Modeling of the initial stage of secondary flows formation for a fluid, the rheological model of which implies a threshold “addition” of the transverse viscosity factor

V N Kolodezhnov
Air Force Academy, Voronezh, 394064, Russia

E-mail : kvn117@mail.ru

Abstract. This paper proposes a rationale for the rheological model of a fluid, which implies a threshold “addition” of the transverse viscosity factor. A modeling of the secondary flows generation in a small neighborhood of the spatial point of the fluid flow region, the mechanical behavior of which is described by the rheological model of this kind, is carried out. The equations of the fluid dynamics for an index arbitrary value in the case of the transverse viscosity power dependence on the modulus exceedance of the strain rate tensor second invariant of the critical level in question are obtained. Interpretation of various scenarios of the flow development in a small neighborhood of the flow region point in which the modulus of the second invariant exceeds the threshold value is presented.

1. Introduction
The so-called secondary flows that appear in fluid flows are of considerable interest, since they are capable of changing the pattern of the main flow. At the same time, it was deduced from the experiments that secondary flows are formed and do not exist in all modes.

The best-known example is the fluid flow in the gap between two coaxial cylinders, one of which — the inner one — rotates, and the other — the outer one — remains stationary [1-4]. With a relatively slow rotation of the inner cylinder, the streamlines in the gap between the cylinders form concentric circles. However, when the angular velocity exceeds a certain critical value, an interesting effect occurs and the flow pattern changes. The velocity components transverse to the previously existing flow lines are “generated” in the fluid particles, and vortex structures begin to form in the gap.

The streamlines for such structures form spirals wound on a torus.

Another important example for applications is the initial stage of the laminar-turbulent transition [5,6]. Indeed, the turbulence transition is characterized by the mandatory formation of the transverse velocity components with respect to the streamlines of the initial laminar flow. Then it can be assumed that at the initial stage of the transition it should be preceded by the secondary flows “generation”, characterized by the presence of transverse velocity components with respect to the streamlines of the primary (main) laminar flow. In other words, the initial stage of the laminar-turbulent transition, presumably, can be considered the result of an incipient secondary flow.

Based on the examples cited, it can be preliminary noted that secondary flows are not formed under relatively “slow” modes of the main flow. It should also be pointed out that in these examples
secondary flows have a “threshold” nature, they appear when a process parameter exceeds a certain critical value. In case of a rotational flow in the gap between the coaxial cylinders, this parameter, with other parameters fixed, is the angular velocity of the internal cylinder rotation. In the example for a laminar-turbulent transition in the channels, the beginning of the transition, with all other parameters being equal, is determined by the average flow velocity.

The process of “generating” the velocity transverse components cited in the examples above should be determined by a certain “mechanism”, which, it is not improbable, is determined by the fluid internal “structure”. At the level of continuum mechanics models, this internal “structure” can be taken into account within the framework of the fluid rheology.

2. A possible rheological model hypothesis

In the most general form, the relationship between the components of the stress tensor $\tau_{ij}$ and the strain rate tensor $\varepsilon_{ij}$ and can be represented in the form [7–9]

$$\tau_{ij} = -P \cdot \delta_{ij} + 2 \cdot \varphi_1(I_1, I_2, I_3) \cdot \varepsilon_{ij} + 4 \cdot \varphi_2(I_1, I_2, I_3) \cdot \xi_{ij};$$

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right); \quad \xi_{ij} = \sum_{k=1}^{3} \varepsilon_{ik} \cdot \varepsilon_{kj}; \quad i, j = 1, 2, 3;$$

where $P$ is hydrostatic pressure; $u_i$ are velocity vector components; $\delta_{ij}$ is Kronecker symbol; $x_i$ are coordinates in the Cartesian frame of reference, $\varphi_1(I_1, I_2, I_3)$, $\varphi_2(I_1, I_2, I_3)$ are given functions of three independent invariants $I_1, I_2, I_3$ of strain rate tensor, which will be taken in the following form [9]

$$I_1 = \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2;$$

$$I_2 = \varepsilon_{11}^2 \cdot \varepsilon_{22} + \varepsilon_{22}^2 \cdot \varepsilon_{33} + \varepsilon_{33}^2 \cdot \varepsilon_{11} - \varepsilon_{12}^2 - \varepsilon_{23}^2 - \varepsilon_{31}^2;$$

$$I_3 = \varepsilon_{11} \cdot \varepsilon_{22} \cdot \varepsilon_{33} + 2 \cdot \varepsilon_{12} \cdot \varepsilon_{23} \cdot \varepsilon_{31} - \varepsilon_{11} \cdot \varepsilon_{22}^2 - \varepsilon_{23} \cdot \varepsilon_{31}^2 - \varepsilon_{33} \cdot \varepsilon_{12}^2.$$  

