LINEAR AND FULLY NONLINEAR ELLIPTIC EQUATIONS WITH $L_d$-DRIFT

N.V. KRYLOV

Abstract. In subdomains of $\mathbb{R}^d$ we consider uniformly elliptic equations $H(v(x), Dv(x), D^2v(x), x) = 0$ with the growth of $H$ with respect to $|Dv|$ controlled by the product of a function from $L_d$ times $|Dv|$. The dependence of $H$ on $x$ is assumed to be of BMO type. Among other things we prove that there exists $d_0 \in (d/2, d)$ such that for any $p \in (d_0, d)$ the equation with prescribed continuous boundary data has a solution in class $W^2_{p, \text{loc}}$. Our results are new even if $H$ is linear.

1. Introduction and main results

In this article we consider elliptic equations

$$H[v](x) := H(v(x), Dv(x), D^2v(x), x) = 0$$

(1.1)

in subdomains $\Omega$ of $\mathbb{R}^d$, where $H(u, x)$ is a function given for $x \in \mathbb{R}^d$ and $u = (u', u'')$, $u' = (u'_0, u'_1, ..., u'_d) \in \mathbb{R}^{d+1}$, $u'' \in \mathcal{S}$, where $\mathcal{S}$ is the set of symmetric $d \times d$-matrices. The “coefficients” of the first order derivatives of $v$ in (1.1) are assumed to be in $L_d(\Omega)$ and we take $p \in (d_0, d)$ for certain $d_0 < d$. We present some results about a priori estimates and the solvability in $W^2_{p, \text{loc}}(\Omega)$ of (1.1). These results are new even for linear equations (see Section 2 and Example 1.3) although in the linear case results somewhat close to ours can be found in [9] under some additional regularity assumptions on the matrix of second order coefficients allowing one to rewrite the equation in divergence form. Also see the references in [9]. Most likely our results are false if $p = d$ even if the equation is linear.

In the literature, the $W^2_{p, \text{loc}}$, $p > d$, estimates like (1.4) with $\tau_0 = 0$ and $\Omega = B_1$ for viscosity solutions of a class of fully nonlinear uniformly elliptic equations of the form

$$H(D^2u, x) = f(x)$$

were first obtained by Caffarelli in [2] (see also [4]). His proof is based on an ingenious application of the Aleksandrov–Bakel’man–Pucci a priori estimate, the Krylov–Safonov Harnack inequality, and a covering result which can be also found in [16] and [17]. Our results are based on ideas and results

2010 Mathematics Subject Classification. 35J60, 35J15.

Key words and phrases. Fully nonlinear equations, interior estimates, solvability, unbounded coefficients.
from [12], which uses the Evans-Krylov, Fang-Hua Lin, and Fefferman–Stein theorems as presented in [13], and results from recent papers [14], and [15]. By exploiting a weak reverse Hölder’s inequality, the result of [2] was sharpened by Escauriaza in [6], who obtained the interior $W^{2,p}_{loc}$ estimate for the same equations allowing $p > d - \varepsilon$, with a small constant $\varepsilon > 0$ depending only on the ellipticity constant and $d$. No terms with $Du$, however, were involved. In the present article we use $p$ which is less than $d$ unlike [13], where $p > d$ and the drift terms are bounded.

The above cited works [2] and [4] are quite remarkable in one respect—they do not suppose that $H$ is convex or concave in $D^2 u$ and relate to any viscosity solution. The assumptions in [2] and [4] are quite different from ours. One of these assumptions is that the equations $H(D^2 u, x_0) = 0$ admit $C^2_{loc}(B_r(x_0))$-solutions for any $B_r(x_0) \subset B_1$ and any continuous boundary data. Until now we only know that, generally, this assumption is satisfied if $H$ is convex or concave with respect to $u''$. Paper [21] and the references there present a few exceptions.

A number of existence results of $W^{2,p}_{loc}$-solutions and a priori estimates in $W^{2,p}_{loc}$ obtained by means of the theory of viscosity solutions can be found in [3], [4], and [5]. In all of them $H$ is supposed to be Lipschitz continuous in $u$ uniformly with respect to $x$ and for any $K > 0$ and $|u| \leq K$ to be sufficiently uniformly close to functions continuous with respect to $x$. Note that these assumptions exclude, for instance, Example 1.2 below, and for, that matter, exclude linear equations even with bounded coefficients and VMO-coefficients in the main part. On the other hand, our results do not cover those from [3], [4], and [5] either, in particular, just because we are not dealing with viscosity solutions.

Note that in Theorem 4.2 of [22] one more interior estimate of type (1.4) is obtained under the assumptions that $H$ is convex in $u''$, Lipschitz continuous in $u$ and satisfies a continuity condition in $x$ similar to the one mentioned above. Again some values of $p < d$ are allowed. Finally, in [5] and [22] the function $H$ is assumed to be nonincreasing with respect to $u'_0$ unlike $H$ in our Theorem 1.2.

The article was motivated by Safonov’s results in [20] where he, in particular, proved the Harnack inequality and established the Hölder continuity for harmonic functions associated with linear elliptic equations with measurable coefficients and drift in $L_d$.

To start the exposition of our results recall that $S$ is the set of symmetric $d \times d$ matrices and, for a fixed $\delta \in (0, 1]$, let

$$
S_\delta = \{ a \in S : \delta |\xi|^2 \leq a^{ij}\xi^i\xi^j \leq \delta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d \}.
$$

We fix a number $\|b\| < \infty$ and fix a nonnegative function $b \in L_d(\mathbb{R}^d)$ such that

$$
\|b\|_{L_d(\mathbb{R}^d)} \leq \|b\|.
$$
Also fix some constants $K_0, K_F \in [0, \infty)$ and fix a nonnegative $\bar{G}$ given on $\mathbb{R}^d$.

Let $\Omega$ be an open bounded subset of $\mathbb{R}^d$ satisfying the exterior ball condition. Quite often we deal with

$$\Omega^\rho = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \rho \},$$

where $\rho > 0$ is a given number. For measurable $\Gamma \subset \mathbb{R}^d$ we denote by $|\Gamma|$ the volume of $\Gamma$ and if $f$ is a real-valued function on $\Gamma$ with finite integral, then we set

$$\int_{\Gamma} f \, dx = \frac{1}{|\Gamma|} \int_{\Gamma} f(x) \, dx.$$ 

The following assumptions contain parameters $\hat{\theta}, \theta \in (0, 1]$ which are specified later in our results.

Assumption 1.1. There are Borel functions $F(u, x) = F(u', u''_0, x)$ and $G(u, x)$ such that

$$H = F + G.$$ 

Furthermore, for all $u'' \in S, u' \in \mathbb{R}^{d+1}$, and $x \in \mathbb{R}^d$, we have

$$|G(u, x)| \leq \hat{\theta}|u''| + K_0|u'_0| + b(x)||u'| + \bar{G}(x), \quad F(0, x) \equiv 0, \quad (1.2)$$

where

$$[u'] := (u'_1, ..., u'_d).$$

Introduce

$$B_r(x) = \{ y \in \mathbb{R}^d : |x - y| < r \}, \quad B_r = B_r(0).$$

Recall that Lipschitz continuous functions are almost everywhere differentiable, thanks to the Rademacher theorem.

Assumption 1.2. (i) The function $F$ is Lipschitz continuous with respect to $u''$ with Lipschitz constant $K_F$.

Moreover, there exist $R_0 \in (0, 1]$ and $\tau_0 \in [0, \infty)$ such that, if $r \in (0, R_0]$, $z \in \Omega, B_{\tau}(z) \subset \Omega$, and $u'_0 \in \mathbb{R}$, then one can find a convex function $F(u'') = \tilde{F}_{z, \tau, u'_0}(u'')$ (independent of $x$) for which

(ii) We have $\tilde{F}(0) = 0$ and $D_{u''}\tilde{F} \in S_\delta$ at all points of differentiability of $F$;

(iii) For any $u'' \in S$ with $|u''| = 1$, we have

$$\int_{B_{\tau}(z)} \sup_{\tau > \tau_0} \tau^{-1}|F(u'_0, \tau u'', x) - \tilde{F}(\tau u'')| \, dx \leq \theta; \quad (1.3)$$

(iv) There exists a continuous increasing function $\omega_F(\tau), \tau \geq 0$, such that $\omega_F(0) = 0$ and for any $u'_0, v'_0 \in \mathbb{R}, x \in \Omega$, and $u'' \in S$ we have

$$|F(u'_0, u'', x) - F(v'_0, u'', x)| \leq \omega_F(|u'_0 - v'_0||u''|).$$
Remark 1.1. It is useful to note that Assumptions 1.1 and 1.2 (iv) imply that $F(u'_0, 0, x) = 0$ for any $u'_0 \in \mathbb{R}$ and $x \in \Omega$. Also observe that, apart from (iv), Lipschitz continuity in $u''$, and measurability, nothing is imposed on $F$ if $|u''| \leq \tau_0$.

