Friend or Foe?
Population Protocols can perform Community Detection

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Abstract

We present a simple distributed algorithm that, given a regular graph consisting of two communities (or clusters), each inducing a good expander and such that the cut between them has sparsity $1/\text{polylog}(n)$, recovers the two communities.

More precisely, upon running the protocol, every node assigns itself a binary label of $m = \Theta(\log n)$ bits, so that with high probability, for all but a small number of outliers, nodes within the same community are assigned labels with Hamming distance $o(m)$, while nodes belonging to different communities receive labels with Hamming distance at least $m/2 - o(m)$. We refer to such an outcome as a community sensitive labeling of the graph.

Our algorithm uses $\Theta(\log^2 n)$ local memory and computes the community sensitive labeling after each node performs $\Theta(\log^2 n)$ steps of local work.

Our algorithm and its analysis work in the (random) population protocol model, in which anonymous nodes do not share any global clock (the model is asynchronous) and communication occurs over one single (random) edge per round. We believe, this is the first provably-effective protocol for community detection that works in this model.

Keywords: Distributed Community Detection, Population Protocols, Probabilistic Algorithms, Spectral Analysis.
1 Introduction

1.1 The model and the problem

Informally, a Population Protocol \[3, 4, 8\] is a dynamic distributed system consisting of a finite set of identical agents that interact over an underlying graph \(G = (V, E)\). In the discrete-time, random setting, at every round, a random edge \((u, v) \in E\) is activated and both endpoints \(u\) and \(v\) apply a fixed update rule (the same for every agent) to respectively compute their next states. Agents are anonymous (so, they have no unique IDs), they do not share any global clock, nor can they define a local labeling of their respective neighbors.

Population protocols received considerable attention in the recent past, mostly because they offer a distributed model that is minimal with respect to communication and computational capabilities. In several network scenarios (like many arising in biological systems or opportunistic networks), one cannot realistically assume the existence of a global clock and/or the simultaneous presence of all neighbors of an agent in a given round. Population protocols thus help capture key aspects of important processes in social networks, biological systems and other domains of interest in network science \[4, 8, 11, 18, 23, 27, 28\]. In particular, research in biology and nanotechnology has shown that population protocol algorithms can be implemented at the molecular level \[19\], at the same time highlighting deep connections between population protocols and important tasks of a computational nature in cell biology \[13\]. The same model is known as Chemical Reaction Networks in the field of molecular computing for fixed volume reactions \[23\].

Most recent studies on population protocols address complexity issues, i.e., the class of computational tasks population protocols can solve \[7, 4\] and/or focus on convergence bounds for key tasks in distributed computing, such as self-stabilizing consensus or majority computations \[3, 6, 8, 15, 43\]. Departing from these lines of work, in this paper we investigate the effectiveness of population protocols for community detection in graphs, a task which is not trivial even in a centralized setting \[25, 39\]. In one of the simplest versions of this problem, we want to design an algorithm that, given a regular clustered graph \[1\] \(G = ((V_1, V_2), E)\), is able to recover the communities \(V_1, V_2\). In a centralized setting, this amounts to computing a coloring \(C : (V_1 \cup V_2) \rightarrow \{\text{Blue}, \text{Red}\}\), so that nodes within the same community are assigned the same color and the two communities receive different colors.

Several notions of community (or cluster) appear in the literature \[15, 26, 42, 39\]. A natural one (including, the popular Stochastic Block Model \[12, 39, 38\] as an important case) refers to the expansion properties of the graph: Essentially, the two clusters \(V_1\) and \(V_2\) are connected over a sparse (or small conductance) cut, while both clusters exhibit large inner expansion/conductance. In the case of regular clustered graphs studied in this paper, if we call \(P\) the transition matrix of the random walk on \(G\), this condition is equivalent to asking that the second largest eigenvalue of \(P\) be \(1 - 1/\mathcal{O}(\log^2 n)\) and that the third largest be \(1 - \mathcal{O}(1)\). Community detection has been investigated extensively using a number of techniques, which include combinatorial algorithms \[25\], spectral-based techniques \[39, 21\], Metropolis approaches \[42\], and semidefinite programming \[1\], among others.

On the other hand, few distributed approaches have been proposed and rigorously analyzed so far \[10, 33, 35, 17\]. While these previous efforts are discussed in detail further in Section \[2\] here we emphasize that, to the best of our knowledge, no community detection algorithm working with provable guarantees in the population protocol model is currently available for any class of graphs.

1See Definition 2.

2In this paper, the two terms will be used interchangeably.
1.2 Our contribution

A first, non-obvious issue is the notion itself of a distributed solution for the community detection problem in the population protocol framework. Intuitively, a population protocol for this task should enable every node to tell “friend” from “foe”. A natural solution, and the one pursued in this paper, is a protocol that assigns each node a binary word, i.e., a signature or label, so that nodes from the same community are assigned labels with (very) small Hamming distance, while nodes from different communities receive labels that have (very) large Hamming distance. Such binary labels introduce a notion of similarity between nodes of the graph, in fact behaving like “profiles” that reflect community membership, hence the phrase Community Sensitive Labeling we use to refer to our approach. Actually, given the very-restrictive computational model we consider and given its probabilistic nature, we slightly relax the above requirements in two ways: i) We allow the existence of a small (i.e. $o(n)$) subset of “ill-labelled” nodes that may be assigned labels not reflecting their community membership and ii) We accept a small (i.e. $O(1/poly(n))$) probability that the (randomized) protocol might fail. More formally, if $\Delta(x,y)$ denotes the Hamming distance between two binary strings $x$ and $y$, we introduce the following notion of distributed community detection.

Definition 1 (Community sensitive labeling). Let $G = (V,E)$ be a graph, $V_1, V_2$ a partition of $V$ and let $\delta \in (0,1]$. A function $h : V_1 \cup V_2 \to \{0,1\}^m$, for some $m \in \mathbb{N}$, is a $\delta$-community sensitive labeling for partition $(V_1,V_2)$ of $G$ if, for a subset $\tilde{V} \subseteq V$, with $|\tilde{V}| \geq (1-\delta)|V|$, of the nodes, two constants $0 \leq c_1 < c_2 \leq 1$ exist, such that for all $u,v \in \tilde{V}$ we have

$$\Delta(h_u,h_v)\begin{cases} \leq c_1 m & \text{if } i_u = i_v \text{ (Case (i))}, \\ \geq c_2 m & \text{otherwise (Case (ii))}, \end{cases}$$

where $i_u = 1$ if $u \in V_1$ and $i_u = 2$ if $u \in V_2$.

Our algorithm is thus simple, lightweight and communication-efficient: the nodes exchange $O(n \log(n))$ bits of information, which is sublinear in the size of the graph if the graph is dense. Moreover, CSL meets all requirements of the (random) discrete-time population protocol model (see for instance [3]): In particular, nodes are anonymous and the updating rule requires neither a global clock nor a local labeling of link ports.

We analyze protocol CSL over the class of $(n,d,b)$-clustered regular graphs $G = ((V_1,V_2), E)$ [10] [17].

Definition 2 (Clustered regular graphs). Let $n \geq 2$ be an even integer and $d$ and $b$ two positive integers such that $b < d < n$. An $(n,d,b)$-clustered regular graph $G = ((V_1,V_2), E)$ is a graph

\[\text{where } i_u = 1 \text{ if } u \in V_1 \text{ and } i_u = 2 \text{ if } u \in V_2.\]
over node set \( V = V_1 \cup V_2 \), with \(|V_1| = |V_2| = n/2\) and such that: (i) Every node has degree \( d \) and (ii) Every node in \( V_1 \) has \( b \) neighbors in \( V_2 \) and every node in \( V_2 \) has \( b \) neighbors in \( V_1 \).

Our main result can be informally stated as follows (see Theorem 4 for a rigorous claim). Assume \( G = ((V_1, V_2), E) \) is an \((n, d, b)\)-clustered regular graphs, such that \( d \epsilon^2 / \log^2 n \gg b \) for some \( \epsilon = \epsilon(n) > 0 \) and such that the subgraphs induced by \( V_1 \) and \( V_2 \) are good expanders (equivalently, that the third largest eigenvalue of the transition matrix is at most a fixed constant bounded away from 1). Then, we prove that, if we run CSL on \( G \) for \( \Theta(\epsilon^{-1} n \log^2 n) \) rounds, the labels \( h_u \) are, with high probability\(^4\) (w.h.p.), a \( \delta \)-community sensitive labeling with \( c_1 = O(\epsilon) \), \( c_2 = 1/2 - O(\epsilon) \) and \( \delta = O(\sqrt{\epsilon}) \).

To implement the protocol, nodes require \( \Theta(\epsilon^{-1} \log^2 n) \) local memory and they set their respective labels within \( \Theta(\epsilon^{-1} \log^2 n) \) activations.

The class of graphs to which our analysis applies includes w.h.p. graphs sampled from the regular stochastic block model \([10, 17, 41]\) with appropriate parameters. The regular stochastic block model is defined as follows: given parameters \( a(n) \) and \( b(n) \), a random graph on \( n \) nodes is obtained by partitioning nodes into two equal-sized communities \( V_1 \) and \( V_2 \) and then sampling a random \( a(n)\)-regular graph over each of \( V_1 \) and \( V_2 \). A random \( b(n)\)-regular graph is then sampled between \( V_1 \) and \( V_2 \). The final graph is the union of the two edge sets. This model can be instantiated in different ways depending on how we sample the random regular graph (for example, via the uniform distribution over regular graphs, or by taking the disjoint union of random matchings). Thus, by definition, every graph sampled from the regular stochastic block model with parameters \( a(n) \) and \( b(n) \) is a \((n, a(n)+b(n), b(n))\)-clustered regular graph according to our definition. If \( a(n)/b(n) \gg \log^2 n \) then the condition that we require on the third largest eigenvalue is satisfied w.h.p. (see \([13, 17]\)), and in fact much stronger conditions are known.

An overview of the main new technical contribution of our analysis is deferred to Section 4.4. Here, we briefly discuss the two major technical challenges we need to tackle, namely, the (random) sparseness of the averaging process and the “asynchronicity” of the node labeling process.

The averaging process implemented by our protocol is a time-inhomogeneous Markov chain, which can be described by a sequence of multiplications between (very-sparse) stochastic matrices (see Section 4.3); that is, the state after \( t \) rounds is of the form \( W_1 \cdots W_1 \cdot x \) where \( x \) is the initial state and the \( W_i \) are i.i.d. samples from a certain distribution of sparse matrices. For a given initial state \( x \), it is easy to compute the expectation, over the choice of the matrices, of \( W_1 \cdots W_1 \cdot x \), but this calculation is not useful without a second-moment bound. Our first contribution, discussed in Section 4.3, is such a second-moment calculation, which proceeds via a double induction in which we keep track of the projection of the state on the indicator vector of the cut and the projection of the state on the space orthogonal to the cut.

This calculation shows that for every round \( t \) of the order of \( n \log n \) there is a good probability that most nodes, if they set their label according to the state at round \( t \), would set a label consistent with the community that they belong to. If we had a synchronous protocol, we would be done.

Nodes, however, are not aware of how many (global) rounds have elapsed, and they fix their label according to a count of the number of times they have been activated. All we can say is that w.h.p. all nodes will set their label in a round, say, between \( t_1 \) and \( 2t_1 \), where \( t_1 = \Theta(n \log n) \).

Our next result is to show that, during such a time window, it is true at every round that most nodes are “good” at that round, meaning that if they were to set their label at that round, they would do so consistently with the community they belong to. This cannot be obtained

\(^4\)A sequence of events \( \mathcal{E}_n, n = 1, 2, \ldots \) holds with high probability if \( P(\mathcal{E}_n) = 1 - \mathcal{O}(1/n^\gamma) \) for some positive constant \( \gamma > 0 \).
by taking a union bound over rounds, because the probability that a particular round has the
above property is only $1 - \epsilon$ for constant $\epsilon$, and the number of rounds is order of $n \log n$, and a
more subtle argument is required.

