Abstract. This is the first of a series of papers in which we initiate and develop the theory of reflection monoids, motivated by the theory of reflection groups. The main results identify a number of important inverse semigroups as reflection monoids, introduce new examples, and determine their orders.

Introduction

The symmetric group $S_n$ comes in many guises: as the permutation group of the set $\{1, \ldots, n\}$; as the group generated by reflections in the hyperplanes $x_i - x_j = 0$ of an $n$-dimensional Euclidean space; as the Weyl group of the reductive algebraic group $GL_n$, or (semi)simple group $SL_{n+1}$, or simple Lie algebra $sl_{n+1}$; as the Coxeter group associated to Artin’s braid group, and so on.

If one thinks of $S_X$ as the group of (global) symmetries of $X$, then the partial symmetries naturally lead one to consider the symmetric inverse monoid $I_X$, whose elements are the partial bijections $Y \to Y'$ ($Y, Y' \subset X$). It too has many other faces. It arises in its incarnation as the “rook monoid” as the so-called Renner monoid of the reductive algebraic monoid $M_n$ (see §4.2 for the definitions). An associated Iwahori theory and representations have been worked out by Solomon [27,29]. There is a braid connection too, with $I_n$ naturally associated to the inverse monoid of “partial braids” defined recently in [7].

But what is missing from all this is a realization of $I_n$ as some kind of “partial” reflection monoid, or indeed, a definition and theory of partial mirror symmetry and the monoids generated by partial reflections that generalizes the theory of reflection groups.

Such is the purpose of the present paper. Reflection monoids are defined as monoids generated by certain partial linear isomorphisms $\alpha : X \to Y$ ($X, Y$ subspaces of $V$), that are the restrictions (to $X$) of reflections. Initially one is faced with many possibilities, with the challenge being to impose enough structure for a workable theory while still encompassing as many interesting examples as possible. It turns out that a solution is to consider monoids of partial linear isomorphisms whose domains form a $W$-invariant semilattice for some reflection group $W$ acting on $V$.

Two pieces of data will characterise a reflection monoid: a reflection group and a collection of well behaved domain subspaces (see §3 for the precise definitions). What results is a theory of reflection monoids for which our main theorems determine their orders, presentations and identify the natural examples (it turns out that much of the general theory works when an arbitrary group is fed into the input data, but at various crucial stages the reflection group structure will be used in an essential way to obtain results in specific examples).

For instance, just as $S_n$ is the reflection group associated to the type $A$ root system, so now $I_n$ becomes the reflection monoid associated to the type $A$ root system, and where the domains...
form a Boolean lattice (see §4.1). The analogy continues: the group of signed permutations of \{1, \ldots, n\} is the Weyl group of type \(B\), and the inverse monoid \(\mathcal{J}_n\) of partial signed permutations becomes the reflection monoid of type \(B\), with again this Boolean lattice of domain subspaces.

By the “rigidity of tori”, a maximal torus \(T\) in a linear algebraic group \(G\) has automorphisms a finite group, the Weyl group of \(G\), and this is a reflection group in the space \(X(T) \otimes \mathbb{R}\), where \(X(T)\) is the character group of the torus. A similar role is played in the theory of linear algebraic monoids by the Renner monoid (see §4.2 for the definitions). One might hope that the Renner monoids are examples of reflection monoids, but in fact it turns out to be more complicated than this. We construct a reflection monoid in \(X(T) \otimes \mathbb{R}\), where the extra piece of data, the semilattice of domain spaces, comes from the character semigroup \(X(T)\) of the Zariski closure of \(T\). This reflection monoid then maps homomorphically onto the Renner monoid, with the two isomorphic in some cases.

Another interesting class of examples arises from the theory of hyperplane arrangements. The reflection arrangement monoids have as their input data a reflection group and for the domains, the intersection lattice of the reflecting hyperplanes. These intersection lattices possess many beautiful combinatorial and algebraic properties (see [18]). Thus, the reflection arrangement monoids tie up reflection groups and the intersection lattices of their reflecting hyperplanes in one very natural algebraic object.

This first paper has been written so as to include in its readership workers in both reflection groups and semigroups, and is organized as follows: §1 contains background material on reflection groups; §2 introduces the semilattice of subspaces forming the domains of our partial isomorphisms, and discusses in some detail two classes of examples arising from hyperplane arrangements. Reflection monoids proper are defined in §3, along with basic concepts in semigroup theory, and a number of their basic properties are considered. The final section gives three families of examples: the Boolean, Renner and reflection arrangement monoids along with their orders in a number of cases.

In the sequel [8] to this paper, a general presentation is derived (among other things) using the factorizable inverse monoid structure, and interpreting the various ingredients of a presentation for such given recently in [6]. This presentation is determined explicitly (and massaged a little more) for the Boolean and arrangement monoids associated to the classical Weyl groups. The benchmark here is provided by a classical presentation [19] for the symmetric inverse monoid \(\mathcal{J}_n\), which we rederive in its new guise as the “Boolean monoid of type \(A\”).

1. Preliminaries from reflection groups

Before venturing into partial mirror symmetry, we summarize the results we will need from (full) mirror symmetry, ie: from the theory of reflection groups. A number of these will not be needed until the sequel [8] to this paper, but we place them here for convenience. Standard references are [2,13], and more recently [14].

Let \(F\) be a field, \(V\) an \(F\)-vector space and \(GL(V)\) the group of linear isomorphisms \(V \to V\). A reflection is a non-trivial element of finite order in \(GL(V)\) that is semisimple and leaves pointwise invariant a hyperplane \(H \subset V\). A subgroup \(W \subset GL(V)\) is a reflection group when it is generated by reflections.

The most commonly studied examples arise in the cases \(F = \mathbb{R}, \mathbb{C}, \mathbb{F}_q\) and \(\mathbb{Q}_p\) (\(p\)-adics), and as all but one of the eigenvalues of an order \(n\) reflection are equal to 1, the last must be a primitive \(n\)-th root of unity in \(F\). Thus \(F\) plays a role in the kinds of orders that reflections may have: they are involutions in the reals and 2-adics, can have arbitrary finite order in the complexes, order dividing \(p - 1\) in \(\mathbb{Q}_p\) for \(p\) an odd prime, and so on.

There are classical and celebrated classifications due to Coxeter [4,5] in the reals, Shephard-Todd [26] (complexes), Clark-Ewing [3] (\(p\)-adics) and Wagner [33,34], Zalesski˘i-Serežkin [35] (\(\mathbb{F}_q\)). In this paper, more for concreteness than any other reason, we will restrict ourselves to \(F = \mathbb{R}\) and \(\mathbb{C}\), and to reflection groups \(W\) that are finite.
Any finite subgroup of $GL(V)$ for $V$ a complex space leaves invariant a positive definite Hermitian form, obtained in the usual way by an averaging process. Two such reflection groups $W_i \subset GL(V_i)$ are isomorphic if and only if there is a vector space isomorphism $V_1 \rightarrow V_2$ conjugating $W_1$ to $W_2$ (and from which one can obtain an isomorphism with these properties that preserves the forms, also by an averaging process; see [14, §14.1]). A reflection group is reducible if it has the form $W_1 \times W_2$ for non-trivial reflection groups $W_i \subset GL(V_i)$, and is essential if only the origin is left fixed by all $g \in W$.

The Shephard-Todd classification (up to this isomorphism) of the finite essential irreducible complex reflection groups then contains three infinite families and 34 exceptional cases (see for instance, [14, §15]). The infinite families are the cyclic and symmetric groups, and the groups $G(m,n,p)$ of $n \times n$ monomial matrices whose non-zero entries $\omega_1, \ldots, \omega_n$ are $m$-th roots of unity with $(\omega_1 \ldots \omega_n)^{m/p} = 1$.

If $X \subset V$ is a subspace, then the isotropy group $W_X$ consists of those elements of $W$ that fix $X$ pointwise. Possibly the most significant property of reflection groups for us, at least in this paper, is that $W_X$ is then also a reflection group, generated by reflections in those hyperplanes containing $X$ [32].

Among the complex groups are the real ones, with the transition from a real group $W \subset GL(V_\mathbb{R})$ to a complex one coming about by passing to reflections with hyperplanes $H \otimes \mathbb{C} \subset V_\mathbb{R} \otimes \mathbb{C}$. A finite real reflection group leaves invariant an inner product $(\cdot, \cdot)$, so that $V$ has the structure of a Euclidean space.

Traditionally, the finite real groups are studied via the combinatorics of their root systems: an (abstract) root system $\Phi$ in a Euclidean space $V$ is a finite set of non-zero vectors such that, (i), if $v \in \Phi$ then $\lambda v \in \Phi$ if and only if $\lambda = \pm 1$, and (ii), if $u, v \in \Phi$ then $(u)v = \Phi$, where $s_v$ is the reflection in the hyperplane $v^\perp$. The system is essential if the $\mathbb{R}$-span of $\Phi$ is $V$; reducible if $V = V_1 \perp V_2$ and $\Phi = \Phi_1 \cup \Phi_2$ for (non-empty) root systems $\Phi_i \subset V_i$ (in which case we write $\Phi = \Phi_1 \cup \Phi_2$), and crystallographic if

$$\langle u, v \rangle := \frac{2\langle u, v \rangle}{(v, v)} \in \mathbb{Z},$$

for all $u, v \in \Phi$. The associated reflection group is $W(\Phi) = \langle s_v \mid v \in \Phi \rangle$, and every finite reflection group arises from some root system in this way, with the essential, irreducible groups arising from essential, irreducible systems. The $W(\Phi)$ for $\Phi$ crystallographic are the Weyl groups.

Root systems $\Phi_i \subset V_i$ are isomorphic if there is an inner product preserving linear isomorphism $V_1 \rightarrow V_2$ sending $\Phi_1$ to $\Phi_2$, and are stably isomorphic if the isomorphism is between...
the subspaces spanned by the $\Phi_i$. In particular, every root system is stably isomorphic to an essential one. The corresponding groups $W(\Phi_i)$ are stably isomorphic if there is a vector space isomorphism between the spans of the $\Phi_i$ conjugating one group to the other.

The irreducible crystallographic root systems have been classified, up to stable isomorphism: there are four infinite families $A, B, C$ and $D$ (the classical systems), and five exceptional ones of types $E, F$ and $G$. The resulting reflection groups $W(\Phi)$ provide a list of almost all the finite reflection groups up to stable isomorphism, with the only omissions being the dihedral groups and the symmetry groups of the 3-dimensional dodecahedron/icosahedron and the 4-dimensional 120/600-cell.

Table 1 shows the classical crystallographic $\Phi \subset V$. The root systems of types $B$ and $C$ have the same symmetry, but different lengths of roots; nevertheless the associated Weyl groups are identical, and it is these that ultimately concern us. We have thus given just the type $B$ system in the table (type $C$ has roots $\pm \mathbf{x}_i$ rather than the $\mathbf{x}_i$).

The last column gives the Coxeter symbol, whose nodes are labelled by the vectors in a simple system $\Delta \subset \Phi$: a basis for the $\mathbb{R}$-span of $\Phi$ such that each root is a linear combination of $\Delta$ with coefficients all of the same sign. The Weyl group $W(\Phi)$ is then generated by the reflections $s_v$ for $v \in \Delta$ simple. The $i$-th and $j$-th nodes of the symbol are connected by an edge labelled $m_{ij}$, where $\langle u, v \rangle = m_{ij} - 2$, for the simple roots $u, v$ labelling the nodes, and the rotation $s_us_v$ has order $m_{ij}$ in $W(\Phi)$. It is traditional to omit labels $m_{ij} = 3$, and to remove completely the edges labelled by $m_{ij} = 2$. For convenience in expressing some of the formulae of §4, we adopt the additional conventions $A_{-1} = A_0 = \emptyset$, $B_0 = \emptyset$, $B_1 = \{\pm \mathbf{x}_1\}$, and $D_0 = D_1 = \emptyset$, $D_n = \{\pm \mathbf{x}_i, \pm \mathbf{x}_j \mid 1 \leq i < j \leq n\}$ for $n = 2, 3$. In Table 2 we have given just the Coxeter symbols for the exceptional Weyl groups. See [13, §2.10] for their root systems.

The Coxeter symbol also gives the reflectional representation of the Weyl group: let $S$ be the set of nodes of the symbol and $V$ the real space with basis $\{v_s \mid s \in S\}$ and symmetric bilinear form defined by,

$$B(v_s, v_t) = -\cos \frac{\pi}{m_{st}}.$$ 

For $u \in V$, define $\sigma_u : V \to V$ by $v \sigma_u = v - 2B(v, u)u$; then the map $s_v \mapsto \sigma_{v_s}$, where $v \in \Delta$ is the label of $s \in S$, extends to a faithful irreducible representation $\sigma : W(\Phi) \to \text{GL}(V)$. We will abbreviate $v(\sigma(g))$ to $vg$. Any faithful representation of $W(\Phi)$ with the $s_v (v \in \Delta)$ acting as reflections is equivalent to the direct sum of the reflectional representation and a trivial representation.

