DG-ALGEBRAS AND DERIVED A∞-ALGEBRAS

STEFFEN SAGAVE

Abstract. A differential graded algebra can be viewed as an A∞-algebra. By a theorem of Kadeishvili, a dga over a field admits a quasi-isomorphism from a minimal A∞-algebra. We introduce the notion of a derived A∞-algebra and show that any dga A over an arbitrary commutative ground ring k is equivalent to a minimal derived A∞-algebra. Such a minimal derived A∞-algebra model for A is a k-projective resolution of the homology algebra of A together with a family of maps satisfying appropriate relations.

As in the case of A∞-algebras, it is possible to recover the dga up to quasi-isomorphism from a minimal derived A∞-algebra model. Hence the structure we are describing provides a complete description of the quasi-isomorphism type of the dga.

1. Introduction

An A∞-algebra over a commutative ring k is a Z-graded k-module A with structure maps m_j: A^⊗j → A[2 − j] satisfying certain relations. A∞-algebras were introduced by Stasheff in the 1960’s [21]. We refer to Keller’s survey [13] for background and for references to the various applications A∞-algebras have found in different fields of mathematics.

A differential graded algebra over k is an A∞-algebra with m_1 the differential, m_2 the multiplication, and m_j = 0 for j ≥ 3. If k is a field, a theorem of Kadeishvili [11] asserts that every k-dga A admits a quasi-isomorphism of A∞-algebras from a minimal A∞-algebra, where minimal means m_1 = 0. The underlying graded k-algebra of such a minimal model for A is the homology algebra H_*(A). The quasi-isomorphism type of A can be recovered from its minimal model. Therefore, minimal A∞-algebras provide an alternative description for quasi-isomorphism classes of dgas [13] §3.3, and the minimal A∞-structure on H_*(A) specifies precisely the additional information needed to reconstruct the quasi-isomorphism type of a dga from its homology [13] §2.1.

From now on let k be a commutative ring. Then the conclusion of Kadeishvili’s theorem will in general not hold if the homology H_*(A) is not k-projective since for example a k-linear cycle selection morphism may not exist.

The aim of the present paper is to generalize the notion of an A∞-algebra to a context in which a dga A admits a minimal model without restrictions on k or H_*(A), and thereby to provide a different description of the quasi-isomorphism type of A.

Our approach is motivated by the following two observations. On the one hand, A∞-algebras are closely related to Hochschild cohomology. For example, the m_3 of the minimal A∞-structure on H_*(A) is a cocycle in the Hochschild complex of H_*(A) whose cohomology class encodes relevant information about A; see [2]. On the other hand, Hochschild cohomology is for many purposes the ‘wrong’ cohomology theory if we apply it to k-algebras which are not k-projective. In this case,
one should rather consider the derived Hochschild cohomology, which is also known as Shukla cohomology. Following Baues and Pirashvili\[1\], the latter cohomology theory can be defined as the Hochschild cohomology of a $k$-projective dga resolving the algebra.

Our strategy for finding minimal models in the general context is therefore to first take a $k$-projective resolution of $H_\ast(A)$ in the direction of a new grading, and then to look for the analog to the minimal $A_\infty$-structure on this bigraded $k$-module. To take the additional grading into account, we introduce the notion of a derived $A_\infty$-algebra (d$A_\infty$-algebra for short). A d$A_\infty$-algebra is a $(\mathbb{N}, \mathbb{Z})$-bigraded $k$-module $E$ with structure maps $m_{ij}^E \colon E^{(2)} \to E[i, 2 - (i + j)]$ for $i \geq 0$ and $j \geq 1$ satisfying appropriate relations. It is minimal if $m_{01}^E = 0$. Both dgas and $A_\infty$-algebras can be considered as d$A_\infty$-algebras concentrated in degree 0 of the $\mathbb{N}$-grading. The notion of equivalence suitable for our purposes is that of an $E_2$-equivalence of d$A_\infty$-algebras, which is detected on the iterated homology with respect to $m_{01}^E$ and $m_{11}^E$.

**Theorem 1.1.** Let $A$ be a differential graded algebra over a commutative ring $k$. There exists a $k$-projective minimal d$A_\infty$-algebra $E$ together with an $E_2$-equivalence $E \to A$ of d$A_\infty$-algebras. This minimal d$A_\infty$-algebra model $E$ of $A$ is well defined up to $E_2$-equivalences between $k$-projective minimal d$A_\infty$-algebras. Particularly, the bigraded $k$-module $E$ together with the differential $m_{ij}^E$ and the multiplication $m_{02}^E$ is a $k$-projective resolution of the graded $k$-algebra $H_\ast(A)$.

The first step in the proof of the theorem is to construct a d$A_\infty$-algebra $B$ with $m_{ij}^B = 0$ if $i + j \geq 3$ together with a map $B \to A$ of d$A_\infty$-algebras such that the induced map $H_\ast(B, m_{01}^B) \to H_\ast(A)$ is a $k$-projective resolution. To find such a $B$, we employ the cofibrant replacement in a resolution model category structure on the category of simplicial dgas. The second step is to replace $B$ by a minimal d$A_\infty$-algebra, which is now possible since $H_\ast(B, m_{01}^B)$ is $k$-projective.

If $E$ is a minimal d$A_\infty$-algebra model of the dga $A$, the sum $m_{01}^E + m_{12}^E + m_{21}^E$ is a cocycle in a complex computing the derived Hochschild cohomology of $H_\ast(A)$. Its cohomology class does not depend on the choice of the minimal model. This derived Hochschild cohomology class $\gamma_A \in \text{dHH}^{1,-1}(H_\ast(A))$ associated with $A$ generalizes the Hochschild cohomology class of a dga over a field studied by Benson, Krause, and Schwede in \[2\]. For example, $\gamma_A$ determines all triple matric Massey products in $H_\ast(A)$.

Let $k = \mathbb{Z}/p^2$ with $p$ an odd prime, and let $A$ be the dga which has a copy of $k$ in degrees 0 and 1 and multiplication by $p$ as the differential $A_1 \to A_0$. Then $H_\ast(A)$ is an exterior algebra over $\mathbb{F}_p$ on a class in degree one. In Example 5.1 we describe a minimal d$A_\infty$-algebra model of $A$. The derived Hochschild cohomology class $\gamma_A$ defined by means of this minimal model is non-zero.

The information encoded in a minimal d$A_\infty$-algebra model is not restricted to the resolution of $H_\ast(A)$ and the triple Massey products. In fact, we can recover the dga from a minimal d$A_\infty$-algebra model:

**Theorem 1.2.** For a d$A_\infty$-algebra $E$, there is an associated differential graded algebra $\text{Tot} \text{Hom}_E(E, E)$. If $E$ is a minimal d$A_\infty$-algebra model of a dga $A$, the dga $\text{Tot} \text{Hom}_E(E, E)$ is quasi-isomorphic to $A$.

Therefore, we have given an answer to the question of which additional structure is needed to reconstruct the quasi-isomorphism type of a dga over a commutative ground ring from its homology algebra: A $k$-projective resolution of its homology equipped with a minimal d$A_\infty$-algebra structure provides the necessary information.

A different approach to describe quasi-isomorphism types of dgas is to use Postnikov systems and $k$-invariants. This approach is used by Dugger and Shipley in
Our main object of study is the following generalization of an \( A_\infty \)-algebra whose definition was already outlined in the Introduction.

**Definition 2.1.** A derived \( A_\infty \)-algebra (\( dA_\infty \)-algebra for short) is a \((\mathbb{N}, \mathbb{Z})\)-bigraded \( k \)-module \( E \) with \( k \)-module maps \( m_{ij} : E^{|i-j|} \to E[i, 2 - (i + j)] \) and a unit map \( \eta_E : k \to E \) for \( i \geq 0, j \geq 1 \) satisfying

\[
(2.2)_{uv} \sum_{i+p=q, j+t=r} (-1)^{r+q+t} m_{ij} E^r v \cdot m_{pq} E^q u = 0 \quad \text{for } i \geq 0, j \geq 1, r, q \geq 0
\]
for all \(u \geq 0, v \geq 1\) and the unit condition:
\[
\begin{align*}
    m_{01}^E \eta_E = 0, \quad m_{02}^E (\eta_E \otimes 1) &= 1 = m_{02}^E (1 \otimes \eta_E), \\
    m_{ij}^E (1^{\otimes r-1} \otimes \eta_E \otimes 1^{\otimes j-r}) &= 0 \text{ if } i + j \geq 3 \text{ with } 1 \leq r \leq j
\end{align*}
\]

(2.3)

**Example 2.4.** A derived \(A_\infty\)-algebra concentrated in horizontal degree 0 is the same as a (strictly unital) \(A_\infty\)-algebra with the \(m_0\) as structure maps: the equations \([2.2]_{uv}\) for \(u = 0\) are the \(A_\infty\)-relations. Since a \(k\)-dga \(A\) is a special instance of an \(A_\infty\)-algebra, it is a \(dA_\infty\)-algebra concentrated in horizontal degree 0 for which only \(m_0^1\) and \(m_0^2\) may be non-zero.

**Definition 2.5.** A map of \(dA_\infty\)-algebras from \(E\) to \(F\) consists of a family of \(k\)-module maps \(f_{pq}: E^{\otimes q} \to F[p, 1 - (p + q)]\) with \(p \geq 0\) and \(q \geq 1\) satisfying
\[
\begin{align*}
    \sum_{i + p = v, j + q = w + 1} \sum_{r + t = j, q \geq 1, i, r, p, t \geq 0} (-1)^{r + t + p j} f_{ij}(1^{\otimes r} \otimes m_{pq}^E \otimes 1^{\otimes t})
\end{align*}
\]

(2.6)\(_{uv}\)

\[
\sigma = u + \sum_{w=1}^{i-1} \left( j p_w + w(q_{j-w} - p_w) + q_{j-w} \left( \sum_{s=j-w+1}^{i} p_s + q_s \right) \right)
\]

(2.8)

for \(u \geq 0, v \geq 1\) and the unit condition
\[
\begin{align*}
    f_{01}^E \eta_E = \eta_F \quad \text{and} \quad f_{pq}(1^{\otimes r-1} \otimes \eta_E \otimes 1^{\otimes j-r}) = 0 \text{ if } p + q \geq 2 \text{ with } 1 \leq r \leq q.
\end{align*}
\]

The \(\sigma\) governing the sign in \((2.6)_{uv}\) is

**Example 2.9.** Maps between \(dA_\infty\)-algebras concentrated in horizontal degree 0 are maps of \(A_\infty\)-algebras.

In Section 4 we will give an equivalent but more economic description of \(dA_\infty\)-algebras and their maps in terms of a structure on the reduced tensor algebra of \(E[0, 1]\). Amongst others, we employ it to define the composition of \(dA_\infty\)-algebra maps and the notion of a module over a \(dA_\infty\)-algebra, and it will also help to explain the signs. We use the less streamlined Definition 2.1 here since both our examples and our notion of equivalence fit in more naturally.

We single out an interesting special case of a \(dA_\infty\)-algebra:

**Definition 2.10.** A bidga is a \(dA_\infty\)-algebra \(B\) with \(m_{ij}^B = 0\) if \(i + j \geq 3\). A map \(f: A \to B\) of bidgas is a map of \(dA_\infty\)-algebras with \(f_{pq} = 0\) for \(p + q \geq 2\). We write bidga\(_k\) for the resulting category.

Of course, dgas give rise to bidgas concentrated in horizontal degree 0.

**Remark 2.11.** The category bidga\(_k\) admits a different description. Let biCh\(_k\) be the category of \((\mathbb{N}, \mathbb{Z})\)-graded bicomplexes of \(k\)-modules. Objects are bidigraded \(k\)-modules \(E\) with differentials \(d_0: E \to E[0, 1]\) and \(d_1: E \to E[1, 0]\) satisfying \(d_0 d_0 = 0, d_1 d_1 = 0,\) and \(d_0 d_1 = d_1 d_0\). The tensor product of bidigraded \(k\)-modules induces a symmetric monoidal product on biCh\(_k\). A monoid in biCh\(_k\) is a bicomplex \(B\) with a multiplication \(\mu: B \otimes B \to B\) satisfying associativity, two Leibniz rules, and a unit condition. Setting \(m_{ij}^B = d_0, m_{11}^B = d_1\) and \(m_{02}^B = \mu\), we observe that the six non-trivial formulas \([2.2]_{uv}\) for \(u + v \leq 3\) are the associativity law, the two Leibniz rules, and the three identities for the differentials. After comparing the unit conditions, we see that bidga\(_k\) is the category of monoids in biCh\(_k\). This is similar to the characterization of dgas as the monoids in the category of chain complexes with respect to the tensor product of chain complexes.
Definition 2.12. A twisted dga (tdga for short) is a $dA_{\infty}$-algebra $T$ for which only $m^T_{i2}$ and $m^T_{i1}$ with $i \geq 0$ may be non-zero. A map $f: T \rightarrow S$ of tdgas is a map of $dA_{\infty}$-algebras with $f_{pq} = 0$ if $q \geq 2$. We write tdga$_k$ for the resulting category.

Every bidga is a tdga. The name twisted dga is chosen in view of

Definition 2.13. [8] A twisted chain complex $E$ is an $(\mathbb{N}, \mathbb{Z})$-graded $k$-module with differentials $d^E_i: E \rightarrow E[i, 1-i]$ for $i \geq 0$ satisfying

$$(2.14)_u \sum_{i+p=u} (-1)^i d^E_i d_p = 0$$

for $u \geq 0$. A map of twisted complexes $E \rightarrow F$ is a family of maps $f_i: E \rightarrow F[i, -i]$ satisfying

$$(2.15)_u \sum_{i+p=u} (-1)^i f_i d^E_p = \sum_{i+p=u} d^F f_p.$$ 

The composition of two maps $f: E \rightarrow F$ and $g: F \rightarrow G$ is defined by $(gf)_u = \sum_{i+p=u} g_i f_p$. We write tCh$_k$ for the resulting category.

A class of examples arises from the fact that biCh$_k$ is a subcategory of tCh$_k$. The signs in Definition 2.13 differ from those in [8]. We explain our choice in Remark 4.4. Some authors use the term ‘multicomplex’ instead of ‘twisted chain complex’.

Remark 2.16. Similarly as in Remark 2.11 the tensor product of graded modules defines a symmetric monoidal product on tCh$_k$. Monoids in tCh$_k$ are tdgas. The resulting composition tdga maps is of course a special case of the composition of $dA_{\infty}$-algebra maps to be given in Definition 4.5.

Let $E$ be a $dA_{\infty}$-algebra. Identifying (2.2)$_u$ for $v = 1$ with (2.14)$_u$ we see that the $m^E_{i1}$ with $i \geq 0$ specify the underlying twisted chain complex of $E$. For a map $f: E \rightarrow F$ of $dA_{\infty}$-algebras, the $f_i$ form a map of the underlying twisted chain complexes. We thus occasionally write $d_i$ for $m^E_{i1}$ when we refer to the underlying twisted chain complex of a $dA_{\infty}$-algebra $E$.

Remark 2.17. The underlying twisted chain complex is one instance in which twisted chain complexes behave to $dA_{\infty}$-algebras as unbounded chain complexes behave to $A_{\infty}$-algebras. This analogy turns out to be fruitful, and we frequently exploit it in Sections 4 and 6.

