Tripartite Bell inequality, random matrices and trilinear forms

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Abstract

In this seminar report, we present in detail the proof of a recent result due to J. Briët and T. Vidick, improving an estimate in a 2008 paper by D. Pérez-García, M. Wolf, C. Palazuelos, I. Villanueva, and M. Junge, estimating the growth of the deviation in the tripartite Bell inequality. The proof requires a delicate estimate of the norms of certain trilinear (or d-linear) forms on Hilbert space with coefficients in the second Gaussian Wiener chaos. Let $E_n^\vee$ (resp. $E_n^{\text{min}}$) denote $\ell_1^n \otimes \ell_1^n \otimes \ell_1^n$ equipped with the injective (resp. minimal) tensor norm. Here $\ell_1^n$ is equipped with its maximal operator space structure. The Briët-Vidick method yields that the identity map $I_n$ satisfies (for some $c > 0$)

$$\|I_n : E_n^\vee \to E_n^{\text{min}}\| \geq c n^{1/4} (\log n)^{-3/2}.$$ 

Let $S_n^2$ denote the (Hilbert) space of $n \times n$-matrices equipped with the Hilbert-Schmidt norm. While a lower bound closer to $n^{1/2}$ is still open, their method produces an interesting, asymptotically almost sharp, related estimate for the map $J_n : S_n^2 \otimes S_n^2 \otimes S_n^2 \to \ell_2^3 \otimes \ell_2^3 \otimes \ell_2^3$ taking $e_{i,j} \otimes e_{k,l} \otimes e_{m,n}$ to $e_{[i,k,m]} \otimes e_{[j,l,n]}$.

1. Tripartite Bell inequality

We will prove the following theorem due to J. Briët and T. Vidick, improving an estimate in Junge et al. The proof in [1] was kindly explained to me in detail by T. Vidick. The improvements below (improving the power of the logarithmic term) are routine refinements of the ideas in [1].

**Theorem 1.1.** Let $Y^{(N)}$ be $N \times N$ a Gaussian random matrix with Gaussian entries all i.i.d. of mean zero and $L_2$-norm equal to $N^{-1/2}$. Let $Y_j^{(N)}$ ($j = 1, 2, \cdots$) be a sequence of independent copies of $Y^{(N)}$. There is a constant $C$ such that with large probability we have for all scalars $a_{ij}$ with $1 \leq i, j \leq N$

$$\| \sum_{i,i'=1}^N a_{ii'} Y_i^{(N)} \otimes Y_{i'}^{(N)} \| \leq C (\log N)^{3/2} \left( \sum_{i,i'=1}^N |a_{ii'}|^2 \right)^{1/2}$$

Equivalently, let $g_{i,k,l}$ be i.i.d. Gaussian normal random variables, with $i, k, l \leq N$. Let $g_{i,k,l}'$ be an independent copy of the family $g_{i,k,l}$. Then the norm $\|T\|_\vee$ of the tensor

$$T = \sum g_{i,k,l} g_{i',k',l'} e_{ii'} \otimes e_{kk'} \otimes e_{ll'}$$

in the triple injective tensor product $\ell_2^{N^2} \vee \ell_2^{N^2} \vee \ell_2^{N^2}$ satisfies for some $C$

$$\mathbb{E} \|T\|_\vee \leq C (\log N)^{3/2} N$$

**Remark 1.2.** Note that if we replace in $T$ the random coordinates by a family of i.i.d. Gaussian normal variables indexed by $N^6$, then by well known estimates (e.g. the Chevet inequality) the corresponding random tensor, denoted by $\hat{T}$, satisfies $\mathbb{E} \|\hat{T}\|_\vee \leq CN$. 

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Remark 1.3. Let \( g \) be a Gaussian vector in a finite dimensional (real) Hilbert space \( H \) and let \( g' \) be an independent copy of \( g \). We assume (for simplicity) that the distribution of \( g \) and \( g' \) is the canonical Gaussian measure on \( H \). Let \( u_i \) \((i = 1, \cdots, M)\) be operators on \( H \). Let \( Z_i = \langle u_i g, u_i g' \rangle \), and let \( \hat{Z}_i = \langle u_i g, u_i g \rangle - \mathbb{E}(u_i g, u_i g) \). We have then for any \( p \geq 1 \)

\[
2^{-1} \| \sup_{i \leq M} |\hat{Z}_i| \|_p \leq \| \sup_{i \leq M} |Z_i| \|_p \leq \| \sup_{i \leq M} |\hat{Z}_i| \|_p, \tag{1.1}
\]

and hence

\[
\| \sup_{i \leq M} \langle u_i g, u_i g \rangle \|_p \leq 2 \| \sup_{i \leq M} |Z_i| \|_p + \sup_{i \leq M} \| u_i \|_{S_2}^2, \tag{1.2}
\]

where \( \| \cdot \|_{S_2} \) denotes the Hilbert-Schmidt norm. Indeed, (denoting \( \approx \) equality in distribution) we have

\[
\langle g, g' \rangle \approx (2^{-1/2}(g + g'), 2^{-1/2}(g - g')).
\]

Therefore

\[
\langle u_i g, u_i g \rangle - \langle u_i g', u_i g' \rangle \approx 2 \langle u_i g, u_i g' \rangle.
\]

Thus, if \( \{ \hat{Z}_i' \} \) is an independent copy of \( \{ \hat{Z}_i \} \), we have

\[
\hat{Z}_i - \hat{Z}_i' \approx 2Z_i.
\]

From this (1.1) follows easily and (1.2) is an immediate consequence.

