Perron-Frobenius operators and representations of the Cuntz-Krieger algebras for infinite matrices

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Abstract

In this paper we extend work of Kawamura, see \cite{3}, for Cuntz-Krieger algebras $O_A$ for infinite matrices $A$. We generalize the definition of branching systems, prove their existence for any given matrix $A$ and show how they induce some very concrete representations of $O_A$. We use these representations to describe the Perron-Frobenius operator, associated to an nonsingular transformation, as an infinite sum and under some hypothesis we find a matrix representation for the operator. We finish the paper with a few examples.

1 Introduction

The interactions between the theory of dynamical systems and operator algebras are one of the main venues in modern mathematics. Exploring this interplay Kawamura, see \cite{3}, recently showed that the

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theory of representations of the Cuntz-Krieger algebras is closely related to the theory involving the Perron-Frobenius operator. The work of Kawamura is done for the Cuntz-Krieger algebras $O_A$, for finite matrices $A$. In this paper we generalize many of the results in [3] for the Cuntz-Krieger algebras for infinite matrices (a concept introduced by Exel and Laca in [1]). For example, under some mild assumptions, we are able to give a explicit characterization of the Perron-Frobenius operator, associated to a nonsingular transformation, as a infinite sum, using a representation of an infinite Cuntz-Krieger algebra. In our efforts to generalize the notions of [3] we found two problems with the work done in there that we believe are worth mentioning. First is the necessity of an extra hypothesis in the definition of a branching function system given in [3]. The other problem is in the statement of theorem 1.2 of [3], where $BA$ should read $A^T B$. We will deal with both these cases when introducing our generalized versions of the theory of [3].

We organize the paper in the following way: In the remaining of the introduction we quickly recall the reader the main definitions of [3] and show the need for an extra hypothesis in the definition of a branching function system. In section 2, we define branching systems for infinite matrices $A$, which we denote by $A_\infty$. We deal with the existence of $A_\infty$ branching systems for any given matrix $A$ (infinite or not) and show how they induce representations of $O_A$ in section 3. Next, in section 4, we use the representations introduced in section 3 to describe the Perron-Frobenious operator as an infinite sum; we also present the generalizde and corrected version of theorem 1.2 of [3] in this section. We finish the paper in section 5 with a few examples.

Given a measure space $(X, \mu)$, let $L_\mu(X, \mu)$ be the set of all complex valued measurable functions $f$ such that $\|f\|_p < \infty$. For a nonsingular transformation $F : X \to X$ (that is, $\mu(F^{-1}(A)) = 0$ if $\mu(A) = 0$) let $P_F : L_1(X, \mu) \to L_1(X, \mu)$ be the Perron-Frobenius operator, that is, $P_F$ is such that

$$\int_A P_F \psi(x) d\mu = \int_{F^{-1}(A)} \psi(x) d\mu$$
for each measurable subset \( A \) of \( X \), for all \( \psi \in L_1(X,\mu) \). Notice that, for \( \psi \in L_1(X,\mu) \), \( P_F(\psi) \) is the Radon-Nikodym derivative of the measure \( \mu_{P_F} \), given by \( \mu_{P_F}(A) = \int_{F^{-1}(A)} \psi(x)d\mu \), with respect to \( \mu \) (see [4] for more details about the Perron-Frobenius operator).

In order to describe the Perron-Frobenius operators and representations of the Cuntz-Krieger algebras, Kawamura, in [3], introduces an \( A \)-branching function system on a measure space \((X,\mu)\): a family \( \{\{f_i\}_{i=1}^{N},\{D_i\}_{i=1}^{N}\} \) of measurable maps and measurable subsets of \( X \) respectively, together with a nonsingular transformation \( F : X \to X \) such that \( f_i : D_i \to f_i(D_i) = R_i, \mu(X \setminus \bigcup_{i=1}^{N} R_i) = 0, \mu(R_i \cap R_j) = 0 \) for all \( i \neq j \), there exists the Radon-Nikodym derivative \( \Phi_{f_i} \) of \( \mu \circ f_i^{-1} \) with respect to \( \mu \) and \( \Phi_{f_i} > 0 \) almost everywhere in \( D_i \) for \( i = 1,..,N \), \( F \circ f_i = id_{D_i} \) for each \( i \in \mathbb{N} \) and \( \mu(D_i \setminus \bigcup_{j:a_{ij}=1} R_i) = 0 \), where \( a_{ij} \) are the entries of the matrix \( A \) defining \( O_A \).

