On independent GKM-graphs without nontrivial extensions

Grigory Solomadin

Received: 26 August 2022 / Accepted: 3 October 2023 / Published online: 24 October 2023
© Sociedad Matemática Mexicana 2023

Abstract
In this paper, a new obstruction to an extension for GKM-graphs (in sense of Guillemin and Zara) is given. For a $j$-independent GKM-graph, we give a comparison result between the subposets of rank $< j$ in the face posets for the orbit space of the GKM-action and for the GKM-graph. An example of a $k$-independent $(n, k)$-type GKM-graph is constructed for any $n \geq k \geq 2$ by taking a quotient of a certain periodic GKM-graph. As an application of the above results, we prove that such a GKM-graph has no nontrivial extensions (for any $n \geq k \geq 2$) and cannot be realized by a GKM-manifold (for any $n = k = 3$ or $n \geq k \geq 4$).

Keywords Torus action · GKM-theory · Graph · Poset

Mathematics Subject Classification Primary: 57S12 · 55N91 · 06A06; Secondary: 55U10

1 Introduction

GKM-theory [9, 11] provides with a useful method for computation of the equivariant cohomology ring for a wide class of manifolds equipped with a torus action (having only isolated fixed points) called GKM-manifolds (named after Goresky, Kottwitz and MacPherson). The main tool of this theory is a graph that is naturally associated with the orbit space of the equivariant 1-skeleton for a GKM-manifold. The labels on the edges of this graph are given by the weights of the isotropy representations at the tangent spaces at fixed points of the torus action. By abstracting from the torus action one obtains the definition of a GKM-graph axiomatized in [11]. Apart from the equivariant cohomology ring many other important geometric objects and topological
invariants related to a GKM-manifold can be studied in terms of the respective GKM-graph, such as Betti numbers [11], invariant almost complex structures [10], invariant symplectic and Kähler structures [8], etc.

A GKM-manifold $M^{2n}$ is called a GKM-manifold \textit{in $j$-general position} [2, 3] if any $j$ weights of the tangential representation at any fixed point of this action on $M^{2n}$ are linearly independent. A $j$-independent GKM-graph is defined similarly. Properties of a GKM $T^k$-action on $M^{2n}$ in $j$-general position were studied in several papers (in a wider context of equivariantly formal smooth actions with isolated fixed points). The number $n-k$ is called the \textit{complexity} of the $T^k$-action on $M^{2n}$. In [15] it was shown that the equivariant cohomology ring of a torus manifold with vanishing odd cohomology groups (that is, a $2n$-dimensional GKM-manifold of complexity 0 which is automatically $n$-independent) is isomorphic to the Stanley–Reisner ring of the face poset of the respective GKM-graph. (Also see [14] for treatment of equivariant topology for torus manifolds from GKM-perspective.) Furthermore, any GKM-manifold $M^{2n}$ of complexity 1 with a GKM-action in general position (that is, in $(n-1)$-general position) has a description [4] of the equivariant cohomology ring in terms of face rings similar to [15]. The poset of faces $S_M$ in the orbit space $M/T$ was associated to a GKM-manifold $M$ with the $T$-action in [4]. We remark that in the case of complexity 0 this object was studied in earlier papers [15] (for a locally standard action, so that $M/T$ is a manifold with corners) and in [3] (in the more general case, namely, when $M/T$ is a homological cell complex). It was shown in [4] that for a GKM-manifold $M$ in $j$-general position the subposets $(S_M)_{<s}$ and $(S_M)_r$ are $\min\{\dim s - 1, j + 1\}$- and $\min\{r - 1, j + 1\}$-acyclic, respectively, for any $r > 0$ and $s \in S_M$. This implies by [3, 15] that the corresponding orbit space $M/T$ of $M$ is $(j+1)$-acyclic. This result was proved earlier in [15] for $j = n$ ($n$-independent case) and in [3] for the case of arbitrary $j$. On the other hand, by dropping the condition of general position, an arbitrary $T^{n-1}$-action on a smooth manifold $M^{2n}$ (of complexity 1) with isolated fixed points shows much more complicated behaviour of the orbit space homology [2]. Therefore, GKM-actions in $j$-general position form a particularly nice class of GKM-actions exhibiting many useful properties (depending on the value of $j$).

A natural question (called an extension problem in [13]) is how to determine whether a given GKM-action extends to an effective GKM-action of a torus of greater dimension (so that the complexity of the new action is lower) on the same manifold. This question leads to a search of a $k$-independent $(n, k)$-type GKM-action that is nonextendible for arbitrary $n \geq k \geq 2$. Homogeneous GKM-manifolds [10] supply with many interesting examples of GKM-graphs, including those of positive complexity. Recall that if the Euler characteristic of a homogeneous complex manifold $G/H$ is nonzero then it is a GKM-manifold by [10], where $G$ is a compact connected semisimple Lie group, $H$ is an arbitrary parabolic subgroup in $G$ and $T$ is any maximal torus of $H \subset G$. Two phenomena occur for homogeneous GKM-manifolds related to the search problem mentioned above. First, S. Kuroki proved the following gap property for $G$ of type $A_n$ with the natural GKM $T$-action on $G/H$ as above (unpublished): if the respective GKM-graph is $j$-independent for some $j \geq 4$ then it is $n$-independent, where $n$ is the rank of $G$. This leads to a conjecture that any homogeneous GKM-manifold satisfies such a gap property. Second, the natural $T^{n-1}$-action on the Grassmanian $Gr_k(\mathbb{C}^n)$ of $k$-planes in $\mathbb{C}^n$ is a GKM-action admitting no
nonisomorphic extensions for any \( k \). This can be shown by modifying the proof for \( k = 2 \) from [13] in a simple way. This result leads to another conjecture that most of the homogeneous GKM-manifolds are nonextendible with possibly a few exceptions determined by the condition that the respective automorphism group identity component \( \text{Aut}^0(G/H) \) is greater than \( G \). (The description of \( \text{Aut}^0(G/H) \) is given in [1], for instance.)

Therefore, the extension problem seems to be a bit challenging. Instead, one can study a rather simplistic extension problem for arbitrary GKM-graphs by replacing a GKM-action of \( T^k \) on \( M^{2n} \) with an \((n, k)\)-type \( k \)-independent GKM-graph \( \Gamma \) (not necessarily coming from a GKM-action). An occurring new realization problem here is to determine whether a given GKM-graph \( \Gamma \) is a GKM-graph of some GKM-manifold. We remark that some results on realization problem were obtained in dimensions \( n = 1, 2 \) in [6] for GKM-graphs (in terms of conditions given by the ABBV localization formula), and for GKM-orbifolds in dimension \( n = 2 \) in [7, Remark 4.13].

In this paper, we give new examples of non-realizable and non-extendible actions of different complexity. The precise statement is given in the following main theorem of this paper (see Theorem 5.4).

**Theorem 1** For any \( n \geq k \geq 2 \) there exists an \((n, k)\)-type GKM-graph \( \Gamma \) satisfying the following properties:

(i) The GKM-graph \( \Gamma \) is \( k \)-independent;
(ii) The GKM-graph \( \Gamma \) has no nontrivial extensions;
(iii) The GKM-graph \( \Gamma \) is not realized by a GKM-action if \( n = k = 3 \) or \( n \geq k \geq 4 \).

The necessary tools for the proof of Theorem 1 are twofold. First, in Sect. 2 we give a sufficient criterion (Corollary 2.14) of non-extendibility of any GKM-graph satisfying some condition formulated in terms of chords for a face in the GKM-graph. Second, in Sect. 3 for any GKM-manifold \( M \) in \((j + 1)\)-general position with the corresponding GKM-graph \( \Gamma \), we compare (Proposition 3.9) some subposets in the face poset \( S_\Gamma \) of a GKM-graph \( \Gamma \) with the subposets in the face poset \( S_M \) of \( M \). It is important to notice that this comparison result gives a simple approach to the face subposets of the torus action in terms of the corresponding GKM-graph. This implies (by [4]) partial acyclicity for some subposets in \( S_\Gamma \). For a GKM-action in \((j + 1)\)-general position the poset \((S_\Gamma)^{op}_{\leq \Xi}\) turns out to be a simplicial poset for any face \( \Xi \) of dimension not exceeding \( j \) in \( \Gamma \) (Proposition 3.14). As a corollary of this fact, the respective Euler characteristic of the order complex for the poset \((S_\Gamma)^{op}_{\leq \Xi}\) is computed in terms of the corresponding \( f \)-numbers (Proposition 3.13) by P. Hall’s formula.

In Sect. 4, we construct an example suitable for the proof of Theorem 1. Here is a brief outline of the corresponding construction. First, we introduce an infinite \((d + 1)\)-regular graph embedded in \( \mathbb{R}^d \times \mathbb{R} \). We endow it with an axial function in order to obtain a torus graph \( \Gamma(d, 0) \). We add to it edges and define the axial function agreeing to that of \( \Gamma(d, 0) \). This causes the increase of the complexity for the GKM-graph. The resulting \((d + 1 + r, d + 1)\)-type GKM graph \( \Gamma(d, r) \) is \((d + 1)\)-independent. We remark that the definition of the axial function on \( \Gamma(d, r) \) is obtained by applying the decision method of Tarski [19] to some Vandermonde matrices (Lemma 4.12), and therefore, it is implicit. The infinite GKM graph \( \Gamma(d, r) \) is periodic (invariant) with respect to
the subgroup $2^{r+2} \cdot \mathbb{Z}^d \subset \mathbb{Z}^d \times \{0\} \subset \mathbb{Z}^d \times \mathbb{Z}$ in the group of parallel translations in $\mathbb{R}^d \times \mathbb{R}$. The quotient $I_{d+1}^{d+1+r} := \Gamma^{2^{r+2}}(d, r)$ of the GKM-graph $\Gamma(d, r)$ by the group $2^{r+2} \cdot \mathbb{Z}^d$ is shown to be a well-defined GKM-graph.

