Abstract. In this paper, first we introduce dense unital magmas and magma-valued metric spaces. The density property of the ordered unital magmas, monoids, and hemirings helps us to generalize a couple of classical results related to the convergence and Cauchy property of sequences and series. For example, we prove that the Cauchy Condensation Test holds for Cauchy complete fields. We also introduce hemiring-valued pseudonormed rings which are a generalization of pseudonormed rings and valuation domains. Then, we prove that finite-dimensional algebras over hemiring-valued pseudonormed fields can be pseudonormed which is a generalization of a result by Albert.

0. Introduction

The main purpose of this paper is to generalize some classical results in abstract mathematical analysis from the uniqueness of the limit of a sequence and Cauchyness of a convergent sequence to the Cauchy Condensation Test and Albert’s theorem for finite-dimensional real algebras in real analysis. In the current section, first we introduce some terminologies and notations, and then briefly report what we do in the rest of the paper.

Let \( M \) be a nonempty set. If \( \star : M \times M \to M \) is a function, then it is said that \((M, \star)\) is a magma \([27, \text{Definition 1.1}]\). Let \((M, \star)\) be a magma. It is said that a (binary) relation \( R \) on \( M \) is compatible with the magma \((M, \star)\) if \( aRb \) implies \((a \star c)R(b \star c)\) and \((c \star a)R(c \star b)\), \( \forall a, b, c \in M \).

Let us recall that a relation on a set is partial if it is reflexive, antisymmetric, and transitive. A relation \( R \) on a set \( M \) is linear if either \( aRb \) or \( bRa \) for all \( a, b \in M \). In this paper, partial relations, also known as partial orderings, are denoted by \( \leq \). If a partial ordering \( \leq \) on a set \( M \) is linear, it is said that \( \leq \) is a total ordering on \( M \) or \((M, \leq)\) is totally ordered.

If \( \leq \) is a partial ordering on \( M \), then \((M, \star, \leq)\) is said to be an ordered magma if \( \leq \) is compatible with the magma \((M, \star)\). As usual, if \( \leq \) is a relation on \( M \), then by \( a < b \) it is meant that \( a \leq b \) and \( a \neq b \). Note that by the sentence “\((M, \star, <)\) is an ordered magma” \((M, \star)\) is a magma (or any other group-like algebraic structure), we mean that < is compatible with \( \star \). This already means that if \( a < b \), then \( a \star c < b \star c \) and \( c \star a < b \star a \), for all \( a, b, c \in M \).

A magma \((M, \star)\) is a unital magma \([1]\) if there is an element \( 0 \in M \) such that \( m \star 0 = 0 \star m = m, \quad \forall m \in M \).

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In any ordered unital magma \((M,\ast,0,\leq)\), an element \(m \in M\) is non-negative, if \(0 \leq m\). Also, \(m \in M\) is positive, if \(0 < m\), i.e. \(0 \leq m\) and \(m \neq 0\).

Let us recall that \((H,+,\cdot,0,\leq)\) is an ordered hemiring if the following properties are satisfied:

1. \((H,+,0,\leq)\) is a commutative ordered monoid;
2. \((H,\cdot)\) is a semigroup, and \(a \leq b\) and \(0 \leq c\) imply that \(ac \leq bc\) and \(ca \leq cb\), for all \(a,b,c \in H\).
3. The multiplication distributes over addition from both sides and \(0\) is an absorbing element of \(H\), i.e. \(0h = h0 = 0\), for all \(h \in H\).

A hemiring \(H\) is said to be commutative if its multiplication is commutative, i.e. \(ab = ba\) for all \(a\) and \(b\) in \(H\). We say \((H,+,\cdot,0,\leq)\) - for short, \((H,\leq)\) - is an ordered hemiring if \((H,+,\cdot,0,\leq)\) is an ordered hemiring and \(\leq\) is compatible with addition and \(a < b\) and \(0 < c\) imply that \(ac < bc\) and \(ca < cb\) for all \(a,b,c \in H\).

It is evident that if \(\leq\) is a total ordering on \(H\) and \((H,\leq)\) is an ordered hemiring, then \(H\) is entire, i.e. \(ab = 0\) implies that either \(a = 0\) or \(b = 0\), for all \(a,b \in H\).

An algebraic structure \((S,+,\cdot,0,1,\leq)\) is an ordered semiring if \((S,+,\cdot,0,\leq)\) is an ordered hemiring, and \(1 \neq 0\) is an identity element of the semigroup \((S,\cdot)\), i.e. \(1s = s1 = s\), for all \(s \in S\).

In this paper, by “\(R\) is an ordered ring”, we mean that \(R\) is an ordered hemiring and each element of \(R\) has an additive inverse. Note that rings do not have necessarily multiplicative identity, unless explicitly stated. Also, note that if \((H,\leq)\) is an ordered hemiring, then \(P = \{x \in H : 0 < x\}\) is called to be the positive cone of \(H\). For more on hemirings and semirings see [13, 14, 15, 16].

The first section of the paper is devoted to a property called density. We define an ordered unital magma \((M,\ast,0,\leq)\) to be dense if

- for any positive element \(\epsilon\) in \(M\), there are two positive elements \(\beta\) and \(\gamma\) with \(\beta \ast \gamma < \epsilon\).

In order to explain our motivation for the density property, we should consider the following:

Let \((M,\ast,0,\leq)\) be an ordered unital magma. As a generalization to monoid-valued metric spaces introduced and discussed in [6] and [9], we say \((X,d)\) is an \(M\)-metric space (see Definition 2.1) if \(d : X \times X \to M\) is a function such that for all \(x,y,z \in X\), the following properties hold:

1. \(d(x,y) \geq 0\), and \(d(x,y) = 0\) if and only if \(x = y\).
2. \(d(x,y) = d(y,x)\).
3. \(d(x,z) \leq d(x,y) \ast d(y,z)\).

By definition, a sequence \((x_n)\) in an \(M\)-metric space \(X\) is convergent to \(x \in X\), denoted by

\[\lim_{n \to +\infty} x_n = x,\]

if for any positive element \(\epsilon\) in \(M\), there is a natural number \(N\) such that \(n \geq N\) implies \(d(x_n,x) < \epsilon\).

In Proposition 2.1 we prove that if \((M,\ast,0,\leq)\) is a dense unital magma and \(X\) is an \(M\)-metric space, then a convergent sequence has a unique limit.

Not only that but also density property is useful for generalizing a couple of other classical results in abstract analysis. For example, with the help of density, we prove that Cauchyness is implied by convergence as we explain in the following:

Let \(M\) be an ordered unital magma and \(X\) an \(M\)-metric space. It is natural to define that a sequence \((x_n)\) in \(X\) is a Cauchy sequence if for any positive element \(\epsilon\)
in $M$ there is a natural number $N$ such that $m, n \geq N$ implies $d(x_m, x_n) < \epsilon$ (see Definition 2.12). In Theorem 2.14 we prove that if $M$ is a dense unital magma and $X$ an $M$-metric space, then any convergent sequence in $X$ is a Cauchy sequence.

Also, note that in Theorem 2.15 we show that if $M$ is a dense unital magma and $X$ is an $M$-metric space, then any convergent sequence in $X$ is a Cauchy sequence.

Surprisingly, the definition of density property given above is equivalent to the definition of density property for ordered rings given in [17], and also the definition of density in set theory given in [18]. As a matter of fact, we prove that a totally ordered ring with 1 is dense if and only if its additive monoid is dense. Even more, we generalize this for near-rings.

Let us recall that an algebraic structure $(N, +, \cdot, 0)$ is a near-ring (see Definition 1.1 in [24]) if the following conditions are satisfied:

1. $(N, +)$ is a (not necessarily abelian) group and 0 is the identity element of the group $N$.
2. $(N, \cdot)$ is a semigroup.
3. The right-distributive law holds, i.e.
   \[(x + y)z = xz + yz, \quad \forall x, y, z \in N.\]

A near-ring $N$ is with 1 if $(N, \cdot, 1)$ is a monoid.

It is said that $(N, +, 0, \leq)$ is an ordered near-ring if $(N, \leq)$ is a partial ordering on $N$ and $(N, +, 0)$ is a near-ring such that the following properties hold:

1. $(N, +, 0, \leq)$ is an ordered group.
2. If $0 \leq x$ and $0 \leq y$, then $0 \leq xy$, for all $x, y \in N$.

We say $(N, +, \cdot, 0, <)$ - for short, $(N, <)$ - is an ordered near-ring if $(N, \leq)$ is an ordered near-ring and $0 < x$ and $0 < y$ imply $0 < xy$, for all $x, y \in N$.

Note that in Theorem 1.8 we prove that if $\leq$ is a total ordering on $N$ and $(N, <)$ is an ordered near-ring with $1 \neq 0$, then the following statements are equivalent:

1. The near-ring $N$ is dense, i.e. the positive cone $P$ of $N$ has no least element [17].
2. There is a positive element $\alpha$ in $N$ smaller than 1.
3. The ordered monoid $(N, +, 0, \leq)$ is a dense monoid.
4. For any positive element $\epsilon$ in $N$ and a positive integer number $n$, we can find $n$ positive elements $\{\epsilon_i\}_{i=1}^n$ in $N$ satisfying the following inequality:
   \[\sum_{i=1}^n \epsilon_i < \epsilon.\]
5. The ordered set $N$ is dense, i.e. for $r < t$ in $N$ there is an $s$ in $N$ with $r < s < t$ [18, Definition 4.2].

Here is a brief sketch of the contents of the other sections of our paper:

In §3 we introduce magma-valued norms in the following way (see Definition 3.1):

Let $(M, +, 0, \leq)$ be an ordered unital magma. A group $(G, +)$ is, by definition, an $M$-normed group if there is a function $\| \cdot \| : G \to M$ with the following properties:

1. $\|g\| \geq 0$, and $\|g\| = 0$ if and only if $g = 0$, for all $g \in G$.
2. $\|g - h\| \leq \|g\| + \|h\|$, for all $g, h \in G$. 
These norms evidently induce magma-valued metric spaces (check Proposition 3.4). By considering this, in Theorem 3.6 we prove that if \( M \) is a dense unital magma and \( G \) an \( M \)-normed group, then Zero\((G)\) is a subgroup of Conv\((G)\) and 
\[
\text{Conv}(G)/\text{Zero}(G) \cong G,
\]
where Conv\((G)\) is the group of all convergent sequences in \( G \) and Zero\((G)\) is the set of all sequences in Conv\((G)\) convergent to 0 \( \in G \).

In 4 we introduce hemiring-valued pseudonormed rings which include real pseudonormed rings [2] and valuation rings explained in the classical book [3] by Artin. Then, we generalize a result by Albert [1, Theorem 4] proving that if \( \leq \) is a total ordering on \( H \) and \((H, +, \cdot, 0, <)\) is an ordered commutative hemiring, also, if \( F \) is a field and an \( H \)-pseudonormed ring, then every finite-dimensional \( F \)-algebra can be considered as an \( H \)-pseudonormed ring (see Theorem 4.3).

