DEFORMATION THEORY OF COHEN-MACaulay APPROXIMATION

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Abstract. In [20] we established axiomatic parametrised Cohen-Macaulay approximation which in particular was applied to pairs consisting of a finite type flat family of Cohen-Macaulay rings and modules. In this sequel we study the induced maps of deformation functors and deduce properties like smoothness and injectivity under general, mainly cohomological conditions on the module.

1. Introduction

In this article we study local properties of flat families of Cohen-Macaulay approximations by homological methods.

Let \( A \) be a Cohen-Macaulay ring of finite Krull dimension with a canonical module \( \omega_A \). Let \( \text{MCM}_A \) and \( \text{FID}_A \) denote the categories of maximal Cohen-Macaulay modules and of finite modules with finite injective dimension, respectively. M. Auslander and R.-O. Buchweitz proved in [3] that for any finite \( A \)-module \( N \) there exists short exact sequences

\[ 0 \to L \to M \to N \to 0 \]
\[ 0 \to N \to L' \to M' \to 0 \]

with \( M \) and \( M' \) in \( \text{MCM}_A \) and \( L \) and \( L' \) in \( \text{FID}_A \). The maps \( M \to N \) and \( N \to L' \) in (1.0.1) are called a maximal Cohen-Macaulay approximation and a hull of finite injective dimension, respectively, of the module \( N \).

In [20] we noted some of the developments since [3], as the study of new invariants, e.g. [7, 4, 16, 32] and various characterisations and applications [24, 36, 25, 18, 27, 8]. In his book [15] M. Hashimoto gave several new examples of the axiomatic Cohen-Macaulay approximation in [3]. However, the ‘relative’ and continuous aspects have received surprisingly little attention. It seems only [15, IV 1.4.12] and [35] touch upon this.

In [20, 5.1] we proved the following result. Let \( h : S \to A \) be a Cohen-Macaulay map and \( N \) an \( S \)-flat finite \( A \)-module. Then there exists short exact sequences of \( S \)-flat finite \( A \)-modules

\[ 0 \to \mathcal{L} \to \mathcal{M} \to N \to 0 \]
\[ 0 \to N \to \mathcal{L}' \to \mathcal{M}' \to 0 \]

such that the fibres give sequences as in (1.0.1) and any base change gives short exact sequences of the same kind. In the local case there are minimal sequences (1.0.2) which are unique up to non-canonical isomorphisms [20, 6.2]. By Lemma 3.1 there are induced maps of deformation functors of pairs (algebra, module)

\[ \sigma_X : \text{Def}_{(A,N)} \to \text{Def}_{(A,X)} \quad \text{for} \quad X = M, M', L \text{ and } L'. \]

There are corresponding maps \( \text{Def}_N^A \to \text{Def}_{X}^A \) of deformation functors of the modules where the algebra \( A \) only deforms trivially. To our knowledge these maps have not been defined before. They are the principal objects of study in this article.

2010 Mathematics Subject Classification. Primary 13C60, 14B12; Secondary 13D02, 13D10.

Key words and phrases. Cohen-Macaulay map, versal deformation, Artin approximation.
Rather weak conditions imply the injectivity, respectively the formal smoothness of $\sigma_X$, e.g. $\text{Hom}_A(N, M') = 0$, respectively $\text{Ext}^1_A(N, M') = 0$ for $X = L'$. Moreover, if there is a closed subscheme $A$ is cocartesian. The fibre sum is given by the henselisation of the tensor product $\text{Alg}$.

Define the first map is of finite typ and the second is the henselisation in a maximal ideal.

In Corollary 5.12 we prove that $\text{dim}$ with a projective formation $M$ and essential.

Consider a quotient ring $S$ such that the resulting commutative square $\Lambda$.

In Proposition 5.14 we prove that $\sigma_{M}: \text{Def}_{(A, N)} \to \text{Def}_{(A, M)}$ is smooth if $A$ is Gorenstein and $\text{depth} N = \text{dim} A - 1$, extending A. Ishii’s [23, 3.2] to deformations of the pair.

In Section 2 we define the fibred categories and in Section 3 the maps $\sigma_X$. We give some relevant obstruction theory for deforming modules in Section 4. The main results about the maps $\sigma_X$ with some general consequences are found in Section 5. Section 6 concludes after several auxiliary technical results with the proof of Theorem 6.6.

Many results have analogous parts with similar arguments and the policy has been to give a fairly detailed proof of one case and leave the other cases to the reader. All rings are commutative with 1-element. Subcategories are usually full and essential.

2. Preliminaries

2.1. Fibred categories. Fix a finite ring map $A \to k$ where $A$ is assumed to be excellent (in particular noetherian) and $k$ a field. The kernel of $A \to k$ is denoted $m_A$. Put $k_0 = A/m_A$. Define $\mathcal{H}_k$ to be the category of surjective maps of $A$-algebras $S \to k$ where $S$ is a noetherian, henselian, local ring. A morphism is a local ring map of $A$-algebras $S_1 \to S_2$ commuting with the given maps to $k$. A map $h: S \to A$ of local henselian rings is algebraic if $h$ factors as $S \to A^k \to A$ where the first map is of finite typ and the second is the henselisation in a maximal ideal. Define $\text{Alg}$ to be the category where an object is an object $A \to S \to k$ in $\mathcal{H}_k$ together with a map of local henselian rings $S \to A$ which is flat and algebraic. A morphism $(S_1 \to A_1) \to (S_2 \to A_2)$ is a morphism $g: S_1 \to S_2$ in $\mathcal{H}_k$ together with a local $S_1$-algebra map $f: A_1 \to A_2$ such that the resulting commutative square is cocartesian. The fibre sum is given by the henselisation of the tensor product $A = A_1 \otimes_{S_1} S_2$ in the maximal ideal $m_{A_1} A + m_{S_2} A$, denoted by $A_1 \otimes_{S_1} S_2$ or by $(A_1)_{S_2}$. It has the same closed fibre as $S_1 \to A_1$ and it follows that the forgetful $\text{Alg} \to \mathcal{H}_k$ is a fibred category fibred in groupoids. In general a flat and local ring map $S \to A$ will be called Cohen-Macaulay if $A \otimes_S S/m_S$ is a Cohen-Macaulay ring. There is a subcategory $\text{CM}$ in $\text{Alg}$ with objects Cohen-Macaulay maps. The forgetful $\text{CM} \to \mathcal{H}_k$ is a fibred category.
Let $\text{mod}$ denote the category of pairs $(h: S \to \mathcal{A}, \mathcal{N})$ with $h$ in $\text{Alg}$ and $\mathcal{N}$ a finite $\mathcal{A}$-module. A morphism $(h_1: S_1 \to \mathcal{A}_1, \mathcal{N}_1) \to (h_2: S_2 \to \mathcal{A}_2, \mathcal{N}_2)$ in $\text{mod}$ is a morphism $(g: S_1 \to S_2, f: \mathcal{A}_1 \to \mathcal{A}_2)$ in $\text{Alg}$ and an $f$-linear map of modules $\alpha: \mathcal{N}_1 \to \mathcal{N}_2$. Let $(\mathcal{N}_1)_{S_2}$ denote the base change $\mathcal{A}_2 \otimes_{\mathcal{A}_1} \mathcal{N}_1$. The forgetful functor $\text{mod} \to \mathcal{A}_k$ is a fibred category. There is a fibred subcategory $\text{mod}^\text{fl} \subseteq \text{mod}$ of modules flat over the base and a fibred subcategory $\text{MCM} \subseteq \text{mod}^\text{fl}$ where $S \to \mathcal{A}$ is in $\text{CM}$ and $\mathcal{N} \otimes_{\mathcal{A}_k} S$ is a maximal Cohen-Macaulay $\mathcal{A} \otimes_{\mathcal{A}_k} S$-module.

Any local CM map $h: S \to \mathcal{A}$ has a dualising module $\omega_h$ obtained by base change from the dualising module $\omega_{h^n}$ as defined in [6, Sec. 3.5] where $h^n$ is a finite type representative for $h$. In particular, $(h, \omega_h)$ is in $\text{CM}$. Two finite type representatives for $h$ factor through a common étale neighbourhood which is Cohen-Macaulay relative to $S$. The dualising module commutes with base change for finite type $\text{CM}$ maps and so does $\omega_h$. Let $D$ denote the subcategory of $\text{CM}$ of objects $(h, D)$ with $D$ in $\text{Add}(\omega_h)$ and $D^\text{fl}$ the subcategory of $\text{mod}^\text{fl}$ of objects $(h: S \to \mathcal{A}, \mathcal{N})$ such that $h$ is in $\text{CM}$ and $\mathcal{N}$ has a finite resolution by modules in $\text{Add}(\omega_h)$. The forgetful maps make $D$ and $D^\text{fl}$ fibred categories over $\mathcal{A}_k$.

There is also a version for a fixed flat algebra. With $A \to k$ as above, fix a flat ring map $A \to A$ which is the composition of two ring maps $A \to A^\text{fl} \to A$ where the first is of finite type and the second is the henselisation at a maximal ideal. We call such an $A$ a flat and algebraic $A$-algebra. There is a section $\mathcal{A}_k \to \text{Alg}$ given by $S \mapsto (S \to \mathcal{A}_S)$ where $\mathcal{A}_S = \mathcal{A} \otimes_A S$. Let $\text{Alg}^A$ denote the resulting fibred subcategory of $\text{Alg}$ and $\text{mod}^A$, respectively $\text{mod}^A_\text{fl}$, the restriction of the fibred categories $\text{mod}$ and $\text{mod}^\text{fl}$ to $\text{Alg}^A$. Put $A = \mathcal{A} \otimes_A k$. If $A$ is Cohen-Macaulay then the section $\mathcal{A}_k \to \text{Alg}$ factors through $\text{CM}$. Let $\text{MCM}_A$ denote the induced fibred subcategory of $\text{MCM}$.

2.2. Deformation functors. If $A \to k = S \to A$ is an object in $\mathcal{A}_k$ we define $\mathcal{A}_k/S$ as the comma category $\mathcal{A}_k/\mathcal{S}$ of maps to $S$ in $\mathcal{A}_k$. If $h: S \to \mathcal{A}$ is an element in $\text{Alg}$ we define $\text{Def}_{\mathcal{A}/S}$ as the comma category $\text{Alg}/(S \to \mathcal{A})$, i.e. the objects are maps $(U \to V) \to (S \to \mathcal{A})$ in $\text{Alg}$ and morphism are morphisms in $\text{Alg}$ commuting with the maps to $S \to \mathcal{A}$. The objects in $\text{Def}_{\mathcal{A}/S}$ are called deformations of $A$. If $a = (h: S \to \mathcal{A}, \mathcal{N})$ is an object in $\text{mod}^\text{fl}$, we define a deformation of $a$ as a cocartesian map to $a$ in $\text{mod}^\text{fl}$. A map of deformations is a cocartesian map in $\text{mod}^\text{fl}$ commuting with the maps to $a$. Let $\text{Def}_{\mathcal{A}/S}^{\text{fl}}(a)$ denote the resulting category of deformations of $a$. The forgetful functors make $\text{Def}_{\mathcal{A}/S}$ and $\text{Def}_{\mathcal{A}/S}^{\text{fl}}(a)$ categories fibred in groupoids over $\mathcal{A}_k$. Similarly, with a fixed flat and algebraic $A$-algebra $A$ as in the previous subsection, let $\text{Def}_{\mathcal{A}/S}^{\text{fl}}(a)$ denote the restriction of $\text{Def}_{\mathcal{A}/S}^{\text{fl}}(a)$ to $\text{Alg}^A/(S \to \mathcal{A}_S)$.

Let the deformation functors $\text{Def}_{\mathcal{A}/S}$, $\text{Def}_{\mathcal{A}/S}^{\text{fl}}(a)$ and $\text{Def}_{\mathcal{A}/S}^{\text{fl}}$ from $\mathcal{A}_k$ to $\text{Sets}$ be the functors corresponding to the associated groupoids of sets obtained by identifying all isomorphic objects in the fibre categories and identifying arrows accordingly. If $S = k$ we write $\text{Def}_A$ for $\text{Def}_{\mathcal{A}/k}$ and so on.

