Second-forbidden nonunique $\beta^-$ decays of $^{24}\text{Na}$ and $^{36}\text{Cl}$ assessed by the nuclear shell model

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We have performed a systematic study of the log$ft$ values, shape factors and electron spectra for the second-forbidden nonunique $\beta^-$ decays of $^{24}\text{Na}(4^+) \rightarrow ^{24}\text{Mg}(2^+)$ and $^{36}\text{Cl}(2^+) \rightarrow ^{36}\text{Ar}(0^+)$ transitions under the framework of the nuclear shell model. We have performed the shell model calculations in the $sd$ model space, using more recent microscopic effective interactions such as Daejeon16, chiral N3LO, and JISP16. These interactions are derived from the no-core shell model wave functions using Okubo-Lee-Suzuki transformation. For comparison, we have also shown the results obtained from the phenomenological USDB interaction. To test the predictive power of these interactions first we have computed low-lying energy spectra of parent and daughter nuclei involved in these transitions. The computed results for energy spectra, nuclear matrix elements, log$ft$ values, shape factors, electron spectra and decomposition of the integrated shape factor are reported and compare with the available experimental data.

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I. INTRODUCTION

The $\beta^-$ decay plays an important role in astrophysics e.g. for the $r$ process [1]. In the nuclear chart, there are selected candidates for double beta decays, but on the other hand, there are several potential candidates known for forbidden beta decay. Out of these, only around 27 possible candidates of second-forbidden nonunique beta decay is observed as reported in Ref. [2]. Recently, a new candidate is observed corresponding to second-forbidden nonunique decay of $^{20}\text{Fe}(2^+) \rightarrow ^{20}\text{Ne}(0^+)$ from ground-state-to-ground-state transition [3–5]. This study could change our understanding of the fate of intermediate-mass stars. The comprehensive review on the theoretical and experimental status of single and double beta decay is recently reported in Ref. [6].

In the beta decay based on the value of angular momentum ($I$) we can characterize any decay as allowed or forbidden. The $l = 0$ decays are called as “allowed” while the $l > 0$ decays are called as “forbidden”. Further, we can divide decays as forbidden unique (FU) and forbidden nonunique (FNU). In the case of FU, the total angular momentum $K = l + 1$, whereas in FNU decay $K = l$. The $\beta^-$ decay half-life of the 4th forbidden nonunique decay of $^{50}\text{V}$ using nuclear shell model is reported in Ref. [7]. The 4th forbidden nonunique ground-state-to-ground-state $\beta^-$ decay branches of $^{113}\text{Cd}$ and $^{115}\text{In}$ using the microscopic quasiparticle-phonon model and the nuclear shell model is reported in Refs. [8, 9]. Also in these references the half-life method [8] and spectrum-shape method (SSM) [9] are reported to extract the value of axial-vector coupling constant $g_A$.

Studies of the forbidden beta decay using the nuclear shell model with phenomenological interactions are available in the literature. With the recent progress in the ab initio approaches for nuclear structure study, it is highly desirable to see how these interactions are able to predict nuclear observables such as forbidden beta decay. Recently, shell model results for allowed beta decay properties of $sd$, $fp$ and $fpg$ shell nuclei are reported by us in Refs. [10–13].

In the present work, our aim is to study second-forbidden nonunique $\beta^-$ transitions of $^{24}\text{Na}(4^+) \rightarrow ^{24}\text{Mg}(2^+)$ and $^{36}\text{Cl}(2^+) \rightarrow ^{36}\text{Ar}(0^+)$ using ab initio interactions. Beta decay transitions in these nuclei have been calculated and compared with the available experimental data to test the quality of the ab initio interaction wave functions. A theoretical attempt has been made in the past to calculate the beta decay transition observable of $^{36}\text{Cl}$ [14]. However, no theoretical estimate is found in the literature for the beta-decay of $^{24}\text{Na}$ and also no experimental shape factors and electron spectra are found in the literature. Thus, our theoretical predictions for the beta decay of $^{24}\text{Na}$ are useful for the future experiments. In this work, we have computed the log$ft$ values, shape factors and electron spectra of these branches. We have constrained the relativistic nuclear matrix element based on conserved vector current (CVC) theory and test the role of this matrix element in the shape factors and electron spectra. In order to test our computed wave functions, first we have computed the low-lying energy spectra of $^{24}\text{Na}$, $^{24}\text{Mg}$, $^{36}\text{Cl}$, and $^{36}\text{Ar}$ and compare them with the available experimental energy spectra [15].

This article is organized as follows. In Sec. II we give a short overview of the theoretical formalism for the
\(\beta^-\) decay and details about microscopic effective interactions. Results and discussions corresponding to low-lying energy spectra, nuclear matrix elements, \(\log ft\) values, shape factors, electron spectra and decomposition of the integrated shape factors are reported in Sec. III. Finally, in Sec. IV we draw the conclusions.