Next we study the passage from twisted chain complexes to chain complexes. There is a total complex functor

$$\text{Tot}: \text{tCh}_k \rightarrow \text{Ch}_k, \quad \text{Tot}_n X = \bigoplus_{s+t=n} X_{st},$$

where the differential $d: \text{Tot}_n X \rightarrow \text{Tot}_{n-1} X$ is $\sum_{i \geq 0} d^X_i$. A map of twisted complexes $f: X \rightarrow Y$ induces a map $\sum_{i \geq 0} f_i$ of total complexes. This is well defined since only finitely many $d^X_i$ or $f_i$ leave a fixed $X_{st}$.

If we consider the total complex as a twisted chain complex concentrated in horizontal degree 0, there is a map $\rho_X: X \rightarrow \text{Tot} X$ of twisted chain complexes. Its component $(\rho_X)_i: X \rightarrow (\text{Tot} X)[i, -i]$ is the inclusion of $X_{i*}$ into $(\text{Tot} X)[i, -i]_{i*}$. We can interpret $\rho$ as a natural transformation from the identity on tCh$_k$ to Tot. The presence of this natural transformation is one advantage of the category of twisted chain complexes over the category of bicomplexes.

Lemma 2.18. The functor $\text{Tot}: \text{tCh}_k \rightarrow \text{Ch}_k$ is strongly monoidal, and $\rho$ is a strong monoidal transformation. In particular, if $E$ is a tdga, then $\text{Tot} E$ is a dga, and $\rho_E: E \rightarrow \text{Tot} E$ is a map of tdgas.
The total complex has a filtration defined by $F^p(Tot X)_n = \bigoplus_{s \leq p} X_{s,n-s}$. The $E_2$-term of the resulting spectral sequence is $E_{2pq}^2 \simeq H^h_{pq}(X)$. Here we write $H^h_{pq}(E)$ for the ‘vertical’ homology of $E$ with respect to the differential $d_0$. By formula (2.14) for $u = 2$, $d_1$ induces a differential on $H^h_{pq}(E)$, and we write $H^h_{pq}(H^h_{pq}(E))$ for the resulting ‘horizontal’ homology group.

Maps of twisted complexes induce maps of spectral sequences: given $f : E \to F$ in tCh$_k$, $f_0$ is a $d_0$-chain map on $E \to F$, and formula (2.15) for $u = 1$ ensures that $H^h_{pq}(f_0)$ induces chain map with respect to the $d_1$-differential $H^h_{pq}(d_1)$. Hence we obtain $H^h_{pq}(H^h_{pq}(f_0))$ on the $E_2$-term, and so on.

**Definition 2.20.** A map $f : E \to F$ of twisted complexes is an $E_2$-equivalence if $H^h_{pq}(H^h_{pq}(f_0))$ is an isomorphism for $s \in \mathbb{N}, t \in \mathbb{Z}$. A map of bicomplexes, bidgas, twisted dgas or derived $A_{\infty}$-algebras is an $E_2$-equivalence if the underlying map of twisted chain complexes is.

Accordingly, $f$ is an $E_1$-equivalence if it induces an isomorphism on $H^h_{pq}$, and every $E_1$-equivalence is an $E_2$-equivalence.

The following lemma is immediate.

**Lemma 2.20.** If $X$ is a twisted chain complex with $E_2$-homology concentrated in horizontal degree 0, then $\rho_X : X \to Tot X$ is an $E_2$-equivalence.

3. Resolving dgas by bidgas

The aim of this section is to construct resolutions of dgas that induce resolutions of their graded homology algebras. We saw that a dga can be considered as a bidga concentrated in horizontal degree 0. Particularly, a graded $k$-algebra is a bidga concentrated in horizontal degree 0 with both differentials trivial.

**Definition 3.1.** A termwise $k$-projective resolution of a graded algebra $\Lambda$ is a termwise $k$-projective bidga $E$ with trivial vertical differential together with an $E_2$-equivalence $E \to \Lambda$ of bidgas.

Here we think of a bidga with trivial vertical differential as a differential graded algebra in graded algebras. If $E \to \Lambda$ is a termwise $k$-projective resolution, $E_{\ast,0}$ is a dga which is $k$-projective and whose homology is isomorphic to $\Lambda_{\ast,0}$ concentrated in degree 0. In other words, a termwise $k$-projective resolution is the graded analog of the resolution of a $k$-algebra by a degreewise $k$-projective dga.

If the graded algebra is the homology of a differential graded algebra, we can ask even more:

**Definition 3.2.** Let $A$ be a $k$-dga. A $k$-projective $E_1$-resolution of $A$ is a bidga $B$ together with an $E_2$-equivalence $B \to A$ of bidgas such that $H^h_{pq}(B)$ is $k$-projective and the map $k \to H^h_{pq}(B)$ induced by the unit $k \to B$ splits as a $k$-module map.

A termwise $k$-projective $E_1$-resolution of a dga $A$ is a termwise $k$-projective resolution of the graded algebra $H^h_{\ast}(A)$ which is induced from a resolution of the dga and satisfies an additional unit condition.

**Example 3.3.** Let $p$ be an odd prime and $k = \mathbb{Z}/p^2$. The $k$-dga $A = \Lambda_{\mathbb{Z}/p^2}(w)$ with $|w| = 1$ and $d(w) = p$ is non-formal and has homology $\Lambda_{\mathbb{Z}/p^2}(\overline{v})$ with $|\overline{v}| = 1$. It is also considered in [11] §5.2. We will give a $k$-projective $E_1$-resolution of $A$.

Consider the $(\mathbb{N}, \mathbb{Z})$-bigraded $k$-algebra

$$V = k[a, b, u, v] / (a^2, b^2, au - bv, auv, bav, u^2, v^2),$$

with $|a| = |b| = (0, 0)$ and $|u| = |v| = (0, 1)$. We set $d_0^h(u) = a$ and $d_0^h(v) = b$. This extends uniquely to a differential on $V$. We think of $V$ as a dga concentrated in
horizontal degree 0. Its homology is \( \Lambda^s_\ast (\pi) \) with \( \pi \) is represented by \( au \). Let \( H \) be the \((\mathbb{N}, \mathbb{Z})\)-bigraded \( k \)-algebra

\[
\Lambda^s_\ast (x) \otimes \Gamma^s_\ast (y) \quad \text{with} \quad |x| = (1,0), |y| = (2,0),
\]

concentrated in vertical degree 0. We define \( B = V \otimes H \), so that \( B_{st} = V_{st} \otimes H_{st} \).

The multiplications on \( V \) and \( H \) turn \( B \) into a bigraded \( k \)-algebra. Since \( d^0_{st} = 0 \) for degree reasons, the \( d^1_{st} \) on \( V \) induces a vertical differential \( d^0_{st} \) satisfying the Leibniz rule. We define \( d^1_{st} (xy_i) = (p - pa + b)y_i \) and \( d^1_{st} (y_i) = (p - b - ab)xy_{i-1} \), where \( y_i \) is the divided power algebra generator in degree \( 2i \). The \( d^0_{st} \) is linear with respect to \( a, b, u, \) and \( v \), and \( d^1_{st} \) and \( d^2_{st} \) commute.

It is easy to see \( H^s_{st} (B) = \Lambda^s_\ast (\pi, \pi) \otimes \Gamma^s_\ast (\pi) \) as well as \( d_1 (\pi y_i) = p \pi y_{i-1} \) and \( d_1 (\pi y_i) = p \pi \). Hence \( B \) is a bigda for which \( H^s_{st} (B) \) is bidegreewise \( k \)-projective.

A map of bidg\( \alpha \): \( B \to A \) is defined by setting

\[
\alpha (y_0) = 1, \alpha (a) = p, \alpha (b) = -p, \alpha (u) = w, \quad \text{and} \quad \alpha (v) = -w. \tag{3.4}
\]

This is multiplicative and compatible with the relations among \( a, b, u, \) and \( v \). Since \( \alpha (d^0_{st} (x)) = 0 \), the \( \alpha \) is also compatible with \( d^1_{st} \). Therefore, it is a bidga map. On vertical homology \( \alpha \) induces \( H^s_{st} (\alpha): H^s_{st} (B) \to H^s_{st} (A) \) sending \( y_0 \) to 1 and \( \pi \) to \( \pi \).

This is easily seen to be a resolution of \( H^s_{st} (A) \).

Returning to the general case, a map of \( k \)-projective \( E_1 \)-resolutions from \( B \to A \) to \( B' \to A \) is a bidga map \( f: B \to B' \) making the obvious triangle commutative.

It is defined to be an \( E_2 \)-equivalence if \( f \) is one.

**Theorem 3.4.** Every \( k \)-dga \( A \) admits a \( k \)-projective \( E_1 \)-resolution \( B \to A \). Two such resolutions can be related by a zig-zag of \( E_2 \)-equivalences between \( k \)-projective \( E_1 \)-resolutions.

The proof of the theorem occupies the rest of the section. It uses Quillen’s language of model categories [15, 9]. The point is to use the cofibrant replacement in an appropriate model structure on the category \( \text{sdg}_{\mathbb{A}} \) of simplicial dgas.

We start by briefly recalling Bousfield’s general setup for resolution model structures [3], or rather the dual version of [10]. Let \( \mathcal{C} \) be a pointed model category. Let \( \mathcal{P} \) be a class of group-objects in \( \text{Ho} (\mathcal{C}) \). A morphism \( p : X \to Y \) in \( \text{Ho} (\mathcal{C}) \) is a \( \mathcal{P} \)-epi if \( p : |P, X|_n \to |P, Y|_n \) is onto for every \( P \in \mathcal{P} \) and \( n \geq 0 \), where \( |X, Y|_n = \text{Ho} (\mathcal{C}) (\Sigma^n X, Y) \). An object \( A \in \text{Ho} (\mathcal{C}) \) is \( \mathcal{P} \)-projective if \( p : [A, X]|_n \to [A, Y]|_n \) is onto for all \( \mathcal{P} \)-epis \( p \) and \( n \geq 0 \). Maps in \( \mathcal{C} \) (resp. objects in \( \mathcal{C} \)) are \( \mathcal{P} \)-epis (resp. \( \mathcal{P} \)-projective) if they are in \( \text{Ho} (\mathcal{C}) \). A map in \( \mathcal{C} \) is a \( \mathcal{P} \)-projective cofibration if it has the left lifting property with respect to all \( \mathcal{P} \)-epi fibrations. A map \( X \to Y \) in \( \mathcal{C} \) is \( \mathcal{P} \)-free if it is a composition of an inclusion \( X \to X \prod P \) with \( P \) cofibrant and \( \mathcal{P} \)-projective and an acyclic cofibration \( X \prod P \to Y \). We say that \( \text{Ho} (\mathcal{C}) \) has enough \( \mathcal{P} \)-projectives if every object \( X \in \text{Ho} (\mathcal{C}) \) admits a \( \mathcal{P} \)-epi \( Y \to X \) with \( \mathcal{P} \)-projective source. In this case, we call \( \mathcal{P} \) a class of projective models in \( \text{Ho} (\mathcal{C}) \).

The next definition makes use of the standard model structure on the category of simplicial groups ([15 II.3.7]), the Reedy model structure on \( \text{sC} \) [7 VII.2.12], and the notion of the latching object \( L_n X \) and its structure map \( L_n X \to X \) for \( X \in \text{sC} \) [7 VII.1.5].

**Definition 3.5.** Let \( f : X \to Y \) be a map in \( \text{sC} \).

(i) \( f \) is a \( \mathcal{P} \)-equivalence if \( f_* : [P, X]|_n \to [P, Y]|_n \) is a weak equivalence in simplicial groups for all \( P \in \mathcal{P} \) and \( n \geq 0 \).

(ii) \( f \) is a \( \mathcal{P} \)-fibration if it is a Reedy-fibration and \( f_* : [P, X]|_n \to [P, Y]|_n \) is a fibration of simplicial groups for all \( P \in \mathcal{P} \) and \( n \geq 0 \).

(iii) \( f \) is a \( \mathcal{P} \)-cofibration if the induced maps \( X_n \prod_{L_n X} L_n Y \to Y_n \) are \( \mathcal{P} \)-projective cofibrations for all \( n \geq 0 \).
Theorem 3.6. [3] Theorem 3.3] Let $\mathcal{C}$ be a right proper pointed model category and let $\mathcal{P}$ be a class of projective models for $\text{Ho}(\mathcal{C})$. With the above $\mathcal{P}$-equivalences, $\mathcal{P}$-fibrations, and $\mathcal{P}$-cofibrations, $s\mathcal{C}$ becomes a right proper model category. It is simplicial with respect to the external simplicial structure.

We frequently use the following feature of this model structure:

Lemma 3.7. A $\mathcal{P}$-cofibration in $s\mathcal{C}$ is termwise a $\mathcal{P}$-projective cofibration in $\mathcal{C}$.

Proof. This is a consequence of [8, Lemma 15.3.9], compare [3, Lemma 5.3]. □

Our main example is the category $\text{Ch}_k$ with the projective model structure [9, Theorem 2.3.11]. Since $\text{Ch}_k$ is stable, all objects in $\text{Ho}(\text{Ch}_k)$ are cogroup objects. Let $\mathcal{P}$ be the set of objects $\{S^n(k)|n \in \mathbb{Z}\}$ in $\text{Ho}(\text{Ch}_k) = \mathcal{D}(k)$, where $S^n(k)_i = k$ if $i = n$ and 0 otherwise. The $\mathcal{P}$-epis are the maps which induce surjections in homology.

Lemma 3.8. With this $\mathcal{P}$, the category $\text{Ho}(\text{Ch}_k)$ has enough $\mathcal{P}$-projectives. An object in $\text{Ho}(\text{Ch}_k)$ is $\mathcal{P}$-projective if and only if it has degreewise $k$-projective homology.

Proof. Since every object is the codomain of a $\mathcal{P}$-epi mapping out of a sum of objects in $\mathcal{P}$, there are enough $\mathcal{P}$-projectives. Applying this to a $\mathcal{P}$-projective object $X$ shows that $X$ is a retract of a sum of objects in $\mathcal{P}$. Hence $X$ has degreewise $k$-projective homology. On the other hand, an object with degreewise $k$-projective homology is a retract of a sum of objects of $\mathcal{P}$, and therefore $\mathcal{P}$-projective. □

All objects in $\text{Ch}_k$ are fibrant, so $\text{Ch}_k$ is right proper. Theorem 3.6 yields the $\mathcal{P}$-model structure on $s\text{Ch}_k$.

Analogous to the classical case of simplicial abelian groups and simplicial rings, simplicial dgas can be considered as the monoids in the category $s\text{Ch}_k$ of simplicial chain complexes with respect to the termwise tensor product.

Proposition 3.9. The forgetful functor $U: \text{sdga}_k \to s\text{Ch}_k$ creates a cofibrantly generated model structure on $\text{sdga}_k$ in which a map $f$ is a $\mathcal{P}$-fibration or a $\mathcal{P}$-equivalence if $U(f)$ is. Cofibrant objects in $\text{sdga}_k$ are termwise $\mathcal{P}$-projective cofibrant in $\text{Ch}_k$. The unit map of a cofibrant replacement in $\text{sdga}_k$ splits in homology.

