Approximation by Lupas-Type Operators and Szász-Mirakyan-Type Operators

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Lupas-type operators and Szász-Mirakyan-type operators are the modifications of Bernstein polynomials to infinite intervals. In this paper, we investigate the convergence of Lupas-type operators and Szász-Mirakyan-type operators on \([0, \infty)\).

1. Introduction and Main Results

For \(f \in C([0, 1])\), Bernstein operator \(B_nf(x)\) is defined as follows: Let

\[
p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leq k \leq n,
\]

and then we define

\[
(B_nf)(x) = \sum_{k=0}^{n} p_{n,k}(x)f\left(\frac{k}{n}\right).
\]

Derriennic [1] gave a modified operator of \(B_nf\) such as

\[
(M_n^* f)(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{0}^{1} p_{n,k}(t)f(t)dt,
\]
and obtained the result that for \( f \in C^2([0,1]) \),
\[
\lim_{n \to \infty} \left( (M_n f)(x) - f(x) \right) = (1 - 2x)f'(x) + x(1-x)f''(x).
\] (1.4)

Lupas investigated a family of linear positive operators which mapped the class of all bounded and continuous functions on \([0, \infty)\) into \(C[0, \infty)\) such that
\[
(L_n f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \in [0, \infty).
\] (1.5)

Moreover, Sahai and Prasad [2] modified Lupas operators as follows: Let \( f \) be integrable on \([0, \infty)\) and let \( n \) be a positive integer. Then we define
\[
(M_n[f])(x) = (n-1) \sum_{k=0}^{\infty} P_{n,k}(x) \int_0^\infty P_{n,k}(y) f(y) dy, \quad x \in [0, \infty),
\] (1.6)

where
\[
P_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.
\] (1.7)

In this paper, we assume that \( n \) is a positive integer. Then they obtained the following:

**Theorem 1.1** (see [2], Theorem 1). If \( f \) is integrable on \([0, \infty)\) and admits its \((r+1)\)th and \((r+2)\)th derivatives, which are bounded at a point \( x \in [0, \infty) \), and \( f^{(r)}(x) = O(x^\alpha) \) (\( \alpha \) is a positive integer \( \geq 2 \)) as \( x \to \infty \), then
\[
\lim_{n \to \infty} n \left( (M_n[f])^{(r)}(x) - f^{(r)}(x) \right) = (r+1)(1 - 2x)f^{(r+1)}(x) + x(1-x)f^{(r+2)}(x).
\] (1.8)

**Theorem 1.1** holds only for bounded \( x \leq K \), so it does not mean the norm convergence on \([0, \infty)\). In this paper, we improve **Theorem 1.1** with respect to the norm convergence on \([0, \infty)\).

Let \( 0 < p \leq \infty \) and let \( w \) be a positive weight, that is, \( w(x) \geq 0 \) for \( x \in \mathbb{R} \). For a function \( g \) on \([0, \infty)\), we define the norm by
\[
\|g\|_{L_p([0,\infty))} := \left\{ \left( \int_{[0,\infty)} |g(t)|^p dt \right)^{1/p} \right\}, \quad 0 < p < \infty
\]
\[
\sup_{[0,\infty)} |g(t)|, \quad p = \infty.
\] (1.9)

For convenience, for nonnegative integers \( n \geq 2, r, \) and \( n-r-2 \geq 0 \), we let
\[
A_{n,r} := \frac{(n-1)!(n-2)!}{(n-r-2)!(n+r-1)!}.
\] (1.10)
Let $0 < p \leq \infty$. Let $\alpha$ and $r$ be nonnegative integers and $n - r - 2 \geq 0$. Let $f \in C^{(r+1)}([0, \infty))$ satisfy
\[
\left| f^{(r)}(x) \right| \leq O(1)(x + 1)^{\alpha}, \quad \left| f^{(r+1)}(x) \right| \leq O(1)(x + 1)^{\alpha + 2}.
\]
(1.11)

Then we have uniformly for $f$ and $n$,
\[
\left| A_{n,r}(M_n[f])^{(r)}(x) - f^{(r)}(x) \right| = O\left(\frac{1}{n^{1/3}}\right)(x + 1)^{\alpha + 2}.
\]
(1.12)

In particular, if $\|(x + 1)^{\alpha + 2}w(x)\|_{L_p([0, \infty))} < \infty$, then we have uniformly for $n$,
\[
\left\| (A_{n,r}(M_n[f])^{(r)}(x) - f^{(r)}(x))w(x) \right\|_{L_p([0, \infty))} = O\left(\frac{1}{n^{1/3}}\right).
\]
(1.13)

Remark 1.3. (a) We see that for nonnegative integers $n \geq 2$, $r$, and $n - r - 2 \geq 0$,
\[
\frac{(n - 1)!(n - 2)!}{(n - r - 2)!(n + r - 1)!} \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty.
\]
(1.14)

(b) The following weight is useful.
\[
w_1(x) = \frac{1}{1 + x^\lambda} \begin{cases} 
\lambda > 1 + \alpha + 2, & 0 < p < \infty, \\
\lambda \geq \alpha + 2, & p = \infty.
\end{cases}
\]
(1.15)

Let
\[
\psi(x) := \frac{1}{1 + x^\lambda}.
\]
(1.16)

Theorem 1.4. Let $r$ and $\beta$ be nonnegative integers and $n - r - 2 \geq 0$. Let $f \in C^{(r+2)}([0, \infty))$ satisfy
\[
\left\| f^{(r+1)}(x)\psi^{2\beta + 1}(x) \right\|_{L_\infty([0, \infty))} < \infty, \quad \left\| f^{(r+2)}(x)\psi^{2\beta}(x) \right\|_{L_\infty([0, \infty))} < \infty.
\]
(1.17)

Then we have uniformly for $f$ and $n$,
\[
\left| A_{n,r}(M_n[f])^{(r)}(x) - f^{(r)}(x) \right| \psi^{2\beta + 2}(x) \leq O\left(\frac{1}{n}\right) \left( \left\| f^{(r+1)}(x)\psi^{2\beta + 1}(x) \right\|_{L_\infty([0, \infty))} + \left\| f^{(r+2)}(x)\psi^{2\beta}(x) \right\|_{L_\infty([0, \infty))} \right).
\]
(1.18)
Let us define the weighted modulus of smoothness by

$$
\omega_k(f; \eta; t) := \sup_{0 \leq h \leq t} \left\| \Delta^k_h f(\cdot) \eta(\cdot) \right\|_{L_\infty([0, \infty))},
$$

where

$$
\Delta^1_h f(x) = f(x + h) - f(x), \quad \Delta^2_h f(x) = f(x) - 2f(x + h) + f(x + 2h).
$$

**Theorem 1.5.** Let $\beta$ and $r$ be nonnegative integers and $n - r - 2 \geq 0$. Let $f \in C^r([0, \infty))$. Then we have uniformly for $f$ and $n$,

$$
\left\| \left( A_{n,r} (M_n [f])^r (x) - f^r (x) \right) q^{2\beta + 2} \right\|_{L_\infty([0, \infty))} \leq C \left( \frac{1}{\sqrt{n}} \omega_1 \left( f^r ; q^{2\beta + 1} ; \frac{1}{\sqrt{n}} \right) + \omega_2 \left( f^r ; q^{2\beta} ; \frac{1}{\sqrt{n}} \right) \right).
$$

The Szász-Mirakyan operators are also generalizations of Bernstein polynomials on infinite intervals. They are defined by:

$$
S_n(f)(x) = \sum_{k=0}^{\infty} S_{n,k}(x) f\left( \frac{k}{n} \right),
$$

where

$$
S_{n,k}(x) = \frac{e^{-nx}(nx)^k}{k!}.
$$

In [3], the class of Szász-Mirakyan operators $S_{n,r,q}(f; x)$ was defined as follows:

$$
S_{n,r,q}(f; x) := \frac{1}{A_r(nx)} \sum_{k=0}^{\infty} (nx)^k \frac{(rk)^r}{(rk)!} f\left( \frac{rk}{n + q} \right), \quad x \in [0, \infty),
$$

where $q > 0$ and

$$
A_r(t) = \sum_{k=0}^{\infty} \frac{t^k}{(rk)!}, \quad t \in [0, \infty).
$$

**Theorem 1.6** (see [3]). Let $q > 0$ and $r \in \mathbb{N}$ be fixed numbers. Then there exists $M_{q,r} = \text{const.} > 0$ depending only on $q$ and $r$ such that, for every uniformly continuous and bounded function $f(x) e^{-qx}$ on $[0, \infty)$, the following inequalities hold;
(a)
\[
\left\| (S_{n,q,r}(f;x) - f(x)) \varphi(x)e^{-\alpha x} \right\|_{L_\infty([0,\infty))} 
\leq M_{q,r} \frac{1}{n+q} \left( \left\| f'(x)e^{-\alpha x} \right\|_{L_\infty([0,\infty))} + \left\| f''(x)e^{-\alpha x} \right\|_{L_\infty([0,\infty))} \right);
\]
(1.26)

(b)
\[
\left\| (S_{n,q,r}(f;x) - f(x)) \varphi(x)e^{-\alpha x} \right\|_{L_\infty([0,\infty))} 
\leq M_{q,r} \frac{1}{n+q} (\delta_{n,q}\omega_1(f;e^{-\alpha x};\delta_{n,q}) + \omega_2(f;e^{-\alpha x};\delta_{n,q})),
\]
(1.27)
where \(\delta_{n,q} := (n+q)^{-1/2}\).

