ALMOST EVERYWHERE CONVERGENCE AND POLYNOMIALS

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Abstract. Denote by $\Gamma$ the set of pointwise good sequences. Those are sequences of real numbers $(a_k)$ such that for any measure preserving flow $(U_t)_{t \in \mathbb{R}}$ on a probability space and for any $f \in L^\infty$, the averages $\frac{1}{n} \sum_{k=1}^{n} f(U_{a_k} x)$ converge almost everywhere.

We prove the following two results.

(1) If $f : (0, \infty) \to \mathbb{R}$ is continuous and if $\left( f(ku + v) \right)_{k \geq 1} \in \Gamma$ for all $u, v > 0$, then $f$ is a polynomial on some subinterval $J \subset (0, \infty)$ of positive length.

(2) If $f : [0, \infty) \to \mathbb{R}$ is real analytic and if $\left( f(ku) \right)_{k \geq 1} \in \Gamma$ for all $u > 0$, then $f$ is a polynomial on the whole domain $[0, \infty)$.

These results can be viewed as converses of Bourgain’s polynomial ergodic theorem which claims that every polynomial sequence lies in $\Gamma$.

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1. Introduction

For $1 \leq p \leq \infty$, a sequence of real numbers $(a_k)_{k \geq 1}$ is said to be $p$-good pointwise if for any measure preserving flow $(U_t)_{t \in \mathbb{R}}$ on a probability space $(X, \mathcal{B}, \mu)$ and for any $f \in L^p(X, \mu)$, the averages $\frac{1}{n} \sum_{k=1}^{n} f(U_{a_k} x)$ converge almost everywhere.

Denote by $\Gamma_p$ the set of $p$-good pointwise sequences and set $\Gamma = \Gamma_\infty$.

J. Bourgain proved in a series of papers (see [?] for the proof and further references) that polynomial sequences lie in $\Gamma_p$, for all $p > 1$. Bourgain, in fact, formulated his result for $\mathbb{Z}^d$ actions instead of $\mathbb{R}$ actions. To see the idea how we can translate the $\mathbb{Z}^d$ results to results on a flow, consider the example $f(x) = \sqrt{2}x^2 + x$. The transformation $T_{u,v} = U_{\sqrt{2}u+v}^\perp$ is a $\mathbb{Z}^2$ action,

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and $U_{f(n)} = T_{n^2,n}$, hence Bourgain’s result applied to $T$ gives the result for the flow $U_t$.

For a simpler proof (in the case $p = 2$) of Bourgain’s result, and for references see [?]. It is a famous open problem whether or not $(k^2) \in \Gamma_1$.

In the paper we prove the following two theorems. (Each can be viewed as a converse of J. Bourgain’s result mentioned above).

Denote $\mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\} = (0, \infty)$, and $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\} = [0, \infty)$.

**Theorem A.** Let $f: \mathbb{R}^+ \to \mathbb{R}$ be a real analytic function such that the set

$$U = \{u > 0 : (f(ku))_{k \geq 1} \in \Gamma\}, \quad \text{where} \quad \Gamma \overset{\text{def}}{=} \Gamma_\infty,$$

is uncountable. Then $f(x)$ must be a polynomial.

**Theorem B.** Let $U, V \subset \mathbb{R}^+$ be two non-empty open subsets. Let $f: \mathbb{R}^+ \to \mathbb{R}$ be a real continuous function such that $(f(ku + v))_{k \geq 1} \in \Gamma$, for all $u \in U, v \in V$. Then there exists a subinterval $J \subset \mathbb{R}^+$ (of positive length) such that $f$ restricted to $J$ is a polynomial.

We remark that, in general, the conclusion of Theorem B cannot be that $f(x)$ agrees with a polynomial on a halfline, say. Indeed, we have

**Proposition 1.1.** The function

$$f(x) = \text{dist}(x, \mathbb{Z}) = \min(x - [x], [x] + 1 - x), \quad x \in \mathbb{R},$$

satisfies the conditions of Theorem A with $U = V = \mathbb{R}^+$.

The above proposition shows that $f(x)$ satisfying the conditions of Theorem B (even with $U = V = \mathbb{R}^+$) may be a periodic non-constant function, so definitely not a polynomial on large subintervals of $\mathbb{R}^+$.

