Generalized Eigenvectors for Resonances in the Friedrichs Model and Their Associated Gamov Vectors

Hellmut Baumgärtel

Abstract

A Gelfand triplet for the Hamiltonian $H$ of the Friedrichs model on $\mathbb{R}$ with multiplicity space $\mathcal{K}, \dim \mathcal{K} < \infty$ is constructed such that exactly the resonances (poles of the inverse of the Livšic-matrix) are (generalized) eigenvalues of $H$. The corresponding eigen(anti-)linearforms are calculated explicitly. Using the wave matrices for the wave (Möller) operators the corresponding eigen(anti-)linearforms on the Schwartz space $\mathcal{S}$ for the unperturbed Hamiltonian $H_0$ are also calculated. It turns out that they are of pure Dirac type and can be characterized by their corresponding Gamov vector $\lambda \to k/(\zeta_0 - \lambda)^{-1}, \zeta_0$ resonance, $k \in \mathcal{K}$, which is uniquely determined by restriction of $\mathcal{S}$ to $\mathcal{S} \cap \mathcal{H}_+^2$, where $\mathcal{H}_+^2$ denotes the Hardy space of the upper half plane. Simultaneously this restriction yields a truncation of the generalized evolution to the well-known decay semigroup for $t \geq 0$ of the Toeplitz type on $\mathcal{H}_+^2$. That is: exactly those pre-Gamov vectors $\lambda \to k/(\zeta - \lambda)^{-1}, \zeta$ from the lower half plane, $k \in \mathcal{K}$, have an extension to a generalized eigenvector of $H$ if $\zeta$ is a resonance and if $k$ is from that subspace of $\mathcal{K}$ which is uniquely determined by its corresponding Dirac type anti-linearform.

Keywords: Friedrichs model, scattering theory, resonances, generalized eigenvectors, Gamov vectors

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1 Introduction

In quantum scattering systems bumps in cross sections often can be described by expressions like $\lambda \to c((\lambda - \lambda_0)^2 + (\frac{\Gamma}{2})^2)^{-1}$, where $\lambda_0$ is the resonance energy, $\Gamma/2$ the halfwidth, called Breit-Wigner formulas (see e.g. Bohm [1, pp. 428 - 429]). Sometimes, if the scattering matrix is analytically continuable into the lower half plane $\mathbb{C}_-$, these bumps can be connected with complex poles $\lambda_0 - i \frac{\Gamma}{2}$ of the scattering matrix in $\mathbb{C}_-$. Then $c((\lambda - \lambda_0) - i \frac{\Gamma}{2})^{-1}$ is called the Breit-Wigner amplitude, if the pole is of first order (see e.g. [1, pp. 428 - 429]). These poles are called resonances (see e.g. Brändas/Elander [2], Albeverio/Ferreira/Streit [3]).

The basic idea is that these points should coincide with eigenvalues for generalized eigenvectors of the evolution which is determined by the Hamiltonian $H$ of the scattering system. Obviously this (first) problem cannot be solved within the Hilbert space $\mathcal{H}$, it requires extension techniques, e.g. the use of Gelfand triplets.
A further (second) problem is to establish a rigorous mathematical framework to derive modified associated states, also corresponding to resonances as eigenvectors, but of a truncated evolution, such that the eigenvectors satisfy the exponential decay law. These vectors are called Gamov vectors in the literature (see e.g. Gamov [4], Bohm/Gadella [5], Bohm/Harshman [6] and further references therein). An obvious suggestion is that also this problem has to be solved by techniques beyond the Hilbert space. Such an approach was presented by Bohm/Gadella and others by using Gelfand triplets (Rigged Hilbert Spaces (RHS) in their terminology) on Hardy subspaces of $H_0$, the Hilbert space of the unperturbed Hamiltonian $H_0$ of the scattering system (see [5, 6], Bohm/Maxson/Loewe/Gadella [7] and papers quoted therein).

Originally, the theory of Gelfand triplets (see e.g. Gelfand/Wilenkin [8], see also Baumgärtel [9]) was developed for selfadjoint operators to generalize eigenvector expansions also for the absolutely continuous spectrum. For this purpose the occurence of complex eigenvalues is only a nuisance.

In this paper it is shown that for the finite-dimensional Friedrichs model the first problem can be solved rigorously by the Gelfand triplet approach, i.e. the construction of a triplet is presented such that exactly the resonances are eigenvalues of the extended Hamiltonian. The corresponding (generalized) eigenvectors are calculated explicitly (a slightly modified triplet was already considered in Baumgärtel [10]). This result confirms the basic idea mentioned above.

On the other hand, recently it turned out that to solve the second problem the use of the triplet approach is not indispensable. On the contrary, the Gamov vectors can be identified as vectors in the Hilbert space $H_0$ resp. $H$, more precisely, they are eigenvectors of the decay semigroup for $t \geq 0$, which is of Toeplitz type and which can be defined by a truncation of the quantum evolution. This insight came into the light and was supported by analogies in the Lax-Phillips scattering theory. This approach has been promoted and emphasized by Strauss [11] (see also Eisenberg/Horwitz/Strauss [12]).

However, if one adopts this point of view then a third problem arises: One has to point out the connection between the generalized eigenvector (the solution of the first problem) and the corresponding Gamov vector, i.e. one has to determine the selection principle which selects the right Gamov vector from the whole collection of all pre-Gamov vectors (eigenvectors of the decay semigroup). Also this problem is solved in this paper: Exactly those eigenvectors of the decay semigroup have extensions to a generalized eigenvector if the eigenvalue is a resonance and which belong to a distinguished subeigenspace, which is calculated explicitly. Vice versa, the restriction of the generalized eigenvector (for $H_0$) which is an eigen(anti-)linearform on the Schwartz space of pure Dirac type to the Hardy subspace for the upper half plane $\mathbb{C}_+$ is (via the Paley-Wiener theorem) characterized by a vector from this Hardy space. This vector is the Gamov vector corresponding to the generalized eigenvector.
2 Preliminaries

2.1 Basic objects of the Friedrichs model

In the following we collect the concepts and denotations for the finite-dimensional Friedrichs model on \( \mathbb{R} \). Let \( \mathcal{H}_0 := L^2(\mathbb{R}, \mathcal{K}, d\lambda) \), where \( \mathcal{K} \) denotes a multiplicity Hilbert space, \( \dim \mathcal{K} < \infty \). Further let \( \mathcal{E} \) be a finite-dimensional Hilbert space, \( \dim \mathcal{E} =: N \) and put \( \mathcal{H} := \mathcal{H}_0 \oplus \mathcal{E} \). The projection onto \( \mathcal{E} \) is denoted by \( P_\mathcal{E} \). \( \mathcal{H}_0 \) is a selfadjoint operator on \( \mathcal{H} \) with reducing projection \( P_\mathcal{E} \), where \( \mathcal{H}_0 \) is the multiplication operator on \( \mathcal{H}_0 \). The selfadjoint operator \( H \) on \( \mathcal{H} \) is given by a perturbation of \( \mathcal{H}_0 \) as

\[
H := \mathcal{H}_0 + \Gamma + \Gamma^*,
\]

where \( \Gamma \) denotes a partial isometry on \( \mathcal{H} \) with the properties

\[
\Gamma^* \Gamma = P_\mathcal{E}, \quad \Gamma \Gamma^* \leq P_{\mathcal{H}_0} := 1 - P_\mathcal{E}.
\]

The operator function

\[
L_{\pm}(z) := (z - \mathcal{H}_0)P_\mathcal{E} - \Gamma^*(z - \mathcal{H}_0)^{-1} \Gamma, \quad z \in \mathbb{C}_\pm,
\]

the so-called Livšic-matrix, is decisive in the following. One has \( L_{\pm}(z) \mathcal{E} \in \mathcal{L}(\mathcal{E}) \) is holomorphic on \( \mathbb{C}_\pm \). For brevity, if there is no danger of confusion, we write \( L_{\pm}(z) \mathcal{E} \) instead of \( L_{\pm}(z) \mathcal{E} \). Further we need the so-called partial resolvent \( P_\mathcal{E}(z - H)^{-1} \mathcal{E} \).

