Diophantine approximation, large intersections and geodesics in negative curvature

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Abstract
In this paper, we prove quantitative results about geodesic approximations to submanifolds in negatively curved spaces. Among the main tools is a new and general Jarník–Besicovitch type theorem in Diophantine approximation. Our framework allows manifolds of variable negative curvature, a variety of geometric targets, and logarithm laws as well as spiraling phenomena in both measure and dimension aspect. Several of the results are new also for manifolds of constant negative sectional curvature. We further establish a large intersection property of Falconer in this context.

MSC 2020
37D40 (primary), 53C23 (secondary)

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1 | INTRODUCTION

This paper has its origins in fundamental work of Patterson [37] on Diophantine approximation and Fuchsian groups, and of Sullivan [44] on logarithm laws for cuspidal excursions of geodesics in hyperbolic manifolds. Our subject here is a quantitative study of certain asymptotic properties
of geodesics in negatively curved spaces. Consider a negatively curved manifold \( M \) and a ‘target’ \( N \) which is a subset of the manifold. Given a point \( p \) in \( M \), what can be said about the size (in terms of measure and dimension) of the set of geodesics starting at \( p \) which

1) infinitely often visit neighbourhoods of \( N \) which are shrinking in volume? We refer to this problem as the shrinking target problem, following Hill and Velani \[24\] as well as Kleinbock and Margulis \[29\].

2) take longer and longer sojourns into a fixed neighbourhood of \( N \)? We refer to this as the spiral trap problem.

When \( N \) is an ‘isolated point at infinity’, (1) is the setting of Sullivan’s celebrated work \[44\]. A Hausdorff dimension estimate in this case is given in the work of Melián–Pestana \[35\]. When \( N \) is a point in \( M \), (1) has been considered in the work of Maucourant \[34\], who provides a zero-one law (i.e. a measure theoretic Borel–Cantelli statement) in the case \( M \) is a finite volume manifold of constant negative sectional curvature. When \( N \) is a closed geodesic bounding a funnel in a surface of constant negative sectional curvature without cusps, a Hausdorff dimension estimate for (1) is provided by Dodson, Melián, Pestana and Velani \[13\]. When \( M \) is a closed manifold of variable, strictly negative sectional curvature, and \( N \) is again a point in \( M \), a zero-one law for (1) is known from the work of Hersonsky, Paulin and Aravinda in \[20\] and a Hausdorff dimension estimate is known from the work of Hersonsky–Paulin \[19\]. Question (2) has been considered in the work of Hersonsky–Paulin \[20\], who give a zero-one law for convex subsets in the more general setting of CAT(\(-1\)) spaces. A necessarily incomplete list of work related to the stated problems includes the papers \[2,3,5,13,36,41,45\].

In this paper, we develop a framework for obtaining comprehensive measure and dimension results for the shrinking target problem and the spiral trap problem. This allow us to treat general and natural geometric targets and our results are valid in a very general setting, allowing for instance, manifolds of variable negative curvature and more generally, quotients of CAT(\(-1\)) metric spaces. Nevertheless, several of our results are new even for closed manifolds of constant negative curvature. Namely, we give sharp Hausdorff dimension estimates for (2). For (1), we relax the requirement that the manifold be closed, and generalise from targets being points to more general convex subsets, and obtain zero-one laws as well as sharp dimension estimates.

We illustrate the type of problems studied here by mentioning the following very special cases of results proved in Section 4. For a smooth manifold \( M \) and \( x \in M \), let \( SM_x \) denote the unit tangent sphere at \( x \). In this paper we write manifold to mean a smooth manifold. Theorem 1.1 is a result about geodesics hitting shrinking targets and a special case of Theorem 4.10.

**Theorem 1.1.** Let \( M \) be a complete connected manifold of finite volume, with dimension \( n \) and constant negative sectional curvature, \( k = -1 \). Let \( N \) be a closed, totally geodesic submanifold of \( M \) of dimension \( 0 \leq s \leq n - 1 \). Let \( \tau \geq 0 \) be fixed. Then given \( x_0 \in M \), we have that the set

\[
E_{N}^{\tau} := \left\{ v \in SM_{x_0} \mid \exists \text{ positive times } t_n \to \infty \text{ such that } \gamma_v(t_n) \subset B(N, e^{-\tau t_n}) \right\}
\]

has Hausdorff dimension

\[
\dim_H(E_{N}^{\tau}) = (n - 1) \cdot \frac{1 + \frac{s\tau}{n - 1}}{1 + \tau},
\]

where \( \gamma_v \) is the geodesic such that \( \gamma_v(0) = x_0 \) and \( \gamma'_v(0) = v \).
Theorem 1.2 is an example of the spiral trap phenomenon and a special case of Theorem 4.9.

Theorem 1.2. Let $\tau \geq 0$ be fixed. Let $M$ be a complete connected manifold of finite volume, with dimension $n$ and constant negative sectional curvature, $k = -1$. Let $N$ be a compact, totally geodesic submanifold of dimension $1 \leq s \leq n - 1$. Let $x_0 \in M$. Then for all $\varepsilon > 0$ small enough, the set

$$T^\tau_N := \left\{ v \in SM_{x_0} \mid \exists \text{ positive times } t_n \to \infty \text{ such that } \gamma_v(t_n, t_n + \tau t_n) \subset B(N, \varepsilon) \right\}$$

has

$$\dim_H(T^\tau_N) = (n - 1) \cdot \frac{1 + \frac{\tau(s - 1)}{n - 1}}{1 + \tau},$$

where $\gamma_v$ is the geodesic such that $\gamma_v(0) = x_0$ and $\gamma'_v(0) = v$.

Theorem 1.3 is a special case of Corollary 4.12.

Theorem 1.3. Let $M$ be a complete connected manifold of finite volume, with dimension $n$ and constant negative sectional curvature, $k = -1$. Let $N$ be a closed, totally geodesic submanifold in $M$ of dimension $0 \leq s \leq n - 1$. Let $x_0 \in M$.

1. For $\varepsilon > 0$ small enough, the set

$$E_N := \left\{ v \in SM_{x_0} \mid \exists \text{ positive times } t_n \to \infty \text{ such that } \gamma_v(t_n, t_n + \tau t_n) \subset B(N, \varepsilon) \text{ for some } \tau > 0 \right\}$$

has

$$\dim_H(E_N) = n - 1.$$

2. Also,

$$T_N := \left\{ v \in SM_{x_0} \mid \exists \text{ positive times } t_n \to \infty \text{ such that } \gamma_v(t_n) \subset B(N, e^{-\tau t_n}) \text{ for some } \tau > 0 \right\}$$

has

$$\dim_H(T_N) = n - 1,$$

where $\gamma_v$ is the geodesic such that $\gamma_v(0) = x_0$ and $\gamma'_v(0) = v$. The set $E_N$ is a null set for the Lebesgue measure in $SM_{x_0}$ for $0 \leq s \leq n - 1$. The set $T_N$ is a null set for the Lebesgue measure if and only if $0 \leq s < n - 1$. 
1.1 | Geodesic flows and Jarník–Besicovitch theorems

Asymptotic properties of geodesics in hyperbolic manifolds, namely spiral traps and shrinking target properties, are closely related to metric Diophantine approximation.

Namely, these phenomena can be recast as Diophantine approximation questions about (subsets of) orbits of the fundamental group in the visual boundary of the universal cover of the manifold. For instance, Khintchine type results in Diophantine approximation follow as a result of mixing of the geodesic flow. On the other hand, Jarník–Besicovitch type results utilising the metric measure structure of the limit set can be used to obtain quantitative estimates on fine asymptotic behaviour of geodesics. There is thus a well-known bilateral correspondence between these two rich areas. In this paper we prove some of the most general results exploring this correspondence and obtain sharp estimates for the problems stated above.

The Hausdorff measure and dimension refinements of Khintchine’s theorem in Diophantine approximation were obtained by Jarník [26], and independently by Besicovitch [7]. Subsequently, there have been numerous developments, especially the mass transference principle of Beresnevich and Velani [6]. We refer the reader to [5] and the references therein. In [30], a more general version of the mass transference principle was obtained and in [38], this theme was further developed using the notion of Riesz energies. The method employed in [38] easily generalises to the setting of Ahlfors-regular metric spaces and provides a useful Hausdorff content lower bound, besides a dimension lower bound for suitable limsup sets. The Hausdorff content bound leads to a large intersection property for countable families of limsup sets which is discussed below.

In Section 3, we develop an abstract approach with a view towards geometric applications. We prove a general, abstract Jarník–Besicovitch Theorem 3.3, using the results in [38]. Our theorem can be used to study the action of a discrete group \( \Gamma \) acting properly on a CAT(-1) space \( X \) and ergodically on its limit set in the visual boundary. We then connect metric aspects of the \( \Gamma \) action on the visual boundary of \( X \) and the asymptotic behaviour of geodesics on \( X/\Gamma \). This leads to a wealth of applications as discussed earlier and in more detail in Section 4.

1.2 | Large intersections

The sets \( E^*_N \) of geodesic directions hitting exponentially shrinking targets in Theorem 1.1, are zero measure sets and so there is no control \( \text{a priori} \) on the size of their intersections which could be trivial. Nevertheless, they have a ‘large intersection’ property which leads to the surprising fact that countable intersections have Hausdorff dimensions bounded below by the infimum of the respective Hausdorff dimensions of the individual sets (see Theorem 1.4).

In [16], Falconer defined a class of subsets, denoted \( G^s \), of \( \mathbb{R}^n \), which form a maximal class of \( G^s \)-sets of dimension at least \( s \) that is closed under countable intersections and under similarity transformations (see also [15]). He named this property the large intersection property. Falconer’s definition unifies several earlier categories of sets with similar properties including the ‘regular systems’ of Baker and Schmidt [4] and the ‘ubiquitous systems’ of Dodson, Rynne and Vickers [14] and consequently these classes play an important role in Diophantine approximation. Theorem 3.14 provides in particular a large intersection property for actions of hyperbolic groups on the visual boundaries of hyperbolic metric spaces. We avoid the use of net measures as we use the ‘coarser’ dyadic decomposition coming from shadows of balls centred at orbits of \( \Gamma \); instead...
we use a variant of the Hausdorff measure. Here, by coarse we mean that the decomposition does not need to have the property that when sets of different 'generations' overlap, then the interior of one has to be completely contained in the other. We note however that more abstract versions (not determined by concrete geometric properties) of dyadic decompositions also exist for Ahlfors-regular metric spaces, see [11, 28]. The abstract dyadic decomposition given in [28] also has the properties we require. The following result is an application of Theorem 3.14 to the problem of spiraling of geodesics. It is a special case of Theorem 4.11.

**Theorem 1.4.** Let $M$ be a closed connected manifold of dimension $n$, with pinched sectional curvature $-a^2 \leq k \leq -1$. Let $\{N_i\}_i$ be a countable collection of closed totally geodesic submanifolds or points of $M$. Let $x_0 \in M$. Let $\tau_i$ be a sequence of positive numbers. Then the set of directions

$$E := \left\{ v \in SM_{x_0} \mid \exists \text{ positive times } t_n^{(i)} \to \infty \text{ for each } i \text{ such that } \gamma_v(t_n^{(i)}) \in B(N_i, e^{-\tau_i t_n^{(i)}}) \text{ for each } i \right\}$$

has Hausdorff dimension

$$\inf_i (n - 1) \frac{1 + \frac{\tau_i \dim(N_i)}{n-1}}{1 + \tau_i} \leq \dim_H(E) \leq \inf_i (n - 1) \frac{1 + \frac{\tau_i \dim(N_i)}{n-1}}{1 + \tau_i/a}.$$  

Finally, we note that a detailed and systematic study of Diophantine approximation in the context of hyperbolic groups has been carried out in the monograph [17] of Fishman, Simmons and Urbanski. They construct ‘partition structures’ (similar to our dyadic decomposition in the case of geometric actions) and prove results about the limit set of discrete group actions in very general settings. The dyadic decomposition we use enjoys stronger properties that we require for establishing the large intersection property. The aforementioned paper has no intersection with our results.

**Structure of the paper**

In Section 2, we collect some of the preliminary definitions and facts and introduce the Whitney decomposition for metric spaces with a dyadic decomposition. In Section 3, we describe an abstract framework for Diophantine approximation and prove a generalised Jarník–Besicovitch theorem in this context as well as a version of Falconer’s large intersection property. In Section 4, we apply the results of Section 3 and study shrinking targets for the geodesic flow and spiraling of geodesics in negatively curved spaces. In this paper, we use the notation $A \geq B$ to mean that there is a constant $c > 0$ such that $A \geq cB$. Dependence of the constant on parameters will be specified. The notation $A \approx B$ will stand for $A \geq B \geq A$.

## 2 | PRELIMINARIES

Let $(X, \rho)$ be a proper, geodesic and hyperbolic metric space.
2.1 | The visual metric

The visual (Gromov) boundary $\partial X$, is the set of equivalence classes of geodesic rays of $X$, where two rays are equivalent if for all times large enough, they lie in a bounded distance of each other. This set has a natural structure of a compact, metrizable topological space.

Indeed, it is a compact metric space with a family of mutually quasisymmetric visual metrics $d_\beta$, $0 < \beta < \beta_X$, for some $\beta_X > 0$ which satisfy

$$\frac{1}{C(\beta)} e^{-\beta (\xi, \eta)} \leq d_\beta (\xi, \eta) \leq C(\beta) e^{-\beta (\xi, \eta)},$$

for $\xi, \eta \in \partial X$. Here

$$(x|y) := (x|y)_{x_0} = \frac{1}{2} (\rho(x, x_0) + \rho(y, x_0) - \rho(x, y))$$

for $x, y, x_0 \in X$, is the Gromov product, which is extended to $\partial X$ by taking limits (linsup in case a limit does not exist). The Gromov product changes up to an additive constant on changing the base point $x_0$, and the visual metrics $d_\beta$ corresponding to different base points are biLipschitz to each other. We fix a $\beta \in (0, \beta_X)$ and write $d = d_\beta$ in what follows. We refer the reader to Bridson–Haefliger [9], for details on the visual boundary.

Geodesics

Let $x, y \in X$. We will use $[x, y]$ to denote any unit-speed geodesic segment starting from $x$ and ending at $y$, that is $[x, y](0) = x$ and $[x, y](\rho(x, y)) = y$. If $x \in \partial X$, $y \in X$, then $[x, y]$ denotes any unit-speed geodesic ray such that $[x, y](-\infty) = x$ and $[x, y](0) = y$. Similarly, if $x \in X$ and $y \in \partial X$, then $[x, y]$ denotes any unit-speed geodesic ray such that $[x, y](0) = x$, $[x, y](\infty) = y$. In case $x, y \in \partial X$, $x \neq y$, then $[x, y]$ denotes any unit-speed geodesic line such that $[x, y](-\infty) = x$, $[x, y](\infty) = y$. When we assume $X$ to be a complete CAT($-1$) space (see Subsection 2.5) and $x \in X$, $y \in X \cup \partial X$, if $t \in \mathbb{R}$ is not in the domain of $[x, y]$, we denote by $[x, y](t)$ to be the point at time $t$ in the unique, unit-speed extension of $[x, y]$ to a geodesic line where $[x, y](0) = x$.

**Definition 2.1** (Shadow). The set

$$S(x, z, R) := \{ \xi \in \partial X \mid \exists [x, \xi], \text{ such that } [x, \xi] \cap B(z, R) \neq \emptyset \}$$

is the shadow of the ball $B(z, R)$ with respect to $x$.

Now let $\Gamma$ be a finitely generated group of isometries of $X$, which acts properly discontinuously. We record the following well-known fact. For a metric space $(Y, d)$ and subset $E \subset Y$, $diam(E)$ means the $d$-diameter of $E$.

**Lemma 2.2.** Let $R > 0$. There is a constant $c = c(R, X) > 0$, such that for all $g \in \Gamma$, we have

$$\frac{1}{c} \cdot e^{-\beta \cdot \rho(x_0, gx_0)} \leq diam(S(x_0, gx_0, R)) \leq c \cdot e^{-\beta \cdot \rho(x_0, gx_0)}.$$
Proof. Let $\xi, \xi' \in S(x_0, g_{x_0}, R)$. Let $x \in \gamma_\xi \cap B(g_{x_0}, R)$ and $x' \in \gamma_{\xi'} \cap B(g_{x_0}, R)$. Then the claim follows from the inequalities

$$(\xi | \xi') \geq \min\{(\xi | x), (x | x'), (x' | \xi')\} - C,$$

and

$$(x | x') \geq \min\{(x | \xi), (\xi | \xi'), (\xi' | x')\} - C,$$

which hold for a constant $C = C(X) > 0$. □

2.2 Patterson–Sullivan (quasiconformal) measures

Let $\Gamma$ be as above. Denote by $\Lambda_\Gamma = \Lambda_\Gamma(X)$, its limit set in $\partial X$. We will henceforth assume that the action of $\Gamma$ is non-elementary, so $\#\Lambda_\Gamma > 2$, and hence uncountable. The action of $\Gamma$ on $\partial X$, induced by its action on $X$, is by homeomorphisms, and $\Lambda_\Gamma$ is the maximal closed set invariant under the action of $\Gamma$ on $\partial X$.

There exist $\Gamma$-equivariant families of finite Borel measures, namely Patterson–Sullivan measures $\{\mu_x\}_{x \in X}$, with mutual Radon–Nikodym derivatives given in terms of Busemann functions. Fix a base-point $x_0$ and denote henceforth by $\mu$, a Patterson–Sullivan measure $\mu_{x_0}$, normalised to be a probability measure. Recall that the critical exponent of the $\Gamma$ action on $X$ is

$$v_\Gamma := \limsup_n \frac{1}{n} \cdot \log(\#\{g \in \Gamma | g_{x_0} \in B(x_0, n)\}).$$

The number $v_\Gamma$ is independent of the choice of $x_0$. We next record two standard but important lemmas below, originally due to Sullivan [43] for $\mathbb{H}^n$, which are proved in Coornaert [12] in the generality that we consider.

Lemma 2.3 (Sullivan’s Shadow Lemma). Let $R > 0$. There exists $a_\Gamma = a_\Gamma(R) \geq 1$, such that

$$\frac{1}{a_\Gamma} \cdot e^{-v_\Gamma \cdot \varphi(x_0, g_{x_0})} \leq \mu(S(x_0, g_{x_0}, R)) \leq a_\Gamma \cdot e^{-v_\Gamma \cdot \varphi(x_0, g_{x_0})}.$$ 

The parameter $R$ in the statement above is usually chosen to be a large number depending on the action of $\Gamma$. We state it for all positive values of $R$ as the constant $a_\Gamma$ may be suitably adjusted.

