Calculus on random integral mappings $I_{(a,b]}^{h,r}$ and their domains*

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October 2, 2013

Abstract. It is proved that the random integral mappings (some type of functionals of Lévy processes) are isomorphisms between convolution semigroups of infinitely divisible measures. However, the inverse mappings are no longer of the random integral form. Domains are characterized in many ways. Compositions (iterated integrals) can be expressed as a single random integral mapping. Finally, obtained results are illustrated by examples.

Mathematics Subject Classifications (2010): Primary 60E07, 60H05, 60B11; Secondary 44A05, 60H05, 60B10.

Key words and phrases: Lévy process; infinite divisibility; Lévy-Khintchine formula; Lévy (spectral) measure; stable measure; Lévy exponent; random integral; Fourier transform; tensor product; image measures; product measures; Banach space.

Abbreviated title: Calculus on random integral mappings

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*Research funded by Narodowe Centrum Nauki (NCN) Dec2011/01/B/ST1/01257
†Part of this work was done when Author was visiting Indiana University, Bloomington, USA in Spring 2013.
For an interval \((a, b]\) in the positive half-line, two deterministic functions \(h\) and \(r\), and a Lévy process \(Y_{\nu}(t), t \geq 0\), where \(\nu\) is the law of random variable \(Y_{\nu}(1)\), we consider the following mapping

\[\nu \mapsto I_{h,r}^{(a,b]}(\nu) := \mathcal{L}\left( \int_{(a,b]} h(t) \, dY_{\nu}(r(t)) \right), \quad (\ast)\]

where \(\mathcal{L}\) denotes the probability distribution of the random (stochastic) integral. One of the problems related to (\(\ast\)) is to show that the mappings \(I_{h,r}^{(a,b]}\) are one-to-one and to characterize their domains. We consider here both questions for fairly general classes of functions \(h\) and \(r\) and measures \(\nu\) (Lévy processes \(Y_{\nu}\)) on a real separable Banach space.

Let us recall that over the past decades the method of describing a given measure as a probability distribution of an integral (\(\ast\)) was successfully applied in many instances. Already in Jurek-Vervaat (1983) it was proved that in order that

\[a_n(\xi_1 + \xi_2 + \ldots + \xi_n) + x_n \Rightarrow \mu, \quad (\ast\ast)\]

for some infinitesimal triangular array \(a_n\xi_j, 1 \leq j \leq n; n \geq 1\), it is necessary and sufficient that

\[\mu = \mathcal{L}\left( \int_{(0,\infty)} e^{-t} dY_{\nu}(t) \right) \equiv I_{(0,\infty)}(\nu), \quad (\ast\ast\ast)\]

where the Lévy process \((Y_{\nu}(t), 0 \leq t < \infty)\) is such that \(\nu\) has finite logarithmic moment. The expression (\(\ast\ast\ast\)) was called a random integral representation of the selfdecomposable (or Lévy class \(L\)) measure \(\mu\) and \(Y_{\nu}\) was referred to as the background driving Lévy process (BDLP) of \(\mu\).

The phenomena of identifying a class of limit distributions with a collection of laws of random integrals (\(\ast\)) was proved for many other limiting schemes. In Jurek (1988),(1989) and Jurek-Schreiber (1992) almost the whole class \(ID\), of all infinitely divisible probability measures, was described as a sum of increasing subsemigroups. More precisely, if

\[U_\beta := \{ I_{(0,1]}^{t,\beta} (\nu) = \mathcal{L}\left( \int_{(0,1]} t \, dY_{\nu}(t^\beta) \right) : \nu \in ID \},\]

then

\[ID = \overline{\bigcup_{\beta_n} U_{\beta_n}}\]

for any sequence \(\beta_n\) increasing to infinity and the bar means closure in the weak topology.

Still later on, many new classes of probability distributions were simply defined as laws of some random integrals analogous to (\(\ast\)) without any
reference to limiting procedures. We illustrate that approach with two papers. Sato (2006) introduced two specific families of random integrals on $\mathbb{R}^d$ by specifying the inner clock $r$ in $(\ast)$. One of them had the time change function

$$r(t) := \int_t^{\infty} u^{-\alpha-1} e^{-u} du,$$

and the space scaling $h(t) = t$. On the other hand, in Maejima, Perez-Abreu and Sato (2012), the authors introduced subclasses of infinitely measures by specifying the map $(\ast)$ not in terms of measures $\nu$ but in terms of their Lévy (spectral) measures $M$; comp. formula 10 (iii) below. Using the arcsine density for the time change $r$, the authors introduced two transforms $A_1$ and $A_2$ and defined the corresponding subclasses of infinitely probability measures. The one given by $A_2$ gives the mapping $(\ast)$ with $h(t) = t$ and $r$ as the cumulative distribution function of the arcsine density on $(0,1)$.

In this paper we propose a quite general approach to random integral representations and mappings. For a more convenient way of navigating the body of research in random representations and an ease of comparing results of different authors we propose here a new and unified form of definitions and notations.

Finally, we will utilize here the notion and properties of image measures, in particular, images under the tensor product of functions. Our results are new on $\mathbb{R}^d$ (Euclidean space). Most of proofs are given in the generality of measures on real separable Banach spaces. However, no essential knowledge of the functional analysis is required.

Last but not least, the proposed calculus on random integral mappings and their domains might be formally viewed as an analogue of the calculus on linear operators on Banach spaces and their domains (in functional analysis).

0. Notations and brief descriptions of results.

0.1. Notations and basic facts

$E$ is a real separable Banach space;
$E'$ is the topological dual of $E$;
$\langle \cdot, \cdot \rangle$ is the dual pair (scalar product) between $E'$ and $E$;
$\Rightarrow$ denotes the weak convergence of probability measures;
$\mathcal{L}(X)$ is the probability distribution of random variable $X$;
$\overset{d}{=} \text{means equality in distribution};$
$D_F[a, b]$ denotes the Skorohod space of $F$-valued functions that are right continuous on $[a, b]$ with left-hand limits on $(a, b)$; in short: $\text{cadlag}$ functions; $F$ is a complete separable metric space;
$(Y_\nu(t), 0 \leq t < \infty)$ denotes a Lévy process such that $\mathcal{L}(Y_\nu(1)) = \nu$;
$ID(E)$ denotes the set of all infinitely divisible Borel measures on $E$; 
$\hat{\mu}$ is the characteristic functional (Fourier transform) of $\mu$;
$\Phi(y) = \log \hat{\mu}(y)$ is the Lévy exponent of $\mu \in ID(E)$, ($y \in E'$);
$\Phi(y) = i<y, z_0> - \frac{1}{2} <y, R_y>$
$+ \int_{E \setminus \{0\}} (e^{i<y, x>} - 1 - i<y, x> - 1 - \frac{1}{2} <y, R_y>) M(dx)$ (Lévy-Khintchine formula);
($z_0 \in E$, $R$ is a Gaussian covariance operator, $M$ is a Lévy (spectral)
measure and $B$ is the unit ball in $E$);
$\nu = [z_0, R, M]$ means $\nu \in ID$ with the triple from its Lévy-Khintchine
formula;
$\nu^c = [z_0, R, M]^c := [c \cdot z_0, c \cdot R, c \cdot M]$ for $c > 0$;
$(f \mu)$ denotes the image of a measure $\mu$ under a measurable mapping $f$;
$(f \otimes g)(s_1, s_2) := f(s_1) \cdot g(s_2)$ (tensor product) for $(s_1, s_2) \in S \times S$
and $f, g : S \rightarrow \mathbb{R}$;
$I^{h,r}_{(a,b)}$ is the random integral mapping with a space transform function $h$,
a deterministic monotone time change $r$ (an inner clock) and the time
interval $(a, b]$; cf. (1) below;
$D^{h,r}_{(a,b)}$ is the domain of the integral mapping $I^{h,r}_{(a,b)}$;
$R^{h,r}_{(a,b)} := I^{h,r}_{(a,b)}(D^{h,r}_{(a,b)}) \subset ID$ denotes the range of the mapping $I^{h,r}_{(a,b)}$;
$I^{h,r}_{(a,b)}(\nu)$ means weak limit of $I^{h,r}_{(a,b)}(\nu)$ as $b \rightarrow \infty$;
$\Phi^{h,r}_{(a,b)}(y) := \log \hat{I^{h,r}_{(a,b)}}(\nu)(y)$ when $\Phi = \log \hat{\nu}(y)$;
$I^{h,r}_{(a,b)}([z, R, M]) := [z^{h,r}_{(a,b)}, R^{h,r}_{(a,b)}, M^{h,r}_{(a,b)}]$; cf. (10);

0.2. Summary of results
In section 1 the random integral and its basic properties are given. In Theorem 1, in section 2, we proved that (some) mappings $I^{h,r}_{(a,b)}$ are isomorphisms,
of the corresponding measure convolution subsemigroups of the semigroup
$ID(E)$, but not always (Remark 3). An alternative approach on $\mathbb{R}^d$, for
retrieving the measures from random integral mappings, is given in Proposition
2. Then, in section 3, we discuss the domains $D^{h,r}_{(a,b)}$ of mappings $I^{h,r}_{(a,b)}$
(Propositions 3 - 6). In section 4, Theorem 2 shows that all compositions
of $I^{h,r}_{(a,b)}$ (iterated integral mappings) can be expressed as a single integral
random mapping. Here the language of tensor products and the notion of
image measures are very convenient. In section 5, the inverse mappings to
$I^{h,r}_{(a,b)}$ are discussed in Theorem 3. However, they are no longer of the random
integral form $I^{h,r}_{(a,b)}$. Section 6, in particular Proposition 7, is devoted to fixed
points of mappings $I^{h,r}_{(a,b)}$ and to the role of the stable distributions. In section
7, the factorization property of measures is discussed (Proposition 8). As a
consequence we get that the selfdecomposable (in other words class L distribu-
tions) measures have the factorization property (Corollary 12). In the last 
section (section 8) we illustrate our results on some new or previously studied 
integral mappings and semigroups.

