A cosmological model describing the early inflation, the intermediate decelerating expansion, and the late accelerating expansion by a quadratic equation of state

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We develop a cosmological model based on a quadratic equation of state $p/c^2 = -(\alpha + 1)^2/\rho p + \alpha p - (\alpha + 1)\rho \Lambda$ (where $\rho p$ is the Planck density and $\rho \Lambda$ the cosmological density) “unifying” vacuum energy and dark energy in the spirit of a generalized Chaplygin gas model. For $\rho \rightarrow \rho p$, it reduces to $p = -\rho c^2$ leading to a phase of early accelerated expansion (early inflation) with a constant density equal to the Planck density $\rho p = 5.16 \times 10^{99} \text{ g/m}^3$ (vacuum energy). For $\rho \Lambda \ll \rho \ll \rho p$, we recover the standard linear equation of state $p = \alpha \rho c^2$ describing radiation ($\alpha = 1/3$) or pressureless matter ($\alpha = 0$) and leading to an intermediate phase of decelerating expansion. For $\rho \rightarrow \rho \Lambda$, we get $p = -\rho c^2$ leading to a phase of late accelerated expansion (late inflation) with a constant density equal to the cosmological density $\rho \Lambda = 7.02 \times 10^{-24} \text{ g/m}^3$ (dark energy). We show a nice “symmetry” between the early universe (vacuum energy + $\alpha$-fluid) and the late universe (dark energy). The cosmological constant “problem” may be a false problem. We study the evolution of the scale factor, density, and pressure. Interestingly, this quadratic equation of state leads to a fully analytical model describing the evolution of the universe from the early inflation (Planck era) to the late accelerated expansion (de Sitter era). These two phases are bridged by a decelerating algebraic expansion ($\alpha$-era). This model does not present any singularity at $t = 0$ and exists eternally in the past (although it may be incorrect to extrapolate the solution to the infinite past). On the other hand, it admits a scalar field interpretation based on a quintessence field or a tachyon field.

I. INTRODUCTION

The evolution of the universe may be divided into four main periods [1]. In the vacuum energy era (Planck era), the universe undergoes a phase of early inflation that brings it from the Planck size $l_p = 1.62 \times 10^{-35} \text{ m}$ to an almost “macroscopic” size $a \sim 10^{-6} \text{ m}$ in a tiniest fraction of a second [2, 3]. The universe then enters in the radiation era and, when the temperature cools down below approximately $10^{10} \text{ K}$, in the matter era [4]. Finally, in the dark energy era (de Sitter era), the universe undergoes a phase of late inflation [5]. The early inflation is necessary to solve notorious difficulties such as the singularity problem, the flatness problem, and the horizon problem [2, 3]. The late inflation is necessary to account for the observed accelerating expansion of the universe [6]. At present, the universe is composed of approximately 5% baryonic matter, 20% dark matter, and 75% dark energy [1]. Despite the success of the standard model, the nature of vacuum energy, dark matter, and dark energy remains very mysterious and leads to many speculations.

The phase of inflation in the early universe is usually described by some hypothetical scalar field $\phi$ with its origin in quantum fluctuations of vacuum [2, 3]. This leads to an equation of state $p = -\rho c^2$, implying a constant energy density, called the vacuum energy. This energy density is usually identified with the Planck density $\rho p = 5.16 \times 10^{99} \text{ g/m}^3$. As a result of the vacuum energy, the universe expands exponentially rapidly on a timescale of the order of the Planck time $t_p = 5.39 \times 10^{-44} \text{ s}$.

The phase of acceleration in the late universe is usually ascribed to the cosmological constant $\Lambda$ which is equivalent to a constant energy density $\rho \Lambda = \Lambda/(8\pi G) = 7.02 \times 10^{-24} \text{ g/m}^3$ called the dark energy [5]. This acceleration can be modeled by an equation of state $p = -\rho c^2$, implying a constant energy density identified with the cosmological density $\rho \Lambda$. As a result of the dark energy, the universe expands exponentially rapidly on a timescale of the order of the cosmological time $t_\Lambda = 1.46 \times 10^{18} \text{ s}$ (de Sitter solution). This leads to a phase of late inflation.

Between the phase of early inflation and the phase of late accelerated expansion, the universe is in the radiation era, then in the matter era [4]. These phases are described by a linear equation of state $p = \alpha \rho c^2$ with $\alpha = 1/3$ for the radiation and $\alpha = 0$ for the pressureless matter (including baryonic matter and dark matter). The scale factor increases algebraically as $t^{2/3(1+\alpha)}$ and the density decreases algebraically as $t^{-2}$. For $\alpha > -1/3$, the universe is decelerating.

In recent works [7], we have proposed to describe the transition between a phase of algebraic expansion ($p = \alpha \rho c^2$) and a phase of exponential expansion ($p = -\rho c^2$) by a generalized polytropic equation of state of the form

$$p = (\alpha \rho + k \rho^{-1+1/n}) c^2. \quad (1)$$

This is the sum of a standard linear equation of state $p = \alpha \rho c^2$ and a polytropic equation of state $p = k \rho^{n} c^2$ with $\gamma = 1 + 1/n$. Polytropic equations of states play an important role in astrophysics [8, 9], statistical physics [10], and mathematical biology [11], and they may also be useful in cosmology. We have studied the equation of state (1) for any values of $\alpha$, $k$, and $n$ and found...
the following structure. Positive indices $n > 0$ describe the early universe where the polytropic component dominates the linear component because the density is high. Negative indices $n < 0$ describe the late universe where the polytropic component dominates the linear component because the density is low. On the other hand, a positive polytropic pressure ($k > 0$) leads to past or future singularities (or peculiarities) while a negative polytropic pressure ($k < 0$) leads to a phase of exponential expansion (inflation) in the past or in the future. In the early universe ($n > 0$, $k < 0$), the generalized polytropic equation of state \( (1) \) leads to a maximum bound for the density that it is natural to identify with the Planck density \( \rho_P = 5.16 \times 10^{69} \text{g/m}^3 \). In the late universe ($n < 0$, $k > 0$), it leads to a minimum bound for the density that it is natural to identify with the cosmological density \( \rho_m = 0.70 \times 10^{-25} \text{g/m}^3 \). These bounds differ by 122 orders of magnitude. Therefore, taking $n = 1, k = -4/3 \rho_P$ and $\alpha = 1/3$ we obtain an equation of state $p/c^2 = -4\rho/3\rho_P + \rho/3$ that describes the transition between the vacuum energy era and the radiation era. On the other hand, taking $n = -1, k = -\rho_\Lambda$ and $\alpha = 0$ we obtain an equation of state $p/c^2 = -\rho_\Lambda$ that describes the transition between the matter era and the dark energy era. More generally, the equation of state $p/c^2 = -\rho/\rho_P + \alpha \rho$ describes the transition from the vacuum energy era to the $\alpha$-era in the early universe and the equation of state $p/c^2 = \alpha \rho - (\alpha+1) \rho_\Lambda$ describes the transition from the $\alpha$-era to the dark energy era in the late universe.²

In this paper, we propose to describe the vacuum energy, the $\alpha$-fluid, and the dark energy in a “unified” manner by a single, quadratic, equation of state of the form

$$p = -(\alpha + 1) \frac{\rho^2}{\rho_P} c^2 + \alpha \rho c^2 - (\alpha + 1) \rho_\Lambda c^2$$

(2)

involving the Planck density and the cosmological density.³ In the early universe, we recover the equation of state $p/c^2 = -\rho/\rho_P + \alpha \rho$ unifying the vacuum energy and the $\alpha$-fluid. In the late universe, we recover the equation of state $p/c^2 = \alpha \rho - (\alpha+1) \rho_\Lambda$ unifying the $\alpha$-fluid and the dark energy. The $\alpha$-fluid may represent radiation ($\alpha = 1/3$) or pressureless dark matter ($\alpha = 0$). Actually, some works [13] indicate that dark matter may be described by an isothermal equation of state $p = \alpha \rho c^2$ with a small value of $\alpha \ll 1$. It is therefore interesting to leave $\alpha$ unspecified and treat the general case of arbitrary $\alpha$. However, in the discussion, we shall usually assume $\alpha \geq 0$. Moreover, for illustrations, we will select the values $\alpha = 1/3$ and $\alpha = 0$ considered in our previous studies [7].

The quadratic equation of state \( (2) \) leads to a fully analytical model describing the evolution of the universe from the initial inflation (Planck era) to the late accelerated expansion (de Sitter era). These two phases are bridged by an algebraic decelerated expansion (\( \alpha \)-era). This model does not present any singularity at $t = 0$ - the phase of early inflation avoids the primordial Big Bang singularity - and exists eternally in the past (although it may be incorrect to extrapolate the solution to the infinite past). On the other hand, this model admits a scalar field interpretation based on a quintessence field or a tachyon field. Some limitations of this model, and some possible generalizations, are discussed in the Conclusion.

The paper is organized as follows. In Sec. [II] we recall the basic equations of cosmology that will be needed in our study. In Sec. [III] we describe the transition between the vacuum energy era and the $\alpha$-era in the early universe. In Sec. [IV] we describe the transition between the $\alpha$-era and the dark energy era in the late universe. In Sec. [V] we introduce the general model where vacuum energy + $\alpha$-era + dark energy are described by the quadratic equation of state \( (2) \). This model reveals a nice “symmetry” between the early universe (vacuum energy + $\alpha$-era) and the late universe ($\alpha$-era + dark energy). These two phases are described by two polytropic equations of state with index $n = +1$ and $n = -1$ respectively. The mathematical formulae in the early and in the late universe are then strikingly symmetric. Furthermore, the cosmological density \( \rho_\Lambda \) in the late universe plays a role similar to the Planck density \( \rho_P \) in the early

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1 We note that the polytropic equation of state \( (1) \) with $k < 0$ and $n < 0$ is equivalent to the generalized Chaplygin gas model $p/c^2 = B_\rho - A/\rho^k$ with $k \geq 1$ that has been proposed to describe the late accelerated expansion of the universe (the original Chaplygin gas model corresponds to $B = 0$ and $a = 1$). Therefore, the polytropic equation of state \( (1) \) with $k < 0$ and $n > 0$ can be seen as an extension of the generalized Chaplygin gas model to describe the early accelerated expansion of the universe (inflation).