It turns out that

\[
L_{\pm}(z) \cdot P_\mathcal{E}(z - H)^{-1} \mathcal{E} = P_\mathcal{E}(z - H)^{-1} \mathcal{E} \cdot L_{\pm}(z) = P_\mathcal{E}, \quad z \in \mathbb{C}_\pm,
\]

(see e.g. Baumgärtel [10]), that is

\[
P_\mathcal{E}(z - H)^{-1} \mathcal{E} = (L_{\pm}(z) \mathcal{E})^{-1}, \quad z \in \mathbb{C}_\pm,
\]

and this equation shows that \( (L_{\pm}(z) \mathcal{E})^{-1} \in \mathcal{L}(\mathcal{E}) \) is holomorphic on \( \mathbb{C}_\pm \).

For \( \mathcal{H} \ni x := f + e, \ f \in \mathcal{H}_0, \ e \in \mathcal{E} \) one has \( \Gamma x = \Gamma e, \ \Gamma^* x = \Gamma^* f \). Therefore

\[
(\Gamma e)(\lambda) = M(\lambda)e, \quad \mathcal{E} \ni \Gamma^* f = \int_{-\infty}^{\infty} M(\lambda)^* f(\lambda)d\lambda,
\]

where \( \lambda \to M(\lambda) \in \mathcal{L}(\mathcal{E} \to \mathcal{K}) \) is a.e. defined on \( \mathbb{R} \).

Assumption 1: \( M(\cdot) \) is a Schwartz function, i.e. \( M(\cdot) \in \mathcal{S}(\mathcal{L}(\mathcal{E} \to \mathcal{K})) \).

For example, this implies

\[
\int_{-\infty}^{\infty} \| M(\lambda)^* M(\lambda) \|_{2, \mathcal{E}}^2 d\lambda < \infty, \quad \int_{-\infty}^{\infty} \| M(\lambda)^* M(\lambda) \|_{2, \mathcal{E}} d\lambda < \infty,
\]

where \( \| \cdot \|_{2, \mathcal{E}} \) denotes the Hilbert-Schmidt norm on \( \mathcal{E} \). Obviously one has

\[
\Gamma^*(z - \mathcal{H}_0)^{-1} \Gamma \mathcal{E} = \int_{-\infty}^{\infty} \frac{M(\lambda)^* M(\lambda)}{z - \lambda} d\lambda, \quad z \in \mathbb{C}_\pm.
\]
Therefore \( \text{s-lim}_{t \to +0} \Gamma^* (\lambda \pm i \epsilon - H_0)^{-1} \Gamma \) exists on \( \mathbb{R} \), hence also \( L_\pm(\lambda) := \text{s-lim}_{t \to +0} L_\pm (\lambda \pm i \epsilon) \) exists and it is infinitely differentiable and polynomially bounded. From (1) we obtain

\[
\frac{\Gamma^* E_0(d\lambda) \Gamma}{d\lambda} \uparrow \mathcal{E} = M(\lambda)^* M(\lambda), \quad \lambda \in \mathbb{R},
\]

where \( E_0(\cdot) \) denotes the spectral measure of \( H_0 \) on \( \mathcal{H}_0 \).

**Assumption 2:** \( H \) has no eigenvalues. This is equivalent to \( \det L_+ (\lambda) \neq 0 \) for all \( \lambda \in \mathbb{R} \) (see e.g. Baumgärtel [10]).

Then \( L_+(\lambda)^{-1} \) exists for all \( \lambda \in \mathbb{R} \), it is infinitely differentiable and 
\( \sup_\lambda \| L_+(\lambda)^{-1} \|_E < \infty \). Furthermore we have

\[
\text{s-lim}_{t \to +0} P_\mathcal{E} (\lambda \pm i \epsilon - H)^{-1} P_\mathcal{E} \uparrow \mathcal{E} = (L_\pm(\lambda) \uparrow \mathcal{E})^{-1}, \quad \lambda \in \mathbb{R}. \tag{2}
\]

\( H \) has no singular continuous spectrum. From (2) we obtain

\[
\frac{P_\mathcal{E} E(d\lambda) P_\mathcal{E}}{d\lambda} \uparrow \mathcal{E} = \frac{1}{2\pi i} (L_-(\lambda)^{-1} - L_+(\lambda)^{-1}) = L_\pm(\lambda)^{-1} M(\lambda)^* M(\lambda) L_\pm(\lambda)^{-1}, \quad \lambda \in \mathbb{R},
\]

where \( E(\cdot) \) denotes the spectral measure of \( H \).

### 2.2 Wave operators and wave matrices

Since \( \Gamma + \Gamma^* \) is a finite-dimensional perturbation the wave operators \( W_\pm = W_\pm (H, H_0) := \text{s-lim}_{t \to +\infty} e^{itH} e^{-itH_0} P_\mathcal{F} \) exist, they are isometric from \( \mathcal{H}_0 \) onto \( \mathcal{H} \). Furthermore, \( W_\pm = W_\pm (H_0, H) = \text{s-lim}_{t \to -\infty} e^{itH_0} e^{-itH} \).

In the following we rewrite the wave operators as limits of operator spectral integrals. We refer to Baumgärtel/Wollenberg [15] for details on operator spectral integrals, where this theory is presented. We use also results of Baumgärtel [13] (see also [14]). Here we mention only the following facts: If \( \mu \to t(\mu) := \sum_{j=1}^m \chi_{\Delta_j}(\mu) t_j, \ t_j \in \mathcal{H}_0, \) is a step function then the spectral integral \( \int_{-\infty}^{\infty} E_0(d\mu) t(\mu) \) is given by

\[
\int_{-\infty}^{\infty} E_0(d\mu) t(\mu) = \sum_{j=1}^m \int_{-\infty}^{\infty} E_0(d\mu) \chi_{\Delta_j}(\mu) t_j = \sum_{j=1}^m \int_{-\infty}^{\infty} \chi_{\Delta_j}(\mu) E_0(d\mu) t_j = \sum_{j=1}^m E_0(\Delta_j) t_j.
\]

The spectral integral \( \int_{-\infty}^{\infty} E_0(d\mu) x(\mu) \) for a more general function \( \mu \to x(\mu) \in \mathcal{H}_0 \) exists if

\[
\int_{-\infty}^{\infty} \frac{(x(\lambda), E_0(d\mu) x(\lambda))}{d\mu} \big|_{\mu=\lambda} d\lambda < \infty.
\]

Note that \( \frac{(g, E_0(d\mu) g)}{d\mu} \) exists a.e. on \( \mathbb{R} \) for all \( g \in \mathcal{H}_0 \) because the spectral measure \( E_0(\cdot) \) is absolutely continuous.

