On the Riesz Summability of the Eigenfunction Expansions on a Closed Domain

Abdumalik Rakhimov\textsuperscript{a}, Erkin Saydamatov\textsuperscript{b}, Shuhrat Sheraliev\textsuperscript{b}

\textsuperscript{a}Department of Science in Engineering, IIUM
\textsuperscript{b}Lomonosov named MSU Tashkent branch

Abstract. In the present paper we study the Riesz summability on a closed domain expansions in eigenfunctions of the first boundary value problem for the Schrodinger operator with a weak singular potential.

1. Introduction and preliminaries

Let $\Omega$ be a finite two dimensional domain with a smooth boundary $\partial \Omega$. By $\overline{\Omega}$ denote the closer of the domain $\Omega$. Let $u_n(x)$ be an eigenfunction of the first boundary value problem corresponding to the eigenvalue $\lambda_n$:

$$
\Delta u(x) + (\lambda + q(x))u(x) = 0, \quad x \in \Omega 
$$

$$
u |_{\partial \Omega} = 0, \quad u \in C^1(\overline{\Omega}) \cap W^2_2(\Omega),
$$

where a potential function $q(x)$ is from the Sobolev space $W^1_2(\Omega)$ and may have singularities at one point of the domain $\Omega$ (or in finite numbers of the points) and enough smooth out of this point.

A system of eigenfunctions $\{u_n(x)\}$ is a complete orthogonal in $L^2(\Omega)$ corresponding to the sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$. For the $\mu_0 \to \infty$ the following estimation for the eigenfunctions $u_n(x)$ is valid [13]

$$
\sum_{|\sqrt{\mu} - \mu_0| \leq 1} u_n^2(x) = O(\mu_0 \ln^2 \mu_0)
$$

uniformly on the closed domain $\overline{\Omega}$.

Let $f \in L^2(\Omega)$. Recall the Riesz means of order $s$ of the partial sums of the Fourier series

$$
E^s_{\lambda} f(x) = \sum_{\lambda_n < A} \left(1 - \frac{\lambda_n}{A}\right)^s f_n u_n(x),
$$

2010 Mathematics Subject Classification. Primary 42B10 ; Secondary 42A24
Keywords. Eigenfunction expansions, summability, Riesz means
Received: 08 August 2019; Accepted: 04 October 2019
Communicated by Ivana Djolović
First author would like to thank IIUM and Lomonosov named MSU Tashkent branch for the support.
Email addresses: abdumalik0@iium.edu.my (Abdumalik Rakhimov), saydamatov@mail.ru (Erkin Saydamatov)
here \( \lambda > 0 \), \( f_n = (f, u_n) \) are the Fourier coefficients of the function \( f \) with respect to the system \( \{u_n(x)\} \). Note that if \( s = 0 \), then (4) is just the partial sum of the Fourier series of the function \( f \).

Further in this paper we consider convergence of \( E^s_\lambda f(x) \) in the domains that are separated from the singular point(s) of \( q(x) \). Also we will suppose that the support of the function \( f \) is not containing the singular point(s).

When \( q(x) = 0 \), then precise conditions of uniform convergence of the Riesz means (4) of the functions from the Sobolev spaces \( W^s_p(\Omega) \) on compact subsets of the domain \( \Omega \) as follows [7]

\[
\alpha + s \geq \frac{1}{2}, \quad ap > 2, \quad s \geq 0, \quad p \geq 1.
\]

The conditions (5) are sufficient for the functions from the Nikol’skii spaces \( H^s_p(\Omega) \) [4]. If the second condition in (5) is replaced by \( ap = 2 \), then it is necessary additional to assume that the function \( f \) is continuous (see [3]) and the inequalities (5) will be as follows:

\[
\alpha + s > \frac{1}{2}, \quad ap = 2, \quad s \geq 0, \quad p \geq 1.
\]

In this case the first condition \( \alpha + s > \frac{1}{2} \) in (6) is also precise meaning that if \( x_0 \in \Omega \) and

\[
\alpha + s = \frac{1}{2}, \quad ap = 2, \quad s \geq 0, \quad p \geq 1,
\]

then there exists a continuous in \( \Omega \) function \( f \in \hat{W}^{ap}_p(\Omega) \) and such that [3]

\[
\lim_{\lambda \to \infty} E^s_\lambda f(x_0) = +\infty.
\]