Definition 1.1. For a function $u \in C(\Omega)$ set
\[
\omega_u(\Omega, \rho) = \sup \{ |u(x_1) - u(x_2)| : x_1, x_2 \in \Omega, |x_1 - x_2| \leq \rho \},
\]
and in the formulations of a theorem, lemma,... let us say that a certain constant depends only on A,B,..., and the function $\omega_{F,u,\Omega}$ if it depends only on A,B,..., and on the maximal solution of an inequality like $N_0 \omega_{F,u,\Omega}(\rho) \leq 1/2$, where the range of $\rho$ and the value of $N_0$ depending only on A,B,... could be always traced down in our arguments.

To finish the setting, take $d_0 = d_0(d, \delta, \|b\|) \in (d/2, d)$ from [15] and take $p \in (d_0, d)$. In the statement of the following theorem we use the function $\hat{R}(p)$, which is introduced before Lemma 3.4 (see (3.7)).

Theorem 1.1. Under the above assumptions there exist constants $\hat{\theta}, \theta \in (0, 1)$, depending only on $d, \rho, \delta$, and $K_F$, such that, if Assumptions 1.2 and 1.1 are satisfied with these $\theta$ and $\hat{\theta}$, respectively, then, for any $u \in W^2_{p,\text{loc}}(\Omega) \cap C(\Omega)$ that satisfies (1.1) in $\Omega$ (a.e.) and $0 < \rho < \rho_{\text{int}}(\Omega) \wedge 1 \wedge \hat{R}(p)$, where $\rho_{\text{int}}(\Omega)$ is the interior radius of $\Omega$, we have
\[
\|u\|_{W^2_p(\Omega)} \leq N\|\hat{G}\|_{L_p(\Omega)} + N\rho^{-2}\|u\|_{C(\Omega)} + N\tau_0,
\]
where the constants $N$ depend only on $K_0, K_F, d, \rho, \delta, \|b\|, R_0, \text{diam}(\Omega)$, and the function $\omega_{F,u,\Omega}$.

This theorem is proved in Section 2 after we develop necessary results in Section 3.

To state an existence result we need the following additional assumptions.

Assumption 1.3. The function $H(u, x)$ is continuous in $u$ for any $x$, is Lipschitz continuous with respect to $u''$, and $D_{u''}H \in S_\delta$ at all points of differentiability of $H$ with respect to $u''$.

Assumption 1.4. There exists $n_0 \geq 0$ such that for any $x \in \{G > n_0\}$ we have $D_{u''}F(u'_0, u'', x) \in S_\delta$ at all points of differentiability of $F(u'_0, u'', x)$ with respect to $u''$.

Assumption 1.5. For all values of the arguments,
\[
H(u'_0, 0, x) \text{ sign } u'_0 \leq b(x)|u'_{0}| + \hat{G}(x) \quad (\text{sign } 0 := \pm 1).
\]

Here is our result concerning the solvability of (1.1) in Sobolev spaces. We fix $p \in (d_0, d)$ and a function $g \in C(\partial \Omega)$.
**Theorem 1.2.** There exist constants \( \hat{\theta}, \theta \in (0, 1] \), depending only on \( d, p, \delta, \|b\| \), and \( K_F \), which are, generally, smaller than \( \theta_0, \theta \) from Theorem 1.1 and such that, if Assumptions 1.2 and 1.1 are satisfied with these \( \theta \) and \( \hat{\theta} \), respectively, and Assumptions 1.3, 1.4, and 1.5 are also satisfied and \( \bar{G} \in L_p(\Omega) \), then there exists \( u \in W^{2,p}_{\text{loc}}(\Omega) \cap C(\bar{\Omega}) \) satisfying (1.1) in \( \Omega \) (a.e.) and such that \( u = g \) on \( \partial \Omega \). Furthermore, in \( \Omega \)

\[
|u| \leq N\|\bar{G}\|_{L_p(\Omega)} + \sup_{\partial \Omega} |g|,
\]

(1.6)

where \( N \) depends only on \( p, d, \delta, \|b\| \), and the diameter of \( D \).

The proof of this theorem is given in Section 4.

**Remark 1.2.** Since none of characteristics of \( \Omega \), apart from \( \rho_{\text{int}}(\Omega) \) and \( \text{diam}(\Omega) \) enters Theorem 1.1, one can use Theorem 1.2 to prove the solvability in much worse domains than those satisfying the exterior ball condition. Usually one does it by approximating from inside a given domain, say with smooth ones. For instance, it would suffice to have

\[
\lim \inf_{\rho \downarrow 0} \inf_{x \in \partial \Omega} \frac{|B_\rho(x) \cap \Omega^c|}{\rho^d} > 0,
\]

(1.7)

see, for instance, Theorem 3.1 of [19] or Theorem 2.4.

We are not pursuing this path and leave it to the interested reader.

**Remark 1.3.** Observe that generally there is no uniqueness in Theorem 1.2. For instance, in the one-dimensional case the (quasilinear) equation

\[
D^2u + \sqrt{12}|Du| = 0
\]

for \( x \in (-1, 1) \) with zero boundary data has two solutions: one is identically equal to zero and the other one is \( 1 - |x|^3 \).

Another example is given by the (semilinear) equation

\[
D^2u + 2u(1 + \sin^2 x + u^2)^{-1} = 0
\]

on \((-\pi/2, \pi/2)\) with zero boundary condition. Again there are two solutions: one is \( \cos x \) and the other one is identically equal to zero.

To have uniqueness we need different assumptions (see, for instance, Section 4.1:2 in [13]).

**Example 1.1.** Let \( d = 3, f, \bar{G} \in L_p(\Omega), b \in L_d(\Omega), \alpha \in (0, 1] \). Let \( w(t), t \in [0, \infty) \), be a continuously differentiable function with sufficiently small derivative. Then the equation

\[
H(Du, D^2u, x) := \bar{G}(x) \wedge |D_{12}u| + \bar{G}(x) \wedge |D_{23}u| + \bar{G}(x) \wedge |D_{31}u| + \Delta u + w(|D^2u|) + b(x)|Du|^\alpha - f(x) = 0
\]

(1.8)

satisfies our assumptions and Theorem 1.2 is applicable.

Observe that \( H \) in (1.8) is neither convex nor concave with respect to \( D^2u \). Also note that we can replace \( \Delta u \) with \( a^{ij}(x)D_{ij}u \) if \( a(x) = (a^{ij}(x)) \) is an \( \mathbb{S}_\delta \)-valued VMO-function such that \( a(x) \geq (\delta ij) \).
Assumption 1.2 with $\bar{\rho} > 0$, this theorem in combination with Theorem 1.1 shows that for all sufficiently small $\rho$.

Consider equation (1.1), where $\bar{a} \in S_{\delta}$ is sufficiently small (to accommodate Theorem 1.2).

As in Example 10.1.24 of [13] one easily sees that Theorem 1.2 is applicable.

Example 1.3. A further specification of Example 1.2 is given by linear equations. Suppose that we are given an $S_{\delta}$-valued measurable function $a(x)$ and an $\mathbb{R}^d$-valued function $b(x)$ such that $b \in L_d(\Omega)$.

Next assume that there is an $R_0 \in (0, \infty)$ such that for any $z \in \Omega$, $r \in (0, R_0]$, and $u_0' \in \mathbb{R}$ one can find $\bar{\alpha} \in S_{\delta}$ (independent of $x$) such that

$$\int_{B_{r}(z)} \sup_{\alpha \in A} \left| a^\alpha(u_0', x) - \bar{\alpha} \right| dx \leq \theta,$$

where $\theta$ is sufficiently small (to accommodate Theorem 1.2).

Consider equation (1.1), where

$$H(u, x) := \inf_{\beta \in B} \sup_{\alpha \in A} \left[ \sum_{i,j=1}^{d} a_{ij}^\alpha(u_0', x)u''_{ij} + b^\alpha(u', x) \right].$$

As in Example 10.1.24 of [13] one easily sees that Theorem 1.2 is applicable.

Example 1.3. A further specification of Example 1.2 is given by linear equations. Suppose that we are given an $S_{\delta}$-valued measurable function $a(x)$ and an $\mathbb{R}^d$-valued function $b(x)$ such that $b \in L_d(\Omega)$.

