HARMONIC MAPS FROM $\mathbb{C}^n$ TO KÄHLER MANIFOLDS

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Abstract. In this paper, we shall prove that a harmonic map from $\mathbb{C}^n$ ($n \geq 2$) to any Kähler manifold must be holomorphic under an assumption of energy density. It can be considered as a complex analogue of the Liouville type theorem for harmonic maps obtained by Sealey.

1. Introduction

The classical Liouville theorem says that a bounded harmonic function on $\mathbb{R}^n$ (or holomorphic function on $\mathbb{C}^n$) has to be constant. Sealey (see [3] or [6]) gave an analogue for harmonic maps. He proved that a harmonic map of finite energy from $\mathbb{R}^n$ ($n \geq 2$) to any Riemannian manifold must be a constant map. In this paper we consider the complex analogue of Sealey’s result, that is the following

**Question:** Must a harmonic map with finite $\bar{\partial}$-energy from $\mathbb{C}^n$ ($n \geq 2$) to any Kähler manifold be holomorphic?

On the other hand, from Siu-Yau’s proof of Frankel conjecture [4] (the key is to prove a stable harmonic map from $S^2$ to $\mathbb{C}P^n$ is holomorphic or conjugate holomorphic), we know that it is very important to study the holomorphicity of harmonic maps. So the above question is obviously interesting. We hope that it is true. But we do not know how to prove it. Our partial result can be stated as follows

**Theorem 1.1.** Let $f$ be a harmonic map from $\mathbb{C}^n$ ($n \geq 2$) to any Kähler manifold. Let $e(f)$ be the energy density and $e''(f)$ be the $\bar{\partial}$-energy density. If

\[(1.1) \quad e(f)e''(f)(p) = O\left(\frac{1}{R^{4n+\alpha}}\right),\]

for some $\alpha > 0$, where $R$ denotes the distance from origin to $p$, then $f$ is a holomorphic map.

The condition (1.1) implies that the $\bar{\partial}$-energy is finite. Since

\[(e''(f))^2 \leq e(f)e''(f) = O\left(\frac{1}{R^{4n+\alpha}}\right),\]

one has

\[e''(f) = O\left(\frac{1}{R^{4n+\alpha+2}}\right).\]
This leads to
\[ \int_{C^m} e^{''}(f) dv < \infty. \]

Note that we do not have any curvature assumption for the target manifold.

We should mention some other related holomorphicity of harmonic maps. For instance, Dong [1] established many holomorphicity under the assumption of target manifolds with strongly semi-negative curvature. In [7] Xin obtained some holomorphicity of harmonic maps from a complete Riemann surface into \( \mathbb{C}\mathbb{P}^n \).

If the target manifold is \( \mathbb{C}^m \) (in this case every component of the map is a harmonic function), then the answer of above question is positive (see [5]).

The main idea of the proof of theorem 1.1 is to consider a one parameter family of maps and study the \( \bar{\partial} \)-energy variation.

The rest of the paper is organized as follows: Section 2 contains some basic materials of harmonic maps; In section 3 we study the first variation of \( \bar{\partial} \)-energy; Theorem 1.1 is proved in the last section.

2. PRELIMINARIES

The materials in this section may be found in the book of Xin [6].

2.1. Basic concepts of harmonic maps. Let \( f \) be a smooth map between two Riemannian manifolds \( (M, g) \) and \( (N, h) \). We can define the energy density of \( f \) by
\[ e(f) = \frac{1}{2} \text{trace} |df|^2 = \frac{1}{2} \sum_{i=1}^{m} \langle f_* e_i, f_* e_i \rangle, \]
where \( \{e_i\} \) (\( i = 1, \ldots, m = \dim M \)) is a local orthonormal frame field of \( M \). The energy integral is defined by
\[ E(f) = \int_M e(f) dv. \]

If we choose local coordinates \( \{x^i\} \) and \( \{y^\alpha\} \) in \( M \) and \( N \), respectively, the energy density can be written as
\[ e(f)(x) = \frac{1}{2} g^{ij}(x) \frac{\partial f^\alpha(x)}{\partial x^i} \frac{\partial f^\beta(x)}{\partial x^j} h_{\alpha\beta}(f(x)). \]

The tension field of \( f \) is
\[ \tau(f) = (\nabla_e df)(e_i), \]
where \( \nabla \) is the induced connection on the pull-back bundle \( f^{-1}TN \) over \( M \) from those of \( M \) and \( N \).

