Reconstruction of inhomogeneous media by iterative reconstruction algorithm with learned projector

Kai Li* Bo Zhang† Haiwen Zhang‡

Abstract

This paper is concerned with the inverse problem of scattering of time-harmonic acoustic waves from an inhomogeneous medium in two dimensions. We propose a deep learning-based iterative reconstruction algorithm for recovering the contrast of the inhomogeneous medium from the far-field data. The proposed algorithm is given by repeated applications of the Landweber method, the iteratively regularized Gauss-Newton method (IRGNM) and a deep neural network. The Landweber method is used to generate initial guesses for the exact contrast, and the IRGNM is employed to make further improvements to the estimated contrast. Our deep neural network (called the learned projector in this paper) mainly focuses on learning the a priori information of the shape of the unknown contrast by using a normalization technique in the training process and is trained to act like a projector which is expected to make the estimated contrast obtained by the Landweber method or the IRGNM closer to the exact contrast. It is believed that the application of the normalization technique can release the burden of training the deep neural network and lead to good performance of the proposed algorithm. Furthermore, the learned projector is expected to provide good initial guesses for IRGNM and be helpful for accelerating the proposed algorithm. Extensive numerical experiments show that our inversion algorithm has a satisfactory reconstruction capacity and good generalization ability.

Keywords: inverse medium scattering problem, far-field data, Landweber method, iteratively regularized Gauss-Newton method, deep learning method.

1 Introduction

In this paper, we consider the inverse problem of scattering of time-harmonic acoustic waves from inhomogeneous media in two dimensions. This kind of problem occurs in many applications such as sonar detection, remote sensing, geophysical exploration, medical imaging and nondestructive testing (see, e.g., [5,11,12,29]).

It is well-known that inverse medium scattering problems are nonlinear and severely ill-posed. Therefore, many iterative algorithms with various regularization strategies have been

* Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (likai98@amss.ac.cn)
†LSEC and Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (b.zhang@amt.ac.cn)
‡Corresponding author. Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (zhanghaiwen@amss.ac.cn)
developed for recovering inhomogeneous media (or contrasts of inhomogeneous media) from a knowledge of the far-field data or scattered-field data. A continuation method was proposed in [3] to reconstruct the inhomogeneous medium from multi-frequency scattering data. In [3] the Born approximation was first employed to compute the initial guess of the inhomogeneous medium from the measured data with the lowest frequency, and the Landweber method was then recursively applied to obtain the reconstructed result from the measured data with multiple frequencies. In [3], the author proposed the iteratively regularized Gauss-Newton method (IRGNM) with preconditioning techniques, which was applied to solve inverse medium scattering problems. The author in [23] proposed the regularized Newton method with a preconditioner for the inverse medium scattering problems, which reduces the computational costs compared with standard regularized Newton methods. The contrast source inversion method (CSI) was proposed in [42] for the inverse medium scattering problems. The basic idea of CSI is to minimize the cost functional by alternatively updating the contrast and the contrast source. [8] proposed the subspace-based optimization method, which shares similar properties with CSI. For a comprehensive discussions of regularization methods, see the monographs [15, 25] and the references quoted therein. Recently, non-iterative algorithms have attracted much attention in inverse medium scattering problems, such as linear sampling method [13], gap functional method [6], factorization method [28], singular sources method [36] and approximate factorization method [37]. Non-iterative methods do not need to solve the forward problem, and therefore they are computationally fast. However, the reconstruction results of non-iterative algorithms are usually less accurate than those of iterative algorithms.

In recent years, deep learning methods have been promoting the development of various fields (see, e.g., [19, 20, 32, 40]). Motivated by these successes, researchers have begun to apply the convolutional neural networks (CNNs) to solve inverse problems, such as computed tomography (CT) [7], [17], magnetic resonance imaging (MRI) [45], optical diffraction tomography (ODT) [44] and electrical impedance tomography (EIT) [43]. We refer to [1, 35] for good surveys of deep learning-based methods for various inverse problems. It is worth noting that some researchers have developed various iterative regularization methods based on deep learning methods. The authors in [45] proposed a novel deep learning approach for MRI, which combines the traditional compressive sensing method and the deep learning method. The basic idea of [45] is to unroll the alternating direction method of multipliers (ADMM) as two deep learning architectures, which automatically learn regularization strategies from training data and can produce promising results with low computational costs. [17] developed a reconstruction approach for CT, which combines the projected gradient descent method and deep learning method. The approach in [17] first trains the CNN to act like a projector onto the set of desirable solutions and then produces the reconstructed results by incorporating such CNN into the projected gradient descent method. This approach was extended to the case of ODT in [44].

For inverse medium scattering problems, deep learning methods have also been extensively studied. In [46], the authors proposed a two-step enhanced deep learning approach, where the first step is to retrieve initial contrasts from the scattered fields by a CNN and the second step is to employ a residual CNN to refine the initial contrasts. The authors in [26] proposed a novel deep neural network called SwitchNet for solving the inverse medium scattering problems under the assumption that the contrasts of inhomogeneous media are small, where SwitchNet was elaborately designed by analyzing the inherent low-rank structure of the scattering problems and was trained to map the scattered-field data to the unknown scatterers. [33, 47] first retrieved the initial contrasts of inhomogeneous media by non-CNN-based methods and then employed
well-trained CNNs to refine the reconstruction results. The CNN in [33] was built based on a conventional iterative regularized algorithm. [39] first employed a well-trained CNN to get a good approximation of the contrast source and then applied subspace optimization methods with total variation regularization for further refinement. For a recent review of deep learning-based methods for inverse medium scattering problems, see [9].

In this paper, we propose an iterative reconstruction algorithm based on deep learning method for recovering the contrast of an inhomogeneous medium from the far-field data. The proposed algorithm is given by repeated applications of the Landweber method, the IRGNM and a deep neural network. It is known that one Landweber step is very cheap and the convergence of Landweber method is very slow (see, e.g., [15, 22, 23]). Thus we use the Landweber method in the proposed algorithm to generate initial guess for the exact contrast. Moreover, since it is observed in the numerical experiments of [22] that the convergence of IRGNM is faster than that of the Landweber method, we employ IRGNM in the proposed algorithm to make further improvements to the estimated contrast. Furthermore, motivated by the deep learning-based projector in [17], our deep neural network (called the learned projector in this paper) is trained to act like a projector which is expected to make the estimated contrast obtained by the Landweber method or IRGNM closer to the exact contrast. Note that the deep learning-based projector in [17] was trained to learn the distribution of the set of desirable solutions. However, our deep neural network mainly focuses on learning the a priori information of the shape of the unknown contrast by using a normalization technique in the training process (see Section 4.1.2). It is believed that the application of the normalization technique could release the burden of training our deep neural network and lead to good performance of the proposed algorithm. Moreover, it is known that the choice of initial guesses of the unknown contrast has an important impact on the convergence of IRGNM (see the discussions in Remark 3.4). Based on this observation, we will add the samples obtained both by the Landweber method and by IRGNM into the training dataset. By doing so, our deep neural network is expected to provide good initial guesses for IRGNM and be helpful for accelerating the proposed algorithm. Further, we will introduce the multi-frequency strategy and the multi-resolution strategy to make our algorithm more practical and efficient. Extensive numerical experiments show that our algorithm has a satisfactory reconstruction capacity and good generalization ability.

The remaining part of the paper is organized as follows. In Section 2, we present the considered direct and inverse medium scattering problems. In Section 3, we introduce the Landweber method, IRGNM and a combined iterative algorithm based on these two methods. Motivated by these approaches, we propose an iterative reconstruction algorithm with the learned projector in Section 4. Numerical experiments are carried out in Section 5 to illustrate the effectiveness of our algorithm. Some conclusions and remarks are given in Section 6.

2 Problem formulation

In this section, we introduce the direct and inverse medium scattering problems considered in this paper. Precisely, let $B_\rho := \{x \in \mathbb{R}^2 : |x| < \rho\}$ with $\rho > 0$. Assume that the whole space $\mathbb{R}^2$ is filled with an inhomogeneous medium characterized by the piecewise smooth refractive index $n(x) > 0$. Define $m(x) := n(x) - 1$ to be the contrast of the inhomogeneous medium and suppose that $\text{supp}(m) \subset B_\rho$. We illuminate the inhomogeneous medium by the incident plane wave $u^i = u^i(x, d) := e^{ikx \cdot d}$, where $k > 0$ is the wave number and $d \in S^1 := \{x \in \mathbb{R}^2 : |x| = 1\}$.
denotes the incident direction. Then the scattering problem by the inhomogeneous medium is modeled by the Helmholtz equation

$$\Delta u(x) + k^2 n(x) u(x) = 0 \quad \text{in } \mathbb{R}^2,$$  \hspace{1cm} (2.1)

where the total field \( u := u^i + u^s \) is the sum of the incident field \( u^i \) and the scattered field \( u^s \), and \( u^s \) is required to satisfy the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = 0, \quad r = |x|. \hspace{1cm} (2.2)$$

Moreover, it is known that the scattered field \( u^s \) has the asymptotic behavior \[12\]

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ u^\infty(\hat{x}) + \mathcal{O}\left( \frac{1}{|x|} \right) \right\}, \quad |x| \to \infty,$$

uniformly for all directions \( \hat{x} := x/|x| \in \mathbb{S}^1 \), where \( u^\infty \) is the far-field pattern of \( u^s \). To illustrate the dependence on the direction \( d \in \mathbb{S}^1 \), we write the far-field pattern, the scattered field and the total field as \( u^\infty(x, d), u^s(x, d) \) and \( u(x, d) \), respectively. We refer to \[12\] for the well-posedness of the direct scattering problem (2.1)–(2.2). In this paper, we consider the following inverse problem.