2 For definiteness, we shall select the indices $n = +1$ and $n = -1$ to describe the early and late universe respectively. The index $n = -1$ is consistent with the observations because it leads to a model that is equivalent to the $\Lambda$CDM model at late times. However, more general models can be constructed by letting the index $n$ free. They may be used to describe deviations to the standard $\Lambda$CDM model. General results valid for arbitrary values of $n, \alpha,$ and $k$ have been given in [7].

3 It is oftentimes argued that the dark energy (cosmological constant) corresponds to the vacuum energy. This leads to the so-called cosmological problem [13] since the cosmological density \( \rho_\Lambda = 7.02 \times 10^{-24} \text{g/m}^3 \) and the Planck density \( \rho_P = 5.16 \times 10^{69} \text{g/m}^3 \) differ by about 122 orders of magnitude. We think it is a mistake to identify the dark energy with the vacuum energy. In this paper, we shall regard the vacuum energy and the dark energy as two distinct entities. We shall call vacuum energy the energy associated with the Planck density and dark energy the energy associated with the cosmological density. The vacuum energy is responsible for the inflation in the early universe and the dark energy for the inflation in the late universe. In this viewpoint, the vacuum energy is due to quantum mechanics and the dark energy is an effect of general relativity. The cosmological constant $\Lambda$ is interpreted as a fundamental constant of nature applying to the cosmophysics in the same way the Planck constant $\hbar$ applies to the microphysics.
universe. They represent fundamental upper and lower density bounds differing by 122 orders of magnitude. Interestingly, these densities \( \rho_p \) and \( \rho_\Lambda \) (together with \( \alpha \)) appear as the coefficients of the equation of state \( p \). Therefore, this equation of state provides a “unification” of vacuum energy and dark energy.

II. BASIC EQUATIONS OF COSMOLOGY

In a space with uniform curvature, the line element is given by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

\[
ds^2 = c^2 dt^2 - a(t)^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\},
\]

where \( a(t) \) represents the radius of curvature of the 3-dimensional space, or the scale factor. By an abuse of language, we shall sometimes call it the “radius of the universe”. On the other hand, \( k \) determines the curvature of space. The universe may be closed \((k > 0)\), flat \((k = 0)\), or open \((k < 0)\).

If the universe is isotropic and homogeneous at all points in conformity with the line element \(g\), and contains a uniform perfect fluid of energy density \( \epsilon(t) = \rho(t)c^2 \) and isotropic pressure \( p(t) \), the energy-momentum tensor \( T^i_{\ j} \) is

\[
T^0_{\ 0} = \rho c^2, \quad T^1_{\ 1} = T^2_{\ 2} = T^3_{\ 3} = -p.
\]

The Einstein equations

\[
R^i_{\ j} - \frac{1}{2} g^i_{\ j} R - \Lambda g^i_{\ j} = -\frac{8\pi G}{c^2} T^i_{\ j}
\]

relate the geometrical structure of the spacetime \(g_{ij}\) to the material content of the universe \(T_{ij}\). For the sake of generality, we have accounted for a possibly non-zero cosmological constant \( \Lambda \). Given Eqs. (3) and (4), these equations reduce to

\[
8\pi G \rho + \Lambda = 3 \frac{\dot{a}^2 + kc^2}{a^2},
\]

\[
\frac{8\pi G}{c^2} p - \Lambda = -2\ddot{a} + \frac{\dot{a}^2 + kc^2}{a^2},
\]

where dots denote differentiation with respect to time. These are the well-known cosmological equations describing a non-static universe first derived by Friedmann [4].

The Friedmann equations are usually written in the form

\[
\frac{d\rho}{dt} + 3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right) = 0,
\]

\[
\frac{\ddot{a}}{a} = -4\pi G \left( \rho + \frac{3p}{c^2} \right) + \frac{\Lambda}{3},
\]

where we have introduced the Hubble parameter \( H = \dot{a}/a \). Among these three equations, only two are independent. The first equation, which can be viewed as an “equation of continuity”, can be directly derived from the conservation of the energy momentum tensor \( \partial_i T^{ij} = 0 \) which results from the Bianchi identities. For a given barotropic equation of state \( p = p(\rho) \), it determines the relation between the density and the scale factor. Then, the temporal evolution of the scale factor is given by Eq. [10].

In this paper, we consider a flat universe \((k = 0)\) in agreement with the observations of the cosmic microwave background (CMB) [1]. On the other hand, we set \( \Lambda = 0 \) because the effect of the cosmological constant will be taken into account in the equation of state. The Friedmann equations then reduce to

\[
\frac{d\rho}{dt} + 3 \frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right) = 0,
\]

\[
\frac{\ddot{a}}{a} = -4\pi G \left( \rho + \frac{3p}{c^2} \right),
\]

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{a^2} + \frac{\Lambda}{3},
\]

The deceleration parameter is defined by

\[
q(t) = -\frac{\ddot{a}a}{\dot{a}^2}.
\]

The universe is decelerating when \( q > 0 \) and accelerating when \( q < 0 \). Introducing the equation of state parameter \( w = p/\rho c^2 \), and using the Friedmann equations [12] and [13], we obtain for a flat universe

\[
q(t) = -\frac{1 + 3w(t)}{2}.
\]

We see from Eq. [15] that the universe is decelerating if \( w > -1/3 \) (strong energy condition) and accelerating if \( w < -1/3 \). On the other hand, according to Eq. [11], the density decreases with the scale factor if \( w > -1 \) (null dominant energy condition) and increases with the scale factor if \( w < -1 \).

III. THE EARLY UNIVERSE

In Ref. [7], we have proposed to describe the transition between the vacuum energy era and the \( \alpha \)-era in the early universe by a single equation of state of the form [4] with \( k < 0 \) and \( n > 0 \). It can be written as

\[
p/c^2 = \alpha \rho - |k| \rho^{1+1/n\alpha}.
\]
The continuity equation \[\rho = \frac{\rho_p}{1 + (a/a_1)^{3(1+\alpha)/|n_c|}|n_c|},\] (17)
where $\rho_p = [(\alpha + 1)/k]|n_c|$ and $a_1$ is a constant of integration. We see that $\rho_p$ corresponds to an upper bound (maximum value) for the density reached for $a \to 0$. Since this solution describes the early universe, it is natural to identify $\rho_p$ with the Planck density $\rho_p = c^3/G^2h = 5.16 \times 10^{99} \text{g/m}^3$. As a result, the equation of state \[\frac{p}{c^2} = \alpha \rho - (\alpha + 1) \frac{\rho}{\rho_p} \left(\frac{\rho}{\rho_p}\right)^{1/|n_c|}.\] (18)

For the sake of simplicity, and for definiteness, we shall select the index $n_c = 1$. The general case $n_c > 0$ has been treated in [7] and leads to qualitatively similar results. Therefore, we propose to describe the transition between the vacuum energy era and the $\alpha$-era in the early universe by a single equation of state of the form

\[p = \alpha \rho c^2 - (\alpha + 1) \frac{\rho}{\rho_p} c^2,\] (19)

where $\rho_p$ is the Planck density. This equation of state corresponds to a generalized polytropic equation of state \[p \sim \rho^{1+\alpha},\] with $n = 1$ and $k = -(\alpha + 1)/\rho_p$. For $\rho \ll \rho_p$, we recover the linear equation of state $p \sim \alpha \rho c^2$. For $\rho \to \rho_p$, we get $p \to -\rho_p c^2$ corresponding to the vacuum energy. The relation \[(20)\] between the density and the scale factor becomes

\[\rho = \frac{\rho_p}{1 + (a/a_1)^{3(1+\alpha)}}.\] (20)

The characteristic scale $a_1$ marks the transition between the vacuum energy era and the $\alpha$-era. The equation of state \[p \sim \rho^{1+\alpha}\] interpolates smoothly between the vacuum energy era ($p = -\rho c^2$ and $\rho = \rho_p$) and the $\alpha$-era ($p = \alpha \rho c^2$ and $\rho \sim \rho_p/(a/a_1)^{3(1+\alpha)}$). It provides therefore a “unified” description of the vacuum energy (Planck) era and of the $\alpha$-era in the early universe. This amounts to summing the inverse of the densities of these two phases. Indeed, Eq. (20) may be rewritten as

\[\frac{1}{\rho} = \frac{1}{\rho_p} + \frac{1}{\rho_\alpha}.\] (21)

At $a = a_1$ we have $\rho_\alpha = \rho_p$ so that $\rho_1 = \rho_p/2$.

The equation of state parameter $w = p/\rho c^2$ and the deceleration parameter $q$ are given by

\[w = \alpha - (\alpha + 1) \frac{\rho}{\rho_p}, \quad q = \frac{1 + 3\alpha}{2} - \frac{3}{2} (\alpha + 1) \frac{\rho}{\rho_p}.\] (22)

The velocity of sound $c_s^2 = p'(\rho)$ is given by

\[\frac{c_s^2}{c^2} = \alpha - 2(\alpha + 1) \frac{\rho}{\rho_p}.\] (23)

As the universe expands from $a = 0$ to $a = +\infty$, the density decreases from $\rho_p$ to 0, the equation of state parameter $w$ increases from $-1$ to $\alpha$, the deceleration parameter $q$ increases from $-1$ to $(1 + 3\alpha)/2$, and the ratio $(c_s/c)^2$ increases from $-\alpha - 2$ to $\alpha$.

### A. The vacuum energy era: early inflation

When $a \ll a_1$, the density tends to a maximum value

\[\rho = \rho_{\text{max}} = \rho_p\] (24)

and the pressure tends to $p = -\rho_p c^2$. The Planck density $\rho_p = 5.16 \times 10^{99} \text{g/m}^3$ (vacuum energy) represents a fundamental upper bound for the density. A constant value of the density $\rho \approx \rho_p$ gives rise to a phase of early inflation. From the Friedmann equation \[H^2 = \frac{1}{3} (\rho - 4\pi G \rho_0)\] (15), we find that the Hubble parameter is constant $H \approx (8\pi/3)^{1/2} t_p^{-1}$ where we have introduced the Planck time $t_p = 1/(G \rho_p)^{1/2} = (\hbar G/c^3)^{1/2}$. Numerically, $H = 5.37 \times 10^{43} \text{s}^{-1}$. Therefore, for $a \ll a_1$, the scale factor increases exponentially rapidly with time as

\[a(t) \sim l_p e^{(8\pi/3)^{1/2}/t_p}.\] (25)

The timescale of the exponential growth is the Planck time $t_p = 5.39 \times 10^{-44} \text{s}$. We have defined the “original” time $t = 0$ such that $a(0)$ is equal to the Planck length $l_p = ct_p = (G \hbar /c^3)^{1/2} = 1.62 \times 10^{-35} \text{m}$. Mathematically speaking, the universe exists at any time in the past ($a \to 0$ and $\rho \to \rho_p$ for $t \to -\infty$), so there is no primordial singularity (Big Bang). However, when $a \to 0$, we cannot ignore quantum fluctuations associated with the spacetime. In that case, we cannot use the classical Einstein equations anymore and a theory of quantum gravity is required. It is not known whether quantum gravity will remove, or not, the primordial singularity. Therefore, we cannot extrapolate the solution \[a(t) \sim l_p e^{(8\pi/3)^{1/2}/t_p}\] to the infinite past. However, this solution may provide a semi-classical description of the phase of early inflation when $a > t_p$.

