A NOVEL LYAPUNOV FUNCTIONAL WITH APPLICATION TO STABILITY ANALYSIS OF NEUTRAL SYSTEMS WITH NONLINEAR DISTURBANCES

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Abstract. It is well-known that the global asymptotic stability analysis of neutral systems is an important concept in designing the appropriate controllers or filters for this class of systems. This paper carries out a delay-independent stability analysis of neutral systems possessing discrete time delays in the states and discrete neutral delays in the time derivative of the states in the presence of nonlinear disturbances. Some new global asymptotic stability criteria are proposed by introducing a novel Lyapunov functional. The obtained delay-independent stability criteria establish some simple and easily verifiable mathematical expressions involving the elements of the system matrices and the disturbance parameters of the neutral system. Different from the most of the previously reported stability results for neutral systems, the conditions obtained in this paper are not expressed in terms the Linear Matrix Inequalities (LMIs). Therefore, the criteria presented in this paper can be considered as the alternative results to previously published stability results stated in the LMI forms. A comparison between the results of this paper and some of previously published corresponding stability results is made to substantiate the significant improvement of the proposed results. A constructive numerical example is also presented to show applicability and the effectiveness of the proposed stability condition.

1. Introduction. In recent years, the analysis of the stability problem for dynamical systems have been given a great deal of attention due to the requirement of the stability property in various engineering problems such as traffic system modelling, network control systems, neuro-controller design, neural networks, and robotics [1]-[8]. Time delays in the states of dynamical systems may have some significant effects on the dynamical behaviors of these systems. For instance, when designing a stable dynamical system for solving a specific problem, the existence of time delays may change this stable system to an unstable system that possesses some undesired complex behaviors such as oscillation and instability. On the other hand, time delays may not only exist in the states, but they can also exist in the time derivatives of the states. This class of delayed systems is known as neutral systems. Neutral systems have found a wide range application areas such as population dynamic modelling, lossless transmission lines, robotic systems, rolling and milling machines [9]-[11].
addition, dynamical systems can also be affected by some nonlinear disturbances due to the external noises or uncertainties. Thus, it is of crucial importance to establish the stability conditions for neutral systems by considering the effects of time delays in the presence of nonlinear disturbances.

In the recent literature, many different techniques and methodologies combining with the Lyapunov approach have been employed to study the stability problem of various classes of neutral systems [12]-[33]. These papers have basically employed the Lyapunov functionals possessing double or triple integral terms to obtain delay-independent or delay-dependent stability criteria with or without nonlinear perturbations in the form of linear matrix inequalities (LMIs). The main drawback with the LMI approach for the stability of neutral systems is that it requires to check the negative definiteness of the high dimensional matrices formed by the system matrices. Therefore, it is always of great interest to finding the easily verifiable sufficient algebraic stability conditions by employing new approaches exploiting the Lyapunov stability techniques. It is known that the Lyapunov stability theorems are only suitable to derive sufficient conditions for the stability of nonlinear dynamical systems. Therefore, there is always a space for searching for modified and improved novel Lyapunov functionals to derive milder stability conditions, which has recently been a very important research topic in the area of the analysis of nonlinear dynamical systems.

This paper first introduces a novel modified suitable Lyapunov functional to analyze the stability of delayed neutral systems and obtains new sufficient conditions for global asymptotic stability of this class of neutral systems with or without nonlinear perturbations. Then, the condition derived in this paper will be compared with the previously published corresponding stability result to show the significant improvement of our proposed result. Finally, a constructive numerical example will be given to demonstrate applicability and the effectiveness of the proposed stability condition for neutral-type systems.

2. Problem statement and preliminaries. In this paper, we will carry out a global stability analysis of the neutral system involving discrete time delays and discrete neutral delays with nonlinear disturbances which is defined by following set of nonlinear differential equations:

\[
\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t - \tau_j) + \sum_{j=1}^{n} c_{ij} \dot{x}_j(t - \zeta_j) + f_i(x(t)) + g_i(x(t - \tau)) + h_i(\dot{x}(t - \zeta)), i = 1, 2, \cdots, n
\]

whose equivalent mathematical expression in the matrix-vector form can be stated as follows:

\[
\dot{x}(t) = Ax(t) + Bx(t - \tau) + C\dot{x}(t - \zeta) + f(x(t)) + g(x(t - \tau)) + h(\dot{x}(t - \zeta))
\]

