Determining the optimal coefficient of the spatially periodic Fisher-KPP equation that minimizes the spreading speed

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Abstract

This paper is concerned with the spatially periodic Fisher-KPP equation

\[ u_t = (d(x)u_x)_x + (r(x) - u)u, \quad x \in \mathbb{R}, \]

where \( d(x) \) and \( r(x) \) are periodic functions with period \( L > 0 \). We assume that \( r(x) \) has positive mean and \( d(x) > 0 \). It is known that there exists a positive number \( c_0(r) \), called the minimal wave speed, such that a periodic traveling wave solution with average speed \( c \) exists if and only if \( c \geq c_0(r) \). In the one-dimensional case, the minimal speed \( c_0(r) \) coincides with the “spreading speed”, that is, the asymptotic speed of the propagating front of a solution with compactly supported initial data. In this paper, we study the minimizing problem for the minimal speed \( c_0(r) \) by varying \( r(x) \) under a certain constraint, while \( d(x) \) arbitrarily. We have been able to obtain an explicit form of the minimizing function \( r(x) \). Our result provides the first calculable example of the minimal speed for spatially periodic Fisher-KPP equations as far as the author knows.

Keywords: KPP equation; traveling wave; minimal speed; spreading speed

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1 Introduction

Propagation phenomena appear in various fields of natural science, including population genetics, epidemiology, ecology and so on. The Fisher-KPP equation is among the classical models that describe propagation phenomena. From the viewpoint of ecology, this equation describes the expansion of the territory of invading alien species in a given habitat.

In this paper, we investigate the spatially periodic Fisher-KPP equation:

\[ u_t = (d(x)u_x)_x + (r(x) - u)u, \quad x \in \mathbb{R}, t > 0, \quad (E) \]

where \( d(x) \) and \( r(x) \) are periodic functions with period \( L > 0 \). While we always assume \( d > 0 \), we will allow \( r(x) \) to change sign: so long as its mean \( \langle r \rangle_a \) is positive (see (2.1)).
The solution \( u(x, t) \) represents the population density of an invading species, while \( r(x) \) denotes the intrinsic growth rate and \( d(x) \) is the diffusion coefficient. These periodic coefficients represent an environment in which favorable zones and less favorable zones appear alternately in a periodic manner.

The Fisher-KPP equation was introduced by Fisher [6] and Kolmogorov, Petrovsky and Piskunov [9] in 1937 in the content of population genetics. In 1951, Skellam [16] used this equation as a model for biological invasion in ecology. The above works were focused on the spatially homogeneous equation. Shigesada, Kawasaki and Teramoto [15] in 1986 considered the case where the coefficients are spatially periodic and studied the influence of periodic environments on the invasion speed. The paper [15] introduced a notion of traveling wave solutions in the periodic setting, while they called “traveling periodic solution”.

Berestycki and Hamel [1] proved the existence of periodic traveling waves for the positive coefficient \( r(x) \). They also proved that the slowest traveling wave exists. We call its speed the minimal traveling wave speed (or minimal speed in short) and it is denoted by \( c^*_d(r) \), that is, the traveling wave with average speed \( c \) exists if and only if \( c \geq c^*_d(r) \).

Weinberger [17] also studied periodic traveling waves together with the “spreading speed” in a rather abstract setting that include reaction-diffusion equation of the form (E) as a special case. The term “spreading speed” refers to the asymptotic speed of the propagating front of a solution with compactly supported initial data. Under the assumption that \( u \equiv 0 \) is unstable the existence of the spreading speed and periodic traveling waves was proved. He also derived that the spreading speed coincides with the minimal speed in the one-dimensional case and the spreading speed is characterized by using the principal eigenvalue of the corresponding linearized operator.

Berestycki-Hamel-Nadirashvili [2] proved that the minimal speed \( c^*_d(r) \) is characterized by the following formula:

\[
    c^*_d(r) = \min_{\lambda > 0} \left( - \frac{k_{\lambda}(d, r)}{\lambda} \right),
\]

where \( k_{\lambda}(d, r) \) is the principal eigenvalue of a certain operator \(-\mathcal{L}_{\lambda, d, r}\). The variational characterization of the principal eigenvalue \( k_{\lambda}(d, r) \) has been derived by Nadin [13]. Hence we can analyze the minimal speed by using variational method. See also subsection 2.1.

