A Second Look at Counting Triangles in Graph Streams

Graham Cormode, Hossein Jowhari

Abstract

In this paper we present improved results on the problem of counting triangles in edge streamed graphs. For graphs with \( m \) edges and at least \( T \) triangles, we show that an extra look over the stream yields a two-pass streaming algorithm that uses \( O\left(\frac{m}{\epsilon^4 \sqrt{T}}\right) \) space and outputs a \( (1 + \epsilon) \) approximation of the number of triangles in the graph. This improves upon the two-pass streaming tester of Braverman, Ostrovsky and Vilenchik, ICALP 2013, which distinguishes between triangle-free graphs and graphs with at least \( T \) triangle using \( O\left(\frac{m}{T^{1/3}}\right) \) space. Also, in terms of dependence on \( T \), we show that more passes would not lead to a better space bound. In other words, we prove there is no constant pass streaming algorithm that distinguishes between triangle-free graphs from graphs with at least \( T \) triangles using \( O\left(\frac{m}{T^{1/2+\rho}}\right) \) space for any constant \( \rho \geq 0 \).

1. Introduction

Many applications produce output in form of graphs, defined an edge at a time. These include social networks that produce edges corresponding to new friendships or other connections between entities in the network; communication networks, where each edge represents a communication (phone call, email, text message) between a pair of participants; and web graphs, where each edge represents a link between pages. Over such graphs, we wish to answer questions about the induced graph, relating to the structure and properties.

One of the most basic structures that can be present in a graph is a triangle: an embedded clique on three nodes. Questions around counting the number of triangles in a graph have been widely studied, due to the inherent interest in the problem, and because it is a necessary stepping stone to answering questions around more complex structures in graphs. Triangles are of interest within social networks, as they indicate common friendships: two friends of an individual are
themselves friends. Counting the number of friendships within a graph is therefore a measure of the closeness of friendship activities. Another application for triangle counting is as a parameter for large graph models [LBKT08].

For these reasons, and for the fundamental nature of the problem, there have been numerous studies of the problem of counting or enumerating triangles in various models of data access: external memory [LWZW10, HTC13]; map-reduce [SV11, PT12, TKMF09]; and RAM model [SW05, Tso08]. Indeed, it seems that triangle counting and enumeration is becoming a de facto benchmark for testing “big data” systems and their ability to process complex queries. The reason is that the problem captures an essentially hard problem within big data: accurately measuring the degree of correlation. In this paper, we study the problem of triangle counting over (massive) streams of edges. In this case, lower bounds from communication complexity can be applied to show that exactly counting the number of triangles essentially requires storing the full input, so instead we look for methods which can approximate the number of triangles. In this direction, there has been series of works that have attempted to capture the right space complexity for algorithms that approximate the number of triangles. However most of these works have focused on one pass algorithms and thus, due to the hard nature of the problem, their space bounds have become complicated, suffering from dependencies on multiple graph parameters such maximum degree, number of paths of length 2, number of cycles of length 4, and etc.

In a recent work by Braverman et al. [BOV13], it has been shown that at the expense of an extra pass over stream, a straightforward sampling strategy gives a sublinear bound that depends only on $m$ (number of edges) and $T$ (a lower bound on the number of triangles). More precisely [BOV13] have shown that one extra pass yields an algorithm that distinguishes between triangle-free graphs from graphs with at least $T$ triangles using $O\left(\frac{mT}{\ell^5}\right)$ word of space. Although their algorithm does not give an estimate of the number of triangles and more important is not clearly superior to the $O\left(\frac{m\Delta}{\ell}\right)$ one pass algorithm by [PT12, PTTW13] (especially for graphs with small maximum degree $\Delta$), it creates some hope that perhaps with the expense of extra passes one could get improved and cleaner space bounds that beat the one pass bound for a wider range of graphs. In particular one might ask is there a $O\left(\frac{m}{\ell}\right)$ multi-pass algorithm? In this paper, while we refute such a possibility, we show that a more modest bound is possible. Specifically here we show modifications to the sampling strategy of [BOV13] along with a different analysis results in a 2-pass $(1 + \epsilon)$ approximation algorithm that uses only $O\left(\frac{m}{\ell^{4.5}\sqrt{T}}\right)$ space. We also observe that this bound is attainable in one pass–if we can make the string assumption that the order of edge arrivals is random.
Additionally, via a reduction to a hard communication complexity problem, we demonstrate that this bound is optimal in terms of its dependence on $T$. In other words there is no constant pass algorithm that distinguishes between triangle-free graphs from graphs with at least $T$ triangles using $O(\frac{m}{T^{1/2+\rho}})$ for any constant $\rho > 0$. We also give a similar two pass algorithm that has better dependence on $\epsilon$ but sacrifices the optimal dependence on $T$. Our results are summarized in Figure 2 in terms of the problem addressed, bound provided, and number of passes.

