A note on quantum odometers

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Abstract We discuss various aspects of noncommutative geometry of smooth subalgebras of Bunce-Deddens-Toeplitz algebras.

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1 Introduction

In noncommutative geometry, it is often necessary to consider dense *-subalgebras of C*-algebras, in particular, in connection with cyclic cohomology or with the study of unbounded derivations on C*-algebras [5]. Smooth subalgebras of noncommutative spaces are also naturally present in studying spectral triples. If C*-algebras are thought of as generalizations of topological spaces, then dense subalgebras may be regarded as specifying additional structures on the underlying space, like a smooth structure. In analogy the algebras of smooth functions on a compact manifold, such a smooth subalgebra should be closed under holomorphic functional calculus of all the elements and under smooth functional calculus of self-adjoint elements. It should also be complete with respect to a locally convex algebra topology [1].

The purpose of this paper is to study smooth subalgebras \( A^\infty_S \) of Bunce-Deddens-Toeplitz C*-algebras \( A_S \) associated with a supernatural number \( S \), objects that capture their smooth structure. This work is a continuation of, and heavily relies on, our previous papers on the subject of smooth subalgebras, in particular, [8, 9] which investigated smooth structures on Bunce-Deddens algebras, the algebras of compact operators, and the Toeplitz algebra.

Bunce-Deddens algebras \( B_S \) [3,4] are crossed product C*-algebras obtained from odometers and Bunce-Deddens-Toeplitz algebras \( A_S \) are their extensions by compact operators \( K \):

\[
0 \to K \to A_S \to B_S \to 0.
\]

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Due to the topology of odometers [6], which are Cantor sets with a minimal action of a homeomorphism, the smooth subalgebras are naturally equipped with inductive limit Fréchet (LF) topology. Using a version of the Toeplitz map [10], we build smooth subalgebras $A_S^\infty$ from Toeplitz operators with smooth symbols and from smooth compact operators. Smooth compact operators, introduced in [11], were studied in detail in [9]. Smooth Bunce-Deddens algebras $B_S^\infty$, the symbols of Toeplitz operators, were introduced in [9]. We explicitly construct appropriate LF structures on $A_S^\infty$ and prove that those algebras are closed under holomorphic calculus so that they have the same K-theory as their corresponding C*-algebra closures, and we verify that they are closed under smooth functional calculus of self-adjoint elements.

We also focus on describing continuous derivations [14] on smooth subalgebras $A_S^\infty$. In particular, using the results from [8,9], we classify derivations on $A_S^\infty$ and show that up to inner derivations with a compact range, they are lifts of derivations on $B_S^\infty$, the factor algebra of $A_S^\infty$ modulo the ideal $K_0^\infty$ of smooth compact operators. Since many derivations on $B_S^\infty$ are themselves inner, the factor space of continuous inner derivations on $A_S^\infty$ modulo inner derivations turns out to be one-dimensional. Additionally, we shortly describe the K-theory and K-homology of $A_S$.

The rest of this paper is organized as follows. Section 2 contains our notation and a short review of relevant results from [8,10]. In Section 3, we review smooth compact operators and introduce and study the smooth Bunce-Deddens-Toeplitz algebra. Section 4 contains a detailed discussion of stability of $A_S^\infty$ under both the holomorphic functional calculus and the smooth calculus of self-adjoint elements. In Sections 5, we investigate the structure and classifications of derivations. Finally, Section 6 contains remarks on the K-theory and K-homology.

2 Preliminaries

2.1 Supernatural numbers

A supernatural number $S$ is defined as the formal product:

$$S = \prod_{p\text{-prime}} p^{\varepsilon_p}, \quad \varepsilon_p \in \{0, 1, \ldots, \infty\}.$$  

We assume that $\sum \varepsilon_p = \infty$ so that $S$ is an infinite supernatural number. We define the following S-adic ring:

$$\mathbb{Z}/S\mathbb{Z} = \prod_{p\text{-prime}} \mathbb{Z}/p^{\varepsilon_p}\mathbb{Z}.$$  

Here if $S = p^\infty$ for a prime $p$, then $\mathbb{Z}/S\mathbb{Z}$ is equal to $\mathbb{Z}_p$, the ring of $p$-adic integers.

If the ring $\mathbb{Z}/S\mathbb{Z}$ is equipped with the Tychonoff topology, it forms a compact, Abelian topological ring with unity, though only the group structure is relevant for this paper. In addition, if $S$ is an infinite supernatural number, then $\mathbb{Z}/S\mathbb{Z}$ is a Cantor set.

The ring $\mathbb{Z}/S\mathbb{Z}$ contains a dense copy of $\mathbb{Z}$ by the following identification:

$$\mathbb{Z} \ni k \leftrightarrow \{k \mod p^{\varepsilon_p}\} \in \prod_{p\text{-prime}} \mathbb{Z}/p^{\varepsilon_p}\mathbb{Z}. \quad (2.1)$$

2.2 Hilbert spaces

We use two concrete Hilbert spaces for this paper: $H = \ell^2(\mathbb{Z})$ and $H_+ = \ell^2(\mathbb{Z}_{\geq 0})$. Let $\{E_l\}_{l \in \mathbb{Z}}$ and $\{E_k^+ : k \geq 0\}$ be the canonical bases for $H$ and $H_+$, respectively. We need the following shift operator $V : H \to H$ on $H$ and the unilateral shift operator $U : H_+ \to H_+$ on $H_+$:

$$VE_l = E_{l+1} \quad \text{and} \quad UE_k^+ = E_{k+1}^+.$$  

Notice that $V$ is a unitary while $U$ is an isometry. We have $[U^*, U] = P_0$, where $P_0$ is the orthogonal projection onto the one-dimensional subspace spanned by $E_0^+$. 
For a continuous function $f \in C(\mathbb{Z}/S\mathbb{Z})$, we define two operators $m_f : H \rightarrow H$ and $M_f : H_+ \rightarrow H_+$ via formulas, i.e.,

$$m_f E_l = f(l)E_l \quad \text{and} \quad M_f E^+_k = f(k)E^+_k.$$ 

In those formulas, we consider the integers $k$ and $l$ as elements of $\mathbb{Z}/S\mathbb{Z}$ using the identification (2.1). Since $\mathbb{Z}$ is a dense subgroup of $\mathbb{Z}/S\mathbb{Z}$, we immediately obtain

$$\|m_f\| = \|M_f\| = \sup_{l \in \mathbb{Z}} |f(l)| = \sup_{k \in \mathbb{Z}_{>0}} |f(k)| = \sup_{x \in \mathbb{Z}/S\mathbb{Z}} |f(x)| = \|f\|_{\infty}.$$ 