**Inverse problem (IP).** Determine the contrast \( m \) from the measured data \( u^\infty(\hat{x}, d) \) for \( \hat{x}, d \in \mathbb{S}^1 \).

For the inverse problem (IP), we introduce the far-field operator \( \mathcal{F} : L^2(B_\rho) \to L^2(\mathbb{S}^1 \times \mathbb{S}^1) \) mapping the contrast \( m(x) \) to its corresponding far-field pattern \( u^\infty(\hat{x}, d) \), that is,

$$\mathcal{F}(m) = u^\infty.$$

Note that this equation is nonlinear and severely ill-posed. For the uniqueness result of the inverse problem (IP), we refer to \[12\]. In practical applications, only the noisy measured data \( u^{\infty, \delta} \) is available, where \( \delta > 0 \) denotes the noise level (see Section 5.1.1 for the choice of \( u^{\infty, \delta} \)).

The present work consists in solving the perturbed equation

$$\mathcal{F}(m) \approx u^{\infty, \delta} \hspace{1cm} (2.3)$$

for the unknown contrast \( m \).

It is known that the scattering problem (2.1)–(2.2) is equivalent to the problem of solving the well-known Lippmann-Schwinger equation \[12\]

$$u(x, d) = u^i(x, d) + k^2 \int_{\mathbb{R}^2} \Phi(x, y)m(y)u(y, d)dy, \quad x \in \mathbb{R}^2,$$  \hspace{1cm} (2.4)

where \( \Phi(x, y) := (i/4)H_0^{(1)}(k|x - y|), \ x \neq y \), denotes the fundamental solution to the Helmholtz equation in two dimensions and \( H_0^{(1)} \) denotes the Hankel function of the first kind of order zero. It can be deduced from (2.4) that the corresponding far-field pattern \( u^\infty \) is given by

$$u^\infty(\hat{x}, d) = \frac{k^2 e^{ik\hat{x} \cdot y}}{\sqrt{8\pi}} \int_{\mathbb{R}^2} e^{-ik\hat{x} \cdot y}m(y)u(y, d)dy \hspace{1cm} (2.5)$$
for \( \hat{x} = x/|x| \) on \( S^1 \). It is known that if \( k \) is sufficiently small, (2.5) can be approximated by the following equation

\[
 u^\infty(\hat{x}, d) \approx \sqrt{8\pi} \frac{k^{\frac{3}{2}} e^{\frac{i\pi}{4}}}{\sqrt{\pi}} \int_{\mathbb{R}^2} e^{-i\hat{x} \cdot y} m(y) u^i(y, d) dy, 
\]

which is called the Born approximation (see [12] Chapter 8.4) for an analogous result in the 3D case. Accordingly, (2.3) can be approximated by the following linear equation

\[
 \mathcal{F}_b(m) \approx \hat{u}^\infty, 
\]

where \( (\mathcal{F}_b(m))(\hat{x}, d) := \sqrt{8\pi} \int_{\mathbb{R}^2} e^{-i\hat{x} \cdot y} m(y) u^i(y, d) dy \) is a linear operator of the contrast \( m \).

In this paper, Newton type methods are used in our numerical algorithm as backbone. In doing so, we need the Fréchet derivative of the far-field operator \( \mathcal{F} \). [12] Theorem 11.6] derived the Fréchet derivative of the far-field operator in the 3D case. By a similar argument as in [12], it can be proved that the far-field operator \( \mathcal{F} : m \mapsto u^\infty \) is Fréchet differentiable, and the derivative is given by \( \mathcal{F}'(m)(q) = v^\infty \), where \( q \in L^2(B_\rho) \) and \( v^\infty := v^\infty(\hat{x}, d) \) is the far-field pattern of the scattered field \( v \) satisfying the following inhomogeneous scattering problem

\[
 \begin{cases} 
 \triangle v + k^2 nv = -k^2 uq & \text{in } \mathbb{R}^2, \\
 \lim_{r \to \infty} \sqrt{r} \left( \frac{\partial v}{\partial r} - ikv \right) = 0, & r = |x|. 
\end{cases} 
\] (2.6)

Here, \( u = u(x, d) \) is the total field corresponding to the contrast \( m \). From this, it can be seen that one needs to compute the numerical solution of the scattering problem (2.6) in order to numerically solve the Fréchet derivative of \( \mathcal{F} \).

### 3 Iterative algorithms for solving inverse scattering problem

In this section, we begin by introducing two iterative reconstruction algorithms for solving the problem (IP). We will also present a combined iterative reconstruction algorithm based on these two algorithms. These iterative methods will motivate us to develop the reconstruction algorithm in combination of iterative method and deep learning method in the next section.

For the sake of numerical reconstruction, it is necessary to discretize the contrast \( m \). Precisely, let the square area \( C_\rho := [-2\rho, 2\rho] \times [-2\rho, 2\rho] \subset \mathbb{R}^2 \) and we discretize \( C_\rho \) into evenly distributed \( N \times N \) pixels which are denoted as \( x_{ij} \) \( (i, j = 1, \ldots, N) \). Then the contrast \( m \) can be approximately represented by a discrete matrix \( m_N = (m_N^{(i,j)}) \in \mathbb{C}^{N \times N} \) with \( m_N^{(i,j)} := m(x_{ij}) \), which is also called the contrast matrix in the rest of the paper. Conventionally, the inhomogeneous medium is illuminated by \( Q \) incident plane waves \( u^i(x, d_q) \) with distinct incident directions \( d_q \) \( (q = 1, \ldots, Q) \) uniformly distributed on \( S^1 \), and we measure far-field data at \( P \) distinct observation directions \( \hat{x}_p \) \( (p = 1, \ldots, P) \) uniformly distributed on \( S^1 \). The noisy far-field pattern \( u^{\infty, \delta} \) now can be discretized as a measurement matrix \( u^{\infty, \delta} := (a^\delta_p) \in \mathbb{C}^{P \times Q} \) with \( a^\delta_p := u^{\infty, \delta}(\hat{x}_p, d_q), \) \( p = 1, \ldots, P, \) \( q = 1, \ldots, Q \). Note that \( \text{supp}(m) \subset B_\rho \subset C_\rho \). Then the formula (2.3) can be approximated as follows

\[
 \mathbf{F}_N(m_N) \approx u^{\infty, \delta}, 
\] (3.1)

where \( \mathbf{F}_N \) denotes the discrete form of the far-field operator \( \mathcal{F} \). We denote the discrete form of \( \mathcal{F}' \) by \( \mathbf{F}'_N \).
Remark 3.1. The scattering problem (2.1)–(2.2) can be numerically computed by employing fast Fourier transformation to solve the well-known Lippmann-Schwinger equation in a disk containing the support of contrast \( m \), which was suggested by Vainikko (see [23, 41]). In the present paper, \( F_N \) and \( F'_N \) is implemented by using this method with the disk to be \( B_\rho \).

3.1 Landweber method

The Landweber iteration has been extensively studied for linear ill-posed problems and nonlinear ill-posed problems. For the comprehensive introduction of the Landweber iteration, we refer the reader to [18, 22, 29, 30]. To solve (3.1), the Landweber method has the following form in each iterative step:

\[
m_{\delta,i+1} = m_{\delta,i} + \mu_i [F'_N(m_{\delta,i}^\delta)]^*(u_{\infty,\delta} - F_N(m_{\delta,i}^\delta)), \tag{3.2}
\]

where \( m_{\delta,i} \) and \( m_{\delta,i+1} \) are the approximations of the unknown contrast at the \( i \)-th and \( (i+1) \)-th iterations, respectively. Here, the superscript \( \delta \) indicates the dependence on the noise level, \([F'_N(m_{\delta,i}^\delta)]^*\) denotes the adjoint of \( F'_N(m_{\delta,i}^\delta) \) and \( \mu_i > 0 \) is the stepsize at the \( i \)-th iteration. It should be noted that usually (3.2) is called standard Landweber iteration when \( \mu_i \) equals to each other for all \( i \in \mathbb{N} \). Here, for practical consideration, we choose \( \mu_i := 1/||F'_N(m_{\delta,i}^\delta)||_2^2 \) at the \( i \)-th iteration with \( || \cdot ||_2 \) denoting the \( L^2 \) norm of a matrix. Let \( N_{La} \) be the total iteration number and \( m_{\delta,0}^L \) be the initial guess of \( m_N \). Then we can present the Landweber method in Algorithm 1 for the problem (IP).