II. THEORETICAL FORMALISM

In the section (IIA), we discuss the theory of forbidden \(\beta^-\) decay, and the shape of the electron spectra. Section (IIB), give the details about the valence space and microscopic effective interactions used in the present work.

A. Beta Decay Theory:

The full details of formalism for both allowed and forbidden types of the \(\beta^-\) decay are available in the literature by Behrens and Bühring [16] (see also Ref. [17]). The generalized framework of the forbidden nonunique \(\beta\) decay theory is available in the Refs. [9, 18, 19]. When the beta decay process is described as a point-like interaction vertex with an effective Fermi coupling constant \(G_F\), the probability of the electron emission in the kinetic energy interval \(W_e\) and \(W_e + dW_e\) is expressed as

\[
P(W_e)dW_e = \frac{G_F^2}{(\hbar c)^{3}} \left( \frac{1}{2\pi^3}\epsilon \right) C(W_e) \times p_e c W_e (W_0 - W_e)^2 F_0 (Z, W_e) dW_e.
\]

Where the \(C(W_e)\) is the shape factor containing the nuclear structure information, and \(W_0\) is the endpoint energy of the \(\beta\) spectrum. The factor \(F_0 (Z, W_e)\) is the Fermi function, which takes into account Coulomb interaction between the daughter nucleus and \(\beta\) particle, and \(Z\) is the proton number of the final nucleus. Furthermore, \(p_e\) and \(W_e\) are the momentum and energy of the emitted electron, respectively.

The partial half-life of the \(\beta\) decay is expressed as

\[
t_{1/2} = \frac{\ln(2)}{\int_{m_e c^2}^{W_0} P(W_e) dW_e},
\]

where \(m_e\) is the mass of the electron. For the convenience, Eq. (2) can be expressed in the form

\[
t_{1/2} = \frac{\kappa}{C},
\]

where \(\tilde{C}\) is the unitless integrated shape factor, and the constant \(\kappa\) has the value

\[
\kappa = \frac{2\pi^3 h}{m_e^2 c^2 (G_F \cos \theta_C)^2} = 6147 s,
\]

where \(\theta_C\) is the Cabibbo angle and the usual dimensionless kinematics quantities are defined as \(w_0 = W_0/m_e c^2, \ w_e = W_e/m_e c^2, \) and \(p = p_e c/m_e c^2 = \sqrt{(w_e^2 - 1)}\), then the dimensionless integrated shape factor \(\tilde{C}\) can be expressed as

\[
\tilde{C} = \int_{w_0}^{w_e} C(w_e) p w_e (w_0 - w_e)^2 F_0 (Z, W_e) dW_e.
\]

The comparative half-life, or the \(ft\) values, is obtained by multiplying the partial half-life with the following dimensionless integrated Fermi function

\[
f_0 = \int_{w_0}^{w_e} p w_e (w_0 - w_e)^2 F_0 (Z, W_e) dW_e,
\]

but \(ft\) values are usually large, so it is normally expressed in term of “\(\log ft\)” values [20]. The \(\log ft\) values is defined as

\[
\log ft = \log_{10} \left( f_0 t_{1/2} [s] \right).
\]

The shape factor \(C(w_e)\) in Eq. (5) for pure Gamow-Teller transition is defined as

\[
C(w_e) = \left( \frac{g_A^2}{2J_1} \right) |M_{GT}|^2,
\]

where the \(J_1\) is the angular momentum of the initial state, \(g_A\) is the axial-vector coupling constant, and the \(M_{GT}\) is the Gamow-Teller nuclear matrix element [20]. Which is defined as

\[
M_{GT} = \frac{1}{2} j_f |s_{(1)}|, \|s_{(1)} J_1\|,
\]

where \(M_{GT}(pm)\) is the single particle matrix elements (SPMEs). In case of forbidden nonunique beta decay the form of the shape factor \(C(w_e)\) in Eq. (5) is defined as

\[
C(w_e) = \sum_{k_e, k_e', K} \lambda_{k_e} [M_K(k_e, k_e')^2 + m_K(k_e, k_e')^2 - 2\gamma_{k_e} M_K(k_e, k_e') m_K(k_e, k_e')].
\]

The comparative half-life, or the \(ft\) values, is obtained by multiplying the partial half-life with the following dimensionless integrated Fermi function

\[
f_0 = \int_{w_0}^{w_e} p w_e (w_0 - w_e)^2 F_0 (Z, W_e) dW_e,
\]

but \(ft\) values are usually large, so it is normally expressed in term of “\(\log ft\)” values [20]. The \(\log ft\) values is defined as

