s-homogeneous algebras via s-homogeneous triples

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Abstract

To study s-homogeneous algebras, we introduce the category of quivers with s-homogeneous corelations and the category of s-homogeneous triples. We show that both of these categories are equivalent to the category of s-homogeneous algebras. We prove some properties of the elements of s-homogeneous triples and give some consequences for s-Koszul algebras. Then we discuss the relations between the s-Koszulity and the Hilbert series of s-homogeneous triples. We give some application of the obtained results to s-homogeneous algebras with simple zero component. We describe all s-Koszul algebras with one relation recovering the result of Berger and all s-Koszul algebras with one dimensional s-th component. We show that if the s-th Veronese ring of an s-homogeneous algebra has two generators, then it has at least two relations. Finally, we classify all s-homogeneous algebras with s-th Veronese rings $k\langle x, y \rangle/(xy, yx)$ and $k\langle x, y \rangle/(x^2, y^2)$. In particular, we show that all of these algebras are not s-Koszul while their s-homogeneous duals are s-Koszul.

1 Introduction

All the algebras under consideration are graded algebras of the form $\Lambda = kQ/I$, where the grading is induced by the path length and, in particular, $I$ is a homogeneous ideal. The Ext-algebra of the algebra $\Lambda$ is the graded algebra $\bigoplus_{i \geq 0} \text{Ext}_\Lambda^i(\Lambda_0, \Lambda_0)$, where $\Lambda_0$, as usually, denotes the right $\Lambda$-module $\Lambda/\Lambda_{>0}$.

The notion of a Koszul algebra was introduced by S. Priddy in [8]. All Koszul algebras are quadratic and they appear in pairs. The Ext-algebra of a Koszul algebra $\Lambda$, is another time a Koszul algebra and its Ext-algebra is the original one. There is a duality between the category of Koszul modules over a Koszul algebra and the category of Koszul modules over its Ext-algebra. If we know that an algebra $\Gamma$ is the Ext-algebra of a Koszul algebra $\Lambda$ then one may recuperate $\Lambda$ from $\Gamma$ by taking its quadratic dual. An interesting result states that an algebra is Koszul if and only if its Ext-algebra is generated by its components of degrees zero and one. Note that even if a quadratic algebra is not Koszul, then it can be recovered from its quadratic dual algebra that in the non Koszul case is not anymore isomorphic to the Ext-algebra. Note also that if we have a quadratic algebra, then it is an Ext-algebra of some quadratic algebra if and only if it is Koszul.

Suppose now that the algebra $\Lambda = kQ/I$ is s-homogeneous, i.e. $I$ is generated by elements of degree $s$. The notion of s-Koszul algebra was the first generalization of the notion of a Koszul algebra, and it was given for the first time in [2] for a quiver with one vertex. Later the definition was rewritten for the case of an arbitrary quiver in [6]. It was shown in the last mentioned work that if an algebra is s-homogeneous, then it is s-Koszul if and only if its
Ext-algebra is generated by its components of degree less than or equal to two. The notion of s-Koszul algebra is important. For example, it was shown in [3] that an Artin-Shelter regular algebra of global dimension 3 is 3-Calabi-Yau if and only if it is s-Koszul.

The idea of the current paper appeared from the following question. Can we recover the algebra $\Lambda$ from its Ext-algebra if $\Lambda$ is s-Koszul? Note that, by the results of [3], the Ext-algebra of $\Lambda$ is isomorphic in the s-Koszul case to the semidirect product of $(\Lambda^i)_0$ and $(\Lambda^i)_1$ after some regrading, where $\Lambda^i$ is the s-homogeneous dual of the algebra $\Lambda$. In other words, Ext-algebra of $\Lambda$ consists of the quadratic algebra $(\Lambda^i)_0$ and $(\Lambda^i)_0$-bimodule $(\Lambda^i)_1$. At this moment new interesting questions appear. Assume that we have a quadratic algebra $A$ and an $A$-bimodule $M$. Is any s-homogeneous algebra $\Lambda$ such that the pair $(A,M)$ coincides with the pair $((\Lambda^i)_0,(\Lambda^i)_1)$? Can $\Lambda$ or $\Lambda^1$ be s-Koszul in this case? Should $\Lambda$ or $\Lambda^1$ be s-Koszul in this case?

Essentially, these questions appear, because the algebra structure of the Ext algebra does not contain all the $A_\infty$-structure in the s-Koszul case for $s > 2$. Indeed, there is a missing map from $(\Lambda^i)^\otimes (\Lambda^i)_0$ to $(\Lambda^i)_0$ that is included in the algebra structure in the quadratic case. The notion of s-homogeneous triple comes from this argument and becomes a good alternative to the s-homogeneous dual in the s-homogeneous case. While an s-homogeneous algebra can be presented by a quiver with s-homogeneous relations, we will show that an s-homogeneous triple can be naturally presented by a quiver with s-homogeneous corelations. We will show that the pair $(A,M)$ has to satisfy some restrictive conditions to have a complement to an s-homogeneous triple. Moreover, we will show that in many cases the pair $((\Lambda^i)_0,(\Lambda^i)_1)$ determines the algebras $\Lambda$ and $\Lambda^1$.

The connections between the s-Koszulity and Hilbert series of an algebra was discussed in [4]. In particular, it was shown that Hilbert series of s-Koszul algebra and its Koszul dual satisfy some condition. In this paper we give a further discussion of this. We rewrite the condition on Hilbert series of algebras in terms of Hilbert series of s-homogeneous triples and show that some part of this condition is equivalent to the extra condition introduced in [2].

In the last part of our work we show some applications of our technique. In our examples we consider only quivers with one vertex, i.e. algebras of the form $\Lambda = k\langle x_1, \ldots, x_n \rangle/I$, where $I$ is an ideal generated by elements of degree $s$. Firstly, we consider the case where $\dim_k I_s = 1$. We discuss the s-Koszulity of $\Lambda$ in this case recovering and clarifying the results of [3] and show that $\Lambda^i$ is s-Koszul only in the case $n = 1$ (i.e. in the case $\Lambda = k[x]/(x^s)$). Next we consider the case $\dim_k I_s = 2$. We show that in this case the $s$-th Veronese ring of $\Lambda^1$ is a quadratic algebra with two generators and at least two relations and classify all the algebras $\Lambda$, for which this ring is isomorphic to $k\langle x, y \rangle/(xy, yx)$ or $k\langle x, y \rangle/(x^2, y^2)$. It occurs that all such $\Lambda$ are s-Koszul, while $\Lambda^1$ is not s-Koszul in all cases.

2 s-homogeneous relations and corelations

We fix some notation during the paper. First of all, we fix some ground field $k$. Secondly, we fix some semisimple $k$-algebra $O$. Moreover, we assume that $O$ is isomorphic to a direct sum of copies of $k$ as an algebra and fix some basis $e_1, \ldots, e_D$ of $O$ such that $e_ie_j = \delta_{i,j}e_j$. Everything in this paper is over $O$. So we write simply $\otimes$ instead of $\otimes_O$. All modules in this paper are right modules if the opposite is not stated. If $U$ is an $O$-bimodule, then
$T(U) = \oplus_{i\geq 0} U^{\otimes i}$ denotes the tensor algebra of $U$ over $O$. For convenience we set everywhere $U^{\otimes i} = 0$ for $i < 0$. If $A$ is an algebra, $M$ is a right $A$-module, and $X$ is a subset of $M$, then $\langle X \rangle_A$ denotes the right $A$-submodule of $M$ generated by the set $X$.

**Definition 2.1.** An $O$-quiver with relations is a triple $(U, V, \iota)$, where $U$ and $V$ are $O$-bimodules and $\iota : V \hookrightarrow \oplus_{i\geq 2} U^{\otimes i}$ is an $A$-bimodule monomorphism. In this situation $U$ is called an $O$-quiver and $\text{Im} \iota$ is called a set of relations. The $O$-quiver $U$ is called finite if $U$ is a finitely generated $O$-bimodule. The set of relations $\text{Im} \iota$ is called $s$-homogeneous if $\text{Im} \iota \subset U^{\otimes s}$. If $(U', V', \iota')$ is another $O$-quiver with relations, then a morphism from $(U, V, \iota)$ to $(U', V', \iota')$ is a pair $(f, g)$, where $f : U \to U'$ and $g : V \to V'$ are such $O$-bimodule homomorphisms that $\iota'g = (\oplus_{i\geq 2} f^{\otimes i})\iota$. We denote by $\text{RQuiv}(O, s)$ the category of finite $O$-quivers with $s$-homogeneous sets of relations.

Since $O$ is fixed, we will write simply quiver instead of $O$-quiver. Now we are going to define a quiver with corelations. Since the present paper is devoted to $s$-homogeneous case, we define only $s$-homogeneous corelations.

**Definition 2.2.** A quiver with $s$-homogeneous corelations is a triple $(U, W, \pi)$, where $U$ and $W$ are $O$-bimodules and $\pi : U^{\otimes s} \to W$ is an $A$-bimodule epimorphism. In this situation $U$ is called a quiver as before and $W$ is called a set of corelations. If $(U', W', \pi')$ is another quiver with $s$-homogeneous corelations, then a morphism from $(U, W, \pi)$ to $(U', W', \pi')$ is a pair $(f, g)$, where $f : U \to U'$ and $g : W \to W'$ are such $O$-bimodule homomorphisms that $g\pi = \pi'f^{\otimes s}$. We denote by $\text{coRQuiv}(O, s)$ the category of finite quivers with $s$-homogeneous sets of corelations.

Note that $D = \text{Hom}_k(-, k)$ is a contravariant endofunctor of the category of finitely generated $O$-bimodules. Moreover, $D^2$ is isomorphic to the identity functor, i.e. $D$ is a duality. We will write $(-)^*$ instead of $D(-)$. Note that, for finitely generated $O$-bimodules $M$ and $L$, $(M \otimes L)^*$ can be identified with $M^* \otimes L^*$ via the canonical embedding $M^* \otimes L^* \hookrightarrow (M \otimes L)^*$. Thus, we obtain also the contravariant functor $D_R : \text{RQuiv}(O, s) \to \text{coRQuiv}(O, s)$ defined by the equalities $D_R(U, V, \iota) = (U^*, V^*, \iota^*)$ and $D_R(f, g) = (f^*, g^*)$ for an object $(U, V, \iota)$ and a morphism $(f, g)$ of the category $\text{RQuiv}(O, s)$. Analogously one can construct $D_{\text{coR}} : \text{coRQuiv}(O, s) \to \text{RQuiv}(O, s)$. We will write $(-)^*$ instead of $D_R(-)$ and $D_{\text{coR}}(-)$.

Now we define also the functors $\text{Coker} : \text{RQuiv}(O, s) \to \text{coRQuiv}(O, s)$ and $\text{Ker} : \text{coRQuiv}(O, s) \to \text{RQuiv}(O, s)$ in the following way. We simply define $\text{Coker}(U, V, \iota) = (U, \text{Coker} \iota, \pi)$, where $\pi : U^{\otimes s} \to \text{Coker} \iota$ is the canonical projection and $\text{Ker}(U, W, \pi) = (U, \text{Ker} \pi, \iota)$, where $\iota : \text{Ker} \pi \hookrightarrow U^{\otimes s}$ is the canonical inclusion. The definition on morphisms is the natural one, we simply define $\text{Coker}(f, g) = (f, \text{Coker}(g, f^{\otimes s}))$ and $\text{Ker}(f, g) = (f, \text{Ker}(f^{\otimes s}, g))$ for a morphism $(f, g)$ in the corresponding category. We have the following lemma.

**Lemma 2.3.** $(D_R, D_{\text{coR}})$ is a pair of quasi inverse dualities and $(\text{Coker}, \text{Ker})$ is a pair of quasi inverse equivalences. Moreover, there is an isomorphism of contravariant functors $\text{Ker} \circ D_R \cong D_{\text{coR}} \circ \text{Coker}$.

