The thermal dissipation in two dimensional relativistic Fermi gases with a relaxation time model

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The thermal transport properties of a two dimensional Fermi gas are explored, for the full range of temperatures and densities. The heat flux is established by solving the Uehling-Uhlebeck equation using a relaxation approximation given by Marle's collisional kernel and considering the temperature and chemical potential gradients as independent thermodynamic forces. It is shown that the corresponding transport coefficients are proportional to each other, which leads to the possibility of defining a generalized thermal force and a single transport coefficient. The behavior of such conductivity with the temperature and chemical potential is analyzed and a discussion on its dependence with the relaxation parameter is also included. The relevance and applications of the results are briefly addressed.

I. INTRODUCTION

After the discovery of graphene [1] the study of two-dimensional statistical systems has acquired particular importance. Indeed, such material is considered to be the most perfect two-dimensional electronic system attainable, having exactly one thick atomic monolayer. All the dynamics and transport processes are carried out confined to that 2D layer [2], so it can be studied as a two-dimensional gas of relativistic fermions, where the Fermi energy imposes a limit scale on the system [3]. Moreover, this phenomenon has been successfully applied to the study of certain processes in relativistic gases that can not be otherwise explored, in particular in an experimental setting [4, 5].

Theoretical and experimental studies of non-relativistic fermion gases in two or quasi-two dimensions, indicate that dimensionality plays a role in the transport properties of the system. The corresponding modifications have been explored by several authors [6–12]. Certainly, by controlling the size of the systems it has been possible to effectively manipulate the electronic properties of some materials [13].

The thermodynamic features of relativistic fermionic systems have been also shown to be affected by a reduction in the number of dimensions [14, 15]. For instance, different behaviors of the thermal variables have been found at different temperature regimes depending on the dimensionality [16]. Indeed it can be seen by inspection of the distribution function of a relativistic non-degenerate gas in any dimension $d$, that for even values of $d$ the modified Bessel functions that appear are of semi-integer index [17], such that it admits polynomial expansions [18].

Moreover, for relativistic gases there is a threshold temperature that depends on both dimensionality and statistics, from which relativistic effects begin to be relevant in the description of the gas. This describes a transition in the shape of the relativistic distribution: from unimodal in the non-relativistic case, to bimodal in the ultra relativistic limit [19]. This process has been characterized within the theory of phase transitions and the corresponding critical exponents, critical temperature and diverse thermodynamic properties have been studied [19, 20].

In spite of recent developments and vast motivation for the study of non-equilibrium two-dimensional relativistic gases only a reduced number of theoretical works on the matter can be currently found in the literature. Indeed, relativistic kinetic theory for 2D systems was first applied to specific cosmological models [21], and has also served to explore computational methods for heat flux [22]. Hydrodynamic electron transport models have been recently studied for graphene and other quasi-two-dimensional materials (see for example [23, 24]).

The transport coefficients of a 2D ultra-relativistic fluid were calculated by Mendoza et. al. in [25] by introducing BGK-like relaxation approximations for the collision operator and considering the Jüttner distribution function for the equilibrium state. Whereas in [26], the constitutive equation for the heat flux and the corresponding thermal conductivity were obtained for a non-degenerate gas by using the complete collision term of the 2D relativistic Boltzmann equation within the well-known Chapman-Enskog method.

Also, and since in the hydrodynamic regime the electrons in graphene can be thought of as a two-dimensional relativistic gas, a numerical scheme has been constructed to study such electronic transport [27]. The condition for the existence of a Rayleigh-Benard instability was studied in this framework and the thermal conductivity and the shear viscosity were numerically established.
A lattice Boltzmann method has also been recently developed for 2D relativistic fluids with different statistics based on a fifth order expansion. With this scheme the transport coefficients were numerically obtained and corroborated against theoretical results and previous developments [31, 32].

It is noteworthy that even in one and three dimensions, relativistic dissipation and transport processes are still not fully understood. Recent theoretical and numerical works in relativistic kinetic theory strive to gain a deeper comprehension of such processes. These range from comparisons between the Chapman-Enskog and Grad solution methods, to the development and application of lattice Boltzmann methods and other hydrodynamic solvers in the relativistic regime [32–35].