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The shape factor \(C(w_e)\) in Eq. (5) for pure Gamow-Teller transition is defined as

\[
C(w_e) = \left( \frac{g_A^2}{2J_1} \right) |M_{GT}|^2,
\]

where the indices \(k_e\) and \(k_e'\) \((k_e, k_e'=1,2,3,...)\) are positive integers, which are emerging from the partial-wave expansion of the lepton wave functions and \(K\) is the order of forbiddenness of the transition. The nuclear structure information is contained in the quantities \(M_K(k_e, k_e')\) and \(m_K(k_e, k_e')\), which are complicated combinations of different nuclear matrix elements (NMEs) and leptonic phase-space factors. The factor \(\lambda_{k_e}\) is the Coulomb function and expressed as

\[
\lambda_{k_e} = \frac{F_{k_e - 1}(Z, w_e)}{F_0 (Z, w_e)},
\]

where \(F_{k_e - 1}(Z, w_e)\) is the generalized Fermi function [9, 18, which is expressed as

\[
F_{k_e - 1}(Z, w_e) = 4^k e^{-1}(2k_e)(k_e + \gamma_{k_e})!(2k_e - 1)!\}^2 2^e \pi \nu \times \left( \frac{2p_e R}{\hbar} \right)^{2\gamma_{k_e} - k_e} \left( \frac{1}{1+2\gamma_{k_e}} \right)^2.
\]
The auxiliary quantities are defined as \( \gamma_{\kappa} = [k_{\kappa}^2 - (\alpha Z)^2]^{1/2} \) and \( y = (\alpha Z w_e / p_e c) \), where \( \alpha = 1/137 \) is the fine structure constant.

The nuclear matrix elements (NMEs) are given by

\[
V/A_{M}^{(N)}_{KLS}(pn)(k_{e}, m, n, \rho) = \frac{1}{\sqrt{2J_{i} + 1}} \times \sum_{pn} V/A_{m}^{(N)}_{KLS}(pn)(k_{e}, m, n, \rho)(\psi_{f}||c_{n}^\dagger c_{m}||\psi_{i}). \tag{13}
\]

The nuclear matrix elements are divided in two parts: first part \( V/A_{m}^{(N)}_{KLS}(pn)(k_{e}, m, n, \rho) \) is called the single-particle matrix element and second part \( (\psi_{f}||c_{n}^\dagger c_{m}||\psi_{i}) \) is the reduced one-body transition density (OBTD) between the initial \((i)\) and final \((f)\) nuclear states. The single-particle matrix elements characterize the properties of the transition operators, so they are the same for all nuclear models. But the OBTDs are the nuclear model dependent. In the present work the SPMEs are calculated using harmonic-oscillator wave functions (see Refs. [9, 18]). The summation of Eq. (13) runs over the proton \((p)\) and neutron \((n)\) single-particle states.

The shape factor \( C(w_{e}) \) \((10)\) can be decomposed into vector, axial-vector, and mixed vector-axial-vector components \([8, 9, 21–23]\) in the form

\[
C(w_{e}) = g_{V}^{2}C_{V}(w_{e}) + g_{A}^{2}C_{A}(w_{e}) + g_{V}g_{A}C_{VA}(w_{e}). \tag{14}
\]

After the integration of Eq. (14) with respect to electron kinetic energy, we get the analogous expression to Eq. (5) for the integrated shape function \( \tilde{C} \).

\[
\tilde{C} = g_{V}^{2}\tilde{C}_{V} + g_{A}^{2}\tilde{C}_{A} + g_{V}g_{A}\tilde{C}_{VA}. \tag{15}
\]

In Eq. (14) the shape factors \( C_{i} \) are functions of the electron kinetic energy, while the integrated shape factors \( \tilde{C}_{i} \) in Eq. (15) are just constant numbers.

B. ADOPTED MODEL SPACE AND HAMILTONIANS

In the present work shell model calculations for the low-lying energy spectra, log\(ft\) values, shape factors, electron spectra and decomposition of the integrated shape factors for the second-forbidden nonunique \( \beta^- \) transitions of \( ^{24}\text{Na}(4^+) \rightarrow ^{24}\text{Mg}(2^+) \) and \( ^{36}\text{Cl}(2^+) \rightarrow ^{36}\text{Ar}(0^+) \).

Previously, the log\(ft\) values and shape factors of the second-forbidden beta decay of \( ^{36}\text{Cl} \) \([14]\) have been reported by applying two different nuclear models: with the pure \( 1d_{3/2} \rightarrow 1d_{5/2} \) transitions and shell-model with \( sd \) shell configuration space.

