THE STOKES PROBLEM IN FRACTAL DOMAINS:
ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS

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Abstract. We study a Stokes problem in a three dimensional fractal domain
of Koch type and in the corresponding prefractal approximating domains. We
prove that the prefractal solutions do converge to the limit fractal one in a
suitable sense. Namely the approximating velocity vector fields as well as the
approximating associated pressures converge to the limit fractal ones respec-
tively.

1. Introduction. The study of viscous flow in rough microchannels is object of
many papers due to the quick development of Micro Electro-Mechanical Systems
such as micro-motors micro-turbines. Flow characteristics have remarkable effect
on the design and process control of MEMS and heat transfer processes [16], [14]
and [12] and they are modeled by Stokes equations. Fractal geometry are a good
tool to model such irregular geometries. The numerical approximation of BVPs in
fractal domains is a crucial issue (see e.g. [2] and [3]).

In order to achieve this ambitious goal in the framework of Stokes problems in
fractal domains a first step is to study how to approximate the solution of an un-
steady Stokes problem in a cylindrical domain with a fractal boundary in terms
of the smoother solutions of the associated Stokes problems in the corresponding
prefractal domains. A second step will be to consider the numerical approxima-
tion of Stokes problems in prefractal domains which will be object of a further
investigation.

More precisely, in the present paper, we consider unsteady Stokes problem (P)
and (P_h) in a cylindrical domain Q with a Koch-type cross section with no-slip
boundary conditions and in the corresponding approximating prefractal domains
Q_h (see figure 2.)

\[ \begin{align*}
\mathbf{u}_t(t,P) - \Delta \mathbf{u}(t,P) + \nabla p &= f(t,P) & \text{in } [0,T] \times Q \\
\nabla \cdot \mathbf{u}(t,P) &= 0 & \text{on } [0,T] \times Q \\
\mathbf{u}(t,P) &= 0 & \text{on } [0,T] \times \partial Q \\
\mathbf{u}(0,P) &= \mathbf{u}_0(P) & \text{in } Q,
\end{align*} \]

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\[ (P_h) \begin{cases} 
\mathbf{u}^h(t, P) - \Delta \mathbf{u}^h(t, P) + \nabla p_h = f_h(t, P) & \text{in } [0, T] \times Q_h \ (i) \\
\nabla \cdot \mathbf{u}^h(t, P) = 0 & \text{in } [0, T] \times Q_h \\
\mathbf{u}^h(t, P) = 0 & \text{on } \partial Q_h \\
\mathbf{u}^h(0, P) = \mathbf{u}_0^h(P) & \text{in } Q_h.
\end{cases} \]

Our aim is to prove that the velocity vector fields \( \mathbf{u}_h \) converge to \( \mathbf{u} \), that the pressure gradients \( \nabla p_h \) converge to the pressure gradient of \( p \) as well as the pressures \( p_h \) converge to \( p \). Here \( f, f_h \) are given source terms belonging to suitable functional spaces as well as the initial data \( \mathbf{u}_0 \) and \( \mathbf{u}_0^h \).

The main difficulty in this asymptotic analysis is due to the irregular geometry of the boundary of the domain \( Q \) and to the fact that the approximating polyhedral domains \( Q_h \) change accordingly to \( h \), namely they are an increasing sequence of domains invading \( Q \).

To prove the existence and uniqueness of a weak solution \( \mathbf{u} \) and \( \mathbf{u}_h \) of problem \((P)\) and \((P_h)\) respectively in the sense of Definition 3.9, we consider the associated abstract Cauchy problems \( \bar{P} \) and \( \bar{P}_h \) (see Section 3). In both cases, we prove existence and uniqueness of a mild solution via a standard semigroup approach (see Theorems 3.7 and 3.8), which turns out to be a weak solution of problem \((P)\) and \((P_h)\) respectively for \( p \in L^2(0, T), L^2_{\text{loc}}(Q) \) and \( p_h \in L^2(0, T), L^2(Q) \) respectively follows straightforward (see [13] and [9]).

Our main result is to prove that the velocity vector fields \( \mathbf{u}_h \) converge to \( \mathbf{u} \) in \( L^2([0, T] : H^1_0(Q)) \) see Theorem 4.6. This is achieved via the M-convergence of the approximating prefractal energy forms \( a_h \) associated to \((P_h)\) to the limit fractal energy form \( a \) associated to \((P)\), see Theorem 4.4 which in turn implies the convergence of the associated semigroups see Theorem 4.5.

In Theorem 4.8 we prove that the gradients of the pressures \( p_h \) weakly converge in \( L^2((0, T), H^{-1}(Q)) \) to the gradient of the pressure \( p \) and in Theorem 4.9 we prove that \( p_h \) weakly converge to \( p \) in \( L^2((0, T) \times Q) \). The proof of theorem 4.8 follows from the weak convergence of the approximating Stokes operators \( A^h \) to \( A \) and of the time derivatives given in Theorem 4.6. The convergence of the pressures is a delicate issue, it deeply relies on some recent results on the solutions of the divergence operator on John domains such as our domain \( Q \).

The plan of the paper is the following. In section 2 we introduce the geometry and the relevant functional spaces. In Section 3 we state the problems \((P)\) and \((P_h)\). We consider the abstract problems \( \bar{P} \) and \( \bar{P}_h \), we prove the existence of a unique mild solution which in turn is a weak solution (see Theorem 3.11). Finally we construct the associated pressures. In Section 4 we adapt the results in [13] to the present case. We prove the M-convergence of the energy forms in Theorem 4.4 and we deduce the convergence of the associated semigroups. In Theorems 4.6, 4.8 and 4.9 we prove the convergence of the velocity vector fields \( \mathbf{u}_h \), of the associated gradient pressures and of the pressures.

2. Preliminaries. We denote by \( |P - P_0| \) the Euclidean distance in \( \mathbb{R}^3 \). By the snowflake \( F \) we denote the union of three coplanar Koch curves \( K_i \) (see [5]).

