Profinite semigroups

Revekka Kyriakoglou, Dominique Perrin
Université Paris-Est, LIGM

March 30, 2017

Abstract

We present a survey of results on profinite semigroups and their link with symbolic dynamics. We develop a series of results, mostly due to Almeida and Costa and we also include some original results on the Schützenberger groups associated to a uniformly recurrent set.

Contents

1 Introduction 2
2 p-adic numbers 4
3 The Fibonacci morphism 5
4 Topological spaces, groups and semigroups 6
  4.1 Topological spaces 6
  4.2 Topological semigroups 8
  4.3 Topological groups 8
5 Profinite semigroups 9
  5.1 Projective limits 9
  5.2 Profinite semigroups 11
  5.3 Profinite groups 12
  5.4 Endomorphisms of profinite semigroups 13
  5.5 The \(\omega\) operator 13
  5.6 The free profinite monoid 14
  5.7 The free profinite group 15
  5.8 Recognizable sets 16
  5.9 The natural metric 17
6 Profinite codes 18
  6.1 Finite codes 18
  6.2 Codes and submonoids 19
7 Uniformly recurrent sets
7.1 Uniformly recurrent pseudowords ............................... 20
7.2 The $J$-class $J(F)$ .................................................. 23
7.3 Fixed points of substitutions ...................................... 24

8 Sturmian sets and tree sets ........................................... 26
8.1 Sturmian sets ............................................................. 26
8.2 Tree sets ................................................................. 28

9 Return words .............................................................. 29
9.1 Left and right return words ....................................... 29
9.2 Limit return sets ....................................................... 32

10 Schützenberger groups ................................................ 33
10.1 Groups of tree sets .................................................. 34
10.2 Groups of fixed points of morphisms ......................... 35
10.3 Groups of bifix codes ............................................... 37

1 Introduction

The theory of profinite groups originates in the theory of infinite Galois groups and of $p$-adic analysis (see [33]). The corresponding theory for semigroups was considerably developed by Jorge Almeida (see [2] for an introduction). The initial motivation has been the theory of varieties of languages and semigroups, put in correspondence by Eilenberg’s theorem. Later, Almeida has initiated the study of the connexion between free profinite semigroups and symbolic dynamics (see [1]). He has shown in [7] that minimal subshifts correspond to maximal $J$-classes of the free profinite monoid. Moreover, the Schützenberger group of such a $J$-class is a dynamical invariant of the subshift [4]. Finally, it is shown in [5] that if a minimal system satisfies the tree condition (as defined in [11]), the corresponding group is a free profinite group.

In these notes, we give a gentle introduction to the notions used in profinite algebra and develop the link with minimal sets.

Our motivation to explore the profinite world is the following. We are interested in the situation where we fix a uniformly recurrent set $F$ on an alphabet $A$ (in general a non rational set, like the set of factors of the Fibonacci word). We want to study sets of the form $F \cap L$ where $L$ is a rational set on $A$, that is the inverse image through a morphism $\varphi : A^* \to M$ of a subset of a finite monoid $M$. The aim is thus to develop a theory of automata observing through the filter of a non rational set.

We are particularly interested in sets $L$ of the form

1. $L = wA^*w$ for some word $w \in F$ (linked to the complete return words to $w$)

2. $L = h^{-1}(H)$ where $h : A^* \to G$ is a morphism onto a finite group $G$ and $H$ is a subgroup of $G$. 
This question has been studied successfully in a number of cases starting with a Sturmian set \( F \) in [9] and progressively generalizing to sets called tree sets in [11].

The framework of profinite semigroups allows one to work simultaneously with all rational sets \( L \). This is handled, as we shall see, both through the definition of an inverse limit and through a topology on words. This topology is defined by introducing a distance on words: two words are close if one needs a morphism \( \varphi \) on a large monoid \( M \) to distinguish them. For example, for any word \( x \), the powers \( x^n \) of \( x \) will not be distinguished by a monoid with less than \( n \) elements. Thus one can consider a limit, called a pseudoword, and denoted \( x^\omega \) which has the same image by all morphisms \( \varphi \) onto a finite monoid \( M \).

The moment of inspiration was the derivation by Ameida and Costa [5] of a result on profinite groups (the Schützenberger group of a tree set is free) which uses the main result of [11]. This result gives a global version of properties like the Finite Index Basis Property, as defined in [12]. One of the goals of this paper is to develop this connexion.

We begin with two motivating examples: the first one concerns \( p \)-adic and profinite numbers (Section 2) and the second one the Fibonacci morphism (Section 3).

We give in Section 5 an introduction to the basic notions concerning profinite semigroups. We have chosen a simplified presentation which uses the class of all semigroups instead of working inside a pseudovariety of semigroups. It simplifies the statements but the proofs work essentially in the same way.

In Section 6 we describe results concerning codes in profinite semigroups. The main result, from [29], is that any finite code is a profinite code (Theorem 6.1).

In Section 7 we introduce uniformly recurrent pseudowords. We prove a result of Almeida (Theorem 7.2) showing that uniformly recurrent pseudowords can be characterized in algebraic terms, as the \( J \)-maximal elements of the free profinite monoid.

In Section 8 we recall some basic properties of Sturmian sets and their generalization, the tree sets, introduced in [11].

In Section 9 we prove several results due to Costa and Almeida concerning the presentation of the Schützenberger group of a uniformly recurrent set.

In Section 10 we prove a new result (Theorem 10.18) concerning the Schützenberger groups of uniformly recurrent sets.

**Acknowledgements** We would like to thank Jorge Almeida and Alfredo Costa for their help in the preparation of this manuscript. It was written in connexion with a workshop held in Marne la Vallée on January 20–22 2016 and gathering around them Marie-Pierre Béal, Valérie Berthé, Francesco Dolce, Pavel Heller, Julien Leroy, Jean-Eric Pin and the authors.
2 p-adic numbers

We begin with a motivating example (see [24] for an introduction to this subject). Let $p$ be a prime number and let $\mathbb{Z}_p$ denote the ring of $p$-adic integers, namely the completion of $\mathbb{Z}$ under the $p$-adic metric. This metric is defined by the norm

$$|x|_p = \begin{cases} p^{-\ord_p(x)} & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\ord_p(x)$ is the largest $n$ such that $p^n$ divides $x$. For the topology defined by this metric, $\mathbb{Z}_p$ is compact.

Any element $\gamma \in \mathbb{Z}_p$ has a unique $p$-adic expansion

$$\gamma = c_0 + c_1p + c_2p^2 + \ldots = (\ldots c_3c_2c_1c_0)_p$$

Note that infinite expansions may represent ordinary integers. For example

$$(\ldots111)_2 = -1.$$  

We can express the expansion of the elements in $\mathbb{Z}_p$ as

$$\mathbb{Z}_p = \lim \mathbb{Z}/p^n\mathbb{Z}$$

$$= \{(a_n)_{n \geq 1} \in \prod_{n \geq 1} \mathbb{Z}/p^n\mathbb{Z} \mid \text{for all } n, a_{n+1} \equiv a_n \mod p^n \}$$

This expresses the ring $\mathbb{Z}_p$ as a projective limit of the rings $\mathbb{Z}/p^n\mathbb{Z}$. The direct product of all rings $\mathbb{Z}_p$

$$\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$$

over all prime numbers $p$ is the ring of profinite integers. Its topology is that induced by the product. As a product of compact spaces, $\hat{\mathbb{Z}}$ is itself compact. It does not depend on a particular number $p$ and shares with all rings $\mathbb{Z}_p$ the property of being a compact topological space. Thus it is a compact covering of all rings $\mathbb{Z}_p$.

One may define it equivalently as the projective limit of all cyclic groups

$$\hat{\mathbb{Z}} = \prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z}$$

$$= \{(a_n)_{n \geq 1} \in \prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z} \mid \text{for all } m, a_m \equiv a_n \mod n \}$$

or as the projective limit of the cyclic groups $\mathbb{Z}/n!\mathbb{Z}$

$$\hat{\mathbb{Z}} = \prod_{n \geq 1} \mathbb{Z}/n!\mathbb{Z}$$

$$= \{(a_n)_{n \geq 1} \in \prod_{n \geq 1} \mathbb{Z}/n!\mathbb{Z} \mid \text{for all } n, a_{n+1} \equiv a_n \mod n! \}$$
The last representation corresponds to an expansion of the form
\[ γ = c_1 + c_2 2! + c_3 3! + \ldots = (\ldots c_3 c_2 c_1)! \]
with digits \( 0 \leq c_i \leq i \). This expansion forms the \textit{factorial number system} (see [23]).

Note that this time
\[ -1 = (\ldots 321)! \]
which holds because \( 1 + 2.2! + \ldots + n.n! = (n + 1)! - 1 \), as one may verify by induction on \( n \). The profinite topology on \( \hat{\mathbb{Z}} \) can also be defined directly by the norm \( |x|! = 2^{-r(x)} \) where \( r(x) \) is the largest \( n \) such that \( n! \) divides \( x \). A sequence converges with respect to this topology if the expansions converge in the usual sense, that is the number of equal digits, starting from the right, tends to infinity.

It is possible to define profinite Fibonacci numbers (see [26]). Indeed, Fibonacci numbers are defined by \( F_0 = 0, F_1 = 1 \) and \( F_n = F_{n-1} + F_{n-2} \). The definition can be extended to negative \( n \) by \( F_n = (-1)^{n-1} F_{-n} \). Then one may extend the function \( n \mapsto F_n \) to a continuous function from \( \hat{\mathbb{Z}} \) into itself. We note that since \( n! \) tends to 0 in \( \hat{\mathbb{Z}} \), the sequence \( F_{n!} \) tends to \( F_0 = 0 \). Similarly \( F_{n!+2} \) and \( F_{n!+1} \) tend to 1. We come back to this in the next section.

3 The Fibonacci morphism

In the last example, we have seen how a linear recurrent sequence (the Fibonacci numbers) can interestingly be extended to a topological limit. As well known, Fibonacci numbers are the lengths of the Fibonacci words defined inductively by \( w_0 = b, w_1 = a \) and satisfying the recurrence relation \( w_{n+1} = w_n + w_{n-1} \) for \( n \geq 1 \). Then \( |w_n| = F_{n+1} \). In the same way as one may embed the ring of ordinary integers into the compact ring of profinite integers, we will see that one may extend the Fibonacci sequence to a converging sequence of pseudowords whose lengths are profinite integers.

The \textit{Fibonacci morphism} is the morphism \( \varphi : A^* \to A^* \) with \( A = \{a, b\} \) defined by \( \varphi(a) = ab \) and \( \varphi(b) = a \). The sequence of \( \varphi^n(a) \)
\[
\varphi(a) = ab \\
\varphi^2(a) = aba \\
\varphi^3(a) = abaab \\
\varphi^4(a) = abaababa \\
\vdots
\]
is the Fibonacci sequence of words of length equal to the Fibonacci numbers. One has \( F_n = |\varphi^{n-2}(a)| \) for \( n \geq 2 \) (and for \( n \in \mathbb{Z} \) with appropriate extensions).

The Fibonacci sequence of words converges in the space \( A^\mathbb{N} \) to the Fibonacci infinite word
\[ x = abaababa \cdots \]
It is a fixed-point of \( \varphi \) in the sense that \( \varphi(x) = x \). From another point of view, this sequence is not convergent. Indeed, the terms of the sequence end alternately with \( a \) or \( b \) and thus can be distinguished by a morphism from \( A^* \) into a monoid with 3 elements. A sequence converging in this stronger sense is \( \varphi^n(a) \)

\[
\begin{align*}
\varphi(a) &= ab \\
\varphi^2(a) &= aba \\
\varphi^6(a) &= abaababaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaab
The notion of limit point can also be defined for a sequence. A point \( x \) is said to be a limit point of the sequence \( (x_n)_{n \geq 0} \) if any open set containing \( x \) contains all but a finite number of terms of the sequence. It is said to be an accumulation point of the sequence if every open set containing \( x \) contains an infinite number of elements of the sequence. A limit point of the sequence \( (x_n)_{n \geq 0} \) is a limit point of the set \( \{x_n \mid n \geq 0\} \).

A map \( \varphi : X \to Y \) between topological spaces \( X,Y \) is continuous if for any open set \( U \subset Y \), the set \( \varphi^{-1}(U) \) is open in \( X \).

A basis of the family of open sets is a family \( B \) of sets such that any open set is a union of elements of \( B \).

Given a family of topological spaces \( X_i \) indexed by a set \( I \), the product topology on the direct product \( X = \prod_{i \in I} X_i \) is defined as the coarsest topology such that the projections \( \pi_i : X \to X_i \) are continuous. A basis of the family of open sets is the family of open boxes, that is sets of the form \( \prod_{i \in I} U_i \) where the \( U_i \) are open sets such that \( U_i \neq X_i \) only for a finite number of indices \( i \).

**Example 4.1** The set \( A^\mathbb{N} \) of infinite words over \( A \) is a topological space for the discrete topology on \( A \). It is not a semigroup (a big motivation for introducing pseudowords!). However, there is a well defined product of finite words and infinite words. The open sets are the sets of the form \( XA^\mathbb{N} \) for \( X \subset A^* \).

Metric spaces form a vast family of topological spaces. A metric space is a space \( S \) with a function \( d : S \times S \to \mathbb{R} \), called a distance, such that for all \( x,y,z \in S \),

(i) \( d(x,y) = 0 \) if and only if \( x = y \),

(ii) \( d(x,y) = d(y,x) \)

(iii) \( d(x,z) \leq d(x,y) + d(y,z) \) (triangle inequality).

Any metric space can be considered as a topological space, considering as open sets the unions of open balls \( B_\varepsilon(x) = \{ y \in S \mid d(x,y) < \varepsilon \} \) for \( x \in S \) and \( \varepsilon \geq 0 \).

For example, the set \( \mathbb{R}^n \) is a metric space for the Euclidean distance.

A topological space is a Hausdorff space if any two distinct points belong to disjoint open sets. A metric space is a Hausdorff space. In a Hausdorff space, every limit point of a set is an accumulation point.

A topological space is compact if it is a Hausdorff space and if from any family of open sets whose union is \( S \), one may extract a finite subfamily with the same property.

A closed subset of a compact space is compact and, by Tychonoff’s theorem, any product of compact spaces is compact.

One may verify that in a compact space every infinite set has an accumulation point (the converse is true in metric spaces). Indeed, if \( X \) is an infinite subset of a compact space \( S \) without accumulation point, there is for every \( x \in S \) an open set containing \( x \) which contains only a finite number of elements of \( X \). For a finite set \( F \subset X \), denote by \( O_F \) the union of the \( O_x \) such that \( O_x \cap X = F \). Then the sets \( O_F \) form a family of open sets whose union is \( S \).
every finite subfamily of which intersects at most finitely many points in $X$. Thus no finite subfamily may cover $S$, a contradiction.