Recently, much progress has been achieved in developing modern effective interactions for the shell model calculations. Thus we have revisited calculation for \( ^{36}\text{Cl} \) and also the first time for \( ^{24}\text{Na} \) with recently developed microscopic (DJ16A, N3LO, and JISP16) and phenomenological (USDB) interactions in the \( sd \) model space. Our results for \( ^{24}\text{Na} \) will be useful when compared with upcoming experimental data.

Below we have presented low-lying energy spectra ([Figs. 1-2]), nuclear matrix elements ([Table I and III]), log\(ft\) values ([Table II and IV]), shape factors and electron spectra ([Figs. 3-4]). The low-lying energy spectra are discussed in Sec. III.A. The \( \beta^- \) decay nuclear matrix elements and log\(ft\) values are discussed in Sec. III.B. Results of the shape factors and electron spectra are presented in Sec. III.C. Decomposition of the integrated shape factor are discussed in Sec. III.D.

A. Low-lying energy spectra

In Fig. 1, we show the low-lying energy spectra of \( ^{24}\text{Na} \) and \( ^{24}\text{Mg} \). In the case of \( ^{24}\text{Na} \), the ground state (g.s.) 4\(^+\) is correctly reproduced by USDB interaction, while the other microscopic effective interactions N3LO, and JISP16 give the 2\(^+\) as a g.s., and DJ16A predict g.s. as 1\(^+\). The low-energy spectrum of the well known \( sd \)-shell rotor nucleus \( ^{24}\text{Mg} \) is already shown in Ref. [24] for all the interactions that we have used in the present work. For \( ^{24}\text{Mg} \), the 0\(^+_g \), 2\(^+_g \), and 2\(^+_h \) are relatively well described by all the interactions. The computed 2\(^+_h \) state is obtained at 1.213, 1.310, 1.231, and 1.502 MeV corresponding to DJ16A, JISP16, N3LO, and USDB, respectively, while the corresponding experimental value is 1.369 MeV. The theoretical low-lying energy spectra of \( ^{36}\text{Cl} \) and \( ^{36}\text{Ar} \) are shown in Fig. 2 in comparison with the experimental data. The g.s. is correctly reproduced by the microscopic (DJ16A, JISP16, and N3LO) and USDB interactions for
FIG. 1. Comparison of calculated and experimental [15] low-lying energy spectra for positive parity states of $^{24}\text{Na}$ and $^{24}\text{Mg}$ from microscopic and USDB interactions.

FIG. 2. Comparison of calculated and experimental [15] low-lying energy spectra for positive parity states of $^{36}\text{Cl}$ and $^{36}\text{Ar}$ from microscopic and USDB interactions.

$^{36}\text{Cl}$ and $^{36}\text{Ar}$. For $^{36}\text{Cl}$, the order of $3^+_1$ and $1^+_1$ states are correctly reproduced from the JISP16, N3LO, and USDB interactions as in the experimental data, while the DJ16A interaction invert the order of these states. In the case of $^{36}\text{Ar}$, the calculated $2^+_1$ state from the DJ16A, JISP16, N3LO, and USDB interactions are close to the experimental data. So, in general, the comparison of the computed low-lying energy levels are in good agreement with the experimental data for $^{24}\text{Na}$, $^{36}\text{Cl}$, and $^{36}\text{Ar}$. In the present work we have taken $Q$ values from the experimental data [15] for further calculations listed in the Table II and IV.

B. Nuclear matrix elements and log$t$ values

The nuclear matrix elements contain the nuclear-structure information. The Gamow-Teller matrix elements $M_{GT}$ calculated from the microscopic and USDB interactions for the allowed $\beta^-$ decays of
$^{24}\text{Na}(4^+)\rightarrow^{24}\text{Mg}(3^{+}, 4^{+})$ transitions are presented in the Table I with comparison to the experimental data. The experimental $M_{\text{GT}}$ value is obtained from the log $ft$ [15] values corresponding to the axial-vector coupling constant $g_A = 1.00$. In the present work, we have calculated these matrix elements by using OBTDs corresponding to all microscopic and USBD interactions. After that, we compare the calculated $M_{\text{GT}}$ with the experimental data. For the both allowed transitions, the calculated $M_{\text{GT}}$ values from USBD are close to the experimental data as compared to the microscopic interactions. In the case of $^{24}\text{Na}(4^+)\rightarrow^{24}\text{Mg}(4^{+})$ transition, our calculated value of $M_{\text{GT}}$ (0.0441) from the DJ16A is very small in comparison with the experimental data.

The calculated log $ft$ values of allowed $\beta^-$ decays of $^{24}\text{Na}(4^+)\rightarrow^{24}\text{Mg}(3^{+}, 4^{+})$ transitions are presented in Table II in comparison to the experimental data. For the calculation, we have used the axial-vector coupling constant $g_A = 1.00$ and $g_A = 1.27$. For the transition $4^+ \rightarrow 3^+_1$, the calculated log $ft$ values for $g_A = 1.00$ are in nice agreement with the experimental values corresponding to USBD, also all other microscopic effective interactions are in a reasonable agreement. However, in the case of $4^+ \rightarrow 4^+_1$ transition, the calculated log $ft$ value from DJ16A is larger in comparison with the experimental data, but from other interactions they are close to the experimental data with both $g_A$ values.

