Multiqubit quantum teleportation

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Abstract

We provide a class of six-qubit states for three-qubit perfect teleportation. These states include the six-qubit cluster states as a special class. We generalize this class of six-qubit states to $2^n$-qubit pure states for $n$-qubit teleportation, $n \geq 1$. These states can be also used for $2^n$-bit classical information transmission in dense coding.

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1. Introduction

Quantum teleportation, employing classical communication and shared resource of entanglement, allows us to transmit an unknown quantum state from a sender to a receiver that are spatially separated. Let $|\phi\rangle$ be an arbitrary unknown pure state that is to be sent from Alice to Bob, and $|\psi\rangle$ the entangled state shared by Alice and Bob. To carry out teleportation, Alice needs to perform projective measurements on her two particles: one in the state $|\phi\rangle$ and one part of the entangled state $|\psi\rangle$. Learning the measurement results from Alice via the classical communication channel, Bob applies a corresponding unitary transformation on the other part of the entangled state $|\psi\rangle$, so as to transform the state of this part to the unknown state $|\phi\rangle$. In this scenario, Bennett et al [1] first demonstrate the teleportation of an arbitrary qubit state in terms of an entangled Einstein–Podolsky–Rosen pair. Then the three-qubit Greenberger–Horne–Zeilinger (GHZ) state and a class of W states are revealed to be the ideal resource for faithful teleportation of one-qubit state [2, 3]. For two-qubit teleportation, the tensor product of two Bell states [4], the genuine four-qubit entangled state [5] and one five-qubit state in [6] are shown to have the ability for faithful teleportation. Lee et al [7] have also proposed a scheme for two-qubit teleportation. For three-qubit teleportation, Choudhury et al [8] and Yin et al [9] have investigated the teleportation by one genuine entangled six-qubit state. Generally, Chen et al [10] provide the necessary and sufficient condition that the genuine
2n-qubit entanglement channels must satisfy to teleport an arbitrary n-qubit state, and Cheung and Zhang [11] analyze the criterion of multiqubit states for n-qubit teleportation. Then one kind of entangled 2n-qubit states has been presented for n-qubit teleportation [12]. Besides potential applications in quantum communication, quantum error correction and one way quantum computation, the cluster states are also of importance in quantum teleportation and dense coding [13].

In this paper, we propose a class of 2n-qubit states for n-qubit \( n \geq 1 \) perfect teleportation, which are also the ideal resource for transmission of 2n-bit classical information in dense coding. This class of states is not equivalent to the tensor product of Bell states. For three-qubit and two-qubit cases, this class of states includes the cluster states as a special case.

2. Three-qubit teleportation

We first consider the quantum teleportation of an arbitrary unknown three-qubit state \( |\phi\rangle^{(3)} \). Suppose Alice and Bob share \textit{a priori} three pairs of qubits in the state

\[
|\xi^{(3,3)}_{AB}\rangle = \frac{1}{2\sqrt{2}} \sum_{K=0}^{7} |\vec{K}^{(3)}_A\rangle \otimes |\vec{K}'^{(3)}_B\rangle,
\]

where \( |\vec{K}^{(3)}_A\rangle \) constitute an orthonormal basis,

\[
|\vec{0}^{(3)}_A\rangle = |00\rangle \otimes (\cos \theta_1 |0\rangle + \sin \theta_1 |1\rangle),
|\vec{1}^{(3)}_A\rangle = |00\rangle \otimes (-\sin \theta_1 |0\rangle + \cos \theta_1 |1\rangle),
|\vec{2}^{(3)}_A\rangle = |01\rangle \otimes (\cos \theta_2 |0\rangle + \sin \theta_2 |1\rangle),
|\vec{3}^{(3)}_A\rangle = |01\rangle \otimes (-\sin \theta_2 |0\rangle - \cos \theta_2 |1\rangle),
|\vec{4}^{(3)}_A\rangle = |10\rangle \otimes (\cos \theta_3 |0\rangle + \sin \theta_3 |1\rangle),
|\vec{5}^{(3)}_A\rangle = |10\rangle \otimes (-\sin \theta_3 |0\rangle + \cos \theta_3 |1\rangle),
|\vec{6}^{(3)}_A\rangle = |11\rangle \otimes (\cos \theta_4 |0\rangle + \sin \theta_4 |1\rangle),
|\vec{7}^{(3)}_A\rangle = |11\rangle \otimes (-\sin \theta_4 |0\rangle + \cos \theta_4 |1\rangle),
\]