**Algorithms for Triangle Counting in Graph Streams.** The triangle counting problem has attracted particular attention in the model of graph streams: there is now a substantial body of study in this setting. Algorithms are evaluated on the amount of space that they require, the number of passes over the input stream that they take, and the time taken to process each update. Different variations arise depending on whether deletions of edges are permitted, or the stream is ‘insert-only’; and whether arrivals are ordered in a particular way, so that all edges incident on one node arrive together, or the edges are randomly ordered or adversarially ordered.

The work of Jowhari and Ghodsi [JG05] first studies the most popular of these combinations: insert-only, adversarial ordering. The general approach, common to many streaming algorithms, is to build a randomized estimator for the desired quantity, and then repeat this sufficiently many times to provide a guaranteed accuracy. Their approach begins by sampling an edge uniformly from the stream of $m$ arriving edges on $n$ vertices. Their estimator then counts the number of triangles on a sampled edge. Since the ordering is adversarial, the estimator has to keep track of all edges incident on the sampled edge, which in the worst case is bounded by $\Delta$, the maximum degree. The sampling process is repeated $O(\frac{1}{\epsilon^2} \frac{m\Delta}{T})$ times (using the assumed bound on the number of triangles, $T$), leading to a total space requirement proportional to $O(\frac{1}{\epsilon^2} \frac{m\Delta^2}{T})$ to give an $\epsilon$ relative error estimation of $t$, the number of triangles in the graph. The parameter $\epsilon$ ensures that the error in the count is at most $\epsilon t$ (with constant probability, since the algorithm is randomized). The process can be completed with a single pass over the input. Jowhari and Ghodsi also consider the case where edges may be deleted, in which case a randomized estimator using “sketch” techniques is introduced, improving over a previous sketch algorithm due to Bar-Yossef et al. [BYKS02].

The work of Buriol et al. [BFL+06] also adopted a sampling approach, and built a one-pass estimator with smaller working space. An algorithm is proposed which samples uniformly an edge from the stream, then picks a third node, and scans the remainder of the stream to see if the triangle on these three nodes is
present. Recall that \( n \) is the number of nodes in the graph, \( m \) is number of edges, and \( T \leq t \) is lower bound on the (true) number of triangles. To obtain an accurate estimate of number of triangles in the graph, this procedure is repeated independently \( O\left(\frac{mn}{\varepsilon^2 T}\right) \) times to achieve \( \varepsilon \) relative error.

Recent work by Pavan et al. [PTTW13] extends the sampling approach of Buriol et al.: instead of picking a random node to complete the triangle with a sampled edge, their estimator samples a second edge that is incident on the first sampled edge. This estimator is repeated \( O\left(\frac{m\Delta}{\varepsilon^2 T}\right) \) times, where \( \Delta \) represents the maximum degree of any node. That is, this improves the bound of Buriol et al. by a factor of \( n/\Delta \). In the worst case, \( \Delta = n \), but in general we expect \( \Delta \) to be substantially smaller than \( n \).