2.3. Linear approximation. A proof of the following known result is provided in [20, 6.1].

**Lemma 2.1.** Let $S \to \mathcal{A}$ be a homomorphism of noetherian rings and $a$ an ideal in $S$ such that $I = a^2$ is contained in the Jacobson radical of $\mathcal{A}$. Let $M$ and $N$ be finite $\mathcal{A}$-modules. Let $A_n = A/I^{n+1}$, $M_n = A_n \otimes M$ and $N_n = A_n \otimes N$. Suppose there exists a tower of surjections $\{\varphi_n: M_n \to N_n\}$. Fix any non-negative integer $n_0$. Then there exists an $\mathcal{A}$-linear surjection $\psi: M \to N$ such that $\varphi_{n_0} \psi = \varphi_{n_0}$. If the $\varphi_n$ are isomorphisms and $N$ is $S$-flat then $\psi$ is an isomorphism.
3. Cohen-Macaulay approximation of deformations

We extend the Cohen-Macaulay approximation over henselian local base rings given in [20, 5.7] to deformations.

For each object \(a_v = (h_v : S_v \rightarrow A_v, N_v)\) in \(\text{mod}^\text{fl}\) with \(h_v\) in CM we fix a minimal MCM-approximation and a minimal \(\mathcal{D}^0\)-hull

\[
(3.0.1) \quad \pi_v: 0 \rightarrow \mathcal{L}_v \rightarrow \mathcal{M}_v \xrightarrow{\pi_v} N_v \rightarrow 0 \quad \text{and} \quad \iota_v: 0 \rightarrow N_v \xrightarrow{\iota_v} \mathcal{L}_v' \rightarrow \mathcal{M}_v' \rightarrow 0
\]

which exist by [20, 5.7, 6.3]. For each deformation \(a_v \rightarrow a\) we choose extensions to commutative diagrams of deformations

\[
(3.0.2) \quad \mathcal{L}_v \xrightarrow{\lambda_v} \mathcal{M}_v \xrightarrow{\pi_v} N_v \quad \text{and} \quad \mathcal{N}_v \xrightarrow{\iota_v} \mathcal{L}_v' \xrightarrow{\pi_v'} \mathcal{M}_v' \quad \text{as follows: By [20, 6.3] a base change of } \pi_v \text{ by } S_v \rightarrow S \text{ gives a minimal MCM-approximation } \mathcal{M}_v \otimes_{S_v} S \rightarrow \mathcal{N}_v \otimes_{S_v} S \xrightarrow{\cong} N. \text{ By minimality it is isomorphic to } \pi. \text{ Choose an isomorphism. Let } \mu \text{ be the composition } \mathcal{M}_v \rightarrow \mathcal{M}_v \otimes_{S_v} S \xrightarrow{\cong} \mathcal{M} \text{. It is} \text{ cocartesian. Do similarly for the } \mathcal{D}^0\text{-hull. Let these choices be fixed.}
\]

Lemma 3.1. There are four maps

\[
\sigma_X : \text{Def}_{(A/S, N)} \rightarrow \text{Def}_{A/S, X} \text{ of functors } A \text{H}_S \rightarrow \text{Sets}
\]

where \(X\) can be \(M, L, L'\) and \(M'\) given by \([(h_v \rightarrow h, \nu)] \mapsto [(h_v \rightarrow h, x)]\) for \(x\) equal to \(\mu, \lambda, \lambda'\) and \(\mu'\) in (3.0.2) respectively.

For a flat and algebraic \(A\)-algebra \(A\) the same formulas induces well-defined maps of deformation functors of \(A\)-modules

\[
\sigma^A_X : \text{Def}^A_{N} \rightarrow \text{Def}^A_X.
\]

The following lemma implies that these maps are well defined and independent of choices and thus proves Lemma 3.1.

Lemma 3.2. Given two deformations

\[
((f_j, g_j), \nu_j) : (h_{v_j} : S_{v_j} \rightarrow A_{v_j}, N_{v_j}) \rightarrow (h_j : S_j \rightarrow A_j, N_j), \quad j = 1, 2,
\]

in \(\text{mod}^\text{fl}\) over CM. Consider the minimal MCM-approximations \(\pi_{v_j}\) and \(\pi_j\) (respectively the \(\mathcal{D}^0\)-hulls \(\iota_{v_j}\) and \(\iota_j\)) defined in (3.0.1) and the corresponding maps of short exact sequences \(\pi_{v_j} \rightarrow \pi_j\) (respectively \(\iota_{v_j} \rightarrow \iota_j\)) which extends \(\nu_j\) defined in (3.0.2).

Given

- a map \((f, g) : h_1 \rightarrow h_2\) in CM and an \(f\)-linear map \(\alpha : N_1 \rightarrow N_2\),
- maps of short exact sequences \(\pi_1 \rightarrow \pi_2\) and \(\iota_1 \rightarrow \iota_2\) which extends \(\alpha\),
- a map \((\tilde{f}, \tilde{g}) : h_{v_1} \rightarrow h_{v_2}\) in CM which lifts \((f, g)\), and
- an \(\tilde{f}\)-linear map \(\tilde{\alpha} : N_{v_1} \rightarrow N_{v_2}\) which lifts \(\alpha\).

In particular the following two diagrams of solid arrows are commutative:
Then there exists \( f \)-linear maps \( \gamma: M_{i_1} \to M_{i_2} \) and \( \gamma': L'_{i_1} \to L'_{i_2} \) such that the induced diagrams are commutative. If \( \tilde{\alpha} \) is cocartesian, so are \( \gamma \) and \( \gamma' \).

**Proof.** Consider the MCM-approximation case. By applying base changes to the front diagram, we can reduce the problem to the case \( h_{i_1} \to h_1 \) equals \( h_{i_2} \to h_2 \).

Then, by \([20, 5.7]\), there is a lifting \( \gamma_1: M_{i_1} \to M_{i_2} \) of \( \tilde{\alpha} \). We would like to adjust \( \gamma_1 \) so that it lifts \( \beta \) too. We have that \( \mu_2 \gamma_1 - \beta \mu_1 \) factors through \( L_2 \) by a map \( \tau: M_{i_2} \to L_2 \). It induces a unique map \( \overline{\tau}: M_1 \to L_2 \) since \( \mu_1 \) is cocartesian. If \( D_* \) is a finite \( D \)-resolution, then base change gives a finite \( D \)-resolution \( D_0 \otimes_S S_2 \to L_2 \) and \( \tau \) factors through a \( \overline{\sigma}: M_1 \to D_0 \otimes_S S_2 \) by \([20, 5.7]\). Since \( \text{Hom}(M_{i_1}, D_0) \) is a deformation of \( \text{Hom}(M_1, D_0 \otimes_S S_2) \) (cf. \([20, 2.4]\)) there is a \( \sigma: M_{i_1} \to D_0 \) lifting \( \overline{\sigma} \). Subtracting the induced map \( M_{i_1} \to M_{i_2} \) from \( \gamma_1 \) gives our desired \( \gamma \). If \( \alpha \) is an isomorphism so is \( \gamma \) by minimality of the approximations \( \pi_{i_1} \). The argument for the \( D^0 \)-case is similar.

\( \square \)

4. **Obstructions**

We summarise some obstruction theory for deformations of modules which will be used to study the maps in Lemma 3.1.

**Definition 4.1.** Suppose \( \beta: 0 \to J \to B' \to B \to 0 \) is an extension of rings with \( J^2 = 0 \), \( N \) and \( J_1 \) are \( B \)-modules, and \( \varphi: N \otimes J \to J_1 \) a \( B \)-linear isomorphism. Then a short exact sequence of \( B' \)-modules \( \nu: 0 \to J_1 \to N' \to N \to 0 \) with induced map \( N \otimes J \to J_1 \) equal to \( \varphi \) is called a lifting of \( N \) along \( \beta \) (or to \( B' \)). Two liftings \( \nu_1 \) and \( \nu_2 \) along \( \beta \) are equivalent if they are isomorphic as extensions.

Note that \( N \otimes J \cong J_1 \) implies \( N' \otimes B \cong N \) and \( \text{Ext}^1_B(N', B) = 0 \) and vice versa. There is an obstruction theory for liftings of modules in terms of \( \text{Ext} \) groups.

**Proposition 4.2.** Given an extension \( \beta \), a \( B \)-module \( N \), and an isomorphism \( N \otimes J \cong J_1 \) as in Definition 4.1.

(i) There is an element \( \text{ob}(\beta, N) \in \text{Ext}^2_B(N, N \otimes J) \) such that \( \text{ob}(\beta, N) = 0 \) if and only if there exists a lifting of \( N \) along \( \beta \).

(ii) If \( \text{ob}(\beta, N) = 0 \) then the set of equivalence classes of liftings of \( N \) along \( \beta \) is a torsor for \( \text{Ext}^1_B(N, N \otimes J) \).

(iii) The set of automorphisms of a given lifting \( \nu \) is canonically isomorphic to \( \text{Hom}_B(N, N \otimes J) \).

**Proof.**

(i) Pick a \( B \)-free resolution of \( N: \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\delta_0} N \to 0 \). Lift the differentials to maps \( \delta_n: \tilde{F}_n \to \tilde{F}_{n-1} \) of \( B' \)-free modules of the same rank. Denote the map \( \tilde{F} \to F \) by \( \pi \). Then \( \delta_1 d_2 \) is induced by a map \( \eta_2: F_2 \to F_0 \otimes J \) and \( \eta_2 := (\varepsilon \otimes \text{id}) \eta_2 \) is a 2-cocycle in the complex \( \text{Hom}_B(F, N \otimes J) \) since \( \eta_2 d_3 \) equals \( (d_1 \otimes \text{id}) \eta_3 \) where \( \eta_3: F_3 \to F_1 \otimes J \) is inducing \( d_2 \). The class of \( \eta_2 \) defines \( \text{ob}(\beta, N) \). It is independent of the chosen resolution and liftings.

If there is a lifting \( N' \) of \( N \) along \( \beta \) we can choose a \( B' \)-free resolution \( F' \) of \( N' \) first. Then \( \text{Ext}^2(B \otimes F') \cong N \) and \( \text{Ext}^2(B \otimes F) = 0 \). A \( B \)-free resolution \( F' \) of \( N \) is obtained by adding terms in degree \( \geq 3 \). It follows that \( \text{ob}(\beta, N) = 0 \).

Suppose \( \text{ob}(\beta, N) = 0 \). Then there is a \( \xi: F_1 \to N \otimes J \) with \( \xi := \xi \pi \). Let \( \xi_1: F_1 \to F_0 \otimes J \) be a lifting of \( \xi \) and let \( \iota \) denote the inclusion \( F \otimes J \to \tilde{F} \). Let \( \xi_2: F_2 \to F_1 \otimes J \) be a lifting of \( \iota \xi_1 \). Then \( (d_1 - \iota \xi_1 \pi)(d_2 - \iota \xi_2 \pi) = 0 \) which implies that \( N' := \text{coker}(d_1 - \xi_1 \pi) \) with its natural map to \( N \) gives a lifting of \( N \) along \( \beta \).

(ii) Given two liftings \( N_1' \) and \( N_2' \) of \( N \) along \( \beta \). By what we did above there are maps \( d_{i,2}, d_{i,1}: N_i' \to N_{i-1}' \) for \( i = 1, 2 \) such that \( B \otimes_B d_{i,1} = d_i \). Then \( d_{i,2} - d_{i,1} \) equals \( \xi_i \pi \) for some \( \xi_i \). One calculates

\[
(4.2.1) \quad \iota(\xi_1 d_2 + (d_1 \otimes \text{id}) \xi_2) \pi = (\iota \xi_1 \pi) d_{2,2} + d_{1,1}(\iota \xi_2 \pi) = 0
\]
The obstruction element \( \text{ob}(\beta,N) \) is called the obstruction of \((\beta,N)\).

**Lemma 4.3.** Given a map of extensions

\[
\begin{array}{ccc}
\rho: & 0 & \rightarrow H \\
\downarrow & & \downarrow \\
\sigma: & 0 & \rightarrow I
\end{array}
\]

\[
\begin{array}{ccc}
& R' & \rightarrow R \\
\downarrow & & \downarrow \\
& S' & \rightarrow S
\end{array}
\]

in \( \mathcal{A}_k \) with \( H^2 = 0 = I^2 \). Suppose \( \beta: 0 \rightarrow J \rightarrow B' \rightarrow B \rightarrow 0 \) is an extension of deformations of algebras over \( \rho \) and let \( \beta_{S'} \) denote the base change of \( \beta \) along \( R' \rightarrow S' \). Suppose \( N \) is an \( R \)-flat \( B \)-module. Then:

(i) Up to natural isomorphisms \( \beta_{S'} \) is the extension of deformations

\[
0 \rightarrow B_S \otimes_S I \rightarrow B'_{S'} \rightarrow B_S \rightarrow 0.
\]

(ii) There is a natural isomorphism

\[
\text{Ext}^n_{B_S}(N_S,N_S \otimes_{B_S}(B_S \otimes_S I)) \cong \text{Ext}^n_B(N,N_S \otimes_S I)
\]

for all \( n \).

