Power fluctuations in a finite-time quantum Carnot engine

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Stability is an important property of small thermal machines with fluctuating power output. We here consider a finite-time quantum Carnot engine based on a degenerate multilevel system and study the influence of its finite Hilbert space structure on its stability. We optimize in particular its relative work fluctuations with respect to level degeneracy and level number. We find that its optimal performance may surpass those of nondegenerate two-level engines or harmonic oscillator motors. Our results show how to realize high-performance, high-stability cyclic quantum heat engines.

The Carnot engine is one of the most emblematic examples of a thermal machine. Since its introduction in 1824, it has become a representative model for all heat engines. The Carnot cycle simply consists of two isothermal (expansion and compression) steps and of two adiabatic (expansion and compression) processes \[1\]. In its ideal reversible limit of infinite cycle duration, the Carnot motor is the most efficient heat engine. The corresponding Carnot efficiency, \(\eta_C = 1 - \frac{T_c}{T_h}\), where \(T_c\) and \(T_h\) are the respective temperatures of the cold and hot heat baths, is regarded as the first formulation of the second law of thermodynamics \[1\]. The first experimental realization of a Carnot engine using a colloidal particle in an optical harmonic trap has been reported lately \[2\]. In addition, the finite-time properties of the Carnot cycle have been well investigated theoretically both in the classical \[3–10\] and in the quantum \[11–19\] regimes. Strong emphasis has been put on the optimization of the performance of the engine for finite cycle durations, in particular on its average power output at the expense of its efficiency.

For classical microscopic heat engines, like the one implemented in the experiment of Ref. \[2\], thermal fluctuations are no longer negligible as is the case for macroscopic motors \[20\]. As a result, key performance measures, such as efficiency \[21–24\] and power \[25–28\], are stochastic variables. In that context, attention has recently been drawn to power fluctuations as a limiting factor for the practical usefulness of thermal machines: heat engines should indeed ideally have high efficiency, large power output but small power fluctuations \[25–28\].

A new figure of merit, the constancy, defined as the product of the variance of the power and time, has been introduced to characterize the stability of heat engines \[25–28\]. While a strict trade-off between efficiency, power and constancy has been established for steady-state heat engines, implying that power fluctuations diverge at maximum efficiency and finite power \[27\], they may remain finite for quasistatic cyclic thermal machines \[28\]. On the other hand, quantum motors are not only dominated by thermal fluctuations but also by quantum fluctuations.

In this paper, we investigate the generic features of the power fluctuations in a finite-time quantum Carnot engine in the quasistatic limit. We specifically study the interplay between power fluctuations, the finite dimensionality of the Hilbert space of the working medium and the degree of degeneracy of its levels. Degenerate finite level structures commonly appear in atomic \[29\], molecular \[30\] and condensed-matter physics \[31\]. An understanding of their influence on the stability of quantum heat engines is therefore essential for future experimental realizations of thermal devices in these systems \[32\].

An important illustration of the effect of the finiteness of quantum systems on thermodynamic fluctuations is provided by the Schottky anomaly \[33\]: the corresponding increase of the heat capacity at low temperatures does not occur in infinite dimensional systems like the harmonic oscillator, in which the energy is not bounded; it is, furthermore, strongly affected by level degeneracy and level number \[34\]. We mention, however, that our results are not directly related to the Schottky anomaly.

In the following, we compute the inverse coefficient of variation for work, defined as the ratio of the mean work and its standard deviation \[35\], for a quasistatic quantum Carnot engine whose working medium is described by a homogeneous Hamiltonian of degree \(-2\), \(\mathcal{H}(b r) = b^{-2} \mathcal{H}(r)\). Such Hamiltonians characterize a large class of single-particle, many-body and nonlinear systems that exhibit equidistant spectra \[36–40\]. We obtain a general formula that only depends on the heat capacity and on the entropy variation during the hot isotherm. We use this expression to maximize the inverse coefficient of variation for work, with respect to the degree of degeneracy and the number of levels of the system, in order to attain optimal cyclic quantum engines that operate close to the Carnot efficiency with large power output and small power fluctuations. We illustrate our results by analyzing (i) a two-level system with arbitrary degeneracy, (ii) a nondegenerate system with arbitrary level number, and (iii) a three-level system with generic degree of degeneracy. In all cases the optimal inverse coefficient of variation for work may be numerically determined by solving a transcendental equation.