We will write $(-)^!$ instead of $\text{Ker} \circ D_R(-)$ and $\text{Coker} \circ D_{\text{coR}}(-)$.
3 s-homogeneous algebras

In this section we introduce the notion of s-homogeneous algebra. Usually, it is defined as an algebra corresponding to an element of $\text{RQuiv}(\mathcal{O}, s)$ as will be explained, but we prefer to give an intrinsic definition that does not use any presentation of an algebra.

**Definition 3.1.** An $\mathcal{O}$-algebra is a graded algebra $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ such that $\Lambda_0 = \mathcal{O}$. Given an $\mathcal{O}$-algebra $\Lambda$, we denote by $\Lambda^{(r)}$ the $r$-Veronese ring of $\Lambda$, i.e. a graded algebra $\Lambda^{(r)} = \bigoplus_{i \geq 0} \Lambda_i^{(r)}$, where $\Lambda_i^{(r)} = \Lambda_{ri}$ and the multiplication of $\Lambda^{(r)}$ is induced by the multiplication of $\Lambda$.

Also we define the $(r, t)$-Veronese bimodule $\Lambda^{(r, t)}$ as a graded $\Lambda^{(r)}$-bimodule $\Lambda^{(r, t)} = \bigoplus_{i \geq 0} \Lambda_i^{(r, t)}$, where $\Lambda_i^{(r, t)} = \Lambda_{ri+t}$ and the $\Lambda^{(r)}$-bimodule structure on $\Lambda^{(r, t)}$ is induced by the multiplication of $\Lambda$.

Note that the multiplication of $\Lambda$ induces a $\Lambda^{(r)}$-bimodule homomorphism from $(\Lambda^{(r, 1)})^{\otimes \Lambda^{(r)}}$ to $\Lambda^{(r)}$. We denote this homomorphism by $\phi_A^{(r)}$. Also we denote by $\phi_A^r$ the zero component of $\phi_A^{(r)}$, that is a map from $\Lambda_1^{(r)}$ to $\Lambda_r$. A morphism of $\mathcal{O}$-algebras is by definition a graded homomorphism of algebras identical on the zero component $\mathcal{O}$. Any such a morphism induces isomorphisms between all Veronese rings and all Veronese bimodules of the algebras under consideration. Moreover, the induced morphisms are compatible with the maps $\phi_A^{(r)}$.

**Definition 3.2.** Given an $\mathcal{O}$-algebra $\Lambda$, we call it s-homogeneous if $\phi_A^r$ is surjective and

$$\text{Ker} \phi_A^r = \sum_{i=0}^{r-s} \Lambda_1^{(r, i)} \otimes \text{Ker} \phi_A^s \otimes \Lambda_1^{(r-s-i)}$$

for any integer $r \geq 1$. In particular, if $\Lambda$ is s-homogeneous, then $\phi_A^r$ is bijective for $r < s$. We denote by $\text{HAlg}(\mathcal{O}, s)$ the category of s-homogeneous $\mathcal{O}$-algebras. We will write sometimes quadratic algebras instead of 2-homogeneous algebras.

There is another definition of an s-homogeneous algebra in terms of its Veronese modules. We give it in our next statement.

**Proposition 3.3.** Let $\Lambda$ be an $\mathcal{O}$-algebra, and, for $n, r, t \geq 1$,

$$\phi_n^{r, t}: \Lambda^{(n, r)} \otimes \Lambda^{(n, t)} \rightarrow \Lambda^{(n, r+t)}$$

be the $\Lambda^{(n)}$-bimodule homomorphism induced by the multiplication of $\Lambda$. Then the following conditions are equivalent:

1. $\Lambda \in \text{HAlg}(\mathcal{O}, s)$.

2. $\phi_n^{r, t}$ is surjective for all $n, r, t \geq 1$; $\phi_n^{r, t}$ is bijective if $n \geq s - 1$ and $r + t < s$; $\phi_n^{r, t}$ splits and $\text{Ker} \phi_n^{r, t}$ is concentrated in degree 0 if $n \geq s - 1$ and $r + t \geq s$.

3. $(\phi_s^{1, 1})_0$ is surjective for any $r \geq 1$ and is bijective for $r < s - 1$; $(\phi_n^{1, 1})_1$ is bijective for $n \geq s - 1$. 

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Proof. For any integer $i, j, k \geq 0$ such that $j + k = i$ and $n, r, t \geq 1$, we have a map

$$
\mu_{n,j,k}^{r,t} : \Lambda_{n+j+r} \otimes \Lambda_{n+k+t} \to (\Lambda^{(n,r)} \otimes_{\Lambda(n)} \Lambda^{(n,t)})_i.
$$

Then we have the commutative diagram

$$
\begin{array}{ccc}
\Lambda_1^{\otimes (n+i+r+t)} & \xrightarrow{\phi^{n+i+r+t}_\Lambda} & \Lambda_{n+i+r+t} \\
\phi^{n+i+r}_\Lambda \otimes \phi^t_\Lambda & \downarrow & \Lambda_{n+i+r} \otimes \Lambda_t \\
\Lambda_{n+i+r} \otimes \Lambda_t & \xrightarrow{\mu_{n,j,k}^{r,t}} & (\Lambda^{(n,r)} \otimes_{\Lambda(n)} \Lambda^{(n,t)})_i
\end{array}
$$

“1 $\implies$ 2” The surjectivity of $(\phi^{r,t}_n)_i$ follows from the surjectivity of $\phi^{n+i+r+t}_\Lambda$ and the just mentioned commutative diagram.

Suppose that $j > 0$, $k \geq 0$. It is clear that $\Lambda_1^{\otimes (n+i+r+t)}$ is generated as a $k$-linear space by the elements of the form $a \otimes b \otimes c$, where $a \in \Lambda_1^{\otimes (n+j-1+r)}$, $b \in \Lambda_1^{\otimes m}$ and $c \in \Lambda_1^{\otimes (n+k+t)}$. But for such an element we have

$$
\mu_{n,j,k}^{r,t}(\phi^{n+j+r}_\Lambda(a \otimes b) \otimes \phi^{n+k+t}_\Lambda(c)) = \mu_{n,j,k}^{r,t}(\phi^{n+j-1+r}_\Lambda(a) \otimes \phi^{n+k+t}_\Lambda(c)) = \mu_{n,j-1,k+1}^{r,t}(\phi^{n+j-1+r}_\Lambda(a) \otimes \phi^{n(k+1)+t}_\Lambda(b \otimes c)).
$$

Thus, we have $\mu_{n,j,k}^{r,t}(\phi^{n+j+r}_\Lambda \otimes \phi^{n+k+t}_\Lambda) = \mu_{n,j,k}^{n,i,0}$. Since $\phi^{n+j+r}_\Lambda \otimes \phi^{n+k+t}_\Lambda$ is surjective, this means, in particular, that $\mathrm{Im} \mu_{n,j,k}^{r,t}$ depends only on the sum of $j$ and $k$, but does not depend on $j$ and $k$. Thus, we have $(\Lambda^{(n,r)} \otimes_{\Lambda(n)} \Lambda^{(n,t)})_i = \sum_{j+k=i} \mathrm{Im} \mu_{n,j,k}^{r,t} = \mathrm{Im} \mu_{n,j,k}^{n,i,0}$. In particular, $\mu_{n,j,k}^{r,t}$ is surjective for any values of indices.

From now we assume that $n \geq s - 1$. Suppose that $(\phi^{r,t}_n)_i(u) = 0$ for some $u \in (\Lambda^{(n,r)} \otimes_{\Lambda(n)} \Lambda^{(n,t)})_i$. Since $\phi^{n+i+r}_\Lambda \otimes \phi^t_\Lambda$ and $\mu_{n,i,0}^{r,t}$ are surjective, we have $u = \mu_{n,i,0}^{r,t}(\phi^{n+i+r}_\Lambda \otimes \phi^t_\Lambda)(v)$ for some $v \in \Lambda_1^{\otimes (n+i+r+t)}$. It follows from the commutative diagram above that $\phi^{n+i+r+t}_\Lambda(v) = 0$. Thus,

$$
v \in \ker \phi^{n+i+r+t}_\Lambda = \sum_{l=0}^{n-i-s-r+t} \Lambda_1^{\otimes l} \otimes \ker \phi^s_\Lambda \otimes \Lambda_1^{\otimes (n-i-s-r+t-l)}.
$$

Let us prove that $\mu_{n,i,0}^{r,t}(\phi^{n+i+r}_\Lambda \otimes \phi^t_\Lambda)(\Lambda_1^{\otimes l} \otimes \ker \phi^s_\Lambda \otimes \Lambda_1^{\otimes (n-i-s-r+t-l)}) = 0$ if $i > 0$. Let $m$ denote $ni - s + r$. If $l \leq m$, then we have

$$
(\phi^{m+s}_\Lambda \otimes \phi^t_\Lambda)(\Lambda_1^{\otimes l} \otimes \ker \phi^s_\Lambda \otimes \Lambda_1^{\otimes (m+s-r+t-l)}) = \phi^l_\Lambda(\Lambda_1^{\otimes l}) \phi^s_\Lambda(Ker \phi^s_\Lambda)(\Lambda_1^{\otimes (m-l)}) \otimes \phi^t_\Lambda(\Lambda_1^{\otimes t}) = 0.
$$

If $l \geq m + 1$, then we have $\mu_{n,i,0}^{r,t}(\phi^{m+s}_\Lambda \otimes \phi^t_\Lambda) = \mu_{n,i-1,1}^{r,t}(\phi^{m+s-n}_\Lambda \otimes \phi^{n+t}_\Lambda)$ and

$$
(\phi^{m+s-n}_\Lambda \otimes \phi^{n+t}_\Lambda)(\Lambda_1^{\otimes l} \otimes \ker \phi^s_\Lambda \otimes \Lambda_1^{\otimes (m+t-l)}) = \phi^{l-m-s+n}_\Lambda(\Lambda_1^{\otimes l-m-s+n}) \otimes \phi^s_\Lambda(Ker \phi^s_\Lambda)(\Lambda_1^{\otimes (m+t-l)}) = 0.
$$

Thus, $u = \mu_{n,i,0}^{r,t}(\phi^{n+i+r}_\Lambda \otimes \phi^t_\Lambda)(v) = 0$ and $(\phi^{r,t}_n)_i$ is bijective if $i > 0$.

If $r + t < s$, then $\ker \phi^{r+t}_\Lambda = 0$, i.e. $(\phi^{r,t}_n)_0$ is also bijective.
It remains to prove that $\phi_n^r$ splits if $r + t \geq s$. Let us denote $\Lambda^{(n,r)} \otimes_{\Lambda^{(s)}} \Lambda^{(n,t)}$ by $M$. It is clear that $M$ a graded $\Lambda^{(s)}$-bimodule concentrated in nonnegative degrees. Let $\pi : M \to M/M_{>0}$ be the canonical projection and $\iota : \text{Ker} \phi_n^r \hookrightarrow M$ be the canonical inclusion. It is enough to show that $\iota$ splits. Since Ker $\phi_n^r$ is concentrated in degree 0, the map $\pi \iota$ is a monomorphism. Since $M/M_{>0}$ is semisimple, $\pi \iota$ splits. Consequently, $\iota$ splits too.

"2 $\implies$ 3" is clear.

"3 $\implies$ 1" Note that $\mu_{r, t}^{n, 0, 0} : \Lambda_r \otimes \Lambda_t \to (\Lambda^{(n, r)} \otimes_{\Lambda^{(s)}} \Lambda^{(n, t)})_0$ is an isomorphism for any $n, r, t \geq 1$.

Using the commutative diagram above and the fact that $(\phi_s^{r, 1})_0$ is surjective for any $r \geq 1$, we get surjectivity of $\phi_n^r$ for any $n \geq 1$ by induction on $n$. Using the commutative diagram above and the fact that $(\phi_s^{r, 1})_0$ is bijective for any $r < s - 1$, we get bijectivity of $\phi_n^r$ for any $n < s$ by induction on $n$.