(c) for every fixed \(x \in [0,\infty)\), we have for every continuous \(f\) with \(f^{(j)}(x)e^{-\alpha x}, j = 0,1,2,\) bounded on \([0,\infty)\),
\[
\lim_{n \to \infty} n (S_{n,q,r}(f;x) - f(x)) = -qx f'(x) + \frac{x}{2} f''(x).
\]
(1.28)

Now, we modify the Szász-Mirakyan operators as follows: let \(f\) be integrable on \([0,\infty)\), then we define
\[
(Q_{n,\beta}[f])(x) = (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k}(y)f(y)dy, \quad x \in [0,\infty),
\]
(1.29)
where \(\beta\) is a nonnegative integer. Then we have the following results:

**Theorem 1.7.** Let \(\alpha, \beta\) and \(r\) be nonnegative integers. Let \(f \in C^{(r+1)}([0,\infty))\) satisfies
\[
\left| f^{(r)}(x) \right| \leq O(1)e^{\beta x}(x+1)^\alpha, \quad \left| f^{(r+1)}(x) \right| \leq O(1)e^{\beta x}(x+1)^{\alpha+2}.
\]
(1.30)

Then one has uniformly for \(f\) and \(n\),
\[
\left| \left( \frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right| = O\left( \frac{1}{n^{1/3}} \right) e^{\beta x}(x+1)^{\alpha+2}.
\]
(1.31)

In particular, let \(0 < p \leq \infty\). If one supposes \(\|e^{\beta x}(x+1)^{\alpha+2}w(x)\|_{L_p([0,\infty))} < \infty\), then one has uniformly for \(f\) and \(n\),
\[
\left\| \left( \frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right\|_{L_p([0,\infty))} = O\left( \frac{1}{n^{1/3}} \right).
\]
(1.32)
Remark 1.8. (a) We note that for nonnegative integers $\beta$ and $r$,

\[
\left(\frac{n + \beta}{n}\right)^r \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty.
\] (1.33)

(b) The following weight is useful.

\[
\omega_{1,\beta}(x) = e^{-\beta x}\omega_1(x),
\] (1.34)

where $\omega_1(x)$ is defined in Remark 1.3.

**Theorem 1.9.** Let $\beta$, $\gamma$, and $r$ be nonnegative integers. Let $f \in C^{(r+2)}([0, \infty))$ satisfies

\[
\left\| f^{(r+1)}(x)e^{-\beta x}q^{r+1}(x) \right\|_{L_s([0, \infty))} < \infty, \quad \left\| f^{(r+2)}(x)e^{-\beta x}q^{2r+2}(x) \right\|_{L_s([0, \infty))} < \infty.
\] (1.35)

Then one has uniformly for $f$ and $n$,

\[
\left\| \left(\frac{n + \beta}{n}\right)^r (Q_n[f])^{(r)}(x) - f^{(r)}(x) \right\| e^{-\beta x}q^{2r+2}(x) \leq \mathcal{O}\left(\frac{1}{n}\right) \left( \left\| f^{(r+1)}(x)e^{-\beta x}q^{r+1}(x) \right\|_{L_\infty([0, \infty))} + \left\| f^{(r+2)}(x)e^{-\beta x}q^{2r}(x) \right\|_{L_\infty([0, \infty))} \right).
\] (1.36)

**Theorem 1.10.** Let $\beta$, $\gamma$, and $r$ be nonnegative integers. Then one has for $f \in C^r([0, \infty))$,

\[
\left\| \left(\frac{n + \beta}{n}\right)^r (Q_n[f])^{(r)}(x) - f^{(r)}(x) \right\| e^{-\beta x}q^{2r+2}(x) \leq C\left(\frac{1}{\sqrt{n}}\omega_1(f; e^{-\beta x}q^{r+1}(x); \frac{1}{\sqrt{n}}) + \omega_2(f; e^{-\beta x}q^{2r}(x); \frac{1}{\sqrt{n}}) \right).
\] (1.37)

2. **Proofs of Results**

First, we will prove results for Lupas-type operators such as Theorems 1.2, 1.4, and 1.5. To prove theorems, we need some lemmas.

**Lemma 2.1.** Let $m$ and $r$ be nonnegative integers and $n > m + r + 1$. Let

\[
T_{n,m,r}(x) := (n - r - 1) \sum_{k=0}^{\infty} P_{n-r,k}(x) \int_0^\infty P_{n-r,k+r}(y)(y-x)^{m} \, dy.
\] (2.1)

Then

(i) $T_{n,0,r}(x) = 1$, 

(ii) $T_{n,r,0}(x) = 1$, 

(iii) $T_{n,m,r}(x)$ is defined for all $m, r, n$.
(ii)

\[ T_{n,1,r}(x) = \frac{(r + 1)(1 + 2x)}{(n - r + 2)}, \]

\[ T_{n,2,r}(x) = \frac{2(n - 1)x(1 + x)}{(n - r - 2)(n - r - 3)} + \frac{(r + 1)(r + 2)(1 + 2x)^2}{(n - r - 2)(n - r - 3)}; \]  

(2.2)

(iii) for \( m \geq 1 \),

\[ (n - m - r - 2)T_{n,m+1,r}(x) \]

\[ = x(1 + x)(T'_{n,m,r}(x) + 2mT_{n,m-1,r}(x)) + (m + r + 1)(1 + 2x)T_{n,m,r}, \]  

(2.3)

where \( n > m + r + 2 \);

(iv) for \( m \geq 0 \),

\[ T_{n,m,r}(x) = O\left(\frac{1}{n^{[m+1/2]}}\right)q_{n,m,r}(x), \]  

(2.4)

where \( q_{n,m,r}(x) \) is a polynomial of degree \( \leq m \) such that the coefficients are bounded independently of \( n \) and they are positive for \( n > m + r + 1 \).

**Proof.** (i), (ii), and (iii) have been proved in [2, Lemma 1]. So we may show only the part of (2.4). For \( m = 1, 2 \), (2.4) holds. Let us assume (2.4) for \( m \geq 2 \). We note

\[ T_{n,m,r}(x) = O\left(\frac{1}{n^{[m+1/2]}}\right)q_{m,r}(x), \quad q_{m,r}(x) \in \mathcal{P}_{m-1}. \]  

(2.5)

So, we have by the assumption of induction,

\[ (n - m - r - 2)T_{n,m+1,r}(x) \]

\[ = x(1 + x)(T'_{n,m,r}(x) + 2mT_{n,m-1,r}(x)) + (m + r + 1)(1 + 2x)T_{n,m,r} \]

\[ \leq x(1 + x) \left( C \frac{1}{n^{[m+1/2]}} q_{m,r}(x) + 2m \frac{1}{n^{[m/2]}} q_{m-1,r}(x) \right) \]

\[ + (m + r + 1)(1 + 2x) \frac{1}{n^{[m+1/2]}} q_{m,r}(x). \]  

(2.6)

Here, if \( m \) is even, then

\[ \left\lceil \frac{m + 1}{2} \right\rceil + 1 = \frac{m + 2}{2}, \quad \left\lfloor \frac{m}{2} \right\rfloor + 1 = \frac{m + 2}{2}, \quad \left\lceil \frac{m}{2} \right\rceil + 1 = \frac{m + 2}{2}, \quad \left\lfloor \frac{m}{2} \right\rfloor + 1 = \frac{m + 2}{2}. \]  

(2.7)
and if $m$ is odd, then
\[
\left[\frac{m+1}{2}\right] + 1 = \frac{m+1}{2} + 1 = \left[\frac{m+2}{2}\right], \quad \left[\frac{m}{2}\right] + 1 = \frac{m-1}{2} + 1 = \frac{m+1}{2} = \left[\frac{m+2}{2}\right]. \tag{2.8}
\]
Hence, we have
\[
T_{n,m+1,r}(x) = O\left(\frac{1}{n^{(m+2)/2}}\right)q_{n,m+1,r}(x), \tag{2.9}
\]
and here we see that $q_{n,m+1,r}(x)$ is a polynomial of degree $\leq m+1$ such that the coefficients of $q_{n,m+1,r}(x)$ are bounded independently of $n$. Moreover, we see from (2.6) that the coefficients of $q_{n,m+1,r}(x)$ are positive for $n > m + r + 2$.

