Motivic Euler Characteristic of Nearby Cycles and a Generalized Quadratic Conductor Formula

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Abstract

We compute the motivic Euler characteristic of Ayoub’s nearby cycles in terms of strata of a semi-stable reduction, for a degeneration to multiple isolated quasi-homogeneous singularities resolved by a blow-up. This allows us to compare the local picture at the singularities with the global conductor formula for hypersurfaces developed by Levine, Pepin Lehalleur and Srinivas, revealing that the formula is local in nature, thus extending it to the more general setting considered in this paper. This gives a quadratic refinement for the Milnor number formula with multiple singularities.

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1 Introduction

1.1 The Milnor fibre, nearby cycles and the Euler characteristic

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a non-zero polynomial function. Suppose that $X_t := f^{-1}(t)$ is smooth for $0 < |t| < 1$, and that $X_0$ has an isolated singularity at $0 \in \mathbb{C}^{n+1}$. Take a small $\varepsilon > 0$, and even smaller $t$. Take $p \in X_0$ and consider $B_{p,\varepsilon}$, the open ball with radius $\varepsilon$. The Milnor fibre is defined by

$$M_{f,p} = B_{p,\varepsilon} \cap X_t.$$ 

$M_{f,p}$ is homotopically equivalent to a wedge of spheres, the number of spheres is defined to be the Milnor number, $\mu_{f,p}$ [Mil, Theorem 6.5]; we may also consider the sheaf on $X_0$ defined by $x \mapsto H^*(M_{f,p}, \mathbb{Z})$; or we can just compare the Euler characteristics of the fibres $\chi^{top}(X_\eta), \chi^{top}(X_\sigma)$.

Conductor formulas express the difference of Euler characteristics in the case of a proper map $f$ in terms of local invariants around the singular points of the special fibre:

$$\chi(X_t) - \chi(X_0) = \text{invariants related to the singular points in } X_0.$$ 

In the setting of complex geometry, this was investigated by Milnor. Consider a complex manifold $X$ of dimension $n+1$ and a proper holomorphic map $f : X \to D$, with $D$ the open unit disk in $\mathbb{C}$; as above, let $X_t$ denote the complex analytic variety $f^{-1}(t)$. Suppose that $f$ is a submersion outside of a finite subset $\{p_1, \ldots, p_n\}$ of $X_0$. At each singular point $p$, a choice of local coordinates $s_0, \ldots, s_n$ for a neighbourhood of $p$ gives us a description of the Milnor number $\mu_{f,p}$ by local terms, as the dimension of the Jacobian ring of $f$ at $p$, that is ([Mil, Theorem 7.2]),

$$\mu_{f,p} = \dim \mathcal{O}_{X,p}/(\partial f/\partial s_0, \ldots, \partial f/\partial s_n).$$ 

As an immediate consequence, we have the conductor formula with multiple isolated singularities,

$$\chi^{top}(X_t) - \chi^{top}(X_0) = (-1)^n \sum \mu_{f,p}. \tag{1.1}$$ 

Milnor and Orlik computed the value of $\mu_{f,p}$ explicitly in [MO] for the case of a $f$ a weighted homogeneous polynomial. In this paper we develop a quadratic refinement for this formula in the setting of algebraic geometry. By a quadratic refinement we mean an identity of quadratic forms over a common base field, instead of an identity of integers, from which the corresponding identity on integers follows over the complex numbers and other relevant fields. These quadratic refined invariants arise when considering motivic analogues for the relevant concepts in algebraic geometry. Our first main result is a formula for the motivic Euler characteristics of the motivic nearby cycles functor 1.4. Combined with the conductor formula of [LPS] for projective hypersurfaces, we obtain a ‘local’ conductor formula for a scheme with several singularities 1.6. We elaborate in the rest of this introduction.

First, the concepts considered above around the Milnor fibre in the complex setting can all be developed in the setting of algebraic geometry and étale cohomology. Let $f : X \to B$ be a flat family of schemes. We assume that we have a distinguished closed point $\sigma \hookrightarrow B$, with complement $\eta = B \setminus \sigma \hookrightarrow B$, allowing us to talk about a generic fibre $X_\eta \to \eta$, which we assume to be smooth, and a special fibre $X_\sigma$, which may have singular points. The definition of the Milnor number in terms of the Jacobian ring carries naturally to this case, the Euler characteristics can be defined also using l-adic étale cohomology, and the sheaf structure of the cohomology of the Milnor fibre can be realised through the formalism of the nearby cycles functor

$$\Psi_f : D^b_{cons}(X_\eta) \to D^b_{cons}(X_\sigma),$$ 

defined in [SGA I, Exposé 1, 2].

The Deligne-Milnor conjecture [SGA7 II, Exposé XVI, Conjecture 1.9] is concerned with an algebraic version of Milnor’s computation of the Milnor number, without restriction to characteristic zero. Let $X \to B$ be a separated, finite type, flat morphism of relative dimension $n$, where $B$ is a henselian trait. Suppose that $X$ is regular, that the general fibre $X_\eta$ is smooth over $\eta$, and that $X_\sigma$ has a unique singular closed point $p$. Let $l$ be a prime number which is invertible on $\mathcal{O}_B$. Then

$$\chi^{l-adic}(X_\eta) - \chi^{l-adic}(X_\sigma) + \dim Sw(\Phi^n(\mathbb{F}_l)_p) = (-1)^n \mu_{f,p} \tag{1.2}$$
with the Swan conductor $Sw(\Phi)$ being an additional term, adjusting for the case of positive characteristic. The formula is proven in the case of equal characteristics ([SGA7 II, Exposé XVI, Théorème 2.4]), and in the cases of relative dimension 1 and of a simple normal crossing divisor $(X_\sigma)_{\text{red}}$ ([Blo], [KS, Theorem 6.2.3], and [Or, Théorème 0.8] for the statement with Milnor number as appearing here). The global difference of Euler characteristics comes form considering $\dim \Phi^n(F_{\ell})_p$ at the singularity $p$, where $\Phi$ is the vanishing cycles functor, closely related to the nearby cycles functor $\Psi$. The local formula yields as in the complex analytic case a conductor formula for a flat proper map $f : X \to B$ as above, but allowing the special fibre to have multiple isolated singularities.

### 1.2 Motivic refinements

In the context of motivic homotopy, the nearby functor cycles formalism was developed by Ayoub in [Ay07]. Here bounded complexes are replaced by motivic spectra over the generic and special fibres, constructing a functor $\Psi_f : \text{SH}(X_0) \to \text{SH}(X_\sigma)$. As a somewhat parallel concept, this time with a motivic integration approach, Denef and Loeser [DL98], [DL00] constructed a motivic Milnor fibre in the Grothendieck ring of varieties. It is expressed in terms of certain étale coverings for components of the special fibre and their intersections. Using rigid analytic motives Ayoub, Ivorra and Sebag [AIS] show that for a semi-stable scheme $X$, the class of the motivic nearby cycles in $K_0(\text{SH}(X_\sigma))$ is equal to the one computed by those covers, the formula has the form

$$[\Psi_f] = \text{alternating sum of étale coverings of intersections of strata of } X_\sigma.$$  

Within the setting of stable $A^1$-homotopy theory, we can refine the topological Euler characteristic as well to a motivic setting. The motivic Euler characteristic of a smooth and proper scheme is defined as the categorical trace of the identity morphism of the motive of the scheme in the category of motivic spectra. A variant definable over singular schemes is the compactly supported Euler characteristic. Working in the motivic stable homotopy category $\text{SH}(k)$ over a perfect field $k$, for every finite type $k$-scheme $X$ we get an element $\chi_c(X/k)$ in the Grothendieck-Witt group $GW(k)$. That is, instead of integers we use quadratic forms, which encode more information.

Let $\mathcal{O}$ be a discrete valuation ring with residue field $k$, fraction field $K$ and a fixed uniformizer $t \in \mathcal{O}$. Let $F(T_0, \ldots, T_n) \in k[T_0, \ldots, T_n]$ be a homogeneous (or weighted-homogeneous) polynomial of degree $e$, defining a smooth projective (or weighted projective) hypersurface. The hypersurface $H^F$ defined as an $\mathcal{O}$-scheme by $F(T_0, \ldots, T_n) - tT_{n+1}^e$ thus gives a family of hypersurfaces that degenerates to the cone over the section defined by $F$. In this setting, Levine, Pepin Lehalleur and Srinivas [LPLS, Theorem 5.2] develop a quadratic conductor formula that takes the form

$$\Delta_e(F) := \text{sp}_t \chi_c(H^F_t) - \chi_c(H^F_0) = (1) - \langle e \rangle + \langle (e) \rangle e \cdot \mu^e_{F,0}$$

in the homogeneous case. They also develop a similar formula for a weighted homogeneous $F$ [LPLS, Theorem 5.3]. Since $\chi_c(H^F_t) \in GW(K)$, $\chi_c(H^F_0) \in GW(k)$ live in different rings, one has to use the specialization map $\text{sp}_t : GW(K) \to GW(k)$ to compare them; the term $\mu^e_{F,0} \in GW(k)$ in the right hand side is a quadratic refinement of the Milnor number $\mu_{F,0} \in \mathbb{Z}$. It can be defined in algebraic terms in the Jacobian ring $J(F,0)$, by a certain quadratic form on this ring corresponding to a distinguished element in the ring, defined by Scheja-Storch. The main goal of this paper is to formulate and prove a generalization of this result, for a more general scheme, and with multiple singularities.

### 1.3 Main results

We compute the quadratic Euler characteristic of the motivic nearby cycles functor in the case of a flat morphism $f : X \to S$ with a few quasi-homogeneous singularities. We compute it by strata of the special fibre, using the étale coverings of the strata as in the Ayoub-Ivorra-Sebag formula. Then we compare the result with the Levine-Pepin Lehalleur-Srinivas formula to obtain our generalized conductor formula.

First we give a proof for a special case of a formula proved in greater generality by Ayoub-Ivorra-Sebag, for the case of no triple intersections, which suffices for our purposes. The coverings called here Denef-Loeser are defined in Section 3.4, but are obtained in the case considered here as strata of the special fibre of a semi-stable reduction for $X$.  

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Proposition 1.1. (Proposition 3.15, [AIS, Theorem 8.6]) Let $f : X \rightarrow \text{Spec} \mathcal{O}$ be a flat, quasi-projective morphism, with $X$ smooth over its residue field $k$ and with generic fibre $X_\eta$ smooth over $\eta$. Suppose that the special fibre $X_\sigma$ is a normal crossing divisor $X_\sigma = \sum a_i D_i$; if $\text{char}(k) = p > 0$, we suppose in addition that $p \not\mid \prod a_i$. Assume that for all $i \neq j$ $\text{gcd}(a_i, a_j) = 1$, and that there are no triple intersections.

Denote by $D_i$, $D_i^\circ$, $\tilde{D}_{ij}$ the Denef-Loeser coverings. Then

$$\chi_c(\Psi_f(\mathbb{I}_X)) = \sum_i \chi_c(D_i^\circ) - \sum_{i<j} \chi_c(G_m \times \tilde{D}_{ij}).$$

Using our method of proof we can get the same formula also in some cases in which the pieces $D_i$ are not smooth, see remark 3.17. We need this to address the quasi-homogeneous case for the main formula we prove in the paper.

Below we state most of the results in the homogeneous case, for the clarity of the introduction. We refer the reader to the main text for the more general quasi-homogeneous case, where one has to take the weights into account. The type of singularities we deal with when we say the homogeneous case is defined as follows:

Definition 1.2. Let $f : X \rightarrow \text{Spec} \mathcal{O}$ be a flat quasi-projective morphism of schemes over a discrete valuation ring $\mathcal{O}$ with quotient field $K$, residue field $k$ and parameter $t$, with $X$ a regular scheme and with $X_\sigma$ smooth. Let $p \in X_\sigma$ be an isolated singular point and let $F \in k[p][T_0, \ldots, T_n]$ be a homogeneous polynomial of degree $e$. Let $\mathcal{O}_{X,p}$ be the stalk at $p$, and $m_p \subset \mathcal{O}_{X,p}$ the maximal ideal. We say that $X_\sigma$ looks like the homogeneous singularity defined by $F$ at $p$ if there is a regular sequence of generators $s_0, \ldots, s_n$ for $m_p$ such that

$$f^*(t) = F(s_0, \ldots, s_n) \mod m_p^{e+1}.$$ 

Remark 1.3. This definition includes the case of a homogeneous singularity, but allows for additional higher degree terms. It is closely related to the notion of semi-homogeneous singularity appearing in the literature.

Similarly we treat the more general quasi-homogeneous case, where the defining polynomial $F$ at each singular point is a weighted homogeneous polynomial with respect to a sequence of positive integer weights $a_* = (a_0, \ldots, a_n)$. The projective space $\mathbb{P}^n$ is replaced by the $a_*$-weighted projective space $\mathbb{P}(a_*)$ and its presentation as a finite group quotient of $\mathbb{P}^n$ is used to reduce to the homogeneous case. For the precise definition of when a point of $X_\sigma$ looks like a quasi-homogeneous singularity see Definition 5.1. For the precise assumption in this case, see Assumption 5.5.

Now let $p \in X_\sigma$ be a singularity of the special fibre of $f : X \rightarrow \text{Spec} \mathcal{O}$ that looks like a homogeneous or an $a_*$-weighted homogeneous singularity defined by $F$. We compute Ayoub’s nearby cycles functor $\Psi_f$ for the scheme $X_\sigma$ locally at a point $p$ piece by piece, using a semi-stable reduction $Y$ for $X$ constructed by a blow-up, base change and normalisation. Using the formalism of the nearby cycles functor and the Euler characteristics of the pieces - the strata of $Y_\sigma$ - we obtain a global formula for the Euler characteristic of nearby cycles.

Theorem 1.4 (Corollary 4.4). Let $f : X \rightarrow \text{Spec} \mathcal{O}$ be as in definition 1.2, and suppose that the special fibre $X_\sigma$ has finitely many singular points $p_1, \ldots, p_r$, and for each $i$, $f$ looks at $p_i$ like the singularity defined by a homogeneous polynomial $F_i \in k(p_i)[T_0, \ldots, T_n]$ of degree $e_i$, with $V(F_i) \subset \mathbb{P}^n_{k(p_i)}$ a smooth hypersurface, and with $\prod e_i$ prime to the exponential characteristic of $k$. Let $X_\sigma^0 = X_\sigma \setminus \{p_1, \ldots, p_r\}$. Then

$$\chi_c(\Psi_f(\mathbb{I}_{X^0})) = \chi_c(X_\sigma^0) + \sum_{i=1}^r \chi_c(V(F_i - T_{n+1}^c)) - (-1)^r \sum_{i=1}^r \chi_c(V(F_i)).$$

We write here the homogeneous case only not to overload notation, but we prove a similar formula in the more general case of singular points that look like quasi-homogeneous singularities, see Corollary 5.8. We then deduce the following formula.

Theorem 1.5 (Theorem 7.2). Let $f : X \rightarrow \text{Spec} \mathcal{O}$ be as in Definition 1.2, with $k$ of characteristic $0$. Assume that an isolated singular point $p$ of the special fibre $X_\sigma$ looks like an $a_*$-weighted homogeneous singularity defined by a polynomial $F \in k[p][T_0, \ldots, T_n]$ of weighted degree $e$, and with $V(F) \subset \mathbb{P}^n_{k(p)}(a_*)$ smooth over $k(p)$. Then

$$\chi_c(\Psi_f(\mathbb{I}_{X^0}|_p)) = \text{Tr}_{k(p)/k}(\Delta_e(F/k(p)) + \langle 1 \rangle).$$
Note that this includes the case of a homogeneous $F$ by taking the weights $a_\ast = (1, \ldots, 1)$. Assume for simplicity $k = k(p)$. The formula reads
\[ \chi_c(\Psi_f(\mathbb{1}_{X_p})) - \langle 1 \rangle = \Delta_t(F/k); \]
we may think of the left hand side as enumerating vanishing cycles for $X$ around $p$, and the right hand side as doing the same for the hypersurface $H^F$. So this gives us a comparison between the two schemes, and allows us to use the main result of [LPLS] for $H^F$, in order to get a formula for the scheme $X$ at $p$. Using the formalism of Ayoub’s functor, we can consequently extend it to a global formula on a scheme $X$ with several isolated homogeneous or quasi-homogeneous singularities. We state now the quasi-homogeneous version.

**Theorem 1.6** (Generalized quadratic conductor formula for quasi-homogeneous singularities, Corollary 7.4, Corollary 8.3). Let $f : X \to \text{Spec } k$ be as in Definition 1.2, of relative dimension $n$ with $f$ proper and $k$ of characteristic 0. Suppose that $X_p$ satisfies Assumption 5.5 with singular points $\{p_1, \ldots, p_s\}$. Let $e_i$ denote the weighted-homogeneous degree of the corresponding polynomial $F_i$. Then
\[ \sum_{i} \text{Tr}_{k(p_i)/k} \left( \prod_j a_{j}^{(i)} \cdot e_i - \langle 1 \rangle \right) + (-\langle e_i \rangle)^n \cdot \mu_{j,p_i}^q. \]

This settles Conjecture 5.4 in [LPLS] for the case of characteristic zero and singularities resolved by a single blow-up with a smooth exceptional divisor (satisfying Assumption 4.1 or 5.5); in fact, our result handles cases not covered by Conjecture 5.4, as the types of singularities treated above are not necessarily homogeneous or weighted-homogeneous in the sense of *loc. cit.* This is a generalization of the formula in [LPLS, Theorem 5.3] even for the case of a single singularity, as it does not assume $X$ is the hypersurface $H^F$. An interesting aspect in the quadratic formula, is that besides generalizing the classical formulas over the complex and real numbers, the summands $\text{Tr}_{k(p_i)/k} ((\prod_j a_{j}^{(i)} \cdot e_i) - \langle 1 \rangle)$ for each $p_i$ vanish in the classical cases and hence make appearance only ’motivically’. For more on that last point see [LPLS, Section 1 and Remark 5.5].

The quadratic Milnor number $\mu_{j,p_i}^q$ is the same as the $A^1$-local Euler class for $X$ at $p_i$, $e_{p_i}(\Omega_{X/k})$. By a homotopy invariance argument this class is the same as the local Euler class for $H^F_i$ (Corollary 6.10), which also equals to the quadratic Milnor number $\mu_{F_i,0}^q$, defined purely algebraically in terms of $F_i$, this is dealt with in section 6.

As an application of the main formula, we deduce a quadratic formula for curves.

**Corollary 1.7** (Corollary 8.4). Let $C$ be a reduced curve on a smooth projective surface $S$ over a field $k$ of characteristic zero, let $\mathcal{C}$ be its normalisation. Suppose that $\mathcal{O}_S(C)$ admits a section $s$ with smooth divisor $C_1$ that intersects $C$ transversely. Suppose in addition that each singular point $p$ of $C$ is a homogeneous singularity, let $e_p$ denote the homogeneous degree at $p$. Then
\[ \sum_{p \in C_{\text{sing}}} \text{Tr}_{k(p)/k} \left( (e_p) - (e_p)[\mu_{F,p}^q] - (\sum_{q \in p^{-1}(p)} \text{Tr}_{k(q)/k(p)}(1)) \right). \]

Again we write here the homogeneous version for simplicity, the more general weighted homogeneous formula appears in the text. This formula refines a formula in integers that can be deduced from the Jung-Milnor formula for curves.

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2 The motivic Euler characteristic with compact supports

A construction of central importance in this article is the motivic Euler characteristic with compact supports, \( \chi_c \). For a finite type separated \( k \)-scheme, \( \chi_c(X/k) \) is an element in the Grothendieck-Witt ring GW(\( k \)) of \( k \). Before going into a detailed discussion of \( \chi_c(X/k) \) and related notions, we first give sketch of the main ideas that go into its construction. We use the notation and properties of the unstable and stable motivic homotopy categories to be found in [Ay07], [CD], and [Hoy17], including the six-functor formalism for SH(-).

Let \( p : X \to \text{Spec } k \) be a smooth and proper scheme over a field \( k \). As we shall see below, its motive with compact supports \( p_! \mathbb{1}_X \) is a strongly dualisable object in the symmetric monoidal category (SH(\( F \)), \( \otimes \)), with dual \( (p_! \mathbb{1}_X)^\vee = p_! \mathbb{1}_X \). The Euler characteristic with compact supports of \( X/k \) is the trace of the identity endomorphism for \( p_! \mathbb{1}_X \in \text{SH}(k) \), this being defined as the composition

\[
\text{tr}(id_{p_! \mathbb{1}_X}) : \mathbb{1}_k \xrightarrow{\delta} p_! \mathbb{1}_X \otimes (p_! \mathbb{1}_X)^\vee \xrightarrow{\tau} (p_! \mathbb{1}_X)^\vee \otimes p_! \mathbb{1}_X \xrightarrow{ev} \mathbb{1}_k
\]

where \( \delta \) and \( ev \) are respectively the co-evaluation and evaluation maps of the dualising data, and \( \tau \) is the non-trivial permutation. Here \( \mathbb{1}_X, \mathbb{1}_k \) denote the respective unit objects in SH(\( X \)) and SH(\( k \)). This yields an element in \( \text{End}_{\text{SH}(k)}(\mathbb{1}_k) \), which is isomorphic as a ring to GW(\( k \)) via the Morel isomorphism [Mo, Lemma 6.3.8, Theorem 6.4.1]. We denote the corresponding element of GW(\( F \)) by \( \chi_c(X/k) \); we omit \( k \) when it is obvious from the context. For more details on the motivic Euler characteristic see [Le20, §2], and [AMBOWZ, §1] for a nice introduction on the compactly supported version.

Here are some useful notations and definitions.

**Notation 2.1.** For a field \( k \), we usually let \( p \) denote the exponential characteristic of \( k \), that is, \( p \) is the characteristic of \( k \) if this is positive, and is 1 if the characteristic is zero. We will always assume that the characteristic is different than 2.

For \( X \) a separated noetherian scheme, we let \( \text{Sch}_X \) denote the category of separated finite type schemes over \( X \) and let \( \text{Sm}_X \) be the full subcategory of smooth (separated and finite type) schemes over \( X \). We will refer to an object \( Y \to X \) of \( \text{Sch}_X \) as an \( X \)-scheme or scheme over \( X \) and similarly refer to an object \( Y \to X \) of \( \text{Sm}_X \) as a smooth \( X \)-scheme or smooth scheme over \( X \).

**Definition 2.2.** [Sch, Definition 1.9] Let \( k \) be a field. Let \( M(k) \) be the monoid of equivalence classes of non-degenerate quadratic forms on \( k \), with the operation induced by direct sum of quadratic forms, \( \oplus \). Define GW(\( k \)) to be the Grothendieck group completion of \( M(k) \). Concretely, elements of GW(\( k \)) are formal differences of classes of quadratic forms on \( k \). The tensor product of quadratic forms \( \otimes \) induces a well-defined multiplication on \( M(k) \) that extends to GW(\( k \)), making (GW(\( k \)), \( \oplus \), \( \otimes \)) a ring that we call the Grothendieck-Witt ring of \( k \).

For \( a \in k^\times \), we denote by \( \langle a \rangle \in \text{GW}(k) \) the class corresponding to the quadratic form \( x \mapsto ax^2 \).

**Definition 2.3.** Let \( (\mathcal{C}, \otimes, \mathbb{1}_C) \) be a symmetric monoidal category, and take \( x \in \text{Ob}(\mathcal{C}) \). We say that \( x \) is strongly dualisable if there exists an object \( x^\vee \in \text{Ob}(\mathcal{C}) \) and morphisms \( \delta_x : \mathbb{1}_C \to x \otimes x^\vee \) and \( ev_x : x^\vee \otimes x \to \mathbb{1}_C \), called respectively co-evaluation and evaluation, such that

\[
x \simeq \mathbb{1}_C \otimes x \xrightarrow{\delta_x \otimes id} x \otimes x^\vee \otimes x \xrightarrow{id \otimes ev_x} x \otimes \mathbb{C} \simeq x
\]

and

\[
x^\vee \simeq x^\vee \otimes 1 \xrightarrow{id \otimes \delta_{x^\vee}} x^\vee \otimes x \otimes x^\vee \xrightarrow{ev_x \otimes id} 1 \otimes x^\vee \simeq x^\vee
\]

are the identity morphisms. We call the object \( x^\vee \) the dual of \( x \).