2. Main theorem

Let \( f \) is a function with compact support from the space \( W^\frac{1}{2}_p(\Omega), \quad p > 4 \), such that the series

\[
\sum_{n=1}^{\infty} \lambda_n^\frac{3}{2} (\ln \lambda_n)^{2+s} f_n^2
\]

converges. Then its expansion via eigenfunctions associated with the elliptic operators of second order for the first boundary value problem converges uniformly on the closed domain \( \overline{\Omega} \) [8].

For the continuous functions \( f(x) \) from the Nikol’skii space \( H^s_p(\Omega) \) with compact support in the domain \( \Omega \) (where \( H^s_p(\Omega) \) is the closure of the space \( C^s_0(\Omega) \) with the respect to the norm of the space \( H^s_p(\Omega) \)) the conditions for the uniformly on the closed domain \( \overline{\Omega} \) convergence of the Riesz means (4) are [5]

\[
\alpha + s > \frac{1}{2}, \quad ap \geq 2, \quad s \geq 0, \quad p \geq 1.
\]

Note that if \( ap = 2 \) the condition \( \alpha + s > (1/2) \) is precise. In the case \( ap > 2 \) a problem of the sharpness of the inequality \( \alpha + s > (1/2) \) is still open.

Recall \( ||f||_{L_p(\Omega)} \) the norm in the Lebesgue spaces with mixed norm [20] \( L_{pq}(\Omega) \) and define \( H^s_{pq} \) the Banach space of all measurable functions \( f \) with respect to the norm

\[
||f||_{H^s_{pq}(\Omega)} = ||f||_{L_p(\Omega)} + \sum_{|\ell| = \ell} \sup_{|z| = r} ||z^{-s} \Delta^2 f(z)||_{L_{pq}(\Omega)},
\]

\( \ell \in \mathbb{N} \).
where \( h > 0 \) such that the set \( \Omega_h = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > h \} \) is proper subset of the domain \( \Omega \), \( \alpha = \ell + \kappa \), \( \ell \) is a non negative integer, \( 0 < \kappa \leq 1 \), \( p, q \geq 1 \), \( k = (k_1, k_2) \) multi-index, \( |k| = k_1 + k_2 \), and \( \partial^y f \) denotes the weak derivative

\[
\partial^y f(y) = \frac{\partial^{\ell+2\kappa} f(y)}{\partial y_1^{\ell} \partial y_2^{2\kappa}},
\]

and the symbol \( \Delta^2 \partial^y f(y) \) denotes the second difference of the function \( \partial^y f(y) : \)

\[
\Delta^2 \partial^y f(y) = \partial^y f(y + z) - 2 \partial^y f(y) + \partial^y f(y).
\]

By \( \bar{H}^p_{pq}(\Omega) \) denote the closure of the space \( C_0^\infty (\Omega) \) with respect to the norm of the space \( H^p_{pq}(\Omega) \).

In [13] it is established te following sufficient conditions for the uniform convergence on the closed domain of the eigenfunction expansions associated with the first boundary value problems of the functions from the space \( \bar{H}^p_{pq}(\Omega) \)

\[
\alpha > \frac{1}{2} - s, \quad \alpha = \frac{1}{q} + \frac{1}{p}, \quad 2 \leq p < q.
\] (10)

The main result of the present paper is to prove the following

**Theorem 2.1.** Let the conditions (10) are valid and let \( f \) be a continuous function with compact support in the domain \( \Omega \) belonging to the space \( \bar{H}^p_{pq}(\Omega) \). Then uniformly on \( \bar{\Omega} \)

\[
\lim_{\lambda \to \infty} E^r_{\lambda} f(x) = f(x).
\]

As can be seen in this paper we study convergence in the spaces of smooth functions. One can study this problem in the spaces of singular functions. Some results for the spectral expansions of the singular distributions can be found in [9], [10], [11], [15], [16] and [12]. These questions for the expansions associated with the Laplace Beltrami operator on the sphere studied in [1], [6], [17].

