$USp(2k)$ Matrix Model:
Schwinger-Dyson Equations and
Closed-Open String Interactions

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Abstract

We derive the Schwinger-Dyson/loop equations for the $USp(2k)$ matrix model which close among the closed and open Wilson loop variables. These loop equations exhibit a complete set of the joining and splitting interactions required for the nonorientable $TypeI$ superstrings. The open loops realize the $SO(2n_f)$ Chan-Paton factor and their linearized loop equations derive the mixed Dirichlet/Neumann boundary conditions.

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I. Introduction

Intensive study has been recently made for the nonperturbative formulation of superstrings, trying to uncover the properties which are not accessible in the first quantized perturbative formulation of superstrings. The study in this direction appears to be imperative in order to confront string physics with the world we observe in nature. One such approach toward this goal starts with a model as the constructive definition of the type $IIB$ or the type $I$ superstrings in the form of zero dimensional reduced model $[1, 2, 3, 4]$ and this direction, in particular, the case of the $USp$ matrix model $[3, 4]$ for the type $I$ superstrings is the focus of the present paper.

The reduced model is a nonabelian counterpart of the first quantized critical superstring theory in the Schild gauge $[5]$ and the large $k$ limit makes this connection clear. From this point alone, it is certain that the model is for unification of all forces including gravity and is not limited to low energy phenomena mediated by open strings. Another aspect of the model is that the matrix degrees of freedom in fact generate manybody effects of strings albeit the fact that the model is originally in its first quantized form. We would appreciate these points better if we are able to formulate the model in the second quantized form. The Schwinger-Dyson equations representing loop dynamics accomplish this.

The Schwinger-Dyson equations for the case of the $IIB$ matrix model have already been examined in $[2]$. Here we focus on the derivation of the Schwinger-Dyson equations for the case of the $USp$ matrix model. The properties and implications of the $USp$ matrix model have been elaborated in $[3, 4, 6, 7]$. Among other things, this model introduces open string degrees of freedom in an explicit way in contrast to the ones associated with D-objects $[8]$ as classical solutions. This point translates into the open loop variables of our paper. In the next section, we specify a set of loop variables adopted for the Schwinger-Dyson equations of the $USp(2k)$ matrix model. The $SO(2n_{(f)})$ Chan-Paton factor emerging from the open loop is observed. In section three, we derive the Schwinger-Dyson equations and a complete set of the joining and splitting interactions required for the nonorientable $TypeI$ superstrings is exhibited. Comparison with the string field theory of $[9, 10]$ is made. In section four, we study these equations at the linearized level. In addition to the Virasoro condition for the closed loops noted before at $[2]$, we find that the open loops satisfy the appropriate mixed Dirichlet/Neumann boundary conditions. The final section is devoted to outlook and open questions for the reduced model in general.

In Appendix A, we present the action of the $USp$ matrix model in a more compact component form than is presented in $[3, 4]$, so that the derivation in section three becomes a more manageable procedure. Readers are advised to look at some of the notation established
here before going into the text. In Appendix B, we list kinetic terms of the Schwinger-Dyson equations.

II. Choice of variables

Let us first introduce a discretized path-ordered exponential which represents a configuration of a string in momentum superspace:

$$U[p^M, \eta; n_1, n_0] \equiv P \exp(-i \sum_{n=n_0}^{n_1} (p^M_n v_M + \bar{\eta}_n \Psi)) = \prod_{n=n_0}^{n_1} \exp(-i p^M_n v_M - i \bar{\eta}_n \Psi) \ . \quad (2.1)$$

where $p^M_n$ and $\eta_n$ are respectively the sources or the momentum distributions for $v_M$ and those for $\Psi$. The closed loop is then defined by

$$\Phi[p^M, \eta; n_1, n_0] \equiv TrU[p^M, \eta; n_1, n_0] \ . \quad (2.2)$$

To consider an open loop, let us introduce $\Xi = (\xi, \xi^*)$ as bosonic sources for $Q_{(f)}$ and $Q^*_{(f)}$, and $\Theta = (\theta, \bar{\theta})$ as Grassmannian ones for $\psi_{Q_{(f)}}$ and $\psi_{Q^*_{(f)}}$: $\left(\Xi \Omega_{(f)}\right) = \xi Q_{(f)} + F^{-1} \xi^* Q^*_{(f)}$, $\left(\Theta \Upsilon_{(f)}\right) = \theta \psi_{Q_{(f)}} + F^{-1} \bar{\theta} \psi_{Q^*_{(f)}}$. We write these collectively as

$$\left(\Lambda \Pi_{(f)}\right) \equiv \left(\Xi \Omega_{(f)}\right) + \left(\Theta \Upsilon_{(f)}\right) \ . \quad (2.3)$$

The open loop is defined by

$$\Psi_{f f'}[k^m, \zeta; l_1, l_0; \Lambda', \Lambda] \equiv \left(\Lambda' \Pi_{(f')}\right) FU[k^m, \zeta; l_1, l_0] \left(\Lambda \Pi_{(f)}\right) \ , \quad (2.4)$$

where $f$ and $f'$ are the Chan-Paton indices. In view of the notion of the macroscopic loop in the one and multi matrix models of random surfaces, it is clear that these loops are the appropriate nonabelian generalization to the reduced model for string unification and that they generate all of the observables in the theory under question.

Let us see how the nonorientability of the closed and the open strings is realized in the loops we introduced. Using the eqs. (A.1), (A.2), namely, $v'_M = \mp F v_M F^{-1}$, $\Psi' = \mp F \Psi F^{-1}$, and $F^t = -F$ , we readily obtain

$$\Phi[p^M, \eta; n_1, n_0] = Tr\left( \prod_{n=n_0}^{n_1} \exp(-i p^M_n v'_M - i \bar{\eta}_n \Psi') \right) = \Phi[\mp p^M, \mp \eta; n_0, n_1] \ , \quad (2.5)$$

and

$$\Psi_{f f'}[k^M, \zeta; l_1, l_0; \Lambda', \Lambda] = -\Psi_{f f'}[\mp k^M, \mp \zeta; l_0, l_1; \Lambda, \Lambda'] \ . \quad (2.6)$$

These equations relate a string configuration to the one with its orientation reversed. The Chan-Paton factor is reversed as well for the case of the open loops. The minus signs in
front of \( p^m, \eta, k^m \) and \( \zeta \) in eq. (2.5) and eq. (2.6) reflect the orientifold structure of the \( USp(2k) \) matrix model. The overall minus sign in the last line of (2.6) comes from \( F' = -F \) of the \( usp \) Lie algebra and corresponds to the \( SO(2n_f) \) gauge group. (Clearly we obtain the plus sign for the case of the \( so \) Lie algebra: \( F' = F \).) We see that the infrared stability of perturbative vacua \([11, 12]\) tells that the original matrices must be based on the \( usp \) as opposed to the \( so \) Lie algebra and that \( n_f = 16 \). This latter property also follows from the anomaly cancellation of the T-dualized representation of the theory by the 6D worldvolume gauge theory \([4]\).