For \( x \) a strongly dualisable object of \( \mathcal{C} \) and \( f : x \to x \) an endomorphism, the trace of \( f \) is the element \( \text{tr}(f) \in \text{End}_\mathcal{C}(\mathbb{1}_C) \) defined as the composition

\[
\mathbb{1}_C \xrightarrow{\delta_x} x \otimes x^\vee \xrightarrow{\tau \otimes id} x^\vee \otimes x \xrightarrow{ev_x} \mathbb{1}_C.
\]

In particular, taking \( f = id_x \), we have the categorical Euler characteristic \( \chi_c(x) := \text{tr}_x(id_x) \).

**Remark 2.4.** It follows directly from the definitions that for \( x, y \) strongly dualisable objects of \( \mathcal{C} \), we have

\[
\chi_c(x \otimes y) = \chi_c(x) \otimes \chi_c(y).
\]
Definition 2.5 ([CD, Definition 4.2.1]). Define $\text{SH}_c(X)$, the subcategory of constructible objects in $\text{SH}(X)$, as the thick triangulated subcategory generated by the objects $\sum^n_{\alpha} f_\# \mathbb{1}_Y$, where $f : Y \to X$ is a smooth $X$-scheme and $n \in \mathbb{Z}$. An object in this category is called a constructible object.

Proposition 2.6. Constructible objects are stable under $f^*$ for any morphism $f$, under $f_\#$ for a smooth $f$, under $f'$ for a proper $f$, and under $f_i$ for a separated $f$ of finite type [CD, Proposition 4.2.4, 4.2.11, 4.2.12]

In addition, for $i : Z \to X$ a closed immersion and $j : U \to X$ its open complement, an object $\alpha \in \text{SH}(X)$ is constructible if and only if $i^* \alpha$ and $j^* \alpha$ are constructible. [CD, Proposition 4.2.10]

A result of May [May, Theorem 0.1] about additivity of trace maps in triangulated categories has the following consequences for $\text{SH}(k)$.

Proposition 2.7. Let $n > 0$ be an integer and let

$$\alpha \to \beta \to \gamma \to \alpha[1]$$

be a distinguished triangle in $\text{SH}(k)[1/n]$. Then

1. If any two of $\alpha$, $\beta$, $\gamma$, is strongly dualisable, so is the third (the subcategory of strongly dualisable objects in $\text{SH}(k)[1/n]$ is thick).

2. $\chi_{\text{SH}(k)[1/n]}(\beta) = \chi_{\text{SH}(k)[1/n]}(\alpha) + \chi_{\text{SH}(k)[1/n]}(\gamma)$.

Proposition 2.8. Take $\alpha \in \text{SH}(k)[1/p]$, with $k$ a perfect field of exponential characteristic $p$. If $\alpha$ is constructible then it is strongly dualisable.

Proof. By Proposition 2.7 (1), the subcategory of strongly dualisable objects in $\text{SH}(k)$ is thick. By [LYZR, Appendix B, Cor. B2], for every smooth, separated, and finite type morphism $Y \to k$ in $\text{Sm}_k$, $\sum^n_{\alpha} f_\# \mathbb{1}_Y$ is strongly dualisable in $\text{SH}(k)$. As elements of this type generate the thick subcategory of constructible objects we get the result. $\square$

As a consequence we can now make the following definition.

Definition 2.9. Let $k$ be a perfect field of exponential characteristic $p$, $q : X \to k$ a $k$-scheme and $\alpha \in \text{SH}(X)[1/p]$ a constructible object. Then $\chi_c(\alpha/k)$ is defined to be the categorical Euler characteristic of $q_\alpha$ in $\text{SH}(k)[1/p]$:

$$\chi_c(\alpha/k) := \chi_{\text{SH}(k)[1/p]}(q_\alpha).$$

This is well defined as $q_\alpha \in \text{SH}(k)$ is constructible by Proposition 2.6 and is strongly dualisable (in $\text{SH}(k)[1/p]$) by Proposition 2.8.

In particular we define

$$\chi_c(X/k) := \chi_c(\mathbb{1}_X/k) = \chi_{\text{SH}(k)[1/p]}(q_\mathbb{1}_X) := tr(id_{q_\mathbb{1}_X})$$

for every $k$-scheme $q : X \to \text{Spec} k$. We write $\chi_c(\alpha)$ for $\chi_c(\alpha/k)$ when the base-field $k$ is clear from the context.

Remark 2.10. In the case $k = \mathbb{C}$, the rank homomorphism of quadratic forms gives an isomorphism, $rk : \text{GW}(\mathbb{C}) \cong \mathbb{Z}$. We recover the topological Euler characteristic under this identification

$$\chi_c(X/\mathbb{C}) = \chi_c^{\text{top}}(X(\mathbb{C})) = \chi^{\text{top}}(X(\mathbb{C})).$$

For the first equality see [Le20, remark 1.5]. The second equality is true for every complex algebraic variety.

A useful property of the compactly supported motivic Euler characteristic is the cut-and-paste property, which is formulated in the following proposition.

Proposition 2.11. Let $q : X \to \text{Spec} k$ be a $k$-scheme. Let $\alpha \in \text{SH}(X)$ be a constructible object, and let

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\leftarrow} U$$

be a closed embedding and its open complement. Then

$$\chi_c(\alpha) = \chi_c(i^* \alpha) + \chi_c(j^* \alpha)$$

and

$$\chi_c(\alpha) = \chi_c(i^! \alpha) + \chi_c(j_* j^* \alpha).$$

7
Proof. The distinguished triangle of endofunctors on $SH(X)$

$$j_! j^! \to \text{id}_{SH(X)} \to i_* i^* \to$$

gives a distinguished triangle of endofunctors on $SH(k)$ after composing with $q_i$,

$$q_i j_! j^! \to q_i \to q_i i_* i^* \to$$

Applying each of these terms to $\alpha$ gives a constructible object in $SH(k)$ by Proposition 2.6, which is therefore strongly dualisable in $SH(k)[1/p]$ (Proposition 2.8) so we can apply $\chi_c$ and use additivity (Proposition 2.7 (2)) to get

$$\chi_c(\alpha) = \chi_c(j_! j^! \alpha) + \chi_c(i_* i^* \alpha).$$

Since $i_* = i_!$, $j^* = j^!$, we have

$$\chi_c(\alpha) = \chi_c(j^! \alpha) + \chi_c(i^* \alpha).$$

Doing the same with the distinguished triangle

$$i_! i^! \to \text{id}_{SH(X)} \to j_* j^* \to$$

we get

$$\chi_c(\alpha) = \chi_c(i^! \alpha) + \chi_c(j_* j^* \alpha).$$

Remark 2.12. Let $k$ be a perfect field and let $X$ be a $k$-scheme, $Y \subset X$ a closed subscheme and $U$ the open complement $X \setminus Y$, then from Proposition 2.11 applied to $\alpha = 1_X$ we get

$$\chi_c(X) = \chi_c(Y) + \chi_c(U).$$

From this relation it follows that the motivic Euler characteristic factorises through the Grothendieck ring of $k$-varieties $K_0(\text{Var}_k)$. In other words we have the following commutative diagram

$$\xymatrix{\text{Var}_k \ar[rr]^\chi_{c(-/k)} \ar[d]_{X \mapsto [X]} \ar[rd]_{K_0(\text{Var}_k)} & & \text{GW}(k) \ar[d] \\ & & },$$

The cut-and-paste relation yields in the standard way a Mayer-Vietoris property with respect to a Zariski open cover for $\chi_c(-)$.

Proposition 2.13. Let $k$ be a perfect field, let $X$ be a $k$-scheme with a Zariski open cover $X = U_1 \cup U_2$ and let $\alpha \in SH(X)$ be a constructible object. Let $U_{12} = U_1 \cap U_2$ and let $j_1 : U_1 \to X$, $j_2 : U_2 \to X$, and $j_{12} : U_{12} \to X$ be the inclusions. Then

$$\chi_c(\alpha) = \chi_c(j_1^* \alpha) + \chi_c(j_2^* \alpha) - \chi_c(j_{12}^* \alpha).$$

Proof. Let $Z = X \setminus U_1 = U_2 \setminus U_{12}$, with reduced scheme structure, and with closed immersions $i : Z \to X$, $i_2 : Z \to U_2$. We have the canonical isomorphism $i_2^* j_{12}^* \alpha \cong i^* \alpha$, whence the identity

$$\chi_c(i^* \alpha) = \chi_c(i_2^* j_{12}^* \alpha).$$

By Proposition 2.11, we have the identities

$$\chi_c(\alpha) = \chi_c(j_1^* \alpha) + \chi_c(i^* \alpha)$$

and

$$\chi_c(j_2^* \alpha) = \chi_c(j_{12}^* \alpha) + \chi_c(i_2^* j_{12}^* \alpha).$$

Putting these together gives the desired result. \qed
Proposition 2.14 (Purity). Let \( i : Z \to X \) be a closed immersion of smooth \( k \)-schemes, or pure codimension \( c \), let \( f : X \to \text{Spec} \ k \), \( g : Z \to \text{Spec} \ k \) be the structure morphisms. Then for \( \alpha \) a constructible object of \( \text{SH}(k) \), we have
\[
\chi_c(i^! f^* \alpha) = (-1)^c \cdot \chi_c(g^* \alpha)
\]
In particular,
\[
\chi_c(i^! 1_X) = (-1)^c \chi_c(Z/k).
\]
Proof. We use the notation from [Hoy17]. Let \( f : Z \to \text{Spec} \ k \), \( g : Z \to \text{Spec} \ k \) be the structure morphisms, let \( \Omega_f, \Omega_g \) be the respective sheaves of relative differentials, and let \( N_i \) be the conormal sheaf of \( i \). We have the purity isomorphism (see [Hoy14, Appendix A])
\[
i^! \circ f^* \cong \Sigma^{-N_i} \circ g^*.
\]
Using the Mayer-Vietoris property Proposition 2.13 for \( \chi_c(-) \), we reduce to the case of trivial conormal sheaf, \( N_i \cong \mathcal{O}_Z \), inducing the natural isomorphism \( \Sigma^{-N_i} \cong \Sigma^{-c} \), and giving the purity isomorphism
\[
i^! \circ f^* \cong \Sigma^{-c} \circ g^*.
\]
We have the projection formula [Hoy17, Theorem 6.18(7)]
\[
g_!(\Sigma^{-c} \beta) = \Sigma^{-c} g_!(\beta)
\]
for \( \beta \in \text{SH}(Z) \). Since \( \Sigma^{-c} \gamma \cong S^{-2c-c} \wedge \gamma \) for \( \gamma \in \text{SH}(k) \), it follows from Remark 2.4 and [Le20, Lemma 2.2] that
\[
\chi_{\text{SH}(k)}(\Sigma^{-c} \gamma) = (-1)^{-c} \cdot \chi_{\text{SH}(k)}(\gamma) = (-1)^c \cdot \chi_{\text{SH}(k)}(\gamma)
\]
for \( \gamma \in \text{SH}(k) \) strongly dualisable. Thus
\[
\chi_c(i^! f^* \alpha) = \chi_{\text{SH}(k)}(\Sigma^{-c} g_!(g^* \alpha)) = (-1)^c \cdot \chi_c(g^* \alpha).
\]

The special case \( \chi_c(i^! 1_X) = (-1)^c \chi_c(Z/k) \) follows by taking \( \alpha = 1_k \).

\[\square\]

Remark 2.15 (Non-perfect fields). Let \( F \) be a field of characteristic \( p > 2 \), and with perfect closure \( F^{\text{perf}} \supset F \). The base-extension \( \text{GW}(F)[1/p] \to \text{GW}(F^{\text{perf}})[1/p] \) is an isomorphism. For a constructible object \( \beta \in \text{SH}(F) \), the base-extension \( \beta^{\text{perf}} \in \text{SH}(F^{\text{perf}}) \) is constructible. Moreover, for an \( F \)-scheme \( q : X \to \text{Spec} \ F \) and an element \( \alpha \in \text{SH}(X) \), we have the base-change \( q^{\text{perf}} : X \times_{\text{Spec} \ F} \text{Spec} \ F^{\text{perf}} \to \text{Spec} \ F^{\text{perf}} \) and \( \alpha^{\text{perf}} \in \text{SH}(X) \), with \( q^! (\alpha^{\text{perf}}) \) canonically isomorphic to the base-change \( q^! (\alpha) \) of \( q^! (\alpha) \). Thus, we may define \( \chi_c(\alpha/F) \) by
\[
\chi_c(\alpha/F) \coloneqq \chi_c(\alpha^{\text{perf}}/F^{\text{perf}}) \in \text{GW}(F^{\text{perf}})[1/p] = \text{GW}(F)[1/p].
\]
Having done this, it is easy to show that all the properties of \( \chi_c(-/k) \) described above extend to non-perfect base-fields \( F \), and we will use this extension to non-perfect \( F \) without further mention.

Another useful formula concerns change of base-field. For \( k_1 \subset k_2 \) a finite separable field extension, we have the transfer map on the Grothendieck-Witt rings
\[
\text{Tr}_{k_2/k_1} : \text{GW}(k_2) \to \text{GW}(k_1).
\]
This is the so-called Scharlau transfer\(^1\) with respect to the trace map \( \text{Tr}_{k_2/k_1} : k_2 \to k_1 \) and is defined as follows. For a finite-dimensional \( k_2 \)-vector space \( V \) and a non-degenerate symmetric \( k_2 \)-bilinear map \( b : V \times V \to k_2 \), one considers \( V \) as a (finite-dimensional) \( k_1 \)-vector space, giving the symmetric \( k_1 \)-bilinear map \( \text{Tr}_{k_2/k_1} \circ b : V \times V \to k_1 \); the fact that \( k_2 \) is separable over \( k_1 \) implies that \( \text{Tr}_{k_2/k_1} \) is surjective and hence \( \text{Tr}_{k_2/k_1} \circ b \) is non-degenerate. Sending \( b \) to \( \text{Tr}_{k_2/k_1} \circ b \) defines the map \( \text{Tr}_{k_2/k_1} : \text{GW}(k_2) \to \text{GW}(k_1) \).

\[
\text{Proposition 2.16.} \text{ Let } k_1 \subset k_2 \text{ be a finite separable extension of fields, let } \pi : \text{Spec} \ k_2 \to \text{Spec} \ k_1 \text{ be the induced morphism, and let } f : X \to \text{Spec} \ k_2 \text{ be a } k_2 \text{-scheme, which we consider as } k_1 \text{-scheme via composition with } \pi. \text{ For a constructible object } \alpha \in \text{SH}(X) \text{ we have}
\]
\[
\chi_c(\alpha/k_1) = \text{Tr}_{k_2/k_1}(\chi_c(\alpha/k_2)) \in \text{GW}(k_1).
\]
\[
\text{Proof. } \text{This is [Hoy14, Proposition 5.2] combined with the canonical isomorphism } (\pi \circ f)_! \cong \pi_! \circ f_! \text{.} \]
3 Motivic nearby cycles

3.1 Ayoub’s motivic nearby cycles functor

For a noetherian separated scheme $X$, we let $\mathbf{QProj}_X$ denote the category of quasi-projective $X$-schemes and $\mathbf{SmQProj}_X$ the full subcategory of smooth, quasi-projective $X$-schemes.

Throughout the paper we fix a discrete valuation ring $\mathcal{O}$ with residue field $k$, fraction field $K$ and fixed uniformizer $t \in \mathcal{O}$; $\sigma$ will denote the closed point $\text{Spec} k$ and $\eta$ the generic point $\text{Spec} K$. We define $B$ to be $\text{Spec} \mathcal{O}$. We will assume in addition that $\mathcal{O}$ contains a subfield $k_0$ such that $B$ is smooth and essentially of finite type over $k_0$, and the field extension $k_0 \to k$ is finite and separable.

Let $f : X \to B$ be a flat, quasi-projective $B$-scheme. We have the closed immersion $i$ and the open immersion $j$

$$\sigma \mapsto B \leftarrow j.\eta.$$  

We denote the respective pullbacks by $X_\sigma$, $X_\eta$ (‘the special and the generic fibre’) and denote the maps induced by $f$ according to the following diagram

$$X_\sigma \longrightarrow X \leftarrow X_\eta$$

For the construction of the motivic nearby cycles functor

$$\Psi_f : \text{SH}(X_\eta) \to \text{SH}(X_\sigma)$$

see [Ay07, 3.2.1]. Fixing the parameter $t$ defines a map $t : \text{Spec} \mathcal{O} \to \text{Spec} k_0[t]$. By abuse of notation we will use $\Psi_f$ also to denote $\Psi_{t \circ f}$, with the base being $k_0$. In the paper we use some of the functorial properties satisfied by $\Psi(-)$, among the we have the following.

**Property 3.1** (see [Ay07, Definition 3.1.1]). For each morphism $g : Y \to X$, of flat quasi-projective $B$-schemes, there are well-defined natural transformations

$$\alpha_g : g_\sigma^* \circ \Psi_f \to \Psi_{f \circ g} \circ g_\eta^*$$

and

$$\beta_g : \Psi_f \circ g_\eta^* \to g_\sigma^* \circ \Psi_{f \circ g}$$

such that:

(a) If $g$ is smooth $\alpha_g$ is a natural isomorphism.

(b) if $g$ is projective then $\beta_g$ is a natural isomorphism.

These natural transformations satisfy some compatibility conditions, for details check [Ay07, 3.1.1, 3.1.2]

The next result describes a very useful property for computing $\Psi_f$.

**Notation 3.2.** Let $X$ be a smooth $k_0$-scheme, $D$ a reduced normal crossing divisor on $X$ with irreducible components $D_1, \ldots, D_r$. For $I \subset \{1, \ldots, r\}$, let $D_I := \cap_{i \in I} D_i$, $D_\mathcal{O}_I := \cup_{i \in I} D_i \setminus \cup_{j \notin I} D_j$, $D_{(I)} := \cup_{i \in I} D_i$, and $D_{(I)} := D_{(I)} \setminus \cup_{j \in I} D_j$.

**Proposition 3.3** ([Ay07, Théorème 3.3.44]). Let $f : X \to B$ be a flat quasi-projective $B$-scheme. Suppose that $X$ is smooth over $k_0$ and that $X_\sigma := f^{-1}(0)$ is a reduced normal crossing divisor with irreducible components $D_1, \ldots, D_r$. Fix a non-empty subset $I \subset \{1, \ldots, r\}$, let $D_{(I)} \xrightarrow{v} D_{(I)} \xrightarrow{w} X_\sigma$ denote the respective open and closed immersions. Then composing $u^*\Psi_f f_\eta^*$ with the unit map $id \to v_! v^*$ of the adjunction, induces a natural isomorphism

$$u^*\Psi_f f_\eta^* \simeq v_! v^* u^* \Psi_f f_\eta^*$$

For the rest of the section we fix $I$ and let $D := D_{(I)}$, $D^\sigma := D_{(I)}$. For $i : Z \to Y$ the inclusion of a locally closed subscheme, and $\alpha \in \text{SH}(Y)$, we sometimes write $\alpha|_Z$ for $i^*(\alpha) \in \text{SH}(Z)$.
Remark 3.4. We retain the notation from Proposition 3.3 Evaluating at \( \mathbb{1}_\eta \in \text{SH}(\eta) \) and formulating the statement slightly differently, we have

\[
(\Psi_f(\mathbb{1}_{X_\eta}))|_D = v_*(\Psi_f(\mathbb{1}_{X_\eta}))|_{D^\circ}.
\]

Here \( (\Psi_f(\mathbb{1}_{X_\eta}))|_D \) denotes the pullback \( u^*\Psi_f(\mathbb{1}_{X_\eta}) \in \text{SH}(D) \) via the inclusion \( u : D \to X_\sigma \), and similarly \( \Psi_f(\mathbb{1}_{X_\eta})|_{D^\circ} := v^*u^*\Psi_f(\mathbb{1}_{X_\eta}) \in \text{SH}(D^\circ) \).

Moreover, taking \( I = \{i\} \), we get

\[
(\Psi_f(\mathbb{1}_{X_\eta}))|_{D^\circ} = w^*\Psi_{id}(\mathbb{1}_B) = w^*(\mathbb{1}_\sigma) = \mathbb{1}_{D^\circ}
\]

where \( w : D^\circ \to \sigma \) is the structure morphism. This last statement follows from the compatibility of \( \Psi(-) \) with smooth pullback, Property 3.1, applied to the open immersion \( X \setminus \cup j \neq i D_j \to X \) and then to the smooth morphism \( X \setminus \cup j \neq i D_j \to B \). In addition, the identity \( \Psi_{id}(\mathbb{1}_B) = \mathbb{1}_\sigma \) follows from [Ay07, Proposition 3.4.9, Lemma 3.5.10].

Remark 3.5. The statement of the theorem appears in [Ay07, Théorème 3.3.10, Remarque 3.3.12] for the case \( X = Sp_1 := B[T_1, \ldots, T_k]/(T_1 \cdot \ldots \cdot T_k - t) \) and \( f \) the obvious morphism to \( B \). In [Ay07, Théorème 3.3.44] the statement is essentially the same as in our Proposition 3.3, with the assumption \( I = \{i\} \). This special case is in fact all we need to use later on.

3.2 The Euler characteristic of nearby cycles

Retain the notation of \( \mathcal{O} \) and \( B = \text{Spec} \mathcal{O} \) as in Section 3.1. Let \( f : X \to B \) be a flat quasi-projective morphism with \( X \) smooth over \( k_0 \) and \( X_\eta \) smooth over \( \eta \). We make here some first computations of \( \chi_c(\Psi_f(\mathbb{1}_{X_\eta})) \).

Proposition 3.6.

(1) \( f_\sigma ! \Psi_f(\mathbb{1}_{X_\eta}) \) is a strongly dualisable object in \( \text{SH}(k) \).

(2) \( \chi_c(\Psi_f(\mathbb{1}_{X_\eta})) \in \text{GW}(k) \) is well-defined.

Proof. For the first assertion, \( \Psi_f \) sends constructible objects to constructible objects [Ay07, Théorème 3.5.14] and constructibles are stable under the exceptional pushforward functor \( (-)_{\!f} \) [Ay07, Corollaire 2.2.20], hence \( f_\sigma ! \Psi_f(\mathbb{1}_{X_\eta}) \) is constructible and therefore strongly dualisable (Proposition 2.8).

(2) follows from (1) and Definition 2.9. \( \square \)

By the following formal consequence of the properties of \( \Psi_f \) one can compute \( \chi_c(\Psi_f) \) by just investigating \( \Psi_f \) around isolated singularities.

Proposition 3.7. [LPLS, Proposition 8.3] Assume \( P = \{p_1, \ldots, p_s\} \) is the (finite) set of singular points in \( X_\sigma \). Then

\[
\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) = \sum_i \chi_c(\Psi_f(\mathbb{1}_{X_\eta}))_{|P_i} + \chi_c(X_\sigma \setminus P)
\]

Proof. Denote by \( j : X \setminus P \to X \), then by Property 3.1,

\[
\Psi_f(\mathbb{1}_{X_\eta})|_{X_\sigma \setminus P} \simeq \Psi_{f_{j} !}(j_{\eta}^* \mathbb{1}_X) = \Psi_{f_{j} !}((\mathbb{1}_{X \setminus P})_{\eta}) = \mathbb{1}_{X_\sigma \setminus P}
\]

the last equality being since \( X \setminus P \) is smooth (e.g. by 3.3).

