Flattening a non-degenerate CR singular point of real codimension two

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Dedicated to Ngaiming Mok on the occasion of his 60th birthday

Abstract: This paper continues the previous studies in two papers of Huang-Yin [HY3-4] on the flattening problem of a CR singular point of real codimension two sitting in a submanifold in $\mathbb{C}^{n+1}$ with $n + 1 \geq 3$, whose CR points are non-minimal. Partially based on the geometric approach initiated in [HY3] and a formal theory approach used in [HY4], we are able to provide a very general flattening theorem for a non-degenerate CR singular point. As an application, we provide a solution to the local complex Plateau problem and obtain the analyticity of the local hull of holomorphy near a real analytic definite CR singular point in a general setting.

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1 Introduction

Let $M \subset \mathbb{C}^{n+1}$ be a smooth real submanifold. For a point $q \in M$, there is an immediate first order holomorphic invariant $Rk(q) := \dim_{\mathbb{C}} T_q^{1,0} M$ for the germ of $M$ at $q$. $Rk(q)$ is an upper semi-continuous function. When $Rk(q)$ is constant for $q \approx p \in M$, we call $p$ a CR point of $M$. Otherwise, $p$ is called a CR singular point. The study of the geometric and analytic properties for $M$ near a CR singular point has attracted considerable attentions since the celebrated paper of Bishop in 1965 [Bis]. Bishop considered the case when $M$ is a real $(n+1)$-manifold in $\mathbb{C}^{n+1}$ with a CR singular point at $p$ and with $Rk(p) = 1$. Bishop discovered that under a certain natural non-degeneracy assumption and a certain holomorphically invariant convexity of $M$ near $p$, $M$ has a non-trivial local hull of holomorphy $\hat{M}$ and has a very rich holomorphically invariant geometric structure. Bishop conjectured that $\hat{M}$ is a Levi-flat submanifold which has more or less the same regularity as $M$ does even up to $M$ near $p$. Bishop’s problem was confirmed in a sequence of papers by Kenig-Webster [KW1-2], Moser-Webster [MW], Huang-Krantz [HK], and finally in Huang [Hu1]. The global version of the Bishop problem was investigated in the work of Bedford-Gaveau [BG] and Bedford-Klingenberg [BK]. Other closely related work at least includes the papers by Gong [Gong1-2], Gong-Lebl [GL], Gong-Stolotvich [GS1-2], Lebl [Leb], Burcea [Bur1-2], Coffman [Co], Huang-Yin [HY1-2], Lebl-Noell-Ravisankar [LNR1-2], etc.

In 2010, Dolbeault, Tomassini and Zaitsev [DTZ1-2] initiated the study of the generalized Bishop problem for a real codimension two submanifold $M \subset \mathbb{C}^{n+1}$ with $n + 1 \geq 3$. In this setting, the CR singularity must have complex codimension one and CR points have CR dimension $n - 1 \geq 1$. For $M$ to bound a Levi-flat submanifold $\hat{M}$, a CR point must be a CR non-minimal point, namely, for each CR point $q \in M$, there is a proper CR submanifold in $\mathbb{C}^{n+1}$ passing through $q$, that is contained in $M$ and has CR dimension $n - 1$. (This CR submanifold is in fact the intersection of a leaf of the Levi-foliation in $\hat{M}$ with $M$). Moreover, more restricted geometric assumptions have to be imposed at the CR singular points. A solution in certain cases for this generalized Bishop problem was obtained in Huang-Yin [HY3]. In a subsequent paper of Huang-Yin [HY4], a formal version of problems similar to that considered in [HY3] has been studied.
To be more detailed, we assume that \( p \in M \) is a CR singular point of \( M \). We write \((z_1, \cdots, z_n, w)\) for the coordinates of \( \mathbb{C}^{n+1} \). After a holomorphic change of coordinates, we assume that \( p = 0 \), \( T_p^{(1,0)} M = \{ w = 0 \} \). Then \( M \) near \( p = 0 \) is the graph of a function of the form:

\[
w = F(z, \overline{z}) = q^{(2)}(z, \overline{z}) + o(|z|^2),
\]

where \( q^{(2)}(z, \overline{z}) \) is a polynomial of degree two in \((z, \overline{z})\). In the classical case, namely, \( n + 1 = 2 \), after a holomorphic change of variables, we can always make \( q^{(2)}(z, \overline{z}) \)-real-valued. However, this is no longer the case for \( n + 1 \geq 3 \). Indeed, after a simple holomorphic change of coordinates, if needed, we can write

\[
q^{(2)}(z, \overline{z}) = 2\Re(z \cdot A \cdot z^t) + z \cdot B \cdot \overline{z} \tag{2}
\]

with \( A \) and \( B \) being two \((n \times n)\)-matrices. Suppose that \( z = \overline{z} \cdot P + \bar{a}w + O(|(z, w)|^2) \); \( w = \mu \bar{w} + z \cdot b^t + O(|(z, w)|^2) \) is a holomorphic transformation preserving the form as in (1) and (2). Then \( b = 0, \mu \neq 0 \), and \( P \) is an \((n \times n)\)-invertible matrix. Moreover, if \((M, 0)\) is defined in the new coordinates by

\[
\bar{w} = \tilde{q}^2(\bar{z}, \overline{z}) + o(|\overline{z}|^2),
\]

with \( \tilde{q}^{(2)}(\bar{z}, \overline{z}) = 2\Re\bar{z} \cdot \tilde{A} \cdot \bar{z}^t + \overline{z} \cdot \widetilde{B} \cdot \overline{z} \). Then

\[
\widetilde{B} = \frac{1}{\mu} P \cdot B \cdot \bar{P}^t, \quad \tilde{A} = \frac{1}{\mu} P \cdot A \cdot P^t. \tag{4}
\]

When there do not exist a \( \mu \neq 0 \) and an invertible \( P \) such that \( \frac{1}{\mu} P \cdot B \cdot \bar{P}^t \) is Hermitian, one can never make \( \tilde{q}^{(2)}(\bar{z}, \overline{z}) \) real-valued. Also notice that the non-degeneracy of the matrix \( B \) is a holomorphic invariant property. More general, we make the following definition:

**Definition 1.1.** (A). \( M \) is said to have a non-degenerate CR singularity at \( p \) if there is a holomorphic change of variables such that in the new coordinates, \( p = 0 \), \( M \) is defined by an equation of the form as in (1) and (2) with \( \det B \neq 0 \). If there is a holomorphic change of variables such that \( B \) is a definite Hermitian matrix, we call \( p \) a definite CR singular point of \( M \). (B). Let \( M \) be a real-codimension two real submanifold with \( p \in M \) a CR singular point. We say that \( M \) is quadratically flattenable if there is a change of coordinates such that in the new coordinates, \( p = 0 \), \( M \) near \( p = 0 \) is defined by an equation of the form as in (1) with \( q^{(2)}(z, \overline{z}) \) real-valued. One says that \( M \) can be holomorphically flattened at \( p \) if there is a holomorphic change of variables such that in the new coordinates, \( p = 0 \), \( M \) is defined by an equation of the form as in (1) with \( \Im (F(z, \overline{z})) \equiv 0 \).

In the above mentioned work of Dolbeault-Tomassini-Zaitsev [DTZ1-2] and Huang-Yin [HY3-4], the starting point is to define a generalized notion of the Bishop non-degeneracy and generalized Bishop invariants at a CR singular point \( p \). For that purpose, one needs to assume that \( M \) near \( p \) is quadratically flattenable. However, in the setting considered in [DTZ1-2] and [HY3-4], \( M \) is always CR non-minimal at its CR points. This raises a natural question ([Zat]) to understand the implication of CR non-minimality to the quadratic flattenability of
Let $M$ be a real codimension two smooth submanifold in $\mathbb{C}^{n+1}$ with $p \in M$ a non-degenerate CR singular point. Assume that $M$ is CR non-minimal at its CR points near $p$. Then $M$ is quadratically flattenable.

In the case of $n+1 = 3$, we will give a much more detailed result (see Theorem 3.1) in terms of the normal form for the pair $\{A, B\}$ given by Coffman in [Co], even if $B$ is degenerate. This result, due to its technical nature, will be stated as Theorem 3.1 in §3.

Let $M \subset \mathbb{C}^{n+1}$ be a codimension two real submanifold with $p \in M$ a non-degenerate CR singular point. Also assume that after a holomorphic change of variables, $p = 0$ and $M$ is defined as in (1) (2) with $B$ a Hermitian matrix. When $B$ is definite, then by the classical Takagi theorem [HY2] [HK], we can further make $q^{(2)}(z, \overline{z}) = \sum_{j=1}^{n} (|z_j|^2 + \lambda_j(z_j^2 + \overline{z}_j^2))$, where $0 \leq \lambda_1 \leq \cdots, \lambda_n < \infty$. The set $\{\lambda_1, \cdots, \lambda_n\}$ is called the set of generalized Bishop invariants of $M$ at the CR singular point. $\lambda_j$ is called an elliptic, parabolic or hyperbolic Bishop invariant, if $0 \leq \lambda_j < \frac{1}{2}$, $\lambda_j = \frac{1}{2}$ or $\lambda_j > \frac{1}{2}$. This terminology coincides with the classical definition of Bishop when $n+1 = 2$ ([Bis]) ([KW1] [MW] [HK]). However, when $B$ is not a definite matrix, we can not, in general, simultaneously diagonalize $A$ and $B$. In the case of $n + 1 = 3$, Coffman gave a list of the forms that the pair $\{A, B\}$ can be transformed to. (See the list given at the beginning of §3). Two cases in his list are geometrically quite special, in which the corresponding quadratic term takes one of the following forms after a holomorphic change of coordinates:

$$q^{(2)} = |z_1|^2 + |z_2|^2 + \frac{1}{2}(z_1^2 + \overline{z}_1^2) + \frac{1}{2}(z_2^2 + \overline{z}_2^2) \quad \text{or}$$

$$q^{(2)} = |z_1|^2 - |z_2|^2 + \lambda(z_1^2 + \overline{z}_1^2) + \lambda(z_2^2 + \overline{z}_2^2), \quad \lambda \geq \frac{1}{2} \quad (6)$$

In the case of (5), the two generalized Bishop invariants of the CR singular point at the origin are both parabolic. A consequence of this is that the set of CR singular points may have real dimension $n = 2$, which does create a lot of problems for the geometric studies of $M$ near $0$.

To explain the speciality of (6), we recall a definition from [HY3]: Let $(M, p)$ be a codimension two real submanifold in $\mathbb{C}^{n+1}$ with $p \in M$ a CR singular point. We say $(M, p)$ possesses an elliptic complex tangent direction if there is an affine complex plane $\mathcal{H}$ that passes through $p$ and is transversal to the complex tangent space of $M$ at $p$ such that $M \cap \mathcal{H}$ is an elliptic Bishop surface inside $\mathcal{H}$ in the classical sense ([Bis]). (See Definition 6.1). Now, a simple algebraic computation shows that a codimension two real submanifold $M \subset \mathbb{C}^{n+1}$ with a non-degenerate quadratically flattenable CR singular point at $p$ has no elliptic directions at $p$ if and only if $n+1 = 3$ and after a holomorphic change of variables sending $p$ to $0$, $M$ near $p = 0$ is defined by an equation of the form as in (1) with $q^{(2)}$ being given by (6). (See the paper by Lebl-Noell-Ravisankar[LNR]).
A major part of the paper continues the study in [HY3] and [HY4], which is devoted to the understanding of the holomorphically flattening problem near a CR singular point when $M$ is real analytic. This problem is equivalent to finding a real analytic Levi-flat hypersurface with $M$ as part of its real analytic boundary and with leaves moving along a transversal direction to the complex tangent space of the CR singular point. This has an immediate application to the study of the precise description of the local hull of holomorphy of $M$. Our purpose is to provide the following general holomorphic flattening theorem:

**Theorem 1.3.** Let $M$ be a real analytic real-codimension two submanifold in $\mathbb{C}^{n+1}$ with $n \geq 2$ and with $p \in M$ a non-degenerate CR singular point. Assume that $M$ is CR non-minimal at its CR points near $p$. Then $(M, p)$ can be holomorphically flattened if $M$ has an elliptic direction at $p$. More precisely, $(M, p)$ can be holomorphically flattened if either $n + 1 \geq 4$ or $n + 1 = 3$ but $(M, p)$ is not holomorphically equivalent to a submanifold $(M', 0)$ whose quadratic term takes the form in (6).

**Corollary 1.4.** Let $M$ be a real analytic real-codimension two submanifold in $\mathbb{C}^{n+1}$ with $n \geq 2$ and with $p \in M$ a non-degenerate CR singular point. Assume that either $n + 1 \geq 4$ or $n + 1 = 3$ but $(M, p)$ is not holomorphically equivalent to a submanifold $(M', 0)$ whose quadratic term takes the form in (6). Then there is a real analytic Levi-flat hypersurface $\hat{M}$, which has $M$ near $p$ as part of its real analytic boundary and is foliated by complex hypersurfaces shrinking down to $p$ along the normal direction of $M$ in $\hat{M}$ at $p$. Moreover, when $p$ is a definite CR singular point, then there is a small $\epsilon_0 > 0$ such that for any $0 < \epsilon << \epsilon_0$, $\hat{M} \cap B_p(\epsilon)$ is a connected open piece containing the origin of the hull of holomorphy of $M \cap B_p(\epsilon_0)$. Here $B_p(\epsilon)$ denotes the ball centered at $p$ of radius $\epsilon$.

Theorem 1.3 is contained in Huang-Yin [HY3] when $p$ is a definite CR singular point with one of the generalized Bishop invariants elliptic.

We next say a few words about the organization of the paper. In §2, we review a fundamental identity for $n + 1 = 3$ from the non-minimality condition first obtained in [HY3]. In §3, we first give a list of the normal form for the quadratic terms when $n + 1 = 3$. We then state Theorem 3.1 which gives an understanding about the quadratic flattening problem when $n + 1 = 3$. We also show that the result in Theorem 3.1 is optimal by presenting several examples. In §4, we give a proof of Theorem 3.1 by making an extensive use of the identity discussed in §2. In §5, we give a proof of Theorem 1.2. §6 – §7 will be devoted to the proof of Theorem 1.3. In §6, we use a geometric argument initiated in [HY3] to approach Theorem 1.3. The nice feature for this approach is that we do not need to know much about the quadratic normal form, which is almost impossible to obtain when $n + 1 > 3$. However this argument, though very general, needs the real dimension of the real analytic set of CR singular points of $M$ is no more than $2n - 2$. (And there is an example in Remark 6.5 showing that this approach fails when the singular set is too large). This excludes the case when $n + 1 = 3$ and the quadratic term of the defining equation of the manifold takes the normal form as in (5). In §7, we give a proof of Theorem 1.3 in this exceptional case. A good thing about this exceptional case is that the
associated quadratic term has the simplest possible symmetric form. This makes the formal argument developed in [HY4] to be very much adaptable to this setting. Indeed, we can first prove that in this case $M$ can always be formally flattened by a special form of holomorphic transformations, which together with the Huang-Krantz construction of holomorphic disks give a convergent flattening as in the other cases considered in [HY4].

2 Implication of integrability conditions

Let $(M, 0)$ be a smooth submanifold of codimension two in $\mathbb{C}^3$ with $0 \in M$ as a CR singular point. Assume that all CR points are non-minimal with CR dimension one. Use $(z, w) = (z_1, z_2, w)$ for the coordinates of $\mathbb{C}^3$. Assume that, after a holomorphic change of coordinates, $M$ near 0 is defined by an equation of the form:

$$w = q^{(2)}(z, \overline{z}) + p(z, \overline{z}) + iE(z, \overline{z}),$$

where $q^{(2)}(z, \overline{z}) = 2\Re(z \cdot A \cdot \overline{z}^t) + z \cdot B \cdot \overline{z}^t, p(z, \overline{z}), E(z, \overline{z}) = O(|z|^3)$ and both $p(z, \overline{z})$ and $E(z, \overline{z})$ are real-valued smooth functions. For convenience of notation, we also write $F(z, \overline{z}) = p(z, \overline{z}) + iE(z, \overline{z})$ and $G(z, \overline{z}) = q^{(2)}(z, \overline{z}) + p(z, \overline{z}).$

Then we have

$$w = q^{(2)}(z, \overline{z}) + F(z, \overline{z}) = G(z, \overline{z}) + iE(z, \overline{z}).$$

In what follows, we write $\chi_\alpha = \frac{\partial \chi}{\partial z_\alpha}, \chi_{\overline{\alpha}} = \frac{\partial \chi}{\partial \overline{z}_\alpha}$ with $\alpha = 1, 2$ for a smooth function $\chi(z, \overline{z})$ in $z$. We define

$$L := (G_2 - iE_2) \frac{\partial}{\partial z_1} - (G_1 - iE_1) \frac{\partial}{\partial z_2} + 2i(G_2E_1 - G_1E_2) \frac{\partial}{\partial w}$$

$$= A \frac{\partial}{\partial z_1} - B \frac{\partial}{\partial z_2} + C \frac{\partial}{\partial w}. $$

Then $L$ is a complex tangent vector field of type $(1, 0)$ along $M$ near 0. (See [§2, HY4]). Moreover, a straightforward computation shows that

$$T := [L, \overline{L}] = \left[ A \frac{\partial}{\partial z_1} - B \frac{\partial}{\partial z_2} + C \frac{\partial}{\partial w}, A^t \frac{\partial}{\partial z_1} - B^t \frac{\partial}{\partial z_2} + C^t \frac{\partial}{\partial w} \right]$$

$$= \lambda_1 \frac{\partial}{\partial z_1} + \lambda_2 \frac{\partial}{\partial z_2} + \lambda_3 \frac{\partial}{\partial w} + \lambda_4 \frac{\partial}{\partial z_1} + \lambda_5 \frac{\partial}{\partial z_2} + \lambda_6 \frac{\partial}{\partial w},$$

where

$$\lambda_1 = A \cdot (\overline{A})_1 - B \cdot (\overline{A})_2, \quad \lambda_2 = A \cdot (\overline{B})_1 + B \cdot (\overline{B})_2, \quad \lambda_3 = A \cdot (\overline{C})_1 - B \cdot (\overline{C})_2, \quad \lambda_4 = -\overline{A} \cdot A_T + \overline{B} \cdot A_{\overline{T}}, \quad \lambda_5 = \overline{A} \cdot B_T - \overline{B} \cdot B_{\overline{T}}, \quad \lambda_6 = -\overline{A} \cdot C_T + \overline{B} \cdot C_{\overline{T}}.$$
Notice that
\[ \lambda(1) = -\lambda(4), \quad \lambda(2) = -\lambda(5), \quad \lambda(3) = -\lambda(6). \]

Write \([L, T] =: (18)\) can be written as:
\[
\Gamma(1) \frac{\partial}{\partial T} + \Gamma(2) \frac{\partial}{\partial \bar{T}} + \Gamma(3) \frac{\partial}{\partial \bar{T}} + \Gamma(4) \frac{\partial}{\partial \bar{T}} + \Gamma(5) \frac{\partial}{\partial \bar{T}} + \Gamma(6) \frac{\partial}{\partial \bar{T}}. \]
By a direct computation as in [HY4], we have the following explicit expressions for \(\Gamma(1), \cdots, \Gamma(6)\):
\[
\Gamma(1) = A \cdot (\lambda(1))_1 - B \cdot (\lambda(1))_2, \quad \Gamma(2) = A \cdot (\lambda(2))_1 - B \cdot (\lambda(2))_2, \quad \Gamma(3) = A \cdot (\lambda(3))_1 - B \cdot (\lambda(3))_2,
\Gamma(4) = A \cdot (\lambda(4))_1 - B \cdot (\lambda(4))_2 - \lambda_1 \cdot A - A \bar{\lambda} - A \bar{\lambda} - \lambda_1 \cdot A_1 - \lambda_5 \cdot A_2,
\Gamma(5) = A \cdot (\lambda(5))_1 - B \cdot (\lambda(5))_2 + \lambda_1 \cdot B - A_b + \lambda_2 \cdot B_1 + \lambda_5 \cdot B_2,
\Gamma(6) = A \cdot (\lambda(6))_1 - B \cdot (\lambda(6))_2 - \lambda(1) \cdot C - \lambda(2) \cdot C_1 - \lambda(5) \cdot C_2.
\]
(11)

Since we assumed that all CR points are non-minimal with CR dimension one, \([L, T]\) is spanned by \(\{L, \bar{L}, T\}\) over a dense subset of \(M\) near 0. Hence, at these points, there are complex numbers \(k, \sigma, \tau, \) such that
\[
[L, T] = kL + \sigma \bar{L} + \tau T.
\]

Comparing the coefficients of both sides, we have as in [HY4]:
\[
\Gamma(1) = \sigma A + \tau \lambda(1), \quad (12)
\Gamma(2) = -\sigma B + \tau \lambda(2), \quad (13)
\Gamma(4) = kA + \tau \lambda(1), \quad (14)
\Gamma(5) = -kB + \tau \lambda(5). \quad (15)
\]

An elementary algebraic computation gives the following equality by eliminating \(\sigma\) from (12) and (13):
\[
B \Gamma(1) + A \Gamma(2) = B(\sigma A + \tau \lambda(1)) + A(-\sigma B + \tau \lambda(2)) = \tau(\lambda(1)B + \lambda(2)A). \quad (16)
\]

Similarly we derive the following equality by eliminating \(k\) from (14) and (15):
\[
B \Gamma(4) + A \Gamma(5) = B(kA + \tau \lambda(4)) + A(-kB + \tau \lambda(5)) = \tau(\lambda(4)B + \lambda(5)A). \quad (17)
\]

Combining (16) and (17), we obtain an identity in an open dense subset whose closure contains 0 \(\in \mathbb{C}^2\), which will be fundamentally used for the proof of Theorem 1.2: (Hence, in the real analytic case, the identity holds in a neighborhood of 0. And in the smooth category, it holds as a germ at 0)
\[
(B \Gamma(1) + A \Gamma(2))(\lambda(4)B + \lambda(5)A) = (B \Gamma(4) + A \Gamma(5))(\lambda(1)B + \lambda(2)A). \quad (18)
\]

For convenience, we introduce the following notation:
\[
X_1 := B \Gamma(1) + A \Gamma(2), \quad X_2 := \lambda(4)B + \lambda(5)A, \quad Y_1 := B \Gamma(4) + A \Gamma(5), \quad Y_2 := \lambda(1)B + \lambda(2)A. \quad (19)
\]
Then (18) can be written as:
\[
X_1X_2 = Y_1Y_2. \quad (20)
\]
3 Quadratic flattening in the 3-dimensional case

Let $M$ be defined by an equation of the form in (7). Recall that $q(z, \bar{z}) = 2\Re(z \cdot A \cdot z^t) + z \cdot B \cdot \bar{z}^t$ is the quadratic term in the defining function (7). When $B = 0$, $M$ near 0 is already quadratically flattened. Hence, we assume that $B \neq 0$. Then the CR points must have CR dimension one.

