Influence of geometry variations on the gravitational focusing of timelike geodesic congruences

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We derive a set of equations describing the linear response of the convergence properties of a geodesic congruence to arbitrary geometry variations. It is a combination of equations describing the deviations from the standard Raychaudhuri-type equations due to the geodesic shifts and an equation describing the geodesic shifts due to the geometry variations. In this framework, the geometry variations, which can be chosen arbitrarily, serve as probes to investigate the gravitational contraction processes from various angles.

We apply the obtained framework to the case of conformal geometry variations, characterized by an arbitrary function \( f(x) \), and see that the formulas get simplified to a great extent. We investigate the response of the convergence properties of geodesics in the latest phase of gravitational contractions by restricting the class of conformal geometry variations to the one satisfying the strong energy condition. We then find out that in the final stage, \( f \) and \( D \cdot D f \) control the overall contraction behavior and that the contraction rate gets larger when \( f \) is negative and \( |f| \) is so large as to overwhelm \( |D \cdot D f| \). (Here \( D \cdot D \) is the Laplacian operator on the spatial hypersurfaces orthogonal to the geodesic congruence in concern.)

To get more concrete insights, we also apply the framework to the time-reversed Friedmann-Robertson-Walker model as the simplest case of the singularity formations.

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I. INTRODUCTION

It has been known that the spacetime manifolds with high symmetries often contain singularities. Typical examples are the initial big-bang singularity in the spatially homogeneous and isotropic universe models and the curvature singularities in the black-hole solutions. The singularity theorems [1–4] have shown that, however, the spacetime singularities do not result from high symmetries, but are quite general features of spacetimes satisfying reasonable physical conditions, such as the energy condition and no closed timelike curves.

Though the singularity theorems show the generality of singularities in physically reasonable spacetimes, it is also true that their statements are too universal and general to get detailed information on gravitational collapses.

Among these processes needed to be clarified, the black-hole formations are especially important ones. Though the spacetime structures after black-hole formations are quite well-understood by the uniqueness theorem for the Kerr solution and its related theorems [2, 3], little is clarified about the black-hole formation processes themselves. Indeed, the cosmic censorship hypothesis [5, 6] and the hoop conjecture [7], which are the two central conjectures for the black-hole formations, have not been satisfactorily proved so far even though there have been no physically reasonable model found manifestly contradicting with these conjectures [8]. One of the reasons for this situation might be the fact that there is no established framework for analytically describing the black-hole formation processes.

Here let us pay attention to the Raychaudhuri equation [2, 3, 9–11], which describes the focusing property of a given geodesic congruence. The equation indicates that, once the expansion \( \theta \) along a geodesic in the congruence gets negative, it approaches to \( -\infty \) within some finite proper-time along the geodesic provided that the strong energy condition is satisfied (see Sec.VI). This phenomenon signals the occurrence of the conjugate point (the focal point) in the future which in turn implies the singularity (in the sense of the timelike geodesic incompleteness) in the future within a finite proper-time provided some other conditions are also satisfied [2–4].

Considering the above fact, we here choose the strategy to pay attention to the convergence properties of the geodesic congruence as the starting point for the analytical description of the gravitational contractions. More specifically, we here aim at constructing a theoretical framework describing how the convergence properties of a given time-like geodesic congruence are influenced by the slight variation in geometrical properties around the geodesic congruence. In other words, we study the linear response of the convergence properties of the geodesic congruence to the arbitrary geometry variations. The origins of the geometry variations are not specified and they can either be real physical processes or virtual displacements. Since the geometry variations can be arbitrarily chosen by hand, then, we might be able to use them as probes to investigate the gravitational contraction processes from various angles.

We now show the plan for constructing the framework. The outline of the physical process in question is as follows: When the geometry is varied, geodesics are also shifted accordingly, which in turn changes the convergence properties of the geodesics. The construction of the framework, thus, naturally consists of two steps. As the first step, we establish

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a key equation relating the geometry variation to the geodesic shift. As the second step, then, we investigate the variations in the convergence properties of the geodesic congruence caused by these geodesic shifts. Combining these two steps, we can construct a framework which describes the changes in the convergence properties of the geodesic congruence due to the geometry variations.

One key point of the above plan resides in the first step. Since the geodesic is the integral curve of a tangent vector field, it is a global object, which clearly causes some difficulty in pursuing our plan. We shall tackle this problem by introducing a suitable vector describing the geodesic shift. Furthermore, it turns out that using solely the component of this geodesic shift vector orthogonal to the geodesic is essential, making the expressions much simpler as well as giving much more transparent interpretations of the equations. In this way, we shall find out the key equation.

Another key point is to introduce a 1-parameter family of geometries on a fixed manifold $\mathcal{M}$, by means of which we can mathematically handle the geometry variations and derive a set of equations describing their influence on the convergence properties of a given geodesic congruence.

After establishing general formulas, we shall then pay special attention to the case of the conformal geometry variations. This case not only makes all the expressions much simpler, but also is quite important for several applications. We shall analyze the influence on the convergence properties of geodesics in the final phase of gravitational contractions by restricting the class of conformal geometry variations to the one satisfying the strong energy condition. We then find out that, when $f(s)$ is the function characterizing the conformal geometry variation, $f$ and $D \cdot D f$ control the overall contraction behavior in the final stage (where $D \cdot D$ is the Laplacian operator induced on the spatial hypersurfaces orthogonal to the timelike geodesic congruence in question), and that the contraction rate gets larger when $f$ is negative and $|f|$ is so large as to overwhelm $|D \cdot D f|$. This paper is organized as follows. In Sec.II, we shall review the established results regarding the deformation properties of timelike geodesic congruences needed in the paper. The basic quantities of our analysis - the expansion $\Theta$, the shear $\sigma_{ab}$, the twist $\omega_{ab}$, and the Raychaudhuri-type equations shall be introduced here. This section is also served to fix the notations and the definitions of the terms adopted throughout the paper. The main parts of the paper are Sec.III and Sec.IV. In Sec.III, we shall derive a key equation describing the geodesic shift caused by a geometry variation. In Sec.IV, then, we shall derive a system of equations describing changes in the deformation properties of the geodesic congruences due to geometry variations. In Sec.V, we shall focus on the case of the conformal variations and shall see all the formulas reduce to much simpler counterparts in this case. We shall then apply our framework to one concrete, explicitly analyzable model; the conformal variations applied to the time-reversed Friedmann-Robertson-Walker model. Based on the results of the previous section, we shall analyze in Sec.VI the changes in the contraction properties due to the conformal geometry variations in some detail. Section VII is devoted for the summary and several discussions.

II. BASIC FORMULAS

A. Timelike geodesic congruences

Let $(\mathcal{M}, g)$ be a smooth $n$-dimensional spacetime manifold with signature $(- + + + + +)$. Let $\nabla_a$ be the standard covariant derivative on $(\mathcal{M}, g)$, satisfying the metricity condition $(\nabla_a g_{bc} = 0)$ and the torsion-free condition $(\nabla_a \nabla_b - \nabla_b \nabla_a f = 0$ for any smooth scalar function $f$).

From now on, any timelike geodesic $\gamma$ is assumed to be affine-parametrized by the proper-time $\tau$ along it. Thus the tangent vector of $\gamma$, $\xi^a(\tau) \equiv (\partial/\partial \tau)^a$, satisfies [12]

$$\xi^b \nabla_b \xi^a = 0 \quad \text{with} \quad \xi^a \xi_a = -1 . \quad (1)$$

Let us take a timelike geodesic congruence $\mathcal{C}$ over an open set $\Omega$ in $\mathcal{M}$; it means that we consider some family $\mathcal{C}$ of timelike geodesics among which there is one and only one geodesic passing through any given point $p$ in $\Omega$.

We note that, for a given $\Omega$, one can choose continuously infinite number of timelike geodesic congruences over $\Omega$ as is easily seen for the case of the Minkowski spacetime for $(\mathcal{M}, g)$.

For a fixed timelike geodesic congruence $\mathcal{C}$, let $\{\gamma_s(\tau)\} \in \{s \in \mathcal{s}_1, s_2\}$ be a smooth 1-parameter family of geodesics in $\mathcal{C}$: Then $\{\gamma_s(\tau)\} \in \{s \in \mathcal{s}_1, s_2\}$ forms a smooth 2-dimensional submanifold embedded in $\Omega$ with $\{s, \tau\}$ providing a system of smooth coordinates on it [13]. By suitable parameterizations w.r.t. (respect to) $s$ and suitable synchronizations w.r.t. $\tau$ (i.e., choosing suitably the $\tau = 0$ point on each member of the family), one can assume that the tangent vector $\xi^a_s(\tau)$ of $\gamma_s(\tau)$ and the deviation vector $\eta^a_s(\tau)$, defined by $\eta^a_s(\tau) \equiv (\partial/\partial s)^a$, are orthogonal to each other for a fixed $s$. One can thus assume that geodesics in the 1-parameter family $\{\gamma_s(\tau)\} \in \{s \in \mathcal{s}_1, s_2\}$ are synchronized to all the cases $\tau = const$ curves drawn within $\{\gamma_s(\tau)\} \in \{s \in \mathcal{s}_1, s_2\}$ are orthogonal to the geodesics themselves.

The deviation vector $\eta^a(\tau)$ along a fixed geodesic $\gamma$ is measuring the deviation of the nearby geodesics among the family from being parallel to $\gamma$.

Here we note that, for a fixed geodesic in the family, the pseudo-norm of $\eta^a, \eta^a \eta_a$, is not constant along the geodesic in general, contrary to the case of $\xi^a(\tau)$ (Eq.(1)). This fact is important since it allows us to describe the convergence properties of the geodesic congruence in terms of $\eta^a$ (see the arguments after Eq.(6)).

B. Expansion $\Theta$, shear $\sigma_{ab}$ and twist $\omega_{ab}$

Let $\gamma$ be some timelike geodesic and $\xi^a$ be its tangent vector as before. Then the symmetric covariant tensor $h_{ab}$ defined by

$$h_{ab} \equiv g_{ab} + \xi_a \xi_b , \quad (2)$$
satisfies $h_{ab} \xi^b = 0$ so that it is regarded as a spatial tensor in the following sense.

