The Wigner function associated to the Rogers-Szegő polynomials

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We show here that besides the well known Hermite polynomials, the q-deformed harmonic oscillator algebra admits another function space associated to a particular family of q-polynomials, namely the Rogers-Szegő polynomials. Their main properties are presented, the associated Wigner function is calculated and its properties are discussed. It is shown that the angle probability density obtained from the Wigner function is a well-behaved function defined in the interval $-\pi \leq \theta < \pi$, while the action probability only assumes integer values $m \geq 0$. It is emphasized the fact that the width of the angle probability density is governed by the free parameter $q$ characterizing the polynomial.

I. INTRODUCTION

Deformed algebras, quantum groups and quantum spaces have been thoroughly investigated since their introduction in physics [1–5] and non-trivial commutation relations have been proposed and analyzed by many authors [6–10]. Following this trend, the $q$-deformed harmonic oscillator was introduced in the articles [11,12] and has been the recipient of further investigation [13–38]. Many kinds of 3-generator algebras $\{A, A^\dagger, N\}$ were proposed, whose commutation relations take the general form

$$AA^\dagger - f_1(q, \alpha, \beta, \gamma, \ldots)A^\dagger A = f_2(q, \alpha, \beta, \gamma, \ldots; N)$$ (1)

$$[N, A] = -A, \quad [N, A^\dagger] = -A^\dagger,$$ (2)

where $f_1(q, \alpha, \beta, \ldots)$ is a c-number function of the parameters $q, \alpha, \beta, \gamma, \ldots$ (real or complex) and $f_2(q, \alpha, \beta, \ldots; N)$ is an operator function since it also depends on $N$. Many realizations or representations are possible for $\{A, A^\dagger, N\}$, however, when $f_1(q, \alpha, \beta, \ldots) = 1$ and $f_2(q, \alpha, \beta, \ldots; N) = 1$, the linear harmonic oscillator algebra is recovered. Each realization allows a specific space of functions on which the operators act; different spaces show quite different properties and different physics. For instance, setting $f_1(q, \alpha, \beta, \ldots) = 1$ and $f_2(q, \alpha, \beta, \gamma, \ldots; N) = q^N$, two different realizations for the space of functions are possible: (a) the $L^2$ space of Hermite polynomials, with a Gaussian weight [10,38,39], that have the line $\mathbb{R} = \{x \in (-\infty, \infty)\}$ as the domain; and the less familiar (b) Rogers-Szegő polynomials which have the Jacobi $\vartheta_3$-function as weight function, the independent variable is an “angle”, $\phi$, with compact domain $S = \{\phi \in [0, 2\pi)\}$.

Though the properties of the Wigner function associated to the Hermite polynomial are quite known, the same can not be said about the Rogers-Szegő polynomials, so it is the aim of the present paper to study their properties, to derive the associated Wigner function and the angle distribution function.

We remind that the Weyl-Wigner transformation associated to the well-known translational degree of freedom for Cartesian variables and moments of a particle has long been established and widely discussed in the literature [40–45].
On the other hand, in what refers to the Weyl-Wigner transformation, the rotational degree of freedom has been scarcely touched upon. In this connection, the treatment of this case was directly inferred from the previous one by means of symmetry arguments \([46,47]\), by the continuous limit of finite-dimensional Weyl-Wigner mappings \([48]\), or by the implementation of the appropriate kinematics relations \([49]\). It is clear in all these cases that one is dealing with functions of angular variable that have period \(2\pi\) and the measure of this function space is simply the unity.