Proof. First notice that the $\mathcal{P}$-model structure on $s\text{Ch}_k$ is cofibrantly generated. To obtain the generating cofibrations $I$ and generating acyclic cofibrations $J$, one has to add the generating acyclic cofibrations of the Reedy model structure [8, Proposition 15.6.24] to the two sets of maps described in [11, Lemma 2.7].

The projective model structure on $\text{Ch}_k$ creates a model structure on $\text{dga}_k$ [18, §5], which gives rise to a Reedy model structure on $\text{sdga}_k$. As the matching object functor is defined as a limit and the forgetful functor $U$ commutes with limits, a map $f$ in $\text{sdga}_k$ is a Reedy fibration or a Reedy weak equivalence if and only if $U(f)$ is.

Since all objects in simplicial abelian groups are fibrant, the $\mathcal{P}$-fibrant objects in $s\text{Ch}_k$ coincide with the Reedy fibrant objects. Therefore, the Reedy fibrant replacement functor in $\text{sdga}_k$ is a fibrant replacement functor for the $\mathcal{P}$-model structure to be constructed.

The cotensor of the external simplicial structure on $s\text{Ch}_k$ is defined as a limit [7, p. 371] and therefore extends to a functor

$$\text{hom}_{\text{sdga}_k}: \mathcal{S}^{op} \times \text{sdga}_k \to \text{sdga}_k.$$ 

As the $\mathcal{P}$-model structure on $s\text{Ch}_k$ is simplicial, $\text{hom}_{\text{sdga}_k}(-, Y)$ maps weak equivalences to $\mathcal{P}$-equivalences and cofibrations to $\mathcal{P}$-fibrations provided $Y$ is $\mathcal{P}$-fibrant.
Hence the simplicial path object for a fibrant object qualifies as a path object for the \( \mathcal{P} \)-model structure on \( \text{sCh}_k \) we are heading for.

Since the \( \mathcal{P} \)-model structure on \( \text{sCh}_k \) is cofibrantly generated, the verification of the axioms for the \( \mathcal{P} \)-model structure on \( \text{sCh}_k \) now follows from applying \cite[Lemma 2.3(2)]{18} to the left adjoint of \( U \). As in Quillen’s original argument \cite[II.4.9]{15}, one can reduce the assumptions to presence of a path object and the fibrant replacement functor as established above.

The generating cofibrations \( T(I) \) of the \( \mathcal{P} \)-model structure on \( \text{sCh}_k \) are obtained by applying the free associative algebra functor \( T \) to \( I \). If \( X \) is \( \mathcal{P} \)-projective and cofibrant, \( - \otimes X \) preserves \( \mathcal{P} \)-free maps. Cobase change also preserves \( \mathcal{P} \)-free maps. The maps in \( I \) are either termwise acyclic cofibrations in the projective model structure on \( \text{Ch}_k \), or termwise of the form \( X \to X \coprod P \) with \( P \in \mathcal{P} \). By the careful analysis of the free associative algebra functor carried out in \cite[Lemma 6.2]{18}, it follows that for a termwise \( \mathcal{P} \)-projective cofibrant \( X \) in \( \text{sCh}_k \), the map \( X \to Y \) obtained from attaching a generating cofibration \( f \in T(I) \) to \( X \) is isomorphic to the inclusion of \( X \) into the colimit of a countable sequence of termwise \( \mathcal{P} \)-free maps: inspecting the proof the second statement of the cited lemma, we see that it is enough to provide the last four statements and Lemma \[3.10\].

The initial object of \( \text{sCh}_k \) is itself termwise \( \mathcal{P} \)-projective. Hence the map from the initial object \( k \) to a cofibrant replacement constructed by means of the small object argument applied to \( T(I) \) is termwise a transfinite composition of \( \mathcal{P} \)-free maps. Therefore, it splits in homology. In particular, any cofibrant replacement is termwise \( \mathcal{P} \)-projective cofibrant. By the retract argument, all cofibrant objects are termwise \( \mathcal{P} \)-projective cofibrant.

Let \( f : X \to Y \) and \( g : X' \to Y' \) be maps in \( \text{Ch}_k \) or \( \text{sCh}_k \). We define the pushout product map \( f \Box g \) to be the induced map \( Y \otimes X' \coprod_{X \otimes X'} X \otimes Y' \to Y \otimes Y' \).

**Lemma 3.10.** Let \( K \) be the class of termwise acyclic cofibrations in \( \text{sCh}_k \), and let \( L \) be the class of maps in \( \text{sCh}_k \) which are termwise of the form \( X \to X \coprod P \) with \( P \) a \( \mathcal{P} \)-projective cofibrant object. Then \((K \cup L) \Box (K \cup L) \subset (K \cup L)\).

**Proof.** The pushout product axiom \cite[Definition 3.1]{19} holds in \( \text{Ch}_k \). It states that if \( f \) and \( g \) are cofibrations in \( \text{Ch}_k \), then the pushout product map \( f \Box g \) is a cofibration which is acyclic if \( f \) or \( g \) is. The axiom immediately implies \( K \Box K \subset K \).

Since objects in \( \mathcal{P} \) are cofibrant, it also shows that for \( P \in \mathcal{P} \) and \( f \in K \), the map \( P \otimes f \) is again in \( K \). This argument implies \( K \Box L \cup L \Box K \subset K \). If both maps are in \( L \), it is enough to show that for two \( \mathcal{P} \)-projective cofibrant objects \( P, P' \in \mathcal{P} \), their product \( P \otimes P' \) is \( \mathcal{P} \)-projective cofibrant. Cofibrancy is clear. Since both have \( k \)-projective homology, the \( E_2 \)-term of the bicomplex spectral sequence computing the homology of \( P \otimes P' \) consists only of a single non-trivial line in which all entries are \( k \)-projective. By Lemma \[3.8\] \( P \otimes P' \) is \( \mathcal{P} \)-cofibrant.

We need an instance of the Dold-Kan correspondence to go back and forth between simplicial chain complexes and bicomplexes. The version in \cite[III Theorem 2.5]{7} is sufficiently general to apply to our context.

The associated chain complex and normalized chain complex are functors

\[
C : \text{sCh}_k \to \text{biCh}_k \quad \text{and} \quad N : \text{sCh}_k \to \text{biCh}_k.
\]

The first is defined by \( C(X)_{\ast \ast} = X_{\ast \ast} \) with horizontal differential given by the alternating sum of the simplicial face maps, and \( N(X) \) is the quotient of \( C(X) \) by the subobject generated by the image of the degeneracies. The functor \( N \) is an equivalence of categories. Its inverse \( \Gamma : \text{biCh}_k \to \text{sCh}_k \) can be defined by

\[
\Gamma(E)_n = \bigoplus_{[n] \to [p]} E_{p \ast}.
\]
with the simplicial structure maps described on [7, p. 148].

The monoidal properties of these functors are as in the classical case discussed in [19, §2.3, §2.4]. We have a shuffle map \( \nabla \) and the Alexander-Whitney map \( AW \),

\[
\nabla : CX \otimes CY \to C(X \otimes Y) \quad \text{and} \quad AW : C(X \otimes Y) \to CX \otimes CY,
\]

\( \nabla \) is a lax monoidal transformation, \( AW \) is a lax comonoidal transformation, and both maps preserve the subcomplexes of degenerate simplicies and induce

\[
\nabla : NX \otimes NY \to N(X \otimes Y) \quad \text{and} \quad AW : N(X \otimes Y) \to NX \otimes NY.
\]

For a simplicial dga \( X \), the associated chain complex of the underlying simplicial chain complex is a bidga with multiplication defined by

\[
CX \otimes CX \xrightarrow{\nabla} C(X \otimes X) \to C(X).
\]

Similarly, \( NX \) is a bidga. For a bidga \( B \), the Alexander-Whitney map induces the structure of a simplicial dga on \( \Gamma(B) \) [19, (2.9)]. As explained in [19, Remark 2.14], the functors \( \Gamma \) and \( N \) do not induce an equivalence between sdga\(_k\) and bidga\(_k\), due to the failure of the unit of the adjunction to be monoidal. However, the adjunction counit isomorphism \( \eta : N \Gamma \cong \text{id}_{\text{biCh}_k} \) is monoidal on the level of complexes [19, Lemma 2.11], which is sufficient for our purposes.

**Lemma 3.11.** (i) The functor \( C \) maps \( \mathcal{P} \)-equivalences of simplicial dgas to \( E_2 \)-equivalences of bidgas.

(ii) For a \( k \)-bidga \( B \), there is a canonical \( E_2 \)-equivalence \( C(\Gamma(B)) \to B \) of bidgas which is natural in \( B \).

**Proof.** It is enough to check (i) additively. Since \( C \) commutes with taking vertical homology, the assertion follows from the definition of \( \mathcal{P} \)-equivalences and \( E_2 \)-equivalences.

The map in (ii) is the one inducing the counit \( N(\Gamma(B)) \to B \). It is a map of bidgas since the counit is monoidal and the shuffle map is compatible with the degeneracies. Both \( C \) and \( \Gamma \) (but not \( N \)) commute with taking vertical homology, so the Dold-Kan correspondence for simplicial \( k \)-modules and \( \text{Ch}_k^+ \) applied in each vertical degree shows that \( C(\Gamma(B)) \to B \) is an \( E_2 \)-equivalence. \( \square \)

**Proof of Theorem 3.4.** We consider \( \Gamma(A) \), the constant simplicial dga \( A \), and its cofibrant replacement \( \Gamma(A)^{\text{cof}} \). Lemma 3.8 and Proposition 3.9 show that \( \Gamma(A)^{\text{cof}} \) has \( k \)-projective vertical homology. The unit is the composite \( k \to C(\Gamma(k)) \to C(\Gamma(A)^{\text{cof}}) \), which splits in vertical homology by the last part of Proposition 3.9. Lemma 3.11 then shows that the composite

\[
C(\Gamma(A)^{\text{cof}}) \to C(\Gamma(A)) \to A
\]

is a \( k \)-projective \( E_1 \)-resolution. The second statement is the \( A = A' \) case of the next lemma. \( \square \)

**Lemma 3.12.** Given a map \( A \to A' \) of \( k \)-dgas and \( k \)-projective \( E_1 \)-resolutions \( B_1 \to A_1 \) and \( B_1' \to A' \), there is a commutative diagram

\[
\begin{array}{cccccc}
B_2 & \to & B_2' & \leftarrow & B_2'' \\
\downarrow & & \downarrow & & \downarrow \\
B_1 & \to & A & \leftarrow & B_1'
\end{array}
\]

of bidgas in which all maps \( B_i \to A \) and \( B_i' \to A' \) are \( k \)-projective \( E_1 \)-resolutions.
Proof. We apply the cofibrant replacement functor in the \( \mathcal{P} \)-model structure on \( \text{sdga}_k \) to \( \Gamma(B_1) \rightarrow \Gamma(A) \rightarrow \Gamma(A') \rightarrow \Gamma(B'_1) \) and obtain
\[
\begin{array}{ccccccccc}
C(\Gamma(B_1)_{\text{cof}}) & \rightarrow & C(\Gamma(A)_{\text{cof}}) & \rightarrow & C(\Gamma(A')_{\text{cof}}) & \leftarrow & C(\Gamma(B'_1)_{\text{cof}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C(\Gamma(B_1)) & \rightarrow & C(\Gamma(A)) & \rightarrow & C(\Gamma(A')) & \leftarrow & C(\Gamma(B'_1)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_1 & \rightarrow & A & \rightarrow & A' & \leftarrow & B'_1.
\end{array}
\]
By Lemma 3.11 the upper and the lower row assemble to the desired diagram. \( \square \)

4. Minimal \( dA_\infty \)-algebras

We start by giving an alternative description of \( dA_\infty \)-algebras and their maps. Disregarding unit conditions, an \( A_\infty \)-algebra structure on a \( \mathbb{Z} \)-graded \( k \)-module \( M \) can be encoded by giving a differential on the reduced tensor algebra of the suspension of \( A \) \([3, 21]\). Next we explain how the structure of a twisted chain complex on the reduced tensor algebra (in bigraded \( k \)-modules) describes a \( dA_\infty \)-algebra.

Let \( S \) be the shift \( [0, 1] \) of bigraded \( k \)-modules. There is a canonical isomorphism \( \sigma : S \rightarrow \text{id}[0, 1] \) of degree \( [0, 1] \). It induces an isomorphism
\[
\sigma : \text{Homm}(E^{\otimes i}, F) \rightarrow \text{Homm}((SE)^{\otimes j}, SF) \text{ with } \sigma \sigma(E(\sigma(f)(g))) = (-1)^{\langle q, q \rangle} \sigma f \sigma E.
\]
For a \( k \)-linear map \( m_{ij}^E : E^{\otimes j} \rightarrow E[i, 2-(i+j)] \), we define \( \tilde{m}_{ij}^{E,1} : SE^{\otimes j} \rightarrow SE[i, 1-i] \) to be \( \sigma_2(m_{ij}^E) \).

Let \( T \colon \text{biMod-}k \rightarrow \text{biMod-}k \) be the reduced tensor algebra functor of bigraded \( k \)-modules, i.e., \( \tilde{T}E = \bigoplus_{j \geq 1} E^{\otimes j} \). Let \( m_{ij}^r : E^{\otimes j} \rightarrow E[i, 2-(i+j)] \) be a family of \( k \)-linear maps with \( i \geq 0 \) and \( j \geq 1 \). For fixed \( i \), the \( \tilde{m}_{ij}^{E,1} \) assemble to a map \( \tilde{m}_{ij}^{E,1} : \tilde{T}E \rightarrow SE[i, 1-i] \). More generally, we define
\[
\tilde{m}_{ij}^{E,q} := \sum_{r+t=q} 1^{\otimes r} \otimes \tilde{m}_{ij}^{E,1} \otimes 1^{\otimes t} : SE^{\otimes q} \rightarrow SE[i, 1-i]
\]
for \( q \leq j \) and write \( \tilde{m}_{ij}^E : \tilde{T}SE \rightarrow \tilde{T}SE[i, 1-i] \) for the map whose component mapping \( SE^{\otimes q} \) to \( SE^{\otimes q}[i, 1-i] \) is \( \tilde{m}_{ij}^{E,q} \). The process of building the \( \tilde{m}_{ij}^E \) from the \( \tilde{m}_{ij}^{E,1} \) may also be described using the universal property of \( \tilde{T}SE \) as a (cocomplete) tensor coalgebra.

Lemma 4.1. Let \( E \) be a bigraded \( k \)-module and let \( m_{ij}^E : E^{\otimes j} \rightarrow E[i, 2-(i+j)] \) be a family of \( k \)-linear maps with \( i \geq 0 \) and \( j \geq 1 \) satisfying the unit condition 2.3. The following are equivalent:

(i) The \( m_{ij}^E \) specify a \( dA_\infty \)-algebra structure on \( E \).
(ii) \( \sum_{i+p=u} (-1)^i \tilde{m}_{ij}^E \tilde{m}_{ij}^E = 0 \) for all \( u \geq 0 \).
(iii) \( \sum_{i+p=u} (-1)^i \tilde{m}_{ij}^{E,1} \tilde{m}_{ij}^E = 0 \) for all \( u \geq 0 \).
(iv) \( \tilde{T}SE \) is a twisted chain complex with \( d_i = \tilde{m}_{ij}^E \).