Now it is enough to show that Ker $\phi_n^r = \text{Ker} \phi_n^{n-1} \otimes \Lambda_1 + \Lambda_1 \otimes \text{Ker} \phi_n^{n-1}$ for $n > s$. Let us take some $n > s$ and $u \in \text{Ker} \phi_n^r$. We have $(\phi_n^{r, 1})_1 \mu_{n-2, 1}^{n-2, 1, 0}(\phi_n^{n-1} \otimes \phi_1^n)(u) = 0$. Since $(\phi_n^{r, 1})_1$ is bijective, we have $\mu_{n-2}^{n-2, 1, 0}(\phi_n^{n-1} \otimes \phi_1^n)(u) = 0$. It is easy to see that the kernel of the canonical map from $(\Lambda^{n-1} \otimes \Lambda_1) \oplus (\Lambda_1 \otimes \Lambda^{n-1})$ to $(\Lambda^{n-1} \otimes \Lambda^{n-1})$ is $(\phi_n^{n-1} \otimes \phi_1^n - \phi_1^n \otimes \phi_n^{n-1})(\Lambda_1^{\otimes s})$. Thus, $(\phi_n^{r, 1} - \phi_1^n)(u) = (\phi_n^{r, 1} - \phi_1^n)(u')$ for some $u' \in \Lambda_1 \otimes \text{Ker} \phi_n^{n-1}$ and we have $u = (u - u') + u' \in \text{Ker} \phi_n^{n-1} \otimes \Lambda_1 + \Lambda_1 \otimes \text{Ker} \phi_n^{n-1}$.

□

Let us define the representing functor $\mathcal{ARK} : \text{HAlg}(\mathcal{O}, s) \to \text{RQuiv}(\mathcal{O}, s)$ and the algebraizing functor $\mathcal{A} : \text{RQuiv}(\mathcal{O}, s) \to \text{HAlg}(\mathcal{O}, s)$ in the following way. Given $\Lambda \in \text{HAlg}(\mathcal{O}, s)$, we define $\mathcal{ARK}(\Lambda) = (\Lambda_1, \text{Ker} \phi_1^n, \iota)$, where $\iota : \text{Ker} \phi_1^n \hookrightarrow \Lambda_1^{\otimes s}$ is the canonical inclusion. If $\Lambda'$ is another $s$-homogeneous $\mathcal{O}$-algebra and $f : \Lambda \to \Lambda'$ is a morphism of $\mathcal{O}$-algebras, then we define $\mathcal{ARK}(f) = (f|_{\Lambda_1}, (f|_{\Lambda_1})^{\otimes s}|_{\text{Ker} \phi_1^n})$.

Let us now consider $(U, V, \iota) \in \text{RQuiv}(\mathcal{O}, s)$. We define $\mathcal{A}(U, V, \iota) = T(U)/I_\iota$, where $I_\iota$ is the ideal of $T(U)$ generated by $\text{Im} \iota \subset U^{\otimes s}$. It is clear that $T(U)/I_\iota \in \text{HAlg}(\mathcal{O}, s)$. If $(f, g) : (U, V, \iota) \to (U', V', \iota')$ is a morphism of quivers with relations, then it induces the morphism $T(f) : T(U) \to T(U')$ such that $T(f)|_{U^{\otimes s}} = f^{\otimes i}$ for all $i \geq 0$. It is easy to see that $T(f)|_{I_\iota} \subset I_{\iota'}$, i.e. $T(f)$ induces a well defined morphism of $\mathcal{O}$-algebras

$$\mathcal{A}(f, g) : \mathcal{A}(U, V, \iota) = T(U)/I_\iota \to T(U')/I_{\iota'} = \mathcal{A}(U', V', \iota').$$

The proof of the next proposition is standard.

**Proposition 3.4.** $(\mathcal{ARK}, \mathcal{A})$ is a pair of quasi inverse equivalences.

Thus, we have a duality $(-)^! : \mathcal{A} \circ (-)^! \circ \mathcal{ARK} : \text{HAlg}(\mathcal{O}, s) \to \text{HAlg}(\mathcal{O}, s)$. Given $\Lambda \in \text{HAlg}(\mathcal{O}, s)$, we will call the algebra $\Lambda'$ the $s$-homogeneous dual algebra for $\Lambda$ or $s$-dual algebra for $\Lambda$ for short. Note that the definition depends on $s$, i.e. if $\Lambda \in \text{HAlg}(\mathcal{O}, s)$ and $\Lambda' \in \text{HAlg}(\mathcal{O}, s')$, then the $s$-dual and $s'$-dual algebras of $\Lambda$ are not isomorphic. This will not cause any confusion, because we always fix $s$. 

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4 \textit{s-homogeneous triples}

Another object that we are going to study in this paper is the $s$-homogeneous triples. In this section we introduce their definition and show that they can be naturally represented by quivers with corelations.

Given an $O$-algebra $A$, the category of graded $A$-modules is the category whose objects are graded $A$-modules and whose morphisms are degree preserving homomorphisms of $A$-modules. A graded $A$-module $M$ is called \textit{linear until the $n$-th degree} if there exists a projective resolution of $M$ in the category of graded $A$-modules

$$M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_n \leftarrow \cdots$$

such that $P_i$ is generated in degree $i$, i.e. $P_i = (P_i)_i A$, for $0 \leq i \leq n$.

\textbf{Definition 4.1.} An $s$-\textit{homogeneous triple} is a triple $(A, M, \varphi)$, where $A$ is a quadratic $O$-algebra, $M$ is a graded $A$-bimodule which is linear until the first degree as left and as right $A$-module, and $\varphi : M \otimes_A A \to A(1)$ is a homomorphism of graded $A$-bimodules such that

1. $\text{Im} \, \varphi = A_{\geq 0}(1)$;
2. $1_M \otimes_A \varphi = \varphi \otimes_A 1_M : M^{\otimes_A 1} \to M(1)$;
3. $\text{Ker}(1_M \otimes_A \varphi) = \text{Ker} \, \varphi \otimes_A M + M \otimes_A \text{Ker} \, \varphi$;
4. $\text{Ker}(\varphi \otimes_A \varphi) = \sum_{i=0}^s M^{\otimes_A i} \otimes_A \text{Ker} \, \varphi \otimes_A M^{\otimes_A i}$.

Let $(A, M, \varphi)$ and $(B, L, \psi)$ be $s$-homogeneous triples. A \textit{morphism} from $(A, M, \varphi)$ to $(B, L, \psi)$ is a pair $(f, g)$, where $f : A \to B$ is a morphism of graded $O$-algebras and $g : M \to L$ is a morphism of graded $A$-bimodules such that $f \varphi = \psi g^{\otimes_A}$. Here the $A$-bimodule structure on $L$ is induced by the map $f$. Let $\text{HTrip}(s, \mathcal{O})$ denote the category of $s$-homogeneous triples.

Note that if $(A, M, \varphi)$ is an $s$-homogeneous triple and $n \geq s$ is an integer, then the map $1_{M^{\otimes_A i} \otimes_A \varphi} \otimes_A 1_M : M^{\otimes_A n} \to M^{\otimes_A n}(1)$ does not depend on $0 \leq i \leq n - s$ due to the second point of Definition 4.1. For simplicity we denote this map by $\varphi$ too. So, if $n \geq ks$, then we have a graded $A$-bimodule homomorphism $\varphi^k : M^{\otimes_A A} \to M^{\otimes_A s}(s)$. As usually, $M^{\otimes_A} = A$ everywhere.

Let us define the \textit{representing functor} $\text{TRep} : \text{HTrip}(O, s) \to \text{coRQuiv}(O, s)$ and the \textit{tripling functor} $\text{Trip} : \text{coRQuiv}(O, s) \to \text{HTrip}(O, s)$ in the following way. Given $(A, M, \varphi) \in \text{HTrip}(O, s)$, we define $\text{TRep}(A, M, \varphi) = (M_0, A_1, \varphi_0)$. If $(A', M', \varphi')$ is another $s$-homogeneous triple and $(f, g) : (A, M, \varphi) \to (A', M', \varphi')$ is a morphism in $\text{HTrip}(O, s)$, then we define $\text{TRep}(f, g) = (g_0, f_1)$.

Let us now consider the quiver with corelations $Q = (U, W, \pi)$. Let us define $A_Q = T(W)/I_\pi$, where $I_\pi$ is the ideal of $T(W)$ generated by

$$\pi \otimes \pi) \left( \sum_{i=0}^s U^{\otimes i} \otimes \text{Ker} \, \pi \otimes U^{\otimes(s-i)} \right) \subset W \otimes W.$$

Let us equip all the elements of $U$ with degree 0 grading and consider the graded projective $A_Q$-bimodule $P_Q = A_Q \otimes U \otimes A_Q$. Since $(A_Q)_1 = W$, the sets $W \otimes U$ and $U \otimes W$ can be
naturally identified with subspaces $(A_Q)_1 \otimes U \otimes O$ and $O \otimes U \otimes (A_Q)_1$ of $P_Q$. Let us define $M_Q = P_Q/N_Q$, where \( N_Q = ((\pi \otimes 1_U - 1_U \otimes \pi)(U^{\otimes(s+1)}))_{A^m Q A} \subset P_Q \). For any \( m \geq 1 \) let us consider the projective right $A_Q$-module $P^R_m = U^{\otimes m} \otimes A_Q$. There is a right $A_Q$-module homomorphism $\beta_m : P^R_m \to M^\text{opp}_Q \otimes Q$ that sends $u_1 \otimes \cdots \otimes u_m \otimes a \in P^R_m$ to the class of the element

\[
(1_Q \otimes u_1 \otimes 1_Q) \otimes A_Q \cdots \otimes A_Q \left( 1_Q \otimes u_{m-1} \otimes 1_Q \right) \otimes A_Q \left( 1_Q \otimes u_m \otimes a \right).
\]

It easily follows from the definition of $N_Q$ that $\beta_m$ is surjective for any $m \geq 1$. Let us denote by $\rho_i : U^{\otimes(is)} \to (A_Q)_i$, the composition of $\pi^{\otimes i}$ and the natural projection $W^{\otimes i} \to (A_Q)_i$.

**Lemma 4.2.** The kernel of the map $\beta_1$ is $((1_U \otimes \pi)(\ker \pi \otimes U))_1$. In particular, $M_Q$ is linear until the first degree as a right $A_Q$-module.

**Proof.** Let $M$ denote the right $A_Q$-module $(U \otimes A_Q)/((1_U \otimes \pi)(\ker \pi \otimes U))_1$. Direct verification shows that $\beta_1$ induces an epimorphism $\beta : M \to M_Q$. We define $\alpha : M_Q \to M$ in the following way. Let us consider the map from $P_Q$ to $M$ that sends the element of the form $a \otimes u \otimes b \in (A_Q)_i \otimes U \otimes A_Q$, to the class of the element $(1_U \otimes \rho_i)(v \otimes u)a$, where $v \in U^{\otimes(is)}$ is some element such that $\rho_i(v) = b$. Since two elements $v, v' \in U^{\otimes(is)}$ such that $\rho_i(v) = \rho_i(v') = b$ differ by an element from $\sum_{j=0}^{s+i} U^{\otimes i} \otimes \ker \pi \otimes U^{\otimes(is-s-j)}$, it is easy to see that $(1_U \otimes \rho_i)(v \otimes u - v' \otimes u)$ belongs to $(1_U \otimes \pi)(\ker \pi \otimes U)A_Q$. Thus, the map just defined is a well-defined homomorphism of right $A_Q$-modules. It is not difficult to see also that this map vanishes on $(\pi \otimes 1_U - 1_U \otimes \pi)(U^{\otimes(s+1)})$, and hence induces the required homomorphism $\alpha$. It is not difficult also to see that $\alpha \beta = \text{Id}_M$, and hence $\beta$ is an isomorphism.