In this contribution we intend first of all to briefly present a family of \(q\)-polynomials that also have period \(2\pi\), but are orthonormalized on the circle with respect to a measure function \(\mu(\theta; q)\) that is the Jacobi \(\vartheta_3\)-function, namely, the Rogers-Szegö polynomials \([50–54]\). This set of polynomials has been shown to be associated to a realization of the \(q\)-deformed harmonic oscillator \([12,55]\), and are then the functions describing the states of that system for the deformation parameter ranging from 0 to 1. Therefore, these functions are characterized by a discrete variable \(n\) and a continuous angle variable \(\theta\), defined on the circle, besides depending on the deformation parameter \(q\). We then write the Weyl-Wigner transformation for this class of \(q\)-polynomials from which we also extract the angle and action probability distribution functions. The simplest cases of the ground state and first state are directly obtained and discussed.

In section II we present a brief review of definitions and relations that are relevant in the present context for the study of the Rogers-Szegö polynomials \([50–54]\). This set of polynomials has been shown to be associated to a realization of the \(q\)-deformed harmonic oscillator \([12,55]\), and are then the functions describing the states of that system for the deformation parameter ranging from 0 to 1. Therefore, these functions are characterized by a discrete variable \(n\) and a continuous angle variable \(\theta\), defined on the circle, besides depending on the deformation parameter \(q\). We then write the Weyl-Wigner transformation for this class of \(q\)-polynomials from which we also extract the angle and action probability distribution functions. The simplest cases of the ground state and first state are directly obtained and discussed.

In section II we present a brief review of definitions and relations that are relevant in the present context for the study of the Rogers-Szegö polynomials which are also presented. The Wigner function for these polynomials are calculated in section III, where the probability distributions for \(n\) and \(\theta\) are also presented. Finally, section IV is devoted to the summary and a discussion of what has been obtained.

II. Q-SERIES, ROGERS-SZEGÖ POLYNOMIALS AND Q-DEFORMED ALGEBRA

A. Brief review of some results from q-series

In order to set the stage for the introduction of the \(q\)-polynomials we are interested in, let us first introduce some basic concepts \([54,56]\). The basic notation will be, for \(|q| < 1\),

\[(x)_{n+1} \equiv (x; q)_{n+1} \equiv (1 - x)(1 - xq)\ldots(1 - xq^n)
\]
such that

\[(x)_{\infty} \equiv (x; q)_{\infty} \equiv \lim_{n \to \infty} (x; q)_n;
\]
and

\[(x)_0 \equiv 1.
\]
The particular case \(x = q\) should be noted

\[(q)_{n+1} \equiv (q; q)_{n+1} \equiv (1 - q)(1 - q^2)(1 - q^3)\ldots(1 - q^{n+1}).
\]

Furthermore, we can recognize that, for any real \(n\), we can write

\[(x)_n = \frac{(x)_{\infty}}{(xq^n)_{\infty}} = \frac{(1 - x)(1 - xq)(1 - xq^2)\ldots(1 - xq^{n+1})\ldots}{(1 - xq^n)(1 - xq^{n+1})\ldots} = (1 - x)(1 - xq)(1 - xq^2)\ldots(1 - xq^{n-1}). \tag{3}\]

The \(q\)-binomial is defined as

\[
\begin{bmatrix} n \\ j \end{bmatrix} = \frac{(1 - q)(1 - q^2)\ldots(1 - q^n)}{(1 - q)(1 - q^2)\ldots(1 - q^n)} = \frac{(q)_n}{(q)_j(q)_{n-j}} \tag{4}\]

for \(j\) and \(n\) integers, with \(0 \leq j \leq n\) and \((0)_n = 1\), and has the following properties

\[
\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1, \tag{5}\]
\[
\binom{n}{j} = \binom{n}{n-j},
\]  
(6)

\[
\lim_{q \to 1} \binom{n}{j} = \binom{n}{j} = \frac{n!}{j!(n-j)!}.
\]  
(7)

The so-called \(q\)-number can be directly realized from the definition (4),

\[
\binom{n}{1} = \frac{1 - q^n}{1 - q} \equiv [n].
\]

