1. Introduction

A characteristic feature of standard plasmas is the domination of collective forces over single particle forces. This scaling is equivalent to saying that there should be a large number of particles in a Debye sphere. For this regime, as opposed to strongly coupled plasmas where the opposite condition holds, classical equations of motion are usually thought to be adequate. While this certainly applies to some of the well-known quantum effects, it is not true in general. In this article we will focus on plasmas that are classical in the sense that there are large numbers of particles in a Debye sphere, which implies the importance of collective processes. But at the same time we will include a number of different quantum plasma effects, that have been of much interest recently see e.g. Refs.,\textsuperscript{1–13} starting . The interest in quantum plasma effects has several different origins, for example recent progress in nanoscale technology,\textsuperscript{14} various astrophysical applications,\textsuperscript{15–17} high intensity effects made relevant by the continuous increase of laser powers,\textsuperscript{18,19} as well as a general theoretical motives.\textsuperscript{20–24} As indicated above, the combined focus on collective and quantum plasma effects are to some extent contradictory, as in many cases these effects are important in different regimes. Nevertheless there are several important reasons to treat them...
simultaneously:

(1) **Unification:** Before a detailed calculation has been done, it can be difficult to know whether quantum or collective effects will be dominant in a specific problem. In this case it is useful to be able to start from a set of equations including both types of phenomena.

(2) **Different scalings:** Certain quantum effects, in particular those due to quantum electrodynamics (QED) and particle spin does not necessarily become insignificant even if there is a large number of particles in a Debye sphere. For such plasmas, collective and quantum effects can simultaneously be important.

(3) **Symmetry dependent effects:** The symmetry properties of the standard and quantum terms differ to some extent in the equations of motion. Thus for a problem with a specific geometry, a classical effect may sometimes vanish due to a symmetry, while a small quantum effect survives and dominate the dynamical picture.

(4) **Extreme regimes:** In certain extreme regimes, as for example found in astrophysics, the formal conditions for collective and quantum effects to be important simultaneously can be fulfilled.

In this paper we will describe a number of different quantum phenomena that can be fit into the standard Maxwell-Fluid model by adding various terms to the classical equations. In particular we will be dealing with particle dispersion,

\[
m_{c}^{2}c^{3}/\hbar e \approx 10^{16} \text{ V/cm}^{-1}.
\]

Here \(m_{c}\) is the electron mass, \(c\) is the speed of light in vacuum, \(e\) is the elementary charge and \(\hbar\) is Plank’s constant. The applicability of the presented models will be discussed, and a number of phenomena induced by the quantum terms will be shown. Finally, various applications to specific plasmas will be pointed out.

### 2. Particle dispersion and Fermi pressure

Naturally the most basic quantum effect is that particles are described by wave functions rather than classical point particles. Following e.g. Ref., the particles are described by the statistical mixture of \(N\) states \(\psi_{i}\), \(i = 1, 2, \ldots, N\) where the index \(i\) sums over all particles independent of species. We then take each \(\psi_{i}\) to satisfy a single particle Schrödinger equation where the potentials \((A, \phi)\) is due to the collective charge and current densities,
i.e. \( \varepsilon_0 \nabla^2 \phi = -\sum_{i=1}^{N} q_i p_i |\psi_i|^2 \), etc., where \( p_i \) is the occupation probability of state \( \psi_i \). This model amounts to assume that all entanglement between particles are neglected. To derive a fluid description we make the ansatz \( \psi_i = \sqrt{n_i} \exp(i S_i/\hbar) \) where \( n_i \) is the particle density, \( S_i \) is real, and the velocity of the \( i' \)th particle is \( \mathbf{u}_i = \nabla S_i/m_i - (q_i/m_i c) \mathbf{A} \). Next we define the global density and velocity as \( n = \sum_j p_j n_j \) and \( \mathbf{u} = \sum_j p_j n_j \mathbf{u}_j/n \), where \( j \) runs over all particles. Separating the real and the imaginary part in the Schrödinger equation, we obtain the continuity equation

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{u}) = 0,
\]

and the momentum equation

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{q}{m} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{1}{mn} \nabla P + \frac{\hbar^2}{2m^2} \nabla \left( \frac{1}{\sqrt{n}} \nabla^2 \sqrt{n} \right).
\]

The last term is the gradient of the so called Bohm potential, and the tendency to smoothen a density profile naturally reflects the dispersive tendencies of a localized wave packet. Furthermore, we stress that the pressure term contains both the fermion pressure, \( P_F \), and the thermal pressure, \( P_T \). For low temperature plasmas, where the Fermi pressure is of most significance, \( P_F \) can be written as \( P_F = (4\pi^2 \hbar^2/5m)(3/8\pi)^{2/3} n^{5/3} \).

