LINEAR QUANTUM ADDITION RULES

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To Ron Graham on his 70th birthday

Abstract. The quantum integer \([n]_q\) is the polynomial \(1 + q + q^2 + \cdots + q^{n-1}\). Two sequences of polynomials \(U = \{u_n(q)\}_{n=1}^{\infty}\) and \(V = \{v_n(q)\}_{n=1}^{\infty}\) define a linear addition rule \(\oplus\) on a sequence \(F = \{f_n(q)\}_{n=1}^{\infty}\) by \(f_m(q) \oplus f_n(q) = u_n(q)f_m(q) + v_m(q)f_n(q)\). This is called a quantum addition rule if \([m]_q \oplus [n]_q = [m+n]_q\) for all positive integers \(m\) and \(n\). In this paper all linear quantum addition rules are determined, and all solutions of the corresponding functional equations \(f_m(q) \oplus f_n(q) = f_{m+n}(q)\) are computed.

1. Multiplication and addition of quantum integers

We consider polynomials \(f(q)\) with coefficients in a commutative ring with 1. A sequence \(F = \{f_n(q)\}_{n=1}^{\infty}\) of polynomials is nonzero if \(f_n(q) \neq 0\) for some integer \(n\). For every positive integer \(n\), the quantum integer \([n]_q\) is the polynomial \([n]_q = 1 + q + q^2 + \cdots + q^{n-1}\).

These polynomials appear in many contexts. In quantum calculus (Cheung-Kac [2]), for example, the \(q\) derivative of \(f(x) = x^n\) is

\[f'(x) = \frac{f(qx) - f(x)}{qx - x} = [n]_q x^{n-1}.\]

The quantum integers are ubiquitous in the study of quantum groups (Kassel [3]).

Let \(F = \{f_n(q)\}_{n=1}^{\infty}\) be a sequence of polynomials. Nathanson [5] observed that the multiplication rule

\[f_m(q) \ast f_n(q) = f_m(q)f_n(q^n)\]

induces a natural multiplication on the sequence of quantum integers, since

\([m]_q \ast [n]_q = [mn]_q\)

for all positive integers \(m\) and \(n\). He asked what sequences \(F = \{f_n(q)\}_{n=1}^{\infty}\) of polynomials, rational functions, and formal power series satisfy the multiplicative functional equation

\[(1) \quad f_m(q) \ast f_n(q) = f_{mn}(q)\]

for all positive integers \(m\) and \(n\). Borisov, Nathanson, and Wang [1] proved that the only solutions of (1) in the field \(Q(q)\) of rational functions with rational coefficients

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are essentially quotients of products of quantum integers. More precisely, let \( F = \{ f_n(q) \}_{n=1}^{\infty} \) be a nonzero solution of (1) in \( \mathbb{Q}(q) \), and let \( \text{supp}(F) \) be the set of all integers \( n \) with \( f_n(q) \neq 0 \). They proved that there is a finite set \( R \) of positive integers and a set \( \{ t_r \}_{r \in R} \) of integers such that, for all \( n \in \text{supp}(F) \),

\[
  f_n(q) = \lambda(n)q^{t_0(n-1)} \prod_{r \in R} [n]_{q^r}^{t_r},
\]

where \( \lambda(n) \) is a completely multiplicative arithmetic function and \( t_0 \) is a rational number such that \( t_0(n-1) \in \mathbb{Z} \) for all \( n \in \text{supp}(F) \). Nathanson \([6]\) also proved that if \( F = \{ f_n(q) \}_{n=1}^{\infty} \) is any solution of the functional equation (1) in polynomials or formal power series with coefficients in a field, and if \( f_n(0) = 1 \) for all \( n \in \text{supp}(F) \), then there exists a formal power series \( F(q) \) such that

\[
  \lim_{n \to \infty} f_n(q) = F(q).
\]

Nathanson \([7]\) also defined the addition rule

\[
  f_m(q) \oplus f_n(q) = f_m(q) + q^m f_n(q)
\]

on a sequence \( F = \{ f_n(q) \}_{n=1}^{\infty} \) of polynomials, and considered the additive functional equation

\[
  f_m(q) \oplus q^m f_n(q) = f_{m+n}(q).
\]

He noted that

\[
  [m]_q \oplus [n]_q = [m + n]_q
\]

for all positive integers \( m \) and \( n \), and proved that every solution of the additive functional equation \([3]\) is of the form

\[
  f_n(q) = h(q)[n]_q,
\]

where \( h(q) = f_1(q) \). This implies that if a nonzero sequence of polynomials \( F = \{ f_n(q) \}_{n=1}^{\infty} \) satisfies both the multiplicative functional equation \([1]\) and additive function equation \([3]\), then

\[
  f_n(q) = [n]_q
\]

for all positive integers \( n \).