Cauchy theorem plays an essential role in this context and states that, for \(|t| < 1\), the following equality holds

\[
\prod_{n=0}^{\infty} \frac{1 - xtq^n}{1 - tq^n} = \sum_{n=0}^{\infty} \frac{(x; q)_n t^n}{(q; q)_n}
\]

from which follows the important result

\[
(x; q)_n = \prod_{s=0}^{n-1} (1 - q^s x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} q^{j(1-j)/2} x^j.
\]

B. Rogers-Szegő polynomials

Let us introduce the Rogers-Szegő polynomials in the general form through their definition [50,52,53]

\[
H_n (y) \equiv H_n (y; q) = \sum_{r=0}^{n} \binom{n}{r} y^r.
\]  
(8)

Among the several properties they satisfy, we call attention on the three-term recurrence relation,

\[
H_{n+1} (y; q) = (1 + y) H_n (y; q) - (1 - q^n) yH_{n-1} (y; q),
\]  
(9)

and the \(q\)-differentiation relation,

\[
D_q H_n (y; q) = \frac{H_n (y; q) - H_n (yq; q)}{y (1 - q)} = [n] H_{n-1} (y; q),
\]  
(10)

where \(D_q\) is known as Jackson’s \(q\)-derivative, which goes to the usual derivative in the limit \(q \to 0\).

From the definition (8) and properties (5) and (6), the first two polynomials are

\[
H_0 (y; q) = 1,
\]  
(11)

\[
H_1 (y; q) = 1 + y.
\]  
(12)

The other ones can be obtained through the use of the recurrence relation (9). In the limit \(q \to 1\) we have

\[
\lim_{q \to 1} H_n (y; q) = \sum_{r=0}^{n} \binom{n}{r} y^r = (1 + y)^n.
\]

Another important property of the Rogers-Szegő polynomials is their orthogonality on the circle when the Jacobi \(\vartheta_3 (y; q)\) function is taken as the measure function [51]. In order to explicitly verify this, we should perform a proper choice for the variable \(y\), \(y = -q^{-1/2} e^{i \phi}\), such that

\[
H_n (y; q) = H_n \left( -q^{-1/2} e^{i \phi}; q \right).
\]  
(13)
In this form, the orthonormalization integral is written as

\[
I_{mn}(q) = \int_{-\pi}^{\pi} H_m\left(-q^{-1/2}e^{i\varphi}; q \right) H_n\left(-q^{-1/2}e^{-i\varphi}; q \right) \vartheta_3(\varphi; q) \frac{d\varphi}{2\pi}
\]

with the explicit form

\[
\vartheta_3(\varphi; q) = \sum_{m=-\infty}^{\infty} q^{m^2/2} e^{im\varphi} = \sum_{m=-\infty}^{\infty} e^{-\mu m^2 + im\varphi},
\]

with \( \mu = -(\ln q)/2 \), which is the measure function \([57]\). Using the definition of the Rogers-Szegö polynomials, Eq. (8), we see that

\[
I_{mn}(q) = \sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{r+s} \left[ \begin{array}{c} m \\ r \end{array} \right] \left[ \begin{array}{c} n \\ s \end{array} \right] q^{r(r-1)/2} s^{(s-1)/2} q^{-rs},
\]

this is a result discussed by Carlitz \([53]\) (see Appendix for the proof), and shown to give

\[
I_{mn}(q) = \frac{(q,q)_{n}}{q^{n}} \delta_{m,n}.
\]

From that result we get what is sometimes known as the Rogers-Szegö functions

\[
R_n(\varphi; q) = \frac{q^{n/2}}{[(q,q)_{2}]^{1/2}} H_n\left(-q^{-1/2}e^{i\varphi}; q \right).
\]

It is worth noting that the Jacobi \( \vartheta_3(\varphi; q) \) function is associated to a sum of Gaussians on the circle. To see this, let us recall the basic relation obeyed by that function, namely, in a general form \([57,58]\)

\[
\sum_{n=-\infty}^{\infty} \exp\left[-\alpha (y + n)^2\right] = \sqrt{\frac{\pi}{\alpha}} \sum_{k=-\infty}^{\infty} \exp\left(-\pi^2 \frac{k^2}{\alpha} + 2\pi i ky\right).
\]