In this paper, the constitutive equation for the heat flux and the corresponding coefficient of thermal conductivity for a two-dimensional relativistic fermionic gas are established by solving the relativistic Boltzmann equation for a degenerate Fermi gas considering a relaxation approximation. In order to accomplish such task, the rest of the work is structured as follows: In Section 2 we review the equilibrium solution to the relativistic Uehling-Uhlenbeck equation for fermions. The Fermi energy is introduced by means of the statistical definition of the number density and the corresponding relevant limits are briefly described. In Section 3 the balance equations are addressed and the Chapman-Enskog procedure is introduced, by means of which the local equilibrium equations are obtained and the general expression for the heat flux is established in terms of the first order out of equilibrium solution. The first non-equilibrium correction to the distribution function is established in Section 4 by replacing the collision operator of the relativistic Uehling-Uhlenbeck equation by a relaxation term, corresponding specifically to the so-called Marle model. With such solution, the heat flux is obtained in Section 5 by considering the temperature and chemical potential gradients as independent thermodynamic forces. The discussion section, Section 6, includes both a comparison with the 3D case and relevant comments concerning the choice of relaxation time while Section 7 is devoted to the concluding remarks.

II. THE UEHLING-UHLENBECK EQUATION AND EQUILIBRIUM SOLUTION FOR FERMIONS

The kinetic description of a quantum system is given by the relativistic Uehling-Uhlenbeck equation [38, 39]

\[ p^\alpha \frac{\partial f}{\partial x^\alpha} = \int \left[ \tilde{f} f' \left( 1 - \frac{h^2}{gs} f \right) \left( 1 - \frac{h^2}{gs} \tilde{f} \right) - \tilde{f} \left( 1 - \frac{h^2}{gs} f \right) \left( 1 - \frac{h^2}{gs} \tilde{f} \right) \right] F \sigma d\Omega \frac{d^2 \tilde{p}}{\tilde{p}_0}, \]  

which specifies the time evolution of the distribution function \( f \), i.e., the energy level occupation number. The particle coordinates are here denoted by \( (x^\alpha) = (ct, x^1, x^2) \) and the corresponding momentum tensor by \( (p^\alpha) = (p^0, p^1, p^2) \). Space-time is specified by the metric \( ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta \), where \( \alpha, \beta = 0, 1, 2 \) and \( \eta_{0\beta} = \text{diag}(1, -1, -1) \). In Eq. (1), \( F \) is the so-called invariant flux, \( \sigma \) the differential cross section and \( \Omega \) is the solid angle. Also, \( h \) is Planck’s constant and \( g_s \) is the spin degeneracy factor. Primed and unprimed quantities indicate values for the particles before and after a collision respectively.

The local equilibrium solution to the relativistic Uehling-Uhlenbeck equation for particles that obey Pauli’s exclusion principle [11] is the relativistic Fermi-Dirac distribution function, given by

\[ f^{(0)} = \frac{g_s}{h^2} (\Sigma + 1)^{-1}, \]  

where we have introduced \( \Sigma = \exp \left( -\mu_e/kT + u^\alpha p^\alpha/kT \right) \) in order to simplify the notation. Also here \( k \) is the Boltzmann constant, \( u^\alpha \) the hydrodynamic three-velocity, \( T \) the temperature of the gas and \( \mu_e \) the equilibrium chemical potential. The particle number density is given by

\[ n = \int \frac{u^\alpha p^\alpha}{c^2} f^{(0)} dp^*, \]  

where \( dp^* = cd^2p/p_0 \), and can be calculated as follows

\[
\begin{align*}
    n &= 2\pi m^2 c^2 \frac{g_s}{h^2} \left[ \int_{1}^{x_F} x \left( e^{-\frac{\mu_e}{kT} - \xi} \right) e^{\xi(x-1)} + 1 \right]^{-1} dx \\
    &\quad + \int_{x_F}^{\infty} x \left( e^{-\frac{\mu_e}{kT} - \xi} \right) e^{\xi(x-1)} + 1 \right]^{-1} dx,
\end{align*}
\]

where we have introduced the dimensionless variable \( x = u^\alpha p^\alpha/mc^2 \) as well the standard relativistic parameter \( \xi = mc^2/kT \) with which the limiting cases can easily be identified.