We assume that the junction points \( A_1, A_2, A_3 \) are the vertices of a regular triangle with unit side length, that is \( |A_1 - A_3| = |A_1 - A_5| = |A_3 - A_5| = 1 \). One can define, in a natural way, a finite Borel measure \( \mu_F \) supported on \( F \) by

\[ \mu_F := \mu_1 + \mu_2 + \mu_3 \]

where \( \mu_i \) denotes the normalized \( d_f \)-dimensional Hausdorff measure, restricted to \( K_i, i = 1, 2, 3 \).
We denote by $F_h$ the closed polygonal curve approximating $F$ at the $h$–th step. By $S_h$ we denote $F_h \times I$, where $F_h$ is the prefractal approximation of $F$ at the step $h$, $I = [0,1]$. $S_h$ is a surface of polyhedral type. We give a point $P \in S_h$ the Cartesian coordinates $P = (x, x_3)$, where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of $P$ on the plane containing $F_h$ and $x_3$ is the coordinate of the orthogonal projection of $P$ on the $x_3$-line containing the interval $I$.

By $\Omega_h$ we denote the open bounded two-dimensional domain with boundary $F_h$. By $Q_h$ we denote the domain with $S_h$ as lateral surface and $\Omega_h := \Omega_h \times \{0\} \cup \Omega_h \times \{1\}$ as bases of $Q_h$.

The measure on $S_h$ is

$$d\sigma = dl \times dx_3,$$

where $dl$ is the arc-length measure on $F_h$ and $dx_3$ is the one-dimensional Lebesgue measure on $I$. We introduce the fractal surface $S = F \times I$ given by the Cartesian product between $F$ and $I$. It can be defined on $S$ the finite Borel measure

$$dg = d\mu_F \times dx_3$$

supported on $S$. By $\Omega$ we denote the two-dimensional domain whose boundary is $F$. By $Q$ we denote the open cylindrical domain where $S = F \times I$ is the lateral surface and where the sets $\Omega \times \{0\}$, $\Omega \times \{1\}$ are the bases and $\tilde{\Omega}$ is their union.
We note that the sequence \( \{ Q_h \} \) is an increasing sequence invading \( Q \), that is the \( \mathcal{L}(Q \setminus Q_h) \to 0, h \to \infty \) where \( \mathcal{L} \) is the Lebesgue measure in \( \mathbb{R}^3 \).

Let \( K \) be a compact set of \( \mathbb{R}^N \), by \( C(K) \) we denote the space of continuous functions on \( K \) and by \( C^\infty_0(K) \) the space of continuous infinitely differentiable functions with compact support in \( K \). Let \( M \) be an open set of \( \mathbb{R}^3 \). By \( L^2(M) \) we denote the Lebesgue space with respect to the Lebesgue measure \( \mathcal{L} \). By \( H^1(M) \), we denote the usual Sobolev space by \( \| \cdot \|_{H^1(M)} \) its norm. By \( H^1_0(M) = \overline{C^\infty_0(M)}^{\| \cdot \|_{H^1(M)}} \) and by \( H^{-1}(M) \) its dual.

In the following we identify by \( H^1_0(Q_h) \) the closure in \( H^1(Q) \) of all functions \( \phi \in C^\infty_0(Q) : \phi = 0 \) on \( Q \setminus Q_h \). We denote the relative capacity of a compact set \( T \) in \( Q \) by

\[
\text{cap}(T) = \inf \{ \| \nabla \phi \|_{L^2(Q)}^2 : \phi \in C^\infty_0(Q), \phi \geq 1 \text{ on } T \}.
\]

We define \( L^2(M)^3 = \{ \mathbf{u} = u_1, u_2, u_3 : u_i \in L^2(M) \} \) and \( L^2_0(M) = \overline{(C^\infty_0(M))}^{L^2(M)^3} \) where \( L^2_0(M) \) is endowed with the \( L^2 \)- scalar product, and \( C^\infty_0(M) = \{ \mathbf{v} \in C^\infty_0(M)^3 : \nabla \cdot \mathbf{v} = 0 \} \). By \( H^1_{0,\sigma}(M) = \overline{(C^\infty_{0,\sigma}(M))}^{H^1(M)} \) we denote the closed subspace of \( H^1_0(M)^3 \) moreover the following result holds.

**Proposition 2.1.** The space \( H^1_{0,\sigma}(M) \) is dense in \( (H^1_0(M))^3 \) (see Proposition 2.2. in [9]).

3. The Stokes problem.

\[
(P) \quad \begin{cases}
\mathbf{u}_t(t, P) - \nabla \cdot \mathbf{u}(t, P) + \nabla p = \mathbf{f}(t, P) & \text{in } [0, T] \times Q \\
\nabla \cdot \mathbf{u}(t, P) = 0 & \text{on } [0, T] \times \partial Q \\
\mathbf{u}(0, P) = \mathbf{u}_0(P) & \text{in } Q, \\
\end{cases}
\]

\[
(P_h) \quad \begin{cases}
\mathbf{u}_h(t, P) - \nabla \cdot \mathbf{u}_h(t, P) + \nabla p_h = \mathbf{f}_h(t, P) & \text{in } [0, T] \times Q_h (i) \\
\nabla \cdot \mathbf{u}_h(t, P) = 0 & \text{on } [0, T] \times \partial Q_h \\
\mathbf{u}_h(t, P) = 0 & \text{on } [0, T] \times \partial Q_h \\
\mathbf{u}_h(0, P) = \mathbf{u}_0^h(P) & \text{in } Q_h, \\
\end{cases}
\]

