Gröbner basis and Anick’s resolution for $\mathfrak{U}_{\mathbb{F}_2}(sl_3^+)$.  \\
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1 Introduction

Despite extension groups between modules over an algebra are very easy to define and taught nowadays in every standard course in homological algebra, it is still to be very difficult to compute them explicitly for a given pair of modules. One of such problems is a computation of extension groups between Weyl modules over the Schur algebra $S(n,r)$. It was shown in the joint work [3] of the author with Ana Paula Santana that this problem is closely related to the construction of a minimal projective resolution of the trivial module $K$ over Kostant form $\mathfrak{U}_K(sl_n^+)$ of the universal enveloping algebra of the Lie algebra $sl_n^+$.  

In this paper we compute the first three steps of a minimal projective resolution of $K$ in the case $p = 2$ and $n = 3$. For this we use Anick’s resolution constructed in [1]. Our result depends on the knowledge of a Gröbner basis for $\mathfrak{U}_K(sl_n^+)$. In the last section we give several conjectures about the Gröbner basis for $\mathfrak{U}_K(sl_n^+)$. It should be noted that with this conjectures proved it would be easy to extend the result of Theorem 3 to the cases $p \geq 3$, $n = 3$ and $p = 2$, $n \geq 4$.

In the Section 2 we recall the definition of Gröbner basis and in the Section 3 the construction of the Anick’s resolution. Then we proceed with the definition of $\mathfrak{U}_K(sl_n^+)$ in Section 4. The Sections 5, 6, 7 contain new results. Note that all the results of Section 5 are proved for an arbitrary $p$ and $n$, and they will be used in the subsequent papers.

2 Gröbner basis

Let $X$ be a set. We denote by $X^*$ the set of all words with letters in $X$. Then $X^*$ is a free monoid generated by $X$ with the multiplication given by concatenation of words and the unity $e$ given by the empty word. There is a partial order $\prec$ on $X^*$ given by the incusion of words. Note that $\prec$ is the coarsest partial order

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on $X^*$ such that $X^*$ is an ordered monoid with $e$ the least element of $X^*$. A monoidal order on $X^*$ is a total order that refines $\prec$.

Let $\mathbb{K}$ be a field. We denote by $\mathbb{K}\langle X^*\rangle$ a vector space spanned by $X^*$. A vector space $\mathbb{K}\langle X^*\rangle$ is a free associative algebra generated by $X$. We will call the elements of $X^*$ monomials, and the elements of $\mathbb{K}\langle X^*\rangle$ polynomials. Define the support of $p \in \mathbb{K}\langle X^*\rangle$ to be the set of element in $X^*$ with non-zero coefficients in $p$. If $\leq$ is a monoidal order on $X^*$ then we define the leading monomial $\text{lm}(p)$ of $p \in \mathbb{K}\langle X^*\rangle$ to be the maximal element of support of $p$ with respect $\leq$. Define the leading term $\text{lt}(p)$ of $p$ to be the leading monomial of $p$ with coefficient it enters in $p$. A monoidal order $\leq$ on $X^*$ can be extended to a partial order $\preceq$ on $\mathbb{K}\langle X^*\rangle$ by the rule

$$p \preceq q \iff \text{lm}(p) \prec \text{lm}(q) \text{ and } \text{lt}(p) = \text{lt}(q) \text{ and } p - \text{lt}(p) \preceq q - \text{lt}(q).$$

Note that in the case $\text{lm}(p) = \text{lm}(q)$ but $\text{lt}(p) \neq \text{lt}(q)$ the polynomials $p$ and $q$ are incompatible.

The pair $(m, f)$, where $m$ is a monomial and $f$ an element of $\mathbb{K}\langle X^*\rangle$, is called a rewriting rule if $m \succ f$. Note that every element $p \in \mathbb{K}\langle X^*\rangle$ gives a rewriting rule $r(p) = (\text{lm}(p), f)$ where $f = (p - \text{lt}(p))/\lambda$ and $\lambda$ is the leading coefficient of $p$. We will say that $h$ is a result of application of $(m, f)$ to $g$ if there is $m' \in \text{supp}(g)$ such that $m' = umv$ for some $u, v \in X^*$, and $h = g - \lambda m' + \lambda u f v$, where $\lambda$ is the coefficient of $m$ in $g$. We will write in this situation $g \to_r h$. If $r = r(p)$ for some $p \in \mathbb{K}\langle X^*\rangle$ then we write $g \to_f h$ instead of $g \to_{r(p)} h$. Let $S$ be a collection of rewriting rules or polynomials. Then $g \to_S h$ denotes that there is $r \in S$ such that $g \to_r h$. Formally, $\to_S$ is a set relation on $\mathbb{K}\langle X^*\rangle$. We denote by $\to_S^*$ the reflexive and transitive closure of $\to_S$. An element $g$ of $\mathbb{K}\langle X^*\rangle$ is called non-reducible with respect to the set of rewriting rules or polynomials $S$ if $g$ is a minimal element of $\mathbb{K}\langle X^*\rangle$ with respect to $\to_S^*$.

**Definition 1.** Let $A$ be an algebra over a field $\mathbb{K}$ and $X = \{a_i | i \in I\}$ a set of generators of $A$. Denote by $\pi$ the canonical projection from $\mathbb{K}\langle X^*\rangle$ to $A$. We say that a subset $S$ of $\ker(\pi)$ is a Gröbner basis of $\ker(\pi)$ if $\pi$ restricted on the vector space of non-reducible elements with respect $\{r(p) | p \in S\}$ is an isomorphism of $\mathbb{K}$-vector spaces. A Gröbner basis $S$ is called reduced if elements $p \in S$ are non-reducible with respect to $S \setminus \{p\}$.

Suppose that $\leq$ is an artinian monoidal order on $X^*$, that is every descending chain in $X^*$ stabilizes. Let $f \in \mathbb{K}\langle X^*\rangle$. If $f$ is reducible with respect to a Gröbner basis then there is $f_1$ such that $f \to_S f_1$. By definition of Gröbner basis $f_1 < f$ with respect to the induced ordering on $\mathbb{K}\langle X^*\rangle$. If $f_1$ is reducible we can find $f_2$ such that $f_1 \to_S f_1, f_1 > f_2$ and so on. Thus we get a descending sequence $f > f_1 > f_2 > \ldots$. As we assumed that the ordering $\leq$ is artinian this sequence have to break. Thus there is $f'$ that is non-reducible with respect to $S$ and $f \to_S f'$. We call $f'$ the normal form of $f$ with respect to $S$ and denote it by $\text{NF}(f, S)$. Note that the use of the article “the” is justified by the fact that $f'$ is unique. In fact suppose there are $f'$ and $f''$ such that $f \to_S f'$ and
$f \rightarrow_S f''$. Then $f' - f'' = (f' - f) + (f - f'') \in \ker(\pi)$ is an element of the kernel of the natural projection $\pi: \mathbb{K}(X^*) \rightarrow A$. Moreover, all monomials in $f' - f''$ are non-reducible with respect to $S$. Since the images of non-reducible monomials with respect to $S$ give a basis of $A$ under the map $\pi$ it immediately follows that $f' - f'' = 0$.

