Schwinger-Dyson Equations in 2D Induced Gravity in Covariant Gauges

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Abstract

We formulate the Schwinger-Dyson equations in the ladder approximation for 2D induced quantum gravity with fermions using covariant gauges of harmonic type. It is shown that these equations can be formulated consistently in a gauge of Landau type (for negative cosmological constant). A numerical analysis of the equations hints towards the possibility of chiral symmetry breaking, depending on the value of the coupling constant.
A very useful tool for the study of dynamical symmetry breaking and dynamical mass generation in quantum field theory are the Schwinger-Dyson equations. The standard approach, which consists in working with this infinite set of integral equations has been developed in the pioneering works [1, 2], where the example of quantum electrodynamics was considered. By investigating the truncated version of the Schwinger-Dyson equations (the ladder approximation), the possibility of chiral symmetry breaking and dynamical fermion mass generation in QED could be demonstrated [1, 2] (for a review, see the proceedings [3]). The critical coupling constant in the Landau gauge has been found also. However, if dynamical fermion mass generation is taking place, then the non-perturbative Ward-Takahashi identity of QED is not satisfied. As a result, the dynamical fermion mass and the critical coupling constant in QED are very much gauge-dependent (for a recent discussion in an arbitrary covariant gauge, see [4, 5] and references therein).

If one is interested in further developing the Schwinger-Dyson formalism then more complicated models to study such equations, like those of quantum gravity, are to be considered. An example of an investigation of this kind for the case of 4D Einstein gravity coupled to fermions on a flat background has been presented in ref. [6], where by means of numerical estimations the possibility of chiral symmetry breaking has again been shown.

In the present work we will study the Schwinger-Dyson equations in 2D quantum gravity [7] with fermions, on a flat background. The covariant gauge with two gauge parameters will be chosen and chiral symmetry breaking in 2D quantum gravity will be investigated numerically.

The action of the theory under discussion is given by

\begin{align}
S &= S_g + S_f, \\
S_g &= -\frac{1}{2\gamma} \int d^2 x \sqrt{-g} \left( R \frac{1}{\Box} R + \Lambda \right), \\
S_f &= \int d^2 x \sqrt{-g} i \bar{\Psi} \gamma^\mu D_\mu \Psi, \quad (1)
\end{align}

where \( R \) is the two-dimensional curvature, \( \Psi \) the 2D spinor, \( \Lambda \) the cosmological constant, and \( D_\mu = \partial_\mu - (i/4) \omega_\mu^{bc} \sigma_{bc} \) is the 2D covariant derivative for spinors, where \( \sigma_{ab} = \)...
(i/2)[γ_a, γ_b], ω^bc_µ is spin-connection, and e^a_µ will denote the vierbein.

In the standard approach to string theory [7], one can start from a pure-matter theory (as given by $S_f$) in an external gravitational field, integrate then over spinors (this is easy to do in the conformal gauge) and get finally as a result 2D induced gravity, $S_g$, where γ is then specified.

Here we are going to employ a more traditional approach, in which we will start from the theory (1) and use the background field method on the flat background

$$g_{µν} = η_{µν} + h_{µν}, \quad e_{aµ} = η_{aµ} + \frac{1}{2} h_{aµ}. \quad (2)$$

Hence, γ in (1) is some given constant and we do not integrate over spinors. Expanding $S_f$ on the flat background and working in the momentum representation, one easily finds that the interaction Lagrangian has the following form

$$L_{int} = \frac{1}{4} \bar{\Psi}(p') \left[ 2\bar{p}\eta_{µν} - 2γ(µp_ν) + \left( \bar{k}\eta_{µν} - γ(µk_ν) \right) \right] \Psi(p) h^{µν}(k), \quad (3)$$

what corresponds to the fermion-graviton vertex

$$Γ_{µν}(p, k) = \frac{1}{4} (2p + k)^\lambda γ^\sigma I_{λσµν},$$

$$I_{λσµν} = \frac{1}{4} (2η_{λσ}η_{µν} - η_{λµ}η_{σν} - η_{λν}η_{σµ}). \quad (4)$$

