Comparison of analytic and numerical results in the XY-model

Janos Balog\textsuperscript{a}, Francesco Knechtl\textsuperscript{b}, Tomasz Korzec\textsuperscript{b}, and Ulli Wolff\textsuperscript{b}

\textsuperscript{a}Research Institute for Particle and Nuclear Physics
1525 Budapest 114, Pf. 49, Hungary

\textsuperscript{b}Institut für Physik, Humboldt Universität
Newtonstr. 15
12489 Berlin, Germany

We study the two-dimensional XY-model with high precision Monte Carlo techniques and investigate the continuum approach of the step-scaling function of its finite volume mass gap. The continuum extrapolated results are found consistent with analytic predictions for the finite volume energy spectrum based on the equivalence with sine-Gordon theory. To come to this conclusion it was essential to use an also predicted form of logarithmic decay of lattice artifacts for the extrapolation.

1. Introduction

Interacting quantum field theories in four space time dimensions are in all relevant cases impossible to solve exactly and results are obtained only in suitable approximation schemes. The situation is different with two dimensional systems which often may be (partially) solved exactly. Such a system is for instance the sine-Gordon model for which detailed predictions for the finite volume energy spectrum exist \cite{1,2,3}. These predictions can be extended to the two dimensional $O(2)$ non-linear $\sigma$-model which is believed to lie in the same universality class as the sine-Gordon model at a special value of its coupling. For the derivation of these exact results one adopts some not rigorously provable conjectures at intermediate steps, therefore a numerical confirmation is desirable.

We study the massive phase of the XY-model with Monte Carlo techniques and extract the finite volume mass gap from a time-slice (zero momentum) correlation function. The continuum extrapolation is performed according to a prediction by J. Balog \cite{4}. In the continuum limit some points of the step-scaling function of the Lüscher-Weisz-Wolff (LWW) coupling \cite{5} are compared to values obtained from solutions of the Destri-de Vega (DdV) equation. A more detailed account on the subject is given in \cite{6}.

2. Sine-Gordon theory in finite volume

A complete description of the exact spectrum of the sine-Gordon model in finite volume is provided by the DdV non-linear integral equations. Following \cite{7} we solve the equations iteratively and refer to \cite{6} for further details of this procedure. We are interested in the point where the sine-Gordon model and the $O(2)$ non-linear $\sigma$-model coincide, that is at $\beta_{SG} = \sqrt{8\pi}$. For a chosen volume $ML$ (expressed in units the infinite volume mass gap $M$) we solve the equations for the ground state and for the first excited state hence the LWW coupling

\begin{equation}
M(L) = E_1(L) - E_0(L)
\end{equation}

and hence the LWW coupling

\begin{equation}
\bar{g}^2 = 2M(L)L.
\end{equation}

At the doubled volume a point of the step scaling function

\begin{equation}
\sigma(2,\bar{g}^2) = \bar{g}^2(2L)
\end{equation}

is obtained. A number of such results is given in the second column of table \cite{11}.
### 3. Numerical work

We simulate the two dimensional XY-model with standard action
\[ S = -\beta \sum_{\langle k,l \rangle} \vec{s}_k \cdot \vec{s}_l. \] (4)
on \( L/a \times 5L/a \) lattices. We apply periodic boundary conditions in the spatial direction and free boundary conditions in the temporal direction. As Monte Carlo algorithm we choose the highly efficient single cluster algorithm [8] which does practically not suffer from critical slowing down.

To measure the time slice correlation function
\[ G(\tau) = \langle \vec{S}(t) \cdot \vec{S}(t + \tau) \rangle, \quad \vec{S}(t) = \frac{1}{L} \sum_x \vec{s}(x, t), \] (5)
we employ Hasenbusch’s improved estimator [9]. From the correlation function we extract the finite volume mass gap
\[ M(L)a \xrightarrow{\tau \to \infty} \ln \left[ \frac{G(\tau)}{G(\tau + 1)} \right] \] (6)
and obtain a value of the LWW-coupling \( \bar{g}^2(L) \).

