MODULE SUPER-AMENABILITY FOR SEMIGROUP ALGEBRAS

ABASAL BODAGHI AND MASSOUD AMINI

Abstract. Let $S$ be an inverse semigroup with the set of idempotents $E$. In this paper we define the module super-amenability of a Banach algebra which is a Banach module over another Banach algebra with compatible actions, and show that when $E$ is upward directed and acts on $S$ trivially from left and by multiplication from right, the semigroup algebra $\ell^1(S)$ is $\ell^1(E)$-module super-amenable if and only if an appropriate group homomorphic image of $S$ is finite.

1. Introduction

The second author in [1] introduced the concept of module amenability and showed that for an inverse semigroup $S$, the semigroup algebra $\ell^1(S)$ is module amenable as a Banach module on $\ell^1(E)$, where $E$ is the set of idempotents of $S$, if and only if $S$ is amenable (see also [2]). This is the semigroup analog of Johnson’s theorem for locally compact groups [8]. In this paper, we find a similar result for module super-amenability of the semigroup algebra of an inverse semigroup which is the semigroup analog of Selivanov’s theorem for locally compact groups [9].

Recall that a Banach algebra $A$ is called super-amenable (contractible) if $H^1(A, X) = \{0\}$ for every Banach $A$-bimodule $X$, where the left hand side is the first cohomology group of $A$ with coefficient in $X$ (see [4, 10]). A Banach space $E$ has the approximation property if there is a net $(T_j)_j$ in $\mathcal{F}(E)$, the space of the bounded finite rank operators on $E$ such that $T_j \to \text{id}_E$ uniformly on compact subsets on $E$. It is shown in [10, Theorem 4.1.5] if $A$ is a super-amenable Banach algebra and has the approximation property, then $A$ is finite dimensional (see also Propositions 5.1 and 5.2 of [12]). In particular, since $\ell^1(S)$ has the approximation property [4], so it is not super-amenable, when $S$ is infinite. For groups, Selivanov showed in [11] that for any locally compact group $G$, $L^1(G)$ is super-amenable if and only if $G$ is finite (see also [10, Exercise 4.1.3]). In this paper, we develop the concept of module super-amenability for a class of Banach algebras, and prove that for an inverse semigroup $S$ with an upward directed subsemigroup of idempotents $E$ which acts on $S$ trivially from left and by multiplication from right, the semigroup algebra $\ell^1(S)$ is $\ell^1(E)$-module super-amenable if and only if an appropriate group homomorphic image of $S$ is finite.

The paper is organized as follows: Section 2 is devoted to the concept of module super-amenability. The main result of this section asserts that module super-amenability is equivalent to the existence of a module diagonal. In section 3 we show that for an inverse semigroup $S$ with an upward directed set of idempotents $E$, $\ell^1(S)$ is $\ell^1(E)$-module super-amenable (with respect to the above action) if and only if the group homomorphic image $S/\approx$ is finite, where $s \approx t$ whenever $\delta_s - \delta_t$ belongs to the closed linear span of the set

$$\{\delta_{se} - \delta_{te} : s, t \in S, e \in E\}.$$

Examples of semigroups with an upward directed set of idempotents include all unital inverse semigroups, the bicyclic semigroup, and the semigroup of natural numbers with max operation.

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The free inverse semigroup on two generators is an example of an inverse semigroup whose set of idempotents is not upward directed.

2. Module Super Amenability

Throughout this paper, $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \ (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \ a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \ (a \cdot x) \cdot a = a \cdot (x \cdot a) \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in \mathcal{X})$$

and the same for the right or two-sided actions. Then we say that $\mathcal{X}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover

$$\alpha \cdot x = x \cdot \alpha \quad (\alpha \in \mathfrak{A}, x \in \mathcal{X})$$

then $\mathcal{X}$ is called a commutative $\mathcal{A}$-$\mathfrak{A}$-module. If $\mathcal{X}$ is a (commutative) Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is $\mathcal{X}^*$, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on $\mathcal{X}^*$ are defined by

$$\langle (f \cdot x), \alpha \rangle = \langle f, x \cdot \alpha \rangle \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in \mathcal{X}, f \in \mathcal{X}^*)$$