Let $T_p \mathcal{M}$ be the tangent space at a point $p$ in $\gamma$. Let $T_p \mathcal{M}^\perp$ be the $(n-1)$-dimensional subspace of $T_p \mathcal{M}$ consisting of all the vectors orthogonal to $\xi^a$. Then $T_p \mathcal{M}^\perp$ represents all the vectors in the spatial directions for an observer moving along $\gamma$, so that vectors and tensors constructed by $T_p \mathcal{M}^\perp$ are regarded as spatial objects for the observer. In particular, when $h_{ab}$ in Eq. (2) is regarded as restricted on $T_p \mathcal{M}^\perp$, the former plays the role of the spatial, positive definite induced metric on the latter.

Furthermore, the $(1,1)$-tensor $h_{ab}$, obtained by raising one index of $h_{ab}$ by $g^{ab}$, satisfies $h_{ab} h^c_b = \delta^c_a$, playing the role of the projection operator

$$ h : T_p \mathcal{M} \rightarrow T_p \mathcal{M}^\perp. \quad (3) $$

Now, for a given 1-parameter family $\{ \gamma_t \}_{t \in (s_1,s_2)}$ in a timelike geodesic congruence $\mathcal{C}$, we can estimate $\frac{d}{d\tau} \eta^a \equiv \nabla_\gamma \eta^a$, the rate of change in the deviation vector $\eta^a$ along a geodesic $\gamma (\equiv \gamma_s = 0)$ in $\{ \gamma_t \}_{t \in (s_1,s_2)}$. One can easily get [3]

$$ \frac{d}{d\tau} \eta^a = B^a_{\ b} \eta^b, \quad (4) $$

where we have introduced

$$ B_{ab} \equiv \nabla_b \xi^a. \quad (5) $$

To get Eq. (4), we have used $[\xi, \eta]^a \equiv \nabla_\xi \eta^a - \nabla_\eta \xi^a = 0$, which follows due to the fact that $\xi^a$ and $\eta^a$ are vectors associated with the coordinate functions $\tau$ and $s$, respectively. Here we should note the positions of indices $a$ and $b$ in the definition of $B_{ab}$, describing $\frac{d}{d\tau} \eta^a$ as a linear transformation in the conventional form as the R.H.S. (right-hand side) of Eq. (4). We also note that, even though we have chosen the 1-parameter family $\{ \gamma_t \}_{t \in (s_1,s_2)}$ in $\mathcal{C}$ for introducing $B_{ab}$, the latter describes the deformation properties of $\mathcal{C}$ as a whole independent of the choice of the 1-parameter family $\{ \gamma_t \}_{t \in (s_1,s_2)}$.

It is important that $B_{ab}$ is a spatial tensor, satisfying

$$ B_{ab} \xi^b = 0 , \quad \xi^a B_{ab} = 0 , \quad (6) $$
as is shown with the help of Eq. (1).

The tensor $B_{ab}$ plays the central role in our analysis below. It is convenient to decompose $B_{ab}$ into the following form;

$$ B_{ab} = \frac{1}{n-1} \theta h_{ab} + \sigma_{ab} + \omega_{ab}. \quad (7) $$

Here the first term on the R.H.S. is the trace-part of $B_{ab}$ with $\theta \equiv B^{a}{^a}$; the second and the third terms are both trace-free with the former and the latter being symmetric and anti-symmetric in the indices, respectively. Accordingly it follows that $\sigma_{ba} = \sigma_{ab}$ with $\sigma_{a}^{a} = 0$ and $\omega_{ba} = -\omega_{ab}$.

It is conventional to call $\theta$, $\sigma_{ab}$ and $\omega_{ab}$ the expansion, the shear and the twist, respectively.

Due to Eq. (4), which is in the standard form describing deformations in the linear transformations, one can convince oneself of the interpretations for $\theta$, $\sigma_{ab}$ and $\omega_{ab}$. For a given timelike geodesic congruence $\mathcal{C}$ (synchronized as discussed in Sec. II A), the set $\mathcal{T}_s$ of the points with the affine-parameter $\tau$ forms an $(n-1)$-dimensional section of $\mathcal{C}$. When comparing $\mathcal{T}_s$ and $\mathcal{T}_s + \Delta \tau$, then, $\theta$, $\sigma_{ab}$ and $\omega_{ab}$ are interpreted as the rate of the dilation of the $(n-1)$-volume, the shape-change and the rotation, respectively, averaged during the time-interval $\Delta \tau$.

By the Frobenius’s theorem, $\omega_{ab} = 0$ holds iff the timelike geodesic congruence $\mathcal{C}$ is hypersurface orthogonal, i.e. the whole of $\mathcal{C}$ can be foliated by smooth $(n-1)$-dimensional spatial sections orthogonal to every timelike geodesic contained in $\mathcal{C}$ [3]. In such cases, $B_{ab}$ reduces to a spatial symmetric tensor, coinciding with the extrinsic curvature $K_{ab}$ for the smooth spatial sections orthogonal to all timelike geodesics in $\mathcal{C}$. At the same time, $\omega_{ab} = 0$ holds iff the spatial covariant derivative $D_a$ is torsion-free (see Eq. (A12) and arguments following it in Appendix A). Here $D_a$ is the spatial derivative operator on $T_p \mathcal{M}^\perp$ induced from $\nabla_a$ through the projection $h$ (Eq. (3)), defined as

$$ D_a T^{\nu^r}_{\nu^s} : = h_{a \nu} T^{\nu^r}_{\nu^s} \cdots \nabla_p T^{\nu^r}_{\nu^s}. \quad (7) $$

for any tensor $T^{b^r}_{b^s}$.

### C. Notations

At this point, let us summarize the notations employed below.

(i) We often adopt the standard notations for the products among matrices and vectors when they save indices. For instance, second-rank tensors $A^a_b$ and $B^a_b$ can be regarded as matrices. Then

$$ (AB)^a_b \equiv A^a_c B^c_b , \quad (Au)^a \equiv A^a_b u^b. $$

We also allow the remaining indices to be raised or lowered freely as usual tensorial indices. For instance,

$$ (AB)_{ab} \equiv A_{ac} B^c_b. $$

(ii) For vectors $u^a$ and $v^a$, we define

$$ u \cdot v \equiv u^a v^a. $$

Similarly, for tensors $\alpha_{ab}$ and $\beta_{ab}$, we define (note the index-positions)

$$ (\alpha \cdot \beta)_{ab} \equiv \alpha_{ac} \beta^c_b, $$

and

$$ \alpha \cdot \beta \equiv (\alpha \cdot \beta)^a_a \equiv \alpha_{ab} \beta^{ab}. $$

Note the difference between $(\alpha \cdot \beta)_{ab}$ defined here and $(\alpha \beta)_{ab} \equiv \alpha_{ac} \beta_{cb}$ defined in (i).
(iii) Overlines (underlines) attached to more than two indices indicate symmetrization (anti-symmetrization) of these indices. For instance,
\[ \alpha_{\underline{ab}} = \frac{1}{2} (\alpha_{ab} + \alpha_{ba}) , \quad \alpha_{\underline{a}bc} = \frac{1}{2} (\alpha_{abc} + \alpha_{bca}) , \]
\[ \beta_{\underline{ab}} = \frac{1}{2} (\beta_{ab} - \beta_{ba}) , \quad \beta_{\underline{ abc}} = \frac{1}{2} (\beta_{abc} - \beta_{bac}) . \]

(Anti-)symmetrizations are operated prior to contractions when they appear simultaneously. For instance,
\[ g^{\underline{ac}}(\gamma B)^{\underline{d}}_{\underline{ab}} = \frac{1}{2} g^{\underline{ac}} (\gamma c d B^d - \gamma d B^c) . \]

(iv) Any index \( a \) is replaced by a dot ("·") when it is contracted with a geodesic tangent vector \( \xi^a \). For instance,
\[ u_a \equiv u_{\xi a} \xi^a , \quad \gamma^a \equiv \gamma_{\xi a} \xi^a , \quad \gamma . \equiv \gamma_{\xi a} \xi^a \xi^b . \]

When no confusions are caused, these dots may often be omitted. For instance, \( \gamma^a \) can be used as a shorthand for \( \gamma^a \xi^a \) provided that it is obvious in the context.

(v) A second rank spatial tensor with the tilde-symbol ("\( \tilde{\cdot} \)) indicates its trace-free part. For instance, when \( \alpha_{ab} \) is a spatial tensor,
\[ \alpha_{\tilde{ab}} \equiv \alpha_{ab} - \frac{1}{n-1} \alpha^c h_{ab} . \]

Furthermore, \( (\alpha \cdot \alpha)_{ab} \) is also a spatial tensor. Then,
\[ \tilde{(\alpha \cdot \alpha)}_{ab} \equiv (\alpha \cdot \alpha)_{ab} - \frac{1}{n-1} (\alpha \cdot \alpha) h_{ab} . \]

(vi) The projection map Eq.(3) induces maps from tensors to their corresponding spatial tensors. These maps are denoted by the “underline” symbol attached to the main letter representing the mapped object. For instance,
\[ \underline{\gamma} \equiv h_a^b v_b , \quad \underline{\gamma}_{\underline{ab}} \equiv h_a^c h_b^d \gamma_{cd} . \]

We also note that, for instance, \( \tilde{R}_{\xi a b} \) unambiguously indicates the trace-free part of the spatial projection of \( R_{\xi ab} \), noting that "\( \tilde{\cdot} \)" is meaningful only for the spatial objects ((v)).