This is not yet enough for our purposes, because the time at which a node sets its label is
not independent of its state, and so it could be that even though at every round between $t_1$ and
$2t_2$ most nodes are “good” (and so most nodes are good in most rounds) all nodes choose a label
at one of the few rounds in which they are not good. To overcome this difficulty, we have to
show that most nodes are good at all times between $t_1$ and $2t_1$, which requires an additional
idea.

2 Related work

Several rigorous analysis of community detection algorithms focus over the so called stochastic
block models. Such models offer a popular framework for the probabilistic modelling of graphs
that present good clustering or community properties. They have been studied in a number of
areas, including computer science [12, 39, 38], probability [41], statistical physics [22] and social
sciences [30]. In the simplest version of the stochastic block model, we have a random graph
$G_{n,p,q}$ consisting of $n$ nodes and an edge probability distribution defined as follows: The node
set is partitioned into two subsets $V_1$ and $V_2$, each of size $n/2$; edges linking nodes belonging to
the same partition appear in $E$ independently at random with probability $p = p(n)$, while edges
connecting nodes from different partitions appear with probability $q = q(n) < p$.

As described in the previous subsection, we consider a natural variant of the stochastic block
models, the so called regular stochastic block model [10, 17, 41], where nodes have same inner
and out degree.

The literature related to community detection and stochastic block models is quite vast.
Thus, after a concise review of the general results for the reconstruction problem, we mainly
focus on those contributions concerning distributed approaches.

**General results for block reconstruction.** In general, the reconstruction problem has been
studied extensively using a multiplicity of techniques, which include combinatorial algorithms
[25], belief propagation [22] and variants of it [42], spectral-based techniques [39, 21], Metropolis
approaches [32], and semidefinite programming [11], among others. Spectral methods typically
consider the eigenvector associated to the second eigenvalue of the adjacency matrix $A$ of $G$, or
the eigenvector corresponding to the largest eigenvalue of the matrix $A - \frac{d}{n} J$ [12, 20, 21, 39],
since these are correlated with the hidden partition. More recently spectral algorithms have been
proposed [2, 21, 40, 36, 15] that find a weak reconstruction even in the sparse, tight regime.
Unlike the distributed setting, where the existence of light-weight protocols [29] is the main
issue (even in non-sparse regimes), in centralized setting strong attention has been devoted to
establishing sharp thresholds for weak and strong reconstruction. Define $a = np$ as the expected
internal degree (the number of neighbors that each node has on the same side of the partition)
and $b = nq$ as the expected external degree (the number of neighbors that each node has on the
opposite side of the partition). Decelle et al. [22] conjectured that weak reconstruction is possible
if and only if $a - b > 2\sqrt{a + b}$. This was proved by Massoulie and Mossel et al. [40, 38, 41].
Strong recovery is instead possible if and only if $a - b > 2\sqrt{a + b + \log n}$ [4]. Versions of the
stochastic block model in which the random graph is regular have also been considered [41, 17].
In particular Brito et al. [17] show that strong reconstruction is possible in polynomial-time
when $a - b > 2\sqrt{a + b - 1}$.

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[A] $A$ is the adjacency matrix of $G$, $J$ is the matrix having all entries equal to 1, $d$ is the average degree and $n$ is the number of nodes.
Distributed eigenvector computation. In [33], Kempe and McSherry propose a synchronous distributed algorithm for spectral graph analysis. The algorithm works for any graph and, when executed over a clustered graph with a fixed planted partition (such as the random graphs sampled from $G_{n,p,q}$), it also computes the correct bipartition. On the other hand, this protocol is far from simple, it definitely does not work in the population protocol model, and, moreover, its convergence time is \textit{linear in the mixing time of the input graph}. This implies that, when the protocol is run over clustered graphs with a small cut between the two unknown clusters, its parallel convergence time can be superlogarithmic in $n$, or even $n^{\Omega(1)}$.

Label propagation algorithms (LPA). LPA [46] are simple protocols based on majority updating rules [6] and have been applied to several problems including community detection. Several papers present experimental results for such protocols on specific classes of clustered graphs [9, 37, 40]. To the best of our knowledge, the only rigorous analysis of LPA on planted partition graphs is the one presented in [33], where the authors propose and analyze an LPA on the stochastic block model $G_{n,p,q}$ for highly-dense topologies. The proposed algorithm requires every node to exchange messages with all its neighbors in the same round and its analysis considers $p = \Omega(1/n^{1/4-\varepsilon})$ and $q = O(p^2)$, a parameter range in which dense clusters of constant diameter separated by a sparse cut occur w.h.p. (for short \textit{w.h.p}). In this restricted setting characterized by a polynomial gap between $p$ and $q$, simple combinatorial and concentration arguments show that the protocol converges in constant expected time.

\textit{Averaging dynamics}. In [10], Becchetti et al. present a synchronous \textit{averaging dynamics} for community detection on the class of $(n,d,b)$-clustered regular graphs and that of almost-regular clustered graphs (a more general class including the stochastic block model). As in our protocol, every node $v$ has an initial value randomly set to $-1$ or $+1$ with equal probability. In each subsequent round $t \geq 1$, every node $v \in V$ updates its value to the average of the values of its neighbors at the end of the previous round. Running the averaging dynamics on any $(n,d,b)$-clustered regular graphs with $1 - 2b/d \gg \max \{\lambda_3, \lambda_n\}$ (where $\lambda_i$ denotes the $i$-th eigenvalue of the graph), the authors prove that if nodes are also equipped with a simple cluster coloring criterion, the process converges to a stable regime where all nodes are properly colored, w.h.p.: Nodes in the same cluster get the same color while any two nodes from different clusters get different colors. The convergence time is $O(\log n)$. They also prove that a slightly weaker cluster coloring is guaranteed by the same protocol in the class of almost-regular graphs including the stochastic block model with parameters $p(n) \gg \log n$ and $q(n)$ so that $(p - q) \gg n^{-2} \sqrt{p + q}$. Observe that the kind of community detection achieved by the averaging dynamics is stronger than community sensitive labeling obtained by our population protocol since the latter achieves a proper 2-coloring of the two hidden communities. Moreover, the class of clustered graphs they consider is wider than ours. On the other hand, in order to perform the local averaging rule, the required interaction model is the \textit{synchronous LOCAL} one [44]: At every round, each node in parallel must exchange informations with all their respective neighbors. Hence, despite its simplicity, the averaging dynamics does definitely not work in the population protocol model and, especially in non-sparse graphs, it is not efficient in terms of communication costs. We here remark that in several network scenarios (like the ones arising from biological systems and from opportunistic networks), it is not reasonable to assume the existence of a global clock and/or the simultaneous “presence” of all neighbors of an agent in a given round: Such features are in fact some of the main reasons to adopt population protocols for such applications. As a rough comparison between the work in [10] and the one presented here, one may consider

\footnote{Recall that we say that a sequence of events $\mathcal{E}_n$, $n = 1, 2, \ldots$ holds \textit{with high probability} if $P(\mathcal{E}_n) = 1 - O(1/n^\gamma)$ for some positive constant $\gamma > 0$.}

\footnote{Informally speaking, a dynamics is a lightweight protocol based on an elementary updating rule.}

\footnote{We write $f(n) \gg g(n)$ for $f(n) = \Omega(g(n))$.}
the latter population protocol as a (random) “sparsification” and “de-synchronization” of the averaging dynamics analyzed in [10]. As remarked in the previous subsection, the combination of “sparsification” and “de-synchronization” in the averaging process yields a set of issues the analysis of which represents our major technical contribution. In this regard, in Subsection 2.1 we review some previous analyses which are, in some respects, related to our.

Community detection through load balancing. In [47], Sun and Zanetti provided a first important step toward the design of a light-weight protocol for community detection to clustered graphs that works for a “sparse” interaction model: At every round, there is a random perfect matching (locally computed by the nodes) that defines the active links where communication can take place. As in [10] and in this paper, the protocol is based on averaging the values of the nodes \( u \) and \( v \) for every link of the current random matching and it makes use of a more sophisticated local coloring criterion. They consider a wider class of clustered graphs than ours and, for this class, they prove that their protocol construct a good coloring with probability larger than an absolute constant; we remark that, unlike ours, their current analysis gives no performance results with good concentration of probability. We believe that running \( \Theta(\log n) \) independent instances of their protocol and using a labeling criterion similar to ours, their protocol can be adapted to achieve community sensitive labeling for clustered regular graphs, w.h.p. Nonetheless, the latter vectorial version of their protocol would still leave the following crucial differences between their work and ours. Their algorithm does not work in the asynchronous population protocol model, since nodes need to share a global clock and require a local neighborhood labeling (for instance to get a random perfect matching at every round). The asynchronous behaviour of the nodes is one of the main technical issues we solved in the analysis of our protocol (see Section 4). Sun et al. prove that the product of \( O(\log n) \) random matrices behave similarly to its expectation, however this does not suffice to prove that the projection on the second eigenvector keeps sufficiently aligned to the its initial value after multiplying it for \( O(n \log n) \) random matrices: in order to achieve this goal we have to take a much deeper look at the behaviour of the averaging population protocol, as explained in Section 4.

2.1 Other potential approaches

The major technical problem one encounters in trying to sparsify the AVERAGING dynamics of [10] and to adapt it to the population protocol model is the fact that the dynamics evolves according to a product of random matrices. While nowadays the theory of random matrices provides some powerful tools for dealing with sums of matrices [13], the situation is still technically poor when dealing with products of singular matrices. Hence, our ad-hoc analysis in Section 4 carefully exploits the dynamics’ linear structure. In the following we discuss similar issues that have been tackled in previous works. Whether these previous approaches can be adapted or combined to improve our analysis is an intriguing open problem.

Khandekar et al. In [34], the following centralized algorithm is analyzed. A sequence of perfect matching \( (M(t))_{t \geq 1} \) is given, which satisfies a certain property (that we omit for brevity’s sake). At the outset, the vector of nodes’ values \( x \) is initialized to a random unitary vector orthogonal to \( \mathbf{1} \). Then, at each following (synchronous) round \( t \), each pair of nodes matched in \( M(t) \) averages their values. In [34] the authors leverage on a simple potential function argument to show that the nodes’ average value is close to the projection of \( x \) on \( \mathbf{1} \). The use of an analogous potential function for estimating the projection \( x \) on the eigenspace of the first two eigenvectors may circumvent some of the technical difficulties which arises, for example, in trying to adapt the analysis of [10].

Jain et al. Another related analysis is that of Oja’s algorithm given in [31]. Oja’s algorithm is a popular streaming algorithm for PCA of a matrix \( A \) where the initial vector \( w^{(0)} \) is iteratively
updated according to the rule \( \mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} + \eta^{(t)} A^{(t)} \mathbf{w}^{(t)} \) where \( \eta^{(t)} \) is a suitable coefficient and \( \mathbf{E}[A^{(t)}] = A \). In our case, let us consider the Laplacian \( L \) of the underlying communication graph \( G = (V, E) \) expressed in terms of the incidence matrix \( B \) (in formulas, \( L = BB^{\top} \)), and let \( B_{(i,j)} \) be the adjacency matrix associated to edge \( (i, j) \) (i.e., \( B_{(i,j)} \) is zero everywhere excepts that at the entries \( i, j \) and \( j, i \)). If we consider the random matrix \( B \) sampled uniformly at random from \( \{ B_{(i,j)} \}_{(i,j)\in E} \) and set \( \eta^{(t)} = \frac{1}{t} \) for every \( t \), we get exactly the averaging population protocol we consider. However, again, it’s not clear how to adapt the analysis in [31] to obtain sufficiently good bounds on the projection of the initial random vector \( \mathbf{x} \) on the space of the first and second eigenvectors.