where \( n \) is the number of the states in the system, \( x_i(t), (i = 1, 2, \cdots, n) \), are the states of the system, \( A = (a_{ij})_{n \times n}, B = (b_{ij})_{n \times n}, \) and \( C = (c_{ij})_{n \times n} \) are constant system matrices, the \( \tau_j, (j = 1, 2, \cdots, n) \), are the time delays and \( \zeta_j, (j = 1, 2, \cdots, n) \), are the neutral delays, and

\[
\begin{align*}
{x}(t) &= [x_1(t), x_2(t), \cdots, x_n(t)]^T \\
{x}(t - \tau) &= [x_1(t - \tau_1), x_2(t - \tau_2), \cdots, x_n(t - \tau_n)]^T \\
{\dot{x}}(t - \zeta) &= [{\dot{x}}_1(t - \zeta_1), {\dot{x}}_2(t - \zeta_2), \cdots, {\dot{x}}_n(t - \zeta_n)]^T
\end{align*}
\]
In system (1), functions from representing the nonlinear disturbances, satisfy the following inequalities:

\[ f(x(t)) = [f_1(x(t)), f_2(x(t)), \ldots, f_n(x(t))]^T \]
\[ g(x(t - \tau)) = [g_1(x(t - \tau)), g_2(x(t - \tau)), \ldots, g_n(x(t - \tau))]^T \]
\[ h(\dot{x}(t - \zeta)) = [h_1(x(t - \zeta)), h_2(x(t - \zeta)), \ldots, h_n(x(t - \zeta))]^T \]
\[ f_i(x(t)) = f_i(x_1(t), x_2(t), \ldots, x_n(t)), \forall i \]
\[ g_i(x(t - \tau)) = g_i(x_1(t - \tau_1), x_2(t - \tau_2), \ldots, x_n(t - \tau_n)), \forall i \]
\[ h_i(\dot{x}(t - \zeta)) = h_i(\dot{x}_1(t - \zeta_1), \dot{x}_2(t - \zeta_2), \ldots, \dot{x}_n(t - \zeta_n)), \forall i \]

In system (1), \( \eta = \max\{\tau_j\}, \kappa = \max\{\zeta_i\}, 1 \leq j \leq n, \) and \( \delta = \max\{\eta, \kappa\} \) Accompanying the system (1) are the initial conditions of the form: \( x_i(t) = \phi_i(t) \) and \( \dot{x}_i(t) = \psi_i(t) \in C([-\delta, 0], R) \), where \( C([-\delta, 0], R) \) denotes the set of all continuous functions from \([-\delta, 0]\) to \( R \).

It is usually assumed that the functions \( f(x(t)), g(x(t - \tau)) \) and \( h(\dot{x}(t - \zeta)) \), representing the nonlinear disturbances, satisfy the following inequalities:

\[ ||f(x(t))|| \leq \alpha||x(t)|| \]
\[ ||g(x(t - \tau))|| \leq \beta||x(t - \tau)|| \]

and

\[ ||h(\dot{x}(t - \zeta))|| \leq \gamma||\dot{x}(t - \zeta)|| \]

where \( \alpha, \beta \) and \( \gamma \) are some positive constants. In this paper, we will impose more relaxed and less restrictive conditions on the functions \( f(x(t)), g(x(t - \tau)) \) and \( h(\dot{x}(t - \zeta)) \). These new assumptions are given below:

\( A_1 \) : The function \( f(x(t)) = [f_1(x(t)), f_2(x(t)), \ldots, f_n(x(t))]^T \) satisfies the condition

\[ \sum_{i=1}^{n} |f_i(x(t))| \leq \sum_{i=1}^{n} \alpha_i|x_i(t)| \]

where \( \alpha_i \) are positive constants.

\( A_2 \) : The function \( g(x(t - \tau)) = [g_1(x(t - \tau)), g_2(x(t - \tau)), \ldots, g_n(x(t - \tau))]^T \) satisfies the condition

\[ \sum_{i=1}^{n} |g_i(x(t - \tau))| \leq \sum_{i=1}^{n} \beta_i|x_i(t - \tau_i)| \]

where \( \beta_i \) are some positive constants.

\( A_3 \) : The function \( h(\dot{x}(t - \zeta)) = [h_1(\dot{x}(t - \zeta)), h_2(\dot{x}(t - \zeta)), \ldots, h_n(\dot{x}(t - \zeta))]^T \) satisfies the condition

\[ \sum_{i=1}^{n} |h_i(\dot{x}(t - \zeta))| \leq \sum_{i=1}^{n} \gamma_i|\dot{x}_i(t - \zeta_i)| \]

where \( \gamma_i \) are some positive constants.