**Definition 2.1.** We say that \( f \) is a harmonic map if \( \tau(f) = 0 \).

From the variation point of view, a harmonic map can be seen as the critical point of energy integral functional. Let \( f_t \) be a one parameter family of maps. We can regard it as a smooth map from \( M \times (-\epsilon, \epsilon) \to N \). Let \( f_0 = f \), \( \frac{df}{dt}|_{t=0} = v \). Then we have the first variation formula (see [6])
\[ \frac{d}{dt} E(f_t)|_{t=0} = \int_M \text{div}W dv - \int_M \langle v, \tau(f) \rangle dv, \]
where \( W = \langle v, f_* e_j \rangle e_j \). If \( M \) is compact, then \( \int_M \text{div}W dv = 0 \). We know that a harmonic map is the critical point of energy functional.
2.2. $\bar{\partial}$-energy. Let us consider the complex case.

Let $f$ be a smooth map from $\mathbb{C}^n$ to a Kähler manifold $N$. Let $J$ be the standard complex structure of $\mathbb{C}^n$ and $J'$ be the complex structure of $N$. $\omega$ and $\omega^N$ are the corresponding Kähler forms of $\mathbb{C}^n$ and $N$ (i.e. $\omega(\cdot, \cdot) = \langle J \cdot, \cdot \rangle$ and $\omega^N(\cdot, \cdot) = \langle J' \cdot, \cdot \rangle$).

The $\bar{\partial}$-energy density is defined by

$$e''(f) = |\bar{\partial}f|^2 = |f_* J - J' f_*|^2 = \frac{1}{4} (|f_* e_i|^2 + |f_* J e_i|^2 - 2 \langle J' f_* e_i, f_* J e_i \rangle) = \frac{1}{2} (e(f) - \langle f^* \omega^N, \omega^M \rangle),$$

where $\{e_i, Je_i\}$ $(i = 1, ..., n)$ is the Hermitian frame of $\mathbb{C}^n$ and $\langle f^* \omega^N, \omega \rangle$ denotes the induced norm. We call that $f$ is holomorphic if $f_* J = J' f_*$. Obviously $f$ is holomorphic if and only if $|\bar{\partial}f|^2 \equiv 0$.

It is well known that a holomorphic map between two Kähler manifolds must be harmonic (c.f. [6]).

We denote $\bar{\partial}$-energy by

$$E(\bar{\partial})(f) = \int_{\mathbb{C}^n} |\bar{\partial}f|^2 dv.$$ 

3. $\bar{\partial}$-energy variation

Let us consider the one parameter family of maps $f_t(x) = f(tx) : \mathbb{C}^n \to N, t \in (1 - \epsilon, 1 + \epsilon)$ and $f_1 = f$. Let $B_R$ denote the Euclid ball in $\mathbb{C}^n$ of radius $R$ around 0. We write

$$E(R, t) = \int_{B_R} |\bar{\partial}f_t|^2 dv.$$

Lemma 3.1. $E(R, t) = t^{2 - 2n} E(Rt, 1)$.

