Convergence of the EM method for NSDEs with time-dependent delay in the $G$-framework

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ABSTRACT

Consider a neutral stochastic differential equation (NSDE) with time-dependent delay $\delta(t)$ ($0 \leq \delta(t) \leq \tau$) in the $G$-framework

\[ d(x(t) - D(x(t - \delta(t)))) = f(x(t), x(t - \delta(t))) \, dt + g(x(t), x(t - \delta(t))) \, dB(t) + h(x(t), x(t - \delta(t))) \, dB(t), \]

where $B(t)$ denotes a $G$-Brownian motion and $(B(t))$ the quadratic variation process of $B(t)$. We introduce an Euler–Maruyama (EM) method for solving this equation and prove that the EM approximate solution converges to the exact solution with a strong order of the mean square closeness equal to one under the global Lipschitz condition. A numerical example is provided to illustrate the effectiveness of our method.

1. Introduction

Motivated by the problems of asset pricing, risk measures, financial decisions under model uncertainty, Peng established the framework of $G$-expectation ($G$-framework), $G$-Brownian motion and related stochastic calculus of Itô type (see Peng, 2007, 2010). Since then, the theory of stochastic differential equations driven by $G$-Brownian motion ($G$-SDEs) has been extensively studied (see Denis, Hu, & Peng, 2011; Faizullah, 2016; Fei & Fei, 2019; Luo & Wang, 2014; Ren, Yin, & Sakthivel, 2018; Wei, Zhang, & Luo, 2017; Zhang & Chen, 2012). Luo and Wang (2014) proved that the integration of $G$-SDEs in $\mathbb{R}$ can be reduced to the integration of ordinary differential equation parameterized by a variable in $(\Omega, \mathcal{F})$. Fei and Fei (2019) investigated the consistency of the least squares estimator (LSE) of the parameter for SDEs under distribution uncertainty and developed an algorithm for estimating the $G$-expectation. Wei et al. (2017) gave the asymptotic estimates for the solution of $G$-SDEs. Yang and Zhao (2016) introduced a numerical method for simulating $G$-Brownian motion. Fei et al. explored the stability and boundedness of solutions to highly nonlinear $G$-SDEs in Fei, Fei, and Yan (2019). These results lay the theoretical foundation for further research on the $G$-SDEs.

We know that the stability of the classical stochastic differential equations is an important topic in the study of stochastic systems (see Fei, Fei, Mao, Shen, & Yan, 2019; Fei, Hu, Mao, & Shen, 2019; Fei, Shen, Fei, Mao, & Yan, 2019; Shen, Fei, Fei, & Mao, 2019; Shen, Fei, & Liang, 2018). Recently, many researchers have showed great interests to the stochastic stability in the $G$-framework (see Hu, Ren, & Xu, 2014; Li, Lin, & Lin, 2016; Ren, Jia, & Sakthivel, 2016; Zhang & Chen, 2012; Zhu, Li, & ZHU, 2017). Zhang and Chen (2012) discussed the exponential stability for $G$-SDEs. Li et al. (2016) studied the solvability and the stability of $G$-SDEs under Lyapunov-type conditions. Using the $G$-Lyapunov function technique to investigate the $p$-the moment stability of solutions to $G$-SDEs can be found in the references Hu et al. (2014), Ren et al. (2016), and Yin and Ren (2017). However, in the applications of $G$-SDEs, most of these equations can not be analytically solved, we have to resort to the numerical methods. Unfortunately, the research and papers on this issue are quite few. Li and Yan (2018) considered the stability of the EM method for solving the delayed Hopfield neural networks under the $G$-framework. Yang and Li (2019) proved that under global Lipschitz assumption a $G$-SDE is $\rho$-th ($0 < \rho < 1$) moment exponentially...
stable if and only if the stochastic $\theta$-method is also $p$-th moment exponentially stable for sufficiently small step size. In Li and Yang (2018), they also showed that the stochastic $\theta$-numerical solution converges to the exact solution for the neutral stochastic delay differential equation driven by G-Brownian motion (G-NSDDE) if the coefficients of the equation satisfy the global Lipschitz assumption. Moreover, Deng, Fei, Fei, and Mao (2019) discussed the stability equivalence between the stochastic differential delay equations driven by G-Brownian motion and the EM method.

Based on the above discussions, we are interested in designing a numerical approach for solving the G-NSDE with time-dependent delay such that the numerical solution converges to the true solution in the sense of mean square. In the case where the SDEs are driven by the classical Brownian motion, numerical methods for SDEs or NSDEs have been discussed by many authors. We refer the reader to Deng, Fei, Liu, and Mao (2019), Feng, Qiu, Meng, and Rong (2019), Liu and Mao (2016), Liu, Li, and Deng (2018), Mo, Deng, and Zhang (2017), Mao, Zhu, and Mao (2015), Tan, Wang, Guo, and Zhu (2014), and Zong and Wu (2016), and the literature cited therein. It is worth mentioning that Milošović (2011) established the convergence in probability of the EM approximate solution for a highly nonlinear NSDEs with time-dependent delay under the Khasminskii-type conditions. Naturally, we follow the train of his thought in this paper. However, under the G-framework, distribution uncertainty of G-Brownian motion brings about difficulties for estimating the error between the numerical and the exact solutions, we develop new technique to overcome these by virtue of stochastic analysis technique. Thus, this paper is not simply a trivial extension of the existing results to the more complex models. In reference to the existing results in the literature, we make the following contributions.

1. A computational method is, for the first time, developed for a class of neutral stochastic differential equations with variable delay under the G-framework.
2. A comprehensive system model is proposed to account for the phenomena of time-dependent delay and distribution uncertainty of stochastic disturbances.
3. New mathematical techniques are well applied to solve the difficulties due to G-Brownian motion and variable delay.
4. A numerical example including G-expectation simulation is provided to show the convergence order.

The rest of the paper is organized as follows. In Section 2, we present the essential notations, definitions and propositions which are necessary for the whole work. In Section 3, we introduce the EM method for NSDEs with time-dependent delay in the G-framework. The convergence results are shown in Section 4. In Section 5, we give a numerical example to support our theory. At the end, we conclude the paper and points out some future research.

2. Preliminaries

Let us begin with the notion of a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, where $\Omega$ is a given set and $\mathcal{H}$ is a linear space of real-valued functions defined on $\Omega$. The space $\mathcal{H}$ can be viewed as the space of random variables.

**Definition 2.1:** A sublinear expectation $\hat{\mathbb{E}}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying:

1. Monotonicity: $\hat{\mathbb{E}}[X_1] \geq \hat{\mathbb{E}}[X_2]$ if $X_1 \geq X_2$,
2. Constant preserving: $\hat{\mathbb{E}}[C] = C$ for $C \in \mathbb{R}$,
3. Sub-additivity: $\hat{\mathbb{E}}[X_1 + X_2] \leq \hat{\mathbb{E}}[X_1] + \hat{\mathbb{E}}[X_2]$ for $X_1, X_2 \in \mathcal{H}$,
4. Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space. Let $B(t)$ be a one-dimensional G-Brownian motion on the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with

$$G(\alpha) := \frac{1}{2} \hat{\mathbb{E}}[\alpha (B(\tau)^2)] = \frac{1}{2} (\hat{\sigma}^2 \alpha^+ - \hat{\sigma}^2 \alpha^-), \quad \alpha \in \mathbb{R},$$

where $\hat{\sigma}^2 = \hat{\mathbb{E}}[\sigma^2(1)^2]$, $\sigma^2 = -\hat{\mathbb{E}}[\sigma^2(1)^2]$, $0 < \hat{\sigma} < \infty$.

