Homogeneous Models of Nonlinear Circuits

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Abstract

This paper develops a general approach to nonlinear circuit modelling aimed at preserving the intrinsic symmetry of electrical circuits when formulating reduced models. The goal is to provide a framework accommodating such reductions in a global manner and without any loss of generality in the working assumptions; that is, we avoid global hypotheses imposing the existence of a classical circuit variable controlling each device. Classical (voltage/current but also flux/charge) models are easily obtained as particular cases of our general homogeneous model. Our approach extends the results introduced for linear circuits in a previous paper, by means of a systematic use of global parametrizations of smooth planar curves. This makes it possible to formulate reduced models in terms of homogeneous variables also in the nonlinear context: contrary to voltages and currents (and also to fluxes and charges), homogeneous variables qualify as state variables in reduced models of uncoupled circuits without any restriction in the characteristics of devices. The inherent symmetry of this formalism makes it possible to address in broad generality certain analytical problems in nonlinear circuit theory, such as the state-space problem and related issues involving impasse phenomena, as well as index analyses of differential-algebraic models. Our framework applies also to circuits with memristors. Several examples illustrate the results.

Keywords: nonlinear circuit, smooth device, state-space reduction, planar curve, closed characteristic, hysteresis, homogeneous coordinates, regular set, impasse set, Van der Pol circuit, Murali-Lakshmanan-Chua circuit, memristor.

1 Introduction

We extend in this paper the approach of [19] to the nonlinear circuit context. Our main goal is to introduce and exploit, for analytical purposes, circuit models of the form

\[ A_c \psi'_c(u_c)u_c' + A_l \psi_l(u_l) + A_r \psi_r(u_r) = 0 \]
\[ B_c \zeta'_c(u_c) + B_l \zeta'_l(u_l)u_l' + B_r \zeta_r(u_r) = 0, \]

where we use the prime \( \prime \) to denote differentiation (with respect to time when no argument is given, as e.g. in \( u'_c \)). This model is formulated in terms of certain vector-valued homogeneous...
variables, namely \( u_c, u_l \) and \( u_r \) for (smooth, possibly nonlinear) capacitors, inductors and resistors. Independent sources and memristors can be easily included in the model and will be considered later. To get a brief overview before going into details, the reader may think of the matrices \( A = (A_c A_l A_r) \) and \( B = (B_c B_l B_r) \) as describing the circuit topology (with Kirchhoff laws reading as \( Ai = 0, Bv = 0 \)), whereas the maps \( \psi_c, \zeta_c \) etc. comprise the characteristics of the circuit devices. Solutions in terms of classical circuit variables (current, voltage, charge and flux) are explicitly obtained from those of (1) by means of the relations 
\[
  i_r = \psi_r(u_r), \quad v_r = \zeta_r(u_r), \quad \sigma_c = \psi_r(u_c),
\]
each. Details in this regard are carried out in subsection 2.4: cf. the relations (12), (14), (15) and the derivation of the model (17).

The basic idea behind our approach is writing Ohm’s law in parametric form, that is,
\[
  i = pu \quad \text{(2a)}
\]
\[
  v = qu \quad \text{(2b)}
\]
Here we are dealing with an individual device (a linear resistor) so that all variables and parameters in (2) are scalar. We deliberately avoid the current-controlled form \( v = zi \) (\( z \) is either the impedance or the resistance, depending on the context) and the voltage-controlled one \( i = yv \), because both lack generality: indeed, the former does not accommodate an open-circuit (governed by the relation \( i = 0 \)), and the latter excludes a short-circuit (for which \( v = 0 \)). However, all cases are covered in terms of the parameters \( p \) and \( q \) in (2), which are assumed not to vanish simultaneously and therefore define homogeneous coordinates of a projective line (cf. [19]); under the obvious non-vanishing assumptions, we get either the impedance/resistance in the form \( z = q/p \) or the admittance/conductance as \( y = p/q \). In (2), \( u \) is an abstract (so-called homogeneous) variable which will qualify as a state variable in all possible parameter scenarios, by contrast to both \( i \) and \( v \), in light of the excluded configurations resulting from the aforementioned classical forms of Ohm’s law.

The extension of this idea to the nonlinear context proceeds through the nonlinear counterpart of (2); that is, we would now describe the characteristic of a nonlinear resistor as 
\[
  i = \psi(u) \quad \text{(3a)}
\]
\[
  v = \zeta(u) \quad \text{(3b)}
\]
for certain nonlinear functions \( \psi, \zeta \) and a given parameter \( u \). The key fact here is that this description is feasible in a global sense for (smooth and uncoupled) nonlinear devices, as a result of the classification theorem for smooth planar curves. This way we will describe the characteristic of each individual device, under a smoothness assumption to be made precise later, in terms of a globally defined parameter \( u \), lying either on the real line \( \mathbb{R} \) or on the 1-sphere (circle) \( S^1 \); this parameter brings to the nonlinear context the idea of a homogeneous variable discussed above. Here we are assuming that the device is a resistor (in other words, that its characteristic relates current and voltage), but the same applies in a natural manner to reactive devices, whose characteristics involve either the electrical charge or the magnetic flux, and also to memristors.
These ideas are presented in Section 2 where, going from the device level of the last two paragraphs to the circuit level, we derive and discuss in detail the model (1). In the absence of coupling effects, the vector-valued maps $\psi_c$, $\zeta_c$, etc. are guaranteed to exist in a global sense by the classification theorem mentioned above, having a (say) diagonal form (that is, the $k$-th component of each map depends only on the $k$-th component of its argument); note also that coupling effects may be naturally accommodated in (1) by deflating this diagonal requirement. Independent sources can be included within the maps $\psi_r$ and $\zeta_r$ (provided that they depend also on $t$ in cases beyond the DC setting), something that we assume throughout the document, in most cases without explicit mention; dependent sources can be handled in a similar manner to coupled devices.

As detailed in Section 2 homogeneous models of the form (1) are of interest from two different perspectives. On the one hand, they make it possible to handle, in a global manner, reduced models (involving one state variable per branch) in situations in which a global explicit description in terms of a classical circuit variable (current, voltage, charge or flux) does not exist: an example of this, involving a hysteresis loop, can be found in subsection 2.5. On the other, and we emphasize this second feature, this formalism provides a truly general and flexible circuit modelling framework, of interest even if such classical global descriptions are eventually used. These classical descriptions are easily accommodated as particular cases of (1): for instance, focusing (for simplicity) on an individual nonlinear resistor, a global current-controlled description $v = \gamma(i)$ is included in (3), and therefore in (1), simply by setting $\psi \equiv \text{id}$ and $\zeta \equiv \gamma$; in this case the homogeneous variable $u$ amounts to the current $i$. Similar remarks will apply to all devices and all possible controlling variables. This means that classical (voltage/current and flux/charge) formulations are included in the general model (1), and that our results apply to classical contexts in a straightforward manner: we get such particular models simply by choosing the maps $\psi_c$, $\zeta_c$, etc. in an appropriate way. To summarize this second feature, by using (1) we get rid of assumptions on the existence of global explicit descriptions in terms of classical circuit variables, much as in the linear case the homogeneous formalism avoids the need to impose an impedance (current-controlled) or an admittance (voltage-controlled) description of each individual device.

In Section 3 we apply the framework sketched above to address certain analytical problems in nonlinear circuit theory, involving the state-space problem and also the structure of the so-called regular and impasse sets. For simplicity we restrict the analysis to topologically nondegenerate problems, the homogeneous formalism paving the way for a completely general characterization of the so-called regular set in graph-theoretic terms (specifically, in terms of the family of spanning trees in the circuit) and, subsequently, of the regular manifold where the circuit equations define a smooth flow. Our framework also yields a nice distinction between linear and (in a strict and local sense) nonlinear circuits with regard to the structure of the regular and impasse sets. We extend in less detail the results to the memristive context in Section 4. Several examples are discussed through all these sections. Finally, concluding remarks can be found in Section 5.
2 Homogeneous modelling

2.1 Linear circuits

The homogeneous formalism in the linear setting is developed in [19], and naturally drives parametric analyses of linear circuits to the context of projective geometry (related ideas can be found in [1, 5, 16]). This framework leads to a completely general reduction of linear circuits, without any restriction on the controlling variables of individual devices, and to a compact way of writing the equations of any uncoupled circuit. In the linear setting, this reduction has the form

\[
\begin{pmatrix} AP \\ BQ \end{pmatrix} u = \begin{pmatrix} AQ \\ -BP \end{pmatrix} \bar{s},
\]

where \( u \) is a vector of homogeneous variables, one for each circuit branch; \( A \) and \( B \) are digraph matrices describing the circuit topology, \( P \) and \( Q \) comprise the parameters \( p, q \) (cf. (2)) of individual devices, either in the real or in the complex setting, and finally \( \bar{s} \) captures the contribution of sources. Find details in [19], where different analytical properties of linear circuits are examined from this perspective. Worth emphasizing if the fact that classical reductions (not only the branch-voltage and branch-current models [6] but also nodal and loop analysis models) can be derived from (4) by defining regions of the parameter space which capture different types of working assumptions. For instance, a voltage-control assumption, key to the formulation of branch-voltage and nodal models, is captured in (4) in terms of the nonsingularity of the \( Q \) matrix; in such regions, the model (4) can be naturally recast in terms of the voltage vector \( v \), or (further) in terms of node potentials. Note that it is also possible to combine the homogeneous approach with classical methods by using a homogeneous formalism only for certain branches, yielding so-called partially homogeneous models.

2.2 Global implicit descriptions of smooth curves. Associate submersions

In the linear context, the formalism above can be understood to rely on the homogeneous version of Ohm’s law, namely

\[ pv - qi = 0. \]

Here we are ignoring sources for the sake of simplicity, even if they can be easily accommodated in the right-hand side of (5). As detailed in [19], a resistor governed by (5) can be identified with a class of equivalent linear forms, namely those which yield the zero set in \((i, v)\)-space defined by (5). The key idea is that the \( p, q \) parameters in (5) are defined only up to a non-vanishing factor: this naturally frames the linear form in the left-hand side of (5), and the resistor itself, in a projective line, \((p : q)\) hence being homogeneous coordinates of a projective point.