### III. Schwinger-Dyson equations

To proceed to the loop equations, let us first introduce abbreviated notation:

\[
\Phi[(i)] \equiv \Phi[p^{(i)}, \eta^{(i)}; n_1^{(i)}, n_0^{(i)}], \quad \Psi[(i)] \equiv \Psi_{f(i)\bar{f}(i)}[k^{(i)}, \zeta^{(i)}; l_1^{(i)}, l_0^{(i)}; \Lambda^{(i)}, \Lambda^{(i)}],
\]

\[
\int d\mu \cdots \equiv \int [dv][dQ][dQ^*][d\Psi_Q][d\Phi^q] \cdots.
\]

We begin with the following set of equations consisting of \( N \) closed loops and \( L \) open loops:

\[
0 = \int d\mu \frac{\partial}{\partial X^r} \left\{ Tr(U[p^{(i)}, \eta^{(i)}; n_2^{(i)}, n_1^{(i)}] + 1)Tr^U[p^{(i)}, \eta^{(i)}; n_1^{(i)}, n_0^{(i)}] \right\}
\]

\[
\int d\mu \cdots \equiv \int [dv][dQ][dQ^*][d\Psi_Q][d\Phi^q] \cdots.
\]

\[
\Phi[(2)] \cdots \Phi[(N)][\Psi[(1)] \cdots \Psi[(L)] e^{-S} \right\}, \quad (3.1)
\]

\[
0 = \int d\mu \frac{\partial}{\partial X^r} \left\{ (\Lambda^{(i)})^{\top} \Pi^{(0)(1)}(f(1)) \right) FU[k^{(i)}, \zeta^{(i)}; l_2^{(i)}, l_1^{(i)}] + 1)|Tr^U[k^{(i)}, \zeta^{(i)}; l_1^{(i)}, l_0^{(i)}] (\Lambda^{(i)})^{\top} \Pi^{(0)(1)}(f(1)) \right) \]

\[
\Phi[(1)] \cdots \Phi[(N)][\Psi[(2)] \cdots \Psi[(L)] e^{-S} \right\}, \quad (3.2)
\]

\[
0 = \int d\mu \frac{\partial}{\partial Z_{(j)\bar{j}}} \left\{ (U[k^{(i)}, \zeta^{(i)}; l_1^{(i)}, l_0^{(i)}]) (\Lambda^{(i)})^{\top} \Pi^{(0)(1)}(f(1)) \right) \]

\[
\Phi[(1)] \cdots \Phi[(N)][\Psi[(2)] \cdots \Psi[(L)] e^{-S} \right\}, \quad (3.3)
\]

where \( X^r \) denotes \( v_M^r \) or \( \Psi_\alpha^r \) while \( Z_{(j)\bar{j}} \) denotes \( Q_{(f)\bar{i}} \) or \( \psi_{Q_{(f)\bar{i}}} \).

In what follows, we will exhibit eqs. (3.1) \~ (3.3) in the form of loop equations (3.10) \~ (3.19). We will repeatedly use

\[
\sum_{r=1}^{2k^2+k} (T^r)^i_j (T^r)^l_k = \frac{1}{2} (\delta^i_l \delta^j_k + F_{ik}^{-1} F^{ij} ), \quad (3.4)
\]

which is nothing but the expression for the projector \((A.3)\). In these equations below,

\[
P_n^{(i)} = \begin{cases} 
p_n^{(i)M} & \text{if } X^r = v_M^r, \\
-\eta_n^{(i)} & \text{if } X^r = \Psi^r,
\end{cases} \quad K_n^{(i)} = \begin{cases} 
k_n^{(i)M} & \text{if } X^r = v_M^r, \\
-\eta_n^{(i)} & \text{if } X^r = \Psi^r,
\end{cases} \quad (3.5)
\]
and $\Lambda^{(i)}$ not multiplied by $\Pi$ represents either $\Xi^{(i)}$ or $\Theta^{(i)}$. The symbol $\hat{b}$ denotes an omission of the $b$-th closed or open loop.

\[ \bullet \tag{3.1} \Rightarrow \quad 0 = (1) \text{ kinetic term (Fig. [1], [2])} + (2) \text{ splitting and twisting (Fig. [3])} \quad (3.6) \]

\[ + (3) \text{ joining with a closed string (Fig. [4])} + (4) \text{ joining with an open string (Fig. [5])} \quad . \]

Here

\[ (1) = \frac{1}{g^2} \langle (\delta_{X} \Phi[(1); X^{r}]) \Phi[(2)] \cdots \Phi[(N)] \Psi[(1)] \cdots \Psi[(L)] \rangle , \quad (3.7) \]

\[ (2) = \left\langle -\frac{i}{2} \sum_{n=n^{(1)}_{0}}^{n^{(1)}_{1}} P^{(1)} \right\rangle \left\{ \Phi[p^{(1)}; \eta^{(1)}; n^{(1)}, n^{(1)} + 1] \Phi[p^{(1)}; \eta^{(1)}; n, n^{(1)} + 1] \right\} \]

\[ + Tr(U[p^{(1)}; \eta^{(1)}; n^{(1)} + 1] U[\mp p^{(1)}, \mp \eta^{(1)}; n^{(1)} + 1]) \Phi[(2)] \cdots \Phi[(N)] \Psi[(1)] \cdots \Psi[(L)] \rangle , \quad (3.8) \]

\[ (3) = \left\langle -\frac{i}{2} \sum_{b=2}^{N} \sum_{n=n^{(b)}_{0}}^{n^{(b)}_{1}} P^{(b)} \right\rangle \left\{ Tr(U[p^{(1)}; \eta^{(1)}; n^{(1)} + 1] U[p^{(b)}, \eta^{(b)}; n, n + 1]) \right\} \]

\[ + Tr(U[\mp p^{(1)}, \mp \eta^{(1)}; n^{(1)} + 1, n^{(1)}] U[p^{(b)}, \eta^{(b)}; n, n + 1]) \}

\[ \Phi[(2)] \cdots \hat{b} \cdots \Phi[(N)] \Psi[(1)] \cdots \Psi[(L)] \rangle , \quad (3.9) \]

\[ (4) = \left\langle -\frac{i}{2} \sum_{b=1}^{L} \sum_{l=l^{(b)}_{0}}^{l^{(b)}_{1}} K^{(b)} \right\rangle \]

\[ \left\{ \left( \Lambda^{(b)} \Pi_{(f^{(b)})} \right) FU[k^{(b)}; \zeta^{(b)}; l^{(b)}, l + 1] U[p^{(1)}, \eta^{(1)}; n^{(1)}, n^{(1)} + 1] U[k^{(b)}, \zeta^{(b)}; l, l^{(b)}] \left( \Lambda^{(b)} \Pi_{(f^{(b)})} \right) \right\} \]

\[ + \left( \Lambda^{(b)} \Pi_{(f^{(b)})} \right) FU[k^{(b)}; \zeta^{(b)}; l^{(b)}, l + 1] U[\mp p^{(1)}, \mp \eta^{(1)}; n^{(1)} + 1, n^{(1)}] U[k^{(b)}; \zeta^{(b)}; l, l^{(b)}] \left( \Lambda^{(b)} \Pi_{(f^{(b)})} \right) \}

\[ \Phi[(2)] \cdots \Phi[(N)] \Psi[(1)] \cdots \hat{b} \cdots \Psi[(L)] \rangle . \quad (3.10) \]

We spell out the explicit form of $\delta_{X} \Phi[(1); X^{r}]$ in Appendix B. (This term comes from the variation of the action and contains terms representing closed-open transition.)

\[ \bullet \tag{3.2} \Rightarrow \quad 0 = (1) \text{ kinetic term (Fig. [1], [2])} + (2) \text{ splitting and twisting (Fig. [3])} \quad (3.11) \]
Figure 1: infinitesimal deformation of a closed string

Figure 2: closed-open transition

Figure 3: splitting and twisting of a closed string

Figure 4: joining of two closed strings

Figure 5: joining of a closed string and an open string
+ (3) joining with a closed string (Fig. 3) + (4) joining with an open string (Fig. 8) .