1. A path-wise random integral mappings.

1.1. Integrals on finite intervals

For an interval \((a, b]\) in a positive half-line, a real-valued continuous of bo-
und variation function \(h\) on \((a, b]\), a positive non-decreasing right-continuous 
(or non-increasing left-continuous) time change function \(r\) on \((a, b]\) and a ca-
dlag Lévy stochastic processes \((Y_\nu(t), 0 \leq t < \infty)\), let us define via a formal 
integration by parts formula the following random integral

\[
\int_{(a,b]} h(t) dY_\nu(r(t)) := h(b)Y_\nu(r(b)) - h(a)Y_\nu(r(a)) - \int_{(a,b]} Y_\nu(r(t) -) dh(t) \in E, \quad (1)
\]

and the corresponding random integral mapping

\[
\nu \to I^{h,r}_{(a,b]}(\nu) := \mathcal{L}\left( \int_{(a,b]} h(t) dY_\nu(r(t)) \right) \in ID, \quad (2)
\]

with \(\nu\) in its domain \(\mathcal{D}^{h,r}_{(a,b]}\) being a subset of the class ID consisting of those 
measures \(\nu\) for which the integral (1) is well defined. In that case, the law in 
(1) is infinitely divisible; cf. Jurek-Vervaat (1983), Lemma 1.1.

Remark 1. (i) Note that if the random integral \(\int_{(a,b]} h(t) dY_\nu(r(t))\) is well 
defined then so are random integrals \(\int_{[c,d]} h(t) dY_\nu(r(t))\), where \(a \leq c < d \leq b\).

(ii) Since the random integral \(\int_{(a,b]} h(t) dY_\nu(r(t))\) is a functional of the pro-
cess on \((a, b]\), thus if two Lévy processes \(\widetilde{Y}_\nu\) and \(Y_\nu\) have the same probability 
distribution, that is, \((\widetilde{Y}_\nu(t) : t \geq 0) \overset{d}{=} (Y_\nu(t) : t \geq 0)\), then

\[
\mathcal{L}\left( \int_{(a,b]} h(t) dY_\nu(r(t)) \right) = \mathcal{L}\left( \int_{(a,b]} h(t) d\widetilde{Y}_\nu(r(t)) \right).
\]

(iii) Since Lévy processes are semi-martingales the random integral (1) 
can be defined as an Ito stochastic integral. However, for our purposes we 
do not need that generality of stochastic calculus.

1.2. Lévy exponents (characteristic functions) of random mappings
If \( \nu \in \mathcal{D}_{[a,b]}^{h,r} \) and \( I_{[a,b]}^{h,r}(\nu) \) have the Lévy exponents \( \Phi \) and \( \Phi_{[a,b]}^{h,r} \), respectively, then, from already mentioned in Lemma 1.1 in Jurek-Vervaat (1983), we get
\[
\Phi_{[a,b]}^{h,r}(y) = \int_{[a,b]} \Phi(h(t)y)dr(t), \quad y \in \mathbb{E}' \quad (\text{for non-decreasing } r).
\]
Similarly we have that
\[
\Phi_{[a,b]}^{h,r}(y) = \int_{[a,b]} \Phi(-h(t)y)|dr(t)|, \quad y \in \mathbb{E}' \quad (\text{for non-increasing } r),
\]
because for \( 0 < u < w \), we have \( \mathcal{L}(Y_{\nu}(u) - Y_{\nu}(w)) = (\nu^{-})^{(w-u)} \) where \( \nu^{-} := \mathcal{L}(-Y_{\nu}(1)) \). In other words, \( (-Y_{\nu}(t), t \geq 0) \equiv (Y_{\nu^{-}}(t), t \geq 0) \).

### 1.3. Improper random integrals
Integrals over intervals \((a,b)\) or \((a,\infty)\) or \([a,b]\) and others are defined as week limits of integrals over intervals \((a,b]\) in (1). Thus, the random integral \( \int_{(a,\infty)} h(t)dY_{\nu}(r(t)) \) is well-defined if and only if the function
\[
\mathbb{E}' \ni y \rightarrow \int_{(a,\infty)} \Phi(h(t)y)dr(t) \in \mathbb{C}
\]
is a Lévy exponent.

Or equivalently, the three parameters \( z_{(a,\infty)}^{h,r}, R_{(a,\infty)}^{h,r} \) and \( M_{(a,\infty)}^{h,r} \) in Lévy-Khintchine formula are well-defined; cf. (12) below.

### 1.4. Different graphic notations
Note that we have
\[
I_{[a,b]}^{h,r}(\nu) = \mathcal{L} \left( \int_0^{\infty} 1_{(a,b]}(r^*(s))h(r^*(s))dY_{\nu}(s) \right) = I_{(0,\infty)}^{\tilde{h}(s), \nu} \equiv \tilde{I}^{\nu},
\]
where \( \tilde{h}(s) := 1_{(a,b]}(r^*(s))h(r^*(s)) \) and \( r^* \) is the inverse function of \( r \).

However, instead of that above graphicaly simpler notation, for a greater flexibility of our considerations, we will keep the three parameters: the time interval \((a, b]\), the space-valued normalization \( h \) and the inner time change \( r \) in symbols and notions related to the random integral mappings (1).

For improper random integrals with decreasing \( r \) with \( r(a+) < \infty \) we have
\[
I_{[a,b]}^{h,r}(\nu) = I_{[a,b]}^{-h,r(a+)-r}(\nu) = I_{[a,b]}^{h,r(a+)-r}(\nu),
\]
that is,
\[
\int_{[a,b]} h(t)dY_{\nu}(r(t)) \equiv \int_{[a,b]} h(t)dY_{\nu^{-}}(r(a+) - r(t)),
\]
and \( t \to r(a+) - r(t) \) is a positive increasing function.

### 2. Properties of random integral mappings.

#### 2.1. The isomorphism property
THEOREM 1. Assume that \( h(a) := h(a^+) \), \( r(a) := r(a^+) \) exists in \( \mathbb{R} \), \( r \) is continuously differentiable in \((a, b)\) and \( h \neq 0 \) on \([a, b]\). Then the mapping

\[
D_{(a, b)}^{h,r} \ni \nu \mapsto I_{(a, b)}^{h,r}(\nu) \in \mathcal{R}_{(a, b)}^{h,r}
\]

is a continuous isomorphism between the corresponding measure convolution semigroups.

Proof. Let \( \rho := I_{(a, b)}^{h,r}(\nu) \) and let \( \Phi^{h,r}_{(a, b)} \) and \( \Phi \) be the Lévy exponents of \( \rho \) and \( \nu \), respectively. Note that for fixed \( y \in E' \), the function \( [a, b] \ni s \mapsto \Phi(h(s)y)r'(s) \) is continuous. By the Mean Value Theorem there exist \( c_y \in (a, b) \) such that

\[
\Im \Phi^{h,r}_{(a, b)}(s) = \int_{[a, b]} \Im \Phi(h(s)y)dr(s) = \int_{[a, b]} \Im \Phi(h(s)y)r'(s)ds = (b - a)\Im \Phi(h(c_y)y)r'(c_y),
\]

where \( \Im \) stands for the imaginary part of a complex number. Therefore we have that \( \Im \Phi(y) = [(b-a)r'(c_y)]^{-1}\Im \Phi^{h,r}_{(a, b)}(y/h(c_y)) \). Analogous equality holds for the real parts of \( \Re \Phi \) and \( \Re \Phi^{h,r}_{(a, b)} \). Consequently, we get that \( \rho \) uniquely determines \( \nu \), which proves the one-to-one property.

The homomorphism property of \( I_{(a, b)}^{h,r} \), that is, the equality

\[
I_{(a, b)}^{h,r}(\nu_1 \ast \nu_2) = I_{(a, b)}^{h,r}(\nu_1) \ast I_{(a, b)}^{h,r}(\nu_2),
\]

in terms of the corresponding Lévy exponents, follows from (3) or (4).

For the continuity, let us note that \( 0 \leq |r(b) - r(a)| < \infty \) and the cadlag property imply that functions \( t \mapsto Y(r(t)) \) are bounded and with at most countable many discontinuities; cf. Billingsley (1968), Chapter 3, Lemma 1. Furthermore, the mapping

\[
D_E[a, b] \ni y \mapsto \int_{(a, b]} h(t)dy(r(t)) := h(b)y(r(b)) - h(a)y(r(a)) - \int_{[a, b]} y(r(t) -)dh(t) \in E,
\]

is continuous in Skorohod topology (for details see Billingsley (1968), p. 121.). Furthermore, if \( \nu_n \Rightarrow \nu \) then \( (Y_{\nu_n}(t), a \leq t \leq b) \Rightarrow (Y_{\nu}(t), 0 \leq t \leq b) \) in \( D_E[a, b] \). Consequently, we have

\[
\mathcal{L} \left( \int_{(a, b]} h(t)dY_{\nu_n}(r(t)) \right) \Rightarrow \mathcal{L} \left( \int_{(a, b]} h(t)dY_{\nu}(r(t)) \right)
\]

which proves the continuity of \( I_{(a, b)}^{h,r} \) and completes the proof of Theorem 1.
Remark 2. (i) In some specific cases as $I_{e^{-t} \cdot}^{c,1}$ or $I_{L^{1}}^{t,1}$, the one-to-one property can be proved by Fourier or Laplace transforms; cf. Jurek-Vervaat (1983) or Jurek (1988), (2007).