In Sect. 5 we prove that $I_{d+1}^{d+1+r}$ satisfies all conditions of Theorem 1 (where we put $n = d + 1 + r$, $k = n + 1$). We apply the results on acyclicity of face subposets in GKM-manifolds (in the case of complexity 0) from [15] in order to prove non-realizability of the constructed GKM-graph $I_{d+1}^{d+1+r}$ by studying the respective Euler characteristic (by the comparison results mentioned above). This argument relies on the explicit computation of face numbers in the torus graph $\Gamma^a(d, 0)$ (Lemma 5.2). The nonextendibility is proved by using the method of chords mentioned above. Section 6 finishes this paper with some related concluding remarks.

2 An obstruction to a GKM-graph extension

In this section we recall some definitions from GKM-theory (we follow the notation from [11, 12]). We introduce an obstruction to have extensions for a given GKM-graph in terms of chords.

Definition 2.1 [11] A GKM-graph $\Gamma$ is a triple $((V, E), \nabla, \alpha)$ consisting of:

- a graph $(V, E)$ with the set of vertices $V \neq \emptyset$ and with the set of edges $E \subseteq V \times V \setminus \Delta(V)$, where $\Delta(V) := \{(v, v) | v \in V\} \subseteq V \times V$. In addition, it is required that for an edge $e = (i(e), t(e)) \in E$ one has $\bar{e} := (t(e), i(e)) \in E$, where $i(e)$ and $t(e)$ are called the initial and terminal vertices of the edge $e$, respectively;

- a collection of bijections

$$\nabla = \{\nabla_e : \text{star}_\Gamma i(e) \to \text{star}_\Gamma t(e)\},$$

such that the identity

$$\nabla_e(e) = \bar{e},$$

holds for any $e \in E$, where $\text{star}_\Gamma i(e) := \{e \in E | i(e) = v\}$ is the star of $\Gamma$ at $i(e)$;

- a function $\alpha : E \to \mathbb{Z}^k$ satisfying the rank condition

$$\alpha(\Gamma) := \mathbb{Z}\{\alpha(e) | e \in \text{star}_\Gamma(v)\} = \mathbb{Z}^k,$$

the opposite sign condition

$$\alpha(\bar{e}) = -\alpha(e),$$

and the congruence condition

$$\alpha(\nabla_e(e')) = \alpha(e') + c_e(e')\alpha(e),$$

$\copyright$ Birkhäuser
for some integer $c_e(e') \in \mathbb{Z}$ and for any $e, e' \in E$ with a common source, and $v \in V$. Here $\mathbb{Z}(\alpha(e) \mid e \in \text{star}_\Gamma(v))$ denotes the $\mathbb{Z}$-linear span of vectors $\alpha(e)$, where $e$ runs over star$_\Gamma(v)$.

Additionally it is required that the values of $\alpha$ at any two distinct elements of star$_\Gamma(v)$ are linearly independent, and that $\alpha(e)$ is a primitive vector of the lattice $\mathbb{Z}^k$ for any $e \in E$. The collection $\mathbb{V}$ is called a connection of $\Gamma$ and the function $\alpha$ is called an axial function of $\Gamma$.

**Remark 2.2** In this paper (e.g., see Definition 2.1) graphs with possibly infinite sets of vertices and edges are studied. However, we restrict to the class of $n$-valent graphs, where $n \in \mathbb{Z}_{\geq 0}$ is any non-negative integer. Notice that the star of any $n$-valent graph in any its vertex is a finite set. In what follows we consider only graphs with finitely many vertices and edges, unless explicitly stated otherwise. (Graphs with infinite sets of vertices and edges are only necessary in Sect. 4.)

In this section, we fix a GKM-graph $\Gamma$ with the corresponding connected $n$-valent graph $(V, E) = (V_\Gamma, E_\Gamma)$, axial function $\alpha: E \rightarrow \mathbb{Z}^k$ on $\Gamma$ and a connection $\nabla$ on it.

**Remark 2.3** We use a restricted Definition 2.1 of a GKM-graph, say, $\Gamma$ that differs from the one given in [11] in the following sense. First, the underlying graph of $\Gamma$ is simple, i.e., has neither loops nor multiple edges. Second, the opposite sign condition holds for any edge of $\Gamma$.

**Definition 2.4** [11] A connected $r$-regular subgraph $\mathcal{S}$ of $\Gamma$ is called an $r$-face (or a face) of $\Gamma$, if $\nabla_e(e') \in E_\mathcal{S}$ holds for any $e, e' \in E_\mathcal{S}$ such that $i(e) = i(e')$ (in [11] it is called a totally geodesic subgraph). Any edge $e \in \text{star}_\Gamma v \setminus \text{star}_\mathcal{S} v$ is called a transversal edge to a face $\mathcal{S}$ in $\Gamma$, where $v \in V_\mathcal{S}$. Let

$$\alpha(\mathcal{S}) = \alpha_v(\mathcal{S}) := \mathbb{Z}\langle\alpha(e) \mid e \in \text{star}_\mathcal{S}(v)\rangle \subseteq \mathbb{Z}^k,$$

be a span of a face $\mathcal{S}$ in $\Gamma$. The GKM-graph $\Gamma$ is called a GKM-graph of $(n, k)$-type, if $\Gamma$ is $n$-valent and the rank $\text{rk}\, \alpha := \text{rk}\, \alpha(\Gamma)$ of $\alpha$ is equal to $k$.

**Remark 2.5** A face $\mathcal{S}$ of a GKM-graph $\Gamma$ becomes a well-defined GKM-graph by taking restrictions of the connection and of the axial function from $\Gamma$ to $\mathcal{S}$. The span of the face $\mathcal{S}$ is well defined because of the identity $\alpha_p(\mathcal{S}) = \alpha_q(\mathcal{S})$ for every $p, q \in V_\mathcal{S}$ which immediately follows from the congruence condition.

**Definition 2.6** A GKM-graph $\Gamma$ is called $j$-complete if for any $v \in V$, any integer $i \leq j$ and any distinct edges $e_1, \ldots, e_i \in \text{star}_\Gamma(v)$ there exists an $i$-face $\mathcal{S}$ of $\Gamma$, such that $\text{star}_\mathcal{S}(v) = \{e_1, \ldots, e_i\}$ holds. An $n$-regular $n$-complete GKM-graph is called a complete GKM-graph. A GKM-graph $\Gamma$ is called $j$-independent if for any $v \in V$ and any distinct edges $e_1, \ldots, e_j \in \text{star}_\Gamma v$ the values $\alpha(e_1), \ldots, \alpha(e_j)$ of the axial function $\alpha$ on $\Gamma$ are linearly independent in $\mathbb{Z}^k$.

**Definition 2.7** Let $\mathcal{S}$ be a face of $\Gamma$. We call a transversal edge $e \in E_\Gamma$ to $\mathcal{S}$ a chord of the face $\mathcal{S}$, if $i(e), t(e) \in V_\mathcal{S}$. If the face $\mathcal{S}$ in $\Gamma$ admits no chords then we call $\mathcal{S}$ a chordless face of $\Gamma$. 
**Example 2.8** The standard $T^2$-action on the flag manifold $Fli(3)$ is a GKM-action whose associated GKM-graph $Gamma$ (see next section) is of $(3, 2)$-type. The respective underlying graph is the bipartite graph $K_{3,3}$ [10, p.40]. The GKM-graph $Gamma$ satisfies the opposite sign condition. It has five 2-faces (three 4-cycles and two 6-cycles). For any such 6-cycle 2-face $Xi$ the remaining 3 transversal edges in $Gamma$ are chords of $Xi$.

**Proposition 2.9** Let $Xi$ be a j-face of a $(j+1)$-complete GKM-graph $Gamma$. Then for any chord $e in E_G$ of $Xi$ and any edge path $gamma = (e_1, \ldots, e_r)$ in $Xi$, such that $i(gamma) = i(e)$, $t(gamma) = t(e)$, one has $Pi_{gamma} e = overline{e}$, where by definition (see [18]):

$$\Pi_{gamma} : star_G(i(gamma)) \rightarrow star_G(t(gamma)), \quad \Pi_{gamma}(e) := \nabla_{e_r} \circ \cdots \circ \nabla_{e_1}(e).$$

**Proof** By the $(j+1)$-completeness condition, there exists a $(j+1)$-face $Phi$ of $Gamma$, such that $Xi$ and $e$ belong to $Phi$. Notice that $Pi_{gamma}(e) = overline{e} in Phi holds by the definition. The definition of invariance also implies that $Pi_{gamma}(e) in star_G(t(gamma)) \setminus star_G(t(gamma))$ holds. However, $star_G(t(gamma)) \setminus star_G(t(gamma))$ has cardinality one, since $Xi$ and $Phi$ are $j$- and $(j+1)$-faces, respectively. Hence

$$star_G(t(gamma)) \setminus star_G(t(gamma)) = \{overline{e}\},$$

holds. Observe that $Pi_{gamma}(e) in star_G(t(gamma)) \setminus star_G(t(gamma))$ holds by the definition of invariance, because $gamma$ belongs to $Xi$ by the condition. Therefore, we conclude that $Pi_{gamma}(e) = overline{e}$ holds. This proves the claim of the proposition. $\square$

**Proposition 2.10** For a chord $e in E_G$ of a face $Xi$ in $Gamma$ suppose that there exists an edge path $gamma$ in $Xi$, such that $i(gamma) = i(e)$, $t(gamma) = t(e)$ and $Pi_{gamma} e = overline{e}$ hold. Then, one has $2alpha(e) in alpha(Xi)$.