Let \( H \) be an ordered hemiring. The following property is often useful:

- For the given positive elements \( \alpha \) and \( M \) in \( H \), there are positive elements \( \alpha_l \) and \( \alpha_r \) such that
  \[ M \alpha_r < \alpha \text{ and } \alpha_l M < \alpha. \]

We call such hemirings shrinkable (see Definition 4.5). Examples of shrinkable hemirings include dense ordered division semirings (see Proposition 4.6), DeMarr division rings (see Definition 1.12 and Examples 4.7) and the following ring (check Proposition 4.8):
\[
\mathbb{Z}[1/p] = \{m/p^n : m \in \mathbb{Z}, n \in \mathbb{N}\}.
\]

In Theorem 4.10 we prove that if \((S, \leq)\) is a dense and shrinkable ring, and a join-semilattice, also, if \( R \) is an \( S \)-pseudonormed ring, then Conv\((R)\) is a ring and the function \( \varphi: \text{Conv}(R) \to R \) defined by \( (x_n) \mapsto \lim_{n \to +\infty} x_n \) is a ring epimorphism with ker\((\varphi) = \text{Zero}(R)\), where by Zero\((R)\), we mean the set of all sequences in \( R \) convergent to 0. In particular, we show that if \( R \) is a commutative ring with identity, then the ideal Zero\((R)\) is prime (maximal) if and only if \( R \) is an integral domain (a field).

In §5 we introduce hemiring-valued pseudonormed rings which include real pseudonormed rings [2] and valuation rings explained in the classical book [3] by Artin. Then, we generalize a result by Albert [1, Theorem 4] proving that if \( \leq \) is a total ordering on \( H \) and \((H, +, \cdot, 0, <)\) is an ordered commutative hemiring, also, if \( F \) is a field and an \( H \)-pseudonormed ring, then every finite-dimensional \( F \)-algebra can be considered as an \( H \)-pseudonormed ring (see Theorem 4.3).

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In §5 we discuss infinite series in \( M \)-normed groups and generalize some of the classical results of mathematical analysis reported in [23]. For example, in Theorem 5.5 we prove that if \((R, <)\) is a dense and Cauchy complete ordered ring and \((x_n), (y_n), (z_n)\) are sequences in \( R \) such that \( n \geq N_1 \) implies
\[
x_n \leq y_n \leq z_n,
\]
and \( \sum_{n=1}^{\infty} x_n \) and \( \sum_{n=1}^{\infty} z_n \) are convergent, then so is the sequence \( \sum_{n=1}^{\infty} y_n \). We also generalize the Cauchy Condensation Test for Cauchy complete fields. In other words, we show that if \( \leq \) is a total ordering on \( F \) and \((F, <)\) a Cauchy complete ordered field, and \( (x_n) \) is a positive and decreasing sequence in \( F \), then \( \sum_{i=1}^{\infty} x_i \) is convergent in \( F \) if and only if \( \sum_{i=0}^{\infty} 2^i x_{2^i} \) is convergent in \( F \) (see Corollary 5.8).

In §6 we discuss the convergence of geometric series in dense and shrinkable commutative rings. In fact, in Theorem 6.1 we prove that if \( \leq \) is a total ordering on \( R \) and \((R, <)\) an ordered commutative ring with 1 such that \( R \) is a dense and shrinkable ring, then for \( r \neq 1 \), the geometric series \( \sum_{n=0}^{\infty} r^n \) is convergent in \( R \) if and only if the sequence \( (r^n) \) converges to zero and \( 1 - r \) is invertible in \( R \). We use this to prove that with the same properties for \( \leq \) and \((R, <)\), if \((G, +)\) is a
Cauchy complete $R$-normed abelian group and $(x_n)_{n=0}^{\infty}$ is a sequence in $G$ such that $\|x_{n+1}\| \leq r\|x_n\|$ for all $n$, then $\sum_{n=0}^{\infty} x_n$ is convergent in $G$ (see Theorem 6.8). Note that in the same section, we generalize Bernoulli’s inequality for ordered semirings (check Theorem 6.2) and prove that if $(S, \leq)$ is an ordered semiring and $(x_i)_{i=1}^{n}$ is a sequence of $n$ elements in $S$ such that $x_i \geq 0$ and $1 + x_i \geq 0$, for each $1 \leq i \leq n$, then the following inequality holds:

$$\prod_{i=1}^{n} (1 + x_i) \geq 1 + \sum_{i=1}^{n} x_i.$$  

In this paper, a binary operation of a magma $M$ is sometimes denoted by “$+$”, though “$+$” is not necessarily associative or commutative unless explicitly stated. Also, if the binary operation of a monoid and a group is denoted additively, similar to near-ring theory [24], it does not mean that the addition is necessarily commutative unless explicitly stated. Whenever we say an algebraic structure is ordered, we mean that its ordering is partial, otherwise we explicitly assert that the ordering is total (linear). For the general theory of ordered algebraic structures see [11]. For ordered groups consult with [12]. Some of the results in this paper are generalizations of their counterparts in Ovchinnikov’s 2021 book [23].

1. Dense magmas

**Definition 1.1.** Let $(M, *, 0, \leq)$ be a unital magma. We say $M$ is a dense unital magma if for any positive element $\epsilon$ in $M$, there are two positive elements $\beta$ and $\gamma$ with $\beta * \gamma < \epsilon$.

**Remark 1.2.** Let $(M, *, 0, \leq)$ be an ordered group-like algebraic structure such that $0$ is an identity element of $M$. We say $M$ is dense if $(M, *, 0, \leq)$ as an ordered unital magma is dense.

**Examples 1.3.** In the following, we give a couple of examples:

1. Let $K$ be any subfield of the field of real numbers. Evidently, $(K, +, 0, \leq)$ is a dense unital magma because for any positive number $\epsilon \in K$, we have

$$2\epsilon/5 + 2\epsilon/5 < \epsilon.$$

2. The additive group of integer numbers $(\mathbb{Z}, +, 0, \leq)$ is not dense.

3. Let $M$ be the set of all functions of the form $f : \mathbb{R} \to \mathbb{R}$. Let addition be component-wise on $M$ and define $f \leq g$ in $M$, if $f(x) \leq g(x)$, for all $x \in \mathbb{R}$. Then, $M$ is a dense unital magma because for the given $0 < f$, the function $g = 2f/5$, we have $0 < g$ and $g + g < f$.

Let us recall that if $(P, \leq)$ is a poset with no least element, one may annex an element $-\infty$ to $P$ and extend $\leq$ as follows:

$$-\infty < x \quad \forall x \in P.$$  

Then, $-\infty$ is the least element of the new poset $(P \cup \{-\infty\}, \leq)$.

**Proposition 1.4.** Let $(T, \leq)$ be a totally ordered set with no least element. Annex the least element $-\infty$ to $T$. Then, $(T \cup \{-\infty\}, \max)$ is a dense monoid.
Proof. It is evident that \((T \cup \{-\infty\}, \max)\) is a commutative monoid and its neutral element is \(-\infty\). Note that by definition, for any \(a \in T\), we have \(-\infty < a\). It is also easy to see that \(a \leq b\) implies \(\max\{a, c\} \leq \max\{b, c\}\) for all \(a, b, c \in T \cup \{-\infty\}\). Therefore, \(T \cup \{-\infty\}\) is an ordered monoid.

Since \(T\) has no least element, for the given element \(\epsilon \in T\), there is an element \(\beta \in T\) such that \(\beta < \epsilon\). Now, if we set \(\gamma = \beta\), we see that \(\max\{\beta, \gamma\} = \beta < \epsilon\). Hence, \(T\) is a dense monoid, as required. \(\square\)

Remark 1.5. The monoid \((\mathbb{R} \cup \{-\infty\}, \max)\) which is an example of a dense monoid (Proposition 1.4) is extensively used in idempotent analysis. For more, refer to [20].

In the following, we give an example of a dense but non-commutative monoid:

Theorem 1.6. The set \(\mathbb{R} \times \mathbb{R}\) equipped with the following binary operation and the lexicographical ordering is a non-abelian totally ordered dense group:

\[
(r_1, r_2) * (s_1, s_2) = (r_1 + s_1, r_2 e^{s_1} + s_2).
\]

Proof. It is routine to see that \((\mathbb{R} \times \mathbb{R}, *, (0, 0), <)\) is a non-abelian totally ordered group, where \(<\) is the lexicographical ordering on \(\mathbb{R} \times \mathbb{R}\) (see p. 140 in [29]). Let \((r, s)\) be a positive element of \(\mathbb{R} \times \mathbb{R}\), i.e. \((0, 0) < (r, s)\). Then, we have two cases: Either \(0 < r\), or \(0 = r\) and \(0 < s\).

Assume that \(0 < r\). So, there are two positive real numbers \(r_1\) and \(r_2\) with \(r_1 + r_2 < r\). Now, it is obvious that

\[
(0, s_1) < (r, s).
\]

Assume that \(0 = r\) and \(0 < s\). Therefore, there are two positive real numbers \(s_1\) and \(s_2\) such that \(s_1 + s_2 < s\). Observe that

\[
(0, s_1) < (0, s), \quad (0, s_2) < (0, s)
\]

and

\[
(0, s_1) * (0, s_2) = (0, s_1 + s_2) < (0, s).
\]

Hence, \(\mathbb{R} \times \mathbb{R}\) is dense and the proof is complete. \(\square\)

Lemma 1.7. Let \((M, +, 0, \leq)\) be a dense (not necessarily commutative) monoid. Then, for each \(\epsilon > 0\) and \(n \in \mathbb{N}\), there are \(n\) positive elements \(\epsilon_i\) in \(M\) with

\[
\sum_{i=1}^{n} \epsilon_i < \epsilon.
\]

Proof. The proof is by induction on \(n\). By Definition 1.1, the cases \(n = 1\) and \(n = 2\) hold evidently. Now, assume that \(n = k\) holds. Therefore, for the given \(\epsilon > 0\), we have

\[
\sum_{i=1}^{k} \epsilon_i < \epsilon.
\]

Since \(\epsilon_k\) is positive, we can find two positive elements \(\beta_1\) and \(\epsilon_{k+1}\) such that

\[
\beta_1 + \epsilon_{k+1} < \epsilon_k.
\]

Now, observe that

\[
\epsilon_1 + \cdots + \epsilon_{k-1} + \beta_1 + \epsilon_{k+1} \leq \sum_{i=1}^{k} \epsilon_i < \epsilon.
\]

This completes the proof. \(\square\)
Let $\leq$ be a total ordering on $R$ and $(R, <)$ an ordered ring with 1. It is easy to see that the positive cone $P$ of $R$ has the least element if and only if 1 is the least element of $P$ [17]. Heuer [17] calls an ordered ring discrete if its positive cone has the least element; otherwise dense. Note that in order theory, a totally ordered set $(S, \leq)$ is dense if for each $a, b \in S$ with $a < b$ there is an element $c \in S$ such that $a < c < b$ [18, Definition 4.2]. The following result supports the use of the term “dense unital magma” in Definition 1.1:

**Theorem 1.8.** Let $\leq$ be a total ordering on $N$. Also, let $(N, <)$ be an ordered near-ring with $1 \neq 0$. Then, the following statements are equivalent:

1. The near-ring $N$ is dense, i.e. the positive cone $P$ of $N$ has no least element.
2. There is a positive element $\alpha$ in $N$ smaller than 1.
3. The ordered monoid $(N, +, 0, \leq)$ is a dense monoid.
4. For any positive element $\epsilon$ in $N$ and a positive integer number $n$, we can find $n$ positive elements $\{\epsilon_i\}_{i=1}^n$ in $N$ satisfying the following inequality:
   \[ \sum_{i=1}^n \epsilon_i < \epsilon. \]
5. The ordered set $N$ is dense, i.e. for $r < t$ in $N$ there is an $s$ in $N$ with $r < s < t$.