Stability of equilibrium points is mainly characterized in the sense of Lyapunov where a positive energy function is defined and then its time derivative is examined to draw conclusions about the stability properties of the equilibrium points. The Lyapunov stability theorem is mathematically stated by the following theorem [34]:

**Theorem 2.1.** Let \( x = 0 \) be an equilibrium point of the system given by \( \dot{x}_i(t) = f_i(x_1(t), x_2(t), \ldots, x_n(t)), (i = 1, 2, \ldots, n) \) or \( \dot{x}(t) = f(x(t)) \). Let \( V(x(t)) : \mathbb{R}^n \to \mathbb{R} \)
be a continuously differentiable positive definite function. The time derivative of $V(x(t))$, denoted by $\dot{V}(x(t))$, is given by

$$
\dot{V}(x(t)) = \sum_{i=1}^{n} \frac{\partial V(x(t))}{\partial x_i(t)} \dot{x}_i(t) = \sum_{k=1}^{n} \frac{\partial V(x(t))}{\partial x_i(t)} f_i(x(t)) = \frac{\partial V(x(t))}{\partial x(t)} f(x(t))
$$

If $\dot{V}(x(t)) \leq 0$, $\forall x(t) \in \mathbb{R}^n$, then $x = 0$ is stable. If $\dot{V}(x(t)) < 0$, $\forall x(t) \neq 0$ and $\dot{V}(x(t)) = 0$ only at $x = 0$, then $x = 0$ is asymptotically stable. If $x = 0$ is asymptotically stable and $V(x(t))$ is radially unbounded, that is $V(x(t)) \to \infty$ as $||x(t)|| \to \infty$, then $x = 0$ is globally asymptotically stable.

3. Stability analysis. This section is devoted to obtaining the main stability result of this paper, which is expressed by the following theorem:

**Theorem 3.1.** For system (1), assume that the conditions $A_1$, $A_2$ and $A_3$ hold. Then, system (1) is globally asymptotically stable if the following conditions are satisfied:

$$
\mu_i = \sum_{j=1}^{n} |c_{ji}| + \gamma_i = c_i + \gamma_i < 1, \quad i = 1, 2, \ldots, n
$$

$$
\xi_i = a_{ii} + \sum_{j=1}^{n} |a_{ji}| + \sum_{j \neq i}^{n} |b_{ji}| + \alpha_i + \beta_i < 0, \quad i = 1, 2, \ldots, n
$$

**Proof.** In order to prove the result of Theorem 3.1., the Lyapunov stability approach will be used. To this end, we employ the following modified and improved version of the positive definite Lyapunov functional introduced in [35]:

$$
V(t) = V_1(t) + V_2(t) + V_3(t)
$$

where

$$
V_1(t) = \sum_{i=1}^{n} \left( 1 - \mu_i \text{sgn}(x_i(t)) \text{sgn}(\dot{x}_i(t)) \right) |x_i(t)|
$$

$$
V_2(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{t-\zeta_i}^{t} |c_{ij}| |\dot{x}_j(s)| ds + \sum_{i=1}^{n} \gamma_i \int_{t-\zeta_i}^{t} |\dot{x}_i(s)| ds
$$

and

$$
V_3(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{t-\tau_j}^{t} |b_{ij}| |x_j(\zeta)| d\zeta + \sum_{i=1}^{n} \beta_i \int_{t-\tau_i}^{t} |x_i(\zeta)| d\zeta
$$

$$
+ \varepsilon \sum_{i=1}^{n} \int_{t-\tau_i}^{t} |x_i(\zeta)| d\zeta
$$

where $\varepsilon$ is a positive constant to be determined later. The time derivatives of the functions $V_1(t)$, $V_2(t)$ and $V_3(t)$ along the trajectories of the system (1) are obtained as follows:

$$
\frac{dV_1(t)}{dt} = \sum_{i=1}^{n} \left( 1 - \mu_i \text{sgn}(x_i(t)) \text{sgn}(\dot{x}_i(t)) \right) \text{sgn}(x_i(t)) \dot{x}_i(t)
$$

$$
= \sum_{i=1}^{n} \left( \text{sgn}(x_i(t)) - \mu_i \left( \text{sgn}(x_i(t)) \right)^2 \text{sgn}(\dot{x}_i(t)) \right) \dot{x}_i(t)
$$
We first note the following

