Laplacian eigenvalues of equivalent cographs

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ABSTRACT
Let $G$ and $H$ be equivalent cographs with reductions $R_G$ and $R_H$, and suppose the vertices of $R_G$ and $R_H$ are labelled by the twin numbers $t_i$ of the $k$ twin classes they represent. In this paper, we prove that $G$ and $H$ have at least $k + \sum_{i \in I} (t_i - 1)$ Laplacian eigenvalues in common, where $I \subseteq \{1, 2, \ldots, k\}$ is the indices of the twin classes whose types are identical in $G$ and $H$. This confirms the conjecture proposed by Abrishami [A combinatorial analysis of the eigenvalues of the Laplacian matrices of cographs [Master's thesis]. Johns Hopkins University; 2019. Available from: http://jscholarship.library.jhu.edu/bitstream/handle/1774.2/61684/ABRISHAMI-THESIS-2019.pdf]. We also show that no two nonisomorphic equivalent cographs are cospectral with relation to the Laplacian matrix.

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1. Introduction
Let $G = (V, E)$ be an undirected graph with vertex set $V$ and edge set $E$. For $v \in V, N(v)$ denotes the open neighbourhood of $v$, that is, $\{w \mid \{v, w\} \in E\}$. The closed neighbourhood $N[v] = N(v) \cup \{v\}$. If $|V| = n$, the adjacency matrix $A = [a_{ij}]$ is the $n \times n$ matrix of zeros and ones such that $a_{ij} = 1$ if and only if $v_i$ is adjacent to $v_j$. Let $\delta(G)$ be the diagonal matrix of vertex degrees of $G$. The Laplacian matrix of $G$ is defined as $L(G) = \delta(G) - A(G)$. The Laplacian matrix has non-negative eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. Denote by $\text{Spec}(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$ the spectrum of $L(G)$, i.e. the Laplacian spectrum of the graph $G$.

Two nonisomorphic graphs with the same spectrum are called cospectral. In recent years, there has been a growing interest to find families of cospectral graphs. There are many constructions in the literature [1,2]. This notion is originally defined for the adjacency matrix of the graph $G$, but a natural extension of the problem is to find families of graphs that are cospectral with relation to the Laplacian matrix.

For a graph $G$, we say two vertices $v$ and $w$ are twins if $N(v) - w = N(w) - v$. A twin partition of a graph $G$ is the partition of the vertices into their equivalence classes under the relation of being twins, denoted by $V(G) = T_1 \cup T_2 \cup \ldots \cup T_k$. The twin numbers $t_1, t_2, \ldots, t_k$ of a graph $G$ are the sizes of twins classes.
Let $G$ be a graph with twin classes $T_1, T_2, \ldots, T_k$. The twin reduction of graph $G$, denoted $R_G$, is the subgraph induced by \{u_1, \ldots, u_k\}, where $u_i \in T_i$ is a representative of class $T_i$. The Figure 1 shows a graph $G$ and its twin reduction $R_G$. The twins classes $T_1$, $T_2$, $T_3$, and $T_4$, where $T_1$ is the green vertices, $T_2$ is the grey vertices, $T_3$ is the white vertex and $T_4$ is the red vertices. We say two graphs $G$ and $H$ are equivalent, if their reduction $R_G$ and $R_H$ are isomorphic and if the twin numbers of the vertices $V(R_G)$ and $V(R_H)$ are identical.

This paper is concerned with cographs. A cograph is a simple graph which contains no path on four vertices an induced subgraph, namely it is a $P_4$-free graph. An equivalent definition (see [3]) is that cographs can be obtained recursively by the following rules: (i) a graph on a single vertex is a cograph, (ii) the union and join of two cographs are cographs. This allows us to represent this class of graphs through an unique rooted tree $T_G$, called the cotree. For more details, see Section 2.

An important subclass of cographs are the threshold graphs. Threshold graphs also can be defined in terms of forbidden subgraphs, namely they are \{$P_4, 2K_2, C_4$\}-free graphs. For an account on different characterizations and properties of threshold graphs, one can see [4] and the references therein.

Our motivation for considering cographs comes from spectral graph theory. There is a considerable body of knowledge on the spectral properties of cographs and threshold graphs related to adjacency matrix [5–19]. However, the literature does not seem to provide many articles about the Laplacian matrices of cographs. One of those sporadic works and very well known is the paper of Russel Merris [20] which shows that the nonzero Laplacian eigenvalues of threshold graphs are equal to Ferrer’s conjugate of its degree sequence.