Indeed, \( \xi_F = mc^2(x_F - 1) \) corresponds to the highest energy level occupied at zero absolute temperature where all particles fill the lower energy levels (up to such value) due to Pauli’s exclusion principle. It is worthwhile to point out that the number 1 that is subtracted from \( x_F \) corresponds to the rest energy. This allows one to clearly isolate the relativistic limit which in the degenerate case requires special care because of the interplay of the parameters that characterize the system, particularly the energies [40].
The degenerate relativistic behavior is characterized by \( e^{-\mu_{\alpha}/kT} \ll 1 \), which implies that the distribution function on the first integral of Eq. (4) is bounded and therefore can be replaced by a step-function, while the second integral becomes negligible. Therefore the particle number density can be calculated in such scenario as follows

\[
n = 2\pi m^2 e^2 g_s \int_1^{x_F} x dx = \pi m^2 c^2 g_s \frac{g_s}{h^2} \left( x_F^2 - 1 \right). \tag{5}
\]

If we introduce the quantum 2D density \( n_0 = \pi m^2 c^2 g_s / h^2 \), then the relativistic Fermi energy can be written as a function of the number density as

\[
\epsilon_F = \sqrt{\frac{n}{n_0}} + 1 - 1. \tag{6}
\]

Notice that since the non-relativistic case is characterized by \( e^{-\mu_{\alpha}/kT} \gg 1 \), by considering that for a degenerate gas \( \mu_{\alpha} \sim \epsilon_F \), one can write the condition for a non-relativistic degenerate system as \( \epsilon_F \ll mc^2 \). When this condition is introduced in Eq. (5), the non relativistic Fermi energy is recovered

\[
\epsilon_F \approx \frac{n}{2n_0}. \tag{7}
\]

All these cases are summarized in Fig. 1 where the existence of four regimes, corresponding to the inter-lapping of degenerate, not degenerate, ultra relativistic and non-relativistic cases is shown. This is in accordance with what is known for 3D electrons [11].

The boundary of the non-relativistic and relativistic regimes is determined by the relative magnitude of the rest energy to the thermal energy. In particular, consider \( mc^2 \) compared with some factor of the thermal energy, say \( akT \). The constant \( \alpha \) in the non-degenerate case can be taken as \( d/2 \) by considering the value of the thermal energy corresponding to the equipartition theorem. Thus, in the bidimensional case here addressed, the line \( T = mc^2 / k \) separates non-relativistic and relativistic regimes. However, as mentioned in [5], relativistic effects can also be present at lower temperatures with \( \alpha = d + 2 = 4 \), and by considering also the effects of degeneracy one can consider \( \alpha = d + 2 + W \left( (d+2)e^{-(d+2)} \right) \approx 4.0684 \), where \( W \) is the so-called Lambert-W function. This yields a temperature \( T = mc^2 / (\alpha k) \) separating non-relativistic and relativistic regimes, together with \( \epsilon_F = mc^2 \), which in terms of density is given by \( n = 3n_0 \).

On the other hand, the boundary between the degenerate and non-degenerate cases is given by the relative magnitude of thermal and Fermi energies, i.e. \( akT = \epsilon_F \), which can be expressed as follows

\[
T = \frac{mc^2}{ak} \sqrt{\frac{n}{n_0}} + 1 - 1. \tag{8}
\]

In Fig. 1 the particular case given by the equipartition theorem was considered, since it gives the wider range in temperature and density for the non-relativistic case. However, the effects of considering the transition temperature have not yet been fully explored [5]. It should be noted that the above expressions can be written as a function of the dimension showing the influence of the confinement on the characteristic energies of the system.

### III. BALANCE EQUATIONS AND DISSIPATIVE FLUXES

The kinetic equation described above (Eq. 11), allows for the establishment of the hydrodynamic equations. Indeed, since the state variables are given as moments of the distribution function, the corresponding dynamics can be obtained through the evolution of \( f \), given by Eq. 11. In the relativistic case, the state variables number density \( n \) and internal energy \( \varepsilon \) are related with the conserved fluxes through the following equations

\[
n = \frac{1}{c^2} N^\alpha u_\alpha, \tag{9}
\]

\[
n \varepsilon = \frac{1}{c^2} T^{\alpha \beta} u_\alpha u_\beta \tag{10}
\]

where \( N^\alpha \) is the particle four-flux

\[
N^\alpha = \int f p^\alpha dp^\star, \tag{11}
\]

and \( T^{\mu \nu} \) the energy-momentum tensor

\[
T^{\alpha \beta} = \int f p^\alpha p^\beta dp^\star. \tag{12}
\]

