Liouville and Toda dyonic branes: regularity and BPS limit

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Abstract

We reconsider dyonic p-brane solutions derivable from Liouville and Toda integrable systems and investigate their geometric structure. It is shown that the non-BPS non-black dyonic branes are not regular on the horizon.
1 Introduction

Einstein equations with the antisymmetric form and dilaton sources describing singly charged branes with the metric possessing \( SO(p, 1) \times SO(D - p - 1) \) isometries are fully integrable via reduction of the system to uncoupled Liouville equations [1, 2, 3]. For an even dimension of the space-time the similar metric ansatz leads to Liouville or Toda integrable systems also in the case of dyonic branes [1], now for certain discrete values of the dilaton coupling constant. Some subclasses of these solutions are also encountered within the case of the intersecting branes [4, 2, 5, 6, 7, 8, 9, 10, 11]. Integration of the system leads to a generic solution containing the number of integration constants. It was claimed in the literature that extra parameters other than charges and the event horizon radius may be associated with additional physical structures such as tachyon on the brane [5]. Meanwhile, the detailed investigation of the geometric structure of the solution in the case of a single charge revealed that the additional parameters lead to naked singularities [12, 13]; this does not invalidate these solutions, but raises the question of resolution of singularities in more general context. The purpose of the present paper is to investigate the singularity structure of dyonic branes. Note that branes with both electric and magnetic charges may exist in any space-time with electric and magnetic branes having different dimensions (branes within branes of the type of Ref. [14]. In even dimensions and with the antisymmetric form of a suitable rank, both electric and magnetic branes may have the same dimensions [15]. Here we will be interested by dyonic branes of this latter type, which are derivable from Liouville and Toda systems. Due to complete integrability of the equations one is able to obtain the solution with the maximal number of free parameters and then to determine the values of parameters suitable for asymptotically flat solutions without naked singularities. In particular, we reproduce the solutions of the Ref. [1] and reveal the presence of naked singularities in the case of non-BPS non-black branes.

2 General setting

We consider the standard (Einstein frame) action describing classical branes in supergravities which contains the metric, the antisymmetric form \( F_{[q]} \), and the dilaton \( \phi \) with the coupling constant \( a \):

\[
S = \int d^d x \sqrt{-g} \left( R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2 q!} e^{q \phi} F_{[q]}^2 \right),
\]

the corresponding equations of motion being

\[
R_{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{e^{q \phi}}{2(q-1)!} \times \left[ F_{\mu\nu\rho\tau} F^{\rho\tau} - \frac{q-1}{q(d-2)} F_{[q]}^2 g_{\mu\nu} \right] = 0,
\]

\[
\partial_\mu \left( \sqrt{-g} e^{q \phi} F^{\mu\nu\rho\tau} \right) = 0,
\]

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \partial^\mu \phi \right) - \frac{a}{2 q!} e^{q \phi} F_{[q]}^2 = 0.
\]

Consider the space-time consisting of the \((p + 1)\)-dimensional brane world-volume and the transverse \( q = D - p - 1 \)-dimensional space \( \Sigma_{k, \sigma} \times \mathbb{R}_{q-k} \):

\[
ds^2 = -e^{2B} dt^2 + e^{2D} (dx_1^2 + \ldots + dx_p^2) + e^{2A} dr^2 + e^{2C} d\Sigma_{k, \sigma}^2 + e^{2E} (dz_1^2 + \ldots + dz_{q-k}^2),
\]
where five metric functions $A(r)$, $B(r)$, $C(r)$, $D(r)$ and $E(r)$ depend on the single radial variable $r$. Here $\Sigma_{k,\sigma}$ with $\sigma = 0, +1, -1$ is the $k$-dimensional flat, spherical or hyperbolic space with isometries $SO(k-1, 1)$, $ISO(k)$ and $SO(k)$ respectively:

$$d\Sigma_{k,\sigma}^2 = \bar{g}_{ab}dy^a dy^b = \begin{cases} 
  d\varphi^2 + \sinh^2 \varphi d\Omega_{(k-1)}^2, & \sigma = -1, \\
  d\varphi^2 + \varphi^2 d\Omega_{(k-1)}^2, & \sigma = 0, \\
  d\varphi^2 + \sin^2 \varphi d\Omega_{(k-1)}^2, & \sigma = +1,
\end{cases} \tag{2.6}$$

The corresponding Ricci tensor reads

$$\bar{R}_{ab} = \sigma(k-1)\bar{g}_{ab}. \tag{2.7}$$

The reparametrization of the radial variable allows to choose arbitrarily the gauge function $F$:

$$\ln F = -A + B + kC + pD + (q-k)E, \tag{2.8}$$

while the equations of motion can be solved for any $F$. Using this notation, one can present the Ricci tensor for the metric (2.5) as follows:

\begin{align*}
R_{tt} &= e^{2B-2A} \left[ B'' + B'(\ln F)' \right], \\
R_{\alpha\beta} &= -e^{2D-2A} \left[ D'' + D'(\ln F)' \right] \delta_{\alpha\beta}, \\
R_{rr} &= -B'' - B'(B' - A') + k(C'' + C'^2 - A' B') \\
&\quad - (q-k)(E'' + E'^2 - A'E') - p(D'' + D'^2 - A'D'), \\
R_{ab} &= -e^{2C-2A} \left[ C'' + C'(\ln F)' \right] - \sigma(k-1) \bar{g}_{ab}, \\
R_{ij} &= -e^{2E-2A} \left[ E'' + E'(\ln F)' \right] \delta_{ij}, \tag{2.9}
\end{align*}

where prime denotes the derivative with respect to $r$.