The algebras of operators generated by $m_f$ or by $M_f$ are thus isomorphic to $C(\mathbb{Z}/S\mathbb{Z})$, so they carry all the information about the space $\mathbb{Z}/S\mathbb{Z}$, while the operators $U$ and $V$ reflect the odometer dynamics $\varphi$ on $\mathbb{Z}/S\mathbb{Z}$ given by

$$\varphi(x) = x + 1. \quad (2.2)$$

The relation between those operators is

$$V^{-1}m_f V = m_{f\circ\varphi}. \quad (2.3)$$

Similarly, we have

$$M_f U = U M_{f\circ\varphi}. \quad (2.4)$$

There is also another, less obvious relation between $U$ and $M_f$, namely,

$$M_f P_0 = P_0 M_f = f(0)P_0. \quad (2.5)$$

### 2.3 Algebras

Following [10], we define the Bunce-Deddens and Bunce-Deddens-Toeplitz algebras, $B_S$ and $A_S$, respectively, to be the following $C^*$-algebras: $B_S$ is generated by the operators $V$ and $m_f$, i.e.,

$$B_S = C^*\{V, m_f : f \in C(\mathbb{Z}/S\mathbb{Z})\},$$

while $A_S$ is generated by the operators $U$ and $M_f$, i.e.,

$$A_S = C^*\{U, M_f : f \in C(\mathbb{Z}/S\mathbb{Z})\}.$$ 

The algebra $A_S$ contains the projection $P_0$ and in fact all the compact operators $K$ and the quotient $A_S/K$ can be naturally identified with $B_S$ [8]. Let $\tau$ be the natural homomorphism $\tau : A_S \rightarrow B_S$.

The algebra $B_S$ is isomorphic to the crossed product algebra, i.e.,

$$B_S \cong C(\mathbb{Z}/S\mathbb{Z}) \rtimes \varphi \mathbb{Z},$$

and is simple [8]. Consequently, it is isomorphic to the universal $C^*$-algebra with the generators $v$ and $f$, where $v$ is unitary, and $f \in C(\mathbb{Z}/S\mathbb{Z})$ with relations (compared with (2.3))

$$v^{-1}fv = f \circ \varphi.$$ 

Interestingly, the algebras $A_S$ can also be described in terms of generators and relations as follows.

**Proposition 2.1.** The universal $C^*$-algebra $A$ with the generators $u$ and $f$ such that $u$ is an isometry, and $f \in C(\mathbb{Z}/S\mathbb{Z})$ with relations (compared with (2.3) and (2.5))

$$fu = u(f \circ \varphi) \quad \text{and} \quad fp_0 = f(0)p_0,$$

where $[u^*, u] = p_0$, is isomorphic to $A_S$. 

Proof. We show that any irreducible representation of $A$ either factors through $B_S$ or is isomorphic to the defining representation of $A_S$. Since $B_S \cong A_S/\mathcal{K}$ is a factor algebra, the defining representation of $A_S$ dominates the factor representation, so by universality, $A$ is isomorphic to $A_S$.

Consider an irreducible representation of $A$, and let $U$ represent $u$ and $M_f$ represent $f$. Notice that $P_0 := I - UU^*$ is the orthogonal projection onto the kernel of $U^*$. If that kernel is zero, then $U$ is unitary and $U$ and $M_f$ give a representation of $B_S$ by universality, since they satisfy the crossed product relations.

If the kernel of $U^*$ is not zero, pick a unit vector $E_0^+$ such that $U^*E_0^+ = 0$. Since $U$ is an isometry, the set $\{E_k^+\}$ ($k = 0, 1, \ldots$) is orthonormal, where $E_k^+ := U^kE_0^+$. Moreover, we have by using relations,

$$M_fE_0^+ = M_fP_0E_0^+ = f(0)E_0^+,$$

and similarly,

$$M_fE_k^+ = M_fU^kE_0^+ = U^kM_fE_0^+ = f(k)U^kE_0^+ = f(k)E_k^+.$$

It follows that the vectors $\{E_k^+\}$ span an invariant subspace, so by irreducibility, $\{E_k^+\}$ is an orthonormal basis. Since $U$ is the unilateral shift in this basis, we reproduce the defining representation of $A_S$, finishing the proof.

\section{Toeplitz maps}

Next, we discuss the key relation between the two algebras $A_S$ and $B_S$. Let $P_{\geq 0} : H \to H_+$ be the following map from $H$ onto $H_+$ given by

$$P_{\geq 0}E_k = \begin{cases} E_k^+, & \text{if } k \geq 0, \\ 0, & \text{if } k < 0. \end{cases}$$

We also need another map $s : H_+ \to H$ given by

$$sE_k^+ = E_k.$$

Define the map $T : B(H) \to B(H_+)$ between the spaces of bounded operators on $H$ and $H_+$ in the following way: given $b \in B(H)$, we set

$$T(b) = P_{\geq 0}bs.$$

$T$ is known as a Toeplitz map. It has the following properties [7]:

1. $T(1_H) = 1_{H_+}$.
2. $T(bV^n) = T(b)U^n$ and $T(V^{-n}b) = (U^*)^nT(b)$ for $n \geq 0$ and all $b \in B(H)$.
3. $T(bm_f) = T(b)M_f$ and $T(m_f b) = M_fT(b)$ for all $f \in C(\mathbb{Z}/S\mathbb{Z})$ and all $b \in B(H)$.
4. $T(b^*) = T(b)^*$ for all $b \in B(H)$.

