Estimating Information Gain in Measurements in Suboptimal Bases for Quantum State Tomography

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Abstract

It is known that mutually unbiased bases, whenever they exist, are optimal in an information theoretic sense for the determination of unknown state of a quantum ensemble. These bases may not exist in most dimensions and some suboptimal choices have to be made. The present paper deals with estimates of the information loss in suboptimal choice of bases. The information is calculated directly in terms of transition probabilities. I give estimates for the information content of measurement in some approximate MUBs proposed recently.

1 Introduction

The state of a quantum system is completely specified by a ray in a complex Hilbert space $\mathcal{H}$, or more generally by a density matrix. A density matrix is a positive operator on $\mathcal{H}$ with unit trace. Thus, a density matrix has nonnegative eigenvalues whose sum equals 1. In particular, it is hermitian. The Hilbert space $\mathcal{H}$ is in general infinite dimensional. However, if we confine our attention to some physical quantities like spin or polarisation then the corresponding space is finite dimensional. The complete space of the system is the tensor product of this finite dimensional space and an infinite dimensional space which correspond to physical quantities like momentum, energy, angular momentum etc. As long as the interaction between these two types of quantities is negligible we may consider them separately since the complete state is product or “unentangled” state. Henceforth, we will consider finite dimensional spaces mostly. The finite dimensional case is of paramount interest in quantum computing and information.

Often the state of the quantum system is not known and has to be determined by certain tests. For this, we need an ensemble. Imagine for

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example, a preparation apparatus designed to prepare quantum systems in some arbitrary dimension $n$ (qunits!) in some specified pure state. Noise in the apparatus will distort the qunits and what we will get is distribution over pure states: a density matrix. Similarly, in the context of tomographic quantum cryptography[1] we have to choose a set of positive-operator valued measures (POVM) for measurement. What is the optimal choice of such POVMs? This has been answered for projection valued measures (PVM) by an information theoretic analysis[2]. Let us distinguish two problems concerned with general measurements. The first is the problem of estimation or hypotheses testing[3]: given a measurement procedure and the data find the best estimates of the state that produced the data, assuming some prior distribution on the state. A recent thorough analysis may be found in [4]. The second problem may termed as a design problem. Given a class of measurement to determine the parameters characterizing the state, find the "best measurement". Now, the best measurement will most likely depend upon the state, so we look for the best measurement on the average. That is we optimize the average information gain corresponding to the different measurements from the given family. The present paper is concerned with the second problem.

Let $\mathcal{H}$ be a Hilbert space of dimension $n$. Let $\mathcal{V}(\mathcal{H})$ be the set of hermitian operators on $\mathcal{H}$. The dimension of $\mathcal{V}(\mathcal{H})$ as a real vector space is $n^2$. Let $\Omega(\mathcal{H}) \subset \mathcal{V}(\mathcal{H})$ be the set of positive operators with trace 1. It is a convex set. The map $L : \mathcal{V}(\mathcal{H}) \to \mathcal{V}(\mathcal{H})$ such that $L(T) = T - 1/n \text{Tr}(T) \cdot I$ is linear and the image $l(\mathcal{H})$ is the space of hermitian operators with trace 0. It has dimension $n^2 - 1$. Here $I$ denotes the identity operator. The affine space $l(\mathcal{H}) + I/n$ consists of all hermitian operators with trace 1. Therefore, a density matrix is completely specified by $n^2 - 1$ parameters.

The state of a quantum system is not directly measurable. A measurement yields only probabilities. We assume for simplicity that all hermitian operators are observable. Thus, let $\rho \in \Omega(\mathcal{H})$ is a state and $A$ be Hermitian operator. Let $A = \sum_i c_i |\alpha_i\rangle\langle\alpha_i|$ be the spectral decomposition of $A$. Here we use the familiar Dirac notation: $\langle\alpha|$ is the unique dual vector (via the inner product) of the vector $|\alpha\rangle$ in $\mathcal{H}$. A measurement of $A$ will record one of the eigenvalues $c_i$ with probability $\text{Tr}(|\alpha_i\rangle\langle\alpha_i|\rho) = \langle\alpha_i|\rho|\alpha_i\rangle$. The space of linear operators on $\mathcal{H}$ becomes a Hilbert space of complex dimension $n$ by defining the inner product