Consider now the even-dimensional space-time $d = 2n$ and assume the rank of the form to be $q = n$, so that $p = n - 2$. Then the equations for the form field can be solved dyonically as follows:

$$F_{[n]} = b_1 \text{vol}_n + b_2 e^{-a\phi} \ast \text{vol}_n, \tag{2.10}$$

where

$$\text{vol}_n = \text{vol}(\Sigma_{k,\sigma}) \wedge dz_1 \wedge ... \wedge dz_{n-k}.$$

The form is self-dual with respect to S-duality transformation with magnetic and electric charges $b_1, b_2$ interchanged.

## 3 Liouville case

Introducing a new independent variable $\tau$ via

$$\frac{d\tau}{dr} = \frac{(k-1)}{F}, \tag{3.11}$$

and denoting the derivative with respect to $\tau$ by a dot, we obtain the following system of equations:

$$\dot{B} = \frac{b_1^2 e^{G_1} + b_2^2 e^{G_2}}{4(k-1)^2} , \tag{3.12}$$
\[ \ddot{D} = \frac{b_1^2 e^{G_1} + b_2^2 e^{G_2}}{4(k-1)^2}, \quad (3.13) \]
\[ \ddot{E} = -\frac{b_1^2 e^{G_1} + b_2^2 e^{G_2}}{4(k-1)^2}, \quad (3.14) \]
\[ \ddot{C} = -\frac{b_1^2 e^{G_1} + b_2^2 e^{G_2}}{4(k-1)^2} + \frac{\sigma}{k-1} e^{2(A-C)}, \quad (3.15) \]
\[ \ddot{\phi} = \frac{a}{2(k-1)^2} \left[ b_1^2 e^{G_1} - b_2^2 e^{G_2} \right], \quad (3.16) \]

where

\[ G_{1,2} = \pm a\phi + 2B + 2(n-2)D, \]

together with the constraint

\[ -\dot{A}^2 + \dot{B}^2 + k\dot{C}^2 + (n-2)\dot{D}^2 + (n-k)\dot{E}^2 + \frac{1}{2} \dot{\phi}^2 \leq \frac{b_1^2 e^{G_1} + b_2^2 e^{G_2}}{2(k-1)^2} - \frac{\sigma}{k-1} e^{2(A-C)}. \quad (3.17) \]

Here \( A = A + \mathcal{F} \). For the linear combination of the metric functions \( H = 2(A-C) \) one then finds the separated equation:

\[ \ddot{H} = 2\sigma e^H, \quad (3.18) \]

which has a solution:

\[ H = \begin{cases} \ln \left[ \frac{\sigma^2}{4} \right] - \ln \left[ \sinh^2 \left( \frac{\sigma}{2}(\tau - \bar{\tau}) \right) \right], & \sigma = 1, \\ \beta(\tau - \bar{\tau}), & \sigma = 0, \\ \ln \left[ \frac{\sigma^2}{4} \right] - \ln \left[ \cosh^2 \left( \frac{\sigma}{2}(\tau - \bar{\tau}) \right) \right], & \sigma = -1. \end{cases} \quad (3.19) \]

The functions \( G_1 \) and \( G_2 \) satisfy the equations:

\[ \ddot{G}_1 = \frac{b_1^2 \Delta_1 e^{G_1} + b_2^2 \Delta_2 e^{G_2}}{2(k-1)^2}, \quad (3.20) \]
\[ \ddot{G}_2 = \frac{b_2^2 \Delta_1 e^{G_1} + b_2^2 \Delta_2 e^{G_2}}{2(k-1)^2}, \quad (3.21) \]

with

\[ \Delta_1 = a^2 + (n-1), \quad \Delta_2 = -a^2 + (n-1). \]

For arbitrary values of the coupling parameter, the system for \( G_1 \) and \( G_2 \) can not be separated. However, in two particular cases it reduces to separate equations. Namely, for \( a = 0 \) one obtains the coinciding equations

\[ G_1 = G_2 = G, \quad \Delta_1 = \Delta_2 = \Delta = n - 1, \]
\[ \ddot{G} = \frac{(b_1^2 + b_2^2) \Delta e^{G}}{2(k-1)^2}, \quad (3.22) \]

the solution being:

\[ G = \ln \left[ \frac{\alpha^2(k-1)^2}{\Delta(b_1^2 + b_2^2)} \sinh^2 \left( \frac{\alpha}{2}(\tau - \bar{\tau}) \right) \right]. \quad (3.23) \]
In another particular case $a^2 = n - 1$ one has $\Delta_2 = 0$, $\Delta_1 = 2(n - 1)$ and the equations for $G_1$ and $G_2$ split

\[ \ddot{G}_{1,2} = \frac{b_{1,2}^2 \Delta_1}{2(k-1)^2} e^{G_{1,2}}, \quad (3.24) \]

the solutions being

\[ G_{1,2} = \ln \left[ \frac{\alpha^2 (k-1)^2}{\Delta_1 b_{1,2}^2} / \sinh^2 \left( \frac{\alpha_{1,2}}{2} (\tau - \tau_{1,2}) \right) \right]. \quad (3.25) \]

