ALMOST UNIVERSALITY OF SUM OF NORMS

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Abstract. Let $k_0, k_1, k_2 > 1$ and $K, E$ be abelian number fields of degree $k_1, k_2$ with $(\delta_K, \delta_E) = 1$. It is shown that every sufficiently large natural number $n$ can be written in the form $a^{k_0} + N_K(\alpha) + N_E(\beta)$ where $a$ is an integer and $\alpha, \beta$ are algebraic integers in $K, E$.

Introduction

Let $f$ be an integer-valued polynomial in many variables that is locally universal, meaning that $f(\zeta) = n$ is soluble over $\mathbb{Z}_p$ for all $n$ and $p$. It is quite often the case that $f$ is not universal or even almost universal as indicated by the Brauer-Manin obstruction. The failure of Hasse principle, though, can be in some sense recovered if we allow more variables in the equation; it is a general phenomena that if a surface $S(n)$ of higher dimension defined by $F(\vec{z}) = n$ contains the original surface $S_0(n)$ given by $f(\zeta) = n$ as a section, then it becomes almost universal, provided their codimension is sufficiently large. The Hardy-Littlewood method gives a powerful tool to conquer mostly the best results about the number of variables necessary for this.

Although this analytic method is powerful for many problems of additive nature, it has some technical limitations and a factor of $\log(\deg f)$ is invincible in general; most of the results satisfy $\dim(S(n))/\dim(S_0(n)) \gg \log(\deg f)$. The term $\log(\deg f)$ arises from the technical difficulty to handle the minor arcs. If one considers a specific polynomial that is irrelevant of this difficulty, it is provable that $\log(\deg f)$ reduces to $O(1)$.

In the 1960s, Birch, Davenport and Lewis already observed this and considered a specific type of $k$-forms. Let $K$ be a number field of degree $k$, and let $\omega_1, \ldots, \omega_k$ be an integral basis for $K$. Denote the norm of $x_1\omega_1 + \cdots + x_k\omega_k$ by $N_K(x_1, \ldots, x_k)$. In [3] they considered $z^k + N_K(x_1, \cdots, x_k) +$...

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under some provisos, whose main result was following theorem.

**Theorem 0.1.** Suppose $K$, $E$ are not both totally complex, and let $\delta_K$, $\delta_E$ denote their discriminants. Suppose that, for every prime $p$ dividing both $\delta_K$ and $\delta_E$, the equation

\[
(0.1) \quad n = z^k + N_K(x_1, \ldots, x_k) + N_E(y_1, \ldots, y_k)
\]

has a non-singular solution in the $p$-adic field. Then (0.1) has infinitely many solutions in integers.

The strategy in their work was to follow the routine procedure of the circle method, except the treatment of the singular series. A typical way of dealing the singular series is to obtain an estimate of exponential sums $S_{a,q}$, but a norm form $N_K(x_1, \ldots, x_k)$ has many off-diagonal terms which make it not easy to do that directly. In [3] instead, a technique invented by Birch was used to get an auxiliary lemma of the type $\sum_{q \leq X} q^{-2k} \sum_{\substack{a=1 \\ (a,q)=1}} \left| S_{a,q} \right|^2 \ll X^\epsilon$.

They also showed the positivity of the singular series, but without giving an estimate of its range.

In this memoir we prove almost universality of the polynomial $z^{k_0} + N_K(x) + N_E(y)$ in a different setting that enlightens several new aspects. We treat abelian number fields $K$, $E$ whose degrees $k_1$, $k_2$ may be different and can be totally imaginary as well. We also provide an effective bound for the singular series. At the end of the memoir we include a section which shows the optimality of our conclusion; the sum of two norms can fail to be almost universal even when it is locally universal, as we will give an example in section 8.

The techniques in [3] seems to be applicable to deduce some of our results but will not exhaust the contents of this research. On a technical point of view, the key features of this memoir are the estimation of the exponential sum $S_{a,q}$ using local class field theory and the use of restricted sum which gives an estimate of the singular series, by removing solutions that are $p$-adically singular for a bad prime $p$.

In section 2 we reduce the integration over major arcs into the singular series and singular integral. The estimation of the singular series comes in three subsequent sections. Section 3 covers the exponential sums over bad primes. Section 4 gives some basics about algebraic number theory, which will be used in section 5 to design a particular system of representatives of $O_K/p^tO_K$ and by doing so to obtain a successful bound of $S_{a,p^t}$ for nice primes $p$. Section 6 gives a fairly classic argument on the singular integral. We omit the estimation on the minor arcs because the reasoning in [3] carries over verbatim.
1. Notations and Settings

Let $K$ and $E$ be abelian number fields with ring of integers $O_K$, $O_E$ and integral bases $\{\varphi_1, \cdots, \varphi_{k_1}\}$ and $\{\psi_1, \cdots, \psi_{k_2}\}$ whose discriminants $\delta_K$, $\delta_E$ are relatively prime. Since $(\delta_K, \delta_E) = 1$, for each rational prime $p$ at least one of the inertial groups $\Xi_K(p)$, $\Xi_E(p)$ is trivial. Suppose $\Xi_K(p)$ is trivial. It is the cokernel of the norm map in a part of the statements of local class field theory, and we know that $N_{K|Q}(O_K^*)$ is dense in $\mathbb{Z}_p^\times$. Therefore $N_K(\vec{x}) + N_E(\vec{y}) = N_{K|Q}(\vec{x} \cdot \vec{x}^{'}) + N_E(\vec{y} \cdot \vec{y}^{'})$ is locally universal.

Throughout this paper we write $F(\vec{z}) = F(z_0, z_1, \cdots, z_{k_1+k_2}) = F(z_0, x_1, \cdots, x_{k_1}, y_1, \cdots, y_{k_2})$ where $\vec{z} = (z_0, \vec{x}, \vec{y}) \pmod{q}$ with $\vec{x} \in \tilde{R}_q^K$, $\vec{y} \in \tilde{R}_q^E$. Note that $|\tilde{R}_q| = |\tilde{R}_q||\tilde{R}_r|$ if $(q, r) = 1$. For brevity in the sequel we write $\vec{z} \equiv \tilde{R}_q$ for an integral vector $\vec{z}$ if $\vec{z} \equiv \vec{z} \pmod{q}$ for some $\vec{z} \in \tilde{R}_q$. Observe that $\vec{z} \equiv \tilde{R}_q$ for all $q \geq 1$ if and only if $\vec{z} \equiv \tilde{R}_Q$.

Throughout this article we let $k = \max\{k_0, k_1, k_2\}$ and $X_i = n^{1/k_i}$ in the definition of major and minor arcs, we consider the Farey dissection of order $n^{1-\nu}$ where $\nu$ is a fixed small positive number, in particular, less than $\frac{1}{\text{Max}}$. For a technical reason in the estimation on the major arcs, we need to choose small boxes $\mathcal{B}_i$ in $[0, 1]$, $[0, 1]^{k_1}$ and $[0, 1]^{k_2}$ that will be determined later. Let $\mathcal{B} = \mathcal{B}_0 \times \mathcal{B}_1 \times \mathcal{B}_2$ and let $X_i \mathcal{B}_i = \{\vec{t} : \frac{1}{X_i} \vec{t} \in \mathcal{B}_i\}$. The restricted sum is defined over

$$\hat{\mathcal{B}}(n) = \{\vec{t} \in (X_0 \mathcal{B}_0 \times X_1 \mathcal{B}_1 \times X_2 \mathcal{B}_2) \cap \mathbb{Z}^{1+k_1+k_2} : \vec{t} \equiv \tilde{R}_Q\}.$$ 

$\hat{\mathcal{B}}^K(n)$ is defined similarly using $X_i \mathcal{B}_i \cap \mathbb{Z}^{k_i}$ and $\tilde{R}_Q^K$ and so is $\hat{\mathcal{B}}^E(n)$. We want to estimate the number of solutions to $F(\vec{z}) = n$ with $\vec{z} \in \hat{\mathcal{B}}(n)$, viz,

$$\hat{r}(n) = \int_{1+c}^{1+c} \sum_{\vec{z} \in \hat{\mathcal{B}}(n)} e(\alpha F(\vec{z})) e(-n\alpha) d\alpha$$
for some real number \( c \).

For simplicity we write \( \sum_{a \mod^* q} \) to denote the summation over \( a \) that runs through 1 to \( q \) under the condition \( (a, q) = 1 \). We choose the major arcs for \( 1 \leq a \leq q \leq n^\nu \) and \( (a, q) = 1 \), given by

\[
\mathcal{M}(q, a) = \{ \alpha : |\alpha - a/q| \leq n^{-1+\nu} \}
\]

\[
\mathcal{M} = \bigcup_{q \leq n^\nu} \bigcup_{a \mod^* q} \mathcal{M}(q, a).
\]

and the minor arcs \( m = (n^{\nu-1}, 1 + n^{\nu-1}] - \mathcal{M} \). Note that \( \mathcal{M} \) is a disjoint union of the segments \( \mathcal{M}(q, a) \).

2. The Major Arcs

For each rational prime \( p \) and \( (a, p) = 1 \), let

\[
\mathcal{S}_{a,p}^K = \sum_{x \mod p^\nu} e\left( \frac{a}{p^\nu} N_K(\overrightarrow{x'}) \right), \quad \mathcal{S}_{a,p}^E = \sum_{x \in R_{p^\nu}^K} e\left( \frac{a}{p^\nu} N_K(\overrightarrow{x'}) \right).
\]

\( \mathcal{S}_{a,p}^E \) and \( \mathcal{S}_{a,p}^K \) are defined in the same manner. For \( (a, q) = 1 \) we also put

\[
\mathcal{S}_{a,q} = \sum_{\overrightarrow{z} \in \mathcal{R}_q} e\left( \frac{a}{q} F(\overrightarrow{z}) \right), \quad I(\beta) = \int_{X_0 \mathcal{B}_0 \times X_1 \mathcal{B}_1 \times X_2 \mathcal{B}_2} e\left( \beta F(\overrightarrow{r}) \right) d\overrightarrow{r}.
\]

Let \( \kappa(q) \) be the squarefree kernel of \( (q, Q) \).