This idea is extended to the nonlinear context in [20], where a smooth planar curve defining the characteristic of a nonlinear resistor is shown to be defined by a family of equivalent submersions. The equivalence relation defining these so-called associate submersions,
which extends the projective one above, is made precise in [20]. Given a smooth planar curve, any such submersion \( f \) can be defined on some open subset of \( \mathbb{R}^2 \) including the whole characteristic; it may happen, though, that \( f \) cannot be defined on the whole of \( \mathbb{R}^2 \).

Let \( f \) be any representative of the aforementioned equivalence class, that is, consider a smooth planar characteristic defined by

\[
f(i, v) = 0,
\]

for some smooth submersion \( f : U \to \mathbb{R} \) defined on an open set \( U \subseteq \mathbb{R}^2 \). We may define the \textit{homogeneous incremental resistance} at any point of this characteristic as the pair of homogeneous coordinates

\[
\left( \frac{\partial f}{\partial v}(i, v) : -\frac{\partial f}{\partial i}(i, v) \right),
\]

whose ratio can be proved independent of the choice of \( f \) (find details in [20]). The key aspect of this idea is its global nature: \( f \) can be defined globally (on some open subset of \( \mathbb{R}^2 \) including the characteristic) and the homogeneous incremental resistance so-defined applies at \textit{any} point of the curve, in a way which in essence is independent of the choice of the submersion \( f \) describing the characteristic. In the linear case, this definition of the homogeneous resistance amounts to the aforementioned description as a pair of homogeneous coordinates \((p : q)\). Note also that we are focusing for simplicity on characteristics relating current and voltage but the same applies to those involving charge and/or flux, so that the same ideas apply to capacitors, inductors and memristors.

Of course, locally we can always describe a smooth current-voltage characteristic either in terms of the voltage \( v \) or the current \( i \). Indeed, since \( f \) is a submersion (meaning that the differential \( df \) does not vanish identically anywhere), at every point of the curve at least one of the partial derivatives in (6) does not vanish. Fix e.g. a point where the partial derivative \( f_v(i, v) \) does not vanish (here we use subscripts for the partial derivatives for notational simplicity). A local current-controlled description \( v = \gamma(i) \) and the expression \( \gamma'(i) = -(f_v(i, \gamma(i)))^{-1}f_i(i, \gamma(i)) \) for the classical incremental resistance follow naturally from the implicit function theorem. The same holds for the classical incremental conductance \( \xi'(v) = -(f_i(\xi(v), v))^{-1}f_v(\xi(v), v) \), which is well defined on regions where the partial derivative \( f_i(i, v) \) does not vanish, allowing for a local voltage-controlled description \( i = \xi(v) \) of the curve. But the point is that the homogeneous definition (7), formulated in terms of the globally-defined submersion \( f \), holds at \textit{any} point of the characteristic.

### 2.3 Global parametrization of smooth curves and homogeneous descriptions of nonlinear devices

A key question arises at this point, namely, how to reduce the implicit description \( f(i, v) = 0 \) (cf. (6)) of a smooth characteristic in terms of a single variable? Needless to say, this should be relevant in the formulation of reduced circuit models. We indicated above that this is always feasible in terms of either \( i \) or \( v \) \textit{in a local sense}, as a consequence of the implicit function theorem, but the goal is to perform such a reduction in a global sense. In what
follows we show how to do this without the need for additional assumptions (that is, we will not impose additional conditions supporting e.g. global versions of the implicit function theorem). A homogeneous variable $u$ will play the intended global role in the reduction.

As in [20], we assume that the characteristics of the different circuit devices will be defined by smooth, connected 1-manifolds in $\mathbb{R}^2$ (more precisely, they will be regular submanifolds of $\mathbb{R}^2$, cf. [24]). For simplicity we assume that “smooth” means $C^\infty$. In this context, the key result making it possible to extend to the nonlinear context the homogeneous description (2) presented above for linear devices is the classification theorem for smooth 1-manifolds (see [13, Appendix]). This theorem says that any smooth, connected 1-manifold (without boundary) is diffeomorphic either to the real line $\mathbb{R}$ or to the 1-sphere $S^1$. This means that any smooth planar curve (throughout the document we will assume all curves to be connected, without further explicit mention) can be globally parametrized in the form $x = \Gamma(u)$, with $u$ taking values either on the real line $\mathbb{R}$ or on the 1-sphere $S^1$. The parameter $u$ will play the role of a homogeneous variable in the nonlinear context and we refer the reader to subsection 2.6 below for further remarks in this regard.

Later on we will write $\Gamma(u)$ as $(\psi(u), \zeta(u))$ where, for any $u$, either $\psi'(u)$ or $\zeta'(u)$ (or both) is (are) non-zero. Note also that, above, we are letting $x$ denote generically a point in $\mathbb{R}^2$: for different types of devices $x$ will stand either for $(i, v)$ (for resistors) or for other pairs of variables involving the charge $\sigma$ and/or the flux $\varphi$ (for reactive devices and, eventually, memristors), as detailed in what follows.

**Resistors.** Let us first focus the attention on a resistor defined by a smooth planar characteristic. The classification theorem above implies that there exists a global parametrization of this characteristic curve of the form

$$i = \psi(u) \quad (8a)$$
$$v = \zeta(u) \quad (8b)$$

with $\psi'(u), \zeta'(u)$ not vanishing simultaneously for any value of the homogeneous variable $u$. As indicated above, this variable takes values either on $\mathbb{R}$ or on $S^1$.

The homogeneous incremental resistance, as defined in subsection 2.2, can be naturally recast in terms of the homogeneous description (8), as shown below.

**Proposition 1.** The homogeneous incremental resistance of a smooth resistor at a given point $(i, v) = (\psi(u), \zeta(u))$ of the characteristic can be written as

$$\left(\psi'(u) : \zeta'(u)\right).$$

Indeed, let $f(i, v) = 0$ stand for the characteristic of the smooth resistor. By writing $f(\psi(u), \zeta(u)) = 0$ we get, by the chain rule,

$$f_i(\psi(u), \zeta(u))\psi'(u) + f_v(\psi(u), \zeta(u))\zeta'(u) = 0,$$
so that
\[(\psi'(u) : \zeta'(u)) = (f_\psi(\zeta(u), \psi(u)) : -f_\zeta(\zeta(u), \psi(u))),\]
meaning that the ratios are the same; in other words, both pairs of homogeneous coordinates
describe the same projective point. The claim then follows from (7).

We introduce in the nonlinear context the incremental parameters \(p, q\) as
\[p(u) = \psi'(u), \quad q(u) = \zeta'(u),\]
so that the homogeneous incremental resistance reads, at any point of the characteristic, as
\[(p(u) : q(u)).\]
In the linear context these amount to the linear coefficients \(p, q\) arising in (9). In these
terms, the (classical) incremental resistance and the incremental conductance at a given \(u\)
read as \(q(u)/p(u)\) and \(p(u)/q(u)\) (under a nonvanishing assumption on \(p\) or \(q\), respectively).

**Reactive devices.** Nonlinear capacitors and inductors defined by smooth characteristics
also admit descriptions in terms of homogeneous variables. A capacitor with a smooth
charge-voltage characteristic admits, in light of the aforementioned classification theorem, a
global parametrization of the form
\[
\begin{align*}
\sigma &= \psi(u) \\
v &= \zeta(u).
\end{align*}
\]
We will set \(p(u) = \psi'(u), q(u) = \zeta'(u)\) also for capacitors. Note that, formally, \(p\) and \(q\) will
stand for the derivatives \(\psi'\) and \(\zeta'\) (cf. (9)) for all types of devices; the difference is made
by the fact that e.g. \(\psi(u)\) defines the current in the resistive case described in (8) but the
charge in the capacitive setting (cf. (10)). Near points where \(q(u) \neq 0\), the capacitor can
be locally described in a voltage-controlled form, with incremental capacitance \(p(u)/q(u)\).
Dually, a charge-controlled description is locally feasible if \(p(u) \neq 0\).

Analogously, for smooth inductors there exists a global parametrization of the form
\[
\begin{align*}
i &= \psi(u) \\
\varphi &= \zeta(u).
\end{align*}
\]
Again, by setting \(p(u) = \psi'(u), q(u) = \zeta'(u)\) we get the incremental inductance in the form
\(q(u)/p(u)\) near points of the characteristic where \(p(u) \neq 0\), allowing for a local current-
controlled description of the device; as before, local flux-controlled descriptions exist near
points where \(q(u) \neq 0\).

**Classical descriptions.** As indicated in the Introduction, in addition to accommodating
devices which do not admit a global description in terms of a classical circuit variable (cur-
rent, voltage, charge or flux; an example can be found in subsection 2.5), the formalism above
can also be useful in such classical contexts, specifically when one does not wish to specify in advance which one is the controlling variable (e.g. for theoretical purposes or symbolic analysis) even if a global classical description is eventually used. For instance, for nonlinear resistors one can use \( \mathbb{R} \) generically, even in the understanding that, when needed, the description may amount to a current-controlled one (just by setting \( \psi \equiv \text{id} \), so that \( u \) amounts to the current \( i \) and \( \zeta \) stands for the current-to-voltage function) or to a voltage-controlled one (by taking \( \zeta \equiv \text{id} \), with \( u \) standing now for the voltage \( v \) and \( \psi \) for the voltage-to-current function). This way the homogeneous formalism avoids (or delays) unnecessarily restrictive modelling assumptions on the characteristics of devices.

### 2.4 Homogeneous models of nonlinear circuits

**Homogeneous description of uncoupled devices.** Extending the framework above from the device level to the circuit level can be performed in a natural manner under the assumption that the different group of devices (resistors, capacitors and inductors) do not exhibit coupling effects. As before, we assume that all devices are smooth.

