A brief note on the compatibility of quantum instruments: A conceptual problem

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The compatibility of quantum instruments is defined as the existence of a joint instrument through the implementation of which one can simultaneously implement the individual instruments by doing classical post-processing. In this paper, we discuss the concept of parallel compatibility of instruments and claim that the traditional definition of compatibility of instruments is conceptually incomplete and provide a re-definition of the compatibility of quantum instruments. We discuss the proper reason for the re-definition. We show that the modified definition of the compatibility of quantum instruments can capture the idea of observable-observable compatibility, observable-channel compatibility and channel-channel compatibility.

I. INTRODUCTION

Incompatibility is a basic feature of quantum mechanics \cite{1}. A set of quantum devices is compatible if those devices are simultaneously implementable given a single input state. Incompatibility of observables is necessary for the demonstration of several quantum information theoretic tasks or to get an advantage in different quantum information theoretic tasks \cite{2,3}. On the other hand, from the incompatibility of quantum channel, one can arrive at no-cloning theorem \cite{4,5}. Similarly, incompatibility between the identity channel and any non-trivial observable is consistent with the uncertainty principal \cite{6}.

In this paper, we study the compatibility of quantum instruments. We investigate whether the traditional definition of the compatibility of quantum instruments is conceptually complete. The rest of this paper is organised as follows. In Sec. \textbf{II} we discuss the preliminaries. In Sec. \textbf{III A} we discuss the main results. More specifically, in Sec. \textbf{III A} we discuss the concept of parallel compatibility of instruments and show that this concept is different than the traditional definition of compatibility of instruments. In Sec. \textbf{III B} we claim that the traditional definition of compatibility of instruments is conceptually incomplete and provide a re-definition of the compatibility of quantum instruments and we discuss the proper reason for the re-definition. In Sec. \textbf{III C} we show that the modified definition of compatibility of quantum instruments can capture the idea of observable-observable compatibility, observable-channel compatibility and channel-channel compatibility. Finally, in Sec. \textbf{IV} we summarize our work and discuss the future outlook.

II. PRELIMINARIES

In this section, we discuss the preliminaries.

A. Observables

In the quantum mechanics, an observable $A$ which is acting on Hilbert space $\mathcal{H}$, is defined by a set of positive Hermitian matrices $A = \{A(x)\}$ such that $\sum_{x} A(x) = 1$ where $I$ is a $d \times d$ identity matrix where $d$ is the dimension of the Hilbert space $\mathcal{H}$. The probability of obtaining a outcome $x$ is $p_A(x) = \text{tr}[\rho A(x)]$ for any quantum state $\rho \in \mathcal{S}(\mathcal{H})$ where $\mathcal{S}(\mathcal{H})$ is the state space (i.e., the set of all density matrices). We denote the outcome set of an observable $A$ as $\Omega_A$.

B. Quantum channels

A quantum channel $\Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K})$ is a completely positive trace preserving map (CPTP map) \cite{6,10}. Any quantum channel $\Lambda$ can be written as $\Lambda(\rho) = \sum_{i} K_i \rho K_i^\dagger$ where $K_i$'s are the Kraus operators and satisfies $\sum_{i} K_i^\dagger K_i = I$. This is known as the Kraus representation of $\Lambda$.

