The Influence of One Strategic Agent on The Matching Market

Ron Kupfer

The Hebrew University of Jerusalem Ron.Kupfer@mail.huji.ac.il

Abstract. Consider a matching problem with \( n \) men and \( n \) women, with preferences drawn uniformly from the possible \((n!)^2\) full ranking options. We analyze the influence of one strategic agent on the quality of the other agents’ matchings in the Gale-Shapley Algorithm. We show that even though the Gale Shapley algorithm is famous for being optimal for men, one small change in the reported preferences is enough for the women to get a near optimal match. In this case, the quality of the matching dramatically improves from the women’s perspective. The expected women-rank is \( O(\log^4(n)) \) and almost surely the average women-rank is \( O(\log^{2+\epsilon}(n)) \) rather than a rank of \( O(n \log(n)) \) in both cases under truthful regime.

1 Introduction

The stable matching problem concerns a scenario where we must find a matching between two disjoint sets of agents that satisfies natural stability constraints. This problem has received an enormous amount of attention, starting with the seminal work of Gale and Shapley [7], and has been used as a paradigm in a host of applications ranging from matching doctors to hospitals [23] to matching kidney donors to recipients [25]. The basic formalism considers a matching between a set of \( n \) men and \( m \) women. Each man has a preference order over all women and the option of being unmatched, and the same goes for women. A matching between the set of men and the set of women is called stable if there exist no “blocking pair”: a man and a woman who prefer each other to their current matching.

The Men-Proposing Deferred-Acceptance Algorithm (also known as the Gale-Shapley Algorithm [7]) is an algorithm for finding a stable matching and its proof of correctness shows that such a matching always exists. The algorithm works in an iterative way. At each round, every unmatched man proposes to his most preferred woman who has not previously rejected him. Each woman chooses her most preferred man out of those who proposed to her and release all the other proposers to continue with their lists. The algorithm terminates when all men are matched or reached to the end of their list.

It is known [7] that the algorithm is optimal for men in the following sense: For each man \( m \), there is no other stable matching in which \( m \) is matched with someone whom he prefers better. On the other hand, as shown in [17], the algorithm yields the worst stable matching for any woman. That is, for each woman \( w \), there is no other stable matching in which \( w \) is matched with someone who she prefers less.

Under the Gale-Shapley mechanism it is a dominant strategy for men to report their preferences truthfully [5]. This is not the case for women. Here is a simple example with two men \((m_1,m_2)\) and two women \((w_1,w_2)\) demonstrating this. Suppose their preference rankings are as follows:

\[
\begin{align*}
& m_1 \text{ prefers } w_1 > w_2 \\
& m_2 \text{ prefers } w_2 > w_1
\end{align*}
\]

\[
\begin{align*}
& w_1 \text{ prefers } m_2 > m_1 \\
& w_2 \text{ prefers } m_1 > m_2.
\end{align*}
\]

and for all of the agents the least preferred option is to stay single. The algorithm matches \( w_1 \) to \( m_1 \). If \( w_1 \) falsely reports that she prefers staying single rather than being matched with \( m_1 \), the algorithm will match her to \( m_2 \), whom she prefers to \( m_1 \).

Strategic behavior in stable matching algorithms has been a topic of vast research. In [22] it is shown that there is no algorithm for which reporting the true preferences is a dominant strategy for both men and women. A partial list of works on strategic behavior by women in the Gale-Shapley Algorithm includes [4, 5, 8, 11, 16, 24, 28].

Notice that strategic behavior by a woman affects also the outcome of the other women, e.g. in the example above perhaps surprisingly, the strategic behavior of \( w_1 \) improves the matching of \( w_2 \). It is known [3, 9] that in general, in the Gale-Shapley Algorithm, strategic behavior by any set of women can only benefit the other women in the sense that if none of the former is worse off, then neither is any of the latter.

A natural and well known way to manipulate the algorithm is to truncate the list of the reported preferences i.e. setting a threshold so that only spouses from some given rank and above are acceptable. This strategy is
optimal for any woman when all the other agents’ preferences are known to her, assuming all other agents report the truth [24]. In the same work it is shown that this is also true for a wide range of partial information structures.

In this work, we analyze the effect of a selfish reporting strategy by one (or more) woman, on the quality of the matching obtained by the other agents, both men and women.

Specifically, we explore the characteristics of such strategies in the commonly studied setting of a balanced market with \( n \) men and \( n \) women (e.g. [12,13,29]). As studied in [4,15,19,21] and others, the set of preferences is drawn uniformly at random from the set of all possible \((n!)^{2n}\) full ranking options. A simple measure of the quality of a match from a perspective of a given agent is the rank of its match (e.g. [13]). We say that a person has a rank \( k \) in some matching is they are matched with their \( k \)th favorite mate (where rank 1 denotes being match to the most preferred mate and rank \( n \) denotes being match to the least preferred mate). In this model, under truthful regime, it is known by [20] that the expected rank of any woman \( w \) is of order \( n \log(n) \). The expected rank of men is of order \( \log(n) \).

Our main theorem shows that one strategic agent is expected to dramatically affects the entire outcome of the matching. In particular, the average women-rank is polylogarithmic in \( n \) compare to an expected average rank of order \( n \log(n) \) under a truthful reporting regime. These results hold in expectation and with high probability. Formally -

**Theorem 1.** In a random uniformly ranked balanced matching market with \( n \) men and \( n \) women, where a single woman uses her optimal strategy and all the other agents report truthfully, we have that

1. Almost surely, \( 1 - o(1) \) fraction of the women get a match from their top \( O(\log^{2+\epsilon}(n)) \) men. Furthermore, the expected average women-rank is of order of \( O(\log^4(n)) \).
2. Almost surely, the average men-rank is no better than \( \Omega(\frac{n}{\log^{2+\epsilon}(n)}) \).

