Relativistic Burgers and Nonlinear Schrödinger Equations

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Abstract

Relativistic complex Burgers-Schrödinger and Nonlinear Schrödinger equations are constructed. In the non-relativistic limit they reduce to the standard Burgers and NLS equations respectively and are integrable at any order of relativistic corrections.

1 General Burgers-Schrödinger Hierarchy

The relativistic linear Schrödinger equation has been discussed at the early years of quantum mechanics but was dismissed promptly by the Klein-Gordon and the Dirac equations. Recently, relativistic versions of the Schrödinger equation have been considered in the study of relativistic quark-antiquark bound states \[1\], and gravitational collapse of a boson star [2]. A nonlinear version of the model has appeared in the form of semi-relativistic Hartree-Fock equation [3]. But none of those models is known to be integrable. In the present paper we construct an integrable relativistic nonlinear Schrödinger equation, preserving integrability at any order of \(1/c\) approximation.

We start from the Schrödinger equation in \(1 + 1\) dimensions

\[
\frac{i\hbar}{\partial t} \frac{\partial \Psi}{\partial t} = \mathcal{H}(P_1)\Psi
\]

for a free particle with classical dispersion of the general analytic form \(E = E(p)\). Here \(P_0 = i\hbar \frac{\partial}{\partial t}\) and \(P_1 = -i\hbar \frac{\partial}{\partial x}\) are operators of the time and space translations respectively, commuting with the Schrödinger operator \(S = i\hbar \frac{\partial}{\partial t} - \mathcal{H}(P_1)\): \([P_\mu, S] = 0, \mu = 0,1\). The general boost operator, defined as \(K = x - t \mathcal{H}'(P_1)\), is also commuting with \(S\), \([K, S] = 0\). Commuting it with space and time translations we have the algebra of symmetry operators

\[
[P_0, P_1] = 0, \quad [P_0, K] = -\hbar \mathcal{H}'(P_1), \quad [P_1, K] = -i\hbar .
\]
Then, if $\Psi$ is a solution of $\mathbf{1}$ and $W$ is an operator from this algebra, so that $[W,S] = 0$, then $S\Psi$ is also solution of $\mathbf{1}$.

For a given classical dispersion $E = E(p)$, $E_0 \equiv E(0)$ we define $E$-polynomials $H_n^{(E)}(x,t)$ by the generating function

$$e^{i\hbar(px - (E(p) - E_0)t)} = \sum_{n=0}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{p^n}{n!} H_n^{(E)}(x,t).$$

(3)

It is equivalent to

$$H_n^{(E)}(x,t) = e^{-\frac{\hbar}{2} \left(\mathcal{H} - E_0\right)t} x^n$$

(4)

so that $H_n^{(E)}(x,t)$ is a solution of

$$i\hbar \frac{\partial}{\partial t} H_n^{(E)}(x,t) = (\mathcal{H} - E_0) H_n^{(E)}(x,t)$$

(5)

with the initial value $H_n^{(E)}(x,0) = x^n$. From commutativity $[S,K] = 0$, time evolution of the operator $K$ satisfies

$$i\hbar \frac{\partial K}{\partial t} = [\mathcal{H}, K]$$

(6)

and has the form $K(t) = e^{-\frac{\hbar}{2} \mathcal{H}t} K(0) e^{\frac{\hbar}{2} \mathcal{H}t} = e^{-\frac{\hbar}{2} \mathcal{H}t} x e^{\frac{\hbar}{2} \mathcal{H}t}$. Then as follows, operator $K$ generates the infinite hierarchy of polynomials according to

$$KH_n(x,t) = Ke^{\frac{\hbar}{2} \left(\mathcal{H} - E_0\right)t} x^n = e^{\frac{\hbar}{2} \left(\mathcal{H} - E_0\right)t} x^{n+1} = H_{n+1}(x,t)$$

(7)

