A Simple, Direct Finite Differencing of the Einstein Equations

Travis M. Garrett∗
Louisiana State University
(Dated: February 4, 2009)

We investigate a simple variation of the Generalized Harmonic method for evolving the Einstein equations. A flat space wave equation for metric perturbations is separated from the Ricci tensor, with the rest of the Ricci tensor becoming a source for these wave equations. We demonstrate that this splitting method allows for the accurate simulation of compact objects, with gravitational field strengths less than or equal to those of neutron stars. This method could thus provide a straightforward path for general relativistic effects to be added to astrophysics simulations, such as in core collapse, accretion disks, and extreme mass ratio systems.

I. INTRODUCTION

The wave equation is one of the best known and most studied partial differential equations, and it is solved by a variety of numerical methods in computer simulations involving radiative problems. Wave equations also appear as the leading order terms within the Einstein equations of General Relativity (GR), when selecting coordinates that follow the harmonic coordinate condition $\square x^a = 0$. The Generalized Harmonic (GH) formulation of GR expands on the harmonic coordinate condition by setting the d’Alembertian of the coordinates to be a general function: $\square x^a = H^a$. This preserves the Einstein equations in a wave equation form, but now allows for any coordinate system to be used. The GH formulation of GR was the first to enable the simulation of the inspiral, merger and ringdown of two black holes [1] (and was quickly followed by groups using the BSSN formalism [2, 3] – for a recent comparison of the two methods see [4]).

We report here on a simple rewriting of the GH formulation that further exploits the wave equation nature of the Einstein equations. Flat space wave equations for metric perturbations are split off from the Ricci tensor, and the rest of the Ricci tensor is converted into sources for these wave equations. The complexities of the Einstein field equations are thus packaged within these new source terms, which we find to be well behaved when the gravitational field strengths are on the order of those for a neutron star or weaker. Note that similar methods for splitting off flat space wave equations have existed for a long time, dating back to de Donder: [5, 6] (and see also [7, 8]).

To determine whether this splitting is useful numerically we perform a simple simulation of a binary neutron star inspiral and merger. We find that the gravitational fields evolve stably, that we can extract the expected gravitational waves, and we verify that the Hamiltonian constraint violations converge to zero. This test shows that this splitting method provides a fairly simple way for general relativistic effects and gravitational wave production to be added to astrophysics simulations that have traditionally only used Newtonian gravity or simplified versions of GR. As an example, some current core collapse codes (see e.g. [9]) use the ADM formulation of GR, along with the approximation of a conformally flat spatial metric [10, 11]. Our splitting method provides a simple alternative way to add GR, without making any approximations. It should also be possible to split off and evolve waves equations about a curved background (such as Schwarzschild or Kerr), which would be useful in simulating accretion disks or white dwarfs and neutron stars orbiting a supermassive black hole.

II. FIELD EQUATIONS

We will briefly review the GH formulation of GR – see e.g. [12, 13] for more details. We start with the metric $g_{ab}$:

$$ds^2 = g_{ab} dx^a dx^b.$$  (1)

The Einstein equations $G_{ab} = 8\pi T_{ab}$ can be written in trace reversed form:

$$R_{ab} = 4\pi (2T_{ab} - g_{ab} T),$$  (2)

with the Ricci tensor $R_{ab}$ given in terms of the connection coefficients $\Gamma^c_{ab}$:

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{cb,a} + \Gamma^d_{ab} \Gamma^c_{dc} - \Gamma^d_{cb} \Gamma^c_{da}.$$  (3)

The lowered-indices connection coefficients are given by derivatives of the metric:

$$\Gamma_{abc} = \frac{1}{2} (g_{ab,c} + g_{ac,b} - g_{bc,a}),$$  (4)

and the metric can be used to raise the first index or get a contracted form:

$$\Gamma^{bc}_{a} = g^{ad} \Gamma_{dbc},$$  (5)

$$\Gamma^c = g^{bc} \Gamma^c_{bc},$$  (6)

$$\Gamma_a = g^{bc} \Gamma_{abc}.$$  (7)