A recent and interesting work about the Laplacian eigenvalues of cographs is the Master’s thesis presented by Tara Abrishami [21]. In this work, a characterization of the Laplacian eigenvalues of a cograph is given. In particular, the following conjecture is posed in the same reference:

**Conjecture 1.1:** Let $G$ and $H$ be equivalent cographs with reductions $R_G$ and $R_H$, and suppose the vertices of $R_G$ and $R_H$ are labelled by the twin numbers $t_i$ of the $k$ twin classes they represent. Then $G$ and $H$ have at least $k + \sum_{i \in I}(t_i - 1)$ Laplacian eigenvalues in common, where $I \subseteq \{1, 2, \ldots, k\}$ is the indices of the twin classes whose types are identical in $G$ and $H$. 

**Figure 1.** A graph and its reduction representation.
In this paper, we prove the conjecture holds true. As an immediate result, we show that no two nonisomorphic equivalent cographs are cospectral. The main tool used to prove the conjecture, and some known results are reviewed in Section 2. In Section 3, we characterize equivalent cographs in terms of their cotrees. In Section 4, we confirm the conjecture holds true. In the final section, we show how to find the cospectral linear size families of cographs with relation to the Laplacian matrix, from a pair of two nonisomorphic cospectral cographs.

2. Notations and preliminaries

2.1. Cotrees

A cotree $T_G$ of a cograph $G$ is a rooted tree in which any interior vertex $w$ is either of $\cup$-type (corresponds to union) or $\otimes$-type (corresponds to join). The terminal vertices (leaves) are typeless and represent the vertices of the cograph $G$. We say that depth of the cotree is the number of edges of the longest path from the root to a leaf. To build a cotree for a connected cograph, we simply place a $\otimes$ at the tree’s root, placing $\cup$ on interior vertices with odd depth, and placing $\otimes$ on interior vertices with even depth. All interior vertices have at least two children. In [11] this structure is called minimal cotree, but throughout this paper we call it simply a cotree. The Figure 2 shows a cograph and its cotree with depth equals to 4.

Two vertices $u$ and $v$ are duplicate if $N(u) = N(v)$ and coduplicate if $N[u] = N[v]$. In fact, any collection of mutually coduplicate (resp. duplicate) vertices, e.g. with the same neighbours and adjacent (resp. not adjacent), have a common parent of $\otimes$-type (resp. $\cup$-type).

Let $u$ and $v$ be leaves in $T_G$ which are in the same branch with the leaf $v$. We say that $u$ succeeds (resp. precedes) $v$ if $u$ is on level below (resp. above) to $v$.

**Remark:** We note that vertices in a twin class $T_i$ of $G$ correspond to coduplicate (resp. duplicate) vertices in $T_G$, if they are pairwise adjacent (resp. not adjacent) twins.
Algorithm Diagonal \((T_G, x)\)

initialize \(d_i := \delta(v_i) + x\), for \(1 \leq i \leq n\)
while \(T_G\) has \(\geq 2\) leaves

select a pair \((v_k, v_l)\) (co)duplicate of maximum depth with parent \(w\)

\(\alpha \leftarrow d_k, \beta \leftarrow d_l\)

if \(w = \emptyset\)

if \(\alpha + \beta \neq -2\) \hspace{1cm} // subcase 1a

\(d_l \leftarrow \frac{\alpha - 1}{\alpha + \beta + 2}; \hspace{0.5cm} d_k \leftarrow \alpha + \beta + 2; \hspace{0.5cm} T_G = T_G - v_k\)

else if \(\beta = -1\) \hspace{1cm} // subcase 1b

\(d_l \leftarrow -1; \hspace{0.5cm} d_k \leftarrow 0; \hspace{0.5cm} T_G = T_G - v_k\)

else \hspace{1cm} // subcase 1c

\(d_l \leftarrow -1; \hspace{0.5cm} d_k \leftarrow (1 + \beta)^2; \hspace{0.5cm} T_G = T_G - v_k; \hspace{0.5cm} T_G = T_G - v_l\)

else if \(w = \emptyset\)