The notion of Gröbner basis is closely connected with the notion of critical pairs. We say that two monomials $m_1, m_2 \in X^*$ overlap if there are $u, v, w \in X^*$ such that $m_1 = uw$ and $m_2 = vw$. Note that two given monomials can have different overlappings. To make things more convenient we define an overlapping as a triple $(m, m_1, m_2)$, such that there are $u, v \in X^*$ such that $m = m_1v$ and $m = um_2$.

**Definition 2.** A critical pair is a triple $(w, r_1, r_2)$, where $w$ is a word and $r_1 = (m_1, f_1), r_2 = (m_2, f_2)$ are rewriting rules such that there are $u, v \in X^*$ with the property

$$w = um_1 = m_2v \text{ or } w = um_1v = m_2.$$

A word $w$ is called the tip of the critical pair $(w, r_1, r_2)$.

Let $(w, r_1, r_2)$ be a critical pair with $r_1, r_2 \in S$ and $u, v \in X^*$ such that $w = um_1 = m_2v$ (or $w = um_1v = m_2$). It is called reducible if $uf_1 - f_2v \rightarrow^*_S 0$ (respectively $uf_1v - f_2 \rightarrow^*_S 0$). The set of rewriting rules $S$ is called complete if all critical pairs $(w, r_1, r_2)$ with $r_1, r_2 \in S$ are reducible.

**Theorem 1.** Suppose $\leq$ is an artinian monoidal ordering on $X^*$. A subset $S$ of $\mathbb{K}(X^*)$ is a Gröbner basis of a two-sided ideal $I \subset \mathbb{K}(X^*)$ if and only if the set of rewriting rules $\{ r(p) \mid p \in S \}$ is complete.

We shall need the following proposition

**Proposition 1.** Suppose $R$ is a complete rewriting system in variables $X$ and $Y$ is a subset of $X$. We denote by $R(Y)$ the subset of $R$ that consist from all the rules $(m, p)$ such that $m \in Y^*$. If for all $(m, p) \in R(Y)$ we have $p \in \mathbb{K}(Y^*)$ then $R(Y)$ is a complete rewriting system.

**Proof.** Suppose $f \in \mathbb{K}(Y^*)$ and $f \rightarrow_R g$ then $f \rightarrow_{(m, p)} g$ for some $(m, p) \in R$. Since $m \preceq m'$ for some $m' \in \text{supp}(f)$ and $m' \in Y^*$ we get that $(m, p) \in R(Y)$. By assumption of the proposition we get $p \in \mathbb{K}(Y^*)$. Therefore $g \in \mathbb{K}(Y^*)$ and $f \rightarrow_{R(Y)} g$. Now by repetition we get that $f \in \mathbb{K}(Y^*)$ and $f \rightarrow^*_R g$ implies that $f \rightarrow^*_R g$.

Suppose that $(w, r_1, r_2)$ is an overlap of two rules from $R(Y)$ and $u, v \in Y^*$ are such that $w = m_1v = um_2$ (or $w = um_1v = m_2$). Then $p_1v - up_2 \in \mathbb{K}(Y^*)$ and $p_1v - up_2 \rightarrow_R 0$ (or $p_1v - p_2 \rightarrow_R 0$), since $R$ is complete. But then $p_1v - up_2 \rightarrow_{R(Y)} 0$ (or $p_1v - p_2 \rightarrow_{R(Y)} 0$), which shows that $R(Y)$ is complete. □
3 Anick resolution

The anick resolution was introduced in [1]. Let $A$ be an algebra over a field $K$ and $\varepsilon: A \to K$ a homomorphism of algebras. Let $X = a_1, \ldots$ be a set of generators of $A$ and $S \subset K \langle X^* \rangle$ a reduced Gröbner basis with respect to a monomial ordering $\le$ on $X^*$. For this set of data Anick constructed a free resolution of $K$ over $A$, which is nowadays called anick resolution. We will describe only the first four steps of Anick’s construction under additional assumption that $\varepsilon(x) = 0$ for all $x \in X$.

First we define sets $T_k$, $k = -1, 0, 1, 2$, that will serve as bases of $A$-free modules $P_k$. Denote by $T_{-1}$ the set $\{e\}$ with one element $e$ and by $T_0$ the set $X$. The set $T_1$ is the set of all leading monomials in $S$. Denote by $T_2$ the set of all possible overlaps of elements of $T_1$. Every element of $T_2$ is a triple $(w, r_1, r_2)$. We say that an overlap $(w, r_1, r_2)$ is minimal if there is no overlap $(w', r_1', r_2')$ such that $w' \prec w$. Note that if an overlap $(w, r_1, r_2)$ is minimal then the rules $r_1$ and $r_2$ are uniquely determined by $w$. In fact, suppose that $(w, r_1, r_2)$ $(w, r_1', r_2') \in T_2$. Then $w = m_1v = m_1'v'$. But this means that $m_1 \le m_1'$ or $m_1' \le m_1$. Since $S$ is a reduced Gröbner basis it follows that $r_1 = r_1'$. Similarly $r_2 = r_2'$. Denote by $T_2$ the set of monomials in $X^*$ such that there is a minimal overlap $(w, r_1, r_2)$. Denote for $k = -1, 0, 1, 2$ by $P_k$ the $A$-linear span of $T_k$. Let $M$ be the set of all non-reducible monomials with respect to $S$. Then for $k = -1, 0, 1, 2$ the set

$$N_k = \{m.t \mid m \in M, t \in T_k\}$$

is the basis of $P_k$ over $K$.

The sets $N_k$ have a full ordering induced by the ordering $\le$ on $X^*$ via the map $m.t \mapsto mt$. We define maps $\delta_0 : P_n \to P_n$ and $j_n : P_{n-1} \to P_n$ as follows.

$$\delta_0(m.x) := NF(mx, S).e$$

$$j_0(u.x.e) := u.x$$

$$\delta_1(m.t) := NF(mt, S).x, \text{ where } t = t'x$$

Now let $m \in M$ and $x \in X$. Suppose there are $u, v \in M$ such that $m = uv$ and $vx \in T_1$. Then we define $j_1(m.x) = u.vx$. Otherwise we let $j_1(m.x) = 0$. Note that $j_1$ is well-defined as $m = uv = u'v'$ would imply that $v \le v'$ or $v' \le v$ and therefore $vx \le v'x$ or $v'x \le v'x$. But since $S$ is reduced Gröbner basis any two different elements of $T_1$ are incompatible with respect to $\le$ (in other words $T_1$ is an anti-chain in the Anick’s terminology).

Let $w \in T_2$ be such that $w = m_1v = um_2$ with $m_1, m_2 \in T_1$. Define $\delta_2(m.w) = NF(mu, S).m_2$.

Suppose $t \in T_1$ and $m \in M$. If $m = uv$ for some $u, v \in M$ such that $vt \in T_2$ then we define $j_2(m.t) = u.vt$. Note that if such $u$ and $v$ exist then they are unique as $S$ is a reduced Gröbner basis. If there is no $u$ and $v$ with the above property then we let $j_2(m.t) = 0$.