$$\langle B, C \rangle = \text{Tr}(B^\dagger C).$$  \hspace{1cm} (1)$$

The corresponding norm is the Frobenius norm. Its restriction to $\mathcal{V}(\mathcal{H})$ makes the latter a real inner product space of dimension $n^2$. Let us assume that $A$ is nondegenerate. The probability of obtaining the $i^{th}$ outcome is

$$p_i = \text{Tr}(|\alpha_i\rangle\langle\alpha_i|\rho) = \langle\alpha_i|\langle\alpha_i|, \rho\rangle.$$

$$\hspace{1cm} (2)$$
The probabilities may be interpreted as projections of the state $\rho$ onto the corresponding “coordinate” vector $|\alpha_i\rangle\langle\alpha_i|$. Of course, here a vector means an element in the space $V(\mathcal{H})$. It is more convenient to consider the traceless hermitian operators $\rho - I/n$ instead of $\rho$. In any case a measurement on an ensemble in some basis can at best give us an estimate of the $n^2$ probabilities of the possible outcomes. We may thus characterize a measurement of a nondegenerate observable $A$ by the corresponding orthonormal basis in the spectral decomposition. We shall henceforth simply refer to the bases of measurement. Of the $n^2$ probabilities obtained by measurement in some basis $\mathcal{B}$ only $n^2 - 1$ are independent since $\sum_i p_i = 1$. Let $P_i = |\alpha_i\rangle\langle\alpha_i|$ projection operators corresponding to $\mathcal{B}$. They satisfy $P_i P_j = \delta_{ij} P_i$ and $\sum_i P_i = I$. As vectors in $V(\mathcal{H})$ they are linearly independent. To avoid confusion we call the projection operators corresponding to some basis in the ambient space $\mathcal{H}$ projection vectors or simply projectors when they are considered as elements of $V(\mathcal{H})$. But there are only $n$ of them. Further, if we have two bases $\mathcal{B}_1$ and $\mathcal{B}_2$ then at most $n - 1$ projectors from $\mathcal{B}_2$ can be independent of those in $\mathcal{B}_1$ due to the relation $\sum_i P_i = I$. Hence to get the $n^2 - 1$ coordinates of $\rho$ we need $n + 1$ bases such that they are independent in the following sense. Pick the first $n - 1$ projectors each from each of the basis. If they are independent after the affine transformation described above they constitute a basis in $l(\mathcal{H})$. We call such a set of projectors a complete set of measurement bases (CSMB for short). Suppose we have two CSMBs $\mathcal{S}_1$ and $\mathcal{S}_2$. If all other conditions remain same which one should we pick for determining the unknown state of a quantum ensemble? We may assume ideal conditions- perfect preparation procedures, perfect detectors and measuring devices etc.- to compare the two. In a classic paper [2] Fields and Wootters proved that a set of mutually unbiased bases (MUBs) is an optimal choice. Two sets of orthonormal bases $\{|\alpha_i\rangle\}$ and $\{|\beta_j\rangle\}$ are said to be mutually unbiased if $|\langle\alpha_i|\beta_j\rangle|^2 = 1/n$. They further go on to show that such bases exist whenever the dimension $n$ is a prime power, extending earlier work of Ivanovic [5] who proved the existence of MUBs in prime dimension. These works however left open the question of the existence of MUBs for $n$ which divides two or more distinct primes e.g. 6. It is now widely believed that MUBs do not exist in such dimensions. However, we can expect CSMBs which approximate MUBs. Then it is natural to ask: how much do we lose in the approximation process. This question is relevant even in the cases where MUBs are known to exist because in more realistic situations the measurement apparatus will only approximately implement the MUBs. However, to answer such questions we must have an appropriate framework in which these questions may be posed and answered precisely and quantitatively. The natural candidate seems to be information theory.

