AUTOMORPHISMS OF THE SEMIGROUP OF ENDOMORPHISMS OF FREE ASSOCIATIVE ALGEBRAS

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Abstract. Let $A = A(x_1, \ldots, x_n)$ be a free associative algebra in the variety of associative algebras $A$ freely generated over $K$ by a set $X = \{x_1, \ldots, x_n\}$, $End A$ be the semigroup of endomorphisms of $A$, and $Aut End A$ be the group of automorphisms of the semigroup $End A$. We investigate the structure of the group $Aut End A$ and $Aut A^0$, where $A^0$ is the category of finitely generated free algebras from $A$. We prove that the group $Aut End A$ is generated by semi-inner and mirror automorphisms of $End F$ and the group $Aut A^0$ is generated by semi-inner and mirror automorphisms of the category $A^0$. This result solves an open Problem formulated in [14].

1. Introduction

Let $\Theta$ be a variety of linear algebras over a commutative-associative ring $K$ and $F = F(X)$ be a free algebra from $\Theta$ generated by a finite set $X$. Here $X$ is supposed to be a subset of some infinite universe $X^0$. Let $G$ be an algebra from $\Theta$ and $K_\Theta(G)$ be the category of algebraic sets over $G$. Here and below we refer to [15, 16] for Universal Algebraic Geometry (UAG) definitions used in our work.

The category $K_\Theta(G)$ can be considered from the point of view of the possibility to solve systems of equations in the algebra $G$. Algebras $G_1$ and $G_2$ from $\Theta$ are categorically equivalent if the categories $K_\Theta(G_1)$ and $K_\Theta(G_2)$ are correctly isomorphic. Algebras $G_1$ and $G_2$ are geometrically equivalent if

$$T''_{G_1} = T''_{G_2}$$

holds for all finite sets $X$ and for all binary relations $T$ on $F$ and $'$ is Galois correspondence between sets in $Hom(F, G)$ and the binary relations on $F$.

It has been shown in [16] that categorical and geometrical equivalences of algebras are related and their relation is determined by the structure of the group $Aut \Theta^0$, where $\Theta^0$ is the category of free finitely generated algebras of $\Theta$. There is a natural connection between a structure of the groups $Aut End F$, $F \in \Theta$, and $Aut \Theta^0$.

Let $\mathcal{A}$ be the variety of associative algebras with (or without) 1, $A = A(x_1, \ldots, x_n)$ be a free associative algebra in $\mathcal{A}$ freely generated over $K$ by a set $X = \{x_1, \ldots, x_n\}$. One of our aim here is to describe the group $Aut End A$ and, as a consequence, to obtain a description of the group $Aut A^0$ for the variety of associative algebras over a field $K$.

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We prove that the group $\text{Aut}_{\text{End}} A$ is generated by semi-inner and mirror automorphisms of $\text{End} A$ and the group $\text{Aut} A^\circ$ is generated by semi-inner and mirror automorphisms of the category $A^\circ$.

Earlier, the description of $\text{Aut} A^\circ$ for the variety $A$ of associative algebras over algebraically closed fields has been given in [11] and, over infinite fields, in [3]. Also in the same works, the description of $\text{Aut}_{\text{End}} F(x_1, x_2)$ has been obtained.

Note that a description of the groups $\text{Aut}_{\text{End}} F$, $F \in \Theta$, and $\text{Aut} \Theta^\circ$ for some other varieties $\Theta$ has been given in [2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 18].

2. Automorphisms of the semigroup $\text{End} F$ and of the category $\Theta^0$

We recall the basic definitions we use in the case of the variety $A$ of associative algebras over a field $K$.

Let $F = F(x_1, ..., x_n)$ be a finitely generated free algebra of a variety $\Theta$ of linear algebras over $K$ generated by a set $X = \{x_1, ..., x_n\}$.

Definition 2.1. [2] An automorphism $\Phi$ of the semigroup $\text{End} F$ of endomorphisms of $F$ is called quasi-inner if there exists a bijection $s : F \to F$ such that $\Phi(\nu) = s\nu s^{-1}$, for any $\nu \in \text{End} F$; $s$ is called adjoint to $\Phi$.