Braverman et al. [BOV13] take a different approach to sampling. Instead of building a single estimator and repeating, their algorithms sample a set of edges, and then look for triangles induced by the sampled edges. Specifically, an algorithm which takes two passes over the input stream distinguishes triangle-free graphs from those with \( T \) triangles in space \( O\left(\frac{m}{T^{1/3}}\right) \).

For graphs with \( W \geq m \) where \( W \) is the number of wedges (paths of length 2), Jha et al. [JSP13] have shown a single pass \( O\left(\frac{1}{\varepsilon^2} m/\sqrt{T}\right) \) space algorithm that returns an additive error estimation of the number of triangles.

Pagh and Tsourakakis [PT12] propose an algorithm in the MapReduce model of computation. However, it can naturally be adapted to the streaming setting. We conceptually “color” each vertex randomly from \( C \) colors (this can be accomplished, for example, with a suitable hash function). We then store each monochromatic edge, i.e. each edge from the input such that both vertices have the same color. Counting the number of triangles in this induced graph, and scaling up by a factor of \( C^2 \) gives an estimator for \( t \). The space used is \( O(m/C) \) in expectation. Setting \( C \) appropriately yields a one-pass algorithm with space \( \tilde{O}\left(\frac{m}{T} J + \frac{m}{\sqrt{T}}\right) \), where \( J \) denotes the maximum number of triangles incident on a single edge.

**Lower bounds for triangle counting.** A lower bound in the streaming model is presented by Bar-Yossef et al. [BYKS02]. They argue that there are (dense) families of graphs over \( n \) nodes such that any algorithm to approximate the number of triangles must use \( \Omega(n^2) \) space. The construction essentially encodes \( \Omega(n^2) \) bits of information, and uses the presence or absence of a single triangle to recover a single bit. Braverman et al. [BOV13] show a lower bound of \( \Omega(m) \) by demonstrating a family of graphs with \( m \) chosen between \( n \) and \( n^2 \). Their construction encodes \( m \) bits in a graph, then adds \( T \) edges such that there are either \( T \) triangles
or 0 triangles, which reveal the value of an encoded bit.

For algorithms which take a constant number of passes over the input stream, Jowhari and Ghodsi [JG05] show that still $\Omega(n/t)$ space is needed to approximate the number of triangles up to a constant factor, based on a similar encoding and testing argument. Specifically, they create a graph that encodes two binary strings, so that the resulting graph has $T$ triangles if the strings are disjoint, and $2T$ if they have an intersection. In a similar way, Braverman et al. [BOV13] encode binary strings into a graph, so that it either has no triangles (disjoint strings) or at least $T$ triangles (intersecting strings). This implies that $\Omega(m/T)$ space is required to distinguish the two cases. In both cases, the hardness follows from the communication complexity of determining the disjointness of binary strings.

2. Preliminaries and Results

In this section, we define additional notation and define the problems that we study.

As mentioned above, we use $t(G)$ to denote the number of triangles in graph $G = (V, E)$. Let $J(G)$ denote the maximum number of triangles that share an edge in $G$, and $K(G)$ the maximum number incident on any vertex. We use $t$, $J$, and $K$ when $G$ is clear from the context.

Problems Studied. We define some problems related to counting the number of triangles in a graph stream. These all depend on a parameter $T$ that gives a promise on the number of triangles in the graph.
Dist\((T)\): Given a stream of edges, distinguish graphs with at least \(T\) triangles from triangle-free graphs.

Estimate\((T, \epsilon)\): Given the edge stream of a graph with at least \(T\) triangles, output \(s\) where \((1-\epsilon)t(G) \leq s \leq (1+\epsilon) \cdot t(G)\).

Observe that any algorithm which promises to approximate the number of triangles for \(\epsilon < 1\) must at least be able to distinguish the case of 0 triangles or \(T\) triangles. Consequently, we provide lower bounds for the Dist\((T)\) problem, and upper bounds for the Estimate\((T, \epsilon)\) problem. Our lower bounds rely on the hardness of well-known problems from communication complexity. In particular, we make use of the hardness of Disj\(_p^r\):

**Problem 1** The Disj\(_p^r\) problem involves two players, Alice and Bob, who each have binary vectors of length \(p\). Each vector has Hamming weight \(r\), i.e. \(r\) entries set to one. The players want to distinguish non-intersecting inputs from inputs that do intersect.