Then by cut-and-paste (2.11)

\[
\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) = \sum_i \chi_c(\Psi_f(\mathbb{1}_{X_\eta}))_{|P_i} + \chi_c(\Psi_f(\mathbb{1}_{X_\eta})|_{X_\sigma \setminus P})
\]

and we get the result. \( \square \)

The following example illustrates how we can use Proposition 3.3 to compute \( \chi_c(\Psi_f) \) on a reduced normal crossing divisor stratum by stratum.
Example 3.8. Suppose $X_\sigma$ is a reduced normal crossing divisor on $X$ that can be written as $X_\sigma = D_1 + D_2$ with $D_1$ and $D_2$ smooth over $\sigma$ and with transverse intersection $D_{12} := D_1 \cap D_2$. Let $D_i^0 := D_i \setminus D_{12}$, $i = 1, 2$.

We have the close-open complements

$$D_1 \hookrightarrow X_\sigma \overset{j}{\twoheadrightarrow} D_2^0.$$ 

Then by Proposition 2.11

$$\chi_c(\Psi_f(1_{X_\eta})) = \chi_c(\Psi_f(1_{X_\eta})|_{D_1}) + \chi_c(\Psi_f(1_{X_\eta})|_{D_2}).$$

Using Proposition 3.3

$$\chi_c(\Psi_f(1_{X_\eta})) = \chi_c(v_1 \cdot 1_{D_1^0}) + \chi_c(1_{D_2^0}),$$

applying both equations of Proposition 2.11 to $1_{D_1}$ and the close-open complements

$$D_{12} \overset{i}{\hookrightarrow} D_1 \overset{\nu}{\twoheadrightarrow} D_1^0$$

gives

$$\chi_c(1_{D_1}) = \chi_c(i^*1_{D_1^0}) + \chi_c(v_1^*1_{D_1}) = \chi_c(1_{D_{12}}) + \chi_c(1_{D_1^0})$$

and

$$\chi_c(v_1 \cdot 1_{D_1}) = \chi_c(v_1v_1^*1_{D_1}) = \chi_c(1_{D_1}) - \chi_c(i^*1_{D_1}).$$

Applying Proposition 2.14, we have

$$\chi_c(v_1 \cdot 1_{D_1}) = \chi_c(1_{D_1}) - (-1) \chi_c(1_{D_{12}}).$$

Combining the equations we get

$$\chi_c(\Psi_f(1_{X_\eta})) = \chi_c(1_{D_{12}}) + \chi_c(1_{D_1^0}) - (-1) \chi_c(1_{D_{12}}) + \chi_c(1_{D_2^0}).$$

We obtain the nice formula

$$\chi_c(\Psi_f(1_{X_\eta})) = \chi_c(D_1^0) + \chi_c(D_2^0) - (-1) \cdot \chi_c(D_{12}).$$

This exhibit how Proposition 3.3 enables us to compute the Euler characteristic of the nearby cycles functor of the unit when the special fibre is a reduced normal crossing divisor. We would like to be able to reduce the general case to that case, also when the special fibre is not reduced.

3.3 Semi-stable reduction

Let $O$ be still be a discrete valuation ring, and $B = \text{Spec} O$ as in Section 3.1. Let $f : X \to B$ be as in Section 3.2 a flat quasi-projective morphism with $X$ smooth over $k_0$ and $X_\eta$ smooth over $\eta$. Let $O_e := O[s]/(se - t)$, $B_e := \text{Spec} O_e$ and $b_e : B_e \to B$ the projection. Let $X_e := X \times_B B_e$. Note that $\sigma_e = \sigma$ as the residue field does not change by adding a root, but $\eta_e \to \eta$ may not be trivial.

Definition 3.9. A semi-stable reduction datum for $f$ consists of a natural number $e$ and a projective birational map $p_e : Y \to X_e$, such that $Y$ is smooth over $k_0$, $Y_\sigma$ is a reduced normal crossings divisor and $p_{\eta e} : Y_\eta \to X_{\eta e}$ is an isomorphism. In addition, we will require that the cover $B_e \to B$ is tame, that is, that $e$ is prime to the exponential characteristic of $k$. 
We have the following diagram describing the morphisms

A theorem by Kempf, Knudsen, Mumford, and Saint-Donat [KKMSD] asserts that over a field of characteristic 0, and base $B$ a smooth curve, every variety $X$ admits a semi-stable reduction.

**Proposition 3.10.** Assume $f : X \to B$ admits a semi-stable reduction $Y \xrightarrow{p_e} X_e \xrightarrow{f_e} B_e$ for some $e$. Let $\pi : X_e \to X$ be the projection, and let $f_Y = f_e \circ \pi_e$. Then

$$
\Psi_f(\mathbb{I}_{X_e}) = (\pi \circ p_e)_\sigma \circ \Psi_{f_Y}(\mathbb{I}_{Y_e})
$$

**Proof.** By [Ay07, Proposition 3.5.9], we have the natural isomorphism $\Psi_f \simeq p_{\sigma^*} \circ \Psi_{f_e} \circ p_\eta^*$. Since $p_{\eta^*}$ is an isomorphism, the natural map $\text{id}_{\text{SH}(X_{x_0})} \to p_{\eta^*} \circ p_{\eta^*}$ is an isomorphism. This together with the pushforward property of $\Psi$ for projective maps, Property 3.1(b), gives the sequence of isomorphisms

$$
\Psi_f(\mathbb{I}_{X_e}) \simeq p_{\sigma^*} \circ \Psi_{f_e}(\mathbb{I}_{X_{x_0}}) \simeq p_{\sigma^*} \circ \Psi_{f_e} \circ p_{\eta^*} \circ p_{\eta^*}(\mathbb{I}_{X_{x_0}})
$$

$$
\simeq p_{\sigma^*} \circ p_{\eta^*} \circ \Psi_{f_Y}(\mathbb{I}_{Y_e}) \simeq (\pi \circ p_e)_\sigma \circ \Psi_{f_Y}(\mathbb{I}_{Y_e}).
$$

As a consequence we can compute $\chi_c(\Psi_f)$ on a semi-stable reduction.

**Corollary 3.11.** $\chi_c(\Psi_f(\mathbb{I}_{X_e})) = \chi_c(\Psi_{f_Y}(\mathbb{I}_{f_Y}))$.

**Proof.** $(\pi \circ p_e)_\sigma$ is proper, so $(\pi \circ p_e)_{\sigma^*} = (\pi \circ p_e)_{\sigma^*}$. Since $\sigma = \sigma$, we thus have

$$
\chi_c(\Psi_f(\mathbb{I}_{X_e})) = \chi_{\text{SH}(X_e)}(f_{\sigma^*} \circ \Psi_{f}(\mathbb{I}_{X_e})) = \chi_{\text{SH}(X_e)}(f_{\sigma^*} \circ (p \circ p_e)_{\sigma^*} \circ \Psi_{f_Y}(\mathbb{I}_{Y_e}))
$$

$$
= \chi_{\text{SH}(X_e)}(f_{\sigma^*} \circ \Psi_{f_Y}(\mathbb{I}_{y_e})) = \chi_c(\Psi_{f_Y}(\mathbb{I}_{f_Y})).
$$

**Example 3.12.** Assume $X$ has a normal crossing special fibre with two components which are not necessarily reduced,

$$
X_\sigma = aD_1 + bD_2,
$$

and that $X$ admits a semi-stable reduction $Y$, with a (reduced) special fibre $Y_\sigma = \widehat{D}_1 + \widehat{D}_2$; let $\widehat{D}_{12}$ denote the intersection $\widehat{D}_1 \cap \widehat{D}_2$. Then by Example 3.8 and Corollary 3.11 we get

$$
\chi_c(\Psi_f(\mathbb{I}_{X_e})) = \chi_c(\Psi_{f_Y}(\mathbb{I}_{f_Y})) = \chi_c(\widehat{D}_1^\sigma) + \chi_c(\widehat{D}_2^\sigma) - (-1) - (1) \cdot \chi_c(\widehat{D}_{12}).
$$

Since $\chi_c(\mathbb{G}_m) = \chi_c(\mathbb{A}^1) - \chi_c(pt) = -1 - (1)$, the formula can be rewritten as

$$
\chi_c(\Psi_f(\mathbb{I}_{X_e})) = \chi_c(\widehat{D}_1^\sigma) + \chi_c(\widehat{D}_2^\sigma) - \chi_c(\mathbb{G}_m \times \widehat{D}_{12}).
$$

The computations of Example 3.12 can be extended for a fibre having more than two components, with no triple intersections. In the next section we describe how to construct a semi-stable reduction in such case.
3.4 Expressing $\chi_c(\Psi_f \mathbb{1})$ by coverings of the strata

In the course of their work on motivic integration and motivic Zeta functions, Denef and Loeser define a motivic Milnor fibre of morphism $f : X \to \mathbb{A}^1$ [DL00, 3.3], [DL98, 4] as an element in the Grothendieck ring of varieties, defined by certain coverings of the strata of the special fibre of a resolution of $f$. Ayoub, Ivorra and Sebag prove that the class of Ayoub’s functor at the identity in this ring can be computed by an alternating sum involving these coverings [AIS, Thm. 8.6]; their proof relies on the use of motivic stable homotopy categories for rigid analytic sheaves. We treat here a simple case in which semi-stable reduction can be achieved by a simple construction, and then the formula can be proven by purely geometric means, relying on the properties of the nearby cycles functor developed in the previous section. We take $\mathcal{O}$ and $B = \text{Spec } \mathcal{O}$ as in Section 3.1. Let $\sigma \mapsto B \leftarrow \eta$ be the closed point and generic point of $B$.

We recall the construction of the covering maps, that we call here the Denef-Loeser covers following the description in [IS, 3.1]:

Let $f : X \to B$ be a flat quasi-projective morphism with $X$ smooth over $k_0$ and $X_\eta$ smooth over $\eta$, and suppose $X_\sigma$ is a simple normal crossing divisor. We write $X_\sigma = a_1D_1 + \ldots + a_rD_r$ with $D_1, \ldots, D_r$ the reduced irreducible components and assume that if $\text{char } k = p > 1$, then $p \nmid a_i$ for each $i$.

Let $I$ be a non-empty subset of $\{1, \ldots, r\}$, giving us the closed stratum $u_I : D_I \to X_\sigma$ and open substratum $v_I : D_I^0 \to D_I$. $f$ may be described on some affine open neighbourhood $U$ of some point of $D_I$ as

$$f = u \cdot \prod_{i \in I} t_i^{a_i}$$

with $t_i \in \mathcal{O}_X(U), u \in \mathcal{O}_X(U)^\times$, and $D_I$ being $V(t_i)$ in $U$.

Let $N_I = \gcd_{i \in I}(a_i)$. We have the finite étale cover

$$\tilde{D}_{I,U} := \text{Spec}(\mathcal{O}_{D_I \cap U}[T]/(T^{N_I} - u)) \to D_I^0 \cap U.$$  

The finite morphism $\tilde{D}_I \to D_I$ is defined as the normalisation of $D_I$ in $\tilde{D}_{I,U}$ and $\tilde{D}_I^0 \subset D_I$ is defined to be the open subscheme $\tilde{D}_I \times_{D_I} D_I^0 \to \tilde{D}_I$. One shows that this construction is independent of the choice of $U$ and that $\tilde{D}_I \to D_I$ is étale.

We call the coverings $D_I \to D_I, \tilde{D}_I \to D_I^0$ the Denef-Loeser coverings of $D_I, D_I^0$, respectively. These coverings are well-defined up to isomorphism and do not depend on the choice of open neighbourhood and local coordinates.

In some cases semi-stable reduction can be achieved by taking $p : Y \to X_\sigma$ to be the normalisation of a base change $X_\epsilon$ of $X$, and the components of the special fibre $Y_\eta = D_1 + \ldots + D_r$ which lie above $D_1, \ldots, D_r$ give indeed the Denef-Loeser coverings described here. We address such a situation in the following proposition.

**Proposition 3.13.** Let $f : X \to B = \text{Spec } \mathcal{O}$ be a flat morphism, essentially of finite type. We assume that $X$ is smooth over $k_0$, with the generic fibre $X_\eta$ smooth over $\eta$. Suppose $X_\sigma$ is a normal crossing divisor, $X_\sigma = aD_1 + bD_2$, with each $D_i$ smooth over $\sigma$. Suppose in addition that $\gcd(a, b) = 1$, and if $\text{char } k = p > 0$ then $p \nmid a, b$. Let $e = ab$.

Form the base-change $X_\epsilon$ as defined above and let $Y \to \text{Spec } \mathcal{O}_\epsilon$ be the normalisation of $X_\epsilon$, with the induced morphism $h : Y \to X$. Let $E_i = h^{-1}(D_i)_{\text{red}}, i = 1, 2$. Then

1. $Y$ is a smooth $k_0$-scheme.
2. $E_1$ and $E_2$ are smooth divisors on $Y$, intersecting transversally. In particular, $Y_\sigma = E_1 + E_2$ is a reduced normal crossing divisor and $Y$ is a semi-stable reduction of $X$.
3. The maps $E_i \to D_i, \emptyset \neq I \subset \{1, 2\}$, are isomorphic to the Denef-Loeser covers $\tilde{D}_I \to D_I$.

**Proof.** Let $m, n$ be integers such that $1 = ma + nb$.

For the first assertion, take $q \in Y$; we will show that $Y$ is smooth over $k_0$ at $q$. If $q \in Y_\eta \simeq X_\eta$, then as $B$ is smooth over $k_0$ and $B_\epsilon \to B$ is tame, $B_\epsilon$ is also smooth over $k_0$. Since $X_\eta$ is smooth over $\eta$, we see that $Y$ is smooth over $k_0$ at $q$.

If $q$ is a point of $Y_\sigma$, let $p = h(q)$. We deal separately with the cases $p \in D_{12}, p \in D_1^0, p \in D_2^0$.

**Case 1** For $p \in D_{12}$, $f$ may be locally described on some affine open $U \ni p$ by $t = ux^ay^b, x, y \in \mathcal{O}_X(U)$ local coordinates on $U$ with $V(x) = D_1 \cap U, V(y) = D_2 \cap U$ and $u \in \mathcal{O}_X(U)^\times$. We may assume $u = 1$ as $ux^ay^b = u^{ma+nb}x^ay^b = (u^ax)^a(u^by)^b$ and so by replacing $x$ and $y$ by unit multiples we can get rid of $u$.

In the $e$-base change scheme $X_\epsilon$, where we take $s$ with $s^e = t$, the defining equation on $U_\epsilon$ becomes $s^e = x^ay^b$. 

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Normalisation can be achieved by adjoining roots $z^b = x$, $w^a = y$ as follows. Set $z = \frac{x^m}{y^n}$, $w = \frac{y^n}{x^m}$ and let $V = h^{-1}(U)$. Then $z$ and $w$ are in $\text{Frac}(O_X(U_c))$ and satisfy the integral equations above, so $z$ and $w$ are in the normalisation $O_Y(V)$, and satisfy the equation $z \cdot w = s$.

Now consider the ring $O_X(U)[s, z, w] \subset O_Y(V)$. We claim that in fact $O_X(U)[s, z, w] = O_Y(V)$ and that $V$ is smooth over $k$. Indeed, since $x, y$ are local coordinates they define an étale map $\text{Spec} O_X(U) \to \mathbb{A}^2_k$. This gives the étale ring extension $k[X, Y] \to O_X(U)$. The algebraic picture after adjoining $s, z, w$ to the ring $O_X(U)$ is described by the following commutative diagram:

$$
k_0[X, Y] \longrightarrow O_X(U)$$

$$k_0[X, Y, S, Z, W]/(S - ZW, S^n - X^nY^b, Z^b - X, W^a - Y) \longrightarrow O_X(U)[s, z, w]$$

which induces a surjective homomorphism

$$\phi : O_X(U) \otimes_{k_0[X, Y]} k_0[X, Y, S, Z, W]/(S - ZW, S^n - X^nY^b, Z^b - X, W^a - Y) \to O_X(U)[s, z, w].$$

We claim that $\phi$ is an isomorphism. To see this, denote the quotient ring in the left lower corner by $C$. Of the equations defining $C$, the second is redundant as it follows from the other three, the first makes the variable $S$ redundant, and the last make $X$ and $Y$ redundant, so we can write $C \simeq k_0[Z, W]$. Since $k_0[X, Y] \to O_X(U)$ is smooth, the homomorphism $k_0[Z, W] \to O_X(U) \otimes_{k_0[X, Y]} k_0[Z, W]$ is smooth as well, hence $O_X(U) \otimes_{k_0[X, Y]} k_0[Z, W]$ is smooth over $k_0$, of Krull dimension equal to the Krull dimension of $O_X(U)$. From the equations defining $C$ we can deduce the further relations

$$ZY^m = X^nS^m, WX^a = S^nY^m. \tag{3.1}$$

From the relations $S^n = X^nY^b$, $S = ZW$, and $t = x^n y^b$, we see that canonical map $O_X(U) \to O_X(U) \otimes_{k_0[X, Y]} k_0[Z, W]$ extends to $O_X(U)[s] \to O_X(U) \otimes_{k_0[X, Y]} k_0[Z, W]$ by sending $s$ to $1 \otimes ZW$. After inverting $x$ and $y$, the relations (3.1) and the universal property of the localization yield an extension of this homomorphism to

$$\psi : O_X(U)[x^{-1}, y^{-1}][s, z, w] \to O_X(U)[x^{-1}, y^{-1}] \otimes_{k_0[X, Y]} k_0[Z, W]$$

sending $z$ to $1 \otimes Z$, $w$ to $1 \otimes W$; $\psi$ then defines an inverse to $\phi$, after inverting $x$ and $y$. Furthermore, the extension $k_0[X, Y] \to k_0[Z, W]$ is flat, so $O_X(U) \to O_X(U) \otimes_{k_0[X, Y]} k_0[Z, W]$ is flat as well, and thus $x$ and $y$ are non-zero divisors on $O_X(U) \otimes_{k_0[X, Y]} k_0[Z, W]$. As $O_X(U) \otimes_{k_0[X, Y]} k_0[Z, W]$ and $O_X(U)[s, z, w]$ have the same Krull dimension and both rings are finite type $k_0$-algebras, the surjective, birational $k_0$-algebra homomorphism $\phi$ has zero kernel (by Krull’s principal ideal theorem), hence is an isomorphism, as claimed.

In addition, this shows that $O_X(U)[s, z, w]$ is a smooth $k_0$-algebra. Since $O_X(U)[s, z, w]$ contains $O_X(U_c)$ and is contained in the normalisation $O_Y(V)$ we have the desired equality $O_X(U)[s, z, w] = O_Y(V)$ and hence $V \subset Y$ is smooth. This also verifies that $Y_s \cap U$, defined by $s = 0 = z \cdot w$, is a reduced divisor, $Y_s \cap U \simeq \text{Spec} \ O_X(U)[z, w]/(zw)$, with $V_s \cap D_1 = V(z)$ and $V_s \cap D_2 = V(w)$.

By definition of the Denef-Loeser covers, since $\gcd(a, b) = 1$, $D_1 \simeq D_2 \xrightarrow{id} D_{12}$. But also

$$E_{12} \cap V \simeq \text{Spec} \ O_X(U)[z, w]/(z, w) \simeq \text{Spec} \ O_X(U) \otimes_{k[X, Y]} k[Z, W]/(Z, W) \simeq \text{Spec} \ O_X(U)/(x, y) \simeq D_{12} \cap U.$$

Thus $E_{12}$ coincides with the Denef-Loeser cover $\tilde{D}_{12} \simeq D_{12} \simeq E_{12}$.

**Case 2** Consider the case $p \in D_2^0$: the case $p \in D_2^0$ is handled the same way. There is a neighbourhood $U \ni p$ on which $f$ is described as $f^*(t) = t = u \cdot x^a$ with $u \in \Gamma(U, O_X)^\times$ and $U \cap D_1 = V(x)$. After $e$-base change we have the equation $s^e = u \cdot x^a$. Set $v = \frac{1}{v^a}$, then $v^a = u$, so $v$ is in $O_Y(V)$. In a similar manner to the previous case we wish to describe the ring $O_Y(V)$, to ascertain that $V \subset Y$ is smooth. We have to show that the inclusion $O_X(U)[s, v] \subset O_Y(V)$ is an equality. For this, we define a commutative square

$$k_0[W, W^{-1}, X] \longrightarrow O_X(U)$$

$$k_0[W, W^{-1}, X, V, S]/(V^a - W, S^b - VX) \longrightarrow O_X(U)[v, s]$$
We have the isomorphism
\[ k_0[W,W^{-1}, X, V, S]/(V^n - W, S^k - VX) \simeq k_0[V, V^{-1}, X, S]/(S^k V^{-1} - X) \simeq k_0[V, V^{-1}, S]. \]

As in the previous case, one shows that the square induces an isomorphism
\[ \mathcal{O}_X(U) \otimes_{k_0[W,W^{-1},X]} k_0[V, V^{-1}, S] \simeq \mathcal{O}_X(U)[v, s], \]
so \( \mathcal{O}_X(U)[v, s] \) is a smooth \( k_0 \)-algebra and is therefore equal to the normalisation \( \mathcal{O}_Y(V) \). Thus \( V \subset Y \) is smooth and \( Y_\sigma \cap V \), being defined by \( s = 0 \), is a smooth divisor on \( V \).

We can now show that \( \tilde{D}_1 \simeq E_1 \) over \( D_1 \). Let \( \pi : \tilde{D}_1 \to D_1 \) be the Denef-Loeser covering, \( U \) being the same neighbourhood of \( p \in D_1^\circ \) as above. Then by definition \( \pi^{-1}(D_1 \cap U) = \text{Spec}(\mathcal{O}_X(U)[T]/(T^n - u))/x) \simeq \mathcal{O}_X(U)(v)/(x) \). On the other hand
\[ E_1 \cap V = \text{Spec} \mathcal{O}_X(U)[v, s]/(s) \simeq \text{Spec} \mathcal{O}_X(U)(v)/(x). \]

We get \( E_1 \cap V \simeq \pi^{-1}(D_1 \cap U) \). Since \( E_1 \) is normal and \( \tilde{D}_1 \) is the normalisation of \( D_1 \) in \( \pi^{-1}(D_1 \cap U) \), we get \( \tilde{D}_1 \simeq E_1 \). In the same way \( \tilde{D}_2 \simeq E_2 \). This completes the proof of (1), (2) and (3).

**Remark 3.14.** With \( f : X \to B = \text{Spec} \mathcal{O} \) and \( a, b \) and \( e = ab \) as in Proposition 3.13, suppose that \( X \) is irreducible and that \( a = 1 \). We retain the notation of Proposition 3.13. We claim that the base-change \( X_e \) is integral. To see this, let \( x \) be a generic point of \( D_1 \). Since \( X \) is smooth, \( D_1 \) is a Cartier divisor on \( X \) and thus the local ring \( \mathcal{O}_{X,x} \) is a dvr. Moreover, since \( a = 1, t \) is a parameter for \( \mathcal{O}_{X,x} \). Let \( y \in X_e \) be the unique point lying over \( x \). Then \( \mathcal{O}_{X_e,y} = \mathcal{O}_{X,x} \otimes \mathcal{O}[s]/s^t - t = \mathcal{O}_{X,x}[s]/s^t - t. \)

Since \( e \) is prime to the characteristic, \( \mathcal{O}_{X_e,y} \) is smooth over \( k \), so \( \mathcal{O}_{X_e,y} \) is a normal local ring, hence integral. Since \( X_e \to X \) is finite and flat, each irreducible component of \( X_e \) dominates \( X \), and thus \( X_e \) is irreducible and is also reduced in a neighbourhood of \( y \). Since \( X_e \) is a hypersurface in the smooth \( k \)-scheme \( X \times_k \text{Spec} \mathcal{O}[s] \), \( X \) is Cohen-Macaulay, and the fact that \( X_e \) is irreducible and generically reduced then implies that \( X_e \) is integral.

