Electrical Networks and Hyperplane Arrangements

Bob Lutz

Mathematical Sciences Research Institute

June 7, 2019

Partially supported by NSF grants DMS-1401224 and DMS-1701576
Section I: Electrical Networks
What is an electrical network?

- A connected graph $G = (V, E)$ (edges = wires)
- A set $\partial V \subseteq V$ of at least 2 boundary nodes
- A (real or complex) voltage $v_j$ at every boundary node $j$
The Dirichlet problem

- Electrical current flows from higher voltages to lower voltages
- Consider the interior \( V^\circ = V \setminus \partial V \)
- *What are the voltages at the interior nodes?*
The Dirichlet solution

Every wire $ij \in E$ has a (real or complex) conductance $c_{ij}$

Voltages and conductances satisfy

$$\sum_{j \sim i} c_{ij}(v_i - v_j) = 0$$

at every interior node $i \in V^\circ$

“The current across every interior node is 0”

Uniquely determines the interior voltages (for generic $c$)
Example

Network with conductances labeled:  

\[ \sum_{j \sim i} c_{ij} (v_i - v_j) = 1 \left( \frac{55}{71} - 1 \right) + 3 \left( \frac{55}{71} - \frac{52}{71} \right) + 4 \left( \frac{55}{71} - 0 \right) = 0 \quad \checkmark \]
**Energies**

**Definition**

The **energy** dissipated by an edge $ij \in E$ is

$$e_{ij} = c_{ij}(v_i - v_j)^2$$

**Example**

Let $\Delta = (c_1 + c_3 + c_4)(c_2 + c_3 + c_5) - c_3^2$. We have

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix} = \frac{1}{\Delta^2} \begin{pmatrix} c_1(c_3c_5 + c_4(c_2 + c_3 + c_5))^2 \\ c_2(c_3c_4 + (c_1 + c_3 + c_4)c_5)^2 \\ c_3(c_2c_4 - c_1c_5)^2 \\ c_4(c_2c_3 + c_1(c_2 + c_3 + c_5))^2 \\ c_5(c_1c_3 + c_2(c_1 + c_3 + c_4))^2 \end{pmatrix}$$
Question

- The map $c \mapsto e$ is rational (polynomial/polynomial)
- What do the fibers look like?
- Equivalently, which conductances and interior voltages produce the energies $e$?

Example

Let $\Delta = (c_1 + c_3 + c_4)(c_2 + c_3 + c_5) - c_3^2$. We have

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  e_4 \\
  e_5 \\
\end{pmatrix}
= \frac{1}{\Delta^2}
\begin{pmatrix}
  c_1(c_3c_5 + c_4(c_2 + c_3 + c_5))^2 \\
  c_2(c_3c_4 + (c_1 + c_3 + c_4)c_5)^2 \\
  c_3(c_2c_4 - c_1c_5)^2 \\
  c_4(c_2c_3 + c_1(c_2 + c_3 + c_5))^2 \\
  c_5(c_1c_3 + c_2(c_1 + c_3 + c_4))^2 \\
\end{pmatrix}
\]
Definition
Fix a graph $G$, boundary voltages $v$, and (real or complex) edge energies $e$. A function $h \in \mathbb{C}^V$ is $e$-harmonic on $(G, v)$ if

1. There are conductances $c$ with respect to which $h$ is the voltage function for the network
2. The energies dissipated are $e$.

Interesting Problem
Describe the set of $e$-harmonic functions for a given $e$. 
Let all edge energies $e$ be 1. Fix boundary voltages 0 and 1. There are two $e$-harmonic functions, with conductances labeled:

$$
\begin{align*}
25a - 5 & \quad 25b - 5 \\
1 & \quad 5 & \quad 0 & \quad 25b - 5 & \quad 25a - 5 \\
25b - 5 & \quad 25a - 5 & \quad 25a - 5 & \quad 25b - 5
\end{align*}
$$

where $a = \frac{1}{10}(5 - \sqrt{5})$ and $b = \frac{1}{10}(5 + \sqrt{5})$. Can check energy of middle edge $ij$:

$$
e_{ij} = c_{ij}(v_i - v_j)^2 = 5 \left( \frac{1}{10}(5 + \sqrt{5}) - \frac{1}{10}(5 - \sqrt{5}) \right)^2 = 1 \quad \checkmark$$
Section II: Dirichlet Arrangements
An **arrangement** is a set of affine hyperplanes (in \( \mathbb{R}^n \) or \( \mathbb{C}^n \)).

Arrangements in \( \mathbb{R}^n \) divide the space into **chambers**.