Next, a family \( \{S(f_i)\}_{i=1}^{N} \) of partial isometries in \( L_2(X,\mu) \) is defined by \( S(f_i)(\phi) = \chi_{R_i} \cdot (\Phi_F)^{1/2} \cdot \phi \circ F \), where \( \chi_{R_i} \) denotes the characteristic function of \( R_i \), and a representation of \( O_A \) in \( L_2(X,\mu) \) is obtained by defining \( \pi_f(s_i) = S(f_i) \) (where \( s_i \) is one of the generating partial isometry in \( O_A \)), and using the universal property of \( O_A \). But it happens that the definition given above for an \( A \)-branching function system is not enough to guarantee that we get a representation of \( O_A \), in fact, it is not enough to prove most of the theorems in [3]. For example, let \( X = [0,2] \), \( \mu \) be the Lebesgue measure, \( R_1 = [0,1] = D_1, R_2 = [1,2] = D_2 \), \( F : X \to X \) defined by \( F(x) = x \) for each \( x \in [0,2] \) (so, \( f_1(x) = x \) for each \( x \in D_i \)) and \( A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \). Following [3], \( \{f_i\}_{i=1}^{2},\{D_i\}_{i=1}^{2} \) is an \( A \)-branching function system, but \( S(f_1)^*S(f_1)(\phi) = \chi_{[0,1]} \cdot \phi \) and \( (S(f_1)S(f_1)^* + S(f_2)S(f_2)^*)(\phi) = \chi_{[0,2]} \cdot \phi \), for each \( \phi \in L_2(X,\mu) \), so that \( S(f_1)^*S(f_1) \neq \sum_{i=1}^{2} S(f_i)S(f_i)^* \). Therefore, the existence of a representation of \( O_A \) in \( L_2([0,2],\mu) \) is not guaranteed.

As we seen, we need to add some extra hypothesis to the definition of an \( A \)-branching function system. Namely, we also have to ask that
\[ \mu \left( \bigcup_{j : a_{ij} = 1} R_j \setminus D_i \right) = 0, \text{ for each } i = 1, \ldots, N. \] We should mention that this extra condition is satisfied in all the examples given in [3]. With this new definition of an A-branching function system in mind, we are now able to generalize it to the countable infinite case.

## 2 \( A_\infty \)-branching systems

For a measure space \((X, \mu)\) and for measurable subsets \(Y, Z\) of \(X\), we write \(Y \, \mu-a.e. \, Z\) if \(\mu(Y \setminus Z) = 0 = \mu(Z \setminus Y)\) or equivalently, if there exists \(Y', Z' \subset X\) such that \(Y \cup Y' = Z \cup Z'\) with \(\mu(Y') = 0 = \mu(Z')\).

Let \(A\) be an infinite matrix, with entries \(A(i, j) \in \{0, 1\}\), for \((i, j) \in \mathbb{N} \times \mathbb{N}\), and let \((X, \mu)\) be a measurable space. For each pair of finite subsets \(U, V\) of \(\mathbb{N}\) and \(j \in \mathbb{N}\) define

\[ A(U, V, j) = \prod_{u \in U} A_{uj} \prod_{v \in V} (1 - A_{vj}). \]

**Definition 2.1** An \( A_\infty \)-branching system on a \( \sigma \)-finite measure space \((X, \mu)\) is a family \((\{f_i\}_{i=1}^\infty, \{D_i\}_{i=1}^\infty)\) together with a nonsingular transformation \(F : X \to X\) such that:

1. \( f_i : D_i \to R_i\) is a measurable map, \(D_i, R_i\) are measurable subsets of \(X\) and \(f_i(D_i) \, \mu-a.e. \, R_i\) for each \(i \in \mathbb{N}\);
2. \( F \) satisfies \( F \circ f_i = \text{Id}_{D_i} \, \mu-a.e.\) in \(D_i\) for each \(i \in \mathbb{N}\);
3. \( \mu(R_i \cap R_j) = 0 \) for all \(i \neq j\);
4. \( \mu(R_i \cap D_i) = 0 \) if \(A(i, j) = 0\) and \(\mu(R_j \setminus D_i) = 0 \) if \(A(i, j) = 1\);
5. for each pair \(U, V\) of finite subsets of \(\mathbb{N}\) such that \(A(U, V, j) = 1\) only for a finite number of \(j\)'s,

\[ \bigcap_{u \in U} D_u \bigcap_{v \in V} (X \setminus D_v) \, \mu-a.e. \bigcup_{j \in \mathbb{N} : A(U, V, j) = 1} R_j. \]

6. There exists the Radon-Nikodym derivatives \( \Phi_{f_i}\) of \(\mu \circ f_i\) with respect to \(\mu\) in \(D_i\) and \( \Phi_{f_i^{-1}}\) of \(\mu \circ f_i^{-1}\) with respect to \(\mu\) in \(R_i\).