The purpose of this work is to analyze the influence of periodic environment on the invasion speed. Specifically, in this paper, we consider the problem of finding a minimizing function of \( c^*_d(r) \) varying \( r(x) \), where \( d \in C^{1+\delta}_{\text{per}}(\mathbb{R}) \) is fixed and a minimizer \( r(x) \) is sought in

\[
    \Lambda(\alpha) := \{ r \in C^{\delta}_{\text{per}}(\mathbb{R}) \mid \frac{1}{L} \int_0^L r(x) dx = \alpha \}.
\]

Here \( \delta > 0 \) is given positive constant. In other words, we consider the following
minimizing problem.

Minimize $c_d^*(r)$ 

For $r \in \Lambda(\alpha)$

From the ecological point of view, the spreading speed describes the invasion speed of alien species. Hence the problem means seeking the best disposition of environment to prevent the invasion of alien species.

A minimizing problem associated with the minimal speed is partially discussed in Shigesada-Kawasaki-Teramoto [15]. They studied the dependence of the period $L > 0$ to $c_d^*(r)$ under the certain assumption, and they proved that $L \mapsto c_d^*(r)$ is nondecreasing. Their work was partly unrigorous from mathematical point of view because their analysis was based on a formal asymptotic representation of traveling waves. Nadin [13] gave the rigorous proof of this research by dealing with much more general equations.

In the case where $d(x)$ is a constant, Berestycki-Hamel-Roques [4] derived that a constant function minimizes the minimal speed, and Liang-Lin-Matano [10] proved that the principal eigenfunction is a constant function if $r(x)$ is constant. These results are derived by using the eigenvalue problem associated with the operator $-\mathcal{L}_{\lambda,d,r}$.

These previous works suggest that the minimal speed will be slower if environments are more homogenized, and the most averaged environment minimizes the spreading speed.

In the case of sinusoidal diffusion and growth coefficient, N. Kinezaki, K. Kawasaki and N. Shigesada [8] computed the minimal speed varying the phase of the diffusion coefficient. By numerically solving the equation, they concluded that the minimal speed attains its minimum (maximum) when the diffusion and the growth coefficient have same (opposite) phases. Nadin [13] formulated this numerical result about maximizing the speed as a conjecture using the Schwarz rearrangement, and he studied the influence of the concentrating effect on the minimal speed when the diffusion coefficient is not constant. A maximizing problem is also investigated by some researchers. See also [13, 10, 11, 12, 7]. The effect of temporal averaging on the minimal speed is also considered by Nadin [14].

In this paper, we consider the case where $d(x)$ is a periodic function. The main difficulty with this problem is that the eigenvalue problem is more complicated than the constant case. As we will see later, in the periodic case, the principal eigenfunction is not a constant function. See subsection 2.2. The mathematical motivation of this work is to analyze how the optimal growth coefficient depends on the fixed diffusion coefficient.

In 2010, Nadin derived the following inequality

$$c_d^*(r) \geq 2\sqrt{\langle d \rangle_h \langle r \rangle_a}$$

(1.1)

Here $\langle r \rangle_a$ is the spatial arithmetic mean of $r(x)$, and $\langle d \rangle_h$ is the spatial har-
monic mean of $d(x)$, that is, the symbols $\langle r \rangle_a$, $\langle d \rangle_h$ are defined by

\[
\langle r \rangle_a = \frac{1}{L} \int_0^L r(x) \, dx, \quad \langle d \rangle_h = \frac{1}{L} \int_0^L \frac{1}{d(x)} \, dx
\]  

(1.2)

for any $r \in C_{\text{per}}(\mathbb{R})$ and $d \in C^1_{\text{per}}(\mathbb{R})$. We solve the minimizing problem $(P)_d$ by finding out a condition under which equality holds in the inequality (1.1).

We will see equality in (1.1) holds if and only if $d(x)$ and $r(x)$ satisfy the following relational expression.

\[
\frac{r}{\langle r \rangle_a} + \frac{\langle d \rangle_h}{d(x)} = 2
\]  

(1.3)

By the condition (1.3), we see that the minimizing problem $(P)_d$ has the solution for any $d \in C^{1+\delta}_{\text{per}}(\mathbb{R})$ with $\inf d > 0$, that is,

\[
r_d(x) = \alpha \left(2 - \frac{\langle d \rangle_h}{d(x)}\right), \quad x \in \mathbb{R}
\]  

(1.4)

is the solution for the minimizing problem $(P)_d$. The condition (1.3) is first introduced by El Smaily-Hamel-Roques [5] in a study on an approximate value of the spreading speed. See also subsection 2.1. In this paper, we will rediscover this condition to find the optimal coefficient.