D. Raychaudhuri-type equations

Now, to estimate the rate of change of \( B_{ab} \) along a geodesic, one takes its derivative w.r.t. \( \tau \). Noting that \( \frac{d}{d\tau} = \nabla_\xi \) and taking into account Eq.(1), it is then straightforward to get
\[ \frac{d}{d\tau} B_{ab} = - (BB)_{ab} - R_{\xi a b} , \]
where we adopt the standard definitions for the curvatures[3],
\[ (\nabla_a \nabla_b - \nabla_b \nabla_a) u_c = R_{abc} d u_a \quad \text{for} \forall u^a , \]
\[ R_{ab} = R_{\xi a b} , \quad R = R^a \]
We note that \( \frac{d}{d\tau} B_{ab} \) is a spatial tensor as is seen by Eq.(8). In general, \( \frac{d}{d\tau} A_{abc} \) is spatial when \( A_{abc} \) is spatial, which is easily shown as
\[ \xi^a \frac{d}{d\tau} A_{abc} = \frac{d}{d\tau} (\xi^a A_{abc}) = 0 . \]

By decomposing the both sides of Eq.(8) into the trace-part and the trace-free part, and by decomposing the latter into the symmetric and anti-symmetric parts, we finally obtain
\[ \frac{d\theta}{d\tau} = - \frac{1}{n-1} \theta^2 - \sigma \cdot \sigma + \omega \cdot \omega - R_{\xi} , \]
\[ \frac{d\sigma_{ab}}{d\tau} = - \frac{2}{n-1} \theta \sigma_{ab} - (\sigma \cdot \sigma)_{ab} + (\omega \cdot \omega)_{ab} - \tilde{R}_{\xi a b} , \]
\[ \frac{d\omega_{ab}}{d\tau} = - \frac{2}{n-1} \theta \omega_{ab} - 2 (\omega \cdot \sigma)_{ab} , \]
where \( \tilde{R}_{\xi a b} \) is the trace-free part of \( R_{\xi a b} \) regarded as a second-rank spatial tensor (see (iv) and (v) in Sec.II.C). Furthermore, it is easy to see that \( \tilde{R}_{\xi a b} \) is expressed as
\[ \tilde{R}_{\xi a b} = C_{\xi a b} - \frac{1}{n-2} \tilde{R}_{\xi} , \]
where \( C_{\xi a b} \) is the Weyl tensor.

The first equation in Eq.(10) is often called the Raychaudhuri equation [9–11], which plays the central role for proving the singularity theorems. For brevity let us call the three equations in Eq.(10) the Raychaudhuri-type equations.

III. DEVIATIONS OF RAYCHAUDHURI-TYPE EQUATIONS DUE TO GEOMETRICAL VARIATIONS

A. One-parameter family of geometries on \( \mathcal{M} \)

Let us consider a 1-parameter family of \( n \)-dimensional spacetime geometries on \( \mathcal{M} \). \( \{ (\mathcal{M}, g^{(\lambda)}) \} \lambda \in \mathbb{A} \). Let \( V^{(\lambda)} \) be the covariant derivative compatible with \( g^{(\lambda)} \).

For some quantity \( A^{(\lambda)} \) regarded as a smooth function of \( \lambda \), let \( \tilde{A} \) indicate the derivative of \( A^{(\lambda)} \) w.r.t. \( \lambda \) at \( \lambda = 0 \) and call the \( \circ \)-derivative of \( A^{(\lambda)} \) for brevity.

For notational simplicity, we shall write just \( A \) for \( A^{(0)} \), such as \( V \) and \( g_{ab} \) for \( V^{(0)} \) and \( g_{ab}^{(0)} \), respectively.

We here introduce two quantities in connection with the \( \circ \)-derivative for later analysis,
\[ \gamma_{ab} \equiv g_{ab} , \quad \chi^a \equiv \xi^a . \]

The former represents the geometrical variation while the latter describes the shift of a geodesic due to the geometrical variation.

By taking the \( \circ \)-derivative on both sides of \( g_{ab} \delta^{bc}_{\xi a} = \delta^c_a \) and \( \xi^a = g_{ab} \xi^b \), we also get
\[ \delta^{ab}_{\circ} = - \nabla^{ab} , \quad \xi^a = \chi^a + \gamma^a . \]
Taking the $o$-derivative on both sides of $\xi^a \xi_a = -1$, we get a useful formula

$$\xi \cdot \chi = -\frac{1}{2} \gamma \, ,$$

(14)

where Eq.(12) and Eq.(13) have been used.

Now, according to general properties of covariant derivatives [3], for any vector $w_a$, the difference between $\nabla_a w_b$ and $\nabla^a \nabla_b w_b$ should be represented as $g^c_{ab}(\lambda) w_c$ with $g^c_{ab}(\lambda)$ being some function of $\lambda$,

$$\nabla_a w_b = \nabla_a w_b - g^c_{ab}(\lambda) w_c \, .$$

(15)

Applying Eq.(15) to the identity $\nabla_a \xi^b(\lambda) \xi^c(\lambda) = 0$ and following the standard procedure, $g^c_{ab}(\lambda)$ can be expressed as

$$g^c_{ab}(\lambda) = \frac{1}{2} g^{cd}(\lambda) \left( \nabla_a b_d(\lambda) + \nabla_d b_a(\lambda) - \nabla_a b_d(\lambda) \right) \, .$$

(16)

Here $\nabla_a$ is the covariant derivative compatible to $g_{ab}$ so that it is independent of $\lambda$.

Taking the $o$-derivative on both size of Eq.(16), then, it follows that

$$g^c_{ab} \xi^a \xi^b = \frac{d}{d\tau} \gamma^c - \frac{1}{2} (\nabla_a \gamma_b) \xi^a \xi^b \, .$$

(17)

From Eq.(17), we get a useful formula

$$g^c_{ab} \xi^a \xi^b = \frac{d}{d\tau} \gamma^c - \frac{1}{2} (\nabla_a \gamma_b) \xi^a \xi^b \, .$$

(18)

**B. Key equation for geodesic shift due to geometrical variations**

When a spacetime geometry is varied, geodesics should be shifted accordingly. We now derive a key equation describing such a geodesic shift caused by geometry variations.

Consider a geodesic $\gamma^i$ in the spacetime $\mathscr{M}$. Its tangent vector, $\dot{\gamma}^a(\lambda)$, should satisfy

$$\nabla^a \dot{\gamma}^b(\lambda) \dot{\gamma}^a(\lambda) = 0 \, .$$

(19)

Taking the $o$-derivative on both sides of Eq.(19), we get

$$\chi^b B^a + \dot{\gamma}^b (\nabla_b \chi^a + g^c_{ab} \dot{\gamma}^c) = 0 \, ,$$

which along with Eq.(18) yields

$$\frac{d}{d\tau} \chi^a + B^a \chi^b = -\frac{d}{d\tau} \gamma^a + \frac{1}{2} (\nabla^a \gamma_b) \xi^b \xi^c \, .$$

(20)

What we are concerned with is, however, the deviation of the geodesic from the original geodesic $\gamma$ caused by geometry variations $\gamma_{ab}$, so that it is more appropriate to use $\chi^a$, the component of $\chi^a$ orthogonal to $\xi^a$, rather than $\dot{\gamma}^a$ itself. Indeed, it turns out that the formulas below become much simpler and more transparent in meaning in terms of $\chi^a$ [14]. On the other hand, the component $\chi^a_{\parallel}$ which is parallel to $\xi_a^a$ does not shift $\gamma$, so that it may be regarded as the “gauge-freedom” in the present description. In effect, we fix the gauge-freedom by $\chi^{a}_{\parallel} = 0$.

Let us then introduce

$$v^a \equiv \chi^a_{\parallel} = \chi^a \, ,$$

(21)

which satisfies with the help of Eq.(14)

$$v^a = \chi^a - \frac{1}{2} \gamma^a \xi^a \, ,$$

$$\chi \cdot v = \xi \cdot \frac{dv}{d\tau} = 0 \, .$$

(22)

Considering Eq.(12) along with Eq.(22), we note

$$g^c_{ab} = v_a + \gamma_a + \frac{1}{2} \gamma^a \xi^a \, .$$

(23)

Now Eq.(20) can be represented in terms of $v^a$ by means of Eq.(22);

$$\frac{d}{d\tau} v^a + B^a v^b = \frac{1}{2} D^a \gamma^a - \frac{d}{d\tau} \chi^a \, ,$$

(24)

where $D_a$ is the spatial derivative operator induced from $\nabla_a$ (Eq.(7)).

It is obvious that Eq.(24), the relation solely among spatial quantities, is much more desirable than Eq.(20).

Equation (24) is our key equation, telling us the linear response of the geodesic shift $v^a$ to the geometry variation $\gamma_{ab}$. Equation (24) is also expressed as

$$v^a = \frac{1}{2} D^a \gamma^a - \frac{d}{d\tau} \chi^a \, ,$$

(25)

with

$$v^a = \frac{d}{d\tau} B^a + B^a \chi^b \, .$$

(26)

Since $v^a$ (rather than $\xi_a^a$) contains the pure geometrical information on the geodesic shift, it should hold $v^a \equiv 0 \iff \gamma_{ab} \equiv 0$, so that $v^a B^a$ should be invertible. Thus one can formally solve Eq.(25) as

$$v^a = (L^{-1})^b \left( \frac{1}{2} D^a \gamma^a - \frac{d}{d\tau} \chi^a \right)^b \equiv \left[ \left( \frac{d}{d\tau} B^a + B^a \right)^{-1} \left( \frac{1}{2} D^a \gamma^a - \frac{d}{d\tau} \chi^a \right) \right]^a \, ,$$

(27)

which formally expresses the geodesic shift $v^a$ in terms of the geometry variation $\gamma_{ab}$.

**IV. CHANGES IN GEODESIC CONVERGENCE PROPERTIES DUE TO GEOMETRY VARIATIONS**

**A. Deviations of $\theta$, $\sigma_{ab}$, and $\omega_{ab}$ due to geometry variations**

Now taking the $o$-derivative of

$$B(ab(\lambda)) = \nabla_b \xi_a(\lambda) = \nabla_b \xi_a(\lambda) - g^c_{ab}(\lambda) \xi_c(\lambda)$$

we get after some straightforward calculations,

\[
B_{ab} = (\gamma B)_{\gamma a} + \frac{1}{2} \gamma \dot{B}_{ba} + \frac{1}{2} \frac{d}{d\tau} \gamma \dot{B}_{ab} + D_b \nu_a + (Bv)_a \xi_b + \xi_a (vB)_b + D_b \gamma_a + \xi_a (\gamma B)_a .
\] (28)

Here we note that, according to Sec.II.C, all the terms are unambiguously defined in shorthand notations. Namely, 

\[
(\gamma B)_{\nu a} = \gamma B^b_{\nu} , \quad (Bv)_a = B_{ac} \nu^c , \quad (vB)_b = \nu^c B_{cb} , \quad \xi_a (\gamma B)_b = \xi_a \gamma_{0d} \xi_d B_{eb} .
\]

Contrary to the case of \( \frac{d}{d\tau} B_{ab} \) (see Eq.(8)), \( \dot{B}_{ab} \) is not spatial any more as is seen in the second line on the R.H.S. of Eq.(28). This result turns out to be a reasonable one, repeating the similar argument after Eq.(9) along with \( \xi^a \neq 0 \).