### B. The $\alpha$-era

When $a \gg a_1$, we recover the equation

\[\rho \sim \left(\frac{\rho_p}{a_1} \right)^{3(1+\alpha)}\] (26)

corresponding to the pure linear equation of state $p = \alpha \rho c^2$. When $a \gg a_1$, the Friedmann equation \[H^2 = \frac{1}{3} (\rho - 4\pi G \rho_0)\] yields

\[a/a_1 \sim \left[\frac{3}{2} (\alpha + 1) \left(\frac{8\pi}{3}\right)^{1/2} \frac{t}{t_p}\right]^{2/[3(\alpha + 1)]}.\] (27)

We then have

\[\rho \rho_p \sim \left[\frac{3}{2} (\alpha + 1) \left(\frac{8\pi}{3}\right)^{1/2} \frac{t}{t_p}\right]^2.\] (28)
During the $\alpha$-era, the scale factor increases algebraically as $t^2/[3(\alpha + 1)]$ and the density factor decreases algebraically as $t^{-2}$.

C. The general solution

Setting $R = a/a_1$, the general solution of the Friedmann equation \cite{Friedman1922} with Eq. \ref{eq:20} is \cite{Weinberg2008}:

$$\sqrt{R^{3(\alpha+1)} + 1} - \ln\left(\frac{1 + \sqrt{R^{3(\alpha+1)} + 1}}{R^{3(\alpha+1)/2}}\right) = \frac{3}{2}(\alpha + 1)\left(\frac{8\pi}{3}\right)^{1/2} t/l_P + C,$$

where $C$ is a constant of integration. It is determined such that $a = l_P$ at $t = 0$. Setting $\epsilon = l_P/a_1$, we get

$$C(\epsilon) = \sqrt{3(\alpha+1)} + 1 - \ln\left(\frac{1 + e^{3(\alpha+1)/2}}{e^{3(\alpha+1)/2}}\right).$$

For $t \to -\infty$, we have the exact asymptotic result

$$a(t) \sim a_1e^{(8\pi/3)^{1/2}t/l_P + D}$$

with $D = 2(C + \ln 2 - 1)/[3(1 + \alpha)]$. In general, $\epsilon \ll 1$, so a very good approximation of $C$ and $D$ is given by $C \approx 1 - \ln 2 + (3/2)(\alpha + 1)\ln \epsilon$ and $D \approx \ln \epsilon$. With this approximation, Eq. \ref{eq:31} returns Eq. \ref{eq:29}. The time $t_1$ marking the end of the inflation, corresponding to $a = a_1$ and $\rho = \rho_P/2$, is obtained by substituting these expressions in Eq. \ref{eq:29}. On the other hand, according to Eq. \ref{eq:22}, the universe is accelerating when $a < a_c$ (i.e. $\rho > \rho_c$) and decelerating when $a > a_c$ (i.e. $\rho < \rho_c$) where $\rho_c/\rho_P = (1 + 3\alpha)/[3(\alpha + 1)]$ and $\rho_c/a_1 = [2/(1 + 3\alpha)]^{1/[3(\alpha + 1)]}$. The time $t_c$ at which the universe starts decelerating is obtained by substituting these expressions in Eq. \ref{eq:29}. This corresponds to the time at which the curve $a(t)$ presents a first inflexion point. For $\alpha = 1/3$ (radiation) this inflexion point $a_c$ coincides with $a_1$, so it also marks the end of the inflation ($t_c = t_1$). For $\alpha \neq 1/3$ the two points differ.

D. The pressure

The pressure is given by Eq. \ref{eq:19}. Using Eq. \ref{eq:20}, we get

$$p = \frac{\alpha a/a_1}{[(a/a_1)^{3(\alpha+1)} + 1]^{1/2}} \rho_P c^2.$$ \hspace{1cm} (32)

The pressure starts from $p = -\rho_P c^2 = -4.6410^{116}\text{g/m}^2\text{s}^2$ at $t = -\infty$, remains approximately constant during the inflation, becomes positive, reaches a maximum value $p_c$, and decreases algebraically during the $\alpha$-era. At $t = t_c = 0$, $p_c \simeq -\rho_P c^2$. The point at which the pressure vanishes ($w = 0$) is $\rho_w/\rho_P = (1/\alpha)^{1/[3(\alpha + 1)]}$. On the other hand, the pressure reaches its maximum ($e_\alpha = 0$) when $\rho_c/\rho_P = \alpha/[2(\alpha + 1)]$ and $\alpha_c/a_1 = [(\alpha + 2)/\alpha]^{1/[3(\alpha + 1)]}$. The maximum pressure is $p_c/\rho_P c^2 = \alpha^2/[4(\alpha + 1)]$. At $t = t_1$, we have $p_1/(\rho_P c^2) = -(1 - \alpha)/4$. At $t = t_c$, we have $p_c/\rho_P c^2 = -(1 + 3\alpha)/[9(1 + \alpha)]$.

E. The evolution of the early universe

In this model, the universe “starts” at $t = -\infty$ with a vanishing radius $a = 0$, a finite density $\rho = \rho_P = 5.1610^{99}\text{g/m}^3$, and a finite pressure $p = -\rho_P c^2 = -4.6410^{116}\text{g/m}^2\text{s}^2$. The universe exists at any time in the past and does not present any singularity. For $t < 0$, the radius of the universe is less than the Planck length $l_P = 1.6210^{-35}\text{m}$. In the Planck era, quantum gravity should be taken into account and our semi-classical approach is probably not valid in the infinite past. At $t = t_i = 0$, the radius of the universe is equal to the Planck length $a_i = l_P = 1.6210^{-35}\text{m}$. The corresponding density and pressure are $\rho_i \simeq \rho_P = 5.1610^{99}\text{g/m}^3$ and $p_i \simeq -\rho_P c^2 = -4.6410^{116}\text{g/m}^2\text{s}^2$. We note that quantum mechanics regularizes the finite time singularity present in the standard Big Bang theory. This is similar to finite size effects in second order phase transitions (see Sec. \ref{sec:E}). The Big Bang theory is recovered for $h = 0$. The universe first undergoes a phase of inflation during which its radius increases exponentially rapidly while its density and pressure remain approximately constant. The inflation “starts” at $t_i = 0$ and ends at $t_1$. The timescale of the inflation corresponds to the Planck time $t_P = 5.3910^{-44}\text{s}$ (for the radiation $\alpha = 1/3$, we find $t_1 = 23.3 t_P = 1.2510^{-42}\text{s}$ \cite{Weinberg2008}).
During this very short lapse of time, the scale factor grows from $a_i = l_P = 1.62 \times 10^{-35}$ m to $a_1$ (for $\alpha = 1/3$, we find $a_1 = 2.61 \times 10^{-6}$ m). By contrast, the density and the pressure do not change significantly: they go from $\rho_i \simeq \rho_P$ and $p_i \simeq -\rho_P c^2$ to $\rho_1 = \rho_P/2$ and $p_1 = -[(1-\alpha)/4] \rho_P c^2$ (for $\alpha = 1/3$, we find $p_1 = 0.5 \rho_P = 2.58 \times 10^{99}$ g/m$^3$ and $p_1 = -0.167 \rho_P c^2 = -7.73 \times 10^{115}$ g/m$^3$). The pressure passes from negative values to positive values at $t_w$ (for $\alpha = 1/3$, we find $t_w = 23.4 t_P = 1.26 \times 10^{-42}$ s, $\rho_w = 0.25 \rho_P = 1.29 \times 10^{99}$ g/m$^3$, $a_w = 1.32 a_1 = 3.43 \times 10^{-6}$ m). After the inflation, the universe enters in the $\alpha$-era. The radius increases algebraically as $a \propto t^{2/[3(\alpha+1)]}$ while the density decreases algebraically as $\rho \propto t^{-2}$. The pressure achieves its maximum value $p_e$ at $t_e$ (for $\alpha = 1/3$, we find $p_e = 2.08 \times 10^{-2} \rho_P c^2 = 9.66 \times 10^{114}$ g/m$^3$, $t_e = 23.6 t_P = 1.27 \times 10^{-42}$ s, $\rho_e = 0.125 \rho_P = 6.44 \times 10^{98}$ g/m$^3$, $a_e = 1.63 a_1 = 4.24 \times 10^{-6}$ m). During the inflation, the universe is accelerating and during the radiation era it is decelerating. The transition (marked by an inflexion point) takes place at a time $t_1$ (for $\alpha = 1/3$ it coincides with the end of the inflation $t_1$). The evolution of the scale factor and density as a function of time are represented in Figs. 2-5 in logarithmic and linear scales (the figures correspond to the radiation $\alpha = 1/3$).
F. Analogy with phase transitions

The standard Big Bang theory is a classical theory in which quantum effects are neglected. In that case, it exhibits a finite time singularity: the radius of the universe is equal to zero at \( t = 0 \) while its density is infinite. For \( t < 0 \), the solution is not defined and we may take \( a = 0 \). For \( t > 0 \) the radius of the universe increases as \( a \propto t^{2/(3(\alpha+1))} \). This is similar to a second order phase transition if we view the time \( t \) as the control parameter (e.g. the temperature \( T \)) and the scale factor \( a \) as the order parameter (e.g. the magnetization \( M \)). For \( \alpha = 1/3 \) (radiation), the exponent \( 1/2 \) is the same as in mean field theories of second order phase transitions (i.e. \( M \sim (T_c - T)^{1/2} \)) but this is essentially a coincidence.