By (1.4), the spreading speed attains its minimum when $r(x)$ is large in the area where $d(x)$ is large and $r_d(x)$ is small in the area where $d(x)$ is small. See also subsection 2.2 and subsection 3.1.

The interpretation of our main result from the ecological point of view is that the invasion speed of alien species reaches its minimum when the species quickly disperse in their favorable areas and slowly disperse in their less favorable areas.

Our result provides the influence of a non-trivial relation between the shape of the diffusion coefficient and the growth coefficient on the spreading speed. In some sense, our result formulated a numerical result computed by Kinezaki-Kawasaki-Shigesada [8] in a different way from Nadin.

By the effect of the surrounding environment, the most averaged function is not the minimizing function. As far as the minimizing problem associated with the spreading speed (or the minimal traveling wave speed), this work provides the first example finding out the influence of the shape of the heterogeneity of the diffusion coefficient on the optimal growth coefficient.

Our result means that $c^*_2(r) = 2\sqrt{\langle d \rangle_h \langle r \rangle_a}$ when $(d, r)$ satisfy the condition (1.3). An approximate value of the spreading speed is known when $L \to 0$ and $L \to \infty$, but the exact value is only known in the case where $d$ and $r$ are constant as far as the author knows. This is the first calculable example of the spreading speed (or the minimal traveling wave speed) for the spatially periodic Fisher-KPP equation.

This paper is organized as follows: In section 2, we state our main results, and we introduce known results of the spreading speed. In section 3, we give the proofs of our main results.
2 Main results

In this section, we explain the minimal speed of traveling waves and state the main results as well as the explanation of related works.

2.1 Formulation of the problem

In this subsection, we recall some known results of the spatially periodic Fisher-KPP equation. We consider the following Cauchy problem:

\[
\begin{aligned}
& u_t = (d(x)u_x)_x + (r(x) - u)u, \quad x \in \mathbb{R}, t > 0, \\
& u(x,0) = u_0(x) \geq 0, \quad x \in \mathbb{R},
\end{aligned}
\]

where \( u_0 \in C_c(\mathbb{R}) , u_0 \geq 0 , u_0 \not\equiv 0 \). In what follows, we assume that

\[
\inf d > 0, \langle r \rangle_a > 0.
\]

In this case, a stationary problem of \((E)\) has the positive periodic solution \( p(x) \), that is, there exists the positive function \( p(x) \) that satisfies

\[
(d(x)p_x)_x + (r(x) - p)p = 0.
\]

See Berestycki-Hamel-Roques [3].

Weinberger [17] and Berestycki-Hamel-Roques [3, 4] proved that \((E)\) has traveling wave solutions.

**Definition 2.1** (Periodic traveling waves). A solution \( u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) of \((E)\) is called a periodic traveling wave solution in the positive direction if the following conditions hold:

1. \( \lim_{x \to -\infty} (u(x,t) - p(x)) = 0, \lim_{x \to \infty} u(x,t) = 0 \) locally uniformly in \( t \in \mathbb{R} \);

2. There exists a constant \( T > 0 \) such that

\[
(u(x-L,t) - u(x,t+T)) (x,t) \in \mathbb{R} \times \mathbb{R}.
\]

Here we call the quantity \( c := L/T \) the average speed of the traveling wave \( u(x,t) \) (or “speed” for simplicity).

They also proved that there exists the minimal traveling wave speed \( c^*_d(r) \) (or “minimal speed” for simplicity), that is, the traveling wave with speed \( c \) exists if and only if \( c \geq c^*_d(r) \).

The Cauchy problem \((E_0)\) has the classical global solution \( u(x,t) \) for any \( u_0 \in C_c(\mathbb{R}) , u_0 \geq 0 , u_0 \not\equiv 0 \). Furthermore, trivial solution 0 is unstable under the assumption (2.1). It means that the solution \( u(x,t) \) goes to the positive function \( p(x) \) as \( t \to \infty \). The speed of an expanding front of \( u(x,t) \) asymptotically approaches to a certain value as \( t \to \infty \).
Definition 2.2 (Spreading speed). A quantity $\hat{c} > 0$ is called the spreading speed if for any nonnegative initial data $u_0 \in C_c(\mathbb{R})$ with $u_0 \geq 0$, $u_0 \not\equiv 0$, the solution $u(x, t)$ of the Cauchy problem with initial data $u_0$ satisfies that

1. $\lim_{t \to \infty} u(ct, t) = 0$ if $c > \hat{c}$,

2. $\liminf_{t \to \infty} u(ct, t) > 0$ if $0 < c < \hat{c}$.