Next assume that there is an $R_0 \in (0, \infty)$ such that for any ball $B \subset \mathbb{R}^d$ of radius smaller than $R_0$

$$\int_{B} |a(x) - \bar{a}_B| dx \leq \theta, \quad \bar{a}_B = \int_{B} a(x) dx.$$

By using $d$, $\delta$, and $\|b\|_L_d(\Omega)$ find $d_0$ as before Theorem 1.1 and take $p \in (d_0, d)$.

Suppose that we are given $f \in L_p(\Omega)$, nonnegative bounded $c$ on $\Omega$, and $g \in C(\partial \Omega)$. Consider the equation

$$a^{ij}D_{ij}u + b^jD_iu - cu + f = 0$$

in $\Omega$ with boundary condition $u = g$ on $\partial \Omega$.

In this situation one can obviously take $F(u'', x) = a^{ij}(x)u''_{ij}$ and satisfy Assumption 1.2 with $\tilde{F}(u) = \tilde{a}_{ij}(x)u''_{ij}$ and $\tau_0 = 0$. Assumptions 1.1 (with $\bar{\theta} = 0$, $K_0 = \sup c$, $b = |b|$, $G = |f|$), 1.3, 1.4, and 1.5 are also satisfied. Therefore, by Theorem 1.2, if $\theta$ is sufficiently small, depending only on $d, p, \delta$, and $\|b\|_L_d(\Omega)$, the above boundary value problem has a solution in $u \in W^2_{p, \text{loc}}(\Omega) \cap C(\bar{\Omega})$. Owing to Theorem 2.1 this solution is unique and this theorem in combination with Theorem 1.1 shows that for all sufficiently small $\rho > 0$

$$\|u\|_{W^2_p(\Omega)} \leq N\rho^{-2}(\|f\|_{L_p(\Omega)} + \|g\|_{C(\partial \Omega)}).$$
ELLiptic Equations With $L_d$-Drift

Just in case, observe that how small $\rho$ is depends on the function $|b|$ and not only on its $L_d$-norm. The main novelty in this example is that $b \in L_d(\Omega)$, and even if $a$ is continuous the result was not known before.

We finish the section with a general comment. In the proofs of various results we use the symbol $N$ to denote finite nonnegative constants which may change from one occurrence to another and we do not always specify on which data these constants depend. In these cases the reader should remember that, if in the statement of a result there are constants called $N$ which are claimed to depend only on certain parameters, then in the proof of the result the constants $N$ also depend only on the same parameters unless specifically stated otherwise. Of course, if we write $N = N(...)$, this means that $N$ depends only on what is inside the parentheses. Another point is that when we say that certain constants depend only on such and such parameters we mean, in particular, that the dependence is such that these constants stay bounded as the parameters vary in compact subsets of their ranges.

2. Some results from [14] and [15]

The proofs of Theorems 1.1 and 1.2 is based on some results from [14] and [15] which we collect here.

Let $F(u'')$ be a convex function defined for $u'' \in S$ such that at all points of its differentiability we have

$$D_{u''}F(u'') \in S_\delta,$$

where $\delta \in (0,1]$ is a fixed number. Introduce $\mathcal{L}(\delta, ||b||)$ as the set of operators

$$L = a^{ij}D_{ij} + b^iD_i,$$

where $a = (a^{ij})$ is a measurable $S_\delta$-valued function on $\mathbb{R}^d$, $b = (b^i)$ is a measurable $\mathbb{R}^d$-valued function such that

$$||b||_{L_d(\mathbb{R}^d)} \leq ||b||.$$

We need the following which for bounded $b$ is found in [1] and for $b \in L_{d+\epsilon}(\Omega)$ in [8]. This is Corollary 3.1 of [14].

Theorem 2.1. There is a constant $d_0 = d_0(d, \delta, ||b||) \in (d/2, d)$ such that if $p \in [d_0, \infty)$, $\Omega$ is a bounded domain in $\mathbb{R}^d$, and $u \in W^2_{p,loc}(\Omega) \cap C(\bar{\Omega})$, then for any nonnegative measurable function $c$ on $\Omega$ and $L \in \mathcal{L}(\delta, ||b||)$ we have in $\Omega$

$$u \leq N||(Lu - cu)_-||_{L_p(\Omega)} + \sup_{\partial \Omega} u_+,$$

where $N$ depends only on $p, d, \delta, ||b||$, and the diameter of $\Omega$.

Here is Theorem 3.2 of [14], which is useful while passing to the limit in our nonlinear equations.
Theorem 2.2. Let \( p \geq d_0, R \in (0, \infty], \) and \( L \in \mathcal{L}(\delta, \|b\|). \) Then there exists a constant \( N = N(p, d, \delta, \|b\|) \geq 0 \) such that for any \( \lambda > 0 \) and \( u \in W^2_{p, \text{loc}}(B_R) \cap C(\bar{B}_R) \) \( (B_\infty = \mathbb{R}^d, C(\mathbb{R}^d) \) is the set of bounded continuous functions on \( \mathbb{R}^d) \) we have

\[
\lambda \|u\|_{L^p(B_R/2)} \leq N(\lambda u - Lu)_{+} \|L_{\partial B_R} + N \lambda R^{d/p} e^{-R \sqrt{\gamma/N} \sup_{\partial B_R} u},
\]

where the last term should be dropped if \( R = \infty. \)

We also need the following Theorem 4.5 of [14], which is similar to the Fanghua Lin theorem and is used as one of the main tools in the way the theory of fully nonlinear elliptic equations is developed in [13].

Theorem 2.3. Let \( R \in (0, \infty), p \in [d_0, \infty), u \in W^2_{p, \text{loc}}(B_R) \cap C(\bar{B}_R), \) \( L \in \mathcal{L}(\delta, \|b\|), \) and \( c \in L_{d_0}(B_R), \) \( c \geq 0. \) Then

\[
\left( \int_{B_R} |D^2u|^\gamma dx \right)^{1/\gamma} \leq N \left( \int_{B_R} |Lu - cu|^p dx \right)^{1/p} + NR^{-2} \sup_{\partial B_R} |u|,
\]

where \( \gamma = \gamma(d, \delta, \|b\|) \in (0, 1) \) and \( N \) depends only on \( d, \delta, \|b\|, p, \) and \( R^{2-d/d_0} \|c\|_{L_{d_0}(B_R)}. \)

The following is Corollary 4.11 of [14] about the boundary behavior of solutions of linear equations which easily carries over to the nonlinear case.

Theorem 2.4. Let \( D \) be a bounded domain in \( \mathbb{R}^d, 0 \in \partial D, \) and assume that for some constants \( \rho, \gamma > 0 \) and any \( r \in (0, \rho) \) we have \( |B_r \cap D| \geq \gamma|B_r|. \) Suppose that we are given a function \( u \in W^2_{d_0, \text{loc}}(D) \cap C(D) \) and let \( u(r) \) be a concave continuous function on \( [0, \infty) \) such that \( u(0) = 0 \) and \( |u(x) - u(0)| \leq w(|x|) \) for all \( x \in \partial D. \) Then for \( x \in D \) we have

\[
|u(x) - u(0)| \leq N|x|^\beta \|Lu\|_{L_{d_0}(D)} + \omega(N|x|^{\beta/2}),
\]

where \( L \in \mathcal{L}(\delta, \|b\|) \) and \( N \) depends only on \( d, \delta, \|b\|, \gamma, \rho, \) and the diameter of \( D. \)

The following is Corollary 6.8 of [15] about estimates of the H"older constant of solutions.

Theorem 2.5. Let \( R \in (0, \infty), p \geq d_0, \) and let \( u \in W^2_p(B_{2R}) \) and \( L \in \mathcal{L}(\delta, \|b\|). \) Define \( f = Lu. \) Then there exists a constant \( N, \) which depends only on \( p, d, \|b\|, \) and \( \delta, \) such that

\[
|u(x_1) - u(x_2)| \leq NR^{-\alpha}|x_1 - x_2|^{\alpha} \left( \sup_{B_{2R}} |u| + R^{2-d/p} \|f\|_{L_p(B_{2R})} \right)
\]

for \( x_1, x_2 \in B_R \) with \( \alpha = \alpha(d, \delta, \|b\|) \in (0, 1). \)

We also need the following result by Safonov (see [18], [19], or Section 10.3 in [13]). This is another building block in the way the theory of fully nonlinear elliptic equations is developed in [13].
Theorem 2.6. There exists a constant $\alpha_0 = \alpha_0(\delta, d) \in (0, 1)$ such that for any $g \in C(\partial B_2)$ there exists a unique $v \in C(B_2) \cap C^{2+\alpha_0}_{\text{loc}}(B_2)$ satisfying
\begin{equation}
F(D^2v) = 0 \quad \text{in} \quad B_2, \quad v = g \quad \text{on} \quad \partial B_2. \tag{2.2}
\end{equation}
Furthermore,
\[
|D^2v(x) - D^2v(y)| \leq N|x-y|^\alpha_0 \sup_{\partial B_2} |g - p|
\]
as long as $x, y \in B_1$, where $p$ is an arbitrary polynomial of degree 2 on $\mathbb{R}^d$ and $N$ depends only on $\delta$ and $d$.