3 The averaging-based population protocol

In this section we propose a population protocol, called \( \text{CSL}(T, m) \), for community sensitive labelling. The simple structure of the protocol, described in Algorithm 1, executes \( m \) interleaved independent copies of an elementary procedure called \( \text{Averaging}(T) \), shown in Algorithm 2.

In \( \text{Averaging}(T) \), each node is equipped with a community sensitive label vector \( \mathbf{h}_u \in \{-1, 1\} \), a value (also called state) \( \mathbf{x}_u \in [-1, 1] \) (see Algorithm 2), a local time counter \( t_u \) which stores the node’s current number of activations and, finally, nodes keep a time upper bound \( T \). At each round, the two interacting nodes \( u \) and \( v \) exchange their values \( \mathbf{x}_u \) and \( \mathbf{x}_v \). The first time node \( u \) interacts, it initializes its value \( \mathbf{x}_u \) to a value chosen independently and u.a.r. in \( \{-1, +1\} \) while its label \( \mathbf{h}_u \) is set to \text{null}. After exchanging \( \mathbf{x}_u \) and \( \mathbf{x}_v \), \( u \) and \( v \) set \( \mathbf{x}_u, \mathbf{x}_v := (\mathbf{x}_u + \mathbf{x}_v)/2 \). Moreover, each of the two active nodes, say \( u \), checks whether it has made \( T \) activations, namely whether \( t_u = T \) (clearly, this event may not happen at the same round for the two active nodes): If this is the case (say this happens for node \( u \)), then \( \mathbf{h}_u \) takes \(-1 \) or \( +1 \) depending on the sign of the current value of \( \mathbf{x}_u \).

**Algorithm 1:** The pseudo-code of Protocol \( \text{CSL}(T, m) \) given for node \( u \in V \).

```plaintext

CSL(T, m)
Let \((u, v)\) be the active edge:
if \( u \) is not initialized then
  for \( j = 1 \) to \( m \) do
    set \( t_u(j) = 0 \), \( \mathbf{h}_u(j) := \text{null} \) and
    \( \mathbf{x}_u(j) := \text{rnd}(-1, +1) \)
  Jointly\(^4\) (with \( v \)) choose comp.
  \( j := \text{rnd}(1, \ldots, m) \)
  Receive \( \mathbf{x}_v \)
  \( \mathbf{x}_u(j) := (\mathbf{x}_u(j) + \mathbf{x}_v(j))/2 \)
  if \( t_u(j) < T \) then
    \( t_u(j) := t_u(j) + 1 \)
  if \( t_u(j) = T \) then
    \( \mathbf{h}_u(j) := \text{sgn}(\mathbf{x}_u(j)) \)
```

**Algorithm 2:** \( \mathbf{x}_u \) is the value that is updated by node \( u \). \( t_u \) is a clock counter local to \( u \). It is updated until the value \( T \). At this point, the bit \( \mathbf{h}_u \) is set. From this point onward, \( \mathbf{x}_u \) will continue to be updated, but \( \mathbf{h}_u \) is no longer modified.

```plaintext

Averaging(T)
Let \((u, v)\) be the active edge:
if \( u \) is not initialized then
  set \( t_u = 0 \), \( \mathbf{h}_u := \text{null} \) and
  \( \mathbf{x}_u := \text{rnd}(-1, +1) \)
Receive \( \mathbf{x}_v \)
\( \mathbf{x}_u := (\mathbf{x}_u + \mathbf{x}_v)/2 \)
if \( t_u < T \) then
  \( t_u := t_u + 1 \)
if \( t_u = T \) then
  \( \mathbf{h}_u := \text{sgn}(\mathbf{x}_u) \)
```

\(^4\)This instruction can be implemented using a simple strategy: \( u \) and \( v \) generate random values \( j_u \in [m] \) and \( j_v \in [m] \) exchange them and then set \( j = j_u + j_v \mod m \).

The reader may notice that the components \( \mathbf{x}_u \) are updated at every activation of \( u \) while the components of \( \mathbf{h}_u \) (yielding the community sensitive labeling) are set once and for all after \( T \) updates.

7
In CSL\((T, m)\) instead, the memory of node \(u\) contains \(m\) copies of AVERAGING\((T)\): So, each node is equipped with a community sensitive label vector \(h_u \in \{-1, 1\}^m\), an averaging state vector \(x_u \in [-1, 1]^m\), a local time vector counter \(t_u \in \mathbb{N}^m\) and a time upper bound \(T\) (see Algorithm \(\ref{algorithm:averaging}\)). At each round, additionally to exchanging their values \(x_u\) and \(x_v\), the two interacting nodes \(u\) and \(v\) also agree on what random component \(j \in [m]\) to update: \(u\) and \(v\) set \(x_u(j), x_v(j) := (x_u(j) + x_v(j))/2\) and store in their respective counters \(t_u(j)\) and \(t_v(j)\) the current number of activations on component \(j\). Also, the first time node \(u\) interacts, it independently initializes every component \(j \in [m]\) of vectors \(x_u\) and \(h_u\) as in AVERAGING\((T)\).

In our proposed solution (Theorem \(\ref{theorem:main}\)), we have \(m = 10\varepsilon^{-1}\log n\) and \(T = \alpha \log n\) where \(\alpha\) is an absolute constant which is determined by the bound assumed on \(\lambda_3\) (see Definition \(\ref{definition:good-expansion}\)), and \(\varepsilon\) is a parameter which determines how many nodes will be misclassified by the algorithm.

### 4 Protocol analysis and the main result

In this section, we analyze protocol CSL\((T, m)\) on the class of regular clustered graphs defined in the introduction. Our analysis applies to clustered regular graphs having a relatively small cut between \(V_1\) and \(V_2\), meaning that \(b/d\) is small, and a good expansion inside the two clusters, a property that is equivalent to a bound on the third largest eigenvalue of the transition matrix of \(G\).

If \(A\) is the adjacency matrix of \(G\), let \(P = \frac{1}{d} A\) be the transition matrix of the random walk of \(G\). Then \(P\) is a symmetric matrix with \(n\) real eigenvalues \(1 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\), and it is easy to see (cf. \(\ref{lemma:eigenvectors}\)) that, if \(\lambda_3 < 1 - b/d\), then the indicator vector of the cut \(\chi := 1_{V_1} - 1_{V_2}\) is an eigenvector of \(\lambda_2\), and \(\lambda_2 = 1 - b/d\). We are interested in graphs with a large gap between \(\lambda_3\) and \(\lambda_2\) and, specifically, we will refer to the following definition.

**Definition 3** (Clustered regular graphs with good inner expansion). An \((n, d, b)\)-clustered regular graph \(G = ((V_1, V_2), E)\) is said to have good inner expansion if, considering of the spectrum of its transition matrix \(P\), it holds that \(\lambda_3 \leq 1/2\).

### 4.1 The main theorem and an overview of the analysis

In this section we prove the main result of our paper. Specifically, Theorem \(\ref{theorem:main}\) below follows from Theorem \(\ref{theorem:inductive}\) (Section 4.2) and from Lemma \(\ref{lemma:inductive}\) (Section 4.3).

**Theorem 4.** Assume \(G\) is an \((n, d, b)\)-clustered regular graph with good inner expansion, such that \(\frac{d\epsilon}{\log n} \gg b\) for some \(\epsilon = \epsilon(n) > 0\). Then, w.h.p., protocol CSL\((10\log n, 10\epsilon^{-1}\log n)\) performs a \(\delta\)-community sensitive labeling of \(G\) (according to Definition \(\ref{definition:community-sensitive}\)), with \(c_1 = 4\epsilon\), \(c_2 = 1/6\) and \(\delta = \sqrt{\epsilon}\).

We next outline the flow and discuss the main ingredients of the analysis that brings to the above result. In Section 4.2 we define the goal that we wish the basic averaging protocol (Algorithm \(\ref{algorithm:averaging}\)) to achieve: To have each node \(u\) assign itself a binary value \(h_u\), such that, except for a small fraction of nodes, nodes in the same cluster have \(1 - o(1)\) probability of having the same value, and nodes in different clusters have \(\Omega(1)\) probability of having different values. By running \(\mathcal{O}(\log n)\) interleaved, mutually-independent copies of a process that has such properties (that is, by running Algorithm \(\ref{algorithm:averaging}\), each node \(u\) assigns itself an \(\mathcal{O}(\log n)\)-bit label \(h_u\) such that, except for a small fraction of nodes, w.h.p. nodes in the same cluster have labels with relative

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\(\frac{d\epsilon}{\log n} \gg b\) for some \(\epsilon = \epsilon(n) > 0\). Then, w.h.p., protocol CSL\((10\log n, 10\epsilon^{-1}\log n)\) performs a \(\delta\)-community sensitive labeling of \(G\) (according to Definition \(\ref{definition:community-sensitive}\)), with \(c_1 = 4\epsilon\), \(c_2 = 1/6\) and \(\delta = \sqrt{\epsilon}\).

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Hamming distance $o(1)$ and nodes in different clusters have relative Hamming distance $\Omega(1)$. According to Definition 1, we refer to such an outcome as community sensitive labeling.

As a warm-up to the analysis of Algorithm 2 in Section 1.3, we calculate the expected state after $t$ global steps given a fixed initial state $x \in \{-1, 1\}^n$. We see that for most choices of the initial state, the expected state after $n \log n$ steps is such that most nodes in the same cluster have the same sign; furthermore, for almost half of the initial states, most states in different clusters have different signs. This calculation is helpful to gain intuition about the probabilistic process, but it does not have any implication for what we aim to prove, in the absence of a second-moment bound. To see why the expected state can be quite far from the typical state, consider that if we take the expectation over the initial state $x$ as well, then the expected state is always the all-zero vector, at each time $t$.

In Section 4.3, we study the expected distance of the state at time $t$ from the projection of the initial state onto the all-one vector and to the indicator of the cut, and we show that, after $t \gg n \log n$ steps, the expected distance of this distance is at most $O(tb/nd)$, which is small if $b \ll d/\log n$. This bound is enough to argue that, if $b \ll \varepsilon^2 d/\log n$, there is a time $t_1 = \Theta(n \log n)$ at which there is a probability at least $1 - \varepsilon$ that, for all but an $\varepsilon$ fraction of nodes, nodes on the same clusters have the same sign; and there is a probability at least $1/2 - \varepsilon$ that, for all but an $\varepsilon$ fraction of nodes, nodes in different clusters have different signs. This would be enough to complete our analysis if we could make sure that all nodes set their value of $h$ at the same time $t_1$. Because of the asynchronous nature of the protocol, however, different nodes will set their respective value of $h$ at different times, in a window of width $O(n \log n)$.

We show that there is indeed a window of width $O(n \log n)$ during which most nodes are “good”, where being “good” is defined in such a way that if most nodes are good at the time in which they set their values of $h$ then our analysis is complete. Unfortunately, this is not yet enough: the time at which a node sets its value of $h$ is correlated to the random choices of the previous steps, and hence with the states at various times, thus it is not trivial to rule out that, although, at all times in the window of interest, most nodes are good, each node sets its value of $h$ at a time in which it is not good. To complete our analysis, we need to prove the stronger statement that most nodes are good at all times in the window of interest. We are able to prove this as well, under the stronger assumption that $b \ll d/\log^2 n$. Combining these ingredients allows us to complete the analysis of Algorithm 2.