The following result due to the Rham [4] holds

**Theorem 3.1.** Let \( M \) be an open set of \( \mathbb{R}^3 \) and \( \mathbf{f} \in ((C^\infty_0(M))^3)' \) a necessary and sufficient condition that \( \mathbf{f} = \nabla p \) for some \( p \in (C^\infty_0(M))^3 \) is that \( < \mathbf{f}, \mathbf{w} > = 0 \) for every \( \mathbf{w} \in C^\infty_{0,\sigma}(M) \).
Remark 1. (see Temam [15] p. 20) If $f \in H^{-1}(M)$ and $< f, w > = 0$ for every $w \in C_{0, \alpha}^\infty(M)$ then $f = \nabla p$ for some $p \in L^2_{loc}(M)$. If $M$ is Lipschitz the same result holds with $p$ in $L^2(M)$. If $f$ depends also on time, following [13] Lemma 1.4.2 cap. IV pag.202 we have that if $f \in L^2((0, T), H^{-1}_{loc}(M))$ and it is such that
\[
\int_0^T f(w) dt = 0
\]
for all $w \in C_{0}^\infty((0, T), C_{0, \alpha}^\infty(M))$. Then there exists a unique $p$ in $L^2((0, T), L^2_{loc}(M))$ satisfying $f = \nabla p$ in the sense of distributions in $(0, T) \times Q$ and $\int_{M_0} p(t) d\mathcal{L} = 0$ for almost all $t \in [0, T)$ and for any bounded subdomain $M_0$ with $\overline{M_0} \subseteq M$.

Moreover it holds (see Theorem 4.1 in [1]):

**Theorem 3.2.** Let $M$ be a bounded John domain in $\mathbb{R}^N$, given $f \in L^2((0, T) \times M)$ such that for almost every $t \in (0, T)$, $\int_M f d\mathcal{L} = 0$ then there exists, for almost every $t \in (0, T)$, $v(t) \in \{H^1_0(M)\}^3$ such that $\text{div} v(t) = f(t)$ and there exists a constant $c$ depending on $M$ such that
\[
\|v(t)\|_{(H^1_0(M))^3} \leq c\|f(t)\|_{L^2(M)}
\]

We recall the Hodge decomposition that $L^2(M)^3 = L^2_\sigma(M) \oplus G$ where $G = \{\nabla p; p \in L^2_{loc}(M) \text{ with } \nabla p \in L^2(M)^3\}$

We denote by $\mathcal{P} : L^2(M)^3 \rightarrow L^2_\sigma(M)$ the Leray-Helmoltz projection and by $J$ the canonical injection $L^2_\sigma(M) \hookrightarrow L^2(M)^3$, by $J' = \mathcal{P}$ the adjoint of $J$ and $\mathcal{P}J$ is the identity on $L^2_\sigma(M)$. The canonical injection $J : H^1_{0, \sigma}(M) \hookrightarrow H^1_0(M)^3$ is the restriction of $J$ to $H^1_{0, \sigma}(M)$. We denote by $\hat{\mathcal{P}}$ the adjoint of $\hat{J}$ since $\hat{J}$ is the restriction of $J$ to $H^1_{0, \sigma}(M)$, $\hat{\mathcal{P}}$ is an extension of $\mathcal{P}$ to $(H^1_{0, \sigma}(M))^\prime$, the dual of $H^1_{0, \sigma}(M)$.

3.1. The abstract problem. We introduce the bilinear symmetric form $a(u, v) : H^1_{0, \sigma}(Q) \times H^1_{0, \sigma}(Q) \rightarrow \mathbb{R}$:
\[
a(u, v) = \int_Q \nabla u \nabla v d\mathcal{L}.
\]

We note that it is coercive in $H^1_{0, \sigma}(Q)$ thanks to Poincaré inequality and closed in $L^2_\sigma(Q)$. By Kato’s theorem [7] there exists a unique non positive self-adjoint operator $A_0 : H^1_{0, \sigma}(Q) \rightarrow (H^1_{0, \sigma}(Q))^\prime$ such that:
\[
a(u, v) = -< A_0 u, v >_{(H^1_{0, \sigma}(Q))^\prime, H^1_{0, \sigma}(Q)}
\]

The following proposition holds (see [9]):

**Proposition 3.3.** Let $\Delta : H^1_0(Q) \rightarrow H^{-1}(Q)$ denote the Dirichlet Laplacian there holds.
\[
A_0 = \hat{\mathcal{P}} \circ (\Delta) \hat{J}
\]

We call the Stokes operator $-A$, the part of $-A_0$ in $H^1_{0, \sigma}(Q)$ i.e. $D(A) = \{u \in H^1_{0, \sigma}(Q) : A_0 u \in L^2_\sigma(Q)\}$ and $Au = A_0 u$.

**Theorem 3.4.** The operator $A$ is self-adjoint in $L^2_\sigma(Q)$ and generates an analytic contraction semigroup $T(t) = e^{+tA} : L^2_\sigma(Q) \rightarrow D(A)$ with
\[
D(A) = \{u \in H^1_{0, \sigma}(Q) : \exists p \in (D(Q))^\prime, \nabla p \in H^{-1}(Q), -\Delta u + \nabla p \in L^2_\sigma(Q)\}
\]
and
\[
-Au = -\Delta u + \nabla p \in L^2_\sigma(Q).
\]
From Lemma 2.2.1 in [13] we have that there exists a unique positive self-adjoint operator \((-A)^\frac{1}{2} : D((-A)^{1/2}) \to L^2(Q)\) with domain \(D((-A)^{1/2})\) such that \((-A) \subset D((-A)^{1/2}) \subset L^2(Q)\) which enjoys the following property:

\[
D((-A)^{1/2}) = H^{1}_{0,\sigma}(Q)
\]

and

\[
< (-A)^{1/2}u, (-A)^{1/2}v > = \nabla u \cdot \nabla v
\]