Now we define homomorphisms of left $A$-modules $d_n : P_n \to P_{n-1}$ and homomorphisms of $K$-vector spaces $i_n : \ker(d_{n-1}) \to P_n$ for $n = 0, 1, 2$ by induction.
Since $d_n$ is a homomorphism of free $A$-modules it is enough to define $d_n$ on the basis elements $t$, where $t \in T_n$. On the other hand $i_n$ is a homomorphism of $K$ vector spaces, moreover we do not have any convenient basis for $\ker(d_{n-1})$. We will define $i_n$ by induction on the leading term of $f \in \ker(d_{n-1})$.

\[
\begin{align*}
d_0(t) & := \delta_0(t) \\
i_0(m.e) & := j_0(m.e) \\
d_{n+1}(t) & := \delta_{n+1}(t) - i_n d_n(\delta_{n+1}(t)) \\
i_n(f) & := j_n(\ellt(f)) + i_n(f - d_n(j_n(\ellt(f))).
\end{align*}
\]

Note that it is not obvious that $d_n$ and $i_n$ are well-defined. This a part of the claim of Proposition 2. The following proposition is proved in [1]. Note that Anick [1] constructed modules $P_n$ and maps $d_n$ for all $n \in \mathbb{N}$.

**Proposition 2.** The sequence of left $A$-modules

$$
P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_2} P_0 \xrightarrow{d_2} P^{-1} \epsilon \longrightarrow K \longrightarrow 0$$

is an exact complex.

4 Konstant form of universal enveloping algebra

Denote by $sl_n^+$ the Lie algebra of upper triangular nilpotent matrices. Let $\mathcal{U}_n(\mathbb{C})$ be its universal enveloping algebra over $\mathbb{C}$. We shall consider $sl_n^+$ with the standard basis $\{e_{ij} \mid 1 \leq i < j \leq n\}$. Then $\mathcal{U}_n(\mathbb{C})$ is generated as an algebra by the elements $e_{1,2}, e_{2,3}, \ldots, e_{n-1,n}$.

Let $\ll$ be an arbitrary full ordering on the set $\{e_{ij} \mid i < j\}$. We always assume that in the product $\prod_{i<j}^* e_{ij}$ the generators increase from the left to right, with respect to the ordering $\ll$. It follows from the Poincare-Birkhoff-Witt Theorem, that the set

$$\mathbb{B}_n = \left\{ \prod_{1 \leq i < j \leq n} e_{ij}^{k_{ij}} \left| k_{ij} \in \mathbb{N} \right. \right\}$$

is a $\mathbb{C}$-basis of $\mathcal{U}_n(\mathbb{C})$. Denote by $e_{ij}^{(k)}$ the element $\frac{1}{k!} e_{ij}^k$ of the algebra $\mathcal{U}_n(\mathbb{C})$.

We define $\mathcal{U}_n(\mathbb{Z})$ to be the $\mathbb{Z}$-sublattice of $\mathcal{U}_n(\mathbb{C})$ generated by the set

$$\mathbb{B}_n = \left\{ \prod_{i<j}^* e_{ij}^{(k_{ij})} \left| k_{ij} \in \mathbb{N} \right. \right\}.$$  

**Proposition 3.** The set $\mathcal{U}_n(\mathbb{Z})$ is a subring of $\mathcal{U}_n(\mathbb{C})$. In other words, $\mathcal{U}_n(\mathbb{Z})$ is a $\mathbb{Z}$-algebra. It is called the Kostant form of the universal enveloping algebra $\mathcal{U}_n(\mathbb{C})$ over $\mathbb{Z}$.

**Proof.** For a proof see [2, Lemma 2 after Proposition 3] and [2, Remark 3] thereafter. \qed

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Definition 3. For any field $K$, the algebra $\mathcal{U}_n(K) := K \otimes \mathbb{Z} \mathcal{U}_n(\mathbb{Z})$ is called Kostant form of the algebra $\mathcal{U}_n(\mathbb{C})$ over $K$.

Define a grading on $\mathfrak{sl}_n^+(\mathbb{C})$ by $\deg(e_{ij}) = j - i$.

This grading extends to the grading of $\mathcal{U}_C(\mathfrak{sl}_n^+)$ such that

$$\mathcal{U}_C(\mathfrak{sl}_n^+)_d = \left\{ \prod_{i<j} e_{ij}^{k_{ij}} \left| \sum_{i<j} k_{ij}(j-i) = d \right. \right\}.$$ 

Since the intersection of $\mathcal{U}_C(\mathfrak{sl}_n^+)_d$ with $\mathcal{U}_C(\mathfrak{sl}_n^+)$ is a lattice in $\mathcal{U}_C(\mathfrak{sl}_n^+)_d$, this grading downgrades to $\mathcal{U}_C(\mathfrak{sl}_n^+)$. After tensoring with $K$ we get a grading on $\mathcal{U}_K(\mathfrak{sl}_n^+)$ such that

$$\mathcal{U}_K(\mathfrak{sl}_n^+)_d = \left\{ \prod_{i<j} e_{ij}^{(k_{ij})} \left| \sum_{i<j} k_{ij}(j-i) = d \right. \right\}.$$ 

5 Big Gröbner basis

In this section we describe a Gröbner basis of the algebra $\mathcal{U}_n(\mathbb{K})$ with respect to the generating set $X = \left\{ e_{ij}^{(k)} \left| i < j, \ k \in \mathbb{N} \right. \right\}$. We will consider deglex ordering on $X^*$ with respect to the degree function defined in the previous section and the ordering $\ll$ on $X$.

Theorem 2. Let $X$ and the ordering on $X$ be as above. Then the following set of rewriting rules is complete:

1. $e_{ij}^{(k)} e_{ij}^{(r)} \rightarrow \binom{k+r}{k} e_{ij}^{(k+r)}$ 

2. $e_{ij}^{(k)} e_{st}^{(r)} \rightarrow e_{st}^{(r)} e_{ij}^{(k)}$, if $i, j, s, t$ are different and $(s, t) \ll (i, j)$

3. $e_{ij}^{(k)} e_{jt}^{(r)} \rightarrow \sum_{s=0}^{\min(k, r)} e_{jt}^{(r-s)} e_{i,s}^{(s)} e_{ij}^{(k-s)}$, if $(i, j) \gg (j, t)$

4. $e_{ij}^{(k)} e_{si}^{(r)} \rightarrow \sum_{t=0}^{\min(k, r)} (-1)^t e_{si}^{(r-t)} e_{s,i}^{(t)} e_{ij}^{(k-t)}$, if $(i, j) \gg (s, i)$

Note that the corresponding Gröbner basis is not reduced in general, since it can happen that $(i, t)$ doesn’t lie between $(j, t)$ and $(i, j)$ in $[3]$ or that $(s, j)$ doesn’t lie between $(s, i)$ and $(i, j)$ in $[3]$.

Proof. It is clear that the set

$$B = \left\{ \prod_{i<j} e_{ij}^{(k_{ij})} \left| i < j, \ k_{ij} \in \mathbb{N} \right. \right\}$$
is the set of non-reducible words with respect to the given rewriting system. By definition the natural image of $B$ in $U_K(sl_n^+)$ is a basis of $U_K(sl_n^+)$. Therefore it is enough to check that for every rule the left hand side and the right hand side are equal in $U_K(sl_n^+)$. This is obvious for (1) and (2). Thus we have only to check the claim for (3) and (4). We shall do this only for (3) as the case (4) is similar.

