, \mathcal{L}) := -\frac{1}{N_k} \log Z_{N_k}$, where $Z_{N_k}$ is the canonical partition function defined above (which by the previous theorem tends to the height of $(\mathcal{X}, \mathcal{L})$ with respect to the metric on $L(= K_X)$ induced by the Kähler-Einstein metric $\omega_{KE}$). Inspired by Manin’s remark in [11] that the Kähler-Einstein metric $\omega_{KE}$ appears to be a reasonable analog “at infinity” of the minimality of an arithmetic model, it is tempting to conversely try to find a “minimal arithmetic model $(\mathcal{X}_k, \mathcal{L}_k)$ at level $k$” by maximizing $h_k(\mathcal{X}, \mathcal{L})$ over all arithmetic models for $(X_\mathbb{Q}, K_{X_\mathbb{Q}})$. If such a maximizer $(\mathcal{X}_k, \mathcal{L}_k)$ exists one could then let $k$ tend to infinity, hoping that the models $(\mathcal{X}_k, \mathcal{L}_k)$ stabilize or converge to a unique arithmetic “minimal model” (in some generalized sense). This is certainly very speculative and should be backed-up by concrete examples, before judging whether it is a fruitful path.

3One motivation for this procedure is that $\omega_{KE}$ may be characterized as the unique maximizer of the height $h_{\omega}(\mathcal{X}, \mathcal{L})$, when $\omega$ ranges over all Kähler metrics in $c_1(K_X)$ with negative Ricci curvature, for a fixed arithmetic model $(\mathcal{X}, \mathcal{L})$. 44
In this section we will outline a conjectural general picture concerning the case when $\beta = -1$ in the Kähler-Einstein equation, i.e. the case of Kähler-Einstein metrics with positive Ricci curvature. In other words, this the case when the dual $-K_X$ of the canonical line bundle is ample, which means that $X$ is a Fano manifold. We will establish a weak form of the conjecture, by we leave the general case for the future.

If a Kähler-Einstein metric exists on a Fano manifold $X$ then, by the Bando-Mabuchi theorem, it is uniquely determined modulo the action of automorphism group generated by holomorphic vector fields on $X$. But in general there are obstructions to the existence of a Kähler-Einstein metric on $X$ and according to the Yau-Tian-Donaldson conjecture $X$ admits a Kähler-Einstein metric precisely when $X$ is K-polystable. This latter notion of stability is of an algebro-geometric nature and can be formulated as an asymptotic version of stability in the sense of Geometric Invariant Theory (GIT). Recently, the conjecture was finally settled by Chen-Donaldson-Sun [24] and Tian independently [49]. Here we will introduce a probabilistic/statistical mechanical version of the Yau-Tian-Donaldson where the notion of K-stability is replaced by a notion that we will call Gibbs stability. To explain this first observe that to be able to define an analog of the probability measure appearing in formula 1.2 in the introduction of the paper to the Fano setting we have to replace $K_X$ with its dual $-K_X$ to ensure the existence of holomorphic sections. But this forces us to replace the exponent $1/k$ with $-1/k$, in order to get a well-defined density on $X$. However, there is then no guarantee that the corresponding normalization constant

$$Z_{N_k} := \int_{X^N} \left| (\det S^{(k)})(z_1, \ldots, z_{N_k}) \right|^{-2/k} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k} \wedge d\bar{z}_{N_k}$$

is finite, since the corresponding integrand is singular along the zero-locus of $\det S^{(k)}$. Accordingly, we say that a Fano manifold $X$ is Gibbs stable at level $k$ if $Z_{N_k}$ is finite (which is independent of the choice of generator $\det S^{(k)}$) and asymptotically Gibbs stable if it is Gibbs stable at level $k$ for any $k$, sufficiently large.

**Conjecture 7.1.** Let $X$ be Fano manifold. Then $X$ admits a unique Kähler-Einstein metric $\omega_{KE}$ if and only if $X$ is asymptotically Gibbs stable. Moreover, if $X$ is asymptotically Gibbs stable, then the empirical measures of the corresponding point processes converge in probability towards the normalized volume form of $\omega_{KE}$ and the corresponding canonical sequence of curvature forms $\omega_k$ converge weakly towards $\omega_{KE}$.