Weinberger [17] and Berestycki-Hamel-Nadirashvili [2] proved that the minimal speed $c^*_d(r)$ is the spreading speed in the one-dimensional case.

Any traveling wave solution $u(x, t)$ with speed $c > c^*_d(r)$ in the negative direction has the following asymptotic expression if $(x, t)$ satisfies $u(x, t) \approx 0$:

$$u(x, t) \sim e^{\lambda(x+ct)}\psi(x), \quad (2.2)$$

where $\psi > 0$ is some $L$-periodic function and $\lambda > 0$ is some constant.

In $u(x, t) \approx 0$, $(r(x) - u)$ is practically equal to the intrinsic growth rate $r(x)$. Substituting (2.2) into the equation (E), we have

$$-(d(x)\psi'(x))' - 2\lambda d(x)\psi'(x) - (\lambda^2 d(x) + \lambda d'(x) + r(x))\psi(x) = -\lambda c\psi(x), \quad x \in \mathbb{R}$$

Set the operator $-L_{\lambda, d, r}$ on $C^2_{\text{per}}(\mathbb{R})$ for any constant $\lambda > 0$ as follows

$$-L_{\lambda, d, r}\psi(x) := -(d(x)\psi'(x))' - 2\lambda d(x)\psi'(x) - (\lambda^2 d(x) + \lambda d'(x) + r(x))\psi(x).$$

and we denote by $k_{\lambda}(d, r)$ the principal eigenvalue of the operator $-L_{\lambda, d, r}$, that is,

$$\begin{cases} -L_{\lambda, d, r}\psi = k_{\lambda}(d, r)\psi, \\ \psi(x + L) \equiv \psi(x), \end{cases} \quad (2.3)$$

and the eigenfunction $\psi$ is positive.

It is expected by the above formal calculation that $-\lambda c$ is the principal eigenvalue of the operator $-L_{\lambda, d, r}$, that is,

$$-\lambda c = k_{\lambda}(d, r).$$

Recall that $c^*_d(r)$ is the minimal speed, we expect

$$c^*_d(r) = \min_{\lambda > 0} \left(-\frac{k_{\lambda}(d, r)}{\lambda}\right), \quad (2.4)$$

and this formula was established by Berestycki-Hamel-Nadirashvili [2].

In the case where $d(x)$ is a constant, Liang-Lin-Matano [10] proved that equality in the inequality (1.1) holds if and only if $r(x)$ is a constant. They also derived that the principal eigenfunction of the operator $-L_{\lambda_0, d, r}$ is constant, where $\lambda_0 = \sqrt{\langle r(x) \rangle/d}$ satisfies

$$c^*_d(r) = -\frac{k_{\lambda_0}(d, r)}{\lambda_0}.$$
These results are proved by using the eigenvalue problem (2.3).

However, if $d(x)$ is not constant, the eigenvalue problem (2.3) is more complicated than the constant case. As we will see later (Theorem 2.10), the principal eigenfunction is not constant. That is why we use the formula about the principal eigenvalue derived by Nadin which is simpler than (2.3).

The first eigenvalue of $-\Delta$ is represented by an integral functional (the Rayleigh characterization). Nadin [13] gives the following representation of $k_{\lambda}(d, r)$ that is the principal eigenvalue of the non-symmetric operator $-L_{\lambda,d,r}$.

**Proposition 2.3** (Nadin [13]). Set $E_L := \{ \varphi \in C^1_{\text{per}} | \varphi > 0, \int_0^L \varphi^2 dx = 1 \}$. The principal eigenvalue $k_{\lambda}(d, r)$ of $-L_{\lambda,d,r}$ is characterized as follows:

$$
k_{\lambda}(d, r) = \min_{\varphi \in E_L} \left\{ \int_0^L d|\varphi'|^2 dx - \int_0^L r \varphi^2 dx - \frac{\lambda^2 L^2}{\int_0^L 1 d\varphi^2 dx} \right\}.
$$

By using this formula, Nadin studied the dependence of the period $L$ on the minimal speed $c^*_L(r)$, and derived the lower estimate of the minimal speed which we investigate as a corollary. Define

$$
d_L(x) = d(x/L), r_L(x) = r(x/L),
$$

where $d(x)$ and $r(x)$ are 1-periodic functions. Set $c^*_L = c^*_L(r_L)$.