Remark 1.4. Let us denote by \( S_{2,1}(H) \) the class of operators \( u \) on \( H \) such that the eigenvalues \( \lambda_j \) of \( |u| \) (rearranged as usual with multiplicity in non-increasing order) satisfy

\[
\sum j^{-1/2} \lambda_j < \infty,
\]

equipped with the quasi-norm (equivalent to a norm)

\[
\| u \|_{2,1} = \sum j^{-1/2} \lambda_j < \infty.
\]

It is clear that by Cauchy-Schwarz (for some constant \( C \))

\[
\| u \|_{2,1} \leq C (\log \text{rk}(u))^{1/2} \| u \|_2. \tag{1.3}
\]

It is well known that the unit ball of \( S_{2,1}(H) \) is equivalent (up to absolute constants) to the closed convex hull of the set formed by all operators of the form \( u = k^{-1/2} P \) where \( P \) is a (orthogonal) projection of rank \( k \). In particular, for any \( u \) we have (for some constant \( C > 0 \))

\[
C^{-1} \| u \|_2 \leq \| u \|_{2,1}.
\]

Therefore, if \( Z \) is a trilinear form on \( S_{2,1}(H) \), we have (for some constant \( C \))

\[
\sup \{|Z(r, s, t)| \mid \| r \|_{2,1} \leq 1, \| s \|_{2,1} \leq 1, \| t \|_{2,1} \leq 1 \} \leq C \sup_{k,l,m} (klm)^{-1/2} \sup |Z(R, S, T)| \tag{1.4}
\]

where the second supremum runs over all integers \( k, l, m \) and the third one over all projections \( R, S, T \) of rank respectively \( k, l, m \). Let us denote by \( P(k) \) the set of projections of rank \( k \). By (1.3) letting \( d = \text{dim}(H) \), this implies (for some constant \( C \))

\[
\sup_{\| x \|_2 \leq 1, \| y \|_2 \leq 1, \| z \|_2 \leq 1} |Z(x, y, z)| \leq C (\log d)^{3/2} \sup_{k,l,m} (klm)^{-1/2} \sup_{(R, S, T) \in P(k) \times P(l) \times P(m)} |Z(R, S, T)|. \tag{1.5}
\]
Proof of Theorem. We identify $\ell_2^{N^2}$ with the Hilbert-Schmidt class $S_2^{N^2}$. Then, viewing $R$ as an operator (or matrix) acting on $\ell_2^N$, we denote by $\|R\|_2$ and $\|R\|_\infty$ respectively its Hilbert-Schmidt norm and operator norm. Let

$$Z(R, S, T) = \sum g_{i,k,l}^i g_{i',k',l'}^i R_{ii'} S_{kk'} T_{ll'}$$

The norm of $T$ is the supremum of $Z(R, S, T)$ over $R, S, T$ in the unit ball of $\ell_2^{N^2}$.

Note that $Z(R, S, T) = (\langle R \otimes S \otimes T \rangle g, g')$ where $g, g'$ are independent canonical random vectors on $\ell_2^{N^2}$, and $u = R \otimes S \otimes T$ is an operator on $\ell_2^{N^3} = \ell_2^N \otimes \ell_2^N \otimes \ell_2^N$.

Fix integers $r, s, t$ and $\delta > 0$. Let $\mathcal{P}(r, s, t) = \mathcal{P}(r) \otimes \mathcal{P}(s) \otimes \mathcal{P}(t)$. Let $\mathcal{P}_\delta(r)$ be a $\delta$-net in $\mathcal{P}(r)$ with respect to the norm in $S_2$. It is easy to check that we can find such a net with at most $\exp\{c(\delta)rN\}$ elements, so we may assume that $|\mathcal{P}_\delta(r)| \leq \exp\{c(\delta)rN\}$. Let

$$\mathcal{P}_\delta(r, s, t) = \mathcal{P}_\delta(r) \otimes \mathcal{P}_\delta(s) \otimes \mathcal{P}_\delta(t).$$

Note that $\mathcal{P}_\delta(r, s, t)$ is a $3\delta$-net in $\mathcal{P}(r, s, t)$ and $|\mathcal{P}_\delta(r, s, t)| \leq \exp\{3c(\delta)(r + s + t)N\}$.

Let

$$\|Z\| = \sup_{\|x\|_2 \leq 1, \|y\|_2 \leq 1, \|z\|_2 \leq 1} |Z(x, y, z)|,$$

and

$$\|Z\|_* = \sup_{\|x\|_2,1 \leq 1, \|y\|_2,1 \leq 1, \|z\|_2,1 \leq 1} |Z(x, y, z)|.$$

Recall that by (1.5)

$$\|Z\|_* \leq C(\log N)^{3/2} \|Z\|.$$

Claim: We claim that for some constant $C_\delta$ we have

$$\sup_{(R,S,T) \in \mathcal{P}_\delta(r,s,t)} (rst)^{-1/2} |Z(R, S, T)|_N \leq C_\delta N.$$

We will use a bound of Latala (actually easy to prove in the bilinear case) that says that for some absolute constant $c$ we have for all $p \geq 1$

$$\|Z(R, S, T)\|_p \leq c(p^{1/2}\|R\|_2\|S\|_2\|T\|_2 + p\|R\|_\infty\|S\|_\infty\|T\|_\infty).$$

Let us record here for further reference the obvious inequality

(1.6) \quad \|\sup_{i \leq M} \|Z(R_i, S, T_i)\|_q \leq M^{1/q} \sup_{i \leq M} \|Z(R_i, S, T_i)\|_q.$$