Similarly we also get

\[
\dot{B}^a_b = -8^{ac}(\gamma B)_{cb} + \frac{1}{2} \gamma \dot{B}^a_b + \frac{1}{2} \frac{d}{d\tau} \gamma \dot{B}^a_b + D_b \nu^a + (Bv)^a \xi_b + \xi^a (vB)_b + 8^{ac} (D_b \gamma_a + \xi_a (\gamma B)_b) .
\] (29)

Here the difference between the first terms in Eq.(28) and Eq.(29) should be noted. (Note the signs and the (anti)symmetrization.)

By contracting both sides of Eq.(28) with \( \gamma \) along with simple manipulations, we get the formula for \( \dot{\gamma} \) as

\[
\dot{\gamma} = \frac{1}{2} \gamma^2 \theta + \frac{1}{2} \frac{d}{d\tau} \gamma^2 + D \cdot \nu ,
\] (30)

where \( \gamma \) is the shorthand notation for \( h_{ab} \gamma_{ab} \), the trace of \( \gamma_{ab} \). Equation (30) is one of our basic formulas describing changes in geodesic convergence properties caused by geometry variations.

To get similar basic formulas for \( \dot{\sigma}_{ab} \) and \( \dot{\omega}_{ab} \), some more considerations are helpful. Firstly we note that

\[
\dot{\sigma}_{ab} - \dot{\sigma}_{ba} = (\sigma_{ab} - \sigma_{ba}) = 0 ,
\]

\[
\dot{\omega}_{ab} + \dot{\omega}_{ba} = (\omega_{ab} + \omega_{ba}) = 0 ,
\] (31)

due to the linearity of the \( \sigma \)-derivative. Thus, \( \dot{\sigma}_{ab} \) and \( \dot{\omega}_{ab} \) are symmetric and anti-symmetric in the indices, respectively just as \( \sigma_{ab} \) and \( \omega_{ab} \).

Now taking the \( \sigma \)-derivative on both sides of Eq.(6), we see

\[
\ddot{\sigma}_{ab} + \dot{\omega}_{ab} = \ddot{B}_{ab} - \frac{1}{n-1} \ddot{\theta} h_{ab} - \frac{1}{n-1} \ddot{\theta} h_{ab} .
\] (32)

Considering Eq.(31), then, we can get the formulas for \( \dot{\sigma}_{ab} \) and \( \dot{\omega}_{ab} \) by decomposing the R.H.S. of Eq.(32) into the symmetric part and the anti-symmetric part, respectively. Taking into account Eq.(28), Eq.(30) and Eq.(A13), we finally obtain

\[
\dot{\sigma}_{ab} = \frac{1}{2} \gamma^2 \sigma_{ab} + \frac{1}{n-1} \gamma \sigma_{h_{ab}} + (\gamma \omega)_{\mu a} + (\gamma \omega)_{\mu b} + \frac{1}{2} \frac{d}{d\tau} \gamma \omega + D \cdot \nu + 2 \xi^a (\sigma \nu^b) - (54)
\]

\[
\dot{\omega}_{ab} = -2 \gamma \omega_{ab} - 2 \xi^a (\sigma \nu^b) - D \cdot \nu + 2 \xi^a (\sigma \nu^b) + (55)
\]

\[
\dot{\omega}_{ab} = \frac{1}{2} \gamma \omega_{ab} - \frac{1}{2} \gamma \omega_{ab} ,
\]

where the second term on the R.H.S. of Eq.(33) is the only trace-part of \( \dot{\sigma}_{ab} \), which vanishes for the case of the conformal geometry variation \( \gamma_{ab} \). The last lines of Eq.(33) and Eq.(34) also vanish for the case of the conformal geometry variation \( \gamma = 0 \). The case of conformal geometry variations shall be analyzed in detail in Sec.V.)

Let us note that \( \omega_{ab} = 0 \) iff the timelike geodesic congruence \( \zeta \) in question allows smooth \( (n-1) \)-dimensional spatial sections (see Appendix A for more details). Once \( \omega_{ab} = 0 \) is satisfied, it is satisfied all the way along the geodesic \( \gamma \) as is seen by the third equation in Eq.(10). When discussing gravitational collapses, thus, it is mostly assumed \( \omega_{ab} = 0 \), since we are usually interested in gravitational collapses started from ordinary, mild initial conditions satisfying \( \omega_{ab} = 0 \).

When considering the geometry variations, it is thus a reasonable assumption that the class of variations we consider retains the property \( \omega_{ab} = 0 \). As discussed above, it corresponds to restricting the variations within collapsing geometries started from ordinary, mild initial conditions. From Eq.(34), this condition for the geometry variation \( \gamma_{ab} \) should be \( \omega_{ab} = 0 \), yielding

\[
\ddot{D}_{\nu \nu} - \xi^a \left( \frac{\theta}{n-1} \nu_{ab} + (\sigma \nu^b)_{ab} = 0 .
\] (35)

With the help of Eq.(27), then, Eq.(35) is understood as the condition restricting the geometry variation \( \gamma_{ab} \) to those that preserve the well-behaved nature of the geodesic congruence described by \( \omega_{ab} = 0 \).

\section{B. Deviation of \( \frac{d}{d\tau} B_{ab} \) due to geometry variations}

Noting that

\[
R_{acb}^{d}(\lambda) = R_{acb}^{d} - 2\nu_{d} \xi^{c} \xi^{d}(\lambda) + 2 \xi_{d} \xi^{c} (\lambda) \xi^{d}(\lambda) ,
\]

we get

\[
\frac{\ddot{R}_{acb}^{d}}{d^{d}} = -2 \nu_{d} \ddot{\xi}^{c} \xi^{d} = -2 \nu_{d} \Gamma^{d} \xi^{d} \xi^{d}(\gamma) ,
\] (36)

where

\[
\Gamma_{cab}(\gamma) \equiv \frac{1}{2} (\nu_{a} \nu_{bc} + \nu_{b} \nu_{ac} - \nu_{c} \nu_{ab} ) ,
\]

\[
\Gamma^{c}_{ab}(\gamma) \equiv \frac{1}{2} \gamma^{c} \Gamma^{d} \xi^{d} \Gamma^{d}(\gamma) .
\]
Now taking the -derivative on both sides of Eq.(8) with \(g_{ab}\) replaced by \(\gamma_{ab}^{(\lambda)}\), we get

\[
\left( \frac{d}{d\tau} B_{ab} \right)^{\circ} = -\circ B_{ac} B_{bc}^\circ - \circ B_{ab} B_{bc}^\circ - \circ R_{acbd}^{\circ} \xi_d^c \cdot (37)
\]

With the help of Eq.(23), Eq.(28), Eq.(29) and Eq.(36), we can rewrite Eq.(37), yielding

\[
\left( \frac{d}{d\tau} B_{ab} \right)^{\circ} = 2\nabla_a \Gamma_{bc} \xi^c - \frac{1}{2} \left( \frac{d}{d\tau} B_{ab} \right) \]

\[-B_{ac} (D_b v^c + \xi_b (B v)^c) - B_{ab} (D^c v_a + \xi_a (B v)^c) \]

\[-R_{acbd} (2\nabla^c \xi_d + \xi^c \xi_d) \]

\[-(\gamma B)_{ac}^{\circ} + \frac{1}{2} \gamma_a B_{bc} \]

\[+ \{ B, D_{ab} \xi^a + \xi_b (\gamma B)_{ab} \}_{ab} \cdot (38) \]

Here \(\Gamma_{bc} = \Gamma_{dcb} \xi^d\) (see Sec.II.C (iv)); \(\{ A, B \}_{ab} = (AB + BA)_{ab}\), the anti-commutator between \(A\) and \(B\), so that \(\{ B, D_{ab} \xi^a + \xi_b (\gamma B)_{ab} \}_{ab}\) is the anti-commutator between \(B_{ab}\) and \(D_{ab} \xi^a + \xi_b (\gamma B)_{ab}\).

With some computations, it is straightforward to get

\[
2 \nabla_a \Gamma_{bc} (\gamma B)^c = \frac{1}{2} \left( \frac{d^2}{d\tau^2} B_{ab} \right) + \frac{1}{2} \left( \frac{d}{d\tau} B_{ab} \right) \]

\[+ (\gamma B)_{ac}^{\circ} + \frac{1}{2} \gamma_a B_{bc} \]

\[-2D_{ab} \gamma^a_{\circ} B_{b}^c - \gamma_a \epsilon_{\circ} \left( \frac{d}{d\tau} B_{ab} \right) \xi_b \]

\[-(\gamma B)_{ab}^{\circ} - \frac{d}{d\tau} D_{ab} \gamma^a \cdot (39) \]

It is worth noting that, in case of the conformal geometry variation described by \(\gamma_{ab} = 2f g_{ab}\) (\(f\) is any smooth function), it follows that \(\gamma_{ab} + \gamma_. h_{ab} = 0\) and \(\gamma_{ab} = 0\) so that all the lines but the first one on the R.H.S. of Eq.(39) vanish and the expression gets greatly simplified.