**Proposition 2.4** (Nadin [13]). The following statements hold:

1. The function $L \mapsto k_{\lambda}(d_L, r_L)$ and $L \mapsto c^*_L$ are nondecreasing. Moreover,

$$
\lim_{L \to 0} k_{\lambda}(d_L, r_L) = -\langle r \rangle_a - \lambda^2 \langle d \rangle_h,
$$

$$
\lim_{L \to 0} c^*_L = 2\sqrt{\langle d \rangle_h \langle r \rangle_a}.
$$

2. For any $L > 0$,

$$
c^*_L \geq 2\sqrt{\langle d \rangle_h \langle r \rangle_a}.
$$

The condition (1.3) is first introduced by El Smaily-Hamel-Roques.

**Proposition 2.5** (El Smaily-Hamel-Roques [5]). For some $L_0 > 0$, the map $L \mapsto c^*_L$ is in $C^\infty(0, L_0)$. Moreover,

$$
\lim_{L \to 0} \frac{dc^*_L}{dL} = 0, \lim_{L \to 0} \frac{d^2 c^*_L}{dL^2} \geq 0.
$$

Finally, the following two statements are equivalent:

1. $\lim_{L \to 0} \frac{d^2 c^*_L}{dL^2} > 0$.

2. $\frac{r}{\langle r \rangle_a} + \frac{\langle d \rangle_h}{d} \neq 2$. 

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2.2 Main results

For any \( \varphi \in E_L \), \( \lambda > 0 \), \( r \in C_{\text{per}}(\mathbb{R}) \), \( d \in C^1_{\text{per}}(\mathbb{R}) \), set

\[
I(\varphi; \lambda, d, r) := \int_0^L d|\varphi'|^2 \, dx - \int_0^L r\varphi^2 \, dx - \frac{\lambda^2 L^2}{\int_0^L \frac{1}{d\varphi^2} \, dx}
\]

By Proposition 2.4, we can rewrite the principal eigenvalue \( k_{\lambda}(d, r) \) as

\[
k_{\lambda}(d, r) = \min_{\varphi \in E_L} I(\varphi; \lambda, d, r).
\]

Hence the spreading speed \( c^*_d(r) \) is also rewritten as

\[
c^*_d(r) = -\max_{\lambda > 0} \min_{\varphi \in E_L} \lambda^{-1} I(\varphi; \lambda, d, r)
\]

from the formula (2.4). Now we state our main results.

**Theorem 2.6.** The following statements are equivalent:

1. \( c^*_d(r) = 2\sqrt{\langle d \rangle_h \langle r \rangle_a} \).
2. \( \frac{r}{\langle r \rangle_a} + \frac{\langle d \rangle_h}{d} = 2 \).
3. Set \( \lambda_0 = \sqrt{\langle r \rangle_a / \langle d \rangle_h} \) and \( \varphi_0 \equiv 1/\sqrt{L} \). Then

\[
c^*_d(r) = -\lambda_0^{-1} I(\varphi_0; \lambda_0, d, r).
\]

Moreover, \( (\lambda_0, \varphi_0) \) is unique pair that satisfy (2.6).

**Remark 2.7.** This theorem is a generalization of the result proved by Berestycki-Hamel-Roque [4] in some sense. In the case where \( d \) is a constant, we can easily see that \( \langle d \rangle_h = d \). Thus the condition (1.3) is rewritten as follows:

\[
r(x) = \langle r \rangle_a
\]

for any \( x \in \mathbb{R} \). It means that equality in (1.1) holds if and only if \( r \) is a constant.

Theorem 2.6 implies that the problem \( (P)_d \) has the solution. Furthermore, we see that the solution does not depend on size of the diffusion coefficient. It depends only on the shape of the diffusion coefficient.

**Corollary 2.8.** For any \( d \in C^1_{\text{per}}(\mathbb{R}) \) with inf \( d > 0 \), the minimizing problem \( (P)_d \) has the solution \( r_d(x) \) and it is defined by

\[
r_d(x) = \alpha \left( 2 - \frac{\langle d \rangle_h}{d(x)} \right)
\]

for any \( x \in \mathbb{R} \). Moreover, for any \( k > 0 \), \( r_d \) is the unique solution for \( (P)_{kd} \).
The right-hand side of (2.5) can be defined even if \( d \in C^1_{\text{per}}(\mathbb{R}) \) with \( \inf d > 0 \) and \( r \in C^1_{\text{per}}(\mathbb{R}) \) with \( \langle r \rangle_a > 0 \). For any \( d \in C^1_{\text{per}}(\mathbb{R}) \) and \( r \in C^1_{\text{per}}(\mathbb{R}) \), our results still hold if we formally define the spreading speed by the right-hand side of (2.4).