**Lemma 2.2** (see [2, Lemma 2]). Let $r$ be a nonnegative integer and $n - r - 2 \geq 0$. Then one has for $f \in C^r([0, \infty))$:
\[
(M_n[f])^{(r)}(x) = \frac{(n-r-1)!}{(n-1)!(n-2)!} P_{n+r,k}(x) \int_0^\infty P_{n-r,k+r}(y) f^{(r)}(y) dy. \tag{2.10}
\]

Let
\[
(\overline{M}_n[f])^{(r)}(x) = (n-r-1) \sum_{k=0}^\infty P_{n+r,k}(x) \int_0^\infty P_{n-r,k+r}(y) f^{(r)}(y) dy. \tag{2.11}
\]
Then we have
\[
(\overline{M}_n[f])^{(r)}(x) = A_{n,r}(M_n[f])^{(r)}(x), \tag{2.12}
\]
where $A_{n,r}$ is defined by (1.10).

**Proof of Theorem 1.2.** Let $|y-x| \leq 1$. By the second inequality in (1.11),
\[
|f^{(r)}(y) - f^{(r)}(x)| \leq |y-x| |f^{(r+1)}(\xi)| \leq O(1)|y-x| |f^{(r+1)}(\xi)| \leq O(1)|y-x| |f^{(r+1)}(\xi)| \leq O(1)|y-x| |f^{(r+1)}(\xi)| \leq O(1)|y-x| (x+1)^{a+2}. \tag{2.13}
\]
Let \( \varepsilon := n^{-\gamma}, 0 < \gamma < 1, \)

\[
\left| \left( \frac{M_n[f]}{\varepsilon} \right)^{(r)}(x) - f^{(r)}(x) \right|
\]

\[
= \left| (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \times \left( \int_{|y-x| < \varepsilon} P_{n-r,k+r}(y) \left| f^{(r)}(y) - f^{(r)}(x) \right| dy + \int_{|y-x| \geq \varepsilon} P_{n-r,k+r}(y) \left| f^{(r)}(y) - f^{(r)}(x) \right| dy \right) \right|
\]

\[
=: A + B.
\]  

(2.14)

First, we see by (2.13) and Lemma 2.1,

\[
A = O(1) \left| (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x| < \varepsilon} P_{n-r,k+r}(y) |y-x|(x+1)^{a+2} dy \right|
\]

\[
\leq O(1) \varepsilon |T_{n,0,r}(x)|(x+1)^{a+2} = O(1) \varepsilon (x+1)^{a+2}.
\]  

(2.15)

Next, we estimate \( B \). By the first inequality in (1.11),

\[
B \leq C \left| (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x| < \varepsilon} P_{n-r,k+r}(y) \left( |f^{(r)}(y)| + |f^{(r)}(x)| \right) dy \right|
\]

\[
\leq C \left| (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x| \geq \varepsilon} P_{n-r,k+r}(y) ((y+1)^a + (x+1)^a) dy \right|.
\]  

(2.16)

Here, using

\[
(y+1)^a = ((y-x) + x+1)^a = \sum_{i=0}^{a} \binom{a}{i} (y-x)^i (x+1)^{a-i}
\]  

(2.17)

and the notation:

\[
\langle i \rangle = \begin{cases} 
1, & (i : \text{odd}) \\
0, & (i : \text{even}), 
\end{cases}
\]  

(2.18)

we have

\[
B \leq C |(n-r-1) | \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x| \geq \varepsilon} P_{n-r,k+r}(y) \times \left( \sum_{i=1}^{a} \binom{a}{i} (y-x)^i (x+1)^{a-i} + 2(x+1)^a \right) dy
\]
\[
\leq C(n - r - 1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x| > \epsilon} P_{n-r,k+r}(y) \times \left( \sum_{i=1}^{a} \binom{a}{i} (y-x)^{(i)} \right) (x+1)^{a-i} dy
\]

\[
+ C(n - r - 1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x| > \epsilon} P_{n-r,k+r}(y) \left( \frac{y-x}{\epsilon} \right)^2 (x+1)^a dy
\]

\[
:= B_1 + B_2.
\]

Then, we obtain

\[
B_1 \leq C \left( \sum_{i=1}^{a} \binom{a}{i} |T_{n,i+1,r}(\frac{1}{\epsilon})| (x+1)^{a-i} \right)
\]

\[
\leq C \sum_{i=1}^{a} \binom{a}{i} O\left( \frac{n^{(i)}}{n[(i+1)/2]} \right) |q_{n,i+1,r}(x)| (x+1)^{a-i}
\]

\[
\leq O\left( \frac{1}{n[(i+1)/2] - \gamma(i)} \right) (x+1)^{a+(i)} \leq O\left( \frac{1}{n^{1-\gamma}} \right) (x+1)^{a+1}.
\]

Here, we used the following that for \( i \geq 1 \),

\[
\left[ \frac{i+(i)+1}{2} \right] - \gamma(i) \geq 1 - \gamma,
\]

because

\[
\left[ \frac{i+(i)+1}{2} \right] - \gamma(i) = \begin{cases} 
\frac{i+1}{2} - \gamma, & i : \text{odd}, \\
\frac{i}{2}, & i : \text{even}.
\end{cases}
\]

And we know that

\[
B_2 \leq C|T_{n,2,r}(x)|\left( \frac{1}{\epsilon} \right)^2 x^a \leq O\left( \frac{1}{n^{[3/2]}} \right) |q_{n,2,r}(x)|\left( \frac{1}{\epsilon} \right)^2 x^a
\]

\[
\leq O\left( \frac{n^{2r}}{n^{[3/2]}} \right) |q_{n,2,r}(x)| x^a \leq O\left( \frac{1}{n^{1-2r}} \right) (x+1)^{a+2}.
\]

Thus, we obtain

\[
B \leq O\left( \frac{1}{n^{1-2r}} \right) (x+1)^{a+2}.
\]
Therefore, we have uniformly on $n$,
\[
\left| \left( \mathcal{M}_n[f] \right)^{(r)}(x) - f^{(r)}(x) \right| \leq O \left( \frac{1}{n^\gamma} \right) (x + 1)^{a+2} + O \left( \frac{1}{n^{1/3}} \right) (x + 1)^{a+2}.
\]
(2.25)

Here, if we let $\gamma = 1/3$, then we have
\[
\left| \left( \mathcal{M}_n[f] \right)^{(r)}(x) - f^{(r)}(x) \right| = O \left( \frac{1}{n^{1/3}} \right) (x + 1)^{a+2},
\]
(2.26)
that is, (1.12) is proved. So, we also have a norm convergence (1.13) \(\Box\)

**Proof of Theorem 1.4.** We know that for $f \in C^{(r+2)}([0, \infty))$,
\[
f^{(r)}(t) = f^{(r)}(x) + f^{(r+1)}(x)(t-x) + \int_x^t (t-u)f^{(r+2)}(u)du,
\]
(2.27)
\[
\left| \int_x^t (t-u)f^{(r+2)}(u)du \right| \leq C \left\| f^{(r+2)}(x)\varphi^{2\beta}(x) \right\|_{L^\infty([0,\infty))} \left( (1+x)^{2\beta} + (1+t)^{2\beta} \right)(t-x)^2,
\]
(2.28)
where $\varphi(t) = 1/(1+x)$. Then we obtain from (2.10) and (2.27),
\[
\left( \mathcal{M}_n[f] \right)^{(r)}(x)
= f^{(r)}(x) + f^{(r+1)}(x)T_{n,1,r}(x) + (n-r-1)\sum_{k=0}^\infty P_{n+r,k}(x) \int_0^\infty P_{n-r,k+r}(y) \int_x^y (y-u)f^{(r+2)}(u)du du dy
\]
(2.29)
and from (2.28),
\[
\left| (n-r-1)\sum_{k=0}^\infty P_{n+r,k}(x) \int_0^\infty P_{n-r,k+r}(y) \int_x^y (y-u)f^{(r+2)}(u)du du dy \right|
\leq \left\| f^{(r+2)}(x)\varphi^{2\beta}(x) \right\|_{L^\infty([0,\infty))}
\times \left( (1+x)^{2\beta}|T_{n,2,r}(x)| + (n-r-1)\sum_{k=0}^\infty P_{n+r,k}(x) \int_0^\infty P_{n-r,k+r}(y) (1+y)^{2\beta}(y-x)^2 dy \right).
\]
(2.30)

Using $(1+y)^{2\beta} \leq C((y-x)^{2\beta} + (1+x)^{2\beta})$, we have
\[
\left| (n-r-1)\sum_{k=0}^\infty P_{n+r,k}(x) \int_0^\infty P_{n-r,k+r}(y) (1+y)^{2\beta}(y-x)^2 dy \right|
\leq C \left( |T_{n,2\beta+2,r}(x)| + (1+x)^{2\beta}|T_{n,2,r}(x)| \right).
\]
(2.31)
Therefore, we have

\[
\left| \left( \tilde{M}_n[f] \right)^{(r)}(x) - f^{(r)}(x) \right| \psi^{2\beta+2}(x) \\
\leq \left| f^{(r+1)}(x) \psi^{2\beta+1}(x) \right| |T_{n,1,r}(x)| \psi(x) \\
+ C \left\| f^{(r+2)}(x) \psi^{2\beta}(x) \right\|_{L^\infty([0,\infty))} (1 + x)^{2\beta} |T_{n,2,r}^{(2)}(x)| \psi^{2\beta+2}(x) \\
+ C \left\| f^{(r+2)}(x) \psi^{2\beta}(x) \right\|_{L^\infty([0,\infty))} |T_{n,2\beta+2,r}(x)| \psi^{2\beta+2}(x)
\] (2.32)