A Weyl group $W(\Phi)$ is of $(-1)$-type if in the reflectional representation there is an element $g \in W(\Phi)$ acting on $V$ as the antipodal map, ie: $vg = -v$ for all $v \in V$. They are precisely the groups with non-trivial center; a reducible Weyl group $W_1 \times W_2$ is of $(-1)$-type iff each $W_i$ is of $(-1)$-type, and the irreducible Weyl groups of $(-1)$-type are the $W(\Phi)$ for $\Phi = A_1, B_n, D_n (n \text{ even})$, and $E_6$.

It turns out that the classical Weyl groups have alternative descriptions as certain permutation groups, and we will use these extensively in this paper. This is very much in the spirit of the historical development of the theory of reflection groups, where a number of the classical theorems were initially proved on a case by case basis, using such descriptions and the classification of Coxeter, and while many now have uniform proofs that intrinsically use the reflection group structure, some still do not.

Firstly then, the map $(i, j) \mapsto s_{\mathbf{x}_i - \mathbf{x}_j}$ induces an isomorphism $\mathfrak{S}_n \to W(A_{n-1})$, and indeed the $W(A_{n-1})$-action on the basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is just permutation of coordinates.
There are two descriptions that prove useful for the Weyl group $W(B_n)$. Let $I$ be a set and $\Omega(I)$ the collection of all subsets of $I$, which forms an Abelian group under symmetric difference $X \triangle Y := (X \cup Y) \setminus (X \cap Y)$. Writing $\prod_I \mathbb{Z}_2$ for the (unrestricted) direct product, we have an isomorphism $\prod_I \mathbb{Z}_2 \to \Omega(I)$ given by the map $x = (x_i)_{i \in I} \mapsto X = \{ i \in I \mid x_i = 1 \}$, and this makes it easy to see that $\Omega(I)$ is generated by the singletons. The symmetric group $\mathfrak{S}_I$ acts on $\Omega(I)$ via the obvious $X \mapsto X\sigma$, and thus we may form the semi-direct product $\mathfrak{S}_I \ltimes \Omega(I)$, in which every element has a unique expression as a pair $\sigma X, \sigma \in \mathfrak{S}_I, X \subset I$, and with $\sigma X \tau Y = \sigma \tau (X \triangle Y)$. Write $\mathfrak{S}_n \ltimes \Omega(n)$ if $I = \{1, \ldots, n\}$, in which case the map $(i, j) \mapsto s_{x_i-x_j}, \{ i \} \mapsto s_{x_i}$ induces an isomorphism $\mathfrak{S}_n \ltimes \Omega(n) \to W(B_n)$.

The second viewpoint is to consider the group $\mathfrak{B}_I$ of signed permutations of $I$, ie: $\mathfrak{B}_I = \{ \sigma \in \mathfrak{S}_I \cup (-I) \mid (-x)\sigma = -(x\sigma) \}$. We then have an isomorphism $\mathfrak{B}_n \to W(B_n)$ induced by $(i, j)(-i, -j) \mapsto s_{x_i-x_j}$ and $(i, -i) \mapsto s_{x_i}$ (cf. Proposition 12(ii)).

Finally, $\Omega(n)$ has a subgroup $\Omega^+(n)$ consisting of those $X$ with $|X|$ even, and the $\mathfrak{S}_n$ action restricting to an action on $\Omega^+(n)$. The map $(i, j) \mapsto s_{x_i-x_j}, \{ i, j \} \mapsto s_{x_i-x_j}s_{x_i+x_j}$ induces an isomorphism $\mathfrak{S}_n \ltimes \Omega^+(n) \to W(D_n)$. There is also a description of $W(D_n)$ in terms of even signed permutations, but this will be of no use to us.

2. Systems of subspaces for reflection groups

Partial mirror symmetry describes the phenomenon of restricting the linear isomorphisms of a reflection group to “local isomorphisms” between certain subspaces. In this section we place a modest amount of structure on these subspaces that still allows for a large number of interesting examples.

Let $G \subset GL(V)$ be a group. A collection $\mathcal{B}$ of subspaces of $V$ is a system of subspaces for $G$ if and only if

(S1). $V \in \mathcal{B}$,
(S2). $\mathcal{B}G = \mathcal{B}$, ie: $Xg \in \mathcal{B}$ for any $X \in \mathcal{B}$ and $g \in G$, and
(S3). if $X, Y \in \mathcal{B}$ then $X \cap Y \in \mathcal{B}$.

If $\mathcal{B}_1, \mathcal{B}_2$ are systems for $G$ then clearly $\mathcal{B}_1 \cap \mathcal{B}_2$ is too, and thus for any set $\Omega$ of subspaces we write $\langle \Omega \rangle_G$ for the intersection of all systems for $G$ containing $\Omega$, and call this the system for $G$ generated by $\Omega$.

A system $\mathcal{B}$ can be partially ordered by inclusion (respectively, reverse inclusion) and both will turn out to be useful for us. The result is a meet (resp. join) semilattice with $\hat{1}$ (resp. $\hat{0}$), indeed a lattice if $\mathcal{B}$ is finite (see [31, §3.1] for basic facts concerning lattices). It is an elementary fact in semigroup theory [11, Proposition 1.3.2] that a meet semilattice with $\hat{1}$ is a commutative monoid $E$ of idempotents and vice-versa. For any $e \in E$, let $E e = \{ x \in E \mid x \leq e \}$. The Munn semigroup [11, §5.4] $\mathcal{T}_E$ of $E$ is then defined to be the set of all isomorphisms $E e \to Es$ where $e, f$ range over all elements of $E$ with. The following is then easily proved:

**Proposition 1.** $\mathcal{B}$ is a system in $V = \mathbb{F}\text{-}\text{span}\{\mathcal{B}\}$ for $G \subset GL(V)$ if and only if $E = (\mathcal{B}, \cap)$ is a commutative monoid of idempotents and the mapping $g \mapsto \theta_g$ where $X \theta_g = Xg$ for $X \in \mathcal{B}$ and $g \in G$ is a (monoid) homomorphism $G \to \mathcal{T}_E$ to the Munn semigroup of $G$.

Recall that in a poset $(P, \leq)$, if $x < y$ and there is no $z$ with $x < z < y$ then we say that $y$ covers $x$, and write $x <_c y$. $\mathcal{P}$ is graded of rank $n$ if every chain $x_1 <_c \cdots <_c x_n$, maximal under inclusion of such chains, has the same length $n$. There is then a unique rank function $\mathbb{R}k: \mathcal{P} \to \{ 0, 1, \ldots, n \}$ with $\mathbb{R}k(x) = 0$ if and only if $x$ is minimal, and $\mathbb{R}k(y) = \mathbb{R}k(x) + 1$ whenever $x < y$. The rank $1$ (resp. rank $n - 1$) elements of $\mathcal{P}$ are the atoms (resp. coatoms) and $\mathcal{P}$ is atomic (resp. coatomic) if every element is a join of atoms (resp. meet of coatoms). A Boolean lattice on a finite set $X$ is a lattice isomorphic to the lattice of all subsets of $X$ under inclusion.

In particular, if we order a system $\langle \Omega \rangle_G$ of subspaces for $G \subset GL(V)$ by inclusion (resp. reverse inclusion), then every element is a meet (resp. join) of the $Xg$ for $X \in \Omega$ and $g \in G$; if
V is finite dimensional and all the $X \in \Omega$ have the same dimension, then we have a coatomic poset with coatoms (resp. atomic poset with atoms) the $Xg$.

A hyperplane arrangement $\mathcal{A}$ is a finite collection of hyperplanes in $V$. General references are [18, 36], where the hyperplanes are allowed to be affine, but we will restrict ourselves to arrangements where the hyperplanes are linear (hence subspaces of $V$). An important combinatorial invariant for $\mathcal{A}$ is the intersection lattice $L(\mathcal{A})$—the set of all possible intersections of elements of $\mathcal{A}$, ordered by reverse inclusion, and with the null intersection taken to be the ambient space $V$. What results is a graded atomic lattice of rank codim $\bigcap_{X \in \mathcal{A}} X$ [18, §2.1], with $\emptyset$ the space $V$, atoms the hyperplanes in $\mathcal{A}$ and rank $\text{codim} X$.

If $G \subset GL(V)$ is finite and $\mathcal{A} \subset V$ a hyperplane arrangement, then $AG$ is also a hyperplane arrangement, for which the following is then obvious,

**Lemma 1.** The system $\langle \mathcal{A} \rangle_G$ for $G$ generated by $\mathcal{A}$ is the intersection lattice $L(AG)$, and the $G$-action on $\langle \mathcal{A} \rangle_G$ is rank preserving.

In general $L(\mathcal{A}) \subset L(AG)$, but we will often have $AG = \mathcal{A}$, hence equality of the intersection lattices.

### 2.1. Boolean systems

Specializing now to reflection groups, a simple but nevertheless interesting example of a system arises if $V$ is a Euclidean space with orthonormal basis $\{x_1, \ldots, x_n\}$ and $W = W(\Phi)$ a Weyl group as in Tables 1-2. The Boolean (or orthogonal) hyperplane arrangement [18, §1.2] $\mathcal{A} = \{x_1^\perp, \ldots, x_n^\perp\}$ consists of the coordinate hyperplanes, and we call the system $\langle \mathcal{A} \rangle_W$ for $W$ generated by $\mathcal{A}$ a Boolean system. The name stems from the fact that $L(A)$ is a Boolean lattice, although it should be noted that the system $\langle \mathcal{A} \rangle_W$ itself will only be Boolean when we have $\mathcal{A}W = \mathcal{A}$.

Consider a Weyl group $W = W(\Phi)$ with $\Phi$ a classical root system as in Table 1. Then $\mathcal{A}W = \mathcal{A}$, and hence the Boolean system $\langle \mathcal{A} \rangle_W = L(\mathcal{A})$ is a Boolean lattice with the map $x_{i_1}^\perp \cap \cdots \cap x_{i_k}^\perp \mapsto \{i_1, \ldots, i_k\}$ being a lattice isomorphism from $L(\mathcal{A})$ to the lattice of subsets of $I = \{1, 2, \ldots, n\}$.

The rank $k$ elements of $L(A)$ are the intersections $x_{i_1}^\perp \cap \cdots \cap x_{i_k}^\perp$ of $k$ distinct hyperplanes, and as the symmetric group $S_X$ for $X = \{x_1, \ldots, x_n\}$ is a subgroup of $W(\Phi)$ for classical $\Phi$, the action of $W(\Phi)$ on the rank $k$ elements is transitive.

If $\Phi$ is a root system for one of the exceptional groups in Table 2, then $\mathcal{A} \subset \mathcal{A}W$, but the system for $W(\Phi)$ will have more elements than the intersection lattice $L(\mathcal{A})$. If for instance $\Phi$ is the $F_4$ root system of [13, §2.10] and $W = W(\Phi)$, then the system $\langle \mathcal{A} \rangle_W$ has atoms the hyperplanes $x_1^\perp$ and $\frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4)^\perp$, ie: the reflecting hyperplanes of $W(F_4)$ corresponding to the short roots, and as such is a subsystem of the intersection lattice of the type $F_4$ reflection arrangement of §2.2. If $\Phi = E_6, E_7$ or $E_8$, then a description of the Boolean system is possible, but messier.

### 2.2. Intersection lattices of reflection arrangements

A more natural example of a system of subspaces for a reflection group $W$ is given by the intersection lattice $L(A)$ of the reflecting hyperplanes $A$ of $W$. If $X \in A$ and $s_X \in W$ is the reflection in $X$, then for $g \in W$ we have $s_X g = g^{-1}s_X g$, and so $Xg \in A$. Thus $\mathcal{A}W = \mathcal{A}$, and we have,

**Lemma 2.** If $W \subset GL(V)$ is a reflection group and $\mathcal{A}$ the hyperplane arrangement consisting of the reflecting hyperplanes of $W$, then $\langle \mathcal{A} \rangle_W = L(\mathcal{A})$.