Adapting to our case, we take \( q = \exp(-2\mu) \), and using Eq. (14), we have

\[
\sum_{m=-\infty}^{\infty} \exp\left(-\mu m^2 - im\varphi\right) = \sqrt{\frac{\pi}{\mu}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{1}{4\mu} (\varphi - 2\pi n)^2\right];
\]

the RHS of this relation shows the measure function as a sum of Gaussians of the angle variable \( \varphi \).

From a different point of view, the Jacobi \( \vartheta_3 \)-function has been proposed as a valuable function to describe particular limiting situations in quantum optics \([59]\), and also as a coherent state for a particle on a circle where the angular variable plays an essential role \([60,61]\). In this case, the algebra is given in terms of the angular momentum and an unitary operator so that the commutation relation is

\[
[J, U] = U,
\]

being \( U \) a unitary operator associated to angular momentum shifts. Such a commutation relation has been discussed long ago in the literature \([62,63]\), and was also obtained as the limiting case in finite dimensional phase space representation of quantum mechanics \([48]\). It is worth noticing that the Jacobi \( \vartheta_3 \)-function with integer argument was also proposed as a coherent state for the case of any finite-dimensional degrees of freedom \([64]\), since in these cases the eigenvalue problem associated to the discrete Fourier matrix in the discrete basis \([65]\) gives a solution which is directly expressed in terms of that Jacobi’s function. In this sense we see that the Jacobi \( \vartheta_3 \)-function plays a wider role in connection with coherent states, and, in particular, with the rotational or action-angle degrees of freedom.

Here, instead, the Jacobi \( \vartheta_3 \)-function will be considered the measure function for the Rogers-Szegö polynomials, in the same way as the Gaussian is the measure function for the standard Hermite polynomials associated to the one-dimensional harmonic oscillator.

The angle probability distribution constructed out of the \( \vartheta_3 \)-function and the Rogers-Szegö polynomials is in fact a good candidate for describing action-angle degrees of freedom, since if we consider the Rogers-Szegö functions, the angle probability distribution associated to these states is immediately recognized as
\[ \Omega^{(n)}(\varphi; q) d\varphi = |R_n(\varphi; q)|^2 \vartheta_3(\varphi; q) \frac{d\varphi}{2\pi}. \]

In the same form we have for the harmonic oscillator functions on the line

\[ \Phi^{(n)}(x; b) dx = |H_n(x; b)|^2 \exp \left( -\frac{x^2}{2b^2} \right) dx, \]

where \( b \) in the second case stands for the harmonic oscillator width and \( H_n(x; b) \) is the standard Hermite polynomial.

As a further remark we observe that with the particular choice of the deformation parameter, namely, \( q = \exp (-2\mu) \), associated to the range \( 0 \leq q \leq 1 \), we have \( 0 \leq \mu \leq \infty \). Obviously in the limit \( q \to 1 \) (\( \mu \to 0 \)) we verify that the \( \vartheta_3 \)-function is simply the usual Fourier series while

\[ \lim_{q \to 1} H_n(y; q) = \sum_{r=0}^{n} \binom{n}{r} y^r = (1 + y)^n. \]

**C. The \( q \)-deformed algebra**

Now, from Eqs. (9) and (10) we show that the Rogers-Szegö polynomials make the function space realization of the one-parameter, three-element \( q \)-deformed algebra,

\[ [A, A^\dagger] = q^N; \quad [N, A^\dagger] = A, \quad [N, A] = -A \]

and

\[ A^\dagger A = \frac{1 - q^N}{1 - q} = [N], \]

such that \( \lim_{q \to 1} A^\dagger A = N. \) Under certain requirements [66] the commutation relation \([A, A^\dagger] = q^N\) can be shown to be cast in the form [7,14]

\[ AA^\dagger - qA^\dagger A = 1. \]