Here

\begin{align}
(1) & = \frac{1}{g^2} \langle (\delta_X \Psi[(1); X']) \Phi[(1)] \cdots \Phi[(N)] | \Psi[(2)] \cdots \Psi[(L)] \rangle, \quad (3.12) \\
(2) & = \langle \left( -\frac{i}{2} \sum_{l=t^{(1)}_0}^{t^{(1)}_1} K^{(1)}_l \right) \\
& \left\{ \left( \Lambda^{(1)'} \Pi^{(f'(1)')} \right) FU[k^{(1)}, \zeta^{(1)}; l^{(1)}_2, t^{(1)}_1 + 1] U[k^{(1)}, \zeta^{(1)}; l, l^{(1)}_0] \left( \Lambda^{(1)} \Pi^{(f(1))} \right) \Phi[k^{(1)}, \zeta^{(1)}; l^{(1)}_1, l + 1] \\
& \pm \left( \Lambda^{(1)'} \Pi^{(f'(1)')} \right) FU[k^{(1)}, \zeta^{(1)}; l^{(1)}_2, t^{(1)}_1 + 1] U[k^{(1)}, \zeta^{(1)}; l + 1, l^{(1)}_1] U[k^{(1)}, \zeta^{(1)}; l, l^{(1)}_0] \left( \Lambda^{(1)} \Pi^{(f(1))} \right) \right\} \Phi[(1)] \cdots \Phi[(N)] | \Psi[(2)] \cdots \Psi[(L)] \rangle \\
& + \langle \left( -\frac{i}{2} \sum_{l=t^{(1)}_1+1} K^{(1)\bar{m}}_l \right) \\
& \left\{ \left( \Lambda^{(1)'} \Pi^{(f'(1)')} \right) FU[k^{(1)}, \zeta^{(1)}; l^{(1)}_2, t^{(1)}_1 + 1] U[k^{(1)}, \zeta^{(1)}; l^{(1)}_1, l^{(1)}_0] \left( \Lambda^{(1)} \Pi^{(f(1))} \right) \Phi[k^{(1)}, \zeta^{(1)}; l^{(1)}_1, l + 1] \\
& \pm \left( \Lambda^{(1)'} \Pi^{(f'(1)')} \right) FU[k^{(1)}, \zeta^{(1)}; l^{(1)}_2, t^{(1)}_1 + 1] U[k^{(1)}, \zeta^{(1)}; l^{(1)}_1, l^{(1)}_0] \left( \Lambda^{(1)} \Pi^{(f(1))} \right) \right\} \Phi[(1)] \cdots \Phi[(N)] | \Psi[(2)] \cdots \Psi[(L)] \rangle, \quad (3.13) \\
(3) & = \langle \left( -\frac{i}{2} \sum_{b=1}^{N} \sum_{n=r^{(b)}_0} P^{(b)}_n \right) \\
& \left\{ \left( \Lambda^{(1)'} \Pi^{(f'(1)')} \right) FU[k^{(1)}, \zeta^{(1)}; l^{(1)}_2, t^{(1)}_1 + 1] U[p^{(b)}, n^{(b)}; n, n + 1] U[k^{(1)}, \zeta^{(1)}; l^{(1)}_1, l^{(1)}_0] \left( \Lambda^{(1)} \Pi^{(f(1))} \right) \right\} \Phi[(1)] \cdots \hat{b} \cdots \Phi[(N)] | \Psi[(2)] \cdots \Psi[(L)] \rangle, \quad (3.14) \\
(4) & = \langle \left( -\frac{i}{2} \sum_{b=2}^{l^{(b)}_1} \sum_{l=t^{(b)}_0}^{l^{(b)}_1} K^{(b)}_l \right) \\
& \left\{ \left( \Lambda^{(1)'} \Pi^{(f'(1)')} \right) FU[k^{(1)}, \zeta^{(1)}; l^{(1)}_2, t^{(1)}_1 + 1] U[k^{(1)}, \zeta^{(1)}; l, l^{(1)}_0] \left( \Lambda^{(1)} \Pi^{(f(1))} \right) \right\} \Phi[k^{(1)}, \zeta^{(1)}; l^{(1)}_0] U[l^{(b)}_2, l^{(b)}_1, l + 1] U[k^{(1)}, \zeta^{(1)}; l^{(1)}_1, l^{(1)}_0] \left( \Lambda^{(1)} \Pi^{(f(1))} \right) \\
& \pm \left( \Lambda^{(1)'} \Pi^{(f'(1)')} \right) FU[k^{(1)}, \zeta^{(1)}; l^{(1)}_2, t^{(1)}_1 + 1] U[k^{(1)}, \zeta^{(1)}; l^{(1)}_1, l^{(1)}_0] \left( \Lambda^{(1)} \Pi^{(f(1))} \right) \\
& \times \left( \Lambda^{(b)} \Pi^{(f(1))} \right) FU[k^{(1)}, \zeta^{(1)}; l^{(1)}_2, t^{(1)}_1 + 1] U[k^{(1)}, \zeta^{(1)}; l^{(1)}_1, l^{(1)}_0] \left( \Lambda^{(1)} \Pi^{(f(1))} \right) \right\} \Phi[(1)] \cdots \Phi[(N)] | \Psi[(2)] \cdots \hat{b} \cdots \Psi[(L)] \rangle. \quad (3.15)
\end{align}
The form of $\delta_X \Psi[(1); X^r]$ is similar in spirit to that of $\delta_X \Phi[(1); X^r]$. Space permits us to write this explicitly only for the case $X^r = v^r_m$ in Appendix B.

\( \bullet (3.3) \Rightarrow 0 = (1) \) kinetic term (Fig. [10]) + (2) open-closed transition (Fig. [11])

\[ + \quad (3) \text{ joining with an open string (Fig. [12]).} \]

Here

\[ (1) = \frac{1}{g^2} \left\langle \left( \delta Z \Psi[(1); Z] \right) \Phi[(1)] \cdots \Phi[(N)] \Psi[(2)] \cdots \Psi[(L)] \right\rangle, \quad (3.17) \]

\[ (2) = \delta_{f^{(1)}} \Lambda^{(1)} \left\langle \Phi[k^{(1)}_1, \zeta^{(1)}_1; l^{(1)}_0, l^{(1)}_1] \Phi[(1)] \cdots \Phi[(N)] \Psi[(2)] \cdots \Psi[(L)] \right\rangle, \quad (3.18) \]

\[ (3) = \left\langle \sum_{b=2}^{L} \left\{ \delta_{f^{(b')}} \Lambda^{(b')} \times \left( \Lambda^{(1)} \Pi_{(f^{(1)})} \right) FU[k^{(b)}_{1}, \zeta^{(b)}_{1}; l^{(b)}_0, l^{(b)}_1] U[k^{(b)}; \zeta^{(b)}; l^{(b)}; l^{(b)}_0] \left( \Lambda^{(1)} \Pi_{(f^{(1)})} \right) \right\} \right. \]

\[ + \quad \delta_{f^{(b)}} \Lambda^{(b)} \left( \Lambda^{(1)} \Pi_{(f^{(1)})} \right) FU[k^{(b)}_1, \zeta^{(b)}_1; l^{(b)}_0, l^{(b)}_1] U[k^{(b)}; \zeta^{(b)}; l^{(b)}; l^{(b)}_0] \left( \Lambda^{(1)} \Pi_{(f^{(1)})} \right) \}

\[ \Phi[(1)] \cdots \Phi[(N)] \Psi[(2)] \cdots \hat{b} \cdots \Psi[(L)] \right\rangle. \quad (3.19) \]

The form of $\left( \delta Z \Psi[(1); Z] \right)$ is in Appendix B.
Figure 9: joining of two open strings: case one

Figure 10: infinitesimal deformation of an open string: case two

Figure 11: open-closed transition

Figure 12: joining of two open strings: case two
We have checked that all of the terms in (3.10) \(\sim\) (3.19) are expressed by the closed and open loops \(\Phi\) and \(\Psi\) and their derivatives with respect to the sources introduced. For example, the expression 

\[ \psi Q \ast \bar{\sigma}^m U[p^{(1)}, \eta^{(1)}; n_1^{(1)}, n_1^{(1)} + 1] \psi Q \]

in eq. (A.30) for \( (\delta_X \Phi[(1); X']) \) in eq. (3.7) is represented as

\[ \sum_{f=1}^{2n_f} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \bar{\theta}^\dot{\alpha}} \Psi_{ff}[p^{(1)}, \eta^{(1)}; n_1^{(1)}, n_1^{(1)} + 1; \Lambda', \Lambda] \] .