Suppose $\Phi : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K})$ is a completely positive map (CP map). Then if $\Phi^* : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$ is the dual (i.e., in Heisenberg picture) of the CP map $\Phi$ then $\text{tr}[\Phi(\rho)X] = \text{tr}[\rho \Phi^*(X)]$ for all quantum state $\rho \in \mathcal{S}(\mathcal{H})$ and operator $X \in \mathcal{L}(\mathcal{K})$ where $\mathcal{L}(\mathcal{K})$ is the set of the linear operators in the Hilbert space $\mathcal{K}$ and $\mathcal{L}^+(\mathcal{K})$ is the set of the positive linear operators in the Hilbert space $\mathcal{K}$. Then clearly, for any CPTP map $\Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K})$, $\Lambda^*(\mathbb{I}_K) = \mathbb{I}_\mathcal{H}$ \cite{10}. If $\Gamma : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K}_1 \otimes \mathcal{K}_2)$ is a CP map and $\Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K}_1)$ is a CP map such that $\Lambda(\rho) = \text{tr}_{\mathcal{K}_2}[\Gamma(\rho)]$ for all quantum state $\rho$ then $\text{tr}_{\mathcal{K}_1} [\Lambda(\rho)] = \text{tr}[\Gamma(\rho)]$ for all quantum state $\rho$ which implies $\text{tr}_{\mathcal{K}_1} [\rho \Lambda^*(\mathbb{I}_{\mathcal{K}_1})] = \text{tr}[\rho \Gamma^*(\mathbb{I}_{\mathcal{K}_1} \otimes \mathbb{I}_{\mathcal{K}_2})]$ for all quantum state $\rho$. Therefore, $\Lambda^*(\mathbb{I}_{\mathcal{K}_1}) = \Gamma^*(\mathbb{I}_{\mathcal{K}_1} \otimes \mathbb{I}_{\mathcal{K}_2})$.

C. Quantum Instruments

A quantum instrument $\mathcal{I}$ is defined as a set of completely positive maps (CP maps) $\{\Phi_x : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K})\}$.
such that \( \sum_x \Phi_x = \Phi^T \) is a CPTP map \([10]\). An observable \( A \) can be measured through implementing an instrument if \( \text{tr}[\Phi_x(\rho)] = \text{tr}[\rho A(x)] \). Such instruments are known as \( A \)-compatible instrument. In this case, if \( \Phi_x(\rho) = \sum_j K_{xj}\rho K_{xj}^\dagger \), then \( \sum_j K_{xj}^\dagger K_{xj} = A(x) \) and therefore, clearly \( \sum_j K_{xj}^\dagger K_{xj} = I \). A lot of works have been already done to understand the different properties of the quantum instruments \([11][23]\).

D. Three kinds of compatibility in quantum mechanics

Depending on the types of the quantum devices, there are mainly three kinds of compatibility in quantum mechanics \([3]\). Those are-

1. **Observable-Observable compatibility**: Two observables \( A = \{A(x)\} \) and \( B = \{B(y)\} \) are compatible if there exists an observable \( \mathcal{G} = \{G(x,y)\} \) with outcome set \( \Omega_G = \Omega_A \times \Omega_B \) such that

\[
A(x) = \sum_y G(x,y); \quad B(y) = \sum_x G(x,y)
\]

for all \( x \in \Omega_A \) and \( y \in \Omega_B \). Through measuring the observable \( \mathcal{G} \), one can simultaneously measure both of the observables \( A \) and \( B \).

2. **Observable-Channel compatibility**: An observable \( A = \{A(x)\} \) acting on the Hilbert space \( \mathcal{H} \) and a quantum channel \( \Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K}) \) are compatible if there exists a quantum instrument \( I = \{\Phi_x : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K})\} \) such that \( \text{tr}[\Phi_x(\rho)] = \text{tr}[\rho A(x)] \) for all \( x \in \Omega_A \) and \( \rho \in \mathcal{S}(\mathcal{H}) \) and \( \sum_x \Phi_x = \Lambda \). Through implementing the quantum instrument \( I \), one can simultaneously measure the observable \( A \) and implement the channel \( \Lambda \).

3. **Channel-Channel compatibility**: Two quantum channels \( A_1 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K}_1) \) and \( A_2 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K}_2) \) are compatible if there exists a quantum channel \( \Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K}_1 \otimes \mathcal{K}_2) \) such that \( A_1(\rho) = \text{tr}_{\mathcal{K}_2}[A(\rho)] \) and \( A_2(\rho) = \text{tr}_{\mathcal{K}_1}[\Lambda(\rho)] \) for all \( \rho \in \mathcal{S}(\mathcal{H}) \). Through implementing the channel \( \Lambda \), one can simultaneously implement both of the channels \( A_1 \) and \( A_2 \).