As can be seen from the Eqs. (9) and (10), the description of the fluid is here given within Eckart’s representation (considering the fluid’s frame) in order for the analysis and physical interpretation to be more straightforward, particularly in the non-relativistic limit [39, 42, 43].
hydrodynamic equations are thus given by the conservation of both tensor quantities:

\[ N^\alpha_{\alpha} = 0, \quad (13) \]

\[ T^{\alpha\beta} = 0 \quad (14) \]

where here, and throughout this work, a comma indicates a covariant derivative. When \( N^\nu \) and \( T^{\mu\nu} \) are calculated introducing \( f^{(0)} \) in Eqs. (11) and (12), one obtains

\[ N^\alpha_{(0)} = 2\pi m^2 c^2 g_s u^\alpha I_1, \quad (15) \]

\[ T^{\alpha\beta}_{(0)} = 2\pi m^3 c^4 g_s \left( \frac{I_2}{c^2} \left( \frac{u^\alpha u^\beta}{c^2} + \frac{1}{2} (I_0 - I_2) h^{\alpha\beta} \right) \right), \quad (16) \]

from which the dynamic equations in a local equilibrium, non-dissipative, scenario are obtained, i.e. Euler’s regime. Such equations can be readily obtained and are here written as

\[ \theta I_1 + \frac{\mu_c u^\beta}{kT} \left[ \frac{\mu_c}{\mu_e} I_1 - \frac{T_{\alpha\beta}}{T} \left( \frac{I_1}{\mu_c} + \frac{m_c^2}{\mu_e} I_2 \right) \right] = 0, \quad (17) \]

\[ h^{\alpha\beta}_{\alpha,\beta} - \frac{c^2 \mu_c}{3(I_2 - I_0) kT} h^{\nu\alpha} \left[ \frac{\mu_c}{\mu_e} (I_2 - I_0) - \frac{T_{\alpha\beta}}{T} \left( \frac{(I_2 - I_0) - \frac{m_c^2}{\mu_e} (I_3 - I_1)}{} \right) \right] = 0, \quad (18) \]

and

\[ \frac{1}{2} \theta (3I_2 - I_0) + \frac{\mu_c u^\beta}{kT} \left[ \frac{\mu_c}{\mu_e} I_2 - \frac{T_{\alpha\beta}}{T} \left( I_2 - \frac{m_c^2}{\mu_e} I_3 \right) \right] = 0, \quad (19) \]

which correspond with the particle, momentum and energy balances, respectively. Here \( \theta = u^\alpha_{,\alpha} \) and the integrals \( I_i \) and \( I_i \) above are given by

\[ I_n = \frac{1}{2\pi h^2} \frac{1}{m^{n+1}_c e^{2n+2}} \int_{-\infty}^{\infty} \frac{(u^\alpha p_\alpha)^n}{\Sigma + 1} dp^*, \quad (20) \]

\[ I_n = \frac{1}{2\pi h^2} \frac{1}{m^{n+1}_c e^{2n+2}} \int_{-\infty}^{\infty} \frac{(u^\alpha p_\alpha)^n}{\Sigma + 1} dp^*. \quad (21) \]

It is worthwhile to point out that Eqs. (18) and (19) are obtained by projecting Eq. (14) in the direction orthogonal and parallel to \( u_\alpha \) respectively. More precisely, (18) is obtained by contracting Eq. (14) with

\[ h^{\mu\gamma} = \eta^{\mu\gamma} - \frac{u^\mu u^\gamma}{c^2}, \quad (22) \]

which is the so-called spatial projector and represents the direction orthogonal to the time direction, given by \( u^\alpha \). It is worthwhile to recall the reader that in this representation space and time directions are fixed by \( h^{\alpha\beta} \) and \( u^\alpha \) in the fluid’s comoving frame.