In this sense, the set of loop equations we have derived is closed. It is noteworthy that all of the terms in the above loop equations are either an infinitesimal deformation of a loop or a consequence from the two elementary local processes of loops which are illustrated in Fig. 13.

It is interesting to discuss the system of loop equations we have derived in the light of string field theory. In addition to the lightcone superstring field theory constructed earlier in [1], there is now gauge invariant string field theory for closed-open bosonic system [10]. We find that the types of the interaction terms of our equations are in complete agreement with the interaction vertices seen in [9] and the second paper of [10]. In particular, Figures 2 \(\sim\) 5, 7 \(\sim\) 9, 11 \(\sim\) 12 for the interactions of our equations are in accordance with \(U, V_\infty, V_3^e, U_\Omega, V_3^0, V_\alpha, V_4^0\) of [10]. While BRS invariance determines the coefficients of the interaction vertices in [10], the (bare) coefficients are already determined in our case from the first quantized action. This may give us insight into properties of the model which are not revealed.
IV. Linearized loop equations and a free string

Let us consider the all three loop equations eqs. \( (3.1) \), \( (3.2) \) and \( (3.3) \) in the linearized approximation, namely, ignoring the joining and splitting of the loops. Let us first introduce a variable conjugate to \( p_{An} \) or \( k_{An} \) and that to \( \eta_n \) or \( \zeta_n \) by

\[
\hat{X}^A_n = i \frac{\delta}{\delta p_{An}} \text{ or } i \frac{\delta}{\delta k_{An}} \quad (4.1)
\]

\[
\hat{\Psi}_n = i \frac{\delta}{\delta \eta_n} \text{ or } i \frac{\delta}{\delta \zeta_n} \quad (4.2)
\]

By acting \( \hat{X}^A_n \) and \( \hat{\Psi}_n \) on a loop, we obtain respectively an operator insertion of \( v^A \) and that of \( \Psi \) at point \( n \) on the loop.

Now consider eq. \( (3.1) \) and eq. \( (3.2) \) for the case \( X^r = v^r_M \), multiplying them by \( p^{(1)}_{nM} \) and \( k^{(1)}_{nM} \) respectively. Consistency requires that, for these terms, we must take into account the term from the interactions which represents splitting of a loop with infinitesimal length. This in fact occurs when the splitting point \( n \) coincides with the point \( n^{(1)}_1 \) at which \( T^r \) is inserted. We obtain

\[
0 = \frac{1}{g^2 p^{(1)}_{nM}} \langle (\delta_X \Phi[(1); v^r_M]) \Phi[(2)] \cdots \Phi[(N)] \Psi[(1)] \cdots \Psi[(L)] \rangle - \frac{i}{2} 2k^{(1)}_{nM} \langle \Phi[(1)] \Phi[(2)] \cdots \Phi[(N)] \Psi[(1)] \cdots \Psi[(L)] \rangle \quad (4.3)
\]

\[
0 = \frac{1}{g^2 k^{(1)}_{nM}} \langle (\delta_X \Psi[(1); v^r_M]) \Phi[(1)] \cdots \Phi[(N)] \Psi[(1)] \cdots \Psi[(L)] \rangle - \frac{i}{2} 2k^{(1)}_{nM} \langle \Phi[(1)] \Phi[(2)] \cdots \Phi[(N)] \Psi[(1)] \cdots \Psi[(L)] \rangle \quad (4.4)
\]

These equations lead to the half of the Virasoro conditions \([4]\):

\[
0 = (p^{(1)}_{n} + (\hat{X}^A_n)^{\prime 2} + \text{fermionic part}) \langle \Phi[(1)] \Phi[(2)] \cdots \Phi[(N)] \Psi[(1)] \cdots \Psi[(L)] \rangle \quad (4.5)
\]

\[
0 = (k^{(1)}_{n} + (\hat{X}^A_n)^{\prime 2} + \text{fermionic part}) \langle \Phi[(1)] \Phi[(2)] \cdots \Phi[(N)] \Psi[(1)] \cdots \Psi[(L)] \rangle \quad (4.6)
\]

where \( t \) implies taking a difference between two adjacent points \( n \) and \( n+1 \). The reparametrization invariance of the Wilson loops leads to the remaining half of the Virasoro conditions:

\[
0 = (p^{(1)M}_{\hat{n}} \, \hat{X}^{(1)'}_{\hat{n}M} + \text{fermionic part}) \langle \Phi[(1)] \Phi[(2)] \cdots \Phi[(N)] \Psi[(1)] \cdots \Psi[(L)] \rangle \quad (4.7)
\]
\begin{equation}
0 = (k_n^{(1)M} X_n^M + \text{(fermionic part)})
\langle \Phi[(1)]\Phi[(2)] \cdots \Phi[(N)]\Psi[(1)] \cdots \Psi[(L)] \rangle .
\end{equation}

Next, let us consider eq. (3.3), ignoring joining and splitting of the loops. Again consistency appears to require that we drop the cubic terms consisting of \(Q\) and \(Q^*\) in \(\delta Z\Psi[(1); Z]\). To write explicitly, the following expression must vanish
\begin{equation}
\begin{align*}
\{Q(f)(v_\nu v^\nu + [\Phi, \Phi]) + (Q \Sigma)(f) F[\Phi_2^I, \Phi_3^I] + (Q^* M^2)(f) + 2(Q^* M)(f) v_4 - i\sqrt{2} \psi_{Q(f)}^* \bar{\lambda} \\
- \sqrt{2}(\psi_Q \Sigma)(f) F\psi_{\Phi_1} \} U[k_1^{(1)}, \zeta_1^{(1)}; l_1^{(1)}, l_0^{(1)}](\Lambda^{(1)} \Pi_{f(1)}) \approx 0 ,
\end{align*}
\end{equation}
when inserted in
\begin{equation}
\langle \Phi[(1)]\Phi[(2)] \cdots \Phi[(N)]\Psi[(2)] \cdots \Psi[(L)] \rangle .
\end{equation}

As we stated before, the lefthand sides of eqs. (4.9) and (4.10) are expressible as an open loop with some functions of \(\hat{X}_{K_1}^A\) and \(\hat{\Phi}_{l(1)}\) acting on the loop. Let us see by inspection how eqs. (4.9) and (4.10) are satisfied by the source functions alone. Consider the following configuration of \(\hat{X}_n\) and \(\hat{\Phi}_n\), \(n = l_1^{(1)}:
\begin{align*}
\hat{X}_\mu &\approx 0 \text{ for } \mu = 0, 1, 2, 3, 7 , \quad \hat{X}_4 = \pm m_f \quad ,
\hat{X}_I^I &\approx 0 \text{ for } I = 5, 6, 8, 9 \\
\hat{\Gamma}_3 \hat{\Phi} &\approx -\hat{\psi}_{\Phi_1} \quad .
\end{align*}
\end{equation}
where \(\hat{\Gamma}_3 \equiv \Gamma_5 \Gamma_6 \Gamma_8 \Gamma_9\). Again these equations should be understood in the sense of an insertion at the end point of the open loop.