if \(\alpha + \beta \neq 0\) \hspace{1cm} // subcase 2a

\(d_l \leftarrow \frac{\alpha - \beta}{\alpha + \beta + 2}; \hspace{0.5cm} d_k \leftarrow \alpha + \beta; \hspace{0.5cm} T_G = T_G - v_k\)

else if \(\beta = 0\) \hspace{1cm} // subcase 2b

\(d_l \leftarrow 0; \hspace{0.5cm} d_k \leftarrow 0; \hspace{0.5cm} T_G = T_G - v_k\)

else \hspace{1cm} // subcase 2c

\(d_l \leftarrow \beta; \hspace{0.5cm} v_k \leftarrow -\beta; \hspace{0.5cm} T_G = T_G - v_k; \hspace{0.5cm} T_G = T_G - v_l\)

end loop

Figure 3. Diagonalization algorithm.

2.2. Diagonalization

For comparing the Laplacian eigenvalues of two cographs \(G\) and \(H\), we use a straightforward translation of an algorithm due to Jacobs et al. [14] to the context of Laplacian matrices of cographs. The original algorithm constructs a diagonal matrix congruent to \(A + xI_n\), where \(A\) is the adjacency matrix of a cograph, and \(x\) is an arbitrary scalar, using \(O(n)\) time and space.

One of the advantages of this method is that it can be slightly modified in such a way that we can determine, for any \(-x \in \mathbb{R}\), the number of Laplacian eigenvalues of a cograph \(G\) that are larger than \(x\), equal to \(x\), and smaller than \(x\), respectively. The algorithm’s input is the cotree \(T_G\) and \(x \in \mathbb{R}\). Each leaf \(v_i, i = 1, \ldots, n\) have a value \(d_i\) that represents the diagonal element of \(L(G) + xI_n\). It initializes all entries \(d_i\) with \(\delta(v_i) + x\), where \(\delta(v_i)\) denotes the degree of vertex \(v_i\). In each iteration, a pair \(\{v_k, v_l\}\) of duplicate or coduplicate vertices with maximum depth is selected. Then they are processed, that is, assignments are given to \(d_k\) and \(d_l\), such that either one or both rows (columns) are diagonalized. When a \(k\) row (column) corresponding to vertex \(v_k\) has been diagonalized then \(v_k\) is removed from the \(T_G\), it means that \(d_k\) has a permanent final value. Then the algorithm moves to the cotree \(T_G - v_k\). The algorithm is shown in Figure 3.

It is worth mentioning that for each iteration, the algorithm executes one of the six subcases. It should be noted that subcase 1a and subcase 2a are the ordinary cases, and the other remaining four subcases represent singularities. Executing subcase 1b requires \(\beta = -1\), executing subcase 2b requires \(\beta = 0\), executing subcase 1c requires \(\alpha + \beta = -2\), and executing subcase 2c requires \(\alpha + \beta = 0\).

Now, we will present a few results which the proofs are similar to work [14]. The following theorem is based on Sylvester’s Law of Inertia.

**Theorem 2.1:** Let \(G\) be a cograph and let \((d_v)_{v \in T_G}\) be the sequence produced by Diagonalize \((T_G, -x)\). Then the diagonal matrix \(D = \text{diag}(d_v)_{v \in T_G}\) is congruent to \(L(G) + xI_n\), so
that the number of (positive – negative – zero) entries in \((d_v)_{v \in T_G}\) is equal to the number eigenvalues of \(L(G)\) that are (greater than \(x\) - small than \(x\) – equal to \(x\)).

The following two lemmas show that if a vertex \(\otimes\)-type or \(\cup\)-type, in the cotree, have leaves with the same value, then, we can use the following routines.