The contracted connection coefficients are equivalent to minus a wave operator acting on the coordinates:

$$g^{bc} \nabla_b \nabla_c x^a = \square x^a = -\Gamma^a.$$  (8)
Choquet-Bruhat showed [14] that if one picks harmonic coordinates \( \Box x^a = 0 \), then the Ricci tensor can be expanded and the Einstein equations [12] transformed into:

\[
g^{cd}g_{ab,cd} + 2g^{cd}(a\nabla_b)d + 2\Gamma^d_{cb}\Gamma^c_{da} = -8\pi(2T_{ab} - g_{ab}T). \tag{9}
\]

The highest order term in Eq. (9) is the curved space wave equation \( g^{cd}g_{ab,cd} \), so this places the Einstein equations in a manifestly hyperbolic form.

A wider range of options are available however – one has the freedom in General Relativity to pick any set of coordinates. As shown by Friedrich [15] and Garfinkle [16] this freedom can be alternatively expressed by picking source functions \( H^a(x,t) \) to drive the wave equations for the coordinates:

\[
\Box x^a = H^a(x,t) = -\Gamma^a. \tag{10}
\]

Using these source functions we can decompose Eq. (2) in the standard GH formulation:

\[
g^{cd}g_{ab,cd} + 2g^{cd}(a\nabla_b)d + 2H_{(ab)} - 2H_c\Gamma^c_{ab} + 2\Gamma^d_{cb}\Gamma^c_{da} = -8\pi(2T_{ab} - g_{ab}T). \tag{11}
\]

We will now modify this equation in order to explicitly separate off the wave equations, and convert the rest of the Ricci tensor into sources. We will proceed generally at first by considering metric perturbations \( f_{ab} \) away from a general background metric \( \bar{g}_{ab} \), (such as a Minkowski or Schwarzschild spacetime):

\[
g_{ab} = \bar{g}_{ab} + f_{ab}. \tag{12}
\]

We then refer to the raised indices perturbations by \( h^{ab} \):

\[
g^{ab} = \bar{g}^{ab} + h^{ab} \tag{13}
\]

(note that the \( h^{ab} \) perturbations are determined by the background metric \( \bar{g}_{ab} \) and the lowered indices perturbations \( f_{ab} \)).

One can then split the second order piece of Eq. (11) into:

\[
g^{cd}g_{ab,cd} = (\bar{g}^{cd} + h^{cd})(\bar{g}_{ab,cd} + f_{ab,cd}). \tag{14}
\]

We want to separate out a scalar wave equation for each of the metric perturbations \( f_{ab} \) on the background geometry \( \bar{g}_{ab} \):

\[
\bar{g}^{cd}\nabla_c\nabla_d f_{ab} = \bar{g}^{cd}f_{ab,cd} - f_{ab,\bar{c}}\bar{\Gamma}^c. \tag{15}
\]

We thus need to subtract \( f_{ab,\bar{c}}\bar{\Gamma}^c \) from both sides of Eq. (11) in order to get \( \Box f_{ab} \) on the LHS. We collect the rest of the terms on the LHS of Eq. (11) into a source \( S_{ab} \) which we will move to the RHS:

\[
S_{ab} = -\bar{g}^{cd}g_{ab,cd} - h^{cd}(\bar{g}_{ab,cd} + f_{ab,cd}) - 2g^{cd}(a\nabla_b)d,c - 2H_{(ab)} + 2H_c\bar{\Gamma}^c_{ab} - 2\Gamma^d_{cb}\Gamma^c_{da} - f_{ab,\bar{c}}\bar{\Gamma}^c. \tag{16}
\]

This allows us to rewrite the standard GH equations (11) in the simple form:

\[
\Box f_{ab} = S_{ab} - 8\pi(2T_{ab} - g_{ab}T). \tag{17}
\]

In Eqs. (16) and (17) we give a splitting for metric perturbations on a general background geometry \( \bar{g}_{ab} \). Here we specialize to the case examined in this paper: we set the background to be flat and use Cartesian coordinates \( \eta_{ab} = \text{diag}(-1,1,1,1) \):

\[
g_{ab} = \eta_{ab} + f_{ab}. \tag{18}
\]

In this case the coordinate source functions \( H^a \) are zero and the \( S_{ab} \) source terms simplify to:

\[
S_{ab} = -h^{cd}f_{ab,cd} - 2g^{cd}(a\nabla_b)d,c - 2\Gamma^d_{cb}\Gamma^c_{da} \tag{19}
\]

(where all of the metric derivatives now stem from the \( f_{ab} \) functions).