The existence of the Radon-Nikodym derivative \( \Phi_{f_i}\) of \(\mu \circ f_i\) with respect to \(\mu\) in \(D_i\) together with the fact that \( F \circ f_i = \text{Id}_{D_i} \, \mu-a.e.\).
imply that \( f_i \circ F_{|R_i} = Id_{R_i} \) \( \mu - a.e. \). So, the function \( f_i \) is \( \mu - a.e. \) invertible, with inverse \( f_i^{-1} := F_{|R_i} \). These are the functions that appear in condition 6 above. If follows from the same condition that \( \Phi_{f_i} \) and \( \Phi_{f_i^{-1}} \) are measurable functions in \( D_i \) and \( R_i \), respectively. We will also consider these functions as measurable functions in \( X \), defining it as being zero out of \( D_i \) and \( R_i \), respectively.

The functions \( \Phi_{f_i} \) and \( \Phi_{f_i^{-1}} \) are nonnegative \( \mu \)-a.e., because \( \mu \) is a (positive) measure. It is possible to show (by using the following proposition) that \( \Phi_{f_i}(x) \Phi_{f_i^{-1}}(f_i(x)) = 1 \) \( \mu \)-almost everywhere in \( D_i \). This equality will be used in the next section.

**Proposition 2.2** Let \((X, \mu)\) be a \( \sigma \)-finite measure space and \( Y, Z \) measurable subsets of \((X, \mu)\). Consider two measurable maps \( f : Y \rightarrow Z \) and \( g : Z \rightarrow X \), and suppose that there exists the Radon-Nikodym derivatives \( \Phi_f \) of \( \mu \circ f \) with respect to \( \mu \) in \( Y \) and \( \Phi_g \) of \( \mu \circ g \) with respect to \( \mu \) in \( Z \). Suppose also that \( \mu \circ f \) and \( \mu \circ g \) are \( \sigma \)-finite. Then there exists the Radon-Nikodym derivative \( \Phi_{g \circ f} \) of \( \mu \circ (g \circ f) \) with respect to \( \mu \) in \( Y \) and \( \Phi_{g \circ f}(x) = \Phi_g(f(x)) \Phi_f(x) \) \( \mu - a.e. \) in \( Y \).

**Proof:**

First note that \( \mu \circ (g \circ f) \) is a \( \sigma \)-finite measure in \( Y \).

Now, for each \( E \subseteq Y \),

\[
\int_E \Phi_f(x) \Phi_g(f(x)) d\mu = \int_{f(E)} \Phi_g(f(x)) d(\mu \circ f) = \int_{f(E)} \Phi_g(x) d\mu = \int_{f(E)} d(\mu \circ g) = \int_E d(\mu \circ g \circ f).
\]

The first and the third equality are a consequence of the Radon-Nikodym derivative. The other two follow by the change of variable theorem. So for each \( E \subseteq Y \),

\[
\int_E \Phi_f(x) \Phi_g(f(x)) d\mu = \int_E d(\mu \circ g \circ f) = (\mu \circ g \circ f)(E).
\]
So, if $\mu(E) = 0$ then $(\mu \circ g \circ f)(E) = 0$. By [2] there exists the Radon-Nikodym derivative $\Phi_{gof}$ of $\mu \circ f \circ g$ with respect to $\mu$ in $Y$ and the equality $(\mu \circ f \circ g)(E) = \int_E \Phi_{gof}(x)d\mu$ holds, for each $E \subseteq Y$. So, for each $E \subseteq Y$,

$$\int_E \Phi_f(x)\Phi_g(f(x))d\mu = \int_E \Phi_{gof}(x)d\mu,$$

and therefore, $\Phi_f(x)\Phi_g(f(x)) = \Phi_{gof}(x) \mu - a.e.$ □

### 3 Representations of Cuntz-Krieger algebras for infinite matrices.

Representations of the Cuntz-Krieger algebras are of great importance, having applications both to operator algebras and to dynamical systems. In this section we show that for each $A_\infty$-branching system, there exists a representation of the unital Cuntz-Krieger C*-algebra $O_A$ on $B(L_2(X,\mu))$, the bounded operators on $L_2(X,\mu)$.