(iii) The obstruction element \( \text{ob}(\beta,N) \) maps to \( \text{ob}(\beta_{S'},N_S) \) along the map

\[
\tau^2: \text{Ext}^2_B(N,N \otimes_R H) \rightarrow \text{Ext}^2_B(N,N_S \otimes_S I)
\]

induced by the natural map \( \tau: N \otimes_R H \rightarrow N_S \otimes_S I \).

(iv) The torsor action commutes with base change: If \( N' \rightarrow N \) is a deformation of \( N \) along \( \beta \) and \( \xi \in \text{Ext}^1_B(N,N \otimes_R H) \) then the base change of the deformation \( (N' + \xi) \rightarrow N \) along \( R' \rightarrow S' \) is equivalent to \( (N'_S + \tau^2(\xi)) \rightarrow N_S \).

**Proof.** (i) Note that the base change \( \beta_{S'} \) equals \( B'_{S'} \otimes_{S'}(I \rightarrow S' \rightarrow S) \) which gives a short exact sequence. Moreover, \( B'_{S'} \otimes_{S'} I \cong B_S \otimes_S I \). Since \( N_S \otimes_{B_S}(B_S \otimes_S I) \cong N_S \otimes_S I \), (ii) follows by a change of rings spectral sequence (cf. [20, 2.3.1]).

(iii) If \( (F,d) \rightarrow N \) is a \( B \)-free resolution of \( N \), then the base change \( (F_S,d_S) \rightarrow N_S \) is a \( B_S \)-free resolution of \( N_S \). Then the base change \( \tilde{d}_{S'} \) of a lifting \( \tilde{d} \) of the differential \( d \) is a lifting of the differential \( d_{S'} \). The obstruction \( \text{ob}(\beta_{S'},N_S) \) is induced by \( (\tilde{d}_{S'})^2 \) which equals \( (\tilde{d}^2)_{S'} \), i.e. the base change of the map which induces \( \text{ob}(\beta,N) \). The map in (ii) then takes \( \text{ob}(\beta_{S'},N_S) \) to \( \tau^2 \text{ob}(\beta,N) \). (iv) is similar. □

If \( m_R H = 0 \) then \( \text{Ext}^n_B(N,N \otimes_R H) \cong \text{Ext}^n_{B \otimes_R k}(N \otimes_R k,N \otimes_R k) \otimes_k H \), i.e. in this case there are fixed \( k \)-vector spaces which classify obstruction and give the torsor action.

**Lemma 4.4.** Given an extension \( \beta \), a \( B \)-module \( N \), and an isomorphism \( N \otimes J \cong J_1 \) as in Definition 4.1. Suppose \( \varepsilon: (F,d) \rightarrow N \) is a \( B \)-free resolution of \( N \). Put \( N'_1 = \ker \varepsilon \), \( N'_1 = B \otimes_B N'_1 \) and \( F = B \otimes_B F \). Applying \( B \otimes_B - \) to the short exact sequence \( 0 \rightarrow N'_1 \rightarrow F_0 \rightarrow N \rightarrow 0 \) gives a 4-term exact sequence

\[
0 \rightarrow N \otimes B J \rightarrow N'_1 \rightarrow F_0 \rightarrow N \rightarrow 0
\]

which represents \( \text{ob}(\beta,N) \) in \( \text{Ext}^2_B(N,N \otimes J) \).

**Proof.** The 4-term exact sequence is obtained since \( \text{Tor}^1_B(N,B) \cong N \otimes B J \). Choose a surjection \( \gamma: E \rightarrow J \) where \( E \) is \( B' \)-free. Since \( NJ = 0 \), the composition with
the multiplication map $F_0 \otimes E \to F_0 \otimes J \to F_0$ factors through a $B'$-linear map
\[ \psi: F_0 \otimes E \to F_1. \]

Similarly, let $\tilde{\gamma}, \tilde{\eta}$ be given by $\tilde{\gamma}: F_0 \otimes E \to N \otimes J$. Since the upper row in the commutative diagram
\[ (4.4.1) \]
\[ \begin{array}{ccccccc}
F_0 & \stackrel{\gamma}{\longrightarrow} & F_1 & \longrightarrow & F_2 & \ddots & \\
\tilde{\gamma}, & \tilde{\eta} & \tilde{\eta} & \tilde{\eta} & \tilde{\eta} & \tilde{\eta} & \\
0 & \longrightarrow & N & \longrightarrow & \tilde{N} & \longrightarrow & 0 \end{array} \]
gives a $B'$-free 2-presentation of $N$; cf. [19, Lemma 3]. Following the proof of Proposition 4.2, $\tilde{\eta}$ can be given by $\tilde{\gamma}: F_0 \otimes E \to N \otimes J$. Since the upper row in the commutative diagram
\[ (4.4.2) \]
\[ \begin{array}{ccccccc}
F_0 & \stackrel{\gamma}{\longrightarrow} & F_1 & \longrightarrow & F_2 & \ddots & \\
\tilde{\gamma}, & \tilde{\eta} & \tilde{\eta} & \tilde{\eta} & \tilde{\eta} & \tilde{\eta} & \\
0 & \longrightarrow & N & \longrightarrow & \tilde{N} & \longrightarrow & 0 \end{array} \]
is the beginning of a $B'$-free resolution of $N$, $\tilde{\gamma}$ also defines the image of the 4-term exact sequence in $\text{Ext}^2_B(N, N \otimes J)$.

**Lemma 4.5.** Let $k$ be a field and $A$ a local algebraic $k$-algebra. Given a small surjection $p: R \to S$ in $\mathcal{M}_k$ and a deformation $R \to A$ of $k \to A$. Put $q = \text{id} \otimes 1: A \to A \otimes k S = A_S$ and $I = \ker p$. Given short exact sequences of finite $A_S$-modules which are deformations mapping to short exact sequences of $A$-modules:

\[ 0 \longrightarrow N \longrightarrow L' \longrightarrow M' \longrightarrow 0 \]
\[ 0 \longrightarrow N \longrightarrow \tilde{N} \longrightarrow \tilde{M} \longrightarrow 0 \]
\[ and \]
\[ 0 \longrightarrow L \longrightarrow M \longrightarrow \pi_0 N \longrightarrow 0 \]
\[ 0 \longrightarrow L \longrightarrow \rho_0 M \longrightarrow \pi_0 N \longrightarrow 0 \]

Then:

(i) $\iota_* \text{ob}(q, N) = \iota_* \text{ob}(q, L')$ in $\text{Ext}^2_A(N, L') \otimes k I$

(ii) $\pi_* \text{ob}(q, N) = \pi_* \text{ob}(q, M)$ in $\text{Ext}^2_A(M, N) \otimes k I$

Furthermore, assume we have short exact sequences of finite $A$-modules:

\[ 0 \longrightarrow \tilde{N}_i \longrightarrow \tilde{L}_i \longrightarrow \tilde{M}_i \longrightarrow 0 \]
\[ for \ i = 1, 2 \text{ which are deformations mapping to the upper sequences above. Let } \delta, \zeta \text{ and } \xi \text{ denote the differences of the deformations } \tilde{N}_i, \text{ the } \tilde{L}_i \text{ and the } \tilde{M}_i \text{ respectively (cf. Proposition 4.2). Then:} \]

(iii) $\iota_* \delta = \iota_* \zeta$ in $\text{Ext}^1_A(N, L') \otimes k I$ and $\pi_* \delta = \pi_* \zeta$ in $\text{Ext}^1_A(M, N) \otimes k I$

**Proof.** (i) Let $(F, d)$ be an $A_S$-free resolution of $N$ and put $F := F \otimes_S k$ which is an $A$-free resolution of $N$:

\[ (4.5.1) \]
\[ \begin{array}{ccccccc}
0 & \longrightarrow & N & \longrightarrow & A_{S_0} & \longrightarrow & A_{S_1} & \longrightarrow & A_{S_2} & \longrightarrow & \cdots \\
\epsilon & \longrightarrow & d_1 & \longrightarrow & d_2 & \longrightarrow & d_3 & \longrightarrow & \cdots \\
0 & \longrightarrow & N & \longrightarrow & A_{0} & \longrightarrow & A_{1} & \longrightarrow & A_{2} & \longrightarrow & \cdots \\
\epsilon & \longrightarrow & d_1 & \longrightarrow & d_2 & \longrightarrow & d_3 & \longrightarrow & \cdots \\
\end{array} \]

Similarly, let $(G, d')$ be a free resolution of $M'$ and put $G = G \otimes_S k$. Then one can make $F \oplus G$ a free resolution of $L'$ with a differential of the form $(\begin{array}{cc} d' & 0 \\ \epsilon & d \end{array})$. To find the obstruction we lift the differentials to maps of free $A$-modules: $d_1: A^0 \to A_{S_1}$ lifts $d_1$ and so on. Then the obstruction for lifting $N$ to $A$ is induced by $d^2$ which factors through a degree two cocycle $a: F \to F \otimes k I$ in the Yoneda complex $\text{End}_A(F) \otimes k I$ which represents $\text{ob}(q, N')$.

In the case of $L'$ the obstruction is induced by $d^2$ which factors through a degree two cocycle $(\begin{array}{cc} d & 0 \\ \epsilon & d' \end{array})$ in $\text{End}_A(G \oplus F) \otimes k I$.
which represents $\text{ob}(q, \mathcal{L}')$. Since $i$ is represented by the inclusion of resolutions $F \to G \oplus F$ we find that $\xi \alpha = \left( \begin{smallmatrix} 0 & 0 \\ 0 & \xi \end{smallmatrix} \right) = \iota^*(\begin{smallmatrix} 0 & 0 \\ 0 & \xi \end{smallmatrix})$ which in cohomology gives $i_* \text{ob}(q, \mathcal{N}) = \iota^* \text{ob}(q, \mathcal{L}')$. A similar argument gives (ii).

(iii), first part: We can assume that $\tilde{\mathcal{L}}_i$ has a resolution with differential $\frac{\partial}{\partial z_i ^* \sqrt{a}}$ for $i = 1, 2$ lifting the resolution of $\mathcal{L}'$ given above. Then the difference of the two differentials factors through a degree one cocycle $\left( \begin{smallmatrix} 0 & 0 \\ 0 & \xi \end{smallmatrix} \right)$ in $\text{End}_\mathbb{R}(G \oplus F) \otimes \mathbb{K} I$ which represents $\xi$. Then the rest is analogous to (ii). The second part is similar. \hfill $\square$

5. Maps of deformation functors induced by Cohen-Macaulay approximation

After two lemmas relating to the Schlessinger-Rim conditions in Artin’s [2] we state several results about various maps of deformation functors induced by Cohen-Macaulay approximation.

For any fibred category $\mathcal{F}$ over $\mathcal{A}_H_k$ (or over the subcategory $\mathcal{A}_k$ of artin rings) we will in the following assume that $\mathcal{F}(k)$ is equivalent to a one-object, one-morphism category. Furthermore, for all maps $f: R \to S$ in $\mathcal{A}_H_k$ and for all objects $a$ in $\mathcal{F}(R)$ we choose a push forward $f_* a$ in $\mathcal{F}(S)$. Let $F \equiv F$ denote the functor associated to $\mathcal{F}$.

**Definition 5.1.** Assume that $\mathcal{F}$ and $\mathcal{G}$ are fibred categories over $\mathcal{A}_H_k$ which are locally of finite presentation. A map $\varphi: \mathcal{F} \to \mathcal{G}$ is smooth (formally smooth) if, for all surjections $f: S' \to S$ in $\mathcal{A}_H_k$ (respectively in $\mathcal{A}_k$), the natural map

$$ (f_*, \varphi(S')): F(S') \to F(S) \times_{G(S)} G(S') $$

is surjective. Put $h_R = \text{Hom}_{\mathcal{A}_H_k}(R, -)$. Let $v$ be an object in $\mathcal{F}(R)$ and let $c_v: h_R \to F$ denote the corresponding Yoneda map. If $R$ is algebraic as $A$-algebra and $c_v$ is smooth (an isomorphism) then $v$ is versal (respectively universal). Moreover, $v$ (or a formal element $v = \{v_n\}$ in $\lim_{\leftarrow} F(R/\mathfrak{m}_R^{n+1})$) is formally versal if $c_v$ restricted to $\mathcal{A}_k$ is formally smooth.

