Abstract

Recently, the synchronization of coupled dynamical systems has been widely studied. Synchronization is referred to as a process wherein two (or many) dynamical systems are adjusted to a common behavior as time goes to infinity, due to coupling or forcing. Therefore, before discussing synchronization, a basic problem on continuation of the solution must be solved: For given initial conditions, can the solution of coupled dynamical systems be extended to the infinite interval $[0, +\infty)$? In this paper, we propose a general model of coupled dynamical systems, which includes previously studied systems as special cases, and prove that under the assumption of QUAD, the solution of the general model exists on $[0, +\infty)$.

Coupled dynamical systems, Synchronization, Existence, Uniqueness, Continuation.

1 Introduction

In past years, collective behaviors of coupled dynamical systems have been widely studied. In particular, synchronization in networks of coupled dynamical systems, as one of the simplest and most striking behaviors, has attracted increasing attention in mathematical and physical literatures because of its potential applications in various fields, such as communication [1], seismology [2], and neural networks [3].

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The word “synchronization” comes from a Greek word, which means “share time”. Today, in science and technology, it has come to be considered as “time coherence of different processes”. Since the first observation of synchronization phenomenon was made by Huygens [4] in the 17th century, many different types of synchronization phenomena have been found, e.g., phase synchronization, lag synchronization, full synchronization, partial synchronization, almost synchronization, and so on. In mathematics, synchronization can be defined as a process wherein two (or many) dynamical systems adjust a given property of their motion to a common behavior as time goes to infinity, due to coupling or forcing (see [5]). For example, full synchronization requires that the difference between any two nodes converges to zero as time goes to infinity. Therefore, it is natural to raise following question: For given initial conditions, can the solution be extended to the infinite interval $[0, +\infty)$?

For example, in the paper [11], the following coupled systems with a delay is considered:

$$\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1,j\neq i}^{m} a_{ij} \Gamma \left[ x_j(t-\tau) - x_i(t) \right],$$

$$i = 1, \ldots, m,$$

where $x_i(t) = [x_{i1}(t), \ldots, x_{in}(t)]^T \in \mathbb{R}^n$ denotes the $n$-dimensional state variable of the $i$-th node, $i = 1, \ldots, m$; $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a differential function of the intrinsic system; $c$ is the coupling strength; $\Gamma = \text{diag}\{\gamma_1, \ldots, \gamma_n\}$ is the inner connection diagonal matrix with $\gamma_i \geq 0$, $i = 1, \ldots, n$; $a_{ij} \geq 0$, for all $i \neq j$, is the coupling coefficient from node $j$ to node $i$; and $\tau \geq 0$ is the coupling delay. It is assumed that $\sum_{j=1,j\neq i}^{m} a_{ij} = 1$, $a_{ii} = -1$, for all $i = 1, \ldots, m$. And the following theorem was proved.

**Proposition 1** Suppose that there are a positive definite diagonal matrix $P = \text{diag}\{p_1, \cdots, p_n\}$ and a diagonal matrix $D = \text{diag}\{d_1, \cdots, d_n\}$, such that

$$(x - y)^T P [f(x) - f(y) - Dx + Dy] \leq -\alpha (x - y)^T (x - y)$$

holds for some $\alpha > 0$, any $x, y \in \mathbb{R}^n$. Then, for sufficiently large coupling strength $c$ and sufficiently small delay $\tau$, the coupled system (1) will be globally synchronized.
Here, a prerequisite condition in discussing synchronization is that the solution \( x^i(t), \)
\( i = 1, \cdots, m, \) can be extended to the infinite interval \([0, +\infty)\). However, in most papers on
synchronization of coupled systems, such as \([6, 7, 8, 11]\) and others, it is always assumed
that for each initial condition, the coupled system under consideration has a unique solution
for all time \( t \geq 0 \) without any theoretical justification.

In this short paper, we address this issue and propose a general model of coupled dynamical systems, which includes previously studied systems as special cases. We prove that
under the assumption of QUAD (Assumption (A5) in Section 2), the solution of the general
model exists on \([0, +\infty)\). The assumption of QUAD is often used when using a Lyapunov
function with a quadratic form to investigate the global synchronization (e.g., in Proposition 1, and in \([6, 9, 10]\)). Therefore, the theorem proved in this paper provides a theoretical basis
for the discussion of synchronization of the coupled systems.