Since we know that for \( x \in [0, \infty) \),

\[
|g_{n,1,r}(x)| \psi(x) \leq C, \quad (1 + x)^{2\beta} |g_{n,2,r}(x)| \psi^{2\beta+2}(x) \leq C, \quad |g_{n,2\beta+2,r}(x)| \psi^{2\beta+2}(x) \leq C,
\] (2.33)

we have

\[
\left| \left( \tilde{M}_n[f] \right)^{(r)}(x) - f^{(r)}(x) \right| \psi^{2\beta+2}(x) \\
\leq O\left( \frac{1}{n} \right) \left( \left\| f^{(r+1)}(x) \psi^{2\beta+1}(x) \right\|_{L^\infty([0,\infty))} + \left\| f^{(r+2)}(x) \psi^{2\beta}(x) \right\|_{L^\infty([0,\infty))} \right).
\] (2.34)

\[\square\]

**Lemma 2.3.** Let \( r \) and \( \beta \) be nonnegative integers and \( n - r - 2 \geq 0 \). Let \( f \in C^r([0, \infty)) \) satisfies

\[
\left\| f^{(r)} \psi^{2\beta} \right\|_{L^\infty([0, \infty))} < \infty.
\] (2.35)

Then one has uniformly for \( n \), \( f \) and \( x \in [0, \infty) \),

\[
\left| \left( \tilde{M}_n[f] \right)^{(r)}(x) \right| \psi^{2\beta}(x) \leq C \left\| f^{(r)} \psi^{2\beta} \right\|_{L^\infty([0, \infty))}.
\] (2.36)
Proof. Using \((1 + y)^{2\beta} \leq C((y - x)^{2\beta} + (1 + x)^{2\beta})\), we have

\[
\left| (n - r - 1) \sum_{k=0}^{\infty} P_{n-r,k}(x) \int_0^\infty P_{n-r,k+y}(y)(1 + y)^{2\beta} \, dy \right| \leq C \left( \phi^{-2\beta}(x) + |T_{n,2\beta,r}(x)| \right).
\]  

(2.37)

The assumption (2.35) means

\[
|f^{(r)}(y)| \leq C(1 + y)^{2\beta}.
\]  

(2.38)

Then we can obtain by (2.10),

\[
\left| \mathcal{M}_n[f]^{(r)}(x) \right| \\
\leq C \left| (n - r - 1) \sum_{k=0}^{\infty} P_{n-r,k}(x) \int_0^\infty P_{n-r,k+y}(y)(1 + y)^{2\beta} \, dy \right| \left\| f^{(r)} \phi^{2\beta} \right\|_{L^\infty([0,\infty))} \\
\leq C \left( \phi^{-2\beta}(x) + |T_{n,2\beta,r}(x)| \right) \left\| f^{(r)} \phi^{2\beta} \right\|_{L^\infty([0,\infty))} \\
\leq C \left( \phi^{-2\beta}(x) + O \left( \frac{1}{n^2} \right) |q_{n,2\beta,r}(x)| \right) \left\| f^{(r)} \phi^{2\beta} \right\|_{L^\infty([0,\infty))}.
\]  

(2.39)

Consequently, since \(|q_{n,2\beta,r}(x)|\phi^{2\beta}(x)\) is uniformly bounded on \([0,\infty)\), we have the result. \(\square\)

The Steklov function \([f]_h(x)\) for \(f \in C([0,\infty))\) is defined as follows:

\[
[f]_h(x) := \frac{4}{h^2} \int_0^{h/2} [2f(x + s + t) - f(x + 2(s + t))] \, ds \, dt, \quad x \geq 0, h > 0.
\]  

(2.40)

Then for the Steklov function \([f]_h(x)\) with respect to \(f \in C([0,\infty))\), we have the following properties.

Lemma 2.4 (cf.[4]). Let \(f(x) \in C([0,\infty))\) and \(\eta(x)\) be a positive and nonincreasing function on \([0,\infty)\). Then (i) \([f]_h(x) \in C^2([0,\infty))\);\n
(ii) \[\left\| ([f]_h(x) - f(x))\eta(x) \right\|_{L^\infty([0,\infty))} \leq \omega_2 \left( f; \eta; \frac{h}{2} \right);\]  

(2.41)

(iii) \[\left\| [f]_h'(x)\eta(x) \right\|_{L^\infty([0,\infty))} \leq \frac{4}{h} \omega_1 \left( f; \eta; \frac{h}{2}, \eta(x) \right) \frac{\eta(x)}{\eta(x + (h/2))} + \frac{1}{h} \omega_1 \left( f; \eta; h \right) \frac{\eta(x)}{\eta(x + h)};\]  

(2.42)
\[ (iv) \]
\[
\| [f]_h''(x) \eta(x) \|_{L^\infty([0,\infty))} \leq \frac{4}{h^2} \left[ 2\omega_2 \left( f; \eta; \frac{h}{2} \right) + \frac{1}{4} \omega_2(f; \eta; h) \right]. \quad (2.43)
\]

**Proof.** (i) For \( f \in C([0,\infty)) \), we have the Steklov functions \([f]_h'(x)\) and \([f]_h''(x)\) as follows. We note

\[
[f]_h(x) = \frac{4}{h^2} \int_0^{h/2} \left( \int_x^{x+h/2} 2f(u+t)du - \int_x^{x+h} \frac{1}{2}f(u+2t)du \right) dt, \quad x \geq 0, \ h > 0. \quad (2.44)
\]

Then, we can see from (2.44),

\[
[f]_h'(x) = \frac{4}{h^2} \int_0^{h/2} \left[ \left( \int_x^{x+h/2} f(u+t)du - f(x+t) \right) - \frac{1}{2} \left( f(x+h+2t) - f(x+2t) \right) \right] dt \\
= \frac{4}{h^2} \int_0^{h/2} \left[ 2\Delta_{h/2}^1 f(x+t) - \frac{1}{2} \Delta_h^1 f(x+2t) \right] dt. \quad (2.45)
\]

Similarly to (2.44), we know

\[
[f]_h''(x) = \frac{4}{h^2} \left[ \int_x^{x+h/2} \left( f(u+h) - f(u) \right)du - \frac{1}{4} \int_x^{x+h} \left( f(u+h) - f(u) \right)du \right]. \quad (2.46)
\]

Therefore, we have from (2.46),

\[
[f]_h''(x) = \frac{4}{h^2} \left[ 2 \left( f(x+h) - 2f \left( \frac{x+2h}{2} \right) + f(x) \right) - \frac{1}{4} \left( f(x+2h) - 2f(x+h) + f(x) \right) \right] \\
= \frac{4}{h^2} \left[ 2\Delta_{h/2}^2 f(x) - \frac{1}{4} \Delta_h^2 f(x) \right]. \quad (2.47)
\]

Therefore, (i) is proved.

(ii) We easily see from (2.44) that

\[
| (f(x) - [f]_h(x)) \eta(x) | = \left| \frac{4}{h^2} \int_0^{h/2} \Delta_{h/2}^2 f(x) \eta(x)ds dt \right| \\
\leq \omega_2 \left( f; \eta; \frac{h}{2} \right). \quad (2.48)
\]
From (2.46), we have

\[
\left| [f]_h^{(r)}(x)\eta(x) \right| \leq \frac{4}{h^2} \left| \int_0^{h/2} \left( \Delta_{h/2} f(x + t) \eta(x + t) \right) \frac{\eta(x)}{\eta(x + t)} dt \right| + \frac{4}{h^2} \left| \int_0^{h/2} \frac{1}{2} \left( \Delta_h f(x + 2t) \eta(x + 2t) \right) \frac{\eta(x)}{\eta(x + 2t)} dt \right| + \frac{1}{h} \omega_1(f; \eta; h) \frac{\eta(x)}{\eta(x + h)}.
\]  

(2.49)

(iv) From (2.47), we have

\[
\left| [f]_h^{(r)}(x)\eta(x) \right| \leq \frac{4}{h^2} \left[ 2\omega_2 \left( f; \eta; \frac{h}{2} \right) + \frac{1}{4} \omega_2(f; \eta; h) \right].
\]  

(2.50)

**Proof of Theorem 1.5.** We know that for \( f(x) \in C^{r}([0, \infty)) \),

\[
[f]_h^{(r)}(x) = [f]_h^{(r)}(x), \quad [f]_h^{(r+1)}(x) = [f]_h^{(r+1)}(x), \quad [f]_h^{(r+2)}(x) = [f]_h^{(r+2)}(x).
\]  

(2.51)

