Amount of quantum coherence needed for measurement incompatibility

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A pair of quantum observables diagonal in the same “incoherent” basis can be measured jointly, so some coherence is obviously required for measurement incompatibility. Here we first observe that coherence in a single observable is linked to the diagonal elements of any observable jointly measurable with it, leading to a general criterion for the coherence needed for incompatibility. Specialising to the case where the second observable is incoherent (diagonal), we develop a concrete method for solving incompatibility problems, tractable even in large systems by analytical bounds, without resorting to numerical optimisation. We verify the consistency of our method by a quick proof of the known noise bound for mutually unbiased bases, and apply it to study emergent classicality in the spin-boson model of an N-qubit open quantum system. Finally, we formulate our theory in an operational resource-theoretic setting involving “genuinely incoherent operations” used previously in the literature, and show that if the coherence is insufficient to sustain incompatibility, the associated joint measurements have sequential implementations via incoherent instruments.

I. INTRODUCTION

Coherence typically refers to nonzero off-diagonal elements in a quantum state, and is an essential resource for quantum information tasks [1–5]. Coherence in measurements (observables) [6–8] is equally fundamental, with an obvious relation to the non-commutativity of projective measurements, which has recently been refined [9–10]. It is therefore natural to ask how coherence is related to incompatibility of general observables – positive operator valued measures (POVMs). Incompatibility is a resource as well [9–11], specifically for steering [12–17] and state discrimination [18–20], and clearly requires non-commutativity, hence coherence.

As usual [2], we define coherence relative to a fixed “incoherent” basis (there is also a basis-independent approach [21]). Our key observation is the following: while incompatibility of POVMs is not linked to the overall coherence in their matrices, there is an asymmetric entry-wise relation: coherences in one POVM are linked to the corresponding diagonal probabilities of any POVM jointly measurable with it. Heuristically, an observable that sharply distinguishes a pair of basis elements is incompatible with observables detecting coherence between that pair. An extreme case is any basis observable mutually unbiased [22] to the incoherent basis – it is both complementary and maximally coherent.

We warm up in section II by formalising the above observation into a simple but completely general inequality, the violation of which witnesses incompatibility. Combined with a sufficient condition for incompatibility, this leads to an analytical method for tackling the incompatibility problem, generalising the usual robustness idea [23–34], and easily reproducing the known noise bound for incompatible mutually unbiased bases (MUB) [19,23–26]. In section III we specialise to the physically motivated setting where measurement coherence is given by a fixed “pattern matrix” describing decoherence [27,29], and subsequently use it to study emergent classicality in the spin-boson model [29,34–36], including the role of decoherence-free subspaces [35–38]. Unlike existing results on incompatibility in open systems [39], our method works for arbitrary system size. Finally, in Section IV we formulate the idea in general operational terms motivated by resource theory, including genuinely incoherent operations [40,41] and introducing incoherent instruments, which turn out to provide sequential implementations for any joint measurement in an instance of channel-observable compatibility [40–42].

II. GENERAL FORMULATION

Let H be a Hilbert space of dim H = d < ∞, and { |n⟩}ₙ=1ᵈ its incoherent basis [2]. An observable (POVM) M with a finite outcome set Ω consists of positive semi-definite (PSD) matrices M(i) ≥ 0, for which ∑ᵢ∈Ω M(i) = 1 (the identity matrix). For any POVM M we define the entry-wise coherence

cohₙₘ(M) := ∑ᵢ∈Ω |⟨|n⟩|M(i)|m⟩|, for each n, m.

We note that 0 ≤ cohₙₘ(M) ≤ 1, and call M maximally coherent if cohₙₘ(M) = 1 for all n ≠ m. We now observe (Appendix A) that any maximally coherent M with d outcomes is mutually unbiased to the incoherent basis { |n⟩ }, i.e. M(i) = |ψᵢ⟩⟨ψᵢ| with |⟨ψᵢ|n⟩|² = d⁻¹ for all n, i. This reflects the importance of MUBs in the context of measurement coherence.

For any M we let pₙⁱ(j) := ⟨|n⟩|M(j)|n⟩ be the outcome distribution in state |n⟩. The ability of M to distinguish |n⟩ from |m⟩ can be quantified by f-divergences [43] between pⁿⁱ and pᵐⁱ; we use the Hellinger distances [44]

d²ₙₘ(M) := 1 − √(pⁿⁱ(j)pᵐⁱ(j)).

Finally, an observable M is jointly measurable with an observable F, if there is a joint observable G = (Gᵢ,j)ᵢ,j, where


with \( \sum_i G(i,j) = M(i) \) for all \( i \), and \( \sum_j G(i,j) = F(j) \) for all \( j \); otherwise \( M \) and \( F \) are incompatible \[45\].

A. Joint measurability criteria

The following observation provides a simple tradeoff between distinguishability and coherence, under the assumption of joint measurability:

**Proposition 1.** If \( M \) and \( F \) are jointly measurable, then
\[
\text{coh}_{nm}(M) + d_{nm}^2(F) \leq 1 \quad \text{for all } n,m. \tag{1}
\]

**Proof.** We have \( \text{coh}_{nm}(M) \leq \sum_i |\langle n|G(i,j)|m\rangle| \) for any joint POVM \( G \) of \( M \) and \( F \). But \( G(i,j) \) is PSD, and hence \( |\langle n|G(i,j)|m\rangle| \leq \sqrt{p^*_n(i,j)p_n^m(i,j)} \) so \( \sum_i |\langle n|G(i,j)|m\rangle| \leq \sqrt{\sum_i p^*_n(i,j)\sum_j p_n^m(i,j)} = \sqrt{p_n^m(j)} \) by the Schwarz inequality.

Hence, \( M \) and \( F \) are incompatible if \( 1 \) is violated for at least one pair \( n,m \) – the result is an upper bound for the coherence needed for incompatibility. The interpretation is that coherence between \( |n\rangle \) and \( |m\rangle \) cannot be precisely detected by a measurement capable of distinguishing these states. In particular, if \( \text{coh}_{nm}(M) = 1 \) then \( M \) is incompatible with any \( F \) having \( p_n^F \neq p_n^m \).

Necessary conditions for incompatibility require finding joint observables, equivalent to hidden variable models for quantum steering \[12\,17\]. This is hard to tackle analytically, and often restricted to single qubits or highly symmetric cases. Surprisingly, we now obtain a very general result using the Schur product theorem \[17\], which states that the entry-wise (Hadamard / Schur) product \( A * B \) of PSD matrices \( A \) and \( B \) is also PSD. We call an observable \( P \) incoherent if \( P(i) = \sum_{n=1}^d p_n^i(i)|n\rangle\langle n| \) for all \( i \in \Omega_p \), and define a matrix \( \mathcal{S}(P) \) by \( \mathcal{S}_{nm}(P) = (1 - d_{nm}^2(P))^{-1} \) if \( d_{nm}^2(P) < 1 \) for all \( n,m \).

**Proposition 2.** If \( P \) is incoherent and \( \mathcal{S}(P) * M(i) \geq 0 \) for all \( i \in \Omega_m \), then \( M \) and \( P \) are jointly measurable.

**Proof.** Define \( C_P(j) \geq 0 \) by \( c_{nm}^P(j) = \sqrt{p_n^i(j)p_m^j(i)} \). Then \( G(i,j) := \mathcal{S}(P) * M(i) * C_P(j) \geq 0 \) by the assumption and the Schur product theorem. But \( \sum_j G(i,j) = M(i) \), and \( \mathcal{S}(P) * I = \mathcal{S}(P) * C_P(j) = P(j) \) as \( P \) is incoherent. Hence \( G \) is a joint observable for \( M \) and \( P \).

To appreciate how this result describes the coherence needed for incompatibility, note that the diagonal elements of \( P \) enter into the matrix \( \mathcal{S}(P) \), while the positivity condition describes the (lack of) coherence in \( M \). More specifically, when the coherences \( \text{coh}_{nm}(M) \) are small enough relative to \( 1 - d_{nm}^2(P) \), then the off-diagonal elements \( |\langle n|M(i)|m\rangle| (1 - d_{nm}^2(P))^{-1} \) of the matrix \( \mathcal{S}(P) * M(i) \) are small relative to the unit diagonal, and hence (e.g. by the Sylvester determinant criterion), the positivity condition \( \mathcal{S}(P) * M(i) \) of Prop. \[2\] will hold. This ensures the existence of a joint observable, showing that the (collective) coherence in \( M \) is not enough for incompatibility. In examples with suitable parametrisation, this then translates into a lower bound for the coherence needed for incompatibility.

B. Basic examples

We now link the above results to the noise bounds for incompatibility \[5\,23\,32\,33\]: consider
\[
P_\alpha(j) = \alpha|j\rangle\langle j| + (1 - \alpha)d^{-1}I, \quad 0 \leq \alpha \leq 1. \tag{2}
\]
Let \( \alpha_M \) be the minimal \( \alpha \) for which a given observable \( M \) is incompatible with \( P_\alpha \); this is a way of quantifying incompatibility-robustness of \( P_1 \) relative to \( M \) \[32\]. Now \( P_\alpha \) has only one Hellinger distance; \( d_{nm}^2(P_\alpha) = 1 - gd(\alpha) \) and \( \text{coh}_{nm}(P_\alpha) = 1 - gd(\alpha) \) for \( n \neq m \), with \( gd(\alpha) = \frac{\alpha}{2} \left( (d - 2)(1 - \alpha) + 2\sqrt{1 - \alpha} \sqrt{1 + (d - 1)\alpha} \right) \).

Here \( \alpha \mapsto d_{nm}^2(P_\alpha) \) is monotone increasing, setting up a correspondence between \( \alpha_M \) and the Hellinger distance, the latter providing a link to coherence via Prop. \[1\] and Prop. \[2\]
\[
\max_{n,m \neq n} \text{coh}_{nm}(M) \leq gd(\alpha_M) = \max_{n,i} \sum_m |\langle n|M(i)|m\rangle| \quad \text{for all } n,m \neq n.
\]

The upper bound follows from Prop. \[2\] as \( \mathcal{S}(P_\alpha) * M(i) \) is diagonally dominant \[48\], hence PSD, if \( gd(\alpha) \) exceeds this bound. As a simple example take a qubit with \( \sigma_z \)-basis as the incoherent basis. Then \( P_\alpha(0) = \frac{1}{2}(I + \alpha \sigma_z) \), so any observable \( M \) with \( \text{coh}_{01}(M) > gd(\alpha) \) is incompatible with \( P_\alpha \). If \( M \) is binary with \( M(0) = \frac{1}{2}(I + m \cdot \sigma) \), we have
\[
1 \leq \frac{gd(\alpha_M)^2}{m_1^2 + m_2^2} \leq \max \left\{ \frac{1}{1 + m_3}, \frac{1}{1 - m_3} \right\}.
\]
We can check this using the standard qubit criterion \[40\], according to which \( gd(\alpha_M)^2 = (m_1^2 + m_2^2) / (1 - m_3^2) \); hence our bounds are exact if \( m_3 = 0 \).

Next we obtain a quick proof for the known noise bound for the incompatibility of MUBs \[19\,23\,26\]:

**Proposition 3.** Let \( M(i) = \lambda Q_0(i) + (1 - \lambda)d^{-1}I \), where \( Q_0 \) is mutually unbiased to the incoherent basis. Then \( \alpha_M = gd(\lambda) \) for any \( \lambda \in [0,1] \).

**Proof.** The crucial observation is that \( M = C * Q_0 \), where \( C \) has unit diagonal and \( c_{nm} = \lambda \) for \( n \neq m \). Hence \( \text{coh}_{nm}(M) = \lambda, \) so \( \lambda \leq gd(\alpha) \) by Prop. \[1\] if \( M \) and \( P_\alpha \) are jointly measurable. Conversely, if \( \lambda \leq gd(\alpha) \) then \( \mathcal{S}(P_\alpha) * C \geq 0 \), so \( \mathcal{S}(P_\alpha) * M(i) = \mathcal{S}(P_\alpha) * C * Q_0(i) \geq 0 \), so \( P_\alpha \) and \( M \) are jointly measurable by Prop. \[2\], therefore \( \alpha_M = gd(\lambda) \) as \( gd(\lambda) \) is the bound obtained in the cited literature by other methods. We will return to this example later.
III. INCOMPATIBILITY DUE TO A COHERENCE PATTERN

Here we specialise to the physically relevant class of observables, introducing first their general structure, and then focusing on the spin-boson model.

A. General consideration of coherence matrices

Starting with a brief motivation, we consider again the above qubit example: when $m_3 = 0$, we have $M = C \ast Q_0$ where $C$ is a PSD matrix with $c_{00} = c_{11} = 1$, $c_{01} = m_1 - i m_2$, and $Q_0(0) = \frac{1}{2}(I + \sigma_x)$. Note that $\text{coh}_{01}(Q_0) = 1$ (maximal coherence), and $M$ is a “noisy” version of $Q_0$ obtained via pure decoherence [27, 29, 31]. The same structure appears in the case of MUBs (proof of Prop. 3), in any dimension. Accordingly, we call any PSD matrix $C$ with unit diagonal a coherence (pattern) matrix, and consider noisy observables $M(i) = C \ast Q(i)$. In the special case where $Q_0$ is maximally coherent, we have $\text{coh}_{nm}(M) = |c_{nm}|$ so the coherence in $M$ is “imprinted” by $C$. This setting captures a remarkable interplay of maximal incompatibility and coherence. Indeed, in Appendix C we use dilation theory to prove the following result:

**Theorem 1.** Let $C$ be a coherence matrix and $Q_0$ a maximally coherent observable. If an incoherent observable $P$ is jointly measurable with $C \ast Q_0$, it is jointly measurable with $C \ast Q$ for every observable $Q$.

Hence, any maximally coherent observable (such as one mutually unbiased to the incoherent basis) is also maximally incompatible in this setting (which is not true in general [23]). Incompatibility arising from coherence is now described as follows:

**Definition 1.** For a coherence matrix $C$, we denote by $C_C$ the set of incoherent observables $P$ jointly measurable with $C \ast Q$ for every observable $Q$. If $P \notin C_C$ we say that $P$ has incompatibility due to the coherence (pattern) $C$.

Note that to find out whether a given $P$ lies in $C_C$, it suffices (by Thm. 1) to check whether $P$ is jointly measurable with $C \ast Q_0$ for some fixed maximally coherent observable $Q_0$. The general results of the preceding section have the following useful corollaries:

**Corollary 1.** Let $C$ be a coherence matrix. Any $P \in C_C$ has $d_{nm}^2(P) \leq 1 - |c_{nm}|$ for all $n,m$.

**Proof.** Follows by Prop. 1 as $\text{coh}_{nm}(C \ast Q_0) = |c_{nm}|$.

**Corollary 2.** Let $C$ be a coherence matrix and $P$ an incoherent observable. If $C \ast \mathcal{S}(P) \geq 0$ then $P \in C_C$.

**Proof.** Let $Q$ be an arbitrary observable, and $M = C \ast Q$. Then $\mathcal{S}(P) \ast M(i) = (C \ast \mathcal{S}(P)) \ast Q(i) \geq 0$ by the Schur product theorem, so $P$ is jointly measurable with $M$ by Prop. 2, hence $P \in C_C$.

The coherence matrix model is strongly motivated by open quantum systems. In fact, quantum coherence is notoriously fragile against noise, and one of the basic mechanisms by which it decays is pure decoherence (i.e. no dissipation), typically arising as subsystem dynamics from a unitary evolution on a larger system which leaves the incoherent basis unchanged [29 Chapt. 4]. While incompatibility seems rarely tractable under general dynamics (see [29] for a qubit case), our theory applies neatly to this type of dynamics. Each incoherent observable represents a conserved quantity, whose incompatibility with all other system observables is lost when the decaying coherence fails to sustain it: this characterises the emergent classicality of the open system in a more operational way than the decoherence itself.

