Fitting method of rotating paraboloid reflector

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Abstract. In order to get the actual spatial position and orientation and evaluate surface accuracy of rotating paraboloid reflector under working condition, a spatial transforming model is established by analyzing relationship between local coordinate system and global measuring coordinate system. Two optimization fitting methods, eigenvalue parameter method and rigid body displacement method, were promoted based on the principle of least squares. Two fitting methods were analyzed and compared by comparing their fitting results. Optimum fitting parameters and the best fitting paraboloid surface are obtained, which shows that parabolic surface of reflector can be effectively fitted by two methods. During calculation, rigid body displacement method performs better for its stability. This could be a useful reference for surface accuracy fitting and detecting of different reflectors.

1. Introduction

Paraboloid reflectors with high surface accuracy, which can send and receive electrical signal, are widely used for communication, navigation and exploration. Reflectors are carefully designed to obtain high surface accuracy by choosing proper structure parameters. However, surface errors are unavoidable due to several reasons, such as manufacturing and assembly tolerance, structure deformation caused by gravity, influence of radiation and temperature changes, loads from wind or snow etc. Sometimes, it may cause high surface deviation and lead to serious problems or disable the reflector.

Actual surface contour under working condition can be got by measuring positions of points on reflector surface using total station instrument. The coordinates of all points are measured in global measuring coordinate system $o_s - x_s,y_s,z_s$. However, the target standard paraboloid surface should be seated at its original point in local coordinate system $o_e - x_e,y_e,z_e$, as shown in Fig. 1. The target standard paraboloid surface should be fitted and surface accuracy should be checked.

![Figure 1. Paraboloid position diagram](image-url)
To fit rotating paraboloid, Chen J.H. [1] studied the selection of minimum objective function value in least square method for fitting paraboloid, and analyzed the results of different algorithms. Li G. [2] proposed a fitting method based on robust estimation and weighted least squares, and compared it with traditional least square method. Cheng X.J. [3] fitted paraboloid by using general quadratic surface equation and rotating parabolic restrictive condition. Wang J.X. [4] proposed a fitting method with parameters of eigenvalue, rotating angle and translation. Parameters obtained by above methods are indirect parameters and need to be furtherly converted to paraboloid parameters finally. Wang C.S. [5] proposed a method which can solve optimum fitting parameters directly. However, it only can be applied to transforming with small rotating angles. Ma Z.X. [6] proposed a new method based on rigid body displacement principle which could directly solve optimum fitting parameters and could be applied to any rotating angle values. Wang L.H. [7] improved genetic algorithm to solve nonlinear least squares error equation. Späth H. [8] and Dai M. [9] introduced additional variables for different solving algorithms to get direct solution of fitting parameters.

Two fitting algorithms, eigenvalue parameter method and rigid body displacement method, were promoted to find the target fitting paraboloid surface. The best fitting paraboloid surface was obtained based on least squares principle [10] and the shape, position, orientation and surface error of rotating paraboloid reflector were calculated too.

2. Paraboloid Surface Fitting Model

According to spatial coordinate transforming theory [11], points $X_i = (x_i, y_i, z_i)^T$ ($i = 1, 2, ..., n$, $n$ is the number of measure points) in global coordinate system can be transformed to points $X_i' = (x_i', y_i', z_i')^T$ in local coordinate system by applying relation

$$
\begin{pmatrix}
x_i \\
y_i \\
z_i
\end{pmatrix} = R(\alpha, \beta, \gamma) \begin{pmatrix}
x_i - \Delta x \\
y_i - \Delta y \\
z_i - \Delta z
\end{pmatrix}
$$

(1)

where $\alpha, \beta, \gamma$ and $\Delta x, \Delta y, \Delta z$ represent rotating angles around axes $x, y, z$ and translations along axes $x, y, z$ in global coordinate system. $R(\alpha, \beta, \gamma)$ is standard rotating transforming matrix. $R(\alpha, \beta, \gamma) = R_z(\gamma) R_2(\beta) R_1(\alpha)$

(2)

Where $R_1(\alpha) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos(\alpha) & \sin(\alpha) \\
0 & -\sin(\alpha) & \cos(\alpha)
\end{pmatrix}$; $R_2(\beta) = \begin{pmatrix}
\cos(\beta) & 0 & -\sin(\beta) \\
0 & 1 & 0 \\
\sin(\beta) & 0 & \cos(\beta)
\end{pmatrix}$; $R_z(\gamma) = \begin{pmatrix}
\cos(\gamma) & \sin(\gamma) & 0 \\
-\sin(\gamma) & \cos(\gamma) & 0 \\
0 & 0 & 1
\end{pmatrix}$.