1.1 Non-relativistic Schrödinger equation

The non-relativistic dispersion $E(p) = p^2 / 2m$ implies the Hamiltonian operator

$$\mathcal{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

(8)

and the Galilean boost operator

$$K = x + it \frac{\hbar}{m} \frac{\partial}{\partial x}.$$}

(9)

From the generating function

$$e^{\frac{\hbar}{2} \left(px - \frac{x^2}{2m}\right)t} = \sum_{n=0}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{p^n}{n!} H_n^{(S)}(x,t)$$

(10)

we have the Schrödinger polynomials

$$H_n^{(S)}(x,t) = e^{\frac{\hbar}{2} \left(\mathcal{H} - E_0\right)t} x^n.$$}

(11)

If $H_n^{(KF)}(x,t) = \exp(t \frac{\hbar^2}{2m}\partial^2) x^n$ is the Kampe de Feriet polynomial, then $H_n^{(S)}(x,t) = H_n^{(KF)}(x, \frac{\hbar}{2m} t)$ or in terms of the Hermite polynomial

$$H^{(S)}(x,t) = \left(\frac{-i\hbar}{2m} t\right)^{n/2} H_n \left(\frac{x}{\sqrt{-2i\hbar t/m}}\right).$$

(12)
1.2 Semi-relativistic Schrödinger equation

The relativistic dispersion \( E(p) = \sqrt{m^2c^4 + c^2p^2} \) implies the Hamiltonian

\[
\mathcal{H} = mc^2 \sqrt{1 - \frac{\hbar^2}{m^2c^2} \frac{\partial^2}{\partial x^2} - 1}
\]

and the semi-relativistic boost operator

\[
K = x + \frac{i\hbar}{m} t \frac{\partial}{\partial x} \sqrt{1 - \frac{\hbar^2}{m^2c^2} \frac{\partial^2}{\partial x^2}}.
\]

In the non-relativistic limit, when \( c \to \infty \), it reduces to the Galilean boost \((9)\).

The generating function in the form of relativistic plane wave

\[
e^{\frac{i\hbar}{m} (px - (\sqrt{m^2c^4 + c^2p^2} - mc^2)t)} = \sum_{n=0}^{\infty} \left( \frac{i}{\hbar} \right)^n \frac{p^n}{n!} H_n^{(SRS)}(x, t)
\]

then gives semi-relativistic polynomials

\[
H_n^{SRS}(x, t) = e^{-\frac{\hbar}{m} mc^2 t} (1 - \frac{\hbar^2}{m^2c^2} \frac{\partial^2}{\partial x^2} - 1)^n x^n.
\]

In the non-relativistic limit \( H_n^{SRS} \to H_n^S \). The first three polynomials coincide exactly with the Schrödinger polynomials \( H_1^{SRS}(x, t) = x, H_2^{SRS}(x, t) = x^2 + i \frac{\hbar}{m} t, H_3^{SRS} = x^3 + i \frac{\hbar}{m} 3xt \), while starting from the fourth one

\[
H_4^{(SRS)}(x, t) = x^4 + i \frac{\hbar}{m} 3x^2t - \frac{\hbar^2}{m^2c^2} 3t^2 + i \frac{\hbar^3}{m^3c^2} 3t
\]

we have relativistic corrections of order \( 1/c^2 \). For complex valued space coordinate \( x \), as it appears in 2+1 dimensional Chern-Simons theory \[[4] \], zeros of these polynomials describe a motion of point vortices in the plane. Equations of motion for \( N \) vortices are \((k = 1, \ldots, N)\):

\[
\dot{x}_k = \frac{i}{\hbar} \text{Res}_{x=x_k} \mathcal{H} \left( \frac{\hbar}{t} \left( \frac{\partial}{\partial x} + \sum_{i=1}^{N} \frac{1}{x - x_i} \right) \right) \cdot 1
\]