The clopen sets in a product of compact spaces $S_i$ are the finite unions of clopen boxes, that is the sets of the form $\prod_{i\in I} K_i$ where each $K_i$ is clopen in $S_i$ and $K_i = S_i$ for all but a finite number of indices $i$.

4.2 Topological semigroups

A semigroup is a set with an associative operation. A monoid is a semigroup with a neutral element.

A topological semigroup is a semigroup $S$ endowed with a topology such that the semigroup operation $S \times S \to S$ is continuous. A topological monoid is a topological semigroup with identity.

Example 4.2 A finite semigroup can always be viewed as a topological semigroup under the discrete topology.

Example 4.3 As a less trivial example, the set $\mathbb{R}$ of nonnegative real numbers is a topological semigroup for the addition and the interval $[0, 1]$ is a topological semigroup for the multiplication.

A compact monoid is a topological monoid which is compact (as a topological space). Note that we assume a compact space to satisfy Hausdorff separation axiom (any two distinct points belong to disjoint open sets). Note also the following elementary property of compact monoids. Recall that, if $M, u \in M$ is a factor of $v \in M$ if $v \in MuM$.

4.3 Topological groups

A topological group is a group with a topology such that the multiplication and taking the inverse are continuous operations. It is in particular a topological semigroup.

Example 4.4 The set $\mathbb{R}$ of real numbers with the usual topology is a topological group under addition.

Any closed subgroup is a topological group for the induced topology. Moreover, since multiplication is continuous, the cosets $Hg$ of an open (resp. closed) subgroup are open (resp. closed).

Every open subgroup $H$ of a topological group $G$ is also closed since its complement is the union of all cosets $Hg$ for $g \in G \setminus H$ which are open.

The following is [33, Lemma 2.1.2].

Proposition 4.5 In a compact group, a subgroup is open if and only if it is closed and of finite index.
Proof. Assume that $H$ is an open subgroup of $G$. We have already seen that $H$ is also closed. The union of the cosets of $H$ form a covering by open sets. Since $G$ is compact, there is a finite subfamily covering $G$ and thus $H$ has finite index.

Conversely, if $H$ is a closed subgroup of finite index, then the complement of $H$ is the union of the cosets $Hg$ for $g \notin H$ and thus $H$ is open. □

5 Profinite semigroups

In this section, we introduce the notions of profinite semigroup and of profinite group. We begin with the notion of projective limit.

5.1 Projective limits

We want to define profinite semigroups as some kind of limit of finite semigroups in such a way that properties true in all finite semigroups will remain true in profinite semigroups. For this we need the notion of projective limit.

A projective system (or inverse system) of semigroups is given by

(i) a directed set $I$, that is a poset in which any two elements have a common upper bound,

(ii) for each $i \in I$, a topological semigroup $S_i$,

(iii) for each pair $i, j \in I$ with $i \geq j$, a connecting morphism $\psi_{i,j} : S_i \to S_j$ such that $\psi_{i,i}$ is the identity on $S_i$ and for $i \geq j \geq k$, $\psi_{i,k} = \psi_{i,j} \circ \psi_{j,k}$.

Example 5.1 Let $I$ be the set of natural integers ordered by divisibility: $n \geq m$ if $m|n$. The family of cyclic groups $(\mathbb{Z}/n\mathbb{Z})_{n \in I}$ forms a projective system for the morphisms $\psi_{n,m}$ defined by $\psi_{n,m}(x) = x \mod m$.

In the same way, the family of cyclic groups $(\mathbb{Z}/n!\mathbb{Z})_{n \geq 0}$ indexed by the set $I$ of natural integers with the natural order is a projective system.

The projective limit (or inverse limit) of this projective system is a topological semigroup $S$ together with morphisms $\Phi_i : S \to S_i$ such that for all $i, j \in I$ with $i \geq j$, $\psi_{i,j} \circ \Phi_i = \Phi_j$, and for any topological semigroup $T$ and morphisms $\Psi_i : T \to S_i$ such that for all $i, j \in I$ with $i \geq j$, $\psi_{i,j} \circ \Psi_i = \Psi_j$, there exists a morphism $\theta : T \to S$ such that $\Phi_i \circ \theta = \Psi_i$ for all $i \in I$.

The uniqueness of the projective limit can be verified (“as a standard diagram chasing exercise” [2]). The existence can be proved by considering the subsemigroup $S$ of the product $\prod_{i \in I} S_i$ consisting of all $(s_i)_{i \in I}$ such that, for all $i, j \in I$ with $i \geq j$,

$$\psi_{i,j}(s_i) = s_j$$

endowed with the product topology. The maps $\Phi_i : S \to S_i$ are the projections, that is, if $s = (s_i)_{i \in I}$, then $\Phi_i(s) = s_i$. 

9
One defines in the same way a projective system of monoids or groups and a projective limit of monoids or groups. For a projective system of monoids, one has to take all morphisms as monoid morphisms and similarly for groups (actually a monoid morphism between groups is already a group morphism).

Example 5.2 The projective limit of the family of cyclic groups (Example 5.1) is the group of profinite integers.

A variant of this construction allows to specify a fixed generating set for all semigroups. An \(A\)-\textit{generated topological semigroup} is a topological semigroup \(S\) together with a mapping \(\varphi : A \rightarrow S\) whose image generates a subsemigroup dense in \(S\). A morphism between \(A\)-generated topological semigroups \(\varphi : A \rightarrow S\) and \(\psi : A \rightarrow T\) is a continuous morphism \(\theta : S \rightarrow T\) such that \(\theta \circ \varphi = \psi\). We denote \(\theta : \varphi \rightarrow \psi\) such a morphism.

A projective system in this category of objects is given by a directed set \(I\) and for each \(i \in I\), an \(A\)-generated topological semigroup \(\varphi_i : A \rightarrow S_i\), and for each pair \(i, j \in I\) with \(i \geq j\), a connecting morphism \(\psi_{i,j} : \varphi_i \rightarrow \varphi_j\) such that \(\psi_{i,i}\) is the identity on \(S_i\) and for \(i \geq j \geq k\), \(\psi_{i,k} = \psi_{i,j} \circ \psi_{j,k}\).

The projective limit of this projective system is an \(A\)-generated topological semigroup \(\Phi : A \rightarrow S\) together with morphisms \(\Phi_i : \Phi \rightarrow \varphi_i\) such that for all \(i, j \in I\) with \(i \geq j\), \(\psi_{i,j} \circ \Phi_i = \Phi_j\), and for any \(A\)-generated topological semigroup \(\Psi : A \rightarrow T\) and morphisms \(\Psi_i : \Psi \rightarrow \varphi_i\) such that for all \(i, j \in I\) with \(i \geq j\), \(\psi_{i,j} \circ \Psi_i = \Psi_j\), there is a morphism \(\theta : \Psi \rightarrow \Phi\) such that \(\Phi_i \circ \theta = \Psi_i\) for all \(i \in I\).
5.2 Profinite semigroups

A **profinite semigroup** is a projective limit of a projective system of finite semigroups.

**Example 5.3** The ring of profinite integers is a profinite group.

A profinite semigroup is compact. Indeed, let $S$ be the projective limit of a family $S_i$ of finite semigroups. Then $S$ is a closed submonoid of a direct product of the finite and thus compact semigroups $S_i$ and thus is compact.

A topological space is **connected** if it is not the union of two disjoint open sets. A subset of a topological space is connected if it is connected as a subspace, that is it cannot be covered by the union of two disjoint open sets. Every topological space decomposes as a union of disjoint connected subsets, called its connected components.

A topological space is

(i) **totally disconnected** if its connected components are singletons,

(ii) **zero-dimensional** if it admits a basis consisting of clopen sets.

The term zero-dimensional is by reference to a notion of dimension in topological spaces (the Lebesgue dimension). The following result, from [2] gives a possible direct definition of profinite semigroup without using projective limits. A topological semigroup is **residually finite** if for any $u, v \in S$ there exists a continuous morphism $\varphi : S \to M$ into a finite semigroup $M$ such that $\varphi(u) \neq \varphi(v)$.

**Theorem 5.4** The following conditions are equivalent for a compact semigroup $S$.

(i) $S$ is profinite,

(ii) $S$ is residually finite as a topological semigroup,

(iii) $S$ is a closed subsemigroup of a direct product of finite semigroups,
(iv) $S$ is totally disconnected,
(v) $S$ is zero-dimensional.

The explicit construction of the projective limit shows that (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) results from the definitions. For (iii) $\Rightarrow$ (i), see [2]. Since a product of totally disconnected spaces is totally disconnected, we have (iii) $\Rightarrow$ (iv). The equivalence (iv) $\Leftrightarrow$ (v) holds for any compact space. Finally, the implication (v) $\Rightarrow$ (ii) results from Hunter’s Lemma (see [2]).

**Corollary 5.5** The class of profinite semigroups is closed under taking closed subsemigroups and direct product.

The following is [2, Proposition 3.5]. A subset $K$ of a semigroup $S$ is recognized by a morphism $\phi : S \to M$ if $K = \phi^{-1}(\phi(K))$.

**Proposition 5.6** Let $S$ be a profinite semigroup. A subset $K \subset S$ is clopen if and only if it is recognized by a continuous morphism $\phi : S \to T$ into a finite semigroup $T$.

**Proof.** The condition is sufficient since the set $K$ is the inverse image under a continuous function of a clopen set. Conversely assume that $K$ is clopen. Since $S$ is profinite, it is by Theorem 5.4 a closed subsemigroup of a direct product $\prod_{i \in I} S_i$ of finite semigroups $S_i$. Since $K$ is clopen, it is a finite union of clopen boxes, that is a finite union of sets $\prod_{i \in F} K_i$ with $K_i = S_i$ for all but a finite number of indices $i$. Thus there is a finite set $F \subset I$ such that being in $K$ only depends on the coordinates in $F$. Then the projection $\phi : S \to \prod_{i \in F} S_i$ is a continuous morphism into a finite semigroup which recognizes $K$. Indeed, for $s \in \phi^{-1}(\phi(K))$, there is $t \in K$ such that $\phi(s) = \phi(t)$. Since $s$ and $t$ agree on their $F$-coordinates, we have $s \in K$. ■

### 5.3 Profinite groups

A **profinite group** is a projective limit of a projective system of finite groups.

A topological group which is a profinite semigroup is actually a profinite group. Indeed, let $S$ be the projective limit of a family $S_i$ of finite semigroups. We may assume that the morphisms $\Phi_i : S \to S_i$ are surjective. If $S$ is a group, since the image of a group by a semigroup morphism is a group, each $S_i$ is a finite group and thus $S$ is a profinite group.

Any profinite group is a compact group. Indeed, it is a closed subgroup of a direct product of finite and thus compact groups.

A simple example of a compact group which is not profinite is the multiplicative group $[0,1]$, which has no nontrivial image which is a finite group.

A closed subgroup of a profinite group is profinite. Indeed, let $H$ be a closed subgroup of a profinite group $G$. Then $H$ is a closed subsemigroup of $G$ and thus it is a profinite semigroup by Corollary 5.5, whence a profinite group.
A group $G$ is Hopfian if every endomorphism of $G$ which is onto is an isomorphism. The following is [33, Proposition 2.5.2]. It is an analogue property for profinite groups.

**Proposition 5.7** Let $G$ be a finitely generated profinite group, and let $\varphi : G \to G$ be a continuous surjective morphism. Then $\varphi$ is an isomorphism.

*Proof.* We show that $\text{Ker}(\varphi)$ is contained in any subgroup of finite index of $G$. Since $G$ is profinite, it will imply that $\text{Ker}(\varphi) = \{1\}$.

For each finite group $F$, since $G$ is finitely generated, there is a finite number of morphisms from $G$ into $F$. Thus there is only a finite number of morphisms from $G$ into a finite group of order $n$ and thus also a finite number of normal subgroups of index $n$ of $G$.

Let $U_n$ be the family of normal subgroups of $G$ of index $n$. Let $\Phi : U_n \to U_n$ be defined by $\Phi(U) = \varphi^{-1}(U)$. Then $\Phi$ is an injective map from a finite set into itself and thus it is a bijection.

Let $U$ be a normal subgroup of index $n$ of $G$. Then, since $\Phi$ is surjective, we have $U = \varphi^{-1}(V)$ for some $V \in U_n$. This implies that $\text{Ker}(\varphi) \subset U$, which was to be proved. □

### 5.4 Endomorphisms of profinite semigroups

For a topological semigroup, we denote by $\text{End}(S)$ the monoid of all its continuous endomorphisms. We consider $\text{End}(S)$ as a topological monoid for the pointwise convergence.

The following result is [2, Theorem 4.14].

**Theorem 5.8** Let $S$ be a finitely generated profinite semigroup. Then $\text{End}(S)$ is a profinite monoid.

### 5.5 The $\omega$ operator

Recall that an idempotent in a semigroup $S$ is an element $e \in S$ such that $e^2 = e$.

In a finite semigroup $S$, the semigroup generated by $s \in S$ can be represented as in Figure 4 (the frying pan) with the index $i$ and the period $p$ such that $s^{i+p} = s^i$. It contains a unique idempotent, which is of the form $s^{np}$ with $n$ such that $np \geq i$. Thus, $\mathbb{Z}/p\mathbb{Z}$ is a maximal subgroup of the semigroup generated by $s$, which coincides with its minimal ideal.

In a compact semigroup $S$, just as in a finite semigroup, the closure of the semigroup generated by an element $s \in S$ contains a unique idempotent, denoted $s^\omega$. If $S$ is profinite, it is the limit of the sequence $s^n$.

We note that in any profinite group, one has $x^\omega = 1$ since the neutral element is the only idempotent of $G$.
In general, the index can be a finite integer $i$, in which case we have $s^{\omega+i} = s^\omega$. It can also be infinite and equal to $\omega$, as in the semigroup $\hat{\mathbb{N}}$. Independently, the period can also be finite or infinite.

We also note that for any endomorphism $\varphi$ of a profinite monoid $S$, the endomorphism $\varphi^\omega$ is a well defined endomorphism of $S$.

5.6 The free profinite monoid

Consider the projective system formed by representatives of isomorphism classes of all $A$-generated finite monoids (the finite monoids are considered as topological monoids for the discrete topology). For $\varphi : A \to M$ and $\psi : A \to N$, one has $\varphi \geq \psi$ if there is a morphism $\mu : M \to N$ such that $\mu \circ \varphi = \psi$. Note that $\varphi, \psi, \mu$ have to be surjective.

The free profinite monoid on a finite alphabet $A$, denoted $\hat{A}^*$ is the projective limit of this family. It has the following universal property (see Figure 5).

**Proposition 5.9** The natural mapping $\iota : A \to \hat{A}^*$ is such that for any map $\varphi : A \to M$ into a profinite monoid there exists a unique continuous morphism $\hat{\varphi} : \hat{A}^* \to M$ such that $\hat{\varphi} \circ \iota = \varphi$.

