Self-dual compact gauged baby skyrmions in nonlinear media

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We investigate a class of topological structures with compact support (called compactons) into a nonlinear dielectric medium driven by a real scalar field in an extended version of the restricted gauged baby Skyrme model. To obtain stable compactons, we use the Bogomol’nyi procedure that provides a lower bound for the energy and the respective self-dual equations whose solutions saturate such a bound. We analyze two dielectric media and examine how the compacton profiles change for different values of the parameters controlling the effects of the media nonlinearity.

I. INTRODUCTION

The Skyrme model [1] is a (3+1)-dimensional nonlinear field theory conceived initially to study some nonperturbative aspects of quantum Chromodynamics (QCD); for instance, it provides physical properties of hadrons and nuclei [2] compatible with the low-energy regime of QCD. The hadrons described by the Skyrme model emerge as topological solitons, so-called skyrmions. The (3+1)-dimensional or planar version [3], known as the baby Skyrme model, is a laboratory to study many aspects of the original Skyrme model. Besides, planar versions have attracted the community’s attention because they are used or come up in the description of various physical systems, e.g., in condensed matter [4], in the quantum Hall effect [5], Bose-Einstein condensates [6], superconductors [7], and magnetic materials [8], including recent investigations with the Dzyaloshinsky–Moriya interaction [9, 10].

Like the original version, the baby Skyrme model consists of a $O(3)$ nonlinear sigma term (a quadratic kinetic term), the Skyrme term (a quartic kinetic term), and a potential $V(\vec{\phi}) = V(\vec{\phi} \cdot \hat{n})$ (a non-derivative one). The presence of such a potential is obligatory because it stabilizes the solitons [11]. The Skyrme field $\vec{\phi}$ is a three-component vector of scalar fields $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ which complies with $\vec{\phi} \cdot \vec{\phi} = 1$. This field describes a spherical surface denoted by $S^2$. The vector $\hat{n} \in S^2$ gives preference to a direction in the internal space of the sphere. The Skyrme model is invariant under the $SO(3)$ symmetry, while the auto-interaction potential partially breaks this symmetry, keeping the $SO(2)$ symmetry unchanged (which is isomorphic to the $U(1)$ one). The self-interaction potential must satisfy the following condition $V(\phi_n) \to 0$ when $\phi_n \to 1$, is known as the single vacuum configuration. Although the baby Skyrme model describes stable solitons, it does not support a Bogomol’nyi–Prasad-Sommerfield (BPS) structure [12], i.e., an energy lower-bound (the Bogomol’nyi bound) and a set of self-dual equations providing solitons that saturate this energy bound. On the other hand, in the absence of the sigma term arises the so-called restricted baby Skyrme model [13], which supports BPS or self-dual configurations [14].

A natural physical extension for the Skyrme model is to couple it with a $U(1)$ gauge field [15], which allows us to investigate the electric and magnetic properties. In this context, Ref. [16] provides the first study showing that the restricted gauge baby Skyrme model supports a BPS structure. Afterward, BPS or self-dual solutions carrying both magnetic flux and electric charge were obtained [17–19]. Such studies in the restricted gauge baby Skyrme model have shown different BPS soliton profiles, i.e., it supports compact and noncompact topological structures. In general terms, a compacton is a soliton solution with compact support which reaches the vacuum at a finite distance. The noncompactons attain the vacuum at the infinite by following a Gaussian decay law or a power-law decay. Further, since such compactons are indeed localized solutions in the sense of being solitons with finite extension, they could allow extracting relevant information about the effects of topological objects on finite laboratory samples. Thus, although the proposal to be presented by us supports compactons and noncompactons, we are interested in studying only defects with compact support, i.e., the compact skyrmions.

In the context of compact skyrmions, such topological structures have been investigated firstly in the non-gauged case [13], and later studies in the gauged version also shown that it supports purely magnetic compactons [16, 20]. Afterward, in Ref. [18] were found those carrying both the magnetic flux and electric charge. In the present manuscript, we go beyond these studies and show the existence of BPS compact baby skyrmions immersed in nonlinear magnetic media. For such a proposal, we have promoted the extension of the symmetry $SO(3)$ to $SO(3) \times Z_2$, being $Z_2$ an extra symmetry ruling a neutral scalar field coupled to the system by mean of a generalized dielectric function. Research in this direction, where a specific model becomes enlarged to accommodate an additional symmetry, have been performed in other scenarios, including new vortex solutions in the Maxwell-Higgs models [21, 22], magnetic monopoles [23], new solitons in gauged $CP(2)$ model [24], and gauged $O(3)$ sigma model [25].

In what follows, the additional symmetry allows us to modify the compacton profiles keeping them into geomet-
rical constrained regions, i.e., entrapping the skyrmions into a limited plane region. Such an approach can be related to a geometric constriction inducing modifications in localized structures, such as nanometrical solitons, possibly giving rise to new effects and providing information about geometrical constraints at the nanometric scale. Among the current studies of interest concerning skyrmions at the nanometric scale, it is interesting to highlight the following cases: the reversible conversion between a skyrmion and domain-wall by using a geometric junction [26]; the critical role concerning geometric constriction in the description of the current-driven transformation from stripe domains to magnetic skyrmion bubbles [27], which could to give a progress glimmer in the skyrmion-based spintronics; the pattern formation of geometrically confined skyrmions [28], which is directly related to the challenges on the path of developing skyrmion-based room-temperature applications.

We presented the results in the following way: In Sec. II, we introduce our model and present the corresponding Euler-Lagrange equations. In Sec. III, by considering rotationally symmetric configurations, we implement the BPS technique that provides the system energy lower bound and the corresponding self-dual or BPS equations. In Sec. IV, we study two scenarios describing different dielectric media and solve the BPS system numerically; then, we highlight the main new features presented by the new compacton profiles. Lastly, we make our final comments and conclusions in Sec. V.