Let us first focus on the description of the resistive devices of a given circuit. Assume that there are \( m_r \) smooth uncoupled resistors, and let \( i_r \in \mathbb{R}^{m_r} \) and \( v_r \in \mathbb{R}^{m_r} \) stand for the vectors of currents and voltages in the set of resistive branches. In the terms detailed in subsection 2.2, the \( k \)-th resistor has a characteristic which can be written as \( f_{r_k}(i_{r_k}, v_{r_k}) = 0 \), that is, as the zero set of a submersion \( f_{r_k} : U_k \to \mathbb{R} \), with \( U_k \) open in \( \mathbb{R}^2 \). Altogether, the whole set of resistive characteristics defines a manifold \( C_r \) of dimension \( m_r \) in \( \mathbb{R}^{2m_r} \), which is simply the zero set \( f_r(i_r, v_r) = 0 \), with the components of \( f_r \) being the aforementioned individual submersions \( f_{r_k} \). Note that the domain of \( f_r \) can be written as \( U_1 \times \ldots \times U_{m_r} \) after an obvious permutation of variables. Be aware of the fact that the absence of coupling effects confers \( f_r \) a simple structure, since its \( k \)-th component depends only on the \( k \)-th components of the arguments \( i_r \) and \( v_r \). Note also that independent voltage and current sources can be included in this group of devices in a straightforward manner, extending the domains of the corresponding functions \( f_{r_k} \) to include time if necessary.

Analogously, the characteristics of the capacitors and inductors define two manifolds \( C_c \) and \( C_l \), of dimensions \( m_c \) and \( m_l \), which can be written as the zero sets of certain maps \( f_{c}(\sigma_c, v_c) \) and \( f_{l}(i_l, \varphi_l) \). In the absence of coupling effects, these maps amount to a product of individual submersions, as in the resistive case.

Now, the homogeneous description of individual devices displayed in (8), (10) and (11) can be naturally extended to apply to the different sets of devices, yielding global parametrizations of the aforementioned manifolds \( C_r, C_c \) and \( C_l \). In the resistive case we may write

\[
\begin{align*}
  i_r &= \psi_r(u_r) \quad (12a) \\
  v_r &= \zeta_r(u_r), \quad (12b)
\end{align*}
\]

the \( k \)-th entries of \( \psi_r \) and \( \zeta_r \) defining the parametrization (8) of the \( k \)-th resistor. The \( k \)-dimensional homogeneous variable \( u_r \) lies on the space

\[
\mathbb{H}_r = \mathbb{R}^{r_1} \times \mathbb{T}^{r_2}.
\]
The first factor in $H_r$ accommodates the domains of resistors whose characteristic are not closed curves, so that each such characteristic is diffeomorphic to the real line $\mathbb{R}$ (w.l.o.g. we order the resistive branches in a way such that these are the first ones). In turn, $T^{r_2}$ denotes the torus $S^1 \times \{r_2\} \times S^1$ and defines the domain of the homogeneous description of the set of resistors whose characteristics define closed curves (to be termed loops in the sequel). In the absence of loops $H_r$ amounts to $\mathbb{R}^{m_r}$; this is very often the case in circuit theory and is always met in the linear setting. Note also that both $\psi_r$ and $\zeta_r$ are smooth maps $H_r \to \mathbb{R}^{m_r}$, and that the manifold $C_r$ accommodating the characteristics of all resistors is the image of the map $(\psi_r, \zeta_r) : H_r \to \mathbb{R}^{2m_r}$, which provides a global parametrization of $C_r$.

Analogously, the reactive homogeneous variables $u_c$ and $u_l$ lie on the spaces $H_c = \mathbb{R}^{c_1} \times T^{c_2}$ and $H_l = \mathbb{R}^{l_1} \times T^{l_2}$, respectively, with the same splitting of variables in both cases. For capacitors, we get a global parametrization of $C_c$ by joining together the parametrizations (10) of the individual devices to get

$$\sigma_c = \psi_c(u_c)$$
$$v_c = \zeta_c(u_c).$$

(14a)
(14b)

and the same goes for inductors, for which the individual parametrizations (11) define the maps

$$i_l = \psi_l(u_l)$$
$$\phi_l = \zeta_l(u_l).$$

(15a)
(15b)

As before, $\psi_c$ and $\zeta_c$ are smooth maps $H_c \to \mathbb{R}^{m_c}$ and, analogously, $\psi_l$ and $\zeta_l$ are maps $H_l \to \mathbb{R}^{m_l}$. We are denoting by $m_c$ and $m_l$ the number of capacitors and inductors, respectively; that is, $m_c = c_1 + c_2$, $m_l = l_1 + l_2$. Note also that the manifolds $C_c$ and $C_l$ are the images of the maps $(\psi_c, \zeta_c) : H_c \to \mathbb{R}^{2m_c}$ and $(\psi_l, \zeta_l) : H_l \to \mathbb{R}^{2m_l}$.

**Kirchhoff laws and homogeneous model.** In order to derive the full homogeneous model we need to add the electromagnetic relations

$$\sigma_c' = i_c, \quad \phi_l' = v_l,$$

and also Kirchhoff laws. These can be written as

$$Ai = 0, \quad Bv = 0,$$

where $i$ and $v$ denote the $m$-dimensional vectors of currents and voltages (with $m = m_c + m_l + m_r$ denoting the total number of branches), whereas $A$ and $B$ are reduced cut and cycle matrices (find details e.g. in [3, 17, 19]). By splitting these matrices and, as before, the current/voltage vectors in terms of the capacitive, inductive or resistive nature of the circuit devices, Kirchhoff laws read as $A_c i_c + A_l i_l + A_r i_r = 0$ and $B_c v_c + B_l v_l + B_r v_r = 0$, respectively.
 Altogether, these relations and the parametrizations (12), (14) and (15) make it possible to write the equations of any uncoupled, smooth, possibly nonlinear RLC circuit as

\[ \psi_c'(u_c) u_c' = i_c \quad \text{(16a)} \]
\[ \zeta_l'(u_l) u_l' = v_l \quad \text{(16b)} \]
\[ 0 = A_c i_c + A_l \psi_l(u_l) + A_r \psi_r(u_r) \quad \text{(16c)} \]
\[ 0 = B_c \zeta_c(u_c) + B_l v_l + B_r \zeta_r(u_r) \quad \text{(16d)} \]

We may further eliminate the variables \( i_c \) and \( v_l \) by means of the first two equations, to get the homogeneous model

\[ A_c \psi_c'(u_c) u_c' + A_l \psi_l(u_l) + A_r \psi_r(u_r) = 0 \quad \text{(17a)} \]
\[ B_c \zeta_c(u_c) + B_l \zeta_l'(u_l) u_l' + B_r \zeta_r(u_r) = 0. \quad \text{(17b)} \]

This approach yields a description of the circuit dynamics on the \( m \)-dimensional homogeneous space \( H = H_c \times H_l \times H_r \) where the homogeneous variables \( u = (u_c, u_l, u_r) \) lie. We emphasize that only one variable per branch is involved in the model but, at the same time (as far as all devices are assumed to be smooth and uncoupled), there is no loss of generality in the formulation of this reduced model. The compactness and generality of (17) makes it suitable for different analytical purposes and we will exploit this in Section 3. Remember that the classical circuit variables are obtained from the solutions of this model via (12), (14) and (15).

Also worth recalling is the fact that this model encompasses in particular classical ones (formulated in terms of currents, voltages, charges and/or fluxes), which are simply obtained by choosing appropriately the \( \psi \) and \( \zeta \) maps (e.g. if all resistors are assumed to be voltage-controlled we simply fix \( \zeta_r = \text{id} \), so that \( u_r = v_r \), and \( \psi_r \) amounts to the voltage-to-current characteristic). With this in mind, (17) provides a general model where all possible controlling relations can be accommodated. A simple example illustrating this, in the memristive context, can be found in subsection 4.2.

2.5 Example: Van der Pol’s system with a closed characteristic in the inductor

We show in what follows how the models above can be used in practice, focusing on a low-scale example. In particular we will illustrate how the homogeneous model (17) naturally accommodates trajectories evolving on regions where classical (current/voltage, or even charge/flux) descriptions do not hold globally, whereas homogeneous ones do; this way we avoid the need to resort to piecewise descriptions of the reduced dynamics. We also illustrate how partially homogeneous models, combining classical variables with homogeneous ones, provide a useful simplification in practice, based on the fact that for many devices a global description in terms of one of the classical variables is often justified by physical reasons.

To this end, consider the well-known Van der Pol system, defined by a (parallel, in the present case and without loss of generality) connection of a capacitor, an inductor and a
An admissible choice for the reduced cut and cycle matrices is

\[ A = (A_c \ A_l \ A_r) = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}, \quad B = (B_c \ B_l \ B_r) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}. \]

If we avoid imposing a specific control variable for each device (that is, if we do not assume the resistor to be either current-controlled or voltage-controlled, etc.) we get a completely general model of the Van der Pol circuit dynamics in terms of homogeneous variables \( u_c, u_l, u_r \), which are scalar in this example since there is exactly one device of each type. This is made possible by the global parametric descriptions (8), (10) and (11). With the above choice for \( A, B \), the model (17) reads for our example as

\[
\begin{align*}
\psi'_c(u_c) & = \psi_l(u_l) - \psi_r(u_r) \quad (18a) \\
\zeta'_l(u_l) & = -\zeta_c(u_c) \quad (18b) \\
0 & = \zeta_c(u_c) - \zeta_r(u_r). \quad (18c)
\end{align*}
\]

Several simplified versions of this model will be derived for different purposes and under certain assumptions on the devices. First, assuming that the capacitor is linear and voltage-controlled, the variable \( u_c \) can be simply taken to be \( v_c \) (that is, \( \zeta_c \) amounts to the identity), with \( \psi_c(v_c) = Cv_c \), \( C \) being the capacitance. This yields a partially homogeneous model, namely

\[
\begin{align*}
Cv'_c & = \psi_l(u_l) - \psi_r(v_c) \quad (19a) \\
\zeta'_l(u_l) & = -v_c \quad (19b) \\
0 & = v_c - \zeta_r(u_r). \quad (19c)
\end{align*}
\]

Additionally, the resistor will be assumed to be voltage-controlled by a relation of the form \( i_r = -v_r + v_r^3 \), as in the parallel version of the classical Van der Pol system (which would be obtained after an additional linear assumption on the inductor). This implies that we may further take \( u_r \) to be the voltage \( v_r \) (equivalently, \( \zeta_r \) amounts to the identity), with \( \psi_r(v_r) = -v_r + v_r^3 \). This results in

\[
\begin{align*}
Cv'_c & = \psi_l(u_l) - \psi_r(v_c) \quad (20a) \\
\zeta'_l(u_l) & = -v_c, \quad (20b)
\end{align*}
\]

where we have eliminated \( v_r \) in light of the identity \( v_r = v_c \).

