$L_\infty$-Formality check for the Hochschild Complex of certain Universal Enveloping Algebras

Martin Bordemann$^1$, Olivier Elchinger$^2\ast$, Simone Gutt$^3$, and Abdenacer Makhlouf$^1$

$^1$ Laboratoire de Mathématiques, Informatique et Applications, Université de Haute-Alsace, Mulhouse, Martin.Bordemann@uha.fr, Abdenacer.Makhlouf@uha.fr
$^2$ Mathematics Research Unit, University of Luxembourg, Olivier.Elchinger@uha.fr
$^3$ Département de Mathématiques, Université Libre de Bruxelles, sgutt@ulb.ac.be

July 10, 2018

Abstract

We study the $L_\infty$-formality problem for the Hochschild complex of the universal enveloping algebra of some examples of Lie algebras such as Cartan-3-regular quadratic Lie algebras (for example semisimple Lie algebras and in more detail $\mathfrak{so}(3)$), and free Lie algebras. We show that for these examples formality in Kontsevich’s sense does NOT hold, but we compute the $L_\infty$ structure on the cohomology given by homotopy transfer in certain cases.

Introduction

Since Maxim Kontsevich’s seminal paper [Kon03] on deformation quantization on any Poisson manifold, his concept of $L_\infty$-formality of the Hochschild complex of an associative algebra with values in the algebra turned out to be

\ast This author has been fully supported in the frame of the AFR scheme of the Fonds National de la Recherche (FNR), Luxembourg with the project QUHACO 8969106
extremely useful for the deformation theory of that algebra. More precisely the Hochschild complex seen as a differential graded Lie algebra (by means of the Hochschild differential and the Gerstenhaber bracket) is called formal if it is quasi-isomorphic in the $L_\infty$-sense to its Hochschild cohomology. If this is the case, first order deformations (seen as 2-cocycles) having induced Gerstenhaber bracket equal to zero (so-called Maurer-Cartan elements) always integrate to formal deformations.

Kontsevich’s basic example is the symmetric algebra of a finite dimensional vector space (over a field $K$ of characteristic 0) whose Hochschild complex he showed is formal. An interesting playground for formality checks seems to be the class of universal enveloping algebras of Lie algebras which are very close to symmetric algebras. Two of us (M.B. and A.M.) have already looked at the Lie algebra of all infinitesimal affine transformations of $\mathbb{K}^n$ where we found formality, see [BM08]. One of us (O.E.) has studied the three-dimensional Heisenberg algebra and found that the corresponding Hochschild complex was NOT formal, see [Elc12] and [Elc14].

The aim of the present work is to check formality of the Hochschild complex of the universal enveloping algebras of two classes of Lie algebras: on one hand some finite-dimensional quadratic Lie algebras which we call Cartan-3-regular (e.g. semisimple Lie algebras and in more detail $\mathfrak{so}(3)$), and on the other hand free Lie algebras over any vector space. The second aim is –if possible– to explicitly compute the higher brackets of order $\geq 3$ on the cohomology which then will ensure an $L_\infty$-quasi-isomorphism with the Hochschild complex by homotopy transfer.

Our first main result is that the Hochschild complex of the universal enveloping algebra of a nonabelian reductive Lie algebra is NOT formal. In fact, we show a more general result for Cartan-3-regular quadratic Lie algebras which are quadratic Lie algebras whose Cartan 3-cocycle defines a nontrivial cohomology class. Moreover, in the case of $\mathfrak{so}(3)$, one just has to add one higher bracket $d_3$ of order 3 to restore the $L_\infty$-quasi-isomorphism with the Hochschild complex which we can describe explicitly.

The second main result consists in showing that the Hochschild complex of the universal enveloping algebra of any free Lie algebra generated by a vector space of dimension $\geq 2$ is NOT formal by explicit computations. Again by adding one higher order bracket $d_3$ of order 3 we can restore the $L_\infty$-quasi-isomorphism with the Hochschild complex.

On the other hand note that the universal enveloping algebras of semisimple and free Lie algebras are well-known to be rigid, hence every first order
deformation integrates to a deformation which is equivalent to the trivial deformation: it follows that these associative algebras provide examples where the deformation problem can always be solved, but where formality does not hold.

The main tool for formality checks is a characteristic 3-class $c_3$ in the graded Chevalley-Eilenberg cohomology of the graded Lie algebra given by the Hochschild cohomology equipped with a graded Lie bracket induced by the Gerstenhaber Lie bracket: this is well-known in the literature in order-by-order computations, and provides the first obstruction to $L_\infty$-formality. In order to deal with finite-dimensional Lie algebras we use a result already sketched in [Kon03, Secs. 8.3.1,8.3.2] and further explicit in [BM08] (based on the work [BMP05]) that the Hochschild complex of the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$ as a differential graded Lie algebra is quasi-isomorphic (in the $L_\infty$-sense) to the much ‘easier’ Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in the symmetric algebra $S\mathfrak{g}$ of the adjoint module $\mathfrak{g}$ equipped with the Chevalley-Eilenberg differential and the Schouten bracket for poly-vector fields (on the manifold $\mathfrak{g}^*$). The proof we know of involves Kontsevich’s formality map. For quadratic Lie algebras, we compute the Schouten brackets of polynomials in the quadratic Casimir (degree 0) with the Cartan 3-cocycle (degree 3) where the Lie bracket (degree 2) and the Euler field (degree 1) appear, and evaluate a representing graded 3-cocycle for the 3-class $c_3$ on combinations of these classes.

For the free Lie algebra generated by a vector space $V$ we use the classical result that its universal enveloping algebra is simply isomorphic to the free algebra $TV$ generated by $V$ whose Hochschild cohomology is also well-known to be concentrated in degree 0 and 1. Here one of the main tools is to construct an explicit complement to the space of all inner derivations inside the space of all derivations in case $V$ is finite-dimensional.

Another general tool which we shall use several times is the homotopy perturbation lemma (in its $L_\infty$-form) to compute the higher order brackets if necessary.

The paper is organized as follows: Section 1 recalls graded coalgebraic structures, definitions of $L_\infty$ algebras and morphisms, and of formality, and the characteristic 3-class. Section 2 presents the Perturbation Lemma for chain complexes and its well-known extension to $L_\infty$-algebras. We add a seemingly less known observation, see Theorem 2.1 (for which the proof will be in [BE18]), that the ordinary geometric series formulas from the ‘unstructured’ perturbation lemma automatically preserve the underlying graded
coalgebra structures. In particular, this result allows to transfer the $L_\infty$-structure of the Hochschild cochain complex to its cohomology and allows to construct quasi-inverses. In Section 3 and Section 4 we show that the characteristic 3-class cannot be zero for certain examples by computing a representing 3-cocycle on well-chosen elements. Moreover, by means of the perturbation lemma we show that for the Lie algebra $\mathfrak{so}(3)$, and for free Lie algebras, the $L_\infty$ structure on the cohomology does not come from the Gerstenhaber bracket only, but involves a computable map of arity 3.

Acknowledgements

The authors would like to thank B. Hurle, B. Valette, F. Wagemann, and P. Xu for fruitful discussions, and T. Petit for making us aware of Sections 8.3.1 and 8.3.2 in Kontsevich’s article [Kon03].

1 Kontsevich formality

1.1 Generalities

The material of the following Section is mostly contained in [FHT01], [Kon03], [AMM02], the Appendix of [BGH+05], [BM08 §4.1], [LV12], [Elc12], [Bor15]).

Let $\mathbb{K}$ be a field of characteristic 0. We will use the framework of graded bialgebras. Unless explicitly specified, vector spaces $V,W,\ldots$ and algebras will be graded over $\mathbb{Z}$, and the degree in $\mathbb{Z}$ of a homogeneous element $x$ will be denoted by $|x|$. Hom$(V,W)$ will always denote the subspace of the space of all linear maps $V \to W$ generated by all homogeneous linear maps $V \to W$. Tensor products of graded vector spaces $V$ and $W$ are graded as usual.

We recall the notations used: there is the graded transposition $\tau: V \otimes W \to W \otimes V$ defined by $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$ on homogeneous elements $x \in V$ and $y \in W$, and the Koszul rule of signs

\[
(\phi \otimes \psi)(x \otimes y) := (-1)^{|\psi||x|} \phi(x) \otimes \psi(y),
\]

for homogeneous linear maps $\phi$ and $\psi$ between graded spaces; and in the tensor product of two graded algebras $\mathcal{A}$ and $\mathcal{B}$ the graded multiplication

\[
(a \otimes b)(a' \otimes b') := (-1)^{|b'||a'|}(aa' \otimes bb'),
\]
for elements $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$.

Shifted graded vector spaces are noted $V[j]$ for any integer $j$, with $V[j]^i := V^{i+j}$ for all integers $i$. The suspension map $s : V \to V[-1]$, defined by the identity of the underlying vector spaces, is of degree one. Multilinear maps $\phi : V^\otimes k \to W^\otimes l$ can be shifted, i.e. $\phi[j] : V[j]^\otimes k \to W[j]^\otimes l$ by setting $\phi[j] := (s^\otimes l)^{-j} \circ \phi \circ (s^\otimes k)^j$. Note that $(s^\otimes k)^j = (-1)^{j\frac{(j(k-1)}{2}} (s^\otimes k)^\otimes 1$. The degree of the shifted map $\phi[j]$ is given by $|\phi[j]| = j(k - l) + |\phi|$. For $k = l$, the maps $\phi$ and its shift $\phi[j]$ have the same degree and the same action on the underlying ungraded vector spaces. We clearly have the following rules

$$\phi[j][j'] = \phi[j + j'], \quad (\psi \circ \phi)[j] = (\psi[j]) \circ (\phi[j]), \quad \text{and} \quad (1.1)$$

$$(\phi \otimes \phi')[j] = (-1)^{\frac{j(j-1)}{2} kl + \frac{j(j+1)}{2} ll' + j(2l' + k)|\phi'| + |\phi||\phi'[j]) \otimes (\phi'[j]). \quad (1.2)$$

where $j, j'$ are integers and $\phi'$ is a homogeneous $K$-linear map from $V'^\otimes k'$ to $W'^\otimes l'$, $V'$ and $W'$ being graded vector spaces.

Recall the graded symmetric bialgebra of $V$ $(SV, \mu_{sh} = \bullet, \Delta_{sh}, 1, \varepsilon)$, with $\mu_{sh}$ the graded commutative multiplication and $\Delta_{sh}$ the graded cocommutative (shuffle) comultiplication: it is defined by the free algebra $TV$ modulo the two-sided graded ideal generated by $x \otimes y - (-1)^{|x||y|} y \otimes x$ for any homogeneous elements $x, y$ in $V$. Note that we shall keep the notation $\Lambda V$ (used in the framework of rational homotopy theory for the graded symmetric algebra, see e.g. [FHT01]) for another object, the graded Grassmann algebra which will be explained further down. $SV$ is well-known to be graded cocommutative connected as a coalgebra, the canonical filtration (see Appendix B) simply being given by $\oplus_{s=0}^{\infty} S^*V$ for all $r \in \mathbb{N}$. Moreover, $SV$ is free as a graded commutative associative algebra, and cofree as a graded cocommutative coassociative connected coalgebra. This means that for any graded cocommutative coassociative connected coalgebra $(C, \Delta_C, \varepsilon_C, 1_C)$ and any linear map of degree 0, $\varphi : C \to V$ such that $\varphi(1_C) = 0$ there is a unique morphism of graded connected coalgebras $\Phi : C \to SV$ such that $pr_V \circ \Phi = \varphi$ where $pr_V : SV \to V$ denotes the obvious projection. Conversely, every such morphism is given this way. Likewise, for any given morphism of graded connected coalgebras $\Psi : C \to SV$, and any linear map $d : C \to V$ there is a unique graded coderivation of graded counital coalgebras $D : C \to SV$ along $\Psi$ such that $pr_V \circ D = d$. Again every graded coderivation of unital coalgebras $D : C \to SV$ along $\Psi$ is given that way. These induced maps can be computed as $\Phi = e^*\varphi$ (which we shall use henceforth as notation), and $D = d \ast \Psi$ (for which we shall use the abridged notation $\delta$) where $\ast$ is
the corresponding convolution multiplication in $\text{Hom}(C, SV)$ (see Appendix B) for the definition of convolution, and see e.g. [Hel89] (or [Bor15, App.A]) for the given formula. The convolution exponential converges since $\phi$ is of filtration degree $-1$ if $C$ carries the canonical filtration and $SV$ is equipped with the trivial discrete filtration. In the case $C = SU$ which we shall mostly encounter in this paper the maps $D$ and $e^*\phi$ are uniquely determined by the sequence of restrictions $d_n := d|_{S^n U} \rightarrow V$ and $\phi_n := \phi|_{S^n U} \rightarrow V$ (also called Taylor coefficients). It is not entirely necessary for this paper, but we would like to mention that the graded commutator $[d_1, d_2]$ of two coderivations $d_1, d_2$ of $SV$ (along the identity) is again a graded coderivation of $SV$ (along the identity) which is induced by the so-called Nijenhuis-Richardson multiplication $\circ_{NR}$ and Nijenhuis-Richardson bracket $[, ]_{NR}$ on the space $\text{Hom}(SV, V)$ defined by

\[ d_1 \circ_{NR} d_2 = d_1 \circ d_2, \quad \text{and} \quad [d_1, d_2]_{NR} = d_1 \circ_{NR} d_2 - (-1)^{|d_1||d_2|} d_2 \circ_{NR} d_1 \]

whence $[d_1, d_2] = [d_1, d_2]_{NR}$. The Nijenhuis-Richardson bracket is well-known to be a graded Lie bracket, and the Nijenhuis-Richardson multiplication is a non-associative multiplication called graded pre-Lie, see [Ger63].

We shall also have to use the graded exterior algebra, $\Lambda V$, defined as the free algebra $TV$ modulo the two sided ideal generated by $x \otimes y + (-1)^{|x||y|} y \otimes x$ for any homogeneous elements $x, y$ in $V$. The induced multiplication is denoted by $\wedge$. The exterior algebra is $\mathbb{Z} \times \mathbb{Z}$-graded (or bigraded) where an element $x_1 \wedge \cdots \wedge x_k$ (where $x_1, \ldots, x_k$ are homogeneous elements of $V$) has bidegree $(k, |x_1| + \cdots + |x_k|)$. Assigning to a pair of bidegrees $(k, i)$ and $(l, j)$ the grading sign $(-1)^{kl + ij}$ the graded Grassmann algebra becomes a bigraded (co)commutative bialgebra. Note that for the comultiplication to make sense a bigraded Koszul rule has to be used. Recall that the shift $\phi \mapsto \phi[j]$ of multilinear maps switches from graded symmetric to graded exterior algebras for odd $j$: if $\phi$ is a homogenous linear map from $SV \rightarrow V'$ then its shift $\phi = \phi[j]$ can be viewed as a bihomogeneous linear map $\Lambda^k(V[j]) \rightarrow V'[j]$ of bidegree $(1 - k, |\phi| + (k - 1)j)$ if $j$ is odd. In particular, a graded antisymmetric bilinear map $c : \Lambda^2 W \rightarrow W$ of bidegree $(-1, 0)$ will shift to a graded symmetric bilinear map $c[1] : \Lambda^2 W[1] \rightarrow W[1]$ of degree 1.

We note the following well-known

**Lemma 1.1.** Let $\Phi := e^*\phi : SU \rightarrow SV$ be a morphism of graded connected coalgebras. Then it is a bijection if and only if the component $\phi_1 = e^*\phi|_U : U \rightarrow V$ is a $\mathbb{K}$-linear bijection.
$A$ converges by the completeness of $\Psi$ that $\Phi$ has an inverse $\Psi = (id_{SU} + A)^{-1} \circ e^{*\psi_1}$.

Conversely, suppose that $\Phi|_U = \varphi_1 : U \rightarrow V$ is bijective with inverse $\psi_1 : V \rightarrow U$. By evaluating the projections $pr_U$ and $pr_V$, respectively, it follows that $e^{*\psi_1} \circ e^{*\varphi_1} = id_{SU}$, and $e^{*\varphi_1} \circ e^{*\psi_1} = id_{SV}$ (equation of maps of graded connected coalgebras). Set $\varphi = \varphi_1 + \varphi'$ with $\varphi' = \sum_{k=2}^{\infty} \varphi_k$ where $\varphi_k : S^k U \rightarrow V$. We can write

\[
e^{*\varphi} = e^{*(\varphi_1 + \varphi')} = e^{*\varphi_1} \ast e^{*\varphi'} = e^{*\varphi_1} \ast \left( e^{*\varphi'} - 1 \epsilon \right) = e^{*\varphi_1} \circ (id_{SU} + A),
\]

where $A = e^{*\varphi_1} \circ (e^{*\varphi_1} \ast (e^{*\varphi'} - 1 \epsilon))$ clearly is of filtration degree $-1$. Hence the inverse of $id_{SU} + A$ is given by the geometric series $\sum_{r=0}^{\infty} (-A)^r$ which converges by the completeness of $\text{Hom}(SU, SU)$, see Appendix A. It follows that $\Phi$ has an inverse $\Psi = (id_{SU} + A)^{-1} \circ e^{*\psi_1}$. \hfill \QED

See also [LV12, Sec. 10.4, Thm. 10.4.1] for the more general operadic version.

Let $W = \bigoplus_{j \in \mathbb{Z}} W^j$ a $\mathbb{Z}$-graded vector space. Let $V = W[1]$ be the shifted graded vector space.

**Definition 1.2.** A $L_\infty$-structure on $W$ is defined to be a graded coderivation $\mathcal{D} = \overline{D}$ of $S(W[1])$ of degree 1 satisfying $\mathcal{D}^2 = 0$ and $\mathcal{D}(1_{SW[1]}) = 0$. The pair $(W, \mathcal{D})$ is called an $L_\infty$-algebra.

Note that $D = \sum_{r=1}^{\infty} D_r$ is a $\mathbb{K}$-linear map $S(W[1]) \rightarrow W[1]$ of degree 1, and the first component $\mathcal{D}|_W[1] = D_1 : W[1] \rightarrow W[1]$ of $\mathcal{D}$ is a differential, i.e. $D_1^2 = 0$. $L_\infty$-algebras with $D_1 = 0$ are called minimal. In this paper we shall encounter particular $L_\infty$-algebras for which $D_n = 0$ for all integers $n \geq n_0$ for some nonnegative integer $n_0$.

A $L_\infty$-morphism from a $L_\infty$-algebra $(W, \mathcal{D})$ to a $L_\infty$-algebra $(W', \mathcal{D}')$ is a morphism of differential graded connected coalgebras $\Phi : (S(W[1]), \mathcal{D}) \rightarrow (S(W'[1]), \mathcal{D}')$, i.e. $\Phi = e^{*\varphi}$ is a morphism of graded connected coalgebras (see Appendix B) intertwining differentials,

\[
\Phi \circ \mathcal{D} = \mathcal{D}' \circ \Phi.
\]  

(1.3)
Moreover, a $L_\infty$-map $\Phi$ is called an $L_\infty$-\textit{quasi-isomorphism} if its first component $\Phi_1 = \Phi|_{W[1]} = \varphi_1 : W[1] \to W'[1]$—which is a chain map $(W[1], D_1) \to (W'[1], D'_1)$—induces an isomorphism in cohomology. It can be shown that every $L_\infty$-quasi-isomorphism has a \textit{quasi-inverse}, i.e. a $L_\infty$-morphism $\Psi = e^*\psi$ going from $(S(W'[1]), D' = D')$ to $(S(W[1]), D = D)$ such that the chain map $\psi_1 = \Psi|_{W'[1]} : (W'[1], D'_1) \to (W'[1], D'_1)$ induces an isomorphism in cohomology, see e.g. \cite{Kon03}, \cite{AMM02, Prop. V2}, or \cite{LV12, Thm. 10.4.4}. It even follows that $\Phi$ and $\Psi$ induce isomorphism of the cohomologies with respect to the entire differentials $D$ and $D'$ but we shall not need this statement.