**Definition 5.2.** Suppose $\mathcal{F} \to \mathcal{A}_H_k$ is a fibred category satisfying the Schlessinger-Rim condition (S1') in [2, 2.2] with associated functor $F$. Let $a$ be an object in $\mathcal{F}(S)$ and $I$ a finite $S$-module. Let $\mathcal{F}_a(S[I])$ denote the groupoid of maps $a' \to a$ above the projection $p: S[I] \to S$ and let $D^F_a(I)$ denote the $S$-module of isomorphism classes in $\mathcal{F}_a(S[I])$. Define the condition on $F$:

$$(S2) \quad D^F_a(I) \text{ is a finite } S\text{-module}$$

for all reduced $S$ in $\mathcal{A}_H_k$, objects $a$ and finite $S$-modules $I$.

If $A$ is a local algebraic $k$-algebra and $N$ a finite $A$-module then by standard arguments $\text{Def}_{(A,N)}$ is locally of finite presentation and satisfies (S1'); cf. [22, 4.1], and likewise for $\text{Def}^A_N$ where $A$ is a flat and algebraic $A$-algebra.

**Lemma 5.3.** Suppose $F$ satisfies (S1') and has a versal object $v$ in $\mathcal{F}(R)$. Then $F$ satisfies (S2).

**Proof.** We use the assumptions in Definition 5.2. By versality there is a $g$ in $h_R(S)$ with $g_* v \cong a$ in $\mathcal{F}(S)$. If $a'$ is a lifting of $a$ along $p$ then there is a $g'$ lifting $g$ with $g'_* v \cong a'$ by versality. I.e. the $S$-linear map $D^F_{g'}(I) \to D^F_a(I)$ is surjective. Now $D^F_{g'}(I) \cong \text{Hom}_R(\Omega_{R/A}, I)$. Since $R$ is algebraic, $\Omega_{R/A}$ is a finite $R$-module and so is $D^F_a(I)$. \hfill $\square$

Let $A$ be a Cohen-Macaulay local algebraic $k$-algebra and $N$ a finite $A$-module. Fix a minimal MCM$_A$-approximation $0 \to L \to M \to N \to 0$ and a minimal $\tilde{\mathcal{D}}_A$-hull $0 \to N \to L' \to M' \to 0$. 
Lemma 5.4. Suppose (S2) holds for Def\textsubscript{A(N,N)}.

(i) If Ext\textsubscript{1}^A(N,M') = 0 then (S2) holds for Def\textsubscript{A(L,L')}.

(ii) If Ext\textsubscript{1}^A(L,N) = 0 then (S2) holds for Def\textsubscript{A(M,M')}.

Proof. (i) We use the assumptions in Definition 5.2. Suppose \( \mathcal{M} \neq 0 \). As Ext\textsubscript{1}^A(M',L') = 0, base change theory implies that Ext\textsubscript{1}^A(M',L' \otimes I) = 0; cf. [28, 5.1]. Then the natural map \( \sigma_L^1 : Ext\textsubscript{1}^A(M',L' \otimes I) \rightarrow Ext\textsubscript{1}^A(M',L' \otimes I) \) is an isomorphism. Composing the surjection Ext\textsubscript{1}^A(N,N \otimes I) \rightarrow Ext\textsubscript{1}^A(N,L' \otimes I) with the inverse of \( \sigma_L^1 \) gives a natural map \( \eta^1 : Ext\textsubscript{1}^A(N,N \otimes I) \rightarrow Ext\textsubscript{1}^A(L',L' \otimes I) \). Base change theory and the assumption implies as above that Ext\textsubscript{1}^A(N,M' \otimes I) = 0. From the long exact sequence it follows that \( \eta^1 \) is surjective and \( \eta^2 \) is injective.

Put \( D_0(I) = Def\textsubscript{A(S)}(S[I]) \) and \( b = (A/S, L') \). Then, by [22, 2.10], there is a natural map of exact sequences of \( A \)-modules:

\[
\begin{array}{c}
\text{Ext}^1_A(N,N \otimes I) \\
\downarrow \eta^1 \\
\text{Ext}^1_A(L',L' \otimes I) \\
\downarrow \delta \\
\text{Ext}^2_A(N,N \otimes I) \\
\downarrow \eta^2
\end{array}
\]

By a diagram chase it follows that \( \delta \) is surjective and (S2) holds for Def\textsubscript{A(L,L')}.

Similarly for (ii).

Theorem 5.5. Consider the map \( \sigma_L^1 : Def\textsubscript{A(N,N)} \rightarrow Def\textsubscript{A(L,L')} \) in Lemma 3.1.

(i) If Hom\textsubscript{A}(N,M') = 0 then \( \sigma_L^1 \) is injective.

(ii) If Ext\textsubscript{1}^A(N,M') = 0 then \( \sigma_L^1 \) is formally smooth.

(iii) Suppose Def\textsubscript{A(N,N)} has a versal element \( v = (R \rightarrow \mathcal{A}, \mathcal{N}) \) and Ext\textsubscript{1}^A(N,M') = 0. Then \( (R \rightarrow \mathcal{A}, \sigma_L^1(\mathcal{N})) \) is a versal element for Def\textsubscript{A(L,L')} and \( \sigma_L^1 \) is smooth.

Analogous statements hold for \( \sigma_L^2 : Def\textsubscript{A(L,L')} \rightarrow Def\textsubscript{A(M,M')} \) with Ext\textsubscript{1}^A(N,M) = 0 in (i) and Ext\textsubscript{2}^A(N,M) = 0 in (ii)-(iii).

Example 5.6. If grade \( N \geq n + 1 \) then Ext\textsubscript{4}^A(N,M) = 0 for all \( i \leq n \) and any \( M \) in MCM\textsubscript{A}.

Proof. (i) Given \( S \) in \( \mathcal{A}\mathcal{H}_k \) and deformations \( (h : S \rightarrow \mathcal{A}, \mathcal{N}) \) of \( (A,N) \) to \( S \) for \( i = 1, 2 \). Assume that the images \( (h : \mathcal{L}', \mathcal{N}) \) under \( \sigma_L^1 \) are isomorphic, identify \( h : S \rightarrow \mathcal{A} \) with \( \bar{h} : S \rightarrow \mathcal{A} = \mathcal{A} \), and let \( \beta : \mathcal{L}' \rightarrow \mathcal{L} \). Since \( \beta : \mathcal{L}' \rightarrow \mathcal{L} \) is a homomorphism, \( Hom\textsubscript{A}(N,M') = 0 \). By Proposition 4.2 the \( \mathcal{N}' \) are isomorphic to \( \mathcal{N}_n \). Since \( \beta \) is injective by assumption, \( \gamma = 0 \). By Proposition 4.2 the \( \mathcal{N}_n \) are isomorphic by an isomorphism \( \alpha_n \) compatible with \( \alpha_{n-1} \). Then \( \beta_n^{-1} \gamma_n - \gamma_n \alpha_n \) by the induction hypothesis factors through a \( \delta_n : N \rightarrow \mathcal{L}_n \otimes I \) which (since \( Hom\textsubscript{A}(N,M') = 0 \)) factors through a map \( \eta : N \rightarrow N \otimes I \). Adding the map induced from \( \eta \) to \( \alpha_n \) gives \( \alpha_n \) which commutes with \( \beta_n \).

(ii) Let \( S \rightarrow \mathcal{A} \) in \( \mathcal{A}\mathcal{H}_k \) be surjective with kernel \( I \), \( b = (h : S \rightarrow \mathcal{A}, \mathcal{L}') \) a deformation of \( (A,L') \) to \( S \) and let \( b = (\bar{h} : S \rightarrow \mathcal{A}, \mathcal{N}) \) denote the base change of \( b \) to \( S \). Suppose there is a deformation \( (h^* : S \rightarrow \mathcal{A}, \mathcal{N}') \) of \( (A,N) \) which \( \sigma_L^1 \) maps to \( b \). As above we can assume that \( h^* = \bar{h} \). By induction on the length of \( S \) we can assume that \( I \cdot m_S = 0 \). By Lemma 4.5, \( \text{ob}(A \rightarrow \mathcal{A}, \mathcal{N}) \) maps to \( \text{ob}(A \rightarrow \mathcal{A}, \mathcal{N}) \) under \( \text{Ext}^3(N,N) \otimes I \rightarrow \text{Ext}^3(L',L') \otimes I \) which by the assumption is injective. Since \( \mathcal{L}' \) lifts \( \mathcal{L} \) to \( \mathcal{A} \), \( \text{ob}(A \rightarrow \mathcal{A}, \mathcal{N}) = 0 \). By Proposition 4.2 there exists a lifting \( \mathcal{N} \) of \( \mathcal{N} \).
to $\mathcal{A}$. Put $\mathcal{L}' = \sigma_L(\mathcal{N})$. The difference of $\mathcal{L}'$ and $\mathcal{L}$ gives a $\theta \in \Ext^1_A(L', L') \otimes I$. By assumption $\Ext^1_A(N, N) \otimes I$ maps surjectively to $\Ext^1_A(L', L') \otimes I$ and a lifting of $\theta$ perturbs $\mathcal{N}$ to a lifting $\mathcal{N}'$ with $\sigma_L(\mathcal{N}') = \mathcal{L}'$ by Lemma 4.5.

(iii) By Lemma 5.3, the versality of $\nu$ implies (S2) for $\Def_{(A, N)}$. Then (S2) follows for $\Def_{(A, L')}$ by Lemma 5.4. Put $\mathcal{L}' = \sigma_L(\mathcal{N})$ and $\nu' = (R \to \mathcal{A}, \mathcal{L}')$. By (ii), $\nu'$ is formally versal. To test $\nu'$ for versality, let $S \to S_0$ in $\mathcal{A}_{H_k}$ be surjective with kernel $I$ and $b_0 = (b_0: S_0 \to A_0, L_0')$ a deformation of $(A, L')$ to $S_0$ induced from $\nu'$ by a map $f_0: R \to S_0$. Let $b = (b: S \to A, L')$ be a lifting of $b_0$ to $S$. Put $S_n = S/I^n$ and $b_n = b_{S_n}$. As noted by H. van Essen [34, p. 416], H. Flenner's [10, 3.2] (where (S2) is needed) implies that a lifting $f: R \to S$ of $f_0$ with $f_*\nu' \cong b$ above $b_0$ exists in the case $I$ is nilpotent; cf. [22, 3.3]. This implies that we can find a projective system of maps $\{f_n: R \to S_n\}$ and isomorphisms $\{(f_n)_*\nu' \cong b_n\}$. Let $\tilde{f}: R \to \lim S_n =: S'$ denote the induced map. The isomorphism $\lim S_n \cong \lim S_n$ implies that the completions in maximal ideals are isomorphic too; $\mathcal{A}' \cong \mathcal{A}$. Any $S$ in $\mathcal{A}_{H_k}$ is a direct limit of a filtering system of algebraic $A$-algebras in $\mathcal{A}_{H_k}$. Since $\Def_{(A, L')}$ is locally of finite presentation it is sufficient to prove the lifting property for $S$ algebraic. Since $A$ is excellent, so is $S$ by [12, 7.8.3] and [13, 18.7.6].

An analogous proof gives:

**Theorem 5.7.** Consider the map $\sigma_M: \Def_{(A, N)} \to \Def_{(A, M)}$ in Lemma 3.1.

(i) If $\Hom_A(L, N) = 0$ then $\sigma_M$ is injective.

(ii) If $\Ext^1_A(L, N) = 0$ then $\sigma_M$ is formally smooth.

(iii) Suppose $\Def_{(A, N)}$ has a versal element $(R \to \mathcal{A}, \mathcal{N})$ and $\Ext^1_A(L, N) = 0$. Then $(R \to \mathcal{A}, \sigma_M(\mathcal{N}))$ is a versal element for $\Def_{(A, M)}$ and $\sigma_M$ is smooth.

The analogous statements hold for $\sigma_{M'}: \Def_{(A, N)} \to \Def_{(A, M')}$ with $\Ext^1_A(L', N) = 0$ in (i) and $\Ext^1_A(L', N) = 0$ in (ii)-(iii).