Here $\omega_k$ is defined by

$$\omega_k := -\frac{i}{2\pi} \partial \bar{\partial} \log \int_{X^N} \left| (\det S^{(k)})(z_1, \ldots, z_{N_k-1}) \right|^{-2/k} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k-1} \wedge d\bar{z}_{N_k-1}.$$

It follows from applying Berndtsson’s positivity of direct images [10] $N-1$ times (i.e. for each factor of $X^{N-1}$) that $\omega_k$ is in fact positive in the sense of currents and hence a Kähler form for $k$ sufficiently large. One can also define a weaker notion of Gibbs stability which takes automorphisms into account, but for simplicity we will focus here on the case when $X$ admits no automorphisms.

Interestingly, the notion of Gibbs stability introduced above can also be given the following purely algebro-geometric formulation in the spirit of the Minimal Model
Program: let $\mathcal{D}_k$ be the effective divisor in $X^{N_k}$ cut out by the section $\det S^{(k)}$. Geometrically, $\mathcal{D}_k$ may be represented as the following incidence divisor in $X^{N_k}$:

$$\mathcal{D}_k := \{(x_1, \ldots, x_N) \in X^{N_k} : \exists s \in H^0(X, -kK_X) : s(x_i) = 0, i = 1, \ldots, N_k\}$$

Gibbs stability at level $k$ amounts to saying that $\mathcal{D}_k/k$ is mildly singular in the sense of MMP (i.e. its singularities are Kawamata Log Terminal) or more precisely that

$$\text{lct}(\mathcal{D}_k/k) > 1 \quad \text{for } k \gg 1,$$

for $k \gg 1$, where $\text{lct}(\mathcal{D}_k/k)$ denotes the the log canonical threshold (lct) of the anti-canonical $\mathbb{Q}$–divisor divisor $\mathcal{D}_k/k$ on $X^{N_k}$. The equivalence with the original definition follows directly from the analytic definition of the lct of a divisor $D = \{s = 0\}$ as the sup of all $t$ such that $1/|s|^{2t}$ is locally integrable (also using the “openness property” for the lct of algebraic singularities). It also seems natural to say that $X$ is uniformly Gibbs stable, if

$$\gamma(X) := \liminf_{k \to \infty} \text{lct}(\mathcal{D}_k/k) > 1.$$

There is also an even stronger notion of Gibbs stability which we call strong Gibbs stability and which demands that

$$\lim_{k \to \infty} \frac{1}{N_k} \log Z_{N_k, \beta} < \infty$$

for some $\beta < 1$, where $Z_{N_k, \beta}$ is the partition function at inverse temperature $\beta$ (formula 4.7) defined with respect to a metric $\| \cdot \|$ on $-K_X$, a volume form $dV$ (for example the one defined by the metric $\| \cdot \|$) and the corresponding generator $\det S^{(k)}$.

**Proposition 7.2.** Suppose that $X$ admits non-trivial holomorphic vector fields. Then $\gamma(X) \leq 1$. More precisely, for $k$ sufficiently large the canonical partition function $Z_{N_k}$ is non-finite.

**Proof.** Assume to get a contradiction that $Z_{N_k}$ is finite. Then we may define a probability measure $\nu_k$ on $X$ as the corresponding one-point correlation measure. But, by construction, $\nu_k$ is invariant under the action of the automorphism group of $X$. Moreover, for $k$ sufficiently large $H^0(X, -kK_X)$ is basepoint free and hence $\nu_k$ then defines a volume form on $X$. But it follows from general principles that if a complex manifold $X$ admits a non-trivial holomorphic vector field $v$ than it cannot admit an automorphism invariant volume form. This is proved in a standard way by taking the Lie derivative of $\nu_k$ along the vector field $v$ and using Cartan’s formula. \qed

Next, we will show the following weak partial confirmation of the conjecture above.

**Theorem 7.3.** Suppose that the Fano manifold $X$ is strongly Gibbs stable. Then it admits a unique Kähler-Einstein metric. More generally, the corresponding result holds in the setting of (possibly singular) log Fano varieties.