A very important example of a $L_\infty$-algebra (motivating the whole structure) is a \textit{differential graded Lie algebra} $(\mathfrak{g}, b, [ , ]_H)$, i.e. $(\mathfrak{g}, [ , ])_H$ is a graded Lie algebra and the $\mathbb{K}$-linear map $b : \mathfrak{g} \to \mathfrak{g}$ is of degree $1$, $b^2 = 0$, and $b$ is a graded derivation of the graded Lie bracket $[ , ]$. In this case, on the shifted space $V = \mathfrak{g}[1]$, one sets $D_1 = b[1]$, and $D_2 = [ , [ , ]]_H[1]$, $D_n = 0$ for all $n \geq 3$, and the structure of a differential graded Lie algebra ensures that $D^2 = 0$. Hence one gets a $L_\infty$-structure on $\mathfrak{g}$. Moreover, it is well-known that its cohomology $\mathfrak{h}$ with respect to $b$ carries a canonical graded Lie bracket $[ , ]_{\mathfrak{h}}$ induced from $[ , ]_H$. Likewise, on the shifted space $\mathfrak{h}[1]$ the map $d = d_2 = [ , ]_{\mathfrak{h}}[1]$ is the Taylor coefficient of order $2$ of a coderivation $\bar{d}$ of square zero on $S(\mathfrak{h}[1])$.

The formality problem for differential graded Lie algebras is the question whether there is an $L_\infty$-quasi-isomorphism $\Phi = e^*\varphi$ from $(S(\mathfrak{g}[1]), [ , ]_{\mathfrak{g}}[1])$ to $(S(\mathfrak{g}[1]), (b[1] + [ , ]_{\mathfrak{h}}[1]))$. This is the analog of D. Sullivan’s formality for differential graded associative algebras, see e.g. \cite{FHT01, p.156}.

We shall not need this in the sequel, but recall that a stronger and more classical notion is a \textit{quasi-isomorphism of differential graded Lie algebras} which is a morphism of differential graded Lie algebras whose induced morphism of graded Lie algebras on cohomologies is an isomorphism. Clearly, for a given quasi-isomorphism there is in general not a quasi-isomorphism in the other direction inducing the inverse morphism on cohomology. There is the notion of two differential graded Lie algebra being \textit{weakly quasi-isomorphic} if there is a finite zig-zag of quasi-isomorphisms of intermediate differential graded Lie algebras with ends at the two given Lie algebras. This notion turns out to be equivalent to $L_\infty$-quasi-isomorphism, see e.g. \cite[p.423, Thm.11.4.9]{LV12}.

There are several important differential graded Lie algebras describing the identities, the (co)homology, and the algebraic deformation theory of
certain classes of algebras, see e.g. the book [LV12]. The one of interest in
this paper concerns the class of associative algebras and has been invented
by M. Gerstenhaber, [Ger63]:

Let \((A, \mu)\) be an associative (not necessarily unital and trivially graded)
algebra over the field \(K\) of characteristic 0. On the Hochschild complex
\(C_H := \bigoplus_{n \in \mathbb{N}} C^n_H(A, A)\) of \(A\) with values in the bimodule \(A\) (considered
with its natural grading by number of arguments), recall the Gerstenhaber
multiplication \(\circ_G : C_H \times C_H \to C_H\) which is a bilinear map of degree \(-1\)
defined for any nonnegative integers \(k, l\) and any \(f \in C^k_H(A, A)\) and any
\(g \in C^l_H(A, A)\) by (for any \(a_1, \ldots, a_{k+l-1} \in A\)
\[
(f \circ_G g)(a_1, \ldots, a_{k+l-1}) = \\
\sum_{i=1}^{k+l} (-1)^{(i-1)(l-1)} f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+l-1}), a_{i+l}, \ldots, a_{k+l-1}).
\]

(1.4)

This multiplication can be considered on the shifted space \(G(A) = G := C_H[1]\) and the graded commutator,
\[
[f, g]_G = f \circ_G g - (-1)^{(k-1)(l-1)} g \circ_G f,
\]
(1.5)

(where of course \(k-1\) is the shifted degree of a \(k\)-cochain \(f\)) is called the Gerstenhaber bracket and turns out to be a graded Lie bracket on \(G\). The proof
of the graded Jacobi-identity is largely simplified by the classical observation
that \(G\) is isomorphic to the space of all coderivations of the tensor algebra
\(T(A[1])\) equipped with the usual deconcatenation comultiplication (for which
it is connected and cofree), but this is not important for the sequel.

Note that the algebra multiplication \(\mu\) of \(A\) did not enter in the defi-
nition of the Gerstenhaber multiplication and bracket. For the shifted version
\(G = C_H[1]\) any bilinear map \(\mu : A \times A \to A\) is of degree 1, and
gives rise to an associative multiplication iff \([\mu, \mu]_G = 0\). Moreover, for
any such \(\mu\) the square of \(b := [\mu, \ ]_G\) vanishes and defines, up to a global
sign, the Hochschild coboundary operator on the complex \(C_H[1]\). Hence
\((G(A), [\ , \ ]_G, b = [\mu, \ ]_G)\) is a differential graded Lie algebra associated to
any associative algebra \((A, \mu)\). It follows that the shifted Hochschild coho-
mology, \(\mathfrak{H}(A) = H_H(A, A)[1]\) is a graded Lie algebra with the induced Lie
bracket \([\ , \ ]_H\).
Maxim Kontsevich has linked the deformation problem in deformation quantization of Poisson manifolds to the formality problem of the above differential graded Lie algebra built on the Hochschild cochain complex of the algebra $\mathcal{A}$ of all smooth functions on the underlying differentiable manifold, see [Kon03].

The problem we would like to consider in this paper is contained in the framework of the following slight generalization of the formality problem: let $(\mathfrak{g},[\ ,\ ],b)$ be a differential graded Lie algebra, and let $(\mathfrak{g}_H,[\ ,\ ]_H)$ be its cohomology graded Lie algebra. Again let $D = D_2 = b[1] + [\ ,\ ][1]$ denote the Taylor coefficients of the corresponding coderivation $\overline{D}$ of $S(\mathfrak{g}[1])$, and we fix this $L_{\infty}$-structure on $\mathfrak{g}$. We shall put a general minimal $L_{\infty}$-structure on $\mathfrak{g}_{H}$ whose coderivation $\overline{d}$ (of $S(\mathfrak{g}_H[1])$) is given by a series $d = d_2 + \sum_{k \geq 3} d_k = d_2 + d'$ where $d_2 = [\ ,\ ]_H[1]$. It is well-known (see [Kon03], [AMM02], [LV12]) that it is always possible to find a sequence of ‘higher order brackets’ $d_k : S^k(\mathfrak{g}_H[1]) \to \mathfrak{g}_H[1]$ for $k \geq 3$ and a $L_{\infty}$-quasi-isomorphism $\Phi = e^{*\varphi} : (S(\mathfrak{g}_H[1]),[\ ,\ ]_H[1] + d') \to (S(\mathfrak{g}[1]),b[1] + [\ ,\ ]_H[1])$ which also follows from the homotopy perturbation Lemma, see the next Section. If all the higher order brackets $d_k'$ for $k \geq 3$ vanish there is formality, and if formality holds then they can be transformed away by the conjugation with an invertible chain map of differential graded coalgebras $S(\mathfrak{g}_H[1]) \to S(\mathfrak{g}_H[1])$.

### 1.2 Low orders and a characteristic 3-class

This Subsection is well-known for the description of recursive obstructions to $L_{\infty}$-quis, see e.g. [GH03], Appendix of [BGH+05], [LV12]. We have included it to get explicit formulae. Let $(\mathfrak{g},[,\ ]_G,b)$ be a differential graded Lie algebra, and let $(\mathfrak{g}_H,[\ ,\ ]_H)$ be its cohomology. As before let $D = D_1 + D_2 = b[1] + [\ ,\ ]_G[1]$ and $d = d_2 = [\ ,\ ]_H[1]$ the Taylor coefficients of the corresponding graded coderivations $\overline{D}$ and $\overline{d}$ of degree 1 of $S(\mathfrak{g}[1])$ and $S(\mathfrak{g}_H[1])$, respectively. Clearly $\overline{D}^2 = 0$ and $\overline{d}^2 = 0$. Let $\varphi = \sum_{r=1}^{\infty} \varphi_r$ be a $\mathbb{K}$-linear map of degree 0 from $S(\mathfrak{g}_H[1])$ to $S(\mathfrak{g}[1])$ with $\varphi_r = \varphi_r|S(\mathfrak{g}_H[1])$, and let $\Phi = e^{*\varphi}$ the corresponding morphism of connected coalgebras. Define, as in [BGH+05], Prop.A3,

\[ \overline{P}(\varphi) = \overline{D} \circ \Phi - \Phi \circ \overline{d}, \quad \text{and} \quad P(\varphi) = D \circ e^{*\varphi} - \varphi \circ \overline{d}, \quad (1.6) \]

the latter being the projection to $\mathfrak{g}[1]$. Clearly, $\overline{P}(\varphi)$ is a graded coderivation of degree 1 from $S(\mathfrak{g}_H[1])$ to $S(\mathfrak{g}[1])$ along $\Phi$, and is of course uniquely determined by its Taylor coefficient $P(\varphi)$. We shall call $\Phi$ an $L_{\infty}$-morphism of
order \( r \) if \( P_s(\varphi) = 0 \) for all integers \( 1 \leq s \leq r \). Clearly, \( \Phi \) is an \( L_\infty \) morphism iff it is a \( L_\infty \)-morphism of order \( r \) for each positive integer \( r \). Since \( \Phi \) is filtration preserving it clearly follows that \( \Phi \) being a \( L_\infty \)-morphism of order \( r \) gives only conditions on the maps \( \varphi_1, \ldots, \varphi_r \), the higher orders not being affected. We can ‘regauge’ \( \Phi \) in the following way: let \( \alpha : \mathcal{S}(\mathfrak{H}[1]) \to \mathfrak{H}[1] \) and \( \beta : \mathcal{S}(\mathfrak{G}[1]) \to \mathfrak{G}[1] \) be \( \mathbb{K} \)-linear maps of degree 0 vanishing on the unit elements, and write \( \mathcal{A} = e^\circ \alpha \) and \( \mathcal{B} = e^\circ \beta \) for the corresponding morphisms of connected coalgebras. Supposing that \( \mathcal{A} \circ \delta = \delta \circ \mathcal{A} \) and \( \mathcal{B} \circ \overline{\delta} = \overline{\delta} \circ \mathcal{B} \) it is straight-forward to see that the regauged \( \varphi \), i.e. \( \varphi' = \beta \circ \Phi \circ \mathcal{A} \), satisfies

\[
P(\varphi') = \beta \circ \mathcal{P}(\varphi) \circ \mathcal{A},
\]

whence for each positive integer \( r \) it follows that \( e^\circ \varphi' = \mathcal{B} \circ \Phi \circ \mathcal{A} \) is a \( L_\infty \)-morphism of order \( r \) if \( \Phi \) is.

We compute \( P_r(\varphi) \) for \( r = 1, 2, 3 \): for any homogeneous elements \( x_1, x_2, x_3 \in \mathfrak{H}[1] \)

\[
P_1(\varphi) = b[1] \circ \varphi_1,
\]

\[
P_2(x_1 \bullet x_2) = b[1](\varphi_2(x_1 \bullet x_2)) + D_2(\varphi_1(x_1) \bullet \varphi_1(x_2))
\]

\[
- \varphi_1(d_2(x_1 \bullet x_2)),
\]

\[
P_3(x_1 \bullet x_2 \bullet x_3) = b[1](\varphi_3(x_1 \bullet x_2 \bullet x_3)) + D_2(\varphi_1(x_1) \bullet \varphi_2(x_2 \bullet x_3)) + (-1)^{|x_1||x_2|}D_2(\varphi_1(x_2) \bullet \varphi_2(x_1 \bullet x_3)) + (-1)^{|x_3|(|x_1| + |x_2|)}D_2(\varphi_1(x_3) \bullet \varphi_2(x_1 \bullet x_2)) - \varphi_2(d_2(x_1 \bullet x_2 \bullet x_3) - (-1)^{|x_2||x_3|}\varphi_2(d_2(x_1 \bullet x_3) \bullet x_2) - (-1)^{|x_1|(|x_2| + |x_3|)}\varphi_2(d_2(x_2 \bullet x_3) \bullet x_1).
\]

Let \( Z\mathfrak{G} \) and \( B\mathfrak{G} \) denote the graded vector space of all cocycles and coboundaries of the complex \( (\mathfrak{G}, b) \), respectively, let \( \pi : Z\mathfrak{G} \to \mathfrak{H} \) be the canonical projection which obviously is a morphism of graded Lie algebras. Then the cohomology of the shifted complex \( (\mathfrak{G}[1], b[1]) \) can be identified with the shifted cohomology \( \mathfrak{H}[1] \). An \( L_\infty \)-morphism \( \Phi = e^\circ \varphi : \mathcal{S}(\mathfrak{H}[1]) \to \mathcal{S}(\mathfrak{G}[1]) \) of order \( r \) is called a \( L_\infty \)-quis of order \( r \) if \( \varphi_1 : \mathfrak{H}[1] \to \mathfrak{G}[1] \) induces an isomorphism in cohomology. The latter condition is equivalent to stating that \( b[1] \circ \varphi_1 = 0 \) and \( \pi[1] \circ \varphi_1 : \mathfrak{H}[1] \to \mathfrak{H}[1] \) is invertible. More specifically, we shall call any linear map map of degree 0 \( i : \mathfrak{H}[1] \to \mathfrak{G}[1] \) a \textit{section} if all the values of \( i \) are in \( Z\mathfrak{G}[1] \) and if

\[
\pi[1] \circ i = \text{id}_{\mathfrak{H}[1]}.
\]
The following statements seem to be well-known:

**Lemma 1.3.** With the above notations one has:

1. There is a $L_\infty$-quis $\Phi = e^\varphi : S(\mathfrak{H}[1]) \to S(\mathfrak{G}[1])$ of order 2 such that $\varphi_1$ is a section.

2. Let $\Phi = e^\varphi$ and $\Psi = e^\psi$ be two $L_\infty$-quis of order 2 where $\varphi_1$ is a section.

Then there are linear maps $\alpha_1 : \mathfrak{H}[1] \to \mathfrak{H}[1]$ and $\beta : S(\mathfrak{G}[1]) \to \mathfrak{G}[1]$ of degree 0 (where $\beta$ vanishes on the unit), and $\chi_2 : S^2(\mathfrak{H}[1]) \to \mathfrak{G}[1]$ of degree 0 such that the morphisms of connected coalgebras $e^*\alpha_1$ and $e^*\beta$ commute with the corresponding differentials $d$ and $D$, respectively, and such that the regauged morphism $e^*\psi' = \Psi' = e^*\beta \circ \Psi \circ e^*\alpha_1$ is a $L_\infty$-quis of order 2 with

$$\psi_1' = \varphi_1, \quad b[1] \circ \chi_2 = 0, \quad \text{and} \quad \psi_2' = \varphi_2 + \chi_2.$$  \hspace{1cm} (1.12)

3. Let $\varphi : S(\mathfrak{H}[1]) \to \mathfrak{G}[1]$ be a linear map of degree 0 vanishing on 1. Then $\Phi = e^\varphi$ is a $L_\infty$-quis of order 2 if and only if the shifted maps $\phi_1 = \varphi_1[1]$ and $\phi_2 = \varphi_2[1]$ satisfy for all $y, y' \in \mathfrak{H}$

$$0 = b \circ \phi_1, \quad \pi \circ \phi_1 \text{ invertible, and} \quad 0 = b(\phi_2(y, y')) + [\phi_1(y), \phi_1(y')]_G - \phi_1([y_1, y_2]_H).$$  \hspace{1cm} (1.13)

**Proof.** 1. Choose a vector space complement $\mathcal{H}$ of $B\mathfrak{H}$ in $Z\mathfrak{G}$, then the restriction of $\pi$ to $\mathcal{H}$ is clearly invertible. Let $\phi_1$ be the inverse of this map followed by the inclusion of cocycles. Then $\pi \circ \phi_1 = \text{id}_{\mathfrak{G}}$, and the shift $\varphi_1 = \phi_1[1]$ gives the desired section. Moreover, since $\pi$ is a morphism of graded Lie algebras, it follows that for all $y_1, y_2 \in \mathfrak{H}$ the difference $[\phi_1(y_1), \phi_1(y_2)]_G - \phi_1([y_1, y_2]_H)$ (which obviously is in $Z\mathfrak{G}$) lies in the kernel of $\pi$ and therefore is in $B\mathfrak{G}$, thus proving the existence of a linear map $\varphi_2$ of degree $-1$ such that $P_2(\varphi) = 0$.

2. Since the component $\psi_1$ is a quis, it follows that $\pi[1] \circ \psi_1$ is a linear isomorphism of $\mathfrak{H}[1]$ which intertwines $d_2$ which can be seen by applying $\pi[1]$ to the right hand side of the equation $0 = P_2(\psi)$. Hence $\alpha = \alpha_1$ can be defined as the inverse of this map. Then $\tilde{\Psi} = \Psi \circ e^*\alpha_1$ is a $L_\infty$-quis such that $\tilde{\psi}_1$ is a section. It follows that the difference $\tilde{\psi}_1 - \varphi_1$ is a linear map from $\mathfrak{H}[1]$ into the space of all coboundaries $B\mathfrak{G}[1]$. Upon choosing a graded vector space
complement \( W' \) of \( Z\mathfrak{g}[1] \) in \( \mathfrak{g}[1] \) we can define a linear map \( \chi_1 : \mathfrak{g}[1] \to \mathfrak{g}[1] \) of degree \(-1\) vanishing on the coboundaries and on \( W' \) and having all its values in \( W' \) such that \( \psi_1 - \varphi_1 = b[1] \circ \chi_1 \circ \varphi_1. \) Thanks to the definition of \( \chi_1 \) it follows that \( \chi_1 \circ \chi_1 = 0, \chi_1 \circ b[1] = 0, \) and \( b[1] \circ \chi_1 \circ \psi_1 = b[1] \circ \chi_1 \circ \varphi_1. \) Moreover the graded linear map \( T \) of degree 0 defined by \( T = -[\overline{D}, \chi_1] \) is a graded coderivation of degree 0 of the graded connected coalgebra \( \mathcal{S}(\mathfrak{g}[1]) \) which can be computed (in a straight-forward way) to be locally nilpotent in the sense that for each element \( c \in \mathcal{S}(\mathfrak{g}[1]) \) there is a positive integer \( N \) such that \( T^{\circ N}(c) = 0. \) Therefore the composition exponential \( B = e^{\circ T} \) is a well-defined morphism of connected coalgebras \( \mathcal{S}(\mathfrak{g}[1]) \to \mathcal{S}(\mathfrak{g}[1]) \) commuting with \( \overline{D} \) since obviously \([\overline{D}, T] = 0. \) Hence \( \Psi' = B \circ \Psi \) is a \( L_\infty \)-quas, (and hence of the form \( e^{\psi'} \)), and a straight-forward computation gives \( \psi'_1 = \varphi_1. \) Finally equation \( P_2(\psi') = 0 = P_2(\varphi) \) implies that \( 0 = b[1] \circ (\psi'_2 - \varphi_2) \) proving the second statement of the Lemma.