Following [1], recall that the unital Cuntz-Krieger algebra of an infinite matrix $A$, with $A(i,j) \in \{0,1\}$ and $(i,j) \in \mathbb{N} \times \mathbb{N}$ is the unital universal C*-algebra generated by a family $\{S_i\}_{i \in \mathbb{N}}$ of partial isometries that satisfy:

1. $S_iS_i^*S_jS_j^* = 0$ if $i \neq j$;
2. $S_i^*S_i$ and $S_j^*S_j$ commute, for all $i,j$;
3. $S_i^*S_jS_jS_i^* = A(i,j)S_jS_j^*$, for all $i,j$;
4. $\prod_{u \in U} S_u S_u^* \prod_{v \in V} (1 - S_v S_v^*) = \sum_{j=1}^{\infty} A(U,V,j)S_jS_j^*$, for each pair of finite subsets $U,V \subseteq \mathbb{N}$ such that $A(U,V,j) := \prod_{u \in U} A(u,j) \prod_{v \in V} (1 - A(v,j))$ vanishes for all but a finite number of $j$'s.

**Theorem 3.1** For a given $A_\infty$-branching system (see [2.1]), there exist a *-homomorphism $\pi : O_A \to B(L_2(X,\mu))$ such that $\pi(S_i)\phi = \chi_{R_i} \cdot (\Phi_{F_{i-1}})^{1/2} \cdot \phi \circ F$ for each $\phi \in L_2(X,\mu)$.

**Proof:**

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First notice that for a given \( \phi \in L_2(X, \mu) \) we have that

\[
\int_X |\chi_{R_i}(x)\Phi_{f_i^{-1}}(x)\tfrac{1}{2}\phi(F(x))|^2 d\mu = \int_{R_i} \Phi_{f_i^{-1}}(x)|\phi(f_i^{-1}(x))|^2 d\mu = \]

\[
= \int_{R_i} |\phi(f_i^{-1}(x))|^2 d(\mu \circ f_i^{-1}) = \int_{D_i} |\phi(x)|^2 d\mu \leq \int_X |\phi(x)|^2 d\mu.
\]

To obtain the second equality we have considered the Radon-Nikodym derivative of \( \mu \circ f_i^{-1} \) with respect to \( \mu \) in \( R_i \) and the last equality is an application of the change of variable theorem.

So, we define the operator \( \pi(S_i) : L(L_2(X, \mu)) \to L(L_2(X, \mu)) \) by

\[
\pi(S_i) \phi = \chi_{R_i} \cdot (\Phi_{f_i^{-1}})^{\frac{1}{2}} \cdot (\phi \circ F),
\]

for each \( \phi \in L_2(X, \mu) \). By using the above computation, we see that \( \pi(S_i) \in \mathcal{B}(L_2(X, \mu)) \).

Our aim is to show that \( \{ \phi(S_i) \}_{i \in \mathbb{N}} \) satisfies the relations 1-4 which define the Cuntz-Krieger algebra \( O_A \). With this in mind, let us first determine the operator \( \psi(S_i)^* \).

For each \( \phi, \psi \in L_2(X, \mu) \),

\[
\langle \pi(S_i) \phi, \psi \rangle = \int_X \chi_{R_i}(x)\Phi_{f_i^{-1}}(x)^{\frac{1}{2}}\phi(F(x))\overline{\psi(x)}d\mu = \int_{R_i} \Phi_{f_i^{-1}}(x)^{\frac{1}{2}}\phi(f_i^{-1}(x))\overline{\psi(x)}d\mu = ...
\]

...by using the change of variable theorem...

\[
... = \int_{D_i} \Phi_{f_i^{-1}}(x)^{\frac{1}{2}}\phi(x)\overline{\psi(f_i(x))}d(\mu \circ f_i) = ...
\]

...considering the Radon derivative \( \Phi_{f_i} \) of \( \mu \circ f_i \)...
...by proposition 2.2...

\[
\begin{align*}
... &= \int_{D_i} \Phi_{f_i}(x)^{\frac{1}{2}} \phi(x) \psi(f_i(x)) \, d\mu = \int_X \phi(x) \Phi_{f_i}(x)^{\frac{1}{2}} \psi(f_i(x)) \, d\mu = \left\langle \phi, \chi_{D_i} \cdot \Phi_{f_i}^{\frac{1}{2}} \cdot (\psi \circ f_i) \right\rangle.
\end{align*}
\]

Then

\[
\pi(S_i)^* \psi = \chi_{D_i} \cdot \Phi_{f_i}^{\frac{1}{2}} \cdot (\psi \circ f_i).
\]

Using proposition 2.2 again, it is easy to show that

\[
\pi(s_i)^* \pi(S_i) \psi = \chi_{D_i} \cdot \psi = M_{\chi_{D_i}}(\psi)
\]

for each \( \psi \in L^2(X, \mu) \) (that is, \( \pi(S_i)^* \pi(S_i) \) is the multiplication operator by \( \chi_{D_i} \)). In the same way \( \pi(s_i)^* \pi(S_i)^* = M_{\chi_{R_i}} \).