We will call such an $L(A)$ a (reflection) arrangement system, and for the remainder of this section we focus on these systems (ordered by reverse inclusion) when $W$ is a Weyl group as
in Tables 1-2, summarizing the necessary results of [18, §6.4]. Recall that a partition of \( I = \{1, 2, \ldots, n\} \) is a collection \( \Lambda = \{A_1, \ldots, A_p\} \) of nonempty pairwise disjoint subsets \( A_i \subseteq I \) whose union is \( I \). If \( \lambda_i = |A_i| \) then \( \lambda = \|\Lambda\| = (\lambda_1, \ldots, \lambda_p) \) is a partition of \( n \), ie: the integers \( \lambda_i \geq 1 \) with \( \sum \lambda_i = n \), and we order the \( A_i \) so that \( \lambda_1 \geq \cdots \geq \lambda_p \geq 1 \). Order the set \( \Pi(n) \) of partitions of \( I \) by refinement, ie: \( \Lambda \leq \Lambda' \) if and only if for every \( A_i \) there is a \( A'_i \) with \( A_i \subseteq A'_i \). The result is an atomic graded lattice with \( \text{rk} \, \Lambda = \sum (\lambda_i - 1) \) and atoms the \( \Lambda \) with \( \lambda_i = 2 \) and \( \lambda_i = 1 \) for \( i > 1 \). The following is [18, Proposition 2.9]:

**Proposition 2.** Let \( A \) be the hyperplane arrangement consisting of the reflecting hyperplanes of the Weyl group \( W(A_{n-1}) \). Then the map that sends the atomic partition with \( \Lambda_1 = \{i, j\} \) to the hyperplane \( (x_i - x_j)\perp \) extends to a lattice isomorphism \( \Pi(n) \to L(A) \).

Indeed, writing \( X(A) \in L(A) \) for the image of \( A \), we have

\[
X(A) = \bigcap_{\lambda_k > 1} \bigcap_{i,j \in A_k} (x_i - x_j)^\perp.
\]

For a partition \( \Lambda \), let \( b_i > 0 \) be the number of \( \lambda_j \) equal to \( i \), and

\[
b_\Lambda = b_1!b_2! \cdots (1!)^{b_1}(2!)^{b_2} \cdots
\]

If \( \sigma \mapsto g(\sigma) \) is the isomorphism \( \mathfrak{S}_n \to W(A_{n-1}) \) of §I, then the action of \( W(A_{n-1}) \) on \( L(A) \) is given by \( X(A)g(\sigma) = X(\Lambda\sigma) \), where \( \Lambda\sigma = \{\Lambda_1\sigma, \ldots, \Lambda_p\sigma\} \). The following is [18, Proposition 6.72]:

**Proposition 3.** In the action of the Weyl group \( W(A_{n-1}) \) on \( L(A) \), two subspaces \( X(\Lambda) \) and \( X(\Lambda') \) lie in the same orbit if and only if \( \|\Lambda\| = \|\Lambda'||. \) The cardinality of the orbit of the subspace \( X(\Lambda) \) is \( n! / b_\Lambda \).

Turning now to the Weyl group \( W(B_n) \), let \( \mathcal{T}(I) \) be the set of triples \( (\Delta, \Gamma, A) \) where \( \Delta \subset I \), \( \Gamma \subset J := I \setminus \Delta \) and \( \Lambda = \{A_1, \ldots, A_p\} \) is a partition of \( J \). There is then [18, Proposition 6.74] a surjective mapping \( \mathcal{T}(I) \ni (\Delta, \Gamma, A) \mapsto X(\Delta, \Gamma, A) \in L(A) \), with

\[
X(\Delta, \Gamma, A) = \bigcap_{\lambda_k > 1} \bigcap_{i,j \in A_k} (x_i + \varepsilon_j x_j)^\perp \cap \bigcap_{i \in \Delta} x_i^\perp \quad \text{and} \quad \text{rk} \, X(\Delta, \Gamma, A) = |\Delta| + \sum (\lambda_i - 1),
\]

where \( \varepsilon_j = 1 \) if \( j \in \Gamma \) or \( \varepsilon_j = -1 \) if \( j \not\in \Gamma \). Moreover, \( X(\Delta, \Gamma, A) = X(\Delta', \Gamma', A') \) if and only if \( \Delta = \Delta', A = A' \) and for each \( 1 \leq i \leq p \), \( \Gamma_i' = \Gamma_i \) or \( A_i \setminus \Gamma_i \), where \( \Gamma_i = \Gamma \cap A_i \) and \( \Gamma_i' \) is defined similarly.

If \( \sigma T \mapsto g(\sigma, T) \) is the isomorphism \( \mathfrak{S}_n \times \mathcal{Q}(n) \to W(B_n) \) of §I, then the action of \( W(B_n) \) on \( L(A) \) is given by

\[
X(\Delta, \Gamma, A)g(\sigma, T) = X(\Delta\sigma, (T \cap \Gamma)\sigma, A\sigma),
\]

where \( T \cap J = T \cap J \).

**Proposition 4.** In the action of the Weyl group \( W(B_n) \) on \( L(A) \), two subspaces \( X(\Delta, \Gamma, A) \) and \( X(\Delta', \Gamma', A') \) lie in the same orbit if and only if \( |\Delta| = |\Delta'| \) and \( \|\Lambda\| = \|\Lambda'||. \) The cardinality of the orbit of the subspace \( X(\Delta, \Gamma, A) \) is

\[
2^{j-p} \binom{n}{j} \frac{j!}{b_\Lambda},
\]

where \( j = |J| \) and \( A = \{A_1, \ldots, A_p\} \).
| g2  | 1a0:3a1:3a1:1g2 |
| f4  | 1a0:12a1:12a1:72a12:16a2:16a2:1.18b2:12b3:12b3:48a1a2:48a1a2:1f4 |
| e6  | 1a0:36a1:270a12:120a2:540a13:720a1a2:270a3:1080a1a2:120a22 |
| e7  | 540a1a3:216a4:45d4:360a1a22:216a1a4:36a5:27d5:1e6 |
| e8  | 1a0:63a1:945a12:336a2:315a13:3780a13:720a1a2:270a3:3780a1a2:720a1a2 |

Table 3. Orbit data for the exceptional arrangement systems [18, Appendix C]: each orbit is encoded in a string consisting of the number of subspaces in the orbit followed by their common stabilizer written in the form \( x_{nmypq} \ldots \) to indicate the product of Weyl groups \( X^n_m \times Y^n_p \ldots \). Different orbits of subspaces of the same rank are separated by a period and orbits of different ranks by a colon.

(See [18, Proposition 6.75]. What Orlik and Terao actually describe is the corresponding result for the full monomial group \( G(r, 1, n) \), where we have contented ourselves with \( G(2, 1, n) \equiv W(B_n) \).

For the Weyl group \( W(D_n) \) and its reflecting hyperplanes \( A \), let \( S(I) \) be the subset of \( T(I) \) consisting of those triples \( (\Delta, \Gamma, \Lambda) \) with \( |\Delta| \neq 1 \). Then by [18, Proposition 6.78] there is a surjective mapping \( S(I) \rightarrow L(A) \), where

\[
X(\Delta, \Gamma, \Lambda) = \begin{cases} \lambda \lambda > 1 \bigcup_{i,j \in A} (x_i + e_j x_j) & \text{if } \Delta = \emptyset, \\ \lambda > 1 \bigcup_{i,j \in A} (x_i + e_j x_j) \cap \bigcup_{i,j \in A} (x_i + x_j) \cap (x_i - x_j) & \text{if } |\Delta| \geq 2, \end{cases}
\]

and \( X(\Delta, \Gamma, \Lambda) = X(\Delta', \Gamma', \Lambda') \) if and only if \( \Delta = \Delta', \Lambda = \Lambda' \) and for each \( 1 \leq i \leq p, \Gamma_i = \Gamma_i' \) or \( A_i \setminus \Gamma_i \) where \( \Gamma_i = \Gamma \cap A_i \) and \( \Gamma_i' \) is defined similarly. [18, Proposition 6.79] then gives,

**Proposition 5.** If \( X(\Delta, \Gamma, \Lambda) \) and \( X(\Delta', \Gamma', \Lambda') \) lie in the same orbit of the action of \( W(D_n) \) on \( L(A) \), then \( |\Delta| = |\Delta'| \) and \( ||\Lambda|| = ||\Lambda'|| \). Conversely, suppose that \( |\Delta| = |\Delta'| \) and \( ||\Lambda|| = ||\Lambda'|| \).

1. If \( |\Delta| \geq 2 \) then \( X(\Delta, \Gamma, \Lambda) \) and \( X(\Delta', \Gamma', \Lambda') \) lie in the same orbit, which has cardinality as in Proposition 4.

2. If \( \Delta = \emptyset \), then the \( W(B_n) \) orbit determined by \( ||\Lambda|| = (\lambda_1, \ldots, \lambda_p) \) forms a single \( W(D_n) \) orbit, except when each \( \lambda_i \) is even, in which case it decomposes into two \( W(D_n) \) orbits of size

\[
\frac{2n-p-1}{b_{\lambda}}.
\]

In part 2 of Proposition 5, and when all the \( \lambda_i \) are even, one of the \( W(D_n) \) orbits consists of the \( X(\emptyset, \Gamma, \Lambda) \) with \( |\Gamma| \) even, and the other with the \( |\Gamma| \) odd (again, Orlik and Terao deal with the monomial group \( G(r, r, n) \), while we consider only \( G(2, 2, n) \equiv W(B_n) \), with the decomposition of the second part of Proposition 5 being into \( d W(D_n) \)-orbits, for \( d \) the greatest common divisor of \( \{r, \lambda_1, \ldots, \lambda_p\} \).

If \( W \) is an exceptional Weyl group then a convenient description of \( L(A) \) is harder, but an enumeration of the orbits of the \( W \)-action on \( L(A) \) suffices for our purposes. We summarize some of the results of [16, 17] (see [18, Appendix C]) in Table 3. For example, the orbit data for the Weyl group \( W(E_6) \), which starts as,

\[
1a0:36a1:270a12:120a2:540a13.720a1a2.270a3
\]
indicates a single rank 0 orbit with stabilizer the Weyl group \( A_0 \cong 1 \) (corresponding to the ambient space \( V \)), a single rank 1 orbit of size 36 with stabilizer \( A_1 \cong \mathbb{Z}_2 \) (corresponding to the reflecting hyperplanes, or the 72 roots in the \( E_6 \) root system arranged in \( 36 \pm \) pairs), two orbits of rank 2 subspaces of sizes 270 and 120 with stabilizers \( A_1 \times A_1 \) and \( A_2 \) respectively, and so on. There are distinct rank one orbits with isomorphic stabilizers in types \( G_2 \) and \( F_4 \), corresponding to the two conjugacy classes of generating reflections (this phenomenon not arising in type \( E \) where all the generating reflections are conjugate).

We have stuck to the Weyl groups, as promised in §1, but the data in Table 3 could just as easily be read off [18, Appendix C] for all 34 exceptional finite complex reflection groups.

3. Inverse Monoids and Reflection Monoids

We are now ready for reflection monoids and some of their elementary properties, but first we recall some of the basic concepts of inverse monoids. For more on the general theory of inverse monoids see [11, Chapter 5] and [15].

An inverse monoid is a monoid \( M \) such that for all \( a \in M \) there is a unique \( b \in M \) such that \( aba = a \) and \( bab = b \). The element \( b \) is the inverse of \( a \) and is denoted by \( a^{-1} \). It is worth noting that \((a^{-1})^{-1} = a \) and \((ab)^{-1} = b^{-1}a^{-1} \) for all \( a, b \in M \). The set of idempotents \( E(M) \) of \( M \) forms a commutative submonoid, referred to as the semilattice of idempotents of \( M \). We denote the group of units of \( M \) by \( G(M) \). An inverse submonoid of an inverse monoid \( M \) is simply a submonoid \( N \) closed under taking inverses; it is full if \( E(N) = E(M) \).

The archetypal example of an inverse monoid is the symmetric inverse monoid defined as follows. For a non-empty set \( X \), a partial permutation is a bijection \( \sigma : Y \to Z \) for some subsets \( Y, Z \) of \( X \). We allow \( Y \) and \( Z \) to be empty so that the empty function is regarded as a partial permutation. The set of all partial permutations of \( X \) is made into a monoid by using the usual rule for composition of partial functions; it is called the symmetric inverse monoid on \( X \) and denoted by \( \mathcal{I}_X \) (if \( X = \{1, 2, \ldots, n\} \), we write \( \mathcal{I}_n \) for \( \mathcal{I}_X \)). That it is an inverse monoid follows from the fact that if \( \sigma \) is a partial permutation of \( X \), then so is its inverse (as a function) \( \sigma^{-1} \), and this is the inverse of \( \sigma \) in \( \mathcal{I}_X \) in the sense above. Clearly, the group of units of \( \mathcal{I}_X \) is the symmetric group \( \mathfrak{S}_X \), and \( E(\mathcal{I}_X) \) consists of the partial identities \( \varepsilon_Y \) for all subsets \( Y \) of \( X \) where \( \varepsilon_Y \) is the identity map on the subset \( Y \). It is clear that, for \( Y, Z \subset X \), we have \( \varepsilon_Y \varepsilon_Z = \varepsilon_{Y \cap Z} \) and hence that \( E(\mathcal{I}_X) \) is isomorphic to the Boolean algebra of all subsets of \( X \).