3. This follows directly from the shifted versions of eqs (1.8) and (1.9) upon using the rules (1.1) and (1.2). \[\square\]

In order to prepare the grounds for the characteristic 3-class, we recall one variant of graded Chevalley Eilenberg cohomology of a graded Lie algebra \( (\mathfrak{a}, [\ , \ ], \mathfrak{a}) \): the shifted Lie bracket \( d_2 = [\ , \ ]_{\mathfrak{a}[1]} \) is an element of \( \text{Hom} \left( \mathcal{S}(\mathfrak{a}[1]), \mathfrak{a}[1] \right) \) of degree 1 satisfying \([d_2, d_2]_{\mathcal{N}R} = 0. \) Hence the linear map of degree 1 from \( \text{Hom} \left( \mathcal{S}(\mathfrak{a}[1]), \mathfrak{a}[1] \right) \) to itself, sending \( g \) to \([d_2, g]_{\mathcal{N}R} \) is a coderivation. Applying the shift \([-1]\) yields a coderivation \( \delta_{\mathfrak{a}} \) on the space \( \text{Hom}(\Lambda \mathfrak{a}, \mathfrak{a}). \) The following formula can be computed in a straight-forward manner for any homogeneous \( \phi_k \) in \( \text{Hom}(\Lambda^k \mathfrak{a}, \mathfrak{a}) \) and homogeneous elements \( y_1, \ldots, y_{k+1} \in \mathfrak{a} \) by computing the shift using the formulas (1.1) and (1.2):

\[
(\delta_{\mathfrak{a}} \phi_k)(y_1, \ldots, y_{k+1}) \\
= \sum_{i=1}^{k+1} (-1)^{i-1}(-1)^{|\phi_k||y_i|}(\phi_k)(|y_i|(|y_1|+\cdots+|y_{i-1}|)) \\
\quad - \bigg[ y_i, \phi_k(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{k+1}) \bigg]_{\mathfrak{a}} \\
+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j}(-1)^{(|y_i|+|y_j|)(|y_1|+\cdots+|y_{i-1}|)}(-1)^{|y_j|(|y_{i+1}|+\cdots+|y_{j-1}|)} \\
\quad \phi_k \bigg[ y_i, y_j, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{k+1} \bigg]. \tag{1.14}
\]

where we have left out an inessential global factor of \(-1 \frac{k(k-1)}{2}\) appearing in the computation. In case all arguments \( y_1, \ldots, y_{k+1} \) are of even degree
the usual formula –going back to the exterior derivative of $k$-forms– is easily
recognised. Note that $\delta_a$ increases the number of arguments by one, but is
of degree zero since the graded Lie bracket is.

We state the following well-known fact in order to have a concrete formula
which can be computed in examples:

**Proposition 1.4.** With the above-mentioned definitions let $\varphi : S(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ be a linear map of degree 0 vanishing on 1, and suppose that $e^\varphi$ defines
a $L_\infty$-quis of order 2 where $\varphi_1$ is a section. Let $\phi = \varphi[-1] : \Lambda \mathfrak{g} \to \mathfrak{g}$ be the
shift of $\varphi$. Then the following holds:

1. The linear map $w_3 = w_3(\phi) : \Lambda^3 \mathfrak{g} \to \mathfrak{g}$ of degree $-1$ defined on homo-
genous elements $y_1, y_2, y_3 \in \mathfrak{g}$ by

$$w_3(\phi)(y_1, y_2, y_3) =$$

$$= (-1)^{|y_1|}[\phi_1(y_1), \phi_2(y_2, y_3)]_G - (-1)^{|y_2|}(-1)^{|y_2||y_1|}\phi_1(y_2, \phi_2(y_1, y_3))_G$$

$$+ (-1)^{|y_3|}(-1)^{|y_3|(|y_1|+|y_2|)}[\phi_1(y_3), \phi_2(y_1, y_2)]_G$$

$$- \phi_2([y_1, y_2]_H, y_3) + (-1)^{|y_3||y_2|}\phi_2([y_1, y_3]_H, y_2)$$

$$- (-1)^{|y_2|+|y_3|}|y_1|\phi_2([y_2, y_3]_H, y_1)$$

(1.15)

satisfies $b \circ w_3 = 0$.

2. The linear map

$$z_3 = z_3(\phi) = \pi \circ w_3$$

(1.16)

from $\Lambda^3 \mathfrak{g}$ to $\mathfrak{g}$ is a well-defined graded Chevalley-Eilenberg 3-cocycle
of degree $-1$, i.e. $\delta_{\mathfrak{g}}z_3 = 0$.

3. The graded Chevalley-Eilenberg 3-class $c_3 = c_3(\mathfrak{g}, \mathfrak{b}, [\ , ]_G)$ of $z_3(\phi)$
does not depend on the chosen $\phi_1, \phi_2$ satisfying eqn (1.13).

4. There is a $L_\infty$-quis of order 3 between $\mathfrak{g}$ and its cohomology $\mathfrak{h}$ if and
only if $c_3 = 0$.

**Proof.** Note that the following equation is trivially satisfied for any linear
map of degree zero $\varphi : S(\mathfrak{h}[1]) \to \mathfrak{g}[1]$ vanishing on the unit:

$$\overline{D} \circ \overline{P}(\varphi) + \overline{P}(\varphi) \circ \overline{\partial}_2 = 0.$$

(1.17)

1. Projecting the preceding identity (1.17) to $\mathfrak{g}[1]$ and evaluating it on
$S^3(\mathfrak{g}[1])$ we derive the first statement $b \circ w_3 = 0$ upon using $P_1(\varphi) = 0$ and
2. By the preceding part, the values of $w_3$ are cocycles (w.r.t. $b$) whence the application of $\pi$ is legal, and $z_3$ is well-defined. Evaluating eqn (1.17) on $S(I[1])$, projecting onto $\mathfrak{g}[1]$, and applying the shift $[1]$ we get the graded 3-cocycle equation $\delta_3 z_3 = 0$, compare eqn (1.14) for $k = 3$.

3. Choose a different linear map of degree zero $\psi$ from $S(H[1])$ to $G[1]$ vanishing on the unit with $\psi_1$ a section such that $P_1(\psi) = 0$ and $P_2(\psi) = 0$.

According to the second statement of Lemma 1.3 there is a locally nilpotent coderivation $T$ of degree 0 commuting with $D$ such that for the regauged map $\psi' = \text{pr}_{\mathfrak{g}[1]} \circ e_\ast T \circ e_\ast \phi$ we have $\psi_1' = \phi_1$. Moreover, using eqn (1.7) and using $P_1(\psi) = 0$ and $P_2(\psi) = 0$ we get

$$P_3(\psi') = \text{pr}_{\mathfrak{g}[1]} \circ e_\ast T \circ P(\psi) |_{S(I[1])} = P_3(\psi) + b[1] \circ \text{something},$$

showing that $\pi[1] \circ P_3(\psi) = \pi[1] \circ P_3(\psi')$. According to Lemma 1.3 the map $\psi_2' - \phi_2 = \chi_2$ takes its values in the $b[1]$-cocycles. It follows that the projection $\pi$ applied to the difference $w_3(\psi') - w_3(\phi)$ gives the graded 3-coboundary $\delta_3(\chi_2[-1])$ because the $b$-cocycles form a graded Lie subalgebra, and $\pi$ is a morphism of graded Lie algebras. Hence modulo graded 3-coboundaries the expression $c_3$ is independent on $\phi_1, \phi_2$.

4. If there is a $L_\infty$-quis of order three there is a regauged one, $e_\ast \phi$, where $\phi_1$ is a section according to the proof of Lemma 1.3. But then $P_3(\varphi) = 0$, hence $z_3(\varphi) = 0$, and the class $c_3$ vanishes.

Conversely, suppose the class $c_3$ vanishes. Choose any $L_\infty$-quis $e_\ast \varphi$ of order 2 with $\varphi_1$ a section, which exists by Lemma 1.3. We can then add to $\varphi_1$ a linear map $\chi_2[-1]$ taking values in the $b$-cocycles such that for $\varphi'_2 = \phi_2 + \chi_2[-1]$ the projection $z_3(\varphi') = 0$. This means that $w_3(\varphi')$ is a map into the $b$-coboundaries, whence after a shift we get a linear map $\varphi'_3$ such that $P_3(\varphi') = 0$ whence $e_\ast \varphi$ is a $L_\infty$-quis of order 3.

We shall call the above-mentioned graded cohomology 3-class $c_3(\mathfrak{g}, b, [, ]_G)$ the characteristic 3-class of the differential graded Lie algebra $(\mathfrak{g}, b, [, ]_G)$. It obviously is the first obstruction to $L_\infty$-formality and can be computed with any $\phi_1$ and $\phi_2$ satisfying eqs (1.11) and (1.13).

2 $L_\infty$-Perturbation Lemma

We refer to Appendix C for homotopy contractions and the usual Perturbation Lemma.
In case the corresponding complexes carry additional algebraic structures it is interesting to see whether the maps in the perturbation lemma can be modified such that these structures are preserved. This has been done by an inductive procedure in the $A_\infty$ and $L_\infty$ cases, and on other operads, see e.g. \cite{Hue10, Hue11, BGH+05, PropA.3, Man10, DSV16}. We would like to present the observation that in the $L_\infty$-case the usual geometric series will already give maps preserving the graded coalgebra structures.

To approach the $L_\infty$-perturbation Lemma we consider an arbitrary homotopy contraction (C.2). One can pass to the graded coderivations $b_{U[1]}$ of $S(U[1])$ and $b_{V[1]}$ of $S(V[1])$, respectively, and – upon writing $\varphi_1$ for $i[1]$ and $\psi_1$ for $p[1]$ – consider the morphisms of graded coalgebras $e^*\varphi_1 : S(U[1]) \to S(V[1])$ and $e^*\psi_1 : S(V[1]) \to S(U[1])$, respectively. By applying the corresponding projections $pr_{U[1]}$ and $pr_{V[1]}$ it can be seen that $b_{U[1]}$ and $b_{V[1]}$ are differentials, and $e^*\varphi_1$ and $e^*\psi_1$ are chain maps satisfying $e^*\psi_1 \circ e^*\varphi_1 = id_{S(U[1])}$. In order to extend the chain homotopy $h$ from $V$ to $S(V[1])$, the simple choice $h[1]$ (as a graded coderivation and derivation) will not be enough. Recall that $P = [h, b_V]$ is an idempotent $\mathbb{K}$-linear map $V \to V$. Let $V_U$, be the kernel, and $V_{acyc}$ be the image of $P$. Clearly $V = V_U \oplus V_{acyc}$, and $S(V[1]) \cong S(V_U[1]) \otimes S(V_{acyc}[1])$ as graded bialgebras. Define the $\mathbb{K}$-linear map $\beta$ of degree 0 from $S(V[1])$ to $S(V[1])$ for all $y_1, \ldots, y_k \in V_U[1]$ and $w_1, \ldots, w_l \in V_{acyc}[1]$ where $k, l \in \mathbb{N}$:

$$
\beta(y_1 \ldots \cdot y_k \cdot w_1 \cdot \ldots \cdot w_l) = \begin{cases} 
\frac{1}{l}(y_1 \cdot \ldots \cdot y_k \cdot w_1 \cdot \ldots \cdot w_l) & \text{if } l \neq 0, \\
0 & \text{if } l = 0,
\end{cases}
$$

(2.1)

and set

$$
\eta = \overline{h[1]} \circ \beta = \beta \circ \overline{h[1]}.
$$

(2.2)

It is then easy to see that

$$
(S(U[1]), b_{U[1]}) \xrightarrow{e^*\varphi_1} (S(V[1]), b_{V[1]}) \xrightarrow{e^*\psi_1} (S(V[1]), b_{V[1]}) \xrightarrow{\eta}
$$

(2.3)

is a homotopy contraction. Now the following Theorem is quite useful since it allows to transfer $L_\infty$-structures via homotopy contraction from $V$ to $U$. 

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**Theorem 2.1.** Let \((U,b_U)\) and \((V,b_V)\) be two chain complexes. Suppose that there is a homotopy contraction \(\tilde{\eta}\). In the corresponding shifted symmetric algebra version \([2.3]\), suppose that there is a \(K\)-linear map \(D_V = \sum_{k \geq 2} D_k^V : S(V[1]) \to V[1]\) of degree 1 such that \(D = b_V[1] + D_V\) defines an \(L_\infty\)-structure, i.e. the coderivation \(\delta\) and \(\delta^*\) of \(\tilde{\eta}\) will automatically preserve the structure of graded connected coalgebras, and \(\delta_{S(U[1])}\) will be a graded coderivation of degree 1.

More explicitly, defining the \(\mathbb{K}\)-linear maps \(e^{*\varphi_1}, e^{*\psi_1}, \delta_{S(U[1])}\), and \(\tilde{\eta}\) of the Perturbation Lemma \([C.4]\) – which define a homotopy contraction between complexes,

\[
(S(U[1]), b_U[1] + \delta_{S(U[1])}) \xrightarrow{e^{*\varphi_1}} (S(V[1]), b_V[1] + D_V) \xrightarrow{\tilde{\eta}} (S(V[1]), b_V[1] + D_V)
\]

will automatically preserve the structure of graded connected coalgebras, i.e. \(e^{*\varphi_1}\) and \(e^{*\psi_1}\) are morphism of graded differential connected coalgebras, and \(\delta_{S(U[1])}\) will be a graded coderivation of degree 1.

More explicitly, defining the \(\mathbb{K}\)-linear maps of degree 0, \(\varphi = \varphi_1 + \sum_{k \geq 2} \varphi_k : S(U[1]) \to V[1]\) and \(\psi = \psi_1 + \sum_{k \geq 2} \psi_k : S(V[1]) \to U[1]\) and the \(\mathbb{K}\)-linear map \(d_U' = \sum_{k \geq 2} d_k^U : S(U[1]) \to U[1]\) of degree 1 by

\[
\varphi = \text{pr}_{V[1]} \circ (id_{S(V[1])} + \eta \circ D_V)^{-1} \circ \varphi_1, \quad (2.4a)
\]
\[
\psi = \psi_1 \circ \text{pr}_{V[1]} \circ (id_{S(V[1])} + D_V \circ \eta)^{-1}, \quad (2.4b)
\]
\[
d_U' = \psi_1 \circ \text{pr}_{V[1]} \circ (id_{S(V[1])} + D_V \circ \eta)^{-1} \circ D_V \circ e^{*\varphi_1} \quad (2.4c)
\]

we get

\[
e^{*\varphi} = e^{*\varphi_1} = (id_{S(V[1])} + \eta \circ D_V)^{-1} \circ e^{*\varphi_1}, \quad (2.5a)
\]
\[
e^{*\psi} = e^{*\psi_1} = e^{*\psi_1} \circ (id_{S(V[1])} + D_V \circ \eta)^{-1} = e^{*\psi_1} \circ (id_{S(V[1])} + D_V \circ \eta)^{-1} \circ D_V \circ e^{*\varphi_1}, \quad (2.5b)
\]
\[
\overline{d}_U = \delta_{S(U[1])} = e^{*\psi_1} \circ (id_{S(V[1])} + D_V \circ \eta)^{-1} \circ D_V \circ e^{*\varphi_1}, \quad (2.5c)
\]

and of course the perturbed chain homotopy

\[
\tilde{\eta} = (id_{S(V[1])} + \eta \circ D_V)^{-1} \circ \eta. \quad (2.5d)
\]

This entails in particular that \(e^{*\varphi}\) is a \(L_\infty\)-quasi-isomorphism with quasi-inverse \(e^{*\psi}\).

A proof of this is given in \(\text{[BE18]}\).

The preceding Theorem 2.1 has the following well-known corollary:
Corollary 2.2. Suppose that in the homotopy contraction \((C.2)\) the differential \(b_U = 0\) whence \(U\) is isomorphic to the cohomology of \((V, b_V)\). Then under the hypothesis of the preceding Theorem 2.1 we have the following:

1. There is a minimal \(L_\infty\)-structure \((U, d)\) quasi-isomorphic to \((V, b_V[1] + D'_V)\). In case the latter \(L_\infty\)-structure comes from the structure of a differential graded Lie algebra, \((V, b, [\ , ])\), then the term \(d_2\) is isomorphic to the shift of the induced Lie bracket \([\ , ]_H\) on cohomology, \([\ , ]_H[1]\).

2. Suppose there is another homotopy contraction \((U', 0) \xrightarrow{\psi_1'} (V, b_V) \xrightarrow{D'} h'\).

Then, writing \(\varphi'_1 = i'[1], \psi'_1 = p'[1]\) under the hypothesis of the preceding Theorem 2.1 the two \(L_\infty\) structures \((U, d)\) and \((U', d')\) are conjugated, i.e. there is an isomorphism of coalgebras \(e^*\chi : S(U[1]) \to S(U'[1])\) such that

\[
\overline{d'} = e^*\chi \circ \overline{d} \circ (e^*\chi)^{-1}.
\]

(2.6)

Proof. 1. The first statement is immediate from the preceding Theorem 2.1. For the second, according to equation (2.4c) we have \(d'_2 = \psi_1 \circ D'_2 \circ (\varphi_1 \bullet \varphi_1)\), and for \(D'_2 = [\ , ][1]\) it is clear that this is thus the shifted induced bracket on cohomology with isomorphism given by \(\varphi_1\).

2. Using restrictions, since \(e^*\psi'|_{V[1]} = \psi'_1\) induces an isomorphism \(V[1] \to U'[1]\) and \(e^*\varphi'|_{U[1]} = \varphi_1\) induces an isomorphism \(U[1] \to V[1]\), we have that \((e^*\psi' \circ e^*\varphi)|_{V[1]} = \psi'_1 \circ \varphi_1\) is an isomorphism \(U[1] \to U'[1]\), which implies that the morphism of graded connected coalgebras \(e^*\chi := e^*\psi' \circ e^*\varphi\) is invertible using Lemma 1.1 above. We have 

\[
e^*\psi' \circ e^*\varphi \circ \overline{d} = e^*\psi' \circ \overline{D} \circ e^*\varphi = \overline{d'} \circ e^*\psi' \circ e^*\varphi
\]

proving the Corollary.

\[\square\]

We get the following \(L_\infty\)-analogue of Remark C.2.4 which is quite useful:
Corollary 2.3. Let \((U, b_U)\) and \((V, b_V)\) be complexes and suppose there is a homotopy contraction (C.2). Let furthermore \((U, b_U[1] + D_U)\) and \((V, b_V[1] + D_V)\) be \(L_{\infty}\)-structures and 

\[
\Phi : (S(U[1]), b_U[1] + D_U) \to (S(V[1]), b_V[1] + D_V)
\]

an \(L_{\infty}\)-map such that (writing \(\psi'_1 = p[1]\) and \(\varphi'_1 = i[1]\)) 

\[
\psi'_1 \circ \Phi|_{U[1]} : U[1] \to U[1]
\]
is invertible. Then \(\Phi\) is a \(L_{\infty}\)-quasi-isomorphism. Moreover, if both \(L_{\infty}\)-structures come from differential graded Lie algebra structures on \(U\) and \(V\), respectively, then the corresponding graded Lie structures on the cohomologies of \(U\) and \(V\) with respect to \(b_U\) and \(b_V\), respectively, are isomorphic.