This problem is “hard” in the (randomized) communication complexity setting: it requires a large amount of communication between the players in order to provide a correct answer with sufficient probability [KN97]. Specifically, Disj\(_p^r\) requires \(\Omega(r)\) bits of communication for any \(r \leq p/2\), over multiple rounds of interaction between Alice and Bob.

**Our Results.** We summarize the results for this problem discussed in Section\([1]\) and include our new results, in Figure\([2]\). We observe that, in terms of dependence on \(T\), we achieve tight bounds for 2 passes: Theorem\([8]\) shows that we can obtain a dependence on \(T^{-1/2}\), and Theorem\([7]\) shows that no improvement for constant
passes as a function of $T$ can be obtained. It is useful to contrast to the results of [PT12], where a one pass algorithm achieves a dependence of $m/T^{1/2}$, but has an additional term of $mJ/T$. This extra term can be large: as big as $m$ in the case that all triangles are incident on the same edge; here, we show that this term can be avoided at the cost of an additional pass.

Our results improve over the 2-pass bounds given in [BOV13]. We show that the Estimate($T$, $\epsilon$) can be solved with dependence on $T^{-1/3}$ (not just the decision problem Dist($T$)), and that the dependence on $T$ can be improved to $T^{-1/2}$, at the expense of higher dependence on $\epsilon$.

Comparing with the additive estimator of [JSP13], while our sampling strategy is somewhat similar, using an extra pass over the stream we return a relative error estimation of the number of triangles. Moreover our biased estimator (Algorithm I) has enabled us to obtain an unconditional result, although this is achieved at the expense of higher dependence on $\epsilon$.

3. Upper bounds

In this section, we provide our two upper bounds. The first provides a simple sampling-based unbiased estimator, which has a low dependence on $\epsilon$, but scales with $T^{-1/3}$. The second uses a similar sampling procedure, and provides a biased estimator, whose dependence is improved to $T^{-1/2}$, but with higher cost based on $\epsilon$.

Algorithm I (unbiased estimator).
Let $p \in (0, 1]$. The value of $p$ will be determined later.
In the first pass, the algorithm stores each edge independently at random with probability $p$. Let $G' = (V, E')$ be the sampled subgraph.
In the second pass, the algorithm, upon reading the edge $e_i \notin E'$, counts the number of new triangles in $(V, E' \cup \{e_i\})$ and adds it to a global counter $s$. At the end of the second pass, the algorithm outputs $Y = \frac{s}{3p^2(1-p)}$ as the estimate for $t(G)$.

**Theorem 1** Algorithm I is a 2-pass randomized streaming algorithm for Estimate($T$, $\epsilon$) that uses $O\left(\frac{m}{\epsilon^{4/3} \sqrt{\log(n)}} T^{1/3}\right)$ space.

**Proof:** Let $T$ represent the set of triangles in the graph. For the analysis, we partition $T$ into several groups through the following process. Fix an $L \in [1, t]$
We notionally assign the triangles on \( (determined below) \). Pick an arbitrary edge \( e \in E \) with at least \( L \) triangles on it. We notionally assign the triangles on \( e \) to the edge \( e \). Let this be the set \( \mathcal{T}_e \subseteq \mathcal{T} \). Continue this process until all the remaining edges participate in fewer than \( L \) unassigned triangles. Let \( \mathcal{T}' \) be the unassigned triangles.

Let \( X_i \) be the indicator random variable associated with the \( i \)-th triangle in \( \mathcal{T} \). We have \( X_i = 1 \) with probability \( 3p^2(1 - p) \). For each edge \( e \), let \( s_e = \sum_{i \in \mathcal{T}_e} X_i \) and define \( s_{\mathcal{T}'} = \sum_{i \in \mathcal{T}'} X_i \). We have \( s = \sum_{e \in E} s_e + s_{\mathcal{T}'} \) and the expectation of \( s \) is \( \mathbb{E}(s) = 3p^2(1 - p)t \).