(ii) Weak convergence continuity of the mappings $\nu \rightarrow I_{(a,b)}^{h,r}(\nu)$, for measures on finite dimensional linear spaces, easily follows by the characteristic functional argument.

**COROLLARY 1.** For any $s > 0$, $\nu \in D_{(a,b)}^{h,r}$ if and only if $\nu^{**} \in D_{(a,b)}^{h,r}$, and
$$I_{(a,b)}^{h,r}(\nu^{**}) = (I_{(a,b)}^{h,r}(\nu))^{**} = (I_{(a,b)}^{h,r}(\nu)).$$

For $u \in \mathbb{R}$ and the dilation operator $T_{u}$, $\nu \in D_{(a,b)}^{h,r}$ if and only if $T_{u}\nu \in D_{(a,b)}^{h,r}$, and
$$T_{u}(I_{(a,b)}^{h,r}(\nu)) = I_{(a,b)}^{h,r}(T_{u}\nu) = I_{(a,b)}^{u+h,r}(\nu).$$

For bounded linear operator $A$ on $E$ and $\nu \in D_{(a,b)}^{h,r}$ we have that $A\nu \in D_{(a,b)}^{h,r}$ and
$$A(I_{(a,b)}^{h,r}(\nu)) = I_{(a,b)}^{h,r}(A\nu).$$

These are consequences of the formula (3) and (4).

2.2. **Convolution factors**

We say that probability measures $\mu$ on $E$ is a convolution factor of a measure $\rho$ if there exists a measure $\nu$ such that $\mu \ast \nu = \rho$; in symbols we write $\mu \prec \rho$.

**PROPOSITION 1.** For $\nu \in D_{(a,b)}^{h,r}$, the family $\{I_{(a,x]}^{h,r}(\nu) : a < x < b\}$ is sequentially shift convergent for $x \uparrow c \leq b$ or $x \downarrow c \geq a$.

**Proof.** Note that if $a < x_{1} < x_{2} < \ldots < x_{n} < \ldots \uparrow c \leq b$ then
$$I_{(a,x_{1})}^{h,r}(\nu) < I_{(a,x_{2})}^{h,r}(\nu) < \ldots < I_{(a,b]}^{h,r}(\nu),$$
and by Theorem 5.3 in Parthasarathy (1968) there exist sequence $\delta_{n}$ and a measure $\rho$ such that $I_{(a,x_{n})}^{h,r}(\nu) \ast \delta_{n} \Rightarrow \rho$ as $n \rightarrow \infty$. Similarly we argue in the remaining case.

2.3. **Retrieving the measure $\nu$**

Knowing integrals (1) over a family of intervals $(c, x]$, with $x \downarrow c$, we can retrieve the measure $\nu$, as it is seen below.

**PROPOSITION 2.** Let $\nu \in ID(R^{d})$, $r$ is continuously differentiable and there exists $c \in (a, b)$ such that $h(c) \neq 0$ and $r'(c) \neq 0$. Then
$$L\left(\int_{c}^{x} \frac{h(t)}{h(c)} dY_{\rho}(r(t)\frac{r'(c)}{r'(c)}) \right)^{1-c} \Rightarrow \nu, \quad as \ x \downarrow c. \quad (7)$$
Proof. The weak convergence in (6), for measures \( \nu \) on \( \mathbb{R}^d \), in terms of Lévy exponents is equivalent to the following claim

\[
\lim_{x \to c} \frac{1}{x - c} \int_c^x \Phi\left( \frac{h(t)}{h(c)} y \right) \frac{r'(t)}{r'(c)} dt = \Phi(y) \quad \text{for all } y,
\]

that is obviously true because of de l’Hospital rule.

COROLLARY 2. For a measure \( \nu \in ID(\mathbb{R}^d) \),

\[
\mathcal{L}\left( \int_a^{a+1/n} h(t) dY_\nu(r(t)) \right)^* \Rightarrow T_{h(a+)} \nu^{* r'(a+)} \text{ as } n \to \infty.
\]

Note that \( \nu \) is uniquely determined whenever \( h(a+) \neq 0 \) and \( r'(a+) \neq 0 \).

3. Domains \( \mathcal{D}^{h,r}_{(a,b)} \) of integral mappings \( I^{h,r}_{(a,b)} \).

3.1. Domains on Banach spaces \( E \).

PROPOSITION 3. In order that \( \mathcal{D}^{h,r}_{(a,b)} = ID(E) \) it is necessary and sufficient that integrals \( \int_{(a,b)} y(r(t) -) dh(t) \) exists for all \( y \in D_E[a,b] \). In particular, if \( |r(b) - r(a+) | < \infty \) then \( \mathcal{D}^{h,r}_{(a,b)} = ID(E) \).

Proof. Because Lévy processes \( Y \) are cadlag and the random integrals (1) are defined by the formal integration by parts, we infer the claim concerning the first part.

Since the range of \( r \) is bounded then using the fact that cadlag functions, on bounded intervals, are integrable (cf. Billingsley (1968), p.121) we get that the integral in (1) is well-defined. This concludes the second claim.

In terms of the parameters in Lévy-Khintchine representation, domains of random integrals are characterized as follows:

PROPOSITION 4. A measure \( \nu = [z, R, M] \) is in the domain \( \mathcal{D}^{h,r}_{(a,b)} \) if and only if the following holds

\[
\int_{(a,b)} |h(t)||dr(t)| < \infty, \text{ if } z \neq 0; \quad \int_{(a,b)} h^2(t)|dr(t)| < \infty, \text{ if } R \neq 0,
\]

and for the \( \sigma \)-algebra \( \mathcal{B}_0 \) of Borel subsets of \( E \setminus \{0\} \), the set function

\[
\mathcal{B}_0 \ni A \mapsto \int_{(a,b)} [T_{h(t)} M(A)]|dr(t)| \text{ is a Lévy spectral measure on } E.
\]
Moreover, if \( t_{(a,b)}^{h,r}(\nu) \) is determined by the triple \([z_{(a,b)}^{h,r}, R_{(a,b)}^{h,r}, M_{(a,b)}^{h,r}]\) and \( r \) is nondecreasing then

\[
(i) \quad z_{(a,b)}^{h,r}(a,b) = (\int_{(a,b]} h(t) dr(t)) \cdot z + \int_{(a,b]} h(t) \int_{E \setminus \{0\}} [1_B(h(t)x) - 1_B(x)] x M(dx) dr(t);
(ii) \quad R_{(a,b)}^{h,r} = (\int_{(a,b]} h(t)^2 dr(t)) \cdot R;
(iii) \quad M_{(a,b)}^{h,r}(A) = \int_{(a,b]} [T_{h(t)} M(A)] dr(t) = \int_{(a,b]} \int_{E \setminus \{0\}} 1_A(h(t)x) M(dx) dr(t),
\]

(10)

**Proof.** From formulas in Section 0.1, (2) and from the uniqueness of the triple (shift vector, Gaussian covariance and Lévy spectral measure), in the Lévy-Khintchine formula, we get the above claims and the three formulas in (9).

**COROLLARY 3.** If \( M_{(a,b)}^{h,r} \) is a Lévy spectral measure on \( E \) then

\[
\int_{(a,b]} (1 \land h^2(s)) |dr(s)| < \infty.
\]

**Proof.** For \( y \in E' \) and the mapping \( \pi_y(x) := <y, x> \ (x \in E) \), the image measure \( \pi_y(M_{(a,b)}^{h,r}) \) is a Lévy spectral measure on \( \mathbb{R} \). Since for positive \( s \) and \( t \) we have \((1 \land s)(1 \land t) \leq (1 \land st)\) therefore we have that

\[
(\int_{(a,b]} (1 \land h^2(s)) |dr(s)|) \cdot (\int_{E} (1 \land <y, x>^2) dM(x)) \\
\leq \int_{(a,b]} \int_{E} (1 \land <y, h(s)x>^2) dM(x) |dr(s)| \\
= \int_{E} (1 \land <y, u>^2) M_{(a,b]}^{h,r}(du) = \int_{E} (1 \land w^2) (\pi_y M_{(a,b]}^{h,r})(dw) < \infty,
\]

which concludes the proof.

**COROLLARY 4.** Let \( h \) be any real-valued function on \( (a, b] \), let \( r \) be monotone on the interval \( (a, b] \) such that \(|r(b) - r(a^+)\) < \( \infty \) and let Lévy (spectral) measures \( M \) and \( N \) be such that \([0, 0, M]\) and \([0, 0, N]\) are in the domain \( D_{(a,b]}^{h,r} \). Then \( M_{(a,b]}^{h,r} = N_{(a,b]}^{h,r} \) implies that \( M = N \).