### III. NUMERICAL SIMULATION

#### A. Matter EOM

In order to test our splitting (16,17) of the Einstein equations and see if they allow for stable and accurate numerical evolutions, we build a simplistic model of a binary neutron star inspiral and merger. We construct and evolve a stress energy tensor \( T_{ab} \) by fiat during the simulation, which sources the gravitational fields \( f_{ab} \) (which in turn give rise to the corrective sources \( S_{ab} \)). Note that we are not using \( T_{ab} = 0 \) to generate realistic equations of motion for the matter, as the main goal for this (single processor) code is to test the response of the gravitational fields \( f_{ab} \) to the rapid accelerations of dense matter sources. Later versions could use quasiequilibrium binary neutron star initial data as in [17, 18], or fully evolve the matter equations of motion as in [19].

The neutron stars are constructed as rigid polytropes with a compactness in the range of: \( M/R \sim 0.1-0.3 \). We choose to drive these density profiles on a quasi-circular inspiral path as found by Peters and Mathews [20, 21], which captures the leading order radiation reaction effects. The inspiral is parameterized by:

\[
a(t) = a_0 \left( 1 - \frac{t}{t_{\text{decay}}} \right)^{1/4}, \tag{20}
\]

where \( a(t) \) is the semimajor axis as a function of time, \( a_0 \) is the initial semimajor axis (between the centers of masses of the bodies), and the decay time is given by:

\[
t_{\text{decay}} = \frac{5}{64} \frac{a_0^3}{M}, \tag{21}
\]

where \( M \) is the total mass of the (equal mass) neutron stars. The instantaneous coordinate velocities of the stars are simply given by the Keplerian velocities of a
binary with the same mass and separation. Note that eventually the separation $a(t)$ will decrease enough that the stars begin to overlap: here the stress energy tensor is determined by a simple superposition of the individual stars densities, with the separation quickly falling to zero and the velocities also modulated to zero. This is, of course, not a particularly realistic model of a binary neutron star merger, the point is just to demonstrate that the gravitational fields $f_{ab}$ still evolve stably in this strong field and highly dynamical setting.

Having determined the bulk motion of the stars, we need the distributions of their fluid variables in order to build the stress energy tensor. We choose the standard perfect fluid form:

$$ T^{ab} = (\rho(1 + \varepsilon) + p)u^a u^b + pg^{ab}, \quad (22) $$

with rest mass density $\rho$, specific internal energy $\varepsilon$, pressure $p$, and four velocities $u^a$. The internal energy and pressure are given in terms of the density through a polytropic equation of state:

$$ p = \kappa \rho \Gamma, \quad (23) $$
$$ \varepsilon = \frac{\kappa}{\Gamma - 1} \rho^{\Gamma - 1}, \quad (24) $$

where we choose $\Gamma = 2$, and a value for $\kappa$ such that the internal energy density is about 5% of $\rho$ in the center of the star.

The density $\rho$ is in turn derived from a conserved “baryonic” density $\rho^*$ (see e.g. [22]):

$$ \rho^* = \rho(-g)^{1/2}w^0. \quad (25) $$

This density has a number of convenient properties, including a flat space conservation law:

$$ \partial_t \rho^* = \partial_i (\psi^i \rho^*), \quad (26) $$

which ensures that the total integrated baryonic mass:

$$ M^* = \int \rho^* d^3x, \quad (27) $$

is constant throughout the evolution. We thus choose initial radial profiles for $\rho^*$ for each star and hold these constant during the evolution.