II. THE RESTRICTED GAUGED BABY SKYRME MODEL IN A DIELECTRIC MEDIUM

We investigate an extension of the restricted baby Skyrme model immersed in a dielectric medium driven by a real scalar field defined by the Lagrangian

$$ L = E_0 \int d^2 \mathbf{x} \mathcal{L}, $$

where $E_0$ is a common factor that sets the energy scale (which hereafter we will take as $E_0 = 1$) and $\mathcal{L}$ the Lagrangian density given by

$$ \mathcal{L} = -\frac{1}{4g^2} \Sigma(\chi) F_{\mu \nu} F^{\mu \nu} - \frac{\lambda^2}{4} (D_\mu \bar{\phi} \times D_\nu \bar{\phi})^2 + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - V(\phi_n, \chi), $$

with $\Sigma(\chi)$ being a non-negative real function (a dielectric function) of the neutral real scalar field $\chi$. $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the strength-tensor of the Abelian gauge field $A_\mu$, $g$ is the electromagnetic coupling constant, and $\lambda$ the Skyrmie coupling constant. In what follow, the Skyrmie and gauge fields are coupled minimally through the covariant derivative

$$ D_\mu \bar{\phi} = \partial_\mu \bar{\phi} + A_\mu \hat{n} \times \bar{\phi}, $$

where $\hat{n}$ is a unitary vector in the internal space. Further, we have the potential $V(\phi_n, \chi)$ which stands for some appropriated interaction between the Skyrmie field $\phi$ and the neutral scalar field $\chi$. It is important to point out that in the absence of such an interaction, we must recover the corresponding standard configurations investigated in [16].

Here, for the $(2 + 1)$-dimensional Minkowski space we adopt the metric with signature $(+, −, −)$. Moreover, we take the natural units $\hbar = c = 1$. Thus, the scalar fields $\phi$ and $\chi$ are dimensionless, the gauge field and the electromagnetic coupling constant $g$ possess mass dimension $1$, and the Skyrmie coupling constant $\lambda$ has mass dimension $-1$.

The Euler-Lagrange equations obtained from the Lagrangian density (2) are

$$ \partial_\mu \partial^\mu \phi + \frac{1}{4} \Sigma(\chi) F_{\mu \nu} F^{\mu \nu} + V_\chi = 0, $$

$$ D_\mu \bar{J}^\mu + \frac{\partial V}{\partial \phi_n} \hat{n} \times \bar{\phi} = 0, $$

$$ \partial_\mu (\Sigma F^{\mu \nu}) = g^2 j^\nu, $$

where $\Sigma = \partial \Sigma_\chi / \partial \chi$, $V_\chi = \partial V / \partial \chi$ and $j^\mu = \hat{n} \cdot \bar{J}^\mu$ is the conserved current density with

$$ \bar{J}^\mu = \lambda^2 [\bar{\phi} \cdot (D^\mu \bar{\phi} \times D^\nu \bar{\phi})] D_\nu \bar{\phi}. $$

In the present study, we are interested in stationary soliton solutions of the model described by the Lagrangian density (2). In this sense, from Eq. (6), we extract a piece of important information via the stationary Gauss law

$$ \partial_i (\Sigma \partial_i A_0) = g^2 \lambda^2 A_0 (\hat{n} \cdot \partial_i \bar{\phi})^2, $$

which is identically satisfied by gauge condition $A_0 = 0$, allowing us to choose such a condition henceforth; consequently, the resulting configurations only carry magnetic flux. Furthermore, from Eq. (6), we also obtain the stationary Ampère law, which is given by,

$$ \partial_i (\Sigma B_i) + g^2 \lambda^2 (\hat{n} \cdot \partial_i \bar{\phi}) Q = 0, $$

where $B_{12} = \epsilon_{ij} \partial_i A_j$ is the magnetic field and $Q \equiv \bar{\phi} \cdot (D_1 \bar{\phi} \times D_2 \bar{\phi})$ which can still be expressed as

$$ Q = \bar{\phi} \cdot (\partial_1 \bar{\phi} \times \partial_2 \bar{\phi}) + \epsilon_{ij} A_i (\hat{n} \cdot \partial_j \bar{\phi}). $$

It is interesting to point out that the term $\bar{\phi} \cdot (\partial_1 \bar{\phi} \times \partial_2 \bar{\phi})$ present in this last equation is related to the topological charge or topological degree (also called the winding number) of the Skyrmie field,

$$ \text{deg} [\bar{\phi}] = -\frac{1}{4\pi} \int d^2 \mathbf{x} \bar{\phi} \cdot (\partial_1 \bar{\phi} \times \partial_2 \bar{\phi}) = k, $$

where $k \in \mathbb{Z} \setminus 0$. 

We now consider rotationally symmetric solitons where, without loss of generality, we set \( \bar{n} = (0, 0, 1) \) such that \( \phi_\alpha = \phi_3 \) and we assume the standard ansatz for the Skyrme field

\[
\phi(r, \theta) = \begin{pmatrix}
\sin f(r) \cos(N\theta) \\
\sin f(r) \sin(N\theta) \\
\cos f(r)
\end{pmatrix},
\]

(12)

where \( r \) and \( \theta \) are polar coordinates, \( N = \deg[\tilde{\phi}] \) is the winding number presented in Eq. (11) and \( f(r) \) is a well behaved function obeying the boundary conditions

\[
f(0) = \pi, \quad \lim_{r \to \infty} f(r) = 0.
\]

(13)

By convenience, we introduce the following field redefinition [16]:

\[
\cos f(r) = 1 - 2h(r),
\]

with the field \( h(r) \) obeying

\[
h(0) = 1, \quad \lim_{r \to \infty} h(r) = 0.
\]

(15)