When quantum mechanics effects are taken into account, as in our simple semi-classical model, the singularity at \( t = 0 \) disappears and the curves \( a(t) \) and \( \rho(t) \) are regularized. In particular, we find that \( a = l_P > 0 \) at \( t = 0 \), instead of \( a = 0 \), due to the finite value of \( \hbar \). This is similar to the regularization due to finite size effects (e.g. the system size \( L \) or the number of particles \( N \)) in ordinary phase transitions. In this sense, the classical limit \( \hbar \to 0 \) is similar to the thermodynamic limit \( (L \to +\infty \) or \( N \to +\infty \)) in ordinary phase transitions.

The convergence towards the classical Big Bang solution when \( \hbar \to 0 \) is shown in Fig. 6 for \( \alpha = 1/3 \) (radiation).

\[ -10 0 10 20 30 40 50 60 70 \]
\[ -10 0 10 20 30 40 50 60 70 \]
\[ -20 0 20 40 60 80 100 \]
\[ t/t_P \]
\[ 0 \]
\[ 1 \]
\[ 2 \]
\[ 3 \]
\[ 4 \]
\[ 5 \]
\[ 6 \]
\[ 7 \]
\[ \psi \]
\[ 0 \]
\[ 0.2 \]
\[ 0.4 \]
\[ 0.6 \]
\[ 0.8 \]
\[ 1 \]
\[ V/\rho P_c^2 \]
\[ 0 \]
\[ 0.5 \]
\[ 1 \]
\[ 1.5 \]
\[ 2 \]
\[ 2.5 \]
\[ 3 \]
\[ \psi \]
\[ 0 \]
\[ 0.2 \]
\[ 0.4 \]
\[ 0.6 \]
\[ 0.8 \]
\[ 1 \]
\[ V/\rho P_c^2 \]
\[ 0 \]
\[ 0.5 \]
\[ 1 \]
\[ 1.5 \]
\[ 2 \]
\[ 2.5 \]
\[ 3 \]
\[ \psi \]
\[ 0 \]
\[ 0.2 \]
\[ 0.4 \]
\[ 0.6 \]
\[ 0.8 \]
\[ 1 \]
\[ V/\rho P_c^2 \]
\[ 0 \]
\[ 0.5 \]
\[ 1 \]
\[ 1.5 \]
\[ 2 \]
\[ 2.5 \]
\[ 3 \]
\[ \psi \]
\[ 0 \]
\[ 0.2 \]
\[ 0.4 \]
\[ 0.6 \]
\[ 0.8 \]
\[ 1 \]

**FIG. 6:** Effect of quantum mechanics (finite value of the Planck constant) on the regularization of the singular Big Bang solution (\( \hbar \to 0 \), dashed line) in our simple semi-classical model. The singularity at \( t = 0 \) is replaced by an inflationary expansion from the vacuum energy era to the \( \alpha \)-era. We can draw an analogy with second order phase transitions where the Planck constant plays the role of finite size effects (see the text for details).

G. Scalar field theory

The phase of inflation in the very early universe is usually described by a scalar field [3]. The ordinary scalar field minimally coupled to gravity evolves according to the equation

\[ \ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0, \]

where \( V(\phi) \) is the potential of the scalar field. The scalar field tends to run down the potential towards lower energies. The density and the pressure of the universe are related to the scalar field by

\[ \rho c^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2} \dot{\phi}^2 - V(\phi). \]

Using standard technics [5], we find that the potential corresponding to the equation of state (19) is [7]:

\[ V(\psi) = \frac{1}{2} \rho c^2 (1 - \alpha) \cosh^2 \psi + \alpha + 1 \frac{1}{\cosh^4 \psi} \quad (\psi \geq 0), \]

\[ \left( \frac{a}{a_1} \right)^{3(\alpha+1)/2} = \sinh \psi, \]

\[ \rho = \frac{8\pi G \rho c^2}{3} \quad (\psi \geq 0). \]
ψ = \left( \frac{8\pi G}{3c^2} \right)^{1/2} \frac{3\sqrt{\alpha + 1}}{2} \phi. \quad (37)

In the vacuum energy era (t \to -\infty), using Eq. (25), we get

ψ \sim \left( \frac{t}{a_1} \right)^{3(\alpha+1)/2} e^{\frac{2}{3}(\alpha+1)(\frac{2\pi}{3})^{1/2} t/t_P} \to 0. \quad (38)

In the α-era (t \to +\infty), using Eq. (27), we get

ψ \simeq \ln \left( \frac{t}{t_P} \right) + \frac{1}{2} \ln \left( \frac{8\pi}{3} \right) + \ln [3(\alpha + 1)] \to +\infty. \quad (39)

More generally, using Eq. (29), the evolution of the scalar field ψ(t) in the early universe is given by

\cosh ψ - \ln \left( \frac{1 + \cosh ψ}{\sinh ψ} \right) = \frac{3}{2} (\alpha + 1) \left( \frac{8\pi}{3} \right)^{1/2} \frac{t}{t_P} + C. \quad (40)

These results are illustrated in Figs. 7 and 8 for α = 1/3 (radiation).

IV. THE LATE UNIVERSE

In Ref. [7], we have proposed to describe the transition between the α-era and the dark energy era in the late universe by a single equation of state of the form \[ p = \rho c^2 - (\alpha + 1)\rho \Lambda c^2, \quad (44) \] where \( \rho \Lambda \) is the cosmological density. This equation of state corresponds to a generalized polytropic equation of state \[ p = \rho c^2 - (\alpha + 1)\rho \Lambda c^2, \quad (44) \]

The velocity of sound \( c_s^2 = p/ρ \) is given by \( c_s^2/c^2 = \alpha \). As the universe expands from \( a = 0 \) to \( a = +\infty \), the density decreases from \( +\infty \) to \( \rho \Lambda \), the equation of state parameter \( w \) decreases from \( -1 \) to \( -1/3 \), the deceleration parameter \( q \) decreases from \( 1 + 3\alpha/2 \) to \(-1\), and the ratio \( (c_s/c)^2 \) remains constant with the value \( \alpha \).

A. The dark energy era: late inflation

When \( a \gg a_2 \), the density tends to a minimum value

\[ \rho = \rho_{\text{min}} = \rho \Lambda \]

and the pressure tends to \( p = -\rho \Lambda c^2 \). The cosmological density \( \rho \Lambda = \Lambda/8\pi G = 7.02 \times 10^{-24} \text{g/m}^3 \) (dark energy) represents a fundamental lower bound for the density. A constant value of the density \( \rho \approx \rho \Lambda \) gives rise to a phase of late inflation (accelerated expansion). It is convenient to define a cosmological time \( t_\Lambda = 1/\sqrt{(8\pi G/\Lambda)^{1/2}} = (8\pi/\Lambda)^{1/2} = 1.46 \times 10^{18} \text{s} \) and a cosmological length \( l_\Lambda = c t_\Lambda = (8\pi c^2/\Lambda)^{1/2} = 4.38 \times 10^{26} \text{m} \). These are the counterparts of the Planck scales for the late universe. From the Friedmann equation (19), we find that the Hubble parameter is constant \( H = (8\pi/3)^{1/2} t_\Lambda^{-1} \). Numerically,

\[ \rho \Lambda \approx \rho \rho_f \left( \frac{a_2}{a} \right)^{3(\alpha+1)/2}. \]

4 Comparing Eqs. (20) and (45) in the α-era \( a_1 \ll a \ll a_2 \) (or \( \rho \Lambda \ll \rho \ll \rho_f \)) where the two equations of state overlap, we get \( \rho \Lambda a_2^{3(\alpha+1)} = \rho_f a_1^{3(\alpha+1)} \).
\( H = 1.98 \times 10^{-18} \text{ s}^{-1} \). Therefore, the scale factor increases exponentially rapidly with time as
\[
a(t) \propto e^{(8\pi/3)t/\lambda},
\]
\( a(t) \) increases in time for \( t \to +\infty \). This exponential growth corresponds to the de Sitter solution. The timescale of the exponential growth is the cosmological time \( t_\Lambda = 1.46 \times 10^{18} \text{ s} \). This solution exists at any time in the future (\( a \to +\infty \) and \( \rho \to \rho_\Lambda \) for \( t \to +\infty \)), so there is no future singularity. This is not the case of all cosmological models. In a "phantom universe" \([15]\), violating the null dominant energy condition \( w < -1 \), the density increases as the universe expands. This may lead to a future singularity called Big Rip (the density becomes infinite in a finite time). The possibility that we live in a phantom universe is not ruled out by observations.

### B. The \( \alpha \)-era

When \( a \ll a_2 \), we recover the equation
\[
\rho_\alpha \sim \frac{\rho_\Lambda a_2^{3(1+\alpha)}}{a^{3(1+\alpha)}}
\]
corresponding to the pure linear equation of state \( p = \alpha \rho c^2 \). When \( a \ll a_2 \), the Friedmann equation \([13]\) yields
\[
\frac{a}{a_2} \sim \left[\frac{3}{2} (\alpha + 1) \left(\frac{8\pi}{3}\right)^{1/2} \frac{t}{t_\Lambda}\right]^{2/3(\alpha+1)}.
\]
We then have
\[
\frac{\rho}{\rho_\Lambda} \sim \left[\frac{3}{2} (\alpha + 1) \left(\frac{8\pi}{3}\right)^{1/2} \frac{t}{t_\Lambda}\right]^{-2/3(\alpha+1)}.
\]
During the \( \alpha \)-era, the scale factor increases algebraically as \( t^{2/3(\alpha+1)} \) and the density decreases algebraically as \( t^{-2} \).

### C. The general solution

The general solution of the Friedmann equation \([13]\) with Eq. \([45]\) is
\[
\frac{a}{a_2} = \sinh^{2/3(\alpha+1)} \left[\frac{3}{2} (\alpha + 1) \left(\frac{8\pi}{3}\right)^{1/2} \frac{t}{t_\Lambda}\right],
\]
\[
\frac{\rho}{\rho_\Lambda} = \tanh^{2/3(\alpha+1)} \left[\frac{3}{2} (\alpha + 1) \left(\frac{8\pi}{3}\right)^{1/2} \frac{t}{t_\Lambda}\right].
\]
For \( t \to 0 \) it reduces to Eq. \([51]\) and for \( t \to +\infty \) we obtain Eq. \([49]\) with a prefactor \( a_2/2^{2/3(\alpha+1)} \).