Using (7) and (8) in (5) results in

\[
\frac{dV_2(t)}{dt} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} ||\dot{x}_j(t)|| - \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}||\dot{x}_j(t - \zeta_j)|
\]

\[
+ \sum_{i=1}^{n} \gamma_i |\dot{x}_i(t)| - \sum_{i=1}^{n} \gamma_i |\dot{x}_i(t - \zeta_i)|
\]

\[
\frac{dV_3(t)}{dt} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} ||x_j(t)|| - \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}||x_j(t - \tau_j)|
\]

\[
+ \sum_{i=1}^{n} \beta_i |x_i(t)| - \sum_{i=1}^{n} \beta_i |x_i(t - \tau_i)|
\]

\[
+ \varepsilon \sum_{i=1}^{n} |x_i(t)| - \varepsilon \sum_{i=1}^{n} |x_i(t - \tau_i)|
\]

We first note the following

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} ||\dot{x}_j(t)|| = \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}||\dot{x}_j(t)|| = \sum_{i=1}^{n} c_i |\dot{x}_i(t)|
\]

\[
= \sum_{i=1}^{n} c_i sgn(\dot{x}_i(t))\dot{x}_i(t)
\]

and

\[
\sum_{i=1}^{n} \gamma_i |\dot{x}_i(t)| = \sum_{i=1}^{n} \gamma_i sgn(\dot{x}_i(t))\dot{x}_i(t)
\]

Using (7) and (8) in (5) results in

\[
\frac{dV_2(t)}{dt} = \sum_{i=1}^{n} (c_i + \gamma_i) sgn(\dot{x}_i(t))\dot{x}_i(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}||\dot{x}_j(t - \zeta_j)|
\]

\[
- \sum_{i=1}^{n} \gamma_i |\dot{x}_i(t - \zeta_i)|
\]

\[
= \sum_{i=1}^{n} \mu_i sgn(\dot{x}_i(t))\dot{x}_i(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}||\dot{x}_j(t - \zeta_j)|
\]

\[
- \sum_{i=1}^{n} \gamma_i |\dot{x}_i(t - \zeta_i)|
\]

The sum of (4) and (9) leads to

\[
\frac{dV_1(t)}{dt} + \frac{dV_2(t)}{dt} = \sum_{i=1}^{n} sgn(x_i(t))\dot{x}_i(t) + \sum_{i=1}^{n} \mu_i sgn(\dot{x}_i(t))\dot{x}_i(t)
\]

\[
- \sum_{i=1}^{n} \mu_i \left( sgn(x_i(t)) \right)^2 sgn(\dot{x}_i(t))\dot{x}_i(t)
\]

\[
- \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}||\dot{x}_j(t - \zeta_j)| - \sum_{i=1}^{n} \gamma_i |\dot{x}_i(t - \zeta_i)|
\]

\[
\text{(10)}
\]
Let
\[ \rho_i(t) = \text{sgn}(x_i(t))\dot{x}_i(t) - \mu_i \left( \text{sgn}(x_i(t)) \right)^2 \text{sgn}(\dot{x}_i(t))\dot{x}_i(t) + \mu_i sgn(\dot{x}_i(t))\dot{x}_i(t), \quad \forall i \]

Then, (10) can be written as follows:
\[ \frac{dV_1(t)}{dt} + \frac{dV_2(t)}{dt} = \sum_{i=1}^{n} \rho_i(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}| |\dot{x}_j(t - \zeta_j)| - \sum_{i=1}^{n} \gamma_i |\dot{x}_i(t - \zeta_i)| \quad (11) \]

If \( x_i(t) \neq 0 \), then, \( \left( \text{sgn}(x_i(t)) \right)^2 = 1 \). In this case, we obtain
\[ \rho_i(t) = \text{sgn}(x_i(t))\dot{x}_i(t) \]
\[ = \sum_{j=1}^{n} \text{sgn}(x_i(t))a_{ij}x_j(t) + \sum_{j=1}^{n} \text{sgn}(x_i(t))b_{ij}x_j(t - \tau_j) \]
\[ + \sum_{j=1}^{n} \text{sgn}(x_i(t))c_{ij}\dot{x}_j(t - \zeta_j) + \text{sgn}(x_i(t))f_i(x(t)) + \text{sgn}(x_i(t))g_i(x(t - \tau)) \quad (12) \]
\[ \leq a_{ii}|x_i(t)| + \sum_{j=1}^{n} |a_{ij}||x_j(t)| + \sum_{j=1}^{n} |b_{ij}||x_j(t - \tau_j)| + \sum_{j=1}^{n} |c_{ij}||\dot{x}_j(t - \zeta_j)| \]
\[ + |f_i(x(t))| + |g_i(x(t - \tau))| + |h_i(\dot{x}(t - \zeta))| \]