In this paper we consider other binary operations \( f_m(q) \oplus f_n(q) \) on sequences of polynomials that induce the natural addition of quantum integers or, equivalently, that satisfy \([4]\). The goal of this paper is to prove that the addition rule \([2]\) is essentially the only linear quantum addition rule, and to find all solutions of the associated additive functional equation.

2. Linear addition rules

A general linear quantum addition rule is defined by two doubly infinite sequences of polynomials \( U = \{ u_{m,n}(q) \}_{m,n=1}^{\infty} \) and \( V = \{ v_{m,n}(q) \}_{m,n=1}^{\infty} \) such that

\[
  [m + n]_q = u_{m,n}(q)[m]_q + v_{m,n}(q)[n]_q
\]

for all positive integers \( m \) and \( n \). If the sequences \( U \) and \( V \) satisfy \([5]\), then \( U \) determines \( V \), and conversely. It is not known for what sequences \( U \) there exists a complementary sequence \( V \) satisfying \([5]\).
A linear zero identity is determined by two sequences of polynomials \( S = \{s_{m,n}(q)\}_{m,n=1}^\infty \) and \( T = \{t_{m,n}(q)\}_{m,n=1}^\infty \) such that
\[
s_{m,n}(q)[m]_q + t_{m,n}(q)[n]_q = 0
\]
for all positive integers \( m \) and \( n \).

We can construct new addition rules from old rules by adding zero identities and by taking affine combinations of addition rules. For example, the simplest quantum addition rule is
\[
[m + n]_q = [m]_q + q^m[n]_q.
\]
Then
\[
[m]_q + q^m[n]_q = [m + n]_q = [n + m]_q = [n]_q + q^n[m]_q
\]
for all positive integers \( m \) and \( n \), and we obtain the zero identity
\[
(1 - q^n)[m]_q + (q^m - 1)[n]_q = 0.
\]
Adding (6) and (7), we obtain
\[
[m + n]_q = (2 - q^n)[m]_q + (2q^m - 1)[n]_q.
\]
An affine combination of (6) and (8) gives
\[
[m + n]_q = (4 - 3q^n)[m]_q + (4q^m - 3)[n]_q.
\]

We can formally describe this process as follows.

**Theorem 1.** For \( i = 1, \ldots, k \), let \( U^{(i)} = \{u_{m,n}(q)\}_{m,n=1}^\infty \) and \( V^{(i)} = \{v_{m,n}(q)\}_{m,n=1}^\infty \) be sequences of polynomials that determine a quantum addition rule. If \( \alpha_1, \ldots, \alpha_k \) are elements of the coefficient ring such that
\[
\alpha_1 + \cdots + \alpha_k = 1,
\]
and if the sequences \( U = \{u_{m,n}(q)\}_{m,n=1}^\infty \) and \( V = \{v_{m,n}(q)\}_{m,n=1}^\infty \) are defined by
\[
u_{m,n}(q) = \sum_{i=1}^k \alpha_i u_{m,n}^{(i)}(q)
\]
and
\[
v_{m,n}(q) = \sum_{i=1}^k \alpha_i v_{m,n}^{(i)}(q)
\]
for all positive integers \( m \) and \( n \), then the sequences \( U \) and \( V \) determine a quantum addition rule.