Just as \( \mathfrak{S}_n \) is isomorphic to the group of permutation matrices, so \( \mathcal{I}_n \) is isomorphic to the monoid of partial permutation matrices, or rook monoid: the \( n \times n \) matrices having 0, 1 entries with at most one non-zero entry in each row and column (and so called as each element represents an \( n \times n \) chessboard with the 0 squares empty and the 1 squares containing rooks, with the rooks mutually non-attacking).

We observe that if \( M \) is an inverse submonoid of \( \mathcal{I}_X \), then

\[
E(M) = M \cap E(\mathcal{I}_X) = \{ \varepsilon_Y \mid Y = \text{dom } \sigma \text{ for some } \sigma \in M \}.
\]

Equally, \( E(M) = \{ \varepsilon_Y \mid Y = \text{im } \sigma \text{ for some } \sigma \in M \} \) since \( \text{im } \sigma = \text{dom } \sigma^{-1} \) for all \( \sigma \in M \). Putting

\[
\mathcal{B} = \{ \text{dom } \sigma \mid \sigma \in M \},
\]

we see that \( \mathcal{B} \) is a meet semilattice isomorphic to \( E(M) \). Moreover, \( X \in \mathcal{B} \) since \( M \) is a submonoid, and finally, if \( Y \in \mathcal{B} \) and \( g \in G(M) \), then \( Yg = \text{im } (\varepsilon_Yg) \in \mathcal{B} \). Thus \( \mathcal{B} \) satisfies analogues of (S1)-(S3) in §2 for a system of subspaces for a subgroup of \( GL(V) \), so we say that it is a system of subsets for the group \( G(M) \).

Every inverse monoid \( M \) has a faithful representation (called the Wagner-Preston representation) \( \rho_M : M \to \mathcal{I}_M \) by partial permutations given by partial right multiplication [11,15], and the significance of the symmetric inverse monoid is due partly to this fact.

Another example of an inverse monoid that we will encounter in §4.1 is the monoid of partial signed permutations of a non-empty set \( X \). Let \( -X = \{ -x \mid x \in X \} \) be disjoint from \( X \) such
that \( x \mapsto -x \) is a bijection, and define

\[
J_X := \{ \sigma \in J_{X \cup -X} \mid (-x)\sigma = -(x\sigma) \text{ and } x \in \text{dom } \sigma \iff -x \in \text{dom } \sigma \},
\]

where we write \( J_n \) when \( X = \{1, 2, \ldots, n\} \) and in this case \(-x\) has its usual meaning. The group of units of \( J_X \) is the group \( \mathbb{B}_X \) of partial signed permutations of \( X \).

We shall be particularly interested in factorizable inverse monoids, where an inverse monoid \( M \) is factorizable if \( M = E(M)G(M) (= G(M)E(M)) \). See [15] for more details regarding factorizable inverse monoids. For \( \sigma \in M \) where \( M \) is an inverse submonoid of \( J_X \), we have \( \sigma \in E(M)G(M) \) if and only if \( \sigma \) is a restriction of a unit of \( M \), so that factorizable inverse submonoids of \( J_X \) are those in which every element is a restriction of some unit of \( M \). For example, \( J_n \) is factorizable, since any partial permutation of \( \{1, \ldots, n\} \) can be extended (not necessarily uniquely) to an element of \( \mathfrak{S}_n \). However, if \( X \) is infinite, then \( J_X \) is not factorizable since, for example, an injective map from \( X \) to itself (with domain \( X \)) which is not a restriction of a permutation of \( X \). Similarly, \( J_n \) is factorizable, but \( J_X \) is not when \( X \) is infinite.

Let \( \mathcal{B} \) be a system of subsets for a subgroup \( G \) of \( \mathfrak{S}_X \) and define

\[
F = M(G, \mathcal{B}) = \{ g_Y \mid g \in G, \ Y \in \mathcal{B} \}
\]

where \( g_Y \) is the restriction of \( g \) to the subset \( Y \). Note that \( F \subset J_X \) and that if \( g_Y, h_Z \in F \), then \((g_Y)^{-1} = (g^{-1})_Y \in F \) and \( g_Y h_Z = (gh)_T \) with \( T = Y \cap Zg^{-1} \), so that \( F \) is an inverse submonoid of \( J_X \). Clearly, \( G \) is the group of units of \( F \), and \( E(F) = \{ \varepsilon_Y \mid Y \in \mathcal{B} \} \). Moreover, every element of \( F \) is a restriction of a unit, so \( F \) is factorizable.

Now let \( M \) be an inverse submonoid of \( J_X \) and \( G \) be its group of units. Let \( \mathcal{B} \) be the system of subsets for \( G \) described above, that is,

\[
\mathcal{B} = \{ \text{dom } \sigma \mid \sigma \in M \}.
\]

Put \( F_M = M(G, \mathcal{B}) \) and note that \( F_M \) is a factorizable inverse submonoid of \( M \), and, in fact, it is the largest such submonoid (cf. [15, Proposition 2.2.1]).

Thus if \( M \) is actually factorizable, then \( M = F_M \), and since every inverse monoid can be embedded in some \( J_X \), we have a description of all factorizable inverse monoids. As an illustration, we note that \( J_n \) can be realised as \( M(\mathfrak{S}_n, \mathcal{B}) \) where \( \mathcal{B} \) is the power set of \( \{1, \ldots, n\} \).

Another class of inverse monoids of interest to us are the fundamental inverse monoids. On any inverse monoid \( M \), define the relation \( \mu \) by the rule:

\[
a \mu b \text{ if and only if } a^{-1}ea = b^{-1}eb \text{ for all } e \in E(M).
\]

It is easy to see that \( \mu \) is a congruence on \( M \); it is idempotent-separating in the sense that distinct idempotents in \( M \) are not related by \( \mu \), and, in fact, it is the greatest idempotent-separating congruence on \( M \). We say that \( M \) is fundamental if \( \mu \) is the equality relation, and mention that for any \( M \), the monoid \( M/\mu \) is fundamental. The Munn semigroup \( T_E \) of a semilattice \( E \) that we introduced in §2 plays a crucial role in describing fundamental inverse monoids. First, we note that \( T_E \) is an inverse submonoid of \( J_E \) whose semilattice of idempotents is isomorphic to \( E \) (see [11, Theorem 5.4.4] or [15, Theorem 5.2.7]).

Given any inverse monoid \( M \) and \( a \in M \), define an element \( \delta_a \in T_{E(M)} \) as follows. The domain of \( \delta_a \) is \( Eaa^{-1} \) and \( x\delta_a = a^{-1}xa \) for \( x \in Eaa^{-1} \). Note that \( \text{im } \delta_a = Eaa^{-1}a \). The main results are the following, for which one should consult [11, Theorem 5.4.4], [15, Theorem 5.2.8] and [11, Theorem 5.4.5], [15, Theorem 5.2.9].

**Proposition 6.** If \( M \) is an inverse monoid, then the mapping \( \delta : M \to T_{E(M)} \) given by \( a\delta = \delta_a \) is a homomorphism onto a full inverse submonoid of \( T_{E(M)} \) such that \( a\delta = b\delta \) if and only if \( a \mu b \).

**Proposition 7.** An inverse monoid \( M \) is fundamental if and only if \( M \) is isomorphic to a full inverse submonoid of \( T_{E(M)} \).
The homomorphism \( \delta: M \to \mathcal{J}_{E(M)} \) of Proposition 6 is called the fundamental or Munn representation of \( M \). Note that \( M \) is fundamental if and only if \( \delta \) is one-one.

It is well known that \( \mathcal{J}_X \) is fundamental for any set \( X \) (see, for example, [11, Chapter 5, Exercise 22]). In contrast, for any nonempty set \( X \), it is easy to see that \( \mathcal{J}_X \) is not fundamental: a simple calculation shows that the identity of \( \mathcal{J}_X \) and the transposition \((x, -x)\) are \( \mu \)-related. In the next section we see that \( \mathcal{J}_n \) is a reflection monoid, so there are non-fundamental reflection monoids.

We now describe fundamental factorizable inverse monoids in terms of semilattices and their automorphism groups, a point of view that will prove useful in §4.2. We remark that the principal ideals of a semilattice \( E \) regarded as a monoid are precisely the principal order ideals of \( E \) regarded as a partially ordered set. It will be convenient to write \( \varepsilon_x \) for the partial identity with domain \( Ex \).

**Proposition 8.** If \( E \) is a semilattice with greatest element \( \hat{1} \) and \( G \) is a subgroup of the automorphism group \( \text{Aut}(E) \), then the collection

\[
\mathcal{B} = \{ Ex \mid x \in E \}
\]

of all principal ideals of \( E \) forms a system of subsets (of \( E \)) for \( G \), and the resulting \( M(G, \mathcal{B}) \) is the submonoid of \( \mathcal{J}_E \) generated by \( G \) and \( E \).

Conversely, any fundamental factorizable inverse monoid \( M \) is isomorphic to a submonoid of \( \mathcal{J}_{E(M)} \) generated by a group \( G \) of automorphisms of \( E(M) \) and \( E(M) \).

**Proof.** Given \( E \) and \( G \) we observe that \( \mathcal{B} \) does form a system of subsets (in \( E \)) for \( G \) since \( E = \mathcal{E}1 \), \( Ex \cap Ey = Exy \) and the image under \( g \in G \) of \( Ex \) is \( E(xg) \). We can thus define the factorizable inverse monoid \( M(G, \mathcal{B}) \subset \mathcal{J}_E \) as above. As \( G \) is a subgroup of \( \text{Aut}(E) \), it is a subgroup of the group of units of \( \mathcal{J}_E \), and hence if \( \varepsilon_x g \in M(G, \mathcal{B}) \) with \( g \in G \), then \( \varepsilon_x g \in \mathcal{J}_E \). Thus \( M(G, \mathcal{B}) \subset \mathcal{J}_E \); in fact, it is clearly a full inverse submonoid of \( \mathcal{J}_E \) and so it is fundamental. Identifying \( E(\mathcal{J}_E) \) with \( E \), it is also clear that \( M(G, \mathcal{B}) \) is generated as a submonoid by \( G \) and \( E \).

For the converse, let \( F \) be a fundamental factorizable inverse monoid and write \( E \) for \( E(F) \). Then \( F \) is isomorphic to a full submonoid of \( \mathcal{J}_E \) which we identify with \( F \). The group \( G = G(F) \) of units of \( F \) is a subgroup of the group of units of \( \mathcal{J}_E \), that is, of \( \text{Aut}(E) \). As above \( \mathcal{B} = \{ \text{dom } \sigma \mid \sigma \in F \} \) is a system of subsets (of \( E \)) for \( G \) and since \( F \) is factorizable, \( F = M(G, \mathcal{B}) \). Thus \( F \) is generated by \( G \) and \( E \) (identifying \( E \) with \( E(\mathcal{J}_E) \)). \( \square \)

If the semilattice \( E \) has a least element \( 0 \) (in particular, if \( E \) is a lattice), then the principal ideal \( Ex \) of \( E \) is just the interval \([0, x] = \{ z \in E \mid 0 \leq z \leq x \} \), so that the system described above is the collection of intervals \( \mathcal{B} = \{ [0, x] \mid x \in E \} \).

We now turn to reflection monoids. Throughout the rest of this section, \( V \) is a vector space over a field \( F \). A partial linear isomorphism of \( V \) is a vector space isomorphism \( \alpha: X \to Y \) between vector subspaces \( X, Y \) of \( V \). Thus the set \( ML(V) \) of all partial isomorphisms of \( V \) is a subset of \( \mathcal{J}_V \). In fact, it is an inverse submonoid of \( \mathcal{J}_V \) since the composition of two partial isomorphisms is easily seen to be a partial isomorphism, and the inverse of an isomorphism is again an isomorphism. The group of units of \( ML(V) \) is \( GL(V) \), and the semilattice of idempotents consists of all the partial identities on subspaces of \( V \).

If \( V \) has finite dimension and \( X \) is a subspace, then by extending a basis of \( X \), any partial isomorphism with domain \( X \) can be extended to a (not necessarily unique) full isomorphism of \( V \). Thus every element of \( ML(V) \) is a restriction of a unit, so that \( ML(V) \) is factorizable. Of course, this is not the case if \( V \) has infinite dimension. We record these observations in the next result.

**Lemma 3.** The set \( ML(V) \) of all partial isomorphisms of the vector space \( V \) is an inverse submonoid of \( \mathcal{J}_V \). Moreover, \( ML(V) \) is factorizable if and only if \( V \) is finite dimensional.
A system of subspaces \( \mathcal{B} \) for a subgroup \( G \) of \( GL(V) \) is a special case of a system of subsets for \( G \) regarded as a subgroup of of \( G \), so as above we can construct a factorizable inverse submonoid \( M(G, \mathcal{B}) \) of \( ML(V) \) with group of units \( G \) and idempotents, the partial identities \( \varepsilon_X \) for \( X \in \mathcal{B} \).