Now we verify if \( \{ \pi(S_i) \}_{i \in \mathbb{N}} \) satisfies the relations 1-4, which define the C*-algebra \( \mathcal{O}_A \). The first relation follows from the fact that \( \mu(R_i \cap R_j) = 0 \) for \( i \neq j \). The second one is trivial.

To see that the third relation is also satisfied, recall that if \( A(i,j) = 0 \) then \( \mu(R_j \cap D_i) = 0 \) and hence

\[
\pi(S_i)^* \pi(S_j) \pi(S_j)^* = M_{\chi_{D_i}} M_{\chi_{R_j}} = M_{\chi_{D_i} \cap R_j} = 0,
\]

and if \( A(i,j) = 1 \) then \( \mu(R_j \setminus D_i) = 0 \) and hence

\[
\pi(S_i)^* \pi(S_j) \pi(S_j)^* = M_{\chi_{D_i}} M_{\chi_{R_j}} = M_{\chi_{D_i} \cap R_j} = M_{\chi_{R_j}} = \pi(S_j) \pi(S_j)^*.
\]

So, for each \( i, j \in \mathbb{N} \)

\[
\pi(S_i)^* \pi(S_i) \pi(S_j) \pi(S_j)^* = \pi(i,j) \pi(S_j) \pi(S_j)^*.
\]

To verify the last relation, let \( U, V \) be finite subsets of \( \mathbb{N} \) such that \( A(U, V, j) = 1 \) only for finitely many \( j \)'s.

Then, by definition 2.15,

\[
M_X \left( \bigcap_{u \in U} D_u \cap \bigcap_{v \in V} (X \setminus D_v) \right) = M_X \left( A(U, V, j) = 1 \right) R_j.
\]

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Note that
\[
M_k \left( \bigcap_{u \in U} D_u \cap (X \setminus D_v) \right) = \prod_{u \in U} M_{\chi D_u} \prod_{v \in V} (Id - M_{\chi D_v}) = \\
= \prod_{u \in U} \pi(S_u)^* \pi(S_u) \prod_{v \in V} (Id - \pi(S_v)^* \pi(S_v)).
\]

On the other hand,
\[
M_X \left( \bigcup_{j \in \mathbb{N} : A(U,V,j) = 1} R_j \right) = \sum_{j \in \mathbb{N} : A(U,V,j) = 1} M_{\chi R_j} = \sum_{j \in \mathbb{N} : A(U,V,j) = 1} \pi(S_j) \pi(S_j)^*.
\]
This shows that the last relation defining $O_A$ is also verified.

So, there exist a $*$-homomorphism $\pi : O_A \to B(L_2(X,\mu))$ satisfying $\pi(S_i) \phi = \chi_{R_i} \cdot (\Phi_{f_i})^{\frac{1}{2}} \cdot \phi \circ F$. \qed

The previous theorem applies only if an $A_\infty$-branching system is given. Our next step is to guarantee the existence of $A_\infty$-branching systems for any matrix $A$. First we prove a lemma, which will be helpful in some situations.

**Lemma 3.2** Let $A$ be an infinite matrix with entries in $\mathbb{N} \times \mathbb{N}$ having no identically zero rows, $(X,\mu)$ be a measure space, and let $\{R_j\}_{j=1}^\infty$ and $\{D_j\}_{j=1}^\infty$ be families of measurable subsets of $X$ such that

a) $\mu(R_i \cap R_j) = 0$ for all $i \neq j$;

b) $X = \bigcup_{j=1}^\infty R_j$;

c) $D_i = \bigcup_{j \in \mathbb{N} : A(i,j) = 1} R_j$;

Then conditions 4 and 5 of [21] are satisfied.

**Proof:** Condition 4 follows from a) and b). To show 5 firs we note that $X \setminus D_v = \bigcup_{j \in \mathbb{N} : A(v,j) = 0} R_j$. Then, given $U,V$ finite subsets of $N$, we have that
\[
\bigcap_{u \in U} D_u \cap (X \setminus D_v) = \bigcap_{v \in V} R_j \cap \left( \bigcup_{j \in \mathbb{N} : A(u,j) = 1 \forall u \in U} R_j \right) \cap \left( \bigcup_{j \in \mathbb{N} : A(v,j) = 0 \forall v \in V} R_j \right).
\]
\[
\left( \bigcup_{j \in \mathbb{N}}: \prod_{u \in U} A(u,j) = 1 \right) \cap \left( \bigcup_{j \in \mathbb{N}}: \prod_{v \in V} (1 - A(v,j)) = 1 \right) \mu^{-a.e} \equiv \bigcup_{j \in \mathbb{N}: A(U,V,j) = 1} R_j.
\]

\[\square\]

**Theorem 3.3** For each infinite matrix \( A \), without identically zero rows, there exists an \( A_\infty \)-branching system in the measure space \((0, \infty), \mu\), where \( \mu \) is the Lebesgue measure.