By a result of Coffman in [Co], after a holomorphic change of coordinates, the pair $\{A, B\}$ can be transformed into one of the forms $\{A', B'\}$ listed below. Namely, there is a biholomorphic change of coordinates such that in the new coordinates $(z', w')$, $M$ is defined by an equation of the form:

$$w' = 2\Re(z' \cdot A' \cdot z'^t) + z' \cdot B' \cdot \bar{z'}^t + o(|z'|^2)$$

where the pair $\{A', B'\}$ takes one of the following forms:

(1a). $B' = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, $0 < \theta < \pi$; $A' = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $a > 0, d > 0$. (21)

(1b). $B' = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, $0 < \theta < \pi$; $A' = \begin{pmatrix} 0 & b \\ b & d \end{pmatrix}$, $b \geq 0, d \geq 0$. (22)

(1c). $B' = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, $0 < \theta < \pi$; $A' = \begin{pmatrix} a & b \\ b & 0 \end{pmatrix}$, $a > 0, b \geq 0$. (23)

(2a). $B' = \begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix}$, $0 < \tau < 1$; $A' = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $b > 0, |a| = \frac{1}{2}$. (24)

(2b). $B' = \begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix}$, $0 < \tau < 1$; $A' = \begin{pmatrix} 0 & b \\ b & d \end{pmatrix}$, $b > 0, |d| = \frac{1}{2}$. (25)

(2c). $B' = \begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix}$, $0 < \tau < 1$; $A' = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$, $b > 0$. (26)

(2d). $B' = \begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix}$, $0 < \tau < 1$; $A' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$. (27)

(2e). $B' = \begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix}$, $0 < \tau < 1$; $A' = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. (28)

(2f). $B' = \begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix}$, $0 < \tau < 1$; $A' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. (29)

(3a). $B' = \begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$; $A' = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $a > 0, b \in \mathbb{R}$. (30)

(3b). $B' = \begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$; $A' = \begin{pmatrix} 0 & b \\ b & d \end{pmatrix}$, $b > 0, d \in \mathbb{R}$. (31)

(3c). $B' = \begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$; $A' = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$, $d \geq 0$. (32)
We remark that we made the change on the forms in (3a) and (3b) for the $B'$ part (compared with the original list in [Co]). (See [p950, Co]). In the cases of (5)-(9), the manifold is already quadratically flattened. Therefore we restrict our discussion in this and the next sections to the cases in (1)-(4). We will prove the following theorem in §4.

**Theorem 3.1.** Let $M$ be a real codimension two submanifold in $\mathbb{C}^3$ with $0 \in M$ a CR singular point. Suppose that $M$ near $p = 0$ is defined by an equation as in (7). Assume that $M$ is CR
non-minimal at its CR points. Then $M$ can be quadratically flattened, possibly except in the cases when the normal form for the pair of the matrices $\{A, B\}$ takes the following form:

\[(i) \quad A' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \text{ or } (ii) \quad A' = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

or (iii) $A' = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, B' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

In particular, if $B$ is non-degenerate, namely $\det(B) \neq 0$, then $M$ is always quadratically flattenable.

The following examples show that Theorem 3.1 is optimal.

Example 3.1: ([Hu2]): Let $M \subset \mathbb{C}^3$ with coordinates $(z_1, z_2, w)$ be defined by $w = z_1 \overline{z}_2 + z_1 z_2 + \overline{z}_1 \overline{z}_2$. It is easy to see that $0 \in M$ is a CR singular point and that the quadratic term in the defining function takes the following normal form:

$$A' = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$  

Now, following [Hu2], we verify that $M$ is CR non-minimal at its CR points. For a CR point $p_0 \in M$, if $p_0 \in \{z_2 = w = 0\}$, $M$ must be non-minimal at $p_0$. Otherwise $p_0 \notin \{z_2 = w = 0\}$ and we define $L = (z_1 + \overline{z}_1) \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} + (\overline{z}_1 z_2 + \overline{z}_2 z_1 + \overline{z}_2 \overline{z}_1) \frac{\partial}{\partial w}$. Write $h = -w + z_1 \overline{z}_2 + z_1 z_2 + \overline{z}_1 \overline{z}_2$.

Then we have $L(h) = L(h) = 0$ along $M$. Hence $L$ is a holomorphic tangent vector field along $M$, which is non-vanishing at any CR point of $M$. Define $\chi = (z_1 + \overline{z}_1)|z_2|^2$, which is real valued. Then one computes that $L(\chi) = 0$. As shown in [Hu2], $\{h = 0, \overline{h} = 0, \chi = 0\}$ defines a submanifold $X_{p_0}$ in $M$ of real dimension 3 near $p_0$. Since $L$ is tangent to $X_{p_0}$, $X_{p_0}$ has to be a CR submanifold with CR dimension one near $p_0$. By definition, $M$ is non-minimal at $p_0$. However $(M, 0)$ can not be quadratically flattened.

Example 3.2: Let $M \subset \mathbb{C}^3$ with coordinates $(z_1, z_2, w)$ be defined by $w = z_1 \overline{z}_2 + \frac{1}{2} z_2^2 + \frac{1}{2} \overline{z}_2^2$. It is easy to see that $0 \in M$ is a CR singular point and that the quadratic term in the defining function takes the following normal form:

$$A' = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$ 

Define $L = (z_2 + \overline{z}_2) \frac{\partial}{\partial z_2} + (z_2 \overline{z}_2 + \overline{z}_2 z_2) \frac{\partial}{\partial w}$ and $\chi = |z_2|^2$. Write $h = -w + z_1 \overline{z}_2 + \frac{1}{2} z_2^2 + \frac{1}{2} \overline{z}_2^2$. Then one computes that $L(h) = L(h) = L(\chi) = 0$. Through the similar argument as above, we verify that $M$ is CR non-minimal at its CR points. However $(M, 0)$ can not be quadratically flattened.

Example 3.3: Let $M \subset \mathbb{C}^3$ with coordinates $(z_1, z_2, w)$ be defined by $w = z_1 \overline{z}_2$. It is easy to see that $0 \in M$ is a CR singular point and that the quadratic term in the defining function
We further compute (52), (10) and (11) to get the following:

\[
\mathcal{A}' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Define \( L = \overline{z_1} \frac{\partial}{\partial z_1} + \overline{z_2} \frac{\partial}{\partial z_2} \) and \( \chi = |z_2|^2 \). Write \( h = -w + z_1 \overline{z_2} \). Then one computes that \( L(h) = L(h) = L(\chi) = 0 \). Through a similar argument as above, one verifies that \( M \) is CR non-minimal at its CR points. However \((M,0)\) can not be quadratically flattened.

### 4 Proof of Theorem 3.1

In this section, we give a proof of Theorem 3.1. Assume that \( M \) is defined by (7) with \( p = 0 \in M \) a CR singular point. We adapt the notations we have set up so far. We assume that \( \{ \mathcal{A}, \mathcal{B} \} \) already takes the normal form listed in the above section. Our argument is through a computation based on the fundamental identity (18) obtained in §2. Since \( \mathcal{B} \neq 0 \), it is apparent that any CR point in \( M \) has CR dimension one. Hence, the fundamental identity (18) or (20) can be applied.

#### 4.1 Case (1a)

We start by assuming that the pair \( \{ \mathcal{A}, \mathcal{B} \} \) takes the normal form in (1a). A direct computation gives the following:

\[
w = (a \bar{z}_2^2 + a \bar{z}_1 z_2 + 2 \bar{b} z_1 z_2 + d z_2^2 + d \bar{z}_2^2) + z_1 \bar{z}_1 + \cos \theta |z_2|^2 + i(\sin \theta |z_2|^2) + o(|z|),
\]

\[
G = (a \bar{z}_2^2 + a \bar{z}_1 z_2 + 2 \bar{b} z_1 z_2 + d z_2^2 + d \bar{z}_2^2) + z_1 \bar{z}_1 + \cos \theta |z_2|^2 + o(|z|),
\]

\[
E = \sin \theta |z_2|^2 + o(|z|), \quad G_1 = 2a z_1 + 2b z_2 + \bar{z}_1 + o(|z|), \quad G_2 = \cos \theta \bar{z}_2 + 2b z_1 + 2d z_2 + o(|z|),
\]

\[
E_1 = o(|z|), \quad E_2 = \sin \theta \bar{z}_2 + o(|z|).
\]

We further compute (52), (10) and (11) to get the following:

\[
A = e^{-i\theta} \bar{z}_2 + 2b z_1 + 2d z_2 + o(|z|), \quad B = 2a z_1 + 2b z_2 + \bar{z}_1 + o(|z|),
\]

\[
\lambda_1 = -(2a z_1 + 2b z_2 + \bar{z}_1)e^{i\theta} + o(|z|), \quad \lambda_2 = -(e^{-i\theta} \bar{z}_2 + 2b z_1 + 2d z_2) + o(|z|),
\]

\[
\lambda_3 = (2a \bar{z}_2 + 2 \bar{b} z_2 + z_1)e^{-i\theta} + o(|z|), \quad \lambda_4 = (e^{i\theta} z_2 + 2 \bar{b} \bar{z}_1 + 2d z_2) + o(|z|),
\]

\[
\Gamma_1 = -2a \bar{z}_2 + (4b^2 - 4ad) e^{i\theta} \bar{z}_2 + 2a \bar{z}_1 e^{i\theta} + o(|z|),
\]

\[
\Gamma_2 = -2b e^{-i\theta} \bar{z}_2 + (4ad - 4b^2) z_1 + 2d \bar{z}_1 + o(|z|),
\]

\[
\Gamma_3 = 2b e^{-i\theta} z_1 - (4a e^{-i\theta} + 4 \bar{b} d) \bar{z}_1 + (4d e^{-i\theta} - 2 \bar{b} d) z_2 + (2e^{-2i\theta} - 4|b|^2 e^{-i\theta} - 4d^2) \bar{z}_2 + o(|z|),
\]

\[
\Gamma_4 = (2a e^{-i\theta} - 4ae^{i\theta}) z_1 + (4d^2 e^{-i\theta} + 4|b|^2 - 2e^{i\theta}) \bar{z}_1 - 2b e^{i\theta} z_2 + (4a b e^{i\theta} + 4bd) \bar{z}_2 + o(|z|).
\]
Substituting the above quantities into (19), we get $X_1, X_2, Y_1$ and $Y_2$:

$$X_1 = (2be^{i\theta} + \overline{b}(4ad - 4b^2))z_1\overline{z}_1 + (-2a + 2d(4ad - 4b^2))z_1\overline{z}_2 + (4abe^{i\theta} + 4\overline{bd})z_1\overline{z}_1 + (2a(4b^2 - 4ad)e^{i\theta} + 2de^{i\theta})\overline{z}_1\overline{z}_2 + (-4a^2 + 4d^2 + 4|b|^2e^{i\theta} - 4|b|^2e^{-i\theta})z_1\overline{z}_2 + (2b(4b^2 - 4ad)e^{i\theta} - 2b)\overline{z}_2\overline{z}_2 + (-4a\overline{b} - 4bde^{-i\theta})\overline{z}_2\overline{z}_2 + o(|z|^2),$$

$$X_2 = (2ae^{-i\theta})z_1z_1 + (4|b|^2 + 4a^2e^{-i\theta} + e^{-i\theta})z_1\overline{z}_1 + (2be^{i\theta} + 2b^{-i\theta})z_1z_2 + (2ae^{-i\theta})z_1\overline{z}_1 + (4bd + 4a\overline{b}e^{-i\theta})z_1\overline{z}_2 + (4\overline{b}e^{-i\theta})z_1\overline{z}_2 + (2de^{i\theta})z_2z_2 + (4d^2 + 2 + 4|b|^2e^{-i\theta})z_2\overline{z}_2 + (2de^{-i\theta})\overline{z}_2\overline{z}_2 + o(|z|^2),$$

$$Y_1 = (8abe^{-i\theta} - 8abe^{i\theta})z_1z_1 + (8b|b|^2 - 8abd + 2be^{-i\theta} - 4be^{i\theta})z_1\overline{z}_1 + (-4ab - 4\overline{bd})\overline{z}_1\overline{z}_1 + (8a^2e^{-i\theta} + 4de^{-i\theta} - 6de^i - 8a^2e^{-i\theta})z_1\overline{z}_2 + (4a^2e^{-i\theta} - 4d^2 + 2e^{-2i\theta} - 2)\overline{z}_1\overline{z}_2 + (8bde^{-i\theta} - 8bdde^{i\theta})z_2z_2 + (8\overline{b}e^{-i\theta} - 4bdde^{-i\theta} - 8b|b|^2e^{-i\theta} - 2b)\overline{z}_2\overline{z}_2 + (4\overline{b}e^{-i\theta} - 4bde^{i\theta})\overline{z}_2\overline{z}_2 + o(|z|^2),$$

$$Y_2 = (-2ae^{i\theta})z_1z_1 + (-4a^2e^{i\theta} + 4d^2 - e^{i\theta})z_1\overline{z}_1 + (-4ae^{i\theta})z_1z_2 + (-4ae^{i\theta} - 4\overline{bd})z_1\overline{z}_2 + (2ae^{i\theta})z_1\overline{z}_2 + (2ae^{i\theta})z_2z_2 + (4d^2 + 1 + 4|b|^2e^{i\theta})z_2\overline{z}_2 + (2de^{i\theta})\overline{z}_2\overline{z}_2 + o(|z|^2).$$

Now we substitute the above formulas into (20): $X_1X_2 = Y_1Y_2$. Comparing the coefficients for the $z_1^4$ terms, we have:

$$(4abe^{i\theta} + 4\overline{bd})(2ae^{-i\theta}) = (-4ab - 4\overline{bd})(-2ae^{i\theta}).$$

Hence, we see that $8\overline{bd}(e^{i\theta} - e^{-i\theta}) = 0$.

Because $a > 0, d > 0, 0 < \theta < \pi$, we must have $b = 0$. Substituting $b = 0$ back into $X_1, X_2, Y_1$ and $Y_2$, we obtain:

$$X_1 = (4d^2 - 4a^2)z_1\overline{z}_2 + (2de^{-i\theta} - 8a^2de^{i\theta})\overline{z}_1\overline{z}_2 + (-2a + 8ad^2)z_1\overline{z}_2 + o(|z|^2),$$

$$X_2 = (2ae^{-i\theta})z_1z_1 + (4a^2e^{-i\theta} + e^{-i\theta})z_1\overline{z}_1 + (2ae^{i\theta})z_1\overline{z}_2 + (2de^{i\theta})z_2z_2 + (4d^2 + 1)z_2\overline{z}_2 + (2de^{-i\theta})\overline{z}_2\overline{z}_2 + o(|z|^2),$$

$$Y_1 = (12ade^{-i\theta} - 12ade^{i\theta})z_1z_2 + (6a^2e^{-i\theta} - 4a - 8ad^2)z_1\overline{z}_2 + (8a^2de^{-i\theta} + 4de^{-i\theta} - 6de^i)\overline{z}_1\overline{z}_2 + (4a^2e^{-2i\theta} - 2 + 2e^{-2i\theta} - 4d^2)\overline{z}_1\overline{z}_2 + o(|z|^2),$$

$$Y_2 = (-2ae^{i\theta})z_1z_1 + (-4a^2e^{i\theta} - e^{i\theta})z_1\overline{z}_1 + (2ae^{i\theta})z_1\overline{z}_2 + (2ae^{i\theta})z_2z_2 + (4d^2 + 1 + 4|b|^2e^{i\theta})z_2\overline{z}_2 + (2de^{i\theta})\overline{z}_2\overline{z}_2 + o(|z|^2).$$

Now compare the coefficients for the $z_2^3\overline{z}_1$ terms in (20) after being substituted by (46). We then have:

$$(2de^{i\theta} - 8a^2de^{i\theta})(2de^{-i\theta}) = (8a^2de^{-i\theta} + 4de^{-i\theta} - 6de^i)(-2de^{i\theta}).$$

Thus, we see that $(16a^2d^2 + 8d^2)(1 - e^{2i\theta}) = 0$. However this can not hold since $a > 0, d > 0$ and $0 < \theta < \pi$. Thus, we proved that, under the assumptions in Theorem 3.1 the pair \{\mathcal{A}, \mathcal{B}\} can not take the normal form in (1a).
4.2 Case (1b)

We now assume that the pair \( \{A, B\} \) takes the normal form in (1b). We modify the computation we just did for Case (1a) by substituting \( a = 0 \) into \( X_1, X_2, Y_1 \) and \( Y_2 \) and by requiring that \( b \geq 0, d \geq 0 \). We then have:

\[
X_1 = (2be^{i\theta} - 8b^3)z_1\overline{z}_1 - (8b^2d)z_1\overline{z}_2 + (4bd)z_1\overline{z}_2 + (2de^{i\theta})z_1\overline{z}_2 + (8b^3e^{i\theta} - 2b)z_2\overline{z}_2 + (4b^2e^{i\theta} - 4b^3e^{-i\theta} + 4d^2)z_1\overline{z}_2 + (-4bde^{-i\theta})z_2\overline{z}_2 + o(|z|^2),
\]

\[
X_2 = (4b^2 + e^{-i\theta})z_1\overline{z}_1 + (2be^{i\theta} + 2be^{-i\theta})z_1\overline{z}_2 + (4bd)z_1\overline{z}_2 + (4be^{-i\theta})z_1\overline{z}_2 + (2de^{i\theta})z_2\overline{z}_2 + (4b^2e^{i\theta} - 4b^3e^{-i\theta})z_2\overline{z}_2 + o(|z|^2),
\]

\[
Y_1 = (8b^3 + 2be^{-i\theta} - 4be^{i\theta})z_1\overline{z}_1 + (4b^2e^{-i\theta} - 4b^2e^{i\theta})z_1\overline{z}_2 + (8b^2d)z_1\overline{z}_2 + (-4bd)z_1\overline{z}_1 + (4de^{-i\theta} - 6de^{i\theta})z_1\overline{z}_2 + (2e^{-2i\theta} - 4d^2 - 2)z_1\overline{z}_2 + (8bde^{-i\theta} - 8bde^{i\theta})z_2\overline{z}_2 + (4b^2e^{-i\theta} - 2b - 8b^3e^{-i\theta})z_2\overline{z}_2 + o(|z|^2),
\]

\[
Y_2 = (-e^{-i\theta} - 4b^2)z_1\overline{z}_1 + (-4be^{i\theta})z_1\overline{z}_2 + (-4bd)z_1\overline{z}_2 + (-2e^{i\theta} - 2be^{i\theta})z_1\overline{z}_2 + (-4bd)z_2\overline{z}_2 + (-2de^{-i\theta})z_2\overline{z}_2 + o(|z|^2).
\]

Substituting \( X_1, X_2, Y_1 \) and \( Y_2 \) into (20) and comparing the coefficients for the \( z_2^2 \) terms, we have:

\[
(8bde^{-i\theta} - 8bde^{i\theta})(-2de^{i\theta}) = 0.
\]

Since \( 0 < \theta < \pi \), we either have \( b = 0 \) or \( d = 0 \). If \( b = 0 \), we have:

\[
X_1 = (2de^{i\theta})z_1\overline{z}_2 + 4d^2\overline{z}_1\overline{z}_2 + o(|z|^2),
\]

\[
X_2 = e^{-i\theta}z_1\overline{z}_1 + (2de^{i\theta})z_2\overline{z}_2 + (4d^2 + 1)z_2\overline{z}_2 + (2de^{i\theta})z_2\overline{z}_2 + o(|z|^2),
\]

\[
Y_1 = (4de^{-i\theta} - 6de^{i\theta})z_1\overline{z}_2 + (2e^{-2i\theta} - 4d^2 - 2)\overline{z}_1\overline{z}_2 + o(|z|^2),
\]

\[
Y_2 = -e^{-i\theta}z_1\overline{z}_1 + (-2de^{i\theta})z_2\overline{z}_2 + (-1 - 4d^2)\overline{z}_2\overline{z}_2 + (-2de^{-i\theta})\overline{z}_2\overline{z}_2 + o(|z|^2).
\]

