Bound states of spatial optical dark/gray solitons in nonlocal media

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It is shown that three or more dark/gray solitons can form bound states in nonlocal media. More over dark/gray solitons can form bound states in several balance distances. Numerical simulations indicate that some of such bound states are unstable and will decay into a group of fundamental solitons, while others may be stable. There exist degenerate bound states with the same velocity, Hamiltonian, particle numbers and momentum but decaying in different ways and having different lifetimes.

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I. INTRODUCTION

As were pointed out previously by N. I. Nicolov, et. al. [1] and Y. V. Kartashov, et. al. [2], two dark/gray solitons can form bound states due to their long-range attraction in nonlocal nonlinear media. In this paper we will show that three or more dark/gray solitons can form bound states also. As were indicated previously [3], nonlocal dark/gray solitons have exponentially decaying oscillatory tails which in turn give rise to widely distributed exponentially decaying oscillatory light-induced perturbed refractive index. As a result, in this paper, we will find that dark/gray solitons can form bound states in several balance distances. Numerical simulation shows that fundamental dark/gray solitons are stable. Some of such bound states are unstable and will decay into a group of fundamental solitons, while others may be stable. Bound states of no central symmetry or asymmetry will have degenerate states with the same non-vanishing velocity, Hamiltonian, particle numbers and momentum. Some of such degenerate states are unstable and will decay in different ways and have different lifetimes.

II. BOUND STATES OF MULTIPLE DARK/GRAY SOLITONS

The propagation of a paraxial optical beam in a medium with a self-defocusing spatially nonlocal nonlinearity can be described by the dimensionless nonlocal nonlinear Schrödinger equation (NNLSE) [1, 3]:

\[ \frac{\partial u}{\partial z} + \frac{i}{2} \frac{\partial^2 u}{\partial x^2} - u \int R(|x-x'|)|u(x',z)|^2 dx' = 0 \]  

(1)

where \( u(x, z) \) is the complex amplitude envelop of the light beam, \( I(x, z) = |u(x, z)|^2 \) is the light intensity, \( x \) and \( z \) are transverse and longitude coordinates respectively, \( R(x) \) is the real nonlocal response function and satisfies the normalization condition \( \int R(x) dx = 1 \), \( n(x, z) = -\int R(|x-x'|)|u(x', z)|^2 dx' \) is the light-induced perturbed refractive index. Note that not stated otherwise all integrals in this paper will extend over the whole \( x \) axis. It is easy to prove if \( u(x, z) \) satisfies Eq. (1) with a nonlocal response function \( R(x) \), the scale-transformed function \( \tilde{u}(x, z) = \frac{1}{\alpha} u \left( x, \frac{z}{\alpha} \right) \) satisfies Eq. (1) with another nonlocal response function \( \tilde{R}(x) = \frac{1}{\alpha^2} R \left( \frac{x}{\alpha^2} \right) \), where \( \alpha \) is an arbitrary positive real number. So it is adequate to consider dark/gray soliton having asymptotic behavior \( |u(x, z)|^2 \xrightarrow{x\to\pm\infty} 1 \). Not stated otherwise all dark/gray solitons in this paper have this asymptotic behavior.

By introducing a transformation \( u(x, z) = \psi(x, z)e^{-iz} \), equation (1) turns into

\[ \frac{i}{2} \frac{\partial^2 \psi}{\partial x^2} - \psi \left[ \int R(|x-x'|)|\psi(x')|^2 dx' - 1 \right] = 0 \]  

(2)

which has three integrals of motion [4, 5], namely, the number of particles

\[ N = \int (1 - |\psi|^2) dx, \]  

(3)

the momentum

\[ P = i \int \left( \psi \frac{\partial \psi}{\partial x} - \psi^* \frac{\partial \psi^*}{\partial x} \right) \left( 1 - \frac{1}{|\psi|^2} \right) dx, \]  