Naturally, for incompressible fluid, the following condition is always satisfied

$$I_1 = 0.$$  

Relationships (1) with regard to (2) represent Reiner-Rivlin’s rheological model [7]. This model with regard to (3) depending on the specific type of functions $\varphi_1(I_1, I_2, I_3)$ and $\varphi_2(I_1, I_2, I_3)$, makes it possible to describe the mechanical behavior of various nonlinear-viscous incompressible continuum equations. In this case, the function $\varphi_2(I_1, I_2, I_3)$ is often called transverse viscosity.

It should be noted that the secondary flows modeling is quite simply carried out when considering the liquids flow, the rheological model of which takes into account the transverse viscosity factor. It is known that consideration of flow problems in fluid channels, the rheology of which involves taking into account the transverse viscosity, leads to solutions demonstrating the presence of secondary flows [8]. These flows are characterized by transverse velocity components with respect to the streamlines which could be obtained by modeling the same flow, but without taking into account the transverse viscosity.

Considering, on the one hand, the contribution of transverse viscosity to the secondary flows formation, and, on the other hand, the threshold nature of their “generation” for real flows, we can formulate a hypothesis, the essence of which runs as follows: “Newtonian incompressible fluid rheological model has a limited applicability area beyond which the transverse viscosity factor is added.”
Let us consider conditions and assumptions to which the rheological model corresponding to this hypothesis must satisfy.

Firstly, we should point out that in a certain range of flow characteristics (those defining the main mode without secondary flows) the rheological model (1) must be in agreement with the well-known classical model of Newtonian fluid.

Note that the functions \( \varphi_1(I_2, I_3) \) and \( \varphi_2(I_2, I_3) \), determining the rheological model (1), depend taking into account (3) only on the second and third invariants of the strain rate tensor. In this regard, it can be assumed that in the hypothetical plane of these functions’ arguments there is a certain area \( \{I_2, I_3\} \equiv I_{\text{main}} \) within which the upgraded rheological model must be identically reduced to the well-known Newtonian fluid classical model with constant dynamic viscosity \( \mu \). In this case, of course, the following conditions must be met

\[
\varphi_1(I_2, I_3) = \mu = \text{const}; \quad \varphi_2(I_2, I_3) = 0; \quad \{I_2, I_3\} \in I_{\text{main}}.
\]

Secondly, note the following condition. When crossing the boundary \( \partial(I_{\text{main}}) \) of the region \( I_{\text{main}} \) in a rheological model, a certain “mechanism” responsible for “generating” the transverse velocity components should be added. In this regard, we assume that for the classical Newtonian fluid the following condition must be met

\[
\varphi_2(I_2, I_3) \neq 0; \quad \{I_2, I_3\} \not\in I_{\text{main}}.
\]

The third condition concerns a possible behavior of the function \( \varphi_1(I_2, I_3) \) outside the region \( I_{\text{main}} \). At present, we are not aware of the experimental data (or such data was not available) on this issue. Thereupon, we will assume the following. For that part of the hypothetical plane of the second and third invariants of the strain rate tensor, in which \( \{I_2, I_3\} \not\in I_{\text{main}} \), we assume that this function still takes a constant value \( \mu \) equal to the fluid dynamic viscosity in the region \( \{I_2, I_3\} \in I_{\text{main}} \) of the main flow. Note that it is this assumption, a priori, that is accepted in hydrodynamics when considering secondary flows. This also applies to the secondary flows formation, which determine the laminar-turbulent transition initial stage.

The following condition concerns the effect of the function \( I_3 \) on transverse viscosity. Formally, in the general case, the function \( \varphi_2(I_2, I_3) \) outside the region \( I_{\text{main}} \) should depend on both the second and third strain rate tensor invariants. However, due to the fact that we currently lack reliable experimental data on the nature of such a dependency, we propose to adopt a simplifying assumption as a first approximation

\[
\frac{\partial \varphi_2(I_2, I_3)}{I_3} = 0; \quad \{I_2, I_3\} \not\in I_{\text{main}}.
\]

This assumption in the framework of the considered approximation means the independence of this function from \( I_3 \), which allows us to further represent it as

\[
\varphi_2(I_2, I_3) \equiv \eta_c(I_2), \quad (4)
\]

where \( \eta_c(I_2) \) is a function expressing the transverse viscosity dependence on the strain rate tensor second invariant.