The elements of $\hat{A}^*$ are called pseudowords and the elements of $\hat{A}^* \setminus A^*$ are called infinite pseudowords.

The free profinite monoid on one generator is commutative. Its image by the map $a^n \mapsto n$ is the monoid of profinite natural numbers, denoted $\hat{\mathbb{N}}$. 

14
The topology induced $A^*$ by the profinite topology on $\hat{A}^*$ is discrete. Indeed, if $u \in A^*$ is a word of length $n$, the quotient of $A^*$ by the ideal formed by the words of length greater than $n$ is a finite monoid. The congruence class of $u$ for the corresponding quotient is reduced to $u$.

The length $|x|$ of a pseudoword $x \in \hat{A}^*$ is a profinite natural number. The map $x \in \hat{A}^* \to |x| \in \hat{\mathbb{N}}$ is the continuous morphism $\lambda$ such that $\lambda(a) = 1$ for every $a \in A$.

**Example 5.10** The length of $xy^\omega$ is $|x| + \omega$.

### 5.7 The free profinite group

Likewise, the **free profinite group** , denoted $\hat{FG}(A)$ is the projective limit of the projective system formed by the isomorphism classes of $A$-generated finite groups.

The topology on the free group $FG(A)$ induced by the topology of $\hat{FG}(A)$ is not discrete. Indeed, for any $x$ in $FG(A)$, the sequence $x^n$ tends to 1. Thus $A^*$ is dense in $FG(A)$ and there is an onto homomorphism from $\hat{A}^*$ onto $\hat{FG}(A)$.

The topology induced on $FG(A)$ by the topology of $\hat{FG}(A)$ is also called the **Hall topology**. It has been indeed introduced by M. Hall in [20]. Note that, since $A^*$ is embedded in $FG(A)$, we actually have two topologies on $A^*$ respectively induced by the topologies of $\hat{A}^*$ and $\hat{FG}(A)$. To distinguish them, the first one is called the **pro-M topology** and the second one the **pro-G topology**. The first one is strictly stronger than the second one. The pro-G topology on $A^*$ was introduced by Reutenauer in [32].

The image of the free profinite group on one generator $a$ by the map $a^n \mapsto n$ is the group $\hat{\mathbb{Z}}$ of profinite integers (see Section 2). The length $|x|$ of an element $x$ of $\hat{FG}(A)$ is a profinite integer. The map $x \in \hat{FG}(A) \to |x| \in \hat{\mathbb{Z}}$ is the unique continuous morphism such that $|a| = 1$ for every $a \in A$. In particular $|a^{-1}| = -1$.

The following result is from [20, p. 131]. The property is not true for non finitely generated subgroups. The classical example is the commutator subgroup $F'$ of a finitely generated free group $F$, which is known to be a free group of countable infinite rank. In the topology induced on $F'$ by the profinite topology on $F$, there are only countably many open subgroups while the number of open subgroups in the profinite topology of $F'$ is uncountable.

**Proposition 5.11** Any finitely generated subgroup $H$ of $FG(A)$ is closed for the topology induced on $FG(A)$ by the pro-G topology.

The proof relies on the following result (see [19, Theorem 5.1] or [28, Proposition 3.10]). Recall that a **free factor** of a group $G$ is a subgroup $H$ such that $G$ is a free product of $H$ and a subgroup $K$ of $G$. When $G$ is a free group, this is equivalent to the following property: for some basis $X$ of $H$ there is a subset $Y$ of $G$ such that $X \cup Y$ is a basis of $G$. 


Theorem 5.12 (Hall) For any finitely generated subgroup \( H \) of \( FG(A) \) and any \( x \in FG(A) \setminus H \), there is a subgroup of finite index \( K \) such that \( H \) is a free factor of \( K \) and \( x \not\in K \).

To deduce Proposition 5.11 from Hall’s Theorem, consider a sequence \( (x_n) \) of elements of \( H \) converging to some \( x \in FG(A) \setminus H \). Suppose that \( x \not\in H \). By Theorem 5.12, there is a subgroup \( K \) of finite index in \( FG(A) \) containing \( H \) such that \( x \not\in K \). Thus \( (x_n) \) cannot converge to \( x \).

It follows from Theorem 5.12 that one has the following result [14, Corollary 2.2]

Corollary 5.13 Any injective morphism \( \varphi : FG(B) \to FG(A) \) between finitely generated free groups extends to an injective continuous morphism \( \hat{\varphi} : \hat{FG}(B) \to \hat{FG}(A) \).

Proof. Let \( H = \varphi(FG(B)) \). Then \( \varphi \) is an isomorphism between \( FG(B) \) and \( H \) which extends to an isomorphism between \( \hat{FG}(B) \) and \( \hat{H} \), the completion of \( H \) with respect to the profinite metric. But, by Proposition 5.11 the subgroup \( H \) is closed in \( FG(A) \) and thus \( \hat{H} \) is the same as the closure \( \bar{H} \) in \( \hat{FG}(A) \), which shows that \( \hat{\varphi} \) is an injective morphism from \( \hat{FG}(B) \) into \( \hat{FG}(A) \).

5.8 Recognizable sets

A subset \( X \) of a monoid \( M \) is recognizable if it is recognized by a morphism into a finite monoid, that is, if there is a morphism \( \varphi : M \to N \) into a finite monoid \( N \) which recognizes \( X \). We also say that \( X \) is recognized by \( \varphi \).

Proposition 5.14 The following conditions are equivalent for a set \( X \subset A^* \).

(i) \( X \) is recognizable.

(ii) the closure \( \bar{X} \) of \( X \) in \( \hat{A}^* \) is open and \( X = \bar{X} \cap A^* \).

(ii) \( X = K \cap A^* \) for some clopen set \( K \subset \hat{A}^* \).

Proof. Assume that \( X \) is recognized by a morphism \( \varphi : A^* \to S \) from \( A^* \) into a finite monoid \( S \). By the universal property of \( \hat{A}^* \), there is a unique continuous morphism \( \hat{\varphi} \) extending \( \varphi \). Then \( X = \hat{\varphi}^{-1}(\varphi(X)) \) is open and satisfies \( X = \bar{X} \cap A^* \). Thus (i)\( \Rightarrow \) (ii). The implication (ii)\( \Rightarrow \) (iii) is trivial. Finally, assume that (iii) holds. By Proposition 5.6 there exists a continuous morphism \( \psi : \hat{A}^* \to S \) into a finite monoid \( S \) which recognizes \( K \). Let \( \varphi \) be the restriction of \( \psi \) to \( A^* \). Then \( X = A^* \cap K = A^* \cap \psi^{-1}(\psi(K)) \) and so \( X \) is recognizable.

As an example, the sets of the form \( \hat{A}^* w \hat{A}^* = \hat{A}^* w A^* \) are clopen sets and so are the sets \( w \hat{A}^* = w A^* \). This shows that a pseudoword has a well-defined set of finite factors and a well-defined prefix of every finite length.
The analogue of Proposition 5.6 for the pro-$G$ topology is also true (it actually holds in the pro-$V$ topology for any pseudovariety $V$). Thus a set $X$ is recognizable by a morphism on a finite group if and only if $X = K \cap A^*$ for some clopen set $K \subset \hat{F}G(A)$. In condition (ii), one has to add that $\bar{X} \cap A^* = X$, a condition always satisfied for the closure with respect to the pro-$M$ topology.

5.9 The natural metric

The natural metric on a profinite monoid $M$ is defined by

$$d(u, v) = \begin{cases} 2^{-r(u, v)} & \text{if } u \neq v \\ 0 & \text{otherwise} \end{cases}$$

where $r(u, v)$ is the minimal cardinality of a monoid $N$ for which there is a continuous morphism $\varphi : M \to N$ such that $\varphi(u) \neq \varphi(v)$.

It is actually an ultrametric since it satisfies the condition

$$d(u, w) \leq \min(d(u, v), d(v, w))$$

stronger than the triangle inequality.

Proposition 5.15 For a finitely generated profinite semigroup $S$, the topology is induced by the natural metric.

Proof. Denote $B_{2\varepsilon}(u) = \{v \in S \mid d(u, v) < \varepsilon\}$. Let $K$ be a clopen set in $S$. By Proposition 5.6 there is a continuous morphism $\varphi : S \to T$ into a finite semigroup $T$ which recognizes $K$. Let $n = \text{Card}(T)$ and set $\varepsilon = 2^{-n}$. For $s \in S$ and $t \in B_{2\varepsilon}(s)$, we have $d(s, t) < \varepsilon$ and thus $r(s, t) > n$. It implies that $\varphi(s) = \varphi(t)$. Since $\varphi$ recognizes $K$, we conclude that $t \in K$. Thus the ball $B_{2\varepsilon}(s)$ is contained in $\varphi^{-1}(t)$ and thus $K$ is a union of open balls. Thus, since the clopen sets form a basis of the topology, any open set is a union of open balls.

Conversely, consider the open ball $B = B_{2^{-n}}(u)$. Since there is a finite number of isomorphism types of semigroups with at most $n$ elements, and since $S$ is finitely generated, there are finitely many kernels of continuous morphisms into such semigroups and so their intersection is a clopen congruence on $S$. It follows that there exists a continuous morphism $\varphi : S \to T$ into a finite semigroup such that $\varphi(u) = \varphi(v)$ if and only if $r(u, v) > n$. Hence $B = \varphi^{-1}(\varphi(B))$ so that $B$ is open.

This leads to an alternative definition of the free profinite monoid.

Theorem 5.16 For a finite set $A$, the completion of $A^*$ for the natural metric is the free profinite monoid $\hat{A}^*$. 

17
Presentations of profinite semigroups A congruence of a profinite semigroup is called *admissible* if its classes are closed and the quotient is profinite. In other terms, admissible congruences are the kernels of continuous homomorphisms into profinite monoids.

Given a set $X$ and a binary relation $R$ on the monoid $\hat{X}^*$, the profinite semigroup $\langle X | R \rangle$ is the quotient of $\hat{X}^*$ by the admissible congruence generated by $R$. It is also said to have the presentation $\langle X | R \rangle$.

The same notion holds for groups instead of semigroups and use the notation $\langle X | R \rangle_S$ or $\langle X | R \rangle_G$ to specify if the presentation is as a profinite semigroup or as a profinite group.

The following is [4, Lemma 2.2].

Proposition 5.17 Let $S$ be a profinite semigroup and let $\varphi$ be an automorphism of $S$. Let $\pi$ be a continuous homomorphism from $\hat{A}^*$ onto $S$ and let $\Phi$ be a continuous endomorphism of $\hat{A}^*$ such that the diagram below commutes. Then $S$ has the presentation $\langle A | R \rangle$ with $R = \{(\Phi^a(a), a) | a \in A\}$.

6 Profinite codes

We will expore the notion of a code in the free profinite monoid.

6.1 Finite codes

Let $A, B$ be finite alphabets. Any morphism $\beta : B^* \to A^*$ extends uniquely by continuity to a continuous morphism $\hat{\beta} : \hat{B}^* \to \hat{A}^*$. A finite set $X \subset \hat{A}^*$ is called a *profinite code* if the continuous extension $\hat{\beta}$ of any morphism $\beta : B^* \to A^*$ inducing a bijection from $B$ onto $X$ is injective.

The following statement is from [29].

Theorem 6.1 Any finite code $X \subset A^+$ is a profinite code.

Proof. Let $\beta : B^* \to A^*$ be a coding morphism for $X$. We have to show that for any pair $u, v \in \hat{B}^*$ of distinct elements, we have $\hat{\beta}(u) \neq \hat{\beta}(v)$, that is, there is a continuous morphism $\hat{\alpha} : \hat{A}^* \to M$ into a finite monoid $M$ such that $\hat{\alpha}\hat{\beta}(u) \neq \hat{\alpha}\hat{\beta}(v)$. For this, let $\psi : \hat{B}^* \to N$ be a continuous morphism into a finite monoid $N$ such that $\psi(u) \neq \psi(v)$. Let $P$ be the set of proper prefixes of $X$ and let $T$ be the prefix transducer associated to $\beta$ (see [10]). Let $\alpha$ be the morphism from $A^*$ into the monoid of $P \times P$-matrices with elements in $N \cup 0$.
defined as follows. For \( x \in A^* \) and \( p, q \in P \), we have

\[
\alpha(x)_{p,q} = \begin{cases} 
\psi(y) & \text{if there is a path } p \xrightarrow{x|y} q \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( M = \alpha(A^*) \) is a finite monoid and \( \alpha \) extends to a continuous morphism \( \hat{\alpha} : \hat{A}^* \to M \). Since, by \[10\] Proposition 4.3.2, the transducer \( T \) realizes the decoding function of \( X \), we have \( \alpha\beta(y)_{1,1} = \psi(y) \) for any \( y \in B^* \). By continuity, we have \( \hat{\alpha}\hat{\beta}(y)_{1,1} = \psi(y) \) for any \( y \in B^* \). Then \( \hat{\alpha} \) is such that \( \hat{\alpha}\hat{\beta}(u) \neq \hat{\alpha}\hat{\beta}(v) \).

Indeed \( \hat{\alpha}\hat{\beta}(u)_{1,1} = \psi(u) \neq \psi(v) = \hat{\alpha}\hat{\beta}(v)_{1,1} \).

**Example 6.2** Let \( A = \{a, b\} \) and \( X = \{a, ab, bb\} \). The set \( X \) is a suffix code. It has infinite deciphering delay since \( abb \cdots = a(bb)(bb) \cdots = (ab)(bb)(bb) \cdots \). Nonetheless, \( X \) is a profinite code in agreement with Theorem 6.1. Note that, in \( \hat{A}^* \), the pseudowords \( a(bb)^\omega \) and \( ab(bb)^\omega \) are distinct (the first one is a limit of words of odd length and the second one of words of even length).

Theorem 6.1 shows that the closure of the submonoid generated by a finite code is a free profinite monoid. This has been extended to rational codes in [6]. Actually, for any rational code \( X \), the profinite submonoid generated by \( X \) is free with basis the uncoutable set \( \bar{X} = \{a^v b \mid v \in \hat{N}\} \).

**Example 6.3** Let \( A = \{a, b\} \) and \( X = a^* b \). Then the profinite monoid \( \hat{X}^* \) is free with basis the uncoutable set \( \hat{X} = \{a^v b \mid v \in \hat{N}\} \).

### 6.2 Codes and submonoids

A submonoid \( N \) of a monoid \( M \) is called **stable** if for any \( u, v, w \in M \), whenever \( u, uv, uw, w \in N \) then \( v \in N \). It is called **right unitary** if for every \( u, v \in A^* \), \( u, uv \in N \) implies \( v \in N \). It is well known that a submonoid of \( A^* \) is stable if and only if it is generated by a code and it is unitary if and only if it is generated by a prefix code.

The following statement extends these notions to pseudowords.

