Continuum Gauge Fields from Lattice Gauge Fields

M. Göckeler\textsuperscript{1,2}, A. S. Kronfeld\textsuperscript{3}, G. Schierholz\textsuperscript{2,4} and U.-J. Wiese\textsuperscript{5}

\textsuperscript{1}Institut für Theoretische Physik, RWTH Aachen, Sommerfeldstraße, D-5100 Aachen, Germany
\textsuperscript{2}Gruppe Theorie der Elementarteilchen, Höchstleistungsrechenzentr um HLRZ, c/o Forschungszentrum Jülich, D-5170 Jülich, Germany
\textsuperscript{3}Theoretical Physics Group, Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510, U.S.A.
\textsuperscript{4}Deutsches Elektronen-Synchrotron DESY, Notkestraße 85, D-2000 Hamburg 52, Germany
\textsuperscript{5}Institut für Theoretische Physik, Universität Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland

Abstract

On the lattice some of the salient features of pure gauge theories and of gauge theories with fermions in complex representations of the gauge group seem to be lost. These features can be recovered by considering part of the theory in the continuum. The prerequisite for that is the construction of continuum gauge fields from lattice gauge fields. Such a construction, which is gauge covariant and complies with geometrical constructions of the topological charge on the lattice, is given in this paper. The procedure is explicitly carried out in the $U(1)$ theory in two dimensions, where it leads to simple results.
1 Introduction

At present the most attractive nonperturbative regulator of field theories is the lattice. In this approach, matter fields are located on the lattice sites, and gauge fields are represented by parallel transporters on the links connecting the sites. Usually it is most practical to work on a hypercubic lattice.

For many questions it is productive to envision and define properties of an underlying continuum field. The topological charge is a well-known example \cite{1}. The geometrical object which determines the topological charge is a principal fiber bundle. To construct this bundle one needs to define transition functions throughout the boundary of each elementary hypercube, i.e. in the interior of each cube, using the lattice gauge field as input. Under certain continuity assumptions on the lattice gauge field, the bundle reconstructed is unique \cite{1, 2}. In other circumstances, one would like to define the gauge field itself in the interior of the cubes and even the hypercubes. Let us give a few examples:

As is well known, the lattice formulation of chiral gauge theories faces great difficulties, which stem from the doubling problem \cite{3}. To overcome this problem we have proposed a method \cite{4} which keeps the fermionic degrees of freedom in the continuum. The motivation comes from considering the interplay between chiral symmetries, anomalies, and the doubling problem. Anomalies may arise when the ultraviolet regulator breaks some of the action’s symmetries. For vector-like lattice theories chiral symmetry-breaking terms in the lattice action solve the doubling problem, and produce the abelian chiral anomaly \cite{6}. (As for the nonabelian anomaly, see ref. \cite{7}.) Without these terms, e.g. for supposed-to-be chiral fermions, the doublers cancel the anomalies in Noether currents, to which gauge fields can be coupled. Usually anomaly cancellation is a desired feature, but here it is unacceptable for several reasons. First, the pattern is wrong, because the cancellation is among doublers and not among different fermion species. Second, physically desirable anomalies in global symmetries might also be eliminated. Finally, the doublers behave like a multiplet of particles with an equal number of right- and left-handed low-energy fermions; they make the lattice theory vector-like.

In the continuum path integral formalism, anomalous symmetries are broken by the fermion measure \cite{8}. This formalism is a more flexible approach. Gauged symmetries can be rendered non-anomalous either through anomaly cancellation among different flavors (as in the standard model) or by choosing anomaly free

\footnote{For similar ideas see also ref. \cite{5}.}
representations (as in GUT’s). When the measure is correctly defined \cite{9}, it is
also possible to obtain consistent anomalies for global symmetries. In ref. \cite{4} we
suggest treating fermions in this way, even when the gauge fields are regulated by
the lattice. The most transparent way to proceed is to construct continuum gauge
fields from lattice gauge fields.

In vector theories, a related problem arises in the context of the Atiyah-Singer
index theorem \cite{10}. This theorem, which holds for smooth gauge fields, says that the
number of right-handed minus left-handed zero modes of the Dirac operator equals
the topological charge of the gauge field, thus connecting the topology of gauge fields
with the chiral properties of the fermions. On the lattice the index theorem is lost
whenever the discretized Dirac operator has no geometric meaning. For example,
both the staggered and Wilson formulations have no exact zero modes and one must
be satisfied with a remnant of the index theorem \cite{11}. On the other hand, the index
theorem holds for continuum fermions in the continuum gauge fields constructed in
this paper. The zero modes could be studied numerically by re-introducing finer
and finer sublattices.

Another observable, which has become of interest recently in the context of
baryon number violating processes \cite{12} in the standard model, is the Chern-Simons
number \cite{13}. One of its characteristic features is that it changes by an integer
under “large” gauge transformations, the integer being the winding number of the
gauge transformation. A lattice construction of the Chern-Simons number should
share this feature of the continuum expression. This is not the case for the naively
discretized expression, so that a more refined treatment is necessary. Again, knowing
the continuum gauge fields, we may define the Chern-Simons number exactly as in
the continuum.\footnote{The price is a separate ultraviolet regulator for the fermions.}

In this paper we shall construct a continuum gauge field, $A_\mu$, from a lattice gauge
field. Our continuum gauge field fulfills the following constraints: (i) The parallel
transporters derived from $A_\mu$ agree with those of the original lattice gauge field. (ii)
A lattice gauge transformation results in a continuum gauge transformation of $A_\mu$,
i.e. the construction of $A_\mu$ is gauge covariant. (iii) The gauge field $A_\mu$ is a connection
in the fiber bundle \footnote{For different constructions see, however, refs. \cite{14,15,16}.}
that has been constructed from the lattice gauge field. We
consider these three criteria essential for consistency. In particular, the third feature
 guarantees that one obtains the same value for the topological charge, independent
of whether one computes it from the transition functions or by integrating the Chern-
Pontryagin density.
The paper is organized as follows. In sec. 2 we derive the continuum gauge field in four dimensions. Some extensions of older results, which we need here, are described in Appendix A. We then apply our results to the $U(1)$ gauge theory in two dimensions in sec. 3. Finally, in sec. 4 we conclude with some remarks.