In local coordinate system, target rotating paraboloid is circular symmetric about axis $z'$, as shown in Fig. 1. So, the angle $\gamma$ around axis $z'$ can be set to $0$ ($\gamma = 0$).

In order to get standard paraboloid equation $x'^2 + y'^2 = 4fz'$ ($f$ is focal length) of measure points in local coordinate system, six optimum fitting parameters: $\alpha, \beta, \Delta x, \Delta y, \Delta z$ and $f$, need to be determined by fitting method. The key of fitting is to determine the best values of these six parameters to obtain the best fitting paraboloid surface of measure points. Two fitting methods are promoted to solve this model.

3. Eigenvalue Parameter Method

When fitting rotating paraboloid surface, eigenvalue of coefficient matrix is added as a surface fitting calculation parameter in eigenvalue parameter method [12].

The basic equation of general quadric surface can be expressed as
where \( a_0, a_1, a_2, \ldots, a_g \) are indetermined coefficients.

Eq. 3 can be rewritten as following matrix form

\[
a_0 + (a_1, a_2, a_3)x_i + X_i^TDX_i = 0
\]  

(4)

where \( X_i = (x_{i1}, y_{i1}, z_{i1})^T \) is the coordinate matrix of measure points in global coordinate system;

\[
D = \begin{pmatrix}
a_2 & a_4/2 & a_6/2 \\
a_4/2 & a_8 & a_6/2 \\
a_6/2 & a_8/2 & a_g
\end{pmatrix}
\]

is coefficient matrix.

If the eigenvalue of \( D \) is \( \Lambda \), the eigenvector of \( D \) is \( R \), \( \Lambda \) can be expressed as

\[
\Lambda = R^T DR
\]

(5)

According to Eq. 5, Eq. 4 can be expanded as

\[
a_0 + (a_1, a_2, a_3)x_i + X_i^T RAR^T X_i = 0
\]

(6)

Constant parameter \( a_0 \) is assumed to be 1. There are six unknown parameters: \( a_1, a_2, a_3, \lambda, \alpha \) and \( \beta \) in Eq. 6 with initial given values \( a_1^{(0)} = a_2^{(0)} = a_3^{(0)} = \lambda^{(0)} = \alpha^{(0)} = \beta^{(0)} = 0 \).

During calculation, error equation for point \( i \) is listed as

\[
v_i = a_0 + (a_1, a_2, a_3)x_i + X_i^T RAR^T X_i
\]

(7)

By applying newton method, \( v_i \) can be expressed as

\[
v_i = \frac{\delta v_i}{\delta a_1} \delta a_1 + \frac{\delta v_i}{\delta a_2} \delta a_2 + \frac{\delta v_i}{\delta a_3} \delta a_3 + \frac{\delta v_i}{\delta \lambda} \delta \lambda + \frac{\delta v_i}{\delta \alpha} \delta \alpha + \frac{\delta v_i}{\delta \beta} \delta \beta - l_i
\]

(8)

where

\[
l_i = -a_0 - (a_1, a_2, a_3)x_i - X_i^T RAR^T X_i.
\]

Inconsistent equations can be obtained from Eq. 8 and can be solved by generalized inverse theory [13]. The termination condition for calculation iteration is

\[
tol_i = \delta a_1^2 + \delta a_2^2 + \delta a_3^2 + \delta \lambda^2 + \delta \alpha^2 + \delta \beta^2 \leq \varepsilon_i
\]

(9)

where \( \varepsilon_i \) is given error limit, take \( \varepsilon_i = 10^{-12} \).

After getting values of all parameters, all measure points could be rotated, and the coordinates of points are changed to

\[
X_{mi} = R^T X_i = (x_{mi}, y_{mi}, z_{mi})^T
\]

, Eq. 6 becomes

\[
a_0 + (b_1, b_2, b_3)x_i + X_{mi}^T D X_{mi} = 0
\]

(10)

where \( (b_1, b_2, b_3) = (a_1, a_2, a_3)R \).

Then \( X_{mi} \) can be expressed as
Substitute Eq. 11 into Eq. 10, standard paraboloid equation is as following

$$x_{\alpha}^2 + y_{\alpha}^2 = -\frac{b_1}{\lambda} z_{\alpha}$$

(12)

The focal length $f$ of rotating paraboloid is

$$f = -\frac{b_1}{4\lambda}$$

4. **Rigid Body Displacement Method**

All points on reflector are assumed to be rigid during translating and rotating, that is no relative motion among these points [6]. In this method, Eq. 1 can be written as

$$X_{\alpha} = R^T (X_{\alpha} - \Delta X) = R^T X_{\alpha} - R^T \Delta X$$

(13)