**Proposition 6.4** Let \( N \) be a recognizable submonoid of \( A^* \). If \( N \) is stable (resp. unitary) its closure in \( \hat{A}^* \) is a stable (resp. unitary) submonoid of \( \hat{A}^* \).

**Proof.** The set \( \bar{N} \) is clearly a submonoid of \( \hat{A}^* \). Assume that \( N \) is stable. Let \( u, v, w \in \hat{A}^* \) be such that \( uv, w, u, vw \in \bar{N} \). Let \((u_n),(v_n)\) and \((w_n)\) be sequences of words converging to \( u, v \) and \( w \) respectively with \( u_n, w_n, v_n \in N \). By Proposition 5.14 the closure \( \bar{N} \) of the submonoid \( N \) is open in \( \hat{A}^* \) and \( \bar{N} = \bar{N} \cap A^* \). Since \( \bar{N} \) is open, we have \( u_n v_n, v_n w_n \in \bar{N} \) for large enough \( n \) and thus also \( u_n v_n, v_n w_n \in \bar{N} \). Since \( N \) is stable, this implies \( v_n \in N \) for large
enough $n$, which implies $v \in \bar{N}$. Thus $\bar{N}$ is stable. The proof when $N$ is unitary is similar.

When $X$ is a prefix code, every word $w$ can be written in a unique way $w = xp$ with $x \in X^*$ and $p \in A^* \setminus XA^*$, a word without any prefix in $X$. When $X$ is a maximal prefix code, a word which has no prefix in $X$ is a proper prefix of a word in $X$ and thus every word $w$ can be written in a unique way $w = xp$ with $x \in X^*$ and $p$ a proper prefix of $X$. This extends to pseudowords as follows.

**Proposition 6.5** Let $X \subset A^*$ be a finite maximal prefix code and let $P$ be its set of proper prefixes. Any pseudoword $w \in \hat{A}^*$ has a unique factorization $w = xp$ with $x \in X^*$ and $p \in P$.

**Proof.** Let $(w_n)$ be a sequence of words converging to $w$. For each $n$, we have $w_n = x_n p_n$ with $x_n \in X^*$ and $p_n \in P$. Taking a subsequence, we may assume that the sequences $(x_n)$ and $(p_n)$ converge to $x \in X^*$ and $p \in P$. Since $X$ is finite, $P$ is finite and thus $\bar{P} = P$. This proves the existence of a factorization. To prove the uniqueness, consider $xp = x'p'$ with $x, x' \in \hat{X}^*$ and $p, p' \in P$. Then, assuming that $p$ is longer than $p'$, we have $p = up'$ and $xu = x'$ for some $u \in A^*$. Since $\hat{X}^*$ is unitary, we have $u \in X^*$ and thus $u = \epsilon$. □

## 7 Uniformly recurrent sets

In this section, we study the closure in the free profinite monoid of a uniformly recurrent set.

### 7.1 Uniformly recurrent pseudowords

Let $A$ be a finite alphabet. A set of finite words on $A$ is *factorial* if it contains the alphabet $A$ and all the factors of its elements.

A factorial set $F$ is *recurrent* if for any $x, z \in F$ there is some $y \in F$ such that $xyz \in F$.

Recall that an infinite factorial set $F$ of finite words is said to be *uniformly recurrent* or *minimal* if for any $x \in F$ there is an integer $n \geq 1$ such that $x$ is a factor of every word in $F$ of length $n$.

A uniformly recurrent set is obviously recurrent.

Note that a uniformly recurrent set is actually minimal for inclusion among the infinite factorial sets. Indeed, assume first that $F$ is uniformly recurrent. Let $F' \subset F$ be an infinite factorial set. Let $x \in F$ and let $n$ be such that $x$ is factor of any word of $F$ of length $n$. Since $F'$ is factorial infinite, it contains a word $y$ of length $n$. Since $x$ is a factor of $y$, it is in $F'$. Thus $F$ is minimal. Conversely consider $x \in F$ and let $T$ be the set of words in $F$ which do not contain $x$ as a factor. Then $T$ is factorial. Since $F$ is minimal among infinite factorial sets, $T$ is finite. Thus there is an $n$ such that $x$ is a factor of all words of $F$ of length $n$.
A factorial set $F$ is periodic if it is the set of factors of a finite word $w$. One may always assume $w$ to be primitive, that is not a power of another word. In this case, the length of $w$ is called the period of $F$. A periodic set is obviously uniformly recurrent.

For an infinite pseudoword $w$, we denote by $F(w)$ the set of finite factors of $w$. It is an infinite factorial set.

An infinite pseudoword $w$ is uniformly recurrent if $F(w)$ is uniformly recurrent. This is the same definition as the definition commonly used for infinite words.

The following is [7, Lemma 2.2].

**Proposition 7.1** An infinite pseudoword is uniformly recurrent if and only if all its infinite factors have the same finite factors.

**Proof.** Let $w$ be an infinite pseudoword. Assume first that $w$ is uniformly recurrent. Let $u$ be an infinite factor of $w$. Let us show that $F(u) = F(w)$. The inclusion $F(u) \subset F(w)$ is clear. Conversely, let $x \in F(w)$. Since $w$ is uniformly recurrent, $x$ is a factor of every long enough finite factor of $w$. In particular, $x$ is a factor of every long enough finite prefix of $u$. Thus $x \in F(u)$.

Conversely, assume that $F(w) = F(u)$ for all infinite factors $u$ of $w$. Let $v$ be a finite factor of $w$. Arguing by contradiction, assume that there are arbitrary long factors of $w$ which do not have $v$ as a factor. This infinite set contains, by Proposition ?, a subsequence converging to some infinite pseudoword $u$ which is also a factor of $w$, and such that $v \notin F(u)$, a contradiction.

Recall that the $J$-order in a monoid $M$ is defined by $x \leq_J y$ if $x$ is a factor of $y$. Two elements $x, y$ are $J$-equivalent if each one is a factor of the other (this is one of the Green’s relations , see [10]).

Replacing the notion of factor by prefix (resp. suffix), one obtains the $R$-order (resp. $L$-order). Thus, two elements $x, y$ of a monoid $M$ are $R$-equivalent (resp. $L$-equivalent) if $xM = yM$ (resp. $Mx = My$). The $H$-equivalence is the intersection of $R$ and $L$. In any monoid, one has $RL = LR$ and one denotes $D$ the equivalence $RL = LR$ which is the supremum of $R$ and $L$. In a compact monoid, one has $D = J$. The proof is the same as in a finite monoid. It uses the fact that a compact monoid satisfies the stability condition: if $x \leq_R y$ and $xJy$, then $xRy$ and dually for $L$.

The following result is [7, Theorem 2.6]. It gives an algebraic characterization of uniform recurrence in the free profinite monoid.

**Theorem 7.2** An infinite pseudoword is uniformly recurrent if and only if it is $J$-maximal.

The proof uses three lemmas. An element $s$ of a semigroup $S$ is regular if there is some $x \in S$ such that $sx = s$. In a compact semigroup, a $J$-class contains a regular element if and only if all its elements are regular, if and only if it contains an idempotent.
For a pseudoword $w$, we denote $X(w)$ the set of all infinite pseudowords which are limits of sequences of finite factors of $w$.

**Lemma 7.3** Let $w$ be uniformly recurrent pseudoword over a finite alphabet $A$.
1. Every element of $X(w)$ is a factor of $w$.
2. All elements of $X(w)$ lie in the same $\mathcal{J}$-class of $\hat{A}^*$.
3. Every element of $X(w)$ is regular.

**Proof.** Assertion 1 results from the fact that $F(w)$ is closed. Suppose that $u, v \in X(w)$. By Proposition 7.1, they have the same set of finite factors. Thus, by Assertion 1, they are $\mathcal{J}$-equivalent.

Assume that $u$ is the limit of a sequence $(u_n)_{n \geq 0}$ of finite factors of $w$. Since $w$ is recurrent, there are finite words $v_n$ such that $u_nv_nv_n$ is a factor of $w$. If $v$ is an accumulation point of the sequence $v_n$, then $uvu$ is a factor of $w$ which belongs to $X(w)$. By Assertion 2, it is $\mathcal{J}$-equivalent to $u$. In a compact monoid, by the stability condition, this implies that $u$ and $uvu$ are $\mathcal{H}$-equivalent and thus that $u$ is regular (indeed, $uHuvu$ implies $uH(uv)^\omega$ and thus $(vu)^\omega$ is an idempotent in $J(u)$).

**Lemma 7.4** Let $w$ be a uniformly recurrent pseudoword. Each $\mathcal{H}$-class contained in the $\mathcal{J}$-class of $w$ contains some element of $X(w)$.

**Proof.** Let $u \in J(w)$. Denote by $x_n$ and $y_n$ the prefix and the suffix of $u$ of length $n$. Since $w$ is uniformly recurrent by Proposition 7.1, there is a factor $t_n$ of $w$ of length at least $2n$ having $x_n$ as a prefix and $y_n$ as a suffix. Taking a subsequence, we may assume that the sequences $(x_n)$, $(y_n)$ and $(t_n)$ converge to $x, y, t$. Then $x, y, t \in J(w)$ by Lemma 7.3(2). Since $x \geq_R t$ and $t \leq_L y$, by stability, we obtain $uHt$.

**Lemma 7.5** Let $u$ be a uniformly recurrent pseudoword and suppose $v$ is a pseudoword such that $uv$ is still uniformly recurrent. Then $u$ and $uv$ are $\mathcal{R}$-equivalent.

**Proof.** Suppose first that $v$ is finite. Let $u_n$ be the suffix of $u$ of length $n$. Since $u$ is an infinite factor of $uv$ which is uniformly recurrent, they have the same finite factors by Lemma 7.1. Hence for every $n$ there is some $m(n)$ such that $u_{m(n)} = x_n u_n y_n$. By compactness, we may assume by taking subsequences that the sequences $x_n, y_n, u_n$ converge to $x, y, u'$ respectively. Then by continuity of the multiplication, the sequence $u_{m(n)}$ converges to $xu'vy$. Since the limits of two convergent sequences of suffixes of the same pseudoword are $\mathcal{L}$-equivalent, we obtain that $xu'vyLu$ and thus $uRu'v$ by stability. Since $u'$ is the limit of a sequence of suffixes of $u$, there is some factorization of the form $u = zu'$. Since the $\mathcal{R}$-equivalence is a left congruence, we finally obtain $uv = zvRzu' = u$.

Assume next that $v$ is infinite. We assume by contradiction that $u >_R uv$. Let $v_n$ be a sequence of finite words converging to $v$. Taking a subsequence, we
may assume that $uv_n >_R u$ for all $n$. Thus for each $n$, we have a factorization $v_n = x_n a_n y_n$ with $a_n \in A$ such that $u R u x_n >_R u x_n a_n \geq_R u v$. Since the alphabet is finite and $A^*$ is compact we may, up to taking a subsequence, assume that the letter sequence $a_n$ is constant and the sequences $x_n, y_n$ converge to $x$ and $y$ respectively. Thus we have $p = x a y$ with $a \in A$ and $u R u x >_R u x a \geq_R u v$.

On the other hand, since $u x$ and $u x a$ are infinite factors of $u v$, they are both uniformly recurrent by Proposition 7.1. By the first part, we have $u x R u x a$, a contradiction.

A dual result holds for the $L$-order.

Proof of Theorem 7.2. Suppose first that $w$ is $J$-maximal as an infinite pseudoword. If $v$ is an infinite factor of $w$, it is $J$-equivalent to $w$. Hence $v, w$ have the same factors and, in particular, the same finite factors. By Proposition 7.1 $w$ is uniformly recurrent.

Suppose conversely that $u, w \in \hat{A}^- \setminus A^*$ are such that $u \geq_J w$ uniformly recurrent. Set $w = puq$ with $p, q \in \hat{A}^+$. By the dual of Lemma 7.5 we have $pu L u$. And by Lemma 7.5 we have $pu R puq$. Thus $u$ and $w$ are $J$-equivalent.

Example 7.6 The $J$-class of $a^\omega$ in $\hat{A}^+$ is made of one $H$-class. The $J$-class of $(ab)^\omega$ has four $H$-classes. It is represented in Figure 6.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{jclass_ab.png}
\caption{The $J$-class of $(ab)^\omega$.}
\end{figure}

7.2 The $J$-class $J(F)$

Let $F$ be a uniformly recurrent set of finite words on the alphabet $A$. The closure $\bar{F}$ of $F$ in $\hat{A}^*$ is also factorial (see [3, Proposition 2.4]). The proof relies on the following useful lemma from [3, Lemma 2.5].

Lemma 7.7 For every $u, v \in \hat{A}^*$ and every sequence $(w_n)$ converging to $uv$, there are sequences $(u_n), (v_n)$ such that $\lim u_n = u$, $\lim v_n = v$ and $(u_n v_n)$ is a subsequence of $(w_n)$.

The set of two-sided infinite words with all their factors in $F$ is denoted $X(F)$. It is closed for the product topology of $A^\mathbb{Z}$. It is also invariant by the shift $\sigma : A^\mathbb{Z} \to A^\mathbb{Z}$ defined by $y = \sigma(x)$ if $y_n = x_{n+1}$ for any $n \in \mathbb{Z}$. Such a
closed and shift invariant set is called a subshift. It is classical that a subshift is of the form $X(F)$ for some uniformly recurrent set if and only if it is minimal (see [27] for example).

By the results of Section 7, all the infinite pseudowords in the closure $F$ of $F$ are $J$-equivalent. We denote by $J(F)$ their $J$-class.

**Example 7.8** The Fibonacci morphism $\varphi$ is primitive. The set $F$ of factors of the words $\varphi^n(a)$ for $n \geq 1$ is called the Fibonacci set. It contains the infinitely recurrent pseudowords $\varphi^w(a)$ and $\varphi^w(b)$.

For a uniformly recurrent set $F$, the $J$-class $J(F)$ can be described as follows.

**Proposition 7.9** For a uniformly recurrent set $F$, two words $u,v \in J(F)$ are $R$-equivalent if and only if $\rightarrow u = \rightarrow v$ and $L$-equivalent if and only if $\leftarrow u = \leftarrow v$

It follows from this that $w \in J(F)$ belongs to a subgroup if and only if the two-sided infinite word $\overrightarrow{w} \cdot \overleftarrow{w}$ has all its factors in $F$. Indeed, the finite factors of $w^2$ are those of $w$ plus the products $uv$ where $u$ is a finite suffix of $w$ and $v$ is a finite prefix of $w$ [3, Lemma 8.2].

Thus the maximal subgroups of $J(F)$ are in bijection with the elements of the set $X(F)$ of two-sided infinite words with all their factors in $F$. For $x \in X(F)$, we denote by $H_x$ the maximal subgroup corresponding to $x$.