The gauge-fixing action will be chosen in the following form

$$S_{gf} = -\frac{1}{2γ} \int d^2x \sqrt{-g} \frac{1}{α} \left( ∇_µ h^{µρ} - β ∇_ρ h \right) \left( ∇_ν h^{νρ} - β ∇_ρ h \right), \quad (5)$$

where α and β are the gauge parameters. (For a discussion of 2D induced gravity in a covariant gauge of the harmonic type, see also [8, 9]).

The quadratic part of the total action $S = S_g + S_{gf}$ on a flat background is found to be

$$S^{(2)} = -\frac{1}{2γ} \int d^2x h^{µν} H_{µνρσ} h^{ρσ}, \quad (6)$$

where

$$H_{µνρσ} = \frac{∇_µ ∇_ν ∇_ρ ∇_σ}{□} + \frac{1}{2} ξ_1 (η_{ρσ} ∇_µ ∇_ν + η_{µσ} ∇_ρ ∇_ν) + ξ_2 η_{µρ} η_{νσ} □$$

$$+ \frac{1}{4} ξ_3 (η_{µσ} ∇_ν ∇_ρ + η_{µρ} ∇_ν ∇_σ + η_{νσ} ∇_µ ∇_ρ + η_{νρ} ∇_µ ∇_σ) + \frac{Λ}{4} I_{µνρσ}, \quad (7)$$
being $\xi_1 = 2(\beta/\alpha - 1)$, $\xi_2 = 1 - \beta^2/\alpha$ and $\xi_3 = -1/\alpha$. The graviton propagator is given by the inverse operator $H^{-1}_{\mu\nu\rho\sigma}$ and can be found using the algorithm of refs. [10], which yield

$$G_{\mu\nu\rho\sigma}(k) = -\gamma H^{-1}_{\mu\nu\rho\sigma}(k) = -\frac{4}{\Lambda} L_{\mu\nu\rho\sigma} + \frac{1}{\alpha - (2\beta - 1)^2} \left( \frac{[\alpha + \beta(1 - 2\beta)]^2}{(\beta - 1)^2(k^2 - m^2)} - \frac{\alpha}{k^2} \right) P_{\mu\nu\rho\sigma}$$

$$+ \frac{2\alpha}{k^2 - m^2} M_{\mu\nu\rho\sigma} + \frac{1}{\alpha - (2\beta - 1)^2} \left( \frac{(2\beta - 1)[\alpha + \beta(1 - 2\beta)]}{(\beta - 1)(k^2 - m^2)} - \frac{\alpha}{k^2} \right) (L_{\mu\nu} P_{\rho\sigma} + L_{\mu\rho} P_{\nu\sigma})$$

$$+ \left[ \frac{4}{\Lambda} + \frac{1}{\alpha - (2\beta - 1)^2} \left( \frac{(1 - 2\beta)^2}{(k^2 - m^2)} - \frac{\alpha}{k^2} \right) \right] L_{\mu\nu} L_{\rho\sigma}. \quad (8)$$

Here

$$L_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \quad L_{\mu\nu\rho\sigma} = \frac{1}{2} (L_{\mu\rho} L_{\nu\sigma} + L_{\mu\sigma} L_{\nu\rho}), \quad P_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}, \quad (9)$$

$$P_{\mu\nu\rho\sigma} = \frac{k_\mu k_\nu k_\rho k_\sigma}{k^4}, \quad M_{\mu\nu\rho\sigma} = \frac{1}{2} (L_{\mu\rho} P_{\nu\sigma} + L_{\mu\sigma} P_{\nu\rho} + L_{\nu\rho} P_{\mu\sigma} + L_{\nu\sigma} P_{\mu\rho}), \quad m^2 = \frac{\alpha \Lambda}{2}.$$  

The exact spinor propagator has the following form

$$S^{-1}(p) = A(p)\hat{p} - B(p^2), \quad (10)$$

where $A$ and $B$ are some unknown functions. Now we have at hand all the Feynman diagram elements: the exact spinor propagator, the free spinor propagator, $S_0^{-1}(p) = \hat{p}$, the free graviton propagator [8] and the vertex [4].