Further, for the gauge field components and neutral scalar field we consider

\[
A_i = -\epsilon_{ij}x_j \frac{Na(r)}{r^2}, \quad \chi = \chi(r),
\]

(16)

respectively. At the origin, the functions \( a(r) \) and \( \chi(r) \) are regular functions satisfying the conditions

\[
a(0) = 0, \quad \chi(0) = \chi_0,
\]

(17)

whereas for the asymptotic limit we require

\[
\lim_{r \to \infty} a(r) = a_\infty, \quad \lim_{r \to \infty} \chi(r) = \chi_\infty,
\]

(18)

where \( \chi_0, \chi_\infty \) and \( a_\infty \) are finite constants. Now, under the considerations presented until here, the equation of motion for the Skyrme field \( h(r) \) assume the form

\[
\frac{4\lambda^2 N^2}{r} \frac{d}{dr} \left[ \frac{(1 + a)^2}{r} \frac{dh}{dr} \right] = V_h,
\]

(19)

where \( V \equiv V(h, \chi) \), with \( V_h = \partial V/\partial h \), while for the neutral scalar field \( \chi(r) \) we get

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{d\chi}{dr} \right) - \frac{B^2}{2g^2} \Sigma \chi = V_\chi.
\]

(20)

Already for the stationary Ampère law we obtain

\[
\frac{d}{dr}(\Sigma B) = \frac{4Ng^2\lambda^2}{r^2} (1 + a) \left( \frac{dh}{dr} \right)^2,
\]

(21)

being \( B(r) \) the magnetic field given by

\[
B = \frac{Na}{r} \frac{da}{dr}.
\]

(22)

By using this latest relation, we can express the magnetic flux as

\[
\Phi = 2\pi \int_0^R Br\,dr = 2\pi Na(R) = 2\pi Na_R,
\]

(23)

The \( a_R \) is a real constant, whereas \( R > 0 \) defines the maximum size of the topological defect characterizing the type of soliton, i.e., \( R \) is finite for compactons and infinite for noncompact solitons (see, e.g., Refs. [16, 18, 29]). As well as their counterpart model [16], the magnetic flux is nonquantized since \( a_R \) and \( a_\infty \) (when \( R \to \infty \)) belong to the interval \((-1, 0]\). A known result is that for sufficiently large values of the electromagnetic coupling \( g \), the quantities \( a_R \) or \( a_\infty \) tend to \(-1\) and, in such a limit, the magnetic flux becomes “quantized” in units of \( 2\pi \).

In the next section, we implement the Bogomol’nyi procedure [12] to investigate the conditions under which the model (2) engenders self-dual or BPS structures that minimize the energy of the system.

### III. BOGOMOLO’NYI FRAMEWORK

The starting point is the stationary energy density of the model defined by the Lagrangian density (2), which reads as

\[
\varepsilon = \frac{1}{2g^2} \Sigma B^2 + \frac{\lambda^2}{2} Q^2 + \frac{1}{2} \left( \frac{d\chi}{dr} \right)^2 + V(h, \chi),
\]

(24)

where \( Q^2 = \frac{1}{2} (D_i \tilde{\phi} \times D_j \tilde{\phi})^2 \) written in polar coordinates gives

\[
Q^2 = \frac{4N^2}{r^2} (1 + a)^2 \left( \frac{dh}{dr} \right)^2.
\]

(25)

In what follows, to implement the Bogomol’nyi procedure, we introduce three auxiliary functions, \( W \equiv W(h) \), \( Z \equiv Z(h) \) and \( W \equiv W(\chi) \), that allows us to express total energy as

\[
\frac{E}{2\pi} = \int_0^\infty \varepsilon(r) r\,dr
\]

\[
= \int_0^\infty \left[ \frac{1}{2g^2} \Sigma B^2 + g^2 \lambda^2 W^2 + \frac{\lambda^2}{2} (Q \pm Z)^2 + \frac{1}{2} \left( \frac{d\chi}{dr} \right)^2 \right] + V(h, \chi)
\]

\[
+ \frac{1}{2} \left( \frac{d\chi}{dr} \right)^2 + \frac{1}{r} \frac{\partial W}{\partial \chi} \right)^2 + \lambda^2 (BW + QZ) \leq \frac{1}{2} \frac{dW}{dr} + \frac{1}{2} \left( \frac{d\chi}{dr} \right)^2 \right] dr.
\]

(26)

Under the considerations for the fields presented in the previous section, we express the quantity \( BW + QZ \) in the form

\[
BW + QZ = \frac{N}{r} \left[ \frac{d(1 + a)}{dr} W + (1 + a) \left( \frac{2Z}{dr} \right)^2 \right].
\]

(27)
becoming a total derivative by setting
\[ Z = \frac{1}{2} \partial W \frac{\partial W}{\partial h}. \] (28)

Further, from the third row of Eq. (26), we require the potential to be defined as
\[ V(h, \chi) = V^{(2)} + \frac{1}{2r^2} W^2, \] (29)

where
\[ V^{(2)} = \frac{g^2 \lambda^4}{2 \Sigma} W^2 + \frac{\lambda^2}{8} W^2, \] (30)

with \( W_h = \partial W/\partial h \) and \( \mathcal{W}_\chi = \partial W/\partial \chi \).