III. COMPATIBILITY OF QUANTUM INSTRUMENTS

A. Definitions and concepts

**Definition 1** (Traditional compatibility). Two quantum instruments \( I_1 = \{\Phi^1_x : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K})\} \) and \( I_2 = \{\Phi^2_y : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K})\} \) are (traditionally) compatible if there exists an instrument \( I = \{\Phi_{xy} : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K})\} \) such that \( \sum_x \Phi_{xy}(.) = \Phi^1_x(.) \) and \( \sum_y \Phi_{xy}(.) = \Phi^2_y(.) \) for all \( x, y \).

The definition \([11]\) is the traditional definition of the compatibility of quantum instruments and is given in the Sec. 2.2.2 in page no. 33 in \([13]\) as Definition 2.5. The same definition is given in the Sec. 2 in page no. 15 in \([10]\), but the term "coexistence" has been used instead of the term "compatibility". This means that for traditionally compatible instruments, there exists an joint instrument, implementation of which is equivalent to the simultaneous implementation of both the instruments and individual instruments can be obtained using the classical post-processing of that joint instrument. Fig. \([11a]\) shows the traditional compatibility of the instruments \( I_1 \) and \( I_2 \).

**Definition 2** (Weak compatibility). Two quantum instruments \( I_1 = \{\Phi^1_x : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K})\} \) and \( I_2 = \{\Phi^2_y : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K})\} \) are weakly compatible if \( \sum_x \Phi^1_x = \Lambda \) and \( \sum_y \Phi^2_y = \Lambda \).

It is known that if a set of instrument is compatible then it is also weakly compatible, but it is easily understood that converse is not true, in general \([15]\).

Now we provide a new concept of compatibility which we call as "the parallel compatibility". The definition of parallel compatibility is given below-

**Definition 3** (Parallel compatibility). Two quantum instruments \( I_1 = \{\Phi^1_x : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_1)\} \) and \( I_2 = \{\Phi^2_y : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_2)\} \) are parallelly compatible if there exists an instrument \( I = \{\Phi_{xy} : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2)\} \) such that \( \sum_x \text{tr}_{\mathcal{K}_2}\Phi_{xy}(.) = \Phi^1_x(.) \) and \( \sum_y \text{tr}_{\mathcal{K}_1}\Phi_{xy}(.) = \Phi^2_y(.) \) for all \( x, y \).

Two instruments \( I_1 \) and \( I_2 \) which are parallelly compatible, can be implemented parallelly (i.e., there will be two output subsystems (on \( \mathcal{S}(\mathcal{K}_1) \) and on \( \mathcal{S}(\mathcal{K}_2) \) respectively) and effectively \( I_1 \) will be implemented on the first subsystem and effectively \( I_2 \) will be implemented on the second subsystem) . Clearly, if \( I_1 = \{\Phi^1_x : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_1)\} \) and \( I_2 = \{\Phi^2_y : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_2)\} \) are parallelly compatible, then the channels \( \Phi^1 = \sum_x \Phi^1_x \) and \( \Phi^2 = \sum_y \Phi^2_y \) are compatible (the joint channel is \( \Phi = \sum_{xy} \Phi_{xy} \)). Next example (and also Fig. \([11b]\) will clarify this concept.

**Example 1** (An example of parallelly compatible instruments). Consider two compatible quantum channels \( A_1 : \mathcal{S}(\mathcal{H}_1) \rightarrow \mathcal{S}(\mathcal{H}_1) \) and \( A_2 : \mathcal{S}(\mathcal{H}_2) \rightarrow \mathcal{S}(\mathcal{H}_2) \) with a joint channel \( \Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2) \). Therefore, since from the No-signaling principle, implementation of local quantum channel on one side does not change the density matrix on the other side, for any arbitrary quantum channel \( \Gamma_1 : \mathcal{S}(\mathcal{H}_1) \rightarrow \mathcal{S}(\mathcal{K}_1) \) and \( \Gamma_2 : \mathcal{S}(\mathcal{H}_2) \rightarrow \mathcal{S}(\mathcal{K}_2) \),

\[
\text{tr}_{\mathcal{K}_2}(\Gamma_1 \otimes \Gamma_2) \circ A_1(.) = \Lambda_1(.) \quad \text{and} \quad \text{tr}_{\mathcal{K}_1}(\Gamma_1 \otimes I) \circ A_2(.) = \Lambda_2(.)
\]