To get one point of the lattice step-scaling function
\[ \Sigma(2, \bar{g}^2(L), a/L) = \bar{g}^2(2L) \] (7)
we keep \( \beta \) fixed and measure the LWW coupling on a lattice of doubled size. The procedure is repeated several times for the same value of the coupling but different lattice resolutions \( a/L \). An extrapolation to \( a/L = 0 \) yields one point of the continuum step-scaling function that can be compared to the predicted value [10]. For that purpose the numerical value of \( ML \) has to be determined first, at which the DdV equations produce the same coupling.

For models with a Kosterlitz-Thouless phase transition [10][11] like the XY-model, J. Balog has predicted the leading lattice artifacts to be universal and to decay very slowly, i.e. proportional to inverse powers of the logarithm of the infinite volume correlation length. Lattice artifacts of the step scaling function have the form
\[ \Sigma(2, \bar{g}^2, a/L) = \sigma(2, \bar{g}^2) + \frac{c}{(\ln \xi + U)^2} + \ldots, \] (8)
where \( \xi \) is the infinite volume correlation length, \( c \) is universal and can be calculated for each volume and \( U \) is a non-universal constant \( (U = 1.3(1) \) for the standard action [12]). Corrections to this formula are of order \( \ln(\xi)^{-1} \).

### 4. Results

We have performed our calculations at four different values of the LWW coupling. Table 1 summarizes our results. The lattice artifacts prediction includes information about the constant \( c \) which is also listed in the table. Fig. 1 corresponds to the second line of the table, plots for the other points look similar. The theoretical predictions are compatible with continuum extrapolated lattice results. The small differences between \( c_{th} \) and \( c_{MC} \) may be explained by subleading cutoff effects in the Monte Carlo data. The knowledge of the form of lattice artifacts was essential to obtain this result to the precision that is reached.

![Figure 1. Comparison of MC-data with a numerical solution of the DdV equation at \( \bar{g}^2 = 1.7865 \). The spatial extents \( L/a \) of the lattices were 10, 20, 40, 80, 120 and 160. The smallest lattice was discarded in the fit.](image-url)
Table 1

At different values of the LWW-coupling theoretical predictions (DdV) for the step-scaling function are compared with numerical results (MC). Also the slope of the fit \( c \) as predicted by theory (th) is compared to the numerical value. The last column lists the \( \chi^2 \) values of the extrapolations.

| \( g^2 \) | \( \sigma_{\text{DMV}}(2, g^2) \) | \( \sigma_{\text{MC}}(2, g^2) \) | \( c_{\text{th}} \) | \( c_{\text{MC}} \) | \( \chi^2 / \text{dof} \) |
|--------|-----------------|-----------------|--------|--------|------------------|
| 3.0038 | 4.3895          | 4.40 ± 0.02     | 2.6176 ± 0.0002 | 2.4 ± 0.6 | 2.51/3            |
| 1.7865 | 1.8282          | 1.829 ± 0.007   | 5.30 ± 0.01     | 4.8 ± 0.5 | 0.73/3            |
| 1.6464 | 1.6515          | 1.657 ± 0.003   | 5.4 ± 0.2       | 4.3 ± 0.3 | 0.35/3            |
| 1.6020 | 1.6029          | 1.608 ± 0.004   | 5.5 ± 1.5       | 4.4 ± 0.5 | 0.90/3            |

5. Conclusions

We have investigated the massive scaling limit of the XY-model by means of the step-scaling function of the LWW coupling and found it to be consistent with continuum predictions based on the equivalence with sine-Gordon theory. A predicted form of lattice artifacts was also confirmed and has been essential to find the agreement. An extrapolation with powers of \( a \) would have led to a significantly different continuum result in spite of a perfectly reasonable looking fit. We would like to recall here that deviations from this Symanzik behavior have also been found in the asymptotically free \( O(3) \) non-linear \( \sigma \)-model where they have not yet been understood [13,14].

Our results fit well into the picture that was drawn in [12], where among other things the renormalized 4-point coupling and 2-point correlation functions in the continuum sine-Gordon model were compared to their lattice counterparts in the XY-model.

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