Weyl correspondence method to construct multipartite entangled quantum state

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Via the Weyl correspondence approach, we construct multipartite entangled state which is the common eigenvector of their center-of-mass coordinate and mass-weighted relative momenta. This approach is concise and effective for setting up the Fock representation of continuous multipartite entangled states. The technique of integration within an ordered product (IWOP) of operators is also essential in our derivation.

I. INTRODUCTION

By inventing the symbolic method, establishing quantum mechanical representations and transformation theory in 1926, Dirac laid the mathematical-physical foundation of quantum mechanics [1]. The original work of comprising entanglement in quantum mechanics is introduced in 1935 by Einstein, Podolsky, and Rosen, who formulated the EPR paradox, a quantum-mechanical thought experiment designed to show that the theory is incomplete. Now quantum entanglement, which is now considered as the feature other than the hole of quantum mechanics, is widely studied as a physical resource, like energy, in quantum communication and quantum information [2, 3, 4, 5, 6, 7, 8]. Introducing the entangled representation will certainly help the study of entangled states. The EPR state denoted as $|\eta\rangle$, as one of the simplest bipartite entangled state is constructed in Ref. [9, 10, 11, 12], which was enlightened by Einstein-Podolsky-Rosen’s argument that two particles’ relative coordinate operator $Q_1 - Q_2$ (we use capital letter and small letter to represent operator and number separately, and henceforth) and the total momentum $P_1 + P_2$ are commutable and can be simultaneously measured [13]. It seems that introducing entangled state representation was inevitable since many entangled problems can only be clearly explained by it. An important question thus naturally arises: how to concisely obtain the explicit form of multipartite entangled state in the Fock representations? Do we have a convenient approach for it? The answer is affirmative. In this work with the help of method of integration within an ordered product of operators (IWOP) [14, 15], we shall adopt Weyl correspondence (Weyl quantization scheme) to realize our goal, and this approach is concise and effective for setting up the Fock representation. To illustrate our approach clearly, in Sec. II after briefly introducing the Weyl correspondence rule, we demonstrate how the Fock representation of bipartite entangled state $|\eta\rangle$ can be derived via
the Weyl correspondence approach. In Sec. III, IV and V we discuss the bipartite, tripartite case and multipartite EPR case respective. Via the Weyl correspondence approach, we concentrate on the setting up the Fock representation of continuous multipartite entangled states in this paper, however, this is not to say the approach is limited to this scene.

II. THE WEYL CORRESPONDENCE APPROACH AND VIA WHICH DERIVING BIPARTITE EPR STATE $|\eta\rangle$

Weyl correspondence is a quantization scheme which quantizes a classical function $h(q,p)$ as an operator by the following integration

$$H(P,Q) = \int_{-\infty}^{+\infty} dp dq h(p,q) \Delta(q,p),$$

where $\Delta(q,p)$ is the Wigner operator $[16]$, its original form in the coordinate representation is

$$\Delta(q,p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx' e^{-ipx'} |q-x'|^2 \langle x + x' | x + x' |$$

(2)

In Ref. $[17]$ we have performed this integral using $IWOP$ and get the normally ordered form of $\Delta(q,p)$,

$$\Delta(q,p) = \frac{1}{\pi} \exp \left[ - (q - Q)^2 - (p - P)^2 \right] :$$

(3)

where the symbol $: :$ denotes normal ordering.

In Ref. $[19]$ Fan introduced the Weyl ordering by symbol $: :$, which is defined through the Weyl quantization scheme to quantize classical quantity $q^m p^n$ as

$$q^m p^n \rightarrow \left( \frac{1}{2} \right)^m \sum_{l=0}^{m} \binom{m}{l} Q^{m-l} P^n Q^l \rightarrow : Q^m P^n :,$$

(4)

and the Weyl ordering possesses three remarkable properties $[19]$:

(a) the order of Bose operators within a Weyl ordered product (or within the Weyl ordering symbol $: :$) can be permuted; the right-hand side of (4) exhibits the definition of Weyl ordering, so

$$\left( \frac{1}{2} \right)^m \sum_{l=0}^{m} \binom{m}{l} Q^{m-l} P^n Q^l = : \left( \frac{1}{2} \right)^m \sum_{l=0}^{m} \frac{m!}{l! (m-l)!} Q^{m-l} P^n Q^l : = : Q^m P^n :,$$

(5)
which means

\[ Q^m P^n := \int_{-\infty}^{\infty} dp dq \eta^m p^n \Delta (q,p). \] (6)

It then follows the second property,

(b) Comparing (6) with (1) we derive the Weyl ordered form of the Wigner operator is

\[ \delta (p - P) \delta (q - Q) \] (7)

Moreover, we can have the technique of integration within the Weyl ordered product (IWWOP) of operators:

(c) a Weyl ordered product can be integrated with respect to a \( c \)-number provided that the integration is convergent.

Thus the Weyl quantization rule for a classical function \( h(p,q) \) transiting to its quantum operator is

\[
\begin{align*}
&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dp dq h(p,q) \Delta (q,p) \\
&\quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dp dq \delta (p - P) \delta (q - Q) = h(P,Q),
\end{align*}
\] (8)

which means that a Weyl ordered operator \( h(P,Q) \)'s classical correspondence is just \( h(p,q) \).