In this work I elaborate on the information content of a quantum measurement process. This was partly done in [2]. I then give estimates for the information content of measurements in CSMBs which approximate MUBs.
Even in the case where MUBs are known to exist there is always a margin of error. So even here it is reasonable to estimate the information content of CSMBs.

I point out that information optimization appears in a different context in estimation theory, namely, hypothesis testing. Given some prior information about the distribution of states and the outcomes of some experiment we seek for optimal choice of state form the experimental data.

2 Information content of a measurement

In this section we follow [6] to define the information content of a measurement and apply it to the case of measurements for the determination of the quantum state of an ensemble. Let $\mathcal{M}$ be a measurement on some system $S$. We should use the term “experiment” rather than measurement since the latter seems to imply a single measurement. Let $S$ be characterized by some parameters denoted by $\theta$ which will usually be drawn from some subset $\Theta$ of $\mathbb{R}^k$, the $k$-dimensional Euclidean space. Let $p(\theta)$ represent the a priori probability distribution of the parameters $\theta$. Corresponding to every value of $\theta$ there is a probability measure on $X$ - the set of possible measurement data which is again a subset of some Euclidean space. We assume that this measure is given by $p(x|\theta)dx$. $\int_B p(x|\theta)dx$ is the conditional probability of getting the outcome $x$ in $B \subset X$ given the state $\theta$. Let $p(X) = \int_X p(x|\theta)p(\theta)d\theta$ be the probability density of the random variable $x$. Note that, we have used the same symbol $p$ for the probability densities of different random variables. This does not of course imply that they are the same functions. The notation is more convenient and unambiguous if taken in proper context. Moreover, we do not differentiate between a random variable and its values. In an experiment we often are often interested in the posterior probability $p(\theta|x)$. That is, given the measured values $x$ the probability density for $\theta$ which in turn gives us the probability of the state. This is the primary problem in estimation theory and hypothesis testing. The information content of the measurement $\mathcal{M}$ is defined as

$$I(\mathcal{M}, p(\theta), x) \equiv \int p(\theta|x) \log p(\theta|x) d\theta - \int p(\theta) \log p(\theta) d\theta. \quad (3)$$

If $p(\theta|x) = 0$ then the integrand is defined to be zero and the logarithm is taken over an arbitrary but fixed base. The justification for this definition is as follows. Consider the term

$$I_0 \equiv \int p(\theta) \log p(\theta) d\theta.$$

It is supposed to represent the prior information about the state $\theta$. Let us take a simple example to illustrate an important property. Suppose it is
known that the state $\theta$ is found in $\Theta' \subset \Theta$ with probability $q$. Let $I_1$ be a measure of information corresponding to the knowledge whether $\theta$ is in $\Theta'$ or its complement. Let $I_2$ and $I_3$ be the amount of information gained in the next phase when get the value of $\theta$ in $\Theta'$ or its complement respectively. Then a fundamental additive property required of the information measure is that the total information

$$I = I_1 + qI_2 + (1 - q)I_3$$  \hspace{1cm} (4)

Then it is not difficult to show that the information measure $I_0$ is unique up to a constant multiple. We do not discuss these points further but refer the reader to any good source on basic information theory e.g. [7] and [6] for a discussion in the context of experiments. The difference between the posterior information $I_1 = \int p(\theta|x) \log p(\theta|x) d\theta$ and the prior information $I_0 = \int\int p(\theta) \log p(\theta) d\theta$ is the net information gain. It depends upon the experiment and the distribution of the data $x$. Thus we may say that one experiment or measurement is more informative than other. Let us calculate information content for some simple measurements in the quantum domain. Let the dimension $n = 2$. Suppose we have prior information that the state is a pure state $|0\rangle$ or $|1\rangle$ with probability $1/2$. We may therefore model the parameter space as $\Theta = \{0, 1\}$ with $p(0) = p(1) = 1/2$. Then $I_0 = 1/2 \log (1/2) + 1/2 \log (1/2) = -1$. The logarithm is taken to the base 2. Now suppose that we choose to make measurement $M_1$ in the basis $\{ |0\rangle, |1\rangle \}$ which is natural, given the prior information. Then the conditional probabilities may be conveniently written in the matrix form, for $i, j \in \{0, 1\}$