The rest of the paper is organized as follows: In Section 2, we propose a general model
of coupled dynamical systems. In Section 3, we present some fundamental theorems of
retarded functional differential equations with infinite delay, which are taken from \([14]\). In
Section 4, the main theorem is proved. We conclude the paper in Section 5

2 Model descriptions

In this section, we investigate the coupled dynamical systems described by the following
retarded functional integro-differential equations:

\[
\dot{x}^i(t) = f(t, x^i(t)) \\
+ \sum_{j=1}^{m} a_{ij}(t) \int_{0}^{\infty} g(t, x^j(t - \tau_{ij}(t) - s))dK_{ij}(s), \quad i = 1, 2, \ldots, m,
\]

(2)

where “\(\dot{}\)” represents the right-hand derivative, \(m\) is the network size, \(x^i(t) \in \mathbb{R}^n\) is the
state variable of the \(i\)-th node, \(t \in [0, +\infty)\) is a continuous time, \(f : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n\)
describes the dynamical behavior of each uncoupled system, \( A(t) = (a_{ij}(t)) \in \mathbb{R}^{m \times m} \) is the time-varying coupling matrix, which is determined by the topological structure of the network, \( g : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n \) is the output function, \( dK_{ij}(s) \) is a Lebesgue-Stieljes measure for each \( i,j = 1, \ldots, m \), and satisfies \( \int_0^\infty |dK_{ij}(s)| < +\infty \).

In addition, the following assumptions are necessary in discussion of retarded systems:

(A1) \( f(t,u) \) is continuous, and locally Lipschitz continuous with respect to \( u \), i.e., in each compact subset \( W \) of \([0, +\infty) \times \mathbb{R}^n\), there exists a constant \( l(W) > 0 \) such that
\[
\| f(t,u_1) - f(t,u_2) \| \leq l(W) \| u_1 - u_2 \| \quad \text{for any } (t,u_k) \in W, k = 1, 2;
\]
(A2) \( A(t) = (a_{ij}(t))_{i,j=1}^m \) is continuous;
(A3) \( g(t, u) \) is continuous, and there exists a continuous function \( \kappa(t) : [0, +\infty) \to \mathbb{R}^+ \), such that \( \| g(t, u_1) - g(t, u_2) \| \leq \kappa(t) \| u_1 - u_2 \| \) for any \( t \in [0, +\infty) \) and \( u_1, u_2 \in \mathbb{R}^n \);
(A4) For each \( i,j = 1, \ldots, m \), \( \tau_{ij}(t) \) is continuous and nonnegative;
(A5) There are a symmetric positive definite matrix \( P \) and a diagonal matrix \( \Delta = \text{diag}\{\delta_1, \ldots, \delta_n\} \) such that \( f(t,u) \in \text{QUAD}(\Delta, P) \), where \( \text{QUAD}(\Delta, P) \) denotes a class of continuous functions \( h(t, u) : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n \) satisfying
\[
(u_1 - u_2)^\top P \{[h(t, u_1) - h(t, u_2)] - \Delta [u_1 - u_2]\} 
\leq -\epsilon (u_1 - u_2)^\top (u_1 - u_2) \tag{3}
\]
for some \( \epsilon > 0 \), all \( u_1, u_2 \in \mathbb{R}^n \) and \( t \in [0, +\infty) \).

Here, \( \| \cdot \| \) can be any norm in \( \mathbb{R}^n \) (Without loss of generality, in this paper we assume that \( \| \cdot \| \) is 2-norm).

The model (2) includes many previously studied systems as special cases. In the following, we present several examples.