Eqs. (4.12), (4.13), and (4.14) tell us that the open loop \(\Psi[(1)]\) obeys the Dirichlet boundary conditions for 0, 1, 2, 3, 4, 7 directions and the Neumann boundary conditions for 5, 6, 8, 9 directions. Eq. (4.14) is seen in [13]. Note that eq. (4.14) is equivalent to
\begin{equation}
\hat{\lambda} \approx \hat{\bar{\lambda}} \approx \hat{\psi}_{\Phi_1} \approx \hat{\bar{\psi}}_{\Phi_1} \approx 0 .
\end{equation}
Also note that
\begin{equation}
[\Phi_i, \Phi_j] \approx \hat{X}_i^I \hat{X}_j^J \text{ or } \hat{X}_j^J \hat{X}_i^I .
\end{equation}

We find that the configuration given by eqs. (4.12), (4.13) and (4.14) solves the linearized loop equations (4.9), (4.10). This configuration clearly tells us the existence of \(n_f D3\) branes.
and their mirrors each of which is at a distance $\pm m_f$ away from the orientifold surface in the fourth direction. There has been positive evidence in favour of this both from the connection of the T dualized (the 4D worldvolume gauge theory) representation of our model [3, 4] with Sen’s scaling limit [14] for F theory [15] and from the configuration emerging from the fermionic integration [6, 7]. (See also [16].) The result in this section consolidates our picture.

V. Discussion

We have been able to formulate the USp matrix model in the second quantized form in which the manybody effects of the model as string theory are manifest. From our discussion, it is clear that the closed and open Wilson loop variables serve as string fields. It is satisfying to see that the linearized equations translate into the classical Virasoro condition of the closed loops/string fields and the boundary conditions of the open loops/string fields. It is encouraging to us for a further pursuit of the model that the simple completeness relation of the $usp$ Lie algebra is able to capture the complete set of the joining and splitting interactions required.

While our paper supplies several satisfactory features of the model as unified theory of all forces including gravity and matter, it provides us with a host of open questions many of which are shared by the type IIB case. Let us discuss some of them. The theory is still formulated in bare variables and the proper scaling limit is yet to be determined. This limit in the USp case is closely related to the problem of the field/loop redefinition and therefore relative strengths of the string interactions among the closed and open string fields and that of the typeI-heterotic duality [17] of the USp matrix model. A related but different question is how, given a model, we find perturbative vacuum on which string perturbation theory is based. This is a nontrivial problem in reduced models as perturbative vacuum is neither the true vacuum realized by the scaling limit nor simple theory of loops as bare variables which ignores the joining and splitting of the loops. As we discussed at the beginning, the connection of the reduced model action in the large $k$ limit with the first quantized string action in the Schild gauge ensures that string perturbation theory is somewhere in the model. The nonrenormalizability of the worldsheet action and the absence of free field technique, however, prevent us from the direct study.

Turning to physical consequences, the reduced matrix model, in particular, their second quantized formulation provides an opportunity to answer questions which are difficult to address in the conventional first quantized string theory. These are, for example, the size and the shape of spacetime which the model predicts and the issue of spontaneous breaking
of gauge symmetry. These can be studied within the model by numerical as well as analytical method.

VI. Acknowledgements

We thank Satoshi Iso for a helpful discussion on this subject.
Appendix

A. the action of the USp matrix model

The action of the $USp(2k)$ reduced matrix model can be obtained from the dimensional reduction of $\mathcal{N} = 2, d = 4$ $USp(2k)$ supersymmetric gauge theory with one hypermultiplet in the antisymmetric representation and $n_f$ hypermultiplets in the fundamental representation. This makes manifest the presence of the eight dynamical supercharges. In the $\mathcal{N} = 1$ superfield notation with spacetime dependence all dropped, we have a vector superfield $V$ and a chiral superfield $\Phi \equiv \Phi_1$ which are $usp$ Lie algebra valued

$$V^t = -FF^{-1} , \quad \Phi^t = -F\Phi F^{-1} , \quad V^\dagger = V \ , \quad \text{with} \quad F = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} , \quad (A.1)$$

and the two chiral superfields $\Phi_I$ for $I = 2, 3$ in the antisymmetric representation which obey

$$\Phi_I^t = F\Phi_I F^{-1} \quad \text{for} \quad I = 2, 3 \ . \quad (A.2)$$

We will suppress the $USp$ indices in the rest of our discussion.

It is often expedient to introduce the projector acting on $U(2k)$ matrices:

$$\hat{\rho}_+ \bullet \equiv \frac{1}{2} \left( \bullet \mp F^{-1} \bullet^t F \right) . \quad (A.3)$$

The action of $\hat{\rho}_-$ and that of $\hat{\rho}_+$ take any $U(2k)$ matrix into the matrix lying in the adjoint representation of $USp(2k)$ and that in the antisymmetric representation respectively. We can therefore write $V = \hat{\rho}_- V$, $\Phi_1 = \hat{\rho}_- \Phi_1$, $\Phi_I = \hat{\rho}_+ \Phi_I$ for $I = 2, 3$, where the symbols with underlines lie in the adjoint representation of $U(2k)$. The total action is written as

$$S = \frac{1}{4g^2} Tr \left( \int d^2 \theta W^{\alpha} W_{\alpha} + h.c. \right) + 4 \int d^2 \theta d^2 \bar{\theta} \Phi_I^\dagger e^{2V} \Phi_I e^{-2V} \right) + \frac{1}{g^2} \left( \int d^2 \theta W(\theta) + h.c. \right) , \quad (A.4)$$

where the superpotential

$$W(\theta) = \sqrt{2} Tr \left( \Phi_1 [\Phi_2, \Phi_3] \right) + \sum_{f=1}^{n_f} \left( m_{(f)} \bar{Q}_{(f)} Q_{(f)} + \sqrt{2} \bar{Q}_{(f)} \Phi_1 Q_{(f)} \right) . \quad (A.5)$$