### B. Field Evolution

Having determined the form of the matter stress energy tensor, we now concentrate on how to evolve the gravitational fields $f_{ab}$ in response via (17) (which has been simplified by Eqs. (18)(19)). We proceed in a slightly nonstandard way: given previous successes we choose to finite difference the scalar wave equation $\Box f_{ab}$ using implicit methods, while we use the standard explicit methods to construct the source $S_{ab}$ (note that this is a particular choice, but any other stable scheme to evolve wave equations should work as well).

First consider the implicit finite differencing of $\Box f_{ab}$. We use operator splitting to divide the 3+1 wave equation into three 1+1 wave equations (see e.g. [23]). For conciseness replace $f_{ab}$ with $\psi$ and the entire source $S_{ab} = -8\pi(2T_{ab} - g_{ab}T)$ with $\tau$. Adopting an operator splitting strategy we divide:

$$ -\partial_t^2 \psi + \partial_x^2 \psi + \partial_y^2 \psi + \partial_z^2 \psi = \tau \quad (28) $$

into a set of 1+1 problems:

$$ -\partial_t^2 \psi + \partial_x^2 \psi = 1/3\tau, \quad (29) $$
$$ -\partial_t^2 \psi + \partial_y^2 \psi = 1/3\tau, \quad (30) $$
$$ -\partial_t^2 \psi + \partial_z^2 \psi = 1/3\tau, \quad (31) $$

which are sequentially finite-differenced implicitly. We use a variation of the Crank-Nicholson method, modified for second time derivatives, to evolve these – for instance Eq. (29) becomes:

$$ \frac{1}{\Delta t^2} \left( \psi_{i}^{N+1} - 2\psi_{i}^{N} + \psi_{i}^{N-1} \right) = c_{N+1} \frac{1}{\Delta x^2} \left( \psi_{i+1}^{N+1} - 2\psi_{i}^{N+1} + \psi_{i-1}^{N+1} \right) + c_{N-1} \frac{1}{\Delta x^2} \left( \psi_{i+1}^{N-1} - 2\psi_{i}^{N-1} + \psi_{i-1}^{N-1} \right) - \frac{1}{3} \tau_{i}^{N}, \quad (32) $$

where the field is discretized at time step $N$ and spatial mesh location $i$. We have also included generalized coefficients $c_{N+1}$ and $c_{N-1}$ (with $c_{N+1} + c_{N-1} = 1$) so that we can transition from Crank-Nicholson $c_{N+1} = c_{N-1} = 1/2$ to a fully implicit method $c_{N+1} = 1, c_{N-1} = 0$, or somewhere in between. More implicit splitting allows us to dissipate high frequency noise if needed. Boundary conditions are added to Eq. (32) which is then solved by inverting the resulting tri-diagonal matrix. We note that in general implicit methods allow one to take as large a time step $\Delta t$ as desired and still maintain stability, however for reasons of accuracy we take time steps that are about the same size an explicit scheme takes in order to satisfy the Courant stability condition.

This operator-splitting implicit method is combined with a Fixed Mesh Refinement (FMR) grid structure so that a high resolution mesh is available near the origin to resolve the stars, while also extending through successive lower resolution meshes out into the wave zone. Sommerfeld outgoing wave boundary conditions are applied to the largest and coarsest mesh, which is updated first. Successively finer meshes are then updated, with their boundary conditions found via interpolation from the next coarsest mesh. After all the meshes are updated one time step the coarser mesh points are replaced with the finer mesh’s field values wherever there is overlap (all meshes are updated each time step – no subcycling in time is used).

This leaves the evaluation of the metric source $S_{ab}$ given by Eq. (19). The spatial derivatives are computed in the standard second order explicit way, but evaluating the time derivatives is more subtle as the updated field
values $f_{ab}^{N+1}$ will not be known until $\Box f_{ab}$ is calculated. We choose to solve this iteratively: for the first pass the first and second time derivatives contained in (19) are evaluated using the current and two previous time steps $f_{ab}^N, f_{ab}^{N-1}, f_{ab}^{N-2}$ – this allows for a preliminary value of $\hat{f}_{ab}^{N+1}$ to be found. The second iterative pass then uses the preliminary $\hat{f}_{ab}^{N+1}$ to calculate centered time derivatives in (19) which are used to complete the time step.