**TABLE I.** Calculated Gamow-Teller matrix elements of the allowed $\beta^-$ decays from the g.s. $(4^+)$ of $^{24}\text{Na}$ to the excited states in $^{24}\text{Mg}$ from microscopic and USBD effective interactions.

| Transitions | USBD | DJ16A | N3LO | JISP16 | Expt |
|-------------|------|-------|------|--------|------|
| $4^+ \rightarrow 3^+_1$ | 0.1859 | 0.1982 | 0.2274 | 0.2108 | 0.1179 |
| $4^+ \rightarrow 4^+_1$ | 0.2663 | 0.0441 | 0.1069 | 0.0839 | 0.2072 |

For the second-forbidden nonunique $\beta^-$ decays of $^{24}\text{Na}(4^+)\rightarrow^{24}\text{Mg}(2^+_1)$ and $^{36}\text{Cl}(2^+)\rightarrow^{36}\text{Ar}(0^+)$, the computed NMEs from different microscopic and USBD effective interactions are presented in Table III. The relativistic matrix element $V_{M_{211}^{(0)}}$ is becoming identically zero due to limitation of our $0\hbar\omega$ sd-shell calculations for harmonic-oscillator wave functions. To get the value of $V_{M_{211}^{(0)}}$ matrix element non-zero we need to perform shell model calculations in the multi-$\hbar\omega$ excitations. However, here we follow a different approach to calculate the $V_{M_{211}^{(0)}}$ matrix element. We have used a approach based on CVC theory, since we have an experimental partial half-life, so we keep the value of coupling constants $g_V = g_A = 1.0$ and try to reproduce the value of the experimental partial half-life by varying the matrix element $V_{M_{211}^{(0)}}$. The $V_{M_{211}^{(0)}}$ matrix element obtained with this approach is labeled as $i^V M_{211}^{(0)} (\text{CVC})$ in Table III.

The axial-vector matrix elements $A M_{221}^{(0)}$, $A M_{221}^{(0)}(1, 1, 1, 1)$, $A M_{221}^{(0)}(2, 1, 1, 1)$, and $A M_{321}^{(0)}$ could be affected by the quenching of axial-vector coupling constant $g_A$. The affected value of the Gamow-Teller transition matrix element by the quenching of axial coupling constant was observed in [33]. In the recent study of the second-forbidden nonunique beta decay of $^{20}\text{F}$, the effect of the quenching of axial-vector coupling constant in axial-vector matrix elements is reported in Refs. [3, 4]. Here, we will use the value of the axial-vector coupling constant for the two different cases, either the bare value of $g_A = 1.27$ or the quenched value of $g_A = 1.00$.

In the Table IV, we presented the log $ft$ values for the second-forbidden nonunique $\beta^-$ decays of $^{24}\text{Na}$ and $^{36}\text{Cl}$ calculated with different microscopic and phenomenological interactions in comparison with the experimental data, and the value of coupling constants are taken as $g_A = 1.27$ and $g_V = 1.00$ for the calculations. The results with pure shell-model labeled as “SM”, and those constrained from experimental information, labeled “SM + CVC”. The prediction of log $ft$ values with SM is far from the experimental data. However, the agreement between the calculation with “SM+CVC” and the experimental value came out to be very satisfactory.

C. Shape Factors and Electron Spectra

In Fig. 3 and 4, we have shown the shape factors (left panel) and $\beta$ spectra (right panel) of the second forbidden nonunique $\beta^-$ decays of $^{36}\text{Cl}$ and $^{24}\text{Na}$. The second-forbidden nonunique beta decay of $^{36}\text{Cl}$ is predicted with strong branching ratio 98.1%, while that of $^{24}\text{Na}$ is predicted with a weak branching ratio less than 1%. These figures represent the shape factor of Eq. (10) and $\beta$ spectrum corresponding to the integrand of Eq. (5) as a function of electron kinetic energy for different microscopic and USBD effective interactions. For all these calculations of second-forbidden nonunique beta decay of $^{24}\text{Na}$ and $^{36}\text{Cl}$, we have used the experimentally measured $Q$ value 4147 KeV and 709,547 KeV, respectively. We have calculated the shape factor by including only the leading-order terms, and the value of vector coupling constant $g_V = 1.00$ were adopted by CVC hypothesis. We presented in figures the purely theoretical results from the shell model interactions, labeled “name of interactions,” and those constrained from experimental information labeled “name of interactions and CVC theory” with quenched ($g_A = 1.00$) or bare ($g_A = 1.27$) cases. The areas under both the theoretical and experimental curves are normalized to unity.

