Tunneling in fractional quantum mechanics

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Abstract

We study tunneling through delta and double delta potentials in fractional quantum mechanics. After solving the fractional Schrödinger equation for these potentials, we calculate the corresponding reflection and transmission coefficients. These coefficients have a very interesting behavior. In particular, we can have zero energy tunneling when the order of the Riesz fractional derivative is different from 2. For both potentials, the zero energy limit of the transmission coefficient is given by $T_0 = \cos^2(\pi/\alpha)$, where $\alpha$ is the order of the derivative (1 $<$ $\alpha$ $\leq$ 2).

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1. Introduction

In recent years, the study of fractional integrodifferential equations applied to physics and other areas has grown. Some examples are [1–3], among many others. More recently, the fractional generalized Langevin equation has been proposed to discuss the anomalous diffusive behavior of a harmonic oscillator driven by a two-parameter Mittag–Leffler noise [4].

Fractional quantum mechanics (FQM) is the theory of quantum mechanics based on the fractional Schrödinger equation (FSE). In this paper, we consider the FSE as introduced by Laskin in [5, 6]. It was obtained in the context of the path integral approach to quantum mechanics. In this approach, path integrals are defined over Lévy flight paths, which is a natural generalization of the Brownian motion [7].

There are some papers in the literature studying solutions of FSE. Some examples are [8–10]. However, recently Jeng et al [11] have shown that some claims to solve the FSE have not taken into account the fact that the fractional derivation is a nonlocal operation. As a consequence, all those attempts based on local approaches are intrinsically wrong. Jeng et al pointed out that the only correct approach is the one found in [12] involving the delta potential. However, in [12] the FSE with delta potential was studied only in the case of negative energies.
This has been generalized in [13], where we solved the FSE for the delta and double delta potentials for positive and negative energies.

The objective of this paper is to study tunneling through delta and double delta potentials in the context of the FSE. As a result, we found some very interesting properties that are not observed in the usual $\alpha = 2$ quantum mechanics. Probably the most interesting is the presence of tunneling through delta and double delta potentials even at zero energy. Moreover, in the case of the double delta potential, this zero energy tunneling is independent of the relation of the two delta functions. In Lin et al [14], the problem of calculating the transmission coefficient in FQM for the double delta potential has been addressed; however, the authors used the same local approach that Jeng et al [11] showed to be wrong. As expected in this case, our results differ from theirs.

We organized this paper as follows. Firstly, and for the sake of completeness, we reproduce the solution of the FSE for the delta and double delta potentials, as given in [13], presenting their respective solutions in terms of Fox’s $H$-function. Some calculations and properties of Fox’s $H$-function are given in appendices. Then we study the asymptotic behavior of those solutions, calculate the reflection and transmission coefficients, and study some of their properties. The limit $\alpha \to 2$ for these coefficients and the boundary conditions satisfied by the solutions are also discussed in two other appendices.

2. The fractional Schrödinger equation

The one-dimensional FSE is

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = D_\alpha (-\hbar^2 \Delta)^{\alpha/2} \psi(x,t) + V(x) \psi(x,t),$$

(1)

where $1 < \alpha \leq 2$, $D_\alpha$ is a constant, $\Delta = \partial_x^2$ is the Laplacian, and $(-\hbar^2 \Delta)^{\alpha/2}$ is the Riesz fractional derivative [15], that is,

$$(-\hbar^2 \Delta)^{\alpha/2} \psi(x,t) = \frac{1}{2\pi \hbar} \int_{-\infty}^{+\infty} e^{ipx/\hbar} |p|^\alpha \phi(p,t) dp,$$

(2)

where $\phi(p,t)$ is the Fourier transform of the wavefunction:

$$\phi(p,t) = \int_{-\infty}^{+\infty} e^{-ipx/\hbar} \psi(x,t) dx, \quad \psi(x,t) = \frac{1}{2\pi \hbar} \int_{-\infty}^{+\infty} e^{ipx/\hbar} \phi(p,t) dp.$$

(3)

The time-independent FSE is

$$D_\alpha (-\hbar^2 \Delta)^{\alpha/2} \psi(x) + V(x) \psi(x) = E \psi(x),$$

(4)

In the momentum representation, this equation is written as

$$D_\alpha |p|^\alpha \phi(p) + \frac{(W * \phi)(p)}{2\pi \hbar} = E \phi(p),$$

(5)

where $(W * \phi)(p)$ is the convolution

$$(W * \phi)(p) = \int_{-\infty}^{+\infty} W(p - q) \phi(q) dq,$$

(6)

and $W(p) = \mathcal{F}[V(x)]$ is the Fourier transform of the potential $V(x)$.

The solutions of the FSE for delta and double delta potentials are given in [13] in the situations of bound and scattering states. Since we need to study the asymptotic behavior of these solutions in order to find the transmission coefficients, and for the sake of completeness, we will reproduce the calculations of the wavefunctions in the case of scattering states.
2.1. FSE for delta potential

Let us consider the case
\[ V(x) = V_0 \delta(x), \tag{7} \]
where \( \delta(x) \) is the Dirac delta function and \( V_0 \) is a constant. Its Fourier transform is \( W(p) = V_0 \) and the convolution \( (W * \phi)(p) \) is
\[ (W * \phi)(p) = V_0 K, \tag{8} \]
where the constant \( K \) is
\[ K = \int_{-\infty}^{\infty} \phi(q) \, dq. \tag{9} \]
The FSE in the momentum representation (5) is
\[ \left( \left| p \right|^\alpha - \frac{E}{D_\alpha} \right) \phi(p) = -\gamma K, \tag{10} \]
where
\[ \gamma = \frac{V_0}{2\pi \hbar D_\alpha}. \tag{11} \]