Comparing the coefficients for the \( z_1\overline{z}_1 \overline{z}_2 \) terms in (20), we obtain:

\[
e^{-i\theta}4d^2 = -e^{-i\theta}(2e^{-2i\theta} - 4d^2 - 2).
\]

This contradicts with the fact \( 0 < \theta < \pi \). If \( d = 0 \),

\[
X_1 = (2be^{i\theta} - 8b^3)z_1\overline{z}_1 + (4b^2(e^{i\theta} - e^{-i\theta}))z_1\overline{z}_2 + (8b^3e^{i\theta} - 2b)z_2\overline{z}_2 + o(|z|^2),
\]

\[
X_2 = (2be^{i\theta} + 2be^{-i\theta})z_1\overline{z}_2 + (4b^2 + e^{-i\theta})z_1\overline{z}_1 + (4be^{-i\theta})z_1\overline{z}_2 + (1 + 4b^2e^{-i\theta})z_2\overline{z}_2 + o(|z|^2),
\]

\[
Y_1 = (4b^2e^{-i\theta} - 4b^2e^{i\theta})z_1\overline{z}_1 + (8b^3 - 4be^{i\theta} + 2be^{i\theta})z_1\overline{z}_2 + (2e^{-2i\theta} - 2)\overline{z}_1\overline{z}_2 + (-2b + 4be^{-2i\theta} - 8b^3e^{-i\theta})z_2\overline{z}_2 + o(|z|^2),
\]

\[
Y_2 = (-e^{i\theta} - 4b^2)z_1\overline{z}_1 + (-4be^{i\theta})z_1\overline{z}_2 + (-4b^2e^{i\theta} - 1)z_2\overline{z}_2 + (-2be^{i\theta} - 2be^{i\theta})\overline{z}_1\overline{z}_2 + o(|z|^2).
\]

Comparing the coefficients for the \( z_1\overline{z}_1^2 \) terms in (20), we have:

\[
(2be^{i\theta} - 8b^3)(4b^2 + e^{-i\theta}) = (8b^3 - 4be^{i\theta} + 2be^{i\theta})(-e^{-i\theta} - 4b^2).
\]

Therefore \( 4b(1 - e^{2i\theta}) = 0 \). Since \( 0 < \theta < \pi \), we conclude \( b = 0 \) and thus we are in the setting of the previous case, which is impossible as just shown. Thus, we proved that, under the assumptions in Theorem 3.1, the pair \( \{A, B\} \) can not take the normal form in (1b).
4.3 Case \((1c)\)

Assume that the pair \(\{A, B\}\) takes the normal form in \((1c)\). We modify our argument in \((1a)\) by substituting \(d = 0\) into \(X_1, X_2, Y_1, Y_2\) and by requiring that \(b \geq 0, a > 0\). We have:

\[
X_1 = (2be^{i\theta} - 8b^3)z_1\overline{z}_1 + (-2a)z_1z_2 + (4abe^{i\theta})\overline{z}_1z_1 + (-4a^2 + 4b^2e^{i\theta} - 4b^2e^{-i\theta})z_1\overline{z}_2 + (8ab^2e^{i\theta})z_1z_2 + (8b^3e^{i\theta} - 2b)z_2\overline{z}_2 + (-4ab)z_2\overline{z}_2 + o(|z|^2),
\]

\[
X_2 = (2ae^{-i\theta})z_1z_1 + (2be^{i\theta} + 2be^{-i\theta})z_1z_2 + (4b^2 + 4a^2e^{i\theta} + e^{-i\theta})z_1\overline{z}_1 + (4abe^{-i\theta})z_1\overline{z}_2 + (4abe^{-i\theta})z_2\overline{z}_1 + (4be^{-i\theta})z_1\overline{z}_2 + (4be^{-i\theta})z_2\overline{z}_1 + (1 + 4b^2e^{-i\theta})z_2\overline{z}_2 + o(|z|^2),
\]

\[
Y_1 = (8abe^{-i\theta} - 8abe^{i\theta})z_1z_1 + (8b^3 - 4be^{i\theta} + 2be^{-i\theta})z_1\overline{z}_1 + (4b^2e^{-i\theta} - 4b^2e^{i\theta})z_1z_2
\]

\[
+ (6ae^{-2i\theta} - 4a)z_1\overline{z}_2 + (-4abe^{-i\theta})\overline{z}_1z_1 + (-8a^2e^{i\theta} - 2 + 2e^{-2i\theta})\overline{z}_1z_2 + (4a^2e^{-2i\theta} - 2 + 2e^{2i\theta})z_1\overline{z}_2 + (-2b + 4be^{-2i\theta} - 8b^3e^{-i\theta})z_2\overline{z}_2 + (4abe^{-2i\theta})z_2\overline{z}_2 + o(|z|^2),
\]

\[
Y_2 = (-2ae^{i\theta})z_1z_1 + (-4abe^{i\theta})z_1\overline{z}_2 + (-4be^{i\theta})z_1z_2 + (-4a^2e^{i\theta} - e^{-i\theta} - 4b^2)z_1\overline{z}_1
\]

\[
+ (-2ae^{i\theta})\overline{z}_1z_1 + (-4abe^{i\theta})\overline{z}_1z_2 + (-2be^{i\theta} - 2be^{-i\theta})\overline{z}_1\overline{z}_2 + (-4b^2e^{i\theta} - 1)z_2\overline{z}_2 + o(|z|^2) + o(|z|^2).
\]

Compare the coefficients for the \(z_3^2\overline{z}_2^2\) terms in \((20)\). We get

\[
(-4ab)(1 + 4b^2e^{-i\theta}) = (4abe^{-2i\theta})(-4b^2e^{i\theta} - 1).
\]

Therefore, \(4ab(e^{2i\theta} - 1) = 0\). Since \(0 < \theta < \pi\), we conclude that either \(a = 0\) or \(b = 0\). The case for \(a = 0\) can be included in Case \((1b)\), which has been shown to be impossible. When \(b = 0\), we have:

\[
X_1 = (-2a)z_1\overline{z}_2 + (-4a^2)\overline{z}_1\overline{z}_2 + o(|z|^2),
\]

\[
X_2 = (2ae^{-i\theta})z_1z_1 + (4a^2e^{-i\theta} + e^{-i\theta})z_1\overline{z}_1 + (2ae^{-i\theta})\overline{z}_1\overline{z}_1 + z_2\overline{z}_2 + o(|z|^2),
\]

\[
Y_1 = (6ae^{-2i\theta} - 4a)z_1\overline{z}_2 + (4a^2e^{-2i\theta} - 2 + 2e^{-2i\theta})\overline{z}_1\overline{z}_2 + o(|z|^2),
\]

\[
Y_2 = (-2ae^{i\theta})z_1z_1 + (-4a^2e^{i\theta} - e^{-i\theta})z_1\overline{z}_1 + (-2ae^{i\theta})\overline{z}_1\overline{z}_1 + z_2\overline{z}_2 + o(|z|^2).
\]

Comparing the coefficients for the \(z_3^2\overline{z}_2\) in \((20)\), we get

\[
(-2a)(2ae^{-i\theta}) = (6ae^{-2i\theta} - 4a)(-2ae^{i\theta}).
\]

Hence, \(8a^2(1 - e^{2i\theta}) = 0\). Since \(0 < \theta < \pi\), \(a = 0\), which is reduced to the case discussed above. Thus, we proved that, under the assumptions in Theorem \(3.1\), the pair \(\{A, B\}\) can not take the normal form in \((1c)\).

4.4 Case \((2a)\)

Assume that the pair \(\{A, B\}\) takes the normal form in \((2a)\). Notice that in Case \((2a) - (2f)\), \(b\) is required to be a real number. A direct computation gives the following:

\[
w = (az_1^2 + \overline{a}\overline{z}_1^2 + 2bz_1z_2 + 2b\overline{z}_1\overline{z}_2 + d\overline{z}_2^2 + d\overline{z}_2^2) + z_1\overline{z}_2 + r\overline{z}_1z_2 + o(|z|^2),
\]

\[
G = (az_1^2 + \overline{a}\overline{z}_1^2 + 2bz_1z_2 + 2b\overline{z}_1\overline{z}_2 + d\overline{z}_2^2 + d\overline{z}_2^2) + \frac{1 + r}{2}(z_1\overline{z}_2 + \overline{z}_1z_2) + o(|z|^2),
\]

\[
E = \frac{1 + r}{2}(z_1\overline{z}_2 + \overline{z}_1z_2) + o(|z|^2),
\]

\[
G_1 = 2az_1 + 2bz_2 + \frac{1 + r}{2}\overline{z}_2 + o(|z|),
\]

\[
G_2 = 2bz_1 + 2dz_2 + \frac{1 + r}{2}\overline{z}_1 + o(|z|),
\]

\[
E_1 = \frac{1 + r}{2}\overline{z}_2 + o(|z|),
\]

\[
E_2 = \frac{1 + r}{2}(\overline{z}_1) + o(|z|).
\]
We further compute (52), (10) and (11) to derive the following:

\[ A = 2bz_1 + 2d\bar{z}_2 + \tau + o(|z|), \quad B = 2az_1 + 2bz_2 + \tau \bar{z}_2 + o(|z|), \]
\[ \lambda(1) = 2bz_1 + 2d\bar{z}_2 + \tau + o(|z|), \quad \lambda(2) = (2az_1 + 2bz_2 + \tau \bar{z}_2) + o(|z|), \]
\[ \lambda(4) = -(2b\bar{z}_1 + 2d\bar{z}_2 + z_1) + o(|z|), \quad \lambda(5) = - (2a\bar{z}_1 + 2b\bar{z}_2 + \tau z_2) + o(|z|), \]
\[ \Gamma(1) = (4b^2 - 4ad)z_1 + 2b\bar{z}_1 - 2\tau \bar{z}_2 + o(|z|), \]
\[ \Gamma(2) = (4ad\tau - 4b^2\tau)z_2 + (2a\tau)\bar{z}_1 - 2b\tau^2 \bar{z}_2 + o(|z|), \]
\[ \Gamma(4) = -2bz_1 + (4b^2 + 4\alpha d\tau - 2)\bar{z}_1 + (2d\tau^2 - 4d)z_2 + (4b\bar{d} + 4b\tau d)\bar{z}_2 + o(|z|), \]
\[ \Gamma(5) = (4a\tau^2 - 2a)z_1 + (-4ab - 4\alpha \bar{b} \tau)\bar{z}_1 + (2b\tau^2)z_2 + (2\tau^3 - 4b^2\tau - 4ad)\bar{z}_2 + o(|z|). \]

Substituting the above quantities into (19), we have

\[ X_1 = (2\bar{a}(4b^2 - 4ad) + 2a\tau)z_1\bar{z}_1 + (4\bar{a}b + 4\alpha b \tau)\bar{z}_1\bar{z}_2 + (2b(4b^2 - 4ad) - 2b\tau^2)z_1\bar{z}_2 \]
\[ + (2b\tau + 2(4ad\tau - 4b^2\tau))\bar{z}_1 z_2 + (4\tau a\bar{d} - 4\alpha \bar{d} a + 4b^2 - 4b^2\tau^2)\bar{z}_1 \bar{z}_2 \]
\[ + (2\bar{d}(4ad\tau - 4b^2\tau) - 2\tau^2 d)z_2 \bar{z}_2 + (-4\tau b d - 4\alpha \tau d^2)\bar{z}_2 \bar{z}_2 + o(|z|^2), \]
\[ X_2 = -2a\bar{z}_1 z_1 + (4\tau b - 4ab)z_1 \bar{z}_1 + (-2b\tau^2 - 2b)z_1 z_2 + (-4b^2\tau - 4ad\tau - \tau)z_1 \bar{z}_2 \]
\[ + (2\tau \bar{z}_1 \bar{z}_1 + (-4ad\tau - \tau^2 - 2b^2)\bar{z}_1 z_2 + (4\tau b - 4ab)\bar{z}_1 \bar{z}_2 + (2\tau^2)z_2 \bar{z}_2 \]
\[ + (4\alpha b - 4d\tau)z_2 \bar{z}_2 + (-2\bar{d}\tau - 2\tau d)\bar{z}_2 \bar{z}_2 + o(|z|^2), \]
\[ Y_1 = (8\alpha b^2 - 8ab)z_1 z_1 + (4\alpha r^2 - 6a + 8\alpha a^2 d\tau - 8\alpha b d^2 \tau)z_1 \bar{z}_1 + (4\alpha d^2 + 4b\alpha \tau^2)z_2 \bar{z}_2 \]
\[ + (4b^3 - 8b^2 \tau + 8ab \alpha d\tau - 2b)z_1 z_2 + (-4ab - 4\alpha b \tau)z_1 \bar{z}_1 + (8\alpha d^2 - 8bd)z_2 \bar{z}_2 \]
\[ + (2b(4b^2 + 4\alpha d\tau - 2) + 2b\tau^2 - 2d(4ab + 4\alpha b \tau))\bar{z}_1 \bar{z}_2 + (2\tau^2 - 4d + 4\alpha b \tau - 4ad)\bar{z}_1 \bar{z}_2 \]
\[ + (6\alpha d - 8a|d|^2 + 8b^2 d - 4d\tau)z_2 \bar{z}_2 + (12\alpha d^2 - 12ad + 4b^2 \tau^2 - 4b^2)z_1 z_2 + o(|z|^2), \]
\[ Y_2 = (2a\tau)z_1 z_1 + (4\alpha b + 4ab)z_1 \bar{z}_1 + (4b\tau)z_1 z_2 + (4b^2 + 4b \alpha \tau + \tau^2)z_1 \bar{z}_2 + (2\tau)z_2 \bar{z}_2 \]
\[ + (4\alpha d + \tau + 4b^2 \tau)\bar{z}_1 \bar{z}_2 + (2b + 2b\tau^2)\bar{z}_1 \bar{z}_2 + (2\tau)z_2 \bar{z}_2 + (4bd + 4\alpha d\tau)z_2 \bar{z}_2 \]
\[ + (2\tau^2)\bar{z}_2 \bar{z}_2 + o(|z|^2). \]

Now comparing the coefficients for the \(z_1^4\) terms in (20), we have:

\[ (8ab\tau^2 - 8ab)(2a\tau) = 0. \]

Since \(a, b \neq 0\) and \(0 < \tau < 1\), the above equation has no solution. Thus, we proved that, under the assumptions in Theorem 3.1, the pair \(\{A, B\}\) cannot take the normal form in (2a).

### 4.5 Case (2b)

We now assume that the pair \(\{A, B\}\) takes the normal form in (2b). Modifying the formulas in Case (2a) by substituting \(a = 0\) into \(X_1, X_2, Y_1, Y_2\) and by requiring that \(b, d \neq 0\) and \(0 < \tau < 1\),
we have the following:

\[
X_1 = (8b^3 - 2b^2)z_1 \overline{z}_2 + (2b \tau - 8b^3 \tau)z_1 \overline{z}_2 + (4b^2 - 4b^2 \tau^2)z_1 \overline{z}_2 + (-8bd^2 \tau - 2\tau^2 d)z_2 \overline{z}_2
\]
\[+ (-4\tau bd - 4bd^2 \tau^2)z_2 \overline{z}_2 + o(|z|^2),
\]
\[
X_2 = (-2b^2 - 2b)z_1 z_2 + (-4b^3 - 8b^3 \tau - 2b \tau)z_1 \overline{z}_2 + (-4br)z_1 \overline{z}_2
\]
\[+ (-2b\tau^2)z_2 \overline{z}_2 + (-4bd\tau - 4b \overline{d} \tau)z_2 \overline{z}_2 + o(|z|^2),
\]
\[
Y_1 = (4b^2 \tau^2 - 4b^2)z_1 z_2 + (4b^3 - 8b^3 \tau - 2b \tau)z_1 \overline{z}_2 + (2\tau^3 - 2\tau)z_1 \overline{z}_2 + (8b^3 - 4b + 2b\tau^2)z_1 z_2
\]
\[+ (8bd\tau^2 - 8bd)z_2 \overline{z}_2 + (6d\tau^3 + 8b^2 \overline{d} - 4d\tau)z_2 \overline{z}_2 + (4b \overline{d} \tau + 4bd^2 \tau^2)z_2 \overline{z}_2 + o(|z|^2),
\]
\[
Y_2 = (4b\tau)z_1 z_2 + (4b^2 + \tau^2)z_1 \overline{z}_2 + (\tau + 4b^2 \tau)z_1 \overline{z}_2 + (2b + 2b\tau^2)z_1 \overline{z}_2 + (2d\tau)z_2 \overline{z}_2 + (4bd + 4b \overline{d} \tau)z_2 \overline{z}_2 + (2\overline{d} \tau^2)z_2 \overline{z}_2 + o(|z|^2).
\]

Now substitute the above formulas for \(X_1, X_2, Y_1\) and \(Y_2\) into (20). By comparing the coefficients for the \(z_1^2\) terms, we have:

\[(8bd\tau^2 - 8bd)2d\tau = 0,
\]

which is impossible because \(b, d \neq 0\) and \(0 < \tau < 1\). Thus, we proved that, under the assumptions in Theorem 3.1, the pair \(\{A, B\}\) can not take the normal form in (2b).

### 4.6 Case (2c)

Assume that the pair \(\{A, B\}\) takes the normal form in (2c). Modifying the formulas in the Case (2a) and (2b) by substituting \(a = 0, d = 0\) into \(X_1, X_2, Y_1, Y_2\), we have the following:

\[
X_1 = (8b^3 - 2b^2 \tau)z_1 \overline{z}_2 + (2b \tau - 8b^3 \tau)z_1 \overline{z}_2 + (4b^2 - 4b^2 \tau^2)z_1 \overline{z}_2 + o(|z|^2),
\]
\[
X_2 = (-2b^2 - 2b)z_1 z_2 + (-4b^3 - 8b^3 \tau - 2b \tau)z_1 \overline{z}_2 + (-4b \tau)z_1 \overline{z}_2 + o(|z|^2),
\]
\[
Y_1 = (4b^2 \tau^2 - 4b^2)z_1 z_2 + (4b^3 - 8b^3 \tau - 2b \tau)z_1 \overline{z}_2 + (2\tau^3 - 2\tau)z_1 \overline{z}_2
\]
\[+ (8b^3 - 4b + 2b\tau^2)z_1 \overline{z}_2 + o(|z|^2),
\]
\[
Y_2 = (4b\tau)z_1 z_2 + (4b^2 + \tau^2)z_1 \overline{z}_2 + (\tau + 4b^2 \tau)z_1 \overline{z}_2 + (2b + 2b\tau^2)z_1 \overline{z}_2 + o(|z|^2).
\]

Comparing the coefficients for the \(z_1^2 z_2^2\) terms in the identity \(X_1X_2 = Y_1Y_2\), we have:

\[(4b^2 \tau^2 - 4b^2)4b\tau = 0.
\]

This is not possible because \(b \neq 0\) and \(0 < \tau < 1\). Thus, under the assumptions in Theorem 3.1, the pair \(\{A, B\}\) can not take the normal form in (2c).
4.7 Case (2d), (2e), (2f)

Use the computation we derived in the Case (2a) and substitute $b = 0$ into $X_1, X_2, Y_1, Y_2$. We have the following:

$$X_1 = (2a\tau - 8|a|^2d)z_1\bar{z}_1 + (4\tau\Re - 4\tau\Re d)\bar{z}_1\bar{z}_2 + (8a|d|^2\tau - 2\tau^2d)z_2\bar{z}_2 + o(|z|^2),$$

$$X_2 = (-2a)z_1z_1 + (-4ad - \tau)z_1\bar{z}_2 + (-2a\tau)\bar{z}_1\bar{z}_1 + (-4ad\tau - \tau^2)\bar{z}_1z_2 + (-2d\tau^2)z_2\bar{z}_2 + o(|z|^2),$$

$$Y_1 = (4ar^2 - 6a + 8|a|^2d\tau)z_1\bar{z}_1 + (12ad\tau^2 - 12ad)z_1z_2 + (2\tau^3 - 2\tau + 4\bar{a}\tau^2 - 4a\bar{d})\bar{z}_1\bar{z}_2 + (6a\tau^3 - 8|a|^2d - 4a\bar{d})z_1z_2 + o(|z|^2),$$

$$Y_2 = (2ar)z_1z_1 + (4ad\tau + \tau^2)z_1\bar{z}_2 + (2\bar{a})\bar{z}_1\bar{z}_1 + (4\bar{a}d + \tau)\bar{z}_1z_2 + (2d\tau)z_2z_2 + (2\bar{d}\tau^2)\bar{z}_2\bar{z}_2 + o(|z|^2).$$

Comparing the coefficients for the $z_1\bar{z}_1$ terms in the identity (20), we have

$$(2a\tau - 8|a|^2d)(-2\bar{a}\tau) = (4ar^2 - 6a + 8|a|^2d\tau)(2\bar{a}).$$

Hence $12|a|^2(r^2 - 1) = 0$, from which it follows that $a = 0$ by taking into consideration the fact $0 < \tau < 1$.