Now consider a pair of arbitrary quantum instruments
I

\[ \rho \]

Classical post-processing of outcomes

\[ \mathcal{I}_1 \quad \mathcal{I}_2 \]

(a) Traditional compatibility: This figure shows the traditional way to implement two instruments simultaneously which is mentioned in the Definition 1. This can be done through two steps: (i) implementing the joint instrument \( \mathcal{I} \) on the state \( \rho \) and (ii) then performing the post-processing of outcomes i.e., taking the marginal over the outcomes either \( x \) or \( y \) as mentioned in the Definition 1.

\[ \mathcal{I}_1 \quad \mathcal{I}_2 \]

\[ \Lambda \]

Joint channel \( \Lambda \) (Approx. asymm. cloning)

\[ \mathcal{J}_1 \quad \mathcal{J}_2 \]

(b) Parallel compatibility: This figure shows one specific way of parallel simultaneous implementation (defined in the Definition 2) of two instruments which is mathematically described in Example 1. This can be done through the following steps: (i) implementing the channel \( \Lambda \) on the state \( \rho \) which is the joint channel of the compatible channels \( \Lambda_1 \) and \( \Lambda_2 \), (where \( \Lambda_1(\rho) \) and \( \Lambda_2(\rho) \) can be considered as the approximate unequal clones (unless \( \Lambda_1 = \Lambda_2 \)) of the state \( \rho \), in general and therefore, it can be considered as approximate asymmetric cloning), in general and then (ii) applying the instruments \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) on \( \Lambda_1(\rho) \) and \( \Lambda_2(\rho) \) respectively such that \( \mathcal{J}_1 \circ \Lambda_1 = \mathcal{I}_1 \) and \( \mathcal{J}_2 \circ \Lambda_2 = \mathcal{I}_2 \). The existences such channel \( \Lambda \) and such instruments \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) implies the parallel compatibility of the instruments \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \).

FIG. 1: The Fig. 1a and Fig. 1b shows two ways of simultaneous implementation of two instruments. These two ways are inequivalent which is evident from the Proposition 1 and the Proposition 2.

\[ \mathcal{J}_1 = \{ \Phi^1_y : \mathcal{S}(\mathcal{H}_1) \rightarrow \mathcal{L}^+(\mathcal{K}_1) \}; \quad \sum_y \Phi^1_y = \Gamma_1 \} \]  
\[ \mathcal{J}_2 = \{ \Phi^2_y : \mathcal{S}(\mathcal{H}_2) \rightarrow \mathcal{L}^+(\mathcal{K}_2) ; \quad \sum_y \Phi^2_y = \Gamma_2 \} \]

Consider another pair of instruments \( \mathcal{I}_1 = \mathcal{J}_1 \circ \Lambda_1 \) = \{ \Phi^1_x \circ \Lambda_1 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_1) \} \) and \( \mathcal{I}_2 = \mathcal{J}_2 \circ \Lambda_2 \) = \{ \Phi^2_y \circ \Lambda_2 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_2) \} \). Now again consider a giant instrument \( \mathcal{I} = \{ \Phi_{xy} = (\Phi^1_x \circ \Phi^2_y) \circ \Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2) \} \).

Clearly, for all \( y \)

\[ \Phi^1_x(\cdot) = \Phi^1_x \circ \Lambda_1(\cdot) = \Phi^1_x \circ \text{tr}_{\mathcal{K}_2}(\mathbb{I} \otimes \Gamma_2) \circ \Lambda(\cdot) = \text{tr}_{\mathcal{K}_2} \big( \sum_y \Phi^2_y \circ \Phi^1_x \big) \circ \Lambda(\cdot) = \text{tr}_{\mathcal{K}_2} \Phi_{xy}(\cdot) \]

Similarly, \( \Phi^2_y = \sum_x \text{tr}_{\mathcal{K}_1} \Phi_{xy} \) for all \( x \). Hence, \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) are parallelly compatible.

Next Lemma relates observable-observable compatibility and parallel compatibility of instruments.