Here it is interesting to point out that both \( W(h) \) and \( W(\chi) \) play the role of superpotential functions, the first one for the Skyrme field \( h(r) \) and the second for the scalar field \( \chi \). The choosing of both the superpotentials must consider that the resultant self-dual potential (29) ensures the finite-energy requirement, i.e., the energy density (24) must be null when the fields attain their vacuum values (in \( r = R \) or \( r \to \infty \)). These considerations allow us to establish boundary conditions to be satisfied by the BPS fields of the system. Specifically, the self-dual potential must to satisfy the boundary condition
\[ \lim_{r \to \infty} V(h, \chi) = 0, \] (31)

which implies
\[ \lim_{r \to \infty} \frac{W}{\sqrt{\Sigma}} = 0, \quad \lim_{r \to \infty} W_h = 0, \quad \lim_{r \to \infty} \mathcal{W}_\chi = 0. \] (32)

Further, these superpotentials are functions well behaved obeying the following boundary conditions in \( r = 0 \):
\[ W(h(0)) = W(1) = W_0, \] (33)
\[ W(\chi(0)) = W(\chi_0) = W_0, \] (34)

whereas for the asymptotic limit we get
\[ \lim_{r \to \infty} W(h) = W(0) = 0, \] (35)
\[ \lim_{r \to \infty} W(\chi) = W(\chi_\infty) = W_\infty, \] (36)

where \( W_0, W_\infty \) and \( W_\infty \) are constants, and they are in accordance with the boundary conditions (15), (17) and (18). We also highlight that the presence of the neutral scalar field \( \chi \) demands the insertion of an additional contribution (the second term on the right-hand side of Eq. (29)) that is an explicit function of the radial coordinate. This term guarantees the existence of kinklike solutions in the \( \chi \) field sector [30], having already been previously considered in literature in different scenarios [21–25].

We return to the development of the BPS framework by using the considerations hitherto presented. Accordingly, the total energy (26) becomes written as
\[ E = 2\pi \int_0^{\infty} r \varepsilon(r) \, dr = \tilde{E} + E_{\text{BPS}}, \] (37)

where \( \tilde{E} \) is a nonnegative quantity composed for the remaining quadratic terms,
\[ \tilde{E} = 2\pi \int_0^{\infty} \left[ \frac{1}{2g^2\Sigma}(\Sigma B + g^2 \lambda^2 W)^2 \right. \]
\[ + \frac{\lambda^2}{2} \left( Q + \frac{1}{2} W_h \right)^2 + \frac{1}{2} \left( \frac{d\chi}{dr} + \frac{1}{r} W_\chi \right)^2 \] \[ \left. dr \right], \] (38)

and \( E_{\text{BPS}} \) defines the energy’s lower bound (or Bogomol’nyi bound) that is given by
\[ E_{\text{BPS}} = 2\pi \int_0^{\infty} \left[ \frac{\lambda^2 N d}{r} (1 + a) W \right] + \frac{1}{r} dW \] \[ \left. dr \right] = \pm 2\pi (\lambda^2 N W_0 + \Delta W) \geq 0, \] (39)

where to perform the integration, we have used the boundary conditions (33), (34), (35), and (36). Furthermore, we have defined the quantity \( \Delta W = W_\infty - W_0 \), such that the upper (lower) sign in (39) describes the self-dual solitons (antisolitons) corresponding to \( N \) and \( \Delta W \) positive (negative) quantities.

On the other hand, the total energy (37) is bounded from below by \( E_{\text{BPS}} \), i.e., \( E \geq E_{\text{BPS}} \). Therefore, the Bogomol’nyi’s bound is saturated when \( \tilde{E} = 0 \), or equivalently when the field profiles obey the following set of differential equations
\[ Q = \frac{2N}{r}(1 + a) \frac{dh}{dr} = \mp \frac{W_h}{2}, \] (40)
\[ B = \frac{N \, da}{r \, dr} = \mp g^2 \lambda^2 \frac{W}{\Sigma}, \] (41)

and
\[ \frac{d\chi}{dr} = \pm \frac{W_\chi}{r}. \] (42)

The set above defines the model’s self-dual or BPS equations (2). In addition, a straightforward calculation checks that the BPS equations reproduces the Euler-Lagrange equations (19), (20), and (21).

We now observe that the third BPS equation (42) has no explicit dependence on the functions \( h(r) \) and \( a(r) \) allowing it to be solved separately by adequately choosing the superpotential \( W(\chi) \). The kinklike solution obtained from Eq.(42) allows defining the dielectric function \( \Sigma(\chi) \) appearing explicitly in the BPS equation (41). Consequently, once \( \Sigma(\chi) \) is known, we proceed to solve the two remaining BPS equation (40) and (41).
In the remainder of the manuscript, without loss of generality, we only study soliton solutions by adopting the upper sign, i.e., \( N > 0 \) and \( \Delta W > 0 \).

We next show the BPS energy density, obtained by rewriting the energy density (24) using the BPS equations, expressed as the sum of two contributions,

\[
\varepsilon_{\text{BPS}} = \varepsilon_\Sigma + \varepsilon_\chi, \tag{43}
\]

where we have defined

\[
\varepsilon_\Sigma = \frac{g^2 \lambda^4}{\Sigma} W^2 + \frac{\lambda^2}{4} W_h^2 \quad \text{and} \quad \varepsilon_\chi = \frac{W^2}{r^2}, \tag{44}
\]

respectively. The density \( \varepsilon_\Sigma \) represents the energy density associated purely with the new skyrmion configurations, while \( \varepsilon_\chi \) is the contribution associated with the kinklike solution engendered by scalar field \( \chi \).