1.3 Relativistic Burgers-Schrödinger Equation

Using Schrödinger’s log \( \Psi \) transform \[[5] \], \( \Psi = e^{i\ln \Psi} \), and identity

\[
e^{-i\ln \Psi} \frac{\partial^n}{\partial x^n} e^{i\ln \Psi} = \left( \frac{\partial}{\partial x} + \frac{\partial \ln \Psi}{\partial x} \right)^n \cdot 1
\]

the Schrödinger equation \[[11] \] can be rewritten in the form

\[
i \hbar \frac{\partial}{\partial t} \ln \Psi = \mathcal{H} \left( -i \hbar \left( \frac{\partial}{\partial x} + \frac{\partial \ln \Psi}{\partial x} \right) \right) \cdot 1
\]
For complex function $\Psi = e^{iF} = e^{R+iS}$ we introduce a new complex function $V = -i\frac{\hbar}{m} \frac{\partial}{\partial x} \ln \Psi = \frac{1}{m} \frac{\partial}{\partial x} F_x$, with dimension of velocity. Then $F = S - i\hbar R$, is the complex potential, real and imaginary part of which are the classical and quantum velocities $V = V_c + iV_q = \frac{1}{m} S_x - i\frac{\hbar}{m} R_x$. Hence \(20\) becomes of the complex Hamilton-Jacobi form (quantum Hamilton-Jacobi equation)

$$\frac{\partial F}{\partial t} + \mathcal{H} \left( -i\hbar \frac{\partial}{\partial x} + F_x \right) \cdot 1 = 0 \quad (21)$$

In the classical (dispersionless) limit $\hbar \to 0$, the quantum velocity $V_q$ vanishes and complex potential $F$ reduces to the real velocity potential $S$, playing the role of Hamilton’s principal function. In this case \(21\) becomes the classical Hamilton-Jacobi equation $\frac{\partial S}{\partial t} + \mathcal{H} \left( \frac{\partial S}{\partial x} \right) = 0$. By differentiation of \(20\) we have equation for the complex velocity

$$i\hbar \frac{\partial V_c}{\partial t} = -\frac{\hbar}{m} \frac{\partial}{\partial x} \left[ \mathcal{H} \left( -i\hbar \frac{\partial}{\partial x} + mV \right) \right] \cdot 1 \quad (22)$$

This equation is the Madelung fluid type representation of the Schrödinger equation \(1\). In the classical limit it gives the Newton equation $m \frac{\partial V_c}{\partial t} = -\frac{\partial}{\partial x} \left[ \mathcal{H}(mV_c) \right]$ or in the hydrodynamic type form

$$\frac{\partial V_c}{\partial t} + \mathcal{H}'(mV_c) \frac{\partial V_c}{\partial x} = 0 \quad (23)$$

which is just differentiation of the classical Hamilton-Jacobi equation. Equation \(23\) has implicit general solution as $V_c(x,t) = f(x - \mathcal{H}'(mV_c)t)$, where $f$ is an arbitrary function, and it develops shock at a critical time when derivative $(V_c)_x$ is blowing up.

### 1.3.1 Non-relativistic Burgers-Schrödinger equation

In this case the Schrödinger equation \(1\) with non-relativistic Hamiltonian \(8\) is equivalent to the nonlinear equation for complex velocity $V$

$$i\hbar \frac{\partial V}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 V}{\partial x^2} - i\hbar V \frac{\partial V}{\partial x} \quad (24)$$

which we call the Burgers-Schrödinger equation. In terms of the real and imaginary parts it gives the Madelung fluid with density $\rho = e^R$ and velocity $V_c$. In the classical limit it reduces to the one real equation for classical velocity $V_c$, namely the ordinary dispersionless Burgers equation.