Proceeding similarly, we define for every \(h \in \mathbb{N}\), the bilinear symmetric form \(a_h(u,v) : H^1_{0,\sigma}(Q_h) \times H^1_{0,\sigma}(Q_h) \to \mathbb{R}\):

\[
a_h(u,v) = \int_{Q_h} \nabla u \nabla v dL. \tag{2}
\]

We note that it is coercive in \(H^1_{0,\sigma}(Q_h)\) thanks to Poincaré inequality and closed in \(L^2(Q_h)\). By Kato’s theorem there exists a unique non positive self-adjoint operator \(A_h^1 : H^1_{0,\sigma}(Q_h) \to (H^1_{0,\sigma}(Q_h))'\) and

**Proposition 3.5.** Let \(A_h : H^1_{0,\sigma}(Q_h) \to H^{-1}(Q_h)\) denote the Dirichlet Laplacian there holds

\[
A_h^1 = \tilde{\mathcal{P}} \circ (\Delta_h)\tilde{\mathcal{J}}
\]

We call the Stokes operator \(-A_h\), the part of \(-A_h^1\) in \(H^1_{0,\sigma}(Q_h)\) i.e. \(D(A_h) = \{ u \in H^1_{0,\sigma}(Q_h) : A_h^1 u \in L^2(Q_h) \}\) and \(A_h u = A_h^1 u\).

**Theorem 3.6.** The operator \(A_h\) is self-adjoint in \(L^2(Q_h)\), generates an analytic contraction semigroup \(T_h(t) = e^{+tA_h} : L^2(Q_h) \to D(A_h)\) with

\[
D(A_h) = \{ u \in H^1_{0,\sigma}(Q_h) \exists p \in (D(Q_h))', \nabla p \in H^{-1}(Q_h), -\Delta_h u + \nabla p \in L^2(Q_h) \}\]

and

\[
-A_h u = -\Delta_h u + \nabla p \in L^2(Q_h)
\]

As above we introduce \((-A_h)^{1/2}\) with domain \(D((-A_h)^{1/2})\).

We can now consider the abstract problems associated to Problems (P) and (P_h) respectively.

\[
\tilde{\mathcal{P}} \left\{
\begin{array}{ll}
\mathbf{u}_t = A \mathbf{u} + P \mathbf{f}, & \text{in } [0,T] \\
\mathbf{u}(0) = \mathbf{u}_0,
\end{array}
\right.
\]

\[
\tilde{\mathcal{P}}_h \left\{
\begin{array}{ll}
\mathbf{u}^h_t = A_h \mathbf{u}^h + P_h \mathbf{f}, & \text{in } [0,T] \\
\mathbf{u}(0) = \mathbf{u}_0^h,
\end{array}
\right.
\]

where \(-A\) and \(-A_h\) are the Stokes operators in \(Q\) and \(Q_h\) respectively. The following existence results hold.

**Theorem 3.7.** Let \(f \in L^2([0,T];L^2(Q)^3)\) and \(u_0 \in D((-A)^{1/2})\) and let

\[
u(t) = T(t)(u_0) + \int_0^t T(t-s)Pf(s)ds, \tag{3}
\]

where \(T(t)\) is the analytic semigroup generated by \(\tilde{A}\). Then \(u\) is the unique mild solution of \((\tilde{P})\) , moreover

\[
\mathbf{u} \in H^1((0,T);L^2(Q)) \cap L^2((0,T);D(A)), \\
\mathbf{u}_t(t) = A \mathbf{u}(t) + P \mathbf{f}(t), \text{ for almost every } t \in [0,T], \mathbf{u}(0) = \mathbf{u}_0.
\]

and there exists \(c\) such that the following inequality holds:

\[
\|u\|_{H^1((0,T);L^2(Q))} + \|u\|_{L^2((0,T);D(A))} \leq c\|f\|_{L^2((0,T);L^2(Q)^3)} + \|u_0\|_{D((-A)^{1/2})}. \tag{4}
\]
\textbf{Theorem 3.8.} Let $f_h \in L^2((0,T);L^2(Q_h)^3)$ and $u_h^0 \in D((-A)^\frac{1}{2})$ let
\[ u_h(t) = T_h(t)(u_h^0) + \int_0^t T_h(t-s)Pf_h(s)ds, \forall h \in \mathbb{N} \]
where $T_h(t)$ is the analytic semigroup generated by $A_h$. Then $u_h$ is the unique mild solution of $(P_h)$, moreover
\[ u_h \in H^1((0,T);L^2(Q_h)^3) \cap L^2((0,T);D(A_h)), \]
\[ u_h^t(t) = A_hu_h(t) + Pf_h(t), \text{ for almost every } t \in [0,T], \quad u_h(0) = u_h^0, \]
and there exists $C$, independent from $h$, such that the following inequality holds:
\[ \|u_h\|_{H^1((0,T);L^2(Q_h)^3)} + \|u_h\|_{L^2((0,T);D(A_h))} \leq C\|f_h\|_{L^2((0,T);L^2(Q_h)^3)} + \|u_h^0\|_{D((-A)^\frac{1}{2})}. \]

The proofs of Theorems 3.7 and 3.8 follow from Theorem 1.5.2 and Lemma 1.6.1 - 1.6.2 in Ch. IV (for $s=2$) [13]. We remark that in Theorem 3.8 the independence of $C$ from $h$ in (5) follows from estimates 1.6.20-1.6.22 In Ch. IV of [13].

\subsection{3.2. Weak formulation.}
In this section we prove that the solution of the abstract Cauchy problems $(\bar{P})$ and $(P_h)$ are weak solutions of the Stokes system $(P)$ and $(P_h)$ respectively. Adapting [13] (see chapter IV 2) we now give the definition of weak solution for problem $(P)$.