On the other hand, if \( F \) is a factorizable inverse submonoid of \( ML(V) \), then we know that \( F = M(G, \mathcal{B}) \) where \( \mathcal{B} = \{ \operatorname{dom} \sigma \mid \sigma \in M \} \); now the domain of every element in \( F \) is a subspace of \( V \), so \( \mathcal{B} \) is, in fact, a system of subspaces.

A partial reflection of a vector space \( V \) is defined to be the restriction of a reflection \( s \in GL(V) \) to a subspace \( X \) of \( V \). We denote this partial reflection by \( s_X \). A reflection monoid is defined to be a factorizable inverse submonoid of \( ML(V) \) generated by partial reflections.

It is easy to see that the non-units in a reflection monoid \( M \subseteq ML(V) \) form a semigroup, and hence every unit of \( M \) must be a product of (full) reflections, that is, the group of units of \( M \) is a reflection group \( W \). Indeed, if \( S \) is the set of generating partial reflections for \( M \), let \( S' \subseteq S \) be the subset of full reflections. Then \( W = \langle S' \rangle \). Also, since \( M \) is factorizable, it follows that \( M = M(W, \mathcal{B}) \) for a system of subspaces for \( W \).

If we choose a (subspace) system \( \mathcal{B} \) for a reflection group \( W \subseteq GL(V) \), the units of \( M(W, \mathcal{B}) \) are generated by reflections. Any other element has the form \( \varepsilon_X g \) for some \( X \in \mathcal{B} \) and \( g \in W \). Now \( g = s_1 \ldots s_k \) for some reflections \( s_1, \ldots, s_k \) and \( \varepsilon_X s_1 \) is a partial reflection, so \( \varepsilon_X g = (\varepsilon_X s_1) s_2 \ldots s_k \) is a product of partial reflections. Thus \( M(W, \mathcal{B}) \) is a reflection monoid.

Most of the elementary properties of reflection monoids appear in the above discussion. For emphasis, we list them in the following result.

**Proposition 9.** Every reflection monoid \( M \subseteq ML(V) \) has the form \( M(W, \mathcal{B}) \) where \( W \) is the reflection group of units and \( \mathcal{B} = \{ \operatorname{dom} \sigma \mid \sigma \in M \} \). Conversely, if \( W \subseteq GL(V) \) is a non-trivial reflection group and \( \mathcal{B} \) is a system of subspaces for \( W \), then \( M(W, \mathcal{B}) \) is a reflection monoid with group of units \( W \).

In \( M = M(W, \mathcal{B}) \) we have:

1. \( \mathcal{B} = \{ \operatorname{dom} \sigma \mid \sigma \in M \} \) = \{ im \sigma \mid \sigma \in M \},
2. \( E(M) = \{ X \mid X \in \mathcal{B} \} \), and
3. the inverse of \( g_X \) is \((g^{-1})_X\).

Finally, \( M(W, \mathcal{B}) \) is finite if and only if \( W \) and \( \mathcal{B} \) are finite.

Recall that in any monoid \( M \), Green’s relation \( \mathcal{R} \) is defined by the rule that \( a\mathcal{R}b \) if and only if \( aM = bM \). The relation \( \mathcal{L} \) is the left-right dual of \( \mathcal{R} \); we define \( \mathcal{H} = \mathcal{R} \cap \mathcal{L} \) and \( \mathcal{D} = \mathcal{R} \lor \mathcal{L} \). In fact, by [11, Proposition 2.1.3], \( \mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} \). Finally, a \( J \) if and only if \( \operatorname{dom} M = \operatorname{im} M \). In an inverse monoid, \( a\mathcal{R}b \) if and only if \( a^{-1} \mathcal{R} b^{-1} \) and similarly, \( a\mathcal{L}b \) if and only if \( a^{-1} a = b^{-1} b \). More information on Green’s relations can be found in [11, 15].

**Proposition 10.** Let \( \rho, \sigma \) be elements of the reflection monoid \( M = M(W, \mathcal{B}) \) with \( \rho = g_X \) and \( \sigma = h_Y \), where \( g, h \in W \) and \( X, Y \in \mathcal{B} \). Then

1. \( \rho\mathcal{R}\sigma \) if and only if \( X = Y \);
2. \( \rho\mathcal{L}\sigma \) if and only if \( X g = Y h \);
3. \( \rho\mathcal{D}\sigma \) if and only if \( Y \in XW \);
4. \( \rho\mathcal{F}\sigma \) if and only if \( \mathcal{F} \) consists of finite dimensional spaces, then \( \rho\mathcal{D}\sigma \).

**Proof.** (1) and (2) follow from [11, Proposition 2.4.2] and the well known fact that in \( ML(V) \) we have \( \rho\mathcal{L}\sigma \) if and only if \( \operatorname{dom} \rho = \operatorname{dom} \sigma \), and \( \rho\mathcal{R}\sigma \) if and only if \( \operatorname{im} \rho = \operatorname{im} \sigma \).

If \( \rho\mathcal{D}\sigma \), then \( \rho\mathcal{D}\mathcal{L}\mathcal{D}\sigma \) for some \( \tau \in M \), and it follows from (1) and (2) that \( Y \in XW \). On the other hand, if \( Y \in XW \), say \( Y = X k \) where \( k \in W \), then \( Yh = Xkh \) so that \( \sigma\mathcal{L}(kh)_X \) by (2), and \( (kh)_X \mathcal{R}\rho \) by (1), whence \( \rho\mathcal{D}\sigma \).

Certainly, \( \mathcal{D} \subseteq \mathcal{F} \). If \( \rho\mathcal{F}\sigma \), then \( \rho = \alpha\mathcal{D}\beta \) and \( \sigma = \gamma\mathcal{D}\delta \) for some \( \alpha, \beta, \gamma, \delta \in M \). Comparing domains gives \( X \subseteq Ya \) and \( Y \subseteq Xb \) for some \( a, b \in W \). If the dimensions are finite, we get \( Y = Xb \) so that \( \rho\mathcal{D}\sigma \) by (3). \( \square \)
We remark that although we have stated this result for reflection monoids, an entirely analogous result holds for factorizable monoids in general.

The power of realising reflection monoids in the form $M(W, B)$ will be seen in the next section where we produce a wealth of examples and calculate their orders. For now, we use the idea to give an example of a non-fundamental reflection monoid in which the restriction of the Munn representation to the group of units is one-one. (Of course, we have seen that $f_X$ is not fundamental, but in this case there are distinct units which are $\mu$-related.) First, note that if $M = M(W, B)$ is any reflection monoid, and $\alpha \in M$ has domain $X$, then for any $Y \in B$ we have

$$\alpha^{-1} \varepsilon_Y \alpha = \varepsilon_{(Y \cap X)\alpha}.$$  

(4)

Now let $V = \mathbb{R}^2$ and $\Phi \subset V$ the root system shown (stably isomorphic to the crystallographic $G_2 \subset \mathbb{R}^3$ of Table 2) and for $W$ take the subgroup of $W(\Phi)$ generated by $\rho$ and $\tau$ where $\rho$ is a rotation through $2\pi/3$ and $\tau$ is the reflection in the $y$-axis. Thus $W \cong \mathbb{S}_3$. The $\mathbb{R}$-spans of these roots, together with $V$ and 0, form a system (of subspaces) for $W$. The $\mu$-class of the identity $\varepsilon_Y$ is a normal subgroup of $W$ and so to show that $\mu$ is trivial on $W$, it is enough to show that $\rho$ and $\varepsilon_Y$ are not $\mu$-related. This is clear from (4) using any of the six lines for $Y$. On the other hand, letting $X$ be the $x$-axis, we see that $\tau_X$ and $\varepsilon_X$ are distinct but $\mu$-related.

We now consider when two reflection monoids are isomorphic. Let $W \subset GL(V), W' \subset GL(V')$ be reflection groups and $B, B'$ systems of subspaces for $W, W'$ respectively. We say that a vector space isomorphism $f : V \to V'$ induces an isomorphism of reflection monoids $f : M = M(W, B) \to M' = M(W', B')$ if $M' = f^{-1}Mf$. It is easy to see that the map $\alpha \mapsto f^{-1}\alpha f$ is a monoid isomorphism $M \to M'$.

**Proposition 11.** $M$ and $M'$ are isomorphic reflection monoids if and only if there is a vector space isomorphism $f : V \to V'$ with $W' = f^{-1}Wf$ and $B f = B'$, ie: $f : W \to W'$ is an isomorphism of reflection groups with $B f = B'$.

In particular, if the systems are the intersection lattices of hyperplane arrangements, then as $\text{rk} X = \text{codim} X$, an isomorphism of reflection monoids will induce a bijection between the rank $k$ elements of the two systems.

**Proof.** If $f$ is an isomorphism of reflection monoids then the monoid isomorphism $\alpha \mapsto f^{-1}\alpha f$ sends units to units, hence $W' \subset f^{-1}Wf$ with $f^{-1}$ giving the reverse. If $X \in B$ then $\varepsilon_X \in M$, hence $Xf = f^{-1}\varepsilon_X f \in M'$ giving $Xf \in B'$. Thus $B f \subset B'$ and $f^{-1} \alpha f \in M'$. Conversely, if $f$ an isomorphism of the reflection groups $W$ and $W'$ with $B f = B'$ and $\alpha \in M$ then $\alpha = g_X$ for $g \in W$ and $X \in B$ hence $f^{-1}\alpha f = (f^{-1}g f)_X f$ with $Xf \in B'$ and $f^{-1}g f \in W'$ giving $f^{-1}\alpha f \in M'$.

We now proceed to find the orders of our reflection monoids, for which the following result is straightforward but crucial.

**Theorem 1.** Let $W \subset GL(V)$ be a reflection group and $B$ a system for $W$. Then

$$|M(W, B)| = \sum_{X \in B} |W : W_X|,$$

where $W_X \subset W$ is the isotropy group of $X \in B$.

**Proof.** For $X \in B$ let $M(X)$ be the set of $\alpha \in M(W, B)$ with $\text{dom}(\alpha) = X$. Then $M(W, B)$ is the disjoint union of the $M(X)$ and so $|M(W, B)| = \sum_{X \in B} |M(X)|$. The elements of $M(X)$ are the partial isomorphisms obtained by restricting the elements of $W$ to $X$, and $w_1, w_2 \in W$ yield the same partial isomorphism if and only if they lie in the same coset of the isotropy subgroup $W_X$. Thus, $|M(X)| = |W : W_X|$ and the result follows. $\square$
Observe that the $M(X)$ of the proof is the $R$-class containing the partial identity $\varepsilon_X$, so that
the sum of Theorem 1 can be interpreted as a sum over $R$-classes. The proof also shows that the result is true for an arbitrary $G \subseteq GL(V)$, however, when $G$ is not a reflection group, it may not be so easy to calculate the number of orbits and their sizes, and dealing with the isotropy group $G_X$ may be difficult.

If $X, Y \in B$ lie in the same orbit of the $W$-action on $B$, then their isotropy groups $W_X, W_Y$ are conjugate, and the sum in Theorem 1 becomes

$$|M(W, B)| = |W| \sum_{X \in \Omega} ^{n_X} W_X,$$

where $\Omega$ is a set of orbit representatives for the $W$-action on $B$, and $n_X$ is the size of the orbit containing $X$. Most of our applications of Theorem 1 will use the form (5).

### 4. Examples

In this section we identify some important monoids pre-existing in the literature as reflection monoids, and introduce some new examples. In some cases the choices are motivated by reflection groups that can be identified with other common or garden variety groups.

#### 4.1. Boolean monoids

We saw in §1 that the classical Weyl groups have alternative descriptions as groups of permutations, with $W(A_{n-1}) \cong S_n$, $W(B_n) \cong B_n = S_n \times \Sigma(n)$ and $W(D_n) = S_n \times \Sigma^+(n)$.

Much the same happens in the partial case. Let $W = W(\Phi)$ be a Weyl group as in Tables 1-2, and $B = \langle A \rangle_W$ the Boolean system of §2.1. Then the the resulting reflection monoid $M(W, B) = M(\Phi, B)$ is called a Boolean (reflection) monoid. Both $M(A_{n-1}, B)$ and $M(B_n, B)$ can be identified with naturally occurring permutation monoids.