Similarly, if \( U = \{u_{m,n}(q)\}_{m,n=1}^\infty \) and \( V = \{v_{m,n}(q)\}_{m,n=1}^\infty \) are sequences of polynomials that determine a quantum addition rule, and if \( S = \{s_{m,n}(q)\}_{m,n=1}^\infty \) and \( T = \{t_{m,n}(q)\}_{m,n=1}^\infty \) are sequences of polynomials that determine a zero identity, then the sequences \( U + S = \{u_{m,n}(q) + s_{m,n}(q)\}_{m,n=1}^\infty \) and \( V + T = \{v_{m,n}(q) + t_{m,n}(q)\}_{m,n=1}^\infty \) determine a quantum addition rule.
3. The fundamental quantum addition rule

In this paper we consider sequences $\mathcal{U}$ and $\mathcal{V}$ that depend only on $m$ or $n$. We shall classify all linear zero identities and all linear quantum addition rules.

**Theorem 2.** Let $\mathcal{S} = \{s_n(q)\}_{n=1}^{\infty}$ and $\mathcal{T} = \{t_m(q)\}_{m=1}^{\infty}$ be sequences of polynomials. Then

\[(10)\quad s_n(q)[m]_q + t_m(q)[n]_q = 0\]

for all positive integers $m$ and $n$ if and only if there exists a polynomial $z(q)$ such that

\[(11)\quad s_n(q) = z(q)[n]_q \quad \text{for all } n \geq 1\]

and

\[(12)\quad t_m(q) = -z(q)[m]_q \quad \text{for all } m \geq 1.\]

If

\[(13)\quad s_m(q)[m]_q + t_m(q)[n]_q = 0\]

for all positive integers $m$ and $n$, or if

\[(14)\quad s_m(q)[m]_q + t_n(q)[n]_q = 0\]

for all positive integers $m$ and $n$, then $s_n(q) = t_n(q) = 0$ for all $n$.

**Proof.** If there exists a polynomial $z(q)$ such that the sequences $\mathcal{S}$ and $\mathcal{T}$ satisfy identities (11) and (12), then

\[s_n(q)[m]_q + t_m(q)[n]_q = z(q)[n]_q[m]_q - z(q)[m]_q[n]_q = 0\]

for all $m$ and $n$.

Conversely, suppose that the sequences $\mathcal{S}$ and $\mathcal{T}$ define a linear zero identity of the form (13). Letting $m = n = 1$ in (10), we have

\[s_1(q) + t_1(q) = s_1(q)[1]_q + t_1(q)[1]_q = 0.\]

Let

\[z(q) = s_1(q) = -t_1(q).\]

For all positive integers $n$ we have

\[s_n(q)[1]_q + t_1(q)[n]_q = s_n(q) - z(q)[n]_q = 0,\]

and so

\[s_n(q) = z(q)[n]_q.\]

Similarly,

\[s_1(q)[m]_q + t_m(q)[1]_q = z(q)[m]_q + t_m(q) = 0,\]

and so

\[t_m(q) = -z(q)[m]_q\]

for all positive integers $m$.

If the sequences $\mathcal{S}$ and $\mathcal{T}$ define a linear zero identity of the form (13), then

\[t_m(q)[n]_q = -s_m(q)[m]_q = t_m(q)[n + 1]_q = t_m(q)([n]_q + q^n),\]

and so

\[t_m(q)q^n = 0.\]

It follows that $t_m(q) = 0$ for all $m$, and so $s_m(q) = 0$ for all $m$. 


Suppose that the sequences $\mathcal{S}$ and $\mathcal{T}$ define a linear zero identity of the form (14). Then

$$s_m(q)[n]_q = -t_n(q)[m]_q$$

for all $m$ and $n$. This implies that if $s_m(q) \neq 0$ for some $m$, then $t_n(q) \neq 0$ for all $n$ and $s_m(q) \neq 0$ for all $m$. If $\mathcal{S}$ and $\mathcal{T}$ are not the zero sequences, then, denoting the degree of a polynomial $f$ by $\deg(f)$, we obtain

$$\deg(s_m) + m - 1 = \deg(t_n) + n - 1 \geq n - 1,$$

and so

$$\deg(s_m) \geq n - m$$

for all positive integers $n$, which is absurd. Therefore, $\mathcal{S}$ and $\mathcal{T}$ are the zero sequences. This completes the proof. \qed

**Theorem 3.** Let $\mathcal{U} = \{u_n(q)\}_{n=1}^{\infty}$ and $\mathcal{V} = \{v_m(q)\}_{m=1}^{\infty}$ be sequences of polynomials. Then

$$[m + n]_q = u_n(q)[m]_q + v_m(q)[n]_q$$

for all positive integers $m$ and $n$ if and only if there exists a polynomial $z(q)$ such that

$$u_n(q) = 1 + z(q)[n]_q$$

and

$$v_m(q) = q^m - z(q)[m]_q$$

for all positive integers $m$ and $n$. Moreover, $z(q) = u_1(q) - 1 = q - v_1(q)$.