\( |\eta\rangle \) is the bipartite EPR state \( \eta = \eta_1 + i\eta_2 \), whose definite expression and eigen-equations are

\[
|\eta\rangle = \exp[-|\eta|^2/2 + \eta a_1 \dagger - \eta^* a_2 \dagger + a_1 \dagger a_2 \dagger] |00\rangle
\]

\[
\begin{align*}
(Q_1 - Q_2) |\eta\rangle &= \sqrt{2} \eta_1 |\eta\rangle \\
(P_1 - P_2) |\eta\rangle &= \sqrt{2} \eta_2 |\eta\rangle
\end{align*}
\]

where in Fork space \( a_1 \dagger, a_2 \dagger \) are generate operators. The classical Weyl correspondence of the projector of the projector operator \( |\eta\rangle \langle \eta| \) writes

\[
|\eta\rangle \langle \eta| \Rightarrow \delta \left[ \sqrt{2} \eta_1 - (q_1 - q_2) \right] \delta \left[ \sqrt{2} \eta_2 - (p_1 + p_2) \right]
\] (9)
that is

\[ |\eta\rangle \langle \eta| \]

\[ \Rightarrow \iiint dq_1 dp_1 dq_2 dp_2 \delta \left[ \sqrt{2} \eta_1 - (q_1 - q_2) \right] \delta \left[ \sqrt{2} \eta_2 - (p_1 + p_2) \right] \Delta_1 (q_1, p_1) \Delta_2 (q_2, p_2) \]

\[ = \iiint dq_1 dp_1 dq_2 dp_2 \delta \left[ \sqrt{2} \eta_1 - (q_1 - q_2) \right] \delta \left[ \sqrt{2} \eta_2 - (p_1 + p_2) \right] \times \frac{1}{\pi^2} \exp \left[ - (q_1 - Q_1)^2 - (p_1 - P_1)^2 - (q_2 - Q_2)^2 - (p_2 - P_2)^2 \right] \]

\[ = \frac{1}{2\pi} \exp \left\{ \frac{1}{2} \left[ - \left( \sqrt{2} \eta_1 - (Q_1 - Q_2)^2 \right) - \left( \sqrt{2} \eta_2 - (P_1 + P_2)^2 \right) \right] \right\} : \] (10)

Due to

\[ Q_i = \left( a_i + a_i^\dagger \right) / \sqrt{2}, \quad P_i = \left( a_i - a_i^\dagger \right) / (i \sqrt{2}), \] (11)

where \( [a_i, a_j^\dagger] = \delta_{ij} \), we rewrite (10) as

\[ |\eta\rangle \langle \eta| = |C|^2 \frac{1}{2\pi} \exp \left\{ \frac{1}{2} \left[ - \left( \sqrt{2} \eta_1 - (Q_1 - Q_2)^2 \right) - \left( \sqrt{2} \eta_2 - (P_1 + P_2)^2 \right) \right] \right\} : \]

\[ = |C|^2 \frac{1}{2\pi} \exp \left\{ - |\eta|^2 + \eta^* a_1 + \eta a_1^\dagger - \eta a_2 - \eta^* a_2^\dagger - a_1^\dagger a_1 - a_2^\dagger a_2 + a_1^\dagger a_2 + a_2^\dagger a_1 \right\} : \]

and by using : \( \exp \left\{ -a_1^\dagger a_1 - a_2^\dagger a_2 \right\} = |00\rangle \langle 00| \), we can decompose the \( |\eta\rangle \langle \eta| \) as following

\[ \exp \left\{ - |\eta|^2 + \eta^* a_1 + \eta a_1^\dagger - \eta a_2 - \eta^* a_2^\dagger - a_1^\dagger a_1 - a_2^\dagger a_2 + a_1^\dagger a_2 + a_2^\dagger a_1 \right\} : \]

\[ = \exp \left[ -|\eta|^2 / 2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_2^\dagger a_2 \right] \exp \left[ -a_1^\dagger a_1 - a_2^\dagger a_2 \right] : \exp \left[ -|\eta|^2 / 2 + \eta^* a_1 - \eta a_2 + a_1 a_2 \right] \]

\[ = f(a_1^\dagger, a_2^\dagger) |00\rangle \langle 00| f^\dagger(a_1^\dagger, a_2^\dagger) \]

and the normalized constant \( C \) can be determined by normalized condition \( \langle \eta | \eta' \rangle = \pi \delta^{(2)} (\eta - \eta') \) up to random phase, so we deducted the expression of \( |\eta| \) as

\[ |\eta| = \exp \left[ -|\eta|^2 / 2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_2^\dagger a_2 \right] |00\rangle \] (12)

This is the bipartite EPR state, it is easy to check its (over)completeness and orthogonal
\[
\langle \eta | \eta' \rangle = \pi \delta^{(2)}(\eta - \eta')
\]
\[
\int \frac{d^2 \eta}{\pi} |\eta\rangle \langle \eta| = 1
\]

So far, we success deduce the expression of \( |\eta\rangle \) by employing the Weyl correspondence approach. In the following sections, we will also employ the same approach to derive the bipartite, tripartite and multipartite representation of their center-of-mass coordinate and mass-weighted relative momentum entangled state.

**III. DERIVING THE COMMON EIGENSTATES OF BIPARTITE’S CENTER-OF-MASS
COORDINATE AND MASS-WEIGHTED RELATIVE MOMENTUM VIA WEYL
CORRESPONDENCE APPROACH**

In this section we will find the common eigenstates of bipartite’s center-of-mass coordinate and mass-weighted relative momentum through the Weyl correspondence and the IWOP technique. Following our previous work, we introduce the bipartite’s center-of-mass coordinate operator \( Q_{cm} = \mu_1 Q_1 + \mu_2 Q_2 \), and the mass-weighted relative momentum operator \( P_r = P_1/\mu_1 - P_2/\mu_2 \) of bipartite system, where \( \mu_i \) is the relative of mass defined \( \mu_i = m_i/(m_1 + m_2) \), and \( m_i \) is the mass of the \( i \)-th paritcle, and we have \( \mu_1 + \mu_2 = 1 \). It is easy to check that the two operators are compatible \([Q_{cm}, P_r] = 0\), so it is meaningfull to construct the simultaneous eigenstate of \( Q_{cm} \) and \( P_r \) in terms of conventional creation and annihilation operators, and we note its common eigenvector is \( |q_{cm}, \varrho\rangle \), which is \( |\xi\rangle \) in Ref. [9]. In this section we shall employ the Weyl correspondence and the IWOP technique to derive the explicit form of \( |\xi\rangle \) in two-mode Fock space, and our starting point are the the eigen-equations write