**Lemma 2.1:** If \(v_1, \ldots, v_m\) have parent \(w = \otimes\), each with the same diagonal value \(y \neq -1\), then the algorithm performs \(m - 1\) iterations of subcase 1a assigning, during iteration \(j\):

\[
d_k \leftarrow \frac{j + 1}{j} (y + 1) \quad d_l \leftarrow \frac{y - (j - 1)}{j + 1}
\]

**Lemma 2.2:** If \(v_1, \ldots, v_m\) have parent \(w = \cup\), each with the same diagonal value \(y \neq 0\), then the algorithm performs \(m - 1\) iterations of subcase 2a assigning, during iteration \(j\):

\[
d_k \leftarrow \frac{j + 1}{j} y \quad d_l \leftarrow \frac{y}{j + 1}
\]

### 3. Equivalent cographs and their cotrees

In this section, we characterize equivalent cographs in terms of their cotrees. Let \(G\) and \(H\) be equivalent cographs with reductions \(R_G\) and \(R_H\). For a cograph \(G\) and its cotree \(T_G\), let \(u, v\) be leaves in the cotree \(T_G\) which have the lowest common ancestor an interior vertex, represented by \(\text{lca}(u, v)\). Clearly, they are adjacent if and only if \(\text{lca}(u, v) = \otimes\). The following result can be verified immediately.

**Lemma 3.1:** Let \(G\) and \(H\) be equivalent cographs with their cotrees \(T_G\) and \(T_H\). Let \(u, v\) be leaves in \(T_G\) which are neither coduplicate nor duplicate vertices, and let \(u', v'\) be their corresponding leaves in \(T_H\). Then \(\text{lca}(u, v)\) and \(\text{lca}(u', v')\) are the same type.

**Definition 3.1:** Let \(G\) be a cograph and \(T_G\) its cotree. For any pair \(u, v \in T_G\), we define the distance between \(u\) and \(v\) in \(T_G\), denoted by \(\text{dist}_{T_G}(u, v)\), as the shortest path of interior vertices between them.

**Definition 3.2:** Let \(G\) and \(H\) be equivalent cographs with reductions \(R_G\) and \(R_H\). Let \(u\) be a representative of the twin class \(T_u \in G\), and let \(u'\) be its corresponding in \(T_{u'} \in H\). We say \(u = u'\), if \(T_u\) and \(T_{u'}\) are twin classes of same type. Otherwise, we say \(u \neq u'\).

**Lemma 3.2:** Let \(G\) and \(H\) be equivalent cographs with their cotrees \(T_G\) and \(T_H\). Let \(u, v\) be the representatives of the twin classes \(T_u, T_v \in G\), and let \(u', v'\) be their respective correspondents in \(T_{u'}, T_{v'} \in H\).

(i) If \(u = u'\) and \(v = v'\) then \(\text{dist}_{T_G}(u, v) = \text{dist}_{T_H}(u', v')\).

(ii) If \(u = u'\) and \(v \neq v'\) then \(\text{dist}_{T_G}(u, v) = \text{dist}_{T_H}(u', v') \pm 1\).

(iii) If \(u \neq u'\) and \(v \neq v'\) then \(\text{dist}_{T_G}(u, v) = \text{dist}_{T_H}(u', v') \pm 2\).
Figure 4. The partial cotrees $T_G$ and $T_H$.

Proof: We prove the item (i). We assume that $\text{lca}(u, v)$ and $\text{lca}(u', v')$ are $\otimes$-type. By contradiction, we suppose that $\text{dist}_{T_G}(u, v) < \text{dist}_{T_H}(u', v')$. Since that $u = u'$ and $v = v'$, we have that

$$\text{dist}_{T_H}(u', v') = \text{dist}_{T_G}(u, v) + 2l$$

for some positive integer $l \geq 1$.

Without loss of generality, we assume that $u$ and $u'$ are in twin classes of $\otimes$-type while that $v$ and $v'$ are in twin classes of $\cup$-type and their partial cotrees $T_G$ and $T_H$ are represented in the Figure 4.

Now consider the respective reduction graphs $R_G$ and $R_H$ of cographs $G$ and $H$. From Equation (3) we get that there are vertices $t', w' \in R_H$ such that $w' \sim v'$ and $t' \sim v'$. Since $R_G$ and $R_H$ are isomorphic graphs, there are vertices $t, w \in R_G$ with the same properties.

We claim the leaves $t, w \in T_G$ are in the same branch with the leaf $v$. Since $t \sim u$ and for any leaf in a different branch with $v$ implies being adjacent to $v$, follows the statement.

Now, if $w$ is in a different branch with $v$ it implies that $w \sim t$, which contradics $w \sim t$. If the leaf $t$ succeeds $v$ and $w \sim v$, it implies $w \sim t$, a contradiction. Now, if the leaf $t$ precedes $v$, and since $w \sim v$, $w \sim t$, it implies that $w$ is between $t$ and $v$, and therefore we must have $l = 0$, in the Equation (3). If $\text{lca}(u, v)$ and $\text{lca}(u', v')$ are $\cup$-type the proof is analogous.