Then, we have

\[
\left\| \left( \widetilde{M}_n[f] \right)^{(r)}(x) - f^{(r)}(x) \right\|_{L_\infty([0, \infty))} \leq \left\| \left( \widetilde{M}_n[f - [f]_h] \right)^{(r)}(x) q^{2\beta+2} \right\|_{L_\infty([0, \infty))} + \left\| \left( \widetilde{M}_n[[f]_h] \right)^{(r)}(x) - [f]_h^{(r)}(x) \right\|_{L_\infty([0, \infty))} \right\|_{L_\infty([0, \infty))}.
\]  

(2.52)

From (2.51) and (2.41) of Lemma 2.4,

\[
\left\| \left( \widetilde{M}_n[f - [f]_h] \right)^{(r)}(x) q^{2\beta+2} \right\|_{L_\infty([0, \infty))} \leq \left\| [f]_h^{(r)}(x) - [f]_h^{(r)}(x) q^{2\beta+2} \right\|_{L_\infty([0, \infty))} = \left\| [f]_h^{(r)}(x) - [f]_h^{(r)}(x) q^{2\beta+2} \right\|_{L_\infty([0, \infty))} \right\|_{L_\infty([0, \infty))}.
\]  

(2.53)
Here, we suppose $0 < h \leq 1$ and then we know that

$$\frac{q(x)}{q(x + h)} \leq 2, \quad \frac{q(x)}{q(x + h/2)} \leq 2. \quad (2.54)$$

From Theorem 1.4, (2.51), (2.42), and (2.43) of Lemma 2.4, we have

$$\left\| \left( \widetilde{M}_n [f] \right)^{(r)}(x) - [f]^{(r)}_h(x) q^{2\beta + 2}(x) \right\|_{L_\infty([0,\infty))} \leq O \left( \frac{1}{n} \right) \left( \left\| [f]^{(r+1)}_h(x) q^{2\beta + 1}(x) \right\|_{L_\infty([0,\infty))} + \left\| [f]^{(r+2)}_h(x) q^{2\beta}(x) \right\|_{L_\infty([0,\infty))} \right) \quad (2.55)$$

$$\leq O \left( \frac{1}{n} \right) \left( \frac{1}{h} \omega_1 \left( f^{(r)}, q^{2\beta + 1}; h \right) + \frac{1}{h^2} \omega_2 \left( f^{(r)}, q^{2\beta}; h \right) \right). \quad (2.56)$$

Therefore, we have

$$\left\| \left( \widetilde{M}_n [f] \right)^{(r)}(x) - f^{(r)}(x) \right\|_{L_\infty([0,\infty))} \leq O \left( \frac{1}{n} \right) \left( \frac{1}{h} \omega_1 \left( f^{(r)}, q^{2\beta + 1}; h \right) + \frac{1}{h^2} \omega_2 \left( f^{(r)}, q^{2\beta}; h \right) \right). \quad (2.57)$$

If we let $h = 1/\sqrt{n}$, then

$$\left\| \left( \widetilde{M}_n [f] \right)^{(r)}(x) - f^{(r)}(x) \right\|_{L_\infty([0,\infty))} \leq C \left( \frac{1}{\sqrt{n}} \omega_1 \left( f^{(r)}, q^{2\beta + 1}; \frac{1}{\sqrt{n}} \right) + \omega_2 \left( f^{(r)}, q^{2\beta}; \frac{1}{\sqrt{n}} \right) \right), \quad (2.58)$$

because $\omega_2 \left( f^{(r)}, q^{2\beta + 2}; 1/\sqrt{n} \right) \leq \omega_2 \left( f^{(r)}, q^{2\beta}; 1/\sqrt{n} \right)$. \qed

From now on, we will prove Theorems 1.7, 1.9, and 1.10, which are the results for the Szász-Mirakyan operators, analogously to the case of Lupas-type operators.

**Lemma 2.5.** Let $r$ be a nonnegative integer. Then one has for $f \in C^r([0,\infty))$,

$$(Q_{n,\beta} [f])^{(r)}(x) = \left( n + \beta \right) \left( \frac{n}{n + \beta} \right)^r \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+k+r,y} \left( y \right) f^{(r)}(y) dy, \quad x \in [0, \infty). \quad (2.59)$$
Lemma 2.6. Let

\begin{equation}
S_{n,k}^{(r)}(x) = \sum_{i=0}^{r} \binom{r}{i} \frac{(e^{-ny})^r (ny)^i}{k!} S_{n,k-i}(x),
\end{equation}

(2.59)

Then one has

\begin{equation}
\binom{n}{a+b} \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^\infty S_{n+a,k}(y) f(y) dy
\end{equation}

Therefore, we have

\begin{equation}
(Q_n \beta [f])^{(r)}(x)
\end{equation}

\begin{equation}
= (n + \beta) \sum_{k=0}^{\infty} S_{n,k}^{(r)}(x) \int_0^\infty S_{n+\beta,k}(y) f(y) dy
\end{equation}

\begin{equation}
= (n + \beta) \sum_{i=0}^{r} \sum_{k=0}^{\infty} \binom{r}{i} (-1)^i (-1)^i n^n S_{n,k-i}(x) \int_0^\infty S_{n+\beta,k}(y) f(y) dy
\end{equation}

\begin{equation}
= (n + \beta) \left( \frac{n}{n + \beta} \right) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^\infty \sum_{i=0}^{r} \binom{r}{i} (-1)^i (-1)^i (n + \beta)^i S_{n+\beta,k+i}(y) f(y) dy
\end{equation}

\begin{equation}
= (n + \beta) \left( \frac{n}{n + \beta} \right) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^\infty S_{n,\beta,k+r}(y) f^{(r)}(y) dy.
\end{equation}

\begin{equation}
(R_{n,m,r}(a,b;x) := (n + b) \sum_{k=0}^{\infty} S_{n+a,k}(x) \int_0^\infty S_{n+b,k+r}(y) (y-x)^m dy.
\end{equation}

Then one has

\begin{enumerate}
\item $R_{n,0,r}(a,b;x) = 1$ and $R_{n,1,r}(a,b;x) = ((a-b)x + r + 1)/(n + b)$;
\item For $m \geq 1$
\end{enumerate}

\begin{equation}
(n + b) R_{n,m+1,r}(a,b;x)
\end{equation}

\begin{equation}
= x R_{n,m,r}^{(1)}(a,b;x) + ((a-b)x + m + r + 1) R_{n,m,r}(a,b;x) + 2x m R_{n,m-1,r}(a,b;x);
\end{equation}

\[\square\]
(iii) 

\[ R_{n,m,r}(a, b; x) = O\left(\frac{1}{n^{(m+1)/2}}\right) g_{n,m,r}(a, b; x), \]  

(2.63) 

where \( g_{n,m,r}(a, b; x) \) is a polynomial of degree \( \leq m \) such that the coefficients of \( g_{n,m,r}(a, b; x) \) are bounded independently of \( n \).

Proof. Let \( R_{n,m,r}(x) := R_{n,m,r}(a, b; x) \). Then (i)

\[ R_{n,0,r}(x) = (n + b) \sum_{k=0}^{\infty} S_{n+a,k}(x) \int_{0}^{\infty} S_{n+b,k+r}(y) dy = 1, \]

\[ R_{n,1,r}(x) = (n + b) \sum_{k=0}^{\infty} S_{n+a,k}(x) \int_{0}^{\infty} S_{n+b,k+r}(y - x) dy \]

\[ = \sum_{k=0}^{\infty} S_{n+a,k}(x) \frac{k + r + 1}{n + b} - x \]

\[ = \sum_{k=0}^{\infty} S_{n+a,k}(x) \frac{k}{n + b} + \frac{r + 1}{n + b} - x \]

\[ = \frac{(n + a)x}{n + b} + \frac{r + 1}{n + b} - x = \frac{(a - b)x + r + 1}{n + b}. \] 

(2.64) 

(ii) Using \( x S_{n,k}^{(1)}(x) = (k - nx) S_{n,k}(x) \), we obtain

\[ x \left( R_{n,m,r}^{(i)}(x) + mR_{n,m-1,r}(x) \right) \]

\[ = (n + b) \sum_{k=0}^{\infty} x S_{n+a,k}^{(1)}(x) \int_{0}^{\infty} S_{n+b,k+r}(y) (y - x)^m dy \]

\[ = (n + b) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{0}^{\infty} (k - (n + a)x) S_{n+b,k+r}(y) (y - x)^m dy. \] 

(2.65) 

Here, we see

\[ (k - (n + a)x) S_{n+b,k+r}(y) \]

\[ = ((k + r - (n + b)y) - (r + (a - b)x) + (n + b)(y - x)) S_{n+b,k+r}(y) \] 

\[ = y S_{n+b,k+r}^{(1)}(y) - (r + (a - b)x) S_{n+b,k+r}(y) + (n + b) S_{n+b,k+r}(y) (y - x). \] 