**Example 1** \( dK_{ij}(s) = \delta(s), \) where \( \delta(s) \) is the Dirac-delta function, i.e., \( \delta(0) = 1 \) and \( \delta(s) = 0 \) for \( s \neq 0 \); \( A(t) = A \) is a constant matrix with zero-sum rows and nonnegative
off-diagonal elements; \( g(t, u) = \Gamma u \), where \( \Gamma \) is a constant matrix; \( \tau_{ij}(t) = 0 \) for each \( i, j = 1, \ldots, m \) and all \( t \geq 0 \). Then, (2) reduces to the system with undelayed, constant and linear coupling discussed in [6, 10]:

\[
\dot{x}_i(t) = f(t, x_i(t)) + \sum_{j=1}^{m} a_{ij} \Gamma_j x_j(t), \quad i = 1, 2, \ldots, m.
\]

**Example 2** \( dK_{ij}(s) = \delta(s) \), where \( \delta(s) \) is the Dirac-delta function; \( A(t) \) is a time-dependent matrix with zero-sum rows and nonnegative off-diagonal elements; \( g(t, u) = \Gamma(t) u \), where \( \Gamma(t) \) is a time-dependent matrix; \( \tau_{ij}(t) = 0 \) for each \( i, j = 1, \ldots, m \) and all \( t \geq 0 \). Then, (2) reduces to the system with undelayed, time-varying and linear coupling discussed in [12, 13]:

\[
\dot{x}_i(t) = f(t, x_i(t)) + \sum_{j=1}^{m} a_{ij}(t) \Gamma(t) x_j(t), \quad i = 1, 2, \ldots, m.
\]

**Example 3** \( dK_{ij}(s) = \delta(s) \), where \( \delta(s) \) is the Dirac-delta function; \( f(t, u) = f(u) \), i.e., \( f \) is independent of \( t \); \( A(t) = A \) is a constant matrix with zero-sum rows and nonnegative off-diagonal elements, and satisfies \( a_{ii} = -c \) for \( i = 1, \ldots, m \); \( g(t, u) = \Gamma u \), where \( \Gamma \) is a diagonal matrix with nonnegative diagonal elements; \( \tau_{ij}(t) = \tau \) for \( i \neq j \) and \( \tau_{ii}(t) = 0 \) for \( i = 1, \ldots, m \). Then, (2) reduces to the system with delayed, constant and linear coupling discussed in [11]:

\[
\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1, j \neq i}^{m} a_{ij} \Gamma [x_j(t - \tau) - x_i(t)], \quad i = 1, 2, \ldots, m.
\]

Besides Examples 1-3, the model (2) includes coupled dynamical systems with nonlinear coupling, time-varyingly delayed coupling, distributedly delayed coupling, etc.
3 Preliminaries

In this section, we present some fundamental results of retarded functional differential equations with infinite delay, which will be used in the sequel.

Firstly, we introduce some notations and definitions.

Denote $BC((−∞, a], \mathbb{R}^N)$ the family of continuous functions $φ$ mapping the interval $(−∞, a]$ into $\mathbb{R}^N$ such that $∥φ∥ =: \sup\{∥φ(θ)∥ : −∞ < θ ≤ a\}$ is finite. Also, denote $C^∞((−∞, a], \mathbb{R}^N) = \{φ ∈ BC((−∞, a], \mathbb{R}^N) : \lim_{θ \to −∞} φ(θ) \text{ exists in } \mathbb{R}^N\}$. When $a = 0$, we generally denote $C^∞ = C^∞((−∞, 0], \mathbb{R}^N)$. For $σ ∈ \mathbb{R}$, $B ≥ 0$, $x ∈ C^∞((−∞, σ + B], \mathbb{R}^N)$, and $t ∈ [σ, σ + B]$, we define $x_t ∈ C^∞$ as $x_t(θ) = x(t + θ), θ ∈ (−∞, 0]$. Assume $Ω$ is an open subset of $\mathbb{R} × C^∞$, $h : Ω → \mathbb{R}^N$ is a given function, and “˙” represents the right-hand derivative; then, we call

$$\dot{x}(t) = h(t, x_t)$$  (4)

a retarded functional differential equation with infinite delay on $Ω$.

**Definition 1** A function $x$ is said to be a solution of Equation (4) on the interval $I = [σ, σ + B]$ if there are $σ ∈ \mathbb{R}$ and $B > 0$ such that $x ∈ C^∞((−∞, σ + B], \mathbb{R}^N)$, $(t, x_t) ∈ Ω$ and $x(t)$ satisfies Equation (4) for $t ∈ I$. For given $σ ∈ \mathbb{R}$, $φ ∈ C^∞$, if a solution $x$ of Equation (4) is defined on an interval $[σ, σ + B)$, $B > 0$, and satisfies $x_σ = φ$, then $x$ is called a solution of Equation (4) with initial value $φ$ at $σ$ or simply a solution through $(σ, φ)$.