Algorithm 1 Landweber method

**Input:** \( m_{\delta,0}^L, F_N, u_{\infty,\delta}, N_{La} \)

**Output:** final estimated contrast for \( m_N \)

**Initialize:** \( i = 0, m_{\delta,0}^L = m_{\delta,0}^L \)

1: while \( i < N_{La} \) do
2: \( \mu_i \leftarrow 1/||F'_N(m_{\delta,i}^\delta)||_2^2 \)
3: \( m_{\delta,i+1} = m_{\delta,i} + \mu_i [F'_N(m_{\delta,i}^\delta)]^*(u_{\infty,\delta} - F_N(m_{\delta,i}^\delta)) \)
4: \( i \leftarrow i + 1 \)
5: end while
6: Set final estimated contrast to be \( m_{\delta,N_{La}}^\delta \).

Remark 3.2. For the case of linear inverse problem, [29, Theorem 2.15] presented a convergence result of the Landweber method under certain assumptions. It should be noted that, in [29, Theorem 2.15], the initial guess is simply chosen to be zero. On the other hand, as mentioned in Section 2, the far-field operator \( F \) can be approximated by the linear operator \( F_b \) when \( k \) is sufficiently small. Based on these observations, it is reasonable to choose the initial guess of Algorithm 1 to be zero for the case of small wave numbers.

Remark 3.3. It is known that one Landweber step is very cheap. However, both theoretical analysis and numerical experiments show that the standard Landweber method has low convergence rate [15, 22, 23]. Hence, in this work, we only employ Algorithm 1 to generate an acceptable initial guess for further refinement.
3.2 Iteratively regularized Gauss-Newton method

Iteratively regularized Gauss-Newton method (IRGNM) was first proposed by Bakushinskii [2], which is an inexact Newton method and incorporates initial guess as an important a priori information for regularization. In this section we present the IRGNM for solving (3.1). Let \( m_{N,0}^{Ir} \) be the initial guess of the unknown contrast matrix \( m_N \). To present the IRGNM iteration, we set \( m_{N,i}^{δ} := m_{N,0}^{Ir} \) and define \( h_{N,i} := m_{N,i+1}^{δ} - m_{N,i}^{δ} \), where \( m_{N,i}^{δ} \) and \( m_{N,i+1}^{δ} \) are the approximations of the unknown contrast at the \( i \)-th and \((i+1)\)-th iterations, respectively. The update \( h_{N,i} \) is computed by solving the following minimization problem:

\[
 h_{N,i} = \arg \min_{h_N \in \mathbb{C}^{N \times N}} \left\{ \| F'_N(m_{N,i}^{δ})h_N + F_N(m_{N,i}^{δ}) - u^{∞,δ}\|^{2} + \alpha_i \| h_N + m_{N,i}^{δ} - m_{N,0}^{Ir} \|^2 \right\}, \tag{3.3}
\]

where \( \{\alpha_i\}_{i=0}^{\infty} \) is a fixed sequence such that

\[
 \alpha_i > 0, \quad \alpha_{i+1} \leq \alpha_i \leq \sigma \alpha_{i+1}, \quad \lim_{i \to \infty} \alpha_i = 0 \tag{3.4}
\]

for some \( \sigma > 1 \) (suggested by [21]). For the convenience of later use, we stop the IRGNM iteration by choosing a fixed total iteration number \( N_{Ir} \). For the IRGNM with an a posteriori stopping rule, we refer to [21]. Now we present the IRGNM in Algorithm 2 for the problem (IP).

**Remark 3.4.** It is known from [22, Chapter 4.5] that the penalty term \( \| h_N + m_{N,i}^{δ} - m_{N,0}^{Ir} \|^2 \) in (3.3) has regularization effect by preventing the iterations moving too far away from the initial guess \( m_{N,0}^{Ir} \). Moreover, Hohage obtained a convergence result of the IRGNM iteration in [21, Theorem 2.3] for general (possibly nonlinear and ill-posed) inverse problem under some appropriate conditions, where it is assumed that the initial guess should be close enough to the ground truth (see [21, formula (2.11)]). These arguments show that the choice of the initial guess \( m_{N,0}^{Ir} \) plays an important role in IRGNM.

**Remark 3.5.** Numerical examples in [22] show that IRGNM has faster convergence rate than the standard Landweber method. Hence, in this work, we try to find an acceptable initial guess for IRGNM and then employ Algorithm 2 to refine the reconstruction result.

**Algorithm 2** Iteratively regularized Gauss-Newton method

**Input:** \( m_{N,0}^{Ir}, F_N, u^{∞,δ}, N_{Ir}, \{\alpha_i\}_{i=0}^{N_{Ir}-1} \)

**Output:** final estimated contrast for \( m_N \)

**Initialize:** \( i = 0, m_{N,0}^{δ} = m_{N,0}^{Ir} \)

1: while \( i < N_{Ir} \) do
2: \( h_{N,i} = \arg \min_{h_N \in \mathbb{C}^{N \times N}} \left\{ \| F'_N(m_{N,i}^{δ})h_N + F_N(m_{N,i}^{δ}) - u^{∞,δ}\|^{2} + \alpha_i \| h_N + m_{N,i}^{δ} - m_{N,0}^{Ir} \|^2 \right\} \)
3: \( m_{N,i+1}^{δ} = m_{N,i}^{δ} + h_{N,i} \)
4: \( i \leftarrow i + 1 \)
5: end while
6: Set final estimated contrast to be \( m_{N,N_{Ir}}^{δ} \).
3.3 Combined iterative reconstruction algorithm

Based on discussions in Remarks 3.2 and 3.3, we present an iterative reconstruction algorithm combining the Landweber method and the IRGNM for the problem (IP). Roughly speaking, we employ the Landweber method to provide an acceptable initial guess and then passes it to the IRGNM for further refinement. In order to balance the trade-off between the reconstruction ability and the computational burden, we first introduce the multi-resolution strategy and the multi-frequency strategy in Section 3.3.1. With the aid of these strategies, we present the combined iterative reconstruction algorithm and give some explanations in Section 3.3.2.

3.3.1 Multi-resolution strategy and multi-frequency strategy

First, we introduce the multi-resolution strategy inspired by [39]. As mentioned before, for the Landweber method, it will be used to provide an acceptable initial guess for later use, hence we adopt a low resolution to reduce the computational burden. On the other hand, for IRGNM, since it will be used to refine the reconstruction result, we apply a high resolution to maintain the reconstruction ability. To be more specific, we set the resolutions of the Landweber method and the IRGNM to be $N_1 \times N_1$ pixels and $N_2 \times N_2$ pixels, respectively, with $N_1 < N_2$ and $N_2 = N_1 d$ for some $d \in \mathbb{N}^+$. In the combined iterative reconstruction algorithm, the Landweber method outputs a low resolution image $\tilde{m}_{N_1} = (b_{ij})_{N_1 \times N_1}$ with $b_{ij} \in \mathbb{C}$, which is the approximation of the exact contrast. To shift the resolution from $N_1 \times N_1$ to $N_2 \times N_2$, we introduce the up-scaling operator $I_U$ mapping $\tilde{m}_{N_1}$ to $\tilde{m}_{N_2} := (B_{ij})_{N_2 \times N_2} \in \mathbb{C}^{N_2 \times N_2}$, where $B_{ij}$ is a $d \times d$ sub-matrix of $\tilde{m}_{N_2}$ with each element to be $b_{ij}$. Then $\tilde{m}_{N_2}$ is set to be the initial guess for IRGNM.

Secondly, we introduce the multi-frequency strategy. The iterative algorithms with multi-frequency data have been extensively studied in the inverse scattering problems (see, e.g., [4][12] and references therein). Roughly speaking, the low frequency measurement data can be applied to recover the large scale features of the unknown contrasts, while the high frequency measurement data lead to the reconstructions of the small scale features of the contrasts. Following the ideas in [4][12], we measure the far-field data at a sequence of wave numbers $k_1 < k_2 < \cdots < k_L$ for the Landweber method to provide an acceptable and robust initial guess for later use. The detailed iterative process for the Landweber method with multi-frequency strategy will be given in Section 3.3.2.