(2.66)
Then substituting (2.66) for (2.65), we consider the following:

\[
x\left(R_{n,m,r}^{(1)}(x) + mR_{n,m-1,r}(x)\right)
= (n + b) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{0}^{\infty} (yS_{n+b,k+r}^{(1)}(y) - (r + (a - b)x)S_{n+b,k+r}(y)
+ (n + b)S_{n+b,k+r}(y)(y - x)(y - x)^m dy
\]

\[
:= \int_{1} + \int_{2} + \int_{3}.
\]

Then, we have:

\[
\int_{1} = (n + b) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{0}^{\infty} yS_{n+b,k+r}^{(1)}(y)(y - x)^m dy
= (n + b) \sum_{k=0}^{\infty} S_{n+a,k}(x) \int_{0}^{\infty} S_{n+b,k+r}^{(1)}(y)(y - x)^{m+1} dy
+ x(n + b) \sum_{k=0}^{\infty} S_{n+a,k}(x) \int_{0}^{\infty} S_{n+b,k+r}^{(1)}(y)(y - x)^m dy
= -(m + 1)R_{n,m,r}(x) - xmR_{n,m-1,r}(x).
\]

Here the last equation follows by parts of integration. Furthermore, we have

\[
\int_{1} + \int_{2} = -(r + (a - b)x)R_{n,m,r}(x) + (n + b)R_{n,m+1,r}(x).
\]

Therefore, we have

\[
(n + b)R_{n,m+1,r}(x) = xR_{n,m,r}^{(1)}(x) + ((a - b)x + m + r + 1)R_{n,m,r}(x) + 2xmR_{n,m-1,r}(x).
\]

(iii) It is proved by the same method as the proof of Lemma 2.1 (iv). \[\square\]

**Proof of Theorem 1.7.** Let \(|y - x| \leq 1\). By the second inequality in (1.30),

\[
\left|f^{(r)}(y) - f^{(r)}(x)\right| = \left|y - x\right|\left|f^{(r+1)}(\xi)\right|
\leq C\left|y - x\right|e^{\beta\xi}(\xi)^{a+2} \leq C\left|y - x\right|e^{\beta x}(x + 1)^{a+2}.
\]
Let $\varepsilon = n^{-\gamma}, 0 < \gamma < 1,$

$$\left| \left( \frac{n + \beta}{n} \right)^r (Q_n \beta [f])^{(r)}(x) - f^{(r)}(x) \right|$$

$$= \left| (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \left( \int_{|y-x| \leq \varepsilon} S_{n+\beta,k+r}(y) \left| f^{(r)}(y) - f^{(r)}(x) \right| dy \right. \right.$$

$$+ \left. \int_{|y-x| \geq \varepsilon} S_{n+\beta,k+r}(y) \left| f^{(r)}(y) - f^{(r)}(x) \right| dy \right) \right|$$

(2.72)

$$=: A + B.$$

First, we see that by (2.71) and Lemma 2.6(i),

$$A \leq C(n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x| \leq \varepsilon} S_{n+\beta,k+r}(y) |y-x| dy \ e^{\beta y} (x+1)^{a+2}$$

(2.73)

$$\leq C \varepsilon e^{\beta x} (x+1)^{a+2} \leq O \left( \frac{1}{n^4} \right) e^{\beta x} (x+1)^{a+2}.$$

Next, to estimate $B$, we split it into two parts:

$$B = \left| (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x| \geq \varepsilon} S_{n+\beta,k+r}(y) \left| f^{(r)}(y) - f^{(r)}(x) \right| dy \right|$$

$$\leq (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x| \geq \varepsilon} S_{n+\beta,k+r}(y) \left( \left| f^{(r)}(y) \right| + \left| f^{(r)}(x) \right| \right) dy$$

(2.74)

$$\leq (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x| \geq \varepsilon} S_{n+\beta,k+r}(y) e^{\beta y} (y+1)^a + e^{\beta x} (x+1)^a dy$$

$$=: B_1 + B_2.$$

First, we estimate

$$B_1 = (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x| \geq \varepsilon} S_{n+\beta,k+r}(y) e^{\beta y} (y+1)^a dy.$$  

(2.75)

Then, using the following facts:

$$S_{n+\beta,k+r}(y) e^{\beta y} = S_{n,k+r}(y) \left( \frac{n + \beta}{n} \right)^{k+r},$$

(2.76)
\[
\left(\frac{n + \beta}{n}\right)^{k+r} S_{n,k}(x) = \left(\frac{n + \beta}{n}\right)^r S_{n+\beta,k}(x)e^{\beta x},
\]
(2.77)

\[\alpha = ((y - x) + x + 1)^\alpha = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (y - x)^i (x + 1)^{\alpha - i},\]
(2.78)

we have

\[
B_1 \leq Ce^{\beta x}(n + \beta) \left(\frac{n + \beta}{n}\right)^r \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \int_{[y-x] > \varepsilon} S_{n,k+r}(y) (y + 1)^\alpha dy
\]
\[
= Ce^{\beta x}n \left(\frac{n + \beta}{n}\right)^{r+1} \sum_{k=0}^{\infty} S_{n+\beta,k}(x)
\]
\[
\times \int_{[y-x] > \varepsilon} S_{n,k+r}(y) \sum_{i=1}^{\alpha} \binom{\alpha}{i} (y - x)^i (x + 1)^{\alpha - i} dy
\]
\[
+ Ce^{\beta x}n \left(\frac{n + \beta}{n}\right)^r \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \int_{[y-x] > \varepsilon} S_{n,k+r}(y) (y + 1)^\alpha dy
\]
\[
=: Ce^{\beta x}(B_{11} + B_{12}).
\]

Then, using (2.18) and Lemma 2.6, we have

\[
B_{11} = n \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \int_{[y-x] > \varepsilon} S_{n,k+r}(y) \sum_{i=1}^{\alpha} \binom{\alpha}{i} (y - x)^i (x + 1)^{\alpha - i} dy
\]
\[
\leq n \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \int_{[y-x] > \varepsilon} S_{n,k+r}(y) \sum_{i=1}^{\alpha} \binom{\alpha}{i} (y - x)^i \left|\frac{y-x}{\varepsilon}\right|^i (x + 1)^{\alpha - i} dy
\]
\[
\leq n \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \int_{[y-x] > \varepsilon} S_{n,k+r}(y) \sum_{i=1}^{\alpha} \binom{\alpha}{i} (y - x)^i \left(\frac{1}{\varepsilon}\right)^i (x + 1)^{\alpha - i} dy
\]
\[
\leq \sum_{i=1}^{\alpha} \binom{\alpha}{i} R_{n,i+1}(\beta,0;x) \left(\frac{1}{\varepsilon}\right)^i (x + 1)^{\alpha - i}
\]
\[
= \sum_{i=1}^{\alpha} \binom{\alpha}{i} O\left(\frac{\epsilon}{n^{(i+1)/2}}\right) q_{n,i+1,r}(\beta,0;x) (x + 1)^{\alpha - i}.
\]

Then by (2.21) we have

\[
B_{11} \leq O\left(\frac{1}{n^{1-\gamma}}\right)(x + 1)^{\alpha+1}.
\]
(2.81)
For $B_{12}$, we have

$$B_{12} = n \sum_{k=0}^{\infty} S_{n^2, k}(x) \int_{|y-x|>\varepsilon} S_{n,k+r}(y)(x+1)^{\alpha} dy$$

$$\leq n \sum_{k=0}^{\infty} S_{n^2, k}(x) \int_{|y-x|>\varepsilon} S_{n,k+r}(y) \left(\frac{y-x}{\varepsilon}\right)^2 dy (x+1)^{\alpha}$$

$$\leq R_{n,2,r}(\beta, 0; x) \left(\frac{1}{\varepsilon}\right)^2 (x+1)^{\alpha} = O\left(\frac{n^{2r}}{n^{[3/2]}}\right) q_{n,2,r}(\beta, 0; x) (x+1)^{\alpha}$$

$$= O\left(\frac{1}{n^{1-2r}}\right) (x+1)^{\alpha+2}.$$ (2.82)

From (2.81), (2.82) and (2.79), we have

$$B_1 \leq O\left(\frac{1}{n^{1-\gamma}}\right) e^{\beta x} (x+1)^{\alpha+2}. \quad (2.83)$$

We estimate

$$B_2 = (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x|>\varepsilon} S_{n^2+k+r}(y)e^{\beta x} (x+1)^{\alpha} dy.$$ (2.84)

Then we can estimate $B_2$ by the same method as $B_{12},$

$$B_2 = (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x|>\varepsilon} S_{n^2+k+r}(y)e^{\beta x} (x+1)^{\alpha} dy$$

$$\leq (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x|>\varepsilon} S_{n^2+k+r}(y) \left(\frac{y-x}{\varepsilon}\right)^2 dy e^{\beta x} (x+1)^{\alpha}$$

$$\leq R_{n,2,r}(0, \beta; x) \left(\frac{1}{\varepsilon}\right)^2 e^{\beta x} (x+1)^{\alpha},$$ (2.85)

so we have

$$B_2 \leq O\left(\frac{n^{2r}}{n^{[3/2]}}\right) q_{n,2,r}(0, \beta; x) e^{\beta x} (x+1)^{\alpha} \leq O\left(\frac{1}{n^{1-2r}}\right) e^{\beta x} (x+1)^{\alpha+2}. \quad (2.86)$$