In what follows, the characteristic of the nonlinear inductor will be assumed to be defined by a closed curve, an assumption which makes it convenient to keep a homogeneous description for this device. Specifically, the current-flux relation will be assumed to lie on the curve depicted in Fig. 1(a). Such loops typically arise in the presence of hysteresis phenomena (see e.g. [9], where a Jiles-Atherton model for ferroresonance in a ferromagnetic core yields a loop such as the one displayed in the figure). We give the loop a parametric description following [12], namely

\[
\begin{align*}
i_l & = \psi_l(u_l) = \alpha \cos^m u_l + \beta \sin^n (u_l + \delta) \quad (21a) \\
\varphi_l & = \zeta_l(u_l) = \gamma \sin u_l, \quad (21b)
\end{align*}
\]
for certain parameters \( m, n, \alpha, \beta, \gamma \) and \( \delta \). Set \( m = n = 3, \alpha = 0.2, \beta = \gamma = 1, \delta = 0.05 \).

Our goal is simply to illustrate the convenience of using a model such as (20) to track trajectories along which a global current- or flux-controlled description of the inductor does not apply, because of the closed nature of the characteristic governing the nonlinear inductor. Note, indeed, that at local extrema of the curve in Fig. 1(a) (where the flux meets local maxima or minima) we have \( \zeta'(u_l) = 0 \) and near such points there is no local flux-controlled description of the characteristic. Similarly, at turning points (points with a vertical tangent) we have \( \psi'(u_l) = 0 \) and there is no local current-controlled description of the curve. In order to describe the dynamics of the circuit in a given region in terms of a state-space model, the flux would be precluded as a model variable for trajectories which reach at least one of the aforementioned extrema and, analogously, the inductor current would be ruled out for trajectories undergoing turning points. Obviously, there is no chance to formulate a single state model in terms of either the flux or the current if we want such a model to cover trajectories reaching both extrema and turning points. Such a trajectory, stemming from the initial point defined by \( v_0 = 0.500, u_0 = -1.805 \) and approaching a limit cycle, is depicted in Fig. 1(b); a zero of \( \zeta'(u_l) \) is met at \( t = 0.100 \), whereas zeroes of \( \psi'(u_l) \) are found at the values \( t = 0.080, 1.206, 1.223, 3.147, 3.161, \) etc. The fact that (20) holds globally is the key for the model to accommodate such trajectories. If needed, the values of the current \( i_l \) and \( \varphi_l \) along the trajectory can be explicitly computed via (21).

Figure 1: (a) Hysteresis loop (21) in the inductor of Van der Pol’s circuit (abscissae: \( i_l \), ordinates: \( \varphi_l \)). (b) A trajectory of (20) undergoing both turning points and extrema of the loop (abscissae: \( v_c \), ordinates: \( u_l \)).

### 2.6 Homogeneous variables and the homogeneous space

We finish this section with a brief remark on the nature of homogeneous variables. The proof of the classification theorem of 1-manifolds (cf. Milnor’s book [13]) makes use of the arc-length to build the global parametrization \( g \) mentioned in subsection 2.5 above; it is then possible, after fixing a distinguished point and an orientation in each individual characteristic curve,
to think of the corresponding scalar variable $u$ as the arclength of the curve, setting $u = 0$ for that distinguished point and defining positive/negative values of $u$ accordingly to the chosen orientation. But there is not really a need to privilege this particular choice; indeed, the map $\Gamma$ referred to there, and the variable $u$ itself, is defined only up to a diffeomorphism of $\mathbb{R}$ or $\mathbb{S}^1$, respectively. This is analogous to what happens in the linear case, where $u$ is defined only up to a (linear) isomorphism of $\mathbb{R}$ (cf. [19]).

This similarity with the linear case supports calling $u$ a homogeneous variable also in the nonlinear setting, and we extend the use of the term to call $\mathbb{H} = \mathbb{H}_c \times \mathbb{H}_l \times \mathbb{H}_r$ the homogeneous space. By construction, this space is diffeomorphic to the manifold $\mathcal{C}_c \times \mathcal{C}_l \times \mathcal{C}_r$ which accommodates the characteristics of all devices, as described in subsection 2.4, and actually provides a convenient way to handle this manifold in broadly general terms.

3 The state-space problem in the homogeneous setting

The formalism introduced above provides a framework to address in full generality different analytical problems in circuit theory. The key remark is that the homogeneous space $\mathbb{H} = \mathbb{H}_c \times \mathbb{H}_l \times \mathbb{H}_r$, where the homogeneous variables $u$ lie, together with the homogeneous model (17), provide a reduced setting for such analyses without the need for any unnecessarily restrictive hypothesis on controlling variables. In this section we apply such framework to a classical problem in nonlinear circuit theory, that is, the state-space problem. We refer the reader to subsection 3.2 for an introduction to this problem.

In order to make the discussion lighter, we impose a restriction on the allowed topologies in the circuit: specifically, we assume that it has neither loops composed exclusively of capacitors, nor cutsets composed only of inductors. It is well known that these topological assumptions imply that the matrices $A_c$ and $B_l$ have maximal column rank; details in this regard can be found in [3, 17, 23] and references therein. Circuits satisfying this restriction are said to be topologically nondegenerate. We also assume throughout that the circuit is connected.

3.1 Splitting the circuit equations into differential equations and constraints

The homogeneous model (17) has a differential-algebraic form. As detailed below, we may rewrite it in a way which splits this system into a set of differential equations and a set of constraints. To do so, denote by $m = m_c + m_l + m_r$ the total number of branches and by $n$ the number of nodes in the circuit. Let $A^\bot_c \in \mathbb{R}^{(n-1-m_c)\times(n-1)}$, $B^\bot_l \in \mathbb{R}^{(m-n+1-m_l)\times(m-n+1)}$ be two full row rank matrices such that $A^\bot_c A_c = 0$, $B^\bot_l B_l = 0$. Allowed by the aforementioned fact that $A_c$ and $B_l$ have maximal column rank, we will choose in addition two matrices $A^-_c \in \mathbb{R}^{m_c\times(n-1)}$, $B^-_l \in \mathbb{R}^{m_l\times(m-n+1)}$ such that $A^-_c A_c = I_{m_c}$ and $B^-_l B_l = I_{m_l}$ (to be specific, set $A^-_c = (A_c^\top A_c)^{-1}A_c^\top$, $B^-_l = (B_l^\top B_l)^{-1}B_l^\top$). By construction, it easy to see that

$$A_0 = \begin{pmatrix} A^-_c \\ A^\bot_c \end{pmatrix}, \quad B_0 = \begin{pmatrix} B^-_l \\ B^\bot_l \end{pmatrix}$$  (22)
are non-singular matrices with orders $n-1$ and $m-n+1$, respectively.

Now, by premultiplying each one of the two equations in (17) by $A_0$ and by $B_0$, respectively, and after an obvious reordering, we get a splitting of the homogeneous model into a set of (so-called quasilinear or linearly implicit) differential equations

\[ \psi_c'(u_c)u_c' = -A_c^- (A_l \psi_l(u_l) + A_r \psi_r(u_r)) \]
\[ \zeta_l'(u_l)u_l' = -B_l^- (B_c \zeta_c(u_c) + B_r \zeta_r(u_r)) \]

and a set of constraints

\[ A_c^\perp (A_l \psi_l(u_l) + A_r \psi_r(u_r)) = 0 \]
\[ B_l^\perp (B_c \zeta_c(u_c) + B_r \zeta_r(u_r)) = 0. \]

3.2 The state-space reduction problem

The circuit equations (23) and (24) will make it possible to tackle under really broad assumptions the state-space modelling problem. To introduce it we drive the attention to a classical nonlinear circuit model, namely the one obtained by writing explicitly Kirchhoff laws and the characteristics of devices together with the elementary electromagnetic laws relating capacitor charges and currents, and inductor fluxes and voltages. This yields

\[ \sigma_c' = i_c \]
\[ \varphi_l' = v_l \]
\[ 0 = A_c i_c + A_i i_l + A_r i_r \]
\[ 0 = B_c v_c + B_i v_l + B_r v_r \]
\[ 0 = f_c(\sigma_c, v_c) \]
\[ 0 = f_l(\varphi_l, i_l) \]
\[ 0 = f_r(i_r, v_r). \]

It is very common in the circuit-theoretic literature to impose assumptions on the controlling variables within the characteristics (25c), (25f) and (25g). Say, for example, that inductors are globally current-controlled in the form $\varphi_l = \gamma_l(i_l)$, and capacitors and resistors globally voltage-controlled by certain maps $\sigma_c = \xi_c(v_c)$, and $i_r = \xi_r(v_r)$. This yields, from (25) and again under a smoothness assumption on the reactive devices, a reduced model of the form

\[ A_c \xi_c'(v_c)v_c' + A_i i_l + A_r \xi_r(v_r) = 0 \]
\[ B_c v_c + B_i \gamma_l'(i_l)i_l' + B_r v_r = 0. \]

Now the problem is how to formulate conditions both on the topology of the circuit and on the characteristics making it possible to derive, from (26), a state-space model of the circuit equations, that is, a system of explicit ordinary differential equations which captures all the dynamics of (26) (and thereby of (25)). The goal is, essentially, to eliminate $v_r$ from (26).
to get a state model in terms of $v_c$ and $i_l$. Note that, whatever the conditions allowing this are, this approach is irremediably restricted by the initial assumptions on the form of the characteristics (namely, the current- and voltage-control assumptions above, or any other analogous ones).

