NON-VANISHING OF JACOBI POINCARÉ SERIES

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ABSTRACT. We prove that under suitable conditions, the Jacobi Poincaré series of exponential type of integer weight and matrix index does not vanish identically. For classical Jacobi forms, we construct a basis consisting of the “first” few Poincaré series and also give conditions both dependent and independent of the weight, which ensures non-vanishing of classical Jacobi Poincaré series. Equality of certain Kloosterman-type sums is proved. Also, a result on the non-vanishing of Jacobi Poincaré series is obtained when an odd prime divides the index.

1. Introduction

In [11] R. A. Rankin has proved that the $m$-th Poincaré series $P^k_m$ of weight $k$, where $k,m$ are positive integers, for the full modular group $SL(2, \mathbb{Z})$ is not identically zero for sufficiently large $k$ and finitely many $m$ depending on $k$. C. J. Mozzochi extended Rankin’s result to integral weight modular forms for congruence subgroups in [9].

In this paper we prove similar results for higher degree Jacobi Poincaré series defined on the full Jacobi group $\Gamma_J = SL(2, \mathbb{Z}) \ltimes (\mathbb{Z}^g \times \mathbb{Z}^g)$, where $g$ is a positive integer and is referred to as the degree of the Jacobi group. The Jacobi group operates on $\mathcal{H} \times \mathbb{C}^g$ and also on functions $\phi: \mathcal{H} \times \mathbb{C}^g \to \mathbb{C}$. We denote the latter action by $|_{k,m}$. (See section 2 for the definitions.)

Let $k,g \in \mathbb{Z}$, $m$ a symmetric, positive-definite, half-integral $(g \times g)$ matrix. The vector space of Jacobi cusp forms of weight $k$, index $m$ and degree $g$, denoted by $J_{cusp}^k,m,g$ is defined to be the space of holomorphic functions $\phi: \mathcal{H} \times \mathbb{C}^g \to \mathbb{C}$ satisfying $\phi|_{k,m} \gamma = \phi$ (where $\gamma \in \Gamma^J_g$) and having a Fourier expansion

$$\phi(\tau, z) = \sum_{n \in \mathbb{N}, r \in \mathbb{Z}^g, 4n > m^{-1}[r]} c_\phi(n, r) e(n \tau + rz)$$

If $g = 1$, we denote $J_{cusp}^k,m,1$ by $J_{cusp}^k,m$.

For $n \in \mathbb{N}$, $r \in \mathbb{Z}^g$ with $4n > m^{-1}[r]$, let $P^k_n,r_m$ be the $(n, r)$-th Poincaré series of weight $k$ and index $m$ (of exponential type) defined for $k > g + 2$ as in [3] (see Section 2 for definition). It is well-known that the Poincaré series $P^k_n,r_m$ $(n \in \mathbb{Z}, r \in \mathbb{Z}^g)$ span $J_{cusp}^{k,m,g}$. It is then natural to ask when such a Poincaré series vanish identically or when it does not. We prove the following theorem, which gives a partial answer to the above question.

Let $D = \det \begin{pmatrix} 2n & r \\ r & 2m \end{pmatrix}$ and define $k' := k - g/2 - 1$.

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Theorem 1.1. Let \( k \) be even when \( 2r \equiv 0 \pmod{2m} \). Then there exist an integer \( k_0 \) and a constant \( B > 3 \log 2 \) such that for all \( k \geq k_0 \) (depending only on \( g \)), the Jacobi Poincaré series \( P_{n,m}^{k,r} \) does not vanish identically for

\[
\frac{\pi D}{\text{det} (2m)} \leq k^{1+\alpha(g)} \exp \left\{ -\frac{B \log k'}{\log \log k'} \right\},
\]

where \( \alpha(g) = \begin{cases} 
\frac{2}{3(g+2)} & \text{if } 1 \leq g \leq 4, \\
\frac{2}{3g} & \text{if } g \geq 5.
\end{cases} \)

We construct a basis of \( J_{k,m}^{\text{cusp}} \) consisting of the “first” \( \dim J_{k,m}^{\text{cusp}} \) Poincaré series (see Theorem 4.1 in section 4). We also give conditions for non-vanishing of Poincaré series independent of the weight for classical Jacobi forms \((g = 1)\).

Define \( M(x) := \exp \left\{ B_1 \log x / \log \log 2x \right\} (x \geq 2, B_1 > \log 2) \) as in \([11]\).

Theorem 1.2. Let \( g = 1 \). For \( D > \frac{m}{\pi} \), we have \( P_{D,r}^{k,m} \neq 0 \) for

\[
M \left( \frac{\pi D}{m} \right) \sigma_0(D) D < \frac{m^{\frac{1}{2}}}{\lambda},
\]

where \( \lambda = (2\sqrt{2\pi}^\frac{3}{2} A)^\frac{3}{2} \), \( A = \frac{1}{\pi} \left( \frac{2}{6\pi} + \frac{54}{2\pi} + \frac{16}{2\pi} \right) \) and \( \sigma_0(D) = \sum_{d|D} 1 \).

Finally following \([11]\), we give conditional statements on the non-vanishing of Jacobi Poincaré series, based on the relation of \( g \)-dimensional Kloosterman sums with corresponding 1-dimensional sums and identities involving them.

Theorem 1.3. Let \( p \) be an odd prime, \( \mu \in \mathbb{N} \). Suppose \( p | (m, r) \), \( p \nmid n \). If \( P_{p^m,n,p^r}^{k,m} \neq 0 \) then

\[
\text{either } P_{n,p^m-1,r,p^r-1}^{k,m} \neq 0 \quad \text{or} \quad P_{n,p^m,p^m}^{k,m} \neq 0 \quad \text{and} \quad P_{n,r,p^m}^{k,m} \neq 0
\]

(Here \( p|m \) means \( p \) divides every entry of \( m \); since \( 2m \) is a \((g \times g)\) matrix with integer entries and \( p \) is odd, this makes sense.)

Remark 1.4. (1) In Section 3 we first prove the trivial case where the Poincaré series \( P_{n,r}^{k,m} \) does not vanish when the ratio \( \frac{\pi D}{\text{det} (2m)} \left( \frac{D}{\text{det} (2m)} = 2n - 2m^{-1}[\frac{1}{2} r] \right) \) by which we measure the non-vanishing of Jacobi Poincaré series, is \( O(k) \), but with explicit range of the weight \( k \) where this is valid. This follows from Proposition 3.1 for arbitrary \( g \) and also from Theorem 4.1 in the case \( g = 1 \) (recall that \( \dim J_{k,1,m}^{\text{cusp}} \sim O \left( \frac{k(m+1)}{12} \right) \) ).