**Example 7.10** Let $F$ be the Fibonacci set and let $w = \varphi^w(a)$. The right infinite word $\overrightarrow{w}$ is the Fibonacci word. The left infinite word $\overleftarrow{w}$ is the word with suffixes $\varphi^{2n}(a)$ (see Section 3). The two sided infinite word $\overrightarrow{w} \cdot \overleftarrow{w}$ is a fixed point of $\varphi^2$.

**Example 7.11** Let $A = \{a, b\}$ and let $\varphi : A^* \to A^*$ be defined by $\varphi(a) = ab$ and $\varphi(b) = a^3b$. Let $F$ be the set of factors of $\varphi^w(a)$. Since $\varphi$ is primitive, $F$ is uniformly recurrent. The pseudowords $\varphi^w(a)$ and $\varphi^w(b)$ belong to the same $H$-class of $J(F)$. Indeed, we have $\varphi^w(a) = \varphi^w(b)$ and $\varphi^w(a) = \varphi^w(b)$.

### 7.3 Fixed points of substitutions

Let $A$ be a finite alphabet. Let $\varphi : A^* \to A^*$ be a morphism, also called a substitution over $A$. Then $\varphi$ extends uniquely by continuity to a morphism still denoted $\varphi : \hat{A}^* \to \hat{A}^*$. The monoid $\text{End}(\hat{A}^*)$ is profinite by Theorem 5.8. Thus the morphism $\varphi^w$ is well defined as the unique idempotent in the closure of the semigroup generated by $\varphi$.

A substitution $\varphi : A^* \to A^*$ is said to be primitive if there is an integer $n$ such that all letters appear in every $\varphi^n(a)$ for $a \in A$. 

24
A fixed point of a substitution $\varphi$ is an infinite word $x \in A^\mathbb{N}$ such that $\varphi(x) = x$. As well known, a fixed point of a primitive substitution is uniformly recurrent (see [17, Proposition 1.3.2] for example).

The following is a particular case of [7, Theorem 3.7] (in which the notion weakly primitive substitution is introduced).

**Theorem 7.12** Let $\varphi$ be a primitive substitution over a finite alphabet $A$. Then the pseudowords $\varphi^n(a)$ with $a \in A$ are uniformly recurrent and are all $J$-equivalent.

**Proof.** Let $u$ be a finite factor of $\varphi^n(a)$ for some $a \in A$. Then there is an integer $N$ such that $u$ is a factor of any factor of length $N$ of $\varphi^n(b)$ for all $b \in A$. Thus $u$ is a factor of any finite factor of length $N$ of any $\varphi^n(b)$. This proves both claims.

**Example 7.13** The Fibonacci morphism $\varphi : a \mapsto ab, b \mapsto a$ is primitive. Thus the pseudowords $\varphi^n(a)$ and $\varphi^n(b)$ are uniformly recurrent and $J$-equivalent.

**Example 7.14** The Thue-Morse substitution is the morphism $\tau : a \mapsto ba, b \mapsto ab$. It is primitive. The unique fixed point $x = abbabaab \cdots$ of $\tau$ beginning with $a$ is called the Thue-Morse infinite word. The set of its factors is called the Thue-Morse set.

Given a substitution $\varphi$ over $A$, we denote by $\varphi_G$ the endomorphism of $\hat{\text{FG}}(A)$ such that the following diagram commutes where $\pi : \hat{A}^* \to \text{FG}(A)$ denotes the canonical projection.

For a substitution $\varphi$ over $A$, the endomorphism $\varphi_G$ is the identity if and only if $\varphi$ is invertible (as a map from $\text{FG}(A)$ into itself).

Indeed, if $\varphi$ is an automorphism of $\text{FG}(A)$, then its extension to $\hat{\text{FG}}(A)$ is also an automorphism. But then $\varphi_G$ is the identity since in a group one has $x^{\varphi} = 1$ for any element $x$.

Conversely, if $\varphi_G$ is the identity, then $\varphi$ is a bijection from $\text{FG}(A)$ onto itself and thus it is an automorphism of $\text{FG}(A)$.

**Example 7.15** Let $A = \{a,b\}$ and let $\varphi : a \mapsto ab, b \mapsto a$ be the Fibonacci morphism. Then $\varphi$ is an automorphism of $\text{FG}(A)$ since $\varphi^{-1} : a \mapsto b, b \mapsto b^{-1}a$. Accordingly $\varphi_G$ is the identity. In particular, one has $\varphi_G^n(a) = a$. This explains in a simple way that the length of $\varphi_G^n(a)$ is equal to 1 (see Section 3).
8 Sturmian sets and tree sets

Let $F$ be a factorial set on the alphabet $A$. For $w \in F$, we denote
\[
L_F(w) = \{ a \in A \mid aw \in F \},
\]
\[
R_F(w) = \{ a \in A \mid wa \in F \},
\]
\[
E_F(w) = \{ (a,b) \in A \times A \mid awb \in F \}
\]
and further
\[
\ell_F(w) = \text{Card}(L_F(w)), \quad r_F(w) = \text{Card}(R_F(w)), \quad e_F(w) = \text{Card}(E_F(w)).
\]

For $w \in F$, we denote
\[
m_F(w) = e_F(w) - \ell_F(w) - r_F(w) + 1.
\]

A word $w$ is called neutral if $m_F(w) = 0$. A factorial set $F$ is neutral if every word in $F$ is neutral.

Example 8.1 The Fibonacci set is neutral as any Sturmian set.

Example 8.2 The Thue-Morse set $T$ is not neutral. Indeed, since $A^2 \subset T$, one has $m_T(\varepsilon) = 1$.

Example 8.3 Let $\varphi : a \mapsto ab, b \mapsto a^3b$ be as in Example 7.11. Let $F$ be the set of factors of $\varphi^\omega(a)$. It is not neutral since $m(a) = 1$ and $m(aa) = -1$.

A neutral set has complexity $kn + 1$ where $k = \text{Card}(A) - 1$ (see [11]).

8.1 Sturmian sets

We recall here some notions concerning episturmian words (see [9] for more details and references).

A word $w$ is right-special (resp. left-special) if $\ell_F(w) \geq 2$ (resp. $r_F(w) \geq 2$). A right-special (resp. left-special) word $w$ is strict if $\ell_F(w) = \text{Card}(A)$ (resp. $r_F(w) = \text{Card}(A)$). In the case of a 2-letter alphabet, all special words are strict.

By definition, an infinite word $x$ is episturmian if $F(x)$ is closed under reversal and if $F(x)$ contains, for each $n \geq 1$, at most one word of length $n$ which is right-special.

Since $F(x)$ is closed under reversal, the reversal of a right-special factor of length $n$ is left-special, and it is the only left-special factor of length $n$ of $x$. A suffix of a right-special factor is again right-special. Symmetrically, a prefix of a left-special factor is again left-special.

As a particular case, a strict episturmian word is an episturmian word $x$ with the two following properties: $x$ has exactly one right-special factor of each length and moreover each right-special factor $u$ of $x$ is strict, that is satisfies the inclusion $uA \subset F(x)$ (see [15]).
For $a \in A$, denote by $\psi_a$ the morphism of $A^*$ into itself, called elementary morphism, defined by

$$
\psi_a(b) = \begin{cases} 
ab & \text{if } b \neq a \\
a & \text{otherwise}
\end{cases}
$$

Let $\psi : A^* \to \text{End}(A^*)$ be the morphism from $A^*$ into the monoid of endomorphisms of $A^*$ which maps each $a \in A$ to $\psi_a$. For $u \in A^*$, we denote by $\psi_u$ the image of $u$ by the morphism $\psi$. Thus, for three words $u, v, w$, we have $\psi_{uv}(w) = \psi_u(\psi_v(w))$. A palindrome is a word $w$ which is equal to its reversal. Given a word $w$, we denote by $w^{(+)}$ the palindromic closure of $w$. It is, by definition, the shortest palindrome which has $w$ as a prefix.

The iterated palindromic closure of a word $w$ is the word $\text{Pal}(w)$ defined recursively as follows. One has $\text{Pal}(1) = 1$ and for $u \in A^*$ and $a \in A$, one has $\text{Pal}(ua) = (\text{Pal}(u)a)^{(+)}$. Since $\text{Pal}(u)$ is a proper prefix of $\text{Pal}(ua)$, it makes sense to define the iterated palindromic closure of an infinite word $x$ as the infinite word which is the limit of the iterated palindromic closure of the prefixes of $x$.

Justin’s Formula is the following. For every words $u$ and $v$, one has

$$
\text{Pal}(uv) = \psi_u(\text{Pal}(v)) \text{Pal}(u).
$$

This formula extends to infinite words: if $u$ is a word and $v$ is an infinite word, then

$$
\text{Pal}(uv) = \psi_u(\text{Pal}(v)).
$$

(1)

There is a precise combinatorial description of standard episturmian words (see e.g. [21, 18]).

**Theorem 8.4** An infinite word $s$ is a standard episturmian word if and only if there exists an infinite word $\Delta = a_0a_1 \cdots$, where the $a_n$ are letters, such that

$$
s = \lim_{n \to \infty} u_n,
$$

where the sequence $(u_n)_{n \geq 0}$ is defined by $u_n = \text{Pal}(a_0a_1 \cdots a_{n-1})$. Moreover, the word $s$ is episturmian strict if and only if every letter appears infinitely often in $\Delta$.

The infinite word $\Delta$ is called the directive word of the standard word $s$. The description of the infinite word $s$ can be rephrased by the equation

$$
s = \text{Pal}(\Delta).
$$

As a particular case of Justin’s Formula, one has

$$
u_{n+1} = \psi_{a_0 \cdots a_{n-1}}(a_n)u_n.
$$

(2)

The words $u_n$ are the only prefixes of $s$ which are palindromes.
Example 8.5 The Fibonacci word is a standard episturmian word with directive word $\Delta = ababa \cdots$. Indeed, by Formula (2) one has $\text{Pal}(\Delta) = \psi_{ab}(\text{Pal}(\Delta))$. Since $\psi_{ab} = \phi^2$ where $\phi$ is the Fibonacci morphism, we have $\text{Pal}(\Delta) = \phi^2(\text{Pal}(\Delta))$. This shows that $\text{Pal}(\Delta)$ is the Fibonacci word.

We note that for $n \geq 1$, one has
\[ |u_{n+1}| \leq 2|u_n|. \quad (3) \]
Indeed, set $u_n = u'_n ba$. If $a_n = a$, the word $u'_n baab\tilde{u}_n$ is palindrome of length at most $2|u_n|$. If $a_n = b$, then $u'_n bab\tilde{u}_n$ is a palindrome of length strictly less than $2|u_n|$.

Example 8.6 As a consequence of Equation (2), when $s$ is the Fibonacci word and $\phi$ the Fibonacci morphism, we have for every $n \geq 0$
\[ u_{n+1} = \phi^n(a)u_n. \quad (4) \]
In view of Equation (2), we need to show that $\phi^n(a) = \psi_{a_0 \cdots a_{n-1}}(a_n)$. By Example 8.5 the directive word of $s$ is $ababa \cdots$. If $n$ is even, then $\psi_{a_0 \cdots a_{n-1}}(a_n) = \psi_{(ab)^n/2}(a) = \phi^n(a)$ since $\psi_{ab} = \phi^2$. If $n$ is odd, then $\psi_{a_0 \cdots a_{n-1}}(a_n) = \psi_{(ab)^{(n-1)/2}a}(b) = \phi^{n-1}(ab) = \phi^n(a)$ and the property is true also.

As a consequence, we have in the prefix ordering for every $n \geq 0$,
\[ u_n < \phi^{n+1}(a). \quad (5) \]
Indeed, both words are prefixes of the Fibonacci word and it is enough to compare their lengths. For $n = 0$, we have $|u_0| = 0$ and $|\phi(a)| = 2$. Next, for $n \geq 1$, we have by (1), $u_n = \phi^{n-1}(a)u_{n-1}$. Arguing by induction, we obtain $|u_n| < |\phi^{n-1}(a)| + |\phi^n(a)| = |\phi^{n+1}(a)|$.

8.2 Tree sets

Let $F$ be a factorial set of words. For $w \in F$, we consider the set $E_F(w)$ as an undirected graph on the set of vertices which is the disjoint union of $L_F(w)$ and $R_F(w)$ with edges the pairs $(a, b) \in E_F(w)$. This graph is called the extension graph of $w$.

A factorial set $F$ is called biextendable if every $w \in F$ can be extended on the left and on the right, that is such that $\ell_F(w) > 0$ and $r_F(w) > 0$.

A biextendable set is a tree set if for every $w \in F$, the graph $E_F(w)$ is a tree. A tree set is neutral.

More generally one also defines a connected (resp. acyclic), as a biextendable set $F$ such that for every $w \in F$, the graph $E_F(w)$ is connected (resp. acyclic). Thus a biextendable set is a tree set if and only if it is both connected and acyclic.

Example 8.7 The Fibonacci set is a tree set. This follows from the fact that it is a Sturmian set (see [11]) and that every Sturmian set is a tree set.
Example 8.8 The Tribonacci set is the set of factors of the fixed point of the morphism $\varphi : a \mapsto ab, b \mapsto ac, c \mapsto a$. It is a also a tree set (see [11]). The graph $E(\varepsilon)$ is represented in Figure 7.

![Figure 7: The extension graph of $\varepsilon$ in the Tribonacci set.](image)

Example 8.9 Let $\varphi : a \mapsto ab, b \mapsto a^3b$ be as in Example 7.11. Let $F$ be the set of factors of $\varphi^\omega(a)$. The graphs $E(a)$ and $E(aa)$ are shown in Figure 8. The first graph has a cycle of length 4 and the second one has two connected components. Thus $F$ is not a tree set (it is not either an acyclic or a connected set, as defined in [11]).

Example 8.10 Let $T$ be the Thue-Morse set (see example 7.14). The set $F$ is not a tree set since $E_T(\varepsilon)$ is the complete bipartite graph $K_{2,2}$ on two sets with 2 elements.

9 Return words

In this section, we introduce return words. We begin with the classical notion of left and right return words in factorial sets. We then develop a notion of limit return sets for pseudowords.

9.1 Left and right return words

Let $F$ be a factorial set. A return word to $x \in F$ is a nonempty word $w \in F$ such that $xw$ begins and ends by $x$ but has no internal factor equal to $x$. We denote by $R_F(x)$ the set of return words to $x$.

For $x \in F$, we denote

$$\Gamma_F(x) = \{w \in F \mid xw \in F \cap A^*x\}.$$
Thus $R_F(x)$ is the set of nonempty words in $\Gamma_F(x)$ without any proper prefix in $\Gamma_F(x)$. Note that $\Gamma_F(x)$ is a right unitary submonoid of $A^*$.

One also defines a left return word to $x \in F$ as a nonempty word such that $wx$ begins and ends with $x$ but has no internal factor equal to $x$. We denote by $R'_F(x)$ the set of left return words to $x$. One has obviously $R'_F(x) = xR_F(x)x^{-1}$.