The effective potential for the composite fields [11] in the ladder approximation [4, 2] can be written as

$$V_{\text{eff}} = -i \text{Sp} \left( \ln S_0^{-1}S - S_0^{-1}S + 1 \right) + V_2, \quad (11)$$

where $V_2$ corresponds to the two-particle irreducible vacuum diagram, which follows from the vertex

$$V_2 = -\frac{i}{2} \int \frac{d^2p}{(4\pi)^2} \int \frac{d^2q}{(4\pi)^2} \text{Tr} \left[ \Gamma(p - q, q)S(q)\Gamma(q - p, p)G(p) \right]. \quad (12)$$

The Schwinger-Dyson equations (in the ladder approximation) correspond to the minimum of the potential [11]

$$S^{-1}(p) - S_0^{-1}(p) = -i \int \frac{d^2q}{(4\pi)^2} \Gamma_{\mu\nu}(q, p - q)S(q)\Gamma_{\rho\sigma}(p, q - p)G^{\mu\nu\rho\sigma}(p - q). \quad (13)$$
Using the explicit form of the spinor and graviton propagators (13), (14), and the vertex (4), one can get —after performing Wick’s rotation and the angular integration (we drop the details of these straightforward but very tedious calculations)

\[ V_{\text{eff}} = -\frac{N_f M^2}{8\pi} \left\{ \int_0^1 dx \left[ \ln \left( \frac{A^2(x) + B^2(x)}{x^2} \right) - 2 \frac{A(x)(A(x) - 1)x + B^2(x)}{x^2 + x A^2(x) + B^2(x)} \right] + g \int_0^1 dx \int_0^1 dy \frac{dy}{y A^2(y) + B^2(y)} [A(x)A(y)K_A(x, y) + B(x)B(y)K_B(x, y)] \right\} , \]

(14)

where \( N_f \) is the dimension of the fermion representation, \( M \) the momentum cutoff, \( x = p^2/M^2 \), \( y = q^2/M^2 \), \( A(x) = A(p^2) \), \( B(x) = B(p^2)/M \), and \( g = \gamma/(64 \pi) \) and \( l = \Lambda/(2M^2) \). The explicit expressions for \( K_A \) and \( K_B \) are very complicated for arbitrary \( \alpha \) and \( \beta \). Moreover, if \( \alpha \neq 0 \) the Schwinger-Dyson equations contain the infrared divergences caused by the graviton zero momentum. Let us give some examples of \( K_A \) and \( K_B \) for different choices of the gauge parameters.

1. Gauge with \( \alpha \) arbitrary, \( \beta = 1/2 \).

\[ K_A(x, y) = \frac{1}{2} \left\{ (1 - 3\alpha)(x + y) + \frac{\alpha[7(x^2 + y^2) + 10xy + 3\alpha(x + y)] + 4(x + y)(x - y)^2/l}{\sqrt{x + y + \alpha l}^2 - 4xy} \right\} , \]

\[ K_B(x, y) = 5\alpha - 1 - \frac{5\alpha[2(x + y) + \alpha l] + 4(x - y)^2/l}{\sqrt{x + y + \alpha l}^2 - 4xy} + \frac{2(x + y) + 4(x - y)^2/l}{|x - y|}. \]

(15)

Here \( \Lambda > 0 \) and \( l = \Lambda/(2M^2) \), and in this gauge one finds infrared divergences in the Schwinger-Dyson equations (i.e., at the lower limit of the integrals in (14)).