More formally, suppose we have a family of coherence matrices $C[\lambda]$ depending on a parameter $\lambda \in [0,1]$. If $\lambda = \lambda(t)$ depends on a time parameter $0 \leq t < \infty$, the map $\Delta_t(p) := C[\lambda(t)] \ast p$ defines a quantum dynamical map, i.e. a family of completely positive trace preserving maps on the set of density matrices. A natural “Markovianity” property in this setting is

$$C[\lambda \lambda'] = C[\lambda] \ast C[\lambda'], \quad \lambda, \lambda' \in [0,1],$$

which leads to the CP-divisibility of the dynamical map if the function $\lambda(t)$ is monotone decreasing. Then the loss of incompatibility is irreversible, and (as the Heisenberg picture evolution has the same form), the above corollaries can be used to bound the critical time at which a given incoherent observable loses its incompatibility by entering the set $C_{\lambda(t)}$. Next we show, by considering a specific model, that this approach is amenable to analytical results even in large systems.

B. Spin-boson model

We consider the spin-boson model with collective interaction [29, 31, 35] – $N$ qubits coupled to bosonic modes $b_k$ via the total spin $S_z = \frac{1}{2} \sum_{i=1}^N \sigma_z^{(i)}$. The total Hamiltonian is

$$H = H_S + \sum_k \omega_k b_k^{\dagger} b_k + \sum_k S_z (g_k b_k^{\dagger} + \overline{g}_k b_k),$$

where $H_S = \omega_0 S_z$ is the system Hamiltonian. The incoherent basis is the $\sigma_z$ basis $\{|\mathbf{m}\rangle\}$, where $\mathbf{m} = (m_1, \ldots, m_N)$, and we let $|\mathbf{m}\rangle = \sum_i m_i$. With the bath initially in a thermal state, the system state at time $t$ is $\rho(t) = C[\lambda(t)] \ast \rho_0$, where $C[\lambda]$ is the coherence matrix $c_{nm}[\lambda] = \lambda^{|m| - |m'|^2}$, and $\lambda(t) \in [0,1]$ is given by the bath temperature and spectral density [29 Sec. 4.2]. In the Heisenberg picture, the dynamics transform the system observables $Q$ into $C[\lambda(t)] \ast Q$, which is precisely of the form considered above, and has the divisibility property [3]. The task is to characterise the set $C_{C[\lambda]}$ for $\lambda \in [0,1]$. 
If $t \rightarrow \lambda(t)$ is monotone decreasing (as in [29], p. 230]), the loss of incompatibility is irreversible due to [3]. However, the model also has decoherence-free subspaces (DFS) $D_j = \text{span} \{ |n| \mid |j| = j \}$ [[35], [36]]; basis elements in the same DFS have $c_{n,m}(\lambda) = 1$ for all $\lambda$. By Cor. 1 each DFS “protects” the incompatibility of any $P$ not proportional to $I$ inside it. Observables $P$ exhibiting a transition to classicality therefore have $p^P_n = p^P_m$ when $|n|, |m|$ lie in the same DFS. In Appendix D we show that for these $P$ the problem reduces to an $N + 1$-dimensional space with incoherent basis $\{|k\}$ indexed by the DFS labels $k = 0, \ldots, N$: we have $P \in C(\lambda)$ iff $\tilde{P} \in C(\lambda)$ where $\tilde{p}^P_n = p^P_n$ for $|n| \in D_k$, and the coherence matrix is

$$
\tilde{C}(\lambda) = \begin{pmatrix}
1 & \lambda & \lambda^2 & \cdots & \lambda^{N^2} \\
\lambda & 1 & \lambda & \cdots & \vdots \\
\lambda^2 & \lambda & 1 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda^{N^2} & \cdots & \cdots & \cdots & 1
\end{pmatrix}.
$$

We further focus on the measurements of the DFS label $j = 0, \ldots, N$ which are covariant for the permutation $j \mapsto N - j$ leaving $C(\lambda)$ invariant, i.e. $p^P_n(j) = p^P_{N-n}(N-j)$ when $|n| = N - |m|$. We denote by $C^{C(\lambda)}$ the set of covariant $P \in C(\lambda)$. This set has affine dimension $\frac{1}{2}N(N + 1)$, and we find it analytically for $N = 2$ in Appendix D; we develop a general theory of covariance systems for coherence matrices.

For larger $N$ we focus on covariant observables $P^{C(\lambda)}_\alpha := \alpha P^{C(\lambda)} + (1 - \alpha)P^{\text{dep}, C(\lambda)}$, $\alpha \in [0, 1]$, where $P^{\text{dep}}$ is the spectral resolution of $H_S$ (so $P^{C(\lambda)}(j)$ is the projector onto the DFS $D_j$), and $P^{\text{dep}, C(\lambda)}(j)$ is the depolarisation of $P^{\text{dep}, C(\lambda)}(j)$:

$$
\frac{1}{2N} \text{tr}[P^{C(\lambda)}(j)] I = c_N(j) I
$$

into the “coin toss” distribution $c_N(j) = \binom{N}{j} \frac{1}{2N}$. The proportion of $\{P^{C(\lambda)}_\alpha \mid \alpha \in [0, 1]\}$ having lost incompatibility at time $t$ is $\alpha_N(\lambda(t))$ where

$$
\alpha_N(\lambda) := \max\{\alpha > 0 \mid P^{C(\lambda)}_\alpha \in C(\lambda)\}.
$$

With the reduction [3] we find $\alpha_N(\lambda) = \max\{\alpha > 0 \mid P^{C(\lambda)}_\alpha \in C(\lambda)\}$, where $P^{C(\lambda)}_\alpha(j) = \alpha |j\rangle \langle j| + (1 - \alpha) c_N(j) I$. The task is to find analytical bounds for $\alpha_N(\lambda)$. For $N = 2$ we can explicitly solve (Appendix D) $\alpha_N(\lambda) = 1 - \frac{4\lambda^2}{3 + \lambda^2 + 2\sqrt{2(1 - \lambda^2)}}$. Crucially, the Hellinger distances are tractable for any $N$: $d_{kk'}(\tilde{P}^{C(\lambda)}_\alpha) = 1 - \beta_{kk'}(\alpha)$ when $k \neq k'$, with $\beta_{kk'}(\alpha) = 1 - \alpha + \beta_{kk'}(\alpha) = \lambda^2 c_N(k) - c_N(k) - c_N(k) - (1 - \alpha)$. Each $\alpha \mapsto \beta_{kk'}(\alpha)$ is decreasing in $\alpha$; using the inverse functions $[\beta_{01}]^{-1}$, $[\beta_{00}]^{-1}$ we set $U_N(\lambda) = [\beta_{01}]^{-1}(\lambda)$ and $L_N(\lambda) = [\beta_{00}]^{-1}(\lambda)$, where $\vartheta_3(x, \lambda) = 1 + 2 \sum_{n=1}^{\infty} \lambda^2 x^n \text{cos}(2kx)$ is the Jacobi Theta function [49]. The following result holds:

**Proposition 4.** $\vartheta_3(\frac{\pi}{2}, \lambda) \leq U_N(\lambda) \leq L_N(\lambda) \leq \alpha_N(\lambda) \leq U(\lambda)$ for all $N = 1, 2, \ldots$, and $\lambda \in [0, 1]$. 

**Proof.** We fix $\lambda \in [0, 1]$ (and hence also $\tilde{C}(\lambda)$, $U_N = U_N(\lambda)$, $L_N = L_N(\lambda)$, $\alpha_N = \alpha_N(\lambda)$). Now if $\alpha$ is such that $P^{C(\lambda)}_\alpha \in C(\lambda)$, then $\lambda \geq \beta_{01}(\alpha)$ by Cor. I i.e. $U_N \geq \alpha$, so $U_N \geq \alpha_N$; this establishes the upper bound. For the lower bounds, define $B = (b_{nm})$ by $b_{nm} = 1/\beta_{nm}$ for each $n, m$. (Notice that $\beta_{nm} \neq 1$, as $1 - \beta_{nm}$, only coincides with the corresponding Hellinger distance on the off-diagonal elements.) Now $B \geq 0$, by $b_{nm} = \int_0^1 x^{\beta_{nm}-1} dx = \int_0^1 x^{\beta_{nm}-1} x^{\alpha-n} \text{d}x$. Now $b_{nm} = (1 - d_{nm}^{-1}(P^{C(\lambda)}_\alpha))^{-1}$ when $n \neq m$, but $b_{nn} = 1/\beta_{nm} \geq 1$, so $\tilde{C} * B = \tilde{C} + B$ with $B$ diagonal, $d_{nm} = 1 - b_{nm}$. Since $B \geq 0$, we get $(\tilde{C} - r I) * B \geq 0$, where $r$ is the bottom eigenvalue of $\tilde{C}$. By the theory of Toeplitz matrices...
(50) p. 194, 211, [19] Lemma 1),

\[ r \geq \min_{x \in [0, 2\pi]} \sum_{k=-\infty}^{\infty} \varepsilon_{k0} e^{ikx} = \min_{x \in [0, 2\pi]} \theta_3(\frac{x}{\sqrt{2}}, \lambda) = \theta_3(\frac{\pi}{\sqrt{2}}, \lambda). \]

Hence, if \( \theta_3(\frac{\pi}{\sqrt{2}}, \lambda) \geq 1 - \beta_{00}(\alpha) \) then \( \tilde{C} \cdot \tilde{S}(\tilde{P}_C^\alpha) \geq r(1 - D) + D \geq (1 - \beta_{00})(1 - D) + D \geq 0 \) as \( 1 - \beta_{00} = \max_n (1 - \beta_{nn}) \), and so \( \tilde{P}_C^\alpha \in C_{\tilde{C}} \) by Cor. 2. Hence \( \tilde{P}_C^\alpha \in C_{\tilde{C}} \) for all \( \alpha \leq N_L \), so \( \alpha_N \geq L_N \). Finally, \( \theta_3(\frac{\pi}{\sqrt{2}}, \lambda) \leq L_N \) since \( \alpha \geq 1 - \beta_{00}(\alpha) \) for all \( \alpha \). This completes the proof. \( \square \)

Fig. 2 shows the bounds for \( N = 2, 10 \), and the analytical curve \( \alpha_2(\lambda) \) with a numerical consistency check (blue dots) computed with a generic joint measurability SDP available in [21] applied to \((C[\lambda] \ast Q_0, P^C_0)\) (see Thm. 1). This SDP is not practical for large \( N \); to compute \( \alpha_0(\lambda) \) we used the efficient SDP [3] adapted to our setting as described in the next section (implemented in Python [32]). Prop. 4 says that at any time \( t \), at least the proportion \( 1 - U_N(\lambda(t)) \) of the line \( \{P^C_0 \mid \alpha \in [0, 1]\} \) has incompatibility due to coherence, while at least the proportion \( L_N(\lambda(t)) \) \( \geq \theta_3(\frac{\pi}{\sqrt{2}}, \lambda(t)) \) has lost it. Remarkably, the last bound is independent of \( N \), i.e., holds for any system size. Finally, the bound \( U_N \) is tight near the classical limit (small \( \lambda \)):

**Proposition 5.** For each fixed \( N \), the curves \( \alpha = \alpha_N(\lambda) \) and \( \alpha = U_N(\lambda) \) have the same asymptotic form: \( \lambda = \frac{1 + \sqrt{2}}{2\sqrt{N}} \sqrt{1 - \alpha + O(1 - \alpha)} as \alpha \to 1 (\lambda \to 0) \).

We prove this (Appendix [4]) by explicitly constructing relevant joint observables in the operational framework of the next section. Note that the square root behaviour near the classical limit is distinct from the “middle” regime for \( \lambda \), where \( U_N(\lambda) \) decreases towards \( 1 - \lambda \) as \( N \) increases. Clearly, incompatibility is much more intricate than coherence which sustains it in this model.

**IV. OPERATIONAL FRAMEWORK FOR COHERENCE AND INCOMPATIBILITY**

Having demonstrated our theory in applications, we now gain further insight by reformulating it in resource-theoretic terms, in the context of quantum measurement theory.

**A. Resource-theoretic aspects**

Here we describe how our theory formally integrates into the resource theory of quantum coherence and especially measurement coherence. We stress that our aim is not to develop a comprehensive joint resource theory for coherence and incompatibility, but rather to focus on the most relevant aspects.

The “free resources” are the incoherent observables \( P(i) = \sum_n p^n_{in}(i)\langle n | n \rangle \) [5, 19] already used above; they are jointly measurable with each other. Their nonclassicality is quantified by the Hellinger distances: \( d_{nm}(P) = 1 \) for all \( n \neq m \), while \( d_{nm}(P) = 0 \) for all \( n, m \) iff each \( P(j) \) is a multiple of the identity, i.e. \( P \) is jointly measurable with every observable. An observable \( P \) is incoherent iff \( \text{coh}_{nm}(P) = 0 \) for all \( n \neq m \).

Any quantum channel, a completely positive (CP) trace preserving map \( \Lambda \), is a “free operation” for incompatibility [51, 11, 13], acting on observables via pre-processing \( \Lambda^* (M) (i) = \Lambda^* (M(i)) \), where \( \Lambda^* \) is defined by \( \text{tr} [\Lambda^* (X) \rho] = \text{tr} [X \Lambda (\rho)] \) for all matrices \( X, \rho \). If \( \Lambda (|n\rangle \langle m|) = |n\rangle \langle m| \) for all \( n, m \) i.e. A leaves each incoherent observable unchanged, then \( \Lambda \) is also “free” for coherence, called a genuinely incoherent operation (GIO) [11, 19]. These have already appeared in our setting: each GIO has the form \( \Lambda^*_C(M)(i) = C^*M(i) \) for some coherence matrix \( C \) [21, 32]. The entry-wise coherence is monotonic in GIOs, as \( \text{coh}_{nm}(\Lambda^*_C(M)) = |c_{nm}||\text{coh}_{nm}(M)| \leq \text{coh}_{nm}(M) \).

Observables are measured by *instruments* \( l = (l_i)_{i \in \mathbb{O}} \), where each \( l_i \) is a CP map such that \( \sum_i l_i \) is a channel [15]. The observable measured by \( l \) is \( \mathbb{M}(i) = l_i^* (\mathbb{1}) \). For a state (density matrix) \( \rho \) the post-measurement state given outcome \( i \) is \( l_i (\rho) \). Instruments are needed for sequential implementation of joint measurements: measuring first \( M \) with \( l_i \), and then an observable \( F \), we get a joint POVM \( G(i,j) = l_i^* (F(j)) \) for \( M \) and \( \Lambda^* (F) \). Joint measurements usually do not have sequential implementations (unless one “cheats” by allowing a larger output space [33, Prop. 2]). We define a genuinely incoherent instrument (GII) as one whose channel \( \sum_i l_i \) is a GIO. It follows (Appendix [5]) that any GII with channel \( \Lambda_C \) has the form \( l_i^* (X) = C(i) * X \) for some PSD matrices \( C(i) \) with \( \sum_i C(i) = C \); we call \( C \) the coherence matrix of \( l \). GII are free operations also for coherence: they cannot create coherent observables from incoherent ones by sequential combination. The observable measured by a GII is incoherent, namely \( P(i) := l_i^* (\mathbb{1}) = \sum_n c_{nn}(i)|n\rangle \langle n| \).