Definition 2.2. [15] A quasi-inner automorphism $\Phi$ of $\text{End} F$ is called semi-inner if its adjoint bijection $s : F \to F$ satisfies the following conditions:

1. $s(a + b) = s(a) + s(b)$,
2. $s(a \cdot b) = s(a) \cdot s(b)$,
3. $s(\alpha a) = \varphi(\alpha)s(a)$,

for all $\alpha \in K$ and $a, b \in F$ and an automorphism $\varphi : K \to K$. If $\varphi$ is the identity automorphism of $K$, we say that $\Phi$ is an inner.

Let $A = A(x_1, ..., x_n)$ be a finitely generated free associative algebra over a field $K$ of the variety $A$. Further, without loss of generality, we assume that associative algebras of $A$ contain $1$.

Definition 2.3. [11] A quasi-inner automorphism $\Phi$ of $\text{End} A$ is called mirror if its adjoint bijection $s : A \to A$ is anti-automorphism of $A$.

Recall the notions of category isomorphism and equivalence [10]. An isomorphism $\varphi : C \to D$ of categories is a functor $\varphi$ from $C$ to $D$ which is a bijection both on objects and morphisms. In other words, there exists a functor $\psi : D \to C$ such that $\psi \varphi = 1_C$ and $\varphi \psi = 1_D$.

Let $\varphi_1$ and $\varphi_2$ be two functors from $C_1$ to $C_2$. A functor isomorphism $s : \varphi_1 \longrightarrow \varphi_2$ is a collection of isomorphisms $s_D : \varphi_1(D) \longrightarrow \varphi_2(D)$ defined for all $D \in \text{Ob} C_1$ such that for every $\nu : D \longrightarrow B$, $\nu \in \text{Mor} C_1$, $B \in \text{Ob} C_1$, holds

$$s_B \cdot \varphi_1(\nu) = \varphi_2(\nu) \cdot s_D,$$

i.e., the following diagram is commutative

$$\begin{array}{ccc}
\varphi_1(D) & \xrightarrow{s_D} & \varphi_2(D) \\
\varphi_1(\nu) & \downarrow & \varphi_2(\nu) \\
\varphi_1(B) & \xrightarrow{s_B} & \varphi_2(B)
\end{array}$$

The isomorphism of functors $\varphi_1$ and $\varphi_2$ is denoted by $\varphi_1 \cong \varphi_2$. 
An equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $\varphi : \mathcal{C} \to \mathcal{D}$ and $\psi : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\psi \varphi \cong 1_{\mathcal{C}}$ and $\varphi \psi \cong 1_{\mathcal{D}}$. If $\mathcal{C} = \mathcal{D}$, then we get the notions of automorphism and autoequivalence of the category $\mathcal{C}$. For every small category $\mathcal{C}$ denote the group of all its automorphisms by $\text{Aut} \mathcal{C}$. We will distinguish the following classes of automorphisms of $\mathcal{C}$.

**Definition 2.4.** An automorphism $\varphi : \mathcal{C} \to \mathcal{C}$ is equinumerous if $\varphi(D) \cong D$ for any object $D \in \text{Ob} \mathcal{C}$; $\varphi$ is stable if $\varphi(D) = D$ for any object $D \in \text{Ob} \mathcal{C}$; and $\varphi$ is inner if $\varphi$ and $1_{\mathcal{C}}$ are naturally isomorphic, i.e., $\varphi \cong 1_{\mathcal{C}}$.

In other words, an automorphism $\varphi$ is inner if for all $D \in \text{Ob} \mathcal{C}$ there exists an isomorphism $s_D : D \to \varphi(D)$ such that

$$\varphi(\nu) = s_{B\nu}s_D^{-1} : \varphi(D) \to \varphi(B)$$

for any morphism $\nu : D \to B$, $B \in \text{Ob} \mathcal{C}$.

Let $\Theta$ be a variety of linear algebras over $K$. Denote by $\Theta^0$ the full subcategory of finitely generated free algebras $F(X), |X| < \infty$, of the variety $\Theta$.