**Lemma 2.1.** Assume \( |\beta| \leq n^{\nu-1} \), \( 1 \leq a \leq q \leq n^\nu \), \( (a, q) = 1 \). Then

\[
\sum_{\overrightarrow{z} \in \mathcal{B}(n)} e\left( \left( \frac{a}{q} + \beta \right) F(\overrightarrow{z}) \right) = \frac{|\mathcal{R}_q|}{Q^{1+k_1+k_2}} \left( \frac{\kappa(q)^{1+k_1+k_2}}{|\mathcal{R}_{\kappa(q)}|} \frac{1}{q^{1+k_1+k_2}} \mathcal{S}_{a,q} I(\beta) \right) + O\left(n^{2+1/k_0-1/k+2\nu}\right)
\]

**Proof.** Let \( q = \text{lcm}(q, Q) = q_{\kappa(q)} \) and write \( \overrightarrow{z} = q \overrightarrow{\alpha} + \overrightarrow{b} \), \( 0 \leq b_i < q \). Let \( \mathcal{A}(q, \overrightarrow{b}) = \{ \overrightarrow{\eta} \in \mathbb{R}^{1+k_1+k_2} : q \overrightarrow{\eta} + \overrightarrow{b} \in X_0 \mathcal{B}_0 \times X_1 \mathcal{B}_1 \times X_2 \mathcal{B}_2 \} \) and
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\[ A(q, \vec{b}) = A(q, \vec{b}) \cap \mathbb{Z}^{1+k_1+k_2} \]. One easily has

\[
\sum_{\vec{z} \in \mathcal{B}(n)} e\left( \left( \frac{a}{q} + \beta \right) F(\vec{z}) \right)
= \sum_{\vec{b} \in R_q} \sum_{\vec{a} \in A(q, \vec{b})} e\left( \frac{a}{q} F(\vec{q} \vec{a} + \vec{b}) \right) e\left( \beta F(\vec{q} \vec{a} + \vec{b}) \right)
= \sum_{\vec{b} \in R_q} e\left( \frac{a}{q} F(\vec{b}) \right) \sum_{\vec{a} \in A(q, \vec{b})} e\left( \beta F(\vec{q} \vec{a} + \vec{b}) \right).
\]

Let \( h : \mathbb{R}^{1+k_1+k_2} \to \mathbb{C} \) be a map defined by \( h(\vec{η}) = e(\beta F(\vec{q} \vec{η} + \vec{b})) \). If \( \vec{a} \) is an integral vector near \( \vec{η} \) so that \( |η_i - a_i| \leq 1/2 \) for all \( 0 \leq i \leq k_1+k_2 \), and \( M \) is the supremum of the absolute value of directional derivatives of \( h(\vec{η}) \) for \( \vec{η} \in A(q, \vec{b}) \), then clearly \( |h(\vec{η}) - h(\vec{a})| \ll M \). Observe that any convex body \( U \subset A(q, \vec{b}) \) can be divided into unit boxes together with at most

\[
O \left( \frac{X_0}{q} \left( \frac{X_1}{q} \right)^{k_1} \left( \frac{X_2}{q} \right)^{k_2} \left( \frac{q}{X_0} + \frac{q}{X_1} + \frac{q}{X_2} \right) \right)
\]
possible broken boxes. Hence

\[
\left| \int_{A(q, \vec{b})} h(\vec{η}) d\vec{η} - \sum_{\vec{a} \in A(q, \vec{b})} h(\vec{a}) \right|
\ll \left| A(q, \vec{b}) \right| M + \frac{X_0 X_1^{k_1} X_2^{k_2}}{q^{k_1+k_2}} \left( \frac{1}{X_0} + \frac{1}{X_1} + \frac{1}{X_2} \right) \sup_{\vec{eta}} |h(\vec{eta})|
\]

where \( \left| A(q, \vec{b}) \right| \) is the Lebesgue measure of \( A(q, \vec{b}) \). Here

\[
\left| A(q, \vec{b}) \right| \ll \frac{X_0 X_1^{k_1} X_2^{k_2}}{q^{k_1+k_2}}, \quad \sup_{\vec{eta}} |h(\vec{eta})| = 1,
\]

and

\[
M \ll \sum_{i=0}^{k_1+k_2} \left| \frac{\partial}{\partial η_i} h(\vec{η}) \right| \ll q |\beta| \left( X_0^{k_0-1} + X_1^{k_1-1} + X_2^{k_2-1} \right) \ll q |\beta| n^{1-1/k}
\]
which gives

\[
\left| \int_{A(q, b)} h(\eta) d\eta - \sum_{\eta \in A(q, b)} h(\eta) \right| \ll \frac{n^{2+1/k_0}}{q^{k_1+k_2}} n^{1-1/k} |\beta| + \frac{n^{2+1/k_0-1/k}}{q^{k_1+k_2}} \frac{n^{2+1/k_0-1/k+\nu}}{q^{k_1+k_2}}.
\]

Writing \( q \eta + b = r \),

\[
\int_{A(q, b)} h(\eta) d\eta = \int_{A(q, b)} e \left( \beta F(q \eta + b) \right) d\eta = \int_{x_0 \mathcal{B}_0 \times x_1 \mathcal{B}_1 \times x_2 \mathcal{B}_2} e \left( \beta F(r) \right) d\tilde{r} \frac{1}{q^{1+k_1+k_2}} I(\beta).
\]

Since \( q = q_{\kappa(q)} \), one has

\[
\sum_{\tilde{b} \in \mathcal{B}_q} e \left( \frac{a}{q} F(\tilde{b}) \right) = |\tilde{R}_{Q/\kappa(q)}| \sum_{\tilde{b} \in \mathcal{B}_q} e \left( \frac{a}{q} F(\tilde{b}) \right) = |\tilde{R}_{Q/\kappa(q)}| \tilde{S}_{a,q}.
\]

It follows that

\[
\sum_{\tilde{z} \in \mathcal{B}(n)} e \left( \left( \frac{a}{q} + \beta \right) F(\tilde{z}) \right) = \sum_{\tilde{b} \in \mathcal{B}_q} \left( \frac{a}{q} F(\tilde{b}) \right) \frac{1}{q^{1+k_1+k_2}} I(\beta) + O \left( \frac{n^{2+1/k_0-1/k+\nu}}{q^{k_1+k_2}} \right) \]

\[
= \frac{|\tilde{R}_{Q/\kappa(q)}|}{(Q/\kappa(q))^{1+k_1+k_2}} \frac{\tilde{S}_{a,q}}{q^{1+k_1+k_2}} I(\beta) + O \left( \frac{|\tilde{R}_{Q/\kappa(q)}|}{(Q/\kappa(q))^{1+k_1+k_2}} \right) \]

\[
= \frac{|\tilde{R}_{Q/\kappa(q)}|}{Q^{1+k_1+k_2}} \frac{\kappa(q)^{1+k_1+k_2}}{|\tilde{R}_{\kappa(q)}|} \frac{\tilde{S}_{a,q}}{q^{1+k_1+k_2}} I(\beta) + O \left( \frac{Q^{2+1/k_0-1/k+\nu}}{q^{k_1+k_2}} \right)
\]

and the lemma follows.

We introduce typical notations for the major arcs now. Compared to the classical ones, the singular series contains some additional terms in its summands as we use the restricted sum over \( \tilde{\gamma} \in \mathcal{B}(n) \).
\[ \tilde{A}_n(q) = \sum_{a \mod^* q} \frac{\tilde{S}_{a,q}}{q^{1+k_1+k_2}} e\left(-\frac{a}{q}n\right). \quad \tilde{S}(X, n) = \sum_{q \leq X} \frac{\kappa(q)^{1+k_1+k_2}}{|\tilde{R}_{\kappa(q)}|} \tilde{A}_n(q) \]

\[ \tilde{\chi}_n(p) = \sum_{l=0}^{\infty} \frac{\kappa(p^l)^{1+k_1+k_2}}{|\tilde{R}_{\kappa(p^l)}|} \tilde{A}_n(p^l) \]

In particular \( \tilde{A}_n(1) = 1 \). Possible issues on the convergence of \( \tilde{\chi}_n(p) \) will be clarified later. We also define the singular integral

\[ \mathfrak{J}(c) = \int_{-c}^{c} \int_{\mathcal{B}} e(\gamma F(\vec{\zeta})) d\vec{\zeta} e(-\gamma) d\gamma. \]

**Theorem 2.2.**

(2.1) \[ \int_{\mathfrak{M}} \sum_{\vec{\gamma} \in \mathfrak{M}(n)} e(\alpha F(\vec{\gamma}')) e(-n\alpha) d\alpha \]

\[ = \frac{|\tilde{R}_Q|}{Q^{1+k_1+k_2}} \tilde{S}(n^\nu, n) \mathfrak{J}(n^\nu) n^{1+1/k_0} + O\left(n^{1+1/k_0-1/k+5\nu}\right). \]