E.g. — Arrangements in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \):

![Diagram of arrangements in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \)]
Dirichlet arrangements

- Recall: $G$ a graph with boundary voltages $v$

**Definition**

The **Dirichlet arrangement** $A_{G,v}$ consists of two types of hyperplanes, corresponding to the edges of $G$, with coordinates indexed by interior nodes:

- A hyperplane $x_i = v_j$ for every edge $ij$ with $j \in \partial V$
- A hyperplane $x_i = x_j$ for every edge $ij$ not meeting $\partial V$
Let $\mathcal{A}$ be an arrangement of $k$ hyperplanes in $\mathbb{C}^n$, defined by affine functions $f_1, \ldots, f_k : \mathbb{C}^n \to \mathbb{C}$.

**Master function** of $\mathcal{A}$ with weights $b \in \mathbb{C}^k$ is multivalued $\mathbb{C}^n \to \mathbb{C}$ given by

$$\Phi_b(x) = \prod_{i=1}^k f_i(x)^{b_i}$$

A critical point $x \in \mathbb{C}^n$ of $\Phi_b$:

$$\frac{\partial}{\partial x_j} \log \Phi_b(x) = \sum_{i=1}^k \frac{\partial f_i}{\partial x_j} \frac{b_i}{f_i(x)} = 0$$

for all $j = 1, \ldots, n$
Arrangement $\mathcal{A}$ in $\mathbb{C}^2$ defined by

$$\mathcal{A} : \quad x = 0, \; y = 0, \; x + y - 1 = 0$$

Master function with weights $b = (b_1, b_2, b_3) \in \mathbb{C}^3$:

$$\Phi_b(x, y) = x^{b_1} y^{b_2} (x + y - 1)^{b_3}$$

Critical point equations:

$$\nabla \log \Phi_b(x, y) = \left( \frac{b_1}{x} + \frac{b_3}{x + y - 1}, \; \frac{b_2}{y} + \frac{b_3}{x + y - 1} \right) = (0, 0)$$

$$(x, y) = \left( \frac{b_1}{b_1 + b_2 + b_3}, \; \frac{b_2}{b_1 + b_2 + b_3} \right)$$
Theorem
Fix edge energies $e \in \mathbb{C}^E$. The $e$-harmonic functions on $(G, \nu)$ are the critical points of the master function of $\mathcal{A}_{G,\nu}$ with weights $e$.

Important example
When $(G, \partial V)$ is complete, $\mathcal{A}_{G,\nu}$ is a discriminantal arrangement:
Critical points of master functions are well studied!

▶ Interior point methods: *logarithmic barrier functions* and *analytic centers*

...especially for discriminantal arrangements.

▶ Real algebraic geometry: solution of B. and M. Shapiro conjecture on Wronskians

▶ Quantum integrable systems: *Bethe ansatz* in the *Gaudin model*

  ▶ Main problem: diagonalize a set of commuting linear operators (*Hamiltonians*)
  ▶ Critical points of master functions parameterize the eigenvectors
Consequences

Corollary

Suppose the boundary voltages are real, so $\mathcal{A}_{G,v}$ lives in real space. If the energies $e$ are all positive, then

- The $e$-harmonic functions take all real values
- There is exactly one $e$-harmonic function in each bounded chamber of $\mathcal{A}_{G,v}$, and no others

Example

Let $a = \frac{1}{10}(5 - \sqrt{5})$ and $b = \frac{1}{10}(5 + \sqrt{5})$ with all energies 1
Current flow

- Real-valued e-harmonic functions induce current flows
- Edges directed from higher voltages to lower
- \# nonzero current flows = \# e-harmonic functions = \# bounded chambers

Example

In the running example:

\[
\begin{align*}
&\text{where } a = \frac{1}{10}(5 - \sqrt{5}) \text{ and } b = \frac{1}{10}(5 + \sqrt{5}).
\end{align*}
\]
Counting bounded chambers

- Chromatic polynomial $\chi_G$ counts proper vertex colorings of $G$
- The beta invariant is $\beta(G) = |\chi'_G(1)|$
- $\beta(G) = 0$ iff $G$ is disconnected by removing single vertex

Theorem (L.)

Obtain a graph $\hat{G}$ by adding edges between all boundary vertices. Then for generic energies $e$ (including all positive) we have

$$\frac{\beta(\hat{G})}{(|\partial V| - 2)!} = \# \text{ current flows} = \# \text{ e-harmonic functions} = \# \text{ bounded chambers}$$

- This formula also has applications to a visibility problem
Definition
Let $\mathcal{P}$ be a convex $n$-polytope in $\mathbb{R}^n$. The **visibility arrangement** $\text{vis}(\mathcal{P})$ is the set of affine spans of the top-dimensional faces of $\mathcal{P}$.
Visibility sets

Chambers of $\text{vis}(\mathcal{P}) \leftrightarrow$ Sets of top-dimensional faces of $\mathcal{P}$ visible from different points in $\mathbb{R}^n$
Definition
Let $P$ be a finite poset. The order polytope $\mathcal{O}(P)$ is the convex polytope in $\mathbb{R}^P$ of all order-preserving functions $P \rightarrow [0, 1]$

Example
If every pair in $P = \{x_1, \ldots, x_n\}$ is incomparable, then
$\mathcal{O}(P) = [0, 1]^n$ is the unit hypercube in $\mathbb{R}^n$

Example
If $P = \{x_1, \ldots, x_n\}$ is totally ordered, then $\mathcal{O}(P)$ is an $n$-simplex in $\mathbb{R}^n$
**Proposition (Stanley)**

Let $P$ be a finite poset. The visibility arrangement $\text{vis}(\mathcal{O}(P))$ of the order polytope of $P$ is a Dirichlet arrangement $\mathcal{A}_{G,v}$. 