Lemma 2.4. Let $R > 0$. There exists $a_\Gamma = a_\Gamma(R) \geq 1$, such that

$$\frac{1}{a_\Gamma} \cdot \text{diam}(S(x_0, g_{x_0}, R))^{v_\Gamma / \beta} \leq \mu(S(x_0, g_{x_0}, R)) \leq a_\Gamma \cdot \text{diam}(S(x_0, g_{x_0}, R))^{v_\Gamma / \beta}.$$ 

Proof. This follows from Lemmas 2.2 and 2.3. □

By a metric measure space, we mean a metric space $(Y, d)$, equipped with a Borel measure $\mu$. 
Definition 2.5 (Ahlfors regularity). For $D > 0$, $a \geq 1$ we say that a metric measure space $(Y, d, \mu)$ is $(a, D)$-Ahlfors regular, if for each $x \in Y$ and $0 < r < \text{diam}(Y)$,

$$\frac{1}{a} \cdot r^D \leq \mu(B(x, r)) \leq a \cdot r^D.$$ 

If a constant $a > 0$ exists such that $Y$ is $(a, D)$-Ahlfors regular, we call $Y$ $D$-Ahlfors regular. A relevant example of such a space for us is the limit set, equipped with a Patterson–Sullivan measure, of a convex-cocompact Kleinian group acting on $\mathbb{H}^3$.

2.3 Dyadic and Whitney decompositions

We require the notions of dyadic and Whitney decompositions for the results on dimensions of large intersections of limsup-type sets.

As above, let $\Gamma$ be a finitely generated group which acts on $X$ properly discontinuously by isometries. We will say that the action of $\Gamma$ is convex-cocompact, if $\Gamma$ also acts coboundedly on the convex hull in $X$ of its limit set $\Lambda_{\Gamma}$. Here a $\Gamma$-action on a metric space $Y$ is cobounded, if there is $R > 0$ such that for any $y \in Y$, $Y \subset \bigcup_{g \in \Gamma} B(qy, R)$. If the $\Gamma$ action on $(X, \rho)$ is convex-cocompact then the space $(\Lambda_{\Gamma}, d, \mu)$ is $\nu_{\Gamma}/\beta$-Ahlfors regular where $d = d_{\rho}$ is a visual metric, corresponding to the parameter $\beta > 0$ (recall that $\mu$ is a fixed measure in a Patterson–Sullivan family of $\Gamma$ and $\nu_{\Gamma}$ is the critical exponent of the action of $\Gamma$ on $X$). Moreover, the space $(\Lambda_{\Gamma}, d, \mu)$ also admits a decomposition similar to the standard dyadic decomposition of $\mathbb{R}^n$ (see [28] and also [11]).

Definition 2.6 (Dyadic decomposition). Let $Y$ be a $D$-regular metric space, for $D > 0$. A dyadic decomposition $D$ of $Y$ is a countable collection $\{Q_i\}_{i \in I}$, of subsets $Q_i$ of $Y$, such that there exist constants $C \geq 1$, $n_0 \in \mathbb{N}$, and a decomposition $D = \bigcup_{n \geq n_0} \mathcal{W}_n$, so that the following hold.

(2.11.1) For each $n \in \mathbb{N}$, $n \geq n_0$, $\# \mathcal{W}_n < \infty$ and

$$Y = \bigcup_{\mathcal{W}_n} Q_i := \bigcup_{Q_i \in \mathcal{W}_n} Q_i.$$ 

(2.11.2) For each $Q \in D$, there exists $x \in Q$, such that

$$B(x, \text{diam}(Q)/C) \subset Q \subset B(x, C \cdot \text{diam}(Q)).$$ 

(2.11.3) Given $n \in \mathbb{N}$, $n \geq n_0$, $l \in \mathbb{N} \cup \{0\}$ and $Q_i \in \mathcal{W}_n$,

$$\#\{Q_j \in \mathcal{W}_{n+l} \mid Q_j \cap Q_i \neq \emptyset\} \leq C^{l \cdot D} \text{ and } Q_i \setminus \bigcup_{\mathcal{W}_n \setminus \{Q_i\}} Q_j \neq \emptyset.$$ 

(2.11.4) For $n \in \mathbb{N}$, $n \geq n_0$ if $Q \in \mathcal{W}_n$, then

$$C^{-(n+1)} \leq \text{diam}(Q) \leq C^{-n}.$$
The data associated to $D$ mean the constants $C, n_0$ and the collection $\{W_n\}_{n \geq n_0}$.

The following lemma is folklore and we omit the proof. The main ingredients are the Milnor–Svarč lemma and Sullivan’s shadow lemma. The action of a group on a metric space is called geometric if it is properly discontinuous, by isometries and cobounded.

**Lemma 2.7.** Suppose that $X$ is a proper, geodesic, hyperbolic metric space with a geometric action of a group $\Gamma$. Fix $x_0 \in X$. Then there exist constants $R = R(\Gamma, X) > 0$ and $R' = R'(\Gamma, X) > 0$ such that

$$D = \{ S(x_0, gx_0, R) \mid \rho(x_0, gx_0) \geq R' \},$$

is a dyadic decomposition for the visual boundary $\partial X$, equipped with a visual metric $d$. The data associated to $D$ (namely, the constants $C, n_0$ and the decomposition $D = \bigcup_{n \geq n_0} W_n$) depend only on $X$ and $\Gamma$.

In the above lemma, the role of $W_n$ is played by collections of the form $\{ S(x_0, gx_0, R) \mid \rho(x_0, gx_0) \in (c \cdot n, c \cdot (n + 1)) \}$, for a suitable constant $c = c(\Gamma, X) > 0$. An instance when it applies is thus a group acting convex-cocompactly on a proper, geodesic, hyperbolic metric space $Y$; in this case $X$ is the convex hull in $Y$ of $\Lambda_\Gamma = \partial X \subset \partial Y$.

A dyadic decomposition $D$ has the following property from definition.

**Lemma 2.8.** Let $Y$ be a proper $(a, D)$-regular metric space and $D$ be a dyadic decomposition of $Y$, with data $C, n_0, \{W_n\}_{n \geq n_0}$. Then given $R \geq 1$ there exists $M = M(C, n_0, a, D, R) > 0$, such that for all $Q_i \in D$,

$$\# \left\{ Q_j \in D \left\| \frac{1}{R} \cdot \text{diam}(Q_i) \leq \text{diam}(Q_j) \leq R \cdot \text{diam}(Q_i) \text{ and } Q_j \cap B(Q_i, R \cdot \text{diam}(Q_i)) \neq \emptyset \right\} \leq M.$$

Given a dyadic decomposition we also have the notion of a Whitney decomposition, which will be useful in Subsection 3.2 on the large intersection property.

**Definition 2.9** (Whitney decomposition). Let $Y$ be an Ahlfors-regular metric space and $D$ be a dyadic decomposition of $Y$, with data $C, n_0, \{W_n\}_{n \geq n_0}$. Then given $a > 1$ and let $U \subset Y$ be an open set. A sub-collection $W$ of $D$ is an $a$-Whitney decomposition of $U$ if $U = \bigcup_{W} Q$ and the following hold.

1. Given $Q \in W$, there exists $x \in Q$ such that $Q' \in W$ and $x \in Q'$ implies $Q' = Q$.
2. For each $Q \in W$,

$$\text{diam}(Q) \leq \frac{1}{a} \cdot \min\{\text{dist}(Q, Y \setminus U), \text{diam}(U)\} \leq a \cdot \text{diam}(Q).$$

As an easy consequence of the definitions of the dyadic and Whitney decompositions we get the following.

**Lemma 2.10.** Suppose the proper, $(a, D)$-regular metric space $Y$ has a dyadic decomposition $D$ with data $C, n_0, \{W_n\}_{n \geq n_0}$. For any proper open set $U \subset Y$, there exists a constant $a = a(C, n_0, a, D) > 1$, such that for any $a \geq a$, there is an $a$-Whitney decomposition of $U$. 

Proof. Pick $a' > 20 \cdot C$. For $\xi \in U$, choose $Q_\xi$ to be an element in $D$ of maximal diameter, satisfying the properties $\xi \in Q_\xi \subset U$ and

$$\text{diam}(Q_\xi) \leq \frac{1}{a'} \cdot \min\{\text{dist}(\xi, Y \setminus U), \text{diam}(U)\}.$$  

This exists because of property (2.11.1) of $D$ from Definition 2.6. Then by the maximality of $Q_\xi$ and property (2.11.4) of $D$, there exists $b' = b'(C, n_0) > 0$, such that

$$\frac{1}{a'} \cdot \min\{\text{dist}(\xi, Y \setminus U), \text{diam}(U)\} \leq b' \cdot \text{diam}(Q_\xi).$$

Then,

$$\text{diam}(Q_\xi) \leq \frac{1}{a' - 1} \cdot \min\{\text{dist}(Q_\xi, Y \setminus U), \text{diam}(U)\} \leq \frac{a' \cdot b'}{a' - 1} \cdot \text{diam}(Q_\xi).$$

We have to modify the uncountable collection $\{Q_\xi\}_{\xi \in U}$ to obtain a Whitney decomposition of $U$, which is done using property (2.11.2) of $D$ as follows. Consider for each $\xi \in U$, a ball $B(x_\xi, C \cdot \text{diam}(Q_\xi))$ containing $Q_\xi$, using (2.11.2). Then by the 5r-covering theorem there exist countably many points $\{\xi_i\}_{i \in \mathbb{N}}$, such that $U = \bigcup_{i \in \mathbb{N}} B(x_{\xi_i}, 5 \cdot C \cdot \text{diam}(Q_{\xi_i}))$. For each $i \in \mathbb{N}$, let $n_i \in \mathbb{N}$ be the smallest integer greater than $n_0 - 1$, such that $5 \cdot C \cdot \text{diam}(Q_{\xi_i}) \geq C^{-n_i}$. Set

$$Q_i := \{Q \in \mathcal{W}_{n_i} \mid Q \cap B(x_{\xi_i}, 5 \cdot C \cdot \text{diam}(Q_{\xi_i})) \neq \emptyset\}.$$  

Note that by volume comparison using Ahlfors-regularity, $\#Q_i \leq m$, where $m = m(C, n_0, a, D) > 0$ is a constant. Also, by the choice of $a'$ and construction of $\{Q_\xi\}_{\xi \in U}$,

$$U = \bigcup_{i \in \mathbb{N}} Q_i.$$  

Note moreover that by minimality of $n_i$, there is $c_1(C, n_0, a, D) > 0$ such that for each $i \in \mathbb{N}$, and $Q \in Q_i$,

$$\frac{1}{c_1} \cdot \text{diam}(Q) \leq \text{diam}(Q_{\xi_i}) \leq c_1 \cdot \text{diam}(Q).$$

Then, by a simple computation involving only the triangle inequality, for $a' > 1 + \sqrt{1 + c_1}$, we get

$$\text{diam}(Q) \leq \frac{1}{c_2} \cdot \min\{\text{dist}(Q, Y \setminus U), \text{diam}(U)\} \leq \frac{c_1 + a' b' c_1 + 1}{c_2} \cdot \text{diam}(Q),$$

where $c_2^{-1} = \frac{c_1}{a' - 1} \cdot \max\{1, \frac{a'^{-1}}{(a'^{-1})^2 - c_1^{-1}}\}$. A constant $\bar{a}$ as needed for requirement (2) in the definition of a Whitney decomposition may finally be specified by choosing $a'$ large enough that $\frac{c_1 + a' b' c_1 + 1}{c_2} < c_2$. Such a choice is determined only by the numbers $C, n_0, a$ and $D$. Thus, the elements of $\mathcal{W}' := \{Q \in Q_i \mid i \in \mathbb{N}\}$ satisfy requirement (2) in the definition of a Whitney decomposition.
Take any \( \hat{a} \geq \bar{a} \), so that (2) still holds. To ensure property (1), consider the decomposition \( D = \bigcup_{n \geq n_0} \mathcal{W}_n \) and let \( N_1 \in \mathbb{N} \) be the smallest integer greater than \( n_0 - 1 \) such that \( \mathcal{W}_{N_1} \cap \mathcal{W}' \neq \emptyset \). Note that by property (2.11.1), \( \mathcal{W}_{N_1} \) is finite. We may then exclude a collection \( \mathcal{V}_{N_1}' \) of finitely many elements from \( \mathcal{W}_{N_1} \cap \mathcal{W}' \), such that requirement (1) is satisfied for each \( Q \in (\mathcal{W}_{N_1} \cap \mathcal{W}') \setminus \mathcal{V}_{N_1} \), and setting \( \mathcal{W}'_1 = \mathcal{W}' \setminus \mathcal{V}_{N_1} \), we still have that \( U = \bigcup_{n \geq n_0} \mathcal{W}_n \). Next let \( N_2 \in \mathbb{N} \) be the smallest integer greater than \( n_0 - 1 \) such that \( \mathcal{V}_{N_1} \cap \mathcal{V}_{N_2}' \neq \emptyset \). Repeat as above to obtain a collection \( \mathcal{V}_{N_2}' \) from \( \mathcal{W}_{N_2} \) and \( \mathcal{V}_{N_1} \) such that requirement (1) is satisfied for each \( Q \in (\mathcal{W}_{N_1} \cup \mathcal{W}_{N_2}) \cap \mathcal{W}'_1 \setminus \mathcal{V}_{N_2} \), where \( \mathcal{V}_{N_2} = \mathcal{V}_{N_1} \cup \mathcal{V}_{N_2}' \) and setting \( \mathcal{W}'_2 = \mathcal{W}'_1 \setminus \mathcal{V}_{N_2} \), we have \( U = \bigcup_{n \in \mathbb{N}} \mathcal{W}_n \). We may define recursively a sequence \( \{\mathcal{V}_{N_k}\}_{k \in \mathbb{N}} \) such that for \( \mathcal{W} := \mathcal{W}' \setminus \bigcup_{k \in \mathbb{N}} \mathcal{V}_{N_k} \) we have for each \( Q \in \mathcal{W}, Q \) satisfies requirement (1). Let us now check that \( U = \bigcup_{W} Q \). First claim that for any \( \xi \in U \),

\[
\#\{Q \in \mathcal{W}' \mid \xi \in Q\} \leq M,
\]

for a constant \( M \), depending only on \( C, n_0, a, D, \hat{a} \). This is because by a computation involving only the triangle inequality and requirement (2) which has been verified, we have \( \text{diam}(Q) \leq \frac{\hat{a}^{a+1}}{\hat{a}^{a-1}} \cdot \text{diam}(Q') \), whenever \( \xi \in Q \cap Q' \), with \( Q, Q' \in \mathcal{W}' \), so Lemma 2.8 implies the claim. Now if there is \( \xi \notin U \setminus \bigcup_{W} Q \), then there are \( k \in \mathbb{N} \) and \( Q \in \mathcal{V}_{N_k} \) such that \( \xi \in Q \). Choose the largest such \( k \), which is possible by finiteness of the constant \( M \) in the inequality (1). Then for any \( j > k \), \( \xi \not\in \bigcup_{j \leq k} \mathcal{V}_{N_j} \). However by construction, \( \xi \in \bigcup_{W} Q \). Thus, \( \xi \in \bigcup_{j \leq k} \mathcal{V}_{N_j} \setminus \bigcup_{j > k} \mathcal{V}_{N_j} = \bigcup_{W} Q \), which contradicts \( U \neq \bigcup_{W} Q \). This concludes the proof.

## 2.4 Hausdorff content and dimension

Let \( Y \) be a metric space and let \( t \in [0, \infty) \). The \( t \)-Hausdorff content of a set \( E \subset Y \) is defined as

\[
\mathcal{H}^t_{\infty}(E) := \inf \left\{ \sum_i \text{diam}(E_i)^t \mid E \subset \bigcup_i E_i \right\}.
\]

The Hausdorff dimension of \( E \) is defined as

\[
\text{dim}_{H}(E) := \inf \{ t \in [0, \infty) \mid \mathcal{H}^t_{\infty}(E) = 0 \}.
\]

The \( t \)-Hausdorff content is an outer measure. It is finite for bounded sets. It is not a Borel measure in general.

## 2.5 CAT(−1) spaces

For the results on the spiraling of geodesics in Section 4, we will have to assume curvature bounds for our hyperbolic spaces. Indeed, we consider a complete connected manifolds \( M \) of pinched negative sectional curvature, \(-\alpha^2 \leq k \leq -1\), whose universal covers are thus CAT(−1). Note that any CAT(−1) space is also hyperbolic. Our approach in Section 4 however does not crucially depend on the smooth structure; see Remark 4.13.
In Section 4, we will use the Alexandrov ‘thin’-CAT(−1) inequality for triangles in CAT(−1) spaces and the ‘fat’-CBB(−\(a^2\)) inequality for spaces with curvature bounded below by \(-a^2\); see, for example, [9] or [1]. A CAT(−1) space is uniquely geodesic and hence simply connected. Another fact we frequently use is that for a CAT(−1) space \(X\),

\[ d_X(\xi, \eta) = e^{-(\xi|\eta)_x}, \]

for any point \(x \in X\) is a visual metric on the visual boundary called the Bourdon metric (see [8]), where we recall that \((\xi|\eta)_x\) is the Gromov product with base-point \(x\), introduced in Subsection 2.1. We will use the following notion frequently in Section 4. Let \(X\) be a complete connected CAT(−1) space.

**Definition 2.11 (Trail).** Given \(x \in X\) and \(E \subset X\), define the **trail** of \(E\)

\[ T(x, E) = \{ y \in X \mid [x, y] \cap E \neq \emptyset \}. \]

**Counting estimates:** For a complete connected CAT(−1) space \((X, \rho)\) and a finitely generated group \(\Gamma\) of isometries of \(X\) acting properly, we describe two properties in the form of estimates on the distribution of orbits. These properties will be assumed in Section 4 for the action of fundamental groups of complete connected manifolds of strictly negative sectional curvature on their metric universal covers.

The properties are as follows. There exist positive reals \(n_0 = n_0, C\), such that for all \(n \in \mathbb{N}\), \(x \in X\),

\[ \# \{ g \in \Gamma \mid gx \in B(x, n + n_0) \setminus B(x, n) \} \in \left( \frac{1}{C}, C \right) \cdot e^{n \cdot v_{\Gamma}}, \]  

(C1)

and for any \(x, y_0 \in \bar{M}, L > 0\), there is \(n_1 = n_1(x, \rho(x, y_0), L)\) such that for all \(n \in \mathbb{N}, n \geq n_1\),

\[ \# \{ g \in \Gamma \mid gx \in T(x, B(y_0, L)) \cap (B(x, n + n_0) \setminus B(x, n)) \} \]

\[ \in \left( \frac{1}{C}, C \right) \cdot \mu_x(S(x, y_0, L) \cdot e^{n \cdot v_{\Gamma}}, \]  

(C2)

where \(B(x, n)\) is a ball of radius \(n\) and centre \(x\) in \(X\), and the notation \(E \cdot x\) for a set \(E \subset (0, \infty)\) and a number \(x \in (0, \infty)\), means the set \(\{e \cdot x \mid e \in E\}\). The second inequality says that the density of orbit points in the intersection of an annulus with a cone is proportional to the measure of the shadow of the cone. It is well-known that these conditions hold for example in the case of a complete connected manifold \(M\) of strictly negative sectional curvature, for the action of \(\Gamma = \pi_1(M)\) on \(\bar{M}\) when it is geometrically finite or more generally, when the quotient by \(\Gamma\) of the unit tangent bundle of \(\bar{M}\) admits a finite Bowen–Margulis measure (see [39, Theorem 4.1.1 and Corollary 2]). When \(\pi_1(M)\) acts geometrically on \(M\), the topological entropy of the geodesic flow in \(M, v_M\), is the critical exponent \(v_{\pi_1(M)}\) (see [31]).