### 3. Proof of the theorem

First we obtain estimation of the Riesz means of the spectral expansions \( E^r_{\lambda} f(x) \). Let’s choose a small positive number \( h \) such that a set \( \Omega_h \) is a non-empty proper subset of \( \Omega \), and distance \( r = |x - y| \) between \( x \in \Omega_h \) and \( y \in \partial \Omega \) less than \( h \). Let \( y \in \bar{\Omega} \) and consider the following function of the distance \( r \) :

\[
V(r, \lambda) = \begin{cases} \frac{\Gamma((s+1)\lambda)J_0(t \sqrt{\lambda}))}{\pi^{1/2} t^{(s+1)}} & r \leq R, \\ 0, & r > R, \end{cases}
\] (11)

where \( 0 < R < h/4 \) and \( J_0(t) \) the Bessel function of the first kind of order \( \nu \).

Let

\[
S_{\lambda}(\phi) = \int_0^{2\pi} \phi(x_1 + \tau \cos \theta, x_2 + \tau \sin \theta) d\theta.
\]

We use the following mean value formula for the eigenfunctions \( u_n(x) \) [7]:

\[
S_{\lambda}(u_n) = 2\pi J_0(r \sqrt{\lambda_0}) u_n(x) + \frac{\pi}{2} \int_0^r \left[ J_0(t \sqrt{\lambda_0}) Y_0(r \sqrt{\lambda_0}) - Y_0(t \sqrt{\lambda_0}) J_0(r \sqrt{\lambda_0}) \right] S_{\lambda}(\sqrt{\lambda_0}) J_0(\sqrt{\lambda_0}) dt,
\]

where \( x = (x_1, x_2) \in \Omega_h \).

Using this mean value formula we get the following representation for the Fourier coefficients of the function \( V(r, \lambda) \)

\[
v_0^r(x) = 2^{s+1} \Gamma(s+1)\sqrt{\lambda_0} u_n(x) \int_0^r J_{1+\kappa}(\sqrt{\lambda_0}) J_0(\sqrt{\lambda_0}) r^{-\kappa} dr +
\]
Lemma 3.1. For any function \( f \in L_2(\Omega) \) the integral

\[
\int_{\Omega} f(x)V(x, y, \lambda)dx
\]

is continuous in \( y \) on the closed domain \( \overline{\Omega} \).

Proof. From estimate (3) it follows that for any positive number \( \varepsilon \) uniformly with respect to \( y \in \overline{\Omega} \)

\[
\sum_{\lambda_n < \lambda} u_n^2(y)\lambda_n^{n-1} = O(\lambda^{-\varepsilon} \ln^2 \lambda),
\]

\[
\sum_{\lambda_n > \lambda} u_n^2(y)\lambda_n^{n-1} = O(\lambda^{1-\varepsilon} \ln^2 \lambda).
\]

This can be achieved if the sum in the left-hand side of the second formula in (16) represent as

\[
\sum_{\lambda_n > \lambda} u_n^2(y)\lambda_n^{n-1} \leq \sum_{k=0}^{\infty} \sum_{\sqrt{\lambda + k} < \sqrt{\lambda_n} < \sqrt{\lambda + k + 1}} u_n^2(y)\lambda_n^{n-1} \leq \sum_{k=0}^{\infty} (\sqrt{\lambda} + k)^{-2\varepsilon - 2} \sum_{\sqrt{\lambda + k} < \sqrt{\lambda_n} < \sqrt{\lambda + k + 1}} u_n^2(y).
\]

Similarly from the estimate of integral (15) (see [19])

\[
|I_0(\lambda, \lambda_n)| \leq \frac{c}{1 + |\sqrt{\lambda} - \sqrt{\lambda_n}|}
\]
(where \( c > 0 \) is independent of \( \lambda \) and \( n \)) and estimate (16) with the application of Cauchy-Schwarz inequality it follows that series

\[
\sum_{n=1}^{\infty} f_n u_n(y) \lambda_n^{-\frac{1}{2}} I_0(\lambda, \lambda_n)
\]

converges uniformly on the closed domain \( \overline{\Omega} \). \( \square \)

