A comment on the Outgoing Radiation Condition for the gravitational field and the Peeling Theorem.

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Abstract
The connection between the Bondi-Sachs (BS) and the Newman-Penrose (NP) framework for the study of the asymptotics of the gravitational field is done. In particular the coordinate transformation relating the BS luminosity parameter and the NP affine parameter is obtained. Using this coordinate transformation it is possible to express BS quantities in terms of NP quantities, and to show that if the Outgoing Radiation Condition is not satisfied then the spacetime will not decay in the way prescribed by the Peeling theorem.

1 Introduction.

In the pioneering work by Bondi et al. and Sachs on the asymptotic behaviour of the gravitational field of an isolated body, the following metric was used

\[ ds^2 = \frac{Ve^{2\beta}}{\tilde{r}} du^2 + 2e^{2\beta} dud\tilde{r} - \tilde{r}^2 h_{ij} (dx^i - U^i du)(dx^j - U^j du), \]

where

\[ h_{ij} = \begin{pmatrix} e^{2\gamma} \cosh 2\delta & \sinh 2\delta \sin \theta \\ \sinh 2\delta \sin \theta & e^{-2\gamma} \cosh 2\delta \sin^2 \theta \end{pmatrix}, \]

\[ h^{ij} = \begin{pmatrix} e^{-2\gamma} \cosh 2\delta & -\sinh 2\delta \csc \theta \\ -\sinh 2\delta \csc \theta & e^{2\gamma} \cosh 2\delta \csc^2 \theta \end{pmatrix}. \]

A nice property of this metric is that the field equations form a hierarchy (4 hypersurface equations, 2 standard equations, 1 trivial equation, and 3 supplementary conditions). So if the the initial values of the functions \( \gamma \) and \( \delta \) are given...
on a null hypersurface, then it is possible to solve the hypersurface equations so that the values of the remaining functions \((\beta, V, U^i)\) on that hypersurface can be obtained. Then, using the evolution equations we can obtain their values for previous retarded times. The asymptotic study of Bondi et al. and Sachs used the following Ansatz for \(\gamma\) and \(\delta\):

\[
\begin{align*}
\gamma &= cr^{-1} + \gamma_3 r^{-3} + ..., \\
\delta &= dr^{-1} + \delta_3 r^{-3} + ...
\end{align*}
\]  

(4)  

(5)

The important fact to realize here is the absence of the \(r^{-2}\) term in both expansions. When solving the hypersurface equations, two integrations with respect to \(r\) will be carried out; hence if the \(r^{-2}\) term is present, then terms of the form \(r^{-i}\ln r\) will arise in the functions \(V\) and \(U^i\) (see [5] for an example). This Ansatz was known by the misleading name of the Outgoing Radiation Condition of the gravitational field (ORC), in analogy to the Sommerfeld condition for the electromagnetic field.

Now, the ORC does not rule out the existence of incoming gravitational radiation travelling infinitely long distances (i.e. radiation coming from null infinity). In order to do so, a condition on the news function at past null infinity should be imposed [6]. The presence of incoming radiation of finite duration is not a problem, as it may describe the phenomena of gravitational wave scattering and gravitational wave tails [2], [10], [1] that die off suitably in a neighborhood of \(\mathcal{J}\).

If we keep the \(r^{-2}\) terms in our expansions, then we enter into the realm of the polyhomogeneous spacetimes; spacetimes that can be expanded asymptotically in a combination of powers of \(1/r\) and \(\ln r\). Some study on these spacetimes has been done [5], [3], [11].

Another framework for the treatment of the gravitational radiation is the Newman-Penrose formalism [7]. In the NP framework we find that the field equations are naturally adapted for the study of the characteristic initial value problem. The equations also form a (more lengthy) hierarchy of first order differential equations. In this case the initial data that has to be prescribed on the initial hypersurface is contained in the \(\Psi_0\) of the Weyl tensor. The crucial assumption in the NP framework is that the components of the Weyl tensor fall off in the way prescribed by the Peeling Theorem:

\[
\Psi_k = O(r^{k-5}),
\]

(6)

and in particular the data on the initial hypersurface (\(\Psi_0\)) should decay as \(O(r^{-5})\) [9]. The objective of this note is to find the connection between the Bondi-Sachs quantities (the coefficients in \(\gamma\) and \(\delta\)) and the Newman-Penrose ones (the coefficients in \(\Psi_0\)); with these tools in hand it will be shown that the if the Outgoing Radiation Condition is not satisfied then the Peeling theorem does not hold.
2 The coordinates.

2.1 The Bondi coordinates.

The coordinate $u$ in the line element of equation (1) is a retarded time that parametrizes outgoing null hypersurfaces. The angular coordinates $\theta$ and $\varphi$ are constructed in such a way that they remain constant along the generators of the null hypersurfaces, and $\tilde{r}$ is a luminosity parameter that satisfies

$$
\tilde{r}^4 \sin^2 \theta = \det h_{ij} = (\det h^{ij})^{-1} = g_{22}g_{33} - (g_{23})^2.
$$

(7)

2.2 The NP coordinates.

The coordinates used in the NP treatment can be constructed in a similar way. The key difference lies in the choice of the radial coordinate. Newman and Penrose use as radial coordinate the affine parameter $r$ of the generators of the null hypersurfaces $u = const$. There is some freedom left in the choice of this coordinate. The change

$$
r' = ar + b.
$$

(8)

can always be performed. The scaling of the affine parameter is chosen such that the contravariant metric tensor has the form

$$
g_{ij}^{NP} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 2Q & C^\theta & C^\varphi \\
0 & C^\theta & -2\xi^\theta & -\varepsilon^\theta \\
0 & C^\varphi & -\xi^\theta & -2\varepsilon^\theta
\end{pmatrix},
$$