Finite-time quantum Carnot cycle. We consider a general quantum system with time-dependent Hamiltonian, \(H_t = \omega_t \mathcal{H}_t\), as the working fluid of a finite-time quantum Carnot engine, with driving parameter \(\omega_t\). The quantum Carnot cycle consists of the following four steps \[11–19\]: (1) hot isothermal expansion from \(\omega_1\) to \(\omega_2\), at temperature \(T_h\) in time \(\tau_1\), during which work \(W_1\) is produced by the system and heat \(Q_h\) is absorbed; (2) adiabatic expansion from \(\omega_2\) to \(\omega_3\), in time \(\tau_2\), during which work \(W_2\) is performed and the entropy remains constant. The
The whole Carnot cycle is hence quasistatic. The adiabatic driving such that the system is in a thermal state at the end of each adiabat, with energy at the beginning and at the end of each step [41]. Without level transitions, we obtain for process (2), where the average total work, $W = \langle w \rangle = \sum_i W_i$, directly follows from the combination of the first and second law [1].

$$W = -Q_h - Q_c = (T_c - T_h)\Delta S,$$

(1)

where $\Delta S$ denotes the entropy change during the hot isotherm. Meanwhile, the total work fluctuations are characterized by the probability distribution $P(w)$,

$$P(w) = \langle \delta(w - \sum_i w_i) \rangle,$$

(2)

where the average is taken over the joint probability distribution given by the convolution of the work densities of the four branches of the cycle, $P(w_1, w_2, w_3, w_4) = P_1(w_1) \ast P_2(w_2) \ast P_3(w_3) \ast P_4(w_4)$ [28]. The two isotherms (1) and (3) are assumed to be slower than the (fast) relaxation induced by the baths. The system thus remains in a thermal state and the two finite-time processes are quasi-static. In this case, the work distributions are sharp (with no fluctuations) and work is deterministic [42],

$$P_{1,3}(w_{1,3}) = \delta(w_{1,3} - W_{1,3}).$$

(3)

On the other hand, since no heat is exchanged during the two unitary adiabats (2) and (4), the corresponding work distributions can be obtained via the usual two-point-measurement scheme by projectively measuring the energy at the beginning and at the end of each step [41]. Without level transitions, we obtain for process (2),

$$P_2(w_2) = \sum_n \delta[w_2 - (E_n^m - E_n^s)]p_n^2,$$

(4)

where $E_n^m$ and $E_n^s$ denote the respective eigenvalues of the Hamiltonians $H_2 = H_{\tau_1}$ and $H_3 = H_{\tau_1 + \tau_2}$. The initial thermal distribution reads $p_n^2 = \exp(-\beta_n E_n^m)/z_2$ with inverse hot temperature $\beta_n$ and partition function $z_2$. We have similarly for transformation (4),

$$P_4(w_4) = \sum_m \delta[w_4 - (E_m^m - E_m^s)]p_m^4,$$

(5)

with $p_m^4 = \exp(-\beta_m E_m^m)/z_4$. In order to ensure that the system is in a thermal state at the end of each adiabat, and thus at the beginning of each isotherm, we adjust the adiabatic driving such that $\omega_3/\omega_2 = \omega_4/\omega_1 = \beta_h/\beta_c$ [18]. The whole Carnot cycle is hence quasi-static.

Combining the contributions of all the four branches of the cycle, we find the work output distribution,

$$P(w) = \langle \delta(w - (W - \Delta H_2 - \Delta H_4)) \rangle,$$

(6)

where $W$ is given by Eq. (1). We have furthermore defined the (stochastic) difference $\Delta H_i = \langle \Delta H_i \rangle - \Delta H_i$ and used the cycle condition $\sum_i(\Delta H_i) = 0$. The average in Eq. (1) may be computed using the Boltzmann distributions at the beginning of each adiabat [12].