\textbf{Definition 3.9.} Let $f \in L^2((0,T),L^2(Q)^3)$ and $u_0 \in L^2_0(Q)$, a function $u \in L^2((0,T),H^1_{0,s}(Q))$ is called a weak solution of the Stokes system $(P)$ if and only if
\[ -\int_0^T \int_Q (u \cdot v_t + \nabla u \cdot \nabla v) d\mathcal{L}dt = \int_Q u_0 \cdot v(0)d\mathcal{L} + \int_0^T \int_Q f \cdot v d\mathcal{L}dt \]
for all $v \in C_0^\infty([0,T),C_0^\infty(Q))$. A Distribution $p$ in $[0,T) \times Q$ is called the associated pressure of a weak solution if and only if
\[ u_t(t,P) - \Delta u(t,P) + \nabla p = f(t,P) \]
is satisfied in the sense of distributions.

The definition of weak solution for problem $(P_h)$ can be obtained from 3.9 with the obvious changes.

\textbf{Remark 2.} We note that if $u$ is the solution of $(\bar{P})$ then the following identity holds:
\[ \int_0^T \int_Q (u_t \cdot v + \nabla u \cdot \nabla v) d\mathcal{L}dt = \int_0^T \int_Q f \cdot v d\mathcal{L}dt \]
An analogous statement holds for the solution $u_h$ of $(\bar{P}_h)$.

We preliminary state this result (see [13], Chapter IV Lemma 2.2.1).

\textbf{Lemma 3.10.} Let $f \in L^2((0,T),(L^2(Q)^3)$ and $u_0 \in L^2_0(Q)$. It results that $u \in L^2((0,T),H^1_{0,s}(Q))$ is a weak solution of the Stokes system if and only if there exists a dense subspace $\Gamma \subset H^1_{0,s}(Q)$ such that
\[ -\int_0^T \int_Q (u \cdot w \phi_t + \nabla u \cdot \nabla w \phi) d\mathcal{L}dt = \int_Q u_0 \cdot w \phi(0)d\mathcal{L} + \int_0^T \int_Q f \cdot w \phi d\mathcal{L}dt \]
for all $w \in \Gamma$ and $\phi \in C_0^\infty([0,T))$.

From Theorem 2.4.1 in [13] in Chapter IV we deduce:
Theorem 3.11. Let \( T > 0 \), \( u_0 \in L^2_0(Q) \) and \( f \in L^2((0, T), (L^2(Q))^3) \) then (3) is a weak solution of the Stokes system \((P)\) with data \( f \) and \( u_0 \). Similarly, for each \( h \in \mathbb{N} \) if \( u^h_0 \in L^2_0(Q_h) \) and \( f^h_0 \in L^2((0, T), (L^2(Q_h))^3) \) then (4) is a weak solution of the Stokes system \((P_h)\) with data \( f^h_0 \) and \( u^h_0 \).

For the sake of completeness we sketch the proof.

Proof. In order to prove our thesis we use Lemma 3.10 by setting \( \Gamma = C_{0,\sigma}^\infty(Q) \). We choose as test function in (6) the function \( v(x, t) = w(x)\phi(t) \) where \( w \in C_{0,\sigma}^\infty((0, T)) \) and \( \phi \in C_{0,\sigma}^\infty([0, T]) \).

Since \( u \) is a solution and \( P \) is self-adjoint we get:

\[
- \int_0^T \int_Q (u \cdot \phi_t + \nabla u \cdot \nabla \phi) \, d\mathcal{L} dt
= - \int_0^T \int_Q \left( (u \cdot \phi)_t + (u \cdot \nabla \phi) \right) \, d\mathcal{L} dt
= \int_Q u_0 \cdot \phi(0) d\mathcal{L} + \int_0^T \int_Q \left( (Au + Pf) \cdot \phi + \nabla u \cdot \nabla \phi \right) \, d\mathcal{L} dt
= \int_Q u_0 \cdot \phi(0) d\mathcal{L} + \int_0^T Pf \cdot \phi \, d\mathcal{L}
= \int_Q u_0 \cdot \phi(0) d\mathcal{L} + \int_0^T f \cdot \phi \, d\mathcal{L}
\]

One can proceed analogously for \( u^h \).

3.3. Associated pressure. We construct the pressure \( p \) and \( p_h \) associated to the weak solution of the Stokes system \((P)\) and \((P_h)\) respectively. That is a distribution \( p \) in \((0, T) \times Q\) satisfying \( u_t - \Delta u + \nabla p = f \) and \( p_h \) in \((0, T) \times Q_h\) satisfying \( u^h_t - \Delta u^h + \nabla p_h = f_h \). We will carry out the construction of \( p \), the one of \( p_h \) is analogous with the obvious changes.

Under the hypothesis of Theorem 3.7, let \( u \) be the solution (3) that is \( u \) is the solution with data \( f \) and \( u_0 \), by Remark 2 the following functional is well defined (see [13] page 252):

\[
G : v \rightarrow< G, v >_{(C_0^\infty((0, T) \times Q))^3, (C_0^\infty((0, T) \times Q))^3}
\]

In the following we denote by \([G, v]|_{T,Q} = < G, v >_{(C_0^\infty((0, T) \times Q))^3}$, $(C_0^\infty((0, T) \times Q))^3$,

\[
[G, v]|_{T,Q} = [f - u_t + \Delta u, v]|_{T,Q} = \int_0^T ((f, v)_{L^2(Q)^3} - (u_t, v)_{L^2(Q)^3} - (\nabla u, \nabla v)_{L^2(Q)^3}) \, dt.
\]

We now prove that \( G \in L^2((0, T), H^{-1}(Q)) \).

\[
||G, v||_{T,Q} \leq ||f + u_t||_{L^2((0, T), L^2(Q)^3)} ||v||_{L^2((0, T), L^2(Q)^3)}
+ ||\nabla u||_{L^2((0, T), L^2(Q)^3)} ||\nabla v||_{L^2((0, T), L^2(Q)^3)}
\leq C(||f||_{L^2((0, T), L^2(Q)^3)} + ||u_0||_{D((-A)^{1/2})}) ||v||_{L^2((0, T), H^1_0(Q)^3)}
\]

where the last inequality follows from the a priori estimate (4).