**Proposition 3.15.** [AIS, Theorem 8.6] Let \( f : X \to \text{Spec} \mathcal{O} \) be a flat, quasi-projective morphism, with \( X \) smooth over \( k \) and with generic fibre \( X_\eta \) smooth over \( \eta \). Suppose that the special fibre \( X_\sigma = \sum a_i D_i \); if \( \text{char} k = p > 0 \), we suppose in addition that \( p \nmid \prod a_i \). Assume that for all \( i \neq j \) \( \gcd(a_i, a_j) = 1 \), and that there are no triple intersections, i.e. for each triple of distinct indices \( i, j, k \), \( D_i \cap D_j \cap D_k = \emptyset \).

Denote by \( D_i, \tilde{D}_i, \tilde{D}_ij \) the Denef-Loeser coverings.

Then
\[ \chi_c(\Psi f(1_{X_\eta})) = \sum_i \chi_c(\tilde{D}_i) - \sum_{i<j} \chi_c(\mathbb{G}_m \times \tilde{D}_{ij}). \]

**Remark 3.16.** This is a special case of the formula by Ayoub-Ivorra-Sebag [AIS, Theorem 8.6] which is proven there in a more general setting, relying on the theory of rigid analytic motives. The case considered here suffices for our use in this paper and follows from the same general construction as in our main result so we include it here.

**Proof.** To analyse each intersection separately consider
\[ X_{ij} := X \setminus \bigcup_{k \neq i,j} D_k, \]
and set \( D_i' := D_i \setminus \bigcup_{k \neq i,j} D_k \). Then \( X_{ij,e} = a_i D_i' + a_j D_j' \). Define \( f_{ij} = f|_{X_{ij}} : X_{ij} \to B \).

By Proposition 3.13, \( X_{ij} \) admits a semi-stable reduction \( Y_{ij} \) with components of the special fibre giving the Denef-Loeser coverings \( \tilde{D}_i' \to D_i', \tilde{D}_j' \to D_j' \) and \( \tilde{D}_{ij} = D_{ij} \). Note that \( (D_i')^o = D_i' \) and \( (D_j')^o = D_j' \), so \( \tilde{D}_i^o = \tilde{D}_i' \) and \( \tilde{D}_j^o = \tilde{D}_j' \).

We can use Example 3.12 to get
\[ \chi_c(\Psi f_{ij}(1_{X_{ij,e}})) = \chi_c(\tilde{D}_i') + \chi_c(\tilde{D}_j') - \chi_c(\mathbb{G}_m \times \tilde{D}_{ij}). \]
By the same argument applied to \(X_{ij} \setminus D_{ij}\), we find
\[
\chi_c(\Psi_{f_{ij}}(\mathbbm{1}_{X_{ij}\eta}|_{X_{ij}\setminus D_{ij}})) = \chi_c(\widetilde{D}_i^\eta) + \chi_c(\widetilde{D}_j^\eta),
\]
and by cut and paste, we have
\[
\chi_c(\Psi_{f_{ij}}(\mathbbm{1}_{X_{ij}\eta}|_{D_{ij}})) = \chi(\Psi_{f_{ij}}(\mathbbm{1}_{X_{ij}\eta})) - \chi_c(\Psi_{f_{ij}}(\mathbbm{1}_{X_{ij}\eta}|_{X_{ij}\setminus D_{ij}})),
\]
so
\[
\chi_c(\Psi_{f_{ij}}(\mathbbm{1}_{X_{ij}\eta}|_{D_{ij}})) = -\chi_c(\mathbb{G}_m \times \widetilde{D}_{ij}).
\]
Similarly,
\[
\chi_c(\Psi_{f_{ij}}(\mathbbm{1}_{X_{ij}\eta}|D^\eta_j)) = \chi_c(\widetilde{D}^\eta_j).
\]
Since \(X_{ij}\) is an open neighbourhood of \(D_{ij}\) in \(X\), the compatibility of \(\Psi_{(-)}\) with respect to the smooth morphism \(X_{ij} \hookrightarrow X\) (Property 3.13) implies
\[
\Psi(f(\mathbbm{1}_{X\eta}|_{D_{ij}}) = \Psi_{f_{ij}}(\mathbbm{1}_{X_{ij}\eta}|_{D_{ij}}).
\]
Similarly,
\[
\Psi(f(\mathbbm{1}_{X\eta}|_{D^\eta_i}) = \Psi_{f_{ij}}(\mathbbm{1}_{X_{ij}\eta}|_{D^\eta_i}).
\]
And by cut and paste along \(X_\sigma = \coprod^\iota D^\eta_i \coprod^\alpha D_{ij}\) we have
\[
\chi_c(\Psi(f(\mathbbm{1}_{X\eta}))) = \sum_i \chi_c(\Psi(f(\mathbbm{1}_{X\eta})|D^\eta_i)) + \sum_{ij} \chi_c(\Psi(f(\mathbbm{1}_{X\eta})|D_{ij})) = \sum \chi_c(\widetilde{D}^\eta_i) - \sum \chi_c(\mathbb{G}_m \times \widetilde{D}_{ij}).
\]

**Remark 3.17.** Suppose that we drop the hypothesis that the components \(D_i\) are smooth, but assume that the same construction as in Proposition 3.13 applied to the schemes \(X_{ij}\) yield a semi-stable reduction \(Y_{ij}\) with smooth components as \((Y_{ij})_\sigma, D_i, D_j, D_{ij}\) (so \(Y_{ij}\) is smooth, \((Y_{ij})_\sigma\) is a smooth normal crossing divisor and \((Y_{ij})_\eta \simeq (X_{ij})_\eta\)). In that case all the arguments in the proof of Proposition 3.15 still hold, and so the concluded formula does as well; however the terms \(D_i\) are no longer Denef-Loeser covers. We use this for the weighted homogeneous case in section 5.

**3.5 Nearby cycles at the base**

We continue to use our discrete valuation ring \(\mathcal{O}\), with subfield \(k_0\), residue field \(k\), fraction field \(K\) and parameter \(t\), and let \(B = \text{Spec} \mathcal{O}\), as in Section 3.1; in this section, however, we assume in addition that \(k_0\) has characteristic zero.

We have a ring homomorphism \(sp_t\) (see [LPLS, Remark 5.1]) from the Grothendieck-Witt ring of the fraction field \(K\) to that of the residue field \(k\), characterised as the unique map
\[
sp_t : \text{GW}(K) \to \text{GW}(k)
\]
satisfying:

1. \(sp_t(t) = \langle 1 \rangle\) for the uniformizer \(t\).
2. \(sp_t(u) = \langle \bar{u} \rangle\) for all invertible elements \(u \in O^\times\) where \(\bar{u}\) denotes the image of \(u\) under the quotient map \(O \to k\).

Given a strongly dualisable object \(\alpha \in \text{SH}(K)\), the motivic Euler characteristic \(\chi(\alpha)\) is an endomorphism of \(\text{SH}(K)\), and so the functor \(\Psi_{id} : \text{SH}(K) \to \text{SH}(k)\) can be applied to it and produce an endomorphism of the unit in \(\text{SH}(k)\). Via the Morel isomorphism we get an object in \(\text{GW}(k)\). We state results from [LPLS], which follows from the fact that \(\Psi_{id}\) is a monoidal functor in characteristic 0 [Ay07, Corollaire 3.5.19].

**Proposition 3.18 ([LPLS, Lemma 8.1]).** For \(\alpha \in \text{SH}(K)\), we have \(\Psi_{id}^*(\chi(\alpha)) = \chi(\Psi_{id}(\alpha))\).

In fact, \(sp_t\) computes Ayoub’s functor \(\Psi_{id}\).
Proposition 3.19 ([LPLS, Proposition 8.2]). The following diagram commutes.

\[
\begin{array}{ccc}
\text{End}_{\text{SH}(\mathbb{K})}(\mathbb{I}_K) & \xrightarrow{\Psi_{\text{ide}}} & \text{End}_{\text{SH}(k)}(\mathbb{I}_k) \\
\downarrow & & \downarrow \\
\text{GW}(K) & \xrightarrow{\psi_p} & \text{GW}(k)
\end{array}
\]

Here the vertical arrows are Morel’s isomorphisms.

4 The homogeneous case

We continue to use our discrete valuation ring \(\mathcal{O}\) and base-scheme \(B := \text{Spec} \mathcal{O}\), and retain the notations and assumptions from Section 3.1.

Let \(f : X \to B\) be a flat quasi-projective morphism with \(X\) smooth over \(k_0\) and with \(X_\eta\) is smooth over \(\eta\). We make the following assumption on the special fibre.

Assumption 4.1. The reduced special fibre \(X_\sigma\) has only isolated singularities \(p_1, \ldots, p_r\). Moreover, if \(X = Bl_p(X)\) is the blow up of \(X\) at \(P := \{p_1, \ldots, p_r\}\), \(E = E_{p_1} \cup \cdots \cup E_r\) the exceptional divisor and \(\pi^{-1}[X_\sigma] := \pi^{-1}(X_\sigma \setminus\{p_1, \ldots, p_r\})\) the proper transform, then \(\pi^{-1}[X_\sigma]\) is smooth over \(k\) and intersects each \(E_i\) transversally.

Now consider function on stalks \(f^*_p : \text{Spec} \mathcal{O} \to \text{Spec} \mathcal{O}_{X,p}\) (which we sometimes denote just by \(f^*\)). We show that Assumption 4.1 is equivalent to having an ‘analytic expansion’ of \(f\) at each singular point \(p\) of the form

\[f^*(t) = F(s_0, \ldots, s_n) + h\]

with \(s_0, \ldots, s_n\) local coordinates at \(p\), \(F\) a homogeneous polynomial of degree \(e\) defining a smooth projective hypersurface over \(k(p)\), and \(h \in m_p^{e+1}\), where \(m_p\) is the maximal ideal in \(\mathcal{O}_{X,p}\).

We say then that at \(p\), \(f\) looks like the homogeneous singularity defined by \(F\) (see Definition 1.2).

Proposition 4.2. Assumption 4.1 above is equivalent to the following two conditions:

1. The special fibre \(X_\sigma\) has only isolated singularities.
2. At each singular point \(p\), let \(\mathcal{O}_{X,p}\) denote the local ring at \(p\), with maximal ideal \(m_p\), let \(e_p\) be the maximal integer with \(f^*(t) \in m_p^{e_p}\), and let \(\overline{F^*(t)}_p\) be the image of \(f^*(t)\) in \(m_p^{e_p}/m_p^{e_p+1}\). Then \(\overline{F^*(t)}_p\) defines a smooth hypersurface in \(\text{Proj} \text{Sym}^*(m_p/m_p^2) \cong \mathbb{P}^n_{k(p)}\).

Moreover, if Assumption 4.1 is satisfied then for each singular point \(p\) there is a neighbourhood \(U\) such that, letting \(U \to U\) denote the blow-up of \(U\) at \(p\), the special fibre \(U_\sigma\) decomposes as \(U_\sigma = e_pD_1 + D_2\) with \(D_1 \cong \mathbb{P}^n_{k(p)}\) the reduced exceptional divisor and \(D_2 = \pi^{-1}[U_\sigma]\) the strict transform of \(U_\sigma\). Both \(D_1\) and \(D_2\) are smooth and intersect transversely, with \(D_1 \cap D_2 \subset D_1\) the hypersurface defined by \(\overline{F^*(t)}_p\).

Proof. Let \(p\) be a singularity and let \((s_0, \ldots, s_n) = m_p\) be a regular sequence of parameters on the maximal ideal \(m_p\) of \(\mathcal{O}_{X,p}\). We write

\[f^*(t) = F(s_0, \ldots, s_n) + h\]

with \(F\) a homogeneous polynomial of degree \(e\) with coefficients in \(\mathcal{O}_{X,p}\), and \(h \in m_p^{e+1}\).

\(\overline{F^*(t)}_p = F(s_0, \ldots, s_n)\) is a homogeneous equation defining an hypersurface in \(\mathbb{P}^n_{k(p)}\), \(k(p)\) the residue field of \(\mathcal{O}_{X,p}\). We show that this hypersurface is isomorphic to the intersection \(D_{12}\).

Define

\[
\tilde{X} = Bl_p(\text{Spec} \mathcal{O}_{X,p}) = \text{Proj} \mathcal{O}_{X,p}[T_0, \ldots, T_n]/(s_iT_j - s_jT_i)_{i < j}
\]

Let \(\tilde{X} = \bigcup U_i\) be the standard covering of the blow up, where \(U_i\) is defined by \(T_i \neq 0\).

For simplicity of notation we describe \(U_0\) but the argument is similar for each of the \(U_i\). Use \(s_0, t_1 = T_1/T_0, \ldots, t_n = T_n/T_0\) as coordinates on \(U_0\).

\[
U_0 = \text{Spec} \mathcal{O}_{X,p}[T_1/T_0, \ldots, T_n/T_0]/(s_iT_j - s_jT_i)_{i,j} = \text{Spec} \mathcal{O}_{X,p}[t_1, \ldots, t_n]/(s_0t_1 - s_1, \ldots, s_0t_n - s_n)
\]

We may write now

\[f^*(t) = s_0^\sigma \cdot (F(1, t_1, \ldots, t_n) + s_0h) =: s_0^\sigma \cdot y_0\]
with \( h \in m_p \). Then \( D_1 \cap U_0 = V_{U_0}(s_0) \), \( D_2 \cap U_0 = V_{U_0}(g_0) \) and \( D_{12} = V_{U_0}(s_0, g_0) \); We have \((U_0)_\sigma = e \cdot (D_1 \cap U_0) + D_2 \cap U_0\) and similarly for all \( i \), so \( X_\sigma = e \cdot D_1 + D_2 \).

So \( D_1 \cap U_0 \simeq \text{Spec } \mathcal{O}_{X,p}[t_1, \ldots, t_n]/(s_0, s_1, \ldots, s_n) \simeq \text{Spec } k(p)[t_1, \ldots, t_n] \). We have a similar computation for each \( i \). This shows that the \( D_1 \cap U_i \) form the standard affine chart for the projective space \( \mathbb{P}^n_{k(p)} \), giving the isomorphism \( D_1 \simeq \mathbb{P}^n_{k(p)} = \text{Proj } k(p)[T_0, \ldots, T_n] \), with \( D_1 \cap U_i \) defined as usual as the open subscheme \( T_i \neq 0 \).

\( D_{12} \cap U_0 \) is defined then by \( F(1, t_1, \ldots, t_n) = 0 \) inside \( D_1 \cap U_0 \); making the same construction for general \( i \) shows that \( D_1 \cap U_i \) is defined by \( F(t_1, \ldots, t_{i-1}, 1, t_i, \ldots, t_n) = 0 \) inside \( D_1 \cap U_i = \text{Spec } k(p)[t_1, \ldots, t_n] \), with \( t_j = T_{j-1}/T_i \) for \( j = 1, \ldots, i \) and \( t_j = T_j/T_i \) for \( j = i+1, \ldots, n \). This shows that \( D_{12} \) is globally defined in \( D_1 \simeq \mathbb{P}^n_{k(p)} \) by \( F \), as claimed. Thus the condition in the statement of the proposition is equivalent to the smoothness of \( D_{12} \).

Now, since the blow-up of \( X \) is smooth, \( \text{codim}(D_1) = \text{codim}(D_2) = 1 \) in the blow-up, and \( \text{codim}(D_{12}) = 2 \) being a hypersurface in \( D_1 \), the condition of the proposition is equivalent to Assumption 4.1.

In the following theorem we compute explicitly the strata of the special fibre of a semi-stable reduction, constructed according to Proposition 3.13.

**Theorem 4.3.** Let \( f : X \to \text{Spec } \mathcal{O} \) be a flat quasi-projective morphism with \( X \) smooth of dimension \( n + 1 \) over \( k_0 \). Suppose that \( X_\sigma \) has a single singular point \( p \) and that at \( p, f \) looks like the homogeneous singularity defined by \( F \in k(p)[T_0, \ldots, T_n] \) of degree \( e \), and that \( V(F) \subset \mathbb{P}^n_{k(p)} \) is a smooth hypersurface. We suppose in addition that \( e \) is prime to the exponential characteristic of \( k_0 \).

Let \( q : \tilde{X} \to X \) be the blow-up of \( X \) and let \( \mathcal{O}_e = \mathcal{O}[s]/(s^e - t) \). Let \( D_1 \subset \tilde{X} \) be the exceptional divisor and let \( D_2 \subset \tilde{X} \) be the proper transform of \( X_\sigma \). Then there exists a quasi-projective morphism \( Y \to \text{Spec } \mathcal{O}_e \) and a morphism \( \pi : Y \to X \) over \( \text{Spec } \mathcal{O}_e \to \text{Spec } \mathcal{O} \) such that

1. \( \pi \) defines a semi-stable reduction of \( X \).
2. The special fibre \( Y_\sigma \) is of the form \( \tilde{D}_1 + \tilde{D}_2 \) with \( \tilde{D}_1 \) and \( \tilde{D}_2 \) smooth, with intersection \( \tilde{D}_{12} \), and with \( \pi \) mapping \( \tilde{D}_1 \) to \( D_1 \), and \( \tilde{D}_2 \) to \( D_2 \).
3. We have \( \tilde{D}_1 \simeq V(F - T_0^{e+1}) \subset \mathbb{P}^n_{k(p)} \), and \( \tilde{D}_{12} \simeq V(F) \subset \mathbb{P}^n_{k(p)} \);
4. The maps \( \pi : \tilde{D}_2 \to D_2 \), \( \pi : \tilde{D}_{12} \to D_{12} := D_1 \cap D_2 \) are isomorphisms; the morphism \( \tilde{D}_1 \to D_1 = \mathbb{P}^n_{k(p)} \) is the evident cyclic cover, induced by the projection \( \mathbb{P}^n_{k(p)} \setminus \{(0, \ldots, 0, 1)\} \to \mathbb{P}^n_{k(p)} \) from \( (0, \ldots, 0, 1) \).

**Proof.** By Proposition 4.3, \( \tilde{X}_\sigma = e D_1 + D_2 \) with \( D_1 \simeq \mathbb{P}^n \) and \( D_2 \to X_\sigma \) a resolution of singularities of \( X_\sigma \), and so \( f \circ q : \tilde{X} \to \text{Spec } \mathcal{O} \) satisfies the requirements of Proposition 3.13 (with \( a = e, b = 1 \)). Then we have the scheme \( Y \) constructed by first forming the base-change by \( \mathcal{O} \to \mathcal{O}_e \), and then taking the normalisation. By Proposition 3.13, \( Y \) is a semi-stable reduction for \( \tilde{X} \). That is, \( Y \) is smooth over \( k_0 \) and \( Y_\sigma = \tilde{D}_1 + \tilde{D}_2 \) is a reduced simple normal crossing divisor. Also if we denote by \( h \) the composition

\[
h : Y \to \tilde{X}_e \to \tilde{X} \to X,
\]

then \( \tilde{D}_1 = h^{-1}(D_1) \to D_1 \) are the Denef-Loeser coverings for all \( \emptyset \neq I \subset \{1, 2\} \). The only thing we have left to do is to give the explicit description of those coverings.

By definition of Denef-Loeser covers and since \( b = 1 \), \( \tilde{D}_{12} \simeq D_{12} \) and \( \tilde{D}_{12} \simeq D_2 \). By Proposition 4.2 then, \( D_{12} \simeq V(F) \subset \mathbb{P}^n_{k(p)} \). In the remaining part of the proof we shall describe \( D_{12} \).

We only need to check the explicit description of the covering \( D_{12} \to D_1 \) after restriction over some neighbourhood of \( p \) in \( X \). Thus, we may replace \( X \) with the local scheme \( \text{Spec } \mathcal{O}_{X,p} \); we change notation and assume that \( X = \text{Spec } \mathcal{O}_{X,p} \) is local. Take the standard covering of the blow-up \( X = \bigcup U_i \), where \( U_i \) is defined by \( T_i \neq 0 \). Write again \( f^*(t) = F(s_0, \ldots, s_n) + h \) with \( F \) a homogeneous polynomial of degree \( e \) and \( h \in m_p^{e+1} \). Take \( s_0, t_1 = T_1/T_0, \ldots, t_n = T_n/T_0 \) as coordinates on \( U_0 \). Then

\[
U_0 \simeq \text{Spec } \mathcal{O}_{X,p}[t_1, \ldots, t_n]/(s_0 - s_0 t_i).
\]
On $U_0$, $f^*(t) = s_0 \cdot (F(1, t_1, \ldots, t_n) + s_0 \tilde{h}) = s_0 \cdot g_0$ with $\tilde{h} \in m_p$ and $g_0 = F(1, t_1, \ldots, t_n) + s_0 \tilde{h}$. After the base change, on $U_{0,e} = U_0 \times_{\mathcal{O}[t]/(t^n - t)}$ we have

$$U_{0,e} \simeq \mathcal{O}_{X,p}[t_1, \ldots, t_n, t^e]/(s_i - s_0 t_i, s_0 \cdot g_0 - (t^e)^e).$$

Normalising amounts to adjoining $t_{n+1} = t'/s_0$, which is an integral element as $t_{n+1}^e = g_0$ [see the proof of Proposition 3.13]. So on $V_0$, the inverse image of $U_0$ in $Y$, we have

$$V_0 = \text{Spec}(\mathcal{O}_{X,p}[t_1, \ldots, t_n, t_{n+1}]/(s_i - s_0 t_i, s_0 \cdot g_0 - (t_{n+1})^e)).$$

The special fibre $V_0$ then is covered by the $V_i = h^{-1}(U_i)$. The exceptional divisor $\widetilde{D}_1$ is the fibre along Spec $k(p) \hookrightarrow \text{Spec} \mathcal{O}_{X,p}$, defined by $s_0 = 0$ on $V_0$, and so

$$\widetilde{D}_1 \cap V_0 = \text{Spec} k(p)[t_1, \ldots, t_{n+1}]/(\tilde{g}_0 - t_{n+1}^e),$$

where $\tilde{g}_0 = F(1, t_1, \ldots, t_n)$. Set $\mathbb{P}^{n+1}_{k(p)} = \text{Proj} k(p)[T_0, \ldots, T_{n+1}] = \bigcup_{i=0}^{n+1} W_i$ to be the standard affine covering, with $W_i$ corresponding to $T_i \neq 0$, and identify $\widetilde{D}_1 \cap V_0$ as embedded in the affine space $W_0 = \text{Spec} k(p)[t_1, \ldots, t_{n+1}]$ with $t_j = T_j/T_0$.

In order to describe the cover $\widetilde{D}_1 \cap V_0 \rightarrow D_1 \cap U_0$, we also use the identification $D_1 = \mathbb{P}^n_{k(p)} = \text{Proj} k(p)[T_0, \ldots, T_n]$ as in Proposition 4.2, with $D_1 \cap U_0$ being Spec $k(p)[t_1, \ldots, t_n]$, still with $t_j = T_j/T_0$. We then get the restriction of the cover $\widetilde{D}_1 \rightarrow D_1$ to $V_0$ to be

$$\text{Spec} k(p)[t_1, \ldots, t_n, t_{n+1}]/(F(1, t_1, \ldots, t_n) - t_{n+1}^e) \longrightarrow \text{Spec} k(p)[t_1, \ldots, t_n]$$

with $t_j := T_j/T_i$ as in the proof of Proposition 4.2. Here we are considering $V_i \cap \widetilde{D}_1$ as a closed subscheme of $W_i$. We get $\widetilde{D}_1 \cap V_i = V(F - T_{n+1}^e) \cap W_i$ in $\mathbb{P}^{n+1}_{k(p)}$. These restrictions of $\widetilde{D}_1 \rightarrow D_1$ to $V_i$, patch together then to give exactly the desired cover

$$\mathbb{P}^{n+1}_{k(p)}(F(T_0, \ldots, T_n) - T_{n+1}^e) \rightarrow \mathbb{P}^n_{k(p)}.$$

To be precise, the open subschemes we described here are $V(F - T_{n+1}^e) \cap W_i$ for $i = 0, \ldots, n$, and in principle we should also consider the remaining open $V(F - T_{n+1}^e) \cap W_{n+1}$. This open is defined by $F(y_0, \ldots, y_n) - 1 = 0$ on $W_{n+1} = \text{Spec} k(p)[y_0, \ldots, y_n]$ with $y_i = T_i/T_{n+1}$, $i = 0, \ldots, n$. But since $F$ is homogeneous, $y_0, \ldots, y_n$ satisfying this equation cannot be all 0, so at least one $T_i \neq 0$, $i < n + 1$, and the point falls in some $W_i$, $i < n + 1$. So this remaining open is contained in the union of the others, and is therefore redundant for our covering describing $V(F - T_{n+1}^e)$.