When we take the sup over a family $R_i, S, T_i$ indexed by $i = 1, \ldots, M$ in the unit ball of $S_2^N$ and such that for all $i$ we have $\|R_i\|_\infty \leq r^{-1/2}$, $\|S_i\|_\infty \leq s^{-1/2}$, $\|T_i\|_\infty \leq t^{-1/2}$, this gives us

$$\|\sup_{i \leq M} \|Z(R_i, S, T_i)\|_p \leq cM^{1/p}(p^{1/2} + p(rst)^{-1/2}).$$

Choosing $p = \log M$ and $p \geq q$ we find a fortiori

(1.7) \quad \|\sup_{i \leq M} \|Z(R_i, S, T_i)\|_q \leq c'(\log M)^{1/2} + (\log M)(rst)^{-1/2}).
To prove the claim we may reduce to triples \((r, s, t)\) such that \(t = \max(r, s, t)\). Indeed exchanging the roles of \((r, s, t)\), we treat similarly the cases \(r = \max(r, s, t)\) and \(s = \max(r, s, t)\) and the desired result follows with a tripled constant \(C_\delta\).

We will treat separately the sets

\[
A = \{(r, s, t) \mid rs > t, \ t = \max\{r, s\}\} \quad \text{and} \quad B = \{(r, s, t) \mid rs \leq t, \ t = \max\{r, s\}\}.
\]

Note that both sets have at most \(N^3\) elements.

- Fix \((r, s, t) \in A\). By (1.7) with \(q = N\), we have

\[
\| \sup_{(R, S, T) \in \mathcal{P}_3(r, s, t)} (rs)^{-1/2} |Z(R, S, T)| \|_N \leq c'(\log |\mathcal{P}_\delta(r, s, t)|)^{1/2} + (\log |\mathcal{P}_\delta(r, s, t)|)(rs)^{-1/2},
\]

and hence

\[
(1.8)
\]

\[
\leq c'(3c(\delta)(r + s + t)N)^{1/2} + c'(3c(\delta)(r + s + t)N)(rs)^{-1/2}.
\]

Now on the one hand \((r + s + t)N)^{1/2} \leq 3^{1/2}N\) and on the other hand, since we assume \(t = \max\{r, s\}\) and \(rs > t\), we have \((r + s + t)N)(rs)^{-1/2} \leq 3tN(rs)^{-1/2} \leq 3N\). So assuming \((r, s, t) \in A\) we obtain

\[
\| \sup_{(R, S, T) \in \mathcal{P}_3(r, s, t)} (rs)^{-1/2} |Z(R, S, T)| \|_N \leq 3c'(3^{1/2} + 3)N.
\]

- Now assume \((r, s, t) \in B\) and in particular \(rs \leq t\). By Cauchy-Schwarz if \(R, S, T\) are projections we have \(|Z(R, S, T)| = |\langle g, (R \otimes S \otimes T)g' \rangle| \leq |\langle g, (R \otimes S \otimes T)g' \rangle|^{1/2}(g', (R \otimes S \otimes T)g')^{1/2}\). We may write a fortiori

\[
\|Z(R, S, T)\|_p \leq |\langle g, (R \otimes S \otimes T)g' \rangle|_p.
\]

Therefore by (1.2) and since \(T \leq I\) (and hence \(R \otimes S \otimes T \leq R \otimes S \otimes I\)) we have

\[
\|Z(R, S, T)\|_p \leq 2\|Z(R, S, I)\|_p + rsN.
\]

Similarly we find

\[
(1.9)
\]

\[
\| \sup_{(R, S, T) \in \mathcal{P}_3(r, s, t)} |Z(R, S, T)| \|_p \leq 2\| \sup_{(R, S, I) \in \mathcal{P}_\delta(r, s, N)} Z(R, S, I)\|_p + rsN,
\]

and hence by (1.7) again (we argue as for (1.8) above with \(q = N\), but note however that in the present case \(T = I\) is fixed so the supremum over \((R, S, I) \in \mathcal{P}_\delta(r, s, N)\) runs over at most \(\exp\{3c(\delta)(r + s)N\}\) elements and we may use \(p = 3c(\delta)(r + s)N\) we find

\[
\| \sup_{(R, S, I) \in \mathcal{P}_\delta(r, s, N)} (rs)^{-1/2} |Z(R, S, I)| \|_N \leq c'(3c(\delta)(r + s)N)^{1/2}(N/t)^{1/2} + c'(3c(\delta)(r + s)N)(rs)^{-1/2}.
\]

But now, since we assume \(rs \leq t\), \(r + s \leq 2rs \leq 2t\) so that \((r + s)(rs)^{-1/2} \leq 2(rs/t)^{1/2} \leq 2\), and hence we find \(\| \sup_{(R, S, I) \in \mathcal{P}_\delta(r, s, N)} (rs)^{-1/2} Z(R, S, I)\|_N \leq c_3(\delta)N\). Substituting this in (1.9) yields

\[
\| \sup_{(R, S, T) \in \mathcal{P}_\delta(r, s, t)} (rs)^{-1/2} |Z(R, S, T)| \|_N \leq 2c_3(\delta)N + (rs/t)^{1/2}N \leq (2c_3(\delta) + 1)N.
\]

This completes the proof of the above claim.