and \( |\vec{K}'^{(3)}_A\rangle \) constitute another orthonormal basis,

\[
|\vec{0}'^{(3)}_B\rangle = |00\rangle \otimes (\cos \theta'_1 |0\rangle + \sin \theta'_1 |1\rangle),
|\vec{1}'^{(3)}_B\rangle = |00\rangle \otimes (-\sin \theta'_1 |0\rangle + \cos \theta'_1 |1\rangle),
|\vec{2}'^{(3)}_B\rangle = |01\rangle \otimes (\cos \theta'_2 |0\rangle + \sin \theta'_2 |1\rangle),
|\vec{3}'^{(3)}_B\rangle = |01\rangle \otimes (-\sin \theta'_2 |0\rangle + \cos \theta'_2 |1\rangle),
|\vec{4}'^{(3)}_B\rangle = |10\rangle \otimes (\cos \theta'_3 |0\rangle + \sin \theta'_3 |1\rangle),
|\vec{5}'^{(3)}_B\rangle = |10\rangle \otimes (-\sin \theta'_3 |0\rangle + \cos \theta'_3 |1\rangle),
|\vec{6}'^{(3)}_B\rangle = |11\rangle \otimes (\cos \theta'_4 |0\rangle + \sin \theta'_4 |1\rangle),
|\vec{7}'^{(3)}_B\rangle = |11\rangle \otimes (-\sin \theta'_4 |0\rangle + \cos \theta'_4 |1\rangle),
\]

with \( 0 \leq \theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta'_4 \leq \frac{\pi}{2} \). Here these orthonormal basis \( \{|\vec{K}^{(3)}_A\rangle\}_{K=0}^{7} \) and \( \{|\vec{K}'^{(3)}_B\rangle\}_{K=0}^{7} \) can be viewed factually as the generalizations of the computational basis, and they will be shown to be able to give more resource for quantum teleportation. If we choose the basis for \( A \) and \( B \) as the three-qubit computational basis, then the resource for quantum teleportation in the form of equation (1) is naturally the tensor product of Bell states.

\[
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M-J Zhao et al
The arbitrary unknown three-qubit state \( |\phi_{AB}^{(3)}\rangle \) can be expressed as

\[
|\phi_{AB}^{(3)}\rangle = \sum_{K=0}^{7} a_K |K_{AB}^{(3)}\rangle
\]

with \( \sum_{K=0}^{7} |a_K|^2 = 1 \). Noting that \(|\xi_{AB}^{(3)}\rangle\rangle\) is a maximally entangled state, we may construct the following basis of 64 orthonormal states:

\[
|\Pi_{ijk}^{(3)}\rangle = (\sigma^{(i)} \otimes \sigma^{(j)} \otimes \sigma^{(k)})|\Pi_{000}^{(3)}\rangle, \quad (2)
\]

where \(|\Pi_{000}^{(3)}\rangle\rangle = \sum_{K=0}^{7} |K_{AB}^{(3)}\rangle \otimes |K_{AB}^{(3)}\rangle\rangle \), \( \sigma^{(i)} \) is the 2 \times 2 identity matrix and \( \sigma^{(1)}, \sigma^{(2)} \) and \( \sigma^{(3)} \) are three Pauli operators. If Alice performs a complete projective measurement jointly on \( A' \) in the above basis in equation (2) with the measurement outcome \( ijk \), then Bob’s sequences of qubits will be in the state \( \sigma^{(i)} \otimes \sigma^{(j)} \otimes \sigma^{(k)} |\phi_{AB}^{(3)}\rangle \). Bob will always succeed in recovering an exact replica of the original unknown state upon receiving 8 bits of classical information about measurement results from Alice. Namely

\[
|\phi_{AB}^{(3)}\rangle \otimes |\xi_{AB}^{(3)}\rangle = \frac{1}{8} \sum_{ijk} |\Pi_{ijk\rangle}^{(3)}\rangle (\sigma^{(i)} \otimes \sigma^{(j)} \otimes \sigma^{(k)})|\phi_{AB}^{(3)}\rangle. \quad (3)
\]

These equations follow from the result given below, which also guarantees the success of the protocol.