Combing Eq.(38) with Eq.(39), we finally obtain

\[
\left( \frac{d}{d\tau} B_{ab} \right)^{\circ} = \frac{1}{2} \left( \frac{d^2}{d\tau^2} B_{ab} \right) - \frac{1}{2} \left( \frac{d}{d\tau} B_{ab} \right) \]

\[-(B_{ac} D_{bc} v^c + D_{bc} v_c B_{bc}^{\circ}) - (\xi_a (B v) B_{bc}^{\circ}) + (BB v)_a \xi_b \]

\[-R_{ab} (2\nabla^c \xi_d + \xi^c \xi_d) \]

\[+ (\gamma B + \gamma_. h_{ab} D_{bc} B_{bc}^{\circ} + \frac{d}{d\tau} (\gamma B)_{bc} + \frac{1}{2} \gamma_. B_{bc}) \]

\[+ \{ B, D_{ab} \xi^a + \xi_b (\gamma B)_{ab} \}_{ab} \cdot (40) \]

Here we note that \((\gamma B)_{ab} = \gamma_a B_{bc}^{\circ} B^c_{\circ a}\) and \((B^{\circ} v B)_{ab} = B_{ab} (\gamma B)_{bc}^{\circ} \gamma_. B_{bc}^{\circ}\), faithfully following the notation rules introduced in Sec.II.C.

We note that \((d/d\tau B_{ab})^{\circ}\) as well as \(B_{ab}^{\circ}\) is not a spatial tensor.

In the case of the conformal geometry variation, \(\gamma_{ab} = 2f g_{ab}\), only the first three lines on the R.H.S. of Eq.(40) remain.

To get explicit expressions for \((\frac{d}{d\tau} \theta)\) and \((\frac{d}{d\tau} \sigma_{ab})\) from Eq.(40), one can follow the similar procedures to get Eq.(30), Eq.(33) and Eq.(34) from Eq.(28). However, Eq.(40) might be enough for the moment and we shall give the explicit expressions for the case of the conformal geometry variations below.

V. CONFORMAL VARIATIONS TO THE GEOMETRY

A. Basic formulas for the conformal variations to the geometry

From now on, we confine ourselves to the cases of the conformal variations to the geometry described as

\[
g^{(\lambda)}_{ab}(x) = e^{2\lambda f(x)} g_{ab}(x) \cdot (41)\]

where \(f(x)\) is an arbitrary smooth function.

According to the first equation in Eq.(12), we then get

\[
\gamma_{ab}(x) = 2f(x) g_{ab}(x) \cdot (42)\]

Based on Eq.(42), we further get

\[
\gamma_{ab} = 2f h_{ab} \quad \gamma_{ab} = 2f \xi_{ab} \quad \gamma_{ab} = 2f \xi_{ab} + \gamma_. h_{ab} = 0 \cdot (43)\]

\[
\gamma_{ab} = \gamma_a = 2f \quad \gamma_{ab} = \gamma_a = 2f \xi_{ab} \quad \gamma_{ab} + \gamma_. h_{ab} = 0 \cdot
ddottedline
Then, the key equation Eq.(24) reduces to

$$\frac{dv^a}{d\tau} + B^a_b v^b = -D^a f$$ ,

(44)

or, in the form of Eq.(25),

$$\mathcal{L} v^a + B^a_b v^b = -D^a f$$ ,

(45)

where $\mathcal{L}^a_b$ is given by Eq.(26), which is invertible as discussed just after Eq.(26). Thus Eq.(44) or Eq.(45) can be formally solved for $v^a$ as

$$v^a = -\left(\mathcal{L}^{-1}\right)^a_b D^b f + \left[\left(\frac{d}{d\tau} + B\right)^{-1}\right]^a D f$$ .

(46)

Now, when $\gamma_{ab} = 2f g_{ab}$, Eq.(28) reduces to

$$\theta_{ab} = f b + D_b v_a + \frac{df}{dt} h_{ab} + (Bv)^a_b \xi_b + \xi_a (Bv)_b$$ .

(47)

From Eq.(47) one can extract equations for $\theta, \theta_{ab}$ and $\omega_{ab}$, as before, which of course coincide with the equations reduced directly from Eq.(30), Eq.(33) and Eq.(34) for the case $\gamma_{ab} = 2f g_{ab}$. In order to reduce Eq.(33) to the corresponding equation (Eq.(48) below), for instance, we note Eq.(43) along with $\gamma^0 = 2f \gamma^{ab} = 0, (\tilde{\gamma})^a_b = 2f \delta_{ab}, (\tilde{\gamma}^a) = 2f \omega_{ab} = 0$ and so on. Thus we get

$$\theta = -f \theta + (n-1) \frac{df}{d\tau} + D \cdot v$$ ,

$$\omega_{ab} = f \omega_{ab} - D_a v_b - 2 \xi_a (\omega v)_b$$ .

(48)

As for $\theta_{ab}$, comparing Eq.(33) and Eq.(48), we see that the only trace-part on the R.H.S. of Eq.(33), $\frac{1}{n-1} \gamma \cdot h_{ab}$, vanishes in the conformal cases, so that the latter equation becomes completely trace-free. We also note that the last lines in Eq.(33) and Eq.(34) vanish in the conformal cases, due to $\gamma_a = 0$.

Furthermore the condition for preserving $\omega_{ab} = 0$ (Eq.(35)) reduces to a simple form

$$D_a v_a = 0$$ ,

(49)

which means that the geodesic shift $v^a$ should not form any rotation in the $(n-1)$-dimensional section orthogonal to the flow-line. A typical situation is that $v^a$ is described as a gradient of some scalar function as is the case analyzed in the next subsection Sec.V B (see Eq.(57) below).

As for the equation for $\left(\frac{d}{d\tau} B_{ab}\right)$, Eq.(40) also simplifies drastically for the conformal cases. Indeed, only the first three lines on the R.H.S. of Eq.(40) remain, yielding

$$\left(\frac{d}{d\tau} B_{ab}\right) = \frac{d^2 f}{d\tau^2} h_{ab} - D_a D_b f - 2 \frac{df}{d\tau} B_{ab} + \frac{df}{d\tau} B_{ba}$$

$$- \left( B_{ab} D_c v^c + D_c v_a B^c_b \right) + \left( \xi_a (v BB)_b + (BBv)_a \xi_b \right)$$

$$- 2 R_{abcd} v^c \xi^d$$ .

(50)

From Eq.(50), we can extract the formulas for $\left(\frac{d}{d\tau} \theta\right)$, $\left(\frac{d}{d\tau} \sigma_{ab}\right)$ and $\left(\frac{d}{d\tau} \omega_{ab}\right)$.

First of all, we note that

$$\left(\frac{d}{d\tau} \theta\right) = \frac{d}{d\tau} \theta (\gamma)$$

$$= g_{ab} \frac{d}{d\tau} B_{ab} + \frac{d}{d\tau} \left(\frac{d}{d\tau} B_{ab}\right)$$ .

(51)

Noting that $\theta_{ab} = -2 f g_{ab}$ due to Eq.(13) and Eq.(42), the first term in the last line reduces to $-2 f \frac{d}{d\tau} B_{ab}$. With the help of Eq.(10) and Eq.(50), then, we get

$$\left(\frac{d}{d\tau} \theta\right) = \left(\frac{d}{d\tau} \frac{d}{d\tau} \theta\right)$$

$$+ (n-1) \frac{df}{d\tau} - 2 f R_{ab} - 2 f B_{ab}$$

$$+ 2 f \sigma \cdot \sigma - 2 f \omega \cdot \omega - 2 \sigma \cdot D v - 2 \omega \cdot D v$$ .

(52)

By the similar argument as in Eq.(31), we see that $\left(\frac{d}{d\tau} \sigma_{ab}\right)$ and $\left(\frac{d}{d\tau} \omega_{ab}\right)$ are symmetric and anti-symmetric in the indices, respectively, due to the linearity of the $\sigma$-derivative and $\omega$ derivative. Thus $\left(\frac{d}{d\tau} \sigma_{ab}\right)$ is manifestly trace-free. Furthermore $\left(\frac{d}{d\tau} \sigma_{ab}\right)$ is also trace-free in the conformal cases, which is shown as

$$\left(\frac{df}{d\tau} \sigma_{ab}\right) = \left(\frac{df}{d\tau} \sigma_{ab}\right)$$

$$= g_{ab} \frac{d}{d\tau} \sigma_{ab} - \frac{d}{d\tau} \sigma_{ab}$$

$$= 2 f g_{ab} \frac{d}{d\tau} \sigma_{ab} = 0$$ ,

where we have used the fact that $\left(\frac{d}{d\tau} \sigma_{ab}\right)$ is trace-free (as is seen by Eq.(10)).

Thus, by extracting the trace-free part of Eq.(50) and dividing it into the symmetric part and the anti-symmetric part, we get

$$\left(\frac{d}{d\tau} \sigma_{ab}\right) = \frac{d}{d\tau} \sigma_{ab} - D_a D_b f$$

$$- \frac{2 \theta}{n-1} D_{ab} \gamma - \{\sigma, D \gamma\}_{ab} + \{\omega, D \gamma\}_{ab}$$

$$- \frac{2 \theta^2}{(n-1)^2} \xi_{ab} - \frac{2 \theta}{n-1} \xi_{ab} \sigma - 2 \xi_a \sigma \gamma - 2 \xi_a \sigma \gamma$$

$$- 2 R_{ab} - \frac{1}{n} R_{abcd} v^c \xi_d$$ .

(53)

and

$$\left(\frac{d}{d\tau} \omega_{ab}\right) = -2 \frac{df}{d\tau} \omega_{ab}$$

$$+ \left(\frac{2 \theta}{n-1} D_{ab} v_a + \{\sigma, D_{ab} v_a\}_{ab} - \{\omega, D_{ab} v_a\}_{ab} \right.$$}

$$+ \frac{4 \theta}{n-1} \xi_a (\omega v)_a + 2 \xi_a \{\{\sigma, \omega\} v\}$$ .

We see from Eq.(53) and the third equation in Eq.(48) that one can set $\omega_{ab} \equiv 0$ provided that the ‘twist-less’ condition Eq.(49) is satisfied for the geometry variations.
We note that the arbitrary function \( f(x) \) determines the geometry variation \( \mathcal{L}_{ab} \) (Eq.\((42)\)), which causes the geodesic shift (Eq.\((46)\)). Then the two sets of equations (the equations in Eq.\((48)\) and the equations Eq.\((51)\)-Eq.\((53)\)) describe the linear responses of \( \theta, \, \sigma, \) and \( \omega \) and those of \( \frac{\partial}{\partial t}, \frac{d}{\partial x}, \sigma_{ab}, \) and \( \frac{d}{\partial x} \omega_{ab} \) to the geometry variation caused by the arbitrary function \( f \).