Since we are interested in scattering states, we will consider that \( E > 0 \) and write
\[ E = \frac{\lambda^\alpha}{D_\alpha}, \tag{12} \]
where \( \lambda > 0 \). Since \( f(x)\delta(x) = f(0)\delta(x) \), the solution of equation (10) in this case is
\[ \phi(p) = \left( \frac{-\gamma K}{\left| p \right|^\alpha - \lambda^\alpha} + 2\pi \hbar C_1 \delta(p - \lambda) + 2\pi \hbar C_2 \delta(p + \lambda), \right. \tag{13} \]
where \( C_1 \) and \( C_2 \) are arbitrary constants and the constant \( 2\pi \hbar \) was introduced for later convenience. Using this in equation (9) gives that
\[ K = -\gamma K \int_{-\infty}^{\infty} \frac{dp}{\left| p \right|^\alpha - \lambda^\alpha} + 2\pi \hbar C_1 + 2\pi \hbar C_2, \tag{14} \]
where the integral is interpreted in the sense of the Cauchy principal value, and it gives
\[ \int_{-\infty}^{\infty} \frac{dp}{\left| p \right|^\alpha - \lambda^\alpha} = 2\lambda^{-1-\alpha} \int_{0}^{\infty} \frac{dq}{q^\alpha - 1} = -2\lambda^{-1-\alpha} \frac{\pi}{\alpha} \cot \frac{\pi}{\alpha}, \tag{15} \]
where we have used formula 3.241.3 (page 322) of [16]—see equation (B.3). The constant \( K \) is therefore
\[ K = \frac{2\pi \hbar (C_1 + C_2) \alpha \lambda^{\alpha-1}}{\alpha \lambda^{\alpha-1} - 2\pi \gamma \cot (\pi/\alpha)}, \tag{16} \]
and we have
\[ \phi(p) = 2\pi \hbar C_1 \delta(p - \lambda) + 2\pi \hbar C_2 \delta(p + \lambda) - \frac{2\pi \hbar \gamma (C_1 + C_2) \alpha \lambda^{\alpha-1}}{(\alpha \lambda^{\alpha-1} - 2\pi \gamma \cot (\pi/\alpha)) (\left| p \right|^\alpha - \lambda^\alpha)}. \tag{17} \]

Next we need to calculate the inverse Fourier transform of \( \phi(p) \) to obtain \( \psi(x) \), that is,
\[ \psi(x) = C_1 e^{i\lambda x/\hbar} + C_2 e^{-i\lambda x/\hbar} - \frac{2\pi \gamma \alpha (C_1 + C_2)}{(\alpha \lambda^{\alpha-1} - 2\pi \gamma \cot (\pi/\alpha))} \Im \left( \frac{\lambda x}{\hbar} \right), \tag{18} \]
where \( \Im(w) \) is the Cauchy principal value of the integral
\[ \Im(w) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos wq}{q^\alpha - 1} \, dq, \tag{19} \]
and such that

$$\int_{-\infty}^{\infty} \frac{e^{ipx/\hbar}}{|p|^\alpha - \lambda^\alpha} \, dp = 2\pi \lambda^{1-\alpha} \Omega_\alpha(\lambda x/\hbar).$$

(20)

The above integral is calculated in appendix B, and is given by equation (B.5). Using this result, and the definition of $\gamma$ in equation (11) and $\lambda$ in equation (12), we can write that

$$\psi(x) = C_1 e^{ix\lambda/\hbar} + C_2 e^{-ix\lambda/\hbar} + \frac{\Omega_a (C_1 + C_2)}{2} \Phi_a \left( \frac{\lambda|x|}{\hbar} \right),$$

(21)

where

$$\Phi_a \left( \frac{\lambda|x|}{\hbar} \right) = \frac{a \hbar}{\lambda|x|} \left[ H_{2,3}^{1,1} \left[ \left( \frac{\lambda|x|}{\hbar} \right)^\alpha \begin{bmatrix} (1, 1), (1, (2 + \alpha)/2) \\ (1, \alpha), (1, 1), (1, (2 + \alpha)/2) \end{bmatrix} \right]
- H_{2,3}^{2,1} \left[ \left( \frac{\lambda|x|}{\hbar} \right)^\alpha \begin{bmatrix} (1, 1), (1, (2 - \alpha)/2) \\ (1, \alpha), (1, 1), (1, (2 - \alpha)/2) \end{bmatrix} \right] \right).$$

(22)

with $H_{2,3}^{i,j}$ denoting Fox’s $H$-function (see appendix A), and

$$\Omega_a = \left[ \left( \frac{E}{U} \right)^{\frac{\alpha - 1}{a - 1}} - \cot \frac{\pi}{\alpha} \right]^{-1},$$

(23)

and

$$U = \left( \frac{V_0}{a \hbar D_{1/a}^\alpha} \right)^{\alpha/(\alpha - 1)}.$$  

(24)

2.2. FSE for double delta potential

Now let the potential be given by

$$V(x) = V_0 \left[ \delta(x + R/2) + \mu \delta(x - R/2) \right].$$

(25)

with $\mu$, $V_0$ and $R$ being real constants. When $V_0 < 0$ this potential can be seen as a model for the one-dimensional limit of the molecular ion $H^+_2$ [17]. The parameter $R$ is interpreted as the internuclear distance and the coupling parameters are $V_0$ and $\mu V_0$. Its Fourier transform is

$$W(p) = V_0 e^{ipR/2\hbar} + V_0 \mu e^{-ipR/2\hbar}.$$  

(26)

and for the convolution

$$(W * \phi)(p) = V_0 e^{ipR/2\hbar} K_1(R) + V_0 \mu e^{-ipR/2\hbar} K_2(R),$$

(27)

where $K_1(R)$ and $K_2(R)$ are the constants given by

$$K_1(R) = K_2(-R) = \int_{-\infty}^{\infty} e^{-iRq/2\hbar} \phi(q) \, dq.$$  

(28)

The FSE in momentum space is

$$\left( |p|^\alpha - \frac{E}{D_\alpha} \right) \phi(p) = -\gamma e^{iRp/2\hbar} K_1(R) - \gamma \mu e^{-iRp/2\hbar} K_2(R),$$

(29)

where we used the notation introduced in equation (11).

Since we are interested in scattering states, we have $E > 0$ and we write $\lambda$ as in equation (12) and for the solution of equation (29) we have

$$\phi(p) = 2\pi \hbar C_1 \delta(p - \lambda) + 2\pi \hbar C_2 \delta(p + \lambda) - \frac{\gamma e^{iRp/2\hbar} K_1(R)}{|p|^\alpha - \lambda^\alpha} - \frac{\mu \gamma e^{-iRp/2\hbar} K_2(R)}{|p|^\alpha - \lambda^\alpha}. $$

(30)
Using this expression for $\phi(p)$ in equation (28) of definition of $K_1(R)$ and $K_2(R)$ we have

\[ (1 + 2\pi \gamma \lambda^{1-\alpha})J_u(0))K_1(R) + \mu 2\pi \gamma \lambda^{1-\alpha}J_u(\lambda R/h)K_2(R) = 2\pi h C'_1, \]

\[ 2\pi \gamma \lambda^{1-\alpha}J_u(\lambda R/h)K_1(R) + (1 + \mu 2\pi \gamma \lambda^{1-\alpha}J_u(0))K_2(R) = 2\pi h C'_2, \]

where

\[ C'_1 = C_1 e^{-iR\lambda/2h} + C_2 e^{iR\lambda/2h}, \quad C'_2 = C_1 e^{iR\lambda/2h} + C_2 e^{-iR\lambda/2h}. \]