The assumption (4) made within the modeling framework can be considered quite admissible. Moreover, many rheometric flows realized in experiments, as a rule, obviously imply the fulfillment of the condition \( I_3 = 0 \). This means that the factor of the third invariant influence on the determined rheological characteristics simply does not manifest itself in such experiments.

Then we come to the conclusion that the belonging of a point \( \{I_2, I_3\} \) to a region \( I_{\text{main}} \) can, in fact,
be traced to the fulfillment of the condition
\[ |I_2| \leq I_{2\eta}, \]
where \(I_{2\eta}\) is some critical (threshold) value of the strain rate tensor second invariant modulus, below which a laminar flow mode is realized without any secondary flows formation.

In the case of the opposite condition
\[ |I_2| > I_{2\eta}, \]  
the transverse viscosity factor is added and the prerequisites for the transverse (relative to the initially laminar streamlines of the main flow) velocity components generation arise.

As noted above, we lack the reliable experimental data on the transverse viscosity identification. Therefore, it is rather common either to neglect it, or to consider it the constant value [10]. There are also representation variants \(\eta_c(I_2)\) in the form of a specific dependence of the type of a power function [8, 11], but for the entire range of values \(|I_2| \geq 0\) of the second invariant modulus.

For further consideration we take the approximation of the function \(\eta_c(I_2)\) at \(|I_2| \geq I_{2\eta}\) in the form of the power function
\[ \eta_c(I_2) = \eta_q \cdot (|I_2| - I_{2\eta})^q, \]
where \(\eta_q, q\) are the empirical parameters of the model.

Since the condition \(I_2 < 0\) [7] is satisfied for the second invariant of the strain rate tensor, in the particular case \(q = 1\) it is convenient to present the last relationship as follows
\[ \eta_c(I_2) = -\eta_q \cdot (I_2 + I_{2\eta}). \]  

Considering all the observations and assumptions mentioned above, we can offer the following version of the rheological model with a threshold “addition” of the transverse viscosity factor [12]
\[ \tau_{ij} = -p \cdot \delta_{ij} + 2 \cdot \mu \cdot \varepsilon_{ij} + 4 \cdot \eta_c(I_2) \cdot \xi_{ij}; \quad i, j = 1, 2, 3; \]  
\[ \eta_c(I_2) = \begin{cases} 0; & |I_2| < I_{2\eta}; \\ \eta_q \cdot (|I_2| - I_{2\eta})^q; & |I_2| \geq I_{2\eta}; \end{cases} \]

The particular case of the rheological model (7), but with a constant value of the transverse viscosity (\(q = 0\)), was analyzed in [10].

The following results from (7). Let the condition \(|I_2| < I_{2\eta}\) be satisfied at some spatial point of the flow region. Then, in a small neighborhood of this point, the behavior of the fluid corresponds to the classical Newtonian model. If the condition \(|I_2| > I_{2\eta}\) is fulfilled at this spatial point, then the transverse viscosity factor “adds up” and in its small neighborhood, there are prerequisites for “generating” the transverse velocity components, and, therefore, the secondary flows formation.

3. Secondary flows formation modeling
In [12], without taking into account the influence of bulk forces, the setting of the following problem was considered.

Suppose that in a given region of fluid flow with the rheological model (7), at some point in time, which we will conditionally as take as the initial \((t = 0)\), the corresponding velocity and pressure fields have been formed. We will assume that the initial distributions of velocity and pressure exactly
satisfy the Navier-Stokes equations and the continuity condition. Nevertheless, we will additionally assume that at a certain spatial point of the flow region and its small neighborhood at the initial time, condition (5) is satisfied.

This condition means that immediately after the initial moment of time, in accordance with (7), the transverse viscosity factor begins to manifest itself in a small neighborhood of the spatial point in question.

Let us write down the equations that describe the secondary flow formation dynamics for a fluid with the rheological model (7) in the vicinity of the spatial point in question, taking into account the external bulk forces.