These results are not hinged on the strategic woman having a full information of the preferences or on taking the exact optimal truncation but are rather more robust as it discussed later.

Since the Gale-Shapley algorithm yields the worst stable matching for women, any upper bound for the women-rank holds for any matching algorithm which outputs a stable matching in the true preferences. Assume the strategic woman considers only truncation strategies. The optimal truncation strategy is indeed an optimal strategy for her among all the possible strategies. Since optimal truncation strategy yields a stable matching in the true preferences for any matching algorithm which guarantees to outputs a stable matching in the reported preferences, our results hold for large family of algorithms. Such algorithms are discussed for example in [1,6,10,13,14,26].

We present simulation results in the setting described above, for different market sizes. These simulation results reinforce the theorems conclusions and hint that the actual effect might be even stronger than the formally proven effect.

One interpretation of the results is that although the Gale-Shapley Algorithm is optimal for men, in fact a strategic behavior from only one woman is sufficient in order to almost completely eliminate the men’s advantage. Observe that strategic behavior narrows the set of possible stable matchings attainable by any algorithm. Hence, one strategic agent may strongly influence the matching in favor for its sex, either men or women.

## 2 Influence Under Full Information

### 2.1 The Model

**The Process** We analyze the effect of one strategic agent in a balanced market of \( n \) men and \( n \) women where the set preferences are chosen uniformly from the possible \((n!)^{2n}\) full ranking options. In this section, we assume one woman, \( g \), gets to look at all the other agents’ preference lists before running the algorithm. Then \( g \) acts according to a preference list chosen in a strategic way in order to maximize her utility from the matching. This maximization is done with respect to her original preferences.

It is shown in [17] that the outcome of the Gale-Shapley Algorithm is independent of the order of proposals made by the men. By using this fact and the principle of deferred decisions we follow [15] and describe the algorithm as follows: a man needs to choose his \( i \)th preferences only after the first \( i - 1 \) women already rejected him. The fact that the preferences are chosen uniformly allows another simplification. When a man chooses his
ith woman he chooses with “amnesia”, i.e. the realization is done over all \( n \) possibilities, allowing him to choose a woman who already rejected him and propose to her a redundant proposal. In addition, each woman also reveals her preferences in an online manner and she accepts her \( k \)th (non redundant) proposal with probability \( \frac{1}{k} \). It is convenient to look at the running of the algorithm under \( g \)'s strategy as a process in which women keep getting proposals and \( g \) keeps rejecting all of the offers she get. At some point, \( g \) decides to accept one of the offers and the process terminates. The stopping point is selected optimally given the preferences of all the agents. As we will show, the outcome is a stable matching in the original preferences. Combining those observations, we follow [15] and describe the process using the following algorithm. In order to account for the property that \( g \)'s strategy is optimal, we add an oracle to \( g \) that tells her when to stop the rejection process. That is, the oracle is exposed to the realization of all the players’ preferences yet to come. This process is equivalent to the original Gale-Shapley algorithm with \( g \) acting in a strategically optimal way.

\[ A_1, ..., A_n = \text{sets representing the women proposed to so far by men 1 to n.} \]
\[ l = \text{the number of men who have proposed so far.} \]
\[ p = \text{the man who is currently proposing.} \]
\[ h = \text{the woman who is currently being proposed to.} \]
\[ x_1, ..., x_n = \text{men who made the best offer so far to women 1 to n.} \]
\[ x_i = 0 \text{ if the woman } i \text{ has received no offer.} \]
\[ k_1, ..., k_n = \text{the number of proposals received by women 1 to n.} \]

1. Let \( A_j = \emptyset, x_j = 0, \) and \( k_j = 0, \) for \( 1 \leq j \leq n; \) also let \( l = 0. \)
2. If \( l < n, \) increase \( l \) by 1 and let \( p = l. \) Otherwise, if \( g \)'s oracle tells her to stop, the process terminates and this is the final matching. Otherwise, let \( g \) reject \( x_g \) and set \( p = x_g. \)
3. Let \( h \) be a random matching, uniformly chosen between 1 and \( n. \) We say that man \( p \) has proposed to woman \( h. \) If \( h \in A_p \) (i.e., if \( p \)'s proposal is redundant), repeat this step. Otherwise replace \( A_p \) with \( A_p \cup \{h\} \) and go on to step 4.
4. Increase \( k_h \) by one. With probability \( 1 - \frac{1}{k_h}, \) return to step 3 (in this case we say that woman \( h \) rejects the proposal). Otherwise interchange \( p \leftrightarrow x_h \) (that is \( h \) accepts the proposal and her former partner is the next proposer). If the new value of \( p \) is zero, or if \( h = g, \) go back to step 2; otherwise continue with step 3.

**Probabilistic Notations** Some asymptotic notations and probabilistic bounds are noted. We say that an event occurs almost surely (a.s.) or high probability or (whp), if the probability that it doesn’t happen is \( o(1), \) i.e., if the probability of nonoccurrence approaches zero as \( n \) goes to infinity.