**Proof.**

(1) $\implies$ (2): Since $\leq$ is a total ordering on the near-ring $N$, by Remark 9.134 in [24], 1 is a positive element of $N$. The set of positive elements of $N$ has no least element. Therefore, there must be an element between 0 and 1.

(2) $\implies$ (3): Let there be an $\alpha \in N$ with $0 < \alpha < 1$. Since $(N, <)$ is an ordered near-ring, we have
   \[ 0 < -\alpha + 1. \]
   This implies that $0 < (-\alpha + 1)\alpha$. By the right-distributive law, we have $0 < -\alpha^2 + \alpha$. On the other hand, since $0 < \alpha$, we have $0 < \alpha^2$. Now, let the positive element $\epsilon$ be given and set
   \[ \epsilon_1 = \alpha^2 \epsilon \quad \text{and} \quad \epsilon_2 = (-\alpha^2 + \alpha)\epsilon. \]
   Observe that $\epsilon_1$ and $\epsilon_2$ are positive elements and in view of Proposition 1.5 in [24], we have
   \[ \epsilon_1 + \epsilon_2 = \alpha \epsilon < \epsilon. \]

(3) $\implies$ (4): Lemma [17]

(4) $\implies$ (1): For any positive element of $N$, we can find a smaller positive element in $N$. This means that the positive cone of $N$ has no least element.

Up to now, we have proved that the statements (1), (2), (3), and (4) are equivalent. Now, we show that the statement (5) is also equivalent to any of them:

(5) $\implies$ (2): Since $0 < 1$, we can find an $\alpha \in N$ with $0 < \alpha < 1$.

(4) $\implies$ (5): Let $r < t$. This implies that $0 < -r + t$. So, by assumption, there is a positive element $\epsilon$ in $N$ with
   \[ 0 < \epsilon < -r + t. \]
   Evidently this implies that $r < r + \epsilon < t$ showing that $(N, <)$ is a dense ordered set and the proof is complete. □

**Corollary 1.9.** Let $\leq$ be a total ordering on $N$. Also, let $(N, <)$ be an ordered near-ring with $1 \neq 0$. Assume that $S$ is a sub-near-ring of $N$ such that $1 \in S$. If $S$ is a dense near-ring, then so is the near-ring $N$. 
Proof. Since $S$ is a dense near-ring, by Theorem [1.8] there is an element $\alpha$ in $S$ with $0 < \alpha < 1$. Since $S \subseteq N$, we have $\alpha \in N$. Thus by Theorem [1.8] $N$ is also a dense near-ring and the proof is complete.

Corollary 1.10. Let $(F, <)$ be a totally ordered field. Then, $(F, +, 0, \leq)$ is a dense monoid.

Let $(G, +)$ be a group and 0 be its neutral element. Assume that $M_0(G)$ is the set of all functions $f$ from $G$ into $G$ with $f(0) = 0$. Then, $M_0(G)$ equipped with component-wise addition and composition of functions is a near-ring (see Example 1.4 in [24]). It is clear that the set of all increasing functions $\mathcal{I}_0(\mathbb{R})$ in $M_0(\mathbb{R})$ is a sub-near-ring of $M_0(\mathbb{R})$.

Proposition 1.11. Define $\leq$ on $M_0(\mathbb{R})$ as follows:

$$f \leq g \text{ if } f(x) \leq g(x), \ \forall x \in \mathbb{R}.$$ 

Then, $(\mathcal{I}_0(\mathbb{R}), \leq)$ is a dense ordered near-ring with $0 < I$, where by $I$, we mean the identity function on $\mathbb{R}$.

Proof. It is easy to show that $\leq$ is a partial ordering on $\mathcal{I}_0(\mathbb{R})$ and $f \leq g$ implies that $f + h \leq g + h$, for all $f, g, h$ in $\mathcal{I}_0(\mathbb{R})$. In order to prove that $(\mathcal{I}_0(\mathbb{R}), \leq)$ is an ordered near-ring is to prove that if $f \geq 0$ and $g \geq 0$, then $f \circ g \geq 0$. By assumption, we have $g(x) \geq 0$ for all $x \in \mathbb{R}$. Since $f$ is an increasing function and passes through origin, i.e. $f(0) = 0$, we have

$$(f \circ g)(x) = f(g(x)) \geq f(0) = 0.$$ 

Now, we proceed to prove that $\mathcal{I}_0(\mathbb{R})$ is dense. Observe that if $0 < f$, then there is at least one point $x$ in $\mathbb{R}$ such that $0 < f(x)$. Set $g = f/3$. It is evident that $0 < g$ and

$$g + g = 2f/3 < f.$$ 

This completes the proof.

Definition 1.12. Let $(D, +, \cdot, 0, 1)$ be a division ring and $\leq$ a partial ordering on $D$. We say $(D, \leq)$ is a DeMarr division ring if the following conditions are satisfied:

1. $(D, +, 0, \leq)$ is an ordered monoid.
2. If $0 \leq x$ and $0 \leq y$, then $0 \leq xy$, for all $x, y \in D$.
3. $0 < 1$.
4. If $0 < x$, then $0 < x^{-1}$, for all $x \in D$.

Theorem 1.13. Let $(D, \leq)$ be a DeMarr division ring. Then, $(D, +, 0, \leq)$ is a dense monoid.

Proof. By definition, $0 < 1$. Set $n = n \cdot 1 = \sum_{i=1}^{n} 1$. This implies that $0 < n$ and also, $0 < n^{-1}$, for each positive integer $n$. Evidently, since $D$ is a division ring, we obtain that

$$0 < m \cdot n^{-1} \text{ and } 0 < n^{-1} \cdot m \quad (m, n \in \mathbb{N}).$$ 

In particular, $2 \cdot 5^{-1}$ is positive. On the other hand, since $4 \cdot 1 < 5 \cdot 1$, by multiplying both sides of the inequality by $5^{-1}$, we obtain that

$$2 \cdot 5^{-1} + 2 \cdot 5^{-1} = (2 + 2) \cdot 5^{-1} = 4 \cdot 5^{-1} < 1.$$
Now, assume that \( 0 < x \) in \( D \) is given. Set \( y = z = (2 \cdot 5^{-1})x \) and observe that
\[
0 < y + z = (4 \cdot 5^{-1})x < x.
\]
Hence, \((D, +, 0, \leq)\) is a dense monoid, as required.

In the following, we give an example of a DeMarr field which is not a totally ordered field.

**Example 1.14.** Define \( \leq \) on the field of complex numbers \( \mathbb{C} \) as \( z_1 \leq z_2 \) if \( z_2 - z_1 \) is a non-negative real number. Then, \((\mathbb{C}, \leq)\) is a DeMarr field which is not a totally ordered field [10 Example III].

Let \((R, <)\) be an ordered ring with 1, \( P \) its positive cone, and \( M \) an \( R \)-module. It is said that \( M \) is an ordered \( R \)-module ordered by \( N \) if \( N \) is a nonempty subset of \( M \) satisfying the following properties:
- If \( m_1, m_2 \in N \) then \( m_1 + m_2 \in N \), for all \( m_1, m_2 \in M \).
- \( N \cap -N = \{0\} \), where \( -N = \{-x : x \in N\} \).
- If \( r \in R \setminus -P \) and \( m \in N \), then \( rm \in N \).

**Proposition 1.15.** Let \((R, \leq)\) be an ordered ring, and \( M \) an ordered \( R \)-module ordered by \( N \). Define \( \leq \) on \( M \) by \( m_2 \leq m_1 \) if \( m_1 - m_2 \in N \). Then, the following statements hold:

1. \((M, \leq)\) is a partially ordered set.
2. If \( x \leq y \) then, \( x + z \leq y + z \) for all \( x, y, z \in M \).
3. If \( x \leq y \) and \( 0 \leq r \), then \( rx \leq ry \), for all \( r \in R \) and \( x, y \in M \).

**Proof.** Straightforward. \( \square \)

Let us recall that if \( R \) is an integral (commutative) domain and \( M \) a unital \( R \)-module, then by definition, \( M \) is said to be a torsion-free \( R \)-module if \( rm = 0 \) implies either \( r = 0 \) or \( m = 0 \) for all \( r \in R \) and \( m \in M \) [25 p. 134].

**Theorem 1.16.** Let \( \leq \) be a total ordering on \( R \). Also, let \((R, <)\) be an ordered dense ring and \( M \) an ordered torsion-free \( R \)-module. Then, \((M^{\geq 0}, +, 0, \leq)\) is a dense monoid, where by \( M^{\geq 0} \), we mean the set of non-negative elements of \( M \).

**Proof.** Since \( R \) is a dense ring, by Theorem [10,11] there is an element \( \alpha \in R \) with \( 0 < \alpha < 1 \). It is, then, clear that \( 0 < \alpha^2 < \alpha < 1 \). Take \( m \in M \) with \( 0 < m \). Since \( M \) is ordered and torsion-free, we obtain that
\[
0 < \alpha^2 m < \alpha m < m.
\]
Similarly, since \( 0 < \alpha - \alpha^2 < 1 \), we have \( 0 < (\alpha - \alpha^2)m < m \). Set
\[
m_1 = (\alpha - \alpha^2)m \text{ and } m_2 = \alpha^2 m.
\]
Observe that \( 0 < m_1 < m \) and \( 0 < m_2 < m \) and we have
\[
m_1 + m_2 = (\alpha - \alpha^2)m + \alpha^2 m = \alpha m < m.
\]
Hence, \((M^{\geq 0}, +, 0, \leq)\) is a dense monoid, as required. \( \square \)

**Example 1.17.** Let \( C[0,1] \) be the set of all continuous functions from \([0,1]\) into \( \mathbb{R} \). Set
\[
N = \{ f \in C[0,1] : f(x) \geq 0, \forall x \in [0,1]\}.
\]
It is easy to see that $C[0, 1]$ is an ordered $\mathbb{R}$-vector space ordered by $N$. By Theorem 1.16, $M \geq 0$ is dense.