**B. Sequential measurement setting**

Given the above concepts, the operational scheme in Fig. 3 naturally emerges; in this setting, coherence needed for incompatibility can now be characterised as follows:

**Theorem 2.** Let \( C \) be a coherence matrix and \( P \) an incoherent observable. The following are equivalent:

(i) \( P \in C_C \);

(ii) There exists a GII with coherence matrix \( C \) and observable \( P \), that is, matrices \( C(j) \) satisfying

\[ C(j) \geq 0, \quad \sum_j C(j) = C, \quad c_{nn}(j) = p^n_C(j). \quad (5) \]

In that case a joint measurement of \( P \) and \( C \ast Q \), for any observable \( Q \), can be implemented sequentially by first measuring \( P \) using the GII in (ii), and subsequently \( Q \).
Proof. \(P\) implies \(P\). Indeed, \(P\) is jointly measurable with \(Q\) if and only if \(P\) and \(Q\) are jointly measurable. By Thms. 1 and 2, the existence of \(I\) is equivalent to joint measurability in (A) when \(Q\) is a maximally coherent.

The crucial part of the proof of this result is the construction of the special joint observable given in the proof of Thm. 1 (see Appendix C). The result is surprising, as joint observables with a GII implementation are quite special, requiring channel-observable compatibility [10–22]. We note that [3], as a SDP [3], is more efficient than a generic joint measurability SDP due to lower dimensionality, but still not analytically solvable except in simple cases; see Appendix D for a d = 3 example. However, there is a useful class of GIs: for each incoherent observable \(P\) define \(C(P)\) as in the proof of Prop. 2 this is a GII with coherence matrix \(C(P)\) given by the Hellinger distances: \(C_{nm} = \sum_{j} C_{nm}(j) = 1 - d_{nm}(P)\). Following the proof of Prop. 2 we obtain a GII for the setting in Cor. 2.

**Proposition 6.** If \(C \ast \mathcal{S}(P) \geq 0\) then \(C(j) = C \ast \mathcal{S}(P) * C(P)\) defines a GII with coherence matrix \(C\) and observable \(P\).

Proof. Now \(\sum_j C(j) = C \ast \mathcal{S}(P) \ast \sum_j C(P) = C\), and \(C_{nn}(j) = C_{nn}(j) = C_{nn}(j) = \rho_{nn}(j)\). If \(C \ast \mathcal{S}(P) \geq 0\) then \(C(j) \geq 0\) by the Schur product theorem, so [3] holds for \(C(j)\).

As an example of a GII we consider the qubit case:

\[
C = \begin{pmatrix} 1 & c \\ \bar{c} & 1 \end{pmatrix}, \quad P(j) = \begin{pmatrix} p_j & 0 \\ 0 & q_j \end{pmatrix}, \quad j \in \mathbb{N},
\]

with \(c \in \mathbb{C}, |c| \leq 1, (q_j), (p_j)\) being probability distributions. The Hellinger distance is \(d_{H}(P) = 1 - \gamma\) with \(\gamma = \sum_j \sqrt{p_j q_j}\), and we obtain \(P \in \mathcal{C}\) iff \(|c| \leq \gamma\).

Indeed, \(P \in \mathcal{C}\) implies \(|c| \leq \gamma\) by Cor. 1 while \(|c| \leq \gamma\) implies \(P \in \mathcal{C}\) by Cor. 2 with the GII

\[
C(j) = \begin{pmatrix} p_j & c \gamma^{-1} \sqrt{q_j} \\ \bar{c} \gamma^{-1} \sqrt{q_j} p_j & q_j \end{pmatrix}.
\]

Revisiting the noisy MUB example (see Prop. 3), we note that any \((d-1)^{-1} \leq \lambda \leq 1\) defines a valid coherence matrix \(C\), and \(P_\alpha\) is an incoherent observable for the same range of \(\alpha\). When \(\lambda < 0\) the bounds of Cor. 1 and 2 do not coincide, but our general method applies (Appendix E), reproducing the result in [21], and (additionally) yielding a general GII implementation for all jointly measurable cases, including the “corner” \(\lambda = \alpha = -(d-1)^{-1}\) where the Lüders instrument fails. We consider this interesting exceptional case here, and postpone the rest of the proof to the Appendix. We have \(P_\alpha(j) = (d-1)^{-1}(1 - (1/|\phi_d|)),\)

\[
C = d(d-1)^{-1}(1 - |\phi_d|)(|\phi_d|),
\]

where \(\phi_d = d^{-\frac{1}{2}} \sum_k |k\rangle\). In this case the two corollaries do not tell us anything: the tradeoff in Cor. 1 is not violated, and the matrix in Cor. 2 is not positive semidefinite. However, now \(P_\alpha \in \mathcal{C}\) directly by Thm. 2 as we can construct a GII \(C(j)\) satisfying [4]:

\[
C(j) := (d - 2)^{-1}(1 - |\phi_d|)(|\phi_d| - |\phi_d^\prime|(|\phi_d^\prime|)),
\]

where \(\phi_d^\prime := \sqrt{d(d-1)^{-1}(1 - d^{-\frac{1}{2}} \phi_d.\) Indeed, positivity follows from the fact that \(\phi_d^\prime\) is orthogonal to \(\phi_d\) for each \(j\) (so each \(C(j)\) is a multiple of a projection), and it is easy to check that \(\sum_j C(j) = C\) and \(c_{nm}(j) = (d-1)^{-1}(1 - \lambda_{nm}) = \rho_{nm}(j)\).

In addition to these examples, in Appendix H we explicitly construct a GII for the spin-boson model: it establishes the tight bound for the coherence needed for incompatibility near the classical limit, and proves Prop. 4.

**C. Reduction by symmetry**

One of the main obstacles in our joint measurability problem is the difficulty of finding the form of a suitable GII. Symmetries in the coherence pattern \(C\) can be used to simplify the search, and also single out relevant incoherent observables. We outline this reduction here; detailed derivations are given in Appendix D.

First note that the matrix \(D\) of a unitary GIO \(\Lambda_D(\rho) = U^\dagger \rho U\) has \(d_{nm} = u_n u_m\) with \(|u_n| = 1\) for all \(n;\) then \(\Lambda_D\) changes neither incompatibility nor coherence. For coherence matrices \(C, C'\), we write \(C \simeq C'\) if \(C = D \ast C'\) for a unitary GIO \(\Lambda_D\). Clearly, \(C_C = C_C\).

Second, we call an incoherent \(P\) adapted to \(C\) if \(P_m = \rho_m\) (\(P\) does not distinguish \(n\) from \(m\)) whenever \((n, m)\) has maximal coherence \(|c_{nm}| = 1\). Crucially, each \(P \in \mathcal{C}\) is adapted to \(C\), as \(|c_{nm}| = 1\) implies \(d_{nm}(P) = 0\) by Cor. 1. So every \(P\) not adapted to \(C\) has incompatibility due to coherence. Now \(\{1, \ldots, d\}\) splits into \(n_{C} \leq d\) disjoint equivalence classes \(I^C_k, k \in \mathcal{C}\) := \{1, \ldots, n_{C}\}, such that \(I^C_k = \{m \mid |c_{nm}| = 1\}\) for any \(n \in I^C_k\). Let \(H\) be the Hilbert space with incoherent basis \(|k\rangle\) for \(k \in \mathcal{C}\) and define \(L: H \to H\) by \(L|k\rangle = \sum_{n \in I^C_k} |n\rangle\). Then \(C \simeq (L^C L)^1\) for a “reduced” coherence matrix \(C\) acting on \(H\).
As an example, consider the spin-boson model: we have the coherence matrix \( C = C[\lambda] \) with \( n_C = N + 1 \) equivalence classes \( I^C_k = \{ n \mid \sum_n n_k = k \} \), \( k \in \Omega_C = \{ 0, 1, \ldots, N \} \) of size \( |I^C_k| = \binom{N}{k} \), corresponding to the decoherence-free subspaces \( \text{span}(\{|n\rangle \mid n \in I^C_k\}) \). The resulting reduced coherence matrix \( \tilde{C} \) is easily seen to be the one given by (4).

Next, let \( S_d \), the group of permutations of \( \{1, \ldots, d\} \), act on \( \mathcal{H} \) via \( U_r|n\rangle = |\pi(n)\rangle \). We define

\[
G_C := \{ \pi \in S_d \mid \cup_k^d C U_r \pi \cong C \}.
\]

In Appendix D we show that \( G_C \) is a permutation group, i.e. a subgroup of \( S_d \), and we call it the symmetry group of \( C \). It moves each class \( I^C_k \) as a whole, and hence gives rise to a map \( \phi : G_C \rightarrow G_C \) through \( \pi(I^C_k) = I^C_{\phi(\pi)(k)} \). Now let \( G \) be any subgroup of \( G_C \). We then say that an incoherent observable \( P \) is \( G \)-covariant if it has outcome set \( \Omega_P \) and \( p^p_{\phi^{-1}(n)(j)} = p^P_n(\phi(n)(j)) \) for \( \pi \in G \), \( j \in \Omega_C \). These observables have their outcomes directly linked to the equivalence classes of the basis labels. The joint measurability problem reduces considerably when restricted to them; we set

\[
C_C[G] := \{ P \in C_C \mid P \text{ is } G \text{-covariant} \}.
\]

The case of the full symmetry group is denoted by \( C^\text{sym}_C := C_C[G_C] \). In Appendix D we show that for \( P \in C^\text{sym}_C \), the SDP (5) can be constrained by a corresponding covariance condition at the GII level without any loss. Furthermore, we also link the incoherent observables on \( \mathcal{H} \) and \( \mathcal{H} \) as \( \mathcal{L}(\ket{k}\bra{k}) = \sum_n \ket{n}\bra{n} \); the reduction by symmetry is then given by

\[
C^\text{sym}_C = \{ \mathcal{L}(P(\cdot)) \mid P \in C_C[\phi(G_C)] \}.
\] (6)

We note that \( \phi(G_C) \) may be different from \( G_C \). However, they coincide when \( |I^C_k| = |I^C_{\pi(n)k}^C| \) for each \( \pi \in G_C \) and \( k \in \Omega_C \), i.e. the equivalence classes linked by permutations in the reduced symmetry group have equal size. In that case we have the straightforward reduction \( C^\text{sym}_C = \{ \mathcal{L}(P(\cdot)) \mid P \in C^\text{sym}_C \} \). An example is provided by the spin-boson model, where the reduced coherence matrix \( \tilde{C} = \tilde{C}[\lambda] \) is invariant under the exchange permutation \( \pi_0 \) defined by \( \pi_0(k) = N - k \) for \( k \in \Omega_C \). In fact, \( G_C = \{ e, \pi_0 \} \) for all \( \lambda \), and \( |I^C_k| = |I^C_{\pi_0(k)}| \) for each \( k \), so \( \phi(G_C) = \{ e, \pi_0 \} \). Hence, by (6), the set \( C^\text{sym}_C \) is isomorphic to \( C^\text{sym}_{\tilde{C}} \), as stated in Section III B.

Finally, we give a simple example involving also complex phase factors, and demonstrating the case \( \phi(G_C) \neq G_C \):

For \( \lambda \in [0, 1) \), and consider the following:

\[
C \mathbin{:=} \begin{pmatrix} 1 & i & \lambda \\ -i & 1 & \lambda \\ -\lambda i & \lambda & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 1 & \lambda \\ 1 & 1 & \lambda \\ \lambda & \lambda & 1 \end{pmatrix} \rightarrow \tilde{C} \mathbin{:=} \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}.
\]

Here \( d = 3 \), \( n_C = 2 \), \( \Omega_C = \{ 1, 2 \} \), \( I^C_1 = \{ 1 \} \), \( I^C_2 = \{ 2 \} \), \( G_C = \{ e, (12) \} \), \( G_C = S_2 \), \( \phi(G_C) = \{ e \} \). Note that the symmetry of \( C \) is “revealed” after factoring out a unitary GIO in the first step. As a further subtlety, the exchange symmetry of \( \tilde{C} \) is excluded as \( |I^C_1| \neq |I^C_2| \). The point of the reduction is that we can use the simpler two-dimensional case to solve the original three-dimensional joint measurability problem. Indeed, we first characterise \( C^\text{sym}_C[\{ e \}] \) as described in the qubit case after Prop. 6 above, and then use (5). \( C^\text{sym}_C \) consists of the binary observables \( P \) of the form

\[
P(1) = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & q \end{pmatrix}, \quad P(2) = \mathbb{1} - P(1),
\]

where \( p, q \in [0, 1] \) and \( \sqrt{pq} + \sqrt{(1-p)(1-q)} \geq \lambda \).

V. CONCLUSION

We considered a general operational setting where quantum coherence is tightly linked to measurement incompatibility. We derived two explicit conditions for the coherence needed for incompatibility, and demonstrated that these are amenable to analytical calculations even in large open quantum systems. Topics of further study include the infinite-dimensional case and adaptation to quantum steering.

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Appendix A: Properties of the entry-wise coherence

Recall that an observable is a POVM $M$ on a Hilbert space $\mathcal{H} = \mathbb{C}^d$ with (finite) outcome set $\Omega_M$, i.e. $M(i) \geq 0$ for each $i \in \Omega_M$, and $\sum_{i \in \Omega_M} M(i) = I$. We assume that the outcome set is taken minimal, i.e. $M(i) \neq 0$ for each $i \in \Omega_M$. We fix a basis $\{n \}$ for all $i \in \Omega_M$ with $|n| = 1, \ldots, d$ and call it the incoherent basis. For any observable we denote $p^M_n(i) := \langle n | M(i) | n \rangle$ for all $n \in \{1, \ldots, d\}$, $i \in \Omega_M$.

An observable $M$ is incoherent if each POVM element $M(i)$ is diagonal, i.e. $M(i) = \sum_{n \in \Omega_M} p^M_n(i) | n \rangle \langle n |$ for all $i \in \Omega_M$. It is mutually unbiased to the incoherent basis if $M(i) = \frac{1}{\sqrt{d}} | \psi_i \rangle \langle \psi_i |$ where $\{ \psi_i \}$ is a basis of $\mathcal{H}$ such that $\langle n | \psi_i \rangle^2 = \frac{1}{d}$ for each $n \in \{1, \ldots, d\}$ and $i \in \Omega_M$. Note that incoherent observables can have arbitrary outcome set $\Omega_M$, while $|\Omega_M| = d$ for any if $M$ is mutually unbiased to the incoherent basis.

In the main text we introduced the entry-wise coher-
ence and Hellinger distances for each \( n, m \in \{1, \ldots, n\} \):
\[
\text{coh}_{nm}(M) = \sum_{i \in \Omega_M} |\langle n|M(i)|m \rangle|,
\]
\[
d^2_{nm}(M) = 1 - \sum_{i \in \Omega_M} \sqrt{p^M_n(i)p^M_m(i)}.
\]

We are not aware of the entry-wise coherence having appeared in the literature as such, but it has been used recently in the construction of overall \( L_p \)-type measures [5].

Hellinger distance is a known \( f \)-divergence [34], but (as far as we know) has not been used in the present context before. It is an actual metric in the space of probability distributions; in particular, if \( d^2_{nm}(M) = 0 \) for some \( n, m \), then \( p^M_n(i) = p^M_m(i) \) for all \( i \in \Omega_M \). Furthermore, \( 0 \leq d^2_{nm}(M) \leq 1 \) by a simple application of the classical Schwarz inequality.

We say that \( M \) is maximally coherent if \( \text{coh}_{nm}(M) = 1 \) for all \( n, m = 1, \ldots, d \). The following proposition summarises the basic properties of the entry-wise coherence:

**Proposition A.1.** Let \( M \) be an observable.

(a) (Bounds). \( 0 \leq \text{coh}_{nm}(M) \leq 1 - d^2_{nm}(M) \leq 1 \) for all \( n, m \).

(b) (Zero coherence). \( \text{coh}_{nm}(M) = 0 \) for all \( n \neq m \), if and only if \( M \) is incoherent.