At this level, to define the model entirely, we observe that it is necessary to choose the functions \( W(h) \) and \( W(\chi) \). Thus, for our analysis, we consider functions \( W(h) \) already studied in other BPS Skyrme models (see, e.g., Refs. [16, 18, 19, 29]). For the function \( W(\chi) \) associated with the neutral scalar field \( \chi \), we will consider the ones engendering kinklike solutions. Furthermore, such as in other Skyrme’s models [16, 18, 29], we have verified that the BPS models here studied support compact and noncompact skyrmions, too. For a given superpotential \( W(h) \), the type of skyrmion engendered is determined by analyzing how the fields approach their respective vacuum values. Then, for our purposes, we consider a superpotential \( W(h) \) behaving near the vacuum as

\[
W(h) \approx W^{(\sigma)} h^\sigma, \tag{45}
\]

where \( W^{(\sigma)}_R > 0 \) is a real constant and \( \sigma > 1 \). Thus, the analysis is performed by taking now the boundary conditions

\[
h(r) = 0, \quad a(r) = a_R, \quad W(h(r)) = 0, \tag{46}
\]

being \( R \) and \( a_R \), the quantities already introduced in Eq. (23). The \( \sigma \) values characterize the soliton solutions. Indeed, for \( 1 < \sigma < 2 \), we have the soliton called compacton, whose vacuum value is reached in a finite radius \( R \) (the compacton’s radius) and remains in the vacuum for all \( r > R \), see, e.g., Refs. [18, 29]. Otherwise, for \( \sigma \geq 2 \), the radius \( R \rightarrow \infty \), so we have extended or noncompact configurations that result in localized or delocalized solitons, see, e.g., Refs. [18, 19, 29].

Nevertheless, in the remainder of the paper, we are interested in the study of the topological defects type compacton.

IV. BPS SKYRMIONS IN NONLINEAR MEDIA: COMPACTONS

Let us now investigate BPS compactons engendered by superpotentials that behave as (45) with \( 1 < \sigma < 2 \), as already mentioned before. This way, we fix \( \sigma = 3/2 \) and choose the superpotential as being

\[
W(h) = W_0 h^{3/2}, \tag{47}
\]

where \( W_0 \) is a constant. An important detail is that this superpotential provides a potential \( V^{(2)} \propto h \) when \( r \rightarrow R \), this being analogous to the so-called “old baby Skyrme potential” [16].

The next step is to consider some specific functional form to both the superpotential \( W(\chi) \) and the dielectric function \( \Sigma(\chi) \); the last one will characterize the nonlinear medium of interest. Below, let us consider two different superpotentials \( W(\chi) \) (the ones engendering the \( \chi^4 \) and \( \chi^6 \) models, respectively) supplying new nonlinear effects to the gauged BPS baby Skyrme model [16]. This way, we shall analyze how the new nonlinearities modify the shape of the compact BPS skyrmions.

A. \( \chi^4 \) nonlinear medium

In our first scenario, we consider a superpotential \( W(\chi) \) engendering a \( \chi^4 \) model, i.e.,

\[
W(\chi) = \alpha \chi - \frac{\alpha}{3} \chi^3, \tag{48}
\]

where \( \alpha \) is a nonnegative real parameter. This type of superpotential is approached in the literature in different contexts, for instance, in the study of vortices with internal structures [21, 22, 24, 25], magnetic monopoles [23], skyrmion-like configurations [31, 32], and the behavior of massless Dirac fermions in a skyrmion-like background [33]. Then, by using (48) in (42) and solving the BPS equation, we obtain the exact kinklike solution

\[
\chi(r) = \frac{r^{2\alpha} - r_0^{2\alpha}}{r^{2\alpha} + r_0^{2\alpha}}, \tag{49}
\]

with \( r_0 \) being an arbitrary positive constant such that \( \chi(r_0) = 0 \). This solution satisfies the boundary conditions \( \chi(0) = \chi_0 = -1 \) and \( \chi(\infty) = \chi_\infty = 1 \). In this case, the BPS bound (39) for the energy becomes

\[
E_{\text{BPS}} = 2\pi \lambda^2 NW_0 + \frac{8}{3} \alpha \pi, \tag{50}
\]

where the last term is the contribution from the neutral scalar field \( \chi \).

We now need to select the dielectric function defining the medium surrounding the compactons; it is done by setting

\[
\Sigma(\chi) = \frac{1}{\sin^2(m\pi\chi)}, \tag{51}
\]

where \( m \in \mathbb{N} \). We have chosen such a function motivated by previous applications, for instance, in the study of kinklike solutions [34] and vortexlike structures [22, 25].
Then, the magnetic field reads of both the magnetic field and the energy density

In the sequel, we present the behavior, close to the origin, $C$ where we have defined the constant $m$ with

FIG. 1. Depiction by assuming the dielectric function (51) with $r_0 = 0.5$, $m = 1$ and distinct values for $\alpha$. Left: compacton radius $R$ vs. $\alpha$ (solid red line) and the compacton radius of the standard case (black dot line). Right: the present Skyrme field (color lines) is depicted for $\alpha = 1$ (solid line), $\alpha = 2$ (dashed line) and $\alpha = 5$ (dot line), and the solid black line represents the profile without the dielectric medium.

This way, the energy density $\varepsilon_\Sigma$ results be

$$\varepsilon_\Sigma = g^2 \lambda^4 W_0^2 \sin^2 (m \pi \chi) + \frac{9}{16} \lambda^2 W_0^2 h.$$  \hspace{1cm} (52)

Already the BPS equations (41) and (40) to be investigated assume the form

$$\frac{N}{r} \left(1 + \alpha\right) \frac{dh}{dr} + \frac{3}{8} W_0 h^{1/2} = 0,$$  \hspace{1cm} (53)

where we must solve with suitable field boundary conditions to provide compact skyrmions in the current scenario.