**Coefficient of variation for work.** In statistics, the Fano factor (the ratio of the variance $\sigma^2$ and the mean) and the coefficient of variation (the ratio of the standard deviation $\sigma$ and the mean) are two measures of the dispersion of a probability distribution [28]. For heat engines, the Fano factor for work, $\sigma_w^2/W$, is equal to the quotient of the constancy $\sigma_w^2/\tau$ and the average power $P = W/\tau$ (defined over one cycle time) [28], while the corresponding coefficient of variation for work describes the relative work fluctuations. All the moments of the total work can be evaluated from Eq. (6) by integration $\langle w^n \rangle = \int dw P(w)w^n$. The variance then reads [12],

$$\sigma_w^2 = (T_c - T_h)^2 \left[C(\beta_h, \omega_2) + C(\beta_c, \omega_4)\right],$$

(7)

where we have introduced the heat capacity of the system, $C(\beta, \omega_i) = d(H_i)/dT_i$, at the beginning of each adiabat [12]. We accordingly obtain the Fano factor,

$$\sigma_w^2/W = \frac{(T_c - T_h)^2 \left[C(\beta_h, \omega_2) + C(\beta_c, \omega_4)\right]}{\Delta S},$$

(8)

and the corresponding coefficient of variation,

$$\sigma_w/W = \frac{\sqrt{C(\beta_h, \omega_2) + C(\beta_c, \omega_4)}}{\Delta S}.$$

(9)

Equations (8) and (9) describe similar physics. However, in contrast to the Fano factor (3), the coefficient of variation (9) has the advantage that (i) it is a dimensionless quantity that (ii) depends solely on the heat capacities of the system (since the entropy variation can be written as an integral of the heat capacities [22]). We shall therefore focus on that quantity in the following.

A finite-time quantum Carnot engine with large work output and small work output fluctuations is characterized by a large inverse coefficient of variation $|W|/\sigma_w$. We will thus next optimize the inverse of Eq. (9) with respect to the degree of degeneracy and with respect to the number of levels of the working medium.

**Degenerate two-level system.** We begin by considering a degenerate qubit with Hamiltonian $H_1 = \omega_0 g_1 |1\rangle \langle 1|$, where $g_0$ and $g_1$ are the respective degeneracies of the ground $|0\rangle$ and excited $|1\rangle$ states. The partition function at inverse temperature $\beta$ and frequency $\omega$ is $Z_0 = g_0 + g_1 \exp(-\beta \omega)$ [33]. The heat capacity then follows as,

$$C_2(\beta, \omega) = \frac{\gamma(\beta \omega)^2}{(e^{\beta \omega/2} + e^{-\beta \omega/2})^2},$$

(10)
with the degeneracy ratio $\gamma = g_1/g_0$. The entropy difference during the hot isotherm further reads,

$$\Delta S_2 = \frac{\beta_c \omega_4}{1 - e^{\beta_c \omega_4}} - \frac{\beta_h \omega_2}{1 - e^{\beta_h \omega_2}} + \ln \left[ \frac{1 + \gamma e^{-\beta_h \omega_2}}{1 + \gamma e^{-\beta_c \omega_4}} \right].$$

(11)

The average of the work output $\langle W \rangle$ (blue), its variance $\sigma_w^2$, and the corresponding inverse coefficient of variation $\gamma$ (red) are shown in Fig. 1 as a function of the degeneracy ratio $\gamma$. We observe that, for given frequencies and bath temperatures, both mean and variance first increase with increasing values of $\gamma$, before they both decrease as a result of the finiteness of the Hilbert space of the qubit. However, the mean augments and decays faster than the variance. As a consequence, the inverse coefficient of variation for work exhibits a clear maximum (green arrow) for an optimal degeneracy value $\bar{\gamma}$. Remarkably, the degenerate quantum Carnot engine here outperforms its nondegenerate counterpart (orange dashed). The blue (red) arrow indicates the maximum value of the average (variance) of the work output. The vertical black dotted lines mark the respective maxima of the heat capacity (Schottky anomaly), Eq. (10), at the hot and cold bath temperatures. Parameters are $\beta_c = 1$, $\beta_h = 0.1$, $\omega_2 = 2$ and $\omega_4 = 1$.