Since the function \( f(x) \) is arbitrary, then, we can analyze in detail the relation between the geometry variations and the changes in the focusing properties of the geodesic congruence with the help of these sets of equations.

B. Application to the time-reversed Friedmann-Robertson-Walker model

To get some concrete insights for the present framework, we here apply it to the case of the time-reversed Friedmann-Robertson-Walker (FRW) model. The FRW model describes the expanding universe with an initial singularity, so that its time-reversed version can be regarded as a simple spacetime model describing a gravitational collapse with a final singularity. (See Appendix B for the basic results of the FRW spacetime.)

The metric for the FRW spacetime is given in the co-moving coordinates as

\[
ds^2 = -dt^2 + a^2(t) \left( F^2(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2 \right),
\]

\[
F(r) := \frac{1}{\sqrt{1 - kr^2}} \quad (k = -1, 0, 1).
\]

Let us choose as a geodesic congruence \( \xi \) the set of standard geodesics compatible with the co-moving coordinates, described by \( x^a(t) = f(t, \vec{r}_0, \varphi_0) \) with \( \vec{r}_0 \) and \( \varphi_0 \) being some constants. Then in the present frame of coordinates it follows

\[
\xi^a = T(1 \ 0 \ 0 \ 0), \quad \xi_a = T(-1 \ 0 \ 0 \ 0),
\]

\[
h_{ab} = \text{diag}(0, a^2F, a^2r^2, a^2r^2\sin^2\theta).
\]

It is then straightforward to get

\[
B_{ab} = T^0_{ab} = \frac{1}{2} h_{ab} = \frac{\partial}{\partial \varphi} h_{ab},
\]

which yields

\[
\theta = \frac{\partial}{\partial \varphi}, \quad \sigma_{ab} = \omega_{ab} = 0.
\]

From Eq.\((B6)\) and Eq.\((B8)\) in Appendix B, we see that

\[
R_{a,b} = -\frac{\partial}{\partial \varphi} h_{ab}, \quad \bar{R}_{a,b} = 0, \quad R_{..} = -3 \frac{\partial}{\partial \varphi}.
\]

As far as \( \partial < 0 \) (a time-reversal invariant relation), thus, it follows that \( R_{..} > 0 \).

In the context of the generic condition for the singularity theorems [2, 3], which is stated as “each timelike geodesic contains at least one point at which \( R_{a,b} \neq 0 \),” the trace-part of \( R_{a,b} \) (i.e. \( R_{..} \)) is non-zero rather than the trace-free part \( (\bar{R}_{a,b}) \). Because of this fact, \( \frac{\partial}{\partial \varphi} \lesssim 0 \) is guaranteed even though \( \sigma_{ab} \equiv 0 \) (see Eq.\((10)\)).

Then Eq.\((44)\) in this case becomes

\[
\psi^a + \frac{\partial}{\partial \varphi} \psi^a = -D^a f,
\]

which can be explicitly solved in the form of Eq.\((46)\) as

\[
\psi^a(t, \vec{x}) = \frac{a(t_0)}{a(t)} \psi^a(t_0, \vec{x}) - a(t) D^a \psi(t, \vec{x}),
\]

\[
\psi(t, \vec{x}) = \int_{t_0}^t f(t', \vec{x}) dt'.
\]

Here we should note that the operator \( D^a \) is \( t \)-dependent through \( a(t) \), which is clearly seen in the co-moving coordinates as

\[
D^a = T \left( \frac{1}{a^2(t)} \right) F^2(\tau) \frac{\partial}{\partial \tau} - \frac{1}{a^2(t)} \tau^2 \frac{\partial}{\partial \varphi}.
\]

Thus Eq.\((57)\) is expressed in a more explicit form

\[
\psi^a(t, \vec{x}) = \frac{a(t_0)}{a(t)} \psi^a(t_0, \vec{x}) - \frac{1}{a(t)} D^a_{[t]} \psi(t, \vec{x}),
\]

where \( D^a_{[t]} = D^a \) with \( a(t) \equiv 1 \) so that it is \( t \)-independent. (These two operators are related by \( D^a_{[t]} = a(t) D^a \). ) One can then easily check that Eq.\((58)\) is the solution for Eq.\((56)\).

For simplicity, we shall set \( \psi^a(t_0, \vec{x}) = 0 \) below. Then Eq.\((48)\) becomes

\[
\begin{align*}
\theta &= -3 \frac{\partial}{\partial \varphi} + 3 \frac{\partial}{\partial \tau} - \frac{a(t) D \cdot D \psi(t, \vec{x})}{a^2(t)} , \\
\sigma_{ab} &= -a(t) \bar{D}_D \bar{D}_b \psi(t, \vec{x}) , \\
\omega_{ab} &= 0.
\end{align*}
\]

We note that Eq.\((59)\) describes how the focusing of geodesics starts to deviate from the one in the exact FRW spacetime: The conformal geometry variation induces the variations in \( \theta \) and \( \sigma_{ab} \). In particular \( \sigma_{ab} \) obtains a non-zero value even though \( \sigma_{ab} = 0 \) at the beginning. Once \( \sigma_{ab} \) gets non-zero, the first and the third terms on the R.H.S. of the second equation in Eq.\((48)\) also cause the variation in \( \sigma_{ab} \). On the other hand, \( \omega_{ab} = 0 \) is preserved as far as the condition Eq.\((49)\) is satisfied.

Furthermore Eq.\((51)\), Eq.\((52)\) and Eq.\((53)\) become

\[
\begin{align*}
\left( \frac{d\theta}{d\tau} \right)_{\phi} &= \frac{\partial}{\partial \varphi} + 6 \left( \frac{\partial}{\partial \varphi} \right)^2 - \frac{\partial}{\partial \tau} \frac{\partial}{\partial \varphi} f + 2aD \cdot D \psi - D \cdot D \psi f, \\
\left( \frac{d\sigma_{ab}}{d\tau} \right)_{\phi} &= -\bar{D}_D \bar{D}_b \psi f + 2a \bar{D}_D \bar{D}_\psi \psi + 2a \left( \frac{\partial}{\partial \varphi} \right)^2 + \frac{\partial}{\partial \tau} \frac{\partial}{\partial \varphi} \bar{D}_D \bar{D}_\psi \psi, \\
\left( \frac{d\omega_{ab}}{d\tau} \right)_{\phi} &= 0.
\end{align*}
\]

Here let us pay attention to the first equations in Eq.\((59)\) and Eq.\((60)\) for \( a < 0, \bar{a} < 0 \) and \( f < 0 \);
The function $f(t, \vec{x})$ can be chosen arbitrarily. The above two equations serve as concrete examples for the general cases of the conformal variations, which is analyzed in the next section.

VI. INFLUENCE OF CONFORMAL GEOMETRY VARIATIONS ON THE FOCUSING PROPERTIES OF GEODESIC CONGRUENCES

We now investigate the response of the gravitational focusing properties of geodesic congruences to conformal geometry variations. For this purpose, let us focus on the behavior of the expansion $\theta$.

We first recall how $\theta$ behaves during the latest phase of gravitational contractions. We pay attention to the Raychaudhuri equation, the first equation in Eq.(10). As discussed at the end of Sec.II.B, one can set $\omega_{ab} = 0$ when the timelike geodesic congruence $\mathcal{G}$ is hypersurface orthogonal, which is usually assumed. Furthermore $R_{..} > 0$ is guaranteed assuming that the strong energy condition is satisfied for the matter content. Thus

\[ \frac{d\theta}{d\tau} + \frac{1}{n-1} \theta^2 \leq 0, \tag{61} \]

which is equivalent to

\[ \frac{d\theta^{-1}}{d\tau} \geq \frac{1}{n-1}, \]

or

\[ \theta(\tau) \leq \left( \theta(\tau_0)^{-1} + \frac{1}{n-1} (\tau - \tau_0) \right)^{-1}. \]

Thus once $\theta$ becomes negative at some $\tau_0$ (i.e. $\theta(\tau_0) < 0$), then $\theta \to -\infty$ as the proper-time tends to some finite value $\tau$ satisfying $\tau_0 < \tau \leq \tau_0 + \frac{\Delta \tau_0}{n-1}$ [3].

The function $f(x)$ is arbitrary as far as the consistency of arguments are retained. In the present context of gravitational collapses, it is reasonable to enforce the strong energy condition for the consistency with the argument just before Eq.(61). Indeed, taking into account the energy condition for restricting the class of functions to be considered is sometimes vital for describing the gravitational contractions correctly [8].

Let us then pay attention to the strong energy condition, translated as $R_{..} > 0$, and derive a formula for $(R_{..})^\circ$. By the conformal transformation Eq.(41), the Ricci tensor is transformed as [3],

\[ R^\lambda_{ab} = R_{ab} - \lambda (n-2) \nabla_a \nabla_b f - \lambda g_{ab} \Delta f + O(\lambda^2). \]

On the other hand, the tangent vector of a geodesic is transformed as

\[ \xi^{(\lambda)} = (1 - \lambda f) \xi + \lambda v^a + O(\lambda^2), \]

where Eq.(23) and Eq.(43) have been used. Thus we obtain

\[ (R_{..})^\circ = -2 f R_{..} + D \cdot D f - \theta \frac{df}{d\tau} - (n-1) \frac{d^2 f}{d\tau^2} + 2 v \cdot R, \tag{62} \]

where Eq.(A11) has been used and $v \cdot R \equiv v^a R_{ab} \xi^b$ (following the notation rules in Sec.II.C).

To retain the strong energy condition, it is reasonable to choose the function $f(x)$ which does not greatly depart from the condition $(R_{..})^\circ \geq 0$.