Changing the notation for the case $a = 0$ as follows $b_{1,2}^2 = \frac{b_1^2 + b_2^2}{2}$, $\tau_{1,2} = \tau_0$, $\alpha_{1,2} = \alpha$, one can present the solution in both cases in a unique way

\[ G_{1,2} = \ln \left[ \frac{\alpha^2 (k-1)^2}{2(n-1)b_{1,2}^2} / \sinh^2 \left( \frac{\alpha_{1,2}}{2} (\tau - \tau_{1,2}) \right) \right]. \quad (3.26) \]

One has to consider also the case of zero $\alpha$ and $\beta$. Taking into account the first integrals of the Liouville equations

\[ \ddot{G}_{1,2} - \frac{b_{1,2}^2(n-1)}{2(k-1)^2} e^G = \alpha_{1,2}^2, \quad \dot{H}^2 - 4\sigma e^H = \beta^2, \quad (3.27) \]

one finds:

\[ \begin{aligned}
G_{1,2} &= \ln \left[ \frac{2(k-1)^2}{b_{1,2}^2(n-1)(\tau - \tau_{1,2})^2} \right], \\
H &= \begin{cases} \\
H_0, & \sigma = 0, \\
\ln \left[ \frac{1}{\sigma(\tau - \tau_0)} \right], & \sigma = -1, 1.
\end{cases} \\ & (3.28)
\end{aligned} \]

The constant $\bar{\tau}$ can be set zero since the system is autonomous in terms of $\tau$.

For both particular cases of $a$, the metric functions and the dilaton will be expressed through $H$, $G_1$, $G_2$, as follows:

\[ \begin{aligned}
B &= \frac{G_1 + G_2}{4(n-1)} - \frac{n-2}{n-1} (d_1 \tau + d_0), \\
D &= \frac{G_1 + G_2}{4(n-1)} + \frac{1}{n-1} (d_1 \tau + d_0), \\
E &= -B + \epsilon_1 \tau + \epsilon_0 = - \frac{G_1 + G_2}{4(n-1)} + \epsilon_1 \tau + \epsilon_0, \\
A &= \frac{kH}{2(k-1)} - \frac{G_1 + G_2}{4(n-1)} + c_1 \tau + c_0, \\
C &= \frac{H}{2(k-1)} - \frac{G_1 + G_2}{4(n-1)} + c_1 \tau + c_0, \\
\phi &= \frac{G_1 - G_2}{2a} = \frac{1}{a} \ln \left[ \frac{b_2 \sinh \left( \frac{\alpha_{1,2}}{2} (\tau - \tau_0) \right)}{b_1 \sinh \left( \frac{\alpha_{1,2}}{2} (\tau - \tau_1) \right)} \right], \\
\end{aligned} \quad (3.29) \]

with free parameters $d_{0,1}$, $\epsilon_{0,1}$ and

\[ \begin{aligned}
\epsilon_{0,1} &= \frac{n-2}{n-1} d_{0,1} + \epsilon_{0,1}, \\
c_{0,1} &= \frac{n-k}{k-1} \epsilon_{0,1}.
\end{aligned} \quad (3.30) \]
From the constraint equation one finds the following relation between the parameters:

\[-\frac{k}{4(k-1)}\beta^2 + \frac{1}{4(k-1)}(\alpha_1^2 + \alpha_2^2) + (n-k)\epsilon_1^2 + (k-1)\epsilon_1^2 + \frac{n-2}{n-1}\epsilon_1^2 = 0. \tag{3.36}\]

To reveal the location of singularities it is convenient first to analyze the scalar curvature. Using the equations of motion one can present it as follows

\[R = \frac{(k-1)^2}{2}e^{-2A}\phi^2. \tag{3.37}\]

Our solution (3.29) this reduces to

\[R = \frac{1}{2}\frac{(k-1)^2}{(\sqrt{\pi}a)^{k+1}} \left[ \sum_{i} l_i (l_i - 1) Y_i^4 - \sigma \frac{k}{k-1} C^2 e^{2A-2C} + \frac{1}{2} \sum_{i \neq j} l_i l_j Y_i^2 Y_j^2 \right]. \tag{3.39}\]

We will also need the Kretchmann scalar

\[K = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = 4(k-1)^4 e^{-4A} \left[ \sum_i l_i (\bar{Y}_i - \bar{Y}_i^2 + \bar{A} Y_i^2)^2 + \frac{1}{2} \sum_{i \neq j} l_i l_j \bar{Y}_i \bar{Y}_j \right] + \frac{1}{2} \sum_i l_i (l_i - 1) Y_i^4 - \frac{1}{2} \sum_{i \neq j} l_i l_j Y_i^2 Y_j^2. \tag{3.40}\]

where for brevity the following multicomponent notation is introduced \(Y_i = \{B, D, C, E\}\), \(l_i = \{1, n-2, k, n-k\}\).

Our solution has the following special points: \(\tau = \tau_{1,2}, \pm \infty, 0\). At the points \(\tau = \tau_{1,2}\), which locate at finite geodesic distance (we will consider the null geodesics) from any non-special point, the Ricci scalar diverges. These points mark the curvature singularities.

The point \(\tau = 0\) locates at an infinite null geodesic distance, and both the Ricci and Kretchmann scalars are zero there. This point will be identified with the asymptotic infinity.