Returning to the inverse monoids $I_n$ of the previous section, let $X = \{1, \ldots, n\}$. If $\{i, i + 1\} \subset Y \subset X$, let $\sigma_{i,Y}$ be the partial permutation with domain and image $Y$, and whose effect on $Y$ is as the transposition $(i, i + 1)$, ie: $\sigma_{i,Y}$ interchanges $i$ and $i + 1$, fixes the remaining points of $Y$, and is undefined on $X \setminus Y$. Similarly, let $\tau_{i,Y}$ have domain and image $Y \cup -Y \subset X \cup -X$ with $i \in Y$ and effect $(i, -i)$ on $Y \cup -Y$; let $\mu_{i,Y}$ have effect $(i, i + 1)(-i, -(i + 1))$ on $Y \cup -Y$ for $\{i, i + 1\} \subset Y$.

**Lemma 4.** Let $n \geq 3$. (1). The symmetric inverse monoid $I_n$ is generated by the partial transpositions $\sigma_{i,Y}$ for $1 \leq i \leq n - 1$ and $Y \subset X$.

(2). The monoid of partial signed permutations $J_n$ is generated by the $\tau_{i,Y}$ and $\mu_{j,Y}$ for $1 \leq i \leq n$, $1 \leq j \leq n - 1$, $Y \subset X$.

**Proof.** For (1), we note that $S_n$ is generated by the full transpositions $\sigma_{i,X}$, and that by (the proof of) [10, Theorem 3.1], $I_n$ is generated by any generating set for $S_n$ together with any partial permutation of rank $n - 1$.

For (2), we recall that $J_n$ is factorizable so that every element can be written as $\varepsilon_{Y \cup -Y} \tau$ for some (full) signed permutation $\tau$. Certainly $B_n$ is generated by the $\tau_{i,X}$ and $\mu_{i,X}$, so it suffices to express $\varepsilon_{Y \cup -Y}$ in terms of the proposed generating set. Writing $\varepsilon_{i_1 \ldots i_k}$ for $\varepsilon_{Y \cup -Y}$ where $Y = X \setminus \{i_1, \ldots, i_k\}$, we have $\varepsilon_{i_1 \ldots i_k} = \varepsilon_{i_1} \ldots \varepsilon_{i_k}$; hence it is enough to show that $\varepsilon_i$ (for $1 \leq i \leq n$) can be expressed in terms of the proposed generators. As $n \geq 3$, we have $\varepsilon_n = \tau_{i,Y}$, where $Y = X \setminus \{n\}$, and $\varepsilon_j = \mu_{j,Y} \varepsilon_{j+1} \mu_{j,Y}$ for $j < n$; hence $\varepsilon_1, \ldots, \varepsilon_n$ are generated by the $\tau_{i,X}$ and $\mu_{i,X}$ as required. \(\square\)

Let $V$ be a Euclidean space with orthonormal basis $\{x_1, \ldots, x_n\}$, and for $Y = \{i_1, \ldots, i_k\} \subset X = \{1, \ldots, n\}$, write $\langle Y \rangle$ for the span of $\{x_{i_1}, \ldots, x_{i_k}\}$. If $x \in V$ let $s_x$ be the reflection in $x^\perp$ and $(s_x)_{(Y)}$ the corresponding partial reflection.
Proof. As mentioned in §1, it is well known that when restricted to full reflections, the map in (1) induces an isomorphism \( \varphi : W(A_{n-1}) \to S_n \). For \( g, h \in W(A_{n-1}) \) and \( Y \subset X \), it is clear that \( g_{(Y)} = h_{(Y)} \) if and only if \( (g\varphi)_Y = (h\varphi)_Y \). Hence there is a bijection \( \overline{\varphi} : M(A_{n-1}, B) \to \mathcal{J}_n \) extending \( \varphi \) and given by \( g_{(Y)}\overline{\varphi} = (g\varphi)_Y \). It is easy to verify that \( \overline{\varphi} \) is an isomorphism which restricts to the map given in (1). It follows that this map induces \( \overline{\varphi} \) since the \( \sigma_{i,Y} \) generate \( \mathcal{J}_n \).

The proof of (2) is similar. \( \square \)

Unlike the Weyl group \( W(D_n) \), there seems to be no nice interpretation of the reflection monoid \( M(D_n, B) \) as a group of partial permutations. Now to the orders:

**Theorem 2.** Let \( \Phi_n \) be a root system of type \( A_{n-1}, B_n \) or \( D_n \) as in Table 1 and \( B \) the Boolean system for \( W(\Phi_n) \). Then the Boolean reflection monoids have orders,

\[
|M(\Phi_n, B)| = |W(\Phi_n)| \sum_{k=0}^{n} \binom{n}{k} \frac{1}{|W(\Phi_k)|}.
\]

**Proof.** The \( W \)-action on \( B \) is rank preserving and transitive on the rank \( k \) elements, with \( \text{rk}(X = x_{i_1}^+ \cap \cdots \cap x_{i_k}^+) = \text{rk}\{i_1, \ldots, i_k\} = k \) (see §2.1). Thus the \( X = x_{i_1}^+ \cap \cdots \cap x_{i_k}^+ \) for \( 0 \leq k \leq n \) are orbit representatives, with \( n_X \) the number of \( k \) element subsets of \( I \), and \( W_X \) generated by the reflections \( s_v \) for \( v \in \Phi_n \cap X^+ \cong \Phi_k \). The result now follows from (5). \( \square \)

By the conventions of §1 we have \( |W(A_k)| = (k+1)! \), \( |W(B_k)| = 2^k k! \), \( |W(D_0)| = 1 \), and \( |W(D_k)| = 2^{k-1} k! \) for \( k > 1 \), thus giving,

| \( \Phi_n \) | \( A_{n-1} \) | \( B_n \) | \( D_n \) |
|---|---|---|---|
| \( |M(\Phi_n, B)| \) | \( \sum_{k=0}^{n} \binom{n}{k} \frac{2^k}{k!} \) | \( \sum_{k=0}^{n} 2^k \binom{n}{k} \frac{2^k}{k!} \) | \( 2^{n-1} n! + \sum_{k=1}^{n} 2^k \binom{n}{k} \frac{2^k}{k!} \) |

Notice that the given orders gel with the isomorphisms \( M(A_{n-1}, B) \cong \mathcal{J}_n \) and \( M(B_n, B) \cong \mathcal{J}_n \) of Proposition 12 and the well known order of \( \mathcal{J}_n \) (see eg: [11, Chapter 5, Exercise 3]): one can independently choose a domain and image of size \( k \) for a partial permutation \( \sigma \in \mathcal{J}_n \), with there then being \( k! \) partial permutations having the given domain and image; similarly for \( \mathcal{J}_n \), there being \( 2^k k! \) partial signed permutations with a given domain and image. One can also show, by thinking in terms of partial signed permutations, that the non-units of \( M(B_n, B) \) and \( M(D_n, B) \) coincide, which is why the orders of these reflection monoids are identical except for the \( k = 0 \) terms (recall that \( \varnothing \subset I \) corresponds to the ambient space \( V \in B \)).

### 4.2. The Renner monoids

The theory of linear algebraic monoids was developed independently, and then subsequently collaboratively, by Mohan Putcha and Lex Renner during the 1980’s. Among the chief achievements of the theory is the classification [24,25] of the reductive monoids, and the formulation of a Bruhat decomposition [23] for a reductive algebraic monoid, with the role of the Weyl group being played by a certain finite factorizable inverse monoid, coined the Renner monoid by Solomon [29].

Thus the Renner monoids play the same role for algebraic monoids that the Weyl groups play for algebraic groups, and in this section we investigate to what extent the analogy continues further. Standard references on algebraic groups are [1,12,30], and on algebraic monoids,
Throughout, \( \mathbb{F} \) is an algebraically closed field. An affine (or linear) algebraic monoid \( M \) over \( \mathbb{F} \) is an affine algebraic variety together with a morphism \( \varphi : M \times M \to M \) of varieties, such that the product \( xy = \varphi(x, y) \) gives \( M \) the structure of a monoid (ie: \( \varphi \) is an associative morphism of varieties and there is a two-sided unit \( 1 \in M \) for \( \varphi \)). We will assume that the monoid \( M \) is connected, that is, the underlying variety is irreducible, in which case the group \( G \) of units is a connected algebraic group with \( \overline{G} = M \) (Zariski closure). Adjectives normally applied to \( G \) are then transferred to \( M \); thus we have semisimple monoids, reductive monoids, simply connected monoids, and so on.

From now on, let \( M \) be reductive. The key players, just as they are for algebraic groups, are the maximal tori \( T \subset G \) and their closures \( T' \subset M \). Let \( \mathcal{X}(T) \) be the character group of all morphisms of algebraic groups \( \chi : T \to G_m \) (with \( G_m \) the multiplicative group of \( \mathbb{F} \)) and \( \mathcal{X}(T) \) similarly the commutative monoid of morphisms of \( T' \). Then \( \mathcal{X}(T) \) is a free \( \mathbb{Z} \)-module, and restriction (together with the denseness of \( T \) in \( T' \)) embeds \( \mathcal{X}(T) \to \mathcal{X}(T) \).

The Weyl group \( W_G = N_G(T)/T \) of automorphisms of \( T \) acts faithfully on \( \mathcal{X}(T) \) via \( \chi^g(t) = \chi(g^{-1}tg) \), thus realizing an injection \( W_G \hookrightarrow GL(V) \) for \( V = \mathcal{X}(T) \otimes \mathbb{R} \). We will write \( W \) for both the Weyl group and its image in \( GL(V) \). The non-zero weights \( \Phi := \Phi(G, T) \) of the adjoint representation \( G \to GL(g) \) form a root system with the Weyl group \( W \) generated by reflections \( s_\alpha \) for \( \alpha \in \Phi \) (with respect to a \( W \)-invariant bilinear form).

The Renner monoid [23] \( R_M \) of \( M \) is defined to be \( R_M = N_M(T)/T \), which turns out (although this is not obvious) to be \( N_M(T)/T \), where \( N_M = \{ x \in M \mid xT = Tx \} \). Just as \( \mathcal{A}_n \) is the archetypal inverse monoid, and as \( M(A_{n-1}, \text{Boolean}) \) is the archetypal reflection monoid, so in its incarnation as the root monoid it is the standard example of a Renner monoid, namely for \( M = M_n(\mathbb{F}) \), the algebraic monoid of \( n \times n \) matrices over \( \mathbb{F} \). These monoids have been explicitly described in some other cases, for example, when \( M \) is the “symplectic monoid” \( \text{MSp}_n(\mathbb{F}) = \mathbb{F}^* \text{Sp}_n(\mathbb{F}) \subset M_n(\mathbb{F}) \) [38].

Suppose now \( M \) has a zero, and let \( E = E(T) \) be the lattice of idempotents of \( T \), for \( T \) a maximal torus in \( G \). Then by the results of [20, Chapter 6], \( E \) is a graded lattice with \( 0 \) and \( 1 \). Moreover, by [20, Theorem 10.7], the Weyl group \( W \) is the automorphism group of \( E \), via \( e^g = g^{-1}eg \), and by [20, Remark 11.3(i)] the Renner monoid \( R_M \) is the submonoid \( \langle E, W \rangle \subset \mathcal{TE} \), of the Munn semigroup \( \mathcal{TE} \) of \( E \). Thus by Proposition 8, the Renner monoid has the form \( M(\mathcal{W}, \mathcal{E}) \) where \( \mathcal{E} = \{ Ex \mid x \in E \} \) is a system of subsets in \( E \).

Before proceeding we summarize some basic facts about cones from [9, §1.2]. If \( V \) is a real space and \( v_1, \ldots, v_s \) a finite set of vectors, then the convex polyhedron cone with generators \( \{ v_i \} \) is the set \( \sigma = \sum \lambda_i v_i \) where \( \lambda_i \geq 0 \). The dual cone \( \sigma^\vee \subset V^\vee \) consists of those \( u \in V^\vee \) taking non-negative values on \( \sigma \). A face \( \tau \subset \sigma \) is the intersection with \( \sigma \) of the kernel \( u^\perp \) of a \( u \in \sigma^\vee \), and the faces form a meet semilattice \( \mathcal{F}(\sigma) \) under inclusion. If \( \tau \in \mathcal{F}(\sigma) \), let \( \overline{\tau} \) be the \( \mathbb{R} \)-span in \( V \) of \( \tau \), so that if \( \tau = \sigma \cap u^\perp \) for \( u \in \sigma^\vee \), then \( \sigma \cap \overline{\tau} = \tau \). In particular, if \( \bigcap \overline{\tau}_j = \bigcap \overline{\tau}_j \) in \( \mathcal{F}(\sigma) \) are faces of \( \sigma \) then we have \( \overline{\tau} \subset \bigcap \overline{\tau}_j \) for \( \tau = \bigcap \overline{\tau}_j \).