We introduce a Cartesian coordinate system, the origin of which is placed at the point in question. The orientation of the coordinate axes is associated with the initial streamline passing through the selected origin. In particular, we will direct the axis \( O\hat{x}_1 \) tangentially to the streamline at this flow region point.

Then, using the approach proposed in [12], taking into account (7), it can be shown that, in the dimensionless form, the equations of dynamics for the particular case of constant transverse viscosity (\( \eta = 0 \)) and the continuity condition are written as

\[
\frac{\partial u'_i}{\partial t'} = -K_3 \cdot \frac{\partial P'}{\partial x'_i} + K_5 \cdot F'_i + \frac{3}{2} \sum_{j=1}^{3} \left\{ \frac{2}{K_4} \cdot \frac{\partial e'_{ij}}{\partial x'_j} - u'_j \cdot \frac{\partial u'_i}{\partial x'_j} \right\} + 4 \cdot \frac{K^{(0)}_4}{K_4} \cdot \frac{\partial}{\partial x'_j} \sum_{k=1}^{3} e'_{ik} \cdot e'_{kj}; \quad i = 1, 2, 3; \quad (8)
\]

\[
\frac{\partial u'_i}{\partial x'_1} + \frac{\partial u'_2}{\partial x'_2} + \frac{\partial u'_3}{\partial x'_3} = 0.
\]

In this case, condition (5) in the dimensionless form of the record can be represented as

\[
[|I'_2|] = \left[ \frac{I_2}{I_2} \right] = K_2 > K_{2g}; \quad (9)
\]

\[
I'_2 = K_{inv}^2 \cdot (e'_{11} \cdot e'_{22} + e'_{22} \cdot e'_{33} + e'_{33} \cdot e'_{11} - e'_{12}^2 - e'_{23}^2 - e'_{31}^2); \quad e'_{ij} = \frac{1}{2} \left( \frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i} \right);
\]

\[
i, j = 1, 2, 3.
\]

Here

\[
t' = \frac{t}{t_s}; \quad P' = \frac{P}{P_s}; \quad x'_i = \frac{x_i}{L_s}; \quad F'_i = \frac{F_i}{F_s}; \quad u'_j = \frac{u_j}{u_s}; \quad t_s = \frac{L_s}{u_s}; \quad L_s = \frac{\mu \cdot E_s}{\rho \cdot u_s \cdot D_s}; \quad P_s = \frac{\rho^2 \cdot u_s^5}{\mu \cdot E_s};
\]

\[
I_s = \left( \frac{E_s}{\rho \cdot u_s} \right)^2; \quad K_4 = \frac{K^{(0)}_4}{D_s}; \quad K_{2g} = \frac{12 \eta}{I_s}; \quad K_3 = \frac{K_{inv}^2}{K_4} \cdot \frac{\rho \cdot u_s^2 \cdot D_s^2}{\mu^2 \cdot E_s^2};
\]

\[
\bar{E} = \text{grad} \left\{ P + \frac{\rho \cdot u_s^2}{2} \right\}; \quad \bar{D} = \text{grad} \left\{ 2 \cdot \mu \cdot \sqrt{|I'_2|} \right\}; \quad u = \sqrt{u_1^2 + u_2^2 + u_3^2}.
\]
where \( u_s \) is taken as a scale value of the fluid velocity at the origin and at the initial time; \( t_s \); \( P_s \); \( L_s \); \( F_s \) are scale values for time, pressure, geometrical dimensions and body forces; \( \rho \) is fluid density; \( E_s \), \( D_s \) are vector modules \( \bar{E}, \quad \bar{D} \), respectively; \( K_1 \), \( K_{28} \), \( K_3 \), \( K_4^{(0)}, K_5 \) are independent dimensionless complexes.

It should be noted that all the dimensionless complexes introduced above are composed exclusively of invariant values. In this case, no specific geometrical dimensions of the initial flow region are used. This applies, among other things, to the choice of a scale value \( L_s \) for the linear coordinates characteristic.

The subscript \( s \) in the last relationships indicates that the respective functions have been calculated at the considered spatial point (the origin) for \( t = 0 \).