## 2 Continuum gauge field

We now turn to the construction of the continuum gauge field $A_\mu = -A_\mu^+$. For definiteness we consider the gauge group $SU(N)$. We shall restrict ourselves to a hypercubic, periodic lattice of period $L$, i.e. the underlying continuum manifold is the four-torus $T^4$. The lattice is then defined by

$$\Lambda = \{ s \in T^4 \mid s_\mu \in \mathbb{Z}, \forall \mu \}. \quad (2.1)$$

We denote the parallel transporters by $U(s, \mu)$.

The construction proceeds in two steps. In the first step we construct in each hypercube,

$$c(s) = \{ x \in T^4 \mid s_\mu \leq x_\mu \leq s_\mu + 1, \forall \mu \}, \quad (2.2)$$

a vector field $A_\mu^{(s)}$ such that on the intersection of two adjacent hypercubes the corresponding fields are connected by a gauge transformation given by Lüscher’s transition functions. In the second step all these gauge fields will be brought into the same gauge such that one (possibly singular) gauge field on the whole torus emerges.

Since the gauge transformation of a field involves derivatives, it turns out to be necessary to enlarge the hypercubes $c(s)$ to open sets

$$\tilde{c}(s) := \{ x \in T^4 \mid -\varepsilon < x_\mu - s_\mu < 1 + \varepsilon, \forall \mu \} \quad (2.3)$$

with $0 < \varepsilon < 1/2$. The intersection of two neighboring hypercubes is denoted by

$$\tilde{f}(s, \mu) := \tilde{c}(s) \cap \tilde{c}(s - \hat{\mu}) = \{ x \in \tilde{c}(s) \mid -\varepsilon < x_\mu - s_\mu < \varepsilon \}. \quad (2.4)$$

Furthermore, we define for $\mu \neq \nu$

$$\tilde{p}(s, \mu, \nu) := \tilde{c}(s) \cap \tilde{c}(s - \hat{\mu}) \cap \tilde{c}(s - \hat{\nu}) \cap \tilde{c}(s - \hat{\mu} - \hat{\nu})$$

$$= \{ x \in \tilde{c}(s) \mid -\varepsilon < x_\mu - s_\mu < \varepsilon, -\varepsilon < x_\nu - s_\nu < \varepsilon \}. \quad (2.5)$$
From Lüscher’s transition functions

\[ v_{s,\mu} : f(s, \mu) \rightarrow SU(N), \]  

(2.6)
defined on the faces \( f(s, \mu) = c(s) \cap c(s - \hat{\mu}) \), we construct “smeared” transition functions

\[ \tilde{v}_{s,\mu} \rightarrow SU(N). \]  

(2.7)

Let \( \phi : \mathbb{R} \rightarrow [0, 1] \) be smooth and such that

\[ \phi(t) = 0 \] if \( t < \eta \), \( \phi(t) = 1 \) if \( t > 1 - \eta \),

(2.8)

and \( \varepsilon < \eta < 1/2 \). We then define

\[ \tilde{v}_{s,\mu}(x) = v_{s,\mu}(s_1 + \phi(x_1 - s_1), \ldots, s_\mu, \ldots, s_4 + \phi(x_4 - s_4)) \]  

(2.10)

(Here and in the following \( f(\cdots, s_\alpha, \cdots) \) means that the function \( f \) is to be evaluated with \( x_\alpha \) put equal to \( s_\alpha \), irrespective of the order in which the arguments are written.)

From the fact that Lüscher’s transition functions satisfy the cocycle condition

\[ v_{s-\hat{\mu},\nu}(x)f(s,\mu)(x) = v_{s-\hat{\nu},\mu}(x)f(s,\nu)(x) \]  

(2.11)
on \( p(s,\mu,\nu) = c(s) \cap c(s - \hat{\mu}) \cap c(s - \hat{\nu}) \cap c(s - \hat{\mu} - \hat{\nu}) \) it follows that on \( \tilde{p}(s,\mu,\nu) \) the cocycle condition for \( \tilde{v}_{s,\mu} \) is fulfilled. Therefore the transition functions \( \tilde{v} \) define an \( SU(N) \) principal bundle over \( T^4 \).

Now we introduce the abbreviations

\[ Z_\lambda(s,\alpha,\beta,\gamma | x) := (\tilde{v}_{s+\gamma,\gamma}\tilde{v}_{s+\gamma+\beta,\beta}\tilde{v}_{s+\gamma+\beta+\alpha,\alpha} \times \partial_\lambda \tilde{v}_{s+\gamma+\beta+\alpha,\alpha}^{-1} \tilde{v}_{s+\gamma+\beta+\alpha,\alpha}^{-1} \tilde{v}_{s+\gamma+\beta+\alpha,\alpha}^{-1} (x), \]  

(2.12)

\[ Z_\lambda(s,\alpha,\beta | x) := (\tilde{v}_{s+\alpha,\alpha}\tilde{v}_{s+\alpha+\beta,\beta}\partial_\lambda \tilde{v}_{s+\alpha+\beta,\beta}^{-1} \tilde{v}_{s+\alpha+\beta,\beta}^{-1} \tilde{v}_{s+\alpha+\beta,\beta}^{-1} (x), \]  

(2.13)

\[ Z_\lambda(s,\alpha | x) := (\tilde{v}_{s+\alpha,\alpha}\partial_\lambda \tilde{v}_{s+\alpha,\alpha}^{-1} \tilde{v}_{s+\alpha,\alpha}^{-1} \tilde{v}_{s+\alpha,\alpha}^{-1} (x), \]  

(2.14)

\[ \text{Ad}(\tilde{v})M := \tilde{v}M\tilde{v}^{-1}. \]  

(2.15)