Below we fix $\alpha \in (0, \alpha_0]$. Here is Lemma 10.3.2 of [13].

Lemma 2.7. Let $r \in (0, \infty)$, $\nu \geq 2$ and let $\phi \in C(\partial B_{\nu r})$. Then there exists a unique $v \in C(B_{\nu r}) \cap C^{2+\alpha}_{\text{loc}}(B_{\nu r})$ such that
\[
F(D^2v) = 0 \quad \text{in} \quad B_{\nu r}, \quad v = \phi \quad \text{on} \quad \partial B_{\nu r}.
\]
Furthermore,
\[
\int_{B_r} \int_{B_r} |D^2v(x) - D^2v(y)| \, dx \, dy \leq \int_{\partial B_{\nu r}} |\phi|.
\]

Finally, we will use the following, which allows us to use a version of the Fefferman-Stein theorem.

Lemma 2.8. Let $r \in (0, \infty)$ and $\nu \in [2, \infty)$. Then for any $u \in W^{2}_d(B_{\nu r})$ we have
\[
\left( \int_{B_r} \int_{B_r} |D^2u(x) - D^2u(z)|^\gamma \, dx \, dy \right)^{1/\gamma} \leq N\nu^{d/\gamma} \left( \int_{B_{\nu r}} |F[u]|^{d_0} \, dx \right)^{1/d_0} + N\nu^{-\alpha} \left( \int_{B_{\nu r}} |D^2u|^{d_0} \, dx \right)^{1/d_0}, \tag{2.3}
\]
where $N$ depends only on $d$, $\delta$, and $\|b\|$ and $\gamma$ is taken from Theorem 2.3.

Proof. Define $v$ to be a unique $C(B_{\nu r}) \cap C^{2+\alpha}_{\text{loc}}(B_{\nu r})$-solution of the equation $F[v] = 0$ in $B_{\nu r}$ with boundary condition $v = u$ on $\partial B_{\nu r}$. Such a function exists by Lemma 2.7. Furthermore, $v(x) - b^i x^i - c$ satisfies the same equation for any constants $b^i, c$. Hence by Lemma 2.7 and Hölder’s inequality
\[
I_r := \left( \int_{B_r} \int_{B_r} |D^2v(x) - D^2v(y)|^\gamma \, dx \, dy \right)^{1/\gamma} \leq N\nu^{-2-\alpha} \sup_{x \in \partial B_{\nu r}} |u(x) - (D_iu)_{B_{\nu r}} x^i - u_{B_{\nu r}}|.
\]
By Poincaré’s inequality (recall that $d_0 > d/2$) the last supremum is dominated by a constant times
\[
\nu^2 \sup_{x \in \partial B_{\nu r}} |u(x) - (D_iu)_{B_{\nu r}} x^i - u_{B_{\nu r}}|.
\]
It follows that
\[ I_r \leq N \nu^{-\alpha} \left( \int_{B_{ \nu r}} |D^2 u|^{d_0} \, dx \right)^{1/d_0}. \tag{2.4} \]

Next, the function \( w = u - v \) is of class \( C(\overline{B_{\nu r}}) \cap W^{2, d_0}_{d_0, \text{loc}}(B_{\nu r}) \) and for an operator \( L \in L_{d_0} \) we have
\[ F[u] - F[v] = L(u - v), \quad Lw = F[u] \]
in \( B_{\nu r} \) (a.e.). Moreover, \( w = 0 \) on \( \partial B_{\nu r} \). Therefore, by Theorem 2.3
\[ \int_{B_{\nu r}} |D^2 w| \, dx \leq \nu^{d_0} \int_{B_{\nu r}} |D^2 u| \, dx \leq N \nu^{d_0} \left( \int_{B_{\nu r}} |F[u]|^{d_0} \, dx \right)^{1/d_0}. \]

Upon combining this result with (2.4) we come to (2.3) and the lemma is proved. \( \square \)

3. PROOF OF THEOREM 1.1

Here we suppose that Assumptions 1.1 and 1.2 are satisfied with \( \theta, \hat{\theta} \) to be specified later. Thus, we suppose that all assumptions stated before Theorem 1.1 are satisfied.

First we recall the following Lemma 10.4.1 of [13].

**Lemma 3.1.** For any \( q \in [1, \infty) \) and \( \mu > 0 \) there is a \( \theta = \theta(d, \delta, K_F, \mu, q) > 0 \) such that, if Assumption 1.2 is satisfied with this \( \theta \), then the following holds:
\[ \text{for any } u_0' \in \mathbb{R}, \ r \in (0, R_0] \text{ and } z \in \Omega \text{ such that } B_r(z) \subset \Omega \text{ we have} \]
\[ \int_{B_r(z)} \sup_{u'' \in \mathbb{S}, \ |u''| > \tau_0} \left| F(u_0', u'', x) - \bar{F}(u'') \right| \frac{q}{|u''|^q} \, dx \leq \mu^q, \]
where \( \bar{F} = \bar{F}_{z, r, u_0'} \).

Below \( \gamma \) is taken from Theorem 2.3.

**Lemma 3.2.** Let \( r \in (0, \infty) \) and \( \nu \geq 2 \) be such that \( \nu r \leq R_0 \) and \( \Omega^{\nu r} \neq \emptyset \). Take
\[ \mu \in (0, \infty), \quad \beta \in (1, \infty), \]
and suppose that the assertion of Lemma 3.1 holds with \( q = \beta d_0 \). Take a function \( u \in W^2_{d_0}(\Omega) \), and for \( x_0 \in \Omega^{\nu r} \) denote
\[ I_r(x_0) = \left( \int_{B_r(x_0)} \int_{B_r(x_0)} |D^2 u(x_1) - D^2 u(x_2)|^\gamma \, dx_1 \, dx_2 \right)^{1/\gamma}. \]

Then for any \( x_0 \in \Omega^{\nu r} \)
\[ I_r(x_0) \leq N \nu^{d/\gamma} \left( \int_{B_{\nu r}(x_0)} |F[u]|^{d_0} \, dx \right)^{1/d_0} + N \tau_0 \nu^{d/\gamma} \]
\[ + N \left[ (\mu + \omega_{F,u,\Omega(\nu r)}) \nu^{d/\gamma} + \nu^{-\alpha} \right] \left( \int_{B_{\nu r}(x_0)} |D^2 u|^{\beta d_0} \, dx \right)^{1/(\beta' d_0)}, \tag{3.1} \]
where \( \beta' = \beta / (\beta - 1) \) and \( N \) depends only on \( d, K_F, \delta, \) and \( \| b \| \).

This lemma is proved in the same way as Lemma 10.4.2 of [13], basically, using only Hölder’s inequality and Lemmas 3.1 and 2.8. By the way the term \( \omega_{F,u,\Omega}(\nu r) \) appears because of Assumption 1.2 (iv).

Lemma 3.2 allows us to follow the proof of Lemma 10.4.3 of [13], which we prefer here to split into two parts. Here is the first part.