Denote by $\mathcal{H}_t$ the filtration generated by G-Brownian motion $(B(t))_{t \geq 0}$ and $(\langle B \rangle(t))$ the quadratic variation process of $B(t)$.

Let $| \cdot |$ be the Euclidean norm in $\mathbb{R}^d$. If $A$ is a vector or matrix, its norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$, where $A^T$ is the $A$'s transpose. If $a$ is a real number, its integer part is denoted by $[a]$. Let $\tau > 0$. Denote by $BC([-\tau, 0]; \mathbb{R}^d)$ the family of all bounded continuous $\mathbb{R}^d$-valued functions $\psi$ defined on $[-\tau, 0]$ to $\mathbb{R}^d$ with norm $\|\psi\| = \sup_{-\tau \leq \theta \leq 0} |\psi(\theta)|$. If $x(t)$ is a continuous $\mathbb{R}^d$-valued stochastic process on $t \in [-\tau, \infty)$, we let $x_t = (x(t + \theta) : -\tau \leq \theta \leq 0)$, which is regarded as a $C([-\tau, 0]; \mathbb{R}^d)$-valued stochastic process. For $x_t \in L^2_{\mathcal{H}_t}((-\tau, T]; \mathbb{R}^d)$, define

$$\|x_t\|_{\hat{\mathbb{E}}}^2 = \sup_{-\tau \leq \theta \leq 0} \hat{\mathbb{E}}[x(t + \theta)^2] < \infty.$$

For more details on G-Brownian motion, Itô integral and G-SDEs, one can refer the reference Peng (2010). Before stating the main results, we present two useful propositions.
Proposition 2.2 (Peng, 2010): Let \( \zeta \in M^2_0([-\tau,T];\mathbb{R}^d) \). Then
\[
\mathbb{E}\left[ \int_0^T \zeta_t \, dB(t) \right] = 0, \tag{1}
\]
\[
\mathbb{E}\left[ \int_0^T |\zeta_t|^2 \, dt \right] \leq \int_0^T \mathbb{E}\left[ |\zeta_t|^2 \right] \, dt, \tag{2}
\]
\[
\mathbb{E}\left[ \left( \int_0^T \zeta_t \, dB(t) \right)^2 \right] = \mathbb{E}\left[ \int_0^T |\zeta_t|^2 \, d\langle B \rangle(t) \right] \leq \sigma^2 \mathbb{E}\left[ \int_0^T |\zeta_t|^2 \, dt \right]. \tag{3}
\]

Proposition 2.3 (Fei & Fei, 2013): Let \( 0 \leq t \leq T < \infty \). Then
\[
\sigma^2(T - t) \leq \langle B(t) - B(t) \rangle \leq \sigma^2(T - t) \quad \text{q.s.}
\]
Moreover, \( \sigma^2 \, dt \leq d\langle B \rangle(t) \leq \sigma^2 \, dt \) q.s.

By Propositions 2.2 and 2.3, we deduce
\[
\mathbb{E}\left[ \left( \int_0^T \zeta_s \, d\langle B \rangle(s) \right)^2 \right] \leq \sigma^4 \mathbb{E}\left[ \int_0^T |\zeta_s|^2 \, ds \right]. \tag{4}
\]

Consider a G-NSDE with time-dependent delay described by the following form:
\[
d(x(t) - D(x(t - \delta(t)))) = f(x(t), x(t - \delta(t))) \, dt + g(x(t), x(t - \delta(t))) \, dB(t)
\] + h(x(t), x(t - \delta(t))) \, d\langle B \rangle(t), \quad t \geq 0, \tag{5}
\]
with initial data \( x_0 = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\} \in L^2_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^d) \), where \( f, g, h : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \), as well as \( f, g, h \in M^2_0([-\tau,T];\mathbb{R}^d), \forall T \geq 0 \). In this paper, we assume that delay function \( \delta : \mathbb{R}^+ \rightarrow [0,\tau] \) is continuous and there is a positive constant \( \rho \) such that
\[
|\delta(t) - \delta(s)| \leq \rho|t - s|, \quad \forall \, t, s \geq 0. \tag{6}
\]
Now, we rewrite G-NSDE (5) as the following integral form:
\[
x(t) = D(x(t - \delta(t))) - D(\xi(-0)) + \xi(0)
\] + \( \int_0^t f(x(s), x(s - \delta(s))) \, ds \)
\[
+ \int_0^t g(x(s), x(s - \delta(s))) \, dB(s)
\] + \( \int_0^t h(x(s), x(s - \delta(s))) \, d\langle B \rangle(s) \). \tag{7}

For the purpose of the following consideration, we impose the following assumptions:

**Assumption 2.4** (Global Lipschitz condition): There is a positive constant \( L \) such that \( \forall x_1, x_2, y_1, y_2 \in \mathbb{R}^d \),
\[
|f(x_1, y_1) - f(x_2, y_2)|^2 \lor |g(x_1, y_1) - g(x_2, y_2)|^2
\] \[ \lor |h(x_1, y_1) - h(x_2, y_2)|^2 \leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2). \tag{8}
\]

**Assumption 2.5** (Contractive mapping condition): There is a constant \( \nu \in (0,1) \) such that \( \forall x, y \in \mathbb{R}^d \),
\[
|D(x) - D(y)| \leq \nu|x - y|.
\] \[ \tag{9}
\]
Moreover, we suppose that \( D(0) = 0 \) which, together with (9), means that
\[
|D(x)| \leq \nu|x|, \quad \forall \, x \in \mathbb{R}^d. \tag{10}
\]

**Assumption 2.6**: There is a positive constant \( \beta \) such that
\[
\sup_{s,t \in [-\tau,0], s \neq t \leq \Delta} \mathbb{E}[|\xi(s) - \xi(t)|^2] \leq \beta \Delta.
\]

From Assumption 2.4, we can deduce that \( f, g \) and \( h \) satisfy the linear growth condition, that is, for any \( x, y \in \mathbb{R}^d \),
\[
|f(x, y)|^2 \lor |g(x, y)|^2 \lor |h(x, y)|^2 \leq K(1 + |x|^2 + |y|^2), \tag{11}
\]
where \( K = 2(L \lor |f(0,0)|^2 \lor |g(0,0)|^2 \lor |h(0,0)|^2) \).