The time \( t_2 \) corresponding to the transition between the \( \alpha \)-era and the dark energy era, corresponding to \( a = a_2 \) and \( \rho_2 = 2\rho_\Lambda \), is obtained by substituting these relations in Eq. \([53]\). On the other hand, according to Eq. \([47]\), we find that the universe is decelerating when \( a < a_c \) (i.e. \( \rho > \rho_c \)) and accelerating when \( a > a_c \) (i.e. \( \rho < \rho_c \)) where \( \rho_c = 3(\alpha + 1)/(1 + 3\alpha) \) and \( \rho_c/\rho = [(1 + 3\alpha)/2]^{1/3(\alpha+1)} \). The time \( t_c \) at which the universe starts accelerating is obtained by substituting these expressions in Eq. \([53]\). This corresponds to the time at which the curve \( a(t) \) presents a second inflexion point. For \( \alpha = 1/3 \) (radiation) this inflexion point \( a_c \) coincides with \( a_2 \). For \( \alpha \neq 1/3 \) the two points differ.

### D. The pressure

The pressure is given by Eq. \([44]\). Using Eq. \([45]\), we get
\[
p = \left[\rho \left(\frac{a_2}{a}\right)^{3(\alpha+1)} - 1\right] \rho_\Lambda c^2.
\]
For \( \alpha > 0 \), the pressure decreases algebraically during the radiation era and tends to a constant negative value \( p = -\rho_\Lambda c^2 \) for \( t \to +\infty \). The point at which the pressure vanishes \( (w = 0) \) is \( \rho_c/\rho = (\alpha + 1)/\alpha \), \( \rho_c/\rho = a_c/2 = 1/3(\alpha+1/3\alpha) \). For \( \alpha = 0 \), the pressure is a constant \( p = -\rho_\Lambda c^2 \). At \( t = t_2 \), we have \( p_2/(\rho_\Lambda c^2) = -(1 - \alpha) \). At \( t = t_c \), we have \( p_c/(\rho_\Lambda c^2) = -(1 + \alpha)/(1 + 3\alpha) \).

![Graph showing scale factor evolution](image-url)

**FIG. 9:** Evolution of the scale factor \( a \) with the time \( t \) in logarithmic scales. This figure clearly shows the transition between the \( \alpha \)-era and the dark energy era (de Sitter). In the \( \alpha \)-era, the radius increases as \( t^{2/3(\alpha+1)} \). In the dark energy era, the radius increases exponentially rapidly on a timescale of the order of the cosmological time \( t_\Lambda = 1.46 \times 10^{18} \text{ s} \). This corresponds to a phase of late inflation. The universe is decelerating for \( a < a_c \) and accelerating for \( a > a_c \). The transition between the \( \alpha \)-era and the dark energy era takes place at \( a_2 \).
In the α-era \((a \ll a_2)\), the radius increases algebraically as \(t^2/3(\alpha+1)\) while the density decreases algebraically as \(\propto t^{-2}\). When \(a \gg a_2\), the universe enters in the dark energy era. It undergoes a late inflation (de Sitter) during which its radius increases exponentially rapidly while its density remains constant and equal to the cosmological density \(\rho_\Lambda\). The transition takes place at \(t_2\). In the α-era, the universe is decelerating and in the dark energy era it is accelerating. The time at which the universe starts accelerating is \(t'\). The evolution of the scale factor and density as a function of time are represented in Figs. 9-12 in logarithmic and linear scales (the figures correspond to the pressureless matter \(\alpha = 0\)). Some numerical values are given at the end of Sec. V C.

\[ V(\psi) = \frac{1}{2} \rho_\Lambda c^2 \left[(1 - \alpha) \cosh^2 \psi + \alpha + 1\right] \quad (\psi \leq 0), \]
\[ \left(\frac{a_2}{a}\right)^{3(\alpha+1)/2} = -\sinh \psi, \]
\[ \psi = \left(\frac{8\pi G}{3c^2}\right)^{1/2} \sqrt{\frac{3\alpha + 1}{2}} \phi. \]

For \(t \to 0\), using Eq. (51), we get
\[ \psi \simeq \ln \left(\frac{t}{t_\Lambda}\right) + \frac{1}{2} \ln \left(\frac{3\pi}{2}\right) + \ln(1 + \alpha) \to -\infty. \]

For \(t \to +\infty\), using Eq. (49) with a prefactor \(a_2/2^{3/5(1+\alpha)}\), we get
\[ \psi \sim -2e^{-\frac{2}{3}(1+\alpha)(\frac{2t}{t_\Lambda})^{1/2}} t^{1/2} \to 0. \]

More generally, using Eq. (53), the evolution of the scalar field is given by
\[ \psi = -\sinh^{-1}\left\{1/\sinh \left[\frac{3}{2}(1 + \alpha) \left(\frac{8\pi G}{3}\right)^{1/2} \frac{t}{t_\Lambda}\right]\right\}. \]

A tachyon field [17] has an equation of state \(p = w\rho c^2\) with \(-1 \leq w \leq 0\). This scalar field evolves according to the equation
\[ \frac{\dot{\phi}}{1 - \dot{\phi}^2} + 3H \dot{\phi} + \frac{1}{V} \frac{dV}{d\phi} = 0. \]
In the dark energy era ($t \to +\infty$), using Eq. (49) with a prefactor $a_2/2^{2/3}$, we get

$$
\psi \sim -2e^{-\sqrt{6\pi} / t}\rightarrow 0.
$$

(67)

More generally, using Eq. (53), the evolution of the tachyon field $\psi(t)$ is given by

$$
\psi = -\text{Arctan} \left( \frac{1}{\sinh \left( \sqrt{6\pi} \frac{t}{t_\Lambda} \right)} \right).
$$

(68)

These results are illustrated in Figs. 13 and 14 for $\alpha = 0$ (pressureless matter).

V. THE GENERAL MODEL

A. The quadratic equation of state

We propose to describe the vacuum energy, the $\alpha$-fluid, and the dark energy by a unique equation of state

$$
p/c^2 = -(\alpha + 1) \rho \left( \frac{\rho}{\rho_p} \right)^{1/n_e} + \alpha \rho - (\alpha + 1) \rho \left( \frac{\rho}{\rho_p} \right)^{1/n_i},
$$

(69)

where $n_e > 0$ and $n_i < 0$ are the polytropic indices of the early and late universe, respectively. For the sake of simplicity, and for definiteness, we shall select the indices $n_e = +1$ and $n_i = -1$. Therefore, we consider the quadratic equation of state

$$
p = -(\alpha + 1) \frac{\rho^2}{\rho_p} c^2 + \alpha \rho c^2 - (\alpha + 1) \rho \Lambda c^2.
$$

(70)

For $\rho \to \rho_p$, $p \to -\rho c^2$ (vacuum energy); for $\rho \gg \rho_p$, $p \sim \rho c^2$ ($\alpha$-fluid); for $\rho \rightarrow \rho_\Lambda$, $p \rightarrow -\rho_\Lambda c^2$ (dark energy). This quadratic equation of state combines the properties of the equation of state (19) valid in the early universe, and of the equation of state (41) valid in the late universe. A nice feature of this equation of state is that both the Planck density (vacuum energy) and the cosmological density (dark energy) explicitly appear. Therefore, this equation of state reproduces both the early inflation and the late inflation, described by an equation of state $p = -\rho c^2$, in which the scale factor increases exponentially rapidly. They are connected by the $\alpha$-era, with a linear equation of state $p = \alpha \rho c^2$, in which the scale factor increases algebraically. For $\alpha > -1/3$, the universe is decelerating during this period.

Using the equation of continuity (11), we get

$$
\rho = \frac{\rho_p}{(a/a_1)^{3(\alpha+1)} + 1} + \rho_\Lambda.
$$

(71)

To obtain this expression, we have used the fact that $\rho_p \gg \rho_\Lambda$ so that $p/c^2 + \rho \simeq [(\alpha + 1)/\rho_p] \rho - \rho_\Lambda \rho - \rho$. When $a \rightarrow 0$, $\rho \rightarrow \rho_p$ (vacuum energy); when $\rho_\Lambda \ll \rho \ll \rho_p$, $\rho \simeq \rho_p (a_1/a)^{3(\alpha+1)}$ ($\alpha$-era); when $a \rightarrow +\infty$, $\rho \rightarrow \rho_\Lambda$ (dark energy). In the early universe, the contribution of
dark energy is negligible and we recover Eq. (20). In the late universe, the contribution of vacuum energy is negligible and we recover Eq. (45) with \( \rho \alpha a_0^3(1+\alpha) = \rho_\Lambda a_0^3(1+\alpha). \)

Using \( \rho \rho_\alpha a_0^3(1+\alpha) = \rho_{0,0} a_0^3(1+\alpha) \), where \( \rho_{0,0} \) is the present-day density of the \( \alpha \)-fluid and \( a_0 = c/H_0 = 1.3210^{26} \) is the present distance of cosmological horizon determined by the Hubble constant \( H_0 = 2.2710^{-18} \) (the Hubble time is \( H_0^{-1} = 4.4110^{17} \)).

The Friedmann equation \( \text{(73)} \) becomes
\[
\rho = \frac{\rho_{0,0}}{(a/a_0)^3(1+\alpha)} + \frac{(a_1/a_0)^3(1+\alpha)}{\rho_\Lambda}. \tag{72}
\]
Substituting this relation in the Friedmann equation \( \text{(71)} \), and writing \( \rho_{0,0} = \Omega_{\alpha,0} \rho_0 \) and \( \rho_\Lambda = \Omega_{\Lambda,0} \rho_0 \), where \( \rho_0 = 3H_0^2/8\pi G = 9.2010^{-24} \) is the present-day density of the universe, we obtain
\[
\frac{H}{H_0} = \sqrt{\frac{\Omega_{\alpha,0}}{(a/a_0)^3(1+\alpha)} + \frac{(a_1/a_0)^3(1+\alpha)}{\Omega_{\Lambda,0}}}. \tag{73}
\]
Finally, using Eqs. (70) and (71), the pressure can be written in very good approximation as
\[
p = \frac{\alpha(a_1/a_0)^{3(\alpha+1)} - 1}{[a/a_0]^{3(\alpha+1)} + 1} \rho \alpha c^2 - \rho_\Lambda c^2. \tag{74}
\]
It can be useful to re-express the previous results in terms of the present-day variables. To that purpose, we use the following relations \( a_1/a_0 = (\rho_{0,0}/\rho_\alpha)^{1/[3(1+\alpha)]}, a_2/a_0 = (\Omega_{\alpha,0}/\Omega_{\Lambda,0})^{1/[3(1+\alpha)]}, H_0 t_\gamma = (8\pi/3)^{1/2}(\rho_0/\rho_\alpha)^{1/2}, \) and \( H_0 t_\Lambda = (8\pi/3)^{1/2}(\Omega_{\alpha,0})^{-1/2}. \)