A cone is simplicial if it has a set \( A = \{ v_i \} \) of linearly independent generators. If \( \tau \) is the cone on \( \{ v_1, \ldots, \widehat{v_i}, \ldots, v_s \} \), then \( \tau_1 = \sigma \cap u^\perp_i \), where \( u_i \) is the vector corresponding to \( v_i \) in the dual basis for \( V^\vee \). Thus \( \tau_1 \) is a face of \( \sigma \), and the face lattice \( \mathcal{F}(\sigma) \) is isomorphic to the Boolean lattice on the \( 1 \)-dimensional faces \( \mathbb{R}^+ \cdot v_i \) of \( \sigma \). If \( \tau \in \mathcal{F}(\sigma) \) corresponds to \( A_\tau \subset A \) then \( \tau_1 \cap \tau_2 \) corresponds to \( A_{\tau_1} \cap A_{\tau_2} \), and \( \overline{\tau} = \mathbb{R} \)-span of \( A_\tau \). In particular, \( \mathbb{R} \)-span\( \{ A_{\tau_1} \} = \bigcap \{ \mathbb{R} \text{span} A_{\tau_j} \} \), and we have \( \overline{\tau} = \bigcap \overline{\tau}_j \) when \( \tau = \bigcap \overline{\tau}_j \) for \( \sigma \) simplicial. Finally, a cone is strongly convex if the dual \( \sigma^\vee \) spans \( V^\vee \). Simplicial cones are strongly convex. On the other hand, if \( \dim V = 2 \), then any strongly convex cone is simplicial [9, 1.2.13].

Returning to algebraic monoids, we may assume, by conjugating suitably, that the maximal torus \( T \) is a subgroup of the group \( T_n \) of invertible diagonal matrices, where \( n \) is the rank of \( G \). If \( \chi_j \) is the restriction to \( T \) of the \( j \)-th coordinate function on \( T_n \), then the cone \( \sigma = \sum \mathbb{R}^+ \chi_i \subset \mathcal{X}(T) \otimes \mathbb{R} \) is strongly convex. The dual cone \( \sigma^\vee \) lives in the group of \( 1 \)-parameter subgroups of \( T \).
This \( \sigma \) has a number of nice properties. Firstly, the character monoid \( X(T) = \sigma \cap X(T) \). Secondly, the Weyl group \( W \), in its reflectional action on \( V \), acts on \( \sigma \), and this induces an action \( \tau \mapsto \tau g \) of \( W \) on \( \mathcal{F}(\sigma) \). Finally, the face lattice \( \mathcal{F}(\sigma) \) models the idempotents: there is a lattice isomorphism \( \mathcal{F}(\sigma) \to E(T) \), with \( \tau \mapsto e_{\tau} \), that is \( W \)-equivariant with respect to the Weyl group actions, i.e.: for any \( g \in W \), the diagram on the left of Figure 1 commutes. In short, \( e_{g}^{2} = e_{\tau g} \) (Solomon [28, Corollary 5.5], working with the dual cone, has a lattice anti-isomorphism \( \mathcal{F}(\sigma) \to E(T) \)).

We can now define a reflection monoid using the Weyl group of \( G \) and the convex polyhedral cone \( \sigma \subset X(T) \cap Q \). Let \( B = \langle \tau | \tau \in \mathcal{F}(\sigma) \rangle \) be the system for \( W \) generated by the subspaces \( \tau \). As \( W \) acts on the face lattice \( \mathcal{F}(\sigma) \), each \( X \in B \) has the form \( X = \cap \tau_{j} \) for \( \tau_{j} \in \mathcal{F}(\sigma) \). Call \( M(W, B) \) the reflection monoid associated to \( M \).

Figure 1 depicts the situation for \( M = M_{3} \). The system \( B \) is just the Boolean one generated by the coordinate hyperplanes \( \chi_{i} \), and the reflection monoid \( M(W, B) \) is the symmetric inverse monoid on the vertices of the 2-simplex (hence, in this case, isomorphic to the Renner monoid \( R_{M} \)).

If \( X = \cap \tau_{j} \in B \), then the idempotents of \( M(W, B) \) are products \( e_{X} = \prod e_{\tau_{j}} \) where \( e_{\tau_{j}} \) is the partial identity on \( \tau_{j} \), hence any element of the reflection monoid has the form \( e_{\tau} = \prod e_{\tau_{j}} \cdot g \) for \( g \in W \). Define a mapping \( f : M(W, B) \to M(W, \mathcal{C}) = R_{M} \) by \( f(e_{\tau}) = \prod e_{\tau_{j}} \cdot g \), where \( e_{\tau_{j}} := e_{\tau_{j}} \in E \).

**Theorem 3.** Let \( M \) be connected reductive with 0, \( R_{M} \) its Renner monoid, and \( M(W, B) \) the associated reflection monoid. Then \( f : M(W, B) \to R_{M} \) is a surjective homomorphism, which is injective if and only if \( \sigma \subset X(T) \cap Q \) is a simplicial cone.

**Proof.** Let \( X = \cap \tau_{j} \), \( Y = \cap \tau_{j} \) and \( e_{X}g_{1} = e_{Y}g_{2} \) in the reflection monoid. Then \( X = Y \) and \( g_{2}g_{1}^{-1} \) is in the isotropy group \( W_{X} \) of \( X \). By intersecting the expressions for \( X \) and \( Y \) with \( \sigma \) we get \( \cap \tau_{j} = \cap \mu_{j} \) in \( \mathcal{F}(\sigma) \), and so \( \prod e_{\tau_{j}} = \prod e_{\mu_{j}} \) in \( E \), as these are the images under the lattice isomorphism \( \mathcal{F}(\sigma) \cong E \).

Writing \( e_{\tau} := e_{\tau_{j}} \) and \( \tau = \cap \tau_{j} \) from now on, it suffices, for \( f \) to be well defined, to show that the elements \( \prod e_{\tau_{j}} \cdot g_{i} \) (\( i = 1, 2 \)), give the same partial permutations in \( \mathcal{F}(\sigma) \), and this follows if \( g_{2}g_{1}^{-1} \) fixes the ideal \( E(\prod e_{\tau_{j}}) \) pointwise. Let \( e_{\kappa} \) be in this ideal for some \( \kappa \in \mathcal{F}(\sigma) \), so that \( \kappa \subset \tau \) by the isomorphism \( \mathcal{F}(\sigma) \cong E \), hence \( \kappa \subset \tau \subset \tau \subset \cap \tau_{j} = X \). Thus, as \( g_{2}g_{1}^{-1} \) fixes \( X \) pointwise, it fixes \( \tau \) pointwise, giving

\[
e_{\tau}g_{2}g_{1}^{-1} = e_{\tau}g_{1}^{-1} = e_{\tau},
\]

as the isomorphism \( \mathcal{F}(\sigma) \cong E \) is \( W \)-equivariant. Thus \( f \) is well defined. To see that it is a homomorphism, observe that \( e_{X}g_{1}^{-1} = e_{X}g_{1}^{-1} \cdot X \) and \( Xg_{1}^{-1} = (\cap \tau_{j})^{-1} \) and \( X \) is as above. If

![Figure 1](image-url)
The homomorphism $f$ of Theorem 3 need not be injective: if $\mathcal{M} = \text{Ad}(G)\mathbb{F}^*$ with $G$ the adjoint simple group of type $B_2$, then the lattice of idempotents of the associated reflection monoid (left) contains non-zero elements mapping via $f$ to zero in the lattice of idempotents of the Renner monoid (right).

$\tau_j \mapsto e_j$ via $\mathcal{F}(\sigma) \cong E$, then $\tau_j g \mapsto e_{\tau_j} g = e_{\sigma_j}^g$, and so

$$e_X^g \mapsto \prod_j e_{\tau_j}^{g^{-1}} = \left( \prod_j e_{\tau_j} \right)^{g^{-1}}$$

under $f$. We then have

$$e_X g_1 \cdot e_Y g_2 = e_X e_Y g_1 g_2 \mapsto f \prod e_{\tau_j} \left( \prod e_{\mu_j} \right) g_1 g_2 = \prod e_{\tau_j} \cdot g_1 \cdot \prod e_{\mu_j} \cdot g_2.$$

Surjectivity is clear.

For the second part of the Theorem, let $\overline{\tau}$ be the $\mathbb{R}$-span in $V$ of $\sigma$, where we must have $\dim \overline{\tau} > 2$ if $\sigma$ is not simplicial. There are then maximal faces $\tau_1, \tau_2 \in \mathcal{F}(\sigma)$ with $\tau_1 \cap \tau_2 = \{0\}$. As the $\tau_i$ are hyperplanes in $\overline{\tau}$, the intersection $\overline{\tau}_i \cap \overline{\tau}_j$ has codimension 2 in $\overline{\tau}$, hence is non-zero. If $\varepsilon_i$ is the partial identity on $\overline{\tau}_i$ and $e_i : = e_{\tau_i}$, then this translates into $e_1 e_2 \neq 0$ in $M(W, B)$, but $e_1 e_2 = 0$ in $E$. In particular, the injectivity of $f$ fails, even on the idempotents.

On the other hand, if $\sigma$ is simplicial, let $e_X g_1 \mapsto \prod e_{\tau_j} \cdot g_1$, $e_Y g_2 \mapsto \prod e_{\mu_j} \cdot g_2$ with $\prod e_{\tau_j} \cdot g_1 = \prod e_{\mu_j} \cdot g_2$. As elements of $\mathcal{F}_E$ we have $\prod e_{\tau_j} = \prod e_{\mu_j}$ and $g_1^{-1} g_2$ fixing the ideal $E(\prod e_{\tau_j})$ pointwise. The lattice isomorphism then gives $\tau = \bigcap \tau_j = \bigcap \mu_j = \mu$ and thus $X = \bigcap \overline{\tau}_j = \overline{\tau} = \bigcap \overline{\mu}_j = Y$. If $\bigcap \tau_j$ is generated by the (independent) vectors $v_1, \ldots, v_t$ and $v_i = \mathbb{R}_+ \cdot v_i$, then $e_{\tau_j} \subseteq E(\prod e_{\tau_j})$ and so fixed by $g_1^{-1} g_2$. Thus $\nu_i$ is also fixed, hence $X$ too, as it is spanned by such $\nu_i$. Thus $g_1 \varepsilon_X = g_2 \varepsilon_Y$, and $f$ is injective.

As an illustration of the phenomenon in the last part of the proof, let $\mathcal{M}$ be the (normalization of) $\text{Ad}(G)\mathbb{F}^*$ for $G$ the adjoint simple group of type $B_2$. Then [24, Example 3.8.3], $\dim(X(T) \otimes \mathbb{R}) = 3$ with $\sigma$ a cone on a square (see [24, Figure 6]). If $\tau_i$, ($i = 1, 2$) are the cones on opposite, non-intersecting faces of the square, then $\tau_1 \cap \tau_2 = \{0\}$, whereas $\overline{\tau}_1 \cap \overline{\tau}_2$ is a 1-dimensional subspace. Figure 2 gives the lattice of idempotents of the reflection monoid associated to $\mathcal{M}$ (left) with a pair a $\varepsilon_1 e_2 \neq 0$ marked, mapping via $f$ to $e_1 \land e_2 = 0$ (right).

Not only does the above homomorphism fail to be injective in this case, but we can also show quite easily that $R_{\mathcal{M}} B_2$ cannot be isomorphic to a reflection monoid. For, suppose that $R_{\mathcal{M}} \cong M(W, B)$ where $B$ is a system of subspaces of a Euclidean space $V$ on which $W$ acts as a reflection group. Since $W$ must be isomorphic to the group of units of $R_{\mathcal{M}}$, we have $W = W(B_2)$. Hence four of the elements of order 2 in $W$ must be reflections. Also, the lattice $B$ must be isomorphic to the lattice shown on the right in Figure 2. Moreover, if the bottom element of $B$ is a non-zero subspace, we can factor it out to obtain a lattice of subspaces with bottom element $\{0\}$.

Reading from left to right, let the atoms and coatoms of $B$ be $U_0, U_1, U_2, U_3$ and $X_0, X_1, X_2, X_3$ respectively. The intersection of any two $U_i$’s is zero, as is the intersection of $X_0$ and $X_2$. Hence for any choice of non-zero vectors $u_i \in U_i$ ($i = 0, 1, 2, 3$), the set $\{u_0, \ldots, u_3\}$ is linearly independent.
The group of units of $R_\mathbb{M}$ is the automorphism group of $E(R_\mathbb{M})$ where the action is by conjugation. Hence $W$ acting by conjugation on $\{\varepsilon_Y \mid Y \in \mathcal{B}\}$ gives all automorphisms of $E(M(W, \mathcal{B}))$ and since $\varepsilon_Y g = g^{-1} \varepsilon_Y g$ for all $Y \in \mathcal{B}$ and $g \in W$, the same is true of the induced action of $W$ on $\mathcal{B}$.