Also, it is immediate to verify that for any analytic function \( f(x) \) one has the relations,

\[ Af(N) = f(N + 1)A, \quad A^\dagger f(N) = f(N - 1)A^\dagger. \]

Now setting a representation to the operators \( A \) and \( A^\dagger \) [55],

\[ A = D_q, \quad A^\dagger = (1 + y) - (1 - q) yD_q, \]

one verifies that

\[ A^\dagger H_n(y; q) = H_{n+1}(y; q), \quad AH_n(y; q) = [n] H_{n-1}(y; q) \]

and

\[ NH_n(y; q) = nH_n(y; q), \quad A^\dagger AH_n(y; q) = [n] H_n(y; q). \]

It must be remarked that the Rogers-Szegö polynomials have also appeared in a different way in the treatment of the \( q \)-deformed harmonic oscillator by MacFarlane [12]: Doing a nonlinear transformation on \( A \),

\[ B = (q - q^{-1})^{1/2} Aq^{-N/2} = (1 - q^{-2})^{1/2} q^{-N/2} A, \]

the new operators obey the \( q \)-commutation relation

\[ q^2BB^\dagger - B^\dagger B = q^2 - 1, \quad (21) \]

having the differential representation,
\[ B = \exp(-i\varphi) - \exp(-i\varphi/2) \exp(-4i\mu \frac{\partial}{\partial \varphi}) \]  

(22)

and

\[ B^+ = \exp(i\varphi) - \exp(4i\mu \frac{\partial}{\partial \varphi}) \exp(i\varphi/2)， \]  

(23)

where \( q = \exp(-2\mu) \).

On this basis, we may think of the Rogers-Szegő polynomials as a particularly interesting candidate to describe phase properties of the deformed harmonic oscillator, with the additional advantage of automatically allowing for the introduction of a parameter that can control the width of the angle distribution function. In other words, we can see that the \( \mu \)-parameter present in the Rogers-Szegő polynomials variable – may stand for a squeezing parameter which controls the width of the angle distribution function.

### III. THE WIGNER FUNCTION FOR THE ROGERS-SZEGŐ FUNCTIONS

Although some proposals have been advanced in the past for defining the Wigner function for the Abelian case of an angular momentum - angle degree of freedom [46–49], in the present case we must draw our attention to the fact that the Rogers-Szegő polynomials are orthonormalized with respect to the Jacobi \( \vartheta_3 (\varphi; q) \) measure function, in contrast to the cases discussed before when that function is simply a constant.

Since \( \vartheta_3 (\varphi; q) \) is an even function of \( \varphi \) and guided by previous results [46–48], we define the Weyl-Wigner mapped expression for an operator by taking the Fourier transform namely,

\[ O(m, \theta) = \int_{-\pi}^{\pi} e^{im\tilde{\theta}} (\theta - \tilde{\theta} \vartheta_3 (\varphi - \tilde{\varphi}/2; q) \frac{d\tilde{\theta}}{2\pi}. \]  

(24)

This expression defines the quantum phase space representative of operator \( \hat{O} \). It must be noted that the choice for the \( \vartheta_3 (\varphi; q) \) argument could be \( \theta + \frac{\varphi}{2} \) as well; the change would result in a change of sign in the \( \tilde{\theta} \) variable that leads, however, to the same final expression.