The following two results have very similar proofs to Theorems 5.5 and 5.7.

**Corollary 5.8.** Consider the map $\sigma^A_{L'}: \Def^A_{\mathcal{N}} \to \Def^A_{L'}$ in Lemma 3.1.

(i) If $\Hom_A(N, M') = 0$ then $\sigma^A_{L'}$ is injective.

(ii) If $\Ext^1_A(N, M') = 0$ then $\sigma^A_{L'}$ is formally smooth.

(iii) Suppose $\Def^A_{\mathcal{N}}$ has a versal element $(R, \mathcal{N})$ and $\Ext^1_A(N, M') = 0$. Then $(R, \sigma^A_{L'}(\mathcal{N}))$ is a versal element for $\Def^A_{L'}$ and $\sigma^A_{L'}$ is smooth.

The analogous statements hold for $\sigma^A_{L'}: \Def^A_{\mathcal{N}} \to \Def^A_{L'}$ with $\Ext^1_A(N, M) = 0$ in (i) and $\Ext^1_A(N, M) = 0$ in (ii) and (iii).

**Corollary 5.9.** Consider the map $\sigma^A_M: \Def^A_{\mathcal{N}} \to \Def^A_M$ in Lemma 3.1.

(i) If $\Hom_A(L, N) = 0$ then $\sigma^A_M$ is injective.

(ii) If $\Ext^1_A(L, N) = 0$ then $\sigma^A_M$ is formally smooth.

(iii) Suppose $\Def^A_{\mathcal{N}}$ has a versal element $(R \to \mathcal{A}_R, \mathcal{N})$ and $\Ext^1_A(L, N) = 0$. Then $(R \to \mathcal{A}_R, \sigma^A_M(\mathcal{N}))$ is a versal element for $\Def^A_{(A, M)}$ and $\sigma^A_M$ is smooth.

The analogous statements hold for $\sigma^A_M: \Def^A_{\mathcal{N}} \to \Def^A_M$, with $\Ext^1_A(L', N) = 0$ in (i) and $\Ext^1_A(L', N) = 0$ in (ii) and (iii).
Proposition 5.10. Put $Q' = \text{Hom}_A(\omega_A, L')$ and $Q = \text{Hom}_A(\omega_A, L)$. Then:

(i) $Q'$ and $Q$ have finite projective dimension.
(ii) $\text{Def}_{A,L'}(A, Q') \cong \text{Def}_{A,Q}(A, L)$ and $\text{Def}_{A,L}(A, L) \cong \text{Def}_{A,Q'}(A, Q')$.
(iii) There are natural maps

\[ s: \text{Def}_{A,M}(A,M) \rightarrow \text{Def}_{A,M'}(A,M') \quad \text{and} \quad t: \text{Def}_{A,L'}(A,L) \rightarrow \text{Def}_{A,L}(A,L) \]

commuting with the maps $\sigma_X: \text{Def}_{A,N}(A,N) \rightarrow \text{Def}_{A,X}(A,X)$ for $X$ equal to $M$ and $M'$, and to $L'$ and $L$, respectively. If $A$ is a Gorenstein ring, then $s$ is an isomorphism.

If $A$ is a flat and algebraic $A$-algebra with $A \otimes A_k \cong A$, the analogous statements hold for the deformation functors $\text{Def}_{X'}^A$. 

Proof. (i) Applying $\text{Hom}_A(\omega_A, -)$ to a finite $D$-resolution of $L'$ gives a finite projective resolution of $Q'$, see [20], 6.10 which also gives (ii).

(iii) There is a short exact sequence $0 \rightarrow M \rightarrow \omega_A^\oplus n \rightarrow M' \rightarrow 0$ such that the last map is without a common $\omega_A$-summand, corresponding (through dualisation) to a short exact sequence $0 \leftarrow M' \leftarrow A^\oplus n \leftarrow (M')^\vee \leftarrow 0$ where $n$ is minimal. The map $s$ is the composition $\text{Def}_{A,M}(A,M) \cong \text{Def}_{A,M'}(A,M') \rightarrow \text{Def}_{A,M'}(A,M') \cong \text{Def}_{A,M'}(A,M')$ where the middle map is obtained by taking the syzygy of the deformation. If $A$ is a Gorenstein ring then $\omega_A \cong A$ and there is an inverse $\text{Def}_{A,M'}(A,M') \rightarrow \text{Def}_{A,M}(A,M)$ given by the syzygy map.

Note that the pushout of $M \rightarrow \omega_A^\oplus n$ with $M \rightarrow N$ gives $N \rightarrow L'$. Consider the induced short exact sequence $0 \rightarrow L \rightarrow \omega_A^\oplus n \rightarrow L' \rightarrow 0$. For a deformation $(h, L')$ in $\text{Def}_{A,L'}(A,L')$ with structure map $\lambda: L' \rightarrow L$ there is a lifting of $\mu$ to a map $\tilde{\mu}: \omega_A^\oplus n \rightarrow L'$. If $L \cong L'$ commutes with $\omega_A^\oplus n \rightarrow \omega_A^\oplus n$. By Lemma 3.2, $(h, \lambda') \mapsto (h, \lambda)$ gives a well defined map of deformation functors $t: \text{Def}_{A,L'}(A,L') \rightarrow \text{Def}_{A,L}(A,L)$.

Given a deformation $(h, N)$ in $\text{Def}_{A,N}(A,N)$, let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ and $0 \rightarrow N \rightarrow L' \rightarrow M' \rightarrow 0$ be the minimal sequences in (3.0.1). There is a commutative diagram of short exact sequences with (co)cartesian square (cf. [3])

\[
\begin{array}{c}
0 \\
\downarrow \\
L \\
\downarrow \\
M \\
\downarrow \\
N \\
\downarrow \\
0 \\
\end{array}
\quad \begin{array}{c}
0 \\
\downarrow \\
\omega_A^\oplus n \\
\downarrow \\
L' \\
\downarrow \\
M' \\
\downarrow \\
0 \\
\end{array}
\]  

where $\omega_A^\oplus n \rightarrow L'$ is given as above. The stated commutativity of maps of deformation functors follows.

Corollary 5.11. Suppose $A$ has residue field $k$ and $\dim A \geq 2$. Then there exists finite $A$-modules $L'$ and $Q'$ with inj.dim$L' = \dim A = p\dim Q'$ and universal deformations $L' \in \text{Def}_{A,L'}^A(A)$ and $Q' \in \text{Def}_{A,L'}^A(A)$.

Proof. Let $h = 1 \otimes \text{id}: A \rightarrow A \otimes A_k = A$ and $N = A$ be the cyclic $A$-module defined through the multiplication map. Then $A \otimes A_k \cong A$ and $N \otimes A_k \cong k$ and this gives a deformation $N \rightarrow k$ of the residue field of $A$ which is universal. If $L'$ is the minimal $D_A$-hull of the residue field $k$ then $L' = \sigma_{L'}(N') \in \text{Def}_{A,L'}^A(A)$ is universal by Corollary 5.8. If $Q' = \text{Hom}_A(\omega_A, L')$ then $\text{Hom}_A(\omega_A, L') \in \text{Def}_{A,L'}^A(A)$ is universal by Proposition 5.10.
Corollary 5.12. Put $X = \text{Spec} \, A$. Let $Z$ be a closed subscheme of $X$ such that the complement $U$ is contained in the regular locus. Assume $\mathcal{N}_U$ is locally free, depth$_U N \geq 2$ and $H^2_U(\text{Hom}_A(L, N)) = 0$. Then $\text{Ext}^1_A(L, N) = 0$ and so

$$
\sigma_M : \text{Def}_i(A, N) \longrightarrow \text{Def}_i(A, M) \quad \text{and} \quad \sigma_M^A : \text{Def}_4^A \longrightarrow \text{Def}_4^A \quad \text{are formally smooth.}
$$

Proof. We show that $\text{Ext}^1_A(L, N) = 0$ and apply Theorem 5.7 and Corollary 5.9.

By Théorème 1.6 in [11, Exposé VI] there is a cohomological spectral sequence

$$
E_2^{p, q} = \text{Ext}^i_A(L, H^p_Z(N)) \Rightarrow \text{Ext}^{i+q}_{Z}(X; L, N).
$$

Since $H^2_Z(N) = 0$ for $i = 0, 1$ the restriction map $\text{Ext}^1_A(L, N) \to \text{Ext}^1_U(X; L, N)$ in the long exact sequence is injective. Since $U$ is contained in the regular locus, $M_U$ and hence $\mathcal{L}_U$ are locally free. It follows that $\text{Ext}^1_U(X; L, N)$ is isomorphic to

$$
\text{Ext}^1_U(\mathcal{L}_U, \mathcal{N}_U) \cong H^1(U, \mathcal{H}om_{\mathcal{O}_A}(L, N)) \cong H^2_Z(\text{Hom}_A(L, N))
$$

which is zero by assumption.

Example 5.13. The condition $H^2_Z(\text{Hom}_A(L, N)) = 0$ is implied by $\mathcal{N}_U = 0$ and also by depth$_U(\text{Hom}_A(L, N)) \geq 3$.

The following result extends A. Ishii’s [23, 3.2] to deformations of the pair.

Proposition 5.14. Assume $A$ is Gorenstein. If depth $N = \dim A - 1$ then

$$
\sigma_M : \text{Def}_i(A, N) \longrightarrow \text{Def}_i(A, M) \quad \text{and} \quad \sigma_M^A : \text{Def}_4^A \longrightarrow \text{Def}_4^A \quad \text{are smooth.}
$$

Proof. Let $S_2 \to S_1$ be a surjection in $\mathcal{A}_k$ and $(h_2 : S_2 \to B_2, M_2)$ an element in $\text{Def}_i(A, M) \setminus (S_2)$ which maps to $(h_1 : S_1 \to B_1, M_1)$ in $\text{Def}_i(A, M)(S_1)$. Suppose $\sigma_M$ maps $(h', N_1)$ in $\text{Def}_i(A, N)(S)$ to $(h_1, M_1)$. By assumption $L \cong A^{\oplus r}$ for some $r$. We can assume that $h' = h_1$ and that the minimal MCM-approximation of $N'$ is $0 \to L_1 \to M_1 \to N_1 \to 0$ where $L_1 \cong B_1^{\oplus r}$. Put $L_2 := B_2^{\oplus r}$ and choose a lifting $\rho_2 : L_2 \to M_2$ of $\rho_1$. Put $N_2 := \text{coker} \rho_2$ with its natural map to $N_1$. Then $N_2$ is $S_2$-flat ($\rho_2 \otimes S_1 = \rho_1$) and $\sigma_M(h_2, N_2) = (h_2, M_2)$.

Remark 5.15. If $A$ is a Gorenstein domain and $M$ is an MCM $A$-module there is a short exact sequence $0 \to A^{\oplus r} \to M \to N \to 0$ with $N$ a codimension one Cohen-Macaulay module; cf. [5, 1.4.3]. This sequence is an MCM-approximation and Proposition 5.14 applies. However, it is not always possible to continue this reduction. Assume $A$ is a normal Gorenstein complete local ring. Then all MCM $A$-modules are MCM-approximations of codimension $2$ Cohen-Macaulay modules up to stable isomorphism if and only if $A$ is a unique factorisation domain; see [36, 25].

Let $A$ be a Gorenstein normal domain of dimension $2$ and $0 \to A^{\oplus r - 1} \to M \to I \to 0$ the minimal MCM approximation of a torsion-free rank $1$ module $I$. Let $U$ denote the regular locus in $X = \text{Spec} \, A$. If $A = A \otimes_k S$ for $S$ in $kH_k$ there is a natural section $A \to A$. Let $U_A$ denote $U \times_X \text{Spec} \, A$. Consider the subfunctor $\text{Def}_i^{A, ^{\Lambda}} \subseteq \text{Def}_i$ of deformations $M$ such that $^{\Lambda}M_{U_A} \cong \mathcal{O}_{U_A}$. Note that $H^0(U, ^{\Lambda}M)$ is isomorphic to the MCM $A$-module $T := H^0(U, I)$. Proposition 5.14 implies that the resulting map from the (local) functor of quotients $\text{Quot}_{T \subseteq I}^T \to \text{Def}_i^{A, ^{\Lambda}}$ is smooth; cf. [23, 3.2]. In particular, if $E_A$ is the fundamental module (see (5.16.3) below) and $A/m_A \cong k$ then $h_A \cong \text{Quot}^{A, ^{\Lambda}}_{m_A, A} \cong \text{Def}_4^A$ gives a versal family for $\text{Def}_i^{A, ^{\Lambda}}$ by the MCM approximation in [20, 7.4]; see [23, 3.4].