**Proof:** Consider \([0, \infty)\) with the Lebesgue measure \( \mu \). Define \( R_i = [i, i + 1] \) and \( D_i = \bigcup_{j : A(i,j) = 1} R_j \). Note that \( \mu(R_i \cap R_j) = 0 \) for \( i \neq j \).

Then, by the previous lemma, conditions 4 and 5 of definition 2.1 are satisfied. So, it remains to define maps \( f_i : D_i \to R_i \) and \( F : [0, +\infty) \to [0, +\infty) \) satisfying the conditions of definition 2.1. For a fixed \( i_0 \in \mathbb{N} \) we define \( f_{i_0} \) as follows. First divide the interval \( R_{i_0} \) (where \( \overset{o}{R}_{i_0} \) denotes the interior of \( R_{i_0} \)) in \( \# \{ j : A(i_0,j) = 1 \} \) intervals \( I_j \). Then, define \( \tilde{f}_{i_0} : \bigcup_{j : A(i_0,j) = 1} \overset{o}{R}_j \to \bigcup_{j : A(i_0,j) = 1} \overset{o}{I}_j \) such that \( \tilde{f}_{i_0} : \overset{o}{R}_j \to \overset{o}{I}_j \) is a \( C^1 \)-diffeomorphism. We now define \( f_{i_0} : D_{i_0} \to R_{i_0} \) by

\[
f_{i_0}(x) = \begin{cases} 
\tilde{f}_{i_0}(x) & \text{if } x \in \bigcup_{j : A(i_0,j) = 1} \overset{o}{R}_j \\
i_0 & \text{if } x \in D_{i_0} \setminus \bigcup_{j : A(i_0,j) = 1} \overset{o}{R}_j
\end{cases}
\]

and \( F : [0, \infty) \to [0, \infty) \) by

\[
F(x) = \begin{cases}
\tilde{f}_{i_0}^{-1}(x) & \text{if } x \in \bigcup_{j : A(i_0,j) = 1} \overset{o}{I}_j \\
0 & \text{if } x \in R_{i_0} \setminus \bigcup_{j : A(i_0,j) = 1} \overset{o}{I}_j
\end{cases}
\]

Note that \( f_i \) and \( F \) are measurable maps. Moreover, \( \mu \circ f_i \) and \( \mu \circ f_i^{-1} \) are \( \sigma \)-finite measures in \( D_i \) and \( R_i \). Next we show that there exists the Radon-Nikodym derivatives \( \Phi_{f_i} \) of \( \mu \circ f_i \) with respect to \( \mu \) in \( D_i \). Let \( E \subseteq D_i \) be such that \( \mu(E) = 0 \). To show that \( \mu \circ f_i(E) = 0 \)
it is enough to show that $\mu \circ f_i (E \cap ( \bigcup_{j:A(i,j)=1} \overset{\circ}{R}_j ) ) = 0$, and this equality is true by [5]. Then, by [2], there exist the desired nonnegative Radon-Nikodym derivative $\Phi_{f_i}$. In the same way there exists the (nonnegative) Radon-Nikodym derivative $\Phi_{f_i^{-1}}$ of $\mu \circ f_i^{-1}$ with respect to $\mu$ in $R_i$. We still need to show that $F$ is nonsingular. For this, let $A \subseteq [0, \infty)$ be such that $\mu(A) = 0$. Notice that it is enough to prove that $\mu(F^{-1}(A) \cap R_j) = 0$ for each $j$. Now $\mu(F^{-1}(A) \cap R_j) = \mu(f_j(A \cap D_j)) = 0$, (where the last equality follows from the fact that $\mu \circ f_j \ll \mu$ in $D_j$), and hence $\mu(F^{-1}(A)) = 0$ as desired.

\[ \square \]

**Corollary 3.4** Given an infinite matrix $A$, there exists a representation of $O_A$ in $L_2([0, \infty), \mu)$ where $\mu$ is the Lebesgue measure. If $A$ is $N \times N$ then there exists a representation of $O_A$ in $L_2([0, N), \mu)$ where $\mu$ is the Lebesgue measure.