Our point is that we can do the same in terms of (17), except for the fact that now we get to an equivalent scenario without any control assumptions on the characteristics. Incidentally, it is not by chance that (17) and (26) have the same structure: we can get (26) as a particular case of the general model (17) in light of the assumptions above just by setting $u_r = v_r$ (that is, $ξ_r(u_r) = u_r$) and then $ψ_r(v_r) = ξ_r(v_r)$, etc. But, as indicated above, the difference between both approaches is that (17) does not require any a priori control assumptions on the characteristics.

Actually, in the homogeneous framework we can now easily formulate the state-space problem as the chance to express $u_r$ in terms of $u_c$ and $u_l$ from (24), so that the insertion of the resulting expressions in (23) would yield the desired state-space reduction. Needless to say, once the trajectories are computed in terms of the homogeneous variables $u$, we get the corresponding values of the classical electromagnetic variables simply via $ψ_c$, $ζ_c$ in (14), etc., which can be understood to be output maps (in the terminology of control theory).

In this setting, the state-space reduction problem actually involves three different aspects which we present in the sequel and tackle in later subsections. First, since the solutions of the circuit equations (17) (or, equivalently, of (23)-(24)) are explicitly bound to lie on the set defined by (24), it is important in practice to examine when these equations define a smooth manifold. Borrowing the term from the differential-algebraic literature, we will call the set defined by (24) the constraint set and denote it by $M$. Second, as indicated above, the most natural approach to address the state-space problem is to express the variables $u_r$ in terms of $u_c$, $u_l$. Because of the linearity of Kirchhoff laws, we will be able to assess the conditions for this independently of the constraint set requirement above, specifically by examining a non-singularity condition on the matrix of partial derivatives of the equations in the left-hand side of (24) with respect to the variables $u_r$. This will be the key ingredient in the definition of the regular set $R$. Finally, the intersection of the constraint set $M$ and the regular set $R$, which by construction is guaranteed to be a manifold, will be termed the regular manifold and denoted by $M_{reg}$. The circuit equations yield a well-defined flow (in the usual sense of dynamical systems theory: see e.g. [1]) on $M_{reg}$. In our context, this set would correspond to the index one set in the differential-algebraic literature (cf. [11][17]); be aware of the fact that the index one context is due to the topological nondegeneracy hypothesis. A closely related problem involves the structure of the intersection of the constraint set and the singular set, which defines the so-called impasse set.

Note that the sets defined above lie on the homogeneous space $H$. Again, via the maps (12), (14) and (15) these sets are easily recast in terms of the classical circuit variables.
3.3 The constraint set, the regular set and the regular manifold

As indicated above, the subset $\mathcal{M}$ of $\mathbb{H} = \mathbb{H}_c \times \mathbb{H}_l \times \mathbb{H}_r$ defined by (24) is called the constraint set. In general, this set is defined by $m_r = m - (m_c + m_l)$ equations on the $m = m_c + m_l + m_r$ variables $u$. Note that in degenerate cases this may be an empty set (think e.g. of a circuit with two diodes in series which are oriented in opposite directions). When this is not the case, the state-space problem (bound to the topological nondegeneracy hypothesis) may now be formulated in general terms as the formulation of conditions on (24) under which the variables $u_r$ can be expressed (at least locally) in terms of $u_c, u_l$; this locally makes $\mathcal{M}$ a manifold which can be parametrized in terms of these homogeneous reactive variables. This will make it possible to recast (23) as a (quasilinear) differential system on $u_c, u_l$, providing an explicit state-space model for the dynamics on the subset of $\mathbb{H}_c \times \mathbb{H}_l$ where the leading coefficients of (23) do not vanish. It is worth indicating, however, that there are other contexts in which $\mathcal{M}$ may be guaranteed to be a manifold: cf. subsection 3.5 in this regard.

Certainly, the natural way to describe locally $\mathcal{M}$ in terms of the reactive homogeneous variables $u_c, u_l$ involves characterizing the set of points where the matrix of derivatives of the equations in the left-hand side of (24) w.r.t. the variables $u_r$, that is,

$$
\begin{pmatrix}
A'_c A_r \psi'_c (u_r) \\
B'_l B_r \zeta'_l (u_r)
\end{pmatrix},
$$

is non-singular. Note that the structure of (24) (or, in essence, the linearity of Kirchhoff laws) makes this matrix of partial derivatives dependent only on $u_r$ and not on $u_c, u_l$. Together with the fact that the coefficients of $u'_c$ and $u'_l$ on (23) depend only on $u_c$ and $u_l$, respectively, this will yield a Cartesian product structure on the regular set defined below.

**Definition 1.** We define the regular set $\mathcal{R} \subseteq \mathbb{H} = \mathbb{H}_c \times \mathbb{H}_l \times \mathbb{H}_r$ of the homogeneous model (17) as the Cartesian product $\mathcal{R}_c \times \mathcal{R}_l \times \mathcal{R}_r$, where

- $\mathcal{R}_c$ and $\mathcal{R}_l$ are the sets of values of $u_c \in \mathbb{H}_c$ and $u_l \in \mathbb{H}_l$ where all the components of $\psi'_c (u_c)$ and $\zeta'_l (u_l)$ are non-null; and

- $\mathcal{R}_r$ is the set of values of $u_r \in \mathbb{H}_r$ where the matrix (27) is non-singular.

The set $\mathbb{H} - \mathcal{R}$ is called the singular set.

Mind the terminological abuse: $\psi'_c (u_c)$ and $\zeta'_l (u_l)$, as matrices of partial derivatives, are diagonal because of the absence of coupling effects, and by their components we mean the diagonal entries of such matrices, namely, the derivatives $\psi'_c$ and $\zeta'_l$ (depending on $u_{c_i}$ and $u_{l_j}$, respectively), $i$ and $j$ ranging over the sets of capacitors and inductors, respectively.

The only factor in the regular set which is not explicitly characterized in Definition 1 is the (say) “resistive” regular set $\mathcal{R}_r$. More precisely, the problem here is to characterize this set in structural terms, that is, in terms of the topology of the circuit graph and the electrical features of the devices. In Theorem 1 below, these circuit-theoretic terms involve
the structure of the circuit spanning trees: specifically, we make use of the notion of a proper tree (a notion which can be traced back to [2]), which is a spanning tree including all capacitors and no inductor. The existence of at least one proper tree is a well-known consequence of the topological nondegeneracy hypothesis. The set of proper trees of a given circuit will be denoted by $T_p$, whereas $T$ denotes the family of all spanning trees. In Theorem 1 we denote by $E_r$ the index set of resistive branches, and assume w.l.o.g. that these branches are the first $m_r$ ones: this way, $T \cap E_r$ and $\overline{T} \cap E_r$ stand, respectively, for the index sets of the resistive branches within a given tree $T$ and of those in the corresponding co-tree, whereas $p_{ri}$ and $q_{ri}$ consistently denote the derivatives of the $i$-th component of $\psi_r$ and $\zeta_r$ in (12); note that both derivatives depend only on $u_{ri}$.

**Theorem 1.** The set $R_r \subseteq \mathbb{H}_r$ is explicitly characterized by the non-vanishing of the function

$$K(u_r) = \sum_{T \in T_p} \left( \prod_{i \in T \cap E_r} p_{ri}(u_{ri}) \prod_{j \in \overline{T} \cap E_r} q_{rj}(u_{rj}) \right). \quad (28)$$

The proof will be based on the following auxiliary result (cf. [19, Theorem 1]), which can be understood as a projectively-weighted version of the matrix-tree theorem.

**Lemma 1.** Assume that $A$ and $B$ are two given reduced cut (or incidence) and cycle matrices of a connected digraph. Let $P$, $Q$ be arbitrary diagonal matrices, with $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_m)$ the vectors of diagonal entries of $P$ and $Q$. Then

$$\det \begin{pmatrix} AP & BQ \end{pmatrix} = k_{AB} \sum_{T \in T} \left( \prod_{i \in T} p_i \prod_{j \in T} q_j \right), \quad (29)$$

for a certain non-vanishing constant $k_{AB}$.

Disregarding the constant $k_{AB}$, the function in the right-hand side of (29) is the so-called multihomogeneous Kirchhoff (or tree-enumerator) polynomial of a connected graph, to be denoted by $\tilde{K}(p, q)$, in which every spanning tree $T$ sets up a monomial which includes $p_i$ (resp. $q_i$) as a factor if the $i$-th branch belongs to $T$ (resp. to $\overline{T}$) [5, 19]; regarding this concept, the example discussed below can be of help for the reader at this point.

**Proof of Theorem 1.** With the splitting $A = (A_c \ A_l \ A_r), \ B = (B_c \ B_l \ B_r)$, and by setting $P = \text{block-diag}(I_c, \ 0_l, \ \psi'_r(u_r)), \ Q = \text{block-diag}(0_c, \ I_l, \ \zeta'_r(u_r))$, the matrix in the left-hand side of (29) reads as

$$\begin{pmatrix} AP & BQ \end{pmatrix} = \begin{pmatrix} A_c & 0_l & A_r \psi'_r(u_r) \\ 0_c & B_l & B_r \zeta'_r(u_r) \end{pmatrix}. \quad (30)$$

By Lemma 1 the determinant of this matrix is defined by the polynomial in the right-hand side of (29). Because of the definition of the $P$ matrix, all values of $p$ corresponding to inductors do vanish, whereas for capacitors we have $p_{ci} = 1$; dually, values of $q$ which correspond to capacitors are null, and for inductors we have $q_{li} = 1$. This means that any
in a tree annihilates the corresponding term in the Kirchhoff polynomial, because of the vanishing of $p_i$; analogously, any capacitor in a co-tree renders the term for that tree null, since $q_i = 0$. Therefore, the only (possibly) non-null terms in the polynomial must correspond to proper trees, namely, trees including all capacitors and no inductor. Note, additionally, that within these trees we have $p_i = 1$ and $q_i = 1$, so that only the resistive terms actually contribute a (possibly) nontrivial factor within each monomial. Altogether, this means that the determinant of (30) equals $k_{AB} K(u_r)$, with the latter function defined in (28).