The dual argument shows that $M_Q$ is linear until the first degree as a left $A_Q$-module.

Let us define the homomorphism of graded bimodules $\varphi_Q : M^\text{opp}_Q \to A(1)$ in the following way. For a homogeneous element

\[
(a_1 \otimes u_1 \otimes b_1) \otimes_A \cdots \otimes_A (a_s \otimes u_s \otimes b_s) \in (A_Q)_i \otimes U \otimes (A_Q)_j \otimes_A \cdots \otimes_A (A_Q)_k \otimes U \otimes (A_Q)_l,
\]

we choose elements $v_1, \ldots, v_{s-1}$ in such a way that $\rho_{j_k+i_{k+1}}(v_k) = b_k a_{k+1}$ for $1 \leq k \leq s - 1$ and set

\[
\varphi_Q((a_1 \otimes u_1 \otimes b_1) \otimes_A \cdots \otimes_A (a_s \otimes u_s \otimes b_s)) = a_1 \rho_{j_1+i_{2}}(v_1) \cdots \rho_{j_{k}+i_{k+1}}(v_k) b_s.
\]

Analogously to the proof of Lemma 4.2, one can show that $\varphi_Q : P^\text{opp}_Q \to A(1)$ is a well-defined homomorphism of graded $A_Q$-bimodules. Moreover, it is clear that $\operatorname{Im} \varphi_Q = A_{>0}(1)$. Let $\iota_Q : N_Q \hookrightarrow P_Q$ be the canonical inclusion. Since the kernel of the projection $P^\text{opp}_Q \to M^\text{opp}_Q$ belongs to the image of

\[
\sum_{i=1}^{s} 1_{P^\text{opp}_Q}^{i-1} \otimes_A \iota_Q \otimes_A 1_{P^\text{opp}_Q}^{i-1} : \sum_{i=1}^{s} P^\text{opp}_Q^{i-1} \otimes_A N_Q \otimes_A P^\text{opp}_Q^{i-1} \to P^\text{opp}_Q
\]

and it is easy to check that $\varphi_Q \left( \sum_{i=1}^{s} 1_{P^\text{opp}_Q}^{i-1} \otimes_A \iota_Q \otimes_A 1_{P^\text{opp}_Q}^{i-1} \right) = 0$, we get the homomorphism $\varphi_Q$. 

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Lemma 4.3. \((A_Q, M_Q, \varphi_Q)\) is an \(s\)-homogeneous triple.

Proof. It remains to prove Conditions 2-4 from the definition of an \(s\)-homogeneous triple. It is not difficult to verify Condition 2. In particular, we have

\[
\ker \varphi_Q \otimes_{A_Q} M_Q + M_Q \otimes_{A_Q} \ker \varphi_Q \subset \ker (1_{M_Q} \otimes_{A_Q} \varphi_Q) \quad \text{and} \quad \sum_{i=0}^{s+1} M_Q^{s+1-i} \otimes_{A_Q} \ker \varphi_Q \subset \ker (\varphi_Q \otimes_{A_Q} \varphi_Q).
\]

Let us prove the reverse inclusions. Suppose that \(x \in M_Q^{s+1} \otimes_{A_Q} \varphi_Q\) is such that \((1_{M_Q} \otimes_{A_Q} \varphi_Q)(x) = 0\). We may assume that \(x\) is a homogeneous element of degree \(k\). Let us represent \(x\) in the form \(x = \beta_{s+1}(1_U^{s+1} \otimes \rho_k)(y)\) for \(y \in U^{(s+1+ks)}\). We have

\[
y \in \sum_{i=0}^{ks+1} U^{s+i} \otimes \ker \pi \otimes U^{(ks+1-i)}
\]

by Lemma 4.2. Now it is not difficult to show that

\[
\beta_{s+1}(1_U^{s+1} \otimes \rho_k) \left( \sum_{i=1}^{ks+1} U^{s+i} \otimes \ker \pi \otimes U^{(ks+1-i)} \right) \subset M_Q \otimes_{A_Q} \ker \varphi_Q
\]

and \(\beta_{s+1}(1_U^{s+1} \otimes \rho_k) (\ker \pi \otimes U^{(ks+1)}) \subset \ker \varphi_Q \otimes_{A_Q} M_Q\). Thus, Condition 3 holds. Analogously, using the surjectivity of the map \(\beta_{2s}\), one can prove Condition 4.

We define \(\mathfrak{Trip}(U, W, \pi) = (A_Q, M_Q, \varphi_Q)\). If \((f, g) : Q = (U, W, \pi) \rightarrow (U', W', \pi') = Q'\) is a morphism of quivers with corelations, then it induces the morphism \(T(g) : T(W) \rightarrow T(W')\) such that \(T(g)|_{W^{s+i}} = g^{s+i}\) for all \(i \geq 0\). It is easy to see that \(T(g)(I_\pi) \subset I_{\pi'}\), i.e. \(T(g)\) induces a well defined morphism of \(O\)-algebras \(T(g) : A_Q \rightarrow A_{Q'}\). Now, it is easy to see that the map \(f : (M_Q)_0 = U \rightarrow U' = (M_{Q'})_0\) can be proceeded to an \(A_Q\)-bimodule homomorphism \(T(f) : M_Q \rightarrow M_{Q'}\) in a unique way. Thus, we can define \(\mathfrak{Trip}(f, g) = (T(g), T(f))\). One can check that \(\mathfrak{Trip}(f, g)\) is really a morphism of \(s\)-homogeneous triples. The proof of the next proposition is standard.

Proposition 4.4. \((\mathfrak{Rep}, \mathfrak{Trip})\) is a pair of quasi inverse equivalences.

As in the case of algebras, we have a duality \((-)^! = \mathfrak{Trip} \circ (-)^! \circ \mathfrak{Rep} : H\text{Trip}(O, s) \rightarrow H\text{Trip}(O, s)\). Given \(T \in H\text{Alg}(O, s)\), we will call the \(s\)-homogeneous triple \(T^!\) the \(s\)-dual triple for \(T\).

5 \(s\)-homogeneous algebras and \(s\)-homogeneous triples

In this section we study the relations between \(s\)-homogeneous algebras and \(s\)-homogeneous triples. Our main theorem follows easily from the discussion of the previous sections.
**Theorem 5.1.** The categories $\text{HAlg}(s, \mathcal{O})$ and $\text{HTrip}(s, \mathcal{O})$ are equivalent.

**Proof.** The assertion of the theorem follows from Lemma 2.3 and Propositions 3.4 and 4.4.

Really, $\text{Trip} \circ \text{Coker} \circ \text{TRep} : \text{HAlg}(s, \mathcal{O}) \rightarrow \text{HTrip}(s, \mathcal{O})$ and $\mathcal{A} \circ \text{Alg} \circ \text{Rep} : \text{HTrip}(s, \mathcal{O}) \rightarrow \text{HAlg}(s, \mathcal{O})$

are quasi inverse equivalences.

Now, it is not difficult to describe $\mathcal{F} = \text{Trip} \circ \text{Coker} \circ \text{TRep}$ and $\mathcal{G} = \text{Alg} \circ \text{Alg} \circ \text{Rep}$.

We describe them using the notation introduced in Section 3.

If $\Lambda$ is an object of $\text{HAlg}(s, \mathcal{O})$, then $\mathcal{F}(\Lambda) = (\Lambda^{(s)}, \Lambda^{(s,1)}, \phi^{(s)}_{\Lambda})$. If $\Gamma \in \text{HAlg}(s, \mathcal{O})$ and $\theta : \Lambda \rightarrow \Gamma$ is a morphism of graded $\mathcal{O}$-algebras, then $\mathcal{F}(\theta) := (\theta|_{\Lambda^{(s)}}, \theta|_{\Lambda^{(s,1)}}) : \mathcal{F}(\Lambda) \rightarrow \mathcal{F}(\Gamma)$.

Given $(A, M, \varphi) \in \text{HTrip}(s, \mathcal{O})$, one has $\mathcal{G}(A, M, \varphi) = T(M_0) / (\ker \varphi)_0$. If $(B, L, \psi)$ is another $s$-homogeneous triple and $(f, g) : (A, M, \varphi) \rightarrow (B, L, \psi)$ is a morphism of $s$-homogeneous triples, then $\mathcal{G}(f, g) : \mathcal{G}(A, M, \varphi) \rightarrow \mathcal{G}(B, L, \psi)$ is the map induced by $g|M_0 : M_0 \rightarrow L_0$.

Now we have several corollaries of Theorem 5.1 and the description of $\mathcal{F}$ and $\mathcal{G}$.

**Corollary 5.2.** If $\Lambda \in \text{HAlg}(s, \mathcal{O})$, then $(\Lambda^{(s)}, \Lambda^{(s,1)}, \phi^{(s)}_{\Lambda}) \in \text{HTrip}(s, \mathcal{O})$.

**Corollary 5.3.** If $(A, M, \varphi)$ is an $s$-homogeneous triple, then there exists a graded $A$-bimodule $S$ concentrated in degree 0 and an isomorphism of graded $A$-bimodules $\theta : M^{\otimes s} \cong S \oplus A_{>0}(1)$ such that $\varphi$ equals to the composition

$$M^{\otimes s} \theta \rightarrow S \oplus A_{>0}(1) \rightarrow A_{>0}(1) \rightarrow A(1),$$

where the second map is the canonical projection on the second summand and the third arrow is the canonical inclusion.

**Proof.** There exists $\Lambda \in \text{HAlg}(s, \mathcal{O})$ such that $(A, M, \varphi) \cong \mathcal{F}(\Lambda) = (\Lambda^{(s)}, \Lambda^{(s,1)}, \phi^{(s)}_{\Lambda})$. In the notation of Proposition 3.3 we have $\phi^{(s)}_{\Lambda} = t_{s,\Lambda} \phi_{\varphi - 2,1} \phi_{\varphi - 1,1} \cdots \phi_{\varphi - 1,1} \otimes 1_{\Lambda^{(s,1)}}$, where $t_{s,\Lambda} : A_{>0} \cong \Lambda^{(s)} \hookrightarrow A^{(s)} \cong A$ is the canonical inclusion. Now the assertion of the corollary follows from the second item of Proposition 3.3.

**Corollary 5.4.** Let $(A, M, \varphi)$ and $(A, M, \varphi')$ be $s$-homogeneous triples. If the $A$-bimodule $A_{>0}(1)$ does not contain nonzero direct summands concentrated in degree 0, then $(A, M, \varphi) \cong (A, M, \varphi')$.

**Proof.** By Corollary 5.3 there are graded $A$-bimodules $S$ and $S'$ concentrated in degree 0 and isomorphisms of graded $A$-bimodules $\theta : M^{\otimes s} \rightarrow S \oplus A_{>0}(1)$ and $\theta' : M^{\otimes s} \rightarrow S' \oplus A_{>0}(1)$ such that $\ker \varphi = \theta^{-1}(S)$ and $\ker \varphi' = (\theta')^{-1}(S')$. Suppose that $(\theta')^{-1}(S') \not\subseteq \theta^{-1}(S)$. Then $\theta(\theta')^{-1}(S')$ is a graded $A$-subbimodule of $S \oplus A_{>0}(1)$ that does not lie in $S$, i.e., $T = (\theta(\theta')^{-1}(S') + S) \cap A_{>0}(1) \neq \{0\}$. Now it is clear that the monomorphism $T \hookrightarrow A_{>0}(1)$ splits and, hence, $A_{>0}(1)$ contains nonzero direct summand concentrated in degree 0. The obtained contradiction proves that $\ker \varphi' \subset \ker \varphi$. The inverse inclusion can be proved in the same way, i.e., $\ker \varphi' = \ker \varphi$. Thus, $\mathcal{G}(A, M, \varphi) = \mathcal{G}(A, M, \varphi')$. Since $\mathcal{G}$ is an equivalence, we have $(A, M, \varphi) \cong (A, M, \varphi')$. 