(2) Theorem 1.1 therefore improves the trivial case mentioned in the previous remark. However, achieving the “order of \( k^{2-\epsilon} (\epsilon > 0) \)” as in \([11]\) in the case of Jacobi Poincaré series using Rankin’s methods seems difficult mainly because of the presence of the factor \((c, D)\) instead of \((c, D)^{\frac{1}{2}}\) in the estimate of Kloosterman sums of degree \( g \) (even for small \( g \), see section 3).
The condition that \( k \) be even when \( 2r \equiv 0 \pmod{Z^g \cdot 2m} \) in Theorem 1.1 is necessary, as the \((n, r)\)-th Poincaré series vanish when \( k \) is odd and \( 2r \equiv 0 \pmod{Z^g \cdot 2m} \). The restriction \( k' \leq \frac{\pi D}{\det (2m)} \) in Theorem 1.1 is natural since we know the result in the complement (see Proposition 3.1). Same is true for the condition \( D > m \) in Theorem 1.2.

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2. Notations and Preliminaries

The Jacobi group \( \Gamma^J_g \) operates on \( \mathcal{H} \times \mathbb{C}^g \) in the usual way by

\[
\left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), (\lambda, \mu) \right) \circ (\tau, z) = \left( \frac{a\tau + b}{c\tau + d}, (c\tau + d)^{-1}(z + \lambda\tau + \mu) \right).
\]

Let \( k \in \mathbb{Z}, m \) a symmetric, positive-definite, half-integral \((g \times g)\) matrix. Then we have the action of \( \Gamma^J_g \) on functions \( \phi: \mathcal{H} \times \mathbb{C}^g \to \mathbb{C} \) given by:

\[
\phi|_{k,m}(\tau, z) := (c\tau + d)^{-k}e(-c(c\tau + d)^{-1}m[z + \lambda\tau + \mu] + m[\lambda] + 2\lambda^t m z) \phi(\gamma \circ (\tau, z)).
\]

(Here \( A[B] = B^t A B \) for matrices \( A, B \) of appropriate sizes, \( B^t \) is the transpose of the matrix \( B \).)

The vector space of Jacobi cusp forms of weight \( k \), index \( m \) and degree \( g \), denoted by \( J^\text{cusp}_{k,m,g} \), is defined to be the space of holomorphic functions \( \phi: \mathcal{H} \times \mathbb{C}^g \to \mathbb{C} \) satisfying \( \phi|_{k,m} = \phi \) and having a Fourier expansion

\[
\phi(\tau, z) = \sum_{n \in \mathbb{N}, r \in \mathbb{Z}^g, 4n > m^{-1}[r]} c_\phi(n, r)e(n\tau + rz)
\]

For \( n \in \mathbb{N}, r \in \mathbb{Z}^g \) with \( 4n > m^{-1}[r] \), let \( P_{n, r}^{k,m} \) be the \((n, r)\)-th Poincaré series of weight \( k \) and index \( m \) (of exponential type) defined for \( k > g + 2 \) by

\[
P_{n, r}^{k,m}(\tau, z) := \sum_{\gamma \in \Gamma^J_g \setminus \Gamma^J_g} e(n\tau + rz)|_{k,m}(\tau, z) \quad (\tau \in \mathcal{H}, z \in \mathbb{C}^g),
\]

where \( \Gamma^J_g \setminus \Gamma^J_g := \left\{ \left( \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right), (0, \mu) \mid n \in \mathbb{Z}, \mu \in \mathbb{Z}^g \right\} \).

It is well known that \( J^\text{cusp}_{k,m,g} \) is finite dimensional and the family of Poincaré series \( P_{n, r}^{k,m} \) \((n \in \mathbb{N}, r \in \mathbb{Z}^g)\) form a basis of \( J^\text{cusp}_{k,m,g} \). In [3, Lemma 1], S. Böcherer and W. Kohnen obtained the Fourier expansion of \( P_{n, r}^{k,m} \):

**Proposition 2.1.** (1) The function \( P_{n, r}^{k,m} \) is in \( J^\text{cusp}_{k,m,g} \). The Fourier expansion of the Poincaré series is given by
\[ P_{n,r}^{k,m}(\tau, z) = \sum_{n' \in \mathbb{N}, r' \in \mathbb{Z}, 4n' > m^{-1}r'^2} c_{n',r'}^{k,m}(n', r') e(n'\tau + r'z), \]

where

\[ c_{n,r}^{k,m}(n', r') = \delta_m(n, r, n', r') + (-1)^k \delta_m(n, r, n', -r') + 2\pi i^k \text{det}(2m)^{-1/2} \cdot (D'/D)^{k/2-g/4-1/2} \]

\[ \times \sum_{c \geq 1} \left( H_{m,c}(n, r, n', r') + (-1)^k H_{m,c}(n, r, n', -r') \right) J_{k-g/2-1} \left( \frac{2\pi\sqrt{DD'}}{\text{det}(2m) \cdot c} \right) \]

where \( D' = \text{det} \left( \begin{array}{cc} 2n' & r' \\ r' & 2m \end{array} \right) \), \( \delta_m(n, r, n', r') := \begin{cases} 1 & \text{if } D = D', r \equiv r' \pmod{Z^g \cdot 2m}, \\ 0 & \text{otherwise}, \end{cases} \)

and \( H_{m,c}(n, r, n', r') := c^{-g/2-1} \sum_{x(c), g(c)^r} e_c \left( (m[nx] + r x + n)y + n'y + r'x \right) e_{2e}(r' y - m^{-1} r'^2) \)

where in the summation \( x \) (resp. \( y \)) run over a complete set of representatives for \( \mathbb{Z}^g / c\mathbb{Z}^g \) (resp. \( \mathbb{Z} / c\mathbb{Z} \)), \( y \) denotes an inverse of \( y \) (mod \( c \)), \( e_c(a) := e^{2\pi ia/c} (a \in \mathbb{Z}) \), and \( J_{k-g/2-1} \) denotes the Bessel function of order \( k - g/2 - 1 \).

\[ \langle \phi, P_{n,r}^{k,m} \rangle = \lambda_{k,m,D} c_{\phi}(n, r), \quad \text{where } c_{\phi}(n, r) \text{ denotes the } (n, r)\text{-th Fourier coefficient of } \phi \text{ and } \]

\[ \lambda_{k,m,D} = 2^{(g-1)(k-g/2-1)-g} \cdot \Gamma(k-g/2-1) \cdot 2^{-k+g/2+1} \cdot (\text{det } m)^{k-(g+3)/2} \cdot D^{-k+g/2+1} \]

(2) \( \langle \phi, P_{n,r}^{k,m} \rangle \) is the Petersson inner product on \( J_{k,m,g}^{\text{cusp}} \).