Finally initial data is needed – we choose very simple initial data: $f_{ab}(t = 0) = 0$. There is thus a large amount of noise initially as the fields respond to the matter and then settle down to their correct physical solutions, with the transients propagating off the grid within a fraction of an orbit. For values of neutron star compactness much larger than $M/R \sim 0.1$ it is useful to also add a large amount of dissipation initially, and then linearly decrease the dissipation to a small or zero value over a fraction of an orbit. Otherwise the large amplitudes of the initial transient spikes can lead to infinities in the $S_{ab}$ sources (for example if $|f_{00}| \geq 1$), thus crashing the code. Using physical initial data could also solve this, but it is encouraging that the code still rapidly converges to the correct solution when given initial data which completely ignores the self gravity of the stars (as has been seen elsewhere, see e.g. [24]).

C. Results

We find that our method of splitting off flat space wave equations for metric perturbations and turning the rest of the Ricci tensor into sources provides an effective method for simulating systems with neutron star strength or weaker gravitational fields. The results for an example evolution are given: we pick two stars, each with baryonic mass $M^* = 0.15$ and coordinate radius $R = 1$, and with an initial coordinate separation of 4 (which gives a value of $t_{\text{decay}} \sim 751$ via Eq. (21)). The finest mesh spacing is $\Delta x = 0.2$, giving about 10 grid points across the diameter of the stars.

Initially the Hamiltonian constraint:

$$3R - K_{ij}K^{ij} + K^2 - 16\pi \rho_H = 0$$

is highly violated as all the terms except for the Hamiltonian density $\rho_H$ are zero. As mentioned, after a short while the initial transients propagate off the grid and the Ricci scalar $3R$ grows to counterbalance $\rho_H$ (with the $K_{ij}K^{ij}$ and $K^2$ terms giving small corrections). The constraint (33) is then satisfied to within several percent, which is improved further with increasing resolution.

In figure (1) the real and imaginary parts of the $l = 2, m = 2$ mode of the gravitational waveform $\Psi_4$ are shown (as measured at a radial distance of 50 or $\sim 167M^*$). The initial noise propagates through, and the waveform then settles down approximately to the standard chirp profile. After the merger the waveform dies away quickly, due to the abrupt cessation of motion for the artificially driven sources (there is a small amount of residual noise visible, which converges to zero with increasing resolution).

The simple binary inspirals we have evolved show that our direct method in equation (17) successfully solves the Einstein equations, at least in the case of neutron star type systems with a flat Minkowski background (18)-(19). This method thus provides a simple way for general relativistic effects to be added to astrophysical simulations.

Acknowledgments

We would like to thank Luis Lehner, Carlos Palenzuela, and Charles Evans for useful discussions. This work was supported in part by NSF grants PHY-0803629, PHY-0653375, PHY-0653369, and CCF-0833193. Computations were done at LSU and LONI, and on Teragrid through grants MCA09X003, and MCA02N014.

[1] F. Pretorius. Evolution of binary black hole spacetimes. *Phys.Rev.Lett.*, 95:121101, (2005), gr-qc/0507014.

[2] M. Campanelli, C. O. Lousto, P. Marronetti, and Y. Zlochower. Accurate evolutions of orbiting black-hole bina-
ries without excision. Phys. Rev. Lett., 96:111101, (2006), gr-qc/0511048.

[3] J. G. Baker, J. Centrella, D. I. Choi, M. Koppitz, and J. van Meter. Gravitational wave extraction from an inspiraling configuration of merging black holes. Phys. Rev. Lett., 96:111102, (2006), gr-qc/0511103.