### 4.2 A sufficient condition for community sensitive labeling

We first consider the behavior of algorithm AVERAGING($T$) on a graph $G = (V, E)$ where $V$ has a sparse cut $(V_1, V_2)$ that we wish to discover. For every node $u$, we define

$$p^T(u) = \begin{cases} P(h_u \neq \text{sgn}(\sum_{v \in V_1} x_v)) & \text{if } u \in V_1, \\ P(h_u \neq \text{sgn}(\sum_{v \in V_2} x_v)) & \text{if } u \in V_2, \end{cases}$$

where the randomness is over the choice of $x \sim \{-1, 1\}^n$ and over the execution of Algorithm 2 starting from the state $x$. We omit parameter $T$ from $p^T(u)$ when clear from context. We remark that $p(u)$ depends solely on the graph $G$ and the partition $(V_1, V_2)$.

To understand the point of this definition, consider the extreme case in which the cut $(V_1, V_2)$ is empty, and $V_1$ and $V_2$ induce connected graphs. Then the averaging process AVERAGING($T$) will converge to a global state in which all nodes in $V_1$ have a local state close to the average $\frac{2}{n} \sum_{v \in V_1} x_v$ and all nodes in $V_2$ have a state close to the average $\frac{2}{n} \sum_{v \in V_2} x_v$. Since $h_u$ is equal to the sign of the state of $u$ at a (random) round $\tau_u$, then if $T$ is chosen so that $\tau_u$ is large enough, we would expect $h_u$ to be typically equal to the sign of $\frac{2}{n} V_1$ if $u \in V_1$ and to the sign
of $\frac{2}{\pi} V_2$ if $u \in V_2$, and so $p(u)$ is small for all $u$. Then, a natural intuition here is that the above scenario is true for graphs with a sparse, rather than empty, cut, provided that the subgraphs induced by $V_1$ and $V_2$ are good expanders.

Now, considering $m$ independent copies of AVERAGING($T$), i.e. protocol CSL($T,m$) (for a sufficiently large $m$), the following theorem shows that the protocol performs community-sensitive labeling that works on the set of lucky nodes for which $p(u)$ is smaller than $\varepsilon$. We thus define the corresponding set of unlucky nodes

$$U^{\varepsilon,T}_{G, (V_1, V_2)} := \{ u \mid p^T(u) \geq \varepsilon \}$$

(we omit the parameters $T$, $G$ and $(V_1, V_2)$ from $U^{\varepsilon,T}_{G, (V_1, V_2)}$ when clear from context).

**Theorem 5.** Let $G = (V, E)$ be a graph, $V_1, V_2$ be a partition of $V$ and fix $\varepsilon \in (0, \frac{1}{12}]$. Then, CSL($T, 10\varepsilon^{-1} \log n$) computes a $\delta$-community sensitive labeling for partition $(V_1, V_2)$ of $G$ (according to Definition 1), with $c_1 = 4\varepsilon$, $c_2 = 1/6$ and $\delta = |U^{\varepsilon,T}|/n$.

**Proof.** The proof of the theorem relies on the mutual independence among the components of any label $h(\cdot)$ and some standard arguments. We remark that the aforementioned independence crucially depends on the fact that CSL($T, m$) updates one component per round (i.e. per interaction): The evolution of $x(j_1)$ and $x(j_2)$ depends solely on the respective initial vector values (which are independent), and on the sequence of sampled edges which update component $j_1$ and $j_2$ (which are independent conditional on their number).

Call $m := \frac{12}{\varepsilon} \log n$ and denote

$$h_{V_i} := (h_{V_i}(1), \ldots, h_{V_i}(m)), \quad \text{where } h_{V_i}(j) := \text{sgn} \left( \sum_{x \in V_i} x_w(j) \right).$$

We first claim that w.h.p. for every node $u \in V_1 \setminus U^\varepsilon$, $\Delta(h_u, h_{V_1}) \leq 2\varepsilon m$. Indeed, observe that by definition of $U^\varepsilon$,

$$E[\Delta(h_u, h_{V_1})] = p(u)m \leq \varepsilon m.$$ 

Since the $m$ components are mutually independent, the Chernoff bound [21] implies that $\Delta(h_u, h_{V_1}) \leq 2\varepsilon m$, w.h.p. A union bound over nodes in $V_1 \setminus U^\varepsilon$ implies the claim.

Henceforth, assume that $\Delta(h_u, h_{V_1}) \leq 2\varepsilon m$ for each $u \in V_1 \setminus U^\varepsilon$ and a similar claim for all nodes $v \in V_2 \setminus U^\varepsilon$.

Let us first consider the case $u, v \in V_1 \setminus U$ (the case $u, v \in V_2 \setminus U$ is similar). By triangle inequality, we get Case (i) of Definition 1

$$\Delta(h_u, h_v) \leq \Delta(h_u, h_{V_1}) + \Delta(h_{V_1}, h_v) \leq 4\varepsilon m.$$

Now, let us consider the case $u \in V_1 \setminus U$ and $v \in V_2 \setminus U$ (the symmetric case is similar). Since the initial values of $x_w(j)$ ($w \in V_1 \cup V_2$) are chosen independently and uniformly at random in $\{-1, 1\}$, a simple symmetry argument shows that the probability of the event “$\text{sgn}(\sum_{x \in V_1} x_w(j)) = \text{sgn}(\sum_{x \in V_2} x_w(j))$” is $1/2$. Hence, $E[\Delta(h_{V_1}, h_{V_2})] = m/2$ and from Chernoff bounds we get that

$$\Delta(h_{V_1}, h_{V_2}) \geq \frac{m}{3},$$

with all but a probability exponentially small in $m$. Henceforth, conditioning on the event “$\Delta(h_{V_1}, h_{V_2}) \leq \frac{m}{3}$”, by triangle inequality, we get

$$\Delta(h_u, h_v) \geq \Delta(h_{V_1}, h_{V_2}) - \Delta(h_u, h_{V_1}) - \Delta(h_{V_2}, h_v)$$

10
\[ \geq \frac{m}{3} - 2\varepsilon m - 2\varepsilon m \geq \frac{m}{6}. \]

We thus have Case (ii) of Definition 1 concluding the proof. \hfill \Box

In the following subsections we will prove that if \( G = (V_1, V_2) \) is a \((n, d, b)\)-clustered regular graph with good inner expansions (i.e. \( \lambda_3 \leq 1/2 \) - see Definition 3), \( b \leq \varepsilon^3 d / \log^2 n \), and the protocol parameter \( T \) is fixed to \( 10 \log n \), then at least \( (1 - \mathcal{O}(\sqrt{\varepsilon})) \cdot n \) nodes \( u \) are lucky, i.e. \( |U^\varepsilon| = \mathcal{O}(\sqrt{\varepsilon} n) \), and thus the protocol finds a \((1 - \mathcal{O}(\sqrt{\varepsilon}))\)-community sensitive labeling for \( G \).

### 4.3 The evolution of the state: Expected state after \( t \) rounds

We here analyze the expected behaviour of Algorithm 2 over any fixed \((n, d, b)\)-clustered regular graph \( G = ((V_1, V_2), E) \) whose transition matrix \( P \) is such that \( \lambda_3 < 1 - \frac{b}{d} \) and, thus, \( \chi \) is an eigenvector of \( \lambda_2 = 1 - \frac{b}{d} \).

Let \( x \in \{-1, 1\}^n \), and let \( x^{(t)} \) be the global state of Algorithm 2 after \( t \) rounds, if initialized with the starting state \( x^{(0)} = x \). In this subsection we will compute \( \mathbb{E}[x^{(t)}] \) as a function of \( x \), and we will see that, for \( t = \Theta(n \log n) \), for most choices of \( x \), most nodes \( u \in V_1 \) satisfy

\[ \text{sgn}(\mathbb{E}[x^{(t)}]) = \text{sgn}(\sum_{v \in V_1} x_v) \]

and the same for most nodes in \( V_2 \).

Although this will not be directly helpful to bound the number of lucky nodes (see Theorem 5), this calculation will allow us to introduce various linear-algebraic facts that will be useful later, and it will provide intuition for the more complicated calculations to follow.

Observe that \( \{x^{(t)} : t \in \mathbb{N}\} \) is a Markov chain defined by the recursion

\[ x^{(t+1)} = W_t \cdot x^{(t)}, \]

where \( W_t \) is the appropriate random matrix that can be defined as follows. Namely, if we define \( A(i, j) \) as the matrix

\[ A(i, j) = I - \frac{(e_i - e_j)(e_i - e_j)^T}{2}, \]

where \( e_i \) and \( e_j \) are the column vectors of the standard base (see (21) and (22) in [16]), then \( W_t \) is the random matrix taking value \( A(i, j) \) with probability \( (1/n)P_{i,j} \).

Notice that random matrices \( \{W_t\}_{t \geq 0} \) are independent and identically distributed and they do not depend on the initial vector. Hence, since from (3) it follows that

\[ x^{(t)} = W_t \cdots W_1 x^{(0)}, \]

when we compute the expectation of \( x^{(t)} \) conditional on the initial vector being \( x \) we get

\[ \mathbb{E}[x^{(t)} | x^{(0)} = x] = \overline{W}^t \cdot x, \]

where we set \( \overline{W} = \mathbb{E}[W_0] \). In what follows, we will omit the conditioning on the initial vector when clear from context.

Since \( P \) is doubly stochastic we have that (see (24) and (25) in [16]),

\[ \overline{W} = \frac{1}{n} P + \left( 1 - \frac{1}{n} \right) I. \]

Observe that \( P \) and \( \overline{W} \) have the same eigenvectors. In particular, if \( (x, \lambda) \) is an eigenvector pair of \( P \), \( 1 - \frac{1 - \lambda}{n} \) is the eigenvalues associated to \( x \) in \( \overline{W} \), meaning that \( 1 \) is an eigenvector of \( \overline{W} \) with eigenvalue 1, \( \chi \) is an eigenvector of eigenvalue \( 1 - \frac{1 - \lambda}{n} \), and all other eigenvectors are orthogonal to \( 1 \) and \( \chi \), and have eigenvalues that are at most \( 1 - \frac{1 - \lambda}{n} \).
We write the initial state as
\[ x = x_\parallel + y + z, \]
where \( x_\parallel = \frac{1}{n} x^T 1 \cdot 1 \) is the component parallel to \( 1 \), \( y = \frac{1}{n} x^T \chi \cdot \chi \) is the component parallel to \( \chi \), and \( z \) is the component orthogonal to \( 1 \) and to \( \chi \). Then, from (1) and the spectrum of \( \mathcal{W} \) discussed above, we get
\[ \mathbb{E} \left[ x^{(t)} \mid x^{(0)} = x \right] = x_\parallel + \left( 1 - \frac{b}{dn} \right)^t y + \mathcal{W}' z \]
and, in particular,
\[ \| \mathbb{E} \left[ x^{(t)} \mid x^{(0)} = x \right] - x - y \| \leq \frac{tb}{dn} \| y \| + \left( 1 - \frac{1 - \lambda_3}{n} \right)^t \| z \|. \]
If \( t \gg \frac{1}{1 - \lambda_3} n \log n \) the second term on the right-hand side is small, and if \( b \ll dn/t \) so is the first term. Under such conditions, for most nodes \( u \), \( \mathbb{E} \left[ x^{(t)} \right]_u \) is close to \( (x_\parallel + y)_u \). Now notice that
\[ (x_\parallel + y)_u = \begin{cases} \frac{1}{n} x^T 1 + \frac{1}{n} x^T \chi = \frac{2}{n} \sum_{v \in V_1} x_v & \text{if } u \in V_1, \\ \frac{1}{n} x^T 1 - \frac{1}{n} x^T \chi = \frac{2}{n} \sum_{v \in V_2} x_v & \text{if } u \in V_2. \end{cases} \]
So, under conditions that make the right-hand-side of (7) small, and provided that \( \sum_{v \in V_1} x_v \) and \( \sum_{v \in V_2} x_v \) are sufficiently large in absolute value, we have that for most nodes \( u \in V_i \) (where \( i = 1, 2 \)) \( \text{sgn}(\mathbb{E} \left[ x^{(t)} \right]_u) = \text{sgn}(\sum_{v \in V_i} x_v) \), which looks similar to our goal of proving that for most nodes \( u \) there is a high probability that \( \text{sgn}(x_u^{(t)} \tau_u) = \text{sgn}(\sum_{v \in V_i} x_v) \), where \( \tau_u \) is the global round at which node \( u \) sets its value \( h(u) \).