(c) (Maximal coherence). The following are equivalent:

(i) \( M \) is maximally coherent;

(ii) \( M \) has rank one, and \( p^M_n \) is the same probability distribution for each \( n = 1, \ldots, d \);

(iii) There is a probability distribution \( i \mapsto p(i) \) on \( \Omega_M \), and a sequence of unit vectors \( \psi_i \in \mathcal{H} \), with \( |\langle n|\psi_i \rangle|^2 = d^{-1} \) for each \( n = 1, \ldots, d \), \( i \in \Omega_M \), such that \( M(i) = p(i)\langle \psi_i|\psi_i \rangle \) for each \( i \in \Omega_M \).

If \( M \) has exactly \( d \) outcomes, then \( M \) is maximally coherent if and only if \( M \) is a MUB to the incoherent basis.

**Proof.** Denote \( \psi_i^n = \sqrt{M(i)|n \rangle} \), for each \( n = 1, \ldots, d \), \( i \in \Omega_M \). Then \( |\langle n|\psi_i^n \rangle|^2 = p^M_n(i) \), and hence \( \sum_{i \in \Omega_M} |\langle n|\psi_i^n \rangle|^2 = \sum_{i \in \Omega_M} p^M_n(i) = 1 \) for each \( n \) by the normalisation of the observable \( M \). To prove (a) we use the Cauchy-Schwarz inequality:
\[
\text{coh}_{nm}(M) = \sum_i |\langle n|M(i)|m \rangle| = \sum_i |\langle \psi_i^n|\psi_i^m \rangle| \\
\leq \sum_i |\psi_i^n||\psi_i^m| = 1 - d^2_{nm}(M) \leq 1.
\]

If \( \text{coh}_{nm}(M) = 0 \) for all \( n \neq m \) we have \( \langle n|M(i)|m \rangle = 0 \) for all \( n \neq m \), and hence \( M \) is incoherent; this proves (b).

To prove (c), assume (i), so that \( \text{coh}_{nm}(M) = 1 \) for each pair \( (n, m) \). Then the second inequality in the above calculation is saturated, so \( d^2_{nm}(M) = 0 \) for all \( n \neq m \), which implies \( p(i) := p^M_n(i) = p^M_m(i) \) for each pair \( n, m = 1, \ldots, d \), and all \( i \in \Omega_M \), so \( \|\psi_i^n\|^2 = p(i) \) for each \( n, i \) (i.e. the norm only depends on \( i \)). Note that \( p(i) > 0 \) for each \( i \) (since otherwise \( M(i) = 0 \)). Also the first inequality is saturated, that is, \( \sum_i (|\langle \psi_i^n|\psi_i^m \rangle|^2 - |\langle \psi_i^n|\psi_i^m \rangle|) = 0 \), so \( |\langle \psi_i^n|\psi_i^m \rangle| = \|\psi_i^n\||\psi_i^m\| \) for all \( n, m, i \), as each term in the sum is nonnegative. Hence the Cauchy-Schwarz inequality is saturated for each pair \( \psi_i^n, \psi_i^m \), so \( \psi_i^n = c_i^n \psi_i^m \) for some constants \( c_i^n \) which must have modulus one as \( \|\psi_i^n\|^2 = \|\psi_i^m\|^2 = p(i) \).

Define \( \psi_i := p(i)^{-\frac{1}{2}}\psi_i^m \) for each \( i \in \Omega_M \). Then \( \|\psi_i\| = 1 \), and \( \sqrt{M(i)|n \rangle = \psi_i \} \}
\]

Appendix B: Dilation theory

We review here briefly some well-known aspects of dilation theory of quantum channels and observables (see, e.g., [13]), applied to our framework introduced in the main text.

First recall that the Naimark dilation of an observable \( M = \{M(i)\}_{i \in \Omega} \) on a Hilbert space \( \mathcal{H} \) is a projection valued observable \( A \) on a larger Hilbert space \( \mathcal{H}_\otimes \) such
that $M(i) = J^i A(i) J$ for all $i$, where $J : \mathcal{H} \to \mathcal{H}_\oplus$ is an isometry, i.e. $J^* J = 1$. The dilation is minimal, if $\mathcal{H}_\oplus = \text{span}\{A(i)J\varphi \mid \varphi \in \mathcal{H}, i \in \Omega_M\}$. The following is a basic joint measurability result:

**Theorem B.1** (\textbf{\textit{\[53\]}}). Let $F = (F(j))_{j \in \Omega_F}$ be any observable jointly measurable with $M$ and $(\mathcal{H}_\oplus, A, J)$ a minimal Naimark dilation of $M$. Then each joint observable $G$ of $M$ and $F$ is of the form $G(i,j) = J^i A(i) B(j) J$ where $B$ is a unique POVM of $\mathcal{H}_\oplus$ such that $[A(i), B(j)] = 0$ for all $i \in \Omega_M$, $j \in \Omega_F$.

Any quantum channel $\Lambda$ of $\mathcal{H} \simeq \mathbb{C}^d$ has a minimal Stinespring dilation, i.e. its Heisenberg picture (a completely positive unital map on the matrix algebra $M_d(\mathbb{C})$) can be written in the form $\Lambda^\dagger(X) = J^\dagger (X \otimes 1) J$, $X \in M_d(\mathbb{C})$, where $J : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ is an isometry, $\mathcal{K}$ a Hilbert space (an ancilla) and the vectors $(X \otimes 1) \psi$, $X \in M_d(\mathbb{C})$, $\psi \in \mathcal{H}$, span $\mathcal{H} \otimes \mathcal{K}$ \textbf{\textit{\[17\]}}. It follows from the Radon-Nikodym theorem of completely positive maps \textbf{\textit{\[55\]}} that any instrument $(l_i)_{i \in \Omega}$ whose channel is $\Lambda$ has the form $l_i^\dagger(X) = J^\dagger (X \otimes F(i)) J$ where $F = (F(i))_{i \in \Omega}$ is a (unique) POVM.

We now specialise to our case, with a channel $\Lambda_C$ given by a PSD matrix $C$ with unit diagonal, through Hadamard multiplication $\Lambda_C^\dagger(X) = C \ast X$. Since $C \succeq 0$, we may write $c_{nm} = \langle \eta_n | \eta_m \rangle$ where $\eta_n$ are unit vectors in a Hilbert space $\mathcal{K}$ with dimension equal to the rank of $C$, that is, $\mathcal{K} = \text{span}\{\eta_n \mid n = 1, \ldots, d\}$ (see, e.g. \textbf{\textit{\[21\]}}). These vectors constitute the minimal Stinespring dilation $\Lambda_C^\dagger(X) = J^\dagger (X \otimes 1^C) J$ of $\Lambda_C$, where the isometry is defined by $J|n\rangle = |n\rangle \otimes \eta_n$ (note that $|n\rangle \otimes \eta_n = (X \otimes 1^C) |n\rangle$ where $X = |m\rangle \langle n|.$ Then any GII $(l_i)_{i \in \Omega}$ with channel $\Lambda_C$ has the form

$$l_i^\dagger(X) = J^\dagger (X \otimes F(i)) J = \sum_{n,m} (n|X|m) \langle \eta_n | F(i) \eta_m \rangle |n\rangle \langle m|$$

where $F = (F(j))_{j \in \Omega}$ is a POVM of $\mathcal{K}$. This gives us the Hadamard form $l_i^\dagger(X) = C(j) \ast X$ used in the main text, with the matrix $C(j)$ given by

$$c_{nm}(j) = \langle \eta_n | F(j) \eta_m \rangle.$$

We stress that every instrument with channel $\Lambda_C$ has this form. In particular, if $\{C(j)\}$ is any collection of PSD matrices with $\sum_j C(j) = C$, then $C(j)$ can be written as \textbf{\textit{\[B1\]}} for some (unique) $F$.

**Appendix C: Proofs of Theorems \textbf{\textit{\[1\]}} and \textbf{\textit{\[2\]}}**

Recall that in Theorem \textbf{\textit{\[1\]}} we let $Q_0$ be any fixed maximally coherent observable, with outcome set $\Omega_0$ (assumed to be minimal). By Prop. \textbf{\textit{\[A.1\]}} we can write

$$Q_0(i) = p(i) |d\psi_i\rangle \langle \psi_i|$$

where $i \mapsto p(i)$ is a probability distribution on $\Omega_0$, and $\psi_i$ are unit vectors such that $| \langle \psi_i | n \rangle |^2 = d^{-1}$ for all $n, i$. Let $M$ be a copy of $\Omega_0|\mathcal{H}$, and note that $\langle n| Q_0(i) |n \rangle = p(i)$ for all $n = 1, \ldots, d$.

The proofs are based on the dilation theory described above; let $\eta_n$ be vectors such that $c_{nm} = \langle \eta_n | \eta_m \rangle$ as in Appendix \textbf{\textit{\[B\]}}.

**Proof of Theorem \textbf{\textit{\[1\]}}** Fix a basis $\{ |i\rangle \mid i \in \Omega_0 \}$ of $M$, and define an isometry $V : \mathcal{H} \to \mathcal{M} \otimes \mathcal{K}$ via

$$V|n\rangle = \sum_{i \in \Omega_0} \sqrt{p(i)d} \langle \psi_i | n \rangle |i\rangle \otimes \eta_n.$$ 

Then $C \ast Q_0(i) = V^\dagger (|i\rangle \langle i| \otimes 1_\mathcal{K}) V$. Hence, this is a Naimark dilation of the observable $C \ast Q_0$. Since $\langle \sum_i (|i\rangle \langle i| \otimes 1_\mathcal{K}) V|n\rangle | n \in \{1, \ldots, d\}, i \in \Omega_0 \rangle = \sum_i (|i\rangle \otimes \eta_n \mid n n \in \{1, \ldots, d\}, i \in \Omega_0 \rangle = \mathcal{M} \otimes \mathcal{K}$, the dilation is minimal.

Now assume that $C \ast Q_0$ is jointly measurable with an incoherent observable $P$. Since the above dilation is minimal, Thm. \textbf{\textit{\[B.1\]}} applies: $P$ must be of the form $P(j) = V^\dagger (F(j)) V$, $j \in \Omega_P$, where $|F(j)| = |i\rangle \langle i| \otimes 1_\mathcal{K}$ is zero for all $i, j$. This implies that for each $i$ there is a POVM $F_i = (F_i(j))_{j \in \Omega}$ on $\mathcal{K}$, such that $F(j) = \sum_j |i\rangle \langle i| \otimes F_i(j)$. Furthermore, according to Thm. \textbf{\textit{\[B.1\]}} $P$ and $C \ast Q_0$ have a joint observable

$$\tilde{G}(i,j) = V^\dagger (F(j) \langle i| \otimes 1_\mathcal{K}) V = V^\dagger (|i\rangle \langle i| \otimes F_i(j)) V = \sum_{n,m} \langle n| Q_0(i) | n \rangle \langle n|m\rangle |n\rangle \langle m|$$

and hence $P$ must have the form

$$P(j) = \sum_i \tilde{G}(i,j) = \sum_i \sum_{n,m} \langle n| F_i(j) \eta_n \rangle \langle n| Q_0(i) | m\rangle \langle n|m\rangle$$

$$= \sum_n \sum_i \langle n| F_i(j) \eta_n \rangle \langle n| Q_0(i) | n \rangle \langle n|m\rangle$$

$$= \sum_n \langle \eta_n | \sum_i p(i) F_i(j) \eta_n \rangle |n\rangle \langle n|.$$ (C2)

In the third step we have used the assumption that $P$ is incoherent (so there are no off-diagonal elements), and in the fourth step the maximal coherence condition $\langle n| Q_0(i) | n \rangle = p(i)$. We now define, for each $j$, a matrix $C(j)$ by $c_{nm}(j) := \langle \eta_n | A(j) \eta_m \rangle$, where $A := \sum_i p(i) F_i$ is a POVM by convexity. Therefore we have $C(j) \succeq 0$ and $\sum_j c_{nm}(j) = \langle \eta_n | \eta_m \rangle = c_{nm}$, that is, $C(j)$ form a GII whose channel is $\Lambda_C$. Finally, by the computation \textbf{\textit{\[C2\]}} $P(j) = \sum_n \langle \eta_n | A(j) \eta_n \rangle |n\rangle \langle n| = \sum_n c_{nm}(j) |n\rangle \langle n|$, showing that the observable of this GII is precisely $P$. Applying the GII to any observable $Q$ we get a joint observable $G(i,j) = C(j) \ast Q$ for $C \ast Q$ and $P$, as $\sum_i G(i,j) = C(j) \ast 1 = P(j)$ and $\sum_i G(i,j) = C \ast Q(i)$. Hence $C \ast Q$ and $P$ are jointly measurable. This completes the proof of Thm. \textbf{\textit{\[1\]}}.

The crucial point of the proof is the computation \textbf{\textit{\[C2\]}}; one can readily see how the two strong assumptions: $P$ incoherent and $Q_0$ maximally coherent, fit together rather neatly to form the single dilation POVM $A$.

**Proof of Theorem \textbf{\textit{\[2\]}}** If (i) holds then $C \ast Q$ is jointly measurable with $P$ for all $Q$, so in particular for $Q_0$. By
the above proof we obtain matrices $C(j)$ satisfying [3], so (ii) holds. Conversely, if such matrices exist (that is, (ii) holds), the observable $G(i, j) = C(j) * Q(i)$ defined in the above proof is a joint observable for $P$ and $C * Q$ for any observable $Q$, hence $P \in C_C$, i.e., (i) holds.

**Appendix D: Reduction by symmetry**

Here we develop in detail the theory of covariance systems for a $d \times d$ coherence matrix $C$. Recall that the aim is to characterise the set $C_C$ of incoherent observables $P$ for which there is a GII with GIO $C$ and observable $P$. The idea is that symmetries in the coherence pattern can be used to simplify the problem, and single out relevant incoherent observables.

As above, we make use of the dilation $c_{nm} = \langle \eta_n | \eta_m \rangle$ (see Appendix B). For each pair $(n, m)$ we write $n \sim_c m$ when $|c_{nm}| = 1$. This implies that $|\langle \eta_n | \eta_m \rangle| = 1 = |\eta_n||\eta_m|$, i.e. the Cauchy-Schwarz inequality is saturated for this pair of unit vectors, and hence $\eta_n = e^{i\theta_n} \eta_n$ and $\eta_m = e^{i\theta_m} \eta_m$, whenever $|c_{nm}| = 1$. Hence the relation $n \sim_c m$ is transitive, and since $|c_{nm}| = |c_{mn}|$ for each pair $(n, m)$, it is also symmetric, so an equivalence relation on the set $\{1, \ldots, d\}$. Hence the set splits into a union $\bigcup_{k=1}^{n_C} I^C_k$ of $n_C$ equivalence classes $I^C_k$ (unique up to ordering). We let $\Omega_C = \{1, \ldots, n_C\}$.

Now we pick from each equivalence class $I^C_k$ one fixed representative $\tilde{\eta}_k \in I^C_k$, and let $\tilde{\eta}_{nk} := \eta_{nk}$ for each $k$; then for each $n = 1, \ldots, d$ there is a unique phase factor $e^{i\theta_n}$ so that $\eta_n = e^{i\theta_n} \tilde{\eta}_n$ where $I^C_n$ is the class of $n$. Now define an $n_C \times n_C$ matrix $\tilde{C}$ by $\tilde{c}_{kk'} := \langle \tilde{\eta}_k | \tilde{\eta}_{k'} \rangle$. By construction, $\tilde{C}$ is a structure matrix, and we observe that $c_{nm} = \langle \eta_n | \eta_m \rangle = e^{-i(\theta_n - \theta_m)} \tilde{c}_{kk}$ whenever $n \in I^C_k$ and $m \in I^C_k$. Let $D$ be the matrix $d_{nm} = e^{-i(\theta_n - \theta_m)}$; this is symmetric rank-1, hence the structure matrix of a unitary GIO $\Delta_C(X) = U_{\delta}XU_{\delta}^*$, where $U_{\delta}$ is the diagonal matrix with phases $e^{-i\theta_n}$ on the diagonal. Then we let $H$ be the Hilbert space with incoherent basis $\{|k\}_{k \in \Omega_C}$ and define $L : H \to H$ by

$$L|k\rangle = \sum_{n \in I^C_k} |n\rangle.$$ 

We then obtain the decomposition $C = D \ast (L \tilde{C} L^*)$, so that $C \simeq LCL^*$, and see that the $(n, m)$ entry of the structure matrix $LCL^*$ is equal to $\tilde{c}_{nk}$ for all $n \in I^C_k$, $m \in I^C_k$, that is, only depends on the classes of $n$ and $m$. In other words, after the unitary GIO is factored out, the remaining channel compresses into the GIO $\Delta_C$ on a $n_C$-dimensional system. By construction, this channel is unique up to diagonal unitaries, corresponding to different choices of the representatives $\eta_k$.