### 3. The EM method for NSDEs with time-dependent delay in the \( G \)-framework

Let us propose the EM scheme for the G-NSDE (5). For any given time \( T \geq 0 \), there are sufficiently large integers \( m, n \geq 0 \) such that \( \Delta = \tau/m = T/n \). Set \( t_k = k\Delta \) and \( \delta_k = [\delta(k\Delta)/\Delta] \). Now, we introduce the discrete EM approximate solution \( Y \) to (5) as follows:
\[
Y_k = \xi(t_k), \quad k = -m, -m + 1, \ldots, 0,
\]
\[
Y_{k+1} - D(Y_{k+1} - \delta_{k+1}) = Y_k - D(Y_k - \delta_k) \Delta + g(Y_k, Y_{k-\delta_k}) \Delta B_k
\] + \( h(Y_k, Y_{k-\delta_k}) \Delta \langle B \rangle_k \),
\[
k = 0, 1, 2, \ldots, \tag{12}
\]
where \( \Delta B_k = B(t_{k+1}) - B(t_k), \quad \Delta \langle B \rangle_k = \langle B(t_{k+1}) - \langle B \rangle(t_k) \). The step processes of EM solution are defined by
\[
z_1(t) = \sum_{k=-m}^{\infty} Y_k \mathbb{I}_{[k\Delta, (k+1)\Delta)}(t), \quad t \geq -\tau,
\]
\[
z_2(t) = \sum_{k=0}^{\infty} Y_{k-\delta_k} \mathbb{I}_{[k\Delta, (k+1)\Delta)}(t), \quad t \geq 0, \tag{13}
\]
where \( I_A \) denotes the indicator function of the set \( A \). Then, we define the continuous EM approximate solution

\[
y(t) = \xi(t), \quad -\tau \leq t \leq 0, \\
y(t) = \xi(0) + D(z_2(t)) - D(-\xi(0)) \\
+ \int_0^t f(z_1(s), z_2(s)) \, ds \\
+ \int_0^t g(z_1(s), z_2(s)) \, dB(s) \\
+ \int_0^t h(z_1(s), z_2(s)) \, dB(s), \quad t \geq 0.
\]

Letting \( t = t_k \) in (14), then we get that

\[
y(t_k) = \xi(0) + D(Y_{k-\delta_k}) - D(-\xi(0)) \\
+ \int_0^{t_k} f(z_1(s), z_2(s)) \, ds \\
+ \int_0^{t_k} g(z_1(s), z_2(s)) \, dB(s) \\
+ \int_0^{t_k} h(z_1(s), z_2(s)) \, dB(s) \\
= Y_k, \tag{15}
\]

which means that the continuous and discrete EM solutions coincide at the grid points. Moreover, on the basis of (14) and (15), we get that, for any \( t \in [t_k, t_{k+1}) \),

\[
y(t) - z_1(t) = y(t) - Y_k = y(t) - y(t_k) \\
= f(Y_k, Y_{k-\delta_k})(t - t_k) \\
+ g(Y_k, Y_{k-\delta_k})(B(t) - B(t_k)) \\
+ h(Y_k, Y_{k-\delta_k})(B(t) - B(t_k)). \tag{16}
\]

4. Main results

In this section, we prove that the order of the mean square closeness of the exact solution \( x(t) \) for system (5) and the corresponding EM solution \( y(t) \) is equal to one. The following theorem shows the convergence and convergence rate.

**Theorem 4.1:** Suppose that Assumptions 2.4–2.6 and \( v(3 + \| \rho \|) < 1 \) are satisfied. Then, for any \( T > 0 \),

\[
\sup_{-\tau \leq t \leq T} \mathbb{E}|y(t) - y(t_k)|^p \leq C_6 \Delta,
\]

where \( C_6 = C_6(\sigma, L, \rho, v, \beta, T, \| \xi \|^2_{L^2}) \) is a positive constant independent of \( \Delta \).

In order to establish this theorem, we need to show some useful lemmas.

**Lemma 4.2:** Let \( p > 1 \) and Assumption 2.5 hold. Then

\[
\sup_{0 \leq s \leq t} \mathbb{E}|x(s)|^p \\
\leq \frac{v}{1 - v} \sup_{-\tau \leq s \leq 0} \mathbb{E}|x(s)|^p \\
+ \frac{1}{(1 - v)^p - 1} \sup_{0 \leq s \leq t} \mathbb{E}|x(s) - D(x(s - \delta(s)))|^p.
\]

**Proof:** Recall the elementary inequality that for any \( a, b > 0, p > 1 \) and \( \eta > 0 \),

\[
(a + b)^p \leq (1 + \eta^{1/(p-1)})(a^p + b^p). \tag{17}
\]

Then, Assumption 2.5 implies that for any \( \eta > 0 \),

\[
|x(s)|^p = |D(x(s - \delta(s))) + x(s) - D(x(s - \delta(s)))|^p \\
\leq (1 + \eta^{1/(p-1)}) \\
\times \left( \frac{|D(x(s - \delta(s)))|^p}{\eta} + |x(s) - D(x(s - \delta(s)))|^p \right) \\
\leq (1 + \eta^{1/(p-1)}) \\
\times \left( \frac{\nu|x(s - \delta(s))|^p}{\eta} + |x(s) - D(x(s - \delta(s)))|^p \right).
\]

Letting \( \eta = \nu/(1 - \nu) \) yields that

\[
|x(s)|^p \leq \nu|x(s - \delta(s))|^p \\
+ \frac{1}{(1 - \nu)^p - 1} |x(s) - D(x(s - \delta(s)))|^p.
\]

Therefore,

\[
\sup_{0 \leq s \leq t} \mathbb{E}|x(s)|^p \\
\leq \nu \sup_{-\tau \leq s \leq 0} \mathbb{E}|x(s) - D(x(s - \delta(s)))|^p \\
+ \frac{1}{(1 - \nu)^p - 1} \sup_{0 \leq s \leq t} \mathbb{E}|x(s) - D(x(s - \delta(s)))|^p. \
\]
Rearranging this gives

\[(1 - \nu) \sup_{0 \leq s \leq t} \hat{E}|\xi(s)|^p \leq \nu \sup_{-r \leq s \leq 0} \hat{E}|\xi(s)|^p + \frac{1}{(1 - \nu)^{p-1}} \sup_{0 \leq s \leq t} \hat{E}|\xi(s) - D(x(s - \delta(s)))|^p,\]

which means the desired assertion. Thus, we complete the proof. ■

**Lemma 4.3:** Suppose that Assumptions 2.4–2.6 are satisfied. Then, for any \(T > 0\),

\[
\sup_{-r \leq t \leq T} \hat{E}|x(t)|^2 \leq C_1,
\]

where

\[
C_1 = (a_1 \|\xi\|^2_E + a_2) \exp(2a_2),
\]

\[
a_1 = \frac{1}{1 - \nu} + \frac{8(1 + \nu^2)}{(1 - \nu)^2},
\]

\[
a_2 = \frac{4KT(T + \sigma^2 + \sigma^4T)}{(1 - \nu)^2}.
\]

**Proof:** For any \(t \in [0, T]\), set \(\hat{x}(t) = x(t) - D(x(t - \delta(t)))\), then Equation (7) becomes

\[
\hat{x}(t) = \hat{x}(0) + \int_0^t f(x(s), x(s - \delta(s))) \, ds + \int_0^t g(x(s), x(s - \delta(s))) \, dB(s) + \int_0^t h(x(s), x(s - \delta(s))) \, dB(s).
\]

The elementary inequality gives

\[
\hat{E}|\hat{x}(t)|^2 \leq 4\hat{E}|\hat{x}(0)|^2 + 4\hat{E} \left| \int_0^t f(x(s), x(s - \delta(s))) \, ds \right|^2
\]

\[
+ 4\hat{E} \left| \int_0^t g(x(s), x(s - \delta(s))) \, dB(s) \right|^2 + 4\hat{E} \left| \int_0^t h(x(s), x(s - \delta(s))) \, dB(s) \right|^2.
\]