The proof is similar for the items (ii) and (iii). ■

Given a cograph $G$, we note that its reduction $R_G$ is obtained by taking only one representative of each twin classes of $G$. In terms of cotree, it means, if we remove all exceed leaves of $T_G$, we have a cotree which represents $T_{RG}$. If $R_G$ and $R_H$ are isomorphic graphs by Lemma 3.2 we have $|\text{dist}_{T_G}(u, v) - \text{dist}_{T_H}(u', v')| \leq 2$, for any leaves $u, v \in T_G$ and their corresponding $u', v' \in T_H$. This allows us to claim that $T_H$ can be obtained from $T_G$, as in the following result:

Theorem 3.1: Let $G$ and $H$ be cographs with reductions $R_G$ and $R_H$. If $T_G$ and $T_H$ are the cotrees of $G$ and $H$, then $R_G \cong R_H$ if and only if fixed $T_G$ and for some interior vertex $w_i \in T_G$, having leaves $t_i \geq 2$, then $T_H$ is obtained from $T_G$ by one of following operations:

(i) adding an interior vertex, one level below to $w_i$, and taking the leaves $t_i$ to which belong $w_i$. 

(ii) **removing the interior vertex** \(w_i\) **which has no interior vertex as successor, whose father has no leaves and taking the leaves** \(t_i\) **to which belong** \(w_i\).

**Proof:** Let \(G\) and \(H\) be equivalent cographs with reductions \(R_G\) and \(R_H\). If \(T_G\) and \(T_H\) are the cotrees of \(G\) and \(H\), respectively, according to Lemma 3.2, for each pair of leaves \(u\) and \(v\) in \(T_G\) which are neither coduplicate nor duplicate vertices and their corresponding \(u'\) and \(v'\) in \(T_H\), we have that

\[
|\text{dist}_{T_G}(u, v) - \text{dist}_{T_H}(u', v')| \leq 2
\]

If the distance is preserved and since the lca\((u, v)\) and lca\((u', v')\) are the same type in both cotrees then \(T_G\) and \(T_H\) are the same. If the distance increased or decreased by one, and taking into account that the lca\((u, v)\) = \(w_i\) and lca\((u', v')\) = \(w'_i\) are the same type, then either of following situations can be occurs: adding a new interior vertex which succeeds \(w_i\) and taking the leaves \(t_i\) to which belong \(w_i\), or removing the vertex \(w_i\), which has no interior vertex as successor, whose father has no leaves and taking the leaves \(t_i\) to which belong \(w_i\). Finally, if the distance increase or decrease by two, then we repeat one of the operations described above.

Now, let \(T_G\) and \(T_H\) be the cotrees of equivalent cographs \(G\) and \(H\), respectively. We just need to check that reductions \(R_G\) and \(R_H\) are isomorphic. We assume that \(T_H\) is obtained from \(T_G\) under the operations (i) and (ii). First, we note that the number of twin classes are preserved, since that the only operation allowed is to become a coduplicate vertices into duplicate vertices or vice versa. Second, from the operations (i) and (ii) the lca\((u, v)\) in \(T_G\) and its corresponding lca\((u', v')\) in \(T_H\) are the same type, which implies that the adjacencies of \(R_G\) and \(R_H\) are preserved. Therefore, thus \(R_G\) and \(R_H\) are isomorphic. ■

**4. The proof of conjecture**

The next lemmas will be used to prove the main results of this section:

**Lemma 4.1:** Let \(G\) and \(H\) be equivalent cographs. Let \(u\) be a representative of twin class \(T_i\) of \(G\) with twin number \(t_i\), and let \(u'\) be its corresponding in a twin class \(T'_i\) of \(H\). If \(\delta(u)\) denotes the degree of vertex \(u\) then

\[
\delta(u') = \begin{cases} 
\delta(u) & \text{if } u' = u \\
\delta(u) + (t_i - 1) & \text{if } u \neq u' \text{ and } T_i \text{ is a clique set.}
\end{cases}
\]