**Example 1.18.** Let $R$ be a commutative ring with 1. Then, $(\text{Id}(R), +, \cdot, \subseteq)$ is an ordered ring, where the addition and multiplication of ideals are defined as follows:

$$I + J = \{a + b : a \in I, b \in J\}, \text{ and}$$

$$IJ = \left\{ \sum_{i=1}^{n} a_i b_i : a_i \in I, b_i \in J, i \in \mathbb{N} \right\}.$$  

Note that $(\text{Id}(R), +, (0), \subseteq)$ is never a dense monoid because the set of maximal ideals of $R$ is always nonempty and there is no ideal properly between a maximal ideal $m$ of $R$ and the ring $R$.

2. Magma-valued metric spaces

As a generalization of group-valued metric spaces defined in Definition 2.10 in [22] and monoid-valued metric spaces defined in [9], we introduce magma-valued metric spaces as follows:

**Definition 2.1.** Let $(M, \ast, 0, \leq)$ be an ordered unital magma. We say $(X, d)$ is an $M$-metric space if $d : X \times X \to M$ is a function such that for all $x, y, z \in X$, the following properties hold:

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) \ast d(y, z)$.

**Remark 2.2.** In [6], $L$-metric spaces have been discussed, where $L$ is a complete lattice. Note that if $X$ is an $L$-metric space, the triangle inequality is as follows:

$$d(x, y) \leq d(x, z) \lor d(z, y), \quad \forall x, y, z \in X.$$  

We recall that a lattice $L$ is complete if, by definition, any subset $P$ of $L$ has infimum and supremum (see Definition 4.1 in [7]). It is obvious that if $L$ is complete and $0 = \inf(L)$, then $0$ is the least element of $L$ and identity element of $\lor$, so that $(L, \lor, 0, \leq)$ is an ordered monoid.

**Proposition 2.3.** Let $(G, +, \leq)$ be a totally ordered (but not necessarily abelian) group. Define $| \cdot | : G \to G$ by $|x| = \max\{x, -x\}$ and set $d(x, y) = |x - y|$. Then, $(G, d)$ is a $G$-metric space.

**Proof.** The proof is similar to the case of the absolute value function over real numbers (see [28]) and so, omitted. $\square$

**Example 2.4.** An example of a non-abelian group satisfying the condition of Proposition 2.3 is the group explained in Theorem 1.6.

**Proposition 2.5.** Let $(M, +, 0, \leq)$ be an ordered commutative monoid and $(X_i, d_i)$ be an $M$-metric space for each $1 \leq i \leq n$. Define a function

$$d : \prod_{i=1}^{n} X_i \times \prod_{i=1}^{n} X_i \to M$$

by

$$d((x_i)_{i=1}^{n}, (y_i)_{i=1}^{n}) = \sum_{i=1}^{n} d_i(x_i, y_i).$$
Then, \( \prod_{n=1}^{\infty} X_i, d \) is an \( M \)-metric space.

**Proof.** Straightforward. \( \square \)

**Definition 2.6.** A sequence \((x_n)\) in an \( M \)-metric space \( X \) is convergent to \( x \in X \), denoted by

\[
\lim_{n \to +\infty} x_n = x,
\]

if for any positive element \( \epsilon \) in \( M \), there is a natural number \( N \) such that \( n \geq N \) implies \( d(x_n, x) < \epsilon \).

**Proposition 2.7.** Let \((M, *, 0, \leq)\) be a dense unital magma. In an \( M \)-metric space \( X \), if a sequence is convergent in \( X \), then its limit is unique.

**Proof.** Let \((x_n)\) be convergent to \( a \) and \( b \). If \( a \neq b \), then \( d(a, b) > 0 \). Set \( \epsilon = d(a, b) \).

Since \( M \) is dense, we can find two positive elements \( \beta \) and \( \gamma \) such that \( \beta * \gamma < \epsilon \).

For \( \beta \), we find a natural number \( N_1 \) such that if \( n \geq N_1 \), then \( d(x_n, a) < \beta \). For \( \gamma \), we find a natural number \( N_2 \) such that if \( n \geq N_2 \), then \( d(x_n, b) < \gamma \). Now, set \( N = \max\{N_1, N_2\} \) and observe that

\[
\epsilon = d(a, b) \leq d(x_n, a) * d(x_n, b) \leq \beta * \gamma < \epsilon,
\]

a contradiction. Thus the limit of any sequence is unique if it exists and the proof is complete. \( \square \)

**Example 2.8.** Let \((M, *, 0, \leq)\) be an ordered unital magma and \( X \) an \( M \)-metric space and assume that there is a natural number \( N \) such that \( x_n = c \in X \) for all \( n \geq N \). Then, \((x_n)\) is convergent to \( c \). In particular, a constant sequence converges to its constant value.

**Proposition 2.9.** Let \( k \) be a natural number and \((x_n)\) a sequence in an \( M \)-metric space \( X \). Then, \((x_n)\) is convergent if and only if \((x_n+k)\) is convergent.

**Proof.** Straightforward. \( \square \)

**Definition 2.10.** Let \((M, *, 0, \leq)\) be an ordered unital magma and \( X \) an \( M \)-metric space. A sequence \((x_n)\) is bounded if there is an element \( a \in X \) and there is a positive element \( \epsilon \) in \( M \) such that

\[
d(x_n, a) < \epsilon \quad \forall \ n \in \mathbb{N}.
\]

**Proposition 2.11.** Let \((M, *, 0, \leq)\) be an ordered unital magma such that \((M, \leq)\) is a join-semilattice. In an \( M \)-metric space \( X \), a convergent sequence is bounded.

**Proof.** Let \((x_n)\) be convergent to \( a \in X \). Fix a positive element \( \epsilon \) of \( M \). Then, there is a natural number \( N \) such that \( n \geq N \) implies \( d(x_n, a) < \epsilon \). Set

\[
R = \sup\{d(x_1, a), \ldots, d(x_{N-1}, a), \epsilon\}.
\]

It is now clear that \( d(x_i, a) \leq R \) for each \( i \in \mathbb{N} \) showing that \((x_n)\) is bounded and the proof is complete. \( \square \)

**Definition 2.12.** Let \((M, *, 0, \leq)\) be an ordered unital magma and \( X \) an \( M \)-metric space. A sequence \((x_n)\) in \( X \) is a Cauchy sequence if for any positive element \( \epsilon \) in \( M \) there is a natural number \( N \) such that \( m, n \geq N \) implies \( d(x_m, x_n) < \epsilon \).

**Proposition 2.13.** Let \((M, *, 0, \leq)\) be an ordered unital magma such that \((M, \leq)\) is a join-semilattice. In an \( M \)-metric space \( X \), a Cauchy sequence is bounded.
Then, we see that we can find two positive elements ≥m such that if m ≥ N. Then, the following statements hold:

**Theorem 2.14.** Let (M, *, 0, ≤) be a dense unital magma and X an M-metric space. Then, any convergent sequence in X is a Cauchy sequence.

**Proof.** Let (x_n) be convergent to a. Then, for each positive element ϵ ∈ M, there is a natural number N such that n ≥ N implies d(x_n, a) < ϵ. For the positive element ϵ, one can find two positive elements β and γ such that β * γ < ϵ. On the other hand, for β and γ, one can find natural numbers N_1 and N_2, respectively, such that if m ≥ N_1 and n ≥ N_2, then

\[ d(x_m, a) < β \text{ and } d(x_n, a) < γ. \]

Now, observe that

\[ d(x_m, x_n) ≤ d(x_m, a) * d(x_n, a) ≤ β * γ < ϵ \]

whenever n ≥ max\{N_1, N_2\}. This shows that (x_n) is a Cauchy sequence and the proof is complete. □

**Theorem 2.15.** Let (M, *, 0, ≤) be a dense unital magma. In an M-metric space X, a Cauchy sequence having a convergent subsequence is convergent.

**Proof.** Let (x_n) be a Cauchy sequence in X. Also, assume that a subsequence (x_{n_k}) of (x_n) is convergent to x ∈ X. Let ϵ be a positive element in M. Since M is dense, we can find two positive elements β and γ with β * γ < ϵ. For β, there is a natural number N_1 such that for m, n ≥ N_1, we have d(x_m, x_n) < β. Also, for γ, there is a natural number k_1 such that k ≥ k_1 implies that d(x_{n_k}, x) < γ. Note that since n_k is a strictly increasing sequence of positive integers, we have n_k ≥ k for each positive integer k. Set N = max\{N_1, k_1\} and observe that for any m ≥ N and k ≥ N, we have

\[ d(x_m, x) ≤ d(x_m, x_{n_k}) * d(x_{n_k}, x) ≤ β * γ < ϵ. \]

Hence, (x_n) is convergent, as required. □

3. **M-normed groups**

**Definition 3.1.** Let (M, +, 0, ≤) be an ordered unital magma. A (not necessarily abelian) group (G, +) is an M-normed group if there is a function \| \cdot \| : G → M with the following properties:

1. \|g\| ≥ 0, and \|g\| = 0 if and only if g = 0, for all g ∈ G,
2. \|g - h\| ≤ \|g\| + \|h\|, for all g, h ∈ G.

**Proposition 3.2.** Let M be an ordered unital magma and G an M-normed group. Then, the following statements hold:

1. \|-g\| = \|g\|, for all g ∈ G,
2. \|g\| ≤ \|g - h\| + \|h\|, for all g, h ∈ G.
Proof. (1): For any \( g \in G \), observe that
\[
\| -g \| = \| 0 - g \| \leq \| 0 \| + \| g \| = 0 + \| g \| = \| g \|.
\]
On the other hand, from this we obtain that
\[
\| g \| = \| - ( -g ) \| = \| - ( -g ) \| \leq \| -g \|.
\]
This proves the first statement.

(2): Let \( g \) and \( h \) be elements of \( G \). In view of the first statement, observe that
\[
\| g \| = \| g - 0 \| = \| ( g - h ) - ( -h ) \| \leq \| g - h \| + \| h \|.
\]
This finishes the proof. \( \square \)

Corollary 3.3. Let \( M \) be an ordered group and \( G \) an \( M \)-normed group. Then,
\[
\| g \| - \| h \| \leq \| g - h \|, \quad \forall \, g, h \in G.
\]
Proof. Observe that \( \| g \| \leq \| g - h \| + \| h \| \). Since \( M \) is a group, by adding \( -\| h \| \) to the both sides of the latter inequality (from the right side), the desired inequality is obtained. \( \square \)

Proposition 3.4. Let \((M, +, 0, \leq)\) be an ordered unital magma and \((G, +)\) an \( M \)-normed group. Then, \((G, d)\) is an \( M \)-metric space, where
\[
d(g, h) = \| g - h \|.
\]
Proof. Let \( g, h, \) and \( k \) be elements of the group \( G \). Observe that
\[
d(g, k) = \| g - k \| = \| (g - h) + (h - k) \| \leq \| g - h \| + \| h - k \|.
\]
The other properties are straightforward. Hence, \((G, d)\) is an \( M \)-metric space, as required. \( \square \)

Proposition 3.5. Let \( M \) be an ordered monoid, \((M, \leq)\) a join-semilattice, and \((x_n)\) a sequence in an \( M \)-normed group \( G \). Then, the following statements hold:

1. If \((x_n)\) is convergent to \( 0 \in G \), then there is a positive element \( s \in M \) such that \( \| x_n \| \leq s \), for all \( n \in \mathbb{N} \).
2. If \( M \) is a group and \((x_n)\) is convergent to \( a \in G \), then there is a positive element \( t \) in \( M \) such that \( \| x_n \| \leq t \).