From Assumption 2.5, we have

\[
\hat{E}|\xi(0)|^2 = \hat{E}|\xi(0) - D(\xi(-\delta(0)))|^2
\]

\[
\leq 2\hat{E}|\xi(0)|^2 + 2\nu^2 \hat{E}|\xi(-\delta(0))|^2
\]

\[
\leq 2(1 + \nu^2)\|\xi\|^2_E.
\]

By Assumption 2.4 and estimate (4), we have

\[
\hat{E} \left| \int_0^t h(x(s), x(s - \delta(s))) \, dB(s) \right|^2
\]

\[
\leq \sigma^4T\hat{E} \int_0^t |h(x(s), x(s - \delta(s)))|^2 \, ds
\]

\[
\leq K\sigma^4T\hat{E} \int_0^t (1 + |x(s)|^2 + |x(s - \delta(s))|^2) \, ds
\]

\[
\leq K\sigma^4T^2 + 2K\sigma^2 \int_0^T \hat{E} \left( \sup_{-r \leq s \leq 0} |x(s)|^2 \right) \, ds.
\]

With the help of Proposition 2.2 and Assumption 2.4, we get

\[
\hat{E} \left| \int_0^t g(x(s), x(s - \delta(s))) \, dB(s) \right|^2
\]

\[
\leq \sigma^2\hat{E} \int_0^t |g(x(s), x(s - \delta(s)))|^2 \, ds
\]

\[
\leq \sigma^2K\hat{E} \int_0^t (1 + |x(s)|^2 + |x(s - \delta(s))|^2) \, ds
\]

\[
\leq KT\hat{E} \int_0^t (1 + |x(s)|^2 + |x(s - \delta(s))|^2) \, ds.
\]

By the H"{o}lder inequality and Assumption 2.4, we have

\[
\hat{E} \left| \int_0^t f(x(s), x(s - \delta(s))) \, ds \right|^2
\]

\[
\leq t\hat{E} \int_0^t |f(x(s), x(s - \delta(s)))|^2 \, ds
\]

\[
\leq KT\hat{E} \int_0^t (1 + |x(s)|^2 + |x(s - \delta(s))|^2) \, ds
\]

\[
\leq KT^2 + 2KT\int_0^T \hat{E} \left( \sup_{-r \leq s \leq 0} |x(r)|^2 \right) \, ds.
\]

Inserting (20)–(23) into (19), we have

\[
\sup_{0 \leq t \leq T} \hat{E}|\hat{x}(t)|^2 \leq K_1 + K_2 \int_0^T \hat{E} \left( \sup_{-r \leq s \leq 0} |x(r)|^2 \right) \, ds,
\]

where \(K_1 = 8(1 + \nu^2)\|\xi\|^2_E + 4KT^2 + 4K\sigma^2T + 4K\sigma^4T^2\), and \(K_2 = 8K(\sigma^2 + \sigma^4T) + 2\). Using Lemma 4.2 gives

\[
\sup_{0 \leq t \leq T} \hat{E}|x(t)|^2 \leq \frac{1}{1 - \nu} \|\xi\|^2_E.
\]

Note that

\[
\sup_{-r \leq t \leq T} \hat{E}|x(t)|^2 \leq \|\xi\|^2_E + \sup_{0 \leq t \leq T} \hat{E}|x(t)|^2.
\]
Hence, inserting (25) into (26), we get
\[ \sup_{-r \leq t \leq T} \mathbb{E}|x(t)|^2 \leq \frac{1}{1 - \nu} \|\xi\|^2_{E} \]
\[ + \frac{1}{(1 - \nu)^2} \sup_{0 \leq s \leq T} \mathbb{E}|x(s)|^2. \]  
(27)
Substituting (24) into (27), we have
\[ \sup_{-r \leq t \leq T} \mathbb{E}|x(t)|^2 \leq \left( \frac{\|\xi\|^2_{E}}{1 - \nu} + \frac{K_1}{(1 - \nu)^2} \right) \exp \left( \frac{K_2 T}{(1 - \nu)^2} \right), \]
where
\[ C_1 = \left( \frac{\|\xi\|^2_{E}}{1 - \nu} + \frac{K_1}{(1 - \nu)^2} \right) \exp \left( \frac{K_2 T}{(1 - \nu)^2} \right), \]
\[ K_1 = 8(1 + \nu)^2 \|\xi\|^2_{E} + 4KT^2 + 4K\sigma^2 T + 4K\sigma^4 T, \]
\[ K_2 = 8K(\sigma^2 + \sigma^4 T + T). \]
Rearranging \( C_1 \) gives the desired assertion. Thus, we complete the proof. \qed

**Lemma 4.4:** Suppose that Assumptions 2.4–2.6 are satisfied. Then, for any \( T > 0 \),
\[ \sup_{-r \leq t \leq T} \mathbb{E}|y(t)|^2 \leq C_2, \]
where
\[ C_2 = \left( (a_1 + a_2) \|\xi\|^2_{E} + a_2 \right) \exp(2a_2), \]
a1 and a2 were defined in Lemma 4.3.

**Proof:** Recall the elementary inequality
\[ (a + b)^2 \leq \frac{a^2}{\varepsilon} + \frac{b^2}{1 - \varepsilon}, \quad \forall a, b \in (0, \infty), \ \varepsilon \in (0, 1). \] (28)
From Equation (14), we have that for \( 0 \leq t \leq T \),
\[ \mathbb{E}|y(t)|^2 \leq \frac{1}{\varepsilon} \mathbb{E}|D(z_2(t))|^2 + \frac{4}{1 - \varepsilon} \mathbb{E}|\xi(0) - D(\xi(\varepsilon(0)))|^2 \]
\[ + \frac{4}{1 - \varepsilon} \mathbb{E} \left| \int_0^t f(z_1(s), z_2(s)) \, ds \right|^2. \]
Using contractive mapping and linear growth conditions as well as Proposition 2.2, we have
\[ \mathbb{E}|y(t)|^2 \leq \frac{1}{\varepsilon} \mathbb{E}|z_2(t)|^2 + \frac{8(1 + \nu^2)}{1 - \varepsilon} \sup_{-r \leq t \leq 0} \mathbb{E}|\xi(t)|^2 \]
\[ + \frac{4T}{1 - \varepsilon} \mathbb{E} \left| \int_0^t (1 + \mathbb{E}|z_1(s)|^2 + \mathbb{E}|z_2(s)|^2) \, ds \right|^2 \]
\[ + \frac{4T^2}{1 - \varepsilon} \mathbb{E} \left| \int_0^t (1 + \mathbb{E}|z_1(s)|^2 + \mathbb{E}|z_2(s)|^2) \, ds \right|^2 \]
\[ + \frac{4\sigma^2 K}{1 - \varepsilon} \mathbb{E} \left| \int_0^t (1 + \mathbb{E}|z_1(s)|^2 + \mathbb{E}|z_2(s)|^2) \, ds \right|^2. \] (30)
For any \( t \in [t_k, t_{k+1}) \), we get that \( z_1(t) = Y_k = y(t_k) \) and \( z_2(t) = Y_k - \delta_k = y(t_k - \delta_k \Delta) \). Hence, we obtain that
\[ \mathbb{E}|z_1(t)|^2 \leq \sup_{0 \leq s \leq t} \mathbb{E}|y(s)|^2 \] (31)
and
\[ \mathbb{E}|z_2(t)|^2 \leq \sup_{-r \leq s \leq 0} \mathbb{E}|\xi(s)|^2 + \sup_{0 \leq s \leq t} \mathbb{E}|y(s)|^2 \]
\[ = \|\xi\|^2_{E} + \sup_{0 \leq s \leq t} \mathbb{E}|y(s)|^2. \] (32)
Then, (31) and (32) together with (30) give
\[ \sup_{0 \leq s \leq t} \mathbb{E}|y(s)|^2 \leq \frac{1}{\varepsilon} \left( \|\xi\|^2_{E} + \sup_{0 \leq s \leq t} \mathbb{E}|y(s)|^2 \right) + \tilde{K}_1 T \]
\[ + \frac{8(1 + \nu^2)}{1 - \varepsilon} \|\xi\|^2_{E} + \tilde{K}_1 \int_0^t \left( 2 \sup_{0 \leq u \leq s} \mathbb{E}|y(u)|^2 + \|\xi\|^2_{E} \right) \, ds. \] (33)
where $\bar{K}_1 = 4K(T + \sigma^2 + \sigma^4T)/(1 - \nu)$. Choosing $\nu = \nu$ and rearranging this give
\[
(1 - \nu) \sup_{0 \leq u \leq r} \hat{E} |y(u)|^2 \leq \left( \nu + \frac{8(1 + \nu^2)}{1 - \nu} + \bar{K}_1 T \right) \|\xi\|^2_E + \bar{K}_1 T + 2\bar{K}_1 \int_0^r \sup_{0 \leq u \leq s} \hat{E} |y(u)|^2 \, \text{ds}. \tag{34}
\]