Proof. Write \(\Phi|_{U[1]} =: \varphi_1 : U[1] \to V[1]\), \(A := \psi'_1 \circ \varphi_1\) the invertible \(\mathbb{K}\)linear map which clearly is a chain map \((U[1], b_U[1]) \to (U[1], b_U[1])\). Set \(\hat{\psi}_1 := A^{-1} \circ \psi'_1\). Then \(\hat{\psi}_1\) is a chain map \((V[1], b_V[1]) \to (U[1], b_U[1])\) and clearly 

\[
\hat{\psi}_1 \circ \varphi_1 = A^{-1} \circ \psi'_1 \circ \varphi_1 = id_{U[1]}.
\]

On the other hand \(\varphi_1 \circ \hat{\psi}_1\) commutes with \(b_V[1]\), and thanks to eqn (C.1c) we get 

\[
\varphi_1 \circ \hat{\psi}_1 = (\varphi'_1 \circ \psi'_1 + [h[1], b_V[1]]) \circ \varphi_1 \circ \hat{\psi}_1
= \varphi'_1 \circ \psi'_1 + [h[1] \circ \varphi_1 \circ \hat{\psi}_1, b_V[1]]
= id_{V[1]} - [h[1] \circ (id_{V[1]} - \varphi_1 \circ \hat{\psi}_1), b_V[1]],
\]

whence the \(\mathbb{K}\)-linear maps \(\hat{\psi}_1, \varphi_1, h'[1] := h[1] \circ (id_{V[1]} - \varphi_1 \circ \hat{\psi}_1)\) define a homotopy contraction (C.2) for the complexes \((U[1], b_U[1])\) and \((V[1], b_V[1])\), the check of the side conditions for \(h'[1]\) being straight-forward. In particular, \(\varphi_1\) induces an isomorphism in cohomology whence it is a \(L_{\infty}\)-quasi-isomorphism. For the second statement let \(D_U = D_2 = [\ , \ ][1]\) and \(D_V = D'_2 = [\ , \ ][1]\) where \([\ , \ ]\) and \([\ , \ ]'\) denote the graded Lie brackets on \(U\) and \(V\), respectively. Pick a quasi-inverse \(\Psi : S(V[1]) \to S(U[1])\) which exists, see e.g. [AMM02, Thm. V1, V2], then the fact that both \(\Phi = e^{\ast \varphi}\) and \(\Psi = e^{\ast \psi}\) are chain maps read when evaluated on two elements and projected to the primitive space:

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for all $y_1, y_2 \in U[1]$ and $z_1, z_2 \in V[1]$ we get with $d_U = b_U[1]$ and $d_V = b_V[1]$

$$
\varphi_2(d_U(y_1) \cdot y_2 + (-1)^{|y_1|} y_1 \cdot d_U(y_2)) + \varphi_1(D_2(y_1 \cdot y_2))
= d_V(\varphi_2(y_1 \cdot y_2)) + D'(\varphi_1(y_1) \cdot \varphi_1(y_2))
$$

(2.7)

and

$$
\psi_2(d_V(z_1) \cdot z_2 + (-1)^{|z_1|} z_1 \cdot d_U(z_2)) + \psi_1(D_2(z_1 \cdot z_2))
= d_U(\psi_2(z_1 \cdot z_2)) + D'(\psi_1(z_1) \cdot \psi_1(z_2)).
$$

(2.8)

Hence if $y_1, y_2, z_1, z_2$ are cocycles, it follows that $\varphi_1$ and its inverse $\psi_1$ preserve Lie brackets up to coboundaries, and upon projecting onto the corresponding cohomology, we get the desired isomorphism of graded Lie brackets.

For later use, in the case $b_U = 0$ we give the shifted formula for the linear map $\phi_2 = \varphi_2[-1]$ (of degree $-1$) from $\Lambda^2 g$ to $S g$ in case of a differential graded Lie algebra $(\mathfrak{g}, b, [\ , ]_G)$ (whence $D = b[1] + [\ , ]_G[1]$), i.e.\[\phi_2(y_1, y_2) = -h([\phi_1(y_1), \phi_1(y_2)]_G).\]

(2.9)

which can be used to compute the characteristic 3-class $c_3$ from $w_3$, see eqn (1.15).

## 3 Finite-dimensional Lie algebras

In this Section we shall consider different examples of finite-dimensional (trivially graded) Lie algebras $(\mathfrak{g}, [\ , ]_G)$ and study the formality of the Hochschild complex $C_H(U\mathfrak{g}, U\mathfrak{g})$ of their universal enveloping algebra $U\mathfrak{g}$.

Recall the well-known Chevalley-Eilenberg complex $(C_{CE}(\mathfrak{g}, S\mathfrak{g}), \delta)$ of the Lie-algebra $\mathfrak{g}$ taking values in the symmetric algebra $S\mathfrak{g}$ seen as a $\mathfrak{g}$-module via the adjoint representation: Since $\mathfrak{g}$ is finite-dimensional it is canonically isomorphic to the tensor product $S\mathfrak{g} \otimes \Lambda\mathfrak{g}^*$ and is $\mathbb{Z}$-graded by the ‘Grassmann degree’, i.e. the form degree of the second factor $\Lambda\mathfrak{g}^*$. With this grading, it clearly is a graded commutative algebra by means of the tensor product of the commutative multiplication in $S\mathfrak{g}$ and the usual exterior multiplication in $\Lambda\mathfrak{g}^*$ which we shall also denote by $\wedge$. Considering an element $f \in S\mathfrak{g}$ as a polynomial function on the dual space $\mathfrak{g}^*$ we can consider $C_{CE}(\mathfrak{g}, S\mathfrak{g})$ as the space of all polynomial poly-vector-fields on $\mathfrak{g}^*$ equipped with the
usual Schouten bracket $[,]_S$, see e.g. [BM08, eq.(5.1)] or [BMP05]; we recall the definition: let $e_1, \ldots, e_n$ be a fixed basis of $\mathfrak{g}$, let $\epsilon^1, \ldots, \epsilon^n$ denote the dual basis of $\mathfrak{g}^*$, and recall the structure constants of the Lie algebra $\mathfrak{g}$, $c^i_{jk} = \epsilon^i([e_j, e_k]) \in \mathbb{K}$. Then each $x \in \mathfrak{g}^*$ can be written as a sum $x = \sum_{i=1}^n x_i \epsilon^i$. For each $\xi \in \mathfrak{g}$ we have the usual interior product graded derivation $\iota_\xi : \Lambda \mathfrak{g}^* \to \Lambda \mathfrak{g}^*$, and for each $y \in \mathfrak{g}^*$ we have the corresponding derivation $\iota_y : S \mathfrak{g} \to S \mathfrak{g}$. For a dual basis vector $e_i$ we shall sometimes write the more suggestive way $\iota_\epsilon \epsilon_i(f) = \partial_i f$ for each $f \in S \mathfrak{g}$. We extend these derivations to the tensor product $C_{CE}(\mathfrak{g}, S \mathfrak{g})$ in the obvious way and write $\wedge$ for the combined multiplication. With these conventions the Schouten bracket of two elements $F, G \in C_{CE}(\mathfrak{g}, S \mathfrak{g})[1]$ reads

$$[F, G]_S = \sum_{i=1}^n \epsilon_i(F) \wedge \partial_i G - (-1)^{|F|-1} \sum_{i=1}^n \epsilon_i(G) \wedge \partial_i F$$

(3.1)

where the degree $|F|$ is the original unshifted ‘Grassmann degree’ to which we have stucked for computational reasons. Recall that $(C_{CE}(\mathfrak{g}, S \mathfrak{g}), \wedge, [\cdot, \cdot],_S)$ is a Gerstenhaber algebra, i.e. there is a graded Leibniz rule

$$[F, G \wedge H]_S = [F, G]_S \wedge H + (-1)^{|F|-1}|G| G \wedge [F, H]_S$$

(3.2)

Denoting by

$$\pi = [\cdot, \cdot] = \frac{1}{2} \sum_{i,j,k} c^i_{jk} e_i \otimes (e^j \wedge e^k)$$

(3.3)

the so-called linear Poisson structure of $\mathfrak{g}^*$ we of course have $[\pi, \pi]_S = 0$, and we can use the coboundary operator $\delta = \delta_\mathfrak{g} = [\pi, \cdot]_S$ for the (shifted) Chevalley Eilenberg coboundary operator which differs from the historical definition by an unessential global sign. It thus follows that $(C_{CE}(\mathfrak{g}, S \mathfrak{g})[1], \delta_\mathfrak{g}, [\cdot, \cdot],_S)$ is a differential graded Lie algebra. The following Theorem shows that in order to check formality of the Hochschild complex it suffices to check it for the ‘easier’ Chevalley-Eilenberg complex. Since we shall need Kontsevich’s formality theorem we shall assume for the rest of this Section that the field $\mathbb{K}$ is equal to $\mathbb{C}$:

**Theorem 3.1.** Let $(\mathfrak{g}, [\cdot, \cdot])$ be a finite-dimensional complex Lie-algebra.

1. There is a $L_\infty$-quasi-isomorphism between the differential graded Lie algebra $(C_{CE}(\mathfrak{g}, S \mathfrak{g})[1], \delta_\mathfrak{g}, [\cdot, \cdot],_S)$ and the differential graded Lie algebra

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In particular, this induces an isomorphism of graded Lie algebras of their cohomologies (with respect to $\delta_g$ and $b$, respectively).\footnote{This answers a question asked by F. Wagemann for the case of finite-dimensional Lie algebras.}

2. The $L_\infty$-formality of $(C_{CE}(\mathfrak{g}, \mathcal{S}_g)[1], \delta_g, [\ , \ ]_g)$ is equivalent to the $L_\infty$-formality of $(C_H(\mathcal{U}g, \mathcal{U}g)[1], b, [\ , \ ]_G)$.

Proof. In [BM08, Theorem 6.2], a morphism of differential graded coalgebras $e^*\varphi'$ from $\mathcal{S}(C_{CE}(\mathfrak{g}, \mathcal{S}_g)[2])$ to $\mathcal{S}(C_H(\mathcal{U}g, \mathcal{U}g)[2])$ had been constructed by twisting the well-known Kontsevich formality quasi-isomorphism $e^*\varphi$ from $\mathcal{S}(C_{CE}(\mathfrak{g}, \mathcal{S}_g)[2])$ to $\mathcal{S}(C_H(\mathcal{S}_g, \mathcal{S}_g)[2])$ by the formal exponential $e^{\hbar\pi}$ of the linear Poisson structure $\pi$ defined by the Lie bracket of $\mathfrak{g}$ and observing that the resulting map converges for $\hbar = 1$ on polynomials. Here the fact that the universal enveloping algebra of $\mathfrak{g}$ can be seen as a converging Kontsevich deformation of the symmetric algebra (sketched in [Kon03, Secs. 8.3.1, 8.3.2]) has also been used, see [BMP05] for more details. Now $e^*\varphi'$ is even a $L_\infty$-quasi-isomorphism which has just been stated without proof in [BM08, Theorem 6.2]. We shall indicate the proof of it which is relatively straight-forward. Recall the standard quasi-isomorphism of the Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in $\mathcal{S}_g$ with the Hochschild cohomology complex $C_H(\mathcal{U}g, \mathcal{U}g)$ of its enveloping algebra which we can write in the following way as a contraction of complexes:

\[
(C_{CE}(\mathfrak{g}, \mathcal{S}_g), \delta_g) \xrightarrow{\psi_{CE}} (C_H(\mathcal{U}g, \mathcal{U}g), b) \xrightarrow{\phi_{CE}} h.
\]

Here $\psi_{CE}$ is given by $\psi_{CE}(F) = \omega^{-1} \circ F \circ F$ where the map $F : \Lambda^* \mathfrak{g} \to \mathcal{U}g^{\otimes}$ is already given in [CE56, p.280] and consists of evaluation of a Hochschild cochain on the antisymmetrization of its arguments restricted to $\mathfrak{g} \subset \mathcal{U}g$, and $\omega : \mathcal{S}_g \to \mathcal{U}g$ is the canonical symmetrization map which is an isomorphism of $\mathfrak{g}$-modules, see e.g. [Dix77, p.78]. $\phi_{CE}$ is a quasi-inverse of $\psi_{CE}$ and $h$ is a chain homotopy which are much harder to describe explicitly: it can be done in terms of Eulerian idempotents and iterated integrals, see the PhD-thesis of S. Rivière [Riv12]. By Corollary 2.3 we just have to check whether $\psi_{CE}[2] \circ \varphi'_1$ is an invertible map: according to eqn (6.1) of Thm 6.1 of [BM08], $\varphi'_1$ takes the following form: for any polyvector-field $F$ of rank $m$ in $C_{CE}(\mathfrak{g}, \mathcal{S}_g)$ and
m polynomials \( f_1, \ldots, f_m \) in \( Sg \) (seen as polynomial functions on \( g^* \)) we have (the image of the Kontsevich formality map \( \varphi \) are poly-differential operators)

\[
\varphi'_1(F)(f_1, \ldots, f_m) = \varphi_1(F)(f_1, \ldots, f_m) + \sum_{r=1}^{\infty} \frac{1}{r!} \varphi_{r+1}(\pi^{\star r} \bullet F)(f_1, \ldots, f_m).
\]

Let \( x_1, \ldots, x_m \in g \) seen as linear functions on the dual space \( g^* \). Then

\[
(\psi_{CE}[2] \circ \varphi'_1)(F)(x_1, \ldots, x_m) = \text{Alt} \left( \varphi_1(F)(x_1, \ldots, x_m) + \sum_{r=1}^{\infty} \frac{1}{r!} \varphi_{r+1}(\pi^{\star r} \bullet F)(x_1, \ldots, x_m) \right)
\] (3.4)

where Alt denotes antisymmetrization in the arguments \( x_1, \ldots, x_m \). The first term on the right hand side of eqn (3.4) is up to a nonzero constant factor equal to \( \xi(x_1, \ldots, x_m) \). In the second term on the right hand side of eqn (3.4) involving the sum \( \sum_{\infty}^{\infty} \) we check the polynomial degree of the corresponding polyvector-field: due to Kontsevich’s universal formula [Kon03] it follows that \( \varphi_{r+1}(\pi^{\star r} \bullet F) \) is a polydifferential operator acting on \( m \) functions: it is a finite sum parametrised by certain graphs where in each term \( 2r + m \) partial derivatives are distributed over the \( r \) linear Poisson structures \( \pi \), the polynomial polyvector-field \( F \) (where \( \delta \) denotes the maximal polynomial degree of its coefficients), and the \( m \) functions. In eqn (3.4) we need to consider only \( m \) linear functions \( x_1, \ldots, x_m \), hence \( 2r + m \) derivatives meet a polynomial of degree \( r + \delta + m \), whence the resulting polynomial degree is \( \delta - r \): it follows that the above sum in the second part on the right hand side of (3.4) has at most \( \delta \) terms, and the polynomial degree of the resulting polyvector-field is strictly lower than \( \delta \). By a simple filtration argument in the polynomial degree it follows that \( \psi_{CE}[2] \circ \varphi'_1 \) is equal to an invertible map plus lower order terms and is thus invertible proving the first part of the Theorem.

The second statement is immediate. \( \square \)

In the following subsections we check formality for the Chevalley-Eilenberg complex of certain finite-dimensional Lie algebras, hence we look at the following differential graded Lie algebras

\[
(\mathfrak{g}, b, [\ , \ ]_G) = (C_{CE}(g, Sg)[1], \delta_g, [\ , \ ]_s) \quad \text{and} \quad (\mathfrak{h}, [\ , \ ]_H) = (H_{CE}(g, Sg)[1], [\ , \ ]_H).
\] (3.5)

We shall denote the \( \delta_g \)-cohomology classes of a cocycle \( F \) in \( C_{CE}(g, Sg) \) by \([F]\).
3.1 Abelian Lie algebras

In case the Lie algebra $\mathfrak{g}$ is abelian, $\mathcal{U}\mathfrak{g} = S\mathfrak{g}$, and then there is nothing to prove since the Chevalley-Eilenberg differential is zero, whence $C_{CE}(\mathfrak{g}, S\mathfrak{g}) \cong H_{CE}(\mathfrak{g}, S\mathfrak{g})$, and formality of $C_{CE}(\mathfrak{g}, S\mathfrak{g})$ is the content of the Kontsevich formality theorem.

3.2 Lie algebra of the affine group of $K^m$

**Theorem 3.2** ([BM08, Theorem 6.3]). Let $\mathfrak{g}$ be the affine Lie algebra i.e. the semidirect sum

$$\mathfrak{gl}(m, K) \oplus K^m.$$ 

Then the differential graded Lie algebra $(C_{CE}(\mathfrak{g}, S\mathfrak{g})[1], \delta_{\mathfrak{g}}, [\ , \ ]_s)$ is formal.

Here the cohomology is represented by certain ‘constant poly-vector fields’, i.e. by elements of $\Lambda^* \mathfrak{g}$ whose Schouten brackets all vanish, hence the cohomology injects as a graded Lie sub-algebra of the complex which gives the formality.