Lemma 1. If an A-compatible quantum instrument \( \mathcal{I}_A = \{ \Phi^1_x : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_1) \} \) and a B-compatible quantum instrument \( \mathcal{I}_B = \{ \Phi^2_y : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_2) \} \) are parallelly compatible then A and B are compatible.

Proof. Suppose an A-compatible quantum instrument \( \mathcal{I}_A = \{ \Phi^1_x : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_1) \} \) and a B-compatible quantum instrument \( \mathcal{I}_B = \{ \Phi^2_y : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_2) \} \) are parallelly compatible. Then there exists an instrument \( \mathcal{I} = \{ \Phi_{xy} : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2) \} \) such that \( \sum_y \text{tr}_{\mathcal{K}_2} \Phi_{xy}(\cdot) = \Phi^1_x(\cdot) \) and \( \sum_x \text{tr}_{\mathcal{K}_1} \Phi_{xy}(\cdot) = \Phi^2_y(\cdot) \) for all \( x, y \). Clearly, as \( \mathcal{I}_A \) an A-compatible quantum instrument \( \text{tr}[\rho \mathcal{A}(x)] = \text{tr}[\Phi^1_x(\rho)] = \text{tr}[\rho(\Phi^1_x)^*(\mathcal{I}_{\mathcal{K}_1})] \) all quantum state \( \rho \) and all outcomes \( x \) of the observable \( A \). This implies \( \sum_y \Phi_{xy}^*(\mathcal{I}_{\mathcal{K}_1}) = (\Phi^1_x)^*(\mathcal{I}_{\mathcal{K}_1}) = \mathcal{A}(x) \) for all outcomes \( x \) of the observable \( A \). Similarly \( \sum_x \Phi_{xy}^*(\mathcal{I}_{\mathcal{K}_2}) = (\Phi^2_y)^*(\mathcal{I}_{\mathcal{K}_2}) = \mathcal{B}(y) \) for all outcomes \( y \) of the observable \( B \).

Now we define the observable \( \mathcal{G} = \{ G(x, y) = \Phi_{xy}^*(\mathcal{I}_{\mathcal{K}_1} \otimes \mathcal{K}_2) \} \) where \( x, y \) are the outcomes of \( A \) and \( y \) are the outcomes of \( B \). As, \( p_{ij} = \text{tr}[\Phi_{xy}(\rho)] = \text{tr}[\rho \Phi_{xy}(\mathcal{I}_{\mathcal{K}_1 \otimes \mathcal{K}_2})] \geq 0 \) (since those are probabilities) for all quantum state \( \rho \) for all \( x, y \) implies \( G(x, y) \geq 0 \) for all \( x, y \). As \( \Phi(\rho) = \sum_{xy} \Phi_{xy}(\rho) \) is a quantum channel (a CPTP map), \( \sum_{xy} G(x, y) = \sum_{xy} \Phi_{xy}^*(\mathcal{I}_{\mathcal{K}_1 \otimes \mathcal{K}_2}) = \mathcal{I}_{\mathcal{H}} \). Therefore, \( \mathcal{G} \) is indeed an observable acting on the Hilbert space \( \mathcal{H} \). Now, \( \sum_y G(x, y) = \sum_y \Phi_{xy}^*(\mathcal{I}_{\mathcal{K}_1 \otimes \mathcal{K}_2}) = (\Phi^1_x)^*(\mathcal{I}_{\mathcal{K}_1}) = \mathcal{A}(x) \) and \( \sum_x G(x, y) = \sum_x \Phi_{xy}^*(\mathcal{I}_{\mathcal{K}_1 \otimes \mathcal{K}_2}) = (\Phi^2_y)^*(\mathcal{I}_{\mathcal{K}_2}) = \mathcal{B}(y) \). Therefore, \( \mathcal{G} = \{ G(x, y) \} \) is joint observables. Therefore, \( A \) and \( B \) are compatible.

Next, we show that the traditional compatibility of instrument and the parallel compatibility of instruments are conceptually different.