[4] H. Hannam, S. Husa, J. G. Baker, M. Boyle, B. Bruegmann, T. Chu, N. Dorband, F. Herrmann, I. Hinder, B. J. Kelly, L. E. Kidder, P. Laguna, K. D. Matthews, J. R. van Meter, H. P. Pfeiffer, D. Pollney, C. Reisswig, M. A. Scheel, and D. Shoemaker. The samurai project: verifying the consistency of black-hole-binary waveforms for gravitational-wave detection. arXiv:0901.2437v2 [gr-qc], (2009).

[5] T. DeDonder. La gravifique einsteinienne. Gauthier-Villars, Paris, (1921).

[6] T. DeDonder. The mathematical theory of relativity. Massachusetts Institute of Technology, Cambridge, MA, (1927).

[7] V. Fock. The Theory of Space Time and Gravitation. Pergamon Press, New York, (1959).

[8] K. S. Thorne. Multipole expansion of gravitational radiation. Rev. Mod. Phys., 52:341–392, (1980).

[9] H. Dimmelmeier, C. D. Ott, A. Marek, and H.-T. Janka. The gravitational wave burst signal from core collapse of rotating stars. arXiv:0806.4955v2 [astro-ph], (2008).

[10] J. A. Isenberg. Waveless approximation theories of gravity. arXiv:gr-qc/0702113v1, (1978).

[11] J. R. Wilson, G. J. Mathews, and P. Marronetti. Relativistic numerical method for close neutron star binaries. Phys. Rev. D, 54:1317–1331, (1996), arXiv:gr-qc/9601017.

[12] L. Lindblom, M. A. Scheel, L. E. Kidder, R. Owen, and O. Rinne. A new generalized harmonic evolution system. Class. Quant. Grav., 23:S447–S462, (2006), gr-qc/0601203.

[13] F. Pretorius. Simulation of binary black hole spacetimes with a harmonic evolution scheme. Class. Quant. Grav., 23:S529–S552, (2006), gr-qc/0602115.

[14] Y. Fourès-Bruhat. Théorème d’existence pour certains systèmes d’équations aux dérivées partielles non linéaires. Acta Math., 88:141–225, (1952).

[15] H. Friedrich. On the hyperbolicity of einstein’s and other gauge field equations. Commun. Math. Phys., 100:525–543, (1985).

[16] D. Garfinkle. Harmonic coordinate method for simulating generic singularities. Phys. Rev. D., 65:044029, (2002), gr-qc/0110013.

[17] K. Uryu and Y. Eriguchi. New numerical method for constructing quasiequilibrium sequences of irrotational binary neutron stars in general relativity. Phys. Rev. D, 61:124023, (2000), gr-qc/9908059v2.

[18] T. W. Baumgarte, G. B. Cook, M. A. Scheel, S. L. Shapiro, and S.A. Teukolsky. General relativistic models of binary neutron stars in quasiequilibrium. Phys. Rev. D, 57:7299–7311, (1998), gr-qc/9709026.

[19] M. Anderson, E. W. Hirschmann, L. Lehner, S. L. Liebling, P. M. Motl, D. Neilson, C. Palenzuela, and Tohline J. E. Simulating binary neutron stars: dynamics and gravitational waves. Phys. Rev. D, 77:024006, (2008) arXiv:0708.2720.

[20] P. C. Peters and J. Mathews. Gravitational radiation from point masses in a keplerian orbit. Phys. Rev. 131,435, (1963).

[21] P. C. Peters. Gravitational radiation and the motion of two point masses. Phys. Rev. 136,1224, (1964).

[22] C. M. Will. Theory and Experiment in Gravitational Physics Revised Edition. Cambridge University Press, (1993).

[23] W. H. Press, S. A. Teukolsky, Vetterling W. T., and B. P. Flannery. Numerical Recipes in C 2nd Ed. Cambridge University Press, (1992).

[24] N. T. Bishop, R. Gomez, L. Lehner, M. Maharaj, and J. Winicour. On characteristic initial data for a star orbiting a black hole. Phys. Rev. D, 72:024002, (2005) arXiv:gr-qc/0412080.