**Example 9.1** Let $F$ be the Fibonacci set. The sets of right and left return words to $a$ are $R_F(a) = \{a, ba\}$ and $R'_F(a) = \{a, ab\}$. Similarly, $R_F(b) = \{ab, aab\}$ and $R'_F(b) = \{ba, baa\}$.

**Example 9.2** Let $F$ be a periodic set. Let $w$ be a primitive word of length $n$ such that $F = F(w^*)$. Then, for any word $x \in F$ of length at least $n$, the set $R_F(w)$ is reduced to one word of length $n$.

The following is [13, Equation (4.2)].

**Proposition 9.3** Let $F$ be a factorial set. For any $x \in F$, one has $\Gamma_F(x) = R_F(x)^* \cap x^{-1}F$.

*Proof.* If a nonempty word $w$ is in $\Gamma_F(x)$ and is not in $R_F(x)$, then $w = uv$ with $u \in \Gamma_F(x)$ and $v$ nonempty. Since $\gamma_F(x)$ is right unitary, we have $v \in \Gamma_F(x)$, whence the conclusion $w \in R_F(x)^*$ by induction on the length of $w$. Moreover, one has $xw \in F$ and thus $w \in x^{-1}F$.

Conversely, assume that $w$ is a nonempty word in $R_F(x)^* \cap w^{-1}F$. Set $w = uv$ with $u \in R_F(x)$ and $v \in R_F(x)^*$. Then $xw = xuv \in A^*xv \subset A^*x$ and $xw \in F$. Thus $w \in \Gamma_F(x)$.

Note that, as a consequence, for $x, y \in F$ such that $xy \in F$, we have

$$R_F(xy) \subset R_F(y)^*.$$  \hspace{1cm} (6)

Indeed, if $w \in R_F(xy)$, then $xyw \in F \cap A^*xy$ implies $yw \in F \cap A^y$ and thus the result follows since $\Gamma_F(y) \subset R_F(y)^*$ by Proposition 9.3.

The dual of Proposition 9.3 and of Equation (6) hold for left return words.

By a result of [3], in a uniformly recurrent neutral set, one has

$$\text{Card}(R_F(x)) = \text{Card}(A)$$  \hspace{1cm} (7)

for every $x \in F$.

**Example 9.4** Let $\varphi : a \mapsto ab, b \mapsto a^3b$ be as in Example 7.11. Let $F$ be the set of factors of $\varphi^n(a)$. One has $R_F(a) = \{a, ba\}$ but $R_F(aa) = \{a, babaa, babababaab\}$. Thus the number of return words is not constant in a uniformly recurrent set which is not neutral.

There is an explicit form for the return words in a Sturmian set (see [21, Theorem 4.4, Corollaries 4.1 and 4.5]).
Theorem 9.7 Let $L$ be a uniformly recurrent tree set. For any $x \in L$, the set $R_F(x)$ is a basis of $FG(A)$.

The proof uses Equation (7) and the following result [11, Theorem 4.5].

Theorem 9.8 Let $F$ be a uniformly recurrent connected set. For any $w \in L$, the set $R_F(w)$ generates the free group $FG(A)$.

Example 9.9 Let $F$ be the Tribonacci set on $A = \{a, b, c\}$. Then $R_F(a) = \{a, ba, ca\}$, which is easily seen to be a basis of $FG(A)$. 

31
9.2 Limit return sets

Let \( k \geq 1 \) be an integer. A recurrent set \( F \) is \( k \)-bounded if every \( x \in F \) has at most \( k \) return words. The set \( F \) is bounded if it is \( k \)-bounded for some \( k \).

Thus, by Equation (7), a neutral set is \( k \)-bounded with \( k = \text{Card}(A) \).

Clearly, any bounded set is uniformly recurrent. There exist uniformly recurrent sets which are not bounded (see [16, Example 3.17]).

Let \( F \) be a \( k \)-bounded set. Consider an element \( x \) which belongs to a group in \( J(F) \). Let \( r_n \) be a sequence of finite prefixes of \( x \) strictly increasing for the prefix order. Similarly, let \( \ell_n \) be a sequence of finite suffixes of \( x \) strictly increasing for the suffix order. Since \( x^2 \in J(F) \), we have \( \ell_n r_n \in F \). Let

\[
\mathcal{R}_n = r_n \mathcal{R}_F(\ell_n r_n) r_n^{-1}.
\]

Up to taking a subsequence, we may assume that the set \( \mathcal{R}_n \) has a fixed number \( \ell \leq k \) of elements \( r_{n,1}, \ldots r_{n,\ell} \) and that the sequence \( (r_{n,1}, \ldots, r_{n,\ell}) \) is convergent in \( \hat{A}^\ell \). Its limit \( \mathcal{R} \) is called a limit return set to \( x \). The sequence \( (\ell_n, r_n, \mathcal{R}_n) \) is called an approximating sequence for \( \mathcal{R} \).

We note that

\[
\mathcal{R}_{n+1} \subset \mathcal{R}_n^*.
\]

Indeed, set \( r_{n+1} = r_n s_n \). Since \( \ell_{n+1} r_n \) is a prefix of \( \ell_{n+1} r_{n+1} \), we have by the dual of (8) the inclusion \( \mathcal{R}_F(\ell_{n+1} r_{n+1}) \subset \mathcal{R}_F(\ell_{n+1} r_n)^* \). Thus

\[
\mathcal{R}_F(\ell_{n+1} r_{n+1}) = r_{n+1}^{-1} \ell_{n+1}^{-1} \mathcal{R}_F(\ell_{n+1} r_{n+1}) \ell_{n+1} r_{n+1},
\]

\[
\subset r_{n+1}^{-1} \ell_{n+1}^{-1} \mathcal{R}_F(\ell_{n+1} r_n)^* \ell_{n+1} r_{n+1},
\]

\[
\subset s_n^{-1} \mathcal{R}_F(\ell_{n+1} r_n)^* s_n.
\]

Since \( \ell_n r_n \) is a suffix of \( \ell_{n+1} r_n \), we have by (7) the inclusion \( \mathcal{R}_F(\ell_{n+1} r_n) \subset \mathcal{R}_F(\ell_n r_n)^* \). Thus we obtain

\[
\mathcal{R}_{n+1} = r_{n+1} \mathcal{R}_F(\ell_{n+1} r_{n+1}) r_{n+1}^{-1},
\]

\[
\subset r_{n+1} s_n^{-1} \mathcal{R}_F(\ell_{n+1} r_n)^* s_n r_{n+1}^{-1},
\]

\[
\subset r_n \mathcal{R}_F(\ell_n r_n)^* r_n^{-1} = \mathcal{R}_n^*.
\]

**Example 9.10** Let \( \varphi \) be the Fibonacci morphism and \( F \) be the Fibonacci set. We will show that \( \{\varphi^n(a), \varphi^n(ba)\} \) is a limit return set to \( x = \varphi^\omega(a) \).

For this, consider the sequence \( \ell_n = r_n = \varphi^{2n}(a) \). The sequence is increasing both for the prefix and the suffix order and its terms are both prefixes and suffixes of \( x \). The set \( \mathcal{R}_n = r_n \mathcal{R}_F(\ell_n r_n) r_n^{-1} = \ell_n^{-1} \mathcal{R}_F(\ell_n r_n) \ell_n \) is by (8)

\[
\mathcal{R}_n = \varphi^{2n}(a)^{-1} \{\varphi^{2n}(a), \varphi^{2n+1}(a)\} \varphi^{2n}(a) = \{\varphi^{2n}(a), \varphi^{2n}(b) \varphi^{2n}(a)\}
\]

\[
= \{\varphi^{2n}(a), \varphi^{2n}(ba)\}.
\]

The sequence \( \{\varphi^{2n}(a), \varphi^{2n}(ba)\} \) converges to \( \{\varphi^\omega(a), \varphi^\omega(ba)\} \), which proves the claim.
The following example shows that in degenerated cases, a limit return set may contain finite words.

**Example 9.11** Let \( F \) be a periodic set of period \( n \). By Example 9.2, a return set to any word of length larger than \( n \) is formed of one word of length \( n \). Thus any return set to a pseudoword \( x \in J(F) \) is formed of one word of length \( n \).

### 10 Schützenberger groups

Let \( M \) be a topological monoid. For an element \( x \in M \), we denote by \( H(x) \) the \( H \)-class of \( x \).

Let \( H \) be an \( H \)-class of \( M \). Set \( T(H) = \{ x \in M \mid Hx = H \} \). Each \( x \in T(H) \) defines a map \( \rho_x : H \to H \) defined by \( \rho_x(h) = hx \). The set of the translations \( \rho_x \) for \( x \in T(H) \) is a topological group acting by permutations on \( H \), denoted \( \Gamma(H) \). The groups corresponding to different \( H \)-classes contained in the same \( J \)-class \( J \) are continuously isomorphic and the equivalence called the Schützenberger group of \( J \), denoted \( G(J) \).

If \( J \) is a regular \( J \)-class, any \( H \)-class of \( J \) which is a group is isomorphic to \( G(J) \). Indeed, \( H \subset T(H) \) and the restriction to \( H \) of the mapping \( \rho : x \in T(H) \to \rho_x \in \Gamma(H) \) is an isomorphism (see [23] for a more detailed presentation).

The following is [1, Proposition 5.2].

**Theorem 10.1** Let \( F \) be a non-periodic bounded set. Let \( x \in J(F) \) be such that \( H(x) \) is a group and let \( R \) be a limit return set to \( x \). Then \( H(x) \) is the closure of the semigroup generated by \( R \).

Note that Theorem 10.1 does not hold without the hypothesis that \( F \) is non-periodic (see Example 9.11).

This fundamental result is the key to understand the role played by the group generated by return words. Actually, let \((\ell_n, r_n, R_n)\) be an approximating sequence for \( R \). Then, since \( R_{n+1} \subset R_n \) by (12), we have \( R^* = \cap_{n \geq 0} R_n \). Thus the group \( H(x) \) is the intersection of the submonoids \( R_n \) and each of them is the closure of the submonoid generated by \( R_n \).

**Example 10.2** Let \( \varphi \) be the Fibonacci morphism and let \( F \) be the Fibonacci set. The \( H \)-class of the pseudoword \( x = \varphi^\omega(a) \) is a group. Indeed, \( H(x) \) contains the idempotent \( \varphi^\omega(a^\omega) \). The group \( H(x) \) is the closure of the semigroup generated by \( x \) and \( y = \varphi^\omega(ba) \), that is, isomorphic to \( \hat{F}(A) \).

The following statement is a generalization of [5, Theorem 6.5]. We denote by \( G(F) \) the Schützenberger group of \( J(F) \).

**Theorem 10.3** Let \( F \) be a non-periodic bounded set and let \( f : \hat{A}^* \to G \) be a continuous morphism from \( \hat{A}^* \) onto a profinite group \( G \). The following conditions are equivalent.
(i) The restriction of $f$ to $G(F)$ is surjective.

(ii) For every $w \in F$, the submonoid $f(\mathcal{R}_F(w)^*)$ is dense in $G$.

**Proof.** Let $x \in J(F)$ be such that $H(x)$ is a group. Let $(\ell_n, r_n)$ be a sequence of pairs of suffixes and prefixes of $x$ of strictly increasing length. Taking a subsequence, we may assume that $\mathcal{R}_n = r_n \mathcal{R}_F(\ell_n r_n)^{-1}$ converges to the limit return set $\mathcal{R}$. Since, by (12) we have $\mathcal{R}_{n+1} \subset \mathcal{R}_n^*$, the semigroup generated by $\mathcal{R}$ is $\cap_{n \geq 0} \mathcal{R}_n$.

(i) implies (ii). Assume by contradiction that $f(\mathcal{R}_F(w)^*)$ is not dense in $G$. Since $w$ is a factor of $x$, we may assume that $r_0$ ends with $w$. Then $\mathcal{R}_F(\ell_0 r_0) \subset \mathcal{R}_F(w)^*$. This implies that $f(\mathcal{R}_F(\ell_0 r_0)^*)$ is not dense in $G$. Since $f(\mathcal{R}_0^*)$ is conjugate to $f(\mathcal{R}_F(\ell_0 r_0)^*)$, the same holds for $f(\mathcal{R}_0^*)$. Thus $f(\mathcal{R}^*)$ is not dense in $G$. But by Theorem 10.1, $H(x)$ is the closure of the semigroup generated by $\mathcal{R}$. We conclude that $f(H(x))$ is not dense in $G$.

(ii) implies (i). Since $H(x) = \mathcal{R}^* = \cap_{n \geq 0} \overline{\mathcal{R}_n}$, we have $f(H(x)) = \cap_{n \geq 0} f(\overline{\mathcal{R}_n})$. But $\mathcal{R}_n$ is a subset of $\mathcal{R}_k(r_n r_{n+1})$ and thus by (ii), each $f(\mathcal{R}_n^*)$ is dense in $G$. Thus $f(H(x)) = G$.

**Corollary 10.4** Let $F$ be a non-periodic bounded set on the alphabet $A$. The following conditions are equivalent.

(i) The restriction to any maximal subgroup of $J(F)$ of the natural projection $p_G : \hat{A}^* \to \hat{F}G(A)$ is surjective.

(ii) For each $w \in F$ the set $\mathcal{R}_F(w)$ generates the free group $FG(A)$.

**Proof.** We apply Theorem 10.3 with $f$ being the identity. Let $x \in J(F)$ be such that $H(x)$ is a group.

(i) implies (ii). Let $w \in F$. There is a maximal subgroup of $J(F)$ contained in the topological closure of $R_F(w)^*$ in the free profinite monoid (indeed, $RR_F(\ell_n r_n)^*$ is a subset of $R_F(w)^*$, for some suitable sequence of words $\ell_n r_n$ as defined some pages before, for infinitely many $n$).

It follows that the topological closure of $R_F(w)^*$ in the free profinite group generated by $A$ is the whole free profinite group. This implies that $\mathcal{R}_F(w)$ generates $FG(A)$.

(ii) implies (i). By Theorem 10.3 the restriction to $H(x)$ of the projection $p_G : \hat{A}^* \to \hat{F}G(A)$ is surjective.

**10.1 Groups of tree sets**

We now consider uniformly recurrent tree sets. Note that a uniformly recurrent tree set $F$ is non-periodic. Indeed, if $F$ is the set of factors of $w^*$ with $w$ primitive, then $\mathcal{R}_F(w) = \{w\}$ since $w$ does not overlap nontrivially $w^2$. Thus $F$ is not a tree set by Theorem 9.7.

**Theorem 10.5** Let $F$ be a uniformly recurrent tree set. Then the following assertions hold.
1. The group $G(F)$ is the free profinite group on $A$. More precisely, the restriction to any maximal subgroup of $J(F)$ of the natural projection $p_G : \hat{A}^* \to \hat{F}(A)$ is an isomorphism.