**Definition 2.5.** Let $A_1$ and $A_2$ be algebras from $\Theta$, $\delta$ be an automorphism of $K$ and $\varphi : A_1 \to A_2$ be a ring homomorphism of these algebras. A pair $(\delta, \varphi)$ is called semimomorphism from $A_1$ to $A_2$ if

$$\varphi(\alpha \cdot u) = \alpha^\delta \cdot \varphi(u), \ \forall \alpha \in K, \forall u \in A_1.$$  

Define the notion of a semi-inner automorphism of the category $\Theta^0$.

**Definition 2.6.** An automorphism $\varphi \in \text{Aut} \Theta^0$ is called semi-inner if there exists a family of semi-isomorphisms $(s_{F(X)} = (\delta, \varphi)) : F(X) \to \tilde{\varphi}(F(X)), F(X) \in \text{Ob} \Theta^0$, where $\delta \in \text{Aut} K$ and $\varphi$ is a ring isomorphism from $F(X)$ to $\tilde{\varphi}(F(X))$ such that for any homomorphism $\nu : F(X) \to F(Y)$ the following diagram

$$
\begin{array}{ccc}
F(X) & \xrightarrow{s_{F(X)}} & \tilde{\varphi}(F(X)) \\
\nu \downarrow & & \downarrow \varphi(\nu) \\
F(Y) & \xrightarrow{s_{F(Y)}} & \tilde{\varphi}(F(Y))
\end{array}
$$

is commutative.

Now we define the notion of a mirror automorphism of the category $A^\circ$.

**Definition 2.7.** An automorphism $\varphi \in \text{Aut} A^\circ$ is called mirror if it does not change objects of $A^\circ$ and for every $\nu : A(X) \to A(Y)$, where $A(X), A(Y) \in \text{Ob} A^\circ$, it holds

$$\varphi(\nu) : A(X) \to A(Y)$$

such that $\varphi(\nu)(x) = \delta(\nu(x)), \forall x \in X$,

where $\delta : A(Y) \to A(Y)$ is the mirror automorphism of $A(Y)$.

Further, we will need the following

**Proposition 2.8.** For any equinumerous automorphism $\varphi \in \text{Aut} \mathcal{C}$ there exists a stable automorphism $\varphi_S$ and an inner automorphism $\varphi_I$ of the category $\mathcal{C}$ such that $\varphi = \varphi_S \varphi_I$. 
3. Quasi-inner automorphisms of the semigroup $End F$ for associative and Lie varieties

We will need the standard endomorphisms of free algebra $F = F(x_1, \ldots, x_n)$ of the variety $\Theta$.

**Definition 3.1.** [$\Psi$] Standard endomorphisms of $F$ in the base $X = \{x_1, \ldots, x_n\}$ are the endomorphisms $e_{ij}$ of $F$ which are determined on the free generators $x_k \in X$ by the rule: $e_{ij}(x_k) = \delta_{jk}x_i$, $x_i \in X$, $i, j, k \in [1n]$, $\delta_{jk}$ is the Kronecker delta.

Denote by $S_0$ a subsemigroup of $End F$ generated by $e_{ij}$, $i, j \in [1n]$. Further, we will use the following statements

**Proposition 3.2.** [$\Psi$] Let $\Phi \in Aut End F(X)$. Elements of the semigroup $\Phi(S_0)$ are standard endomorphisms in some base $U = \{u_1, \ldots, u_n\}$ of $F$ if and only if $\Phi$ is a quasi-inner automorphism of $End F$.

The description of quasi-inner automorphisms of $End A(X)$, where $A(X)$ is a free associative algebra with or without 1 over a field $K$, is following

**Proposition 3.3.** [$\Phi$] Let $\Phi \in Aut End A(X)$ be a quasi-inner automorphism of $End A(X)$. Then $\Phi$ is either a semi-inner or a mirror automorphism, or a composition of them.

Let us investigate the images of standard endomorphisms under automorphisms of $End A$. To this end we introduce endomorphisms of rank 1.