(4)

and the Hamiltonian

\[ H = \int \frac{1}{2} \left( \frac{\partial \psi}{\partial x} \right)^2 dx + \int (1 - |\psi|^2) dx 
+ \frac{1}{2} \int R(|x-x'|)[|\psi(x)|^2 |\psi(x')|^2 - 1] dx dx' \]  

(5)

In this paper we numerically solve Eq. (2) to find dark/gray soliton solutions

\[ \psi(x, z) = \phi(x-vz)e^{i\theta(x-vz)}, \]  

(6)

where \( v \) is the velocity of the dark/gray soliton relative to the cw background, and real functions \( \phi \) and \( \theta \) asymptotically approach \( \phi(x) \xrightarrow{x\to\pm\infty} \phi_0(x) \), \( \theta(x) \xrightarrow{x\to\pm\infty} \theta_0 \), where \( \theta_0 \) is a real constant. By inserting Eq. (6) into (2), we get

\[ \phi'' + \frac{v^2 + \theta''}{2} \phi - \frac{\theta^2}{2\phi^2} - \phi \int R(x-x')\phi^2(x') dx' = 0 \]  

(7a)

\[ \frac{\theta''}{2} + \frac{v^2 + \theta''}{2} \phi - \frac{\theta^2}{2\phi^2} - \phi \int R(x-x')\phi^2(x') dx' = 0 \]  

(7b)
As an example, we consider this following nonlocal case in which the light-induced perturbed refractive index is governed by

$$n - w^2 \frac{\partial^2 n}{\partial x^2} = -|u|^2,$$

which results in

$$n(x, z) = -\int R(|x - x'|)|u(x', z)|^2 dx',$$

where $R(x) = \frac{1}{2w} \exp \left(-\frac{|x|}{w}\right)$ and $w$ is the characteristic nonlocal response length of the media. Numerically solving Eqs. (7), we can find the dark and gray soliton solutions. The intensity profiles of dark solitons $\psi_0$ and $\psi_1, \psi_{1,1,1}$ and their counterpart evolutions are shown in Fig. (1) when $w = 5, v = 0$. (Similarly, bound states $\psi_1, \psi_{1,1,1}$ can form on the interface of two media.) No observable changes in the intensity profile of the fundamental soliton $\psi_0$ can be found during its propagation. Numerical simulation (not shown here) indicate that an initially broadened beam $\psi(x, 0) = \psi_0(x, 0)$ will converge into $\psi_0$ quickly during its propagation. So the fundamental soliton $\psi_0$ is stable. However, from Fig. (1), with no initial perturbation bound states $\psi_1, \psi_{1,1}, \psi_{1,1,1}$ are all unstable and will decay into a group of fundamental solitons. Numerical simulations (not shown here) also indicate that initially broadened bound states, instead of converging into bound solitary states, will ultimately decay into a group of fundamental solitons.

As has been shown previously, when $w > 1/2$ the maximal velocity of gray soliton is $v_{\text{max}} = \sqrt{\frac{4w-1}{4w}}$. So when $w = 5$, we have $v_{\text{max}} = 0.436$. As shown in Fig. (1b), when $w = 5$ the bound state $\psi_1$ with an initial velocity $v = 0.4$ is unstable. Numerical simulations (not shown here) also indicate that $\psi_1$ with other velocities $v = 0.1, 0.2, 0.3$ are all unstable. So we strongly bound state $\psi_1$ with any velocity are unstable when $w = 5$. Similarly, bound state $\psi_{1,1}, \psi_{1,1,1}$ with any velocity are all unstable when $w = 5$. When $w = 5$, the functional dependence of momentum $P$, Hamiltonian $H$ and particle numbers $N$ on the velocity $v$ of $\psi_0$ and $\psi_1$ are shown in Fig. (2), from which, both for $\psi_0$ and $\psi_1$, we get $dP/dv < 0$ for all velocity. $\psi_0$ is stable but $\psi_1$ is not. So the criterion of dark soliton instability $dP/dv > 0$ may be a sufficient but not necessary condition. Namely, we cannot tell a dark/gray soliton whether stable or not if $dP/dv < 0$.