In order to write the solution of these equations it is convenient to define

\[ \varepsilon = \frac{\lambda^{\alpha-1}}{2\pi \gamma} = \frac{1}{\alpha} \left( \frac{E}{U} \right)^{\frac{1}{\alpha}}, \]

where $U$ was defined in equation (24), in such a way that we have

\[ K_1(R) = \frac{2\pi h E}{W} [\varepsilon \mu^{-1} + J_u(0)]C'_1 - J_u(\lambda R/h)C'_2], \]

\[ K_2(R) = \frac{2\pi h E}{\mu W} [\varepsilon + J_u(0)]C'_2 - J_u(\lambda R/h)C'_1], \]

where

\[ W = (\varepsilon + J_u(0)) \varepsilon \mu^{-1} + J_u(0) - (\lambda R/h)^2. \]

Using $K_1(R)$ and $K_2(R)$ in equation (30) gives $\phi(p)$. Then, for $\psi(x)$, we have

\[ \psi(x) = C_1 e^{i\lambda x/\hbar} + C_2 e^{-i\lambda x/\hbar} + \frac{1}{2\alpha W} [\varepsilon \mu^{-1} + J_u(0)]C'_1 - J_u(\lambda R/h)C'_2 \Phi_u \left( \frac{\lambda |x + R/2|}{\hbar} \right) \]

\[ + \frac{1}{2\alpha W} [\varepsilon + J_u(0)]C'_2 - J_u(\lambda R/h)C'_1 \Phi_u \left( \frac{\lambda |x - R/2|}{\hbar} \right), \]

where we have expressed the result in terms of the function $\Phi_u$ defined in equation (22).

### 3. Calculation of the transmission coefficients

In order to calculate the transmission coefficients, we need to know the asymptotic behavior of the solutions. The asymptotic behavior of Fox’s $H$-function is given, if $\Delta > 0$, by equation (A.8) or equation (A.10) according to $\Delta^* > 0$ or $\Delta^* = 0$, respectively—see equation (A.7). In $\Phi_u(\lambda|x|/\hbar)$ we have the difference between two Fox’s $H$-functions of the form

\[ H_{2,3}^{\alpha,1} \left[ w^\alpha \right] \left( \begin{array}{c} (1, 1), (1, \mu) \\ (1, \alpha), (1, 1), (1, \mu) \end{array} \right), \]

for $\mu = (2 + \alpha)/2$ and $\mu = (2 - \alpha)/2$. In both cases we have $\Delta = \alpha > 0$, but $\Delta^* = 0$ for $\mu = (2 + \alpha)/2$ and $\Delta^* > 0$ for $\mu = (2 - \alpha)/2$. Therefore, using equation (A.8) when $\mu = (2 - \alpha)/2$ and equation (A.10) when $\mu = (2 + \alpha)/2$ we have, respectively, that

\[ H_{2,3}^{\alpha,1} \left[ w^\alpha \right] \left( \begin{array}{c} (1, 1), (1, (2 + \alpha)/2) \\ (1, \alpha), (1, 1), (1, (2 + \alpha)/2) \end{array} \right] = \frac{2w}{\alpha} \sin w + o(1), \quad |w| \to \infty, \]

\[ H_{2,3}^{\alpha,1} \left[ w^\alpha \right] \left( \begin{array}{c} (1, 1), (1, (2 - \alpha)/2) \\ (1, \alpha), (1, 1), (1, (2 - \alpha)/2) \end{array} \right] = o(1), \quad |w| \to \infty, \]

and then

\[ \Phi_u \left( \frac{\lambda |x|}{\hbar} \right) = 2 \sin \frac{\lambda |x|}{\hbar} + o(|x|^{-1}), \quad |x| \to \infty. \]
3.1. Transmission coefficient for the delta potential

The behavior of the wavefunction $\psi(x)$ given by equation (21) for $x \to \pm \infty$ is therefore

$$\psi(x) = C_1 e^{i\omega x/\hbar} + C_2 e^{-i\omega x/\hbar} + \Omega_\alpha (C_1 + C_2) \sin \frac{\lambda x}{\hbar} + o(x^{-1}), \quad x \to \pm \infty. \quad (42)$$

or

$$\psi(x) = A e^{i\omega x/\hbar} + B e^{-i\omega x/\hbar} + o(x^{-1}), \quad x \to -\infty, \quad (43)$$

$$\psi(x) = C e^{i\omega x/\hbar} + D e^{-i\omega x/\hbar} + o(x^{-1}), \quad x \to + \infty, \quad (44)$$

where we defined

$$A = C_1 + i(C_1 + C_2)\Omega_\alpha/2, \quad B = C_2 - i(C_1 + C_2)\Omega_\alpha/2, \quad (45)$$

$$C = C_1 - i(C_1 + C_2)\Omega_\alpha/2, \quad D = C_2 + i(C_1 + C_2)\Omega_\alpha/2. \quad (46)$$

Now let us consider the situation of particles coming from the left and scattered by the delta potential. In this case $D = 0$ (no particles coming from the right) and $B = rA$ and $C = tA$, where the reflexion $R$ and transmission $T$ coefficients are given by $R = |r|^2$ and $T = |t|^2$ (see, for example, [18]). The result is

$$r = \frac{-i\Omega_\alpha}{1 + i\Omega_\alpha}, \quad t = \frac{1}{1 + i\Omega_\alpha}, \quad (47)$$

and then

$$R = \frac{\Omega_\alpha^2}{1 + \Omega_\alpha^2}, \quad T = \frac{1}{1 + \Omega_\alpha^2}. \quad (48)$$

In figure 1 we show the behavior of these coefficients for different values of $\alpha$. This plot and the other ones this paper have been done by means of numerical integration of equation (19) using Mathematica 7.

We must note that the transmission coefficient has a very interesting behavior at zero energy. If we take the limit $E \to 0$ in the expression for $\Omega_\alpha$ in equation (23) we see that

$$\lim_{E \to 0} \Omega_\alpha = - \tan \frac{\pi}{\alpha} \quad (49)$$
and then for the transmission coefficient \( T \) we have
\[
\lim_{E \to 0} T = \cos^2 \frac{\pi}{\alpha}.
\]
(50)
This is an unexpected and very interesting effect, which demands further interpretation (see conclusions).