**Proof** Let $gamma = (e_1, \ldots, e_r)$. One deduces the identity

$$alpha(Pi_{gamma} e) = alpha(e) + \sum_{i=1}^r c_{e_i} (Pi_{gamma - 1} e) \cdot alpha(e_i), \quad (2.1)$$

from the congruence condition. This identity implies that

$$alpha(Pi_{gamma} e) - alpha(e) in alpha(Xi) \setminus alpha(Xi), \quad (2.2)$$

is true. Notice that the identities $alpha(overline{e}) = -alpha(e)$, $alpha(overline{e}) = alpha(Pi_{gamma} e)$ hold. Together with the inclusion (2.2) this implies the claim of the proposition. $\square$

**Proposition 2.11** Suppose that $Gamma$ is $(j+1)$-independent, where $j in Z$. Then $Gamma$ is $j$-complete.

**Proof** Fix a nonzero integer $s leq j$. Let $E' := \{e_1, \ldots, e_s\}$ be an $s$-element set of some mutually different edges in $Gamma$ with a common origin $v$. In order to prove the claim it is enough to construct an $s$-face $Xi$ in $Gamma$, such that the inclusion:

$$E' \subseteq E_Xi, \quad (2.3)$$

holds.
holds. We give the inductive definition as follows:

\[ P_{i+1} := P_i \cup \{ \Pi e' \mid e, e' \in P_i \}, \quad P_0 := E', \quad i \geq 0. \]

By the definition, the filtration \( P_0 \subseteq P_1 \subseteq \cdots \) is bounded by the finite set \( E_\Gamma \) from above. Hence, there exists \( N \in \mathbb{N} \), such that \( P_i = P_N \) holds for any \( i \geq N \). Define the subgraph \( \Xi \) in \( \Gamma \) by the formulas:

\[ V_\Xi := \{ i(e) \mid e \in P_N \}, \quad E_\Xi := P_N. \]

The set \( P_N \) is closed under reversion of an edge operation, because \( \Pi e = e \in P_{i+1} \) holds for any \( e \in P_i \). By the condition, for any \( e, e' \in P_N \) there exists an edge path \( \gamma \subseteq \Xi \), such that \( i(\gamma) = i(e) \) and \( t(\gamma) = i(e') \) holds. Hence, \( \Xi \) is a connected subgraph in \( \Gamma \). It follows from the definition that \( \Xi \) is a face of \( \Gamma \).

It remains to show that \( \Xi \) is an \( s \)-face. Assume the contrary. Then, there exists \( e \in \text{star}_\Xi (v) \setminus E' \). It follows from the definition that there exist \( i = 1, \ldots, s \) and \( \gamma \subseteq \Xi \), such that the identities \( i(\gamma) = t(\gamma) = v \) and

\[ \Pi e(b) = e, \]

hold. It follows from the formula (2.1) that

\[ \alpha(\Pi e(b)) \in \mathbb{Z} \langle \alpha(e_j) \rangle \mid j = 1, \ldots, s \}. \]

Hence, the collection of \( s + 1 \) vectors \( \alpha(e), \alpha(e_j), \) \( j = 1, \ldots, s \), is linearly dependent. However, this contradicts the condition of \((j + 1)\)-independency of \( \Gamma \), because \( s \leq q \). We conclude that \( \Xi \) is an \( s \)-face, which proves the claim of the proposition. \( \square \)

**Corollary 2.12** Suppose that \( \Gamma \) is a \((j + 2)\)-independent \((n\text{-independent, respectively})\) GKM-graph, where \( j \in \mathbb{Z} \). Then any \( r \)-face \((\text{face, respectively})\) of \( \Gamma \) is chordless, where \( r = 1, \ldots, j \).

**Proof** Assume the contrary. Then there exist an \( r \)-face \( \Xi \) of \( \Gamma \) and its chord \( e \), where \( r \leq j \). One has \( 2\alpha(e) \notin \alpha(\Xi) \), because \( \Gamma \) is \((j + 2)\)-independent and \( \Xi \) is \( r \)-regular, where \( r \leq j \). By Proposition 2.11 \( \Gamma \) is \((j + 1)\)-complete. Then one can apply Propositions 2.9 and 2.10 in order to obtain \( 2\alpha(e) \in \alpha(\Xi) \). This contradiction proves the first claim of the corollary. The proof of the second claim is similar to the proof of the first claim. \( \square \)

**Definition 2.13** Let \( \Gamma', \Gamma \) be two GKM-graphs with the same underlying graph \( (V, E) \), the same connection \( \nabla \), with the axial functions \( \alpha', \alpha \) taking values in \( \mathbb{Z}^{k'} \) and \( \mathbb{Z}^k \), respectively. The GKM-graph \( \Gamma' \) is called an extension of \( \Gamma \) (see [13]), if there exists an epimorphism \( p: \mathbb{Z}^{k'} \rightarrow \mathbb{Z}^k \), such that \( p(\alpha'(e)) = \alpha(e) \) holds for any \( e \in E \). We say that an \((n, k)\)-type GKM-graph \( \Gamma \) has no nontrivial extensions if for any \( s > 0 \) it does not admit an extension to an \((n, k + s)\)-type GKM-graph. (This terminology was proposed by S. Kuroki.)
The next corollary is a principal tool for the proof of Theorem 1.

**Corollary 2.14** Let $\Gamma$ be a $(k + 1)$-complete GKM-graph. Suppose that there exists a $k$-face $\Xi$ of $\Gamma$, such that any transversal edge $e \in E_\Gamma$ to $\Xi$ is a chord for $\Xi$. Then $\Gamma$ has no nontrivial extensions.

**Proof** Suppose that there exists an extension of $\alpha$ to an axial function $\tilde{\alpha}$ of rank $k + s$ for some $s > 0$. Choose a vertex $v \in V_\Xi$. Then it follows from the condition by Propositions 2.9 and 2.10 that $2\tilde{\alpha}(e) \in \tilde{\alpha}(\Xi)$ holds for any edge $e \in \text{star}_\Gamma(v)$. Hence, $k + s = \text{rk} \tilde{\alpha} = \text{rk} \tilde{\alpha}(\Xi)$. However, by definition $\text{rk} \tilde{\alpha}(\Xi) \leq k$. This contradiction proves the claim. $\square$

### 3 Face posets of a GKM-graph and of a GKM-manifold

In this section, we continue to recall some basic notions of GKM-theory and of the related [4, 14] posets $S_M$, $S_\Gamma$ of faces arising from the orbit space $M/T$ and from the GKM-graph $\Gamma$ of a given GKM-manifold $M$ with the $T$-action, respectively. We compare some specific simplicial subposets in $S_M$, $S_\Gamma$ under assumption of $j$-general position for $M$. After that we recall the P.Hall formula for the Euler characteristic of an order complex for a finite simplicial poset which is used later in the text.

**Definition 3.1** [4, 14] For a GKM-graph $\Gamma$ the collection $S_\Gamma$ of all faces in $\Gamma$ is called a face poset of the GKM-graph $\Gamma$ with the partial order given by inclusion of faces in $\Gamma$.

Due to [15, Lemma 2.1] one can give the following definition of a GKM-manifold that is equivalent to the standard one (e.g., see [11]).

**Definition 3.2** [9, 11, 15] A smooth manifold $M^{2n}$ with an effective action of $T^k = (S^1)^k$ is called a GKM-manifold if the following conditions hold:

- The set of $T^k$-fixed points $M^T$ in $M$ is finite and nonempty;
- The weights of the isotropy tangential representation at any $x \in M^T$ are pairwise linearly independent;
- All odd cohomology groups of $M$ vanish, i.e., one has $H^{\text{odd}}(M; \mathbb{Z}) = 0$, where $H^{\text{odd}}(M; \mathbb{Z})$ denotes the sum of odd-degree singular cohomology groups of $M$ with $\mathbb{Z}$-coefficients.

**Remark 3.3** To any complex GKM-manifold one associates a GKM-graph, e.g., see [12]. We notice that the definition of a GKM-manifold varies in literature. For example, for an arbitrary GKM-manifold (e.g., in sense of [13]) the opposite sign condition is in general satisfied only up to a sign. We also remark that it is possible to have loops and multiple edges for a GKM-action (e.g., in sense of [13]). In this paper we restrain from considering such torus actions and we use a restricted definition above of a GKM-action and of a GKM-graph (where it is a simple graph).

Let $T'$ and $T$ be two GKM-actions of tori on the same manifold $M$. Let $\Gamma'$, $\Gamma$ be the respective GKM-graphs. Let $\alpha'$, $\alpha$ be the axial functions of $\Gamma'$, $\Gamma$, respectively.
Definition 3.4 The action of \( T' \) is called an **extension** of the action \( T \) on \( M \) if there is a group monomorphism \( \pi : T \to T' \) that is equivariant with respect to these torus actions.

Remark 3.5 In other words, the \( T \)-action is the restriction of the \( T' \)-action in the above definition. The epimorphism

\[
p : \mathbb{Z}^{k'} \cong \text{Hom}(T', S^1) \to \text{Hom}(T, S^1) \cong \mathbb{Z}^{k},
\]
corresponding to \( \pi \) induces the extension of the GKM-graphs \( \Gamma' \), \( \Gamma \) corresponding to the \( T' \)- and the \( T \)-action, respectively. Notice that the connection \( \nabla' \) of the GKM-graph \( \Gamma' \) is a connection for the GKM-graph \( \Gamma \). Therefore, the GKM-graph \( \Gamma' \) is an extension of \( \Gamma \) if \( \nabla' = \nabla \) holds. However, in general the GKM-graph \( \Gamma' \) is not an extension of \( \Gamma \) due to the freedom of choice for the connection \( \nabla \) on \( \Gamma \).