**Proof.** Let \( \mathfrak{J}(c, n) = \int_{-c}^{c} I(\beta) e(-n\beta) d\beta \). Note that there are \( O(n^{2\nu}) \) pairs \((q, a)\) in the major arcs and each interval \( \mathfrak{M}_{(q, a)} \) is of length \( 2n^{\nu-1} \). From lemma 2.1

\[ \sum_{q \leq n^{\nu}} \sum_{a \mod^* q} \int_{\mathfrak{M}_{(q, a)}} \sum_{\vec{\gamma} \in \mathfrak{M}(n)} e(\alpha F(\vec{\gamma}')) e(-n\alpha) d\alpha \]

\[ = \sum_{q \leq n^{\nu}} \sum_{a \mod^* q} \frac{|\tilde{R}_Q|}{Q^{1+k_1+k_2}} \frac{\kappa(q)^{1+k_1+k_2}}{|\tilde{R}_{\kappa(q)}|} \frac{\tilde{S}_{a,q}}{q^{1+k_1+k_2}} e\left(-\frac{a}{q}n\right) \mathfrak{J}(n^{\nu-1}, n) \]

\[ + O\left(n^{2\nu n^{\nu-1}-2+1/k_0-1/k+2\nu}\right) \]

\[ = \frac{|\tilde{R}_Q|}{Q^{1+k_1+k_2}} \tilde{S}(n^\nu, n) \mathfrak{J}(n^{\nu-1}, n) + O\left(n^{1+1/k_0-1/k+5\nu}\right). \]
Put $\vec{r} = (X_0\zeta_0, X_1\zeta_1, \ldots, X_1\zeta_{k_1}, X_2\zeta_{k_1+1}, \ldots, X_2\zeta_{k_1+k_2})$ and $\gamma = n\beta$ so that $\beta F(\vec{r}) = \gamma F(\vec{\zeta})$. Then

$$J(n\nu^{-1}, n) = \int_{-n^{-1}}^{n^{-1}} \int_{X_0\mathbb{B}_0 \times X_1\mathbb{B}_1 \times X_2\mathbb{B}_2} e(\beta F(\vec{r})) d\vec{r} e(-n\beta) d\beta$$

$$= \int_{-n}^{n} X_0 X_1^{k_1} X_2^{k_2} \int_{\mathbb{B}_0} e(\gamma F(\vec{\zeta})) d\vec{\zeta} e(-\gamma) \frac{d\gamma}{n}$$

and (2.1) follows. \qed

3. The singular series I

Lemma 3.1. If $(q, r) = (a, q) = (b, r) = 1$ then $\tilde{S}_{\alpha q} \tilde{S}_{b r} = \tilde{S}_{\alpha r + b q r}$.

Proof. Write $\vec{z} = (z_0, \vec{x}, \vec{y})$ and $\vec{\zeta} = (\zeta_0, \vec{\alpha}, \vec{\beta})$.

$$\sum_{\tilde{\alpha} \in \tilde{R}_q} e\left(\frac{a}{q} F(\vec{z})\right) \sum_{\tilde{\zeta} \in \tilde{R}_r} e\left(\frac{b}{r} F(\vec{\zeta})\right)$$

$$= \sum_{\tilde{\alpha} \in \tilde{R}_q} \sum_{\tilde{\zeta} \in \tilde{R}_r} e\left(\frac{a}{q} F(\vec{z}) + \frac{b}{r} F(\vec{\zeta})\right)$$

$$= \sum_{z_0=1}^{q} \sum_{\tilde{x} \in \tilde{R}_q} \sum_{\tilde{\alpha} \in \tilde{R}_q} \sum_{\tilde{\alpha} \in \tilde{R}_q} \sum_{\tilde{\beta} \in \tilde{R}_r} e\left(\frac{a}{q} F(\vec{z}) + \frac{b}{r} F(\vec{\zeta})\right)$$

Let $s_0 = rz_0 + q\zeta_0$ so that $s_0$ takes every value modulo $qr$. Since $q \in O_K$ is a unit modulo $r O_K$, for any $\tilde{\alpha} \in \tilde{R}_K$ one has $N_K(q\tilde{\alpha}) \neq 0 \pmod{p}$ for all $p \in \tilde{P}$ that divides $r$, i.e., $q\tilde{\alpha} \equiv \tilde{\alpha}$ (mod $p$). Because the map $\tilde{\alpha} \cdot \tilde{\zeta} \mapsto q\tilde{\alpha} \cdot \tilde{\zeta}$ gives a bijection between systems of representatives of $O_K$ modulo $r O_K$, we obtain a bijection $q\tilde{R}_K \rightarrow \tilde{R}_K$. In the same manner, the set

$$\{ \vec{u} = r \vec{x} + q \tilde{\alpha} : \vec{x} \in \tilde{R}_q, \tilde{\alpha} \in \tilde{R}_K \}$$

is bijective to $\tilde{R}_K$ and so is

$$\{ \vec{u} = r \vec{y} + q \tilde{\beta} : \vec{y} \in \tilde{R}_K, \tilde{\beta} \in \tilde{R}_K \}$$
to $\tilde{R}_E^{qr}$. Therefore

$$
\sum_{z_0=1}^{q} \sum_{\xi_0=1}^{r} \sum_{\gamma_0=1}^{K} \sum_{\beta_0=1}^{E} e \left( \frac{a}{q} F(\tilde{z}) + \frac{b}{r} F(\tilde{\xi}) \right)
$$

$$
\sum_{\bar{z} \in \tilde{R}_E^{qr}} \sum_{\bar{\xi} \in \tilde{R}_E^{qr}} e \left( \frac{a}{q} N_K(\bar{\xi}) + \frac{b}{r} N_K(\bar{\xi}) \right)
$$

which is $\sum_{\bar{z} \in \tilde{R}_E^{qr}} e \left( \frac{ar+br}{qr} F(\tilde{z}) \right) = \tilde{S}_{ar+br,qr}$. \hfill \Box$

An immediate implication of lemma 3.1 is following

**Corollary 3.2.** If $(q, r) = 1$, then $\tilde{A}_n(q)\tilde{A}_n(r) = \tilde{A}_n(qr)$.

Observe that we obviously have $\frac{\kappa(q)^{k_1+k_2} \kappa(r)^{k_1+k_2}}{|R_{\kappa(q)}|} = \frac{\kappa(qr)^{k_1+k_2}}{|R_{\kappa(qr)}|}$ for $(q, r) = 1$. Hence if we assume the convergence of $\lim_{t \to \infty} \tilde{S}(t, n) = \tilde{S}(\infty, n)$ which will be established later, we get

$$
\tilde{S}(\infty, n) = \prod_{p \in \mathbb{P}} \tilde{\chi}_n(p).
$$

Let $\tilde{M}_n(q)$ be the number of solutions to $F(\tilde{z}) \equiv n \pmod{q}$ with $\tilde{z} \in \tilde{R}_q$.

**Lemma 3.3.** For $L \geq 1$

$$
\frac{\tilde{M}_n(p^L)}{p^{k_1+k_2}L} = \frac{|\tilde{R}_p|}{p^{k_1+k_2}} + \tilde{A}_n(p) + \tilde{A}_n(p^2) + \cdots + \tilde{A}_n(p^L).
$$

**Proof.** Let $q = p^L$. It is obvious that $\tilde{R}_{p^m}$ can be constructed from $\tilde{R}_p$ for any $m \geq 1$, namely

$$
\tilde{R}_{p^m} := \{ \tilde{z} \pmod{p^m} : \tilde{z} \equiv \tilde{R}_p \}
$$

so that $\frac{|\tilde{R}_{p^m}|}{p^{m(1+k_1+k_2)}} = \frac{|\tilde{R}_p|}{p^{k_1+k_2}}$. For $d = p^l$, $l \leq L$, write

$$
\tilde{S}_{a/d} \tilde{S}_{d/q} = \sum_{\tilde{z} \in \tilde{R}_q} e \left( \frac{a/d}{q/d} F(\tilde{z}) \right) = \begin{cases} 
\frac{d^{1+k_1+k_2} \tilde{S}_{a/d,q/d}}{|\tilde{R}_q|} & \text{if } d \neq q \\
\frac{d^{1+k_1+k_2} \tilde{S}_{a/d,q/d}}{|\tilde{R}_q|} & \text{if } d = q.
\end{cases}
$$
It follows that

\[ M_n(q) = \sum_{\zeta \in \mathbb{Z}} \frac{1}{q} \sum_{a=1}^{q} e\left(\frac{a}{q} (F(\zeta) - n)\right) \]

\[ = \frac{1}{q} \sum_{d|q} \sum_{a=1}^{q} \tilde{S}_a \frac{1}{d^2} \frac{1}{d} e\left(-\frac{a/d}{d} n\right) \]

\[ = \frac{1}{q} \left( L-1 \sum_{l=0}^{L-1} \sum_{a=1}^{q} \frac{\tilde{S}_a}{p^l \cdot p^{L-l}} \frac{1}{p^{1+k_1+k_2}} e\left(-\frac{a/p^l}{p^{L-l}} n\right) + |\tilde{R}_q| \right) \]

\[ = q^{k_1+k_2} \left( \sum_{l=0}^{L-1} \sum_{a=1}^{q} \frac{\tilde{S}_a/p^{l}}{p^{1+k_1+k_2}} e\left(-\frac{a/p^l}{p^{L-l}} n\right) + |\tilde{R}_q| \right) \]

\[ = q^{k_1+k_2} \left( \frac{|\tilde{R}_p|}{p^{1+k_1+k_2}} + \tilde{A}_n(p) + \tilde{A}_n(p^2) + \cdots + \tilde{A}_n(p^L) \right) . \]

The estimation of \( \tilde{A}_n(p^L) \) for \( p \in \mathbb{P} \) uses next lemma.

**Lemma 3.4.** Let \( f(\zeta) \) be an integral polynomial in \( t+1 \) variables for which \( \zeta = (\zeta_0, \zeta_1, \cdots, \zeta_t) \) is a solution to \( f(\zeta) \equiv n \) (mod \( p^M \)). Assume there is some component \( \zeta_i \) such that \( u_i = v_p\left(\frac{\partial f(\zeta)}{\partial \zeta_i}\big|_{\zeta = \zeta} \right) \leq \frac{M-1}{2} \). Let \( N_i(M) \) be the number of solutions \( \zeta \) modulo \( p^M \) such that \( \zeta_j \equiv \zeta_j \) (mod \( p^{2u_j+1} \)) for \( j \neq i, \xi_i \equiv \zeta_i \) (mod \( p^{2u_i+1} \)) and \( f(\zeta) \equiv n \) (mod \( p^M \)). Then \( N_i(M) = p^{i(M-2u_i-1)+u_i} \).