3.2. Transmission coefficient for the double delta potential

Let us introduce the following notations:
\[
U = \frac{\varepsilon \mu^{-1} + \nu_0(0)}{aW}, \quad V = \frac{\varepsilon + \nu_0(0)}{aW}, \quad \chi = \frac{\nu_0(\lambda R/h)}{aW}.
\]
(51)
The asymptotic behavior of the wavefunction \( \psi(x) \) given by equation (38) for \( x \to \pm \infty \) is therefore
\[
\psi(x) = C_1 e^{i\lambda x/h} + C_2 e^{-i\lambda x/h} + (UC_1' - XC_2') \sin \frac{\lambda|x + R/2|}{\hbar} + (VC_2' - XC_1') \sin \frac{\lambda|x - R/2|}{\hbar} + o(x^{-1}), \quad x \to \pm \infty,
\]
or
\[
\psi(x) = A' e^{i\lambda x/h} + B' e^{-i\lambda x/h} + o(x^{-1}), \quad x \to -\infty,
\]
\[
\psi(x) = C' e^{i\lambda x/h} + D' e^{-i\lambda x/h} + o(x^{-1}), \quad x \to +\infty,
\]
where we defined
\[
A' = C_1 + M_1, \quad B' = C_2 + M_2,
\]
\[
C' = C_1 - M_1, \quad D' = C_2 - M_2,
\]
and
\[
M_1 = i(\rho C_1 + \sigma C_2 + i\tau C_1),
\]
\[
M_2 = -i(\rho C_1 + \rho C_2 - i\tau C_1),
\]
with
\[
\rho = \frac{(U + V)}{2} - \chi \cos \frac{\lambda R}{h}, \quad \sigma = \frac{(U + V)}{2} \cos \frac{\lambda R}{h} - \chi,
\]
\[
\tau = \frac{(U - V)}{2} \sin \frac{\lambda R}{h}.
\]
(59)
As in the case of the delta potential, let us consider the situation of particles coming from the left and scattered by the double delta potential. In complete analogy we have \( D' = 0 \) (no particles coming from the right), \( B' = rA' \) and \( C' = tA' \), where the reflexion \( \mathcal{R} \) and transmission \( T \) coefficients are given by \( \mathcal{R} = |r|^2 \) and \( T = |t|^2 \). The result is
\[
r = \frac{2(\tau + i\sigma)}{(\rho^2 - \sigma^2 - \tau^2 + 1 - 2i\rho),} \quad t = -\frac{(\rho^2 - \sigma^2 - \tau^2 + 1)}{(\rho^2 - \sigma^2 - \tau^2 + 1 - 2i\rho)}.
\]
(60)

and \( \mathcal{R} \) and \( T \) can be written as
\[
\mathcal{R} = \frac{4(\sigma^2 + \tau^2)}{(\rho^2 - \sigma^2 - \tau^2 + 1)^2 + 4(\sigma^2 + \tau^2)}, \quad T = \frac{(\rho^2 - \sigma^2 - \tau^2 + 1)^2}{(\rho^2 - \sigma^2 - \tau^2 + 1)^2 + 4(\sigma^2 + \tau^2)}.
\]
(61)
Figure 2. Transmission coefficients as function of $\lambda R/2\hbar$, as given by equation (63), for different values of $\alpha$, when $2\pi\gamma = 10$ and $\mu = 1$.

Figure 3. Transmission coefficients as function of $\lambda R/2\hbar$, as given by equation (63), for different values of $\mu$, when $2\pi\gamma = 20$ and $\alpha = 1.8$.

We can simplify these expressions a little once we note that

$$\rho^2 - \sigma^2 - \tau^2 = \frac{\sin^2 (\lambda R/\hbar)}{\alpha^2 W},$$

(62)

and then

$$R = \frac{\Delta_\alpha^2}{1 + \Delta_\alpha^2}, \quad T = \frac{1}{1 + \Delta_\alpha^2},$$

(63)

with

$$\Delta_\alpha^2 = \frac{4\alpha^2 W^2 (\sigma^2 + \tau^2)}{(\alpha^2 W + \sin^2 (\lambda R/\hbar))^2}.$$  

(64)

In figure 2 we show the behavior of the transmission coefficient for different values of $\alpha$, and in figure 3 for different values of $\mu$. 

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As in the case of the delta potential, we have a very interesting behavior for these coefficients when \( E \to 0 \). Firstly, using equation (B.13) in appendix B, we have
\[
J_\alpha(\lambda R/\hbar) = J_\alpha(0) + A_1 \lambda^{\alpha-1} + A_2 \lambda^2 + O(\lambda^{3\alpha-1}),
\]
where
\[
A_1 = \frac{(R/\hbar)^{\alpha-1}}{2\Gamma(\alpha) \cos(\pi \alpha/2)}, \quad A_2 = \frac{(R/\hbar)^2 \cot(3\pi/\alpha)}{2\alpha}.
\]
Using this, the expression equation (37) for \( W \) gives
\[
W = J_\alpha(0)(H(1 + \mu^{-1}) - 2A_1)\lambda^{\alpha-1} + (H^2 \mu^{-1} - A_1^2)\lambda^{2(\alpha-1)} - 2J_\alpha(0)A_2 \lambda^2 + O(\lambda^{3\alpha-1}),
\]
where \( H = 1/2\pi \gamma \). Then, with some calculations, we obtain that
\[
4\alpha^2 W^2(\sigma^2 + \tau^2) = B_1 \lambda^{2(\alpha-1)} + B_2 \lambda^{\alpha+1} + O(\lambda^{2\alpha}),
\]
with
\[
B_1 = (H(1 + \mu^{-1}) - 2A_1)^2, \quad B_2 = -2(H(1 + \mu^{-1}) - 2A_1)(2A_2 + J_\alpha(0)(R/\hbar)^2),
\]
and
\[
\alpha^2 W^2(\rho^2 - \sigma^2 - \tau^2 + 1) = B'_1 \lambda^{2(\alpha-1)} + B'_2 \lambda^{4(\alpha-1)} + B'_3 \lambda^{\alpha+1} + O(\lambda^{2\alpha}),
\]
with
\[
B'_1 = \alpha^2 J_\alpha(0)(H(1 + \mu^{-1}) - 2A_1)^2, \quad B'_2 = \alpha^2(H^2 \mu^{-1} - A_1)^2, \quad B'_3 = -2\alpha J_\alpha(0)(H(1 + \mu^{-1}) - 2A_1)(2\alpha^2 J_\alpha(0)A_2 - (R/\hbar)^2).
\]
Using these results we can easily see that, for \( E/D_\alpha = \lambda^{\alpha} \to 0 \),
\[
\lim_{E \to 0} \Delta_\alpha^2 = \frac{1}{\alpha^2 J_\alpha^2(0)},
\]
and, since \( J_\alpha(0) = -(1/\alpha) \cot(\pi/\alpha) \), that
\[
\lim_{E \to 0} T = \cos^2 \frac{\pi}{\alpha}.
\]
This is the same result we obtained for the zero energy limit of the transmission coefficient for a single delta potential. Moreover, this limit does not depend on \( \mu \), which is the parameter that relates the two delta functions in the potential in equation (25). Again, this is a very interesting and unexpected result.