Example 3.6 The natural \( T^2 \)-action on \( Fl_3 \) has no nontrivial extensions. One can prove this by applying Corollary 2.14 to the corresponding GKM-graph (see Example 2.8). This fact may also be easily obtained by the results of [13], or by studying the automorphism group of the homogeneous space \( Fl_3 \) (in a different category of complex-analytic torus actions).

Consider a GKM-action of \( T = T^k \) on \( M = M^{2n} \). For a smooth \( T \)-action on \( M \), consider the canonical projection \( p : M \to Q := M/T \) to the respective orbit space, and let

\[
Q_0 \subset Q_1 \subset \cdots \subset Q_k = Q
\]

be the filtration on the orbit space \( Q \), where \( Tx \) denotes the \( T \)-orbit of \( x \) in \( M \).

Definition 3.7 [4] The closure of a connected component of \( Q_i \setminus Q_{i-1} \) is called an \( i \)-face (or a face) \( F \) of \( Q \) if it contains at least one fixed point. The poset of faces for the GKM-manifold \( M \) is the poset \( S_M \) of faces of non-negative dimension ordered by inclusion in the orbit space \( Q \) of the \( T \)-manifold \( M \).

Lemma 3.8 [4, p.5, Lemma 2.9] The full preimage \( M_F = p^{-1}(F) \) of any face \( F \subseteq Q \) is a smooth submanifold in \( M \) called a face submanifold in \( M \).

Let \( S_M \) be the poset of faces for a GKM-manifold \( M \). Let \( S_F \) be the poset of faces of non-negative dimension (ordered by inclusion) of the GKM-graph \( \Gamma \) associated to the GKM-manifold \( M \). Recall that any GKM-graph of \( (n, n) \)-type (see Definition 2.4) is called a torus graph [15].

Proposition 3.9 For a GKM-manifold \( M \) in \( (j + 1) \)-general position for some \( j \geq 1 \) the following claims hold.
(i) For any $q \leq j$, any $q$-face $\Xi$ in $\Gamma$ is an equivariant 1-skeleton of a face submanifold in $M$ and the GKM-graph $\Xi$ is a torus graph.

(ii) The posets $(S_M)_{\leq s(\Xi)}$ and $(S_{\Gamma})_{\leq \Xi}$ are isomorphic for any face $\Xi$ of $\Gamma$, such that $\dim \Xi \leq j$, where $s(\Xi)$ is the face in $M$ corresponding to $\Xi$ by (i).

**Proof** Choose $v \in V_\Xi$ and let $\star_\Xi(v) = \{e_1, \ldots, e_q\}$. Consider the sublattice $L := \mathbb{Z}(\alpha_i \mid i = 1, \ldots, q)$ in $\mathbb{Z}^k$, where $\alpha_i := \alpha(e_i)$. Choose the minimal sublattice $L_0 \subseteq \mathbb{Z}^k$, such that $L \subseteq L_0$ holds. Then the natural lattice monomorphism $L_0 \rightarrow \mathbb{Z}^k$ splits. The natural lattice embedding $i_\ast : L_0 \rightarrow L$ has a finite index. Recall that assigning the character lattice gives the bijection between split monomorphisms to $\mathbb{Z}$ and closed connected subgroups in $T$. By this correspondence, let $G_0$ be the closed subgroup in $T$ corresponding to the sublattice $L_0 \subseteq \mathbb{Z}^k$. Consider the connected component $Y$ of $M^{G_0}$, such that $v \in Y$ holds. The space $Y$ is a smooth manifold with effective $T'$-action, where $T' := T/G_0$ (see [5]). Notice that $Y^{T'} \subseteq M^{T'}$ holds. One deduces the identity dim $T_x Y = 2 \dim L_0$ from the independence condition for the weights of the tangential representation at $x \in Y \subseteq M$. Therefore, $Y$ is a $2q$-dimensional manifold. The projection $p : T \rightarrow T'$ induces the lattice map $i_\ast$ on the corresponding character lattices.

Let $\beta_1, \ldots, \beta_q$ be the collection of the $T'$-weights of the tangential representation at $x \in Y$. The embedding $Y \rightarrow M$ is equivariant with respect to the projection $p : T \rightarrow T'$. In particular, the induced linear map $dp_x : T_x Y \rightarrow T_x M$ of the tangent spaces is equivariant and sends the linear $G_0$-representation $V(\beta_i) \subseteq T_x Y$ to the linear $T$-representation $V(i_\ast(\beta_i)) \subseteq T_x M$, where $\beta \in L_0, i = 1, \ldots, q$. One has $\mathbb{Z}(i_\ast(\beta_i))(i = 1, \ldots, q) = L_0$. Notice that $\mathbb{Q}$-spans of $L$ and $L_0$ coincide by the construction. Hence, $\beta_1, \alpha_1, \ldots, \alpha_q$ are linearly dependent over $\mathbb{Q}$ for any $i = 1, \ldots, q$. Then by linear independence condition we conclude that the collection $\beta_1, \ldots, \beta_q$ coincides with the collection $\alpha_1, \ldots, \alpha_q$ up to an ordering. Therefore, $Y$ is a GKM-manifold, its equivariant 1-skeleton $Y_1$ is a GKM-graph $\Xi$ of type $(q, q)$ and $\Xi$ is a face of the GKM-graph $\Gamma$. This proves (i). By the definition, one has $(S_M)_{\leq s} \subseteq (S_{\Gamma})_{\leq s}$. The claim of (ii) then follows trivially from (i). \hfill \Box

Let $P$ be a finite poset [17]. Recall the following definitions.

**Definition 3.10** [17] The order complex of a finite poset $P$ is the simplicial complex:

$$\Delta(P) := \{\sigma = \{I_1, I_2, \ldots, I_{k+1}\} \in 2^P \mid I_1 < I_2 < \cdots < I_{k+1}, \ k \geq 0\},$$

on the vertex set $P$ consisting of chains of increasing elements in $P$.

**Definition 3.11** [17] The poset $P$ with the least element $\hat{0}$ is called a simplicial poset if the subposet $[\hat{0}, x]$ of $P$ is a Boolean lattice for any $x \in P$. For any element $x$ of a simplicial poset $P$ the length $l(x)$ of $x$ is the length of a maximal chain in $[\hat{0}, x]$. Here $l(\hat{0}) := 0$. Define the dimension of a simplicial poset $P$ to be the number $\dim P := \dim \Delta(P) = \max_{x \in P} l(x)$. For a simplicial poset $P$ let

$$f_i(P) := |\{x \in P \mid l(x) = i + 1\}|,$$

be the number of elements in $P$ of length $i + 1$, where $i \geq 0$. In particular, $f_{-1}(P) = 1$. 

© Birkhäuser
Remark 3.12 The poset $S^\text{op}_M$ has the least element $\Gamma$ by the definition. The poset $S^\text{op}_M$ has the least element $M$ by principal orbit theorem, see [5, p.179]. Therefore, these posets are both acyclic. The relation between $S_M$ and the orbit space components $Q_r \subseteq Q$ is given by formula:

$$Q_r = \bigcup_{F \in (S_M)_r} F.$$  

The Leray-Serre spectral sequence of the filtration on $Q_r$ by $Q_i$, where $i \leq r$, describes the relation between cohomology groups of such spaces. In some special cases this spectral sequence was studied in different works, for example, see [3, 4, 15]. In general such a spectral sequence is non-degenerate, and the relation is non-trivial.

In the following, we need the following well-known Philip Hall’s theorem.

Proposition 3.13 [16, Proposition 6] Let $S$ be a simplicial poset of dimension $d$. Then the Euler characteristic $\tilde{\chi}(\Delta(\mathcal{S}))$ of the order complex for $\mathcal{S} := S \setminus \mathcal{0}$ in the reduced simplicial homology $\tilde{H}_s(\Delta(\mathcal{S}))$ is given by the formula:

$$\tilde{\chi}(\Delta(\mathcal{S})) = \sum_{i=-1}^{d-1} (-1)^i f_i(S).$$

The computation of Euler characteristic for certain face subposets in $S_{\Gamma}$ for a $j$-complete GKM-graph $\Gamma$ is possible (by using Proposition 3.13) due to the following proposition.

Proposition 3.14 Let $\Gamma$ be a $j$-complete GKM-graph for some $j \geq 1$. Then for any $j$-face $\mathcal{E}$ of $\Gamma$ the poset $(S_{\Gamma})^\text{op}_{\leq \mathcal{E}}$ is a simplicial poset of dimension $\dim \mathcal{E}$. In particular, for any $\Omega \in (S_{\Gamma})^\text{op}_{\leq \mathcal{E}}$ one has $l(\Omega) = j - \dim \Omega$ in $(S_{\Gamma})^\text{op}_{\leq \mathcal{E}}$, and $f_i((S_{\Gamma})^\text{op}_{\leq \mathcal{E}})$ is equal to the number of $(j - i - 1)$-dimensional faces in $(S_{\Gamma})^\text{op}_{\leq \mathcal{E}}$.

Proof To prove the claim of the proposition it is enough to show that for any face $\Omega$ in $\mathcal{E}$ the poset $[\Omega, \mathcal{E}] := \{ \Phi \in S_{\Gamma} \mid \Omega \subseteq \Phi \subseteq \mathcal{E} \}$ is isomorphic to the poset of faces in the simplex $\Delta^{j-\dim \Omega}$ of dimension $j - \dim \Omega$. Let $v \in V_{\mathcal{E}}$. Since the connection of $\Gamma$ is $j$-independent, any face $\Phi \in [\Omega, \mathcal{E}]$ is uniquely determined by the collection $C(\Phi)$ of $\dim \Phi - \dim \Omega$ mutually different elements from $\operatorname{star}_{\mathcal{E}} v \setminus \operatorname{star}_{\mathcal{E}} v$, and vice versa. Moreover, for any $\Phi_1, \Phi_2 \in [\Omega, \mathcal{E}]$ one has $\Phi_1 \subseteq \Phi_2$ iff $C(\Phi_1) \subseteq C(\Phi_2)$. This implies the necessary claims of the proposition. □

Recall that a topological space $X$ is called $j$-acyclic if $\tilde{H}^i(X) = 0$ holds for any $i \leq j$, and acyclic, if $H^*(X) = 0$ holds. We need the following particular case of a theorem from [4].