3.3.2 Description of the combined iterative algorithm

Now we describe the combined iterative reconstruction algorithm in detail. Let $N_1 \times N_1$ and $N_2 \times N_2$ be the resolutions of the Landweber method and IRGNM, respectively, where $N_1$ and $N_2$ satisfy the conditions in Section 3.3.1. In what follows, for any wave number $k$, we rewrite the discrete far-field operator $F_N$ and the noisy far-field data $u^{\infty,\delta}$ in (3.1) as $F_{N,k}$ and $u^{\infty,\delta}_k$, respectively, to indicate the dependence on the wave number. For the Landweber method, we measure the noisy far-field data $u_k^{\infty,\delta}$, for $l = 1, 2, \ldots, L$, at a sequence of wave numbers $k_1 < k_2 < \cdots < k_L$. For IRGNM, we measure the noisy far-field data $u_k^{\infty,\delta}$ at the wave number $k_{L+1}$. Here, we choose $k_1$ to be small and $k_{L+1}$ is not necessarily greater than $k_L$. Then the combined iterative reconstruction algorithm can be divided into the following three steps.

Step 1. We apply the Landweber method with multi-frequency strategy. For $l = 1, \ldots, L$, we recursively employ Algorithm 1 with $m_{N_0}^{a} = \tilde{m}_{N_1,l-1}^{a}$, $F_N = F_{N_1,k_l}$ and $u^{\infty,\delta} = u^{\infty,\delta}_k$ to
obtain an estimated contrast matrix \( \tilde{m}_{N_1,l} \). Here, based on Remark \ref{rem:choice_k1} and the choice of \( k_1 \), we just choose \( \tilde{m}_{N_1,0}^{La} = 0 \). Then we set \( \tilde{m}_{N_1} := \tilde{m}_{N_1,L} \). Given the above far-field data used in the Landweber method, let the mapping from \( \tilde{m}_{N_1,0}^{La} \) to \( \tilde{m}_{N_1} \) be denoted by \( \mathcal{L} \).

**Step 2.** We up-scale the output \( \tilde{m}_{N_1} \) of the mapping \( \mathcal{L} \) by setting \( \tilde{m}_{N_2} := \tilde{m}_{N_1}^{U} \) with \( \tilde{m}_{N_1}^{U} \) given in Section \ref{sec:projection}. This combined iterative reconstruction algorithm for the problem (IP) is visualized in Figure 1 and presented in Algorithm 3. For simplicity, we also call Algorithm 3 as Combined Algorithm in the rest of the paper. See Section \ref{sec:performance} for the performance of this algorithm.

**Algorithm 3 Combined iterative reconstruction algorithm**

**Input:** Parameters I (Landweber iteration): \( F_{N_1,k_1}, u_{k_1}^{\infty,\delta}, N_{La}, l = 1, 2, \ldots, L \)  
Parameters II (IRGNM iteration): \( F_{N_2,k_{L+1}}, u_{k_{L+1}}^{\infty,\delta}, N_{Ir}, \{\alpha_i\}_{i=0}^{N_{Ir}-1} \)  
Parameters III: \( I_{U} \)

**Output:** final estimated contrast \( \hat{m}_{N_2} \)

**Initialize:** \( l = 1, \tilde{m}_{N_1,0}^{La} = 0 \)

1: while \( l \leq L \) do
2: \hspace{1em} Employ Algorithm 1 with \( m_{N,0}^{La} = \tilde{m}_{N_1,0}^{La}, F_N = F_{N_1,k_1} \) and \( u^{\infty,\delta} = u_{k_1}^{\infty,\delta} \) to obtain an estimated contrast \( \tilde{m}_{N_1,L}^{La} \).
3: \hspace{1em} \( l \leftarrow l + 1 \)
4: end while
5: Set \( \tilde{m}_{N_1} = \tilde{m}_{N_1,L}^{La} \) and \( \tilde{m}_{N_2} = I_{U}(\tilde{m}_{N_1}) \).
6: Employ Algorithm 2 with \( m_{N,0}^{Ir} = \tilde{m}_{N_2}, F_N = F_{N_2,k_{L+1}}, u^{\infty,\delta} = u_{k_{L+1}}^{\infty,\delta} \) and \( \{\alpha_i\}_{i=0}^{N_{Ir}-1} \) to obtain an estimated contrast \( \tilde{m}_{N_2}^{Ir} \).
7: Set the final estimated contrast \( \hat{m}_{N_2} = \tilde{m}_{N_2}^{Ir} \).

**4 Iterative reconstruction algorithm with learned projector**

For the inverse problem (IP), the combined iterative reconstruction algorithm in Section \ref{sec:combine} may be improved in the following aspects.
(i) In practice, the unknown contrasts usually share some common a priori information (such as shape information) which is hard to be directly applied to inversion algorithms. We hope to utilize such information for reconstruction.

(ii) In general, the number of iterations should be large enough in order to obtain an accurate reconstruction result. In particular, the Landweber method has a low convergence rate \[15,22\]. Moreover, one need to solve direct problems at every iteration, which means that the computational costs will be larger as the total iteration number increases. Hence, we hope to accelerate the iterative process.

(iii) Based on Remark 3.4 and the references therein, it can be seen that the initial guess plays an essential role in IRGNM. Hence, it is better to choose the initial guess of IRGNM in a suitable way.

Based on these perspectives, we propose an iterative reconstruction algorithm based on deep neural network for the problem (IP). In order to encode some a priori information of the unknown contrasts into the proposed algorithm, we will introduce a learned projector \(P_\Theta\). Here, \(P_\Theta\) is parameterized by a deep convolutional neural network and \(\Theta\) are some parameters which need to be determined during the training process. We call \(P_\Theta\) the learned projector because we will train \(P_\Theta\) such that it may act as a projector to make the estimated contrasts, which are obtained by the Landweber method or IRGNM, closer to the corresponding exact contrasts. After the training process, we will incorporate such trained neural network \(P_\Theta\) into the combined iterative reconstruction algorithm in Section 3.3. Roughly speaking, we first generate estimated contrasts by Landweber method, and then repeatedly apply the trained neural network \(P_\Theta\), which is followed by IRGNM, to update the estimated contrasts. For this algorithm, we hope that \(P_\Theta\) can be helpful for improving the estimated contrasts and providing good initial guesses for IRGNM. Moreover, we also expect \(P_\Theta\) could accelerate the combined iterative reconstruction algorithm. The architecture and the training strategy of \(P_\Theta\) will be given in Section 4.1. Based on the learned projector \(P_\Theta\), we present the proposed algorithm in Section 4.2.

4.1 Learned projector \(P_\Theta\)

This section is devoted to introducing the architecture of the learned projector \(P_\Theta\) and the training strategy for finding a suitable \(P_\Theta\).

4.1.1 Network architecture

We parameterize \(P_\Theta\) by a convolutional neural network called U-Net, which has a U-shaped structure. The original version of U-net was first proposed in \[38\] for biomedical image segmentation. In this work, we adopt a modified version of U-net (see Figure 2), which is similar to the one used in \[44\]. To be more specific, the input and the output of \(P_\Theta\) are the volumes with sizes \((N_1 \times N_1 \times 2)\) and \((N_1 \times N_1 \times 1)\), respectively, where \(N_1\) is given as in Section 3.3. For our proposed algorithm in Section 4.2 we choose \(N_1 = 64\). As shown in Figure 2 each red and blue item represents a volume (also called multichannel feature map \[44\]), the number of channels is shown at the top of the volume, and the length and width are provided at the lower-left edge of the volume. The left part and the right part of \(P_\Theta\) are the contracting path and the expansive path, respectively. For each convolutional layer of both these two paths, we employ a \((3 \times 3)\) convolution with zero-padding and \((1 \times 1)\) convolution stride, batch normalization (BN), and
rectified linear unit (ReLU) (see yellow right arrow in Figure 2). For each down-sampling layer in the contracting path, we apply a $(2 \times 2)$ max pooling layer (see green downward arrow in Figure 2). For each up-sampling layer in the expansive path, we use a $(2 \times 2)$ transposed convolution (see purple upward arrow in Figure 2). Each up-sampled output in the expansive path is concatenated with the corresponding multichannel feature map from the contracting path (see gray right arrow in Figure 2). Moreover, the first channel of the input is added to the output of the penultimate layer (see the eternal skip connection in Figure 2). At last, we add a Leaky rectified linear unit (LeakyReLU) \cite{leakyrelu} behind the $(1 \times 1)$ convolution with $(1 \times 1)$ convolution stride to obtain the final output (see red right arrow in Figure 2). For more details of U-net, we refer to \cite{unet, unetplusplus}.

Figure 2: The architecture of $\mathcal{P}_\Theta$. Each red and blue item represents a volume (also called multichannel feature map). The number of channels is shown at the top of the volume, and the length and width are provided at the lower-left edge of the volume. The arrows denote different operations, which are explained at the lower-right corner of the figure.