Finally, we state that the principal eigenfunction is not constant in the periodic case.

**Theorem 2.10.** Assume that (1.3) holds and \( \lambda_0 = \sqrt{\langle r \rangle_a / \langle d \rangle_h} \). Set \( \psi_c \equiv C \) for some constant \( C \neq 0 \). Then the following statements are equivalent:

1. \( \psi_c \) is a principal eigenfunction of the operator \(-L_{\lambda_0,d,r} \).
2. The diffusion coefficient \( d(x) \) is a constant function.

## 3 Examples and proof

### 3.1 Specific examples of the main results

In this subsection, we give some examples of our main result. We notice that the exact value of the minimal traveling wave speed is calculable if the condition (1.3) establish. One example is as follows. This is the first calculable example of the minimal speed for spatially periodic Fisher-KPP equations.

**Example 1.** Define \( r(x) \) and \( d(x) \) by

\[
\begin{align*}
  r(x) &= 1 + \frac{1}{2} \sin x, \\
  d(x) &= \frac{1}{1 - \frac{1}{2} \sin x}
\end{align*}
\]

for any \( x \in \mathbb{R} \). Then, \( r \) and \( d \) satisfy the condition (1.3). By Theorem 2.6, we can calculate the minimal traveling wave speed \( c^*_d(r) \) for the equation \((E)\) as follows:

\[
c^*_d(r) = 2\sqrt{\langle d \rangle_h \langle r \rangle_a} = 2.
\]

Then we obtain the exact value of the minimal speed.

**Remark 3.1.** We notice that \( r_d(x) \) is large in the area where \( d(x) \) is large and \( r_d(x) \) is small in the area where \( d(x) \) is small. See (1.4).

If \( r(x) \) is fixed, equality in (1.1) may not hold by varying \( d(x) \).

**Example 2.** Define \( r(x) \) by

\[
r(x) = 1 + 2 \sin x
\]

for any \( x \in \mathbb{R} \). Then

\[
d_r^{-1}(x) = 2 - r(x) = 1 - 2 \sin x
\]

and \( r \) satisfy the condition (1.3). However, \( d_r(x) < 0 \) on \((\pi/3, 2\pi/3)\). We conclude that

\[
c^*_d(r) > 2\sqrt{\langle d \rangle_h \langle r \rangle_a}
\]

for any \( d \in C^1_{\text{per}}(\mathbb{R}) \) with \( \inf d > 0 \).
3.2 Proof of the lower estimate

For the readers’ convenience, we give the proof of the inequality (1.1) in this subsection. This inequality was first proved by Nadin [13].

Proof of (1.1). By Nadin’s formula, we have

\[ k(\lambda, d, r) = \min_{\varphi \in E_L} \left\{ \int_0^L d|\varphi'|^2 \, dx - \int_0^L r\varphi^2 \, dx - \frac{\lambda^2 L^2}{\int_0^L \frac{1}{d\varphi^2} \, dx} \right\} = \min_{\varphi \in E_L} I(\varphi; \lambda, d, r). \]

Taking \( \varphi_0(x) = 1/\sqrt{L} \) as a test function, we obtain

\[ k(\lambda, d, r) \leq \int_0^L d|\varphi'|^2 \, dx - \int_0^L r\varphi^2 \, dx - \frac{\lambda^2 L^2}{\int_0^L \frac{1}{d\varphi_0^2} \, dx} \]

\[ = -\langle r \rangle_a - \lambda^2 \langle d \rangle_h. \]

Therefore

\[ c_d^*(r) = \min_{\lambda > 0} - \frac{k(\lambda, d, r)}{\lambda} \geq \min_{\lambda > 0} \left( \frac{\langle r \rangle_a}{\lambda} + \lambda \langle d \rangle_h \right) \]

\[ = 2\sqrt{\langle d \rangle_h \langle r \rangle_a}. \]

The inequality (1.1) is proved. \( \square \)

3.3 Proof of the main results

Proof of Theorem 2.6. We first prove that (1) \( \Rightarrow \) (2). Assume that the equality holds in (1.1). Define \( \lambda^* > 0 \) and \( \varphi^*_\lambda \in E_L \) by

\[ c_d^*(r) = -\frac{k(\lambda^*, d, r)}{\lambda^*}, \quad k(\lambda, d, r) = I(\varphi^*_\lambda; \lambda, d, r) \]

for any \( \lambda > 0 \). As in the proof of (1.1), for any \( \lambda > 0 \), we have

\[ k(\lambda, d, r) = I(\varphi^*; \lambda, d, r) \leq I(\varphi_0; \lambda, d, r) = -\langle r \rangle_a - \lambda^2 \langle d \rangle_h. \]