To render the action to its component form, let us first list some formulas. $W_{\alpha} = -\frac{1}{8} D \bar{D} e^{-2V} D_{\alpha} e^{2V}$, $\Phi_I = \Phi_I + \sqrt{2} \theta \psi_{\Phi_I} + \theta \Phi_{\Phi_I}$, $Q = Q + \sqrt{2} \theta \psi_{\Phi_I} + \theta \Phi_{\Phi_I}$, $V = -\theta \sigma^m \bar{\theta} v_m +$
\[ i \partial \theta \bar{\lambda} - i \partial \bar{\theta} \lambda + \frac{1}{2} \partial \bar{\theta} \partial \lambda \bar{D}, \quad D_\alpha = \frac{\partial}{\partial \theta^\alpha}, \quad \bar{D}_\dot{\alpha} = -\frac{\partial}{\partial \bar{\theta}^\dot{\alpha}}. \] Solving the equation for the \( D \) term, we obtain
\[ D = [\Phi^+_I, \Phi_I] - \rho - \sum_{j=1}^{n_f} (Q_{(j)} Q^*_f - \bar{Q}^*_f \bar{Q}_{(j)}) , \] (A.6)
where we have placed the \( USp \) vectors \( Q_{(j)}, \bar{Q}_{(j)} \) and their complex conjugates in the form of dyad. The \( F \) terms are such that
\[ -\delta W = \sum_{I=1,2,3} tr F_{\Phi_I}^+ \delta \Phi_I + F_{Q_{(j)}}^* \delta Q_{(j)} + F_{\bar{Q}_{(j)}}^* \delta \bar{Q}_{(j)} . \] (A.7)
Explicitly
\[ F_{\Phi_1}^+ = -\sqrt{2} [\Phi_2, \Phi_3] - \sqrt{2} \rho - \left( \sum_{j=1}^{n_f} Q_{(j)} \bar{Q}_{(j)} \right), \quad F_{\Phi_2}^+ = -\sqrt{2} [\Phi_3, \Phi_1], \quad F_{\Phi_3}^+ = -\sqrt{2} [\Phi_1, \Phi_2], \]
\[ F_{Q_{(j)}}^* = - \left( m_{(j)} \bar{Q}_{(j)} + \sqrt{2} Q_{(j)} \Phi_1 \right), \quad F_{\bar{Q}_{(j)}}^* = - \left( m_{(j)} Q_{(j)} + \sqrt{2} \Phi_1 \bar{Q}_{(j)} \right) . \] (A.8)
As for the Yukawa couplings, they can be read off from
\[ \delta^2 W \equiv \sum_{A,B} \frac{\partial^2 W}{\partial A \partial B} \delta A \delta B , \] (A.9)
where the summation indices \( A, B \) are over all chiral superfields \( \Phi_I, I = 1, 2, 3 \), and \( Q_{(j)}, \bar{Q}_{(j)}, f = 1, \cdots, n_f \). The component expression for the total action is
\[ S = \frac{1}{g^2} Tr \left\{ -\frac{1}{4} v_m v^m - [D_m, \Phi_I]^+ [D^m, \Phi_I] - i \lambda \sigma^m [D_m, \lambda] - i \bar{\psi}_{\Phi_I} \sigma^m [D_m, \psi_{\Phi_I}] - i \sqrt{2} [\lambda, \psi_{\Phi_I}] \Phi_I^+ \right. \]
\[ \left. - i \sqrt{2} [\lambda, \bar{\psi}_{\Phi_I}] \right\} \]
\[ + \frac{1}{g^2} \sum_{f=1}^{n_f} \left\{ - (D_m Q_{(f)})^* (D^m Q_{(f)}) - \bar{\psi}_{Q_{(f)}} \sigma^m D_m \psi_{Q_{(f)}} + i \sqrt{2} \bar{Q}_{(f)} \lambda \psi_{Q_{(f)}} - i \sqrt{2} \psi_{Q_{(f)}} \lambda \bar{Q}_{(f)} \right\} \]
\[ + \frac{1}{g^2} \sum_{f=1}^{n_f} \left\{ - (D_m \bar{Q}_{(f)}) (D^m \bar{Q}_{(f)})^* - \bar{\psi}_{\bar{Q}_{(f)}} \sigma^m D_m \psi_{\bar{Q}_{(f)}} - i \sqrt{2} \bar{\psi}_{\bar{Q}_{(f)}} \lambda \bar{Q}_{(f)} + i \sqrt{2} \psi_{\bar{Q}_{(f)}} \lambda \bar{Q}_{(f)} \right\} \]
\[ - \frac{1}{g^2} Tr \left( \frac{1}{2} DD + F_{\Phi_I}^+ F_{\Phi_I} \right) - \frac{1}{g^2} \left( \sum_{f=1}^{n_f} F_{Q_{(f)}} F_{Q_{(f)}}^* + \sum_{f=1}^{n_f} F_{\bar{Q}_{(f)}} F_{\bar{Q}_{(f)}}^* \right) \]
\[ - \frac{1}{g^2} \left( \sum_{A,B} \frac{\partial^2 W}{\partial A \partial B} \psi_A \psi_B + h.c. \right) . \] (A.10)
Here \( D_m = iv_m \) in the fundamental representation.
Let us denote by $S_0$ the part in $S$ which does not contain the fundamental hypermultiplet. We split the total action into $$S = S_0 + \Delta S.$$ The part $S_0$ is expressible in terms of the type $IIB$ matrix model. This is stated as

$$S_0(v_m, \Phi_I, \lambda, \psi_{\Phi_I}, \bar{\lambda}, \bar{\psi}_{\Phi_I}) = S_{IIB}(\hat{\rho}_{b\mp} v_M, \hat{\rho}_{f\mp} \Psi).$$  \hfill (A.11)

Here

$$S_{IIB}(v_M, \Psi) = \frac{1}{g^2} Tr \left( \frac{1}{4} [v_M, v_N] [v^M, v^N] - \frac{1}{2} \Psi \Gamma^M [v_M, \Psi] \right),$$ \hfill (A.12)

and

$$\Phi_i = \frac{1}{\sqrt{2}} (v_{3+i} + iv_{6+i}),$$

$$\Psi = (\lambda, 0, \psi_{\Phi_4}, 0, \psi_{\Phi_5}, 0, 0, \bar{\lambda}, 0, \bar{\psi}_{\Phi_1}, 0, \bar{\psi}_{\Phi_2}, 0, \bar{\psi}_{\Phi_3})^t.$$ \hfill (A.13)

This latter one $\Psi$ is a thirty two component Majorana-Weyl spinor satisfying

$$C \bar{\Psi}^t = \Psi, \quad \Gamma_{11} \Psi = \Psi.$$ \hfill (A.14)

With regard to eqs. (A.13) and (A.14), the same is true for objects with underlines. The ten dimensional gamma matrices have been denoted by $\Gamma^M$. The projector $\hat{\rho}_{b\mp}$ is a diagonal matrix with respect to Lorentz indices while $\hat{\rho}_{f\mp}$ is to spinor indices:

$$\hat{\rho}_{b\mp} = \text{diag}(\hat{\rho}_-, \hat{\rho}_-, \hat{\rho}_-, \hat{\rho}_-, \hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_+, \hat{\rho}_+),$$

$$\hat{\rho}_{f\mp} = \hat{\rho}_{-1(4)} \otimes \begin{pmatrix} 1_{(2)} & 0 \\ 0 & 1_{(2)} \end{pmatrix} + \hat{\rho}_{+1(4)} \otimes \begin{pmatrix} 0 & 1_{(2)} \\ 1_{(2)} & 0 \end{pmatrix}. \hfill (A.15)$$

The proof of the equivalence (A.11) is sketched here for this appendix to be self-contained. (See also [15].) The only nontrivial term in the bosonic part of this equivalence is

$$\frac{1}{2} Tr D^{(0)} D^{(0)} + Tr F^{(0)} F^{(0)} = -\frac{1}{4} Tr [v_r, v_{r'}] [v^r, v^{r'}],$$ \hfill (A.16)

where $r, r'$ run over $4 \sim 9$ and the superscript $(0)$ implies omission of the parts containing the fundamental scalars in eq. (A.8). The left hand side is written as

$$tr \left\{ \frac{1}{2} \left( [\Phi_1, \Phi_1]^2 + [\Phi_2, \Phi_2]^2 + [\Phi_3, \Phi_3]^2 \right) + \sum_{(K,J)} \left( [\Phi_K, \Phi_K] [\Phi_J, \Phi_J] - 2 [\Phi_K, \Phi_J] [\Phi_K^+ \Phi_J^+] \right) \right\},$$ \hfill (A.17)
where the sum \((K, J)\) runs over the pairs \((1, 2), (2, 3), (3, 1)\). Using the Jacobi identity, we convert the first term of the summand into another expression. The summand becomes

\[
- [\Phi_K, \Phi_J^\dagger] [\Phi_K^\dagger, \Phi_J] - [\Phi_K, \Phi_J] [\Phi_K^\dagger, \Phi_J^\dagger].
\]

Substituting eq. (A.13) into eq. (A.17), we confirm eq. (A.16). As for the fermion bilinear, we check the equivalence eq. (A.11) by finding an explicit representation of the ten \(\Gamma^M\) matrices in the bases (A.13). They are

\[
\Gamma^m = \gamma^m \otimes I_8 \quad \text{for} \quad m = 0, 1, 2, 3,
\]

\[
\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix},
\]

\[
\Gamma^I = \gamma_5 \otimes \hat{\Gamma}^I \quad \text{for} \quad I = 4 \sim 9,
\]

\[
\gamma_5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},
\]

where

\[
\hat{\Gamma}^4 = \begin{pmatrix}
0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\
0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{\Gamma}^5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{\Gamma}^6 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{\Gamma}^7 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{\Gamma}^8 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{\Gamma}^9 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
We have checked that they in fact form the Clifford algebra.