![Fig. 1. Average work output $\langle W \rangle$, Eq. (1) (blue), its variance $\sigma_w^2$, Eq. (6) (red), and inverse coefficient of variation (COV), $\langle W \rangle/\sigma_w$, Eq. (9) (green) (inset), for a degenerate two-level quantum Carnot engine, as a function of the degeneracy ratio $\gamma$. The maximum COV (green arrow) outperforms its nondegenerate counterpart (orange dashed line). The blue (red) arrow indicates the maximum value of the average (variance) of the work output. The vertical black dotted lines mark the respective maxima of the heat capacity (Schottky anomaly), Eq. (10), at the hot and cold bath temperatures. Parameters are $\beta_c = 1$, $\beta_h = 0.1$, $\omega_2 = 2$ and $\omega_4 = 1$.](image)

In the high-temperature limit, $H_t = \omega t \sum_{n=0}^{-1} |n\rangle \langle n|$, as appearing in homogeneous Hamiltonians of degree $-2$ [36][40]. The partition function at inverse temperature $\beta$ and frequency $\omega$ is given by $Z_N = [1 - \exp(-N\beta \omega)]/[1 - \exp(-\beta \omega)]$ [42]. The explicit (and lengthy) expressions for the heat capacity $C_N(\beta, \omega)$ and the entropy difference $\Delta S_N$ are given in the Supplemental Material [42]. Compact expressions for the inverse coefficient of variation for work may be obtained in the limit of a harmonic oscillator ($N \to \infty$),

$$\left( \frac{\langle W \rangle}{\sigma_w} \right)_\infty = \frac{\Delta S_\infty}{\sqrt{(\text{sech}(y/2)y)^2 + (\text{sech}(x/2)x)^2}},$$

(15)

and for the case a (nondegenerate) qubit ($N = 2$),

$$\left( \frac{\langle W \rangle}{\sigma_w} \right)_2 = \frac{\Delta S_2}{\sqrt{y^2(1 - \tanh(y)^2) + x^2[1 - \tanh(x)^2]}},$$

(16)

In the high-temperature limit, $\beta_c \omega_4, \omega_2 \ll 1$. Eq. (15) reduces to the result obtained for the classical harmonic Carnot heat engine in Ref. [28],

$$\left( \frac{\langle W \rangle}{\sigma_w} \right)_\infty = \frac{\Delta S_\infty}{\sqrt{2}},$$

(17)

On the other hand, the high-temperature limit of Eq. (16) exhibits a completely different $(x, y)$-dependence, which reflects the finite Hilbert space of the qubit,

$$\left( \frac{\langle W \rangle}{\sigma_w} \right)_2 = \frac{\Delta S_2}{\sqrt{2x[1 - x^2] + y[1 - y^2]}}.$$

(18)
Such behavior can be traced back to the properties of the heat capacity in Eq. (9): while it reaches a constant value for the (infinite-dimensional) harmonic oscillator in the classical limit (Dulong-Petit law), it vanishes for the (finite-dimensional) two-level system.

Figure 2 displays the mean work output $\langle W \rangle$, the corresponding inverse variance $1/\sigma_w^2$, and inverse coefficient of variation (COV) $1/\sigma_w$, as a function of the level number $N$. The maximum COV outperforms both that of the two-level engine and that of the harmonic oscillator motor (violet dotted-dashed line). Same parameters as in Fig. 1.