**Proof.** Without loss of generality, assume \( i = 0 \) and let \( \Delta_0 = \frac{\partial f(\zeta)}{\partial \zeta_0} \big|_{\zeta = \zeta} \),

\[ \Delta_0(a, b, c, \cdots) = \frac{\partial f(\zeta)}{\partial \zeta_0} \big|_{\zeta = (\zeta_0 + p^{u_0+1} a + p^{u_0+2} b + \cdots, \zeta_1, \cdots, \zeta_t)} \].

Note that

\[ v_p(\Delta_0(a, b, c, \cdots)) = a_0 \]
for all \(a, b, c, \cdots \in \mathbb{Z}\). The Hensel’s lemma can be applied to this situation in a form that starts from a higher power of \(p\):

\[
\begin{align*}
\tilde{f}(\tilde{z}_0 + p^{u_1+1}z_0, \cdots, \tilde{z}_\ell) & \equiv \tilde{f}(\tilde{z}) + \Delta_0 p^{u_0+2}z_0 \mod p^{2u_0+2}, \\
\tilde{f}(\tilde{z}_0 + p^{u_1+1}z_0 + p^{u_0+2}z_0', \cdots, \tilde{z}_\ell) & \equiv \tilde{f}(\tilde{z}_0 + p^{u_0+1}z_0, \cdots, \tilde{z}_\ell) + \Delta_0(p_0) p^{u_0+2}z_0' \mod p^{2u_0+3},
\end{align*}
\]

And this process goes on. Thus for any \(\xi, \cdots, \xi\) modulo \(p^M\) with \(\xi_j \equiv \xi_j \mod p^{2u_0+1}\), there exists a unique \(r\) modulo \(p^{M-2u_0-1}\) such that

\[
\tilde{f}(\tilde{z}_0 + p^{u_0+1}r, \cdots, \tilde{z}_\ell) \equiv n \mod p^M.
\]

Observe that this \(r\) can be lifted to a number modulo \(p^{M-u_0-1}\) in \(p^u\) distinct ways. The value of \(N_0(M)\) easily follows by counting the choices of \(\xi, \cdots, \xi\) and \(\tilde{z}_0 + p^{u_0+1}r\) modulo \(p^M\). □

Let \(\gamma = \gamma(p) = \min\{v_p(k_1), v_p(k_2)\}\).

**Lemma 3.5.** If \(p \in \mathbb{P}\) then \(\tilde{A}_n(p^L) = 0\) for all \(L > 2\gamma + 1\).

**Proof.** Note that \(p^{(k_1+k_2)2} \tilde{A}(p^L) = \tilde{M}_n(p^L) - p^{k_1+k_2} \tilde{M}_n(p^{L-1})\) for \(L \geq 2\). We show that all of the solutions that are counted in \(\tilde{M}_n(p^L)\) and \(\tilde{M}_n(p^{L-1})\) cancel out if \(L \geq 2\gamma + 2\). Assume \(\gamma = v_p(k_1)\). Suppose \(\tilde{z} = (z_0, \vec{x}, \vec{y}) \in \tilde{R}_{p^m}\) is a solution counted in \(\tilde{M}_n(p^m)\) where \(p \in \mathbb{P}\). Let \(G = \text{Gal}(K|\mathbb{Q})\). Then \(N_{K}(\vec{x}) = \prod_{\sigma \in G} \left(x_1\varphi_1^\sigma + \cdots + x_{k_1}\varphi_{k_1}^\sigma\right)\), and we have

\[
\frac{\partial N_{K}(\vec{x})}{\partial x_i} = \sum_{\sigma \in G} \varphi_i^\sigma \prod_{\tau \in G, \tau \neq \sigma} \left(x_1\varphi_1^\tau + \cdots + x_{k_1}\varphi_{k_1}^\tau\right)
\]

\[
= \text{Tr}_{K|\mathbb{Q}} \left(\varphi_i \prod_{\sigma \in G, \sigma \neq \text{id}} \left(x_1\varphi_1^\sigma + \cdots + x_{k_1}\varphi_{k_1}^\sigma\right)\right)
\]

\[
= N_{K}(\vec{x}) \text{Tr}_{K|\mathbb{Q}} \left(\frac{\varphi_i}{x_1\varphi_1 + \cdots + x_{k_1}\varphi_{k_1}}\right).
\]

Hence

\[
x_1\frac{\partial N_{K}(\vec{x})}{\partial x_1} + \cdots + x_{k_1}\frac{\partial N_{K}(\vec{x})}{\partial x_{k_1}} = N_{K}(\vec{x}) \text{Tr}_{K|\mathbb{Q}}(1) = k_1N_{K}(\vec{x}).
\]

But \(\vec{x} \in \tilde{R}_{p^m}\), i.e., \(N_{K}(\vec{x}) \equiv 0 \mod p\) which implies \(x_i\frac{\partial N_{K}(\vec{x})}{\partial x_i} \equiv 0 \mod p^{\gamma+1}\) for some \(i\). As described in the proof of lemma 3.4, this solution \(\vec{x}\) is one of the lifts of \(\tilde{z} \in \tilde{R}_{p^{2\gamma+1}}\) when \(m \geq 2\gamma + 1\) and so the solutions \(\vec{x} \in \tilde{R}_{p^L}\) and \(\vec{z} \in \tilde{R}_{p^{L-1}}\) that are counted in \(\tilde{M}_n(p^L)\) and \(\tilde{M}_n(p^{L-1})\) are all lifts of the solutions \(\vec{x} \in \tilde{R}_{p^{2\gamma+1}}\). Lemma 3.4 shows that the number of such lifts
grows by a factor of $p^{k_1+k_2}$ for each increment of $m$ in the modulus $p^m$ for $m \geq 2\gamma + 1$. Thus all of them are canceled in $\tilde{M}_n(p^L) - p^{k_1+k_2}\tilde{M}_n(p^{L-1})$. □

The estimation of $\tilde{A}_n(p^L)$ for $p \notin \tilde{\Phi}$ is a bit more complicated. Contrary to the classical problems in additive number theory for which the exponential sum $S_{a,p^L}$ splits into the product of many exponential sums over single variable, $S^K_{a,p^L}$ does not behave in the same way because $N_K(\overrightarrow{a})$ has many off-diagonal terms of the form $x_1^{a_1}x_2^{a_2} \cdots x_k^{a_k}$. Instead of obtaining a bound of exponential sums over a single variable, therefore, we focus on the properties of the norm map $N_K|Q$ and it is here that the class field theory plays a role. A successful bound for $S^K_{a,p^L}$ comes in following sections.

4. ALGEBRAIC LEMMAS FOR THE SINGULAR SERIES

For the estimation of the exponential sum $S_{a,q}$, we translate the summands $\overrightarrow{a}$ in $S^K_{a,p^L}$ to a system of well-chosen representatives of the quotient ring $O_K/p^L O_K$.

Lemma 4.1. Let $e, f, g$ be the ramification index, inertial degree and decomposition number of $p$ in $K|Q$ and write $pO_K = p_1^{e_1} \cdots p_g^{e_g}$. Let

$$\alpha^{(m)} \in \left( \prod_{i \in \{1, \ldots, g\} \setminus \{m\}} p_i^{e_i L} \right) \setminus \mathfrak{p}_m$$

and for each $m$ with $1 \leq m \leq g$, let $\{r_1^{(m)}, r_2^{(m)}, \ldots, r_{p^L}^{(m)}\}$ be a system of representatives of $O_K$ modulo $\mathfrak{p}_m^{p^L}$. Then

$$S^K_{a,p^L} = \sum_{i_1=1}^{p^{e_1 L}} \cdots \sum_{i_g=1}^{p^{e_g L}} e \left( \frac{a}{q} N_K|Q(\alpha^{(1)} r_1^{(1)} + \cdots + \alpha^{(g)} r_g^{(g)}) \right).$$

Proof. Let $G = Gal(K|Q)$. We first recall that $N_K(\overrightarrow{a}) \in \mathbb{Z}[x_1, \ldots, x_k]$. Indeed, the coefficient of each term $x_1^{a_1} \cdots x_k^{a_k}$ in $N_K(\overrightarrow{a}) = \prod_{\sigma \in G}(x_1^{\varphi_1^\sigma} + \cdots + x_k^{\varphi_k^\sigma})$ is an algebraic integer which is invariant under every $\sigma \in G$, so it is a rational integer. Hence for any integral vector $\overrightarrow{v}$ that is congruent to $\overrightarrow{a}$ modulo $p^L$ one has $N_K(\overrightarrow{v}) \equiv N_K(\overrightarrow{a}) \pmod{p^L}$. Since $x_1 \varphi_1 + \cdots + x_k \varphi_k \equiv v_1 \varphi_1 + \cdots v_k \varphi_k \pmod{p^L}$ if and only if $x_i \equiv v_i \pmod{p^L}$ for all $i$, clearly

$$S^K_{a,p^L} = \sum_{\gamma \in R} e \left( \frac{a}{p^L} N_K|Q(\gamma) \right)$$

for any system $R$ of representatives of $O_K/p^L O_K$. 


We may work on a more general situation like the following. Let $I, J$ be integral ideals of $O_K$ that are relatively prime and $q, r$ be their absolute norms. Note that $q, r$ need not be relatively prime. Let $\{t_1, \cdots, t_q\}$ and $\{u_1, \cdots, u_r\}$ be systems of representatives of $O_K/I$ and $O_K/J$. Suppose we put $v_{i,j} = \beta t_i + \alpha u_j$ for some $\alpha \in I$, $\beta \in J$. Then $v_{i,j} \equiv \beta t_i \mod I$, and hence $v_{i,j} \equiv v_{i',j'} \mod I \iff \beta(t_i - t_{i'}) \in I$. If $\beta + I$ is not a zero divisor of $O_K/I$, this is equivalent to $t_i \equiv t_{i'} \mod I$. Thus $v_{i,j}$ for $1 \leq i \leq q$, $1 \leq j \leq r$ form a system of representatives of $O_K/\{IJ\}$ if and only if $\beta + I$ and $\alpha + J$ are not zero divisors in $O_K/I$ and $O_K/J$ respectively.