Theorem 3.15 [4, Theorem 1] For any GKM-manifold $M$ of complexity 0 in $n$-general position with the half-dimensional torus action the $(n - 1)$-dimensional poset $S^\text{op}_M$ is $(n - 2)$-acyclic, where $n \geq 2$. 

© Birkhäuser
4 A periodic GKM-graph and its quotient

In this section, we give a detailed construction of the GKM-graph suitable for the proof of Theorem 1 and study some of its properties.

Construction 4.1 [Graph $\Gamma'$] Let $X_s$ be the graph with the set of vertices $\mathbb{Z} \times \{s\}$ and the set of edges $(2n+s, 2n+s+1)$, where $s = 0, 1$. For any $d \geq 1$ we define the graph $\Gamma' = \Gamma'(d)$ with the set of vertices $\mathbb{Z}^d \times \{0, 1\}$ as the union of the following graphs:

(i) The Cartesian product $X_0^d$ of $d$ copies of $X_0$;
(ii) The Cartesian product $X_1^d$ of $d$ copies of $X_1$;
(iii) Edge $D(u, 0) = ((u, 0), (u, 1))$ and its inverse $D(u, 1)$, where $u$ runs over $\mathbb{Z}^d$.

Here, we identify $(\mathbb{Z} \times \{s\})^d$ with $\mathbb{Z}^d \times \{s\}$ by the bijection

$$((u_1, s), \ldots, (u_d, s)) \mapsto ((u_1, \ldots, u_d), s), \ s = 0, 1.$$ 

We call any edge of $(X_0)^d, (X_1)^d$ a horizontal edge, and any edge of the form $D(u, s)$ a vertical edge, $s = 0, 1$. For any vertex $v \in V_{\Gamma'}$ denote by $\text{Cube}(v)$ the unique connected component of $(X_0)^d \cup (X_1)^d$ containing $v$. The graph $\Gamma'(d)$ is embedded in $\mathbb{R}^d \times \mathbb{R}$. The graph $\text{Cube}(u, s)$ is isomorphic to an edge graph of the $d$-dimensional cube in $\mathbb{R}^d$. Notice that the graph $\Gamma'$ is a $(d + 1)$-regular connected graph with infinite set of vertices.

Construction 4.2 [Functions $\varepsilon_j^i$] In what follows it is necessary to define additional functions on the set of vertices for the graph $\Gamma$, with values in $\{\pm 1\}$. For any $d \geq 1$ define the functions $\varepsilon_j^i : V_{\Gamma'} \to \{\pm 1\}$, where $i = 1, \ldots, d + 1$ and $j \in \mathbb{N}$. For any $y = (y_1, \ldots, y_d) \in \mathbb{Z}^d$ let

$$\varepsilon_j^i(y, s) := (-1)^{\frac{j-1}{2^i}}; \ i = 1, \ldots, d; \ j \in \mathbb{N}; \ s = 0, 1. \quad (4.1)$$

By definition, put

$$\varepsilon_j^{i+1}(y, s) := \varepsilon_j^i(y, s). \quad (4.2)$$

Construction 4.3 [Graph $\Gamma$] For any $d \geq 1$ and $r \geq 0$ let $\Gamma = \Gamma(d, r)$ be the graph obtained from $\Gamma'(d)$ by adding for any $j = 1, \ldots, r$ the edges:

$$E_j(u) := (u, v),$$

where $u, v$ are any elements from $\mathbb{Z}^d \times \{0, 1\}$, such that $\varepsilon_i^{j+1}(u) = \varepsilon_i^{j+1}(v)$ holds for any $i = 1, \ldots, d + 1$ and $u - v \in 2^j \cdot \{\pm 1\}^d \times \{0\}$.

Lemma 4.4 The graph $\Gamma$ is a $(d + 1 + r)$-valent connected graph with infinite set of vertices, and that $V_{\Gamma} = V_{\Gamma'}$ holds (see Figs. 1, 2).
Proof For any $u \in V_{\Gamma'}$, there is the equality

$$\text{star}_{\Gamma} u = \text{star}_{\Gamma'} u \sqcup X(u),$$

where $X(u)$ is the set of transversal edges to $\Gamma'$ at $u$. To prove regularity of $\Gamma'$ it is enough to check that $|X(u)| = r$ holds for any $u \in V_{\Gamma}$. Clearly, for any $n = 2^{j+1}a + b$, where $0 \leq b < 2^{j+1}$, $a \in \mathbb{Z}$,

$$\left\lfloor \frac{n + (-1)^c2^j}{2^{j+1}} \right\rfloor = a,$$

where $c$ is an integer.
Fig. 3 Graph $\Gamma(2, 1)$ (chords omitted) with signs of values for $(\varepsilon_1^1, \varepsilon_1^2)$, viewed along $z$-axis

holds for either $c = 0$ or 1. Applying this component-wisely to $u$ we obtain that there exists a unique $w \in 2^{d+1} \cdot \{\pm 1\}^d$, such that $\varepsilon_i^{j+1}(u + w) = \varepsilon_i^{j+1}(u)$ holds for any $i = 1, \ldots, d + 1$. Hence, $E_j(u) \in X(u)$ for any $j = 1, \ldots, r$, and therefore, $|X(u)| = r$ holds, as claimed. The proof is complete (Fig. 3).

Construction 4.5 [Axial function $\alpha$ on $\Gamma$] Fix a collection of integers $t_1, \ldots, t_{d+1} \in \mathbb{Z}$. Let $\alpha: E_\Gamma \to \mathbb{Z}^{d+1}, \alpha = \alpha(d, r, t_1, \ldots, t_{d+1})$ be the function defined as follows. Let

$$\alpha(u, v) := (-1)^{u_{d+1}v_{d+1}}(v - u), \quad (4.3)$$

where $u = (u_1, \ldots, u_{d+1}), v = (v_1, \ldots, v_{d+1}) \in \mathbb{Z}^{d+1}$ (See Fig. 4.) For any $v \in V_\Gamma$ let

$$\alpha(E_j(v)) := \sum_{i=1}^{d+1} \varepsilon_i^j(v) t_i^{j-1}w_i(v), \quad (4.4)$$

for any $j \in \mathbb{N}$, where $\{w_1(v), \ldots, w_{d+1}(v)\}$ are the values of $\alpha$ on star $\Gamma^*(v)$ denoted in such a way that $w_i(v) = (-1)^{v_i}e_i$ holds for $i = 1, \ldots, d + 1$. More precisely, a brief computation shows that

$$w_i(v) = (-1)^{v_i}e_i, \quad (4.5)$$

holds, where $i = 1, \ldots, d + 1$. 
Denote by $\alpha'$ the restriction of the axial function $\alpha$ to the subgraph $\Gamma'$ of $\Gamma$. Our next task is to define the connections $\nabla', \nabla$ on $\Gamma'$ compatible with $\alpha'$ and $\alpha$ by describing the corresponding facets, respectively. We do this by listing all facets in the corresponding graphs in the next two definitions. One can easily check that the facets given below are compatible with $\alpha'$ and $\alpha$. (If $d \geq 2$ then the graph $\Gamma(d, r)$ is shown to be 3-independent below, and, therefore, admits a unique connection compatible with the axial function $\alpha(d, r)$.)

Denote by $Ax$ the bijective map from $\mathbb{Z}^d \times \{0, 1\}$ to itself given by $(y, s) \mapsto (x + y, s)$ in $\mathbb{Z}^{d+1}$, where $x, y \in \mathbb{Z}^d, s = 0, 1$.

**Definition 4.6** [Facets of $\Gamma'$] For any $v \in V_{\Gamma'}$ let $F_0(v) := \text{Cube}(v)$ be the subgraph in $\Gamma'$. Denote by $\text{Cube}_i(v)$ the edge graph in $\text{Cube}(v)$ of a unique facet of the respective $d$-dimensional cube with the normal vector $\pm e_i$, such that $v \in \text{Cube}_i(v)$, where $i = 1, \ldots, d$. Let $(u, v)$ be any vertical edge of $\Gamma'$. For any $i = 1, \ldots, d$ let $F^d_i(v)$ be the $d$-valent subgraph of $\Gamma'$ that is the union of subgraphs $A_{2aej} \text{Cube}_i(u)$, $A_{2aej} \text{Cube}_i(v)$ and $A_{aej}e$, where $e$ runs over $2^{d-1}$ vertical edges of $\Gamma'$ incident to $\text{Cube}_i(v)$, $j$ runs over $\{1, \ldots, d\}\{i\}$ and $a$ runs over $\mathbb{Z}$. The list of all pairwise distinct facets in $\Gamma'(d)$ is as follows:

$$F_0(x, s), \ x \in 2 \cdot \mathbb{Z}^d, \ s = 0, 1; \ F_i(ae_i, 0), \ a \in \mathbb{Z}, \ i = 1, \ldots, d.$$  