**Proof:** Let \(u\) and \(u'\) be the representatives of twin classes \(T_i\) of \(G\) and \(T'_i\) of \(H\), respectively. If \(u = u'\) then obvious we have that \(\delta(u') = \delta(u)\). Now, we assume that \(u \neq u'\), and \(T_i\) is a clique set. Taking into account that vertex \(u'\) will be disconnected only of the \(t_i - 1\) vertices of same class \(T'_i\), follows that \(\delta(u') = \delta(u) + (t_i - 1)\), as desired. ■

**Lemma 4.2:** Let \(G\) be a cograph with twin classes \(T_1, T_2, \ldots, T_k\), and twin numbers \(t_1, t_2, \ldots, t_k\). Let \(u_i\) be a representative of twin class \(T_i\). If \(\delta(u_i)\) denotes the degree of \(u_i\), then

\[
\mu(G) = \begin{cases} 
\delta(u_i) & \text{if } T_i \text{ is a coclique set} \\
\delta(u_i) + 1 & \text{if } T_i \text{ is a clique set}
\end{cases}
\]

is a Laplacian eigenvalue of \(G\) with multiplicity at least \(t_i - 1\), for \(i = 1, 2, \ldots, k\).
Proof: Let $G$ be a cograph and let $u_i$ be a representative of twin class $T_i$ with twin number $t_i$, for $i = 1, 2, \ldots, k$. Now, we consider the Diagonalization of $(T_G, x)$, with $x = -\delta(u_i)$, if $T_j$ is a coclique set and $x = -\delta(u_i) - 1$, if $T_j$ is a clique set, for $i = 1, 2, \ldots, k$. Since that coduplicate (resp. duplicate) vertices of $T_G$ correspond to clique (resp. coclique) set in $G$, then after initialization ($d_i = \delta(u_i) + x$), we have the following values for the leaves of $T_G$

$$\begin{cases} 
-1 & \text{for coduplicate vertices} \\
0 & \text{for duplicate vertices.}
\end{cases}$$

From this, it is easy to see that for coduplicate vertices the subcase 1b occurs, while that for the duplicate vertices the subcase 2b occurs. In both cases, the algorithm assigns a zero as a permanent value. Since each twin class $T_i$ has $t_i$ vertices, follows for each iteration ($i = 1, 2, \ldots, k$), we have at least $t_i - 1$ zeros, as desired. \hfill \qed

**Theorem 4.1:** Let $G$ and $H$ be equivalent cographs with reductions $R_G$ and $R_H$, and suppose the vertices of $R_G$ and $R_H$ are labelled by the twin numbers $t_i$ of the $k$ twin classes they represent. Then $G$ and $H$ have at least $k + \sum_{i \in I} (t_i - 1)$ Laplacian eigenvalues in common, where $I \subseteq \{1, 2, \ldots, k\}$ is the indices of the twin classes whose types are identical in $G$ and $H$.

Proof: Let $G$ and $H$ be equivalent cographs with reductions $R_G$ and $R_H$, and suppose the vertices of $R_G$ and $R_H$ are labelled by the twin numbers $t_i$ of the $k$ twin classes they represent. Let $I \subseteq \{1, 2, \ldots, k\}$ be the indices of the twin classes whose types are identical in $G$ and $H$. Assuming that $G$ and $H$ are nonisomorphic graphs, we have that: $0 \leq |I| < k$.

In order for proving the conjecture, we note that a cograph $G$ of order $n = \sum_{i=1}^{k} t_i$, each Laplacian eigenvalue $\mu(G)$ of $G$ belongs one of following subsets:

$$\sum_{i \in I} (t_i - 1) \cup (|I|) \cup \left( \bigcup_{i=1}^{k-|I|} t_i \right)$$

(5)

We first will show that $G$ and $H$ have $\sum_{i \in I} (t_i - 1)$ Laplacian eigenvalues in common. Let $T_G$ and $T_H$ be the cotrees of $G$ and $H$, respectively. They have $\sum_{i \in I} (t_i - 1) \cup (|I|)$ leaves in common that correspond to duplicate and coduplicate vertices. Since $R_G \cong R_H$, and $u_i, u'_i$ are the respective representatives of twin classes which are identical in $G$ and $H$ having the same degree $\delta(u_i) = \delta(u'_i)$, hence by Lemma 4.2, we have at least

$$\sum_{i \in I} (t_i - 1)$$

(6)

Laplacian eigenvalues in common.