Substituting $a = 0$ into the above expressions (47), we have:

$$X_1 = -2r^2dz_2\bar{z}_2 + o(|z|^2),$$

$$X_2 = -\tau z_1\bar{z}_2 - \tau^2z_1z_2 + (-2d\tau)z_2z_2 + (-2d\tau)\bar{z}_2\bar{z}_2 + o(|z|^2),$$

$$Y_1 = (2\tau^3 - 2\tau)\bar{z}_1\bar{z}_2 + (6d\tau^3 - 4d\tau)z_2\bar{z}_2 + o(|z|^2),$$

$$Y_2 = \tau^2z_1\bar{z}_2 + \tau z_1z_2 + (2d\tau)z_2z_2 + (2d\tau^2)\bar{z}_2\bar{z}_2 + o(|z|^2).$$

Compare the coefficients for the $z_1\bar{z}_1\bar{z}_2^2$ terms. Then $(2\tau^3 - 2\tau)\tau^2 = 0$, which is impossible. Thus, we proved that, under the assumptions in Theorem 3.1, the pair $\{A, B\}$ can not take the normal form in (2d), (2e), and (2f).

4.8 Case (3)

We study the cases in (3a), (3b) and (3c) in this subsection. Notice that in (3a), (3b) and (3c), $b$ is required to be a real number. By a computation, we derive that

$$w = (az_1^2 + az_1^2 + 2bz_1z_2 + 2b\bar{z}_1\bar{z}_2 + dz_2^2 + \bar{d}z_2^2) + z_1\bar{z}_2 + \bar{z}_1z_2 + i\bar{z}_2\bar{z}_2 + o(|z|^2),$$

$$G = (az_1^2 + az_1^2 + 2bz_1z_2 + 2b\bar{z}_1\bar{z}_2 + dz_2^2 + \bar{d}z_2^2) + z_1\bar{z}_2 + \bar{z}_1z_2 + o(|z|^2),$$

$$E = z_2\bar{z}_2 + o(|z|^2), \quad G_1 = 2az_1 + 2bz_2 + \bar{z}_2 + o(|z|), \quad G_2 = 2bz_1 + 2dz_2 + \bar{z}_1 + o(|z|),$$

$$E_1 = o(|z|), \quad E_2 = \bar{z}_2 + o(|z|).$$
We substitute the above quantities into (52), (10) and (11), to derive the following:

\[ A = 2b z_1 + 2d z_2 + \overline{z}_1 - \overline{z}_2 + o(|z|), \quad B = 2a z_1 + 2b z_2 + \overline{z}_2 + o(|z|), \]
\[ \lambda_{(1)} = (2b - 2ai) z_1 + (2d - 2bi) z_2 + \overline{z}_1 - 2i \overline{z}_2 + o(|z|) \]
\[ \lambda_{(2)} = 2a z_1 + 2b z_2 + \overline{z}_2 + o(|z|), \]
\[ \lambda_{(4)} = -(2b + 2ai) \overline{z}_1 - (2d + 2bi) \overline{z}_2 - z_1 - 2iz_2 + o(|z|) \]
\[ \lambda_{(5)} = -(2a z_1 + 2b \overline{z}_2 + \overline{z}_2) + o(|z|), \]
\[ \Gamma_{(1)} = (4b^2 - 4ad) z_1 + (2b - 2ai) z_1 + i(4b^2 - 4ad) z_2 + (-2a - 2d) \overline{z}_2 + o(|z|), \]
\[ \Gamma_{(2)} = (4ad - 4b^2) z_2 + 2a \overline{z}_1 - (2ai + 2b) \overline{z}_2 + o(|z|), \]
\[ \Gamma_{(4)} = (8ai - 2b) z_1 + (4b^2 + 4abi + 4ad - 2) \overline{z}_1 + (12bi - 2d) z_2 + (4bi - 4bd + 4b^2i + 6i) \overline{z}_2 + o(|z|), \]
\[ \Gamma_{(5)} = 2a z_1 + (-4ab - 4bi - 4a|z|^2) \overline{z}_1 + (2b - 4ai) z_2 + (2 - 4b^2 - 4ad - 4abi) \overline{z}_2 + o(|z|). \]

Further we substitute the above quantities into (19) to obtain:

\[ X_1 = (8ib^3 - 8a|b|^2 d + 2a) z_1 \overline{z}_1 + (8b^3 - 8abd - 2ai - 2b) z_1 \overline{z}_2 + (4ab + 4b - 4|a|^2 i) \overline{z}_1 \overline{z}_1 + (8ib^2 i - 8a|b|^2 d + 2b + 8abd - 8b^3) z_1 z_2 + (4ad - 4a\overline{d} - 4|a|^2 - 8abi) \overline{z}_1 \overline{z}_2 + (8ib^3 - 8abi + 8a|b|^2 d - 2a - 2bi) \overline{z}_2 \overline{z}_2 + \overline{z}_2 \overline{z}_2 + (4ad - 4d\overline{a} - 4\overline{d} + 4d - 4ad - 8\overline{d}i - 4\overline{d}i) \overline{z}_2 \overline{z}_2 + o(|z|^2), \]
\[ X_2 = (-2a) z_1 z_1 + (-4|a|^2 b - 4ab - 4a|b|^2) \overline{z}_1 \overline{z}_1 + (4ad - 4d\overline{a} - 4\overline{d} + 4d - 4ad - 8\overline{d}i - 4\overline{d}i) \overline{z}_1 \overline{z}_1 \]
\[ + (2b - 8b^3 + 8abd + 18ai) \overline{z}_1 \overline{z}_2 + (4ad - 4\overline{d}d - 4|a|^2 - 8\overline{a}b) \overline{z}_1 \overline{z}_2 + (32abi) z_1 z_2 \]
\[ + (4ad - 4\overline{a}b - 4|a|^2 i) \overline{z}_1 \overline{z}_1 + (8b^3 - 8abd - 2b - 8a|b|^2) \overline{z}_2 \overline{z}_2 + o(|z|^2), \]
\[ Y_1 = (16a^2 i) z_1 z_1 + (-8\overline{a}b^2 - 2a + 8|a|^2 d) z_1 \overline{z}_1 + (4ad + 4b + d + 4i + 8b^2 i + 4a\overline{d} - 4ab) \overline{z}_2 \overline{z}_2 \]
\[ + (2b - 8b^3 + 8abd + 18ai) \overline{z}_1 \overline{z}_2 + (4ad - 4\overline{d}d - 4|a|^2 - 8\overline{a}b) \overline{z}_1 \overline{z}_2 + (32abi) z_1 z_2 \]
\[ + (4ad - 4\overline{a}b - 4|a|^2 i) \overline{z}_1 \overline{z}_1 + (8b^3 - 8abd - 2b - 8a|b|^2) \overline{z}_2 \overline{z}_2 + o(|z|^2), \]
\[ Y_2 = (2a) z_1 z_1 + (4\overline{a}b + 4ab + 4a|b|^2 i) z_1 \overline{z}_1 + (4ad + 4b + d + 4i - 4abi) z_1 \overline{z}_2 + (4ad - 4\overline{d}i) \overline{z}_1 \overline{z}_2 + (2d) z_2 z_2 + (2\overline{d}) \overline{z}_2 \overline{z}_2 \]
\[ + (4bd + 4\overline{a}d - 4\overline{d} - 4bi) \overline{z}_2 \overline{z}_2 + (2\overline{d} - 4bi) \overline{z}_2 \overline{z}_2 + o(|z|^2). \]

Substituting the above formulas into (20) and comparing the coefficients for the $z_1^4$ terms, we have: $32a^3 = 0$. Hence $a = 0$ and thus, the pair \{A, B\} cannot take the normal form in (3a).

Substituting $a = 0$ into $X_1, X_2, Y_1, Y_2$, we have:

\[ X_1 = (8b^3 - 2b) z_1 \overline{z}_1 + (8ib^3 - 8\overline{d}b^2 - 2d - 2bi) z_2 \overline{z}_2 + (-4bd - 4\overline{d}b) \overline{z}_2 \overline{z}_2 \]
\[ + (2b - 8b^3) \overline{z}_1 \overline{z}_2 + o(|z|^2), \]
\[ X_2 = (-4b) z_1 \overline{z}_2 + (-4b) \overline{z}_1 \overline{z}_2 + (-8b^3 - 2d - 2b) \overline{z}_2 \overline{z}_2 + (2d - 2b) \overline{z}_2 \overline{z}_2 \]
\[ + (2b - 8b^3) \overline{z}_1 \overline{z}_2 + (-4bd - 4\overline{d} - 4bi - i) \overline{z}_2 \overline{z}_2 + (2\overline{d} - 4bi) \overline{z}_2 \overline{z}_2 + o(|z|^2), \]
\[ Y_1 = (4bd + 4bd + 4i + 8\overline{b}i) \overline{z}_2 \overline{z}_2 + (2b - 8\overline{b}i) \overline{z}_1 \overline{z}_2 + (8b^3 - 2b) \overline{z}_1 \overline{z}_2 \]
\[ + (24bi) \overline{z}_2 \overline{z}_2 + (2d + 2bi + 8\overline{b}i) \overline{z}_2 \overline{z}_2 + o(|z|^2), \]
\[ Y_2 = (4b) z_1 z_2 + (4b^3 + 1) z_1 \overline{z}_2 + (4b) \overline{z}_1 \overline{z}_2 + (2d) z_2 z_2 \]
\[ + (4bd + 4\overline{d} - 4bi) \overline{z}_2 \overline{z}_2 + (2\overline{d} - 4bi) \overline{z}_2 \overline{z}_2 + o(|z|^2). \]

Substituting the above formulas into (20) and comparing the coefficients for the $z_2^3 \overline{z}_2$ terms, we have:

\[ (8ib^3 - 8\overline{a}b^2 - 2d - 2bi)(-2d - 4bi) = (24bi)(4bd + 4\overline{d} - 4 - 4bi) + (2d + 8\overline{b}^2 + 22bi + 8\overline{b}^3)2d \]
Noticing that, in (3b) and (3c), \( b \) and \( d \) are both real numbers, we can simplify the above equality to derive the following:

\[
(64b^4 + 32b^2) + i(192b^3d + 32bd) = 0.
\]

Hence \( b = 0 \). Substituting \( b = 0 \) further into (18), we have

\[
X_1 = -2d z_2 \overline{z}_2 + o(|z|^2),
\]
\[
X_2 = -z_1 \overline{z}_2 - z_1 z_2 + (2d) z_2 \overline{z}_2 + (4i) \overline{z}_2 \overline{z}_2 + o(|z|^2),
\]
\[
Y_1 = (2d) z_2 \overline{z}_2 + (4i) \overline{z}_2 \overline{z}_2 + o(|z|^2),
\]
\[
Y_2 = z_1 \overline{z}_2 + z_1 z_2 + (2d) z_2 \overline{z}_2 + (4i) \overline{z}_2 \overline{z}_2 + o(|z|^2).
\]

Substituting the above formulas into (20) and comparing the coefficients for the \( z_2 \overline{z}_2 \) terms, we get \( 4i = 0 \), which is a contradiction.

Therefore, we proved that, under the assumptions in Theorem 3.1, the pair \( \{A, B\} \) cannot take the normal form in (3).

### 4.9 Case (4)

We modify the computation we did for (2a) by substituting \( \tau = 0 \) into \( X_1, X_2, Y_1, Y_2 \). Further noticing that \( b \) and \( d \) are real numbers, we obtain:

\[
X_1 = (8ab^3 - 8|a|^2d) z_1 \overline{z}_1 + (4ab) z_1 \overline{z}_1 + (8b^3 - 8abd) z_1 \overline{z}_2 + (4b^3) z_1 \overline{z}_2 + o(|z|^2),
\]
\[
X_2 = (-2ab) z_1 \overline{z}_1 + (2b) z_1 \overline{z}_2 + (-4ad) z_1 \overline{z}_2 + (-4bd) z_2 \overline{z}_2 + o(|z|^2),
\]
\[
Y_1 = (-8ab) z_1 \overline{z}_1 + (2ab) z_1 \overline{z}_2 + (-12ad - 4b^2) z_2 \overline{z}_2 + (4bd) z_1 \overline{z}_2 + (4ad) \overline{z}_1 \overline{z}_2
\]
\[
+ (8b^3 - 4b - 8abd) \overline{z}_1 \overline{z}_2 + (8bd) z_2 \overline{z}_2 + (-8ad^2 + 8b^2d) z_2 \overline{z}_2 + o(|z|^2),
\]
\[
Y_2 = (4ab) z_1 \overline{z}_1 + (4b^2) z_1 \overline{z}_2 + (2b) z_1 \overline{z}_2 + (4bd) z_2 \overline{z}_2 + (4ad) \overline{z}_1 \overline{z}_2 + o(|z|^2).
\]

Comparing the coefficients for the \( z_1 \overline{z}_1 \overline{z}_2^2 \) terms in \( X_1X_2 = Y_1Y_2 \), we have: \( (-8bd)(4ad) = 0 \). If \( a = 0 \), then we substitute it into \( X_1, X_2, Y_1, Y_2 \) to derive:

\[
X_1 = (8b^3) z_1 \overline{z}_1 + (4b^2) z_1 \overline{z}_2 + o(|z|^2),
\]
\[
X_2 = (-2b) z_1 \overline{z}_2 + (-4b^3) z_1 \overline{z}_2 + (4bd) z_2 \overline{z}_2 + o(|z|^2),
\]
\[
Y_1 = (-4b) z_1 \overline{z}_2 + (8b^3 - 4b) z_1 \overline{z}_2 + (-8bd) z_2 \overline{z}_2 + (8b^2d) z_2 \overline{z}_2 + o(|z|^2),
\]
\[
Y_2 = (4b^2) z_1 \overline{z}_2 + (2b) z_1 \overline{z}_2 + (4bd) z_2 \overline{z}_2 + o(|z|^2).
\]

Comparing the coefficients for the \( \overline{z}_1^2 z_2 \overline{z}_2 \) terms in \( X_1X_2 = Y_1Y_2 \), we derive that

\[
(-4b^2)(4b^2) = (8b^3 - 4b)(2b).
\]

Hence \( b = 0 \) or \( b = \frac{1}{2} \). If \( a = 0, b = 0 \), then \( d = \frac{1}{3} \) or \( d = 0 \) by making use of the normal form. If \( a = 0, b = \frac{1}{2} \), by comparing the coefficients for the \( \overline{z}_1^2 \overline{z}_2^2 \) terms, we have \( (8b^2d)(4bd) = 0 \). Thus, \( d = 0 \). Therefore we have the following possibilities:
1. $a = 0, b = 0, d = 0$ ;
2. $a = 0, b = 0, d = \frac{1}{2}$ ;
3. $a = 0, b = \frac{1}{2}, d = 0$.

If $a \neq 0, b = 0$, then we have:

$$X_1 = (-8|a|^2d)z_1\bar{z}_1 + o(|z|^2),$$
$$X_2 = (-2a)z_1z_1 + (-4ad)z_1\bar{z}_2 + o(|z|^2),$$
$$Y_1 = (-6a)z_1\bar{z}_1 + (-12ad)z_1z_2 + (-4ad)\bar{z}_1\bar{z}_2 + (-8ad^2)z_2\bar{z}_2 + o(|z|^2),$$
$$Y_2 = (2\bar{a})z_1\bar{z}_1 + (4\bar{a}d)\bar{z}_1z_2 + o(|z|^2).$$

Comparing the coefficients for the $\bar{z}_2^iz_1$ terms in $X_1X_2 = Y_1Y_2$, we have $(-6a)(2\bar{a}) = 0$, which is impossible.

If $a \neq 0, b \neq 0, d = 0$, then by the normal form in §3, we have $a = \frac{1}{2}, b > 0$. Therefore,

$$X_1 = (4b^2)z_1\bar{z}_1 + (2b)\bar{z}_1z_1 + (8b^3)z_1\bar{z}_2 + (4b^2)\bar{z}_1\bar{z}_2 + o(|z|^2),$$
$$X_2 = -z_1z_1 + (-2b)z_1\bar{z}_1 + (-2b)z_1z_2 + (-4b^2)\bar{z}_1z_2 + o(|z|^2),$$
$$Y_1 = (-4b)z_1z_1 + (-3)z_1\bar{z}_1 + (-4b^2)z_1z_2 + (-2b)\bar{z}_1\bar{z}_1 + (8b^3 - 4b)\bar{z}_1z_2 + o(|z|^2),$$
$$Y_2 = (2b)z_1\bar{z}_1 + (4b^2)z_1\bar{z}_2 + \bar{z}_1z_1 + (2b)\bar{z}_1\bar{z}_2 + o(|z|^2).$$

Comparing the coefficients for the $\bar{z}_1z_3^2$ terms in $X_1X_2 = Y_1Y_2$, we have $(4b^2)(-1) = (-4b)(2b)$, which contradicts the fact $b > 0$.

Combing all the above, we finally achieved a proof for Theorem 3.1.

5 Proof of Theorem 1.2

In this section, we give a proof for Theorem 1.2. Recall that $M$ is assumed to be a real codimension two submanifold in $\mathbb{C}^{n+1}$ with $0 \in M$ a CR singular point. Near $p = 0$, $M$ is defined by an equation as in (1) and (2). Let $M$ be CR non-minimal at its CR points. Furthermore, we assume that the matrix $\mathcal{B}$ is non-degenerate.

It is clear that, under the above setting, $M$ is quadratically flattenable if and only if there exists an $n$ by $n$ invertible matrix $P$ and a $\mu \neq 0$ such that

$$\tilde{\mathcal{B}} = \frac{1}{\mu}P \cdot \mathcal{B} \cdot \overline{P}$$

is Hermitian. (See (1) in §2). We will prove Theorem 1.2 by slicing $M$ with some special three-dimensional complex submanifolds and thus reducing the proof to that of Theorem 3.1.
Proof of Theorem 1.2: By Schur’s Lemma, there is a unitary matrix $U$ such that $U \cdot B \cdot U^t$ is an upper triangular matrix:

$$B' = U \cdot B \cdot U^t = \begin{pmatrix}
\mu_1 & \cdots & \cdots & \cdots \\
\mu_2 & \cdots & \cdots & \cdots \\
\vdots & \cdots & b_{ij} & \cdots \\
0 & \cdots & \cdots & \mu_n
\end{pmatrix}.$$  \hfill (50)

Since we assumed that $\det(B) \neq 0$, $\mu_1, \mu_2, \cdots, \mu_n \neq 0$. Choose $\mu = \mu_1$, $P = U$ to be as in (49). Without lost of generality, we can just assume $B$ is $B'$ with the above form and $\mu_1 = 1$, $\mu_2, \cdots, \mu_n \neq 0$.

Now if $0 \in M$ is not a quadratically flattenable CR singular point, then either we have some $\mu_j / \in \mathbb{R}$ or $\mu_2, \cdots, \mu_n \in \mathbb{R}$ but some element $b_{ij}(j > i) \neq 0$.

In the first situation, we let $E = \{z_k = 0, k \neq 1, j\}$ and $M' = M \cap E$. Notice that $M'$ is a real codimension two submanifold in $\mathbb{C}^3$ with 0 as a CR singular point and satisfies all the assumptions made in Theorem 3.1. Moreover, the defining function for $M'$ in $\mathbb{C}^3$ (with coordinates $(z_1, z_j, w)$) takes the form that is similar to (7):

$$w = q(z, \overline{z}) + p(z, \overline{z}) + iE(z, \overline{z}),$$

where $q(z, \overline{z}) = 2\Re(h^{(2)}(z)) + (z_1, z_j) \cdot B_1 \cdot (\overline{z}_1, \overline{z}_j)^t$, and $B_1 = \begin{pmatrix} 1 & b_{ij} \\ 0 & \mu_2 \end{pmatrix}$.

On the one hand, by applying Theorem 3.1 we know $M'$ is quadratically flattenable at the point 0; for $\det(B_1) \neq 0$. However, we can not find $\mu \neq 0$ and an invertible 2 by 2 matrix $P$ such that

$$\overline{B}_1 = \frac{1}{\mu}P \cdot B_1 \cdot \overline{P}^t,$$

is Hermitian due to the fact $\mu_2 \notin \mathbb{R}$. This contradicts Theorem 3.1.

We turn to the second situation where $\mu_2, \cdots, \mu_n \in \mathbb{R}$ but some element $b_{ij}(j > i) \neq 0$. In this case we let $E = \{z_k = 0, k \neq i, j\}$ and $M' = M \cap E$. Here $M'$ satisfies all the assumptions made in Theorem 3.1 as well. Similar to (7) and the discussion above, we derive the defining function for $M'$ in $\mathbb{C}^3$ (with coordinates $(z_i, z_j, w)$) as

$$w = q + O(3),$$

where $q = 2\Re(h^{(2)}(z_i, z_j)) + (z_i, z_j) \cdot B_2 \cdot (\overline{z}_i, \overline{z}_j)^t$, and $B_2 = \begin{pmatrix} \mu_i & b_{ij} \\ 0 & \mu_j \end{pmatrix}$.