and the same for the right actions. Let $\mathcal{Y}$ be another $\mathcal{A}$-$\mathfrak{A}$-module, then a $\mathcal{A}$-$\mathfrak{A}$-module morphism from $\mathcal{X}$ to $\mathcal{Y}$ is a norm-continuous map $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ with $\varphi(x \pm y) = \varphi(x) \pm \varphi(y)$ and

$$\varphi(\alpha \cdot x) = \alpha \cdot \varphi(x), \ \varphi(x \cdot \alpha) = \varphi(x) \cdot \alpha, \ \varphi(a \cdot x) = a \cdot \varphi(x), \ \varphi(x \cdot a) = \varphi(x) \cdot a,$$

for $x, y \in \mathcal{X}, a \in \mathcal{A}, \alpha \in \mathfrak{A}$.

Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach $\mathcal{A}$-$\mathfrak{A}$-module, as we have not assumed the compatibility condition

$$a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b \quad (\alpha \in \mathfrak{A}, a, b \in \mathcal{A}).$$

If $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.

If $\mathcal{A}$ is a Banach $\mathfrak{A}$-module with compatible actions, then so are the dual space $\mathcal{A}^*$ and the second dual space $\mathcal{A}^{**}$. If moreover $\mathcal{A}$ is a commutative $\mathfrak{A}$-module, then $\mathcal{A}^*$ and the $\mathcal{A}^{**}$ are commutative $\mathcal{A}$-$\mathfrak{A}$-modules.

Consider the projective tensor product $\mathcal{A} \hat{\otimes} \mathfrak{A}$. It is well known that $\mathcal{A} \hat{\otimes} \mathfrak{A}$ is a Banach algebra with respect to the canonical multiplication map defined by

$$(a \otimes b)(c \otimes d) = (ac \otimes bd)$$

and extended by bi-linearity and continuity [4]. Then $\mathcal{A} \hat{\otimes} \mathfrak{A}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module with canonical actions. Let $I$ be the closed ideal of the projective tensor product $\mathcal{A} \hat{\otimes} \mathfrak{A}$ generated by elements of the form $\alpha \cdot a \otimes b - a \otimes b \cdot \alpha$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$. Consider the map $\omega \in \mathcal{L}(\mathcal{A} \hat{\otimes} \mathfrak{A}, \mathcal{A})$ defined by $\omega(a \otimes b) = ab$ and extended by linearity and continuity. Let $J$ be the closed ideal of $\mathcal{A}$ generated by $\omega(I)$. Then the module projective tensor product $\mathcal{A} \hat{\otimes}_{\mathfrak{A}} \mathfrak{A} \cong (\mathcal{A} \hat{\otimes} \mathfrak{A})/I$ and the quotient Banach algebra $\mathcal{A}/J$ are Banach $\mathfrak{A}$-modules with compatible actions. Also the map $\tilde{\omega} \in \mathcal{L}(\mathcal{A} \hat{\otimes}_{\mathfrak{A}} \mathfrak{A}, \mathcal{A}/J)$ defined by $\tilde{\omega}(a \otimes b + I) = ab + J$ extends to an $\mathfrak{A}$-module morphism.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and $\mathcal{X}$ be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A bounded map $D : \mathcal{A} \rightarrow \mathcal{X}$ is called a module derivation if

$$D(a \pm b) = D(a) \pm D(b), \ D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}),$$
and 
\[ D(a \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}). \]

Note that \( D : \mathcal{A} \to \mathcal{X} \) is bounded if there exist \( M > 0 \) such that \( ||D(a)|| \leq M||a|| \), for each \( a \in \mathcal{A} \).

Although \( D \) is not necessarily linear, but still its boundedness implies its norm continuity (since \( D \) preserves subtraction). When \( \mathcal{X} \) is commutative, each \( x \in \mathcal{X} \) defines a module derivation
\[ D_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}). \]

These are called \textit{inner} module derivations.

**Definition 2.1.** The Banach algebra \( \mathcal{A} \) is called module super-amenability (as an \( \mathfrak{A} \)-module) if for any commutative Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-module \( \mathcal{X} \), each module derivation \( D : \mathcal{A} \to \mathcal{X} \) is inner.