Now, let \( x_i(t) = 0 \). Then, we have
\[ \rho_i(t) = \mu_i sgn(\dot{x}_i(t))\dot{x}_i(t) \]
\[ = \mu_i \sum_{j=1}^{n} \text{sgn}(\dot{x}_i(t))a_{ij}x_j(t) + \mu_i \sum_{j=1}^{n} \text{sgn}(\dot{x}_i(t))b_{ij}x_j(t - \tau_j) \]
\[ + \mu_i \sum_{j=1}^{n} \text{sgn}(\dot{x}_i(t))c_{ij}\dot{x}_j(t - \zeta_j) + \mu_i \text{sgn}(\dot{x}_i(t))f_i(x(t)) + \mu_i \text{sgn}(\dot{x}_i(t))g_i(x(t - \tau)) \]
\[ + \mu_i \text{sgn}(\dot{x}_i(t))h_i(\dot{x}(t - \zeta)) \quad (13) \]

Since \( \mu_i < 1 \), it follows from (13) that
\[ \rho_i(t) \leq a_{ii}|x_i(t)| + \sum_{j=1}^{n} |a_{ij}||x_j(t)| + \sum_{j=1}^{n} |b_{ij}||x_j(t - \tau_j)| + \sum_{j=1}^{n} |c_{ij}||\dot{x}_j(t - \zeta_j)| \]
\[ + |f_i(x(t))| + |g_i(x(t - \tau))| + |h_i(\dot{x}(t - \zeta))| \quad (14) \]

Thus, in the light of (12) and (14), we can see that, for all \( x_i(t) \in R \), the following holds:
\[ \rho_i(t) \leq a_{ii}|x_i(t)| + \sum_{j=1}^{n} |a_{ij}||x_j(t)| + \sum_{j=1}^{n} |b_{ij}||x_j(t - \tau_j)| + \sum_{j=1}^{n} |c_{ij}||\dot{x}_j(t - \zeta_j)| \]
\[ + |f_i(x(t))| + |g_i(x(t - \tau))| + |h_i(\dot{x}(t - \zeta))| \quad (15) \]
Using (15) in (11) yields

\[
\frac{dV_1(t)}{dt} + \frac{dV_2(t)}{dt} \leq \sum_{i=1}^{n} a_{ii}|x_i(t)| + \sum_{j=1}^{n} |a_{ij}| |x_j(t)| + \sum_{i=1}^{n} \sum_{j \neq i}^{n} |b_{ij}| |x_j(t - \tau_j)| + \sum_{i=1}^{n} |c_{ij}| |\dot{x}_j(t - \zeta_j)|
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} |f_i(x(t))| + \sum_{i=1}^{n} |g_i(x(t - \tau))| + \sum_{i=1}^{n} |h_i(\dot{x}(t - \zeta))| - \sum_{i=1}^{n} \gamma_i |\dot{x}_i(t - \zeta_i) |
\]

Under the conditions \(A_1 - A_3\), (16) leads to

\[
\frac{dV_1(t)}{dt} + \frac{dV_2(t)}{dt} \leq \sum_{i=1}^{n} a_{ii}|x_i(t)| + \sum_{j=1}^{n} |a_{ij}| |x_j(t)| + \sum_{j=1}^{n} |b_{ij}| |x_j(t - \tau_j)|
\]

\[
+ \sum_{i=1}^{n} \alpha_i |x_i(t)| + \sum_{i=1}^{n} \beta_i |x_i(t - \tau_i)|
\]

Taking the sums of the both sides of (6) and (17) results in

\[
\frac{dV(t)}{dt} = \frac{dV_1(t)}{dt} + \frac{dV_2(t)}{dt} + \frac{dV_3(t)}{dt}
\]

\[
\leq \sum_{i=1}^{n} a_{ii}|x_i(t)| + \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| |x_j(t)| + \sum_{j=1}^{n} |b_{ij}| |x_j(t - \tau_j)|
\]

\[
+ \sum_{i=1}^{n} \alpha_i |x_i(t)| + \sum_{i=1}^{n} \beta_i |x_i(t - \tau_i)|
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| |x_j(t)| - \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| |x_j(t - \tau_j)|
\]

\[
+ \sum_{i=1}^{n} \beta_i |x_i(t)| - \sum_{i=1}^{n} \beta_i |x_i(t - \tau_i)|
\]