The claim of Proposition 1.1 follows from the results in [?].

Note that our proof of Theorem A depends on the fact that $f$ is analytic at 0. (See Remark 2.1 in the next section).

The following result (first proved in [?]) plays a central role in the proofs of both theorems.

**Lemma 1.2 ([?]).** Any sequence $(a_k) \in \Gamma$ must be linearly dependent over the field $\mathbb{Q}$ of rational constants.

For a simpler proof of this lemma (avoiding the use of Bourgain’s entropy method) we refer the reader to [?].

2. Proof of Theorem A

**Proof of Theorem A.** In view of Lemma 1.2, for every $x \in U$, there are an integer $n(x) \geq 1$ and an $n(x)$-tuple of rationals

$$\mathbf{q}(x) = (q_1(x), \ldots, q_{n(x)}(x)) \in \mathbb{Q}^{n(x)}, \quad q_{n(x)} = 1,$$
such that
\[ \sum_{k=1}^{n(x)} q_k(x) f(kx) = 0. \]

Since the set of possible pairs \((n(x), q(x))\) is countable while the set of \(x \in U\) is uncountable, there is a pair \((n, q)\) corresponding to an uncountable set \(U' \subset U\) of \(x\):
\[ (n(x), q(x)) = (n, q), \quad \text{for } x \in U'. \]

Since \(f(x)\) is analytic, the identity
\[ \sum_{k=1}^{n} q_k f(kx) = 0, \quad q_n = 1, \]
extends (from \(x \in U'\)) to all \(x \in [0, \infty)\) (the set \(U'\) being uncountable has an accumulation point).

Set \(f(x) = \sum_{r \geq 0} c_r x^r\) to be the series expansion of \(f\) at 0. Then from the identity
\[ 0 = \sum_{k=1}^{n} q_k f(kx) = \sum_{r \geq 0} \left( \sum_{k=1}^{n} q_k k^r \right) c_r x^r \]
one deduces that, for all integers \(r \geq 0\), either \(c_r = 0\) or \(\sum_{k=1}^{n} q_k k^r = 0\). It follows that
\[ \sum_{k=1}^{n} q_k k^r = 0, \]
for all \(r\) in the set \(K = \{ r \in \mathbb{Z}^+ : c_r \neq 0 \}\).

We observe that
\[ \lim_{r \to +\infty} \sum_{k=1}^{n} q_k k^r = +\infty \]
(the last term \(q_n n^r = n^r\) is dominant in the sum).

We conclude that the set \(K = \{ r \in \mathbb{Z}^+ : c_r \neq 0 \}\) is finite, and \(f(x)\) is a polynomial. \(\square\)

Remark 2.1. We don’t know whether Theorem A holds if the domain of \(f\) is assumed to be \((0, \infty)\) rather than \([0, \infty)\). The recurrence relation (2.1) still holds in this setting. The conclusion of Theorem can be derived under the assumptions that \(f\) is analytic on \((0, \infty)\) and that either 0 or \(\infty\) is an isolated singularity of the analytic extension of \(f\).

3. Proof of Theorem B

Proof of Theorem B In view of Lemma 1.2 for every \(u \in U\) and \(v \in V\) there is an integer \(n(u, v) \geq 1\) and an \(n(u, v)\)-tuple of rationals
\[ q(u, v) = (q_1(u, v), q_2(u, v), \ldots, q_{n(u, v)}(u, v)) \in \mathbb{Q}^{n(u, v)}, \quad q(u, v) \neq 0, \]
such that
\[ \sum_{k=1}^{n} q_k(u, v) f(ku + v) = 0. \]

For every integer \( n \geq 1 \) and an \( n \)-tuple of rationals \( q \in \mathbb{Q}^n \), denote
\[ K(n, q) = \{(u, v) \in U \times V \mid n(u, v) = n \text{ and } q(u, v) = q\}. \]

Since \( U \times V \) is a countable union of its closed subsets \( K(n, q) \), by the Baire category theorem there is a choice of \( n \) and \( q = (q_1, q_2, \ldots, q_n) \in \mathbb{Q}^n \), with not all \( q_k = 0 \), such that the set \( K(n, q) \) contains a non-empty interior, say the set \( U' \times V' \) where \( U' \subset U \) and \( V' \subset V \) are non-empty open subintervals of \((0, \infty)\). We conclude that
\[ (3.1) \quad \sum_{k=1}^{n} q_k f(ku + v) = 0, \quad \text{for } u \in U', \ v \in V'. \]