We use the notations \( Z_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}) \) and \( B_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}) \) for the set of all module derivations and inner module derivations from \( \mathcal{A} \) to \( \mathcal{X} \), respectively. The quotient group (called the first relative -to \( \mathfrak{A} \)-cohomology group of \( \mathcal{A} \) with coefficients in \( \mathcal{X} \)) is denoted by \( H^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}) \). Hence \( \mathcal{A} \) is module super-amenability if and only if \( H^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}) = \{0\} \), for each commutative Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-module \( \mathcal{X} \).

**Proposition 2.2.** If \( \mathfrak{A} \) has an identity for \( \mathcal{A} \), then super-amenability of \( \mathcal{A} \) implies its module super-amenability.

**Proof.** Let \( \varepsilon \in \mathfrak{A} \) be a identity for \( \mathcal{A} \), that is \( \varepsilon \cdot a = a \cdot \varepsilon = a \), for each \( a \in \mathcal{A} \), and \( \mathcal{X} \) be a commutative \( \mathcal{A} \)-\( \mathfrak{A} \)-module. Assume that \( D : \mathcal{A} \to \mathcal{X} \) is a module derivation, then obviously \( D(a \cdot \lambda \varepsilon) = D(\lambda a) \), for each \( a \in \mathcal{A} \) and \( \lambda \in \mathbb{C} \). On the other hand,
\[ D(a \cdot \lambda \varepsilon) = D(a) \cdot \lambda \varepsilon = \lambda D(a) \cdot \varepsilon = \lambda D(a \cdot \varepsilon) = \lambda D(a). \]

Thus \( D \) is \( \mathbb{C} \)-linear, and so inner. \( \square \)

As we will see later in section 3, there are module super-amenable Banach algebras that are not super-amenable, so the converse of the above Proposition is false. It is known that every super-amenable Banach algebra has an identity. Also a Banach algebra is super-amenable if and only if it has a diagonal \([10]\). Recall that a \textit{diagonal} for \( \mathcal{A} \) is an element \( M \in \mathcal{A} \hat{\otimes} \mathcal{A} \) satisfying
\[ a \cdot \omega(M) = a, \quad a \cdot M = M \cdot a \quad (a \in \mathcal{A}). \]

We start this section by showing that similar results hold \( \mathcal{A} \) is a commutative \( \mathcal{A} \)-\( \mathfrak{A} \)-module.

**Proposition 2.3.** Let \( \mathcal{A} \) be a commutative Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-module. If \( \mathcal{A} \) is module super-amenable, then it is unital.

**Proof.** Let’s consider \( \mathcal{X} = \mathcal{A} \) as an \( \mathcal{A} \)-bimodule, with actions
\[ a \cdot b := ab, \quad b \cdot a := 0 \quad (a \in \mathcal{A}, b \in \mathcal{X}). \]

Let \( D : \mathcal{A} \to \mathcal{X} \) be the identity map, it is clear that \( D \in Z_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}) = B_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}) \). This means that there is \( a_0 \in \mathcal{A} \) such that \( a_0 a_0 = a \), for all \( a \in \mathcal{A} \). Therefore \( a_0 \) is a right identity for \( \mathcal{A} \).

Similarly \( \mathcal{A} \) has a left identity. The left and right identities now have to coincide. \( \square \)

**Proposition 2.4.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be Banach algebras and Banach \( \mathfrak{A} \)-modules with compatible actions. If \( \mathcal{A} \) is \( \mathfrak{A} \)-module super-amenable and \( \varphi : \mathcal{A} \to \mathcal{B} \) is a continuous Banach algebra homomorphism with dense range, then \( \mathcal{B} \) is also \( \mathfrak{A} \)-module super-amenable.
Proof. Let X be a commutative $\mathcal{B}\mathfrak{A}$-module, then it is a commutative $\mathcal{A}\mathfrak{A}$-module with the following actions

$$a \cdot x := \varphi(a) \cdot x, \quad x \cdot a := x \cdot \varphi(a) \quad (a \in \mathcal{A}, x \in X).$$

If $D : \mathcal{B} \rightarrow X$ is a module derivation, then $D \circ \varphi : \mathcal{A} \rightarrow X$ is a module derivation, which is inner. By density of the range of $\varphi$ and continuity of $D$, $D$ is inner.