4.4 The evolution of the state: Analysis in probability

In this subsection, we first derive a bound in probability analogous to (7), but with the actual vector \( x^{(t)} \) replacing the expected vector \( \mathbb{E} \left[ x^{(t)} \right] \) on the left-hand side (see Lemma 6); according to the previous notation, at each round \( t \), we write
\[ x^{(t)} = x_\parallel + y^{(t)} + z^{(t)}, \]
where \( x_\parallel \) is the component of \( x \) (and of \( x^{(t)} \)) parallel to \( 1 \), \( y^{(t)} \) is the component of \( x^{(t)} \) parallel to \( \chi \) and \( z^{(t)} \) is the component orthogonal to \( 1 \) and \( \chi \). Unlike the calculations in the above subsection, the bound in probability is quite non-trivial.

Then, in order to say that w.h.p. for most of the nodes \( u \) the value of \( x_u^{(t)} \) is close to \( (x_\parallel + y)_u \) when node \( u \) fixes its label, we will have to derive a bound in probability that holds for a whole window of global rounds, rather than for a fixed global round, so that we can replace a fixed round \( t \) with different rounds \( \tau_u \) for the various nodes (see Claim 11 and Lemma 9).

4.4.1 A bound on the second moment of \( \| y^{(t)} + z^{(t)} - y^{(0)} \| \)

We recall that, according to Theorem 4 throughout the section we consider a graph \( G \) which is an \((n, d, b, \varepsilon)\)-clustered regular graph with good inner expansion (Definition 3), such that \( d \gg \varepsilon^3 b \log^2 n \).

Lemma 6 (Second moment analysis). For every round \( t \gg \frac{n}{1 - \lambda_3} \log n \), it holds
\[ \mathbb{E} \left[ \| y^{(t)} + z^{(t)} - y^{(0)} \|^2 \right] \leq O \left( \frac{bt}{dn} \right). \]
The reader may notice that, if \( t \gg n^{1-\lambda_3 \log n} \) and \( b \ll \varepsilon d n/t \), then the previous lemma implies that
\[
\mathbb{E} \left[ \|x(t) - x\| - y(0)\|^2 \right] \overset{(a)}{=} \mathbb{E} \left[ \|y(t) + z(t) - y(0)\|^2 \right] \leq O(\varepsilon),
\]
where in \((a)\) we substituted \((9)\). So, we obtain the goal we declared at the end of the previous section since for example, using Markov inequality, we get
\[
P \left( \|x(t) - x\| - y(0)\| < \varepsilon^{1/4} \right) \geq 1 - O(\sqrt{\varepsilon}),
\]
that is a form of \((7)\) in probability.

The proof of Lemma 6 requires two technical claims and further work: all this is given after the following short discussion on the difficulties that come up in the proof.

We separately bound \( \mathbb{E} \left[ \|W_t \cdots W_1 y(0) - y(0)\|^2 \right] \) and \( \mathbb{E} \left[ \|W_t \cdots W_1 z(0)\|^2 \right] \). As for the former, there is a relatively simple proof by induction on \( t \); as for the latter, one can use the fact that \( z(0) \) is orthogonal to \( \chi \) to see that
\[
\mathbb{E} \left[ \|W_1 z(0)\|^2 \right] \leq \left( 1 - \frac{1 - \lambda_3}{n} \right) \|z(0)\|,
\]
which also suggests an inductive approach. Unfortunately, however, \( W_1 z(0) \) is not orthogonal to \( \chi \) anymore, and so applying \( W_2 \) to it will shrink on average the component orthogonal to \( \chi \) but preserve the component parallel to \( \chi \). To account for it, we set up a more complex induction, in which we keep track, at every round \( t \), of the expectation of the norm squared of the component of \( z(t) \) parallel to \( \chi \) and of the expectation of the norm squared of the component of \( z(t) \) orthogonal to \( \chi \).

**Claim 7.** Let \( W_1, \ldots, W_t \) be a sequence of rounds of AVERAGING(T) which includes \( c \) cross edges and \( t - c \) internal edges. Then, it holds that
\[
\|W_t \cdots W_1 \chi - \chi\|^2 \leq 4c.
\]
Furthermore, if \( W_1, \ldots, W_t \) are chosen randomly according to the AVERAGING(T) process, then it holds that
\[
\mathbb{E} \left[ \|W_t \cdots W_1 \chi - \chi\|^2 \right] \leq 4 \frac{tb}{d} \quad \text{and} \quad P \left( \|W_t \cdots W_1 \chi - \chi\|^2 \geq 8t \frac{b}{d} \right) \leq e^{-\Omega(bt/d)}.
\]

**Proof.** We first observe that
\[
\|W_t \cdots W_1 \chi - \chi\|^2 = \|W_t \cdots W_1 \chi\|^2 - 2\chi^T W_t \cdots W_1 \chi + \|\chi\|^2 \\
\leq 2n - 2\chi^T W_t \cdots W_1 \chi.
\]
Then to complete the proof, notice that if in a round \( \ell \in \{1, \ldots, t\} \) an internal edge \((u, v)\) is selected then
\[
\chi^T W_\ell w = \chi^T w
\]
While, in general, for every vector \( w \) such that \( \|w\|_\infty \leq 1 \), it holds
\[
\chi^T W_\ell w = \chi^T \left( w + \frac{1}{2}(w_u - w_v)1_v + \frac{1}{2}(w_v - w_u)1_u \right) \geq \chi^T w - 2.
\]

From the lemma hypothesis, we can use induction and get that
\[ \chi^T W_t \cdots W_1 \chi \geq n - 2c, \]
which implies the first part of the claim. The furthermore part follows by noting that the expected value of \( c \) after \( t \) rounds of the process is \( bt/d \), and that, for a sufficiently large \( t \), Chernoff bounds easily imply that \( c \) is concentrated around its expectation. \( \square \)

**Claim 8.** Let \( z \) be a vector orthogonal to \( 1 \) and \( \chi \). Then for every \( 0 \leq t \leq nd/b \) it holds that
\[ \mathbb{E} \left[ \| W_t \cdots W_1 z \|^2 \right] \leq \left( 5t \frac{b}{n^2 d} + \left( 1 - \frac{1 - \lambda_3}{n} \right)^t \right) \| z \|^2. \]

**Proof.** Let \( P_\chi \) be the projector on \( \chi \), that is \( P_\chi = \frac{1}{n} \chi \chi^T \) and let \( P_\perp = I - \frac{1}{n} \chi \chi^T \) be the projector on the space orthogonal to \( \chi \). Let us write
\[ W_t \cdots W_1 z = \tilde{y}^{(t)} + \tilde{z}^{(t)} \]
where
\[ \tilde{y}^{(t)} = P_\chi W_t \cdots W_1 z \quad \text{and} \quad \tilde{z}^{(t)} = P_\perp W_t \cdots W_1 z. \]
We will argue by induction that
\[ \mathbb{E} \left[ \| \tilde{y}^{(t)} \|^2 \right] \leq \frac{2t}{d} \| z \|^2, \quad (10) \]
and
\[ \mathbb{E} \left[ \| \tilde{z}^{(t)} \|^2 \right] \leq \left( \frac{3t^2 b}{d^2 n^2} + \left( 1 - \frac{1 - \lambda_3}{n} \right)^t \right) \| z \|^2. \quad (11) \]
The claim will then follow from Pythagoramas’s theorem and the hypothesis \( \frac{b}{md} \leq 1 \).

We will often use the observation that
\[ \chi^T \tilde{z}^{(t)} = 0 \quad \text{and} \quad 1^T \tilde{z}^{(t)} = 1^T P_\perp W_1, \ldots, W_t z = 1^T z = 0. \quad (12) \]
We have
\[ \mathbb{E} \left[ \| \tilde{y}^{(t+1)} \|^2 \right] = \mathbb{E} \left[ \| P_\chi W \tilde{y}^{(t)} + \tilde{z}^{(t)} \|^2 \right] \]
\[ = \mathbb{E} \left[ \| P_\chi W \tilde{y}^{(t)} \|^2 + 2 \tilde{y}^{(t)^T} P_\chi W \tilde{z}^{(t)} + \| P_\chi W \tilde{z}^{(t)} \|^2 \right]. \quad (13) \]
In the following we bound the three terms on the right hand side of the previous equation. As for the first term we have
\[ \mathbb{E} \left[ \| P_\chi W \tilde{y}^{(t)} \|^2 \right] \leq \mathbb{E} \left[ \| \tilde{y}^{(t)} \|^2 \right], \quad (14) \]
since \( P_\chi \) and \( W \) are matrices of spectral norm at most 1.

As for the second term, from (12) and the fact that \( \chi^T \overline{W} = \lambda_2 \cdot \chi^T \), we have
\[ \mathbb{E} \left[ 2 \tilde{y}^{(t)^T} P_\chi W \tilde{z}^{(t)} \right] = 2 \left( \mathbb{E} \left[ \tilde{y}^{(t)} \right] \right)^T \cdot \frac{1}{n} (\chi \chi^T) \cdot \overline{W} \cdot \mathbb{E} \left[ \tilde{z}^{(t)} \right] = 0. \quad (15) \]
As for the third term, observe that if \( W_{u,v} \) is the transition matrix corresponding to the choice of the edge \((u,v)\), then
\[ P_\chi W_{u,v} \tilde{z}^{(t)} = P_\chi \left( \tilde{z}^{(t)} + \frac{1}{2} \left( (\tilde{z}^{(t)}_u - \tilde{z}^{(t)}_v) 1_v + (\tilde{z}^{(t)}_v - \tilde{z}^{(t)}_u) 1_u \right) \right) \]
is an internal edge, and it is equal to $\text{sgn}_n(\chi_u - \chi_v)$. Plugging (14), (15) and (17) into (13) we can prove (10) as follows:

$$ \| \chi \|^2 = \frac{2b}{n^2d} \| \tilde{z} \|^2,$$

where in the last equality we used that $(\tilde{y}^{(t)} + \tilde{z}^{(t)})^T W^T W (\tilde{y}^{(t)} + \tilde{z}^{(t)}) = (\tilde{y}^{(t)} + \tilde{z}^{(t)})^T W (\tilde{y}^{(t)} + \tilde{z}^{(t)})$

because the last expectation is the expectation of the quadratic form of the Laplacian of the cross edges, which is at most $\frac{2}{n} \| \tilde{z} \|^2$ (each node $u$ is selected with probability $\frac{2}{n}$). Therefore, by plugging (14), (15) and (17) into (13) we can prove (10) as follows:

\[
\mathbf{E}\left[\|P_\chi W \tilde{z}^{(t)}\|^2\right] = \frac{1}{4n^2} \mathbf{E}\left[(\tilde{z}_u^{(t)} - \tilde{z}_v^{(t)})^2 :(\chi_u - \chi_v)^2\right] \| \chi \|^2
\]

\[
= \frac{b}{nd} \mathbf{E}\left[(\tilde{z}_u^{(t)} - \tilde{z}_v^{(t)})^2 \mid (u, v) \in E(V_1, V_2)\right]
\]

\[
\leq \frac{2b}{n^2d} \| \tilde{z}^{(t)} \|^2,
\]

where in the last equality we used that

$$ \| \tilde{z}^{(t)} \|^2 = \| P_\perp W_1 \cdots W_t z \|^2 \leq \| z \|^2$$

since the matrices $P_\chi$ and $W_1, \ldots, W_t$ are matrices of spectral norm at most 1.