3.3 Quadratic Lie algebras

Recall that a symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to K$ is called **invariant** if for all $\xi, \xi', \xi'' \in \mathfrak{g}$ we have

$$\kappa([\xi, \xi'], \xi'') = \kappa([\xi, \xi'], [\xi', \xi'']).$$  \hspace{1cm} (3.6)

A triple $(\mathfrak{g}, [\ , \ ], \kappa)$ is called a **quadratic Lie algebra** if the symmetric bilinear form $\kappa$ is **invariant and nondegenerate**. Examples are abelian Lie algebras (with any nondegenerate symmetric bilinear form), or semisimple Lie algebra equipped with their Killing form $\kappa(\xi, \xi') = \text{trace}(\text{ad}_\xi \circ \text{ad}_{\xi'})$. See e.g. [MR85], [AB93], or [Bor97] for more details on these algebras. We shall call any Lie algebra admitting a nondegenerate invariant symmetric bilinear form metrisable, see [Bor97]. Let $q \in S^2 \mathfrak{g}$ be the ‘inverse’ of $\kappa$: if $\kappa^* : \mathfrak{g} \to \mathfrak{g}^*$ denotes the canonical map $\xi \mapsto (\eta \mapsto \kappa(\xi, \eta))$, take its inverse $\kappa^* : \mathfrak{g}^* \to \mathfrak{g}$, and consider it as an element $q$ in $S^2 \mathfrak{g}$, or using a base $e_1, \ldots, e_n$ of $\mathfrak{g}$, and $q = \sum_{i,j=1}^n q^{ij} e_i \otimes e_j$ where $q^{ij} \in K$ are the components of the symmetric bivector $q$, then for all $\xi \in \mathfrak{g}$: $\sum_{i,j=1}^n \kappa(\xi, e_i) q^{ij} e_j = \xi$. Then $q$ is invariant under the adjoint representation of $\mathfrak{g}$. Consider the morphism of commutative associative unital algebras $K[t] \to S\mathfrak{g}$ induced by $t \mapsto q$. Since every
symmetric power $q^m \in S^{2m}g$ is easily seen to be nonzero (because the free trivially graded commutative algebra $Sg$ does not have nilpotent elements), and since the subspaces $S^i g$ are independent, the above morphism is injective, and we denote its image by $\mathbb{K}[q] \subset Sg$. Clearly every polynomial $\alpha \in \mathbb{K}[q] \subset Sg$ is invariant. Next, there are three more important elements of $C_{CE}(g, Sg)$, the linear Poisson structure $\pi$ (see eqn 3.3), the Euler field $E \in g \otimes g^* \cong \text{Hom}(g, g)$, defined by the identity map $g \to g$, $E = \sum_{i=1}^n e_i \otimes e^i$, and the Cartan 3-cocycle $\Omega \in \Lambda^3 g^*$ defined by

$$\Omega(\xi, \xi', \xi'') = \kappa(\xi, [\xi', \xi''])$$  \hspace{1cm} (3.7)

for all $\xi, \xi', \xi'' \in g$. Upon using formula (3.1) we easily compute the following Schouten brackets where $\alpha, \beta, \gamma \in \mathbb{K}[q]$ and $\alpha'$ denotes the derivative of the polynomial $\alpha$:

$$[\alpha, \beta]_s = 0,$$  \hspace{1cm} (3.8)

$$\delta_g(\alpha) = [\pi, \alpha]_s = 0,$$  \hspace{1cm} (3.9)

$$[E, \alpha]_s = 2q\alpha',$$  \hspace{1cm} (3.10)

$$\delta_g(\alpha \wedge E) = [\pi, \alpha \wedge E]_s = \alpha \wedge \pi,$$  \hspace{1cm} (3.11)

$$\delta_g(\alpha \wedge \Omega) = [\pi, \alpha \wedge \Omega]_s = 0,$$  \hspace{1cm} (3.12)

$$[E, \Omega]_s = -3\Omega,$$  \hspace{1cm} (3.13)

$$[\beta \wedge \Omega, \alpha]_s = 2(\beta \alpha') \wedge \pi = \delta_g(2(\beta \alpha') \wedge E),$$  \hspace{1cm} (3.14)

$$[\beta \wedge \Omega, \gamma \wedge \Omega]_s = 2(\beta \gamma' - \gamma \beta') \wedge \pi \wedge \Omega = \delta_g(2(\beta \gamma' - \gamma \beta') \wedge E \wedge \Omega).$$  \hspace{1cm} (3.15)

We can now compute a representing graded 3-cocycle $z_3$ (see eqn (1.16)) for the characteristic 3-class $c_3$ of $C_{CE}(g, Sg)$ on certain elements of $C_{CE}(g, Sg)$. For this purpose it seems to be interesting to single out a subclass of finite-dimensional quadratic Lie algebras: We call a quadratic Lie algebra $(g, [\ , \ ], \kappa)$ a Cartan-3-regular quadratic Lie algebra if the cohomology class of the Cartan cocycle $\Omega, [\Omega]$, is nonzero. A metrisable Lie algebra will be called Cartan-3-regular if there is a nondegenerate symmetric invariant bilinear form $\kappa$ such that $(g, [\ , \ ], \kappa)$ is Cartan-3-regular. Semisimple Lie algebras are well-known to be Cartan-3-regular.

**Lemma 3.3.** Let $(g, [\ , \ ], \kappa)$ be a quadratic Lie algebra of finite dimension, and let $(Sg)^g$ denote the subspace of all ad-invariant elements of $Sg)^g$
1. Then $\mathbb{K}[q] \subset (S\mathfrak{g})^g \cong H^0_{CE}(\mathfrak{g}, S\mathfrak{g})$.

2. $(\mathfrak{g}, [ , ], \kappa)$ is Cartan-3-regular iff there is no derivation of $\mathfrak{g}$ whose $\kappa$-symmetric part is a nonzero multiple of the identity.

3. If $(\mathfrak{g}, [ , ], \kappa)$ is Cartan-3-regular then the linear map $\mathbb{K}[q] \rightarrow H^3_{CE}(\mathfrak{g}, S\mathfrak{g})$ defined by $\alpha \mapsto \alpha \wedge \Omega$ is an injection.

Proof. 1. The last isomorphy is true for any Lie algebra since the 0-coboundaries vanish. As $q$ is $ad$-invariant, every polynomial of $q$ is also $ad$-invariant.

2. $(\mathfrak{g}, [ , ], \kappa)$ is not Cartan-3-regular iff there is a 2-form $\theta : \Lambda^2 \mathfrak{g} \rightarrow \mathbb{K}$ such that $\Omega = -\delta_q \theta$. The space of all 2-forms is isomorphic to the space of all $\kappa$-antisymmetric linear maps $C : \mathfrak{g} \rightarrow \mathfrak{g}$ via $C \mapsto ((\xi, \eta) \mapsto \kappa(C(\xi), \eta))$. Thanks to nondegeneracy and invariance of $\kappa$ the condition $\Omega = -\delta_q \theta$ is easily be computed to be equivalent to

$$[\xi, \eta] = C[\xi, \eta] - [C(\xi), \eta] - [\xi, C(\eta)]$$

for all $\xi, \eta \in \mathfrak{g}$. The above equation is equivalent to $D = C + I$ being a derivation of the Lie algebra $\mathfrak{g}$. The $\kappa$-symmetric part of $D$ is clearly the identity map $I$.

3. Note that for any nonnegative integer $n$ the symmetric power $\kappa^n$ can be seen as a nonzero linear form on $S^{2n} \mathfrak{g}$ where in particular $\kappa^n(q^n) \neq 0$. Thanks to the invariance of $\kappa$ we have $\kappa^n([\xi, T]) = 0$ for any $\xi \in \mathfrak{g}$ and $T \in S^{2n-1} \mathfrak{g}$. Denoting by $K_{2n}$ the linear map $S^{2n} \mathfrak{g} \otimes \Lambda^g \rightarrow \Lambda^g$ sending $S \otimes \xi$ to $\kappa^n(S)\xi$ and by $K$ the sum over all even degrees (on the odd degrees $K$ being defined to be zero) we see that $K$ intertwines Chevalley-Eilenberg differentials w.r.t. the usual representation on $S\mathfrak{g}$ induced by the adjoint representation and those w.r.t. the trivial representation on $\mathbb{K}$. It suffices to show that each $q^n \wedge \Omega$ gives a non-zero class in cohomology. If there was $\theta \in S\mathfrak{g} \otimes \Lambda^g$ such that $q^n \wedge \Omega = \delta_q(\theta)$, then –upon applying $K$ to this equation– a non-zero multiple of $\Omega$ would be an exact form which would be in contradiction with $\mathfrak{g}$ being Cartan-3-regular.

We have the following central result:

**Theorem 3.4.** Let $(\mathfrak{g}, [ , ], \kappa)$ be a finite-dimensional Cartan-3-regular quadratic Lie algebra. Then the Hochschild complex of its universal enveloping algebra is NOT $L_\infty$-formal.
Proof. We shall show that the characteristic 3-class $c_3$ is nontrivial: Let $\alpha, \beta, \gamma \in \mathbb{K}[q]$, upon writing $[\alpha]$ or $[\beta \wedge \Omega]$ for the $\delta_0$-cohomology classes represented by $\alpha$ and $\beta \wedge \Omega$, respectively. From the Schouten brackets in (3.8), (3.14), and (3.15) which all give $\delta$-coboundaries it is clear that the following graded Lie brackets in cohomology vanish:

$$[[\alpha], [\beta]]_H = 0, \quad [[\alpha], [\beta \wedge \Omega]]_H = 0, \quad [[\beta \wedge \Omega], [\gamma \wedge \Omega]]_H = 0. \tag{3.16}$$

Next, we choose any graded vector space complement of the $\delta_0$-coboundaries in the $\delta_0$-cocycles which includes the space of all $\alpha \in \mathbb{K}[q]$ and all $\beta \wedge \Omega$, we get the resulting section $\phi_1 : \mathfrak{H} \to \mathfrak{G}$ satisfying the natural condition $\phi_1([\alpha]) = \alpha$ and $\phi_1([\alpha \wedge \Omega]) = \alpha \wedge \Omega$ for all $\alpha \in \mathbb{K}[q] \subset (\mathfrak{S}g)^q$. Then, according to eqn (1.13), eqs (3.8), (3.14), and (3.15) we can choose a $\mathbb{K}$-linear map $\phi_2 : \Lambda^2 \mathfrak{H} \to \mathfrak{G}$ of degree $-1$ satisfying

$$\phi_2(\alpha, \beta) = 0 \quad \text{and} \quad \phi_2(\alpha, \beta \wedge \Omega) = 2(\alpha' \beta) \wedge E. \tag{3.17}$$

For later use we also note the following fact which will not be necessary in this proof:

$$\phi_2(\beta \wedge \Omega, \gamma \wedge \Omega) = 2(\beta \gamma' - \gamma \beta') \wedge E \wedge \Omega. \tag{3.18}$$

It follows that the graded Chevalley-Eilenberg $\delta_0$-3-cocycle $z_3$ (which represents the characteristic 3-class $c_3$ of the differential graded Lie algebra $\mathfrak{G} = C_{CE}(\mathfrak{g}, \mathfrak{S}g)[1]$ and depends on $\phi_1$ and $\phi_2$, see eqs (1.15) and (1.16) takes the following values: $z_3([\alpha], [\beta], [\gamma]) = 0$, and, most importantly,

$$z_3([\alpha], [\beta], [\gamma \wedge \Omega]) = 8[q\alpha' \beta' \gamma]. \tag{3.19}$$

Again, for later use and not necessary for this proof we note that

$$z_3([\alpha], [\beta \wedge \Omega], [\gamma \wedge \Omega]) = -8[q\alpha' (\beta \gamma' - \gamma \beta')] \wedge \Omega]. \tag{3.20}$$

Finally, in case $c_3$ vanished there would be a graded 2-form $\theta : \Lambda^2 \mathfrak{H} \to \mathfrak{H}$ (where $\mathfrak{H} = H_{CE}(\mathfrak{g}, \mathfrak{S}g)[1]$) of degree $-1$ (since $z_3$ is of degree $-1$) such that $z_3 = \delta_3 \theta$. We evaluate $\delta_3 \theta$ on the three elements $[\alpha], [\beta]$, and $[\gamma \wedge \Omega]$ of $\mathfrak{H}$. According to formula (1.14) we need to know $\theta([\alpha], [\beta])$ (which must vanish since both $[\alpha]$ and $[\beta]$ are of degree $-1$ as is $\theta$) and $\theta([\alpha], [\gamma \wedge \Omega])$ which has to be of degree $0$, hence in $H^1_{CE}(\mathfrak{g}, \mathfrak{S}g)[1]$. We consider the particular case $\alpha = q = \beta$ and $\gamma = 1$. Let $D \in \text{Hom}(\mathfrak{g}, \mathfrak{S}g)$ be a $\delta_0$-1-cocycle such that
\[ [D] = \theta([q], [\Omega]). \] Then the equation \( z_3([q], [q], [\Omega]) = (\delta_3\theta)([q], [q], [\Omega]) \) would give (using the above eqn (3.19) and formula (1.14))

\[ 8[q] = -2[[q], [D]]_H = 2[D(q)], \tag{3.21} \]

since the Schouten bracket of a vector field with a function, \([f, X]_S\), equals \(-X(f)\), and since \([\pi, q]_s = 0\) we get \([[\pi, f]_s, q]_s = 0\) for all \(f \in Sg\) showing that the last term in the above equation is well-defined on the class \([D]\). Hence we would get the equation

\[ D(q) = 4q. \tag{3.22} \]

Write \( D = \sum_{r=0}^N D_r \) where for each \( r \in \mathbb{N} \) the component \( D_r \in \text{Hom}(g, S^r g) \). Clearly, each \( D_r \) must be a \( \delta_g \)-1-cocycle, hence \( D_1 : g \rightarrow g \) would be a derivation of the Lie algebra \( g \), and comparing symmetric degrees we must have \( D_1(q) = 4q \). Elementary linear algebra (e.g. expressing the previous equation in coordinates w.r.t. a chosen base of \( g \)) gives for all \( \xi, \xi' \in g \) the equation

\[ \kappa(D_1(\xi), \xi') + \kappa(\xi, D_1(\xi')) = 4\kappa(\xi, \xi'). \]

This would show that the derivation \( D_1 : g \rightarrow g \) has a \( \kappa \)-symmetric part equal to 2 times the identity which is in contradiction to the hypothesis of \((g, [ , ], \kappa)\) being Cartan-3-regular, see the second statement of the preceding Lemma \[3.3\]. Hence \( c_3 \) is a nontrivial class whence there is no formality. \( \square \)

The subclass of all Cartan-3-regular quadratic Lie algebras includes also non semisimple Lie algebras whose derivations are all antisymmetric, see e.g. [AB93], for which there is NO formality according to the preceding Proposition \[3.4\].

### 3.3.1 Reductive Lie algebras

Let \((g, [ , ])\) be a finite-dimensional reductive Lie algebra: recall that such a Lie algebra decomposes into a direct sum \( g = \mathfrak{z} \oplus [g, g] \) where \( \mathfrak{z} \) is its centre and the derived ideal \( I = [g, g] \) is a semisimple Lie algebra, see e.g. [Jac79] for definitions. Recall that every reductive Lie algebra is quadratic: pick any nondegenerate symmetric bilinear form on \( \mathfrak{z} \) and the Killing form \((\xi, \xi') \mapsto \text{trace}(\text{ad}_\xi \circ \text{ad}_{\xi'})\) on \( I \), and define the nondegenerate invariant symmetric bilinear form \( \kappa \) to be the orthogonal sum of the two preceding ones. Note that the Cartan 3-cocycle \( \Omega \) of \( g \) is given by \( \Omega(z_1 + l_1, z_2 + l_2, z_3 + l_3) = \Omega(l_1, l_2, l_3) \) for
any \( z_1, z_2, z_3 \in \mathfrak{g} \) and \( l_1, l_2, l_3 \in \mathfrak{l} \) where \( \Omega \) is the Cartan 3-cocycle of \( \mathfrak{l} \) which is well-known to be a nontrivial 3-cocycle for \( \mathfrak{l} \) if \( \mathfrak{l} \neq \{0\} \) since all derivations of a semisimple Lie algebra are well-known to be inner hence antisymmetric w.r.t. the Killing form. It clearly is also a nontrivial 3-cocycle for \( \mathfrak{g} \). Hence \( (\mathfrak{g}, [\ , \ ], \kappa) \) is Cartan-3-regular, and according to Theorem 3.4 we get

**Proposition 3.5.** Let \( \mathfrak{g} \) be a nonabelian reductive Lie algebra. Then the Chevalley-Eilenberg complex of \( \mathfrak{g} \) with values in \( S\mathfrak{g} \) (and hence the Hochschild complex of its universal enveloping algebra) is NOT formal in the \( L_\infty \) sense.

In case \( \mathfrak{g} \) is semisimple it can be shown (by the Whitehead Lemma and some standard representation theory) that the induced graded Lie bracket on cohomology vanishes.

### 3.3.2 Lie algebra \( \mathfrak{so}(3) \)

The smallest semisimple Lie algebra is the Lie algebra \( \mathfrak{so}(3) \) (isomorphic to \( \mathfrak{sl}(2, \mathbb{K}) \)) which can be spanned by a basis \( e_1, e_2, e_3 \) subject to the brackets

\[
[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_1,
\]

where all other brackets are clear from antisymmetry. The Killing form is given by \( \kappa(e_i, e_j) = -2\delta_{ij} \). From the Whitehead Lemma it is clear that

\[
\mathfrak{H} = H_{CE}(\mathfrak{g}, S\mathfrak{g}) \cong \mathbb{K}[q] \mathbf{1} \oplus \{0\} \oplus \{0\} \oplus \mathbb{K}[q] \Omega \quad (3.23)
\]

where \( \Omega \) is the Cartan 3-cocycle.

As in the general semisimple case, the cohomology \( H_{CE}(\mathfrak{g}, S\mathfrak{g}) \) does not inject in the Chevalley-Eilenberg complex \( \mathfrak{G} = C_{CE}(\mathfrak{g}, S\mathfrak{g}) \) as a graded Lie subalgebra, but we can define a smaller graded Lie subalgebra of \( \mathfrak{G} \) which contains the cohomology, viz

\[
\mathfrak{G}_{\text{red}} := \mathbb{K}[q] \mathbf{1} \oplus \mathbb{K}[q] E \oplus \mathbb{K}[q] \pi \oplus \mathbb{K}[q] \Omega \quad (3.24)
\]

where \( E \) is the Euler field, \( \pi \) is the linear Poisson structure.

**Proposition 3.6.** \( \mathfrak{G}_{\text{red}} \) is a differential graded Lie subalgebra of \( (\mathfrak{G}, \delta, [\ , \ ]_\mathfrak{s}) \), and there is a contraction

\[
\mathfrak{H} \overset{i}{\longrightarrow} \mathfrak{G}_{\text{red}}, \delta \quad \circlearrowright \quad h \quad (3.25)
\]

29
where $i$ is the natural injection, $p$ the natural projection (with kernel $\mathbb{K}[q]E \oplus \mathbb{K}[q]\pi$), and the map $h$ is given by $h = h^1 : \mathbb{K}[q] \pi \to \mathbb{K}[q]E$, $h^1(\alpha \wedge \pi) = \alpha \wedge E$, for $\alpha \in \mathbb{K}[q]$, and is defined to vanish in degree $-1, 0, 2$.

The injection $\mathfrak{G}_{\text{red}} \to \mathfrak{G}$ is a quasi-isomorphism of differential graded Lie algebras.

**Proof.** This follows from the identities (3.8) – (3.15). $\square$

For this simple example we can use the $L_\infty$-Perturbation Lemma to compute an explicit $L_\infty$-structure on the cohomology $\mathfrak{H}$ and an explicit $L_\infty$-quasi-isomorphism $\varepsilon^*\varphi$ from $S(\mathfrak{H}[1]) \to S(\mathfrak{G}[1])$ with $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 = i[1]$:

**Theorem 3.7.** With the above notation we have the following:

1. The Chevalley-Eilenberg complex of $\mathfrak{so}(3)$ is NOT formal.

2. There is a $L_\infty$ structure $d$ on $S(\mathfrak{H}[1])$ whose only nonvanishing Taylor coefficient is $d_3$ (which can be given by the shifted characteristic 3-cocycle $z_3$, see eqn (1.16), i.e. $d_3 = z_3[-1]$) for its only nonvanishing component (up to permutations).

3. There is a $L_\infty$-quasi-isomorphism $\varepsilon^*\varphi$ from $(S(\mathfrak{H}[1]), d_3)$ to $(S(\mathfrak{G}_{\text{red}}[1]), \delta[1] + \overline{D}_2)$ (where $D_2$ denotes the shifted Schouten bracket).

The only nonvanishing Taylor coefficients of $\varepsilon^*\varphi$ are $\varphi_1 = i[1]$ and $\varphi_2$ which can explicitly be given.