4. Conclusions

The tunneling effect in fractional quantum mechanics has some very interesting properties which are not observed in the usual \( \alpha = 2 \) quantum mechanics. The most interesting is the presence of tunneling through delta and double delta potentials even at zero energy. Moreover, in the case of the double delta potential, this zero energy tunneling is independent of the relation of the two delta functions. Let us give a possible explanation of these results.

FQM was defined from the point of view of path integrals. As is well known, in this approach, when the sum is taken over paths of the Brownian motion type, we have the standard quantum mechanics. On the other hand, in FQM the sum is taken over the paths of Lévy flights [5], which are generalizations of Brownian motion, and such that the corresponding probability distribution has infinite variance. By means of Lévy flights, there is a non-negligible probability of a particle reaching faraway points in a single jump, in contrast to a random walk of the Brownian motion type [19].
In FQM, an uncertainty principle still holds, but with an appropriate modification, that is, we have [6]
\[ \langle |\Delta x|^\mu \rangle > \frac{\hbar}{(2\alpha)^{1/\mu}}, \quad \mu < \alpha, \quad 1 < \alpha \leq 2, \]  
(74)
and such that in the standard quantum mechanics limit we can also have \( \mu = \alpha = 2. \) Thus, even when \( E = 0, \) the particle can have energy \( \Delta E = \langle |\Delta p|^\mu \rangle / 2m \) and momentum \( \langle |\Delta p|^\mu \rangle / \mu. \) This may not be enough for tunneling through a delta potential in the standard case, but in the fractional one, where long jumps of Lévy flights enter the sum in the path integral, this may be responsible for the tunneling with probability \( \cos^2(\pi/\alpha). \)

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Appendix A. Fox’s \( H \)-Function

Fox’s \( H \)-function, also known as \( H \)-function or Fox’s function, was introduced in the literature as an integral of the Mellin–Barnes type [20].

Let \( m, n, p \) and \( q \) be integer numbers. Consider the function
\[
\Lambda(s) = \frac{\prod_{i=1}^{m} \Gamma(b_i + s) \prod_{i=1}^{n} \Gamma(1 - a_i - A_i s)}{\prod_{i=m+1}^{p} \Gamma(1 - b_i - B_i s) \prod_{i=n+1}^{q} \Gamma(a_i + A_i s)},
\]  
(A.1)
with \( 1 \leq m \leq q \) and \( 0 \leq n \leq p. \) The coefficients \( A_i \) and \( B_i \) are positive real numbers; \( a_i \) and \( b_i \) are complex parameters.

Fox’s \( H \)-function, denoted by \( H_{m,n}^{p,q}(x) \), is defined as the inverse Mellin transform, i.e.
\[
H_{m,n}^{p,q}(x) = \frac{1}{2\pi i} \int_{L} \Lambda(s) x^{-s} ds,
\]  
(A.3)
where \( \Lambda(s) \) is given by equation (A.1), and the contour \( L \) runs from \( L - i\infty \) to \( L + i\infty \) separating the poles of \( \Gamma(1 - a_i - A_i s), (i = 1, \ldots, n) \) from those of \( \Gamma(b_i + B_i s), (i = 1, \ldots, m). \) The complex parameters \( a_i \) and \( b_i \) are taken with the imposition that no poles in the integrand coincide.

There are some interesting properties associated with Fox’s \( H \)-function. We consider here the following ones.

P.1. Change the independent variable. Let \( c \) be a positive constant. We have
\[
H_{m,n}^{p,q}(x) = x^{c} H_{m,n}^{p,q}(c x).
\]  
(A.4)
To show this expression one introduce a change of variable \( s \rightarrow c s \) in the integral of the inverse Mellin transform.

P.2. Change the first argument. Set \( \alpha \in \mathbb{R}. \) Then we can write
\[
x^{\alpha} H_{m,n}^{p,q}(x) = x^{\alpha} H_{m,n}^{p,q}(x^{\alpha}).
\]  
(A.5)
To show this expression first we introduce the change \( a_p \rightarrow a_p + \alpha A_p \) and take \( s \rightarrow s - \alpha \) in the integral of the inverse Mellin transform.
\textbf{P.3. Lowering of order.} If the first factor \((a_1, A_1)\) is equal to the last one, \((b_q, B_q)\), we have

\[
H_{p,q}^{m,n} \left[ \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_{q-1}, B_{q-1}) \end{array} \right] (a_1, A_1) \\
= H_{p-1,q-1}^{m,n-1} \left[ \begin{array}{c} (a_2, A_2), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_{q-1}, B_{q-1}) \end{array} \right].
\]

(A.6)

To show this identity is sufficient to simplify the common arguments in the Mellin–Barnes integral.

\textbf{P.4. Asymptotic expansions.} The asymptotic expansions for Fox’s \(H\)-functions have been studied in \cite{[21]}. Let \(\Delta\) and \(\Delta^*\) be defined as

\[
\Delta = \sum_{i=1}^{p} B_i - \sum_{i=1}^{q} A_i, \quad \Delta^* = \sum_{i=1}^{p} B_i + \sum_{i=1}^{m} A_i - \sum_{i=1}^{q} B_i.
\]

(A.7)

If \(\Delta > 0\) and \(\Delta^* > 0\), we have \cite{[22]}

\[
H_{p,q}^{m,n} (x) = \sum_{r=1}^{n} [h_r x^{(a_r-1)/A_r} + o(x^{(a_r-1)/A_r})], \quad |x| \to \infty
\]

(A.8)

where

\[
h_r = \frac{1}{A_r} \prod_{j=1}^{m} \Gamma(b_j + (1 - a_r) B_j / A_r) \prod_{j=1}^{n} \Gamma(1 - a_j - (1 - a_r) A_j / A_r) \prod_{j=m+1}^{n} \Gamma(1 - b_j - (1 - a_r) B_j / A_r),
\]