Let us turn to the remaining part $\Delta S$ of the action. We are interested in presenting this part in a way $SO(2n_f)$ flavour symmetry is easily seen. To establish this, we introduce complex $2n_f$ dimensional vectors

$$Q \equiv \begin{cases} Q_{(f)} , & f = 1 \sim n_f \\ F^{-1}Q_{(f-n_f)} , & f = n_f + 1 \sim 2n_f \end{cases}, \quad Q^* \equiv \begin{cases} Q^*_{(f)} , & f = 1 \sim n_f \\ Q^*_{(f-n_f)}F , & f = n_f + 1 \sim 2n_f \end{cases}. \quad (A.19)$$

Similarly,

$$\psi_Q \equiv \begin{cases} \psi_{Q(f)} , & f = 1 \sim n_f \\ F^{-1}\psi_{Q(f-n_f)} , & f = n_f + 1 \sim 2n_f \end{cases}, \quad \psi_Q^* \equiv \begin{cases} \psi^*_{Q(f)} , & f = 1 \sim n_f \\ -\psi^*_{Q(f-n_f)}F , & f = n_f + 1 \sim 2n_f \end{cases}. \quad (A.20)$$

We denote the $f$-th components of these vectors by $Q_{(f)}$ etc. in the text. After some algebras, we find

$$\Delta S = \Delta S_b + \Delta S_f = (S_{g-s} + V_{\text{scalar}} + S_{\text{mass}}) + (S_{g-f} + S_{\text{Yukawa}}) , \quad (A.21)$$

$$S_{g-s} = -\frac{1}{g^2} tr \left( \sum_{\nu=0,1,2,3,4,7} v_\nu v^\nu + \sum_{i=2,3} [\Phi_i, \Phi_i^\dagger] \right) Q \cdot Q^*$$

$$+ \frac{1}{g^2} tr [\Phi_2, \Phi_3] F^{-1}Q^* \cdot \Sigma Q^* - \frac{1}{g^2} tr [\Phi_2, \Phi_3^\dagger] Q \cdot \Sigma F Q , \quad (A.22)$$

$$S_{\text{mass}} = \frac{1}{g^2} tr \left( Q \cdot M^2 Q^* \right) - \frac{2}{g^2} tr (v_4 Q \cdot MQ^*) , \quad (A.23)$$

$$V_{\text{scalar}} = \frac{1}{2g^2} tr Q \cdot \Sigma Q Q^* \cdot \Sigma Q^* - \frac{1}{8g^2} tr \left[ Q \cdot Q^* - F^{-1}Q^* \cdot QF \right]^2 , \quad (A.24)$$

$$S_{g-f} = \frac{1}{g^2} \left\{ \psi_Q^* \bar{\sigma}^m v_m \cdot \psi_Q + i\sqrt{2}Q^* \lambda \cdot \psi_Q - i\sqrt{2}\psi_Q^* \lambda \cdot Q \right\} , \quad (A.25)$$

$$S_{\text{Yukawa}} = -\frac{1}{g^2} \left\{ \sum_{(c_1,c_2)=(Q,Q),(Q,\Phi),(\Phi,\bar{Q})} \frac{\partial^2 W_{\text{matter}}}{\partial C_1 \partial C_2} \psi_{C_2} \psi_{C_1} + h.c. \right\}$$

$$= \frac{1}{g^2} \left( \frac{1}{2} \psi_Q \cdot \Sigma F \left( \sqrt{2} \Phi_1 + M \right) \psi_Q + \sqrt{2}Q \cdot \Sigma F \psi_Q + h.c. \right) . \quad (A.26)$$
Here
\[
\Sigma \equiv \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (A.27)
\]
\[
M \equiv \text{diag} \left( m_1, \cdots, m_{n_f} - m_1, \cdots, -m_{n_f} \right), \quad (A.28)
\]
\[
W_{\text{matter}} = \sum_{f=1}^{n_f} \left( m_f \tilde{Q}_f Q_f + \sqrt{2} \tilde{Q}_f \Phi(f) \right), \quad (A.29)
\]
and \cdot implies the standard inner product with respect to the 2n_f flavour indices.

**B. The kinetic terms of Schwinger-Dyson equations**

- \( \delta_X \Phi[(1); X'] \)
  \[
  X' = v_4^r:
  \]
  \[
  Tr(\{[v_4, v^M] - \frac{1}{2} \{ \bar{\Psi} \Gamma^m, \Psi \} U[p, \eta; n_1, n_1 + 1]) \]
  \[
  -\frac{1}{2} Q U[p, \eta; n_1, n_1 + 1] v^m \cdot Q + \frac{1}{2} Q^* U[\mp p, \mp \eta; n_1, n_1 + 1] v^m \cdot Q
  \]
  \[
  -\frac{1}{2} Q^* v^m U[p, \eta; n_1, n_1 + 1] \cdot Q + \frac{1}{2} Q^* v^m U[\mp p, \mp \eta; n_1, n_1 + 1] \cdot Q
  \]
  \[
  +\frac{1}{2} \psi Q^* \cdot \bar{\sigma}^m U[p, \eta; n_1, n_1 + 1] \cdot \psi Q - \frac{1}{2} \psi Q^* \cdot \bar{\sigma}^m U[\mp p, \mp \eta; n_1, n_1 + 1] \cdot \psi Q,
  \]
  \[
  X' = v_4^r:
  \]
  \[
  Tr(\{[v_4, v^M] - \frac{1}{2} \{ \bar{\Psi} \Gamma^4, \Psi \} U[p, \eta; n_1, n_1 + 1]) \]
  \[
  -\frac{1}{2} Q U[p, \eta; n_1, n_1 + 1] v^m \cdot Q + \frac{1}{2} Q^* U[\mp p, \mp \eta; n_1, n_1 + 1] v^m \cdot Q
  \]
  \[
  -\frac{1}{2} Q^* v^m U[p, \eta; n_1, n_1 + 1] \cdot Q + \frac{1}{2} Q^* v^m U[\mp p, \mp \eta; n_1, n_1 + 1] \cdot Q
  \]
  \[
  -Q U[p, \eta; n_1, n_1 + 1] \cdot MQ + \frac{1}{2} Q^* U[\mp p, \mp \eta; n_1, n_1 + 1] \cdot MQ
  \]
  \[
  +\frac{1}{4} \psi Q F U[p, \eta; n_1, n_1 + 1] \cdot \Sigma Q - \frac{1}{4} \psi Q F U[\mp p, \mp \eta; n_1, n_1 + 1, n_1] \cdot \Sigma F Q
  \]
  \[
  +\frac{1}{4} \psi Q^* U[p, \eta; n_1, n_1 + 1] F^{-1} \cdot \Sigma Q^* - \frac{1}{4} \psi Q^* U[\mp p, \mp \eta; n_1, n_1 + 1, n_1] F^{-1} \cdot \Sigma Q^*,
  \]
\( X^r = v_1^r : \)

\[
Tr(((v_M, [v^7, v^M]) - \frac{1}{2}\{\bar{\psi}\Gamma^7, \Psi\})U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1]) \tag{A.32}
\]

\[
-\frac{1}{2} Q^*U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1]v^m \cdot Q + \frac{1}{2} Q^*U[\mp p^{(1)},\mp \eta; n^{(1)},n^{(1)} + 1]v^m \cdot Q
\]

\[
-\frac{1}{2} Q^*v^mU[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1] \cdot Q + \frac{1}{2} Q^*v^mU[\mp p^{(1)},\mp \eta; n^{(1)},n^{(1)} + 1] \cdot Q
\]

\[
+ \frac{i}{4} \psi_Q F U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1] \cdot \Sigma \psi_Q - \frac{i}{4} \psi_Q F U[\mp p^{(1)},\mp \eta^{(1)};n^{(1)} + 1,n^{(1)}] \cdot \Sigma F \psi_Q
\]

\[
- \frac{i}{4} \psi_Q^*U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1]F^{-1} \cdot \Sigma \psi_Q^* + \frac{i}{4} \psi_Q^*U[\mp p^{(1)},\mp \eta^{(1)};n^{(1)} + 1,n^{(1)}]F^{-1} \cdot \Sigma \psi_Q^*
\]