□
The next corollary follows directly from Corollary 5.5.

**Corollary 5.5.** Suppose that $\Lambda, \Gamma \in \text{HAlg}(O, s)$ are such that $\Lambda^{(s,1)} \cong \Gamma^{(s,1)}$ as $\Lambda^{(s)}$-bimodules, where $\Lambda^{(s)}$-bimodule structure on $\Gamma^{(s,1)}$ is induced by some isomorphism $\Lambda^{(s)} \cong \Gamma^{(s)}$. If $\Lambda_{>0}$ does not contain direct $\Lambda^{(s)}$-bimodule summand concentrated in degree 0, then $\Lambda \cong \Gamma$.

**Example 1.** Suppose that $\Lambda \in \text{HAlg}(O, s)$ and $\Lambda^{(s)} \cong S(W)$ is the symmetric algebra of the space $W$. Then, due to Corollary 5.5, $\Lambda$ can be uniquely recovered from the $S(W)$-bimodule $\Lambda^{(s,1)}$.

**Example 2.** Suppose that $\Lambda \in \text{HAlg}(O, s)$ and $\Lambda^{(s)} \cong \Lambda(W)$ is the exterior algebra of the space $W$ with $\dim_k W \geq 2$. Then, due to Corollary 5.5, $\Lambda$ can be uniquely recovered from the $\Lambda(W)$-bimodule $\Lambda^{(s,1)}$. On the other hand, we will show later that it is not true in the case $\dim_k W = 1$.

Note that the functors $\mathcal{F}$ and $\mathcal{G}$ respect the duality $(-)^!$, i.e., for $\Lambda \in \text{HAlg}(O, s)$ and $T \in \text{HTrip}(O, s)$, one has $\mathcal{F}(\Lambda^!) \cong \mathcal{F}(\Lambda)^!$ and $\mathcal{G}(T^!) \cong \mathcal{G}(T)^!$.

### 6 s-Koszulity

In this section we discuss the notion of an $s$-Koszul algebra.

**Definition 6.1.** The $s$-homogeneous algebra $\Lambda$ is called $s$-Koszul if $\text{Ext}_\Lambda^i(O, O)$ is concentrated in degree $-\chi_s(i)$, where

$$\chi_s(i) = \begin{cases} \frac{i}{2}, & \text{if } 2 \mid i, \\ \frac{i-1}{2} + 1, & \text{if } 2 \nmid i. \end{cases} \quad (6.1)$$

The $2$-Koszul algebras are called simply *Koszul algebras*.

Let $\Lambda$ be an $s$-homogeneous algebra and $(V, U, \iota) = \mathfrak{ARep}(\Lambda)$. Let us define the components of the graded vector space $R$ by the equality $R_n = \cap_{i+j=n} U^\otimes i \otimes \text{Im} \iota \otimes U^\otimes j$. The inclusion $R_{n+m} \hookrightarrow R_n \otimes R_m$ induces a map $R_n^* \otimes R_m^* \rightarrow R_{n+m}^*$ that gives a graded $O$-algebra structure on $R^* = \oplus_{n \geq 0} R_n^*$. In fact, it is well known that $R^* \cong \Lambda^!$. We also introduce the graded space $I$ with $n$-th component $I_n = \sum_{i+j=n} U^\otimes i \otimes \text{Im} \iota \otimes U^\otimes j$. Note that, in fact, $I$ is an ideal in $T(U)$ such that $\mathcal{G}(V, U, \iota) = T(U)/I$.

For $n > m$, the inclusion $R_m \hookrightarrow R_m \otimes U^\otimes (n-m)$ induces a $\Lambda$-module homomorphism $d_m^n : R_n \otimes \Lambda \rightarrow R_m \otimes \Lambda$. Then we can define the *generalized Koszul complex* $K$ of $\Lambda$ in the following way. Its $n$-th member is $K_n = R_{\chi_s(n)} \otimes \Lambda$. The differential is defined by the equality $d(K)_n = d_{\chi_s(n+1)} : K_{n+1} \rightarrow K_n$. Note that there is a surjective homomorphism $\mu_K : K_0 \rightarrow O$ induced by the composition of the isomorphism $O \otimes \Lambda \cong \Lambda$ and the canonical projection $\Lambda \rightarrow \Lambda_0 = O$. It is proved in [2] that $\Lambda$ is $s$-Koszul if and only if $K$ is exact in
positive degrees (though \( \mathcal{O} = k \) there, it is not difficult to transfer the arguments of Berger to our case). Direct calculations (which are fulfilled in [2] for \( \mathcal{O} = k \)) show that \( K \) is exact in positive degrees if and only if the following conditions are satisfied for all \( k, n \geq 0 \):

\[
R_{ns+1} \otimes U^{\otimes k} \cap U^{\otimes (ns)} \otimes I_{k+1} = R_{(n+1)s} \otimes U^{\otimes (k-s+1)} + R_{ns+1} \otimes I_k, \\
R_{(n+1)s} \otimes U^{\otimes k} \cap U^{\otimes (ns+1)} \otimes I_{k+s-1} = R_{(n+1)s+1} \otimes U^{\otimes (k-1)} + R_{(n+1)s} \otimes I_k.
\] (6.2)

It is shown in [2] that (6.2) is satisfied if and only if

\[
I_s \otimes U^{\otimes (s-1)} \cap U \otimes I_{2s-2} = R_{s+1} \otimes U^{\otimes (s-2)}
\] (6.3)

and for all \( k, n \geq 0 \) we have

\[
R_{ns+1} \otimes U^{\otimes k} \cap U^{\otimes (ns)} \otimes I_{k+1} = R_{ns+1} \otimes U^{\otimes k} \cap U^{\otimes (ns)} \otimes I_s \otimes U^{\otimes (k-s+1)} + R_{ns+1} \otimes I_k, \\
R_{(n+1)s} \otimes U^{\otimes k} \cap U^{\otimes (ns+1)} \otimes I_{k+s-1} = R_{(n+1)s} \otimes U^{\otimes k} \cap U^{\otimes (ns+1)} \otimes I_{2s-2} \otimes U^{\otimes (k-1)} + R_{(n+1)s} \otimes I_k.
\] (6.4)

We call the condition (6.3) the **extra condition** and call the conditions (6.4) the **distributivity conditions**.

If \( s = 2 \), then it is well known that \( \Lambda \) is Koszul if and only if \( \Lambda' \) is. For \( s > 2 \) there are examples where it is not so. One can show that the conditions for the \( s \)-Koszulity of the algebra \( \Lambda' \) are conditions (6.3) and (6.4) with \( R \) and \( I \) interchanged.

If \( s = 2 \) and \( \Lambda \) is Koszul, then \( \text{Ext}^*_\Lambda(\mathcal{O}, \mathcal{O}) \cong (\Lambda')^{\text{op}} \) as a graded algebra. For \( s > 2 \) the situation changes a little. Let \( \mathcal{F}(\Lambda') = (A, M, \varphi) \). It is proved in [3] that if \( \Lambda \) is \( s \)-Koszul, then \( A \) is a Koszul algebra and \( M \) is linear as left and right \( A \)-module. If \( \Lambda \) is \( s \)-Koszul and \( s > 2 \), then \( \text{Ext}^*_\Lambda(\mathcal{O}, \mathcal{O}) \cong A \rtimes M \) as an algebra, where \( A \rtimes M \) is the trivial extension of \( A \) by \( M \), i.e. its underlying space is \( A \oplus M \) and the multiplication is given by the equality \((a, x)(b, y) = (ab, ay + xb)\) for \( a, b \in A \) and \( x, y \in M \). If we define the grading on \( A \rtimes M \) by the equalities \((A \rtimes M)_{2n} = A_n \) and \((A \rtimes M)_{2n+1} = M_n \) for \( n \geq 0 \), then the isomorphism above will become degree preserving. From our previous results we get the following easy corollary.

**Corollary 6.2.** Suppose that \( \Lambda \) and \( \Gamma \) are \( s \)-Koszul algebras such that \( \text{Ext}^*_\Lambda(\mathcal{O}, \mathcal{O}) \cong \text{Ext}^*_\Gamma(\mathcal{O}, \mathcal{O}) \). If \( \Lambda \not\equiv \Gamma \), then there exists nonzero \( \theta \in \text{Ext}^2_\Lambda(\mathcal{O}, \mathcal{O}) \) such that \( \theta \text{Ext}^2_\Lambda(\mathcal{O}, \mathcal{O}) = \text{Ext}^2_\Lambda(\mathcal{O}, \mathcal{O}) \theta = 0 \).

**Proof.** Let us set \((A, M, \varphi) = F(\Lambda')\) and \((B, L, \psi) = F(\Gamma')\). Then \( \text{Ext}^*_\Lambda(\mathcal{O}, \mathcal{O}) \cong A \rtimes M \) and \( \text{Ext}^*_\Gamma(\mathcal{O}, \mathcal{O}) \cong B \rtimes L \) with the gradings defined above. By the degree argument this means that there is an isomorphism of graded algebras \( A \cong B \) and an isomorphism of graded \( A \)-bimodules \( M \cong L \), where the \( A \)-bimodule structure on \( L \) is induced by the just mentioned isomorphism of algebras. Then \((B, L, \psi) \cong (A, M, \varphi')\) for some \( \varphi' : M^{\otimes A} \to A(1) \). If \( \Lambda \not\equiv \Gamma \), then we have \((A, M, \varphi) \not\equiv (A, M, \varphi')\), and hence \( A_{>0}(1) \) contains nonzero \( A \)-bimodule summand concentrated in degree zero by Corollary 5.3. i.e. there is a nonzero element \( a \in A_1 \) such that \( aA_1 = A_1a = 0 \). Since \( A_1 \) can be identified with \( \text{Ext}^2_\Lambda(\mathcal{O}, \mathcal{O}) \), this is exactly the required assertion.

\(\square\)
7 Hilbert series

Let us now discuss the notion of Hilbert series and its relations with the notion of s-Koszulity. For an \( \mathcal{O} \)-bimodule \( W \) we will denote by \( \text{dim}_W \) the endomorphism of \( \mathcal{O} \) defined by the equality

\[
\text{dim}_W(e_j) = \sum_{i=1}^D (\text{dim}_W e_i) e_i \quad \text{for } 1 \leq j \leq D.
\]

**Definition 7.1.** Let \( W = \bigoplus_{k \geq 0} W_k \) be a nonnegatively graded \( \mathcal{O} \)-bimodule. The Hilbert series of \( W \) is the map \( \mathcal{H}_W(t) : \mathcal{O} \to \mathcal{O}[[t]] \) defined by the equality

\[
\mathcal{H}_W(t) = \sum_{k=0}^{\infty} t^k \text{dim}_W k.
\]

If \( \Lambda \in \text{HAlg}(\mathcal{O}, s) \) is \( s \)-Koszul, then, by the results of [4], one has

\[
\left( \mathcal{H}_A(t^s) - t \mathcal{H}_M(t^s) \right) \mathcal{H}_\Lambda(t) = \text{Id}_\mathcal{O},
\]

where \( (A, M, \varphi) = \mathcal{F}(\Lambda) \). To prove this it is enough to note that the left part of the equality is \( \sum_{k=0}^{\infty} (-1)^k \mathcal{H}_K(t) \), where \( K \) is the generalized Koszul complex of \( \Lambda \). We will denote by \( O(t^n) \) the set \( t^n \text{Hom}_k(\mathcal{O}, \mathcal{O}[[t]]) \subset \text{Hom}_k(\mathcal{O}, \mathcal{O}[[t]]) \). In what follows, \( f = g + O(t^n) \) means \( f - g \in O(t^n) \). Then we have the following result

**Lemma 7.2.** \( \Lambda \in \text{HAlg}(\mathcal{O}, s) \) satisfies the extra condition if and only if

\[
\left( \mathcal{H}_A(t^s) - t \mathcal{H}_M(t^s) \right) \mathcal{H}_\Lambda(t) = \text{Id}_\mathcal{O} + O(t^{2s}).
\]