Now, let $\mu(G)$ be a Laplacian eigenvalue of $G$ which $\mu(G) \in (|I|) \cup \left( \bigcup_{i=1}^{k-|I|} t_i \right)$. We will show that $\mu(G)$ is also one of the $k$ Laplacian eigenvalues of $H$. For this, we apply the Diagonalization algorithm simultaneously in both cotrees $T_G$ and $T_H$ with $x = -\mu(G)$. It is sufficient to show when the algorithm assigns a zero in $\text{Diag}(T_G, x)$, we also must have a zero in $\text{Diag}(T_H, x)$.

Let $\alpha_j$ and $\alpha'_j$ be the assignments given in the $j$th iteration during execution of $\text{Diag}(T_G, x)$ and $\text{Diag}(T_H, x)$, respectively. Obviously, we have $\alpha_j = \alpha'_j$, if $T_G$ and $T_H$ are identical.
Figure 5. The partial cotrees $T_G$ and $T_H$.

It remains to be seen when $T_G$ and $T_H$ have different types of cotrees but $G$ and $H$ are equivalent cographs.

Suppose that $T_G$ has an interior vertex $w_i$ having $t_i \geq 2$ coduplicate vertices and a pendant vertex with assignment $\alpha_j$, while that $T_H$ has an interior vertex $w'_i$ having no leaves but the same pendant vertex with same assignment $\alpha_j$ and an interior vertex as successor having $t_i \geq 2$ duplicate vertices, as the Figure 5 has shown.

Let $v_i$ and $v'_i$ be the respective representatives of twin classes which are not identical in $G$ and $H$. Obviously, by Lemma 4.2, we can assume that $\mu(G)$ differs of $\delta(v_i) + 1$ and $\delta(v'_i)$. Applying the algorithm in coduplicate vertices of $T_G$, since all $t_i$ leaves have the same value $y = \delta(v_i) + x$, by Lemma 2.1, after $t_i - 1$ iterations, we have a pendant vertex with value

$$d_l = \frac{y - (t_i - 1)}{t_i + 1} = \frac{\delta(v_i) - \mu(G) - (t_i - 1)}{t_i + 1} \tag{7}$$

Now, we apply the algorithm in duplicate vertices of $T_H$, since all $t_i$ leaves have the same value $y' = \delta(v'_i) + x$, by Lemma 2.2, after $t_i - 1$ iterations, we have a pendant vertex with value

$$d'_l = \frac{y'}{t_i + 1} = \frac{\delta(v'_i) - \mu(G)}{t_i + 1} \tag{8}$$

We claim that $d_l = d'_l$. From Equations (7) and (8), follows

$$\frac{\delta(v_i) - \mu(G) - (t_i - 1)}{t_i + 1} = \frac{\delta(v'_i) - \mu(G)}{t_i + 1} \iff \delta(v_i) - \mu(G) - (t_i - 1) = \delta(v'_i) - \mu(G)$$

$$\delta(v'_i) = \delta(v_i) + (t_i - 1) \tag{9}$$

which accords with the Lemma 4.1. This shows that the algorithm will assign the same value to both $T_G$ and $T_H$, after to process the leaves with assignments $\alpha_j$ and $d_l$. Since $-x = \mu(G)$ is a Laplacian eigenvalue of $G$ and a zero should be assigned during execution of $\text{Diag}(T_G, x)$ then one of situations occurs: a zero is assigned previously to $\alpha_j$, what contradicts $\mu(G) \in (|I|) \cup (\bigcup_{i=1}^{k-|I|} t_i)$, or a zero is assigned exactly after to process the values $\alpha_j$ and $d_l$, since $\mu(G) \neq \delta(v_i) + 1, \delta(v'_i)$. Thus, a zero must be assigned during execution of $\text{Diag}(T_H, x)$ too.

The proof is similar if the $w_i$ and $w'_i$ are of $\cup$-type. Therefore, thus follows the result as desired. ■
Corollary 4.1: Let $G$ and $H$ be equivalent cographs with reductions $R_G$ and $R_H$ having $k \geq 2$ vertices. If $G$ and $H$ are cospectral graphs with relation to Laplacian matrix then $G \cong H$.