Using the claim, we conclude the proof as follows: To pass from \(\mathcal{P}_\delta(r, s, t)\) to \(\mathcal{P}(r, s, t)\) we first note that if (say) \(P, P'\) are both projections of rank \(r\), by (1.3) \(\|P - P'\|_2 \leq \delta\) implies (for some constant \(c_1\)) that \(r^{-1/2}\|P - P'\|_{2,1} \leq c_1\delta\). Thus, using this for \(r, s, t\) successively, we find

\[
\sup_{(R, S, T) \in \mathcal{P}(r, s, t)} (rs)^{-1/2} |Z(R, S, T)| \leq \sup_{(R, S, T) \in \mathcal{P}_\delta(r, s, t)} (rs)^{-1/2} |Z(R, S, T)| + 3c_1\delta\|Z\|_\bullet.
\]
This implies
\[ \| \sup_{(R,S,T) \in \mathcal{P}(r,s,t)} (rst)^{-1/2} |Z(R,S,T)|_{\lambda N} \| \leq \| \sup_{(R,S,T) \in \mathcal{P}(r,s,t)} (rst)^{-1/2} |Z(R,S,T)|_{\lambda N} + 3c_1 \| |Z|_{\lambda N}. \]

Using (1.6) to estimate the sup over the \( N^3 \) integers \( r, s, t \) we find (recalling (1.4))
\[ \| |Z|_{\lambda N} \| \leq CN^{3/N} \sup_{r,s,t} (rst)^{-1/2} |Z(R,S,T)|_{\lambda N} + 3c_1 \| |Z|_{\lambda N}, \]
and hence by the claim
\[ \leq 8CC_6 N + 24C_1 \delta \| |Z|_{\lambda N} \|
\]
from which follows that
\[ \| |Z|_{\lambda N} \| \leq (1 - 24C_1 \delta)^{1/2} 8CC_6 N. \]
Observe that \( E[Z|N|^{1/2}] E[Z|\bullet|_N \leq (\log N)^{3/2} |Z|_{\lambda N} \). Thus, if \( \delta \) is small enough, chosen so that say \( 24C_1 \delta = 1/2 \), we finally obtain a fortiori
\[ E[Z|N|^{1/2}] \leq 16CC_6 (\log N)^{3/2} N. \]

Actually since we obtain the same bound for \( (E[Z|N|^{1/2}) \) we also obtain for suitable positive constants \( c_2, c_3 \) that
\[ \mathbb{P} \{ |Z| > c_3 N (\log N)^{3/2} \} \leq \exp -c_2 N. \]

Theorem 1.5. Consider the following two norms for an element \( t = \sum_{ijk} t_{ijk} e_i \otimes e_j \otimes e_k \) in the triple tensor product \( \ell_1^n \otimes \ell_1^p \otimes \ell_1^q \):
\[ (1.10) \quad \| |t|_{\min} = \sup \{ \| \sum_{ijk} t_{ijk} u_i \otimes v_j \otimes w_k \|_{\mathcal{B}(H \otimes_2 H \otimes_2 H)} \} \]
where the sup runs over all possible Hilbert spaces \( H \) and all possible unitary operators \( u_i, v_j, w_k \) acting on \( H \), and also:
\[ (1.11) \quad \| |t|_{\lambda N} = \sup \{ \sum_{ijk} t_{ijk} x_i y_j z_k \} \]
where the sup runs over all unimodular scalars \( x_i, y_j, z_k \) or equivalently the sup is as before but restricted to \( \dim(H) = 1 \). Let
\[ (1.12) \quad C_3(n) = \sup \{ |t|_{\min} \| |t|_{\lambda N} \leq 1 \}. \]
Then we have for some constant \( C' > 0 \) (independent of \( n \))
\[ C_3(n) \geq C' n^{1/4} (\log n)^{-3/2}. \]

Remark 1.6. It is well known that the supremum in (1.10) is unchanged if we restrict the supremum to finite dimensional spaces \( H \). Moreover, we have also
\[ \| |t|_{\min} = \sup \{ \| \sum_{ijk} t_{ijk} u_i v_j w_k \|_{M_N} \}, \]
where the supremum runs over all \( N \) and all \( N \times N \)-unitary matrices \( u_i, v_j, w_k \) such that \( u_i v_j = v_j u_i, u_i w_k = w_k u_i \) and \( w_k v_j = v_j w_k \) for all \( i, j, k \). Note that (1.11) corresponds again to restricting this sup to \( N = 1 \).
Proof of Theorem 1.5. Let \( n = N^2 \). We again identify \( \ell_2^N \) with the space of \( N \times N \) matrices equipped with the HS norm. Let \( \{ u_j \mid j \leq N^2 \} \) be an orthogonal basis in \( \ell_2^N \) consisting of unitaries (this is called an EPR basis in quantum information). Note that \( \| u_j \|_2 = \sqrt{N} \) for all \( j \).

Then \( u_i \otimes u_j \otimes u_k \) \((i, j, k \leq N^2)\) forms an orthogonal basis in \( \ell_2^N \otimes \ell_2^N \otimes \ell_2^N \).

Consider now \( T \in \ell_2^{N^2} \otimes \ell_2^{N^2} \otimes \ell_2^{N^2} \) and let

\[
T = \sum \hat{T}_{ijk} u_i \otimes u_j \otimes u_k
\]

be its development on that orthogonal basis, so that \( \hat{T}_{ijk} = N^{-3} \langle T, u_i \otimes u_j \otimes u_k \rangle \). Consider now \( t = \sum_{ijk} \hat{T}_{ijk} e_i \otimes e_j \otimes e_k \). Then for any unimodular scalars \( x_i, y_j, z_k \) we have

\[
\sum \hat{T}_{ijk} x_i \otimes y_j \otimes z_k = \langle T, X \otimes Y \otimes Z \rangle
\]

with \( X = \sum x_i u_i, Y = \sum y_j u_j, Z = \sum z_k u_k \) and hence

\[
\| t \|_\nu \leq N^{3/2} \sup\{ \| \langle T, X \otimes Y \otimes Z \rangle \| \mid X, Y, Z \in B_{\ell_2^{N^2}} \}.
\]