By the same token, we conclude that $M'$ is quadratically flattenable at the point 0; for $\det(B_2) \neq 0$. However for this matrix $B_2$, it is clear that there do not exist $\mu \neq 0$ and an invertible 2 by 2 matrix $P$ such that

$$\overline{B}_2 = \frac{1}{\mu}P \cdot B_2 \cdot \overline{P}^t.$$
is Hermitian, due to the fact $\mu_i, \mu_j, b_{ij} \neq 0$. This is again a contradiction. As a conclusion, we complete the proof Theorem 1.2.

**Remark 5.1.** It is clear from our proof, Theorem 1.2 and Theorem 3.1 hold when the submanifold $M$ is just assumed to be $\mathbb{C}^3$-smooth.

### 6 Holomorphic flattening: a geometric approach

#### 6.1 A general approach

In this section, we apply Theorem 1.2 and the work in [HY3] to give a proof of Theorem 1.3 when the set of CR singular points has real dimension less than $(2n - 2)$. Our approach here is more along the lines of the geometric argument used in [HY3]. The great benefit of this argument, compared with the formal argument in [HY4], is that we do not need to know the precise structure of the quadratic normal form which is almost impossible to obtain when $n + 1 > 3$. The reason we want to have the set of CR singular points has real dimension less than $(2n - 2)$ is because we need to find a good elliptic complex tangency for the sliced manifolds.

We recall the definition of elliptic directions first introduced in the paper of Huang-Yin [HY3] (see [Theorem 2.3, HY3] and also the papers [Bur1-2] and [LNR]):

**Definition 6.1.** Let $M$ be a smooth submanifold with $0 \in M$ being its CR singular point. Suppose that there is a holomorphic change of coordinates preserving the origin such that $M$ in the new coordinates (which, for simplicity, we still write as $(z, w) \in \mathbb{C}^n \times \mathbb{C}$) is defined by an equation of the form:

$$w = G(z, \overline{z}) + iE(z, \overline{z}) = O(|z|^2), \quad (G + iE)(z_1, 0, \overline{z}_1, 0) = |z_1|^2 + \lambda_1(z_1^2 + \overline{z}_1^2) + o(|z_1|^2).$$

(51)

Here the constant $\lambda_1$ is such that $0 \leq \lambda_1 < \frac{1}{2}$. We say that $M$ has an elliptic direction along the $z_1$-direction in the new coordinates.

Suppose $M$ in Definition 6.1 has an elliptic direction along the $z_1$-direction. By the stability of elliptic complex tangency, then $M_t := M \cap \{(z_1, z'' = t, w)\}$ is also an elliptic Bishop surface in the affine plane $\{(z_1, z'' = t, w)\}$ for each fixed $t = (t_2, \cdots, t_n) \approx 0$ and has a unique elliptic CR singular point, when regarded as a real surface in the affine complex plane defined above. Denote by $p(t)$ the elliptic CR singular point in $M_t$, which is regarded as a real surface in a complex affine plane. We need the following result whose proof is contained in [HY3, Theorems 2.1, 2.2, 2.3] for our purpose here:

**Theorem 6.2.** (Huang-Yin [HY3]) Let $M$ be a real analytic submanifold of codimension two with $0 \in M$ a CR singular point. Suppose that $M$ has an elliptic direction along the $z_1$-direction and let $p(t)$ be the elliptic CR singular point of $M_t$ defined above. Suppose for some sufficiently small $|t|$, there is a real-analytic Levi-flat hypersurface $H_{p(t)}$ containing $p(t)$ such that all small
holomorphic disks attached to $M$ near $p(t)$ stay inside $H_{p(t)}$. Then $(M,0)$ can be holomorphically flattened. Namely, there is a holomorphic change of coordinates preserving the origin such that in the new coordinates, $M$ can be defined by a function near the origin of the form: $w = \rho(z, \overline{z})$ with $\Im \rho \equiv 0$.

Applying Theorem 1.2 and a holomorphic change of coordinates, we have the following corollary:

**Corollary 6.3.** Let $(M,0)$ be a real analytic real codimension two submanifold with $0 \in M$ a CR singular point. Then $(M,0)$ can be holomorphically flattened if the following two conditions hold: (A) $M$ has an elliptic direction $\vec{c} = (c_1, \cdots, c_n) \neq 0$ at 0, namely, $M \cap \{z_j = c_j \xi, \xi \in \mathbb{C}\}$, when regarded as a real surface in the $(\xi, w)$-plane, has an elliptic complex tangency at 0; and (B) there is a real analytic Levi-flat hypersurface $H_{p_c,\tau}$ passing through $p_c,\tau$ for some sufficiently small $|\tau|$, which contains all small holomorphic disks attached to $M$ near $p_c,\tau$. Here $p_c,\tau$ is the elliptic complex tangent point of $M_{a,c,\tau} := M \cap \{z_j = \vec{c}\xi + \tau_1 \vec{a}_1 + \cdots + \tau_{n-1} \vec{a}_{n-1}\}$ when regarded as a real surface in the complex plane with coordinates $(\xi, w)$, and $(\vec{c}, \vec{a}_1, \cdots, \vec{a}_{n-1})$ forms a linear independent system of vectors in $\mathbb{C}^n$ and $\tau = (\tau_1, \cdots, \tau_{n-1}) \in \mathbb{C}^{n-1}$ is the parameter.

Now we assume that $(M,p)$ is as in Theorem 1.3. After a holomorphic change of coordinates, we assume that $p = 0$ and $M$ is defined by a real analytic function whose quadratic term is as in (1) and (2). The rest of this section is to verify that the hypotheses in Corollary 6.3 hold when $n + 1 \geq 4$ or when $(M,0)$ is not equivalent to a surface $(M',0)$ whose quadratic term is either as in (5) or as in (6). We first modify an argument in [HY3] to give a proof of the existence of a Levi-flat piece as in Corollary 6.3 assuming the existence of a generic good elliptic complex point. The existence of these good points will be verified later this section.

**Proposition 6.4.** Let $M \subset \mathbb{C}^{n+1}$ be a real analytic real codimension two submanifold with $0 \in M$ a CR singular point and with all its CR points non-minimal. Assume that $M$ is elliptic along the $\vec{c} = (c_1, \cdots, c_n)$-direction. Let $\{\vec{a}_j \in \mathbb{C}^n\}_{j=1}^{n-1}$ be such that $\{\vec{c}, \vec{a}_1, \cdots, \vec{a}_{n-1}\}$ are linearly independent. Write, for a parameter $\tau = (\tau_1, \cdots, \tau_{n-1})$ with $|\tau| \ll 1$, $p_c,\tau$ for the elliptic complex tangent point of $M \cap \{z = \vec{c}\xi + \vec{a}_1 \tau_1 + \cdots + \vec{a}_{n-1} \tau_{n-1}\}$, when regarded as a real surface in the $(\xi, w)$-complex plane. Assume that for some $\tau \approx 0$, $p_c,\tau$ is a CR point of $M$, and $M$ does not contain any complex analytic variety of dimension $(n-1)$ passing through $p_c,\tau$. Then $(M,0)$ can be holomorphically flattened.

**Remark 6.5.** Before we proceed to the proof of Proposition 6.4, we mention that it is crucial to have some $p_c,\tau$ to be a CR point of $M$. For instance, the 4-manifold $M$ defined by $w = z_1 \overline{z_2}$ in Example 3.3 has an elliptic direction $\vec{c} = (1,1)$. However, the CR singular point of $M$ is of real dimension two, defined by $z_1 = 0$. Hence, no matter how $\vec{a}$ is chosen, $p_c,\tau$ will be a CR singular point of $M$. Hence, the assumption for the existence of $H_{p_c,\tau}$ can not hold. Indeed, as we know in Example 3.3, $M$ is not even quadratically flattenable at 0, though all CR points of $M$ are non-minimal.
Proof of Proposition 6.4 Assume, without loss of generality, that \( c = (0, \cdots, 0, 1) \) is the \( z_n \)-direction. By Theorem 6.2, we need only to show that there is a Levi-flat hypersurface \( H_{c,a}(\tau) \) passing through \( p_{c,a}(\tau) \) as in the proposition such that any small holomorphic disks attached to \( M \) near \( p_{c,a}(\tau) \) are contained inside \( H_{c,a}(\tau) \). First, since \( p_{c,a}(\tau) \) is a CR point of \( M \), by the assumption, \( M \) is non-minimal for any point near \( p_{c,a}(\tau) \) and for each point \( q(\in M) \approx p_{c,a} \) there is a CR submanifold \( X_q \) of \( M \) of dimension \((n - 1)\) through \( q \). Since we assumed that there is no complex analytic variety of dimension \((n - 1)\) passing through \( p_{c,a}(\tau) \), \( X_p \) is of hypersurface type and is of finite type in the sense of Bloom-Graham \([BG]\). Hence, we conclude that \( X_q \subset M \) is a CR submanifold of hypersurface type of CR dimension \((n - 1)\), that is also of Bloom-Graham finite type. Notice that for each \( q \approx p_{c,a}(t) \) sufficiently close to 0, the tangent vector fields of type \((1,0)\) of \( X_q \) near \( q \) are spanned by \( \{L_1, \cdots, L_{n-1}\} \) with

\[
L_j = (G_n - iE_n) \frac{\partial}{\partial z_j} - (G_j - iE_j) \frac{\partial}{\partial z_n} + 2i(G_nE_j - G_jE_n) \frac{\partial}{\partial w} \tag{52}
\]

where \( G, E \) are similarly defined as in \([7]\) and \([8]\). Also, a certain fixed iterated Lie bracket \( T \) from elements in \( \{L_1, \cdots, L_{n-1}, T_1, \cdots, T_{n-1}\} \) is a non-zero tangent vector field of \( M \) near \( p_{c,a}(\tau) \) such that the span of the vector fields \( \{\Re L_j, \Im L_j, \Re T\} \) defines a real analytic distribution of real codimension one near \( p_{c,a}(\tau) \). After a linear (holomorphic) change of coordinates, we assume that \( p_{c,a}(\tau) = 0 \) and \( T_0M = \{y_1 = v = 0\} \) and \( T_0^{(1,0)}M = \{z_1 = w = 0\} \), here \( z_j = x_j + \sqrt{-1}y_j \). Performing a linear transformation, we also assume that \( \frac{\partial}{\partial x_1}|_0 \) is tangent to the CR foliation at 0 and \( \frac{\partial}{\partial y_n}|_0 \) is transversal to the CR foliation of \( M \) near \( q = 0 \). Notice here that \( \frac{\partial}{\partial x_1}|_0 \) and \( \frac{\partial}{\partial y_n}|_0 \) are tangential to \( M \) at 0. Now, from the foliation theory, we can find a real valued real analytic function \( t(Z) \) defined over a certain neighborhood \( U_0 \) of \( p_{c,a}(\tau) = 0 \) in \( M \) such that for each \( t_0 \in I_{\delta_0} = (-\delta_0, \delta_0) \) with a certain small \( 0 < \delta_0 << 1 \), \( M_{t_0} = \{Z \in U_0, t(Z) = t_0\} \) is a connected real analytic CR submanifold of hypersurface type of CR dimension \((n - 1)\). Moreover \( dt|_{U_0} \neq 0 \). We assume that \( 0 \in M_{t_0 = 0} \). Define \( \Psi : U_0 \to \mathbb{C}^n \times \mathbb{R} \) by sending \( Z = (z_1, z_2, \cdots, z_n, w) \in U_0 \) to \( \Psi(Z) = (z_1, z_2, \cdots, z_n, t(Z)) \). After shrinking \( U_0 \) and \( \delta_0 \) if needed, we can assume that \( \Psi \) is a real analytic embedding. Write \( M^* \) for \( \Psi(M_{t}) \) for each \( t \in I_{\delta_0} \). Since each component of \( \Psi \) is the restriction of a holomorphic function over \( M_t \), \( \Psi \) is a CR diffeomorphism from \( M_t \) to \( M^*_t \). Write \( M^* = \Psi(M) \). Then \( M^* \) is a real analytic hypersurface in \( \mathbb{C}^n \times \mathbb{R} \) and \( \Psi \) is a real analytic diffeomorphism from a neighborhood of \( p_{c,a}(\tau) \) in \( M \) to a certain neighborhood of \( p^* := \Psi(p_{c,a}(\tau)) = (p^{**}, 0) \) in \( M^* \). Also \( \Phi := \Psi^{-1} \) is a CR diffeomorphism when restricted to each \( M^*_t \) near \( p^* = \Psi(p_{c,a}(\tau)) \). Now, the real analytic CR function \( \Phi \) extends holomorphically to a certain fixed neighborhood of \( p^{**} \) in \( \mathbb{C}^n \) for each fixed \( t \). Also the holomorphic extension depends real analytically on its parameter \( t \) from the way the holomorphic extension is constructed as demonstrated by the following lemma:

**Lemma 6.6.** Let \( M_t \subset \mathbb{C}^n \) be a real analytic family of real analytic hypersurfaces with a real parameter \( t \) defined by \( 3w = \rho(z, \overline{z}, \Re w; t) \). Here \( \rho(z, \overline{z}, \Re w; t) \) is a real analytic function in \((z, \overline{z}, \Re w, t)\) in a neighborhood of the origin with \( \rho(0, 0, 0; 0) = 0, \, dp|_0 = 0 \). Suppose that \( f \) is a real analytic function in \((z, w, t)\) near the origin and is a CR function when restricted to each
$M_t$ near 0. Then $f$ extends to a real analytic function in a neighborhood of $(0', 0) \in \mathbb{C}^n \times \mathbb{R}$ which is holomorphic for each fixed small $t$.

**Proof of Lemma 6.6** We first notice that after applying a holomorphic transformation of the form: $(z, w) = H(z', w'; t)$, where $H(z', w'; t)$ is real analytic in $(z', w'; t)$ in a small neighborhood of the origin and is biholomorphic for each fixed $t$, and by applying an implicit function theorem (see §3, BR, for instance), we can assume that $M_t$ is defined near the origin by an equation of the form: $\Im w = \rho(z, \overline{z}, Rw; t)$ or $\overline{\tau} = \overline{\rho}(z, \overline{z}, w; t)$. Here, $\rho$, $\overline{\rho}$ can be expanded to a Taylor series at the origin in $(z, \overline{z}, Rw, t)$ (or $(z, \overline{z}, w, t)$, respectively) and $\rho|_0 = \overline{\rho}|_0 = 0, d\rho_0 = d\overline{\rho}_0 = 0$, $\rho(0, 0, Rw; t) = \rho(0, \overline{z}, Rw; t) = 0$, $\overline{\rho}(0, 0, w; t) = \overline{\rho}(0, \overline{z}, w; t) = 0$.

Now, let $f(z, \overline{z}, w, \overline{\tau}, t)$ be a real analytic function near 0 and is CR when restricted to each $M_t$. Notice that the CR vector field of $M_t$ is given by $L_j = \frac{\partial}{\partial z_j} + \overline{\rho_{\overline{z}j}}$, $j = 1, \cdots, n - 1$. Define $\tilde{f}(z, \overline{z}, w, t) = f(z, \overline{z}, w, \overline{\rho}(z, \overline{z}, w; t); t)$. Since $f$ is CR along each $M_t$, we obtain

$$\frac{\partial \tilde{f}}{\partial \overline{z}_j} = \frac{\partial f}{\partial \overline{z}_j} + \frac{\partial f}{\partial \overline{\tau}} \overline{\rho_{\overline{z}j}} = 0, \quad j = 1, \cdots, (n - 1) \quad \text{along each } M_t.$$  

Notice that $\overline{\rho_{\overline{z}j}}$ is holomorphic in $(z, \overline{z}, w, t)$. $\frac{\partial \tilde{f}}{\partial \overline{z}_j} = F_j(z, \overline{z}, w, \overline{\tau}, t)|_{\overline{\tau} = \overline{\rho}(z, \overline{z}, w; t)}$ for a certain $F_j$ holomorphic in its variables. Hence we see, in the same way, that $\frac{\partial^2 \tilde{f}}{\partial \overline{z}_j \partial \overline{z}_j} = 0$ along $M_t$. Inductively, we see that for each multi-indices $\alpha = (\alpha_1, \cdots, \alpha_{n - 1})$ and $\beta = (\beta_1, \cdots, \beta_{n - 1})$ with $|\beta| \geq 1$, we have $L_j^\alpha \frac{\partial^{|\beta|} \tilde{f}}{\partial \overline{z}_j^{|\beta|}} = 0$ along each $M_t$. Evaluating this at $(z, w) = (0, Rw)$ and making use of the normalization condition of $\overline{\rho}$, we obtain that

$$\frac{\partial^{(|\alpha| + |\beta|)} \tilde{f}}{\partial z^\alpha \overline{z}^\beta}(0, 0, Rw; t) = 0, \quad (0, Rw; t) \in M_t, \quad |\beta| \geq 1.$$  

Hence, from the facts that $\frac{\partial^{(|\alpha| + |\beta|)} \tilde{f}}{\partial z^\alpha \overline{z}^\beta}(0, 0, Rw; t) = 0$ and it is holomorphic in $w$, it follows that $\frac{\partial^{(|\alpha| + |\beta|)} \tilde{f}}{\partial z^\alpha \overline{z}^\beta}(0, 0, w; t) \equiv 0$ for $|\beta| \geq 1$. From this, one can easily conclude that in the Taylor expansion of $\tilde{f}(z, \overline{z}, w; t)$ in $(z, \overline{z}, w, t)$ at 0, there are no $\overline{z}$-terms. Hence, $\tilde{f}$ gives the desired holomorphic extension of $f$ to a certain fixed neighborhood of the origin and is real analytic as a function in the joint variables $(z, w, t)$.

We now continue the proof of the proposition. Complexifying $t$, we then get a holomorphic extension of $\Psi$ to a neighborhood of $p^*$ in $\mathbb{C}^n \times \mathbb{C}$, that is biholomorphic near $p^*$. Then define $X_p$ to be the biholomorphic image of $\mathbb{C}^n \times \mathbb{R}$ near $p^*$. Then $X_p$ is a Levi flat hypersurface containing a small piece of $M$ near $p_{c.a}(\tau)$, which certainly contains all small holomorphic disks attached $M$ near $p_{c.a}(\tau)$. By Theorem 6.2 we thus complete the proof of Proposition 6.4.

## 6.2 The case of $n + 1 \geq 4$

For the rest of this section, we will verify the hypotheses stated in Proposition 6.4 and thus complete the proof of Theorem 1.3 except for one case which will be handled in the next section.
We now assume that \( p \in M \) is a non-degenerate CR singular point. By Theorem 1.2, there is a holomorphic change of variables such that in the new coordinates, \( p = 0 \) and \( M \) is defined by an equation as in (1). Moreover \( B \) is a non-degenerate Hermitian matrix. Hence, after a linear change of variables, we can further assume that \( B = \text{diag}(1, \cdots, 1, -1, \cdots, -1) \). Therefore, \( M \) is defined by a real analytic equation of the form:

\[
w = F(z, \bar{z}) = \sum_{j=1}^{\ell} |z_j|^2 - \sum_{j=\ell+1}^{n} |z_j|^2 + 2\Re \left( \sum_{j,k=1}^{n} a_{jk} z_j z_k \right) + O(|z|^3). \tag{53}
\]

Here we can always assume, without loss of generality, that \( \ell \geq n/2 \). Write \( S \) for the CR singular points of \( M \). Then over \( S \), we have \( \frac{\partial F}{\partial \overline{z}} = 0 \). From the implicit function theorem, we then can solve that \( z = \phi(\overline{z}) \). Hence, \( S \) is a totally real analytic variety of real dimension at most \( n \). We next give the following lemma:

**Lemma 6.7.** Suppose that \( M \) is a real codimension two smooth submanifold in \( \mathbb{C}^{n+1} \) with coordinates \((w, z_1, \cdots, z_n)\), that is defined by \( w = \sum_{j=1}^{l} |z_j|^2 - \sum_{j=l+1}^{n} |z_j|^2 + 2\Re(z \cdot A \cdot z^') + E \) with \( E = O(|z|^3) \). Then there is a small neighborhood \( U \subset M \) of the origin such that there is no \( r \) dimensional complex analytic subvariety contained in \( U \) for any \( r > \frac{n}{2} \).