2. Let $H$ be a subgroup of finite index $n$ in $FG(A)$. For any maximal group $G$ in $J(F)$, $G \cap p_G^{-1}(H)$ is a subgroup of index $n$ of $G$.

Proof. 1. This results directly from Corollary 10.4 since by Theorem 9.7, $R_F(w)$ is a basis of $FG(A)$ for every $w \in F$ when $F$ is a uniformly recurrent tree set. Since $H(x)$ is the closure of a semigroup generated by Card($A$) elements, there is a continuous morphism $\psi$ from $FG(A)$ onto $H(x)$. Thus $p_G \circ \psi$ is continuous surjective morphism from $FG(A)$ onto itself. By Proposition 5.7, it implies that it is an isomorphism. This proves the first assertion.

2. This results from Corollary 10.4 since the restriction $\alpha$ of $p_G$ to $G$ is an isomorphism from $G$ onto $\hat{F}(A)$ and $G \cap p_G^{-1}(H) = \alpha^{-1}(H)$. 

Example 10.6 Let $F$ be the Fibonacci set. We have seen that $G(F)$ is the free profinite group on $A$ (Example 9.10).

10.2 Groups of fixed points of morphisms

Let $\varphi : A^* \to A^*$ be a primitive substitution and let $F(\varphi)$ be the set of factors of a fixed point of $\varphi$. We denote by $J(\varphi)$ the closure of $F(\varphi)$ and by $G(\varphi)$ the Schützenberger group of $J(\varphi)$.

A connexion for $\varphi$ is a word $ba$ with $b, a \in A$ such that $ba \in F(\varphi)$, the first letter of $\varphi^\omega(a)$ is $a$ and the last letter of $\varphi^\omega(b)$ is $b$. Every primitive substitution has a connexion [4 Lemma 4.1]. A connective power of $\varphi$ is a finite power $\tilde{\varphi}$ of $\varphi$ such that the first letter of $\tilde{\varphi}(a)$ is $a$, the last letter of $\tilde{\varphi}(b)$ is $b$. We denote $X_\varphi(a,b) = aR_F(ba)a^{-1}$. The set $X_\varphi(a,b)$ is a code.

Example 10.7 Let $\tau : a \mapsto ab, b \mapsto ba$ be the Thue-Morse morphism. The word $aa$ is a connection for $\tau$ and $\tilde{\tau} = \tau^2$ is a connecting power of $\tau$. The set $X = X_\tau(a,a)$ has four elements $x = abba, y = ababba, z = abababa$ and $t = ababababa$.

The following is [4 Theorem 5.6].

Theorem 10.8 Let $\varphi$ be a non periodic primitive substitution. Consider a connexion $ba$ for $\varphi$ and a connective power $\tilde{\varphi}$. The intersection $H_{ba}$ of the $R$-class of $\varphi^\omega(a)$ with the $L$-class of $\varphi^\omega(b)$ is a group and $H_{ba} = \tilde{\varphi}(H_{ba}) = \varphi^\omega(H_{ba})$.

The proof uses the notion of recognizability of a substitution. We give the definition in the following form (see [22] for the equivalence with equivalent forms). Given a morphism $\varphi : A^* \to A^*$, a pair $(q,r)$ of words in $A^*$ is synchronizing if for any $p,s,t \in A^*$ such that $\varphi(t) = pQRS$, one has $t = uv$ with
\( \varphi(u) = pq \) and \( \varphi(v) = rs \) (see Figure 9). Let \( F \) be the set of factors of a fixed point of \( \varphi \). The morphism \( \varphi \) is recognizable if there is an integer \( n \geq 1 \) such that for any \( x, y \in F \cap A^n \) such that \( xy \in F \), the pair \((\varphi(x), \varphi(y))\) is synchronizing.

By a result of Mossé [30], any non-periodic primitive substitution is recognizable (see [22] for a new version of the proof).

The following is the main result if [4] (Theorem 6.2).

**Theorem 10.9** Let \( \varphi \) be a non-periodic primitive substitution over the alphabet \( A \). Let \( ba \) be a connexion of \( \varphi \) and let \( X_{\varphi} = X_{\varphi}(a, b) \). Then \( G(\varphi) \) admits the presentation

\[
\langle X \mid \tilde{\varphi}_G^0(x) = x, x \in X \rangle.
\]

**Example 10.10** Let \( \tau : a \mapsto ab, b \mapsto ba \) be the Thue-Morse morphism. We have seen in Example 10.7 that the word \( aa \) is a connection for \( \tau \) and \( \tilde{\tau} = \tau^2 \) is a connecting power of \( \tau \). The set \( X = X_{\tau}(a, a) \) has four elements \( x = abba, y = ababba, z = abbaba \) and \( t = ababbaba \). By Theorem 10.9, the group \( G(\tau) \) is generated by \( X \) with the relations \( \tau^0 \omega(x) = \tau^0 \omega(y) = \tau^0 \omega(z) = \tau^0 \omega(t) \), the relation \( xy^{-1}z = t \) is a consequence of the relations above and thus \( G(\varphi) \) is generated by \( x, y, z \).

Let \( f : A^* \to G \) be a morphism from \( A^* \) into a finite group \( G \) and let \( \varphi : A^* \to A^* \) be a morphism. We denote by \( \varphi_G \) the map from \( G^A \) into itself defined as follows. Consider \( h \in G^A \). We may naturally extend \( h \) to a map from \( A^* \) into \( G \). For \( a \in A \), we define the image of \( a \) by \( \varphi_G(h) \) as \( \varphi_G(h)(a) = h(\varphi(a)) \).

We say that \( \varphi \) has finite \( f \)-order if there is an integer \( n \geq 1 \) such that \( \varphi^n_G(f) = f \). The least such integer is called the \( f \)-order of \( \varphi \).

Any substitution \( \varphi \) which is invertible in \( FG(A) \) is of finite \( h \)-order for any morphism \( h \) into a finite group. Indeed, since \( G \) is finite, there are integers \( n, m \) with \( n < m \) such that \( \varphi^{n+m}_G = \varphi^n_G \). Since \( \varphi \) is invertible, \( \varphi^n_G \) is the identity.

**Example 10.11** Let \( \varphi : a \mapsto ab, b \mapsto a \) be the Fibonacci substitution and let \( h : A^* \to \mathbb{Z}/2\mathbb{Z} \) be the parity of the length, that is the morphism into the additive group of integers modulo 2 sending each letter to 1. Then \( \varphi \) is of \( h \)-order 3.

The following is a consequence of Theorem 10.9 using [4] Proposition 3.2.

**Corollary 10.12** Let \( \varphi \) be a non-periodic primitive substitution over \( A \) and let \( h : A^* \to G \) be a morphism onto a finite group. The restriction of \( \tilde{h} : \tilde{A^*} \to G \) to any maximal subgroup of \( J(\varphi) \) is surjective if and only if \( \varphi \) has finite \( h \)-order.
Example 10.13 Let $\varphi$ be as in Example 8.9, let $G = \mathbb{Z}/2\mathbb{Z}$ and let $h : A^* \to G$ be the parity of the length. Then $\varphi_G(h) = (0, 0)$ and $\varphi_G(0, 0) = (0, 0)$. Thus $\varphi$ does not have finite $h$-order. Actually, any pseudoword in $J(\varphi)$ which is in the image of $\hat{\varphi}$ has even length and thus is mapped by $h$ to $0$. Thus, by Theorem 10.8 there is a maximal group $G$ in $J(F)$ which contains only pseudowords of even length and therefore $\hat{h}(G) = \{0\}$, showing that the restriction of $\hat{h}$ to $G$ is not surjective.

The following example is from [1, Section 7.2].

Example 10.14 Let $\varphi : a \mapsto ab, b \mapsto a^3b$ be as in the previous example and let $h : A^* \to A_5$ be the morphism from $A^*$ onto the alternating group $A_5$ defined by $h : a \mapsto (123), b \mapsto (345)$. One may verify that $\varphi$ has $h$-order 12. Thus $A_5$ is a quotient of $G(\varphi)$. It is not known whether any finite group is a quotient of $G(\varphi)$.

Proper substitutions A substitution $\varphi$ over $A$ is proper if there are letters $a, b \in A$ such that for every $d \in A$, $\varphi(d)$ starts with $a$ and ends with $b$. Theorem 10.9 takes a simpler form for proper substitutions. The following is [1, Theorem 6.4].

Theorem 10.15 Let $\varphi$ be a non-periodic proper primitive substitution over a finite alphabet $A$. Then $G(\varphi)$ admits the presentation

$$\langle A \mid \varphi_G^\omega(a) = a, a \in A \rangle_G.$$ 

The proof uses Proposition 5.17 applied with the diagram of Figure 11.

![Figure 10: A commutative diagram](image)

Example 10.16 Let $A = \{a, b\}$ and let $\varphi : a \mapsto ab, b \mapsto a^3b$. The morphism $\varphi$ is proper. Thus, by Theorem 10.15 the Schützenberger group of $J(\varphi)$ has the presentation $(a, b \mid \varphi_G^\omega(a) = a, \varphi_G^\omega(b) = b)$. Since the image of $FG(A)$ by $\varphi$ is included in the subgroup generated by words of length 2, the relations $\varphi_G^\omega(a) = a$ and $\varphi_G^\omega(b) = b$ are nontrivial and thus $G(F)$ is not a free profinite group of rank two (it is actually not a free profinite group, see [1, Example 7.2]).

### 10.3 Groups of bifix codes

For an automaton $A$, we denote by $\varphi_A$ the natural morphism from $A^*$ onto the transition monoid of $A$. 
Let $F$ be a recurrent set. For any finite automaton $A = (Q, i, T)$, we denote by $\text{rank}_A(F)$ the minimum of the ranks of the maps $\varphi_A(w)$ for $w \in F$. By Proposition 3.2, the set of elements of $\varphi_A(F)$ of rank $\text{rank}_A(F)$ is included in a regular $J$-class, called the $F$-minimal $J$-class of the monoid $\varphi_A(A^*)$ and denoted $J_A(F)$. The structure group of this $J$-class is denoted $G_A(F)$.

Let $X \subset A^+$ be a code. A parse of a word $w \in A^*$ with respect to $X$ is a triple $(p, x, q)$ with $w = pxq$ such that $p$ has no suffix in $X$, $x \in X^*$ and $q$ has no prefix in $X$.

A parse of a profinite word $w \in \tilde{A}^*$ with respect to $X$ is a triple $(p, x, q)$ with $w = pxq$ such that $p$ has no suffix in $X$, $x \in \tilde{X}^*$ and $q$ has no prefix in $X$.

The number of parses of $w \in \tilde{A}^*$ with respect to a finite maximal prefix code $X$ is equal to the number of its prefixes which have no suffix in $X$. Indeed the map $(p, x, q) \mapsto p$ assigning to each parse its first component is bijective by Proposition 6.5.

Let $F$ be a recurrent set. A bifix code $X \subset F$ is $F$-maximal if it is not properly contained in any bifix code $Y \subset F$.

A bifix code $X \subset F$ is $F$-thin if there is a word $w \in F$ which is not a factor of $X$. When $F$ is uniformly recurrent, a set $X \subset F$ is $F$-thin if and only if it is finite.

Let $F$ be a recurrent set. The $F$-degree of a bifix code $X$, denoted $d_X(F)$, is the maximal number of parses of a word in $F$. A bifix code $X$ is $F$-maximal and $F$-thin if and only if its $F$-degree is finite. In this case a word of $F$ has $d_X(F)$ parses if and only if it is not an internal factor of a word of $X$ (see [9, Theorem 4.2.8]).

**Proposition 10.17** Let $F$ be a uniformly recurrent set and let $X$ be a finite $F$-maximal bifix code. The number of parses of any element of $F$ is equal to $d_X(F)$.

**Proof.** Let $w \in \tilde{F}$ and let $(u_n)$ be a sequence of elements of $F$ converging to $w$. Since each long enough $u_n$ has $d_X(F)$ parses, we may assume that all $u_n$ have $d_X(F)$ parses. We may then number the parses of $u_n$ as $(p_{n,i}, x_{n,i}, q_{n,i})$ in such a way that for fixed $i$, each sequence converges to $(p_i, x_i, q_i)$. Since $X$ is finite, the sequences $(p_{n,i})$ and $q_{n,i}$ are ultimately constant and the sequence $(x_{n,i})$ converges to some $x_i \in \tilde{A}^*$. Thus $w$ has $d_X(F)$ parses. There cannot exist more than $d_X(F)$ parses of a word in $\tilde{F}$ since the number of parses is equal to the number of suffixes which are prefixes of $X$. 

Let $X$ be an $F$-thin and $F$-maximal bifix code. The $F$-degree of $X$ is equal to the $F$-minimal rank of the minimal automaton $A$ of $X^*$. We denote by $\varphi$ the morphism $\varphi_A$, by $J_A(F)$ the $J$-class $J_A(F)$ and by $G_A(F)$ the group $G_A(F)$, called the $F$-group of $X$. It is a permutation group of degree $d_X(F)$.

We recall that for any uniformly recurrent tree set $F$ a finite bifix code $X \subset F$ is $F$-maximal of $F$-degree $d$ if and only if it is a basis of a subgroup of index $d$ (Finite Index Basis Theorem, see [12, Theorem 4.4]).
We prove the following result. It has the interesting feature that the hypothesis made on profinite objects has a consequence on finite words.

**Theorem 10.18** Let \( F \) be a uniformly recurrent set, let \( Z \) be a group code of degree \( d \) and let \( X = Z \cap F \). Let \( h : A^* \to G \) be the morphism from \( A^* \) onto the syntactic monoid of \( Z^* \). The restriction of \( h \) to a maximal subgroup of \( J(F) \) is surjective if and only if the following properties hold.

(i) \( X \) is an \( F \)-maximal bifix code of \( F \)-degree \( d \).

(ii) \( G_X(F) \) is isomorphic to \( G \).

(iii) The morphism \( \hat{\phi}_X \) maps each \( \mathcal{H} \)-class of \( J(F) \) which is a group onto \( G_X(F) \).

**Proof.** Set \( \varphi = \varphi_X \) and \( M = \varphi(A^*) \). Let \((Q, i, i)\) be the minimal automaton of \( Z^* \). Thus \( G \) is a transitive permutation group on the set \( Q \) which has \( d \) elements and \( h(Z^*) \) is the stabilizer of \( i \in Q \).

By [9, Theorem 4.2.11], since \( F \) is recurrent, the set \( X \) is an \( F \)-thin \( F \)-maximal bifix code of \( F \)-degree at most \( d \). Since \( F \) is uniformly recurrent, \( X \) is finite.

Let \( x \in J(F) \) be such that \( H = H(x) \) is a group such that the restriction of \( \hat{h} \) to \( H(x) \) is surjective.