Before we solve the BPS equations numerically, let us show the behavior of the field profiles close to the boundary values. Thus, near to the origin, by taking the more relevant terms, the Skyrme and gauge field profiles behave as,

$$h(r) = 1 - \frac{3}{2} \frac{W_0}{N} r^2 + \frac{3^2}{2^4} \frac{W_0^2}{N^2} r^4 - \frac{3}{2^4} \frac{C_0 W_0}{(\alpha + 1)(2\alpha + 1) N^2} r^{4\alpha + 4} + \cdots,$$  \hspace{1cm} (55)

$$a(r) = -\frac{2C_0}{(\alpha + 1) N^2} r^{4\alpha + 2} + \frac{3^2}{2^5} \frac{C_0 W_0}{r_0^{4\alpha}} r^{4\alpha + 4} - \frac{4C_0}{(3\alpha + 1) N} r^{6\alpha + 2} + \cdots,$$  \hspace{1cm} (56)

where we have defined the constant $C_0 = \pi^2 m^2 g^2 \lambda^2 W_0$. In the sequel, we present the behavior, close to the origin, of both the magnetic field and the energy density $\varepsilon_\Sigma(r)$. Then, the magnetic field reads

$$B(r) = -4C_0 \frac{r^{4\alpha}}{r_0^{4\alpha} +} - \frac{3^2}{2^8} \frac{C_0 W_0}{r_0^{4\alpha}} r^{4\alpha + 2} - \frac{8C_0}{r_0^{4\alpha}} r^{6\alpha} + \cdots.$$  \hspace{1cm} (57)

FIG. 2. The gauge field profiles $a(r)$ (top panels), absolute value of the magnetic field $|B(r)|$ (middle panels) and energy density $\varepsilon_\Sigma(r)$ (bottom panels). We depict for $r_0 = 0.5$, $m = 1$ with $\alpha = 1$ (left) and $\alpha = 2$ (right).

FIG. 3. The gauge field profiles $a(r)$ (top panels), absolute value of the magnetic field $|B(r)|$ (middle panels) and energy density $\varepsilon_\Sigma(r)$ (bottom panels). We depict for $r_0 = 0.7$, $m = 2$ with $\alpha = 1$ (left) and $\alpha = 2$ (right).
showing that it is null in \( r = 0 \), whereas the energy density \( \varepsilon_\Sigma(r) \) reading as

\[
\varepsilon_\Sigma = \frac{3^2}{24} \frac{\lambda^2 W_0^2}{N} r^2 + \frac{3^3}{28} \frac{\lambda^2 W_0^3}{N} r^4 + \frac{4}{214} \frac{\lambda^2 W_0^4}{N^2} r^6 + \cdots,
\]

(58)

presents a nonnull value at the origin.

On the other hand, near the vacuum values, i.e., when \( r \to R \), the field profiles possess the following behavior:

\[
h(r) \approx \frac{3^2}{24} \frac{R^2 W_0^2}{(1 + a_R)^2 N^2} (r - R)^2,
\]

(59)

\[
a(r) \approx a_R + \frac{1}{2} \frac{R^2}{N^2} C_R (r - R)^4,
\]

(60)

with the constant \( C_R \) defined as

\[
C_R = \frac{3^3}{2^{12}} \frac{g^2 \lambda^2 W_0^4 R^2}{(1 + a_R)^3 N^2} \sin^2 \left( \frac{m \pi R^{2\alpha} - \frac{2\alpha}{R \bar{\alpha}}}{R^{2\alpha} - \frac{2\alpha}{R \bar{\alpha}}} \right).
\]

(61)

Similarly, the first relevant terms of the magnetic field and energy density \( \varepsilon_\Sigma \), when \( r \to R \), are given by

\[
B(r) = -\frac{R}{N} C_R (r - R)^3 + \cdots,
\]

(62)

\[
\varepsilon_\Sigma = \frac{3^4}{2^{12}} \frac{\lambda^2 W_0^4 R^2}{(1 + a_R)^2 N^2} (r - R)^2 + \cdots
\]

\[
+ \frac{3^3}{2^{12}} \frac{\lambda^2 W_0^3 R^4}{(1 + a_R)^3 N^2} C_R (r - R)^6 + \cdots.
\]

(63)

We next solve the BPS system, Eqs. (54) and (53), numerically to investigate how the dielectric function (51) modifies the soliton profiles. For such a purpose, we fix \( N = 1, \quad W_0 = 1, \quad \lambda = 1, \quad g = 1 \), by considering distinct values of both \( \alpha \) and \( m \), besides, \( r_0 \) will have a specified value in each case. It is important to point out that although the fixed parameters also could be explored (e.g., by running different values), we have used only \( \alpha \) and \( m \) to control the profile shape of the fields. Both parameters are enough to exhibit the main features of the compact skyrmions immersed in the medium defined by (51).

We have begun our analysis by investigating how the compacton radius \( R \) changes as a function of the \( \alpha \) parameter for a fixed \( r_0 \) and \( m \), shown in the left panel of Fig. 1. Likewise, we have verified that exists an interval \( 0 < \alpha \leq \bar{\alpha} \) (here \( \bar{\alpha} \approx 0.6 \)) that provides global extreme values for the compacton radius. Namely, for a sufficiently small \( \alpha \) (close to zero) we have the maximum compacton radius \( R_{\text{max}} \approx 3.26 \) which decreases until it reaches the minimum value \( R_{\text{min}} \approx 2.25 \) when \( \alpha = \bar{\alpha} \). From this point on, i.e., for \( \alpha > \bar{\alpha} \), the values of \( R \) begins to increase in a way that \( R \to R_{\text{max}} \). We also observe that the dielectric modifies the compacton radius that achieves higher values than the value acquired without the medium (we will call \( R_0 \approx 1.99 \), see the horizontal black dotted line). Alike, in the right panel of Fig. 1, we illustrate this effect on the Skyrmie field profiles by depicting \( h(r) \) for some values of \( \alpha > \bar{\alpha} \). So, we observe how the parameter \( \alpha \) modifies the profiles by showing the increase of the compacton radius as \( \alpha \) grows.