### B. The early universe

In the early universe, we can neglect the contribution of dark energy in the density. We only consider the contribution of the vacuum energy and of the \( \alpha \)-fluid. Eq. (72) then reduces to
\[
\rho = \frac{\rho_{0,0}}{(a/a_0)^3(1+\alpha)} + \frac{(a_1/a_0)^3(1+\alpha)}{\rho_\Lambda}. \tag{75}
\]
The Friedmann equation (73) becomes
\[
\frac{H}{H_0} = \sqrt{\frac{\Omega_{\alpha,0}}{(a/a_0)^3(1+\alpha)} + (a_1/a_0)^3(1+\alpha)}. \tag{76}
\]
It has the analytical solution given by Eq. (29). The transition between the vacuum energy era and the \( \alpha \)-era corresponds to \( \rho_\alpha = \rho_\Lambda \). This yields \( a_1/a_0 = (\rho_{0,0}/\rho_\alpha)^{1/[3(1+\alpha)]}, \rho_1/\rho_\alpha = 1/2. \) The inflation ends at the time \( t_1 \). The first inflexion point of the curve \( a(t) \) corresponds to \( a_{\gamma}/a_0 = \{2\rho_{0,0}/[(1 + \alpha)\rho_\alpha]\}^{1/[3(1+\alpha)]} \) and
\[
\rho_\gamma/\rho_\alpha = (1 + 3\alpha)/[3(\alpha + 1)]. \tag{77}
\]
The universe is accelerating for \( t < t_1 \) and decelerating for \( t > t_1. \) For \( \alpha = 1/3, \) \( t_\gamma = t_1. \)

In the vacuum energy era \( (a \ll a_1) \), the density is constant:
\[
\rho \simeq \rho_\Lambda. \tag{78}
\]
The Friedmann equation (76) reduces to
\[
\frac{H}{H_0} = \sqrt{\frac{\Omega_{\alpha,0}}{(a/a_0)^3(1+\alpha)}}, \tag{79}
\]
and yields
\[
\frac{a}{a_0} \sim \left[\frac{3}{2}(\alpha + 1)^{1/2}\Omega_{\alpha,0} H_0 t\right]^{2/[3(1+\alpha)]}, \tag{80}
\]
\[
\frac{\rho}{\rho_0} \sim \left[\frac{2}{3}(\alpha + 1)H_0 t\right]^{-2}. \tag{81}
\]

### C. The late universe

In the late universe, we can neglect the contribution of vacuum energy in the density. We only consider the contribution of the \( \alpha \)-fluid and of the dark energy. Eq. (72) then reduces to
\[
\rho = \frac{\rho_{0,0}}{(a/a_0)^3(1+\alpha)} + \rho_\Lambda. \tag{82}
\]
The Friedmann equation (73) becomes
\[
\frac{H}{H_0} = \sqrt{\frac{\Omega_{\alpha,0}}{(a/a_0)^3(1+\alpha)} + (a_1/a_0)^3(1+\alpha)}. \tag{83}
\]
It has the analytical solution
\[
\frac{a}{a_0} = \left(\frac{\Omega_{\alpha,0}}{\Omega_{\Lambda,0}}\right)^{1/[3(1+\alpha)]} \sinh^{2/[3(1+\alpha)]} \left[\frac{3}{2}(\alpha + 1)^{1/2}\Omega_{\Lambda,0} H_0 t\right]. \tag{84}
\]
The density evolves as
\[
\frac{\rho}{\rho_0} = \tanh^2 \left[\frac{3}{2}(\alpha + 1)^{1/2}\Omega_{\Lambda,0} H_0 t\right]. \tag{85}
\]
Setting \( a = a_0 \) in Eq. (84), we find the age of the universe\(^5\)

\[
t_0 = \frac{1}{H_0} \frac{2}{3(1 + \alpha)} \frac{1}{\sqrt{\Omega_{\Lambda,0}}} \sinh^{-1} \left[ \left( \frac{\Omega_{\Lambda,0}}{\Omega_{\ast,0}} \right)^{1/2} \right]. \tag{86}
\]

The present values of the equation of state parameter and deceleration parameter are

\[
w_0 = \alpha - (\alpha + 1)\Omega_{\Lambda,0}, \quad q_0 = \frac{1 + 3\alpha}{2} - \frac{3}{2}(\alpha + 1)\Omega_{\Lambda,0}. \tag{87}
\]

The transition between the \( \alpha \)-era and the dark energy era corresponds to \( \rho_{\alpha} = \rho_{\Lambda} \). This yields \( a_2/a_0 = (\Omega_{\alpha,0}/\Omega_{\Lambda,0})^{1/\left[3(1+\alpha)\right]} \) and \( \rho_{2}/\rho_{\Lambda} = 2 \). The universe starts accelerating when \( a_{\prime}/a_0 = [(1 + 3\alpha)\Omega_{\alpha,0}/2\Omega_{\Lambda,0}]^{1/\left[3(1+\alpha)\right]} \) and \( \rho_{\prime}/\rho_0 = 3(\alpha + 1)\Omega_{\Lambda,0}/(1 + 3\alpha) \). This corresponds to a second inflexion point of the curve \( a(t) \). The universe is decelerating for \( t < t_c \) and accelerating for \( t > t_c \). For \( \alpha = 1/3 \), \( t_c = t_2 \).

In the dark energy era \( (a \gg a_2) \), the density is constant:

\[
\rho \simeq \rho_{\Lambda}. \tag{88}
\]

Therefore, the Hubble parameter is also a constant \( H = H_0/\sqrt{\Omega_{\Lambda,0}} \). Using Eq. (84), we find that the scale factor increases exponentially rapidly as

\[
a/a_0 \sim \left( \frac{\Omega_{\alpha,0}}{4\Omega_{\Lambda,0}} \right)^{1/\left[3(1+\alpha)\right]} e^{\sqrt{\Omega_{\Lambda,0}}H_0t}. \tag{89}
\]

This corresponds to the de Sitter solution.

In the \( \alpha \)-era \( (a \ll a_2) \), we recover Eqs. (78)-(81).

\textit{Numerical application \[7\].} The Hubble radius, the density and the Hubble time of the present universe are \( a_0 = 0.302\Omega_\Lambda = 1.32 \times 10^{26} \text{m}, \rho_0 = 1.31\rho_\Lambda = 9.20 \times 10^{-24} \text{g/m}^3 \), and \( H_0^{-1} = 4.41 \times 10^{17} \text{s} \). We assume that \( \alpha = 0 \) (pressureless matter) and use \( \Omega_{\text{m},0} = 0.237 \) and \( \Omega_{\Lambda,0} = 0.763 \). The age of the universe is \( t_0 = 1.03 H_0^{-1} = 3.101\Omega_\Lambda = 4.54 \times 10^{17} \text{s} \sim 14 \text{Gyrs} \) (\( t_0 = 13.7 \text{Gyrs} \) if we use a more precise value of \( H_0 \) \[7\]). The present values of the deceleration parameter and of the equation of state parameter are \( q_0 = -0.645 \) and \( w_0 = -0.763 \).

The transition between the matter era and the dark energy era takes place at \( a_2 = 0.677a_0 = 0.204\Omega_\Lambda = 8.95 \times 10^{25} \text{m}, \rho_2 = 1.52\rho_0 = 2\rho_\Lambda = 1.40 \times 10^{-23} \text{g/m}^3 \), and \( t_2 = 0.674H_0^{-1} = 0.203\Omega_\Lambda = 2.97 \times 10^{17} \text{s} \). The time at which the universe starts accelerating is \( t_c = 0.504H_0^{-1} = 0.152\Omega_\Lambda = 2.22 \times 10^{17} \text{s} \), corresponding to a radius \( a_c = 0.538a_0 = 0.162\Omega_\Lambda = 0.794a_2 = 7.11 \times 10^{23} \text{m} \), and a density \( \rho_c = 2.29\rho_0 = 3\rho_\Lambda = 2.11 \times 10^{-23} \text{g/m}^3 \).

\textbf{D. The general solution}

For the quadratic equation of state \[70\], the density is related to the scale factor by Eq. \[71\]. It is possible to solve the Friedmann equation \[19\] with the density-radius relation \[71\] analytically. Introducing \( R = a/a_1 \) and \( \lambda = \rho_\Lambda/\rho_p \ll 1 \), we obtain

\[
\int \frac{1 + \sqrt{1 + R^3(\alpha + 1)}}{R\sqrt{1 + \lambda R^3(\alpha + 1)}} \frac{dR}{R} = \left( \frac{8\pi}{3} \right)^{1/2} t/t_p + C \tag{90}
\]

which can be integrated into

\[
\frac{1}{\sqrt{\lambda}} \ln \left[ 1 + 2\lambda R^\kappa + 2\sqrt{\lambda(1 + R^\kappa + \lambda R^{2\kappa})} \right] - \ln \left[ 2 + R^\kappa + 2\sqrt{1 + R^\kappa + \lambda R^{2\kappa}} \right] \frac{R^\kappa}{R} = \kappa \left( \frac{8\pi}{3} \right)^{1/2} \frac{t}{t_p} + C, \tag{91}
\]

where \( \kappa = 3(\alpha + 1) \) and \( C \) is a constant determined such that \( a = l_p \) at \( t = 0 \). This solution is interesting mathematically because it describes analytically a phase of early inflation and a phase of late inflation (or late accelerated expansion) connected by a power-law evolution corresponding to the \( \alpha \)-era. The corresponding scalar field theory is developed in Appendix A

\textbf{E. The whole evolution of the universe}

In our model, the evolution of the scale factor is given by Eq. \[73\]. For \( a_1 = 0 \), we obtain the same equation as in the CDM model (slightly generalized to account for a non-zero value of \( \alpha \)) where the contributions of matter and dark energy are added individually \[11\]. However, the CDM model does not describe the phase of early inflation and presents a singularity at \( t = 0 \) (Big Bang).

For \( a_1 \neq 0 \), we obtain a generalized model which does not presents a primordial singularity and which displays a phase of early inflation. In this model, the universe always existed in the past but, for \( t < 0 \), it has a very small radius, smaller than the Planck length.\(^6\) At \( t = 0 \), it undergoes an inflationary expansion in a very short lapse

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\(^5\) Of course, for the determination of the age of the universe, we can neglect the contribution of vacuum energy in the early universe and take \( a_1 = 0 \) (strictly speaking, the age of the universe is infinite since it has no origin; however, we define the age of the universe from the time \( t = 0 \) at which \( a = l_p \)).