As is shown in Fig. (3), it needs a longer and longer propagation distance for the decaying of bound states $\psi_1$ as the characteristic nonlocal response length decreases from (a) $w = 4$ to (b) $w = 2$, to (c) $w = 1.5$, and to (d) $w = 1.2$.
III. BALANCE DISTANCES BETWEEN DARK/GRAY SOLTIONS

As was indicated previously, due to the nonlocal nonlinear response of the media the dark/gray solitons have exponentially decaying oscillatory tails when \(|v| \leq \sqrt{\frac{w-1}{w+1}}\) for \(w > \frac{1}{2}\). Let \(\phi(x) = 1 - \chi(x)\), then when \(|x| \to \infty\), we have

\[
\chi(x) \approx c_1 \exp(-\lambda_1|x|) \cos(\lambda_2|x| + c_2)
\]

(9)

where \(c_1\) and \(c_2\) are two constants, and

\[
\lambda_1 = \sqrt{\frac{1 - 4 w^2 v^2 + 4 w \sqrt{1 - v^2}}{4 w^2}},
\]

(10)

\[
\lambda_2 = \sqrt{\frac{1 + 4 w^2 v^2 + 4 w \sqrt{1 - v^2}}{4 w^2}},
\]

(11)

and \(2\pi/\lambda_2\) is the oscillatory spatial period. As shown in Fig. 4(a), when \(w = 5, v = 0.4\), the fundamental soliton \(\psi_0\) has a serial of maximums and minimums interdigitally located at \(x_0 = 0, x_1 = 4.57, x_2 = 10.29, x_3 = 15.61, x_4 = 21.10, x_5 = 26.52, x_6 = 31.97, x_7 = 37.41, x_8 = 42.85\). Such exponentially decaying oscillatory tails can form bound states in not only one but several balance distances. For example, as shown in Fig. 11, when \(w = 5, v = 0.4\), the balance distances (the distance between two deepest dips) of bound states \(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8\) are \(d_i = 4.88, 10.48, 15.97, 21.33, 26.82, 32.24, 37.69, 43.13\), where index \(i\) runs from 1 to 8. Obviously we have \(d_i \approx x_i\). Let \(\Delta d_i \equiv d_{i+1} - d_i\), we have \(\Delta d_i = 5.6, 5.49, 5.36, 5.49, 5.42, 5.45, 5.44\). On the other hand, from Eq. 11, the half of the oscillatory spatial period \(\pi/\lambda_2 = 5.44\) very close to \(\Delta d_i\). Other cases with different \(w\) and \(v\) also show the same relation between \(\pi/\lambda_2\) and \(\Delta d_i\).

To study the interaction of two fundamental solitons \(\psi_0\) separated by a distance \(d\), we introduce a coupled state \(\psi_d(x) = \phi_d(x) \exp[i\theta_d(x)]\), where

\[
\phi_d(x) = 1 - \chi_0 \left(x - \frac{d}{2}\right) - \chi_0 \left(x + \frac{d}{2}\right),
\]

(12a)

\[
\theta'_d = v \left(1 - \frac{1}{\sigma_d^2}\right),
\]

(12b)