**Multiplicative Chernoff Bound.** Suppose \( X_1, ..., X_n \) are independent random variables taking values in \( \{0, 1\}. \) Let \( X \) denote their sum and let \( \mu = \mathbb{E}[X] \) denote the sum’s expected value. Then,

\[
\Pr(X \leq (1 - \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}}, \quad 0 \leq \delta \leq 1.
\]
\[
\Pr(X \geq (1 + \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{8}}, \quad 0 \leq \delta \leq 1.
\]
\[
\Pr(X \geq (1 + \delta)\mu) \leq e^{-\frac{\delta \mu}{3}}, \quad 1 \leq \delta.
\]

### 2.2 Main Results

The main result in this section is that in the above scenario, one strategic agent is expected to dramatically affect the entire outcome of the matching. In particular, the expected woman rank is polylogarithmic in \( n \) compared to an expected rank of \( \frac{n}{\log(n)} \) under truthful regime. Similarly, with a probability that goes to 1 as \( n \) grows, the rank of almost all the women is polylogarithmic in \( n. \)

Duo to the proposing nature of the Gale-Shapley algorithm, any optimal strategy by a single woman \( w \) output the same matching not only for her but to all other women as well.

**Proposition 1.** The process described above is equivalent to \( g \) reporting in a strategically optimal in the Gale-Shapley algorithm.

Another desired property would be that the process yields a stable matching and all women are matched.

**Lemma 1.** The process terminates with a stable matching where all the women are matched.
Proof. Let $M$ be the resulting matching from the process. It is stable in the reported preferences. For any possible pair not including $g$, the stability in the reported preferences implies stability in the true preferences. For any pair which includes $g$ and a man $b$, either $g$ rejected him and got a better match, or $g$ didn’t get an offer from $b$. In the first case this is not a blocking pair due to $g$’s preferences, and in the latter it is not a blocking pair due to $b$’s preferences. Hence, proving that the matching is stable. Since there is a matching where all agents are matched (for example, the women optimal matching), by the lone wolf theorem (see for example [18]), in any stable matching all agents are matched. □

By the work of [20], it is known that $g$’s best stable match is whp of polylogarithmic order. We denote the event where $g$ gets one of her top $a$ preferences by $C_a$.

Lemma 2. For any function $\tau(n)$ that goes to infinity and for any large enough $n$, the following inequalities hold:

\[
P(C_{\tau(n)} \log^{2}(n)) > 1 - \frac{1}{n},
\]

\[
P(C_{\tau(n)} \log(n)) \to 1.
\]

Proof. The first inequality is a corollary of Theorem 6.1 in [20]. The second term is due to the fact that the expected rank of the best stable matching is of order $\log(n)$ and by Markov’s inequality. □

From the previous lemma, it is follows that the probability of the matching process terminating before $g$ gets one of her top $\log^{1+\epsilon}(n)$ preferences goes to zero as $n$ grows. The next lemma shows that the number of proposals made to $g$ conditioned on this event is not very different from the distribution of the proposals without this condition.

Lemma 3. Let $a > 2\log(n)$, and $M$ be the total number of distinct (i.e., non-redundant) proposals made to $g$ before the termination of the algorithm. Then, $P(M = m | C_a) < \frac{2a}{n}$ for large enough $n$ and any $m \in \mathbb{N}$.

Proof. First, we examine $P(M = m)$ without restricting to the event $C_a$. We start by considering a random order of proposers and looking for the first time $g$ got a proposal from a special man $b$. Since the preference list of $g$ is determined online and independently of the proposals she got, the probability that $m$ proposals are needed is exactly $\frac{1}{n}$. Using the union bound, we deduce that $P(M = m) < \frac{a}{n}$ when counting the proposals till $g$ got a proposal from one of her $a$ most desired men. By Bayes’ theorem, and conditioning we are in the event $C_a$, we get:

\[
P(K = m | C_a) = \frac{P(K = m, C_a)}{P(C_a)} \leq \frac{P(K = m)}{P(C_a)} \leq \frac{2a}{n},
\]

where the last inequality holds due to Lemma 2. □

The next step is to show that all women get a similar number of (not necessarily distinct) proposals. This lemma is not directly connected to the termination of the process by $g$ but rather a general statement on the uniform nature of proposals.

Lemma 4. Let $k$ be the total number of proposals made by a man $b$. If $k \geq \frac{mn}{2}$ then with a probability of at least $1 - n \cdot e^{-\frac{m}{16}}$, all of the women get at least $\frac{m}{4}$ proposals.

Proof. First we estimate the probability of an arbitrary woman getting less then $\frac{m}{4}$ proposals. Since in each proposal the woman gets proposed is chosen uniformly and independently from previous proposals, we may use the Chernoff bound with parameters $\mu = \frac{mn}{2}$ and $\delta = \frac{1}{2}$. Thus, the probability that such an event is less than $e^{-\frac{m}{16}}$. By the union bound, the probability of at least one woman not getting enough proposals is bounded by $n \cdot e^{-\frac{m}{16}}$. □

So far, we know the number of proposals distributed approximately in an equal way among the women. But this is true only because we allowed redundant proposals. For distinct proposals, this does not necessarily hold. In the next lemma we show that when conditioning on the event that the process terminates with all men matched, we can get a similar result, albeit a slightly less tight one. This conditioning is reasonable due to lemma 1.

1 Exactly $\frac{1}{2}$ of all possible orders of $n$ proposals will have the special item exactly in location $m$. 

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Lemma 5. With probability greater than \(1 - \frac{2}{n}\), no men proposed to the same woman more than \(20\log(n)\) times.

Proof. Assume some man, \(b\), proposed \(r\) proposals in total and let \(w\) be some woman.