\[
(\mu_1 Q_1 + \mu_2 Q_2) |q_{cm}, \varrho\rangle = q_{cm} |q_{cm}, \varrho\rangle ,
\]
\[
\left(\frac{P_1}{\mu_1} - \frac{P_2}{\mu_2}\right) |q_{cm}, \varrho\rangle = \varrho |q_{cm}, \varrho\rangle ,
\]

from which we can write the classical Weyl correspondence of the projector \( |q_{cm}, \varrho\rangle \langle q_{cm}, \varrho| \),

\[
|q_{cm}, \varrho\rangle \langle q_{cm}, \varrho| \Longrightarrow \delta |q_{cm} - (\mu_1 q_1 + \mu_2 q_2)| \delta \left[ \varrho - \left(\frac{p_1}{\mu_1} - \frac{p_2}{\mu_2}\right) \right]
\]

According to (14), we know the classical Weyl correspondence of the projector \( |q_{cm}, \varrho\rangle \langle q_{cm}, \varrho| \) is
\[ |q_{cm}, \varrho \rangle \langle q_{cm}, \varrho | \]
\[ \Rightarrow \iint \int \int dq_1 dq_2 dp_1 dp_2 \delta [q_{cm} - (\mu_1 q_1 + \mu_2 q_2)] \delta \left[ q - \left( \frac{p_1}{\mu_1} - \frac{p_2}{\mu_2} \right) \right] \Delta_1 (q_1, p_1) \Delta_2 (q_2, p_2) \]
\[ = \iint \int \int dq_1 dq_2 dp_1 dp_2 \delta [q_{cm} - (\mu_1 q_1 + \mu_2 q_2)] \delta \left[ q - \left( \frac{p_1}{\mu_1} - \frac{p_2}{\mu_2} \right) \right] \]
\[ \times \frac{1}{\pi^2} : \exp \left[ -(q_1 - Q_1)^2 - (p_1 - P_1)^2 - (q_2 - Q_2)^2 - (p_2 - P_2)^2 \right] : \]
\[ = \int dq_1 dp_1 \frac{1}{\pi^2} : \exp \left[ -(q_1 - Q_1)^2 - (p_1 - P_1)^2 - \left( \frac{q_{cm} - \mu_1 q_1}{\mu_2} - Q_2 \right)^2 - \left( \frac{p_2}{\mu_1} - \mu_2 \varrho - P_2 \right)^2 \right] : \]
\[ = \frac{\mu_1 \mu_2}{\pi (\mu_1^2 + \mu_2^2)} : \exp \left\{ \frac{1}{\left( \mu_1^2 + \mu_2^2 \right)} \left[ -(q_{cm} - (\mu_1 Q_1 + \mu_2 Q_2))^2 - \mu_1^2 \mu_2^2 \left( \varrho - \left( \frac{P_1}{\mu_1} - \frac{P_2}{\mu_2} \right) \right)^2 \right] \right\} \quad (15) \]

Due to (14) we rewrite (15) as

\[ |q_{cm}, \varrho \rangle \langle q_{cm}, \varrho | = |C|^2 \frac{\mu_1 \mu_2}{\pi \lambda} : \exp \left\{ \frac{1}{\lambda} \left[ -(q_{cm} - (\mu_1 Q_1 + \mu_2 Q_2))^2 - (\mu_1 \mu_2 \varrho - (\mu_2 P_1 - \mu_1 P_2))^2 \right] \right\} : \quad (16) \]

where \( \lambda \equiv \mu_1^2 + \mu_2^2 \), using \( : \exp \left( -a_1^\dagger a_1 - a_2^\dagger a_2 \right) : = |00\rangle \langle 00| \), and \( C \) is the normalized constant that can be determined by

\[ \langle q_{cm}, \varrho | q'_{cm}, \varrho' \rangle = \delta_q (q' - q) \delta (q'_{cm} - q_{cm}) \quad (\mu_1 = \mu_2) \]

up to a random phase, and this can be evaluated easily by insert overcompleteness of coherent state

\[ \int d^2 z \left| z \right\rangle \langle z | = \pi , \text{ where } \left| z \right\rangle = D (z) \left| 0 \right\rangle , \text{ and } D (z) = \exp \left[ za_1^\dagger - z^* a \right] \text{ is displacement operator,} \]

\[ \langle z' | z \rangle = \exp \left[ z'^* z - \left( |z|^2 + |z'|^2 \right)/2 \right] , \text{ so } (z_i = x_i + i y_i , \text{ where } x_i \text{ and } y_i \text{ are real and image part of } z_i) \]