Consequently, it follows that $T$ is a $*$-preserving map from $B_S$ to $A_S$. If $\tau$ is the natural homomorphism from $A_S$ to $B_S$, then we have

$$\tau T(b) = b$$

for all $b \in B_S$. It follows that for any $a$ in $A_S$, there is a compact operator $c$ such that we have a decomposition

$$a = T(b) + c, \quad (2.6)$$

where $b = \tau(a) \in B_S$. One can verify that if $b$ is an element in $B_S$, then $T(b)$ is compact if and only if $b = 0$. This implies the uniqueness of the above decomposition (2.6).
2.5 Fourier series

There are natural one-parameter groups of automorphisms of $B_S$ and $A_S$, respectively. They are given by the formulas

$$\rho^1_\theta(b) = e^{2\pi i \theta} b e^{-2\pi i \theta} \quad \text{for } b \in B_S \quad \text{and} \quad \rho^\infty_\theta(a) = e^{2\pi i \theta} a e^{-2\pi i \theta} \quad \text{for } a \in A_S,$$

where $\theta \in \mathbb{R}/\mathbb{Z}$. Here, we use the following diagonal label operators on $H$ and $H_+$, respectively:

$$\mathbb{L}E_l = lE_l \quad \text{and} \quad \mathbb{K}E^+_k = kE^+_k.$$

We have the following relations:

$$\rho^1_\theta(V) = e^{2\pi i \theta} V \quad \text{and} \quad \rho^\infty_\theta(m_f) = m_f.$$

Automorphisms $\rho^\infty_\theta$ satisfy analogous relations and the extra relation on $U^*$, namely, $\rho^\infty_\theta(U^*) = e^{-2\pi i \theta} U^*$.

Define $E : B_S \to C^*\{m_f : f \in C(\mathbb{Z}/\mathbb{SZ})\} \cong C(\mathbb{Z}/\mathbb{SZ})$ via

$$E(b) = \int_0^1 \rho^1_\theta(b) d\theta.$$

It is easily checked that $E$ is an expectation on $B_S$. For a $b \in B_S$, we define the $n$-th Fourier coefficient $b_n$ by

$$b_n = E(V^{-n} b) = \int_0^1 \rho^1_\theta(V^{-n} b) d\theta = \int_0^1 e^{-2\pi in \theta} V^{-n} \rho^1_\theta(b) d\theta.$$

From this definition, it is clear that $b_n \in C^*\{M_f : f \in C(\mathbb{Z}/\mathbb{SZ})\}$, so we can write $b_n = m_{f_n}$ for some $f_n \in C(\mathbb{Z}/\mathbb{SZ})$.

We define an expectation $E$ on $A_S$ in a similar way, $E : A_S \to (A_S)_{\text{diag}}$, the subalgebra of diagonal elements in $A_S$ by

$$E(a) = \int_0^1 \rho^\infty_\theta(a) d\theta.$$

The following isomorphism was observed in [10, Proposition 2.4]:

$$(A_S)_{\text{diag}} \cong \{M_f + c(\mathbb{K}) : f \in C(\mathbb{Z}/\mathbb{SZ}), \{c(k)\} \in c_0(\mathbb{Z}_{\geq 0})\}.$$

For an $a \in A_S$, its $n$-th Fourier coefficient $a_n$ is given by the following formulas:

$$a_n = \begin{cases} E((U^*)^n a) & \text{for } n \geq 0, \\ E(aU^{-n}) & \text{for } n < 0. \end{cases}$$

Since $a_n \in (A_S)_{\text{diag}}$ for all $n$, we have $a_n = M_{f_n} + c_n(\mathbb{K})$ for some $f_n \in C(\mathbb{Z}/\mathbb{SZ})$ and some $\{c_n(k)\} \in c_0(\mathbb{Z}_{\geq 0})$. Additionally, since $c_n(\mathbb{K}) = 0$ for all $n$ in the Fourier coefficients of an element of the form $a = T(b)$, we have the following relation with the Toeplitz map:

$$(T(b))_n = T(b_n) \quad \text{for all } n.$$

3 Smooth subalgebras

3.1 Smooth compact operators

We begin with reviewing properties of smooth compact operators from [9]. Let $K$ be the algebra of compact operators on $H_+$. The orthonormal basis $\{E^+_k\}_{k \geq 0}$ of $H_+$ determines a system of units $\{P_{ks}\}_{k,s \geq 0}$ in $K$ that satisfy the following relations:

$$P^*_k = P_{sk} \quad \text{and} \quad P_{ks} P_{rt} = \delta_{sr} P_{kt},$$
where \( \delta_{sr} = 1 \) for \( s = r \) and is equal to zero otherwise. The set of smooth compact operators with respect to \( \{ P_k \} \) is the set of operators of the form \( c = \sum_{k,s \geq 0} c_{ks} P_{ks} \) so that the coefficients \( \{ c_{ks} \}_{k,s \geq 0} \) are rapidly decaying (RD). We denote the set of smooth compact operators by \( \mathcal{K}^\infty \).

Next, we review smooth Bunce-Deddens algebras.

### 3.2 Smooth Bunce-Deddens algebras

**Proposition 3.2.** Let \( a \) and \( b \) be bounded operators in \( H_+ \). Then

1. \( a \in \mathcal{K}^\infty \) if and only if \( \|a\|_{M,N} < \infty \) for all nonnegative integers \( M \) and \( N \);
2. \( \|a\|_{M+1,N} = \|a\|_{M,N} + \|d_k(a)\|_{M,N} \);
3. \( \|a\|_{M,N} \leq \|a\|_{M+1,N+1} \);
4. \( \|ab\|_{M,N} \leq \|a\|_{M,0}\|b\|_{M,N} \leq \|a\|_{M,N}\|b\|_{M,N} \);
5. \( \|d_k(a)\|_{M,N} \leq \|a\|_{M+1,N} \);
6. \( \|a^*\|_{M,N} \leq \|a\|_{M+N,N} \);
7. \( \mathcal{K}^\infty \) is a complete topological vector space.

This proposition implies that \( \mathcal{K}^\infty \) is a Fréchet \( * \)-algebra with respect to the norms \( \| \cdot \|_{M,N} \).

### 3.2 Smooth Bunce-Deddens algebras

Next, we review smooth Bunce-Deddens algebras \( B_S^\infty \) from [8]. We need the following terminology. We say a family of locally constant functions on \( \mathbb{Z}/2\mathbb{Z} \) is uniformly locally constant (ULC), if there exists a divisor \( l \) of \( S \) such that for every \( f \) in the family, we have \( f(x+l) = f(x) \) for all \( x \in \mathbb{Z}/2\mathbb{Z} \).

We define the space of smooth elements of the Bunce-Deddens algebra, \( B_S^\infty \), to be the space of elements in \( B_S \) whose Fourier coefficients are ULC and whose norms are RD. Using Fourier series, we see that those conditions can be written as:

\[ B_S^\infty = \left\{ b = \sum_{n \in \mathbb{Z}} V^n m_{f_n} : \| f_n \| \text{ is RD, there is an } l \mid S, V^l b V^{-l} = b \right\}. \]

It is immediate that \( B_S^\infty \) is indeed a nonempty subset of \( B_S \) and it was proved in [8] that \( B_S^\infty \) is a \( * \)-subalgebra of \( B_S \).