\[
+ \varepsilon \sum_{i=1}^{n} |x_i(t)| - \varepsilon \sum_{i=1}^{n} |x_i(t - \tau_i)|
\]
which can also be written as
\[
\frac{dV(t)}{dt} = \frac{dV_1(t)}{dt} + \frac{dV_2(t)}{dt} + \frac{dV_3(t)}{dt}
\]
\[
\leq \sum_{i=1}^{n} a_{ii} |x_i(t)| + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} |a_{ij}| |x_j(t)| + \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| |x_j(t)|
\]
\[
+ \sum_{i=1}^{n} \alpha_i |x_i(t)| + \sum_{i=1}^{n} \beta_i |x_i(t)| + \varepsilon \sum_{i=1}^{n} |x_i(t)|
\]
\[
= \sum_{i=1}^{n} a_{ii} |x_i(t)| + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} |a_{ij}| |x_j(t)| + \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| |x_j(t)|
\]
\[
+ \sum_{i=1}^{n} \alpha_i |x_i(t)| + \sum_{i=1}^{n} \beta_i |x_i(t)| + \varepsilon \sum_{i=1}^{n} |x_i(t)|
\]
\[
= \sum_{i=1}^{n} \left( a_{ii} + \sum_{j=1, j \neq i}^{n} |a_{ji}| + \sum_{j=1}^{n} |b_{ji}| + \alpha_i + \beta_i \right) |x_i(t)| + \varepsilon \sum_{i=1}^{n} |x_i(t)|
\]
\[
= \sum_{i=1}^{n} \xi_i |x_i(t)| + \varepsilon \sum_{i=1}^{n} |x_i(t)|
\]
\[
(18)
\]
Since \( \xi_i < 0, \ i = 1, 2, \ldots, n, \) from (18), we obtain
\[
\frac{dV(t)}{dt} \leq \sum_{i=1}^{n} \xi_i |x_i(t)| + \varepsilon \sum_{i=1}^{n} |x_i(t)|
\]
\[
\leq \sum_{i=1}^{n} \xi_m |x_i(t)| + \varepsilon \sum_{i=1}^{n} |x_i(t)|
\]
\[
= (-\xi_m - \varepsilon) \sum_{i=1}^{n} |x_i(t)|
\]
\[
= (-\xi_m - \varepsilon) \|x(t)\|_1
\]
where \( \xi_m = \min \{\xi_i\} < 0, \ i = 1, 2, \ldots, n. \) Clearly, the choice \( 0 < \varepsilon < -\xi_m \) directly implies that
\[
\frac{dV(t)}{dt} < 0, \forall x(t) \neq 0
\]
Now, we consider the case where \( x(t) = 0. \) In this case,
\[
\rho_i(t) = \mu_i \text{sgn}(\dot{x}_i(t)) \dot{x}_i(t) = \mu_i |\dot{x}_i(t)|, \forall i
\]
\[
(19)
\]
Since \( \mu_i < 1, \ i = 1, 2, \ldots, n, \) it follows from (19) that
\[
\sum_{i=1}^{n} \rho_i(t) = \sum_{i=1}^{n} \mu_i |\dot{x}_i(t)| \leq \sum_{i=1}^{n} |\dot{x}_i(t)|
\]
\[
(20)
\]
For \( x(t) = 0, \) system (2) is of the form :
\[
\dot{x}_i(t) = \sum_{j=1}^{n} b_{ij} x_j(t - \tau_j) + \sum_{j=1}^{n} c_{ij} \dot{x}_j(t - \zeta_j)
\]
\[
+ g_i(x(t - \tau)) + h_i(\dot{x}(t - \zeta)), \ i = 1, 2, \ldots, n
\]
from which we can obtain
\[ \sum_{i=1}^{n} |\dot{x}_i(t)| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| |x_j(t - \tau_j)| + \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}| |\dot{x}_j(t - \zeta_j)| + \sum_{i=1}^{n} |g_i(x(t))| + \sum_{i=1}^{n} |h_i(\dot{x}(t - \zeta))| \]

\[ \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| |x_j(t - \tau_j)| + \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}| |\dot{x}_j(t - \zeta_j)| + \sum_{i=1}^{n} \beta_i |x_i(t - \tau_i)| + \sum_{i=1}^{n} \gamma_i |\dot{x}_i(t - \zeta_i)| \]