We consider two cases:

\[
B_0 = \{ r > 4n\log(n) \} \\
B_1 = \{ r \leq 4n\log(n), \ w \text{ got at least } 20\log(n) \text{ proposals from } b \}.
\]

We show that these events happen with negligible probabilities. The probability that some woman has got no proposal from \(b\) is \((1 - \frac{1}{n})^r\). For \(r > 4n\log(n)\) this probability is no more than \(\frac{1}{n}\). Because only full matchings are considered, we know that no men reached to the end of his preference list (up to maybe one proposal to least preferred spouse). Therefore the probability for the event \(B_0\) is no more than \(\frac{1}{n}\).

In case \(B_1\), \(b\) offers at most \(4n\log(n)\) proposals. Using the Chernoff bound, with \(\mu = 4\log(n)\), \(\delta = 4\). The probability that more than \(20\log(n)\) proposals were made to the same woman \(w\) is at most

\[
P(B_1) < e^{-4\log(n)} = \frac{1}{n^4}.
\]

By summing over all men and women and using the union bound, we get that with probability of at least \(1 - \frac{2}{n}\), no man proposed to no woman more than \(20\log(n)\) times.

\[\square\]

Corollary 1. If a woman got \(k\) proposals then with probability at least \(1 - \frac{2}{n}\) she got at least \(\frac{k}{20\log(n)}\) distinct proposals.

Lemma 6. Let \(a \in \mathbb{R}\) a constant, \(w\) a women and \(k\) the number of different proposals made to \(w\). Then, with probability at least \(1 - e^{-a}\), \(w\) got matched with a man who ranked among her \(\frac{a}{k}\) most-preferred men.

Proof. Since \(w\)’s preferences are independent of the proposals she got, the probability for her not getting a proposal from any subset of \(\frac{an}{k}\) men out of \(n\) men is \((1 - \frac{a}{k})^k < e^{-a}\).

\[\square\]

Lemma 7. Denote by \(X_i\) the random variable denoting woman \(i\)’s rank, given that she got at least \(k\) distinct proposals. Let \(Y = \frac{1}{n} \sum_i X_i\) be the average rank of the women. Then, \(EY \leq \frac{2n}{k}\) and with probability \(1 - \frac{1}{n}\), \(Y < \frac{2n}{k} + 15\log(n)\).

Proof. We start by estimating \(E X_i\) using the following equality:

\[
E X_i = \sum_{j=0}^{n} P(X_i > j),
\]

letting \(j = \frac{an}{k}\) (and thus \(a = \frac{jk}{n}\)) and, by Lemma 6

\[
\leq \sum_{j=0}^{n} e^{-\frac{jk}{n}} = \sum_{j=0}^{n} \left(e^{-\frac{k}{n}}\right)^j \leq \sum_{j=0}^{\infty} \left(e^{-\frac{k}{n}}\right)^j \leq \frac{1}{1 - e^{-\frac{k}{n}}}
\]

where the last inequality is due to the convergence of the sum of geometric series. This term is bounded in the following way:

\[
\frac{1}{1 - e^{-\frac{k}{n}}} \leq \frac{1}{\frac{k}{n} - \frac{1}{2} \left(\frac{k}{n}\right)^2} \leq \frac{2n}{k}
\]

where the first inequality is due to the approximation \(e^{-x} \leq 1 - x + \frac{x^2}{2}\) and the second is true for \(\frac{k}{n} < 1\), which is indeed our case. By the linearity of the expectation, we got that \(EY \leq \frac{2n}{k}\).

Next we show that with high probability \(Y\) is small. We start by rearranging \(Y\)’s summing order:

\[
Y = \frac{1}{n} \sum_{a=1}^{k} \sum_{\frac{an}{k} < X_i \leq \frac{an}{k}} X_i
\]

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This can easily be bounded by
\[ \leq \frac{1}{n} \sum_{a=1}^{k} \sum_{\frac{an}{k} < X_i \leq \frac{an}{k}} \frac{an}{k} \]

For \( a = 1 \), the number of women with rank smaller than \( \frac{an}{k} \) is of course no more than \( n \), getting:
\[ Y \leq \frac{1}{n} \left( n \cdot \frac{1}{k} \cdot \frac{n}{k} + \sum_{a=2}^{k} \sum_{\frac{an}{k} < X_i \leq \frac{an}{k}} \frac{an}{k} \right) \]

On the other hand, if \( a \) is larger we can show that only small number of women a rank far from \( \frac{an}{k} \).

By using the Chernoff bound, with \( \mu = ne^{-a} \), \( \delta = 4 \) we get that the probability of more than \( n5e^{-a} \) of the women getting a worse rank than \( \frac{an}{k} \) is no greater than \( e^{-\frac{4ne^{-a}}{3}} \).

For \( a = \log(n) - 3\log(\log(n)) \), this probability is less than \( \frac{1}{n^a} \) and we can neglect the event that more than \( 5ne^{-\log(n)-3\log(\log(n))} \) = 15\log(n) women are with rank larger than \( \frac{n(\log(n)-3\log(\log(n)))}{k} \). We assume the rank of any of those women to by \( n \).