Moreover it turns out that \([G, v]|_{T,Q} = 0\) for every \( v \in C_0^\infty((0, T) \times Q)^3\) hence by Remark 1 we have there exists a unique \( p \) in \( L^2((0, T), L^2_{loc}(Q)) \) with \( \int_Q p(t) \, d\mathcal{L} = 0 \) for almost every \( t \) such that \( G = \nabla p \) in the sense of distributions.
As to the problem $P_h$ we introduce the functional
\[ [G_h, \mathbf{v}]_{T, Q_h} = \langle G_h, \mathbf{v} \rangle_{(C_0^{\infty}(0,T) \times Q_h)^3, C_0^{\infty}(0,T) \times Q_h)^3} \]
by proceeding as above, we can prove that $G_h \in L^2((0,T), H^{-1}(Q_h))$ and $[G_h, \mathbf{v}]_{T, Q_h} = 0$ for every $\mathbf{v} \in C_0^{\infty}(0,T) \times Q_h)^3$ then since $Q_h$ is a Lipschitz domain there exists a unique $p_h \in L^2((0,T), L^2(Q_h))$ such that $G_h = \nabla p_h$ in the sense of distributions, and with $\int_{Q_h} p_h(t)d\mathcal{L} = 0$ for almost every $t \in [0,T]$.

4. The convergence of forms and semigroups. In this Section we study the convergence of the approximating energy forms $a_h$ to the fractal energy $a$. The convergence of functional is here intended in the sense of the M-convergence which we define below. In the following for any given function $v_h$ defined in $Q_h$, we denote by $\tilde{v}_h$ its trivial extension to $(0,T) \times Q$ that is $\tilde{v}_h = v_h$ in $(0,T) \times Q_h$ and $\tilde{v}_h = 0$ in $(0,T) \times (Q \setminus Q_h)$.

4.1. The M-convergence of forms. We recall, for the sake of completeness, the definition of M-convergence of forms introduced by Mosco in [10].

We extend the forms $a$ defined in $H^1_{0,\sigma}(Q)$ and $a_h$ defined in $H^1_{0,\sigma}(Q_h)$ on the whole space $L^2_\sigma(Q)$ by defining
\[ a(|u|) = +\infty \text{ for every } u \in L^2_\sigma(Q) \setminus H^1_{0,\sigma}(Q) \]
and
\[ a_h(|u|) = +\infty \text{ for every } u \in L^2_\sigma(Q) \setminus H^1_{0,\sigma}(Q_h) \]

We note that taking into account the definition of the space $H^1_{0,\sigma}(Q_h)$ the forms $a_h$ turn out to be well defined in the whole $L^2_\sigma(Q)$ because the functions $u_h$ in the domain of $a_h$ can be trivially extended to $H^1_{0,\sigma}(Q)$ that is $\tilde{u}_h(P) = 0$ in $Q \setminus Q_h$.

**Definition 4.1.** A sequence of form $\{a_h\}$ M-converges to a form $a$ in $L^2_\sigma(Q)$ if
(a) for every $\{v_h\}$ converging weakly to $u$ in $L^2_\sigma(Q)$
\[ \lim_{h \to \infty} a_h(v_h) \geq a(u), \quad \text{as} \quad h \to \infty \]
(b) for every $u \in L^2_\sigma(Q)$ there exists $\{w_h\}$ converging strongly to $u$ in $L^2_\sigma(Q)$ such that
\[ \overline{\lim}_{h \to \infty} a_h(w_h) \leq a(u), \quad \text{as} \quad h \to \infty \]

According to Definition 2.3.1 in [10], we say that

**Definition 4.2.** The sequence of forms $\{a_h\}$ is asymptotically compact in $L^2_\sigma(Q)$ if every sequence $\{u_h\}$ with
\[ \lim\ a_h(|u_h|) + \int_Q |u_h|^2d\mathcal{L} < \infty \]
has a subsequence strongly convergent in $L^2_\sigma(Q)$.

**Proposition 4.3.** The sequence of forms $a_h$ is asymptotically compact in $L^2_\sigma(Q)$.

**Remark 3.** We point out that, as the sequence of forms $a_h$ is asymptotically compact in $L^2_\sigma(Q)$, M-convergence is equivalent to the $\Gamma$-convergence (see Lemma 2.3.2 in [10]), thus we can take in (a) $v_h$ strongly converging to $u$ in $L^2_\sigma(Q)$.

We can now state the main theorem of this section.

**Theorem 4.4.** The sequence of forms $a_h$ defined in (2) M-converges in the space $L^2_\sigma(Q)$ to the form $a$ defined in (1).
Proof. We start by proving condition a). We can suppose that \( v_h \in H^1_{0,\sigma}(Q_h) \) and that \( \lim a_h[v_h] < \infty \) then there exists a subsequence still denoted by \( v_h \) such that \( a_h(v_h) \) converges to its liminf; therefore there exists a \( c \) independent from \( h \) such that \( \|v_h\|_{H^1_{0,\sigma}(Q_h)} \leq c \). We consider the trivial extension to \( Q \) of \( v_h \). Therefore \( \|\tilde{v}_h\|_{H^1_{0,\sigma}(Q)} = \|v_h\|_{H^1_{0,\sigma}(Q_h)} \) from this we can deduce that \( \|\tilde{v}_h\|_{H^1_{0,\sigma}(Q)} \leq c \) with \( c \) independent from \( h \). There exists a subsequence still denoted by \( \tilde{v}_h \) weakly converging to \( \tilde{w} \) in \( H^1_0(Q) \) and strongly in \( L^2(Q) \) from the uniqueness of the limit we deduce that \( u = w \) a.e. in \( L^2(Q) \) and in particular \( \nabla\tilde{v}_h \) weakly converges to \( \nabla u \) in \( L^2(Q) \), the thesis follows from the lower semicontinuity of the \( L^2 \) norm of \( \nabla u \).