**Proof.** Assume that \( r \) is non-decreasing and \( r(b) - r(a^+) < \infty \). Then for any Borel subset \( A \subset E \setminus \{0\} \) and bounded away from zero, i.e., \( 0 \notin A \)
(closure),
\[
M_{(a,b)}^{h,r}(A) - N_{(a,b)}^{h,r}(A) = \int_{E \setminus \{0\}} \left[ \int_{(a,b]} 1_A(h(t)x)dr(t) \right] ((M - N)(dx) = 0,
\]
which can be extended (note that \(r(b) - r(a+)<\infty\)) to the following
\[
\int_{E \setminus \{0\}} \left[ \int_{(a,b]} g_0(h(t)x)dr(t) \right] ((M - N)(dx) = 0, \text{ for all } g_0 \in C_{b0}^+(E),
\]
where \(C_{b0}^+(E)\) stands for the family of all functions that are positive, continuous bounded and vanishing in some neighbourhoods of zero. Since the expression in the square bracket is always positive we conclude that \(M=N\), which completes the proof.

**Remark 3.** Choose two different Lévy spectral measures \(M\) and \(N\) concentrated on \((0,\infty)\) such that
\[
\int_{(0,\infty)} x^2 M(dx) = \int_{(0,\infty)} x^2 N(dx) < \infty,
\]
and the time change function \(r(t) = t^{-2}\). Then for each \(v>0\),
\[
M_{(0,\infty)}^{t,t^{-2}}(x>v) = 2 \int_{0}^{\infty} \int_{(0,\infty)} 1_{(x>v)}(tx)t^{-3}dtM(dx) \quad (w := tx)
\]
\[
= \int_{(0,\infty)} x^2 M(dx) 2 \int_{v}^{\infty} w^{-3}dw = \int_{(0,\infty)} x^2 M(dx) v^{-2}
\]
\[
= \int_{(0,\infty)} x^2 N(dx) v^{-2} = N_{(0,\infty)}^{t,t^{-2}}(x>v).
\]
So, for different Lévy measures \(M\) and \(N\) we got equality \(M_{(0,\infty)}^{t,t^{-2}} = N_{(0,\infty)}^{t,t^{-2}}\). However, the functions \(h(t) = t\) and \(r(t) = t^{-2}\) do not satisfy the integrability condition from Corollary 3. Thus \(M_{(0,\infty)}^{t,t^{-2}}\) is not a Lévy (spectral) measure.

Here are some sufficient conditions (for symmetrized measures \(\gamma^\circ\)) to be in a domain of a random integral mapping.

**Proposition 5.** (i) For \(0 < p \leq 2\), if the integral \(\int_{(a,b]} |h(t)|^p|dr(t)|\) exists then all symmetric \(p\)-stable measures \(\gamma^\circ_p\) on \(E\) are in the domain \(D_{(a,b]}^{h,r}\).

(ii) If a positive Borel measure \(N\) on \(E\) integrates the function \(||x||\) then \(N\) is a Lévy spectral measure and \(\nu = [z,0,N]\) has finite first moment. Moreover, if \(\int_{(a,b]} |h(t)| |dr(t)| < \infty\) then \([z,0,N] \in D_{(a,b]}^{h,r}\).
Proof. For (i), recall that Lévy exponents of symmetric p-stable non-Gaussian measures for $0 < p < 2$ are of the form
\[ \Phi(y) \equiv -\log \gamma_p(y) = \int_{\{||x||=1\}} |y, x|^p m(dx) \]
for some finite measure $m$ on the unit sphere; cf. Araujo and Giné (1980), Chapter III, Theorem 6.16. Hence and from (3)
\[ y \rightarrow \int_{(a,b]} \Phi((h(t)y)dr(t) = \int_{(a,b]} |h(t)|^p dr(t) \int_{\{||x||=1\}} |y, x|^p m(dx) \]
is also a Lévy exponent and thus the random integral is well defined. The case of symmetric Gaussian ($p = 2$) follows from Corollary 4 (ii).

For part (ii), since $\int_{E \{0\}} (1 \wedge ||x||) N(dx) < \infty$, therefore $N$ is a Lévy measure by Araujo-Gine (1980), Chapter 3, Theorem 6.3. Since also $\int_{\{||x|| > 1\}} ||x|| N(dx) < \infty$ we conclude that $\nu$ has finite first moment. Furthermore for measure $N^{h,r}_{(a,b]}$ given by (14) we have
\[ \int_{E \{0\}} (1 \wedge ||x||) N^{h,r}_{(a,b]}(dx) = \int_{(a,b]} \int_{E \{0\}} (1 \wedge |h(t)||x||) N(dx)|dr(t)| \]
\[ \leq \left( \int_{(a,b]} |h(t)||dr(t)| \right) \left( \int_{E} ||x|| N(dx) \right) < \infty, \quad (11) \]
and again by Theorem 6.3 in Chapter 3 in Araujo-Giné (1980) we conclude that $N^{h,r}_{(a,b]}$ is a Lévy spectral measure. Thus $\nu \in D^{h,r}_{(a,b]}$ and the proof is completed.

3.2. Domains on Hilbert space $H$

On real separable Hilbert spaces we have complete characterization of covariance operators and more importantly, for the considerations here, we know that
\[ M \text{ is a Lévy measure on } H \text{ iff } M\{0\} = 0 \text{ and } \int_{H} (1 \wedge ||x||^2) M(dx) < \infty, \]
and by Parthasarathy (1968), Chapter VI. With the above and Proposition 4 we have

COROLLARY 5. In order that a measure $\nu = [z, R, M] \in D^{h,r}_{(a,b]}(H)$ it is necessary and sufficient that
\begin{enumerate}
  \item[(i)] $\int_{(a,b]} |h(t)||dr(t)| < \infty$, provided $z \neq 0$,
  \item[(ii)] $\int_{(a,b]} h^2(t)|dr(t)| < \infty$, provided $R \neq 0$,
  \item[(iii)] $\int_{(a,b]} \int_{E \{0\}} (1 \wedge |h(t)|^2||x||^2) M(dx)|dr(t)| < \infty$, provided $M \neq 0$.
\end{enumerate}
Remark 4. Since for all positive s and t, \((1 \wedge s)(1 \wedge t) \leq (1 \wedge st)\), from the above condition (iii) we infer that if \(M_{(a,b)}^{h,r}\) is a Lévy spectral measure (on H) then so is \(M\) and
\[
\int_{(a,b)} (1 \wedge |h(t)|^2) |dr(t)| < \infty.
\]
Therefore, if \([0,0,M] \in D_{(a,b)}^{h,r}(H)\) then it is necessary that the triple: an interval \((a,b)\) and functions \(h\) and \(r\), satisfies the above integrability condition.

**Proposition 6.** For triples \((a,b), h\) and \(r\) satisfying the conditions (i) and (ii) from Corollary 4, all infinitely divisible measures with finite second moment are in their domains, that is, \(ID_2(H) \subset D_{(a,b)}^{h,r}\), for arbitrary Hilbert space \(H\).

**Proof.** In view of Jurek-Smalara (1981) or Proposition 1.18.13 in Jurek-Mason (1993) or Theorem 25.3 in Sato (1999) we know that \(\nu = [z,R,M] \in ID_2(H)\) if and only if \(\int_{||x||>1} ||x||^2 M(dx) < \infty\). Since
\[
\int_{(a,b)} \int_{H} (1\wedge|h(t)|^2||x||^2) M(dx) |dr(t)| \leq \int_{(a,b)} h^2(t)|dr(t)| \int_{H} ||x||^2 M(dx) < \infty,
\]
( on \(H\), Lévy measure \(M\) always integrates \(||x||^2\) in the unit ball), we conclude that \(\nu \in D_{(a,b)}^{h,r}\), which completes the proof.

4. **Compositions of random integral mappings** \(I_{(a,b)}^{h,r}\)

4.1. **Equivalent mappings**

We say that two integral mappings \(I_{(a,b)}^{h,r}\) and \(I_{(a_1,b_1)}^{h_1,r_1}\) are equivalent if
\[
D_{(a,b)}^{h,r} = D_{(a_1,b_1)}^{h_1,r_1} \quad \text{and} \quad I_{(a,b)}^{h,r}(D_{(a,b)}^{h,r}) = I_{(a_1,b_1)}^{h_1,r_1}(D_{(a_1,b_1)}^{h_1,r_1}),
\]
and we write \(I_{(a,b)}^{h,r} = I_{(a_1,b_1)}^{h_1,r_1}\). In terms of Lévy exponents the above means that
\[
\int_{(a_1,b_1)} \Phi(h_1(t)y)dr_1(t) = \int_{(a_2,b_2)} \Phi(h_2(t)y)dr_2(t), \text{ for all } y \in E',
\]
for Lévy exponents \(\Phi\) (measures) in appropriate domains.

**Remark 5.** Mappings \(I_{(0,\infty)}^{s^{-1},t}\) and \(I_{(0,1)}^{s,-\log s}\) are equivalent. Similarly, \(I_{(0,1)}^{t,\beta}\) and \(I_{(0,1)}^{s^{1/\beta},t}\), for \(\beta > 0\). This follows from above without specifying the domains.
4.2. **Iterated random integral mappings**

Below let the time change \( r(t), a < t \leq b \), be either \( \rho \{ s : s > t \} \) or \( \rho \{ s : s \leq t \} \) for some positive, possibly infinite, measure \( \rho \) on a positive half-line.

For functions \( h_1, ..., h_m \), intervals \( (a_1, b_1], ..., (a_m, b_m] \) and measures \( \rho_1, ..., \rho_m \) let us define

\[ h := h_1 \otimes ... \otimes h_m, \]  
(tensor product of functions)

i.e. \( h(t_1, t_2, ..., t_m) := h_1(t_1) \cdot h_2(t_2) \cdot ... \cdot h_m(t_m), \) where \( a_i < t_i \leq b_i; \)

\[ (a, b] := (a_1, b_1] \times ... \times (a_m, b_m], \ \rho := \rho_1 \times ... \times \rho_m \) (product measure) \hspace{1cm} (13)

**THEOREM 2.** Let functions \( h_i \), measures \( \rho_i \) (given by increments of functions \( r_i \)) and intervals \( (a_i, b_i] \), for \( i = 1, 2, ..., m \), be as above.