### 3 DIOPHANTINE APPROXIMATION

In this section, we describe a fine Diophantine approximation theory in a general setting, which we use later to study geodesic approximation in manifolds with negative curvature.
3.1 Generalised Jarník–Besicovitch

Let $Q$ be a countable set and fix $(A, d, \mu)$ a compact metric measure space with a Borel $D$-regular measure $\mu$ for $D > 0$. Let $F : Q \to P(A)$ be a subset-valued map. Then the set

$$E_F = \left\{ \xi \in A \mid \exists \text{ infinitely many } x \in Q \text{ such that } \xi \in F(x) \right\}$$

will be called $F$-approximable. In the classical situation $Q$ is a subset of a group acting on the space $A$ by homeomorphisms and the sets $F(x)$ are balls. In our applications however, they will be more complicated sets (see Section 4).

We will prove a general Jarník–Besicovitch theorem. Let us first recall what we mean by a ball. By a ball $B$ in $X$, we mean a set $B \neq \emptyset$ and a choice of $x \in B$, such that $B = \{y \in X : d(x, y) \leq r, \text{ for some } r > 0\} =: B(x, r)$. Given a ball $B$, we will write centre($B$), for the chosen point $x$ and set

$$\text{rad}(B) := \inf \{r > 0 : B = B(\text{centre}(B), r)\}.$$

The notation $\lambda B$ for $\lambda > 0$, means the set $B(\text{centre}(B), \lambda \cdot \text{rad}(B))$.

**Dirichlet functions.** Let

$$\text{Dir} : Q \to P(A)$$

be a function such that $x \in Q, \text{Dir}(x) \neq A$ is a ball and,

1. $\lim_{y \to x} \text{rad} (\text{Dir}(y)) \to 0$ (along every distinct sequence in $Q$),
2. $\mu(A \setminus E_{\text{Dir}}) = 0$,
3. $\text{rad}(\text{Dir}(x)) \leq 1$ for each $x \in Q$.

We will call the function $\text{Dir}$ satisfying (1)-(3) a Dirichlet function and the balls $\text{Dir}(x)$, Dirichlet balls in analogy with the classical theory of Diophantine approximation.

**Jarník–Besicovitch functions.** Let

$$F : Q \to P(A)$$

be a function such that $\emptyset \neq F(x) \subset \text{Dir}(x)$ are open sets in $A$, for a given Dirichlet function $\text{Dir}$. We will refer to $F$ as a Jarník–Besicovitch function for $\text{Dir}$. We will show that the Hausdorff dimension of $E_F$ can be obtained in terms of the numbers introduced below which depend only on functions $\text{Dir}, F$ and the exponent of regularity $D$ of $\mu$.

**Lower approximation.** Let $C \geq 1, a_1 > 0$ and $0 < \lambda < D$ be given. For $x \in Q$, let $0 \leq \alpha_x < \infty$ be a number such that there is a finite collection of mutually disjoint balls $\{B_i\}_{i=1}^{n_x}$ satisfying

1. $B_i \subset F(x)$ for each $i$,
2. $\frac{1}{C} \cdot \text{rad}(\text{Dir}(x))^{\alpha_x} \leq \text{rad}(B_i) \leq C \cdot \text{rad}(\text{Dir}(x))^{\alpha_x}$, for each $i$,
3. $\frac{1}{C} \cdot \mu(F(x)) \leq \sum_i \mu(B_i) \leq C \cdot \mu(F(x))$, and,
4. for any $0 < R < e \cdot \text{rad}(\text{Dir}(x))^{1-\alpha_x}$, and any $\xi_i$ such that $B_i = B(\xi_i, \text{rad}(\text{Dir}(x))^{\alpha_x})$ for some $1 \leq i \leq n_x$, the cardinality of any disjoint collection of balls from $\{B_i\}_{i=1}^{n_x}$ contained in $B(\xi_i, R \cdot \text{rad}(\text{Dir}(x))^{\alpha_x})$ belongs to $\left(\frac{1}{\alpha_i} \cdot R^1, a_1 \cdot R^1\right)$. 

Given a sequence \( \{\alpha_x\}_{x \in \mathbb{Q}} \) with the above properties, for each \( x \in \mathbb{Q} \), set

\[
\beta_x := D \cdot \left( \alpha_x - \frac{\log(\mu(F(x)))}{\log(\mu(Dir(x)))} \right).
\]

A sequence of pairs \( \{\alpha_x, \beta_x\}_{x \in \mathbb{Q}} \) with the above properties will be called a lower \( F \)-approximation.

Note that for each \( x \in \mathbb{Q}, 0 \leq \beta_x < \infty \), and as \( \mu \) is a finite \( D \)-regular measure, there exists an \( 0 < \alpha_x < \infty \), by standard covering arguments, such that the requirements (1)–(3) are satisfied. Indeed, to check this, note that the sets

\[
(F(x))_\epsilon := \{ y \in F(x) \mid \text{dist}(y, A \setminus F(x)) > \epsilon \}
\]

converge in the measure \( \mu \) to \( F(x) \) and are precompact in \( F(x) \), which means they may be covered by finitely many balls of fixed radius which are contained in \( F(x) \). The requirement (4) is special and satisfied in examples of interest to us.

**Upper approximation.** Let \( C \geq 1 \) be given. Let \( \{\alpha'_x, \beta'_x\}_{x \in \mathbb{Q}} \) be a sequence of pairs of numbers such that for any \( x \in \mathbb{Q} \), there is a finite collection of mutually disjoint balls \( \{B_i\}_{i=1}^{n'_x} \) satisfying

1. \( F(x) \subset \bigcup_{i} B_i \),
2. \( \frac{1}{C} \cdot \text{rad}(Dir(x))^{\alpha_x} \leq \text{rad}(B_i) \leq C \cdot \text{rad}(Dir(x))^{\alpha_x} \), for each \( i \) and
3. \( n'_x \leq C \cdot \text{rad}(Dir(x))^{-\beta'_x} \).

Such a sequence always exists by precompactness of the sets \( F(x) \).

The setup above is motivated by the classical Jarník–Besicovitch theorem in Diophantine approximation. In this example, the metric space is the interval [0,1] with the Euclidean metric, the countable subset \( \mathbb{Q} \) comprises the rational numbers in [0,1] and the Dirichlet balls are intervals of the form \( (p/q - 1/q^2, p/q + 1/q^2) \). The sets \( F(p/q) \) here are just the intervals \( (p/q - 1/q^\tau, p/q + 1/q^\tau) \), where \( \tau > 2 \). Note that in this case \( \alpha_{p/q} = \tau / 2 \), for all \( p/q \in \mathbb{Q} \), and for any sequence \( \{p_i/q_i\}_{i} \in \mathbb{Q}, \lim_{i \to \infty} \beta_{p_i/q_i} = 0 \). The set \( E_F \) is the limsup set of ‘\( \tau \)-well-approximable’ real numbers.

Now set for a lower \( F \)-approximation \( \{\alpha_x, \beta_x\}_{x \in \mathbb{Q}} \),

\[
\overline{\alpha}_F := \lim \sup_x \alpha_x, \quad \underline{\beta}_F := \lim \inf_x \beta_x,
\]

and for an upper \( F \)-approximation \( \{\alpha'_x, \beta'_x\}_{x \in \mathbb{Q}} \),

\[
\underline{\alpha}_F = \lim \inf_x \alpha'_x, \quad \overline{\beta}_F = \lim \sup_x \beta'_x.
\]

Let us next state the main tool we use in proving our main result in this section. It is a general mass transference principle in euclidean spaces proved in [38]. The same approach is shown to work in the more general setting of Ahlfors-regular metric spaces in [27].

For a metric space \((A, d, \mu)\), with Borel measure \( \mu \), and any \( t > 0 \) and Borel subset \( U \subset A \), define

\[
I_t(U) = \int_U \int_U \frac{1}{d(x, y)^t} \, d\mu(y) d\mu(x).
\]
Theorem 3.1 (Mass Transference Principle). Let \((A, d, \mu)\) be a compact Ahlfors-regular metric space. Let \(\{B(\xi_i, r_i)\}_{i \in \mathbb{N}}\) be a sequence of balls such that \(r_i \to 0\) and \(\mu(A \setminus \limsup_i B(x_i, r_i)) = 0\). For each \(i \in \mathbb{N}\), let \(U_i \subset B(x_i, r_i)\) be an open set. Then for

\[
s := \sup \left\{ t \geq 0 \mid \sup_i \frac{I_t(U_i) \cdot \mu(B(x_i, r_i))}{\mu(U_i)^2} < \infty \right\},
\]

and \(E := \limsup_i U_i\), it holds that \(\dim_H(E) \geq s\). Moreover, for any \(0 < t < s\), there is a constant \(C_t > 0\) such that for any ball \(B \subset A\),

\[
\mathcal{H}_{\infty}^t(E \cap B) \geq C_t \cdot \text{diam}(B)^t.
\]

Remark 3.2. In the references cited for Theorem 3.1, it is actually shown that when the hypothesis is satisfied for a limsup set \(E\) of a collection \(\{U_i\}_{i \in \mathbb{N}}\) as above, then

\[
\inf \left\{ \sum_i \text{diam}(Q_i)^t \mid Q_i \in \mathcal{D}, E \cap Q \subset \bigcup_i Q_i \right\} \geq \text{diam}(Q)^t,
\]

for every \(Q \in \mathcal{D}\), where \(\mathcal{D}\) is a suitable dyadic decomposition. We remark that exactly the same argument (e.g. by the computation proving Lemma 2.1 in [38]) works for a set \(Q \in \mathcal{D}\) replaced with any ball \(B \in A\) and coverings of \(E \cap B\) by balls as we need, giving the content lower bound.

We are now ready to state the main result of this section.

Theorem 3.3 (Generalised Jarník–Besicovitch). Let \(Q\) be a countable set. Let \((A, d, \mu)\) be a compact \((a, D)\)-Ahlfors regular metric measure space. Let a Dirichlet function \(\text{Dir}\) and a Jarník–Besicovitch function \(F\) be given.

- (Lower bound). Let \(\{\alpha_x, \beta_x\}_{x \in Q}\) be a lower \(F\)-approximation. Assume

\[
\beta_F > 0, \, \alpha_F < \infty, \, \text{and} \, 0 < \lambda < \left(\frac{\beta_F + D}{\alpha_F}\right).
\]

Then, for \(\alpha' > \alpha_F, \beta' < \beta_F\), and \(t' := \frac{\beta' + D}{\alpha'}\), there is a constant \(c = c(a, a_1, \lambda_F, D, d', t') > 0\) such that we have

\[
\mathcal{H}_{\infty}^{t'}(E_F \cap B) \geq c \cdot \text{diam}(B)^{t'},
\]

for any ball \(B \subset A\). In particular,

\[
\dim_H(E_F) \geq \left(\frac{\beta_F + D}{\alpha_F}\right).
\]

- (Upper bound). Let \(\{\alpha'_x, \beta'_x\}_{x \in Q}\) be an upper \(F\)-approximation. Then

\[
\dim_H(E_F) \leq \frac{\beta_F + D}{\alpha_F}.
\]

Proof (Dimension lower bound). We prove the Hausdorff content lower bound for \(E_F\) in the claim from which the claimed dimension lower bound follows.
Let \((\alpha_x, \beta_x)\) be a lower \(F\)-approximation. Then there exists a (finite) collection of distinct points \(\{\xi_i\}_i \subset F(x)\), \(d(\xi_i, \xi_j) > \operatorname{rad}(\text{Dir}(x))^{\alpha_x}\) such that

\[
\bigcup_i B(\xi_i, \operatorname{rad}(\text{Dir}(x))^{\alpha_x}) \subset F(x),
\]

and it holds that,

\[
\mu(F(x)) \approx_C \sum_i \mu(B(\xi_i, \operatorname{rad}(\text{Dir}(x))^{\alpha_x})).
\]

Write \(n_x := \#\{\xi_i\}_i\), the cardinality of the set of centres in the above covering of \(F(x)\). Let \(\tilde{\beta}_x\), be such that

\[
n_x = \operatorname{rad}(\text{Dir}(x))^{-\tilde{\beta}_x}.
\]

Then an elementary computation reveals that \(\beta_F = \liminf_x \beta_x = \liminf_x \tilde{\beta}_x\). Write

\[
I_x = \{B(\xi_i, \operatorname{rad}(\text{Dir}(x))^{\alpha_x})\}_{i=1}^{n_x},
\]

for the disjoint collection above.

Now let \(x \in Q\), and \(i \in \{1, \ldots, n_x\}\). Set

\[
k_x := (1 - \alpha_x) \cdot \log(\operatorname{rad}(\text{Dir}(x))),
\]

and for \(0 \leq k \leq k_x\), set

\[
A^x_i(k) := B(\xi_i, e^{k+1} \cdot \operatorname{rad}(\text{Dir}(x))^{\alpha_x}) \setminus B(\xi_i, e^{k} \cdot \operatorname{rad}(\text{Dir}(x))^{\alpha_x}),
\]

and

\[
I_x(i, k) := \left\{ B(\xi_j, \operatorname{rad}(\text{Dir}(x))^{\alpha_x}) \in I_x \mid B(\xi_j, \operatorname{rad}(\text{Dir}(x))^{\alpha_x}) \cap A^x_i(k) \neq \emptyset \right\}.
\]

Then note that by property (4) of lower approximation of \(F\),

\[
\#I_{\delta}(i, k) \leq a_1 \cdot e^{(k+2)\lambda_F}.
\]

Set

\[
U_x = \bigcup_{I_x} B \subset F(x) \quad \text{and} \quad B_i := B(\xi_i, \operatorname{rad}(\text{Dir}(x))^{\alpha_x}) \in I_x.
\]

Then we may write

\[
I_l(U_x) = \sum_{B_i, B_j \in I_x} \int_{B_i} \int_{B_j} \frac{1}{d(\xi, \eta)^l} \, d\mu(\eta) \, d\mu(\xi)
\]

\[
\approx \sum_{B_i \in I_x} \sum_{k=1}^{k_x} \sum_{B_j \in I_x(i, k)} \int_{B_i} \int_{B_j} \frac{1}{d(\xi, \eta)^l} \, d\mu(\eta) \, d\mu(\xi).
\]
Next set $B'_i := B(\xi_i, e^2 \cdot \text{rad}(\text{Dir}(x))^\alpha_x)$ and for $\xi \in B'_i$, define for $1 \leq l \leq l_x$ where $l_x := \alpha_x \cdot \log(2 \cdot \text{rad}(\text{Dir}(x))) - 1$,

$$C_l(\xi) := B(\xi, e^{-l}) \setminus B(\xi, e^{-(l+1)}).$$

We compute then for $B_i \in I_x$ and $\lambda_F < t < D$,

$$\sum_{B_j \in I_x(i,0)} \int_{B_j} \int_{B_i} \frac{1}{d(\xi, \eta)^t} d\mu(\eta) d\mu(\xi) \leq \int_{B'_i} \int_{B'_i} \frac{1}{d(\xi, \eta)^t} d\mu(\eta) d\mu(\xi) \leq a \cdot e^t \cdot \int_{B'_i} \left( \sum_{l=1}^{\infty} e^{-l} \cdot e^{-l} \cdot D \right) d\mu(\xi) \leq \text{rad}(B_i)^{2D - t}. \tag{4}$$

Next we compute using (2) for $\lambda_F < t < D$,

$$\sum_{k=1}^{k_x} \sum_{B_j \in I_x(k,0)} \int_{B_j} \int_{B_i} \frac{1}{d(\xi, \eta)^t} d\mu(\eta) d\mu(\xi) \approx a \cdot a_1 \cdot e^{2\lambda_F} \cdot \sum_{k=1}^{k_x} e^{k \cdot \text{rad}(B_i)^{-t}} \cdot e^{k \lambda_F} \cdot \text{rad}(B_i)^D d\mu(x) \approx a^2 \cdot a_1 \cdot e^{2\lambda_F} \cdot \sum_{k=1}^{\infty} e^{-k(1 - \lambda_F)} \cdot \text{rad}(B_i)^D \approx C(t) \cdot \text{rad}(B_i)^{2D - t}, \tag{5}$$

where $C(t) = a^2 \cdot a_1 \cdot e^{2\lambda_F} \cdot \sum_{k=1}^{\infty} e^{-k(1 - \lambda_F)}$. Therefore, for $\lambda_F < t < D$, by (3), (4), (5),

$$I_t(U_x) \approx \sum_{i \in I_x} \text{rad}(B_i)^{2D - t}$$

Finally we note,

$$I_t(U_x) \cdot \mu(\text{Dir}(x)) \approx \frac{\text{rad}(\text{Dir}(x))^{-\beta_x} \cdot \text{rad}(\text{Dir}(x))^\alpha_x \cdot (2D - t)}{\text{rad}(\text{Dir}(x))^{-2\beta_x} \cdot \text{rad}(\alpha_x \cdot \text{Dir}(x)^{2D}} \approx \text{rad}(\text{Dir}(x))^{D + \beta_x - \alpha_x t}.$$
from which it follows by Theorem 3.1, that for any \( \beta' < \beta_F, \alpha' > \alpha_F \), such that \( 0 < t' := \frac{D + \beta'}{\alpha'} < \frac{D + \beta}{\alpha F} \), and 

\[ E'_F := \limsup \{ U_x \mid x \in Q \}, \]

we have 

\[ \mathcal{H}_\infty (E'_F \cap Q) \geq \text{diam}(Q)' \].

The required lower bound follows by noting that \( E'_F \subset E_F \).