Due to the cocycle condition, \( Z_\lambda(s,\alpha,\beta,\gamma | x) \) and \( Z_\lambda(s,\alpha,\beta | x) \) are symmetric in \( \alpha,\beta,\gamma \) and \( \alpha,\beta \), respectively. Furthermore, one has relations of the type

\[ Z_\lambda(s,\alpha,\beta,\gamma | x) = \text{Ad}(\tilde{v}_{s+\gamma,\gamma}(x))Z_\lambda(s + \gamma,\alpha,\beta | x) + Z_\lambda(s,\gamma | x). \]  

(2.16)
For \( x \in \tilde{c}(s) \) and \( \alpha, \beta, \gamma \) the indices complementary to \( \mu \) we define the field \( A^{(s)}_{\mu}(x) \) by

\[
A^{(s)}_{\mu}(x) = \phi(x_{\alpha} - s_{\alpha})\phi(x_{\beta} - s_{\beta})\phi(x_{\gamma} - s_{\gamma})\{Z_{\mu}(s, \alpha, \beta, \gamma \mid s_{\alpha} + 1, s_{\beta} + 1, s_{\gamma} + 1, x_{\mu})
+ \sum_{\text{cyc. perm.}} [-\text{Ad}(\tilde{v}_{s+\hat{\alpha},\alpha}(s_{\alpha} + 1, x_{\beta}, x_{\gamma}, x_{\mu})\tilde{v}_{s+\hat{\alpha}+\hat{\beta},\beta}(s_{\alpha} + 1, s_{\beta} + 1, x_{\gamma}, x_{\mu}))
\times Z_{\mu}(s + \hat{\alpha}, \hat{\beta}, \gamma \mid s_{\alpha} + 1, s_{\beta} + 1, s_{\gamma} + 1, x_{\mu})
\times Z_{\mu}(s + \hat{\alpha} + \hat{\gamma}, \beta \mid s_{\alpha} + 1, s_{\beta} + 1, s_{\gamma} + 1, x_{\mu})
\times Z_{\mu}(s + \hat{\alpha} + \hat{\gamma} + \hat{\beta}, \gamma \mid s_{\alpha} + 1, s_{\beta} + 1, s_{\gamma} + 1, x_{\mu})
- \text{Ad}(\tilde{v}_{s+\hat{\alpha},\alpha}(s_{\alpha} + 1, x_{\beta}, x_{\gamma}, x_{\mu}))
+ \sum_{\text{cyc. perm.}} [\phi(x_{\alpha} - s_{\alpha})\phi(x_{\beta} - s_{\beta})Z_{\mu}(s, \alpha, \beta \mid s_{\alpha} + 1, s_{\beta} + 1, x_{\gamma}, x_{\mu})
+ \phi(x_{\alpha} - s_{\alpha})\phi(x_{\beta} - s_{\beta})\text{Ad}(\tilde{v}_{s+\hat{\alpha},\alpha}(s_{\alpha} + 1, x_{\beta}, x_{\gamma}, x_{\mu}))
\times Z_{\mu}(s + \hat{\alpha}, \beta \mid s_{\alpha} + 1, s_{\beta} + 1, x_{\gamma}, x_{\mu})
+ \phi(x_{\alpha} - s_{\alpha})\phi(x_{\gamma} - s_{\gamma})\text{Ad}(\tilde{v}_{s+\hat{\alpha},\alpha}(s_{\alpha} + 1, x_{\beta}, x_{\gamma}, x_{\mu}))
\times Z_{\mu}(s + \hat{\alpha}, \gamma \mid s_{\alpha} + 1, x_{\beta}, s_{\gamma} + 1, x_{\mu})
- \phi(x_{\alpha} - s_{\alpha})Z_{\mu}(s, \alpha \mid s_{\alpha} + 1, x_{\beta}, x_{\gamma}, x_{\mu})].
\]

(2.17)

It is straightforward to verify that this field has the desired properties:

\[
A^{(s-\hat{\mu})}_{\lambda}(x) = \tilde{v}_{s,\mu}(x)A^{(s)}_{\lambda}(x)\tilde{v}_{s,\mu}^{-1}(x) + \tilde{v}_{s,\mu}(x)\partial_{\lambda}\tilde{v}_{s,\mu}^{-1}(x)
\]

(2.18)

for \( x \in \tilde{f}(s, \mu) \) and

\[
\mathcal{P} \exp \left\{ \int_{0}^{1} dt A^{(s)}_{\mu}(x + (1 - t)\hat{\mu}) \right\} = u^{s}_{x,x+\hat{\mu}},
\]

(2.19)

where \( x \) and \( x + \hat{\mu} \) are corners of \( c(s) \) and \( u^{s}_{x,x+\hat{\mu}} \) are the link variables in the complete axial gauge defined in ref. [1].

The field (2.17) can be constructed step by step starting from the value zero on the neighborhood \( \{x \in \tilde{c}(s) \mid -\varepsilon < x_{\lambda} - s_{\lambda} < \varepsilon, \forall \lambda\} \) of the lattice point \( s \). Equation (2.18) then gives the field on the analogous neighborhoods of the other corners of
Since according to eq. (2.10) \( \tilde{v} \) is constant on these neighborhoods, one gets again zero. In the next step one puts the field equal to zero on \( \varepsilon \)-neighborhoods of the links in \( c(s) \) originating from \( s \) and applies (2.18) again. Then one defines \( A_\lambda^{(s)} \) on the six \( \tilde{p}(s, \mu, \nu) \) by interpolating the expressions already obtained. By means of (2.18) one finds \( A_\lambda^{(s)} \) on all the \( \tilde{p} \)'s contained in \( c(s) \). Continuing in this fashion one arrives at eq. (2.17).

It should be remarked that our construction works only if the transition functions \( v_{s,\mu} \) are well defined. This is not the case for the exceptional configurations as defined by Lüscher. However, since these form only a set of measure zero in the functional integral, they can be ignored. Furthermore, covariance under hypercubic rotations and reflections is already violated by the construction of the transition functions, hence also by our expression for the continuum gauge field \( A_\mu^{(s)} \).