**Definition 4.7** [Facets of $\Gamma$] For any $i = 0, \ldots, d$ let $G_i(v)$ be the union of the subgraphs $F_i(v), F_i(A_{q, 2j-1, \ldots, 1}(v))$ and edges $\text{E}_j(x)$ (and their inverses), where $j$ runs over $1, \ldots, r$, $q$ runs over $\mathbb{Z}$ and $x$ runs over the union of all vertices of these
graphs. For any \( j = 1, \ldots, r \) define the subgraph \( G_{d+j}(v) \) of \( \Gamma \) to be obtained by omitting all edges \( E_j(v) \) in \( \Gamma \), where \( v \) runs over \( V \). The list of all pairwise distinct facets in \( \Gamma(d, r) \), \( r > 0 \), is as follows:

\[
G_i(2^{r+1}ae_i, 0), \quad G_i((2^{r+1}a + 1)e_i, 0), \quad a \in \mathbb{Z},
\]

\[
G_i(0, 0), \quad i = d + 1, \ldots, d + r.
\]

**Proposition 4.8** Let \( e \in E_{\Gamma^v} \) be any edge, such that \( \alpha(e) = \pm e_i \) holds for some \( i \in \{1, \ldots, d + 1\} \). Then, one has

\[
\varepsilon^q_j(i(e)) = \varepsilon^q_j(t(e)), \quad q \in \mathbb{N}; \quad j = 1, \ldots, d + 1; \quad j \neq i.
\]

**Proof** The claim follows trivially from the definition of \( \varepsilon^q_j \), because the vertices of \( e \) have the same \( j \)-th coordinates. \( \square \)

**Lemma 4.9** For any \( E_j(u) = (u, v) \), there is the identity

\[
\varepsilon^q_j(v) = \begin{cases} 
-\varepsilon^q_i(u), & q = j, \\
\varepsilon^q_i(u), & q \neq j,
\end{cases}
\]

where \( i = 1, \ldots, d + 1, \) \( q \in \mathbb{N} \).

**Proof** By the condition, \( v = u + w \) holds for some \( w = (z, 0), z \in 2^j \cdot \{\pm 1\}^d \). Therefore, for any \( j \leq q \),

\[
\varepsilon^q_j(v) = \varepsilon^q_i(u + w) = (-1)^{\lfloor \frac{x_i + 1 \cdot 2^j}{2^{j+1}} \rfloor}.
\]

(4.8) holds for some \( k = 0, 1 \). If \( j \geq q \) then (4.8) is equal to \( (-1)^{2^j - q} \varepsilon^q_i(u) \). Assume that \( j < q \) holds. The condition implies that \( \lfloor \frac{x_i + 1 \cdot 2^j}{2^{j+1}} \rfloor = \lfloor \frac{x_i}{2^{j+1}} \rfloor \) holds, similar to the proof of Lemma 4.4. Therefore

\[
[\frac{x_i + (-1)^k 2^j}{2q}] = \lfloor \frac{x_i + (-1)^k 2^j}{2j+1} \rfloor \rightarrow \lfloor \frac{x_i + (-1)^k 2^j}{2^{j+1}} \rfloor = \lfloor \frac{x_i}{2^{j+1}} \rfloor.
\]

Here, the second and fourth equalities hold by the elementary property of the floor function. Hence, \( \varepsilon^q_j(v) = \varepsilon^q_i(u) \). This implies (4.7). \( \square \)

**Lemma 4.10** The function \( \alpha \) satisfies the rank, opposite and congruence conditions with respect to \( \Gamma \) and \( \nabla \) (see Definition 2.1).

**Proof** Notice that the rank condition is satisfied for \( \alpha \) by the construction. Let \( e = (u, v) \in E_{\Gamma^v} \). Consider the following cases.

1. Let \( e \in E_{\Gamma^v} \). The opposite sign condition is easily deduced for \( \alpha \) along the edge \( e \). In terms of Construction 4.5 of \( \alpha \), one has \( \alpha(e) = \pm e_q \) for some \( q = 1, \ldots, d + 1 \).
Then one has \( w_i(u) = w_i(v) \) for any \( i \neq q \) and \( w_q(u) = w_q(v) \equiv 0 \pmod{e_q} \). Hence, by Proposition 4.8, and by (4.4), one has

\[
\alpha(E_j(v)) = \sum_{i=1}^{d+1} \varepsilon_i^j(v)t_i^{j-1}w_i(v) \equiv \sum_{i=1}^{d+1} \varepsilon_i^j(v)t_i^{j-1}w_i(u) \\
\equiv \sum_{i=1}^{d+1} \varepsilon_i^j(u)t_i^{j-1}w_i(u) = \alpha(E_j(u)) \pmod{e_q}, \tag{4.9}
\]

where \( j = 1, \ldots, r \). Hence, the congruence condition holds for \( \alpha \) along the edge \( e \).

(2) Let \( e \notin E_{r'} \). Then \( e = E_q(u) \) for some \( q = 1, \ldots, r \). By Construction 4.5, the equality \( w_i(u) = w_i(v) \) holds for any \( i = 1, \ldots, d + 1 \). By (4.7) and (4.4) then one has

\[
\alpha(E_q(u)) = \alpha(E_q(v)) = \sum_{i=1}^{d+1} \varepsilon_i^q(v)t_i^{q-1}w_i(v) \\
= \sum_{i=1}^{d+1} \varepsilon_i^q(v)t_i^{q-1}w_i(u) = -\sum_{i=1}^{d+1} \varepsilon_i^q(u)t_i^{q-1}w_i(u) = -\alpha(E_q(u)).
\]

Hence, the opposite sign condition holds for \( \alpha \) along the edge \( e \). The congruence condition along \( e \) for \( E_j(u), q \neq j \), is shown similar to (4.9) by using Lemma 4.9. The congruence condition along \( e \) for the remaining edges (i.e. from star \( r' \) \( u \)) is clear. This implies that the congruence condition holds for \( \alpha \) along the edge \( e \). The proof is complete. \( \square \)

**Example 4.11** The following identities hold for the respective axial function on the graph \( \Gamma(2, 2) \) in the basis \( e_1, e_2, e_3 \) of \( \mathbb{Z}^3 \):

\[
\alpha(E_1(1, 2, 0)) = (-1, -1, -1) = \alpha(E_1(5, 6, 0)),
\]

\[
\alpha(E_1(3, 0, 0)) = (1, 1, 1) = \alpha(E_1(7, 4, 0)),
\]

\[
\alpha(E_2(1, 2, 0)) = (-t_1, t_2, -t_3) = \alpha(E_2(3, 0, 0)),
\]

\[
\alpha(E_2(5, 6, 0)) = (t_1, -t_2, t_3) = \alpha(E_2(7, 4, 0)).
\]

**Lemma 4.12** Let \( r \geq 1 \). Then there exist integers \( t_1, \ldots, t_{d+1} \in \mathbb{Z} \), such that the axial function \( \alpha(d, r, t_1, \ldots, t_{d+1}) \) is \((d + 1)\)-independent.

**Proof** For any vertex \( v \in V_{r'} \) the values of the axial function \( \alpha \) on star\((v)\) are given by the columns of the following \((d + 1) \times (d + 1 + r)\)-matrix:

\[
\begin{pmatrix}
(-1)^{i_1} & \cdots & 0 & (-1)^{i_d} \varepsilon_1^1(v) & \cdots & (-1)^{i_d} \varepsilon_1^r(v)t_1^{r-1} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
0 & \cdots & (-1)^{i_{d+1}} & (-1)^{i_{d+1}} \varepsilon_{d+1}^1(v) & \cdots & (-1)^{i_{d+1}} \varepsilon_{d+1}^r(v)t_{d+1}^{r-1}
\end{pmatrix}, \tag{4.10}
\]
where \( w_q(v) = (-1)^q e_q \) for \( q = 1, \ldots, d + 1 \) in terms of Construction 4.5, and \( i_1, \ldots, i_{d+1} \) depend on \( v \). By slightly abusing the notation let \( M = M(v; j_1, \ldots, j_{d+1}) \) be the \((d + 1) \times (d + 1)\)-minor of the above matrix (4.10) corresponding to the columns with indices \( 1 \leq j_1 < \cdots < j_{d+1} \leq d + 1 + r \) in (4.10) (from left to right). For any integers \( 1 \leq j_1 < \cdots < j_{d+1} \leq d + 1 + r \) there exists an integer \( q \in \{0, \ldots, d + 1\} \), such that the inequalities \( j_1, \ldots, j_q \leq d + 1 \) and \( j_{q+1}, \ldots, j_{d+1} > d + 1 \) hold, where \( j_0 := 0 \). If \( q = d + 1 \) then

\[
\det M = \prod_{p=1}^{d+1} (-1)^{j_p}.
\]

Let \( q \leq d \). The ordering \( t_1 < \cdots < t_{d+1} \) of the variables induces the lexicographical ordering on the polynomials from the ring \( \mathbb{Z}[t_1, \ldots, t_{d+1}] \). In this ordering the maximal monomial in \( \det M \) is equal to

\[
\prod_{p=1}^{q} (-1)^{j_p} \cdot \prod_{s=q+1}^{d+1} (-1)^{j_s} e_s^j(v) t_s^{j_s-1}.
\]

In particular, \( \det M \) is a nonzero polynomial in \( t_1, \ldots, t_{d+1} \). Hence, the left-hand sides in the system of non-equalities \( \det M(v; j_1, \ldots, j_{d+1}) \neq 0 \), where \( (j_1, \ldots, j_{d+1}) \) exhausts all \((d + 1)\)-subsets of \( \{1, \ldots, d + r + 1\} \) and \( v \) runs over \( V_{\Gamma} \), includes no zero polynomials. The set \( X \) of real solutions for this system is the complement to the union of closed subsets with empty interior (namely, zero sets for nonzero polynomials in \( t_1, \ldots, t_{d+1} \)) in \( \mathbb{R}^{d+1} \).