\(^6\) As already stated in Sect. \[11\] our model certainly breaks down in this period since it does not take quantum fluctuations into account. The Planck era may not be described in terms of an equation of state \( p(\rho) \), or even in terms of the Einstein equations, as we have assumed. It probably requires the development of a theory of quantum gravity that does not exist for the moment. An interesting description of the early inflation has been given by Monerat et al. \[18\] in terms of a quantized model based on a simplified Wheeler-DeWitt equation. In that model, a quantum tunneling process explains the birth of the universe with a well defined size after tunneling. Therefore, other inflationary scenarios are possible in addition to the one based on the generalized equation of state \[70\].
of time of the order of the Planck time. For $t \gg t_P$, this model gives the same results as the ΛCDM model [see Eq. (83)]: the universe first undergoes an algebraic expansion in the α-era, then an exponential expansion (second inflation) in the dark energy era. A nice feature of this model is its simplicity since it incorporates a phase of early inflation in a very simple manner. We just have to add a term $\alpha/(a_0^3(1+\alpha))$ in the standard equation [83] of the ΛCDM model. Therefore, the modification implied by Eq. (73) to describe the early inflation is very natural. On the other hand, this model gives the same results in the late universe as the standard model, so this does not bring any modification to the usual equation [83]. Therefore, our simple model completes the standard ΛCDM model by incorporating the phase of early inflation in a natural manner. This is an interest of this description since the standard model gives results that agree with observations at late times.

![FIG. 15: Evolution of the scale factor $a$ as a function of time in logarithmic scales. The universe first undergoes a phase of early inflation (Planck era) due to the vacuum energy during which the scale factor increases exponentially rapidly on a timescale of the order of the Planck time $t_P$. This is followed by the α-era during which the scale factor increases algebraically. The dashed line corresponds to $\alpha = 1/3$ (radiation) and the dotted line corresponds to $\alpha = 0$ (matter). Without the early inflation, the universe would exhibit a primordial singularity (Big Bang). Finally, the universe undergoes a phase of late inflation (de Sitter era) due to the dark energy during which the scale factor increases exponentially rapidly on a timescale of the order of the cosmological time $t_\Lambda$. The universe exhibits two types of inflation: an early inflation corresponding to the Planck density $\rho_P$ (vacuum energy) due to quantum mechanics (Planck constant) and a late inflation corresponding to the cosmological density $\rho_\Lambda$ (dark energy) due to general relativity (cosmological constant). The evolution of the early and late universe is remarkably symmetric. The expansion of the universe is accelerating during the phases of inflation and decelerating during the α-era. We have also represented the location of the present universe (bullet). It happens to be just at the transition between the matter era and the dark energy era (see Sec. V F).](image)

![FIG. 16: Evolution of the density $\rho$ as a function of time in logarithmic scales. The density goes from a maximum value $\rho_{\text{max}} = \rho_P$ determined by the Planck constant (quantum mechanics) to a minimum value $\rho_{\text{min}} = \rho_\Lambda$ determined by the cosmological constant (general relativity). These two bounds, which are fixed by fundamental constants of physics, are responsible for the early and late inflation of the universe. In between, the density decreases as $t^{-2}$.](image)

F. The cosmic coincidence problem

Let us consider the transition between the matter era ($\alpha = 0$) and the dark energy era described by Eq. (83).

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7 We note that these bounds come from the properties of the polytropic equation of state [122]. In this sense, they may be connected to the bound on the mass of white dwarf stars (Chandrasekhar’s mass) which also comes from the properties of a polytropic equation of state with index $n = 3$ describing an ultra-relativistic degenerate fermionic gas [19].
It is striking to note that the present size of the universe 
\( a_0 = 0.302 \Lambda = 1.32 \times 10^{26} \) m is precisely of the order of the 
scale \( a_2 = 0.204 \Lambda = 8.95 \times 10^{25} \) m (\( a_0 = 1.48 a_2 \)). We have 
\( a_0 \sim a_2 \sim l_\Lambda \) and \( t_0 \sim t_2 \sim t_\Lambda \). Therefore, we live just at 
the transition between the matter era and the dark energy era (see bullets in Figs. 9 and 10). In the context of the standard \( \Lambda \)CDM model, the way to state this observation is to say that the present ratio \( \Omega_{\Lambda,0}/\Omega_{m,0} = 3.22 \) between 
dark energy and matter is of order unity. Since the matter density changes as \( a^{-3} \) the ratio between matter and dark energy is of order unity only during a “brief” period 
of time in the evolution of the universe. It turns out that we live precisely in this period. This coincidence 
of time in the evolution of the universe. It turns out that dark energy is of order unity only during a “brief” period 
for density changes as dark energy and matter is of order unity. Since the mat-

\[ \rho = \rho_m + \rho_{\Lambda}, \]

resulting from observations is \( \rho_\Lambda = 7.02 \times 10^{-24} \) g/m³. It is oftentimes argued that the cosmological density \( \rho_\Lambda \) 
should correspond to the vacuum energy density due to quantum fluctuations. However, according to 
particle physics and quantum field theory, the vacuum energy density is of the order of the Planck density 
\( \rho_P = 5.16 \times 10^{99} \) g/m³ which is \( 10^{122} \) times larger than the 
cosmological density. This leads to the so-called cosmological constant problem. 

Actually, as illustrated in Fig. 10 the Planck density and the cosmological density represent fundamental upper 
and lower density bounds acting in the early and late universe, respectively. It is not surprising therefore that 
they are so different: \( \rho_\Lambda \ll \rho_P \). Because of these bounds, 
the universe undergoes two phases of inflation. The inflation in the early universe is due to quantum mechanics 
(Planck constant) and is related to the Planck density \( \rho_P \) (vacuum energy). The inflation in the late universe 
is due to general relativity (cosmological constant) and is related to the cosmological density \( \rho_\Lambda \) (dark energy). Quantum mechanics is negligible in the late universe. Therefore, we should not identify the dark energy (or 
the cosmological constant) with the vacuum energy. The cosmological constant should be interpreted as a new fund-
amental constant of physics. It applies to the very large universe (cosmophysics) exactly like the Planck constant 
Applies to the very small universe (microphysics). Actually, there is a complete symmetry between the small 
and large universe where \( \hbar \) and \( \Lambda \) play symmetric roles. Therefore, we propose to interpret the cosmological 
constant as a fundamental constant of physics describing the 
cosmophysics (late universe) in the same sense that the 
Planck constant describes the microphysics (early universe). 

If this interpretation is correct, the origin of the dark 
energy density \( \rho_\Lambda \) should not be sought in quantum me-

\[ \rho = \rho_m + \rho_{\Lambda}, \]

chánics, but in pure general relativity. In this sense, the 
cosmological constant “problem” may be a false problem. 
If \( \Lambda \) is a fundamental constant of physics, independent 
from the others, its value should not cause problem. It is 
fixed by nature, just like the values of \( G, c, \) and \( \hbar \). One 
can just expect that \( \rho_P \) is “very large” and \( \rho_\Lambda \) is “very 
small”. Of course, the origin of the cosmological constant 
still needs to be understood by developing a theory of 
cosmophysics. In addition, it would be important to 
understand why \( \rho_P \) and \( \rho_\Lambda \) represent upper and lower 
bounds, and if these bounds are as fundamental as, for 
example, the bound on the velocity fixed by the speed of light. 

VI. CONCLUSION 

We have constructed a cosmological model based on the 

\[ \rho = \rho_m + \rho_{\Lambda}, \]

quadratic equation of state \( \rho/\rho_m = \rho_{\Lambda}/\rho_m = \Omega_{\Lambda}/\Omega_m \). This equation of 
state describes the evolution of a universe presenting a 
phase of early inflation (Planck era), a phase of deceler-
ating expansion (\( \alpha \)-era), and a phase of late accelerated 
expansion (de Sitter era). An interest of this model is its 
simplicity (while being already quite rich) and the fact 
that it admits a fully analytical solution. It provides a 
particular solution of the Friedmann equations. In ad-
in, it is in qualitative agreement with the evolution of 
of our own universe. Finally, it admits a scalar field in-

\[ \rho = \rho_m + \rho_{\Lambda}, \]

terpretation based on a quintessence field or a tachyon 
field. 

This model does not present any singularity and exists 
“eternally” in the past and in the future, so it has no ori-
gin nor end (aioniotic universe). In particular, the phase 
of early inflation avoids the Big Bang singularity and re-
places it by a sort of second order phase transition where 
the non-zero value of the Planck constant \( \hbar \) plays the role
of finite size effects. Of course, it is probably incorrect to extrapolate the results in the infinite past since our model is purely classical, or semi-classical, and does not take quantum gravity into account. In our model, the universe starts from $t = -\infty$ but, for $t < 0$, its size is less than the Planck length $l_P = 1.62 \times 10^{-35}$ m. In the Planck era, the classical Einstein equations may be incorrect and should be replaced by a theory of quantum gravity that still has to be constructed.

Another interest of this model is that it describes the early universe and the late universe in a symmetric manner. The early universe is described by a polytrope $n = 1$ and the late universe by a polytrope $n = -1$. The mathematical formulae are then strikingly symmetric (we sum the inverse of the densities in the early universe and the densities in the late universe). Furthermore, the Planck density in the early universe plays the same role as the cosmological density in the late universe. They represent fundamental upper and lower density bounds differing by 122 orders of magnitude. They lead to phases of early or late inflation with very different timescales $t_P = 1/(G\rho_{P})^{1/2} = 5.39 \times 10^{-44}$ s and $t_\Lambda = 1/(G\rho_\Lambda)^{1/2} = 1.46 \times 10^{38}$ s. The densities $\rho_\Lambda$ and $\rho_\Lambda$ (together with $\alpha$) appear as the coefficients of the equation of state (2). Therefore, this equation of state provides a “unification” of vacuum energy and dark energy.

A drawback of this model is that it cannot describe the transition from the radiation era to the matter era because they are both described by a linear equation of state $p = \alpha \rho c^2$ with $\alpha = 1/3$ and $\alpha = 0$ respectively. Therefore, the equation of state (2) with $\alpha = 1/3$ describes a universe without matter while the equation of state (4) with $\alpha = 0$ describes a universe without radiation. A better model could be achieved by letting $\alpha(t)$ vary with time and decrease smoothly from $1/3$ to 0. Alternatively, we could “unify” vacuum energy + radiation + dark energy by the equation of state (2) with $\alpha = 1/3$ and treat baryonic matter and dark matter as independent species described by the equations of state $p_b = 0$ and $p_\Lambda = 0$. This more general approach, allowing to describe the complete evolution of the universe (inflation, radiation, matter, acceleration) will be developed in another work.