First we analyse the concentration of \( s_{\mathcal{T}'} \). We have \( \mathbb{E}(s_{\mathcal{T}'}) = 3p^2(1 - p)|\mathcal{T}'| \). We also compute

\[
\text{Var}(s_{\mathcal{T}'}) = \mathbb{E}(s_{\mathcal{T}'}^2) - \mathbb{E}^2(s_{\mathcal{T}'}) \\
\leq \sum_{i \in \mathcal{T}'} \mathbb{E}(X_i^2) + \sum_{i \in \mathcal{T}' \neq j \in \mathcal{T}'} \mathbb{E}(X_iX_j) - \mathbb{E}^2(s_{\mathcal{T}'}) \\
\leq 3p^2(1 - p)|\mathcal{T}'| + (4p^3(1 - p)^2 + p^4(1 - p))|\mathcal{T}'|L.
\]

The final term derives from considering pairs of triangles \( i, j \). We break these into those which share an edge, and those which are disjoint. For those sharing an edge, both are sampled if either (a) the shared edge and exactly one other edge in each triangle is sampled, with total probability \( 4p^3(1 - p)^2 \) or (b) if all edges except the shared edge are sampled, which occurs with probability \( p^4(1 - p) \). There are at most \( |\mathcal{T}'|L \) such triangle pairs. For pairs of triangles which do not share any edge, their contribution to the sum is outweighted by the term \(-\mathbb{E}(s_{\mathcal{T}'})^2 \).

Since \( (1 - p) < 1 \) and \( p < 1 \) we simplify this expression to \( \text{Var}(s_{\mathcal{T}'}) < 3p^2|\mathcal{T}'| + 5p^3|\mathcal{T}'|L \). By the Chebyshev inequality,

\[
\Pr[|s_{\mathcal{T}'} - \mathbb{E}(s_{\mathcal{T}'})| \geq \epsilon p^2t] \leq \frac{\text{Var}(s_{\mathcal{T}'})}{\epsilon^2p^4t^2} \leq \frac{3|\mathcal{T}'|}{\epsilon^2p^4t^2} + \frac{5|\mathcal{T}'|L}{\epsilon^2p^4t^2} \quad (1)
\]

To bound the deviation of each \( s_e \), we use the Chernoff bound. Let \( Z_e \) be the event corresponding to \( e \notin \mathcal{E}' \). Since the edges are sampled independently, conditioned on \( Z_e \), the random variables \( \{X_i\}_{i \in \mathcal{T}_e} \) are independent. Moreover we have \( \mathbb{E}(X_i|Z_e) = p^2 \). From the Chernoff bound, we get

\[
\Pr[|s_e - \mathbb{E}(s_e)| \geq \epsilon \mathbb{E}(s_e) \mid Z_e] \leq e^{-\frac{\epsilon^2p^2|\mathcal{T}_e|}{2}} \leq e^{-\frac{\epsilon^2t^2}{2}} \quad (2)
\]

Similarly, conditioned on \( \overline{Z_e} \), the random variables \( \{X_i\}_{i \in \mathcal{T}_e} \) are independent and \( \mathbb{E}(X_i|\overline{Z_e}) = 2p(1 - p) \).
Pr\(|s_e - E(s_e)| ≥ \epsilon E(s_e)| ≥ \epsilon E(s_e)\] \leq e^{-p(1-p)|T_e|/2} \leq e^{-p(1-p)Ls^2} \tag{3}

From (2) and (3), for each \(e \in E\), we get

\[\Pr[|s_e - E(s_e)| ≥ \epsilon E(s_e)] ≤ e^{-\frac{s^2Ls^2}{2}} \tag{4}\]