Because $I$ and $J$ are relatively prime, $I + J = O_K$ and there exist $\alpha \in I$ and $\beta \in J$ such that $\alpha \equiv 1 \mod J$ and $\beta \equiv 1 \mod I$. $1 + I$ is clearly a unit in $O_K/I$ (in particular, not a zero divisor). The existence of $\alpha, \beta$ that satisfies the conditions mentioned above follows from this.

As an obvious generalization, assume $I^{(1)}, \cdots, I^{(g)}$ are integral ideals that are relatively prime and let $\nu_m = N(I^{(m)})$. If $\{r_{i_1}^{(m)}, r_{i_2}^{(m)}, \cdots, r_{i_{\nu_m}}^{(m)}\}$ is a system of representatives of $O_K/I^{(m)}$, there exist $\alpha^{(1)}, \cdots, \alpha^{(g)}$ with $\alpha^{(m)} \in \prod_{i \neq m} I^{i \pm 1}$ for which $\alpha^{(m)} + I^{(m)}$ is not a zero divisor in $O_K/I^{(m)}$ for all $m$.

Writing $v_{i_1, \cdots, i_g} = \alpha^{(1)} r_{i_1}^{(1)} + \cdots + \alpha^{(g)} r_{i_g}^{(g)}$, the set

$$\{v_{i_1, \cdots, i_g} : 1 \leq i_1 \leq \nu_1, \cdots, 1 \leq i_g \leq \nu_g\}$$

forms a system of representatives of $O_K/I^{(1)} \cdots I^{(g)}$. With the substitution $I^{(m)} = p_m^{eL}$ and the choice of $\alpha^{(m)}$ as stated in the lemma, $\alpha^{(m)}$ is trivially not a zero divisor of $O_K/p_m^{eL}$. This completes the proof. \qed

The following is a well-known fact in algebraic number theory (for example, see chapter 8 of [8].)

**Proposition 4.2.** Let $p$ be a nonzero prime ideal of $O_K$ and $m \geq 1$. Let $\Gamma$ be a system of representatives of $O_K$ modulo $p$ containing $0$. Let $t \in p \setminus p^2$. Then $\Delta = \{s_0 + s_1 t + \cdots + s_{m-1} t^{m-1} : s_i \in \Gamma\}$ is a system of representatives of $O_K$ modulo $p^m$.

Let $e, f, g$ be as in lemma [4.1]. We want to choose a system of representatives of $O_K$ modulo $p^m$ in a nice way. Consider the ideal class group $\mathcal{C}$ of $K$ and a class $[I] \in \mathcal{C}$ containing $I$. An analogue of Dirichlet’s theorem on primes in arithmetic progression is that the prime ideals of $O_K$ are equi-distributed among ideal classes in $\mathcal{C}$ on a probabilistic point of view (chapter 11 of [7]). We simply take a weaker form of this for granted that every ideal class contains infinitely many prime ideals.

Firstly, consider $[p_1] \in \mathcal{C}$. Then there is a prime ideal $\mathfrak{a}_1$ such that $[\mathfrak{a}_1] = [p_1]^{-1}$ in $\mathcal{C}$ and $gcd(\mathfrak{a}_1, p_1 p_2 \cdots p_g) = O_K$. Now $p_1 \mathfrak{a}_1$ is principal, say,
Let $\mathcal{R}$ be a system of representatives of $O_K/pL \mathcal{O}_K$ and $\mathcal{R}^* = \{ r \in \mathcal{R} : p \nmid N_{K/Q}(r) \}$. Let $e$, $f$, $g$ be as in lemma 4.1. Note that $\tilde{S}^K_{a,pL} = S^K_{a,pL}$ if $p \not\in \mathcal{P}$. 

5. The singular series II
Lemma 5.1. Write $S^K_{a,p^L} = S^{K,1}_{a,p^L} + S^{K,2}_{a,p^L}$ where

$$S^{K,1}_{a,p^L} = \sum_{\gamma \in \mathcal{R}^*} e\left(\frac{a}{p^L} N_K\mathcal{O}(\gamma)\right), \quad S^{K,2}_{a,p^L} = \sum_{\gamma \in \mathcal{R} \setminus \mathcal{R}^*} e\left(\frac{a}{p^L} N_K\mathcal{O}(\gamma)\right).$$

(1) If $L \leq f$ then $S^{K,2}_{a,p^L} = \sum_{i=1}^{g} (-1)^{i-1}(\gamma^i)p^{k_1-L-if}$
(2) If $L > 1$ and $p \nmid k_1$ then $S^{K,1}_{a,p^L} = 0$
(3) If $L = 1$ and $p$ is unramified in $K|\mathbb{Q}$ then $S^{K,1}_{a,p^L} = -\frac{(p^f-1)^g}{p-1}$.

Proof. Assume $L \leq f$. Since $p \mid N_K|\mathbb{Q}(\gamma)$ if and only if $p^f \mid N_K|\mathbb{Q}(\gamma)$, in this case $S^{K,2}_{a,p^L}$ merely counts the number of nonunits of $O_K/p^LO_K = O_K/p^{1L} \cdots p^{eL}$. Let $T$ be the number of nonunits in $O_K/pO_K$. The number of units in $O_K/pO_K$ is $(p^{ef} - p^{(e-1)f})^g$, so

$$T = p^{k_1} - (p^{ef} - p^{(e-1)f})^g = p^{k_1} - p^{efg} (1 - p^{-f})^g$$

$$= \sum_{i=1}^{g} (-1)^{i-1}\left(\frac{g}{i}\right)p^{k_1-1}$$

and $S^{K,2}_{a,p^L} = p^{(L-1)k_1}T = \sum_{i=1}^{g} (-1)^{i-1}\left(\frac{g}{i}\right)p^{k_1-1}$. As for $S^{K,1}_{a,p^L}$, first assume $L > 1$ and $p \nmid k_1$. Let $\overrightarrow{\gamma}$ be given by $\gamma = \overrightarrow{x} \cdot \overrightarrow{\varphi}$. Since $p \nmid k_1$, $p \nmid N_K(\overrightarrow{x})$ implies that there exists $i$ such that $\frac{\partial N_K(\overrightarrow{x})}{\partial x_i} \equiv 0 \pmod{p}$ as shown in the proof of lemma 3.5. If $m$ is any integer such that $m \equiv N_K(\overrightarrow{x}) \pmod{p}$, lemma 3.4 shows that the number of $\overrightarrow{x}$ modulo $p^{L-1}$ satisfying $N_K(\overrightarrow{x} + p\overrightarrow{u}) \equiv m \pmod{p^L}$ is $p^{(k_1-1)(L-1)}$. Thus

$$S^{K,1}_{a,p^L} = p^{(k_1-1)(L-1)} \sum_{\overrightarrow{x} \equiv \overrightarrow{m} \pmod{p}} \sum_{\overrightarrow{z} \equiv 0 \pmod{p \mid N_K(\overrightarrow{x})}} e\left(\frac{a}{p^L} (N_K(\overrightarrow{x}) + pz)\right)$$

$$= 0.$$

Now assume $L = 1$ and $p$ is unramified in $K|\mathbb{Q}$ so that $N_K|\mathbb{Q}(r)$ takes every nonzero value modulo $p$. Considering $N_K|\mathbb{Q}(ru)$ for $u \in \mathcal{R}^*$, it is easy to see that the number of $r \in \mathcal{R}^*$ satisfying $N_K|\mathbb{Q}(r) \equiv m \pmod{p}$ is the
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same for each value of \( m = 1, 2, \cdots, p - 1 \). It follows that

\[
\sum_{r \in \mathcal{R}^*} e \left( \frac{a}{p} N_{K|Q}(r) \right) = \frac{(p^e f - p^{(e-1)f})^g}{p - 1} \sum_{m=1}^{p-1} e \left( \frac{a}{p} m \right)
\]

\[
= -p^{k_1-f} (p^f - 1)^g \frac{1}{p - 1}
\]

\[
= - \frac{(p^f - 1)^g}{p - 1}.
\]

\[\square\]

We include a classical bound of an exponential sum for convenience. Let \( S_{a,q}^0 = \sum_{m=1}^{q} e \left( \frac{a}{q} m^{k_0} \right) \).

**Lemma 5.2** (Theorem 4.2 of [9]). For \((a, q) = 1\), \( S_{a,q}^0 \ll q^{1/k_0} \).

Let \( e, f, g \) be as in lemma 4.1, \( t_1, \cdots, t_g \) and \( \alpha^{(1)}, \cdots, \alpha^{(g)} \) as described in section 4. Choose a system \( \mathcal{R} \) of representatives of \( O_K/p^L O_K \) whose elements are of the form given by (4.1).