Matter is usually described by a pressureless equation of state $p = 0$. This leads to the famous Einstein-de Sitter (EdS) model. However, there are indications that dark matter may be described by an isothermal equation of state $p = \alpha \rho c^2$ with $\alpha \ll 1$. Therefore, it is useful to provide general results valid for arbitrary values of $\alpha$ as we have done here. This allows us to either describe radiation ($\alpha = 1/3$), pressureless matter ($\alpha = 0$), or isothermal matter ($\alpha \neq 0$).

As we have seen, the equation of state (69) with $n_\ell = -1$ leads to results that coincide with the $\Lambda$CDM model in the late universe. This equivalence is not trivial since our approach is fundamentally different from the $\Lambda$CDM model (in particular this coincidence is not true anymore for $n_\ell \neq -1$). Since the $\Lambda$CDM model provides a good description of the universe at late times, this implies that observations tend to favor the value $n_\ell = -1$ of the polytropic index. This is a reason why we have selected this index. However, we can obtain more general models by taking $n_\ell$ different from $-1$. In particular, a value close to $-1$ may be consistent with observation and improve upon the $\Lambda$CDM model. Similarly, we have selected the index $n_e = +1$ to describe the inflation in the early universe. More general models of inflation can be obtained by selecting other values of $n_e$. The value of $n_e$ could be determined from the constraints on the CMB power spectrum. Therefore, possible extensions of our study would be to consider the generalized equation of state (69) with arbitrary values of $n_e$ and $n_\ell$. General results valid for the asymptotic expressions (18) and (43) of the equation of state (69) in the early and late universe are given in [7].

A weakness of our work is that we have not given a justification of the quadratic equation of state (2). However, since this equation of state is connected to the Chaplygin gas model, there is some hope that this equation of state could arise from fundamental theories (the original Chaplygin gas model was motivated by some works on string theory). We may also argue that this equation of state (and its generalization (69)) is the “simplest” equation of state that we can imagine in order to obtain a non singular model of universe (see the discussion in [7]). This “simplest” model is in agreement with the known properties of our universe. It is interesting that these properties can be reproduced by an equation of state of the form $ax^2 + bx + c$ where the three coefficients are related to $\hbar$, $\alpha$, and $\Lambda$.

Appendix A: Scalar field theory

In this Appendix, we determine the quintessence potential and the tachyon potential corresponding to the quadratic equation of state (70). Interestingly, this scalar field theory describes the whole evolution of the universe, from the early inflation to the late acceleration, passing through a phase of algebraically decelerating expansion.

1. Quintessence field

We consider a scalar field defined by Eqs. (33)-34. As shown in [7], the relation between the scalar field and the scale factor is given by

$$\frac{d\phi}{da} = \frac{(3c^2)}{8\pi G} \sqrt{1 + \omega} \frac{\sqrt{1 + \omega}}{a}. \quad (A1)$$

On the other hand, the potential of the scalar field is given by

$$V = \frac{1}{2} (1 - \omega) \rho c^2. \quad (A2)$$
For the quadratic equation of state \( w = \frac{(\alpha + 1)}{\rho} \), we have
\[
w = -\left(\alpha + 1\right) \frac{\rho}{\rho p} + \alpha = -\left(\alpha + 1\right) \lambda \frac{\rho}{\rho p}
\] (A3)
and
\[
\frac{\rho}{\rho p} = \frac{1}{R^{3(\alpha+1)}} + \lambda,
\]
where \( R = a/a_0 \) and \( \lambda = \rho_\Lambda/\rho_p = 1.36 \times 10^{-123} \). With the change of variables
\[
x = R^{3(\alpha+1)/2}, \quad \psi = \left(\frac{8\pi G}{3c^2}\right)^{1/2} \frac{3\sqrt{\alpha + 1}}{2} \phi,
\]
we obtain after simple calculations (using the fact that \( \lambda \ll 1 \) to simplify some terms):
\[
\psi = \int \frac{dx}{(\lambda x^2 + 1)^{1/2}(x^2 + 1)^{1/2}} = -iF(i \sin^{-1}(x), \lambda),
\]
\[
V = \frac{1}{2} \rho_pc^2 \frac{2 + (1 - \alpha)x^2 + 2\lambda x^4}{(x^2 + 1)^2},
\]
where \( F(a, \lambda) = \int_0^a (1 - \lambda \sin^2 \theta)^{-1/2} d\theta \) is the Elliptic integral of the first kind. We have taken \( \psi = 0 \) when \( x = 0 \). Eqs. (A6)-(A7) define the potential \( V(\psi) \) in parametric form (with parameter \( x \) going from 0 to \( +\infty \)). In the early universe, we can neglect the dark energy (\( \lambda = 0 \)) and the foregoing equations reduce to
\[
\psi = \sinh^{-1}(x),
\]
\[
V = \frac{1}{2} \rho_pc^2 \left(\frac{2 + (1 - \alpha)x^2}{x^2 + 1}\right),
\]
This returns Eqs. (36)-(37). In the late universe, we can neglect the vacuum energy (which amounts to taking \( x \gg 1 \)) and Eqs. (A6)-(A7) reduce to
\[
\psi - \psi_{\text{max}} = -\sinh^{-1} \left(\frac{1}{\sqrt{\lambda x}}\right),
\]
\[
V = \frac{1}{2} \rho_pc^2 \left(\frac{1 - \alpha}{x^2} + 2\lambda\right),
\]
where \( \psi_{\text{max}} \) is a constant of integration. This returns Eqs. (50)-(58) where we have arbitrarily taken \( \psi_{\text{max}} = 0 \).

The constant \( \psi_{\text{max}} \) can be obtained by matching the solutions (A8)-(A9) and (A10)-(A11) in the intermediate region of algebraic expansion. First, we note that \( V = \rho pc^2 \) for \( \psi = 0 \) (\( x = 0 \) and \( V = \rho_\Lambda c^2 \) for \( \psi = \psi_{\text{max}} \) (\( x \to +\infty \)). On the other hand, for \( x \gg 1 \), Eqs. (A8)-(A9) reduce to \( \psi \sim \ln(2x) \) and \( V \sim (1/2)\rho pc^2(1 - \alpha)x^{-2} \) yielding
\[
V(\psi) \approx 2\rho_pc^2(1 - \alpha)e^{-2\psi}, \quad (\psi \gg 1).
\]
Comparing Eqs. (A12) and (A13), we find that
As shown in [7], the relation between the tachyon field and the scale factor is given by

$$\frac{d\phi}{da} = \left(\frac{3c^2}{8\pi G}\right)^{1/2} \frac{\sqrt{1+w}}{\sqrt{\rho c^2 a}}. \quad (A14)$$

On the other hand, the potential of the scalar field is given by

$$V^2 = \rho \psi^2 c^4. \quad (A15)$$

The tachyon field is defined provided that $w < 0$. This is the case for all times with the quadratic equation of state [70] when $\alpha = 0$. With the change of variables

$$x = R^{3(\alpha+1)/2}, \quad \psi = \sqrt{\rho c^2} \left(\frac{8\pi G}{3c^2}\right)^{1/2} \frac{3\sqrt{\alpha} + 1}{2} \phi, \quad (A16)$$

we obtain after simple calculations (using the fact that $\lambda \ll 1$ to simplify some terms):

$$\psi = \int \frac{dx}{1 + \lambda x^2} = \frac{1}{\sqrt{\lambda}} \tan^{-1} \left(\sqrt{\lambda} x\right), \quad (A17)$$

$$V = \rho \psi^2 \sqrt{1 - \alpha x^2 - (\alpha - 1)\lambda x^4 + \lambda^2 x^6}. \quad (A18)$$

We have taken $\psi = 0$ when $x = 0$. Eqs. [A17]-[A18] define the potential $V(\psi)$ in parametric form (with parameter $x$ going from 0 to $+\infty$). In the early universe, we can neglect the dark energy ($\lambda = 0$) and the foregoing equations reduce to

$$\psi = x, \quad V = \rho \psi^2 \sqrt{1 - \alpha x^2 (x^2 + 1)^{3/2}}. \quad (A19)$$

For $\alpha = 0$, we get

$$V(\psi) = \frac{\rho \psi^2}{(\psi^2 + 1)^{3/2}}. \quad (A20)$$

For $\psi = x = 0$, we have $V = \rho \psi^2$. For $x \gg 1$, we obtain $V \sim \rho \psi^2 / \psi^3$. In the late universe, we can neglect the vacuum energy (which amounts to taking $x \gg 1$) and Eqs. [A17]-[A18] reduce to

$$\psi = \frac{1}{\sqrt{\lambda}} \tan^{-1} \left(\sqrt{\lambda} x\right), \quad (A21)$$

$$V = \rho \psi^2 \sqrt{-\alpha - (\alpha - 1)\lambda x^2 + \lambda^2 x^4}. \quad (A22)$$

For $\alpha = 0$, we get

$$V(\psi) = \frac{\rho \Lambda c^2}{\sin(\sqrt{\lambda} \psi)} \quad (A23)$$

which is equivalent to Eqs. [A21]-[A25], where we have arbitrarily shifted $\sqrt{\lambda} \psi$ by $-\pi/2$. For $\psi_{\text{max}} = \pi / (2\sqrt{\lambda}) = 4.2610^{31} (x \to +\infty)$, we have $V = \rho \Lambda c^2$. For $\psi \ll \psi_{\text{max}} (x \ll 1)$, we have $\psi \sim x$ and $V \sim \rho \psi^2 / \psi$. Using Eqs. [A17] and [A18], we can obtain the evolution of the scalar field with time. The tachyon field evolves as $\psi \simeq (l_P / a_1)^{3/2} \exp[(6\pi)^{1/2} t / t_P]$ in the vacuum energy era, as $\psi \simeq (6\pi)^{1/2} t / t_P$ in the $\alpha$-era, and as $\psi \simeq \psi_{\text{max}} - (2 / \sqrt{\lambda}) \exp[-(6\pi\lambda)^{1/2} t / t_P]$ in the dark energy era.

Some representative curves are shown in Figs. [17-20] for $\alpha = 0$. 

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