INSTABILITY OF AN EQUILIBRIUM WITH NEGATIVE DEFINITE LINEARIZATION

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ABSTRACT. A nonlinear parabolic differential equation is presented which has at least one equilibrium. This equilibrium is shown to have a negative definite linearization, but a spectrum which includes zero. An elementary construction shows that the equilibrium is not stable.

1. INTRODUCTION

This note demonstrates that in infinite-dimensional settings, negative definiteness of an equilibrium of a dynamical system is not sufficient to ensure that the equilibrium is stable. This is in stark contrast to the situation in finite-dimensional settings, where negative definiteness implies stability of the equilibrium. (See [1], for instance.)

The particular problem we study is the Cauchy problem

\[
\begin{align*}
\frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x) - 2f(x)u(t,x) - u^2(t,x) \\
u(0,x) &= h(x) \in C^\infty(\mathbb{R}^n)
\end{align*}
\]

for \( t > 0, x \in \mathbb{R}^n \) for \( n \geq 1 \),

where \( f \in C^\infty_0(\mathbb{R}^n) \) is a positive function. Since the linear portion of the right side of (1) is a sectorial operator, we can use (1) to define a nonlinear semigroup. [5] This turns (1) into a dynamical system, the behavior of which is largely controlled by its equilibria. This problem evidently has as an equilibrium, \( u(t,x) \equiv 0 \) for all \( t,x \). Depending on the exact choice of \( f \), there may be other equilibria, however, they will not concern us here. The spectrum of the equilibrium \( u \equiv 0 \) includes zero, even though the linearization of (1) about it is negative definite. We show this using an elementary construction akin to that of [11]. Additionally, we show by a direct construction that this equilibrium is not stable when \( n = 1 \).

2. MOTIVATION

The equation (1) arises as a transformation of a related equation, namely

\[
\begin{align*}
\frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x) - u^2(t,x) + \phi(x) \\
u(0,x) &= w(x) \in C^\infty(\mathbb{R}^n)
\end{align*}
\]

for \( t > 0, x \in \mathbb{R}^n \) for \( n \geq 1 \),

with \( \phi \in C^\infty_0(\mathbb{R}^n) \). This equation describes a reaction-diffusion equation [3], or a diffusive logistic population model with a spatially-varying carrying capacity. The spatial inhomogeneity of \( \phi \) makes the analysis of (2) much more complicated than that of typical reaction-diffusion equations. The existence of the equilibria for (2)
is a fairly difficult problem, which depends delicately on $\phi$. We will not treat the existence of equilibria for (2) here, but assume that $f$ is a positive equilibrium for (2). Then we can look at the behavior of perturbations near $f$, for instance

$$\frac{\partial (f + u)}{\partial t} = \Delta (f + u) - (f + u)^2 + \phi,$$

$$\frac{\partial u}{\partial t} = \Delta f + \Delta u - f^2 - 2fu - u^2 + \phi,$$

$$\frac{\partial u}{\partial t} = \Delta u - 2fu - u^2,$$

which is (1). Notice that this transforms the equilibrium $f$ of (2) to the zero function in (1). The situation of (1) is considerably easier to examine.

3. Properties of the spectrum

We need to linearize (1) in order to examine the spectrum of the equilibrium. In doing so, we roughly follow the outline given in [5]. Recall the following definition of the derivative map in a Banach space:

**Definition 1.** Suppose $R : B_1 \to B_2$ is a map from one Banach space to another. The derivative map of $R$ at $u \in B_1$ is the unique linear map $D : B_1 \to B_2$ such that for each sequence $\{h_n\}_{n=1}^{\infty}$ with $\|h_n\| \to 0$,

$$\lim_{n \to \infty} \left\| \frac{D(h_n) - R(u + h_n) + R(u)}{\|h_n\|} \right\| = 0.$$

Of course, such a map may not exist. If it does, we say $R$ is differentiable at $u$. The linearization $L$ of $R$ is the affine map given by the formula $L(h) = R(u) + D(h)$.

