Bounded Orbits of Quadratic Collatz-type Recursions

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Abstract

We characterize all bounded orbits of two similar Collatz-type quadratic mappings of the set of non-negative integers. In one case, where cycles of all possible lengths may occur, an orbit is bounded if and only if it reaches a cycle. For the other map we prove that every bounded orbit must reach 0 (in particular, there are no cycles).

1 Introduction

Let \( \mathbb{N} \) be the set of all positive integers and \( \mathbb{N}_0 = \{0, 1, 2, 3, \ldots \} \) be the set of all non-negative integers. Consider the function \( Q : \mathbb{N}_0 \to \mathbb{N}_0 \) that is defined as:

\[
Q(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
\frac{n(n-1)}{2} & \text{if } n \text{ is odd}
\end{cases}
\tag{1}
\]

where

\[
\binom{n}{2} = \frac{n(n-1)}{2}
\]

is the binomial coefficient. We call the function \( Q \) in \((1)\) the divide-or-choose-2 rule.

Note that for odd \( n \) the function \( Q \) is a multiplicative version of the Collatz-type map

\[
F(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
\frac{3n-1}{2} & \text{if } n \text{ is odd}
\end{cases}
\]

in the following sense: Since \( n - 1 \) is even for odd \( n \) we may write

\[
\binom{n}{2} = n \left( \frac{n-1}{2} \right), \quad \frac{3n-1}{2} = n + \frac{n-1}{2}
\]

Equivalently, \( F \) is the additive or “linear” version of \( Q \). It is a variant of the better known (compressed) Collatz function

\[
T(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
\frac{3n+1}{2} & \text{if } n \text{ is odd}
\end{cases}
\]

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While all orbits of $T$ are conjectured to reach the base cycle $\{1, 2, 1, 2, \ldots \}$ from any initial value $n \in \mathbb{N}$ the variant $F$ generates nontrivial cycles [1].

The variant of $Q$ that represents a multiplicative version of the Collatz function $T$ is

$$
\begin{cases}
  n/2 & \text{if } n \text{ is even} \\
  \frac{n+1}{2} & \text{if } n \text{ is odd}
\end{cases}
$$

The orbits that are generated by this function are qualitatively similar to those generated by $Q$ so we need not consider this map in detail. Instead, we study the following variant of $Q$

$$
S(n) = \begin{cases}
  n/2 & \text{if } n \text{ is even} \\
  \frac{n^2-1}{4} & \text{if } n \text{ is odd}
\end{cases}
$$

The function $S$ has a symmetric expression in the sense that

$$
\frac{n^2 - 1}{4} = \left( \frac{n - 1}{2} \right) \left( \frac{n + 1}{2} \right)
$$

so we call it the symmetric rule.

Our goal in this paper is two-fold: We show that the orbits of $Q$ may reach cycles of all possible lengths. All cycles contain an odd number of type $2^m + 1$ for some $m \in \mathbb{N}_0$ and an orbit of $Q$ that does not reach a cycle is shown to be unbounded. This result characterizes all bounded orbits that may be generated by iterating $Q$. We similarly show that the orbits of $S$ must either reach 0 or be unbounded and each orbit that reaches 0 contains a number of type $2^m \pm 1$ for some $m \in \mathbb{N}_0$. The existence of an unbounded orbit (i.e. an orbit that does not satisfy the preceding conditions and “escapes to infinity”) for either $Q$ or $S$ is a separate problem and not discussed in this paper.

A note about the domains of the above quadratic maps: Both may be extended to the set $\mathbb{Z}$ of all integers rather trivially in the sense that every orbit with a negative initial value enters $\mathbb{N}_0$ after a finite number of steps and stays there since $\mathbb{N}_0$ is invariant under both mappings. We do not consider extensions to $\mathbb{Z}$ here.