Finally, at the limiting points \(\tau = \pm \infty\) one can have (with some parameter choice) the vanishing of the metric component \(g_{tt} = 0\), which is the necessary condition for existence of the event horizon.

### 3.1 Asymptotically flat solutions with a regular horizon

Consider the case of the spherical symmetry \(\sigma = 1\) (for other values of \(\sigma\) no asymptotically flat solutions exist). Based on the above analysis of special points, we can interpret \(\tau = 0\) as spatial infinity, and \(\tau = -\infty\) as the event horizon. To clarify the geometry near the horizon consider the radial null geodesics curves with the affine parameter \(\lambda\). From the geodesic equation one finds:

\[d\lambda = e^{A+B}d\tau = e^{A+B} \frac{d\tau}{k-1}. \tag{3.40}\]

The regular horizon should be geodetically traversable, so from an analyticity argument we have to require

\[e^{2B} \sim \lambda^n. \tag{3.41}\]
with integer \( n \), namely \( n = 1 \) for a non-degenerate and \( n = 2 \) for a degenerate regular horizons. Near \( \tau = -\infty \) our solution gives

\[
e^{2B} \sim \exp \left[ \frac{1}{(n-1)} \left( \frac{\alpha}{2} - (n-2)d_1 \right) \tau \right],
\]

so \( g_{tt}|_{\tau = -\infty} = 0 \) provided

\[
\alpha > 2(n-2)d_1.
\]

The geodesic equation for a non-degenerate horizon \( n = 1 \) leads to

\[
\frac{d}{d\lambda} e^{2B} = e^{-(A+B)} \frac{d}{d\tau} e^{2B} \to \text{const},
\]

and therefore

\[
e^A \sim e^B.
\]

Then the first two terms in the left hand side of the constraint equation (3.17) cancel, while the remaining expression becomes positive definite. Since the right hand side vanishes, this means vanishing of all the derivatives at the left hand side separately. We obtain the system of equations which has the unique solution

\[
\frac{1}{2}(\alpha_1 + \alpha_2) = \beta = 2e_1 = -2d_1, \ \alpha_1 = \alpha_2 = \alpha.
\]

With this, the constraint (3.36) will be satisfied for all \( \tau \). Imposing now the asymptotic flatness conditions

\[
B \approx 0, \ D \approx 0, \ E \approx 0,
\]

we obtain three more conditions on the parameters

\[
d_0 = 0, \ e_0 = 0, \ \left( \sinh \frac{\alpha}{2} \tau_1 \cdot \sinh \frac{\alpha}{2} \tau_2 \right)^2 = \left( \frac{\alpha^2(k-1)^2}{2b_1b_2(n-1)} \right)^2.
\]

The dilaton at infinity will have the finite value

\[
\phi_\infty = \phi_0, \ \left| \frac{b_2 \sinh \frac{\alpha}{2} \tau_2}{b_1 \sinh \frac{\alpha}{2} \tau_1} \right| = e^{a\phi_0}.
\]

From these equations we obtain the parameters \( \tau_1, \tau_2 \). We also need the asymptotic condition for the conformal factor \( e^{2C} \sim r^2 \) in terms of the “curvature” radial variable. This can be achieved choosing the gauge function \( F \) as follows:

\[
F = r^k f_+ f_-, \quad f_\pm = 1 - \frac{\xi_\pm}{\xi},
\]

\[
\xi = r^{k-1}, \quad \xi_\pm = r^{k-1}_\pm, \quad \xi_- < \xi_+.
\]

Here two new parameters \( r_\pm \) are introduced marking the event horizon and the internal horizon, such that \( r_+ > r_- \), and

\[
\alpha = x_+ - x_-.
\]

The corresponding coordinate transformation reads:

\[
\tau = \frac{1}{\alpha} \ln \frac{f_+}{f_-}.
\]
In terms of this new radial coordinate the solution will read

\[ H = \ln \left[ \xi^2 f_+ f_- \right], \]
\[ G_{1,2} = \ln \frac{f_+ f_-}{f_{1,2}^2} + \ln \left[ \frac{2(k-1)^2}{b^2_{1,2}(n-1)} \xi_{1,2}^2 f_{1,2}^{1,2} \right], \quad (3.53) \]
\[ f_{1,2} = 1 - \frac{\xi_{1,2}}{\xi}, \quad f_{1,2}^{1,2} = 1 - \frac{\xi_{\pm}}{\xi_{1,2}}, \quad \xi_{1,2} = r_{1,2}^{k-1}, \]

where \( r_1(r_2) \) are the images of \( \tau_1(\tau_2) \). The interval and the dilaton exponent then will read:

\[ ds^2 = \left[ \frac{f_2^2}{f_1 f_2} \right]^\frac{1}{n-1} \left( -\frac{f_+ f_-}{f_-} dt^2 + d\mathbf{x}^2 \right) \]
\[ + \left[ \frac{2^{n+k}}{f_-^{2^k+k-1}} f_1 f_2 \right]^\frac{1}{n-1} \left( \frac{dt^2}{f_+ f_-} + r^2 d\Sigma_{k,1}^2 \right) + \left[ \frac{f_1 f_2}{f_2^2} \right]^\frac{1}{n-1} dz^2, \quad (3.54) \]
\[ e^{2a\phi} = \frac{f_2^2}{f_1^2} e^{2a\phi_0}, \quad (3.55) \]
\[ e^{2a\phi_0} = \frac{b_2^2}{b_1^2} (\xi - \xi_+) (\xi_1 - \xi_+) (\xi_2 - \xi_+). \]

For \( a = 0 \) the metric coincides with that of a singly charged brane with the parameters \( p = n - 2, \quad q = n, \quad \Delta = n - 1 \).