The purpose of \( \varepsilon \) and the function \( \phi \) is only to give a well defined meaning to \( v_{s,\mu} \partial_\lambda v_{s,\mu}^{-1} \) even for \( \lambda = \mu \) in the course of the construction. In the final expression (2.17) for \( A_\mu^{(s)} \) one can perform the limit \( \varepsilon, \eta \to 0 \) and put \( \phi(t) = t \) for \( 0 \leq t \leq 1 \) without problems.

We now come to the second step, i.e. we construct a “global” (but possibly singular) field \( A_\mu \). To do so one can make use of the section \( w^s \) constructed in ref. [18] on the boundaries of the hypercubes. In Appendix A we extend \( w^s \) into the interior of \( c(s) \). In this procedure one will, in general, encounter singularities. Nevertheless, one can define on \( c(s) \)

\[
A_\mu(x) = w^s(x)^{-1}A_\mu^{(s)}(x)w^s(x) + w^s(x)^{-1}\partial_\mu w^s(x). \tag{2.20}
\]

Under a lattice gauge transformation

\[
U(s, \mu) \to \bar{U}(s, \mu) = g(s)U(s, \mu)g(s + \tilde{\mu})^{-1} \tag{2.21}
\]

the gauge field \( A_\mu^{(s)} \) transforms as

\[
\bar{A}_\mu^{(s)}(x) = g(s)A_\mu^{(s)}(x)g(s)^{-1}, \tag{2.22}
\]

and the section \( w^s \) transforms as

\[
\bar{w}^s(x) = g(s)w^s(x)g(x)^{-1}, \tag{2.23}
\]

where \( g(x) \) is the interpolation of \( \{g(s)\} \) to all of \( c(s) \) which we give in appendix A. This entails the desired behavior

\[
\bar{A}_\mu(x) = g(x)(A_\mu(x) + \partial_\mu)g(x)^{-1}. \tag{2.24}
\]
3 \textbf{\textit{U(1) gauge theory in two dimensions}}

Since our construction of the continuum gauge field is rather involved, it may be useful to repeat the procedure for a \textit{U(1)}-theory in two dimensions, where it is much simpler. In particular, it will be possible to discuss the “global” field \textbf{(2.20)} explicitly.

Using the obvious analogues of the concepts introduced in sec. 2 for the case of four dimensions we get for Lüscher’s transition functions

\begin{align}
v_{s,1}(x) &= U(s - \hat{1}, 1), \quad x \in f(s, 1), \\
v_{s,2}(x) &= U(s - \hat{2}, 2)P(s - \hat{2})^{x_1 - s_1}, \quad x \in f(s, 2). \tag{3.1}
\end{align}

Here \(P(s)\) denotes the plaquette variable

\begin{equation}
P(s) = U(s, 1)U(s + \hat{1}, 2)U(s + \hat{2}, 1)^{-1}U(s, 2)^{-1}. \tag{3.2}
\end{equation}

The noninteger power of a group element \(e^{i\alpha}\) is defined by

\begin{equation}
(e^{i\alpha})^{t} = e^{it\alpha}, \tag{3.3}
\end{equation}

where \(0 \leq t \leq 1\) and \(\alpha\) is restricted to the interval \(-\pi < \alpha < \pi\). If \(e^{i\alpha} = -1\), the power \((e^{i\alpha})^{t}\) is not well-defined. Configurations with at least one plaquette variable equal to \(-1\) are exceptional in the sense of Lüscher and do not allow our constructions to be performed.

Using the same smearing function \(\phi\) as in four dimensions one obtains the transition functions

\begin{equation}
\tilde{v}_{s,\mu} : \tilde{f}(s, \mu) \to U(1). \tag{3.4}
\end{equation}

The step-by-step construction indicated in sec. 2 leads to the field

\begin{align}
A_1^{(s)}(x) &= i\phi'(x_1 - s_1)\phi(x_2 - s_2)\log P(s), \\
A_2^{(s)}(x) &= 0 \tag{3.5}
\end{align}

for \(x \in \tilde{c}(s)\), where in accordance with our definition of powers we have defined

\begin{equation}
\log(e^{i\alpha}) = i\alpha, \quad -\pi < \alpha < \pi. \tag{3.6}
\end{equation}

Note that the field has been chosen real as is commonly done for gauge group \(U(1)\).

It is now easy to check that

\begin{equation}
A^{(s-\hat{\mu})}_{\lambda}(x) = A^{(s)}_{\lambda}(x) + i\tilde{v}_{s,\mu}^{-1}(x)\partial_{\lambda}\tilde{v}_{s,\mu}(x) \tag{3.7}
\end{equation}
for $x \in \tilde{f}(s, \mu)$. For the field strength in $\tilde{c}(s)$ one finds
\[
F_{12}(x) = -i\phi'(x_1 - s_1)\phi'(s_2 - s_2) \log P(s)
\] (3.8)
so that in the limit $\eta \to 0$, $\phi(t) = t$, one ends up with a constant field strength in each plaquette as was already considered by Flume and Wyler [15]. In this limit the field is given by
\[
A_1^{(s)}(x) = i(x_2 - s_2) \log P(s),
A_2^{(s)}(x) = 0.
\] (3.9)

In order to obtain a “global” field we have to construct the section $w^s$ and to discuss its singularities in the interior of $c(s)$. Straightforward adaptation of the appropriate formulas of ref. [13] to the case of two dimensions yields on the boundary of $c(s)$:
\[
\begin{align*}
w^s(s_1, x_2) &= U(s, 2)^{x_2 - s_2}, \\
w^s(s_1 + 1, x_2) &= U(s + 1, 2)^{x_2 - s_2}U(s, 1), \\
w^s(x_1, s_2) &= U(s, 1)^{x_1 - s_1}, \\
w^s(x_1, s_2 + 1) &= P(s)^{x_1 - s_1}U(s + 2, 1)^{x_2 - s_2}U(s, 2).
\end{align*}
\] (3.10)
Again, certain configurations (those with at least one link variable equal to $-1$) have to be excluded. But this should cause no harm because they form a set of measure zero in the functional integral. The interpolation into the interior of $c(s)$ is however more subtle, since one will necessarily encounter singularities if the winding number of the map
\[
w^s|_{\partial c(s)} : \partial c(s) \to U(1)
\] (3.11)
is nonzero. The corresponding configurations can no longer be ignored because, for example, all configurations with nonvanishing topological charge belong to this class. Nevertheless, proceeding as outlined in appendix A one obtains
\[
w^s(x_1, x_2) = w^s(s_1, x_2)\left[w^s(s_1, x_2)^{-1}w^s(s_1 + 1, x_2)(P(s)^{-1})^{x_2 - s_2}\right]^{x_1 - s_1}
\times [P(s)^{x_2 - s_2}]^{x_1 - s_1}. 
\] (3.12)
This expression is ill-defined whenever
\[
w^s(s_1, x_2)^{-1}w^s(s_1 + 1, x_2)(P(s)^{-1})^{x_2 - s_2} = -1. 
\] (3.13)