**Lemma 3.3.** Take \( p \in (d_0,d), \ R \in (0,1], \) and \( u \in W^2_p(B_{2R}) \). Take \( \mu \in (0,\infty) \) and suppose that the assertion of Lemma 3.1 holds with \( q = \beta p, \) where \( \beta \) is so large that \( \beta'd_0 < p. \) Take \( \varepsilon \in (0,1] \) and let \( 0 < R_1 < R_2 \leq 2R \) be such that

\[
R_2 - R_1 \leq \varepsilon R_0, \quad R_2 \leq 2R_1. \tag{3.2}
\]

Assume that \( B_{2R} \subset \Omega. \) Then there exist constants \( N, N_1, \) and \( N_2, \) depending only on \( d, p, K_F, \delta, \) and \( \| b \|, \) such that

\[
\| D^2u \|_{L_p(B_{R_1})} \leq N_1 \| F[u] \|_{L_p(B_{R_2})} + N_2 R_1^{d/p}
+ \left[ \frac{N_2 (\mu + \omega_{F,u,B_{2R}}(\varepsilon R_0)) + 1/16}{1} \right] \| D^2u \|_{L_p(B_{R_2})}
+ N(R_2 - R_1)^{-\chi_1} R_1^{-\chi_2 + \chi_1} \| D^2u \|_{L_1(B_{R_2})}^{1/\gamma}, \tag{3.3}
\]

where

\[
\chi_1 = (d + 2)/\gamma, \quad \chi_2 = d/\gamma - d/p. \tag{3.4}
\]

Proof. For \( \rho > 0 \) and \( x \in \mathbb{R}^d \) introduce

\[
h_{\gamma,\rho}(x) = \sup_{r \in (0,\rho], \ B_r(x) \ni x} \left( \int_{B_r(x)} \int_{B_r(x)} |h(x_1) - h(x_2)|^\gamma \, dx_1 \, dx_2 \right)^{1/\gamma},
\]

\[
M(h)(x) = \sup_{r > 0, \ B_r(x) \ni x} \int_{B_r(x)} |h(y)| \, dy, \tag{3.5}
\]

whenever these definitions make sense.

Then take \( \nu \geq 2 \) and set

\[
r_0 = (R_2 - R_1)/(\nu + 1).
\]

Next, take \( x, x_0, \) and \( r > 0 \) such that

\[
r \leq r_0, \quad x \in B_{R_1}, \quad x \in B_r(x_0)
\]

and observe that, since \( R_2 - \nu r_0 = R_1 + r_0, \) we have \( x_0 \in B_{R_2-\nu r_0} \) and \( B_{\nu r}(x_0) \subset B_{R_2}. \) Also \( \nu r \leq \nu r_0 \leq R_0. \) Therefore, by Lemma 3.2 applied to \( \Omega = B_{R_2}, \) we have (note \( x_0 \) on the left and \( x \) on the right)

\[
I_r(x_0) \leq N \nu^{d/\gamma} M^{1/d_0}(\| F[u] \|_{d_0}^d I_{B_{R_2}})(x) + N \tau_0 \nu^{d/\gamma}
+ N \left[ (\mu + \omega_{F,u,B_{2R}}(\nu \tau_0)) \nu^{d/\gamma} + \nu^{-\alpha} \right] M^{1/(\beta d_0)}(\nu \beta d_0 I_{B_{R_2}})(x).
\]
with $N$ depending only on $d$, $K_F$, and $\delta$. It follows that in $B_{R_1}$
\[(D^2 u)^{2s}_{2s, r_0} \leq N|u|^{d/\gamma}M_1^{1/d_0}(|F|^{d_0}I_{B_{R_2}}) + N\n\]
\[+ N\left[(\mu + \nu_{r, u, B_{2R}}(\varepsilon R_0)\nu^{d/\gamma} + \nu^{-\alpha}\right)\right]\times \left[M_1^{1/(\beta'd_0)}(|D^2 u|^{\beta'd_0}I_{B_{R_2}})\right].\]

By Theorem C.2.6 of [13] (which is similar to the Fefferman-Stein theorem) with $\kappa = r_0/R_1 \leq 1/3$ and $\chi_1, \chi_2$ from (3.4) and the Hardy-Littlewood maximal function theorem (recall that $p > \beta'd_0$), we obtain
\[
\|D^2 u\|_{L^p(B_{R_1})} \leq N\nu^{d/\gamma}\|F[u]\|_{L^p(B_{R_2})} + N\nu^{d/\gamma}R_1^{d/p} + N\left[(\mu + \nu_{r, u, B_{2R}}(\varepsilon R_0)\nu^{d/\gamma} + N\nu^{-\alpha}\right)\right]\times \left[M_1^{1/(\beta'd_0)}(\|D^2 u\|_{L^p(B_{R_2})})\right] + N\nu^{\delta_{1}}(R_2 - R_1)^{-\delta_{1}}R_1^{-\delta_{2}+\delta_{1}}\|\|D^2 u\|\|_{L^1(B_{2R})},\]
where the constants $N, N_1$ depend only on $d, p, K_F, \|b\|, \text{and} \delta$. Now we take and fix $\nu \geq 2$ so that
\[N\nu^{-\alpha} \leq 1/16.\]

Then (3.6) becomes (3.3). The lemma is proved. 

The constant $N_1$ in (3.3) depends only on $d, p, \beta, K_F, \|b\|, \text{and} \delta$, and $\beta$ can be easily made to depend only on $p$ and $d_0$. Therefore, the constant $N_1$ in (3.3) depends only on $d, p, K_F, \|b\|, \text{and} \delta$: $N_1 = N_1(d, p, K_F, \|b\|, \delta)$. Another constant we need to proceed is the following. For $p \in [1, d]$ and $q = pd/(d - p)$ by interpolation inequalities there is a constant $N(p, d)$ such that for any $R \in (0, 1)$ and $u \in W^2_p(B_R)$ we have
\[
\|D^2 u\|_{L^q(B_R)} \leq N(p, d)\|D^2 u\|_{L^p(B_R)} + N(p, d)R^{-2}\|u\|_{L^p(B_R)}.
\]

Now, we define $\bar{R}(p)$ by requiring that $\bar{R}(p) \in (0, 1)$ and for any $x \in \Omega$, and for any $R \in (0, \bar{R}(p))$
\[N_1(d, p, K_F, \|b\|, \delta)\|b\|_{L^q(B_{2R}(x))}N(p, d) \leq 1/8.\]

**Lemma 3.4.** Take $p \in (d_0, d), R \in (0, \bar{R}(p))$, and $u \in W^2_p(B_{2R})$. Assume that $B_{2R} \subset \Omega$. Then there exist constants $\hat{\theta}, \theta \in (0, 1)$, depending only on $d, p, \delta, \|b\|, \text{and} K_F$, such that, if Assumptions 1.1 and 1.2 are satisfied with these $\theta$ and $\theta$, respectively, then there is a constant $N$, depending only on $R_0, d, p, K_0, K_F, \delta, \|b\|$, and the function $\omega_{r, u, B_{2R}}$, such that
\[
\|D^2 u\|_{L^p(B_{R})} \leq N\|H[u]\|_{L^p(B_{2R})} + N\|G\|_{L^p(B_{2R})} + N\theta R^{d/p} + N\theta R^{d/p - d/\gamma}\|D^2 u\|^{1/\gamma}_{L^1(B_{2R})} + N\theta R^{-2}\|u\|_{L^p(B_{2R})},\]
\[
\|D^2 u\|_{L^p(B_{R})} \leq N\theta R^{d/p} + N\theta R^{d/p - 2}\sup_{B_{2R}}|u| + N\left(\|H[u]\|_{L^p(B_{2R})} + \|G\|_{L^p(B_{2R})}\right).\]

ELLiptic equations with $L_d$-driFT

Proof. Take $\varepsilon \in (0, 1]$ and let $0 < R_1 < R_2 \leq 2R$ be as in Lemma 3.3. Also take $\mu \in (0, \infty)$ and suppose that Assumption 1.2 holds with $\theta = \theta(d, \delta, K_F, \mu, \beta d_0)$ (see Lemma 3.1), where $\beta = \beta(d_0, p)$ is so large that $\beta' d_0 < p$. Then (3.3) holds. We estimate $F[u]$ by observing that

$$|F[u]| \leq |H[u]| + K_0 |u| + \theta |D^2 u|$$

and that by Hölder’s and interpolation inequalities with $q = pd/(d - p)$ and by (3.7)

$$N_1 \|bD u\|_{L_p(B_{R_2})} \leq N_1 \|b\|_{L_d(B_{R_2})} \|D u\|_{L_q(B_{R_2})} \leq \frac{1}{8} \|D^2 u\|_{L_p(B_{R_2})} + \frac{1}{8} R_2^{\delta} \|u\|_{L_p(B_{2R_2})}.$$ 

Then we take $\hat{\theta}$ and $\mu$ so small that

$$N_1 \hat{\theta} \leq 1/8, \quad N_2 \mu \leq 1/8,$$

and, finally, take the largest $\varepsilon \leq 1$ such that

$$N_2 \omega_{F, u, B_{2R}}(\varepsilon R_0) \leq 1/8.$$ 

This $\varepsilon$, which depends only on $d, p, K_F, R_0$, the function $\omega_{F, u, B_{2R}}$, $\|b\|$, and $\delta$, will appear later in our arguments and this is the way how the constant $N$ in the statement of the lemma depends on $\omega_{F, u, B_{2R}}$.