**Definition 3.1.** Let \( U \subset \mathbb{R} \) be an open set, and let \( X \subset U \) be a finite subset, \( \text{card}(X) \geq 1 \). Let \( f : U \to \mathbb{R} \), \( g : X \to \mathbb{R} \) and \( h : X \to \mathbb{R} \) be three functions such that \( f \) is continuous and \( g \) is injective. The quintuple \((f, g, h, X, U)\) is called balanced if
\[ (3.2) \quad \sum_{x \in X} h(x) f(x + s + t g(x)) = 0, \]
provided that \(|t|\) and \(|s|\) are small enough.

**Example 3.2.** Let \( U', V', n \geq 1 \) and \( q = (q_1, q_2, \ldots, q_n) \in \mathbb{Q}^n \) be such as described in the paragraph preceding (3.1). Pick \( u_0 \in U', v_0 \in V' \) and let \( X \) be the set \( X = \{x_k \mid 1 \leq k \leq n\} \) where \( x_k = ku_0 + v_0 \). Define \( g, h : X \to \mathbb{R} \) as follows: \( g(x_k) = k \), \( h(x_k) = q_k \). With these choices, it follows from (3.1) that the quintuple \((f, g, h, X, U)\) is balanced.

Indeed, by setting \( u = u_0 + t \) and \( v = v_0 + s \), we obtain
\[
\sum_{x \in X} h(x) f(x + s + t g(x)) = \sum_{k=1}^{n} h(x_k) f(x_k + s + t g(x_k)) = \sum_{k=1}^{n} q_k f(x_k + s + t k) = \sum_{k=1}^{n} q_k f(x_k + s + t k) = \sum_{k=1}^{n} q_k f((v_0 + s) + (u_0 + t)k) = \sum_{k=1}^{n} q_k f(v + uk) = 0,
\]
since \( u \in U' \), \( v \in V' \) if both \( t, s \) are close to 0.

Now the claim of Theorem 3.3 follows from the following result.

**Proposition 3.3.** Let \((f, g, h, X, U)\) be a balanced quintuple in the sense of Definition 3.1 with \( \text{card}(X) = n \geq 1 \). Let \( x_0 \in X \) be such that \( h(x_0) \neq 0 \). Then \( f(x) \) is a polynomial of degree \( \leq n - 2 \) in a neighborhood of \( x_0 \).

By definition, the zero constant is a polynomial of degree \(-1\). \( \square \)
4. Proof of Proposition 3.3: smooth case

Proof of Proposition 3.3: smooth case. First we provide proof under the additional assumption that $f \in C^\infty(U)$. Without loss of generality, we may assume that $0 \notin h(X)$. The proof is by induction on $n = \text{card}(X)$.

If $n = 1$, then by setting $t = 0$ we get $f(x_0 + s) = 0$, for all $|s|$ small enough, i.e. $f(x)$ vanishes in a neighborhood of $x_0$. The case $n = 1$ is validated.

Next we assume that $n \geq 1$, that $X = \{x_0, x_1, \ldots, x_n\}$ and that the claim of proposition is validated if $\text{card}(X) \leq n$. Observe that for an arbitrary real constant $c$ a quintuple $(f, g, h, X, U)$ is balanced if and only if a quintuple $(f, g + c, h, X, U)$ is. This is because $f(x + s + t g(x)) = f(x + s - ct + (g(x) + c)t)$, and the pair $(s, t)$ is close to $(0, 0)$ (in the $\mathbb{R}^2$ metric) if and only if $(s - ct, t)$ is.

Thus we may assume that $g(x_n) = 0$ (after replacing $g(x)$ by $g(x) - g(x_n)$). Thus

$$0 = \sum_{x \in X} h(x)f(x + s + t g(x)) = \sum_{k=0}^{n-1} h(x_k)f(x_k + s + t g(x_k)) + h(x_n)f(x_n + s).$$

Taking the partial derivative $\frac{\partial}{\partial t}$ we get

$$\sum_{k=0}^{n-1} h(x_k)g(x_k)f'(x_k + s + t g(x_k)) = 0$$

and conclude that the quintuple $(f', g|_{X'}, h g|_{X'}, X', U)$ is balanced where $X' = \{x_0, \ldots, x_{n-1}\} = X \setminus \{x_n\}$. Moreover, $g|_{X'}$ is injective, and $0 \notin h g(X')$.