**Definition 3.4.** We say that an endomorphism $\varphi$ of $A$ has rank 1, and write this as $rk(\varphi) = 1$, if its image $Im \varphi$ is a commutative subalgebra of $F$.

Note that according to Bergman’s theorem [1], the centralizer of any non-scalar element of $A$ is a polynomial ring in one variable over $K$. Thus, an endomorphism $\varphi$ of $A$ is of rank 1 if and only if $\varphi(A) = K[z]$ for some element $z \in A$.

**Proposition 3.5.** An endomorphism $\varphi$ of $A$ is of rank 1 if and only if there exists a non-zero endomorphism $\psi \in End A$ such that for any $h \in End A$

$$\varphi \circ h \circ \psi = 0$$

**Proof.** Let $\varphi \in End A$ be an endomorphism of rank 1. Let us take the endomorphism $\psi \in End A$ such that

$$\psi(x_1) = [x_1, x_2] \quad \text{and} \quad \psi(x_i) = 0 \quad \text{for all} \ i \neq 1.$$ 

Since $\varphi(A)$ is a commutative subalgebra of $F$, the condition (3.1) is fulfilled for any $h \in End A$.

Conversely, let the condition (3.1) is fulfilled for the endomorphism $\varphi$. Assume, on the contrary, that $Im \varphi$ is not a commutative algebra. Without loss of generality, it can be supposed that $[\varphi(x_1), \varphi(x_2)] \neq 0$, $x_1, x_2 \in X$. Denote by $R = K[\varphi(x_1), \varphi(x_2)]$ a subalgebra of $A$ generated by $\varphi(x_1)$ and $\varphi(x_2)$. It is well known (see [1]) that $R$ is a free non-commutative subalgebra of $A$.

Since $\psi$ is a non-zero endomorphism of $A$, there exists $x_i \in X$ such that $\psi(x_i) \neq 0$. Set $P = P(x_1, \ldots, x_n) = \psi(x_1)$. We wish to show that $P$ is an identity of the algebra $R$. Assume, on the contrary, that there exist elements $z_1, \ldots, z_n \in R$ such that $P(z_1, \ldots, z_n) \neq 0$. Consider sets $\varphi^{-1}(z_i)$, $i \in [1n]$, and choose elements $z_1, \ldots, z_n \in R$ such that $P(z_1, \ldots, z_n) \neq 0$. Consider sets $\varphi^{-1}(z_i)$, $i \in [1n]$, and choose elements $z_1, \ldots, z_n \in R$ such that $P(z_1, \ldots, z_n) \neq 0$. Consider sets $\varphi^{-1}(z_i)$, $i \in [1n]$, and choose elements $z_1, \ldots, z_n \in R$ such that $P(z_1, \ldots, z_n) \neq 0$. Consider sets $\varphi^{-1}(z_i)$, $i \in [1n]$, and choose elements $z_1, \ldots, z_n \in R$ such that $P(z_1, \ldots, z_n) \neq 0$. Consider sets $\varphi^{-1}(z_i)$, $i \in [1n]$, and choose elements
$y_i \in \varphi^{-1}(z_i), i \in [1n]$, from them. We may construct an endomorphism $h$ of $A$ such that $h(x_i) = y_i, i \in [1n]$. Then we have

$0 = \varphi \circ h \circ \psi(x_i) = P(\varphi \circ h(x_1), ..., \varphi \circ h(x_n)) = P(\varphi(y_1), ..., \varphi(y_n)) = P(z_1, ..., z_n)$.

We arrived at a contradiction. Therefore, $P$ is an identity of $R$. Since $R$ is a free non-commutative subalgebra of $A$, it has no non-trivial identities. Thus, $P = 0$. We get a contradiction again. Therefore, $Im \varphi$ is a commutative algebra and Proposition is proved.