Example 5.16. Assume $A/m_A \cong k$ and let $M$ denote the minimal MCM approximation of $k$. It is given as $M \cong \text{Hom}_A(\text{Syz}_d^A(k^\nu), \omega_A)$ where $d = \dim A$; cf. [20, 5.6]. One has $k^\nu = \text{Ext}_A^d(k, \omega_A) \cong k$. We apply $\text{Hom}_A(-, \omega_A)$ to the short exact sequence
0 \to \text{Syz}^A(m_A) \to A^{\oplus \beta_1} \xrightarrow{(2)} m_A \to 0. \text{ Assume } \dim A = 2. \text{ Since } \text{Ext}^1_A(m_A, \omega_A) \cong k \text{ we obtain the MCM approximation of } k \text{ from the exact sequence }

\begin{equation}
(5.16.1)
0 \to \omega_A \xrightarrow{(2)_{ir}} \omega_A^{\oplus \beta_1} \to M \to k \to 0.
\end{equation}

In particular \( \text{rk}(M) = \beta_1 - 1 \) and \( \mu(M) = t(A) \cdot \beta_1 + 1 \) where \( t(A) \) is the Cohen-Macaulay type of \( A \). If \( A = A(m) = k[u^m, u^{m-1}v, \ldots, v^m] \), the vertex of the cone over the rational normal curve of degree \( m \), the indecomposable MCM \( A \)-modules are \( M_i = (v^i, u^{-i}v, \ldots, v^i) \) for \( i = 0, \ldots, m-1 \). One finds that \( M = M_{m-1}^{\oplus m} \) and

\begin{equation}
(5.16.2)
dim_k \text{Def}^A_{\mathcal{M}}(k[\varepsilon]) = \dim_k \text{Ext}^1_A(M, M) = (m - 1) \cdot m^2
\end{equation}

while \( \dim_k \text{Def}^A_{\mathcal{M}}(k[\varepsilon]) = m + 1 \). Even in the Gorenstein case \( (m = 2) \) the tangent map is not surjective and so Proposition 5.14 cannot in general be extended to depth \( N \geq \dim A - 2 \). See [14] for a detailed description of the strata of the reduced versal deformation space of \( M \) defined by Ishii in [23].

If \( \dim A = 2 \) the MCM\( A \)-approximation of \( m_A \) is a short exact sequence

\begin{equation}
(5.16.3)
0 \to \omega_A \to E_A \to m_A \to 0
\end{equation}

where \( E_A \) is called the fundamental module; cf. [20, 7.1.1]. Applying \( \text{Hom}_A(k, -) \) to \( 0 \to m_A \to A \to k \to 0 \) gives an exact sequence

\begin{equation}
(5.16.4)
0 \to \text{Ext}^1_A(k, k) \to \text{Def}^A_{\mathcal{M}}(k[\varepsilon]) \to k^{\oplus t(A)} \to \text{Ext}^2_A(k, k)
\end{equation}

since \( \text{Ext}^1_A(m_A, m_A) \cong \text{Ext}^2_A(k, m_A) \) and \( \dim A = 2 \). If \( A = A(m) \) then \( E_A \) is isomorphic to \( M_{m-1}^{\oplus m} \) with \( \dim_k \text{Def}^A_{\mathcal{M}}(k[\varepsilon]) = 4(m - 1) \). Hence the conclusion in Proposition 5.14 cannot hold in the non-Gorenstein case \( m > 2 \).

6. Deforming maximal Cohen-Macaulay approximations of Cohen-Macaulay modules

Several definitions and results are given to prepare the statement of Theorem 6.6 and then to prove it.

**Definition 6.1.** A functor \( F: \mathcal{A}_k \to \text{Sets} \) has an obstruction theory if there is a \( k \)-linear functor \( H^F_\bullet: \text{mod}_k \to \text{mod}_A \) and for each small surjection \( R \to S \) in \( \mathcal{A}_k \) (i.e. with kernel \( I \) such that \( m_R I = 0 \)) and each \( a \in F(S) \) there is an element \( \text{ob}(R/S, a) \in H^F_2(I) \) which is zero if and only if there exists a \( b \in F(R) \) mapping to \( a \). The obstruction should be functorial with respect to such lifting situations. Cf. [2, 2.6].

**Example 6.2.** Consider the functor \( F = \text{Def}^A_{\mathcal{M}}: \mathcal{H}_k \to \text{Sets} \) defined in Section 2.2 where \( N \) is an \( A = A \otimes A \)-module. For a small surjection \( R \to S \) with kernel \( I \), let \( J \) denote the kernel of \( A_R \to A_S \). If \( N \) is a deformation of \( N \) to \( S \), there is an obstruction element \( \text{ob}(R/S, N) \) in \( \text{Ext}_A^2(N, N \otimes I) \cong \text{Ext}_A^2(N, N) \otimes I \) which is natural for the lifting situation by Proposition 4.2 and Lemma 4.3. Then \( H^F_2(-) := \text{Ext}_A^2(N, N) \otimes \text{ob}(a) \) with the obstruction \( \text{ob}(R/S, N) \) gives an obstruction theory for \( \text{Def}^A_{\mathcal{M}} 

**Lemma 6.5.** Given a map \( f: R \to S \) in \( \mathcal{A}_k \) with both rings being algebraic over \( A \) (or complete) such that the induced map

\[ m_R/(m_R^2 + \text{im } m_A R) \to m_S/(m_S^2 + \text{im } m_A S) \]

is surjective. Then \( f \) is a surjection.

**Proof.** Let \( t_{R/A} \) denote the relative Zariski tangent space \( [m_R/(m_R^2 + \text{im } m_A R)]^\ast \). There is a \( A \)-algebra map \( f_{lc}: R_{lc} \to S_{lc} \) which is the Zariski localisation in \( A \)-points of a map of finite type \( A \)-algebras such that the henselisation of \( f_{lc} \) is \( f \).
The induced map \( t_{R_0/A}^* \to t_{R/A}^* \) is an isomorphism, and likewise for \( S \). Then \( t_{R_0/A}^* \to t_{S_0/A}^* \) surjective implies \( m_{R_0}/(m_{R_0})^2 \to m_{S_0}/m_{S_0}^2 \) surjective; cf. [33, Tag 06GB]. Then \( \text{im } m_{R_0}/m_{S_0} = m_{S_0} \) by Nakayama’s lemma. In particular, \( S_0 \) is the Zariski localisation of a finite \( R_0 \)-algebra \( S \) by [33, Tag 052V]. Since \( R_0 \to S_0 \cong S \) is surjective by [33, Tag 00M0], \( R_0 \to S_0 \) is surjective by faithfully flatness of completion. Since henselisation preserves surjections \( f \) is surjective.

**Example 6.4.** Suppose \( F: \mathcal{A}_H \to \text{Sets} \) is a functor with versal elements in \( F(R) \) and \( F(S) \) such that the induced maps \( h_R(k[\varepsilon]) \to F(k[\varepsilon]) \) are bijective. Then \( R \cong S \). Indeed, by versality there are maps \( f: R \to S \) and \( g: S \to R \) which are surjections by Lemma 6.3. Then \( g f \) is an automorphism since \( R \) is noetherian.

Put \( t_{F/A} = F(k[\varepsilon]) \). In the case \( k_0 \to k \) is a separable field extension, we will call a base ring \( k \) a (formally) versal family for \( \mathcal{A}_H \) which have minimal formally versal formal elements. Then every \( \xi \) denotes a formal versal family for \( \mathcal{A}_H \) and let \( G \) be the ideal in \( \mathcal{A}_H \) which have minimal formally versal formal elements.

**Lemma 6.5.** Suppose \( k_0 \to k \) is a separable field extension and \( \phi: F \to G \) is a map of set-valued functors on \( \mathcal{A}_H \) which have minimal formally versal formal families with base rings \( R^F \) and \( R^G \) which are algebraic over \( A \) (or complete). Put \( V = \text{ker}(t_{G/A}^* \to t_{F/A}^*) \). Assume:

(i) The map \( t_{F/A} \to t_{G/A} \) is injective.

(ii) There are obstruction theories for \( F \) and \( G \) such that \( \text{ob}(p, \phi_S(\xi)) = 0 \) implies \( \text{ob}(p, \xi) = 0 \) for any small surjection \( p: R \to S \) in \( \mathcal{A}_H \) and element \( \xi \in F(S) \). Then every \( f: R^G \to R^F \) in \( \mathcal{A}_H \) lifting \( \phi \) is surjective and the ideal \( \text{ker } f \) is generated by a lifting of a \( k \)-basis for \( V \). In particular \( \text{ker } f \) is generated by ‘linear forms’ modulo \( \text{im } m_A \cdot R^G \).

**Proof.** The Jacobi-Zariski-sequence of an object \( A \to R \to k \) in \( \mathcal{A}_H \) gives the exact sequence (cf. [33, Tag 06S9])

\[
(6.5.1) \quad \frac{(m_A/m_A^2) \otimes_{k_0} k}{m_R/m_R^2} \xrightarrow{d} \frac{\Omega_{R/A} \otimes_R k}{\Omega_{k/A}} \xrightarrow{0} 0
\]

where \( \Omega_{k/A} \cong \Omega_{k/k_0} \) which equals 0 by separability. Then

\[
(6.5.2) \quad \Omega_{R/A} \otimes_R k \cong \frac{m_R/(m_R^2 + \text{im } m_A \cdot R)}{t_{R/A}^*}
\]

Hence

\[
(6.5.3) \quad h_R(k[\varepsilon]) \cong \text{Der}_A(R, k) \cong \text{Hom}_k(\Omega_{R/A} \otimes_R k, k) \cong t_{R/A}.
\]

Then the surjective map \( \phi(k[\varepsilon])^* : t_{G/A}^* \to t_{F/A}^* \) by minimality (cf. [33, Tag 06LI]) is canonically isomorphic to the map \( t_{R^G/A} \to t_{R^F/A} \) induced by \( f \) so \( f \) is surjective by Lemma 6.3. Moreover, \( \text{ker } f \) maps surjectively to \( V \) inducing the natural surjective \( k \)-linear map

\[
(6.5.4) \quad g: (\text{ker } f)/(\text{im } m_A \cdot R^G \cap \text{ker } f) \to V.
\]

Lift a \( k \)-basis for \( V \) to elements in \( \text{ker } f \) and let \( J \) be the ideal in \( R^G \) generated by these elements. Then \( g \) is an isomorphism if and only if \( J = \text{ker } f \). Put \( \bar{R} = R^G/J \), \( \bar{R}_n = \bar{R}/(m_R^{n+1} + \text{im } m_A \cdot m_R^{n-1}) \). Let \( \xi_n \in G(\bar{R}_n) \) for \( n = 1, 2, \ldots \) denote the images of a formal versal family \( (\xi_n) \) for \( G \). Similarly, put \( R^F_n = \bar{R}/(m_R^{n+1} + \text{im } m_A \cdot m_R^{n-1}) \) and let \( (\xi_n) \), \( \xi_n \in F(R^F_n) \), denote a formal versal family for \( F \). We prove that the maps \( \bar{R}_n \to R^F_n \) are isomorphisms by induction on \( n \). Surjectivity and isomorphic completions imply that \( \bar{R} \to R^F \) is an isomorphism also in the algebraic case. Put \( K_1 = \text{ker } R^G_1 \to \bar{R}_1 \). Then \( K_1 \) is contained in \( V \), but since \( J \to V \) is surjective and factors through \( K_1 \) we have \( K_1 = V \). This is equivalent to \( t_{R_1/A}^* \cong t_{F/A}^* \) and to \( \bar{R}_1 \cong R^F_1 \). Let \( f_n: \bar{R}_n \to R^F_n \) be the map induced from \( f \). Assume \( f_{n-1}: \bar{R}_n \cong
The additive subcategory of projective modules in $\mathcal{A} \mathcal{M}_M$ induces an endo-functor for all $0$. Assume $\mathcal{A} \mathcal{M}_M$ is Cohen-Macaulay and $\mathcal{J} = (f_1, \ldots, f_n)$ is an $\mathcal{A}$-sequence. By [21, 2.5] $\mathcal{J}$ is an $\mathcal{A}$-sequence if and only if $\mathcal{J}$ is an $\mathcal{A}$-sequence and $\mathcal{A}/\mathcal{J}$ is $\mathcal{S}$-flat. I.e. $\mathcal{J}$ is a transversally $\mathcal{A}$-regular sequence relative to $\mathcal{S}$ as defined in [13, 19.2.1].

