From short-time diffusive to long-time ballistic dynamics: the unusual center-of-mass motion of quantum bright solitons

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Both for small classical particles and for macromolecules, Brownian motion is ballistic on short time scales and diffusive on long time scales. Our theoretical investigations indicate that one can observe the exact opposite — initially diffusive motion that becomes ballistic on longer time scales — in an ultracold atom system with a size comparable to macromolecules. This system is a quantum matter wave bright soliton subject to decoherence via three-particle losses. Using a stochastic approach to model the decoherence-induced dynamics, we investigate the center-of-mass motion. Our simulations show that such unusual center-of-mass dynamics of the quantum bright solitons should be observable on experimentally accessible time scales.

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I. INTRODUCTION

Bright solitons, waves that do not change their shape, were discovered in the 19th century in a water canal [1]. Classical solitons like those in water are good examples for ballistic motion (the distance from the initial position grows linearly with time) as the velocity can stay constant. Bright solitons can be experimentally generated from attractively interacting ultracold cold atomic gases [2–9]. On the mean-field level via the Gross-Pitaevskii equation, these matter wave bright solitons are non-spreading solutions of a non-linear equation. For \( N \) ultracold attractively interacting atoms in a (quasi-)one-dimensional wave guide, the quantum matter wave solitons can be described as a many-particle bound state, the ground state [10–12] of an exactly solvable many-particle quantum model, the Lieb-Liniger(-McGuire) model [13] with attractive interactions [10]. The matter wave bright solitons investigated for attractively interacting atoms thus have some similarities with large (weakly bound) molecules or — when described on the mean-field level via the Gross-Pitaevskii equation [14] — even small classical particles. While measuring one particle reveals all other particles to be close to the first, prior to measurement the center-of-mass of the ground-state soliton is completely delocalized. Excited states of the center-of-mass wave functions are plane waves, which can be used to form wave-packets describing the center-of-mass motion of a quantum bright soliton.

Diffusive motion (for which the root-mean-square fluctuations of the position grows with the square root of time) of both macromolecules and small classical particles often occurs through interactions with the environment: Free Brownian motion [15–21] shows the generic behavior that the short-time dynamics is ballistic whereas on longer time-scales the motion becomes diffusive. While there are models that, depending on the choice of parameters, can behave either diffusively or ballistically [22], in the current paper we show the surprising result that the motion of the center of mass of quantum bright solitons, under the influence of decoherence via three-particle losses, behaves diffusively on short time scales and ballistically on long time scales.

In order to probe the fascinating “middle-ground” between quantum and classical physics [23, 24] with matter wave bright solitons, it is essential to include both mean-field effects via the Gross-Pitaevskii equation and beyond-mean-field aspects of quantum bright solitons. Both interferometry with bright solitons [8, 25–30] and soliton trains [3, 31–35] are examples for which combining the mean-field approach with beyond-mean-field investigations on the \( N \)-particle quantum level lead to physical insights. Further examples of investigations of bright solitons on the beyond-mean-field level can be found in Refs. [12, 36–42]. Diffusive behavior in Bose-Einstein condensates has been observed in the presence of disorder in Ref. [43].

Even for a perfect vacuum and by shielding all external influence, decoherence via three-particle losses will always be present. For matter wave bright solitons made of absolute ground-state atoms such as Li\(^7\) [2], there are no two-particle losses [44]; single-particle losses can be discarded if the vacuum is chosen to be particularly good (cf. [45]). Therefore it is justified to focus on three-particle losses as the only decoherence mechanism.

The paper is organized as follows: We first introduce the physics involved in opening a weak initial trap in which a bright soliton made from an attractive Bose-Einstein condensate has been prepared (Sec. II). We then introduce the decoherence mechanism which will always be present in such a case — atom losses via three-body recombination (Sec. III A) which is modeled via a stochastic approach using piecewise deterministic processes [46] in Sec. III B. Section IV presents
the results of our Monte Carlo simulation with the surprising transition from short-time diffusive to long-time ballistic behavior, and the paper ends with a conclusion and outlook (Sec. V).