Now put \( \mathcal{H}_{E_0} := \text{clo spa}(E_0(\Delta) f, f \in \mathcal{E}) \) and \( \mathcal{H}_E := \text{clo spa}(E(\Delta) e, e \in \mathcal{E}) \). It is not hard to see that \( \mathcal{H}_{E_0} \) and \( \mathcal{H}_E \) have natural spectral representations w.r.t. \( E_0(\cdot), E(\cdot) \), respectively, which are explicitly given by spectral integrals:

\[
\mathcal{H}_{E_0} \ni x = \int_{-\infty}^{\infty} E_0(d\mu) \Gamma f(\mu), \quad \mathcal{H}_E \ni y = \int_{-\infty}^{\infty} E(d\lambda) g(\lambda), \tag{3}
\]
where \( \mu \rightarrow f(\mu) \in \mathcal{E}, \lambda \rightarrow g(\lambda) \in \mathcal{E} \) are vector functions with values in \( \mathcal{E} \) such that the integrals (3) exist. Note that \( \int_{-\infty}^{\infty} E_0(d\mu) \Gamma f(\mu) \) exists iff \( \int_{-\infty}^{\infty} \| M(\mu)f(\mu) \|_K^2 d\mu < \infty \), i.e. iff the function \( \mu \rightarrow M(\mu)f(\mu) \) is an element of \( \mathcal{H}_0 \). The integral \( \int_{-\infty}^{\infty} E(d\lambda)g(\lambda) \) exists iff \( \int_{-\infty}^{\infty} \| M(\lambda)L_+(\lambda)^{-1}g(\lambda) \|_K^2 d\lambda < \infty \), i.e. iff the function \( \lambda \rightarrow M(\lambda)L_+(\lambda)^{-1}g(\lambda) \) is an element of \( \mathcal{H}_0 \). The function \( f(\cdot) \) is called the representer of \( x \) and \( g(\cdot) \) the representer of \( y \) w.r.t. the corresponding spectral representation.

Note further that

\[
\left( \int_{-\infty}^{\infty} E_0(d\mu) \Gamma f(\mu) \right)(\lambda) = (\Gamma f(\lambda))(\lambda) = M(\lambda)f(\lambda)
\]

and

\[
\mathcal{H}_0 \ominus \mathcal{H}_{E_0} = \{ f \in \mathcal{H}_0 : M(\lambda)^*f(\lambda) = 0 \text{ a.e. on } \mathbb{R} \}.
\]

The wave operators \( W_\pm, W_\pm^* \) can be written as strong limits of certain spectral integrals (see [13]):

\[
\mathcal{H}_0 \ni f \rightarrow W_\pm f = \lim_{\epsilon \rightarrow +0} \int_{-\infty}^{\infty} E(d\lambda) (\mathbb{1} - \Gamma^* R_0(\lambda \pm i\epsilon)) f,
\]

\[
\mathcal{H} \ni g \rightarrow W_\pm^* g = \lim_{\epsilon \rightarrow +0} \int_{-\infty}^{\infty} E_0(d\lambda) (\mathbb{1} + (\Gamma + \Gamma^*) R(\lambda \pm i\epsilon)) g,
\]

where \( R_0(z) := (z - H_0)^{-1} \), \( R(z) := (z - H)^{-1} \) denote the resolvent of \( H_0, H \) on \( \mathcal{H}_0, \mathcal{H} \), respectively. From (4) we get immediately

\[
W_\pm f = f, \quad f \in \mathcal{H}_0 \ominus \mathcal{H}_{E_0}.
\]

\( W_\pm \) on \( \mathcal{H}_{E_0} \) and \( W_\pm^* \) on \( \mathcal{H}_E \) can be calculated explicitly.

**LEMMA 1.** The wave operators are given by the following expressions:

\[
W_\pm \left( \int_{-\infty}^{\infty} E_0(d\mu) \Gamma f(\mu) \right) = \int_{-\infty}^{\infty} E(d\lambda)L_\pm(\lambda)f(\lambda),
\]

\[
W_\pm^* \left( \int_{-\infty}^{\infty} E(d\lambda)g(\lambda) \right) = \int_{-\infty}^{\infty} E_0(d\lambda)\Gamma L_\pm(\lambda)^{-1}g(\lambda).
\]

**Proof.** (7): First we calculate \( W_\pm(\Gamma e), e \in \mathcal{E} \). From (4) we obtain

\[
W_\pm(\Gamma e) = \lim_{\epsilon \rightarrow +0} \int_{-\infty}^{\infty} E(d\lambda)(\Gamma e - \Gamma^* R_0(\lambda \pm i\epsilon)\Gamma e)
\]

\[
= \lim_{\epsilon \rightarrow +0} \int_{-\infty}^{\infty} E(d\lambda)(\Gamma e + L_\pm(\lambda \pm i\epsilon)e - ((\lambda \pm i\epsilon) - H_0)e)
\]

\[
= \lim_{\epsilon \rightarrow +0} \int_{-\infty}^{\infty} E(d\lambda)(L_\pm(\lambda \pm i\epsilon)e - \lambda e \mp i\epsilon e + H_0e + \Gamma e)
\]

\[
= \lim_{\epsilon \rightarrow +0} \int_{-\infty}^{\infty} E(d\lambda)(L_\pm(\lambda \pm i\epsilon)e + (H - \lambda)e),
\]

but \( \int_{-\infty}^{\infty} E(d\lambda)(H - \lambda)e = 0 \), i.e.

\[
W_\pm(\Gamma e) = \lim_{\epsilon \rightarrow +0} \int_{-\infty}^{\infty} E(d\lambda)L_\pm(\lambda \pm i\epsilon)e.
\]
Now the spectral integral \( \int_{-\infty}^{\infty} E(d\lambda)L_\pm(\lambda)e \) exists and it turns out by straightforward calculation that one can interchange s-lim and integral, i.e. finally we have

\[
W_{\pm}(\Gamma e) = \int_{-\infty}^{\infty} E(d\lambda)L_\pm(\lambda)e.
\]

Straightforward extension to the spectral integrals \( \int_{-\infty}^{\infty} E_0(d\mu)\Gamma f(\mu) \) yields (7).

(8) Correspondingly, first we calculate \( W^*_\pm e \). According to (5) we have

\[
W^*_\pm e = \lim_{\epsilon \to +0} \int_{-\infty}^{\infty} E_0(d\lambda)P_\mp_e(\Gamma + \Gamma^*)R(\lambda \pm i\epsilon)e
\]

\[
= \lim_{\epsilon \to +0} \int_{-\infty}^{\infty} E_0(d\lambda)\Gamma P_\mp_e R(\lambda \pm i\epsilon)P_\mp_e e
\]

\[
= \lim_{\epsilon \to +0} \int_{-\infty}^{\infty} E_0(d\lambda)\Gamma L_\pm(\lambda \pm i\epsilon)^{-1}e.
\]

Again, the spectral integral \( \int_{-\infty}^{\infty} E_0(d\lambda)\Gamma L_\pm(\lambda)^{-1}e \) exists and we can interchange s-lim and integral, i.e. we arrive at

\[
W^*_\pm e = \int_{-\infty}^{\infty} E_0(d\lambda)\Gamma L_\pm(\lambda)^{-1}e.
\]

Extension to the spectral integrals \( \int_{-\infty}^{\infty} E(d\lambda)g(\lambda) \) gives (8). \( \square \)

Therefore \( W_0(\mathcal{H}_{E_0}) = \mathcal{H}_E \) and \( W_0(\mathcal{H}_0 \ominus \mathcal{H}_{E_0}) = \mathcal{H} \ominus \mathcal{H}_E \). Using (6) we get \( \mathcal{H}_0 \ominus \mathcal{H}_{E_0} = \mathcal{H} \ominus \mathcal{H}_E \). Note that this is compatible with \( \mathcal{E} \subset \mathcal{H}_E \). Thus, the wave operators act nontrivially only on \( \mathcal{H}_{E_0}, \mathcal{H}_E \).