Proof. The signs in (2.2a,b) arise from interchanging \( \sigma \) with the \( \tilde{m}_{ij}^{E,1} \):
\[
(-1)^i \tilde{m}_{ij}^{E,1} (1^{\otimes r} \otimes \tilde{m}_{pq}^{E,1} \otimes 1^{\otimes t}) = (-1)^{p+t+q} \sigma^{-1} m_{ij}^E (1^{\otimes r} \otimes m_{pq}^{E,0} \otimes 1^{\otimes t})
\]
\[
= (-1)^{r+q+t} \sigma^{-1} m_{ij}^E (1^{\otimes r} \otimes m_{pq}^{E,0} \otimes 1^{\otimes t}) \sigma^{\otimes r+q+t}
\]
\( \square \)
The following are equivalent:

Remark 4.4. Because of this, we will frequently switch to the \( \tilde{\cdot} \) and write \( \tilde{\cdot} \) for bidgas. This would be in conflict with the signs resulting from the Dold-Kan \( k \)-splits as a \( k \)-module map, then there exists

\[
\sum_{i+p=u}(-1)^i \tilde{f}_i \tilde{m}^E_p = \sum_{i+p=u} \tilde{m}^F_i \tilde{f}_p \quad \text{for all } u \geq 0.
\]

We define

\[
\tilde{f}_{ij} := \sum_{p_1,\ldots,p_j=q} \tilde{f}_{p_1,q_1} \otimes \ldots \otimes \tilde{f}_{p_j,q_j} : SE^{\otimes q} \to SE^{\otimes j}[p,-p]
\]

and write \( \tilde{f}_p : TSE \to \overline{T}SF[p,-p], p \geq 0, \) for the morphism mapping \( SE^{\otimes q} \) to \( SE^{\otimes j}[p,-p] \) by \( \tilde{f}_{ij} \).

**Lemma 4.3.** Let \( E \) and \( F \) be \( dA_\infty \)-algebras and let \( f_{pq} : E^{\otimes q} \to F[p,1-(p+q)] \) be a family of \( k \)-linear maps with \( p \geq 0 \) and \( q \geq 1 \) satisfying the unit condition (2.7). The following are equivalent:

(i) The \( f_{pq} \) form a \( dA_\infty \)-algebra map from \( E \) to \( F \).

(ii) \( \sum_{i+p=u}(-1)^i \tilde{f}_i \tilde{m}^E_p = \sum_{i+p=u} \tilde{m}^F_i \tilde{f}_p \) for all \( u \geq 0 \).

(iii) \( \sum_{i+p=u}(-1)^i \tilde{f}_i \tilde{m}^E_p = \sum_{i+p=u} \tilde{m}^F_i \tilde{f}_p \) for all \( u \geq 0 \).

(iv) The \( \tilde{f}_i \) form a map of twisted chain complexes.

**Remark 4.4.** One advantage of using the \( \tilde{m}^E_{ij} \) and the \( \tilde{f}_{ij} \) is that their bidegrees \( |\tilde{m}^E_{ij}| = (i,1-i) \) and \( |\tilde{f}_{ij}| = (i,-i) \) do not depend on \( j \). As one can already see by comparing the last lemma with Definition 2.3 this reduces the complexity of signs considerably. Because of this, we will frequently switch to the \( \tilde{m}_{ij} \) for explicit calculations.

However, there is still a choice behind our sign convention. For example, one may omit the sign in the definition of twisted chain complexes. This would change the sign in the definition of \( dA_\infty \)-algebras and, in turn, the sign in the Leibniz rule for bidgas. This would be in conflict with the signs resulting from the Dold-Kan correspondence relating biCh\( k \) and sdga\( k \). In other words, our sign convention is chosen to ensure that the chain complex associated to a simplicial dga is a \( dA_\infty \)-algebra.

At this point we can also make up the promised definition of the composition of \( dA_\infty \)-algebra maps.

**Definition 4.5.** Given two maps \( g : D \to E \) and \( g : E \to F \), the component \( (\tilde{f}g)_u \) of \( fg \) is \( \sum_{i+p=u} \tilde{f}_i \tilde{g}_p \).

An \( A_\infty \)-algebra is minimal if \( m^A_1 = 0 \). This motivates

**Definition 4.6.** A \( dA_\infty \)-algebra \( E \) is minimal if \( m^E_{01} = 0 \).

The device to produce minimal \( dA_\infty \)-algebras is the following proposition. It promotes both the statement and the proof of [11 Theorem 1] about \( A_\infty \)-algebras to the context of \( dA_\infty \)-algebras.

**Proposition 4.7.** Let \( B \) be a bidga, let \( E \) be its homology with respect to \( m^B_{01} \), and let \( d_1 \) and \( \mu \) be the differential and the multiplication induced on \( E \). If \( E \) is \( k \)-projective in every bidegree and the unit map \( \eta_E : k \to E \) induced by \( \eta_B : k \to B \) splits as a \( k \)-module map, then there exists

(i) a minimal \( dA_\infty \)-structure on \( E \) with \( m^E_{11} = d_1 \) and \( m^E_{02} = \mu \) and
(ii) an $E_2$-equivalence of $dA_\infty$-algebras $f : E \to B$.

To construct the structure inductively, we define

$$(4.8)_{ln} \quad \tilde{z}_{ln} = \tilde{m}_{01}^{E,1} f_{ln} + \tilde{m}_{11}^{E,1} f_{l-1,n} - \sum_{i+j \geq 2, j < n} (-1)^j i_{ij} \tilde{m}_{pm}^{E,1}.$$  

The identity $(2.6)_{uv}$ can be rewritten as $\tilde{m}_{01}^{E,1} \tilde{f}_{uv} = \tilde{f}_{01} \tilde{m}_{uv}^{E,1} - \tilde{z}_{uv}$ by Lemma 4.3.

**Proof of Proposition 4.7.** Since $E$ is $k$-projective, we can choose a $k$-linear cycle selection homomorphism $f_{01} : E \to B$ in the diagram

$$
\begin{array}{c}
E \\
\downarrow f_{01} \\
H_*(B, m_{01}^B) \# \ker m_{01}^B & \to B
\end{array}
$$

We may adjust $f_{01}$ to ensure $f_{01} \eta_E = \eta_B$. Setting $m_{01}^E = 0$, we observe that $(2.6)_{uv}$ holds for $u + v = 1$.

For the definition of $\tilde{m}_{ln}^{E,1}$ and $\tilde{f}_{ln}$, we assume that we have constructed the $m_{ij}^{E,1}$ and $\tilde{f}_{ij}$ for $i + j \leq l + n$ such that $(2.2)_{uv}$ holds for $u + v \leq l + n$, $(2.6)_{uv}$ holds for $u + v < l + n$, and the two unit conditions $(2.3)$ and $(2.7)$ are satisfied. Then the $\tilde{z}_{ln}$ of $(4.8)_{ln}$ is defined.

By Lemma 7.1, $\tilde{z}_{ln}$ has values in $\ker \tilde{m}_{01}$. We define $\tilde{m}_{01}^{E,1}$ to be the composition of $\tilde{z}_{ln}$ with the quotient map $\ker \tilde{m}_{01} \to SE$. By Lemma 7.5 the $m_{ln}^{E,1}$ together with the $\tilde{m}_{ln}^{E,1}$ for $i + j < l + n$ satisfy $(2.2)_{uv}$ for $u + v \leq l + n + 1$. Checking $(2.3)$ for $m_{ln}^{E}$ is easy.

We need to define the $\tilde{f}_{ln}$. In each bidegree, $SE \otimes \mathbb{C}$ is $k$-projective as a tensor product of $k$-projective modules. Since $\tilde{f}_{01} \tilde{m}_{ln}^{E}$ and $\tilde{z}_{ln}$ both have values in $\ker \tilde{m}_{01}$ and coincide when composed with the quotient map to $SE$, their difference lies in the image of $\tilde{m}_{01}^{B}$, and we can find lifts $\tilde{f}_{ln}$ in

$$(SE) \otimes \mathbb{C}$$

The $\tilde{f}_{ln}$ satisfy $(2.6)_{uv}$ for $u + v = l + n$ by construction. By assumption, the inclusions of the submodules on which the unit condition $(2.7)$ requires $f_{ln}$ to be zero splits. Hence we may assume $f_{ln}$ to be zero there.

The input for the last proposition will be a $k$-projective $E_1$-resolution arising from Theorem 3.3. Since these are only well defined up to $E_2$-equivalence, we need to check that such an $E_2$-equivalence of bidgas induces a map of the associated minimal $dA_\infty$-algebras. This issue does not come up when constructing minimal models in the world of $A_\infty$-algebras, since the underlying graded algebra of a minimal $A_\infty$-model is unique up to isomorphism 13 §8.7.

To formulate the next proposition, we need to introduce the notion of a homotopy of $dA_\infty$-algebra maps. We only define it in a special case.

**Definition 4.9.** Let $f$ and $g$ be $dA_\infty$-algebra maps from a minimal $dA_\infty$-algebra $E$ to a bidga $C$. A homotopy from $g$ to $f$ is a family of maps $h_i : TSE \to SC[i, -1 - i]$
for $i \geq 0$ satisfying 
\[(4.10)_{uv}\]
\[
g_{uv} - f_{uv} = \sum_{i+p+1+j=q=v} \left( (-1)^u \tilde{m}_{02}^E (\tilde{g}^1_{ij} \otimes \tilde{h}_{pq}^1) + (-1)^v \tilde{m}_{11}^E (\tilde{h}_{ij}^1 \otimes \tilde{f}_{pq}^1) \right) + (-1)^u \tilde{m}_{01}^{\tilde{C}} \tilde{h}_{uv}^1 + (-1)^v \tilde{m}_{11}^{\tilde{C}} \tilde{h}_{uv}^1 + \sum_{i+p+1 \leq j \leq n} \tilde{h}_{ij}^{E,j} \eta_{pv}^{E,j}
\]
for $u \geq 0$ and $v \geq 1$ and the unit condition
\[(4.11)\]
\[
\tilde{h}_{ij}^1 (1^{\otimes r-1} \otimes \sigma_y E^{-1} \eta E \otimes 1^{\otimes j-r}) = 0 \text{ with } 1 \leq r \leq j \text{ for all } i,j.
\]

**Proposition 4.12.** Let $\alpha: B \to C$ be a map of bigdgs with $k$-projective vertical homology algebras $E$ and $F$. Let $E$ and $F$ be equipped with $dA_\infty$-structure and $dA_\infty$-maps $f: E \to B$ and $g: F \to C$ arising from Proposition 4.7. Then there exist $dA_\infty$-map $\beta: E \to F$ with $\beta_0 = H_*^v(g)$ and a homotopy from $g\beta$ to $\alpha f$.

The strategy for the proof is the same as for the last proposition. We define
\[
\tilde{g}_{ln} = \tilde{m}_{01}^{\tilde{C}} \tilde{h}_{l-1,n}^1 + (-1)^l (\alpha f) l_1 - (-1)^l \sum_{i+j \geq 2, i+p = 1} \tilde{g}_{ij}^1 \tilde{h}_{pn}^1 + (-1)^l \sum_{i+p l \leq j \leq n} \tilde{h}_{ij}^1 \tilde{m}_{pn}^E j
\]
and observe that the identity $(4.10)_{uv}$ can be rewritten as
\[
\tilde{m}_{01}^E \tilde{f}_{uv} = (-1)^u \tilde{g}_{01}^{\tilde{C}} \tilde{f}_{uv} - \tilde{g}_{uv}.
\]

**Proof of Proposition 4.12.** We set $\beta_0 = H_*^v(g)$. Since $E$ is $k$-projective, there exists $\tilde{h}_{01}$ with $\tilde{m}_{01}^{\tilde{C}} \tilde{h}_{01} = \tilde{\alpha}^1_0 \tilde{f}_{01} - \tilde{\beta}_0^1 \tilde{g}_{01}$. The $\beta_0$ satisfies $(2.6)_{uv}$ for $u + v \leq 2$ and $(2.7)$, and $\tilde{h}_{01}$ satisfies $(4.10)_{uv}$ for $u + v = 1$ and can be chosen to satisfy $(4.11)$.

Assume that we have constructed $\beta_{ij}$ and $h_{ij}$ for $i + j < l + n$ such that $(2.6)_{uv}$ holds for $u + v < l + n$ and $(4.10)_{uv}$ holds for $u + v < l + n$. Then $y_{ln}$ is defined, and Lemma 7.10 provides $\tilde{m}_{01} \tilde{g}_{ln} = 0$. So we define $\beta_1^0$ to be represented by $\tilde{y}_{ln}$. Lemma 7.17 shows that $(2.6)_{uv}$ holds for $u + v = l + n + 1$, and the unit condition $(2.7)$ can be checked readily. Since $E$ is $k$-projective we can find $\tilde{h}_{ln}$ such that $(4.10)_{uv}$ holds for $u + v = l + n$. The splitting of the unit of $E$ enables us to change $h_{ln}$ such that $(4.11)$ holds.

We can now give the proof of the first main theorem from the introduction.

**Proof of Theorem 4.7.** Given a $k$-dga $A$, we apply Theorem 3.3 to obtain a $k$-projective $E_1$-resolution $B \to A$. Then $B$ satisfies the assumptions of Proposition 4.7 which provides an $E_2$-equivalence $E \to B$. The composition $E \to A$ is the desired minimal $dA_\infty$-algebra model.

The resolution $B \to A$ is well defined up to $E_2$-equivalence of $k$-projective $E_1$-resolutions. Given such an $E_2$-equivalence $B \to C$, Proposition 4.12 provides an $E_2$-equivalence $E \to F$ between the associated minimal $dA_\infty$-algebras.

Lemma 3.12 and Proposition 4.12 imply

**Corollary 4.13.** Minimal $dA_\infty$-algebra models of quasi-isomorphic dgas may be related by a zig-zag of $E_2$-equivalences between minimal $dA_\infty$-algebras.

**Remark 4.14.** We do not claim that every termwise $k$-projective resolution $E$ of $H_*(A)$ may be extended to a minimal $dA_\infty$-algebra model for $A$. We only show that this happens if $E$ is the vertical homology of a bigda resolving $A$. 
Remark 4.15. The underlying twisted chain complex of a minimal \( dA_\infty \)-algebra model of \( A \) is a ‘homological multicomplex resolution’ of the underlying chain complex of \( A \); compare [17].

5. Examples and an Application

In this section, we give two examples for minimal \( dA_\infty \)-algebra structures and apply our theory to define the derived Hochschild cohomology class associated with a dga.