II. OPENING A WEAK HARMONIC TRAP

When attractively interacting Bose-Einstein condensates are used experimentally to generate bright solitons, the bright soliton is in a (quasi-)one dimensional wave guide with additional initial harmonic trapping [2–8]. Important aspects of such bright solitons can be understood by a mean-field approach via the Gross-Pitaevskii equation [14]

\[ i\hbar \frac{\partial}{\partial t} \varphi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \varphi(x, t) + \frac{1}{2} m \omega^2 x^2 \varphi(x, t) + (N - 1) g_{1D} |\varphi(x, t)|^2 \varphi(x, t), \tag{1} \]

where \( m \) is the mass of the particles and \( \omega \) the angular frequency of the harmonic trap and where the interaction \( g_{1D} = hf_s \), \( a \) is set by the \( s \)-wave scattering length \( a \) and the perpendicular trapping-frequency, \( f_\perp \) [47]. For attractive interactions \((g_{1D} < 0)\), Eq. (1) has bright solitons as solutions with densities, normalized to one of

\[ \rho(x) = \frac{1}{4\xi_N} \left| \frac{\cos[x/(2\xi_N)]}{\xi_N} \right|^2, \tag{2} \]

with the soliton length

\[ \xi_N = \frac{\hbar}{m |g_{1D}|(N - 1)}. \tag{3} \]

If the additional harmonic trap is then switched off, this can lead to excitations of the internal degrees of freedom visible in atoms leaving the bright soliton. However, if the initial trap was sufficiently weak, that is, the soliton length small enough against the harmonic oscillator length, hardly any atoms are excited [48].

We can then assume the ground state to be a bright soliton of \( N \) particles of mass \( m \) for which the center-of-mass wave function is described by the text-book example of the Gaussian wave function of a single particle of mass \( M = N m \) [49] in one dimension,

\[ \Psi(X, t) \propto \left( 1 + \frac{\hbar t}{M a^2} \right)^{-1/2} \times \exp \left[ -\frac{X^2 - 2ia^2 MV_0 X/h + ia^2 MV_0^2 t}{2a^2 + i\hbar/2(Ma^2)} \right], \tag{4} \]

where \( X \) is the center-of-mass coordinate and \( V_0 \) the initial velocity. The center-of-mass wave function (4) will then spread, leading to a root-mean-square width of \( |a| \)

\[ \Delta x = \frac{a}{\sqrt{2}} \sqrt{1 + \left( \frac{\hbar t}{N ma^2} \right)^2}. \tag{5} \]

Although the center-of-mass wave function spreads, a single measurement of the atomic density via scattering light off the soliton (cf. [2]) will still yield the density profile of the soliton expected both on the mean-field (Gross-Pitaevskii) level and on the many-particle quantum level for vanishing width of the center-of-mass wave function [11, 12]. By adding harmonic trapping in the perpendicular directions, one has the density in three dimensions [2]

\[ n(x, y, z) = \frac{N}{4\xi_N [\cosh(x/(2\xi_N))]^2} \frac{1}{\lambda_\perp^2} \exp \left( -\frac{y^2 + z^2}{\lambda_\perp^2} \right), \tag{6} \]

where \( \lambda_\perp \) is the perpendicular harmonic oscillator length. In order to experimentally measure the spreading of the center-of-mass density directly, each measurement of the soliton should only record its center-of-mass position when calculating the density from the experimental data. Recording the entire density profile in each measurement yields the single-particle density which can also be obtained on a more formal level as

\[ \rho(x) = \frac{1}{N} \sum_{j=1}^{N} (\delta(x - x_j)). \tag{7} \]

Figure 1 shows the influence of the center-of-mass motion on the single particle density of a quantum bright soliton of 3000 Li atoms (as investigated at high velocities in the experiment [2]). The mean-field (Gross-Pitaevskii) soliton remains stationary (Fig. 1 a); the single particle density is given by Eq. (2) for all times. Panel b of Fig. 1 displays the same situation as panel a but for a quantum bright soliton for which the center-of-mass wave function spreads according to Eq. (5). Thus, for a quantum bright soliton we have a spreading single particle density - although each single measurement yields the mean-field soliton density (Fig. 1 a), shifted from the initial position by some distance.

III. DECOHERENCE VIA THREE-PARTICLE LOSSES

A. Three-particle losses

For three-particle losses, we have [44]

\[ \frac{dN}{dt} = -K_3 \int d^3r n^3(x, y, z), \tag{8} \]

where \( K_3 \) is determined empirically. Combined with Eq. (6), Eq. (8) yields [50]

\[ \frac{dN}{dt} = -\frac{1}{90\pi^2} K_3 \frac{N^3}{\xi_N^3 \lambda_\perp^4} = -\frac{1}{\tau_3} (N - 1)^2 N^3, \tag{9} \]

with the \( N \)-independent time scale:

\[ \tau_3 \equiv \left( \frac{K_3}{90\pi^2} \frac{g_{1D}^2}{\hbar^2 a^4} \right)^{-1}. \tag{10} \]
For large initial particle numbers $N$, the time scale for losing a single particle is given by \[ t_1 \approx \frac{\tau_3}{N^4}. \] (15)

For this equation to be valid at longer time-scales (and not just initially), the form of the wave function must remain of the form \( \psi(x) \) while the width of the wave function in \( x \)-direction increases with decreasing particle numbers. We will show that this assumption is self-consistent, as the times scales involved are long enough such that the changes can take place adiabatically, still allowing us to treat atom losses as point processes within our stochastic approach.