In this form, if we choose \( \hat{O}_n = |n\rangle \langle n| \), the projector for the \( n \)-quanta harmonic oscillator (useful for writing a density operator \( \hat{\rho} = \sum_{n=0}^{\infty} p_n |n\rangle \langle n| \), \( p_n \) being probabilities), we can get the Wigner function associated to the Rogers-Szegő polynomials. Therefore, the Wigner function associated to one Rogers-Szegő function

\[ \langle \theta - \tilde{\theta}/2 | n \rangle = R_n \left( \theta - \tilde{\theta}/2; q \right) \]  

(25)

(and equivalently \( \langle n | \theta + \frac{\tilde{\varphi}}{2} \rangle \) ) will be,

\[ O_n (m, \theta) = \int_{-\pi}^{\pi} e^{im\tilde{\theta}} |n\rangle \langle n| \theta + \frac{\tilde{\varphi}}{2} \vartheta_3 (\varphi - \tilde{\varphi}/2; q) \frac{d\tilde{\theta}}{2\pi}. \]  

Using the convergence of the series defining the \( \vartheta_3 (\theta; q) \) function, we get

\[ O_n (m, \theta) = \sum_{t=-\infty}^{\infty} e^{-\mu t^2 + it\theta} \int_{-\pi}^{\pi} e^{im\tilde{\theta}} e^{-it\tilde{\theta}/2} R_n \left( \theta - \tilde{\theta}/2; q \right) R_n^* \left( \theta + \tilde{\theta}/2; q \right) \frac{d\tilde{\theta}}{2\pi}. \]  

Now, using Eqs. (17) and (8) we get the expression

\[ O_n (m, \theta) = \left[ \frac{q^n}{(q; q)_n} \right] \sum_{t=-\infty}^{\infty} e^{-\mu t^2 + it\theta} \sum_{r,s=0}^{n} (-1)^{r+s} \binom{n}{r} \binom{n}{s} e^{\mu(r+s)} e^{i\theta(r-s)} \]  

\[ \times \int_{-\pi}^{\pi} e^{i\tilde{\theta}(m-(t+r+s)/2)} \frac{d\tilde{\theta}}{2\pi} \]  

from which we obtain the Wigner function for the Rogers-Szegő polynomials.
\[ O_n (m, \theta) = \frac{q^n}{(q, q)_n} \sum_{t=-\infty}^{\infty} e^{-\mu t^2 + i t \theta} \sum_{r, s=0}^{n} (-1)^{r+s} \binom{n}{r} \binom{n}{s} \ e^{\mu(r+s)} e^{i \theta(r-s)} \]

\[
\times \frac{\sin \left( m - \frac{t+r+s}{2} \right) \pi}{\pi} \frac{(m - s)^2}{(m - s)^2}.
\]

(25)

Once we are given the Wigner function, we can extract the general form for the probability distribution both for angle as for angular momentum as well. To this end, we consider the general Wigner function, Eq. (25). First, integrating over the angle variable gives the angular momentum distribution probability, viz.

\[ \Lambda^{(n)} (m) = \int_{-\pi}^{\pi} \rho^{(n)} (m, \theta) \frac{d \theta}{2\pi} \]

\[
= \frac{q^n}{(q, q)_n} \sum_{r, s=0}^{n} (-1)^{r+s} \binom{n}{r} \binom{n}{s} \ e^{\mu(r+s)} e^{-\mu(s-r)^2} \frac{\sin (m - s) \pi}{(m - s) \pi}.
\]

In order to carry out the summations in this expression, we must recall the proof presented in the Appendix for the orthogonality of the Rogers-Szegő polynomials. Making use of expressions (A1, A2, A3), we see that

\[ \Lambda^{(n)} (m) = \frac{q^n}{(q, q)_n} \frac{(-1)^n q^{n-1}}{q} \prod_{r=0}^{n-1} (1 - q^{-r}) \frac{\sin (m - n) \pi}{(m - n) \pi}, \]

and, again from the orthogonality proof, we identify

\[ (-1)^n q^{n-1} \prod_{r=0}^{n-1} (1 - q^{-r}) = \left( \frac{q, q}{q} \right)_n \]

so that

\[ \Lambda^{(n)} (m) = \frac{\sin (m - n) \pi}{(m - n) \pi} = \delta_{m,n}. \]

(26)

Therefore, the angular momentum distribution function only assumes the discrete values associated to the polynomial indices, i.e., \( m \geq 0 \), thus playing the role of an action variable.