We have to prove the equality

$$e_{ij}^{(k)} e_{jt}^{(r)} \rightarrow \sum_{s=0}^{\min(k,r)} e_{jt}^{(r-s)} e_{i,t}^{(k-s)}$$

in $U_K(sl_n^+)$. Clearly it is enough to prove the same equality in $U_Z(sl_n^+)$ and therefore in $U_C(sl_n^+)$. We will do this by induction on the minimum of $k$ and $r$.

The case $\min(k, r) = 1$ splits into two cases $r = 1$ and $k = 1$. The case $k = 1$ we prove by induction on $k$. For $k = r = 1$ we have

$$e_{ij} e_{jt} = e_{jt} e_{ij} + e_{it}.$$  

Suppose we have proved equality for $k = 1$ and $r \leq r_0$. Let us check it for $r = r_0 + 1$.

$$e_{ij} e_{jt}^{(r)} = \frac{1}{r} e_{ij} e_{jt}^{(r-1)} e_{jt} \quad \text{induction assumption}$$  

$$= \frac{1}{r} \left( e_{ij}^{(r-1)} e_{jt} + e_{jt}^{(r-2)} e_{it} \right) e_{jt}$$  

$$= \frac{1}{r} \left( e_{ij}^{(r-1)} e_{jt} e_{ij} + e_{jt}^{(r-1)} e_{it} + e_{jt}^{(r-2)} e_{jt} e_{it} \right)$$  

$$= e_{ij}^{(r)} e_{ij} + \frac{1}{r} (1 + r - 1) e_{jt}^{(r-1)} e_{it}$$  

$$= e_{ij}^{(r)} e_{ij} + e_{jt}^{(r-1)} e_{it}.$$  

Now we prove the equality in the case $r = 1$ and $k \geq 2$. Suppose it is proved for all $k \leq k_0$. Let us show it for $k = k_0 + 1$. We have

$$e_{ij}^{(k)} e_{jt} = \frac{1}{k} e_{ij}^{(k-1)} e_{jt}$$  

$$= \frac{1}{k} \left( e_{ij} e_{jt} e_{ij}^{(k-1)} + e_{ij} e_{it} e_{ij}^{(k-2)} \right)$$  

$$= \frac{1}{k} \left( e_{jt} e_{ij}^{(k-1)} + e_{it} e_{ij}^{(k-1)} + e_{it} e_{ij}^{(k-2)} \right)$$  

$$= e_{jt} e_{ij}^{(k)} + e_{it} e_{ij}^{(k-1)}.$$  

Suppose we have prove equality for all $k$ and $r$ such that $\min(k,r) \leq m_0$. Let us prove it for $\min(k,r) = m_0 + 1$. There are two cases $k = m_0 + 1$ and $r = m_0 + 1$. 

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As the computations are very similar we will treat only the first case.

\[
\begin{align*}
e_{ij}^{(k)} e_{jt}^{(r)} &= \frac{1}{k} e_{ij} e_{ij}^{(k-1)} e_{jt}^{(r)} \\
&= \frac{1}{k} \sum_{s=0}^{k-1} e_{ij} e_{jt}^{(r-s)} e_{it}^{(s)} e_{ij}^{(k-1-s)} \\
&= \frac{1}{k} \sum_{s=0}^{k-1} \left( e_{jt}^{(r-s)} e_{it}^{(s)} e_{ij}^{(k-1-s)} + e_{(jt)}^{(r-s-1)} e_{it}^{(s+1)} e_{ij}^{(k-1-s)} \right) \\
&= \frac{1}{k} \sum_{s=0}^{k} \left( k - s + s \right) e_{jt}^{(r-s)} e_{it}^{(s)} e_{ij}^{(k)} = \sum_{s=0}^{k} e_{jt}^{(r-s)} e_{it}^{(s)} e_{ij}^{(k)}. 
\end{align*}
\]

\[\square\]

**Corollary 1.** Let \( p \) be a characteristic of the field \( \mathbb{K} \) and \( l \geq 0 \). Then the linear span \( U_{\mathbb{K}}(sl_n^+) \) of the set

\[
B' = \left\{ \prod_{i<j} e_{ij}^{(k_{ij})} \left| k_{ij} \leq p^l - 1 \right. \right\}
\]

is a subalgebra of \( U_{\mathbb{K}}(sl_n^+) \).

**Proof.** We claim that \( U_{\mathbb{K}}(sl_n^+) \) is the subalgebra \( A \) of \( U_{\mathbb{K}}(sl_n^+) \) generated by the set

\[
X' = \left\{ e_{ij}^{(k)} \left| k \leq p^l - 1 \right. \right\}.
\]

It is enough to show that the set \( B' \) is a basis of \( A \). Let \( R \) be rewriting system defined in Theorem 2. We claim that \( R(X') \) is complete. To prove this we apply Proposition 1. It is obvious for the rules (2), (3) and (4) that if the left hand side is an element of \( (X')^+ \) then all the monomials on the right hand side are also elements of \( (X')^+ \). Moreover, if \( k + r \leq p^l - 1 \) then the same is true for the rewriting rule (1). Suppose \( k, r \leq p^l - 1 \) and \( k + r \geq p^l \). Then \( k + r choose k = 0 \) in \( \mathbb{K} \). It is well known that the degree of \( p \) in the prime decomposition of \( n! \) is given by the formula

\[
\sum_{j=0}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor.
\]

Therefore the degree of \( p \) in the prime decomposition of \( \binom{k+r}{k} \) is

\[
\sum_{j=0}^{\infty} \left( \left\lfloor \frac{k+r}{p^j} \right\rfloor - \left\lfloor \frac{k}{p^j} \right\rfloor - \left\lfloor \frac{r}{p^j} \right\rfloor \right) = \left\lfloor \frac{k+r}{p^l} \right\rfloor + \sum_{j=0}^{l-1} \left( \left\lfloor \frac{k+r}{p^j} \right\rfloor - \left\lfloor \frac{k}{p^j} \right\rfloor - \left\lfloor \frac{r}{p^j} \right\rfloor \right) \geq \left\lfloor \frac{k+r}{p^l} \right\rfloor = 1 > 0.
\]
Therefore for the rule $\text{(1)}$ and $k + r \geq p^l$ we get
\[ e_{ij}^{(k)} e_{ij}^{(r)} \to 0. \]
This shows that $R(X')$ is complete. Now it is obvious that $B'$ is the set of non-reducible monomials in the alphabet $X'$ with respect to the rewriting system $R(X')$. This shows that $B'$ is a basis of the algebra $A'$.

6 Small Gröbner basis

The Gröbner basis obtained in the previous section is not convenient for the construction of minimal projective resolution of $K$, since the anick resolution is much closer to the minimal resolution if the chosen generating set is minimal.

In this section we will stick to the case $p = 2$ and $n = 3$. The more general case will be considered in other paper. Nevertheless we start with the technical result that is true for an arbitrary $p$ and $n$.