**Definition 2** Suppose $x(t)$ and $y(t)$ are solutions with the same initial condition and satisfies Equation (4) respectively on the intervals $I$ and $J$ whose left end points are $σ$. If $I$ is properly contained in $J$ and $x(t) = y(t)$ for $t ∈ I$, we say $y$ is a continuation of $x$. If $x$ has no continuation, it is called a noncontinuable solution, or a maximal solution.
**Definition 3** We say \( h(t, \phi) \) is Lipschitz in \( \phi \) in a compact subset \( W \) of \( \mathbb{R} \times C^\infty \) if there a constant \( l > 0 \) such that, for any \( (t, \phi_k) \in W, k = 1, 2, \)

\[
\|h(t, \phi_1) - h(t, \phi_2)\| \leq l\|\phi_1 - \phi_2\|. \tag{5}
\]

The following three lemmas on existence, uniqueness, and continuation of the solution of Equation (4), are used in the proof of the main theorem in the next section. The details can be found in [14].

**Lemma 1** (Existence) Suppose \( \Omega \) is an open subset in \( \mathbb{R} \times C^\infty \) and \( h : \Omega \to \mathbb{R}^N \) is continuous. Then, for any \( (\sigma, \varphi) \in \Omega \), there exists a solution of Equation (4) through \( (\sigma, \varphi) \).

**Lemma 2** (Uniqueness) Suppose \( \Omega \) is an open subset in \( \mathbb{R} \times C^\infty \) and \( h(t, \phi) \) is Lipschitz in \( \phi \) in each compact subset of \( \Omega \). Then, for any \( (\sigma, \varphi) \in \Omega \), there exists at most one noncontinuable solution of Equation (4) through \( (\sigma, \varphi) \).

**Lemma 3** (Continuation) Suppose \( \Omega \) is an open subset in \( \mathbb{R} \times C^\infty \), \( h : \Omega \to \mathbb{R}^N \) is continuous, and \( x \) is a noncontinuable solution of Equation (4) defined on \( I = [\sigma, \sigma + B] \). Then, for every compact subset \( W \) of \( \Omega \), there is a \( t_W \) in \( I \) such that \( (t, x_t) \notin W \) for all \( t \in (t_W, \sigma + B) \).

### 4 Main result

In this section, we prove the following theorem.

**Theorem 1** Suppose that Assumptions (A1)-(A5) hold. Then, for any \( \varphi(\theta) = [\varphi^1(\theta)^T, \ldots, \varphi^m(\theta)^T]^T \) with \( \varphi^i(\theta) \in C^\infty((-\infty, 0], \mathbb{R}^n) \), there is a unique noncontinuable solution \( x(t) = [x^1(t)^T, \ldots, x^m(t)^T]^T \) of Equation (2) through \( (0, \varphi) \). Moreover, the interval of existence of the solution \( x \) is \([0, +\infty)\).
Proof: By Assumptions (A1)-(A4) and Lemmas 1-2, it is clear that for the integro-differential system (2), there exists a unique noncontinuable solution \( x(t) \). In the following, we will prove that the interval of existence of the solution \( x(t) \) is \([0, +\infty)\).

We employ “proof by contradiction”, and suppose that the interval of existence of the noncontinuable solution \( x(t) \) is \([0, b)\), where \( b \) is a positive constant.

Firstly, by Assumptions (A1)-(A4), we can find positive constants \( \alpha, \beta \) and \( \gamma \) such that
\[
\| g(t, u_1) - g(t, u_2) \| \leq \alpha \| u_1 - u_2 \|
\]
holds for all \( u_1, u_2 \in \mathbb{R}^n \) and \( t \in [0, b) \), and
\[
|a_{ij}(t)| \leq \beta,
\]
\[
\| f(t, x^i(0)) + \sum_{j=1}^{m} a_{ij}(t) g(t, x^j(0)) \int_{0}^{\infty} dK_{ij}(s) \| \leq \gamma.
\]
hold for all \( i, j = 1, \ldots, m \) and \( t \in [0, b) \).

Now, we will show how the assumption of QUAD (Assumption (A5)) plays an important role in the proof.