**Proof.** Fix a volume form $\mu$ on $X$. By the Gibbs variational principle (Lemma 3.2) combined with Lemma 3.1

$$-\frac{1}{N_k\beta} \log Z_{N_k, \beta} \leq E^{(N)}(\mu \otimes^N) + \frac{1}{\beta} D_\mu(\mu).$$
Setting $\beta = -(1 + \delta)$ for some $\delta > 0$ and using the definition of Gibbs stability thus gives
\[-(1 + \delta)E^{(N)}(\mu \otimes N) + D_{\mu_0}(\mu) \geq -C\]
We will conclude the proof by observing that
\[
\liminf_{N \to \infty} E^{(N)}(\mu \otimes N) \geq E_\theta(\mu).
\]
(7.4)
Accepting this for the moment gives that $(1 + \delta)E(\mu) + D_{\mu_0}(\mu) \geq -C$, which, by definition, means that the canonical free energy functional $F$ (for $\beta = 1$) is coercive.

But since the latter functional may be identified with Mabuchi’s K-energy functional it then follows from a result of Tian that $X$ admits a Kähler-Einstein metric (which by the coercivity has to be unique, since the coercivity rules out automorphisms).

More generally, the existence of a minimizer of the functional $F$, satisfying the corresponding Monge-Ampère equations was shown in [13] in various singular settings, in particular in the setting of log Fano varieties [13]. Finally, let us prove the lower bound (7.4). The proof is similar to the upper bound in the proof of Theorem 4.6. By definition, for any given $u \in C^0(X)$
\[
E^{(N)}(\mu \otimes N) = \int_X^NH^{(N)}_N\mu^\otimes N = \int_X^NH^{(N)}_N + u\mu - \int_X^Nu\mu
\]
Hence, estimating $\frac{H^{(N)}_N + u}{N}$ from below, using the first point in Theorem 4.4 and taking the sup over all $u \in C^0(X)$ proves (7.4) (also using Prop 4.3 in the last step).

It may very well be that in the end all the notions of Gibbs stability introduced above will turn out to be equivalent. For the moment the author has only been able to prove this in the first non-trivial setting of one dimensional log Fano manifolds, where the analog of the Conjecture 7.1 indeed holds (the proof will appear in a separate publication).

It seems also natural to conjecture that the following invariant $\gamma(X)$ defined by
\[
\min\{\gamma(X), 1\} = R(X),
\]
where $R(X)$ is the greatest lower bound on the Ricci curvature [45], i.e. the sup over all $r \in [0, 1]$ such that there exists a Kähler form $\omega \in c_1(-K_X)$ satisfying $\text{Ric } \omega \geq r\omega$. As support for the latter conjecture we note that the inequality $R(X) \geq \min\{\gamma(X), 1\}$ follows from a simple modification of the proof of the previous theorem. Finally, let us point out that these conjectures are related to another conjectural property, namely that the reversed inequality in formula 7.4 holds, i.e.
\[
\lim_{N \to \infty} E^{(N)}(\mu \otimes N) = E_\theta(\mu)
\]
(7.5)
for any volume form on $\mu$. In one dimension this follows from the bosonization formula (a proof will appear elsewhere) and it also holds in the toric setting when $\mu$ is a torus invariant measure (as follows form the results in [9]). The validity of formula (7.5) would imply the following approximation result for the Calabi-Yau equation [54]: given a normalized volume form $dV$ on $X$ and an ample line bundle $L \to X$ the Kähler metrics
\[
\omega_k := \frac{i}{\pi} \frac{1}{k} \int_{X^{N_k-1}} \log |\det S_k(\cdot, x_1, \ldots, x_{N_k-1})| dV^{\otimes N_k-1} \in c_1(L)
\]
converge weakly towards the unique Kähler form $\omega$ in $c_1(L)$ with volume form $dV$. This conjectural approximation result is the Kähler analogue of the approximation for optimal transport maps obtained in [9] in a convex analytical setting.

