REMARKS ON METRIZABILITY OF DUAL GROUPS

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Abstract. We examine sufficient conditions for the dual of a topological group to be metrizable and locally compact, improving on results of [4].

1. Introduction and preliminaries

If $G$ is an abelian topological group, its dual $\hat{G}$ is the set of continuous group homomorphisms into the one-dimensional torus $T$, endowed with the compact-open topology.

Two families $A$ and $B$ of subsets of a set $X$ are said to mesh, in symbols $A \# B$, if $A \cap B \neq \emptyset$ whenever $A \in A$ and $B \in B$. Thus, if the set $\mathcal{F}$ of filters on a set $X$ is ordered by inclusion, two filters $\mathcal{F}$ and $\mathcal{G}$ admit a supremum $\mathcal{F} \vee \mathcal{G}$ if and only if $\mathcal{F} \# \mathcal{G}$. We do not distinguish between a sequence $(x_n)_{n \in \omega}$ and the filter generated by its tails. In particular, if $\mathcal{H}$ is a family of subsets of $X$ and $(x_n)_{n \in \omega}$ is a sequence on $X$, the notation $\mathcal{H} \# (x_n)_{n \in \omega}$ means that $H \cap \{x_n : n \geq k\} \neq \emptyset$ for every $H \in \mathcal{H}$ and $k \in \omega$.

Recall that a topological space $X$ is sequential if every sequentially closed subset is closed. The space is Fréchet-Urysohn, or simply Fréchet, if whenever $x \in X$ and $A \subset X$ with $x \in \text{cl} A$, there is a sequence on $A$ that converges to $x$. It is strongly Fréchet if whenever $(A_n)_{n \in \omega}$ is a decreasing sequence of subsets of $X$ with $x \in \bigcap_{n \in \omega} \text{cl} A_n$ there is $x_n \in A_n$ for each $n$ with $x_n \to_n x$, equivalently if

$$\forall \mathcal{H} \in \mathcal{F}_1, \text{adhd} \mathcal{H} \subseteq \text{adh}_{\text{Seq}} \mathcal{H},$$

where $\mathcal{F}_1$ denotes the class of countably based filters, $\text{adhd} \mathcal{H} := \bigcap_{H \in \mathcal{H}} \text{cl} H$ is the adherence of the filter $\mathcal{H}$, and $\text{adh}_{\text{Seq}} \mathcal{H} := \bigcup_{(x_n)_{n \in \omega} \not\in \mathcal{H}} \text{lim} (x_n)_{n \in \omega}$ is its sequential adherence. We will also denote by $\text{cl}_{\text{Seq}}$ the sequential closure, that is, the closure in the coarsest sequential topology that is finer than the original.

We are interested in metrization results for dual groups akin to:

Theorem 1. [4] Let $G$ be a metrizable topological group. If $\hat{G}$ is Fréchet, then it is metrizable and locally compact.
To this end, let us review its simple proof: A product of a strongly Fréchet space and a metrizable space is (strongly) Fréchet (e.g., [10]), and \( \hat{G} \) is strongly Fréchet as a Fréchet topological group [13], so that \( G \times \hat{G} \) is Fréchet, hence sequential. It is in particular a \( k \)-space \(^{1}\), so that, in view of [9, Proposition 1.2], \( \hat{G} \) is locally compact. But a locally compact topological group with countable tightness (in particular a sequential locally compact one) is metrizable [1]. Hence \( \hat{G} \) is locally compact and metrizable.

Thus, the proof hinges on \( G \times \hat{G} \) being a \( k \)-space, and \( \hat{G} \) being of countable tightness. In this note, which can be considered as an appendix to [4], we will consider sufficient conditions on \( G \) and \( \hat{G} \) to achieve this result that are different and often weaker than those in Theorem 1.

2. Strongly sequential groups

The proof outlined above relies on the following:

**Theorem 2.** [10] Let \( X \) be a topological space. The following are equivalent:

1. \( X \) is strongly Fréchet;
2. \( \lim \mathcal{F} = \bigcap_{H \in \mathcal{F}_1} \text{adh}_{\text{Seq}} \mathcal{H} \) for every filter \( \mathcal{F} \) on \( X \);
3. \( X \times Y \) is strongly Fréchet for every bisequential space \( Y \);
4. \( X \times Y \) is Fréchet for every metrizable atomic topology.

Strongly sequential spaces are in some sense to sequential spaces like strongly Fréchet spaces are to Fréchet spaces:

**Theorem 3.** [11] Let \( X \) be a \( T_1 \) topological space. The following are equivalent:

1. \( X \) is strongly sequential;
2. \( \lim \mathcal{F} = \bigcap_{H \in \mathcal{F}_1} \text{cl}_{\text{Seq}}(\text{adh}_{\text{Seq}} \mathcal{H}) \) for every filter \( \mathcal{F} \) on \( X \);
3. \( X \times Y \) is strongly sequential for every bisequential space \( Y \);
4. \( X \times Y \) is sequential for every metrizable atomic topology.

The proof outlined in the Introduction thus applies virtually unchanged to the effect that:

**Proposition 4.** Let \( G \) be a metrizable topological group. If \( \hat{G} \) is strongly sequential, then it is metrizable and locally compact.

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\(^{1}\)A topological space \( X \) is a \( k \)-space if a subset \( C \) is closed whenever \( C \cap K \) is closed for every compact subset \( K \) of \( X \).
Since Fréchet topological groups are strongly Fréchet and strongly Fréchet spaces are strongly sequential, Proposition 4 formally generalizes Theorem 1.