From Proposition 2.1, we conclude that the Poincaré series \( P_{n,r}^{k,m} \) is non-zero if and only if its \( (n, r)\)-th Fourier coefficient \( c_{n,r}^{k,m} \) is positive. So, it is enough to prove \( c_{n,r}^{k,m} \) is non-zero. For convenience of notation we will drop the \((k, m)\) in the calculations.

**Lemma 2.2.** The Poincaré series \( P_{n,r}^{k,m} \) vanishes if \( k \) is odd and \( 2r \equiv 0 \pmod{Z^g \cdot 2m} \).

**Proof.** In fact the \((n, r)\)-th coefficient \( c(n, r) \) of a general Jacobi form of degree \( g \) is zero if \( k \) is odd when \( 2r \equiv 0 \pmod{Z^g \cdot 2m} \). This is an easy consequence of the transformation property of Jacobi forms. See for instance [5] for \( g = 1 \). \( \square \)

From the Fourier expansion of \( P_{n,r}^{k,m} \) we see that in order to prove that it is non-zero, it is enough to prove \(|S(n, r)| < \frac{1}{2m}\) (noting that \( 2m \) is a positive-definite matrix with integer entries, hence \( \text{det } (2m) \geq 1 \)), where

\[ S(n, r) := \text{det} (2m)^{-1/2} \sum_{c \geq 1} \left( H_{m,c}(n, r, n, r) + (-1)^k H_{m,c}(n, r, n, -r) \right) J_{k-g/2-1} \left( \frac{2\pi D}{\text{det}(2m) \cdot c} \right) \]

(2.2)
We will need the following estimates. (See [14], [15] and [3] respectively for details):

(2.3) (i) \( |J_\nu(x)| \leq \min \left\{ 1, \frac{1}{\Gamma(\nu + 1)} \left( \frac{x}{2} \right)^\nu \right\} \) for \( x > 0 \) and \( \nu \geq 2 \).

(2.4) (ii) \( |H_{m,c}(n, r, n, \pm r)| \leq 2^{\omega(c)} c^{g/2 - 1}(D, c) \),

where \( \omega(c) \) is the number of distinct prime divisors of \( c \), \( (D, c) = \gcd(D, c) \).

3. PROOF OF THEOREM 3.1

3.1. In this section we first obtain the following proposition which follows easily from trivial estimates of Bessel functions.

Proposition 3.1. (1) Let \( k \) be even when \( 2r \equiv 0 \pmod{\mathbb{Z} \cdot 2m} \). Then there exists an integer \( k_0 \) such that the \( (n, r) \)-th Poincaré series \( P_{k,m}^{n,r} \) does not vanish identically for \( k \geq k_0 \) and \( (n, r) \in \mathbb{N} \times \mathbb{Z}^g \) with \( D \leq k' \pi e \cdot \det(2m) \).

If \( k > g + 3 \), then one can take \( k_0 = \max \left( g + 4, \left\lceil \frac{g}{2} \right\rceil + 69 \right) \).

(2) For all \( D < \frac{1}{\pi} \det(2m) \) the Poincaré series \( P_{k,m}^{n,r} \) does not vanish identically, whenever the condition is non-void and \( k > g + 3 \) if \( g \geq 2 \) and \( k > 5 \) if \( g = 1 \).

Lemma 3.2. The condition in Proposition 3.1 (2) is non-void for \( n < \frac{1}{6} + \left( \frac{2m - 3}{144m} \right)^2 \) when \( g = 1 \).

Proof. Suppose that \( D < \frac{1}{\pi}(2m) < \frac{1}{3}(2m) \). We have \( 2m(2n - \frac{1}{3}) < r^2 < 4mn \). Noticing that there is a square in the interval \( [x, y] \), \( (x, y \in \mathbb{R}^+) \) when \( 2\sqrt{x} + 1 < y - x \), we need to have,

\[
2\sqrt{2m(2n - \frac{1}{3}) + 1} < \frac{2m}{3}, \quad \text{or} \quad n < \frac{1}{6} + \left( \frac{2m - 3}{144m} \right)^2.
\]

So, in the case \( g = 1 \), the Poincaré series \( P_{k,m}^{n,r} \) does not vanish identically when \( k > 4 \) and \( n \) satisfies the condition of the lemma.

Proof of Proposition 3.1. In a straightforward manner, using estimates (2.3) and (2.4), we get \( |S(n, r)| \leq \frac{2}{1/(k'+1)} \left( \frac{S}{2} \right)^{k'} \sum_{c} \frac{2^{\omega(c)} c^{g/2 - 1}}{e^{c^{1-g}}} \), where \( S := \frac{2\pi D}{\det(2m)} \). Using this, and Corollary 3.3, the proof follows. We omit the details.

Corollary 3.3. If \( k > g + 3 \), then one can take \( k_0 = \max \left( g + 4, \left\lceil \frac{g}{2} \right\rceil + 69 \right) \) in Proposition 3.1 (\( \lceil \cdot \rceil \) denotes the greatest integer function).

Proof. Examining the proof of Proposition 3.1, we see that when \( k \geq g + 4 \) the series

\[
\sum_{c} \frac{2^{\omega(c)} c^{g/2 - 1}}{e^{c^{1-g}}} < \zeta(2) = \frac{\pi^2}{6}.
\]

using the trivial bound \( 2^{\omega(c)} \leq c \). The rest follows by using Stirling’s formula :

\[
n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\lambda_n}, \quad \text{where} \quad \frac{1}{12n + 1} < \lambda_n < \frac{1}{12n}, \quad \text{for} \quad n \in \mathbb{N}.
\]
3.2. Poincaré series for small weights. For $\text{Re}(s) > \frac{1}{2}(\frac{g}{2} - k + 2)$, using the ‘Hecke trick’, the Jacobi Poincaré series is defined as in [4]

$$P_{n,r}^{k,m}(\tau, z) = \sum_{\gamma \in \Gamma \backslash \Gamma_g} \left( \frac{v}{|c\tau + d|^2} \right)^s e(n\tau + rz)|_{k,m} \gamma(\tau, z)$$

where $\tau = u + iv \in \mathbb{H}$, $z \in \mathbb{C}$, $s \in \mathbb{C}$.

