Pants Decomposition of the Punctured Plane

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Joint work with Sheung-Hung Poon
Surface \equiv 2\text{-manifold}
Pant $\equiv$ a sphere with 3 holes
Pant $\equiv$ a sphere with 3 holes

Every simple cycle on a pant is contractible to a point or to a boundary
Pants decomposition

**Definition:** A set of simple cycles that decompose a surface into disjoint pants

To understand the topology of the surface and to compute its various properties

Every compact orientable surface has at least one pants decomposition*

* except sphere, cylinder, disk, or torus

A surface can have many pants decompositions
Essential cycle

Simple cycle not contractible to a point or to a boundary
Example: decomposing into pants
Example: decomposing into pants

\[ \Sigma \]

\[ \alpha \]

\[ \beta \]

essential cycles on \( \Sigma \)
Example: decomposing into pants

\[ \Sigma' \]

[Diagram of a punctured plane decomposed into pants]

"cut!"
Example: decomposing into pants

The image shows a surface labeled $\Sigma'$ with an essential cycle $\gamma$. The text mentions that this example illustrates decomposing the surface into pants.
Example: decomposing into pants

\[ \pi_1 \quad \text{cut!} \quad \pi_2 \]
The big open problem

Computing an exact or approximate *shortest* pants decomposition of a general combinatorial surface
The big open problem

Computing an exact or approximate shortest pants decomposition of a general combinatorial surface

We consider a variant in the Euclidean plane . . .
Punctured plane

The plane

a  b  c
d  e
f
Punctured plane

Surface $\Sigma$ is the plane minus $n$ points
Decomposing the punctured plane

$\sum$

\[
\begin{array}{c}
a \\
b \\
e \\
f \\
c \\
d
\end{array}
\]
Decomposing the punctured plane

essential cycle on $\Sigma$

\begin{itemize}
  \item $a$
  \item $b$
  \item $c$
  \item $d$
  \item $e$
  \item $f$
\end{itemize}
Decomposing the punctured plane

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Decomposing the punctured plane

\begin{figure}
\centering
\includegraphics[width=\textwidth]{punctured_plane_decomposition}
\end{figure}

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essential cycle on $\Sigma'$
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Properties

\( n - 1 \) disjoint simple cycles

Nested in a binary tree

Decompose the plane into a set of pants and an unbounded component
Shortest pants decomposition

**Input:** A set $P$ of $n$ points in the plane $\mathbb{E}^2$

Remove an $\varepsilon$-disk $D_i$ centered at each point $p_i \in P$

Let $\Pi$ be a pants decomposition of $\mathbb{E}^2 \setminus \bigcup_{i=1}^{n} D_i$

Imagine tightening the cycles of $\Pi$ as $\varepsilon \to 0$

The limit is a *non-crossing* pants decomposition $\Pi'$

**Problem:** Compute a non-crossing pants decomposition $\Pi^*$ of $\mathbb{E}^2 \setminus P$ of minimum total length
 Lemma: Every cycle in a shortest pants decomposition of collinear points encloses an interval of points

Compute shortest pants decomposition in $O(n^2)$ time using dynamic programming with Yao’s speedup
A lower bound

Every cycle in a shortest pants decomposition is a simple polygon with vertices in $P$ (no Steiner points)

A shortest pants decomposition $\Pi^*$ of $\mathbb{E}^2 \setminus P$ contains a TSP tour of $P$

So, $|\Pi^*| \geq |TSP(P)|$
\(O(\log n)\) approximation
Construct an $O(1)$-approximate TSP tour $T$

Start with the $n$ points in order along the tour $T$
$O(\log n)$ approximation

Repeatedly enclose pairs of smaller cycles by a bigger cycle until we have a pants decomposition $\Pi$. 
$O(\log n)$ approximation

Each cycle of $\Pi$ is obtained by doubling the edges of a sub-tour of $T$
$O(\log n)$ approximation

Each edge of $T$ belongs to $O(\log n)$ cycles of $\Pi$

So,

$$|\Pi| \leq O(\log n) |T| \leq O(\log n) |\Pi^*|$$
PTAS

For every $\varepsilon > 0$, compute a $(1 + \varepsilon)$-approximate shortest pants decomposition in polynomial time

Extension of PTAS for Euclidean TSP

Uses Mitchell’s guillotine rectangular subdivisions
Previous work

Allen Hatcher.

**Pants Decompositions of Surfaces.**

arxiv.org/abs/math.GT/9906084

The *pants decomposition complex* of a given surface is simply connected—vertices are isotopy classes of pants decompositions, edges correspond to elementary moves

\[ S\text{-move} \]

\[ A\text{-move} \]
Previous work

Éric Colin de Verdière and Francis Lazarus. Optimal Pants Decompositions and Shortest Homotopic Cycles on an Orientable Surface. Graph Drawing, pp. 478–490, 2003 (+EuroCG’03)

Show how to *shorten* a given pants decomposition

Given a pants decomposition of a general combinatorial surface, they compute a *homotopic* pants decomposition in which each cycle is shortest in its homotopy class
Work in progress

NP-complete for general surfaces? ... I believe so!

NP-complete for the punctured plane? ... I don’t know

David Eppstein recently obtained an $O(n \log n)$-time algorithm to compute an $O(1)$-approximation for the punctured plane using quadtrees [to appear in SODA 2007]
Thank you!