Proposition 1. There exist pairs of quantum instruments which are parallelly compatible, but not traditionally compatible.
In the Example (1), \(\Gamma_1\) and \(\Gamma_2\) can be arbitrary and therefore, \(\Gamma_1 \circ \Lambda_1\) and \(\Gamma_2 \circ \Lambda_2\) can be arbitrary and not necessarily same. Therefore, for the case where \(\Gamma_1 \circ \Lambda_1 \neq \Gamma_2 \circ \Lambda_2\), \(I_1\) and \(I_2\) are parallely compatible and are not weakly compatible and therefore are not traditionally compatible. 

Similarly, we have our next proposition.

**Proposition 2.** There exist pairs of quantum instruments which are traditionally compatible, but not parallely compatible.

**Proof.** Consider two quantum instruments \(I^p = \{\Phi^p = p_1|\mathcal{J}\}, \Phi^q = p_2|\mathcal{J}\}\) and \(I^q = \{\Phi^p = q_1|\mathcal{J}\}, \Phi^q = q_2|\mathcal{J}\}\) with \(|\mathcal{J}\) is an identity channel and \(p_i = \sum_j r_{ij}\) and \(q_j = \sum_i r_{ij}\) where \(r_{ij} \geq 0\) \(i,j=\{1,2\}\) and \(\sum_i r_{ij} = 1\). Clearly, \(I^p\) and \(I^q\) are traditionally compatible and the joint instrument is \(I^p = \{r_{ij}|\mathcal{J}\}_{i,j=\{1,2\}}\). But as discussed before if \(I^p\) and \(I^q\) are parallely compatible, then \(\Phi^p\) and \(\Phi^q\) must be compatible and we know that the identity channel \(\mathcal{J}\) is not compatible with itself (no-cloning theorem).

Therefore as \(\Phi^p = \sum_i \Phi^p_i = |\mathcal{J}\) and \(\Phi^q = \sum_i \Phi^q_i = |\mathcal{J}\) are not compatible, \(I^p\) and \(I^q\) are also not parallely compatible.

B. Reason for re-definition of the compatibility of Quantum Instruments

The basic conceptual idea behind the compatibility of any set of quantum devices, is the simultaneous implementation of those devices, given a single copy of input state. These devices can be measurements, channels or instruments. In the Sec. IIIA it has been clearly pointed out that there are two ways of simultaneous implementation of quantum instruments and those ways are inequivalent. Therefore, one of those two ways alone can not capture the notion of compatibility of quantum instruments. As a result, the traditional definition of compatibility or the definition of parallel compatibility alone is not enough to define the compatibility of quantum instruments. Therefore, the above-said argument suggests us to re-define the compatibility of quantum instruments. The definition is given below.

**Definition 4.** A pair of instruments is compatible if either it is traditionally compatible or it is parallely compatible. Otherwise it is incompatible.

In the next section, we show that the above-said definition of compatibility of quantum instruments is universal i.e., applicable to or can capture incompatibility of all three kinds.

C. Universality of the compatibility of quantum instruments

In this subsection, we show that the compatibility of quantum instruments captures the idea of observable-observable compatibility, observable-channel compatibility and channel-channel compatibility. We start with our next theorem.

**Theorem 1.** We provide three statements here-

1. Two observables \(A\) and \(B\) are compatible iff there exist an \(A\)-compatible instrument \(I_A\) and a \(B\)-compatible instrument \(I_B\) such that \(I_A\) and \(I_B\) are traditionally compatible.

2. Two quantum channels are \(\Phi^1: S(\mathcal{H}) \rightarrow S(\mathcal{K}_1)\) and \(\Phi^2: S(\mathcal{H}) \rightarrow S(\mathcal{K}_2)\) compatible iff there exist two instruments \(I_1 = \{\Phi^1: S(\mathcal{H}) \rightarrow L^+(\mathcal{K}_1)\}\) and \(I_2 = \{\Phi^2: S(\mathcal{H}) \rightarrow L^+(\mathcal{K}_2)\}\) where \(\sum_x \Phi^1_{x} = \Phi^1\) and \(\sum_y \Phi^2_{y} = \Phi^2\) that are parallely compatible.