In perfect analogy, for the particular case of linear dependence (6) of the transverse viscosity \((q = I)\) on the strain rate tensor second invariant, the fluid dynamics equations with the rheological model (7) can be written as

\[
\frac{\partial u'_i}{\partial t'} = -K'_3 \cdot \frac{\partial P'}{\partial x'_i} + K'_5 \cdot F'_i + \frac{3}{2} \sum_{j=1} \left( \frac{2}{K'_1} \frac{\partial \varepsilon''_{ij}}{\partial x'_j} - u'_j \cdot \frac{\partial u'_i}{\partial x'_j} - 4 \cdot K^{(1)}_4 \frac{1}{K'_3} \frac{\partial}{\partial x'_j} \left( K'_2 - K''_{2g} \right) \sum_{k=1}^{3} \varepsilon''_{ik} \cdot \varepsilon''_{kj} \right); \quad i = 1, 2, 3;
\]

\[
K^{(1)}_4 = \frac{\eta_1 \cdot D^2_s}{\rho \cdot \mu^2}.
\]

For the general case of the transverse viscosity power dependence on the strain rate tensor second invariant for an arbitrary value of the parameter \( q \), the fluid dynamics equations acquire the form of

\[
\frac{\partial u'_i}{\partial t'} = -K'_3 \cdot \frac{\partial P'}{\partial x'_i} + K'_5 \cdot F'_i + \frac{3}{2} \sum_{j=1} \left( \frac{2}{K'_1} \frac{\partial \varepsilon''_{ij}}{\partial x'_j} - u'_j \cdot \frac{\partial u'_i}{\partial x'_j} + 4 \cdot K^{(q)}_4 \frac{1}{K'_3} \frac{\partial}{\partial x'_j} \left( K'_2 - K''_{2g} \right) q \cdot \sum_{k=1}^{3} \varepsilon''_{ik} \cdot \varepsilon''_{kj} \right); \quad i = 1, 2, 3;
\]

\[
K_2 > K'_{2g} \quad ; \quad K^{(q)}_4 = \frac{\eta_q \cdot D^2_s \cdot E^2_q}{\rho^{2q-1} \cdot \mu^2 \cdot u^2_q};
\]

The systems of equations presented for particular cases describe the secondary flows development in a small neighborhood of the point in question.

In [12], a basic scheme for constructing an approximate solution of equations of type (8) or (10) was considered in the form of

\[
u_i(x_1, x_2, x_3, t) = U^{(0, i)}(x_1, x_2, x_3) + U^{(1, i)}(x_1, x_2, x_3, t); \quad i = 1, 2, 3;
\]

\[
P(x_1, x_2, x_3, t) = P^0(x_1, x_2, x_3) + P^{(1)}(x_1, x_2, x_3, t),
\]
where \( U^{(0,i)} \), \( p^{(0)} \) are initial distributions of the velocity and pressure vector components in a small neighborhood of the spatial point in question; \( u^{(t,i)} \), \( p^{(t)} \) are non-stationary components for the velocity and pressure fields that “arise” in a small neighborhood of the point in question at the initial stage of flow development from the initial state.

The further approach was based on the representation of the initial velocity components distribution, as well as non-stationary components for pressure and velocity components in the minor neighborhood of the point in question (the origin) through space coordinate power expansion

Let us represent the initial velocity component distributions and non-stationary components for the pressure and velocity components in the small neighborhood of the point in question (zero point) through space coordinate power expansion

\[
U^{(0,i)}(x_1,x_2,x_3) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} U_{kmn}^{(0,i)} x_1^k x_2^m x_3^n;
\]

\[
p^{(t)}(x_1,x_2,x_3,t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{kmn}^{(t)} x_1^k x_2^m x_3^n;
\]

\[
u^{(t,i)}(x_1,x_2,x_3,t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{kmn}^{(t,i)} x_1^k x_2^m x_3^n; \quad i=1,2,3.
\]

where \( U_{kmn}^{(0,i)} \) are the known expansion coefficients; \( u_{kmn}^{(t,i)} \), \( p_{kmn}^{(t)} \) are the expansion coefficients representing the yet unknown time functions and which satisfy zero initial conditions.

Naturally, the non-stationary components for the pressure (11) and the velocity vector component must satisfy the corresponding boundary conditions. It is proposed to place these conditions at certain nodal points, which must be chosen in the vicinity of the flow region spatial point in question. In this case, the fulfillment of the boundary conditions for pressure must be required at three, and for the velocity components - at eight nodal points. The chosen nodal points should form a kind of “final element” which should “cover” the entire flow area. At the same time, it is necessary to fulfill the conditions of velocity and pressure components conjugation in the adjacent node points of the neighboring finite elements.