We require Assumptions 1.1 and 1.2 be satisfied with the above chosen $\hat{\theta}$ and $\theta = \theta(d, \delta, K_F, \mu, \beta d_0)$, respectively. By combining the above, we get

$$\|D^2 u\|_{L_p(B_{R_1})} \leq N \|H[u]\|_{L_p(B_{R_2})} + N \tau_0 R^{d/p} + (5/8) \|D^2 u\|_{L_p(B_{R_2})} + N \|D^2 u\|_{L_p(B_{2R_2})} + \|\tilde{G}\|_{L_p(B_{2R_2})}.$$ 

Now we are going to iterate this estimate by defining $R_1 = R$ and for $k \geq 1$

$$R_{k+1} = R_k + cR(n_0 + k)^{-2},$$

where the constant $c = O(n_0)$ is chosen so that $R_k \uparrow 2R$ as $k \to \infty$, that is

$$c \sum_{k=1}^{\infty} (n_0 + k)^{-2} = 1,$$

and $n_0 > 0$ is chosen so that for $k \geq 1$

$$R_{k+1} - R_k = cR(n_0 + k)^{-2} \leq Rcn_0^{-2} \leq R \leq R_k,$$

which is satisfied if $n_0$ is just an appropriate absolute constant, and

$$R_{k+1} - R_k = cR(n_0 + k)^{-2} \leq cn_0^{-2} \leq \varepsilon R_0$$

(this time we need $n_0^{-1} = o(\varepsilon R_0)$ if $\varepsilon R_0 \to 0$). Also observe that $R \leq R_k \leq 2R$ and

$$(R_{k+1} - R_k)^{-\chi_1} R_k^{-\chi_2 + \chi_1} \leq N(n_0 + k)^{2\chi_1} R^{-\chi_2}.$$ 

Then for $k \geq 1$ we get

$$\|D^2 u\|_{L_p(B_{R_k})} \leq N \|H[u]\|_{L_p(B_{R_{k+1}})} + \tau_0 R^{d/p} + (5/8) \|D^2 u\|_{L_p(B_{R_{k+1}})}.$$
bounded and \((3.7)\).

The solvability result from [13] in which, however, the norm of \(Du\) was assumed to be bounded and \(G \in \mathcal{L}(\delta, \|b\|)\) such that

\[
H[u] = [H(D^2 u, Du, u) - H(0, Du, u)] + H(0, Du, u) = Lu + f,
\]

where \(|f| \leq K_0 \|u\| + \bar{G}\). Therefore, \(Lu = H[u] - f\) and by Theorem 2.3

\[
N^{d/p - d/\gamma} \|D^2 u\|_{L^1(B_{2R})}^{1/\gamma} \
\leq N \|Lu\|_{L^p(B_{2R})} + NR^{d/p - 2} \sup_{B_{2R}} |u|,
\]

where we used that \(\|u\|_{L^p(B_{2R})} \leq NR^{d/p} \sup_{B_{2R}} |u|\) because \(R \leq 1\). Finally, observing that \(\|u\|_{L^p(B_{2R})} \leq NR^{d/p} \sup_{B_{2R}} |u|\) we come from (3.8) to (3.9) and the lemma is proved.

**Proof of Theorem 1.1.** We take the constants \(\hat{\theta}, \theta \in (0, 1]\) from Lemma 3.4. By that lemma, if \(\rho \in (0, \hat{R}(p)]\) and \(\Omega^{2\rho} \neq \emptyset\) and \(z \in \Omega^{2\rho}\), we have \(B_{2\rho}(z) \subset \Omega\) and

\[
\|D^2 u\|^p_{L^p(B_{\rho}(z))} \leq N\tau_0^p |\rho|^d + N\rho^{d - 2p} \sup_{B_{2\rho}(z)} |u|^p + N\|\bar{G}\|^p_{L^p(B_{2\rho}(z))}.
\]

This, Lemma 10.4.4 of [13], implies that, for \(0 < 3\rho < \rho_{int}(\Omega) \wedge 3\),

\[
\int_{\Omega^{2\rho}} |D^2 u(x)|^p dx \leq N \int_{\Omega} |\bar{G}(x)|^p dx + N\tau_0^p + N\rho^{-2p} \sup_{\Omega} |u|^p. \quad (3.10)
\]

Using interpolation inequalities also allows us to estimate the \(L^p(\Omega^{3\rho})\)-norm of \(Du\). The theorem is proved. 

**4. Proof of Theorem 1.2**

We give the proof of Theorem 1.2 after some preparations. First we use the solvability result from [13] in which, however, \(b\) was assumed to be bounded and \(G \in \mathcal{L}(\delta, \|b\|)\).
Lemma 4.1. Assume that $\bar{G} \in L_q(\Omega)$ for some $q > d$ and let $p \in (d_0, d)$. For $n = 1, 2, \ldots$ introduce

$$H^n(u, x) = H(u'_0, n[u']/(n + b(x)), u'', x).$$  \hfill (4.1)

Then there exist constants $\hat{\theta}, \theta \in (0, 1)$, depending only on $d, p, \delta$, and $K_F$, such that, if Assumptions 1.2 and 1.1 are satisfied with these $\theta$ and $\hat{\theta}$, respectively, then for any $n$ there exists a solution $u_n \in W^2_{p, \loc}(\Omega) \cap C(\Omega)$ of the equation

$$H^n[u_n] = 0$$  \hfill (4.2)

(a.e.) in $\Omega$ with boundary data $u_n = g$ on $\partial \Omega$. Furthermore, for any $0 < \rho < \rho_{\text{int}}(\Omega) \wedge 1 \wedge \bar{R}(p)$, we have

$$\|u_n\|_{W^2_p(\Omega)} \leq N\|\bar{G}\|_{L_p(\Omega)} + N\rho^{-2}\|u_n\|_{C(\Omega)} + N\tau_0,$$  \hfill (4.3)

where the constants $N$ depend only on $K_0, K_F, d, p, \delta, \|b\|, R_0, \text{diam}(\Omega)$, and the function $\omega_{F, u, \Omega}$.

Proof. Observe that owing to (1.2) we have

$$|G(u'_0, n[u']/(n + b), u'', x)| \leq \hat{\theta}|u''| + K_0|u'_0| + b_n||u'|| + \bar{G},$$

and $b_n = nb/(n + b)$ is bounded. Therefore, by Theorem 10.1.14 of [13] there exist constants $\hat{\theta}, \theta \in (0, 1)$, depending only on $d, \delta$, and $K_F$, such that, if Assumptions 1.2 and 1.1 are satisfied with these $\theta$ and $\hat{\theta}$, respectively, then equation (4.2) with given boundary data has a solution $u_n \in W^2_{q, \loc}(\Omega) \cap C(\Omega)$. By reducing $\theta$ and $\hat{\theta}$ in order to accommodate those in Theorem 1.1 we prove the second statement. The lemma is proved. \qed

Now, naturally we want to sent $n \to \infty$. For a function $u = u(x)$, for which $Du(x)$ is well defined we set

$$H_{u, Du}(u'', x) = H(u(x), Du(x), u'', x).$$

Lemma 4.2. Let $p \in (d_0, \infty)$, $R \in (0, \infty)$, $u, u_n \in W^2_p(B_R)$, $n = 1, 2, \ldots$.

Suppose that

$$M := \sup_n\|u_n\|_{W^2_p(B_R)} < \infty, \quad u_n \to u \text{ weakly in } W^2_p(B_R).$$

Then there is a subsequence $n' \to \infty$ such that in $B_R$ (a.e.)

$$\lim_{n' \to \infty} H_{u_{n'}, Du_{n'}}(D^2u(x), x) = H[u](x),$$  \hfill (4.4)

$$\sup_{n'} |H_{u_{n'}, Du_{n'}}(u'', x)| \leq N(d, \delta)|u''| + \bar{H},$$  \hfill (4.5)

where the nonnegative $\bar{H}$ is such that

$$\|\bar{H}\|_{L_{d_0}(B_R)} \leq N(d, d_0, p, R)(K_0 + \|b\|)(M + 1) + \|\bar{G}\|_{L_{d_0}(B_R)}.$$
Proof. Let \( q = d_0 d/(d - d_0) \). By embedding theorems \( u_n \to u \) strongly in \( W^1_q(B_R) \) and there exists a subsequence, identified for simplicity with the original one, such that

\[
\|u_{n+1} - u_n\|_{W^1_q(B_R)} \leq 2^{-n}.
\]

Then \( u_n, Du_n \to u, Du \) (a.e.) in \( B_R \) and (4.4) follows since \( H \) is continuous in \( u' \).