Then for $k > \frac{g}{2} + 2$, $P_{n,r,0}^{k,m} \in J_{n,r}^{\text{cusp}}$ and has the same Fourier properties as $P_{n,r}^{k,m}$. We also consider conditions on it’s non-vanishing in the following Proposition.

**Proposition 3.4.** Under the hypotheses of Proposition 3.1 there exists an integer $C(m)$, depending on $m$ such that the Poincaré series $P_{n,r,0}^{k,m}$ does not vanish identically

$$\forall k \in \left[ \max \left\{ C(m), \frac{g+7}{2} \right\}, \infty \right)$$

and $(n, r) \in \mathbb{N} \times \mathbb{Z}^g$ with $D \leq \frac{k'}{\pi e} \cdot \det(2m)$

**Remark 3.5.** Though Proposition 3.1 is applicable here, the above theorem accounts for (possibly) smaller values of $k$.

**Proof.** This theorem again follows from the arguments of the proof of Proposition 3.1. Here we use the following estimate for Kloosterman sums of degree $g$ (see [3, p.508,512]):

$$H_{m,c}(n,r,n \pm r) \leq 2^{\omega(c)}e^{-1/2}(D,c), \forall c \geq C(m)$$

where $C(m)$ is a constant depending on $m$. With the notation of Proposition 3.1 we have for some positive constant $C_1(m)$,

$$|S(n,r)| \leq \sum_{1 \leq c \leq C(m)} \frac{2^{\omega(c)}e^{g/2-1}(D,c)}{\Gamma(k'+1)} \left( \frac{S}{2c} \right)^{k'} + \sum_{c > C(m)} \frac{2^{\omega(c)}e^{g/2-1}(D,c)}{\Gamma(k'+1)} \left( \frac{S}{2c} \right)^{k'}$$

$$\leq C(m)^{(g-1)/2} \sum_{1 \leq c \leq C(m)} \frac{2^{\omega(c)}e^{-1/2}(D,c)}{\Gamma(k'+1)} \left( \frac{S}{2c} \right)^{k'} + \sum_{c > C(m)} \frac{2^{\omega(c)}e^{-1/2}(D,c)}{\Gamma(k'+1)} \left( \frac{S}{2c} \right)^{k'}$$

$$\leq \frac{2C_1(m)}{\Gamma(k'+1)} \left( \frac{S}{2c} \right)^{k'} \sum_c 2^{\omega(c)}e^{-k-(g+3)/2}.$$

The condition $k > \frac{g+7}{2}$ precisely guarantees convergence of the series above. The rest of the proof is identical to that of Proposition 3.1. 

3.3. We now come to the main result of this section.

**Proof of Theorem 1.1.** We use Rankin’s method as in [11]. With $S(n,r)$ as above, we need to prove $|S(n,r)| \leq \frac{1}{2\pi}$. Define

$$\sigma = k'^{-1/6}, \quad Q^* = \frac{2\pi D}{k' \det(2m)}, \quad M(D) = \exp \left\{ \frac{B_1 \log D}{\log \log 2D} \right\}$$
Define $H_{m,c}^\pm(n, r, n, r) = H_{m,c}(n, r, n, r) + (-1)^k H_{m,c}(n, r, n, -r)$. Then we have

$$|S(n, r)| \leq \det (2m)^{-1/2}|S_1(n, r)| + \det (2m)^{-1/2}|S_2(n, r)|,$$

where

$$S_1(n, r) = \sum_{1 \leq c \leq Q^*} |H_{m,c}^\pm(n, r, n, r)||J_{k'} \left( \frac{k'Q^*}{c} \right)|,$$

$$S_2(n, r) = \sum_{c \geq Q^*} |H_{m,c}^\pm(n, r, n, r)||J_{k'} \left( \frac{k'Q^*}{c} \right)|.$$

We get after similar calculations as in [11] (see also [9]) that

$$|S_1(n, r)| \leq A_1 M(D) Q^{s/2 - 1} \sum_{d | D, d < Q^*} 2^{\omega(d)} d \left\{ \frac{Q^* \sigma^3}{d} + 3 \sigma^2 \right\} \leq A_2 M(D)^3 \frac{Q^{g/2}}{k'^{1/2}} + A_3 M(D)^3 \frac{Q^{s/2}}{k'^{1/3}} \leq A_4 M(D)^3 \left( \frac{\pi D}{\det (2m)} \right)^{g/2 + 2} + A_5 M(D)^3 \frac{\pi D}{k'^{g/2 + 1/3}}$$

(3.1)

However, the other sum $S_2(n, r)$ needs to be handled differently. We have

$$|S_2(n, r)| \leq \sum_{Q^* < c < k'Q^*} 2^{\omega(c) + 1} c^{g/2 - 1} (D, c)|J_{k'} \left( \frac{2\pi D}{c \cdot \det (2m)} \right)| + \sum_{c > k'Q^*} 2^{\omega(c) + 1} c^{g/2 - 1} (D, c)|J_{k'} \left( \frac{2\pi D}{c \cdot \det (2m)} \right)| \leq 2M(D) \left( \frac{\pi D}{\det (2m)} \right)^{k'} \sum_{Q^* < c < k'Q^*} \frac{1}{c^{k'-g/2}} + 2 \sum_{c > k'Q^*} c^{g/2 + \epsilon} |J_{k'} \left( \frac{2\pi D}{c \cdot \det (2m)} \right)| \leq 2M(D) \left( \frac{\pi D}{\det (2m)} \right)^{k'} \sum_{Q^* < c < k'Q^*} \frac{1}{c^{k'-g/2}} + 2 \left( \frac{\pi D}{\det (2m)} \right)^{g/2 + 1 + \epsilon} \sum_{c > k'Q^*} \frac{1}{c^{1+\epsilon}} \leq a_0 M(D) \left( \frac{\pi D}{\det (2m)} \right)^{g/2 + \epsilon} + a_1 \left( \frac{\pi D}{\det (2m)} \right)^{g/2 + \epsilon}$$

(3.3)

where $a_i, A_j$ are absolute constants, and $0 < \epsilon, \delta < 1$. Now for any $g \geq 1$, and $\alpha(g) = \frac{2}{3g+2}$; we choose $0 < \epsilon, \delta < \frac{1}{2}$ and find that $S_1(n, r)$ and $S_2(n, r)$ are small if we choose $k$ large. If $g \geq 5$, then we find that a better bound $\alpha(g) = \frac{2}{3g}$ works. This completes the proof. □
4. Explicit basis for $J_{k,m}^{\text{cusp}}$ and proof of Theorem 1.2

4.1. H. Petersson proved that the first $\ell = \dim S_k$ Poincaré series $P_k^1, \ldots, P_k^\ell$ is a basis for the space of cusp forms $S_k$ for $SL(2, \mathbb{Z})$. We prove the corresponding result for Jacobi forms. The proof is based on the dimension formula given in [5].