$\widetilde{D}_{12}$ is given locally on $V_i$ by both $s_i = 0$ and $t_{n+1} = 0$, and so by the description of $\widetilde{D}_1 \rightarrow D_1$ above it is contained in the $\mathbb{P}^n_{k(p)} \subset \mathbb{P}^{n+1}_{k(p)}$ given by $T_{n+1} = 0$. We have

$$\widetilde{D}_{12} \simeq \widetilde{D}_{12} \simeq V(F) \subset \mathbb{P}^n_{k(p)}$$

as we saw in Proposition 4.2.

**Corollary 4.4.** Let $f : X \rightarrow \text{Spec} \mathcal{O}$ be a flat quasi-projective morphism with $X$ smooth over $k_0$ and with $X_\eta$ smooth over $\eta$. Suppose that the special fibre $X_\sigma$ has finitely many singular points $p_1, \ldots, p_r$, and for each $i$, $f$ looks at $p_i$ like the homogeneous singularity defined by a homogeneous polynomial $F_i \in k(p_i)[T_0, \ldots, T_n]$ of degree $e_i$, with $V(F_i) \subset \mathbb{P}^n_{k(p_i)}$ a smooth hypersurface, and with $\prod_i e_i$ prime to the exponential characteristic of $k_0$. Let $X^e_\sigma = X_\sigma \setminus \{p_1, \ldots, p_r\}$. Then

$$\chi_c(\Psi_f(1_{X_\eta})) = \chi_c(X^e_\sigma) + \sum_{i=1}^r \chi_c(V(F_i - T_{n+1}^e)) - (-1)^{\sum_{i=1}^r} \chi_c(V(F_i)).$$
Proof. For the case \( r = 1 \), with notation as in the previous theorem, 
\[
\tilde{D}_1^\eta = \tilde{D}_1 \setminus D_{12} \cong V(F - T_{n+1}^e) \setminus V(F).
\]
Then Theorem 4.3 and Proposition 3.15 tell us that 
\[
\chi_c(\Psi_f(1_X, \eta)) = \chi_c(X^o_\sigma) + \chi_c(V(F - T_{n+1}^e) \setminus V(F)) - \chi_c(G_m \times V(F)).
\]
with \( F = F_1 \). But \( G_m \times V(F) = \mathbb{A}^1 \times V(F) \setminus 0 \times V(F) \), so by cut and paste, we have 
\[
\chi_c(V(F - T_{n+1}^e) \setminus V(F)) - \chi_c(G_m \times V(F)) = \chi_c(V(F)) - \chi_c(\mathbb{A}^1 \times V(F)) + \chi_c(V(F)) = \chi_c(V(F - T_{n+1}^e)) - \chi_c(\mathbb{A}^1 \times V(F)),
\]
which is what we want.

In general, we proceed by induction on \( r \). Let \( U_1 = X \setminus \{ p_1 \} \), \( U_2 = X \setminus \{ p_2, \ldots, p_r \} \) and let \( U_{12} = U_1 \cap U_2 \), with open immersions \( j_1 : U_1 \to X \), \( j_2 : U_2 \to X \) and \( j_{12} : U_{12} \to X \). By our induction hypothesis together with Property 3.1 applied to the smooth morphisms \( j_1, j_2, j_{12} \), we have 
\[
\chi_c(j_1^* \Psi_f(1_X, \eta)) = \chi_c(X^o_\sigma) + \sum_{i=2}^r \chi_c(V(F_i - T_{n+1}^e)) - \sum_{i=1}^r \chi_c(G_m \times V(F_i)),
\]
\[
\chi_c(j_2^* \Psi_f(1_X, \eta)) = \chi_c(X^o_\sigma) + \chi_c(V(F_1 - T_{n+1}^e)) - \chi_c(G_m \times V(F_1)),
\]
and 
\[
\chi_c(j_{12}^* \Psi_f(1_X, \eta)) = \chi_c(X^o_\sigma).
\]
The Mayer-Vietoris property (Proposition 2.13) then yields the result. \( \square \)

5 The quasi-homogeneous case

We can extend the results of the previous section to larger class of singularities for which the defining polynomial is weighted homogeneous. The usual blow-up should be replaced by a weighted blow-up, but treating it is not as straightforward as in the homogeneous case. For example, the exceptional divisor of a weighted blow-up would be a weighted projective space, which is not smooth when the weights are not trivial. Therefore the result of [AIS, Theorem 8.6] (Proposition 3.15), cannot be applied as it is, as the special fibre is not a simple normal crossing divisor. However in view of remark 3.17 and using the construction of Proposition 3.13, if the covering strata are smooth we can still get a similar formula to hold. Presenting the scheme in the weighted case as a quotient of a scheme with a homogeneous singularity modulo a finite group allows us to use the results of the previous section, once we show that the quotient defines a semi-stable reduction of our original degeneration with smooth strata at the special fibre. The end result is completely parallel to the homogeneous case, considering weights into account.

We retain our assumptions on the discrete valuation ring \( O \) with residue field \( k \) and parameter \( t \) from Section 3.1.; as before, we let \( \sigma \mapsto B := \text{Spec} \ O \leftarrow \eta \) denote the closed and generic points of \( B := \text{Spec} \ O \), respectively, and we have the subfield \( k_0 \) of \( O \), with \( B \) smooth and essentially of finite type over \( k_0 \), and with \( k_0 \to k \) finite and separable.

Let \( X \) be a separated \( k_0 \)-scheme, essentially of finite type, and \( p \in X \) a smooth closed point with maximal ideal \( m_p \subset O_{X,p} \). Let \( (s_0, \ldots, s_n) \) be a regular sequence generating \( m_p \) and let \( (a_0, \ldots, a_n) \) be a system of positive integral weights with \( \text{gcd}(a_i, a_j) = 1 \) for every \( i, j \). Define the ideal \( m_p^{(e)} \subset O_{X,p} \) to be the ideal generated by monomials of weighted homogeneous degree \( \ell \), that is, by monomials \( s_0^{a_0} \cdots s_n^{a_n} \) with \( \ell = \sum_j a_j t^j \).

Definition 5.1. Let \( f : X \to \text{Spec} \ O \) a flat proper morphism of schemes with \( X \) smooth over \( k_0 \) and \( X_\eta \) smooth over \( \eta \). Let \( p \in X_p \) be an isolated singular point and let \( F \in k(p)[T_0, \ldots, T_n] \) be a homogeneous polynomial of weighted degree \( e \) for some weights \( a_* = (a_0, \ldots, a_n) \) as above. We say that \( (X_\sigma, p) \) looks like the weighted homogeneous singularity defined by \( F \) if there is a regular sequence of generators for \( m_p \) such that 
\[
f^*(t) \equiv F(s_0, \ldots, s_n) \mod m_p \cdot m_p^{(e)} \cdot m_p^{(e)} \cdot m_p^{(e)} \cdot m_p^{(e)} \cdot m_p^{(e)} \cdot m_p^{(e)} \to k(p).
\]
Here we have implicitly chosen a splitting of \( O_{X,p}/m_p \cdot m_p^{(e)} \cdot m_p^{(e)} \to k(p) \).
5.1 Weighted projective space

First let us review the notion of weighted projective space as in [LPLS]. Let $R$ be a ring and $a = (a_0,\ldots,a_n)$ a sequence of positive integers, which we call weights. Let $R[X_0,\ldots,X_n]$ be the graded ring with $X_i$ having degree $a_i$. Define

$$\mathbb{P}_R(a) = \text{Proj} R[X_0,\ldots,X_n].$$

An alternate description of $\mathbb{P}_R(a)$ is as a quotient of $\mathbb{P}^n$ by the group scheme $\mu_a = \mu_{a_0} \times \cdots \times \mu_{a_n}$. Let $\iota_a : R[X_0,\ldots,X_n] \to R[Y_0,\ldots,Y_n]$ be the graded ring homomorphism mapping $X_i$ to $Y_i^{a_i}$, where the ring $R[X_0,\ldots,X_n]$ is with the $a$-grading, and $R[Y_0,\ldots,Y_n]$ is with the usual grading on a polynomial ring. Let $\mu_a$ act on $R[Y_0,\ldots,Y_n]$ by $Y_i \mapsto \zeta_a Y_i$, for $\zeta_a \in \mu_a$. Then the image of $\iota_a$ can be identified with the fixed ring $R[Y_0,\ldots,Y_n]^{\mu_a}$, hence defining

$$\pi : \mathbb{P}^n \to \mathbb{P}(a)$$

as a quotient $\mathbb{P}(a) \simeq \mathbb{P}^n/\mu_a$.

We may as well view the projective space $\mathbb{P}^n$ at the source of $\pi$ as achieved from $\mathbb{P}(a)$ by adjoining for each $i$ the $a_i$-th root of $X_i$. We now describe a similar construction of a local version of a ‘weighted blow-up’ of our scheme $X$ in Definition 5.1, retaining the notations from that definition.

As our construction will be local around the given point $p \in X$, we replace $X$ with an affine open neighbourhood $U$ of $p$ in $X$, such that the local parameters $s_0,\ldots,s_n$ of Definition 5.1 extend to étale coordinates on $U$, that is, the morphism $(s_0,\ldots,s_n) : U \to \mathbb{A}^{n+1}_k$ is étale. We change notation and suppose $X = U$, and let $A$ denote the ring of functions on the affine scheme $X = \text{Spec} A$. We let $m_p \subset A$ denote the maximal ideal of $p$ and following Definition 5.1, we define $m_p^{(e)} \subset m_p$ as the ideal defined by monomials of weighted degree $e$ in the $s_i$.

Construction 5.2. With $p \in X = \text{Spec} A$, $a_s = (a_0,\ldots,a_n)$ and $s_0,\ldots,s_n \in m_p$ étale coordinates on $X$ as above, define $A[s^{1/a}] := A[\sigma_0,\ldots,\sigma_n]/(\sigma_0^{a_0} - s_0,\ldots,\sigma_n^{a_n} - s_n)$ and let $Z = \text{Spec} A[s^{1/a}]$. Let $\mu_a = \mu_{a_0} \times \cdots \times \mu_{a_n}$. We have the $\mu_a$-action on $A[s^{1/a}]$, where $\zeta \in \mu_a$, acts by

$$\zeta \cdot \sigma_j := \begin{cases} 
\zeta \sigma_i & \text{for } j = i \\
\sigma_j & \text{for } j \neq i.
\end{cases}$$

Then $A$ is equal to the subring of $\mu_a$-invariants in $A[s^{1/a}]$, $A = A[s^{1/a}]^{\mu_a}$, and so the map

$$\pi : Z \to X$$

realises $X$ as the quotient of $Z$ by the action of the group scheme $\mu_a$. Also, there is a unique point $q \in Z$ lying over $Z$, and we have $k(q) = k(p)$. We let $m_q \subset A[s^{1/a}]$ denote the maximal ideal of $q \in Z$.

An argument similar to that given in Remark 3.14 shows that $Z$ is smooth over $k$ and if $X$ is integral, then so is $Z$.

From Definition 5.1, we have

$$f^*(t) = F(s_0,\ldots,s_n) + h.$$ 

After shrinking $X$ if necessary, and changing notation, we may assume that $h$ is in $m_p$, $m_p^{(e)} \subset A$. Letting $g := \pi \circ f : Z \to \text{Spec} \mathcal{O}$, we have

$$g^*(t) = F(\sigma_0^{a_0},\ldots,\sigma_n^{a_n}) + h'$$

with $h' \in m_p^{+1} \subset B$. Let $G(Z_0,\ldots,Z_n) \in k(p)[Z_0,\ldots,Z_n]$ be the degree $e$ polynomial with $G(\sigma_0,\ldots,\sigma_n) = F(\sigma_0^{a_0},\ldots,\sigma_n^{a_n})$.

Definition 5.3 ([LPLS, Def. 4.2]). Let $F$, $G$ be defined as in the above Construction 5.2. We say that $V(F) \subset \mathbb{P}_{k(p)}(a)$ is a smooth quotient hypersurface if the polynomial $G$ defines a smooth hypersurface $V(G) \subset \mathbb{P}_{k(p)}^n$ and in addition $V(F) \subset \mathbb{P}_{k(p)}(a)$ is smooth. Furthermore, letting $v_i \in \mathbb{P}_{k(p)}(a)$ be the point with $i$-th homogeneous coordinate 1 and all other coordinates 0, we require that $F(v_i) \neq 0$ if $a_i > 1$. Finally, we require that the weights $a_i$ are pairwise relatively prime, each $a_i$ divides $e$, and $e$ is prime to the exponential characteristic of $k$.

Remark 5.4. The condition that each $a_i$ divides $e$ implies that $V(F)$ is a Cartier divisor on $\mathbb{P}_{k(p)}(a)$. This being the case, the assumption that $V(F) \subset \mathbb{P}_{k(p)}(a)$ is smooth implies that $V(F)$ does not contain any singular point of $\mathbb{P}_{k(p)}(a)$. If $n \geq 2$, and if the $a_i$ are pairwise relatively prime, then $v_i$ is a singular point of $\mathbb{P}_{k(p)}(a)$ if $a_i > 1$, so in case $n \geq 2$, the last condition in the definition above is superfluous.
5.2  The nearby cycles of a quasi-homogeneous singularity

As before, we take $\mathcal{O}$ and $B = \text{Spec } \mathcal{O}$ as in Section 3.1 and we fix a flat quasi-projective morphism $f : X \to B$ with $X$ smooth over $k_0$ and $X_\eta$ smooth over $\eta$. We formulate conditions for the singularities in the quasi-homogeneous case, similar to Assumption 4.1.

Assumption 5.5.

(1) The special fibre $X_\sigma$ has only isolated singularities $p_1, \ldots, p_r$.

(2) For each $p \in \{p_1, \ldots, p_r\}$ there is a polynomial $F \in k(p)[T_0, \ldots, T_n]$ of weighted degree $e_p$ with respect to some weights $a_\ast$, with $\text{gcd}(a_\ast) = 1$ and $\text{lcm}(a_\ast)$ dividing $e_p$, such that $F$ defines a smooth quotient hypersurface in $P_{k(p)}(a_\ast)$ (Definition 5.3 above), and $(X_\sigma, p)$ looks like the weighted homogeneous singularity defined by $F$ (see Definition 5.1).

For later use we need the following fact:

Lemma 5.6. Let $k$ be a field and let $Y$ be a $k$-scheme, separated and essentially of finite type over $k$. Let $D$ be an effective Cartier divisor on $Y$. Suppose that both $D$ and $Y \setminus D$ are smooth over $k$. Then $Y$ is smooth over $k$.

Proof. Since smoothness is invariant under field extensions we may assume $k$ is algebraically closed. Let $y$ be a point in $Y$. Since $D$ is a closed subscheme of $Y$, if $y \notin D$ then it has a smooth neighbourhood. We have to show that also $y \in D$ is a smooth point in $Y$. Since $D$ is an effective Cartier divisor, there is a neighbourhood $U$ of $y$ in $Y$, and a non-zero divisor $f$ on $U$ such that $D \cap U$ is defined by the vanishing of $f$. The exact sequence of sheaves

$$0 \to \mathcal{O}_U(-D \cap U) \xrightarrow{f} \mathcal{O}_U \to \mathcal{O}_{D \cap U} \to 0$$

gives on stalks at $y$

$$0 \to \mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{D,y} \to 0$$

Let $d = \text{dim } Y$ so $\text{dim } D = d - 1$. $D$ is smooth so $\mathcal{O}_{D,y}$ is a regular local ring of dimension $d - 1$, so we can write the maximal ideal $m_{D,y}$ as generated by a regular sequence, $m_{D,y} = (f_1, \ldots, f_{d-1})$. The $f_i$ lift to $f_1, \ldots, f_{d-1}$ in $m_{Y,y}$. Now since $\text{ker}(\mathcal{O}_{Y,y} \to \mathcal{O}_{D,y}) = (f)\mathcal{O}_{Y,y}$ we get from the exact sequence that $m_{Y,y} = (f, f_1, \ldots, f_{d-1})$, with $(f, f_1, \ldots, f_{d-1})$ a regular sequence. Then $\mathcal{O}_{Y,y}$ is a regular local ring and hence $y$ is a smooth point of $Y$.

Assuming that our only singularity is $p = p_1$, the main result of the section is an analogue of Theorem 4.3. The assumptions in the statement of the theorem are meant to choose a convenient neighbourhood to work with, as it does not matter for the global formula deduced in Corollary 5.8.

Theorem 5.7. Let $f : X \to B$ be a flat quasi-projective morphism such that the generic fibre $X_\eta$ is smooth over $\eta$ and with $X$ smooth over $k_0$, satisfying Assumption 5.5. Suppose in addition that $p \in X_\sigma$ is the only singular point of $X_\sigma$. Let $e = e_p$, let $F$ be as in Assumption 5.5 for $p$, with respect to weights $a_\ast$, and let $\mathcal{O}_e = \mathcal{O}[t]/(t^e - t)$. Finally, we assume that $X = \text{Spec } A$ is affine with a system of étale coordinates $s_0, \ldots, s_n \in m_p$, and that the all the steps in Construction 5.2 can be carried out for $(X, p, F, s_\ast, a_\ast)$ without having to shrink $X$ to a smaller affine neighbourhood of $p$.

Let $\pi : Z \to X \simeq Z/\mu_a$ be the $\mu_a$-quotient map given by Construction 5.2 and let $q \in Z$ be the unique point lying over $p$; note that $k(p) = k(q)$. Let $Z = \text{Bl}_q(Z)$ and let $Y_Z \to \tilde{Z}$ be the normalisation of the base-change $\tilde{Z} := \tilde{Z} \times_{\text{Spec } \mathcal{O}} \text{Spec } \mathcal{O}_e$. Then the $\mu_a$-action on $\tilde{Z}$ extends to a $\mu_a$-action on $Y_Z$. Moreover, letting $Y := Y_Z/\mu_a$ and letting $Y \to X$ be the resulting map on the quotients, we have

(2) $Y$ is smooth over $k$ and $Y \to \text{Spec } \mathcal{O}_e$ is a semi-stable reduction of $X \to \text{Spec } \mathcal{O}$.

(2) Let $F \in k(p)[T_0, \ldots, T_n]$ be the weighted weighted-homogeneous polynomial of weighted degree $e$ as given by Assumption 5.5 for $(X_\sigma, p)$. Then the special fibre $Y_\sigma \subset Y$ is a reduced normal crossing divisor, $Y_\sigma = \overline{D_1} + \overline{D_2}$ with $\overline{D_1}, \overline{D_2}$ smooth. Letting $\overline{D_{12}} := \overline{D_1} \cap \overline{D_2}$, we have

$$\overline{D_1} \simeq V(F - T_{n+1}) \subset \mathbb{P}_{k(p)}(a, 1)$$

$$\overline{D_{12}} \simeq V(F) \subset \mathbb{P}_{k(p)}(a).$$

Moreover, the projection $q : \overline{D_2} \to X_\sigma$ is an isomorphism over $X_\sigma \setminus \{p\}$ and defines a resolution of singularities of $X_\sigma$, with $q^{-1}(p) = \overline{D_{12}}$. 

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We have the identity
\[ \chi_c(\Psi_f(1_{\mathcal{X}_n})) = \chi_c(\hat{D}_1) + \chi_c(\hat{D}_2) - ((-1) - (1)) \cdot \chi(\hat{D}_{12}). \]

Proof. We may assume that \( X \) is integral and we retain the notation from Construction 5.2. Let
\[ A[{s^{1/\alpha}}] = A[\sigma_0, \ldots, \sigma_n]/(\sigma_0^a - s_0, \ldots, \sigma_n^a - s_n). \]
We have \( Z = \text{Spec} A[{s^{1/\alpha}}], Z \) is integral and is smooth over \( k \), and we have a \( \mu_a \)-action on \( Z \) with quotient \( X \). Let \( \pi : Z \to X = Z/\mu_a \) be the quotient map, induced by the inclusion \( A \leftarrow R := A[{s^{1/\alpha}}] \). Let \( q \in Z \) be the unique point lying over \( p \in X \). \((Z, q)\) satisfies Assumption 4.1 for looking like a homogeneous singularity defined by \( G(\sigma_0, \ldots, \sigma_n) := F(\sigma_0^a, \ldots, \sigma_n^a) \) (see Construction 5.2). \( G \) has degree \( e \) and \( V(G) \) is smooth by our assumption on \( F \). We apply Theorem 4.3 and construct the semi-stable reduction \( Y_Z \to \text{Spec} \mathcal{O}_Z \) of \( Z \to \text{Spec} \mathcal{O} \) by forming the blow-up \( \tilde{Z} := Bl_q Z \), and letting \( Y_Z \) be the normalisation of the base-change \( \tilde{Z} := Z \times_{\text{Spec} \mathcal{O}} \text{Spec} \mathcal{O}_e \).

Since the \( \mu_a \)-action on \( Z \) fixes \( q \), this action lifts canonically to an action on \( Z \), which gives a \( \mu_a \)-action on \( \tilde{Z} \). Let \( Y := Y_Z/\mu_a \) and let \( \pi : Y_Z \to Y \) denote the quotient map. Since \( Y_Z \to X \) is proper, it follows that the induced map on the quotients \( Y \to X_e \) is also proper.

Let \( E_1 \subset \tilde{Z} \) be the exceptional divisor, let \( E_2 \subset \tilde{Z} \) be the strict transform of \( Z_\sigma \) and let \( E_{12} = E_1 \cap E_2 \). Denote by \( \tilde{E}_1, \tilde{E}_2, \tilde{E}_{12} \) their respective coverings in \( (Y_Z)_\sigma \), as in the proof of Theorem 4.3. Let \( \tilde{D}_i := \pi(\tilde{E}_i) = \tilde{E}_i/\mu_a \subset Y \).

Since \( Z \) is integral, it follows from Remark 3.14 that \( Y_Z \) is integral and thus the quotient scheme \( Y = Y_Z/\mu_a \) is integral as well.

We use the standard presentation of the blow-up \( \tilde{Z} \) as
\[ \tilde{Z} = \text{Proj} A[{s^{1/\alpha}}][Z_0, \ldots, Z_n]/\{(s_i Z_j - s_j Z_i)_{0 \leq i, j \leq n}\} \]
giving the standard open cover of \( \tilde{Z} \) by the affine open subsets \( Z_i \neq 0 \). This induces the affine open cover \( \{V_0, \ldots, V_n\} \) of \( Y \). As in the proof of Theorem 4.3, we have the explicit description of the \( V_i \), for instance,
\[ V_0 = \text{Spec}(R_c[z_1, \ldots, z_n, z_{n+1}]/(\{s_i - s_0 z_i\}_{1 \leq i \leq n+1}, g_0 - z_{n+1}^e)) \]
with \( R_c := R \otimes_{\mathcal{O}_c} z_i = Z_i/\mathcal{O}_c \) for \( i = 1, \ldots, n, z_{n+1} = t'/\sigma_0 \) and \( g_0 = G(1, z_1, \ldots, z_n) + \sigma_0 h' \) for suitable \( h' \). Letting \( A_c := A \otimes_{\mathcal{O}_c} \mathcal{O}_c \), we can rewrite this as
\[ V_0 = \text{Spec}(A_c[\sigma_0, z_1, \ldots, z_n, z_{n+1}]/\{s_i - s_0^a z_i^a\}_{1 \leq i \leq n+1}, g_0 - z_{n+1}^e, s_0 - s_0^a). \]

Again referring to Theorem 4.3 and its proof, we have the global description of \( \tilde{E}_1 \) as the closed subscheme \( V(G(Z_0, \ldots, Z_n) - Z_{n+1}^e) \) of \( P_{k(q)}^{n+1} := \text{Proj} k(q)[Z_0, \ldots, Z_{n+1}] \), with \( \tilde{E}_{12} \subset \tilde{E}_1 \) defined by \( Z_{n+1} = 0 \). Finally, the projection \( Y_Z \to \tilde{Z} \) restricts to a morphism \( \pi_2 : \tilde{E}_2 \to Z_\sigma \), \( \pi_2 \) is an isomorphism over \( Z_\sigma \setminus \{q\} \) and the reduced inverse image \( \pi_2^{-1}(q) \) is \( \tilde{E}_{12} \).