### 1.3.2 Semi-relativistic Burgers-Schrödinger equation

For Hamiltonian \(13\), the "Burgerization" procedure described above gives the semi-relativistic Burgers-Schrödinger equation

$$\frac{1}{c} \frac{\partial V}{\partial t} + c \frac{\partial}{\partial x} \left[ \sqrt{1 + \frac{1}{m^2c^2} \left( -i\hbar \frac{\partial}{\partial x} + mV \right)^2} \cdot 1 \right] = 0 \quad (25)$$
In the non-relativistic limit it reduces to (24) with relativistic corrections in the lowest order as

\[ i\hbar \frac{\partial V}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 V}{\partial x^2} - i\hbar V \frac{\partial V}{\partial x} + \frac{1}{8m^3c^2} \left[ -i\hbar V_{xxx} \right] \] (26)

\[ + \frac{1}{8m^3c^2} \left[ -i\hbar^3(10V_x V_{xx} + 4VV_{xxx}) + m^2\hbar^2(12V_x^2 + 6V^3V_x) + 4im^3\hbar^3V^3 x \right] \] (27)

In the classical (dispersionless) limit (25) becomes equation of the hydrodynamic type

\[ (V_c)_t + \frac{V_c}{\sqrt{1 + V_c^2/c^2}}(V_c)_x = 0 \] (28)

In the non-relativistic limit it reduces to the dispersionless Burgers equation with the lowest relativistic correction

\[ (V_c)_t + V_c(V_c)_x - \frac{1}{2c^2} V_c^3(V_c)_x = 0 \] (29)

The general implicit solution of (28) is

\[ V_c(x,t) = f \left( x - \frac{V_c t}{\sqrt{1 + V_c^2/c^2}} \right) \] (30)

and it develops shock at a finite time.

1.4 Bäcklund Transformation for Burgers-Schrödinger equation

By the boost transformation of Section 1, from a given solution \( \Psi_1 \) of the Schrödinger equation (1) we can generate another solution as

\[ \Psi_2 = K \Psi_1 = \left[ x - tH' \left( -i\hbar \frac{\partial}{\partial x} \right) \right] \Psi_1. \] (31)

Using identity

\[ \Psi^{-1} G \left( -i\hbar \frac{\partial}{\partial x} \right) \Psi = G \left( -i\hbar \frac{\partial}{\partial x} + mV \right) \cdot 1 \] (32)

for complex velocities \( V_a = -i\frac{\hbar}{m} \ln \Psi_a, \ (a = 1, 2), \) we obtain the Bäcklund transformation

\[ V_2 = V_1 - i \frac{\hbar}{m} \frac{\partial}{\partial x} \ln \left[ x - tH' \left( -i\hbar \frac{\partial}{\partial x} + mV \right) \cdot 1 \right] \] (33)

For the non-relativistic quantum mechanics (8) it gives complex Bäcklund transformation

\[ V_2 = V_1 - i \frac{\hbar}{m} \frac{1 - (V_1)_x t}{x - V_1 t} \] (34)
for the Burgers-Schrödinger equation \(24\).

For the semi-relativistic quantum mechanics \(13\) we have the Bäcklund transformation of the form

\[
V_2 = V_1 - i \frac{\hbar}{m} \frac{\partial}{\partial x} \ln \left( x - \frac{1}{\sqrt{1 + \frac{1}{m^2 c^2} (-i\hbar \frac{\partial}{\partial x} + mV_1)^2}} V_1 t \right)
\]

(35)

It is worth to note that in the classical limit \(\hbar \to 0, V \to V_c\) the above Bäcklund transformations reduce to the trivial identity \(V_{c1} = V_{c2}\).

2 Integrable General NLS Hierarchy

In previous sections we studied the so called C-integrable relativistic Burgers-Schrödinger equation. Now using the AKNS hierarchy for the NLS equation we are going to construct relativistic NLS.