**Lemma 5.3.** Let \( p \notin \mathbb{P} \). For \( L \geq 1 \),

\[
|S_{a,p^L}^K| \ll (p^L)^{k_1-1+2 \log_{p} k_1} \quad \text{and} \quad |S_{a,p^L}^E| \ll (p^L)^{k_2-1+2 \log_{p} k_2}.
\]

**Proof.** We prove the first inequality. Write \( L = uf + v, 1 \leq v \leq f \) and let \( \mathcal{R}_m = \{ r \in \mathcal{R} : r \in p^m \} \). The case \( L = 1 \) easily follows from lemma 5.1 and 5.2 so we assume \( L > 1 \). Since \( p \nmid k_1 \), by lemma 5.1 and the inclusion-exclusion principle

\[
S_{a,p^L}^K = S_{a,p^L}^{K,2}
\]

\[
= \sum_{r \in \bigcup_{m=1}^{g} \mathcal{R}_m} e \left( \frac{a}{p^{L-f}} N_{K|Q}(r) \right)
\]

\[
= \sum_{l=1}^{g} (-1)^{l-1} \sum_{i_1 < \cdots < i_l} \sum_{r \in \bigcap_{m=1}^{g} \mathcal{R}_{im}} E(i_1, \cdots, i_l)
\]

where

\[
E(i_1, \cdots, i_l) = e \left( \frac{a}{p^{L-f}} N_{K|Q}(t_{i_1} t_{i_2} \cdots t_{i_l}) \right) N_{K|Q} \left( \frac{r}{t_{i_1} t_{i_2} \cdots t_{i_l}} \right).
\]

Let \( a_{(i_1, \cdots, i_l)} = a N_{K|Q}(t_{i_1} t_{i_2} \cdots t_{i_l}) \). By the choice of \( \mathcal{R} \), we have \((a_{(i_1, \cdots, i_l)}, p) = 1 \) and \( \frac{r}{t_{i_1} t_{i_2} \cdots t_{i_l}} \in O_K \) when \( r \in \bigcap_{m=1}^{g} \mathcal{R}_{im} \). Observe that \( \left\{ \frac{r}{t_{i_1} t_{i_2} \cdots t_{i_l}} : r \in \bigcap_{m=1}^{g} \mathcal{R}_{im} \right\} \).
\( \bigcap_{m=1}^{l} \mathcal{R}_{i_m} \) runs through a system of representatives modulo \( p^{f-1}fO_K \) by \( p^{f-1}f \) times. Thus we can write
\[
S_{a,p}^{Kj} = \sum_{l=1}^{g} (-1)^{l-1} \sum_{i_1 < \cdots < i_l} p^{f(k_1-1)f} S_{a(i_1,\ldots,i_l),p}^{Kf}.
\]
Let \( M(x) \) be the maximum value of \( |S_{b,p}^{Kj}| \) among all \( b \neq 0 \) (mod \( p \)). Then
\[
|S_{a,p}^{Kj}| \leq \left( \frac{g}{1} \right) p^{f(k_1-1)} M(L - f) + \left( \frac{g}{2} \right) p^{2f(k_1-1)} M(L - 2f) + \cdots
\]
\[
\cdots + \left( \frac{g}{g} \right) p^{gf(k_1-1)} M(L - gf).
\]
Let \( \theta(x) = \theta_K(x) \) be the smallest number for which \( M(x) \leq p^{x(k_1-1+\theta(x))} \). It follows that
\[
|S_{a,p}^{Kj}| \leq g \cdot \max_m \left\{ \left( \frac{g}{m} \right) p^{m f(k_1-1)} M(L - mf) \right\}
\]
\[
\leq g \cdot \max_m \left\{ \left( \frac{g}{m} \right) p^{m f(k_1-1) + (L - mf)(k_1-1+\theta(L-mf))} \right\}
\]
\[
\leq \max_m \left\{ p^{(L-mf)(k_1-1+\theta(L-mf)) + mf(k_1-1+\frac{\log p g}{m f} + \frac{\log p g}{m f})} \right\}
\]
whence for some \( m \)
\[
L(k_1 - 1 + \theta(L)) \leq L(k_1 - 1 + \theta(L-mf)) + mf(-\theta(L-mf) + \frac{m + 1}{mf} \log_p g)
\]
or
\[
L \theta(L) \leq \max_m \left\{ (L - mf) \theta(L-mf) + (m + 1) \log_p g \right\}.
\]
Inductively, one immediately has
\[
(uf + v) \theta((uf+v)) \leq v \theta(v) + 2u \log_p g.
\]
Assume \( v = 1 \) first. By lemma 5.1, \( S_{a,p}^{Kj} = -\frac{(p^f-1)^g}{p-1} + \sum_{i=1}^{g} (-1)^{i-1} \binom{g}{i} p^{k_1-1} \). Here
\[
\frac{(p^f-1)^g}{p-1} = (p^{f-1} + p^{f-2} + \cdots + 1) (p^f-1)^{g-1}
\]
\[
< 2p^{f-1}p^{f(g-1)} = p^{k_1-1+\log_p 2},
\]
and observe that \( 0 < \sum_{i=1}^{g} (-1)^{i-1} \binom{g}{i} p^{k_1-1} \) is not \( \leq gp^{k_1-1} \). Thus
\[
|S_{a,p}^{Kj}| \leq \max \left\{ p^{k_1-1+\log_p 2}, gp^{k_1-1} \right\} \leq k_1 p^{k_1-1} = p^{k_1-1+\log_p k_1}.
\]
Now assume $v > 1$. Then
\[
|S_{a,p}^{K,v}| = \left| \sum_{i=1}^{g} (-1)^{i-1} \binom{g}{i} p^{vk_1-if} \right|
\leq k_1 p^{vk_1-f} = p^{v(k_1-1+1-\frac{\log p k_1}{v})}
\]
and $1 - \frac{\log p k_1}{v} \leq \frac{\log p k_1}{f}$ where the equality holds when $v = f$. We have proved that $v\theta(v) \leq \log p k_1$ in both cases; hence from (5.1)
\[
\theta(u f + v) \leq \frac{\log p k_1 + 2u \log p g}{u f + v} \leq \frac{(2u + 1) \log p k_1}{u + 1} < 2 \log p k_1.
\]
\[
\square
\]

The next one is an immediate corollary.

**Lemma 5.4.** If $p \notin \mathbb{P}$ then there exists an absolute constant $\delta > 0$ such that $\mathbb{A}_n(p^L) \ll \frac{1}{(p^L)^{1+\delta}}$.

**Proof.** Since $S_{a,p}^L = S_{a,p}^L$ for $p \notin \mathbb{P}$, by lemma 5.2 and 5.3
\[
\mathbb{A}_n(p^L) \ll \frac{1}{(p^L)^{1+k_1+k_2}} \cdot p^L \cdot |S_{a,p}^0 S_{a,p}^K S_{a,p}^L| \\
\ll p^L \left(-k_1-k_2+(1-\frac{1}{k_0})+(k_1-1+2 \log p k_1)+(k_2-1+2 \log p k_2)\right) \\
< p^L \left(-1-\frac{1}{k_0}+2 \log p k_1 k_2\right)
\]
and $2 \log p k_1 k_2 < \frac{1}{k_0}$ because $p \notin \mathbb{P}$.  

Now the singular series is estimated.

**Theorem 5.5.** There exist positive absolute constants $c_1$, $c_2$ that depend only on $K$ and $E$ such that $c_1 < \tilde{\mathcal{S}}(n^\nu, n) < c_2$ for all sufficiently large $n$.

**Proof.** Since $1 \leq \frac{r(q)^{1+k_1+k_2}}{|R_{\kappa(q)}|} \leq \frac{Q^{1+k_1+k_2}}{|R_q|} \ll 1$, the absolute convergence of
\[
\lim_{t \to \infty} \tilde{\mathcal{S}}(t, n) = \tilde{\mathcal{S}}(\infty, n)
\]
follows from corollary 3.2, lemma 3.5 and 5.4. More precisely we have
\[
|\tilde{\mathcal{S}}(\infty, n) - \tilde{\mathcal{S}}(n^\nu, n)| \ll \frac{1}{n^{\nu \delta}}
\]
so it suffices to show that $0 < c_1 < \tilde{\mathcal{S}}(\infty, n) = \prod_{p \in \mathbb{P}} \tilde{\chi}_n(p) < c_2$ for some constants $c_1$ and $c_2$. 

For $p \not\in \mathbb{P}$, \(\frac{\kappa(p)^{1+k_1+k_2}}{|R_{\kappa(p)}|} = 1\) so lemma 5.4 gives

\[
|\tilde{\chi}_n(p) - 1| \leq \sum_{L=1}^{\infty} |\tilde{A}_n(p^L)| \ll \sum_{L=1}^{\infty} \frac{1}{(p^L)^{1+\delta}} \ll \frac{1}{p^{1+\delta}}
\]

which implies that there exists a prime $p_0$ depending only on $K$ and $E$ such that

\[
\frac{1}{2} < \prod_{p \leq p_0} \tilde{\chi}_n(p) < \frac{3}{2}.
\]

Suppose $p < p_0$. Recall that every prime $p$ is unramified in at least one of $K|Q$ and $E|Q$, so assume $p$ is unramified in $K|Q$. For $L \geq 2$, $M_n(p^L) \geq N_n(p^L)$ where

\[
N_n(p^L) = \left| \left\{ \vec{z} \in \mathcal{O}_{p^L} : F(\vec{z}) \equiv n \mod p^L, N_K(\vec{z}) \neq 0 \mod p \right\} \right|.
\]

As in the proof of lemma 5.1, the number $s(m)$ of \(\vec{z}\) modulo $p^L$ satisfying $N_K(\vec{z}) \equiv m \mod p^L$ is the same for all $m \neq 0 \mod p$. Since there are $(p^L - 1)^g p^{(L-1)k_1}$ units in $O_K/p^t O_K = p_1^{t_1} \cdots p_r^{t_r}$, it is easy to see that $s(m) = \frac{(p^L - 1)^g p^{(L-1)k_1}}{(p-1)p^{L-1}}$. Let $h(L)$ be the number of \((z_0, \vec{y})\) modulo $p^L$ such that $\vec{y} \in \mathcal{E}_{p^L}$ and $z_0 + N_E(\vec{y}) - n \neq 0 \mod p$. For each $\vec{y} \in \mathcal{E}_{p^L}$, if $N_E(\vec{y}) - n \equiv 0 \mod p$ then there are $p^L - p^{L-1} z_0$’s counted by $h(L)$. Otherwise there are at least $p^{L-1} z_0$’s, so $h(L) \geq |\mathcal{E}_{p^L}| p^{L-1} = |\mathcal{E}_{p^L}| p^{(L-1)(k_2+1)}$. It follows that

\[
N_n(p^L) \geq s(1) h(L) \geq \frac{(p^L - 1)^g}{p^L} p^{(L-1)(k_1-1)+(L-1)(k_2+1)}|\mathcal{E}_{p^L}|
\]

\[
\geq (p-1)^{k_1-1} |\mathcal{E}_{p^L}| p^{(k_2+1)(L-1)}.
\]

Let $e', f', g'$ be the ramification index, inertial degree and decomposition number of $p$ in $E|Q$. Then

\[
|\mathcal{E}_{p^L}| \geq |(O_E/pO_E)^*| = (p^{e'f'} - p^{(e'-1)f'})^{g'} = p^{k_2} \left(1 - \frac{1}{p^{e'}} \right)^{g'} \geq \left(\frac{p}{2}\right)^{k_2},
\]

and hence

\[
\frac{N_n(p^L)}{p^{k_2+1}} \geq \frac{(p-1)^{k_1-1}}{2^{k_2} p^k}.
\]

Now write $\chi_n^{(L)}(p) = \sum_{t=0}^{L} \frac{\kappa(p)^{1+k_1+k_2}}{|R_{\kappa(p)}|} \tilde{A}_n(p^t)$. By lemma 3.3

\[
\tilde{\chi}_n^{(L)}(p) = \frac{\kappa(p)^{1+k_1+k_2}}{|\mathcal{E}_{p^L}| p^{k_2+1}} \tilde{M}_n(p^L) \geq \frac{\kappa(p)^{1+k_1+k_2}}{|\mathcal{E}_{p^L}| p^{k_2+1}} \left(\frac{p-1}{2}\right)^{k_2} =: u_p
\]

and so $\tilde{\chi}_n(p) \geq u_p > 0$. 
Theorem 6.1. We can choose \( \mathfrak{B} \) so that \( J(n^\nu) \to J_0 > 0 \) as \( n \to \infty \).