Lemma 1 says: if \( \lambda \to f(\lambda) \) is the representer of \( x \in \mathcal{H}_{E_0} \) w.r.t. \( E_0 \) then the representer of \( W_\pm x \in \mathcal{H}_E \) w.r.t. \( E \) is given by \( \lambda \to L_\pm(\lambda)f(\lambda) \). Conversely, if \( \lambda \to g(\lambda) \) is the representer of \( y \in \mathcal{H}_E \) w.r.t. \( E \) then the representer of \( W_\pm y \in \mathcal{H}_{E_0} \) w.r.t. \( E_0 \) is given by \( \lambda \to L_\pm(\lambda)^{-1}g(\lambda) \).

In general, operator functions with these properties are called the wave matrices of \( W_\pm, W^*_\pm \) w.r.t. given fixed spectral representations (see Baumgärtel/Wollenberg [15, p. 177] for these concepts). Note that wave matrices are well-defined only if the spectral representations are fixed.

**Lemma 2.** The wave matrices of \( W_\pm, W^*_\pm \) w.r.t. the natural spectral representations in \( \mathcal{H}_{E_0}, \mathcal{H}_E \) are given by

\[
W_\pm(\lambda) = L_\pm(\lambda), \quad W^*_\pm(\lambda) = L_\pm(\lambda)^{-1}, \quad \lambda \in \mathbb{R}.
\]

Note that in the natural spectral representation of \( \mathcal{H}_{E_0} \) the vectors \( \Gamma e, e \in \mathcal{E} \) are considered in some sense as "constants", whereas the corresponding function as a function in \( \mathcal{H}_0 \) w.r.t. the usual \( \mathcal{K} \)-representation is given by \( \lambda \to (\Gamma e)(\lambda) = M(\lambda)e \).

As is well-known (see e.g. Baumgärtel/Wollenberg [15, p. 398 ff.]) the scattering matrix \( S_K(\lambda) := (W^*_+ W_\mp)(\lambda) \) in the usual \( \mathcal{K} \)-representation of \( \mathcal{H}_0 = \mathcal{H}_{E_0} \oplus (\mathcal{H}_0 \ominus \mathcal{H}_{E_0}) \) is given by

\[
S_K(\lambda) = \mathbb{I}_\mathcal{K} - 2\pi i M(\lambda)L_+(\lambda)^{-1}M(\lambda)^*, \quad \lambda \in \mathbb{R}.
\]  

(9)

**LEMMA 3.** On \( \mathcal{H}_{E_0} \) and w.r.t. the natural spectral representation of \( \mathcal{H}_{E_0} \) the scattering matrix \( S_\mathcal{E}(\cdot) \) is given by

\[
S_\mathcal{E}(\lambda) = L_+(\lambda)^{-1}L_-(\lambda) = L_+(\lambda)^{-1}L_+(\lambda)^*.
\]  

(10)
This means if \( f \in \mathcal{H}_{E_0} \) and \( \tilde{f}(\cdot) \) is its representer w.r.t. \( E_0 \), i.e. \( f(\lambda) = M(\lambda)\tilde{f}(\lambda) \) then \( S_\mathcal{E}(\lambda)\tilde{f}(\lambda) \) is the \( E_0 \)-representer of \( Sf \), where \( (Sf)(\lambda) = S_\mathcal{E}(\lambda)f(\lambda) \).

Proof. We have to prove that \( S_\mathcal{K}(\lambda)M(\lambda)\tilde{f}(\lambda) = M(\lambda)S_\mathcal{E}(\lambda)\tilde{f}(\lambda) \). But this is obvious because of

\[
M(\lambda)L_+^{-1}(\lambda) = (\mathbb{1}_\mathcal{K} - 2\pi i M(\lambda)L_+^{-1}(\lambda)^* M(\lambda))M(\lambda) = S_\mathcal{K}(\lambda)M(\lambda), \quad (11)
\]

\( \square \)

REMARK 1. In the following we restrict the consideration to the case that \( \Gamma \mathcal{E} \) is generating for \( H_0 \) and \( \mathcal{E} \) is generating for \( H \), i.e. we assume \( \mathcal{H}_E = \mathcal{H} \) and \( \mathcal{H}_{E_0} = \mathcal{H}_0 \). This implies \( \dim \mathcal{E} = \dim \mathcal{K} \). Moreover, the operator function \( \lambda \rightarrow M(\lambda) \in \mathcal{L}(\mathcal{E} \rightarrow \mathcal{K}) \) is then invertible for all \( \lambda, M(\lambda)^{-1} \in \mathcal{L}(\mathcal{K} \rightarrow \mathcal{E}) \).

3 Gelfand Triplets

3.1 The Schwartz space triplet on \( \mathcal{H}_0 \) and its transformation to \( \mathcal{H} \)

By \( \mathcal{S} \) we denote the space of all Schwartz functions \( \lambda \rightarrow s(\lambda) \in \mathcal{K} \) with values in \( \mathcal{K} \). The canonical norms on \( \mathcal{S} \) are denoted by \( \| \cdot \|_{\sigma} \), where \( \sigma \) labels these norms. \( \mathcal{S} \subset \mathcal{H}_0 \) is dense in \( \mathcal{H}_0 \) w.r.t. the Hilbert space norm of \( \mathcal{H}_0 \). The space of all continuous anti-linearforms on \( \mathcal{S} \) is denoted by \( \mathcal{S}^\times \). Then

\[
\mathcal{S} \subset \mathcal{H}_0 \subset \mathcal{S}^\times
\]

is a Gelfand triplet w.r.t. \( \mathcal{H}_0 \), the Schwartz space triplet. The representer of \( s \) in the \( E_0 \)-representation is denoted by \( \tilde{s} \), \( s(\lambda) = M(\lambda)\tilde{s}(\lambda) \), \( \lambda \rightarrow \tilde{s}(\lambda) \in \mathcal{E} \).

By the wave operator \( W_+ \) the Schwartz space triplet can be transformed to a triplet w.r.t. \( \mathcal{H} \). We put \( \mathcal{D} := W_+\mathcal{S} \) and equip \( \mathcal{D} \) with the topology of \( \mathcal{S} \). Thus we obtain the triplet

\[
\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times, \quad (12)
\]

Note that \( \mathcal{D}^\times = W_+^\times \mathcal{S}^\times \), where \( \mathcal{D}^\times \ni d^\times = W_+^\times s^\times \) is defined by

\[
\langle W_+^\times d \mid s^\times \rangle = \langle d \mid W_+^\times s^\times \rangle, \quad d \in \mathcal{D}.
\]

LEMMA 4. The triplet (12) satisfies the following properties:

(i) \( \mathcal{E} \subset \mathcal{D} \) and \( \mathcal{E} = W_+ \mathcal{T} \) where \( \mathcal{T} := \{ f \in \mathcal{H}_0 : f(\lambda) = M(\lambda)L_+^{-1}(\lambda)e, e \in \mathcal{E} \} \) is an \( N \)-dimensional subspace of \( \mathcal{H}_0 \) with \( \mathcal{T} \subset \mathcal{S} \),

(ii) \( \mathcal{D} = \mathcal{\Phi} \oplus \mathcal{E} \) where \( \mathcal{\Phi} := \{ W_+ s : s \in \mathcal{S} \cap (\mathcal{H}_0 \ominus \mathcal{T}) \} = P_\mathcal{E}^\perp \mathcal{D} \subset \mathcal{H}_0 \),

(iii) \( \mathcal{D}^\times = \mathcal{\Phi}^\times \times \mathcal{E} \) (cartesian product) where \( \mathcal{\Phi}^\times \) is the space of all continuous anti-linearforms on \( \mathcal{\Phi} \),

(iv) if \( d = \phi + e \) and \( d^\times = \{ \phi^\times, e^\times \} \) then \( \langle d \mid d^\times \rangle = \langle \phi \mid \phi^\times \rangle + \langle e, e^\times \rangle_{\mathcal{E}} \).
(v) \( H_0\Phi \subseteq \Phi \) and \( HD \subseteq D \).