and the index \(d\) of \(\psi_d\) denotes the distance. Obviously, when \(d \to \infty\), there will be no overlap between \(\chi_0(x - \frac{d}{2})\) and \(\chi_0(x + \frac{d}{2})\), and \(\psi_d\) will decouple into two well separated fundamental solitons \(\psi_0\). We note, when \(d = x_1, x_3, x_5, \cdots\), the maximums of \(\chi_0(x - \frac{d}{2})\) will overlap with the minimums of \(\chi_0(x + \frac{d}{2})\), and as Fig. 4(b) shows, the Hamiltonian of \(\psi_d\) takes the minimums, while when \(d = x_2, x_4, x_6, \cdots\), the maximums of \(\chi_0(x - \frac{d}{2})\) will overlap with the maximums of \(\chi_0(x + \frac{d}{2})\), and the Hamiltonian of \(\psi_d\) takes the maximums. On the other hand, as indicated by Fig. 4(c),(d), the difference between \(\psi_3\)
and $\psi_d$ is small, and the difference between $\psi_i$ and $\psi_d$, will decrease as $d_i$ increases. So, as Fig. (b) shows, though there is a difference between the Hamiltonian of $\psi_d$ and those of $\psi$, the Hamiltonian of $\psi_i$ assumes a similar behavior of that of $\psi_d$, which may qualitatively explain why $d_i \approx x_i$ and $\Delta d_i \approx \pi/\lambda_2$. Numerical simulations shown in Fig. (b) indicate that $\psi_2, \psi_4, \psi_6, \psi_8$ are all unstable, while $\psi_3, \psi_5, \psi_7$ may be all stable, and $\psi_1$ is weakly unstable when $w = 5, v = 0.4$.

Numerical simulations with initially broadened bound states with vanishing velocity $v = 0$, seeing Fig. (b), also indicate that slightly initial departure from the bound states $\psi_2$ and $\psi_4$ can result in a great different evolutions in the long run. $\psi_2$ and $\psi_4$ are both stable when $w = 5, v = 0$ also. On the other hand the initially broadened bound state of $\psi_3$, instead of converging into $\psi_3$ or decaying into two fundamental solitons, will fall in a seemingly eternal (over a distance larger than $z = 100000$) vibrations around $\psi_3$. So bound state $\psi_3$ may be oscillatory-stable. But it needs a more rigorous proof beyond the numerical simulation method of this paper to judge the stability of $\psi_3, \psi_5, \psi_7$.

Since dark/gray solitons can form bound states at several balance distances, we can construct bound states like $\psi_{1,2,1}$ or $\psi_{3,3,3}$, and so on. Interestingly, as shown in Fig. (b), bound states with no central symmetry or asymmetry will have two degenerate states, like $\psi_{1,2}$ and $\psi_{2,1}$, moving with the same non-vanishing velocity and having the same momentum, Hamiltonian and the particle numbers but decaying in different ways and having different life-times. However, there may still exist stable degenerate states, like $\psi_{3,5}$ and $\psi_{5,3}$.

FIG. 5: (a) Fundamental soliton $\chi_0(x)$ (red line) and its maximums and minimums (black dots). (b) the Hamiltonian of $\psi_d$ (red line) and those of $\psi_3, \psi_4, \psi_5, \psi_7, \psi_8$ (black dots). (c),(d) the intensity profiles of $\psi_2, \psi_3$ (red line) and $\psi_4, \psi_5$ (blue line). Here $w = 5, v = 0.4$.

FIG. 6: (1a),(2a),(3a) are intensity profiles of $\psi_2, \psi_3, \psi_4$; Numerical simulations of initially broadened bound states (1b)$\psi_2(\frac{x}{\lambda_199}), (1c)\psi_2(\frac{x}{\lambda_299})$; (2b)$\psi_3(\frac{x}{\lambda_199}), (2c)\psi_3(\frac{x}{\lambda_299})$; (3b)$\psi_4(\frac{x}{\lambda_199}), (3c)\psi_4(\frac{x}{\lambda_299})$ respectively. Here $w = 5, v = 0$.

FIG. 7: (a) The intensity profile of $\psi_{2,1}$: Two degenerate bound states (b) $\psi_{2,1}$ and (c) $\psi_{1,2}$ decay in different ways. Here $w = 5, v = 0.25$.

IV. CONCLUSION

In nonlocal media, multiple dark/gray solitons can form bound states in several balance distances. Some of such bound states are unstable and will decay into a group of fundamental solitons, while others may be stable. There exist degenerate bound states with the same velocity, Hamiltonian, particle numbers and momentum but decaying in different ways and having different life-times.

Acknowledgments

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