Proof. Choose small positive numbers \( \phi_1, \ldots, \phi_{k_1+k_2} \) so that the real value \( \phi_0 \) that makes \( \phi_0^k + N_K(\phi_1, \ldots, \phi_{k_1}) + N_E(\phi_{k_1+1}, \ldots, \phi_{k_1+k_2}) = 1 \) is positive, not equal to 1, and hence

\[
\frac{\partial F}{\partial \phi_0} \neq 0 \quad \text{and} \quad N_K(\phi_1, \ldots, \phi_{k_1}) + N_E(\phi_{k_1+1}, \ldots, \phi_{k_1+k_2}) \neq 0.
\]

Then \( \vec{\phi} = (\phi_0, \ldots, \phi_{k_1+k_2}) \) is a nonsingular solution to \( F(\vec{\phi}) = 1 \). Let \( \mathfrak{B} \) be a box centered at \( \vec{\phi} \) with side lengths \( 2\lambda \). Write

\[
J(\mu) = \int_{-\mu}^\mu \int_{\mathfrak{B}} e \left( \gamma F(\vec{\zeta}) \right) d\vec{\zeta} e(-\gamma) d\gamma
\]

\[
= \int_{\mathfrak{B}} \frac{\sin 2\pi \mu (F(\vec{\zeta}) - 1)}{\pi (F(\vec{\zeta}) - 1)} d\vec{\zeta}
\]

\[
= \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \frac{\sin 2\pi \mu (F(\vec{\phi} + \vec{\theta}) - 1)}{\pi (F(\vec{\phi} + \vec{\theta}) - 1)} d\vec{\theta}
\]

Write \( F(\vec{\phi} + \vec{\theta}) - 1 = c_0 \theta_0 + \cdots + c_{k_1+k_2} \theta_{k_1+k_2} + P_2(\vec{\theta}) + \cdots + P_k(\vec{\theta}) \) where \( P_m(\vec{\theta}) \) is a homogeneous polynomial of degree \( m \). Note that \( c_0 = \frac{\partial F}{\partial \theta_0} \bigg|_{\vec{\theta} = 0} = k_0 \phi_0^{k_0-1} \neq 0 \). Now for \( r_0, r_1, r_2 \in \mathbb{R} \) consider the equation

\[
r_0^k \phi_0^{k_0} + r_1^k N_K(\phi_1, \ldots, \phi_{k_1}) + r_2^k N_E(\phi_{k_1+1}, \ldots, \phi_{k_1+k_2}) - 1 = 0.
\]
Since both of $\phi_0$ and $N_K(\phi_1, \ldots, \phi_{k_1})+N_E(\phi_{k_1+1}, \ldots, \phi_{k_1+k_2})$ are nonzero, one can choose $r_0, r_1, r_2 > 0$ such that $k_0(r_0\phi_0)^{k_{0-1}} = 1$ and still satisfying

$$F(r_0\phi_0, r_1\phi_1, \ldots, r_1\phi_{k_1}, r_2\phi_{k_1+1}, \ldots, r_2\phi_{k_1+k_2}) - 1 = 0$$
and

$$N_K(r_1\phi_1, \ldots, r_1\phi_{k_1}) + N_E(r_2\phi_{k_1+1}, \ldots, r_2\phi_{k_1+k_2}) \neq 0.$$

So we can assume $c_0 = k_0\phi_0^{k_{0-1}} = 1$ from the beginning.

For $|\overrightarrow{\theta}| < \lambda$, we have $|F(\overrightarrow{\phi} + \overrightarrow{\theta}) - 1| < \sigma$ where $\sigma = \sigma(\lambda)$ is small for small $\lambda$. Put $F(\overrightarrow{\phi} + \overrightarrow{\theta}) - 1 = t$, and consider this as a map from $\theta_0$ to $t$. Then, if $\lambda$ is sufficiently small, the inverse function theorem tells us that $\theta_0$ can be expressed in terms of $t, \theta_1, \ldots, \theta_{k_1+k_2}$ as a power series

$$\theta_0 = t - c_1\theta_1 - \cdots - c_{k_1+k_2}\theta_{k_1+k_2} + \mathcal{P}(t, \theta_1, \ldots, \theta_{k_1+k_2})$$

where $\mathcal{P}$ is a multiple power series whose least degree terms are of degree at least 2. Hence $\partial \theta_0/\partial t = 1 + \mathcal{P}_1(t, \theta_1, \ldots, \theta_{k_1+k_2})$ where $\mathcal{P}_1$ is a multiple power series without a constant term. By taking $\lambda$ sufficiently small, we can make $|\mathcal{P}_1(t, \theta_1, \ldots, \theta_{k_1+k_2})| < 1/2$ for $|\theta_1|, \ldots, |\theta_{k_1+k_2}| < \lambda, |t| < \sigma$. A change of variable from $\theta_0$ to $t$ gives

$$\mathcal{I}(\mu) = \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \frac{\sin 2\pi \mu(F(\overrightarrow{\phi} + \overrightarrow{\theta}) - 1)}{\pi(F(\overrightarrow{\phi} + \overrightarrow{\theta}) - 1)} d\theta_1 \cdots d\theta_{k_1+k_2} d\theta_0$$

$$\sim \int_{-\sigma}^{\sigma} \frac{\sin 2\pi \mu t}{\pi t} V(t) dt$$

where $V(t) = \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \left(1 + \mathcal{P}_1(t, \theta_1, \ldots, \theta_{k_1+k_2})\right) d\theta_1 \cdots d\theta_{k_1+k_2}$ and we wrote $a \sim b$ to mean that the limit of their ratio equals 1.

$V(t)$ is clearly a continuous function of $t$ for $|t|$ sufficiently small. We also observe that $V(t)$ has left and right derivatives at every value of $t$, and these derivatives are certainly bounded for $t$ in a small confined region. Therefore by Fourier integral theorem one has

$$\lim_{\mu \to \infty} \mathcal{I}(\mu) = \lim_{\mu \to \infty} \int_{-\sigma}^{\sigma} \frac{\sin 2\pi \mu t}{\pi t} V(t) dt = V(0) =: \mathcal{I}_0.$$

But

$$|V(0)| = \left| \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} (1 + \mathcal{P}_1(0, \theta_1, \ldots, \theta_{k_1+k_2})) d\theta_1 \cdots d\theta_{k_1+k_2} \right|$$

$$> \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \frac{1}{2} d\theta_1 \cdots d\theta_{k_1+k_2}$$

$$> 0$$

and the theorem follows. $\square$
We merely state the estimation on the minor arcs, which can be easily seen in [3].

**Theorem 6.2.** There exists $\delta > 0$ that depends only on $K$ and $E$ such that
\[
\int_{m} \sum_{\overrightarrow{z} \in \mathcal{B}(n)} e(\alpha F(\overrightarrow{z})) e(-n\alpha) d\alpha \ll n^{1+1/k_0-\delta}.
\]

7. Conclusion

By theorem 2.2, 5.5, 6.1 and 6.2 one has

**Theorem 7.1.** The number of representations, $\hat{r}(n)$, of $n$ in the form $z_0^{k_0} + N_K(\overrightarrow{x}) + N_E(\overrightarrow{y})$ with $\overrightarrow{z} = (z_0, \overrightarrow{x}, \overrightarrow{y}) \in \mathcal{B}(n)$ satisfies
\[
\hat{r}(n) = \frac{|R_Q|}{Q^{1+k_1+k_2}} \tilde{3}_0 \left( \prod_{p \in \mathcal{P}} \tilde{\chi}_n(p) \right) n^{1+1/k_0} + o(n^{1+1/k_0})
\]

where $1 \ll \tilde{3}_0 \prod_{p \in \mathcal{P}} \tilde{\chi}_n(p) \ll 1$.

8. The Optimality of Two Norms and One Variable

In this section we give an example which demonstrates the failure of local-global principle in the equation $N_K(\overrightarrow{x}) + N_E(\overrightarrow{y}) = n$, using an integer valued quadratic form. The famous 15 theorem and 290-theorem classify all of the universal quaternary quadratic forms ([2] and [1]), and Oh and Bochnak provided a complete classification of almost universal ones ([4]). Let $f = f(x, y, z, t)$ be a quaternary quadratic form. $f$ is said to be $p$-isotropic if it represents zero nontrivially over $\mathbb{Z}_p$.