**Definition 2.5.** An element $M \in \mathcal{A}\mathfrak{A}$ is called a module diagonal if $\bar{\omega}(M)$ is an identity of $\mathcal{A}/J$ and $a \cdot M = M \cdot a$, for all $a \in \mathcal{A}$.

**Theorem 2.6.** Let $\mathcal{A}$ be a Banach $\mathcal{A}\mathfrak{A}$-module. Then $\mathcal{A}$ is module super-amenable if and only if $\mathcal{A}$ has a module diagonal.

**Proof.** Assume that $\mathcal{A}$ is module super-amenable, then by Proposition 2.3 it has an identity element $e$. Put $T = e \otimes e + I$, we have $\bar{\omega}(a \cdot T - T \cdot a) = J$. Hence $\bar{\omega}$ vanishes on the range of $D_T$, and $D_T$ could be regarded as a module derivation into $K$. Since $\mathcal{A}\mathfrak{A}$ is always commutative $\mathfrak{A}$-module, so is $K = \text{ker } \bar{\omega}$, hence by module super-amenability of $\mathcal{A}$, there is $N \in K$ such that $D_T = D_N$. Now it is easy to see that $M = N - T$ is a module diagonal.

Conversely suppose that $M = \sum_{n=1}^{\infty} a_n \otimes b_n + I$ is a module diagonal, where $(a_n), (b_n)$ are bounded sequences in $\mathcal{A}$ with $\sum_{n=1}^{\infty} \|a_n\| \|b_n\| < \infty$. Let $\mathcal{X}$ be a commutative Banach $\mathcal{A}\mathfrak{A}$-module, then clearly $J$ acts trivially on $\mathcal{X}$, that is $J : \mathcal{X} = \mathcal{X} : J = \{0\}$. Therefore $\mathcal{X}$ is a Banach $\mathcal{A}/J$-module with the following actions

$$(a + J) \cdot x := a \cdot x, \quad x \cdot (a + J) := x \cdot a \quad (x \in \mathcal{X}, a \in \mathcal{A}).$$

If $D : \mathcal{A} \rightarrow \mathcal{X}$ is a module derivation, then the map $\tilde{D} : \mathcal{A}/J \rightarrow \mathcal{X}$ defined by $\tilde{D}(a + J) = D(a)$, for $a \in \mathcal{A}$, is a module derivation. Consider $x = \sum_{n=1}^{\infty} a_n \cdot \tilde{D}(b_n + J)$, then for each $\phi \in \mathcal{X}^*$ we have

$$\langle \phi, (a + J) \cdot x \rangle = \langle \phi, (a + J) \cdot \sum_{n=1}^{\infty} a_n \cdot \tilde{D}(b_n + J) \rangle$$

$$= \langle \phi, \sum_{n=1}^{\infty} a_n \cdot \tilde{D}(b_n a + J) \rangle$$

$$= \langle \phi, \sum_{n=1}^{\infty} a_n \cdot \tilde{D}(b_n a + J) \cdot (a + J) \rangle + \langle \phi, \sum_{n=1}^{\infty} a_n b_n + J \rangle \cdot \tilde{D}(a + J) \rangle$$

$$= \langle \phi, x \cdot (a + J) \rangle + \langle \phi, \tilde{D}(a + J) \rangle.$$

Hence $\tilde{D}$ is inner. Therefore $D$ is an inner module derivation. □

We say the Banach algebra $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left if for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, $\alpha \cdot a = f(\alpha)a$, where $f$ is a continuous linear functional on $\mathfrak{A}$ (see [3, Lemma 3.1]).

**Lemma 2.7.** Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with trivial left action. If $\mathcal{A}$ is module super-amenable and $\mathcal{A}/J$ is commutative $\mathfrak{A}$-module, then $\mathcal{A}/J$ is super-amenable.