## 4 The Perron-Frobenious Operator

We now describe the Perron-Frobenious operator using the representations introduced in the previous section.

**Theorem 4.1** Let $(X, \mu)$ be a measure space with a branching system as in definition 2.1 and let $\varphi \in L_1(X, \mu)$ be such that $\varphi(x) \geq 0$ $\mu$-a.e.

1. If $\text{supp}(\varphi) \subseteq \bigcup_{i=1}^N R_j$, then

\[
P_F(\varphi) = \sum_{i=1}^N (\pi(S_i) \sqrt{\varphi})^2.
\]

2. If $\text{supp}(\varphi) \subseteq \bigcup_{i=1}^\infty R_j$, then

\[
P_F(\varphi) = \lim_{N \to \infty} \sum_{i=1}^N (\pi(S_i) \sqrt{\varphi})^2,
\]

where the convergence occurs in the norm of $L_1(X, \mu)$. 

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Proof: The first assertion follow from the fact that for each measurable set \( A \subseteq X \), \( \int_A P_F(\varphi)(x)d\mu = \int \sum_{i=1}^{N} (\pi(S_i^*)\sqrt{\varphi(x)})^2 d\mu \). To prove this equality, we will use the Radon-Nikodym derivative of \( \mu \circ f_i \), the change of variable theorem and the fact that \( F^{-1}(A) \cap R_i = f_i(A \cap D_i) \).

Given \( A \subseteq X \) a measurable set we have that

\[
\sum_{i=1}^{N} \int_A (\pi(S_i^*)\sqrt{\varphi(x)})^2 d\mu = \sum_{i=1}^{N} \int_A \chi_{D_i}(x)\Phi_{f_i}(x)\varphi(f_i(x))d\mu = \\
= \sum_{i=1}^{N} \int_{A \cap D_i} \Phi_{f_i}(x)\varphi(f_i(x))d\mu = \sum_{i=1}^{N} \int \varphi(f_i(x))d(\mu \circ f_i) = \sum_{i=1}^{N} \int \varphi(x)d\mu = \\
= \sum_{i=1}^{N} \int_{F^{-1}(A) \cap R_i} \varphi(x)d\mu = \sum_{i=1}^{N} \int_{F^{-1}(A) \cap R_i} \chi_{R_i} \varphi(x)d\mu = \\
= \int_{F^{-1}(A)} \varphi(x)d\mu = \int_{F^{-1}(A)} P_F(\varphi)(x)d\mu.
\]

We now prove the second assertion. For each \( N \in \mathbb{N} \), define \( \varphi_N := \sum_{i=1}^{N} \chi_{R_i} \cdot \varphi \). Note that \( (\varphi_N)_{N \in \mathbb{N}} \) is an increasing sequence, bounded above by \( \varphi \). Then

\[
\lim_{N \to \infty} \int_X P_F(\varphi_N)(x)d\mu = \lim_{N \to \infty} \int_X \varphi_N(x)d\mu = ...
\]

...by the Lebesgue’s Dominated Convergence Theorem...

\[
= \int_X \varphi(X)d\mu = \int_X P_F(\varphi)(x)d\mu.
\]

Moreover, the sequence \( (P_F(\varphi_N))_{N \in \mathbb{N}} \) is \( \mu \) - a. e. increasing and bounded above by \( P_F(\varphi) \).

Then,

\[
\lim_{N \to \infty} \|P_F(\varphi) - P_F(\varphi_N)\|_1 = \lim_{N \to \infty} \int_X |P_F(\varphi)(x) - P_F(\varphi_N)(x)|d\mu = 
\]

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\[
\lim_{N \to \infty} \int_X P_F(\varphi)(x) - P_F(\varphi_N)(x) d\mu = 0.
\]

Therefore, \( \lim_{N \to \infty} P_F(\varphi_N) = P_F(\varphi) \). By the first assertion, \( P_F(\varphi_N) = \sum_{i=1}^{N} (\pi(S_i^*)\sqrt{\varphi_N})^2 \), and a simple calculation shows that

\[
\sum_{i=1}^{N} (\pi(S_i^*)\sqrt{\varphi_N})^2 = \sum_{i=1}^{N} (\pi(S_i^*)\sqrt{\varphi})^2.
\]

So, we conclude that

\[
\lim_{N \to \infty} \sum_{i=1}^{N} (\pi(S_i^*)\sqrt{\varphi})^2 = P_F(\varphi).
\]

\[\square\]