To make the construction more transparent we write
\[
\begin{align*}
U(x, \mu) &= e^{i\psi(x, \mu)}, \quad -\pi < \psi(x, \mu) < \pi, \\
P(s) &= e^{i\omega(s)}, \quad -\pi < \omega(s) < \pi.
\end{align*}
\] (3.14)
Furthermore, we introduce the function \( q(\alpha) \) which shifts an angle into the interval \(-\pi < q(\alpha) < \pi\) by adding integer multiples of \(2\pi\): 

\[
q(\alpha) = \sum_{n=-\infty}^{\infty} (\alpha - 2\pi n) \theta(\alpha - 2\pi n + \pi) \theta(\pi - \alpha + 2\pi n) .
\] (3.15)

So we have 

\[
\omega(s) = q(\psi(s, 1) + \psi(s + \hat{1}, 2) - \psi(s + \hat{2}, 1) - \psi(s, 2))
\] (3.16)

and 

\[
w^s(x_1, x_2) = e^{i\psi(s, 2)(x_2 - s_2)} e^{i\omega(s)(x_1 - s_1)(x_2 - s_2)}
\times e^{i(x_1 - s_1)q((-\psi(s, 2) + \psi(s + \hat{1}, 2) - \omega(s))(x_2 - s_2) + \psi(s, 1))} .
\] (3.17)

The last factor will jump from \( e^{i\pi(x_1 - s_1)} \) to \( e^{-i\pi(x_1 - s_1)} \) (or the other way around) if \( x_2 \) passes through a value with 

\[
(-\psi(s, 2) + \psi(s + \hat{1}, 2) - \omega(s))(x_2 - s_2) + \psi(s, 1) = (2n + 1)\pi , \quad n \text{ integer} .
\] (3.18)

Away from these gauge singularities the field 

\[
A_\mu(x) = A_\mu^s(x) - iw^s(x)^{-1}\partial_\mu w^s(x) , \quad x \in c(s) ,
\] (3.19)

can be written as 

\[
A_1(x) = q(\psi(s, 1) + (-\psi(s, 2) + \psi(s + \hat{1}, 2) - \omega(s))(x_2 - s_2)) ,
\]

\[
A_2(x) = \psi(s, 2) + (-\psi(s, 2) + \psi(s + \hat{1}, 2))(x_1 - s_1) .
\] (3.20)

If the winding number of the map \( [3.11] \) vanishes one has 

\[
\omega(s) = \psi(s, 1) + \psi(s + \hat{1}, 2) - \psi(s + \hat{2}, 1) - \psi(s, 2) ,
\] (3.21)

and consequently 

\[
A_1(x) = \psi(s, 1) + (-\psi(s, 1) + \psi(s + \hat{2}, 1))(x_2 - s_2) ,
\]

\[
A_2(x) = \psi(s, 2) + (-\psi(s, 2) + \psi(s + \hat{1}, 2))(x_1 - s_1) .
\] (3.22)

i.e. in this case the construction of \( A_\mu(x) \) is symmetric with respect to the interchange of the directions 1 and 2, although in general the interpolation violates invariance under 90° rotations.

It is easy to see that integration of the field \( [3.20] \) over the links reproduces the link angles of the lattice gauge field, e.g. 

\[
\int_{s_1}^{s_1 + 1} dx_1 A_1(x_1, s_2) = \psi(s, 1) .
\] (3.23)
One should however note that eq. (3.20) gives $A_\mu(x)$ only within $c(s)$. How do the expressions on neighboring plaquettes fit together? One finds that $A_2(x)$ is continuous as a function of $x_1$ and $A_1(x)$ is continuous as a function of $x_2$. So one gets continuity in the transverse direction whereas in the longitudinal direction the field will in general exhibit jumps. But these can be gauged away and should not cause any trouble.

Let us finally discuss the Chern-Simons number resulting from our continuum gauge field. In the continuum, the Chern-Simons number is given by the line integral

$$n_{CS} = \frac{1}{2\pi} \int dx_\mu A_\mu(x).$$

(3.24)

So using our interpolated field (3.20) we find as the contribution of one link the integral of $A_\mu(x)$ over this link (divided by $2\pi$). As remarked above, this integral reproduces the original link angle:

$$\frac{1}{2\pi} \int_{s_1}^{s_1+1} dx_\mu A_\mu(x) = \frac{1}{2\pi} \psi(s, \mu).$$

(3.25)

On the other hand, starting analogously to Seiberg [14] from the section $w^s(x)$ on $\partial c(s)$ one arrives at the following expressions:

$$-\frac{i}{2\pi} \int_{s_1}^{s_1+1} dx_1 w^s(x)^{-1} \partial_1 w^s(x)|_{x_2=s_2} = \frac{1}{2\pi} \psi(s, 1),$$

(3.26)

$$-\frac{i}{2\pi} \int_{s_1}^{s_1+1} dx_1 w^s(x)^{-1} \partial_1 w^s(x)|_{x_2=s_2+1} = \frac{1}{2\pi} (\omega(s) + \psi(s + \hat{2}, 1)),$$

(3.27)

$$-\frac{i}{2\pi} \int_{s_2}^{s_2+1} dx_2 w^s(x)^{-1} \partial_2 w^s(x)|_{x_1=s_1} = \frac{1}{2\pi} \psi(s, 2),$$

(3.28)

$$-\frac{i}{2\pi} \int_{s_2}^{s_2+1} dx_2 w^s(x)^{-1} \partial_2 w^s(x)|_{x_1=s_1+1} = \frac{1}{2\pi} \psi(s + \hat{1}, 2).$$

(3.29)

Hence one obtains immediately agreement between the two approaches on the links starting from the origin $s$ of the plaquette $c(s)$, whereas on the other links one gets coinciding results in general only in the naive continuum limit. (Re-introduce the lattice spacing $a$ and recall that the plaquette angle $\omega(s)$ should vanish as $a^2$, whereas the link angles $\psi(s, \mu)$ vanish as $a$ in the continuum limit.)