This union is finite because \( \alpha \) is periodic. Indeed, the function \( w_i \) is periodic with respect to \( 2 \cdot \mathbb{Z}^d \) (see (4.5)) for any \( i = 1, \ldots, d + 1 \). It follows directly from the definition that the values of \( e_s^j \) are periodic with respect to \( 2^{j+1} \cdot \mathbb{Z}^d \). Therefore, by (4.4) there are only finitely many pairwise different matrices of the form (4.10), where \( v \) runs over \( V_{\Gamma(d,r)} \).

Therefore, \( X \) is open and dense in \( \mathbb{R}^{d+1} \). Since \( \mathbb{Q}^{d+1} \) is dense in \( \mathbb{R}^{d+1} \), \( X \) has infinitely many rational points with all nonzero coordinates. Choose any such point in \( X \) with the corresponding coordinates \( t'_1, \ldots, t'_{d+1} \in \mathbb{Q} \). Let \( N \) be the least common multiple of the denominators of the reduced fractions \( t'_1, \ldots, t'_{d+1} \in \mathbb{Q} \). After multiplying \( t'_1, \ldots, t'_{d+1} \in \mathbb{Q} \) by \( N \) one obtains \( t''_1, \ldots, t''_{d+1} \in \mathbb{Z} \), such that \( \alpha(d, r, t''_1, \ldots, t''_{d+1}) \) is \((d + 1)\)-independent. Dividing \( t''_1, \ldots, t''_{d+1} \) by the respective greatest common divisor one obtains \( t_1, \ldots, t_{d+1} \in \mathbb{Z} \), such that the columns of (4.10) are primitive vectors in \( \mathbb{Z}^{d+1} \). (Here we use the fact that \( \det M \) is a homogeneous polynomial in \( t_1, \ldots, t_{d+1} \)). This completes the proof.

**Construction 4.13** [GKM-graph \( \Gamma^a \)] For any \( a \in \mathbb{Z} \) define an equivalence relation \( \sim_a \) on \( \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R} \) by putting \( x \sim_a y \) for any \( x, y \in \mathbb{R}^{d+1} \), such that \( x = y + (u, 0) \) for some \( u \in a \cdot \mathbb{Z}^d \). For any \( r \in \mathbb{N} \) and any \( a \in 2^{r+1} \cdot \mathbb{Z} \) define the graph \( \Gamma^a(d, r) \) to be the quotient of \( \Gamma \) by \( \sim_a \). By slightly abusing the notation denote the graph \( \Gamma^a(d, r) \) by \( \Gamma^a \). Define the axial function \( \alpha^a = \alpha^a(d, r, t_1, \ldots, t_{d+1}) \) and the connection.

© Birkhäuser
\n
\n
Example 4.14 For any \(d \geq 1\) the GKM-graph \(\Gamma^2(d, 0)\) is isomorphic to the edge graph of the standard \((d + 1)\)-dimensional cube \(I_{d+1}^d(0)\) with the axial function induced by the embedding of \(I_{d+1}^d(0)\) to \(R^{d+1}\). Here

\[
\mathbb{I}_R^d(x) := \{y = (y_1, \ldots, y_d) \in R^d | |x_i - y_i| \leq R\},
\]

denotes the \(d\)-cube with center at \(x = (x_1, \ldots, x_d) \in R^d\) and with edges of length \(2R\).

By slightly abusing the notation let \([-\] = \([-\]_a: \(R^{d+1} \rightarrow R^{d+1}/ \sim_a\) be the quotient map.

Proposition 4.15 For any non-negative integer \(r \in \mathbb{Z}\), let \(a \in 2^{r+1} \cdot \mathbb{Z}\). Then the GKM-graph \(\Gamma^a(d, r)\) is a well-defined graph with finitely many vertices and edges.

Proof The quotient \((R^d \times \mathbb{R})/ \sim_a\) is obtained as a product of \(R\) with the torus obtained by gluing all pairs of opposite facets of the cube \(I_{d/2}^d(0)\) by respective translations in \(R^d\). Hence, \(V_{\Gamma^a}\) is identified with \([V_{\Gamma} \cap (I_{d/2}^d(0) \times \{0, 1\})]\). It follows from (4.7) that \([\epsilon^j_i]\) is well defined for any \(i = 1, \ldots, d + 1; j = 1, \ldots, r\). The graph \(\Gamma^a\) has neither loops nor multiple edges because the integral distance between distinct vertices of any its edge (with respect to \(Z^d\)) is less or equal to \(2r\) and the integral length of any nonzero element from \(a \cdot Z^d\) is greater or equal to \(2^{r+1}\). Consider any edge in \(\Gamma(d, r)\) of the form \(E_j(u) = (u, v)\) for some \(u = (x, s), v = (y, s), s = 0, 1\). By Construction 4.3, \(\epsilon^{j+1}_i(u) = \epsilon^{j+1}_i(v)\) holds for any \(i = 1, \ldots, d + 1\) and \(y = x + w, w \in 2j \cdot \{\pm 1\}^d\).

By Lemma 4.9, for any \(z \in 2r+1 \cdot Z^d\):

\[
\epsilon^{j+1}_i(A_z(u)) = (-1)^k \epsilon^{j+1}_i(u) = (-1)^k \epsilon^{j+1}_i(v) = \epsilon^{j+1}_i(A_z(v)),
\]

for some \(k \in \mathbb{Z}\). Clearly, \(A_z(v) - A_z(u) = v - u \in 2j \cdot \{\pm 1\}^d \times \{0\}\). Therefore, \(A_z E_j(u)\) is an edge of \(\Gamma(d, r)\). The proof is complete.

5 Euler characteristic of face posets and proof of the main theorem

In this section, we prove Theorem 1 by showing that the GKM-graph constructed in the previous section satisfies all necessary conditions. Nonextendibility is proved by Corollary 2.14 (this would imply (ii) of Theorem 1). The nonrealizability (iii) is proved by using the acyclicity Theorem 3.15 from [15] by comparison results (Proposition 3.9) and by some computations of Euler characteristic for posets given below.

Lemma 5.1 Let \(b \in \mathbb{N}\).

(i) The number of distinct subgraphs \([[\text{Cube}(v)] | v \in V_{\Gamma^{2b(d, 0)}}]\) in \(\Gamma^{2b}(d, 0)\) is equal to \(2bd\).
(ii) The number of $q$-faces in $\Gamma^{2b}(d, 0)$ is equal to

$$2^{d-q+1} \cdot \left( b^d \binom{d}{q} + b^{d-q+1} \binom{d}{q-1} \right).$$

**Proof** The graph $\Gamma^{2b}(d, 0)$ has exactly $2 \cdot (2b)^d$ vertices, and any such vertex $u$ belongs to the unique cube $\text{Cube}(u)$. Since the number of vertices for any such cube is $2^d$, the number of pairwise distinct cubes (of the form $\text{Cube}(u)$) in $\Gamma^{2b}(d, 0)$ is equal to $2b^d$. This proves (i).

Any vertex of $\Gamma^{2b}(d, 0)$ belongs to exactly $\binom{d}{q}$ horizontal $q$-faces. Each cubical (i.e. not containing a vertical edge) $q$-face of $\Gamma^{2b}(d, 0)$ has exactly $2^q$ vertices. Therefore, the number of cubical $q$-faces in the graph is

$$2(2b)^d \binom{d}{q} \frac{1}{2^q} = 2^{d-q+1} b^d \binom{d}{q}.$$

Each vertex of the graph belongs to $\binom{d}{q-1}$ non-cubical (i.e. having a vertical edge) $q$-facets in total. Each of such $q$-faces has exactly $2(2b)^{q-1}$ vertices. Therefore, the number of non-cubical $q$-faces in the graph is equal to

$$2(2b)^d \binom{d}{q} \frac{1}{2(2b)^{q-1}} = (2b)^{d-q+1} \binom{d}{q-1}.$$

This implies the formula from (ii). □

**Lemma 5.2** For any $a = 2b \in 2\mathbb{N}$ one has

$$\tilde{\chi}(\Delta(S^{\text{op}}_{\Gamma^a(d, 0)})) = (-1)^d (2b^d - (2b - 1)^d). \quad (5.1)$$

In particular, $\tilde{\chi}(\Delta(S^{\text{op}}_{\Gamma^a(d, 0)}))$ is nonzero and has sign $(-1)^{d+1}$ for any $b, d \geq 2$.

**Proof** Since the connection on $\Gamma^a(d, 0)$ is complete one may apply Proposition 3.13 by Proposition 3.14 to $\Delta(S^{\text{op}}_{\Gamma^a(d, 0)})$. The computation of the corresponding Euler characteristic is then given as follows by using Lemma 5.1:

$$\tilde{\chi}(\Delta(S^{\text{op}}_{\Gamma^a(d, 0)})) = -1 + \sum_{q=0}^{d} (-1)^q 2^{q+1} \binom{d}{d-q} + b^{q+1} \binom{d}{d-q-1}$$

$$= -1 + 2b^d \sum_{q=0}^{d} (-1)^q \left( \frac{d}{q} \right)^{2^q} + \sum_{q=0}^{d} (-1)^q \left( \frac{d}{q+1} \right)^{(2b)^{q+1}}$$

$$= -1 + 2b^d (1)^d + 1 - (1 - 2b)^d = (-1)^d (2b^d - (2b - 1)^d).$$
The claim about the sign follows from the obvious inequality which holds for any $b, d \geq 2$:

$$2 < \left( \frac{2b - 1}{b} \right)^d.$$  

\[ \square \]

**Example 5.3** Notice that if $a = 2$ or $d = 1$ then $\Delta(S^{\text{op}}_{\Gamma_{n(d,r)}})$ is homeomorphic to a $d$-sphere. In this case Lemma 5.2 implies that its reduced Euler characteristic is equal to $(-1)^d$.