Let \( \delta_L : B_S^\infty \to B_S^\infty \) be given by \( \delta_L(b) = [L, b] \). This derivation is very fundamental below. We have the following simple relations:

\[ \delta_L(V^n) = nV^n \quad \text{and} \quad \delta_L(m_f) = 0. \]

This derivation is in particular used to define the following norms on \( B_S^\infty \) that capture the RD property of the Fourier coefficients of elements of \( B_S^\infty \). They are defined by

\[ \| b \|_P = \sum_{j=0}^{P} \binom{P}{j} \| \delta_L^j(b) \|. \]

The following proposition from [8] states the basic properties of the \( P \)-norms.

**Proposition 3.2.** Let \( b_1 \) and \( b_2 \) be in \( B_S^\infty \). Then

1. \( \| b_1 \|_{P+1} = \| b_1 \|_P + \| \delta_L(b_1) \|_P \) with \( \| b_1 \|_0 := \| b_1 \| ; \)
2. \( \| b_1 b_2 \|_P \leq \| b_1 \|_P \| b_2 \|_P ; \)
3. \( \| \delta_L(b_1) \|_P \leq \| b_1 \|_{P+1} . \)

It follows that we have the following useful way to describe elements in \( B_S^\infty \):

\[ B_S^\infty = \{ b \in B_S : \| b \|_M < \infty \text{ for every } M, \text{ there is an } l \mid S, V^l b V^{-l} = b \}. \]
3.3 Smooth Bunce-Deddens-Toeplitz algebras

Finally, following the similar considerations for the Toeplitz algebra in [9], we define the smooth Bunce-Deddens-Toeplitz algebra $A_S^\infty$ by

$$A_S^\infty = \{a = T(b) + c : b \in B_S^\infty, c \in K^\infty\} \subseteq A_S.$$  

Much like the short exact sequence for $A_S$ and $B_S$, these smooth subalgebras have the following related short exact sequence:

$$0 \to K^\infty \to A_S^\infty \to B_S^\infty \to 0.$$  

Thus, we can view the topology on $A_S^\infty$ as a vector space in the usual way, i.e.,

$$A_S^\infty \cong B_S^\infty \oplus K^\infty.$$  

This gives $A_S^\infty$ its LF topology; moreover, it is a complete topological vector space.

The Toeplitz map $T : B_S \to A_S$ can naturally be restricted to $B_S^\infty$ and considered as a map $T : B_S^\infty \to A_S^\infty$. In addition, the homomorphism $\tau$ can be restricted to $A_S^\infty$ and we have a homomorphism $\tau : A_S^\infty \to B_S^\infty$.

It is easy to verify on generators that we have

$$d_S(T(b)) = T(\delta_S(b)).$$  

As a consequence of continuity of $T$, this formula is true for all $b \in B_S^\infty$.

It remains to verify that $A_S^\infty$ is indeed a subalgebra of $A_S$. This follows from the following two propositions.

**Proposition 3.3.** Let $b$ be in $B_S^\infty$ and $c$ be in $K^\infty$. Then $T(b)c$ and $cT(b)$ are in $K^\infty$.

**Proof.** Because $T(b^*) = T(b)^*$, we only need to prove that $T(b)c$ is in $K^\infty$. Proceeding as in [9], we prove by induction on $M$ that we have the following estimate:

$$\|T(b)c\|_{M,N} \leq \|b\|_M \|c\|_{M,N}. \quad (3.1)$$  

The $M = 0$ case is immediate from the definition of the norms. The inductive step is

$$\|T(b)c\|_{M+1,N} = \|T(b)c\|_{M,N} + \|d_S(T(b))c + T(b)d_S(c)\|_{M,N}$$

$$\leq (\|b\|_M + \|\delta_S(b)\|_M)(\|c\|_{M,N} + \|d_S(c)\|_{M,N})$$

$$= \|b\|_{M+1} \|c\|_{M+1,N}.$$  

Notice also that, again proceeding as in [9], we can obtain the following inequality:

$$\|cT(b)\|_{M,N} \leq \|b\|_{M+N} \|c\|_{M,N}. \quad (3.2)$$  

This completes the proof. □

**Proposition 3.4.** Let $b_1$ and $b_2$ be smooth Bunce-Deddens elements. Then the following expression is a smooth compact element:

$$T(b_1)T(b_2) - T(b_1 b_2).$$  

**Proof.** We follow [9]. Let $b_1$ and $b_2$ be in $B_S^\infty$ with the following decompositions:

$$b_1 = b_1^+ + b_1^- = \sum_{n \geq 0} V^n m_n + \sum_{n < 0} m_n V^n \quad \text{and} \quad b_2 = b_2^+ + b_2^- = \sum_{n \geq 0} V^n m_n + \sum_{n < 0} m_n V^n,$$

where $\{\|m_n\|\}$ and $\{\|g_n\|\}$ are RD sequences and $\{f_n\}$ and $\{g_n\}$ are ULC. Since $T$ is linear, we only need to study the following differences:

$$T(b_1^+)T(b_2^+) - T(b_1^+ b_2^+), \quad T(b_1^-)T(b_2^-) - T(b_1^{-} b_2^-),$$

$$T(b_1^+)T(b_2^-) - T(b_1^+ b_2^-), \quad T(b_1^-)T(b_2^+) - T(b_1^{-} b_2^+).$$
First, we consider the following:

\[ T(b_1^+)T(b_2^+) - T(b_1^+b_2^+) = \sum_{m,n \geq 0} U^m M_{f_m} U^m M_{g_m} - \sum_{m,n \geq 0} T(V^n m_{f_n} V^n m_{g_m}) \]

\[ = \sum_{m,n \geq 0} U^{n+m} M_{f_m} v_m M_{g_m} - \sum_{m,n \geq 0} T(V^{n+m} m_{f_n} v_m m_{g_m}) \]

\[ = \sum_{m,n \geq 0} U^{n+m} M_{f_m} v_m M_{g_m} - \sum_{m,n \geq 0} T(V^{n+m}) M_{f_m} v_m M_{g_m}. \]

Since \( T(V^{n+m}) = U^{n+m} \), the above is zero. A similar argument can be made for \( T(b_1^+)T(b_2^+) - T(b_1^+b_2^+) \).