Assuming $P$ is adapted, we can compress it into an incoherent observable $P$ on the $n_C$-dimensional system, by setting $p^P_k := p^P_n$ for any $n \in I^C_k$, so that $P(j) = L(\tilde{P}(j))$ where $L : \text{span}\{|k\}_{k \in I^C_k} \to \text{span}\{|n\}_{n \in I^C_k}$ links the two diagonal algebras “incoherently”:

$$L|k\rangle\langle k| = \sum_{n \in I^C_k} |n\rangle\langle n|.$$ 

We can now prove our first reduction result:

**Proposition D.1.** There is a GIO $\Delta_P$, acting on $H$ such that $C \simeq LCL^*$. Then $C_C = \{L(P(|i\rangle)) | P \in C_C\}$.

**Proof.** The decomposition $C \simeq LCL^*$ was constructed above. To prove the second claim, assume first that $P' \in C_C$, and let $C(j)$ form a GII with observable $P'$ and GIO $\Delta_C$, so that $c_{nm}(j) = \langle \eta_n | F(j) | \eta_m \rangle$ for some POVM $F$ on the dilation space $K$, with $\langle \eta_n | F(j) | \eta_m \rangle = p^n_{nm}(j)$ for any $n \in I^C_k$. Hence $P'_{nm}$ does not depend on the choice of $n \in I^C_k$, so $P'$ is adapted to $C$, and $P' = L(P)$ where $P$ is defined by $p^n_{nm}(j) := \tilde{c}_{kk}$. This shows that the observable of this GII is $P$, so $P \in C_C$. Conversely, if $P' = L(P)$ with $P \in C_C$ then there is a GII $\tilde{C}(j)$ with $\tilde{c}_{kk'} := \langle \tilde{\eta}_k | \tilde{\eta}_{k'} \rangle$. Now let $c_{nm}(j) = e^{-i(\theta_n - \theta_m)} \tilde{c}_{kk}$ whenever $(n, m)$ is in $I^C_k$. This is a GII for which $\sum_{k, k'} c_{nm}(j) = e^{-i(\theta_n - \theta_m)} \tilde{c}_{kk'} = c_{nm}$, and $c_{nm}(j) = e^{i\theta_n} \tilde{c}_{kk} = p^n_{nm}(j)$ regardless of the choice of $n \in I^C_k$. Hence $P' \in C_C$. This completes the proof.

Now let $S_d$, the group of permutations of $\{1, \ldots, d\}$, act on $\mathcal{H}$ via $U_{\pi}|n\rangle = |\pi(n)\rangle$, and recall from the main text, the symmetry group

$$G_C = \{\pi \in S_d | U_{\pi}CU_{\pi} \simeq C\}.$$ 

By the definition of $\simeq$, $G_C$ consists of exactly those permutations $\pi \in S_d$ for which there exists a unitary GIO with matrix $D$ such that $U_{\pi}CU_{\pi} = D \ast C$. So $\pi \in G_C$ iff there exist phase factors $u_n(\pi), n = 1, \ldots, d$, such that

$$c_{\pi(n), \pi(m)} = u_{\pi(n)}c_{nm}u_{\pi(m)}$$

(D1)

for each $n, m$. In what follows we assume for simplicity that $c_{nm} \neq 0$ for all $n, m$. Then for each $\pi$ the coefficients $u_n(\pi)$ are uniquely determined up to an overall ($\pi$-dependent) phase factor, which we choose by setting $u_{\pi(n)}(1) = 1$ for a fixed $n_0$. We can then construct $u_n(\pi)$ explicitly from the entries of $C$:

$$u_n(\pi) = c_{\pi(n), \pi(n_0)}c_{n_0 n}, \quad n = 1, \ldots, d.$$  

(D2)

We then define, for each $\pi \in G_C$ and $n = 1, \ldots, d$, a unitary operator $W_{\pi}$ on $\mathcal{H}$ by

$$W_{\pi}|n\rangle = u_n(\pi)|\pi(n)\rangle, \quad n = 1, \ldots, d,$$
so that (by (D1)) we may write

\[ G_C = \{ \pi \in S_d \mid W_{\pi}^\dagger C W_\pi = C \}. \]

The following result shows that \( W_\pi \) appropriately reflects the symmetries of \( C \) on the Hilbert space level:

**Proposition D.2.** \( G_C \) is a permutation group (i.e. a subgroup of \( S_d \)), and \( \pi \mapsto W_\pi \) is a projective unitary representation of \( G_C \) with multiplier \( (\pi, \pi') \mapsto u_{\pi'}(n_0)\). If each entry of \( C \) is real positive, then \( W_\pi = U_{\pi} \).

**Proof.** Let \( \pi, \pi' \in G_C \). Then (D1) holds for both, so

\[ c_{\pi' \pi}^{(n)} = u_{\pi'}(n) c_{\pi \pi'}^{(m)} u_{\pi}(m), \]

for all \( n,m \), showing that (D1) holds also for \( \pi \pi' \). Hence \( \pi \pi' \in G_C \), and since \( G_C \) is finite, this implies that \( G_C \) is a subgroup. Taking \( m = n_0 \) and using (D2) we find

\[ u_n(\pi \pi') = u_{\pi'}(n_0) u_n(\pi') u_{\pi'}(n_0) \cdot \]

which reads \( W_\pi W_{\pi'} = u_{\pi'}(n_0) W_{\pi'} \). Since clearly \( W_\pi = I \), the second claim follows. Finally, if each entry of \( C \) is real positive, then \( u_n(\pi) = 1 \) for all \( \pi \) and \( n \), and we have simply \( W_\pi = U_{\pi} \). \( \square \)

It is clear that \( \{ I_k^C \} \) forms a block system for the group \( G_C \): for each \( \pi \in G_C \) we have \( c_{nm} = 1 \) if \( |c_{\pi(n), \pi(m)}| = 1 \), so \( \pi \) moves each class as a whole, \( \pi(I_k^C) = I_k^C \phi(n)(k) \) for a unique \( \phi(\pi) \in G_C \), where \( C \) is the reduced \( n_C \times n_C \) GIO matrix. The map \( \phi: G_C \rightarrow \hat{C} \) is a homomorphism of \( G_C \) consisting of permutations between classes of the same size. This structure is unique up to an irrelevant overall permutation of \( \{ 1, \ldots, n_C \} \), fixed by the labelling of \( I_k^C \).

Next, recall that given any subgroup \( G \leq G_C \), a \( G \)-covariant incoherent observable \( P \) is one with outcome set \( \Omega_C \) satisfying \( p_{\pi \pi'}^{G}(n) = p_{\pi}(\phi(\pi)(j)) \) for each \( \pi \in G \), \( j \in \Omega_C \) and \( n \in \{ 1, \ldots, d \} \). It is convenient to write this condition equivalently using the representation \( W_\pi \) as

\[ W_\pi P(j) W_{\pi}^\dagger = P(\phi(\pi)(j)), \pi \in G, j \in \Omega_C. \]  \( \text{(D3)} \)

Note that the reduced matrix \( C \) obviously does not reduce further, i.e. \( n_C = n_C \) (each equivalence class is a singleton). Hence for any subgroup \( \hat{G} \leq \hat{G} \leq S_n \), the \( \hat{G} \)-covariant observables \( P \) are given by (D3) with \( d \) replaced by \( n_C \) and \( \phi = 1_d \).

We recall that \( C_C[G] \) is the set of all \( G \)-covariant incoherent observables in \( C_C \), and \( C_C^{\text{sym}} = C_C[G] \). The symmetry constraint (D3) can be naturally formulated in the GII level: we call a GII \( C(j) \) \( G \)-covariant if

\[ W_\pi C(j) W_{\pi}^\dagger = C(\phi(\pi)(j)), \pi \in G, j \in \Omega_C. \]  \( \text{(D4)} \)

Notice that here the matrices \( C(j) \) are not diagonal, so we need to state the condition using the representation \( W_\pi \). The following result shows that the SDP (5) in the main text can be supplemented by an extra symmetry constraint if \( P \) is \( G \)-covariant:

**Proposition D.3.** Any \( \mathcal{P} \in \mathcal{C}_C[G] \) has a \( G \)-covariant GII.

**Proof.** To prove the claim, let \( \mathcal{P} \in \mathcal{C}_C[G] \). Then there is a GII \( C(j) \) with \( \sum_j C(j) = C \) and \( c_{\pi nm}(j) = p_{\pi}^k(j) \) for all \( n \). We define

\[ C'(j) := \frac{1}{|G|} \sum_{\pi \in G} W_{\pi}^\dagger C(\pi(\pi)) W_{\pi}. \]

This is essentially the “averaging argument” often used in the context of symmetry constraints for joint measurability \( \{ 56, 57 \}, \) except that now we apply it at the level of structure matrices as opposed to POVM elements. Now \( C'(j) \geq 0 \), so it defines a GII, which is \( G \)-covariant, as

\[ W_{\pi_0} C'(j) W_{\pi_\pi}^\dagger = \frac{1}{|G|} \sum_{\pi \in G} W_{\pi_\pi}^\dagger C(\pi(\pi)) W_{\pi_\pi} \]

for each \( \pi_\pi \in G \). Here we used the fact that \( \pi \mapsto W_\pi \) is a (projective) representation, and \( \phi \) is a homomorphism. Furthermore,

\[ \sum_j C'(j) = \frac{1}{|G|} \sum_{\pi \in G} W_{\pi}^\dagger \sum_j C(\pi(\pi)) W_{\pi} = \frac{1}{|G|} \sum_{\pi \in G} C = C, \]

as \( G \) is a subgroup of \( G_C = \{ \pi \in S_d \mid W_\pi C W_\pi^\dagger = C \} \).

Finally,

\[ c_{\pi mn}(j) = \frac{1}{|G|} \sum_{\pi \in G} c_{\pi mn}(j) \]

because \( P \) is \( G \)-covariant. Therefore, \( C'(j) \) satisfies eq. (5) in the main text, and is \( G \)-covariant. \( \square \)

We also remark that the GII matrices can always be chosen real if \( C \) is a real matrix (independently of permutation symmetry). In fact, if (5) holds for matrices \( C(j) \), we can define \( \tilde{C}(j) = \frac{1}{2} (C(j) + C(j)^T) \); then \( \tilde{C}(j) \geq 0 \) since transpose preserves positivity, and \( C(j) \) is a real symmetric matrix since \( c_{\pi mn}^*(j) = c_{\pi mn}(j) \) (as \( C(j) = C(j)^* \)). Since \( C \) is real we therefore still have
\[
\sum_j C_{\text{vec}}(j) = C, \text{ and since the diagonal of } C(j) \text{ is real in any case, it coincides with the diagonal of } C_{\text{vec}}(j). \text{ Hence the matrices } C_{\text{vec}}(j) \text{ fulfill } [5] \text{ as well.}
\]

We now prove the main reduction result, which in the main text was stated in Eq. [3].

**Proposition D.4.** \(C_{\text{sym}}^C = \{ L(P(\cdot)) \mid P \in C_C[\phi(G_C)] \} \).

**Proof.** Let \( P \in C_C[\phi(G_C)] \). Hence \( P \) is \( \phi(G_C) \)-covariant and \( P \in C_C \). Now define \( P' := L(P) \). Then \( P' \in C_C \) by Prop. [D.1]. Note that \( P' \) still has \( n_C \) outcomes, but lives in dimension \( d \); explicitly, \( P'(j) = \sum_{k=1}^{n_C} p_{jk}^P(j) \sum_{n \in I_C} |n\rangle\langle n| \). The following rearrangement now shows that \( P' \) is \( G_C \)-covariant:

\[
W PW_\pi P'W_\pi^\dagger = \sum_{k=1}^{n_C} p_{jk}^P(j) \sum_{n \in I_C} |\pi(n)\rangle\langle \pi(n) |
\]

\[
= \sum_{k=1}^{n_C} p_{jk}^P(j) \sum_{\pi^{-1}(n) \in I_C} |n\rangle\langle n|
\]

\[
= \sum_{k=1}^{n_C} p_{jk}^P(j) \sum_{n \in I_{\pi(\pi^{-1}(n))}} |n\rangle\langle n|
\]

\[
= \sum_{k=1}^{n_C} p_{jk}^P(\pi^{-1}(k)) \sum_{n \in I_C} |n\rangle\langle n|
\]

\[
= \sum_{k=1}^{n_C} p_{jk}^P(\pi)(j) \sum_{n \in I_C} |n\rangle\langle n| = P'(\phi(\pi)(j)).
\]

Hence \( P' \in C_{\text{sym}}^C \). Conversely, if we pick a \( P' \in C_{\text{sym}}^C \) then by Prop. [D.1] we can write it as \( P' := L(P) \) for some \( P \in C_C \), and reverse the rearrangement to show that \( P_{\alpha^{-1}(k)}(j) = p_{jk}^P(\phi(\pi)(j)) \), i.e. \( P \) is \( \phi(G_C) \)-covariant and hence \( P \in C_C[\phi(G_C)] \). This completes the proof. \(\Box\)

Finally, we modify the robustness index described above to account for symmetry; instead of Eq. [2] we use the “canonical” \( G_C \)-covariant observable \( P_C^C(j) := L(j)(\cdot) \), \( j \in \Omega_C \), and the line \( P_C^\alpha = \alpha P_C + (1 - \alpha)P_{\text{dep}} \), where \( P_{\text{dep}}(j) = d^{-1} \text{tr}(P_C(j)|1\rangle\langle 1|) I = d^{-1}|I_C| I \). Noting that \( P_C \notin c_{\text{sym}}^C \), we use Prop. [D.4] to set

\[
\alpha_C := \max\{\alpha > 0 \mid P_C^\alpha \in C_C\}
\]

\[
= \max\{\alpha > 0 \mid P_C^\alpha \in c_{\text{sym}}^C\}
\]

so \( \alpha_C \) is the proportion of the line where coherence does not sustain incompatibility. Note that in the nondegenerate case \( |c_{nm}| < 1 \) for all \( n \neq m \) we have \( P_C^\alpha = P_\alpha \), so Eq. [D5] is (by Thm. [1]) consistent with \( \alpha_M \) defined after Eq. [2] in the main text, when \( M = C \ast Q_0 \) where \( Q_0 \) is maximally coherent.