If the image \( h((a, b]) = (c, d] \subset \mathbb{R}^+ \) and \( \nu \in ID(E) \) is from an appropriate domain then we have

\[ I_{(a,b]}^{h,\rho}(I_{(a_1,b_1]}^{h_1,\rho_1}(I_{(a_2,b_2]}^{h_2,\rho_2}(... \cdot (I_{(a_m,b_m]}^{h_m,\rho_m}(\nu)))) = I_{(c,d]}^{h,\rho}(\nu) \]  
(14)

where \( h, \rho \) is the image of the product measure \( \rho = \rho_1 \times ... \times \rho_m \) under the mapping \( h := h_1 \otimes ... \otimes h_m \).

**Proof.** For \( \nu \in D_{(a,b]}^{h,\rho} \) and its Lévy exponent \( \Phi \) let us define the (script) mapping \( \mathcal{I}_{(a,b]}^{h,\rho} \) as follows

\[ \mathcal{I}_{(a,b]}^{h,\rho}(\Phi)(y) := \Phi_{(a,b]}^{h,\rho} = \int_{(a,b]} \Phi(\pm h(s)y)ds(\pm r(s), \hspace{1cm} (15) \]

where the sign minus is in the case of decreasing time change \( r \). Then to justify (14) it is enough to notice that

\[ \mathcal{I}_{(a,b]}^{h_1,\rho_1}(\mathcal{I}_{(a_2,b_2]}^{h_2,\rho_2}(... \cdot (\mathcal{I}_{(a_m,b_m]}^{h_m,\rho_m}(\Phi))))(y) \]

\[ = \int_{(a_1,b_1]} \int_{(a_2,b_2]} ... \int_{(a_m,b_m]} \Phi(h_1(t_1) h_2(t_2) ... h_m(t_m) y)) dr_1(t_1) dr_2(t_2) ... dr_m(t_m) \]

\[ = \int_{(a,b]} \Phi(h_1 \otimes ... \otimes h_m(s) y) \rho(ds) = \int_{(c,d]} \Phi(t y)(h \rho)(dt), \hspace{1cm} (16) \]

which follows from the Fubini and the image measure theorems.

In view of the definitions of the tensor product and the product measures we have

\[ h_1 \otimes ... \otimes h_m (\rho_1 \times \rho_2 \times ... \times \rho_m) = h_{\sigma(1)} \otimes ... \otimes h_{\sigma(m)} (\rho_{\sigma(1)} \times \rho_{\sigma(2)} \times ... \times \rho_{\sigma(m)}) \]

for any permutation \( \sigma \) of 1, 2, ...m. Hence
LEMMA 1. Let $h(t) := e^{-t}, r_1(t) := t, r_2(s) := 1 - e^{-s}, 0 < s, t < \infty$. Then the corresponding measures are: $d\rho_1(t) = dt, d\rho_2(s) = e^{-s}ds$ and $d\rho(t,s) = d(\rho_1 \times \rho_2)(t,s) = e^{-s}dt ds$. Finally, for the image measure $h \rho(dw) = (h_1 \otimes h_2)(\rho_1 \times \rho_2)(dw) = \int_0^\infty \int_0^\infty g(s) e^{-s}dw ds$.

Proof. For Borel measurable, bounded and non-negative functions $g$ we have

$$
\int_0^\infty g(u)(h_1 \otimes h_2)(\rho_1 \times \rho_2)(du) = \int_0^\infty \int_0^\infty g((h_1 \otimes h_2)(t,s))\rho_1(dt)\rho_2(ds)
$$

$$
= \int_0^\infty \int_0^\infty g(e^{-t})dt e^{-s}ds = \int_0^\infty (\int_0^\infty g(w)\frac{1}{w}dw) e^{-s}ds = \int_0^\infty g(s)\frac{e^{-s}}{s}ds.
$$

COROLLARY 6. Random integrals $I_{(a_i, b_i)}^{h_i, \rho_i}, i = 1, 2, ..., m$, commute on the domain $D_{(a,b)}^{h, \rho}$, where $\rho = \rho_1 \times ... \times \rho_m$ and $h := h_1 \otimes ... \otimes h_m$.

In case of probability measures $\rho_i$, the time change function $r$ is a cumulative probability distribution and we have

COROLLARY 7. Let assume that $r_i(t) := \rho_i(\{s \in (a_i, b_i) : a < s \leq t\})$, where $\rho_i$ are probability measure on $(a_i, b_i]$ that are distributions of random variables $Z_i$, for $1 \leq i \leq m$. If $Z_1, Z_2, ..., Z_m$ are stochastically independent then

$$
r(t) := h\rho(s \leq t) = P[h_1(Z_1) \cdot ... \cdot h_m(Z_m) \leq t].
$$

The above we can apply, for instance, to $h_i(t) := |t|$ on positive half-line and independent standard normal variable $Z_i$. That case was investigated in $\mathbb{R}^d$ by Aoyama (2009) via polar decomposition of Lévy spectral measures.

4.3 Inclusion of ranges of integral mappings

If a random mapping is a composition of other mappings we may infer some inclusions of their ranges. Namely we have

COROLLARY 8. If an equality $I_{(a,[b])}^{h, r} = I_{(a_1, b_1)}^{h_1, r_1} \circ I_{(a_2, b_2)}^{h_2, r_2}$ (a composition) holds on the domain $D_{(a,b)}^{h,r}$, then we have

$$
R_{(a,b)}^{h,r} = I_{(a_1, b_1)}^{h_1, r_1}(I_{(a_2, b_2)}^{h_2, r_2}(D_{(a,b)}^{h,r})) \subset I_{(a_1, b_1)}^{h_1, r_1}(I_{(a_2, b_2)}^{h_2, r_2}(D_{(a,b)}^{h,r})).
$$

Proof. From the equality of the above mappings we get

$$
I_{(a,b)}^{h,r}(D_{(a,b)}^{h,r}) = I_{(a_1, b_1)}^{h_1, r_1}(I_{(a_2, b_2)}^{h_2, r_2}(D_{(a,b)}^{h,r})) \text{ and hence } I_{(a_2, b_2)}^{h_2, r_2}(D_{(a,b)}^{h,r}) \subset D_{(a,b)}^{h,r}.
$$

Therefore $I_{(a,b)}^{h,r}(D_{(a,b)}^{h,r}) \subset I_{(a_2, b_2)}^{h_2, r_2}(I_{(a_2, b_2)}^{h_2, r_2}(D_{(a,b)}^{h,r})).$ Because of the commutativity we get

$$
I_{(a,b)}^{h,r}(D_{(a,b)}^{h,r}) \subset I_{(a_1, b_1)}^{h_1, r_1}(I_{(a_2, b_2)}^{h_2, r_2}(D_{(a,b)}^{h,r})),
$$

which completes a proof.

4.4. An example of application of Theorem 2
which completes the proof of Lemma 1.

From Theorem 2, Corollary 6, Lemma 1 and Jure-Vervaat (1983) we conclude that

**COROLLARY 9.** For $\nu \in ID_{\log}$ we have

$$I_{(0,\infty)}^{t,1-e^{-t}}(I_{(0,\infty)}^{e^{-s},s}(\nu)) = I_{(0,\infty)}^{e^{-s},s}(I_{(0,\infty)}^{t,1-e^{-t}}(\nu)) = I_{(0,\infty)}^{-w,\Gamma(0;w)}(\nu) = I_{(0,\infty)}^{w,\Gamma(0;w)}(\nu^-)$$

Moreover, $\Gamma(0;w) = (h_1 \otimes h_2)(\rho_1 \times \rho_2)(\{x : x > w\}) = \int_{-w}^{w} \frac{e^{-s}}{s} ds$ for $w > 0$.

**Remark 6.** (a) For the Euler constant $C$ we have

$$-\Gamma(0;w) = Ei(-w) = C + \ln w + \int_{0}^{w} \frac{e^{-t} - 1}{t} dt, \text{ for } w > 0,$$

where $Ei$ is the special exponential-integral function; cf. Gradshteyn-Ryzhik (1994), formulae 8.211 and 8.212.

(b) Recall that the class $I_{(0,\infty)}^{t,1-e^{-t}}(ID) \equiv \mathcal{E}$ was introduced in Jurek (2007), where the mapping $I_{(0,\infty)}^{t,1-e^{-t}}$ was denoted by $K^{(e)}$; ($(e)$ for exponential cumulative distribution function). More importantly, the class $\mathcal{E}$ was related to the class of Voiculescu $\boxplus$ free-infinitely divisible measures; cf. Corollary 6 in Jurek (2007). Note also that $I_{(0,\infty)}^{t,1-e^{-t}} = I_{(0,1]}^{e^{-s},s}$ and thus it coincides with the uppsilon mapping $Y$ studied in Barndorff-Nielsen, Maejima and Sato (2006).

(c) Similarly $I^{e^{-s},s}(ID_{\log}) \equiv L$ coincides with the Lévy class of selfdecomposable probability measures; cf. Jurek-Vervaat (1983), Theorem 3.2 or Jurek-Mason (1993), Theorem 3.6.6.