2.1 NLS hierarchy

We consider the Zakharov-Shabat linear problem

\[
\frac{\partial}{\partial x} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{i}{2} p & -\kappa^2 \bar{\psi} \\ \bar{\psi} & \frac{\kappa^2}{2} p \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},
\]

(36)

for the space evolution, and the generalized AKNS problem \(6\)

\[
\frac{\partial}{\partial t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -iA & -\kappa^2 \bar{C} \\ C & -iA \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J_0 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},
\]

(37)

for the time evolution, where for the real \(A(x,t,p)\) and complex \(C(x,t,p)\) functions, determined by the zero-curvature condition, we substitute

\[
a_N = \sum_{n=0}^{N} A^{(n)} (-\frac{p}{2})^n, \quad C_N = \sum_{n=0}^{N} C^{(n)} (-\frac{p}{2})^n.
\]

It gives the evolution equation

\[
\partial_{tN} \psi = \partial_x C^{(0)} + 2iA^{(0)} \psi \quad \text{and} \quad C^{(N)} = 0, \quad A^{(N)} = a_N = \text{const}.
\]

We fix this constant so that \(a_N = (-2)^{N-1}\). Then we have the recurrence relations

\[
C^{(n)} = \frac{1}{2i} \partial_x C^{(n+1)} + A^{(n+1)} \psi, \quad \partial_x A^{(n)} = i\kappa^2 (\bar{C}^{(n)} \psi - C^{(n)} \bar{\psi}),
\]

where \(n = 0, 1, 2, ..., N - 1\). Integrating the last equation one has

\[
A^{(n)} = -i\kappa^2 \int^x (\bar{\psi} C^{(n)} - \psi \bar{C}^{(n)})
\]

(38)

Substituting \(38\) into recursion formula we find

\[
\begin{pmatrix} C^{(n)} \\ \bar{C}^{(n)} \end{pmatrix} = -\frac{1}{2} \mathcal{R} \begin{pmatrix} C^{(n+1)} \\ \bar{C}^{(n+1)} \end{pmatrix}
\]

(39)

where \(\mathcal{R}\) is the matrix integro-differential operator - the recursion operator of the NLS hierarchy \(3\) -

\[
\mathcal{R} = i\sigma_3 \begin{pmatrix} \partial_x + 2\kappa^2 \bar{\psi} \int^x \bar{\psi} & -2\kappa^2 \bar{\psi} \int^x \psi \\ -2\kappa^2 \bar{\psi} \int^x \bar{\psi} & \partial_x + 2\kappa^2 \bar{\psi} \int^x \psi \end{pmatrix}
\]

(40)
and $\sigma_3$ - the Pauli matrix. Then we get

$$i\sigma_3 \left( \frac{\psi}{\bar{\psi}} \right)_{t_N} = \mathcal{R}_N \left( \frac{\psi}{\bar{\psi}} \right)$$  \hspace{1cm} (41)

where $t_N$, $N = 1, 2, 3, \ldots$ is an infinite time hierarchy. In the linear approximation, when $\kappa = 0$, the recursion operator is just the momentum operator $\mathcal{R}_0 = i\sigma_3 \frac{\partial}{\partial x}$ and the NLS hierarchy (41) becomes the linear Schrödinger hierarchy

$$i\psi_{t_n} = i^n \frac{\partial^n}{\partial x^n} \psi$$  \hspace{1cm} (42)

from Section 1. The Madelung representation for this hierarchy, produced by the complex Cole-Hopf transformation, is given by the complex Burgers hierarchy [4].

Every equation of hierarchy (41) is integrable. The linear problem for the $N$-th equation is given by the Zakharov-Shabat problem (36) for the space part and

$$\frac{\partial}{\partial t_N} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -iA_N & -\kappa^2 \tilde{C}_N \\ \kappa^2 \tilde{C}_N & -iA_N \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J_{0_N} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$  \hspace{1cm} (43)

for the time part. Coefficient functions $C_N$ can be found conveniently as

$$\begin{pmatrix} C_N \\ \tilde{C}_N \end{pmatrix} = \sum_{k=1}^{N} p^{N-k} R^{k-1} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = (p^{N-1} + p^{N-2} R + \ldots + R^{N-1}) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$  \hspace{1cm} (44)