After linearization of the initial dynamics equations and with the involvement of boundary conditions at the corresponding nodal points of the finite element in question, the following can be shown. For the case of the simplest approximation, all the unknown expansions coefficients in (11) in a small neighborhood of the flow region spatial point in question can be found, at least at the principle level, through one of the functions such as \( u_{000}^{(t,2)}(t) \). In turn, this function should be a solution of a linear equation of the following type

\[
\frac{d^2 u_{000}^{(t,2)}}{dt^2} + a_1 \frac{du_{000}^{(t,2)}}{dt} + a_0 u_{000}^{(t,2)} = H(t).
\]

The coefficients \( a_0, a_1 \) in the equation (12) as well as the function \( H(t) \) included in it should be determined by the entire set of boundary and initial conditions, as well as the initial parameters of the problem, including the immeasurable complexes \( K_1, K_2, K_2^{*}, K_3, K_5 \) and \( K_4^{(q)} \) (depending on the selected parameter value \( q \) when representing the transverse viscosity by power dependence on the strain rate tensor second invariant).

The solution of the equation (12) with the corresponding initial conditions within the framework of one “final” element which is a neighborhood of the point in question, assumes, depending on the
coefficients values $a_0$, $a_1$, quite different variants with different interpretations. Therefore, the further evolution of emerging secondary flows which will ultimately be determined by the solution such as $u_{000}^{(1,2)}(t)$, can develop under different scenarios.

Depending on the roots values of the characteristic equation of differential equation (12), there are three main options for the secondary flow development.

First of all, note the situation in which the “generated” transverse velocity components simply fade out.

We cannot exclude the option when the secondary flow forms, depending on the evolving conditions caused by the set of the initial parameters, will then translate the initial laminar flow pattern into some other flow pattern, but still within the laminar flow pattern. It can be expected that such a scenario takes place during the formation of the toroidal vortex structures during the rotational fluid flow in the gap between the two coaxial cylinders [1-4].

However, it can be assumed that a third option is possible, when these “generated” transverse velocity components increase “indefinitely” (at least at the model level) and can lead to the beginning of the laminar - turbulent transition. Apparently, one can talk about only the initial stage of the transition, when a sequential, generally speaking, continuous cascade of velocity field deformation stages is realized with “generating” at each of them the transverse velocity components.

4. Conditions for the secondary flows formation and the beginning of a laminar-turbulent transition

Note that it is difficult to obtain an analytical form of the conditions imposed on the initial process parameters, including the set of dimensionless complexes $K_1$, $K_2$, $K_2$, $K_3$, $K_4(q)$, $K_5$ under which one of the three options for the flow development considered in the previous section is realized. In this regard, in [12], based on the results of processing a sufficiently large number of experimental data, such conditions were proposed in empirical form.

In particular, the condition of the transverse viscosity factor “addition” and, accordingly, “generating” the transverse velocity components was represented as follows

$$K_2 > K_2 = k_0 + \frac{k_1}{K_3} - k_2; \quad k_0 = 0.221; \quad k_1 = 8.572; \quad k_2 = 84.015. \tag{13}$$

As noted above, the secondary flows “generation” does not always lead to the onset of the laminar flow transition to a turbulent one. Therefore, the condition (13) is proposed to be considered as necessary, although not sufficient for the transition occurrence.

Thereupon, in addition to (13), another empirical condition was proposed

$$K_{I_{max}} > q_0 \cdot (K_3)^q_l; \quad q_0 = 9.382 \cdot 10^{-4}; \quad q_l = 2.008. \tag{14}$$

where $K_{I_{max}}$ represents the maximum value of the dimensionless complex $K_1$. Here it is assumed that the value $K_{I_{max}}$ is reached at some spatial point in that part of the flow region where condition (13) is satisfied.

In the case of choosing an a priori whole-number value of the exponent $q_l = 2$, which may be more convenient for further use of this condition in analytical ratios, and after re-processing the experimental data, we arrive at the following value of the proportionality coefficient $q_0 = 9.857 \cdot 10^{-4}$.