**Proof.** Let $\mathcal{X}$ be a commutative $\mathcal{A}/J\mathfrak{A}$-module. Then $\mathcal{X}$ is a commutative $\mathcal{A}\mathfrak{A}$-module with the same actions over $\mathfrak{A}$ and module actions over $\mathcal{A}$ defined by

$$a \cdot x := (a + J) \cdot x, \quad x \cdot a := x \cdot (a + J) \quad (x \in \mathcal{X}, a \in \mathcal{A}).$$
Suppose that $D : A/J \rightarrow \mathcal{X}$ is a module derivation, then $\tilde{D} : A \rightarrow \mathcal{X}$ defined by $\tilde{D}(a) = D(a + J)$, for $a \in A$, is a module derivation. Since $A$ is module super-amenable, $\tilde{D}$, and so $D$ are both inner. Hence $A/J$ is module super-amenable, and since $A/J$ is a commutative $A$-module, by Proposition 2.3, it has an identity. The rest of the proof goes exactly like that of [3 Theorem 3.2], leading to super-amenability of $A/J$. \hfill \Box

**Lemma 2.8.** Let $A$ be an $\mathfrak{A}$-module with trivial left action. If $\mathfrak{A}$ has a bounded approximate identity for $A$ and $A/J$ is commutative $\mathfrak{A}$-module, then super-amenability of $A/J$ implies module super-amenability of $A$.

**Proof.** Let $\mathcal{X}$ be a commutative Banach $A$-$\mathfrak{A}$-module and $D : A \rightarrow \mathcal{X}$ be a module derivation. Since $J \cdot \mathcal{X} = \mathcal{X} \cdot J = 0$, $\mathcal{X}$ is a Banach $A/J$-module with module actions

$$(a + J) \cdot x := a \cdot x, \quad x \cdot (a + J) := x \cdot a \quad (x \in \mathcal{X}, a \in A).$$

Consider $\tilde{D} : A/J \rightarrow \mathcal{X}$, defined by $\tilde{D}(a + J) = D(a)$, for $a \in A$. For each $a, b \in A$ and $\alpha \in \mathfrak{A}$ we have

$$D(\alpha \cdot ab - ab \cdot \alpha) = \alpha \cdot D(ab) - D(ab) \cdot \alpha = 0.$$  

By the above observation, $\tilde{D}$ is also well-defined. Since $A/J$ is a commutative $\mathfrak{A}$-module, its super-amenability implies that it has identity $e + J$. Without loss of generality, we may assume that $\mathcal{X}$ is also a unital $A/J$-bimodule. Now $\mathfrak{A}$ acts on $A$ trivially from left, hence $f(a)a - a \cdot \alpha \in J$, for each $\alpha \in \mathfrak{A}$, where $f$ is a continuous linear functional on $\mathfrak{A}$ [3 Lemma 3.1]. Suppose that $\mathfrak{A}$ has a bounded approximate identity $(\gamma_i)$ for $A$. Since $f$ is bounded, $\{|f(\gamma_i)|\}$ is a bounded net in $\mathbb{C}$. Without loss of generality, we may assume that $f(\gamma_i) \rightarrow 1$. Thus for each $\lambda \in \mathbb{C}$ we have

$$e \cdot (\lambda \gamma_i) - f(\gamma_i)e = (\lambda e) \cdot \gamma_i - f(\gamma_i)e \rightarrow \lambda e - e$$

in norm. Since $J$ is a closed ideal of $A$, $\lambda e - e \in J$. Next, for $\lambda \in \mathbb{C}, a \in A$, we have

$$\tilde{D}(\lambda(a + J)) = \tilde{D}((a + J)(\lambda e + J))$$

$$= (a + J) \cdot \tilde{D}(\lambda e + J) + \tilde{D}(a + J)(\lambda e + J)$$

$$= (a + J) \cdot \tilde{D}(e + J) + \lambda \tilde{D}(a + J) \cdot (e + J)$$

$$= \lambda \tilde{D}(a + J).$$

Thus $\tilde{D}$ is $\mathbb{C}$-linear, and so it is inner. Therefore $D$ is an inner module derivation. \hfill \Box

3. **Module Super-Amenability for Semigroup Algebras**

In this section we find conditions on a (discrete) inverse semigroup $S$ such that the semigroup algebra $\ell^1(S)$ is $\ell^1(E)$-module super-amenable, where $E$ is the set of idempotents of $S$, acting naturally on it. We start this section with the definition of inverse semigroups.