Proof. Under the standard Hermitian metric of $\mathbb{C}^n$, $g^{ij} = \delta_{ij}$, from [2,1] we have

$$e(f_t)(x) = t^2 e(f)(tx).$$

By using the natural coordinates, it is easy to show that

$$\langle f_t^* \omega^N, \omega \rangle(x) = t^2 \langle f^* \omega^N, \omega \rangle(tx).$$

So we get

$$|\bar{\partial}f_t|^2(x) = t^2 |\bar{\partial}f|^2(tx).$$

It is easy to check that

$$\int_{B_R} |\bar{\partial}f_t|^2 dv = t^{2 - 2n} \int_{B_{Rt}} |\bar{\partial}f|^2 dv.$$

Thus we obtain the lemma. \qed

We now prove the following variation formula of $\bar{\partial}$-energy.

Lemma 3.2. $\frac{\partial E(R,t)}{\partial t}|_{t=1} = \frac{R}{2} \int_{\partial B_R} (|f_* \frac{\partial}{\partial r}|^2 - \langle J' f_* \frac{\partial}{\partial r}, f_* J \frac{\partial}{\partial r} \rangle) dv$.

The proof will be separated in two steps.
Proof. Let \( \{ e_1, ..., e_{2n} = \frac{\partial}{\partial r} \} \) be a local orthonormal frame field, where \( \frac{\partial}{\partial r} \) denotes unit radial vector field. By the definition of \( f_t(x) \), it is easy to see that the variation vector field of \( f_t \) at \( t = 1 \) is

\[
v = \frac{df_t}{dt}|_{t=1} = rf_*\frac{\partial}{\partial r}.
\]

**Step 1:** From (2.2) we have

\[
\frac{d}{dt} \int_{B_R} e(f_t)dv|_{t=1} = \int_{B_R} \text{div}(v, f_*e_j) e_j dv - \int_{B_R} \langle v, \tau(f) \rangle dv
\]

\[
= \int_{\partial B_R} \langle v, f_*\frac{\partial}{\partial r} \rangle dv
\]

\[
= R \int_{\partial B_R} |f_*\frac{\partial}{\partial r}|^2 dv.
\]

Since \( f \) is harmonic, we know that the tension field \( \tau(f) = 0 \), and the second "\( = \)" follows from divergence theorem.

**Step 2:** On the other hand, from (6) we know that \( \frac{d}{dt} f_*\omega^N = d\theta_t \), where \( \theta_t = f_t^i(f_*\frac{\partial}{\partial r})^N \). Since \( \frac{df_t}{dt}|_{t=1} = rf_*\frac{\partial}{\partial r} \), we get \( \theta_1 = \theta = rf^i(f_*\frac{\partial}{\partial r})^N \). Then

\[
\frac{d}{dt} \int_{B_R} (f_*^i\omega^N, \omega) dv|_{t=1}
\]

\[
= \int_{B_R} \langle d\theta, \omega \rangle dv
\]

\[
= \int_{B_R} d(\theta \wedge *\omega) + \int_{B_R} \langle \theta, \delta\omega \rangle dv
\]

\[
= \int_{\partial B_R} \theta \wedge *\omega - \int_{B_R} \langle \theta, *d\omega^{n-1} \rangle dv
\]

\[
= \int_{\partial B_R} \theta \wedge *\omega
\]

\[
= - \int_{\partial B_R} \theta(e_i)\omega(e_i, \frac{\partial}{\partial r}) dv
\]

\[
= -R \int_{\partial B_R} \omega^N(\frac{\partial}{\partial r}, f_*e_i)\omega(e_i, \frac{\partial}{\partial r}) dv
\]

\[
= -R \int_{\partial B_R} \langle Jf_*\frac{\partial}{\partial r}, f_*e_i \rangle (Je_i, \frac{\partial}{\partial r}) dv
\]

\[
= R \int_{\partial B_R} \langle Jf_*\frac{\partial}{\partial r}, f_*J\frac{\partial}{\partial r} \rangle dv.
\]

Note that \( (d\theta, \omega) dv = d\theta \wedge *\omega \), the second "\( = \)" follows from the differential rules, where \( \delta \) and \( * \) are the co-differential and star operators. By stokes theorem and the definition of \( \delta \), the third "\( = \)" holds. The fifth "\( = \)" follows from direct computation.