\[
|\Pi_{000}^{(3)}\rangle (|\xi_{AB}^{(3)}\rangle) = \frac{1}{8} (\sigma^{(i)} \otimes \sigma^{(j)} \otimes \sigma^{(k)})|\phi_{AB}^{(3)}\rangle. \quad (4)
\]

It can be verified that the reduced matrix \( \rho_{AB} = tr_{A'B}(|\xi_{AB}^{(3)}\rangle \langle \xi_{AB}^{(3)}|) \) is not a pure state, as it is not rank 1 typically. This can be seen from its nonsingular submatrix spanned by \(|\langle 11|00\rangle, \langle 00|11\rangle, \langle 11|11\rangle\rangle \): [cos\(^2\) (\(\theta_1 - \theta_1') + \cos\^2(\theta_2 + \theta'_2) + \cos\^2(\theta_3 - \theta_3')] + \cos\^2(\theta_4 - \theta_4'] |\langle 00|00\rangle, \langle 00|11\rangle, \langle 11|11\rangle\rangle + \cos\^2(\theta_3 - \theta_3') - \cos\^2(\theta_2 + \theta'_2) + \cos\^2(\theta_4 - \theta_4'] |\langle 00|00\rangle, \langle 00|11\rangle, \langle 11|11\rangle\rangle]. Therefore, \( |\xi_{AB}^{(3)}\rangle \) is not equivalent to the tensor product of Bell states, \( \bigotimes_{i=1}^{3} (|00\rangle + |11\rangle)_{iA'B} \). Furthermore, \( \rho_{A'B} \) is not maximally mixed state generally, so it is not equivalent to the genuine entangled six-qubit state in [8, 9]. When \( \theta_1 = \theta_1', \theta_2 + \theta_2' = \frac{\pi}{2} \), \( \theta_3 = \theta_3', \theta_4 = 0 \) and \( \theta_4' = \frac{\pi}{2} \), the entangled state \( |\xi_{AB}^{(3)}\rangle \) becomes

\[
|\xi_{AB}^{(3)}\rangle = \frac{1}{\sqrt{2}} (|110 111\rangle - |111 110\rangle - |101 101\rangle + |100 100\rangle \\
+ |000 000\rangle + |001 001\rangle + |010 011\rangle + |101 010\rangle)_{AB} |\xi_{AB}^{(3)}\rangle,
\]

which is exactly the six-qubit cluster state [13].

Additionally, \(|\xi_{AB}^{(3)}\rangle\rangle\) is 64-bit classical information. A dense coding scheme using \(|\xi_{AB}^{(3)}\rangle\rangle\) is the following. Because \(|\xi_{AB}^{(3)}\rangle\rangle\) is maximally entangled between \( A \) and \( B \), \(|\xi_{AB}^{(3)}\rangle\rangle \rangle_{A_1,A_2,B_1,B_2} \) are 64 maximally entangled states and constitute an orthonormal basis for the 64-dimensional Hilbert space. The classical information sender Alice encodes her messages by using operators \( \sigma^{(i)}, \sigma^{(j)}, \sigma^{(k)} \) and sends her qubits to Bob. Bob then decodes the messages by performing a joint measurement on all six qubits in the basis in equation (2).
3. Multiqubit quantum teleportation

Now we investigate the teleportation of an arbitrary $n$-qubit state $|\phi_A\rangle^{(n)}$ by extending the above ‘generalized cluster-like’ six-qubit states to 2$n$-qubit ones. A priori sequences of qubits shared by Alice and Bob are in the state

$$|\xi\rangle_{AB}^{(n,n)} = \frac{1}{\sqrt{2^n}} \sum_{K=0}^{2^n-1} |\vec{K}_A^{(n)}\rangle \otimes |\vec{K}_B^{(n)}\rangle,$$

(5)

with $|\vec{K}_A^{(n)}\rangle$ an orthonormal basis,

$$|0\rangle_{A}^{(n)} = |0\rangle_{B}^{(n-1)} \otimes (\cos \theta_1|0\rangle + \sin \theta_1|1\rangle),$$

$$|1\rangle_{A}^{(n)} = |0\rangle_{B}^{(n-1)} \otimes (-\sin \theta_1|0\rangle + \cos \theta_1|1\rangle),$$

$$|2\rangle_{A}^{(n)} = |1\rangle_{B}^{(n-1)} \otimes (\cos \theta_2|0\rangle + \sin \theta_2|1\rangle),$$