Since \( f(t, u) \in \text{QUAD}(\Delta, P) \) (Assumption (A5)), it is clear that there is a constant \( \delta > 0 \) such that for all \( u_1, u_2 \in \mathbb{R}^n \) and \( t \geq 0 \),
\[
(u_1 - u_2)^\top P [f(t, u_1) - f(t, u_2)] \leq \delta (u_1 - u_2)^\top (u_1 - u_2).
\]
Denote
\[
\eta = \frac{2\delta + 2\alpha \beta \| P \| K}{\lambda_{\min}^P} + \frac{2m\gamma \| P \|}{\sqrt{\lambda_{\min}^P}} > 0,
\]
where \( m \) is the number of the nodes, \( \| P \| \) is the 2-norm of the matrix \( P \), \( \lambda_{\min}^P \) is the minimum eigenvalue of the matrix \( P \), and \( K = \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{0}^{\infty} |dK_{ij}(s)| \).

Since the matrix \( P \) is symmetric positive definite, we can define a norm in \( \mathbb{R}^{nm} \):
\[
\| x(t) \|_P = \left( \sum_{i=1}^{m} x^i(t)^\top P x^i(t) \right)^{\frac{1}{2}};
\]
and two nonnegative functions:

\[ V(t) = \frac{1}{2} \| x(t) - x(0) \|_P^2, \]

\[ M(t) = \max \left[ \frac{1}{2}, \sup_{-\infty < s \leq t} \frac{1}{2} \| x(s) - x(0) \|_P^2 \right], \quad t \in [0, b). \]

Clearly, \( V(t) \leq M(t) \). We claim that \( M(t) \leq M(0) e^{nt} \) for all \( t \in [0, b) \).

In fact, at any \( t_0 \in [0, b) \), there are two possible cases:

**Case 1:** \( V(t_0) < M(t_0) \). In this case, by the continuity of \( \| x(t) - x(0) \|_P^2 \), \( M(t) \) is non-increasing at \( t_0 \).

**Case 2:** \( V(t_0) = M(t_0) \).

Calculating the right-hand derivative of \( V \) with respect to time along the trajectories of (2), one has

\[
\dot{V}(t_0) = \sum_{i=1}^{m} (x^i(t_0) - x^i(0))^\top P \dot{x}^i(t_0)
\]

\[
= \sum_{i=1}^{m} (x^i(t_0) - x^i(0))^\top P \left[ f(t_0, x^i(t_0)) + \sum_{j=1}^{m} a_{ij}(t_0) \int_0^{\infty} g(t_0, x^j(t_0) - \tau_{ij}(t_0) - s) dK_{ij}(s) \right]
\]

\[
= \sum_{i=1}^{m} (x^i(t_0) - x^i(0))^\top P \left\{ \left[ f(t_0, x^i(t_0)) - f(t_0, x^i(0)) \right] + \sum_{j=1}^{m} a_{ij}(t_0) \int_0^{\infty} \left[ g(t_0, x^j(t_0) - \tau_{ij}(t_0) - s) - g(t_0, x^j(0)) \right] dK_{ij}(s) \right\}
\]