For the next difference, we have

\[ T(b_1)T(b_2^+) - T(b_1^+b_2^+) = \sum_{m \geq 0, n < 0} M_{f_m} (U^*)^{-m} U^m M_{g_m} - \sum_{m \geq 0, n < 0} M_{f_m} T(V^n V^m) M_{g_m}. \]

However, since \( T(V^{n+m}) = (U^*)^{-n} U^m \) for \( n < 0 \), this difference is also zero. Finally, for the last difference, we have

\[ C := T(b_1^+)T(b_2^+) - T(b_1^+b_2^+) = T(b_1^+) \sum_{m < 0} M_{g_m} (U^*)^{-m} - \sum_{m < 0} T(b_1^+ m_{g_m} V^m) \]

\[ = \sum_{m < 0} (T(b_1^+ m_{g_m}) (U^*)^{-m} - T(b_1^+ m_{g_m} V^m)) \]

\[ = - \sum_{m < 0} T(b_1^+ m_{g_m} V^m) P_{<m}, \]

where we used the following formula for \( m < 0 \):

\[ U^{-m} (U^*)^{-m} = I = -P_{<m}. \]

Clearly, \( C \) is compact but we still need to prove that it is smooth compact. To this end, we prove the \( \| \cdot \|_{M,N} \) norms of \( C \) are finite. A straightforward calculation gives

\[ d_{lg}^j(C) = - \sum_{m < 0} d_{lg}^j(T(b_1^+ m_{g_m} V^m) P_{<m}) = - \sum_{m < 0} T(\delta_1^j(b_1^+ m_{g_m} V^m)) P_{<m}. \]

Next, we estimate norms of \( C \) using \( \| P_{<m} \|_{0,N} = |m|^N \) to obtain

\[ \| d_{lg}^j(C) \|_{0,N} \leq \sum_{m < 0} \sum_{l=0}^{j} \left( \frac{j}{l} \right) |m|^{j-l-N} \| \delta_1^l(b_1^+) \| \| g_m \| \]

\[ \leq \sum_{m < 0} (1 + |m|)^{N+j} \left( \sum_{l=0}^{j} \left( \frac{j}{l} \right) \| \delta_1^l(b_1^+) \| \right) \| g_m \| \]

\[ = \sum_{m < 0} \| b_1^+ \|_j (1 + |m|)^{N+j} \| g_m \| \]

\[ \leq \text{const} \| b_1^+ \|_j \| b_2^+ \|_{N+j+2}. \]

Consequently, since \( b_1 \) and \( b_2 \) are in \( B_{2\infty}^\infty \), we get \( \| C \|_{M,N} < \infty \). This shows that \( T(b_1)T(b_2) - T(b_1 b_2) \) is smooth compact. A more careful analysis following \[9\] yields the following estimate:

\[ \| T(b_1)T(b_2) - T(b_1 b_2) \|_{M,N} \leq \text{const} \| b_1 \|_M \| b_2 \|_{N+M+2}. \]  \hspace{1cm} (3.3) \]

This completes the proof.
4 Stability of the smooth Bunce-Deddens-Toeplitz algebra

The purpose of this section is to establish stability of $A_S^\infty$ under both the holomorphic functional calculus and the smooth calculus of self-adjoint elements. It is well known that showing the former automatically implies that the K-theories of $A_S^\infty$ and $A_S$ coincide [2].

**Proposition 4.1.** The smooth Bunce-Deddens-Toeplitz algebra $A_S^\infty$ is closed under the holomorphic functional calculus.

**Proof.** Since $A_S^\infty$ is a complete topological vector space (see Subsection 3.3), it is enough to check that if $a \in A_S^\infty$ and is invertible in $A_S$, then $a^{-1} \in A_S^\infty$. Consequently, the Cauchy integral representation finishes the proof. To this end, let $a \in A_S^\infty$, and thus $a = T(b) + c$ with $b \in B_S^\infty$ and $c \in K^\infty$. Suppose that $a$ is invertible in $A_S$. Since $\tau$ is a homomorphism, $\tau(a) = b$ is invertible in $B_S^\infty$. It is proved in [8] that if $b \in B_S^\infty$ and is invertible, then $b^{-1} \in B_S^\infty$. Since $K$ is an ideal of $A_S$ and $\tau T$ is the identity map, it follows that

$$a^{-1} = T(b^{-1}) + c'$$

for some $c' \in K$. The proof will be completed if we can show that $c' \in K^\infty$. Notice that

$$c' = a^{-1} - T(b^{-1}) = a^{-1}(I - aT(b^{-1})) = a^{-1}(I - T(b) T(b^{-1}) - cT(b^{-1})).$$

From Propositions 3.3 and 3.4, we have that both $I - T(b) T(b^{-1})$ and $cT(b^{-1})$ are in $K^\infty$. Consequently, there is a $\tilde{c} \in K^\infty$ such that $c' = a^{-1} \tilde{c}$. It follows from the properties of norms on $K^\infty$ that

$$\|c'\|_{0,N} \leq \|a^{-1}\| \|\tilde{c}\|_{0,N} < \infty. \quad (4.1)$$

Computing $d_K$ on $c$, we have

$$d_K(c') = d_K(a^{-1}) \tilde{c} = -a^{-1} d_K(a) a^{-1} \tilde{c} + a^{-1} d_K(\tilde{c}).$$

Similar to the proof of Proposition 3.3, we have, inductively, for any $j$,

$$d^j_K(c') = \sum_i a_i b_i$$

is a finite sum with $a_i$ bounded and $b_i$ smooth compact. Using this and the estimate in (4.1), we see that $\|c'\|_{M,N}$ is finite for all $M$ and $N$. Thus $c' \in K^\infty$, completing the proof. \hfill $\Box$

To prove the closure under the calculus of self-adjoint elements, the approach used in [9] works in this setting as well. Hence, we need results regarding the growth of exponentials of elements of $B_S^\infty$ and $K^\infty$. For $K^\infty$, the exact result needed was proved in [9]. We state it here for convenience.

**Proposition 4.2.** Suppose that $c \in K^\infty$ is a self-adjoint smooth compact operator. Then we have an estimate

$$\|e^{\delta c}\|_{M,0} \leq \prod_{j=1}^M (1 + \|c\|_{j,0})^{2M-j}.$$ 

The second result needed is a minor adaptation of Proposition 3.5 in [9].