4.1.2 Training strategy

Now we describe the training strategy of the learned projector $\mathcal{P}_\Theta$ by using a normalization technique. We wish to obtain a suitable $\mathcal{P}_\Theta$ by training process so that such trained neural network $\mathcal{P}_\Theta$ could project the normalization of estimated contrast matrix $\mathcal{N}(\tilde{m}_{N_1})$ to the normalization of exact contrast matrix $\mathcal{N}(m_{N_1})$, where $m_{N_1}$ denotes the exact contrast matrix with size $(N_1 \times N_1)$ and $\tilde{m}_{N_1}$ denotes the estimated contrast matrix of $m_{N_1}$ with the same size. Here, $\mathcal{N}$ is a normalization operator given by $\mathcal{N}(m) := m/\|m\|_{\infty}$ for any $m \in \mathbb{C}^{N_1 \times N_1}$ with the norm $\|\cdot\|_{\infty}$ denoting the infinity norm of a matrix. We hope that the application of $\mathcal{N}$ could help $\mathcal{P}_\Theta$ learn some a priori information of the shapes of the unknown contrasts we are interested in. In the training stage, we generate the sample set of exact contrast matrices $\{m^{(i)}_{N_1}\}_{i=1}^T$ with $T \in \mathbb{N}^+$ and $m^{(i)}_{N_1} \in \mathbb{R}^{N_1 \times N_1}$ such that $\|m^{(i)}_{N_1}\|_{\infty}, i = 1, \ldots, T$, lie uniformly in the interval $[a, b]$ with $0 < a < b$ (see Section 5.2 for the choice of the interval $[a, b]$). We note that the exact
contrast matrices \( m_N^{(i)} \) \( (i = 1, \ldots, T) \) are all chosen to be real since it is assumed in Section 2 that the exact contrast is real-valued. In what follows, let the input/output pair \((x, y)\) represent the sample of any labeled dataset, where \(x\) and \(y\) denote the input and the output, respectively. For the input/output pair \((x, y)\) used later, \(x\) and \(y\) will be chosen to be a complex matrix and a real matrix, respectively. Moreover, the real part and the imaginary part of the matrix \(x\) will be put into the first channel and the second channel of the input of \(P_{\Theta}\), respectively, and the matrix \(y\) will be used as the output of \(P_{\Theta}\). Now the training process can be divided into the following two parts.

**Part I.** We use the estimated contrast matrices generated by the Landweber method to train \(P_{\Theta}\). Precisely, for each exact contrast matrix \(m_N^{(i)} \), \(i = 1, \ldots, T\), we generate the corresponding output of the mapping \(\mathcal{L}\) (see Step 1 in Section 3.3.2), which is denoted as \(\tilde{m}_N^{(i)}\). Following the training strategy in [17] and applying the normalization operator \(N\), we divide this part into the following three steps.

**Step 1.** Train \(P_{\Theta}\) on the dataset \(S_1\) with Xavier initialization [16], where

\[
S_1 := \{(t_{N,1}^{(i)}, N(m_N^{(i)}))\}_{i=1}^T
\]

with \(t_{N,1}^{(i)} := N(\tilde{m}_N^{(i)})\), \(i = 1, \ldots, T\). Then we obtain \(P_{\Theta_1}\) after \(t_1\) epochs.

**Step 2.** Train \(P_{\Theta}\) on the dataset \(S_1 \cup S_2\) with initial values of \(\Theta\) to be \(\Theta_1\), where

\[
S_2 := \{(t_{N,2}^{(i)}, N(m_N^{(i)}))\}_{i=1}^T
\]

with \(t_{N,2}^{(i)} := P_{\Theta_1}(N(\tilde{m}_N^{(i)}))\), \(i = 1, \ldots, T\). Then we obtain \(P_{\Theta_2}\) after \(t_2\) epochs.

**Step 3.** Train \(P_{\Theta}\) on the dataset \(S_1 \cup S_2 \cup S_3\) with initial values of \(\Theta\) to be \(\Theta_2\), where

\[
S_3 := \{(t_{N,3}^{(i)}, N(m_N^{(i)}))\}_{i=1}^T
\]

with \(t_{N,3}^{(i)} := N(m_N^{(i)})\), \(i = 1, \ldots, T\). Then we obtain \(P_{\Theta_3}\) after \(t_3\) epochs.

For Steps 1, 2 and 3, we use the following error functions

\[
E_M(\Theta) := \sum_{j=1}^M \sum_{i=1}^T \| P_{\Theta}(t_{N,j}^{(i)}) - N(m_N^{(i)}) \|_2^2
\]

with \(M = 1, 2, 3\), respectively. We wish the dataset \(S_2\) used in Steps 2 and 3 could be helpful for training \(P_{\Theta}\) to mimic an important property of any projector \(P\), i.e., \(P \circ P = P\). We also hope that the dataset \(S_3\) used in Step 3 could be helpful for training \(P_{\Theta}\) to approximately project the normalization of the exact contrast matrix to itself.

**Part II.** With the aid of the normalization operator \(N\), we use the estimated contrast matrices generated by the IRGNM to train \(P_{\Theta}\). Precisely, we define \(I_0\) to be the mapping \(I\) in Section 3.3.2 with the resolution \((N_2 \times N_2)\) replaced by \((N_1 \times N_1)\). For each exact contrast matrix \(m_N^{(i)} \), \(i = 1, \ldots, T\), we generate the corresponding output of the mapping \(I_0\) with the initial guess \(\| \tilde{m}_N^{(i)} \|_\infty P_{\Theta_3}(N(\tilde{m}_N^{(i)}))\), which is denoted as \(r_N^{(i)}\). Here we note that, according to the training process in Part I, \(\| \tilde{m}_N^{(i)} \|_\infty P_{\Theta_3}(N(\tilde{m}_N^{(i)}))\) is expected to be closer to the exact contrast matrix \(m_N^{(i)}\), compared with \(m_N^{(i)}\). Next, we train \(P_{\Theta}\) on the dataset \(\bigcup_{j=1}^4 S_j\) with initial values of \(\Theta\) to be \(\Theta_3\), where

\[
S_4 := \{(t_{N,4}^{(i)}, N(m_N^{(i)}))\}_{i=1}^T
\]
Finally, we obtain the learned projector $P_{\Theta}$ after $t_4$ epochs, which will be used for our deep learning-based iterative reconstruction algorithm in Section 4.2.

### 4.2 Description of the proposed algorithm

Now we describe the proposed iterative reconstruction algorithm in detail for the problem (IP). We retain the notations in Sections 3.3 and 4.1, and use the same measured data as in Section 4.1.2. This simplified algorithm can be described by the following four steps.

**Step 1.** Given Parameter I of Algorithm 3, compute $\overline{m}_{N_1,0} := L(\overline{m}_{N_1,0}^{La})$ with $\overline{m}_{N_1,0}^{La} := 0$ (that is, the Landweber method with multi-frequency strategy). Set $i \leftarrow 0$, then go to Step 2.

**Step 2.** If $i > N_O$, go to Step 4 and stop the algorithm. Otherwise, update $\overline{m}_{N_1,i+1}$ by setting $\overline{m}_{N_1,i+1} := ||\overline{m}_{N_1,i}||_\infty P_{\Theta}(\mathcal{N}(\overline{m}_{N_1,i}))$ and up-scale $\overline{m}_{N_2,i+1}$ by setting $\overline{m}_{N_2,i+1} := I_D(\overline{m}_{N_1,i+1})$. Then we compute the error $r_{0,i+1} := ||F_{N_2,k_{L+1}}(\overline{m}_{N_2,i+1}) - u_{\infty,\delta}||_2$ and go to Step 3.

**Step 3.** Given Parameter II of Algorithm 3, compute $\overline{m}_{N_2,i+1} := I(\overline{m}_{N_2,i+1})$ (that is, IRGNM). Then we compute the error $r_{1,i+1} := ||F_{N_2,k_{L+1}}(\overline{m}_{N_2,i+1}) - u_{\infty,\delta}||_2$. If the stopping criterion is satisfied, i.e.,

$$r_{0,i+1} > r_{0,i} \quad \text{and} \quad r_{1,i+1} > r_{1,i},$$

go to Step 4. Otherwise, down-scale $\overline{m}_{N_2,i+1}$ by setting $\overline{m}_{N_1,i+1} := I_D(\overline{m}_{N_2,i+1})$, set $i \leftarrow i + 1$ and go to Step 2.