2. Let us now consider a gauge of Landau type \( (\alpha = 0, \beta \) arbitrary), in which the Schwinger-Dyson equations do not contain infrared divergences. There one finds

\[ K_A(x, y) = \frac{1}{2} \left\{ \frac{\beta^2}{(\beta - 1)^2}(x + y) - \left[ \frac{\beta[4\beta xy + 2(2\beta - 1)(x - y)^2 + \beta(x + y)(x + y + \mu^2)]}{(\beta - 1)^2} \right] \right\} , \]

\[ + \frac{4(4\beta - 1)}{l(2\beta - 1)}(x + y)(x - y)^2 \left[ (x^2 + y^2 + \mu^2)^2 - 4xy \right]^{-1/2} \]

(16)
\[ K_B(x, y) = \left\{ \begin{array}{l}
\frac{\beta^2}{(\beta - 1)^2} [2(x + y) + \mu^2] + \frac{4(x - y)^2}{l(2\beta - 1)} [(x^2 + y^2 + \mu^2)^2 - 4xy]^{1/2} \\
- \frac{4|x - y|}{l(2\beta - 1)} - \frac{\beta^2}{(\beta - 1)^2}.
\end{array} \right. \] 

Here \( \Lambda < 0, \ l = -\Lambda/(2M^2) \) and \( \mu^2 = (4/l)[(2\beta - 1)/(\beta - 1)]^2 \). Notice that for \( \alpha = 0, \ \beta = 1/2 \), the theory contains again infrared divergences, because in this case

\[ K_A(x, y) = \frac{1}{2} \left( x + y - \frac{x^2 + y^2 + 6xy}{|x - y|} \right), \]
\[ K_B(x, y) = 2 \frac{x + y}{|x - y|} - 1. \] 

Observe also that, in principle, one expect to find more complicated covariant gauges which are free of infrared problems in the region where the cosmological constant is positive.

Starting from eqs. (13) and integrating over the angles one can show that the functions \( A \) and \( B \) must obey integral equations of the following form

\[ A(x) = 1 + g \int_0^1 dy \frac{A(y)}{y A^2(y) + B^2(y)} \frac{1}{x} K_A(x, y), \]
\[ B(x) = g \int_0^1 dy \frac{B(y)}{y A^2(y) + B^2(y)} K_B(x, y). \] 

It is not possible to solve these equations analytically. (We will discuss here the case of the physical Landau-type gauge (16) only, where no IR divergences appear in the theory, in order to avoid the introduction of any IR cutoff). We present the result of a numerical calculation, obtained by using an iterative procedure (in close analogy with [3]). We consider two types of trial functions

(a) \( A^0(x) = c_1, \ B^0(x) = 0, \)
(b) \( A^0(x) = c_1, \ B^0(x) = c_2, \)

where \( c_1 \) and \( c_2 \) are some constants between 0 and 1. We will also fix the values of \( g, \ l \) and \( \beta \). The functions \( A^0(x) \) and \( B^0(x) \) can then be taken as the starting point of a
self-consistent iterative calculation of the form

$$A^{i+1}(x) = 1 + g \int_0^1 dy \frac{A^i(y)}{y A^{i2}(y) + B^{i2}(y)} K_A(x, y),$$

$$B^{i+1}(x) = g \int_0^1 dy \frac{B^i(y)}{y A^{i2}(y) + B^{i2}(y)} K_B(x, y).$$  \hspace{1cm} (19)$$

The sequences formed by the \{A^i(x)\} and \{B^i(x)\} are expected to converge towards the functions \(A(x)\) and \(B(x)\), respectively, which are the sought for solutions of (18). In practice one can judge the degree of convergence of these series by the smallness of the squared norms of the differences \(A^{i+1} - A^i\) and \(B^{i+1} - B^i\), which we set at \(10^{-4} - 10^{-6}\) in our calculation. If, for the given \(g\) and \(l\), there are solutions of both types, (a) and (b), only the most stable of both by \(V_{\text{eff}}\) \(\text{(10)}\) is to be chosen as the one corresponding to the true vacuum.