It remains to show that, except for another non-null factor, the determinant of (30) equals that of (27). To check this use the fact that, in light of the topological nondegeneracy hypothesis and the definition of $A_c$ and $B_l$, the matrices

$$
\tilde{A} = \begin{pmatrix} A_c^T \\ A_c^\perp \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} B_l^\perp \\ B_l^T \end{pmatrix}
$$

are non-singular. Now, premultiply the right-hand side of (30) by the matrix block-diag($\tilde{A}$, $\tilde{B}$), which is itself non-singular, to get

$$
\begin{pmatrix} A_c^T & 0 \\ A_c^\perp & 0 \\ 0 & B_l^\perp \\ 0 & B_l^T \end{pmatrix} \begin{pmatrix} A_c & 0_l & A_r \psi_r'(u_r) \\ 0_c & 0_l & A_c^\perp A_r \psi_r'(u_r) \\ 0_c & 0_l & B_l^\perp B_r \zeta_r'(u_r) \\ 0_c & B_l^T B_l & B_l^T B_r \zeta_r'(u_r) \end{pmatrix} = \begin{pmatrix} A_c^T A_c & 0_l & A_c^T A_r \psi_r'(u_r) \\ 0_c & 0_l & A_c^\perp A_r \psi_r'(u_r) \\ 0_c & 0_l & B_l^\perp B_r \zeta_r'(u_r) \\ 0_c & B_l^T B_l & B_l^T B_r \zeta_r'(u_r) \end{pmatrix}
$$

whose determinant equals

$$
\pm \det (A_c^T A_c) \det (B_l^T B_l) \det \begin{pmatrix} A_c^\perp A_r \psi_r'(u_r) \\ B_l^\perp B_r \zeta_r'(u_r) \end{pmatrix}.
$$

Using the fact that $\det (A_c^T A_c) \det (B_l^T B_l) \neq 0$, owing to the absence of C-loops and L-cutsets (which makes the columns of $A_c$ and $B_l$ linearly independent), we get that (27) and (30) actually have (possibly up to a non-null factor) the same determinant and the claim is proved.

\[\square\]

**Example. Murali-Lakshmanan-Chua circuits.** A key role in the result above is played by the polynomial in the right-hand side of (29) and its nonlinear counterpart (28). We illustrate the form that these functions take in practice by means on an example defined by two resistively-coupled Murali-Lakshmanan-Chua (MLC) circuits, depicted in Fig. 2. MLC circuits were introduced in [14], and arrays of these circuits are considered for different purposes e.g. in [10, 15]. We use one of the circuits of the MLC family defined in [10]; to focus on the contribution of resistors we set $C = L = 1$ and annihilate the voltage in voltage sources within the original circuit as defined in that paper.
From the set of proper trees (displayed in Fig. 3) one can easily check that the multihomogeneous Kirchhoff polynomial reads for this circuit as

$$p_1q_2p_3q_4q_5 + p_1q_2q_3q_4q_5 + q_1p_2p_3q_4q_5 + q_1p_2q_3p_4q_5 + p_1q_2q_4p_5 + + q_1p_2q_3q_4p_5 + q_1q_2q_3q_4p_5. \quad (32)$$

The function \(28\), characterizing the set of regular points, is just obtained by letting \(p_i\) and \(q_i\) above depend on the corresponding homogeneous variable \(u_i\). We emphasize the fact that the non-vanishing of this function of the homogeneous variables performs this characterization of the regular set in full generality. It is of interest, however, to show how this general model takes simpler forms and provides additional information in simplified settings which arise from different assumptions on the circuit devices, as we do in the sequel.

Indeed, in each MLC circuit only one of the resistors displays a nonlinear behavior (namely, those labelled with the subindices 2 and 4), whereas numbers 1 and 3, as well as the coupling resistor 5, are typically linear; moreover, we may assume them to be defined by a resistance parameter \(r_i, i = 1, 3, 5\). This is equivalent to saying that \(p_1, p_3\) and \(p_5\) do not vanish and then, by dividing the polynomial above by \(p_1p_3p_5\), we get a partially dehomogenized form which characterizes the regular set of values for the remaining homogeneous variables (namely, \(u_2\) and \(u_4\)). These are defined by the non-vanishing of the function (we group some terms for notational simplicity):

$$p_2(u_2)p_4(u_4)r_1r_3r_5 + p_2(u_2)q_4(u_4)(r_1r_3 + r_1r_5) + q_2(u_2)p_4(u_4)(r_1r_3 + r_3r_5) + + q_2(u_2)q_4(u_4)(r_1 + r_3 + r_5).$$

Note that in the latter formula we retain an homogeneous expression for both nonlinear resistors. Still by way of example, assume now that resistor no. 4 is known to admit a global voltage-controlled expression: \(u_4\) then amounts to the voltage variable \(v_4\) and the expression above may be divided by \(q_4\) to get a description of this device in terms of the incremental conductance \(g_4(v_4)\). For resistor no. 2 we retain, by contrast, the homogeneous form, for instance to be able to model an eventual spurious short-circuit caused by a bridging fault (that is, an unexpected short-circuit). Under these hypotheses, the function characterizing the regular set would be

$$p_2(u_2)g_4(v_4)r_1r_3r_5 + p_2(u_2)(r_1r_3 + r_1r_5) + q_2(u_2)g_4(v_4)(r_1r_3 + r_3r_5) + q_2(u_2)(r_1 + r_3 + r_5). \quad (33)$$
Finally, such a bridging fault in the second resistor would be modeled here by $q_2 = 0$ (implying $p_2 \neq 0$). In this particular setting, the set of singular values for the remaining variable $v_4$ would simply be obtained from annihilating (33), and are given by $g_4(v_4) = -(r_1r_3 + r_1r_5)/(r_1r_3r_5)$. Needless to say, other conclusions could be analogously drawn in other working scenarios from the general form of the multihomogeneous Kirchhoff polynomial (32).

We finish this subsection with the following result, which essentially says that a flow is well-defined on the intersection $M \cap R$. It is an immediate consequence of the non-singularity of (27) and the implicit function theorem, which yields a local description of $M$ in the form $u_r = \eta_r(u_c, u_l)$ near regular points. An elementary example of a state-space model of the form (34) below can be found in (20); note that the homogeneous variable $u_c$ amounts there to $v_c$ because of the working assumptions in that example.

**Theorem 2.** If non-empty, the intersection of the constraint set $M$ defined by (24) and the regular set $R$ in Definition 2 is an $(m_c + m_l)$-dimensional manifold. It is filled by solutions of the circuit equations (17) (or, equivalently, of (23)-(24)), which are defined by the solutions of an explicit state-space model of the form

\[
\begin{align*}
    u'_{c} &= -(\psi'_c(u_c))^{-1}A_c^- (A_l\psi_l(u_l) + A_r\psi_r(\eta_r(u_c, u_l))) \quad (34a) \\
    u'_l &= -(\zeta'_l(u_l))^{-1}B_l^- (B_c\zeta_c(u_c) + B_r\zeta_r(\eta_r(u_c, u_l))) \quad (34b)
\end{align*}
\]

We will call the intersection $M \cap R$ the regular manifold and denote it by $M_{reg}$. This
corresponds to the index one set in the differential-algebraic literature (cf. [11, 17]): note that the focus is restricted to an index one context because of the assumed topological nondegeneracy. The set \( \mathcal{M} - \mathcal{M}_{\text{reg}} \) will be called the impasse set.

### 3.4 The regular set is dense in locally nonlinear problems

In this subsection we elaborate on the structure of the impasse set defined above. In order to motivate the discussion, let us go back to the partially homogeneous form of the Van der Pol system (with a linear capacitor) defined by (19). The regular set in this case is defined by the conditions \( \zeta'_l(u_l) \neq 0 \) and \( \zeta'_r(u_r) \neq 0 \): we note in passing that this parallel configuration has a unique proper tree, just defined by the capacitor; the resistor is therefore in the cotree and hence the latter condition on \( \zeta'_r(u_r) = q_r(u_r) \).

Now, for a generic set of functions \( \zeta_l \) and \( \zeta_r \) (think e.g. of Morse functions, for which the condition \( \zeta'(u) = 0 \) implies \( \zeta''(u) \neq 0 \), making all critical points isolated), the singular set is simply defined by a set of hyperplanes of the form \( u_l = u^*_l \) and \( u_r = u^*_r \), where \( u^*_l \) and \( u^*_r \) denote critical points of \( \zeta_l \) and \( \zeta_r \), respectively. The impasse set is in this case a hypersurface of the constraint set \( \mathcal{M} \) defined by (19c).

By contrast, the nature of the singular set is radically different if the inductor and the resistor in (19) are also assumed to be linear. Indeed, suppose both to be linear and current-controlled, so that \( u_l \) and \( u_r \) amount to the currents \( i_l \) and \( i_r \), with \( \zeta_l(i_l) = L_i \) and \( \zeta_r(i_r) = R_i \). For further simplicity, assume \( C \) and \( L \) not to vanish. In this setting, the assumption \( R \neq 0 \) makes all points regular, whereas when \( R = 0 \) all points would be singular according to Definition 1. In particular, there is no hypersurface of singular points in the whole homogeneous space \( \mathbb{H} \) or of impasse points in the constraint set \( \mathcal{M} \) (which in this case is simply a hyperplane, namely the one defined by the linear relation \( v_c = R_i \), here expressed in terms of classical circuit variables because \( u_c = v_c \) and \( u_r = i_r \)).