By the Gronwall inequality, we get that there is a constant $\bar{K}_2$ such that
\[
\sup_{0 \leq u \leq T} \hat{E} |y(u)|^2 \leq \bar{K}_2,
\]
where
\[
\bar{K}_2 = \left[ \left( \frac{\nu}{1 - \nu} + \frac{8(1 + \nu^2)}{(1 - \nu)^2} + \bar{K}_1 T \right) \|\xi\|^2_E \right] \times \exp \left( \frac{2\bar{K}_1 T}{1 - \nu} \right),
\]
\[
\bar{K}_1 = \frac{4K}{1 - \nu} (T + \sigma^2 + \sigma^4T).
\]

Noting that
\[
\sup_{-\tau \leq u \leq T} \hat{E} |y(u)|^2 \leq \|\xi\|^2_E + \sup_{0 \leq u \leq T} \hat{E} |y(u)|^2,
\]
we get the desired assertion.

The following lemma shows that $y(t)$ and $z_1(t)$ are close to each other in the mean square sense.

**Lemma 4.5:** Suppose that Assumptions 2.4–2.6 are satisfied. Then, for any $T > 0$,
\[
\sup_{-\tau \leq u \leq T} \hat{E} |y(t) - z_1(t)|^2 \leq C_3 \Delta,
\]
where $C_3 = 3K(1 + C_2)(1 + \sigma^2 + \sigma^4)$ and $C_2$ was defined in Lemma 4.4.

**Proof:** For any $t \in [t_k, t_{k+1})$, using the linear growth condition, from (16) we get that
\[
|y(t) - z_1(t)|^2 \leq 3K(1 + |z_1(t)|^2 + |z_2(t)|^2)(t - t_k)^2 + 3K(1 + |z_1(t)|^2 + |z_2(t)|^2)(B(t) - B(t_k))^2 + 3K(1 + |z_1(t)|^2 + |z_2(t)|^2)(B(t) - \langle B \rangle(t_k))^2.
\]

In view of Lemma 4.4, we have that
\[
\hat{E} |z_1(t)|^2 \leq C_2, \quad \hat{E} |z_2(t)|^2 \leq C_2.
\]

Then, for any $t \in [-\tau, T]$ and $\Delta \in (0, 1)$, we get
\[
\sup_{-\tau \leq u \leq T} \hat{E} |y(t) - z_1(t)|^2 \leq 3K(1 + 2C_2)\Delta^2 + 3K(1 + 2C_2)\sigma^2 \Delta + 3K(1 + 2C_2)\sigma^4\Delta^2 \leq C_3 \Delta.
\]

Thus, we complete the proof.

**Lemma 4.6:** Suppose that delay function $\delta(t)$ satisfies condition (6). Then, for any $t \geq 0$,

1. $|\delta_{k+1} - \delta_k| \leq |\rho| + 2$,
2. $|\delta(t) - \delta_k\Delta| \leq (|\rho| + 2)\Delta$,
3. $|[t - \delta(t)]/\Delta - (k - \delta_k)| \leq |\rho| + 4$,

where $k = [t/\Delta], t_k = k\Delta$ and $\delta_k = |\delta(t_k)/\Delta|$.

**Proof:** By the definition of the integer part function, we get that for any $a, b \in \mathbb{R}$,

\[
|a| \leq a,
\]
\[
-|a| \leq -a + 1,
\]
\[
||a| - |b|| \leq |a - b| + 1. \tag{35}
\]

By (6) and (35), we have
\[
|\delta_{k+1} - \delta_k| \leq \frac{|\delta(t_{k+1}) - \delta(t_k)|}{\Delta} + 1 \leq |\rho| + 1 < |\rho| + 2.
\]

On the basis of Assumption 2.6 and the triangle inequality, we get that
\[
|\delta(t) - \delta_k\Delta| \leq |\delta(t) - \delta(t_k)| + |\delta(t_k) - \delta_k\Delta| \leq (|\rho| + 1)\Delta < (|\rho| + 2)\Delta.
\]

Now, we set $k_t = [t - \delta(t)/\Delta]$. When $t_k - \delta_k\Delta \leq k_t \leq t - \delta(t)$, we have
\[
|[k_t \Delta - (t_k - \delta_k\Delta)]| \leq |t - t_k| + |\delta(t) - \delta_k\Delta| \leq |t - t_k| + |\delta(t) - \delta_k\Delta| \leq |t - t_k| + |\delta(t) - \delta_k\Delta| \leq (|\rho| + 2)\Delta < (|\rho| + 3)\Delta. \tag{36}
\]

Otherwise, when $k_t \Delta \leq t - \delta(t) \leq t_k - \delta_k\Delta$, from (35) we have that
\[
|k_t \Delta - (t_k - \delta_k\Delta)| \leq |t - t_k| + |\delta(t) - \delta_k\Delta| \leq |t - t_k| + |\delta(t) - \delta_k\Delta| \leq (|\rho| + 4)\Delta. \tag{37}
\]

Combining (36) and (37), we have
\[
|[t - \delta(t)/\Delta] - (k - \delta_k)| \leq |\rho| + 4.
\]

Thus, we complete the proof.
Lemma 4.7: Suppose that Assumptions 2.4–2.6 and \( \nu(3 + \|\rho\|) < 1 \) are satisfied. Then, for any \( T > 0 \),

\[
\sup_{-\tau \leq t \leq T} \hat{E}|z_1(t + \Delta) - z_1(t)|^2 \leq C_4 \Delta,
\]

where \( C_4 \) is a positive constant independent of \( \Delta \).