**Definition 3.1.** A discrete semigroup $S$ is called an inverse semigroup if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotents of $S$ is denoted by $E$.

There is a natural order on $E$, defined by

$$e \leq d \iff ed = e \quad (e, d \in E).$$

It is easy to see that $E$ is indeed a commutative subsemigroup of $S$. In particular $\ell^1(E)$ could be regard as a subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module.
with compatible canonical actions. However, for technical reasons, here we let \( \ell^1(E) \) act on \( \ell^1(S) \) by multiplication from right and trivially from left, that is
\[
\delta_e \cdot \delta_s = \delta_s, \quad \delta_s \cdot \delta_e = \delta_{se} = \delta_s + \delta_e \quad (s \in S, e \in E).
\]

In this case, \( J \) is the closed linear span of
\[
\{\delta_{se} - \delta_{st} \mid s, t \in S, e \in E\}.
\]
We consider the following equivalence relation on \( S \)
\[
s \approx t \iff \delta_s - \delta_t \in J \quad (s, t \in S).
\]

Recall that \( E \) is called \textit{upward directed} if for every \( e, f \in E \) there exist \( g \in E \) such that \( eg = e \) and \( fg = f \). This is precisely the assertion that \( S \) satisfies the condition \( D_1 \) of Duncan and Namioka \([3]\). It is shown in \([3]\) that if \( E \) is upward directed, then the quotient \( S/\approx \) is a discrete group.

Unital inverse semigroups have an upward directed set of idempotents. Also if \( E \) is totally ordered, it is clearly upward directed. The examples of the latter include the bicyclic semigroup and the semigroup of natural numbers with max operation. On the other hand, the set of idempotents of the free inverse semigroup on two generators is not upward directed. Indeed, if the generators are \( a \) and \( b \), there is no idempotent which is bigger than both \( aa^* \) and \( bb^* \).

With the notations of previous section, \( \ell^1(S)/J \cong \ell^1(S/\approx) \) is a commutative \( \ell^1(E) \)-bimodule with the following actions
\[
\delta_e \cdot (\delta_s + J) = \delta_s + J, \quad (\delta_s + J) \cdot \delta_e = \delta_{se} + J \quad (s \in S, e \in E).
\]

The main theorem of this section is a semigroup analog of the Selivanov’s theorem \([11]\) for groups, characterizing module super-amenability of the semigroup algebra of an inverse semigroup with an upward directed set of idempotents. Indeed we reduce the result for inverse semigroups to that of discrete groups, and use Selivanov’s theorem.

**Theorem 3.2.** Let \( S \) be an inverse semigroup with an upward directed set of idempotents \( E \). Then \( \ell^1(S) \) is module super-amenable, as an \( \ell^1(E) \)-module with trivial left action and canonical right action, if and only if \( S/\approx \) is finite.

**Proof.** Suppose that \( \ell^1(S) \) is module super-amenable, then \( \ell^1(S)/J \cong \ell^1(S/\approx) \) is super-amenable by Lemma 2.7. Since \( S/\approx \) is a (discrete) group, it has to be finite by Selivanov’s theorem \([11]\). Conversely, if \( S/\approx \) is finite, then \( \ell^1(S)/J \) is super-amenable \([11]\). Since \( E \) satisfies condition \( D_1 \) of Duncan and Namioka, so \( \ell^1(E) \) has a bounded approximate identity for \( \ell^1(S) \) \([3]\) \([5]\). Now the result follows from Lemma 2.8 with \( A = \ell^1(S) \) and \( \mathcal{A} = \ell^1(E) \).