**Proposition 4.3.** If $b \in B_S^\infty$ is self-adjoint, then we have an estimate

$$\|e^{ib}\|_M \leq \prod_{j=1}^M (1 + \|b\|_{j})^{2M-j}.$$ 

**Proof.** For $M = 0$, notice that $\|e^{ib}\|_0 = 1$. Utilizing Proposition 3.2(1), we continue by induction,

$$\|e^{ib}\|_{M+1} = \|e^{ib}\|_M + \|\delta_L(e^{ib})\|_M.$$ 

Using

$$\delta_L(e^{ib}) = i \int_0^1 e^{i(1-t)b} \delta_L(b) e^{ibt} dt,$$
we have the following estimate for the inductive step:

\[ \| e^{ib} \|_{M+1} \leq \| e^{ib} \|_M + \int_0^1 \| e^{i(1-t)b} \|_{M} \| \delta_L(b) \|_M \| e^{ib} \|_M dt \]

\[ \leq \prod_{j=1}^M (1 + \| b \|_j)^{2M-j} + \left[ \prod_{j=1}^M (1 + \| b \|_j)^{2M-j} \right]^2 \| \delta_L(b) \|_M. \]

Since \( \| \delta_L(b) \|_M \leq \| b \|_{M+1} \), we have

\[ \| e^{ib} \|_{M+1} \leq \prod_{j=1}^M (1 + \| b \|_j)^{2M-j} \left( 1 + \prod_{j=1}^M (1 + \| b \|_j)^{2M-j} \| b \|_{M+1} \right) \]

\[ \leq \prod_{j=1}^M (1 + \| b \|_j)^{2M-j} \prod_{j=1}^M (1 + \| b \|_j)^{2M-j} (1 + \| b \|_{M+1}) \]

\[ = \prod_{j=1}^{M+1} (1 + \| b \|_j)^{2M+1-j}. \]

This establishes the inductive step and finishes the proof.

\[ \square \]

**Theorem 4.4.** The smooth Bunce-Deddens-Toeplitz algebra \( A_\infty^\infty \) is closed under the smooth functional calculus of self-adjoint elements.

**Proof.** We need to prove that given a self-adjoint element \( a \) of \( A_\infty^\infty \) and a smooth function \( f(x) \) defined on an open neighborhood of the spectrum \( \sigma(a) \) of \( a \), we find that \( f(a) \) is in \( A_\infty^\infty \). It is without loss of generality to assume that \( f(x) \) is smooth on \( \mathbb{R} \) and is \( L \)-periodic: \( f(x + L) = f(x) \) for some \( L \). Then \( f(x) \) admits a Fourier series representation with rapid decay coefficients \( \{ f_n \} \), and hence,

\[ f(a) = \sum_{n \in \mathbb{Z}} f_n e^{2\pi ina/L} \]

for a self-adjoint \( a = T(b) + c \in A_\infty^\infty \). Thus, it remains to establish at most polynomial growth in \( n \) of norms \( \| e^{2\pi ina/L} \|_{M,N} \).

Notice that \( \tau(e^{2\pi ina/L}) \) in \( B_\infty^\infty \) is \( e^{2\pi inb/L} \), which indeed grows at most polynomially in \( n \), by Proposition 4.3. Thus, we only need to show that \( \| \cdot \|_{M,N} \) of the difference

\[ e^{2\pi in(T(b)+c)/L} - T(e^{2\pi inb/L}) \in K^\infty \]

are at most polynomially growing in \( n \).

To analyze the above, we use a version of the Duhamel’s formula, i.e.,

\[ e^{i(T(b)+c)} - T(e^{ib}) = \int_0^1 \frac{d}{dt} e^{i(T(b)+c)} T(e^{i(1-t)b}) dt \]

\[ = i \int_0^1 e^{i(T(b)+c)} T(e^{i(1-t)b}) dt + i \int_0^1 e^{i(T(b)+c)} [T(b)T(e^{i(1-t)b}) - T(be^{i(1-t)b})] dt. \]

Employing Proposition 3.1, we can estimate the norms as follows:

\[ \| e^{i(T(b)+c)} - T(e^{ib}) \|_{M,N} \leq \int_0^1 \| e^{i(T(b)+c)} \|_{M,0} \| c T(e^{i(1-t)b}) \|_{M,N} dt \]

\[ + \int_0^1 \| e^{i(T(b)+c)} \|_{M,0} \| T(b)T(e^{i(1-t)b}) - T(be^{i(1-t)b}) \|_{M,N} dt. \]

All the terms above can now be estimated by using (3.2) and (3.3), as well as Propositions 4.2 and 4.3.
We obtain the following bounds:
\[
\|e^{(T(b)+c)} - T(d^b)\|_{M,N} \leq \prod_{j=1}^{M} (1 + \|b\|_j + \|c\|_{j,0})^{2M-j} \|c\|_{M,N} \prod_{j=1}^{M+N} (1 + \|b\|_j)^{2M+N-j} \\
+ \text{const} \prod_{j=1}^{M} (1 + \|b\|_j + \|c\|_{j,0})^{2M-j} \|b\|_M \prod_{j=1}^{M+N+2} (1 + \|b\|_j)^{2M+N+2-j}.
\]

Clearly, those estimates establish the desired at most polynomial growth, finishing the proof.

\[\square\]

5 Classification of derivations

We begin with recalling the basic concepts from [10]. Let \( A \) be a complete locally compact topological algebra and let \( d : A \to A \) be a continuous derivation on \( A \). Suppose that there is a continuous one-parameter family of automorphisms \( \rho_\theta : A \to A \) of \( A \) and \( \theta \in \mathbb{R}/\mathbb{Z} \).

Given \( n \in \mathbb{Z} \), a continuous derivation \( d : A \to A \) is said to be an \( n \)-covariant derivation if the relation
\[
\rho_\theta^{-1} d\rho_\theta(a) = e^{-2\pi i\theta} d(a)
\]
holds for all \( \theta \). When \( n = 0 \), we say the derivation is invariant. In this definition, \( A \) could be any of the following algebras: \( A_\infty^\infty \), \( B_\infty^\infty \) or \( K^\infty \) with the appropriate one-parameter family of automorphisms \( \rho_\theta^K \) or \( \rho_\theta^K \). With this definition, we point out that \( \delta_L : B_\infty^\infty \to B_\infty^\infty \) is an invariant continuous derivation as \( \delta_L : A_\infty^\infty \to A_\infty^\infty \) and \( \delta_k : K^\infty \to K^\infty \).