As a simple illustration of some effects due to the quantum terms we can study linear wave propagation in a homogeneous plasma that may - or may not - be magnetized. Since both the thermal pressure, the Fermi pressure and the Bohm potential becomes proportional to \( \nabla n_1 \), where \( n_1 \) is the density perturbation of the total density \( n = n_0 + n_1 \), it turns out\(^{25}\) the above quantum effects can be captured by making the following simple substitutions

\[
v_t^2 \rightarrow v_t^2 + \frac{3}{5} v_F^2 + \frac{\hbar^2 k^2}{4m^2}.
\]

in any classical linear dispersion relations. Here \( v_t \) is the thermal velocity, \( v_F \) the Fermi velocity and \( k \) the wavenumber of the perturbation. As a specific example we may consider Langmuir waves in which case the quantum version of the dispersion relation thus becomes

\[
\omega^2 = \omega_p^2 + k^2 \left( v_t^2 + \frac{3}{5} v_F^2 + \frac{\hbar^2 k^2}{4m^2} \right),
\]

where \( \omega_p \) is the plasma frequency. The dispersion relation (2) was recently experimentally verified in X-ray scattering experiments made in Laser produced plasmas.\(^{26}\)
3. Particle spin

The treatment of the previous section can be generalized to include the effects of particle spin. The following modifications are then necessary:\textsuperscript{20,21}

(1) Replace the Schrödinger equation for a scalar wave function with the Pauli equation for the spinors.

(2) Decompose the spinors according to $\psi_i = \sqrt{n_i} \exp(iS_i/\hbar) \varphi_i$, where $\varphi_i$ is a normalized two spinor.

(3) Introduce the velocity $u_i$ and the spin vector $s_i$ as $u_i = (1/m)(\nabla S_i - i\hbar \varphi_i^\dagger \nabla \varphi_i) - (q_i/m_i c) A$ and $s_i = (\hbar/2) \varphi_i^\dagger \sigma \varphi_i$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\sigma_{1,2,3}$ are the Pauli spin matrices.

It is no surprise that the resulting equations are considerably more complicated than the spinless equations in the preceding section. Rather than presenting the full theory (see Refs.\textsuperscript{20,21}) here, we will focus on the leading contributions where a number of terms of higher order in $\hbar$ are neglected. The spin effects can then be captured by a spin force $\mathbf{F}_{sp}$ that is added to the momentum equation, and a magnetization current $\mathbf{j}_m$ associated with the spins. These expressions in turn depend on a macroscopic spin vector $\mathbf{s} = \sum_i s_i/n$, that is described by a separate evolution equation complementing the Maxwell-fluid system. The results for the electrons, denoted by index $e$ are

$$\mathbf{F}_{sp} = \frac{2\mu_B n_e}{\hbar} \mathbf{s}^i \nabla B_j$$  \hspace{1cm} (3)

$$\mathbf{j}_m = \nabla \times \mathbf{M} = \nabla \times \left( \frac{2n_e \mu_B \mathbf{s}}{\hbar} \right)$$  \hspace{1cm} (4)

$$\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{s} = \frac{2\mu_B}{\hbar} \mathbf{B} \times \mathbf{s}$$  \hspace{1cm} (5)

where $\mu_B = e\hbar/2m$ is the Bohr magneton, $\mathbf{B}$ is the magnetic field, $\mathbf{M}$ is the magnetization vector and we use the Einstein summation convention in Eq. (3). The spin effects associated with the ions is usually smaller due to their larger mass. For a generalization including the spin contribution for an arbitrary particle species, see Refs.\textsuperscript{20,21}

There is a rich variety of new dynamical effects associated with the spins, as described by Eqs. (3)-(5). However, in order to start exploring the dynamics, we must first have an expression for the spin vector in thermodynamic equilibrium. The result for spin half particles is\textsuperscript{20,21} $\mathbf{s} = (\hbar/2) \tanh(\mu_B B_0/T)$, where $B_0$ is the unperturbed magnetic field and
$T$ is the temperature given in energy units. A simple example of the results that can be derived from the Maxwell-Fluid results complemented by (3)-(5) is the modification of the Alfvén velocity. Taking the MHD limit, it turns out that the Alfvén velocity $C_A = (B_0^2/\mu_0 \rho_0)^{1/2}$ is modified according to

$$C_A \rightarrow \frac{C_A}{(1 + (h\omega_{ce}^2/2mc^2\omega_{ce}(0)) \tanh(\mu B B_0/T))^{1/2}}$$

where $\omega_{ce}(0) = eB_{0\text{ext}}/m$ is the electron cyclotron frequency due to the external field $B_{0\text{ext}}$ only, i.e. the contribution from the zero order spin magnetization is excluded. The substitution (6) applies both for the shear Alfvén mode as well as for the fast and slow magnetosonic modes.