**Dimension upper bound.** Let \( \{\alpha'_x, \beta'_x\}_{x \in Q} \) be an upper \( F \)-approximation. For \( k \in \mathbb{N} \), write

\[ \alpha'_k = \inf \{ \alpha'_x \mid \text{rad}(\text{Dir}(x)) \leq 1/2^k \}, \]

\[ \beta'_k = \sup \{ \beta'_x \mid \text{rad}(\text{Dir}(x)) \leq 1/2^k \}. \]

For any \( \alpha'' < \alpha_F, \beta'' > \beta_F \), choose \( k_0 \) large enough so that for all \( k \geq k_0 \), \( \alpha'_k > \alpha'' \) and \( \beta'_k < \beta'' \). Now fix \( k \geq k_0 \). It suffices to show that given \( 0 < \delta < 1/2^k \), for all \( d'' > \frac{D + \beta''}{\alpha''} \),

\[ \mathcal{H}_\delta (E_F) = O(\delta \frac{\alpha''d'' - D - \beta''}{\alpha''}). \]

Then for \( x \in Q \), write 

\[ I_x := \{ B_i = B(\xi_i, \text{rad}(\text{Dir}(x))^{\alpha'_x}) \}_{i=1}^{n'_x}, \]

where 

\[ F(x) \subset \bigcup_i B(\xi_i, \text{rad}(\text{Dir}(x))^{\alpha'_x}) \text{ and } n'_x \leq \text{rad}(\text{Dir}(x))^{-\beta'_x}. \]

Let \( 0 < \delta < 1/2^{k_0} \) be fixed now and let \( i_0 \) be the smallest positive integer such that \( 1/2^{i_0} \leq \delta \). Next, given \( \xi \in E_F \), choose \( x^{(i_0)}_\xi \) such that \( \text{Dir}(x^{(i_0)}_\xi) \) has maximal diameter constrained to the conditions \( \xi \in F(x^{(i_0)}_\xi) \) and \( \text{rad}(\text{Dir}(x^{(i_0)}_\xi)) \leq 1/2^{i_0} \). Then define

\[ S_i = \left\{ \xi \in E_F \mid \frac{1}{2^{i+1}} < \text{rad}(\text{Dir}(x^{(i_0)}_\xi)) \leq \frac{1}{2^i} \right\}. \]

Note that 

\[ E_F = \bigcup_{i=i_0}^{\infty} S_i. \]

Apply the 5\( r \)-covering theorem to the collection \( \{\text{Dir}(x^{(i_0)}_\xi)\}_{\xi \in S_i} \), to obtain a countable disjoint subcollection of balls \( S_i \) such that 

\[ S_i \subset \bigcup_{\xi \in S_i} F(x^{(i_0)}_\xi) \subset \bigcup_{5\bar{B} \in S_i} B_5. \]
Apply the covering theorem again to
\[
\bigcup_{\xi \in S_i} F(x_{(i_0)}^\xi) \subset \bigcup_{\xi \in S_i} \bigcup_{B \in I_{(i_0)}^\xi} B,
\]
to obtain an at most countable collection of points \(\xi_l \in S_i\) such that
\[
S_i \subset \bigcup_{\xi \in S_i} F(x_{(i_0)}^\xi) \subset \bigcup_{\xi_l \in S_i} \bigcup_{B \in I_{(i_0)}^\xi} 5B.
\]

Note further that a volume comparison argument provides that for each \(\xi_l\), the set
\[
\mathcal{N}_l = \left\{ B \in S_i \mid 5B \cap \left( \bigcup_{B \in I_{(i_1)}^\xi} 5B \right) \neq \emptyset \right\}
\]
satisfies
\[
1 \leq \# \mathcal{N}_l \leq C,
\]
where \(C\) is an absolute constant depending only on regularity of \(\mu\).

Then
\[
H_{\frac{1}{2^0}}^{d''}(S_i) \leq \sum_{l} \sum_{B \in I_{(i_0)}^\xi} \frac{1}{2^{i\alpha''d''}}
\]
\[
\leq \sum_{l} 2^{i(\beta'' - \alpha''d'')} \left( \text{as } i > k_0 \right)
\]
\[
\leq \frac{1}{C} \cdot \sum_{S_i} 2^{i(\beta'' - \alpha''d'')} \quad \text{(by the estimate on } \# \mathcal{N}_l)\]
\[
\leq \frac{C'}{C} \cdot 2^{i(\beta'' - \alpha''d'' + D)},
\]
where the last inequality followed from \(\# S_i \leq C' \cdot 2^{i-D}\), which uses the regularity of \(\mu\) and disjointedness of the elements of \(S_i\). Thus,
\[
H_{\frac{1}{2^0}}^{d''}(E_F) \leq \sum_{i=i_0}^{\infty} 2^{i(\beta'' - \alpha''d'' + D)}.
\]

The claim follows. \(\square\)
3.2 Large intersections

Fix a proper \((a,D)\)-regular metric space \(A\) with a dyadic decomposition \(D\) with data \(C, n_0, \{\mathcal{W}_n\}_{n \geq n_0}\).

Let \(s > 0\) be given. We define first the class

\[
G_s := \left\{ E \subset A \mid E \text{ is } G_\delta \text{ and } \mathcal{H}_\infty^s (E \cap Q) \geq c_E \cdot \text{diam}(Q)^s \right\}
\]

for some \(c_E > 0\), for all \(Q \in D\).

In this section, we prove a version of Falconer's large intersection property from [16] for \(G_s\) and provide geometric applications in subsequent sections.

**Definition 3.4 (Parent).** Let \(F \subset A\) be such that

\[
\text{diam}(F) \leq \min\{\text{diam}(Q) : Q \in \mathcal{W}_{n_0}\} =: d_0.
\]

By the parent of \(F\), we mean the collection

\[
\Psi(F) := \{Q \in \mathcal{W}_n \mid Q \cap F \neq \emptyset\},
\]

where \(n\) is the maximal such integer that \(\text{diam}(Q) \geq \text{diam}(F)\) for all \(Q \in \mathcal{W}_n\).

**Theorem 3.5.** If \(E \in G_s\) for some \(0 < s < D\), then we have for all \(t \in [0, s)\) and \(U \subset A\) proper, open with \(\text{diam}(U) > 0\), that

\[
\mathcal{H}_\infty^t (E \cap U) \geq \frac{\mathcal{H}_\infty^s (U)}{\text{diam}(U)^s-t}.
\]

**Proof.** Note that to estimate the left-hand side of inequality (6), it suffices to consider coverings by closed sets. Let \(I = \{E_i\}\) be a covering of \(E \cap U\) by closed sets. For \(t \in [0, s)\), we wish to bound from below the sum \(\sum_I \text{diam}(E_i)^t\). We may assume without loss of generality that \(\text{diam}(E_i) \leq \text{diam}(U)\).

Set \(r = \alpha \cdot \text{diam}(U)\) and

\[
I_\alpha := \{E_i \in I : \text{diam}(E_i) \geq r\},
\]

where \(0 < \alpha = \alpha(Q, s, t) < \min\{1, \frac{d_0}{4 \text{diam}(U)^t}\}\) will be specified later.

Let \(\mathcal{W}_\alpha\) be an \(\hat{a}\)-Whitney decomposition (see Definition 2.9) of the open set

\[
U_\alpha := U \setminus \bigcup_{I_\alpha} E_i,
\]

for some \(\hat{a} > 1\) (note that this exists in the case \(I_\alpha = \emptyset\) because \(U \neq A\)).

For every \(Q \in \mathcal{W}_\alpha\), for which there exists \(E' \in I \setminus I_\alpha\), such that the union of sets in the parent of \(B(E', \frac{1}{2} |E'|)\) contains \(Q\), we replace \(Q\) by \(\Psi(B(E', \frac{1}{2} |E'|))\) in the collection \(\mathcal{W}_\alpha\), and call the new collection \(\mathcal{W}_\alpha\). Recall that by definition of parent and by Lemma 2.8, for example,

\[
\bigcup_{\Psi(B(E', \frac{1}{2} |E'|))} \text{diam}(Q)^s \leq c^s \cdot \text{diam}(E)^s,
\]

for a constant \(c > 0\).
Note also that $I_\alpha \cup W_\alpha$ is a cover for $U$. Also note that if $Q \in W_\alpha \cap W'_\alpha$, then for $E' \in I \setminus I_\alpha$ such that $E' \cap Q \neq \emptyset$, we have
\[
\text{diam}(Q) \geq \frac{1}{2} \text{diam}(E').
\]
Moreover, by a property of Whitney decompositions, if $Q, Q' \in W_\alpha \cap W'_\alpha$ are such that $Q' \cap E' \neq \emptyset \neq Q \cap E'$ for some $E' \in I \setminus I_\alpha$, then
\[
\hat{a} \cdot \text{diam}(Q') \leq \min\{\text{dist}(Q', A \setminus U_\alpha), \text{diam}(U)\}
\leq \text{dist}(Q, Q') + \min\{\text{dist}(Q, A \setminus U_\alpha), \text{diam}(U)\}
\leq \text{diam}(Q) + \text{diam}(Q') + \text{diam}(E') + \min\{\text{dist}(Q, A \setminus U_\alpha), \text{diam}(U)\}
\leq \left(\frac{3}{2} + \hat{a}^2\right) \cdot \text{diam}(Q) + \text{diam}(Q').
\]
Thus,
\[
\text{diam}(Q') \leq \frac{3/2 + \hat{a}^2}{\hat{a} - 1} \cdot \text{diam}(Q).
\]
Then by Lemma 2.8, there exists a constant $M$ such that
\[
\#\left\{ Q \in W_\alpha \cap W'_\alpha : Q \cap E' \neq \emptyset \right\} \leq M.
\]
We get
\[
\sum_I \text{diam}(E_i)^t = \sum_{I_\alpha} \text{diam}(E_i)^t + \sum_{I \setminus I_\alpha} \text{diam}(E_i)^t
\geq \text{diam}(U)^{t-s} \sum_{I_\alpha} \text{diam}(E_i)^s + (\alpha \text{diam}(U))^{t-s} \sum_{I \setminus I_\alpha} \text{diam}(E_i)^s
\geq \text{diam}(U)^{t-s} \sum_{I_\alpha} \text{diam}(E_i)^s
\]
\[
+ \frac{1}{2} \cdot \left( \frac{1}{c^s} (\alpha \cdot \text{diam}(U))^{t-s} \sum_{\Psi(B(E_i^{1/2}, |E_i|) \subset W_\alpha} \sum_{\Psi(B(E_i^{1/2}, |E_i|))} \text{diam}(Q)^s \right)
\]
\[
+ \frac{1}{2} \cdot \left( \frac{1}{M} (\alpha \cdot \text{diam}(U))^{t-s} \sum_{Q \in W_\alpha \cap W'_\alpha} \sum_{E_i \cap Q \neq \emptyset} \text{diam}(E_i)^s \right)
\geq \text{diam}(U)^{t-s} \sum_{I_\alpha} \text{diam}(E_i)^s + c' \cdot (\alpha \cdot \text{diam}(U))^{t-s} \sum_{Q \in W_\alpha} \text{diam}(Q)^s
\geq \text{diam}(U)^{t-s} \sum_{I_\alpha \cup W'_\alpha} \text{diam}(Q)^s
\geq \frac{H^s_{\infty}(U)}{\text{diam}(U)^{s-t}}.
\]
for $\alpha > 0$ small enough, where $c' = c_E \cdot \min\left\{ \frac{1}{2c}, \frac{1}{2M} \right\}$. For the third inequality above, one observes that for $Q \in W_\alpha \cap W'_\alpha$,

$$Q \cap E \subset \bigcup \{ E_i \in I \setminus I_\alpha \mid E_i \cap Q \neq \emptyset \},$$

and as $E \in G_s$ then

$$\sum_{E_i \in I \setminus I_\alpha \atop E_i \cap Q \neq \emptyset} \text{diam}(E_i)^s \gtrsim \text{diam}(Q)^s.$$  

Lemma 3.6. For $Q \in D$, $H^s_\infty (Q) \approx \text{diam}(Q)^s$, for $s \in [0, D)$. The comparability constants depend only on $s$, the constants $C, n_0$ associated to $D$ and $a, D$ associated to the Ahlfors regularity of $\mu$.

Proof. Let $\{E_i\}$ be a countable covering of $Q$ where $\text{diam}(E_i) \leq \min\{\text{diam}(Q), d_0\}$ for each $i$. Then the claim is verified as follows

$$\sum_l \text{diam}(E_l)^s \geq \frac{1}{c^s} \cdot \sum_l \sum_{Q_j \in \mathcal{P}(E_l)} \text{diam}(Q_j)^s$$

$$= \frac{1}{c^s} \cdot \sum_l \sum_{Q_j \in \mathcal{P}(E_l)} \sum_j \text{diam}(Q_j)^D \cdot \text{diam}(Q_j)^{s-D}$$

$$\geq \frac{1}{c'^s} \cdot \sum_l \sum_{Q_j \in \mathcal{P}(E_l)} \sum_j \mu(Q_j) \cdot \text{diam}(Q_j)^{s-D} \geq \frac{1}{c'^s} \cdot \mu(Q) \cdot \text{diam}(Q)^{s-D}$$

$$\geq \frac{1}{c''s} \cdot \text{diam}(Q)^s,$$

where $c, c', c''$ are suitable constants depending only on the constants associated to $D$ and the Ahlfors regularity constants. \qed

Remark 3.7. Note that Theorem 3.5 and Lemma 3.6 imply that $G_s \subset G_t$ for $0 \leq t < s$.

Definition 3.8. We define for each $t \geq 0$,

$$\hat{H}^t_\infty (F) := \limsup_{s \downarrow t} H^s_\infty (F),$$

for every $F \subset A$.

Note that $\hat{H}^t_\infty (F) > 0$ implies that $\text{dim}_{H}(F) \geq t$.

Lemma 3.9. Given $s \in (0, D)$ and $E \in G_s$, for all $t \in [0, s)$ we have for any proper open set $U$, that

$$\hat{H}^t_\infty (E \cap U) = \hat{H}^t_\infty (U).$$
Proof. We may assume that diam$(U) > 0$. It is clear that $\hat{H}_\infty^t(E \cap U) \leq \hat{H}_\infty^t(U)$. For the other inequality we note that for any $\varepsilon > 0$, taking $s > t$ small enough, and any $s > s' > t$,

$$\hat{H}_\infty^t(E \cap U) \geq \hat{H}_\infty^{s'}(E \cap U) - \varepsilon \geq \frac{\mathcal{H}_\infty^s(U)}{\text{diam}(U)^{t-s'}} - \varepsilon,$$

where the third inequality is Theorem 3.5. The claim follows by taking the limit of suprema as $s \downarrow t$.  \hfill $\square$

**Definition 3.10** (Metrically dense). Let $t \geq 0$. A set $F \subset A$ is called $t$-metrically dense if for each proper open set $U \subset A$,

$$\hat{H}_\infty^t(F \cap U) = \hat{H}_\infty^t(U).$$

Below we define a version of the increasing sets property suitable to our situation. For details, see for example, Rogers [40] or Howroyd [25].

**Definition 3.11** (Increasing sets property). An outer measure $m$ satisfies the increasing sets property if for any collection $\{F_i\}_i$ of nested increasing sets such that $\bigcup_i F_i$ is bounded, we have that

$$m\left(\bigcup_i F_i\right) = \lim_i m(F_i).$$

**Remark 3.12.** In proper metric spaces, the measure $\mathcal{H}_\infty^s$, for $s > 0$ satisfies the aforementioned version of the increasing sets property. For this, we refer to Howroyd [25, Corollary 8.2, p. 38], where it is shown that for any $s > 0$ and $0 < \delta_1 < \delta_2 < \infty$, $\mathcal{H}_\infty^s\left(\bigcup_i F_i\right) \leq \lim_i \mathcal{H}_\delta_1^s(F_i)$. Thus, taking $\delta_1 > \text{diam}(\bigcup_i F_i)$, we have

$$\lim_i \mathcal{H}_\infty^s(F_i) \leq \mathcal{H}_\infty^s\left(\bigcup_i F_i\right) = \mathcal{H}_\delta_2^s\left(\bigcup_i F_i\right) \leq \lim_i \mathcal{H}_\delta_1^s(F_i) = \lim_i \mathcal{H}_\infty^s(F_i).$$

**Lemma 3.13.** For all $t \in (0, D)$ the following holds. Let $\{F_i\}_{i\in\mathbb{N}}$ be a collection of $t$-metrically dense $G_\delta$ subsets of $A$. Let $U$ be a proper, bounded open set. Then

$$\mathcal{H}_\infty^t\left(\bigcap_i F_i \cap U\right) \geq \hat{H}_\infty^t(U).$$

It follows that for $Q \in D$,

$$\mathcal{H}_\infty^t\left(\bigcap_i F_i \cap Q\right) \geq \mathcal{H}_\infty^t(Q),$$

and for all $U \neq \emptyset$ proper, open bounded,

$$\hat{H}_\infty^t\left(\bigcap_i F_i \cap U\right) = \hat{H}_\infty^t(U).$$
Proof. The proof follows Falconer’s argument in [16, Lemma 4]. We sketch the proof and indicate the required modifications. First assume that \( \{F_i\}_{i \in \mathbb{N}} \) is a sequence of decreasing open sets. Fix \( \varepsilon > 0 \) small. Pick \( s > t \) such that \( \hat{H}_\infty^t(U) - \varepsilon < \hat{H}_\infty^s(U) \). Set \( U_0 := U \). Then by the increasing sets property for \( H_\infty^s \), which we checked above, arguing as in [16], there exists a collection of numbers \( \varepsilon_i \) such that

\[
U_i := \{x \in F_i \cap U_{i-1} : d(x, A \setminus (F_i \cap U_{i-1})) > \varepsilon_i\},
\]

for all \( i \geq 1 \) satisfy

\[
\hat{H}_\infty^s(U_i) > \hat{H}_\infty^s(U) - \varepsilon.
\]

Observe that \( \overline{U}_i \subseteq F_i \cap U \). Let \( \{E_j\}_j \) be a covering of \( \bigcap_i \overline{U}_i \), and we may assume that \( \text{diam}(E_j) \leq \text{diam}(U) \). Then \( \{B(E_j, \text{diam}(E_j/2))\}_j \) is an open covering of \( \bigcap_i \overline{U}_i \). There exists \( k \) such that \( \overline{U}_k \subseteq \bigcup_j B(E_j, \text{diam}(E_j/2)) \). Thus,

\[
\sum_j \text{diam}(E_j)^s \geq \text{diam}(U)^s \sum_j \text{diam}(E_j)^s \geq \hat{H}_\infty^s(U_k) > \hat{H}_\infty^t(U) - 2\varepsilon,
\]

for some \( s > t \). The claim follows by taking infimum over all such coverings of \( \bigcap_i \overline{U}_i \) and letting \( \varepsilon \) go to zero.

The second inequality follows from Lemma 3.6. The third equality follows from Lemma 3.9 and the previous inequalities in the claim. The general case follows as argued in [16, Lemma 4].

We have thus proved the following theorem.

**Theorem 3.14** (Large intersection property). Let \((A, d, \mu)\) be a proper, \( D \)-Ahlfors regular metric space with a dyadic decomposition \( D \). Let \( 0 < s < D \). Let \( F^s \) be a collection of \( G_\delta \) sets. The following are equivalent.

1. For each \( E \in F^s \), each \( 0 \leq t < s \) and \( Q \in D \),

\[
\hat{H}_\infty^t(E \cap Q) \geq c_{t, E} \cdot \hat{H}_\infty^t(Q),
\]

for some \( c_{t, E} > 0 \).

2. For each countable collection \( \{E_i\}_i \subseteq F^s \), each \( 0 < t < s \) and \( U \) a bounded, proper and open set,

\[
\hat{H}_\infty^t\left(\bigcap_i E_i \cap U\right) = \hat{H}_\infty^t(U).
\]

The theorem follows from Lemmas 3.9 and 3.13. Note that the second statement implies in particular that the dense set \( \bigcap_i E_i \) has Hausdorff dimension at least \( s \).

**Remark 3.15.** We remark the following.

1. It is clear that if \( E \in G^s \), then \( \dim_H(E) \geq s \).
2. It also holds that for each \( E \in G^s \), each \( 0 \leq t < s \) and \( U \subseteq A \) bounded, proper and open,

\[
\hat{H}_\infty^t(E \cap U) \geq c'_{t, E} \hat{H}_\infty^t(U),
\]
for some \( c'_{t,E} > 0 \). To see this one compares a given covering of \( E \cap U \) with the Whitney decomposition of \( U \) as in Theorem 3.5. We do not present details as we do not require this later.

### 3.3 A Borel–Cantelli lemma

We have the following version of the classical Borel–Cantelli lemma tailored for our applications. We will assume for the Jarník–Besicovitch sets \( F(x) \), some conditions on their structure, size and mutual separation in this section. The Dirichlet function will also be required to satisfy stronger versions of some of the properties assumed so far. We now describe the properties that will govern the ‘quasi-independence’ phenomena in the Borel–Cantelli lemma.