(A.9)

and if \(\Delta > 0\) and \(\Delta^* = 0\), we have \cite{[22]}

\[
H_{p,q}^{m,n} (x) = \sum_{r=1}^{n} [h_r x^{(a_r-1)/A_r} + o(x^{(a_r-1)/A_r})]
\]

\[
+ A x^{(\nu+1)/\Delta} (c_0 \exp[i(B + C x^{1/\Delta})] - d_0 \exp[-i(B + C x^{1/\Delta})])
\]

\[
+ o(x^{(\nu+2)/|\Delta|}), \quad |x| \to \infty,
\]

(A.10)

where

\[
c_0 = (2\pi i)^{m+n-p} \exp \left[ \pi i \left( \sum_{r=m+1}^{n} a_r \right) \right],
\]

\[
d_0 = (-2\pi i)^{m+n-p} \exp \left[ -\pi i \left( \sum_{r=m+1}^{n} a_r \right) \pi i \right],
\]

\[
A = \frac{1}{2 \pi i \Delta} (2\pi)^{(p-q+1)/2} \Delta^{-\nu} \prod_{r=1}^{p} A_{r-w+1/2} \prod_{j=1}^{q} B_j^{b_j-1/2} \left( \frac{\Delta}{\delta} \right)^{v+1/2} / \Delta,
\]

\[
B = \frac{(2\nu + 1)\pi}{4}, \quad C = \left( \frac{\Delta}{\delta} \right)^{1/\Delta},
\]

\[
\delta = \prod_{l=1}^{p} |A_l|^{1-A_l} \prod_{j=1}^{q} |B_j|^{B_j}, \quad v = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}.
\]
P.5. Series expansion. In [20] we can see that in some cases there is a series expansion for Fox’s \(H\)-function. For example, when the poles of \(\prod_{j=1}^{m} \Gamma(b_j + B_j s)\) are simple, we can write

\[
H_{P,q}^{m,n}(x) = \sum_{j=1}^{m} \sum_{v=0}^{\infty} h_{j,v} x^{(b_j + v)/B_j},
\]

where

\[
h_{j,v} = \frac{(-1)^v}{v!B_j} \prod_{j=1}^{m} \Gamma(b_j) \prod_{v=0}^{\infty} \Gamma(1 - a_i + A_i(b_j + v)/B_j).
\]

Taking the Mellin transform we have that \(\mathcal{M}\)

\[
\mathcal{M}_w[3_\alpha(w)](z) = \frac{1}{\pi} \Gamma(z) \cos \frac{\pi z}{2} \int_0^{+\infty} y^{-z} \frac{y^\mu - 1}{1 - y^\mu} dy.
\]

This last integral is given by formula 3.241.3 (page 322) of [16], that is,

\[
\int_0^{+\infty} x^{\mu - 1} dx = \frac{\pi}{\mu} \cot \frac{\pi \mu}{\alpha},
\]

where the integration is understood as the Cauchy principal value\(^1\). Therefore, we have

\[
\mathcal{M}_w[3_\alpha(w)](z) = \frac{1}{\alpha} \Gamma(z) \sin \frac{\pi (1 - z)}{2} \cot \frac{\pi (1 - z)}{\alpha}.
\]

Using the relation \(2 \sin A \cos B = \sin (A + B) + \sin (A - B)\) and writing the sine function in terms of the product of gamma functions we can write that

\[
\mathcal{M}_w[3_\alpha(w)](z) = -\frac{1}{2\alpha} \Gamma(z) \Gamma\left(\frac{1-z}{\alpha}\right) \Gamma\left(1 - \frac{1-z}{\alpha}\right) = F_2(z).
\]

Taking the inverse Mellin transform and using the definition of Fox’s \(H\)-function we have that

\[
3_\alpha(w) = -\frac{1}{2\alpha} H_{2,3}^{1,1}\left[w^{\alpha} \begin{array}{c}
(1 - 1/\alpha, 1/\alpha), (1 - (2 + \alpha)/2\alpha, (2 + \alpha)/2\alpha) \\
(0, 1), (1 - 1/\alpha, 1/\alpha), (1 - (2 + \alpha)/2\alpha, (2 + \alpha)/2\alpha)
\end{array}\right] + \frac{1}{2\alpha} H_{2,3}^{1,1}\left[w^{\alpha} \begin{array}{c}
(1 - 1/\alpha, 1/\alpha), (1 - (2 - \alpha)/2\alpha, (2 - \alpha)/2\alpha) \\
(0, 1), (1 - 1/\alpha, 1/\alpha), (1 - (2 - \alpha)/2\alpha, (2 - \alpha)/2\alpha)
\end{array}\right].
\]

Using the properties given by equations (A.4, A.5) and replacing \(w\) by \(|w|\) since \(3_\alpha(-w) = 3_\alpha(w)\) we obtain

\[
3_\alpha(w) = -\frac{1}{2|w|} H_{2,3}^{1,1}\left[|w|^\alpha \begin{array}{c}
(1, 1), (1, (2 + \alpha)/2) \\
(1, \alpha), (1, (2 + \alpha)/\alpha)
\end{array}\right] + \frac{1}{2|w|} H_{2,3}^{1,1}\left[|w|^\alpha \begin{array}{c}
(1, 1), (1, (2 - \alpha)/2) \\
(1, \alpha), (1, (2 - \alpha)/\alpha)
\end{array}\right].
\]

\(^1\) We remember that in the inversion of the Fourier transform the integration is to be done in the sense of the Cauchy principal value [23].
Let us see what happens in the particular case $\alpha = 2$. From the definition of Fox’s $H$-function we can see that

$$H_{2,3}^{2,1} \left[ \begin{array}{c|c} w^2 & (1, 1), (1, 2) \\ (1, 2), (1, 1), (1, 0) \end{array} \right] = 0$$

and that

$$H_{2,3}^{2,1} \left[ \begin{array}{c|c} w^2 & (1, 1), (1, 2) \\ (1, 2), (1, 1), (1, 2) \end{array} \right] = H_{2,3}^{1,1} \left[ \begin{array}{c|c} w^2 & (1, 1) \\ (1, 1), (1, 2) \end{array} \right] = w^2 H_{1,2}^{1,1} \left[ \begin{array}{c|c} w^2 & (0, 1) \\ (0, 1), (-1, 2) \end{array} \right].$$