2 The divide-or-choose-2 rule

If $x_0 \in \mathbb{N}_0$ then the numbers

$$
x_0, Q(x_0), Q(Q(x_0)), \ldots
$$

constitute an orbit or trajectory of the quadratic recursion

$$
x_{n+1} = Q(x_n)
$$

(2)
in \( \mathbb{N}_0 \). If \( x_m = x_0 \) for some \( m \in \mathbb{N} \) then the numbers \( x_0, x_1, \ldots, x_{m-1} \) repeat so we have a \textit{periodic orbit with period} \( m \) or equivalently, an \textit{m-cycle}, i.e. a cycle of length \( m \). A number \( x_0 \) that lies on a cycle is called a \textit{periodic point} of \( Q \). If \( m = 1 \) then the cycle is often called a \textit{fixed point} of \( Q \).

The divide-or-choose-2 rule works as follows: If the value of \( x_0 \) is even then we \textit{divide} it by 2 to get the next term:

\[
x_1 = Q(x_0) = \frac{x_0}{2}
\]

On the other hand, if \( x_0 \) has odd value then the next term is \( x_0 \) \textit{choose} 2:

\[
x_1 = \left( \frac{x_0}{2} \right) = x_0 \left( \frac{x_0 - 1}{2} \right)
\]

Let \( m \) be a positive integer and consider \( x_0 = 2^m \). Then

\[
x_1 = 2^{m-1}, \ x_2 = 2^{m-2}, \ldots, x_{m-1} = 2, \ x_m = 1
\]

At this point, since 1 is odd,

\[
x_{m+1} = 0
\]

and with 0 being even, \( x_n = 0 \) for all \( n > m \). The repeating number 0 is a fixed point of \( Q \) since as an even number, \( Q(0) = 0/2 = 0 \). One more fixed point is 3, since

\[
Q(3) = \binom{3}{2} = 3.
\]

By examining the expressions for odd and even numbers it is easy to see that \( Q \) has no other fixed points.

More generally, if \( x_0 = 2^m k \) is an arbitrary even number where \( m \in \mathbb{N}_0 \) and \( k \) is odd then

\[
x_1 = 2^{m-1}k, \ x_2 = 2^{m-2}k, \ldots, x_m = k \quad (3)
\]

Also every odd number larger than 1 can be written as \( 2^m k + 1 \) where \( m \in \mathbb{N}_0 \) and \( k \) is odd. Note that

\[
Q(2^m k + 1) = (2^m k + 1) \left( \frac{2^m k + 1 - 1}{2} \right) = 2^{m-1}k(2^m k + 1) \quad (4)
\]

so if \( x_0 = 2^m k + 1 \) then

\[
x_1 = 2^{m-1}kx_0
\]

If \( m > 1 \) then \( x_2 \) is just half of \( x_1 \) so that \( x_2 = 2^{m-2}kx_0 \). By induction

\[
x_m = kx_0 \quad (5)
\]

This observation has an interesting consequence: if \( k = 1 \) then \( x_m = x_0 \) and we obtain an \textit{m-cycle}.

Notice that since \( m \) is any positive integer in the above argument, we have proved the following.
Lemma 1  For every positive integer $m$ the recursion \((2)\) has an $m$-cycle given by the numbers (in the order shown):

\[
2^m + 1 \to 2^{m-1}(2^m + 1) \to 2^{m-2}(2^m + 1) \to \ldots \to 2(2^m + 1) \to 2^m + 1 \tag{6}
\]

The exceptional value $x_0 = 1$ is mapped to 0 and the 1-cycle 0 is reached. We can also infer from \((3)\) that each number of type $2^j(2^m + 1)$ reaches the $m$-cycle of the above lemma in $j$ steps for every $j \in \mathbb{N}$. But these are not the only numbers that may reach cycles. Suppose that $k > 1$ in \((4)\), say, $k = k_0 \geq 3$. Starting with $x_0 = 2^{m_0}k_0 + 1$, by \((5)\) the orbit reaches

\[
x_{m_0} = k_0x_0 \geq 3x_0
\]

Therefore, on the down-swing the orbit does not reach $x_0$ to form a cycle but instead, it reaches a larger odd number $k_0x_0$. Set $k_0x_0 = 2^{m_1}k_1 + 1$ where $m_1$ is a positive integer and $k_1$ is odd. Calculating as before,

\[
x_{m_0 + m_1} = k_1x_{m_0} = k_1k_0x_0
\]