Note that while Theorems 2 and 3 highlight similarities between strongly Fréchet and strongly sequential spaces, Proposition 4 points to an important difference: While Fréchet-Urysohn topological groups are automatically strongly Fréchet, a sequential topological group does not need to be strongly sequential. Indeed, there are metrizable groups $G$ such that $\hat{G}$ is sequential but not metrizable [4]. In view of Proposition 4, such dual groups are sequential but not strongly sequential.

We will now see that Proposition 4 generalizes Theorem 1 only formally, as both are in fact instances of a general metrization theorem for topological groups that is not specific to dual groups. Indeed, if $G$ is metrizable, then $\hat{G}$ is a hemicompact $k$-space [3], [2]. Thus, Theorem 1 follows from:

**Theorem 5.** [8] A hemicompact Fréchet-Urysohn topological group is locally compact and Polish.

The local compactness (hence also metrization) part of this result follows in turn from a more general fact.

Recall from [12] that a topological space is called a Tanaka space if

\[(\text{Tanaka condition}) \quad \mathcal{H} \in F_1, \text{ adh}\mathcal{H} \neq \emptyset \implies \text{adh}_{\text{Seq}}\mathcal{H} \neq \emptyset.\]

Of course, every strongly sequential space, in particular every strongly Fréchet space, is Tanaka. Thus every Fréchet topological group is Tanaka. Therefore Theorem 5 is an instance of Corollary 8 below, and Proposition 4 follows from Corollary 8 as well, showing that Theorem 1 and Proposition 4 are essentially the same.

**Theorem 6.** A Hausdorff hemicompact Tanaka space has a point with a compact neighborhood.

*Proof.* Let $(K_n)_{n \in \omega}$ be an increasing sequence of compact subsets such that each compact subset of $X$ is contained in some $K_n$. If there is $n$ with $\text{int}K_n \neq \emptyset$, we are done. Otherwise, each $K_n$ has empty interior, so that its complement is dense. Thus $(X \setminus K_n)_{n \in \omega}$ is a decreasing sequence of sets with $\bigcap_{n \in \omega} \text{cl}(X \setminus K_n) = X$, but there is no convergent sequence meshing with $(X \setminus K_n)_{n \in \omega}$. Indeed, if there was such a sequence, there would be a subsequence $(x_n)_{n \in \omega}$ with $x_n \in X \setminus K_n$ for each $n$ that converges to a point $l$. But $K := \{x_n : n \in \omega\} \cup \{l\}$ would then be a compact subset of $X$ that is not contained in any of the $K_n$’s. Thus $X$ is not Tanaka. \qed
Corollary 7. A homogeneous Hausdorff hemicompact Tanaka space is locally compact.

Corollary 8. A Hausdorff hemicompact topological group of countable tightness is locally compact and metrizable whenever it is Tanaka.

Proof. This follows from Corollary 7 and the fact that a locally compact topological group of countable tightness is metrizable [1]. □

3. Productively Fréchet groups

Productively Fréchet spaces are those whose product with every strongly Fréchet space is (strongly) Fréchet [6]. To describe them, we first need to define strongly Fréchet filters. A filter $F$ on $X$ is strongly Fréchet if

$$\mathcal{H} \in F_1, \mathcal{H} \# F \implies \exists G \in F_1 : G \geq \mathcal{H} \vee F.$$ 

Let $F_1^\Delta$ denote the class of strongly Fréchet filters. A topological space is productively Fréchet if

$$\forall \mathcal{H} \in F_1^\Delta, \text{adh}\mathcal{H} \subseteq \text{adh}_{\text{seq}}\mathcal{H}.$$ 

Theorem 9. [6] The following are equivalent:

1. $X$ is productively Fréchet;
2. $X \times Y$ is strongly Fréchet for every strongly Fréchet space $Y$;
3. $X \times Y$ is Fréchet for every strongly Fréchet space $Y$.

Thus it is clear that the proof of Theorem 1 applies with virtually no change to the effect that:

Theorem 10. (1) If $G$ is a productively Fréchet topological group and $\hat{G}$ is Fréchet-Urysohn, then $\hat{G}$ is metrizable and locally compact.

(2) If $G$ is a Fréchet topological group and $\hat{G}$ is productively Fréchet, then $\hat{G}$ is metrizable and locally compact.

It is observed in [6] that the $\Sigma$-product of $\mathfrak{c}$ many copies of $\mathbb{R}$ is a non-metrizable productively Fréchet topological group.

Note that since there are productively Fréchet group whose dual is not hemicompact, Theorem 10 no longer follows from Theorem 5. To see that such groups exist, first note that the $\Sigma$-product of $\mathfrak{c}$ many copies of $\mathbb{R}$ is a locally convex Fréchet-Urysohn topological vector space, in particular a locally quasi-convex (by, e.g., [2, Proposition 6.5]) $k$-group. It is therefore subreflexive, that is,

$$i : G \rightarrow \hat{G}$$

$$g \mapsto \langle g, \cdot \rangle$$
is an embedding.
Moreover, we have:

**Proposition 11.** [5] Proposition 5.10] A subreflexive topological group $G$ is metrizable if and only if $\hat{G}$ is hemicompact.

As already noted, the $\Sigma$-product of $c$ many copies of $\mathbb{R}$ is a productively Fréchet non-metrizable group. In view of Proposition 11, its dual is not hemicompact.

In conclusion, while Theorem 1 and Proposition 4 are not specific to dual groups (but are true for any hemicompact group), the variants given in Theorem 10 seem to be specific to dual groups, even though the proof is essentially identical.

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