For any $d \geq 1$ and $r \geq 0$ let $\Gamma_{d+1}^{d+1+r} := \Gamma_{d+1}^{2r+2}(d, r)$ be the $(d + 1 + r)$-valent GKM-graph of rank $d + 1$ constructed earlier in Sect. 4. Now everything is at hand to prove the main theorem of this paper.

**Theorem 5.4** For any $n \geq k \geq 2$ there exists an $(n, k)$-type GKM-graph $\Gamma$ satisfying the following properties:

(i) The GKM-graph $\Gamma$ is $k$-independent;

(ii) The GKM-graph $\Gamma$ does not have nontrivial extensions;

(iii) The GKM-graph $\Gamma$ is not realized by a GKM-action if $n = k = 3$ or $n \geq k \geq 4$.

**Proof** Lemma 4.10 implies that $\Gamma_{d+1}^{d+1+r}$ is a GKM-graph. We check that the properties (i), (ii) hold for such a GKM-graph. The existence of $t_1, \ldots, t_{d+1}$, such that (i) holds is granted by Lemma 4.12. By the construction, the $(d + 1)$-face $\Xi := \Gamma_{d+1}^{2r+2}(d, 0)$ of $\Gamma_{d+1}^{d+1+r}$ has $r$ chords $E_j(v)$ at any vertex $v \in V_\Xi$, where $j$ runs over $1, \ldots, r$. In particular, any transversal edge to the $(d + 1)$-face $\Xi$ in $\Gamma$ is a chord for the face $\Xi$. Hence, by Corollary 2.14 the GKM-graph $\Gamma_{d+1}^{d+1+r}$ has no nontrivial extensions. This establishes (i), (ii).

Now we prove the nonrealizability (iii) by assuming the contrary and obtaining a contradiction with acyclicity properties given by Theorem 3.15 from [15]. Let $d \geq 3$. Suppose that there exists a GKM-action of $T_{d+1}$ on $M_{d+1+r}$ yielding the GKM-graph $\Gamma_{d+1}^{d+1+r}$. Consider a face $\Omega$ that is isomorphic to $\Gamma_{d+1}^{2r+2}(d - 1, 0)$ in $\Gamma_{d+1}^{d+1+r}$. By Proposition 3.9 the face $\Omega$ is realizable by a torus (face) manifold $F$ in $M$, and $(S_M)^{\text{op}}_{\leq F}$ is isomorphic to $S_{\Gamma_{d+1+r}^{2r+2}(d-1,0)}$. By Theorem 3.15, the poset $(S_M)^{\text{op}}_{\leq F}$ is $(d - 2)$-acyclic. Hence, one has

$$\tilde{\chi}(\Delta((S_{\Gamma_{d+1+r}^{d+1+r}})_{< \Omega})) = \tilde{\chi}(\Delta((S_M)^{\text{op}}_{\leq F})) = (-1)^{d-1} \text{rk } H^{d-1}((S_M)^{\text{op}}_{\leq F}).$$

One has

$$\tilde{\chi}(\Delta((S_{\Gamma_{d+1+r}^{d+1+r}})_{< \Omega})) = \tilde{\chi}(\Delta((S_{\Gamma_{d+1+r}^{d+1+r}})_{< \Omega})).$$

However, by Lemma 5.2 the last expression is nonzero and has sign $(-1)^d$. This contradiction proves (iii). For $d = 2$ and $r = 0$ the proof is conducted in a similar way to the above by taking $\Omega = \Gamma_3^3$. The proof is complete.  

\[ \square \]
6 Concluding remarks

Clearly, 3-independence property ($n > k \geq 3$, or equivalently $d = 2, r > 0$) is not enough to prove nonrealizability of $I_{d+1}^{d+1+r}$ by using the argument from the proof above. A possible strategy to prove nonrealizability of $I_{d+1}^{d+1+r}$ for $d = 1, 2$ is to show that the poset $(S_{I_{d+1}^{d+1+r}})_2$ consisting of faces of dimension not exceeding 2 has nontrivial first homology. Indeed, by the acyclicity theorem of [4], the poset $(S_{M})_2$ of faces of dimension not exceeding 2 in $S_M$ is 1-acyclic. On the other hand, the posets $(S_{I_{d+1}^{d+1+r}})_2$ and $(S_{M})_2$ can be shown to be isomorphic (in the 3-independent case) by using Proposition 3.9. However, we do not pursue such a tedious task here. We also remark that the axial function on $I_{d+1}^{d+1+r}$ depends on the values $t_1, \ldots, t_{d+1} \in \mathbb{Z}$ and, therefore, is defined implicitly and in a non-unique way. So the corresponding GKM-ring depends on $t_1, \ldots, t_{d+1}$, as well.

Define the graph $G = G(d)$ embedded in $\mathbb{R}^d$ as the union of edges

$$
(x, x + \sum_{i=1}^{d} (-1)^{u_i} e_i),
$$

where $x$ runs over $\mathbb{Z}^d$, $u = (u_1, \ldots, u_d)$ runs over $\{\pm 1\}^d$. Let $G^a(d) := G(d)/_a$ be the $2^d$-regular graph embedded in the torus $(S^1)^d$ any even $a \in 2 \cdot \mathbb{Z}$. Let $p: \Gamma^a(d, 0) \to G$ be the graph morphism induced by the composition $\Gamma(d, 0) \to G(d)$ of two morphisms. The first morphism replaces every cubical graph (i.e. a connected component of $X^d_0 \times \{0\} \cup X^d_1 \times \{1\}$) with its vertex having the least coordinates and maps each vertical edge to the corresponding edge of the form (6.1). The second morphism is induced by the projection $\mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ onto the first factor. A face in the obtained graph is by definition a subgraph in $G^a(d)$ obtained as an image of a face under $p$. Define the poset $P(d, a)$ as the poset of all faces in $G^a(d)$. One has $\dim(P(d, a)) = d$. Clearly, the order complexes of $\mathcal{P}(2, 2a)^{op}$ and of $S_{P_G(2, 0)}^{op}$ are homotopy equivalent. Conjecturally, for any $a \geq 2$ one has a homotopy equivalence:

$$
\Delta(\mathcal{P}(2, 2a)^{op}) \simeq (S^1)^{\vee 4a-1} \lor (S^2)^{\vee 2a^2}.
$$

The resulting Euler characteristic agrees with the formula from Lemma 5.2.

Acknowledgements. It is pleasure to acknowledge many fruitful discussions of GKM-theory with Shintarô Kuroki as well as hospitality of Okayama University of Science, where a part of the current paper was written. Some general remarks about the text and useful observations on geometry of torus actions made by A. Ayzenberg positively affected this research. The detailed and insightful remarks of the anonymous referee helped to eliminate several important issues and improve the exposition of the results in the present paper. Finally, the author expresses his gratitude to M. Franz for spotting an error in the early version of the text.

Data availability statement. Data sharing not applicable to this article as no data sets were generated or analysed during the current study.
References

1. Akhiezer, D.: Lie Group Actions in Complex Analysis. Asp. Math., vol. 27, p. vii+204. Vieweg+Teubner Verlag, Berlin (1995)
2. Ayzenberg, A., Cherepanov, V.: Torus actions of complexity one in non-general position. Osaka J. Math. 4, 839–853 (2021)
3. Ayzenberg, A., Masuda, M.: Orbit Spaces of Equivariantly Formal Torus Actions. arXiv:1912.11696
4. Ayzenberg, A., Masuda, M., Solomadin, G.: How is a graph not like a manifold? Sb. Math. 214(6), 41–68 (2023)
5. Bredon, G.: Introduction to Compact Transformation Groups, 2nd edn., p. xiv+459. Academic Press, Cambridge (1972)
6. Carlson, J., Gamse, E.A., Karshon, Y.: Realization of Fixed-Point Data for GKM Actions. https://www.ma.imperial.ac.uk/jcarlson/realization.pdf
7. Darby, A., Kuroki, S., Song, J.: Equivariant cohomology of torus orbifolds. Can. J. Math. 74(2), 299–328 (2022)
8. Goertsches, O., Konstantis, P., Zoller, L.: GKM theory and Hamiltonian non-Kähler actions in dimension 6. Adv. Math. 368, 107–141 (2020)
9. Goresky, M., Kottwitz, R., MacPherson, R.: Equivariant cohomology, Koszul duality, and the localization theorem. Invent. Math. 131, 25–83 (1998)
10. Guillemin, V., Holm, T., Zara, C.: A GKM description of the equivariant cohomology ring of a homogeneous space. J. Algebraic Comb. 23, 21–41 (2006)
11. Guillemin, V., Zara, C.: One-skeleta, Betti numbers, and equivariant cohomology. Duke Math. J. 107, 283–349 (2001)
12. Kuroki, S.: Introduction to GKM-theory. Trends Math. New Ser. 11(2), 111–126 (2009)
13. Kuroki, S.: Upper bounds for the dimension of tori acting on GKM manifolds. J. Math. Soc. Japan 71(2), 483–513 (2019)
14. Maeda, H., Masuda, M., Panov, T.: Torus graphs and simplicial posets. Adv. Math. 212(2), 458–483 (2007)
15. Masuda, M., Panov, T.: On the cohomology of torus manifolds. Osaka J. Math. 43(3), 711–746 (2006)
16. Rota, G.-C.: On the foundations of combinatorial theory I. Theory of Möbius functions. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 2, 340–368 (1964)
17. Stanley, R.P.: Enumerative Combinatorics, vol. I, p. xiii+306. Wadsworth and Brooks/Cole, Monterey (1986)
18. Takuma, S.: Extendability of symplectic torus actions with isolated fixed points. RIMS Kokyuroku 1393, 72–78 (2004)
19. Tarski, A.: A Decision Method for Elementary Algebra and Geometry, 2nd edn., p. iv+63. RAND Corporation, Santa Monica (1951)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.