Eccentric connectivity index of identity graph of cyclic group and finite commutative ring with unity

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Abstract. Research on graph associated with a finite algebraic structure has attracted many attentions. On the other hand, eccentric connectivity index is an interesting topic and many studies have been reported. For simple connected graph $G$, let $e(v)$ denoted the eccentricity of vertex $v$ and $\deg(v)$ denoted the degree of vertex $v$ in $G$. Eccentric connectivity index of $G$ is defined as the sum of all $e(v)\deg(v)$, for any $v$ in $G$. We focus the study on determining eccentricity connectivity index of identity graph of cyclic group and finite commutative ring with unity. We present the exact formula for eccentricity connectivity index of identity graph of these two algebraic structures.

1. Introduction
A topological index of a finite and simple graph, sometimes called a graph-theoretic index or molecular structure descriptor, is a numerical quantity which invariant under isomorphism of graphs [1-4]. Topological index has a wide application in network science, bioinformatics, and chemistry [1]. Topological index of a finite simple graph can be classified into degree-based index such as Zagreb index [5] and Randic index [6], distance-based index such as Wiener index [7] and Harary index [8], spectrum-based index such incidence energy [9] and graphs energies [10-18], and eccentricity-based index such as eccentric connectivity index [19], eccentric distance sum index [20-24], and adjacent eccentric distance sum index [1,25]. Eccentric connectivity index (ECI) has been used as means for modelling molecules structures [19,26-31] and studied extensively for various graphs [3,32-37].

On the other hand, the graph obtained from an algebraic structure also received the great attention from researchers. The examples of a graph from an algebraic structure also received the great attention from researchers. The examples of a graph from an algebraic structure with one binary operation or a group are Cayley graph [38], conjugate graph [39,40], commuting graph [41,42], non-commuting graph [43,44], subgroup graph [45], inverse graphs [46], non-centralizer graph [47], and identity graph [48]. The examples of a graph from an algebraic structure with two binary operations are zero divisor graph [49-51], annihilator graph [52], annihilator ideal graph [53], co-maximal ideal graph [54], and identity or unit graph [48].
In this study, we focus on determining the exact formula of eccentric connectivity index of the identity graph of cyclic group and commutative ring with unity both of finite order.

2. Literature review
Let graph $G = (V(G), E(G))$ be connected of order $|V(G)| = p$. Let $\text{deg}(u)$ and $e(u)$ denoted degree and eccentricity of a vertex $u$ in graph $G$, respectively. The total eccentricity of $G$ is defined [1,55] as

$$
\xi(G) = \sum_{v \in V(G)} e(v).
$$

In 1997, Sharma, Goswami, and Madan [56] defined a new graph invariant which they called the eccentric connectivity index (ECI) as

$$
\xi^c(G) = \sum_{v \in V(G)} e(v) \text{deg}(v).
$$

For finite group $(H, *)$, let $e_H$ denoted the identity element of $H$. Kandasamy and Smarandache [48] defined identity graph $I(H)$ of $H$ as a simple graph with $V(I(H)) = H$ and $h_1h_2 \in E(I(H))$ if and only if $h_1 * h_2 = e_H$. By a convention, the identity element $e_H$ is adjacent to any vertices in $I(H)$. Let $(R, +, \cdot)$ be a finite commutative ring, $0_R$ be identity element under addition, and $1_R$ be identity element under multiplication and $1_R \neq 0_R$. An element $x$ in $R$ and $x \neq 0_R$ is called zero divisor if there exists an element $y$ in $R$ and $y \neq 0_R$ such that $x \cdot y = 0_R$ [57]. An element $u$ in $R$ and $u \neq 0_R$ is called unit if there exists an element $v$ in $R$ and $v \neq 0_R$ such that $u \cdot v = 1_R$ [58]. In other words, a unit is an element that has an inverse under multiplication in $R$. If any non-zero element of $R$ is unit then $R$ is called a field [57]. Identity graph or unit graph $I(R)$ of $R$ is a simple graph with all units of $R$ as its vertices and $xy \in E(I(R))$ if and only if $x \cdot y = 1_R$. The unity $1_R$ is assumed to be adjacent to any vertices in $I(R)$ [48].

3. Method
In order to find the eccentric connectivity index formula of the identity graph of the cyclic group $G$ and the commutative ring $Z_p$ where $p$ is prime, we have done the following steps.

- Drawing the identity graph of the cyclic group $G$ for $|G| = 1, 2, 3, 4, 5, 6$ and the commutative ring $Z_p$ for $p = 3, 5, 7, 11, 13, 17$.
- Determining the total eccentricity and eccentric connectivity index of each graph in step 1).
- Determining the pattern of total eccentricity and eccentric connectivity index and formulating conjectures
- Stating the conjectures as theorems together with their formal proof.

4. Results and discussion
The primary objective of this paper is to determine the ECI of the identity graph of a finite cyclic group $G = \langle x \rangle$ and a finite commutative ring with unity $Z_p$ where $p$ is prime. The total eccentricity of these graphs is also computed. The following are results of our study.

**Theorem 3.1** For even positive integer $n$, let $G$ be a cyclic group of order $n$. Then,

(a) $\xi(I(G)) = 2$ and $\xi^c(I(G)) = 2$ if $n = 2$.

(b) $\xi(I(G)) = 2n - 1$ and $\xi^c(I(G)) = 5n - 7$ if $n > 2$.