Taking the \( \mu_a \)-quotients \( U_i := \tilde{V}_i/\mu_a \) gives the affine open cover \( \{U_0, \ldots, U_n\} \) of \( Y \).
We have the commutative diagram showing some of the schemes in the process

\[
\begin{array}{ccc}
V_i \cap \tilde{E}_1 & \longrightarrow & U_i \cap \tilde{D}_1 \\
\downarrow & & \downarrow \\
E_1 & \longrightarrow & \tilde{D}_1 \\
\downarrow & & \downarrow \\
V_i & \longrightarrow & U_i \\
\downarrow & & \downarrow \\
YZ & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\tilde{Z}_e & \longrightarrow & \tilde{X}_e \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & B
\end{array}
\]

Let us now describe the \( \mu \)-action on \( \hat{Z}_e \) and on \( V_0 \). For \( \zeta \in \mu_{a_i} \), and \( j = 0, \ldots, n \), we have

\[
\zeta \cdot Z_j = \begin{cases} 
\zeta Z_i & \text{for } j = i \\
Z_j & \text{for } j \neq i.
\end{cases}
\]

and

\[
\zeta \cdot \sigma_j = \begin{cases} 
\zeta \sigma_i & \text{for } j = i \\
\sigma_j & \text{for } j \neq i.
\end{cases}
\]

On the affine piece \( V_0 \), and for \( \zeta \in \mu_{a_i} \), \( i = 1, \ldots, n \) and for \( j = 1, \ldots, n+1 \), we thus have

\[
\zeta \cdot z_j = \begin{cases} 
\zeta z_i & \text{for } j = i \\
z_j & \text{for } j \neq i,
\end{cases}
\]

and \( \zeta \cdot \sigma_0 = \sigma_0 \). For \( \zeta \in \mu_{a_0} \), we have \( \zeta \cdot \sigma_0 = \zeta \sigma_0 \) and

\[
\zeta \cdot z_j = \zeta^{-1} z_j
\]

for all \( j = 1, \ldots, n+1 \). The \( \mu_{a_i} \)-action on the other open subschemes \( V_i \) is defined similarly. We also have a global description of the \( \mu_{a_i} \)-action on \( E_1 \subset \mathbb{P}^{n+1}_{k(q)} = \text{Proj} k(q)[Z_0, \ldots, Z_{n+1}] \) by having \( \mu_{a_i} \) act trivially on \( Z_{n+1} \); one can easily check that this restricts to the action on each \( V_i \cap \tilde{E}_1 \) defined above.

We will describe the quotient by \( \mu_{a_i} \) in two steps - first taking the quotient by the subgroup \( \mu_{a_i} > 0 := \mu_{a_1} \times \ldots \times \mu_{a_n} \) and then by the remaining factor \( \mu_{a_0} \).

**Proof of (1).** The assertion (1) is local on \( Y \), so it suffices to prove (1) after restricting to \( U_i \subset Y \); we give the proof for \( U_0 \). We assume at first that \( a_0 > 1 \); the case \( a_0 = 1 \) is easier and will be dealt with at the end of the argument.

Let

\[
C_0 = A_c[\sigma_0, z_1, \ldots, z_n, z_{n+1}] / ((s_i - \sigma^0_{a_1}, s_1)_{1 \leq i \leq n+1}, g_0 - (z_{n+1})^e, s_0 - \sigma^0_{0})
\]

and let \( C \subset C_1 \subset C_0 \) be the rings of invariants

\[
C_1 = C_{0}^{\mu_{a_i} > 0}, C = C_{0}^{\mu_{a_i}} = C_{1}^{\mu_{a_0}},
\]

so \( V_0 = \text{Spec } C_0 \) and \( U_0 = \text{Spec } C \subset Y \). Since \( V_0 \) is smooth over \( k \) and is integral, the invariant subrings \( C, C_1 \) are both integral and normal.

\[
\begin{array}{ccc}
V_0 = \text{Spec } C_0 & \longrightarrow & V_0/\mu_{a_i} > 0 = \text{Spec } C_1 \\
\downarrow & & \downarrow \\
YZ & \longrightarrow & Y = YZ/\mu_{a_i}
\end{array}
\]

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We have $s_0 \in C$ and $\sigma_0 \in C_1$. We first show that $C[s_0^{-1}]$ is a smooth $O_e$-algebra. To see this, note that the special fibre $X_\sigma$ has only $p$ as singular point, so $A[s_0^{-1}]$ is a smooth $O$-algebra. Thus the base extension $A_e[s_0^{-1}] = A[s_0^{-1}] \otimes \mathcal{O}_e$ is a smooth $O_e$-algebra. Moreover, since localization commutes with taking invariants, $A_e[s_0^{-1}]$ is the $\mu_a$-invariants in $R_e[\sigma_0^{-1}]$, and since $\sigma_0$ defines $E_1 \cap V_0$ in $V_0$, $V_0 \to \text{Spec } R_e$ is an isomorphism over $\text{Spec } R_e[\sigma_0^{-1}]$. This shows that $C[s_0^{-1}] = A_e[s_0^{-1}]$ and hence $C[s_0^{-1}]$ is a smooth $O_e$-algebra.

The $\mu_a>0$-invariant subring of $A_e[\sigma_0, z_1, \ldots, z_n, z_{n+1}]/(s_0 - \sigma_0^a)$ is

$$[A_e[\sigma_0, z_1, \ldots, z_n, z_{n+1}]/(s_0 - \sigma_0^a)^{\mu_a>0} = A_e[\sigma_0, t_1, \ldots, t_n, z_{n+1}]/(s_0 - \sigma_0^a),$$

with $t_i = z_\alpha^a_i$. From this it follows that

$$C_1 := C_0^{\mu_a>0} = A_e[\sigma_0, t_1, \ldots, t_n, z_{n+1}]/\{(s_i - \sigma_0^a t_i)_{i=1,\ldots,n}, f_0 - z_n^e, s_0 - \sigma_0^a\},$$

where $f_0 = F(1, t_1, \ldots, t_n) + \sigma_0 \cdot h$ for a suitable $h$. Note that $\mu_a$ now acts by $\zeta \cdot t_i = \zeta^{-a}t_i$.

Our assumption that $F$ defines a smooth quotient hypersurface in $\mathbb{P}(a)$ and our assumption $a_0 > 1$ implies that $F(1, 0, \ldots, 0) \neq 0$, that is

$$\emptyset = V(\sigma_0, t_1, \ldots, t_n, z_{n+1}) \cap V(f_0 - z_n^e + 1) \subset \text{Spec } A_e[\sigma_0, t_1, \ldots, t_n, z_{n+1}]/(s_0 - \sigma_0^a, \{s_i - \sigma_0^a t_i\}_{i=1,\ldots,n}).$$

The $\mu_a$-action on $\text{Spec } A_e[\sigma_0, t_1, \ldots, t_n, z_{n+1}]/(s_0 - \sigma_0^a, \{s_i - \sigma_0^a t_i\}_{i=1,\ldots,n})$ is free outside the origin $V(\sigma_0, t_1, \ldots, t_n, z_{n+1})$. Thus the $\mu_a$-action on $\text{Spec } C_1$ is free and hence the ring extension $C \to C_1$ is étale. In particular, $C_1[\sigma_0^{-1}] = C_0[\sigma_0^{-1}]$ is étale over the smooth $k$-algebra $C[s_0^{-1}]$ and hence $C_1[\sigma_0^{-1}]$ is a smooth $k$-algebra.

Since $\sigma_0$ is $\mu_a>0$-invariant, it follows that $(\sigma_0)C_1$ is the $\mu_a>0$-invariants in $(\sigma_0)C_0$, in other words

$$(\sigma_0)C_1 = C_1 \cap (\sigma_0)C_0.$$

This implies that the evident ring homomorphism $C_1/(\sigma_0) \to C_0/(\sigma_0)$ is injective and since $e$ is prime to the characteristic of $k$, taking $\mu_a$-invariants is an exact functor, and thus

$$C_1/(\sigma_0) = [C_0/(\sigma_0)]^{\mu_a>0}.$$

Explicitly,

$$C_0/(\sigma_0) = k(p)(z_1, \ldots, z_n, z_{n+1})/(G(1, z_1, \ldots, z_n) - z_n^e)$$

Since $G(1, z_1, \ldots, z_n) = F(1, z_1^a, \ldots, z_n^a)$, $G(1, z_1, \ldots, z_n) - z_n^e$ is $\mu_a$-invariant, so as above, we have

$$C_1/(\sigma_0) = [k(p)(z_1, \ldots, z_n, z_{n+1})/(G(1, z_1, \ldots, z_n) - z_n^e)]^{\mu_a>0} = k(p)(t_1, \ldots, t_n, z_{n+1})/(F(1, t_1, \ldots, t_n) - z_n^e).$$

Using again our smoothness assumption on $F$, we see that $C_1/(\sigma_0)$ is a smooth $k$-algebra. By Lemma 5.6, $C_1$ itself is a smooth $k$-algebra and since $C \to C_1$ is étale, $C$ is also a smooth $k$-algebra.

Similarly, to see that $V_0 \to \text{Spec } O_e$ is a semi-stable reduction, it suffices to see that the special fibre $\text{Spec } C_1/(t')C_1$ is a reduced normal crossing divisor on $\text{Spec } C_1$. For this, we have $t' = \sigma_0 z_n + 1$. We have already seen that $C_1/(\sigma_0)$ is a smooth $k$-algebra, in other words, the Cartier divisor $V(\sigma_0)$ on $\text{Spec } C_1$ is smooth. We have

$$C_1/(z_n + 1, \sigma_0) = k(p)(t_1, \ldots, t_n)/(F(1, t_1, \ldots, t_n))$$

which again by our assumption on $F$ is a smooth $k$-algebra. This implies that the Cartier divisors $V(\sigma_0), V(z_{n+1}) \subset \text{Spec } C_1$ intersect transversely on $\text{Spec } C_1$, which implies that $V(z_{n+1})$ is smooth in a neighbourhood of $V(\sigma_0)$ in Spec $C_1$; this also implies that $(t') = (\sigma_0) \cap (z_{n+1})$. We have also shown that $\sigma_0$ is smooth over $O_e$, which implies that $\sigma_0[\sigma_0^{-1}]$ is also smooth over $O_e$, so $V(z_{n+1}) \setminus V(\sigma_0)$ is smooth. Thus the Cartier divisor $V(t')$ on Spec $C_1$ is $V(\sigma_0) + V(z_{n+1})$, which we have just shown is a reduced normal crossing divisor. This completes the proof of (1), and also shows that $Y_\sigma$ is a union of two smooth components, intersecting transversely, proving the first part of (2).

In case $a_0 = 1$, we have $C = C_1$ and a much simpler version of the arguments given above takes care of this case.
Proof of (2). We have just shown that $Y_{\sigma}$ is the Cartier divisor $\tilde{D}_1 + \tilde{D}_2$, with $\tilde{D}_1, \tilde{D}_2$ both smooth and with transverse intersection $\tilde{D}_{12}$. We have the global description of $E_1$ given by Theorem 4.3, namely $E_1$ is the closed subscheme $V(G(Z_0, \ldots, Z_n) - Z_{n+1}^e)$ of $\mathbb{P}^{n+1}_{k(q)}$. We have

$$\tilde{D}_1 = \tilde{E}_1 / \mu_a.$$ 

The $\mu_a$-action on $\tilde{E}_1$ extends to an action on $\mathbb{P}^{n+1}_{k(q)} = \text{Proj} k(q)[Z_0, \ldots, Z_n, Z_{n+1}]$ as described in the proof of (1) by having $\mu_a$ act trivially on $Z_{n+1}$. Then

$$\mathbb{P}^{n+1}_{k(q)}/\mu_a = \mathbb{P}(a_0, \ldots, a_n, 1).$$ 

Let $a_{n+1} = 1$, let $T_i = Z_i^{a_i}$, $i = 0, \ldots, n+1$, and let $K \subset \mathbb{P}(a_0, \ldots, a_n, 1)$ be the hypersurface $V(F - T_n^{e})$. We wish to identify $\tilde{D}_1$ with $K$. Let $W_i \subset \mathbb{P}(a_0, \ldots, a_n, 1)$ be the open subscheme $T_i \neq 0$. Giving $T_i$ weight $a_j$, we have

$$W_i = \text{Spec} k(p)[T_0, \ldots, T_{n+1}][T_i^{-1}]_0.$$ 

We concentrate on the case $i = 0$ to simplify the notation. In the diagram

$$\begin{array}{ccc}
\tilde{E}_1 \cap V_0 = V_{\mathcal{W}_0}(G(1, z_1, \ldots, z_n) - z_{n+1}^e) & \longrightarrow & W_0 = \text{Spec} k(p)[z_1, \ldots, z_{n+1}] \\
| & /_{\mu_a > 0} & | \\
V_{\mathcal{W}_0}(F(1, t_1, \ldots, t_n) - z_{n+1}^e) & \longrightarrow & \tilde{W}_0 = \text{Spec} k(p)[t_1, \ldots, t_n, z_{n+1}] \\
| & /_{\mu_a = 0} & | \\
\tilde{D}_1 \cap U_0 = V_{\mathcal{W}_0}(F(1, t_1, \ldots, t_n) - z_{n+1}^e) & \longrightarrow & W_0 \cong \text{Spec} k(p)[t_1, \ldots, t_n, z_{n+1}]^\mu_a \\
& & \longrightarrow \mathbb{P}(a, 1)
\end{array}$$

The first row describes the restriction of the embedding of $\tilde{E}_1$ in $\mathbb{P}^{n+1}$ to the affine $\mathcal{W}_0$, as in the proof of Theorem 4.3. The objects in the rest of the diagram are defined and discussed below. Let $S_0 := k(p)[T_0, \ldots, T_{n+1}][T_0^{-1}]_0$ and let $S'_0 := [k(p)[t_1, \ldots, t_n, z_{n+1}]^\mu_a$, with the $\mu_a$ action as defined in the proof of (1). A direct computation shows that $S_0 \cong S'_0$. Indeed, a monomial $\prod_j t_j^{b_j} \cdot T_0^{b_0}$ has zero weight if and only if $\sum_j b_j a_j b_j$ is divisible by $a_0$. Similarly, a monomial $\prod_j^{n+1} T_j^{b_j} \cdot T_0^{b_0}$ has weighted degree zero if and only if $\sum_j b_j a_j b_j = a_0 b_0$. So, sending $\prod_j^{n+1} T_j^{b_j} \cdot T_0^{b_0}$ to $\prod_j^{n+1} b_j a_j b_j$ gives an isomorphism of $S_0$ with $S'_0$.

Similarly, recalling that $a_0$ divides $e$, the weighted homogeneous polynomial $F(T_0, \ldots, T_n) - T_n^{e}$ gives the element $F(T_0, \ldots, T_n) / T_0^{a_0} - T_n^{e} / T_0^{a_0}$ in $S_0$, which corresponds to the element $F(1, t_1, \ldots, t_n) - z_{n+1}^e$ of $[k(p)[t_1, \ldots, t_n, z_{n+1}]^\mu_a$.

Let $W_0 := \text{Spec} k(p)[t_1, \ldots, t_n, z_{n+1}]$. The finite extension

$$[k(p)[t_1, \ldots, t_n, z_{n+1}]^\mu_a \to k(p)[t_1, \ldots, t_n, z_{n+1}]$$

defines a finite morphism $p : W_0 \to W_0$. By our computations in the proof of (1) and that given in the previous paragraph, we see that

$$p^{-1}(K \cap W_0) = V(F(1, t_1, \ldots, t_n) - z_{n+1}^e) = \text{Spec} C_1 / (\sigma_0) = (\tilde{E}_1 \cap V_0) / \mu_a > 0,$$

and thus

$$K \cap W_0 = (\tilde{E}_1 \cap V_0) / \mu_a = \tilde{D}_1 \cap W_0.$$ 

An analogous computation shows that $K \cap W_i = \tilde{D}_1 \cap W_i$ for $i = 1, \ldots, n + 1$, so $\tilde{D}_1 = K = V(F - T_{n+1}^{e})$, as desired.

A similar argument shows that $\tilde{D}_{12} = V(F - T_{n+1}^{e}) \cap V(T_{n+1})$, in other words, $\tilde{D}_{12} = V(F) \subset \mathbb{P}(a)$. In the proof of (1), we showed that the projection $U_0 \setminus V(s_0) \to X_0 \setminus V(s_0)$ is an isomorphism; a similar argument shows that $U_i \setminus V(s_i) \to X_0 \setminus V(s_i)$ is an isomorphism for all $i$. This shows that $Y \setminus \tilde{D}_1 \to X_0 \setminus \{p\}$ is an isomorphism. Passing to the fibre over the closed point of Spec $\mathcal{O}_x$, it follows that $D_2 \setminus \tilde{D}_{12} \to X_0 \setminus \{p\}$ is an isomorphism. Since $D_2$ is smooth, $\tilde{D}_2 \setminus \tilde{D}_{12}$ is dense in $\tilde{D}_2$ and $\tilde{D}_2 \to X_0$ is proper, we see that $q : \tilde{D}_2 \to X_0$ is a resolution of singularities of $X_0$, with $q^{-1}(p) = \tilde{D}_{12}$, finishing the proof of (2).

Proof of (3). The formula for $\chi_c(\Psi_f(\mathbb{I}_{X_0}))$ is a consequence of (1), (2) and Proposition 3.15. \qed
**Corollary 5.8.** Let \( f : X \to \text{Spec} \mathcal{O} \) be a quasi-projective flat morphism with \( X \) smooth over \( k_0 \). Suppose that \( X_\eta \) is smooth over \( \eta \) and \( X_\sigma \) has finitely many singular points \( p_1, \ldots, p_r \). Suppose in addition that for each \( i \), \((X_\sigma, p_i)\) looks like the weighted homogeneous singularity defined by a weighted homogeneous polynomial \( F_i \in k(p_i)[T_0, \ldots, T_n] \) of weighted degree \( e_i \) for weights \( a^{(i)}_* \), such that \( f_i \) defines a smooth quotient hypersurface in \( \mathbb{P}_{k(p_i)}(a^{(i)}_*) \). Let \( X_\sigma^0 = X_\sigma \setminus \{p_1, \ldots, p_r\} \). Then

\[
\chi_c(\Psi_f(\mathbb{1}_{X_\eta})) = \chi_c(X_\sigma^0) + \sum_{i=1}^r \chi_c(V(F_i - T_{n+1}^e)) - \langle -1 \rangle \sum_{i=1}^r \chi_c(V(F_i)).
\]

**Proof.** We proceed by induction on \( r \). The induction step is exactly as the proof of Corollary 4.4.

Suppose \( X_\sigma \) has the single singular point \( p = p_1 \), let \( e = e_1 \), \( F = F_1 \). By Theorem 5.7, there is an affine open neighbourhood \( U \) of \( p \) in \( X \) such that the restriction \( f_U : U \to \text{Spec} \mathcal{O} \) admits a semi-stable reduction \( Y \to \text{Spec} \mathcal{O}_e \), with special fibre \( Y_\sigma = D_1 + D_2 \) and where \( D \to U_\sigma \) is an isomorphism over \( U_\sigma^0 := U_\sigma \setminus \{p\} \), with \( D_1 \cong V(F - T_{n+1}^e) \), and with \( D_{12} = V(F) \). By Example 3.8 and Corollary 3.11, we have

\[
\chi_c(\Psi_{f_U}(\mathbb{1}_U)) = \chi_c(U_\sigma^0) + \chi_c(V(F - T_{n+1}^e) \setminus V(F)) - \chi_c(\mathbb{A}^1 \times V(F)).
\]

Using cut and paste, as in the proof of Corollary 4.4, we have

\[
\chi_c(\Psi_{f_U}(\mathbb{1}_U)) = \chi_c(U_\sigma^0) + \chi_c(V(F - T_{n+1}^e)) - \chi_c(\mathbb{A}^1 \times V(F)).
\]

Let \( V = X \setminus \{p\} \) with morphism \( f_V : V \to \text{Spec} \mathcal{O} \). Then \( V_\sigma = X_\sigma^0 \) and since \( V \) is smooth over \( \mathcal{O} \), we have

\[
\chi_c(\Psi_{f_V}(\mathbb{1}_V)) = \chi_c(V_\sigma) = \chi_c(X_\sigma^0).
\]

For \( U \cap V = U_\sigma^0 \), we similarly have

\[
\chi_c(\Psi_{f_{U \cap V}}(\mathbb{1}_{U \cap V})) = \chi_c(U_\sigma^0).
\]

Using Mayer-Vietoris for the cover \( X = U \cup V \), as in the proof of Corollary 4.4, we thus have

\[
\chi_c(\Psi_f(\mathbb{1}_X)) = \chi_c(U_\sigma^0) + \chi_c(V(F - T_{n+1}^e)) - \chi_c(\mathbb{A}^1 \times V(F)) + \chi_c(X_\sigma^0) - \chi_c(U_\sigma^0) = \chi_c(X_\sigma^0) + \chi_c(V(F - T_{n+1}^e)) - \chi_c(\mathbb{A}^1 \times V(F)).
\]

\( \square \)

### 6 Comparison of local Euler classes

In this section we present a motivic local invariant, the \( \mathbb{A}^1 \)-local Euler class, as it is defined in [Le20] and [BW], that gives an effective tool for computing the quadratic Euler characteristic of the nearby cycles. We will show that, for the type of morphism \( f : X \to \text{Spec} \mathcal{O} \) that we have been considering, when \( f \) looks at a point \( p \in X_\sigma \) like a (weighted) homogeneous singularity defined by a (weighted) homogeneous polynomial \( F(T_0, \ldots, T_n) \), the local Euler class at \( p \) for \( df \) is the same as the local Euler class for the map \( F : \mathbb{A}^{n+1} \to \mathbb{A}^1 \) at the origin \( 0 \in \mathbb{A}^{n+1} \) (see Definition 6.7 and Corollary 6.10 for a precise statement).

#### 6.1 The local Euler class

We recall here some preliminary definitions and define \( \mathbb{A}^1 \)-local Euler class with respect to a section of a vector bundle following [BW, 5.1].

**Definition 6.1.** For a vector bundle \( p : V \to X \) with zero section \( s_0 : X \to V \), and dual bundle \( V^* \), define the functor \( \Sigma V^* : \text{SH}(X) \to \text{SH}(X) \) by \( \Sigma V^* := p_\# s_0^* \).

We have the identity \( \Sigma V^* \mathbb{1}_X = V/(V \setminus X) \in \text{SH}(X) \), see [Hoy17, 5.2].

**Definition 6.2.** Let \( S \) be a scheme, let \( E \in \text{SH}(S) \), \( f : X \to S \) an \( S \)-scheme, \( i : Z \hookrightarrow X \) a closed subscheme, and \( p : V \to X \) a vector bundle. We define the \( V \)-twisted \( E \)-cohomology of \( X \) with support on \( Z \), which we denote \( E^V_Z(X) \), to be

\[
E^V_Z(X) = [\mathbb{1}_S, f_* i_! \Sigma^V i^! f^* E]_{\text{SH}(S)} \simeq [X/(X \setminus Z), \Sigma^V f^* E]_{\text{SH}(X)};
\]
see [BW, 4.2.1].