\[ \int \frac{d^2 z_1 d^2 z_2}{\pi^2} \langle q_{cm}, \varrho | z_1 z_2 \rangle \langle z_1 z_2 | q'_{cm}, \varrho' \rangle \]
\[ = |C|^2 \int \frac{d^2 z_1 d^2 z_2}{\pi^2} \frac{\mu_1 \mu_2}{\pi \lambda} \exp \left\{ \frac{1}{\lambda} \left[ \sqrt{2} \mu_1 (q_{cm} + i q' \mu_2^2) z_1^* + \sqrt{2} \mu_2 (q_{cm} - i q' \mu_1^2) z_2^* \right. \right. \]
\[ - \frac{1}{2} (\mu_1^2 - \mu_2^2) (z_1^2 z_1^* + z_2^2 z_2^*) - 2 \mu_1 \mu_2 z_1^* z_2^* \left. \left. - \frac{1}{2} (q_{c_2}^2 + (\mu_1 \mu_2 \varrho^2))^2 \right] \right\} \]
\[ \exp \left[-z_1 z_1^* - z_2 z_2^* \right] \exp \left\{ \frac{1}{\lambda} \left[ \sqrt{2} \mu_1 (q_{cm} - i g \mu_2^2) z_1 + \sqrt{2} \mu_2 (q_{cm} + i g \mu_1^2) z_2 \right. \right. \]
\[ - \frac{1}{2} (\mu_1^2 - \mu_2^2) (z_1 z_1 + z_2 z_2) - 2 \mu_1 \mu_2 z_1 z_2 \left. \left. - \frac{1}{2} (g_2^2 + (\mu_1 \mu_2 \varrho^2))^2 \right] \right\} \]
\[ = |C|^2 \int \frac{dx_2 dy_2}{2 \pi^2} \exp \left[ -i \sqrt{2} \mu_2 (q' - \varrho) x_2 - i \sqrt{2} (q'_{cm} - q_{cm}) \frac{\mu_2}{\mu_1} + C (q', \varrho, q'_{cm}, q_{cm}) \right] \]
\[ = |C|^2 \delta (q' - \varrho) \delta (q'_{cm} - q_{cm}) \exp \left[ C (q', \varrho, q'_{cm}, q_{cm}) \right] \quad (18) \]
where

\[
C(q', \varrho, q_{cm}', q_{cm}) = -\frac{2i\mu_2^2}{\lambda} \left[ \mu_1^2 (q' + \varrho) (q_{cm}' - q_{cm}) - \mu_2^2 (q' - \varrho) (q_{cm}' + q_{cm}) \right] \\
\]

\[
- \frac{\mu_1^2 (q' - \varrho)^2 + (q_{cm}' - q_{cm})^2}{4\mu_2^2 (\mu - i\mu_2) (\mu + i\mu_2)}
\]

When \( \lambda = \mu_1 = \mu_2 = 1/2 \), equation (18) writes

\[
\int \frac{d^2 z_1 d^2 z_2}{\pi^2} \langle q_{cm}, \varrho | z_1 z_2 \rangle \langle z_1 z_2 | q_{cm}'', q' \rangle = |C|^2 \delta (q' - \varrho) \delta (q_{cm}' - q_{cm})
\]

so, we can select \( C = 1 \), and the decomposition yields

\[
|q_{cm}, \varrho \rangle = \sqrt{\frac{\mu_1 \mu_2}{\pi \lambda}} \exp \left\{ \frac{1}{\lambda} \left[ \sqrt{2} \mu_1 (q_{cm} + i\mu_2) a_1^\dagger + \sqrt{2} \mu_2 (q_{cm} - i\mu_2) a_2^\dagger \\
- \frac{1}{2} (\mu_1^2 - \mu_2^2) \left( a_1 a_1^\dagger - a_2 a_2^\dagger \right) - 2\mu_1 \mu_2 a_1^\dagger a_2^\dagger - \frac{1}{2} \left( \mu_1^2 + (\mu_1 \mu_2 \varrho)^2 \right) \right] \right\} |00\rangle.
\]

(19)

If we set \( q_{cm} = \sqrt{\lambda} \xi_q \), \( \varrho = \sqrt{\lambda} \xi_p \), and \( \xi = \xi_q + i \xi_p \), then equation (19) can be rewritten as

\[
|\xi \rangle = \frac{\sqrt{\mu_1 \mu_2}}{\lambda} \exp \left\{ -\frac{1}{2} |\xi|^2 + \frac{1}{\sqrt{2\lambda}} [\xi + (\mu_1 - \mu_2) \xi^*] a_1^\dagger + \frac{1}{\sqrt{2\lambda}} [\xi^* - (\mu_1 - \mu_2) \xi] a_2^\dagger \\
- \frac{1}{2\lambda} (\mu_1 - \mu_2) \left( a_1^\dagger a_2^\dagger - a_2 a_1^\dagger \right) - 4\mu_1 \mu_2 a_1^\dagger a_2^\dagger \right\} |00\rangle.
\]

(20)

When \( \lambda = \mu_1 = \mu_2 = 1/2 \),

\[
|\xi \rangle = \exp \left\{ -\frac{1}{2} |\xi|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger \right\} |00\rangle
\]

This \( |\xi \rangle \) is the conjugate state of \( |\eta \rangle \) in (12). Using the IWOP technique we have

\[
\int \frac{d^2 \xi}{\pi} \langle \xi | \xi \rangle = \int \int \frac{d q_{cm} d \varrho}{\pi \lambda} ; \quad \exp \left\{ \frac{1}{\lambda} \left[ - (q_{cm} - (\mu_1 Q_1 + \mu_2 Q_2))^2 - (\mu_1 \mu_2 \varrho - (\mu_2 P_1 - \mu_1 P_2))^2 \right] \right\}
\]

(21)

This is a convenient approach for finding bipartite entangled state. and its eigen-equations write

\[
(Q_{cm} | \xi \rangle = \sqrt{\lambda} \xi_q | \xi \rangle,
\]

\[
P_r | \xi \rangle = \sqrt{\lambda} \xi_p | \xi \rangle.
\]

(22)
IV. DERIVING THE COMMON ENTANGLED EIGENSTATES OF TRIPARTITE’S CENTER-OF-MASS COORDINATE AND MASS-WEIGHTED RELATIVE MOMENTA