Figure 2 shows the profiles for the gauge field \( a(r) \), magnetic field \( B(r) \) and energy density \( \varepsilon_\Sigma(r) \), all of them well-behaved functions and in accordance with the respective behaviors previously found at the boundary values. The gauge field \( a(r) \) profiles acquire a plateau format around \( r_0 \) that affect directly the profiles of both the magnetic field and \( \varepsilon_\Sigma(r) \), i.e., because of the plateau, they acquire a ringlike shape centered at the origin. We also note that when \( \alpha \) increases, the profiles become more localized around \( r_0 \). On the other hand, the role played by the parameter \( m \) is better understood by examining Fig. 3. We have verified that for a fixed \( m \), the gauge field engenders \( (2m - 1) \)-plateaus that define, in the profiles of both \( B(r) \) and \( \varepsilon_\Sigma(r) \), an equal number of outer rings around its center. For a more precise overview of these features, we depict in Fig. 4 the effects induced in the magnetic field profiles by the dielectric function (51) for different values of \( \alpha \) and \( m \). It is notorious how \( \alpha \) controls the magnetic field distribution around the origin. That is, the size of the inner region to the ringlike structures increases as \( \alpha \) grows; that effect arises because the rings agglomerate around \( r_0 \) when \( \alpha \) grows (see left and middle plots in Fig. 4). In addition, we see the ex-
licit formation of $2m - 1$ outer rings, corresponding to the same quantity of plateaus engendered in the gauge profile.

Therefore, our results show how the dielectric function (51), associated with the extra nonlinearity induced by the $\chi^4$ model, provides interesting new features to the compact skyrmions belonging to the restricted gauged baby Skyrme model.

B. $\chi^6$ nonlinear medium

In this second scenario, we adopt a superpotential that engenders a $\chi^6$ model, so we consider

$$W(\chi) = \frac{\alpha}{2} \chi^2 - \frac{\alpha}{4} \chi^4,$$  

(64)

which has also been used in the study of multilayered structures [22]. Then, by using (64) in the BPS equation (42), we obtain the kinklike solution generated by the $\chi^6$-model that is given by

$$\chi(r) = \frac{\rho^\alpha}{\sqrt{r^{2\alpha} + \rho_0^{2\alpha}}},$$  

(65)

satisfying the following boundary values $\chi_0 = 0$ and $\chi_\infty = 1$. Likewise, now the BPS bound (39) becomes

$$E_{\text{bps}} = 2\pi \lambda^2 N W_0 + \frac{\alpha \pi}{2},$$  

(66)

where the second term is the contribution from the neutral scalar field $\chi$.

To investigate the changes in the shape of the soliton originate from a $\chi^6$ model, we select the following dielectric function,

$$\Sigma(\chi) = \frac{1}{J_0(\gamma \chi)},$$  

(67)

where $J_0$ Bessel function of the first kind and $\gamma \in \mathbb{R}$. This choice is motivated mainly because solitons in Bessel photonic lattices in nonlinear media acquire ringlike shapes [35, 36]. As we shall see in the current scenario, the dielectric function (67) also engenders compact skyrmions with ringlike formats.

With $\Sigma(\chi)$ defined in (67), the BPS equations to be solved are the one in (53) and the second (54) now reads

$$\frac{N da}{dr} + g^2 \lambda^2 W_0 J_0^2 \left( \frac{\gamma r^\alpha}{\sqrt{r^{2\alpha} + \rho_0^{2\alpha}}} \right) h^{3/2} = 0,$$  

(68)

It is also important to write the energy density $\varepsilon_{\Sigma}(r)$ for the current scenario,

$$\varepsilon_{\Sigma} = g^2 \lambda^2 W_0^2 h^3 J_0^2 + \frac{9}{16} \lambda^2 W_0^2 h.$$  

(69)

To continue, we write below the behavior of the field profiles near the boundary values. Like this, around the origin, we obtain

$$h(r) = 1 - \frac{3}{2^4} \frac{W_0}{N} r^2 + \frac{3}{2^{16}} \frac{(16g^2\lambda^2 - 3) W_0^2}{N^2} r^4 + \cdots$$

$$+ \frac{3}{2^6} \frac{\gamma^2 g^2 \chi^2 W_0^2}{(\alpha + 1)(\alpha + 2) N^2} r^{2\alpha+4} + \cdots,$$  

(70)

$$a(r) = -\frac{\lambda^2 g^2 W_0}{2N} r^2 + \frac{3^2 g^2 \lambda^2 W_0^3}{2^7 N^2} r^4 + \cdots$$

$$+ \frac{\gamma^2 g^2 \chi^2 W_0^2}{r^{2\alpha+2}} + \cdots,$$  

(71)

Besides, for the magnetic field and the energy density $\varepsilon_{\Sigma}$ we get

$$B(r) \approx -\lambda^2 g^2 W_0 + \frac{3^2 g^2 \lambda^2 W_0^2}{2^6 N} r^2 + \cdots$$

$$+ \frac{\gamma^2 g^2 \chi^2 W_0^2}{2} \frac{r^{2\alpha}}{r_0^{2\alpha}} + \cdots,$$  

(72)

and

$$\varepsilon_{\Sigma} = g^2 \lambda^4 W_0^2 + \frac{3^2 \gamma^2 \lambda^2 W_0^2}{2^4}$$

$$- \frac{3^2 \lambda^2 (16g^2\lambda^2 + 3) W_0^3}{2^8 N} r^2 + \cdots$$

$$- \frac{\gamma^2 g^2 \chi^2 W_0^2}{2} \frac{r^{2\alpha}}{r_0^{2\alpha}} + \cdots,$$  

(73)

respectively. Moreover, we observe that both are nonnull at the origin.