**Proof.** 1. This is a particular case of Proposition 3.5.

2. and 3. We compute the formulas from the $L_\infty$-Perturbation Lemma, see eqs (2.4c) and (2.4a) which we give in terms of the geometric series:

$$d = \sum_{r=0}^{\infty} \text{pr}_{\delta[1]} \circ \overline{D}_2 \circ (-\eta \circ D_2)^r \circ e^{*\varphi_1},$$

$$\varphi = \sum_{r=0}^{\infty} \text{pr}_{\delta[1]} \circ (-\eta \circ D_2)^r \circ e^{*\varphi_1}.$$

In order to understand –for any nonnegative integer $r$– the iterated product $(-\eta \circ D_2)^r$, we shall apply $-\eta \circ D_2$ to a graded symmetric word containing $k + m + l$ letters or terms of the following kind: $k$ times a term of degree $-2$, i.e. of the form $\alpha \in \mathbb{K}[q]$, $l$ times a term of degree 1, i.e. of the form...
\[ \beta \wedge \Omega \in \mathbb{K}[q] \wedge \Omega, \text{ and } m \text{ times a term of degree } -1, \text{ i.e. of the form } \gamma \wedge E \in \mathbb{K}[q] \wedge E: \text{ the application of the shifted Schouten bracket } \overline{D}_2 \text{ will produce two sums of linear combinations of graded symmetric words; the first type of words containing } k - 1 \text{ terms of degree } -2, m \text{ terms of degree } -1, \text{ one term of degree } 0 \text{ proportional to } \tilde{\gamma} \wedge \pi \text{ with } \tilde{\gamma} \in \mathbb{K}[q] \text{ (which comes from the Schouten bracket of a degree } -2 \text{ term and a degree } 1 \text{-term), and } l - 1 \text{ terms of degree } 1; \text{ the second type of words containing } k \text{ terms of degree } -2, m - 1 \text{ terms of degree } -1, \text{ and } l \text{ terms of degree } 1 \text{ (which come from Schouten brackets involving at least one Euler field). An ensuing application of } \eta \text{ shifts the terms proportional to } \pi \text{ (in the first sum) to a term proportional to } E \text{ and kills the second sum (since it obviously is in the kernel of the graded biderivation } \tilde{h}). \text{ As a result we get a linear combination of graded symmetric words containing } k - 1 \text{ terms of degree } -2, m + 1 \text{ terms of degree } -1, \text{ and } l - 1 \text{ terms of degree } 1. \text{ By induction, and } r \text{-fold iteration yields words with } k - r \text{ terms of degree } -2, m + r \text{ terms of degree } -1, \text{ and } l - r \text{ terms of degree } l - r. \]

In the above formulas for } \varphi \text{ we have } m = 0 \text{ since the expressions are applied to words containing letters in the cohomology. It follows that for all integers } r \geq 2 \text{ there will be } r \geq 2 \text{ factors of type } \tilde{\gamma} \wedge E: \text{ application of the projection } \text{pr}_{\mathfrak{g}[1]} \text{ will kill these terms because there are at least two factors. It follows that there are only two surviving Taylor coefficients of } \varphi: \varphi_1 \text{ (the case } r = 0) \text{ and } \varphi_2 \text{ (the case } r = 1). \text{ Computing on arguments } \alpha \mathbf{1}, \beta \omega \text{ in } \mathfrak{g}[1], \text{ with } \alpha, \beta \in \mathbb{K}[q], \text{ we obtain}

\[
\varphi_2(\alpha \mathbf{1}, \beta \mathbf{1}) = 0, \quad \varphi_2(\alpha \mathbf{1}, \beta \omega) = \alpha'\beta E \quad \text{ and } \quad \varphi_2(\alpha \omega, \beta \omega) = 0.
\]

On the other hand, for each integer } r \geq 2 \text{ an application of the shifted Schouten bracket } \overline{D}_2 \text{ will leave at least one factor of the type } \tilde{\gamma} \wedge E \text{ which is in the kernel of the projection to cohomology, } \text{pr}_{\mathfrak{g}[1]}. \text{ It follows that all Taylor coefficients } d_{r+2} \text{ of } d \text{ vanish for } r \geq 2, \text{ and the shifted induced bracket on cohomology, } d_2 \text{ (the case } r = 0), \text{ vanishes thanks to fact that the induced graded Lie bracket on cohomology vanishes for semisimple Lie algebras). Hence the only remaining Taylor coefficient is } d_3 \text{ (the case } r = 1) \text{ which is of the form}

\[
\begin{align*}
&d_3(\alpha \mathbf{1}, \beta \mathbf{1}, \gamma \wedge \Omega) = 8g\alpha'\beta'\gamma, \\
&d_3(\alpha \mathbf{1}, \beta \wedge \Omega, \gamma \wedge \Omega) = -8(g\alpha'(\beta\gamma' - \gamma\beta')) \wedge \Omega.
\end{align*}
\]
3.4 Heisenberg algebra

We consider the three-dimensional Heisenberg Lie algebra whose underlying vector space is $\mathbb{K}^3$ with basis $x, y, z$, and the only nonvanishing bracket is given by $[x, y] = z = -[y, x]$. Or, writing the Lie bracket as a bivector $[,] = \pi = z\partial_x \wedge \partial_y \in \text{Hom}(\Lambda^2 \mathfrak{g}, \mathfrak{g})$. Although the Lie bracket is simpler than the one of $\mathfrak{so}(3)$, the cohomology is more complex, and is not abelian.

In [Elc14], the cohomology of the Chevalley-Eilenberg complex $C_{CE}(\mathfrak{g}, S\mathfrak{g})$ has been computed and shown that it is not formal.

4 Free Lie algebra

We shall closely follow Chapitre 3 of the thesis [Elc12].

Let $V$ be a vector space over $\mathbb{K}$ and $\mathcal{L}V$ the associated free Lie algebra. Then its universal enveloping algebra is well-known to be isomorphic to the free associative algebra $\mathcal{U}(\mathcal{L}V) \cong TV$.

Here the Hochschild cohomology can be computed using a free resolution, see e.g. [CE56, Chap.IX p.181], and is only concentrated in degree 0 and 1, composed of its centre $TV^TV$ and the space of all outer derivations:

$$H_H(TV, TV) \cong TV^TV \oplus \text{Der}(TV, TV)/\text{Inder}(TV, TV)$$

where, as usual, $\text{Der}(TV, TV)$ denotes the Lie algebra of all derivations of the algebra $TV$, and $\text{Inder}(TV, TV)$ denotes the space of all inner derivations, which are adjoint representations $\text{ad}_x : TV \to TV$ for all $x \in TV$ defined by $\text{ad}_x(y) = xy - yx$ for all $y \in TV$. Note that $\text{ad}_x = b(x)$ for the Hochschild coboundary $b$. We shall sometimes denote the quotient Lie algebra $\text{Der}(TV, TV)/\text{Inder}(TV, TV)$ by $\text{outder}$, and shall again write $\mathfrak{G}$ for the graded Lie algebra $(C_H(TV, TV)[1], [, ],_G, b)$.

For $V$ of dimension 0, $\mathcal{T}\{0\} \cong \mathbb{K}$, the centre is isomorphic to the field $TV^TV \cong \mathbb{K}$, and there is a formality map since the Hochschild cohomology injects as a graded abelian Lie subalgebra in the Hochschild complex

$$\varphi_1 = id_{\mathbb{K}} : \mathbb{K} \to \bigoplus_{k \in \mathbb{N}} \text{Hom}(\mathbb{K}^\otimes k, \mathbb{K}) \cong \bigoplus_{k \in \mathbb{N}} \mathbb{K}, \quad \varphi_k = 0 \text{ for } k \geq 2.$$
For $V$ of dimension 1 we can write $V = \mathbb{K}e$ (having fixed a base vector $e$ of $V$), and $\mathcal{T}(\mathbb{K}e) \cong \mathbb{K}[x]$ is the commutative ring of polynomials in one variable, hence it is also equal to the center $\mathcal{T}V^V = \mathbb{K}[x]$, and all inner derivations vanish. The space $\text{Hom}(\mathbb{K}e, \mathbb{K}[x]) = \{ f \partial_x \mid f \in \mathbb{K}[x] \}$ is the Lie algebra of vector fields. There is again a formality map induced by the inclusion $\varphi_1$ of the Hochschild cohomology into the Hochschild complex as a graded Lie subalgebra, $\varphi_k = 0$ for $k \geq 2$. The map $\varphi_1$ is the identity on the center, and associates to $f \partial_x$ its derivation.

The truly interesting and more involved case is of course given by $V$ of dimension $\geq 2$. The zeroth cohomology group, i.e. the centre of $\mathcal{T}V$, is then well-known to be reduced to $\mathcal{T}V^V = \mathbb{K}$. The graded Lie bracket $[,]_H$ on the cohomology $\mathfrak{H} = \mathbb{K} \oplus \text{outder}$ is readily computed by

$$[(\lambda \mathbf{1}, D), (\lambda' \mathbf{1}, D')]_H = (0, [D, D'])$$

for any $\lambda, \lambda' \in \mathbb{K}$ and $D, D' \in \text{outder}$. Since the cohomology graded Lie algebra is concentrated in degree $-1$ and 0, its shift $\mathfrak{H}[1]$ is concentrated in degree $-2$ and $-1$. Counting degrees we immediately get the following

**Theorem 4.1.** There is a $L_\infty$-structure $d$ on $\mathcal{S}(\mathfrak{H}[1])$ whose Taylor coefficients $d_n$ vanish for all integers $n \geq 4$, and $d_2 = [\ , \ ]_H[1]$. Moreover there is a $L_\infty$-quasi-isomorphism $e^\varphi$ from $(\mathcal{S}(\mathfrak{H}[1]), d_2 + d_3)$ to $(\mathcal{S}(\mathfrak{G}[1], b[1] + D))$ whose Taylor coefficients $\varphi_n$ vanish for all $n \geq 3$. Finally, the map $d_3[-1] : \Lambda^3(\mathfrak{H}) \rightarrow \mathfrak{H}$ (which is of degree $-1$) can be seen as a a scalar 3-cocycle $\sigma$ of the Chevalley-Eilenberg cohomology $C_{CE}(\text{outder}, \mathbb{K})$ of the Lie algebra of all outer derivations of $\mathcal{T}V$.

**Proof.** The $L_\infty$ structure $d = \sum_{n=2}^{\infty} d_n$ on $\mathcal{S}(\mathfrak{H}[1])$ and the $L_\infty$-map $e^\varphi$ exist by the general arguments given in the preceding sections, for instance thanks to the fact that we can always find a chain homotopy to relate $\mathfrak{H}$ and $(\mathfrak{G}, b)$ in context of a deformation retract and using Theorem 2.1. Since $d$ is of degree 1, and $\varphi$ is of degree 0 we get for all integers $k \geq 1$ and $i_1, \ldots, i_k \in \{-2, -1\}$ that $d_k(\mathfrak{H}[1]^{i_1} \cdots \mathfrak{H}[1]^{i_k}) \subset \mathfrak{G}_{\text{red}}[1]^{i_1+\cdots+i_k+1}$ and $\varphi_k(\mathfrak{H}[1]^{i_1} \cdots \mathfrak{H}[1]^{i_k}) \subset \mathfrak{G}_{\text{red}}[1]^{i_1+\cdots+i_k}$. For $k \geq 4$ we have $i_1 + \cdots + i_k + 1 \leq -3$ and for all $k \geq 3$ we have $i_1 + \cdots + i_k \leq -3$, whence $d_k = 0$ for all $k \geq 4$ and $\varphi_k = 0$ for all $k \geq 3$. $d_3[-1]$ is a graded Chevalley-Eilenberg 3-cocycle of degree $-1$: since $\mathfrak{H}^{-1} \cong \mathbb{K} \mathbf{1}$ is central, it follows that all other components of $d_3[-1]$ are reduced to
zero with the possible exception of the restriction of $d_3[-1]$ to three arguments in $\text{outder} = \mathfrak{H}^0$ whose image is in $\mathfrak{H}^{-1} \cong \mathbb{K} \cdot 1$. This surviving component can be seen as an ungraded scalar 3-cocycle $\sigma : \Lambda^3\text{outder} \to \mathbb{K}$ of the ungraded Lie algebra $\text{outder}$. 

In order to check formality we have to check whether the aforementioned 3-cocyle $\sigma$ can be a coboundary, and this requires some more explicit computations:

The Hochschild 1-cocycles of $\mathcal{T}V$ comprise the space of all derivations $\text{Der}(\mathcal{T}V, \mathcal{T}V)$ of $\mathcal{T}V$: since every derivation is uniquely determined by its restriction to the space of generators $V$, and in turn every linear map $\psi : V \to \mathcal{T}V$ uniquely extends to a derivation by the Leibniz rule there is a linear isomorphism $\overline{\psi} : \text{Hom}(V, \mathcal{T}V) \to \text{Der}(\mathcal{T}V, \mathcal{T}V) \subset \text{Hom}(\mathcal{T}V, \mathcal{T}V)$ defined by

$$\overline{\psi}(1) = 0, \quad \overline{\psi}(x_1 \cdots x_n) = \sum_{r=1}^{n} x_1 \cdots x_{r-1} (\psi(x_r)) x_{r+1} \cdots x_n. \quad (4.2)$$

for all $x_1, \ldots, x_n \in V$. We shall sometimes denote $\text{Hom}(V, \mathcal{T}V)$ by $\text{det}$. As for the coderivations we can pull-back the usual Lie bracket of derivations from $\text{Der}(\mathcal{T}V, \mathcal{T}V)$ to a Lie bracket $[\ , \ ]_D$ on the space $\text{det}$ by means of the linear isomorphism $\overline{\psi}$: for any $\psi, \chi \in \text{Hom}(V, \mathcal{T}V)$ we compute

$$[\psi, \chi]_D := \overline{\psi} \circ \chi - \overline{\chi} \circ \psi, \quad (4.3)$$

and $\overline{\psi}$ is a morphism of Lie algebras, i.e. $[\overline{\psi}, \chi]_D = [\overline{\psi}, \chi]$. Moreover we shall write $b' : \mathcal{T}V \to \text{Hom}(V, \mathcal{T}V)$ for the restriction of the adjoint representation $b(x) = \text{ad}_x$ to $V$ (for all $x \in \mathcal{T}V$). We have $\overline{b'({x})} = b(x)$, hence $b'({xy-yx}) = [b'(x), b'(y)]_D$. It follows that $b'\mathcal{T}V$ is an ideal of the Lie algebra $\left( \text{Hom}(V, \mathcal{T}V), [\ , \ ]_D \right)$.

The space $\text{Hom}(V, \mathcal{T}V)$ carries an additional $\mathbb{Z}$-grading (called tensor grading) according to the degree $\text{Hom}(V, \mathcal{T}V)^{(k)} = \text{Hom}(V, V^\otimes k+1)$, for $k \geq -1$, and $\{0\}$ for $k \leq -2$. The tensor grading is auxiliary, no signs are attached. The space $b'\mathcal{T}V^+$ also carries the degree of $\mathcal{T}V^+$. $\mathcal{T}V^+^{(k)} = V^\otimes k$ with $k \geq 0$. In the first degrees, we have $\text{Hom}(V, \mathcal{T}V)^{(-1)} = \text{Hom}(V, \mathbb{K}) = V^*$, $(b\mathcal{T}V^+)^{(-1)} = \{0\}$, and $\text{Hom}(V, \mathcal{T}V)^{(0)} = \text{Hom}(V, V)$, $(b'\mathcal{T}V^+)^{(0)} = \{0\}$. Note that brackets $\sigma'$ are of tensor degree 0, whence the cohomology $\mathbb{K} \cdot 1 \oplus \text{outder}$ is in addition graded by the tensor degree.

As for the Lie algebra $\mathfrak{so}(3)$ we can now define a smaller differential graded Lie algebra $\mathfrak{G}_{\text{red}}$ which injects in the Hochschild complex of $\mathcal{T}V$ as a
differential graded subalgebra, viz.

\[ \mathfrak{g}_{\text{red}} = \mathfrak{g}_{\text{red}}^{-1} \oplus \mathfrak{g}_{\text{red}}^0 := \mathcal{T}V \oplus \text{Hom}(V, \mathcal{T}V) \quad (4.4) \]

equipped with the rather simple graded Lie bracket (where \( x, y \in \mathcal{T}V, \psi, \chi \in \text{der} \) and we write ordered pairs \( (x, \psi) \) for elements of the direct sum \( \mathcal{T}V \oplus \text{Hom}(V, \mathcal{T}V) \))

\[ [(x, \psi), (y, \chi)]_{\text{red}} := (\overline{\psi}(y) - \overline{\chi}(x), [\psi, \chi]_D), \quad (4.5) \]

and differential \( b_{\text{red}}(x, \psi) := (0, b'(x)) \). It is easy to see that the injection \( (x, \psi) \mapsto x + \overline{\psi} \) is a quasi-isomorphism of differential graded Lie algebras \( (\mathfrak{g}_{\text{red}}, [\cdot, \cdot]_{\text{red}}, b_{\text{red}}) \rightarrow (\mathfrak{g}, [\cdot, \cdot]_G, b) \).