The scalar curvature in the new coordinates reads

\[ R = \left( \frac{k-1}{\sqrt{2} n^k} \right)^2 \left[ \frac{\xi_2}{f_2} - \frac{\xi_1}{f_1} \right]^2 \left[ \frac{2^{n+k}}{f_-^{2^k+k-1}} f_1 f_2 \right]^{-\frac{1}{n-1}} f_+ f_- \quad (3.56) \]

One can see that the internal horizon is singular

\[ K \sim \alpha^4 (n-1)(n-2)^2(n-3) e^{2a\tau}, \quad (3.57) \]

except for the case of the purely spherical transverse space \( n = k \) for \( k = 2 \) or 3:

For non-zero dilaton coupling, \( a^2 = n - 1 \), the scalar curvature in the vicinity of \( \tau_{1,2} \) behaves as

\[ R \sim [f_1 f_2]^{-\frac{2a-1}{n-1}}, \quad (3.58) \]

while the Kretschmann scalar for both solutions \( a = 0 \) and \( a^2 = n - 1 \) diverges as

\[ R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \sim [f_1 f_2]^{-2\frac{2a-1}{n-1}}. \quad (3.59) \]

Note that in the case \( a = 0 \) when \( \tau_1 = \tau_2 = \tau_0 \), the divergence of the Kretschmann scalar at \( \tau_0 \) is twice as strong as in the case \( a^2 = n - 1 \). In what follows we choose in both cases the singularities to locate inside the event horizon.

The case of the degenerate horizon corresponds to the following behavior of the metric function and its derivative:

\[ e^{2B} \sim \lambda^2, \quad e^{B-A} \dot{B} \sim \lambda, \quad (3.60) \]

so we have:

\[ e^{-A} \dot{B} \sim O(1). \quad (3.61) \]
Therefore either both factors are finite (non-zero) at the horizon, or both $\dot{B}$ and $e^A$ must vanish there. Assuming the first option for $e^A$, we have to admit $\dot{A} = 0$ at the horizon, and similarly for $\dot{B}$ (from the constraint), thereby coming to contradiction. Then consider the second possibility. If $\dot{B} = 0$ and $e^B = 0$, we find $\alpha = 2(n-2)d_1 = 0$. From the condition $e^A = 0$ and the constraint (3.17) we obtain $e_1 = 0$, and consequently $\beta = 0$. As a result, the degenerate horizon will correspond to the following choice of parameters

$$\alpha = \beta = d_1 = e_1 = 0,$$

(3.62)

which leads (as expected) to the limit

$$r_- \rightarrow r_+$$

in the solution with the non-degenerate horizon. In the degenerate case the metric functions $A$, $B$ according to (3.28) will behave as:

$$A \sim \mu \ln |\tau|, \quad B \sim \nu \ln |\tau|,$$

(3.63)

where

$$\mu = \frac{1}{n-1} - \frac{k}{k-1}, \quad \nu = -\frac{1}{n-1}.$$

From the geodesic equation (3.40) and the condition (3.41) we then find:

$$d\lambda \sim |\tau|^{\mu+\nu} d\tau, \quad e^{2B} \sim |\tau|^{2\nu} \sim \lambda^{\frac{2\nu}{\mu+\nu+1}},$$

(3.64)

therefore, in the degenerate case one more condition has to be satisfied

$$\mu + 1 = 0 \rightarrow n = k,$$

(3.65)

thus the degenerate regular horizon is possible only in the case of purely spherical transverse space. It is worth noting, that contrary to the singly charged brane, now the vanishing of the dilaton coupling constant is not required [13].

In the final solution one can shift the origin to the point $r = r_2$ introducing a new coordinate

$$y^{k-1} = r^{k-1} - r_2^{k-1}.$$

(3.66)

This reduces the number of free parameters by one. Thus we obtain the four-parametric dyon solution characterized by two charges $b_1, b_2$, the asymptotic value of the dilaton and the radius of the event horizon $y_+$. 

4 Toda solution

Consider an open Toda chain described by the lagrangian

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} q_i^2 + \sum_{i=1}^{n-1} g_i e^{2(q_i - q_{i+1})}.$$  

(4.67)

Note, that the number of potentials is less by one than the number of independent variables. One can show that, for a particular dilaton coupling constant, the solution to our system can be derived from that of an open three-dimensional Toda chain. To this end we first introduce the three-dimensional vector

$$X = \{ \sqrt{2}B, \sqrt{2(n-2)}D, \phi/\sqrt{2} \}$$

(4.68)
replacing the three initial variables (note that the function $E$ will differ from $B$ on the linear function, while the equation for $H$ was already separated):

$$\mathcal{L} = \dot{B}^2 + (n - 2)\dot{D}^2 + \frac{\phi^2}{4} + \frac{b_1^2}{4(k - 1)^2}e^{G_1} + \frac{b_2^2}{4(k - 1)^2}e^{G_2}. \tag{4.69}$$