$$p(i|j) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It is simply the unit matrix of order 2. Thus, if we get the measurement outcome 0 we are sure that the state of the system was $|0\rangle$ etc. Then it is easy to see that $I_{1}(M_1, i) = 0$ and hence $I(M_1, i) = I(M_1, i) - I_0 = 1$. Now suppose we perversely choose the basis $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle |+\rangle - |1\rangle |-\rangle)$ for measurement $M_2$. Then the corresponding conditional probability matrix is

$$p(i|j) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Again it is easy to see that the information gain in this case is $I(M_2, i) = 0$. That is we get no information from $M_2$. In fact, the average information, to be defined below, is zero. This is of course intuitively obvious from the choice of basis in $M_2$.

The information measure defined above depends on the state and may be negative. But the average information

$$I(M, p(\theta)) = \int I(M, p(\theta), x)p(x)dx$$ \hspace{1cm} (5)
is independent of the state and is nonnegative \( p(x) = \int p(x|\theta) d\theta \) \( (6) \). Here, the probability density

\[
p(x) = \int p(x|\theta) d\theta
\]

is the mean probability distributions averaged over \( \theta \). It is not difficult to show that

\[
\mathcal{I}(M, p(\theta)) = \int \int p(\theta)p(x|\theta) \log (p(x|\theta)) dx d\theta - \int p(x) \log (p(x)) dx
\]

\( (7) \). This is the formula we use estimate the information gained in quantum measurements.

3 Quantum state tomography and MUBs

Given an \( n \)-dimensional quantum ensemble in an unknown state how do we determine its state? This is the problem of quantum state tomography. The state is not directly observable but we may infer it from the probability distributions observed. As mentioned in the Introduction we need (projective) measurement in \( n + 1 \) bases to determine the state completely from the observed probabilities. Actually, the state tomography problem has broadly two theoretical aspects. The first is a design issue. What is the optimal choice of bases? The second aspect is a problem of decision or estimation theory: for a given measurement what is the best possible estimate of the parameters characterizing the state? In this paper we will be mainly concerned with first aspect. So let us formulate the problem precisely now.

Given an ensemble of quantum systems in some unknown state \( \rho \). By an ensemble we mean an unlimited supply of identically prepared quantum systems. Let \( \mathcal{B}^1, \ldots, \mathcal{B}^{n+1} \) be \( n + 1 \) base in \( \mathcal{H} \) with

\[
\mathcal{B}^k = \{ |\alpha^k_i\rangle \langle \alpha^k_i| \}_{i=1}^n
\]

\( (8) \). As vectors in the \( n^2 \)-dimensional space \( V(\mathcal{H}) \) at most \( n^2 \) of them can be linearly independent. Due to the relations \( \sum_i \mathbb{P}_i^k = I \) for all \( k \) we can only have all the \( n \) vectors from exactly one basis in any independent set and if \( \mathcal{B} = \bigcup_k \mathcal{B}^k \) contains a maximal independent set then we choose the first \( n - 1 \) projectors \( \mathbb{P}_i^k, i = 1 \cdots n - 1 \) and \( \frac{I}{n} \) as a basis for \( V(\mathcal{H}) \). Let

\[
s_{ij}^{kl} = \langle \mathbb{P}_i^k, \mathbb{P}_j^l \rangle = \text{Tr}(\mathbb{P}_i^k \mathbb{P}_j^l).
\]

The nonnegative numbers \( s_{ij}^{kl} \) are the respective transition probabilities among the vectors in the \( k^{th} \) and \( l^{th} \) basis. Note that \( s_{ij}^{kl} = \delta_{ij} \) since each of the basis is orthonormal. It was seen in Section 1 that \( \{\mathbb{P}_i^k - I/n : i = 1, \ldots, n - 1 \text{ and } k = 1, \ldots, n + 1\} \) form a basis for \( l(\mathcal{H}) \), the space of traceless hermitian operators. Thus, for a state \( \rho \) let

\[
\rho - I/n = \sum y_i^k (\mathbb{P}_i^k - I/n) \equiv \sum y_i^k T_i^k
\]