Having experienced how to find the common eigenstates of bipartite’s center-of-mass coordinate and mass-weighted relative momentum via the Weyl correspondence and the IWOP technique, we now search for the common eigenvector of the tripartite. Noticing that the three compatible operators: the center-of-mass coordinate \( \mu_1 Q_1 + \mu_2 Q_2 + \mu_3 Q_3 \), and \( P_1/\mu_1 - P_2/\mu_2 \), \( P_1/\mu_1 - P_3/\mu_3 \), which are mass-weighted relative momentums, where \( \mu_i \) is the relative of mass defined \( \mu_i = m_i/M \), and \( m_i \) is the mass of the \( i \)-th particle, \( M = m_1 + m_2 + m_3 \) is the total mass of all three particles, so \( \mu_1 + \mu_2 + \mu_3 = 1 \). Since the three operators are complete and compatible with each other, there is a representation spaned by their common eigenvector \( |q, q_2, q_3 \rangle \), which satisfies the following eigen-equations

\[
(\mu_1 Q_1 + \mu_2 Q_2 + \mu_3 Q_3) |q, q_2, q_3 \rangle = q |q, q_2, q_3 \rangle. \tag{23a}
\]
\[
(\frac{P_1}{\mu_1} - \frac{P_2}{\mu_2}) |q, q_2, q_3 \rangle = q_2 |q, q_2, q_3 \rangle. \tag{23b}
\]
\[
(\frac{P_1}{\mu_1} - \frac{P_3}{\mu_3}) |q, q_2, q_3 \rangle = q_3 |q, q_2, q_3 \rangle. \tag{23c}
\]

In similar to (14)

\[
|q, q_2, q_3 \rangle \langle q, q_2, q_3 | \implies \delta \left\{ q_2 - \left( \frac{p_1}{\mu_1} - \frac{p_2}{\mu_2} \right) \right\} \delta \left[ q_3 - \left( \frac{p_1}{\mu_1} - \frac{p_3}{\mu_3} \right) \right] \times \delta [q - (\mu_1 q_1 + \mu_2 q_2 + \mu_3 q_3)]. \tag{24}
\]
Accordingly, we have

$$\langle q, q_2, q_3 | \langle q, q_2, q_3 \rangle$$

$$\Rightarrow \int \ldots \int \prod_{i=1}^{3} dq_i dp_i \delta [q_2 - \left( \frac{p_1}{\mu_1} - \frac{p_2}{\mu_2} \right)] \delta [q_3 - \left( \frac{p_1}{\mu_1} - \frac{p_3}{\mu_3} \right)]$$

$$\times \delta [q - (\mu_1 q_1 + \mu_2 q_2 + \mu_3 q_3)] \frac{1}{\pi^3} \text{ exp } \left\{ \sum_{i=1}^{3} \left[ - (q_i - Q_i)^2 - (p_i - P_i)^2 \right] \right\} :$$

$$= \int \ldots \int \prod_{i=1}^{3} dq_i dp_i \left. \frac{\mu_1 \mu_2 \mu_3}{\lambda} \right. \text{ exp } \left\{ - \frac{1}{\lambda} \left[ \mu_1 \mu_2 q_2 - (\mu_2 P_1 - \mu_1 P_2)^2 + \mu_1 \mu_3 q_3 - (\mu_3 P_1 - \mu_1 P_3)^2 \right] \right.$$

$$+ \mu_2 \mu_3 (q_2 - q_3) + (\mu_3 P_2 - \mu_2 P_3)^2 - \left( q - \sum_{i=1}^{3} \mu_i Q_i \right)^2 \right\}$$

(25)

where \( \lambda = \sum_{i=1}^{3} \mu_i^2 \),

$$|q, q_2, q_3 \rangle \langle q, q_2, q_3 |$$

$$= |C|^2 \pi^{-3/2} \frac{\mu_1 \mu_2 \mu_3}{\lambda} \text{ exp } \left\{ - \frac{1}{\lambda} \left[ \right. \right.$$

$$\left. \mu_1 \mu_2 q_2 - (\mu_2 P_1 - \mu_1 P_2)^2 + \mu_1 \mu_3 q_3 - (\mu_3 P_1 - \mu_1 P_3)^2 \right.$$ (26)

$$+ \mu_2 \mu_3 (q_2 - q_3) + (\mu_3 P_2 - \mu_2 P_3)^2 - \left( q - \sum_{i=1}^{3} \mu_i Q_i \right)^2 \right\}$$

(27)

the normalization constant \( C \) is determined by

$$\langle q, q_2, q_3 | q', q_2', q_3' \rangle = \delta (q' - q) \delta (q_2' - q_2) \delta (q_3' - q_3), \quad (\mu_1 = \mu_2 = \mu_3)$$

Similar procedure to the bipartite situation works out \( C = 1 \), So splitting the right hand side of

(27) as the form \( f \left( \hat{a}_i^\dagger \right) |000\rangle \langle 000 | f^\dagger \left( \hat{a}_i^\dagger \right) \), where

$$: \text{ exp } \left( - \sum_{i=1}^{3} \hat{a}_i^\dagger \hat{a}_i \right) : = |000\rangle \langle 000 |,$$ (28)
After the decomposition, we can have [22]

$$|q, q_2, q_3\rangle = \pi^{-3/4} \frac{\mu_1 \mu_2 \mu_3}{\lambda} \exp \left\{ A + \frac{\sqrt{2q}}{\lambda} \sum_{i=1}^{3} \frac{\mu_i a_i^{\dagger}}{3} + \frac{i\sqrt{2\mu_2 q_2}}{\lambda} \left[ \mu_1 \mu_2 a_1^{\dagger} - (\mu_1^2 + \mu_2^2) a_2^{\dagger} + \mu_2 \mu_3 a_3^{\dagger} \right] + \frac{i\sqrt{2\mu_3 q_3}}{\lambda} \left[ \mu_1 \mu_3 a_1^{\dagger} + \mu_2 \mu_3 a_2^{\dagger} - (\mu_1^2 + \mu_3^2) a_3^{\dagger} \right] + S \right\} |000\rangle \tag{29}$$