To rewrite this expression in a compact form we introduce notation of the $q$-number operator

$$1 + q + q^2 + \ldots + q^{N-1} \equiv [N]_q$$  \hspace{1cm} (45)

where $q$ is a linear operator. Hence, with operator $q \equiv \mathcal{R}/p$ we have following finite Laurent form in the spectral parameter $p$

$$1 + \frac{\mathcal{R}}{p} + \left( \frac{\mathcal{R}}{p} \right)^2 + \ldots + \left( \frac{\mathcal{R}}{p} \right)^{N-1} \equiv [N]_{\mathcal{R}/p}$$  \hspace{1cm} (46)

Then we have shortly

$$\begin{pmatrix} C_N \\ \tilde{C}_N \end{pmatrix} = p^{N-1} [N]_{\mathcal{R}/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$  \hspace{1cm} (47)

In a similar way

$$A_N = -\frac{p^N}{2} - i\kappa^2 p^{N-1} \left( \int^x \bar{\psi}_i - \bar{\psi} \right) [N]_{\mathcal{R}/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$  \hspace{1cm} (48)

Equations (43), (47) and (48) give the time part of the linear problem (the Lax representation) for the $N$-th flow of NLS hierarchy (11) in the $q$-calculus form.
2.2 General NLS hierarchy equation

For time \( t \) determined by the formal series \( \partial_t = \sum_{N=0}^{\infty} E_N \partial_{t_N} \) where \( E_N \) are arbitrary constants, the general NLS hierarchy equation is

\[
i \sigma_3 \left( \frac{\psi}{\bar{\psi}} \right)_t = \left( E_0 + E_1 R + ... + E_N R^N + ... \right) \left( \frac{\psi}{\bar{\psi}} \right)
\]  

(49)

Integrability of this equation is associated with the Zakharov-Shabat problem and the time evolution

\[
J_0 = \sum_{N=0}^{\infty} E_N J_{0N} = \begin{pmatrix} -iA & -\kappa^2 \bar{C} \\ C & -iA \end{pmatrix}
\]  

(50)

where

\[
\begin{pmatrix} C \\ \bar{C} \end{pmatrix} = \sum_{N=0}^{\infty} E_N \begin{pmatrix} C_N \\ \bar{C}_N \end{pmatrix} = \sum_{N=1}^{\infty} E_N p^{N-1} [N] R_{/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}
\]  

(51)

In the last equation we have used that for \( N = 0 \), \( C_0 = 0 \). Then we have

\[
A = \sum_{N=0}^{\infty} E_N A_N = -\frac{1}{2} \sum_{N=0}^{\infty} E_N p^N - i\kappa^2 \left( \int_x \bar{\psi}, - \int_x \psi \right) \begin{pmatrix} C \\ \bar{C} \end{pmatrix}
\]  

(52)

The above equation gives integrable nonlinear extension of linear Schrödinger equation with general analytic dispersion considered in Section 1. Let one considers the classical particle system with the energy-momentum relation \( E(p) = E_0 + E_1 p + E_2 p^2 + ... \). Then the corresponding time-dependent Schrödinger wave equation is where the Hamiltonian operator results from the standard substitution for momentum \( p \rightarrow -i\hbar \frac{\partial}{\partial x} \) in the dispersion. Equation together with its complex conjugate can be rewritten as

\[
i\hbar \sigma_3 \frac{\partial}{\partial t} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = H \left( -i\hbar \sigma_3 \frac{\partial}{\partial x} \right) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}
\]  

(53)

The momentum operator here is just the recursion operator in the linear approximation \( R_0 = i\hbar \sigma_3 \frac{\partial}{\partial x} \). Hence, is the linear Schrödinger equation with arbitrary analytic dispersion. The nonlinear integrable extension of this equation appears as \( (49) \), which corresponds to the replacement \( R_0 \rightarrow R \), \((\hbar = 1)\), so that

\[
i\sigma_3 \left( \frac{\psi}{\bar{\psi}} \right)_t = H \left( R \right) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}
\]  