We will multiply equality (14) by \( u_n(y) \) then take summation with the respect \( n, \) get the following representation of the function (11) in the sense of \( L_2 \) space with the variable \( y \)

\[
V(r, \lambda) = \Theta'(x, y, \lambda) - 2^s \Gamma(s + 1) \lambda^{\frac{1-2s}{2}} \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda} I_0(\lambda, \lambda_n) u_n(x) u_n(y) + 2^{s-2} \Gamma(s + 1) \lambda^{\frac{1-2s}{2}} \int_0^R r^{-s} f_{1+s}(\sqrt{\lambda} r) \sum_{n=1}^{\infty} u_n(x) \times
\]

\[
\int_0^1 [J_0(t \sqrt{\lambda_n}) Y_0(r \sqrt{\lambda_n}) - Y_0(t \sqrt{\lambda_n}) J_0(r \sqrt{\lambda_n})] S_i(q u_n) \mu dt dr
\]

where

\[
\Theta'(x, y, \lambda) = \sum_{\lambda_i < \lambda} (1 - \frac{\lambda_i}{\lambda}) f_n(x) u_n(y)
\]

is known as the Riesz means of the spectral function [2].

Let \( f \in L_2(\Omega) \) and \( \text{supp} f \subset \Omega_0. \) Then for \( y \in \overline{\Omega} \) the Riesz means (4) of the partial sums of the Fourier series of the function \( f \) via eigenfunctions \( u_n(x) \) can be written as follows by multiplication both side of (18) and then integration with the respect to \( x \)

\[
E_\lambda^r f(y) = \int_{\Omega} f(x) V(r, \lambda) dx + 2^s \Gamma(s + 1) \lambda^{\frac{1-2s}{2}} \sum_{n=1}^{\infty} f_n u_n(y) \lambda_n^{\frac{1}{2}} I_0(\lambda, \lambda_n) +
\]

\[
+ 2^{s-2} \Gamma(s + 1) \lambda^{\frac{1-2s}{2}} \int_0^R r^{-s} f_{1+s}(\sqrt{\lambda} r) \sum_{n=1}^{\infty} f_n \times
\]

\[
\int_0^1 [J_0(t \sqrt{\lambda_n}) Y_0(r \sqrt{\lambda_n}) - Y_0(t \sqrt{\lambda_n}) J_0(r \sqrt{\lambda_n})] S_i(q u_n) \mu dt dr
\]

(19)

In order to estimate (19) we will study some properties of the eigenfunctions on the closed domain.

Note that from (3), (16) and (17) it follows that uniformly with the respect to \( y \in \overline{\Omega} \) the following estimation is valid [14]

\[
\sum_{n=1}^{\infty} u_n^2(y) \lambda_n^{-\frac{1}{2}} I_0(\lambda, \lambda_n) \leq C \ln^2 \lambda,
\]

(20)

where a constant \( C > 0 \) is independent of \( \lambda \) and \( y. \)

Further we always suppose that \( \alpha \) satisfying (10).

**Lemma 3.2.** Let \( f \in \dot{H}^2_\alpha(\Omega). \) Then uniformly with respect to \( y \in \overline{\Omega} \)

\[
\sum_{n=1}^{\infty} f_n u_n(y) \lambda_n^{\frac{1}{2}} I_0(\lambda, \lambda_n) = O\left(\frac{\ln \lambda}{\lambda^2}\right) ||f||_{\dot{H}^2_\alpha}.
\]

(21)

**Proof.** Note, that for any \( \varepsilon > 0 \) [3]

\[
\sum_{\lambda < \lambda_n < 4\lambda} f_n^2 \lambda_n^\alpha \leq c ||f||_{\dot{H}^2_\alpha}^2,
\]

(22)
where \( c > 0 \) is independent of \( f \) and \( \lambda \).

Then using (17) and (20) obtain

\[
\sum_{\frac{1}{4} < \lambda_n < \frac{3}{4}} f_n u_n(y) \lambda_n^{-\frac{1}{2}} I_0(\lambda, \lambda_n) \leq c \frac{\ln \lambda}{\lambda^2} \|f\|_{H^s},
\]

(23)

where \( c > 0 \) is independent of \( f \) and \( \lambda \).