Proof. (i)-(iv) are obvious because of Lemma 1. (v) is true because \( H_0 \) and \( H \) act on the representers of elements in \( \Phi, D \) by multiplication of the spectral parameters, respectively. \( \square \)

### 3.2 A modified Gelfand triplet

Recall that \( \text{spec}(H_0|E) \) is a finite set of (real) eigenvalues. Let \((a, b) \subset \mathbb{R} \) be an open interval with \( \text{spec}(H_0|E) \subset (a, b) \). Further let \( G_0 \subset \mathbb{C} \) an (open) connected symmetric region (symmetric w.r.t. complex conjugation) such that \( G_0 \cap \mathbb{R} = (a, b) \).

**Assumption 3.** The operator function \( \mathbb{R} \ni \lambda \rightarrow M(\lambda) \in \mathcal{L}(\mathcal{K} \rightarrow E) \) has a holomorphic continuation into \( G_0 \).

Then \( L_+(-) \) is holomorphic in \( \mathbb{C}_+ \cup G_0 \) and \( L_+(-)^{-1} \) is meromorphic there and even holomorphic in \( \mathbb{C}_+ \cup (a, b) \).

We introduce a modified Gelfand triplet: Recall first that the Schwartz functions have the representation \( s(\lambda) = M(\lambda)L_+(-)^{-1}x(\lambda), x(\lambda) \in E \), where the representer in the \( E_0 \)-representation is given by \( \tilde{s}(\lambda) = L_+(-)^{-1}x(\lambda) \). Now let \( S_0 \subset S \) be the following submanifold of the Schwartz space:

\[
S_0 := \{ s \in S : \lambda \rightarrow x(\lambda) \text{ is holomorphic continuable into } G_0 \}.
\]

\( S_0 \) is dense in \( S \) w.r.t. the Schwartz topology. The (stronger) topology in \( S_0 \) is defined by the collection of norms

\[
\|s_0\|_{\sigma,K} := \|s_0\|_{\sigma} + \sup_{z \in K \subset G_0} \|x(z)\|_E,
\]

where \( K \) runs through all compact subsets of \( G_0 \). Then

\[
S_0 \subset H_0 \subset S_0^x
\]

is a modified Gelfand triplet w.r.t. \( H_0 \).

The transformation of \( S_0 \) to \( H \) is given, as before, by \( D_0 := W_+S_0 \). Then

\[
D_0 \subset H \subset D_0^x
\]

is a Gelfand triplet w.r.t. \( H \). Similarly as in Lemma 4 we obtain

**Lemma 5.** The modified Gelfand triplet satisfies the following properties:

(i) \( \mathcal{E} \subset D_0 \).

(ii) \( D_0 = \Phi_0 \oplus \mathcal{E} \), where \( \Phi_0 = P_x^+D_0 \).

(iii) \( D_0^x = \Phi_0^x \times \mathcal{E} \) and for \( d_0 = \phi_0 + e, d_0^x = \{\phi_0^x, e^x\} \) one has

\[
\langle d_0 | d_0^x \rangle = \langle \phi_0 | \phi_0^x \rangle + (e, e^x)_E.
\]

(iv) \( H_0\Phi_0 \subseteq \Phi_0 \) and \( HD_0 \subseteq D_0 \).
Proof. (i) Since the functions $x(\cdot)$ for the elements $f \in \mathcal{T}$ are given by $x(\lambda) = e$ for all $\lambda$, i.e. by constants, the condition of holomorphic continuability is obviously satisfied. (ii)-(iv) are true because of Lemma 4. \(\square\)

REMARK 2. A simple example satisfying assumptions 1-3 is given for multiplicity $N = 1$, i.e. $\mathcal{E} = \mathbb{C}e_0$, then, according to Remark 1 one has also $\mathcal{K} = \mathbb{C}$. Let $\lambda_0 \in \mathbb{R}$ be the eigenvalue of $H_0$, $H_0e_0 = \lambda_0e_0$. Choose $\Gamma e_0(\lambda) := e^{-\lambda^2/2}$. Then

$$
\Gamma^*(z - H_0)^{-1}\Gamma e_0 = \int_{-\infty}^{\infty} \frac{e^{-\lambda^2}}{z - \lambda} d\lambda e_0
$$

and

$$
L_+(z) = z - \lambda_0 + \int_{-\infty}^{\infty} \frac{e^{-\lambda^2}}{\lambda - z} d\lambda,
$$

where we have omitted the factor $e_0$. Let $x_0 \in \mathbb{R}$. The calculation $z \to x_0 + i0$ gives

$$
L_+(x_0) = x_0 - \lambda_0 + i\pi e^{-x_0^2} + \int_{-\infty}^{\infty} \frac{e^{-\lambda^2}}{\lambda - x_0} d\lambda,
$$

where the integral is Cauchy’s mean value. This shows that $L_+(x_0) = 0$ is impossible because Cauchy’s mean value is real. That is, the assumptions 1 and 2 are satisfied. Assumption 3 is satisfied because $\lambda \to e^{-\lambda^2/2}$ is holomorphic in $\mathbb{C}$ hence $z \to L_+(z)$ is also holomorphic in $\mathbb{C}$. The same is true for $L_-(\cdot)$.

### 3.3 Resonances

We define the concept resonance for the Friedrichs model satisfying Assumptions 1, 2, 3 as follows:

The point $\zeta_0 \in \mathcal{G}_0 \cap \mathbb{C}_-$ is called a resonance if $\det L_+(\zeta_0) = 0$.

In other words, $\zeta_0$ is a resonance iff $\zeta_0$ is a pole of $L_+(\cdot)^{-1}$, i.e. a pole of the analytic continuation of the partial resolvent into $G_0 \cap \mathbb{C}_-$. From Lemma 3 we obtain: a point $\zeta_0 \in \mathcal{G}_0 \cap \mathbb{C}_-$ is a pole of $L_+(\cdot)^{-1}$ iff it is a pole of $S_{\mathcal{K}}(\cdot)$ resp. of $S_{\mathcal{E}}(\cdot)$.

### 4 Results

The first result (Theorem 1) says that exactly the resonances are eigenvalues of the extended Hamiltonian $H^\times$ w.r.t. the modified Gelfand triplet for $\mathcal{H}$, if for the corresponding eigenvectors a certain analyticity condition is required.