We close this section by some examples of module super-amenable Banach algebras. Let \( \mathfrak{G} \) be a commutative unital Banach algebra with unit element \( \epsilon \). Consider \( \mathcal{A} = M_n(\mathfrak{G}) \), the Banach algebra of \( n \times n \) matrices with entries from \( \mathfrak{G} \). Then \( \mathcal{A} \) is a unital commutative \( \mathfrak{G} \)-bimodule with the following natural actions
\[
\alpha \cdot [\beta_{ij}] = [\alpha \beta_{ij}], \quad [\beta_{ij}] \cdot \alpha = [\beta_{ij} \alpha] \quad (\alpha \in \mathfrak{G}, [\beta_{ij}] \in \mathcal{A}).
\]
Consider the set of matrix units \( \{E_{ij} \mid i, j = 1, \ldots, n\} \), where \( E_{ij} \) is the matrix having \( \epsilon \) at the \( i^{th} \) row and \( j^{th} \) column, and zero elsewhere. The identity matrix \( E \), which is the unit element of \( \mathcal{A} \), is the matrix whose diagonal entries are \( \epsilon \) and has zero entries elsewhere. Let \( I, J \) be the corresponding
closed ideals, as in section 2. We put

$$M = \sum_{i,j=1}^{n} \frac{1}{n} E_{ij} \otimes E_{ji} + I,$$

we have

$$\tilde{\omega}(M) = \sum_{i=1}^{n} E_{ii} + J = E + J,$$

hence $\tilde{\omega}(M)$ is an identity for $A/J$. Also

$$E_{lk} \cdot M = \sum_{i,j=1}^{n} E_{lk} \frac{1}{n} E_{ij} \otimes E_{ji} + I = \sum_{i=1}^{n} \frac{1}{n} E_{ii} \otimes E_{ik} + I = \sum_{i,j=1}^{n} \frac{1}{n} E_{ij} \otimes E_{ji} E_{ik} + I = M \cdot E_{lk},$$

for each $1 \leq l, k \leq n$. Hence for each $A \in \mathbb{A}$, we have $A \cdot M = M \cdot A$. It follow that $M$ is a module diagonal for $A$, therefore $A$ is module super-amenable by Theorem 2.6. Observe that in this case, $J = \{0\}$, but yet $A$ is not necessarily super-amenable. This shows that the assumption that the action is trivial from one side could not be dropped from Lemma 2.7. As a concrete example, consider $\mathfrak{B} = \ell^1(S)$, where $S = [0,1]$ is a unital commutative semigroup with multiplication $st = \min\{s+t,1\}$, for $s,t \in S$, then $\mathfrak{B} = \ell^1(S)$ and $A = M_n(\mathfrak{B})$ are not even weakly amenable [9], but still $A$ is $\mathfrak{B}$-module super-amenable with $J = \{0\}$.

The last example shows that there is an inverse semigroup $S$ for which $\ell^1(S)$ is module super-amenable but not super-amenable. Let $(\mathbb{N}, \lor)$ be the commutative semigroup of positive integers with maximum operation $m \lor n = \max\{m,n\}$, then each element of $\mathbb{N}$ is an idempotent, that is $E_{\mathbb{N}} = \mathbb{N}$. Hence $\mathbb{N}/\approx$ is the trivial group with one element. Therefore $\ell^1(\mathbb{N})$ is module super-amenable, as an $\ell^1(\mathbb{N})$-module. If $\ell^1(\mathbb{N})$ has a diagonal $M = \sum_{n=1}^{\infty} f_n \otimes g_n$, it should be $M = \delta_1 \otimes \delta_1$. In this case, we have $\delta_p \cdot M = M \cdot \delta_p$ ($p \in \mathbb{N}$), but this equality holds if and only if, $\delta_p \otimes \delta_1 = \delta_1 \otimes \delta_p$, for each $p \in \mathbb{N}$, which is absurd. Therefore $\ell^1(\mathbb{N})$ is not super-amenable by [10, Exercise 4.1.3]. Note that however, in this case, $\ell^1(\mathbb{N})$ has an identity.

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Department of Mathematics, Islamic Azad University, Garmsar Branch, Garmsar, Iran
E-mail address: abasalt.bodaghi@yahoo.com, abasalt.bodaghi@gmail.com

Department of Mathematics, Tarbiat Modares University, Tehran 14115-175, Iran
E-mail address: mamini@modares.ac.ir