It is well known in circuit theory that linear problems do not exhibit impasse phenomena; that is, the behavior described above, with all points having the same (regular or singular) nature, is always found in linear problems. This is a rather obvious consequence of the fact that the eventual singularity of the matrix (27) does not depend on \( u_r \) in linear cases, together with the remark that the leading coefficients of (23) would be constant in a linear setting. But we are now in a position to give much more precise information about this: generically or, more specifically, for the locally nonlinear functions defined below, the regular set is an open dense subset of the homogeneous space, as it was the case for the example (19) mentioned above.

From the theory of parametrized curves we know that the curvature of a (regularly) parametrized curve \((\psi(u), \zeta(u))\) at a given \( u \) is defined as

\[
\kappa(u) = \frac{|\psi'(u)\zeta''(u) - \psi''(u)\zeta'(u)|}{((\psi'(u))^2 + (\zeta'(u))^2)^{3/2}}. \tag{35}
\]

The curvature vanishes at points where \( \psi'(u)\zeta''(u) - \psi''(u)\zeta'(u) = 0 \).

**Definition 2.** A smooth device is said to be **locally nonlinear** if the curvature does not vanish identically on any open portion of its characteristic.
The fact that the curvature does not vanish on any portion of the curve diffeomorphic to an open interval. A device which is not locally nonlinear has at least a portion of the characteristic which is a line segment.

**Theorem 3.** If all devices of a smooth, uncoupled, topologically nondegenerate circuit are locally nonlinear, then the regular set $\mathcal{R}$ is open dense in the homogeneous space $\mathbb{H}$.

**Proof.** The fact that $\mathcal{R}$ is open follows in a straightforward manner from Definition 1. To show that it is also dense, it is enough to show that the sets $\mathcal{R}_c$, $\mathcal{R}_l$ and $\mathcal{R}_r$ are dense in $\mathbb{H}_c$, $\mathbb{H}_l$ and $\mathbb{H}_r$, respectively.

Regarding $\mathcal{R}_c$ and $\mathcal{R}_l$, simply note that these are the sets where all the components of $\psi_c'(u_c)$ and $\zeta_l'(u_l)$ are non-zero. Assuming for instance $\mathcal{R}_c$ not to be dense in $\mathbb{H}_c$, there would exist an open set in $\mathbb{H}_c$, where at least one of the components of $\psi_c'(u_c)$, say $\psi_c'(u_c_i)$, should vanish. By taking a product of open intervals within that open set, not only $\psi_c'$ but also $\psi_c''$ would vanish on an open interval. In light of (35), this would imply that the curvature of the characteristic of the $i$-th capacitor vanishes on an interval, against the local nonlinearity assumption. The same reasoning applies to show that $\mathcal{R}_l$ is dense in $\mathbb{H}_l$.

Assume now that $\mathcal{R}_r$ is not dense in $\mathbb{H}_r$. This is equivalent to the assumption that the identity $K(u_r) = 0$ (cf. (28)) holds on some open set within $\mathbb{H}_r$. Pick any resistive branch (say number 1, w.l.o.g.). By restricting the aforementioned open set if necessary we may guarantee that either $p_{r_1}(u_{r_1}) = \psi_{r_1}'(u_{r_1})$ or $q_{r_1}(u_{r_1}) = \zeta_{r_1}'(u_{r_1})$ (we choose the latter, again w.l.o.g. as detailed later) does not vanish on an interval $I_1$. The key fact is that the Kirchhoff polynomial $\tilde{K}(p, q)$ is homogeneous of degree one in $p_{r_1}$, $q_{r_1}$, and therefore we may divide by $q_{r_1}$ to get

$$K_1(u_r) = \frac{K(u_r)}{q_{r_1}(u_{r_1})} = y_{r_1}(u_{r_1})K_{11}(u_{r_2}, \ldots, u_{r_{mr}}) + K_{12}(u_{r_2}, \ldots, u_{r_{mr}})$$

(36)

with $y_{r_1} = p_{r_1}/q_{r_1}$. Note that either $K_{11}$ or $K_{12}$ (but not both) might be absent in the expression above for topological reasons: e.g. if the first resistor is present in all proper trees then all terms of $K$ include $p_{r_1}$ (and none $q_{r_1}$) as a factor, meaning that the $K_{12}$ term would not be present; in the dual case (namely, when all terms include $q_{r_1}$) the identity (36) would amount to $K_1 = K_{12}$. Including these two scenarios is necessary in order to guarantee that there is no loss of generality in the non-vanishing assumption on $q_{r_1}$ made above.

By construction and with the restriction mentioned above, the quotient in (36) vanishes on the same set as $K(u_r)$ and, therefore, we also have $\partial K_1/\partial u_{r_1} = 0$ on the same set. Now let us first assume that the $K_{11}$ term is indeed present in (36) above. From the vanishing of the first partial derivative we get

$$y_{r_1}'(u_{r_1})K_{11}(u_{r_2}, \ldots, u_{r_{mr}}) = 0.$$  

(37)

If the factor $y_{r_1}'(u_{r_1})$ vanishes on an open interval within the aforementioned $I_1$, we easily get the identity $\psi_{r_1}'\zeta_{r_1}'' - \psi_{r_1}''\zeta_{r_1}' = 0$ there, against the local nonlinearity assumption on the
first resistor. It then follows from (37) that $K_{11}(u_{r_2}, \ldots, u_{r_{mr}})$ must vanish identically on some open set. Should, on the other hand, the $K_{11}$ term be absent from (36), it would follow trivially that $K_1 = K_{12}$ and the latter would vanish on the same (restricted) open set where $K_1$ and $K$ do.

One way or another we get $K_{1i}(u_{r_2}, \ldots, u_{r_{mr}}) = 0$ on some open set, either for $i = 1$ or $i = 2$. But again this is a multihomogeneous polynomial on each pair of variables $p_j$, $q_j$ and the same reasoning applies recursively. This way the argument can be repeated until some $y'_{rk}$ vanishes on some open subinterval, which contradicts the local nonlinearity assumption on all resistors. This shows that $\mathcal{R}_r$ is indeed dense in $\mathbb{H}_r$ and the proof is complete. □

### 3.5 On the manifold structure of the constraint set. Quasilinear reduction

We finish this section with some remarks on the structure of the constraint set $\mathcal{M}$ near impasse points. Let us first emphasize the rather obvious fact that the non-singularity of (27) is not a necessary condition for the constraint set $\mathcal{M}$ defined by (24) to be a manifold. In greater generality, this set would have a manifold structure near a given point if the map in the left-hand side of this equation is (locally) a submersion, that is, if the matrix of partial derivatives

$$
\left( \begin{array}{cccc}
A_t^c A_t \psi'_l(u_l) & 0 & A_t^r A_r \psi'_r(u_r) \\
0 & B_t^l B_c \zeta'_c(u_c) & B_t^r B_r \zeta'_r(u_r) \\
\end{array} \right)
$$

has maximal rank $m_r$.

**Proposition 2.** Assume that, at a given $(u_c, u_l, u_r) \in \mathcal{M}$, all components of $\psi'_l(u_l)$ and $\zeta'_c(u_c)$ do not vanish, and that the matrix

$$
\begin{pmatrix}
A_t \psi'_r(u_r) \\
B_r \zeta'_r(u_r)
\end{pmatrix}
$$

has maximal rank $m_r$. Then $\mathcal{M}$ is locally a manifold near $(u_c, u_l, u_r)$.

**Proof.** The first step of the proof proceeds as in Theorem 1: premultiplying the matrix

$$
\begin{pmatrix}
A_c & A_t \psi'_l(u_l) & 0 & 0 & A_t \psi'_r(u_r) \\
0 & 0 & B_c \zeta'_c(u_c) & B_t & B_r \zeta'_r(u_r)
\end{pmatrix}
$$

by the block-diagonal one block-diag$(\tilde{A}, \tilde{B})$ from (31), we easily get

$$
\begin{pmatrix}
A_t^c A_c & * & * & * & * \\
0 & A_t^l A_t \psi'_l(u_l) & 0 & 0 & A_t^r A_r \psi'_r(u_r) \\
0 & 0 & B_t^l B_c \zeta'_c(u_c) & 0 & B_t^r B_r \zeta'_r(u_r) \\
0 & * & * & B_t^B B_t & *
\end{pmatrix},
$$

where $*$ denotes entries which are not relevant to our present purposes. Using the nonsingular blocks $A_t^c A_c$ and $B_t^B B_t$ one can check that and (38) and (40) have the same corank.
Therefore, the maximal rank condition on $[38]$ can be equivalently examined in terms of $[40]$.

Now let $v = (v_1, v_2, v_3, v_4, v_5)$ be a vector in the kernel of the matrix $[40]$ and write, for notational simplicity in what follows, $P_l = \psi_l(u_l), Q_c = \zeta_c(u_c), P_r = \psi_r(u_r)$ and $Q_r = \zeta_r(u_r)$. In light of the orthogonality of the cycle and cut spaces (see e.g. $[3]$), the relations

\[ v_1 = B_c^T w_1, \quad v_2 = P_l^{-1} B_l^T w_1, \quad v_3 = Q_c^{-1} A_c^T w_2, \quad v_4 = A_l^T w_2 \tag{41} \]

and $P_r v_5 = B_r^T w_1, \quad Q_r v_5 = A_r^T w_2$ must hold for certain vectors $w_1, w_2$. Note that, by construction, the matrix $T = P_r^2 + Q_r^2$ is non-singular and then $v_5$ can be also explicitly written in terms of $w_1$ and $w_2$ as

\[ v_5 = T^{-1} (P_r B_r^T w_1 + Q_r A_r^T w_2). \tag{42} \]

Additionally, since $P_r$ and $Q_r$ are diagonal and hence commute, we have $Q_r P_r v_5 = P_r Q_r v_5$ and then $Q_r B_r^T w_1 = P_r A_r^T w_2$. This means that $(w_1, w_2) \in \ker(-Q_r B_r^T P_r A_r^T)$. The latter matrix has maximal rank by hypothesis, so that its kernel has dimension $m - m_r = m_c + m_l$.