 HASSE diagram of $P$  

 $\text{(} G, v \text{)}$  

 $\hat{G}$
Theorem (L.)

The number of visibility sets of the order polytope \( \mathcal{O}(P) \) is

\[
\frac{1}{2} \alpha(\hat{G}),
\]

where \( \alpha(\hat{G}) \) is the number of acyclic orientations of \( \hat{G} \). Of these visibility sets, all but

\[
\beta(\hat{G})
\]

are visible from far away, where \( \beta(\hat{G}) \) is the beta invariant.
Example

- $P = \{x_1, \ldots, x_n\}$ totally ordered
- $O(P)$ is an $n$-simplex in $\mathbb{R}^n$
- $\frac{1}{2} \alpha(\hat{G}) = 2^{n+1} - 1$ visibility sets (all but full set of faces)
- All but $\beta(\hat{G}) = 1$ visible from far away (the empty set)

Hasse diagram of $P$

$(G, v)$

$\hat{G}$
Section III: The Response Matrix

\[ \Lambda = \frac{1}{S} \begin{bmatrix} c_1(S - c_1) & -c_1 c_2 & \cdots & -c_1 c_n \\ -c_1 c_2 & c_2(S - c_2) & \cdots & -c_2 c_n \\ \vdots & \vdots & \ddots & \vdots \\ -c_1 c_n & -c_2 c_n & \cdots & c_n(S - c_n) \end{bmatrix} \]
Electrical response

- $G$ a graph with boundary $\partial V$ and conductances $c$
- We know current across interior node $i$ is 0:
  $$\sum_{j \sim i} c_{ij}(v_i - v_j) = 0$$
- Current across boundary node not necessarily 0
- Map from boundary voltages $\nu$ to boundary currents is linear
  $$\mathbb{C}^{\partial V} \rightarrow \mathbb{C}^{\partial V}$$

Definition
The response matrix is the matrix $\Lambda = \Lambda(G, \partial V, c)$ of the map from boundary voltages to boundary currents.
Entries of response matrix

- Response matrix $\Lambda$ maps boundary voltages to boundary currents
- Entries of $\Lambda$ are rational functions in the conductances

\[ R = \frac{c_1 c_2 c_3 + c_1 c_2 c_4 + c_1 c_2 c_5 + c_1 c_3 c_4 + c_1 c_3 c_5 + c_2 c_3 c_4 + c_3 c_4 c_5}{c_1 c_3 + c_1 c_4 + c_1 c_5 + c_2 c_3 + c_2 c_4 + c_2 c_5 + c_3 c_5 + c_4 c_5} \]

\[ \Lambda = R \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]
We are interested in the trace of Λ

**Definition**

A *grove* is a forest $F \subseteq E$ in $G$ such that

- $F$ meets every interior node
- Every tree in $F$ meets $\partial V$ at least once
Let $\mathcal{F}_0$ (resp., $\mathcal{F}_1$) be the set of groves containing no (resp., exactly 1) path between boundary nodes.

**Proposition**

The trace of the response matrix can be written as

$$\frac{1}{2} \operatorname{tr} \Lambda = \frac{\sum_{F \in \mathcal{F}_1} \prod_{ij \in F} c_{ij}}{\sum_{F \in \mathcal{F}_0} \prod_{ij \in F} c_{ij}}.$$
Proposition

The trace of the response matrix can be written as

\[
\frac{1}{2} \text{tr } \Lambda = \frac{\sum_{F \in \mathcal{F}_1} \prod_{ij \in F} c_{ij}}{\sum_{F \in \mathcal{F}_0} \prod_{ij \in F} c_{ij}}.
\]
Theorem (L.)

Let $x, y \in \mathbb{R}^E$ with all entries of $y$ positive. The zeros and poles of $\text{tr} \Lambda$ interlace along the line $x + ty$.

Two sequences of points on a line interlace along the line if they alternate like so:
Consider the star network with $n$ boundary nodes and conductances $a = (c_1, \ldots, c_n)$:

$$
\frac{1}{2} \text{tr } \Lambda = \frac{\sum_{i<j} c_i c_j}{\sum_i c_i}
$$

- Consider the line $(-1, \ldots, -1, n-1) + t(1, \ldots, 1)$ in $\mathbb{R}^n$
- The zeros are at $t = \pm 1$ and the sole pole is at $t = 0$
- Case $n = 2$ illustrated on the right