**Step 4.** We set $\overline{m}_{N_1} := \overline{m}_{N_1,i}$ to be the final estimation.

The above algorithm is presented in Algorithm 4. This algorithm is visualized in Figure 3, where it is assumed that the final estimation $\overline{m}_{N_1}$ is the output of $(M + 1)$-th application of the learned projector $P_{\Theta}$. For simplicity, we also call Algorithm 4 as **Learned Combined Algorithm** in the rest of the paper. See Section 5 for the performance of this algorithm. To show the advantage offered by the training dataset $S_4$ for Algorithm 4 in the numerical experiments we also consider a simplified version of Algorithm 4, that is, Algorithm 4 with the learned projector $P_{\Theta}$ replaced by $P_{\Theta_3}$. Here, $S_4$ and $P_{\Theta_3}$ are given as in Section 4.1.2. This simplified algorithm is called **Simplified Learned Combined Algorithm** in the rest of the paper.

![Figure 3: Iterative reconstruction algorithm with learned projector for the problem (IP).](image)
Algorithm 4 Iterative reconstruction algorithm with learned projector

**Input:** Parameters I (Landweber iteration): \( F_{N_1,k_1}, u_{k_1}^{\infty,\delta}, N_{La}, l = 1, 2, \ldots, L \)

Parameters II (IRGNM iteration): \( F_{N_2,k_{L+1}}, u_{k_{L+1}}^{\infty,\delta}, N_{Ir}, \{\alpha_i\}_{i=0}^{N_{Ir}-1} \)

Parameters III: \( \mathcal{P}_{\Theta}, I_U, I_D, N_O \)

**Output:** final estimated contrast \( \hat{m}_{N_1} \)

**Initialize:** \( l = 1, \tilde{m}^{La}_{N_1,0} = 0 \)

1: while \( l \leq L \) do
2: Employ Algorithm 1 with \( m^{La}_{N_1,l-1}, F_N = F_{N_1,k_1} \) and \( u^{\infty,\delta} = u_{k_1}^{\infty,\delta} \) to obtain an estimated contrast \( \tilde{m}^{La}_{N_1,l} \)
3: \( l \leftarrow l + 1 \)
4: end while
5: Set \( i = 0, r_{0,0} = r_{1,0} = 10^4 \) and \( \tilde{m}_{N_1,0} = \tilde{m}^{La}_{N_1,L} \).

6: while \( i \leq N_O \) do
7: Set \( \overline{m}_{N_1,i+1} := ||\tilde{m}_{N_1,i}||_\infty \mathcal{P}_{\Theta}(\mathcal{N}(\tilde{m}_{N_1,i})), \overline{m}_{N_2,i+1} := I_U(\overline{m}_{N_1,i+1}) \) and \( r_{0,i+1} := ||F_{N_2,k_{L+1}}(\overline{m}_{N_2,i+1}) - u_{k_{L+1}}^{\infty,\delta}||_2 \).
8: Employ Algorithm 2 with \( m^{Ir}_{N,0} = \overline{m}_{N_2,i+1}, F_N = F_{N_2,k_{L+1}}, u^{\infty,\delta} = u_{k_{L+1}}^{\infty,\delta} \) and \( \{\alpha_i\}_{i=0}^{N_{Ir}-1} \) to obtain an estimated contrast \( \tilde{m}_{N_2,i+1} \) and set \( r_{1,i+1} := ||F_{N_2,k_{L+1}}(\tilde{m}_{N_2,i+1}) - u_{k_{L+1}}^{\infty,\delta}||_2 \).
9: if \( r_{0,i+1} > r_{0,i} \text{ and } r_{1,i+1} > r_{1,i} \) then
10: Set final estimated contrast \( \hat{m}_{N_1} := \overline{m}_{N_1,i} \) and stop the algorithm.
11: end if
12: Set \( \tilde{m}_{N_1,i+1} := I_D(\tilde{m}_{N_2,i+1}) \) and \( i \leftarrow i + 1 \).
13: end while
14: Set final estimated contrast \( \hat{m}_{N_1} := \overline{m}_{N_1,i} \).
5 Numerical experiments

In this section, we present numerical examples to demonstrate the effectiveness of the iterative reconstruction algorithm with learned projector (i.e. *Learned Combined Algorithm*) for the problem *(IP)*. In Section 5.1 we give the experimental setup for numerical examples. The performance of *Learned Combined Algorithm* is shown in Section 5.2 and we compare this algorithm with other methods in Section 5.3.

5.1 Experimental setup

The training process is performed on COLAB (Tesla P100 GPU, Linux operating system) and is implemented on PyTorch, while the computations of direct scattering problem, the Landweber method and IRGNM are implemented by Python 3.7 on a desktop computer (Intel Core i7-10700 CPU (2.90 GHz), 32 GB of RAM, Ubuntu 20.04 LTS).

5.1.1 Simulation setup for the scattering model

As we mentioned in Sections 2 and 3, the support of the unknown contrast is assumed to lie in a disk \( B_\rho \subset C_\rho \) with \( \rho > 0 \). Without loss of generality, we choose \( \rho = 1 \). We set the number of incident directions \( Q = 16 \) and the number of measured directions \( P = 32 \). To generate the synthetic far-field data, we use the method discussed in Remark 3.1 with \( N = 512 \). The noisy far-field data \( u^\infty,\delta(\hat{x}_p, d_q), \ p = 1, \ldots, P, \ q = 1, \ldots, Q, \) are given as

\[
    u^\infty,\delta(\hat{x}_p, d_q) = u^\infty(\hat{x}_p, d_q)(1 + \delta \xi_{p,q}),
\]

where \( \delta \) is noise level and \( \xi_{p,q} \) is standard normal distribution. In this paper, we choose \( \delta = 5\% \).

5.1.2 Parameter setting for inversion algorithms

For Parameters I in *Learned Combined Algorithm*, we choose \( N_1 = 64, \ N_{La} = 100, \) and \( L = 3 \) with wave numbers \( k_1 = 3, \ k_2 = 5 \) and \( k_3 = 7 \). For Parameters II in *Learned Combined Algorithm*, we choose \( N_2 = 256, \ N_{Ir} = 5, \) wave number \( k_{L+1} = 6 \), and the regularization parameters \( \alpha_i := 10 \times (0.2)^i, \ i = 0, \ldots, N_{Ir} - 1 \) (the choice of \( \alpha_i \) follows the suggestion in (3.4)). For Parameters III in *Learned Combined Algorithm*, we set \( N_O = 20 \). Further, in order to obtain the well-trained neural network \( \mathcal{P}_\Theta \) for *Learned Combined Algorithm*, we train \( \mathcal{P}_\Theta \) by minimizing the error function \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \) and \( \mathcal{E} \) with the epochs \( t_1 = 100, \ t_2 = 20, \ t_3 = 20 \) and \( t_4 = 20 \), respectively, all using the Adam optimizer [27] with batch size of 30 and learning rate of \( 10^{-3} \). Moreover, the parameter settings of other inversion algorithms carried out in the numerical experiments are similar as above with only minor differences (see Section 5.3 for details).

5.1.3 Evaluation criterion for inversion algorithms

In order to quantitatively evaluate the reconstruction performance of *Learned Combined Algorithm*, we introduce an error function to measure the difference between the exact refractive index and the estimated refractive index obtained by *Learned Combined Algorithm*. As mentioned before, for the exact contrast \( m(x), \ m_{N_1} = (m_{N_1}^{(i,j)}) \in \mathbb{C}^{N_1 \times N_1} \) is the exact contrast matrix with \( m_{N_1}^{(i,j)} = m(x_{ij}) \) and \( \hat{m}_{N_1} = (\hat{m}_{N_1}^{(i,j)}) \in \mathbb{C}^{N_1 \times N_1} \) is the output of *Learned Combined Algorithm*.
which is the estimation of $m_{N_1}$. Here $x_{ij}$ ($i, j = 1, 2, \ldots, N_1$) are the points introduced at the beginning of Section 3 with $N = N_1$. Accordingly, $n_{N_1} = (n_{N_1}^{(i,j)}) := m_{N_1} + 1$ is the discretization of the refractive index $n(x) = m(x) + 1$ and $\hat{n}_{N_1} = (\hat{n}_{N_1}^{(i,j)}) := \hat{m}_{N_1} + 1$ is the estimation of $n_{N_1}$. Due to the assumption that the exact contrast is supported in the disk $B_\rho$, we are only concerned with the difference between $n_{N_1}^{(i,j)}$ and $\hat{n}_{N_1}^{(i,j)}$ for all $(i, j) \in I := \{(i, j) : |x_{ij}| \leq \rho, i, j = 1, \ldots, N_1\}$. Hence, we define the relative error function $R_e$ between $n_{N_1}$ and $\hat{n}_{N_1}$ as follows

$$R_e(n_{N_1}, \hat{n}_{N_1}) := \sqrt{\frac{1}{|I|} \sum_{(i,j) \in I} \left| \frac{n_{N_1}^{(i,j)} - \hat{n}_{N_1}^{(i,j)}}{n_{N_1}^{(i,j)}} \right|^2}.$$ 

Moreover, the reconstruction performance of other inversion algorithms carried out in the numerical experiments will be evaluated in the same way as above.