Where

$$A = -\frac{q^2}{2\lambda} - \frac{1}{2\lambda} \left[ -2\mu_2^2 \mu_3 q_2 q_3 + (\mu_1^2 + \mu_2^2) \mu_2^2 q_2^2 + (\mu_1^2 + \mu_2^2) \mu_3^2 q_3^2 \right]$$

$$S = -\frac{1}{\lambda} \sum_{i,j=1}^{3} \left( \mu_i \mu_j a_i a_j - \frac{\lambda}{2} \delta_{ij} \right)$$

When $\lambda = \mu_1 = \mu_2 = \mu_3 = 1/3$, the equation (29) can be rewritten to

$$|q, q_2, q_3\rangle = \frac{1}{\sqrt{3\pi^2/4}} \exp \left\{ -\frac{3}{2} q^2 + \frac{\sqrt{2q}}{\lambda} \left( a_1^{\dagger 2} + a_2^{\dagger 2} + a_3^{\dagger 2} \right) - \frac{2}{3} \left( a_1^{\dagger} a_2^{\dagger} + a_1^{\dagger} a_3^{\dagger} + a_2^{\dagger} a_3^{\dagger} \right) \right\} \tag{30}$$

$$+ \frac{\sqrt{2q}}{9} \left[ (q_2 + q_3) a_1^{\dagger} + (2q_2 - q_3) a_2^{\dagger} + (2q_3 - q_2) a_3^{\dagger} \right] \tag{31}$$

$$- \frac{1}{27} \left( q_2^2 + q_3^2 - q_2 q_3 \right) \right\} |000\rangle \tag{32}$$

Using the IWOP technique and Eq. (27) we have

$$\int\int\int_{-\infty}^{\infty} dq dq_2 dq_3 \ |q, q_2, q_3\rangle \langle q, q_2, q_3|$$

$$= \int\int\int_{-\infty}^{\infty} dq dq_2 dq_3 \pi^{-3/2} \frac{\mu_1 \mu_2 \mu_3}{\lambda} \exp \left\{ -\frac{1}{\lambda} \left[ \mu_1 \mu_2 q_2 - (\mu_2 P_1 - \mu_1 P_2)^2 + \mu_1 \mu_3 q_3 - (\mu_3 P_1 - \mu_1 P_3)^2 \right]ight. + \left. [\mu_2 \mu_3 (q_2 - q_3) + (\mu_3 P_2 - \mu_2 P_3)]^2 - \left( q - \sum_{i=1}^{3} \mu_i Q_i \right)^2 \right\}$$

$$= : \exp (0) : = 1. \tag{33}$$

The Weyl correspondence approach is a very direct way to find the tripartite entangled state $|q, q_2, q_3\rangle$, which make up a complete set.

**V. DERIVING THE ENTANGLED EIGENSTATE OF MULTIPARTITE’S CENTER-OF-MASS COORDINATE AND MASS-WEIGHTED RELATIVE MOMENTA**

Enlightened by the former method, we now search for the mass-dependent multipartite entangled system via the Weyl correspondence and the IWOP technique, we introduce commute
operators: the center-of-mass coordinate $\sum_{i=1}^{n} \mu_i Q_i$, and $P_j/\mu_j - P_i/\mu_i$, ($i, j = 1, 2, \cdots, n$), the mass-weighted and relative momentums, where $\mu_i$ is the relative mass defined $\mu_i = m_i/M$, $m_i$ is the mass of the $i$-th particle, $M = \sum_{i=1}^{n} m_i$ is the total mass of all particles, and $\sum_{i=1}^{n} \mu_i = 1$. Since these operators are commute with each other, there will be common eigenvectors, denoted as $|q, \varrho_2, \varrho_3, \cdots \varrho_n\rangle$ for them, which satisfies the following eigen-equations

$$\sum_{i=1}^{n} \mu_i Q_i |q, \varrho_2, \varrho_3, \cdots \varrho_n\rangle = q |q, \varrho_2, \varrho_3, \cdots \varrho_n\rangle .$$  \hspace{1cm} (34a)$$

$$\left( \frac{p_i}{\mu_i} - \frac{p_j}{\mu_j} \right) |q, \varrho_2, \varrho_3, \cdots \varrho_n\rangle = \varrho_i |q, \varrho_2, \varrho_3, \cdots \varrho_n\rangle , \hspace{0.5cm} i = 2, \cdots, n.$$  \hspace{1cm} (34b)$$

From the above discussion and the eigen-equations of $|q, \varrho_2, \varrho_3, \cdots \varrho_n\rangle$, we immediately write down the classical Weyl correspondence of the projector $|q, \varrho_2, \varrho_3, \cdots \varrho_n\rangle \langle q, \varrho_2, \varrho_3, \cdots \varrho_n|$, we have

$$|q, \varrho_2, \varrho_3, \cdots \varrho_n\rangle \langle q, \varrho_2, \varrho_3, \cdots \varrho_n| \Longrightarrow \delta \left( q - \sum_{i=1}^{n} \mu_i q_i \right) \prod_{j=2}^{n} \delta \left[ \varrho_j - \left( \frac{p_1}{\mu_1} - \frac{p_j}{\mu_j} \right) \right] \prod_{i=1}^{n} \Delta_i (q_i, p_i) ,$$  \hspace{1cm} (35)$$

that is

$$|q, \varrho_2, \varrho_3, \cdots \varrho_n\rangle \langle q, \varrho_2, \varrho_3, \cdots \varrho_n|$$

$$\Longrightarrow \int \cdots \int \prod_{i=1}^{n} dq_i dp_i \delta \left( q - \sum_{i=1}^{n} \mu_i q_i \right) \prod_{j=2}^{n} \delta \left[ \varrho_j - \left( \frac{p_1}{\mu_1} - \frac{p_j}{\mu_j} \right) \right] \prod_{i=1}^{n} \Delta_i (q_i, p_i)$$