If \( \lambda_n > \frac{9}{4} \) we second estimate in (16) and obtain the estimate for such \( n \).

The proof for the numbers \( n \) for which \( \frac{1}{4} < \lambda_n < \frac{9}{4} \) is as follows. Let \( k \) be the least natural number satisfying \( 2^{k+1} \geq \sqrt{\lambda} \). Then using (3), (17) and (22) we get

\[
\left| \sum_{\frac{1}{4} < \lambda_n < \frac{3}{4}} f_n u_n(y) \lambda_n^{-\frac{1}{2}} I_0(\lambda, \lambda_n) \right| \leq \left( \sum_{m=1}^{k} \sum_{2^{m-1} \leq \sqrt{\lambda_n} < 2^m} u_n^2(y) \lambda_n^{-\alpha} [I_0(\lambda, \lambda_n)]^2 \right)^{\frac{1}{2}} \left( \sum_{\frac{1}{4} < \lambda_n < \frac{3}{4}} f_n^2 \lambda_n^{2\alpha} \right)^{\frac{1}{2}} \leq c \lambda^{\frac{s}{2}} \ln \lambda \|f\|_{H^s},
\]

where \( c > 0 \) is independent from \( f \) and \( \lambda \). \( \square \)

From this lemma we get the following estimate for the second term in the expression (19) of the Riesz means \( E_{\alpha} f(y) \)

\[
2^s \Gamma(s+1) \lambda^{\frac{s}{2}} \sum_{n=1}^{\infty} f_n u_n(y) \lambda_n^{-\frac{1}{2}} I_0(\lambda, \lambda_n) \leq c \lambda^{\frac{s}{2}} \ln \lambda \|f\|_{H^s}
\]

(24)

where \( \varepsilon = s + \alpha - \frac{1}{2} \) and from (10) it follows that this number is positive. Third term in (19) can be estimated similarly to (24) using appropriate estimates of the Bessel functions (see for example in [19]).

Thus, for the function \( f \in H^s_\varepsilon(\Omega) \) with \( \text{supp} f \subset \Omega_k \) uniformly with respect to \( y \in \Omega \) we get

\[
E_{\alpha} f(y) = \int_{\Omega_k} f(x) V(r, \lambda) dx + O(\frac{\ln \lambda}{\lambda^{\alpha}}) \|f\|_{H^s_\varepsilon},
\]

(25)

Then choosing \( R < \frac{3}{2} \) from (11) obtain

\[
E_{\alpha} f(y) = c_s \lambda^{\frac{s}{2}} \int_0^R r^{-\alpha} J_{1+s}(r \sqrt{\lambda}) S_c(f) dr + O(\frac{\ln \lambda}{\lambda^{\alpha}}) \|f\|_{H^s_\varepsilon},
\]

(26)

where \( c_s = \frac{\Gamma(s+1)2^{-1}}{\pi} \).

Let \( f \) be a continuous function with compact support in the domain \( \Omega \) belonging to the space \( H^s_{p,q}(\Omega) \) and (10) holds.

Note that because \( 2 \leq p < q \) from the embedding \( H^s_{p,q} \rightarrow H^s_{p,\varepsilon} \) we write (26) as follows

\[
E_{\alpha} f(y) = c_s \lambda^{\frac{s}{2}} \int_0^R r^{-\alpha} J_{1+s}(r \sqrt{\lambda}) S_c(f) dr + O(\frac{\ln \lambda}{\lambda^{\alpha}}) \|f\|_{H^s_\varepsilon},
\]

(27)

Let \( \beta = \frac{1}{2} - s \). Then for the integral in (27) we have [18]

\[
\int_0^R r^{-\alpha} J_{1+s}(r \sqrt{\lambda}) S_c(f) dr \leq C \left( \|f\|_{H^s_{p,\varepsilon}} + \|f\|_{H^s_\varepsilon} \right) \lambda^{\frac{s}{2}-\frac{1}{2}},
\]
where \( C > 0 \) is independent of \( f \) and \( \lambda \).