Example 5.1. Let \( p \) be an odd prime, \( k = \mathbb{Z}/p^2 \), and \( A \) the dga \( (k \rightarrow k) \). In Example 3.3 we gave a \( k \)-projective \( E_1 \)-resolution \( B \rightarrow A \) with \( E := H_*^k(B) = \Lambda^\infty_k(\mathfrak{z}, \mathfrak{z}) \otimes \overline{\Gamma}^*_{TV}(\mathfrak{g}) \). Following the proof of Proposition 4.7 we now extend the multiplication and \( d_1^E \) to a \( dA_\infty \)-algebra structure on \( E \). A cycle selection map \( f_{01} \) can be defined by extending

\[
f_{01}(\mathfrak{z}) = au, f_{01}(\mathfrak{z}) = x, \text{ and } f_{01}(\mathfrak{g}_i) = y_i
\]
multiplicatively. Then \( f_{01}m_{02}^E - m_{02}^B(f_{01} \otimes f_{01}) = 0 \). Consequently, \( m_{03}^E = 0 \), and we can set \( f_{02} = 0 \) and \( f_{03} = 0 \). Since \( m_{11}^E = d_1^E \), we define \( f_{11} : E \rightarrow B[1, -1] \) by

\[
f_{11}(y_i) = (v + av)x_{y_{i-1}} \text{ and } f_{11}(\mathfrak{g}_i) = (pu - v)y_i.
\]
The \( m_{12}^E \) is represented by

\[
z_{12} = f_{11}m_{02}^E - m_{02}^B(f_{01} \otimes f_{11} + f_{11} \otimes f_{01}).
\]

One calculates \( z_{12} = 0 \), hence \( m_{12}^E = 0 \) and we can set \( f_{12} = 0 \). The \( m_{21}^E \) is represented by \( z_{21} = f_{11}^1m_{11}^E + m_{11}^E f_{11} \). One checks

\[
m_{21}^E(\mathfrak{g}_i) = \mathfrak{z} \mathfrak{g}_{i-1} \text{ and } m_{21}^E(\mathfrak{g}_i) = \mathfrak{z} \mathfrak{g}_{i-1}.
\]

Furthermore, we set \( f_{21}(\mathfrak{z} \mathfrak{g}_i) = puvx_{y_{i-1}} \text{ and } f_{21}(\mathfrak{g}_i) = 0 \). The \( m_{ij}^E \) and \( f_{ij} \) for \( i + j \geq 4 \) vanish for degree reasons.

Example 5.7 below provides one way to detect that this \( E \) encodes non-trivial information about \( A \), for example the fact that \( A \) is not quasi-isomorphic to the formal dga \( H_*(A) \) with trivial differential. Of course, Theorem 1.2 provides a different reason this.

The example indicates the general procedure for finding a minimal \( dA_\infty \)-algebra model of a given dga \( A \). Since the resolutions provided by Theorem 3.3 are too large to write down in terms of generators and relations, one has to first guess a \( k \)-projective \( E_1 \)-resolution, for example by lifting a \( k \)-projective resolution of \( H_*(A) \) to a bidga \( B \) coming with an \( E_2 \)-equivalence \( B \rightarrow A \). Then one can use the constructive procedure of Proposition 4.7 to build \( E \) from \( B \).

Remark 5.2. If \( k \) happens to be a field or, more generally, \( H_*(A) \) is \( k \)-projective and the unit \( k \rightarrow H_0(A) \) splits, the minimal model of \( A \) as an \( A_\infty \)-algebra arising from [11, Theorem 1] is one possible choice for the minimal \( dA_\infty \)-algebra model of Theorem 1.1.

Let \( M \) be a \( k \)-module and let \( P \) be a \( k \)-projective resolution of \( M \). The endomorphism dga \( \text{Hom}_k(P, P) \) is defined by \( \text{Hom}_k(P, P) = \text{Hom}_k(P, P) \), where the latter denotes maps of graded \( k \)-modules. Its differential is \( d_{1}^\text{Hom}(f) = d^P f - (-1)^j f d^P \). The quasi-isomorphism type of \( \text{Hom}_k(P, P) \) does not depend on the choice of the resolution \( P \). Its homology algebra is the Yoneda Ext-algebra of \( M \) with a sign shift in the degree, i.e., \( H_* \text{Hom}_k(P, P) = \text{Ext}^{1,*}_{k}(M, M) \).

A minimal \( dA_\infty \)-model for \( \text{Hom}_k(P, P) \) is a resolution \( E \) of \( \text{Ext}^{1,*}_{k}(M, M) \) with structure maps \( m_{ij}^E \). By Theorem 1.2, this structure enables us to recover the quasi-isomorphism type of \( \text{Hom}_k(P, P) \).
Example 5.3. Let $k = \mathbb{Z}$, $p$ a prime, and $M = \mathbb{Z}/p$. We will see that the resolution of $\text{Ext}_2^Z(\mathbb{Z}/p, \mathbb{Z}/p)$ given by $\mathbb{Z} \leftarrow^{p} \mathbb{Z}$ in every degree of the Ext-grading is part of a minimal $d_A\mathfrak{ac}$-algebra.

Let $P$ be the two step resolution $\mathbb{Z} \leftarrow^{p} \mathbb{Z}$ of $\mathbb{Z}/p$. The dga $A = \text{Hom}_Z(P, P)$ may be described as follows (compare [5, §3.1]): Its underlying ungraded algebra is $\text{Mat}_2(\mathbb{Z})$, the algebra of $2 \times 2$-matrices with entries in $\mathbb{Z}$. Viewing elements of $P$ as column vectors with top entry the degree 0 part, $\text{Mat}_2(\mathbb{Z})$ operates on $P$. The resulting grading of $A$ is

$$ A_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}, \quad A_0 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad \text{and} \quad A_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}. $$

On the standard basis, the differential has values

$$ d\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}, \quad d\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}, \quad d\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0. $$

Indeed, $H_*^k(A)$ is an exterior algebra generated by $\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ in degree $-1$.

We give a $k$-projective $E_1$-resolution $A : B \rightarrow A$. Let $B = \text{Mat}_2(\mathbb{Z}[t] \otimes \Lambda^2_*(\lambda))$ with $|t| = (0,0)$ and $|\lambda| = (1,0)$. The horizontal grading is specified through the generators, and the vertical grading comes from the matrix algebra as above.

The vertical differential $d_B^0$ is defined by setting

$$ d_B^0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad d_B^0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad d_B^0\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad d_B^0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0, $$

and requiring $d_B^0$ to be $t$- and $\lambda$-linear. The horizontal differential $d_B^1$ is $t$-linear and maps $\lambda$ to $p - t$. The map $\alpha$ is determined by asking $t \mapsto p$.

One can check that $B \rightarrow A$ has the required properties. The vertical homology $H_*^k(B) = E$ is $\Lambda^2_*(a, b)$ with $|a| = (0, -1)$ and $|b| = (1, 0)$. Here

$$ \iota = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad ab = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, $$

and $f_{01}$ can be chosen to pick the displayed generators. The $m_{12}^B$ provides the multiplicative structure encoded in the exterior algebra. Since $f_{01}$ is multiplicative, we may choose $f_{02} = 0$ (and $f_{0j} = 0, m_{0j}^B = 0$ for $j \geq 3$). Now $z_{11} = m_{11}^B f_{01}$, hence

$$ m_{11}^B(b) = pu \text{ and } m_{11}^B(ab) = pa. $$

Since $m_{01}^B f_{11} = f_{01} m_{11}^B z_{11}$, we may choose

$$ f_{11}(b) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } f_{11}(ab) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. $$

The $m_{12}$ is represented by $z_{12} = f_{11} m_{02}^B - m_{02}^B f_{01} \otimes f_{11} - m_{02}^B (f_{11} \otimes f_{01})$. One checks that

$$ m_{12}^B(a \otimes b) = \iota, \quad m_{12}^B(a \otimes ab) = a, \quad m_{12}^B(ab \otimes b) = -b, \quad \text{and} \quad m_{12}^B(ab \otimes ab) = -ab, $$

and that $m_{12}^B$ is trivial elsewhere. One calculates that, mostly for degree reasons, all other $m_{ij}^B$ vanish. The asymmetry of $m_{12}^B$ comes from the choice of the value of $f_{11}(ab)$. Changing it to $-\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ changes $m_{12}^B$.

Our next aim is to define a variant of derived Hochschild cohomology for graded $k$-algebras. It will generalize the Hochschild cohomology of graded $k$-algebras which is for example used in [2].

We briefly recall from [1, §4] one way to derive the Hochschild cohomology $\text{dH}^\ast$ for ungraded $k$-algebras. (In [1], this theory is called Shakla cohomology.) Considering an ungraded $k$-algebra $A$ as a $\mathbb{N}$-graded dga concentrated in degree 0, one can find a quasi-isomorphism $A \rightarrow A$ from a degreewise $k$-projective dga $A$. The Hochschild complex of $A$ is the bicomplex $C^\ast t : A \otimes A \rightarrow A[t]$. It has the usual Hochschild differential $d_{\text{H}} C^\ast t : C^s \rightarrow C^{s+1,t}$ and a differential $d^t_{\text{H}} : C^s \rightarrow C^{s+1,t+1}$ induced from that of $A$ by considering $C^s t$ as the Hom-complex with source $A^{\otimes s}$ and target $A$. Letting $\text{HH}^\ast(A)$ be the cohomology of the total complex of $C^\ast t$, we define $\text{dH}^\ast(A) := \text{HH}^\ast(A)$. This is well defined since a quasi-isomorphism of $k$-projective dgas $A \rightarrow A'$ over $A$ induces a chain of isomorphisms $\text{HH}^\ast(A) \xrightarrow{\cong} \text{HH}^\ast(A, A') \xrightarrow{\cong} \text{HH}^\ast(A')$; compare [1, 4.1.1 Lemma].
Let \( E \) be a \( k \)-bidga with \( d_E^0 = 0 \), and let \( F \) be an \( E \)-bimodule in the category biCh_k with \( d_F^0 = 0 \). On \( E \) and \( F \) we have differentials \( d_E \) and \( d_F \) and multiplication maps \( E \otimes E \to E \), \( E \otimes F \to F \), and \( F \otimes E \to F \) satisfying appropriate relations.

We define an \((\mathbb{N}, \mathbb{N}, \mathbb{Z})\)-trigraded \( k \)-module \( C^{r,s,t}(E,F) = \text{Hom}_k(E^{(s)}, F^{(t)}) \). It has a Hochschild differential

\[
d^r_{HH}: C^{r,s,t}(E,F) \to C^{r+1,s,t}(E,F)
\]

\[
c \mapsto \mu(1 \otimes c) + (-1)^{r+1} \mu(c \otimes 1) + \sum_{1 \leq j \leq r} (-1)^j c(1^{s-j-1} \otimes \mu \otimes 1^{t-r-j})
\]

and a differential

\[
d^r_{\text{Hom}}: C^{r,s,t} \to C^{r,s+1,t}
\]

\[
c \mapsto d^r_{\text{Hom}} c - (-1)^s d^r_{\text{Hom}}(c) + \sum_{1 \leq j \leq r} c(1^{s-j-1} \otimes d^r_{\text{Hom}} \otimes 1^{t-r-j})
\]

induced by the horizontal differentials of \( E \) and \( F \). It is easy to check \( d_{\text{HH}}d_{\text{HH}} = 0 \), \( d_{\text{Hom}}d_{\text{Hom}} = d_{\text{HH}}d_{\text{Hom}} \) and \( d_{\text{Hom}}d_{\text{Hom}} = 0 \).

We define \( \text{Tot}^q C(E,F) = \bigoplus_{r+s+q} C^{r,s,t}(E,F) \) and

\[
d^q_{\text{Tot}}: \text{Tot}^q C(E,F) \to \text{Tot}^{q+1} C(E,F), \quad c \mapsto d_{\text{Hom}}(c) - (-1)^q d_{\text{HH}}(c)
\]

The relations of \( d_{\text{HH}} \) and \( d_{\text{Hom}} \) imply \( d_{\text{Tot}}d_{\text{Tot}} = 0 \). The graded Hochschild cohomology of \( E \) with coefficients in \( F \) is the homology of this total complex, i.e., \( \text{HH}^q(E,F) = H^q(\text{Tot}^q C(E,F)) \).

Now let \( \Lambda \) be a graded \( k \)-algebra. We view \( \Lambda \) as a \( k \)-dga with trivial differential. Applying vertical homology to the resolution provided by Theorem 3.4, we obtain a termwise \( k \)-projective \( E \to \Lambda \). The resolution is well defined up to \( E_2 \)-equivalence between such \( E \). We define the derived Hochschild cohomology of \( \Lambda \) by

\[
d_{\text{HH}}^q(\Lambda) = \text{HH}^q(E,E).
\]

An \( E_2 \)-equivalence \( \alpha: E \to F \) of termwise \( k \)-projective resolutions induces a chain of isomorphisms \( \text{HH}^q(E,E) \xrightarrow{\alpha} \text{HH}^q(E,F) \xrightarrow{\alpha^*} \text{HH}^q(F,F) \). This ensures that \( d_{\text{HH}}^q(\Lambda) \) is well defined.

**Proposition 5.4.** Let \( A \) be a \( k \)-dga, and let \( E \to A \) be a minimal \( dA_{\infty} \)-algebra model of \( A \). Then the sum \( m_{ij}^E + m_{ij}^E + m_{ij}^F \) is a cocycle in the complex computing \( \text{HH}^{i-1}(E) = \text{HH}^{i-1}(H(A)) \). Its cohomology class \( \gamma_A \in \text{HH}^{i-1}(H(A)) \) does not depend on the choice of \( E \).

**Proof.** The four formulas \([2.2]_{\text{wit}}\) with \( u + v = 4 \) for the \( m_{ij}^E \) can be expressed as \( \text{d}_{\text{HH}}(m_{ij}^E) = 0 \), \( \text{d}_{\text{HH}}(m_{ij}^E) + d_{\text{Hom}}(m_{ij}^E) = 0 \), \( \text{d}_{\text{HH}}(m_{ij}^F) + d_{\text{Hom}}(m_{ij}^F) = 0 \), and \( d_{\text{Hom}}(m_{ij}^F) = 0 \). Hence \( \text{d}_{\text{Tot}}(m_{ij}^E + m_{ij}^E + m_{ij}^F) = 0 \).

Let \( f: E \to F \) be an \( E_2 \)-equivalence of minimal \( dA_{\infty} \)-algebras. We check that the elements \((f_{ij})_*(m_{ij}^E + m_{ij}^E + m_{ij}^F) \) and \((f_{ij})^*(m_{ij}^E + m_{ij}^E + m_{ij}^F) \) of \( \text{Tot}^{i-1} C(E,F) \) represent the same cohomology class. Since \( f \) is map of \( dA_{\infty} \)-algebras, it satisfies the relations \([2.6]_{\text{wit}}\) for \( u + v = 3 \), and these can be rewritten as

\[
(f_{ij})_*(m_{ij}^E + m_{ij}^E + m_{ij}^F) - (f_{ij})^*(m_{ij}^E + m_{ij}^E + m_{ij}^F) = d_{\text{Tot}}(f_{ij}).
\]

\( \square \)

If a dga \( A \) is formal, i.e., quasi-isomorphic to \( H_*(A) \) with trivial differential, then \( \gamma_A = 0 \) since a termwise \( k \)-projective resolution of \( H_*(A) \) provides a minimal \( dA_{\infty} \)-algebra model for \( A \) with \( m_{ij} = i + j \geq 3 \). Hence a non-vanishing \( \gamma_A \) shows that \( A \) is not formal, and that a given minimal \( dA_{\infty} \)-algebra model of \( A \) is not equivalent to one with trivial higher \( m_{ij} \).
Remark 5.5. For a dga over a field, there is a characteristic Hochschild cohomology class \( \gamma_A \in \text{HH}^{3,-1}(H_*(A)) \) which is studied in [2]. It is represented by the \( m_3 \) of a minimal \( A_\infty \)-algebra structure on \( H_*(A) \) and determines by evaluation all triple matric Massey products in \( H_*(A) \). The characteristic dHH-class of a dga over a commutative ring \( k \) given by the previous proposition is the natural generalization. If \( k \) happens to be a field, the \( \gamma_A \) introduced here coincides with the one from [2], as one can easily see from Remark 5.2 and [2, Remark 5.8].