**THEOREM 1.** The point $\zeta_0 \in \mathcal{G}_0 \cap \mathbb{C}_-$ is an eigenvalue of the extended Hamiltonian $H^\times$ w.r.t. the Gelfand triplet $\mathcal{D}_0 \subset \mathcal{H} \subset \mathcal{D}_0^\times$ with eigenanti-linearform $d_0^\times := \{\phi_0^\times(\zeta_0, e_0), e_0\}$ satisfying the eigenvalue equation $H^\times d_0^\times = \zeta_0 d_0^\times$, where $\phi_0^\times(\zeta_0, e_0)$ is the analytic continuation into $G_0 \cap \mathbb{C}_-$ of a holomorphic vector anti-linearform $\phi_0^\times(z, e_0)$ in $\mathbb{C}_+$ iff $\zeta_0$ is a resonance. The anti-linearform $\mathbb{C}_+ \ni z \to \phi_0^\times(z, e)$ is given by

$$
\langle \phi | \phi_0^\times(z, e) \rangle := (\phi, (z - H_0)^{-1}\Gamma e)_{\mathcal{H}_0}, \quad \phi \in \Phi_0, \quad z \in \mathbb{C}_+.
$$
and $e_0$ satisfies $L_+(\zeta_0)e_0 = 0$, i.e. $e_0 \in \ker L_+(\zeta_0)$. That is, the (generalized) eigenspace of $\zeta_0$ is $q$-dimensional, where $q$ is the geometric multiplicity of the eigenvalue 0 of $L_+(\zeta_0)$.

The second result (Theorem 2) concerns the structure of the corresponding eigenanti-linear form $s_0^\times$ of $H_0^\times$ w.r.t. the modified Schwartz space triplet. This anti-linear form is given by

$$s_0^\times(\zeta_0,e_0) = (W_+^*)^x d_0^\times(\zeta_0,e_0).$$

It turns out that $s_0^\times$ is an anti-linear form on $S_0$ of a pure Dirac type w.r.t. the point $\overline{\zeta_0}$ and there is a very simple transformation formula from $e_0$ to the corresponding vector $k_0 \in K$.

**THEOREM 2.** The eigenanti-linear form $s_0^\times$ of $H_0^\times$ w.r.t. the Gelfand triplet $S_0 \subset H_0 \subset S_0^\times$, associated to $d_0^\times$ by $s_0^\times := (W_+^*)^x d_0^\times$ is given by

$$\langle s \mid s_0^\times(\zeta_0,e_0) \rangle = 2\pi i(s(\overline{\zeta_0}),k_0)_K, \quad s \in S_0,$$

where $k_0 := M(\zeta_0)e_0$.

The third result (Corollary 3) connects the eigenanti-linear form $s_0^\times(\zeta_0,e_0)$ with a corresponding Gamov vector which is uniquely determined by $s_0^\times$.

Recall that pre-Gamov vectors are considered (in this paper) as the eigenvectors of the truncated evolution $t \to Q_+ e^{-itH_0} \upharpoonright \mathcal{H}_+^2$, $t \geq 0$, where $\mathcal{H}_+^2 \subset H_0$ is the Hardy subspace for $C_+$ and $Q_+$ the projection onto this Hardy subspace. The truncated evolution is a strongly continuous contractive semigroup on $\mathcal{H}_+^2$ of the Toeplitz type (see e.g. Strauss [11]). As is well-known, each point $\zeta \in \mathbb{C}_+$ is an eigenvalue of the generator of this semigroup and the corresponding eigenspace is given by $\{f \in \mathcal{H}_+^2 : f(\lambda) := k(\lambda - \zeta)^{-1}, k \in K\}$, i.e. the dimension of the eigenspace of $\zeta$ coincides with $\dim K$.

Now the decisive question is which pre-Gamov vectors are connected with eigenanti-linear forms of $H_0^\times$. The first answer is that one has to select the poles of $L_+(\cdot)^{-1}$ resp. of $S_K(\cdot)$. However it remains the question: which values of $k \in K$ have to be chosen such that the pre-Gamov vector given by $k$ is in fact connected to an eigenanti-linear form of $H_0^\times$?

Recall first that $S_0 \cap \mathcal{H}_+^2 \subset \mathcal{H}_+^2$ is dense in $\mathcal{H}_+^2$ w.r.t. the Hilbert space norm of $\mathcal{H}_+^2$. The mentioned connection is then simply given by restriction of $s_0^\times$ to $S_0 \cap \mathcal{H}_+^2$.

**COROLLARY 3.** The restricted eigenanti-linear form $s_0^\times\upharpoonright S_0 \cap \mathcal{H}_+^2$

$$\mathcal{S}_0 \cap \mathcal{H}_+^2 \ni s \to 2\pi i(s(\overline{\zeta_0}),k_0)_K$$

is even continuous w.r.t. the Hilbert space topology of $\mathcal{H}_+^2$, i.e. it can be continuously extended onto $\text{clo}(S_0 \cap \mathcal{H}_+^2) = \mathcal{H}_+^2$. That is, $s_0^\times\upharpoonright \mathcal{H}_+^2$ is realized by the $\mathcal{H}_+^2$-vector $k_0(\zeta_0 - \lambda)^{-1}$ via the relation

$$2\pi i(s(\overline{\zeta_0}),k_0) = \int_{-\infty}^{\infty} \left(s(\lambda),\frac{k_0}{\zeta_0 - \lambda}\right)_K d\lambda. \quad (13)$$

Proof. (13) follows immediately from the Paley-Wiener theorem. □
Corollary 3 means: the restriction on $\mathcal{H}_\pm^\times$ of the eigenanti-linearform $s_0^\times$, which is the back transform $s_0^\times = (W_+^\times)^* d_0^\times$ of $d_0^\times$, associated to the resonance $\zeta_0$ and to the parameter vector $e_0 \in \ker L_+ (\zeta_0)$, to the Hilbert space $\mathcal{H}_0$ resp. the corresponding Gelfand triplet yields the associated Gamov vector $\lambda = k_0 (\zeta_0 - \lambda)^{-1}$, where $k_0 = M (\zeta_0) e_0$. Conversely, exactly the pre-Gamov vectors where $\zeta_0$ is a resonance and $k_0 = M (\zeta_0)$ with $e_0 \in \ker L_+ (\zeta_0)$ have an extension (or "continuation") to an eigenanti-linearform of the extended Hamiltonian $H^\times$ w.r.t. the Gelfand triplet $\mathcal{D}_0 \subset \mathcal{H} \subset \mathcal{D}_0^\times$. That is exactly these pre-Gamov vectors are true Gamov vectors.

The last result presents a simple partial answer to the question, how the parameter space $M (\zeta_0) \ker L_+ (\zeta_0)$ can be derived from the Laurent expansion of the scattering matrix $S_\xi (\cdot)$ at $\zeta_0$.

**PROPOSITION 4.** If $\zeta_0$ is a simple pole of $S_\xi (\cdot)$ then

$$\ker L_+ (\zeta_0) = \text{ima} \{ \text{Res}_{z = \zeta_0} S_\xi (z) \}. \tag{14}$$

Proof. An easy calculation gives

$$\ker L_+ (\zeta_0) = \text{ima} \, L_{-1} = \text{ima} (L_{-1} L_+ (\zeta_0)^*) ,$$

where $L_{-1} = \text{Res}_{z = \zeta_0} L_+ (z)^{-1}$. This gives (14). Note that $L_+ (\zeta_0)^*$ exists. □

**REMARK 3.** The relation between the order $g$ of the pole $\zeta_0$ of $S_\xi (\cdot)$ and $g := \dim \ker L_+ (\zeta_0)$ is complicated. If $m \leq N = \dim \mathcal{E}$ is the algebraic multiplicity of the eigenvalue 0 of $L_+ (\zeta_0)$ and $1 \leq r \leq m$, the order of the zero $\zeta_0$ of $\det L_+ (z)$, then in any case $1 \leq g \leq r$ (see e.g. [16] for details).