If \( d \) is a continuous derivation on \( A \), the \( n \)-th Fourier component of \( d \) is defined as
\[
d_n(a) = \int_0^1 e^{2\pi i\theta} \rho_\theta^{-1} d\rho_\theta(a) d\theta.
\]

We have the following simple observation [10].

**Proposition 5.1.** With the above notation, the \( n \)-th Fourier component \( d_n : A_\infty^\infty \to A_\infty^\infty \) is a continuous \( n \)-covariant derivation.

To classify continuous derivations on \( A_\infty^\infty \), we follow the strategy from [10]. We use the classification of derivations on \( B_\infty^\infty \) from [8] and show how to lift derivations from \( B_\infty \to A_\infty \). We handle the remaining derivations, those with ranges in \( K^\infty \), by using the Fourier decomposition components. This is the heart of the argument and will be described next.

Let \( A_\infty \subseteq A_\infty^\infty \) be the subspace of \( A_\infty^\infty \) consisting of elements \( a = T(b) + c \) such that \( b \) has only finitely many non-zero Fourier components and \( c \) has only finitely many non-zero matrix coefficients (in the standard basis). It was observed in [10] that \( A_\infty \) is a dense subalgebra of \( A_\infty^\infty \). In turn, we note that it is also a dense subalgebra of \( A_\infty^\infty \).

**Theorem 5.2.** If \( d : A_\infty^\infty \to K^\infty \) is a continuous derivation, then there is a \( c \in K^\infty \) such that \( d(a) = [c, a] \) for every \( a \in A_\infty^\infty \). In particular, \( d \) is an inner derivation.

**Proof.** Let \( d : A_\infty^\infty \to K^\infty \) be a continuous derivation. Let \( d_n \) be the \( n \)-th Fourier component of \( d \). From Proposition 5.1, \( d_n \)'s are \( n \)-covariant derivations and \( d_n : A_\infty^\infty \to K^\infty \). We only consider the case \( n \geq 0 \) as the case \( n < 0 \) can be treated similarly. All the \( n \)-covariant derivations \( d_n : A_\infty \to A_\infty \) were classified in [10]. Thus, we know that there exists a sequence \( \{\beta_n(k)\} \) possibly unbounded in \( k \), such that
\[
d_n(a) = [U^n \beta_n(k), a]
\]
for any \( a \in A_\infty \). We are requiring here the range of \( d \) to belong to \( K^\infty \), which places restrictions on \( \{\beta_n(k)\} \).

Let \( \chi \) be a character on \( \mathbb{Z}/SZ \) and since \( d_n(a) \in K^\infty \) for any \( a \in A_\infty^\infty \), we have
\[
\begin{cases}
  d_n(U) = U^{n+1} \beta_n(k) + I - \beta_n(0) = U^{n+1} \alpha_n(k) \in K^\infty & \text{for } n \geq 0, \\
  d_n(M_\chi) = U^n \beta_n(k) M_\chi (1 - \chi(n)) \in K^\infty & \text{for } n \geq 0.
\end{cases}
\]
Since for each $n > 0$, we can choose $\chi$ such that $\chi(n) \neq 1$, we have $\{\alpha_n(k)\}$ and $\{\beta_n(k)\}$ are RD in $k$ for every $n > 0$.

For $n = 0$, the above equation only implies that $\{\alpha_0(k)\}$ is RD in $k$. We have the following difference equation:

$$\alpha_n(k) = \beta_n(k + 1) - \beta_n(k).$$

This equation has a solution of the form

$$\beta_n(k) = -\sum_{r=k}^{\infty} \alpha_n(r).$$

It follows that since $\{\alpha_0(k)\}$ is RD in $k$, so is $\{\beta_0(k)\}$. Thus, $\{\beta_n(k)\}$ is RD for any $n$ and the formula (5.1) extends by continuity to any $a \in A_S^\infty$.

We want to establish that $\{\beta_n(k)\}$ is RD in both $n$ and $k$. Since $d_n(U) \in K^\infty$, we see that $\|d_n(U)\|_{M,N}$ are finite for all $M$ and $N$. So for any $N$ and $j$, there exists a constant $C_{j,N}$ (independent of $n$) such that

$$\|d_n^j(d_n(U))(I + K)^N\| \leq C_{j,N}.$$

On the other hand, consider the following calculation for $n \geq 0$:

$$d_n^j(d_n(U)) = d_n^j(U^{n+1}\alpha_n(K)) = (n + 1)^j U^{n+1}\alpha_n(K)$$

since $\alpha_n(K)$ is diagonal. Therefore, we have $(n + 1)^j\|\alpha_n(K)(I + K)^N\| \leq C_{j,N}$. However,

$$(n + 1)^j\|\alpha_n(K)(I + K)^N\| = (n + 1)^j \sup_k\{(1 + k)^N|\alpha_n(k)|\}.$$

It follows that $(1 + n)^j(1 + k)^N|\alpha_n(k)| \leq C_{j,N}$, and thus $\{\alpha_n(k)\}$ is RD in both $n$ and $k$. Consequently, by (5.2), $\{\beta_n(k)\}$ is RD in both $n$ and $k$. Therefore,

$$d(a) = \sum_{n \in \mathbb{Z}} d_n(a) = \sum_{n \geq 0} U^n \beta_n(K, a) + \sum_{n < 0} |\beta_n(K)(U^*)^{-n}, a|$$

$$= \left[\sum_{n \geq 0} U^n \beta_n(K) + \sum_{n < 0} \beta_n(K)(U^*)^{-n}, a\right] = [c, a],$$

where all the sums converge and $c \in K^\infty$. Thus $d$ is inner, completing the proof.

To analyze general derivations $d : A_S^\infty \to A_S^\infty$, we first notice the following proposition.

**Proposition 5.3.** Let $d : A_S^\infty \to A_S^\infty$ be a continuous derivation. Then $d(K^\infty) \subseteq K^\infty$.

**Proof.** Since $K^\infty$ is generated by the system of units $\{P_{ks}\}$ and $d$ is continuous, we only need to verify that $d(P_{ks})$ is in $K^\infty$. Since $P_{ks} = P_{kr}P_{rs}$, by the Leibniz rule we have

$$d(P_{ks}) = P_{kr}d(P_{rs}) + d(P_{kr})P_{rs}.$$

Since the right-hand side is clearly in $K^\infty$, the claim follows.