**Theorem 4.1.** Let $k \geq m + 12$. Then we have the following classical basis for $J_{k,m,1}^{\text{cusp}}$:

\[(1) \text{If } k \text{ is even, } \left\{ P_{D_\mu + 4m\lambda_\mu, \mu}^{k,m} \right\}, \mu = 0, 1, \ldots, m; \lambda_\mu = 0, 1, \ldots, \dim S_{k+2}\mu - \left[ \frac{\mu^2}{4m} \right] - 1 \text{ where } D_\mu := 4m\left( \left[ \frac{\mu^2}{4m} \right] + 1 \right) - \mu^2.
\]

\[(2) \text{If } k \text{ is odd, } \left\{ P_{D_\mu + 4m\lambda_\mu, \mu}^{k,m} \right\}, \mu = 1, \ldots, m - 1; \lambda_\mu = 0, 1, \ldots, \dim S_k + 2\mu - 1 - \left[ \frac{\mu^2}{4m} \right] - 1.
\]

**Proof.** We prove the Theorem for $k$ even, the other case is analogous. The condition $k \geq m + 12$ ensures $\dim S_{k+2\mu} \geq \left[ \frac{\mu^2}{4m} \right] + 1$ (see [5] p.103)). The proof follows Petersson’s argument in the elliptic case (see [10], [12]). Let $d = \dim J_{k,m}^{\text{cusp}}$ and $\phi_1, \ldots, \phi_d$ be an orthonormal basis. We write

$$\phi_j(\tau, z) = \sum_{D' > 0, D' = -r^2 \pmod{4m}} c_j(D', r)e \left( \frac{D' + r^2}{4m} \tau + rz \right)$$

and

$$P_{D_\mu + 4m\lambda_\mu, \mu}^{k,m} = \lambda_{k,m,D_\mu + 4m\lambda_\mu}^{k,m,1} \sum_{j=1}^{d} c_j(D_\mu + 4m\lambda_\mu, \mu) \phi_j,$$

where $\mu$ and $\lambda_\mu$ vary as in the statement of the Theorem. We get a $d \times d$ matrix indexed by pairs $(D_\mu + 4m\lambda_\mu, \lambda_\mu)$ and $j$. It suffices to prove the matrix is invertible. If not, let there be a linear relation

$$\sum_{j=1}^{d} \xi_j c_j(D_\mu + 4m\lambda_\mu, \mu) = 0, (\xi_1, \ldots, \xi_d) \neq (0, \ldots, 0), \text{ for all } (D_\mu + 4m\lambda_\mu, \mu).$$

**Claim:** Considering the non-zero Jacobi form $\Phi := \sum_{j=1}^{d} \xi_j \phi_j$, we see that the Fourier coefficients $c_\Phi(D_\mu + 4m\lambda_\mu, \mu)$ (and $\lambda_\mu$ as in the Theorem) are zero. This implies that $D_\mu \Phi = 0$ for $\mu = 0, \ldots, m$, (see [5] p.32) for the definition of operators $D_\mu$) which shows that $\Phi = 0$, a contradiction.

**Proof of claim:** Let $\Phi \in J_{k,m}^{\text{cusp}}$. Then we have the following Fourier expansion of the modular form $D_\nu \Phi$ of weight $k + 2\nu$, (cf. [5] p. 32), $k$ even, $\nu = 0, \ldots, m$)

\[(4.1) \quad D_\nu \Phi = A_{k,\nu} \sum_{n \geq 0} \left( \sum_{r^2 < 4mn} \left( \sum_{0 \leq \mu \leq \nu} (k + 2\nu - \mu - 2)! (-mn)^\mu r^{2\nu - 2\mu} \mu! (2\nu - 2\mu)! \right) c_\Phi(n, r) \right) \cdot q^n \]

where, $A_{k,\nu} := (2\pi i)^{-\nu} \frac{(k + 2\nu - 2)! (2\nu)!}{(k + \nu - 2)!}$ and $q := e(\tau)$.

Let $\ell$ be an an positive even integer. Let $d_\ell := \dim S_\ell$. Since an elliptic cusp form $f = \sum_{n=1}^{\infty} a(n, f) q^n$ of weight $\ell$ is determined by the first $d_\ell$ of it’s Fourier coefficients $a(1, f), \ldots, a(d_\ell, f)$, looking at equation (4.1) we need to prove that $c_\Phi(n, r) = 0$ for all $r$, such
that $r^2 < 4mn$, $0 \leq r \leq m$ and all $n$, such that \( \left\lfloor \frac{r^2}{4m} \right\rfloor + 1 \leq n \leq d_{k+2
u} (\nu = 0, \ldots, m) \). From now on let $\ell$ denote one of $k + 2\nu$, $(\nu = 0, \ldots, m)$ and for convenience, we drop the suffix $\nu$. To see this, first, if $|r| > 2m$ in equation (4.1) we can consider $-m \leq r' = r - 2mx \leq m$ for a suitable integer $x$ and an $n' \geq 1$ such that $4mn' - r'^2 = 4mn - r^2$ and use the fact that $c_\phi(n', r') = c_\phi(n, r)$ and that $n \geq n' \geq 1$, so $n'$ also satisfies the same upper bound as that of $n$ (namely, $\left\lfloor \frac{r^2}{4m} \right\rfloor + 1 \leq n' \leq d_\ell$). We can finally reduce to the case $0 \leq r \leq m$ since $c_\phi(n, r) = c_\phi(n, -r)$ as $\ell$ is even.

But any such $n$, can be written as $n = \left\lfloor \frac{r^2}{4m} \right\rfloor + 1 + \lambda = \frac{D + 4n\lambda + r^2}{4m}$ with $0 \leq \lambda \leq m$ and $D_\nu$, $\lambda$ as in the statement of the theorem. This proves the claim.

\[ \square \]

4.2. The Eichler-Zagier map for Jacobi forms of integral weight and index 1 denoted by $Z_1: J_{k,1} \to M_{k-1/2}^+$ is defined by

\[ Z_1: \sum_{D > 0, r \in \mathbb{Z}} c(D)e(D) \left( \frac{D}{4} \tau + rz \right) \mapsto \sum_{0 < D \in \mathbb{Z}} c(D)e(D\tau) \]

where the Fourier coefficient $c(D)$ does not depend on $r$.