**Proof of Lemma 6.7.** If any small neighborhood of \( 0 \in M \), there is a \( r \)-dimensional complex analytic variety contained in \( M \), then we can choose a sequence of points \( \{p_i\}_{i=1}^{\infty} \subset M \) converging to 0 such that at each \( p_i \), there is a \( r \)-dimensional complex submanifold \( V_{p_i} \subset M \) through \( p_i \). We denote the tangent space of \( V_{p_i} \) at \( p_i \) by \( \mathbb{H}_{p_i} \). Write \( M^* \) for the real hypersurface defined by \( \rho := -\frac{1}{2}w - \frac{1}{2}w^* + \sum_{j=1}^{l} |z_j|^2 - \sum_{j=l+1}^{n} |z_j|^2 + 2\Re(z \cdot A \cdot z^') + \Re E \). Then \( M \subset M^* \) and the Levi form of \( M^* \) is given by \( L(Y, Y') = \partial \overline{\partial} \rho(Y, Y') \) when restricted to the holomorphic tangent space of \( M^* \). Notice that the Levi form is zero when restricted to \( \mathbb{H}_{p_i} \). We then extract a subsequence \( \{p'_j\}_{j=1}^{\infty} \) of \( \{p_i\}_{i=1}^{\infty} \) such that \( p'_j \to 0 \) and \( H_{p'_j} \to H_0 \subset T_0^{\ell \cdot 0}M^* \). Here we view \( H_{p'_j} \) and \( H_0 \) as \( r \)-dimensional complex vector spaces in \( \mathbb{C}^{n} \). By continuity, the Levi form \( L \) is zero when restricted to \( H_0 \). On the other hand, the naturally associated matrix for the Levi form of \( M^* \) at the origin is given by

\[
\mathcal{L} = \begin{pmatrix} I_l & -I_{n-l} \\
\end{pmatrix}.
\]

Write a tangent vector as a column vector and represent a basis of \( H_0 \) by an \( n \times r \) matrix \( H \). Then

\[
\overline{H} \mathcal{L} H = 0.
\]

There is a certain invertible \( r \times r \) matrix \( P \) such that \( H' = HP \) has the following form:

\[
H' = \begin{bmatrix} A & B \\
C & 0 \\
\end{bmatrix},
\]

where \( C \) is an \((n-l) \times k\) matrix with rank \( k \) (hence \( k \leq n-l \)). Then since \( \overline{H} \mathcal{L} H = 0 \), we have \( B^* \cdot B = 0 \). Hence \( B = 0 \). This shows that \( r = k \) and thus \( r = k \leq n-l \leq \frac{n}{2} \).
Corollary 6.8. When \( n + 1 \geq 4 \), there is no \((n-1)\)-dimensional complex analytic subvariety that is contained in a small neighborhood of \( 0 \in M \).

We now can complete the proof of Theorem 1.3 for \( n + 1 \geq 4 \). In this case, a simple algebra shows that \( M \) always has an elliptic direction. (See [Lemma 3.1, LNR], for instance). Indeed, setting \( z_j = 0 \) for \( j > \ell \) and diagonalizing the harmonic quadratic part in (53), we need only to show that an \( M \subset \mathbb{C}^3 \) defined by an equation of the form

\[
w = |z_1|^2 + |z_2|^2 + \lambda_1(z_1^2 + \bar{z}_1^2) + \lambda_2(z_2^2 + \bar{z}_2^2) + O(|z|^3), \quad \lambda_1, \lambda_2 \geq 0
\]

has an elliptic direction passing through the origin. Indeed, if \( \lambda_1 = \lambda_2 \), \( M \) is elliptic along the direction \( \vec{c} = (1, \sqrt{-1}) \). Otherwise, assume that \( \lambda_1 < \lambda_2 \). Then the elliptic direction can be simply taken as \( \vec{c} = (\lambda_2, \sqrt{-1} \lambda_1) \). Now when we move along the transversal directions to the elliptic direction, we get a real \( 2(n-1) \)-families of elliptic Bishop surfaces. And the elliptic complex tangent points thus obtained form a smooth manifold of real dimension \( 2(n-1) > \max\{n, 2(n-2)\} \). (See the last equation on page 394 of [HY3]). Hence, a generic elliptic singular point stays at a CR point of \( M \) through which there is no complex analytic subvariety of complex dimension \((n-1)\) contained in \( M \). By Proposition 6.4, we see the proof of Theorem 1.3 when \( n + 1 \geq 4 \).

6.3 The case of \( n + 1 = 3 \)

We next study the existence of good elliptic points when \( n+1 = 3 \). We start with the following:

Lemma 6.9. Let \( M \) be a 4-dimensional real analytic submanifold of \( \mathbb{C}^3 \) defined by

\[
w = 2\Re(z \cdot A \cdot z^t) + z \cdot B \cdot \bar{z}^t + O(|z|^3).
\]

Assume that \( \{A, B\} \) take one of the following forms from part of the list given in §2:

(A). \( B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \lambda_1, \lambda_2 \geq 0 \) and \( \lambda_1 \neq \frac{1}{2} \) or \( \lambda_2 \neq \frac{1}{2} \) (39);

(B). \( B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad 0 \leq \lambda_1 \leq \lambda_2 \) and \( \lambda_1 < \frac{1}{2} \) or \( \frac{1}{2} \leq \lambda_1 < \lambda_2 \) (40);

(C). \( B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \quad \lambda > 0, \) (41);

(D). \( B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \) (42);

(E). \( B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad b > 0, \) (43);

(F). \( B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & d \end{pmatrix}, \quad \text{Im} d > 0, \) (44);

Then \( (M, 0) \) has an elliptic direction \( \vec{c} \) at 0. Also there is a direction \( \vec{a} \) transversal to \( \vec{c} \) such that for the family of complex affine planes \( V(\tau) \subset \mathbb{C}^3 := \{(z, w) : z = \overline{c} \xi + \overline{a} \tau, \xi, \tau \in \mathbb{C}\} \),
being parametrized by $\tau(\approx 0) \in \mathbb{C}$ with $0 \in V(0)$, the elliptic complex tangent point $P(\tau)$ of $V(\tau) \cap M$, for a generic $\tau \approx 0$, is a CR point in $M$. Moreover, there is no holomorphic curve contained in $M$ passing through $P(\tau)$ for a generic $\tau$.

Assuming this lemma, when the quadratic normal form of $M$ is not of the type in (6), then the proof of Theorem 1.3 for $n + 1 = 3$ follows immediately from Proposition 6.4, the classification of the quadratic terms as listed in §3 and Lemma 6.9. We now proceed to the proof of Lemma 6.9.

Proof of Lemma 6.9: We will give the proof of the lemma based on a case by case argument in terms of the normal forms listed in the lemma.

Case (A): $w = |z_1|^2 + |z_2|^2 + \lambda_1(z_1^2 + \overline{z}_1^2) + \lambda_2(z_2^2 + \overline{z}_2^2) + E(z, \overline{z})$ with $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 \neq \frac{1}{2}$ or $\lambda_2 \neq \frac{1}{2}$. It is contained in a strongly pseudoconvex hypersurface and thus there is no non-trivial holomorphic curve contained in $M$. The set $S$ of the complex tangent points in this case are of at most real dimension one. Indeed, the set of CR singular points in $M$ is defined by the equation:

$$0 = \frac{\partial w}{\partial z_1} = z_1 + \lambda_1 \overline{z}_1 + o(|z|), \quad 0 = \frac{\partial w}{\partial z_2} = z_2 + \lambda_2 \overline{z}_2 + o(|z|)$$

Suppose that $\lambda_1 \neq 1/2$. Then one can solve $x_1, y_1$ from the first equation in terms of $x_2, y_2$. Here $x_1 + \sqrt{-1} y_1 = z_j$. Substituting into the second equation, we can always solve $x_2$ in terms of $y_2$ and thus $S$ has real dimension at most one. In fact, if $\lambda_2$ is not $1/2$ neither, then $S$ has an isolated point at 0.

As mentioned in the above subsection, there is an elliptic direction through the origin in this case. Now, we choose a holomorphic family of affine complex planes transversal to the elliptic direction. Then the resulting elliptic complex tangent points form a real dimensional two subset. Hence, we can easily find the good points as required in Proposition 6.4.

Case (B): Now, the manifold is defined by a real analytic equation of the form:

$$w = |z_1|^2 - |z_2|^2 + \lambda_1(z_1^2 + \overline{z}_1^2) + \lambda_2(z_2^2 + \overline{z}_2^2) + E(z, \overline{z}), \quad \text{with } E = O(|z|^3).$$

We first assume that $0 \leq \lambda_1 < \frac{1}{2}$. Then $\vec{c} = (1, 0)$ is an elliptic direction and we construct the family $V(\tau)$ by intersecting $M$ with the affine complex plane: $z_1 = \xi, z_2 = \tau$, where $\tau$ is a complex parameter. (Namely, $\vec{c} = (1, 0)$ and $\vec{a} = (0, 1)$). Then $S$ is defined by the following system of equations:

$$\begin{cases}
0 = \frac{\partial w}{\partial z_1} = z_1 + 2\lambda_1 \overline{z}_1 + o(|z|) \\
0 = \frac{\partial w}{\partial z_2} = -z_2 + 2\lambda_2 \overline{z}_2 + o(|z|) \\
0 = \frac{\partial w}{\partial \xi} = \overline{z}_1 + 2\lambda_1 z_1 + o(|z|) \\
0 = \frac{\partial w}{\partial \overline{\xi}} = -\overline{z}_2 + 2\lambda_2 \overline{z}_2 + o(|z|)
\end{cases} \quad (54)$$

Since $\lambda_1 \neq \frac{1}{2}$, as in case (A), we can solve for $x_1, y_1$ in terms of $x_2, y_2$ through the first and the third equation. Substituting back, we can get at least $x_2$ in terms of $y_2$. Therefore, $S$ has real dimension at most one.
At points where there is a smooth holomorphic curve passing through, it holds that \([L, \overline{L}] \subset \text{Span}\{L, \overline{L}\}\). Hence, there are constants \(\alpha\) and \(\beta\) such that \([L, \overline{L}] = \alpha L + \beta \overline{L}\). By (72) and (10), we have

\[
A = G_2 = -\tau + 2\lambda_2 \tau_2 + o(|z|), \quad B = G_1 = \tau_1 + 2\lambda_1 \tau_1 + o(|z|), \quad C = o(|z|),
\]

\[
\lambda(1) = G_1, \quad \lambda(2) = -G_2, \quad \lambda(4) = -G_T, \quad \lambda(5) = G_T, \quad \lambda(3) = o(|z|), \quad \lambda(6) = o(|z|).
\]

Substituting the above into \([L, \overline{L}] = \alpha L + \beta \overline{L}\), we have

\[
G_1 \frac{\partial}{\partial \tau_1} - G_2 \frac{\partial}{\partial \tau_2} - G_T \frac{\partial}{\partial \tau_1} + G_T \frac{\partial}{\partial \tau_2} = \alpha (G_2 \frac{\partial}{\partial \tau_1} - G_1 \frac{\partial}{\partial \tau_2}) + \beta (G_T \frac{\partial}{\partial \tau_1} - G_T \frac{\partial}{\partial \tau_2}).
\]

In particular, \(G_1 G_T = G_2 G_T\). On the other hand, the set of the elliptic CR singular points of the Bishop surfaces \(V(\tau)\) is given by the equation \(\frac{\partial w}{\partial \xi} = 0\). We substitute \(z_1 = \xi, z_2 = \tau\) in \(\frac{\partial w}{\partial \xi} = 0\) and \(G_1 G_T = G_2 G_T\) to get

\[
\begin{align*}
 (1 + 4\lambda_1^2)|\xi|^2 + 2\lambda_1 (\xi^2 + \tau^2) &= (1 + 4\lambda_1^2)|\tau|^2 - 2\lambda_2 (\tau^2 + \overline{\tau}^2) + o(|\xi|^2 + |\tau|^2) \\
 \xi + 2\lambda_1 \xi + o(\sqrt{|\xi|^2 + |\tau|^2}) &= 0 \\
 \xi + 2\lambda_1 \overline{\xi} + o(\sqrt{|\xi|^2 + |\tau|^2}) &= 0
\end{align*}
\]

Since \(\lambda_1 \neq 1/2\), from the last two equations, we can solve \(\xi = \phi(\tau, \overline{\tau})\) with \(\phi(\tau, \overline{\tau}) = o(\tau)\). Then the first equation can not hold for a generic \(\tau\) by comparing the second order terms in its Taylor expansion at the origin.

We now assume that \(\frac{1}{2} \leq \lambda_1 < \lambda_2\). Then \(\vec{c} = (1, i \sqrt{\frac{\lambda_1}{\lambda_2}})\) is an elliptic direction and we construct the family \(V(\tau)\) by setting \(z_1 = \xi, z_2 = i \sqrt{\frac{\lambda_1}{\lambda_2}} \xi + \tau\). (Namely \(\vec{a} = (0, 1)\)). Then \(S\) is still defined by the system of equations in (10). Since \(\lambda_2 \neq \frac{1}{2}\), by implicit function theorem, \(S\) has an isolated point at 0.

At points where there is a smooth holomorphic curve passing through, following the argument given above, it holds that \(G_1 G_T = G_2 G_T\) as well. Also, the set of the elliptic CR singular points of the Bishop surfaces \(V(\tau)\) is given by the equation \(\frac{\partial w}{\partial \xi} = 0\). We substitute \(z_1 = \xi, z_2 = i \sqrt{\frac{\lambda_1}{\lambda_2}} \xi + \tau\) into \(\frac{\partial w}{\partial \xi} = 0\) and \(G_1 G_T = G_2 G_T\) to get

\[
\begin{align*}
 (1 + 4\lambda_1^2)|\tau|^2 + \frac{\lambda_1}{\lambda_2} |\xi|^2 + i \sqrt{\frac{\lambda_1}{\lambda_2}} (-\tau \xi + \tau \xi) - 2\lambda_2 \{\tau^2 + \overline{\tau}^2 + 2i \sqrt{\frac{\lambda_1}{\lambda_2}} (\tau \xi - \overline{\tau} \xi)\} - (1 + 4\lambda_1^2)|\xi|^2 + o(|\xi|^2 + |\tau|^2) &= 0 \\
 (1 - \sqrt{\frac{\lambda_1}{\lambda_2}}) \xi + i \sqrt{\frac{\lambda_1}{\lambda_2}} \tau - 2i \lambda_1 \lambda_2 \tau + o(\sqrt{|\xi|^2 + |\tau|^2}) &= 0 \\
 (1 - \sqrt{\frac{\lambda_1}{\lambda_2}}) \overline{\xi} - i \sqrt{\frac{\lambda_1}{\lambda_2}} \overline{\tau} + 2i \lambda_1 \lambda_2 \tau + o(\sqrt{|\xi|^2 + |\tau|^2}) &= 0
\end{align*}
\]

Since \(\lambda_1 < \lambda_2\), from the last two equations, we can solve \(\xi = \phi(\tau, \overline{\tau})\) with \(\phi(\tau, \overline{\tau}) = \frac{-i \lambda_1 \sqrt{\lambda_1}}{\sqrt{\lambda_2 - \lambda_1}} \tau + \frac{2i \lambda_1 \lambda_2 \lambda_1}{\sqrt{\lambda_2 - \lambda_1}} \overline{\tau} + o(\tau)\). Substituting \(\xi = \phi\) into the first equation and comparing the coefficient for
the $\tau^2$ term, we get

$$\frac{1}{\sqrt{\lambda_2} - \sqrt{\lambda_1}}\left\{4\lambda_2^2\sqrt{\lambda_2 - \sqrt{\lambda_1}} + (\lambda_1 + 2\lambda_2)(\sqrt{\lambda_2 - \sqrt{\lambda_1}} + 2\lambda_1\sqrt{\lambda_2(1 + 4\lambda_2^2)})\right\}$$

which is not zero due to the fact that the second factor is strictly positive. Hence, we see that the first equation does not hold for a generic $\tau$. Now, as before, the set $\mathcal{P}$ of the elliptic CR singular points associated with the family form a real analytic subset of dimension two, while the set of CR singular points of $M$ is isolated at 0 and for a generic point in $\mathcal{P}$ there is no holomorphic curve in $M$ passing through a generic point in $\mathcal{P}$. Hence a generic point in $\mathcal{P}$ satisfies the property stated in the lemma.

Case (C): In this case, $M$ is defined by $w = |z_1|^2 - |z_2|^2 + \lambda(z_1z_2 + \overline{z}_1\overline{z}_2) + E(z, \overline{z})$. Then $\vec{c} = (1, 0)$ is an elliptic direction and we construct the family $V(\tau)$ by setting $z_1 = \xi, z_2 = \tau$. Then $S$ is defined by the following system of equations:

$$\begin{cases}
0 = \frac{\partial w}{\partial z_1} = z_1 + \lambda \overline{z}_2 + o(|z|) \\
0 = \frac{\partial w}{\partial z_2} = -z_2 + \lambda \overline{z}_1 + o(|z|) \\
0 = \frac{\partial}{\partial \tau} = \overline{\xi} + \lambda \overline{z}_2 + o(|z|) \\
0 = \frac{\partial}{\partial z_2} = -\overline{z}_2 + \lambda z_1 + o(|z|)
\end{cases}$$

Thus the Jacobian matrix at the origin is given by

$$
\begin{pmatrix}
1 & 0 & 0 & \lambda \\
0 & \lambda & -1 & 0 \\
0 & 1 & \lambda & 0 \\
\lambda & 0 & 0 & -1
\end{pmatrix}.
$$

It is clear that the matrix is invertible. Therefore by the implicit function theorem, $S$ consists of a single point near 0.

At points where there is a smooth holomorphic curve passing through, following the argument given for Case (B), we have

$$A = G_2 = -\overline{z}_2 + \lambda z_1 + o(|z|), \quad B = G_1 = \overline{z}_1 + \lambda z_2 + o(|z|), \quad C = o(|z|),$$

$\lambda(1) = G_1, \quad \lambda(2) = -G_2, \quad \lambda(4) = -G_T, \quad \lambda(5) = G_T, \quad \lambda(3) = o(|z|), \quad \lambda(6) = o(|z|),$

and $G_1G_T = G_2G_T$.

Similarly, the intersection of the set of the elliptic CR singular points of the Bishop surfaces $V(\tau)$ with smooth points of holomorphic curves contained in $M$ is given by the following system:

$$\begin{cases}
(1 - \lambda^2)|\tau|^2 = (1 - \lambda^2)|\xi|^2 + 2\lambda(\tau\xi + \overline{\tau}\overline{\xi}) + o(|\xi|^2 + |\tau|^2) \\
\xi + \lambda \tau + o(\sqrt{|\xi|^2 + |\tau|^2}) = 0 \\
\overline{\xi} + \lambda \tau + o(\sqrt{|\xi|^2 + |\tau|^2}) = 0
\end{cases}$$

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From the last two equations, we can solve \( \xi = \phi(\tau, \overline{\tau}) \) with \( \phi(\tau, \overline{\tau}) = -\lambda \tau + o(\tau) \). Substituting this into the first equation, we have \( (1 + \lambda^2) |\tau|^2 + o(|\tau|^2) = 0 \). Thus the system can not hold for \( 0 \neq |\tau| \approx 0 \). It follows that a generic elliptic CR singular point of \( V(\tau) \) is a CR point of \( M \) which no holomorphic curves contained in \( M \) can pass through.

Case (D): Now, \( M \) is defined by \( w = |z_1|^2 - |z_2|^2 + \frac{1}{2}(\overline{z}_1^2 + \overline{z}_1^2 + z_2^2 + \overline{z}_2^2) + (z_1 z_2 + \overline{z}_1 \overline{z}_2) + E(z, \overline{z}) \) with \( E = O(|z|^3) \). Then \( \mathcal{S} = (1, -1 + \epsilon) \) with \( 0 < \epsilon << 1 \) is an elliptic direction and we construct the family \( V(\tau) \) by letting \( z_1 = \xi, z_2 = (-1 + \epsilon)\xi + \tau \). Then \( \mathcal{S} \) is defined by the following system of equations:

\[
\begin{align*}
0 &= \frac{\partial M}{\partial z_1} = z_1 + \overline{z}_1 + \overline{z}_2 + o(|z|) \\
0 &= \frac{\partial M}{\partial \overline{z}_1} = -z_2 + \overline{z}_2 + \overline{z}_1 + o(|z|) \\
0 &= \frac{\partial M}{\partial z_2} = z_1 + \overline{z}_1 + z_2 + o(|z|) \\
0 &= \frac{\partial M}{\partial \overline{z}_2} = -\overline{z}_2 + z_2 + z_1 + o(|z|)
\end{align*}
\]

Its Jacobian matrix at the origin is given by

\[
\begin{pmatrix}
1 & 1 & 0 & 1 \\
0 & 1 & -1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 0 & 1 & -1
\end{pmatrix},
\]

which is invertible. Therefore by the implicit function theorem, \( \mathcal{S} \) consists of a single point near 0.

At points where there is a smooth holomorphic curve passing through, it holds that

\[
A = G_2 = -\overline{z}_2 + z_2 + z_1 + o(|z|), \quad B = G_1 = \overline{z}_1 + z_1 + z_2 + o(|z|), \quad C = o(|z|),
\]

\( \lambda_{(1)} = G_1, \quad \lambda_{(2)} = -G_2, \quad \lambda_{(4)} = -G_\overline{\tau}, \quad \lambda_{(5)} = G_\overline{\tau}, \quad \lambda_{(3)} = o(|z|), \quad \lambda_{(6)} = o(|z|), \)

and \( G_1 G_\overline{\tau} = G_2 G_\overline{\tau} \). Also, the intersection of the set of the elliptic CR singular points of the Bishop surfaces \( V(\tau) \) with smooth points of holomorphic curves contained in \( M \) is given by the following system:

\[
\begin{align*}
\left( \frac{\xi}{2} + \tau \right)^2 + \left( \frac{\overline{\xi}}{2} + \overline{\tau} \right)^2 + |\xi|^2 &= -\frac{1}{2} \xi + \tau |^2 + o(|\xi|^2 + |\tau|^2) \\
3\xi + \overline{\xi} + 2\tau + 2\overline{\tau} + o(\sqrt{|\xi|^2 + |\tau|^2}) &= 0 \\
3\overline{\xi} + \xi + 2\overline{\tau} + 2\tau + o(\sqrt{|\xi|^2 + |\tau|^2}) &= 0
\end{align*}
\]

From the last two equations, we can solve \( \xi = \phi(\tau, \overline{\tau}) \) with \( \phi(\tau, \overline{\tau}) = -\frac{1}{2} \tau - \frac{1}{2} \tau + o(\tau) \). Substituting \( \xi = \phi \) into the first equation, we have \( \frac{1}{10}(9\tau^2 + 9|\tau|^2 - 30|\tau|^2) + o(|\tau|^2) = 0 \). Thus the system can not hold for a generic \( \tau \). It follows that a generic elliptic CR singular point of \( V(\tau) \) stays at a CR point of \( M \) where no holomorphic curves contained in \( M \) can pass.