For this section, we shall work in the Hilbert space $L^2(\mathbb{R}^n)$ with the usual norm (using the fact that $\Delta$ is densely defined wherever necessary). The linearization of (1) at $u \equiv 0$ is easily computed to be

$$\frac{\partial h(t, x)}{\partial t} = \Delta h(t, x) - 2f(x)h(t, x).$$

Suppose $h(x, t) = X(x)T(t)$, then we can separate variables in (3), obtaining

$$T'(t) - \lambda T(t) = 0,$$

$$\Delta X(x) - (\lambda + 2f(x))X(x) = 0.$$

The separation constant $\lambda$ can be determined by examining the eigenvalue problem

$$(\Delta - 2f(x))X(x) = \lambda X(x),$$

which is essentially the computation of the energy levels of a Schrödinger equation. The operator $(\Delta - 2f)$ is a Schrödinger operator with potential $-2f$. Due to its importance in quantum mechanics, much is known about Schrödinger operators (see [10] for a summary).

If $\Re(\lambda) < 0$ over all of the eigenvalues $\lambda$ in (4), we would normally conclude that $h \to 0$ as $t \to \infty$, that $u \equiv 0$ is a stable equilibrium. However, as we shall see in Section 4 this is false. The cause of the instability is that although $\Re(\lambda) < 0$ for all eigenvalues, $\lambda = 0$ is in the spectrum of the operator $(\Delta - 2f)$.

**Lemma 2.** The spectrum of a self-adjoint, negative definite operator $T$ has spectrum which is confined to the closed left half-plane $\{ \lambda \in \mathbb{C} | \Re(\lambda) \leq 0 \}$. 

Proof. This is a standard argument (for instance, see \cite{8}), which we sketch briefly. First, suppose \( \lambda \) is an eigenvalue of \( T \) with an eigenfunction \( \psi \). Then

\[
\lambda = \frac{\langle \psi, T\psi \rangle}{\langle \psi, \psi \rangle} = \frac{\langle T\psi, \psi \rangle}{\langle \psi, \psi \rangle} = \bar{\lambda} \leq 0.
\]

On the other hand, the Fredholm alternative (see \cite{21}) implies that \( T - \lambda \) is surjective for \( \lambda > 0 \).

Finally, we note that for \( \Re(\lambda) > 0 \), \( (T - \lambda)^{-1} \) is bounded:

\[
\langle (T - \lambda)\psi, (T - \lambda)\psi \rangle = \langle T\psi, T\psi \rangle - 2\Re(\lambda) \langle \psi, T\psi \rangle + |\lambda|^2 \langle \psi, \psi \rangle \\
\geq |\lambda|^2 \langle \psi, \psi \rangle,
\]

by the negative definiteness of \( T \). Hence, for \( \Re(\lambda) > 0 \), \( (T - \lambda) \) has a bounded inverse.

\[\Box\]

**Lemma 3.** The self-adjoint operator \( (\Delta - 2f(x)) \) is negative definite if and only if \( f > 0 \) almost everywhere. (See \cite{11} for a generalization.)

**Proof.** It is well-known and easily shown that \( (\Delta - 2f) \) is self-adjoint. See \cite{6}, for example. The self-adjointness of \( (\Delta - 2f) \) follows immediately from that of \( \Delta \). It is also well-known that \( \Delta \) is negative definite: with zero boundary conditions, the divergence theorem gives

\[
\langle u, \Delta u \rangle = \int \bar{u} \Delta u \, dx = -\int \nabla \bar{u} \cdot \nabla u \, dx < 0.
\]

So the only thing that will spoil the negative definiteness is \( f \). Suppose \( f > 0 \) almost everywhere, and \( u \in L^2 \). Then

\[
\langle u, -2fu \rangle = -2 \int \bar{u}fu \, dx = -2 \int f|u|^2 \, dx < 0.
\]

On the other hand, suppose \( A = \{ x \in \mathbb{R}^n | f(x) \leq 0 \} \) has positive measure. Then let \( u = 1_A \) and compute

\[
\langle u, (\Delta - 2f)u \rangle = \langle u, -2fu \rangle = -2 \int \bar{u}fu \, dx = -2 \int f|u|^2 \, dx \geq 0.
\]

So we have that \( (\Delta - 2f) \) is not negative definite in that case. \[\Box\]

**Lemma 4.** Suppose \( f \) is a positive continuous function on \( \mathbb{R}^n \). Then \( (\Delta - 2f) \) is injective on \( C_0^2(\mathbb{R}^n) \).