**Lemma 1.** Let $p$ be the characteristic of $K$ and $k = k_0 p^l + k_{l-1} p^{l-1} + \cdots + k_0$ with $0 \leq k_s \leq p - 1$. Then for any $i < j$
\[ \prod_{s=0}^{l} \left( e_{ij}^{(p^s)} \right)^{k_s} \]
is a non-zero multiple of $e_{ij}^{(k)}$.

**Proof.** We have to check that the integer
\[ n := \frac{k!}{\prod_{s=0}^{l} (p^s)^{k_s}} \]
is non-zero in $K$. The degree of $p$ in the prime decomposition of $n$ is given by
\[ \sum_{l=0}^{l} \left( \left\lfloor \frac{k}{p^l} \right\rfloor - \sum_{s=0}^{l} k_s \left\lfloor \frac{p^s}{p^l} \right\rfloor \right) = \sum_{l=0}^{l} \left( \sum_{s=0}^{l} k_s p^{s-t} \right) - \sum_{s=0}^{l} k_s p^{s-t} = 0. \]
This shows that $n$ is non-zero in $K$. 

Now we note that
\[ e_{ij}^{(p^l)} = e_{j,i+1}^{(p^l)} e_{i+1,j}^{(p^l)} - \sum_{s=0}^{l-1} e_{j,i+1,j}^{(p^{l-s})} e_{i,j}^{(p^{l-s})} e_{i,i+1}^{(p^{l-s})}. \]
In fact it was proved in Theorem $\text{[2]}$. From this equality by induction on $j - i$ and $l$ it follows that the set
\[ \left\{ e_{i,i+1}^{(p^l)} \mid 1 \leq i \leq n - 1, \ l \in \mathbb{N}_0 \right\} \]
generates $\mathfrak{U}_K(sl^+_n)$. Note that the set
\[
\left\{ e^{(p^k)}_{i,i+1} \mid 1 \leq i \leq n-1, \ 0 \leq k \leq l \right\}
\]
generates the subalgebra $\mathfrak{U}_K(sl^+_n)$ of $\mathfrak{U}_K(sl^+_n)$.

From now on we assume that $n = 3$ and $p = 2$. For a convenience we will denote $e^{2k}_{12}$ by $a_k$ and $e^{2k}_{23}$ by $b_k$. We start with the proof of some equalities in $\mathfrak{U}_K(sl^+_3)$.

**Proposition 4.** For any $k$ we have $a_k^2 = 0$.

**Proof.** In the proof of Corollary 1 it was proved that if $r, s \leq pt - 1$ and $r + s \geq pt$ then for any $i < j$ we have $e^{(r)}_{ij}e^{(s)}_{ij} = 0$. We apply this claim to the situation $(i, j) = (1, 2), (2, 3)$ and $r = s = 2k, t = k + 1$. □

**Proposition 5.** For any $l$ and $k$ elements $a_l$ and $a_k$ commutes. Similarly $b_l$ and $b_k$.

**Proof.** Obvious. □

**Proposition 6.** For any $l > k$ we have
\[
a_l b_k + b_k a_l + a_k b_k a_k a_{k+1} \ldots a_{l-1} = 0 \tag{5}
b_l a_k + a_k b_l + b_k a_k b_k b_k + b_{l-1} = 0 \tag{6}
\]
in $\mathfrak{U}_K(sl^+_n)$.

**Proof.** We have
\[
a_l b_k = e^{(2^k)}_{12} e^{(2^k)}_{23} = \sum_{s=0}^{2^k} e^{(2^k-s)}_{23} e^{(s)}_{13} e^{(2^k-s)}_{12} \tag{3}
= b_k a_l + \sum_{s=1}^{2^k} e^{(2^k-s)}_{23} e^{(s)}_{13} e^{(2^k-s)}_{12}
\]
and
\[
a_k b_k a_k \ldots a_{l-1} = e^{(2^k)}_{12} e^{(2^k)}_{23} e^{(2^k)}_{12} \ldots e^{(2^k)}_{12} = e^{(2^k)}_{12} e^{(2^k)}_{23} e^{(2^k)}_{12} \ldots e^{(2^k)}_{12} \tag{4}
= \sum_{s=0}^{2^k} e^{(2^k-s)}_{23} e^{(s)}_{13} e^{(2^k-s)}_{12} e^{(2^k-s)}_{12}.
\]

Since $2^k, 2^l - 2^k \leq 2^l - 1$ and $2^l = 2^k + (2^l - 2^k) \geq 2^l$ by proof in Corollary 1 we have $e^{(2^k)}_{12} e^{(2^l - 2^k)}_{12} = 0$. Now if $1 \leq s \leq 2^k$ then $2^k - s = 2^{i_1} + \cdots + 2^{i_s}$.
for some $0 \leq \sigma \leq k$ and $0 \leq i_1 < i_2 < \cdots < i_\sigma \leq k - 1$. Moreover $2^j - 2^k = 2^k + 2^{k+1} + \cdots + 2^{j-1}$. Therefore applying Lemma \[\text{II}\] twice we get for $1 \leq s \leq 2^k$

$e_{12}^{(2^k-s)} e_{12}^{(2^j-2^k)} = e_{12}^{(2^j-s)}$.

Therefore

$$a_k b_k a_k \ldots a_{i-1} = \sum_{s=1}^{2^k} e_{23}^{(2^k-s)} e_{13}^{(s)} e_{23}^{(2^j-s)}.$$ 

The second equality follows from the obvious duality $a_k \leftrightarrow b_k$.

**Proposition 7.** For all $k \in \mathbb{N}$ we have $b_k b_k a_k a_k + a_k b_k a_k = 0$ in $\mathcal{H}_k(s_l^{-1})$.

**Proof.** We have

$$a_k b_k a_k b_k = e_{12}^{(2^k)} e_{23}^{(2^k)} e_{12}^{(2^k)} e_{23}^{(2^k)} = \sum_{s=0}^{2^k} e_{12}^{(2^k-s)} e_{13}^{(s)} e_{12}^{(2^j-s)} e_{23}^{(2^k-s)}.$$ 

Now if $0 \leq s \leq 2^k - 1$ applying Lemma \[\text{II}\] twice we get $e_{12}^{(2^k)} e_{12}^{(s)} = e_{12}^{(2^k-s)}$. Similarly for $e_{13}$. If $s = 2^k$ we get $e_{12}^{(2^k-s)} = a_k^{2^k} = 0$ by Proposition \[\text{II}\]. Therefore

$$a_k b_k a_k b_k = \sum_{s=0}^{2^k-1} e_{12}^{(2^k+s)} e_{13}^{(2^k-s)} e_{12}^{(2^k+s)}.$$ 

From the duality $e_{12} \leftrightarrow e_{23}$ it follows that

$$b_k a_k b_k a_k = \sum_{t=0}^{2^k-1} e_{23}^{(2^k+t)} e_{13}^{(2^k-t)} e_{12}^{(2^k+t)}.$$ 