(54)

From this point of view, the standard substitution for classical momentum \( p \rightarrow -i\hbar \frac{\partial}{\partial x} \) or equivalently \( p \rightarrow -i\hbar \sigma_3 \frac{\partial}{\partial x} = R_0 \) for the equation in spinor form, gives quantization in the form of the linear Schrödinger equation. While substitution \( p \rightarrow R \) gives ”nonlinear quantization” and the nonlinear Schrödinger hierarchy equation.
The related Lax representation for equation (54) is given by (51), (52). By definition of q-derivative
\[ D_q^\xi \] for operator \( q = \mathcal{R}/p \), we have relation
\[ D_{\mathcal{R}/p}^\xi = [N]_{\mathcal{R}/p} p^{N-1} \]. Then equation (51) can be rewritten as
\[
\begin{pmatrix}
C \\
\bar{C}
\end{pmatrix} = \sum_{N=1}^{\infty} E_N p^{N-1} [N]_{\mathcal{R}/p} \left( \frac{\psi}{\bar{\psi}} \right) = \sum_{N=1}^{\infty} E_N D_{\mathcal{R}/p}^\xi p^N \left( \frac{\psi}{\bar{\psi}} \right)
\] (55)
or using linearity of q-derivative and analytic dispersion form
\[
\begin{pmatrix}
C \\
\bar{C}
\end{pmatrix} = D_{\mathcal{R}/p}^\xi \sum_{N=0}^{\infty} E_N p^N \left( \frac{\psi}{\bar{\psi}} \right) = D_{\mathcal{R}/p}^\xi E(p) \left( \frac{\psi}{\bar{\psi}} \right)
\] (56)
Due to above definition it gives simple formula
\[
\begin{pmatrix}
C \\
\bar{C}
\end{pmatrix} = \frac{E(\mathcal{R}) - E(p)}{\mathcal{R} - p} \left( \frac{\psi}{\bar{\psi}} \right)
\] (57)
Then for \( A \) we obtain
\[
A = -\frac{1}{2} E(p) - i\kappa^2 \left( \int^x \bar{\psi}, - \int^x \psi \right) \frac{E(\mathcal{R}) - E(p)}{\mathcal{R} - p} \left( \frac{\psi}{\bar{\psi}} \right)
\] (58)
Equations (57), (58) give the Lax representation of the general integrable NLS hierarchy model (54) in a simple and compact form. It is worth to note that special form of the dispersion \( E = E(p) \) is fixed by physical problem. In the next Section we will discuss the relativistic form of this dispersion and corresponding semi-relativistic NLS equation.

3 Semi-relativistic NLS

In Section 1.2 we have considered the relativistic dispersion relation \( E(p) = \sqrt{m^2 c^4 + p^2 c^2} \). This may be used to construct a “semi-relativistic” Schrödinger equation with Hamiltonian (13). Then, combining two complex conjugate equations together, we have
\[
i\sigma_3 \left( \frac{\psi}{\bar{\psi}} \right)_t = mc^2 \sqrt{1 + \frac{1}{m^2 c^2} \left( i\sigma_3 \frac{\partial}{\partial x} \right)^2} \left( \frac{\psi}{\bar{\psi}} \right)
\] (59)
We like to emphasize that if \( \psi \) describes the relativistic particle forward in time and with positive energy, \( \bar{\psi} \) corresponds to the backward time or to the negative energy. From this point of view equation (59) is complete since includes both states.