4 Conclusions

The great advantage of lattice gauge theory is that it provides a framework for explicit nonperturbative calculations. Unfortunately, it is not always easy to under-
stand how other nonperturbative information, derived by geometric or topological methods, fits in with the lattice. Our construction of a continuum gauge field from a lattice gauge field should help bridge this gap. Although the expressions appear rather complicated, we have obtained a relatively simple result in the case of the two-dimensional $U(1)$ theory. In the future we hope to apply the four-dimensional results to chiral fermions, as suggested in ref. [4].

A crucial question concerns the influence of short-distance fluctuations on the interpolation and on the quantities constructed from the continuum gauge field. From the example of the topological susceptibility it is known that such fluctuations, the so-called dislocations [17], can make nonperturbative, ultraviolet divergent contributions to physical quantities. In this case the divergence can be controlled by choosing an appropriate gauge field action. It will be important to understand whether analogous problems arise with quantities constructed out of the continuum gauge field, and, if so, how to cope with them.

Having defined a continuum gauge field for each lattice gauge field, one may use the continuum action of this field as the lattice gauge action. Unless there is a construction in which the integrals can be carried out analytically, the use of this action in a numerical simulation is unpractical, but it has some theoretical advantages. For example, one has the usual bound on the action in each topological sector. Then there is no nonperturbative divergence in the topological susceptibility and, perhaps, in other quantities of interest.

In closing we mention some related results that we have obtained, but that we do not present in this paper. Our present discussion is based on Lüscher’s transition functions [1], but we have also carried out the construction for the Phillips-Stone bundle [2], which is defined on simplicial lattices. Furthermore, although this paper focuses on the gauge field, we have obtained corresponding extrapolations for matter fields too.

**Acknowledgements**

This work was supported in part by the Deutsche Forschungsgemeinschaft and by the Schweizerischer Nationalfonds. The paper was completed while one of us (G. S.) was visiting Brookhaven National Laboratory. He wants to thank the theory group for its warm hospitality.
A Interpolation of the section to the interior of the hypercube

Here we describe how the section \( w^s \) is interpolated from the boundary of the hypercube \( c(s) \) to its interior. The interpolation of \( w^s \) from the corners of the hypercube to its boundary was already given in ref. \(^{[8]} \). We also investigate the properties of the section under lattice gauge transformations.

On the corners of the hypercube \( c(s) \) the section is given by

\[
w^s(x) = U(s, 1)^{y_1} U(s + y_1 \hat{1}, 2)^{y_2} U(s + y_1 \hat{1} + y_2 \hat{2}, 3)^{y_3} U(s + y_1 \hat{1} + y_2 \hat{2} + y_3 \hat{3}, 4)^{y_4} , \tag{A.1}
\]

where

\[
x = s + \sum_{\mu=1}^{4} y_\mu \hat{\mu}, \quad y_\mu \in \{0, 1\} \tag{A.2}
\]

are the corners of the hypercube. Denoting the three indices complementary to \( \mu \) by \( \alpha, \beta, \gamma \) (\( \alpha < \beta < \gamma \)) we label the corners of the cube \( f(s, \mu) = c(s) \cap c(s - \hat{\mu}) \) according to

\[
s \doteq 1, \quad s + \hat{\alpha} \doteq 2, \quad s + \hat{\beta} \doteq 3, \quad s + \hat{\alpha} + \hat{\beta} \doteq 4, \quad s + \hat{\gamma} \doteq 5, \quad s + \hat{\alpha} + \hat{\gamma} \doteq 6, \quad s + \hat{\beta} + \hat{\gamma} \doteq 7, \quad s + \hat{\alpha} + \hat{\beta} + \hat{\gamma} \doteq 8, \tag{A.3}
\]

and take the following interpolation of the section on \( f(s, \mu) \)

\[
w^t(s_\mu, s_\alpha, s_\beta, x_\gamma) = w^t(s_\mu, s_\alpha, s_\beta, s_\gamma) [w^t(s_\mu, s_\alpha, s_\beta, s_\gamma)^{-1} w^t(s_\mu, s_\alpha, s_\beta, s_\gamma + 1)]
\times U^+(1, \gamma)^{y_\gamma} U(1, \gamma)^{y_\gamma},
\]

\[
w^t(s_\mu, s_\alpha + 1, s_\beta, x_\gamma) = w^t(s_\mu, s_\alpha + 1, s_\beta, s_\gamma) [w^t(s_\mu, s_\alpha + 1, s_\beta, s_\gamma)^{-1} w^t(s_\mu, s_\alpha + 1, s_\beta, s_\gamma + 1)]
\times w^t(s_\mu, s_\alpha + 1, s_\beta, s_\gamma + 1) U^+(2, \gamma)^{y_\gamma} U(2, \gamma)^{y_\gamma},
\]

\[
w^t(s_\mu, s_\alpha, s_\beta + 1, x_\gamma) = w^t(s_\mu, s_\alpha, s_\beta + 1, s_\gamma) [w^t(s_\mu, s_\alpha, s_\beta + 1, s_\gamma)^{-1} w^t(s_\mu, s_\alpha, s_\beta + 1, s_\gamma + 1)]
\times w^t(s_\mu, s_\alpha, s_\beta + 1, s_\gamma + 1) U^+(3, \gamma)^{y_\gamma} U(3, \gamma)^{y_\gamma},
\]