For the shape factor and $\beta$ spectrum of $^{36}\text{Cl}$, we have done a comparison with the available experimental data due to Rotzinger et al [34] and with the theoretical results of Sadler et al [14]. In the case of $^{36}\text{Cl}$, the shape factor calculated with the matrix element $i^V M_{211}^{(0)} = 0^+$ yields a poor agreement in comparison to the experimental shape factor. After constraining this matrix element with the experimental half-life, the shape factor and electron spec-
FIG. 3. Theoretical shape factors (left panel) and electron spectra (right panel) for second forbidden $\beta^-$ decay of $^{36}$Cl($2^+$) $\rightarrow ^{36}$Ar($0^+$) as functions of electron kinetic energy for different cases. The dashed vertical lines indicate the endpoint energy for forbidden ($Q_{\text{forbidden}}$) decay. The area under each curve are normalized to unity.
FIG. 4. Theoretical shape factors (left panel) and electron spectra (right panel) for second order forbidden $\beta^-$ decay of $^{24}\text{Na}(4^+)\rightarrow^{24}\text{Mg}(2^+)$ as functions of electron kinetic energy for different cases. The dashed vertical lines indicate the end-point energy for forbidden ($Q_{\text{forbidden}}$) decay. The area under each curve are normalized to unity.
TABLE II. Calculated log$ft$ values of the allowed $\beta^-$ decays from g.s. (4$^+$) of $^{24}$Na to the excited states in $^{24}$Mg from the microscopic and USDB effective interactions.

| Transitions | Q(MeV) | BR(%) | log$ft$(gA = 1.00) | log$ft$(gA = 1.27) |
|-------------|--------|-------|-------------------|-------------------|
| $^{4^+} \rightarrow ^3_2P$ | 0.280 | 0.076 | USDB 6.205 DJ16A 6.149 N3LO 6.029 JISP16 6.095 | USDB 5.997 DJ16A 5.941 N3LO 5.822 JISP16 5.888 Expt 6.60(2) |
| $^{4^+} \rightarrow ^4_4P$ | 1.392 | 99.855 | USDB 5.892 DJ16A 7.454 N3LO 6.685 JISP16 6.896 | USDB 5.685 DJ16A 7.247 N3LO 6.478 JISP16 6.688 Expt 6.11(1) |

TABLE III. Calculated leading-order nuclear matrix elements (NMEs) of the second-forbidden nonunique $\beta^-$ decays of $^{24}$Na and $^{36}$Cl are from microscopic and USDB interactions. The Coulomb-corrected NMEs are indicated by $(k_e, m, n, \rho)$, when such elements exist.

| Nuclear Matrix Elements | $^{24}$Na($4^+ \rightarrow ^24$Mg($2^+$)) | $^{36}$Cl($2^+ \rightarrow ^{36}$Ar($0^+$)) |
|-------------------------|---------------------------------|---------------------------------|
| $V \mathcal{M}^{(0)}_{1211}$(CVC) | 0.023790±0.0001 | -0.018446±0.0002 |
| $V \mathcal{M}^{(0)}_{1200}$ | 0.030979 | -0.123441 |
| $V \mathcal{M}^{(0)}_{1120}$ | 0.509588 | -0.110404 |
| $V \mathcal{M}^{(0)}_{1200}$ | -0.430287 | -0.482638 |
| $V \mathcal{M}^{(0)}_{1120}$ | -0.524687 | -0.577264 |
| $V \mathcal{M}^{(0)}_{1200}$ | -0.502493 | -0.550486 |
| $V \mathcal{M}^{(0)}_{1120}$ | -1.459626 | -0.758772 |

TABLE IV. Calculated log$ft$ values of the second-forbidden nonunique $\beta^-$ decays of $^{24}$Na and $^{36}$Cl from shell model and after constrained the matrix element $V \mathcal{M}^{(0)}_{211}$ from experimental data. For the log$ft$ calculations we have used the value of coupling constants $g_V$=1.00 and $g_A$=1.27. The experimental data have been taken from [15].

| Transitions | Type | Q(MeV) | BR(%) | log$ft$(SM) |
|-------------|-----|--------|-------|-------------|
| $^{24}$Na($4^+ \rightarrow ^{24}$Mg($2^+$)) | 2nd non-unique forbidden | 4.147 | 0.064 | USDB 12.237 DJ16A 12.881 N3LO 14.227 JISP16 13.958 Expt 11.340(4) |
| $^{36}$Cl($2^+ \rightarrow ^{36}$Ar($0^+$)) | 2nd non-unique forbidden | 0.710 | 98.1 | USDB 12.635 DJ16A 13.978 N3LO 13.120 JISP16 12.976 Expt 13.321(3) |

Electron spectra from “DJ16A+CVC+Quenched” are perfectly matched with the experimental electron spectra. This means that the shape factor and electron spectra strongly depend on this matrix element $V \mathcal{M}^{(0)}_{211}$. But in the case of JISP16 interaction, we have not obtained a good number of this matrix elements from the experimental half-life method. We have obtained the value of the matrix element $V \mathcal{M}^{(0)}_{211} = -0.007451 \pm 0.0009$ for JISP16 interaction, it is too small as compared to other interactions.