On the other hand, by performing the summation over the angular momentum variable we get the angle probability distribution, namely,

\[ \Omega^{(n)} (\theta, \mu) = \sum_{m=-\infty}^{\infty} \rho^{(n)} (m, \theta) = \frac{q^n}{(q, q)_n} \sum_{t=-\infty}^{\infty} e^{-\mu t^2 + i t \theta} \sum_{r, s=0}^{n} (-1)^{r+s} \binom{n}{r} \binom{n}{s} \ e^{\mu(r+s)} e^{i \theta(r-s)} \]

\[
\times \sum_{m=-\infty}^{\infty} \frac{\sin \left( m - \frac{t+r+s}{2} \right) \pi}{\pi} \frac{(m - s)^2}{(m - s)^2}.
\]

Now, we have to observe that

\[ \sum_{m=-\infty}^{\infty} \frac{\sin \left( m - \frac{t+r+s}{2} \right) \pi}{\pi} = 1 \]

for \( (t + r + s)/2 \) integer or half-integer, therefore we get

\[ \Omega^{(n)} (\theta, \mu) = \sum_{t=-\infty}^{\infty} e^{-\mu t^2 + i t \theta} \left\{ \frac{q^n}{(q, q)_n} \sum_{r, s=0}^{n} (-1)^{r+s} \binom{n}{r} \binom{n}{s} \ e^{\mu(r+s)} e^{i \theta(r-s)} \right\}. \]

The curly bracket can be immediately identified as

\[ \frac{q^n}{(q, q)_n} \sum_{r, s=0}^{n} (-1)^{r+s} \binom{n}{r} \binom{n}{s} \ e^{\mu(r+s)} e^{i \theta(r-s)} = |R_n (\theta; \mu)|^2 \]
so that, finally, the angle distribution probability reads

\[ \Omega^{(n)}(\theta, \mu) = \sum_{t=-\infty}^{\infty} e^{-\mu t^2 + it\theta} |R_n(\theta; \mu)|^2 \]  

as expected.

We can immediately verify that the Wigner function is normalized to unity by just integrating Eq. (27) in the interval \(-\pi \leq \theta < \pi\), and recalling the orthogonalization procedure, or by summing expression (26) over \(m\) in the range \(-\infty \leq m \leq \infty\).

As a first case of study it is now direct to particularize the Wigner function to the lowest Rogers-Szegö function, namely, let us consider \(n = 0\), the vacuum state projector. In this case

\[ O_n(m, \theta) = \sum_{t=-\infty}^{\infty} e^{-\mu t^2 + it\theta} \sin\left(\frac{m - \frac{t}{2}}{\pi}\right) \]  

that gives

\[ \Lambda^{(0)}(m) = \delta_{m,n} \]

for the action probability distribution. On the other hand, the angle probability distribution is

\[ \Omega^{(0)}(\theta, \mu) = \vartheta_3(\theta; \mu), \]

which is expected since for \(n = 0\), \(R_0(\theta; \mu) = 1\). This result coincides with the angle probability distribution as directly obtained from the definition of the Rogers-Szegö functions, Eq.(17). This distribution is shown in Figure 1.

In the same form we can obtain the normalized angle probability distribution for the second polynomial (projector \(O_1\)) that is written as

\[ \Omega^{(1)}(\theta, \mu) = \frac{e^{-2\mu}}{1 - e^{-2\mu}} (1 - 2e^{\mu} \cos \theta + e^{2\mu}) \vartheta_3(\theta; \mu), \]

and is depicted in Figure 2.

It is worth noticing that the angle probability distribution is \(\mu\)-dependent as expected, so that the width of \(\Omega^{(n)}(\theta, \mu)\) is governed by the free parameter \(q\) (or equivalently by \(\mu\)), which is the parameter of the deformed Heisenberg algebra.