By the induction hypothesis, for any $x' \in X'$, the derivative $f'$ is a polynomial of degree $\leq n - 2$ in some neighborhood of $x'$. It follows that $f(x)$ is a polynomial of degree $\leq n - 1 = (n + 1) - 2$ in a neighborhood of $x_0$. This completes the proof of Proposition 3.3 under the added assumption that $f \in C^\infty(\mathbb{R}^+)$. \hfill \Box

5. Proof of Proposition 3.3: continuous case

Proof of Proposition 3.3: continuous case. Since $(f, g, h, X, U)$ is a balanced quintuple, there exists an $\epsilon > 0$ such that (3.2) holds provided that $|s|, |t| < \epsilon$. In the preceding section we proved that, under the additional condition that $f \in C^\infty(\mathbb{R}^+)$, there exists a neighborhood $W$ of a point $x_0$ such that $f|_{W}$ is a polynomial of degree $\leq n - 2$.

Our proof provides slightly more: This neighborhood $W$ depends only on $\epsilon, g, h, X$ and $U$ but not on $f \in C^\infty(\mathbb{R}^+)$. The observation will be used in what follows.

Now we move to the general case of $f \in C(\mathbb{R}^+)$ (rather than $f \in C^\infty(\mathbb{R}^+)$).
Fix any function $\phi \in C^\infty(\mathbb{R})$ which satisfies $\int_{\mathbb{R}} \phi(t) \, dt = 1$ and vanishes outside the interval $[-1, 1]$. For integers $m \geq 1$, denote

$$
\phi_m(x) = m\phi(mx), \quad f_m(x) \overset{\text{def}}{=} (\phi_m \circ f)(x) = \int_{\mathbb{R}} \phi_m(t)f(x-t) \, dt.
$$

The sequence of kernels $\phi_m(x)$ is known to converge the $\delta$-function at $0$ in the sense that

$$
\lim_{m \to \infty} f_m(x) = f(x), \quad \text{for all } x \in U.
$$

Note that all $f_m(x) \in C^\infty(U_m)$ where

$$
U_m \overset{\text{def}}{=} \{ x \in U \mid \text{dist}(x, \partial U) > \frac{1}{m} \},
$$

and dist$(x, \partial U)$ stands for the distance between $x$ and the boundary of $U \subset \mathbb{R}$.

Let $V$ be a neighborhood of the set $X$ such that its closure $\overline{V}$ is compact and is contained in $U$. Then the pointwise convergence in (5.1) is uniform on $V$, and in fact $f_m \in C^\infty(V)$ holds for all $m$ large enough. It is also clear that for $m$ large enough, $(f_m, g, h, X, V)$ forms a balanced quintuple because $(f, g, h, X, V)$ does; moreover, there exists $\epsilon' > 0$ such that

$$
\sum_{x \in X} h(x) f_m(x + s + tg(x)) = 0
$$

holds simultaneously for all large $m$ (say, $m > m_0$) and $s, t \in (-\epsilon', \epsilon')$. It follows that there exists a neighborhood $W \subset V$ of a point $x_0$ such that each function $f_m$ is a polynomial of degree $\leq n - 2$ in it. (Here we use the observation made in the second paragraph of this section). In view of the uniform convergence (5.1), $f|_W$ is also a polynomial of degree $\leq n - 2$, completing the proof of Proposition 3.3. □

6. Concluding remarks

The following is a slightly more general version of Theorem $A$.

**Theorem $A'$.** Let $(r_k)_{k \geq 1}$ be a sequence of distinct positive numbers, let $f: \mathbb{R}^+ \to \mathbb{R}$ be a real analytic function such that the set

$$
U = \{ u > 0 : (f(r_ku))_{k \geq 1} \in \Gamma \}, \quad \text{where } \Gamma \overset{\text{def}}{=} \Gamma_{\infty},
$$

is uncountable. Then $f(x)$ must be a polynomial.

The proof of Theorem $A'$ is very similar to the proof of Theorem $A$.

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