When \( Z = X \), we drop \( Z \) from the notation. We also denote \( E^2_Z(X) = E^\Omega_{Z,X}^2(X) \).

For \( \mathcal{L} \) a line bundle over \( X \), we put \( E^2_Z(X, \mathcal{L}) = E^{2-1+\mathcal{L}}_Z(X) \).

**Definition 6.3.** Let \( E \in SH(S) \) be a motivic ring spectrum. We denote by \((V, \rho)\) pairs consisting of a vector bundle \( p : V \to X \) and an isomorphism \( \rho : det V \to \mathcal{O}_X \).

An \( SL \)-orientation on \( E \) is an assignment of an element \( th(V, \rho) \in E_0^{p+q}(V) \) for each such pair \((V, \rho)\), satisfying the axioms of [LR, Definition 3.4]. An \( SL \)-oriented ring spectrum \( E \) is a motivic ring spectrum \( E \in SH(S) \) together with a given \( SL \)-orientation \( th(-, -) \).

If \( E \) is an \( SL \)-oriented motivic spectrum, and \( p : V \to X \) is a vector bundle of rank \( n \), we have \( E^V_X(X) = E^n_X(X, det V) \).

The motivic ring spectrum we use in this paper is \( H K^{MW} \) representing the Milnor-Witt homotopy module \( K^{MW} \), and admitting a canonical \( SL \)-orientation. For details on the construction of this motivic spectrum and its \( SL \)-orientation see [Le20, Section 2].

Let \( X \) be a smooth scheme over a perfect field \( k \) and \( p \in X \) a closed point. Then we have an isomorphism \( (H K^{MW})^p(X, \omega_{X/k}) \simeq GW(k(p)) \) ([Le20, Cor. 3.3]), so by using classes in cohomology groups defined by this motivic ring spectrum we can express invariants in quadratic forms. We also use the notation \( H^2_p(X, K^{MW}(\mathcal{L})) \) for the group \((H K^{MW})^2(X, \mathcal{L})\).

**Definition 6.4.** Let \( V \to X \) be a vector bundle of rank \( n \), \( s : X \to V \) a section and \( i : Z = Z(s) \hookrightarrow X \) the zero locus of \( s \). The local Euler class of \((V, s)\), also called the refined Euler class, is the element \( e(V, s) = E^2_Z(X) \) defined by the composition

\[
\frac{X}{X \setminus Z} \to V/V \setminus 0 \simeq \Sigma^V \eta_k \to \Sigma^V e|_X \in SH(X).
\]

**Remark 6.5.** In the case of an \( SL \)-oriented theory \( E \), and a rank \( n \) bundle \( V \), we have \( E^*_Z(X) = E^2_Z(X, det^{-1} V) \), giving the local Euler class \( e(V, s) \in E^2_Z(X) = E^2_Z(X, det^{-1} V) \).

We also have the Thom class \( th(V) \in E^{p+q}_{MW}(V) \), defined as the local Euler class \( e(t, p^* V) \), where \( t : V \to p^* V \) is the tautological section (with zero-locus the zero-section in \( V \)). In that case,

\[
e_Z(V, s) = s^* th(V) \in E^2_Z(X),
\]

see [BW, Def. 5.12].

**Example 6.6.** In the case the section \( s \) has as the zero locus \( Z \) a single point \( p \), then for \( E = H K^{MW} \), \( V = \Omega_{X/k} \), we have \( e_p(\Omega_{X/k}, s) \in H^{2p}_{MW}(X, \omega_{X/k}) \). By the purity isomorphism for \( H K^{MW} \), this latter group is canonically isomorphic to \( GW(k(p)) \) and this element can be computed as the Scheja-Storch quadratic form on the Jacobian ring at the point, see [Le20, Cor. 3.3] and below 8.2.

### 6.2 Comparing Euler classes

**Definition 6.7.** Let \( \kappa \) be a field, let \( a_* = (a_0, \ldots, a_n) \) be a sequence of weights and let \( F(T_0, \ldots, T_n) \in \kappa[T_0, \ldots, T_n] \) an \( a_* \)-weighted homogeneous polynomial of weighted degree \( e \). Let \( \mathcal{O}_\kappa = \kappa[t]_{(t)} \); we denote the closed point of \( Spec \mathcal{O}_\kappa \) by \( \sigma_\kappa \) and the generic point by \( \eta_\kappa \).

We assume that the \( a_i \) are pairwise relatively prime, that \( a_i \) divides \( e \) for all \( i \) and that \( V(F) \subseteq \mathbb{P}_\kappa(a_*) \) is a smooth quotient hypersurface; in particular \( e \) is prime to the exponential characteristic of \( \kappa \).

Define \( H^F \subseteq \mathbb{P}_{\mathcal{O}_\kappa}(a_* + 1) \) to be the hypersurface \( V(F - t T_{n+1}) \), and let \( f_F : H^F \to Spec \mathcal{O}_\kappa \) denote the projection.

One can see that \( H^F \) is smooth over \( \kappa \), the generic fibre \( H^F_{\kappa_\eta} \) is smooth over \( \eta = Spec \kappa(t) \) and the special fibre \( H^F_{\kappa_\sigma} \) has a single isolated singular point \( 0 := (0 : \ldots : 0 : 1) \).

We return to our main object of study, a quasi-projective flat map \( f : X \to Spec \mathcal{O} \) with an isolated critical point \( p \in X \). Our goal is to show that, under the assumption that \( f \) looks near \( p \) like a quasi-homogeneous singularity defined by a polynomial \( F \in \kappa[p][T_0, \ldots, T_n] \), the local Euler class \( e_p(\Omega_{X/k_0}, df) \) at the critical point \( p \in X \) is equal to the local Euler class \( e_0(\Omega_{H^F/k(p)}, df_F) \). By \( df \) we mean the section \( d(f^*(t)) \) of \( \Omega_{X/k_0} \), and define \( df_F \) similarly. We first make some elementary simplifications.

First of all, due to the Nisnevich descent enjoyed by all cohomology theories defined by motivic spectra, the local Euler class \( e_p(\Omega_{X/k_0}, df) \in GW(k(p)) \) is unchanged if we replace \( (X, p) \) by a Nisnevich neighbourhood...
Next, blow up $Z = Z \times \mathbb{A}^1$ at $p \times \mathbb{A}^1$ to get $\hat{Z}$ and denote by $\hat{X}$ the quotient by the action of $\mu_a$. Let $q : \hat{X} \to X$ be the natural map. Denote the intersection of the strict transform of $Z_\sigma$ and the exceptional divisor in $\hat{Z}$ by $E_{12} = V_{\mathbb{A}^n}(F) \times \mathbb{A}^1$ (see the paragraph above) and its image under the $\mu_a$-quotient map by $D_{12}$. Then we get $D_{12} = V_{\mathbb{A}^n}(F) \times \mathbb{A}^1$ which is smooth by our Assumption 5.5.
Let $X := f^{-1}_\lambda(\sigma \times \mathbb{A}^1) \subset X$. We have in both cases the proper map $q : \tilde{X} \to X$, which is an isomorphism over $X \setminus p \times \mathbb{A}^1$. Let $q^{-1}[\mathcal{X}_p]$ be the closure of $q^{-1}(X_p) \setminus p \times \mathbb{A}^1$ in $\tilde{X}$. In both cases, the Cartier divisor $D_{12}$ on the reduced scheme $q^{-1}[\mathcal{X}_p]$ is smooth over $\mathbb{A}^1$. Let $r : q^{-1}[\mathcal{X}_p] \to \tilde{X}$ be the morphism induced by $f_\lambda$. Then $r$ is flat and the set $W$ of points $x \in q^{-1}[\mathcal{X}_p]$ such that $x$ is a smooth point of the fibre $r^{-1}(r(x))$ is an open subset of $q^{-1}[\mathcal{X}_p]$, and is equal to the set of points of $q^{-1}[\mathcal{X}_p]$ at which $r$ is a smooth morphism. By Lemma 5.6, $W$ is an open neighbourhood of $D_{12}$ in $q^{-1}[\mathcal{X}_p]$. Letting $F$ be the closed complement of $W$ in $q^{-1}[\mathcal{X}_p]$, and noting the $q$ is proper, $q(F)$ is a closed subset of $X_p$, disjoint from $p \times \mathbb{A}^1$. Set $U := X_p \setminus q(F)$. Then $U$ is open and $U \setminus (p \times \mathbb{A}^1) \approx W \setminus D_{12}$ (via the strict transform identification) is smooth over $\mathbb{A}^1$.

**Proposition 6.9.** Let $X$ be a smooth quasi-projective scheme over a field $k$, with $Z \subset X$ closed, let $p : V \to X$ be a vector bundle, and let $s_1, s_2 : X \to V$ be two sections. Let $E$ be an SL-oriented motivic spectrum with respect to it Euler classes are defined. Consider $\tilde{p} : \pi^*V \to X \times \mathbb{A}^1$ with $\pi$ the projection $\pi : X \times \mathbb{A}^1 \to X$. Define a section $s : X \times \mathbb{A}^1 \to \pi^*V$ by $s = \lambda s_1 + (1 - \lambda)s_2$ and assume that we have an open neighbourhood $U$ of $Z \times \mathbb{A}^1$ in $X \times \mathbb{A}^1$ such that $Z(s) \cap U = Z \times \mathbb{A}^1$. Then

$$e_Z(X, s_1) = e_Z(X, s_2).$$

**Proof.** Let $s_0 : X \to V$ be the zero section. We have the Thom class

$$\text{th}(V) = s_0 \mathbb{I}_X \in E_0^{\pi^*V}(V).$$

We have the two embeddings $i_1 : X \hookrightarrow X \times 0 \subset X \times \mathbb{A}^1$ and $i_2 : X \hookrightarrow X \times 1 \subset X \times \mathbb{A}^1$. By homotopy invariance the two maps

$$i_1^* : E_Z^{\pi^*V}(X \times \mathbb{A}^1) \to E_Z^{\pi^*V}(X)$$

are equal. Using the excision property in cohomology we can remove the piece $(X \times \mathbb{A}^1) \setminus U$ to get the equivalence

$$\alpha : E_Z^{\pi^*V}(U) \simeq E_Z^{\pi^*V}(X \times \mathbb{A}^1).$$

Here $V_U$ is the pullback of $V$ over $U \hookrightarrow X \times \mathbb{A}^1$. Let $s' = s|_U : U \to V_U$ and $\tilde{p} = \tilde{p}|_{V_U} : V_U \to U$. Since $Z(s') = Z(s) \cap U = Z \times \mathbb{A}^1$, we have a map $s'^* : E_0^{\pi^*V}(V_U) \to E_0^{\pi^*V}(U)$. Denote by $\tilde{\pi}$ the pullback map $V_U \to V$ of the vector bundle $V \to X$ along $U \hookrightarrow X \times \mathbb{A}^1 \to X$ and consider the following commutative diagram -

$$
\begin{array}{ccc}
E_0^{\pi^*V}(V_U) & \xleftarrow{\tilde{\pi}^*} & E_0^{\pi^*V}(V) \\
\downarrow{s'^*} & & \downarrow{s_1^*} \\
E_Z^{\pi^*V}(U) & \xleftarrow{\alpha} & E_Z^{\pi^*V}(X) \\
\end{array}
$$

We have

$$s_1^* \text{th}(V) = i_1^* \circ \alpha \circ s'^* \circ \tilde{\pi}^* \text{th}(V) = i_1^* \circ \alpha \circ s'^* \circ \tilde{\pi}^* \text{th}(V) = s_2^* \text{th}(V)$$

which gives the desired equality of local Euler classes.

Let now $E = HK^{MW}$. **Corollary 6.10.** Let $f : X \to \text{Spec} \mathcal{O}$ be a flat quasi-projective morphism with $X$ smooth over $k_0$ and with an isolated critical point $p \in X_\sigma$. Suppose that $f$ looks like $F = F(T_0, \ldots, T_n)$ at $p$ (see 1.2). Then

$$e_p(\Omega_X/k(p)), df = e_0(\Omega_{\mathbb{A}^{n+1}_k/p}, d(F(T_0, \ldots, T_n))) = e_0(\Omega_{\mathbb{A}^n_k/p}, df_F)$$

in $GW(k(p))$. 31
Proof. Proposition 6.8 proves that the assumptions in Proposition 6.9 are satisfied for $E = HK^{MW}$, $Z = \{p\}$, $V = \Omega_{X/k} \rightarrow X$, $s_1 = df$, and $s_2 = dF$. This gives the following identity in $GW(k(p))$,

$$e_p(\Omega_{X/k(p)}, df) = e_p(\Omega_{X/k(p)}, d(F(s_0, \ldots, s_n))).$$

The parameters $s_0, \ldots, s_n \in \mathcal{O}_{X,p}$ define an étale map $\alpha : \text{Spec } \mathcal{O}_{X,p} \rightarrow \mathbb{A}^{n+1}_{k(p)} := \text{Spec } k(p)[t_0, \ldots, t_n]$ which maps $p$ to $0$ and with $\alpha^*F(t_0, \ldots, t_n) = F(s_0, \ldots, s_n)$. Thus $(s_0, \ldots, s_n)$ expresses $(X,p)$ as a Nisnevich neighbourhood of $(\mathbb{A}^{n+1}_{k(p)}, 0)$. Since

$$(s_0, \ldots, s_n)^*(F(t_0, \ldots, t_n)) = F(s_0, \ldots, s_n),$$

we have

$$e_p(\Omega_{X/k(p)}, d(F(s_0, \ldots, s_n))) = (s_0, \ldots, s_n)^*(e_0(\Omega_{\mathbb{A}^{n+1}_{k(p)}}, d(F(t_0, \ldots, t_n))))$$

where $(s_0, \ldots, s_n)^* : GW(k(p))(0) \rightarrow GW(k(p))$ is the isomorphism induced by $(s_0, \ldots, s_n) : p \rightarrow 0$; this is just the identity map on $GW(k(p))$, so we can write this as the identity

$$e_p(\Omega_{X/k(p)}, d(F(s_0, \ldots, s_n))) = e_0(\Omega_{\mathbb{A}^{n+1}_{k(p)}}, d(F(t_0, \ldots, t_n))).$$

The singular point $0 = (0 : \ldots : 0 : 1)$ of $H^F_{\mathbb{A}^{n+1}_{k(p)}}$ is in the affine open subscheme $U_{n+1} \subset \mathbb{P}_{\mathcal{O}_{k(-1)}(a_*)}$. If to compute $e_0(\Omega_{H^F/k(p)}, df_F)$, we can restrict to $U_{n+1}$. We have

$$U_{n+1} = \text{Spec } \mathcal{O}_{k(p)}[T_0, \ldots, T_n, T_{n+1}][T^{-1}_{n+1}]$$

and $\mathcal{O}_{k(p)}[T_0, \ldots, T_n, T_{n+1}][T^{-1}_{n+1}]$ is the polynomial ring $\mathcal{O}_{k(p)}[t_0, \ldots, t_n]$, with $t_i = T_i/T_{n+1}$. On $U_{n+1}$, $H^F$ has defining equation

$$(F(T_0, \ldots, T_n) - tT_{n+1})/T_{n+1} = F(t_0, \ldots, t_n) - t.$$ 

Thus, $H^F \cap U_{n+1}$ is just the graph of the morphism

$$F(t_0, \ldots, t_n) : \mathbb{A}^{n+1}_{k(p)} = \text{Spec } k(p)[t_0, \ldots, t_n] \rightarrow \text{Spec } k[t](t).$$

If we replace the graph $H^F \cap U_{n+1}$ with the isomorphic scheme $\text{Spec } k(p)[t_0, \ldots, t_n]$ via the isomorphism given by the first projection, then $f_F$ transforms to the map $F(t_0, \ldots, t_n)$ and $0$ goes to the origin $(0, \ldots, 0) \in \mathbb{A}^{n+1}_{k(p)}$. In other words,

$$e_0(\Omega_{H^F/k(p)}, df_F) = e_0(\Omega_{\mathbb{A}^{n+1}_{k(p)}}, d(F(t_0, \ldots, t_n))).$$

\[ \square \]

7 The generalized conductor formula

In this section we use the results of the previous sections computing $\chi(\Psi_f(1_{X,n})|_p)$ at a singular point $p$ and reinterpret them in terms of the difference $\Delta_f(F)$ considered in [LPLS]. Using the functorial properties of the functor $\Psi_f$, this allows to generalize the formula proven in [LPLS] to the case of $f : X \rightarrow \text{Spec } \mathcal{O}$ with finitely many isolated critical points, all satisfying Assumptions 4.1 or 5.5, which is our main result in this paper. In particular, this verifies the conjecture formulated in [LPLS, Conjecture 5.4] in characteristic zero, for a somewhat wider class of singularities than what was considered there.

We retain in this section our notations and assumptions for $\mathcal{O}$ and $B = \text{Spec } \mathcal{O}$ as in Section 3.1, assume in addition that the subfield $k_0 \subset \mathcal{O}$ has characteristic zero. We have the characteristic zero residue field $k$ and fraction field $K$ of $\mathcal{O}$. Let $f : X \rightarrow B$ be a flat, quasi-projective morphism such that $X$ is smooth over $k_0$, $X_n$ is smooth over $n$ and such that $X_n$ has finitely many singular points.

Fix a sequence of pairwise relative prime weights $a := (a_0, \ldots, a_n)$ and a field $\kappa$, and let $F \in \kappa[T_0, \ldots, T_n]$ be a degree $e$ $a$-weighted homogeneous polynomial such that $V(F) \subset \mathbb{P}_n(a)$ is a smooth quotient hypersurface, in the sense of Definition 5.3. We have the discrete valuation ring $\mathcal{O}_\kappa := \kappa[t]_t$, the hypersurface $H^F := V(F - tT_{n+1}^e) \subset \mathbb{P}_{\mathcal{O}_\kappa}(a,1)$ with projection $f_F : H^F \rightarrow \text{Spec } \mathcal{O}_\kappa$. $H^F$ is smooth over $\kappa$, $H^F_{\eta_k}$ is smooth over $\eta_k$, and $H^F_{\eta_k}$ has a single singularity at $p := (0 : \ldots : 0 : 1)$. In fact, $H^F_{\kappa}$ is the cone over $V(F,T_{n+1}) \subset V(T_{n+1}) = \mathbb{P}(a)_{\kappa}$ with vertex $p$.
In [LPLS], Levine, Pepin Lehalleur and Srinivas consider the invariant
\[ \Delta_t(F) := \text{sp}_t(\chi_c(H^F_{n_\kappa}/\kappa(t))) - \chi_c(H^F_{\sigma_c}/\kappa) \in GW(\kappa) \]
and derive an expression, which they call a conductor formula, for \( \Delta_t(F) \) in terms of the local Euler class \( e_p(\Omega_{HF/\kappa}, dt) \in GW(\kappa) \). Note that \( f_F : H^F \to \text{Spec} \mathcal{O}_\kappa \) looks at \( p = (0 : \ldots : 0 : 1) \) like the weighted homogeneous singularity defined by \( F \). A generalization of the conductor formulas for \( \Delta_t(F) \) for degenerations with finitely many singularities of a certain type is conjectured in loc. cit. [LPLS, Conjecture 5.4].

We recall the definition of the invariant \( \Delta_t(F) \).

**Definition 7.1.** Let \( \kappa \) be a field, let \( \mathcal{O}_\kappa = \kappa[t]/(t) \), let \( a_\ast = (a_0, \ldots, a_n) \) be a sequence of positive integral weights, and let \( F(T_0, \ldots, T_n) \in \kappa[T_0, \ldots, T_n] \) be an \( a_\ast \)-weighted homogeneous polynomial of weighted degree \( e \). We assume that \( V(F) \subset \mathbb{P}_k(a_\ast) \) is a smooth quotient hypersurface; in particular, the \( a_i \) are pairwise relative prime and \( e \) is prime to the exponential characteristic of \( \kappa \).

Define \( \Delta_t(F/\kappa) \in GW(\kappa) \) by
\[ \Delta_t(F/\kappa) := \text{sp}_t(\chi_c(H^F_{n_\kappa})) - \chi_c(H^F_{\sigma_c}). \]

This includes the homogeneous case by taking \( a = (1, \ldots, 1) \).

For an \( a_\ast \)-weighted homogeneous \( F \) the conductor formula of Levine, Pepin Lehalleur and Srinivas has the form ([LPLS, Theorem 5.3]) -
\[ \Delta_t(F/\kappa) = \langle \prod a_j \cdot e \rangle - \langle 1 \rangle + (-e)^n \cdot e_0(\Omega_{HF/\kappa}, dt) \in GW(\kappa). \]

Here \( e_0(\Omega_{HF/\kappa}, dt) \) is the local Euler class at \( 0 := (0 : \ldots : 0 : 1) \) [LPLS, 5], also see Definition 6.4.

We wish to extend this to a formula in the case of a morphism with isolated critical points that look like homogeneous or quasi-homogeneous singularities. In order to do that we give a comparison between \( \chi_c(\Psi_f|_p) \) of the scheme and the The motivic Euler characteristic of the hypersurface \( H^F \) defined by the polynomial \( F \). Recall from Section 2 that for a finite separable field extension \( k_1 \subset k_2 \), we have the transfer map \( \text{Tr}_{k_2/k_1} : GW(k_2) \to GW(k_1) \).

**Theorem 7.2.** Let \( \mathcal{O} \) and \( B := \text{Spec} \mathcal{O} \) be as in Section 3.1, and assume in addition that the subfield \( k_0 \subset \mathcal{O} \) has characteristic zero.

Let \( f : X \to B \) be a flat quasi-projective morphism with \( X \) smooth over \( k_0 \) and with \( X_\eta \) smooth over \( \eta \), and let \( p \in X_\sigma \) be an isolated critical point of \( f \), satisfying 4.1 or 5.5. Let \( F \in k(p)[T_0, \ldots, T_n] \) be corresponding (weighted) homogeneous polynomial. Then
\[ \chi_c(\Psi_f(\chi_{X_\eta}|_p)|_p) = \text{Tr}_{k(p)/k}(\Delta_t(F/k(p))) + \langle 1 \rangle \in GW(k). \]

**Proof.** The homogeneous case is a special case of the weighted homogeneous case, with all weights equal to 1, so we only need handle the weighted redundant case. Since \( \chi_c(\Psi_f(\chi_{X_\eta}|_p)|_p) \) is determined by a neighbourhood of \( p \) we can assume \( p \) is the only critical point of \( f \).

Note that we have families \( f : X \to \text{Spec} \mathcal{O} \) and \( f_F : H_F \to \text{Spec} k(p)[t]/(t) \) over different bases, so we need to keep track of the base fields for the Euler characteristics and the base schemes for the nearby cycles functors.