Since \( \hat{h} \) maps \( H(x) \) onto \( G \), the pseudoword \( x \) has \( d \) parses with respect to \( Z \) and thus with respect to \( X \). Indeed, for any \( p \in Q \), \( \hat{h}(x) \) sends \( p \) on some \( q \in Q \). Then \( x \) has the interpretation \((u, v, w)\) with \( ph(u) = i \), \( i\hat{h}(v) = i \) and \( ih(w) = q \). This implies that \( d_X(F) = d \) and proves (i).

It is clear that \( \hat{\varphi}(J(F)) \) is contained in \( J_X(F) \) since \( J_X(F) \) contains the image by \( \varphi \) of every long enough word of \( F \). Thus \( \hat{\varphi}(x) \) is in \( J_X(F) \cap \varphi(F) \) and its \( \mathcal{H} \)-class \( K \) is a group.

![Figure 11: The reduction onto G.](image)

Let \( w \) be a word in \( F \) and not a factor of \( X \) such that \( \varphi(w) \in K \). Let \( P \) be the set of suffixes of \( w \) which are proper prefixes of \( X \). Since \( \text{Card}(P) = d_X(F) \), \( K \) is a permutation group on the set \( \{i \cdot p \mid p \in P\} \) which is identified by an isomorphism \( \alpha \) with a subgroup of \( G \).

If the map \( \hat{h} \) is surjective from \( H(x) \) onto \( G \), the commutativity of the diagram in Figure 11 forces \( \alpha \) to be surjective. Moreover \( \hat{\varphi} \) maps \( H(x) \) onto \( K \). The converse is also true. ■
In the case where $F$ is a tree set, the hypothesis of Theorem 10.18 is satisfied by assertion 1 in Theorem 10.5. The conclusion of Theorem 10.18 is implied by assertion 2.

**Example 10.19** Let $A = \{a, b\}$ and let $Z = A^2$. Let $F$ be the Fibonacci set. Then $X = \{aa, ab, ba\}$. The group $G_X(F)$ is the cyclic group of order 2, in agreement with the fact that $X$ generates the kernel of the morphism from $FG(A)$ onto $\mathbb{Z}/2\mathbb{Z}$ sending $a, b$ to 1. The minimal automaton of $X^*$ is shown in Figure 12 on the left and the 0 minimal ideal of its transition monoid $M$ is represented on the right.

![Figure 12: The minimal automaton of $X^*$ and the $F$-minimal $D$-class.](image)

Note that, in the above example, the $F$-minimal $D$-class $D$ of $M$ is the image of the $J$-class $J(F)$. The following example shows that this may be true although the image of $J(F)$ in the monoid $M$ is strictly included in $D$.

**Example 10.20** Let $F$ be the set of factors of the fixed point of the morphism $\varphi : a \mapsto ab, b \mapsto a^2b$ (as in Example 7.11). The set $F \cap A^2$ is the same as in Example 10.19 and the $F$-minimal $D$-class is also the same. However, $J(F)$ contains maximal groups formed of words of even length and thus its image in $M$ is aperiodic, that is has trivial subgroups.

We now deduce from Corollary 10.12 the following statement which gives information on the groups $G_X(F)$ when $X$ is an $F$-maximal bifix code in a set $F$ which is not a tree set. It would be interesting to have a direct proof of this statement which does not use profinite semigroups.

**Theorem 10.21** Let $\varphi$ be a primitive non-periodic substitution over the alphabet $A$ and let $F$ be the set of factors of a fixed point of $\varphi$. Let $Z$ be a group code of degree $d$ on $A$ and let $h$ be the morphism from $A^*$ onto the syntactic monoid of $Z^*$. Set $X = Z \cap F$. If $\varphi$ has finite $h$-order, then $X$ is an $F$-maximal bifix code of $F$-degree $d$ and $G_X(F)$ is isomorphic to $G$.

**Proof.** By Corollary 10.12, the hypothesis of Theorem 10.18 is satisfied and thus the conclusion using conditions (i) and (ii). $\blacksquare$

**Example 10.22** Let $F$ and $\varphi$ be as in Example 8.3. We consider, as in [4], the morphism $h : A^* \to A_5$ from $A^*$ onto the alternating group of degree 5 defined by $h : a \mapsto (123), b \mapsto (345)$. We have seen in Example 10.14 that $\varphi$ has $h$-order
and thus, by Corollary 10.12, \( \hat{h} \) induces a surjective map from any maximal subgroup of \( J(\phi) \) onto \( A_5 \).

Let \( Z \) be the bifix code generating the submonoid stabilizing 1 and let \( X = Z \cap F \). The \( F \)-maximal bifix code \( X \) has 8 elements. It is represented in Figure 13 with the states of the minimal automaton indicated on its prefixes. In agreement with Theorem 10.21, the \( F \)-degree of \( X \) is 5. The \( F \)-minimal \( D \)-class is represented in Figure 14. The word \( a^3 \) has rank 5 and \( R_F(a^3) = \{ babaa, bababaa \} \).

|    | 1, 2, 3, 16, 17 | 1, 4, 5, 14, 15 | 1, 2, 6, 7, 17 | 1, 4, 8, 9, 15 | 1, 2, 6, 10, 11 | 1, 2, 3, 12, 14 |
|----|----------------|----------------|-------------|---------------|---------------|---------------|
| 1/2, 4/3, 6, 15/8/9 | \( a^3 \) | \( a^3b \) | \( a^3ba \) | \( * \) | \( a^3bab \) | \( * \) |
| 1/2, 11, 17/6, 7, 9, 10 | \( ba^3 \) | \( * \) | \( * \) | \( * \) | \( * \) | \( * \) |
| 1/3, 6, 15/9, 14, 5, 8/4 | \( aba^3 \) | \( * \) | \( * \) | \( * \) | \( * \) | \( * \) |
| 1/11, 17/7, 16, 3, 6/2 | \( * \) | \( bab^3 \) | \( bab \) | \( * \) | \( * \) | \( * \) |
| 1/2, 4/9/14, 3, 6/15/5, 12 | \( * \) | \( * \) | \( * \) | \( * \) | \( * \) | \( * \) |
| 1/2, 4/3, 6, 15/10/11 | \( * \) | \( * \) | \( * \) | \( * \) | \( * \) | \( * \) |
\{1, 2, 3, 16, 17\} of \(a^3\) are respectively
\[(1, 2, 16, 3, 17) \quad (1, 17, 16, 2, 3)\]
which generate \(A_5\).

The next example is from \[31\].

**Example 10.23** Let \(F\) be the Thue-Morse set and let \(A\) be the automaton represented in Figure 15 on the left. The word \(aa\) has rank 3 and image \(I = \{1, 2, 10, 3, 17\}\). The next example is from \[31\].

![Figure 15: An automaton of \(F\)-degree 3 with trivial \(F\)-group](image)

{1, 2, 4}. The action on the images accessible from \(I\) is given in Figure 16. All words with image \{1, 2, 4\} end with \(aa\). The paths returning for the first time to \{1, 2, 4\} are labeled by the set \(R_F(aa) = \{b^2a^2, bababa^2, bababa^2, b^2aba^2\}\). Thus \(\text{rank}_A(F) = 3\) by \[31\] Theorem 3.1]. Moreover each of the words of \(R_F(a^2)\) defines the trivial permutation on the set \{1, 2, 4\}. Thus \(G_A(F)\) is trivial.

The fact that \(d_A(F) = 3\) and that \(G_A(F)\) is trivial can be seen directly as follows. Consider the group automaton \(B\) represented in Figure 15 on the right and corresponding to the map sending each word to the difference modulo 3 of the number of occurrences of \(a\) and \(b\). There is a reduction \(\rho\) from \(A\) onto \(B\) such that \(1 \mapsto 0, 2 \mapsto 1, \text{ and } 4 \mapsto 2\). This accounts for the fact that \(d_A(S) = 3\). Moreover, one may verify that any return word \(x\) to \(a^2\) has equal number of \(a\) and \(b\) (if \(x = uaa\) then \(aaaua\) is in \(F\), which implies that \(aua\) and thus \(aua\) have the same number of \(a\) and \(b\)). This implies that the permutation \(\varphi_B(x)\) is the identity, and therefore also the restriction of \(\varphi_A(x)\) to \(I\).
Example 10.24 Consider again the Thue-Morse substitution $\tau$ and the Thue-Morse set $F$ as in Example 10.10. Let $h$ be the morphism $h : a \mapsto (123), b \mapsto (345)$ from $A^*$ onto the alternating group $A_5$ (already used in Example 9.3). One may verify that $\tau$ has $h$-order 6 and thus, by Corollary 10.12, $h$ extends to a surjective continuous morphism from any maximal subgroup of $J(\varphi)$ onto $A_5$.

Let $Z$ be the group code generating the submonoid stabilizing 1 and let $X = Z \cap F$. The $F$-maximal bifix code $X$ is represented in Figure 17. We represent in Figure 17 only the nodes corresponding to right special words, that is, vertices with two sons.

The image of $\tau^4(b)$ is $\{1, 3, 4, 9, 10\}$ and thus it is minimal. The action on its image is shown in Figure 18. The return words to $\tau^4(b)$ are $\tau^4(b), \tau^3(a)$ and $\tau^5(ab)$. The permutations on the image of $\tau^4(b)$ are the 3 cycles of length 5 indicated in Figure 18. Since they generate the group $A_5$, we have $G_X(F) = A_5$.

References

[1] Jorge Almeida. Dynamics of implicit operations and tameness of pseudovarieties of groups. Trans. Amer. Math. Soc., 354(1):387–411, 2002.

[2] Jorge Almeida. Profinite semigroups and applications. In Structural theory of automata, semigroups, and universal algebra, volume 207 of NATO Sci.

43
[3] Jorge Almeida and Alfredo Costa. Infinite-vertex free profinite semi-
groupoids and symbolic dynamics. *J. Pure Appl. Algebra*, 213(5):605–631,
2009.

[4] Jorge Almeida and Alfredo Costa. Presentations of Schützenberger groups
of minimal subshifts. *Israel J. Math.*, 196(1):1–31, 2013.

[5] Jorge Almeida and Alfredo Costa. A geometric interpretation of the
Schützenberger group of a minimal subshift. *Ark. Mat.*, 54(2):243–275,
2016.

[6] Jorge Almeida and Benjamin Steinberg. Rational codes and free profinite
monoids. *J. Lond. Math. Soc. (2)*, 79(2):465–477, 2009.

[7] Zh. Almeıda. Profinite groups associated with weakly primitive substitu-
tions. *Fundam. Prikl. Mat.*, 11(3):13–48, 2005.

[8] Lubomíra Balková, Edita Pelantová, and Wolfgang Steiner. Sequences with
constant number of return words. *Monatsh. Math.*, 155(3-4):251–263, 2008.

[9] Jean Berstel, Clelia De Felice, Dominique Perrin, Christophe Reutenauer,
and Giuseppina Rindone. Bifix codes and Sturmian words. *J. Algebra*,
369:146–202, 2012.

[10] Jean Berstel, Dominique Perrin, and Christophe Reutenauer. *Codes and
Automata*. Cambridge University Press, 2009.

[11] Valérie Berthé, Clelia De Felice, Francesco Dolce, Julien Leroy, Dominique
Perrin, Christophe Reutenauer, and Giuseppina Rindone. Acyclic, con-
nected and tree sets. *Monatsh. Math.*, 176:521–550, 2015.

[12] Valérie Berthé, Clelia De Felice, Francesco Dolce, Julien Leroy, Dominique
Perrin, Christophe Reutenauer, and Giuseppina Rindone. The finite index
basis property. *J. Pure Appl. Algebra*, 219(7):2521–2537, 2015.

[13] Valérie Berthé, Clelia De Felice, Francesco Dolce, Julien Leroy, Dominique
Perrin, Christophe Reutenauer, and Giuseppina Rindone. Maximal bifix
decoding. *Discrete Math.*, 338:725–742, 2015.

[14] Thierry Coulbois, Mark Sapir, and Pascal Weil. A note on the continuous
extensions of injective morphisms between free groups to relatively free
profinite groups. *Publ. Mat.*, 47(2):477–487, 2003.

[15] Xavier Droubay, Jacques Justin, and Giuseppe Pirillo. Episturmian words
and some constructions of de Luca and Rauzy. *Theoret. Comput. Sci.*,
255(1-2):539–553, 2001.
[16] Fabien Durand, Julien Leroy, and Gwenaël Richomme. Do the properties of an $S$-adic representation determine factor complexity? *J. Integer Seq.*, 16(2):Article 13.2.6, 30, 2013.

[17] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[18] Amy Glen and Jacques Justin. Episturmian words: a survey. *Theor. Inform. Appl.*, 43:403–442, 2009.

[19] Marshall Hall, Jr. Coset representations in free groups. *Trans. Amer. Math. Soc.*, 67:421–432, 1949.

[20] Marshall Hall, Jr. A topology for free groups and related groups. *Ann. of Math. (2)*, 52:127–139, 1950.

[21] Jacques Justin and Laurent Vuillon. Return words in Sturmian and episturmian words. *Theor. Inform. Appl.*, 34(5):343–356, 2000.

[22] Karel Klouda and Stepan Starosta. Characterization of circular dol systems. 2014. [http://arxiv.org/abs/1401.0038](http://arxiv.org/abs/1401.0038).

[23] Donald E. Knuth. *The art of computer programming. Vol. 2*. Addison-Wesley, Reading, MA, 1998. Seminumerical algorithms, Third edition [of MR0286318].

[24] Neal Koblitz. *p-adic numbers, p-adic analysis, and zeta-functions*, volume 58 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1984.

[25] Gérard Lallement. *Semigroups and combinatorial applications*. John Wiley & Sons, New York-Chichester-Brisbane, 1979. Pure and Applied Mathematics, A Wiley-Interscience Publication.

[26] Hendrick Lenstra. Profinite Fibonacci numbers. *Nieuw Archief voor Wiskunde*, 6:297–300, 2005.

[27] Douglas Lind and Brian Marcus. *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995.

[28] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[29] S. Margolis, M. Sapir, and P. Weil. Irreducibility of certain pseudovarieties. *Comm. Algebra*, 26(3):779–792, 1998.

[30] Brigitte Mossé. Puissances de mots et reconnaissabilité des points fixes d’une substitution. *Theoret. Comput. Sci.*, 99(2):327–334, 1992.
[31] Dominique Perrin. Codes and automata in minimal sets. In Combinatorics on Words - 10th International Conference, WORDS 2015, Kiel, Germany, September 14-17, 2015, Proceedings, pages 35–46, 2015.

[32] Christophe Reutenauer. Une topologie du monoïde libre. Semigroup Forum, 18(1):33–49, 1979.

[33] Luis Ribes and Pavel Zalesskii. Profinite groups, volume 40 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 2010.

[34] Stephen Willard. General topology. Dover Publications, Inc., Mineola, NY, 2004. Reprint of the 1970 original [Addison-Wesley, Reading, MA; MR0264581].