Case (E) : In this case, \( M \) is defined by \( w = z_1 \overline{z}_2 + z_2 \overline{z}_1 + 2b(z_1 z_2 + \overline{z}_1 \overline{z}_2) + \frac{1}{4}(z_2^2 + \overline{z}_2^2) + o(|z|^3) \). Then \( \mathcal{S} = (1, -4b) \) is an elliptic direction and we construct the family \( V(\tau) \) by setting \( z_1 = \ldots \)
\[ \xi, z_2 = -4b\xi + \tau. \] Then \( S \) is defined by the following system of equations:
\[
\begin{align*}
0 = \frac{\partial w}{\partial \xi} &= z_2 + 2b\overline{z}_2 + o(|z|) \\
0 = \frac{\partial w}{\partial \overline{z}_1} &= z_1 + 2b\overline{z}_1 + \overline{z}_2 + o(|z|) \\
0 = \frac{\partial w}{\partial \xi} &= \overline{z}_2 + 2bz_2 + o(|z|) \\
0 = \frac{\partial w}{\partial z_2} &= \overline{z}_1 + 2bz_1 + z_2 + o(|z|)
\end{align*}
\]

Thus the Jacobian matrix at the origin is given by
\[
\begin{pmatrix}
0 & 0 & 1 & 2b \\
1 & 2b & 0 & 1 \\
0 & 0 & 2b & 1 \\
2b & 1 & 1 & 0
\end{pmatrix},
\]
of which the rank is at least three. Therefore, by the implicit function theorem, \( S \) has real dimension at most one.

Similarly, at the points where there is a holomorphic curve passing, we have
\[
G_2 = \overline{z}_1 + z_2 + 2bz_1 + o(|z|), \quad G_1 = \overline{z}_2 + 2bz_2 + o(|z|), G_1G_\tau = -G_2G_\tau.
\]

At the intersection of the elliptic points of the Bishop surfaces with smooth points of a holomorphic curve, we have
\[
\begin{align*}
\Re((-4b\xi + \tau - 8b^2\xi + 2b\tau)(\xi - 2b\overline{\xi} + \overline{\tau})) &= o(|\xi|^2 + |\tau|^2) \\
-8b\xi + \tau - 2b\overline{\xi} + o(\sqrt{|\xi|^2} + |\tau|^2) &= 0 \\
-8b\overline{\xi} + \overline{\tau} - 2b\tau + o(\sqrt{|\xi|^2} + |\tau|^2) &= 0
\end{align*}
\]

From the last two equations, we can solve \( \xi = \phi(\tau, \overline{\tau}) \) with \( \phi(\tau, \overline{\tau}) = \frac{1}{8b}\tau - \frac{1}{4}\overline{\tau} + o(\tau). \) Substituting \( \xi = \phi \) into the first equation, we have
\[
\frac{1}{4}\{(4b\xi^2 + 4b^2 + b + 2)(\tau^2 + \overline{\tau}^2) + (4b^2 + 5b + \frac{1}{8b})|\tau|^2\} + o(|\tau|^2) = 0.
\]
Thus the system can not hold for a generic \( \tau. \) It follows that a generic elliptic CR singular point of \( V(\tau) \) stays at a CR point of \( M \) where no holomorphic curves contained in \( M \) can pass.

Case (F): We now have \( w = z_1\overline{z}_2 + z_2\overline{z}_1 + \frac{1}{4}(z_1^2 + \overline{z}_1^2) + (d\overline{z}_2 + d\overline{z}_2 + E(z, \overline{z}). \) Then \( \phi = (1, C) \) is an elliptic direction, where \( C \) is a complex number such that \( \frac{1}{2} + dC^2 = 0. \) We construct the family \( V(\tau) \) by letting \( z_1 = \xi, \overline{z}_2 = C\xi + \tau. \) Then \( S \) is defined by the following system of equations:
\[
\begin{align*}
0 = \frac{\partial w}{\partial \xi} &= z_2 + \overline{z}_1 + o(|z|) \\
0 = \frac{\partial w}{\partial \overline{z}_2} &= z_1 + 2d\overline{z}_2 + o(|z|) \\
0 = \frac{\partial w}{\partial \overline{z}_1} &= z_1 + \overline{z}_2 + o(|z|) \\
0 = \frac{\partial w}{\partial z_2} &= \overline{z}_1 + 2dz_2 + o(|z|)
\end{align*}
\]

Thus the Jacobian matrix at the origin is given by
\[
\begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 2d \\
1 & 0 & 0 & 1 \\
0 & 1 & 2d & 0
\end{pmatrix},
\]

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which is invertible due to the fact that \( \text{Im} d > 0 \). Therefore by the implicit function theorem, \( S \) consists of a single point near 0.

Similar to the previous cases, we have

\[
A = G_2 = \overline{z}_1 + 2dz_2 + o(|z|), \quad B = G_1 = z_1 + \overline{z}_2 + o(|z|), \quad C = o(|z|),
\]

\( \lambda_{(1)} = G_2, \quad \lambda_{(2)} = G_1, \quad \lambda_{(4)} = -G_\overline{T}, \quad \lambda_{(5)} = -G_T, \quad \lambda_{(3)} = o(|z|), \quad \lambda_{(6)} = o(|z|),
\]

and

\[
G_2 \frac{\partial}{\partial \overline{z}_1} + G_1 \frac{\partial}{\partial \overline{z}_2} - G_\overline{T} \frac{\partial}{\partial z_1} - G_T \frac{\partial}{\partial z_2} = \alpha(G_2 \frac{\partial}{\partial \overline{z}_1} - G_1 \frac{\partial}{\partial \overline{z}_2}) + \beta(G_\overline{T} \frac{\partial}{\partial z_1} - G_T \frac{\partial}{\partial z_2}).
\]

In particular, \( G_1 G_\overline{T} = -G_\overline{T} G_T \).

On the other hand, the elliptic points of the Bishop surfaces \( V(\tau) \) is given by the equation \( \frac{\partial w}{\partial \xi} = 0 \). We substitute \( z_1 = \xi, z_2 = C\xi + \tau \) into the equations: \( \frac{\partial w}{\partial \xi} = 0 \) and \( G_1 G_\overline{T} = -G_\overline{T} G_T \) to get

\[
\begin{align*}
\Re(\xi^2 + \xi(C\overline{\xi} + \overline{\tau}) + 2\overline{\xi}(C\overline{\xi} + \overline{\tau}) + 2d(C\overline{\xi} + \overline{\tau})^2) &= o(|\xi|^2 + |\tau|^2) \\
(C + \overline{C})\xi + \overline{\tau} + 2dC\tau + o(\sqrt{|\xi|^2 + |\tau|^2}) &= 0 \\
(C + \overline{C})\xi + \overline{\tau} + 2dC\tau + o(\sqrt{|\xi|^2 + |\tau|^2}) &= 0
\end{align*}
\]

From the last two equations, we can solve \( \xi = \phi(\tau, \overline{\tau}) \) with \( \phi(\tau, \overline{\tau}) = \frac{-1}{C + \overline{C}}\tau + \frac{-2d}{C + \overline{C}}\tau + o(\tau) \).

Substituting \( \xi = \phi \) in the first equation, we derive the coefficient of \( \tau^2 \) to be \( \frac{1 - 2d}{(C + \overline{C})^2} (1 + 2d|C|^2) \), which shows that the above system can not hold for a generic \( \tau \). Hence a generic elliptic CR singular point of \( V(\tau) \cap M \) stays at a CR point of \( M \) where no holomorphic curves contained in \( M \) can pass.

Combining what we did above, we complete the proof of Lemma 6.9. Thus we completed the proof of Theorem 1.3 except when \( n + 1 = 3 \) and the quadratic normal form of \( M \) is as in (6).

7  Flattening parabolic CR singular points: a formal argument approach

7.1  A reduction to a formal argument

We now proceed to the proof of the Theorem 1.3 when \( (M, p) \subset \mathbb{C}^3 \) has both Bishop invariants parabolic at \( p \). Namely, we assume that, after a holomorphic change of variables, \( p \) is mapped to 0 and \( M \) is defined by an equation of the form:

\[
w = F(z, \overline{z}) = |z_1|^2 + |z_2|^2 + \frac{1}{2} (z_1^2 + z_2^2 + \overline{z}_1^2 + \overline{z}_2^2) + O(|z|^3). \tag{55}
\]

In this case, \( M \) still has an elliptic direction \( \overrightarrow{c} = (1, i) \). And the same proof as in the previous section shows that \( (M, 0) \) can be holomorphically flattened if the set of CR singular points of \( M \) has real dimension at most 1.
However, the set $S$ of the CR singular points of $M$ might have real dimension two. For instance, if $M$ is defined by $w = q^2(z, \tau)(1 + O(1))$ with $q^2 = |z_1|^2 + |z_2|^2 + \frac{1}{2}(z_1^2 + z_2^2 + \bar{z}_1^2 + \bar{z}_2^2)$, then the set of the CR singular points near the origin is defined by $\mathbb{R}z_1 = \mathbb{R}z_2 = 0$, which is of real dimension two. If this happens, when we consider $M_{a,c,\tau} = M \cap \{(z_1, z_2) = (1, i)\xi + \tilde{a}\tau\}$, regarded as a surface in the $(\xi, w)$ plane, for any vector $\tilde{a} = (a_1, a_2)$ not proportional to $\vec{c}$ and for any $\tau \approx 0$, the elliptic CR singular points $p_{c,a}(\tau)$ of $M_{a,c,\tau}$ will also be the CR singular points of $M$ as a 4-manifold in $\mathbb{C}^3$. Thus the construction of the Levi-flat hypersurface $H_{p_{c,a}(\tau)}$ in the last section will not apply here. Indeed, one would suspect that $(M, 0)$ might be not flattenable as suggested in Remark 6.3. Fortunately, the quadratic term in the defining equation of $M$ in (55) now is in the simplest symmetric form, which made the formal normal form theory developed in [HY4] disposable here. And we are still able to produce a positive flattening result. (Other related work related to formal arguments in CR analysis and geometry can be found, for instance, in [CM] [LM1-2] [KZ].)

We will follow the strategy employed in [HY4]. However, since the formal argument in [HY4] has to exclude the case when both generalized Bishop invariants are parabolic, we need some new ideas to handle the current situation. In what follows, we will be very brief for those arguments already contained in [HY4].

We first choose $\vec{c} = (1, i)$ and $\vec{a} = (0, 1)$. As in [HY4], let $\widehat{M}_{p_{a,c}(\tau)}$ be the Levi-flat submanifold bounded by $M_{a,c,\tau}$ as constructed in Huang-Krantz [HK]. Let $\widehat{M}_{a,c} = \cup_{||\tau|| < 1} \widehat{M}_{p_{a,c}(\tau)}$. Then $\widehat{M}_{a,c}$ is a real analytic hypersurface in $\mathbb{C}^3$ with $M$ near 0 as part of its real analytic boundary [HY3]. Now, suppose we can flatten $(M, 0)$ to order $N$. Then as remarked in (6.1) in [HY4], the Levi-form of $\widehat{M}_{a,c}$ vanishes to order at least $\frac{N}{2} - 3$ at 0. Suppose for $N' > N$, with a holomorphic change of coordinates of the form

$$\Phi : z' = z, w' = w + O(|(z, w)|)^2,$$

$M$ can be further flattened to order $N'$, namely, in the new coordinates $M$ near 0 is now defined by an equation of the form as in (55) with $\Im F = O(N' + 1)$. Notice that with such a transform, $M_{p_{a,c}(\tau)}$ is still mapped to $M'_{p_{a,c}(\tau)} = \Phi(M) \cap \{z' = c\xi + a\tau\}$ and $p'_{a,c}(\tau) = \Phi(p_{a,c}(\tau))$. $\Phi(\widehat{M}_{p_{a,c}(\tau)})$ is now a Levi flat submanifold foliated by attached holomorphic disks shrinking down to $p'_{a,c}(\tau)$. By the unique result of Kenig-Webster [KW], $\Phi((\widehat{M}_{a,c}, 0)) = (\widehat{M}'_{a,c}, 0)$. Thus the Levi form of $\widehat{M}'_{a,c}$ has vanishing order at least $\frac{N'}{2} - 3$. Since the vanishing order of the Levi form is a holomorphic invariant, by the analyticity of $\widehat{M}'_{p_{a,c}(\tau)}$, we see that $\widehat{M}_{a,c}$ is in fact Levi flat itself. Thus we will complete the proof of the flattening of $M$ near 0 if we can verify the following:

**Theorem 7.1.** Let $M$ be a smooth 4-manifold in $\mathbb{C}^3$ defined by (55). Assume all CR points of $M$ are non-minimal. Then for any $N$, there is a holomorphic change of coordinates of the form:

$$z' = z, w' = w + B(z, w) = w + O(|(z, w)|)^2,$$

which flattens $(M, 0)$ to order $N$. 

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For the proof of Theorem 7.1, we basically follow the approach developed in Huang-Yin [HY4]. However, new ideas are needed as the argument in [HY4] had to exclude the case when all generalized Bishop invariants are parabolic.

### 7.2 Proof of Theorem 7.1

Before proceeding to the proof of Theorem 7.1, we recall results from [HY4] which will be needed here. We let $H$ be a real-valued homogeneous polynomial in $(z, \bar{z})$ of order $m$. Define

$$\Phi = (z_2 + \bar{z}_2)H_T - (z_1 + \bar{z}_1)H_Z \quad \text{and} \quad \Psi = (z_2 + \bar{z}_2)^2 \Phi_1 - (z_2 + \bar{z}_2)(z_1 + \bar{z}_1)\Phi_2 + (z_1 + \bar{z}_1) \cdot \Phi,$$

where for a function $\chi(z, \bar{z})$, as before, we write $\chi_\alpha = \frac{\partial \chi}{\partial z_\alpha}$, $\chi_\bar{\alpha} = \frac{\partial \chi}{\partial \bar{z}_\alpha}$. Consider the linear partial differential equation in $H$: (See [Appendix 7, HY4])

$$(z_2 + \bar{z}_2)\Psi_1 - (z_1 + \bar{z}_1)\Psi_2 = 0. \quad (57)$$

We follow [HY4] to introduce the following notation: For any homogeneous polynomial $\chi$ of degree $N$ we write

$$\chi = \sum_{t+s+r+h = N} \chi_{[tsrh]} z_1^t z_2^s \bar{z}_1^r \bar{z}_2^h,$$

where by convention, $\chi_{[tsrh]} = 0$ if one of the indices is a negative integer. Notice that in (56) $H, \Phi$ and $\Psi$ are homogeneous polynomials of degree $m, m$ and $m + 1$ respectively.

From (56), we get:

$$\Phi_{[tsrh]} = (h + 1)H_{[ts(r-1)(h+1)]} + (h + 1)H_{[t(s-1)r(h+1)]} - (r + 1)H_{[t(s-1)(r+1)h]} - (r + 1)H_{[ts(r+1)(h-1)]}, \quad (58)$$

and

$$\Psi_{[tsrh]} = (s + 1)\{ \Phi_{[(t+1)(s-1)r(h-1)]} + 2\Phi_{[(t-1)(s+1)(r-1)h]} + \Phi_{[(t-2)(s+1)r]h]} \} - (t + 1)\Phi_{[(t+1)(s-1)(r-1)h]} - (t + 1)\Phi_{[(t+1)(s-1)(r-1)h]} - t\Phi_{[ts(r+h-1)]} - \Phi_{[t(s-1)r]h]} - \Phi_{[t(s-1)r]h]. \quad (59)$$

Collecting the coefficients of $z_2^t z_1^s \bar{z}_2^r \bar{z}_1^h$ for $t \geq 0, s \geq 0, r \geq 0$ and $h = m + 1 - t + s - r \geq 0$ in (57), we get

$$(s + 1)\Psi_{[(t-1)(s+1)r]h} + (s + 1)\Psi_{[(t+1)(s-1)r]h} - (t + 1)\Psi_{[(t+1)(s-1)r]h} - (t + 1)\Psi_{[(t+1)(s-1)r]h} = 0, \quad (60)$$

or

$$(s + 1)\{ \Psi_{[(t-1)(s+1)r]h} + \Psi_{[(t+1)(s-1)r]h} \} = \mathcal{F}\{ (\Psi_{[t' s' r' h']} )_{s' + h' \leq s + h - 1, s' \leq s, h' \leq h} \}. \quad (60)$$

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Here, for a set of complex numbers (or polynomials) \(\{a_j, b\}_{j=1}^k\), we say \(b \in \mathcal{F}\{a_1, \ldots, a_k\}\) if \(b = \sum_{j=1}^k (c_j a_j + d_j \overline{a_j})\) with \(c_j, d_j \in \mathbb{C}\). Now, we start in (60) with \(r = 0, t = m + 1 - (s + h)\) and notice that \(\Psi_{[\nu-(s-h)(s+1)(-1)h]} = 0\). Letting \(r = 1, 2, \ldots\), we inductively get

\[
\Psi_{[tsrh]} = \mathcal{F}\{(\Psi_{[t's'rh']})_{s'+h' \leq s+h-2, s' \leq s, h' \leq h}\}, \text{ for } s \geq 1, t, r, h \geq 0 \text{ and } t+s+r+h = m+1. \tag{61}
\]

Making use of (59) and (61), we get, for \(s \geq 1, r, t, h \geq 0, r+s+t+h = m+1\), the following:

\[
(s+1)\{\Phi_{[t(s+1)(r-2)h]} + 2\Phi_{[(t-1)(s+1)(r-1)h]} + \Phi_{[(t-2)(s+1)r]h}\} = \mathcal{F}\{(\Phi_{[t's'rh']})_{s'+h' \leq s+h-1, s' \leq s, h' \leq h}\}, \text{ for } s \geq 1, \text{ and } t, r, h \geq 0. \tag{62}
\]

Further substituting (58) into (62), we get, for \(s \geq 2, h \geq 0\), the following:

\[
(h+1)H_{[t(s-1)(r+1)h]} + (h+1)H_{[(t-1)srh]} = \mathcal{F}\{(H_{[t's'rh']})_{s'+h' \leq s+h-1, s' \leq s, h' \leq h+1}\}. \tag{63}
\]

Hence, by an induction argument, we have for \(s, h \geq 2, h \geq 1:\)

\[
H_{[ts(m-t-s-h)h]} = \mathcal{F}\{(H_{[t's'(m-t-s'-h)h']})_{s'+h' \leq s+h-2, s' \leq s, h' \leq h}\}. \tag{63}
\]

Applying this property inductively and noticing that \(H_{[tsrh]} = H_{[rhs]}\), we get

\[
H_{[ts(m-t-s-h)]} = \mathcal{F}\{(H_{[t's'(m-t-s'-h)h']})_{s'+h' \leq s+h-2, s' \leq s, h' \leq h}\}. \tag{63}
\]

A crucial step for the proof of Theorem 7.1 is to prove the following uniqueness result:

**Proposition 7.2.** Assume \(\{H_{[tsrh]}\}\) satisfies the normalization condition in Definition 7.3 and \(H\) satisfies the equations (50) and (57). Suppose \(H_{[t(1(m-t-2)1)]} = H_{[00(m-t)0]} \equiv 0\). Then \(H \equiv 0\).