$$= \frac{1}{\pi^n} \int \cdots \int \prod_{i=1}^{n} dq_i dp_i \delta \left( q - \sum_{i=1}^{n} \mu_i q_i \right) \prod_{j=2}^{n} \delta \left[ \varrho_j - \left( \frac{p_1}{\mu_1} - \frac{p_j}{\mu_j} \right) \right]$$

$$\times \exp \left\{ \sum_{i=1}^{n} \left[ -(q_i - Q_i)^2 - (p_i - P_i)^2 \right] \right\} :$$

$$= \frac{1}{\pi^n} \prod_{i=1}^{n} \mu_i \prod_{i=1}^{n} \mu_n \int \cdots \int dp_1 \prod_{i=1}^{n-1} dq_i : \exp \left[ -\sum_{i=1}^{n-1} (q_i - Q_i)^2 - \left( \frac{q - \sum_{i=1}^{n-1} \mu_i q_i}{\mu_n} - Q_n \right)^2 \right.$$

$$- (p_1 - P_1)^2 - \sum_{i=2}^{n} (-P_i)^2 \right] :$$

$$= \frac{1}{\pi^n} \prod_{i=1}^{n} \mu_i \prod_{i=1}^{n} \mu_n \int dp_1 : \exp \left[ - (p_1 - P_1)^2 - \sum_{i=2}^{n} \left( \frac{\mu_i p_1 - \mu_i q_i - P_i}{\mu_n} \right)^2 \right] :$$

$$\times \int \cdots \int \prod_{i=1}^{n-1} dq_i : \exp \left[ - \left( \frac{q}{\mu_n} - \sum_{i=1}^{n-1} \frac{\mu_i q_i}{\mu_n} - Q_n \right)^2 - \sum_{i=1}^{n-1} (q_i - Q_i)^2 \right] :$$  \hspace{1cm} (36)$$

where

$$\int dp_1 : \exp \left[ - (p_1 - P_1)^2 - \sum_{i=2}^{n} \left( \frac{\mu_i p_1 - \mu_i q_i - P_i}{\mu_n} \right)^2 \right] : = \mu_1 \sqrt{\frac{\pi}{\lambda}} N .$$  \hspace{1cm} (37)$$
and \( \lambda = \sum_{i=1}^{n} \mu_i^2 \), and we introduced \( q_1 = 0 \) for simplification of \( N \)

\[
N = \exp \left\{ -\frac{1}{\lambda} \left[ \sum_{j=2}^{n} [\mu_j \mu_j (q_j - \rho_j)]^2 + \sum_{j<k=2}^{n} [\mu_j \mu_k (q_j - \rho_k) + (\mu_k \rho_j - \mu_j \rho_k)]^2 \right] \right\} = \exp \left\{ -\frac{1}{2\lambda} \left[ \sum_{j,k=1}^{n} [\mu_j \mu_k (q_j - \rho_k) + (\mu_k \rho_j - \mu_j \rho_k)]^2 \right] \right\}
\]

To integrate over the remaining part of equation (36), we resort to the following mathematical formula

\[
\int \cdots \int_{-\infty}^{\infty} d^n \chi \exp[-\chi \mathbf{B} \chi + \chi \mathbf{v}] = \sqrt{\frac{\pi^n}{\det \mathbf{B}}} \exp \left\{ \frac{1}{4} \mathbf{v} \mathbf{B}^{-1} \mathbf{v} \right\}, \quad (38)
\]

where \( \mathbf{B} \) is a symmetric positive-definite invertible covariant tensor of rank \( n \), \( \tilde{\chi} = (\chi_1, \chi_2, \chi_3, \ldots, \chi_n) \) is transpose of \( \chi \), and so we can obtain

\[
\int \cdots \int_{-\infty}^{\infty} d^n q_i \exp \left\{ -\left( \frac{q_i}{\mu_n} - \sum_{i=1}^{n-1} \frac{\mu_i q_i}{\mu_n} - Q_n \right)^2 - \sum_{i=1}^{n-1} (q_i - Q_i)^2 \right\} = \int \cdots \int_{-\infty}^{\infty} d^n q_i \exp \left\{ -(q_1, q_2, \ldots, q_{n-1}) \mathbf{B} \left( \begin{array}{c} q_1 \\ q_2 \\ \vdots \\ q_{n-1} \end{array} \right) + (q_1, q_2, \ldots, q_{n-1}) \left( \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{array} \right) - \text{const} \right\}, \quad (39)
\]

and

\[
\mathbf{B} = (B_{ij})_{(n-1) \times (n-1)}, \quad (40a)
\]

\[
B_{ij} = \frac{\mu_i \mu_j}{\mu_n^2} + \delta_{ij}, \quad (40b)
\]

\[
v_i = \frac{2 \mu_i}{\mu_n^2} q + 2Q_i - \frac{2 \mu_i}{\mu_n} Q_n, \quad (40c)
\]

\[
\det \mathbf{B} = 1 + \sum_{i=1}^{n-1} \left( \frac{\mu_i}{\mu_n} \right)^2 = \frac{\lambda}{\mu_n^2}, \quad (40d)
\]

\[
\text{const} = \left( \frac{q}{\mu_n} - Q_n \right)^2 + \sum_{i=1}^{n-1} Q_i^2 \quad (40e)
\]