If \( B \) is a bigda and \( B \to A \) an \( E_2 \)-equivalence, \( B_{0a} \) is a dga, and \( B \to A \) restricts to a map \( B_{0a} \to A \) of dgas which induces an surjection in homology. If \( E \) is the minimal \( dA_\infty \)-algebra structure on \( H_*(E) \), the restriction of \( m_{01}^E \) to \( H_*(B_{0a}) \) is part of an \( A_\infty \)-structure on \( H_*(B_{0a}) \). By evaluation, it determines triple Massey products of \( B_{0a} \). Moreover, if \( \lambda_1, \lambda_2, \lambda_3 \in H_*(A) \) satisfy \( \lambda_1 \lambda_2 = 0 = \lambda_2 \lambda_3 \), we may lift them to elements of \( H_*(B_{0a}) \), evaluate \( m_{01}^E \) on the lifts, and observe that the image of the evaluation under the map to \( H_*(A) \) is an element of the Massey product \( (\lambda_1, \lambda_2, \lambda_3) \). Hence the dHH-class determines all triple (matric) Massey products in \( H_*(A) \). The difference to the HH-class of [2] is that the dHH-class does not determine the Massey products in a \( k \)-linear fashion, since lifts of elements along a surjection into a not necessarily projective \( k \)-module are involved.

We expect the dHH-class \( \gamma_A \) of a dga to share more properties with the HH-class of [2], like the connection to realizability obstructions. We are not going to elaborate those here. The dHH-class should also be compared to the universal Toda bracket of a ring spectrum studied in [10]. The latter class is an even further generalization of the (derived) Hochschild class and arises when one allows the sphere spectrum of stable homotopy theory as ground ring. This class is an element in a Mac Lane cohomology group and determines triple Toda brackets.

Remark 5.6. Derived Hochschild cohomology classes associated to dgas also appear in [1] and [3]: a dga \( A \) with only \( A_0 \) and \( A_1 \) possibly non-zero gives rise to a crossed extension of \( H_0(A) \) by \( H_1(A) \) and hence to a cohomology class in \( \text{dHH}^3_k(H_0(A), H_1(A)) \) [1, 4.4.1, Theorem]. More generally, for a dga \( A \) with non-negative homology, its first \( k \)-invariant lies in \( \text{dHH}^3_k(H_0(A), H_1(A)) \) [3, §4].

If \( E \to A \) is a minimal \( dA_\infty \)-algebra model of \( A \), \( E_{0a} \) is a \( k \)-projective resolution of \( H_0(A) \), and there is a restriction map \( \text{dHH}^3_k(H_0(A)) \to \text{dHH}^3_k(H_0(A), H_1(A)) \). One can check that the image of \( \gamma_A \) under this map recovers the cohomology classes described above.

Example 5.7. We continue Example 5.1 by applying Proposition 5.4 to the minimal \( dA_\infty \)-algebra \( E \) described there. In this case, the sum representing \( \gamma_A \) consists only of \( m_{11}^E \). The \( m_{11}^E \) cannot be a boundary in the complex computing the derived Hochschild cohomology: \( p \cdot m_{11}^E = 0 \), while \( p \cdot m_{21}^E \) is non-zero. Hence \( \gamma_A \in \text{dHH}^3_{k/p^3}(H_*(A)) \) is non-trivial.

This illustrates that the non-triviality of \( E \) is a consequence of the information encoded in the higher degrees of the horizontal grading.

6. Anti-minimal models

The aim of this section is to prove Theorem 1.2, which provides an ‘anti-minimal’ model of a \( dA_\infty \)-algebra. To this end, we introduce modules over \( dA_\infty \)-algebras and show that the module category is enriched in twisted chain complexes. For a \( dA_\infty \)-algebra \( E \), we thus have an endomorphism tga of the \( E \)-module \( E \). Its total complex is a dga. If \( E \) is the minimal \( dA_\infty \)-algebra model of a dga \( A \), this dga associated with \( E \) recovers the quasi-isomorphism type of \( A \). These constructions are motivated by the Hom-complex of maps between modules over \( A_\infty \)-algebras.
described in Keller’s survey [13 §6.3] and studied in more detail in Lefèvre’s thesis [14 §4].

Let $E$ be a $dA_\infty$-algebra. We write $TSE$ for the unreduced tensor algebra on $SE$, i.e., $TSE = \bigoplus_{j \geq 0} SE^{\otimes j}$. Let $M$ be a bigraded $k$-module. Let $\tilde{m}_{ij}^{M,1} : SM \otimes SE^{\otimes j-1} \to SM[i, 1-i]$ be a family of maps with $i \geq 0$ and $j \geq 1$. These maps assemble to a map $\tilde{m}_{ij}^{M,1} : SM \otimes TSE \to SM[i, 1-i]$. More generally, we define

$$\tilde{m}_{ij}^{M,q} := \tilde{m}_{i,j-q+1}^{M,1} \otimes 1^{\otimes q-1} + 1 \otimes \tilde{m}_{i,j-q-1}^{E,q-1}$$

where the second summand is understood to be zero if $q = 1$. We obtain a map $\tilde{m}_i^M : SM \otimes TSE \to SM \otimes TSE[i, 1-i]$ whose component mapping $SM \otimes SE^{\otimes j-1}$ to $SM \otimes SE^{\otimes q-1}[i, 1-i]$ is $\tilde{m}_{ij}^{M,q}$.

**Lemma 6.1.** Given maps $\tilde{m}_{ij}^{M,1}$ as above, the following are equivalent:

(i) $\sum_{i+p=u} (-1)^{i} \tilde{m}_{i}^{M} \tilde{m}_{j}^{M} = 0$ for all $u \geq 0$.

(ii) $\sum_{i+p=u} (-1)^{i} \tilde{m}_{i}^{M,1} \tilde{m}_{j}^{M} = 0$ for all $u \geq 0$.

(iii) The $\tilde{m}_{ij}^{M}$ turn $SM \otimes TSE$ into a twisted chain complex.

**Definition 6.2.** A module over a $dA_\infty$-algebra $E$ is a bigraded $k$-module $M$ together with maps $\tilde{m}_{ij}^{M,1} : SM \otimes SE^{\otimes j-1} \to SM[i, 1-i]$ for $i \geq 0$ and $j \geq 1$ satisfying the equivalent conditions of the last lemma and the unit condition

$$\tilde{m}_{0,1}^{M,1} (1 \otimes \sigma_{E}^{-1} \eta_{E}) = 1 \quad \text{and} \quad \tilde{m}_{i,j}^{M,1} (1 \otimes 1^{\otimes r-1} \otimes \sigma_{E}^{-1} \eta_{E} \otimes 1^{\otimes j-1-r}) = 0 \text{ if } i + j \geq 3 \text{ with } 1 \leq r < j - 1.$$

**Example 6.4.** The free module of rank 1 is an $E$-module. This isomorphism $TSE \cong SE \otimes TSE$ is used to interpret the structure maps of $E$ as an $dA_\infty$-algebra as those of $E$ as an $E$-module.

Let $f : E \to F$ be a $dA_\infty$-algebra map and let $M$ be an $F$-module. Writing $\tilde{m}_{ij}^{M,F}$ for the $F$-module structure maps of $M$, we define

$$\tilde{m}_{ij}^{M,F,1} = \sum_{i+p=u, 1 \leq j \leq v} \tilde{m}_{ij}^{M,F,1} (1 \otimes \tilde{f}_{p,v}^{-1}) : SM \otimes (SE)^{\otimes v-1} \to SM[u, 1-u]$$

**Lemma 6.5.** The $\tilde{m}_{ij}^{M,E}$ define an $E$-module structure on $M$.

**Example 6.6.** If $f : E \to F$ is a map of $dA_\infty$-algebras, $F$ is an $E$-module.

Let $E$ be a $dA_\infty$-algebra and let $M$ and $N$ be $E$-modules. Given a family of $k$-linear maps $f_{ij}^{1} : SM \otimes SE^{\otimes j-1} \to SN[i, -i]$ with $i \geq 0$ and $j \geq 1$, they induce a map $\tilde{f}_{ij}^{1} : SM \otimes TSE \to SM[i, -i]$. Setting $\tilde{f}_{ij}^{q} = \tilde{f}_{i,j-q+1}^{1} \otimes 1^{\otimes q-1}$ for $1 \leq q \leq j$, we obtain maps $\tilde{f}_{ij} : SM \otimes TSE \to SN \otimes TSE[i, -i]$ whose component mapping $SM \otimes SE^{\otimes j-1}$ to $SN \otimes SE^{\otimes q-1}[i, -i]$ is $\tilde{f}_{ij}^{q}$.

**Lemma 6.7.** Given maps $\tilde{f}_{ij}^{1}$ as above, the following are equivalent:

(i) $\sum_{i+p=u} (-1)^{i} \tilde{f}_{ij}^{1} \tilde{m}_{ij}^{M} = \sum_{i+p=u} \tilde{m}_{ij}^{N} \tilde{f}_{ij}^{1}$ for all $u \geq 0$.

(ii) $\sum_{i+p=u} (-1)^{i} \tilde{f}_{ij}^{1} \tilde{m}_{ij}^{M} = \sum_{i+p=u} \tilde{m}_{ij}^{N} \tilde{f}_{ij}^{1}$ for all $u \geq 0$.

(iii) The $\tilde{f}_{ij}$ form a map of twisted chain complexes.

**Definition 6.8.** Let $E$ be a $dA_\infty$-algebra, and let $M$ and $N$ be $E$-modules. An $E$-module map from $E$ to $F$ is a family of maps $\tilde{f}_{ij}^{1} : SM \otimes SE^{\otimes j-1} \to SN[i, -i]$ for $i \geq 0$ and $j \geq 1$ satisfying the equivalent conditions of the last lemma and

$$\tilde{f}_{ij}^{1} (1 \otimes 1^{\otimes r-1} \otimes \sigma_{E}^{-1} \eta_{E} \otimes 1^{\otimes j-1-r}) = 0 \text{ if } j \geq 2 \text{ with } 1 \leq r < j - 1.$$

Composition of module maps is given by the composition of the $\tilde{f}_{ij}$ as maps of twisted chain complexes.
A map \( f : M \to N \) of \( E \)-modules induces a map of twisted chain complexes between the underlying twisted chain complexes of \( M \) and \( N \). The map \( f \) is an \( E_2 \)-equivalence if this underlying map is.

**Example 6.9.** If \( f : E \to F \) is a map of \( dA_\infty \)-algebras, both \( E \) and \( F \) are \( E \)-modules, and one can check that \( f \) is an \( E \)-module map. The underlying twisted chain complexes of \( E \) and \( F \) as \( dA_\infty \)-algebras and modules coincide. Hence \( f \) is an \( E_2 \)-equivalence of modules if and only if it is an \( E_2 \)-equivalence of \( dA_\infty \)-algebras.

Let \( E \) be a \( dA_\infty \)-algebra and let \( M \) and \( N \) be \( E \)-modules. Let \( X \) be the bigraded \( k \)-module \( \text{Hom}_k(\mathcal{S}\mathcal{M} \otimes TSE, SN) \). For \( s \in \mathbb{N}, t \in \mathbb{Z} \), an element of \( X_{st} \) is a \( k \)-module map \( \tilde{g}^1 : \mathcal{S}\mathcal{M} \otimes TSE[s,t] \to SN \). We shift \( X \) by shifting the second entry \( SN \). As in the definition of module maps, the \( \tilde{g}^1 \) has components \( \tilde{g}^1_{ij} \) which provide \( \tilde{g}^1 = \tilde{g}^1_{ij} \otimes 1^\otimes q^{-1} \) and \( \tilde{g} : \mathcal{S}\mathcal{M} \otimes TSE[s,t] \to SN \otimes TSE \), and giving \( \tilde{g}^1 \) is equivalent to giving the \( \tilde{g} \).

For \( i \geq 0 \), we define \( d_i : X \to X[i, 1 - i] \) by \( d_i(\tilde{g}^1) = \tilde{m}_{i+1}^N \tilde{g} - (-1)^{i+1} \langle \tilde{g}, \tilde{m}_i \rangle \tilde{g} \). Since \( \sum_{i+p=u} (-1)^p d_p(\tilde{g}^1) = 0 \) for \( u \geq 0 \), we may state

**Definition 6.10.** Let \( E \) be a \( dA_\infty \)-algebra and let \( M \) and \( N \) be \( E \)-modules. We define \( \text{Hom}_E(M, N) \) to be the twisted chain complex with underlying bigraded \( k \)-module \( \text{Hom}_k(\mathcal{S}\mathcal{M} \otimes TSE, SN) \) and differentials \( d_i \) as defined above.

Let \( L, M \) and \( N \) are \( E \)-modules. For \( \tilde{f} \in \text{Hom}_E(M, N)_{st} \) and \( \tilde{g} \in \text{Hom}_E(L, M)_{pq} \), its composite \( \tilde{f} \tilde{g} \in \text{Hom}_E(L, M)_{p+q, st} \) is defined. The unit \( k \to \text{Hom}_E(M, M) \) sends 1 to the \( \tilde{g} \in \text{Hom}_E(M, M) \) with \( \tilde{g}^1 = \text{id}_{SM} \) and \( \tilde{g}^1_j = 0 \) for \( j \geq 2 \).

**Proposition 6.11.** The composition induces a pairing of twisted chain complexes

\[
\text{Hom}_E(M, N) \otimes \text{Hom}_E(L, M) \to \text{Hom}_E(L, N)
\]

which is associative and unital. In particular, \( \text{Hom}_E(M, M) \) is a tdga.

**Proof.** Let \( \mu \) be the composition. The only possibly non-trivial part is to verify the Leibniz-rules. If \( i \geq 0 \), \( \tilde{f} \in \text{Hom}_E(M, N)_{st} \) and \( \tilde{g} \in \text{Hom}_E(L, M)_{pq} \), we have

\[
\mu(\mathcal{1} \otimes d_i + d_i \otimes \mathcal{1})(\tilde{f} \otimes \tilde{g}) = (-1)^{i+1} \tilde{f} \tilde{m}_i^M \tilde{g} - (-1)^{\langle \tilde{f}, \tilde{m}_i \rangle} \tilde{g} \tilde{m}_i^N \tilde{f} - (-1)^{\langle \tilde{f}, \tilde{m}_i \rangle} \tilde{g} \tilde{m}_i^M \tilde{f} = d_i \mu(\tilde{f} \otimes \tilde{g}).
\]

\[\square\]

**Remark 6.12.** The last proposition says that the category of modules over a \( dA_\infty \)-algebra is enriched in twisted chain complexes. The morphisms of the ordinary category underlying this \( \text{tCh}_k \)-category are not the \( dA_\infty \)-algebra module morphisms of Definition 6.8.