Therefore using the union bound and the fact that the number of edges with non-empty \(T_e\) is bounded by \(t/L\), we get

\[
\Pr\left[\sum_e |s_e - E(s_e)| ≥ \epsilon \sum_e E(s_e) \right] \leq \frac{t}{L} e^{-\frac{s^2Ls^2}{2}} \tag{5}
\]

Since \(t ≥ T\) and setting \(L = (\epsilon t)^{2/3}\) and \(p = \Omega\left(\frac{1}{\epsilon^{4/3} \sqrt{\log n}} \right)\) with large enough constants, the probabilities in (1) and (5) will be bounded by a small constant. The expected number of edges in the sampled graph \(G'\) is \(pm\), and can be shown to be tightly concentrated around its expectation. so the space usage is as stated above. This proves our theorem. □

We now modify this algorithm to work in the random order streaming model, where all permutations of the input are equally likely [GM09].

**Corollary 2** Assuming the data arrives in random order, there is a one-pass randomized streaming algorithm for Estimate\((T, \epsilon)\) that uses \(O\left(\frac{m}{\epsilon^{4/3} \sqrt{\log n}}\right)\) space.

**PROOF:** The one-pass algorithm collapses the two passes of Algorithm I into one. That is, the algorithm stores each edge into graph \(G'\) with probability \(p\), and also counts the number of triangles completed in \(G'\) by each edge from the stream \(G\).

The analysis follows the same outline as the main theorem, with some modification. First, we now have \(\Pr[X_i = 1] = p^2(1 - p)\), since the unsampled edge must be the last in the stream order, and \(E(s)\) is correspondingly lower by a factor of 3. Then \(E(X_i|Z_i) = p^2/3\), since to count triangle \(i\), we must have that the first two edges are seen before edge \(e\) in the stream. Likewise, \(E(X_i|\bar{Z}_i) = 2p(1-p)/3\), since we must have the unsampled edge appear after the two sampled edges. This causes us to rescale \(p\) by a constant, which does not change the asymptotic cost of the algorithm. □

Note that the requirement of random order is important for the one-pass result. Because we split the analysis based on the particular edges, the order in which these edges appear can affect the outcome. If the edge \(e\) were to always appear

9
after the two other edges in triangle \( i \), then \( \text{E}(X_i | Z_e) \) would be 0. Hence, we need the edges to appear in random order to ensure this one-pass analysis holds.

Our next algorithm builds a similar estimator, but differs in some important ways.

**Algorithm II (biased estimator).**
Repeat the following \( l \geq 16/\epsilon \) times independently in parallel and output the minimum of the outcomes.
In the first pass, pick every edge with probability \( p \) (the value of \( p \) will be determined later.)
In the second pass, count the number of triangles detected: either those where all three edges were sampled in the first pass, or two edges were sampled in the first pass, and the completing edge observed in the second pass. Let \( r \) be the total number of triangles detected. Output \( \frac{r}{3p^2(1-p)+p^3} \).

**Theorem 3** Algorithm II is a 2-pass randomized streaming algorithm for \( \text{Estimate}(T, \epsilon) \) that uses \( O\left(\frac{m}{\epsilon^2 + \sqrt{T}}\right) \) space.

**Proof:** Let \( R \) be the output of the Algorithm II. As in the previous proof, let \( \mathcal{T} \) represent the set of triangles in the graph. Consider one instance of the basic estimator, and let \( X \) be the outcome of this instance. Let \( X_i \) denote the indicator random variable associated with the \( i \)th triangle in \( \mathcal{T} \) being detected. By simple calculation, we have \( \Pr[X_i = 1] = 3p^2(1 - p) + p^3 \) and \( \text{E}(X) = \frac{1}{3p^2(1-p)+p^3} \sum_{i \in \mathcal{T}} X_i = t \). By the Markov inequality, \( \Pr[X \leq (1 + \epsilon)\text{E}(X)] \geq \epsilon \).
Therefore we can conclude,

\[
\Pr[R \leq (1 + \epsilon)t] \geq \Pr[X \leq (1 + \epsilon)t] \geq \frac{7}{8}.
\]