In Fig. 4, we have presented the shape factor and
TABLE V. The dimensionless integrated shape factors $\tilde{C}$ for the studied transitions, and their decompositions to vector $\tilde{C}_V$, axial-vector $\tilde{C}_A$, and vector-axial-vector $\tilde{C}_{VA}$ parts. For the calculation of total integrated shape factor $\tilde{C}$ we have taken $g_v = g_A = 1.0$.

| Interactions | $C_V$ | $C_A$ | $C_{VA}$ | $C$       |
|--------------|--------|--------|----------|-----------|
| USDB         | $1.3982 \times 10^{-6}$ | $3.6829 \times 10^{-6}$ | $1.7841 \times 10^{-6}$ | $6.8653 \times 10^{-6}$ |
| DJ16A        | $9.0878 \times 10^{-8}$ | $1.6952 \times 10^{-6}$ | $5.1005 \times 10^{-7}$ | $1.2833 \times 10^{-6}$ |
| N3LO         | $3.7996 \times 10^{-7}$ | $2.572 \times 10^{-7}$ | $5.0839 \times 10^{-7}$ | $9.7291 \times 10^{-8}$ |
| JISP16       | $2.2998 \times 10^{-7}$ | $3.7051 \times 10^{-7}$ | $5.0759 \times 10^{-7}$ | $9.2903 \times 10^{-8}$ |

| Interactions | $C_V$ | $C_A$ | $C_{VA}$ | $C$       |
|--------------|--------|--------|----------|-----------|
| USDB         | $8.397 \times 10^{-5}$ | $3.6829 \times 10^{-6}$ | $1.3924 \times 10^{-5}$ | $7.2856 \times 10^{-5}$ |
| DJ16A        | $5.8298 \times 10^{-5}$ | $1.6952 \times 10^{-6}$ | $1.2868 \times 10^{-5}$ | $7.2861 \times 10^{-5}$ |
| N3LO         | $6.5792 \times 10^{-5}$ | $2.572 \times 10^{-7}$ | $6.8432 \times 10^{-6}$ | $7.2861 \times 10^{-5}$ |
| JISP16       | $6.3828 \times 10^{-5}$ | $3.7051 \times 10^{-7}$ | $8.6647 \times 10^{-6}$ | $7.2864 \times 10^{-5}$ |

| Interactions | $C_V$ | $C_A$ | $C_{VA}$ | $C$       |
|--------------|--------|--------|----------|-----------|
| USDB         | $6.0691 \times 10^{-9}$ | $3.1198 \times 10^{-10}$ | $2.7292 \times 10^{-9}$ | $3.6519 \times 10^{-9}$ |
| DJ16A        | $2.1890 \times 10^{-9}$ | $8.1048 \times 10^{-10}$ | $2.6419 \times 10^{-9}$ | $3.5761 \times 10^{-10}$ |
| N3LO         | $3.8787 \times 10^{-9}$ | $5.9273 \times 10^{-10}$ | $3.0074 \times 10^{-9}$ | $1.4641 \times 10^{-9}$ |
| JISP16       | $4.4736 \times 10^{-9}$ | $5.4657 \times 10^{-10}$ | $3.1016 \times 10^{-9}$ | $1.9186 \times 10^{-9}$ |

| Interactions | $C_V$ | $C_A$ | $C_{VA}$ | $C$       |
|--------------|--------|--------|----------|-----------|
| USDB         | $4.0126 \times 10^{-10}$ | $3.1198 \times 10^{-10}$ | $7.7968 \times 10^{-11}$ | $6.3528 \times 10^{-10}$ |
| DJ16A        | $1.4791 \times 10^{-10}$ | $8.1048 \times 10^{-10}$ | $3.2311 \times 10^{-10}$ | $6.3528 \times 10^{-10}$ |
| N3LO         | $2.5097 \times 10^{-10}$ | $5.9273 \times 10^{-10}$ | $2.0843 \times 10^{-10}$ | $6.3527 \times 10^{-10}$ |
| JISP16       | $2.2999 \times 10^{-9}$ | $5.4657 \times 10^{-10}$ | $2.2112 \times 10^{-9}$ | $6.3528 \times 10^{-10}$ |