**Proof.** Let $z(q)$ be any polynomial, and define the sequences $\mathcal{U} = \{u_n(q)\}_{n=1}^{\infty}$ and $\mathcal{V} = \{v_m(q)\}_{m=1}^{\infty}$ by (16) and (17). Then

$$u_n(q)[m]_q + v_m(q)[n]_q = (1 + z(q)[n]_q)[m]_q + (q^m - z(q)[m]_q)[n]_q = ([m]_q + q^m[n]_q) + (z(q)[n]_q[q^m - z(q)]_q[m]_q) = [m]_q + q^m[n]_q = [m + n]_q.$$

Conversely, let $\mathcal{U} = \{u_n(q)\}_{n=1}^{\infty}$ and $\mathcal{V} = \{v_m(q)\}_{m=1}^{\infty}$ be a solution of (15). We define

$$z(q) = u_1(q) - 1.$$

Since

$$1 + q = [2]_q = [1 + 1]_q = u_1(q) + v_1(q) = 1 + z(q) + v_1(q),$$

it follows that

$$v_1(q) = q - z(q).$$

For all positive integers $m$ we have

$$[m + 1]_q = u_1(q)[m]_q + v_m(q),$$

and so

$$v_m(q) = [m + 1]_q - u_1(q)[m]_q = q^m + [m]_q - u_1(q)[m]_q = q^m - z(q)[m]_q.$$

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Similarly, for all positive integers \( n \) we have
\[
[n + 1]_q = [1 + n]_q = u_n(q) + v_1(q)[n]_q,
\]
and so
\[
u_n(q) = [n + 1]_q - v_1(q)[n]_q
= 1 + q[n]_q - (q - z(q))[n]_q
= 1 + z(q)[n]_q.
\]
This completes the proof. \( \square \)

For example, we can rewrite the quantum addition rule \( \text{(15)} \) in the form
\[
[m + n]_q = (4 - 3q^m)[m]_q + (4q^m - 3)[n]_q
= (1 + z(q)[n]_q)[m]_q + (q^m - z(q)[m]_q)[n]_q,
\]
where
\[
z(q) = 3 - 3q.
\]

**Theorem 4.** Let \( \mathcal{U} = \{u_m(q)\}_{m=1}^{\infty} \) and \( \mathcal{V} = \{v_n(q)\}_{n=1}^{\infty} \) be sequences of polynomials. Then
\[
[m + n]_q = u_m(q)[m]_q + v_n(q)[n]_q
\]
for all positive integers \( m \) and \( n \) if and only if \( u_m(q) = 1 \) and \( v_m(q) = q^m \) for all \( m \). There do not exist sequences of polynomials \( \mathcal{U} = \{u_m(q)\}_{m=1}^{\infty} \) and \( \mathcal{V} = \{v_n(q)\}_{n=1}^{\infty} \) such that
\[
[m + n]_q = u_m(q)[m]_q + v_n(q)[n]_q
\]
for all positive integers \( m \) and \( n \).

**Proof.** Suppose that for every positive integer \( m \) we have
\[
[m + 1]_q = u_m(q)[m]_q + v_m(q)[1]_q = u_m(q)[m]_q + v_m(q),
\]
and
\[
[m + 2]_q = u_m(q)[m]_q + v_m(q)[2]_q = u_m(q)[m]_q + (1 + q)v_m(q).
\]
Subtracting, we obtain
\[
q^{m+1} = [m + 2]_q - [m + 1]_q = q v_m(q),
\]
and so
\[
v_m(q) = q^m.
\]
Then
\[
u_m(q)[m]_q = [m + 1]_q - v_m(q) = [m + 1]_q - q^m = [m]_q,
\]
and so
\[
u_m(q) = 1
\]
for all \( m \). This proves the first assertion of the Theorem.