**Conclusions.** Two-level systems and harmonic oscillators have been the models of choice for the investigations of quantum heat engines in the past decades due to their simplicity. Such finite-time engines have been mostly optimized by maximizing averaged performance measures, such as mean power, with respect to cycle duration, frequency or temperature. We have here extended these studies to include the effects of work fluctuations and of finite Hilbert space of the working medium, two essential features of small quantum machines. To this end, we have derived a compact expression of the relative work fluctuations of a finite-time quantum Carnot engine, as given by Eq. (9), in terms of the heat capacity of the system. We have shown that the quantum motor can outperform its nondegenerate counterparts, when optimized with respect to level degeneracy or level number. We have additionally found that optimizing the average work output, while ignoring work output fluctuations, generally leads to machines

\[ g_n, \quad (n = 0, 1, 2). \]
with larger instability. Our findings hence enable the analysis and future experimental realization of both high-performance and high-stability cyclic quantum heat engines.

**Acknowledgements.** We acknowledge financial support from the Volkswagen Foundation under project "Quantum coins and nano sensors" and the German Science Foundation (DFG) under project FOR 2724.

**Supplemental Material**

**Work distribution for quasistatic isotherms.** We begin by approximating the Hamiltonian evolution by following the train of thought presented in Ref. [49] to the quantum domain. We here discuss an alternative derivation by extending the classical derivation of Ref. [28] to the quantum case. This result may be derived by extending the classical derivation of Ref. [28] to the quasistatic case. We show that quantum work is delta distributed, and therefore deterministic, during quasistatic isotherms, see Eq. (3) of the main text. This result may be derived by extending the classical derivation of Ref. [28] to the quantum case. We begin by approximating the Hamiltonian evolution $H_i$ during isothermal driving as a set of $K$ discrete steps, $H_0 \rightarrow H_1 \rightarrow \cdots \rightarrow H_K$. We assume the driving to be slow enough that thermal equilibrium is reached after each discrete step, so that the state of the system after the $i$-th step is of the Gibbs form, $\pi_i = e^{-\beta H_i}/Z_i$, at inverse temperature $\beta$. The total work distribution may then be written as a convolution of $K$ independent contributions $P_i(w_i), P(w) = \prod_{i=1}^{K} P_i(w_i)$.

The distribution for the $i$-th quasistatic step may be obtained via the two-point-measurement scheme [41] as,

$$P_i(w_i) = \sum_{E_{i+1}^{(n)} - E_i^{(n)} = w_i} \frac{e^{-\beta H_i}}{Z_i}|E_i^{(n)} \rangle \langle E_i^{(n)}|E_{i+1}^{(n)}\rangle^2;$$

(20)

where $|E_i\rangle$ and $E_i$ denote the respective eigenstate and eigenenergy of the Hamiltonian $H_i$. The cumulant generating function (CGF),

$$G(\lambda) = \ln \int_{-\infty}^{\infty} dw \, P(w) e^{-\beta w},$$

(21)

allows the computation of all the work cumulants via,

$$\kappa_w^l = (-\beta)^{-l} \left. \frac{d^l}{d\lambda^l} G(\lambda) \right|_{\lambda=0}.$$  

(22)

Inserting Eq. (20) into Eq. (21), the CGF becomes,

$$G(\lambda) = \sum_{i=1}^{K-1} \int_{-\infty}^{\infty} dw_i \ln P_i(w_i) e^{-\beta \lambda w_i}$$

$$= \sum_{i=1}^{K-1} \ln \left( \text{Tr} \left[ e^{-\beta \lambda H_{i+1}} e^{\beta \lambda H_i} \pi_i \right] \right)$$

$$= \sum_{i=1}^{K-1} \ln \left( \text{Tr} \left[ e^{-\beta \lambda H_{i+1}} e^{\beta \lambda H_i} \pi_i \right] \frac{Z_{i+1}^\lambda}{Z_i^\lambda} \right)$$

$$= -\beta \lambda \Delta F + \sum_{i=1}^{K-1} (\lambda - 1) S_\lambda (\pi_{i+1}||\pi_i).$$

(23)