The standard paraboloid equation is rewritten as following matrix form

$$(x_{\alpha}, y_{\alpha}, z_{\alpha}) \Lambda_i (x_{\alpha}, y_{\alpha}, z_{\alpha})^T = (0 \ 0 \ 0 \ f) (x_{\alpha} \ y_{\alpha} \ z_{\alpha})^T$$

(14)

Where

$$\Lambda_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Substitute Eq. 13 into Eq. 14, then we have

$$\left(R^T (X_{\alpha} - \Delta X)\right)^T \Lambda_i R^T (X_{\alpha} - \Delta X) = (0 \ 0 \ 0 \ f) R^T (X_{\alpha} - \Delta X)$$

(15)

Six unknown parameters: $\Delta x$, $\Delta y$, $\Delta z$, $\alpha$, $\beta$ and $f$ need to be solved with initial values as $\Delta x^{(0)} = \Delta y^{(0)} = \Delta z^{(0)} = \alpha^{(0)} = \beta^{(0)} = f^{(0)} = 0$.

Error equation for measure point $i$ is

$$v_i = \left(R^T (X_{\alpha} - \Delta X)\right)^T \Lambda_i R^T (X_{\alpha} - \Delta X) - (0 \ 0 \ 0 \ f) R^T (X_{\alpha} - \Delta X)$$

(16)

By applying newton method, $v_i$ can be expressed as

$$v_i = \frac{\partial v_i}{\partial \Delta x} \delta \Delta x + \frac{\partial v_i}{\partial \Delta y} \delta \Delta y + \frac{\partial v_i}{\partial \Delta z} \delta \Delta z + \frac{\partial v_i}{\partial \alpha} \delta \alpha + \frac{\partial v_i}{\partial \beta} \delta \beta + \frac{\partial v_i}{\partial f} \delta f - l_i$$

(17)

where

$$l_i = \left(R^T (X_{\alpha} - \Delta X)\right)^T \Lambda_i R^T (X_{\alpha} - \Delta X) + (0 \ 0 \ 0 \ f) R^T (X_{\alpha} - \Delta X)$$

The termination condition for calculation iteration is

$$tol = \delta \Delta x^2 + \delta \Delta y^2 + \delta \Delta z^2 + \delta \alpha^2 + \delta \beta^2 + \delta f^2 \leq \varepsilon_2$$

(18)

where $\varepsilon_2$ is given error limit, $\varepsilon_2 = 10^{-12}$.

5. **Fitting Example**

For a real paraboloid reflector, spatial positions of 504 measure points were measured, as shown in Fig. 2. Due to deformation of surface, most of points deviate designed paraboloid surface. New paraboloid
surface was calculated by using two fitting methods discussed above. Different error equation leads to different values of fitting parameters for different transforming sequences adopted by two fitting methods which global coordinate system is rotating first and then translating in eigenvalue parameter method but the transforming process is converse in rigid body displacement method. The result is shown in Table 1.

![Figure 2. Spatial position of measure points in global coordinate system](image)

|                  | $f$ [mm] | $\Delta x$ [mm] | $\Delta y$ [mm] | $\Delta z$ [mm] | $\alpha$ [rad] | $\beta$ [rad] | $\gamma$ [rad] | $e_{xys}$ [mm] |
|------------------|----------|-----------------|-----------------|-----------------|----------------|----------------|----------------|----------------|
| Eigenvalue method | -593.43  | 305.90          | 377.84          | 377.84          | 0.6446         | -0.4298        | 0.00           | 1.0925         |
| Rigid body       | -588.91  | 311.19          | 370.42          | 370.42          | 0.6506         | -0.4400        | 0.00           | 1.0928         |

The convergence processes of two fitting methods are shown in Fig. 3. It shows that rigid body displacement method performs better for its faster convergence speed and better stability.

![Figure 3. Convergence process of two methods](image)

To evaluate surface accuracy of fitted paraboloid surface, root mean square surface error [6,14] was obtained by calculating the normal distance between each measure point and fitting surface. Error distribution of all measure points of two methods are shown in Fig. 4. It shows that two methods are consistent for their fitting results.
6. Summary

To fit rotating paraboloid surface, some optimum fitting parameters: $\alpha, \beta, \Delta x, \Delta y, \Delta z$ and $f$ need to be determined to get the best fitting paraboloid surface with discrete measure points. Both of eigenvalue parameter method and rigid body displacement method work well in fitting rotating paraboloid reflector at arbitrary position in space. When using these two methods, generalized inverse theory was used to solve the inconsistent equations of least squares method. Two fitting methods were programmed and an actual example was conducted and analyzed. The result shows that both methods work well in fitting paraboloid surface.

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