On the other hand, by considering the Chapman-Enskog expansion, where the distribution function is written as

\[ f = \sum_{n=0}^{\infty} f^{(n)}, \quad (23) \]

with \( f^{(0)} \) being the local equilibrium solution described in the previous section and \( f^{(n)} \) the non-equilibrium deviation to \( n-th \) order in the Knudsen parameter \( \frac{1}{2}, 12 \), the out of local equilibrium transport equations can be written to any order in \( n \). In particular, for \( n = 1 \), the so-called Navier-Stokes regime is obtained which features both thermal and viscous dissipation. Indeed, in such a regime, thermal dissipation is characterized by a heat flux which is given in terms of the energy-momentum tensor as

\[ q^\gamma = h^\alpha_{\alpha} u^\beta T^{\alpha\beta}_{(1)}, \quad (24) \]

where

\[ T^{\alpha\beta}_{(1)} = \int f^{(1)} p^{\alpha} p^{\beta} dp^*. \quad (25) \]

As mentioned in Section 1, the main goal of this work is precisely to establish a general expression for the heat flux, given by Eq. (24), and to calculate the corresponding transport coefficients within a relaxation time approximation. In order to accomplish such task, the following sections are devoted to the calculation of \( T^{\alpha\beta}_{(1)} \) within Marle’s model.
IV. FIRST ORDER NON-EQUILIBRIUM SOLUTION WITHIN MARLE’S MODEL

In order to establish dissipative fluxes, it is necessary to assess the mathematical structure of the non-equilibrium solution to Eq. (1), which is in general a highly involved task. For this reason, model equations are usually introduced proposing a simpler form of the collision kernel (r. h. s. of Eq. (1)), but retaining its most important basic characteristics. Moreover, relaxation or BGK-like models turn the integrodifferential equation into an algebraic one and allows one to readily obtain valuable approximations for the transport coefficients that relate dissipative fluxes with state variable’s gradients.

In the relativistic case, the most widely employed relaxation models are Marle’s model [46], which has the issue of not being able to treat massless particles, and the Anderson-Witting model [47, 48] which is usually preferred in developments such as relativistic Lattice Boltzmann. However, although such model represents a covariant formulation, it corresponds to the so-called Landau-Lifshitz or energy frame where addressing the relevant limits is somehow not so intuitive as in the Eckart’s or fluid frame here considered. Furthermore, the choice of the appropriate frame for the description of a particular relativistic processes can be a delicate task and it is thus crucial to consider the previous observations, in particular when introducing a relaxation parameter [49, 50].

In this section, we shall determine the first order in the gradients non-equilibrium distribution function of a gas of electrons in 2D by applying Marle’s model

\[ p^\alpha \frac{\partial f}{\partial x^\alpha} = -\frac{m}{\tau} \left( f - f^{(0)} \right), \]  

which is consistent with the fluid’s frame representation here considered. Although, as was said before, the Anderson-Witting model is the most used to describe this kind of systems, this is not the first time that a model of this type has been used to study relativistic degenerate gases [51]. Moreover, there are yet ways to avoid its problems and correct the model [49, 50]. In Eq. (26) \( m \) is the particle’s mass and \( \tau \) is a parameter related with the characteristic relaxation time for the distribution function. In order to obtain the first order correction to the distribution function due to the spatial gradients, we introduce \( f = f^{(0)} + f^{(1)} \) as indicated above from which one obtains, for the Chapman-Enskog method, the following expression

\[ f^{(1)} = -\frac{\tau p^\alpha \frac{\partial f^{(0)}}{\partial x^\alpha}}{m}. \]  

where for the derivatives on the right hand side one makes use of the local equilibrium assumption. Considering a representation where the independent state variables are \( \mu_e, T \) and \( u^\alpha \) yields the following expression for \( f^{(1)} \)

\[ f^{(1)} = \frac{\tau g_s}{m h^2 (\Sigma + 1)^2} \int p^\alpha \frac{\partial f^{(0)}}{\partial x^\alpha} \left( \frac{1}{\mu_e} \right) \times \left( -\frac{\mu_e \mu}{\mu_e} + \frac{\mu_e \mu}{\mu_e} \right) dp^*, \]  

Introducing the distribution function up to first order Eq. (28) into Eqs. (11) and (12) leads to

\[ N^\alpha = N^{(0)} + \frac{\tau g_s}{m h^2 kT} \int p^\alpha \frac{\partial f^{(0)}}{\partial x^\alpha} \left( \frac{1}{\mu_e} \right) \times \left( -\frac{\mu_e \mu}{\mu_e} + \frac{\mu_e \mu}{\mu_e} \right) dp^* \]

and

\[ T^{\alpha\beta} = T^{(0)} \left( \frac{\tau g_s}{m h^2 kT} \int p^\alpha \frac{\partial f^{(0)}}{\partial x^\alpha} \left( \frac{1}{\mu_e} \right) \times \left( -\frac{\mu_e \mu}{\mu_e} + \frac{\mu_e \mu}{\mu_e} \right) dp^* \right) \]