But since we also have \( T = \sum \hat{T}_{ijk} u_i \otimes u_j \otimes u_k \), we have

\[
\| t \|_{\min} \geq \| T \|_{B(\ell_2^N \otimes \ell_2^N \otimes \ell_2^N)}.
\]

Now consider as before

\[
T = \sum g_{i,k,l} g_{i',k',l'} e_i \otimes e_{kk'} \otimes e_{ll'}.
\]

With this choice of \( T \) by the preceding Theorem we find with large probability \( \sup\{ \| \langle T, X \otimes Y \otimes Z \rangle \| \mid X, Y, Z \in B_{\ell_2^{N^2}} \} \leq CN(\log N)^{3/2} \), and hence

\[
\| t \|_\nu \leq C N^{5/2} (\log N)^{3/2},
\]

but also (since \( T \) is a rank one operator) \( \| T \|_{B(\ell_2^N \otimes \ell_2^N \otimes \ell_2^N)} = (\sum |g_{i,k,l}|^2)^{1/2}(\sum |g_{i',k',l'}|^2)^{1/2} \) and the latter is concentrated around its mean and hence with large probability \( \geq N^{3/2} \). Thus we conclude that

\[
C(N^2) \geq (2C)^{-1} N^{1/2} (\log N)^{-3/2}.
\]

\[\square\]

Remark 1.7. The same method works in the \( d \)-dimensional case. Consider

\[
T = \sum g_{i(1), \ldots, i(d)}, g'_{i'(1), \ldots, i'(d)} e_{i(1)i'(1)} \otimes \cdots \otimes e_{i(d)i'(d)} \in \ell_1^{N(1)^2} \otimes \cdots \otimes \ell_1^{N(d)^2}.
\]

Let \( \{ u_{ij} \mid 1 \leq i \leq N(j)^2 \} \) be an orthogonal basis in \( S_2^{N(j)} \) formed of unitary matrices. We will denote by \( \hat{i} \) the elements of the set \( I = [N(1)^2 \times \cdots \times N(d)^2] \). Let

\[
T = \sum \hat{T} \hat{i} \hat{u}_{\hat{i}}
\]

be its orthogonal development according to the basis formed by

\[
\forall \hat{i} = (\hat{i}(1), \ldots, \hat{i}(d)) \in I \quad \hat{u}_{\hat{i}} = u_{\hat{i}(1)}^{(1)} \otimes \cdots \otimes u_{\hat{i}(d)}^{(d)},
\]

so that

\[
\hat{T} \hat{i} = (N(1) \cdots N(d))^{-1} \langle T, \hat{u}_{\hat{i}} \rangle.
\]
Let us denote also by $e_{\underline{i}} = e_{i(1)} \otimes \cdots \otimes e_{i(d)}$ the canonical basis in $\ell_1^{N(1)2} \otimes \cdots \otimes \ell_1^{N(d)2}$. Let then

$$t = \sum \hat{T}(\underline{i})e_{\underline{i}}.$$

As above, on one hand since $T$ appears as an operator of rank one, we have

$$\|t\|_{\text{min}} \geq \|T\|_{B(\ell_1^{N(1)2} \otimes \cdots \otimes \ell_1^{N(d)2})} \geq (\sum |g_i(1),\ldots,i(d)|^2)^{1/2}(\sum |g_i'(1),\ldots,i(d)|^2)^{1/2}.$$

On the other hand using the orthogonality of the $u_{\underline{i}}$'s we have

$$\|t\|_{\vee} \leq (N(1) \cdots N(d))^{1/2}\|T\|_{\ell_2^{N(1)2} \otimes \cdots \otimes \ell_2^{N(d)2}}.$$

Thus we find

$$C_d(N(1)^2,\ldots,N(d)^2) \geq (N(1) \cdots N(d))^{-1/2} \left( \sup_{\ell_2^{N(1)2} \otimes \cdots \otimes \ell_2^{N(d)2}} \|T\| \right)^{-1},$$

where the sup runs over all $T$ of the above form (i.e. of rank one in a suitable sense) such that $(\sum |g_i(1),\ldots,i(d)|^2)^{1/2}(\sum |g_i'(1),\ldots,i(d)|^2)^{1/2} \leq 1$ (i.e. of norm one in a suitable sense). Then using Gaussian variables as above, we obtain $\|t\|_{\text{min}} \geq cN^d$ and $\|t\|_{\vee} \leq cN^{d/2}(N(\log N)^d/2)$

$$C_d(N^2,\ldots,N^2) \geq c_d N^{d/2-1}(\log N)^{-d/2}.$$

2. An almost sharp inequality

Let $(H_j)$ and $(K_j)$ $(1 \leq j \leq d)$ be $d$-tuples of finite dimensional Hilbert spaces. Let

$$J : (H_1 \otimes_2 K_1) \otimes \cdots \otimes (H_d \otimes_2 K_d) \rightarrow (H_1 \otimes_2 \cdots \otimes_2 H_d) \otimes (K_1 \otimes_2 \cdots \otimes_2 K_d),$$

be the natural identification map. After reordering, we may as well assume that the sequence $\{\dim(H_j) \dim(K_j) \mid 1 \leq j \leq d\}$ is non-decreasing.