**Proof.** Let \( u \in C_0^2(\mathbb{R}^n) \) satisfy \( (\Delta - 2f)u = 0 \). Let \( y = \sup_{x \in \mathbb{R}^n} u(x) \). We claim that \( y = 0 \). Suppose the contrary, that \( y > 0 \). Since \( u \in C_0^2(\mathbb{R}^n) \), there is an \( R > 0 \) such that for all \( |x| > R \), \( u(x) < y \). Thus \( M = u^{-1}((y)) \) is compact. By the maximum principle, there exists an \( \epsilon > 0 \) such that the \( \epsilon \)-neighborhood of \( M \),

\[
M_\epsilon = \{ x \in \mathbb{R}^n | \inf_{z \in M} \| z - x \| < \epsilon \}
\]

has \( \Delta u(M_\epsilon - M) < 0 \). On the other hand, \( N = M_\epsilon \cap u^{-1}((0, y)) \) is an open set on which \( u|N > 0 \) and \( \Delta u|N < 0 \). But since \( f \) is positive and \( \Delta u = 2fu \), this is a contradiction. Similar reasoning leads to \( \inf_{x \in \mathbb{R}^n} u(x) = 0 \), so in fact \( u \equiv 0 \). \[\Box\]
Since \( C_0^2(\mathbb{R}^n) \) is dense in \( L^2(\mathbb{R}^n) \), this implies that \( \lambda = 0 \) is not an eigenvalue of \((\Delta - 2f)\) over \( L^2(\mathbb{R}^n) \).

**Lemma 5.** The spectrum of \((\Delta - 2f)\) includes zero when \( f \in C_0^\infty(\mathbb{R}^n) \) is a positive function. (See [11] for the most general result of this kind.)

**Proof.** By Lemma 4 and the Fredholm alternative, \((\Delta - 2f)^{-1}\) exists. We show that \((\Delta - 2f)^{-1}\) is not bounded, by constructing a sequence \( \{\psi_m\} \) such that

\[
\lim_{m \to \infty} \frac{\langle (\Delta - 2f)\psi_m, (\Delta - 2f)\psi_m \rangle}{\langle \psi_m, \psi_m \rangle} = 0.
\]

Let \( \psi_m \) be the function

\[
\psi_m(x) = \left( \frac{1}{2A_m \sqrt{2\pi}} \right)^n e^{-\frac{|x|^2}{4A_m^2}},
\]

where \( A_m \in \mathbb{R} \) and \( B_m \in \mathbb{R}^n \) are constructed as follows. Choose \( A_m \) so that

\[
\langle \Delta \psi_m, \Delta \psi_m \rangle < \frac{1}{2m}
\]

(that this is possible follows from an easy computation). Then select \( B_m \) so that

\[
\langle f \psi_m, f \psi_m \rangle < \frac{1}{2m},
\]

which is possible since \( f \in C_0(\mathbb{R}^n) \). Notice that \( \langle \Delta \psi_m, \Delta \psi_m \rangle \) is independent of \( B_m \), so the second choice does not interfere with the first. Evidently

\[
\lim_{m \to \infty} \frac{\langle (\Delta - 2f)\psi_m, (\Delta - 2f)\psi_m \rangle}{\langle \psi_m, \psi_m \rangle} = 0,
\]

by the Schwarz inequality. On the other hand, \( \|\psi_m\|_2 = 1 \) for all \( m \). As a result, this shows that \((\Delta - 2f)^{-1}\) is not bounded. \( \square \)

As a result of Lemmas 4 and 5, we have three things: (1) that the spectrum is contained in the closed left half plane, (2) the spectrum includes zero, and (3) zero is not an eigenvalue.

### 4. Instability of the equilibrium

Now we construct, for each \( \epsilon > 0 \) and \( 1 \leq p < \infty \), an \( h_\epsilon \in C_0^\infty \cap L^p(\mathbb{R}) \) such that \( \|h_\epsilon\|_p < \epsilon \) which if \( u \) solves (11) with \( h_\epsilon \) as its initial condition, then \( \|u(t, \cdot)\|_p \to \infty \). In particular, this implies that \( u \equiv 0 \) is not a stable equilibrium of (11). We follow the general idea of the first part of [4]. (Additionally, [2] contains a more elementary discussion with a similar construction.)