We introduce also two vectorial parameters

$$\lambda_1 = \{\sqrt{2}, \sqrt{2}(n - 2), \sqrt{2}a\},$$
$$\lambda_2 = \{\sqrt{2}, \sqrt{2}(n - 2), -\sqrt{2}a\} \tag{4.70}$$

and rewrite the lagrangian as

$$\mathcal{L} = \frac{1}{2} \dot{X} \cdot \dot{X} + \sum_{i=1}^{2} a_i e^{<\lambda_i, X>}, \tag{4.71}$$

where $<.,.>$ is the euclidean scalar product with the metric $\delta_{ij}$, and

$$a_1 = \frac{b_1^2}{4(k - 1)^2}, \quad a_2 = \frac{b_2^2}{4(k - 1)^2}. \tag{4.72}$$

The vectors $\lambda_1, \lambda_2$ are not orthogonal:

$$<\lambda_1, \lambda_1> = <\lambda_2, \lambda_2> = 2(a^2 + n - 1),$$
$$<\lambda_1, \lambda_2> = 2(-a^2 + n - 1), \tag{4.73}$$

so it is useful to introduce the orthogonal basis $\{e'_1, e'_3\}$ adding the third vector orthogonal to the first two as follows

$$\lambda_1 = \left[2(a^2 + n - 1)\right]^{\frac{1}{2}}e'_2,$$
$$\lambda_2 = \frac{2(-a^2 + n - 1)}{\sqrt{2(a^2 + n - 1)}}e'_2 + \frac{8a^2(n - 1)}{a^2 + n - 1}e'_3,$$
$$e'_1 = [e'_2, e'_3], \tag{4.74}$$

where $[.,.]$ means the vector product. Then we get:

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{3} \delta_{ij} \dot{X}^i \dot{X}^j + a_1 \exp \left([2(a^2 + n - 1)]^{\frac{1}{2}}\dot{X}^2\right)$$
$$+ a_2 \exp \left(\frac{2(-a^2 + n - 1)}{\sqrt{2(a^2 + n - 1)}}\dot{X}^2 + \frac{8a^2(n - 1)}{a^2 + n - 1}\dot{X}^3\right),$$

where $X^i$ – are components of $X$ in the new basis $X = X^i e'_i$. This system encompass also the two Liouville cases $a^2 = n - 1$ and $a = 0$, when the equations separate. Since the coordinate $X^1$ is cyclic, one obtains for it the linear dependence $X^1 = x_1 \tau + x_0$, while the non-trivial part of the lagrangian reads:

$$\mathcal{L} = \frac{1}{2} \left(\dot{X}^2\right)^2 + \frac{1}{2} \left(\dot{X}^3\right)^2 + \sum_{i=1}^{2} a_i e^{<\lambda_i, X>}. \tag{4.75}$$

The potential term has the form suitable for Toda open chain representation, but to get an appropriate kinetic term one has to introduce an additional variable $X^4$, with the kinetic
term \( \frac{1}{2}X'^2 \), satisfying free equations of motion. It turns out that the canonical form of the kinetic term \( L_{kinetic} \) can be achieved only for a particular coupling constant \( a^2 = 3(n - 1) \), the corresponding transformation to a new set of variables \( \{q_1, q_2, q_3\} \) being in this case:

\[
\begin{align*}
2(a^2 + n - 1) \frac{1}{2} X'^2 &= 2(q_1 - q_2), \\
\frac{2(-a^2 + n - 1)}{\sqrt{2(a^2 + n - 1)}} X'^2 &= \sqrt{\frac{8a^2(n - 1)}{a^2 + n - 1}} X'^3 = 2(q_2 - q_3), \\
X'^4 &= \frac{1}{a}(q_1 + q_2 + q_3).
\end{align*}
\]

(4.76)

The equation for \( X'^4 \) then corresponds to the free motion of the "center of mass" of the Toda chain \( \ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3 = 0 \), and finally the lagrangian will take the form:

\[
\mathcal{L} = \frac{3}{a^2} \left[ \frac{1}{2} \sum_{i=1}^{3} \dot{q}_i^2 + \sum_{i=1}^{2} g_i e^{2(q_i - q_{i+1})} \right],
\]

(4.77)

where

\[
g_1 = \frac{(ab_1)^2}{12(k - 1)^2}, \quad g_2 = \frac{(ab_2)^2}{12(k - 1)^2}.
\]

Following the Ref. [17, 18, 19, 20] we can present the solution of the Toda system as follows:

\[
g_1 e^{2(q_1 - q_2)} = \frac{F_+}{F_-^2}, \quad g_2 e^{2(q_2 - q_3)} = \frac{F_{-sd}}{F_+^2},
\]

\[
F_\pm = \frac{4}{9A_1A_2(A_1 + A_2)} \left[ A_1 e^{\pm(A_1 + 2A_2)\tau \mp B_1} - (A_1 + A_2) e^{\pm(A_1 - A_2)\tau \mp (B_1 - B_2)} + A_2 e^{\mp(2A_1 + A_2)\tau \mp B_2} \right].
\]

(4.78)