In order to obtain the constitutive equation for the heat flux, the integrals in expression (28) need to be evaluated. Such procedure can be performed using the relations in Ref. [28], and leads to

\[ T^{\alpha\beta} = T^{(0)} + \frac{\tau g_s}{m h^2 kT} \int p^\alpha \frac{\partial f^{(0)}}{\partial x^\alpha} \left( \frac{1}{\mu_e} \right) \times \left( -\frac{\mu_e \mu}{\mu_e} + \frac{\mu_e \mu}{\mu_e} \right) dp^* \]

where \( \zeta = m c^2 / k T \). Notice that here we wrote the momentum tensor in its irreducible form in the 2+1 fluid frame representation:

\[ p^\mu = p_\tau h^{\tau\gamma} + \frac{u^\gamma p_\gamma}{c^2} u^\mu, \]  

given by \( u^\alpha \) as the temporal direction and \( h^{\mu\gamma} \) (see Eq. (27)) representing the corresponding orthogonal plane. Such decomposition allows for the separation of proper time and spatial derivatives. The importance of such a step lies on the fact that the existence of the Chapman-Enskog solution of order \( n \) to the kinetic equation relies on the assumption that the time derivatives of the state...
variables, in principle given by \( f^{(n)} \) itself, can be approxi-
mately substituted by their expressions corresponding to
the previous order solution \( f^{(n-1)} \).

In view of the arguments mentioned above, \( \mu \) is

\[
\frac{u^\alpha}{T} \frac{\partial T}{\partial x^{\alpha}} = \theta \frac{kT}{mc^2} \frac{1}{(I_1 I_3 - I_2^2)} \left( I_2 I_1 - \frac{1}{2} I_1 (3I_2 - I_0) \right),
\]
\(\text{(33)}\)

\[
u^\alpha \frac{\partial \mu}{\partial x^{\alpha}} = -\frac{1}{I_1} kT \theta \left[ I_1 + \frac{(I_2 - \mu e)}{mc^2} \frac{1}{I_1 (I_3 - I_2^2)} \left( I_2 I_1 - \frac{1}{2} I_1 (3I_2 - I_0) \right) \right],
\]
\(\text{(34)}\)

\[
u^\alpha \frac{\partial u_m}{\partial x^{\alpha}} = -\frac{c^2}{kT (3I_2 - I_0)} \left[ (I_0 - I_2) \frac{\partial \mu}{\partial x^{\alpha}} + \left[ (mc^2 (I_1 - I_3) + \mu e (I_2 - I_0)) \right] \frac{1}{T} \frac{\partial T}{\partial x^{\alpha}} \right],
\]
\(\text{(35)}\)

Once the substitution is carried out, \( T^{\alpha \beta} \) is obtained in
terms of solely spatial gradients of the state variables
and thus the constitutive equation for the heat flux can
be evaluated by means of Eq. \( \text{(24)} \). Such a calculation is
shown to some detail in the next section.

V. THE HEAT FLUX

As mentioned above, introduction of Eqs. \( \text{(33)}-\text{(35)} \)
in Eq. \( \text{(32)} \) leads to an expression for the heat flux by

\[
q^\alpha = -\pi \gamma g_n \mu_{e, \beta} m^3 \delta \left\{ -\frac{h^{\sigma \beta} \mu_{e, \beta} \mu e}{kT} \frac{\left( I_3 - I_1 \right)}{3I_2 - I_0} \right\},
\]
\(\text{(37)}\)

which can be written as

\[
q^\alpha = -L_T h^{\sigma \beta} \frac{T^{\beta \alpha}}{T} - L_{\mu} h^{\sigma \beta} \frac{\mu_{e, \beta}}{\mu e},
\]
\(\text{(38)}\)

where \( L_{\mu} \) and \( L_T \) are the transport coefficients associated
with heat conduction and can be written as

\[
L_T = \frac{\tau g_m \pi m^3 \delta}{kT} \left\{ \frac{\mu e}{kT} \left( I_3 - I_1 + \frac{\zeta (I_2 - I_0)(I_2 - I_4)}{(3I_2 - I_0)} \right) \right\}
\]
\(-\zeta (I_2 - I_4) \left( \frac{\zeta (I_2 - I_0) - 1}{(3I_2 - I_0)} \right) \}
\]
\(\text{(39)}\)

\[
L_{\mu} = -\frac{\tau g_m \pi m^3 \delta \mu e}{kT} \left( I_3 - I_1 + \frac{\zeta (I_2 - I_0)(I_2 - I_4)}{(3I_2 - I_0)} \right)
\]
\(\text{(40)}\)