Let \((A, d, \mu)\) be a compact metric space with an \((a, D)\)-Ahlfors regular probability measure \( \mu \). Consider the following properties for a Dirichlet function \( \text{Dir} : Q \to \mathcal{P}(A) \) for constants \( b, b' \geq 1 \).

1. **(D1) (Growth)** \( \frac{1}{b} \cdot e^{n-D} \leq \# \{ x \in Q \mid \frac{1}{b'} \cdot e^{-n} \leq \text{rad}(\text{Dir}(x)) \leq b' \cdot e^{-n} \} \leq b \cdot e^{n-D} \),
2. **(D2) (Bounded overlap)** given a point \( \xi \in A \), at most \( b \) of the balls corresponding to \( \{ x \in Q \mid \frac{1}{b'} \cdot e^{-n} \leq \text{rad}(\text{Dir}(x)) \leq b' \cdot e^{-n} \} \) contain it, and
3. **(D3) (Equidistribution)** \( \frac{1}{b} \leq \sum \mu(\text{Dir}(x)) \mid x \in Q, \frac{1}{b'} \cdot e^{-n} \leq \text{rad}(\text{Dir}(x)) \leq b' \cdot e^{-n} \} \leq b \).

And consider the following properties for a Jarník–Besicovitch function \( F : Q \to \mathcal{P}(A) \). With the notation

\[ i_x := -\log(\text{rad}(\text{Dir}(x))) \],

constant \( c \geq 1 \) and positive \( L \)-Lipschitz functions \( n, \varphi : [0, \infty) \to \mathbb{R} \), for \( L > 0 \),

1. **(F1) \( F(x) = \bigcup_{B \in I_x} B \subset \text{Dir}(x) \), and**
2. **(F2) \# I_x \leq c \cdot e^{n(i_x)} \**
3. **(F3) \# I_x \geq \frac{1}{c} \cdot e^{n(i_x)} \**
4. **(F4) \frac{1}{c} \cdot e^{-(i_x + \varphi(i_x))} \leq \text{rad}(B) \leq c \cdot e^{-(i_x + \varphi(i_x))}, \text{ for all } B \in I_x \**
5. **(F5) and if } x \neq y, F(x) \cap F(y) \neq \emptyset, \text{ then } \min\{i_x + \varphi(i_x), i_y + \varphi(i_y)\} - c \leq \max\{i_x, i_y\}, \text{ whenever } \min\{i_x, i_y\} \geq c. \**

We write properties (D1)–(D3) when referring to all the properties (D1), (D2), (D3) together, and similarly (F1)–(F5) for (F1), (F2), (F3), (F4), (F5).

**Lemma 3.16** (Borel–Cantelli). Let \( Q \) be a countable set and \((A, d, \mu)\) be a proper \((a, D)\)-regular probability space, with functions \( F \) and \( \text{Dir} \).

1. Assume that there exist \( b, b', L \geq 1 \) and functions \( n, \varphi \) for which (D1)–(D3) and (F1), (F2), (F4) hold. Then the set \( E_F \) has measure zero if

\[
\int_1^\infty e^{-(\varphi(t)D-n(t))} dt
\]

converges.
(2) Assume that there exist \(b, b', L, c \geq 1\) and functions \(n, \varphi\) for which (D1)–(D3) and (F1)–(F5) hold. Then the set \(E_F\) has positive measure if

\[
\int_1^\infty e^{-(\varphi(t)D-n(t))} dt
\]

diverges.

Proof. For \(k \in \mathbb{N}, k > \log b'\) write

\[
A_k := \bigcup_{x: i_x \in [k-\log b', k+\log b')} F(x) = \bigcup_{x: i_x \in [k-\log b', k+\log b')} \bigcup_{B \in I_x} B.
\]

Then by (F1)

\[
E_F = \limsup_i A_i.
\]

It is clear using (D1)–(D3) and the fact that \(n, \varphi\) are Lipschitz in (F2), (F4) that convergence of the integral implies that \(E_F\) has zero measure.

For the second claim, we show that a suitable subsequence of \(A_i\)'s are quasi-independent. Let \(i, j \in \mathbb{N}\) be such that \(c + L \cdot \log b' \leq i < j, A_i \cap A_j \neq \emptyset\). Then by (F5), \((i + \varphi(i)) - (c - 2L \log b' - 2) < j\). So, given a ball \(B\) defining \(A_i\), namely a ball ‘\(B'\) appearing in (8), the number of balls defining \(A_j\), intersecting \(B\) is bounded above by (a constant multiple of)

\[
e^{n(j)} \cdot e^{-(i+\varphi(i))D} \cdot e^{jD}.
\]

Indeed, by (F4), the ball \(B\) has measure \(e^{-(i+\varphi(i))D}\) up to constant multiples. The cardinality of \(x \in Q\), such that \(F(x) \subseteq \text{Dir}(x)\) intersect \(B\) and \(i_x \in (j - \log(b'), j + \log(b'))\) is by (D2) and (F1), bounded by a constant multiple of \(e^{-(i+\varphi(i))D} \cdot e^{jD}\), and by (F2), the number of balls defining \(A_j\) that intersect \(B\) is bounded as claimed. Thus, by the discussion above and (F3),

\[
\mu(A_i \cap A_j) \leq (e^{n(j)} \cdot e^{-(i+\varphi(i))D} \cdot e^{jD}) \times (e^{-(j+\varphi(j))D}) \times (e^{n(i)} \cdot e^{iD}) \leq \mu(A_i) \cdot \mu(A_j),
\]

and the unstated constants in above inequalities depend only on \(b', b, c\) and \(L\). It follows now by a standard generalisation of the Borel–Cantelli lemma applied to the collection \(\{A_i\}_{i \geq c + L \log b'}\) that \(E_F\) has positive measure when the integral diverges.

Note that a function \(\text{Dir}\) satisfying the hypothesis of Lemma 3.16 is a Dirichlet function (recall definition from Subsection 3.1).

### 3.4 Summary

We summarise the dimension results of this section in the following theorem.

**Theorem 3.17.** Let \((A, d, \mu)\) be a proper \((a, D)\)-regular probability space, for constants \(a, D > 0\) and \(Q\) be a countable set.
Assume given a function \( \text{Dir} : Q \rightarrow \mathcal{P}(A) \), where \( \text{Dir}(x) \) are balls with \( \text{rad}(\text{Dir}(x)) \) accumulating at zero, and the Dirichlet type statement

\[
\mu \left( A \setminus \limsup_{x \in Q} \text{Dir}(x) \right) = 0.
\]

(JB) Given any other function \( F : Q \rightarrow \mathcal{P}(A) \) such that \( F(x) \subset \text{Dir}(x) \) are open for all \( x \in Q \), which has a lower \( F \)-approximation, a Hausdorff-dimension lower bound \( d_F \) for the Jarník–Besicovitch set

\[
E_F = \limsup_{x \in Q} F(x)
\]

can be obtained (in terms of the asymptotic behaviour of \( F \)), as well as the density condition

\[
\mathcal{H}^d_{\infty}(E_F \cap U) = \mathcal{H}^d_{\infty}(U)
\]

for any \( 0 \leq d < d_F \) and open set \( U \subset A \), whenever \( d_F > 0 \), where \( \mathcal{H}^d_{\infty} \) is a suitable modification of the \( d \)-Hausdorff content.

(F) If \( F = \{ F_i \}_i \) is a (countable) sequence of functions such that \( F_i(x) \subset \text{Dir}(x) \) are open for all \( x \in X \), then

\[
\dim_H \left( \bigcap_i E_{F_i} \right) = \inf d_{F_i} := d_F.
\]

The Liouville set \( E_{F} := \bigcap_i E_{F_i} \) also satisfies

\[
\mathcal{H}^d_{\infty}(E_F \cap U) = \mathcal{H}^d_{\infty}(U),
\]

for any \( 0 \leq d < d_F \) if \( d_F > 0 \). If \( d_F = 0 \), but \( d_{F_i} > 0 \) for all \( i \), the same holds for \( d = 0 \) (Baire-category theorem).

### 4 SHRINKING TARGETS AND SPIRAL TRAPS

This section is devoted to establishing fine logarithm law type results for geodesics in negative curvature, especially Hausdorff dimension results for ‘spiraling’ phenomena.

**Definition 4.1.** Let \( X \) be a \( \text{CAT}(-1) \) space and \( \overline{X} \) its compactification with Gromov’s boundary. Let \( \Gamma \) be a discrete group of isometries of \( X \) acting non-elementarily and properly on \( X \). Recall that the limit set of an orbit of \( \Gamma \) in \( \overline{X} \) is denoted \( \Lambda_\Gamma \). We will say the \( \Gamma \) action on \( X \) has property (\( i \)), \( i \in \{1, 2, 3, 4\} \), if it satisfies, respectively,

1. the action of \( \Gamma \) on \( \Lambda_\Gamma \) equipped with the corresponding Patterson–Sullivan density is ergodic;
2. the two orbit counting estimates (C1) and (C2) from Subsection 2.5 hold;
3. the convex hull of the limit set \( \Lambda_\Gamma \) in \( \overline{X} \) admits a cocompact, proper group action by a discrete subgroup of isometries (not necessarily \( \Gamma \)), with critical exponent \( \nu_\Gamma \);
4. \( \Lambda_\Gamma = \partial X \).
By the convex hull of a set $E \subset \partial X$, we mean the convex hull in $X$ of the union of all geodesic lines with their end points in $E$.

### 4.1 The unit tangent space at a point and the visual boundary

We denote by $SM$ the unit tangent bundle of $M$ and by $SM_{x_0}$ the fiber over $x_0$. The unit tangent sphere $SM_{x_0}$ is identified with the visual boundary. Then it is equipped with the visual metric $d_{x_0}$ (see Subsection 2.5) and with the Patterson–Sullivan measure $\mu = \mu_{x_0}$ (normalised to be a probability measure) corresponding to the action of $\pi_1(M)$ and a chosen lift $\tilde{x}_0$ of $x_0$ in $\tilde{M}$. This metric measure structure is better suited for studying the asymptotic properties of geodesics in variable curvature.

We start with a lemma about the distribution of cosets. Recall the definition of trails of sets from Subsection 2.5.

**Lemma 4.2** (Counting cosets). Let $(X, \tilde{\rho})$ be a CAT($-1$) manifold and $\Gamma$ be a discrete group of isometries acting properly, freely and non-elementarily on $X$. Assume property (2) for the action of $\Gamma$ on $X$. Let $\Gamma_N$ be a subgroup of $\Gamma$ acting on $X$ with property (2) and $N_0$ the convex-hull of its limit set $\Lambda_{\Gamma_N}$ in $X$. Suppose $\nu_{\Gamma_N} < \nu_{\Gamma}$. Let $\tilde{x}_0 \in X$ and let $\{N_i\}_{i \in \mathbb{N} \cup \{0\}}$ be the $\Gamma$-orbit of $N_0$. Let $z_i$ be the nearest point projections from $\tilde{x}_0$ to $N_i$. Let $g_i \in \Gamma$ be such that $N_i = g_i N_0$ and

$$\tilde{\rho}(g_i \tilde{x}_0, z_i) = \min \{\tilde{\rho}(g' \tilde{x}_0, z_i) \mid g' \in g_i \Gamma_N\}.$$

Write

$$\mathcal{N} = \{(z_i, g_i)\}_{i \in \mathbb{N}} \subset X \times \Gamma.$$

Then we have the following:

1. For each $y_0 \in \Gamma$, $L > 0$, there exists $k_0 = k_0(y_0, L, X) > 1, R = R(\Gamma_N, \Gamma, X) > 0$, such that for all $k \in \mathbb{N}$, $k \geq k_0$,

$$\# \left\{(z_i, g_i) \mid z_i \in T(\tilde{x}_0, B(y_0, L)) \bigcap (B(\tilde{x}_0, k + R) \setminus B(\tilde{x}_0, k))\right\} \approx \mu_{\tilde{x}_0}(S(\tilde{x}_0, y_0, L)) \cdot e^{\nu_{\Gamma} \cdot k \cdot R}.$$

Moreover, for all $k \geq 1$,

$$\# \{(z_i, g_i) \mid z_i \in B(\tilde{x}_0, (k + 1)R) \setminus B(\tilde{x}_0, kR)\} \approx e^{\nu_{\Gamma} \cdot k \cdot R}.$$

2. Moreover, if $\Gamma$ acts cocompactly, and $\Gamma_N$ convex-cocompactly, there exists $C = C(X) > 0$ such that for all $z \in B(\tilde{x}_0, (k + 1)R) \setminus B(\tilde{x}_0, kR)$, there exists $(z_i, g_i) \in \mathcal{N}$ such that

$$\tilde{\rho}(g_i \tilde{x}_0, z) \leq C.$$
The second assertion in the lemma is a direct geometric consequence of the cocompactness of the action and convexity of $N$.

**Proof.** Towards the proof of part (2), write for each $k \in \mathbb{N}$,

$$\mathcal{N}_k = \{(z_i, g_i) \in \mathcal{N} \mid z_i \in B(\tilde{x}_0, kR)\}.$$  

Now fix $k \in \mathbb{N}$ and fix $g \in \Gamma$, such that

$$g\tilde{x}_0 \in B(\tilde{x}_0, (k + 1)R) \setminus B(\tilde{x}_0, (k + 1/2)R).$$

There is $i \in \mathbb{N}$, such that $g \in g_i\Gamma_N$, where $(z_j, g_j) \in \mathcal{N}_k$. Let $z_g$ be the nearest point projection of $g\tilde{x}_0$ on $g\tilde{N}_0$. Note that in this case $\Gamma\tilde{x}_0$ is an $R_0$-net, for some $R_0 > 0$ (that is $\tilde{M}$ is in an $R_0$ neighbourhood of $\Gamma\tilde{x}_0$).

Let $g' \in \Gamma$ be any other such isometry, that is $g' \in g_j\Gamma_N$,

$$g'\tilde{x}_0 \in B(\tilde{x}_0, (k + 1)R) \setminus B(\tilde{x}_0, (k + 1/2)R)$$

and $(z_j, g_j) \in \mathcal{N}_k$. Define $z_{g'}$ similarly to $z_g$, that is, it is the nearest point projection of $g'\tilde{x}_0$ on $g'\tilde{N}_0$.

Write

$$\xi_g = [z_i, z_g](\infty), \quad \xi_{g'} = [z_j, z_{g'}](\infty).$$

Consider the geodesic triangle $[\tilde{x}_0, \xi_g, \xi_{g'}]$. Note that it follows from the CAT($-1$) inequality that for any $\varepsilon > 0$, a large enough $R > 0$ may be chosen, so that

$$\text{dist}_{\tilde{\rho}}([\tilde{x}_0, \xi_g](kR + R/4), [z_i, z_g]) \leq \varepsilon,$$

and

$$\text{dist}_{\tilde{\rho}}([\tilde{x}_0, \xi_{g'}](kR + R/4), [z_j, z_{g'}]) \leq \varepsilon.$$

Let $w_g$ and $w_{g'}$ be the nearest point projections of $[\tilde{x}_0, \xi_g](kR + R/4)$ on $[z_i, z_g]$ and of $[\tilde{x}_0, \xi_{g'}](kR + R/4)$ on $[z_j, z_{g'}]$, respectively. We apply this observation, along with the fact that there exists a constant $0 < C' = C'(\Gamma_N, X)$, such that

$$\tilde{\rho}(w_g, w_{g'}) \geq C',$$

to deduce that

$$\tilde{\rho}([\tilde{x}_0, \xi_g](kR + R/4), [\tilde{x}_0, \xi_{g'}](kR + R/4)) \geq C'/2,$$

and thus $R$ can be chosen large enough that

$$\tilde{\rho}([\tilde{x}_0, \xi_g](kR + R/2), [\tilde{x}_0, \xi_{g'}](kR + R/2)) \geq 100 \cdot R_0.$$  

From this, we deduce that

$$\min\{\tilde{\rho}(z_g, z_{g'}), \tilde{\rho}(g\tilde{x}_0, g'\tilde{x}_0)\} \geq 50 \cdot R_0.$$  

We will now use the inequality above to obtain part (2) of the claim.
Let \( z \in B(\tilde{x}_0, (k + 1)R) \setminus B(\tilde{x}_0, (k + 1/2)R) \), there exists an isometry \( h \in \Gamma \), such that
\[
\tilde{\rho}(h\tilde{x}_0, z) \leq R_0.
\]

Let \( l \in \mathbb{N} \) be such that \( h \in g_l \Gamma_N \), and \((z_l, g_l) \in S\). If \((z_l, g_l) \notin \mathcal{N}_k\), part (2) of the claim is verified for \( z \). Consider the case \((z_l, g_l) \in \mathcal{N}_k\). Let \( z_h \) be the nearest point projection of \( h\tilde{x}_0 \) on \( hN_0 \). There exists \( C'' = C''(\Gamma_N, X) \) such that
\[
\tilde{\rho}(h\tilde{x}_0, z_h) \leq C''.
\]

Let \( u_h \in X \) be a point at distance \( 10 \cdot R_0 \) from \( hN_0 \) and \( z_h \) (picked from the boundary of a tubular neighbourhood for example). Let \( h' \in \Gamma \) be such that
\[
\tilde{\rho}(h'\tilde{x}_0, u_h) \leq R_0.
\]

Then
\[
\tilde{\rho}(h\tilde{x}_0, h'\tilde{x}_0) \leq 12 \cdot R_0,
\]
and thus
\[
\tilde{\rho}(z, h'\tilde{x}_0) \leq 15 \cdot R_0,
\]

and if \( m \in \mathbb{N} \) is such that \( h' \in g_m \Gamma_N \), where \((z_m, g_m) \in \mathcal{N}_k\), then \((z_m, g_m) \notin \mathcal{N}_k\). This proves part (2).

For the proof of part (1), first write
\[
\mathcal{F}_m = \left\{ g \in \Gamma \mid g\tilde{x}_0 \in T(\tilde{x}_0, B(y_0, L)) \cap (B(\tilde{x}_0, m + R) \setminus B(\tilde{x}_0, m)) \right\},
\]

and
\[
\hat{\mathcal{F}}_m = \left\{ g \in \Gamma \left\mid (z, g) \in \mathcal{N} \text{ for some } z \in M \text{ and,} \right. \right. \\
\left. \left. \left. \left. g\tilde{x}_0 \in T(\tilde{x}_0, B(y_0, L)) \cap (B(\tilde{x}_0, m)) \right. \right. \right. \right. \}
\]

Then note that by \((C2)\) there is \( R > 0 \), and \( k_0 = k_0(y_0, L) \), such that for \( m \in \mathbb{N}, m \geq k_0 \),
\[
c_1 \cdot \mu_{\tilde{x}_0}(S(\tilde{x}_0, y_0, L)) \cdot e^{v_{\Gamma} \cdot m} \leq \#\mathcal{F}_m \leq c_2 \cdot \mu_{\tilde{x}_0}(S(\tilde{x}_0, y_0, L)) \cdot e^{v_{\Gamma} \cdot m},
\]

for \( c_1, c_2 > 0 \) depending only on the action of \( \Gamma \) on \( X \).