(20) and (21) together imply that
\[ \sum_{i=1}^{n} \rho_i(t) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| |x_j(t - \tau_j)| + \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}| |\dot{x}_j(t - \zeta_j)| + \sum_{i=1}^{n} \beta_i |x_i(t - \tau_i)| + \sum_{i=1}^{n} \gamma_i |\dot{x}_i(t - \zeta_i)| \]

Using (22) in (11) results in
\[ \frac{dV_1(t)}{dt} + \frac{dV_2(t)}{dt} = \sum_{i=1}^{n} \rho_i(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}| |\dot{x}_j(t - \zeta_j)| - \sum_{i=1}^{n} \gamma_i |\dot{x}_i(t - \zeta_i)| \]

\[ \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| |x_j(t - \tau_j)| + \sum_{i=1}^{n} \beta_i |x_i(t - \tau_i)| \]

(23)

For \( x(t) = 0 \), (6) is in the form :
\[ \frac{dV_2(t)}{dt} = - \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| |x_j(t - \tau_j)| - \sum_{i=1}^{n} \beta_i |x_i(t - \tau_i)| - \varepsilon \sum_{i=1}^{n} |x_i(t - \tau_i)| \]

(24)

Combining (24) with (23) yields
\[ \frac{dV(t)}{dt} \leq - \varepsilon \sum_{i=1}^{n} |x_i(t - \tau_i)| = - \varepsilon \|x(t - \tau)\|_1 \]

directly implying that
\[ \frac{dV(t)}{dt} < 0, \forall x(t - \tau) \neq 0 \]

Now, we consider the case where \( x(t) = 0 \) and \( x(t - \tau) = 0 \). In this case, system (1) is of the form :
\[ \dot{x}_i(t) = \sum_{j=1}^{n} c_{ij} \dot{x}_j(t - \zeta_j) + h_i(\dot{x}(t - \zeta)), \quad i = 1, 2, \ldots, n \]

from which we can obtain
\[ \sum_{i=1}^{n} |\dot{x}_i(t)| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}| |\dot{x}_j(t - \zeta_j)| + \sum_{i=1}^{n} |h_i(\dot{x}(t - \zeta))| \]

(25)

For this case, we also have
\[ \rho_i(t) = \mu_i |\dot{x}_i(t)| \]
Since $\mu_i < 1, i = 1, 2, \ldots, n$, if there exists at least one index $i$ such that $|\dot{x}_i(t)| \neq 0$, then, we get

$$\sum_{i=1}^{n} \rho_i(t) < \sum_{i=1}^{n} |\dot{x}_i(t)| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |c_{ij}| |\dot{x}_j(t - \zeta_j)| + \sum_{i=1}^{n} |h_i(\dot{x}(t - \zeta))|$$

(26)

Inserting (26) into (11) results in

$$\frac{dV_1(t)}{dt} + \frac{dV_2(t)}{dt} < 0$$

(27)

For $x(t) = 0$ and $x(t - \tau) = 0$, (5) is as follows

$$\frac{dV_3(t)}{dt} = 0$$

(28)

Thus, in the light of (27) and (28), we can conclude that

$$\frac{dV(t)}{dt} < 0, \forall \dot{x}(t - \zeta) \neq 0$$

The above analysis of the time derivative of the Lyapunov functional $V(t)$ indicates that if any of the vectors $x(t)$, $x(t - \tau)$ or $\dot{x}(t - \zeta)$ is a nonzero vector, then

$$\frac{dV(t)}{dt} < 0.$$ 

In addition, it can now easily observed that

$$\frac{dV(t)}{dt} = 0$$

if and only if $x(t) = 0$, $x(t - \tau) = 0$ and $\dot{x}(t - \zeta) = 0$. Thus, from the well-known Lyapunov stability theorems, we can conclude that neutral system (1) is asymptotically stable. In order to show the global asymptotic stability of system (1), we need to prove that the Lyapunov functional $V(t)$ is radial unbounded, namely, $V(t) \to \infty$ if $||x(t)|| \to \infty$. Note that $V(t)$, defined by (3), satisfies

$$V(t) = \sum_{i=1}^{n} \left(1 - \mu_i \text{sgn}(x_i(t)) \text{sgn}(\dot{x}_i(t))\right) |x_i(t)|$$

$$\geq \sum_{i=1}^{n} (1 - \mu_i) |x_i(t)|$$

$$\geq \sum_{i=1}^{n} (1 - \mu_M) |x_i(t)|$$

$$= (1 - \mu_M) \sum_{i=1}^{n} |x_i(t)|$$

$$= (1 - \mu_M) ||x(t)||_1$$

(29)

where $0 < \mu_M = \max\{\mu_i\} < 1, i = 1, 2, \ldots, n$. Thus, (29) directly implies that $V(t) \to \infty$ as $||x(t)|| \to \infty$. $\square$
4. Comparisons and a numerical example. Consider neutral system (1) without nonlinear disturbances:

\[ \dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t - \tau_j) + \sum_{j=1}^{n} c_{ij} \dot{x}_j(t - \zeta_j), i = 1, 2, \ldots, n \]  