Also notice that, using the following identity

\[
I_n (n + 1) = K + \zeta I_{n+1}
\]

for \( K \) a constant and \( n = 0, 1, 2, \ldots \), the thermal conduc-
tivities are equal in magnitude, i.e. \( L_{\mu} = -L_T \) and thus
one can write a Fourier-like constitutive equation:

\[
q^\alpha = \kappa h^{\sigma \beta} \Theta \beta
\]
\(\text{(41)}\)

where \( \kappa = L_T kT/\mu e \) and \( \Theta = \mu e/kT \) can be thought
of as a generalized thermal force in a similar fashion as
some authors propose for the non-degenerate and non-
relativistic case in 3D \([39, 52]\). Figure 2 shows the ther-
mal conductivity \( \kappa \) in Eq. \( \text{(41)} \) as a function of \( \zeta \) for
different values of the chemical potential.
VI. DISCUSSION OF THE RESULTS

In order to compare the results obtained in the previous sections with their 3D counterparts, we begin by rewriting the constitutive equation for the heat flux, Eq. [38], as follows

\[ q^α = \tilde{h} h^{αβ} \left( T_β - \frac{T}{n h_e} p_β \right), \tag{42} \]

Thus, in order to qualitatively compare the present 2D results with the values reported in Ref. [39], the relevant conductivity coefficient is given by

\[ \tilde{\kappa} = \frac{h_e \kappa}{kT^2}, \tag{43} \]

where \( h_e \) is the enthalpy per particle:

\[ h_e = \epsilon + \frac{p}{n} = \frac{mc^2}{2} \left( \frac{3I_2 - I_0}{I_1} \right). \tag{44} \]

Within this approximation the thermal conductivity depends linearly in the relaxation parameter, which must be specified. However, the dependence on temperature, or more precisely on the relativistic parameter \( \zeta \), has not been completely determined for bidimensional systems [27, 31, 32].

As previously discussed in Refs. [49, 50], Marle’s model is proposed as a special relativistic generalization of its non-relativistic counterpart, the BGK model. However, in order to obtain the transport coefficients that coincide with the ones obtained by solving the complete Boltzmann equation, in the relativistic scenario, Marle’s model must be modified. The transport coefficients obtained by model equations are in general proportional to the parameter \( \tau \) in Eq. [24], that plays the role of a relaxation time in the non-relativistic case, this time is proportional to the time between collisions \( \tau_c \), which for hard disks is given by

\[ \tau_c = \frac{1}{bn\mathcal{V}}, \tag{45} \]

where \( b \) is the diameter of the disks and \( \mathcal{V} \) could be the adiabatic sound speed \( v_s \), the mean relative velocity \( \langle v \rangle \) or the mean velocity \( \langle \mathcal{V} \rangle \). A modified parameter \( \tau \), which has been shown to yield more precise results is proposed in Ref. [50], is

\[ \tau \sim \frac{1}{bn\mathcal{V} \langle x \rangle}, \tag{46} \]

where \( \langle x \rangle \) is the mean value of the relativistic parameter \( x = u_d p^α/mc^2 \). Here, the expression given in Eq. [46] is considered, together with \( \mathcal{V} = \sqrt{2} \langle v \rangle \) and the mean velocity being estimated as \( \langle v \rangle = I_e/mc^2 I_0 \), where

\[ \langle v \rangle = \frac{I_0^{-1}}{2\pi mc^2 \bar{h}} \int_{-\infty}^{\infty} \frac{|p|}{m} \frac{\mathcal{V}}{\Sigma + 1} \frac{1}{\mathcal{V}} dp^*, \tag{47} \]

Also notice that \( \langle x \rangle = I_1/I_0 \).