Proof: Let $G$ and $H$ equivalent cographs with reductions $R_G$ and $R_H$. We proceed by induction on the number $k \geq 2$ vertices of $R_G$ and $R_H$. The base case, $k = 2$ is trivial to verify.

We assume that the result holds for any two equivalent cographs $G$ and $H$ with reductions $R_G$ and $R_H$ having $k-1$ vertices. Now let $G'$ and $H'$ be equivalent cographs and cospectral graphs with reductions $R_{G'}$ and $R_{H'}$ having $k$ vertices.

We note that there are vertices $u_i$ and $v_i$ which are representatives of two respective twin classes $T_i \in G'$ and $T'_i \in H'$ of same type and therefore we have $\delta(u_i) = \delta(v_i)$. Otherwise, we have distinct Laplacian eigenvalues for $G'$ and $H'$, according to Lemma 4.2. Then, the cographs $G' - T_i \cong G$ and $H' - T'_i \cong H$ are cospectral graphs. By the induction hypothesis we have $G \cong H$ and therefore follows that $G' \cong H'$.

5. Cospectral cographs

In this last section, we show how to find cospectral linear size families of cographs with relation to the Laplacian matrix, from a pair of two nonisomorphic cospectral cographs.

The next lemma is very well known and it will be used for our construction:

Lemma 5.1: Let $G$ and $H$ be a graphs on $n_1$ and $n_2$ vertices, respectively. If $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n_1}(G) = 0$ and $\mu_1(H) \geq \mu_2(H) \geq \cdots \geq \mu_{n_2}(H) = 0$ are the Laplacian eigenvalues of $G$ and $H$, respectively. Then the Laplacian eigenvalues of $G \otimes H$ are

$$0, n_2 + \mu_2(G), n_2 + \mu_3(G), \ldots, n_2 + \mu_{n_1}(G),$$
$$n_1 + \mu_2(H), n_1 + \mu_3(H), \ldots, n_1 + \mu_{n_2}(H), n_1 + n_2.$$

For each integer $n \geq 3$, we define the following cographs of order $2n + 1$

- $G_{2n+1} = nK_1 \otimes (K_n \cup K_1)$;
- $H_{2n+1} = (((n - 1)K_1 \otimes K_1) \cup K_1) \otimes (K_{n-1} \cup K_1)$;

The Figure 6 shows the reductions of $G_9$ and $H_9$, respectively.

Lemma 5.2: The cographs $G_{2n+1}$ and $H_{2n+1}$ defined above are nonisomorphic and cospectral graphs.

Proof: It is obvious that $G_{2n+1}$ and $H_{2n+1}$ are nonisomorphic graphs. Since the Laplacian eigenvalues of $K_n \cup K_1$ are $n$ and $0$ with multiplicities $n-1$ and $2$, by Lemma 5.1, we have that the Laplacian eigenvalues of $G_{2n+1}$ are

$$0, 0 + (n + 1), \ldots, 0 + (n + 1), 0 + n, 0 + n, \ldots, n + n, (n + 1) + n.$$

Therefore, the Laplacian eigenvalues of $G_{2n+1}$ are $2n + 1, 2n, n + 1, n, 0$ with their respective multiplicities $1, n - 1, n - 1, 1, 1$. By similar calculus, we have $H_{2n+1}$ have the same Laplacian eigenvalues. ■
**Theorem 5.1:** Let $G'$ be a cograph of order $n$. Then $G' \otimes G_{2n+1}$ and $G' \otimes H_{2n+1}$ are nonisomorphic and cospectral cographs.

**Proof:** Let $G'$ and $G_{2n+1}$ be two cographs of order $n$ and $2n + 1$, respectively. If $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0$ are the Laplacian eigenvalues of $G'$, by Lemma 5.1, we have that Laplacian eigenvalues of $G' \otimes G_{2n+1}$ are

$$0, \mu_2 + (2n + 1), \ldots, \mu_n + (2n + 1),$$

$$n + n, n + (n + 1), \ldots, n + (2n), n + (2n + 1), n + (2n + 1).$$

Since $G_{2n+1}$ and $H_{2n+1}$ are cospectral graphs, then the Laplacian eigenvalues of $G' \otimes H_{2n+1}$ are obtained by same procedure as above. Therefore, thus $G' \otimes G_{2n+1}$ and $G' \otimes H_{2n+1}$ are nonisomorphic and cospectral cographs. □

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