But [20]

$$H_{1,2}^{1,1} \left[ \begin{array}{c|c} -z & (0, 1) \\ (0, 1), (1 - b, a) \end{array} \right] = E_{a,b}(z),$$

where $E_{a,b}(z)$ is the two-parameter Mittag–Leffler function. However, it is known [24] that

$$E_{2,2}(z) = \frac{\sin \sqrt{z}}{\sqrt{z}}.$$

Consequently, we have

$$H_{2,3}^{2,1} \left[ \begin{array}{c|c} w^2 & (1, 1), (1, 2) \\ (1, 2), (1, 1), (1, 2) \end{array} \right] = |w|^2 E_{2,2}(-|w|^2) = |w| \sin |w|. \quad (B.9)$$

Then for $\alpha = 2$ we have

$$\tilde{\zeta}_2 \left( \frac{\lambda x}{\hbar} \right) = -\frac{1}{2} \sin \frac{\lambda x}{\hbar}, \quad (B.10)$$

and

$$\int_{-\infty}^{+\infty} \frac{e^{ipx/h}}{|p|^2 - \lambda^2} \, dp = -\frac{\pi}{\lambda} \sin \frac{\lambda x}{\hbar}. \quad (B.11)$$

We are also interested in the expression of $\tilde{\zeta}_\alpha(w)$ for small $w$. From equation (B.5) we see that we need to know the behavior of

$$H_{2,3}^{2,1} \left[ \begin{array}{c|c} |w|^a & (1, 1), (1, \mu) \\ (1, \alpha), (1, 1), (1, \mu) \end{array} \right]$$

for small $w$. This is given by the series expansion from equation (A.11), which gives

$$H_{2,3}^{2,1} \left[ \begin{array}{c|c} z & (1, 1), (1, \mu) \\ (1, \alpha), (1, 1), (1, \mu) \end{array} \right] = \frac{\Gamma(1 - 1/\alpha) \Gamma(1/\alpha)}{\Gamma(1 - \mu/\alpha) \Gamma(\mu/\alpha)} \frac{z^{1/\alpha}}{\alpha} - \frac{\Gamma(1 - 2/\alpha) \Gamma(2/\alpha)}{\Gamma(1 - 2\mu/\alpha) \Gamma(2\mu/\alpha)} \frac{z^{2/\alpha}}{\alpha} + \frac{\Gamma(1 - 3/\alpha) \Gamma(3/\alpha)}{\Gamma(1 - 3\mu/\alpha) \Gamma(3\mu/\alpha) 2\alpha} \frac{z^{3/\alpha}}{\alpha} + O(z^{4/\alpha}) + \frac{\Gamma(1 - 1/\alpha) \Gamma(1)}{\Gamma(1 - \mu/\alpha) \Gamma(\mu/\alpha)} z - \frac{\Gamma(1 - 2/\alpha) \Gamma(2)}{\Gamma(1 - 2\mu/\alpha) \Gamma(2\mu)} z^2 + \frac{\Gamma(1 - 3/\alpha) \Gamma(3)}{\Gamma(1 - 3\mu/\Gamma(3\mu) 2\alpha) \Gamma(3\mu)} z^3 + O(z^4). \quad (B.12)$$

Using this in equation (B.5) we arrive, after some manipulations, at

$$\tilde{\zeta}_\alpha(w) = \tilde{\zeta}_\alpha(0) + \frac{1}{2\Gamma(\alpha) \cos (\pi \alpha/2)} w^{\alpha-1} + \frac{\cot (3\pi/\alpha)}{2\alpha} w^2 + O(w^{3\alpha-1}), \quad (B.13)$$

where

$$\tilde{\zeta}_\alpha(0) = -\frac{1}{\alpha} \cot \frac{\pi}{\alpha}. \quad (B.14)$$
Appendix C. The limit $\alpha = 2$

Let us calculate the transmission coefficients in the standard quantum mechanical limit and see that we recover the usual results.

Firstly, let us consider the delta potential. The transmission coefficient is given by equation (48), so that we need to calculate $\Omega_2$ in this case, where $\Omega_2$ is given by equation (23). The result is that

$$\Omega_2 = \left(\frac{E}{U}\right)^{-1}, \quad U = \frac{mV^2_0}{2\hbar^2}, \quad (C.1)$$

where we used the definition of $U$ in equation (24) and $D^2 = 1/(2m)$, and such that

$$T = \frac{1}{1 + (mV^2_0/2\hbar^2)E}, \quad (C.2)$$

which is the well-known result [18].

Now let us consider the transmission coefficient for the double delta potential. Let us also consider the case $\mu = 1$ since the result in this case is well known [25]. In order to calculate $\Delta_2$ in equation (63) we need equation (B.10), which by the way gives

$$\Delta_2(0) = 0. \quad (C.3)$$

When $\mu = 1$ we have

$$U = V = \frac{1}{2W}, \quad \tau = 0, \quad (C.4)$$

and then

$$\Delta_2^2 = \frac{64W^2\sigma^2}{[4W + \sin^2(\lambda R/\hbar)]^2}. \quad (C.5)$$

But in this case equation (37) together with equation (B.10) gives

$$4W + \sin^2(\lambda R/\hbar) = 4\epsilon^2. \quad (C.6)$$

From equations (51) and (59) we also have that

$$4W^2\sigma^2 = \left(\epsilon \cos \frac{\lambda R}{\hbar} + \frac{1}{2} \sin \frac{\lambda R}{\hbar}\right)^2. \quad (C.7)$$

Then for $\Delta_2$ we have

$$\Delta_2^2 = \left(\frac{1}{\epsilon} \cos \frac{\lambda R}{\hbar} + \frac{1}{2\epsilon^2} \sin \frac{\lambda R}{\hbar}\right)^2. \quad (C.8)$$

Let us change the notation a little in order to compare with the standard result in the literature. Let us define

$$\mu_0 = \frac{\lambda R}{2\hbar}, \quad \beta = \frac{\mu_0}{\epsilon}. \quad (C.9)$$

Using this notation in the above expression for $\Delta_2$, we have for $T$ in equation (63) that

$$T = \frac{\mu_0^4}{\mu_0^4 + [\beta\mu_0 \cos 2\mu_0 + (\beta^2/2) \sin 2\mu_0]^2}. \quad (C.10)$$

which is the result in [25], p 160 (where $\mu_0 = \mu$).
Appendix D. The boundary conditions