\[
w^t(s_\mu, s_\alpha + 1, s_\beta + 1, x_\gamma) = w^t(s_\mu, s_\alpha + 1, s_\beta + 1, s_\gamma) [w^t(s_\mu, s_\alpha + 1, s_\beta + 1, s_\gamma)^{-1} w^t(s_\mu, s_\alpha + 1, s_\beta + 1, s_\gamma + 1)]
\times w^t(s_\mu, s_\alpha + 1, s_\beta + 1, s_\gamma + 1) U^+(4, \gamma)^{y_\gamma} U(4, \gamma)^{y_\gamma},
\]

\[
w^t(s_\mu, s_\alpha, s_\beta, x_\gamma) = w^t(s_\mu, s_\alpha, s_\beta, x_\gamma) [w^t(s_\mu, s_\alpha, s_\beta, x_\gamma)^{-1} w^t(s_\mu, s_\alpha, s_\beta + 1, x_\gamma)]
\times F^+_s(s_\mu, x_\gamma)^{y_\beta} F_{s_\mu}(x_\gamma)^{y_\beta},
\]

\[\]

\[\]

\[\]

\[\]

\[\]
where \( y = x - s \) and \( t = s \) or \( t = s - \mu \). Furthermore

\[
F_{s,\mu}(x_\gamma) = U^+(1, \gamma)^{y_\rho}[U(1, \gamma)U(5, \beta)U^+(3, \gamma)U^+(1, \beta)]^{y_\gamma}U(1, \beta)U(3, \gamma)^{y_\gamma},
\]
\[
G_{s,\mu}(x_\gamma) = U^+(2, \gamma)^{y_\rho}[U(2, \gamma)U(6, \beta)U^+(4, \gamma)U^+(2, \beta)]^{y_\gamma}U(2, \beta)U(4, \gamma)^{y_\gamma},
\]
\[
H_{s,\mu}(x_\gamma) = U^+(1, \gamma)^{y_\rho}[U(1, \gamma)U(5, \alpha)U^+(2, \gamma)U^+(1, \alpha)]^{y_\gamma}U(1, \alpha)U(2, \gamma)^{y_\gamma},
\]
\[
K_{s,\mu}(x_\gamma) = U^+(3, \gamma)^{y_\rho}[U(3, \gamma)U(7, \alpha)U^+(4, \gamma)U^+(3, \alpha)]^{y_\gamma}U(3, \alpha)U(4, \gamma)^{y_\gamma},
\]
\[
L_{s,\mu}(x_\beta, x_\gamma) = F^+_{s,\mu}(x_\gamma)^{y_\beta}[F_{s,\mu}(x_\gamma)K_{s,\mu}(x_\gamma)G^+_{s,\mu}(x_\gamma)H^+_{s,\mu}(x_\gamma)]^{y_\beta} \\
\times H_{s,\mu}(x_\gamma)G_{s,\mu}(x_\gamma)^{y_\beta}.
\]

Now we interpolate the section to the interior of \( c(s) \) as follows:

\[
w^s(x_1, x_2, x_3, x_4) = w^s(s_1, x_2, x_3, x_4)[w^s(s_1, x_2, x_3, x_4)^{-1}w^s(s_1 + 1, x_2, x_3, x_4) \\
\times M^s(x_2, x_3, x_4)]^{y_1}M^s(x_2, x_3, x_4)^{y_1},
\]

where

\[
M^s(x_2, x_3, x_4) = L^+_s(x_3, x_4)^{y_2}[L_s(x_3, x_4)L_{s+\gamma_2}(x_3, x_4)L^+_s(x_3, x_4) \\
\times L^+_s(x_3, x_4)]^{y_2}L_{s,2}(x_3, x_4)L_{s+1,1}(x_3, x_4)^{y_2}
\]

is one more interpolating function.

To investigate the properties of the section \( w^s \) under gauge transformations we also interpolate the gauge transformation \( g \) from the corners of the hypercube to its interior. The interpolation of \( g \) from the corners of \( c(s) \) to its boundary was already given in ref. [18]:

\[
g(s_\mu, s_\alpha, s_\beta, s_\gamma) = g(s_\mu, s_\alpha, s_\beta, s_\gamma)[g(s_\mu, s_\alpha, s_\beta, s_\gamma)^{-1}g(s_\mu, s_\alpha, s_\beta, s_\gamma + 1) \\
\times U^+(1, \gamma)]^{y_\gamma}U(1, \gamma)^{y_\gamma},
\]

13
\[ g(s_\mu, s_\alpha + 1, s_\beta, x_\gamma) = g(s_\mu, s_\alpha + 1, s_\beta, s_\gamma) [g(s_\mu, s_\alpha + 1, s_\beta, s_\gamma)^{-1} \times g(s_\mu, s_\alpha + 1, s_\beta, s_\gamma + 1) U^+(2, \gamma)]^\gamma U(2, \gamma)^\gamma, \]

\[ g(s_\mu, s_\alpha, s_\beta + 1, x_\gamma) = g(s_\mu, s_\alpha, s_\beta + 1, s_\gamma) [g(s_\mu, s_\alpha, s_\beta + 1, s_\gamma)^{-1} \times g(s_\mu, s_\alpha, s_\beta + 1, s_\gamma + 1) U^+(3, \gamma)]^\gamma U(3, \gamma)^\gamma, \]

\[ g(s_\mu, s_\alpha + 1, s_\beta + 1, x_\gamma) = g(s_\mu, s_\alpha + 1, s_\beta + 1, s_\gamma) [g(s_\mu, s_\alpha + 1, s_\beta + 1, s_\gamma)^{-1} \times g(s_\mu, s_\alpha + 1, s_\beta + 1, s_\gamma + 1) U^+(4, \gamma)]^\gamma U(4, \gamma)^\gamma, \]

\[ g(s_\mu, s_\alpha, x_\beta, x_\gamma) = g(s_\mu, s_\alpha, s_\beta, x_\gamma) [g(s_\mu, s_\alpha, s_\beta, x_\gamma)^{-1} g(s_\mu, s_\alpha, s_\beta + 1, x_\gamma) \times F_{s_\mu}(x_\gamma)]^\beta F_{s_\mu}(x_\gamma)^\beta, \]