For large $N$, the approximation

\[ \frac{dN}{dt} \approx -\frac{1}{\tau_3} N^5 \]  (12)

leads to

\[ N(t) \approx N_0 \left(1 + \frac{t}{\tau_{\text{loss}}} \right)^{-1/4}, \]  (13)

\[ \tau_{\text{loss}} = \frac{\tau_3}{4N_0^4}. \]  (14)

For large initial particle numbers $N(t = 0) = N_0$ and experimentally relevant time scales Eq. (13) is a good approximation to the complete time-dependence (which will be shown in Fig. 2 a). For large $N$, the time scale for losing a single particle is given by \[ t_1 \approx \frac{\tau_3}{N^5}. \] (15)

B. Stochastic modeling of decoherence via three-particle losses

For $N$-particle quantum bright solitons, the center-of-mass wave function — both in harmonic potentials and in the wave guide without additional trap — can be treated independently of the relative coordinates.

The internal ground state of $N$ ultracold attractively interacting atoms of mass $m$ in a quasi-one dimensional wave guide has the ground state energy \[ E_0 = -\frac{1}{24} \frac{mg_{1D}^2}{\hbar^2} N(N^2 - 1), \]  (16)

where $g_{1D}$ quantifies the strength of the contact interaction. If changes of the number of particles in a soliton happen on slow enough time scales these changes can be modeled as being adiabatic: The shape of the soliton is protected by an energy gap between ground state and first excited state obtained by calculating the energy difference between the internal ground state of $N - 1$ atoms and the internal ground state of $N$ atoms [Eq. (16)]

\[ E_{\text{gap}} = \frac{mg_{1D}^2 N(N - 1)}{8\hbar^2}. \]  (17)

The energy-time uncertainty yields a characteristic time scale via \( E_{\text{gap}} \tau_{\text{soliton}} \propto \hbar \), where

\[ \tau_{\text{soliton}} = \frac{\hbar^3}{mg_{1D}^2 (N - 1)^2}. \]  (18)

Changes in, for example, particle numbers should happen on time scales longer than this time for the process to be adiabatic and for our approach to treat particle losses as an adiabatic process to be valid. In the absence of particle losses, all times would have to be measured in units of $\hbar^3/(mg_{1D}^2)$, including the $N$-scaling of Eq. (18) this yields the only available time scale on the mean-field level (cf. [39]).

So far, three-particle losses in the experiments have not been observed to destroy solitons on short time scales [2]. We can thus model the particle losses as taking place on time scales longer than the soliton time if [cf. Eq. (15)]

\[ \frac{\tau_3}{\tau_{\text{soliton}}} N^5 > 1. \]  (19)

With $N = 5000$ and the experimental parameters of [2],\(^1\) we have

\[ \frac{\tau_3}{\tau_{\text{soliton}}} \approx 2 \times 10^{19} \]  (20)

\(^1\) While this result is derived on the many-particle quantum level, it remains true in the mean-field limit [51].

\(^2\) The set of parameters used as an example to show that experimentally realistic time scales uses the values given in Ref. [2] for the s-wave scattering length $a = -0.21 \times 10^{-6}$ m, $f_s = 710$ Hz. For this s-wave scattering length we furthermore divide the calculated value [52] for the thermal $K_3$ of $3.6 \times 10^{-4}$ m$/s$ by the factor 3! = 6 for Bose-Einstein condensates and (thus also bright solitons).
The inequality (19) is fulfilled for the parameters of [2] if \( N \leq 5000 \). We can furthermore model the three-particle losses as taking place instantaneously for our stochastic implementation [53–55] of particle losses.