If \( \text{(19)} \) holds for \( n = 1 \) and all \( m \), then
\[
[m + 1]_q = u_m(q)[m]_q + v_1(q)[1]_q = u_m(q)[m]_q + v_1(q),
\]
and so
\[
u_m(q)[m]_q = [m + 1]_q - v_1(q).
\]
We also have
\[
[m + 2]_q = u_m(q)[m]_q + v_2(q)[2]_q = [m + 1]_q - v_1(q) + (1 + q)v_2(q),
\]
and so
\[ q^{m+1} = [m + 2]_q - [m + 1]_q = (1 + q)v_2(q) - v_1(q) \]
for all positive integers \( m \), which is absurd. \( \square \)

Theorems 3 and 4 show that all linear quantum addition rules are of the form 
\[ [m + n]_q = u_n(q)[m]_q + v_m(q)[n]_q. \]
The following result shows that the sequence of quantum integers is essentially the only solution of the corresponding functional equation.

**Theorem 5.** Let \( \mathcal{U} = \{u_n(q)\}_{n=1}^{\infty} \) and \( \mathcal{V} = \{v_m(q)\}_{m=1}^{\infty} \) be sequences of polynomials such that 
\[ [m + n]_q = u_n(q)[m]_q + v_m(q)[n]_q \]
for all positive integers \( m \) and \( n \). Then \( \mathcal{F} = \{f_n(q)\}_{n=1}^{\infty} \) is a solution of the functional equation
\[ f_{m+n}(q) = u_n(q)f_m(q) + v_m(q)f_n(q) \]
if and only if there is a polynomial \( h(q) \) such that 
\[ f_n(q) = h(q)[n]_q \]
for all \( n \geq 1 \).

**Proof.** By Theorem 3 there exists a polynomial \( z(q) \) such that 
\[ u_n(q) = 1 + z(q)[n]_q \]
and 
\[ v_m(q) = q^m - z(q)[m]_q \]
for all positive integers \( m \) and \( n \). The proof is by induction on \( n \). Let \( h(q) = f_1(q) \). Suppose that \( f_n(q) = h(q)[n]_q \) for some integer \( n \geq 1 \). Then 
\[ f_{n+1}(q) = u_1(q)f_n(q) + v_n(q)f_1(q) = (1 + z(q))h(q)[n]_q + (q^n - z(q)[n]_q)h(q) = h(q)([n]_q + q^n) = h(q)[n + 1]_q. \]
This completes the proof. \( \square \)

**Remark.** The only property of polynomials used in this paper is the degree of a polynomial, which occurs in the proof that there is no nontrivial zero identity of the form (14). It follows that Theorems 3 and 4 hold in any algebra that contains the quantum integers, for example, the polynomials, the rational functions, the formal power series, or the formal Laurent series with coefficients in a ring or field.

### 4. Nonlinear addition rules

A. V. Kontorovich observed that the quantum integers satisfy the following two nonlinear addition rules:
\[ [m + n]_q = [m]_q + [n]_q - (1 - q)[m]_q[n]_q \]
and
\[ [m + n]_q = q^n[m]_q + q^n[n]_q + (1 - q)[m]_q[n]_q. \]
These give rise to the functional equations
\[ f_m(q) \oplus f_n(q) = f_m(q) + f_n(q) - (1 - q)f_m(q)f_n(q) \]
and

\[ f_m(q) \oplus f_n(q) = q^m f_m(q) + q^n f_n(q) + (1 - q) f_m(q) f_n(q), \]

whose solutions are, respectively,

\[
\begin{align*}
  f_n(q) &= \frac{1}{q - 1} \sum_{k=1}^{n} \binom{n}{k} ((q - 1) f_1(q))^k \\
  &= \frac{1 - (1 + (q - 1) f_1(q))^n}{1 - q}.
\end{align*}
\]

and

\[
\begin{align*}
  f_n(q) &= \frac{1}{q - 1} \sum_{k=1}^{n} \binom{n}{k} q^{n-k} ((1 - q) f_1(q))^k \\
  &= \frac{(q + (1 - q) f_1(q))^n - q^n}{1 - q}.
\end{align*}
\]

Kontorovich and Nathanson [4] have recently described all quadratic addition rules for the quantum integers. It would be interesting to classify higher order nonlinear quantum addition rules.

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