**Appendix E: Uniform coherence with negative entries**

Let \( C \) be a coherence matrix with full symmetry, i.e. \( G_C = S_d \). Then \( c_{nm} = \lambda \) for all \( n \neq m \), for some \( \lambda \in \mathbb{R} \), i.e. all coherences are equal. From its eigenvalues one sees that \( C \) defines a GIO iff \(-d(d-1)^{-1} \leq \lambda \leq 1 \). Any \( G_C \)-covariant \( P \) has \( p_{jk}^P(j) = q \) for all \( n, j \neq j \), for a fixed \( q \); writing \( q = (1 - \alpha)/d \) we see that \( P = P_C^\alpha = P_\alpha \) for some \(-d(d-1)^{-1} \leq \alpha \leq 1 \). Hence these families are naturally motivated by symmetry considerations. Recall from the main text (Prop. [3]) that for \( \lambda, \alpha > 0 \) we have \( P_\alpha \in C_{\text{sym}}^C \) if and only if \( g_\alpha(0) \geq \lambda \) where

\[
g_\alpha(0) = \frac{1}{4}
\]

\[
(d - 2)(1 - \alpha) + 2\sqrt{1 - \alpha}\sqrt{1 + (d - 1)\alpha},
\]

and we now note that the same argument clearly applies also for \( \alpha < 0 \). If \( \alpha > 0 \) we can write this equivalently as \( \alpha \leq g_\alpha(\lambda) \), the function \( \alpha \mapsto g_\alpha(0) \) is decreasing for \( \alpha \in [0, 1] \) and is its own inverse. If \( \alpha < 0 \) the result still holds, but the inequality cannot be inverted using \( g_\alpha \), as \( \alpha \mapsto g_\alpha(\lambda) \) is increasing for \( \alpha \in [-d(d-1)^{-1}, 0] \) with inverse \( u \mapsto \frac{1}{4}
\]

\[
(d - 2)(1 - u) - 2\sqrt{1 - u}\sqrt{1 + (d - 1)u}.
\]

To summarise the \( \lambda > 0 \) case:

\[
C_{\text{sym}}^C = \{ P_C^\alpha \mid g_\alpha(\lambda) \geq \lambda \} \quad \text{when } \lambda \in [0, 1].
\]

Now if \( \lambda < 0 \), the corollaries Cor. [1] and Cor. [2] do not completely determine \( C_{\text{sym}}^C \). Indeed, Cor. [1] gives the necessary condition \( \lambda \geq -g_\alpha(0) \) for \( P_C^\alpha \in C_{\text{sym}}^C \), and Cor. [2] the sufficient condition \( \lambda \geq g_\alpha(0)(d - 1)^{-1} \), which only coincide in the qubit case. However, since \( C \ast Q_0 \) and \( P_\alpha \) have the exact same form, we can interchange \( \alpha \) and \( \lambda \) above to conclude that for \( \alpha > 0 \) and \( \lambda \in [-d(d-1)^{-1}, 1] \) we have \( P_\alpha \in C_{\text{sym}}^C \) if and only if \( g_\alpha(\lambda) \geq \lambda \). This already gives \( \alpha_C = g_\lambda(\lambda) \) in Eq. [D5]. In order to fully characterise \( C_{\text{sym}}^C \) we need to show that \( P_C^\alpha \in C_{\text{sym}}^C \) for all \( (\alpha, \lambda) \in [-d(d-1)^{-1}, 0] \times [-d(d-1)^{-1}, 0] \); this then gives

\[
C_{\text{sym}}^C = \{ P_C^\alpha \mid -(d - 1)^{-1} \leq \alpha \leq g_\lambda(\lambda) \},
\]

when \( \lambda \in [-d(d-1)^{-1}, 0] \).

To prove the remaining bit it suffices (by convexity) to show that \( P_\alpha \in C_{\text{sym}}^C \) for the “corner” \( \alpha = \lambda = -(d - 1)^{-1} \), which was done in the main text.

**Appendix F: Example – centrosymmetric case in dimension 3**

Here we give a nontrivial example of the theory developed in the main text (and the Appendices above). This example is relevant for the \( N = 2 \) case of the spin-boson model but we work it out slightly more generally.

Let \( C \) be any \( 3 \times 3 \) GIO matrix with real positive entries such that \((13) \in G_C\); that is, the symmetry group contains the permutation which exchanges 1 and 3 and leaves 2 unchanged. Then \( C \) must be centrosymmetric,
i.e., (also) symmetric about the counter-diagonal, so
\[
C = \begin{pmatrix}
1 & \lambda & \gamma \\
\lambda & 1 & \lambda \\
\gamma & 1 & \lambda
\end{pmatrix},
\]
for some \(\lambda, \gamma \in [0,1]\) and \(D := \frac{1}{2}(1+\gamma) - \lambda^2 \geq 0\). The conditions ensure that \(C \geq 0\). This covers both the uniform coherence in dimension 3 (\(\gamma = \lambda\) with \(G_C = S_3\)), and the reduction \(\tilde{C}\) of the spin-boson model for \(N = 2\) (\(\gamma = \lambda^4\) with \(G_C = \{e, (13)\}\)). In the former case \(D = (\lambda + \frac{1}{2})(1 - \lambda)\), and in the latter case \(D = \frac{1}{2}(1 - \lambda^2)^2\), which are indeed both positive for all \(\lambda \in [0,1]\).

If \(G_C = S_3\) (i.e., \(\gamma = \lambda\)) we know from the main text that \(C^\text{sym}_{\gamma} = \{P^\alpha_\gamma \mid \lambda \leq g_3(\alpha)\}\), i.e., has affine dimension one. We now proceed to characterise \(C^\text{sym}_{\gamma}\) assuming \(G_C = \{e, (13)\}\) (i.e., \(\gamma \neq \lambda\)). Denote \(\pi_0 = (13)\) (as in the main text). We first note that each \(\{e, (13)\}\)-covariant incoherent observable \(P\) has \(P(2) = \pi_0^* P(0) \pi_0\) and \(P(1) = \pi_0^* P(1) \pi_0\). Therefore, it is of the form 
\[
P = P(q)
\]
for some \(q = (p, r, s) \in \mathcal{M} := \Delta \times [0, \frac{1}{2}]\) where 
\[
\Delta := \{(q, p) \in [0, 1]^2 \mid p + q \leq 1\},
\]
with \(s = 1 - p - q\). Since the map \(q \mapsto P(q)\) is convex, the convex structure of the set of incoherent observables (including the shape of \(C^\text{sym}_{\gamma}\) inside it) is faithfully represented inside \(\mathcal{M}\). In particular, the incoherent basis observable \(P_0(j) = |j\rangle \langle j|\) and its permutation \(P_0(\pi_0(j))\) are represented by the extremal points \((1, 0, 0)\) and \((0, 1, 0)\), while the trivial observables \(P(j) = \mu(j) \mathbb{1}\) (where \(\mu\) is \(\pi\)-invariant) form the line from the origin (with \(P(1) = \mathbb{1}\)) to \(\frac{1}{2}(1, 1, 1)\) (with \(P(0) = P(2) = \frac{1}{2} \mathbb{1}\)). In particular, the centroid \(\frac{1}{2}(1, 1, 1)\) of \(\mathcal{M}\) is the “coin toss” observable \(P(j) = \frac{1}{2} \mathbb{1}\), while the universal trivial observable \(P(j) = \frac{1}{2} \mathbb{1}\) is \(\frac{1}{2}(1, 1, 1)\). The former appears in the spin-boson model as the depolarisation of the spectral measure of the Hamiltonian (see the main text).

In order to state the result, we define the functions 
\[
\begin{align*}
w_+ : & \Delta \to [0, 2], \\
w_- : & \Delta \to [0, 1], \\
w_0 : & \Delta \to [\gamma - 1, 1], \text{ and} \\
w_0^* : & \Delta \to [0, 1 + \gamma] \text{ by} \\
w_+(p, q) = & (\sqrt{q} \pm \sqrt{p})^2, \quad w_0(p, q) = \gamma - 1 + 2(q + p), \\
w_0^-(p, q) = & \begin{cases} 
\gamma - 1 + 2(q + p), & w_+ \leq 1 - \gamma, \\
w_0, & w_+ \geq 1 - \gamma.
\end{cases}
\end{align*}
\]
Clearly, \(w_- \leq w_+\) (for any \(p, q\)). Moreover, \(w_0 \leq w_-\) when \(w_+ \leq 1 - \gamma\), with \(w_0 = w_-\) when \(w_+ = 1 - \gamma\). Correspondingly, \(w_0 \leq w_+\) when \(w_- \leq 1 - \gamma\), with \(w_0 = w_+\) when \(w_- = 1 - \gamma\). In particular, \(0 \leq w_0 \leq w_+\), and \(w_0^*\) is a continuous function.

Since \(D \geq 0\) we have \(0 \leq 2D \leq 1 + \gamma\) and \(2\gamma^2 \leq 1 + \gamma\). Therefore, we can define the functions
\[
\begin{align*}
h_-([0, 1 + \gamma]) & \to [0, \lambda^2/(1 + \gamma)], \\
h_-(w) & = \begin{cases} 
0, & w \in [0, 2D], \\
\left(\sqrt{\frac{\lambda \sqrt{1 + \gamma - w} - \sqrt{D}}{1 + \gamma}}\right)^2, & w \in [2D, 1 + \gamma],
\end{cases} \\
h_+(w) & = \begin{cases} 
\frac{1}{2} \left(\left(\frac{\lambda \sqrt{1 + \gamma - w} - \sqrt{D}}{1 + \gamma}\right)^2, & w \in [0, 2\lambda^2], \\
\frac{1}{2}, & w \in [0, 2D].
\end{cases}
\end{align*}
\]

One can readily check that these functions are continuous. The following result characterises \(C^\text{sym}_{\gamma}\) explicitly:

**Proposition F.1.**
\[
C^\text{sym}_{\gamma} = \{q \in \mathcal{M} \mid \lambda \leq 1 - r, r \in [h_-(w_0^*), h_+(w_+)]\}.
\]

Before giving a proof, we apply Prop. F.1 to a convex line \(P_\alpha\) of the form \(P_\alpha(j) = |\alpha j\rangle \langle j| + (1 - |\alpha j\rangle \langle j|)\), where \(\mu(\alpha) = \mu(1), \mu(2)\) is a probability distribution, which must satisfy \(\mu(0) = \mu(2) = \frac{t}{2}\) and \(\mu(1) = 1 - 2t\) for \(P_\alpha\) to be \(\{e, (13)\}\)-covariant. Fixing \(t\) we then have \(P_\alpha\) represented by the line
\[
p = \alpha + (1 - \alpha)t, \quad q = r = (1 - \alpha)t,
\]
inside \(\mathcal{M}\). The goal is to find to value of \(\alpha\) at which it intersects the boundary of \(C^\text{sym}_{\gamma}\). The reason for not restricting to the canonical line \(P_\alpha^C\) (i.e., \(t = \frac{1}{2}\)) is that we can cover also the cases where \(C\) is obtained as a reduction from some higher dimension as described in Appendix D. In particular, the case of the spin-boson model for \(N = 2\) corresponds to \(t = \frac{1}{2}\).

Since \(r \) decreases as \(\alpha\) increases, the intersection point must lie on the lower boundary surface \(h_-^\alpha\). We restrict to the case \(t \geq \frac{1}{2}(1 - \gamma)\) for simplicity, because then \(w_+(t, \gamma) = 4t \geq 1 - \gamma\), and hence \(w_+ \geq 1 - \gamma\) on the whole line. Therefore \(w_0^\alpha = w_0 = \gamma + 1 - 2(1 - \alpha)(1 - 2t)\) on the line, and so \(P_\alpha^C \in C^\circ\) if \(\lambda \leq (1 - \alpha)t\), which reads
\[
\lambda\sqrt{w_0^2 - \sqrt{1 + \gamma - w_0\sqrt{D}}} \leq (1 + \gamma)\sqrt{1 - \alpha t}.
\]
Rearranging this yields
\[
\lambda^2 w_0 \leq (1 - \alpha)(\sqrt{2(1 - 2t)\sqrt{D}} + (1 + \gamma)\sqrt{7}),
\]
from which one can conveniently solve \(\alpha\) as
\[
\alpha \leq 1 - \frac{\lambda^2}{1 - t(1 - \gamma) + 2\sqrt{2(1 - 2t)D} \sqrt{1 - \gamma}}.
\]
In particular, for \(t = \frac{1}{2}\) (the centroid of \(\mathcal{M}\) corresponding to \(P(\dot{j}) = \frac{1}{2} \mathbb{1}\)) we get
\[
\alpha \leq 1 - \frac{4\lambda^2}{3 + \gamma + 4\sqrt{D}}.
\]
while the case \( t = \frac{1}{3} \) (the uniform trivial observable) gives instead
\[
\alpha \leq 1 - \frac{3\lambda^2}{2 + \gamma + 2\sqrt{2\lambda}}.
\]
which for uniform decoherence, \( \gamma = \lambda \), reduces to
\[
\alpha \leq 1 - \frac{3\lambda^2}{2 + \lambda + 2\sqrt{(1 + 2\lambda)(1 - \lambda)}}.
\]
One can easily check that the right-hand side is equal to \( g_3(\lambda) \) appearing in the main text. Since \( \alpha \leq g_3(\lambda) \) is equivalent to \( \lambda \leq g_3(\alpha) \) (as \( \alpha, \lambda > 0 \)), the results are consistent.

**Proof of Prop. F.1** As per the reduction method in the main text (proved in Appendix [D]), \( P_q \in C_9^{\text{sem}} \) if and only if there exist real matrices \( C(j) \) satisfying Eqs. (D4) and (D5). This forces the matrices to have the following form, where \( a, b, c, \gamma, \lambda \in \mathbb{R} \):

\[
C(0) = \begin{pmatrix} p & a & c \\ a & r & b \\ c & b & q \end{pmatrix},
\]

\[
C(1) = \begin{pmatrix} s & \lambda - a - b & \gamma - 2c \\ \lambda - a - b & 1 - 2r & \lambda - a - b \\ \gamma - 2c & \lambda - a - b & s \end{pmatrix},
\]

\[
C(2) = \begin{pmatrix} q & b & c \\ b & r & a \\ c & a & p \end{pmatrix}.
\]

The problem is, then, whether we can find \( a, b, c \in \mathbb{R} \) so that the first two of these three matrices are positive semidefinite. (Note that \( C(2) = U_{a_0} C(0) U_{a_0}^\dagger \) is then automatically positive semidefinite.)

In order to further simplify the positivity condition for \( C(1) \) (which is easier of the two), we note that all centrosymmetric matrices (i.e., those commuting with \( U_{a_0} \)) can be brought to a block form by a specific orthogonal matrix \( Q \) only depending on \( U_{a_0} \) [58]; in our case,

\[
Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix},
\]

and letting \( x = a + b, y = a - b \) we obtain

\[
Q^T C(0) Q = \begin{pmatrix} p + q - c & \frac{y}{\sqrt{2}} & \frac{p - q}{\sqrt{2}} \\ \frac{y}{\sqrt{2}} & r & \frac{x}{\sqrt{2}} \\ \frac{p - q}{\sqrt{2}} & \frac{x}{\sqrt{2}} & p + q + c \end{pmatrix},
\]

\[
Q^T C(1) Q = \begin{pmatrix} s - \gamma + 2c & 0 & 0 \\ 0 & 1 - 2r & \sqrt{2}(\lambda - x) \\ 0 & \sqrt{2}(\lambda - x) & s + \gamma - 2c \end{pmatrix}.
\]

Since orthogonal transformations preserve positivity (and determinants), we can extract the conditions for \( C(j) \geq 0 \) from these matrices. First of all, \( \frac{p + q}{\sqrt{2}} + c \geq 0 \) is clearly necessary for \( C(0) \geq 0 \). The \( 2 \times 2 \) principal minors of \( Q^T C(0) Q \) are

\[
d(0) := r(p + q + c) - \frac{p - q}{\sqrt{2}},
\]

\[
d(1) := (p + q - c)(\frac{p + q}{\sqrt{2}} + c) - (\frac{p - q}{\sqrt{2}})^2 = qp - c^2,
\]

\[
d(2) := r(p + q + c) - \frac{p - q}{\sqrt{2}}.
\]

Assuming \( \frac{p + q}{\sqrt{2}} + c > 0 \), we can write

\[
det C(0) = rd(1) - \frac{1}{2}(\frac{p + q}{\sqrt{2}} - c)x^2 - \frac{1}{2}(\frac{p + q}{\sqrt{2}} + c)y^2 + xy\frac{p - q}{\sqrt{2}} = d(0) d(1)/(\frac{p + q}{\sqrt{2}} + c) - \frac{1}{2}(\frac{p + q}{\sqrt{2}} + c)(y - y_0)^2,
\]

where \( y_0 = \frac{1}{\sqrt{2}}x(p - q)/(\frac{p + q}{\sqrt{2}} + c) \). Hence, \( C(0) \geq 0 \) is equivalent to \( det C(0) \geq 0 \) and \( d(1) \geq 0 \) for \( i = 1, 2, 3 \). Since \( d(0), d(1) \) do not depend on \( y \) and imply \( d(2) \geq 0 \) when \( y = y_0 \), it follows that if \( C(0) \geq 0 \) for some choices of \( x, y, c \), it also holds if we take \( y = y_0 \) (as the determinant can only increase). As \( C(1) \) does not depend on \( y \), we may therefore always take \( y = y_0 \). With this choice, \( C(0) \geq 0 \) if and only if

\[
\frac{p + q}{\sqrt{2}} + c \geq 0, \quad |x| \leq \sqrt{2r(\frac{p + q}{\sqrt{2}} + c)}, \quad c^2 \leq qp. \quad (F2)
\]

(In the special case \( \frac{p + q}{\sqrt{2}} + c = 0 \), we have \( C(0) \geq 0 \) only if \( p = q \) and \( x = 0 \), so \( C(0) \geq 0 \) if and only if \( y^2 \leq 4rp \). Hence we can take \( y = y_0 = 0 \), and this case is covered by (F2).)