**Proof.** Let us set \( (V, U, \iota) = \mathcal{ARep}(\Lambda) \),

\[
I_n = \sum_{i+s+j=n} U^{\otimes i} \otimes \text{Im} \iota \otimes U^{\otimes j}, \quad \text{and } R_n = \cap_{i+s+j=n} U^{\otimes i} \otimes \text{Im} \iota \otimes U^{\otimes j}.
\]

Note that \( \mathcal{H}_A(t^s) = \text{Id}_\mathcal{O} + t^s \text{dim}_I + O(t^n) \), \( \mathcal{H}_M(t^s) = t \text{dim}_U + t^{s+1} \text{dim}_{R_{s+1}} + O(t^n) \), and

\[
\mathcal{H}_\Lambda(t) = \sum_{n=0}^{s-1} t^n \text{dim}_{I^\otimes n} + \sum_{n=s}^{2s-1} t^n (\text{dim}_{U^\otimes n} - \text{dim}_{I^\otimes n}) + O(t^{2s}).
\]

Thus, we have

\[
\left( \mathcal{H}_A(t^s) - t \mathcal{H}_M(t^s) \right) \mathcal{H}_\Lambda(t)
\]

\[
= (\text{Id}_\mathcal{O} - t \text{dim}_U + t^s \text{dim}_I - t^{s+1} \text{dim}_{R_{s+1}}) \left( \sum_{n=0}^{2s-1} t^n \text{dim}_{U^\otimes n} - \sum_{n=s}^{2s-1} t^n \text{dim}_{I^\otimes n} \right) + O(t^{2s})
\]

\[
= \text{Id}_\mathcal{O} + \sum_{n=s}^{2s-1} t^n (\text{dim}_I \otimes U^\otimes (n-s) - \text{dim}_I) + \sum_{n=s+1}^{2s-1} t^n (\text{dim}_I \otimes U^\otimes (n-s-1) - \text{dim}_I) + O(t^{2s})
\]

\[
= \text{Id}_\mathcal{O} + \sum_{n=s+1}^{2s-1} t^n (\text{dim}_I \otimes U^\otimes (n-s) \cap U^\otimes I_{n-1} - \text{dim}_{R_{s+1}} + O(t^{2s})).
\]

13
Since \( R_{s+1} \otimes U \otimes (n-s-1) \subseteq I_s \otimes U \otimes (n-s) \cap U \otimes I_{n-1} \), the equality from the assertion of the lemma is satisfied if and only if \( R_{s+1} \otimes U \otimes (n-s-1) = I_s \otimes U \otimes (n-s) \cap U \otimes I_{n-1} \) for any \( s+1 \leq n \leq 2s-1 \). It is clear that the last mentioned equality holds for \( s+1 \leq n \leq 2s-1 \) if and only if it holds for \( n = 2s-1 \), i.e. if and only if the condition (6.3) is satisfied.

\[ \square \]

Note now that the Hilbert series of \( \Lambda \) are fully determined by the first two components of \( \mathcal{F}(\Lambda) \).

**Lemma 7.3.** Let \( \Lambda \in \text{HAlg}(O, s) \) and \( \mathcal{F}(\Lambda) = (B, L, \psi) \). Then \( \mathcal{H}_A(t) = \sum_{k=0}^{s-1} t^k \mathcal{H}_{L \otimes k}(t^s) \).

**Proof.** Follows from the fact that the map \((\Lambda^{(s,1)}) \otimes^{k} \Lambda^{(s)} \rightarrow \Lambda^{(s,k)}\) is bijective for \( 0 \leq k \leq s-1 \) by the second item of Proposition 3.3.

\[ \square \]

**Corollary 7.4.** Let \( \Lambda \in \text{HAlg}(O, s) \) and \( \mathcal{F}(\Lambda) = (B, L, \psi) \). Then \( \Lambda \) satisfies the extra condition if and only if

\[
\left( \text{Id}_O - t \dim_{L_0} + t^s (\dim_{L_0} - \dim_{B_1}) \right) \\
- t^{s+1} \left( \dim_{L_0^{(s+1)} \omega L_1} - \dim_{L_0 \otimes B_1} \right) \sum_{k=0}^{s-1} t^k \mathcal{H}_{L \otimes k}(t^s) - \text{Id}_O \in O(t^{2s}),
\]

In particular, if the algebra \( \Lambda^{(s)} \) and the \( \Lambda^{(s)} \)-bimodule \( \Lambda^{(s,1)} \) are known, then it is known if \( \Lambda \) satisfies the extra condition or not.

**Proof.** We will be free to use the notation of the proof of Lemma 7.2. Since

\[
\dim_{L_0} = \dim_{\Lambda^{(s)}} - \dim_{\Lambda_1} = \dim_{L_0^{(s)}} - \dim_{B_1},
\]

we have \( \mathcal{H}_A(t^s) = \text{Id}_O + t^s (\dim_{L_0^{(s)}} - \dim_{B_1}) + O(t^{2s}) \).

The exact sequence \( R_{s+1} \rightarrow U \otimes I_s \oplus I_s \otimes U \rightarrow I_{s+1} \) gives the equality

\[
\dim_{R_{s+1}} = \dim_{U \otimes I_s \oplus I_s \otimes U} - \dim_{I_{s+1}} = 2 \dim_{L_0^{(s+1)}} - \dim_{L_0 \otimes B_1} \otimes B_1 \otimes L_0 - (\dim_{L_0^{(s+1)}} - \dim_{L_1}) = \dim_{L_0^{(s+1)} \omega L_1} - \dim_{L_0 \otimes B_1} \otimes B_1 \otimes L_0.
\]

Since \( L_0 = U \), we have \( t \mathcal{H}_M(t^s) = t \dim_{L_0} + t^{s+1} (\dim_{L_0^{(s+1)} \omega L_1} - \dim_{L_0 \otimes B_1} \otimes B_1 \otimes L_0) + O(t^{2s}) \).

Now the assertion of the corollary follows from Lemmas 7.2 and 7.3.

\[ \square \]

### 8 Examples and applications

In this section we give some examples showing how the technique of \( s \)-homogeneous triples works. In particular, we will discuss some of the results of [3].

In this section we set \( O = k \). Moreover, we assume for simplicity that \( k \) is algebraically closed. Let, as before, \( \Lambda \) be an \( s \)-homogeneous algebra and \( (A, M, \varphi) = \mathcal{F}(\Lambda') \).
Example 1, $A = k[x]$. As it was mentioned before, in this case $\Lambda^l$ and $\Lambda$ are determined by the bimodule $M$. By Corollary 5.3 we have $M^{\otimes \alpha} \cong k(l \oplus A_{>0}) = k(l \oplus k[x])$ for some $l \geq 0$. From this condition and the fact that $M$ has to have linear presentation, it is not difficult to deduce that $M \cong k^m \oplus k[x]_\alpha$, where $m \geq 0$ and $\alpha$ is an automorphism of $k[x]$ sending $x$ to $\epsilon x$ for some $s$-th root of unit $\epsilon$. Thus, $l = (m+1)^s - 1$ and $\varphi$ is the canonical projection $M^{\otimes \alpha} \cong k(l \oplus (k[x]_\alpha)^{\otimes s}) \twoheadrightarrow (k[x]_\alpha)^{\otimes s} \cong k[x]$. It is easy to verify that the third condition from the definition of an $s$-homogeneous triple holds if and only if $\epsilon = 1$. In this case $\Lambda = k\langle x_1, \ldots, x_m, y \rangle/(y^s)$ is $s$-Koszul and it is not difficult to see that $\Lambda^l$ is $s$-Koszul if and only if either $s = 2$ or $m = 0$.

Example 2, $A = k[x]/(x^2)$. It is not difficult to show that in this case $M \cong k^m$ for some $m \geq 2$. Then $(\Lambda^l)_n = 0$ for $n > s$ and $\Lambda$ is $s$-Koszul if and only if $\Lambda$ has global dimension 2, i.e. the generalized Koszul complex of $\Lambda$ is exact in the second term. It follows from the results of [5] that $\Lambda$ is $s$-Koszul if and only if it satisfies the extra condition, i.e. if and only if $(1 - mt + t^2)H_\Lambda(t) - 1 \in O(t^{2s})$. It is easy to see that the situation under consideration occurs if and only if $\Lambda = k\langle x_1, \ldots, x_m \rangle/(f)$, where $f$ is some homogeneous polynomial in $x_1, \ldots, x_m$ of degree $s$ such that $f \neq g^s$ for any linear polynomial $g$. For example, $\Lambda$ is $s$-Koszul for $f = x_1x_2 - 1$ and $\Lambda$ is not $s$-Koszul for $f = x_1x_2 - x_1$. More detailed description of the situation is given in the next proposition.

Proposition 8.1. Suppose that $\Lambda = k\langle x_1, \ldots, x_m \rangle/(f)$, where $f$ is some homogeneous polynomial in $x_1, \ldots, x_m$ of degree $s$. Then $\Lambda$ is $s$-Koszul if and only if one of the following two conditions holds:

1. $f = g^s$ for some linear polynomial $g$;
2. if $f = gh_1 = h_2g$ for some polynomials $g$, $h_1$ and $h_2$, then $\deg g \in \{0, \deg f\}$.

Proof. If the first condition holds, then the $s$-Koszulity follows from the argument above.

If the first condition does not hold, then it is easy to see that the generalized Koszul complex of $\Lambda$ has the form

$$\mathbf{k}f \otimes \Lambda \xrightarrow{d_1} \bigoplus_{i=1}^m \mathbf{k}x_i \otimes \Lambda \xrightarrow{d_0} \Lambda \to \mathbf{k},$$

where $d_0(x_i \otimes 1) = x_i$ ($1 \leq i \leq m$) and $d_1(f \otimes 1) = (x_1 \otimes f_1, \ldots, x_m \otimes f_m)$ for such $f_1, \ldots, f_m$ that $f = \sum_{i=1}^m x_i f_i$. Then $\Lambda$ is $s$-Koszul if and only if $d_1$ is injective. If $gh_1 = h_2g$ for some $g$, $h_1$ and $h_2$ such that $0 < \deg g < \deg f$, then it is easy to see that $f \otimes h_1 \neq 0$ and $d_1(f \otimes h_1) = 0$,

i.e. $d_1$ is not injective.

Suppose that the second condition holds and $d_1(f \otimes h) = 0$ for some polynomial $h$. This means that $fh = \sum_{j=1}^k u_{1,j} u_{2,j} f$ in $\mathbf{k}\langle x_1, \ldots, x_m \rangle$ for some $u_{1,j}, u_{2,j} \in \mathbf{k}\langle x_1, \ldots, x_m \rangle$ such that $\deg u_{1,j} > 0$. Let us consider the map $T : \mathbf{k}\langle x_1, \ldots, x_m \rangle \to \mathbf{k}\langle x_1, \ldots, x_m \rangle$ defined by the equality $T(y) = \sum_{j=1}^k u_{1,j} y u_{2,j} - y h$. Since $T(f) = 0$ we have

$$T(y) = \sum_{j=1}^r (v_{1,j} v_{2,j} y v_{3,j} - v_{1,j} y v_{2,j} f v_{3,j})$$

for some $v_{i,j} \in \mathbf{k}$.
for some \(v_{1,j}, v_{2,j}, v_{3,j} \in k\langle x_1, \ldots, x_m \rangle\) by [3, Corollary 1.6]. Now it is easy to see that \(h\) has the form \(h = \sum_{j=1}^{m} h_{1,j} f h_{2,j}\) for some \(h_{1,j}, h_{2,j} \in k\langle x_1, \ldots, x_m \rangle\). Thus, \(f \otimes h = 0\) in \(k f \otimes \Lambda\), and hence \(d_1\) is injective.

\[\square\]

On the other hand, we have

\[
\left(1 - t \dim_{M_0} + t^s (\dim_{M_0^\otimes s} - \dim_{A_1})
\right.
\]

\[
- t^{s+1} (\dim_{M_0^\otimes (s+1)} + \dim_{M_0 \otimes A_1} - \dim_{M_0 \otimes A_1 \oplus M_1})
\]

\[\sum_{k=0}^{s-1} t^k \mathcal{H}_{M_0^\otimes A_1} (t^s)
\]

\[
= (1 - mt + (m^s - 1)t^s - (m^{s+1} - 2m)t^{s+1}) \left(\sum_{i=0}^{s-1} (mt)^i + t^s\right) = 1 + t^s \sum_{i=2}^{s-1} (mt)^i + O(t^2s),
\]

i.e. \(\Lambda'\) does not satisfy the extra condition by Corollary [7,4] if \(s > 2\).