8. APPENDIX: THE DIMENSIONAL CONSTANT IN THE CHENG-YAU GRADIENT ESTIMATE

Set $\phi := |\nabla u/u|$ and $F := \phi(a^2 - \rho^2)$. First, Bochner’s identity gives after some calculations [25] that, for any $x$,

$$\frac{\Delta \phi}{\phi} \geq \frac{\phi^2}{(n-1)} - (n-1)k^2 - (2 - \frac{2}{(n-1)})\frac{\nabla\phi}{\phi} \cdot \nabla u$$

(8.1)

Let now $x_1$ be a point in the interior of $B_n(x_0)$ where $F$ attains its maximum and assume that $\rho(:= d(x, x_0))$ is smooth close to $x_1$. Next $\nabla F = 0$ at $x_1$ gives

$$\frac{\nabla\phi}{\phi} = \frac{\nabla\rho^2}{a^2 - \rho^2} = \frac{2\rho\nabla\rho}{a^2 - \rho^2}$$

(8.2)

(in the following all (in-)equalities are evaluated at $x = x_1$) and $\Delta F \leq 0$ at $x_1$ gives

$$\frac{\Delta \phi}{\phi} - \frac{\Delta \rho^2}{a^2 - \rho^2} - \frac{2|\nabla \rho|^2}{(a^2 - \rho^2)^2} \leq 0$$

Next, by the Laplacian comparison

$$\Delta \rho^2 \leq 2 + 2(n-1)(1 + k\rho^2)$$

Substituting this into the previous inequality we get (using $|\nabla \rho| \leq 1$)

$$\frac{\Delta \phi}{\phi} \geq \frac{2 + 2(n-1)(1 + k\rho^2)}{a^2 - \rho^2} - \frac{8\rho^2}{(a^2 - \rho^2)^2} \leq 0$$

(8.3)

By (8.2)

$$-\frac{\nabla \phi}{\phi} \cdot \nabla u \geq -\frac{2\rho\phi}{a^2 - \rho^2}$$

Hence, equation (8.1) combined with equations (8.3) and the previous inequality gives

$$0 \geq \frac{\phi^2}{(n-1)} - (n-1)k^2 - \frac{4(n-2)}{(n-1)}\frac{2\rho\phi}{a^2 - \rho^2} - \frac{2 + 2(n-1)(1 + k\rho^2)}{a^2 - \rho^2} - \frac{8\rho^2}{(a^2 - \rho^2)^2},$$

Equivalently, multiplying by $(a^2 - \rho^2)^2$ gives

$$0 \geq \frac{F^2}{(n-1)} - (n-1)k^2(a^2 - \rho^2)^2 - \frac{4(n-2)}{(n-1)}2\rho F - (2 + 2(n-1))(1 + k\rho^2)(a^2 - \rho^2) - 8\rho^2,$$

Since we are only interested in the large $n$ limit we deduce from the previous inequality that

$$0 \geq \frac{F^2}{(n-1)} - 8\rho F - nk^2(a^2 - \rho^2)^2 - 2n(1 + k\rho^2)(a^2 - \rho^2) - 8\rho^2$$

giving, after multiplication by $n$

$$0 \geq F^2 - 8anF - n^2k^2(a^2 - \rho^2)^2 - 2n^2(1 + ka^2)(a^2 - \rho^2) - 8a^2n,$$

which we write as

$$(4an)^2 + n^2k^2(a^2 - \rho^2)^2 + 2n^2(1 + ka^2)(a^2 - \rho^2) + 8a^2n \geq (F - 4an)^2$$

giving

$$a^2 (16n^2 + n^2k^2a^2 + 2n^2(1 + ka^2) + 8n) \geq (F - 4an)^2$$
Hence,

\[ a^2 n^2 \left( 26 + 3k^2 a^2 \right) \geq (F - 4an)^2 \]

giving

\[ an \left( \left( 26 + 3k^2 a^2 \right)^{1/2} + 4 \right) \geq F := \phi(a - \rho)(a + \rho) \geq \phi(a - \phi)a, \]

so that

\[ n \left( \left( 26 + 3k^2 a^2 \right)^{1/2} + 4 \right) \geq \phi(a - \phi), \]

showing that there exists an absolute constant \( C \) such that

\[ Cn (1 + ka) \geq \phi(a - \phi), \]

as desired.

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