\( X^r = \Phi_2 : \)

\[
Tr(((v_M, [\Phi^+_2, v^M]) - \frac{1}{2}\{\bar{\psi}\Gamma^7, \Phi^+_2\})U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1]) \tag{A.33}
\]

\[
+ \frac{1}{2} Q^*U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1] \Phi_3 F^{-1} \cdot \Sigma Q^* + \frac{1}{2} Q^*U[\mp p^{(1)},\mp \eta; n^{(1)},n^{(1)} + 1] \Phi_3 F^{-1} \cdot \Sigma Q^*
\]

\[
- \frac{1}{2} Q^* \Phi_3 U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1]F^{-1} \cdot \Sigma Q^* - \frac{1}{2} Q^* \Phi_3 U[\mp p^{(1)},\mp \eta; n^{(1)} + 1,n^{(1)}]F^{-1} \cdot \Sigma Q^*
\]

\( X^r = \Phi_3 : \)

\[
Tr(((v_M, [\Phi^+_3, v^M]) - \frac{1}{2}\{\bar{\psi}\Gamma^7, \Phi^+_3\})U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1]) \tag{A.34}
\]

\[
- \frac{1}{2} Q^*U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1] \Phi_2 F^{-1} \cdot \Sigma Q^* - \frac{1}{2} Q^*U[\mp p^{(1)},\mp \eta; n^{(1)},n^{(1)} + 1] \Phi_2 F^{-1} \cdot \Sigma Q^*
\]

\[
+ \frac{1}{2} Q^* \Phi_2 U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1]F^{-1} \cdot \Sigma Q^* + \frac{1}{2} Q^* \Phi_2 U[\mp p^{(1)},\mp \eta; n^{(1)} + 1,n^{(1)}]F^{-1} \cdot \Sigma Q^*
\]

\( X^r = \lambda : \)

\[
Tr((\sigma^m[v_m, \lambda] - i\sqrt{2}[\psi_I, \Phi^+_I])U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1]) \tag{A.35}
\]

\[
+ \frac{i}{\sqrt{2}} Q^*U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1] \cdot \psi_Q - \frac{i}{\sqrt{2}} Q^*U[\mp p^{(1)},\mp \eta; n^{(1)},n^{(1)} + 1] \cdot \psi_Q
\]

\( X^r = \psi_1 : \)

\[
Tr((\sigma^m[v_m, \bar{\psi}^+] - i\sqrt{2}[\lambda, \Phi^+_I])U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1]) \tag{A.36}
\]

\[
+ \frac{1}{\sqrt{2}} Q FU[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1] \cdot \Sigma \psi_Q - \frac{1}{\sqrt{2}} Q FU[\mp p^{(1)},\mp \eta; n^{(1)},n^{(1)} + 1] \cdot \Sigma \psi_Q
\]

\( X^r = \psi_1 (I = 2, 3) : \)

\[
Tr((\sigma^m[v_m, \bar{\psi}^I] - i\sqrt{2}[\lambda, \Phi^+_I])U[p^{(1)},\eta^{(1)};n^{(1)},n^{(1)} + 1]) \tag{A.37}
\]

\[
\]
• $\delta_\lambda \Psi[(1); \Lambda^{(1)}]$ 

\[ X_r = v_m : \]

\[
\left( \Lambda^{(1)\prime} \Pi_{(f^{(1)})} \right) \] 

\[
U[k^{(1)}, \zeta^{(1)}; l^{(1)}, l^{(1)}] \left( \Lambda^{(1)\prime} \Pi_{(f^{(1)})} \right)
\]

\[ \frac{1}{2} \left( \Lambda^{(1)\prime} \Pi_{(f^{(1)})} \right) \] 

\[
-F^{-1} \Psi^* \cdot Q^* U[k^{(1)}, \zeta^{(1)}; l^{(1)}, l^{(1)}] \left( \Lambda^{(1)\prime} \Pi_{(f^{(1)})} \right)
\]

\[ \frac{1}{2} \left( \Lambda^{(1)\prime} \Pi_{(f^{(1)})} \right) \] 

\[
-F \cdot Q^* F U[k^{(1)}, \zeta^{(1)}; l^{(1)}, l^{(1)}] \left( \Lambda^{(1)\prime} \Pi_{(f^{(1)})} \right)
\]

\[ \frac{1}{2} \left( \Lambda^{(1)\prime} \Pi_{(f^{(1)})} \right) \] 

\[
-\psi \cdot \sigma^m \psi^* U[k^{(1)}, \zeta^{(1)}; l^{(1)}, l^{(1)}] \left( \Lambda^{(1)\prime} \Pi_{(f^{(1)})} \right)
\]

\[ \frac{1}{2} \left( \Lambda^{(1)\prime} \Pi_{(f^{(1)})} \right) \] 

\[
F^{-1} \psi \cdot \sigma^m \psi^* F U[k^{(1)}, \zeta^{(1)}; l^{(1)}, l^{(1)}] \left( \Lambda^{(1)\prime} \Pi_{(f^{(1)})} \right)
\].

• $\delta Z \Psi[(1); Z]$ 

\[ Z_{(f)i} = Q_{(f)i} : \]

\[
\left( Q_{(f)} \left( v_\nu v^\nu + [\Phi_f, \Phi_f] \right) + (Q \Sigma)_{(f)} F[\Phi_f, \Phi_f] \right) + \left( Q^* M^2 \right)_{(f)} + 2 (Q^* M)^{(f)} v_4 - i \sqrt{2} \psi Q^* \lambda
\]

\[ -\sqrt{2} \left( \psi Q \Sigma \right)_{(f)} F \psi \Phi_1 + (Q \Sigma)_{(f)} Q^* \cdot \Sigma Q^* + \frac{1}{2} \left( Q_{(f)}^* Q - Q_{(f)} \right) F^{-1} Q^* \cdot Q F \right)
\]

\[ U[k^{(1)}, \zeta^{(1)}; l^{(1)}, l^{(1)}] \left( \Lambda^{(1)} \Pi_{(f)} \right), \]

\[ Z_{(f)i} = \psi Q_{(f)i} \]

\[ \left( \psi Q^* \sigma^m v_m + i \sqrt{2} Q^* \lambda \right) \]

\[ + \left( \psi Q \Sigma F \left( \sqrt{2} \Phi_1 + M \right) \right)_{(f)} + \sqrt{2} (Q \Sigma F \psi \Phi_1)_{(f)} \]

\[ U[k^{(1)}, \zeta^{(1)}; l^{(1)}, l^{(1)}] \left( \Lambda^{(1)} \Pi_{(f)} \right). \]
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