Unlike the case of modules over \( A_\infty \)-algebras and their enrichment in chain complexes, not even all components \( f_{ij} \) of an \( E \)-module map \( f \) are 0-cycles in the \( \text{Hom}_E(M, N) \). The reason is that for \( i \geq 1 \), the \( f_{ij} \) would have to lie in a negative degree with respect to the first grading. We resist from allowing a negative grading there, since this would for example cause problems when forming total complexes. Therefore, the notation \( \text{Hom} \) might be slightly misleading, but we keep it in lack of a better alternative. Moreover, the induced maps on \( \text{Hom}_E(M, N) \) to be defined next are not a consequence of the enrichment.
Given $E$-module maps $f: M \to M'$ and $h: N \to N'$, we define
\[ f^*_i : \text{Hom}_E(M', N) \to \text{Hom}_E(M, N)[i, -i], \quad (f^*_i(g))_i = \sum_{1 \leq j \leq u} (-1)^{(j, \delta)} \tilde{g}_{ij}^1 \tilde{f}_{ij} \]
and
\[ (h_*)_i : \text{Hom}_E(M, N') \to \text{Hom}_E(M, N)[i, -i], \quad ((h_*)_i(g))_i = \sum_{1 \leq j \leq v} \tilde{h}_{ij}^1 \tilde{g}_{ij}. \]
For degree reasons, $f^*_i$ and $(h_*)_i$ are only defined on $\tilde{g}$ in horizontal degree $\geq i$.

**Lemma 6.13.** Both $f^*$ and $h_*$ are maps of twisted chain complexes. If $M' = N$, then $f^*$ is a left $\text{Hom}_E(N, N)$-module map. If $N' = M$, then $h_*$ is a right $\text{Hom}_E(M, M)$-module map.

**Proof.** Treating $f^*$ first, we have to compare the image of $\tilde{g}_1 \in \text{Hom}_E(M', N)$ under $\sum (-1)^j f^*_i d_p$ and $\sum d_p f^*_i$. Restricted to $SM \otimes (SE)^{\otimes v-1}$, the image of $\tilde{g}_1$ under both maps is
\[ \sum_{i+p=u, 1 \leq j \leq u} (-1)^{(j, \delta)} \tilde{m}_{pq} \tilde{g}_{ij} \tilde{f}_{ij} \tilde{g}_{ij} + (-1)^{(j, \delta)} \tilde{m}_{pq} \tilde{g}_{ij} \tilde{f}_{ij} \tilde{g}_{ij}. \]
For $h_*$ we calculate
\[ \sum_{i+p=u} (d_p((h_*)_i(\tilde{g})))_i = \sum_{i+p=u} (-1)^j ((h_*)_i(d_p(\tilde{g})))_i. \]
The statement about the tdga-module structures follows from the associativity of the composition. \(\square\)

For $F$-modules $M$ and $N$, a $dA_{\infty}$-algebra map $f: E \to F$ induces a $k$-linear map $\text{res}_f : \text{Hom}_E(M, N) \to \text{Hom}_E(M, N)$. On $\tilde{g}_1 : SM \otimes TSE[s, t] \to SN$, the restriction of $(\text{res}_f)_*(\tilde{g}_1)$ to $SM \otimes (SE)^{\otimes v-1}$ is defined as follows: for $v = 1$ and $u = 0$, it is $\tilde{g}_1$, for $v = 1$ and $u \neq 0$, it is $0$, and for $v \geq 2$ it is
\[ (\text{res}_f(\tilde{g}_1))_{uv} = \sum_{2 \leq j \leq v} (-1)^{(j, \delta)} \tilde{f}_{ij} \tilde{g}_{ij} (1 \otimes \tilde{f}_{ij}^{-1}) : SM \otimes (SE)^{\otimes v-1}[s, t] \to SN[u, -u]. \]

**Lemma 6.14.** The $\text{res}_f$ is a map of twisted chain complexes and respects the composition pairing. Particularly, it is a tdga map if $M = N$.

**Proof.** Both
\[ \sum_{i+p=u} d_p(\text{res}_f)_i = \sum_{i+p=u} (-1)^j (\text{res}_f)_i d_p \quad \text{and} \quad \sum_{i+p=u} \mu((\text{res}_f)_i \otimes (\text{res}_f)_p) = (\text{res}_f)_u \mu \]
can be checked directly. \(\square\)

Let $E$ be a $dA_{\infty}$-algebra. Viewing $E$ as an $E$-module, $\text{Hom}_E(E, E)$ is an tdga by Proposition 6.11. We want to define a $dA_{\infty}$-algebra map $\psi^E : E \to \text{Hom}_E(E, E)$. By Lemma 4.3, it is enough to give maps $\tilde{\psi}^{E, 1}_{iv} : (SE)^{\otimes v} \to S \text{Hom}_E(E, E)[i, -i]$. For $i \geq 0$ and $v \geq 1$, let $\tilde{\psi}^{E, 1}_{iv}$ be the map with source $SE^{\otimes v}$ being adjoint to the restriction of $\tilde{m}_{ij}^{E, 1}$ to $SE^{\otimes v} \otimes SE \otimes TSE$, and let $\psi^{E, 1}_{iv} = \sigma_{E}^{-1} \tilde{\psi}^{E, 1}_{iv}$.

The analogs in the world of $A_{\infty}$-algebras to the next three lemmas are [13] Lemme 4.1.1.6.b and Lemme 5.3.0.1]

**Lemma 6.15.** The $\tilde{\psi}^{E, 1}_{ij}$ constitute a $dA_{\infty}$-algebra map $\psi^E : E \to \text{Hom}_E(E, E)$.

**Proof.** It is by Lemma 4.3 sufficient to show
\[ \sum_{1 \leq j \leq v, 1 \leq p \leq u} (-1)^{j} \tilde{\psi}^{E, 1}_{ij} \tilde{m}_{pq} \]

In the next three lemmas are [13] Lemme 4.1.1.6.b and Lemme 5.3.0.1].
for $v \geq 1$. This is equivalent to

$$
\sum_{i+p=v} (-1)^{i+p+1} \sigma^{-1}_E dp_{p,v} + \sum_{i_1 + i_2 = v, j_1 + j_2 = v} (-1)^{i_1+1} \sigma^{-1}_E \mu (\sigma^{-1}_E \otimes \rho^{-1}_E)
$$

$$
= \sum_{1 \leq j \leq v, i+p=v} (-1)^i \sigma^{-1}_E \mu \tilde{m}_{ij} \otimes \tilde{m}_{pv}.
$$

We cancel the $\sigma^{-1}_E$ out, restrict along the injection $(SE)^{w} \to SE \otimes TSE$, and consider the adjoint maps $(SE)^{w} \otimes (SE)^{w} \to SE[u, 2-u]$ to get the following equation for $v, w \geq 1$ which is equivalent to the last one.

$$
\sum_{1 \leq j \leq w, i+p=v} (-1)^i \sigma^{-1}_E \mu \tilde{m}_{ij} \otimes \tilde{m}_{wv}.
$$

This holds since $E$ is an $dA_{\infty}$-algebra.

If $M$ is an $E$-module, there is a map $\psi^{M,E} : M \to \text{Hom}_E(M, E)$ of twisted chain complexes: we define $\psi^{M,E}$ to be the adjoint of the map

$$
(-1)^i \tilde{m}^{-1}_{i} (\sigma^{-1}_M \otimes 1 \otimes 1) : M \otimes SE \otimes TSE \to SM[i, -i].
$$

When forming the associated map $\tilde{\psi}^{M,E}$, the $(-1)^i$ cancels, and for $M = E$ we observe that $\tilde{\psi}^{M,E} = \tilde{\psi}^{1,E}$. In particular, the underlying map of twisted chain complexes of the $\psi^{E}$ is $\psi^{E,E}$.

The proof of the last lemma is easily adopted to provide

**Lemma 6.16.** The $\psi^{M,E} : M \to \text{Hom}_E(M, E)$ is a map of twisted chain complexes.

**Lemma 6.17.** The map $\psi^{M,E}$ of Lemma 6.16 is an $E_1$-equivalence.

**Proof.** We show that $\psi^{M,E}_0$ is a chain equivalence. For $\tilde{g}^1 \in \text{Hom}_E(E, M)$,

$$
\text{Hom}_k(SE \otimes TSE, SM) \xrightarrow{\psi^{1,E}} \text{Hom}_k(Sk, Sm) \xrightarrow{\psi^{-1}_1} \text{Hom}_k(k, M) \cong M
$$

can be used to define $\tau(\tilde{g}^1) = (-1)^{[\tilde{g}^1, \psi^1]} \psi^{-1}_1(\tilde{g}^1 \psi^1(\tilde{g}^1)).$ The $\tau$ is a chain map with respect to $d_0$ and $\tau \psi^{M,E}_0 = id$.

Next we define a chain homotopy between $\psi^{M,E}_0$ and id on $\text{Hom}_E(E, M)$. Let $h$ be the composite $SE \otimes TSE[0, 1] \cong Sk \otimes SE \otimes TSE \to SE \otimes TSE$ given by $S\eta$ on $Sk$ and the inclusion $SE \otimes TSE \to TSE$. We define $H : \text{Hom}_E(E, M) \to \text{Hom}_E(E, M)[0, -1]$ by $H(\tilde{g}^1) = (-1)^{[\tilde{g}^1, h]} \tilde{g}^1 h$ and calculate

$$
(d_0 H + H d_0)(\tilde{g}^1) = \tilde{g}^1 - \psi^{M,E}_0(\tilde{g}^1).
$$

Therefore, $\psi^{M,E}$ induces an isomorphism on homology with respect to $d_0$.

**Lemma 6.18.** The maps $(f_*)_0 \psi^E_0$ and $\psi^E_0 f_0$ are chain homotopic as $d_0$-chain maps from $E$ to $\text{Hom}_E(E, F)$.

**Proof.** The chain homotopy is

$$
H : E \to \text{Hom}_E(E, F), \quad (H(x))_j = \tilde{f}_{0,j+1}(\sigma^{-1}_E x \otimes 1^{\text{ot} j}).
$$
Corollary 6.19. Let \( f : E \rightarrow F \) be a map of \( dA_\infty \)-algebras. The induced map \( f_* : \text{Hom}_E(E, E) \rightarrow \text{Hom}_E(E, F) \) is an \( E_2 \)-equivalence of twisted chain complexes if and only if \( f \) is an \( E_2 \)-equivalence.

**Proof.** By Lemma 6.18, the maps induced by \( f, \psi^E \) and \( \psi^{E,F} f \) on \( H_k^b H^*_\psi \) coincide. Both \( \psi^E \) and \( \psi^{E,F} \) are \( E_1 \)-equivalences by Lemma 6.17 and hence \( E_2 \)-equivalences. \( \square \)

**Lemma 6.20.** Let \( f : E \rightarrow F \) be a \( dA_\infty \)-algebra map. Then \( \psi^{E,F} \) and \( f^* \text{res}_f \psi^{F,F} \) coincide as maps of twisted chain complexes from \( F \) to \( \text{Hom}_E(E, F) \).

**Proof.** One can check that

\[
((f^* \text{res}_f)_{ij}(\tilde{g}^1))_j = \sum_{1 \leq q \leq j} (-1)^{\langle i, \tilde{g}^1 \rangle q} \tilde{g}_q \tilde{f}_j
\]

holds for \( j \geq 1 \). This easily implies the assertion. \( \square \)

**Corollary 6.21.** The map \( f^* \text{res}_f \) is an \( E_2 \)-equivalence of twisted chain complexes.

**Proposition 6.22.** Let \( f : E \rightarrow F \) be an \( E_2 \)-equivalence of \( dA_\infty \)-algebras with \( E_2 \)-homology concentrated in horizontal degree \( 0 \). Then the dgas \( \text{Tot} \text{Hom}_E(E, E) \) and \( \text{Tot} \text{Hom}_E(E, F) \) are quasi-isomorphic.

**Proof.** By the Corollaries 6.19 and 6.21 there is a diagram

\[
\begin{array}{ccc}
\text{Hom}_E(E, E) & \xrightarrow{f_*} & \text{Hom}_E(E, F) \\
\downarrow{f^* \text{res}_f} & & \downarrow{f^* \text{res}_f} \\
\text{Hom}_F(F, E) & \xrightarrow{j} & \text{Hom}_F(F, F)
\end{array}
\]

with both maps \( E_2 \)-equivalences. Setting

\[
R = \text{Tot} \text{Hom}_E(E, E), \quad S = \text{Tot} \text{Hom}_F(E, F), \quad \text{and} \quad X = \text{Tot} \text{Hom}_E(E, F),
\]

we obtain dgas \( R \) and \( S \), an \( S \)-\( R \)-bimodule \( X \), a quasi-isomorphism \( \alpha : R \rightarrow X \) of right \( R \)-modules and an quasi-isomorphism \( \beta : S \rightarrow X \) of left \( S \)-modules. This data is compatible with the unit maps in that the square obtained from composing the unit maps of \( R \) and \( S \) with \( \alpha \) and \( \beta \) commutes.

The described data is almost a ‘quasi-equivalence’ of dgas as described in [12] p.31 and generalized in [20] Definition A.2.1. The missing part is an element of \( X \) such that left and right multiplication with the element equals \( \alpha \) and \( \beta \), respectively (compare Remark 6.12). However, we can follow closely the strategy of the proof of [20] Lemma A.2.3, exploiting the (projective) model structures on \( \text{Ch}_k \) and \( \text{Mod-R} \).

We factor \( \alpha \) in the model category of right \( R \)-modules as a composition of an acyclic cofibration \( \alpha' : R \xrightarrow{\sim} Y \) and an acyclic fibration \( \alpha'' : Y \xrightarrow{\sim} X \). We consider the dga \( T = \text{Hom}_R(Y, Y) \), the \( T \)-\( R \)-bimodule \( V = \text{Hom}_R(R, Y) \), and the \( S \)-\( T \)-bimodule \( W = \text{Hom}_R(Y, X) \). The map \( \alpha' \) induces an acyclic fibration \( (\alpha')^* : T \rightarrow V \) and a quasi-isomorphism \( (\alpha')_* : R \cong \text{Hom}_R(R, R) \rightarrow V \). The multiplications on \( R \) and \( T \) induce an associative and unital multiplication on the pullback \( P((\alpha')^* \text{ of } (\alpha')_* \) and \( (\alpha')^* \). Hence the induced maps from \( R \rightarrow P((\alpha')^* \rightarrow \text{Maps of dgas. They are quasi-isomorphisms since } (\alpha')^* \) is an acyclic fibration.