**Proof:** For any \( t \in [0, T] \), the definition of \( z_1(t) \) gives

\[
\sup_{-\tau \leq t \leq T} \hat{E}|z_1(t + \Delta) - z_1(t)|^2 = \sup_{-m \leq k \leq [T/\Delta]} \hat{E}|y_{k+1} - y_k|^2
\]

\[
\leq \sup_{-m \leq k \leq -1} \hat{E}|y_{k+1} - y_k|^2 + \sup_{0 \leq k \leq [T/\Delta]} \hat{E}|y_{k+1} - y_k|^2 = 2 \sup_{-m \leq k \leq -1} \hat{E}|y_{k+1} - y_k|^2. \tag{38}
\]

When \( -m \leq k \leq -1 \), Assumption 2.6 gives

\[
\sup_{-m \leq k \leq -1} \hat{E}|y_{k+1} - y_k|^2 = \sup_{-m \leq k \leq -1} \hat{E}|\xi(t_{k+1}) - \xi(t_k)|^2 \leq \beta \Delta. \tag{39}
\]

When \( k \geq 0 \), using the elementary inequality (28), from (12) we get that

\[
|y_{k+1} - y_k|^2 \leq \frac{\nu^2}{\varepsilon}|y_{k+1} - \delta_{k+1} - y_{k-\delta_k}|^2 + \frac{3}{1 - \varepsilon}|f(y_k, y_{k-\delta_k})|^2 \Delta_k^2
\]

\[
+ \frac{3}{1 - \varepsilon}|g(y_k, y_{k-\delta_k})|^2 |\Delta B_k|^2 + \frac{3}{1 - \varepsilon}|h(y_k, y_{k-\delta_k})|^2 |\Delta (B)k|^2.
\]

On the basis of Lemma 4.4 and the linear growth condition, we have that

\[
\hat{E}|f(y_k, y_{k-\delta_k})|^2 \leq K \hat{E}(1 + |y_k|^2 + |y_{k+1-\delta_{k+1}}|^2) \leq K(1 + 2C_2),
\]

\[
\hat{E}|g(y_k, y_{k-\delta_k})|^2 \leq K(1 + 2C_2),
\]

and

\[
\hat{E}|h(y_k, y_{k-\delta_k})|^2 \leq K(1 + 2C_2).
\]

Moreover, Proposition 2.3 means that

\[
\hat{E}|\Delta B_k|^2 \leq \sigma^2 \Delta \quad \text{and} \quad \hat{E}|\Delta (B)k|^2 \leq \sigma^4 \Delta^2.
\]

Consequently, we obtain

\[
\sup_{0 \leq k \leq [T/\Delta]} \hat{E}|y_{k+1} - y_k|^2
\]

\[
\leq \frac{\nu^2}{\varepsilon} \sup_{0 \leq k \leq [T/\Delta]} \hat{E}|y_{k+1} - \delta_{k+1} - y_{k-\delta_k}|^2 + K_4 \Delta, \tag{40}
\]

where \( K_4 = 3K(1 + 2C_2)(1 + \sigma^2 + \sigma^4)/(1 - \varepsilon) \). From Lemma 4.6, we get

\[
(|k + 1 - \delta_{k+1}) - (k - \delta_k)| \leq |\rho| + 3, \quad k = 0, 1, 2, \ldots. \tag{41}
\]

Then, the elementary inequality gives

\[
\sup_{0 \leq k \leq [T/\Delta]} \hat{E}|y_{k+1} - \delta_{k+1} - y_{k-\delta_k}|^2
\]

\[
\leq (3 + |\rho|)^2 \sup_{0 \leq k \leq [T/\Delta]} \hat{E}|y_{k+1} - y_k|^2. \tag{42}
\]

Substituting (42) into (40) yields

\[
\sup_{0 \leq k \leq [T/\Delta]} \hat{E}|y_{k+1} - y_k|^2 \leq (3 + |\rho|)^2 \sup_{0 \leq k \leq [T/\Delta]} \hat{E}|y_{k+1} - y_k|^2 + K_4 \Delta. \tag{43}
\]

Inserting (39) and (43) into (38) yields

\[
\sup_{-\tau \leq t \leq T} \hat{E}|y_{k+1} - y_k|^2
\]

\[
\leq (3 + |\rho|)^2 \varepsilon \sup_{0 \leq k \leq [T/\Delta]} \hat{E}|y_{k+1} - y_k|^2 + (K_4 + \beta) \Delta. \tag{44}
\]

Recall that condition \((3 + |\rho|)\nu < 1\) which guarantees

\[\nu = (3 + |\rho|)^2 \varepsilon < 1.\]

Consequently, choosing \( \varepsilon \in (\bar{\varepsilon}, 1) \) such that \((3 + |\rho|)^2 \varepsilon < 1\), and rearranging (44), we get

\[
\sup_{-\tau \leq t \leq T} \hat{E}|z_1(t + \Delta) - z_1(t)|^2 \leq \frac{K_4 + \beta}{1 - (3 + |\rho|)^2 \varepsilon}. \Delta.
\]

Thus, we complete the proof. \( \blacksquare \)

Lemma 4.8: Suppose that Assumptions 2.4–2.6 are satisfied. Then, for any \( T > 0 \),

\[
\sup_{0 \leq t \leq T} \hat{E}|y(t - \delta(t)) - z_2(t)|^2 \leq C_5 \Delta,
\]

where \( C_5 = 2(C_3 + \beta + C_4(|\rho| + 4)^2) \).

**Proof:** Fix any \( t \in [0, T] \). Letting \( k = \lfloor t/\Delta \rfloor \) and \( k_t = \lfloor (t - \delta(t))/\Delta \rfloor \), we get that \( t \in [k_t, k_{t+1}) \) and

\[
t - \delta(t) \in [k_t \Delta, (k_t + 1) \Delta).
\]

Bearing in mind Lemma 4.6 and \( \delta_k = \lfloor \delta(k \Delta)/\Delta \rfloor \), we have

\[
|k_t - (k - \delta_k)| \leq |\rho| + 4. \tag{45}
\]
By the elementary inequality, we get
\[
\sup_{0 \leq t \leq T} \mathbb{E}|y(t - \delta(t)) - z_2(t)|^2 \\
\leq 2 \sup_{0 \leq t \leq T} \mathbb{E}|y(t - \delta(t)) - y(k_1\Delta)|^2 \\
+ 2 \sup_{0 \leq t \leq T} \mathbb{E}|y(k_1\Delta) - z_2(t)|^2.
\] (46)

In order to estimate \( \sup_{0 \leq t \leq T} \mathbb{E}|y(t - \delta(t)) - y(k_1\Delta)|^2 \) from (46), we discuss the following two cases.

**Case 1:** If \( k_1 \leq -1 \), using Assumption 2.6 gives
\[
\sup_{0 \leq t \leq T} \mathbb{E} \xi(t - \delta(t)) - \xi(k_1\Delta)|^2 \\
= \sup_{0 \leq t \leq T} \mathbb{E} |\xi(t - \delta(t)) - \xi(k_1\Delta)|^2 \leq \beta \Delta.
\] (47)

**Case 2:** If \( k_1 \geq 0 \), using Lemma 4.5, from (16) we have
\[
\sup_{0 \leq t \leq T} \mathbb{E}|y(t - \delta(t)) - y(k_1\Delta)|^2 \\
\leq \sup_{-\Delta \leq t \leq T} \mathbb{E}|y(t) - z_1(t)|^2 \leq C_3 \Delta.
\] (48)

Consequently, estimates (47) and (48) give
\[
\sup_{0 \leq t \leq T} \mathbb{E}|y(t - \delta(t)) - y(k_1\Delta)|^2 \leq (C_3 + \beta) \Delta.
\] (49)

We then estimate \( \sup_{0 \leq t \leq T} \mathbb{E}|y(k_1\Delta) - z_2(t)|^2 \) from (46). By the definition of \( z_2(t) \) and estimate (45) together with Lemma 4.7, we see that for any \( t \in [t_k, t_{k+1}] \),
\[
\sup_{0 \leq t \leq T} \mathbb{E}|y(k_1\Delta) - z_2(t)|^2 \\
= \sup_{0 \leq t \leq T} \mathbb{E}|Y_{k+1} - Y_{k-\rho_k}|^2 \\
\leq (\rho + 4)^2 \sup_{-\Delta \leq k \leq \lceil T / \Delta \rceil} \mathbb{E}|Y_{k+1} - Y_k|^2 \\
\leq C_4 (\rho + 4)^2 \Delta.
\] (50)

Inserting (49) and (50) into (46) completes the proof.