(30)

The result of Theorem 3.1 obtained for system (1) can be specialized for system (30) as stated by the following theorem:

**Theorem 4.1.** Neutral system defined by (30) is globally asymptotically stable if the following conditions are satisfied:

(i) \( a_{ii} + \sum_{j=1}^{n} |a_{ji}| + \sum_{j=1}^{n} |b_{ji}| < 0, \ i = 1, 2, \ldots, n \)

(ii) \( \sum_{j=1}^{n} |c_{ji}| < 1, \ i = 1, 2, \ldots, n \)

The following result has been given in [36]:

**Theorem 4.2.** Neutral system defined by (30) is asymptotically stable if the following conditions are satisfied:

(i) \( a_{ii} + \sum_{j=1}^{n} |a_{ji}| + \sum_{j=1}^{n} |b_{ji}| < 0, \ i = 1, 2, \ldots, n \)

(ii) \( \sum_{j=1}^{n} |c_{ji}| < 1, \ i = 1, 2, \ldots, n \)

Note that Theorem 4.1 proves the global asymptotic stability of system (30). However, Theorem 4.2 proves only the asymptotic stability of system (30). Therefore, the conditions of Theorem 4.1 improve and generalize the conditions of Theorem 4.2 by addressing the global stability of system (30).

We now the following example to demonstrate the effectiveness of the condition proposed in Theorem 3.1:

**Example.** Assume the the neutral-type system described by (1) have the following system matrices:

\[ A = \begin{bmatrix} -4a & a & a & a \\ a & -4a & a & a \\ a & a & -4a & a \\ a & a & a & -4a \end{bmatrix}, B = \begin{bmatrix} b & b & b \\ b & b & b \\ b & b & b \\ b & b & b \end{bmatrix}, C = \begin{bmatrix} c & c & c & c \\ c & c & c & c \\ c & c & c & c \\ c & c & c & c \end{bmatrix} \]

where \( a, b \) and \( c \) are some real valued positive constants. In this example, also assume that system (1) possesses the following parameters:

\[ \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \frac{1}{2} \]

\[ \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{3} \]

\[ \beta_1 = \beta_2 = \beta_3 = \beta_4 = \frac{2}{3} \]
According to the results obtained in Theorem 3.1, the stability conditions for the system parameters of this example can be expressed as follows:

\[
\mu_i = \sum_{j=1}^{4} |c_{ji}| + \gamma_i = c_i + \gamma_i < 1, \quad i = 1, 2, 3, 4.
\]

\[
\xi_i = a_{ii} + \sum_{j=1 \atop j \neq i}^{4} |a_{ji}| + \sum_{j=1}^{4} |b_{ji}| + \alpha_i + \beta_i < 0, \quad i = 1, 2, 3, 4.
\]

Thus, we obtain

\[
\mu_1 = \mu_2 = \mu_3 = \mu_4 = 4c + \frac{1}{2} < 1,
\]

\[
\xi_1 = \xi_2 = \xi_3 = \xi_4 = -a + 4b + 1 < 0,
\]

from which we determine the global asymptotic stability conditions for the system parameters of this example as \(c < \frac{1}{2}\) and \(4b < a - 1\). Note that, for this example, Theorem 3.1 also requires the condition that \(a > 1\).

5. Conclusions. A new criterion for the global asymptotic stability of neutral systems possessing discrete time delays in the states and discrete neutral delays in the time derivative of the states with nonlinear disturbances has been presented. A delay-independent sufficient condition has been proposed by employing a novel Lyapunov functional. This condition is expressed in the form of linear algebraic equations which involve only the parameters of the neutral system, and therefore, it is easy to test the validity of the proposed criterion. The obtained alternative stability criterion is shown to be a generalization of a previously published corresponding stability result for neutral system. Future works can apply the proposed stability analysis method to further consider multiple time delays and multiple neutral delays in the mathematical model of neutral systems instead of considering discrete time delays and discrete neutral delays.

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