The CGF may thus be split into a protocol-independent part given by the free energy difference, $\Delta F = -\beta^{-1} \ln Z_K/Z_0$, and a protocol-dependent part given by the $\lambda$-Renyi divergence, $S_\lambda (\rho||\sigma) = \ln \text{Tr}[\rho^{\lambda-1}]/(\lambda - 1)$. Taking the limit $K \to \infty$ with $(H_0, H_K)$ fixed and writing the $(i+1)$-th step in terms of the $i$-th one as $H_{i+1} = H_i + \Delta H/K$, with $\Delta H = H_K - H_0$, the last term in Eq. (23) simplifies to,

$$\lim_{K \to \infty} S_\lambda (\pi_{i+1}||\pi_i) = \lim_{K \to \infty} \frac{1}{\lambda - 1} \ln \left( \frac{\text{Tr} \left[ e^{-\beta \lambda \Delta H/K} \pi_i \right]}{\text{Tr} \left[ e^{-\beta (H_i + \Delta H/K)} \pi_i \right] \text{Tr} \left[ e^{-\beta H_i} \pi_i \right]} \right)$$

$$= \lim_{K \to \infty} \frac{1}{\lambda - 1} \ln \left( \text{Tr} \left[ e^{-\beta \lambda \Delta H/K} \pi_i \right] \right) - \ln \left( \frac{\text{Tr} \left[ e^{-\beta H_i} \pi_i \right]}{\text{Tr} \left[ e^{-\beta (H_i + \Delta H/K)} \pi_i \right]} \right)$$

(24)

where we have used $[H_i, H_{i+1}] = 0$ in the first line. All the terms hence vanish in the limit $K \to \infty$. As a result,

$$G(\lambda) = -\beta \lambda \Delta F.$$  

(25)

The work distribution then follows as,

$$P(W) = \delta(W - \Delta F),$$

(26)

indicating that work is deterministic in this case.

**Derivation of the coefficient of variation for work.** In order to derive Eqs. (8) and (9) of the main text, we first need an expression for the second moment. Integration of Eq. (6) of the main text yields,

$$\langle w^2 \rangle = W^2 + \langle \Delta H_2^2 \rangle - \langle \Delta H_2 \rangle^2 + \langle \Delta H_4^2 \rangle - \langle \Delta H_4 \rangle^2.$$  

(27)
So the work variance simply reads,
\[ \sigma^2_w = \langle \Delta H^2_2 \rangle - \langle \Delta H_2 \rangle^2 + \langle \Delta H^2_4 \rangle - \langle \Delta H_4 \rangle^2. \quad (28) \]

Using the condition, \( \omega_3/\omega_2 = \omega_4/\omega_1 = \beta_h/\beta_c \), we can simplify the mean and the square energy difference of the adiabats in terms of the canonical partition functions,
\[ \sigma^2_w = \left( 1 - \frac{\beta_h}{\beta_c} \right)^2 \left[ \frac{1}{z_4} \frac{\partial^2}{\partial \beta_c^2} z_2 - \frac{1}{z_4^2} \left( \frac{\partial}{\partial \beta_h} z_2 \right)^2 \right] \]
\[ + \left( 1 - \frac{\beta_h}{\beta_c} \right)^2 \left[ \frac{1}{z_4} \frac{\partial^2}{\partial \beta_c^2} z_4 - \frac{1}{z_4^2} \left( \frac{\partial}{\partial \beta_h} z_4 \right)^2 \right], \quad (29) \]
where \( z_4 = \text{Tr} \left[ e^{-\beta_c H_4} \right] \) and \( z_2 = \text{Tr} \left[ e^{-\beta_h H_2} \right] \). On the other hand, the heat capacity at inverse temperature \( \beta \) and frequency \( \omega \) reads,
\[ C(\beta, \omega) = \frac{\partial U(\beta, \omega)}{\partial T} = \beta^2 \frac{\partial^2}{\partial \beta^2} \ln z \]
\[ = \beta^2 \left[ \frac{1}{z} \frac{\partial^2}{\partial \beta^2} z - \frac{1}{z} \left( \frac{\partial z}{\partial \beta} \right)^2 \right], \quad (30) \]
Combining Eqs. (29) and (30), we then obtain the coefficient of variation for work,
\[ \frac{\sigma_w}{|W|} = \frac{\sqrt{C(\beta_c, \omega_4) + C(\beta_h, \omega_2)}}{\Delta S}. \quad (31) \]
The Fano factor may be similarly obtained by taking the square of the nominator of Eq. (31).