**Theorem 6.6.** Suppose $k_0 \rightarrow k$ is a separable field extension. Let $h: \mathcal{A} \rightarrow \mathcal{A}$ denote the henselisation of a flat and finite type ring map at a maximal ideal (cf. Section 2.1). Assume $\mathcal{A} = \mathcal{A} \otimes_A k$ is Cohen-Macaulay and $\mathcal{J} = (f_1, \ldots, f_n)$ is an $\mathcal{A}$-sequence. Put $\mathcal{B} = \mathcal{A}/\mathcal{J}$, $\mathcal{B} = \mathcal{B} \otimes_A k$ and let $\mathcal{J}$ be the image of $\mathcal{J}$ in $\mathcal{A}$. Let $\mathcal{N}$ be a maximal Cohen-Macaulay $\mathcal{B}$-module and

\[ 0 \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0 \]

a minimal $\mathcal{M} \mathcal{C} \mathcal{M}_\mathcal{J}$-approximation of $\mathcal{N}$.

Suppose $\text{Def}_\mathcal{N}^\mathcal{B}$ and $\text{Def}_\mathcal{M}^\mathcal{A}$ have formally versal formal families (versal families) for minimal base rings $\mathcal{N}$ and $\mathcal{M}$ which are complete (respectively algebraic over $\mathcal{A}$). If $\text{ob}(\mathcal{A}/\mathcal{J}^2 \rightarrow \mathcal{B}, \mathcal{N}) = 0$ then

\[ \mathcal{R}^N \cong \mathcal{R}^M / \mathcal{J} \]

where $\mathcal{J}$ is generated by elements lifting a $k$-basis of the kernel of the map of dual Zariski vector spaces (cf. Lemma 6.5). In particular $\mathcal{J}$ is generated by ‘linear forms’ modulo $m_{\mathcal{A}} \mathcal{R}^M$.

**Example 6.7.** The existence of a splitting of $\mathcal{A}/\mathcal{J}^2 \rightarrow \mathcal{B}$ implies that $\text{ob}(\mathcal{A}/\mathcal{J}^2, \mathcal{N}) = 0$ for all $\mathcal{B}$-modules $\mathcal{N}$ since $\mathcal{A}/\mathcal{J}^2 \otimes_{\mathcal{B}} \mathcal{N}$ gives a lifting of $\mathcal{N}$ to $\mathcal{A}/\mathcal{J}^2$.

Let $\mathcal{C}$ be a category. Then $\text{Arr} \mathcal{C}$ denotes the category with objects being arrows in $\mathcal{C}$ and arrows being commutative diagrams of arrows in $\mathcal{C}$. An endo-functor $F$ on $\mathcal{C}$ induces an endo-functor $\text{Arr} F$ on $\text{Arr} \mathcal{C}$. Let $\mathcal{B}$ be a noetherian local ring and $\mathcal{P}_\mathcal{B}$ the additive subcategory of projective modules in $\text{mod}_\mathcal{B}$. Let $\text{Hom}_\mathcal{B}(\mathcal{N}, \mathcal{M})$ denote the homomorphisms from $\mathcal{N}$ to $\mathcal{M}$ in the quotient category $\text{mod}_\mathcal{B} = \mathcal{B}/\mathcal{P}_\mathcal{B}$ i.e. $\mathcal{B}$-homomorphisms modulo the ones factoring through an object in $\mathcal{P}_\mathcal{B}$. For each $\mathcal{N}$ in $\text{mod}_\mathcal{B}$ we fix a minimal $\mathcal{B}$-free resolution and use it to define the syzygy modules of $\mathcal{N}$. For each $i$ the association $\mathcal{N} \mapsto \text{Syz}_i^\mathcal{B}(\mathcal{N})$ induces an endo-functor on $\text{mod}_\mathcal{B}$ defined by $\mathcal{A}$. Heller [17]. Define $\text{Ext}_\mathcal{B}^i(\mathcal{N}, \mathcal{M})$ as $\text{Hom}_\mathcal{B}(\text{Syz}_i^\mathcal{B}(\mathcal{N}), \mathcal{M})$ which turns out to be isomorphic to $\text{Ext}_\mathcal{B}^i(\mathcal{N}, \mathcal{M})$ for all $i > 0$.

**Lemma 6.8.** Let $\mathcal{A}$ be a noetherian local ring and $\mathcal{J} = (f_1, \ldots, f_n)$ a regular sequence. Put $\mathcal{B} = \mathcal{A}/\mathcal{J}$ and suppose $\mathcal{N}$ and $\mathcal{N}_j$ ($j = 1, 2$) are finite $\mathcal{B}$-modules. Let $\overline{\mathcal{N}}$ denote $\mathcal{B} \otimes_{\mathcal{A}} \text{Syz}_n^\mathcal{A} \mathcal{N}$.

(i) There is an injective map $u_\mathcal{N}: \mathcal{N} \rightarrow \overline{\mathcal{N}}$ of $\mathcal{B}$-modules which induces a functor $u: \mathcal{mod}_\mathcal{B} \rightarrow \text{Arr} \mathcal{mod}_\mathcal{B}$.

(ii) The functor $u$ commutes with the $\mathcal{B}$-syzygy functor:

\[ \text{Arr} \text{Syz}_i^\mathcal{B}(u_\mathcal{N}) = u_{\text{Syz}_n^\mathcal{A} \mathcal{N}} \]

(iii) Put $\overline{\mathcal{N}}_j = \overline{\mathcal{N}}_{\mathcal{N}_j}$. The endo-functor $\mathcal{B} \otimes_{\mathcal{A}} \text{Syz}_n^\mathcal{A}(-)$ induces a natural map

\[ \text{Ext}_\mathcal{B}^i(\mathcal{N}_1, \mathcal{N}_2) \rightarrow \text{Ext}_\mathcal{B}^i(\overline{\mathcal{N}}_1, \overline{\mathcal{N}}_2) \]
which makes the following diagram commutative for all $i$:

\[
\begin{array}{ccc}
\text{Ex}^i_B(N_1, N_2) & \rightarrow & \text{Ex}^i_B(M_1, M_2) \\
(u_{N_j})_* & \\ & \downarrow \\
\text{Ex}^i_B(N_1, M_2) & \rightarrow & \text{Ex}^i_B(M_1, N_2)
\end{array}
\]

(iv) The inclusion $u_N: N \hookrightarrow B \otimes_A \text{Syz}^A_1 N$ splits $\iff \text{ob}(A/J^2 \rightarrow B, N) = 0$.

Remark 6.9. Lemma 6.8 (iv) strengthens [4, 3.6] (in the commutative case).

Proof. (i) Suppose $F_* \rightarrow N$ is the fixed minimal $A$-free resolution of $N$. Tensoring the short exact sequence $0 \rightarrow \text{Syz}^A_1 N \rightarrow F_{n-1} \rightarrow \text{Syz}^A_{n-1} N \rightarrow 0$ with $B$ gives the exact sequence

\[
(6.9.1) \quad 0 \rightarrow \text{Tor}^1(B, \text{Syz}^A_1 N) \rightarrow M_N \rightarrow B \otimes_A F_{n-1} \rightarrow B \otimes \text{Syz}^A_{n-1} N \rightarrow 0.
\]

We have $\text{Tor}^1(B, \text{Syz}^A_1 N) \cong \text{Tor}^1(B, N) \cong N$. Let $u_N$ be the composition $N \cong \ker(B \otimes A) \subseteq M_N$. Then $N \rightarrow u_N$ gives a functor of quotient categories.

(ii) Let $p: Q \rightarrow N$ be the minimal $B$-free cover and $P_* \rightarrow \text{Syz}^B_1 N$ the minimal $A$-free resolution of the $B$-syzygy $\ker(p)$. Then there is an $A$-free resolution $H_* \rightarrow Q$ which is an extension of $F_*$ by $P_*$. Since $\text{Syz}^A_1 B \cong A$, tensoring the short exact sequence of $A$-free resolutions $0 \rightarrow P_* \rightarrow H_* \rightarrow F_* \rightarrow 0$ by $B$ we obtain by (i) a commutative diagram with exact rows

\[
\begin{array}{cccccccc}
0 & \rightarrow & \text{Syz}^B_1 N & \rightarrow & Q & \rightarrow & N & \rightarrow & 0 \\
& & u_{\text{Syz}^B_1 N} & & & & u_N & \\
0 & \rightarrow & B \otimes \text{Syz}^A_1 (\text{Syz}^B_1 N) & \rightarrow & B^* \otimes Q & \rightarrow & B \otimes \text{Syz}^A_1 N & \rightarrow & 0
\end{array}
\]

which proves the claim.

(iii) By (ii) it is enough to prove this for $i = 0$. The case $i = 0$ follows from the functoriality in (i).

(iv,$\Leftarrow$) For the case $n = 1$ see the proof of [4, 3.2]. Assume $n \geq 2$. We follow the proof of [4, 3.6] closely. Let $A_1 = A/(f_1)$. Then $F_*^{(1)} = A_1 \otimes F_{\geq 1}[1]$ gives a minimal $A_1$-free resolution of $A_1 \otimes \text{Syz}^A_1 N$. We have $\text{ob}(A/J^2 \rightarrow B, N) = 0 \Rightarrow \text{ob}(A/(f_1) \rightarrow A_1, N) = 0$ and hence $N$ is a direct summand of $A_1 \otimes \text{Syz}^A_1 N$. Let $G_* \rightarrow N$ be a minimal $A_1$-free resolution of $N$. Then $G_*$. is a direct summand of $F_*^{(1)}$ and hence $\text{Syz}^A_{n-1} N$ is a direct summand of $\text{Syz}^A_{n-1} (A_1 \otimes \text{Syz}^A_1 N) = A_1 \otimes \text{Syz}^A_1 N$. Tensoring this situation with $B$ (and let $F = B \otimes F$) gives a commutative diagram:

\[
\begin{array}{cccccccc}
N & \rightarrow & B \otimes \text{Syz}^A_1 N & \rightarrow & F_{n-1} & \rightarrow & \cdots & \rightarrow & F_1 & \rightarrow & F_0 & \rightarrow & N \\
& & & & & & & & & & & \\
N & \rightarrow & B \otimes \text{Syz}^A_{n-1} N & \rightarrow & \cdots & \rightarrow & \cdots & \rightarrow & \cdots & \rightarrow & \cdots & \rightarrow & N
\end{array}
\]

Since $\text{ob}(A/J^2 \rightarrow B, N) = 0 \Rightarrow \text{ob}(A_1/(f_2, \ldots, f_n) \rightarrow B, N) = 0$ the map $u_1$ splits by induction on $n$. So $u$ splits. The other direction follows from [4, 3.6]. □

Proposition 6.10. Suppose $h: S \rightarrow A$ is a local Cohen-Macaulay map, $J = (f_1, \ldots, f_h)$ an $h$-sequence, $\tilde{h}: S \rightarrow B = A/J$ the local Cohen-Macaulay map induced from $h$, and $(\tilde{h}, \tilde{N})$ an object in MCM. Let

\[
\xi: \quad 0 \rightarrow \mathcal{L} \rightarrow \mathcal{M} \xrightarrow{\pi} \tilde{N} \rightarrow 0
\]

be the minimal MCM-approximation of $\tilde{N}$ over $h$. Then tensoring $\xi$ by $B$ gives a 4-term exact sequence

\[
0 \rightarrow \mathcal{N} \otimes \mathcal{J}/J^2 \rightarrow \mathcal{E} \rightarrow \overline{\mathcal{M}} \xrightarrow{\pi} \tilde{N} \rightarrow 0
\]
which represents the obstruction class \( \text{ob}(A/J^2 \to B, N) \in \text{Ext}_B^2(N, N \otimes J/J^2) \).

Moreover,

\[
\text{ob}(A/J^2 \to B, N) = 0 \iff \text{ob}(A/J^2 \to B, N^\vee) = 0 \iff \pi \text{ splits}
\]

where \( N^\vee = \text{Ext}_B^1(N, \omega_h) \).