## 5 Proofs

### 5.1 Proof of Theorem 1

The eigenvalue equation for eigenvalues $\zeta_0 \in G_0 \cap \mathbb{C}_-$ of $H^\times$ w.r.t. the triplet $\mathcal{D}_0 \subset \mathcal{H} \subset \mathcal{D}_0^\times$ reads

$$\langle d \mid H^\times d_0^\times \rangle = \langle d \mid \zeta_0 d_0^\times \rangle, \quad d \in \mathcal{D}_0 ,$$

or

$$\langle H d \mid d_0^\times \rangle = \langle \overline{\zeta_0} d \mid d_0^\times \rangle, \quad d \in \mathcal{D}_0 ,$$

where $d = \phi + e$, $\phi \in \Phi_0$, $e \in \mathcal{E}$, $d_0 = \{ \phi_0^\times, e_0 \}$, $\phi_0^\times \in \Phi_0^\times$, $e_0 \in \mathcal{E}$. This is equivalent with

$$(H_0 e - \overline{\zeta_0} e, e_0) + \langle \Gamma e \mid \phi_0^\times \rangle = \langle \overline{\zeta_0} \phi - H_0 \phi \mid \phi_0^\times \rangle - (\Gamma^* \phi, e_0) .$$

Since $e$ and $\phi$ vary independently we obtain two equations:

$$\langle (\overline{\zeta_0} - H_0) e, e_0 \rangle = \langle \Gamma e \mid \phi_0^\times \rangle, \quad e \in \mathcal{E} , \tag{15}$$

and

$$\langle (\overline{\zeta_0} - H_0) \phi \mid \phi_0^\times \rangle = (\Gamma^* \phi, e_0) , \quad \phi \in \Phi_0 . \tag{16}$$

$\phi_0^\times$ depends on $\zeta_0$, the possible eigenvalue (and on $e_0$). According to our analyticity condition for $\phi_0^\times$ this anti-linearform is required to be the analytic continuation of a
holomorphic vector anti-linearform $\mathbb{C}_+ \ni z \to \phi^\lambda_0(z)$. This means that equation (16) has to be valid also on $\mathbb{C}_+$ and it is a vector anti-linearform there:

$$((\mathcal{F} - H_0)\phi, \phi^\lambda_0(z))_{\mathcal{H}_0} = (\Gamma^*\phi, e_0)\varepsilon, \quad z \in \mathbb{C}_+, \quad \phi \in \Phi_0,$$

or

$$(\phi, (z - H_0)\phi^\lambda_0(z))_{\mathcal{H}_0} = (\phi, \Gamma e_0)\varepsilon, \quad z \in \mathbb{C}_+, \quad \phi \in \Phi_0.$$  

This means $(z - H_0)\phi^\lambda_0(z) = \Gamma e_0$ or

$$\phi^\lambda_0(z) = (z - H_0)^{-1}\Gamma e_0, \quad z \in \mathbb{C}_+.$$

Now we have to check that this anti-linearform on $\Phi_0$ is analytically continuable into $\mathbb{C}_+ \cup \mathcal{G}_0$ as a holomorphic anti-linearform according to the requirement in Theorem 1:

We have shown in Subsection 3.2 that the elements $s \in \mathcal{S}_0$ have the representation

$$s(\lambda) = M(\lambda)L_+(\lambda)^{-1}x(\lambda),$$

where $\lambda \to x(\lambda) \in \mathcal{E}$. Then $(W_+s)(\lambda) = x(\lambda)$ and the function $x(\lambda)$ is holomorphic continualeible into $\mathcal{G}_0$. If $\zeta \in \mathbb{C}_+$ we have

$$\langle \phi | \phi^\lambda_0(\zeta) \rangle = \langle P_+^tW_+s, (\zeta - H_0)^{-1}\Gamma e_0 \rangle = \langle W_+s, (\zeta - H_0)^{-1}\Gamma e_0 \rangle = \left( \int_{-\infty}^{\infty} (E(d\lambda)x(\lambda), (\zeta - H_0)^{-1}\Gamma e_0) d\lambda \right)$$

Since $x(\lambda) = \sum_{j=1}^N x_j(\lambda)b_j$, where the $\{b_j\}_j$ form an orthonormal basis of $\mathcal{E}$, we obtain

$$\langle \phi | \phi^\lambda_0(\zeta) \rangle = \sum_{j=1}^N \int_{-\infty}^{\infty} x_j(\lambda) \frac{(E(d\lambda)b_j, R_0(\zeta)\Gamma e_0)}{d\lambda}.$$  

so that we have to calculate the expression

$$\frac{(E(d\lambda)e, R_0(\zeta)\Gamma e_0)}{d\lambda}$$

for any $e \in \mathcal{E}$. This calculation starts with the identity

$$(R(z)e, R_0(\zeta)\Gamma e_0) = (R_0(z)\Gamma L_+(z)^{-1}e, R_0(\zeta)\Gamma e_0), \quad z, \zeta \in \mathbb{C}_+,$$

where for the calculation of the right hand side the explicit expression for the resolvent $R(z) = (z - H)^{-1}$ is used. This implies

$$(R(\mu \pm i0)e, R_0(\zeta)\Gamma e_0) = \frac{1}{\mu - \zeta} \left( (R_0(\zeta)\Gamma L_+^{\pm}(\mu)^{-1}e, \Gamma e_0) - (R_0(\mu \pm i0)\Gamma L_+^{\pm}(\mu)^{-1}e, \Gamma e_0) \right).$$

Using

$$\frac{E(d\mu)}{d\mu} = \frac{1}{2\pi i} \left( R(\mu - i0) - R(\mu + i0) \right).$$
finally after a lengthy but straightforward calculation we obtain

\[
\frac{(E(\lambda) e, R_0(\zeta) \Gamma e_0)}{\lambda} = \frac{1}{\mu - \zeta} \left( L_+(\mu)^{-1} M(\mu)^* M(\mu) L_+(\mu)^1 e, (\zeta - \mu - L_+(\zeta)) e_0 \right).
\]

(18)

Inspection of (18) proves the assertion. Now we know that the anti-linearform \( \phi_0^*(z) \) satisfies the equation (17) for \( z \in \mathbb{C}_+ \). Therefore \( \phi_0^*(\zeta, e_0) \) satisfies the equation (16) for all \( \zeta \in G_0 \cup \mathbb{C}_+ \) (where now we have taken into account the second parameter \( e_0 \)). Since \( z \to \phi_0^*(z, e_0) \) is holomorphic in the whole region \( G_0 \cup \mathbb{C}_+ \) we consider the (second) equation (15) first on \( \mathbb{C}_+ \). Then it reads

\[
(\bar{z} - H_0) e, e_0) = \langle \Gamma e | \phi_0^*(z, e_0) \rangle = \langle \Gamma e, (z - H_0)^{-1} \Gamma e_0 \rangle = \langle e, \Gamma^*(z - H_0)^{-1} \Gamma e_0 \rangle
\]

so that we have

\[
(\bar{e}, (z - H_0) e_0) - \langle \Gamma e | \phi_0^*(z, e_0) \rangle = (e, L_+(z) e_0), \quad e \in E, \quad z \in \mathbb{C}_+,
\]

(19)

and the equation (15) reads simply \( (\bar{e}, L_+(z) e_0) = 0 \) for all \( e \in E \) which obviously has no solution in \( \mathbb{C}_+ \cup (a, b) \). But by analytic continuation the identity (19) is true also in \( \mathbb{C}_- \cap G_0 \). That is, equation (15) is equivalent to

\[
L_+(\zeta_0) e_0 = 0, \quad \zeta_0 \in \mathbb{C}_- \cap G_0.
\]