Following the general procedure described in previous Section one may proceed further: by replacing the derivative operator \( \mathcal{R}_0 = i\sigma_3 \frac{\partial}{\partial x} \) corresponding to linear momenta \( p \) with the full recursion operator \( \mathcal{R} \) (10), one obtains an integrable relativistic nonlinear Schrödinger equation

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\[ i\sigma_3 \left( \frac{\psi}{\bar{\psi}} \right)_t = mc^2 \sqrt{1 + \frac{1}{m^2 c^2} R^2} \left( \frac{\psi}{\bar{\psi}} \right) \] (60)

where the square root operator has meaning of the formal power series so that

\[ i\sigma_3 \left( \frac{\psi}{\bar{\psi}} \right)_t = mc^2 \left( 1 + \frac{1}{2m^2 c^2} R^2 - \frac{1}{8m^4 c^4} R^4 + \frac{1}{16m^6 c^6} R^6 \pm \ldots \right) \left( \frac{\psi}{\bar{\psi}} \right) \] (61)

For the above relativistic dispersion and equation (60), we have the next linear problem

\[ \frac{\partial}{\partial x} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{i}{2} p & -\kappa^2 \bar{\psi} \\ \bar{\psi} & \frac{i}{2} p \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \] (62)

\[ \frac{\partial}{\partial t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -i A & -\kappa^2 \bar{C} \\ \bar{C} & -i A \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \] (63)

where

\[ \begin{pmatrix} C \\ \bar{C} \end{pmatrix} = \sqrt{m^2 c^4 + p^2 c^2 - \sqrt{R^2 c^4 + p^2 c^2} R - p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \] (64)

\[ A = \frac{1}{2} \sqrt{m^2 c^4 + p^2 c^2} - i\kappa^2 \left( \int^x \bar{\psi}, -\int^x \psi \right) \sqrt{m^2 c^4 + R^2 c^2} - \sqrt{m^2 c^4 + p^2 c^2} R - p \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \] (65)

and the spectral parameter \( p \) has meaning of the classical momentum. The model (60), is an integrable nonlinear Schrodinger equation with relativistic dispersion:

\[ i\psi_t = mc^2 \sqrt{1 - \frac{1}{m^2 c^2} \frac{\partial^2}{\partial x^2}} \psi + F(\psi) \] (66)

where the nonlinearity expanded in \( 1/c^2 \) is the infinite sum

\[ F(\psi) = \frac{1}{2m} [-2\kappa^2 |\psi|^2 \psi] \]

\[ -\frac{1}{8m^3 c^4} [2\kappa^2 (2|\psi|^2 \psi + 4|\psi|^2 \psi_{xx} + \bar{\psi}_{xx} \psi^2 + 3\bar{\psi} \psi_x^2) + 6\kappa^4 |\psi|^4 \psi] + O(\frac{1}{c^6}) \]

It is interesting to note that if we expand also the dispersion part in \( 1/c^2 \), then at every order of \( 1/c^2 \) we get an integrable system. It means that we obtain integrable relativistic corrections to the NLS equation at any order. From the known relativistic integrable models like the Sine-Gordon or the Liouville equations, neither one has this property. Finally we note that nonlinear relativistic equations considered in this paper are distinct from those obtained in [1]-[3] and references therein. They might be useful in analyzing relativistic corrections to solitons, Bose-Einstein condensates or other condensed matter systems with an effective equation of relativistic form.
4 Acknowledgements

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References

[1] Nickisch L J, Durand L and Durand B, Salpeter equation in position space: numerical solution for arbitrary confining potentials, Phys. Rev D30(1984) 660-70;

[2] Frohlich J and Lenzmann E, Blowup for nonlinear wave equations describing boson stars, Com. Pure Appl.Math. (2007) 0001-15;

[3] Cho Y and Ozawa T, On the semirelativistic Hartree-type equations, SIAM J. Math. Analys.38(2006) 1060-74

[4] Pashaev O.K. and Gurkan Z.N. Abelian Chern-Simons vortices and holomorphic Burgers hierarchy, Theor Math Phys 152 (2007) 1017-1029

[5] Schrödinger E., Ann. Physik, 79 (1926) 361

[6] Ablowitz M, Kaup D, Newell A and Segur H, Stud. Appl. Math. 53 (1974) 249