Equivalence of neighborhoods of embedded compact complex manifolds and higher codimension foliations

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How to extend an holomorphic object to a neighborhood?

\( C^d \) is a compact complex \( d \)-manifold embedded (holomorphically) in complex manifolds \( M^{n+d} \).

\[ C \hookrightarrow M \rightsquigarrow \text{Normal bundle } N_C := TM/TC. \]

Idea:
- Transform a neighborhood of \( C \) in \( M \) into a neighborhood of \( C \) (the 0th section) in \( N_C \).
- Extend to a neighborhood of 0th in \( N_C \).
- Pull-back the extension to a neighborhood of \( C \) in \( M \).
Embedding of compact complex submanifolds.

$C^d$ is a compact complex $d$-manifold embedded (holomorphically) in complex manifolds $M_1^{n+d}, M_2^{n+d}$

$$C^d \rightarrow \begin{cases} M_1^{n+d} \\ M_2^{n+d} \end{cases}$$

Take a (arbitrary small) neighborhood $U_i$ of $C$ in $M_i$

Question : Is $U_1$ holomorphically equivalent to $U_2$ ?

→ Grauert’s “Formale Prinzip” question : If “$U_1$ is formally equivalent to $U_2$”, is $U_1$ holomorphically equivalent to $U_2$ ?
Some answers

→ If $N_C$ is negative, then Grauert (Hironaka-Rossi) showed

formal equivalence $\Rightarrow$ holomorphic equivalence

→ If $N_C$ is positive, then Griffiths showed

equivalence up to some finite order $\Rightarrow$ holomorphic equivalence

→ Arnold’s construction: **elliptic curve** $\mathbb{T}^1 \hookrightarrow M^2$ s.t. neighborhood of $\mathbb{T}^1$ **formally equiv.** to neighborhood of 0th section in $N_{\mathbb{T}^1}$ and **holomorphic equiv.** based on *small divisors condition*.

→ Ilyashenko-Pyartli: ext. of Arnold’s construction to **direct product** of elliptic curves the normal bdle of which is **direct sum of line bdles** $\Leftarrow$ *small divisors condition*. 
Coordinates Patch

Covering $\mathcal{U} = \{U_i\}$ of $C$ with coordinates $z_i$.
Covering $\mathcal{V} = \{V_i\}$ of a neighborhood of $C$ in $M$: $U_i = V_i \cap C$ with coordinates $(z_i, w_i)$, $U_i := \{w_i = 0\}$. $U_i \cap U_j$:
$\{ z_i = \Psi_{ij}(z_j) \}$

transition set of $C$
Neighborhoods in coordinates

Assumption: $TM|_C = TC \oplus N_C$.

In coordinates, the transitions functions on $V_k \cap V_j$

- of the normal bundle $N_C(M)$ is $N_{kj}(z_j, w_j)$:

$$w_k = t_{kj}(z_j)w_j$$
$$z_k = \phi_{kj}(z_j).$$

- of the neighborhood of the embedding is $\Phi_{kj} := N_{kj}(z_j, w_j) + \phi_{kj}$

$$w_k = t_{kj}(z_j)w_j + \sum_{Q \in \mathbb{N}^d, |Q| \geq 2} \phi^v_{kj,Q}(z_j)w_j^Q$$
$$z_k = \phi_{kj}(z_j) + \sum_{Q \in \mathbb{N}^d, |Q| \geq 2} \phi^h_{kj,Q}(z_j)w_j^Q.$$
Holomorphic (resp. formal) Equivalence Problem

To find $F_j = I + f_j$ holomorphic (resp. formal) on $V_j$ such that

$$F_k \Phi_{k,j} = N_{k,j} F_j$$

$$w_j = v_j + \sum_{Q \in \mathbb{N}^d, |Q| \geq 2} f^v(h_j)v_j^Q$$

$$z_j = h_j + \sum_{Q \in \mathbb{N}^d, |Q| \geq 2} f^h(h_j)v_j^Q.$$
Computations

To simplify exposition: \( N_C \textbf{ flat} \sim t_{kj} = cst. \)

\[ F_j = I + (f^h_j, f^v_j), \quad F_k \Phi_{kj} = N_{kj} F_j \]

We obtain the horizontal equation

\[
\phi_{kj}(h_j) + f^h_k(\phi_{kj}, t_{kj} v_j + \phi^v_{kj}) = \phi_{kj}(h_j + f^h_j(h_j, v_j)) \\
+ f^h_k(h_j + f^h_j, v_j + f^v_j).
\]

\[
f^h_k(\phi_{kj}(h_j), t_{kj} v_j) - D\phi_{kj}(h_j) f^h_j(h_j, v_j) = -\phi^h_{kj}(h_j, v_j) \\
+ \text{nonlinear in the unknowns} \\
= R^h_{kj}
\]