We have

$$\|J\| \leq \prod_{j=1}^{d-1}(\dim(H_j) \dim(K_j))^{1/2}. \quad (2.1)$$

Indeed, it is easy to check that, for any normed space $E$, the identity map $\ell_2^2 \otimes E \rightarrow \ell_2^2(E)$ has norm $\leq \sqrt{n}$. This gives

$$\|(H_1 \otimes_2 K_1) \otimes \cdots \otimes (H_d \otimes_2 K_d) \rightarrow (H_1 \otimes_2 K_1) \otimes_2 \cdots \otimes_2 (H_d \otimes_2 K_d)\| \leq \prod_{j=1}^{d-1}(\dim(H_j) \dim(K_j))^{1/2}.$$

A fortiori we obtain the above bound (2.1) for $J$.

Consider now the case when $\dim(H_j) = \dim(K_j) = N$ for all $j$. In that case the preceding bound becomes

$$\|J\| \leq N^{d-1}.$$
where we identify $H_j \otimes K_j$ with $\ell_2^{N^2}$. The preceding proof yields with large probability
\[ \|T\|_{\ell_2^{N^2} \otimes \cdots \otimes \ell_2^{N^2}} \leq cN(\log N)^{d/2} \]
and also
\[ \|T\|_{(H_1 \otimes \cdots \otimes H_d) \otimes (K_1 \otimes \cdots \otimes K_d)} \geq c'N^d. \]
Thus we obtain the following almost sharp (i.e. sharp up to the log factor)
\[ \|J\| \geq c''N^{d-1}(\log N)^{-d/2}. \]

The argument described in the preceding Remark 1.7 boils down to the estimate
\[ C_d(N^2, \cdots, N^2) \geq N^{-d/2}\|J\|. \]

**Remark 2.1.** Note that the preceding proof actually yields (assuming dim($H_j$) = dim($K_j$) = $N$ for all $j$)
\[ c''N^{d-1}(\log N)^{-d/2} \leq \|(H_1 \otimes K_1) \otimes \cdots \otimes (H_d \otimes K_d) \rightarrow (H_1 \otimes K_1) \otimes \cdots \otimes (H_d \otimes K_d)\| \leq N^{d-1}. \]
However, this norm is much easier to estimate and, actually, we claim it is $\geq c''N^{d-1}$.
Indeed, returning to $H_j$, $K_j$ of arbitrary finite dimension, let $n_j = \dim(H_j) \dim(K_j)$.
Assume $n_1 \leq n_2 \leq \cdots \leq n_d$. Consider then the inclusion
\[ \Phi: \ell_2^{n_1} \otimes \cdots \otimes \ell_2^{n_d} \rightarrow \ell_2^{n_1 \cdots n_d}. \]
The above easy argument shows that $\|\Phi\| \leq (n_1 \cdots n_{d-1})^{1/2}$. Let now $G$ be a random vector with values in $\ell_2^{n_1 \cdots n_d}$ distributed according to the canonical Gaussian measure. We will identify $G$ with $\Phi^{-1}(G)$. Then, by Simone Chevet’s well known inequality we have
\[ \mathbb{E}\|G\|_{\ell_2^{n_1} \otimes \cdots \otimes \ell_2^{n_d}} \leq \sqrt{d} \sum_j \sqrt{n_j} \leq d^{3/2}n_d, \]
while it is clear that $\mathbb{E}\|G\|_{\ell_2^{n_1 \cdots n_d}}^2 = n_1 \cdots n_d$. From this follows that $\|\Phi\| \geq d^{-3/2}(n_1 \cdots n_{d-1})^{1/2}$.
In particular the above claim is established.

### 3. A different method

It is known (due to Geman) that
\[ \lim_{N \to \infty} \|Y^{(N)}\|_{M_N} = 2 \quad \text{a.s.} \]

Let $(Y_1^{(N)}, Y_2^{(N)}, \ldots)$ be a sequence of independent copies of $Y^{(N)}$, so that the family $\{Y_k^{(N)}(i, j) \mid k \geq 1, 1 \leq i, j \leq N\}$ is an independent family of $N(0, N^{-1})$ complex Gaussian.

The next two statements follow from results known to Steen Thorbjørnsen since at least 1999 (private communication). See [2] for closely related results. We present a trick that yields a self-contained derivation of this.
Theorem 3.1. Consider independent copies $Y_i' = Y_i^{(N)}(\omega')$ and $Y_j'' = Y_j^{(N)}(\omega'')$ for $(\omega', \omega'') \in \Omega \times \Omega$. Then, for any $n^2$-tuple of scalars $(\alpha_{ij})$, we have

$$\lim_{N \to \infty} \left\| \sum_{i,j=1}^n \alpha_{ij} Y_i^{(N)}(\omega') \otimes Y_j^{(N)}(\omega'') \right\|_{M_N} \leq 4 \left( \sum_{i,j} |\alpha_{ij}|^2 \right)^{1/2}$$

for a.e. $(\omega', \omega'')$ in $\Omega \times \Omega$.