We have two possible cases: $k_1 = 1$ in which case $x_{m_0 + m_1} = x_{m_0}$ and the orbit has reached an $m_1$-cycle. Otherwise, $k_1 > 1$ and

\[
x_{m_0 + m_1} = k_1x_{m_0} \geq 3x_{m_0} \geq 3^2x_0
\]

This process may be repeated by setting $k_1x_{m_0} = 2^{m_2}k_2 + 1$ as long as the coefficients $k_2$ etc remain larger than 1. We obtain the general expression

\[
x_{m_0 + m_1 + \ldots + m_p} = k_px_{p-1} = k_p \cdots k_1k_0x_0 \geq 3^{p+1}x_0
\]

where $p$ is a positive integer. If $k_p > 1$ for all $p$ then this process generates ever larger values that grow infinitely large. Therefore, either the orbit reaches a cycle or it is unbounded. Since orbits starting with an even number always reach an odd number, the above argument proves the following characterization of the bounded orbits of \((2)\).

Theorem 2  The recursion \((2)\) has cycles of type \((4)\) of all possible lengths. Every orbit of \((2)\) either reaches such a cycle or it is unbounded.

We emphasize that the proof of the above theorem does not establish the existence of orbits that never reach cycles so it does not imply that \((2)\) has any unbounded orbits, i.e. orbits that “escape to infinity”. On the other hand, the theorem gives a complete characterization of all bounded orbits; this much is not known for the classic recursion of Collatz.

It is expected that orbits are generally unbounded given the quadratic growth rate in the odd case and because at each iteration, there is a 50 percent chance that $x_n$ is divided by 2, and if not then it is squared (essentially).
Conjecture 3 The recursion \(2) has an unbounded orbit.

If there is an unbounded orbit then there are infinitely many, for if an (odd) number \(x_0 = n\) leads to an unbounded orbit then so do the numbers \(2^n n\) for all \(m \in \mathbb{N}\). The following makes a more specific proposal:

Conjecture 4 Orbits containing an odd number of type \(2^m - 1\) are unbounded for all \(m \geq 3\).

It is worth a mention that numerical simulations do not prove this statement and they may even lead to false conclusions on digital computers. The reason seems to be that for large \(k\) or \(m\), the crucial distinction between \(2^m k\) and \(2^m k \pm 1\) is typically missed, causing the software to produce a cycle where none exists.

Next, note that for every \(m \in \mathbb{N}\) repeated applications of \(Q\) to a number of type \(2^m k\) where \(k\) is odd leads to \(k\) through a monotonically decreasing chain \(m\) steps long. In particular, decreasing chains of arbitrary length are possible. The following result shows not only that orbits with increasing chains of arbitrary length occur but also gives a type of number that leads to them.

Theorem 5 Let \(x_0 = 2^m + 3\) where \(m \geq 2\). Then for \(j = 1, 2, \ldots, m - 1\)

\[
x_j = 2^m - j(x_0 + 2)(x_1 + 2) \cdots (x_{j-1} + 2) + 3 \tag{7}
\]

\[
x_m = x_{m-1}(x_0 + 2)(x_1 + 2) \cdots (x_{m-2} + 2) + 1 \tag{8}
\]

In particular, \(x_j\) is odd for each \(j\) and \(x_0 < x_1 < \cdots < x_{m-1}\) is an increasing chain reaching the even number \(x_m\).