Since we may choose \( e_1 = J\frac{\partial}{\partial r} \), the last "\( = \)" holds.

Combining step 1 and step 2, we obtain

\[
\frac{d}{dt} \int_{B_R} |\tilde{\theta} f_t|^2 dv|_{t=1} = \frac{R}{2} \int_{\partial B_R} (|f_*\frac{\partial}{\partial r}|^2 - \langle Jf_*\frac{\partial}{\partial r}, f_*J\frac{\partial}{\partial r} \rangle) dv.
\]

This completes the proof of the lemma. \( \square \)
Remark 3.3. If $M$ is a compact manifold, $\int_M (f^*\omega^N,\omega^M)dv$ is a homotopy invariant. This was observed firstly by Lichnerowicz [2].

4. Proof of theorem 1.1

We use the similar trick in [3]. By lemma 3.1, we obtain

$$\frac{\partial E(R,t)}{\partial t}|_{t=1} = (2-2n)E(R,1) + R\frac{\partial E(R,1)}{\partial R}.$$ 

On the other hand, from lemma 3.2 and the condition 1.1 in theorem 1.1, one has

$$\frac{\partial E(R,t)}{\partial t}|_{t=1} \geq -R^2 \int_{\partial B_R} |f_*\frac{\partial f}{\partial r}|^2 - |f_*\frac{\partial f}{\partial r} \cdot |f_* J \frac{\partial f}{\partial r}|dv$$

$$\geq -R^2 \int_{\partial B_R} |f_*\frac{\partial f}{\partial r}| \cdot \|f_* \frac{\partial f}{\partial r} - |f_* J \frac{\partial f}{\partial r}||dv$$

$$\geq -R^2 \cdot R^{2n-1} \cdot \frac{1}{R^{2n+\frac{\alpha}{2}}} \cdot C$$

$$= -\frac{C}{2} R^{-\frac{\alpha}{2}},$$

where $C$ is a positive constant. Hence for any $\epsilon > 0$, there exists an $R_0$ such that

$$\frac{\partial E(R,t)}{\partial t}|_{t=1} \geq -\epsilon$$

for all $R \geq R_0$. Therefore

$$R \frac{\partial E(R,1)}{\partial R} \geq -\epsilon + (2n-2)E(R,1)$$

for $R \geq R_0$.

If $E(\infty,1) = \int_{\mathbb{C}^n} |\bar{\partial}f|^2 dv = E > 0$, then there exists a $R_1$ such that for all $R \geq R_1$, we have $E(R,1) \geq E_0 > 0$. Since $n \geq 2$ we can choose sufficiently small $\epsilon$ such that

$$R \frac{\partial E(R,1)}{\partial R} \geq A = -\epsilon + (2n-2)E_0 > 0,$$

when $R \geq R_2 = \max(R_0,R_1)$. Then

$$E(\infty,1) = \int_{\mathbb{C}^n} |\bar{\partial}f|^2 dv \geq \int_{R_2}^{\infty} \frac{A}{R} dR = \infty.$$ 

It is a contradiction. Therefore $\int_{\mathbb{C}^n} |\bar{\partial}f|^2 dv = 0$. Hence $f$ is a holomorphic map.

Remark 4.1. Compare with the real case [3], lemma 3.2 has the term $\langle J^f f_\cdot \frac{\partial f}{\partial r}, f_* J \frac{\partial f}{\partial r} \rangle$. We need use condition 1.1 to control it.

Remark 4.2. If we consider the $\partial$-energy density $e'(f) = |\bar{\partial}f|^2 = |f_* J + J^f f_\cdot|^2$, the corresponding result of theorem 1.1 also holds, i.e condition 1.1 is replaced by $e(f)e'(f)(p) = O((\bar{\partial}f)^2)$, the conclusion is that $f$ is a conjugate holomorphic map ($|\bar{\partial}f|^2 \equiv 0$).
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