Consequently, we obtain from (2.83) and (2.86),

$$B \leq \left(O\left(\frac{1}{n^{1-\gamma}}\right) + O\left(\frac{1}{n^{1-2r}}\right)\right) e^{\beta x} (x+1)^{\alpha+2} \leq O\left(\frac{1}{n^{1-2r}}\right) e^{\beta x} (x+1)^{\alpha+2}. \quad (2.87)$$
Therefore, from (2.73) and (2.87), we have uniformly on $n$,
\[
\left| \left( \frac{n + \beta}{n} \right)^r Q_{n, \beta} [f]^{(r)} (x) - f^{(r)} (x) \right| \leq O \left( \frac{1}{n^r} \right) e^{\beta x} x^{a+2} + O \left( \frac{1}{n^{1-\gamma}} \right) e^{\beta x} x^{a+2}. \tag{2.88}
\]

Here, if we let $\gamma = 1/3$, then we have
\[
\left| \left( \frac{n + \beta}{n} \right)^r Q_{n, \beta} [f]^{(r)} (x) - f^{(r)} (x) \right| = O \left( \frac{1}{n^{1/3}} \right) e^{\beta x} x^{a+2}, \tag{2.89}
\]
that is, (1.31) is proved. So, we also have a norm convergence (1.32).

**Lemma 2.7.** Let $m$ and $b$ be nonnegative integers. Let
\[
U_{n,m,r} (b; x) := (n + b) \sum_{k=0}^{\infty} S_{n,k} (x) \int_{0}^{\infty} S_{n+b,k+r} (y) (y - x)^m e^{by} dy.
\tag{2.90}
\]
Then one has
\[
U_{n,m,r} (b; x) = \left( \frac{n + b}{n} \right)^{r+1} e^{bx} R_{n,m,r} (b, 0; x). \tag{2.91}
\]

**Proof.** From (2.76) and (2.77) we have
\[
S_{n+b,k+r} (x) e^{bx} = S_{n,k+r} (x) \left( \frac{n + b}{n} \right)^{k+r}, \tag{2.92}
\]
\[
\left( \frac{n + b}{n} \right)^{k+r} S_{n,k} (x) = \left( \frac{n + b}{n} \right)^{r} S_{n+b,k} (x) e^{bx}. \tag{2.93}
\]
We have from (2.92), (2.93), and noting (2.61),
\[
U_{n,m,r} (b; x) = (n + b) \sum_{k=0}^{\infty} S_{n,k} (x) \int_{0}^{\infty} S_{n+b,k+r} (y) (y - x)^m e^{by} dy
\]
\[
= (n + b) \sum_{k=0}^{\infty} \left( \frac{n + b}{n} \right)^{k+r} S_{n,k} (x) \int_{0}^{\infty} S_{n,k+r} (y) (y - x)^m dy
\]
\[
= (n + b) \left( \frac{n + b}{n} \right)^{r} e^{bx} \sum_{k=0}^{\infty} S_{n+b,k} (x) \int_{0}^{\infty} S_{n,k+r} (y) (y - x)^m dy
\]
\[
= \left( \frac{n + b}{n} \right)^{r+1} e^{bx} R_{n,m,r} (b, 0; x). \tag{2.94}
\]
Using the inequality

\[ f^{(r+1)}(x)R_n,1,0(\beta; x) \]

we prove this theorem, similarly to the proof of Theorem 1.4. Using (2.93) and (2.27), we have for \( f \in C^{(r+2)}([0, \infty)) \),

\[
\left( \frac{n + \beta}{n} \right)^{r} (Q_{n, \beta}[f])^{(r)}(x) \\
= (n + \beta)^{r} \sum_{k=0}^{\infty} S_{n,k}(x) \int_{0}^{\infty} S_{n+\beta,k+r}(y) f^{(r)}(y) dy \\
= f^{(r)}(x) + f^{(r+1)}(x) (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{0}^{\infty} S_{n+\beta,k+r}(y) (y - x) dy \\
+ (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{0}^{\infty} S_{n+\beta,k+r}(y) \left( \int_{x}^{y} (y - u) f^{(r+2)}(u) du \right) dy \\
= f^{(r)}(x) + f^{(r+1)}(x) R_n,1,0(\beta; x) \\
+ (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{0}^{\infty} S_{n+\beta,k+r}(y) \left( \int_{x}^{y} (y - u) f^{(r+2)}(u) du \right) dy.
\]

We estimate the last term. We note the given condition:

\[
\left\| f^{(r+2)}(x)e^{-\beta x}q^{2}\gamma(x) \right\|_{L_{\infty}([0, \infty))} < \infty.
\]

Using the inequality \((1 + t)^{2\gamma} \leq C((1 + x)^{2\gamma} + (t - x)^{2\gamma})\) and Lemma 2.7, we have

\[
\sum := (n + \beta)^{r} \sum_{k=0}^{\infty} S_{n,k}(x) \int_{0}^{\infty} S_{n+\beta,k+r}(y) \left( e^{\beta y} q^{-2}\gamma(y) + e^{\beta x} q^{-2}\gamma(x) \right) (y - x)^{2} dy \\
\leq C (n + \beta)^{r} \sum_{k=0}^{\infty} S_{n,k}(x) \int_{0}^{\infty} S_{n+\beta,k+r}(y) \\
\times \left( e^{\beta y} q^{-2}\gamma(y) + e^{\beta x} q^{-2}\gamma(x) \right) (y - x)^{2} dy \\
\leq C \left( q^{-2}\gamma(x) U_{n,2,0}(\beta; x) + U_{n,2,1,0}(\beta; x) + e^{\beta x} q^{-2}\gamma(x) R_{n,2,0}(0, \beta; x) \right) \\
\leq C \left( e^{\beta x} q^{-2}\gamma(x) R_{n,2,0}(0, \beta; x) \left( \frac{n + \beta}{n} \right)^{r+1} \\
+ e^{\beta x} R_{n,2,1,0}(0, \beta; x) \left( \frac{n + \beta}{n} \right)^{r+1} + e^{\beta x} q^{-2}\gamma(x) R_{n,2,0}(0, \beta; x) \right).
\]

Then, we can estimate as follows:

\[
\left\| (n + \beta)^{r} \sum_{k=0}^{\infty} S_{n,k}(x) \int_{0}^{\infty} S_{n+\beta,k+r}(y) \left( \int_{x}^{y} (y - u) f^{(r+2)}(u) du \right) dy \right\|_{L_{\infty}([0, \infty))} < \infty.
\]
\[ \leq \left\| f^{(r+2)}(x)e^{-\beta x}q^{2\gamma}(x) \right\|_{L_{\infty}([0,\infty))} \sum \]
\[ \leq C \left\| f^{(r+2)}(x)e^{-\beta x}q^{2\gamma}(x) \right\|_{L_{\infty}([0,\infty))} e^{\beta x} \]
\[ \times \left( q^{-2\gamma}(x)R_{n,2r}(\beta, 0, x) \left( \frac{n + \beta}{n} \right)^{r+1} + R_{n,2\gamma+2r}(\beta, 0, x) \left( \frac{n + \beta}{n} \right)^{r+1} \ight. \]
\[ + q^{-2\gamma}(x)R_{n,2r}(0, \beta; x). \]
\[ (2.98) \]

Then, we have by (iv) of Lemma 2.6,
\[ \left\| \left( \frac{n + \beta}{n} \right)^r \left( Q_{n,\beta} [f] \right)^{(r)}(x) - f^{(r)}(x) \right\|_{L_{\infty}([0,\infty))} e^{-\beta x}q^{2\gamma}(x) \]
\[ \leq \left\| f^{(r+2)}(x)e^{-\beta x}q^{2\gamma}(x) \right\|_{L_{\infty}([0,\infty))} \]
\[ \times \left( O\left( \frac{1}{n} \right) q^{2}(x)g_{n,2r}(\beta, 0; x) + O\left( \frac{1}{n^2} \right)g_{n,2\gamma+2r}(\beta, 0; x)q^{2\gamma+2}(x) \right. \]
\[ + O\left( \frac{1}{n} q^{2}(x)g_{n,2r}(0, \beta; x) \right) \]
\[ + O\left( \frac{1}{n} \right) \left. \right| f^{(r+1)}(x)e^{-\beta x}q^{2\gamma+1}(x)g_{n,1r}(0, \beta; x)q(x) \right|. \]
\[ (2.99) \]