The vertical equation reads

\[
f^v_k(\phi_{kj}(h_j), t_{kj} v_j) - t_{kj} f^v_j(h_j, v_j) = -\phi^v_{kj}(h_j, v_j) \\
+ \text{nonlinear in the unknowns} \\
= R^v_{kj}
\]
Cohomology operators

Idea: step by step on homogenous degree \( m \geq 2 \) wrt to \( v_j \):

\[
\delta[f]_m = \left( \begin{array}{c}
\delta^h(\{[f^h_j]_m\}) \\
\delta^v(\{[f^v_j]_m\})
\end{array} \right) = \{[R_{k,j}]_m\} = \mathcal{F}_m([f]_2, \ldots, [f]_{m-1}, [\phi^\bullet]_2, \ldots, [\phi^\bullet]_m).
\]

\[
\begin{align*}
\delta^h(\{[f^h_j]_m\}) &= \{[R^h_{k,j}]_m\} \in C^1(\mathcal{U}, TM|_C \otimes S^m(N_C^*)) \\
\delta^v(\{[f^v_j]_m\}) &= \{[R^v_{k,j}]_m\} \in C^1(\mathcal{U}, N_C \otimes S^m(N_C^*))
\end{align*}
\]

\[
C^0(\mathcal{U}, E) \xrightarrow{\delta} C^1(\mathcal{U}, E) \xrightarrow{\delta} C^2(\mathcal{U}, E) \xrightarrow{\delta} \cdots
\]

→ Need to **solve cohomological equations**
→ Need to **estimate** solutions → “small divisors”
Case of torus

In torus case, once can manage just to have 1 equation with 1 unknown

\[ f(\omega + h, \lambda_1 v_1, \cdots, \lambda_n v_n) - \text{diag}(\lambda_1, \ldots, \lambda_n) f(h, z) = \cdots \]

\[ \sim \sum_{k \in \mathbb{Z}^d} \sum_{Q \in \mathbb{N}^n, |Q| \geq 2} \left( e^{2\pi i k \cdot \omega} \lambda_1^{q_1} \cdots \lambda_n^{q_n} - \lambda_i \right) f_{i, k, Q} e^{2\pi i k \cdot h \cdot v} Q = \cdots \]
Estimates of solutions the cohomological equations

→ I.F. Donin : \{U^\varepsilon\} family of covering of \( C \): \( f \in C^1(U^\varepsilon, E) \); \([f] = 0\) in \( H^1 \). then : \( \exists u \in C^0(U^\varepsilon/3, E) \) s.t. \( \delta u = f \) and \( \|u\|_{\varepsilon/3} \leq \frac{K}{\varepsilon^r} \|f\|_{\varepsilon} \)

Proposition ("pumping")
\( \exists \{U^r\}_{r_\ast \leq r \leq r^*} \) family of "polydiscs" coverings of \( C \) s.t.
\[
\|u\|_r \leq K(E)\|f\|_r, \quad \delta u = f
\]
Our ”full linearization” result (flat case)

Let $\eta_0 := 1$ and $d_m := \max (K(N_C \otimes S^m(N_C^*)), K(T_C \otimes S^m(N_C^*)))$

$$\eta_m := d_m \max_{m_1 + \cdots + m_p + s = m} \eta_{m_1} \cdots \eta_{m_p},$$

Theorem (Linearization of neighborhoods)

Assumptions :

- $(M, C) \cong (N_C, C)$
- $N_C$ flat and unitary
- $\eta_m \leq L^m$
- $H^0(C, T_C M \otimes S^\ell(N_C^*)) = 0$, for all $\ell > 1$

Conclusion $(M, C') \equiv (N_C, C')$
Existence of foliation with leaf $C$

If there is a neighborhood s.t.

$$w_k = t_{kj}w_j$$

$$z_k = \phi_{kj}(z_j) + \sum_{Q \in \mathbb{N}^d, |Q| \geq 2} \tilde{\phi}_{kj,Q}^h(z_j)w_j^Q.$$ 

then $w_j = \text{cst}$ gives a foliation.

$\leadsto$ ”vertical” linearization.
Let $\eta_0 := 1$ and $d_m := K(N_C \otimes S^m(N^*_C))$

$$\eta_m := d_m \max_{m_1 + \cdots + m_p + s = m} \eta_{m_1} \cdots \eta_{m_p},$$

**Theorem (Vertical linearization of neighborhoods)**

**Assumptions:**

- $(M, C)$ is "formally vertically linearizable"
- $N_C$ flat and unitary
- $\eta_m \leq L^m$
- $H^0(C, N_C \otimes S^\ell(N^*_C)) = 0$, for all $\ell > 1$

**Conclusion** $(M, C)$ is "holomorphically vertically linearizable"

$\rightarrow$ Ueda : complex curve in a surface $(n = d = 1)$