Now we can use rewriting rules \[\text{II}, \text{III}, \text{IV}\]:

$$\sum_{t=0}^{2^k-1} e_{23}^{(2^k+t)} e_{13}^{(2^k-t)} e_{12}^{(2^k+t)} = \sum_{t=0}^{2^k-1} e_{12}^{(2^k+t-r)} e_{13}^{(2^k-t)} e_{12}^{(2^k+t-r)} = \sum_{s=-2^k}^{2^k-1} \sum_{t=\max(0,s)}^{2^k-1} \binom{2^k-s}{2^k-t} e_{12}^{(2^k+s)} e_{13}^{(2^k-s)} e_{23}^{(2^k+s)}.$$ 

Suppose $-2^k \leq s \leq -1$. Denote $-s$ by $\tilde{s}$. Then

$$\sum_{t=\max(0,s)}^{2^k-1} \binom{2^k-s}{2^k-t} = \sum_{t=0}^{2^k-1} \binom{2^k+s}{2^k-t} = \sum_{j=1}^{2^k} \binom{2^k+s}{j} = 1 + \sum_{j=0}^{2^k} \binom{2^k+s}{j}.$$ 

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Now the sum $\sum_{j=0}^{2^k} (2^{k}+\tilde{s})$ is a coefficient of $x^{2^k}$ in the product

$$\left(\sum_{j=0}^{\infty} x^j\right) \left(\sum_{j=0}^{\infty} x^j\right) = (1 + x)^{2^k}(1 + x)^{-1} = (1 + x)^{2^k+\tilde{s}-1} = (1 + x)^{2^k}(1 + x)^{\tilde{s}-1} = (1 + x)^{2^k}(1 + x)^{\tilde{s}-1}.$$

Since $0 \leq \tilde{s} - 1 \leq 2^k - 1$, this coefficient is 1. Therefore

$$\sum_{t=\max(0,s)}^{2^k-1} \left(\frac{2^k - s}{2^k - t}\right) = 0$$

for $-2^k \leq s \leq -1$. Suppose $0 \leq s \leq 2^k - 1$. Then we get

$$\sum_{t=\max(0,s)}^{2^k-1} \left(\frac{2^k - s}{2^k - t}\right) = \sum_{t=s}^{2^k-1} \left(\frac{2^k - s}{2^k - t}\right) = \sum_{j=1}^{2^k-s} \left(\frac{2^k - s}{j}\right)
= 1 + (1 + 1)^{2^k-s} = 1.$$

Therefore

$$b_k a_k b_k a_k = \sum_{s=0}^{2^k-1} e_{12}^{(2^k+s)} e_{13}^{(2^k-s)} e_{23}^{(2^k+s)}$$

as required. \hfill \Box

Let $X_l = \{ a_k, b_k \mid 0 \leq k \leq l \}$. We order the elements of $X_l$ by

$$a_0 < b_0 < a_1 < b_1 < \cdots < a_l < b_l$$

and define degree $\deg: X_l \to \mathbb{N}$ by $\deg(a_k) = \deg(b_k) = 2^k$. Denote by $\pi$ the natural projection $\mathbb{K} \langle X_l^* \rangle \to U_{\mathbb{K}}^+(sl^+_n))$.

**Proposition 8.** The following set $S_l$ of elements in $\mathbb{K} \langle X_l^* \rangle$

\begin{align*}
    a_l b_k + b_l a_k + a_k b_k a_k a_k + \ldots a_{l-1} & \quad \text{if } l > k \\
    b_l a_k + a_k b_k + b_k a_k b_k + b_k + b_{k+1} + \ldots b_{l-1} & \quad \text{if } l > k \\
    a_k a_k + a_k a_l & \quad \text{if } l > k \\
    b_l b_k & \quad \text{if } l > k \\
    b_k a_k b_k a_k + a_k b_k a_k & \quad \\
    a_k^2 & \quad \\
    b_k^2 & \\
\end{align*}

is a reduced Gröbner basis of $\ker(\pi)$. 

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Proof. If follows from Propositions 4, 5, 6, 7 that $S_l$ is a subset of ker($\pi$). Thus it is enough to show that the images of non-reducible monomials in $X^*_l$ give a basis of $\Omega_k(sl^+_3)$. Since the images of non-reducible monomials in $X^*_l$ generate $\Omega_k(sl^+_3)$ as a vector space and $\Omega_k(sl^+_3)$ is finite dimensional, it is enough to show that the number of non-reducible monomials in $X^*_l$ with respect to $S_l$ is less or equal to the dimension of $U^l_k(sl^+_3)$. From Corollary 1 it follows that the dimension of $U^l_k(sl^+_3)$ is $(2^l)^3 = 2^{3l} = 8^l$.

To find non-reducible monomials with respect to $S_l$ it is enough to find monomials that does not contain submonomials $a_1b_k$, $b_la_k$, $a_lb_k$ for $l > k$, and submonomials $b_kb_ka_kb_k$, $a_ka_kb_k$, $b_ka_kb_k$. If a monomial $m$ does not contain submonomials $a_1b_k$, $b_la_k$, $a_lb_k$ for $l > k$, then the indices of variables in $m$ weakly increase from the left to right. We denote by $m_k$ a submonomial of $m$ that consists from the all variables with index $k$. Then $m = m_0m_1 \ldots m_{l-1}$. Now if $m_k$ does not contain submonomials $a_ka_k$, $b_ka_k$, $b_ka_kb_k$, $a_ka_kb_k$, $a_kbka_kb_ka_kb_k$. Then it is equal to one of the monomials

$$e, a_k, b_k, a_kb_k, b_ka_k, a_kbka_k, b_ka_kb_k, a_kbka_kb_k.$$

Therefore there is no more then $8^l$ non-reducible monomials in $X^*_l$ with respect to $S_l$. \qed

Corollary 2. Let $X = \bigcup_{l \leq 0} X_l$. The set $S = \bigcup_{l \geq 0} S_l$ is a reduced Gröbner basis of ker($\pi$), where $\pi$ is the natural projection $K \langle X^* \rangle \to \Omega_k(sl^+_3)$.

Proof. It is clear that $S \subset$ ker($\pi$). Denote by $R$ the rewriting system $\{ r(p) \mid p \in S \}$. It is enough to show that any critical pair $(w, r_1, r_2)$, with $r_1, r_2 \in R$ is reducible. For a given critical pair $(w, r_1, r_2)$ there is an $l \geq 0$, such that all monomials in $w, r_1, r_2$ lie in $X_l$. By Proposition 8 the set $S_l$ is a Gröbner basis, therefore any critical pair $(w, r_1, r_2)$ with $w \in X^*_l$, $r_1, r_2 \in R_l = \{ r(p) \mid p \in S_l \}$ is reducible. \qed

7 First steps of minimal resolution for $n = 3$ and $p = 2$

We will consider the algebra $\Omega_k(sl^+_3)$ as a graded algebra with the grading induced by degree function $\deg(a_k) = \deg(b_k) = 2^k$. Since the zero component of $\Omega_k(sl^+_3)$ is a field $K$ and $\Omega_k(sl^+_3)$ is a positively graded all graded projective modules are shifts of the regular module $\Omega_k(sl^+_3)$. Suppose $P = \Omega_k(sl^+_3)v$ with $\deg(v) = m$. We define $\text{Rad}(R) = \bigoplus_{d \geq 1} \Omega_k(sl^+_3)v$. This definition of radical can be extended to the arbitrary projective module by additivity. It is well known that a resolution of an $\Omega_k(sl^+_3)$ module $M$

$$\cdots \to P_m \to \cdots \to P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \to M \to 0$$

is minimal up to step $m$ if and only if $\text{Im}(d_k) \subset \text{Rad}(P_k)$ for all $0 \leq k \leq m$. 13
Now we will examine first four steps of the anick resolution of the trivial module $\mathbb{K}$ over $U_{\mathbb{K}}(sl_3^+)$. Then we modify it and get first three steps of a minimal projective resolution.