\( (9) \).
Then,
\[
\text{Tr}((\rho - I/n)T_j^l) = p_j^l - 1/n = \sum_i y_i^k \langle T_i^k, T_j^l \rangle = \sum_{i,k} t_{ij}^{kl} y_i^k
\]  

(10)

It is easy to see that if the original bases are mutually unbiased then \(\langle T_i^k, T_j^l \rangle = 0\) for \(k \neq l\), that is the operators \(T_i^k\) and \(T_j^l\), \(k \neq l\) are orthogonal when considered as vectors. Here
\[
t_{ij}^{kl} \equiv \langle T_i^k, T_j^l \rangle = \text{Tr}(T_i^k T_j^l) = s_{ij}^{kl} - 1/n
\]

(11)

and \(p_i^k\) is the probability of \(i^{th}\) outcome in the measurement in the \(k^{th}\) basis. If we consider the parallelepiped spanned by the vectors \(T_i^k\) then \(t_{ij}^{kl}\) are the angles between the sides \(T_i^k\) and \(T_j^l\). Notice also that the input parameters characterising the state (denoted by \(\theta\) earlier) are the components \(y_i^k\). We will denote these by a vector \(Y\).

By a measurement we mean a collection of several observations in different bases on subensembles of the original ensemble. We picture a massively parallel setup where we have a several measuring devices \(D_k\) for each basis \(B_k\). The original ensemble is divided into large subensembles and tested by each of these \(n + 1\) groups of devices. For each \(k \leq n + 1\) we get frequencies \(m_i^k\) for the \(i^{th}\) outcome, \(1 \leq i \leq n - 1\) in the \(k^{th}\) device group. The numbers \(m_i^k\) constitute the measurement data \(x\) in 5 and 7. What is a reasonable probability distribution for the \(m_i^k\)? Here we appeal to the local limit theorem in probability theory which roughly states that for independent discrete random variables the probability distribution of their frequencies tends to the normal distribution in the limit \(N \to \infty\), \(N\) the number of trials. We must have some prior distribution for the states. Let \(V\) be the volume of the parallelepiped spanned by the vectors \(T_i^k\). Assuming a uniform distribution for the states it can be shown that2 that the information gain in a quantum test is proportional to \(\ln V\) apart from an additive constant. We will in fact take \(\ln V\) as the measure for information content of a quantum test of an ensemble in CSMB and for a CSMB \(C\) write \(\mathcal{I}(C)\) for the information gain and \(V(C)\) for the corresponding volume. The first result I prove was already given in 2 but the present approach is different.

**Theorem 1** Information gain \(\mathcal{I}(C)\) is maximum if and only if \(C\) consists of mutually unbiased bases.

**Proof.** From the preceding discussion, we have to show that the volume \(V(C)\) spanned by the vectors \(T_i^k\) is maximal iff \(T_i^k\) and \(T_j^l\) are orthogonal for \(k \neq l\). Notice first that \(\langle T_i^k, T_j^l \rangle = \delta_{ij} - 1/n\). Consider the \((n^2 - 1) \times (n^2 - 1)\) matrix \(\Gamma(C) = (t_{ij}^{kl}) = \langle T_i^k, T_j^l \rangle\) and assume the ordering defined by the pair \(\{k, i\}\). This simply means that the matrix consists of \(n + 1\) blocks \(\gamma^{kl}\), each
a square matrix of size \((n - 1)\) such that \(\gamma^{kl}(ij) = t^{kl}_{ij}\).

\[
\Gamma(C) = \begin{pmatrix}
\gamma^{11} & \gamma^{12} & \cdots & \gamma^{1n+1} \\
\gamma^{21} & \gamma^{22} & \cdots & \gamma^{2n+1} \\
\vdots & \vdots & \ddots & \vdots \\
& & \cdots & \gamma^{n+1,n+1}
\end{pmatrix}
\]  

(12)