(20)

This means: equation (15) has a solution \( \zeta_0 \) with corresponding parameter \( e_0 \in E \) iff equation (20) is satisfied. Conversely, if \( \zeta_0 \in \mathbb{C}_- \cap G_0 \) and \( e_0 \in E \) satisfy equation (20) then \( \zeta_0 \) is an eigenvalue of \( H^x \) and \( d_0^x := \{ \phi_0^*(\zeta_0, e_0) \} \) is a corresponding eigenanti-linearform. The dimension of the eigenspace of \( \zeta_0 \) is then \( \dim \ker L_+(\zeta_0) \). □

### 5.2 Proof of Theorem 2

To calculate \( s^x_0(\zeta_0, e_0) = (W^x_0)^* d_0^x \) with \( d_0^x = \{ \phi_0^x(\zeta_0, e_0), e_0 \} \) first we consider \( \phi_0^x \) again for \( z \in \mathbb{C}_+ \) and calculate \( s_0^x(z, e_0) = (W^x_0)^* \{ \phi_0^x(z, e_0), e_0 \} \). Later on we consider the analytic continuation into \( G_0 \cap \mathbb{C}_- \). We start with

\[
\langle s | s_0^x(z, e_0) \rangle = \langle W^x_0 s | d_0^x(z, e_0) \rangle
\]

\[
= \langle P^x_0 W^x_0 s | \phi^x_0(z, e_0) \rangle + \langle P^x_0 W^x_0 s, e_0 \rangle
\]

\[
= \langle P^x_0 W^x_0 s, (z - H_0)^{-1} \Gamma e_0 \rangle + \langle P^x_0 W^x_0 s, e_0 \rangle
\]

\[
= \langle W^x_0 s, (z - H_0)^{-1} \Gamma e_0 \rangle + \langle s, W^x_0 e_0 \rangle
\]

We have \( W^x_0 e_0 = \int_{-\infty}^{\infty} E_0(d\lambda) \Gamma L_+(\lambda)^{-1} e_0 d\lambda \) and \( W^x_0 s = \int_{-\infty}^{\infty} E(d\lambda) L_+(\lambda) \tilde{s}(\lambda) \), where \( s = \int_{-\infty}^{\infty} E_0(d\lambda) \Gamma s(\lambda) \), i.e. \( \tilde{s}(\cdot) \) is the representor of \( s \) w.r.t. the \( E_0 \)-representation, \( s(\lambda) = M(\lambda) \tilde{s}(\lambda) \). Then

\[
(W^x_0 s, R_0(z) \Gamma e_0) = \int_{-\infty}^{\infty} \frac{\langle E(d\lambda) L_+(\lambda) \tilde{s}(\lambda), R_0(z) \Gamma e_0 \rangle}{d\lambda} d\lambda.
\]

Again we use (18) for the calculation of this expression and obtain

\[
(W^x_0 s, R_0(z) \Gamma e_0) = \]
\[
\int_{-\infty}^{\infty} \frac{1}{\mu - z} \left( L_- (\mu)^{-1} M(\mu)^* M(\mu) L_+ (\mu)^{-1} L_+ (\mu) \tilde{s}(\mu), (z - \mu - L_+ (z)) e_0 \right) d\mu = \\
- \int_{-\infty}^{\infty} (L_- (\mu)^{-1} M(\mu)^* s(\mu), e_0) d\mu + \int_{-\infty}^{\infty} \frac{1}{z - \mu} \left( L_- (\mu)^{-1} M(\mu)^* M(\mu) \tilde{s}(\mu), L_+ (z) e_0 \right) d\mu.
\]

Furthermore we have

\[
(s, W^*_+ e_0) = \left( s, \int_{-\infty}^{\infty} E_0 (d\lambda) \Gamma L_+ (\lambda)^{-1} e_0 d\lambda \right) \\
= \int_{-\infty}^{\infty} \left( s(\lambda), M(\lambda) L_+ (\lambda)^{-1} e_0 \right)_K d\lambda \\
= \int_{-\infty}^{\infty} \left( L_- (\lambda)^{-1} M(\lambda)^* s(\lambda), e_0 \right)_K d\lambda,
\]

so that we finally obtain

\[
\langle s \mid s_0^\times (z, e_0) \rangle = \left( \int_{-\infty}^{\infty} \frac{1}{\mu - z} L_- (\mu)^{-1} M(\mu)^* s(\mu) d\mu, L_+ (z) e_0 \right)_K.
\]

For the analytic continuation from \( z \in \mathbb{C}_+ \) into \( \mathbb{C}_+ \cup G_0 \) we have to check the integral

\[
\Psi_-(\overline{z}) := \int_{-\infty}^{\infty} \frac{1}{\overline{z} - \mu} L_- (\mu)^{-1} M(\mu)^* s(\mu) d\mu.
\]

Since this integral is the left factor in the scalar product we substitute for the moment \( z \to \overline{z} \), consider

\[
\Psi_-(z) := \int_{-\infty}^{\infty} \frac{1}{z - \mu} L_- (\mu)^{-1} M(\mu)^* s(\mu) d\mu, \quad z \in \mathbb{C}_-, \quad (22)
\]

and check the continuation into \( \mathbb{C}_+ \). Recall that \( z \to \Psi_+(z) \) for \( z \in \mathbb{C}_+ \) is defined by one and the same formula (22). Then we obtain for \( z \in \mathbb{C}_+ \)

\[
\Psi_-(z) = \Psi_+(z) + 2\pi i L_- (z)^{-1} M(z)^* s(z) \\
= \Psi_+(z) + 2\pi i (L_+ (z)^{-1})^* M(z)^* s(z).
\]

Substituting again \( z \to \overline{z} \), i.e. now we have \( \overline{z} \in \mathbb{C}_+ \) and \( z \in \mathbb{C}_- \), we obtain

\[
(\Psi_-(\overline{z}), L_+ (z) e_0) = (\Psi_+(\overline{z}), L_+ (z) e_0) + 2\pi i ((L_+ (z)^{-1})^* M(z)^* s(\overline{z}), L_+ (z) e_0),
\]

where \( \Psi_+(\overline{z}) \) is a holomorphic part such that the first term vanishes for \( z = \zeta_0 \). Then we have

\[
\langle s \mid s_0^\times (z, e_0) \rangle = 2\pi i (M(z)^* s(\overline{z}), L_+ (z)^{-1} L_+ (z) e_0) + (\Psi_+(\overline{z}), L_+ (z) e_0)
\]

and

\[
\langle s \mid s_0^\times (\zeta_0, e_0) \rangle = 2\pi i (M(\zeta_0)^* s(\overline{\zeta_0}), e_0) = 2\pi i (s(\overline{\zeta_0}), M(\zeta_0) e_0)_K,
\]

that is, the anti-linearform \( s_0^\times (\zeta_0, e_0) \) is of pure Dirac type w.r.t. the point \( \overline{\zeta_0} \) and the corresponding vector \( k_0 \in \mathcal{K} \) with

\[
\langle s \mid s_0^\times (\zeta_0, e_0) \rangle = 2\pi i (s(\overline{\zeta_0}), k_0)_K
\]

is given by

\[
k_0 := M(\zeta_0) e_0.
\]

This confirms the fact (which is known from the beginning) that the subspace of the admissible vectors \( k \in \mathcal{K} \) has the dimension \( \dim \ker L_+ (\zeta_0) \), too. \( \square \)
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