### 5.2 Performance of the proposed iterative reconstruction algorithm

We train $P_{\Theta}$ using MNIST dataset [14], which consists of 10 handwritten digits from 0 to 9. To be more specific, we randomly select $T = 3200$ different digits from MNIST dataset to represent the exact contrast matrices $\{m_{N_1}^{(i)}\}_{i=1}^T$ of unknown inhomogeneous media such that $\|m_{N_1}^{(i)}\|_\infty, i = 1, \ldots, T$, lie uniformly in the interval $[1, 5]$. During the training process, 3000 samples are used for training $P_{\Theta}$ with the training strategy in Section 4.1.2 and 200 samples are used to validate the training performance.

First, in order to investigate the influence of values of different contrasts on Learned Combined Algorithm, we choose the exact contrast matrices generated from other MNIST digits for testing, where the infinity norm of these matrices are set to be 2, 4 or 6, and the corresponding reconstructed contrast matrices obtained by Learned Combined Algorithm are illustrated in Figure 4. Each row of Figure 4 presents the reconstruction results (that is, the result after the whole Landweber iterations and the results after one or several applications of the learned projector $P_{\Theta}$) and the ground truth for one sample. The reconstruction results in Figure 4 show that Learned Combined Algorithm could generate satisfactory results even for the case when the values of the contrasts for testing are higher than those for training.

Secondly, we use the exact contrast matrices generated from letters in EMNIST dataset [10] and handwritten Chinese characters for testing. The corresponding reconstruction results of Learned Combined Algorithm are shown in Figure 5, where each row presents the reconstruction results and the ground truth for one sample in the same way as in Figure 4. The reconstructions in Figure 5 demonstrate the good generalization ability of Learned Combined Algorithm.

Thirdly, we evaluate Learned Combined Algorithm quantitatively. To do this, we consider three cases, which are denoted as Cases 1.1, 1.2 and 1.3. In all three cases, we generate 10 samples from the MNIST dataset and 10 samples from the EMNIST dataset to represent the exact contrast matrices. For Cases 1.1, 1.2 and 1.3, we set the infinity norm of each exact contrast matrix to be 2, 4 and 6, respectively. For these three cases, the second row and the third row in Table 1 present the average values of the relative errors $R_e$ for the outputs of Learned Combined Algorithm on the MNIST dataset and the EMNIST dataset, respectively. Notice that we train $P_{\Theta}$ on MNIST dataset. Hence, it can be seen in Table 1 that Learned Combined Algorithm has good performance on EMNIST dataset, which shows the satisfactory generalization ability of this algorithm.
Figure 4: Reconstructions by *Learned Combined Algorithm* with exact contrast matrices generated from MNIST digits. Each row presents the reconstruction results (that is, the result after the whole Landweber iterations and the results after one or several applications of the learned projector $P_{\hat{\Theta}}$) and the ground truth for one sample.
Figure 5: Reconstructions by *Learned Combined Algorithm* with the exact contrast matrices generated from EMNIST dataset and handwritten Chinese characters. Each row presents the reconstruction results (that is, the result after the whole Landweber iterations and the results after one or several applications of the learned projector $P_{\Theta}$) and the ground truth for one sample.
Table 1: The average values of relative errors $R_e$ for the outputs of Learned Combined Algorithm on the MNIST dataset and the EMNIST dataset.

|          | Case 1.1 | Case 1.2 | Case 1.3 |
|----------|----------|----------|----------|
| MNIST    | 9.8%     | 16.9%    | 24.4%    |
| EMNIST   | 11.9%    | 19.1%    | 30.8%    |

5.3 Comparison with other methods

First, in order to appreciate the virtues of the deep learning method in our algorithm, we compare Learned Combined Algorithm with Combined Algorithm given in Section 3.3.2. In our numerical experiments, Parameters I and II in Combined Algorithm are the same as those in Learned Combined Algorithm, except for choosing $N_{Ir} = 10$ and $\alpha_i = 10 \times (0.5)^i$ ($i = 0, \ldots, N_{Ir}$) in Parameters II, where the purpose of choosing such $N_{Ir}$ and $\alpha_i$ is to obtain better reconstruction results. According to the parameter setting in Section 5.1.2, the resolutions of outputs of Learned Combined Algorithm and Combined Algorithm are $64 \times 64$ and $256 \times 256$, respectively. Hence, in order to compare the outputs of these two algorithms, we down-scale the outputs of Combined Algorithm from resolution $256 \times 256$ to resolution $64 \times 64$ by using $I_D$ (see Section 4.2) with $d = 4$.

Secondly, we also compare Learned Combined Algorithm with Simplified Learned Combined Algorithm (see Section 4.2) to illustrate the benefit brought by the training dataset $S_4$ (see Section 4.1.2). Here, Parameters I, II and III in Simplified Learned Combined Algorithm are the same as those in Learned Combined Algorithm, except that the learned projector $P_{\hat{\Theta}}$ is replaced by $P_\Theta$.

In order to present the performance of the above three algorithms, we consider three cases, which are denoted as Cases 2.1, 2.2 and 2.3. We choose the samples in Cases 2.1, 2.2 and 2.3 to be the same 10 samples generated from MNIST dataset in Cases 1.1, 1.2 and 1.3, respectively. For these three cases, we present the average values of the relative errors $R_e$ for the outputs of the above three algorithms in Table 2. Figure 6 presents the reconstruction results of several samples from the three cases. Each row of Figure 6 presents the reconstruction results of the above three algorithms and the ground truth for one sample. It can be seen in Table 2 and Figure 6 that our proposed Learned Combined Algorithm outperforms the other two algorithms, which shows the advantages offered by the deep learning method and the training dataset $S_4$.

Table 2: The average values of relative errors $R_e$ for the outputs of Combined Algorithm, Simplified Learned Combined Algorithm and Learned Combined Algorithm on the MNIST dataset.

|          | Case 2.1 | Case 2.2 | Case 2.3 |
|----------|----------|----------|----------|
| Combined Algorithm | 19.3% | 35.1% | 43.5% |
| Simplified Learned Combined Algorithm | 11.4% | 20.4% | 30.0% |
| Learned Combined Algorithm | 9.8% | 16.9% | 24.4% |

6 Conclusion

In this paper, we considered the inverse problem of scattering of time-harmonic acoustic waves from inhomogeneous media in two dimensions. An iterative reconstruction algorithm based on deep neutral network, which is called Learned Combined Algorithm, is proposed to recover the
Figure 6: Reconstructions by Combined Algorithm, Simplified Learned Combined Algorithm and Learned Combined Algorithm with exact contrast matrices generated from MNIST digits. Each row presents the reconstruction results and the ground truth for one sample.
refractive indices of inhomogeneous media from the far-field data. The proposed algorithm is given by repeated applications of the Landweber iteration, the IRGNM iteration and the learned projector, where the learned projector is a well-trained deep neural network. With the aid of the training strategy in Section 4.1.2 the learned projector is expected to be encoded with some a priori information of unknown contrasts into the proposed algorithm, and is expected to be helpful for improving the estimated contrasts and providing good initial guesses for the IRGNM.

From various numerical experiments, it is shown that Learned Combined Algorithm performs well for the considered problem. First, we observe that the Learned Combined Algorithm has good performance on EMNIST dataset and the dataset of handwritten Chinese characters although we only train $\mathcal{P}_\Theta$ on MNIST dataset. Secondly, we observe that the Learned Combined Algorithm could generate satisfactory results for a large variety of contrasts with different values and even for the case when the values of the contrasts for testing are a little higher than those for training. For this observation, we think the reason is that the learned projector may acquire some a priori information of the shapes of the unknown contrasts since the training datasets for learned projector are obtained by using the normalization operator $\mathcal{N}$ (see Section 4.1.2).

However, it is observed in the numerical experiments that the reconstruction results of Learned Combined Algorithm become worse when the values of the exact contrasts become larger. This may be due to two reasons. The first reason is that if the exact contrasts become larger then the numerical solutions of relevant direct scattering problems in each iterative step will be less accurate, which will deteriorate the final reconstruction results. The second reason is that the deep learning method in our algorithm is lack of interpretability (that is, the underlying a priori information acquired by the learned projector is not well-understood) and thus it is difficult for us to improve the proposed algorithm. Therefore, certain improvements still need to be further investigated. Moreover, it is interesting to extend our method to the case of seismic imaging, which will be considered as a future work.

Acknowledgments

This work was partly supported by the National Key R & D Program of China (2018YFA0702502), Beijing Natural Science Foundation (Z210001), the NNSF of China (11871466) and Youth Innovation Promotion Association of CAS.

References

[1] S. Arridge, P. Maass, O. Öktem and C.-B. Schönlieb, Solving inverse problems using data-driven models, *Acta Numer.* **28** (2019), 1–174.