\[
\leq \delta \sum_{i=1}^{m} (x^i(t_0) - x^i(0))^\top (x^i(t_0) - x^i(0))
\]
\[
+ \sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}(t_0)| \|P\| \|x^i(t_0) - x^i(0)\| \\
\times \int_{0}^{\infty} \|g(t_0, x^j(t_0 - \tau_{ij}(t_0) - s)) - g(t_0, x^j(0))\| dK_{ij}(s) \\
+ \sum_{i=1}^{m} \|P\| \|x^i(t_0) - x^i(0)\| \|f(t_0, x^i(0))\| \\
+ \sum_{j=1}^{m} a_{ij}(t_0) g(t_0, x^j(0)) \int_{0}^{\infty} dK_{ij}(s) \|
\]
\[
\leq \delta \|x(t_0) - x(0)\|^2 + \alpha \beta \|P\| \sum_{i=1}^{m} \sum_{j=1}^{m} \|x^i(t_0) - x^i(0)\| \\
\times \int_{0}^{\infty} \|x^j(t_0 - \tau_{ij}(t_0) - s) - x^j(0)\| dK_{ij}(s) \\
+ \gamma \|P\| \sum_{i=1}^{m} \|x^i(t_0) - x^i(0)\| \\
\leq \delta \|x(t_0) - x(0)\|^2 + \alpha \beta \|P\| \sum_{i=1}^{m} \sum_{j=1}^{m} \|x(t_0) - x(0)\| \\
\times \int_{0}^{\infty} \|x(t_0 - \tau_{ij}(t_0) - s) - x(0)\| dK_{ij}(s) \\
+ \gamma \|P\| \sum_{i=1}^{m} \|x(t_0) - x(0)\| \\
\leq \frac{\delta}{\lambda_{\min}^P} \|x(t_0) - x(0)\|_P^2 \\
+ \frac{\alpha \beta}{\lambda_{\min}^P} \|P\| \sum_{i=1}^{m} \sum_{j=1}^{m} \|x(t_0) - x(0)\|_P \\
\times \int_{0}^{\infty} \|x(t_0 - \tau_{ij}(t_0) - s) - x(0)\|_P dK_{ij}(s) \\
+ \frac{m \gamma \|P\|}{\sqrt{\lambda_{\min}^P}} \|x(t_0) - x(0)\|_P \cdot 1 \\
\leq \frac{\delta}{\lambda_{\min}^P} 2M(t_0) \\
+ \frac{\alpha \beta}{\lambda_{\min}^P} \sqrt{2M(t_0)} \sqrt{2M(t_0)} \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{0}^{\infty} |dK_{ij}(s)|
\]
\[
\begin{align*}
&= \frac{m\gamma}{\sqrt{\lambda_{\min}^P}} \sqrt{2M(t_0)} \sqrt{2M(t_0)} \\
&= \left\{ \frac{2\delta + 2\alpha\beta}{\sqrt{\lambda_{\min}^P}} \left[ \sqrt{2M(t_0)} \right] + \frac{2m\gamma}{\sqrt{\lambda_{\min}^P}} \right\} M(t_0) \\
&= \eta V(t_0)
\end{align*}
\]

In summary, we conclude that \( M(t) \leq M(0)e^{\eta t} \) for all \( t \in [0, b) \), which implies \( V(t) \leq M(0)e^{\eta t} \) and

\[
\|x(t)\| \leq \frac{1}{\sqrt{\lambda_{\min}^P}} \|x(t)\|_P \\
\leq \frac{1}{\sqrt{\lambda_{\min}^P}} (\|x(0)\|_P + \sqrt{2V(t)}) \\
= \frac{1}{\sqrt{\lambda_{\min}^P}} (\|x(0)\|_P + \sqrt{2M(0)e^{\eta t}}) \\
\leq \frac{1}{\sqrt{\lambda_{\min}^P}} (\|x(0)\|_P + \sqrt{2M(0)e^{\eta b}}) \\
\leq \frac{1}{\sqrt{\lambda_{\min}^P}} (\|x(0)\|_P + \sqrt{2M(0)e^{\eta b}}) \\
\leq \frac{1}{\sqrt{\lambda_{\min}^P}} (\|x(0)\|_P + \sqrt{2M(0)e^{\eta b}})
\]

(6)

for all \( t \in [0, b) \).

Now, pick a compact set

\[
W = \left\{ (t, \psi) \in \mathbb{R} \times C^\infty((-\infty, 0], \mathbb{R}^{nm}) \left| 0 \leq t \leq b, \right. \right. \text{ and} \\
\|\psi\| \leq \max \left\{ \frac{1}{\sqrt{\lambda_{\min}^P}} \left( \|x(0)\|_P + \sqrt{2M(0)e^{\eta b}} \right), \|\varphi\| \right\},
\]

where \( \varphi \) is the initial value. By the inequality (6), we conclude that \( (t, x_t) \in W \) for all \( t \in [0, b) \), which contradicts Lemma 3.

Therefore, the interval of existence of the noncontinuable solution \( x \) is \([0, +\infty)\). Theorem is proved completely.
5 Conclusions

In this paper, we propose a general model of coupled dynamical systems, which includes previously studied systems as special cases, and prove that under the assumption of QUAD, the solution of the general model exists on $[0, +\infty)$.

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