$$|3\rangle_{A}^{(n)} = |1\rangle_{B}^{(n-1)} \otimes (-\sin \theta_2|0\rangle + \cos \theta_2|1\rangle),$$

$$|4\rangle_{A}^{(n)} = |2\rangle_{B}^{(n-1)} \otimes (\cos \theta_1|0\rangle + \sin \theta_1|1\rangle),$$

$$|5\rangle_{A}^{(n)} = |2\rangle_{B}^{(n-1)} \otimes (-\sin \theta_1|0\rangle + \cos \theta_1|1\rangle),$$

$$\vdots$$

$$|2^n-2\rangle_{A}^{(n)} = |2^n-1\rangle_{B}^{(n-1)} \otimes (\cos \theta_{2^n-1}|0\rangle + \sin \theta_{2^n-1}|1\rangle),$$

$$|2^n-1\rangle_{A}^{(n)} = |2^n-1\rangle_{B}^{(n-1)} \otimes (-\sin \theta_{2^n-1}|0\rangle + \cos \theta_{2^n-1}|1\rangle),$$

and $|\vec{K}_B^{(n)}\rangle$ another orthonormal basis

$$|0\rangle_{B}^{(n)} = |0\rangle_{B}^{(n-1)} \otimes (\cos \theta_1|0\rangle + \sin \theta_1|1\rangle),$$

$$|1\rangle_{B}^{(n)} = |0\rangle_{B}^{(n-1)} \otimes (-\sin \theta_1|0\rangle + \cos \theta_1|1\rangle),$$

$$|2\rangle_{B}^{(n)} = |1\rangle_{B}^{(n-1)} \otimes (\cos \theta_2|0\rangle + \sin \theta_2|1\rangle),$$

$$|3\rangle_{B}^{(n)} = |1\rangle_{B}^{(n-1)} \otimes (-\sin \theta_2|0\rangle + \cos \theta_2|1\rangle),$$

$$|4\rangle_{B}^{(n)} = |2\rangle_{B}^{(n-1)} \otimes (\cos \theta_1|0\rangle + \sin \theta_1|1\rangle),$$

$$|5\rangle_{B}^{(n)} = |2\rangle_{B}^{(n-1)} \otimes (-\sin \theta_1|0\rangle + \cos \theta_1|1\rangle),$$

$$\vdots$$

$$|(2^n-2)\rangle_{B}^{(n)} = |(2^n-1)\rangle_{B}^{(n-1)} \otimes (\cos \theta_{2^n-1}|0\rangle + \sin \theta_{2^n-1}|1\rangle),$$

$$|(2^n-1)\rangle_{B}^{(n)} = |(2^n-1)\rangle_{B}^{(n-1)} \otimes (-\sin \theta_{2^n-1}|0\rangle + \cos \theta_{2^n-1}|1\rangle),$$

for $0 \leq \theta_1, \theta_2, \ldots, \theta_{2^n-1} \leq \frac{\pi}{2}$. $|\vec{K}_A^{(n)}\rangle_{K=0}^{2^n-1}$ and $|\vec{K}_B^{(n)}\rangle_{K=0}^{2^n-1}$ are arbitrary orthonormal basis of the $n-1$ qubit systems. Hence we may express the state to be teleported $|\phi_A\rangle^{(n)}$ in the basis $|\vec{K}_B^{(n)}\rangle_{K=0}^{2^n-1}$,

$$|\phi_A\rangle^{(n)} = \sum_{K=0}^{2^n-1} a_K |\vec{K}_B^{(n)}\rangle_K,$$

with $\sum_{K=0}^{2^n-1} |a_K|^2 = 1$. By virtue of the fact that $|\xi\rangle_{AB}^{(n,n)}$ is a maximally entangled state between $A$ and $B$, we may construct the following basis of $2^{2n}$ orthonormal states:

$$|\Pi_{i_1i_2\ldots i_n}\rangle_{AA}^{(n,n)} = (\sigma^{(i_1)} \otimes \sigma^{(i_2)} \otimes \ldots \otimes \sigma^{(i_n)})_A |\Pi_{00\ldots 0}\rangle_{AA}^{(n,n)},$$

(6)

where

$$|\Pi_{00\ldots 0}\rangle_{AA}^{(n,n)} = \frac{1}{\sqrt{2^n}} \sum_{K=0}^{2^n-1} |\vec{K}_A^{(n)}\rangle \otimes |\vec{K}_B^{(n)}\rangle,$$

and $i_1, \ldots, i_n \in \{0, 1, 2, 3\}$. 