Before giving a proof of Proposition 7.2 we recall a special case of an initial normalization condition introduced in [HY4]. (See [Theorem 4.2, HY4])

**Definition 7.3.** For a homogeneous real-valued polynomial \(E(z, \overline{z})\) of degree \(m_0 \geq 3\) \((z = (z_1, z_2))\), we say that it satisfies the initial normalization condition if we have

\[
E_{(s_1 e_1 + s_2 e_2, 0)} = E_{(t_2 + t_1 e_1 + t_2 e_2, s_2)} = 0 \text{ for } t \geq s, s_1 + s_2 = m_0, t_1 + t_2 > 0, t + t_1 + t_2 + s = m_0, \tag{64}
\]

and we have the following condition in terms of the remainder of \(m_0\) when divided by 6:

\(\text{(II-3) If } m_0 = 6\tilde{m} - 3, \text{ then}\)

\[
E_{(t_2, s_2)} = 0 \text{ for } 4\tilde{m} - 1 \leq t \leq m_0 - 1, \quad E_{(2m + t_1 e_1 + (m_0 - 2t - 3)e_2 + e_1)} = 0 \text{ for } 2\tilde{m} - 2 \leq t \leq 3\tilde{m} - 3. \tag{65}
\]
Lemma 7.4. Assume that \( \{ H_{[s \tau r h]} \} \) satisfies the normalization condition as in Definition 7.3 and satisfies the condition that \( H_{[1(m-t-2)\ell]} = 0 \) and \( H_{[0(m-t)0]} = 0 \). Assume that there exists an \( h_0 \geq -1 \) such that

\[
\Psi_{[s \tau r h]} = \Phi_{[s \tau r h]} = 0 \text{ for } h \leq h_0 \text{ and any } s, \text{ and } H_{[s \tau r h]} = 0 \text{ for } \max(s, h) \leq h_0 + 1.
\]

Then we have

\[
\Psi_{[s \tau r h]} = \Phi_{[s \tau r h]} = 0 \text{ for } h \leq h_0 + 1 \text{ and any } s, \text{ and } H_{[s \tau r h]} = 0 \text{ for } \max(s, h) \leq h_0 + 2.
\]

\[(II_2) \text{ If } m_0 = 6\hat{m} - 2, \text{ then} \]

\[
E_{(te_2,se_2)} = 0 \text{ for } 4\hat{m} - 1 \leq t \leq m_0 - 1,
\]

\[
E_{((2t+1)e_2+e_1,(m_0-2t-3)e_2+e_1)} = 0 \text{ for } 2\hat{m} - 1 \leq t \leq 3\hat{m} - 3, \\
\Re E_{((4\hat{m}-3)e_2+e_1,(2\hat{m}-1)e_2+e_1)} = 0.
\]

\[(II_1) \text{ If } m_0 = 6\hat{m} - 1, \text{ then} \]

\[
E_{(te_2,se_2)} = 0 \text{ for } 4\hat{m} \leq t \leq m_0 - 1,
\]

\[
E_{((2t+1)e_2+e_1,(m_0-2t-3)e_2+e_1)} = 0 \text{ for } 2\hat{m} - 1 \leq t \leq 3\hat{m} - 2.
\]

\[(II_0) \text{ If } m_0 = 6\hat{m}, \text{ then} \]

\[
E_{(te_2,se_2)} = 0 \text{ for } 4\hat{m} + 1 \leq t \leq m_0 - 1,
\]

\[
E_{((2t+1)e_2+e_1,(m_0-2t-3)e_2+e_1)} = 0 \text{ for } 2\hat{m} - 1 \leq t \leq 3\hat{m} - 2, \\
\Re E_{(4\hat{m}e_2,2\hat{m}e_2)} = 0.
\]

\[(II_1) \text{ If } m_0 = 6\hat{m} + 1, \text{ then} \]

\[
E_{(te_2,se_2)} = 0 \text{ for } 4\hat{m} + 1 \leq t \leq m_0 - 1,
\]

\[
E_{((2t+1)e_2+e_1,(m_0-2t-3)e_2+e_1)} = 0 \text{ for } 2\hat{m} \leq t \leq 3\hat{m} - 1.
\]

\[(II_2) \text{ If } m_0 = 6\hat{m} + 2, \text{ then} \]

\[
E_{(te_2,se_2)} = 0 \text{ for } 4\hat{m} + 2 \leq t \leq m_0 - 1,
\]

\[
E_{((2t+1)e_2+e_1,(m_0-2t-3)e_2+e_1)} = 0 \text{ for } 2\hat{m} \leq t \leq 3\hat{m} - 1, \\
\Re E_{((4\hat{m}+1)e_2,(2\hat{m}+1)e_2)} = 0.
\]

In the above, we write \( E_{(s_1+\ldots+s_2e_2,t_1e_1+t_2e_2)} = E_{[s_2s_1t_2t_1]} \). By an induction argument, to conclude Proposition [7.2] it suffices to prove the following:
In what follows, we make the convention that for two integers $k$ and $t$, we set $(t)_k = 0$ if $t < k$ or if $k \cdot t < 0$ and we set $(0)_k = 1$ for $t \geq 0$. We also recall the convention that $H_{tsrh} = \Phi_{tsrh} = \Psi_{tsrh} = 0$ when one of the indices is negative. Then the following can also be written as:

\[
\begin{align*}
\Psi^{(k)}_{[sh]} &= \sum_{t=-\infty}^{\infty} (-1)^{m+1-t-s-h} \binom{t}{k} \Psi_{[ts(m+1-t-s-h)h]}, \\
\Phi^{(k)}_{[sh]} &= \sum_{t=-\infty}^{\infty} (-1)^{m-t-s-h} \binom{t}{k} \Phi_{[ts(m-t-s-h)h]}, \\
H^{(k)}_{[sh]} &= \sum_{t=-\infty}^{\infty} (-1)^{m-t-s-h} \binom{t}{k} H_{[ts(m-t-s-h)h]}.
\end{align*}
\]

As in [HY4], we would like first to transfer the relations among $\Psi$, $\Phi$ and $H$ into the relations among $\Psi^{(k)}_{[s(h_0+1)]}$, $\Phi^{(k)}_{[s(h_0+1)]}$ and $H^{(k)}_{[s(h_0+2)]}$.

**Lemma 7.5.** Assume that there exists an $h_0 \geq -1$ such that

\[
\Psi_{[tsrh]} = \Phi_{[tsrh]} = 0 \text{ for } h \leq h_0, \quad H_{[tsrh]} = 0 \text{ for } \max(s, h) \leq h_0.
\]

Then from (58) and (59), we have for any $k \geq 0$ the following

\[
\begin{align*}
\Phi^{(k)}_{[s(h_0+1)]} &= (h_0 + 2) H^{(k-1)}_{[s(h_0+2)]} + (m - s - h_0 - k) H^{(k)}_{[(s-1)(h_0+1)]} - (k + 1) H^{(k+1)}_{[(s-1)(h_0+1)]}, \\
\Psi^{(k)}_{[s(h_0+1)]} &= (s + 1) \Phi^{(k-2)}_{[(s+1)(h_0+1)]} - (k - 1) \Phi^{(k)}_{[(s-1)(h_0+1)]}.
\end{align*}
\]

Moreover, by (57), $\Psi^{(k)}_{[s(h_0+1)]}$ satisfies the following equation:

\[
(s + 1) \Psi^{(k-1)}_{[(s+1)(h_0+1)]} = (k + 1) \Psi^{(k+1)}_{[(s-1)(h_0+1)]}.
\]

**Proof of the Lemma 7.5:** The proofs for (76) and (77) are the same as that in [Lemma 5.4, HY4]. For the last one, we modify the method used in [HY4] as follows: Combining (60) and the assumption that $\Psi_{[tsrh_0]} = 0$, we have

\[
(s + 1) \Psi_{[(t-1)(s+1)r(h_0+1)]} + (s + 1) \Psi_{[t(s+1)(r-1)(h_0+1)]} - (t + 1) \Psi_{[(t+1)(s-1)r(h_0+1)]} = 0.
\]
Then using the convention set up before, we obtain the following:

\[
0 = \sum_{t=-\infty}^{\infty} (-1)^t \binom{t}{k} \left\{ (s + 1) \Psi_{[t-1](s+1)r(h_0+1)} + (s + 1) \Psi_{[t+1](r-1)(h_0+1)} - (t + 1) \Psi_{[t+1](s-1)r(h_0+1)} \right\}
\]

\[
= (s + 1) \sum_{t=-\infty}^{\infty} \left\{ (t-1) \binom{t-1}{k} + (t-1) \binom{t-1}{k-1} \right\} (-1)^t \Psi_{[t-1](s+1)r(h_0+1)} + (s + 1) \sum_{t=-\infty}^{\infty} (-1)^t \binom{t}{k} \Psi_{[t+1](r-1)(h_0+1)}
\]

\[
- (k + 1) \sum_{t=-\infty}^{\infty} (-1)^{t+1} \binom{t+1}{k+1} \Psi_{[t+1](s-1)r(h_0+1)}
\]

\[
= (s + 1) \Psi_{[s+1](h_0+1)}^{(k)} + (s + 1) \Psi_{[s+1](h_0+1)}^{(k-1)} - (s + 1) \Psi_{[s+1](h_0+1)}^{(k)} - (k + 1) \Psi_{[s-1](h_0+1)}^{(k+1)}.
\]

**Proof of the Lemma 7.4.** From (61) we have

\[
\Psi_{[t+1](h_0+1)} = F\{ (\Psi_{[t'+(2s'+1)r'(h_0+1)]})_{s' < s}, (\Psi_{[t''s''r''h'']})_{h'' \leq h_0} \}
\]

for \( s \geq 0, h_0 \geq -1, \) and \( t, r \geq 0. \)

In particular, we obtain

\[
\Psi_{[t+1](h_0+1)} = F\{ (\Psi_{[t's'r'h']})_{h' \leq h_0} \} = 0.
\]

The last equality follows from the assumptions in (71). By an induction argument, we obtain

\[
\Psi_{[t+(2s+1)r(h_0+1)]} = 0, \text{ for } s \geq 0, h_0 \geq -1.
\]

(79)

Combining this with (77), we have

\[
(2s + 2) \Phi_{[2s+2](h_0+1)}^{(k-2)} - (k - 1) \Phi_{[2s](h_0+1)}^{(k)} = 0, \text{ for } s \geq 0 \text{ and } h_0 \geq -1.
\]

Taking \( k = 0 \) in the above equality, by the fact that \( \Phi_{[2s+2](h_0+1)}^{(-2)} = 0, \) we have \( \Phi_{[2s](h_0+1)}^{(0)} = 0. \)

Inductively, by setting \( k = 2, 4, \ldots, \) we finally have \( \Phi_{[2k](h_0+1)}^{(k)} = 0 \) for all \( k \geq 0. \)

Combing this with (76), we have \( H_{[0(h_0+2)]}^{(2k-1)} = 0. \) Now by the argument from [(5.75), HY4] to [(5.77), HY4], we know \( H_{[0(h_0+2)]} = 0. \) (This is the place part of the initial normalization conditions are used.) By the assumption \( H_{[t+1]} = 0 \) and the relation in (63), we get \( H_{[ts(t-s)-h_0+2)]} = 0 \) for \( 0 \leq s \leq h_0 + 2. \) Then, by (68), we have \( \Phi_{[s+1](h_0+1)]} = 0 \) for \( s \leq h_0 + 1. \) In particular, \( \Phi_{[0(h_0+1)]} = \Phi_{[1r(h_0+1)]} = 0. \) Thus by (62), it follows that \( \Phi_{[sr(h_0+1)]} = 0. \) Similarly by (59) and (61), we conclude \( \Psi_{[sr(h_0+1)]} \equiv 0. \) Thus we completed the proof of Lemma 7.4.

Now, let \( M \) be as defined by (55). We assume \( M \) is flattened to order \( m - 1. \) We need to find a holomorphic change of coordinates of the form

\[
z' = z, w = w + B(z, w) = w + O(|z, w|^2).
\]

(80)
to flatten $M$ to order $m$. Write $H$ for the homogeneous polynomial of degree $m$ in the Taylor expansion of $\mathfrak{F}$ at 0. By [Appendix 7, HY4], $H$ satisfies the linear equations in (56) and (57). By [Theorem 4.2, HY4], there is a unique transformation of the form $z' = z, w' = w + B(z, w)$, where $B(z, w) = \sum_{|a| + 2j = m} b_{a_j} z^a w^j$ with $b_0 = 0$ if $m$ is even, which transforms $M$ to a new manifold with $H = (\mathfrak{F})^{(m)}$, the homogeneous part of degree $m$ in $\mathfrak{F}$, that satisfies the normalization conditions in Definition 7.3.

**Case I. $m$ is odd:** Step I: We first consider a holomorphic change of coordinates: $(z_1, z_2) \rightarrow (-z_1, z_2)$. It is clear that $H(z_1, z_2)$ and $H(-z_1, z_2)$ both satisfy (56), (57) and the initial normalization conditions in Definition 7.3. Therefore $H(z_1, z_2) - H(-z_1, z_2)$ is also a solution to the linear system (56) and (57) satisfying the initial normalizations in Definition 7.3. Write

$$H(z_1, z_2) = \sum_{s+h \text{ even}} H_{[tsr]} z_1^s \overline{z}_1^{l} z_2^h \overline{z}_2^r + \sum_{s+h \text{ odd}} H_{[tsr]} z_1^s \overline{z}_1^{l} z_2^h \overline{z}_2^r.$$ 

Then

$$H(-z_1, z_2) = \sum_{s+h \text{ even}} H_{[tsr]} z_1^s \overline{z}_1^{l} z_2^h \overline{z}_2^r - \sum_{s+h \text{ odd}} H_{[tsr]} z_1^s \overline{z}_1^{l} z_2^h \overline{z}_2^r$$

$$H(z_1, z_2) - H(-z_1, z_2) = 2 \sum_{s+h \text{ odd}} H_{[tsr]} z_1^s \overline{z}_1^{l} z_2^h \overline{z}_2^r.$$ 

By Lemma 7.2 $H(z_1, z_2) - H(-z_1, z_2) \equiv 0$ or equivalently,

$$H(z_1, z_2) = \sum_{s+h \text{ even}} H_{[tsr]} z_1^s \overline{z}_1^{l} z_2^h \overline{z}_2^r. \quad (81)$$

Step II: In this step, we assume the associated homogeneous polynomial $H$ of the 4-manifold $M$ already take the form as in (81). We then holomorphically change the coordinates by $(\tilde{z}_1, \tilde{z}_2) := (z_2, z_1)$. $\tilde{M}$, the image of $M$ under this map, is defined now by

$$\tilde{w}(\tilde{z}_1, \tilde{z}_2) = \mathcal{F}(\tilde{z}_1, \tilde{z}_2) = F(\tilde{z}_2, \tilde{z}_1) = |\tilde{z}_1|^2 + |\tilde{z}_2|^2 + \frac{1}{2}(\tilde{z}_1^2 + \tilde{z}_2^2 + \overline{\tilde{z}}_1 + \overline{\tilde{z}}_2) + O(|\tilde{z}|^3).$$

In particular, $\tilde{H}$, the homogeneous polynomial of degree $m$ in Taylor expansion of $\mathfrak{F}$ at 0, takes the following form, by the fact that $H_{[tsr]} = 0$ for $t + r$ even:

$$\tilde{H}(\tilde{z}_1, \tilde{z}_2) = \sum_{s+h \text{ odd}} H_{[tsr]} z_1^s \overline{z}_1^{l} z_2^h \overline{z}_2^r = \sum_{s+h \text{ odd}} H_{[tsr]} z_1^s \overline{z}_1^{l} z_2^h \overline{z}_2^r.$$ 

Notice that $\tilde{H}$ may not satisfy the normalization conditions in Definition 7.3. Therefore, we normalize $\tilde{M}$ using a transformation of the following form:

$$z' = \tilde{z}$$

$$w' = \tilde{w} + B(\tilde{z}, \tilde{w}) = \tilde{w} + \sum_{\alpha_1 + \alpha_2 + 2k = m} a_{\alpha_1 \alpha_2 k} \tilde{z}_1^{\alpha_1} \overline{\tilde{z}}_1^{\alpha_2} \tilde{w}^k, \quad a_{0 \alpha} = 0 \text{ if } m \text{ even.} \quad (82)$$
By Theorem 4.2 in [HY4], we know there is a unique \( \tilde{B} \) such that the new \( \tilde{H}' \) satisfies the normalization conditions, (56) and (57). Also from [HY4], we know \( \tilde{H}' = \tilde{H} + \text{Im} \tilde{B}(z, q^{(2)}(z, \overline{z})) \), where \( q^{(2)}(z, \overline{z}) = |\tilde{z}|^2 + |\tilde{z}_2|^2 + \frac{1}{2} (\tilde{z}_1^2 + \tilde{z}_2^2 + \overline{\tilde{z}_1}^2 + \overline{\tilde{z}_2}^2) \).

A crucial observation here is that, when \( \alpha_1, \) the power of \( \tilde{z}_1 \) in (82), is odd, the expansion of \( \tilde{z}_1^\alpha \tilde{z}_2^\alpha \tilde{w}^k \) after substituting \( \tilde{w} = q^{(2)}(z, \overline{z}) \) only contains terms \( \tilde{z}_1^\alpha \tilde{z}_2^\alpha \tilde{w}^r \) with \( s + h \) odd; and when \( \alpha_1 \) is even, the expansion of \( \tilde{z}_1^\alpha \tilde{z}_2^\alpha \tilde{w}^k \) after substituting \( \tilde{w} = q^{(2)}(z, \overline{z}) \) only contains terms \( \tilde{z}_1^\alpha \tilde{z}_2^\alpha \tilde{w}^r \) with \( s + h \) even. Then we split the polynomial \( B \) into two parts depending on whether \( \alpha_1 \) is odd or even:

\[
\tilde{B}(\tilde{z}, \tilde{w}) = \tilde{B}_1(\tilde{z}, \tilde{w}) + \tilde{B}_2(\tilde{z}, \tilde{w}) = \sum_{\alpha_1 + \alpha_2 + 2k = m, \alpha_1 \text{ odd}} a_{\alpha_1 \alpha_2 k} \tilde{z}_1^{\alpha_1} \tilde{z}_2^{\alpha_2} \tilde{w}^k + \sum_{\alpha_1 + \alpha_2 + 2k = m, \alpha_1 \text{ even}} a_{\alpha_1 \alpha_2 k} \tilde{z}_1^{\alpha_1} \tilde{z}_2^{\alpha_2} \tilde{w}^k.
\]

By the uniqueness of \( \tilde{B} \), we have \( \tilde{B}_2 \equiv 0 \) and thus \( \tilde{H}'_{\text{trsh}} = \tilde{H}_{\text{trsh}} = 0 \) for \( s + h \) even. Now, by Lemma 7.3 and what we just argued, we have \( \tilde{H}' \equiv 0 \). Next for the original 4-manifold \( M \) with \( H \) as in (81), we define the transformation of the form:

\[
z' = z
\]

\[
w' = w + B(z, w) = w + \sum_{\alpha_1 + \alpha_2 + 2k = m} a_{\alpha_1 \alpha_2 k} \tilde{z}_1^{\alpha_1} \tilde{z}_2^{\alpha_2} w^k,
\]

where \( \{a_{\alpha_1 \alpha_2 k}\} \) are the same as in (82), or equivalently, \( B(z_1, z_2, w) = \tilde{B}(z_2, z_1, w) \). Then after the transformation, \( H' = H + \text{Im} B(z, q^{(2)}(z, \overline{z})) \equiv 0 \). Thus we prove Theorem 7.1 in the case when \( m \) is odd.

**Case II. \( m \) is even:** In this case, after establishing [(5.47), HY4], the same argument there gives that \( H_{\text{tr} r} = H_{\text{tr} o} \equiv 0 \) for even \( m \). Hence, by Lemma 7.2 we complete the argument for the case of \( m \) even. Now we proceed to establish the following identity:

\[
H^{(2k-2)}_{[11]} + (m + 1 - 2k)H^{(2k-1)}_{[00]} - 2kH^{(2k)}_{[00]} = 0.
\]

which is (5.47) in [HY4] (with \( \theta = 0 \) and \( \xi = 1 \)).

By substituting \( h_0 = -1, k = 2l \) in (78), we have \( (s + 1)\Psi^{(2l-1)}_{[(s+1)0]} = (2l + 1)\Psi^{(2l+1)}_{[(s-1)0]} \).

Substituting \( l = 0, s = 2k + 1 \) in (85), we have \( \Psi^{(1)}_{[(2k)0]} = (2k + 2)\Psi^{(-1)}_{[(2k+2)0]} \equiv 0 \). By setting \( l = i, s = 2k - 2i + 1 \) for \( i = 1, \ldots, k \), inductively, we will have \( \Psi^{(2k+1)}_{[00]} = \cdots = \Psi^{(1)}_{[(2k)0]} = 0 \). Hence we proved that \( \Psi^{(2k+1)}_{[00]} = 0 \) for all \( k \geq 0 \). Next, by substituting \( s = 0, h_0 = -1, k = 2l + 1 \) in (79), we have \( \Phi_{[10]}^{(2l-1)} = 0 \) for \( l \geq 1 \). Finally, substituting \( s = 1, h_0 = -1, k = 2l - 1 \) in (79), we have

\[
H^{(2l-2)}_{[11]} + (m + 1 - 2l)H^{(2l-1)}_{[00]} - 2lH^{(2l)}_{[00]} = 0, \quad \text{for } l \geq 1,
\]

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which is exactly (5.47) in [HY4] by setting the $\theta$ and $\xi$ there to be zero and 1, respectively. Therefore, we proved Theorem 7.1 for $m$ even case.

We thus finally completed the proof of Theorem 1.3. ■

We now complete the proof of Corollary 1.4. Let $(M, p)$ be as in Corollary 1.4 with $p$ now a definite non-degenerate CR singular point. Let $\Phi$ be a biholomorphic map sending a neighborhood of $p$ to a neighbor of 0 with $\Phi(p) = 0$ such that $M' = \Phi(M)$ is defined by an equation of the form $u = |z|^2 + \sum_{j=1}^{n} \lambda_j(z^2 + \bar{z}_j^2) + O(3)$, $v = 0$. Now, let $0 < \varepsilon_0 << 1$. Then $\Phi(B_{\varepsilon_0}(p))$ is a strongly pseudoconvex domain containing the origin. Hence the holomorphic hull of $\Phi(B_{\varepsilon_0}(p)) \cap M'$ near 0 coincides with a neighborhood of 0 in the set defined $u \geq |z|^2 + \sum_{j=1}^{n} \lambda_j(z^2 + \bar{z}_j^2) + O(3)$, $v = 0$ in the Levi-flat hypersurface $v = 0$. From this, one sees the proof of the last statement in Corollary 1.4. Thus, the proof of Corollary 1.4 is complete. ■

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