Let \( k \in \mathbb{N}, k > k_0 + \alpha + 2 \), for \( \alpha \in \mathbb{N} \) to be fixed shortly. Let \( m \in \mathbb{N} \) such that \( k_0 < m \leq k \). For \( g \in \hat{\mathcal{F}}_m \), set
\[
I_g^k = \left\{ h \in \Gamma \mid h\tilde{x}_0 \in g \Gamma_N \tilde{x}_0, h\tilde{x}_0 \in T(\tilde{x}_0, B(y_0, L)) \cap (B(\tilde{x}_0, k + R) \setminus B(\tilde{x}_0, k)) \right\}.
\]

There is \( c_3 > 0 \) depending only on the action of \( \Gamma_N \) on \( X \) such that for \( g \in \hat{\mathcal{F}}_{m+1} \setminus \hat{\mathcal{F}}_m \),
\[
\#I_g^k \leq c_3 \cdot e^{(k-m) \cdot v_{\Gamma_N}}.
\]
Note that,

\[ \# F_k \leq \sum_{m=k_0+1}^{k-\alpha} \# \left( \bigcup \{ F_g^k : g \in \hat{F}_{m+1} \setminus \hat{F}_m \} \right) \]

\[ + \#(F_k \setminus \{ h \in \Gamma \mid h \in I_g^k, \text{ for some } g \in F_m \text{ with } m \leq k - \alpha \}), \]

\[ \leq \sum_{m=k_0+1}^{k-\alpha} \#(\hat{F}_{m+1} \setminus \hat{F}_m) \cdot c_3 \cdot e^{(k-m)\nu_N} \]

\[ + \#(F_k \setminus \{ h \in \Gamma \mid h \in I_g^k, \text{ for some } g \in F_m \text{ with } m \leq k - \alpha \}). \]

Then,

\[ c_1 \cdot \mu_{\bar{x}_0}(S(\bar{x}_0, y_0, L)) \cdot e^{k \nu}\Gamma \]

\[ \leq c_2 \cdot c_3 \cdot \mu_{\bar{x}_0}(S(\bar{x}_0, y_0, L)) \sum_{m=k_0+1}^{k-\alpha} e^{m \nu}\Gamma \cdot e^{(k-m)\nu_N} \]

\[ + \#(F_k \setminus \{ h \in \Gamma \mid h \in I_g^k, \text{ for some } g \in F_m \text{ with } m \leq k - \alpha \}). \]

Then for suitably large \( \alpha = \alpha(\Gamma, \Gamma_N, X) > 0 \) we have

\[ \#(F_k \setminus \{ h \in \Gamma \mid h \in I_g^k, \text{ for some } g \in F_m \text{ with } m \leq k - \alpha \}) \geq \frac{c_1}{2} \cdot \mu_{\bar{x}_0}(S(\bar{x}_0, y_0, L)) \cdot e^{k \nu}\Gamma. \]

This concludes the proof of the first claim of part (1). The second claim uses \((C1)\) and is based on the same argument. \( \square \)

### 4.2 Zero-one laws

We now consider spiraling of long geodesic pieces into fixed neighbourhoods of a fixed totally geodesic submanifold, a phenomenon we refer to as a spiral trap. The next theorem, namely Theorem 4.3 is due to Hersonsky and Paulin [21, Theorem 4.6]. We provide a different argument where we apply the results of Section 3. We will also use part of the argument later for Hausdorff dimension computations.

**Theorem 4.3.** Let \((M, \rho)\) be a complete, connected manifold of pinched negative sectional curvature \(-a^2 \leq k \leq -1, a > 1, \) with \((\bar{M}, \bar{\rho})\) its metric universal cover. Suppose that the covering action of \(\Gamma := \pi_1(M)\) on \(\bar{M}\) has properties (1) and (2). Let \(N\) be a compact, connected, convex submanifold such that \(\Gamma_N := \pi_1(N) \hookrightarrow \Gamma\) is injective, and \(\Gamma_N\) is a non-trivial subgroup of \(\Gamma\) such that \(\nu_{\Gamma_N} < \nu_\Gamma\). Let \(f\) be a positive Lipschitz function. Let \(x_0 \in \bar{M}\). Then for \(\epsilon > 0\) small enough, the set

\[ E_N^f = \left\{ v \in SM_{x_0} \left| \exists \text{ positive times } t_n \to \infty \text{ such that } \gamma_v(t_n, t_n + f(t_n)) \subset B(N, \epsilon) \right\} \]
has full (resp., zero) measure if the integral
\[
\int_{1}^{\infty} e^{-f(t)(v_\Gamma - v_\Gamma N)} dt
\]
diverges (resp., converges), where \( \gamma_v \) is the geodesic such that \( \gamma_v(0) = x_0, \gamma_v(0) = v \).

Proof (Convergence). Let \( \epsilon > 0 \) be small enough that the universal cover of \( B(N, \epsilon) \) is \( B(\tilde{N}, \epsilon) \). Let \( F_M \) be a fundamental domain for the action of \( \Gamma := \pi_1(M) \) on \( \tilde{M} \). Let \( \tilde{x}_0 \) and \( \tilde{N}_0 \) be the components of preimages of \( x_0 \) and \( N \) intersecting \( F_M \) non-trivially. In this proof and below we will use the abbreviations
\[
\rho_g := \rho(\tilde{x}_0, g\tilde{x}_0)
\]
and
\[
\tilde{N}_g := g\tilde{N}_0,
\]
where \( g \in \Gamma \).

Let \( z_0 \) be the nearest point projection from \( \tilde{x}_0 \) to the fundamental domain \( F_N \subset \tilde{N}_0 \) of \( N \) in \( F_M \). With a slight abuse of notation we let \( \Gamma/\Gamma_N \) denote a set of isometries \( g \in \Gamma \), one chosen from each coset of \( \Gamma_N \), such that
\[
\rho(\tilde{x}_0, g\tilde{z}_0) = \rho(\tilde{x}_0, g\Gamma_N\tilde{z}_0).
\]
So, \( g\tilde{z}_0 \) is a nearest point in the \( \Gamma \)-orbit of \( z_0 \) in \( \tilde{N}_g \) to \( \tilde{x}_0 \). We will keep this definition of \( \Gamma/\Gamma_N \) for the rest of this section.

Now let \( \tilde{\gamma}_v \) be the lift of \( \gamma_v \) with \( v \in E^f_N \) such that \( \tilde{\gamma}_v(0) = \tilde{x}_0 \). Let \( t_n \to \infty \) be a sequence of times and \( g_n' \in \Gamma \) be a sequence of isometries such that
\[
\tilde{\gamma}_v(t_n, t_n + f(t_n)) \subset B(g_n'\tilde{N}_0, \epsilon).
\]
Let \( g_n' \) be in the coset \( g_n \in \Gamma/\Gamma_N \). Let \( h_n' \in \Gamma_N \) be such that \( g_n h_n' \tilde{z}_0 \) is a nearest orbit point for \( \tilde{\gamma}_v(t_n) \). Then
\[
\rho(\tilde{\gamma}_v(t_n), g_n h_n' \tilde{z}_0) \leq c
\]
and as \( \epsilon > 0 \) is small, we also have that for some \( h_n \in \Gamma_N \),
\[
\rho(\tilde{\gamma}_v(t_n + f(t_n)), g_n \Gamma_N(\tilde{z}_0)) \leq \rho(\tilde{\gamma}_v(t_n + f(t_n)), g_n h_n \tilde{z}_0) \leq c
\]
(9)
(where \( c \) is a positive constant depending on \( a \) and the Lipschitz constant of \( f \)). Write \( \xi \) for the end point of the geodesic ray \( \tilde{\gamma}_v \). Let \( \xi_{g_n h_n} = [g_n z_0, g_n h_n z_0](\infty) \in \partial \tilde{N}_{g_n} \subset \partial \tilde{M} \), which lies in \( S_{\tilde{N}_{g_n}}(g_n z_0, g_n h_n z_0, R_N) \) for some \( R_N > 0 \). Then there is a constant \( c_2 \) depending on \( a \) such that (as the triangle \( [\tilde{x}_0, g_n z_0, \xi_{g_n h_n}] \) is \((-1)\)-thin and the triangle \( [\xi, \tilde{x}_0, \xi_{g_n h_n}] \) is \((-a^2)\)-fat)
\[
d_{\tilde{x}_0}(\xi, \xi_{g_n h_n}) \leq c_2 e^{-(t_n + f(t_n))},
\]
(10)
where \( S_{\tilde{N}_{g_n}}(\cdot, \cdot) \) denotes shadows in the boundary of the embedded space \( \tilde{N}_{g_n} \subset \tilde{M} \) of balls in \( \tilde{N}_{g_n} \).
Let \( z_n \) be a nearest point from \( \tilde{\gamma}_v(t_n + f(t_n)) \) to the geodesic ray \([g_n z_0, \xi] \). Then the CAT\((-1)\) inequality applied to the triangle \([x_0, \xi, g_n z_0] \) gives

\[
\tilde{\rho}(\tilde{\gamma}_v(t_n + f(t_n)), z_n) \leq c_1,
\]

where \( c_1 \) depends on \( a \) (which follows by noting from hyperbolicity that \( \text{dist}_{\tilde{\rho}}(g_n z_0, g_n^h h_n) \) is bounded above, by thinness of the triangle \([\xi, x_0, g_n^h h_n] \) and using the triangle inequality).

By (9) and (11), we get

\[
\tilde{\rho}(z_n, g_n h_n z_0) \leq c_3
\]

where \( c_3 \) is a positive constant depending on \( a \) and thus (as the triangle \([\xi, g_n z_0, g_n^h h_n] \) is \((-a^2)\)-fat)

\[
d_{g_n z_0}(\xi, g_n^h h_n) \leq c_4 e^{-\rho_{h_n}},
\]

where \( d_{g_n z_0} \) is the visual metric on \( \partial \tilde{M} \) from basepoint \( g_n z_0 \). Then from (12),

\[
d_{g_n z_0}(\xi, g_n^h h_n) \leq c_4 e^{-((\rho_{h_n} + (\rho_{h_n} - \rho_{k_n}))} \leq c_5 e^{-((\rho_{h_n} + f(\rho_{g_h})))},
\]

where \( c_5 \) is a constant depending only on \( a \) and the Lipschitz constant of \( f \).

Therefore, we have that

\[
E^f_N \subset \limsup \left\{ B_{g z_0}(\eta, c_5 e^{-((\rho_{h} + f(\rho_{g_h})))} \mid g \in \Gamma/\Gamma_N, h \in \Gamma_N, \eta \in S(g z_0, g h z_0, R_N) \right\}
\]

where \( B_{g z_0}(\cdot, \cdot) \) is used to denote a ball with visual metric \( d_{g z_0} \).

For \( g \in \Gamma/\Gamma_N \) applying the 5r-covering theorem to the collection

\[
\left\{ B_{g z_0}(\eta, c_5 e^{-((\rho_{h} + f(\rho_{g_h})))) \cap \partial \tilde{N}_g \mid \eta \in S(g z_0, g h z_0, R_N) \right\},
\]

we get a finite collection \( \{ B_{g z_0}(\eta_i, c_5 e^{-((\rho_{h} + f(\rho_{g_h})))} \) of disjoint balls such that concentric balls with five times the radius of the balls in the collection, cover the original collection. By Ahlfors regularity (in the metric space \( \partial \tilde{N}_g \), when equipped with the normalised Patterson–Sullivan measure with base point \( g z_0 \) corresponding to the action by \( \Gamma_N \)), we know that the number of balls in this subcollection lies between constant positive multiples of \( e^{f(\rho_{g_h}) \nu_{TN}} \). Call the set of centres \( J_{g h} \).

Note then that a finite number of balls centred at \( \eta_i \) of radius \( c_7 e^{-((\rho_{g_h} + f(\rho_{g_h})))} \) (in the metric \( d_{g z_0} \)) for large enough \( c_7 \) depending only on \( a \) and the Lipschitz constant of \( f \), with varying \( g \) and \( h \), cover \( E^f_N \) (see (10)), that is

\[
E^f_N \subset \limsup \left\{ B(\eta_i, c_7 e^{-((\rho_{g} + f(\rho_{g_h})))} \mid g \in \Gamma/\Gamma_N, \eta_i \in J_{g} \right\},
\]

where \( \eta_i \) are the centres (up to a constant \( e^{f(\rho_{g}) \nu_{TN}} \) many) obtained from the covering theorem.

Now it follows using the fact that \( f \) is Lipschitz, Lemmas 3.16 and 4.2, that if the integral in the statement of the claim converges, then \( E^f_N \) has measure zero. To check this, first set for \( g \in \Gamma/\Gamma_N \),

\[
F(g) := \bigcup_{\eta_i \in J_{g}} B(\eta_i, c_7 e^{-((\rho_{g} + f(\rho_{g_h})))} \text{ where } \# J_{g} \approx e^{f(\rho_{g}) \nu_{TN}}
\]

(14)
which satisfies properties (F1)–(F4) by construction. Next note that by the first part of Lemma 4.2 and the shadow lemma Lemma 2.3, (D1)–(D3) are satisfied for the Dirichlet function $\text{Dir}(g) = B(\xi_g, q \cdot e^{-\rho_g})$ for $g \in \Gamma / \Gamma_N$, where $\xi_g = [x_0, g\bar{x}_0]((\infty)$ and $q = q(\Gamma, \Gamma_N, M) > 0$ is chosen such that $F(g) \subset \text{Dir}(g)$. This is because the bijective map between $\Gamma / \Gamma_N$ and the set $\mathcal{N}$ in the notation of Lemma 4.2, which sends $g \in \Gamma / \Gamma_N$ to $(z', g') \in \mathcal{N}$ when $g$ and $g'$ are in the same coset of $\Gamma_N$, has the property that $\tilde{\rho}(g\bar{x}_0, g'\bar{x}_0) \leq C$, where $C > 0$ is a constant depending only on $M$.

**Divergence.** For $g \in \Gamma / \Gamma_N$ let $\xi_g \in S(\bar{x}_0, g\bar{x}_0, R) \cap \partial \tilde{N}_g$. Note that $R > 0$ may be chosen large enough so $\partial \tilde{N}_g \subset S(\bar{x}_0, g\bar{x}_0, R)$. There exists a positive number $c_8 > 0$ such that for $t > t_g := \rho_g + c_8$ we have

$$\bar{\rho}([\bar{x}_0, \xi_g](t), \tilde{N}_g) < \epsilon / 2. \quad (15)$$

Now if $\xi \in B(\xi_g, c_9 e^{-(\rho_g + f(\rho_g))})$, (for $c_9 > 0$ small enough depending on $a$ the Lipschitz constant of $f$ and $\epsilon$) then by (15)-thinness of $[\xi, \bar{x}_0, \xi_g]$ we have

$$\bar{\rho}([\bar{x}_0, \xi](t_g + f(t_g)), [\bar{x}_0, \xi_g](t_g + f(t_g))) < \epsilon / 2. \quad (16)$$

Therefore, we have from (15), (16) and the convexity of $\bar{\rho}$ that

$$\limsup \left\{ B_{\bar{x}_0}(\xi_g, c_{10} e^{-(\rho_g + f(\rho_g))}) \mid g \in \Gamma, \xi_g \in S(\bar{x}_0, g\bar{x}_0, R) \cap \partial \tilde{N}_g \right\} \subset E^f_N$$

where $c_{10} > 0$ depends on $a$, the Lipschitz constant of $f$ and $\epsilon$. By a 5$r$-covering argument with $g\bar{x}_0$ as the base point for the visual metric, we can find as before a disjoint collection of balls $\{B(\xi_g, c_{10} e^{-(\rho_g + f(\rho_g))})\}$, indexed by $I_g$ for varying $g$, which form a limsup set contained in $E^f_N$. Here $\text{Dir}$ is defined similarly as above and again

$$F(g) = \bigcup_{I_g} B(\xi_g, c_{10} e^{-(\rho_g + f(\rho_g))}) \text{ where } \# I_g \approx e^{f(\rho_g) \cdot d\Gamma_N}. \quad (17)$$

Lemma 3.16 will be applied to $E^f_N$, once (F5) is verified. Towards that end, suppose that two such balls used in defining $F$, corresponding to $g \in \Gamma / \Gamma_N$ and $g' \in \Gamma / \Gamma_N$, with $\rho_{g'} > \rho_g$, intersect. Let $\xi$ be a common point and $\xi_1, \xi_2$ be the points closest to $\xi$ in $\partial \tilde{N}_g$ and $\partial \tilde{N}_{g'}$, respectively. Again we apply the comparison inequalities from the curvature bounds to the triangles $[x_0, z_g, \xi_1], [x_0, z_g, \xi_2], [x_0, \xi_1, \xi]$ and $[x_0, \xi, \xi_2]$, where $z_g$ and $z_{g'}$ are nearest points from $x_0$ to $\tilde{N}_g$ and $\tilde{N}_{g'}$, respectively. We note that given $\epsilon' > 0$, if $\rho_g$ is large enough (depending only on $a, f$ and $\epsilon'$) there is $C_{\epsilon'} = C(\epsilon', a, f) > 0$ such that for $y \in [z_g, \xi_1]$ with

$$\rho_g + f(\rho_g) - 2 \cdot C_{\epsilon'} < \bar{\rho}(z_g, y) < \rho_g + f(\rho_g) - C_{\epsilon'},$$

we have $\dist_{\bar{\rho}}(y, [\bar{x}_0, \xi]) \leq \epsilon'$, and if $\rho_{g'} < \rho_g + f(\rho_g) - 3 \cdot C_{\epsilon'}$, then, with $y' \in [z_{g'}, \xi_2]$ such that

$$C_{\epsilon'} < \bar{\rho}(z_{g'}, y') < \rho_{g'} + f(\rho_{g'}) - C_{\epsilon'},$$

we have $\dist_{\bar{\rho}}(y', [\bar{x}_0, \xi]) \leq \epsilon'$. Thus, we obtain by convexity of $\bar{\rho}$ that there are $h, h' \in \Gamma_N$ such that $\bar{\rho}(g' h' F_N, gh F_N) < \epsilon'$. As $\epsilon'$ is arbitrary, this is a contradiction. We thus obtain that there is a constant $c_{11} > 0$ depending only on $a, f, N$ such that $\rho_{g'} > (\rho_g + f(\rho_g)) - c_{11}$, which leads to the property (F5) for $F$. Thus, $\mu(E^f_N) > 0$. An argument in the proof of [23, Theorem 5.1, p. 821] can be used to see that the measure of $E^f_N$ is in fact one when the integral diverges. \qed
Remark 4.4. In Theorem 4.3, we do not use the smooth structure of manifolds and the method works in the generality of $\text{CAT}(-1)$ spaces (see [21, Theorem 5.3]).

We now move on to a shrinking target problem for geodesics around totally geodesic submanifolds. This kind of theorem was first proved by Maucourant [34] where he proved a shrinking target theorem for geodesics approximating a point in a finite volume, not necessarily compact, hyperbolic manifold. In particular, Theorem 4.5 generalises it and a theorem of Aravinda, Hersonsky and Paulin (cf. Theorem A.3 in the appendix to [21]). Our methods also apply to cuspidal excursions, see Subsection 4.5.