3. If an \(A\)-compatible instrument \(I_A = \{\Phi^A: S(\mathcal{H}) \rightarrow L^+(\mathcal{K}_1)\}\) and a \(B\)-compatible instrument \(I_B = \{\Phi^B: S(\mathcal{H}) \rightarrow L^+(\mathcal{K}_2)\}\) where \(\sum_x \Phi^A_x = \Phi^A\) and \(\sum_y \Phi^B_y = \Phi^B\), are parallely compatible then \(A\) and \(B\) both are compatible with both \(\Phi^A\) and \(\Phi^B\).

**Proof.** We prove the three statements below-

1. Suppose \(A\)-compatible instrument \(I_A = \{\Phi^A: S(\mathcal{H}) \rightarrow L^+(\mathcal{K}_1)\}\) and \(B\)-compatible instrument \(I_B = \{\Phi^B: S(\mathcal{H}) \rightarrow L^+(\mathcal{K}_2)\}\) are traditionally compatible. Here \(\mathcal{K}_1 = \mathcal{K}_2\). Then there exists an instrument \(I = \{\Phi_{xy}: S(\mathcal{H}) \rightarrow L^+(\mathcal{K}_1)\}\) such that \(\sum_{y} \Phi_{xy}(\cdot) = \Phi^A(\cdot)\) and \(\sum_{x} \Phi_{xy}(\cdot) = \Phi^B(\cdot)\) for all \(x, y\). Let the observable \(G = \{G(x,y)\}\) be associated with \(I\) (i.e., \(I\) is a \(G\)-compatible instrument). Then for any \(\rho \in S(\mathcal{H})\), \(\text{tr}[G(x,y)] = \text{tr}[\Phi^A(\rho)]\) for all \(x, y\) and for all \(\rho \in S(\mathcal{H})\). Therefore, \(\sum_{y} \text{tr}[\rho G(x,y)] = \text{tr}[\Phi^A(\rho)]\) for all \(x\) and for all \(\rho \in S(\mathcal{H})\). Therefore, \(A\) and \(B\) are compatible and the joint observable \(G = \{G(x,y)\}\). Now suppose \(A\) and \(B\) are compatible and the joint observable is \(G = \{G(x,y)\}\) and \(I = \{\Phi_{xy}\}\) be a \(G\)-compatible instrument. Then it is easy to check that the instrument \(I^A = \{\Phi^A = \sum_y \Phi_{xy}\}\) is an \(A\)-compatible instrument and \(I^B = \{\Phi^B = \sum_y \Phi_{xy}\}\) is a \(B\)-compatible instrument.

2. A quantum channel \(\Lambda\) can be considered as a single outcome quantum instrument \(I_\Lambda\). Therefore, compatibility of two quantum channels \(\Phi^1\) and \(\Phi^2\) implies the parallel compatibility of two instruments \(I_1 = \{\Phi^1\}\) and \(I_2 = \{\Phi^2\}\).

From Definition 4 it is trivial to show that parallel compatibility of \(I_1\) and \(I_2\) implies the compatibility of \(\Phi^1\) and \(\Phi^2\).
3. By definition, $A$ is compatible with $\Phi^A$ and $B$ is compatible with $\Phi^B$. Now as $I_A$ and $I_B$ are parallelly compatible, there exists a quantum instrument $I_{AB} = \{\Phi_{xy} : S(H) \rightarrow L^+(K_1 \otimes K_2)\}$ such that $\Phi^A_{xy}(\cdot) = \sum_y \text{tr}_{K_2}[\Phi_{xy}(\cdot)]$ and $\Phi^B_{xy}(\cdot) = \sum_y \text{tr}_{K_1}[\Phi_{xy}(\cdot)]$. Now, as $I_A$ is an $A$-compatible instrument, for all $\rho \in S(H)$ and $x \in \Omega_A$,

$$
\text{tr}_{H}[\rho A(x)] = \text{tr}_{K_1}[\Phi^A_{xy}(\rho)] = \text{tr}_{K_1}[\sum_y \text{tr}_{K_2}[\Phi_{xy}(\rho)]] = \sum_y \text{tr}_{K_1}[\Phi_{xy}(\rho)] = \text{tr}_{K_2}[\Phi^A_{xy}(\rho)]
$$