As for (11), we need two facts. The first fact is that

\[
\mathbf{E}\left[\|\tilde{y}^{(t+1)}\|^2\right] \leq \mathbf{E}\left[\|\tilde{y}^{(t)}\|^2\right] + \mathbf{E}\left[\frac{2b}{n^2d} \| \tilde{z}^{(t)} \|^2\right]
\]

\[
\leq \ldots
\]

\[
\leq \frac{2b}{n^2d} \sum_{i=0}^{t} \mathbf{E}\left[\| \tilde{z}^{(t)} \|^2\right]
\]

\[
\leq \frac{2b}{n^2d} (t + 1) \| z \|^2,
\]

where in the last equality we used that

$$ \| \tilde{z}^{(t)} \|^2 = \| P_\perp W_1 \cdots W_t z \|^2 \leq \| z \|^2$$

since the matrices $P_\chi$ and $W_1, \ldots, W_t$ are matrices of spectral norm at most 1.

As for (11), we need two facts. The first fact is that

\[
\mathbf{E}\left[\|\tilde{y}^{(t+1)}\|^2\right] \geq \mathbf{E}\left[\|P_\chi W \tilde{z}^{(t)}\|^2\right] \geq \left(1 - \frac{4b}{dn}\right) \mathbf{E}\left[\|\tilde{y}^{(t)}\|^2\right],
\]

since with probability $1 - b/d$ we select an internal edge and $W \tilde{y}^{(t)} = \tilde{y}^{(t)}$, and with probability $b/d$ we select an external edge and we have that

$$ \| P_\chi W \tilde{y}^{(t)} \|^2 = \frac{\| \tilde{y}^{(t)} \|^2}{n} \| P_\chi W \chi \|^2
\]

$$ = \frac{\| \tilde{y}^{(t)} \|^2}{n^2} \cdot \chi^T W \chi \chi^T W \chi
\]

$$ = \frac{\| \tilde{y}^{(t)} \|^2}{n^2} \cdot (n - 2)^2 \geq \| \tilde{y}^{(t)} \|^2 \cdot \left(1 - \frac{4}{n}\right).$$

The second fact is that

\[
\mathbf{E}\left[\|\tilde{y}^{(t+1)} + \tilde{z}^{(t+1)}\|^2\right] = \mathbf{E}\left[\|W(\tilde{y}^{(t)} + \tilde{z}^{(t)})\|^2\right]
\]

$$ = \mathbf{E}\left[(\tilde{y}^{(t)} + \tilde{z}^{(t)})^T W^T W (\tilde{y}^{(t)} + \tilde{z}^{(t)})\right]
\]

$$ = (\tilde{y}^{(t)} + \tilde{z}^{(t)})^T W (\tilde{y}^{(t)} + \tilde{z}^{(t)})$$

15
\[ = (\hat{y}(t))^T W \hat{y}(t) + 2(\hat{z}(t))^T W \hat{y}(t) + (\hat{z}(t))^T W \hat{z}(t) \]
\[ \leq \left( 1 - \frac{b}{dn} \right) \|\hat{y}(t)\|^2 + (\hat{z}(t))^T W \hat{z}(t) \]
\[ \leq \left( 1 - \frac{b}{dn} \right) \|\hat{y}(t)\|^2 + \left( 1 - \frac{1 - \lambda_3}{n} \right) \|\hat{z}(t)\|^2. \]  

(20)

Therefore, by combining (19), (20) and the fact that \( \hat{y}(t) \) and \( \hat{z}(t) \) are orthogonal we have

\[ \mathbb{E} \left[ \|\hat{z}(t+1)\|^2 \right] = \mathbb{E} \left[ \|\hat{y}(t+1) + \hat{z}(t+1)\|^2 - \|\hat{y}(t+1)\|^2 \right] \]
\[ \leq \mathbb{E} \left[ \frac{3b}{dn} \|\hat{y}(t)\|^2 + \left( 1 - \frac{1 - \lambda_3}{n} \right) \|\hat{z}(t)\|^2 \right] \]
\[ \leq \mathbb{E} \left[ \frac{6b^2}{d^2 n^2} \|z\|^2 + \left( 1 - \frac{1 - \lambda_3}{n} \right) \mathbb{E} \left[ \|z(t)\|^2 \right] \right] \]
\[ \leq \ldots \]
\[ \leq \left( \frac{3(t + 1)^2 b^2}{d^2 n^3} + \left( 1 - \frac{1 - \lambda_3}{n} \right)^{t+1} \right) \|z\|^2, \]

which concludes the proof.

We can now prove Lemma 6.

**Proof of Lemma 6.** Since \( y(t) \) and \( z(t) \) are orthogonal to \( 1 \) at each round \( t \), we have

\[ y(t) + z(t) = W_t \cdots W_1 y^{(0)} + W_t \cdots W_1 z^{(0)}, \]

which implies

\[ \mathbb{E} \left[ \|y(t) + z(t) - y^{(0)}\|^2 \right] = \mathbb{E} \left[ \|W_t \cdots W_1 y^{(0)} + W_t \cdots W_1 z^{(0)} - y^{(0)}\|^2 \right] \]
\[ \leq 2 \mathbb{E} \left[ \|W_t \cdots W_1 y^{(0)} - y^{(0)}\|^2 + \|W_t \cdots W_1 z^{(0)}\|^2 \right]. \]

It follows that

\[ \mathbb{E} \left[ \|W_t \cdots W_1 y^{(0)} - y^{(0)}\|^2 \right] = \mathbb{E} \left[ \|y^{(0)}\|^2 n^{-1} \cdot \|W_t \cdots W_1 x - x\|^2 \right] \]
\[ = \mathbb{E} \left[ \|y^{(0)}\|^2 n^{-1} \right] \cdot \mathbb{E} \left[ \|W_t \cdots W_1 x - x\|^2 \right] \leq \frac{2b^2}{dn} \]

where in the inequality we used Lemma 7 and

\[ \mathbb{E} \left[ \|y^{(0)}\|^2 \right] = \text{Var} \left( \frac{1}{\sqrt{n}} \sum_{u \in V_1} x_u \right) + \text{Var} \left( \frac{1}{\sqrt{n}} \sum_{u \in V_2} x_u \right) = \frac{1}{2}. \]
Thus, from Claim 8 we finally get
\[
\mathbb{E} \left[ \|W_t \cdots W_1 z^{(0)}\|^2 \right] \leq \left( \frac{3t^2 b^2}{d^2 n^3} + \left( 1 - \frac{1 - \lambda_3}{n} \right)^t \right) \mathbb{E} \left[ \|z^{(0)}\|^2 \right]
\]
\[
\leq \left( \frac{3t^2 b^2}{d^2 n^3} + \left( 1 - \frac{1 - \lambda_3}{n} \right)^t \right) n
\]
\[
\leq O(1) \cdot \left( \frac{bt}{dn} \right)^2 + \frac{1}{n^{O(1)}} \leq O \left( \frac{bt}{dn} \right),
\]
where we used the hypothesis on \( t \) and that
\[
\mathbb{E} \left[ \|z^{(0)}\|^2 \right] = \mathbb{E} \left[ \|x^{(0)}\|^2 \right] - \mathbb{E} \left[ \|x\|^2 \right] - \mathbb{E} \left[ \|y^{(0)}\|^2 \right] = n - \frac{3}{2}.
\]

4.4.2 Dealing with “asynchronicity”

We say that a node \( v \) is \( \varepsilon \)-good at time \( t \) if
\[
(x_v^{(t)} - (x_{\|v\|} + y_v^{(0)}))^2 \leq \frac{\varepsilon^2}{n},
\]
and we call it \( \varepsilon \)-bad otherwise. It follows from the definition that the number of \( \varepsilon \)-bad nodes at time \( t \) is at most \( (x_v^{(t)} - (x_{\|v\|} + y_v^{(0)}))^2 / (\varepsilon^2 / n) \). It thus holds that
\[
\#(\varepsilon\text{-good nodes at time } t) = n - \#(\varepsilon\text{-bad nodes at time } t)
\]
\[
\geq n - \frac{n}{\varepsilon^2} \|x^{(t)} - x\| - \|y^{(0)}\|^2
\]
\[
\overset{(a)}{=} n - \frac{n}{\varepsilon^2} \|y^{(t)} + z^{(t)} - y^{(0)}\|^2.
\]
where in (a) we used (6). If we pick \( t = \mathcal{O} \left( \frac{1}{\varepsilon^2} n \log n \right) \), and our graph is such that \( d \gg b \varepsilon^2 \log n \), then from Lemma 8 it follows that
\[
\mathbb{E} \left[ \|y^{(t)} + z^{(t)} - y^{(0)}\|^2 \right] \leq \varepsilon^4.
\]
Hence
\[
\mathbb{E}[\# \text{ non } \varepsilon\text{-good nodes at time } t] \leq \varepsilon^2 n
\]
and from Markov inequality
\[
\mathbb{P} \left( \text{there are } \geq (1 - \varepsilon)n \text{ nodes that are } \varepsilon\text{-good at time } t \right) \geq 1 - \varepsilon.
\]

If all nodes could set their value of \( h \) at that specific round \( t \), then it would be easy to apply an argument similar to the one that we will use in Lemma 12 below, and prove that most nodes are lucky (according to the definition of (un-)lucky nodes given in Section 4.2).

Unfortunately, the asynchronicity of our model requires us to understand whether nodes are good at all times of a long window. The next lemma gives a bound on the number of nodes that are good in a time-window that is large enough to guarantee that all of them will set the bits of their label within that window (under the hypotheses on \( G \) given at the beginning of the section).
Lemma 9 (Main). If \( t_1 = \Theta\left(\frac{n}{1-\xi_3} \log n\right) \) and \( \varepsilon^4 \gg \frac{b}{(1-\xi_3)\varepsilon} \log^2 n \), then

\[
P\left(\text{there are} \geq (1-\tilde{\varepsilon})n \text{ nodes that are } \varepsilon\text{-good at all times } t_1 \leq t \leq 2t_1 \right) \geq 1 - \varepsilon
\]

where \( \tilde{\varepsilon} = \frac{1}{1-\xi_3} e^{-\frac{2\log(1/\varepsilon)}{\log(1/\varepsilon)}}. \)

The main idea in the proof is to first show that there is probability strictly bigger than \( 1 - \varepsilon \) that, at each time \( t \) between \( t_1 \) and \( 2t_1 \), the number of \( \varepsilon \)-good nodes at time \( t \) is at least \( n(1-\varepsilon)/\log n \) (although the identity of such \( \varepsilon \)-good nodes can change from time to time). Using Lemma 6 we have already seen above that this is true for one fixed time step, but we cannot simply take a union bound, since we have \( n \log n \) time steps and only a \( 1 - \varepsilon \) probability of having the desired outcome in each step. Instead we will reason about how much \( \|y(t) + z(t) - y(0)\|^2 \) can change from time to time, if it is small at time \( t_1 \). Then we will show that this implies that, with probability \( 1 - \varepsilon \), there are at least \( n - \varepsilon n \) nodes that remain \( \varepsilon \)-good through the entire window of time between \( t_1 \) and \( t_2 \).

In order to prove Lemma 9 we need some preliminary results.