In each task in identification of the conditions for the laminar-turbulent transition start, the dimensionless complexes introduced above should be represented by the functions not only of the local reference system spatial coordinates, but also of some additional dimensionless parameters set.
that give a narrower definition of the problem under consideration. In particular, one of these parameters, in addition to all the others, can be the traditional Reynolds number. In other words, the main dimensionless complexes structurally have the form of functions of the following arguments, regardless of the additional dimensionless parameters.

\[ K_i = K_i(x_1, x_2, x_3, Re); \quad i = 1, 2, ..., 5. \]

In this situation, along with the conditions presented above, there may be another additional requirement. In particular, to initiate a laminar-turbulent transition, it is rational to require that, at the complex \( K_i \) maximum point in terms of spatial coordinates, the condition of the maximum value for the following relationship must be satisfied

\[
\left( \frac{K_2}{K_2} \right) \rightarrow \text{max}.
\]

This condition should provide the maximum manifestation of the “generating” the velocity transverse components effect. The latter requirement can be achieved, in particular, by appropriate selection of the Reynolds number or, possibly, other additional dimensionless parameters of the problem in question.

When satisfied at the considered spatial point of the flow region area, taking into account the conditions (13) - (15), this point itself and its small neighborhood begin to act as a startup of the laminar-turbulent transition. If only one condition (13) is satisfied, then in the neighborhood of such spatial point one can only expect transverse velocity components generation and, possibly, the subsequent formation of a new laminar secondary flow.

5. Conclusion

Consideration of the transverse viscosity assumes, at the level of solving the fluid dynamics problems, the transverse velocity components formation relative to the streamlines of the main flow. In this case, we mean the streamlines that could be obtained by solving the same boundary value problem, but without the transverse viscosity factor consideration. “Generation” of such transverse velocity components is proposed to be regarded as the initial stage of the secondary flow. In this case, secondary flows, as a rule, are not observed at sufficiently “slow” modes, but arise in “faster” flows.

Given these circumstances, a rationale for a rheological fluid model is proposed, which implies a threshold "addition" of the transverse viscosity factor. It is proposed to use this rheological model for describing the initial stage of secondary flows development in a small neighborhood area of the spatial point in question.

For the case of the power dependence of the transverse viscosity (with an arbitrary exponent) on the modulus exceedance of the strain rate tensor second invariant of the corresponding critical level, the fluid dynamics equations with the rheological model of this kind were obtained. At a qualitative level, an interpretation of various options for the flow development in a small neighborhood area of the flow region point in question, in which the second invariant modulus exceeds the threshold value is offered.

The previously proposed empirical conditions that can be used in predicting the secondary laminar flows formation are given. The phenomena similar to the secondary flows formation (in the sense of “generating” the transverse velocity components relative to the initial laminar streamlines) also occur at the laminar-turbulent transition initial stage.

References

[1] Taylor G I 1923 Stability of a Viscous Liquid contained between Two Rotating Cylinders *Phil. Trans. Royal Society. London Series A* 223 pp 289-343 Another reference

[2] Coles D 1965 Transition in circular Couette flow *Journal of Fluid Mechanics* 21 Issue 03 pp
385- 425

[3] Gollab J P and Swinney H L 1975 Onset of turbulence in a rotating fluid Phisical Review Letters 36 N 14 pp 927 – 930

[4] Andereck C D, Liu S S and Swinney H L 1986 Flow regimes in circular Couette system with independently rotating cylinders. Journal of Fluid Mechanics 164 March pp 155-183

[5] Goldschtick M A and Stern V N 1977 Hydrodynamic Stability and Turbulence (Novosibirsk: Nauka Publishers) p 367

[6] Zhigulev V N and Tumin A N 1987 Origination of Turbulence. (Novosibirsk: Nauka Publishers) p 283

[7] Astarita G and Marrucci G 1974 Principles of non-Newtonian Fluid Mechanics (London: McGraw-Hill) p 309

[8] Litvinov V G 1982 Motion of Non-linear Viscous Fluid (Moscow: Nauka Publishers) p 376

[9] Vinogradov G V and Malkin A J 1977 Polymer Rheology. (Moscow: Chemistry) p 439

[10] Kolodezhnov V N 2012 On a Possible Model of the Initial Stage of the Laminar-turbulent Transition The Bulletin of Voronezh State Technical University (Voronezh: Voronezh State Technical University Press) 8 N 5 pp 25-30

[11] Huppler J.D. 1965 Experimental determination of the secondary normal stress difference for aqueous polymer solutions Transactions of the Society of Rheology 9 N 2 pp 273 – 286

[12] Kolodezhnov VN 2018. An interpretation of the Laminar-Turbulent Transition Startup against the Consideration of the Transverse Viscosity Factor Journal of Physics Conference Series 973 (012009) pp 1-15