However, proving a lower bound on \( R \) is more complex, and requires a more involved analysis. First, we show that most triangles share an edge with a limited number of triangles. More precisely, let \( L \subseteq E \) denote the set of edges where each \( e \in L \) belongs to at most \( 3\sqrt{t/\epsilon} \) triangles. We call \( L \) the set of light edges and \( H = E \setminus L \) the heavy edges. We claim there exists \( S \subseteq \mathcal{T} \) such that \( |S| \geq (1-\epsilon)t \) and every triangle in \( S \) has at least two light edges. This is true because there can be at most \( \frac{3t}{3\sqrt{t/\epsilon}} = \sqrt{t} \) heavy edges, and moreover every two distinct edges belong to at most one triangle.

For each triangle \( i \in S \), fix two of its light edges. Let \( Y_i \) denote the indicator random variable for the event where the algorithm picks the light edges of \( i \in S \) in
the first pass. We have $E(Y_i) = p^2$ and always $Y_i \leq X_i$. Therefore, finding a lower bound on $Y$ will give a lower bound on $X$. We will argue that the probability of $Y$ being less than $(1 - \epsilon)|S|$ is small, even after taking the minimum of multiple repetitions. Let $Y = \frac{1}{p^2} \sum_{i \in S} Y_i$. We have

$$E(Y) = |S| \geq (1 - \epsilon)t.$$ 

We also have

$$\text{Var}(Y) = E(Y^2) - E^2(Y) \leq \frac{1}{p^2} |S| + \frac{1}{p} |S| \sqrt{t/\epsilon}.$$ 

The first term comes from $\sum_{i \in S} \frac{1}{p^2} E(Y_i^2)$, and the second term arises from pairs of triangles which share a light edge, of which there are at most $|S|^\sqrt{t/\epsilon}$ (since the edge is light), and which are both sampled with probability $p^3$. Using the Chebyshev inequality and assuming $\epsilon < \frac{1}{2}$, we have

$$\Pr[Y < (1 - \epsilon)^2 t] \leq \Pr[Y < (1 - \epsilon)|S|] \leq \frac{\text{Var}(Y)}{\epsilon^2 |S|^2} \leq \frac{1}{\epsilon^2} \left( \frac{1}{p^2 |S|} + \frac{\sqrt{t/\epsilon}}{p |S|} \right) \leq \frac{1}{\epsilon^2} \left( \frac{2}{p^2 t} + \frac{2}{p^3 t} \right).$$

Since $T \leq t$, setting $p > \frac{320}{\epsilon^2 \sqrt{T}}$, allows the above probability to be bounded by $\frac{\epsilon}{160}$. Now the probability that the minimum of $16/\epsilon$ independent trials is below the designated threshold is at most $\frac{16}{160 \epsilon} = 1/10$. Therefore with probability at least $1 - (1/8 + 1/10)$ the output of the algorithm is within the interval $[(1 - 2\epsilon)t, (1 + \epsilon)t]$. This proves the statement of our theorem. \[ \square \]

**Corollary 4** Assuming the data arrives in random order, there is a one-pass randomized streaming algorithm for Estimate$(T, \epsilon)$ that uses $O\left(\frac{m}{\epsilon^2 \sqrt{T}}\right)$ space.

**PROOF:** Under random order, we can combine the first and second passes of algorithm II. We count all triangles formed as $r$: either those with all three edges sampled, or those with two edges sampled and the third observed subsequently in
the stream. The estimator is now \( \frac{2}{3} \), since the probability of counting any triangle is \( p^3 \) (for all three edges sampled) plus \( p^2 (1 - p) \) (for the first two edges in the stream sampled, and the third unsampled). The same analysis as for Theorem 5 then follows: we partition the edges into light and heavy sets, and bound the probability of sampling a subset of triangles. A triangle with two light edges is counted if both light edges are sampled, and the heavy edge arrives last. This happens with probability \( \frac{p^2}{3} \). We can nevertheless argue that we are unlikely to undercount such triangles, following the same Chebyshev analysis as above. This allows us to conclude that the estimator is good. \( \square \)

Again, random order is critical to make this algorithm work in one pass: an adversarial order could arrange the heavy edges to always come last (increasing the probability of counting a triangle under this analysis) or always first (giving zero probability of counting a triangle under this analysis). It remains an open question to understand whether these bounds can be obtained in a single pass without the random order assumption.