4. High field and short wavelength QED effects

The first order QED effects can effectively be modeled through the Heisenberg-Euler Lagrangian density. This Lagrangian describes a vacuum perturbed by a slowly varying electromagnetic field. The effect of rapidly varying fields can be accounted for by adding a derivative correction to the Lagrangian. This correction is referred to as the derivative QED correction or the short wavelength QED correction. The Heisenberg-Euler Lagrangian density with the derivative correction reads

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{HE} + \mathcal{L}_D = \frac{\varepsilon_0}{4} F_{ab} F^{ab} + \frac{\varepsilon_0^2 \kappa}{16} \left( 4 \left( F_{ab} F^{ab} \right)^2 + 7 \left( F_{ab} \tilde{F}^{ab} \right)^2 \right) + \sigma \varepsilon_0 \left[ (\partial_a F^{ab}) (\partial_c F^c_b) - F_{ab} \Box F^{ab} \right],$$

where $\mathcal{L}_0$ is the classical Lagrangian density, while $\mathcal{L}_{HE}$ represents the Heisenberg-Euler correction due to first order strong field QED effects, $\mathcal{L}_D$ is the derivative correction, $\Box = \partial_a \partial^a$ is the d’Alembertian, $F^{ab}$ is the electromagnetic field tensor and $\tilde{F}^{ab} = \epsilon^{abcd} F_{cd}/2$ where $\epsilon^{abcd}$ is the totally antisymmetric tensor. The parameter $\kappa = 2\alpha^2 \hbar^3/45m^4c^5$ gives the nonlinear coupling, $\sigma = (2/15)\alpha \epsilon^2/\omega_c^2$ is the coefficient of the derivative correction and $\alpha = e^2/4\pi\hbar c \varepsilon_0$ is the fine structure constant, where $\varepsilon_0$ is the free space permittivity. We obtain the field equations from the Euler-Lagrange equations

$$(1 + 2\sigma \Box) \partial_a F^{ab} = 2\varepsilon_0 \kappa \partial_a \left[ (F_{cd} F^{cd}) F^{ab} + \frac{\gamma}{4} \left( F_{cd} \tilde{F}^{cd} \right) \tilde{F}^{ab} \right] + \mu_0 j^b,$$

where $j^a$ is the four-current and $\mu_0$ is the free space permeability.
The corresponding sourced Maxwell equations resulting from the derivative corrected field equation then become
\begin{equation}
\left[ 1 + 2\sigma \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \right] \nabla \cdot E = \frac{\rho + \rho_{\text{vac}}}{\varepsilon_0}, \tag{9}
\end{equation}
\begin{equation}
\left[ 1 + 2\sigma \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \right] \left( \nabla \times B - \frac{1}{c^2} \frac{\partial E}{\partial t} \right) = \mu_0 (j + j_{\text{vac}}), \tag{10}
\end{equation}
where the vacuum charge density is \( \rho_{\text{vac}} = -\nabla \cdot P \) and the vacuum current density is \( j_{\text{vac}} = \partial P / \partial t - \nabla \times M \) with the vacuum polarization and magnetization given by
\begin{equation}
P = 2\varepsilon_0^2 \kappa [2(E^2 - c^2 B^2)E + 7c^2(E \cdot B)B] \tag{11}
\end{equation}
\begin{equation}
M = 2c^2 \varepsilon_0^2 \kappa [-2(E^2 - c^2 B^2)B + 7(E \cdot B)E] \tag{12}
\end{equation}
respectively. The source free Maxwell equations are \( \nabla \cdot B = 0 \) and
\begin{equation}
\nabla \times E = -\frac{\partial B}{\partial t}. \tag{13}
\end{equation}

The QED vacuum contribution can give raise to a large number of physical effects.\(^\text{19}\) For example the nonlinear vacuum terms implies processes such as photon-photon scattering. However, in order to keep our examples simple, we here consider just linear wave propagation, and also treat the high field effects proportional to \( \kappa \) and the short wavelength effects proportional to \( \sigma \) separately from now on.