**Proof.** If \( K(f) \) denotes the Koszul complex then \( \text{Tor}_i^A(B, M) = H_i(K(f) \otimes M) = 0 \) for \( i > 0 \) by [28, 5.1.2]; cf. [21, Sec. 2.2]. There is a map from the defining short exact sequence \( 0 \to \text{Syz}^A N \to F_0 \to N \to 0 \) to \( \xi \) extending \( \text{id}_N \). Tensoring with \( B \) gives a map of 4-term exact sequences with outer terms canonically identified. Hence they represent the same element \( \text{ob}(A/J^2 \to B, N) \) in \( \text{Ext}_B^2(N, N \otimes J/J^2) \).

By the argument in [20, 5.6] we can assume that \( \xi \) is given as \( 0 \to \text{im}(d_n^0) \to (\text{Syz}_n^A N^\vee)^\vee \to N^\vee \to 0 \) where \( (F_r, d_r) \) is a minimal \( A \)-free resolution of \( N^\vee \). By Lemma 6.8, \( \text{ob}(A/J^2 \to B, N^\vee) = 0 \) if and only if \( u: N^\vee \to B \otimes \text{Syz}_n^A N^\vee \) splits. But applying \( \text{Hom}_{\text{A}}(-, \omega_f) \) to \( u \) gives \( \pi \) since \( N^\vee \cong \text{Ext}_A^1(N^\vee, \omega_h) \cong \text{Hom}_{\text{A}}(N^\vee, \omega_f) \). \( \Box \)

**Remark 6.11.** In the absolute Gorenstein case with \( n = 1 \) this is given in [4, 4.5].

**Proof of Theorem 6.6.** Let \( \varphi \) denote the composition

\[
F := \text{Def}_N^B \longrightarrow \text{Def}_N^A \longrightarrow \text{Def}_M^A \longrightarrow G
\]

Formal versality in the complete case and versality in the algebraic case implies that there is a lifting \( f: R_M \to R_N \) of \( \varphi \). The theorem follows from Lemma 6.5 once the conditions (i) and (ii) are verified.

(i) By Proposition 4.2, \( t_{F/A} \cong \text{Ext}_A^1(N, N) \) and \( t_{G/A} \cong \text{Ext}_A^1(M, M) \). Let \( \pi: M \to N \) denote the MCM-\( A \)-approximation and \( \tau: M \to B \) the \( B \)-quasicoherent. Then \( \pi \) splits by Proposition 6.10. Let \( \nu: N \to M \) denote a splitting and \( \tau: M \to \overline{M} \) the quotient map. Then \( \tau^*: \text{Ext}_B^1(N, N) \cong \text{Ext}_B^1(M, N) \) splits for any \( n \).

(ii) Suppose \( p: R \to S \) is a small surjection in \( \mathcal{A}_k \), put \( q = \text{id} \circ p: A_R \to A_S \), \( \overline{q} = B \otimes q: B_R \to B_S \), and so on. Suppose \( N \) is in \( \text{Def}_B^S(S) \), consider \( N \) as \( A_S \)-module and put \( M = \sigma_M(N) \to N \); cf. (3.0.1). There is a fixed map to \( \pi \) given in (3.0.2). Then \( \text{ob}(q, M) \) is contained in \( \text{Ext}_B^1(M, M) \otimes I \) by Lemma 4.5 and we prove that it maps to \( \text{ob}(q, N) \) in \( \text{Ext}_B^1(N, N) \otimes I \) along the maps in (6.11.2).

Consider the short exact sequences of \( A_R \)-modules

\[
0 \to \text{Syz}^A R(M) \longrightarrow G' \longrightarrow M \to 0
\]

where \( G' \) is free. Apply \( - \otimes_R S \) and obtain the 4-term exact sequence of \( A_S \)-modules

\[
0 \to M \otimes_R I \longrightarrow \text{Syz}^A S(M) \otimes_R S \longrightarrow G \longrightarrow M \to 0
\]

which represents \( \text{ob}(q, M) \in \text{Ext}_A^1(M, M) \) by Lemma 4.4. It splits into two short exact sequences along \( \text{Syz}^A S(M) \) which is \( S \)-flat. Applying \( \otimes_S k \) to (6.11.4) gives a 4-term exact sequence of \( A \)-modules

\[
0 \to M \otimes_R I \longrightarrow \text{Syz}^A R(M) \otimes_R k \longrightarrow G \longrightarrow M \to 0
\]
which represents $\text{ob}(q, \mathcal{M}) \in \text{Ext}_A^2(M, M \otimes I)$, cf. Lemma 4.5. Since $M$ is MCM and $J$ is a regular sequence, applying $B \otimes_A -$ to (6.11.5) gives another 4-term exact sequence of $B$-modules

\[(6.11.6) \quad 0 \to \overline{M} \otimes_k I \to B \otimes_A \text{Syz}^A_n (\mathcal{M}) \otimes_R k \to \overline{G} \to \overline{M} \to 0\]

which represents $\pi^* \tau_e \text{ob}(q, \mathcal{M})$ in $\text{Ext}_B^2(\overline{M}, \overline{M} \otimes I)$. Pushout by $\pi \otimes \text{id}: \overline{M} \otimes I \to N \otimes I$ and pullback by $\nu: N \to \overline{M}$ gives the image of $\text{ob}(q, \mathcal{M})$ in $\text{Ext}_B^2(N, N \otimes I)$ (cf. (6.11.2)):

\[(6.11.7) \quad 0 \to N \otimes_k I \to E \to Q \to N \to 0\]

With a $B_R$-free cover $F' \to N$, a similar argument gives a 4-term sequence of $B$-modules

\[(6.11.8) \quad 0 \to N \otimes_k I \to \text{Syz}^{B_n}(N) \otimes_R k \to F \to N \to 0\]

which represents $\text{ob}(q, N)$. Since $\overline{M} \cong N \otimes X$ for some $B$-module $X$, we may lift a sum of free covers to a free cover of $\mathcal{M}$ and assume that $G' = G'_1 \oplus G'_2$ with $F' = B_R \otimes G'_1$. Then $Q \cong F \oplus \text{Syz}^B(X)$ and $\text{Syz}^B(\overline{M}) \cong \text{Syz}^B(N) \oplus \text{Syz}^B(X)$. Lifting $\pi$ gives a map from (6.11.6) to (6.11.8). In particular there is a surjection $B \otimes_A \text{Syz}^A_n (\mathcal{M}) \otimes_R k \to \text{Syz}^{B_n}(N) \otimes_R k$ which restricts to the composition $\overline{M} \otimes I \to N \otimes I \to \text{Syz}^{B_n}(N) \otimes_R k$. The induced map from $E$ to $\text{Syz}^{B_n}(N) \otimes_R k$ together with the projection from $Q$ to $F$ gives a map from (6.11.7) to (6.11.8) which is the identity at the end terms. Thus they represent the same class in cohomology.

If $A$ is a local Gorenstein ring and $N$ a Cohen-Macaulay $A$-module of codimension $c$, put $N^c = \text{Ext}_A^c(N, A)$. Assume $Q = k[x_1, \ldots, x_m]_h, f \in m_Q^2$ and put $B = Q/(f)$. Assume $N$ is a MCM $B$-module. Then there are endomorphisms $\varphi$ and $\psi$ of $Q^{\oplus n}$ where $n = \dim_k N/m_N = e(B) \text{rk}(N)$ with $\varphi \psi = f \cdot \text{id} = \psi \varphi$ and $\text{coker} \varphi \cong N$. The pair $(\varphi, \psi)$ is called a matrix factorisation of $f$ which defines $N$. Put $P = Q[t]_h, F = f + t^2 \in P$ and $A = P/(F)$. Define $G(\varphi, \psi) = (\Phi, \Psi)$ where

\[(6.11.9) \quad \Phi = \begin{pmatrix} \varphi & t \\ -t & \psi \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \psi & -t \\ t & \varphi \end{pmatrix}\]

are endomorphisms of $P^{\oplus 2n}$ in block-matrix notation. Then $(\Phi, \Psi)$ is a matrix factorisation of $F$ and thus defines an MCM $A$-module $\text{coker} \Phi$ which we denote by $G(N)$. Indeed, $G$ defines a functor of stable categories $G: \text{mod}_B \to \text{mod}_A$ and was introduced by H. Knörrer in [26].

**Corollary 6.12.** For any $N$ in $\text{MCM}_B$ there is an $\text{MCM}_A$-approximation

\[0 \to A^{\oplus 2n} \to G(N^c)^c \to N \to 0\]

Put $M = G(N^c)^c$. Suppose $\text{Def}_N^B$ and $\text{Def}_M^A$ have formally versal formal families (or versal families). Then the minimal base rings $R^N$ and $R^M$ for $\text{Def}_N^B$ and $\text{Def}_M^A$ satisfy

\[R^N \cong R^M/J\]

where $J$ is generated by elements lifting a $k$-basis of the kernel of the map of dual Zariski tangent vector spaces $\varphi^*_M: \text{Ext}_A^1(M, M)^* \to \text{Ext}_B^1(N, N)^*$; cf. Lemma 6.5.

**Proof.** A minimal $P$-free resolution of $N$ together with a homotopy for the multiplication with $F$ on the resolution is constructed from a minimal matrix factorisation.
(\varphi, \psi) for N:
\begin{equation}
\begin{array}{c}
N \hookrightarrow P^n \twoheadrightarrow P^n \oplus P^n \twoheadrightarrow P^n \twoheadleftarrow \psi \twoheadleftarrow P^n \twoheadrightarrow 0 \\
0 \twoheadleftarrow F \twoheadleftarrow t \varphi \twoheadleftarrow F \twoheadleftarrow \psi - t \twoheadleftarrow F \twoheadleftarrow F \\
N \hookrightarrow P^n \twoheadrightarrow P^n \oplus P^n \twoheadrightarrow P^n \twoheadrightarrow 0
\end{array}
\end{equation}

The Eisenbud construction [9] of an A-free resolution from these data gives:
\begin{equation}
\begin{array}{c}
N \twoheadrightarrow A^n \xrightarrow{[t, \varphi]} A^{2n} \xrightarrow{\Psi} A^{2n} \xrightarrow{\Psi} A^{2n} \xrightarrow{\Psi} \ldots
\end{array}
\end{equation}

In particular there is a short exact sequence $0 \rightarrow N \rightarrow A^n \rightarrow G(N) \rightarrow 0$. Applying $\text{Hom}_A(-, A)$ gives another short exact sequence $0 \rightarrow A^n \rightarrow G(N)^\vee \rightarrow N^\vee \rightarrow 0$. This is then the MCM$_A$-approximation of $N^\vee$. By local duality theory there is a canonical isomorphism $N^\vee \cong N$. Thus the above construction applied to the MCM $B$-module $N^\vee$ gives the MCM$_A$-approximation of $N$.

For the second part, note that $t$ is a non-zero divisor in $A$ and $A/(t)^2 \cong B[t]/(t^2)$, hence $A/(t)^2 \twoheadrightarrow B$ splits and $\text{ob}(A/(t)^2 \twoheadrightarrow B, -) = 0$. Then Theorem 6.6 applies.

**Example 6.13.** Put $\overline{M} = B \otimes M$ and note that $N^\vee \cong \text{Hom}_B(N, B)$. By Proposition 6.10, $\text{ob}(A/(t)^2 \twoheadrightarrow B, N) = 0$ gives a splitting $\overline{M} \cong N \oplus X$ where $X$ is stably isomorphic to $\text{SyZ}^B(N^\vee)^\vee$ (which in the hypersurface case) is isomorphic to $\text{SyZ}^B(N)$. Since $\text{Ext}^1_B(\text{SyZ}^B(N), N) \cong \text{Ext}^2_B(N, N)$, (6.11.2) gives
\begin{equation}
\text{Ext}^1_A(M, M) \cong \text{Ext}^1_B(N, N) \oplus \text{Ext}^2_B(N, N).
\end{equation}

Hence if $\dim_k \text{Ext}^i_B(N, N) < \infty$ for $i = 1, 2$ then Def$_N^B$ and Def$_M^A$ have formally versal formal families for complete base rings; cf. Proposition 4.2 and [31, 2.11].

If Spec $B$ is an isolated singularity and char $k \neq 2$ then Spec $A$ is an isolated singularity. Then Def$_N^B$ and Def$_M^A$ have versal elements over algebraic base rings; cf. [34, 2.4] and [22, 4.5].

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