It is clear that \( \beta < \alpha \) and that is why from \( H^\alpha_{p,q} \to H^\beta_{p,q} \) we obtain the following estimate for the spectral expansions

\[
E^\lambda_\alpha f(y) \leq C \left( \Vert f \Vert_{H^\alpha_{p,q}} + \Vert f \Vert_{C^\Omega} \right).
\]

Note that the Fourier series via eigenfunction expansions of a function \( f \in C^\infty_0(\Omega) \) converges uniformly and absolutely in the closed domain \( \overline{\Omega} \). Due to the density of \( C^\infty_0(\Omega) \) in the space \( \dot{H}^\alpha_{p,q}(\Omega) \), the statement of Theorem 2.1 follows from the inequality (28).

References

[1] AFN bin Rasedee, A. Rakhimov, A Akhmedov, Summation by Riesz Means of the Fourier-Laplace Series, J. FILOMAT, 32(3)(2018) 837-846
[2] Sh. A. Alimov, V. A. Il’in, E. M. Nikishin, Convergence problems of multiple trigonometric series and spectral decompositions. I, Uspekhi Mat. Nauk, 31:6 (192), 26-83.
[3] Sh.A. Alimov, On expanding continuous functions from Sobolev classes in eigenfunctions of Laplace operator, J. Sib. Mat. Zh. 19:1 (1978) 721-734.
[4] Sh.A. Alimov, On spectral decompositions of functions in \( H^\alpha_{p,q} \), J. Math. Anal., USSR, 101:1 (1976) 3-20.
[5] Sh.A. Alimov, A.A. Rakhimov, Localization of spectral expansions of distributions in a closed domain. J. Differential Equations, 33:1 (1997) 80-82.
[6] A. Ahmedov, Nurullah, A.F, A.A. Rakhimov. Localization of fourier-laplace series of distributions, J. Malaysian Journal of Mathematical Sciences 7(2), (2013) 315-326
[7] V.A. Il’in, Spectral theory of the differential operators. Springer Science, 2012.
[8] E.I. Moiseev, The uniform convergence of certain expansions in a closed domain, J. Dokl. Akad. Nauk SSSR, 233:6 (1977) 1042-1045.
[9] A.A. Rakhimov, Spectral decompositions of distributions from negative Sobolev classes, J. Differential Equations 32:7 (1996) 1011-1013.
[10] A.A. Rakhimov, On the summability of multiple trigonometric fourier series of distributions, J. Doklady Mathematics, 62:2 (2000) 163-165.
[11] A.A. Rakhimov, Localization conditions for spectral decompositions related to elliptic operators from the class \( \text{Ar} \), J. Mathematical Notes, 59 (3-4) (1996) 298-302.
[12] A.A. Rakhimov, A. Akhmedov, H. Zainitdin, On the spectral expansions of distributions connected with Schrodinger operators, J. Applied Mathematics Letters, 25:6 (2012), 921-924.
[13] A.A. Rakhimov, K Zakaria, On an Estimation of Eigenfunctions of Schrodinger Operators in a Closed Domain, J. MathDigest, UPM, Malaysia, June Edition, (2010) 23-27.
[14] A.A. Rakhimov, On the uniform convergence of Fourier series on a closed domain, Eurasian Mathematical Journal, 8:3 (2017) 60-69.
[15] A.A. Rakhimov, Localization of the spectral decompositions of distributions, connected with the Schrodinger operator, Mod. Probl. Math. Phys. Inform. Technol. Tashkent 1, (2003) 167-172.
[16] A.A. Rakhimov, On the localization of the Riesz means of multiple Fourier series of distributions, J. Abstract and Applied Analysis, (2011) 1-8.
[17] A.A. Rakhimov, On the uniform convergence of Fourier series, J. Malaysian Journal of Mathematical Sciences 10, (2016) 55-56
[18] V.G. Sozanov, Uniform convergence and Riesz summability of spectral resolutions, J. Mat Zanetki, 29:6 (1981) 887-894.
[19] E.I. Titchmarsh, Decomposition by eigenvalues, related to differential equations of second order, Moscow, Inostr. Liter., 2 1961.
[20] H.trieble, Theory of Function Spaces, Springer, 1992.