We have executed this algorithm to solve (19), starting from the trial functions (a) and (b), for fixed \(l = 4, \beta = 1/3\) and varying \(g\). For very small \(g\)'s, both types lead to curves close to \(A(x) = 1, B(x) = 0\), i.e. the chiral symmetric solution, as was to be expected. As \(g\) increases, the value of \(V_{\text{eff}}\) for the chiral solution of symmetric type (a) appears to be slightly higher than the corresponding one for the non-symmetric solution. In particular, for \(g = 0.1\) (see Fig. 1), the chiral symmetric solution is the preferred one. For \(g = 0.2\) or \(g = 0.3\) (see Fig. 1 again, where typical curves for \(A\) and \(B\) are presented), one can see that the chiral non-symmetric solutions are preferable. Hence, we see clearly that the Schwinger-Dyson equations for 2D gravity with fermions may have chiral symmetry breaking regimes in the covariant gauges.

We will now say a few words about the regime of the Schwinger-Dyson equations corresponding to a theory with positive cosmological constant. In this case, when working in the covariant gauges under discussion, one encounters problems related with infrared divergences. We will use the conformal gauge

$$g_{\mu\nu} = e^{\phi} \eta_{\mu\nu}. \hspace{1cm} (20)$$

In this gauge one finds the Schwinger-Dyson equations \(\text{(13)}\) and \(\text{(18)}\) with the functions
given by

\[
K_A(x, y) = -\frac{4xy + (x + y)(x + y + l - \sqrt{(x + y + l)^2 - 4xy})}{2\sqrt{(x + y + l)^2 - 4xy}},
\]

\[
K_B(x, y) = \frac{2(x + y) + l - \sqrt{(x + y + l)^2 - 4xy}}{\sqrt{(x + y + l)^2 - 4xy}}.
\]

(21)

Numerical solutions of the corresponding Schwinger-Dyson equations can be obtained as above (see ref. [12]). Typical curves for \( l = 0.5 \) and varying \( g \) are shown in Fig. 2. Here we observe again the possibility of chiral symmetry breaking.

Summing up, we have studied the Schwinger-Dyson equations corresponding to 2D gravity coupled with fermions in a covariant (harmonic) gauge. Numerical analysis of the equations show clearly the possibility of chiral symmetry breaking in the region with negative cosmological constant, where the Schwinger-Dyson equations can be consistently formulated in a gauge of Landau type and no infrared divergences appear. In the region of positive cosmological constant, the analysis done in the conformal gauge shows as well the possibility of chiral symmetry breaking. The results of the numerical analysis of the solutions (and the Schwinger-Dyson equations themselves) are certainly gauge dependent. Currently there is no way to solve such a drawback of the Schwinger-Dyson equations, even in the case of renormalizable theories, as QED, where the Ward-Takahashi identities have a quite simple form. We have nothing to add here that can help to resolve this general problem of gauge dependence [4, 5]. Our purpose has been simply to show that chiral symmetry breaking is indeed possible in 2D gravity theories with fermions in different gauges. That this is actually the case has been realized by means of a rather straightforward numerical analysis of the corresponding Schwinger-Dyson equations.

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Figure captions

**Fig. 1.** Plot of the functions $A$ and $B$ obtained as the (a)-type solutions for $g = 0.1$ and (b)-type solutions for $g = 0.2$ and $g = 0.3$ keeping $l = 4$ fixed.

**Fig. 2.** Plot of the functions $A$ and $B$ obtained as the (a)-type and (b)-type solutions for $g = 0.1$, 0.2 and 0.25, keeping $l = 0.5$ fixed. Notice how $B$ deviates more and more from the $g = 0$ solution ($B(x) = 0$) as $g$ increases. Although not shown in the figure, the curve keeps going up for larger values of $g$. 
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