In our situation $T_0 = \{ a_k, b_k \mid k \in \mathbb{N}_0 \}$, and

$$T_1 = \{ a_1 a_k, b_1 b_k, a_1 b_k, b_1 a_k, b_1 a_k b_k a_k b_k, b_1 b_k, a_k b_k, a_k | 0 \leq k < l \}.$$ We start with the computation of $d_1 : P_1 \to P_0$ in the anick resolution. Note that since we are working in characteristic two we can disregard signs. Suppose $l > k$, then

$$d_1(a_1 a_k) = a_1 a_k + i_0 d_0(a_1 a_k) = a_1 a_k + i_0(a_1 a_k)$$

Analogously $d_1(b_1 b_k) = b_1 b_k + b_k b_1$. Now

$$d_1(a_k^2) = a_k a_k + i_0 d_0(a_k a_k) = a_k a_k + i_0(0) = a_k a_k$$

and analogously $d_1(b_k^2) = b_k b_k$.

$$d_1(a_1 b_k) = a_1 b_k + i_0 d_0(a_1 b_k) = a_1 b_k + i_0(a_1 b_k a_k \ldots a_{l-1} e + b_k a_1 e)$$

Analogously

$$d_1(b_1 a_k) = b_1 a_k + b_k a_k b_k \ldots b_{l-2} b_{l-1} + a_k b_k.$$ Now

$$d_1(b_k a_k b_k a_k) = b_k a_k b_k a_k + i_0 d_0(b_k a_k b_k a_k) = b_k a_k b_k a_k + i_0(a_k b_k a_k b_k e) = b_k a_k b_k a_k + a_k b_k a_k b_k.$$ It is readily seen that the image of $d_1$ is a subset of $Rad(P_0)$. Now we examine properties of $d_2 : P_2 \to P_1$ in the anick resolution. In our case

$$T_2 = \{ a_m a_l a_k, b_m b_l b_k \mid m \geq l \geq k \}$$

$$\cup \{ a_m a_l b_k, b_m b_l a_k \mid m \geq l > k \}$$

$$\cup \{ a_m b_l a_k, b_m a_l b_k \mid m > l \geq k \}$$

$$\cup \{ a_l b_k a_k b_k a_k, b_l a_l b_k a_k b_k | l > k \}$$

$$\cup \{ b_k a_k b_k a_k b_k a_k \mid k \in \mathbb{N}_0 \}.$$
We will not compute value of $d_2$ for all elements of $T_2$. Instead we will show that for some elements of $T_2$ the image of $d_2$ lies in $Rad(P_1)$ and for the rest of elements we compute $d_2$. Every element $v$ of $P_1$ can be uniquely written as a sum

$$
\sum_{m \in \mathcal{M}, t \in T_1} \lambda_{m,t} m.t,
$$

where $\lambda_{m,t}$ are elements of $\mathbb{K}$. Now $v \in Rad(P_1)$ if and only if $\lambda_{c,t} = 0$ for all $t \in T_1$. Therefore if $v$ is homogeneous and $\deg(v)$ is not an element of the set $\{ \deg(t) \mid t \in T_1 \}$ then $v$ is an element of $Rad(P_1)$. Now

$$
D = \{ \deg(t) \mid t \in T_1 \} = \{ 2^l + 2^k \mid l \geq k \geq 0 \}.
$$

Note that $d_2$ preserves degree as our Gröbner basis is homogeneous. Therefore if $t \in T_2$ and $\deg(t) \notin D$, then $d_2(t) \in Rad(P_1)$. Now if $m > l > k$ then the degree of elements similar to $a_m a_l a_k$ is $2^m + 2^l + 2^k$ and it is not an element of $D$. Thus we have to consider only the cases when at least two numbers $m, l, k$ are equal.

Note that $d_2(\langle a_k b_k b_k a_k a_k \rangle), d_2(\langle a_k^2 \rangle), d_2(\langle b_k^2 \rangle), d_2(\langle b_k a_k b_k a_k \rangle), d_2(\langle b_k a_k b_k a_k^2 \rangle)$ are linear combinations of monomials that involve only variables with index $k$, since the linear span of such monomials is a subalgebra of $\Omega_k(sl^+_3)$. The set

$$
\{ \deg(t) \mid t \in T_1, \ t \in \{a_k, b_k\}^* \}
$$

contains two numbers: $2^{k+1}$ and $2^{k+2}$. Now $\deg((b_k a_k)^3) = 3 \times 2^{k+1}$, $\deg(a_k^3) = \deg(b_k^3) = 2 \times 2^k$, $\deg(b_k a_k b_k a_k) = \deg(b_k a_k b_k a_k^2) = 5 \times 2^k$ are different from both of them. Therefore $d_2((b_k a_k)^3), d_2(a_k^3), d_2(b_k^3), d_2(b_k a_k b_k a_k), d_2(b_k a_k b_k a_k^2)$ lie in the radical of $P_1$.

Suppose $m > k$. Then the elements $a_m^2 a_k, a_m^2 b_k, b_m^2 b_k, b_m a_m b_m a_m a_k, b_m a_m b_m a_m b_k$ are elements of the subalgebra generated by $X_m$. They all are of degree $2^{m+1} + 2^k$, but the set

$$
\{ \deg(t) \mid t \in T_1, \ t \in X_m^* \} = \{ 2^l + 2^k \mid m \geq l \geq k \geq 0 \}.
$$

does not contain $2^{m+1} + 2^k$, therefore the value of $d_2$ for all of the above mentioned elements lies in the radical of $P_1$.

It is left to compute $d_2$ for $a_m a_k^2, a_m b_k^2, b_m a_k^2, b_m b_k a_k a_k$, and $b_m b_k a_k b_k a_k$. Since $a_m$ and $a_k$ commute we get

$$
d_2(\langle a_l a_k^2 \rangle) = a_l a_k^2 + a_k a_l a_k
$$

and the similar formulae are valid for $d_2(\langle b_l b_k^2 \rangle)$.