In [13], we discussed how the Riesz fractional derivative can be written in terms of the Riesz potentials, that is, for $0 < \alpha < 1$ we have

$$(-\Delta)^{\alpha/2} \psi(x) = \frac{d}{dx} \tilde{R}_1^{1-\alpha} \psi(x),$$  \hspace{1cm} (D.1)

and for $1 < \alpha < 2$ we have

$$(-\Delta)^{\alpha/2} \psi(x) = -\frac{d^2}{dx^2} R_2^{2-\alpha} \psi(x),$$  \hspace{1cm} (D.2)

where $R_\alpha' \psi(x)$ is the Riesz potential of $\psi(x)$ of order $\alpha'$ given by [15]

$$R_\alpha' \psi(x) = \frac{1}{2\Gamma(\alpha')} \frac{\cos(\alpha'\pi/2)}{|x-\xi|^{1+\alpha'}} \int_{-\infty}^{+\infty} \psi(\xi) \frac{1}{1 - \alpha'} d\xi,$$  \hspace{1cm} (D.3)

for $0 < \alpha' < 1$, and $\tilde{R}_\alpha' \psi(x)$ its conjugated Riesz potential given by

$$\tilde{R}_\alpha' \psi(x) = \frac{1}{2\Gamma(\alpha')} \frac{\sin(\alpha'\pi/2)}{|x-\xi|^{1+\alpha'}} \int_{-\infty}^{+\infty} \frac{\text{sign}(x-\xi) \psi(\xi)}{|x-\xi|^{1+\alpha'}} d\xi.$$  \hspace{1cm} (D.4)

If we use these expressions for the Riesz fractional derivative in the FSE for the delta potential $V(x) = V_0 \delta(x)$ and integrate as usual from $-\epsilon$ to $+\epsilon$ and take the limit $\epsilon \to 0$ we obtain that

$$\left. \frac{d}{dx} \tilde{R}_2^{2-\alpha} \psi(x) \right|_{0^+} - \left. \frac{d}{dx} R_2^{2-\alpha} \psi(x) \right|_{0^-} = \frac{V_0}{\hbar^\alpha} \frac{\psi(0)}{D_\alpha}.$$  \hspace{1cm} (D.5)

This condition and the continuity one $\psi(0^-) = \psi(0^+)$ are the boundary conditions to be satisfied by $\psi(x)$.

There is an important point to be noted here: the expression

$$\frac{d}{dx} R_2^{2-\alpha} \psi(x) = \frac{d}{dx} \tilde{R}_1^{1-\alpha} \psi(x)$$

is not the Riesz fractional derivative of order $\alpha - 1$, which is given, for $0 < \alpha - 1 < 1$, by

$$\frac{d}{dx} \tilde{R}_1^{1-(\alpha-1)} \psi(x).$$

Therefore, it is wrong to write condition (D.5) as

$$(-\Delta)^{(\alpha-1)/2} \psi(x)|_{0^+} - (-\Delta)^{(\alpha-1)/2} \psi(x)|_{0^-} = \frac{V_0}{\hbar^\alpha} \frac{\psi(0)}{D_\alpha},$$  \hspace{1cm} (D.6)

with $(-\Delta)^{\alpha/2}$ being the Riesz fractional derivative. Maybe in another version of FQM involving a fractional derivative defined in another sense, it can hold one such condition, but this is not the case when it comes to the Riesz fractional derivative. In [14], the authors have used a boundary condition of the above type in their attempt to solve the problem for the double delta potential. Besides the already discussed problem with their local approach, it also seems that the use of that boundary condition is not justified.

Let us consider the solution in the case of the delta potential and show that it satisfies equation (D.5). In [13], we have seen that

$$\frac{d}{dx} R_2^{2-\alpha} \psi(x) = \frac{i}{2\pi \hbar^\alpha} \int_{-\infty}^{+\infty} e^{-ipx/\hbar} |p|^{\alpha-1} \text{sign}(p) \psi(p) dp.$$  \hspace{1cm} (D.7)
Using $\phi(p)$ given by equation (17) we obtain, after calculating the integrals with the delta functions, that
\[
\frac{d}{dx} \mathcal{R}^{2-a} \psi(x) = 2iC_1 \left( \frac{\lambda}{\hbar} \right)^{\alpha-1} e^{i\lambda x / \hbar} - 2iC_2 \left( \frac{\lambda}{\hbar} \right)^{\alpha-1} e^{-i\lambda x / \hbar} + \frac{\alpha}{\pi} (C_1 + C_2)\Omega_0 \left( \frac{\lambda}{\hbar} \right)^{\alpha-1} \Xi_0 \left( \frac{\lambda x}{\hbar} \right),
\] (D.8)
where
\[
\Xi_0(w) = \int_0^\infty \sin wq \frac{q^{\alpha-1}}{q^\alpha - 1} dq.
\] (D.9)
This integral can be calculated in a way analogous to that in appendix B. The result is
\[
\Xi_0(w) = \frac{\pi}{2} \text{sign}(w) \left( H_{2,3}^{2,1} \left| w \right|^\alpha \begin{bmatrix} (0, 1), (0, (2 + \alpha)/2) \\ (0, \alpha), (0, 1), (0, (2 + \alpha)/2) \end{bmatrix} - H_{2,3}^{2,1} \left| w \right|^\alpha \begin{bmatrix} (0, 1), (0, (2 - \alpha)/2) \\ (0, \alpha), (0, 1), (0, (2 - \alpha)/2) \end{bmatrix} \right).
\] (D.10)
In order to study the limit $x \to 0^\pm$ we need to use
\[
\lim_{w \to 0^\pm} H_{2,3}^{2,1} \left| w \right|^\alpha \begin{bmatrix} (0, 1), (0, \beta) \\ (0, \alpha), (0, 1), (0, \beta) \end{bmatrix} = \frac{\beta}{\alpha},
\] (D.11)
which can be calculated using the definition of Fox’s $H$-function and the residue theorem (the rhs comes from the residue at $w = 0$), and which gives
\[
\lim_{w \to 0^\pm} \Xi_0(w) = \pm \frac{\pi}{2}.
\] (D.12)
Using this result, we have
\[
\lim_{x \to 0^\pm} \frac{d}{dx} \mathcal{R}^{2-a} \psi(x) = \pm \frac{\alpha}{2} (C_1 + C_2)\Omega_0 \left( \frac{\lambda}{\hbar} \right)^{\alpha-1}.
\] (D.13)
On the other hand, $\psi(0)$ can be calculated using equation (B.14) in equation (18), which gives
\[
\psi(0) = \frac{(C_1 + C_2)\alpha\Omega_0 \lambda^{\alpha-1}}{2\pi \gamma} = \frac{\hbar^2 D_0}{V_0} \frac{\alpha(C_1 + C_2)\Omega_0 \lambda^{\alpha-1}}{2\pi \gamma},
\] (D.14)
where we used the definition of $\gamma$ in equation (11). We see, therefore, that equation (D.5) is satisfied.

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