The field profiles in the limit $r \to R$, i.e., when they reach the corresponding vacuum value, possess the following behavior:

$$h(r) \approx \frac{3^2}{2^8} \frac{W_0^2 R^2}{(1 + a_R)^2 N^2} (r - R)^2 + \cdots$$  

(74)

$$a(r) \approx a_R + \frac{1}{2^6} \frac{g^2 R^2}{(1 + a_R)^2 N^2} C_R (r - R)^4 + \cdots,$$  

(75)

where $C_R$ is a constant given by

$$C_R = \frac{3^3}{2^{12}} \frac{\lambda^2 W_0^4 R^2}{(1 + a_R)^2 N^2} J_0^2 \left( \frac{\gamma R^\alpha}{\sqrt{R^{2\alpha} + \rho_0^{2\alpha}}} \right).$$  

(76)

Furthermore, the first relevant terms contributing to both the magnetic field and energy density $\varepsilon_{\Sigma}$ in the limit $r \to R$ are

$$B(r) = -\frac{g^2 R}{(1 + a_R) N} C_R (r - R)^3 + \cdots$$  

(77)

and

$$\varepsilon_{\Sigma} = \frac{3^4}{2^{12}} \frac{\lambda^2 W_0^4 R^2}{(1 + a_R)^2 N^2} (r - R)^2 + \cdots$$

$$+ \frac{3^3}{2^{12}} \frac{g^2 \lambda^2 W_0^4 R^4}{(1 + a_R)^4 N^4} C_R (r - R)^6 + \cdots.$$  

(78)
FIG. 5. Depiction by assuming the dielectric function (67) with $\gamma = 6$ and distinct values for $\alpha$. Left: compacton radius $R$ vs. $\alpha$ (solid red line) and the compacton radius of the standard case (black dot line). Right: the present Skyrme field (color lines) is depicted for $\alpha = 1$ (solid line), $\alpha = 5$ (dashed line) and $\alpha = 15$ (dot line), and the solid black line represents the profile without the dielectric medium.

FIG. 6. The gauge field profiles $a(r)$ (top panels), absolute value of the magnetic field $|B(r)|$ (middle panels) and energy density $\varepsilon_S(r)$ (bottom panels). We depict for $\gamma = 6$ with $\alpha = 1$ (left) and $\alpha = 2$ (right).

FIG. 7. The gauge field profiles $a(r)$ (top panels), absolute value of the magnetic field $|B(r)|$ (middle panels) and energy density $\varepsilon_S(r)$ (bottom panels). We depict for $\gamma = 12$ with $\alpha = 1$ (left) and $\alpha = 2$ (right).

We next present the numerical solutions for the fields by solving the system of BPS equations (53) and (68). Again we have fixing $N = 1$, $W_0 = 1$, $\lambda = 1$, $g = 1$ and $r_0 = 1$ (when adopted other value will be shown), for different values of $\alpha$ and $m$.

To analyze how the compacton radius $R$ changes, we depict $R$ vs. $\alpha$ (see left panel of Fig. 5). We remark the existence of an interval $0 < \alpha \leq \bar{\alpha}$ (here $\bar{\alpha} \simeq 0.9$) where the compacton radius grows until to reach a maximum value $R_{\text{max}} \simeq 3.12$ when $\alpha = \bar{\alpha}$, a behavior unlike from the previous case. Thereafter, for $\alpha > \bar{\alpha}$, the values of the radius monotonically decrease tending asymptotically to a minimum value $R \to R_{\text{min}}$ (here, $R_{\text{min}} \simeq 2.51$). This effect is illustrated in the right panel of Fig. 5 where it is depicted $h(r)$ for some values of $\alpha > \bar{\alpha}$. The picture shows how the profiles are modified, while the compacton radius decreases as $\alpha$ grows.

In Figs. 6 and 7 we have depicted the profiles for the gauge field $a(r)$, magnetic field $B(r)$ and energy density $\varepsilon_S(r)$. They behave according to the analytical expressions calculated previously at the boundaries. We also observe plateaus arising along the gauge field profiles; the inner plateau that shapes the core becomes bigger while $\alpha$ grows. Consequently, the presence of the plateaus engenders ringlike structures in the profiles of both the $B(r)$ and $\varepsilon_S(r)$. Thus, in Fig. 8, we highlight the new effects in the magnetic field profiles: first, one notices that $\alpha$ controls the core size, whose radius increases as $\alpha$ grows (see left and middle plots). Second, the parameter $\gamma$ controls the number of outer rings (surrounding the respective soliton’s core) that increase while $\gamma$ grows.

V. CONCLUSIONS

We have shown the existence of purely magnetic compact skyrmions in an extended restricted gauged baby Skyrme model with $SO(3) \times Z_2$ symmetry. The nonlinearity associated with the $Z_2$-symmetry is related to the presence of a neutral field $\chi$ interacting with the gauge
field through a dielectric function $\Sigma(\chi)$. The extended model describes compact skyrmions whose magnetic field possesses ringlike profiles. Then the successful implementation of the BPS procedure allows obtaining the energy lower bound (Bogomol’nyi bound) and the corresponding self-dual (or BPS) equations whose solutions saturate such a bound. Further, we have verified that the BPS solutions are also solutions of the Euler-Lagrange equations of the model, as expected.

We have verified that the extended gauged baby Skyrme model proposed here also engenders compact and noncompact topological structures. Here, we only focus on compact skyrmions by studying how they are affected by the nonlinearities associated with the supplementary $Z_2$ symmetry introduced via the dielectric function $\Sigma(\chi)$. Specifically, we have investigated two distinct nonlinear scenarios characterized by their respective superpotentials, then analyzed the effects on the compactons immersed in such media separately. Both analyses reveal the impact of the $Z_2$ nonlinearities on the compact skyrmions: (i) they change the compacton radius; (ii) they promote the arising of the ring format, and (iii) they control the inner region size and the number of outer rings surrounding the center.

We enhance that the results obtained here are of current interest and may motivate more investigations about new configurations related to magnetic skyrmions. These topological structures emerge in magnetic materials and have been explored in several contexts [37]. As a perspective, we can consider exploring new configurations to the skyrmions engendered in the context of the Dzyaloshinsky-Moriya interaction [9, 10, 38]. Finally, we are currently studying the existence of compact skyrmions (carrying both the magnetic flux and electric charge) in the presence of dielectric media or magnetic impurities. Advances in these directions we will report elsewhere.

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