$\beta$ spectrum of $^{24}$Na from purely shell model calculation with quenched and unquenched cases. In the pure shell-model calculations, the shape-factor and $\beta$-spectrum curves depend strongly on the quenching value of $g_A$. After CVC constraining the matrix element $V M_{211}^{(0)}$, we find that the shape factor and $\beta$ spectrum are independent of the value of $g_A$. So, we have presented curve for “SM+CVC” only for bare $g_A$ value. For the comparison, there are no experimental data available for shape factor and electron spectra corresponding to the second-forbidden nonunique $\beta^-$ decay of $^{24}$Na. Thus, our theoretical results might be quite useful to compare with a future experimental measurement.

D. Decomposition of the integrated shape factor

In Table V, we present the integrated shape factor $\tilde{C}$ and its decomposition to vector $\tilde{C}_V$, axial-vector $\tilde{C}_A$, and mixed vector-axial-vector $\tilde{C}_{VA}$ components, for the involved transitions using different effective interactions. Hence, we have calculated the value of $\tilde{C}$ and its components with purely shell model labeled “SM” and after putting constrained to the matrix element $V M_{211}^{(0)}$ from experimental information labeled, “SM+CVC”. For all the studied decays transition, the sign of vector $\tilde{C}_V$ and axial-vector $\tilde{C}_A$ components is positive from “SM” and “SM+CVC”, but the sign of mixed-axial-vector $\tilde{C}_{VA}$ component varies. From the pure “SM” for $^{24}$Na, the axial-vector component $\tilde{C}_A$ is dominant in the USDB and DJ16A interactions. For N3LO and JISP16 interactions, the mixed component $\tilde{C}_{VA}$ is roughly the sum of vector and axial-vector components and negative in sign. In “SM+CVC”, the vector component $\tilde{C}_V$ is dominant for all interactions. The mixed component $\tilde{C}_{VA}$ is negative for USDB, while positive for other interactions. In case of $^{36}$Cl, the vector component $\tilde{C}_V$ is dominant for all the interactions in the case of pure “SM”. After applying CVC theory, the vector part is dominant only in USDB and JISP16 interactions and for the other two interactions the axial-vector part is large as compared to other two components. The sign of the mixed complexes $\tilde{C}_{VA}$ are negative in both cases “SM” and “SM+CVC” for all interactions.

IV. CONCLUSIONS

In this article we have calculated log$ft$ values, shape factors and electron spectra for the second-forbidden
nonunique $\beta^-$ transitions of $^{24}$Na($4^+)$→$^{24}$Mg($2^+$) and $^{36}$Cl($2^+$)$→^{36}$Ar($0^+$) using the three microscopic effective interactions (DJ16A, N3LO, and JISP16) obtained from the NCSM wave functions via the OLS transformation. Also, for the comparison, we have used the more popular phenomenological effective USDB interaction.

The low-lying energy spectra of the involved mother and daughter nuclei in $\beta^-$-decay corresponding to different ab initio and phenomenological effective interactions are compared with the available experimental data. The obtained wave functions have been used for further calculations. To calculate the log$ft$ values, shape factors and electron spectra, we have constrained the relativistic matrix element $V\mathcal{M}_{211}^{(0)}$ in the sd model space by experimental information. This matrix element plays an important role in the shape factor and electron spectra. The calculated log$ft$ values are compared with experimental data. In the case of JISP16 interaction, we could not obtain a proper value of this matrix element. In our calculation, we have used two different values of $g_A$, either the bare value of $g_A = 1.27$ or the quenched value of $g_A = 1.00$. For the allowed beta decay of $^{24}$Na, the log$ft$ values are in reasonable agreement with the experimental data. In case of second-forbidden non unique beta decay, we have calculated log$ft$ values corresponding to $g_A = 1.27$ and compared with the experimental data. Before CVC theory the electron spectra of $^{24}$Na depend significantly on the effective value of $g_A$, while after CVC it has become independent. In the case of $^{36}$Cl, the dependency of electron spectra on $g_A$ is opposite from the case of $^{24}$Na for USDB, N3LO, and JISP16 interactions, but in case of DJ16A interaction the electron spectra strongly depend on $g_A$ before and after CVC theory. In case of $^{36}$Cl, the experimental data are available for shape factors and electron spectra. So we have compared our theoretical results with the experimental data to check the role of matrix element $V\mathcal{M}_{211}^{(0)}$. But in the case of $^{24}$Na there are no experimental data available for shape factor and electron spectra. Thus, our calculated results could be quite useful when compared with future experimental data. Also, we have decomposed the integrated shape function $C$ in to vector $C_V$, axial-vector $C_A$, and vector-axial-vector $C_{VA}$ components to see the individual effect of these components.

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