Proof. We use induction. Since \(x_0\) is odd,

\[
x_1 = Q(x_0) = x_0 \left(\frac{2^m + 2}{2}\right) = (2^m + 3)(2^{m-1} + 1)
\]

Multiplying out the last expression and collecting terms

\[
x_1 = (2^m + 3)2^{m-1} + 2^m + 3 = 2^m(2^m + 3 + 2) + 3 = 2^m(x_0 + 2) + 3
\]

This proves (7) for \(j = 1\). If \(m > 2\) and (7) holds for \(k < m - 1\) then

\[
x_{k+1} = x_k \left(\frac{x_k - 1}{2}\right)
\]

\[
= x_k[2^{m-k-1}(x_0 + 2)(x_1 + 2) \cdots (x_{k-1} + 2) + 1]
\]

\[
= x_k2^{m-k-1}(x_0 + 2)(x_1 + 2) \cdots (x_{k-1} + 2) + 2^{m-k}(x_0 + 2)(x_1 + 2) \cdots (x_{k-1} + 2) + 3
\]

\[
= 2^{m-(k+1)}(x_0 + 2)(x_1 + 2) \cdots (x_{k-1} + 2)(x_k + 2) + 3
\]

It follows by induction that (7) is true as long as \(j < m\). For \(j = m - 1\) the application of \(Q\) to the odd number \(x_m = x_{j+1}\) gives (8). \(\blacksquare\)
3 The symmetric rule

For comparison, we now consider the function $S$ which defines the recursion

$$x_{n+1} = S(x_n)$$ (9)

Like $Q$, if $x_0 = 2^n$ for some positive integer $m$ then $x_m = 1$ and $x_{m+1} = 0$. We conclude that the orbit of $2^n$ reaches 0 in $m+1$ steps. It follows that $x_n = 0$ for all $n \geq m+1$. The next lemma extends this observation to similar but odd initial values. The appearance of ± reflects the symmetry in $S$ that was lacking in $Q$.

Lemma 6 For every positive integer $m$ if $x_0 = 2^m \pm 1$ then $x_n = 0$ for all $n \geq \binom{m+1}{2} + 2$.

Proof. First, consider $x_0 = 2^m + 1$. Then

$$x_1 = \left(\frac{2^m}{2}\right)\left(\frac{2^m + 2}{2}\right) = 2^{m-1}(2^m + 1)$$

Further applying $S$ a total of $m-1$ times gives

$$x_m = 2^{m-1} + 1$$

If $m = 1$ then $x_0 = 2 + 1 = 3$ and $x_1 = 2$ from which we obtain $x_2 = 1$ and $x_3 = 0$. If $m > 1$ then repeating the above argument yields

$$x_{m+(m-1)} = 2^{m-2} + 1$$

Continuing this way we obtain

$$x_{m+(m-1)+(m-2)} = 2^{m-3} + 1$$

and so on until

$$x_{m+(m-1)+\ldots+2+1} = 2^0 + 1 = 2$$

Now two more applications of $S$ lead to the fixed value 0. The total number of applications of $S$ is therefore,

$$m + (m-1) + \cdots + 2 + 1 + 2 = \frac{m(m+1)}{2} + 2 = \binom{m+1}{2} + 2$$

Thus for all $n$ larger than the above number, $x_n = 0$. A similar argument shows that if $x_0 = 2^m - 1$ then

$$x_{m+(m-1)+\ldots+2+1} = 2^0 - 1 = 0$$

so $x_n = 0$ for all $n \geq \binom{m+1}{2}$. ■

There are positive integers $x_0$, e.g., $x_0 = 2^p(2^m - 1)$ that reach $2^m \pm 1$ after several iterations of $S$. The orbits of all such initial values reach 0 in a finite number of steps. What happens if the numbers $2^m \pm 1$ are never reached from some initial value $x_0$? The following answers this question.
Theorem 7 Every orbit of the recursion (4) either reaches zero or it is unbounded, i.e. escapes to infinity.