(3)

where $\Phi^A_{xy}(\rho) = \sum_y \text{tr}_{K_2}[\Phi_{xy}(\rho)]$. Clearly, for all $x \in \Omega_A$, $\Phi^A_{xy} : S(H) \rightarrow L^+(K_2)$. Now,

$$
\sum_{x} \Phi^A_{xy}(\cdot) = \sum_{x} \sum_{y} \text{tr}_{K_1}[\Phi_{xy}(\cdot)] = \sum_{y} \Phi^B_{xy}(\cdot) = \Phi^B(\cdot).
$$

(4)

Therefore, $I_A' = \{\Phi^A_{xy} : S(H) \rightarrow L^+(K_2)\}$ is a quantum instrument. Hence, $A$ and $\Phi^B$ are compatible through the instrument $I_A' = \{\Phi^A_{xy} : S(H) \rightarrow L^+(K_2)\}$. Similarly one can prove that $B$ and $\Phi^A$ are compatible.

Hence, we have proved that the traditional compatibility and the parallel compatibility together can capture the compatibility of all three kinds i.e., observable-observable compatibility, observable-channel compatibility and channel-channel compatibility. Next, we discuss that the traditional compatibility is also incomplete in this sense.

The traditional compatibility of quantum instruments can not capture compatibility of quantum channels - If an $A$–compatible instrument $I_A = \{\Phi^A_{xy} : S(H) \rightarrow L^+(K_1)\}$ and a $B$–compatible instrument $I_B = \{\Phi^B_{xy} : S(H) \rightarrow L^+(K_2)\}$ with $\sum_{x} \Phi^A_{xy} = \Phi^A$ and $\sum_{y} \Phi^B_{xy} = \Phi^B$ are traditionally compatible, then $\Phi^A = \Phi^B$. Therefore, it directly can not capture channel-channel compatibility since two different channels can be compatible with each other.

Therefore, it seems that traditional compatibility and parallel compatibility together or in other words Definition[2] is conceptually complete.

From Lemma[1] and Theorem[1] we got that if two instruments are parallelly compatible corresponding observables are compatible as well as corresponding channels are compatible. We provide here one more interesting example-

Example 2 (Two parallelly incompatible instruments associated with the same observable). A trivial observable $J = \{J(x) = p_x\}$ is compatible with any quantum channel $\Lambda$. The corresponding instrument is $I_{J,A} = \{p_x\}$. Now take channel $\Gamma$ which incompatible with $\Lambda$. Now $J$ is also compatible with the quantum channel $\Gamma$ and the corresponding instrument is $I_{J,\Gamma} = \{p_x\}$. Now from the Theorem[2] we know that two instruments $I_{J,\Gamma}$ and $I_{J,A}$ are parallelly compatible the corresponding channels are compatible. But since $\Gamma$ and $\Lambda$ are not compatible, the instruments $I_{J,\Gamma}$ and $I_{J,A}$ are not parallelly compatible.

IV. CONCLUSION

In this paper, we have discussed the concept of parallel compatibility of instruments and have shown that this concept is different than the traditional definition of compatibility of instruments. We have claimed that the traditional definition of compatibility of instruments is conceptually incomplete and have provided a re-definition of the compatibility of quantum instrument. We have discussed the proper reason for the re-definition. We have shown that the modified definition of compatibility of quantum instruments can capture the idea of observable-observable compatibility, observable-channel compatibility and channel-channel compatibility. The parallel compatibility has importance in different information-theoretic tasks. For example- (i) Suppose, Charlie have to simultaneously transfer information to Alice and Bob who are separated by a long distance. The information for Alice is encoded in the probability distribution of the observable A and the information for Bob is encoded in the probability distribution of the observable B. Clearly Fig. [1D] suggests that it can be done through parallel compatibility of $I_A$ and $I_B$, (ii) Fig. [1D] also suggests that if Bob (i.e., $B$) is an eavesdropper then the parallel compatibility of quantum instruments allows him to do cloning attack in Quantum Key Distribution.

In future, it will be interesting to completely characterise the compatibility of quantum instrument.

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