For \( 2 \leq a \leq \log(n) \), using the same bound, the number of women with rank \( \frac{an}{k} \) or better is at least \( 1 - 5e^{-a} \).
If there are more than \( 1 - 5e^{-a} \) such women, we count the extra ones as if they are in next segment (i.e. with rank \( \frac{(a+1)n}{k} \) or better), thus only enlarging our estimation of \( Y \). Thus, the number of women in each segment is \( (1 - 5e^{-a})n - (1 - 5e^{-(a-1)})n = 5n(e^{-(a-1)} - e^{-a}) \). Altogether:
\[ \leq \frac{1}{n} \left( n \cdot \frac{n}{k} + 15 \log(n) \cdot n + \sum_{a=2}^{\log(n)-3\log(\log(n))} 5n(e^{-(a-1)} - e^{-a}) \frac{an}{k} \right) \leq \frac{6n}{k} + 15 \log(n) \]

hence, completing the lemma. □

By now, we have a good knowledge of \( g \)'s rank and see that all women get a similar rank with high probability. We are almost ready for the main result. Denote,
\( P_g \) - the number of (distinct) proposals to \( g \).
\( P_{total} \) - the number of (not necessarily distinct) proposals in total.
\( T_m = (P_g = m, P_{total} = s) \).

Given a parameter \( a \), we consider 3 disjoint events:
\( B_0 = \{ T_m | m \leq 64 \log(n) \} \cap C_a \)
\( B_1 = \{ T_m | m > 64 \log(n), s \leq \frac{mn}{2} \} \cap C_a \)
\( B_2 = \{ T_m | m > 64 \log(n), s > \frac{mn}{2} \} \cap C_a \)

and estimate each event probability and what the average rank is when conditioned on each one. The parameter 64 is adjusted to the requirements of Lemma \[\text{Lemma 4}\]. The event that \( C_a \) didn’t occur is handled separately later on.

**Lemma 8.**
\[ P(B_0) < \frac{128a \log(n)}{n} \]
\[ P(B_0) \cdot \mathbb{E}(\text{average rank}|B_0) \leq 128a \log(n) \]

**Proof.** By Lemma \[\text{Lemma 3}\] the probability that exactly \( m \) proposals were made to \( g \) during the process is no more than \( \frac{2a}{n} \). Using the Union Bound we get that the probability for getting less then 64\log(n) proposals to \( g \) is bounded by \( \frac{128a \log(n)}{n} \).

The outcome of the algorithm is a full matching where all women get at least one proposal. Hence, the worst possible average rank is \( n \). Thus, \( P(B_0) \cdot \mathbb{E}(\text{res}|B_0) \leq 128a \log(n) \). □
Lemma 9.

\[ P(B_1) < \frac{1}{n^2}, \]

\[ P(B_1) \cdot \mathbb{E}(\text{average rank}|B_1) \leq \frac{1}{n^2} \]

Proof. Since \( m > 64 \log(n) \), then in particular \( g \) get at least \( 64 \log(n) \) proposals including redundant ones. The probability the out of the \( s \) proposals made in total, at least \( m \) were made to \( g \) is bounded using the Chernoff bound. Set \( \mu = \frac{m}{n} \leq \frac{\tau}{2} \). Then \( P(B_1) < e^{-\frac{m}{2}} < e^{-64\log(n)} = \frac{1}{n^2} \).

The outcome of the algorithm is a full matching where all women get at least one proposal. Hence, the worst possible average rank is \( n \). Thus, \( P(B_1) \cdot \mathbb{E}(\text{res}|B_1) \leq \frac{1}{n^2} \). \( \square \)

The next lemma implies the first part of Theorem 1 and it is later in use when proving the bound the average men-rank. The bound on the expected average women-rank is proven separately.

Lemma 10. Let \( \tau(n) \) be any function that goes to infinity as a function of \( n \). Then, there exist constants \( c,d > 0 \) such that w.h.p the women average rank is smaller than \( c \cdot \tau(n) \log^2(n) \), and any arbitrary women get a match from their top \( d \cdot \tau(n) \log^2(n) \) options.

Proof. First, we restrict the analysis to the case where \( g \) get a matching from her top \( \sqrt{\tau(n)} \log(n) \) men. By Lemma 2, this event happen with probability approaching 1.

From Lemmas 8 and 9 the events \( B_0 \) and \( B_1 \) are negligible and only \( B_2 \) should be considered.

By Lemma 3 and the Union bound, the probability for \( g \) to get less than \( \frac{n}{\tau(n) \log(n)} \) proposals is no greater than \( \frac{n}{\tau(n) \log(n)} \cdot \frac{m}{\sqrt{\tau(n)}} \).

By Lemma 4 with probability at least \( 1 - \frac{1}{n^2} \), each women get at least \( \frac{m}{2} \) proposals.

By Lemma 5 with probability at least \( 1 - \frac{1}{n^2} \), each women get at least \( \frac{m}{80 \log(n)} \) distinct proposals.

By Lemma 7 it implies that with probability \( 1 - \frac{1}{n^2} \), the average rank of women is

\[ \frac{6n}{80 \log(n)} + 15 \log(n) = \frac{6n}{80 \log(n)} + 15 \log(n). \]

Hence bounding the average rank with \( c' \tau(n) \log^2(n) \) for some constant \( c' \) and large enough \( n \). The events in \( B_2 \) not included in the calculations are of order of \( \frac{1}{n^2} \). Even when assuming worst possible match in those cases, this is only adds a constant factor. Thus, getting the requested.