Proof. (1): Since \((x_n)\) converges to 0, by Proposition 2.11, there is a positive element \( s \in M \) such that
\[
\| x_n \| = \| x_n - 0 \| \leq s, \quad \forall \, n \in \mathbb{N}.
\]
(2): Let \((x_n)\) be a sequence in \( G \) convergent to \( a \in G \). Then, by Proposition 2.11, there is a positive element \( s \in M \) such that
\[
\| x_n - a \| \leq s, \quad \forall \, n \in \mathbb{N}.
\]
Since \( M \) is a group, by Corollary 3.3 we have
\[
\| x_n \| - \| a \| \leq \| x_n - a \| \leq s.
\]
Set \( t = s + \| a \| \). Then, we have \( \| x_n \| \leq t \). This finishes the proof. \( \square \)

Let \((G, +)\) be an abelian group and \( \text{Seq}(G) \) the set of all sequences over \( G \). It is clear that \( \text{Seq}(G) \) with component-wise addition is an abelian group.

Theorem 3.6. Let \((M, \ast, 0, \leq)\) be a dense unital magma and \( G \) an \( M \)-normed abelian group. Then, the following statements hold:
(1) The set of all Cauchy sequences $\text{Cauchy}(G)$ of $G$ is a subgroup of $\text{Seq}(G)$.
(2) If $(x_n)$ and $(y_n)$ are sequences in $G$ convergent to $a$ and $b$ in $G$, respectively, then the sequence $(x_n + y_n)$ is convergent to $a + b$.
(3) The set of convergent sequences, denoted by $\text{Conv}(G)$, is a subgroup of $\text{Cauchy}(G)$.
(4) If $\text{Zero}(G)$ is the set of all sequences in $\text{Conv}(G)$ which are convergent to $0 \in G$, then $\text{Conv}(G)/\text{Zero}(G) \cong G$.

Proof. (1): Let $(x_n)$ and $(y_n)$ be Cauchy sequences. Since $M$ is dense, for the given positive $\epsilon$ in $M$, we can find two positive elements $\beta$ and $\gamma$ such that $\beta \ast \gamma < \epsilon$.
Also, we can find a natural number $N$ such that $m, n \geq N$ implies the following:
$$||x_m - x_n|| < \beta \text{ and } ||y_m - y_n|| < \gamma.$$ 
Now, observe that if $m, n \geq N$, we have
$$||x_m + y_m - (x_n + y_n)|| \leq ||x_m - x_n|| \ast ||y_m - y_n|| \leq \beta \ast \gamma < \epsilon.$$ 
This shows that $(x_n + y_n)$ is a Cauchy sequence. It is obvious that the constant sequence $(0)$, where $0 \in G$ is the identity element of $G$, is Cauchy, and if $(x_n)$ is Cauchy, then so is $(-x_n)$. Thus $\text{Cauchy}(G)$ is a subgroup of $\text{Seq}(G)$.

(2): For the given positive element $\epsilon \in M$, we can find positive elements $\beta$ and $\gamma$ such that $\beta \ast \gamma < \epsilon$. It is evident that one can find a natural number $N$ such that if $n \geq N$, then
$$||x_n - a|| < \beta \text{ and } ||y_n - b|| < \gamma.$$ 
Now, observe that
$$||(x_n + y_n) - (a + b)|| \leq ||x_n - a|| \ast ||y_n - b|| \leq \beta \ast \gamma < \epsilon.$$ 
This shows that $(x_n + y_n)$ is convergent to $a + b$.

(3): By Theorem 2.14, each convergent sequence is a Cauchy sequence. So, $\text{Conv}(G)$ is a subset of $\text{Cauchy}(G)$. By [2], $\text{Conv}(G)$ is closed under addition of sequences. On the other hand, the constant sequence $(0)$ is in $\text{Conv}(G)$, and, if $(x_n)$ is convergent to $a \in G$, then $(-x_n)$ is convergent to $-a$. Consequently, $\text{Conv}(G)$ is a subgroup of $\text{Cauchy}(G)$.

(4): Define $\varphi : \text{Conv}(G) \rightarrow G$ by $(x_n) \mapsto \lim_{n \rightarrow +\infty} x_n$. By [2], $\varphi$ is a group homomorphism. For any $g \in G$, set $g_n = g$. Then, by Example 2.8 $\varphi(g_n) = g$ showing that $\varphi$ is an epimorphism. The kernel of $\varphi$ is the set of all sequences that are convergent to $0 \in G$. Hence, by the fundamental theorem of homomorphisms, we have
$$\text{Conv}(G)/\text{Zero}(G) \cong G,$$
and the proof is complete. \hfill \Box

Let $G$ be an ordered group. We say that $G$ is a dense group if the monoid $(G, +, 0, \leq)$ is dense.

**Theorem 3.7.** Let $(M, +, 0, \leq)$ be a dense and a totally ordered group and $(G, +)$ an $M$-normed group. Let $(x_n)$ be a Cauchy sequence not convergent to zero. Then, there is a positive element $\gamma$ in $M$ and a natural number $N$ such that $n \geq N$ implies $||x_n|| > \gamma$. 

**Proof.** Let $(x_n)$ be a Cauchy sequence not convergent to zero. Then, by Example 2.8, $\varphi(g_n) = g$ showing that $\varphi$ is an epimorphism. The kernel of $\varphi$ is the set of all sequences that are convergent to $0 \in G$. Hence, by the fundamental theorem of homomorphisms, we have
$$\text{Conv}(G)/\text{Zero}(G) \cong G,$$
and the proof is complete. \hfill \Box
Proof. Since \((M, +, 0, \leq)\) is a totally ordered group and \((x_n)\) is not convergent to zero, there is a positive element \(\epsilon\) in \(M\) such that for each \(n \in \mathbb{N}\), there is a natural number \(k\) such that
\[
k \geq n \quad \text{and} \quad \|x_k\| \geq \epsilon. \tag{3.1}
\]
By assumption \(M\) is dense. Therefore, by Lemma 1.7, we can find a positive element \(\beta\) in \(M\) with \(\beta < \epsilon\). Since \((x_n)\) is a Cauchy sequence, we can find a natural number \(N\) such that
\[
\|x_p - x_q\| < \beta \quad \forall \ p, q \geq N. \tag{3.2}
\]
Assume that \(n\) is an arbitrary positive integer with \(n \geq N\). By (3.1) and (3.2), there is a natural number \(k\) such that
\[
\|x_k\| \geq \epsilon \quad \text{and} \quad - \|x_k - x_n\| > -\beta.
\]
Now, observe that for all \(n \geq N\), we have
\[
\|x_n\| \geq -\|x_k - x_n\| + \|x_k\| > -\beta + \epsilon > 0.
\]
Let \(\gamma\) be a positive element of \(M\) smaller than \(\epsilon - \beta\). Hence, there is a natural number \(N\) such that \(n \geq N\) implies \(\|x_n\| > \gamma\), as required. \(\square\)

4. \(H\)-pseudonormed rings

Definition 4.1. Let \(R\) be a ring and \(H\) an ordered hemiring. We say \(R\) is an \(H\)-pseudonormed ring if there is a function \(\| \cdot \| : R \to H\) satisfying the following properties:

(1) \(\|r\| \geq 0, \|r\| = 0\) if and only if \(r = 0\), for all \(r \in R\);
(2) \(\|r - s\| \leq \|r\| + \|s\|\) and \(\|rs\| \leq \|r\|\|s\|\) for all \(r, s \in R\).

In such a case, the function \(\| \cdot \| : R \to H\) is called an \(H\)-pseudonorm. And if in addition, we have \(\|rs\| = \|r\|\|s\|\) for all \(r, s \in R\), we say \(R\) is an \(H\)-normed ring and the function \(\| \cdot \|\) is its \(H\)-norm.

Examples 4.2. In the following, we give a couple of examples for hemiring-valued pseudonormed rings:

(1) Pseudonormed (normed) rings in \([2]\), are \(\mathbb{R}\)-pseudonormed (normed) rings in the sense of Definition 4.1.
(2) Let \(R\) be a ring and \((G, \cdot, 1, <)\) a totally ordered group. Assume that \(0 \notin G\) and set \(G_0 = G \cup \{0\}\) and extend multiplication and inequality for \(G_0\) with the following rules:
(a) \(0 \cdot g = g \cdot 0 = 0\), for all \(g \in G\);
(b) \(0 < g\), for all \(g \in G\).

It is said that \(R\) is a \(G_0\)-valuation ring \([3], \S 10]\) if there is a function \(\| \cdot \| : R \to G \cup \{0\}\) with the following properties:

- \(\|r\| = 0\) if and only if \(r = 0\), for all \(r \in R\).
- \(\|rs\| = \|r\|\|s\|\), for all \(r, s \in R\).
- \(\|r + s\| \leq \max\{\|r\|, \|s\|\}\), for all \(r, s \in R\).

Note that \((G_0, \max, \cdot)\) is a (division) semiring (see Example 4.27 in \([15]\)) and evidently, \(R\) is a \(G_0\)-normed ring.
(3) Let \(\leq\) be a total ordering on \(R\) and \((R, <)\) an ordered ring. Set
\[
|x| = \max\{x, -x\}, \quad \forall x \in R.
\]
It is evident that $R$ is an $R$-normed ring. Note that if we define 
\[d(x, y) = |x - y|, \quad \forall x, y \in R,\]
then $(R, d)$ is an $R$-metric space.

Albert in Theorem 4 in [1] proves that every finite-dimensional real algebra is a normed algebra (see also Proposition 1.1.7 in [8]). Inspired by his proof, we prove the following:

**Theorem 4.3.** Let $\leq$ be a total ordering on $H$ and $(H, +, \cdot, 0, <)$ be an ordered commutative hemiring. Also, let $F$ be a field and an $H$-pseudonormed ring. Then, every finite-dimensional $F$-algebra can be considered as an $H$-pseudonormed ring.

**Proof.** Let $| \cdot | : F \to H$ be the $H$-pseudonorm of $F$. Let $R$ be a finite-dimensional $F$-algebra and $(e_i)_{i=1}^n$ be a basis for the $F$-vector space $R$. For any $a = \sum_{i=1}^n a_i e_i$, where $a_i \in F$, define $\| \cdot \| : R \to H$ as follows:

\[\|a\| = \sum_{i=1}^n |a_i| .\]

It is obvious that $\| \cdot \|$ satisfies the following properties:

- $\|a\| \geq 0$, $\|a\| = 0$ if and only if $a = 0$, for all $a \in R$;
- $\|a - b\| \leq \|a\| + \|b\|$ for all $a, b \in R$.