Now, we estimate the closeness between the exact solution \( x \) and the EM solution \( y \) in the mean square sense.

**Proof:** For any \( t \in [0, T] \), on the basis of (7) and (14), we have
\[
x(t) - y(t) \\
= D(x(t - \delta(t))) - D(z_2(t)) \\
+ \int_0^t (f(x(s), x(s - \delta(s))) - f(z_1(s), z_2(s))) \, ds \\
+ \int_0^t (g(x(s), x(s - \delta(s))) - g(z_1(s), z_2(s))) \, dB(s) \\
+ \int_0^t (h(x(s), x(s - \delta(s))) - h(z_1(s), z_2(s))) \, d[B](s).
\] (51)

By the Hölder and the elementary inequality (28), we have that for any \( \varepsilon \in (0, 1) \)
\[
|\mathbb{E}|x(t) - y(t)|^2 \leq \frac{1}{\varepsilon} |D(x(t - \delta(t))) - D(z_2(t))|^2 \\
+ \frac{3}{1 - \varepsilon} \left( T \int_0^t |f(x(s), x(s - \delta(s))) - f(z_1(s), z_2(s))|^2 \, ds \\
+ \int_0^t |g(x(s), x(s - \delta(s))) - g(z_1(s), z_2(s))|^2 \, dB(s) \right)^2 \\
+ \int_0^t |h(x(s), x(s - \delta(s))) - h(z_1(s), z_2(s))|^2 \, d[B](s) \right)^2.
\] (52)

Using inequality (28) and Assumption 2.5, we get that
\[
|D(x(t - \delta(t))) - D(z_2(t))|^2 \\
\leq \nu^2 |x(t - \delta(t)) - z_2(t)|^2 \\
\leq \frac{\nu^2}{\varepsilon} |x(t - \delta(t)) - y(t - \delta(t))|^2 \\
+ \nu^2 \left( \mathbb{E}|y(t - \delta(t)) - z_2(t)|^2 \right).
\]

Inserting this into (52) and taking expectations give
\[
\mathbb{E}|x(t) - y(t)|^2 \leq \frac{\nu^2}{\varepsilon^2} \mathbb{E}|x(t - \delta(t)) - y(t - \delta(t))|^2 \\
+ \frac{\nu^2}{\varepsilon} \mathbb{E}|y(t - \delta(t)) - z_2(t)|^2 \\
+ \frac{3}{1 - \varepsilon} \left( \mathbb{E} \int_0^t T|f(x(s), x(s - \delta(s))) - f(z_1(s), z_2(s))|^2 \, ds \\
+ \int_0^t |g(x(s), x(s - \delta(s))) - g(z_1(s), z_2(s))|^2 \, dB(s) \right)^2 \\
+ \int_0^t |h(x(s), x(s - \delta(s))) - h(z_1(s), z_2(s))|^2 \, d[B](s) \right)^2.
\]
Due to $x$ and $y$ satisfy the same initial condition, on the basis of Lemma 4.8 and (53), we have that

$$
\sup_{-\tau \leq t} \hat{E}|x(s) - y(s)|^2 \\
\quad = \sup_{0 \leq s \leq t} \hat{E}|x(s) - y(s)|^2 \\
\quad \leq \frac{\nu^2}{\epsilon^2} \sup_{-\tau \leq s \leq t} \hat{E}|y(s - \delta(s)) - z_2(s)|^2 \\
\quad + \frac{3T}{1 - \epsilon} \hat{E} \int_0^t |f(x(s), x(s - \delta(s))) - f(z_1(s), z_2(s))|^2 ds \\
\quad + \frac{3\sigma^2}{1 - \epsilon} \hat{E} \int_0^t |g(x(s), x(s - \delta(s))) - g(z_1(s), z_2(s))|^2 ds \\
\quad + \frac{3\sigma^4}{1 - \epsilon} \hat{E} \int_0^t |h(x(s), x(s - \delta(s))) - h(z_1(s), z_2(s))|^2 ds \\
\quad \leq \frac{\nu^2}{\epsilon^2} \sup_{-\tau \leq s \leq t} \hat{E}|x(s) - y(s)|^2 + \frac{\nu^2C_5}{\epsilon(1 - \epsilon)} \Delta \\
\quad + K_6 \int_0^t (\hat{E}|x(s) - z_1(s)|^2 \\
\quad - \hat{E}|x(s - \delta(s)) - z_2(s)|^2) ds, \tag{54}
$$

where $K_6 = 3L(T + \sigma^2 + \sigma^4T)/(1 - \epsilon)$. Recalling the definition of the step process $z_1(t)$ and applying Lemma 4.5, we have that for $0 \leq s \leq t$,

$$
\hat{E}|x(s) - z_1(s)|^2 \leq 2\hat{E}|x(s) - y(s)|^2 + 2\hat{E}|y(s) - z_1(s)|^2 \\
\quad \leq 2 \sup_{-\tau \leq u \leq s} \hat{E}|x(u) - y(u)|^2 \\
\quad + 2 \sup_{-\tau \leq u \leq s} \hat{E}|y(u) - z_1(u)|^2 \\
\quad \leq 2 \sup_{-\tau \leq u \leq s} \hat{E}|x(u) - y(u)|^2 + 2C_3\Delta. \tag{55}
$$

On the other hand, we have that for $0 \leq s \leq t$,

$$
\hat{E}|x(s - \delta(s)) - z_2(s)|^2 \\
\quad \leq 2\hat{E}|x(s - \delta(s)) - y(s - \delta(s))|^2 \\
\quad + 2\hat{E}|x(s - \delta(s)) - z_2(s)|^2 \leq 2 \sup_{-\tau \leq u \leq s} \hat{E}|x(u) - y(u)|^2 + 2C_5\Delta. \tag{56}
$$

Now, the estimates (55) and (56) together with (54) give

$$
\sup_{-\tau \leq s \leq t} \hat{E}|x(s) - y(s)|^2 \leq \frac{\nu^2}{\epsilon^2} \sup_{-\tau \leq s \leq t} \hat{E}|x(s) - y(s)|^2 + K_7\Delta \\
\quad + K_6 \int_0^t \sup_{-\tau \leq u \leq s} \hat{E}|x(u) - y(u)|^2 ds, \tag{57}
$$

where $K_7 = \nu^2C_5/(\epsilon(1 - \epsilon)) + K_6(2C_3 + 2C_5)$. Rearranging (57) gives

$$
\left(1 - \frac{\nu^2}{\epsilon^2}\right) \sup_{-\tau \leq s \leq t} \hat{E}|x(s) - y(s)|^2 \\
\quad \leq K_6 \int_0^t \sup_{-\tau \leq u \leq s} \hat{E}|x(u) - y(u)|^2 ds + K_7\Delta. \tag{58}
$$

Choosing $\epsilon = \sqrt{\nu}$ such that $1 - \nu^2/\epsilon^2 = 1 - \nu$, then (58) becomes

$$
\sup_{-\tau \leq s \leq t} \hat{E}|x(s) - y(s)|^2 \\
\quad \leq \frac{K_6}{1 - \nu} \int_0^t \sup_{-\tau \leq u \leq s} \hat{E}|x(u) - y(u)|^2 ds + \frac{K_7}{1 - \nu}\Delta.
$$

The application of the Gronwall inequality yields the desired assertion. Thus, we complete the proof. \qed

**Remark 4.9:** Compared with the results in Li and Yang (2018), time-dependent delay is taken into account to investigate the convergence of the Euler-type method for NSDEs in the G-framework, which generalizes the traditional G-model and adapts to more general delay conditions.