It follows from this proposition that any continuous derivation $d : A_S^\infty \to A_S^\infty$ defines a continuous derivation on $B_S^\infty$, which is isomorphic to the factor algebra $A_S^\infty/K^\infty$. We use this observation in the proof of the following main result of this section.

**Theorem 5.4.** Let $d : A_S^\infty \to A_S^\infty$ be any continuous derivation. Then there exist a constant $\gamma$, $b \in B_S^\infty$ and $c \in K^\infty$ such that $d = \gamma d_S + [T(b) + c, -].$

**Proof.** Let $d : A_S^\infty \to A_S^\infty$ be a continuous derivation and define a derivation $\delta : B_S^\infty \to B_S^\infty$ by

$$\delta(a + K^\infty) = d(a) + K^\infty.$$

In other words, $\delta$ is the class of $d$ in the factor algebra $A_S^\infty/K^\infty \cong B_S^\infty$. The continuity of $d$ implies the continuity of $\delta$. But all the continuous derivations $\delta : B_S^\infty \to B_S^\infty$ were classified in [8]. Therefore, by [8],
there exists a constant $\gamma$ such that $\delta = \gamma \delta_L + \tilde{\delta}$, where $\tilde{\delta}$ is inner. Thus, there exists a $b \in B^\infty_S$ such that $\tilde{\delta} = [b, \cdot]$.  

Next, notice that $[T(b), \cdot]$ is an inner derivation on $A_S^\infty$ whose class in $B^\infty_S$ is precisely $[b, \cdot]$. Define a derivation $\tilde{d} : A_S^\infty \to A_S^\infty$ by $\tilde{d} = d - \gamma d_L - [T(b), \cdot]$. Since the class of $d_L$ in $B^\infty_S$ is $\delta_L$, we have $\tilde{d} : A_S^\infty \to K^\infty$, and hence by Theorem 5.2, $\tilde{d} = [c, \cdot]$ for some $c \in K^\infty$. This concludes the proof. $
$

6 K-theory and K-homology

Since $K^\infty$, $A_S^\infty$ and $B^\infty_S$ are closed under the holomorphic functional calculus, each inclusion induces an isomorphism in the K-theory. Using this fact, along with the 6-term exact sequence [12] induced by the short exact sequence of smooth subalgebras, we compute the K-theory of $A_S^\infty$. Then we make use of the universal coefficient theorem [13] to compute the K-homology of $A_S$.

6.1 K-theory

Recall the short exact sequence  

$$0 \to K \to A_S \to B_S \to 0$$

of $C^*$-algebras. This induces the following 6-term exact sequence in the K-theory:

$$
\begin{array}{cccccc}
K_0(K) & \longrightarrow & K_0(A_S) & \longrightarrow & K_0(B_S) \\
\text{ind} & & & & \text{exp} \\
K_1(B_S) & \xrightarrow{\text{K}_1(\tau)} & K_1(A_S) & \longleftarrow & K_1(K).
\end{array}
$$

For details regarding the K-theory of $B_S$, please see [8]. Since the generating unitary $V$ in $B_S$ lifts to the partial isometry $U$, it follows that

$$\text{ind}([V]) = [I - U^*U]_0 - [I - UU^*]_0 = -[P_0],$$

which generates $K_0(K)$. Hence, the index map is an isomorphism. By exactness, it follows that $K_1(\tau)$ is the trivial map. Since $K_1(K) = 0$, by exactness $K_1(\tau)$ is also injective, and hence $K_1(A_S) = 0$. Since exp is trivial, by exactness $K_0(\tau)$ is surjective. But again, since ind is an isomorphism, it follows that the map $K_0(K) \to K_0(A_S)$ is trivial. Hence, $K_0(\tau)$ is injective as well. Using the computation done in [8], we have

$$K_0(A_S) \cong G_S, \quad \text{where } G_S = \{k/l \in \mathbb{Q} : k \in \mathbb{Z}, l \mid S\}.$$  

Due to the fact that the K-theories are stable under the holomorphic functional calculus, we have the following proposition.

**Proposition 6.1.** The K-theory of $A_S$ is given by

$$K_0(A_S) = K_0(A_S^\infty) \cong G_S \quad \text{and} \quad K_1(A_S) = K_1(A_S^\infty) \cong 0.$$

6.2 K-homology

The universal coefficient theorem of Rosenberg and Schochet [13] states that we have two exact sequences

$$0 \to \text{Ext}^1_2(K_1(A_S), \mathbb{Z}) \to K^0(A_S) \to \text{Hom}(K_0(A_S), \mathbb{Z}) \to 0$$

and

$$0 \to \text{Ext}^1_2(K_0(A_S), \mathbb{Z}) \to K^1(A_S) \to \text{Hom}(K_1(A_S), \mathbb{Z}) \to 0,$$

where in the above, we have used the identification $KK^i(A_S, \mathbb{C}) = K^i(A_S)$. From the first sequence, it is clear that $\text{Ext}^1_2(K_1(A_S), \mathbb{Z}) \cong 0$. In [8], it was shown that $\text{Hom}(K_0(A_S), \mathbb{Z}) \cong 0$. Hence, we have $K^0(A_S) = 0$. From the second sequence, it is immediate that

$$K^1(A_S) \cong \text{Ext}^1_2(K_0(A_S), \mathbb{Z}) \cong K^1(B_S),$$

where $\text{Ext}^1_2(\cdot, \mathbb{Z})$ is given by

$$\text{Ext}^1_2(\cdot, \mathbb{Z}) = \frac{\text{Hom}(\cdot, \mathbb{Z})}{\text{Im}([\cdot, \cdot])} = \frac{\text{Hom}(\cdot, \mathbb{Z})}{\text{Im}([\cdot, \cdot])}.$$
where the last isomorphism was derived in [8]. This group was computed in [8] to be isomorphic to \((\mathbb{Z}/S\mathbb{Z})/\mathbb{Z}\). This reference also contains an explicit description of the precise subgroup being modded out. In fact, this subgroup turns out to be the natural dense copy of \(\mathbb{Z} \subseteq \mathbb{Z}/S\mathbb{Z}\). We summarize the above computations in the following proposition.

**Proposition 6.2.** The K-homology of \(A_S\) is given by

\[
K^0(A_S) \cong 0 \quad \text{and} \quad K^1(A_S) \cong (\mathbb{Z}/S\mathbb{Z})/\mathbb{Z}.
\]

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