On the other hand, it is evident that 
\[e_i e_j = \sum_{k=1}^n \gamma_{ijk} e_k ,\]
for some $\gamma_{ijk} \in F$. Set $M = \max \{ |\gamma_{ijk}| \}_{i, j, k \in \mathbb{N}_n}$. Now, assume that 
\[a = \sum_{i=1}^n a_i e_i \text{ and } b = \sum_{j=1}^n b_j e_j ,\]
where $a_i$s and $b_j$s are elements of $F$. Observe that 
\[\|ab\| = \left\| \sum_{1 \leq i, j \leq n} a_i b_j e_i e_j \right\| = \left\| \sum_{1 \leq i, j \leq n} a_i b_j \sum_{k=1}^n \gamma_{ijk} e_k \right\| .\]

\[= \sum_{k=1}^n \left( \sum_{1 \leq i, j \leq n} a_i b_j \gamma_{ijk} \right) e_k \right\| \leq \sum_{k=1}^n \left( \sum_{1 \leq i, j \leq n} a_i b_j \gamma_{ijk} \right) \left( \sum_{k=1}^n \gamma_{ijk} \right) \leq nM \sum_{1 \leq i, j \leq n} |a_i||b_j| = nM \left( \sum_{i=1}^n |a_i| \right) \left( \sum_{j=1}^n |b_j| \right) .\]

This shows that $\|ab\| \leq nM \|a\| \|b\|$. Since $H$ is commutative, we have 
\[nM \|ab\| \leq \left( nM \|a\| \right) \left( nM \|b\| \right) .\]

Therefore, if we define a new function $\| \cdot \|' : A \to H$ by 
\[\|a\|' = nM \|a\| ,\]
then $\| \cdot \|'$ provides a suitable $H$-pseudonorm for $R$ and this completes the proof. \(\Box\)
Corollary 4.4. Let \( \leq \) be a total ordering on \( F \) and \((F, <)\) be an ordered field. Any finite dimensional \( F \)-algebra can be considered as an \( F \)-pseudonormed ring.

Definition 4.5. Let \( H \) be an ordered hemiring. We say \( H \) is shrinkable if for the given positive elements \( \alpha \) and \( M \) in \( H \), there are positive elements \( \alpha_l \) and \( \alpha_r \) such that

\[
M \alpha_r < \alpha \text{ and } \alpha_l M < \alpha.
\]

Let us recall that a semiring \( S \) is a division semiring if any nonzero element of \( S \) is multiplicatively invertible.

Proposition 4.6. Let \( D \) be an ordered division semiring. If \( D \) is a dense semiring, then it is shrinkable.

Proof. Let \( D \) be an ordered and a dense division semiring. Let a positive element \( \alpha \) in \( D \) be given. By Lemma 1.7, there is a positive element \( \beta \in D \) with \( \beta < \alpha \).

Now, let \( M \) be an arbitrary positive element in \( D \). Since \( D \) is a division semiring and \( M \) is nonzero, it is invertible. Assume that \( M^{-1}M = MM^{-1} = 1 \) and set \( \alpha_r = M^{-1}\beta \) and \( \alpha_l = \beta M^{-1} \). Clearly,

\[
M \alpha_r = \alpha_l M = \beta < \alpha.
\]

This completes the proof. \( \square \)

Examples 4.7. In the following, we give a couple of examples for dense ordered division semirings and so, they are shrinkable.

(1) Let \((G, \cdot)\) be a totally ordered group. The semiring \((G_0, \max, \cdot)\) discussed in Examples 4.2 is shrinkable.

(2) By Theorem 1.13, any DeMarr division ring is dense and so, shrinkable.

(3) Let \( F \) be an ordered field (for example, \( F = \mathbb{Z}(X) \) which is the ordered field of rational functions over \( \mathbb{Z} \), Example 1.2). It is clear that the non-negative elements \( F \geq 0 \) of \( F \) is a totally ordered semifield. Note that for any \( \epsilon > 0 \), we can find \( \beta = \gamma = 2\epsilon/5 \) satisfying the following property:

\[
\beta + \gamma = 4\epsilon/5 < \epsilon.
\]

This shows that the monoid of non-negative elements of \( F \), denoted by \( F_{\geq 0} \), is dense.

Proposition 4.8. Let \( p \) be a prime number and \( \mathbb{Z}[1/p] \) the ring of all rational numbers of the form \( m/p^n \), where \( m \in \mathbb{Z} \) and \( n \in \mathbb{N} \). Then, \( \mathbb{Z}[1/p] \) is dense and shrinkable.

Proof. Since \( 1/p \) which is smaller than 1 is an element of \( R \), the ring \( R \) is dense. Now, let \( \alpha = m_1/p^{n_1} \) and \( M = m_2/p^{n_2} \) be given. It is obvious that for a suitable \( n \in \mathbb{N} \), \( \alpha_l = 1/p^n \) which is an element of \( R \) satisfies the inequality \( M \alpha_l < \alpha \). Since \( R \) is commutative, \( R \) is shrinkable and the proof is complete. \( \square \)

Lemma 4.9. Let \( H \) be an ordered hemiring and \( R \) an \( H \)-pseudonormed ring. Also, assume that \( H \) is dense and shrinkable. If \( (x_n) \) is a sequence in \( R \) convergent to 0 and \( (y_n) \) is a sequence in \( R \) such that there is a positive element \( M \) in \( H \) with \( \|y_n\| \leq M \) for all \( n \in \mathbb{N} \), then the sequences \( (x_n y_n) \) and \( (y_n x_n) \) are both convergent to 0 in \( H \).
Proof. By assumption, there is a positive element $M$ in $H$ such that $\|y_n\| \leq M$. Let a positive element $\epsilon$ be given in $H$. Since $H$ is shrinkable, there are positive elements $\epsilon_l$ and $\epsilon_r$ such that

$$M\epsilon_r < \epsilon \text{ and } \epsilon_l M < \epsilon.$$ 

Since $(x_n)$ is convergent to $0$, for $\epsilon_r > 0$, there is a natural number $N$ such that $n \geq N$ implies $\|x_n\| < \epsilon_r$. Since $\|\cdot\|$ is an $H$-pseudonorm on $R$, we see that if $n \geq N$, we have

$$\|x_ny_n\| \leq \|x_n\||y_n\| \leq M\epsilon_r < \epsilon,$$

showing that $(x_ny_n)$ converges to $0 \in H$. In a similar way, it is proved that $(y_nx_n)$ converges to $0 \in H$ and the proof is complete. \qed

**Theorem 4.10.** Let $(S, \leq)$ be an ordered ring and $R$ an $S$-pseudonormed ring. Assume that $S$ is dense and shrinkable such that $(S, \leq)$ is a join-semilattice.

Then, the following statements hold:

1. If $(x_n)$ and $(y_n)$ are sequences in $R$ convergent to $a$ and $b$ in $R$, respectively, then $(x_ny_n)$ is convergent to $ab$.
2. The set $\text{Conv}(R)$ is a ring and the function

$$\varphi : \text{Conv}(R) \rightarrow R \text{ defined by } (x_n) \mapsto \lim_{n \rightarrow +\infty} x_n$$

is a ring epimorphism with $\ker(\varphi) = \text{Zero}(R)$, where by $\text{Zero}(R)$, we mean the set of all sequences in $R$ convergent to $0$.
3. Let $R$ be a commutative ring with $1$. The ideal $\text{Zero}(R)$ is prime (maximal)

if and only if $R$ is an integral domain (a field).

Proof. (1): Let for the moment $\lim_{n \rightarrow +\infty} y_n = b \neq 0$. Let a positive $\epsilon$ in $S$ be given. Since $S$ is dense, we can find two positive elements $\beta$ and $\gamma$ such that $\beta + \gamma < \epsilon$.

Since $(x_n)$ is a convergent sequence, by Proposition 3.5, we can find a positive element $s_1 \in S$ such that

$$\|x_n\| \leq s_1 \quad \forall \ n \in \mathbb{N}.$$ 

Since $S$ is shrinkable, for $\beta > 0$ and $s_1 > 0$, and also for $\gamma > 0$ and $\|b\| > 0$, we can find positive elements $M_l$ and $K_r$ in $S$ such that

$$s_1K_r < \beta \text{ and } M_l\|b\| < \gamma.$$ 

Also, for $K_r > 0$, we can find a natural number $N_1$ such that $n \geq N_1$ implies $\|y_n - b\| < K_r$. Similarly, we can find a natural number $N_2$ such that $n \geq N_2$ implies $\|x_n - a\| < M_l$. Now, set $N = \max\{N_1, N_2\}$ and assume that $n \geq N$.

Observe that

$$\|x_ny_n - ab\| = \|x_ny_n - x_nb + x_nb - ab\|$$

$$\leq \|x_n(y_n - b)\| + \|(x_n - a)b\|$$

$$\leq \|x_n\| \cdot \|y_n - b\| + \|x_n - a\| \cdot \|b\|$$

$$\leq s_1K_r + M_l\|b\| \leq \beta + \gamma < \epsilon.$$ 

This proves that $(x_ny_n)$ is convergent to $ab$. On the other hand, if $\lim_{n \rightarrow +\infty} y_n = 0$, then by Lemma 4.4, $(x_ny_n)$ is convergent to $ab = 0$. In a similar way, one can prove that $(y_nx_n)$ is convergent to $ba$.

(2): In view of Theorem 3.6 and (1), $\varphi$ is a ring epimorphism and its kernel is $\text{Zero}(R)$. 


Then, the following statements hold:

Thus by Proposition 2.9, \((x_n)\) is convergent to zero.

**Corollary 4.11.** Let \(\leq\) be a total ordering on \(R\) and \((R, <)\) be an ordered dense commutative ring with identity. If \(R\) is shrinkable, then \(\text{Conv}(R)/\text{Zero}(R)\) is a ring and \(\text{Zero}(R)\) is a prime ideal with \(\text{Conv}(R)/\text{Zero}(R) \cong R\).