Next we observe that \( C(1) \geq 0 \) if and only if

\[
|c - \frac{\gamma}{2}| \leq \frac{s}{2}, \quad |\lambda - x| \leq \sqrt{(1 - 2r)(\frac{p + q}{\sqrt{2}} - c)}. \quad (F3)
\]

For fixed \( c \), the inequalities (F2) and (F3) force \( x \) into an intersection of two intervals. By the triangle inequality, (F2) and (F3) hold for some \( x \), if and only if \( \lambda \leq g(c) \) where

\[
g(c) := \sqrt{2r(\frac{p + q}{\sqrt{2}} + c)} + \sqrt{(1 - 2r)(\frac{p + q}{\sqrt{2}} - c)},
\]

and the remaining constraints hold for \( c \). These constraints are given by the following set:

\[
\mathcal{D} := \left\{ c \in \mathbb{R} \left| \frac{x}{2} \leq c \leq \frac{\gamma}{2}, |c| \leq \sqrt{pq} \right. \right\}.
\]

Hence, \( q \in \mathcal{C} \) if and only if \( \lambda \leq g(c) \) for some \( c \in \mathcal{D} \). Clearly, this is in turn equivalent to the following:

\[
\mathcal{D} \neq \emptyset \quad \text{and} \quad \lambda \leq \max_{c \in \mathcal{D}} g(c). \quad (F4)
\]

By the triangle inequality, \( \mathcal{D} \neq \emptyset \) if and only if

\[
\frac{\gamma}{2} \leq \sqrt{pq}, \quad (F5)
\]

in which case \( \mathcal{D} \) is the interval

\[
\mathcal{D} := \left[ \max\{-\sqrt{pq}, \frac{\gamma}{2}\}, \min\{\sqrt{pq}, \frac{\gamma}{2}\} \right] \quad (F6)
\]

(where the left boundary does not exceed the right). Hence, \( \frac{\gamma}{2} \) is a necessary (but not sufficient) condition
for \( q \in C_C \). In fact, it is one of the two Hellinger distance conditions given by Cor. 1 of the main text. We note that \( \sqrt{q} \) does not depend on \( r \), and let \( \mathcal{R} \) denote the set of those \((p, q) \in \Delta\) for which it holds. Then

\[
\mathcal{R} = \{(p, q) \in \Delta \mid w_- \leq 1 - \gamma\},
\]

where the function \( w_- = w_-(p, q) \) was defined above. Next we note that the maximal domain of \( g \) (where the square roots are defined) is

\[
\mathcal{D}_{\text{max}} := \left[-\frac{1}{2}(p + q), \frac{1}{2}(s + \gamma)\right],
\]

which clearly contains \( \mathcal{D} \) because \( \sqrt{q} \leq \frac{1}{2}(q + p) \). We readily find the global maximum point of \( g \) within \( \mathcal{D}_{\text{max}} \):

\[
2 \frac{dg}{dc} = \sqrt{\frac{2r}{(s+p)c+1}} - \sqrt{\frac{2r}{(s+p)\gamma - c}},
\]

and hence \( \frac{dg}{dc} \geq 0 \) if \( c \leq c_0 \) where \( c_0 = r(1+\gamma) - \frac{1}{2}(p+q) \). Note that indeed \( c_0 \in \mathcal{D}_{\text{max}} \), as \( c_0 + \frac{r}{2} = r(1+\gamma) \geq 0 \), \( \frac{r}{2} = c = 0 \)(1 + \( \gamma \)). It follows that \( g(c_0) = \sqrt{\frac{1}{2}(1+\gamma)} \), so that \( g(c_0) \geq \lambda \) automatically by the positivity of \( C \), and hence \( c_0 \in \mathcal{D} \) is a sufficient condition for \( q \in C_C \). However, it is not a necessary condition, as \( c_0 \) may fall on either side of \( \mathcal{D} \); in those cases, the maximum is attained at the boundary, and we obtain a constraint in terms of \( \lambda \). In order to find it we consider the three possible cases for \( \mathcal{D} \):

1. \( \mathcal{D}_1 = [-\sqrt{q}, \sqrt{q}] \).
2. \( \mathcal{D}_2 = [\frac{s-\gamma}{2}, \sqrt{q}] \).
3. \( \mathcal{D}_3 = [\frac{s-\gamma}{2}, \frac{s+\gamma}{2}] \).

(The case \( \mathcal{D} = [-\sqrt{q}, \frac{s+\gamma}{2}] \) cannot occur as \( \gamma \geq 0 \).) Since \( \mathcal{D} \) does not depend on \( r \), these cases set up a unique partition of \( \mathcal{R} \); we have \( \mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \) where \( \mathcal{R}_i := \{(p, q) \in \mathcal{R} \mid \mathcal{D} = \mathcal{D}_i\} \). It is then easy to check that

\[
\mathcal{R}_1 = \{(p, q) \in \Delta \mid w_+ < 1 - \gamma\},
\]

\[
\mathcal{R}_2 = \{(p, q) \in \Delta \mid 1 - \gamma \leq w_+ < 1 + \gamma\} \cap \mathcal{R},
\]

\[
\mathcal{R}_3 = \{(p, q) \in \Delta \mid w_+ \geq 1 + \gamma\},
\]

where we have used the function \( w_+ = w_+(p, q) \) defined above. Each of these cases then has three subcases (a)--(c) according to whether \( c_0 \leq \min \mathcal{D} \), \( c_0 \in \mathcal{D} \), or \( c_0 \geq \max \mathcal{D} \), respectively. In order to describe them we also need the functions \( w_0 = w_0(p, q) \), and \( l(w) := \frac{1}{2}w/(1+\gamma) \). We observe that \( q \in C_C \) if and only if \((q, p)\) falls into one of the following eight categories:

1. \((p, q) \in \mathcal{R}_1\) (implying \( 0 \leq l(w_-) \leq l(w_+) \leq \frac{1}{2} \)), and
   (a) \( 0 \leq r \leq l(w_-) \) and \( \lambda \geq g(-\sqrt{q}) \), or
   (b) \( l(w_-) \leq r \leq l(w_+) \), or
   (c) \( l(w_+) \leq r \leq \frac{1}{2} \) and \( \lambda \leq g(\sqrt{q}) \).
2. \((p, q) \in \mathcal{R}_2\) (implying \( 0 \leq l(w_-) \leq l(w_+) \leq \frac{1}{2} \)), and
   (a) \( 0 \leq r \leq l(w_-) \) and \( \lambda \leq g(\frac{\gamma}{s-\gamma}) \), or
   (b) \( l(w_0) \leq r \leq l(w_+) \) or
   (c) \( l(w_+) \leq r \leq \frac{1}{2} \) and \( \lambda \leq g(\sqrt{q}) \).
3. \((p, q) \in \mathcal{R}_3\) (implying \( 0 \leq l(w_0) \leq \frac{1}{2} \)), and
   (a) \( 0 \leq r \leq l(w_0) \) and \( \lambda \leq g(\frac{\gamma}{s-\gamma}) \), or
   (b) \( l(w_0) \leq r \leq \frac{1}{2} \).
   (In this case (c) does not occur.)

We then notice that

\[
g(\pm \sqrt{q}) = \sqrt{rw \pm} + \sqrt{(1-r)(1+\gamma - w \pm)},
\]

\[
g(\frac{s}{2}(\gamma - s)) = \sqrt{rw_0} + \sqrt{(1-r)(1+\gamma - w_0)}; \quad (F7)
\]

hence in each case the condition involving \( \lambda \) is of the form

\[
\lambda \leq \sqrt{rw} + \sqrt{(1-r)(1+\gamma - w)}, \quad (F8)
\]

for \((r, w) \in [0, \frac{s}{2}] \times [0, 1 + \gamma] \). Equivalently,

\[
h_-(w) \leq r \leq h_+(w), \quad (F9)
\]

where \( h_\pm \) were introduced above. We observe that

\[
h_-(w) \leq l(w) \leq h_+(w) \quad (F10)
\]

for all \((r, w) \in [0, \frac{s}{2}] \times [0, 1 + \gamma] \). (In fact, the region defined by \( F8 \) is symmetric about the line \( r = l(w) \).) Using \( F7 \), \( F9 \), and \( F10 \) we can put together subcases (a)--(c) in the above three cases: \( q \in C_C \) if and only if one of the following conditions hold:

1. \((p, q) \in \mathcal{R}_1\) and \( h_-(w_-) \leq r \leq h_+(w_+) \).
2. \((p, q) \in \mathcal{R}_2\) and \( h_-(w_0) \leq r \leq h_+(w_+) \).
3. \((p, q) \in \mathcal{R}_3\) and \( h_-(w_0) \leq r \leq \frac{1}{2} \).

Noting that \( w_+ \geq 1 + \gamma \geq 2\lambda^2 \) when \((p, q) \in \mathcal{R}_3\), we have \( h_+(w_+) = \frac{1}{2} \) when \((p, q) \in \mathcal{R}_3\), and hence the upper bound for \( r \) is always \( h_+(w_+) \). By the definition of \( w_0 \), the lower bound is \( h_-(w_0) \), and the proof of the proposition is complete.

**Appendix G: Spin-boson model with \( N = 2 \)**

Here we present in detail the analytical solution of \( C_C^{\text{sym}} \) for \( N = 2 \) in the spin-boson model. We order the two-qubit incoherent basis in the usual way as \{\(00\), \(01\), \(10\), \(11\)\}, with respective label set \{1, 2, 3, 4\}, on which the symmetric group \( S_3 \) acts. In this basis our matrix \( C[\lambda] \) of the dynamical GIO reads:

\[
C[\lambda] = \begin{pmatrix}
1 & \lambda & \lambda & \lambda^4 \\
\lambda & 1 & \lambda & \lambda \\
\lambda & \lambda & 1 & \lambda \\
\lambda^4 & \lambda & \lambda & 1
\end{pmatrix}.
\]
The equivalence classes of maximal coherence are given by $I^C_0 = \{1\}$, $I^C_1 = \{2,3\}$, $I^C_2 = \{4\}$, so $n_C = 3$. These correspond to the eigenspaces of $S_z$ given by $S_0 = \text{span}\{(00)\}$, $S_1 = \text{span}\{(01),(10)\}$, and $S_2 = \text{span}\{(11)\}$. Note that $S_1$ is a nontrivial two-dimensional decoherence-free subspace, as we can see from the matrix. Now the canonical incoherent observable $P_C$ has three outcomes $\{0,1,2\}$, and is given by $P_C(0) = |00\rangle\langle 00|, P_C(1) = |10\rangle\langle 10| + |01\rangle\langle 01|, P_C(2) = |11\rangle\langle 11|$, which is just the spectral decomposition of $S_z$ as mentioned in the main text. The line $P_C^0$ used to define the quantity $\alpha_C$ is given by $P_C^0(j) = \alpha P_C(j) + (1 - \alpha)\text{tr}[P_C(j)]\frac{1}{4}\mathbb{I} = \alpha P_C(j) + (1 - \alpha)\frac{3}{4}\mathbb{I}$. Finally, the symmetry group $G_C$ leaving $C$ unchanged is the subgroup of $S_4$ generated by the within-class permutation $(23)$ (exchanging 2 and 3), and the order-reversal $(14(23))$ (1 to 4, 2 to 3, 3 to 4, 4 to 1). The task is to characterise the set $C_C^{\text{symm}}$. 

We now carry out the reduction to dimension $n_C = 3$. First, we have

$$
\tilde{C}[\lambda] = \begin{pmatrix}
1 & \lambda & \lambda^4 \\
\lambda & 1 & \lambda \\
\lambda^4 & \lambda & 1
\end{pmatrix},
$$

with $G_{\tilde{C}} = \{e,(13)\}$. The homomorphism $\phi : G_C \to G_{\tilde{C}}$ defined by $\pi(I^C_\phi(\pi)) = I^C_\phi(\pi(k))$ maps as follows: $\phi((23)) = \phi(e) = e, \phi((14)) = \phi((14)(23)) = (13)$, so that $\phi(G_C) = \{e,(13)\} = G_{\tilde{C}}$, i.e. the within-class permutation is mapped to the identity, and the reversal carries over to the reduction. In this way we end up with the case considered above in Appendix F with $\gamma = \lambda^4$, so Prop. F.1 gives the three-dimensional convex set $C_C[\phi(G_C)] = C_C^{\text{symm}}$. Hence every $P \in C_C^{\text{symm}}$ is of the form

$$
P(0) = \begin{pmatrix}
p & 0 & 0 \\
0 & r & 0 \\
0 & 0 & q
\end{pmatrix}, \quad P(1) = \begin{pmatrix}
s & 0 & 0 \\
0 & 1 - 2r & 0 \\
0 & 0 & 1 - 2r
\end{pmatrix}, \quad P(2) = \begin{pmatrix}
0 & 0 & 0 \\
0 & r & 0 \\
0 & 0 & r
\end{pmatrix},
$$

where $(p,q,r) \in [0,1], (r) [0,\frac{1}{2}]$ are such that $w_-(p,q) \leq 1 - \lambda^4, r \in [-w_0(p,q), h_+(w_+(p,q))]$ (see Appendix F). As noted in the main text, $C_C^{\text{symm}}$ is therefore a convex set of affine dimension 3 only depending on $\lambda$, and can conveniently be plotted in the parameterisation $(p,q,r)$, as shown in Fig. 1 in the main text. Furthermore, by substituting $\gamma = \lambda^4$ into Eq. (F1) in Appendix F we immediately deduce that

$$
\alpha_2(\lambda) = 1 - \frac{4\lambda^2}{3 + \lambda^4 + 2\sqrt{2}(1 - \lambda^2)},
$$
as claimed in the main text.