Suppose $m > k + 1$. Then we have

$$
d_2(\langle a_m b_k^2 \rangle) = a_m b_k^2 + i_1(a_m b_k b_k)
$$

where $i_1(b_k a_m b_k) = a_m b_k^2 + b_k a_m b_k + i_1(a_k b_k a_k \ldots a_{m-1} b_k a_k a_{m-1} b_k)
$$

$$
= a_m b_k^2 + b_k a_m b_k + i_1(a_k b_k a_k \ldots a_{m-1} b_k a_k a_{m-2} a_{m-1})
$$

$$
= a_m b_k^2 + b_k a_m b_k + a_k b_k a_k \ldots a_{m-2} a_{m-1} b_k
$$
and the similar formula for $d_2(b_m a_k^2)$. Now suppose $m = k + 1$. Then we get

$$d_2(b_{k+1} a_k^2) = a_{k+1} b_k^2 + i_1 (a_{k+1} b_k b_k)$$

$$= a_{k+1} b_k^2 + i_1 (b_k a_k b_k + a_k b_k a_k, b_k)$$

$$= a_{k+1} b_k^2 + b_k a_{k+1} b_k + i_1 (a_k b_k a_k b_k + b_k a_k b_k a_k)$$

$$= a_{k+1} b_k^2 + b_k a_{k+1} b_k + a_k b_k a_k a_k$$

and

$$d_2(b_{k+1} a_k^2) = b_{k+1} a_k^2 + b_k a_{k+1} a_k + b_k b_k a_k a_k.$$  

Now

$$d_2(a_m b_k a_k b_k a_k) = a_m b_k a_k b_k a_k + i_1 (a_m b_k a_k b_k a_k + a_m b_k a_k a_k)$$

$$= a_m b_k a_k b_k a_k + i_1 (b_k a_k b_k a_m a_k + a_k b_k a_k a_m a_k)$$

$$= a_m b_k a_k b_k a_k + b_k a_k b_k a_m a_k + i_1 (a_k b_k a_k a_m b_k + a_k b_k a_k a_m b_k)$$

$$= a_m b_k a_k b_k a_k + b_k a_k b_k a_m a_k + a_k b_k a_k a_m a_k.$$  

Therefore $d_2(a_m b_k a_k b_k a_k)$ and $d_2(b_m b_k a_k b_k a_k)$ lie in the radical of $P_1$.

Let $T'_1 = T_1 \setminus \{b_k a_k b_k a_k \mid k \in N_0\}$ and $T'_2 = T_2 \setminus \{b_{k+1} a_k^2 \mid k \in N_0\}$. Define $P'_1$ and $P'_2$ to be the submodules of $P_1$ and $P_2$ with $A$-bases $T'_1$ and $T'_2$ respectively. We define $d'_1$ to be the restriction of $d_1$ on $P_1$. The differential $d'_2: P'_2 \to P'_1$ is defined as follows. Let $t \in T'_2$. Then

$$d'_2(t) = f + \sum_{k=0}^{\infty} f_k b_k a_k a_k b_k a_k$$

where $f \in P'_2$ and only finitely many $f_k$ are different from 0. Define

$$d'_2(t) = f + \sum_{k=0}^{\infty} f_k (b_{k+1} a_k^2 + a_k b_{k+1} a_k).$$

By the usual consideration we can see that the complex

$$P'_2 \to P'_1 \to P'_0 \to P'_{-1} \to \mathbb{K} \to 0$$

is exact. Moreover it is minimal up to the term $P'_1$. We get
Theorem 3. Let us denote $\mathcal{U}_K(sl_n^+)$ by $A$ and the free module over $A$, which is generated by an element of degree $s$, by $A[s]$. The trivial module $K$ over $A$ has a minimal projective resolution of the form

$$
\bigoplus_{0 \leq k} A[2k+1]^\oplus 2 \oplus \bigoplus_{0 \leq k < l} A[2l + 2k]^\oplus 4 \to \bigoplus_{k=0}^\infty A[2k]^\oplus 2 \to A \to K \to 0.
$$

8 Conjectures

In this section we formulate several conjectures that were guessed from the excessive computer computations. We will consider the set of generators $X = \{ e_{i,i+1}^{(p^k)} \mid 1 \leq i \leq n - 1, \ k \in \mathbb{N}_0 \}$ of $\mathcal{U}_K(sl_n^+)$, where $p$ is the characteristic of the field $K$. To make formulae more readable we shall write $a_{ik}$ instead of $e_{i,i+1}^{(p^k)}$.

We will assume the ordering $\leq$ on $X$ defined by

$$a_{11} < a_{21} < \cdots < a_{n-1,1} < a_{12} < \cdots < a_{n-1,2} < \cdots$$

and on $\mathcal{U}_K(sl_n^+)$ we consider deglex ordering, where $\deg(a_{ik}) = p^k$.

Conjecture 1. The map $X \to \mathcal{U}_K(sl_n^+)$, $a_{ik} \mapsto a_{i,k+j}$ can be extended to a homomorphism of graded algebras $\mathcal{U}_K(sl_n^+) \to \mathcal{U}_K(sl_n^+)$.

Note that in the case of $n = 3$ and $p = 2$ the claim of the conjecture easily follows from Proposition.

Conjecture 2. Suppose $n = 3$ and $p > 2$. Denote $a_{1k}$ by $a_k$ and $a_{2k}$ by $b_k$. Then the set

$$\{ a_k, b_k, b_k^2a_k - 2b_kak - a_k^2, b_k^2a_k - 2a_kb_kak + a_k^2b_k, (b_k^2a_k - a_kb_k)^p - (a_kb_k)^p \mid k \in \mathbb{N}_0 \}$$

is the Gröbner basis of $\mathcal{U}_K(sl_n^+)$.

Now we formulate a conjecture about Gröbner basis for the case $p = 2$ and $n \geq 4$. For every sequence of integers $i = (i_1, \ldots, i_l)$ we denote by $a_{i,k}$ the product

$$a_{i_1,k} \cdots a_{i_l,k}.$$

We write $i..j$ for a sequence $(i, i+1, \ldots, j)$ if $i < j$ and for a sequence $i, i-1, \ldots, j$ if $i > j$. We denote by $L_l$ the set of all permutations $(i_1, \ldots, i_l)$ of $(1, \ldots, l)$ such that for every $1 \leq \sigma \leq l - 1$ either $i_{\sigma+1} < i_\sigma$ or $i_{\sigma+1} = i_\sigma + 1$. Define for every $m$ such that $m + l \leq n$

$$L_l[m] = \{ (i_1 + m, \ldots, i_l + m) \mid (i_1, \ldots, i_l) \in L_l \}.$$
Conjecture 3. The set

\[
\begin{align*}
&\left\{ a_{ik}^2 \mid 1 \leq i \leq n-1, \; k \in \mathbb{N}_0 \right\} \cup \left\{ \sum_{i \in L[m]} a_{ik}^2 \mid k \in \mathbb{N}_0, l + m \leq n \right\} \\
&\cup \left\{ [a_{i+m-1,k}, a_{i+m,k}] + [a_{i+m-1,k}, a_{i+m}, a_{i+m} \cdots] \mid 1 \leq i \leq n - m - 1, \; k \in \mathbb{N}_0 \right\} \\
&\cup \{ a_i b_k + b_k a_i + a_k b_i k \cdots a_{i-1}, b_i a_k + a_k b_i + b_k a_k b_k \cdots b_{i-1} \}.
\end{align*}
\]

is a Gröbner basis of \( \mathfrak{U}_k(sl^n_+) \).

Note that this conjecture has a very simple proof module Conjecture 1.

References

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