5. **INFINITE SERIES IN \(M\)-NORMED GROUPS**

**Proposition 5.1.** Let \(M\) be a unital magma and \(G\) an \(M\)-normed abelian group. Then, the following statements hold:

1. If \(\sum_{n=1}^{+\infty} x_n\) is convergent, then \((x_n)\) converges to zero.
2. If \(\sum_{n=1}^{+\infty} x_n\) is convergent, then for any positive \(\epsilon\) in \(M\), there is a natural number \(N\) such that \(n \geq m \geq N\) implies \(\sum_{i=m+1}^{n} x_i < \epsilon\).
3. If \(\sum_{n=1}^{+\infty} x_n\) and \(\sum_{n=1}^{+\infty} y_n\) are convergent series in \(G\), then

\[
\sum_{n=1}^{+\infty} (x_n + y_n) = \sum_{n=1}^{+\infty} x_n + \sum_{n=1}^{+\infty} y_n.
\]

**Proof.** (1): Set \(s_n = x_1 + x_2 + \cdots + x_n\). If \((s_n)\) is convergent to \(s \in G\), then its subsequence \((s_{n+1})\) is also convergent to \(s\). Since \(G\) is abelian, we have \(x_{n+1} = s_{n+1} - s_n\). So, we see that by Theorem 3.6 we have

\[
\lim_{n \to +\infty} x_{n+1} = \lim_{n \to +\infty} s_{n+1} - \lim_{n \to +\infty} s_n = s - s = 0.
\]

Thus by Proposition 2.9 \((x_n)\) is convergent to zero.

(2): Since \(M\) is dense, by Theorem 3.6 any convergent sequence is a Cauchy sequence. Therefore, for the given \(\epsilon > 0\), there is a natural number \(N\) such that \(n \geq m \geq N\) implies

\[
\left\| \sum_{i=m+1}^{n} x_i \right\| = \|s_n - s_m\| < \epsilon.
\]

(3): It evidently holds by Theorem 3.6

**Theorem 5.2.** Let \(\leq\) be a total ordering on \(R\) and \((R, <)\) an ordered ring and \((x_n)\) be a strictly decreasing and positive sequence convergent to zero. Then, \(s_n = \sum_{i=1}^{n} (-1)^{i+1} x_i\) is a Cauchy sequence.

**Proof.** Let \(n > m\) and observe that

\[
|s_n - s_m| = \left| \sum_{i=1}^{n} (-1)^{i+1} x_i - \sum_{i=1}^{m} (-1)^{i+1} x_i \right|
= \left| x_{m+1} - x_{m+2} + \cdots + (-1)^{n-m+1} x_n \right|
\]

Since \((x_n)\) is a strictly decreasing and positive sequence, we see that

\[
|s_n - s_m| = \begin{cases} 
  x_{m+1} - \sum_{i=m+2}^{n} (x_i - x_{i+1}) \leq x_{m+1} < x_m, & \text{if } n - m + 1 \text{ is even}, \\
  x_{m+1} - \sum_{i=m+2}^{n} (x_i - x_{i+1}) - x_n \leq x_{m+1} < x_m, & \text{if } n - m + 1 \text{ is odd}.
\end{cases}
\]

Now, since \(\lim_{n \to +\infty} x_n = 0\), \((s_n)\) is a Cauchy sequence and the proof is complete. \(\square\)
Definition 5.3. Let \( \leq \) be a total ordering and \((R, \leq)\) an ordered ring. We define \((R, \leq)\) to be Cauchy complete if each Cauchy sequence in \(R\) is convergent to an element in \(R\).

Corollary 5.4. Let \( \leq \) be a total ordering and \((R, \leq)\) an ordered ring. If \((R, \leq)\) is a Cauchy complete ring and \((x_n)\) is a strictly decreasing and positive sequence convergent to zero, then
\[
s_n = \sum_{i=1}^{n} (-1)^{i+1} x_i
\]
is a convergent sequence.

Theorem 5.5. Let \( \leq \) be a total ordering on \(R\) and \((R, \leq)\) a dense and Cauchy complete ordered ring. Assume that \((x_n), (y_n),\) and \((z_n)\) are sequences in \(R\) and there is a natural number \(N_1\) such that \(n \geq N_1\) implies
\[
x_n \leq y_n \leq z_n.
\]
If \(\sum_{n=1}^{+\infty} x_n\) and \(\sum_{n=1}^{+\infty} y_n\) are convergent, then so is \(\sum_{n=1}^{+\infty} z_n\).

Proof. Since \(\sum_{n=1}^{+\infty} x_n\) and \(\sum_{n=1}^{+\infty} z_n\) are convergent, by Theorem 2.14, they are Cauchy sequences. Therefore, for the given \(\epsilon > 0\), we can find a natural number \(N_2\) such that \(m, n \geq N_2\) implies
\[
\left| \sum_{k=m+1}^{n} x_k \right| < \epsilon \quad \text{and} \quad \left| \sum_{k=m+1}^{n} z_k \right| < \epsilon.
\]
Let \(N = \max\{N_1, N_2\}\). Observe that for \(n \geq N\), we have \(x_n \leq y_n \leq z_n\) which implies that
\[
\left| \sum_{k=m+1}^{n} y_k \right| < \epsilon.
\]
Therefore, the sequence \((\sum_{k=m+1}^{n} y_k)_{n \in \mathbb{N}}\) is Cauchy and so, convergent since \(R\) is Cauchy complete. This completes the proof. \(\square\)

Remark 5.6. The question arises if there is any ring other than the field of real numbers satisfying the conditions of Theorem 5.5. A famous fact in the theory of ordered fields states that “a totally ordered field is Dedekind complete (i.e. isomorphic to the field of real numbers) if and only if it is Cauchy complete and Archimedean (see Theorem 2.5 in [23]). Therefore, any Cauchy complete field which is not Archimedean is an ordered field other than the field of real numbers. For example, let \(F = \mathbb{Z}(X)\) be the ordered field of rational functions over \(\mathbb{Z}\) (see Example 1.2 in [23]). Since \(F\) is not Archimedean, its completion \(\tilde{F}\) is also not Archimedean. Therefore, \(\tilde{F}\) is an example of a Cauchy complete field which is not isomorphic to \(\mathbb{R}\) (see Example 2.3 in [23]). Obviously, \(\tilde{F}\) is a dense ring.

Cauchy in his famous book entitled “Cours d’analyse” proves that whenever each term of the series \(\sum_{i=0}^{+\infty} u_i\) is non-negative and smaller than the one preceding it, that series and \(\sum_{i=0}^{+\infty} 2^i u_{2^i-1}\) are either both convergent or both divergent (see Theorem III on p. 92 in [5]). This is known as the Cauchy Condensation Test in many resources. For example, see Theorem 3.27 on p. 61 and its name on p. 339 in Rudin’s book [26]. There is nothing special for the number 2 in the condensed series \(\sum_{i=0}^{+\infty} 2^i u_{2^i-1}\). For a detailed discussion of the Cauchy Condensation Test see pages 120–122 in [19].
Theorem 5.7 (The Cauchy Condensation Test). Let \( \leq \) be a total ordering on \( R \) and \((R, <)\) a dense and Cauchy complete ordered ring with 1. Assume that \((x_n)\) is a positive and decreasing sequence in \( R \). Then, \( \sum_{i=1}^{\infty} x_i \) is convergent in \( R \) if and only if \( \sum_{i=0}^{\infty} 2^i x_{2^i} \) is convergent in \( R \).

Proof. (\( \Rightarrow \)): Since \((x_n)\) is a decreasing sequence, for all \( n \in \mathbb{N} \), we have

\[
(5.1) \quad 2^n x_{2^n} = 2^{n-1} (x_{2^n} + x_{2^{n+1}}) \leq 2^n \sum_{i=2^{n-1}+1}^{n} x_i.
\]

Now, suppose that \( k \leq l \). Take \( m \) and \( n \) be positive integers such that

\[
m \leq 2^{k-1} + 1 \quad \text{and} \quad 2^l \leq n.
\]

Then, by using the inequality in (5.1), we have

\[
(5.2) \quad \sum_{i=k}^{l} 2^i x_{2^i} \leq 2 \sum_{i=m}^{2^l} x_i \leq 2 \sum_{i=m}^{n} x_i.
\]

Since \( \sum_{i=1}^{\infty} x_i \) is convergent and \( R \) is dense, by Theorem 2.14, \( \sum_{i=1}^{\infty} x_i \) is a Cauchy sequence. Let a positive element \( \epsilon \) in \( R \) be given. Since \( R \) is dense, we can find positive elements \( \beta \) and \( \gamma \) in \( R \) with \( \beta + \gamma < \epsilon \). For \( \beta \) (\( \gamma \)), we can find a natural number \( N_\beta \) (\( N_\gamma \)) such that

\[
N_\beta \leq m \leq n \quad (N_\gamma \leq m \leq n)
\]

implies \( \sum_{i=m}^{n} x_i < \beta \) (\( \sum_{i=m}^{n} x_i < \gamma \)). Now, set \( N_1 = \max \{N_\beta, N_\gamma\} \). Then, for \( N_1 \leq m \leq n \), we have

\[
(5.3) \quad 2 \sum_{i=m}^{n} x_i = \sum_{i=m}^{n} x_i + \sum_{i=m}^{n} x_i \leq \beta + \gamma < \epsilon.
\]

Imagine \( k \) is the smallest natural number with \( m \leq 2^{k-1} + 1 \). If \( k \leq l \), then \( 2^{k-1} + 1 \leq 2^l \). Therefore, for a suitable natural number \( N_2 \), if \( N_2 \leq k \leq l \), then in view of (5.2) and (5.3), we have

\[
\sum_{i=k}^{l} 2^i x_{2^i} \leq 2 \sum_{i=2^{k-1}+1}^{2^l} x_i \leq \beta + \gamma < \epsilon
\]

showing that \( \sum_{i=0}^{\infty} 2^i x_{2^i} \) is a Cauchy sequence. Since \((R, <)\) is a Cauchy complete ordered ring, \( \sum_{i=0}^{\infty} 2^i x_{2^i} \) is convergent.

(\( \Leftarrow \)): Since \((x_n)\) is a decreasing sequence, for all \( n \in \mathbb{N} \), we have

\[
(5.4) \quad x_{2^n} + x_{2^n+1} + \cdots + x_{2^{n+1}-1} \leq 2^n x_{2^n}.
\]

Now, let \( m \leq n \) be positive integers. Imagine \( k \) is the greatest non-negative integer satisfying \( 2^k \leq m \) and \( l \) is the smallest non-negative integer with \( n \leq 2^{l+1} - 1 \). Observe that

\[
(5.5) \quad \sum_{i=m}^{n} x_i \leq \sum_{i=2^k}^{2^{l+1}-1} x_i \leq \sum_{i=k}^{l} 2^i x_{2^i}.
\]

Since \( \sum_{i=0}^{\infty} 2^i x_{2^i} \) is convergent and \( R \) is dense, by Theorem 2.14, \( \sum_{i=0}^{\infty} 2^i x_{2^i} \) is a Cauchy sequence. Therefore, for the given positive \( \epsilon \), there is a natural number \( N_1 \) such that for arbitrary \( k, l \) with \( N_1 \leq k \leq l \) we have \( \sum_{i=k}^{l} 2^i x_{2^i} < \epsilon \). Now, in
view of the inequality in (5.3), we see that there is a natural number \( N_2 \) such that \( N_2 \leq m \leq n \) implies
\[
\sum_{i=m}^{n} x_i < \epsilon
\]
showing that \( \sum_{i=1}^{+\infty} x_i \) is a Cauchy sequence. Since \((R, \prec)\) is a Cauchy complete ordered ring, \( \sum_{i=1}^{+\infty} x_i \) is convergent and the proof is complete.