**Remark 4.10:** It should be pointed out that when $\sigma = \overline{\sigma}$, these results reduce to the corresponding classical stochastic ones that were discussed in the reference Milošović (2011). We extend the convergence results of the EM method for NSDE with variable delay to the case of G-framework.

## 5. Numerical experiment

In this section, we perform a numerical experiment to confirm our theoretical results. We focus on the error between the true solution $x$ and the corresponding EM
solution $y$ at the endpoint $T$ defined by

$$e_{\Delta}^{\text{strong}}(T) = \hat{E}|x(T) - y(T)|.$$ 

We note that Theorem 4.1 implies that

$$e_{\Delta}^{\text{strong}}(T) \leq (\hat{E}|x(T) - y(T)|)^{1/2} \leq \sqrt{C_\delta} \Delta^{1/2}. \quad (59)$$

Now, we use the algorithm introduced by Fei and Fei (2019) to simulate the G-expectation of the absolute error between the exact solution $x$ and EM solution $y$. Let $B(t) \sim \mathcal{N}(0,[\sigma^2 \Delta])$. The partition of the interval $[\sigma^2 \Delta]$ is defined as

$$\sigma_1^2 = \sigma^2 < \cdots < \sigma_k^2 < \cdots < \sigma_M^2 = \sigma^2.$$ 

For a fixed Brownian motion $B^k(t) \sim \mathcal{N}(0,\sigma_k^2 \Delta)(k = 1, 2, \ldots, M)$, we denote the exact solution of (5) driven by $B^k(t)$ as $x^k = x(t;B^k(t))$, that is, for any $t \geq 0$

$$d(x^k(t) - D(x^k(t - \delta(t)))) = f(x^k(t),x^k(t - \delta(t))) \, dt + g(x^k(t),x^k(t - \delta(t)))dB^k(t) + h(x^k(t),x^k(t - \delta(t)))\Delta.$$ 

The corresponding continuous EM solution can be represented as $y^k(t) = y(t;B^k(t))$. Discrete EM solution is defined by

$$y^k_i = \xi(t), \quad i = -m,-m+1,\ldots,0,$$

$$y^k_{i+1} = y^k_i + D(y^k_{i+1-\delta(s)}) - D(y^k_{i-\delta(s)}) + f(y^k_i, y^k_{i-\delta(s)})\Delta$$

$$+ g(y^k_i, y^k_{i-\delta(s)})\Delta B^k_i + h(y^k_i, y^k_{i-\delta(s)})\Delta_2,$$ 

$$i = 0,1,2,\ldots, \quad (60)$$

where $\Delta B^k_i = B^k((i + 1)\Delta) - B^k((i)\Delta) \sim \mathcal{N}(0,\sigma_k^2 \Delta)$. For $k = 1, 2, \ldots, M$, we perform $J$ sampling to estimate the expectations of absolute error $|x^k(t) - y^k(t)|$. In the $j$-th random sampling ($j = 1, 2, \ldots, J$), we represent $x^k_j(t)$ and $y^k_j(t)$ as the solutions defined by $x^k(t)$ and $y^k(t)$, respectively. To approximate $e_{\Delta}^{\text{strong}}$, we define the maximum sample average of absolute error between $x$ and $y$ at time $T$ as follows:

$$e_{\Delta}^{\text{strong}}(T) = \max_{1 \leq k \leq M} \frac{1}{J} \sum_{j=1}^{J} |x^k(T) - y^k(T)|, \quad \forall T \geq 0.$$ 

In our numerical experiment, we take the $J = 500$ and $M = 6$.

**Example 5.1:** Let $B(t) \sim \mathcal{N}(0,[0.6^2,0.8^2])$. The delay function is defined as $\delta(t) = 1 - \frac{1}{4} \sin(t), t \geq 0$. Consider the following one-dimensional NSDE with time-dependent delay driven by G-Brownian motion

$$d(x(t)) = \frac{1}{9}(x(t) - \delta(t))\, dt + x(t - \delta(t))\, dB(t)$$

$$- \sin(x(t)\, dB(t), \quad t \geq 0,$$ 

(61)

with the initial data $\xi(t) = 1$, $t \in [-\tau, 0]$, where $\tau = 2$. Obviously, the coefficients of this system satisfy Assumption 2.4 and $D(x) = x/9$ satisfies Assumption 2.5 with $\nu = 1/9$. Moreover, $|\delta(t) - \delta(s)| \leq |t - s|/4, \forall t, s \geq 0$, which means that (6) holds for $\rho = 1/4$. Hence, we have that $\nu(3 + |\rho_1|) = 1/3 < 1$. Then, the discrete EM scheme can be written as the form (60) with $f(x,y) = x, g(x,y) = y$ and $h(x,y) = -\sin(x)$ for the equidistant partition $\sigma_1 = 0.6 < \cdots < \sigma_k < \cdots < \sigma_M = 0.8$. By Theorem 4.1, the continuous EM solution $y(t)$ converges to the exact solution $x(t)$ with mean square order equal to one. In other words, the root mean square order is equal to 0.5.

On the other hand, as the G-NSDE (61) does not have any explicit solution, the numerical approximation of the EM method with step size $\Delta = 2^{-13}$ is used as a replacement of the unknown exact solution in the numerical simulations. We compute numerical solutions of the EM scheme (60) by using five different step sizes $\Delta = 2^{-8}$, $2^{-9}$, $2^{-10}$, $2^{-11}$ and $2^{-12}$ at $T = 1$. Table 1 and Figure 1 show the results of these calculations.
From Figure 1, we see that there appears to exist positive constants \( C \) and \( \gamma \) such that

\[
\mathcal{E}^{\text{strong}}(T) \leq C \Delta^\gamma
\]

for sufficiently small \( \Delta \). A least squares fit for \( \log C \) and \( \gamma \) is computed, producing the value 0.5752 for \( \gamma \), which is close to the order obtained by (59). Clearly, our simulation results are consistent with the theoretical ones.

6. Conclusion

In this paper, the convergence of the EM method for NSDEs with variable time lag in the \( G \)-framework is studied. The results show that the order of the mean square closeness of the approximate solution \( y \) and the exact solution \( x \) is equal to one. A numerical experiment including \( G \)-expectation simulation is given to demonstrate our theory. Our future research topic will try to relax the global Lipschitz condition to the local one.

Disclosure statement

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