Let $k$ be even. Following the notation in [7], let $P_{k-1,4,D}$ $((-1)^{k-1}D \equiv 0, 1 \pmod{4})$ be the Poincaré series in $M_{k-1/2}^+$. The Fourier development of $P_{k-1,4,D}$ is given below:

**Proposition 4.2** (Kohnen, [7]).

\[ P_{k-1,4,D}(\tau) = \sum_{t \geq 1, (-1)^{k-1}n \equiv 0, 1(4)} g_D(t)e(t\tau), \quad \text{with} \]

\[ g_D(t) = \frac{2}{3} \delta_{D,t} + (-1)^{k/2} \pi \sqrt{2}(t/D)^{k-3/4} \sum_{c \geq 1} H_c(t, D) J_k(\frac{\pi}{c} \sqrt{tD}) \]  

Here $\delta_{t,D}$ is the Kronecker delta, and,

\[ H_c(t, D) = (1 - (-1)^{k-1}i) \left( 1 + \left( \frac{4}{c} \right) \right) \frac{1}{4c} \sum_{\delta \mid (4c)} \left( \frac{4c}{\delta} \right)^{k-1/2} e_{4c}(t\delta + D\delta^{-1}) \]

**Definition 4.3.** (1) Let $w$ be an integer, $c \geq 1$ be even and $u, v \equiv 0, (-1)^w \pmod{4}$. Let $\alpha \in \{1, 2\}$. We define the following exponential sum,

\[ H_{\alpha c}(u, v) := (1 - (-1)^{w}i) \frac{1}{4c} \sum_{1 \leq \delta \leq \alpha c - 1, (\delta, 4c) = 1} \left( \frac{4c}{\delta} \right) \left( \frac{-4}{\delta} \right)^{w+1/2} e_{4c}(u\delta + v\delta^{-1}), \]

where $\delta\delta^{-1} \equiv 1 \pmod{4c}$.

**Lemma 4.4.** Let $w$ be an integer, $c \geq 1$ be even and $u, v \equiv 0, (-1)^w \pmod{4}$.

1. $H_c(u, v) = (1 + (-1)^{u+v+c/2}) H_{2c}(u, v)$ Therefore, when $c \equiv 2 \pmod{4}$, $H_c(u, v)$ vanishes unless $u, v$ have different parity.

2. Let $c \equiv 0 \pmod{4}$. Then $H_c(u, v) = (1 + (-1)^{u+v})(1 + e_4(u - v)) H_{c}(u, v)$. 


Definition 4.5. Let $u, v, w \in \mathbb{Z}$, $c \geq 1$. Define

\begin{equation}
\mathcal{H}_c'(u, v) := \frac{1}{c} \left( \frac{4}{c} \right) \left( \frac{-4}{c} \right)^{-w-1/2} \sum_{\delta \in (c)^*} \left( \frac{\delta}{c} \right) e_c(u \delta + v \delta^{-1})
\end{equation}

\begin{equation}
\mathcal{H}_{4c}(u, v) := \left( 1 + \left( \frac{4}{c} \right) \right) \frac{1}{4c} \sum_{\delta \in (4c)^*} \left( \frac{4c}{\delta} \right) \left( \frac{-4}{\delta} \right)^{w+1/2} e_{4c}(t \delta + D \delta^{-1})
\end{equation}

Remark 4.6. In applications we will always use Lemma 4.4 and the above definitions in the case $w = k - 1$, with $k$ even.

Definition 4.7. We denote by $G(a, b, c)$ the Gauss sum defined by

\begin{equation}
G(a, b, c) := \sum_{n \pmod{c}} e_c(an^2 + bn) \quad \text{where } a, b, c \in \mathbb{Z}.
\end{equation}

Proposition 4.8 ([2]). We have the following:

1. $G(a, b, c_1c_2) = G(c_2a, b, c_1) G(c_1a, b, c_2)$, where $(c_1, c_2) = 1$.

2. Let $(a, c) = 1$.

\begin{equation}
G(a, b, c) = \begin{cases} 
\epsilon_c \sqrt{c} \left( \frac{a}{c} \right) e_c(-\psi(a)b^2) & \text{if } c \equiv 1 \pmod{2}, 4\psi(a) \equiv 1 \pmod{c} \\
2G(2a, b, \frac{c}{2}) & \text{if } c \equiv 2 \pmod{4}, b \equiv 1 \pmod{2} \\
0 & \text{if } c \equiv 2 \pmod{4}, b = 0 \\
(1 + i)\epsilon_a^{-1} \sqrt{c} \left( \frac{a}{c} \right) & \text{if } c \equiv 0 \pmod{4}, b = 0 \\
0 & \text{if } c \equiv 0 \pmod{4}, b \equiv 1 \pmod{2}.
\end{cases}
\end{equation}

Proposition 4.9. Let $c \geq 1$ and $k$ even. Then $H_{1,c}(n, r, n', \pm r') = H_c(D', D)$.

Proof. We distinguish 3 cases, for classes of $c$ modulo 4. Let $\epsilon_\delta = \begin{cases} 
1 & \text{if } \delta \equiv 1 \pmod{4}, \\
i & \text{if } \delta \equiv 3 \pmod{4}.
\end{cases}$
1. \( c \equiv 1 \pmod{2} \)

We use the values of Gauss sums from table (4.9) in Proposition 4.8 and applying the formula from 4.9 we have:

\[
H_{1,c}(n, r, n', r') = c^{-3/2} \sum_{y(c)^*} \left( \frac{\bar{y}}{c} \right) e_c (n\bar{y} + n'y) e_{2c}(rr')
\]

\[
= c^{-3/2} \epsilon c \sqrt{c} \sum_{y(c)^*} \left( \frac{\bar{y}}{c} \right) e_c \left( -4^{-1} y(r\bar{y} + r')^2 n + \bar{y} + n'y \right) e_{2c}(rr')
\]

\[
= \epsilon c \sum_{y(c)^*} \left( \frac{\bar{y}}{c} \right) e_c \left( D'y + 4^{-2}\bar{y}D \right) e_2(DD')
\]

(4.11)

\[= 2(1 + i)\mathcal{H}_{4c}(D', 4^{-2}D) = 2H_c(D', D), \text{ after simplification.}\]

where \( 44^{-1} \equiv 1 \pmod{c} \) and the equality in the last line follows from [7] p. 256, equation (37)].

2. \( c \equiv 2 \pmod{4} \)

Let \( c = 2c' \), with \( c' \) odd. From table (4.9), and Lemma 4.4 we see that

\( H_{1,c}(n, r, n', r') = 0 = H_c(D', D) \) if \( r \) and \( r' \) or equivalently \( D \) and \( D' \) have the same pairity.