Claim 10. If \( t_1 = \Theta\left(\frac{n}{1-\xi_3} \log n\right) \) and \( \varepsilon^4 \gg \frac{b}{(1-\xi_3)\varepsilon} \log^2 n \), then

\[
P\left(\|y(t_1) - y(0)\|^2 \leq \frac{\varepsilon^3}{\log n} \land \|z(t_1)\|^2 \leq \frac{\varepsilon^3}{\log n}\right) \geq 1 - \varepsilon.
\]

Proof. From Lemma 6 and Markov inequality we have

\[
P\left(\|y(t_1) + z(t_1) - y(0)\|^2 \geq \frac{\varepsilon^3}{\log n}\right) \leq \frac{E\left[\|y(t_1) + z(t_1) - y(0)\|^2\right]}{\varepsilon^3/\log n} \leq \frac{bt_1 \log n}{\varepsilon^3} \leq \varepsilon,
\]

where in the last inequality we used the hypothesis on \( \varepsilon \). Since the vectors \( y(t_1) - y(0) \) and \( z(t_1) \) are orthogonal, we have

\[
\|y(t_1) + z(t_1) - y(0)\|^2 = \|y(t_1) - y(0)\|^2 + \|z(t_1)\|^2.
\]

It follows that whenever the left-hand side above is at most \( \varepsilon^3/\log n \) it must be the case that both terms on the right-hand side are at most \( \varepsilon^3/\log n \).

Next, we note that if \( y(t_1) \) is close to \( y(0) \) and \( z(t_1) \) is small, then there is a high probability that, for all \( t_1 \leq t \leq 2t_1 \), \( y(t) \) is close to \( y(0) \) and \( z(t) \) is small. Notice that such a fact cannot be proved using a union bound, because we have a constant probability bound of the event happening at a particular step, and we want it to happen for \( O(n \log n) \) consecutive steps.

Claim 11. If \( t_1 = \Theta\left(\frac{n}{1-\xi_3} \log n\right) \) and \( \varepsilon^4 \gg \frac{b}{(1-\xi_3)\varepsilon} \log^2 n \), then

\[
P\left(\forall t : t_1 \leq t \leq 2t_1 \ | \|y(t) + z(t) - y(0)\|^2 \leq \frac{9}{\log n} \varepsilon^4 - \frac{\log(1/\varepsilon)}{\log(1/\varepsilon)}\right) \geq 1 - 2\varepsilon.
\]

Proof. It suffices to show that, conditioned on the event of Claim 10 happening, the above event happens with probability at least \( 1 - \varepsilon \). Let \( M_t \) be the product of the random matrices \( W_t, \ldots, W_{t+1} \). Observe that \( y(t) + z(t) = M_t(y(t_1) + z(t_1)) \). Then

\[
\|y(t) + z(t) - y(0)\|^2 \leq 3\|y(t_1) - y(0)\|^2 + 3\|y(t) - y(t_1)\|^2 + 3\|z(t)\|^2
\]

\[
\leq 3\|y(t_1) - y(0)\|^2 + 3\|M_t y(t_1) - y(t_1)\|^2 + 3\|M_t z(t_1)\|^2. \quad (22)
\]

18
With probability one,
\[\| M_t z(t_1) \|^2 \leq \| z(t_1) \|^2.\]  
(23)

Further, from Claim 7 with probability \(1 - \exp(-\Omega(t_1 b/d)) = 1 - n^{-\Omega(1)}\) it holds
\[\| M_t y(t_1) - y(t_1) \|^2 \leq \frac{t_1 b}{d} \cdot \frac{\| y(t_1) \|^2}{n}.\]  
(24)

Let us call a starting vector \(x\) **typical** if, \(\|x(0)\|^2 \leq 10 \log(1/\varepsilon)\). By Chernoff bounds, it is easy to see that a starting vector is **typical** with probability at least \(1 - \varepsilon^2\). By assuming that the starting vector \(x\) is **typical**, we can conclude
\[\|y(t_1)\| \leq \|y(0)\| + \|y(t_1) - y(0)\| \leq 4 \sqrt{\log(1/\varepsilon)} + \left(\frac{\varepsilon^3}{\log n}\right)^{1/2} \leq 5 \sqrt{\log(1/\varepsilon)}.\]  
(25)

Substituting (25), (24), and (23) in (22), and using a union bound over \(t \in [t_1, 2t_1]\), we get that for all such \(t\)
\[\|y(t) + z(t) - y(0)\|^2 \leq \frac{9 t_1 b}{d} \cdot \frac{\log(1/\varepsilon)}{n} \leq \frac{9}{\log n} \varepsilon^4 \frac{\log \log(1/\varepsilon)}{\log(t_1/\varepsilon)}\]
with probability at least \(1 - \varepsilon^2\), where in (a) we used the hypotheses on \(t_1\) and \(b\).

We are now ready to complete the proof of the Main Lemma.

**Proof of Lemma 9** For each \(t \in [t_1 + 1, 2t_1]\), define \(B_t\) to be the set of nodes \(v\) that are \(\varepsilon\)-bad at time \(t\).

Call \(A_{t+1} := V - B_{t+1}\) the complement of \(B_{t+1}\) and, for each \(t \in [t_1 + 2, t_2]\), let \(A_t\) denote the set of nodes in \(A_{t+1}\) that have not been averaged along a cross edge or with a node in \(B_t\). Inductively, if \(e_t = (u_t, v_t)\) is the edge chosen at time \(t\) then
\[A_t = \begin{cases} A_{t-1} & \text{if } e_{t-1} \text{ is not a cross edge and } e_{t-1} \cap B_{t-1} = \emptyset, \\ A_{t-1} \setminus \{u_t, v_t\} & \text{otherwise}. \end{cases}\]  
(26)

We make the following facts:

1. Every node \(v\) in \(A_{2t_1}\) was \(\varepsilon\)-good at all times between \(t_1\) and \(2t_1\), because it is a node whose value was good at time \(t_1 + 1\), and then was averaged only with nodes \(u\) on the same side of the partition (that is, such that \(\text{sgn}(y_u) = \text{sgn}(y_v)\)) and at times in which \(u\) was good as well.

2. The cardinality of \(A_{2t_1}\) is at least \((1 - \varepsilon)n\). To see why, observe that by Chernoff bounds and the hypotheses on \(t_1\) and \(b\), among the subsequent \(O(n \log n)\) rounds following \(t_1\), with high probability (i.e., at least \(1 - \varepsilon\)) there are at most \(O(bt_1/d) \leq O\left(\frac{\varepsilon n}{\log n}\right)\) cross edges. Moreover, by Claim 11 and Equation (21), there is a \(1 - O(\varepsilon)\) probability that each \(B_t\) has size at most \(\frac{9n}{\log n} 2^{-\frac{\log \log(1/\varepsilon)}{\log(t_1/\varepsilon)}}\), which implies that at most \(\frac{9n}{\log n} t_1 \leq O(\tilde{\varepsilon} n)\) edges incident to bad nodes are chosen, where \(\tilde{\varepsilon} = \frac{1}{\log n} \varepsilon^2 \frac{\log \log(1/\varepsilon)}{\log(t_1/\varepsilon)}\). By the definition of \(A_t\) (see 25), whenever a cross edge or an edge incident to \(B_t\) is chosen, the cardinality of \(A_t\) decreases by at most 2 compared to \(A_{t-1}\). It follows that
\[\mathbb{P}(\|A_{2t_1}\| \geq (1 - O(\varepsilon))n) \geq 1 - O(\varepsilon).\]

The two observations together imply the main lemma.
4.5 Wrapping up: The number of unlucky nodes

We can now provide an upper bound on the number of $(\sqrt{\varepsilon})$-unlucky nodes.

Lemma 12. If $t_1 = \Theta(\frac{n}{1-\lambda_3} \log n)$, $\varepsilon^4 \gg \frac{b}{(1-\lambda_3)^2} \log^2 n$, then the number of $\sqrt{\varepsilon}$-unlucky nodes is

$$|U^{\sqrt{\varepsilon}}| \leq O \left( 1 + \frac{1}{1-\lambda_3} \varepsilon^{1 - \frac{\log \log(1/\varepsilon)}{\log(1/\varepsilon)}} \right) \sqrt{\varepsilon} n.$$  

Proof. Let $L$ be the set of nodes that freeze their sign $h(v)$ according to the sign of $x_{\|,v} + y_v^{(0)}$,

$$L = \{ v \in V_1 \cup V_2 : \text{sgn} (x_{\|,v}^{(0)} + y_v^{(0)}) = \text{sgn} (x_{\|,v} + y_v^{(0)}) \}.$$  

We first observe that, given any $\varepsilon > 0$, if we have a lower bound on the expected size of $L$, namely $\mathbb{E}[|L|] \geq n - \varepsilon n$, then we have an upper bound on the number of unlucky nodes, namely $|U^{\sqrt{\varepsilon}}| \leq \sqrt{\varepsilon} n$. Indeed,

$$\mathbb{E}[|L|] = \sum_{u \in U^{\sqrt{\varepsilon}}} P (u \in L) + \sum_{u \notin U^{\sqrt{\varepsilon}}} P (u \in L) \leq (1 - \sqrt{\varepsilon}) |U^{\sqrt{\varepsilon}}| + n - |U^{\sqrt{\varepsilon}}| = n - \sqrt{\varepsilon} |U^{\sqrt{\varepsilon}}|. \quad (27)$$

We now give a lower bound on the expected size of $L$.

Let $\mathcal{E}$ be the event

$$\mathcal{E} = \{ |y_v^{(0)}| - |x_{\|,v}| \geq \frac{\varepsilon}{\sqrt{n}} \}.$$  

Notice that:

1. The value $|y_v^{(0)}| - |x_{\|,v}|$ does not depend on the node $v$, only on the initial assignment, and

2. If $\mathcal{E}$ holds then $|x_{\|,v} + y_v^{(0)}| \geq \frac{\varepsilon}{\sqrt{n}}$ and thus any node $u$ that is $\varepsilon$-good at round $\tau_u$ is in $L$.

Hence, for any node $u \in V_1 \cup V_2$ we have that

$$P (u \in L) \geq P (\mathcal{E} \land \{ u \text{ is } \varepsilon \text{-good at round } \tau_u \}) \geq P (\mathcal{E} \land \{ u \text{ is } \varepsilon \text{-good at all rounds } t \in [t_1, 2t_1] \} \land \{ \tau_u \in [t_1, 2t_1] \}) \geq 1 - P (\overline{\mathcal{E}}) - P (\{ u \text{ is not } \varepsilon \text{-good at some round } t \in [t_1, 2t_1] \}) - P (\tau_u \notin [t_1, 2t_1]) = P (u \text{ is } \varepsilon \text{-good at all rounds } t \in [t_1, 2t_1]) - P (\overline{\mathcal{E}}) - P (\tau_u \notin [t_1, 2t_1]). \quad (28)$$

In Appendix A (Lemma 13) we prove that with probability $1 - O(\varepsilon)$ it holds

$$|x \cdot 1 \pm x \cdot \chi| \geq \varepsilon \cdot \sqrt{n}$$

and with probability at least $\frac{1}{2} - O(\varepsilon)$ it holds

$$|x \cdot 1| \leq |x \cdot \chi| - \varepsilon \cdot \sqrt{n}.$$  

From the latter fact it follows that $P (\overline{\mathcal{E}}) \leq O(\varepsilon)$ and it is easy to see that $P (\tau_u \notin [t_1, 2t_1]) \leq 1/n$ (see, e.g., Lemma 1 in Appendix A). Hence, from (28) the expected size of $L$ is

$$\mathbb{E}[|L|] = \sum_u P (u \in L) \geq \mathbb{E}\left[ \# \{ u : u \text{ is } \varepsilon \text{-good at all rounds } t \in [t_1, 2t_1] \} \right] - O(\varepsilon) n - 1. \quad (29)$$
Finally, from Lemma 9 and Markov inequality it follows that

\[
E[\{u : u \text{ is } \varepsilon\text{-good at all rounds } t \in [t_1, 2t_1]\}] \\
\geq (1 - \frac{1}{1 - \lambda_2} \varepsilon^2 \frac{\log(1/\varepsilon^2)}{\log(1/\varepsilon^2)})n \\
\geq P\left(\text{there are } \geq (1 - \tilde{\varepsilon})n \text{ nodes that are } \varepsilon\text{-good at all times } t_1 \leq t \leq 2t_1\right) \geq 1 - \varepsilon,
\]

that is

\[
E[\{u : u \text{ is } \varepsilon\text{-good at all rounds } t \in [t_1, 2t_1]\}] \geq (1 - \varepsilon)(1 - \tilde{\varepsilon})n.
\]

Thus from (29) and the previous inequality we get that \(E[|L|] \geq (1 - \tilde{\varepsilon} - \varepsilon)n\) and the thesis follows from (27).