(d) Finally we get identity $I^{e^{-s},s}(I_{(0,\infty)}^{t,1-e^{-t}}(ID_{\log})) \equiv T$, which is the Thorin class; cf. Grigelionis (2007), Maejima and Sato (2009) or Jurek (2011).

From Corollary 9 and Remark 5(d) we infer that

**COROLLARY 10.** For the three classes: Thorin class $T$, Lévy class $L$ (selfdecomposable measures) and $\mathcal{E}$ we have that $T \subset L \cap \mathcal{E}$

(This inclusion, on $\mathbb{R}^d$, was first noticed in Barndorff-Nielsen, Maejima and Sato (2006) and also in Remark 2.3 in Maejima-Sato (2009) but by using completely different methods.)

5. The identity and the inverse of a random integral mapping.

5.1. **Zero random integral mapping** $I_{(a,b]}^{h,r}$
If \( h(t) \equiv 0 \) or \( r(t) \equiv r_0 \) (constant function) then
\[
\int_{[a,b]} 0 \, dY_\nu(r(t)) = 0 = \int_{[a,b]} h(t) \, dY_\nu(r_0), \text{ for all } \nu \in ID,
\]
are the zero mappings; \( I_{(a,b)}^{0,r}(\nu) = I_{(a,b)}^{h,r_0}(\nu) = \delta_0 \). To exclude that trivial case we assume that the three parameters: an interval \( (a, b) \) and functions \( h, r \), satisfy the basic condition \( 0 < \int_{(a,b)} |h(t)| \, |dr(t)| \).

On the other hand, the condition \( \int_{[a,b]} |h(t)| \, |dr(t)| \prec \infty \), guarantees that the degenerate Lévy process \( Y(t) := ta \) (a fixed) can be used as integrators in the integrals (1); cf. formula (8) in Proposition 4.

5.2. Identity random integral mapping \( I_{(a,b)}^{h,r}(\nu) \)

Note that whenever \( 0 < r_0 := |r(b) - r(a^+)| \prec \infty \) and \( h(t) \equiv 1 \) (constant) then
\[
I_{(a,b)}^{1,r(t)/r_0}(\nu) = \nu \text{ for all } \nu \in ID. \tag{17}
\]

So all mappings \( I_{(a,b)}^{1,r(t)/r_0} \) play a role of the neutral element (identity mapping), under the composition, in the family of all integral mappings. In fact, for \( r(.)/r_0 \) one may take any time change whose increment over the interval \( (a, b) \) is equal to 1.

Similarly, if \( \delta_u(t) := 1_{[u, \infty)}(t) \), \( h(u) \neq 0 \) (\( u \) is fixed) and \( u \in (a, b) \) then from (2) or (3) we have
\[
I_{(a,b)}^{h/h(u),\delta_u}(\nu) = \nu \tag{18}
\]
and thus they also play the role of the identity mapping.

Remark 7. In the case of (17), the time change can be any strictly monotone function while the space change \( h \) is trivial. In the case of (18), the space change is quite arbitrary but time change \( r \) is one point jump function.

Integrals (18) and (19) are equivalent and will be called the identity mappings in the space of all random integral mappings \( I_{(a,b)}^{h,r} \).

5.3. The inverse of a random integral mapping.

Under the conditions in Theorem 1, there exists the inverse of the mapping \( I_{(a,b)}^{h,r} \) for which we have

\textbf{THEOREM 3.} If the mapping \( (I_{(a,b)}^{h,r})^{-1} : R_{(a,b)}^{h,r} \equiv (I_{(a,b)}^{h,r}(D_{(a,b)}^{h,r})) \rightarrow D_{(a,b)}^{h,r} \)

exists then it is an isomorphism between the corresponding subsemigroups of \( ID \). However, it is not of the random integral mapping form, unless it is the identity mapping.
Proof. The isomorphism property of the inverse mapping is a consequence of the fact that \( I_{(a,b)}^{h,r} \) is an isomorphism by Theorem 1.

Now suppose that the inverse of a non-trivial mapping \( I_{(a,b)}^{h,r} \) is, indeed, of an integral form \( I_{(a_1,b_1)}^{h_1,r_1} \). Then by Theorem 2

\[
I_{(a,b)}^{h,r} (I_{(a_1,b_1)}^{h_1,r_1} (\nu)) = I_{(c,d)}^{s,r_2} (\nu) = \nu,
\]

where

\[
dr_2(t) = d(h \otimes h_1)(dr \times dr_1)(t) = d\delta_u(t) \quad \text{for some fixed} \quad u \in (c, d),
\]

and \((c, d) = (h \otimes h_1)((a, b) \times (a_1, b_1))\). In other words, for all continuous functions \( g \) on \([c, d]\) we have

\[
\int_{(a,b)} \int_{(a_1,b_1)} g(h(t)h_1(s))dr(t)dr_1(s) = g(u).
\]

Hence, either \( h(t) \cdot h_1(s) = u \) (constant) for all \( t \in (a, b) \) and \( s \in (a_1, b_1) \) and \( |r(b) - r(a^+)| \delta_1(b_1) - r_1(a_1^+)| = 1 \) or \( dr \times dr_1 = \delta \times \delta \) and \( h(t)h_1(s) = u \). Consequently, in the first case both \( h \) and \( h_1 \) are constant that contradicts the assumption that \( I_{(a,b)}^{h,r} \) is non-trivial mapping. Similarly, in the second case \( r \) and \( r_1 \) are Dirac measures and therefore \( I_{(a,b)}^{h,r} \) is an identity mapping.

Thus this completes a proof of Theorem 3.

6. Fixed points (eigenfunctions) of \( I_{(a,b)}^{h,r} \)

6.1. Definition of fixed points

We will say that an infinitely divisible measure \( \rho \) is a fixed point of an integral mapping \( I_{(a,b)}^{h,r} \), if

\[
I_{(a,b)}^{h,r} (\rho) = \rho \ast c \ast \delta_z,
\]

for some \( c > 0 \) and \( z \in E \). (19)

Equivalently, in terms of Lévy exponents, using (15)

\[
I_{(a,b)}^{h,r} (\Phi)(y) = c \Phi(y) + i < y, z > \quad \text{for all} \quad y \in E'.
\]

(20)

Remark 8. i) Remark 5.2 in Jurek-Vervaat (1983) explains why in the definition (20) we have taken \( \nu \ast c \) instead of the more natural \( T_c \nu \) (multiplying of a corresponding random variable by a constant).

ii) Note that (20) reads that \( \Phi \) is an eigenfunction of the mapping \( I_{(a,b)}^{h,r} \) acting on the positive cone of (symmetric) Lévy exponents, provided we ignore the shift part.
6.2. Stable measures

Let us recall here one of the many equivalent definitions of stable distributions. Namely, we will say that \( \gamma \) is a stable probability measures if there exists a parameter \( 0 < p \leq 2 \) such for each \( t > 0 \) there exists a vector \( z(t) \in E \) such that

\[
t^p \log \hat{\gamma}(y) = \log \hat{\gamma}(ty) + i < y, z(t) > \quad \text{for all } y \in E'; \quad (21)
\]

cf. Jurek (1983), Theorem 3.2. or Linde (1983) or Theorem 4.1 4.2 in Jurek-Mason (1993). We will write \( \gamma \equiv \gamma \) if the above holds and say that it is a \( p \)-stable probability measure. Furthermore, we say that \( \gamma \) is strictly stable, if \( z(t) \equiv 0 \) in (21).

6.3. Fixed points of the mapping \( I_{h,r}^{a,b} \)

**PROPOSITION 7.** In order that \( p \)-stable measure \( \gamma \) be a fixed point of the mapping \( I_{h,r}^{a,b} \) it is necessary and sufficient that \( 0 < \int_{(a,b)} |h(t)|^p |dr(t)| \) \(< \infty \).

**Proof.** Because of the shift \( z \) in (20), if it enough to consider only strictly stable measures. In that case, using (21), we have

\[
\log \widehat{I_{h,r}^{a,b} (\gamma_p)}(y) = \int_{(a,b)} \log \hat{\gamma}_p(h(t)y) \, dr(t) = \left[ \int_{(a,b)} |h(t)|^p \, dr(t) \right] \log \hat{\gamma}_p(y), \quad (22)
\]

that is, \( p \)-stable probability measures \( \gamma \) are fixed points of the mapping \( I_{h,r}^{a,b} \) with the constant \( c := \int_{(a,b)} |h(t)|^p |dr(t)| \), which completes the proof.

Let denote by \( S \) the set of all stable measures. Then we get

**COROLLARY 11.** For the class \( S \) of all stable measures

\[
[I_{h,r}^{a,b}(S) = S] \iff \left[ \int_{(a,b)} |h(t)|^p |dr(t)| < \infty, \quad \text{for all } 0 < p \leq 2 \right]
\]

**Remark** 9. Taking on the unit interval \( (0,1] \) the function \( h(t) = t \) and the time change \( r(t) := t^{-\beta}, \beta > 2 \), we see that the above corollary is not true for the mapping \( I_{t,t^{-\beta}}^{0,1} \) and all \( 0 < p \leq 2 \).