**IV. SUMMARY AND DISCUSSION**

In this contribution we have discussed the main properties of the Rogers-Szegö polynomials, and it was emphasized the fact that they are in close connection with the eigenstates of the \(q\)-deformed harmonic oscillator, as has been also pointed out in the Macfarlane realization. Furthermore, knowing beforehand that these polynomials depend on an angular variable defined on the circle, upon which they are orthonormalized with respect to the Jacobi \(\vartheta_3\)- function, we propose that they can be used as good functions to describe phase states. By making use of their orthonormality requirements, we have also proposed a way to obtain the Wigner function associated to them. Additionally, using the proper trace operation techniques we have obtained the angle and action probability distribution functions as a by-product of the Wigner function. It is seen that the action variable only assumes values \(m \geq 0\), and that the angle distribution function is a well-behaved periodic function in the interval \(-\pi \leq \theta < \pi\). In fact, this distribution function is immediately identified as \(\vartheta_3(\theta; \mu) |R_n(\theta; \mu)|^2\), as expected.

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APPENDIX A: CARLITZ ORTHOGONALITY PROOF OF THE ROGERS-SZEGÖ POLYNOMIALS

Let us first consider

\[ I_{mn} = \sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{r+s} \binom{m}{r} \binom{n}{s} q^{\frac{1}{2}(r-1) + \frac{1}{2}(s-1)} q^{-rs}, \]  

(A1)

and let us show that \( I_{mn} = q^{-n} (q; q)_n \delta_{n,m} \). To this end we recall the result already presented in the text, Eq. (??)

\[ (x; q)_N = \prod_{j=0}^{N-1} (1 - q^j x) = \sum_{j=0}^{N} (-1)^j \binom{N}{j} q^{\frac{1}{2}j(j-1)} x^j \]

and put it in (A1) so that

\[ I_{mn} = \sum_{r=0}^{m} (-1)^r \binom{m}{r} q^{\frac{1}{2}(r-1) - 1} \prod_{s=0}^{n-1} (1 - q^{r-s}). \]  

(A2)

Now, without any loss of generality, we can assume that \( m \leq n \) (the inverse could also be considered). There are two situations to be discussed. First: For \( m < n \), it is evident that the product on the rhs of (A2) will vanish for all \( r \) (the rhs is constituted of a sum of products. Each summand has a product of terms where one of them will give \((1 - q^{r-s}) = 0\), since, as \( m < n \), \( s \) will necessarily assume the value \( r \)). Therefore, the sum only have vanishing summands, since there will always be a zero factor in the products.

Thus

\[ I_{mn} = 0 \quad \text{for} \quad m < n. \]  

(A3)

Second: For \( m = n \) there will be only one term to be considered, namely \( r = m \), that will give

\[ I_{nn} = (-1)^n q^{\frac{1}{2}(n-1)} \prod_{s=0}^{n-1} (1 - q^{s-n}). \]

To calculate this expression, let us explicitly write the product

\[ I_{nn} = (-1)^n q^{\frac{1}{2}(n-1)} \left( 1 - \frac{1}{q^n} \right) \left( 1 - \frac{1}{q^{n-1}} \right) \cdots \left( 1 - \frac{1}{q} \right) \]

\[ = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q)}{q^n} \]

which, upon identifying the numerator, gives

\[ I_{nn} = \frac{(q; q)_n}{q^n}. \]

This contribution together with (A3) gives the final result

\[ I_{mn} = q^{-n} (q; q)_n \delta_{n,m}. \]

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Figure Captions

Figure 1
Angle distribution functions associated to the first Rogers-Szegö polynomial obtained for some values of the parameter of the deformed harmonic oscillator algebra.

Figure 2
Angle distribution functions associated to the second Rogers-Szegö polynomial obtained for some values of the parameter of the deformed harmonic oscillator algebra.
Figure 1
Figure