Now we pay attention to the first equation in Eq.(48), and Eq.(51). We consider the final phase of gravitational contractions where $\theta$ is negative and $|\theta| \Delta \tau \gg 1$ according to the argument after Eq.(61). Here $\Delta \tau$ is the typical time-scale of changes of our concern. Let $\Delta \tau_0$ be the time-scale in which $\theta$ changes substantially, then the above estimation implies $\Delta \tau_0 \ll \Delta \tau$. Next, the behavior of $f(x)$ is quite arbitrary but at least $\Delta \tau_0 \ll \Delta T_j \ll \Delta \tau$ and $\Delta \tau_0 \ll \Delta X_j \ll \Delta \tau$ should satisfy in order for our arguments to be meaningful. In particular, $\Delta \tau$, the typical time-scale of our concern, should be large enough compared to the other scales to investigate the effects of geometry variations. Thus we impose

\[ \Delta \tau_0 \ll (\Delta T_j, \Delta X_j) \ll \Delta \tau, \tag{63} \]

implying that

\[ |\theta| \frac{df}{d\tau} \gg \left( \frac{d^2 f}{d\tau^2}, |D \cdot D f| \right). \]

For simplicity, let us assume $f < 0$ also, which roughly corresponds to the conformal geometry variations in favor of the geodesic focusing.

Now we focus on the first equation in Eq.(48). Let us first analyze the importance of each term on the R.H.S. of the equation. We see from Eq.(46) (along with its counter-part for the time-reversed FRW model, Eq.(57)) that

\[ D \cdot v \sim -\Delta \tau D \cdot D f \sim -\frac{\Delta \tau}{\Delta X_j} f. \tag{64} \]

Then we estimate as

\[ \frac{|D \cdot v|}{|\theta|} \sim \frac{\Delta T_0 \Delta \tau_0}{\Delta X_j^2}, \quad \frac{|df|}{|\theta|} \sim \frac{\Delta \tau_0}{\Delta T_j}, \quad \frac{|df|}{|\theta|} \sim \frac{\Delta X_j^2}{\Delta T_j \Delta \tau}. \]

There is no a priori relation between $\Delta T_j$ and $\Delta X_j$. One reasonable situation is, however, that they are of similar magnitudes, e.g. $\Delta T_j \sim \Delta X_j$, unless quite special situations are considered.

Thus, along with Eq.(63), we see

\[ \left( |\theta|, |D \cdot v| \right) \gg \left| \frac{df}{d\tau} \right|, \tag{65} \]

implying that the term $D \cdot v$ is as important as the term $f \theta$ and they are much more important than the term $\frac{df}{d\tau}$. 

Next let us search for the condition for \( \ddot{\theta} < 0 \) which implies the stronger focusing of geodesics due to the geometry variations. In later phase of the geodesic congruence contractions, the threshold condition \( \ddot{\theta} = 0 \) becomes
\[
\frac{df}{d\tau} + \frac{1}{n-1}f = -\frac{1}{n-1}D \cdot \nu ,
\] (66)
which along with \( f(t_0) = f_0 \) can be solved as
\[
f(\tau) = \left( f_0 - \frac{1}{n-1} \int_{t_0}^{\tau} D(\tau') \cdot \nu e^{A(\tau', t_0)} d\tau' \right) e^{-A(\tau, t_0)} ,
\]
\[
A(\tau, t_0) = \frac{1}{n-1} \int_{t_0}^{\tau} |\theta(\tau')| d\tau' .
\]
We note that
\[
A(\tau, t_0) \sim \frac{|\dot{\theta}|}{n-1} \Delta \tau \sim \frac{\Delta \tau}{\Delta t_0} ,
\]
where \( \Delta \tau = \tau - t_0 \) and \( \dot{\theta} \) is the time-average of \( \theta \) during \( \Delta \tau \). Considering Eq.(63), thus, it follows that \( e^{-A(\tau, t_0)} \sim 0 \), so that,
\[
f(\tau) \sim -\frac{1}{n-1} \int_{t_0}^{\tau} D(\tau') \cdot \nu e^{A(\tau', t_0)} d\tau' e^{-A(\tau, t_0)}
\]
\[
\sim -\Delta t_0 \Delta \theta D \cdot \nu 
\]
\[
\sim \Delta t_0 \Delta \theta D \cdot Df .
\]

Here in the second step of estimations, we have taken into account the time-integral is overwhelmed by the contribution in the final period of order \( \Delta t_0 \) due to the factor \( e^{A(\tau, t_0)} \) along with \( \Delta t_0 \ll \Delta \tau \); in the final step of estimations, Eq.(64) has been used.

Since we are considering the case \( f < 0 \), it follows
\[
\dot{\theta} < 0 \iff |f(\tau)| > \Delta t_0 \Delta \tau |D \cdot Df| .
\]

We have thus found out the following fact: \( f \) and \( D \cdot Df \) are much more important than \( \frac{df}{d\tau} \) in the later phase of gravitational contractions and that when \( f \) is negative and \( |f(\tau)| \) is large enough to overwhelm \( |D \cdot Df| \), the contraction rate gets larger.

Now we turn to Eq.(51), which with the help of Eq.(62) can be recast as
\[
\left( \frac{d\theta}{d\tau} \right) = 2 \left( \frac{\theta^2}{n-1} + \sigma \cdot \sigma \right) f - 2\theta \frac{df}{d\tau} 
\]
\[-\frac{2\theta}{n-1} D \cdot \nu - 2\sigma \cdot D\nu - (R_\nu)^{\circ} \] , (67)
where we have set \( \omega_{ab} \equiv 0 \) by imposing the “twist-less” condition Eq.(49).

Let us search for the condition for \( \left( \frac{d\theta}{d\tau} \right)^{\circ} < 0 \). In later phase of the geodesic congruence contractions, the threshold condition \( \left( \frac{d\theta}{d\tau} \right)^{\circ} = 0 \) becomes
\[
\frac{df}{d\tau} + \left( \frac{1}{n-1} + \frac{\sigma \cdot \sigma}{|\theta|} \right) f
\]
\[-\frac{1}{n-1} D \cdot \nu + \frac{1}{|\theta|} \sigma \cdot D\nu + \frac{1}{2|\theta|} (R_\nu)^{\circ} .\] (68)

It is quite impressive that Eq.(68) is qualitatively similar to Eq.(66) as differential equations. Looking at the first two equations in Eq.(48), one reasonable estimation is \( |\theta| \gg |\sigma_{ab}| \) as far as this inequality holds initially. Provided that the strong energy condition is preserved, the value of \( |(R_\nu)^{\circ}| \) should not be very large and it is a reasonable estimation that \( |\theta| \gg |(R_\nu)^{\circ}| \). Thus we reach almost the same conclusion that \( f \) and \( D \cdot Df \) are important in the later phase of gravitational contractions and that, when \( f \) is negative, the large \( |f(\tau)| \) (compared to \( |D \cdot Df| \)) gives rise to the large contraction rate.

VII. SUMMARY AND DISCUSSIONS

In this paper, we have attempted to construct a mathematical framework for analyzing the later stages of gravitational contractions in terms of the focusing properties of timelike geodesic congruences.

The framework has been formed as a combination of a key equation, which relates the geometry variations to the geodesic shifts, and a set of equations, which relate the changes in the geodesic convergence properties to these geodesic shifts. Then, what we have constructed can be viewed as a set of equations describing the linear response of the geodesic convergence properties to arbitrary geometry variations. Since the geometry variations can be arbitrary and their origins need not be specified, we might be able to use them as probes to investigate the gravitational contraction processes in the present framework.

It has then turned out that the equations get simplified drastically in the case of the conformal variations. We have then studied the latest phase of gravitational contractions in the case of conformal geometry variations, and have found out that in the final stage, \( f \) and \( D \cdot Df \) are much more important than \( \frac{df}{d\tau} \), and furthermore that the contraction rate gets larger when \( f \) is negative and \( |f(\tau)| \) is large enough to overwhelm \( |D \cdot Df| \).

It shall be fruitful to change the classes of geometry variations and see the differences in the focusing properties of geodesics, as has been done to some degree in Sec.V by restricting the class to the one of the conformal variations and in Sec.VI by further restricting the class to the one satisfying the strong energy condition.

The fact that the case of the conformal variations has resulted in enormous simplifications in equations might imply that the non-conformal components of geometry variations contain important information on gravitational contractions which require further investigations. On the other hand, however, it means that we can make full use of the simplifications for the case of the conformal variations, which itself is important and intriguing.

One possible application related to the conformal geometry variations might be the application to the black-hole spacetimes, where the event horizon can be characterized as the set of the zero-points of the conformal factor in the metric by choosing an appropriate system of coordinates. Then the conformal geometry variations in this case correspond to the
various deformations of the event horizon so that investigating their influence on the convergence properties of the geodesic congruence might be of significance.

As a more mathematically oriented application of the conformal variations, there are issues regarding the properties of the conformal mappings. Studying mathematical properties of the Penrose diagrams are a typical example in this category.

It is expected that the present framework serves as a tool to understand more about the late stages of gravitational contractions.

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Appendix A: Useful formulas

Below we enumerate useful formulas frequently used in the body of the paper.

Some among them are formulas regarding the \((n-1)+1\)-decomposition of several quantities. For their derivations, it is helpful to use the relation Eq.(2) and its variations, such as

\[
\delta a^b = h_{ab} - \xi_a \xi^b .
\]

With the help of these relations, any vector \(u_a\) can be decomposed as

\[
u_a = u_a - u \xi_a ,
\]

where \(u \cdot u_b \xi^b\) (see Sec.II C for notations employed in this paper).

For a scalar function \(f\), then, the application of Eq.(A1) to a vector \(\nabla_a f\) yields

\[
\nabla_a f = D_a f - \frac{d f}{d \tau} \xi_a ,
\]

where \(D_a\) is the spatial derivative operator induced from \(\nabla_a\) (Eq.(7)).

For notational brevity, it is convenient to introduce

\[
\tilde{\nabla_a} \equiv D_a - \frac{\xi_a d}{d \tau} .
\]

It is important to note that \(\tilde{\nabla_a}\) coincides with the standard \(\nabla_a\) only when the former is applied to functions (Eq.(A2)).

For a general spacetime vector \(u^a\), it follows

\[
\nabla_b u^a = \tilde{\nabla}_b u^a + \xi^a (u B)_b - \xi^a \tilde{\nabla}_b u^b - u B^b .
\]

For a spatial vector \(u^a\), then, Eq.(A3) reduces to

\[
\nabla_b u^a = \tilde{\nabla}_b u^a + \xi^a (u B)_b .
\]

It is obvious that the contravariant index \(a\) in Eq.(A3) and Eq.(A4) can freely be lowered and the corresponding formulas with the covariant index \(a\) also follow.