**Theorem 4.2** Let \( A \) be a matrix such that each row has a finite number of 1’s and let \((X, \mu)\) be an \( A_\infty \)-branching system. Suppose \( \mu(R_i) < \infty \) for each \( i \) (so that \( \chi_{R_i} \in L_1(X, \mu) \)). Moreover, suppose \( \Phi_{f_i} \) is a constant positive function for each \( i \), say \( \Phi_{f_i} = b_i \) (for example, if \( f_i \) is linear). Let \( W \subseteq L_1(X, \mu) \) be the vector subspace

\[
W = \text{span}\{\chi_{R_i} : i \in \mathbb{N}\},
\]

that is, \( W \) is the subspace of all finite linear combinations of \( \chi_{R_i} \).

Then the Perron-Frobenius operator restricted to \( W \), \( P_F|_W : W \to W \), has a matrix representation given by \( A^T B \), where \( B \) is the diagonal infinite matrix with nonzero entries \( B_{i,i} = b_i \).

Although \( A \) and \( B \) are infinite matrices, we are considering the matrix multiplication \( A^T B \) as the usual multiplication for finite matrices, since \( B \) is column-finite.

**Proof:** Since each row \( z \) of \( A \) has a finite number of 1’s, then, by definition 2.15, taking \( Z = \{z\} \) and \( Y = \emptyset \), we obtain \( D_z^\mu = \bigcup_{j:A(z,j)=1} R_j \) so that \( \chi_{D_z} = \sum_{j:A(z,j)=1} \chi_{R_j} \). Note that

\[
P_F(\chi_{D_z}) = b_z \chi_{D_z} = \sum_{j:A(z,j)=1} b_z R_j,
\]

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and so the element \((j,z)\) of the matrix representation of \(P_{F|_W}\) is \(b_z A(z,j)\).

\[\Box\]

5 Examples

Example 5.1 \(O_\infty\) (\(O_A\) where all entries of the matrix \(A\) are 1).

Consider \(X = [0, 1]\) with Lebesgue measure and define \(D_i = [0, 1]\), for \(i = 1, 2, \ldots\). To define the \(R_i\)’s we first need to define recursively the following sequences in \(X\): Let \(a_1 = 0\), \(a_i = a_{i-1} + \frac{1}{2^i}, i = 2, 3, \ldots\) and let \(b_i = \frac{a_i + a_{i+1}}{2}, i = 1, 2, \ldots\) Now define \(R_i = [a_{i+1}, b_{i+1}]\) for \(i\) odd and \(R_i = [b_1, a_{i+1}]\) for \(i\) even and define a map \(F\) on \(X\) by \(F(x) = \frac{x - b_{i+1}}{b_{i+1} - a_{i+1}} + \frac{a_{i+1} - b_{i+1}}{a_{i+1} - a_{i+2}}\) for \(x \in R_i, i\) odd and \(F(x) = \frac{x - a_{i+1}}{a_{i+1} - b_{i+2}} + \frac{b_{i+2} - a_{i+1}}{b_{i+2} - b_{i+3}}\) for \(x \in R_i, i\) even. Notice that \(F\) is nothing more than an affine transformation that takes the interval \(R_i\) onto \(D_i = [0, 1]\), as shown in the picture below:

![Diagram](image)

Finally, let \(f_i = (F|_{R_i})^{-1}\). Then \((\{f_i\}_{i=1}^\infty, \{D_i\}_{i=1}^\infty)\) is an \(A_\infty\) branching system and hence induces a representation of the Cuntz-Krieger algebra \(O_\infty\).

Example 5.2
Let $X$ be the measure space $[0, \infty)$, with the Lebesgue measure. Consider the map $F : [0, \infty) \to [0, \infty)$ defined by $F(x) = \frac{i}{2}(x - i)^2$ for $x \in [i-1, i]$ and $i$ odd and $F(x) = \left[\frac{i}{2}\right](x - (i-1))^2$ for $x \in [i-1, i]$ and $i$ even ($\left[\frac{i}{2}\right]$ is the least integer greater than or equal to $\frac{i}{2}$). Below we see the graph of $F$.

Define $R_i = [i-1, i]$ for $i = 1, 2, 3, ...$, set $D_i = [0, \left[\frac{i}{2}\right]]$ and let $f_i : D_i \to R_i$ be defined by $f_i = (F_{|R_i})^{-1}$. Then $(\{f_i\}_{i=1}^{\infty}, \{D_i\}_{i=1}^{\infty})$ is an $A_\infty$ branching system. This branching system induces a representation of the C*-algebra $O_A$, for

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & \cdots \\
1 & 1 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}
\]

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