5 Future works

We have shown that there is a simple population protocol that can perform distributed community detection over the class of clustered regular graphs having two balanced communities. We believe that our spectral analysis may be extended to cover more general classes of graphs. For instance, by increasing the length of the labels, we think the protocol can find a good community sensitive labeling for regular clustered graphs with a constant number of balanced communities. Another possible generalization may concern some classes of almost-regular graphs such as that considered in [10] which includes the classic stochastic block model.

Acknowledgements. We thank Aleks Madry and Cameron Musco for pointing out the related analysis discussed in Section 2.

References

[1] Emmanuel Abbe, Afonso S Bandeira, and Georgina Hall. Exact recovery in the stochastic block model. IEEE Trans. on Information Theory, 62(1):471–487, 2014.

[2] Emmanuel Abbe and Colin Sandon. Detection in the stochastic block model with multiple clusters: proof of the achievability conjectures, acyclic bp, and the information-computation gap. arXiv preprint arXiv:1512.09080, 2015.

[3] Dan Alistarh, Rati Gelashvili, and Milan Vojnović. Fast and exact majority in population protocols. In Proc. of the 2015 ACM Symposium on Principles of Distributed Computing (PODC’15), pages 47–56. ACM, 2015.

[4] Dana Angluin, James Aspnes, Zoë Diamadi, Michael J. Fischer, and Peralta René. Computation in networks of passively mobile finite-state sensors. Distributed Computing, 18(4):235–253, 2006.

[5] Dana Angluin, James Aspnes, and David Eisenstat. Stably computable predicates are semilinear. In In Proc. of the 25th Ann. ACM SIGACT-SIGOPS Symp. on Principles of Distributed Computing (PODC’06), pages 292–299, 2006.

[6] Dana Angluin, James Aspnes, and David Eisenstat. A Simple Population Protocol for Fast Robust Approximate Majority. Distributed Computing, 21(2):87–102, 2008. (Preliminary version in DISC’07).
[7] Dana Angluin, James Aspnes, David Eisenstat, and Eric Ruppert. The computational power of population protocols. *Distributed Computing*, 20(4):279–304, 2007.

[8] Dana Angluin, Michael J. Fischer, and Hong Jiang. Stabilizing consensus in mobile networks. In *Proc. of Distributed Computing in Sensor Systems (DCOSS’06)*, volume 4026 of *LNCS*, pages 37–50, 2006.

[9] Michael J. Barber and John W. Clark. Detecting network communities by propagating labels under constraints. *Physical Review E*, 80(2):026129, 2009.

[10] Luca Becchetti, Andrea Clementi, Emanuele Natale, Francesco Pasquale, and Luca Trevisan. Find your place: Simple distributed algorithms for community detection. In *Proc. of the 28th Ann. ACM-SIAM Symp. on Discrete Algorithms (SODA’17)*, pages 940–959. SIAM, 2017.

[11] Ohad Ben-Shahar, Shlomi Dolev, Andrey Dolgin, and Michael Segal. Direction election in flocking swarms. In *Proc. of the 6th Int. Workshop on Foundations of Mobile Computing (DIALM-POMC’10)*, pages 73–80. ACM, 2010.

[12] Ravi B. Boppana. Eigenvalues and graph bisection: An average-case analysis. In *Proc. of the 28th Ann. IEEE Symp. on Foundations of Computer Science (FOCS’87)*, pages 280–285. IEEE, 1987.

[13] Charles Bordenave. A short course on random matrices. Preliminary draft. [https://www.math.univ-toulouse.fr/~bordenave/coursRMT.pdf](https://www.math.univ-toulouse.fr/~bordenave/coursRMT.pdf), 2014.

[14] Charles Bordenave. A new proof of Friedman’s second eigenvalue Theorem and its extension to random lifts. *arXiv preprint arXiv:1502.04482*, 2015.

[15] Charles Bordenave, Marc Lelarge, and Laurent Massoulié. Non-backtracking spectrum of random graphs: community detection and non-regular ramanujan graphs. In *Proc. of 56rd Ann. IEEE Symp. on Foundations of Computer Science (FOCS’15)*, pages 1347–1357. IEEE, 2015.

[16] Stephen Boyd, Arpita Ghosh, Balaji Prabhakar, and Devavrat Shah. Randomized gossip algorithms. *IEEE/ACM Transactions on Networking*, 14:2508–2530, 2006.

[17] Gerandy Brito, Ioana Dumitriu, Shirshendu Ganguly, Christopher Hoffman, and Linh V. Tran. Recovery and rigidity in a regular stochastic block model. In *Proc. of the 26th Ann. ACM-SIAM Symp. on Discrete Algorithms (SODA’15)*, pages 371–390, 2015.

[18] Luca Cardelli and Attila Csikász-Nagy. The Cell Cycle Switch Computes Approximate Majority. *Scientific Reports*, Vol. 2(656), 2012.

[19] Yuan-Jyue Chen, Neil Dalchau, Niranjan Srinivas, Andrew Phillips, Luca Cardelli, David Soloveichik, and Georg Seelig. Programmable chemical controllers made from DNA. *Nature nanotechnology*, 8(10):755–762, 2013.

[20] Amin Coja-Oghlan. *Spectral techniques, semidefinite programs, and random graphs*. Habilitation thesis, Humboldt University Berlin, 2005.

[21] Amin Coja-Oghlan. Graph partitioning via adaptive spectral techniques. *Combinatorics, Probability and Computing*, 19(02):227–284, 2010.
[22] Aurelien Decelle, Florent Krzakala, Cristopher Moore, and Lenka Zdeborová. Asymptotic analysis of the stochastic block model for modular networks and its algorithmic applications. *Physical Review E*, 84(6):066106, 2011.

[23] David Doty. Timing in chemical reaction networks. In *Proc. of 25th Ann. ACM-SIAM Symp. on Discrete Algorithms (SODA’14)*, pages 772–784. SIAM, 2014.

[24] Devdatt P. Dubhashi and Alessandro Panconesi. *Concentration of measure for the analysis of randomized algorithms*. Cambridge University Press, 2009.

[25] Martin E. Dyer and Alan M. Frieze. The solution of some random NP-hard problems in polynomial expected time. *Journal of Algorithms*, 10(4):451–489, 1989.

[26] David Easley and Jon Kleinberg. *Networks, crowds, and markets: Reasoning about a highly connected world*. Cambridge University Press, 2010.

[27] Ofer Feinerman, Bernhard Haeupler, and Amos Korman. Breathe Before Speaking: Efficient Information Dissemination Despite Noisy, Limited and Anonymous Communication. In *Proc. of the ACM Symp. on Principles of Distributed Computing (PODC ’14)*, pages 114–123. ACM, 2014.

[28] Nigel R. Franks, Stephen C. Pratt, Eamonn B. Mallon, Nicholas F. Britton, and David J.T. Sumpter. Information flow, opinion polling and collective intelligence in house–hunting social insects. *Philosophical Transactions of the Royal Society of London B: Biological Sciences*, 357(1427):1567–1583, 2002.

[29] Yehuda Hassin and David Peleg. Distributed probabilistic polling and applications to proportionate agreement. *Information and Computation*, 171(2):248–268, 2001.

[30] Paul W. Holland, Kathryn Blackmond Laskey, and Samuel Leinhardt. Stochastic blockmodels: First steps. *Social networks*, 5(2):109–137, 1983.

[31] Prateek Jain, Chi Jin, Sham M Kakade, Praneeth Netrapalli, and Aaron Sidford. Streaming PCA: Matching Matrix Bernstein and Near-Optimal Finite Sample Guarantees for Oja’s Algorithm. In *Proc. of the 29th Annual Conference on Learning Theory (COLT ’16)*, pages 1147–1164, 2016.

[32] Mark Jerrum and Gregory B. Sorkin. The metropolis algorithm for graph bisection. *Discrete Applied Mathematics*, 82(1):155–175, 1998.

[33] David Kempe and Frank McSherry. A decentralized algorithm for spectral analysis. In *Proc. of the 36th Ann. ACM Symp. on Theory of Computing (STOC’04)*, pages 561–568, 2004.

[34] Rohit Khandekar, Satish Rao, and Umesh Vazirani. Graph partitioning using single commodity flows. *Journal of the ACM*, 56(4):19, 2009.

[35] Kishore Kothapalli, Sriram V. Pemmaraju, and Vivek Sardeshmukh. On the analysis of a label propagation algorithm for community detection. In *Proc. of the 14th Int. Conf. on Distributed Computing and Networking (ICDCN’13)*, pages 255–269, 2013.

[36] Florent Krzakala, Cristopher Moore, Elchanan Mossel, Joe Neeman, Allan Sly, Lenka Zdeborová, and Pan Zhang. Spectral redemption in clustering sparse networks. *Proc. of the National Academy of Sciences*, 110(52):20935–20940, 2013.
lemmas

Lemma 13 (Projection on the first two eigenvectors).

For all $\varepsilon \in (0,1)$, for a random $x \in \{-1,1\}^n$, with probability at least $1 - O(\varepsilon)$ we have,

\[
P(|x \cdot 1 \pm x \cdot \chi| \geq \varepsilon \cdot \sqrt{n}) \geq 1 - O(\varepsilon)
\]

and

\[
P(|x \cdot 1| \leq |x \cdot \chi| - \varepsilon \cdot \sqrt{n} \mid |x \cdot 1 \pm x \cdot \chi| \geq \varepsilon \cdot \sqrt{n}) = \frac{1}{2}.
\]
Proof. Note that $x \cdot (1 + \chi) = 2x \cdot 1_{V_1}$ and $x \cdot (1 - \chi) = 2x \cdot 1_{V_2}$. Using properties of the binomial distribution, it is easy to see that

$$P \left( |x \cdot 1 + x \cdot \chi| \geq \varepsilon \sqrt{n} \land |x \cdot 1 - x \cdot \chi| \geq \varepsilon \sqrt{n} \right) \geq 1 - O(\varepsilon).$$

The above event implies $||x \cdot 1| - |x \cdot \chi|| \geq \varepsilon \sqrt{n}$. Since $x \cdot 1$ and $x \cdot \chi$ are independent sums of Rademacher random variables, they have the same chances of being positive or negative, thus with probability at least $\frac{1}{2}$ we will have $|x \cdot 1| \leq |x \cdot \chi|$. 

Let $\tau_v$ denote the time at which node $v$ freezes its value of $h_v$, i.e., the global round when the node $v$ achieves $T$ activations.

**Lemma 14.** If $T > 72 \log n$ and $t_1 = 3Tn/4$ then

$$P \left( \{\tau_v \mid v \in V\} \subseteq [t_1, 2t_1] \right) \geq 1 - \frac{1}{n}.$$

Proof. For each node $v$, let $X_v^{(i)} = 1_{[v \text{ is activated at round } i]}$. Fix a node $v$. By applying the Chernoff bound on the i.i.d. random variables $\{X_v^{(i)}\}_{i \geq 0}$, we have

$$P \left( \sum_{i=1}^{3Tn/4} X_v^{(i)} \geq T \right) \leq e^{-\frac{T}{36n}} \quad \text{and} \quad P \left( \sum_{i=1}^{3Tn/2} X_v^{(i)} \leq T \right) \leq e^{-\frac{T}{12n}}.$$

The claim follows by applying a union bound over the nodes. \qed

25