**Definition 6.** Let \( H(t, x) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{|x|^2}{4t} \right) \), which is the heat kernel. Let \( v_\epsilon(s, x) = H(t - s + \epsilon, x) \) for fixed \( t \) and \( s < t \).

**Remark 7.** Since \( H \) is the heat kernel, \( v_\epsilon \) will satisfy \( \frac{\partial v_\epsilon}{\partial s} = -\Delta v_\epsilon (s, x) \).

**Lemma 8.** Suppose \( u(t, x) \leq 0 \) satisfies (11), and \( u(t, \cdot) \in L^p(\mathbb{R}) \) for each \( t \). Define

\[
J_\epsilon(s) = \int v_\epsilon(s, x) u(s, x) dx.
\]

Then \( \frac{dJ_\epsilon(s)}{ds} \leq -(J_\epsilon(s))^2 - 2\|f\|_\infty J_\epsilon(s) \).
Proof. First of all, we observe that since \( u \in L^p \), \( v_\epsilon(s,\cdot)u(s,\cdot) \) is in \( L^1(\mathbb{R}) \) for each \( s < t \).

Now suppose we have a sequence \( \{m_n\} \) of compactly supported smooth functions with the following properties: 

- \( m_n \in C^\infty(\mathbb{R}) \),
- \( m_n(x) \geq 0 \) for all \( x \),
- \( \text{supp}(m_n) \) is contained in the interval \( (-n-1, n+1) \), and
- \( m_n(x) = 1 \) for \( |x| \leq n \).

Then it follows that

\[
J_\epsilon(s) = \lim_{n \to \infty} \int v_\epsilon(s, x)u(s, x)m_n(x)dx.
\]

Now

\[
\frac{d}{ds}J_\epsilon(s) = \frac{d}{ds} \lim_{n \to \infty} \int v_\epsilon(s, x)u(s, x)m_n(x)dx
= \lim_{h \to 0} \lim_{n \to \infty} \frac{1}{h} \int (v_\epsilon(s+h, x)u(s+h, x) - v_\epsilon(s, x)u(s, x))m_n(x)dx.
\]

We’d like to exchange limits using uniform convergence. To do this we show that

(6) \[ \lim_{n \to \infty} \lim_{h \to 0} \frac{1}{h} \int (v_\epsilon(s+h, x)u(s+h, x) - v_\epsilon(s, x)u(s, x))m_n(x)dx \]

exists and the inner limit is uniform. We show both together by a little computation, using uniform convergence and LDCT:

\[
\lim_{n \to \infty} \lim_{h \to 0} \frac{1}{h} \int (v_\epsilon(s+h, x)u(s+h, x) - v_\epsilon(s, x)u(s, x))m_n(x)dx
= \lim_{n \to \infty} \int \left( \frac{d}{ds} v_\epsilon(s, x)u(s, x) + v_\epsilon(s, x) \frac{d}{ds} u(s, x) \right) m_n(x)dx
= \lim_{n \to \infty} \int (-\Delta v_\epsilon(s, x)u(s, x) + v_\epsilon(s, x)(\Delta u(s, x) - u^2(s, x) - 2f(x)u(x)))m_n(x)dx
= \lim_{n \to \infty} \int (-v_\epsilon(s, x)u^2(s, x) - 2v_\epsilon(s, x)f(x)u(s, x))m_n(x)dx.
\]