Proof. By (well known) concentration of measure arguments, it is known that (3.1) is essentially the same as the assertion that $\lim_{N \to \infty} \mathbb{E}\|Y^{(N)}\|_{M_N} = 2$. Let $\varepsilon(N)$ be defined by

$$\mathbb{E}\|Y^{(N)}\|_{M_N} = 2 + \varepsilon(N)$$

so that we know $\varepsilon(N) \to 0$. Again by concentration of measure arguments (see e.g. [3, p. 41] or [5, (1.4) or chapter 2]) there is a constant $\beta$ such that for any $N \geq 1$ and $p \geq 2$ we have

$$\mathbb{E}\|Y^{(N)}\|_{M_N}^p \leq \mathbb{E}\|Y^{(N)}\|_{M_N}^p \leq 2 + \varepsilon(N) \leq \beta(p/N)^{1/2}.$$ 

For any $\alpha \in M_n$, we denote

$$Z^{(N)}(\alpha)(\omega', \omega'') = \sum_{i,j=1}^n \alpha_{ij} Y_i^{(N)}(\omega') \otimes Y_j^{(N)}(\omega'').$$

Assume $\sum_{i,j} |\alpha_{ij}|^2 = 1$. We will show that almost surely

$$\lim_{N \to \infty} \|Z^{(N)}(\alpha)\| \leq 4.$$

Note that by the invariance of (complex) canonical Gaussian measures under unitary transformations, $Z^{(N)}(\alpha)$ has the same distribution as $Z^{(N)}(u\alpha v)$ for any pair $u, v$ of $n \times n$ unitary matrices. Therefore, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $|\alpha| = (\alpha^* \alpha)^{1/2}$, we have

$$Z^{(N)}(\alpha)(\omega', \omega'') \stackrel{\text{dist}}{=} \sum_{j=1}^n \lambda_j Y_j^{(N)}(\omega') \otimes Y_j^{(N)}(\omega'').$$

We claim that by a rather simple calculation of moments, one can show that for any even integer $p \geq 2$ we have

$$\mathbb{E} \text{ tr}|Z^{(N)}(\alpha)|^p \leq (\mathbb{E} \text{ tr}|Y^{(N)}|^p)^2.$$

Accepting this claim for the moment, we find, a fortiori, using (3.3):

$$\mathbb{E}\|Z^{(N)}(\alpha)\|_{M_N}^p \leq N^2(\mathbb{E}\|Y^{(N)}\|_{M_N}^p)^2 \leq N^2(2 + \varepsilon(N) + \beta(p/N)^{1/2})^2.$$

Therefore for any $\delta > 0$

$$\mathbb{P}\{\|Z^{(N)}(\alpha)\|_{M_N} > (1 + \delta)4\} \leq (1 + \delta)^{-p} N^2(1 + \varepsilon(N)/2 + (\beta/2)(p/N)^{1/2})^2.$$

Then choosing (say) $p = 5(1/\delta) \log(N)$ we find

$$\mathbb{P}\{\|Z^{(N)}(\alpha)\|_{M_N} > (1 + \delta)4\} \in O(N^{-2})$$

and hence (Borel–Cantelli) $\lim_{N \to \infty} \|Z^{(N)}(\alpha)\|_{M_N} \leq 4$ a.s.. It remains to verify the claim. Let $Z = Z^{N}(\alpha)$, $Y = Y^{(N)}$ and $p = 2m$. We have

$$\mathbb{E} \text{ tr}|Z|^p = \mathbb{E} \text{ tr}(Z^* Z)^m = \sum \bar{\lambda}_{i_1} \lambda_{j_1} \ldots \bar{\lambda}_{i_m} \lambda_{j_m} (\mathbb{E} \text{ tr}(Y_{i_1} Y_{j_1} \ldots Y_{i_m} Y_{j_m}))^2.$$
Note that the only nonvanishing terms in this sum correspond to certain pairings that guarantee that both $\tilde{\lambda}_i \lambda_j \ldots \tilde{\lambda}_m \lambda_j \geq 0$ and $\mathbb{E} \text{tr}(Y_{i_1}^* Y_{j_1} \ldots Y_{i_m}^* Y_{j_m}) \geq 0$. Moreover, by Hölder’s inequality for the trace we have

$$|\mathbb{E} \text{tr}(Y_{i_1}^* Y_{j_1} \ldots Y_{i_m}^* Y_{j_m})| \leq \Pi(\mathbb{E} \text{tr}|Y_{i_k}|^p)^{1/p} \Pi(\mathbb{E} \text{tr}|Y_{j_k}|^p)^{1/p} = \mathbb{E} \text{tr}(|Y|^p).$$

From these observations, we find

$$(3.5) \quad \mathbb{E} \text{tr}|Z|^p \leq \mathbb{E} \text{tr}(|Y|^p)\sum \tilde{\lambda}_i \lambda_j \ldots \tilde{\lambda}_m \lambda_j \mathbb{E} \text{tr}(Y_{i_1}^* Y_{j_1} \ldots Y_{i_m}^* Y_{j_m})$$

but the last sum is equal to $\mathbb{E} \text{tr}(|\sum \lambda_j Y_j|^p)$ and since $\sum \lambda_j Y_j \overset{\text{dist}}{=} Y$ (recall $\sum |\lambda_j|^2 = \sum |\alpha_{ij}|^2 = 1$) we have

$$\mathbb{E} \text{tr}\left(|\sum \alpha_j Y_j|^p\right) = \mathbb{E} \text{tr}(|Y|^p),$$

and hence (3.5) implies (3.4). \hfill \Box

**Corollary 3.2.** For any integer $n$ and $\varepsilon > 0$, there are $N$ and $n$-tuples of $N \times N$ matrices $\{Y''_i \mid 1 \leq i \leq n\}$ and $\{Y''_j \mid 1 \leq j \leq n\}$ in $M_N$ such that

$$(3.6) \quad \sup \left\{ \left\| \sum_{i=j=1}^n \alpha_{ij} Y'_i \otimes Y''_j \right\|_{M_{N^2}} \mid \alpha_{ij} \in \mathbb{C}, \sum |\alpha_{ij}|^2 \leq 1 \right\} \leq (4 + \varepsilon)$$

$$(3.7) \quad \min \left\{ \frac{1}{nN} \sum_1^n \text{tr}|Y'_i|^2, \frac{1}{nN} \sum_1^n \text{tr}|Y''_j|^2 \right\} \geq 1 - \varepsilon.$$