We now prove condition b). Let us assume that \( u \in H^1_{0,\sigma}(Q) \), from Riesz Theorem we have: for any given \( u \in H^1_{0,\sigma}(Q) \) there exists a \( \psi \in (H^1_{0,\sigma}(Q))' \) such that

\[
a(u, v) = \langle \psi, v \rangle_{(H^1_{0,\sigma}(Q))', H^1_{0,\sigma}(Q)}.
\]

Since for every \( h \in \mathbb{N} \), \( \psi \) belongs also to \( (H^1_{0,\sigma}(Q_h))' \) the problem

\[
M_h \left\{ \begin{array}{l}
\text{find } u_h \in H^1_{0,\sigma}(Q_h) : \\
\quad a_h(u_h, v) = \langle \psi, v \rangle_{(H^1_{0,\sigma}(Q_h))', H^1_{0,\sigma}(Q_h)}, \\
\quad \text{for every } v \in H^1_{0,\sigma}(Q_h)
\end{array} \right.
\]

has a unique solution \( u_h \). We set \( E_h := Q \setminus Q_h \). For every compact subset \( Q' \subset Q \) we have that \( \text{cap}(E_h \cap Q') \to 0 \) as \( h \to \infty \). From Corollary 2, Section 3 in [11], \( u_h \) strongly converges to \( u \) in \( H^1_0(Q) \) and hence \( \lim h a_h(u_h) = a(u) \).

\[ \square \]

**Theorem 4.5.** Let \( a_h \) and \( a \) as in Theorem 4.4 then the sequence of semigroups \( \{T_h(t)\} \) associated with the form \( a_h \) converges to the semigroup \( T(t) \) associated with the form \( a \) in the strong operator topology of \( L^2(Q) \) uniformly on every interval \([0, t_1]\).

**Proof.** The proof easily follows from Corollary 4.5 and Theorem 4.2 Chapter 3 in [10].

\[ \square \]

4.2. **Convergence of the solutions.** Let \( f_h \) be as in Theorem (3.8), we denote by \( \tilde{f}_h \) the trivial extension to \( Q \) of \( f_h \).

**Theorem 4.6.** Let \( u \) and \( u_h \) be the solutions of problems \( \tilde{P} \) and \( \tilde{P}_h \) according to Theorems (3.7) and (3.8). If \( \{\tilde{u}_h\} \) converges to \( \tilde{u}_0 \) in \( L^2(Q) \) and \( f_h \) converges to \( f \) in \( L^2(0, T; L^2(Q)^3) \) and there exists a \( c \) independent from \( h \) such that

\[ \|(-A_h)^{1/2} u_h^h\|_{L^2(Q_h)^3} \leq c \]

we have:

i): \( \{\tilde{u}_h\} \) converges to \( \tilde{u} \) in \( L^2((0, T); L^2(Q)) \)

ii): \( \{\frac{\partial \tilde{u}_h}{\partial t}\} \) weakly converges to \( \frac{\partial \tilde{u}}{\partial t} \) in \( L^2([0, T] \times Q)^3 \)

iii): \( \{A_h \tilde{u}_h\} \) weakly converges to \( A \tilde{u} \) in \( L^2([0, T] \times Q)^3 \)

iv): \( \{\tilde{u}_h\} \) converges to \( \tilde{u} \) in \( L^2([0, T]; H^1_0(Q)^3) \)

The proof can be carried out as in Theorem 5.3 in [8]. We only remark that from [iii] one can deduce that

\[
\lim_{h \to \infty} \int_0^T a_h(\tilde{u}_h, \tilde{u}_h) dt = -\lim_{h \to \infty} (A_h \tilde{u}_h, \tilde{u}_h)_{L^2((0, T) \times Q)^3} = -(A \tilde{u}, u)_{L^2(0, T; L^2(Q))^3} = \int_0^T a(u, u) dt
\]
4.3. Convergence of the associated pressure. In this section we prove the convergence of the associated pressures and of their gradients.

Lemma 4.7. Let \( w \in C([0, T), D(A))^{3} \), for each \( v \in H_{0, \sigma}^{1}(Q) \) we have that

\[
\int_{0}^{T} \langle A_{0} w, v \rangle_{L_{x}^{2}(Q), L_{x}^{2}(Q)} dt = \int_{0}^{T} \langle \Delta w, v \rangle_{H^{-1}(Q)^{3}, H_{0, \sigma}^{1}(Q)^{3}} dt
\]

Proof. Since

\[
\int_{0}^{T} \langle Aw, v \rangle_{L_{x}^{2}(Q), L_{x}^{2}(Q)} dt = \int_{0}^{T} \langle A_{0} w, v \rangle_{L_{x}^{2}(Q), L_{x}^{2}(Q)} dt
\]

\[
= \int_{0}^{T} \langle \tilde{P} \circ \Delta \tilde{w}, v \rangle_{(H_{0, \sigma}^{1}(Q))^{3}, (H_{0, \sigma}^{1}(Q))^{3}} dt
\]

\[
= \int_{0}^{T} \langle \Delta \tilde{w}, \tilde{v} \rangle_{(H_{0, \sigma}^{1}(Q))^{3}, (H_{0, \sigma}^{1}(Q))^{3}} dt
\]

\[
= \int_{0}^{T} \langle \Delta w, v \rangle_{H^{-1}(Q)^{3}, H_{0, \sigma}^{1}(Q)^{3}} dt.
\]

\[\square\]

Theorem 4.8. Let \( u, p \) and \( u_{h}, p_{h} \) be the solutions of problems \( P \) and \( P_{h} \) as in Section 3.2 and 3.3. Let \( \tilde{p}_{h} \) denote the trivial extension of \( p_{h} \). Under the hypothesis of Theorem 4.6 we have that:

\[\nabla \tilde{p}_{h} \rightharpoonup \nabla p \ \text{in} \ L^{2}((0, T), H^{-1}(Q))\]