It follows directly from this Proposition

**Corollary 3.6.** Let $\Phi \in Aut \ End A$ and $rk (\varphi) = 1$. Then $rk (\Phi(\varphi)) = 1$.

**Definition 3.7.** A set of endomorphisms $E_c = \{e'_{ij} | e'_{ij} \in End A, i, j \in [1n]\}$ of $A$ is called a subbase of $End A$ if

1. $e'_{ij} e'_{km} = \delta_{jk} e'_{im}, \forall i, j, k, m \in [1n]$,
2. $rk (e'_{ij}) = 1, \forall i, j \in [1n]$, i.e., there exist elements $z_{ij} \in A, i, j \in [1n]$, such that $e'_{ij}(A(X)) = K[z_{ij}]$ for all $i, j \in [1n]$.

Further, for simplicity, we write $z_{ii} = z_i, i \in [1n]$.

**Definition 3.8.** We say that a subbase $E_c$ is a base collection of endomorphisms of $A$ (or a base of $End A$, for short) if $Z = \{z_i | z_i \in A, i \in [1n]\}$ is a base of $A$.

**Proposition 3.9.** A subbase of endomorphisms $E_c$ is a base if and only if for any collection of endomorphisms $\alpha_i : A \to A, \forall i \in [1n]$, and any subbase $E_f = \{f''_{ij} | i, j \in [1n]\}$ of $End A$ there exist endomorphisms $\varphi, \psi \in End A$ such that

$\varphi(x_i) = z_i$ and $\psi(z_i) = \alpha_i(y_i)$, for all $i \in [1n]$

where $y_i = y_{ii}, \forall i \in [1n]$. Since $Z = \{z_i | z_i \in A, i \in [1n]\}$ is a base of $A$, the definition of the endomorphism $\psi$ is correct. Now, it is easy to check that the condition (3.2) is fulfilled for the subbase $E_c$. Let us prove that $Z = \{z_i | z_i \in A, i \in [1n]\}$ is a base of $A$. Choosing in (3.2) $\alpha_i = e_{ii}$ and $f''_{ij} = e_{ij}$ for all $i, j \in [1n]$, we obtain

$e_{ii} = \psi \circ e'_{ii} \circ \varphi$,

i.e., $\psi(e'_{ii} (\varphi(x_i))) = x_i$ for all $i \in [1n]$. Denote $t_i = e'_{ii} (\varphi(x_i))$. We have $\psi(t_i) = x_i$.

Since $A$ is Hopfian, the elements $t_i, i \in [1n]$, form a base of $A$. Taking into account the equality $e'_{ii}(A(X)) = K[z_i]$, we obtain $t_i = \chi_i(z_i) \in K[z_i]$.

By Bergman’s theorem we have $z_i = g_i(t_i)$. Thus, $z_i = g_i(\chi_i(z_i))$. Similarly, $t_i = \chi_i(g_i(t_i))$. Therefore, there exists non-zero elements $a_i$ and $b_i$ in $K$ such that $z_i = a_i t_i + b_i, i \in [1n]$. Thus, $Z = \{z_i | z_i \in F, \forall i \in [1n]\}$ is also a base of $A$ as claimed.

Now we deduce

**Corollary 3.10.** Let $\Phi \in Aut \ End A$. Then $C = \{\Phi(e_{ij}) | i, j \in [1n]\}$ forms a base collection of endomorphisms of $A$. 
Proof. Since $\Phi(e_{ij})\Phi(e_{km}) = \delta_{jk}\Phi(e_{im})$ and by Corollary 3.6 $rk(\Phi(e_{ij})) = 1$, the set $C$ is a subbase of $End\, A$. It is evident that the condition (3.2) is fulfilled for the subbase $C$. By Proposition 3.9 $C$ is a base of $End\, A$. □

**Lemma 3.11.** Let $B_c = \{e_{ij}^c \mid e_{ij}^c \in End\, A, i, j \in [1n]\}$ be a base collection of endomorphisms of $End\, A$. Then there exists a base $S = \{s_k \mid s_k \in A, k \in [1n]\}$ of $A$ such that the endomorphisms $\Phi(e_{ij}^c)$ are standard endomorphisms in $S$.