Proof. We may start with an odd initial value, \( x_0 = 2^{m_0}k_0 - 1 \) where \( k_0 \) is odd. Note that

\[
x_1 = \left( \frac{2^{m_0}k_0 - 2}{2} \right) \left( \frac{2^{m_0}k_0}{2} \right) = 2^{m_0-1}k_0(2^{m_0-1}k_0 - 1)
\]

It follows that

\[
x_{m_0} = k_0(2^{m_0-1}k_0 - 1)
\]

If \( k_0 = 1 \) then Lemma 6 implies that the orbit reaches zero in a finite number of steps. If \( k_0 \geq 3 \) then note that

\[
x_{m_0} \geq 3(2^{m_0-1}k_0 - 1) = 2^{m_0}k_0 + 2^{m_0-1}k_0 - 3 = 2^{m_0}k_0 + 3(2^{m_0-1} - 1) \geq 2^{m_0}k_0
\]

Therefore, if \( k_0 > 1 \) then for all \( m_0 \geq 1 \)

\[
x_{m_0} \geq x_0 + 1
\]

The odd number \( x_{m_0} \) is the lowest point of the down-swing following \( x_1 \) but its value exceeds the initial value \( x_0 \).

Next, let \( m_1 \) and \( k_1 \) be positive integers with \( k_1 \) odd such that \( x_{m_0} = 2^{m_1}k_1 - 1 \). Knowing what happens if \( k_1 = 1 \), assume that \( k_1 \geq 3 \). Repeating the above calculation, we conclude that

\[
x_{m_0+m_1} \geq x_{m_0} + 1 \geq x_0 + 2
\]

This process continues with \( m_2, k_2 \) etc and is stopped only if \( k_i = 1 \) for some positive integer \( i \). Otherwise, for every \( p \in \mathbb{N} \) we have

\[
x_{m_0+m_1+\cdots+m_p} \geq x_0 + p + 1
\]

implying that the orbit is unbounded. ■

Like the earlier case of divide-or-choose-2 map, the above theorem does not state that all orbits reach 0. So the existence of unbounded orbits is not implied even though we expect that they do exist, perhaps abundantly. However, the theorem does characterize all bounded orbits of (4): they must all reach 0.
4 Conclusion

We discussed two quadratic maps of Collatz type and fully characterized their bounded orbits. Proving the expected existence of unbounded orbits for these maps is left as an open problem. Similar ideas and methods may extend to similar types of quadratic maps and help us understand the underlying complexity of these systems a little better.

The maps $Q$ and $S$, as well as the Collatz map $T$ and all other similar maps represent examples of bimodal systems in the sense that each map is divided into two parts: one part is defined on the set of all even integers $\mathcal{E}$ and the other on the set of all odd integers $\mathcal{O}$. Specifically, $Q$ consists of an even part $Q_0 = n/2$ and an odd part $Q_1(n) = \left(\frac{n}{2}\right)$. Importantly, neither of these maps is a self map of its domain; i.e., $\mathcal{E}$ is not invariant under $Q_0$ and $\mathcal{O}$ is not invariant under $Q_1$. These features characterize bimodal systems, which are defined in a general way in [2].

The basic properties of bimodal systems (more generally, polymodal systems) are discussed in [2] where it is also seen that economic and social science models are often bimodal and therefore, provide a rich source of applications for results on bimodal systems. As maps like $T$ and $Q$ illustrate, these systems are capable of generating nontrivial dynamics in the form of orbits that repeatedly enter and exit their domains, namely, $\mathcal{E}$ and $\mathcal{O}$.

With $Q$, all bounded orbits end in $m$-cycles for $m \geq 2$, exhibiting persistent oscillations between $\mathcal{E}$ and $\mathcal{O}$ with the same being possibly true of unbounded orbits. Similarly, if Collatz’s conjecture is true then all orbits of $T$ eventually oscillate between $\mathcal{E}$ and $\mathcal{O}$ as they alternate between 2 and 1. On the other hand, the bounded orbits generated by $S$ always end in 0 and thus remain in $\mathcal{E}$; they do not oscillate persistently between $\mathcal{E}$ and $\mathcal{O}$. The unbounded orbits of $S$ may oscillate persistently between $\mathcal{E}$ and $\mathcal{O}$.

References

[1] Lagarias, J.C., (Editor) The Ultimate Challenge: The 3x+1 Problem, American Mathematical Society, Providence, 2010

[2] Sedaghat, H., Nonlinear Difference Equations: Theory with Applications to Social Science Models, Springer, New York, 2003