**Corollary 8.2.** Let \(\Lambda \in H\text{Alg}(k, s)\) \((s \geq 3)\) be such that \(\dim_k \Lambda_s = 1\). Then \(\Lambda\) is \(s\)-Koszul if and only if \(\Lambda \cong k[x]\).

Now we give an example where the algebra \(A\) has two generators. As a first step in this direction, we will get a restriction on the number of relations for such an algebra.

**Theorem 8.3.** Let \((A, M, \varphi)\) be an \(s\)-homogeneous triple over \(k\). If \(A = k\langle x, y \rangle/I\) for some quadratic ideal \(I\), then \(\dim_k A_2 \leq 2\).

**Proof.** Let \((U, W, \pi) = \mathfrak{S}\text{Rep}(A, M, \varphi)\), i.e. \(U = M_0\), \(W = A_1\), and \(\pi = \varphi_0\). By our assumption, we have \(\dim_k \text{Im } \pi = \dim_k W = 2\). Suppose that \(\dim_k A_2 > 2\). This means that

\[
\dim_k (\pi \otimes \pi) \left(\sum_{i=0}^{s} U^\otimes i \otimes \text{Ker } \pi \otimes U^\otimes (s-i)\right) \leq 1.
\]

Since \(\text{Ker } (\pi \otimes \pi) = \text{Ker } \pi \otimes U^\otimes s + U^\otimes s \otimes \text{Ker } \pi\), we have

\[
\sum_{i=0}^{s} U^\otimes i \otimes \text{Ker } \pi \otimes U^\otimes (s-i) \subseteq \text{Ker } \pi \otimes U^\otimes s + U^\otimes s \otimes \text{Ker } \pi + V
\]

for some \(V \subseteq U^\otimes (2s)\) of dimension not more than 1. For \(S \subseteq U^\otimes l\) we as usually introduce \(S^\perp = \{\alpha \in (U^*)^\otimes l | \alpha(u) = 0 \ \forall u \in S\}\). Then we have

\[
\left(\text{Ker } \pi\right)^\perp \otimes (U^*)^\otimes s \cap (U^*)^\otimes s \otimes (\text{Ker } \pi)^\perp \cap V^\perp = (\text{Ker } \pi \otimes U^\otimes s + U^\otimes s \otimes \text{Ker } \pi + V)^\perp
\]

\[
\subseteq \left(\sum_{i=0}^{s} U^\otimes i \otimes \text{Ker } \pi \otimes U^\otimes (s-i)\right)^\perp = \bigcap_{i=0}^{s} (U^*)^\otimes i \otimes (\text{Ker } \pi)^\perp \otimes (U^*)^\otimes (s-i).
\]

Let us fix some basis \(x_1, \ldots, x_n\) of \(U^*\). Note that any element of \((U^*)^\otimes s\) can be written as a linear combination of words of length \(l\) in letters \(x_1, \ldots, x_n\), i.e. as an element of \(T_n\), where
\[ T = \mathbb{k}\langle x_1, \ldots, x_n \rangle. \] Since \( \dim_\mathbb{k}(\ker \pi)^\perp = 2 \), there are two element \( f_1, f_2 \in T_s \) that form a basis of \((\ker \pi)^\perp\). The condition above can be rewritten in the form

\[
(\mathbb{k}f_1 + \mathbb{k}f_2)T_s \cap T_s(\mathbb{k}f_1 + \mathbb{k}f_2) \cap L \subset \bigcap_{i=0}^{s} T_i(\mathbb{k}f_1 + \mathbb{k}f_2)T_{s-i},
\]

where \( L \subset T_{2s} \) is some subspace of dimension not less than \( n^{2s} - 1 \). Let us introduce the order \( x_1 > x_2 > \cdots > x_n \) on variables and the lexicographic order on the set of monomials of the same length, i.e. \( w_1 > w_2 \) for monomials \( w_1, w_2 \) of the same length if and only if there are monomials \( w, w_1, w_2 \) and integers \( 1 \leq i < j \leq n \) such that \( w_1 = wx_iw_i' \) and \( w_2 = wx_jw_j' \). As usually, for \( f \in T_i \) we denote by \( tip(f) \) the biggest monomial that has a nonzero coefficient in the decomposition of \( f \). Clearly, we may assume that \( tip(f_1) \neq tip(f_2) \). Then it is easy to see that \( tip(w) \in T_i(\mathbb{k}tip(f_1) + \mathbb{k}tip(f_2))T_{s-i} \) for any \( w \in T_i(\mathbb{k}f_1 + \mathbb{k}f_2)T_{s-i} \).

Since \( \dim_\mathbb{k} L \geq n^{2s} - 1 \) and the monomials \( tip(f_1)tip(f_1), tip(f_1)tip(f_2), tip(f_2)tip(f_1), \) and \( tip(f_2)tip(f_2) \) are pairwise not equal, it is easy to see that at least three of the four listed monomials belong to \( T_i(\mathbb{k}tip(f_1) + \mathbb{k}tip(f_2))T_{s-i} \) for any \( 0 \leq i \leq s \). It is clear from our argument that we may assume that \( f_1 \) and \( f_2 \) are monomials.

Since one of the monomials \( f_1f_1 \) and \( f_1f_2 \) belongs to \( T_i(\mathbb{k}tip(f_1) + \mathbb{k}tip(f_2))T_{s-i} \) for any \( 0 \leq i \leq s \), it is clear that, for some letter \( x \), there are maximum two different words of the form \( x\bar{x} \) that occur in \( f_1x \). Analogous assertion can be proved for some word of the form \( y\bar{y} \), where \( y \) is a letter. Then we may assume that either \( f_1 = x_1^s \) or \( f_1 = (x_1x_2)^t \), where \( t = \lceil \frac{s-1}{2} \rceil \). Analogous arguments show that we may set \( f_2 = x_2^s \) in the first case and \( f_2 = (x_2x_1)^tx_2^{s-2t} \) in the second case. Direct verifications show that all the obtained pairs \((f_1, f_2)\) do not satisfy the required conditions.

Thus, it makes sense to consider \( A = \mathbb{k}\langle x, y \rangle/I \), where \( I \) is generated by two, three or four quadratic relations.

It is clear that the only case of four relations is \( A = \mathbb{k}\langle x, y \rangle/(x, y)^2 \). Note that \( s \)-Koszul algebras \( \Lambda \) with \( \mathcal{F}(\Lambda^1) \) of the form \( (\mathbb{k}\langle x_1, \ldots, x_n \rangle)/(x_1, \ldots, x_n)^2, M, \sigma \) correspond exactly to local \( s \)-Koszul algebras with \( n \) relations of global dimensions 2 and 3.

It is not difficult to show that in the case of three relations \( A \) is isomorphic to one of the following algebras:

\[
\mathbb{k}\langle x, y \rangle/(x^2, xy, yx), \mathbb{k}\langle x, y \rangle/(x^2, y^2, xy + qyx) \quad (q \in \mathbb{k}), \quad \text{and} \quad \mathbb{k}\langle x, y \rangle/(x^2, xy + yx, xy + y^2).
\]

Note that \( \mathbb{k}\langle x, y \rangle/(x^2, y^2, xy + qyx) \cong \mathbb{k}\langle x, y \rangle/(x^2, y^2, xy + q'yx) \) for \( q \neq q' \) if and only if \( qq' = 1 \). It is not difficult to show, using the results of [1], that in the case of two relations \( A \) is Koszul if and only if it is isomorphic to one of the following algebras:

\[
\mathbb{k}\langle x, y \rangle/(xy, yx), \mathbb{k}\langle x, y \rangle/(x^2, y^2), \mathbb{k}\langle x, y \rangle/(x^2, yx), \quad \text{and} \quad \mathbb{k}\langle x, y \rangle/(x^2, xy + qyx) \quad (q \in \mathbb{k}).
\]

Note that all the listed algebras except \( \mathbb{k}\langle x, y \rangle/(x^2, xy, yx) \) and \( \mathbb{k}\langle x, y \rangle/(x^2, y^2) \) satisfy the condition of Corollary [5.4]. Thus, if \( \dim_\mathbb{k} A_1 = 2 \), then \( A \) can be recovered from the pair \((A, M)\) except the cases \( A = \mathbb{k}\langle x, y \rangle/(x^2, xy, yx) \) and \( A = \mathbb{k}\langle x, y \rangle/(x^2, y^2) \).

As an example of an application of our technique, we consider in this paper the algebras \( \mathbb{k}\langle x, y \rangle/(xy, yx) \) and \( \mathbb{k}\langle x, y \rangle/(x^2, y^2) \). We believe that our technique can be applied to other
cases to obtain a classification of $s$-Koszul algebras with two homogeneous relations and of $s$-Koszul algebras with two dimensional $s$-th component.

Note that, for any quadratic algebra $A = k\langle x, y \rangle/I$, any finitely generated right $A$-module $M$ linear until the first degree is isomorphic to a direct sum of indecomposable modules linear until the first degree. At the same time, any indecomposable $A$-module linear until the first degree is isomorphic to either a module of the form

$$A^n/\langle \{ f_1 x - f_{i+1} y \} \rangle_{1 \leq i \leq n-1}, a f_1 y, b f_n x \rangle_A \quad (n \geq 1, a, b \in \{0, 1\})$$

or a module of the form

$$A^n/\langle \{ f_1 x - f_{i+1} y \} \rangle_{1 \leq i \leq n-1}, f_n x - q f_1 y \rangle_A \quad (n \geq 1, q \in k^*)$$

Here and further $f_1, \ldots, f_n$ denote standard generators of the free module $A^n$.

**Example 3.** $A = k\langle x, y \rangle/(xy, yx), k\langle x, y \rangle/(x^2, y^2)$. As usually, for $a \in A$, we will denote by $aA$, $Aa$, and $AaA$ respectively the right submodule, left submodule and subbimodule of $A$ generated by $a$. In all cases we have an isomorphism $A_{>0} \cong xA \oplus yA$ of graded right modules and an isomorphism $A_{\geq 0} \cong Ax \oplus Ay$ of graded left modules.

Suppose $(A, M, \varphi)$ is an $s$-homogeneous triple. In particular, $M$ is linear until the first degree as left and right $A$-module and satisfies the condition $M^{\otimes \lambda} \cong A_{>0}(1) \oplus S$ for some $A$-bimodule $S$ concentrated in zero degree by Corollary 5.3.