Minkowski’s inequality has that

\[
\int v_\epsilon u m_n dx \leq \left( \int v_\epsilon m_n dx \right)^{1/2} \left( \int v_\epsilon u^2 m_n dx \right)^{1/2},
\]
since \( v_\epsilon, m_n \geq 0 \). This gives that

\[
\int (-v_\epsilon(s, x)u^2(s, x) - 2v_\epsilon(s, x)f(x)u(s, x))m_n(x)dx
\leq \frac{(\int v_\epsilon u m_n dx)^2}{\int v_\epsilon m_n dx} - 2\|f\|_\infty \int v_\epsilon u m_n dx
\leq \frac{(\int v_\epsilon u dx)^2}{\int v_\epsilon m_1 dx} - 2\|f\|_\infty J_\epsilon(s) < \infty,
\]
hence the inner limit of (6) is uniform. On the other hand,
\[
\lim_{n \to \infty} \int (-v_n(s,x)u^2(s,x) - 2v_n(s,x)f(x)u(s,x))m_n(x)dx \\
\leq \lim_{n \to \infty} \left( -\left( \frac{\int v_n um_n dx}{v_n m_n} \right)^2 - 2\|f\|_\infty \int v_n um_n dx \right) \\
\leq -(J_\epsilon(s))^2 - 2\|f\|_\infty J_\epsilon(s) < \infty,
\]
so the double limit of (6) exists. Hence we conclude that the lemma is true. \(\square\)

**Lemma 9.** Suppose that for some \(t_0 > 0\),
\[
\int H(t_0,x)u(0,x)dx < -2\|f\|_\infty.
\]
Then \(\|u(t,\cdot)\|_p \to \infty\) for \(1 \leq p \leq \infty\).

**Proof.** Note that
\[
J_\epsilon(0) = \int v_\epsilon(0,x)u(0,x)dx \\
= \int H(t + \epsilon,x)u(0,x)dx \\
< -2\|f\|_\infty,
\]

since we may choose \(\epsilon > 0\) and \(t\) such that \(t + \epsilon = t_0\). Thus Lemma 8 implies that \(J_\epsilon(s) \to -\infty\) by elementary ODE theory. \(\square\)

On the other hand,
\[
|J_\epsilon(s)| \leq \int |v_\epsilon(s,x)||u(s,x)|dx \leq \frac{1}{\sqrt{4\pi\epsilon}}\|u(s,\cdot)\|_1 \\
\leq \|u(s,\cdot)\|_\infty.
\]
So we have that \(\|u(s,\cdot)\|_1\) and \(\|u(s,\cdot)\|_\infty\) both blow up. Finally,
\[
\int |v_\epsilon(s,x)||u(s,x)|dx \leq \int |v_\epsilon||u|^p|u|^{1-p}dx \\
\leq \frac{1}{\|u\|_\infty^{p-1}\sqrt{4\pi\epsilon}}\|u\|_p^p \\
\leq \frac{1}{\sqrt{4\pi\epsilon}}\|u\|_p^p
\]
since \(\|u(s,\cdot)\|_\infty \to \infty\). Hence \(\|u(s,\cdot)\|_p \to \infty\). \(\square\)

Finally, we show that \(u \equiv 0\) is unstable. Let \(\epsilon > 0\) be given and \(1 \leq p < \infty\). Take \(h_\epsilon(0) \leq -4\|f\|_\infty\) to be arbitrary. We can construct \(h_\epsilon \in L^p \cap L^\infty \cap C^\infty_c(\mathbb{R})\) such that additionally \(\|h_\epsilon\|_p < \epsilon\), using the smooth Urysohn lemma. \(\square\) Then for sufficiently small \(t > 0\),
\[
\int H(t,x)h_\epsilon(x)dx < -2\|f\|_\infty
\]
by the fact that \( \{H(1/n, \cdot)\} \) is a \( \delta \)-sequence as \( n \to \infty \). Hence by Lemma 2 if \( u \) solves (1) with \( h_e \) as initial condition, then \( \|u\|_p \to \infty \). (Note that this construction fails for \( p = \infty \), since we cannot ensure that both \( h_e(0) \leq -4\|f\|_\infty \) and \( \|h_e\|_\infty < \epsilon \).)

5. Conclusions

As a result of the previous two sections, we conclude that the equilibrium \( u \equiv 0 \) has a real, negative eigenvalues (the set of which may be negative), yet it is not stable. That there exist solutions which start near the equilibrium but blow up to \( \infty \) in any \( p \)-norm indicates that the equilibrium is actually rather unstable. On the other hand, this is precisely the kind of behavior that is expected from a linearization whose spectrum contains zero. This suggests that in infinite-dimensional settings one should be sure to employ the entire spectrum to determine stability.

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