[2] A. B. Bakushinskii, The problem of the convergence of the iteratively regularized Gauss-Newton method, *Comput. Math. Math. Phys.* **32** (1992), 1353–1359.

[3] G. Bao and P. Li, Inverse medium scattering problems for electromagnetic waves, *SIAM J. Appl. Math.* **65** (2005), 2049–2066.

[4] G. Bao, P. Li, J. Lin and F. Triki, Inverse scattering problems with multi-frequencies, *Inverse Problems* **31** (2015), 093001 (21pp).
[5] F. Cakoni and D. Colton, *A Qualitative Approach to Inverse Scattering Theory*, Springer, New York, 2014.

[6] F. Cakoni, M. Di Cristo and J. Sun, A multistep reciprocity gap functional method for the inverse problem in a multilayered medium, *Complex Var. Elliptic Equ.* 57 (2012), 261–276.

[7] H. Chen, Y. Zhang, W. Zhang, P. Liao, K. Li, J. Zhou and G. Wang, Low-dose CT via convolutional neural network, *Biomedical Optics Express* 8 (2017), 679–694.

[8] X. Chen, Subspace-based optimization method for solving inverse-scattering problems, *IEEE Trans. Geosci. Remote Sensing* 48 (2009), 42–49.

[9] X. Chen, Z. Wei, M. Li and P. Rocca, A review of deep learning approaches for inverse scattering problems (invited review), *Prog. Electrom. Research* 167 (2020), 67–81.

[10] G. Cohen, S. Afshar, J. Tapson and A. Van Schaik, EMNIST: Extending MNIST to handwritten letters, *IEEE Int. Joint Conf. on Neural Networks (IJCNN)*, IEEE, 2017, pp. 2921–2926.

[11] D. Colton and R. Kress, Inverse scattering, in: *Handbook of Mathematical Methods in Imaging* (ed. by O. Scherzer), Springer, New York, 2015, pp. 649–700.

[12] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory* (4th edition), Springer, 2019.

[13] D. Colton, M. Piana and R. Potthast, A simple method using Morozov’s discrepancy principle for solving inverse scattering problems, *Inverse Problems* 13 (1997), 1477–1493.

[14] L. Deng, The MNIST database of handwritten digit images for machine learning research, *IEEE Signal Proc. Magazine* 29 (2012), 141–142.

[15] H. W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems*, Kluwer Academic Publisher, Dordrecht, 2000.

[16] X. Glorot and Y. Bengio, Understanding the difficulty of training deep feedforward neural networks, in *Proceedings of the Thirteenth International Conference on Artificial Intelligence and Statistics* (eds. Y. W. Teh and M. Titterington), vol. 9 of Proceedings of Machine Learning Research, PMLR, Chia Laguna Resort, Sardinia, Italy, 2010, 249–256.

[17] H. Gupta, K. H. Jin, H. Q. Nguyen, M. T. McCann and M. Unser, CNN-based projected gradient descent for consistent CT image reconstruction, *IEEE Trans. Med. Imag.* 37 (2018), 1440–1453.

[18] M. Hanke, A. Neubauer and O. Scherzer, A convergence analysis of the Landweber iteration for nonlinear ill-posed problems, *Numer. Math.* 72 (1995), 21–37.

[19] K. He, X. Zhang, S. Ren and J. Sun, Deep residual learning for image recognition, in: *Proc. of the IEEE conf. on Computer Vision and Pattern Recognition* 2016, pp. 770–778.

[20] G. Hinton, L. Deng, D. Yu, et al., Deep neural networks for acoustic modeling in speech recognition: The shared views of four research groups, *IEEE Signal Proc. Magazine* 29 (2012), 82–97.
[21] T. Hohage, Logarithmic convergence rates of the iteratively regularized Gauss-Newton method for an inverse potential and an inverse scattering problem, *Inverse Problems* **13** (1997), 1279–1299.

[22] T. Hohage, *Iterative Methods in Inverse Obstacle Scattering: Regularization Theory of Linear and Nonlinear Exponentially Ill-Posed Problems*, PhD thesis, University of Linz, Austria, 1999.

[23] T. Hohage, On the numerical solution of a three-dimensional inverse medium scattering problem, *Inverse Problems* **17** (2001), 1743–1763.

[24] K. H. Jin, M. T. McCann, E. Froustey and M. Unser, Deep convolutional neural network for inverse problems in imaging, *IEEE Trans. Image Proc.* **26** (2017), 4509–4522.

[25] B. Kaltenbacher, A. Neubauer and O. Scherzer, *Iterative regularization Methods for Nonlinear Ill-Posed Problems*, Walter de Gruyter, Berlin, 2008.

[26] Y. Khoo and L. Ying, SwitchNet: a neural network model for forward and inverse scattering problems, *SIAM J. Sci. Comput.* **41** (2019), A3182–A3201.

[27] D. P. Kingma and J. Ba, Adam: A method for stochastic optimization, *arXiv:1412.6980*, 2014.

[28] A. Kirsch, The MUSIC algorithm and the factorization method in inverse scattering theory for inhomogeneous media, *Inverse Problems* **18** (2002), 1025–1040.

[29] A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems* (3rd edition), Springer, 2021.

[30] R. Kress, *Linear Integral Equations* (3rd edition), Springer, 2014.

[31] S. Langer, Investigation of preconditioning techniques for the iteratively regularized Gauss-Newton method for exponentially ill-posed problems, *SIAM J. Sci. Comput.* **32** (2010), 2543–2559.

[32] Y. LeCun, Y. Bengio and G. Hinton, Deep learning, *Nature* **521** (2015), 436–444.

[33] L. Li, L.G. Wang, F.L. Teixeira, C. Liu, A. Nehorai and T.J. Cui, DeepNIS: Deep neural network for nonlinear electromagnetic inverse scattering, *IEEE Trans. Antennas Propag.* **67** (2018), 1819–1825.

[34] A.L. Maas, A.Y. Hannun and A.Y. Ng, Rectifier nonlinearities improve neural network acoustic models, *Proc. ICML*, vol. 30, 2013.

[35] M.T. McCann, K.H. Jin and M. Unser, Convolutional neural networks for inverse problems in imaging: A review, *IEEE Signal Proc. Magazine* **34** (2017), 85–95.

[36] R. Potthast and I. Stratis, The singular sources method for an inverse transmission problem, *Computing* **75** (2005), 237–255.

[37] F. Qu and H. Zhang, Locating a complex inhomogeneous medium with an approximate factorization method, *Inverse Problems* **35** (2019), 045001 (18pp).
[38] O. Ronneberger, P. Fischer and T. Brox, U-net: Convolutional networks for biomedical image segmentation, in: *Int. Conf. on Medical Image Computing and Computer-Assisted Intervention*, Springer, 2015, pp. 234–241.

[39] Y. Sanghvi, Y. Kalepu and U.K. Khankhoje, Embedding deep learning in inverse scattering problems, *IEEE Trans. Comput. Imag.* **6** (2019), 46–56.

[40] A.W. Senior, R. Evans, J. Jumper, et al., Improved protein structure prediction using potentials from deep learning, *Nature* **577** (2020), 706–710.

[41] G. Vainikko, Fast solvers of the Lippmann-Schwinger equation, in: *Direct and Inverse Problems of Mathematical Physics* (eds. R.P. Gilbert, J. Kajiwara, Y.S. Xu), Kluwer Academic Publisher, Dordrecht, 2000, pp. 423–440.

[42] P.M. van den Berg and R.E. Kleinman, A contrast source inversion method, *Inverse Problems* **13** (1997), 1607–1620.

[43] Z. Wei, D. Liu and X. Chen, Dominant-current deep learning scheme for electrical impedance tomography, *IEEE Trans. Biomed. Eng.* **66** (2019), 2546–2555.

[44] F. Yang, T.-A. Pham, H. Gupta, M. Unser and J. Ma, Deep-learning projector for optical diffraction tomography, *Opt. Express* **28** (2020), 3905–3921.

[45] Y. Yang, J. Sun, H. Li and Z. Xu, ADMM-CSNet: A deep learning approach for image compressive sensing, *IEEE Trans. Pattern Anal. Machine Intell.* **42** (2018), 521–538.

[46] H. Yao, E. Ming Wei and L. Jiang, Two-step enhanced deep learning approach for electromagnetic inverse scattering problems, *IEEE Antennas Wireless Propag. Letters* **18** (2019), 2254–2258.

[47] Y. Zhou, Y. Zhong, Z. Wei, T. Yin and X. Chen, An improved deep learning scheme for solving 2-D and 3-D inverse scattering problems, *IEEE Trans Antennas Propag.* **69** (2020), 2853–2863.

[48] S. Zubair, F. Yan and W. Wang, Dictionary learning based sparse coefficients for audio classification with max and average pooling, *Digital Signal Processing* **23** (2013), 960–970.