Therefore

\[ c_d^*(r) = -\frac{k(\lambda^*, d, r)}{\lambda^*} \geq \frac{\langle r \rangle_a}{\lambda^*} + \lambda^* \langle d \rangle_h \geq 2\sqrt{\langle d \rangle_h \langle r \rangle_a}. \]

From the assumption \( c_d^*(r) = 2\sqrt{\langle d \rangle_h \langle r \rangle_a} \), it follows that

\[ \frac{\langle r \rangle_a}{\lambda^*} + \lambda^* \langle d \rangle_h = 2\sqrt{\langle d \rangle_h \langle r \rangle_a}, \quad k(\lambda, d, r) = I(\varphi_0; \lambda^*, d, r). \quad (3.1) \]
This implies that \( \lambda^* = \lambda_0 = \sqrt{(r)/\langle d \rangle_h} \) and the constant function \( \varphi_0 \) minimizes the functional \( \varphi \mapsto I(\varphi; \lambda^*, d, r) \) on \( E_L \). The constant function \( \varphi_0 \) also minimizes the following functional

\[
I(\varphi) = \frac{1}{\int_0^L \varphi^2 \, dx} \left\{ \int_0^L d|\varphi'|^2 \, dx - \int_0^L r \varphi^2 \, dx - \frac{\lambda_0^2 L^2}{\int_0^L \frac{1}{d \varphi^2} \, dx} \right\}
\]

for \( \varphi \in C^1_{\text{per}} \setminus \{0\} \). We next calculate the Euler-Lagrange equation for the functional \( I \) on \( C^1_{\text{per}} \setminus \{0\} \). Take a minimizing function \( \varphi \in C^1_{\text{per}} \setminus \{0\} \) with \( \|\varphi\|_{L^2} = 1 \). For any \( L \)-periodic function \( \psi \in C^1(\mathbb{R}) \) and sufficiently small \( \varepsilon > 0 \), we see that \( \varphi + \varepsilon \psi \in C^1_{\text{per}} \setminus \{0\} \) and

\[
I(\varphi; \lambda_0, d, r) = I(\varphi) \leq I(\varphi + \varepsilon \psi).
\]

Since

\[
\frac{d}{d \varepsilon} I(\varphi + \varepsilon \psi) \bigg|_{\varepsilon = 0} = 0
\]

and \( \|\varphi\|_{L^2} = 1 \), we obtain

\[
\int_0^L d \varphi' \psi' \, dx - \int_0^L \varphi \psi \, dx - \frac{\lambda_0^2 L^2}{\left( \int_0^L \frac{1}{d \varphi^2} \, dx \right)^2} \int_0^L \frac{\psi}{\varphi^3} \, dx = I(\varphi) \int_0^L \varphi \psi \, dx.
\]

It follows that the minimizing function \( \varphi \) satisfies the following Euler-Lagrange equation in the weak sense:

\[
-(d \varphi')' - r \varphi - \frac{\lambda_0^2 L^2}{\left( \int_0^L \frac{1}{d \varphi^2} \, dx \right)^2} \frac{1}{d \varphi^3} = I(\varphi) \varphi.
\]

Substituting \( \varphi_0 \equiv 1/\sqrt{L} \) into the Euler-Lagrange equation we have

\[
-\frac{1}{\sqrt{L}} \frac{r}{d} - \frac{\lambda_0^2 L^{2/3}}{(\int_0^L \frac{1}{d \varphi^2} \, dx)^2} = \frac{1}{\sqrt{L}} I(\varphi_0).
\]

(3.2)

Since

\[
\langle d \rangle_h = \left( \frac{1}{L} \int_0^L \frac{1}{d(x)} \, dx \right)^{-1}, \quad \lambda_0 = \sqrt{\langle r \rangle_a / \langle d \rangle_h},
\]

(3.3)

we obtain

\[
I(\varphi_0) = -(r) - \lambda_0^2 \langle d \rangle_h = -2 \langle r \rangle_a.
\]

(3.4)

From (3.3) and (3.4), we can rewrite (3.2) as

\[
\frac{r}{\langle r \rangle_a} + \frac{\langle d \rangle_h}{d} = 2.
\]
which is the desired conclusion.