We start by proving that $M \cong S_r \oplus xA(1) \oplus yA(1)$ as a graded right $A$-module for some module $S_r$ concentrated in zero degree. Let $J$ denote the ideal $A_{>0}$ of $A$. For a subset $I \subset A$ and a right $A$-module $X$ let us introduce $l\text{Ann}_J X = \{ u \in X \mid uI = 0 \}$. Note that $l\text{Ann}_JM^{\otimes \lambda} = l\text{Ann}_J M^{\otimes \lambda} = S$. Suppose that $1 \leq k < s$ and $u \in l\text{Ann}_JM^{\otimes \lambda} \setminus l\text{Ann}_J M^{\otimes \lambda}$. We have

$$\left( u \otimes_A M^{\otimes \lambda-I} \right) J^2 = \left( u \otimes_A M^{\otimes \lambda-I} \right) \varphi(M^{\otimes \lambda}) \varphi(M^{\otimes \lambda}) = u \varphi(M^{\otimes \lambda}) \varphi(M^{\otimes \lambda}) \otimes_A M^{\otimes \lambda-I} = u J^2 \otimes_A M^{\otimes \lambda-I} = 0.$$

Hence, $u \otimes_A M^{\otimes \lambda-I} \subset S$, i.e. $\varphi(u \otimes_A M^{\otimes \lambda-I}) = 0$. On the other hand, we have $0 \neq uJ = u \varphi(M^{\otimes \lambda}) = u \otimes_A M^{\otimes \lambda-I}M^{\otimes \lambda}$. The obtained contradiction shows that $l\text{Ann}_JM^{\otimes \lambda} = l\text{Ann}_J M^{\otimes \lambda}$ for any $1 \leq k < s$.

According to the classification of indecomposable $A$-module linear until the first degree given above, direct right $A$-module summands of $M^{\otimes \lambda}$ ($1 \leq k \leq s$) can have the following forms:

$$k, A, xA(1), yA(1), N = (A \oplus A)/\langle(x, -y)\rangle_A.$$

It is easy to show by induction that if $M$ has a direct right $A$-module summand isomorphic to $A$, then $M^{\otimes \lambda}$ has such a summand for any $k \geq 1$. On the other hand, $M^{\otimes \lambda}$ does not have direct summand isomorphic to $A$. Note that $M^{\otimes \lambda}$ has a direct summand isomorphic to $N$ if and only if $l\text{Ann}_JM^{\otimes \lambda}$ does not. Suppose that $M$ has a direct summand isomorphic to $N$. Since $J(l\text{Ann}_JM^{\otimes \lambda}) \subset MJ^2$, $(J(l\text{Ann}_JM^{\otimes \lambda}))J = 0$ and $l\text{Ann}_JM^{\otimes \lambda} = 0$, we have $J(l\text{Ann}_JM^{\otimes \lambda}) = 0$. The argument contained in the previous part of the proof shows that $M$ has a direct left $A$-module summand isomorphic to $N' = (A \oplus A)/\langle(x, -y)\rangle_{A_{\varphi}}$. Direct
calculations show that $\text{Tor}_A(xA(1), N') = \text{Tor}_A(yA(1), N') = 0$. Applying the functor $- \otimes_A M$ to the short exact sequence $k(-1) \hookrightarrow N \twoheadrightarrow xA(1) \oplus yA(1)$ we get the long exact sequence
\[
\cdots \to \text{Tor}(xA(1) \oplus yA(1), M) \xrightarrow{\beta} (M/JM)(-1) \xrightarrow{\alpha} N \otimes_A M \twoheadrightarrow (xA(1) \oplus yA(1)) \otimes_A M.
\]

The argument above shows that $(M/JM)(-1)$ has a direct summand isomorphic to $(N'/JN')(1)$ that does not belong to the image of $\beta$. Thus, $\beta$ is not surjective and $\text{Im} \alpha \subset (\text{Ann}_J N \otimes_A M)_1$ is nonzero. Since $N \otimes_A M$ is a direct summand of $M \otimes_A M$, the argument above shows that $N \otimes_A M$ has a direct summand isomorphic to $N$. Then it is easy to show by induction that $M \otimes_A M$ has a summand isomorphic to $N$ for any $k \geq 1$. The obtained contradiction shows that $M \cong S_r \oplus (xA(1))^{k_1} \oplus (yA(1))^{k_2}$ as a graded right $A$-module for some module $S_r$ concentrated in zero degree and integers $k_1, k_2 \geq 0$.

Since $Mx \neq 0$ and $My \neq 0$, it is easy to see that $k_1, k_2 \geq 1$. Suppose that $k_1 > 1$. Then $(xA(1) \otimes_A M^{s \otimes A})^{k_1}$ is a direct summand of $M^{s \otimes A}$, and hence $xA(1) \otimes_A M^{s \otimes A}$ is concentrated in zero degree. Then $k_2 = 1$ and there exists a subspace $U' \subset M_0$ of codimension 1 such that $J(U' \otimes_A M^{s \otimes A}) = 0$. Applying the dual argument we get a subspace $U \subset M_0$ of codimension 1 such that $(M^{s \otimes A} \otimes_A U)J = 0$. Now it follows from the direct sum right $A$-module decomposition of $M$ that either $M^{s \otimes A}x = 0$ or $M^{s \otimes A}y = 0$. The obtained contradiction shows that $M \cong S_r \oplus xA(1) \oplus yA(1)$ as a graded right $A$-module and finishes our first step.

Since $JS_r \subset M_1$, $(JS_r)J = 0$, and $(\text{Ann}_J(MJ)) = 0$, we have $JS_r = 0$, i.e. $S_r = k^m$ is a direct $A$-bimodule summand of $M$ concentrated in degree 0. It remains to describe left $A$-module structure on $xA(1) \oplus yA(1)$. This step will be fulfilled separately for the two different algebras under consideration. Let us denote by $f_x$ and $f_y$ the degree 0 right $A$-module generators of $xA(1)$ and $yA(1)$ correspondingly.

Suppose that $A = k(\bar{x}, \bar{y})/(\bar{x}, \bar{y})$. Since $(xf_x)y = (yf_x)y = 0$, we have $xf_x = q_x f_x x$ and $yf_x = q_y f_x x$ for some $q_x, q_y \in k$. We have shown before that $xM \cap yM = 0$, i.e. $q_x = 0$ or $q_y = 0$. Analogously, $xf_y = p_x f_y y$ and $yf_y = p_y f_y y$, where one of the elements $p_x, p_y \in k$ is zero and another is nonzero. Moreover, it is clear that either $q_y = p_x = 0$, $q_y \neq 0$, or $q_y = 0$, $q_y \neq 0$, $p_x \neq 0$. Then $M = k^m \oplus \gamma(AxA \oplus AyA)$, where $\gamma$ is an automorphism of $A$ that sends $x$ to $q_x x + q_y y$ and $y$ to $p_x x + p_y y$.

If $q_x = p_x = 0$, then $\gamma(AxA) \otimes_A \gamma(AyA) \cong \gamma(AyA) \otimes_A \gamma(AxA) \cong k$ and the definition of an $s$-homogeneous triple gives us isomorphisms $\alpha : (\gamma(AxA))^{s \otimes A} \to (AxA)$ and $\beta : (\gamma(AyA))^{s \otimes A} \to (AyA)$ that satisfy the equalities $\alpha(x^{s \otimes A}) \ast x = x \ast \alpha(x^{s \otimes A})$ and $\beta(y^{s \otimes A}) \ast y = y \ast \beta(y^{s \otimes A})$, where $\ast$ is the multiplication arising from the $A$-bimodule structure on $\gamma(AxA \oplus AyA)$. Now it is clear that $\gamma = \text{Id}_A$, $M = k^m \oplus AxA \oplus AyA$, and $A \cong k(x_1, \ldots, x_m, y_1, y_2)/(y_1^t, y_2^t)$. This algebra is $s$-Koszul, for example, by the results of [7], or [6].

If $q_x = p_y = 0$, then analogous argument shows that $2 \mid s$ and $q_y = \frac{1}{2} = q$ for some $q \in k^*$. It is not difficult to see that the graded algebra isomorphism from $\tilde{A}$ to $A$ that sends $x$ to $qx$ and $y$ to $y$ induces isomorphism of $s$-homogeneous triples
\[
(A, M, \phi) \cong (A, k^m \oplus \gamma_0(AxA \oplus AyA), \phi_0),
\]
where $\gamma_0$ is the automorphism of $A$ interchanging $x$ and $y$ and $\phi_0$ is the corresponding homomorphism from the definition of an $s$-homogeneous triple. Thus, $A \cong k(x_1, \ldots, x_m, y_1, y_2)/(y_1^t, y_2^t)$ for $t = \frac{s}{2}$ is again $s$-Koszul.

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Suppose now that $A = k\langle x, y \rangle / (x^2, y^2)$. Since $(xf_x)x = (yf_x)x = 0$, we have $xf_x = q_xf_yx$ and $yf_x = q_yf_yx$ for some $q_x, q_y \in k$. Analogously to the previous case we have either $yf_x = xf_y = 0$, $xf_x = q_xf_yx$, and $yf_y = p_yf_xy$ for some $q_x, p_y \in k^*$ or $xf_x = yf_y = 0$, $yf_x = q_yf_yx$, and $xf_y = p_xf_yx$ for some $q_y, p_x \in k^*$. Then it is not difficult to show that $M = k^m \oplus \gamma(A_{>0})$, where $\gamma$ is an automorphism of $A$ that sends $x$ to $q_xx + q_yy$ and $y$ to $p_xx + p_yy$.

If $q_y = p_x = 0$, then $(\gamma(A_{>0}))^\otimes_A \cong k^{2s-2} \oplus L$, where $L$ is the $A$-bimodule generated by the elements $f_x^\otimes_A$ and $f_y^\otimes_A$. It is easy to see that $x(l\text{Ann}_yL) = 0$, and hence $L \not\cong A_{>0}$.

If $q_x = p_y = 0$, then $(\gamma(A_{>0}))^\otimes_A \cong k^{2s-2} \oplus L$, where $L$ is the $A$-bimodule generated by the elements

$$f_x^{s-2} \left[ \frac{\delta_{y}}{\delta_{x}} \right] \otimes \gamma (f_y \otimes_A f_x)^{\otimes_A} \text{ and } f_y^{s-2} \left[ \frac{\delta_{x}}{\delta_{y}} \right] \otimes \gamma (f_x \otimes_A f_y)^{\otimes_A}.$$

If $2 \mid s$, then $L \not\cong A_{>0}$, because $x(l\text{Ann}_yL) = 0$. If $2 \nmid s$, then as before we get $q_y = \frac{1}{p_x} = q$ for some $q \in k^*$ and $(A, M, \varphi) \cong (A, k^m \oplus \gamma_0(A_{>0}), \varphi_0)$, where $\gamma_0$ is the automorphism of $A$ interchanging $x$ and $y$ and $\varphi_0$ is the corresponding homomorphism from the definition of an $s$-homogeneous triple. Thus, $\Lambda \cong k\langle x_1, \ldots, x_m, y_1, y_2 \rangle / ((y_1y_2)^t y_1, y_2(y_1y_2)^t)$ for $t = \frac{s-1}{2}$ is $s$-Koszul.

It is clear that if $s \geq 3$, then $\Lambda^1$ is not $s$-Koszul in all cases. Thus, we obtain the following result.

**Theorem 8.4.** Suppose that $\Lambda \in \mathbb{HAlg}(k, s)$.

1. If $(\Lambda^1)^{(s)} \cong k\langle x, y \rangle / (xy, yx)$, then either

$$\Lambda \cong k\langle x_1, \ldots, x_m, y_1, y_2 \rangle / (y_1^t, y_2)$$

for some $m \geq 0$ or $s = 2t + 1$ and

$$\Lambda \cong k\langle x_1, \ldots, x_m, y_1, y_2 \rangle / ((y_1y_2)^t, (y_2y_1)^t)$$

for some $t \geq 1$ and $m \geq 0$.

2. If $(\Lambda^1)^{(s)} \cong k\langle x, y \rangle / (x^2, y^2)$, then $s = 2t + 1$ and

$$\Lambda \cong k\langle x_1, \ldots, x_m, y_1, y_2 \rangle / ((y_1y_2)^t y_1, y_2(y_1y_2)^t)$$

for some $t \geq 1$ and $m \geq 0$.

In particular, if $(\Lambda^1)^{(s)} \in \{ k\langle x, y \rangle / (xy, yx), k\langle x, y \rangle / (x^2, y^2) \}$, then $\Lambda$ is $s$-Koszul and $\Lambda^1$ is $s$-Koszul only in the case $s = 2$.

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