First we show that the terms in the difference \( \Delta_t(F/k(p)) \) are closely related to the Denef-Loeser covers we computed in Theorem 4.3 and Theorem 5.7. By Property 3.1, Proposition 3.18 and Proposition 3.19, we have
\[ \text{sp}_t(\chi_c(H^F_{n_{k(p)}}/k(p))(t)) = \Psi_{id_{k(p)}|p}(t) \cdot \chi_c(H^F_{n_{k(p)}}/k(p)(t))) \]
\[ = \Psi_{id_{k(p)}|p}(t) \cdot \chi(F_{n_{k(p)}}(\chi_{H^F_{n_{k(p)}}}(t))) \]
\[ = \chi(\Psi_{id_{k(p)}|p}(t) \cdot \chi(F_{n_{k(p)}}(\chi_{H^F_{n_{k(p)}}}(t)))) \]
\[ = \chi(\Psi_{id_{k(p)}|p}(t) \cdot \chi(F_{n_{k(p)}}(\chi_{H^F_{n_{k(p)}}}(t)))) \]
\[ = \chi_c(\Psi_{f_F}(\chi_{H^F_{n_{k(p)}}}(t)) \cdot \chi(\chi_{X_\eta}|_p)|_p(k(p))). \]

On the other hand, we can apply Corollary 5.8 to give
\[ \chi_c(\Psi_{f_F}(\chi_{H^F_{n_{k(p)}}}(t)) \cdot \chi(\chi_{X_\eta}|_p)|_p(k(p))) = \chi_c(V(F - T_{n+1}^c)/k(p)) + \chi_c(H^F_{\sigma_c}|_p(k(p))) - \chi_c(A^1 \times V(F)/k(p)). \]
Corollary 7.3. In the setting of Theorem 7.2, let \( e \) be a homogeneous polynomial of degree \( n \) has isolated singularities at the special fibre \( p \) of \( X \). Suppose that the special fibre \( X_{\eta} \) of \( X \) is smooth over \( \eta \), then
\[
\chi_c(H^{F_0}_{\sigma_k(p)}/k(p)) = \chi_c(V(F)/k(p)) \cdot \chi_c(\mathbb{A}^1/k(p)) = \chi_c(\mathbb{A}^1 \times V(F)/k(p)),
\]
which yields
\[
\chi_c(\Psi_f(\mathbb{A}^1)/k(p)) = \chi_c(V(F - T_{n+1}^e)/k(p)).
\]
Thus
\[
\text{sp}_t \chi_c(H^{F_0}_{\sigma_k(p)}/k(p)(t)) = \chi_c(V(F - T_{n+1}^e)/k(p)) = \chi_c(D_1/k(p)).
\]
Now \( H^{F_0}_{\sigma_k(p)} \oplus (0 : \ldots : 0 : 1)_{k(p)} \) and \( V_{\mathbb{P}^k(\eta)}(F, T_{n+1}) \simeq V_{\mathbb{P}^k(p)}(F) \Rightarrow D_{12}, \) so
\[
\chi_c(H^{F_0}_{\sigma_k(p)}/k(p)) = \chi_c(D_{12}/k(p)) \cdot (-1) + \langle 1 \rangle \in GW(k(p)).
\]
Adding this up (or rather subtracting) we have the formula
\[
\Delta_t(F/k(p)) = \chi_c(D_1/k(p)) - \chi_c(D_{12}/k(p)) \cdot (-1) - \langle 1 \rangle \in GW(k(p)).
\]
Applying Proposition 2.16, this gives
\[
\text{Tr}_{k(p)/k}(\Delta_t(F/k(p))) = \chi_c(D_1/k) - \chi_c(D_{12}/k) \cdot (-1) - \text{Tr}_{k(p)/k}(\langle 1 \rangle) \in GW(k).
\]
On the other hand, by Proposition 3.7 and Theorem 4.3 (5.7), we have
\[
\chi_c(\Psi_f(\mathbb{A}^1)/p/k) = \chi_c(\Psi_f(\mathbb{A}^1)/p/k) - \chi_c(X_\sigma \setminus \{ p \}/k)
\]
\[
= \chi(D_1/k) + \chi(D_2^*/k) - \chi(D_{12}/k) \cdot (-1) - \chi(D_{12}/k) = \chi(D_1/k) - \chi(D_{12}/k) \cdot (-1).
\]
So comparing both terms, we have
\[
\chi_c(\Psi_f(\mathbb{A}^1)/p/k) := \chi_c(\Psi_f(\mathbb{A}^1)/p/k) = \text{Tr}_{k(p)/k}(\Delta_t(F/k(p)) + \langle 1 \rangle),
\]
concluding the proof. 

Corollary 7.3. In the setting of Theorem 7.2, let \((a_0, \ldots, a_n)\) are the weights and \( e \) the weighted degree for \( F \) (where all weights are 1 when \( F \) is homogeneous), then
\[
\chi_c(\Psi_f(\mathbb{A}^1)/p) = \text{Tr}_{k(p)/k}(\prod_j a_j \cdot e) + (-e)^n \cdot e_0(\Omega_{n+1/k(p), dF}) \in GW(k).
\]

Proof. The formula follows from that of Theorem 7.2, together with the formula of [LPLS, Theorem 5.3] mentioned above, and the identity
\[
e_0(\Omega_{n+1/k(p), dF}) = e_0(\Omega_{F^*_{/k}}, dt),
\]
of Corollary 6.10. 

We now proceed to obtain a global formula in the general case, when \( X \) has multiple singular points at the special fibre \( p_1, \ldots, p_s \) satisfying Assumption 4.1 (or 5.5). We state our main result in the weighted homogeneous setting as this also includes the homogeneous case.

Corollary 7.4 (Generalized quadratic conductor formula). Let \( X \to \text{Spec} \, \mathcal{O} \) be a flat projective morphism of relative dimension \( n \), with \( X \) smooth over \( k_0 \) and \( X_\eta \) smooth over \( \eta \). Suppose that the special fibre \( X_\sigma \) has isolated singularities \( p_1, \ldots, p_s \) satisfying Assumption 5.5 with \( F_i \in k(p_i)[T_0, \ldots, T_n] \) an \( a_i^{(i)} \)-weighted homogeneous polynomial of degree \( e_i \). Then
\[
\text{sp}_t(\chi_c(X_\eta/k(\eta))) - \chi_c(X_\eta/k) = \sum_i \text{Tr}_{k(p_i)/k}((\prod_j a_j^{(i)} \cdot e_i) - \langle 1 \rangle + (-e_i)^n \cdot e_p(\Omega_{X/k(p_i)}, dt))
\]

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The Jacobian ring of \( f \) to algebraic invariants of the singularities. We recall here a construction of a distinguished quadratic form

As mentioned in the introduction, the conductor formula is expressed in terms of quadratic forms related

8.1 The Jacobian ring, Milnor number and quadratic refinements

Proof. By applying Proposition 2.16, Proposition 3.7 and Theorem 7.2 we obtain the formula

\[
\chi_c(\Psi_f(1_{X_\eta})) = \sum_i \chi_c(\Psi_f(1_{X_\eta})|_{p_i}) + \chi_c(X_\sigma \setminus \{p_1, \ldots, p_k\}/k)
\]

\[
= \sum_i Tr_{k(p_i)/k}(\Delta_i(F_i/k(p_i)) + (1)) + \chi_c(X_\sigma) - \sum_i Tr_{k(p_i)/k}(1)).
\]

This gives the global formula

\[
\chi_c(\Psi_f(1_{X_\eta})) - \chi_c(X_\sigma) = \sum_i Tr_{k(p_i)/k}(\Delta_i(F_i/k(p_i)))).
\]

Substituting Levine-Pepin Lehalleur-Srinivas’s conductor formula [LPLS, Theorem 5.3] gives

\[
\chi_c(\Psi_f(1_{X_\eta})) - \chi_c(X_\sigma) = \sum_i Tr_{k(p_i)/k}(\chi_c(\Delta_i(F_i/k(p_i))).
\]

But as we proved in Section 6, Corollary 6.10 we can replace \( e_0(\Omega_{H^i/k(p_i)}, dt) \) with \( e_p(\Omega_{X/k(p_i)}, dt) \). Then by [LPLS, Proposition 8.3] which states that \( \chi_c(\Psi_f(1_{X_\eta})) = \text{sp}_t \chi_c(X_\eta/k(\eta)) \), we get the desired result

\[
\text{sp}_t \chi_c(X_\eta/k(\eta)) - \chi_c(X_\sigma/k) = \sum_i Tr_{k(p_i)/k}(\chi_c(\Delta_i(F_i/k(p_i))).
\]

\[
\square
\]

8 Interpretations and applications

8.1 The Jacobian ring, Milnor number and quadratic refinements

As mentioned in the introduction, the conductor formula is expressed in terms of quadratic forms related to algebraic invariants of the singularities. We recall here a construction of a distinguished quadratic form related to the Scheja-Storch element, which gives the local Euler class \( e_p(\Omega_{X/k}, s) \) of Definition 6.4.

Definition 8.1. Let \( k \) be a field and \( X \) be a smooth finite type scheme over \( k \). Let \( p \in X \) be a closed point, take \( f \in \mathcal{O}_{X,p} \), and let \( s_0, \ldots, s_n \) be a regular system of parameters at \( p \). Suppose that \( \sqrt{(\partial f/\partial s_0 \ldots \partial f/\partial s_n)} = m_p \), so \( df \) has an isolated zero at \( p \); note that the ideal \( (\partial f/\partial s_0 \ldots \partial f/\partial s_n) \) does not depend on the choice of the \( s_i \). Let \( k(p) \) be the residue field of \( \mathcal{O}_{X,p} \).

The Jacobian ring of \( f \) at \( p \), \( J(f,p) \), is defined as

\[
J(f,p) := \mathcal{O}_{X,p}/(\partial f/\partial s_0 \ldots \partial f/\partial s_n).
\]

For \( k \) algebraically closed, the dimension of \( J(f,p) \) over \( k \) is the Milnor number \( \mu_{f,p} \).

Since \( \partial f/\partial s_i \) is in \( m_p = (s_0, \ldots, s_n) \), we can write for each \( i \),

\[
\partial f/\partial s_i = \sum a_{ij}s_j
\]

with \( a_{ij} \in \mathcal{O}_{X,p} \). The Scheja-Storch element \( e_{f,p} \in J(f,p) \) is defined as the image of the determinant \( \det(a_{ij}) \) in \( J(f,p) \); \( e_{f,p} \) is independent of the choices made. Since \( J(f,p) \) is an Artinian local \( k \)-algebra, \( J(f,p) \) contains the residue field \( k(p) \).

Let \( Tr : J(f,p) \to k(p) \) be a \( k(p) \)-linear map sending \( e_{f,p} \) to 1. Define

\[
B_{f,p} : J(f,p) \times_{k(p)} J(f,p) \to k(p)
\]

by \( B_{f,p}(x,y) = Tr(xy) \). The class \([B_{f,p}] \in GW(k(p))\) does not depend on the choices of generators \((s_0, \ldots, s_n)\) or the map \( Tr \), see [Le20, Theorem 3.1].

If \( char(k) \neq 2 \) we denote the quadratic form corresponding to the bilinear form \( B_{f,p} \) by \( \mu_{f,p}^q \). By taking the rank of the quadratic form, \( rk(\mu_{f,p}^q) = \dim J(f,p) = \mu_{f,p} \), so the class \([\mu_{f,p}^q] \in GW(k(p))\) can be viewed as a quadratic refinement of the Euler number \( \mu_{f,p} \in \mathbb{Z} \).
Theorem 8.2 ([BW, Proposition 2.32 and Theorem 7.6] and [Le20, Corollary 3.3]). Take $X \in \text{Sm}_k$ and let $\Omega_{X/k}$ be the sheaf of Kähler differentials. Let $f : X \to \text{Spec } O$ be a flat morphism with an isolated critical point $p \in X$, so the section $df \in H^0(X, \Omega_{X/k})$ has zero locus $Z(s) = \{p\}$ in a neighbourhood of $p$. Let $e_p(\Omega_{X/k}, df) \in GW(k(p))$ be the local Euler class as in Example 6.6, and let $[\mu_{f,p}] \in GW(k(p))$ be as defined above. Then

$$e_p(\Omega_{X/k}, df) = [\mu_{f,p}]^q.$$  

Rewriting our main result Corollary 7.4 then, with the same assumptions, we get

Corollary 8.3 (The generalized quadratic conductor formula).

$$\text{sp}_t(\chi_c(X_n/k(\eta))) - \chi_c(X_n/k) = \sum_i Tr_{k(p)/k} \left( \prod_j a_j^{(i)} \cdot e_i - (1) + (-\langle e_i \rangle)^n \cdot [\mu_{f,p}]^i \right).$$

At the same time, $\mu_{f,p}^q$ can be defined purely algebraically in terms of the polynomials $F_i \in k(p_1)[T_0, \ldots, T_n]$ as $\mu_{F_i}^q$, by the Scheja-Storch form.

Notice that this formula refines in quadratic forms the formula by Milnor (1.1) mentioned in the introduction. Assume $k = \mathbb{C}$, and let $f : X \to \mathbb{A}^1 = \text{Spec } \mathbb{C}[t]$ be a flat family of varieties, $X$ being an $n+1$-dimensional smooth $\mathbb{C}$-scheme, and let $X_t = f^{-1}(\mathcal{G}_0)$, $X_0 = f^{-1}(0)$. Suppose that $f|X_t : X_t \to \mathbb{G}_m$ is smooth, and $f/X_0 : X_0 \to \mathbb{C}$ has isolated $F_i$-weighted-homogeneous singular points $p_i$. We can specialize to $X \to \text{Spec } k[t]_{(t)}$ and use our formula above. Then since $\text{rk}[\mu_{F_i}^q] = \dim J(F_i, p_i) = \mu_{F_i, p_i}$, and from Remark 2.10, taking ranks on both sides of the equation in the formula above gives

$$\chi_{\top}(X_t) - \chi_{\top}(X_0) = (-1)^n \sum_i \mu_{F_i, p_i}.$$  

which is Milnor’s formula mentioned in the introduction (1.1). Note that at each point, the difference $\langle \prod_j a_j^{(i)} \cdot e_i \rangle - (1)$ vanishes under the rank map, as a difference of two rank 1 quadratic forms; similarly, the term $(-\langle e_i \rangle)^n$ maps to $(-1)^n$. This simplification also occurs for $k = \mathbb{R}$, as $\prod_j a_j^{(i)} \cdot e_i$ and $e_i$ are squares in $\mathbb{R}$. Thus, these terms are only apparent in the refined formulas; see also [LPLS, Section 1 and Remark 5.5]. Similarly, it refines the Deligne-Milnor formula (1.2) in equal characteristic zero with the type of singularities discussed here, by taking $\ell$-adic realisations.

### 8.2 The case of curves on a surface

As an application of our main theorem, we develop here a formula for the difference between the quadratic Euler characteristic of curves on a surface, refining a formula for complex varieties deduced from the formula of Jung-Milnor.

Let $C$ be a reduced curve on a smooth projective surface $S$ over an algebraically closed field $k$ of characteristic zero. Let $\pi : \tilde{C} \to C$ be the normalisation. Let $p$ be a singular point of $C$. Let $r_p$ be the number of points in $\pi^{-1}(p)$. Let $\delta_p$ be the length of the (finite length) $\mathcal{O}_{C,p}$-module $\pi_* (\mathcal{O}_{\tilde{C}, \pi^{-1}(p)}) / \mathcal{O}_{C,p}$. Let $\mu_p$ be the Milnor number defined above for the local defining equation for $C$, $f_p \in \mathcal{O}_{S,p}$, at $p$. The Jung-Milnor formula [Mil, Chapter 10] states that

$$2\delta_p = \mu_p + r_p - 1.$$  

If $C$ is irreducible, we have $h^0(C, \mathcal{O}_C) = 1 = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}})$ and the short exact sequence

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\tilde{C}} \to \pi_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C \to 0$$

gives

$$h^1(C, \mathcal{O}_C) = h^1(\tilde{C}, \mathcal{O}_{\tilde{C}}) + \sum_{x \in C_{\text{sing}}} \delta_p.$$  

Let $f_0$ be the canonical section of the invertible sheaf $\mathcal{O}_S(C)$ and assume that $\mathcal{O}_S(C)$ has a section $f_1$ whose divisor is a smooth curve $C_1$, such that each point of $C \cap C_1$ is a smooth point of $C$, and that the intersection is transverse. In case $S = \mathbb{P}^2$, and $C$ is a curve of degree $e$, then $\mathcal{O}_S(C) \cong \mathcal{O}_{\mathbb{P}^2}(e)$, the canonical section is
just the section given by the defining equation \( f_0 \) of \( C \), and a general homogeneous polynomial \( f_1 \) of degree \( e \) will have the desired properties.

\( C_1 \) is a smooth deformation of \( C \), and so we have \( g(C_1) = h^1(C, \mathcal{O}_C) \); \( g(\tilde{C}) = h^1(\tilde{C}, \mathcal{O}_{\tilde{C}}) \). The classical formula obtained, relating the genus of \( \tilde{C} \) and of \( C_1 \) in case \( C \) is irreducible, is then

\[
g(\tilde{C}) - g(C_1) = \sum_{p \in C_{\text{sing}}} (1/2)(1 - \mu_p - r_p)
\]

or in terms of the topological Euler characteristic \((= 2 - 2g(-))\) of \( C_1 \) and \( \tilde{C} \)

\[
\chi^{\text{top}}(C_1) - \chi^{\text{top}}(\tilde{C}) = \sum_{p \in C_{\text{sing}}} 1 - \mu_p - r_p,
\]

(8.1)

which holds even if \( C \) is not irreducible. We consider this as the Jung-Milnor formula with several singular points. We show below that our quadratic formula provides a refinement for it.

We can also compare with \( \chi^{\text{top}}(C) \). Since for a curve we have \( C \setminus C_{\text{sing}} \cong \tilde{C} \setminus \pi^{-1}(C_{\text{sing}}) \), we deduce

\[
\chi^{\text{top}}(\tilde{C}) - \sum_{p \in C_{\text{sing}}} r_p = \chi^{\text{top}}(C) - \sum_{p \in C_{\text{sing}}} 1.
\]

Putting this into the genus formula above, we see that this formula is equivalent to

\[
\chi^{\text{top}}(C_1) - \chi^{\text{top}}(C) = \sum_{p \in C_{\text{sing}}} (-\mu_p) = -\sum_{p \in C_{\text{sing}}} \dim J(f_p, p),
\]

(8.2)

where we use some local defining equation \( f_p \in \mathcal{O}_{S,p} \) for \( C \) to define the Jacobian ring.

Using our main result we can deduce a refinement of formulas 8.1, 8.2 with quadratic forms.

**Corollary 8.4.** Let \( C \) be a reduced curve on a smooth projective surface \( S \) over a field \( k \) of characteristic zero. Suppose that \( \mathcal{O}_S(C) \) admits a section \( s \) with smooth divisor \( C_1 \) that intersects \( C \) transversely. Suppose in addition that each singular point \( p \) of \( C \) is a quasi-homogeneous singularity; let \( a^p_0, a^p_1 \) denote the homogeneous weights (with \( a^p_0, a^p_1 \) relatively prime), let \( e_p \) denote the homogeneous degree at \( p \). Let \( \pi : \tilde{C} \to C \) be the normalisation of \( C \). Then

\[
\text{sp}_t(\chi_c(C_\eta) \cap) - \chi_c(C/k) = \sum_{p \in C_{\text{sing}}} \text{Tr}_{k(p)/k}((a^p_0 a^p_1 e_p) - (1) - (e_p)[\mu^q_{f_p,p}])
\]

refining (8.2) by taking the rank; and

\[
\text{sp}_t(\chi_c(C_\eta) \cap) - \chi_c(\tilde{C}/k) = \sum_{p \in C_{\text{sing}}} \text{Tr}_{k(p)/k}((a^p_0 a^p_1 e_p) - (1) - (e_p)[\mu^q_{f_p,p}]) - (\sum_{q \in \pi^{-1}(p)} \text{Tr}_{k(q)/k(p)}(1))
\]

refining (8.1) by taking the rank.

**Proof.** Let \( f_0 \) be the canonical section of \( \mathcal{O}_S(C) \) and \( s \) as in the statement. Let \( B := \text{Spec} k[t]_{(t)} \), let \( H = ts + (1 - t)f_0 \), form the surface \( X := V(H) \subset S \times B \), and let \( f : X \to B \) be the projection. \( H_t = s - f_0 \), the assumption on \( X \cap C_1 \) implies that \( X \) is smooth over \( k \) with generic fibre \( X_\eta \) a smooth curve over \( \eta = \text{Spec} k(t) \), and with special fibre \( C \). Since each singular point \( p \) looks like a weighted homogeneous singularity of degree \( e_p \) with weights \( a^p_0, a^p_1 \), the formula of Corollary 7.4 for \( f : X \to B \) becomes

\[
\text{sp}_t(\chi_c(C_\eta) \cap) - \chi_c(C/k) = \sum_{p \in C_{\text{sing}}} \text{Tr}_{k(p)/k}((a^p_0 a^p_1 e_p) - (1) - (e_p)e_p(\Omega_X/k, dt)).
\]

Note that \( e_p(\Omega_X/k, dt) = e_p(\Omega_{S/k}, df_p) \), where \( f_p \in \mathcal{O}_{S,p} \) is any local expression for \( f_0 \) (this is independent of choice of local expression, since \( \Omega_{S,p} \) has rank 2). Using Theorem 8.2 this is the first formula

\[
\text{sp}_t(\chi_c(C_\eta) \cap) - \chi_c(C/k) = \sum_{p \in C_{\text{sing}}} \text{Tr}_{k(p)/k}((a^p_0 a^p_1 e_p) - (1) - (e_p)[\mu^q_{f_p,p}])
\]

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For the second formula, we just have to recall that since the normalisation of a curve, \( \tilde{C} \to C \), satisfies \( \tilde{C} \setminus \pi^{-1}(C_{\text{sing}}) \approx C \setminus C_{\text{sing}} \), and using cut and paste property, we have
\[
\chi_c(\tilde{C}/k) - \chi_c(C/k) = \chi_c(\pi^{-1}(C_{\text{sing}})/k) - \chi_c(C_{\text{sing}}/k) = \sum_{p \in C_{\text{sing}}} \text{Tr}_{k(p)/k}(\sum_{q \in \pi^{-1}(p)} \text{Tr}(k(q)/k(p))\{1\} - \{1\});
\]
this gives the last formula for the difference
\[
\text{sp}_t(\chi_c(C_{\eta}/\eta)) - \chi_c(\tilde{C}/k) = (\text{sp}_t(\chi_c(C_{\eta}/\eta))) - \chi_c(C/k) - (\chi_c(\tilde{C}/k) - \chi_c(C/k)),
\]
\[
\text{sp}_t(\chi_c(C_{\eta}/\eta)) - \chi_c(\tilde{C}/k) = \sum_{p \in C_{\text{sing}}} \text{Tr}_{k(p)/k}\left(\langle a_0^p a_1^q e_p \rangle - \langle e_p \rangle [\mu_{f,p}] + \sum_{q \in \pi^{-1}(p)} \text{Tr}(k(q)/k(p))\{1\}\right).
\]
To see that those formulas refine the classical formulas over \( \mathbb{C} \) by taking ranks, use remark 2.10, note that \( C_{\eta} \) is a smooth deformation of \( C_1 \), so \( C_{\eta} \) and \( C_1 \) have the same topological Euler characteristic after choosing an embedding of \( k(p) \) into \( \mathbb{C} \), and that \( \text{rk} q_{f,p} = \text{dim}(f,p) = \mu_{f,p} \).

We conclude with the following identity in the Witt ring \( W(k) \).

**Corollary 8.5.** Let \( C \) be a reduced curve on a smooth projective surface \( S \) over a field \( k \) of characteristic zero. Suppose that \( \mathcal{O}_S(C) \) admits a section \( s \) with smooth divisor \( C_1 \) that intersects \( C \) transversely. Suppose in addition that each singular point \( p \) of \( C \) is a quasi-homogeneous singularity; let \( a_0^p, a_1^p \) denote the homogeneous weights (with \( a_0^p, a_1^p \) relatively prime), let \( e_p \) denote the homogeneous degree at \( p \). Then
\[
\sum_{p \in C_{\text{sing}}} \text{Tr}_{k(p)/k}\left(\langle a_0^p a_1^q e_p \rangle - \langle e_p \rangle [\mu_{f,p}] + \sum_{q \in \pi^{-1}(p)} \text{Tr}(k(q)/k(p))\{1\}\right) = 0
\]
in \( W(k) \).

**Proof.** For \( Y \) smooth and projective of odd dimension over \( k \), \( \chi_c(Y/k) = 0 \) in \( W(k) \) (see [Le20, Example 1.7, 2.]).

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