In order for showing almost all women get a good matching, we start with the same arguments as before. Each women get at least \( \frac{n}{80 \tau(n) \log^2(n)} \) distinct proposals. Let \( \tau_0(n) \) be any function that goes to infinity as a function of \( n \). By Lemma 6 the probability that a woman with that amount of proposals will get a rank worse than \( 80 \tau(n) \tau_0(n) \log^2(n) \) is no greater than \( e^{-\tau_0(n)} \) which goes to 0. \( \tau_0(n), \tau(n) \) may be chosen such that \( \tau_0(n) \tau(n) \log^2(n) \leq c' \log^4(n) \), hence finishing the proof. \( \square \)

In 20 it is shown that with probability that goes to 1 any stable matching with an average women-rank of order of \( k \) has an average men-rank which is of order of \( \frac{k}{c} \). Thus, the first part of Lemma 10 gives a bound on the average men-rank.

Corollary 2. For any \( \epsilon > 0 \), w.h.p the average men-rank is of order of \( \tau \left( \frac{n}{\log^{2+\epsilon}(n)} \right) \).

In the last part of this section, we bound the expected woman rank.

Lemma 11. For \( a > 2 \log(n) \), \( \mathbb{E}(\text{average rank}|B_2) < ca \log^2(n) \) for some \( c > 0 \).

Proof. Let \( M \) be the random variable counting the number of distinct proposals made to \( g \). For any realization \( m \) of \( M \), By Lemma 4 with probability at least \( 1 - \frac{1}{n^2} \), all the women get at least \( \frac{m}{2} \) proposals.

By Lemma 5 with probability at least \( 1 - \frac{2}{n^2} \), all of the women get at least \( \frac{m}{80 \log(n)} \) distinct proposals.

For \( a > 2 \log(n) \), Lemma 3 holds.
By Lemma 7 this implies that with probability $1 - \frac{1}{n^2}$, the expected women rank is

$$\leq \frac{1}{n} \left( 15n \log(n) + \sum_{m=64 \log(n)}^{n} \left( \frac{2a}{n} \cdot \left( \frac{6n}{80 \log(n)} \right) \right) \right)$$

$$\leq ca \log^2(n)$$

for some constant $c$ and large enough $n$. 

At this point, it can be deduced that in expected women rank is $O(\log^4(n))$ using $a = 7 \log^2(n)$ and verifying that all “bad” scenarios happen with probability small enough.

Proof. Let $X$ be the random variable for the average women rank. The expected rank is given by

$$P(B_0)E[X|B_0] + P(B_1)E[X|B_1] + P(B_2)E[X|B_2] + P(C_a^c)E[X|C_a^c]$$

for any $a > 2 \log(n)$. Setting $a = 7 \log^2(n)$ and using Lemma 2,

$$\leq 1000 \log^3(n) + \frac{1}{n^2} + 1 \cdot 7c \log^4(n) + \frac{1}{n} \cdot n$$

$$= O(\log^4(n))$$

We conjecture that the actual expected rank is of order $O(\log^2(n))$ and we examine that conjecture in the next parts.

2.3 The Set of Stable Husbands

An alternative point of view for the effect of a single strategic woman on the entire market is given via observation on the set of stable husbands for each woman. A man $m$ is called a stable husband of $w$ if there is a stable matching in which they are matched together.

By the optimality of $g$’s strategy, $g$ got an proposal from her best stable husband $b_0$. Due to the lattice structure of the set of stable matchings, it is known that she is his worst stable wife (e.g. [27]). Thus, we know that at some point in the process $b_0$ proposed to his 2nd worst woman, $g_0$, and eventually got rejected. Since all the women got married eventually, we know that $g_0$ ends up with a proposal from her best stable husband. In the same manner, this event starts a cycle of women all guaranteed to be matched with their best stable husband at the end of the process.

Assume we could reason that the size of this cycle acts like a size of a cycle in a random permutation. Then, with constant probability, at least fraction of the women are matched with their best stable husband. In addition, as the experimental results shows, we conjecture that expected number of women getting their best stable matching is $\frac{n}{2}$.

$g$ received a proposal not only from her second best stable matching but rather from all of her stable matching. Let $g$’s 2nd stable husband $b_1$. If we could reason that she is his second worst stable wife, we would have got a new cycle of women guaranteed to be matched with their second best stable possible husband. Sadly this may not be the case. It will be interesting to see if some close enough results for some markets. Some similar observation on the “median matching” seem to be relevant (see [27]).

It seems that it would be possible to show that a fraction of size $2^{-k}$ of the women is expected to be matched up with their $k$th stable husband. Second, it is known by [20] that the expected rank for a woman who matched with her best stable husband is $O(\log(n))$. Looking at the process described in the previous part, observe that each temporally best proposal in the rejecting part of the process (and thus a stable husband) is randomly located in the proposed woman’s preference list. Hence, the rank being matched with the $k$th stable husband is expected to be twice as good compared to when being matched with the $(k + 1)$th stable husband. Giving an expected rank of $2^{k-1} \log(n)$ to a woman who matched with her $k$th stable husband. Notice that not all women has many stable husbands (see [21]) and it is not clear how to handle this in a rigorous way. Combining these two observations rigorously hopefully yields an improved bound of order $\frac{1}{2} \log^2(n)$ on the expected rank for women.
2.4 Unbalanced Market Perspective

An unbalanced market of $n$ men and $n-1$ women can be described as a balanced market of size $2n$ in which one woman rejects all of the proposals made to her. In this case, the men utility from being matched with her is irreverent and may be ordered arbitrarily. In this market, still assuming uniform and independent preferences, the average rank of the women is order $O(\log(n))$ even in the women-pessimal matching $[2]$. On the other hand, a truthful balanced market can be viewed as a case in which this one woman has decided not to trim her preference list at all. In this case, the average women-rank is of order $O(\frac{n}{\log(n)})$ in the woman-pessimal match.