Next we would like to define a chain homotopy \( h \) in \( \mathfrak{g}_{\text{red}} \). To this end we first choose a complementary subspace \( \mathcal{H}^0 \subset \text{Hom}(V, \mathcal{T}V) \) to the space of all coboundaries \( b'\mathcal{T}V \), i.e. restrictions of inner derivations, in \( \text{Hom}(V, \mathcal{T}V) \) in the following way: we can suppose that it is graded (with respect to the tensor degree), i.e. \( \mathcal{H}^0 = \bigoplus_{n \geq -1} \mathcal{H}^0(n) \), and we set \( \mathcal{H}^0(-1) = V^* \) (dual space of \( V \)), \( \mathcal{H}^0(0) = \text{Hom}(V, V) \), and for \( k \geq 1 \), we choose in each \( \text{Hom}(V, V^{\otimes k+1}) \) a complementary subspace \( \mathcal{H}^0(k) \) to the inner derivations, i.e. \( \text{Hom}(V, V^{\otimes k+1}) = \mathcal{H}^0(k) \oplus (b'\mathcal{T}V^+)^{(k)} \). For the component of degree \(-1\) of \( \mathfrak{g}_{\text{red}}, \mathcal{T}V \), we have the natural decomposition \( \mathcal{T}V = K \mathbb{1} \oplus \mathcal{T}V^+ \), the latter being the augmentation ideal of \( \mathcal{T}V \), i.e. the sum of all elements of strictly positive tensor degree. Hence we set \( \mathcal{H}^{-1} = \mathcal{T}V^+ \) which also carries a second grading according to tensor degree, \( \mathcal{H}^{-1(k)} = \mathcal{T}^kV \) for all integers \( k \geq 1 \). For each integer \( k \geq -1 \) let \( P_k : \text{Hom}(V, V^{\otimes k+1}) \rightarrow \mathcal{H}^0(k) \) be the canonical projection having kernel \( (b'\mathcal{T}V)^k \). We set \( P = \sum_{k \geq -1} P_k \). Hence for each integer \( k \geq -1 \) the linear map \( id_{\otimes^k} - P_k \) (which vanishes for \( k = -1, 0 \)) is a projection onto \( b'\mathcal{T}V \) which is in bijection with \( \mathcal{T}V^+ \) via \( b' \). Using the inverse of this bijection there is, for each integer \( k \geq -1 \), a unique linear map \( Q_k : \text{Hom}(V, V^{\otimes k+1}) \rightarrow V^{\otimes k} \) be such that \( id_{\otimes^k} - P_k = bQ_k \). Setting \( Q_{-1} = 0, Q_0 = 0, Q = \sum_{k \geq 1} Q_k \), we define the chain homotopy \( h : \mathfrak{g}_{\text{red}} \rightarrow \mathfrak{g}_{\text{red}} \)

\[ h(x, \psi) = (Q(\psi), 0) \quad (4.6) \]

for all \( \psi \in \text{Hom}(V, \mathcal{T}V) \) and \( x \in \mathcal{T}V \). Moreover, since the restriction of the natural projection \( \text{Hom}(V, \mathcal{T}V) \rightarrow \text{der} \) to the subspace \( \mathcal{H}^0 \) of \( \text{Hom}(V, \mathcal{T}V) \) is a bijection, its inverse gives an injection \( \text{der} \rightarrow \mathcal{H}^0 \subset \text{Hom}(V, \mathcal{T}V) \) which, combined with the canonical injection \( K \mathbb{1} \rightarrow \mathcal{T}V \), defines an injection
\textit{i} of the cohomology \( \mathfrak{H} \) into \( \mathfrak{G}_{\text{red}} \). On the other hand, the natural projection \( \text{Hom}(V, TV) \to \text{outder} \) combined with the canonical projection \( TV \to K1 \) (having kernel \( TV^+ \)) defines a surjection \( p : \mathfrak{G}_{\text{red}} \to \mathfrak{H} \). Obviously, \( p \circ i = \text{id}_H \) and \( i \circ p \) equals \( P \) on \( \mathfrak{G}^0_{\text{red}} = \text{Hom}(V, TV) \) and the projection onto \( K1 \subset TV = \mathfrak{G}^{-1}_{\text{red}} \). It follows that there is a contraction

\[
\mathfrak{H} \xrightarrow{i} \mathfrak{G}_{\text{red}} \xrightarrow{p} \mathfrak{H}
\]

The graded \( \mathfrak{H} \)-3-cocycle \( \sigma_3 \) (see the preceding Theorem 4.1) is surprisingly simple:

**Proposition 4.2.** The graded \( \mathfrak{H} \)-3-cocyle \( \sigma_3 \) from \( \Lambda^3\text{outder} \to K \) (defined in Theorem 4.1) is of tensor degree zero. For any \( \alpha, \beta \in \text{outder}^{-1} = V^* \), \( A, B, C \in \text{outder}^{(0)} = \text{Hom}(V, V) \), \( \rho \in \text{outder}^{(1)} \), and \( \psi \in \text{outder}^{(2)} \) we get

\[
\begin{align*}
\sigma(A, B, C) &= 0, \\
\sigma(\alpha, B, \rho) &= -\alpha(Q_1([B, \rho]_D)), \\
\sigma(\alpha, \beta, \psi) &= -\alpha(Q_1([\beta, \psi]_D)) + \beta(Q_1([\alpha, \psi]_D)),
\end{align*}
\]

whereas all other components of \( \sigma \) (which are no permutations of the above) vanish.

**Proof.** According to formula (2.9) and formula \( w_3 \) (see eqn (1.15)) we see that \( \sigma \) is of tensor degree 0, hence for any homogeneous \( \psi_1, \psi_2, \psi_3 \in \text{outder} \) it follows that \( \sigma(\psi_1, \psi_2, \psi_3) \) is of tensor degree zero (since \( K1 \subset TV = \mathfrak{G}^{-1} \) is of tensor degree zero), hence \( |\psi_1| + |\psi_2| + |\psi_3| = 0 \) leaving for the only possible non-zero components three possibilities \( |\psi_1| = 0, |\psi_2| = 0, |\psi_3| = 0 \); then \( |\psi_1| = -1, |\psi_2| = 0, |\psi_3| = 1 \); and finally \( |\psi_1| = -1, |\psi_2| = -1, |\psi_3| = 2 \). Identifying \( \text{outder} \) with the complement \( \mathcal{H}^0 \) in \( \text{der} \) and the cohomological Lie bracket \([ , ]_H \) in \( \text{outder} \) with the projection \((\text{id} - Q)([ , ]_D)\) we get

\[
\begin{align*}
\sigma(\psi_1, \psi_2, \psi_3) &= -\epsilon([\psi_1, Q[\psi_2, \psi_3]_D]_{\mathfrak{G}_{\text{red}}} - Q([\psi_1, \psi_2]_H, \psi_3]_{\mathfrak{G}_{\text{red}}}) + \text{cycl.}) \\
&= -\epsilon(\overline{\psi_1}(Q([\psi_2, \psi_3]_D)) + \text{cycl.} \]
\]

where the last three terms vanish thanks to the fact that the double bracket is of tensor degree 0, whence its result is of tensor degree 0, and \( Q_0 = 0 \).
Eqn (4.7) immediately follows since $Q_0 = 0$. Two of the three terms in (4.8) vanish since $Q_{-1} = 0$ and $Q_0 = 0$, and $\pi$ reduces to the application of $\alpha$ applied to the vector which is the value of $Q_1$. In eqn (4.9) the term with the bracket $[\alpha, \beta]_D = 0$ vanishes leaving the other two. \hfill \Box

We shall now construct a more explicit complement $\mathcal{H}$ to the coboundaries in $\partial \mathfrak{c}$ in the case where $V$ is of finite dimension $N$: Let $e_1, \ldots, e_N$ be a base and $\epsilon^1, \ldots, \epsilon^N$ be the corresponding dual base. For $n \in \mathbb{N}$ we can write the applications $\psi \in \text{Hom}(V, V^\otimes n+1)$ as

$$\psi = \sum_{j, i_0, \ldots, i_n = 1}^N \psi_{i_0}^{i_n} e_{i_0} \otimes \cdots \otimes e_{i_n} \otimes \epsilon^j,$$

where $\psi_{i_0}^{i_n} \in \mathbb{K}$ are the components of $\psi$ w.r.t. the base. We have canonically identified $\text{Hom}(V, V^\otimes n+1)$ with $V^\otimes n+1 \otimes V^\ast$. For each integer $n \geq -1$, we consider the following linear map $S_n : \text{Hom}(V, V^\otimes n+1) \to \mathcal{T}^n V$ defined by $S_{-1} = 0, S_0 = 0$, and for all $n \geq 1$, $v_0, v_1, \ldots, v_n \in V$ and $\alpha \in V^\ast$ we set $S_n(v_0 \otimes \cdots \otimes v_n \otimes \alpha)$ equal to $\alpha(v_0) v_1 \otimes \cdots \otimes v_n$ which reads in components

$$S_n(\psi) = \sum_{j, i_1, \ldots, i_n = 1}^N \psi_{i_1}^{i_n} e_{i_1} \otimes \cdots \otimes e_{i_n} \quad \text{if } n \geq 1, \quad (4.11)$$

and can be viewed as a kind of ‘first factor trace’ for $n \geq 1$. Note that each $S_n$ is invariant under the general linear group of $V$. We write $S : \text{Hom}(V, \mathcal{T}V) \to \mathcal{T}V$ for the sum $S := \sum_{n \geq -1} S_n$ whence $S$ is homogenous of degree 0 w.r.t. the tensor grading. For each integer $n \geq 1$ denote by $\zeta_n : V^\otimes n \to V^\otimes n$ the linear map defined by the cyclic permutation where $v_1 \otimes \cdots \otimes v_n$ is sent to $v_2 \otimes \cdots \otimes v_n \otimes v_1$ for all $v_1, \ldots, v_n \in V$. Observing that for each $a \in \mathcal{T}^n V$, $n \geq 1$, the inner derivation $b'(a)$ has components $(\text{ad}_a)^{i_0 i_1 \cdots i_n}$ given by $a^{i_0 i_1 \cdots i_{n-1}} \delta^i_{i_n} - \delta^i_{i_0} a^{i_1 \cdots i_n}$ we get

$$S_n(b'(a)) = \zeta_n(a) - Na. \quad (4.12)$$

Since obviously $\zeta_n^\otimes = \text{id}_{\mathcal{T}^n V}$ it follows that $\zeta_n$ is diagonalizable, and the eigenvalues of $\zeta_n$ are in the set of all $n$th roots of unity, whence $\zeta_n - N \text{id}_{\mathcal{T}^n V}$ is invertible since $N \geq 2$. This shows that for each integer $n \geq 1$ the map $S_n$ is surjective, and the intersection of $\mathcal{H}'(\mathcal{T}^n V)$ with the kernel $\mathcal{H}^{0(n)}$ of $S_n$ is equal to $\{0\}$. By elementary finite-dimensional linear algebra we
conclude that $\mathcal{H} = \bigoplus_{n \geq -1} \text{Ker } S_n$ is a graded complement of $b' TV^+$, and thus defines a section $\text{outer} \to \mathcal{H} \subset \text{der}$. Recall that $\mathcal{H}^{0(-1)} = V^* = \text{der}^{(-1)}$ and $\mathcal{H}^{0(0)} = \text{Hom}(V, V) = \text{der}^{(0)}$. By inverting $\zeta_n - \text{id}_{TV}$ we get the map $Q = \sum_{n \geq -1} Q_n : \text{der} \to TV$ by setting $Q_{-1} = 0$, $Q_0 = 0$, and for each integer $n \geq 1$

$$Q_n = -\frac{1}{N^n - 1} \sum_{r=0}^{n-1} N^{n-r-1} \zeta_n \circ S_n.$$  

(4.13)

**Proposition 4.3.** In the notation of Proposition 4.2 we have

$$\sigma(A, B, C) = 0, \quad \sigma(\alpha, B, \rho) = 0, \quad \text{and}$$

$$\sigma(\alpha, \beta, \psi) = \frac{1}{N - 1} \sum_{k, j, l} \alpha_i \beta_k (\psi_{jk}^l - \psi_{kj}^l),$$  

(4.15)

and there are $\alpha, \beta, \psi$ such that $\sigma(\alpha, \beta, \psi) \neq 0$.

**Proof.** The first eqn in (4.14) follows from (4.7), and the second one from eqn (4.8), from the definition of $S_1$ (4.13) and from the fact that $S_1$ is invariant under the linear group whence $S_1([B, \rho]_D) = B(S_1(\rho)) = 0$ since $\rho \in \mathcal{H}^{0(1)} = \text{Ker}(S_1)$. The last eqn is a straight-forward computation using eqs (4.13) and (4.9). Taking $\alpha = \epsilon^1$, $\beta = \epsilon^2$, and $\psi = e_1 \otimes e_2 \otimes e_2 \otimes \epsilon^2$ we get $\sigma(\alpha, \beta, \psi) = 1$.  

We are now ready to prove the following

**Theorem 4.4.** Let $V$ be a vector space over $\mathbb{K}$ whose dimension is $\geq 2$.

1. The Hochschild complex of the free algebra $TV$ generated by $V$ is NOT formal in the $L_\infty$-sense.

2. There is a $L_\infty$-structure on the Hochschild cohomology of $TV$ whose Taylor coefficients $d_2$ and $d_3$ do not vanish, but $d_n = 0$ for all $n \geq 4$, and there is a $L_\infty$-quis from the Hochschild cohomology (with respect to $d_2 + d_3$) to the Hochschild complex (with respect to the usual structure $b[1] + [\ , \ ]_G[1]$).

**Proof.** We look first at the finite-dimensional case where we shall show the following auxiliary statement:
Let $\theta : \Lambda^2 \text{outder} \to \mathbb{K}$ be a linear map of tensor degree 0 such that the $H$-3-coboundary $\delta_H \theta$ satisfies both eqs of eqn (4.14). Then $\delta_H \theta = 0$.

In the finite-dimensional case this will imply that the 3-cocycle $\sigma$ is nontrivial since it is non-zero showing nonformality whereas the second statement will then follow from Theorem 4.1.

In order to prove the above auxiliary statement $(\ast)$, we first observe that $\theta$ has (up to obvious permutations) only two surviving components, $\theta_{00} : \Lambda^2 \text{Hom}(V,V) \to \mathbb{K}$, and $\theta_{-11} : V^* \otimes \text{Hom}(V, V^\otimes 2) \to \mathbb{K}$ thanks to the fact that we can assume that $\theta$ is of tensor degree zero in order to have that $\delta_H \theta$ is.

The first of the eqs of (4.14) shows that $(\delta_{\mathfrak{gl}} \theta_{00})(A, B, C) = 0$ for three linear maps $A, B, C : V \to V$ where $\mathfrak{gl}$ is short for the Lie algebra $\mathfrak{gl}(N, \mathbb{K})$. It is well-known that the second scalar cohomology group of the Lie algebra $\mathfrak{gl}(N, \mathbb{K})$ vanishes (where Whitehead’s Lemma for $\mathfrak{sl}(n, \mathbb{K})$ and the classical Hochschild-Serre spectral sequence argument are used). Hence there is a linear form $f_0 : \text{Hom}(V,V) \to \mathbb{K}$ such that $\theta_{00} = \delta_{\mathfrak{gl}} f_0$. Upon trivially extending $f_0$ to a linear form (also denoted by $f_0$) of tensor degree 0 to the Lie algebra outder we see that for the modified 2-form $\theta' = \theta - \delta_H f_0$ the component $\theta'_{00}$ vanishes whereas the coboundary remains the same, $\delta_H \theta' = \delta_H \theta$.

Next, with these modifications we look at the second equation in (4.14): using $\theta'_{00} = 0$ we quickly obtain that $\theta'_{-11}$ is invariant under the Lie algebra $\text{Hom}(V,V)$. Since both $H^{0,1} = \text{Ker} S_1$ and $b'V$ are $\text{Hom}(V,V)$-invariant complementary subspaces of $\text{Hom}(V, V^\otimes 2)$ we can extend $\theta'_{-11}$ to a $\text{Hom}(V,V)$-invariant linear map from $V \otimes \text{Hom}(V, V^\otimes 2)$ to $\mathbb{K}$, hence as a $\text{Hom}(V,V)$-invariant element of $V^\otimes 2 \otimes V^* \otimes 2$. Using the Invariant Tensor Theorem (see e.g. the book [KMS93, Thm.20.4, p.214] for a good account) we can conclude that $\theta'_{-11}$ is of the following form with $\lambda, \mu \in \mathbb{K}$ (for all $\alpha \in V^*$ and $\rho \in \text{Ker} S_2 \subset \text{Hom}(V, V^\otimes 2)$)

$$
\theta'(\alpha, \rho) = \lambda \sum_{r,s=1}^N \alpha_r \rho_{rs}^{rs} + \mu \sum_{r,s=1}^N \alpha_r \rho_{rs}^{sr} = \lambda \sum_{r,s=1}^N \alpha_r \rho_{rs}^{rs} \quad (4.16)
$$

since $S_1(\rho) = \sum_{r,s=1}^N \rho_{rs}^{sr} e_r = 0$. We need the computation of the Lie bracket in cohomology of $[\alpha, \psi]_H$ where $\alpha \in V^*$ and $\psi \in \text{Hom}(V, V^\otimes 3)$:

$$
[\alpha, \psi]_H^{i\delta_{ij}}
$$
\[
\sum_{r=1}^{N} \alpha_r \left( \psi_j^{i_0 i_1} + \psi_j^{i_1 i_1} + \psi_j^{i_0 i_1} \right) + \frac{1}{N-1} \sum_{r,s=1}^{N} \alpha_r \left( \psi_s^{r s i_0} \delta_j^{i_1} - \psi_s^{r s i_1} \delta_j^{i_0} \right).
\]

(4.17)

It is then straight-forward to see that \(\delta_\beta \theta'(\alpha, \beta, \psi) = 0\) for all \(\alpha, \beta \in V^*\) and \(\psi \in \text{Hom}(V, V^{\otimes 3})\). Since \(\theta'\) is of tensor degree 0, so is \(\delta_\beta \theta'\), and therefore the only possibly nonzero components are of tensor degrees \((0,0,0), (-1,0,1),\) and \((-1,-1,2)\) which are all zero whence \(\delta_\beta \theta' = 0\). This proves the auxiliary statement and the Theorem in the finite-dimensional case.

Suppose now that \(V\) is a vector space of arbitrary, not necessarily finite dimension greater than 2. Choose a subspace \(W \subset V\) of finite dimension \(N \in \mathbb{N}, N \geq 2\), and a complementary subspace \(X \subset V\), i.e. \(V = W \oplus X\). The inclusion \(W \subset V\) induces an inclusion of associative algebras \(\iota : \mathcal{T}W \rightarrow \mathcal{T}V\). Upon using the above map \((4.13)\) and only possibly nonzero components are of tensor degrees \(\chi\) and the extension \(H\) there is a complement \(T\) of \(b\) in \(V\). Hence the subspace \(\tilde{V}'\) of \(V\) generated by \(V\) and \(b\) is \(\tilde{V}' = \mathcal{T}V\oplus \text{Hom}(W, \mathcal{T}V)\) to \(b_V(\mathcal{T}W)\) where we have written \(b_V\) for the adjoint representation with respect to \(W\). Let \(Q_W : \text{Hom}(W, \mathcal{T}W) \rightarrow \mathcal{T}W\) the above map \((4.13)\). For any linear map \(\chi\) in \(\text{Hom}(W, \mathcal{T}W)\) we define an extension \(\tilde{\chi}\) in \(\text{Hom}(V, \mathcal{T}V)\) by \(\tilde{\chi}(w) = \iota(\chi(w))\) for all \(w \in W\), and \(\tilde{\chi}(x) = \text{ad}_{Q_W(\chi)}(x) = \chi(\iota(Q_W(\chi)))\) for all \(x \in X\). We clearly get for all \(a \in \mathcal{T}W\)

\[
\tilde{b}_V(a) = b_V(\iota(a)),
\]

and the extension \(\chi \mapsto \tilde{\chi}\) is injective. We shall write \(\iota : \mathcal{T}W \oplus \text{Hom}(W, \mathcal{T}W) \rightarrow \mathcal{T}V \oplus \text{Hom}(V, \mathcal{T}V)\) for the injective linear map \((a, \chi) \mapsto (\iota(a), \tilde{\chi})\). It clearly is a morphism of complexes \((\mathcal{G}_{\text{red}}(\mathcal{T}W), b'_V) \rightarrow (\mathcal{G}_{\text{red}}(\mathcal{T}V), b'_V)\). Next, we get the decomposition \(\mathcal{T}V = \iota(\mathcal{T}W) \oplus \mathcal{I}\) where \(\mathcal{I}\) is the two-sided ideal of \(\mathcal{T}V\) generated by \(X\). Note that for any \(c \in \mathcal{T}V\) the adjoint representation \(b'_V(c) = \text{ad}_c\) preserves the subalgebra \(\iota(\mathcal{T}W)\) iff \(c \in \iota(\mathcal{T}W)\), and it follows

\[
\text{Hom}(\tilde{V}', \mathcal{T}V) \cap b'_V(\mathcal{T}V) = b'_V(\iota(\mathcal{T}W)).
\]

Hence the subspace \(\tilde{H}_W^0\) trivially intersects the inner derivations \(b'_V(\mathcal{T}V) = b'_V(\iota(\mathcal{T}W)) \oplus b'_V(\mathcal{I})\), hence we can choose a tensor graded complement \(\tilde{H}_V^0\) of \(b'_V(\mathcal{T}V)\) in \(\text{Hom}(V, \mathcal{T}V)\) such that \(\tilde{H}_W^0 \subset \tilde{H}_V^0\). We denote the projection \(\text{Hom}(V, \mathcal{T}V) \rightarrow \mathcal{T}V^*\) by \(Q_V\). It follows that

\[
Q_V(\tilde{\chi}) = \iota(Q_W(\chi))
\]
hence the linear map $i$ also intertwines the corresponding chain homotopies which we shall call $h_W$ and $h_V$, respectively. The first consequence is that the linear map $\chi \mapsto \tilde{\chi}$, which is not a morphism of Lie algebras, but it is one up to $b_V$-coboundaries, descends to a Lie algebra injection $j : \mathrm{outder}(TW) \to \mathrm{outder}(TV)$ (corresponding to the injection $\mathcal{H}_W^0 \subset \mathcal{H}_V^0$).