4
Now Alice performs a complete projective measurement jointly on $A'$ in the above basis in equation (6). If the measurement outcome is $i_1i_2\ldots i_n$, then Bob’s sequences of qubits will be in the state $\sigma^{(i_1)} \otimes \sigma^{(i_2)} \otimes \cdots \otimes \sigma^{(i_n)} |\phi_B^{(n)}\rangle$. Bob can always recover an exact replica of the original unknown state, since

$$|\phi_A^{(n)}\rangle \otimes |\xi_{AB}^{(n)}\rangle = \frac{1}{2^n} \sum_{i_1i_2\ldots i_n} |\Pi_{i_1i_2\ldots i_n}^{A,A} (\sigma^{(i_1)} \otimes \sigma^{(i_2)} \otimes \cdots \otimes \sigma^{(i_n)}) |\phi_B^{(n)}\rangle,$$

(7)

$$A'A \langle \Pi_{i_1i_2\ldots i_n}^{A,A} (|\phi_A^{(n)}\rangle \otimes |\xi_{AB}^{(n)}\rangle) = \frac{1}{2^n} (\sigma^{(i_1)} \otimes \sigma^{(i_2)} \otimes \cdots \otimes \sigma^{(i_n)}) |\phi_B^{(n)}\rangle.$$  

(8)

These equations follow from the result given below, which also guarantees the success of the protocol.

$$A'A (|\Pi_{\emptyset\ldots\emptyset} (|\xi_{AB}^{(n)}\rangle) = \frac{1}{2^n} \sum_{J, K=0}^{2^n-1} (\mathcal{A}^J (|K\rangle \otimes |A\rangle)(|J\rangle \otimes |A\rangle) = \frac{1}{2^n} \sum_{K=0}^{2^n-1} |K\rangle \otimes |A\rangle.$$  



Therefore, the $2n$-qubit state $|\xi_{AB}^{(n)}\rangle$ is an ideal resource for $n$-qubit teleportation.

As we know, the tensor product of Bell state $|\eta\rangle = \bigotimes_{i=1}^{n} |\psi^+\rangle_{A,B}$ with $|\psi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$ is the ideal resource for the $n$ qubit teleportation. Here one can verify that the reduced matrix $\rho_{A'B} = \mathcal{A}_A \mathcal{B}_B (|\xi_{AB}^{(n)}\rangle \otimes |A\rangle, |\xi_{AB}^{(n)}\rangle \otimes |A\rangle)$ is not a pure state; therefore, $|\xi_{AB}^{(n)}\rangle$ is not equivalent to the tensor product of Bell states. This also shows $|\xi_{AB}^{(n)}\rangle$ is not equivalent to the entangled states used for $n$-qubit teleportation in [12]. Furthermore, when $n = 2$, one can verify that the four-qubit cluster state is a special case of $|\xi_{AB}^{(n)}\rangle$.

The state $|\xi_{AB}^{(n)}\rangle$ in equation (5) can be again used for dense coding and is able to transmit $2n$-bit classical information. Since $|\xi_{AB}^{(n)}\rangle$ is maximally entangled between $A$ and $B$, $|\langle\sigma^{(i_1)} \otimes \sigma^{(i_2)} \otimes \cdots \otimes \sigma^{(i_n)} |\xi_{AB}^{(n)}\rangle_{AB}|^2 = 2^{2n}$ maximally entangled states and constitute an orthonormal basis for the $2^{2n}$-dimensional Hilbert space. Alice can encode the message by using the operators $\sigma^{(i_1)}, \sigma^{(i_2)}, \ldots, \sigma^{(i_n)}$. Upon receiving the qubits from Alice, Bob can decode the message by performing a joint measurement in the basis in equation (6).

4. Conclusions

We have presented a class of $2n$-qubit states which may serve as ideal resources for perfect teleportation of $n$-qubit states ($n \geq 1$). This class of states is also the ideal resource for transmission of $2n$-bit classical information in dense coding. This kind of states is not equivalent to the tensor product of Bell states. For three-qubit and two-qubit cases, it is easy to see that this class of states include the cluster states as a special case. Here we just study the teleportation and dense coding in terms of these kinds of entangled states. It would be also interesting to consider one-way quantum computation according to these ‘generalized cluster-like’ states.

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