Consequently, we have
\[ \left\| \left( \frac{n + \beta}{n} \right)^r \left( Q_{n,\beta} [f] \right)^{(r)}(x) - f^{(r)}(x) \right\|_{L_{\infty}([0,\infty))} e^{-\beta x}q^{2\gamma}(x) \]
\[ \leq O\left( \frac{1}{n} \right) \left( \left\| f^{(r+1)}(x)e^{-\beta x}q^{2\gamma+1}(x) \right\|_{L_{\infty}([0,\infty))} + \left\| f^{(r+2)}(x)e^{-\beta x}q^{2\gamma}(x) \right\|_{L_{\infty}([0,\infty))} \right), \]
\[ (2.100) \]

since we know that \( |g_{n,2r}(\beta, 0; x)q^{2}(x)|, |g_{n,2r}(0, \beta; x)q^{2}(x)|, |g_{n,2\gamma+2r}(\beta, 0; x)q^{2\gamma+2}(x)|, \) and \( |g_{n,1r}(0, \beta; x)q(x)| \) are uniformly bounded on \([0, \infty)\). \( \square \)

**Theorem 2.8.** Let \( \beta \) and \( \gamma \) be nonnegative integers and \( r > 0 \) be a positive integer. Then one has uniformly for \( n, f \) and \( x \in [0, \infty) \),
\[ \left\| \left( Q_{n,\beta} [f] \right)^{(r)}(x) \right\|_{L_{\infty}([0,\infty))} e^{-\beta x}q^{2\gamma}(x) \leq C \left( \frac{n + \beta}{n} \right) \left\| f^{(r)}(x)e^{-\beta x}q^{2\gamma}(x) \right\|_{L_{\infty}([0,\infty))}, \]
\[ (2.101) \]
Proof. Using \((1 + y)^{2r} \leq C((1 + x)^{2r} + (y - x)^{2r})\), we have

\[
\left| (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) e^{\beta y} (1 + y)^{2r} dy \right|
\leq \left| (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) e^{\beta y} (1 + x)^{2r} dy \right|
\leq C \left( U_{n,0,r}(\beta; x) \psi^{2r}(x) + U_{n,2r}(\beta; x) \right).
\]

By Lemma 2.7 and (i) of Lemma 2.6, we know

\[
U_{n,0,r}(\beta; x) = \left( \frac{n + \beta}{n} \right)^{r+1} e^{\beta x} R_{n,0,r}(\beta, 0; x) = \left( \frac{n + \beta}{n} \right)^{r+1} e^{\beta x},
\]

\[
U_{n,2r}(\beta; x) = \left( \frac{n + \beta}{n} \right)^{r+1} e^{\beta x} R_{n,2r}(\beta, 2r; x)
= O \left( \frac{1}{n^r} \right) \left( \frac{n + \beta}{n} \right)^{r+1} e^{\beta x} g_{n,2r}(\beta, 2r; x).
\]

Therefore, we have

\[
\left| (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) e^{\beta y} (1 + y)^{2r} dy \right|
\leq C \left( \frac{n + \beta}{n} \right)^{r+1} e^{\beta x} \left( \psi^{2r}(x) + O \left( \frac{1}{n^r} \right) g_{n,2r}(\beta, 2r; x) \right).
\]

Since \(|g_{n,2r}(\beta, 2r; x)|\) is uniformly bounded on \([0, \infty)\), we have

\[
\left| \left( \frac{n + \beta}{n} \right)^{r} (Q_{n,\beta} [f])^{(r)}(x) e^{-\beta x} \psi^{2r}(x) \right|
\leq \left| (n + \beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) e^{\beta y} (1 + y)^{2r} dy \right| e^{-\beta x} \psi^{2r}(x)
\times \left\| f^{(r)}(x) e^{-\beta x} \psi^{2r}(x) \right\|_{L_\infty([0, \infty))}
\leq C \left( \frac{n + \beta}{n} \right)^{r+1} \left( 1 + O \left( \frac{1}{n^r} \right) \right) \left\| f^{(r)}(x) e^{-\beta x} \psi^{2r}(x) \right\|_{L_\infty([0, \infty))}.
\]

Therefore, we have the result. \(\Box\)
Theorem 1.5. First, we split it as follows:

\[
\left\| \left( \frac{n + \beta}{n} \right)^r \left( Q_{\eta, \beta} [f] \right)^{(r)} (x) - f^{(r)}(x) \right\|_{L_\infty([0, \infty))} \leq \left\| \left( \frac{n + \beta}{n} \right)^r \left( Q_{\eta, \beta} [f - [f]_h] \right)^{(r)} (x) \right\|_{L_\infty([0, \infty))} + \left\| \left( \frac{n + \beta}{n} \right)^r \left( Q_{\eta, \beta} [f]_h \right)^{(r)} (x) - [f]_h^{(r)}(x) \right\|_{L_\infty([0, \infty))} + \left\| [f]_h^{(r)}(x) - f^{(r)}(x) \right\|_{L_\infty([0, \infty))},
\]

(2.106)

Then for the first term, we have, using Theorem 2.8 and (2.41),

\[
\left\| \left( Q_{\eta, \beta} [f - [f]_h] \right)^{(r)} (x) e^{-\beta x \psi_{2r+2}}(x) \right\|_{L_\infty([0, \infty))} \leq C \left( \frac{n + \beta}{n} \right)^r \left\| f^{(r)}(x) - [f]_h^{(r)}(x) \right\|_{L_\infty([0, \infty))} \leq C \omega_2 \left( f^{(r)}; e^{-\beta x \psi_{2r+2}}(x); h \right).
\]

(2.107)

For the second term, we have from Theorem 1.9,

\[
\left\| \left( \frac{n + \beta}{n} \right)^r \left( Q_{\eta, \beta} [f]_h \right)^{(r)} (x) - f^{(r)}_h(x) \right\|_{L_\infty([0, \infty))} \leq O \left( \frac{1}{n} \right) \left( \left\| f^{(r+1)}_h(x) e^{-\beta x \psi_{2r+1}}(x) \right\|_{L_\infty([0, \infty))} + \left\| f^{(r+2)}_h(x) e^{-\beta x \psi_{2r}}(x) \right\|_{L_\infty([0, \infty))} \right).
\]

(2.108)

Here, we suppose \(0 < h \leq 1\) and then we know that \((e^{-\beta x \psi}(x))/(e^{-\beta(x+h)} \psi(x + h))\) and \(e^{-\beta x}/e^{-\beta(x+h)}\) are uniformly bounded on \([0, \infty)\). Therefore, we have from (2.42) and (2.43) of Lemma 2.4,

\[
\left\| f^{(r+1)}_h(x) e^{-\beta x \psi_{2r+1}}(x) \right\|_{L_\infty([0, \infty))} \leq C \frac{1}{h} \omega_1 \left( f; e^{-\beta x \psi_{2r+1}}(x); h \right),
\]

(2.109)

\[
\left\| f^{(r+2)}_h(x) e^{-\beta x \psi_{2r}}(x) \right\|_{L_\infty([0, \infty))} \leq C \frac{1}{h^2} \omega_2 \left( f; e^{-\beta x \psi_{2r}}(x); h \right).
\]

Therefore, we have

\[
\left\| \left( \frac{n + \beta}{n} \right)^r \left( Q_{\eta, \beta} [f]_h \right)^{(r)} (x) - f^{(r)}_h(x) \right\|_{L_\infty([0, \infty))} \leq O \left( \frac{1}{n} \right) \left( \frac{1}{h} \omega_1 \left( f; e^{-\beta x \psi_{2r+1}}(x); h \right) + \frac{1}{h^2} \omega_2 \left( f; e^{-\beta x \psi_{2r}}(x); h \right) \right).
\]

(2.110)
Consequently, we have
\[
\left\| \left( \frac{n + \beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right\| e^{-\beta x} q^{2r+2}(x)
\leq O \left( \frac{1}{n} \left( \frac{1}{h} \omega_1(f; e^{-\beta x} q^{2r+1}(x); h) + \frac{1}{h^2} \omega_2(f; e^{-\beta x} q^{2r}(x); h) \right) + C \omega_2 \left( f^{(r)}; e^{-\beta x} q^{2r+2}(x)(x); h \right). \tag{2.111}
\]

If we let \( h = 1/\sqrt{n} \), then
\[
\left\| \left( \frac{n + \beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right\| e^{-\beta x} q^{2r+2}(x)
\leq C \left( \frac{1}{\sqrt{n}} \omega_1 \left( f; e^{-\beta x} q^{2r+1}(x); \frac{1}{\sqrt{n}} \right) + \omega_2 \left( f; e^{-\beta x} q^{2r}(x); \frac{1}{\sqrt{n}} \right) \right), \tag{2.112}
\]

since \( \omega_2(f^{(r)}; e^{-\beta x} q^{2r+2}(x); 1/\sqrt{n}) \leq \omega_2(f; e^{-\beta x} q^{2r}(x); 1/\sqrt{n}) \).

\[ \square \]

3. Conclusion

In this paper, Lupas-type operators and Szász-Mirakyan-type operators are treated and the various weighted norm convergence on \([0, \infty)\) of these operators are investigated. Moreover, this paper proves theorems on degree of approximation of \( f \in C^r([0, \infty)) \) by these operators using the modulus of smoothness of \( f \).

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