For a NOON state, a Schrödinger cat state [56] that is in a quantum superposition of two “macroscopically” occupied single particle modes, \(|\psi_{\text{NOON}}\rangle \propto |1\rangle^N + |2\rangle^N \), losing three particles leads to a localization in one of the two modes, \(|1\rangle^{(N-3)} \) or \(|2\rangle^{(N-3)} \). Quantum bright solitons are in a spatial quantum superposition given by their center-of-mass wave function (cf. [38]); if the center-of-mass wave function is a delta function (cf. [39]) the corresponding velocity \( V \) can indeed be described by a classical model is justified below.

\[ \Gamma(N) \equiv \int dX' \int dV' W_N(X', V'|X, V) \rho^{N-3} \left[ \frac{1}{2 \sigma_X^2(N)} \exp \left( -\frac{(X - X')^2}{2 \sigma_X^2(N)} \right) \right] \times \left[ \frac{1}{2 \sigma_Y^2(N)} \exp \left( -\frac{(V - V')^2}{2 \sigma_Y^2(N)} \right) \right]. \]

The process \( (X, V) \) is thus a piecewise deterministic process [46] with transition rates

\[ W_N(X', V'|X, V) = \Gamma(N) \left( \frac{1}{2 \sigma_X^2(N)} \exp \left( -\frac{(X - X')^2}{2 \sigma_X^2(N)} \right) \right) \times \left( \frac{1}{2 \sigma_Y^2(N)} \exp \left( -\frac{(V - V')^2}{2 \sigma_Y^2(N)} \right) \right). \]

As we will describe below, both the position and the velocity (via the center-of-mass density) as well as the point of time for this decoherence [via Eq. (9)] are determined via random numbers in a Monte Carlo simulation; the single particle density is averaged over many realizations

\[ \rho(t) = |\langle \psi(t) | \psi(t) \rangle|. \]

The width of the new center-of-mass wave function can be small compared to the root-mean-square width of the soliton

\[ \Delta X_{\text{soliton}} = \frac{\pi \xi_{N-3}}{\sqrt{3}}, \]

thus justifying the delta-function approximation used in [12]. However, if the center-of-mass wave-function collapses to a single product state (23), the root-mean-square width of the new center-of-mass wave function of the soliton consisting of \( N - 3 \) particles is given by the prediction of the central limit theorem (cf. [39])

\[ \Delta X_{\text{CoM}} = \frac{\pi \xi_{N-3}}{\sqrt{3(N-3)}}. \]

In order to describe the stochastic process, we introduce an approach via a classical master equation. While at first glance this approach seems to be impossible, as in between loss events we have a purely quantum mechanical expansion of the center-of-mass wave function, the fact that our system can indeed be described by a classical model is justified below. Within our model the stochastic variables are given by the center of mass coordinate \( X \), the corresponding velocity \( V \) and the particle number \( N \). Introducing the time-dependent probability distribution \( P(X, V, N, t) \) the stochastic process is defined by the master equation

\[ \frac{\partial}{\partial t} P(X, V, N, t) = -V \frac{\partial}{\partial X} P(X, V, N, t) \]

\[ + \int dX' \int dV' \left[ W_{N+3}(X, V|X', V') P(X', V', N + 3, t) - W_N(X', V'|X, V) P(X, V, N, t) \right]. \]

The first term on the right-hand side describes the constant drift of \( X \) with velocity \( V \) while the second term represents the instantaneous random jumps induced by three-particle losses. The process \( (X, V, N) \) is thus a piecewise deterministic process [46] with transition rates

\[ W_N(X', V'|X, V) = \Gamma(N) \left( \frac{1}{2 \sigma_X^2(N)} \exp \left( -\frac{(X - X')^2}{2 \sigma_X^2(N)} \right) \right) \times \left( \frac{1}{2 \sigma_Y^2(N)} \exp \left( -\frac{(V - V')^2}{2 \sigma_Y^2(N)} \right) \right), \]

where \( \sigma_X(N) \) is given by either Eq. (26) or Eq. (25), and \( \sigma_Y(N) = \hbar / [2m(N-3)\sigma_X(N)] \). The total transition rate takes the form

\[ \Gamma(N) \equiv \int dX' \int dV' W_{N+3}(X', V'|X, V) \rho^{N-3} \left[ \frac{1}{2 \sigma_X^2(N)} \exp \left( -\frac{(X - X')^2}{2 \sigma_X^2(N)} \right) \right] \times \left( \frac{1}{2 \sigma_Y^2(N)} \exp \left( -\frac{(V - V')^2}{2 \sigma_Y^2(N)} \right) \right). \]

The width of the following state of the soliton centered at \( (X, V) \) is small compared to its root-mean-square width.

\[ F(N, t) = 1 - \exp \left( -\Gamma(N) t \right). \]

To summarize, for the stochastic simulation of decoherence via three-particle losses [55], the ingredients are:

3. The tensor product power notation \(|1\rangle^N \) describes \( N \) particles occupying the same single-particle mode \(|1\rangle \).