If we choose any orthonormal basis for \(l(H)\) and express \(T^k_i\) in this basis. Let \(T\) be the corresponding real matrix of the coefficients then it is clear that \(TT^t = \Gamma(C)\), where \(A^t\) is the transpose of \(A\). It follows that \(\det \Gamma(C) = (V(C))^2\) and \(\Gamma(C)\) is positive definite. Thus maximizing \(V(C)\) is equivalent to maximising \(\Gamma(C)\). Below we will focus on the latter. From the generalised Fischer-Hadamard inequality \[9\] it follows that

\[
\det \Gamma(C) \leq \det \gamma^{11} \cdots \det \gamma^{n+1,n+1}
\]  

(13)

The rhs is determinant of the product of the diagonal blocks in \(\Gamma C\). Now, if the \(T^k_i\) are orthogonal then the off-diagonal blocks are all zero matrices and the equality holds in eq.\ref{eq:13}. This proves the sufficiency part.

The equality holds in \ref{eq:13} only if the following condition is satisfied \[9\]. Let \(S\) be the \((n + 1) \times (n + 1)\) matrix such that \(S(ij) = 1\) if \(\gamma^{ij} \neq 0\) and \(S(ij) = 0\) otherwise. Then the equality holds if and only there is permutation matrix of order \(n + 1\) such that \(PSP^{-1}\) is triangular. Since \(\Gamma\) is symmetric and \(P^{-1} = P^t\) it follows that if \(PSP^{-1}\) is triangular it must be diagonal. The operation \(S \rightarrow PSP^{-1}\) permutes the diagonal elements of \(S\) among themselves. Hence, \(PSP^{-1}\) is diagonal iff all off-diagonal elements are zero. That is, \(\gamma^{ij} = 0\) for \(i \neq j\). That is the original bases are mutually unbiased. The necessity is proved.

The above theorem gives an upper bound. A natural question is: how tight is the bound. This is related to the estimation of the information content in bases which are complete but not mutually unbiased. We now give an estimate of the relative loss due to such a non-optimal choice. First let us compute the determinant in the case of MUBs. We only have diagonal terms. Recall that a diagonal block \(\gamma^{kk}(ij) = \langle d^k_i - 1/n, d^k_j - 1/n \rangle = 1 - 1/n\).

We write this as \(\gamma^{kk} = I - 1/nT\), \(I\) is the identity matrix of order \(n - 1\) and \(T\) is the matrix with all entries 1. Let \(\Gamma_0\) be the submatrix of \(\Gamma\) consisting of the diagonal blocks. Note that all the diagonal blocks \(\gamma^{kk}\) are identical.

**Lemma 1** \(\det \Gamma_0 = \frac{1}{n^{n+1}}\).

Proof. First note that \(T^2 = (n - 1)T\). The eigenvalues of \(T\) are therefore, \(n - 1\) and 0. The rank of \(T\) is 1. Hence the eigenvalues of \(I - 1/nT\) are \(1/n\) and 1. The determinant of each block is therefore \(1/n\) and since there are \(n + 1\) blocks the result follows.
If $C$ is MUB then $\Gamma(C)$ has all off diagonals zero. Now $\gamma^{-1} = (I + 1/nT)^{-1} = I + T$. Hence in block form we have,

$$
\det \Gamma(C) = \det \Gamma_0 \cdot \det \begin{pmatrix}
I & (I + T)\gamma^{11} & \cdots \\
\vdots & \vdots & \vdots \\
\cdots & \cdots & I
\end{pmatrix}
$$

(14)

That is, the off-diagonal blocks are multiplied by the matrix $I + T$. Consider $\gamma^{kl}$. Recall that $\gamma^{kl}(ij) = s_{ij}^{kl} - 1/n$, $i, j \leq n - 1$, where $s_{ij}^{kl}$ are transition probabilities. An easy calculation shows that $(I + T)\gamma^{kl}(ij) = s_{ij}^{kl} - s_{in}^{kl}$. The term $s_{in}^{kl}$ appears because we omitted the $n$th basis vector from each basis in the state space $\mathcal{H}$. If we had chosen another vector, say, the first then $s_{1n}^{kl}$ would have been subtracted. The point is the information content depends on the differences of probabilities. Only in the case of MUBs are these differences all zero. Next we give an estimate in the general case.