Proof. Since \( u, p \) is a solution of \( P \), for every \( v \in L^{2}((0, T), C_{0}^{\infty}(Q))^{3} \) we have

\[
\int_{0}^{T} \langle \nabla \tilde{p}_{h}, v \rangle_{H^{-1}(Q)^{3}, H_{0, \sigma}^{1}(Q)^{3}} dt
\]

\[
= \int_{0}^{T} \left( \left( \int_{Q} \frac{d\tilde{u}_{h}}{dt} + f_{h} \right) v d\mathcal{L} \right) dt + \int_{0}^{T} \langle \Delta_{h} \tilde{u}_{h}, v \rangle_{H^{-1}(Q)^{3}, H_{0, \sigma}^{1}(Q)^{3}} dt.
\]

We note that from i) of Theorem (4.6) the first term in the right-hand side converges to \( \int_{0}^{T} \int_{Q} (-u_{t} + f)v dL dt \). From Lemma 4.7 we have

\[
\int_{0}^{T} \langle \Delta_{h} \tilde{u}_{h}, v \rangle_{H^{-1}(Q)^{3}, H_{0, \sigma}^{1}(Q)^{3}} dt = \int_{0}^{T} \langle A_{0}^{h} \tilde{u}_{h}, v \rangle_{L^{2}(Q)^{3}, L^{2}(Q)^{3}} dt
\]

\[
= \int_{0}^{T} a_{h}(\tilde{u}_{h}, v) dt.
\]

From iv) of Theorem 4.6 we obtain that

\[
\lim_{h \to \infty} \int_{0}^{T} \langle A_{0}^{h} \tilde{u}_{h}, v \rangle_{L^{2}(Q)^{3}, L^{2}(Q)^{3}} dt = \int_{0}^{T} \langle A_{0} u, v \rangle_{L^{2}(Q)^{3}, L^{2}(Q)^{3}} dt
\]

\[
= \int_{0}^{T} \langle \Delta u, v \rangle_{H^{-1}(Q)^{3}, H_{0, \sigma}^{1}(Q)^{3}} dt,
\]

hence the thesis. \[\square\]

In order to prove the convergence of the pressures a key tool is that the domain \( Q \) is a John domain (for its definition we refer the reader to [6]).

Theorem 4.9. Under the assumptions of Theorem 4.8 we have \( \hat{p}_{h} \) weakly converges to \( p \) in \( L^{2}((0, T) \times Q) \).
Proof. In order to prove our statement we preliminary prove that the sequence \{\hat{p}_h\} is equi-bounded in \(L^2((0, T) \times Q)\). For almost every \(t \in (0, T), \{\hat{p}_h\} \subset L^2(Q)\) and \(\int_Q \hat{p}_h dL = 0\). From Theorem 3.2 we have that there exists a function \(w_h \in H^1_0(Q)^3\) such that \(\text{div} w_h(t) = \hat{p}_h(t)\) for almost every \(t \in (0, T)\) and
\[
\|w_h(t)\|_{(H^1_0(Q))^3} \leq c\|\hat{p}_h(t)\|_{L^2(Q)}.
\]
We multiply equation i) in problem \((P_h)\) by \(w_h\) and we obtain
\[
< \nabla \hat{p}_h, w_h >_{H^{-1}(Q)^3, H^1_0(Q)^3} \; dt = \int_Q \left( -\frac{d\hat{u}_h}{dt} + \hat{f}_h \right) w_h dL + a_h(\hat{u}_h, w_h),
\]
for almost every \(t \in (0, T)\). We have that
\[
\int_Q |\hat{p}_h|^2 dL = \int_Q \left( -\frac{d\hat{u}_h}{dt} + \hat{f}_h \right) w_h dL + a_h(\hat{u}_h, w_h),
\]
by integrating on \([0,T]\) we have
\[
\|\hat{p}_h\|_{L^2((0,T)\times Q)} \leq \|\frac{d\hat{u}_h}{dt}\|_{L^2((0,T)\times Q)} \|w_h\|_{L^2((0,T)\times Q)} + \|\hat{u}_h\|_{L^2((0,T),H^1_0(Q)^3)} \|w_h\|_{L^2((0,T),H^1_0(Q)^3)} + \|\hat{f}_h\|_{L^2((0,T)\times Q)} \|w_h\|_{L^2((0,T),H^1_0(Q)^3)}
\]
Hence
\[
\|\hat{p}_h\|_{L^2((0,T)\times Q)} \leq c\|\hat{p}_h\|_{L^2((0,T)\times Q)} + \|\hat{u}_h\|_{L^2((0,T),H^1_0(Q)^3)} + \|\hat{f}_h\|_{L^2((0,T)\times Q)}
\]
All the terms in the brackets are equibounded from Theorem 4.6 and the assumptions on \(f_h\) hence the claim. There exists a subsequence of \(\hat{p}_h\) still denoted by \(\hat{p}_h\) weakly converging to a function \(\hat{p} \in L^2((0, T) \times Q)\). The subspace \(\mathcal{R} = \{g \in L^2((0, T) \times Q): \int_Q g(t) dL = 0\}\) for a.a. \(t \in [0, T]\) is closed in \(L^2((0, T) \times Q)\) hence \(\hat{p} \in \mathcal{R}\).

Since the distributional gradient of \(\hat{p}_h\) weakly converges to the distributional gradient of \(\hat{p} \in L^2((0, T), H^{-1}(Q))^3\), from Theorem 4.8 and the uniqueness of the limit we have that \(\nabla \hat{p} = \nabla \hat{p}\) in the distributional sense. Hence for almost every \(t \in [0, T]\) \(p(t) = \hat{p}(t)\) almost everywhere in \(Q\). \(\square\)

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