Finally, since the kernel of $[40]$ can be written in terms of $w_1$ and $w_2$ via $[41]$ and $[42]$, and as indicated above $w_1, w_2$ lie on a space of dimension $m_c + m_l$, the dimension of the kernel of $[40]$ cannot be greater than $m_c + m_l$. But the matrix $[40]$ has order $m \times (m + m_c + m_l)$, and therefore its rank may not be less than $m + m_c + m_l - (m_c + m_l) = m$; we then get that this rank indeed attains its maximum possible value, $m$, as we aimed to show. \hfill \Box

The maximal rank assumption on $[39]$ is relevant in practice because it can be shown to express the transversality of the projection $(i_c, v_c, i_l, v_l, i_r, v_r) \rightarrow (i_r, v_r)$ (restricted to the linear space defined by Kirchhoff laws) to the characteristic manifold $C_r$ (find details in this regard in $[21]$). And even if we omit a detailed discussion for the sake of brevity, from a dynamical perspective Proposition $[2]$ is useful because the manifold structure of $\mathcal{M}$ still allows for a quasilinear description of the dynamics. This is no longer possible in terms of $u_c, u_l$ as in $[34]$, but in terms of some $m_c + m_l$ homogeneous variables from within the vector $(u_c, u_l, u_r)$. Just for illustrative purposes, an elementary example can be given in terms of $[19]$: even near an impasse point defined by the condition $\zeta'_r(u_r) = 0$, the constraint set (given by $v_c = \zeta_c(u_c)$) is a manifold where a quasilinear reduction is still feasible, now in terms of $u_l, u_r$. Note that impasse points are captured in the leading coefficients of the reduction, which has the form

\[ C \zeta'_r(u_r) u'_r = \psi_l(u_l) - \psi_r(u_r) \tag{43a} \]

\[ \zeta'_r(u_l) u'_l = -\zeta_r(u_r). \tag{43b} \]

## 4 Memristors

In this section we briefly show how to extend the previous approach to circuits with memristors, a family of devices which has attracted a lot of attention in Electronics in the last decade, following the results reported in the paper $[22]$. By means of a specific example we show the form that the models take and, in particular, how the homogeneous formalism makes it possible to frame in the same context two problems considered in $[7, 8]$.  

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4.1 Homogeneous modelling of circuits with memristors

A memristor is any electronic device characterized by a nonlinear relation between the charge \( \sigma \) and the magnetic flux \( \varphi \). Under the assumption that this relation is smooth (more precisely, that the characteristic is a smooth planar curve) we may proceed as in Section 2 to describe a smooth memristive characteristic in terms of a homogeneous variable \( u \) in the form

\[
\sigma = \psi(u), \quad \varphi = \zeta(u). \tag{44}
\]

Under the obvious nonvanishing assumptions, either the memristance \( \zeta'(u)/\psi'(u) \) or the memductance \( \psi'(u)/\zeta'(u) \) are well-defined at any \( u \). In greater generality, the homogeneous memristance reads as \( (\psi'(u) : \zeta'(u)) \).

With the addition of memristors, the homogeneous model \[ \text{(17)} \] takes the form

\[
A_m \psi'_m(u_m)u'_m + A_c \psi'_c(u_c)u'_c + A_l \psi_l(u_l) + A_r \psi_r(u_r) = 0 \quad \text{(45a)}
\]

\[
B_m \zeta'_m(u_m)u'_m + B_c \zeta'_c(u_c) + B_l \zeta_l(u_l) + B_r \zeta_r(u_r) = 0 \quad \text{(45b)}
\]

with the vector-valued maps \( \psi_m \) and \( \zeta_m \) joining together the contributions of the different memristors. We illustrate below the form that these equations may take in practice.

4.2 Example

The memristor-capacitor circuit displayed in Fig. 4 is analyzed, under different assumptions, in \[ \text{[7, 8]} \]. We show below how the approach introduced in this paper makes it possible to accommodate both analyses in a single, unifying framework, unveiling in addition some symmetry properties which seem to underly this example and possibly other memristive circuits. We assume for simplicity that the capacitor is a linear one with \( C = 1 \).

\[ \text{Figure 4: Memristor-capacitor circuit.} \]

In \[ \text{[7]} \] the memristor is assumed to be flux-controlled, with a cubic characteristic which can be written in the form \( \sigma_m = -\varphi_m + \varphi_m^3 \). Two stability changes are reported in that paper to occur along a line of equilibria and for the flux values \( \varphi_m = \pm \sqrt[3]{1/3} \); more precisely, this circuit can be shown to undergo two transcritical bifurcations without parameters by checking that it satisfies the general requirements characterizing this bifurcation in \[ \text{[18]} \]. By contrast, in \[ \text{[8]} \] the memristor is assumed to have the dual charge-controlled form \( \varphi_m = -\sigma_m + \sigma_m^3 \), which is responsible for the presence of two impasse manifolds, defined by the charge values \( \sigma_m = \pm \sqrt[3]{1/3} \), where trajectories collapse in finite time with infinite speed.
What we want to examine is the reason for the dual characteristics above to yield these two qualitative phenomena. Note that in the framework of [7, 8] two different models must be used, because of the different control variables involved in the memristor; indeed, in the former case the circuit equations are formulated in [7] in terms of the flux, and necessarily in terms of the charge in [8]. Instead, a single reduction applying to both contexts can be obtained from the homogeneous formalism, making it possible to formulate a single model in terms of one and the same homogeneous variable $u_m$ for the memristor (for the capacitor, because of its linear nature, we may choose $v_c$, $\sigma_c$ or even a homogeneous variable $u_c$).

Specifically, the equations for the circuit in Fig. 4 can be written, using an homogeneous description of the memristor (cf. (44)), as

\begin{align}
  p(u_m)u'_m - v'_c &= 0 \quad \text{(46a)} \\
  q(u_m)u'_m &= -v_c, \quad \text{(46b)}
\end{align}

with $p(u_m) = \psi'_m(u_m)$, $q(u_m) = \zeta'_m(u_m)$. Here we need no assumption on controlling variables in the memristor. In particular, denoting $\chi(u_m) = -u_m + u^3_m$, the two cases considered in [7, 8] are accommodated in this model just by setting $\psi_m = \chi$ and $\zeta_m = \text{id}$ (with $p(u_m) = \chi'(u_m) = -1 + 3u^2_m$, $q(u_m) = 1$) to model the flux-controlled context of [7], and $\psi_m = \text{id}$, $\zeta_m = \chi$ (yielding $p(u_m) = 1$, $q(u_m) = \chi'(u_m)$) for the charge-controlled setting of [8].

Regardless of the actual form of the memristor characteristic, it is clear from (46) that this system has a line of equilibria defined by $v_c = 0$. The linearization of (46) at any equilibrium point is defined by the matrix pencil

\[ \lambda \begin{pmatrix} p(u_m) & -1 \\ q(u_m) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \]

whose eigenvalues are given by the roots of the polynomial $\lambda(\lambda q(u_m) + p(u_m))$; these are $\lambda = 0$ and $\lambda = -p(u_m)/q(u_m)$. Worth remarking is the fact that the null eigenvalue reflects that equilibrium points are not isolated but define a line, a phenomenon which is well-known to happen systematically in the presence of a memristor (see [18] and references therein).

Now, the zeros of $p$ and of $q$ in each of the cases defined by the characteristics of [7, 8] are located at $u_m = \pm \sqrt{1/3}$. The zeros of $p$ in the first setting define a second null eigenvalue in the matrix pencil spectrum, which is responsible for the transcritical bifurcation without parameters; in turn, the zeros of $q$ in the second case yield an infinite eigenvalue in the pencil, which results in the aforementioned impasse phenomena. The key remark is that the homogeneous formalism is able to accommodate simultaneously both contexts and capture the intrinsic symmetry of both problems; actually, this framework (and, specifically, the expression for the second eigenvalue) makes it apparent that the eigenvalues are transformed by the relation $\lambda \rightarrow 1/\lambda$ when the expressions defining $p$ and $q$ are interchanged. Note that stability changes in the first setting, due to the transition of an eigenvalue through zero in the transcritical bifurcation without parameters, correspond in the second setting to a sign change in the eigenvalue owing to its divergence through $\pm \infty$. 

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5 Concluding remarks

We have extended in this paper the homogeneous approach of [19] to uncoupled, nonlinear electrical circuits, possibly including memristors, under a smoothness assumption on all devices. The homogeneous framework leads to a new circuit model, displayed in (1) (its detailed derivation can be found in subsection 2.3 cf. (17)), which, involving only one state variable per branch, retains the full generality of larger size model families such as those arising in the tableau approach. From the modelling perspective, worth emphasizing is the fact that the homogeneous model (1) particularizes to classical models in restricted scenarios in which some devices admit global descriptions in terms of the current, voltage, charge or flux; these classical contexts are simply obtained by appropriate choices of the maps $\psi_r$, $\zeta_r$, $\psi_c$, etc. in (1). This way we avoid the need for global current/voltage/charge/flux-controlled descriptions, which entail a loss of generality in the formulation and the reduction of circuit models. Our results make it possible to address in detail certain analytical problems such as the state-space problem in nonlinear circuit theory: in this direction, we have provided a full circuit-theoretic characterization of the so-called regular manifold of topologically nondegenerate (index one) circuits, holding without any restriction on controlling variables of individual devices. We have also proved that the regular set is generically open dense in the so-called homogeneous space, capturing a subtle qualitative distinction between nonlinear (in the strict sense) and linear circuits. The homogeneous approach may be expected to be of help in other analytical problems in the future.

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