As our first example we consider short wavelength linear wave propagation in a magnetized plasma, and keep only the QED terms proportional to \( \sigma \). It turns out that the effects due to a finite \( \sigma \) can be included in a very simple manner, by making the substitution
\begin{equation}
\omega_{ps}^2 \rightarrow \frac{\omega_{ps}^2}{1 - \zeta}. \tag{14}
\end{equation}
everywhere in the susceptibility tensor of a plasma,\(^\text{25}\) where \( \zeta = 2\sigma (\omega^2 / c^2 - k^2) \). In most cases the short wavelength QED corrections is a very small effect. The possibility to confirm such effects in laboratory has been discussed in some detail by Ref.\(^\text{25}\).

As our second QED example we consider the effects of the vacuum polarization a magnetization due to a strong external magnetic field \( B = B_0 \hat{z} \). We note that the term proportional to \( \kappa \) contributes with terms that are linear in the wave field and quadratic in \( B_0 \). Following Ref.\(^\text{31}\) we find
that the susceptibility tensor can be modified to include the QED effect of strong magnetic fields by adding the correction

\[
\chi_{\text{QED}} = -4\xi \begin{pmatrix}
1 - n_\perp^2 & 0 & n_\perp n_\parallel \\
0 & 1 - n^2 - 2n_\perp & 0 \\
n_\perp n_\parallel & 0 & -\frac{5}{2} - n_\perp^2
\end{pmatrix},
\]

(15)

where \(\xi \equiv \kappa \varepsilon_0 c^2 B_0^2 = (\alpha/90\pi)(cB_0/\varepsilon_{\text{crit}})^2\), \(n_\perp = k_\perp c/\omega\), \(n_\parallel = k_\parallel c/\omega\) and \(n = kc/\omega\). Here the indices \(\perp\) and \(\parallel\) denote the directions perpendicular and parallel to the external magnetic field respectively, and we have chosen the wavevector to lie in the \(xz\)-plane. In magnetar environments with extreme magnetic fields, the parameter \(\xi\) can approach unity. As a consequence, wave propagation in the electron-positron plasma surrounding magnetars are likely to be significantly affected by strong field QED effects.

5. Concluding remarks

In this paper we have given a brief review of how the plasma dynamics is modified by various quantum, spin and QED effects. The approach has been to modify the Maxwell-Fluid equations, in order to keep contact with the theoretical results and methods developed for classical systems. In order to illustrate the usefulness of the modified equations, we have presented some simple results for linear wave propagation. It should be stressed, however, that much of the work dealing with quantum and QED effects focuses on nonlinear phenomena. The set of plasmas where the new phenomena tend to be important can be briefly described as follows:

(1) The Bohm potential and Fermi pressure: Low temperature and/or high density plasmas. This includes solid state plasmas, white dwarf stars and to some extent laser produced plasmas. Ultra-cold plasmas generated from Rydberg states are also of interest in this context.

(2) Spin effects: Due to the complexity of the spin dynamics, it is difficult to give simple conditions when these effects are important. However, a few simple rules of thumb can be given: Spin effects are important if the energy difference between the two spin states is larger than the thermal energy. This applies to plasmas in the vicinity of magnetars, and possibly also to ultracold plasmas. Furthermore, spin effects can be important in low-temperature high density plasmas, similarly to the ones described in point 1. Finally spin effects can be important if \(C_A^2 \lesssim \mu_B B_0/m_i\), which contrary to the first conditions tend to be fulfilled in plasmas embedded in a rather weak external magnetic field.
(3) **High field QED effects:** The characteristic scale for this phenomena is the Schwinger critical field. However, since qualitatively new phenomena (i.e. photon-photon scattering in vacuum) occur due to these terms, there is a hope to see such QED effects even before the laser intensities reach this extreme scale. Furthermore, astrophysical plasmas in the vicinity of magnetars can be subject to magnetic field strengths exceeding the critical field.

(4) **Short wavelength QED effects:** These are important when the wavelength approach the Compton wavelength, provided the plasma density is very high, such that the $\frac{c}{\omega_p}$ is comparable or smaller than the Compton wavelength.

The above list should not be taken too literally, as quantum effects certainly can be more important whenever a classical effect vanishes due to some symmetry obeyed by the classical terms only. We conclude this paper by pointing out that the field of quantum plasmas is a very rich one, and that many important aspects remain to be discovered.

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