Automorphisms of $\mathcal{B}$ are determined by their effect on the atoms. Let $g, g' \in W$ be such that their actions give rise to the automorphisms determined by interchanging $U_0$ with $U_3$ and $U_1$ with $U_2$, and interchanging $U_0$ with $U_1$ and $U_2$ with $U_3$ respectively. Choose $u_i \in U_i$ for $i = 0, 1, 2$ such that $u_i g \in U_3$ and $u_1 g \in U_2$, so that $\{u_0, u_1, u_0 g, u_1 g\}$ is a basis for the subspace it spans, say $U$. It is readily verified that $-1$ is an eigenvalue of $g|_U$ of multiplicity 2, so that $-1$ cannot be a simple eigenvalue of $g$ itself. Thus $g$ (which has order 2) is not a reflection. Similarly, $g'$ is not a reflection. This is a contradiction since there is only one element of order 2 in $W$ which is not a reflection.

Our last result in this subsection is a negative one of sorts: if an inverse monoid $M$ is to be a reflection monoid then we must have an injective homomorphism $M \hookrightarrow ML(V)$ with the units of $M$ a reflection group in $V$.

**Proposition 13.** Let $\mathbb{M}$ be connected with 0 and $R_\mathbb{M}$ its Renner monoid. If $\rho : R_\mathbb{M} \to ML(V)$ is faithful with $\rho(W_G)$ a reflection group acting essentially on $V$, then $W_G$ is not of $(-1)$-type.

Thus at least one of the irreducible components of $W_G$ must be $A_n (n > 1)$, $D_n$ ($n$ odd) or $E_6$.

**Proof.** It follows immediately that $W = \rho(W_G)$ is a finite reflection group acting essentially on $V$. In particular, $\rho$ is equivalent to the reflectional representation of a Coxeter system $(W_G, S)$, and if $W_G$ is of $(-1)$-type, there is a $g \neq 1 \in W_G$ with $\rho(g) = -1$ on $V$. By (4), $\rho(g)$ is $\mu$-related to $1 \in \rho(R_\mathbb{M})$, with the resulting reflection monoid not fundamental. □

We conclude the subsection by mentioning that several authors have calculated the orders of certain Renner monoids. The most general results (which include all earlier ones) are in [37].

### 4.3. Reflection arrangement monoids

Let $W \subset GL(V)$ be a reflection group and $\mathcal{H} = L(A)$ the intersection lattice of the arrangement $A$ of the reflecting hyperplanes of $W$. The resulting $M(W, \mathcal{H})$ is called the (reflection) arrangement monoid of $A$.

If $W = W(\Phi)$ we write $M(\Phi, \mathcal{H})$ for the arrangement monoid. If $\Phi \subset V$ and $\Phi' \subset V'$ are essential, then a root system isomorphism $f : \Phi \to \Phi'$ induces an isomorphism of reflection monoids $M(\Phi, \mathcal{H}) \to M(\Phi', \mathcal{H'})$ where $\mathcal{H}, \mathcal{H'}$ are the lattices of the reflection arrangements arising from $\Phi$ and $\Phi'$. Thus we may talk of the arrangement monoids of types $A, B, \ldots$ etc, without reference to the particular choice of root system, although we will usually have in mind the $\Phi$ of §1.

Scrubbing these $\Phi$, we see that in types $B$ and $F$, the Boolean system $\mathcal{B}$ is properly contained in the arrangement system $\mathcal{H}$, thus the Boolean monoid $M(\Phi, \mathcal{B})$ is a proper submonoid of the arrangement monoid $M(\Phi, \mathcal{H})$ in these cases. On the other hand, an isomorphism of reflection monoids $M(\Phi, \mathcal{B}) \to M(\Phi, \mathcal{H})$ would induce, by Proposition 11, a bijection between the rank $k$ subspaces of the Boolean and arrangement systems. For classical $\Phi$, the number of such subspaces in the arrangement systems are

|   |   |   |
|---|---|---|
| $A$ | $B$ | $D$ |
| $S(n, k) \sum_{i=0}^{n} 2^{i-k} \binom{n}{i} S(i, k)$ | $\sum_{i \neq n-1} 2^{i-k} \binom{n}{i} S(i, k)$ |   |
where $S(n, k)$ is a Stirling number of the second kind. As these numbers in the Boolean case are the number of ways of choosing $k$ objects from $n$, there is no isomorphism of reflection monoids between $M(\Phi, B)$ and $M(\Phi, \mathcal{H})$ for these $\Phi$.

We now proceed to compute their orders, which in contrast to the Boolean guys, we can do in both the classical and exceptional cases. Recall from §2.2 that a partition of $n$ is a sequence of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_p)$ with $\sum \lambda_i = n$ and $\lambda_i \geq \lambda_{i+1} \geq 1$, and if $b_i > 0$ is the number of $\lambda_i$ equal to $i$, then $b_\lambda = b_1! b_2! \ldots (1!)^{b_1} (2!)^{b_2} \ldots$

**Theorem 4.** The arrangement monoid $M(A_{n-1}, \mathcal{H})$ has order:

$$|M(A_{n-1}, \mathcal{H})| = (n!)^2 \sum_{\lambda} \frac{1}{b_\lambda b_1! \ldots \lambda_p!},$$

the sum over all partitions $\lambda$ of $n$.

The denominator of the sum in Theorem 4 is largest for the partition $\lambda = (n)$, which contributes $1/(n!)^2$, hence the not a priori obvious fact that the sum is an integer.

**Proof.** This is another application of (5), with by Proposition 3, the partitions of $n$ the orbit representatives, $n_X(A) = n!/b_\lambda$ for $\lambda = ||A||$, and $|W| = n!$. The $W(A_{n-1}) \cong S_n$ action on $\mathcal{H}$ is given by $X(A)g(\sigma) = X(A\sigma)$ for $\sigma \in S_n$, hence $W_X(A) \cong S_{\lambda_1} \times \cdots \times S_{\lambda_p}$. \qed

Proceeding now to the type $B$ case, let $0 \leq m \leq n$ be integers,

$$c_{mn} \overset{\text{def}}{=} \min\{m, n-m\} \sum_{i=0}^{\min\{m, n-m\}} \binom{m}{i} \binom{n-m}{i},$$

and $\delta_{mn} \overset{\text{def}}{=} m!(n-m)!c_{mn}$. The following is more general than we need, but may be of independent interest:

**Proposition 14.** The isotropy group $W_X \subset W(B_n)$ of the subspace $X = X(\Delta, \Gamma, A) \in \mathcal{H}$ has order

$$2^{m+p} m! \prod_{i=1}^{p} \delta_{m, \mu_i},$$

where $m = ||\Delta||, ||\Gamma|| = (\lambda_1, \ldots, \lambda_p)$ for $A = \{A_1, \ldots, A_p\}$, and $\mu_i = ||\Gamma \cap A_i||$.

**Proof.** An element $g(\sigma, T) \in W(B_n)$ stabilizes $X$ precisely when $\Delta \sigma = \Delta, A_i \sigma = A_i$ and if $T_i = \Gamma \cap A_i$ and $T_i = T \cap A_i$, then for each $1 \leq i \leq p$, we have $(T_i \Delta T_i) \sigma = T_i$ or $A_i \setminus T_i$ (see §2.2). We are thus free in the first instance to choose a pair of $T_\Delta = T \cap \Delta$ and $\sigma$ any bijection $\Delta \to \Delta$ (of which there are $2^{m} m!$) and the proof is completed by showing that the number of pairs of a $T_i$ and $\sigma_i$ (which is $\sigma$ restricted to $A_i$) is $2\delta_{m, \lambda_i}$. To have $(T_i \Delta T_i) \sigma_i = T_i$, it is clearly necessary that $T_i \Delta T_i$ and $T_i$ have the same cardinality and conversely, if this is so then $\sigma_i$ can be the extension of any bijection $T_i \Delta T_i \to T_i$. The $T_i \subset A_i$ for which $|T_i \Delta T_i| = |T_i|$ are precisely those subsets that can be partitioned into two equal sized pieces, one contained in $T_i$ and the other in $A_i \setminus T_i$. The number of such is $c_{m, \lambda_i}$ and for each one there are $\mu_i!$ bijections $T_i \Delta T_i \to T_i$, each one in turn extendable to $(\lambda_i - \mu_i)!$ bijections $\sigma_i : A_i \to A_i$.

The other possibility is that $(T_i \Delta T_i) \sigma_i = A_i \setminus T_i$ and as $(A_i \setminus T_i) \Delta T_i = A_i \setminus (T_i \Delta T_i)$, the map $T_i \to A_i \setminus T_i$ is a bijection from the set of $T_i$ with $|T_i \Delta T_i| = k$ to the set of $T_i$ with $|T_i \Delta T_i| = \lambda_i - k$. The result is that there are $c_{m, \lambda_i}$ subsets $T_i$ with $|T_i \Delta T_i| = |A_i \setminus T_i|$, and $(\lambda_i - \mu_i)! \mu_i!$ bijections $\sigma_i : A_i \to A_i$ extending bijections $T_i \Delta T_i \to A_i \setminus T_i$. \qed

For a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$, let $d_\lambda = 4^p b_\lambda \lambda_1! \ldots \lambda_p!$

**Theorem 5.** The arrangement monoid $M(B_n, \mathcal{H})$ has order

$$|M(B_n, \mathcal{H})| = 2^{2n-1} (n!)^2 \sum_{m, \lambda} \frac{1}{4^n d_\lambda},$$

the sum over all pairs $(m, \lambda)$ where $0 \leq m \leq n$ is an integer and $\lambda$ is a partition of $n - m$. 
Proof. Observe by Proposition 4 that the orbit of the subspace $X(\Delta, \Gamma, A)$ is determined by $m = |\Delta|$ and the partition $\lambda = |A|$ of $n - m$, with $\Gamma$ playing no role. We thus choose $\Gamma = \emptyset$ in each orbit, and apply (5) to $X(\Delta, \emptyset, A)$, with $|W| = 2^n n!$,

$$n_X = 2^{n-m-p} \binom{n}{n-m} \frac{(n-m)!}{b_\lambda} \text{ and } |W_X| = 2^{m+p} m! \prod_{i=1}^{p} \lambda_i!,$$

the last by Proposition 14.

For the arrangement monoid of type $D$, the intersection lattice $\mathcal{H}$ of the arrangement of reflecting hyperplanes is a sublattice of the type $B$ one. It then suffices to compare the isotropy groups in $W(B_n)$ and $W(D_n)$ of an $X \in \mathcal{H}$.

**Proposition 15.** If $\mathcal{H}$ is the intersection lattice of the reflection arrangement for $W(D_n)$ and $X = X(\Delta, \Gamma, A) \in \mathcal{H}$, then the isotropy groups $W_X \subset W(D_n)$, $W_X^\prime \subset W(B_n)$ coincide when $\Delta = \emptyset$ and each $\lambda_i$ is even, otherwise $W_X$ has index 2 in $W_X^\prime$.

**Proof.** The index of $W_X$ in $W_X^\prime$ is at most 2 as $W_X = W(D_n) \cap W_X^\prime$ with $W(D_n)$ of index two in $W(B_n)$. Thus either $W_X$ has index 2 in $W_X^\prime$ or the isotropy groups coincide, with the latter happening precisely when $Xg(\sigma, T) = X$ for $g(\sigma, T) \in W(B_n)$ implies that $g(\sigma, T) \in W(D_n)$, i.e., that $|T|$ is even. It is easy to check that this happens if and only if $\Delta = \emptyset$ and each $\lambda_i$ is even. □

**Theorem 6.** The arrangement monoid $M(D_n; \mathcal{H})$ has order,

$$|M(D_n, \mathcal{H})| = 4^{n-1} (n!)^2 \sum_{m,\lambda} \varepsilon_{m,\lambda} 4^m d_\lambda,$$

the sum over all pairs $(m, \lambda)$ where $0 \leq m \leq n$ is an integer $\neq 1$ and $\lambda = (\lambda_1, \ldots, \lambda_p)$ is a partition of $n - m$, with $\varepsilon_{m,\lambda} = 1$ if $m = 0$ and each $\lambda_i$ is even, and $\varepsilon_{m,\lambda} = 2$ otherwise.

**Proof.** Apply Propositions 5, 14 and 15 to (5). □

The orders of the arrangement monoids for the exceptional Weyl groups are calculated directly from (5) and the data in Table 3.

**Proposition 16.** The orders of the exceptional arrangement monoids are

| $\Phi$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|-------|--------|--------|--------|--------|--------|
| $|M(\Phi, \mathcal{H})|$ | $7^2$ | $11 \cdot 4931$ | $2^4 \cdot 5^2 \cdot 40543$ | $3 \cdot 113 \cdot 24667553$ | $11 \cdot 79 \cdot 55099865069$ |

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