(9)

while the freedom in the choice of the origin is generally used to eliminate an arbitrary function of integration that appears in the expansion of the spin coefficient $\rho$. The real functions $Q, C^\theta, C^\varphi$ and the complex functions $\xi^\theta$ and $\xi^\varphi$ depend on all four coordinates. We notice that in the BS treatment there are 6 metric functions whilst in the NP framework there are 7. This difference can be traced back to the choice of the luminosity parameter in the Bondi metric which fixes the form of the determinant of the angular part of the metric. The choice of the radial coordinate as an affine parameter is necessary in the NP formalism. If we were to use a luminosity parameter instead, then the field equations would not give rise to an easy to handle hierarchy. One would have to solve all the NP field equations at once!

2.3 The relation between Bondi’s luminosity parameter and NP affine parameter.

As seen before, the BS coordinates and the NP coordinate differ essentially in the construction of the radial coordinate. The connection between the two coordinates can be found easily by equating the determinants of the “angular part” of the metrics. On one hand one has
\[ \det h_{BS}^{ij} = \frac{1}{r^4 \sin^2 \theta}, \]  
by definition. On the other hand, the determinant of the “angular part” of the NP metric is given by
\[ \det h_{NP}^{ij} = -\left( \xi^0 \xi^\phi - \xi^\phi \xi^0 \right)^2. \]  
Imposing equality of the two determinants one obtains:
\[ \tilde{r}^2 = -i \csc \theta \left( \xi^0 \xi^\phi - \xi^\phi \xi^0 \right), \]  
where the right hand side is a function of the NP coordinates \((u, r, \theta, \phi)\). The minus sign is set in order to have \(\tilde{r}^2 \geq 0\).

3 The relation between the ORC and the Peeling Theorem.

Although the Einstein field equations are consistent with spacetimes that fall off as
\[ \Psi_0 = O(r^{-3} \ln N_3 r) \]  
for our purposes it is sufficient to consider a spacetime such that
\[ \Psi_0 = \Psi_0^{4,0} r^{-4} + O(r^{-5} \ln N_5). \]  
Then using the techniques of reference [11] we see that
\[ \sigma = \sigma_{2,0} r^{-2} - \Psi_0^{4,0} r^{-3} + O(r^{-4} \ln N_5 r), \]  
and
\[ \rho = r^{-1} - \sigma_{2,0} \sigma_{2,0} r^{-3} + \frac{1}{2} \left( \sigma_{2,0} \Psi_0^{4,0} + \sigma_{2,0} \Psi_0^{4,0} \right) r^{-4} + O(r^{-5} \ln N_5 r). \]  
Whence using the commutator equations one finds
\[ \xi^i = \xi_0^i r^{-1} - \xi_0^i \sigma_{2,0} r^{-2} + \left( \xi_0^i \sigma_{2,0} + \frac{1}{2} \Psi_0^{4,0} \xi_0^i \right) r^{-3} + O(r^{-4} \ln N r), \]  
where
\[ \xi_0^\theta = \frac{1}{\sqrt{2}}, \]  
\[ \xi_0^\phi = \frac{-i}{\sqrt{2}} \csc \theta. \]
if the cuts of $\mathcal{I}^-$ are chosen to be $S^2$ metrically.

The substitution of these expansions into equation (12) yields the transformation linking the NP affine parameter and the BS luminosity parameter.

$$\tilde{r} = r - \frac{1}{2}\sigma_{2,0}\bar{r}^{-1} + \frac{1}{6}\left(\sigma_{2,0}\Psi_0^{1,0} + \sigma_{2,0}\Psi_0^{4,0}\right)\bar{r}^{-2} + ...$$

(20)

so that the metric functions $\gamma$ and $\delta$ can be written in terms of the NP quantities as:

$$\gamma = \frac{1}{2}(\sigma_{2,0} + \sigma_{2,0})\bar{r}^{-1} - \left(\Psi_0^{1,0} + \Psi_0^{4,0}\right)\bar{r}^{-2} + ...$$

(21)

$$\delta = \frac{i}{2}(\sigma_{2,0} - \sigma_{2,0})\bar{r}^{-1} - i\left(\Psi_0^{1,0} - \Psi_0^{4,0}\right)\bar{r}^{-2} + ...$$

(22)

We see that the term that breaks the peeling behaviour $\Psi_0^{4,0}$ gives rise to the coefficients forbidden by the outgoing radiation condition. A similar study can be carried out for more general polyhomogeneous space-times giving as a result that the coefficients $\Psi_0^{3,k}$ are related to logarithmic terms in the $1/r$ terms of $\gamma$ and $\delta$, etc. In principle these expansions can be performed up to any desired order.

4 Conclusions.

As we have seen, the Outgoing Radiation Condition and the $\Psi_0 = O(r^{-5})$ condition of Penrose are closely related. If the ORC is not satisfied then Penrose’s condition will not be satisfied. However, the two conditions are not completely equivalent. Penrose’s condition is stronger. For example, if we consider a polyhomogeneous spacetime such that

$$\gamma = cr^{-1} + \gamma_3r^{-3} + \gamma_{31}r^{-3}\ln r + ...$$

(23)

$$\delta = dr^{-1} + \delta_3r^{-3} + \delta_{31}r^{-3}\ln r + ...$$

(24)

then clearly, the ORC will be satisfied, but the leading behaviour of $\Psi_0$ will go as $O(r^{-5}\ln r)$. The null infinity for this spacetime will not be smooth, as the conformally rescaled $\Psi_0$ will blow up there.

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