When they have opposite pairity, using the multiplicative property of Gauss sum in Proposition 4.8 and applying the formula from 4.9 we have:

\[
H_{1,c}(n, r, n', r') = 2c^{-3/2} \sum_{y(2c')^*} G(2\bar{y}, r\bar{y} + r', c') e_{2c'}(n\bar{y} + n'y)e_{4c'}(rr')
\]

\[
= 2c^{-3/2} \epsilon c \sqrt{c'} \sum_{y(2c')^*} \left( \frac{2\bar{y}}{c'} \right) e_{c'} \left( -8^{-1} y(r\bar{y} + r')^2 \right) e_{2c'}(n\bar{y} + n'y) e_{4c'}(rr')
\]

(4.10)

We make a change of variables \( y \mapsto 2y + c' \) and find after simplification that the above sum, (in which \( y \) now varies over a reduced residue system modulo \( c' \) and \( (2y + c')(2 \cdot 4^{-1}\bar{y} + c') \equiv 1 \pmod{2c'}, 44^{-1} \equiv 1 \pmod{c'} \)) :

\[
= \frac{\epsilon c'}{\sqrt{2c'}} \sum_{y(c')^*} \left( \frac{\bar{y}}{c'} \right) e_{c'} \left( D'y + 4^{-3}\bar{y}D \right) e_2(n + n')
\]

\[(4.12) = (1 + i)\mathcal{H}_8(D', D) \mathcal{H}_{c'}(D', 4^{-3}D) = (1 + i)\mathcal{H}_{8c'}(D', D) = H_c(D', D). \]

where the equalities in the last line follows from [7] p. 256, equation (38)].

3. \( c \equiv 0 \pmod{4} \)

From table (4.9), and Lemma 4.4 we see that \( H_{1,c}(n, r, n', r') = 0 = H_c(D', D) \) if \( r \) and \( r' \) or equivalently \( D \) and \( D' \) have opposite pairity. When they have the same pairity, again applying
the formula from [4.9] we have the following:

\[
H_{1,c}(n, r, n', r') = c^{-3/2} \sum_{y(c)^*} G(\bar{y}, r\bar{y} + r', c) e_c(n\bar{y} + n'y)e_{2c}(rr')
\]
\[
= c^{-3/2} \sum_{y(c)^*} G(\bar{y}, 0, c) e_{4c}(D'y + D\bar{y})
\]
\[
= (1 + i)c^{-3/2} \sum_{y(c)^*} \varepsilon_y^{-1} \sqrt{c} \left( \frac{c}{y} \right) e_{4c}(D'y + D\bar{y})
\]

\[(4.13)\]
\[
= 4 \mathbb{H}_c(D', D) = H_c(D', D),
\]
where the equality in the last line follows from Lemma [4.4](2) with \( w = k - 1 \) and the fact that \( D \equiv D' \pmod{4} \).

\[\square\]

**Proposition 4.10.** \( Z_1 \) maps \( P_{D,r} \in J_{k,1}^{\text{cusp}} \) to \( 3 P_{k-1,4,D} \in M_{k-1/2}^+ \).

**Proof.** First trivially we have, \( \delta_1(n, r, n', \pm r') = \delta_{D,D'} \). Therefore comparing the two Fourier developments and noting that \( k \) is even, we see that it is sufficient to prove for all \( c \geq 1 \) that

\[
H_{1,c}(n, r, n', \pm r') = \text{const.} \cdot H_c(D', D).
\]

Combining 1, 2 and 3 from Proposition [4.9] we finally arrive at the conclusion that when \( k \) is even,

\[(4.14)\]

\[
c_{n,r}(n', r') = 3g_D(D') \text{ for all } n, r, n', r',
\]
where \( c_{n,r}(n', r') \) and \( g_D(D') \) are the coefficients on the Fourier expansions of the relevant Poincaré series defined above. This completes the proof of Proposition [4.10]. \[\square\]

**Proposition 4.11.** There exist positive constants \( k_0 \) and \( B \), where \( B > 4 \log 2 \), such that, for all even \( k \geq k_0 \) and all positive integers \( D \leq k^2\exp\{-B \log k/\log \log k\} \), the Poincaré series \( P_{k-1,4,D} \) and hence also the Poincaré series \( P_{D,r}^{k-1,4} \) does not vanish identically.

**Proof.** From the Fourier expansion of \( P_{k-1,4,D} \) given in [7], we see that the proof is the same as in the case of integral weight Poincaré series for congruence subgroups of \( SL(2, \mathbb{Z}) \) given in [9]; so we omit it. \[\square\]

**Proof of Theorem 1.2**. We write \( S(n, r) = S_1(n, r) + S_2(n, r) \), where

\[
S_1(n, r) = i^k \pi \sqrt{2} m^{-1/2} \sum_{1 \leq c \leq \frac{m}{d}} H^\pm_{m,c}(n, r, n, r) J_k\left( \frac{\pi D}{mc} \right)
\]
\[
S_2(n, r) = i^k \pi \sqrt{2} m^{-1/2} \sum_{c > \frac{m}{d}} H^\pm_{m,c}(n, r, n, r) J_k\left( \frac{\pi D}{mc} \right)
\]

We use the following estimate of Bessel functions to estimate \( S_1(n, r) \): \( |J_{\nu}(r)| \leq Ar^{-1/3} \), where \( \nu \geq 0, r \geq 1 \) (cf. [6, Lemma 3.4], the constant \( C \) appearing in the Lemma can be computed to be the constant \( A \) in Theorem 1.2 using [13, p. 333].)
\[ |S_1(n, r)| \leq \frac{2\sqrt{2\pi}}{m^{1/2}} \sum_{1 \leq c \leq \frac{\pi D}{m}} \frac{2^{\omega(c)}(D, c)}{c^{1/2}} |J_{k'} \left( \frac{\pi D}{mc} \right)| \]
\[ \leq \frac{2\sqrt{2}m^{1/2} \pi^{2/3}}{D^{1/3}m^{1/2}} M \left( \frac{\pi D}{m} \right) \sum_{1 \leq c \leq \frac{\pi D}{m}} \frac{(D, c)}{c^{1/6}} \]
\[ \leq \frac{2\sqrt{2}\pi^{2/3}}{D^{1/3}m^{1/6}} M \left( \frac{\pi D}{m} \right) \sum_{d | D, d < \frac{\pi D}{m}} d \]
\[ \leq \frac{2\sqrt{2}m^{1/2} \pi^{2/3}}{D^{1/3}m^{1/6}} M \left( \frac{\pi D}{m} \right) \sigma_0(D) \tag{4.15} \]

\[ |S_2(n, r)| \leq \frac{2\sqrt{2\pi}}{m^{1/2}} \sum_{c > \frac{\pi D}{m}} c^{3/2} |J_{k'} \left( \frac{\pi D}{mc} \right)| \]
\[ \leq \frac{2\sqrt{2}m^{1/2} \pi^{2/3}}{\Gamma(k' + 1)m^{1/2}} \sum_{c > \frac{\pi D}{m}} c^{3/2} \left( \frac{\pi D}{mc} \right)^{3/2+2} \]
\[ \leq \frac{2\sqrt{2}m^{1/2} \pi^{2/3}}{\Gamma(k' + 1)m^{1/2}} \sum_{c > \frac{\pi D}{m}} \frac{1}{c^2} \leq \frac{2\sqrt{2}\pi^{13/2}D^{7/2}}{6 \Gamma(k' + 1)m^4} \tag{4.16} \]