**Theorem 4.5.** Let $(M, \rho)$ be a complete, connected manifold of dimension $n$ with pinched negative sectional curvature $-a^2 \leq k \leq -1$, for $a \geq 1$, and metric universal cover $(\tilde{M}, \tilde{\rho})$. Suppose that the covering action of $\Gamma := \pi_1(M)$ on $\tilde{M}$ satisfies properties (1), (2), (3) and (4). Let $N' \subset N$ be a bounded submanifold of positive Riemannian volume in $N$ of dimension $0 \leq s < n$, such that when $s \neq 0$, $N$ is a complete connected submanifold of positive codimension and the induced action of $\Gamma_N := \pi_1(N)$ on $\tilde{M}$ satisfies property (2) and the homomorphism $\Gamma_N \hookrightarrow \Gamma$ induced by inclusion is non-trivial. Let $f$ be a non-decreasing positive Lipschitz function. Let $x_0 \in M$ be fixed. Then the set

$$T^f_{N'} = \left\{ v \in SM_{x_0} \mid \exists \text{ positive times } t_n \to \infty \text{ such that } \gamma_v(t_n) \subset B(N', e^{-f(t_n)}) \right\}$$

has measure zero if the integral

$$\int_1^\infty e^{-f(t)(v_{\Gamma}/a-s)} dt$$

converges. It has full measure, if the integral

$$\int_1^\infty \frac{1}{f(t)} \cdot e^{-f(t)(v_{\Gamma}-s)} dt$$

diverges, where $\gamma_v$ is the geodesic such that $\gamma_v(0) = x_0$ and $\gamma_v'(0) = v$. If $s = n - 1$, then $T^f_{N'}$ has full measure.

**Proof** (Convergence). First consider the case $0 \leq s < n - 1$.

We carry notation from Theorem 4.3. Let $F_{N'} \subset F_N$ be such that $p(F_{N'}) = N'$ under the covering projection $p : \tilde{M} \to M$. Then there exists $0 < L < \infty$ such that

$$\text{diam}(F_{N'} \cap B(\tilde{x}_0, L)) \geq \min\{1, \text{diam}(N')\},$$

and

$$\text{vol}_{N_0} (F_{N'} \cap B(\tilde{x}_0, L)) \geq \min\{1, \text{vol}(N')\}.$$

For $g \in \Gamma$, consider the collection of balls $\{B(x, e^{-f(\rho_g)/5}) \mid x \in gF_{N'}\}$ and apply the $5r$-covering theorem to obtain a disjoint subcollection of balls $\{B(x^g_j, e^{-f(\rho_g)/5})\}_j$ such that

$$gF_{N'} \subset \bigcup_j B(x^g_j, e^{-f(\rho_g)}),$$
where the cardinality of the set of indices $j$ is (within constant multiples) of $e^{f(\rho)}$. Write

$$J_g = \{x_j^g\}_j$$

for the collection of centres of balls in the cover obtained above. Let $\xi^g = [\tilde{x}_0, g\tilde{x}_0](\infty)$ and $\xi_j^g = [\tilde{x}_0, x_j^g](\infty)$. Consider the functions

$$Dir(g) = B(\xi^g, c_1 \cdot e^{-\rho_g})$$

and

$$F(g) = \bigcup_{x_j^g \in J_g} B\left(\frac{\xi_j^g}{c_2}, c_2 e^{-\left(\frac{\rho_g + f(\rho_g)}{a}\right)}\right)$$

where $\#J_g \lesssim e^{f(\rho_g)}$, (18)

for $g \in \Gamma$ and where $c_2 > 0$ is specified below.

Let $v \in T^f_{N'}$. Then there exist times $t_n \to \infty$ and isometries $g_n \in \Gamma$ such that there exist lifts $\tilde{N}_{g_n} = g_n\tilde{N}_0$ and points $x^u_n \in \tilde{N}_{g_n}$, such that

$$\tilde{\gamma}_v(t_n) \in B(x^u_n, e^{-f(t_n)}).$$

Note that

$$\tilde{\gamma}_v(t_n) \in B(x^u_n, e^{-f(t_n)}) \subset B(x^g_{j_n}, C_1 e^{-f(\rho_{j_n})}),$$

where $C_1$ depends only on the Lipschitz constant of $f$. Let $\xi^v = \tilde{\gamma}_v(\infty)$. Then as the triangle $[\xi^v, \tilde{x}_0, \xi^g_{j_n}]$ is $(-a^2)$-fat, we have

$$d_{\partial M}(\xi^v, \xi^g_{j_n}) \leq c'_2 e^{-\left(\frac{\rho_{j_n} + f(\rho_{j_n})}{a}\right)},$$

where $c'_2$ is a constant depending only on $a$ and the Lipschitz constant of $f$.

So, taking $c_2 = 2c'_2$ in the definition of $F$,

$$T^f_{N'} \subset E_F.$$

Pick $c_1 = c'_1(\Gamma, N, M) > 0$ such that $F(g) \subset Dir(g)$ for all $g \in \Gamma$. Then the first part of the claim follows by Lemma 3.16 as before. The properties (D1)–(D3) for Dir follow from our assumption on the action of $\Gamma$ on $\tilde{M}$. Also, $F$ satisfies (F1), (F2), (F4) by construction. Note that for this we use the assumption of properties (3) and (4) for the action of $\Gamma$ on $\tilde{M}$, as we need to estimate the measure of arbitrary balls in the boundary up to multiplicative constants and we know that by properties (1) and (3), $\Lambda_{\gamma}$ is Ahlfors regular, which is now the whole boundary $\partial X$, by property (4).

**Divergence.** For the next part, we use terminology from Lemma 4.2. The components of the preimages of $N$ in $\tilde{M}$ are denoted $\{\tilde{N}_i\}_{i \in \mathbb{N} \cup \{0\}}$, where $\tilde{N}_0$ is the component nearest to $\tilde{x}_0$. Recall the collection $\mathcal{N}'$ from Lemma 4.2. Write

$$\Gamma_{\mathcal{N}'} := \{g \in \Gamma \mid (z, g) \in \mathcal{N}', \text{ for some } z \in \tilde{M}\}.$$
subcollection of balls \( \{ B(x_j^g, 5\kappa e^{-f(\rho_g)}) \}_j \) such that
\[
gF_{N'} \cap B(g\tilde{x}_0, L) \subset \bigcup_j B(x_j^g, 5\kappa e^{-f(\rho_g)}),
\]
where the cardinality of the set of indices \( j \) is again (within constant multiples depending also on \( \kappa \)) of \( e^{f(\rho_g)} \) (for \( \rho_g \) large enough). Write
\[
J_g = \{ x_j^g \}_j
\]
for the collection of centres of balls in the cover obtained above. We claim that a \( \kappa \geq 1 \) can be chosen such that the shadows of the balls \( B(x_i^g, e^{-f(\rho_g)}) \) are mutually disjoint for \( x_i^g \in J_g \).

Consider \((z, g) \in \mathcal{N} \). Towards the claim, we first observe that the manifold \( g\tilde{N}_0 \) is totally geodesic and
\[
\text{dist}(B(x_l^g, e^{-f(\rho_g)}), B(x_j^g, e^{-f(\rho_g)})) \geq 5\kappa \cdot e^{-f(\rho_g)},
\]
for \( x_l^g \neq x_j^g \). The fact that \( x_l \) and \( x_j \) are within a distance bounded by a constant multiple of \( L \) of the point closest to \( \tilde{x}_0 \) (as \( g \in \Gamma_{\mathcal{N}} \)) can be used to show that
\[
\max\{\text{dist}_\rho(x_l, [\tilde{x}_0, x_j]), \text{dist}_\rho(x_j, [\tilde{x}_0, x_l])\} \geq c(\kappa) \cdot e^{-f(\rho_g)},
\]
where \( c(\kappa) \) is a positive, increasing function of \( \kappa \). For this consider first the geodesic line containing \( x_l \) and \( x_j \), also denoted \([x_j, x_l]\). Let \( z_{jl} \) be the point on \([x_j, x_l]\), closest to \( \tilde{x}_0 \). There are essentially two cases to consider; \( x_j < z_{jl} < x_l \), (where a point to the left of the inequality comes before the point to the right along \([x_j, x_l]\)), and \( z_{jl} < x_j < x_l \).

Let us consider the case \( z_{jl} < x_j < x_l \). Consider the triangle \([\tilde{x}_0, x_l, y_l]\), where \( y_l \) is a point at distance \( \rho^{-a^2}(\tilde{x}_0, z_{jl}) \) along \([x_j, x_l]\) on the direction opposite to that of \( x_l \) and a comparison triangle \([\tilde{x}'_0, x'_l, y'_l]\) in \( \mathbb{H}^{-a^2} \). Note that
\[
\rho(z_l, z_{jl}) \leq L + \delta_M,
\]
where \( \delta_M \) is the constant for thinness of triangles in \( \tilde{M} \) and consequently,
\[
\rho(z_{jl}, x_l) \leq 2L + \delta_M \leq CL,
\]
for a constant \( C = C(L, \delta_M) > 0 \). Let \( z'_{jl} \) be the point corresponding to \( z_{jl} \) in \([\tilde{x}'_0, x'_l, y'_l]\). As (by CBB\((-a^2)\) inequality)
\[
\rho^{-a^2}(\tilde{x}'_0, z'_{jl}) \leq \min\{\rho^{-a^2}(\tilde{x}'_0, x'_l), \rho^{-a^2}(\tilde{x}'_0, y'_l)\},
\]
the point closest to \( \tilde{x}'_0 \) on the geodesic line in \( \mathbb{H}^{-a^2} \) joining \( y'_l \) to \( x'_l \) lies in the segment between them, let us call it \( w'_{jl} \). Note that
\[
\rho^{-a^2}(x'_l, w'_{jl}) \leq \rho^{-a^2}(x'_l, y'_l) \leq CL,
\]
where \( C = C(L, \delta_M) > 0 \) and that the geodesic segment from \( x'_0 \) to \( w'_{jl} \) meets the geodesic joining \( x'_l \) to \( y'_l \) perpendicularly. So, the interior angle at \( x'_l \) in the triangle \([\tilde{x}'_0, x'_l, y'_l]\) is bounded below, depending only on \( L \) (this angle monotonically tends to zero as \( x'_l \) tends to infinity). Consider next
the point \(x'_j\) in the comparison triangle, corresponding to \(x_j\), and the nearest point \(u'_j\) from \(x'_j\) to the side in the comparison triangle joining \(\tilde{x}'_0\) to \(x'_i\). By the angle lower bound at the vertex \(x'_i\), there exists a constant \(c = c(a, L)\), such that

\[
\rho_{-a^2}(x'_j, u'_j) \geq c(a, L) \cdot \rho_{-a^2}(x'_j, x'_i) = c(a, L) \cdot \rho_{-a^2}(x_j, x_i).
\]

Let \(v_j\) be the point on the geodesic joining \(\tilde{x}_0\) to \(x_i\), nearest to \(x_j\), and \(v'_j\), the corresponding point in \([\tilde{x}'_0, x'_i, y'_i]\). Then

\[
\rho_{-a^2}(v'_j, u'_j) \geq \rho_{-a^2}(x'_j, u'_j),
\]

and hence by the CBB\((-a^2)\) inequality again,

\[
\text{dist}_{\rho}(x_j, [\tilde{x}_0, x_i]) = \tilde{\rho}(x_j, v_j) \geq c(a, L) \cdot \tilde{\rho}(x_j, x_i) \geq 2c(a, L) \cdot \kappa \cdot e^{-f(\rho_g)}.
\]

The other case is proved arguing similarly, using the curvature lower bound, we omit the details. The claim follows by choosing \(\kappa\) large enough so that \(2c(a, L) \cdot \kappa \geq 1\).

Then for \(\xi_j^g := [\tilde{x}_0, x_j^g](\infty)\), we have that

\[
\limsup \left\{ B(\xi_j^g, c' e^{-\rho_g + f(\rho_g)}) \mid g \in \Gamma_N', x_j^g \in J_g \right\} \subset T_{f_N'}^f.
\]

Now it follows again from a modification of the argument in the divergence part in the proof of Lemma 3.16, that divergence of the second integral in the statement of the claim implies that \(T_{N'}^f\) has positive measure. Indeed, we have properties (D1)–(D3) satisfied by the function \(Dir\) defined similarly as before and (F1)–(F4) satisfied by \(F\), where

\[
F(g) = \bigcup_{x_j^g \in J_g} B(\xi_j^g, c' e^{-\rho_g + f(\rho_g)}) \quad \text{and} \quad \#J_g \approx e^{f(\rho_g) s}.
\]  

(19)

Now pick a sequence \(\{i_j\}\), such that \(i_{j+1} - i_j = 2 \cdot f(i_j) \cdot (v_{\Gamma} - s)\). Set

\[
A_{i_j} = \bigcup_{i_j - c_0 \leq \rho_g \leq i_j + c_0} F(g),
\]

where \(c_0 > 0\) is a constant suitably chosen so that

\[
\#\{g \mid m - c_0 \leq \rho_g \leq m + c_0\} \approx e^{m \cdot v_{\Gamma}},
\]

for all \(m \in \mathbb{N}\). Then note that a similar argument as in the proof of the divergence part of Lemma 3.16 using (D1)–(D3) and (F1)–(F4) implies that \(A_{i_j}\) are quasi-independent. Finally note that our assumptions on \(f\), namely positive and non-decreasing imply that divergence of the second integral implies that \(\sum_{j} \mu(A_{i_j}) = \infty\), so that \(\mu(T_{N'}^f) > 0\). An argument as mentioned in Theorem 4.3 then shows that the measure of \(T_{N'}^f\) is in fact one, when it is positive.

Finally, we consider the case that \(s = n - 1\). In this case the first integral does not converge, because

\[
v_{\Gamma} = v_M \leq a(n - 1),
\]
where $v_M$ is the critical exponent of a discrete group of isometries acting cocompactly on $M$. Indeed the first equality follows from properties (1) and (4). The second is due to results in [42] and [32] (see also [33]). The second integral may or may not diverge. Nevertheless, the measure is full as we explain next. By Lemmas 4.2 and 3.16 for a fixed $L > 0$ the limsup set of shadows of balls of radius $L$ centred at orbits of $\tilde{x}_0$ by $\Gamma_N$ has positive measure. This set is contained in $T^f_{N'}$ for any Lipschitz $f$ by hyperbolicity of $\tilde{M}$ and the fact that each $\tilde{N}_i$ separates $\tilde{M}$ (by the Cartan–Hadamard theorem, for example), for a suitable value of $L$. Thus, the measure of $T^f_{N'}$ is positive. This completes the proof.

Remark 4.6. We have the following remarks regarding Theorem 4.5.

1. The proof showed that for a function $f$ for which the integral diverges, the geodesic ray in almost every direction hits the target infinitely many times at an ‘angle’ close to $\pi/2$. In fact, in the divergence part of the proof, we could estimate the sizes of the shadows of only those neighbourhoods of the preimages of $N'$, which were hit by the geodesic rays from $\tilde{x}_0$ at such angles (the corresponding translates of the fundamental domains being the ones nearest to $\tilde{x}_0$). Estimating the shadows for this collection of neighbourhoods was sufficient for applying the Borel–Cantelli lemma, because there are sufficiently many such neighbourhoods: a consequence of the non-triviality of $\pi_1(N)$ and property (2) of the $\pi_1(M)$ and $\pi_1(N)$ actions, via Lemma 4.2.

2. As the target is hit by almost every ray, at angles close to $\pi/2$, the geodesic ray spends a time proportional to $e^{-f(t)}$ in the $e^{-f(t)}$ neighbourhood of the target, at a hitting time $t > 0$. This can be checked using the lower curvature bound, with arguments similar to ones used in the proof of Theorem 4.5. The shrinking target phenomenon is complementary to the spiral trap phenomenon (where the time spent is much larger than the thickness of the neighbourhood), in this regard.

3. We used a volume estimate for the target. Such an estimate only requires a local Ahlfors regular measure on $N'$: a measure $m_{N'}$, and $\epsilon_{N'} > 0$, such that for all $0 < \epsilon < \epsilon_{N'}$, and balls $B_\epsilon$ in $N$ of radius $\epsilon$, $m_{N'}(B_\epsilon) \approx \epsilon^{\dim(N')}$. So, the manifold structure is not essential for such phenomena.

### 4.3 Dimension estimates

We begin the section by discussing the flat torus where the spiral trap phenomenon, namely geodesics spending long stretches of time in small neighbourhoods of compact convex submanifolds, is absent.

**Example 4.7.** Let $f(t) := \tau t$, for some $\tau > 0$. Consider the flat $n$-torus. Let $\lambda$ be the closed geodesic which is the projection of the lines $\{ (k_1, ..., k_{n-1}, t) \mid t \in \mathbb{R}, k_i \in \mathbb{Z} \}$ and consider its $\epsilon$ neighbourhood for some $\epsilon > 0$. Then for any other geodesic in the torus, not parallel to $\lambda$ and lying in the $\epsilon$ neighbourhood, the spiral trap problem has no solutions for $\epsilon$ small enough. Indeed, for solutions to exist, $f$ has to be a bounded function.

The next example illustrates the contrast in the geodesic shrinking target problem for flat manifolds and negatively curved manifolds.
Example 4.8. Consider the following shrinking target problem in the 3-torus with the same function \( f \) and geodesic \( \lambda \) as above. Let \( x_0 \) be the image under the covering projection of the origin. A connected component of the preimage of a geodesic passing through \( x_0 \) will be of the form \( t \mapsto (t, \alpha t, \beta t) \) after normalisation (ignoring a set of Hausdorff dimension 1). If the above geodesic is a solution to the exponential shrinking target problem (that is with \( f(t) = \tau t \) for some \( \tau > 0 \)) then it can be seen that there exist distinct \((p_n, q_n) \in (\mathbb{Z} \setminus \{0\})^2\), with \( |q_n| \to \infty \), such that

\[
|\alpha - \frac{p_n}{q_n}| \leq c \cdot e^{-\tau |q_n|},
\]

for some constant \( c = c(\alpha) > 0 \) and all \( n \in \mathbb{N} \). By the Jarník–Besicovitch theorem, the possible values of \( \alpha \) are a set of dimension zero. Thus, the set of directions along which the geodesics are a solution to the exponential shrinking target problem is a set of Hausdorff dimension one (cf. Corollary 4.12; in the case of a hyperbolic 3-manifold, the set of directions satisfying the shrinking target problem for a closed geodesic has dimension two).

We next consider the negative curvature case.

Theorem 4.9. Let \( \tau \geq 0 \) be fixed. Let \((M, \rho)\) be a complete, connected manifold of pinched negative sectional curvature \( -a^2 \leq k \leq -1 \), \( a \geq 1 \), and \((\tilde{M}, \tilde{\rho})\) be its metric universal cover. Suppose that the covering action of \( \Gamma := \pi_1(M) \) on the convex hull of \( \Lambda_{\pi_1(M)} \) in \( \tilde{M} \) has properties (1), (2) and (3). Let \( N \) be a connected, compact, convex submanifold and assume \( \Gamma_N := \pi_1(N) \hookrightarrow \pi_1(M) \) is injective. Let \( x_0 \in M \). Then for \( \varepsilon > 0 \) small enough, the set

\[
E^\tau_N = \{ v \in SM_{x_0} | \exists \text{ positive times } t_n \to \infty \text{ such that } \gamma_v(t_n, t_n + \tau t_n) \subset B(N, \varepsilon) \}
\]

has Hausdorff dimension

\[
\dim_H(E^\tau_N) = \frac{\nu_\Gamma + \tau \cdot \nu_{\Gamma_N}}{1 + \tau},
\]

where \( \gamma_v \) is the geodesic such that \( \gamma_v(0) = x_0 \) and \( \gamma_v(0) = v \).