Adding the results from our work, we get that a strategic woman can affect this range of ranks in an almost continuous way. To see why it is true, observe that as long as all the men are matched in the outcome, all of the lemmas in the previous section hold by choosing a suitable set $C_a$. An interesting corollary is that if $g$’s list is of length of order $\sqrt{n}$ and all the other lists are long enough we get that the women’s rank is around $\sqrt{n}$ up to a logarithmic factor. Due to the hyperbola matching rule, this match catches some of the properties of fairness between the sexes.

Another aspect is in the case of lists which are ordered uniformly but might have different lengths. This scenario can be described as a market with full preferences where each woman decide how she trims her own list. In this case, the expected rank is asymptotically determined by the women with the shortest list. By using different truncations, the agents may force any possible stable matching (see Theorem 4 in [17]). It should be noted that when there are many strategic agents there are more profitable actions that a coalition of agents may achieve.

2.5 Experimental Results

The theoretical bounds from the previous part are quite impressive in asymptomatic notions, but in the practical range of this problem, they are not for much use. For a better understanding of the behavior in typical market sizes, we simulated different markets. We tested our settings for market sizes for $n$ between 100 to 10,000 in leaps of 20. For any market size, the mean of average ranks over 100 iterations was calculated.

Figure 1 shows the mean of the average ranks for men and women, in the truthful and the strategic scenarios. Figure 2 zoom in on the mean of the average ranks for women in the strategic scenario.

For the same settings, we counted the number of women who got their best stable husband in the strategic scenario. As it seems, in expectation half of the women get their best stable husband when another woman act strategically. For comparison, we also count the number of women who got their worst stable husband and the women who got either of them (in order to take into account women with only one stable husbands (see [21]). The results are shown in Figure 3. The simulation gives some justification to the intuition described in section 2.3.

3 Discussion and Future Work

3.1 The Actual Expected Rank?

We’ve shown that the expected women rank is $O(\log^4(n))$. A logarithmic factor was added since the number of distinct proposals to an arbitrary woman was bounded by a fraction of order $O(\frac{1}{\log(n)})$ of the distinct proposals made to $g$. In fact, in most cases, we can show that those two are close up to a constant multiplicative factor. Showing this rigorously will prove that the expected rank is of an order of $O(\log^3(n))$. It will also help to conclude that the expected rank is $\Omega(\log^2(n))$. Some more work is needed in that direction. An alternative direction is to formalize the observations stated in section 2.3.

3.2 Bayesian Information

In the settings of this work, $g$ assumed to has a full information on the market and thus knows how to choose an optimal strategy. This assumption was later relaxed to an access to an oracle who hints when to terminate a running process. Most of the analysis was done under expectation and high probability assumptions and had not used directly $g$’s knowledge of the other preferences. Although it is tempting to deduce that this implies exactly the same results when $g$ knows only the other lists distributions, it is not the case. For example, assume $g$ has
an extremely high utility for not staying single and hardly distinguished between all possible spouses. In that case, it is likely that $g$’s optimal strategy is just to report truthfully and thus guarantees to be match. But now, $g$ has no effect on the rest of the market and the expected rank is unchanged. Note that although $g$ rank isn’t monotone in her truncation and she needs to be careful not to trim her list too much, the average rank of all the other women is indeed monotone in $g$’s decision and they may only benefit from $g$ being too picky.

$g$’s optimal strategy given Bayesian information is discussed in [4, 24]. When the uniform independent assumption holds, truncation strategy is still optimal when maximizing expected utility, and the exact point of truncation is determinant by the agent utilities. Furthermore, [4] shows that reduction in the risk aversion causes the length of $g$’s list after truncation to be negligible compared to the number of agents. If we assume truncation in $\Theta(\log^2(n))$ we get similar results to the full knowledge model: either the strategy was successful and the expected rank is $O(\log^4(n))$. Or, the strategic women is unmatched and we are in the $n$-men $(n-1)$-women case. In this case, it should be reasonable to say the other women get rank $\log(n)$ due to [2]. Some more rigorous analysis is needed since the fact that $g$’s rank when matched with her best stable husband is worst than the truncation, might imply something on the preferences lists of the other agents.

**Other Distributions** The independent uniform distribution on the preferences is widely common assumption in theoretical research, although in real life matching markets it is not always justified to assume such distribution. The following distribution (presented by [11]) presents some of the properties that were used while adding some correlation between the preferences. Men still choose the preferences online as in the process but instead of choosing uniformly between the women, they use an arbitrary distribution $D^n$ over the set of women, thus making some of the women more popular than others while keeping variety in the men’s preferences. It seems the result of this work may be extended to this distribution with minor adjustments assuming $D^n$ is “nice”. The main constrain is that no women is much more popular than others, i.e. constrains on $\max_{x,y \in [n]} \frac{D^n(x)}{D^n(y)}$ or $\mathbb{E}_{x,y \in [n]} \frac{D^n(x)}{D^n(y)}$ will hopefully be sufficient for getting similar results to those received in the uniform case.
Fig. 2. Mean average rank for women with strategic agent compared to log(n) and log^2(n)

3.3 Many Strategic Women

As seen in this work, the fact that one agent is acting strategically effects in a non trivial way on the quality of the matching from the other women’s perspective. This raises some interesting questions regarding a scenario in which number of women act strategically.