So the inverse of the matrix \( \mathbf{B} \) is

\[
\mathbf{B}^{-1} = \left( B'_{ij} \right)_{(n-1) \times (n-1)}, \quad (40f)
\]

\[
B'_{ij} = \frac{1}{\det \mathbf{B}} \left( \delta_{ij} \det \mathbf{B} - \frac{\mu_i \mu_j}{\mu_n^2} \right) \quad (40g)
\]
and using the formula (38), we have

\[
\int \cdots \int \prod_{i=1}^{n-1} dq_i \cdot \exp \left[ -\left( \frac{q}{\mu_n} - \sum_{i=1}^{n-1} \frac{\mu_i q_i}{\mu_n} - Q_n \right)^2 - \sum_{i=1}^{n-1} (q_i - Q_i)^2 \right] : = \pi^{(n-1)/2} \sqrt{\det B} \cdot \exp \left[ -\frac{1}{\lambda} \left( q - \sum_{i=1}^{n} \mu_i Q_i \right)^2 \right] : \quad (41)
\]

Substituting (37) and (41) into (36), we have

\[
|q, q_2, q_3, \cdots q_n \rangle \langle q, q_2, q_3, \cdots q_n| = \pi^{-n/2} \prod_{i=1}^{n} \frac{\mu_i}{\lambda} |C|^2 : N \exp \left[ -\frac{1}{\lambda} \left( q - \sum_{i=1}^{n} \mu_i Q_i \right)^2 \right] : . \quad (42)
\]

Similarly, the \( C \) constant can be select to unit, since

\[
\langle q, q_2, q_3, \cdots q_n | q', q_2', q_3', \cdots q'_n \rangle = \delta (q' - q) \prod_{i=2}^{n} \delta (q'_i - q_i), \quad (\mu_i = \lambda)
\]

and decomposing the right hand side of (42) as the form \( f (a_i^\dagger) |00\cdots0\rangle \langle 00\cdots0| f^\dagger (a_i^\dagger) \), where

\[
: \exp \left[ -\sum_{i=1}^{n} a_i^\dagger a_i \right] : = |00\cdots0\rangle \langle 00\cdots0|,
\]

we get

\[
|q, q_2, q_3, \cdots q_n \rangle = \pi^{-n/4} \sqrt{\prod_{i=1}^{n} \frac{\mu_i}{\lambda}} \exp \left\{ \frac{1}{\lambda} \left[ \frac{M}{2} + \sqrt{2} \sum_{i=1}^{n} A_i a_i^\dagger + \frac{n}{2} \sum_{i,j=1}^{n} K_{ij} a_i^\dagger a_j^\dagger \right] \right\} |00\cdots0\rangle, \quad (44)
\]

where (please attention \( q_1 = 0 \)).

\[
A_i = \mu_i q - \sum_{j=1}^{n} \left[ \mu_i \mu_j^2 (q_i - q_j) \right] \quad (45a)
\]

\[
K_{ij} = -\mu_i \mu_j + \frac{\delta_{ij}}{2} \lambda \quad (45b)
\]

\[
M = -q^2 - \sum_{k=1}^{n} \mu_k^2 \mu_k^2 - \frac{1}{2} \sum_{k,l=1}^{n} \mu_k^2 \mu_l^2 (q_k - q_l)^2
= -q^2 - \frac{1}{2} \sum_{i,j=1}^{n} [\mu_i \mu_j (q_i - q_j)]^2
\]

(45c)

By this concise approach, we find the multipartite EPR entangled state \( |20, 22\rangle \). When \( \lambda = \mu_i = 1/n \ (i = 1, 2, \cdots, n) \), the equation \( 29 \) can be rewritten to \( 21 \).
\[ |q, \varrho_2, \varrho_3, \cdots, \varrho_n \rangle = \pi^{-n/4} n^{(1-n)/2} \exp \left\{ \sum_{j,k=1}^{n} \left[ \left( \frac{1}{2} a_j^\dagger a_k - \frac{q^2}{2} + \sqrt{2} q a_j^\dagger \right) \delta_{jk} \right. \right. \\
\left. \left. - \frac{1}{n} a_j^\dagger a_k^\dagger - \frac{\sqrt{2} i}{n} (\varrho_j - \varrho_k) a_j^\dagger - \frac{1}{4n^2} (\varrho_j - \varrho_k)^2 \right] \right\} |00\cdots0\rangle, \] (46)

It is quite straightforward to demonstrate that equation (44) is the eigenvector of the center-of-mass coordinate \( \sum_{i=1}^{n} \mu_i Q_i \) and mass-weighted relative momentums \( P_i/\mu_1 - P_i/\mu_i \) \( (i = 1, 2, \cdots, n) \) with eigenvalue \( q, \varrho_i \) respective, i.e. (34a) and (34b), and its completeness writes

\[
\int \cdots \int_{-\infty}^{\infty} dq \prod_{i=2}^{n} d\varrho_i \langle q, \varrho_2, \varrho_3, \cdots, \varrho_n | q, \varrho_2, \varrho_3, \cdots, \varrho_n \rangle \\
= \int \cdots \int_{-\infty}^{\infty} dq \prod_{i=2}^{n} \pi^{-n/2} \prod_{i=1}^{n} \mu_i^{1/2} N \exp \left[ -\frac{1}{\lambda} \left( q - \sum_{i=1}^{n} \mu_i Q_i \right)^2 \right] \\
= : \exp (0) : = 1. \] (47)

So, we have derived the multipartite EPR entangled representation of multi-mode via the Weyl correspondence approach.

VI. CONCLUSION

Due to the IWOP technique and the Weyl correspondence (Weyl quantization scheme) we have presented a new concise approach for obtaining the Fock representation of multi-partite entangled states of continuum variables. This is one available approach for finding many new quantum mechanical representations which may enrich Dirac’s representation and transformation theory. In this paper, we employed this new concise approach from Weyl correspondence to derive the extended Fan-Klauder entangled state representation to multipartite case representation. Our derivation itself demonstrate the effective and efficient of the approach for search new representations.

VII. ACKNOWLEDGEMENTS
This work was supported by the President Foundation of Chinese Academy of Science and the National Natural Science Foundation of China under grant 10475056.

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