Proof. Since $(e_{ii}^c)^2 = e_{ii}^c$, we have $e_{ii}^c(z_i) = z_i, i \in [1n]$. The equality $e_{ij}^c e_{ij}^c z_j = e_{ij}^c z_j$ implies the existence of a polynomial $f_j(z_i) \in K[z_i]$ such that $e_{ij}^c z_j = f_j(z_i)$. Similarly, there exists a polynomial $g_i(z_j) \in K[z_j]$ such that $e_{ji}^c z_i = g_i(z_j)$. We have

$$z_j = e_{jj}^c z_j = e_{jj}^c e_{ij}^c z_j = e_{jj}^c(f_j(z_i)) = f_j(g_i(z_j)) \text{ for all } i, j \in [1n].$$

and, in similar way, $z_i = g_i(f_j(z_i))$ for all $i, j \in [1n]$. Thus $f_j$ and $g_i$ are linear polynomials over $K$ in variables $z_i$ and $z_j$, respectively. Therefore,

$$e_{ij}^c z_j = a_j z_i + b_j, \quad a_i, b_i \in K \quad \text{and} \quad a_i \neq 0.$$  

Note that $e_{ij}^c z_k = e_{ij}^c e_{kk}^c z_k = 0$ if $k \neq j$. Now we have for $i \neq j$

$$0 = e_{ij}^c z_j = e_{ij}^c(a_i z_i + b_j) = e_{ij}^c(b_j) = b_j,$$

i.e., $e_{ij}^c z_j = a_j z_i, a_j \neq 0$. Let $V = Span(z_1, ..., z_n)$. Then $V$ is the vector space over $K$ with a basis $Z = \{z_k \mid z_k \in A, k \in [1n]\}$ and $e_{ij}^c, i, j \in [1n]$, are linear operators on $V$. Set

$$S = \{s_i = e_{1i}^c z_1 \mid z_1 \in Z, i \geq 1\}.$$  

Since $s_i = a_1 z_i, a_i \neq 0, i \in [1n]$, we have that $S$ is a base of $A$. In this base we obtain $e_{ij}^c s_k = \delta_{jk} s_i$, $i, j, k \in [1n]$. The proof is complete. □

4. Structure of automorphisms of the semigroup $End\, F$ for associative and Lie varieties

Now we give the description of the groups $Aut\, End\, F$ and $Aut\, A^\circ$.

**Theorem 4.1.** The group $Aut\, End\, A$ is generated by semi-inner and mirror automorphisms of $End\, A$.

Proof. By Corollary 3.10 the set of endomorphisms $C = \{\Phi(e_{ij}) \mid \forall i \in [1n]\}$ is a base collection of endomorphisms of $A$. By Lemma 3.11 there exists a base $S = \{s_k \mid s_k \in A, k \in [1n]\}$ such that the endomorphisms $\Phi(e_{ij})$ are standard endomorphisms in $S$. According to Proposition 3.12 we obtain that $\Phi$ is quasi-inner. By virtue of Proposition 4.3 the group $Aut\, End\, A$ is generated by semi-inner and mirror automorphisms of $End\, A$ as claimed. □

Using Theorem 4.1 we prove

**Theorem 4.2.** The group $Aut\, A^\circ$ of automorphisms of the category $A^\circ$ is generated by semi-inner and mirror automorphisms of the category $A^\circ$.

Proof. Let $\varphi \in Aut\, A^\circ$. It is clear that $\varphi$ is an equinumerous automorphism. By Proposition 2.8 $\varphi$ can be represented as the composition of a stable automorphism $\varphi_S$ and an inner automorphism $\varphi_I$. Since a stable automorphism does not change free algebras from $A^\circ$, we obtain that $\varphi_S \in Aut\, End\, A(x_1, ..., x_n)$. By Theorem 4.1 $\varphi_S$ is generated by semi-inner and mirror automorphisms of $End\, A$. Using this
fact and Reduction Theorem \[7, 13\], we obtain that the group $\text{Aut} \mathcal{A}^\circ$ generated by semi-inner and mirror automorphisms of the category $\mathcal{A}^\circ$. This ends the proof. □

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