4. Lower bounds

We now show lower bounds for the problem \( \text{Dist}(T) \), to distinguish between the case \( t = 0 \) and \( t \geq T \). Our first result builds upon a lower bound from prior work, and amplifies the hardness. We formally state the previous result:

Lemma 5 \([BOV13]\) Every constant pass streaming algorithm for \( \text{Dist}(T) \) requires \( \Omega\left(\frac{m}{T^2}\right) \) space.

Theorem 6 Any constant pass streaming algorithm for \( \text{Dist}(T) \) requires \( \Omega\left(\frac{m}{T^{2/3}}\right) \) space.

PROOF: Given a graph \( G = (V, E) \) with \( m \) edges we can create a graph \( G' = (V', E') \) with \( mT^2 \) edges and \( t(G') = T^3 t(G) \). We do so by replacing each vertex \( v \in V \) with \( T \) vertices \( \{v_1, \ldots, v_T\} \) and replacing the edge \( (u, v) \in E \) with the edge set \( \{u_1, \ldots, u_T\} \times \{v_1, \ldots, v_T\} \). Clearly any triangle in \( G \) will be replaced by \( T^3 \) triangles in \( G' \) and every triangle in \( G' \) corresponds to a triangle in \( G \). Moreover this reduction can be performed in a streaming fashion using \( O(1) \) space. Therefore a streaming algorithm for \( \text{Dist}(T) \) using \( o\left(\frac{m}{T^2}\right) \) (applied to \( G' \)) would imply an \( o(m) \) streaming algorithm for \( \text{Dist}(1) \). But from Lemma 5 we have that \( \text{Dist}(1) \) requires \( \Omega(m) \) space for constant pass algorithms. This is a contradiction and as a result our claim is proved. \( \square \)
Our next lower bound more directly shows the hardness by a reduction to the hard communication problem of $\text{Disj}_{\rho}$.

**Theorem 7** For any $\rho > 0$ and $T \leq n^2$, there is no constant pass streaming algorithm for $\text{Dist}(T)$ that takes $O\left(\frac{m}{T^{1/2} + \rho}\right)$ space.

**Proof:** We show that there are families of graphs with $\Theta(n \sqrt{T})$ edges and $T$ triangles such that distinguishing them from triangle-free graphs in a constant number of passes requires $\Omega(n)$ space. This is enough to prove our theorem.

We use a reduction from the standard set intersection problem, here denoted by $\text{Disj}_{\rho}^n$. Given $y \in \{0, 1\}^n$, Bob constructs a bipartite graph $G = (A \cup B, E)$ where $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_{\sqrt{T}}\}$. He connects $a_i$ to all vertices in $B$ iff $y[i] = 1$. On the other hand, Alice adds vertices $C = \{c_1, \ldots, c_{\sqrt{T}}\}$ to $G$. She adds the edge set $C \times B$. Also for each $i \in [\sqrt{T}]$ and $j \in [n]$, she adds the edge $(c_i, a_j)$ iff $x[j] = 1$. We observe that if $x$ and $y$ (uniquely) intersect there will be precisely $T$ triangles passing through each vertex of $C$. Since there is no edge between the vertices in $C$, in total we will have $T$ triangles. On the other hand, if $x$ and $y$ represent disjoint sets, there will be no triangles in $G$. In both cases, the number of edges is between $2n \sqrt{T}$ and $3n \sqrt{T}$, over $O(n)$ vertices (using the bound $T^2 \leq n$). Considering the lower bound for the $\text{Disj}_{\rho}^n$ (Section 2), our claim is proved following a standard argument: an space efficient streaming algorithm would imply an efficient communication protocol whose messages are the memory state of the algorithm. □

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