7. Factorization property of measures from \( R_{h,r}^{a,b} \)

7.1. A motivating example
Let us recall that for $B_t = (B^1_t, B^2_t)$, Brownian motion on $\mathbb{R}^2$, the process

$$A_t = \int_0^t B^1_s dB^2_s - B^2_s dB^1_s, \quad t > 0,$$

is called Lévy’s stochastic area integral. It is well-known that for fixed $u > 0$, and $a := (\sqrt{u}, \sqrt{u}) \in \mathbb{R}^2$ we have

$$\chi(t) = E[e^{iA_u} | B_u = a] = \frac{tu}{\sinh tu} \cdot \exp\{- (tu \coth tu - 1)\}, \quad t \in \mathbb{R}$$

(cf. Lévy (1951) or Yor (1992), p. 19). If $\mu, \lambda$ and $\nu$ are probability measures corresponding to the characteristic functions $\chi(t)$, $\phi(t) := tu/\sinh tu$ and $\psi(t) := \exp\{- (tu \coth tu - 1)\}$, respectively, then $\lambda = \mathcal{I}_{(0,\infty)}(\nu)$, (cf. Remark 5(c)) and moreover

$$\mu = \mathcal{I}_{(0,\infty)}(\nu) \ast \nu \in \mathcal{L} = \mathcal{I}_{(0,\infty)}(ID_{\log})$$

In other words, the conditional Lévy’s stochastic area integral has selfdecomposable probability distribution $\mu$ that can be factorized by another selfdecomposable measure $\lambda$ and its background driving measure $\nu$; cf. Iksanov, Jurek and Schreiber (2004), p. 1367. That phenomena prompted the introduction of the notion of factorization property for the Lévy class $L$ distributions.

7.2. Definition and a condition for the factorization property

If for $\mu = \mathcal{I}_{(a,b)}^{h,r}(\nu) \in \mathcal{R}_{(a,b)}^{h,r}$ we also have that $\mathcal{I}_{(a,b)}^{h,r}(\nu) \ast \nu \in \mathcal{R}_{(a,b)}^{h,r}$ then we say that $\mu$ has a factorization property.

**PROPOSITION 8.** Suppose that for a given functions $h, r$ and an interval $(a, b]$ there exist function $h', r'$ and an interval $(a', b']$ such that for positive measures $\rho$ and $\rho'$, induced by the monotone functions $r$ and $r'$ respectively, we have the following

$$h((a, b]) \cdot h'((a', b']) = h((a', b')] = h((a, b]) = (c, d], \quad \text{for some } 0 < c < d,$$

and $(h \otimes h')(\rho \times \rho') = (h \rho) - (h' \rho') \geq 0$. \hspace{1cm} (23)

Then $\mathcal{D}_{(a,b]}^{h,r} \subset \mathcal{D}_{(a',b']}^{h',r'}$ and for all $\nu \in \mathcal{D}_{(a,b]}^{h,r}$ putting $\lambda := \mathcal{I}_{(a',b']}^{h',r'}(\nu)$ we have

$$\mathcal{I}_{(a',b']}^{h',r'}(\mathcal{I}_{(a,b]}^{h,r}(\nu) \ast \nu) = \mathcal{I}_{(a,b]}^{h,r}(\lambda) \ast \lambda = \mathcal{I}_{(a,b]}^{h,r}(\nu).$$

In other words, $\mathcal{R}_{(a,b]}^{h,r} = \{ \mathcal{I}_{(a,b]}^{h,r}(\lambda) \ast \lambda : \lambda \in \mathcal{I}_{(a',b']}^{h',r'}(\mathcal{D}_{(a,b]}^{h,r}) \}$
Proof. Since $0 \leq h' \rho' \leq h \rho$ then from Corollary 8 (expressed in terms of measures) we infer the inclusion of the domains.

From (23), Theorem 2 and the formula (3) we get

$$(I^{h, \rho}_{[a, b]} \circ I^{h', \rho'}_{[a', b']})(\nu) = I^{h, \rho}_{[a, b]}(I^{h', \rho'}_{[a', b']}(\nu)) * I^{h', \rho'}_{[a', b']}(\nu)$$

$$= I^{t, (h \otimes h') \rho \times \rho'}_{(c, d)}(\nu) * I^{t, (h' \rho')}_{(c, d)}(\nu) = I^{t, (h \rho)}_{(c, d)}(\nu) = I^{h, \rho}_{(c, d)}(\nu),$$

which completes the proof.

The factorization property of a selfdecomposable measure given by the Levy’s stochastic area integral is not an exception as we have

COROLLARY 12. For the class $L$ of selfdecomposable probability measures on $E$ we have

$$L = \{I^{e^{-t}, t}_{(0, \infty)}(\nu) * \nu : \nu \in I^{s, s}_{(0, 1]}(ID_{\log}(E))\}$$

Proof. We have that $L = I^{e^{-t}, t}_{(0, \infty)}(ID_{\log})$; cf. Jurek and Vervaat (1983). Then taking $h'(s) = s, \rho' = l_1$ (Lebesgue measure on unit interval), $a' = 0$ and $b' = 1$ we check that conditions (23) are fulfilled. Thus Proposition 8 gives the claim in the corollary.

[The above fact was also shown in Jurek (2008), Theorem 3.1 but by a different reasoning.]

8. Some explicit examples.

8.1. Examples of domains of random integral mappings

Here we recall a few examples of domains and in some instances sketch their proofs that rely on Corollary 5.

Example 1.

$$D^{t, -\log t}_{(0, 1]} = ID_{\log}(H) := \{\mu \in ID : \int_H \log(1 + ||x||)M(dx) < \infty\}. \quad (24)$$

For this let us note that

$$\int_H (1 \wedge ||x||^2) M^{t, -\log t}_{(0, 1]}(dx) = \int_0^1 \int_H (1 \wedge t^2 ||x||^2) M(dx) \frac{dt}{t}$$

$$= \int_0^1 t \int_{(||x|| < t^{-1})} ||x||^2 M(dx) + \int_0^1 t \int_{(||x|| > t^{-1})} M(dx) \frac{dt}{t} =$$

$$1/2 \int_H (1 \wedge ||x||^2) M(dx) + \int_{(||x|| > 1)} \log ||x|| M(dx) < \infty,$$
which is equivalent with finite log-moment of $\mu$; cf. Jurek and Smalara (1981) or Proposition 1.8.13 in Jurek and Mason (1993).

Example 1 is valid on any Banach space $E$. However, its proof is completely different from the above for Hilbert space $H$; cf. Jurek and Vervaat (1983).

**Example 2.** (1) $\mathcal{D}_{(0,1]}^{t,0} = ID(E)$, for $\beta > 0$.

(2) $\mathcal{D}_{(0,1]}^{t,\beta} = ID_\beta(H) := \{\nu \in ID(H) : \int_H ||x||^{-\beta} \nu(dx) < \infty\}$, for $-1 < \beta < 0$.

(3) $\mathcal{D}_{(0,1]}^{t,\beta} \cap ID^\circ = ID_\beta(H) \cap ID^\circ$, for $-2 < \beta \leq -1$; where $ID^\circ$ denotes symmetric infinitely divisible measures.

**Remark** 10. Recall that the integral mappings $I_{(0,1]}^{t,\beta}$ and their domains appeared in the context of the classes $\mathcal{U}_{\beta}$ for $-2 \leq \beta < 0$ and $0 \leq \beta < \infty$. The class $\mathcal{U}_0$ coincides with the Lévy class $L = I_{(0,\infty)}^{t,0}(ID_{log})$, while $\mathcal{U}_{-2}(E)$ consists only of Gaussian measures; cf. Jurek (1988), (1989) and Jurek-Schreiber (1992).

We complete this subsection with examples of time changes given by the incomplete Euler function. It is defined as follows

$$\Gamma(\alpha; x) := \int_x^\infty t^{\alpha-1} e^{-t} dt, \quad \alpha \in \mathbb{R}, \quad x > 0. \quad (25)$$

For $\alpha > 0$ the above is just the gamma function and $\Gamma(0+) < \infty$ and thus from Proposition 3 we get

**Example 3.** For $\alpha > 0$, $\mathcal{D}_{(0,\infty)}^{t,\Gamma(\alpha;t)} = ID(E)$.

Further for $\alpha = 0$ we get

**Example 4.**

$$\mathcal{D}_{(0,\infty)}^{t,\int_0^\infty e^{-t} ds} = ID_{log}(H) \quad (26)$$

Similarly as in Example 1,

$$\int_0^\infty \int_H (1 \wedge t^2 ||x||^2) M(dx) \frac{e^{-t}}{t} dt = \int_0^\infty t \int_{(||x|| \leq t^{-1})} ||x||^2 M(dx) e^{-t} dt + \int_0^\infty \int_{(||x|| > t^{-1})} M(dx) \frac{e^{-t}}{t} dt =$$

$$\int_{(||x|| > 0)} ||x||^2 \left[ \int_0^1 te^{-t} dt \right] M(dx) + \int_{(||x|| > 0)} \left[ \int_{(||x|| > 1)} \frac{e^{-t}}{t} dt \right] M(dx). \quad (27)$$
Note that in the first square bracket we get
\[
\int_0^{||x||^{-1}} te^{-t} dt = 1 - e^{-||x||^{-1}}(1 + ||x||^{-1}) \leq 1 \wedge ||x||^{-2}
\]
and hence the first integral in (27) is finite.