**Theorem 2** Let $|s_{ij}^{kl} - s_{ir}^{kl}| < \varepsilon$ for some $\varepsilon > 0$. Let $\Gamma' = \Gamma(C) - I_{n^2-1}$ and let $\lambda_m$ be the minimum eigenvalue of $\Gamma'$. Then

$$
e^{-\frac{(a^2 - n)(a^2 - 1)x^2}{1 + \lambda_m}} \frac{\det \Gamma(C)}{\det \Gamma_0} \geq 1
$$

(15)

Proof. The theorem is a direct consequence of an estimate given in [10]. From its definition $\Gamma(C)$ is positive semidefinite because it is a real matrix of the form $<b_i, b_j>$ for vectors $b_i$ in appropriate dimension. Hence the estimate in [10] is applicable. The upper bound is just the Hadamard-Fischer inequality. The lower estimate in [10] is $e^{-\frac{(a^2 - n)x^2}{1 + \rho}}$, where $\rho = \max\{|\lambda_i| : \lambda_i \text{ an eigenvalue}\}$ is the spectral radius of $\Gamma'$. The fact that $\rho \leq \max\{|R_i|\}$, where $|R_i|$ is the sum of absolute values of the entries in $i$th. row of $\Gamma'$ is easily proved[11]. We get $n^2 - n$ because the diagonal blocks in $\Gamma'$ are zero.

Let $v_d \equiv \frac{\det \Gamma'}{\det \Gamma_0}$. As an illustration let $\varepsilon \leq 1/n^4$ then a simple calculation yields $\det \Gamma(C)/\det \Gamma_0 \geq e^{-1/n^2}$ and the corresponding loss in information is $O(1/n^2)$. In the cases where MUBs are known to exist, that is when $n$ is a prime power it is natural to expect that in some actual designing for testing in these bases there would be errors. If we can bound give an estimate $\varepsilon$ for these errors then the information loss can be estimated. Even in cases where MUBs are not known to exist approximate MUBs may be constructed [12]. However, a direct application of the above estimates to their constructions does not yield very good lower bounds. If $\varepsilon \leq 1/n^3$, as in some cases of [12], then the information loss can be estimated to be less than $a = o(1)$. We now give an exact calculation of the determinant in the second construction in [12].
3.1 Calculation of determinants in special cases

In the case of KRSW construction

\[ s_{ij}^{ab} = \delta_{ij}, \quad a = b \]
\[ = \frac{n + 1}{n^2}, \quad a \neq b, i \neq j \]
\[ = \frac{1}{n^2}, \quad a \neq b, i = j \]

We calculate the determinant of the \((n^2 - 1) \times (n^2 - 1)\) matrix defined by the numbers \(t_{ij}^{ab}, \quad a, b = 1, \ldots, n + 1\) and \(i, j = 1, \ldots, n - 1\). The rows(columns) of the matrix are indexed by pairs \([a, i]/[b, j]\). We do it for a slightly more general case.

It is clear that \(s_{ij}^{aa} = \delta_{ij}\). Let \(\Gamma\) be the matrix whose entries in block form are the \((n - 1) \times (n - 1)\) matrices \(\gamma_{ij}^{ab}\) where

\[ \gamma_{ij}^{ab} = t_{ij}^{ab} \]

Let \(D\) be the matrix which contains only diagonal blocks. Thus,

\[
D = \begin{pmatrix}
\gamma_{11} & 0 & \cdots & 0 \\
0 & \gamma_{22} & \cdots & 0 \\
0 & \vdots & \cdots & 0 \\
0 & \cdots & 0 & \gamma_{n-1,n-1}
\end{pmatrix}
\]