From the bound given in Theorem 1.2, it follows from estimates (4.15) and (4.16) that \( S_1 \) and \( S_2 \) are both less than \( \frac{1}{2} \) in absolute value. Finally, from the expression of the \((n, r)\)-th Fourier coefficient of \( P_{n,r}^{k,m} \) given in Proposition 2.1, we get the Theorem. \( \square \)

5. Further results

Recall the one dimensional Kloosterman sum for a positive integer \( c \),

\[ S(r, m; c) = \sum_{h=1 \atop (h, c)=1}^{c} e_c(rh + mh'), \text{ where } hh' \equiv 1 \pmod{c} \tag{5.1} \]

It is well known that (see [11, §3] for example) the following relation holds for a prime \( p \):

\[ S(rp^\rho, mp^\mu; c) = S(r, mp^{\rho+\mu}; c) + pS(rp^{\rho-1}, mp^{\mu-1}; c/p), \text{ where } p | c, p \nmid r, p \nmid m (\rho, \mu \geq 1). \tag{5.2} \]

**Definition 5.1.** We let

\[ K_{m,c}(n, r, n', r') = \sum_{x(c), y(c)^*} e_c \left( (m[x] + rx + n)y + n'y + r'x \right) (x \in \mathbb{Z}^g/c\mathbb{Z}^9, r \in \mathbb{Z}^9) \tag{5.3} \]

\[ = e^{\theta/2+1} e_{2c} \left( -r'm^{-1}r^t \right) H_{m,c}(n, r, n', r') \tag{5.4} \]
Lemma 5.2. Let \( p \) be an odd prime such that \( p \mid (c, m, r, r') \), \( p \nmid n, p \nmid n' \). Then the following identity holds:

\[
K_{mp^e,c}(p^m n, p^r n', r') = K_{mp^{e+\mu},c}(p^{\mu m} n, p^{\mu r} n', r') + p^2 K_{mp^{e-1},c/p}(p^{\mu-1} n, p^{\mu-1} r, p^{\mu-1} n', r'/p)
\]

Proof. The proof follows by noting that,

\[
K_m(c, n, r, r') = \sum_{x \pmod{c}} e_c(r' x) S(n', m[x] + r x + n; c),
\]

from which the L.H.S. and the first term of the R.H.S. in (5.5) are taken care of by summing both sides of the equation (5.2) with appropriate arguments over \( x \pmod{c} \). For the last term, we split the summation in (5.6) (replacing \( m, n, r, n' ; c \)) by \( (p^{\mu-1} m, p^{\mu-1} n, p^{\mu-1} r, p^{\mu-1} n'; \frac{c}{p}) \) respectively as \( x = \frac{c}{p} x_1 + x_2 \), where \( x_1 \) (resp.) \( x_2 \) range over \( \mathbb{Z} / p \mathbb{Z} \) (resp.) \( \mathbb{Z} / \frac{c}{p} \mathbb{Z} \). We have

\[
\sum_{x \pmod{c}} e_c(r' x) S(p^{\mu-1} n', p^{\mu-1}(m[x] + r x + n); c/p)
\]

\[
= \sum_{x_1, x_2} e_c(r' (c/p x_1 + x_2)) S(p^{\mu-1} n', (c/p x_1 + x_2)^m(c/p x_1 + x_2) + r(c/p x_1 + x_2) + n; c/p)
\]

\[
= \sum_{x_1} e_p(r' c/p x_1) \sum_{x_2} e_{c/p}(r' / p x_2) S(p^{\mu-1} n', p^{\mu-1}(m[x] + r x_2 + n); c/p)
\]

\[
= pK_{mp^{e-1},c/p}(p^{\mu-1} n, p^{\mu-1} r, p^{\mu-1} n', r'/p),
\]

Therefore using (5.2), the lemma follows.

Proof of Theorem 1.3: From Lemma 5.2 we easily deduce that under the conditions of the lemma, in particular, when \( p \mid c \),

\[
H_{mp^e,c}(p^m n, p^r n', r') = H_{mp^{e+\mu},c}(p^{\mu m} n, p^{\mu r} n', r')
\]

\[
+ p^{\frac{1}{2} + 1} H_{mp^{e-1},p}(p^{\mu-1} n, p^{\mu-1} r, p^{\mu-1} n', r'/p).
\]

In the case \( p \nmid c \), we note that we have the equality from the definition,

\[
H_{mp^e,c}(p^m n, p^r n', r') = H_{mp^{e+\mu},c}(p^{\mu m} n, p^{\mu r} n', r')
\]

We sum equation (5.7) over \( c \geq 1 \) such that \( p \mid c \), equation (5.8) over all \( c \geq 1 \) and add them. Gathering all of above and noting that \( \frac{2\pi \sqrt{DD}}{\det(2m) c} \) is the same in all the three sums (putting \( \rho = \mu \) and \( n' = n, r' = p^\mu r \)), we get positive constants \( \alpha_1 \) and \( \alpha_2 \), such that

\[
c^{k,p^e m}(p^m n, p^r n') = \alpha_1 c^{k,p^{e+\mu} m}(p^{2m} n, p^{2r} n; n, p^r) + \alpha_2 c^{k,p^{e-1} m}(p^{\mu-1} n, p^{\mu-1} r)
\]

(\text{where we have used the notation } c^{k,p^e m}(n, r) := c^{k,m}(n, r) = c^{k,m}(n, r)). This immediately implies (1.1) and thus completes the proof of Theorem 1.3.

Remark 5.3. The constants \( \alpha_1, \alpha_2 \) in the above proof can be determined explicitly and may give a better result in the same vein as Theorem 3.4 (see [11]).
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