\[ g(s_\mu, s_\alpha + 1, x_\beta, x_\gamma) = g(s_\mu, s_\alpha + 1, s_\beta, x_\gamma) [g(s_\mu, s_\alpha + 1, s_\beta, x_\gamma)^{-1} \times g(s_\mu, s_\alpha + 1, s_\beta + 1, x_\gamma) G_{s_\mu}(x_\gamma)]^\beta G_{s_\mu}(x_\gamma)^\beta, \]

\[ g(s_\mu, x_\alpha, x_\beta, x_\gamma) = g(s_\mu, s_\alpha, x_\beta, x_\gamma) [g(s_\mu, s_\alpha, x_\beta, x_\gamma)^{-1} g(s_\mu, s_\alpha + 1, x_\beta, x_\gamma) \times L_{s_\mu}(x_\beta, x_\gamma)]^\alpha L_{s_\mu}(x_\beta, x_\gamma)^\alpha, \]

In the interior of \( c(s) \) we obtain

\[ g(x_1, x_2, x_3, x_4) = g(s_1, x_2, x_3, x_4) [g(s_1, x_2, x_3, x_4)^{-1} g(s_1 + 1, x_2, x_3, x_4) \times M^{s_\mu}(x_2, x_3, x_4)]^\mu M^{s_\mu}(x_2, x_3, x_4)^\mu. \]

Using the gauge transformation properties of \( L_{s_\mu}(x_\beta, x_\gamma) \),

\[ \hat{L}_{s_\mu}(x_\beta, x_\gamma) = g(s + y_\beta \hat{\beta} + y_\gamma \hat{\gamma}) L_{s_\mu}(x_\beta, x_\gamma) g(s + \hat{\alpha} + y_\beta \hat{\beta} + y_\gamma \hat{\gamma})^{-1}, \]

we find

\[ \hat{M}^s(x_2, x_3, x_4) = g(s_1, x_2, x_3, x_4) M^s(x_2, x_3, x_4) g(s_1 + 1, x_2, x_3, x_4)^{-1}. \]

Together with the gauge transformation properties of the section at the boundary of \( c(s) \),

\[ \hat{w}^s(s_1, x_2, x_3, x_4) = g(s) w^s(s_1, x_2, x_3, x_4) g(s_1, x_2, x_3, x_4)^{-1}, \]

\[ \hat{w}^s(s_1 + 1, x_2, x_3, x_4) = g(s) w^s(s_1 + 1, x_2, x_3, x_4) g(s_1 + 1, x_2, x_3, x_4)^{-1}, \]

this yields the gauge transformation behavior of the section in the whole hypercube:

\[ \hat{w}^s(x_1, x_2, x_3, x_4) = g(s) w^s(x_1, x_2, x_3, x_4) g(x_1, x_2, x_3, x_4)^{-1}. \]
References

[1] M. Lüscher, Commun. Math. Phys. 85 (1982) 39

[2] A. Phillips and D. Stone, Commun. Math. Phys. 103 (1986) 599

[3] H. B. Nielsen and N. Ninomiya, Nucl. Phys. B185 (1981) 20; Erratum ibid. B195 (1982) 541; ibid. B193 (1981) 173;
D. Friedan, Commun. Math. Phys. 85 (1982) 481

[4] M. Göckeler and G. Schierholz, HLRZ preprint 92-33 (1992), invited talk given at the Workshop on Non-Perturbative Aspects of Chiral Gauge Theories, Rome, March 1992, to appear in Nucl. Phys. B (Proc. Suppl.)

[5] L. Alvarez-Gaumé and S. Della-Pietra, in Recent developments in quantum field theory, eds. J. Ambjørn, B. J. Durhuus and J. L. Petersen (North-Holland, Amsterdam, 1985);
J. Smit, Nucl. Phys. B (Proc. Suppl.) 4 (1988) 451;
A. S. Kronfeld, Nucl. Phys. B (Proc. Suppl.) 4 (1988) 329;
V. Vyas, Phys. Lett. B266 (1991) 453;
A. I. Karanikas and C. N. Ktorides, MIT-preprint CTP#2010

[6] L. Karsten and J. Smit, Nucl. Phys. B183 (1981) 103

[7] A. Coste, C. Korthals Altes and O. Napoly, Phys. Lett. B179 (1986) 125;
A. Coste, C. Korthals Altes and O. Napoly, Nucl. Phys. B289 (1987) 645;
T. Jolicoeur, R. Lacaze and O. Napoly, Nucl. Phys. B293 (1987) 215

[8] K. Fujikawa, Phys. Rev. D21 (1980) 2848

[9] L. Alvarez-Gaumé and P. Ginsparg, Nucl. Phys. B243 (1984) 449

[10] M. Atiyah and I. M. Singer, Ann. Math. 93 (1971) 139

[11] J. Smit and J. Vink, Nucl. Phys. B286 (1987) 485

[12] G. ’t Hooft, Phys. Rev. Lett. 37 (1976) 8; Phys. Rev. D14 (1976) 3432;
for a recent review see:
M. E. Shaposhnikov, CERN preprint TH.6304/91 (1991)

[13] S. S. Chern and J. Simons, Ann. Math. 99 (1974) 48

[14] N. Seiberg, Phys. Lett. B148 (1984) 456

[15] P. Woit, Stony Brook preprint ITP-SB-87-56 (1987)
[16] A. Phillips and D. Stone, Nucl. Phys. B (Proc. Suppl.) 20 (1991) 28

[17] M. Lüscher, Nucl. Phys. B200 [FS4] (1982) 61;
    M. Göckeler, A. S. Kronfeld, M. L. Laursen, G. Schierholz and U.-J. Wiese,
    Phys. Lett. B233 (1989) 192

[18] M. Göckeler, A.S. Kronfeld, M.L. Laursen, G. Schierholz and U.-J. Wiese, Nucl.
    Phys. B292 (1987) 349

[19] R. Flume and D. Wyler, Phys. Lett. B108 (1982) 317