Next set

\[
w_0 = \sum_n |u_{n+1} - u_n| + |u_1|, \quad w_1 = \sum_n |Du_{n+1} - Du_n| + |Du|.
\]

We have that \( w_0, w_1 \in L_q(B_R) \), \( |u|, |u_n| \leq w_0, |Du|, |Du_n| \leq w_1 \), so that

\[
|H_{u_n, Du_n}(u'', x)| \leq N(d, \delta)|u''| + K_0 w + bw_1 + \tilde{G}.
\]

This implies (4.5) because by Hölder’s inequality \( bw_1 \in L_{d_0}(B_R) \). The lemma is proved.

To pass to the limit as \( n \to \infty \) under the sign of \( H \) which is nonlinear we use the following replacement of nonlinear operators with linear ones.

**Lemma 4.3.** Let \( p \in [1, \infty) \), \( u \in W^2_{p, loc}(\Omega) \) satisfy (1.1) in \( \Omega \) (a.e.). Then there exists an \( \mathbb{S}_\delta \)-valued measurable function \( a, \mathbb{R}^d \)-valued measurable \( b \) such that \( |b| \leq b \) and (a.e.) in \( \Omega \)

\[
|a^{ij} D_{ij} u + b^i D_i u| \leq K_0 |u| + \tilde{G}. \tag{4.6}
\]

Furthermore, if \( u \geq 0 \) in \( \Omega \), then (a.e.) in \( \Omega \) (with perhaps different \( b \))

\[
a^{ij} D_{ij} u + b^i D_i u + \tilde{G} \geq 0 \tag{4.7}
\]

and if \( u \leq 0 \) in \( \Omega \), then (a.s.) in \( \Omega \)

\[
a^{ij} D_{ij} u + b^i D_i u - \tilde{G} \leq 0. \tag{4.8}
\]

Proof. By using Assumption 1.3 we get

\[
0 = H[u] = H[u] - H(u, Du, 0, x) + H(u, Du, 0, x) = a^{ij} D_{ij} u + H(u, Du, 0, x),
\]

where by Assumption 1.1

\[
|H(u, Du, 0, x)| \leq K_0 |u| + b |Du| + \tilde{G} = K_0 |u| + be^i D_i u + \tilde{G},
\]

where \( e = Du/|Du| \), which implies that for a function \( t(x) \) with values in \([-1, 1]\)

\[
H(u, Du, 0, x) = tK_0 |u| + tbe^i D_i u + t\tilde{G},
\]

and this yields (4.6).

To prove (4.7) we use the information provided by Assumption 1.5 saying the if \( u \geq 0 \), then

\[
H(u, Du, 0, x) \leq b |Du| + \tilde{G},
\]

which yields (4.7). Similarly (4.8) is obtained. The lemma is proved.

**Lemma 4.4.** The functions \( u_n, n = 1, 2, ..., \) are uniformly continuous and uniformly bounded in \( \bar{\Omega} \) with the estimates of their sup norms and moduluses of continuity involving only the \( L_p(\Omega) \)-norm of \( G \) and not its \( L_q(\Omega) \)-norm.
Proof. Fix $n$ and denote $\Omega_+ = \{ x \in \Omega : u_n(x) > 0 \}$. By Lemma 4.3 we have (4.7) on $\Omega'$. By Theorem 2.1 we have
\[
 u_n \leq N\| G \|_{L_p(\Omega')} + \sup_{\partial \Omega'} (u_n)_+ \leq N\| G \|_{L_p(\Omega)} + \sup_{\partial \Omega \cap \partial \Omega} g_+,
\]
where $N$ depends only on $p, d, \delta, \| b \|$, and the diameter of $D$. Similarly one estimates $u_n$ from below.

To show that $u_n$ are equicontinuous we use (4.6), denote $a^{ij} D_{ij} u_n + b^i D_i u_n = f_n$ and use that, in light of the first assertion, the $L_p(\Omega)$-norms of $f_n$ are uniformly bounded. Then by Theorem 2.5 there is a constant $N$ depending only on $p, d, \| b \|, \| G \|_{L_p(\Omega)}$, $\| g \|$, and the diameter of $\Omega$, such that
\[
 |u_n(x) - u_n(y)| \leq N (\rho(x) \wedge \rho(y))^{-\alpha} |x - y|^\alpha,
\]
where $\rho(z)$ is the distance from $z$ to $\Omega^c$ and $\alpha = \alpha(d, \delta, \| b \|) \in (0, 1)$.

Furthermore, by Theorem 2.4 there is $\beta = \beta(d, \delta, \| b \|) > 0$ and a constant $N$, depending only on $d, \delta, \| b \|$, and $\Omega$, such that
\[
 |u_n(x) - u_n(x_0)| \leq N |x - x_0|^{\beta} \| f_n \|_{L_p(\Omega)} + w (N |x - x_0|^{\beta/2})
\]
whenever $x \in \Omega$ and $x_0 \in \partial \Omega$, where $w$ is the concave modulus of continuity of $g$.

A standard combination of these interior and boundary estimates leads to our assertion. The lemma is proved. 

The main tool allowing us to pass to the limit under the sign of $H$ is given by the following lemma, which is stated for the signs $\pm$ meaning that it holds when one takes everywhere the upper sign and ignores the lower one and also holds when one takes everywhere the lower sign and ignores the upper one. It is worth saying that generally, the results of such kind are taken from Section 3.5 of [11]. They generalize earlier results for elliptic equations by the author [10] (1971) and Evans [7] (1978). The methods in [10] are quite transparent and are based on expressing the solution of the equation $H[u] = -f$ in the form $u = R_\lambda (\lambda u + f)$, $\lambda > 0$, where $R_\lambda$ is a nonlinear integral operator continuous in $L_p$. It is easy to pass to the limit under the sign of $R_\lambda$. In addition, it turns out that if $u \in W^2_p$, then $\lambda [R_\lambda (\lambda u + f) - u] \to F[u] + f$ as $\lambda \to \infty$. Later on it became clear that the above integral representations are equivalent to having (4.9) that possesses the same features as the integral representations.

**Lemma 4.5.** Let $p \in [d_0, \infty)$, $R \in (0, \infty)$, $u \in W^2_p(B_R)$. Suppose that we are given a function $H(u'', x)$ that satisfies Assumption 1.3 and is such that
\[
 |H(u'', x)| \leq N_0 |u''| + \tilde{H}(x),
\]
where $N_0$ is independent of $u''$ and $x$ and nonnegative $\tilde{H}$ belongs to $L_p(B_R)$. Then
(i) there exists a constant $N = N(p, d, \delta) \geq 0$ such that for any $\lambda > 0$ and $\phi \in W^2_p(\partial B_R)$ we have
\[
\lambda \| (\phi - u) \|_{L_p(\partial B_R)} \leq N \| (\lambda (\phi - u) - (H[\phi] - f)) \|_{L_p(\partial B_R)} + N \lambda R^{d/p} e^{-R\sqrt{N}/N} \sup_{\partial B_R} (\phi - u),
\]
(4.9)
for any $f \in L_p(\partial B_R)$ such that $(H[u] - f)_+ = 0$ on $\partial B_R$;
(ii) if there is a constant $N$ such that (4.9) holds for an $f \in L_p(\partial B_R)$ and any sufficiently large $\lambda > 0$ and any $\phi \in W^2_p(\partial B_R)$, then $(H[u] - f)_+ = 0$ on $\partial B_R/2$.

Proof. (i) Observe that for an $S_\delta$-valued $a$ we have $H[\phi] - H[u] = a^{ij} D_{ij}(\phi - u)$. Then the first assertion follows immediately from Theorem 2.2.
(ii) Plug $u + \phi/\lambda$ in (4.9) in place of $\phi$. Then
\[
\| \phi_+ \|_{L_p(\partial B_R/2)} \leq N \| (\phi - (H[u + \phi/\lambda] - f))_+ \|_{L_p(B_R)} + N R^{d/p} e^{-R\sqrt{N}/N} \sup_{\partial B_R} \phi_+.
\]
Letting $\lambda \to \infty$ and using the dominated convergence theorem yields
\[
\| \phi_+ \|_{L_p(\partial B_R/2)} \leq N \| (\phi - (H[u] - f))_+ \|_{L_p(B_R)}.
\]
This is true for any $\phi \in W^2_p(\partial B_R)$ and by continuity for any $\phi \in L_p(\partial B_R)$. Taking $\phi = H[u] - f$ shows that $(H[u] - f)_+ = 0$. This proves (ii) with the sign $+$. Similar argument is valid for $-$. The lemma is proved.