The conductivity coefficient given in Eq. [43] for a gas of electrons is plotted in Fig. 3 considering the modified parameter \( \tau \) as mentioned above. The normalized (to \( \mathcal{V} \)) thermal coefficient is independent of the temperature for large values of \( \zeta \), consistent with the non-relativistic and non-degenerate case. Notice also that the coefficient decreases with \( \mu_e/kT \) and that the result here obtained for a two dimensional gas of electrons is qualitatively similar to the one reported in the three-dimensional scenario in Ref. [39]. Meanwhile, Fig. 4 shows the importance of considering the factor \( \langle x \rangle \) introduced in \( \tau \), which brings the curve of the coefficient to resemble the results in the 3D case.

The non-relativistic and non-degenerate limit can be explored to detail in an analytical fashion. Indeed, considering \( \exp(-\mu_e/mc^2)/kT \gg 1 \), one can approximate the integrals in [20] and [21] as follows

\[ I_n \approx \frac{1}{2\pi h^2} \frac{1}{m^{n+1} c^{2n+2}} \int_{-\infty}^{\infty} \left( \frac{n^α p_α}{\mathcal{V}} \right)^n \Sigma dp^*, \tag{48} \]
and also notice that in this limit $I_n = I_n$. Thus one has

$$\kappa \approx \frac{kc}{\sqrt{2d}} \left[ \exp (-\zeta) \left( 3 + 6\zeta + 2\zeta^2 \right) \right]$$

(49)

which, upon expansion of the Bessel function of the second kind $K_1 (\zeta)$, can be expressed as

$$\kappa \approx 2 \frac{kc}{d\sqrt{\pi}} \zeta^{-1/2} \left[ 1 - \frac{3}{8\zeta} + \ldots \right]$$

(50)

where the leading term is consistent with the classical non-relativistic limit

$$\kappa_{NR} \approx \frac{2kc}{\sqrt{\pi d}} \zeta^{1/2}. $$

VII. FINAL REMARKS

In the present work, an analytical expression for a thermal conductivity coefficient was established based on the relativistic kinetic theory of gases within Marle’s relaxation approximation and the Chapman-Enskog expansion. It is worthwhile to comment on the fact that, in general, the thermal coefficient to which one refers to as the thermal conductivity depends on the thermodynamic forces that are considered as independent sources of dissipation. In Refs. [28, 53], the heat flux is driven both by the temperature and density gradients which are considered as independent forces in the representation where $T$ and $n$ are taken as the scalar state variables. In the present work however, since the gas dealt with is a quantum system, the chemical potential replaces the number density as a relevant state variable. In such a representation, both transport coefficients are identical in magnitude which allows for the identification of a generalized thermodynamic force, namely $\mu_e/kT$, and a single relevant transport coefficient.

On the other hand the authors consider it valuable to emphasize the importance of the role the particular value of the relaxation parameter in BGK-like approximations plays in the magnitude and general behavior of the transport coefficients, in particular for numerical simulations as lattice Boltzmann methods. As is well known, these type of approximations are extremely valuable in order to assess the general structure of constitutive equations as they retain the force-flux relations as obtained from the complete Boltzmann equation in most cases. However, all the details of the collisions and thus the particular molecular interaction, are included in one single parameter instead of various collision integrals. The cost of turning the integrodifferential equation in an algebraic relation is that of having to be extremely careful when addressing the behavior of transport coefficients. In particular, in the relativistic case two BGK-like models, namely Marle and Anderson-Witting, have been proposed which in general lead to different approximations for such quantities. Moreover, within those methods, the choice of the characteristic parameter is not unique. In Marle’s case, $\tau$ can be corrected from its expression in the non-relativistic case in order to resemble more closely the value obtained both with the complete kernel and Anderson-Witting model [49, 50].

The relevance and applications of the results here obtained can be inferred from the comments included in the introduction and the references cited therein. In this final section we would like to add that the results here presented contribute to the understanding of the behavior of bidimensional degenerate relativistic fluids. These systems, which have been shown to have important and novel applications, have been scarcely addressed in the literature from the theoretical point of view. Moreover, the calculation of the viscosity coefficients in order to complete the description of the dissipative properties of these type of gases, is also lacking and will be the focus of a future publication. As a final comment the authors wish to emphasize the importance of addressing these type of calculations using the Chapman-Enskog method, as here presented, and further considering the complete integral kernel, which will be tackled in the near future. Indeed, as pointed out in Ref. [32], the Chapman-Enskog method leads to better results than the simpler and more generally employed Grad moment method in the case of relativistic hydrodynamics.

[1] K. S. Novoselov, A. K. Geim, S.V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, A. A. Firsov, Nature 438, 197-200 (2005).