**Proof.** Fix $\varepsilon > 0$. Let $\mathcal{N}_\varepsilon$ be a finite $\varepsilon$-net in the unit ball of $\ell_2^2$. By Theorem 3.1 we have for almost all $(\omega', \omega'')$

$$(3.8) \quad \lim_{N \to \infty} \sup_{\alpha \in \mathcal{N}_\varepsilon} \left\| \sum_{i,j=1}^n \alpha_{ij} Y'_i \otimes Y''_j \right\|_{M_{N^2}} \leq 4,$$

We may pass from an $\varepsilon$-net to the whole unit ball in (3.8) at the cost of an extra factor $(1 + \varepsilon)$ and we obtain (3.6). As for (3.7), the strong law of large numbers shows that the left side of (3.7) tends a.s. to 1. Therefore, we may clearly find $(\omega', \omega'')$ satisfying both (3.6) and (3.7). \hfill \Box

**Remark 3.3.** A close examination of the proof and concentration of measure arguments show that the preceding corollary holds with $N$ of the order of $c(\varepsilon)n^2$. Indeed, we find a constant $C$ such that for any $\alpha = (\alpha_{ij})$ in the unit ball of $\ell_2^2$ we have (we take $p = N$)

$$\|Z(N)(\alpha)\|_{L_N(M_N)} \leq C$$

from which follows if $A$ is a finite subset of the unit ball of $\ell_2^2$ that

$$\|\sup_{\alpha \in A} Z(N)(\alpha)\|_{M_N} \leq C|A|^{1/N}.$$
Remark 3.4. Using the well known “contraction principle” that says that the variables \((\varepsilon_j)\) are dominated by either \((g_j^R)\) or \((g_j^C)\), it is easy to deduce that Corollary 3.2 is valid for matrices \(Y_j, Y_j''\) with entries all equal to \(\pm N^{-1/2}\), with possibly a different numerical constant in place of \(4\). Analogously, using the polar factorizations \(Y_j = U_j' Y_j' = Y_j'' U_j''\) and noting that all the factors \(U_j', Y_j', U_j'', Y_j''\) are independent, we can also (roughly by integrating over the moduli \(|Y_j'|, |Y_j''|\)) obtain Corollary 3.2 with unitary matrices \(Y_j, Y_j''\), with a different numerical constant in place of \(4\).

Let \(C_3(n, n_2, n_3)\) the supremum appearing in \(11.12\) when \(t\) runs over all tensors in \(\ell_1^n \otimes \ell_1^{n_2} \otimes \ell_1^{n_3}\). Note that \(C_3(n) = C_3(n, n, n)\). Then the proof of Junge et al. as presented in [7] (and incorporating the results of [6]) yields

\[ C_3(n^4, n^8, n^8) \geq cn^{1/2}. \]

Indeed, the latter proof requires an embedding of \(\ell_2^m\) into \(\ell_1^n\) and Junge et al. use the Rademacher embedding, and hence \(m = 2^{n^2}\), but [6] allows us to use \(m = n^4\).

However, if we use the method of Theorem 1.1 together with Corollary 3.2 and the estimate in Remark 3.3, then we find

\[ C_3(n^2, n^4, n^4) \geq cn^{1/2}. \]

The open problems that remain are:

– get rid of the log factor in Theorem 1.1
– improve the lower bound of \(C_3(n)\) in Theorem 1.5 to something sharp, possibly \(cn^{1/2}\), and similar questions for \(C_3(n_1, n_2, n_3)\).
– find explicit non random examples responsible for large values of \(C_3(n)\).

4. Upper bounds

As far as I know the upperbounds for \(C_3(n)\) or \(C_d(n)\) are as follows.

First we have \(C_2(n) \leq K_G\) (here \(K_G\) is the Grothendieck constant).

If \(E\) is any operator space, let \(\min(E)\) be the same Banach space but viewed as embedded in a commutative \(C^*\)-algebra (i.e. the continuous functions on the dual unit ball).

Let \(F\) be an arbitrary operator space, it is easy to show that we have isometric identities

\[ F \otimes \min(E) = F \overset{\vee}{\otimes} E = \min(F) \otimes \min(E). \]

Moreover, it is known (and easy to check) that if \(E = \ell_1^n\), the identity map \(\min(E) \to E\) has cb norm at most \(\sqrt{n}\). A fortiori, we have \(\|F \otimes \min(E) \to F \otimes \min(E)\|_{cb} \leq \sqrt{n}\) and hence

\[ \|F \overset{\vee}{\otimes} E \to F \otimes \min(E)\| \leq \sqrt{n}. \]

This implies that if \(E_d = \ell_1^n \otimes \min \ldots \otimes \min \ell_1^n\) (\(d\) times), then

\[ \|E_d \otimes \min(E) \to E_d\| \leq \sqrt{n}. \]

Iterating we find

\[ \|E \overset{\vee}{\otimes} \cdots \overset{\vee}{\otimes} E \to E_d\| \leq C_{d-1}(n) \sqrt{n}. \]

Thus we obtain

\[ C_d(n) \leq K_G n^{(d-2)/2}. \]

A similar argument yields (note that we can use invariance under permutation to reduce to the case when \(n_1 \leq n_2 \leq \cdots\) whenever \(d \geq 3\):

\[ C_d(n_1, \cdots, n_d) \leq K_G (n_1 \cdots n_{d-2})^{1/2}. \]
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