**Appendix II: Small coherence limit in the spin-boson model**

Here we prove the asymptotic behaviour of the curve $\alpha = \alpha_N(\lambda)$ stated in Prop. 3. We first consider the upper bound $\alpha = U_N(\lambda)$, or, equivalently, $\lambda = \beta_0(\alpha)$, which is an explicit algebraic curve given by the Hellinger distance $d_{\text{D}}(P_C(\alpha))$ corresponding to the dominant coherence $\lambda$ in $C[\lambda]$, as explained in the main text. Hence the asymptotic form is easily obtained: $\beta_0(\alpha) = k_N(1 - \alpha + O(1 - \alpha))$, where $k_N = 2^{-N/2}/(1 + \sqrt{N})$. Here we have used the customary notation where $g(\alpha) = O((1 - \alpha)^k)$ means that the function $\alpha \mapsto |g(\alpha)|/(1 - \alpha)^k$ is bounded on some neighbourhood of $\alpha = 1$. Note that this does not require a convergent series expansion for $g$ at $\alpha = 1$. Indeed, while such an expansion exists for $\beta_0(\alpha)$, the same is not clear for the exact curve $\alpha = \alpha_N(\lambda)$, which nevertheless turns out to have the same behaviour. In order to see this we find a lower bound with the same asymptotic behaviour. We denote $g_k(\alpha) = \left(\frac{N}{12}\right)^{1/2} \alpha^{(1/2)}$, for each $k \in [0,\ldots,N]$, and $u_k(\alpha) = \sqrt{g_k(\alpha)}(\alpha + g_k(\alpha))$ for $k = 0,1$. Then the upper bound reads $\beta_0(\alpha) = u_0(\alpha) + u_1(\alpha) + (1 - 1 - \frac{1}{2}\alpha - (1 + N))$. It turns out that the first two terms form a lower bound:

**Lemma II.1.** If $u_0(\alpha) + u_1(\alpha) \leq \alpha_N^{-1}(\alpha)$ for all $\alpha \in [0,1]$, where $\alpha_N^{-1} : [0,1] \to [0,1]$ is the inverse of the monotone function $\alpha_N$.

As a consequence, we obtain the following result, the first part of which is Prop. 3 in the main text:

**Proposition II.1.** For any fixed $N$,

$$
\alpha_N^{-1}(\alpha) = k_N(1 - \alpha + O(1 - \alpha)) \quad \text{as } \alpha \to 1.
$$

For each $\alpha \in [0,1]$ and $N$, we have the error bound

$$
|\alpha_N^{-1}(\alpha) - k_N(\sqrt{1 - \alpha})| \leq 1 - \alpha + \frac{3}{2}k_N(1 - \alpha)^{3/2}.
$$

**Proof.** Using the bijection $\alpha \mapsto v = \sqrt{1 - \alpha}$ we define

$$
f(v) := u_0(\alpha) + u_1(\alpha)
= v\sqrt{1 - r - rv^2} + v\sqrt{1 - s\sqrt{1 - sv^2}},
$$

where $r = 1 - 2^{-N}$ and $s = 1 - N2^{-2N}$. Now $\beta_0(\alpha) = f(v) + v^2(1 - \frac{1}{2\pi}(1 + N)) \leq f(v) + v^2$. Combined with Lemma II.2, this gives $f(v) \leq \alpha_N^{-1}(\alpha) \leq f(v) + v^2$, that is, $|\alpha_N^{-1}(\alpha) - f(v)| \leq v^2$ for all $v \in [0,1]$. Since $f(0) = 0$, Taylor’s theorem gives $f(v) = k_Nv + z(v)$, where $|z(v)| \leq \frac{1}{2}v^2 \max_{0 \leq \varepsilon \leq v} |\frac{d^2f}{dv^2}(\varepsilon)|$. Now

$$
\left|\frac{d^2f}{dv^2}(\varepsilon)\right| = \frac{rv\sqrt{1 - r - rv^2} + s\sqrt{1 - s\sqrt{1 - sv^2}}}{(1 - vr^2)^{3/2}} \leq 3vr\sqrt{1 - r} - \frac{3us\sqrt{1 - s}}{(1 - sv^2)^{3/2}} \leq \frac{3(1 + \sqrt{N})v}{2N^2(1 - v^2)^{3/2}}.
$$
and hence $|z(v)| \leq \frac{1}{2}k_N v^3 (1 - v^2)^{-\frac{3}{2}}$. Therefore,

$$|\alpha_N^{-1}(\alpha) - k_N v| = |\alpha_N^{-1}(\alpha) - f(v) + z(v)| \leq |\alpha_N^{-1}(\alpha) - f(v)| + |z(v)| \leq v^2 + \frac{3}{2}k_N v^3 (1 - v^2)^{-\frac{3}{2}}.$$ 

Substituting $v = \sqrt{1 - \alpha}$ yields the claim. 

Since $\alpha_N^{-1}(1) = 0$, the validity of the approximation can be quantified by the relative error $\epsilon = |\alpha_N^{-1}(\alpha) - k_N v|/|\alpha_N^{-1}(\alpha)|$. Assuming that the second term in the error bound of the proposition is negligible for this consideration, we get the maximum relative error $\epsilon_{\text{max}} \approx (k_N/\sqrt{1 - \alpha})^{-1}$. This suggests that the asymptotic form becomes valid around $\alpha \approx 1 - ck_N^2$, where $c$ is a constant. For instance, $\alpha \approx 1 - (k_N/24)^2$ corresponds to a maximum error of 4–5%.

We now prove Lemma \[\text{H.1}\] by constructing explicitly a GII satisfying Eqs. \[\text{D.4}\] and \[\text{E.2}\], for $\lambda = u_0(\alpha) + u_1(\alpha)$ and all $\alpha \in [0, 1]$. (This then shows that $P_0^\infty N C$ whenever $\alpha$ satisfies $\lambda \leq u_0(\alpha) + u_1(\alpha)$, and therefore the transition point $\alpha = \alpha_N(\lambda)$ must satisfy $\lambda \geq u_0(\alpha) + u_1(\alpha)$, giving the claimed inequality.) For simplicity, we show the construction for $N \geq 5$ (for smaller $N$ it needs a few modifications).

First define

$$C(0) = \begin{pmatrix} \alpha + q_0 & u_0 & 0 \\ u_0 & q_0 & 0 \\ 0 & 0 & q_0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$C(1) = \begin{pmatrix} q_1 & u_1 & q_1 \\ u_1 & \alpha + q_1 & u_1 \\ q_1 & u_1 & q_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

These matrices are clearly positive semidefinite. (If $N = 5$ the central block is empty.) We then set $C(N) := U_{\pi_0}^\dagger C(0) U_{\pi_0}, C(N - 1) := U_{\pi_0}^\dagger C(1) U_{\pi_0}$, where $U_{\pi_0}$ is the permutation matrix for the order-reversal $\pi_0$. Recalling that $A \mapsto U_{\pi_0} A U_{\pi_0}$ transposes the matrix along the counter-diagonal, we can easily check that $C(0) + C(1) + C(N - 1) + C(N)$ reads

$$\begin{pmatrix} \alpha + r_1 & \lambda & 0 \\ \lambda & \alpha + r_1 & u_1 \\ 0 & u_1 & r_1 \end{pmatrix} \begin{pmatrix} 0 \\ r_1 \\ 0 \end{pmatrix}$$

where $r_k = 2 \sum_{j=0}^k q_j$. Note that $-q_1$ on the lower right block of $C(1)$ cancels out the $q_1$ on the upper left block, and the overlap between $C(0)$ and $C(1)$ produces $\lambda$ on both ends of the diagonal $(n, n + 1)$.

Next, let $j_0 = N/2$ if $N$ is even, and $j_0 = (N - 1)/2$ if $N$ is odd, and define

$$M_j(w) = \begin{pmatrix} q_j & w & 0 \\ w & \alpha + q_j & w \\ 0 & w & q_j \end{pmatrix}$$

for $j = 2, \ldots, j_0$ and $w \in [0, 1]$. Now $M_j(w) \geq 0$ if and only if $2w^2 \leq q_j(\alpha + \alpha)$, which is clearly the case for $w = u_0$ when $j \geq 2$, and for $w = u_1$ when $j \geq 3$, because $q_j$ is increasing in $j$ (up to $j_0$), and $q_2 \geq 2q_1$, $q_3 \geq 2q_1$. (Note that the same is not true for $j = 0, 1$, which is why we defined $C(0)$ and $C(1)$ differently above.) Then let

$$V_j(w) = \begin{pmatrix} q_k j_{j-1} & 0 & 0 \\ 0 & M_j(w) & 0 \\ 0 & 0 & q_k I_{N-j-1} \end{pmatrix}$$

for each $j = 2, \ldots, j_0$.

We now consider even and odd $N$ separately.

If $N$ is even, $j_0$ is the “middle” point of $\{0, \ldots, N\}$, and we can set up $C(2), \ldots, C(j_0 - 1)$ as follows: $C(j) := V_j(w)$ if $j$ is odd, and $C(j) := V_j(u_0)$ if $j$ is even. Notice that the block $M_j(w)$ moves down the diagonal as $j$ increases, with $w$ alternating between $u_1$ and $u_0$. These blocks only overlap between neighbouring matrices $C(j)$, $C(j + 1)$, and hence the sum $\sum_j C(j)$ has $\lambda = u_0 + u_1$ on each $(k, k - 1)$-entry for $k = 0, \ldots, j_0 - 1$. We then set $C(j_0 + j) := U_{\pi_0}^\dagger C(j_0 - j) U_{\pi_0}$ for $j = 1, \ldots, j_0 - 2$, to satisfy the symmetry. Finally, we define the middle element by $C(j_0) := D + G$ where $D$ follows the above pattern, that is, $D := V_j(u_1) (V_j(u_0))$ if $j_0$ is odd (even), $G := C[\lambda - \tilde{C}$, and $\tilde{C}$ denotes the truncation of $C[\lambda$ to second order in $\lambda$, so that $\tilde{C}$ is a tridiagonal matrix. Now $\sum_j C(j) + D = \tilde{C}$, and the remainder $G$ is included in $C(j_0)$ so that $\sum_j C(j) = C[\lambda$.

Note also that $C(j_0)$ satisfies $U_{\pi_0}^\dagger C(j_0) U_{\pi_0} = C(j_0)$ as $D$ has the block $M_{j_0}(w)$ exactly at the centre. We are left to prove that $C(j_0) \geq 0$. To do this we write

$$C(j_0) = D + G = (D - \epsilon I) + (\epsilon I + G),$$
where we pick \( \epsilon > 0 \) small enough so that \( D - \epsilon \mathbb{I} \geq 0 \), but large enough to make \( e \mathbb{I} + G \geq 0 \). We can take, for instance, \( \epsilon = \frac{2}{3} q_1 \). In fact, first note that \( \lambda = u_0 + u_1 \leq k_N \sqrt{1 - \alpha} \), and \( \lambda \leq 0.305 \) for all \( \alpha \in [0, 1] \), \( N \geq 5 \). Also, \( k_N^2 \leq 2.1 N/2^N \) for \( N \geq 5 \) so

\[
\frac{1}{2} \sum_{j \neq i} G_{ij} \leq \sum_{k=2}^{\infty} \lambda^k \leq \lambda^2 \sum_{k=0}^{\infty} \lambda^k = \frac{\lambda^4}{1 - \lambda} \leq 0.14 \lambda^2
\]

\[
\leq 0.14 k_N^2 (1 - \alpha) \leq 0.14 \ast 2.1 q_1 \leq \frac{1}{3} q_1,
\]

and hence \( \epsilon \mathbb{I} + G \geq 0 \) when \( N \geq 5 \) by diagonal dominance. Moreover, \( M_{j_0}(w) - \epsilon \mathbb{I} \geq 0 \) since for \( N \geq 6 \),

\[
(q_{j_0} - \epsilon)(\alpha + q_{j_0} - \epsilon) - 2w^2 \\
\geq (q_3 - \epsilon)(\alpha + q_3 - \epsilon) - 2q_1(\alpha + q_1) \\
\geq (q_3 - 2q_1 - \epsilon)(\alpha + q_3) + 2q_1(q_3 - q_1) - q_3 \epsilon \\
= (q_3 - \frac{5}{2} q_1)(\alpha + q_3) + \frac{3}{2} q_1(q_3 - \frac{3}{2} q_1) \geq 0,
\]

where \( q_3 - \frac{5}{2} q_1 = \frac{2}{2^N}((N - 1)(N - 2) - 16) \frac{q_1}{q_5} \geq 0 \) as \( N \geq 6 \). (Note that the 2 \times 2 principal minors of \( M_{j_0}(w) \) are then automatically positive.) Hence \( D - \epsilon \mathbb{I} \geq 0 \). This completes the construction for even \( N \).

If \( N \) is odd, we have two middle points \( j_0 \) and \( j_0 + 1 \). We now define \( C(2), \ldots, C(j_0 - 1) \) as above, and again set \( C(j_0 + 1) := U_{j_0}^1 C(j_0 - j) U_{j_0}^2 \), for \( j = 1, \ldots, j_0 - 2 \). The sum of these matrices coincides with \( C \) everywhere except in the 4 \times 4 block at the centre of the matrix, and on the main diagonal. The two remaining matrices have to be set up separately. Let \( w = u_1 \) if \( j_0 \) is odd, and \( w = u_0 \) if it is even. We first define \( D \) as the \( (N + 1) \times (N + 1) \) matrix having \( q_{j_0} \) on the main diagonal outside the central 4 \times 4 block,

\[
\begin{pmatrix}
q_{j_0} & w & 0 & 0 \\
w & \alpha + q_{j_0} & \frac{\lambda}{2} & 0 \\
0 & \frac{\lambda}{2} & q_{j_0} & 0 \\
0 & 0 & 0 & q_{j_0}
\end{pmatrix},
\]

(H1)

and the remaining elements zero. Now \( D \geq 0 \) due to \( q_{j_0} \geq q_2 \geq 2q_1 \), which holds as \( N \geq 5 \). Clearly, the sum \( D + U_{j_0}^1 D U_{j_0}^2 \) has \( 2q_{j_0} = q_{j_0} + q_{j_0 + 1} \) on the main diagonal outside the central block, which reads

\[
\begin{pmatrix}
2q_{j_0} & w & 0 & 0 \\
w & \alpha + 2q_{j_0} & \lambda & 0 \\
0 & \lambda & \alpha + 2q_{j_0} & w \\
0 & 0 & w & 2q_{j_0}
\end{pmatrix}.
\]

Added to the previously constructed \( C(j) \), this produces \( \bar{C} \). Now define \( C(j_0) := D + \frac{1}{2} G \) and \( C(j_0 + 1) = U_{j_0}^1 D U_{j_0}^2 + \frac{1}{2} G \), where \( G \) is as before. Then \( C(j_0 + 1) = U_{j_0}^1 C(j_0) U_{j_0}^2 \) and \( \sum_{j=0}^N C(j) = C[\lambda] \). As in the even case, we establish that \( C(j_0) \geq 0 \); we write

\[
C(j_0) = (D - \frac{\epsilon}{2} \mathbb{I}) + \frac{1}{2}(\epsilon \mathbb{I} + G),
\]

with the same \( \epsilon \) as before, so the second term is positive for \( N \geq 5 \). Using \( \frac{1}{4} \lambda^2 \leq \frac{2}{3} (w_0^2 + w_1^2) \) we get

\[
(q_{j_0} - \frac{\epsilon}{2})(\alpha + q_{j_0} - \frac{\epsilon}{2}) - (w/2)^2 \\
\geq (q_2 - \frac{\epsilon}{2})(\alpha + q_2 - \frac{\epsilon}{2}) - \frac{1}{2} (u_0^2 + u_1^2) - q_1(\alpha + q_1) \\
= (q_2 - \frac{1}{4} q_1)(\alpha + q_2 - \frac{1}{4} q_1) - \frac{1}{2} q_1(\alpha + q_1) \\
= (q_2 - \frac{1}{4} q_1)(\alpha + q_2 - \frac{1}{4} q_1) - \frac{1}{2} q_1(\alpha + q_1) \\
= (q_2 - \frac{1}{4} q_1)(\alpha + q_2 - \frac{1}{4} q_1) - \frac{1}{2} q_1(\alpha + q_1) \\
\geq 0 \quad \text{for} \quad N \geq 5,
\]

which implies that \( D - \epsilon \mathbb{I} \geq 0 \). This completes the proof for the odd case, and the proof of Lemma [H1] is complete.