Proof. The proof follows from Theorem 3.3 and (the proof of) Theorem 4.3. Indeed, one can take the Dirichlet function \( \text{Dir} \), and for the function \( f(t) = \tau \cdot t \), the Jarník–Besicovitch function \( F \) as in Theorem 4.3. Consider with \( f(t) = \tau \cdot t \), the function \( F \) constructed in the divergence part of Theorem 4.3, see (17). Then set

\[
Q = \Gamma/\Gamma_N, \quad \alpha_g = 1 + \tau.
\]

Now by Lemma 4.2 and the assumption of property (2), \( \mu(\Lambda_{\Gamma} \setminus E_{\text{Dir}}) = 0 \). Also \( \Lambda_{\Gamma} \) is \( \nu_{\Gamma} \)-regular by assumptions (1) and (3). Moreover, recall that with

\[
\beta_g = \nu_{\Gamma} \cdot \left( \alpha_g - \frac{\log \mu(F(g))}{\log \mu(\text{Dir}(g))} \right),
\]

\( \{\alpha_g, \beta_g\}_{g \in \Gamma/\Gamma_N} \) satisfies properties (1)–(3) of a lower \( F \)-approximation for \( F \) defined using the function \( f(t) = \tau \cdot t \) as in the divergence part of Theorem 4.3. To verify requirement (4) for being a lower \( F \)-approximation, recall again from the divergence part of the proof of Theorem 4.3, that
for any \( g \in \Gamma \setminus \Gamma_N \), the balls \( B_i \) in the collection \( I_g \) have centres in \( \partial N \) and by compactness of \( N \), Patterson–Sullivan measures of \( \Gamma_N \) in \( \partial N \) are \( \nu_{\Gamma_N} \)-regular. Thus, \( \lambda = \nu_{\Gamma_N} \) works. Also note that

\[
\beta_F = \lim_{\rho \to \infty} \beta_{\rho} = \tau \cdot \nu_{\Gamma_N},
\]

and thus \( \nu_{\Gamma_N} < \frac{\nu_{\Gamma} + \beta_F}{\alpha_F} \). As \( E_F \subset E_N \), Theorem 3.3 provides the desired dimension lower bound.

Next note that the constant sequence \( \{ (1 + \tau, \tau \cdot \nu_{\Gamma_N}) \} \) is an upper \( F \)-approximation, for the \( F \) constructed in (14) in the convergence part of Theorem 4.3, with \( f(t) = \tau \cdot t \). Thus, as \( E_N \subset E_F \) we obtain the dimension upper bound as in the claim by Theorem 3.3, which is same as the lower bound and as claimed.

\[ \square \]

We now prove a Hausdorff dimension result for the shrinking target problem. As mentioned earlier, the theorem below was known in the case that \( M \) is a closed manifold and the ‘target’ is a point due to work of Hersonsky and Paulin [19], see also the work of Velani [46] for an analogous statement in the constant curvature case. We generalise the aforementioned results to accommodate more general manifolds \( M \) and more general targets \( N \).

**Theorem 4.10.** Let \( \tau \geq 0 \) be fixed. Let \((M, \rho)\) be a complete, connected manifold with pinched negative sectional curvature, \(-a^2 \leq k \leq -1\), \( a \geq 1 \), and dimension \( n \), with metric universal cover \((\tilde{M}, \tilde{\rho})\).

Suppose that the covering action of \( \Gamma := \pi_1(M) \) on \( \tilde{M} \) satisfies properties (1), (2), (3) and (4). Let \( N' \subset N \) be a bounded submanifold of positive Riemannian volume in \( N \) of dimension \( 0 \leq s < n \), such that when \( s \neq 0 \), \( N \) is a complete, connected submanifold of positive codimension and the induced action of \( \Gamma_N := \pi_1(N) \) on \( \tilde{M} \) satisfies property (2) and the homomorphism \( \Gamma_N \hookrightarrow \Gamma \) induced by inclusion is non-trivial. Then given \( x_0 \in M \), we have that the set

\[
T^{\tau}_{N'} = \left\{ \nu \in SM_{x_0} \mid \exists \text{ positive times } t_n \to \infty \text{ such that } \gamma_{\nu}(t_n) \subset B(N', e^{-\tau t_n}) \right\}
\]

has Hausdorff dimension

\[
\frac{\nu_{\Gamma} + \tau \cdot s}{1 + \tau} \leq \dim_H(T^{\tau}_{N'}) \leq \frac{\nu_{\Gamma} + \tau \cdot s}{1 + \tau/a},
\]

where \( \gamma_{\nu} \) is the geodesic such that \( \gamma_{\nu}(0) = x_0 \) and \( \gamma'_{\nu}(0) = \nu \).

**Proof.** The claim again follows from Theorem 3.3 and the proof of Theorem 4.5. For the limsup set \( E_F \) constructed in (19) in Theorem 4.5, to approximate \( T^{\tau}_{N'} \) from below, for the approximation function \( f(t) = \tau \cdot t \), the Hausdorff dimension can be estimated by Theorem 3.3. Recall notation \( \Gamma_{N'} \) from the proof of Theorem 4.5. Set

\[
Q = \Gamma_{N'}, \quad \alpha_g = 1 + \tau.
\]

By Lemma 4.2 and the assumption of property (2), \( \mu(\Lambda_{\Gamma} \setminus E_{Dir}) = 0 \). Also \( \Lambda_{\Gamma} = \partial X \) is \( \nu_{\Gamma} \)-regular by assumptions (1), (3) and (4). Moreover, note that with

\[
\beta_g = \nu_{\Gamma} \cdot \left( \alpha_g - \frac{\log \mu(F(g))}{\log \mu(\text{Dir}(g))} \right),
\]

We will need the following Lemma:

**Lemma 4.2.** Let \( \nu \in SM_{x_0} \) and \( \gamma_{\nu} \) be a geodesic such that \( \gamma_{\nu}(0) = x_0 \) and \( \gamma'_{\nu}(0) = \nu \). Then for \( t_n \to \infty \),

\[
\mu(\gamma_{\nu}(t_n)) = e^{-\tau t_n} \mu(\gamma_{\nu}(0)) = e^{-\tau t_n} \nu_{\Gamma},
\]

and

\[
\mu(\gamma_{\nu}(t_n) \cap \gamma_{\nu}(t_n + s)) = e^{-\tau s} \mu(\gamma_{\nu}(0)) = e^{-\tau s} \nu_{\Gamma}.
\]

**Proof.** By the definition of \( \nu_{\Gamma} \)-regularity, the desired results follow immediately.

\[ \square \]
\{x_g, \beta_g\}_{g \in \Gamma} \text{ satisfy properties (1)--(3) of a lower } F \text{-approximation. For the fourth requirement, note that the centres of the balls in } J_g \text{ for } g \in \Gamma, \text{ are centred in the manifold } gN. \text{ In this case } 
lambda = s = \dim(N). \text{ Also note that } \lim_{\rho_g \to \infty} \beta_g = \tau \cdot s, \text{ so } \lambda < \frac{v + \beta}{\alpha_F}. \text{ The lower bound then follows by applying Lemma 3.3.}

For the upper bound, set 

\[ Q = \Gamma, \quad \alpha_g = 1 + \tau/a. \]

Again \{1 + \tau/a, \tau \cdot s\}_{g \in \Gamma} \text{ is an upper } F \text{-approximation for } F \text{ defined using the function } f(t) = \tau \cdot t, \text{ in (18) in the convergence part of Theorem 4.5. The upper bound in the claim then follows by applying Lemma 3.3 because } T_{N'} \subset E_F. \]

\section*{4.4 Large intersections and simultaneous spiraling}

Theorem 3.14 applies to the classes of sets considered in Subsection 4.3. We illustrate this with the following theorem.

\textbf{Theorem 4.11.} Let \( M \) be a connected, complete manifold of pinched strictly negative curvature. Suppose that the \( \Gamma := \pi_1(M) \) action on \( \tilde{M} \) satisfies properties (1), (2), (3) and (4). Let \( \{N_i\}_{i \in \mathbb{N}} \) and \( \{N'_i\}_{i \in \mathbb{N}} \) be countable collections of closed totally geodesic submanifolds of \( M \). Let \( x_0 \in M \). Then for any sequence of positive numbers \( \tau_i \) and \( \varepsilon > 0 \) small enough, the set of directions

\[ E = \left\{ v \in SM_{x_0} \mid \exists \text{ positive times } t^{(i)}_n, t'^{(i)}_n \to \infty \text{ for each } i \text{ such that } \gamma_v(t^{(i)}_n, t^{(i)}_n + \tau_i t^{(i)}_n) \subset B(N_i, \varepsilon) \text{ and } \gamma_v(t'^{(i)}_n) \in B(N'_i, e^{-\tau_i t'_n}) \text{ for each } i \right\} \]

has

\[ \dim_H(E) \geq \inf \min_i \left\{ \frac{v_M + \tau_i \cdot v_{N_i}}{1 + \tau_i}, \frac{v_M + \tau_i \dim(N'_i)}{1 + \tau_i} \right\}. \]

We have the following corollary to the previous results.

\textbf{Corollary 4.12.} Let \( M \) be a complete, connected manifold of pinched strictly negative curvature and dimension \( n \). Suppose that the \( \Gamma := \pi_1(M) \) action on \( \tilde{M} \) satisfying properties (1), (2) and (3). Let \( N \) be a closed submanifold in \( M \) of dimension \( s \), such that \( 0 \leq s \leq n - 1 \) and \( \pi_1(N) \hookrightarrow \pi_1(M) \) is non trivial. Let \( x_0 \in M \).

(1) For \( \varepsilon > 0 \) small enough,

\[ E_N = \left\{ v \in SM_{x_0} \mid \exists \text{ positive times } t_k \to \infty \text{ such that } \gamma_v(t_k, t_k + \tau t_k) \subset B(N, \varepsilon) \text{ for some } \tau > 0 \right\} \]

has

\[ \dim_H(E_N) = v_F. \]
(2) If property (4) is also satisfied,

\[ T_N = \left\{ v \in SM_{x_0} \mid \exists \text{ positive times } t_k \to \infty \text{ such that } \gamma_v(t_k) \subset B(N, e^{-\tau t_k}) \text{ for some } \tau > 0 \right\} \]

has

\[ \dim_H(T_N) = v_\Gamma, \]

where \( \gamma_v \) is the geodesic at \( x_0 \) at time zero with direction \( v \). Both \( E_N \) and \( T_N \) have Patterson–Sullivan measures zero if \( 0 \leq s < n - 1 \). If \( s = n - 1 \), \( E_N \) has again measure zero, but \( T_N \) has full measure.

**Proof.** The dimension results follow from Theorems 4.9, 4.10 and 4.11. The measure results follow from Theorems 4.3 and 4.5. \( \square \)

**Remark 4.13.** For any of the dimension results stated above, a smooth manifold structure is not essential; see Remarks 4.4 and 4.6.

### 4.5 The case of cusp excursions

There has been extensive work on the shrinking target problem for cusp excursions following the work of Sullivan. Kleinbock and Margulis [29] generalised Sullivan’s results to locally symmetric spaces of finite volume, and also gave a dynamical proof of Khintchine’s theorem. See also [2, 3] for results in the non-Archimedean setting. We would like to point out that the problem of cusp excursions of geodesics in (variable) negatively curved manifolds also falls within the framework we have developed in this paper. Namely, one can obtain both measure and dimension results for cusp excursions. To do so, one applies the analogue of Lemma 4.2 for horospheres, known in the constant negative sectional curvature case due to the work of Sullivan. To verify such an analogue in general, one again uses mixing of the geodesic flow with respect to the Bowen–Margulis measure. As this result can be obtained by minor and standard modifications of the arguments in [44] and [35], we omit the details. This along with a minor modification of [35, Theorem 2.1] or by Theorem 3.1 in our paper yields a Jarník–Besicovitch theorem for cusp excursions. The Borel–Cantelli statement follows from a standard modification of Sullivan’s method. Theorem 3.3 also yields a Jarník–Besicovitch theorem for Diophantine approximation on the Heisenberg group. Let \( X = \mathbb{H}_c^n \) be complex hyperbolic space endowed with the standard Riemannian metric. Then \( G = \text{PU}(n, 1) \) is the group of holomorphic isometries of \( X \). Then \( \partial X \setminus \{\infty\} \) can be identified with the Heisenberg group equipped with the Carnot–Caratheodory distance and the corresponding Hausdorff measure. This is an Ahlfors regular space. We refer the reader to [22] for background to this problem and relevant definitions. We have that \( \infty \) is a parabolic point of \( \Gamma := \text{PU}(n, 1)(\mathbb{Z}[\sqrt{-1}]) \) and its orbit is a subset of rational points on the Heisenberg group, so

\[ \{ \xi \in \partial X \mid d_{CC}(\xi, g(\infty)) \leq e^{-\tau D(g(\infty))}, \text{ for infinitely many } g \in \Gamma \} \]

corresponds to a Diophantine approximation problem in the Heisenberg group, where \( d_{CC} \) is the Carnot–Caratheodory distance and \( D(g(\infty)) \) is the ‘depth’ of the ‘rational’ geodesic line with end points \( g(\infty) \) and \( \infty \), see [22, p. 208]. In that paper, the authors prove a Khintchine type theorem in the Heisenberg group. Applying Theorem 3.3, we get a Jarník–Besicovitch
theorem for the Carnot–Carathéodory distance. Indeed, the Dirichlet function in this case associates to each orbit point \( g(\infty) \) the ball with centre \( g(\infty) \) and radius \( e^{-D(g(\infty))} \). Moreover, the orbit \( \Gamma(\infty) \) is well-distributed whenever \( \Gamma \) is a lattice in \( PU(n,1) \). See [47], for another approach to a Jarník–Besicovitch theorem on the Heisenberg group using homogeneous dynamics. We also have the following large intersection property for the Heisenberg group which is a consequence of Theorem 3.14.

**Theorem 4.14.** Let \( \{d_i\}_i \subset \mathbb{N} \) be a (possibly infinite) collection of positive square-free integers. Let \( W_i(d_i)(\tau) \) denote the set of points in \( \partial \mathbb{H}^n \setminus \{\infty\} \) which are \( \tau \)-well approximated by parabolic fixed points of \( PU(n,1)\mathbb{Z}[\sqrt{-d_i}] \), simultaneously for each \( i \). Then,

\[
\dim_H \left( \bigcap_i W_i(d_i)(\tau) \right) = \frac{2n}{\tau}.
\]

With regard to the large intersection property, the case of cusp excursions in the constant negative sectional curvature case follows from the work of Falconer [16]. Although large intersection theorems for cusp excursions in the variable negative curvature do not directly follow from the work of Falconer, Theorem 3.14 does apply and we get the following theorem.

**Theorem 4.15.** Let \((X,\rho)\) be a CAT\((-1)\) metric space and let \( \Gamma_i \) be a collection of discrete subgroups of the isometry group of \( X \) such that properties (1), (2),(3) and (4) hold for the action of \( \Gamma_i \) on \( X \). Assume further that each \( \Gamma_i \) has nontrivial parabolic subgroups \( P_i \). Let \( \xi_i \) be corresponding fixed points in \((\partial X,d)\). Given \( \tau > 0 \), the set

\[
E_\tau = \bigcap_i \{ \xi \in \partial X \mid d(\xi, g\xi_i) \leq e^{-\tau \rho(x_0, g x_0)}, \text{ for infinitely many } g \in \Gamma_i \}
\]

has Hausdorff dimension

\[
\dim_H(E_\tau) = \frac{\nu_X}{\tau},
\]

where \( \nu_X \) is the critical exponent of the groups \( \Gamma_i \).

In particular, the theorem above applies to the situation where \( G \) is a rank 1 semisimple Lie group, \( K \) is a maximal compact subgroup of \( G \), the \( \Gamma_i \) are non-uniform lattices in \( G \) and the \( \xi_i \) correspond to points at infinity of the finite volume quotients \( \Gamma_i \backslash G/K \).

Similarly, we have a corollary in the context of Bianchi groups. The following result follows from the large intersection property of Falconer in \( \mathbb{R}^2 \) and the Hausdorff content density estimate from [35] for the action (by Möbius transformations) of the Bianchi subgroups \( \Gamma_d \) on the sphere \( S^2 \) and gives a refinement of the Járnik–Besicovitch theorem in this context.

**Theorem 4.16.** Let \( \{d_i\}_i \subset \mathbb{N} \) be a (possibly infinite) collection of positive square-free integers. Let \( W_{2,d_i}(\tau) \) denote the set of points in \( \mathbb{R}^2 \) which are \( \tau \)-well approximated by the collection \( \{p/q : p, q \in \mathbb{Z}[\sqrt{-d_i}], \text{ Ideal}(p,q) = \mathbb{Z}[\sqrt{-d_i}]\} \), simultaneously for each \( i \). Then \( W_{2,d_i}(\tau) \) is in \( C^{2/\tau} \). In particular,
\[ \dim_H(\mathcal{W}_{2,d_i}(\tau)) = \frac{2}{\tau} \quad \text{and} \quad H^{2/\tau}(\mathcal{W}_{2,d_i}(\tau)) = \infty. \]

Moreover,

\[ \dim_H \left( \bigcap_i \mathcal{W}_{2,d_i}(\tau) \right) = \frac{2}{\tau}. \]

Finally, we note that Theorem 4.15 applies to the situation when \( X \) is a Bruhat-Tits tree attached to a rank one semisimple algebraic group over a local field of characteristic \( p \). The Diophantine problem then becomes one of approximating Laurent series by ratios of polynomials, and one can formulate the corresponding large intersection problem by considering congruence quotients, or indeed even by taking quadratic extensions (cf. [18]) as in the theorem above. However, we refrain from spelling out the details because the boundary of the Bruhat-Tits tree admits net measures and so Falconer’s techniques apply with minor modifications (see [16, Remark (c)]).

**Acknowledgements**

We would like to thank Mahan Mj for helpful comments and encouragement. Anish Ghosh would like to thank Yann Bugeaud and Arnaud Durand for helpful conversations about the large intersection property and François Maucourant for answering some questions. We would also like to thank Frédéric Paulin for his comments on a preliminary version of this paper. Part of this work was completed when both authors were at the International Centre for Theoretical Sciences, Bengaluru as part of the programme Smooth and Homogeneous Dynamics. The hospitality of ICTS is gratefully acknowledged. We thank the referees for a careful reading and several helpful suggestions. Anish Ghosh gratefully acknowledges support from a grant from the Indo-French Centre for the Promotion of Advanced Research; and a MATRICS grant from the Science and Engineering Research Board and a grant from the Infosys Foundation. Debanjan Nandi gratefully acknowledges support from ISF Grant 1149/18. Both authors gratefully acknowledge support from a Department of Science and Technology, Government of India, Swarnajayanti fellowship.

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The *Proceedings of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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