The \( \alpha'' \) induces an acyclic fibration \( (\alpha'')_* : T \rightarrow W \) since \( Y \) is cofibrant as an \( R \)-module. Right multiplication with \( \alpha'' \) induces a map \( S \rightarrow W \). It is a quasi-isomorphism since its composition with the quasi-isomorphism \( W = \text{Hom}_R(Y, X) \rightarrow \text{Hom}_R(R, X) \cong X \) induced by \( \alpha' \) equals \( \beta \). The latter statement exploits the compatibility of \( \alpha \) and \( \beta \) with the units of \( R \) and \( S \). As above, we form the pullback to get a chain \( T \rightarrow P((\alpha'')^* \rightarrow S \) of quasi-isomorphisms. \( \square \)

**Proof of Theorem 1.2** Let \( E \) be a \( dA_\infty \)-algebra. Lemma 2.18 and Proposition 6.11 show that \( \text{Tot} \text{Hom}_E(E, E) \) is a dga. If \( A \) is a dga and \( E \rightarrow A \) an \( E_2 \)-equivalence, Proposition 6.22 implies that \( \text{Tot} \text{Hom}_A(A, A) \) is quasi-isomorphic to...
Tot $\text{Hom}_{E}(E, E)$. Applying Tot to the $E_1$-equivalence $A \to \text{Hom}_{A}(A, A)$ and observing $A \cong \text{Tot } A$ completes the proof.

Over a field, every $A_{\infty}$-algebra is quasi-isomorphic to a dga. However, not every $dA_{\infty}$-algebra $E$ is $E_2$-equivalent to a dga, since $E$ may have $E_2$-homology in horizontal degrees other than 0. From this perspective, the relation of dgas and $dA_{\infty}$-algebras is more like viewing ordinary modules as (positive) chain complexes concentrated in degree 0.

7. FOUR LEMMAS

**Lemma 7.1.** In the situation of the proof of Proposition 4.7, we have $\tilde{m}_{01}^{B,1}\tilde{z}_{ln} = 0$.

**Proof.** We start with calculating

$$m_{01}^{B,1} m_{02}^{B,1} f_{ln} = -m_{02}^{B,1} (1 \otimes m_{01}^{B,1} + m_{01}^{B,1} \otimes 1) f_{ln}$$

(7.2)

$$= m_{11}^{B,1} m_{02}^{B,1} f_{l-1,n} - \sum_{i+p=l; 2 \leq j < n} (-1)^i m_{02}^{B,1} f_{ij} m_{pn}^{E,j}.$$  

The second summand of $\tilde{z}_{ln}$ contributes

$$m_{01}^{B,1} m_{11}^{B,1} f_{l-1,n} = m_{11}^{B,1} m_{01}^{B,1} f_{l-1,n}$$

(7.3)

$$= -m_{11}^{B,1} m_{02}^{B,1} f_{l-1,n} + \sum_{i+p=l; 2 \leq j < n} (-1)^i m_{11}^{B,1} f_{ij} m_{pn}^{E,j}.$$  

Applying $\tilde{m}_{01}^{B}$ to the last summand of $\tilde{z}_{ln}$ gives

$$= \sum_{i+p=l; 2 \leq j < n} (-1)^i \tilde{m}_{01}^{B} f_{ij} \tilde{m}_{pn}^{E,j} - \sum_{i+p=l; 2 \leq j < n} (-1)^i \tilde{m}_{11}^{B} f_{ij} \tilde{m}_{pn}^{E,j}.$$  

(7.4)

Forming the sum of (7.2), (7.3), and (7.4) shows the assertion.

**Lemma 7.5.** In the situation of the proof of Proposition 4.7, the $\tilde{m}_{ln}^{E,1}$ together with the $\tilde{z}_{ij}^{E,1}$ for $i + j < l + n$ satisfy (2.2) at $u + v \leq l + n + 1$.

**Proof.** Since $\tilde{m}_{01}^{E,1} = 0$, we have to verify

$$\sum_{i+j \geq 2; j < v} (-1)^i \tilde{m}_{ij}^{E,1} \tilde{m}_{pv}^{E,j} = 0$$

(7.6)

for $u + v = l + n + 1$. The term $\tilde{m}_{ij}^{E,1} \tilde{m}_{pv}^{E,j}$ is represented by $\tilde{z}_{ij}^{E,1} \tilde{m}_{pv}^{E,j}$ if $i + j > 2$, by $\tilde{m}_{02}^{B,1} (f_{ij} \otimes \tilde{z}_{u,v-1}) + \tilde{m}_{02}^{B,1} (\tilde{z}_{u,v-1} \otimes f_{ij})$ if $i = 0$ and $j = 2$, and by $\tilde{m}_{11}^{B,1} \tilde{z}_{u-1,v}$ if $i = 1$ and $j = 1$. We evaluate these expressions using the already established relations of $\tilde{m}_{pv}$ and the definition of $\tilde{z}_{ij}$. First we observe

$$\tilde{z}_{11}^{E,1} \tilde{z}_{u-1,v} = -\tilde{m}_{11}^{B,1} m_{02}^{B,1} f_{u-1,n} + \sum_{i+p=u-1; i+j \geq 2} (-1)^i \tilde{m}_{11}^{B,1} f_{ij} m_{pv}^{E,j}.$$  

(7.7)
Next we deduce

\[
\tilde{m}_{02}^B (\tilde{f}_{11} \otimes \tilde{z}_{u,v-1}) + \tilde{m}_{02}^B (\tilde{z}_{u,v-1} \otimes \tilde{f}_{11}) + \sum_{i+p=u,j+q=v, i+j \geq 2} (-1)^{i_1+i_2+1} \tilde{m}_{02}^B (\tilde{f}_{i_1,j_1} \otimes \tilde{f}_{i_2,j_2}) (1 \otimes \tilde{m}_{pq}^E) + \tilde{m}_{02}^B (\tilde{f}_{11} \otimes \tilde{m}_{pq}^E) + \tilde{m}_{02}^B (\tilde{m}_{pq}^E \otimes \tilde{f}_{11})
\]

(7.8)

\[
\tilde{m}_{11}^B \tilde{m}_{02}^B \tilde{f}_{u-1,v} + \sum_{i_1+i_2=u,j_1+j_2=v, i_1+i_2 \geq 2} \tilde{m}_{11}^B \tilde{m}_{02}^B (\tilde{f}_{i_1,j_1} \otimes \tilde{f}_{i_2,j_2}).
\]

The last step is to calculate

\[
\sum_{i+p=u,j+2} (-1)^i \tilde{z}_{ij} \tilde{m}_{pq}^E.
\]

(7.9)

Adding up (7.7), (7.8), and (7.9) shows that the sum the representatives for the terms in the desired formula has values in coboundaries.

\[\square\]

**Lemma 7.10.** In the situation of the proof of Proposition 4.12, \(\tilde{m}_{01}^B \tilde{y}_{ln} = 0\) holds.

**Proof.** We apply \(\tilde{m}_{01}^C\) to the six terms of \(\tilde{g}_{ln}\). The first gives

(7.11)

\[
\tilde{m}_{01}^C \tilde{m}_{11}^C \tilde{h}_{i_1}^C \tilde{m}_{ln}/1 = - (1)^i \tilde{m}_{11}^C (g\beta)^1 \tilde{f}_{i_1}^C \tilde{h}_{i_1}^C \tilde{m}_{ln}/1 - \sum_{i+p=-1} \tilde{m}_{11}^C \tilde{m}_{02}^C (\tilde{f}_{i_1,j_1}^C \otimes \tilde{h}_{pq}^1) - \sum_{i+p=-1} (1)^p \tilde{m}_{11}^C \tilde{m}_{02}^C (\tilde{h}_{11}^C \otimes (\tilde{f}_{pq}^C)^1)
\]

The second gives

(7.12)

\[
(1)^p \tilde{m}_{01}^C \tilde{a}_{ln}/1 - (1)^p \tilde{m}_{02}^C \tilde{a}_{ln}/1 + \sum_{i+p=-1} \tilde{m}_{11}^C \tilde{m}_{02}^C (\tilde{a}_{ln}/1)^2.
\]
The third gives

\[
(-1)^{i+1} \sum_{i+p=l} \tilde{m}_{01}^{C_{l-1}} \tilde{g}_{ij} \tilde{g}^j_{p/n} 
\]

\[(7.13)\]

\[= (-1)^{i} \tilde{m}_{11}^{C_{1}} (\tilde{g}^j_{i})_{l-1,n} + (-1)^{i} \tilde{m}_{02}^{C_{1}} (\tilde{g}^j_{i})_{l,n} - \sum_{i+p=l} (-1)^{i} (\tilde{g}^j_{i})_{ij} \tilde{m}_{p/n}^{E_{j}}
\]

The fourth gives

\[
(-1)^{i} \sum_{i+p=l} (-1)^{i} \tilde{m}_{01}^{C_{1}} h_{ij} \tilde{m}_{p/n}^{E_{j}}
\]

\[(7.14)\]

\[= \sum_{i+p=l} (-1)^{i} \tilde{m}_{01}^{C_{1}} h_{ij} \tilde{m}_{p/n}^{E_{j}} - \sum_{i+p=l} (-1)^{i} (\tilde{g}^j_{i})_{ij} \tilde{m}_{p/n}^{E_{j}} - (-1)^{i} \sum_{i+p=l} \tilde{m}_{11}^{C_{1}} h_{ij} \tilde{m}_{p/n}^{E_{j}}
\]

\[= (-1)^{i} \tilde{m}_{02}^{C_{1}} ((\tilde{g}^j_{i})_{ij} \otimes \tilde{h}^{l}_{pq}) + (-1)^{i} \sum_{i+p=l} \tilde{m}_{02}^{C_{1}} ((\tilde{g}^j_{i})_{ij} \otimes (\tilde{f})_{pq})
\]

\[(7.15)\]

\[= (-1)^{i} \tilde{m}_{02}^{C_{1}} (\tilde{g}^j_{i})_{ij} + (-1)^{i} \sum_{i+p=l} \tilde{m}_{02}^{C_{1}} ((\tilde{g}^j_{i})_{ij} \otimes \tilde{h}^{l}_{pq})
\]

\[+ \sum_{i+p+q=l} (-1)^{p} \tilde{m}_{02}^{C_{1}} (1 \otimes \tilde{m}_{02}^{C_{1}}) ((\tilde{g}^j_{i})_{ij} \otimes \tilde{h}^{l}_{pq})
\]

The sixth gives

\[
\sum_{i+p=l} (-1)^{p} \tilde{m}_{01}^{C_{1}} \tilde{m}_{02}^{C_{1}} (h_{ij}^{l} \otimes (\tilde{f})_{pq})
\]

\[= \sum_{i+p=l} (-1)^{p} \tilde{m}_{01}^{C_{1}} \tilde{m}_{02}^{C_{1}} (h_{ij}^{l} \otimes (\tilde{f})_{pq}) - (-1)^{i} \sum_{i+p=l} \tilde{m}_{02}^{C_{1}} ((\tilde{g}^j_{i})_{ij} \otimes (\tilde{f})_{pq})
\]

\[+ (-1)^{i} \tilde{m}_{02}^{C_{1}} (\tilde{f})_{ij}^{l} + \sum_{i+q+q=l} (-1)^{i} \tilde{m}_{02}^{C_{1}} ((\tilde{g}^j_{i})_{ij} \otimes (\tilde{f})_{pq})
\]

\[+ \sum_{i+q+q+q=l} (-1)^{p} \tilde{m}_{02}^{C_{1}} (\tilde{m}_{02}^{C_{1}} \otimes 1) ((\tilde{g}^j_{i})_{ij} \otimes \tilde{h}^{l}_{pq} \otimes (\tilde{f})_{pq})
\]

\[(7.16)\]

Indeed, the sum of \(7.11\), \ldots, \(7.16\) is zero. \(\square\)

**Lemma 7.17.** In the situation of the proof of Proposition 4.12 the \(d_{1}^{l}\) map represented by \(y_{n}\) satisfies \((2.6)_{u}\) for \(u + v = l + n + 1\).

**Proof.** We have to show that \(\beta\) is an \(d_{1}\)-map, that is,

\[
\sum_{i+p=u} (-1)^{i} \beta_{ij}^{l} \tilde{m}_{p}^{E_{j}} = \sum_{i+p=u} \tilde{m}_{ij}^{F_{j}} \beta_{p}^{l}
\]
We use the representing cocycles \((-1)^i \tilde{y}_{ij}\) and \(\tilde{z}_{ij}\). A direct calculation shows

\[
\sum_{i+p=u} (-1)^i \tilde{y}_{ij} \tilde{m}_{p,v}^{E,j} - \sum_{i+p=u} \tilde{z}_{ij} \tilde{h}_{p,v}^{C,1} (\tilde{g} \tilde{\beta})^{1}_{uv} - \sum_{i+p=u} (-1)^i \tilde{m}_{p,v}^{C,1} \tilde{h}_{ij} \tilde{m}_{p,v}^{E,j}
\]

References

[1] H.-J. Baues and T. Pirashvili. Comparison of Mac Lane, Shukla and Hochschild cohomologies. *J. Reine Angew. Math.*, 598:25–69, 2006.
[2] D. Benson, H. Krause, and S. Schwede. Realizability of modules over Tate cohomology. *Trans. Amer. Math. Soc.*, 356(9):3621–3668 (electronic), 2004.
[3] A. K. Bousfield. Cosimplicial resolutions and homotopy spectral sequences in model categories. *Geom. Topol.*, 7:1001–1053 (electronic), 2003.
[4] D. Dugger and B. E. Shipley. Topological equivalences for differential graded algebras. *Adv. Math.*, 212(1):37–61, 2007.
[5] W. G. Dwyer and J. P. C. Greenlees. Complete modules and torsion modules. *Amer. J. Math.*, 124(1):199–220, 2002.
[6] S. I. Gelfand and Y. I. Manin. *Methods of homological algebra*. Springer-Verlag, Berlin, 1996. Translated from the 1988 Russian original.
[7] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.
[8] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
[9] M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.
[10] J. F. Jardine. Bousfield’s $E_2$ model theory for simplicial objects. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 305–319. Amer. Math. Soc., Providence, RI, 2004.
[11] T. V. Kadeishvili. On the theory of homology of fiber spaces. *Uspekhi Mat. Nauk*, 35(3(213)):183–188, 1980. Translated in Russ. Math. Surv., 35(3):231–238, 1980.
[12] B. Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)*, 27(1):63–102, 1994.
[13] B. Keller. Introduction to \(A\)-infinity algebras and modules. *Homology Homotopy Appl.*, 3(1):1–35 (electronic), 2001.
[14] K. Leffèvre-Hasegawa. Sur les \(A_\infty\)-catégories. Thèse de doctorat, Université Denis Diderot – Paris 7, November 2003, arXiv:math/0310337.
[15] D. G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.
[16] S. Sagave. Universal Toda brackets of ring spectra. *Trans. Amer. Math. Soc.*, 360(5):2767–2808 (electronic), 2008.
[17] S. Saneblidze. On derived categories and derived functors. *Extracta Math.*, 22(3):315–324, 2007.
[18] S. Schwede and B. E. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.
[19] S. Schwede and B. E. Shipley. Equivalences of monoidal model categories. *Algebr. Geom. Topol.*, 3:287–334 (electronic), 2003.
[20] S. Schwede and B. E. Shipley. Stable model categories are categories of modules. *Topology*, 42(1):103–153, 2003.
[21] J. D. Stasheff. Homotopy associativity of \(H\)-spaces. I, II. *Trans. Amer. Math. Soc.* 108 (1963), 275–292; ibid., 108:293–312, 1963.

Department of Mathematics, University of Oslo, Box 1053, N-0316 Oslo, Norway

E-mail address: sagave@math.uio.no