4. In the limit of low kinetic energies (low compared to the energy gap (17) between internal ground state and first excited state) this equation could be modified to contain external potentials via the effective potential approach of Refs. [38, 57] thus modeling how the Schrödinger cat proposed in Refs. [38, 58] is destroyed by decoherence.
1. The random variables:

\[ N, X_0, V_0 \]  

(31)

2. Random numbers for the Monte-Carlo process determine:

(a) The time of the next decoherence event via Eq. (9) by choosing an exponential distribution of loss times (30), where the factor 1/3 introduced in Eq. (29) is necessary because three particles are lost in each step: \( N \rightarrow N - 3 \).

(b) The center-of-mass position \( X_0 \) of the new wave-function (23) via the center-of-mass density in real space.

(c) The center-of-mass velocity \( V_0 \) of the new wave-function (23) via the center-of-mass density in momentum space.

3. The center-of-mass wave function corresponding to the product state (23) is chosen to be a Gaussian

\[ \psi_{\text{CoM}} = \exp \left[ -\frac{(X - X_0)^2}{2a^2} + i\frac{(N - 3)mV_0}{\hbar} X \right] \]  

(32)

with a root-mean-square width \( \sigma_X(N) = a/\sqrt{2} \) either

(a) given by Eq. (26)

\[ \sigma_X(N) = \frac{\pi \xi_{N-3}}{\sqrt{3(N-3)}}. \]  

(33)

(b) or by Eq. (25)

\[ \sigma_X(N) = \frac{\pi \xi_{N-3}}{\sqrt{3}}. \]  

(34)

In between loss events, the quantum dynamics is known analytically [Eq. (4)]; rather than solving the Schrödinger equation it is possible to do this in a more classical approach: The truncated Wigner approximation \(^5\) for the center of mass, which was used in Ref. [41] to qualitatively mimic quantum behavior on the mean-field level by introducing classical noise mimicking the quantum uncertainties in both position and momentum, is particularly useful here: both the mean position and the variance calculated via the Truncated Wigner Approximation for the center of mass are identical to the quantum mechanical result. In order to make both results identical, Gaussian noise has to be added independently to both position \( X_0 \rightarrow X = X_0 + \delta X_0 \) and velocity \( V_0 \rightarrow V = V_0 + \delta V_0 \) with \( \langle \delta X_0 \rangle = 0 \) and \( \langle \delta V_0 \rangle = 0 \) and root-mean-square fluctuations \( \sigma_X(N) \) given by either Eq. (33) or Eq. (34) and by the minimal uncertainty relation

\[ \sigma_v(N) = \frac{\hbar}{2(N-3)m \sigma_X(N)} \]  

(35)

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\(^5\) The truncated-Wigner approximation [59] describes quantum systems by averaging over realizations of an appropriate classical field equation (in this case, the GPEs) with initial noise appropriate to either finite [60] or zero temperatures [25].

for the velocity.

The mean position \( \langle X(t) \rangle = X_0 + V_0t \) is thus identical to the quantum mechanical result; the root-mean-square fluctuations \( \Delta X = \sqrt{(\Delta X_0)^2 + (\Delta V_0)^2 t^2} \) coincide with the quantum mechanical equation (5). Thus, in the absence of both the trap in the axial direction and the scattering processes investigated in Ref. [41], TWA for the center of mass gives exact results for both the position of the center of mass and the root-mean-square fluctuations of the center of mass for a quantum bright soliton. While there are other simulation techniques which allow an in principle exact treatment via a stochastic Gross-Pitaevskii-type equation on short time scales [61], the present approximate method works for all times.

Experiments will decide which way to model the width of the wave function after a decoherence event [Eq. (33) or Eq. (34)] is more successful; as we will see in Sec. IV, the main results of this paper are independent of this choice.

IV. RESULTS

Figure 2 shows the influence of decoherence via three-particle losses on quantum bright solitons made out of Li atoms for parameters taken from the experiment [2]. Independent of the choice for the localization after a loss event, the motion is diffusive for small times and ballistic for long times.