**Full Information** - How many women need to act strategically for forcing the women-optimal match? The intuition from section 2.3 suggest this number should be resembling the number of cycles in a random permutation, i.e. order of log(n).

**Bayesian Information** - Almost all of the women benefit from lying given that the other women are truthful. On the other hand, once at least one woman truncates her list, many of the other women already promised themselves a better match and will not gain much from lying (their probability to stay unmatched is indifferent of the truncation of the first woman). It is interesting to examine the dynamics of the strategies in such environments (with some suitable distributions of the preferences and cardinality over the matchings). [4] prove the existence of equilibrium in truncation strategies under incomplete information. Some considerable directions for further research:

- In many real life matchings, the players report their preferences to a centralized mechanism who runs the algorithm for them. Assume women report to the mechanism in a known in advance order. How does it affect the strategies of the agents?
- Assume men may truncate their list as well. What is the set of equilibriums and how a centralized mechanism effects it?

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Fig. 3. Number of women who got best/worst/(best or worst) stable match

References

1. Brian Aldershof, Olivia M Carducci, and David C Lorenc. Refined inequalities for stable marriage. Constraints, 4(3):281–292, 1999.
2. Itai Ashlagi, Yash Kanoria, and Jacob D Leshno. Unbalanced random matching markets: The stark effect of competition. Journal of Political Economy, 125(1):69–98, 2017.
3. Itai Ashlagi and Flip Klijn. Manipulability in matching markets: conflict and coincidence of interests. Social Choice and Welfare, 39(1):23–33, 2012.
4. Peter Coles and Ran Shorrer. Optimal truncation in matching markets. Games and Economic Behavior, 87:591–615, 2014.
5. Lester E Dubins and David A Freedman. Machiavelli and the gale-shapley algorithm. The American Mathematical Monthly, 88(7):485–494, 1981.
6. Piotr Dworczak. Deferred acceptance with compensation chains. In Proceedings of the 2016 ACM Conference on Economics and Computation, pages 65–66. ACM, 2016.
7. David Gale and Lloyd S Shapley. College admissions and the stability of marriage. The American Mathematical Monthly, 69(1):9–15, 1962.
8. David Gale and Marilda Sotomayor. Ms. machiavelli and the stable matching problem. The American Mathematical Monthly, 92(4):261–268, 1985.
9. Yannai A Gonczarowski and Ehud Friedgut. Sisterhood in the gale-shapley matching algorithm. the electronic journal of combinatorics, 20(2):P12, 2013.
10. Dan Gusfield. SIAM Journal on Computing, 16(1):111–128, 1987.
11. Nicole Immorlica and Mohammad Mahdian. Marriage, honesty, and stability. In Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms, pages 53–62. Society for Industrial and Applied Mathematics, 2005.
12. Robert W Irving and Paul Leather. The complexity of counting stable marriages. SIAM Journal on Computing, 15(3):655–667, 1986.
13. Robert W Irving, Paul Leather, and Dan Gusfield. An efficient algorithm for the optimal stable marriage. Journal of the ACM (JACM), 34(3):532–543, 1987.
14. Bettina Klaus and Flip Klijn. Procedurally fair and stable matching. Economic Theory, 27(2):431–447, 2006.
15. Donald E Knuth, Rajeev Motwani, and Boris Pittel. Stable husbands. Random Structures & Algorithms, 1(1):1–14, 1990.
16. Jinpeng Ma. The singleton core in the college admissions problem and its application to the national resident matching program (nrmp). Games and Economic Behavior, 60(1):150–164, 2010.
17. David G McVitie and Leslie B Wilson. The stable marriage problem. Communications of the ACM, 14(7):486–490, 1971.
18. DG McVitie and Leslie B Wilson. Stable marriage assignment for unequal sets. BIT Numerical Mathematics, 10(3):295–309, 1970.
19. Boris Pittel. The average number of stable matchings. SIAM Journal on Discrete Mathematics, 2(4):530–549, 1989.
20. Boris Pittel. On likely solutions of a stable marriage problem. The Annals of Applied Probability, pages 358–401, 1992.
21. Boris Pittel, Larry Shepp, and Eugene Veklerov. On the number of fixed pairs in a random instance of the stable marriage problem. SIAM Journal on Discrete Mathematics, 21(4):947–958, 2007.
22. Alvin E Roth. The economics of matching: Stability and incentives. Mathematics of operations research, 7(4):617–628, 1982.
23. Alvin E. Roth. On the allocation of residents to rural hospitals: A general property of two-sided matching markets. *Econometrica*, 54(2):425–427, 1986.
24. Alvin E Roth and Uriel G Rothblum. Truncation strategies in matching markets in search of advice for participants. *Econometrica*, 67(1):21–43, 1999.
25. Alvin E Roth, Tayfun Sönmez, and M Utku Ünver. Kidney exchange. *The Quarterly Journal of Economics*, 119(2):457–488, 2004.
26. Alvin E. Roth and John H. Vande Vate. Random paths to stability in two-sided matching. *Econometrica*, 58(6):1475–1480, 1990.
27. Chung-Piaw Teo and Jay Sethuraman. The geometry of fractional stable matchings and its applications. *Mathematics of Operations Research*, 23(4):874–891, 1998.
28. Chung-Piaw Teo, Jay Sethuraman, and Wee-Peng Tan. Gale-Shapley stable marriage problem revisited: Strategic issues and applications. *Management Science*, 47(9):1252–1267, 2001.
29. John H Vande Vate. Linear programming brings marital bliss. *Operations Research Letters*, 8(3):147–153, 1989.