**Theorem 8.1 ([4]).** For any positive definite integral quaternary quadratic form $f$, the followings are equivalent:

1. $f$ is almost universal
2. $f$ is locally universal and, either
   a. $f$ is $p$-isotropic for every prime number $p$, or
   b. $f$ is 2-anisotropic and represents every element of $\{16, 32, 48, 80, 96, 112, 160, 224\}$, or
   c. $f$ is equivalent to one of the following four forms
      - $x^2 + 2y^2 + 5z^2 + 10t^2$ (5-anisotropic)
      - $x^2 + 2y^2 + 3z^2 + 5t^2 + 2yz$ (5-anisotropic)
      - $x^2 + y^2 + 3z^2 + 3t^2$ (3-anisotropic)
      - $x^2 + 2y^2 + 4z^2 + 7t^2 + 2yz$ (7-anisotropic)

Here we borrow some of the basic theories about quadratic forms from [5]. Recall that one can readily assume a quadratic form to be diagonal, so we consider only diagonal ones in this section. Let $(a, b) = (a, b)_p = (a, b)_{\mathbb{F}_p} = \pm 1$
be the (Hilbert) norm residue symbol so that \((a, b)_p = 1\) if and only if \(ax^2 + by^2 - z^2\) is isotropic in \(\mathbb{Q}_p\).

**Proposition 8.2** (Lemma 2.6. in [5]). Let \(f = a_1x_1^2 + \cdots + a_nx_n^2\) with \(a_j \in \mathbb{Q}_p\). A necessary and sufficient condition that the quaternary form \(f\) be \(p\)-anisotropic is that it satisfies both of the following conditions:

1. \(d(f) \in (\mathbb{Q}_p^\times)^2\)
2. \(c(f) = -(-1, -1)\)

where \(d(f)\) is the discriminant of \(f\) and \(c(f)\) is the Hasse-Minkowski invariant, given by \(c(f) = c_p(f) = \prod_{i<j}(a_i, a_j)\).

**Proposition 8.3** (Theorem 1.1 in [5]). Let \(n(f)\) be the number of variables in \(f\). Suppose \(p \neq \infty\). Then \(n(f), d(f)\) and \(c(f)\) form a complete set of invariants of the equivalence class of \(f\).

The followings are well known properties of the norm residue symbol.

**Proposition 8.4** (Lemma 2.1. in [5]).

\[
(a, b) = (b, a), \quad (a_1a_2, b) = (a_1, b)(a_2, b),
\]
\[
(a, b_1b_2) = (a, b_1)(a, b_2).
\]
If \(p \neq 2, \infty\) and \(|a|_p = |b|_p = 1\), then \((a, b) = 1\). In particular, if any one of \(a, b\) is 1 then \((a, b) = 1\). (It also worths to include the fact \((-1, -1) = -1\) stated at the same place in [5]).

**Proposition 8.5** (Lemma 1.6 in [5]). Assume \(\alpha \in \mathbb{Q}_p^\times\). If

\[
|\alpha - 1|_p \leq \begin{cases} 1/p & \text{for } p \neq 2 \\ 1/8 & \text{for } p = 2 \end{cases}
\]

then \(\alpha \in (\mathbb{Q}_p^\times)^2\).

The following one asserts that whenever a quadratic form \(f\) is \(p\)-isotropic, it is \(p\)-adically universal.

**Proposition 8.6** (Corollary 2 in [5]). An isotropic space is universal.

Now we give an example. Let \(f_{a,b}(\vec{u}) = u_1^2 + u_2^2 + au_3^2 + bu_4^2\) whose discriminant is \(ab\) and

\[
c_p(f) = \prod_{i<j}(a_i, a_j)_p = (1, 1)_p(1, a)_p(1, b)_p(1, a)_p(1, b)_p(a, b)_p = (a, b)_p.
\]

**Proposition 8.7.** Let \(a, b\) be distinct odd primes and put \(f = f_{a,b}\). Then \(f\) is \(p\)-anisotropic if and only if

1. \(p = \infty\), or
2. \(p = 2\), and \(a \equiv b \equiv 1 \pmod{8}\) or \(a \equiv b \equiv 5 \pmod{8}\).

**Proof.** When \(p = \infty\), clearly the only solution to \(f(\vec{u}) = 0\) is \(\vec{u} = \vec{0}\) so \(f\) is is anisotropic. Hence we assume \(p \neq \infty\).
We first show that $(-1, -1)_p = -1$ only when $p = 2$. The case $p = 2$ is mentioned already. If $p \equiv 1 \pmod{4}$, $-1 \in (\mathbb{Q}_p^*)^2$ so $(-1, -1)_p = 1$. When $p \equiv 3 \pmod{4}$ we have \( \left( \frac{-1}{p} \right) = -1 \). Let $Q = \left\{ x \in \mathbb{Z} : \left( \frac{x}{p} \right) = 1 \right\}$ and $N = \left\{ x \in \mathbb{Z} : \left( \frac{x}{p} \right) = -1 \right\}$. Observe that every element of $Q$ is in $(\mathbb{Q}_p^*)^2$ by Hensel lemma. Now $(-1, -1)_p = 1 \iff -x^2 - y^2 - z^2$ is $p$-isotropic $\iff \{-x^2 - y^2 : x, y \in \mathbb{Z}\} \cap Q \neq \emptyset \iff (Q + Q) \cap N \neq \emptyset$. But $1 \in Q$, so $(Q + Q) \cap N \neq \emptyset$.

Combining this with proposition \( \ref{prop:290} \) we see that $f$ is $p$-anisotropic if and only if $d(f) = ab \in (\mathbb{Q}_p^*)^2$ and $(a, b)_p = 1$ when $p = 2$ and $(a, b)_p = -1$ when $p$ is an odd prime. Assume $p \neq 2$. If $p \mid ab$ then we can choose $x, y$ so that $ax^2 + by^2$ is in $Q$ or $N + N$. Both of them have nonempty intersection with $Q$, so $ax^2 + by^2 - z^2$ is isotropic. Thus $(a, b)_p = 1$ and $f$ is $p$-isotropic. If $p \nmid ab$, say, $p = a$ then $d(f) = ab \not\in (\mathbb{Q}_p^*)^2$ so $f$ is $p$-isotropic.

Now let $p = 2$. Since $|ab|_2 = 1$, a simple observation using proposition \( \ref{prop:congruent forms} \) is that $ab \in (\mathbb{Q}_p^*)^2$ if and only if $ab \equiv 1 \pmod{8}$ which is equivalent to $a \equiv b \pmod{8}$. Assume that this is the case. Then for a solution $x, y, z \in \mathbb{Z}_2$ to $ax^2 + by^2 = z^2$, one has $bz^2 \equiv abx^2 + b^2y^2 \equiv x^2 + y^2 \pmod{8}$. By homogeneity, we may assume max\{|$x|_2, |y|_2$\} = 1, say, 2 $\nmid x$ and $x^2 \equiv 1 \pmod{8}$ so the equation reduces to $1 + y^2 \equiv bz^2 \pmod{8}$. But $1 + y^2$ is congruent to one of \{1, 2, 5\} modulo 8, whereas $bz^2$ is to \{0, 4, b\}. It follows that $b \equiv 1$ or $5 \pmod{8}$. When $b \equiv 1 \pmod{8}$, we may put $y \equiv 0 \pmod{4}$ and $ax^2 + by^2 \equiv 1 + y^2 \equiv 1 \pmod{8}$ so that $ax^2 + by^2 \in (\mathbb{Q}_p^*)^2$, i.e., $(a, b)_2 = 1$ and hence $f$ is 2-anisotropic. When $b \equiv 5 \pmod{8}$, put $y \equiv 2 \pmod{4}$ so that $ax^2 + by^2 \equiv a + 4b \equiv 1 \pmod{8}$ and similarly $f$ is 2-anisotropic.

In the context of proposition \( \ref{prop:congruent forms} \), choose $a, b$ among prime numbers congruent to 1 modulo 8 that are sufficiently large so that $f$ cannot represent all of \{16, 32, 48, 80, 96, 112, 160, 224\} (in particular, 48 = $2^4 \cdot 3 \neq u_1^2 + u_2^2$). Note that $f(\overline{u}) = N_{\mathbb{Q}(\sqrt{-a})/\mathbb{Q}}(u_1 + u_3\sqrt{-a}) + N_{\mathbb{Q}(\sqrt{-b})/\mathbb{Q}}(u_2 + u_4\sqrt{-b})$. $f$ is locally universal, since $N_{\mathbb{Q}(\sqrt{-a})/\mathbb{Q}}$ takes every unit value in $\mathbb{Z}_p$ for $p \neq 2, a$ and the same holds for $N_{\mathbb{Q}(\sqrt{-b})/\mathbb{Q}}$ in $\mathbb{Z}_p$ when $p \neq 2, b$. When $p = 2$, $a$ and $b$ are in $(\mathbb{Z}_2^*)^2$ so $f$ is equivalent to $u_1^2 + u_2^2 + u_3^2 + u_4^2$ over $\mathbb{Z}_2$, which is universal by Lagrange theorem. But the discriminant $d(f) = ab$ is sufficiently large, i.e., $f$ cannot be equivalent to any one of those forms given in theorem \( \ref{thm:290} \)(c). Therefore $f$ is not almost universal.

References

1. M. Bhargava, *Universal quadratic forms and the 290-theorem*.
2. , *On the conway-schneeberger fifteen theorem*, Quadratic forms and their applications (Dublin), Contemporary Mathematics, vol. 272, American Mathematical Society, 1999, pp. 27–37.
3. B. J. Birch, H. Davenport, and D. J. Lewis, *The addition of norm forms*, Mathematika 9 (1962), 75–82.

4. J. Bochnak and B.K. Oh, *Almost universal quadratic forms: an effective solution of a problem of ramanujan*, MPI / Max-Planck-Institut für Mathematik, Bonn, 2007.

5. J.W.S. Cassel, *Rational quadratic forms*, London Mathematical Society Monographs, no. 13, 1978.

6. H. Davenport, *Analytic methods for diophantine equations and diophantine inequalities*, 2nd ed., Cambridge University Press, 2005.

7. J. Esmonde and M.R. Murty, *Problems in algebraic number theory*, 2nd ed., Graduate Texts in Mathematics, vol. 190, Springer, 2005.

8. P. Ribenboim, *Classical theory of algebraic numbers*, Springer, 2001.

9. R.C. Vaughan, *The Hardy-Littlewood method*, 2nd ed., Cambridge University Press, 1997.

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