Here the \( A_{1,2}, B_{1,2} \) are the constant parameters, \( B_1 \) and \( B_2 \) being fully arbitrary, while \( A_1 \) and \( A_2 \) satisfy the sign restriction: \( A_1A_2 > 0 \). For \( a^2 = 3(n - 1) \) one has:

\[
\begin{align*}
X'^2 &= \frac{q_1 - q_2}{\sqrt{2(n - 1)}} = \frac{\sqrt{3}}{a\sqrt{8}} \ln \left[ \frac{1}{g_1} \frac{F_+}{F_-^2} \right], \\
X'^3 &= \frac{(q_1 - q_2) + 2(q_2 - q_3)}{\sqrt{2}a} = \frac{1}{a\sqrt{8}} \ln \left[ \frac{1}{g_1g_2^2} \frac{1}{F_+^3} \right],
\end{align*}
\]

(4.79)

and the initial variables \( B, D, \phi \) are related to \( X'^1, X'^2, X'^3 \) via the relations

\[
\begin{align*}
B &= \frac{1}{\sqrt{8(n - 1)}} \left[ X'^2 + \sqrt{3}X'^3 - \sqrt{4(n - 2)}X'^1 \right], \\
D &= \frac{1}{\sqrt{8(n - 1)}} \left[ X'^2 + \sqrt{3}X'^3 + \sqrt{4(n - 2)}X'^1 \right], \\
\phi &= \sqrt{\frac{3}{2}}X'^2 - \frac{1}{\sqrt{2}}X'^3.
\end{align*}
\]

(4.80)

Substituting the above solution for \( X' \) we obtain:

\[
B = -\frac{\ln [g_1g_2F_+F_-]}{4(n - 1)} - \frac{n - 2}{n - 1}(d_1\tau + d_0),
\]

11
\[ D = -\frac{\ln [g_1g_2F_+F_-]}{4(n-1)} + \frac{1}{n-1}(d_1\tau + d_0), \]
\[ E = \frac{\ln [g_1g_2F_+F_-]}{4(n-1)} + \epsilon_1\tau + \epsilon_0, \]
\[ A = \frac{kH}{2(k-1)} + \frac{\ln [g_1g_2F_+F_-]}{4(n-1)} + c_1\tau + c_0, \]
\[ C = \frac{H}{2(k-1)} + \frac{\ln [g_1g_2F_+F_-]}{4(n-1)} + c_1\tau + c_0, \]
\[ \phi = \frac{1}{2\sqrt{3(n-1)}} \ln \left[ \frac{g_2}{g_1} \left( \frac{F_+}{F_-} \right)^3 \right], \]

where
\[ d_i = \sqrt{\frac{n-1}{2(n-2)}} x_i, \quad \epsilon_i = \frac{n-2}{n-1} d_i + e_i, \quad c_i = -\frac{n-k}{k-1} \epsilon_i. \]

Now the asymptotically flat regular solution for \( a^2 = 3(n-1) \) can be performed similarly to the previous Liouville cases. Introducing a new parameter
\[ \alpha = -\frac{1}{2} \left. \left( \frac{\dot{F}_+}{F_+} + \frac{\dot{F}_-}{F_-} \right) \right|_{\tau=-\infty}, \]
from the constraint \( (3.17) \) we will get the conditions on the event horizon:
\[ \dot{C} = \dot{D} = \dot{E} = \dot{\phi} = 0, \]
the first three giving already known values of parameters:
\[ |\alpha| = \beta = 2e_1 = -2d_1. \]
Vanishing of the dilaton derivative at the horizon means:
\[ \left. \frac{\dot{F}_+}{F_+} \right|_{\tau=-\infty} = \left. \frac{\dot{F}_-}{F_-} \right|_{\tau=-\infty} = -\alpha, \]
implicating \( A_1 = A_2 = \frac{\alpha}{3} \), that leads to simplification:
\[ F_\pm = \frac{2}{\alpha^2} \left[ e^{\pm(\alpha\tau+B_1)} - 2e^{\mp(B_1-B_2)} + e^{\mp(\alpha\tau+B_2)} \right]. \]
An asymptotic flatness condition implies
\[ x_0 = 0, \quad e_0 = 0, \]
\[ F_+F_-|_{\tau=0} = \frac{1}{g_1g_2}, \quad \left. \frac{g_2}{g_1} \left( \frac{F_+}{F_-} \right)^3 \right|_{\tau=0} = e^{2a\phi_\infty}. \]
From here one finds the parameters \( B_1, B_2, \)

The case of the degenerate horizon can be treated along the same lines as before, this leads to following conditions on the parameters:
\[ \alpha = \beta = d_1 = e_1 = 0. \]
In this case, as can be easily seen from the Eq. (4.86), an assumption of finite $|B_{1,2}|$ immediately gives

$$B_1 = B_2 = 0.$$  \hspace{1cm} (4.88)

Then the solution degenerates

$$F_+ = F_-, \hspace{1cm} (4.89)$$

which leads to the constant dilaton and the condition $a = 0$ in contradiction to the assumption $a^2 = 3(n - 1)$. Therefore, for the Toda solution the asymptotically flat extremal configuration does not exist, the Toda dyon exists only in the black version (in this we disagree with the Ref. [1]).