In addition, the entropy difference \( \Delta S \) during the hot isotherm is given by,
\[ \Delta S = \ln \frac{z_2}{z_4} + T_h \frac{\partial \ln z_2}{\partial T_h} - T_c \frac{\partial \ln z_4}{\partial T_c}. \quad (32) \]

or, equivalently, in terms of the heat capacities,
\[ \Delta S = \int_0^{T_h} dT \frac{C(1/T, \omega_4)}{T} - \int_0^{T_c} dT \frac{C(1/T, \omega_2)}{T}. \quad (33) \]

\[ C_N(\beta, \omega) = \frac{\beta^2 \omega^2}{(e^{\beta \omega} - e^{\beta N \omega} - N^2 e^{(N+2) \beta \omega} + 2 (N^2 - 1) e^{(N+1) \beta \omega} + e^{(2N+1) \beta \omega})^2}. \quad (39) \]

On the other hand, the entropy difference during the hot isotherm is given by,
\[ \Delta S_N = -\omega_4 \beta_c \left[ N \left( e^{\omega_4 \beta_h} - e^{N \omega_4 \beta_h} + N - 1 \right) \right] - \ln \left( \frac{1 - e^{-N \omega_4 \beta_h}}{1 - e^{-\omega_4 \beta_h}} \right) \]
\[ + \beta_h \omega_2 \left[ N \left( e^{\omega_2 \beta_h} - e^{N \omega_2 \beta_h} + N - 1 \right) \right] + \ln \left( \frac{1 - e^{-N \beta_h \omega_2}}{1 - e^{-\beta_h \omega_2}} \right). \quad (40) \]

The coefficient of variation for work again follows by inserting Eqs. (39) and (40) into Eq. (31). We emphasize...
that the heat capacity and the entropy difference exhibit a non-trivial and non-monotonous $N$-dependence, which is different from just increasing the number of particles for which the coefficient of variation would stay constant.

**Coefficient of variation for work for a degenerate 3-level system.**

We finally evaluate the heat capacity and the entropy change during the hot isotherm for a degenerate 3-level system ($N = 3$). The partition function is,

$$Z_3 = g_0 + g_1 e^{-\beta \omega} + g_2 e^{-2\beta \omega}. \quad \text{(41)}$$

The heat capacity then reads,

$$C_3 = (\beta \omega)^2 e^{\beta \omega} \frac{2e^{\beta \omega} \gamma_1 + 4e^{\beta \omega} \gamma_2 + \gamma_1 \gamma_2}{(e^{2\beta \omega} + e^{\beta \omega} \gamma_1 + \gamma_2)^2}, \quad \text{(42)}$$

with the degeneracy ratios $\gamma_1 = g_1/g_0$ and $\gamma_2 = g_2/g_0$. The entropy change during the hot isotherm is moreover,

$$\Delta S_3 = \frac{\beta \omega}{e^{\beta \omega} g_1 \gamma_1 + 2 \gamma_2} \frac{\beta \omega \gamma_1 + 2 \gamma_2}{e^{\beta \omega} g_1 \gamma_1 + 2 \gamma_2} - \frac{\beta \omega \gamma_1 + 2 \gamma_2}{e^{\beta \omega} g_1 \gamma_1 + 2 \gamma_2} \ln \left[ \frac{1 + e^{-\beta \omega} (e^{\beta \omega} \gamma_1 + \gamma_2)}{1 + e^{-\beta \omega} (e^{\beta \omega} \gamma_1 + \gamma_2)} \right]. \quad \text{(43)}$$

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