Let \(\gamma_{ab}\) be a symmetric covariant tensor. Then it follows

\[
\gamma_{ab} = \gamma_{ab} - 2 \xi_a \gamma_{b} + \gamma . \xi_a \xi_b ,
\]

\[
= \gamma_{ab} - 2 \xi_a \gamma_{b} - \gamma . \xi_a \xi_b ,
\]

where \(\gamma^b\) is the shorthand notation for the spatial part of \(\gamma^b\). We also note the difference between \(\gamma^b\) and \(\gamma^b\) and the sign difference in the first and the second lines.

Taking the derivative \(\nabla_a\) on both sides of Eq.(A5) with some straightforward computations, one can obtain another formula for \(\gamma_{ab}\).

\[
\nabla_b \gamma_{cd} = \tilde{\nabla}_b \gamma_{cd} + 2 \xi^b (\gamma B)_{cd} - 2 \xi_a B_{cd} b - 2 \xi^b \tilde{\nabla}_b \gamma_{cd} - 2 (\gamma B)_{c} \xi^2 + \nabla_b \gamma . \xi c \xi d + 2 \gamma . \xi^2 - \xi^d B_{\gamma b} ,
\]

which can most easily be derived by applying Eq.(A3) to \(u_a u_b\).

In particular, for a symmetric spatial tensor \(\gamma_{ab}\), Eq.(A6) simplifies to

\[
\nabla_b \gamma_{cd} = \tilde{\nabla}_b \gamma_{cd} + 2 \xi^b (\gamma B)_{cd} b .
\]

There are some useful formulas for the second derivatives of a function \(f\):

\[
\frac{d}{d \tau} D_a f = D_a f - B^a \tau D f ,
\]

and

\[
\nabla_a \nabla_b f = D_a D_b f - B_{ba} \frac{d f}{d \tau} - 2 \xi_a \frac{d}{d \tau} \tau f + \xi_a \xi_b \frac{d^2 f}{d \tau^2} .
\]

By the contraction of Eq.(A9), we get

\[
\Delta f = D \cdot D f - \frac{d f}{d \tau} - \frac{d^2 f}{d \tau^2} .
\]

On the other hand, by the anti-symmetrization of Eq.(A9), we get

\[
(D_a D_b - D_b D_a) f = -2 \omega_{ab} \frac{d f}{d \tau} ,
\]

implying that the induced covariant derivative \(D_a\) is torsion-free iff \(\omega_{ab} = 0\). Thus the following statements are mutually equivalent:

1. \(\omega_{ab} = 0 \iff \nabla_a \xi_b = \nabla_b \xi_a \iff D_a D_b f = D_b D_a f .\)

2. The timelike geodesic congruence \(\xi^b\) in question is hypersurface orthogonal, i.e. \(\xi^b\) can be foliated by smooth \((n-1)\)-dimensional orthogonal sections.

3. \(B_{ab}\) coincides with the extrinsic curvature \(K_{ab}\) for the smooth orthogonal sections of \(\xi^b\).
We here pay attention to a useful formula including the \(\circ\)-derivative. By taking the \(\circ\)-derivative on both sides of \(h_{ab} = g_{ab} + \xi_a \xi_b\) or \(h^{ab} = g^{ab} + \xi^a \xi^b\) and using Eq.(12), Eq.(13), Eq.(22), Eq.(A1) and Eq.(A5), we get
\[
\begin{align*}
\overset{\circ}{h}_{ab} &= \frac{\partial}{\partial t} g_{ab} + 2 \xi_a \gamma_b^t + \gamma_a^t \xi_b^t, \\
\overset{\circ}{h}^{ab} &= -\frac{\partial}{\partial t} g^{ab} + 2 \xi^a \gamma^b_t + \gamma^a_t \xi^b_t.
\end{align*}
\] (A13)

Taking the \(\circ\)-derivative on both sides of \(h_{a}^{\ b} = h_{a}^{\ e} h_{e}^{\ b}\) and using Eq.(A13), it is straightforward to show that
\[
\overset{\circ}{h}_{a}^{\ b} = 0.
\] (A14)

**Appendix B: Basic facts on the Friedmann-Robertson-Walker model**

We here summarize the basic points on the Friedmann-Robertson-Walker (FRW) model discussed in Sec.V B.

The FRW spacetime is a spacetime foliated by maximally symmetric (i.e. homogeneous and isotropic) spatial hypersurfaces. The metric for the FRW spacetime is then given as
\[
d s^2 = -d t^2 + a^2(t) \left( F^2(r) d r^2 + r^2 d \Theta^2 + r^2 \sin^2 \Theta d \Phi^2 \right),
\] (B1)

The world-lines of observers “standing still” relative to the system of coordinates, Eq.(B1), is described by \(x^i(t) = \overline{x}^i(t \ \psi_0 \ \phi_0)\) with \(r_0, \ \psi_0\) and \(\phi_0\) being some constants. They are easily shown to be timelike geodesics. Let us call these geodesics standard geodesics compatible with the system of coordinates, Eq.(B1).

The spatial part of the coordinate system \(\{r, \psi, \phi\}\), thus, is interpreted as the collection of the “names” of the observers free-falling along each standard geodesic. We also note that the time-coordinate \(t\) coincides with the proper-time \(\tau\) for the free-falling observer along a standard geodesic. Hereafter each standard geodesic is assumed to be parametrized by the proper-time \(\tau\), coinciding with \(r\). Based on these facts, we see that the coordinate system \(\{t, r, \psi, \phi\}\) forms a co-moving coordinate system.

It is sometimes convenient to shift from the coordinates \(\{t, r, \psi, \phi\}\) to the local pseudo-orthonormal frame of coordinates \(\{\Theta^0, \Theta^1, \Theta^2, \Theta^3\}\) defined as
\[
\begin{align*}
\Theta^0 &= d t, \quad \Theta^1 = a(t) F(r) d r, \quad \Theta^2 = a(t) r d \theta, \\
\Theta^3 &= a(t) r \sin \theta d \phi,
\end{align*}
\] (B2)

which makes the expression for \(d s^2\) simply
\[
ds^2 = -(\Theta^0)^2 + (\Theta^1)^2 + (\Theta^2)^2 + (\Theta^3)^2.
\] (B3)

Let \(\xi^a\) be the tangent vector of a standard geodesic and let \(h_{ab}\) be given by Eq.(2). We then see that \(\xi^a = \xi^0 = (1 \ 0 \ 0 \ 0), \xi^a = \xi^1 = (-1 \ 0 \ 0 \ 0)\) and \(h_{ab} = \text{diag}(0\ 1\ 1\ 1)\) in the \(\{\Theta^0\}\) frame.

Following the standard procedures for differential forms [15, 16], then, we first get the expression for the connection 1-forms \(\Lambda^a_{\ b}\) as
\[
\begin{align*}
\Lambda^0_{\ k} &= \Lambda^0_{\ 0} = \frac{\dot{a}}{a} \Theta^k \quad (k = 1, 2, 3), \\
\Lambda^1_{\ m} &= -\Lambda^m_{\ 1} = -\frac{1}{a F} \Theta^m \quad (m = 1, 2), \\
\Lambda^2_{\ 3} &= -\Lambda^3_{\ 2} = -\frac{1}{a r} \cot \theta \Theta^3.
\end{align*}
\] (B4)

From Eq.(B4), we can derive the expression for the Riemann curvature as
\[
\begin{align*}
R_{k\ell m} &= \frac{\ddot{a}}{a} \ (k = 1, 2, 3), \\
R^1_{m\ell m} &= \left( \frac{\dot{a}}{a} \right)^2 + \frac{F'}{a^2 r F^3} \quad (m = 1, 2), \\
R^2_{323} &= \left( \frac{\dot{a}}{a} \right)^2 - \frac{1 - F'}{a^2 r^2 F^2},
\end{align*}
\] (B5)

with the other independent components being zero. From Eq.(B5), we get useful coordinate-independent relations
\[
R_{abc} = 2 \frac{\dddot{a}}{a} h_{ab} \xi^c, \quad R_{a\ b\ c\ d} = -\frac{\ddot{a}}{a} h_{ab}.
\] (B6)

Then the Ricci curvature becomes
\[
\begin{align*}
R_{00} &= -3 \frac{\ddot{a}}{a}, \\
R_{11} &= 2 \left( \frac{\dot{a}}{a} \right)^2 + \frac{\dddot{a}}{a} + \frac{2 F'}{a^2 r F^3}, \\
R_{22} &= R_{33} = 2 \left( \frac{\dot{a}}{a} \right)^2 + \frac{\dddot{a}}{a} + \frac{r F' + F^3 - F}{a^2 r^2 F^3},
\end{align*}
\] (B7)

with the other components being zero. From Eq.(B7), we get other useful coordinate-independent relations
\[
R_{a} = 3 \frac{\dddot{a}}{a} \xi_a, \quad R_{..} = -3 \frac{\ddot{a}}{a}.
\] (B8)

Finally the scalar curvature becomes
\[
R = 6 \frac{\dddot{a}}{a} + 6 \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{2 F' F + F^3 - F}{a^2 r^2 F^3}.
\] (B9)

For reference, let us write down the expression for the Einstein tensor also;
\[
\begin{align*}
G_{00} &= 3 \left( \frac{\dot{a}}{a} \right)^2 + \frac{2 r F' F + F^3 - F}{a^2 r^2 F^3}, \\
G_{11} &= - \frac{\ddot{a}}{a} - \frac{1}{a^2} \left( \frac{\ddot{a}}{a} \right)^2 + \frac{1 - F'}{a^2 r^2 F^2}, \\
G_{22} &= G_{33} = - \frac{\ddot{a}}{a} + 2 \frac{\dddot{a}}{a} - \frac{2}{a},
\end{align*}
\] (B10)

with the other components being zero.
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[12] Einstein’s summation convention is adopted throughout this paper. We shall also adopt a suitable system of units in which the light velocity $c$ becomes $c = 1$.
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