With all these preparations we see that the characteristic graded $3$-cocycles $\sigma_V$ (associated to the Hochschild complex of $TV$) and $\sigma_W$ (associated to the Hochschild complex of $TW$) are related by the map $j$, viz.

$$\sigma_V(j(\psi_1), j(\psi_2), j(\psi_3)) = \sigma_W(\psi_1, \psi_2, \psi_3)$$

for all $\psi_1, \psi_2, \psi_3 \in \mathcal{H}_W^0$. If $\sigma_V$ was exact, then by restriction $\sigma_W$ would also be exact in contradiction to the finite-dimensional case. This proves the Theorem.

\[\square\]

## A Filtered vector spaces

Most of the following material can be found e.g. in [Bou89, Ch.III] and [NVO79]. Let $M$ be a vector space. Recall that a family of subspaces $(F_r(M))_{r \in \mathbb{Z}}$ of $M$ is called an (ascending) filtration if $F_r(M) \subset F_{r+1}(M)$ for any integer $r$, and the pair $(M, (F_r(M))_{r \in \mathbb{Z}})$ is called a filtered vector space. Recall that the filtration is called exhaustive if $\bigcup_{r \in \mathbb{Z}} F_r(M) = M$, separated if $\bigcap_{r \in \mathbb{Z}} F_r(M) = \{0\}$, and discrete if there is an integer $r_0$ such that $F_r(M) = \{0\}$ for all integers $r \leq r_0$. Every vector space $M$ can be equipped with the trivial discrete filtration defined by $F_r(M) = \{0\}$ for all integers $r \leq -1$, and $F_r(M) = M$ for all $r \geq 0$. Moreover an exhaustive and separated filtration is well-known to always give rise to a (topological) metric where the distance of two elements $x, y$ of $M$ is defined by $2$ to the power of the minimum of all those integers $r$ such that $x - y \in F_r(M)$ if $x \neq y$, and $0$ iff $x = y$. Hence a filtered vector space whose filtration is exhaustive and separated is called complete if the corresponding metric space is complete in the sense that every Cauchy sequence converges. In such a situation a series $\sum_{n \in \mathbb{N}} x_n$ converges iff $x_n \to 0$, see e.g. [Jac89, p.453]. Note that every filtered vector space whose filtration is exhaustive and discrete is complete. Next, recall that for two filtered vector spaces $(M, (F_r(M))_{r \in \mathbb{Z}})$
and \( (M', (F_r(M'))_{r \in \mathbb{Z}}) \) a linear map \( f : M \to M' \) is called of filtration degree \( m \) iff \( f(F_r(M)) \subseteq F_{r+m}(M') \) for all integers \( r \). It follows that the space \( H = \text{Hom}\,(\text{filt}(M, M')) \) of all linear maps of filtration degree 0 is filtered by declaring \( F_r(H) = H \) for all \( r \geq 0 \) and for all \( r \leq -1 \) \( F_r(H) \) is the subspace of all linear maps of filtration degree \( r \leq -1 \). Note that the filtered vector space \( H = \text{Hom}\,(\text{filt}(M, M')) \) is automatically complete if \( (M, (F_r(M))_{r \in \mathbb{Z}}) \) is exhaustive and separated and \( (M', (F_r(M'))_{r \in \mathbb{Z}}) \) is complete. Finally note that the tensor product of two filtered vector spaces \( M, M' \) is also filtered by \( F_r(M \otimes M') = \sum_{s \in \mathbb{Z}} F_s(M) \otimes F_{r-s}(M') \) for all integers \( r \).

## B Graded Coalgebras

A lot of the following material can be found e.g. in [Qui69, App.B]. Recall that a graded vector space \( C \) equipped with \( \mathbb{K} \)-linear maps \( \Delta_C : C \to C \otimes C \), \( \varepsilon_C : C \to \mathbb{K} \) and an element \( 1_C \) of degree 0 is called a graded coassociative counital coaugmented coalgebra if \( (\Delta_C \otimes \text{id}_C) \circ \Delta_C = (\text{id}_C \otimes \Delta_C) \circ \Delta_C, \)

\[ (\varepsilon_C \otimes \text{id}_C) \circ \Delta_C = \text{id}_C = (\text{id}_C \otimes \varepsilon_C), \]

\( \Delta_C(1_C) = 1_C \otimes 1_C \) and \( \varepsilon(1_C) = 1 \). Recall Sweedler’s notation \( (c) = \sum_{(c)} c^{(1)} \otimes c^{(2)} \) which stands for a nonunique finite sum with homogeneous elements \( c^{(1)} \) and \( c^{(2)} \) in \( C \). Every such coalgebra carries a canonical filtration \( (F_r(C))_{r \in \mathbb{Z}} \) defined by \( F_0(C) = \{0\} \) for all integers \( r \leq -1 \), \( F_0(C) = \mathbb{K} \cdot 1_C \), and recursively \( F_{r+1}(C) = \{c \in C \mid \Delta_C(c) = c \otimes 1_C - 1_C \otimes c \in F_r(C) \otimes F_r(C)\} \). The maps \( \Delta_C \) and \( \varepsilon_C \) are filtration preserving.

A graded coassociative counital coaugmented coalgebra is called a connected coalgebra if the canonical filtration is exhaustive. Most of the graded coalgebras we shall encounter in this paper are graded cocommutative, i.e. \( \tau \circ \Delta_C = \Delta_C \). Recall that a morphism of graded connected coalgebras \( \Phi : C \to C' \) is a \( \mathbb{K} \)-linear map of degree 0 satisfying \( (\Phi \otimes \Phi) \circ \Delta_C = \Delta_{C'} \circ \Phi, \)

\( \varepsilon_{C'} \circ \Phi = \varepsilon_C \), and \( \Phi(1_C) = 1_{C'} \). They are automatically filtration preserving. Moreover, a \( \mathbb{K} \)-linear homogeneous map \( d : C \to C' \) is called a graded coderivation of graded counital coalgebras along the morphism \( \Phi : C \to C' \) iff \( \Delta_{C'} \circ d = (d \otimes \Phi + \Phi \otimes d) \circ \Delta_C \). It follows that \( \varepsilon_{C'} \circ d = 0 \). For any graded associative unital algebra \( (A, \mu_A, 1_A) \) and any graded counital coassociative coalgebra \( (C, \Delta_C, \varepsilon_C) \) the convolution multiplication * on \( \text{Hom}(C, A) \) defined by \( \phi * \psi = \mu_A \circ (\phi \otimes \psi) \circ \Delta_C \) is a graded associative multiplication on \( \text{Hom}(C, A) \) with unit element \( 1_A \varepsilon_C \). The convolution turns out to be very
useful to express combinatorial formulas in the graded symmetric bialgebra.

\section*{C The Perturbation Lemma}

This Appendix is based on work of \cite{Bou89, Hue10, Hue11, Man10}:

\begin{definition}
A \textit{(homotopy) contraction} consists of two chain complexes \((U, b_U)\) and \((V, b_V)\) (the differentials having degree +1) together with chain maps \(i : U \to V\), \(p : V \to U\), \textit{i.e.}

\begin{align}
b_V \circ i &= i \circ b_U, \quad b_U \circ p = p \circ b_V \tag{C.1a}
\end{align}

and a map \(h : V \to V\) of degree \(-1\) such that

\begin{align}
p \circ i &= id_U \tag{C.1b}
id_V - i \circ p &= b_V \circ h + h \circ b_V \tag{C.1c}
h^2 &= 0, \quad h \circ i = 0, \quad p \circ h = 0. \tag{C.1d}
\end{align}

Then \(p\) is a surjection called the \textit{projection}, \(i\) is an injection called the \textit{inclusion} and \(h\) is an \textit{homotopy} between \(id_V\) and \(i \circ p\). We sum up equations \((C.1)\) with the diagram

\begin{equation}
\begin{array}{c}
(U, b_U) \\
\downarrow \quad i \\
(V, b_V) \\
\downarrow \quad p \\
\end{array} \quad \circ \quad h.
\end{equation}

\end{definition}

\begin{remark}
\begin{enumerate}
\item Condition \((C.1c)\) implies that the cohomologies of \(U\) and \(V\) are isomorphic. In the important particular case where the differential \(b_U\) vanishes \(U\) is isomorphic to the cohomology of \(V\).
\item Denoting by \([f, g] := f \circ g - (-1)^{|f||g|} g \circ f\) the graded commutator of two maps, this equation \((C.1c)\) also rewrites \(id_V - i \circ p = [b_V, h]\). Our sign conventions are such that both \(i \circ p\) and \(P := [b_V, h]\) are idempotent linear maps. Note that \(V\) decomposes in the direct sum of two subcomplexes, the kernel \(V_U\) of \(P\) (isomorphic to \((U, b_U)\), and the image \(V_{acyc}\) of \(P\) which is acyclic.
\end{enumerate}
\end{remark}
3. Equations (C.1d) are called side conditions. In case there is a homotopy contraction only satisfying eqs (C.1a), (C.1b), and (C.1c), it is straightforward to see that the ‘polynomially’ modified homotopy

\[ h' = [b_V, h] \circ h \circ b_V \circ h \circ b_V \circ h \circ b_V \circ h \circ b_V [b_V, h] \]

will satisfy all equations of (C.1) including the side conditions (C.1d). Having a homotopy satisfying the side conditions is equivalent to specifying a vector space complement to the coboundaries in \( V^{acyc} \): \( h \) will be zero on \( V_U \) and on that complement and equal to the inverse of the restriction of the differential to that complement.

4. Since we are working in vector spaces, there is a well-known important converse statement: if, for the two above complexes, there is just an injective chain map \( i : (U, b_U) \to (V, b_V) \) inducing an isomorphism in cohomology, then there is a surjective chain map \( p : U \to V \) and a homotopy \( h : V \to V \) satisfying the conditions for a homotopy contraction (C.1c): indeed by some straight-forward linear algebra it can be seen that it suffices to take a vector space complement \( W \) to the subspace \( i(U) + b_V(V) \) of \( V \), then \( V^{acyc} = W \oplus b_V(W) \) will define an acyclic subcomplex of \( (V, b_V) \) complementary to \( i(U) \), which serves as a kernel of an obvious surjective chain map \( p : U \to V \). The chain homotopy \( h \) is constructed as in Remark C.2.

**Definition C.3.** A perturbation of the differential \( b_V \) is a morphism \( \delta_V : V \to V \) of degree +1 such that \( (b_V + \delta_V)^2 = 0 \iff \delta_V^2 + [\delta_V, b_V] = 0 \).

**Lemma C.4 (Perturbation Lemma).** Let be given a contraction (C.2) such that both \( U \) and \( V \) carry exhaustive and separated filtrations with \( V \) complete (see Appendix A for definitions) and such that the linear maps \( b_U, b_V, i, p \) and \( h \) are of filtration degree 0. Moreover, let \( \delta_V : V \to V \) be a perturbation of \( b_V \) and suppose that \( \delta_V \) is of filtration degree \(-1\).

Then the linear maps \( (id_V + h \circ \delta_V) \) and \( (id_V + \delta_V \circ h) \) from \( V \) to \( V \) are invertible, and we define

\[
\begin{align*}
\tilde{i} &= (id_V + h \circ \delta_V)^{-1} \circ i \\
\tilde{h} &= (id_V + h \circ \delta_V)^{-1} \circ h \\
\tilde{p} &= p \circ (id_V + \delta_V \circ h)^{-1} \\
\delta_U &= p \circ (id_V + \delta_V \circ h)^{-1} \circ \delta_V \circ i.
\end{align*}
\]
Then \( \delta_U \) is a perturbation of \( b_U \) of filtration degree \(-1\), and the above maps define a new contraction

\[
(U, b_U + \delta_U) \xrightarrow{\tilde{i}} (V, b_V + \delta_V) \xrightarrow{\tilde{h}} \text{.}
\]

The inverse of \( id_V + \chi \) where \( \chi : V \to V \) is a \( \mathbb{K} \)-linear map of filtration degree \(-1\) is defined by the geometric series \( \sum_{k=0}^{\infty} (-\chi)^k \) which converges. The verification of the above identities is straight-forward, see e.g. [Bro65].

References

[AB93] Eugenios Angelopoulos and Saïd Benayadi, Construction d’algèbres de Lie sympathiques non semi-simples munies de produits scalaires invariants, Comptes rendus de l’Académie des sciences. Série 1, Mathématiques 317 (1993), 741–744. (Cited pages 24 et 28.)

[AMM02] Didier Arnal, Dominique Manchon, and Mohsen Masmoudi, Choix des signes pour la formalité de M. Kontsevich, Pacific Journal of Mathematics 203 (2002), no. 1, 23–66. (Cited pages 4, 8, 10 et 19.)

[BE18] Martin Bordemann and Olivier Elchinger, A remark on the \( L_\infty \) perturbation lemma, preprint Mulhouse and Luxembourg, 2018. (Cited pages 3 et 17.)

[BGH+05] Martin Bordemann, Grégory Ginot, Gilles Halbout, Hans-Christian Herbig, and Stefan Waldmann, Formalité \( G_\infty \) adaptée et star-représentations sur des sous-variétés coïsotropes, arXiv : math/0504276v1 [math.QA], 2005. (Cited pages 4, 10 et 16)

[BM08] Martin Bordemann and Abdenacer Makhlouf, Formality and Deformations of Universal Enveloping Algebras, International Journal of Theoretical Physics 47 (2008), 311–332. (Cited pages 2, 3, 4, 21, 22 et 24)
[BMP05] Martin Bordemann, Abdenacer Makhlouf, and Toukaiddine Petit, *Déformation par quantification et rigidité des algèbres enveloppantes*, Journal of Algebra 285 (2005), no. 2, 623–648. (Cited pages 3, 21 et 22)

[Bor97] Martin Bordemann, *Nondegenerate invariant bilinear forms on nonassociative algebras*, Acta Math. Univ. Comenianae LXVI (1997), 151–201. (Cited page 24)

[Bor15] ——, *An unabelian version of the Voronov higher bracket construction*, Georgian Mathematical Journal 22 (2015), 189–204. (Cited pages 4 et 6)

[Bou89] Nicolas Bourbaki, *Commutative algebra*, Springer Verlag, 1989. (Cited pages 41 et 43)

[Bro65] Ronald Brown, *The twisted Eilenberg-Zilber theorem*, Edizioni Oderisi (Gubbio) (Simposio di Topologia, ed.), Celebrazioni Archimedee del Secolo XX, (Messina, 1964), 1965, pp. 33–37. (Cited page 45)

[CE56] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, 1956. (Cited pages 22 et 32)

[Dix77] Jacques Dixmier, *Enveloping Algebras*, North-Holland mathematical library, Akademie-Verlag, 1977. (Cited page 22)

[DSV16] Vladimir Dotsenko, Sergey Shadrin, and Bruno Vallette, *Pre-Lie deformation theory*, arXiv:1502.03280 [math.QA], 2016. (Cited page 16)

[Elc12] Olivier Elchinger, *Formality related to universal enveloping algebras and study of Hom-(co)Poisson algebras*, Ph.D. thesis, Université de Haute-Alsace, Mulhouse, 2012. (Cited pages 2, 4 et 32)

[Elc14] ——, *Study of formality for the Heisenberg algebra*, arXiv:1409.0175 [math.QA] 2014. (Cited pages 2 et 32)

[FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas, *Rational Homotopy Theory*, Springer-Verlag, 2001. (Cited pages 4, 5 et 8)
Murray Gerstenhaber, *The cohomology structure of an associative ring*, Annals of Mathematics 78 (1963), no. 2, 267–288. (Cited pages 6 et 9)

Grégory Ginot and Gilles Halbout, *A Formality Theorem for Poisson Manifolds*, Letters of Mathematical Physics 66 (2003), 37–64. (Cited page 10)

Jacques Helmstetter, *Série de Hausdorff d’une algèbre de Lie et projections canoniques dans l’algèbre enveloppante*, J. Algebra 120 (1989), 170–199. (Cited page 9)

Hans-Christian Herbig, *Variations on Homological Reduction*, Ph.D. thesis, Johann Wolfgang Goethe-Universität Frankfurt am Main, 2006.

Johannes Huebschmann, *The sh-Lie algebra perturbation Lemma*, Forum Mathematicum 23 (2010), no. 4, 669–691, arXiv:0710.2070 [math.AG]. (Cited pages 16 et 43)

Johannes Huebschmann, *The Lie Algebra Perturbation Lemma*, Higher Structures in Geometry and Physics (Alberto S. Cattaneo, Anthony Giaquinto, and Ping Xu, eds.), Progress in Mathematics, vol. 287, Birkhäuser Boston, 2011, arXiv:0708.3977 [math.AG] pp. 159–179 (English). (Cited pages 16 et 43)

Nathan Jacobson, *Lie Algebras*, Dover, 1979. (Cited page 28)

Nathan Jacobson, *Basic Algebra II*, Freeman, 1989. (Cited page 41)

Christian Kassel, *Quantum Groups*, Springer Verlag, 1995.

Ivan Kólař, Peter Michor, and Jan Slovák, *Natural Operations in Differential Geometry*, Springer Verlag, 1993. (Cited page 39)

Maxim Kontsevich, *Deformation quantization of Poisson manifolds, I*, Letters of Mathematical Physics 66 (2003), no. 3, 157–216. (Cited pages 1, 3, 4, 8, 10, 22 et 23)

Tom Lada and Jim Stasheff, *Introduction to SH Lie algebras for physicists*, International Journal of Theoretical Physics 32 (1993), no. 7, 1087–1103.
[LV12] Jean-Louis Loday and Bruno Vallette, *Algebraic Operads*, Grundlehren der mathematischen Wissenschaften, vol. 346, Springer-Verlag, Berlin, Heidelberg, 2012. (Cited pages 4, 7, 8, 9 et 10)

[Man10] Marco Manetti, *A relative version of the ordinary perturbation lemma*, arXiv:1002.0683 [math.KT], 2010. (Cited pages 16 et 43)

[MR85] Alberto Medina and Philippe Revoy, *Algèbres de Lie et produit scalaire invariant*, Annales Scientifiques de l’Ecole Normale Supérieure 18 (1985), 553–561. (Cited page 24)

[NVO79] Constantin Năstăsescu and Fred Van Oystaeyen, *Graded and filtered rings and modules*, Springer Verlag, Berlin, 1979. (Cited page 41)

[Qui69] Daniel Quillen, *Rational Homotopy Theory*, Annals of Mathematics 90 (1969), no. 2, 205–295. (Cited page 42)

[Riv12] Salim Rivière, *On the isomorphism between Hochschild and Chevalley-Eilenberg cohomologies*, Ph.D. thesis, Université de Nantes, December 2012. (Cited page 22)