**Corollary 5.8.** Let \( \leq \) be a total ordering, and \((F, \prec)\) a Cauchy complete totally ordered field and \((x_n)\) a positive and decreasing sequence in \( F \). Then, \( \sum_{i=1}^{+\infty} x_i \) is convergent in \( F \) if and only if \( \sum_{i=0}^{+\infty} 2^i x_{2^i} \) is convergent in \( F \).

**Definition 5.9.** Let \((M, +, 0, \leq)\) be a dense monoid and \( G \) an \( M \)-normed abelian group.

1. By definition, the \( M \)-normed group \( G \) is Cauchy complete if any Cauchy sequence in \( G \) is convergent.
2. Let \( x_n \in G \) for each \( n \in \mathbb{N} \). It is said that \( \sum_{i=1}^{+\infty} x_n \) is an absolutely convergent series if \( \sum_{i=1}^{+\infty} \|x_n\| \) is convergent.

**Theorem 5.10.** Let \( M \) be a totally ordered abelian group. Also, let \( M \) be a dense group, \( G \) an \( M \)-normed abelian group, and \( x_n \in G \) for each \( n \in \mathbb{N} \). If \( G \) is Cauchy complete and \( \sum_{i=1}^{+\infty} x_n \) is an absolutely convergent series then the series \( \sum_{i=1}^{+\infty} x_n \) is convergent in \( G \).

**Proof.** Since \( M \) is dense, by Theorem 2.14 any convergent sequence in \( G \) is a Cauchy sequence. Since \( \sum_{i=1}^{+\infty} \|x_n\| \) is convergent in \( M \), in view of Proposition 2.3, it is a Cauchy sequence in the \( M \)-normed group \( M \). Therefore, for the given positive \( \epsilon \), we can find a natural number \( N \) such that \( n \geq m \geq N \) implies the following:
\[
\|x_{m+1}\| + \cdots + \|x_n\| < \epsilon.
\]
On the other hand,
\[
\|x_{m+1} + \cdots + x_n\| \leq \|x_{m+1}\| + \cdots + \|x_n\| < \epsilon.
\]
So, the series \( \sum_{i=1}^{+\infty} x_n \) is a Cauchy sequence. Since the \( M \)-normed group \( G \) is Cauchy complete, we see that \( \sum_{i=1}^{+\infty} x_n \) is convergent in \( G \) and the proof is complete.

6. **The geometric series in dense and shrinkable commutative rings**

We start this section with the following result:

**Theorem 6.1.** Let \( \leq \) be a total ordering on \( R \) and \((R, \prec)\) an ordered commutative ring with 1 such that \( R \) is a dense and shrinkable ring. Assume that \( r \neq 1 \). Then, the following statements hold:

1. If the sequence \((r^n)\) is convergent, then it converges to zero.
2. The geometric series \( \sum_{n=0}^{+\infty} r^n \) is convergent in \( R \) if and only if the sequence \((r^n)\) converges to zero and \( 1 - r \) is invertible in \( R \).

**Proof.** (1): If \( r = 0 \), then \((r^n)\) converges to zero. Now, assume that \( r \) is nonzero and \((r^n)\) converges to \( l \in R \). Set \( x_n = r^n \). Evidently, \( x_{n+1} = rx_n \) and \( x_{n+1} \) converges to \( l \) also. Observe that by Theorem 4.10, we have
\[
l = \lim_{n \to +\infty} x_{n+1} = \lim_{n \to +\infty} rx_n = r \lim_{n \to +\infty} x_n = rl.
\]
Since \((R, <)\) is an ordered ring, \(R\) is an integral domain. Now, from \(l(r-1) = 0\) and \(r \neq 1\), we deduce that \(l = 0\).

(2): Assume that \(\sum_{n=0}^{+\infty} r^n\) is to convergent \(\alpha \in R\). By Proposition 5.1, \((r^n)\) converges to zero. Now, set \(s_n = \sum_{i=0}^{n} r^n\). By Theorem 4.10, we have

\[
(1-r)\alpha = \lim_{n \to +\infty} s_n = \lim_{n \to +\infty} (1-r^{n+1}) = 1
\]

showing that \(1-r\) is invertible in \(R\).

Conversely, since \(1-r\) is invertible, we have

\[
1 + r + \cdots + r^n = \frac{1-r^{n+1}}{1-r}.
\]

Now, if \((r^n)\) converges to zero, we have \(\lim_{n \to +\infty} r^{n+1} = 0\) and so, we obtain that

\[
\sum_{n=0}^{+\infty} r^n = \lim_{n \to +\infty} (1+r+\cdots+r^n) = \frac{1}{1-r}
\]

This completes the proof.

Let us recall that if each of the real numbers \(x_i\)s are greater than \(-1\) and either all are positive or negative, then

\[
\prod_{i=1}^{n} (1+x_i) \geq 1 + \sum_{i=1}^{n} x_i \quad \text{(See p. 35 in [21])}.
\]

In the following, we prove this for (not necessarily commutative) ordered semirings:

**Theorem 6.2 (Generalization of Bernoulli’s Inequality for Ordered Semirings).** Let \((S, \leq)\) be an ordered semiring and \((x_i)_{i=1}^{n}\) be a sequence of \(n\) elements in \(S\) such that \(x_i \geq 0\) and \(1+x_i \geq 0\), for each \(1 \leq i \leq n\). Then, the following inequality holds:

\[
\prod_{i=1}^{n} (1+x_i) \geq 1 + \sum_{i=1}^{n} x_i.
\]

**Proof.** The proof is by induction on \(n\). It is evident that the inequality holds for \(n = 1\). Now, let the inequality hold for \(n = k\), i.e.

\[
\prod_{i=1}^{k} (1+x_i) \geq 1 + \sum_{i=1}^{k} x_i.
\]

Note that since either \(x_i \geq 0\) for each \(1 \leq i \leq n\), or \(x_i \leq 0\) for each \(1 \leq i \leq n\), we have \(x_{k+1}x_i \geq 0\), for each \(i\).

Now, multiply the both sides of the inequality \((6.1)\) by \(1+x_{i+1} \geq 0\) (from the left side) and observe that

\[
(1+x_{k+1}) \left( \prod_{i=1}^{k} (1+x_i) \right) \geq (1+x_{k+1}) \left( 1 + \sum_{i=1}^{k} x_i \right) \geq\\
1 + \sum_{i=1}^{k} x_i + x_{k+1} \sum_{i=1}^{k} x_i \geq 1 + \sum_{i=1}^{k+1} x_i.
\]

This completes the proof. □
Corollary 6.3 (Generalization of Bernoulli’s Inequality for Ordered Rings). Let \((R, \leq)\) be an ordered ring and \((x_i)_{i=1}^{n}\) be a sequence of \(n\) elements in \(R\) such that either \(x_i \geq 0\) for each \(1 \leq i \leq n\), or \(x_i \leq 0\) for each \(1 \leq i \leq n\). Also, suppose that \(1 + x_i \geq 0\), for each \(1 \leq i \leq n\). Then, the following inequality holds:

\[
\prod_{i=1}^{n} (1 + x_i) \geq 1 + \sum_{i=1}^{n} x_i.
\]

Proof. The only case that it must be shown is that from \(x_{i+1} \leq 0\) and \(x_i \leq 0\), we obtain that \(0 \leq -x_{i+1}\) and \(0 \leq -x_i\) which implies that \(0 \leq (-x_{i+1})(-x_i) = x_{i+1}x_i\).

The rest of the proof is similar to the proof of Theorem 6.2. \(\square\)

Corollary 6.4. Let \((S, \leq)\) be an ordered semiring such that \(x \geq 0\) and \(1 + x \geq 0\). Then, for each \(n \in \mathbb{N}\), we have

\[(1 + x)^n \geq 1 + nx \quad \text{(Bernoulli’s inequality)}.
\]

Remark 6.5. It is evident that in totally ordered rings we have \(0 < 1\), though this may not hold in (partially) ordered rings. It is not accidental that in the Definition of DeMarr division rings given in Definition 1.12, it is supposed that \(0 < 1\).

Let us recall that a totally ordered field \(F\) is Archimedean if any positive element \(x\) and any element \(y\) in \(F\), there is a natural number \(n\) such that \(nx > y\) \([23\text{, Definition 1.8}]\).

Proposition 6.6. Let \((F, <)\) be a totally ordered and Archimedean field and \(r \in F\) with \(-1 < r < 1\). Then,

\[
\lim_{n \to +\infty} r^n = 0.
\]

Proof. Without loss of generality, we may assume that \(0 < r < 1\). Set \(x = (1/r) - 1\) and let \(\epsilon > 0\) be given. By definition, there is a natural number \(N\) such that \(N(x\epsilon) > 1\). Now, let \(n \geq N\). In view of Corollary \(6.3\), we have

\[
r^n = \frac{1}{(1 + x)^n} \leq \frac{1}{1 + nx} \leq \frac{1}{N\epsilon} < \epsilon.
\]

Hence, \(\lim_{n \to +\infty} r^n = 0\), as required. \(\square\)

In view of Theorem \(6.1\) and Proposition \(6.6\) we have the following:

Corollary 6.7. \((23\text{, Exercise 6.5})\) Let \((F, <)\) be a totally ordered and Archimedean field. Assume that \(r \in F\) with \(-1 < r < 1\). Then,

\[
\sum_{n=0}^{+\infty} r^n = \frac{1}{1 - r}.
\]

Theorem 6.8. Let \(\leq\) be a total ordering on \(R\) and \((R, <)\) an ordered commutative ring with 1 such that \(R\) is dense and shrinkable. Assume that \(1 - r\) is invertible and \((r^n)\) converges to zero in \(R\). Also, let \((G, +)\) be a Cauchy complete \(R\)-normed abelian group and \((x_n)_{n=0}^{+\infty}\) be a sequence in \(G\) such that \(\|x_{n+1}\| \leq r\|x_n\|\) for all \(n\). Then, \(\sum_{n=0}^{+\infty} x_n\) is convergent in \(G\).
Proof. By Theorem 6.1, $\sum_{n=0}^{+\infty} r^n$ is convergent. This implies that the series

$$\sum_{n=0}^{+\infty} (r^n ||x_0||)$$

is also convergent. On the other hand, $R$ is dense. So, by Theorem 2.14,

$$\sum_{n=0}^{+\infty} (r^n ||x_0||)$$

is a Cauchy sequence. It is evident that

$$0 \leq ||x_n|| \leq r^n ||x_0||, \quad \forall \ n \in \mathbb{N}.$$

Similar to the proof of Theorem 5.5, we can see that $\sum_{n=0}^{+\infty} ||x_n||$ is convergent. Also, similar to the proof of Theorem 5.10, it is proved that $\sum_{n=0}^{+\infty} x_n$ is convergent. This completes the proof. □

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