**Proof.** Let $G = \langle x \rangle$.

(a) For $n = 2$, we have $I(G)$ is a complete graph of order 2. So, $e(1) = e(x) = 1$ and $\text{deg}(1) = \text{deg}(x) = 1$. Hence, $\xi(I(G)) = 2$ and $\xi^c(I(G)) = 2$.

(b) For $n > 2$, we have $x^i x^{n-i} = e_G$, $i = 1, 2, 3, ..., n/2$. So, $x^i$ and $x^{n-i}$ are adjacent in $I(G)$, but $x^{n/2}$ is not adjacent to itself. By convention for identity graph, vertex 1 is adjacent to other
vertices in $I(G)$. Identity graph $I(G)$ can be seen in Figure 1. We obtain $\text{deg}(1) = n - 1$, $\text{deg}(x^{n/2}) = 1$ and $\text{deg}(x^j) = 2$ for $j \neq n/2$. We also obtain $e(1) = 1$ and $e(x^j) = 2$ for $j = 1, 2, 3, \ldots, n - 1$. Hence
\[
\xi(I(G)) = \sum_{v \in V(I(G))} e(v) = 1 + (n - 1)2 = 2n - 1
\]
and
\[
\xi^c(I(G)) = \sum_{v \in V(I(G))} e(v)\text{deg}(v) = 1(n - 1) + 2 + (n - 2)4 = 5n - 7. \tag{3}
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{cyclic_group_identity_graph}
\caption{Identity graph of cyclic group $G$ of order $n > 2$.}
\end{figure}

**Theorem 3.2** For odd positive integer $n$, let $G$ be a cyclic group of order $n$. Then,
(a) $\xi(I(G)) = 0$ and $\xi^c(I(G)) = 0$ if $n = 1$.
(b) $\xi(I(G)) = 3$ and $\xi^c(I(G)) = 6$ if $n = 3$.
(c) $\xi(I(G)) = 2n - 1$ and $\xi^c(I(G)) = 2(n - 1) + 4$ if $n > 3$.

Proof. Let $G = \langle x \rangle$.
(a) For $n = 1$, we have $I(G)$ is a trivial graph. Then $e(1) = 0$ and $\text{deg}(1) = 0$. Hence, $\xi(I(G)) = 0$ and $\xi^c(I(G)) = 0$.
(b) For $n = 3$, we obtain $I(G)$ is a complete graph of order 3. So, $\text{deg}(v) = 2$ and $e(v) = 1$ for all vertex $v$ in $I(G)$. Hence, $\xi(I(G)) = 3$ and $\xi^c(I(G)) = 6$.
(c) For $n > 3$, we have $x^i x^{n-i} = e_G$, $i = 1, 2, 3, \ldots, (n-1)/2$. So, $x^i$ and $x^{n-i}$ are adjacent in $I(G)$. Identity graph $I(G)$ can be seen in Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{cyclic_group_identity_graph_odd_order}
\caption{Identity graph of cyclic group $G$ of order $n > 3$.}
\end{figure}

We obtain $\text{deg}(1) = n - 1$ and $\text{deg}(x^j) = 2$ for $j = 1, 2, 3, \ldots, n - 1$. We also obtain $e(1) = 1$ and $e(x^j) = 2$ for $j = 1, 2, 3, \ldots, n - 1$. Hence
\[
\xi(I(G)) = \sum_{v \in V(I(G))} e(v) = 1 + (n - 1)2 = 2n - 1
\]
and
\[
\xi^c(I(G)) = \sum_{v \in V(I(G))} e(v)\text{deg}(v) = 1(n - 1) + (n - 1)4 = 5(n - 1). \tag{3}
\]
Theorem 3.3 For any prime number $p$, let $Z_p$ be a ring of integer modulo $p$ under addition and multiplication. Then,

(a) $\xi(I(Z_p)) = 0$ and $\xi^c(I(Z_p)) = 0$ if $p = 2$.

(b) $\xi(I(Z_p)) = 2$ and $\xi^c(I(Z_p)) = 2$ if $p = 3$.

(c) $\xi(I(Z_p)) = 2p - 3$ and $\xi^c(I(G)) = 5p - 12$ if $p > 3$.

Proof. Since $p$ is prime, then $Z_p$ is a field. So, all non-zero elements of $Z_p$ are units.

(a) If $p = 2$ then $I(Z_p)$ is a trivial graph. The proof is obvious.

(b) If $p = 3$ then $I(Z_p)$ is a path graph of order 2. The desired proof is also obvious.

(c) Let $p > 3$, then $I(Z_p)$ is a simple graph of order $p - 1$, that is $V(I(Z_p)) = \{1, 2, 3, \ldots, p - 1\}$. Graph $I(Z_p)$ can be seen in Figure 3.

![Figure 3. Identity graph of $Z_p$.](image)

Then, we have $deg(1) = p - 2$, $deg(p - 1) = 1$ and $deg(v) = 2$ for $v \neq 1$ or $v \neq p - 1$. We also obtain $e(1) = 1$ and $e(v) = 2$ for $v \neq 1$. Hence

$$\xi(I(Z_p)) = \sum_{v \in V(I(Z_p))} e(v) = 1 + (p - 2)2 = 2p - 3$$

and

$$\xi^c(I(Z_p)) = \sum_{v \in V(I(Z_p))} e(v)deg(v) = 1(p - 2) + 2 + (p - 3)4 = 5p - 12. \square$$

5. Conclusion
The exact formula of eccentric connectivity index of the identity graph of a cyclic group of order $n$ and a commutative ring with unity of order $p$ where $p$ is prime has been calculated in this paper. Further research on the eccentric connectivity index of a graph associated with other algebraic structures still needs to be done.

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