For second integral in (27) let us brake the space \( H \setminus \{0\} \) into two parts. For 
\( 0 < ||x|| \leq 1 \),
\[
\int_{(0<||x||\leq1)} ||x||^2 [||x||^{-1} \int_{||x||^{-1}}^\infty \frac{e^{-t}}{t} dt ] M(dx)
\leq [\sup_{(a\geq1)} a^2 \int_a^\infty \frac{e^{-t}}{t} dt ] \int_{(||x||\leq1)} ||x||^2 M(dx) < \infty.
\]
For the part \( ||x|| > 1 \) we use Remark 4(a) that gives
\[
\int_{(||x||>1)} \frac{e^{-t}}{t} dt = -C + \log ||x|| + \int_0^{||x||^{-1}} \frac{1 - e^{-t}}{t} dt,
\]
where the integral on the right hand side is bounded by \( \int_0^1 (1-e^{-t})t^{-1} dt < \infty \).
All in all the second integral in (27) is finite if and only if \( \int_{(||x||>1)} \log ||x|| M(dx) < \infty \), which completes the proof of Example 4.

**Example 5.** For \(-1 < \alpha < 0\) we have
\[
\mathcal{D}^{t,\int_{(0,\infty)}s^{\alpha-1}e^{-s}ds} = ID_\alpha(\mathbb{R}),
\]
with the notations from Example 2. For the above example and the case \(-2 < \alpha \leq -1\) cf. Sato (2006).

### 8.2. Examples of iterated integral mappings and image measures

**Example 6.** For \( \nu \in ID_{\log^m} \) and \( m = 1, 2, \ldots \)
\[
I_{(0,\infty)}^{e^{-s},s}(I_{(0,\infty)}^{e^{-s},s}(\ldots I_{(0,\infty)}^{e^{-s},s}(\nu))) = I_{(0,\infty)}^{e^{-t},t^m}(\nu)
\]

**Proof.** In view of Theorem 2 it is enough to check that for \( h(t) := e^{-t} \) and \( \rho := t \) (the Lebesque measure on \( \mathbb{R} \)) we have equality
\[
\int_0^\infty g(u)[(e^{-t})^\otimes m](l_1 \times \ldots \times l_1)](du)
= \int_0^\infty \int_0^\infty \ldots \int_0^\infty g(e^{-(s_1+s_2+\ldots+s_m)}u)ds_1ds_m = \int_0^\infty g(e^{-s}u)d\frac{s^m}{m!}
\]
for all \( g \) bounded and measurable. The first equality is just a change of variable argument. For the second, using the induction arguments, we have

\[
\int_0^\infty \left[ \int_0^\infty \ldots \int_0^\infty g(e^{-(s_1+s_2+\ldots+s_{m-1})}e^{-s_m}u)ds_1\ldots ds_{m-1} \right] ds_m = \\
\int_0^\infty \int_0^\infty g(e^{-t}e^{-s}u) \left[ \frac{\Gamma(m-1)}{(m-1)!} \right] ds_m = \int_0^\infty \int_{s_m}^\infty g(e^{-w}u) \left( \frac{w-s_m}{m-2} \right) dw ds_m \\
= \int_0^\infty g(e^{-w}u) \frac{w^{m-1}}{(m-1)!} dw = \int_0^\infty g(e^{-w}u) \left[ \frac{w^m}{m!} \right],
\]

which completes the proof.

**Remark 11.** The class of measures \( I_{(0,\infty)}^{e^{-1},\frac{1}{m}}(ID_{log^m}) \) coincides with the set \( L_m \) of so called \( m \)-times selfdecomposable distributions; cf. Jurek (2011) for the history of those classes and relevant references.

**Example 7.** For \( \beta > 0 \) we have

\[
I_{(0,1]}^{\frac{1}{\beta},t} \circ I_{(0,1]}^{\frac{3}{2\beta},s} = I_{(0,1]}^{w,2w^\beta(1-(1/2)w^\beta)} = I_{(0,1]}^{(1-\sqrt{7})^{1/\beta},t}
\]

Or equivalently, for Lebesgue measure \( l_1 \) on the unit interval and \( 0 < w \leq 1 \) we get

\[
(t^{1/\beta} \otimes s^{1/(2\beta)})(l_1 \times l_1)(dw) = id^{\otimes 2}(\beta^{1-1/2} dt \times 2 \beta s^{2 \beta - 1} dt)(dw) = 2 \beta w^{\beta - 1}(1-w^{\beta}) dw
\]

**Proof.** As in Example 6, it simply follows from Theorem 2 and identity (3) because all time change functions are strictly increasing on the unit interval.

**Example 8.** For \( \beta > 0 \)

\[
I_{(0,1]}^{\frac{1}{\beta},t} \circ I_{(0,\infty)}^{-\gamma,s} = I_{(0,\infty)}^{-w,\beta-1/w - \log w - \beta - 1}
\]

Or equivalently, for \( 0 < w \leq 1 \)

\[
(t^{1/\beta} \otimes e^{-s})(l_1 \times l)(dw) = (\beta^{-1} w^{\beta} - \log w - \beta^{-1}) dw.
\]

This is a consequence of Theorem 2. Also cf. Czyżewska-Jankowska and Jurek (2011), Proposition 2.

**Example 9.** For \( \alpha \in \mathbb{R} \)

\[
I_{(0,\infty)}^{t,\Gamma(\alpha:t)} \circ I_{(0,\infty)}^{-\gamma,s} = I_{(0,\infty)}^{t,\int_0^\infty s^{-1}\Gamma(\alpha:s)ds},
\]

which follows from Theorem 2.
Literatura

[1] T. Aoyama (2009), Nested subclasses of the class of type G selfdecomposable distributions on \( \mathbb{R}^d \), *Probab. Math. Stat.*, vol. 29, pp. 135-154.

[2] A. Araujo, and E. Giné (1980), *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, New York.

[3] O. E. Barndorff-Nielsen, M. Maejima and K. Sato (2006), Some classes of multivariate infinitely divisible distributions admitting integral representations, *Bernoulli* 12, pp. 1-33.

[4] P. Billingsley (1968), *Convergence of probability measures*, Wiley, New York.

[5] A. Czyżewska-Jankowska and Z. J. Jurek (2011), Factorization property of generalized s-selfdecomposable measures and class \( L^f \) distributions, *Theory Probab. Appl.* vol. 55, no 4, pp. 692-698.

[6] I. S. Gradshteyn and I. M. Ryzhik (1965). *Tables of integrals, series, and products*, Academic Press, New York.

[7] B. Grigelionis (2007), Extended Thorin classes and stochastic integrals, *Liet. Matem. Rink.* 47, pp. 497 – 503.

[8] A.M. Iksanov, Z. J. Jurek and B.M. Schreiber (2004), A new factorization property of the selfdecomposable probability measures, *Ann. Probab.* 32, no 2, 1356–1369.

[9] Z. J. Jurek (1983), Limit distributions and one-parameter groups of linear operators on Banach spaces, *J. Multivar. Anal.* 13, pp. 578-604.

[10] Z.J. Jurek (1988), Random integral representations for classes of limit distributions similar to Levy class \( L_0 \), *Probab. Th. Fields* 78, 473-490.

[11] Z. J. Jurek (1989), Random integral representations for classes of limit distributions similar to Lévy class \( L_0 \). II. *Nagoya Math. Journal* 114, pp. 53-64.

[12] Z.J. Jurek (2007), Random integral representations for free-infinitely divisible and tempered stable distributions, *Stat.& Probab. Letters*, 77 no. 4, pp. 417-425.

[13] Z. J. Jurek (2008), A calculus on Lévy exponents and selfdecomposability on Banach spaces, *Probab. Math. Stat.* vol. 28, Fasc. 2, pp. 271-280.
[14] Z. J. Jurek (2011), The Random Integral Representation Conjecture: a quarter of a century later, Lithuanian Math. Journal, 51, no 3, 2011, pp. 362-369.

[15] Z. J. Jurek and J.D. Mason (1993), Operator-limit Distributions in Probability Theory, Wiley Series in Probability and Mathematical Statistics, New York.

[16] Z. J. Jurek and B.M. Schreiber (1992), Fourier transforms of measures from classes $U_\beta, -2 < \beta < -1$. J. Multivariate Analysis 41 (1992), pp. 194-211.

[17] Z. J. Jurek and J. Smalara (1981), On integrability with respect to infinitely divisible measures, Bull. Acad. Polon. Sci. 29, pp. 179-185.

[18] Z.J. Jurek and W. Vervaat (1983), An integral representation for self-decomposable Banach space valued random variables, Z. Wahrsch. verw. Gebiete 62, 247–262.

[19] P. Lévy (1951), Wiener’s random functions, and other Laplacian random functions. In Proc. 2nd Berkeley Symposium Math. Stat. Probab., Univ. California Press, Berkeley, 171–178.

[20] W. Linde (1983), Probability measures in Banach spaces- stable and infinitely divisible distributions, Wiley, New York.

[21] M. Maejima, V.Perez Abreu, K.I. Sato (2012), A class of multivariate infinitely divisible distributions related to arcsine density, Bernoulli, vol. 18 no. 2, pp. 476-495.

[22] M. Maejima and K. Sato (2009), The limits of nested subclasses of several classes of infinitely divisible distributions are identical with closure of the class of stable distributions, Probab. Rel. Fields, vol. 145, pp. 119-142.

[23] K. R. Parthasarathy (1968), Probability measures on metric spaces, Academic Press, New York and London, 1968.

[24] K. Sato (1999), Lévy processes and infinitely divisible distributions, University Press, Cambridge, United Kingdom.

[25] K. Sato (2006), Two families of improper stochastic integrals with respect to Lévy processes, ALEA, Lat. Am. J. Probab. Math. Stat. vol. 1, pp. 47-87.
[26] M. Yor (1992), *Some Aspects of Brownian Motion, Part I: Some Special Functionals*, Birkhäuser, Basel.

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