Then one can show that \(\det \Gamma = \det D \cdot \det \Gamma'\) where

\[
\Gamma' = \begin{pmatrix}
I & \Psi^{12} & \cdots & \Psi^{1,n+1} \\
0 & I & \cdots & 0 \\
0 & \vdots & \cdots & 0 \\
0 & \cdots & 0 & \gamma_{n-1,n-1}
\end{pmatrix}
\]

and \(I\) is the unit matrix of order \(n - 1\). The apparent lack of symmetry in \(\Gamma'\) can be removed by successively subtracting the \((i + 1)^{th}\) row from the \(i^{th}\) in each row of blocks. The result is that typical entries are of the form

\[ s_{ij}^{ab} - s_{i+1,j}^{ab} \]

However, for the present purpose we stay with the first form of \(\Gamma'\). We calculate the determinant for a very special case. Let

\[
s_{ij}^{ab} = \begin{cases}
\frac{c + 1}{n} & \text{if } a \neq b \text{ and } i \neq j \\
\frac{1}{n} - (n - 1)c & \text{if } a \neq b \text{ and } i = j \\
\delta_{ij} & \text{if } a = b.
\end{cases}
\]
Then each of the off-diagonal blocks in $\Gamma'$ is diagonal,

$$
\Psi^{ab} \equiv A = \begin{pmatrix}
-nc & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & \cdots & 0 \\
0 & \cdots & -nc \\
\end{pmatrix}
$$

and

$$
\Gamma' = \begin{pmatrix}
I & A & \cdots & A \\
A & I & \cdots & A \\
\vdots & \vdots & \ddots & \vdots \\
A & A & \cdots & I \\
\end{pmatrix}
$$

First, we calculate the eigenvalues of $\Gamma'$ together with their multiplicities. This will give us the determinant. Let

$$
X = \begin{pmatrix}
x_1 \\
\vdots \\
x_{n+1}
\end{pmatrix}
$$

where $X$ is a $(n^2 - 1)$ column vector and $x_i$ are $(n-1)$ column vectors. This decomposition is made to match the decomposition of the matrix $\Gamma'$. Then if $X$ is an eigenvector with eigenvalue $d$,

$$
\Gamma'X = dX \Rightarrow x_i - nc \sum_{j \neq i}^{n+1} = dx_i, \ i = 1, \ldots, n + 1
$$

Let $y = \sum_{i=1}^{n+1} x_i$ be a $n - 1$ column vector. Then the above equation can be written as

$$
(1 + nc - d)x_i = ncy
$$

Consider two possibilities. First, $y = 0$. Then, $d = 1 + nc$ pro. Now, the subspace of $C^{n^2-1}$ corresponding to the solutions $y = 0$ is $n^2 - n$ dimensional, equal to the multiplicity of the eigenvalue $d = \frac{1}{1+nc}$. The second case is when $y \neq 0$. Then clearly all the $(n-1)$-vectors are equal, i.e.,

$$
x_i = \frac{nc}{1 + nc - d}y = \frac{n(n+1)c}{1 + nc - d}x_i
$$

Hence, in this case $d = 1 - n^2c$. The multiplicity is clearly $n - 1$. Thus,

$$
\det \Gamma' = (1 + nc)^{n^2-n} (1 - n^2c)^{n-1}
$$

For example, if $c = 1/n^2$ then the determinant is zero. This is the case in [KRSW05]. However, notice that the main contribution to the determinant comes from the eigenvalue $1 + nc$ but remembering that $\det \Gamma' \leq 1$. This places restriction on $c$. 

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4 Discussion

We analysed the information content of a quantum state tomography in PVMs. The information seems to depend upon the choice of the basis vector we eliminate initially in constructing $T^k_i$ (see discussion following eq.(8)), in this case, $E^i_n$. However, it is easy to see that the information measures corresponding to different choices differ by an unimportant additive constant. Another point is that the parallelepiped whose volume was used as a measure for information is slightly different from the one given in [2]. But again the corresponding information measures differ by an additive constant. What is perhaps more important and difficult is the weakening of assumption of uniform prior distribution and a characterization of the optimal PVM as a function of the prior distribution function. Another related issue is a similar investigation of optimal POVMs. Two other difficult problems are:1. estimating information content of measurements on infinite dimensional systems and 2. incomplete measurements. These issues will be discussed in a forthcoming paper.

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