The fact that this effect is independent of the chosen parameters is demonstrated in Figs. 3 and 4 by using the scaling for which all curves \( N(t)/N_0 \) lie on top of each other. From Eq. (13) we see that this can be achieved by dividing the time by the loss-time \( \tau_{\text{loss}} \). the time scale on which the particle number starts to forget its initial number of particles. For small times, \( t \ll \tau_{\text{loss}} \), the motion is diffusive, for \( t \gg \tau_{\text{loss}} \) it becomes ballistic. Note that for each of the two Figs 3 and 4, all curves are calculated for the same physical time-interval — the re-scaled curves indicate that some parameters are more suitable to observe the transition from diffusive to ballistic motion than other parameters. For the large localization length (25) used in Fig. 4, the onset of ballistic motion has not reached the \( \propto t \) regime yet [cf. Eq. (5)].

While the total center-of-mass motion only becomes ballistic for long times, between loss-events the motion is ballistic if the localization is modeled via Eq. (33) (Fig. 3). Thus, we can expect both the center-of-mass motion and its root-mean-square width to be proportional to the time passed. After rescaling the time in Fig. 3 b with the factor \( 1/\tau_{\text{loss}} \), the time that has really passed is proportional to \( \tau_{\text{loss}} \). We thus choose a scaling factor \( \propto \tau_{\text{loss}} \),

\[ r = \frac{\tau_3}{\tau_{\text{soliton}} N_0} \times \frac{5000^4}{2 \times 10^{19}}. \]  

(36)

which causes all lines in panel b of Fig. 3 to be parallel to each other; the scaling factor (36) is normalized to one for \( \tau_3/\tau_{\text{soliton}} = 2 \times 10^{19} \) and \( N = 5000 \).
FIG. 2. Influence of decoherence via three-particle losses on the center-of-mass motion of quantum bright solitons using the parameters described in footnote 2. (a) Number of particles $N(t)$ as a function of time (solid black line) with $N(0) = 5000$ obtained numerically. For $5000 - N(t)$ as a function of time, the analytic curve [yellow, Eq. (13)] lies on top of the numeric curve (black dashed line). (b) As soon as particle losses [modeled via Eq. (33)] become important, the root-mean-square width of the single-particle density of a bright soliton (red line) grows like the square-root of time (cf. dashed magenta line) before becoming ballistic at larger times (faster than $\propto t/N(0)$ (lower green line) but slower than $\propto t/N(t)$ (upper green line). (c) Same as previous panel but the blue line is modeled via Eq. (34); green line $\propto t$.

V. CONCLUSION AND OUTLOOK

We have introduced a physically motivated model for the motion of quantum bright solitons which displays short-time diffusive and long-time ballistic behavior, contrary to the usual short-time ballistic and long-time ballistic behavior observed for example in Brownian motion [19]. Bright solitons are investigated experimentally in various groups world-wide. As the ballistic expansion for large times is $\propto t/(Nm)$ [Eq. (5)] the solitons made of thousands of Li atoms [2] are more suitable to observe this motion of the center-of-mass than solitons made of thousands of the more than ten times heavier Rb atoms [6].

While we stochastically model the decoherence events via three-particle losses as point processes, we make sure that we can assume that they happen on time scales longer than the characteristic time for solitons, thus leading to a criterion for the initial upper bound (19) on the number of atoms with which a ground-state soliton can remain a ground-state soliton after many loss events (which for the parameters of [2] is of the same order as the upper limit given there).

There are also cases for which also a lower bound on the number of atoms for which a soliton can be stable, such as the expulsive potential used for the final part of the experiment [2]. Thus, particle losses [Eq. (13)] combined with time scales needed to perform an experiment can also lead to a more restrictive lower bound. While the focus of the current paper was on the center-of-mass motion for light ground-state atoms which do not undergo two-particle losses, the numerical model could be extended to include two-particle losses in order to investigate heavier non-ground atoms such as Rb [62].

The present idea to modify the quantum mechanical mo-
tion by stochastic terms in order to describe instantaneous changes of the wave function to smaller wave packets has formal similarities with well-known stochastic collapse models [63]. However, within our model these random changes describe the decoherence of the center-of-mass wave function which is induced by three-particle losses. It is the decrease of the particle number that leads to a decrease of the rate of particle losses and, hence, to the observed transition from diffusive to ballistic motion. This motion is an effect distinct from both classical [64] and quantum walks cf. [65, 66] as well as anomalous diffusion [64, 67]. As for the classical random walk, our model localizes after each step, but in between steps the motion is given by free expansion of the center-of-mass wave function which will change with decreasing numbers of atoms in the bright soliton.

This unusual behavior of the center-of-mass motion can be observed for experimentally realistic parameters chosen such that both time scales and length scales are accessible experimentally.

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