The curvature scalar (3.37) for the solution obtained reads

$$R = \frac{3(k - 1)^2}{8(n - 1)} \left( \frac{\dot{F}_+}{F_+} - \frac{\dot{F}_-}{F_-} \right)^2 \left( \frac{2}{\beta} \sinh(\beta \tau/2) \right)^{2k-1} (g_{12} g_2 F_+ F_-)^{-\frac{1}{2(n-1)}} e^{-2c_1 \tau - 2c_0}. \hspace{1cm} (4.90)$$

Similarly to the Liouville case we have two singular points defined by the equations

$$F_+|_{\tau = \tau_1} = 0, \hspace{0.5cm} F_+|_{\tau = \tau_2} = 0. \hspace{1cm} (4.91)$$

To satisfy the cosmic censorship conjecture we have to locate them inside the external horizon, thus $\tau_1, \tau_2$ can not lie in the region $(-\infty, 0]$. So we have a four-parametric solution, characterized by the values of charges $b_1, b_2$, the value of the dilaton at infinity $\phi_0$ and the parameter $\alpha$. Contrary to the previous cases of the coupling constant $a = 0$, $a^2 = n - 1$, this solution does not exist in the extremal form.

5 Discussion

In this paper, we reconsidered generic solutions for partially localized black asymptotically flat dyonic branes for three particular values of the coupling constant $a = 0$, $a^2 = n - 1$, $a^2 = 3(n - 1)$. For the first two the system separates in terms of the Liouville equations, while in the third on can construct an open three-dimensional Toda chain which generates the desired solution. In all cases the solutions are four-parametric and possess two curvature singularities (contrary to the singly charged branes, for which the solutions have one singularity). All solutions satisfy the cosmic censorship conjecture, i.e., they do not contain naked singularities. Liouville solutions exist both in black and extremal versions, the extremality being understood as the degeneracy of the event horizon. In the extremal case the brane world-volume possesses the $ISO(p,1)$ isometry and the solution has to be localized (with the spherically symmetric transverse space only). For the Toda solution we did not find a regular extremal limit at all.

Meanwhile, somewhat different conclusions were made in the Ref. [1], and we would like to clarify the situation here. In this paper, only the isotropic localized dyonic branes were considered ($B = D, E = 0$ in our notation), and the solutions obtained were not necessarily extremal (BPS). The solution of the Ref. [1] can be found imposing the following conditions on the parameters of our most general solution (denoting the parameters of [1] by tilde):

$$n = k = \tilde{n}, \hspace{0.5cm} \tau = \tilde{\xi},$$
$$b_{1,2} = \tilde{\lambda}_{1,2}, \hspace{0.5cm} d_{0,1} = e_{0,1} = 0, \hspace{0.5cm} \beta = 4\tilde{k}, \hspace{1cm} (5.92)$$

for $a = 0$, $a^2 = n - 1$

$$\alpha = \tilde{k} \sqrt{8\tilde{n}}, \hspace{0.5cm} \tau_{1,2} = 2\tilde{\alpha}_{1,2} \hspace{1cm} (5.93)$$
for $a^2 = 3(n - 1)$

$$\tilde{n} = 2, \quad \alpha = 4\tilde{k}, \quad B_1 = \ln c_1 c_2, \quad B_2 = \ln c_1. \quad (5.94)$$

It was checked in [1] that this solution satisfies the constraint (3.36) and the curvature scalar is finite on the event horizon. However one can see that the Kretchmann scalar (3.39) diverges there:

$$K \sim \frac{\alpha^4}{8} (n - 1)(n - 2)(2n - 3) e^{-2\alpha\tau}. \quad (5.95)$$

Thus, the non-BPS dyonic branes with the isometry ISO$(p, 1)$ found in the Ref. [1] are not regular on the horizon. In our analysis the regularity of the horizon was ensured from the beginning by imposing conditions of geodesic prolongation through the horizon (see the section 3.1). In the non-extremal case (the non-degenerate horizon) this gives an additional condition on the horizon for the function $\tilde{A}$ of the Ref. [1], namely, $\tilde{A} = 0$ ($\tilde{D} = 0$ in our notation). From this condition one finds

$$\tilde{k} = 0. \quad (5.96)$$

Then in the Liouville case one is led to the regular BPS brane.

For the Toda solution, as we have shown, the analysis of geodesics excludes the possibility of the regular extremal limit. In terms of the Ref. [1], the non-extremal solution with the ISO$(p, 1)$ symmetry must satisfy $\tilde{k} = 0$, which condition leads to $a = 0$ in contradiction with the initially assumed value of $a$.

Thus we conclude that in the case of the symmetry ISO$(p, 1)$ the only regular solutions are the standard BPS ones, and these exist only in the Liouville cases ($a = 0$, $a^2 = n - 1$). In the Toda case ($a^2 = 3(n - 1)$) regular solutions are necessarily black. Of course, relaxing the condition of regularity (admitting naked singularities) one finds larger classes of solutions.

The black dyon solution of the Ref. [21] can be obtained from our solution (3.54) imposing the following conditions on the parameters

$$n = k, \quad \xi_+ = \tilde{k}, \quad \xi_- = 0, \quad \xi_{1,2} = -\tilde{k} \sinh^2 \tilde{\mu}_{1,2}. \quad (5.97)$$

Note that our black solution is more general in the sense of the possibility of the non-spherical transverse space (partial localization). Note also that in the extremal case the solution of the Ref. [21] degenerates: when $\tilde{k} \to 0$, we find $\xi_- = \xi_+ = \xi_1 = \xi_2 = 0$. It is not necessarily so for our general solution.

Here we restricted attention by the asymptotically flat dyonic branes. Relaxing the asymptotic conditions, but still demanding the absence of naked singularities, we are led to another possibility – non-asymptotically flat dyonic branes with the linear dilaton background at spatial infinity [22].

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