**Proof of Theorem 1.2.** Case $G \in L_q(\Omega)$. Take the functions $u_n$ from Lemma 4.1 and extract a subsequence $u_{n(k)}$ such that
(i) it converges weakly in $W^2_{p, \text{loc}}(\Omega)$ to a $u \in W^2_{p, \text{loc}}(\Omega)$, which is possible in light of Lemma 4.1;
(ii) converges uniformly on $\Omega$ to $u$ thus making it belong to $C(\tilde{\Omega})$, which is possible due to Lemma 4.4;
(iii) there is a function $\hat{H} \in L_{d_0, \text{loc}}(\Omega)$ such that in $\Omega$ (a.e.)
\[
\lim_{k \to \infty} H^{n(k)}_{u_n(k), Du_n(k)}(D^2 u(x), x) = H[u](x),
\]
\[
\sup_k |H^{n(k)}_{u_n(k), Du_n(k)}(u'', x)| \leq N|u''| + \hat{H},
\]
which is possible due to Lemma 4.2.

Then for $m = 1, 2, \ldots$ introduce
\[
\hat{H}^m(u'', x) = \sup_{k \geq m} H^{n(k)}_{u_n(k), Du_n(k)}(u'', x).
\]
Obviously, for $k \geq m$ we have $\hat{H}^m[u_{n(k)}] \geq 0$. 

Due to Assumption 1.3, this assumption is also satisfied for $\hat{H}^m$. Also, thanks to (4.11), $\hat{H}^m$ satisfies the assumption of Lemma 4.5 with $p = d_0$. By that lemma (with $f = 0$ and sign $-$)

$$
\lambda \| (\phi - u_n(k)) - \|_{L^{d_0}(B_R(x))} \leq N \| (\lambda (\phi - u_n(k)) - \hat{H}^m[\phi]) - \|_{L^{d_0}(B_R(x))} \\
+ N \lambda R^{d/p} e^{-R \sqrt{\lambda}/N} \sup_{\partial B_R(x)} (\phi - u_n(k))_-, 
$$

whenever $\bar{B}_R(x) \in \Omega$, $\phi \in W^{2,p}_0(B_R(x))$, and $\lambda > 0$. Passing to the limit as $k \to \infty$ and then using Lemma 4.5 again and using the arbitrariness of $B_R(x)$, we obtain $\hat{H}^m[u] \geq 0$ (a.e.) in $\Omega$. Letting $m \to \infty$ and using (4.10) yields $H[u] \geq 0$ (a.e.) in $\Omega$.

One gets that $\bar{H}^m[u] \leq 0$ (a.e.) in $\Omega$ similarly by considering

$$
\bar{H}^m(u'', x) = \inf_{k \geq m} H^{n(k)}_{u_{n}(k), Du_{n}(k)} (u'', x)
$$

and using Lemma 4.5 with sign $+$. Finally estimate (1.6) follows from what is said in the proof of Lemma 4.4. This finishes the proof in our particular case in which Assumption 1.4 was not used.

**General case.** In the above proof for $n \geq n_0$ (see Assumption 1.4) we replace $H^n$ introduced by (4.1) with

$$
H^n(u, x) = H(u, x) I_{G(x) \leq n} + I_{G(x) > n} F(u', u'', x),
$$

keep $F(u', u'', x)$ unchanged, and set $\tilde{G}^n = H^n - F$. Then Assumptions 1.3, 1.2, and 1.1 are obviously satisfied for the new couple $(H^n, F)$ with the new $\tilde{G}^n(x) = G(x) I_{G(x) \leq n}$ which is bounded. Assumption 1.5 is also satisfied with that $\tilde{G}^n$.

After that literally repeating the above proofs with the new $H^n$ proves the theorem also in the general case. □

**Acknowledgment.** The author is very grateful to Hongjie Dong and A.I. Nazarov who read the first draft of the paper and pointed out several glitches in it.

**References**

[1] X. Cabré, *On the Alexandroff-Bakelman-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math., Vol. 48 (1995), 539–570.

[2] L.A. Caffarelli, *Interior a priori estimates for solutions of fully non-linear equations*, Ann. of Math., Vol. 130 (1989), 189–213.

[3] L.A. Caffarelli, *Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation*, Ann. of Math. (2), Vol. 131 (1990), No. 1, 135–150.

[4] L.A. Caffarelli and X. Cabré, “Fully nonlinear elliptic equations”, American Mathematical Society, Providence, 1995.

[5] L. Caffarelli, M. G. Crandall, M. Kocan, and A. Święch, *On viscosity solutions of fully nonlinear equations with measurable ingredients*, Comm. Pure Appl. Math., Vol. 49 (1996), No. 4, 365–397.

[6] L. Escauriaza, *$W^{2,n}$ a priori estimates for solutions to fully non-linear equations*, Indiana Univ. Math. J., Vol. 42 (1993), No. 2, 413–423.
20 N.V. KRYLOV

[7] L.C. Evans, A convergence theorem for solutions of nonlinear second order elliptic equations, Indiana University Math. J., Vol. 27 (1978), 875–887.
[8] K. Fok, A nonlinear Fabes-Stroock result, Comm. PDEs, Vol 23 (1998), No. 5-6, 967–983.
[9] Byungsoo Kang and Hyunseok Kim, On $L^p$-resolvent estimates for second-order elliptic equations in divergence form, Potential Anal., Vol. 50 (2019), No. 1, 107–133.
[10] N.V. Krylov, On uniqueness of the solution of Bellman’s equation, Izvestiya Akademii Nauk SSSR, seriya matematicheskaya, Vol. 35 (1971), No. 6, 1377–1388 in Russian; English translation in Math. USSR Izvestija, Vol. 5 (1971), No. 6, 1387–1398.
[11] N.V. Krylov, “Nonlinear elliptic and parabolic equations of second order”, Nauka, Moscow, 1985 in Russian; English translation: Reidel, Dordrecht, 1987.
[12] N.V. Krylov, On the existence of $W^2_p$ solutions for fully nonlinear elliptic equations under relaxed convexity assumptions, Comm. Partial Differential Equations, Vol. 38 (2013), No. 4, 687–710.
[13] N.V. Krylov, “Sobolev and viscosity solutions for fully nonlinear elliptic and parabolic equations”, Mathematical Surveys and Monographs, 233, Amer. Math. Soc., Providence, RI, 2018.
[14] N.V. Krylov, On stochastic equations with drift in $L_d$, http://arxiv.org/abs/2001.04008
[15] N.V. Krylov, On diffusion processes with drift in $L_d$, http://arxiv.org/abs/2001.04950
[16] N.V. Krylov and M.V. Safonov, A certain property of solutions of parabolic equations with measurable coefficients, Izvestiya Akademii Nauk SSSR, seriya matematicheskaya, Vol. 44 (1980), No. 1, 161–175 in Russian; English translation in Math. USSR Izvestija, Vol. 16 (1981), No. 1, 151–164.
[17] M. V. Safonov, Harnack inequalities for elliptic equations and Hölder continuity of their solutions, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI), Vol. 96 (1980), 272–287 in Russian; English translation in Journal of Soviet Mathematics, Vol. 21 (March 1983), No. 5, 851–863.
[18] M. V. Safonov, On the classical solutions of Bellman’s elliptic equations, Dokl. Akad. Nauk SSSR, Vol. 278 (1984), 810–813 in Russian; English translation in Soviet Math. Dokl., Vol. 30 (1984), No. 2, 482–485.
[19] M. V. Safonov, On the classical solutions of nonlinear elliptic equations of second order, Izvestija Acad. Nauk SSSR, ser. matemat., Vol. 52 (1988), No. 6, 1272–1287 in Russian; English translation in Math. USSR Izvestija, Vol. 33 (1989), No. 3, 597–612.
[20] M.V. Safonov, Non-divergence elliptic equations of second order with unbounded drift, Nonlinear partial differential equations and related topics, 211–232, Amer. Math. Soc. Transl. Ser. 2, 229, Adv. Math. Sci., 64, Amer. Math. Soc., Providence, RI, 2010.
[21] J. Streets and M. Warren, Evans-Krylov estimates for a nonconvex Monge-Ampère equation, Math. Ann., Vol. 365 (2016), No. 1-2, 805–834.
[22] N. Winter, $W^{2,p}$ and $W^{1,p}$-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations, Z. Anal. Anwend., Vol. 28 (2009), No. 2, 129–164.

E-mail address: nkrylov@umn.edu

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455