We next prove that (1) ⇐ (2). Since we know $c_a^*(r_d) \geq 2\sqrt{\langle d \rangle_h \langle r_d \rangle_a}$, it is sufficient to prove that the converse inequality. We have

$$c_a^*(r_d) = \min_{\lambda > 0} \left( -\frac{k_\lambda(d, r_d)}{\lambda} \right) \leq -\frac{k_{\lambda_0}(d, r_d)}{\lambda_0},$$

where $\lambda_0 = \sqrt{\langle r_d \rangle_a / \langle d \rangle_h}$. Nadin’s formula and (1.4) show that

$$k_{\lambda_0}(d, r_d) = \min_{\varphi \in E_L} \left\{ \int_0^L d|\varphi'|^2 \, dx - \int_0^L r_d \varphi^2 \, dx - \frac{\lambda_0^2 L^2}{\int_0^L \frac{1}{d} \varphi^2 \, dx} \right\} \geq -\frac{2\langle r_d \rangle_a}{\lambda_0} + \min_{\varphi \in E_L} \left\{ \langle d \rangle_h \langle r_d \rangle_a \int_0^L \frac{1}{d} \varphi^2 \, dx - \frac{\lambda_0^2 L^2}{\int_0^L \frac{1}{d} \varphi^2 \, dx} \right\}.$$  

By the Cauchy-Schwarz inequality, we have

$$\left( \int_0^L \frac{1}{d} \varphi^2 \, dx \right) \left( \int_0^L \frac{1}{d} \varphi^2 \, dx \right) \geq \left( \int_0^L \frac{1}{d} \varphi^2 \, dx \right)^2 = L^2 \langle d \rangle_h^{-2}.$$  

We thus get

$$\langle d \rangle_h \langle r_d \rangle_a \int_0^L \frac{1}{d} \varphi^2 \, dx - \frac{\lambda_0^2 L^2}{\int_0^L \frac{1}{d} \varphi^2 \, dx} \geq \frac{L^2 \langle d \rangle_h^{-1} \langle r_d \rangle_a - \lambda_0^2}{\int_0^L \frac{1}{d} \varphi^2 \, dx} = 0$$

for any $\varphi \in E_L$. This gives

$$k_{\lambda_0}(d, r_d) \geq -2\langle r_d \rangle_a.$$  

Combining (3.6) with (3.5), we obtain

$$c_a^*(r_d) \leq 2\langle r_d \rangle_a = 2\langle r_d \rangle_a \sqrt{\langle d \rangle_h \langle r_d \rangle_a} = 2\sqrt{\langle d \rangle_h \langle r_d \rangle_a},$$

which completes the proof of (1) ⇐ (2).

It remains to prove that (1) ⇔ (3). We first assume the statement (1). As in the proof of Theorem 2.6, we obtain (3) and $\lambda_0$ only attains the minimum. See (3.1). We next assume the statement (3). Since the constant function $\varphi_0$ is the minimizer for $\varphi \mapsto I(\varphi; \lambda_0, d, r)$, we can substituting the constant function into the Euler-Lagrange equation. As in the proof of Theorem 2.6, we obtain

$$\frac{r}{\langle r \rangle_a} + \frac{\langle d \rangle_h}{d} = 2,$$
which implies that (1) establishes. Finally, we prove that \( \varphi_0 \) only attains 
\[
k_{\lambda_0}(d, r) = \min_{\varphi \in E} I(\varphi; \lambda_0, d, r).
\]
Assume that (1) holds. Since (2) and (3) hold, we have \( r = r_d \) and \( k_{\lambda_0}(d, r) = I(1/\sqrt{L}; \lambda_0, d, r) = -2\langle r \rangle_a \). Take any \( \varphi \in E \) minimizing \( \varphi \mapsto I(\varphi; \lambda_0, d, r) \).

As in the proof of Theorem 2.6, we obtain
\[
-2\langle r \rangle_a = k_{\lambda_0}(d, r) = \int_0^L d|\varphi'|^2 \, dx - \int_0^L r\varphi^2 \, dx - \frac{\lambda_0^2 L^2}{\int_0^L \frac{1}{d\varphi^2} \, dx}
\geq -\int_0^L r\varphi^2 \, dx - \frac{\lambda_0^2 L^2}{\int_0^L \frac{1}{d\varphi^2} \, dx}
\geq -2\langle r \rangle_a.
\]

It implies that
\[
\int_0^L d|\varphi'|^2 \, dx = 0.
\]
This clearly forces \( \varphi \equiv 1/\sqrt{L} \), which is our claim.

**Proof of Theorem 2.10.** Set \( d \in C_{\text{per}}(\mathbb{R}) \) with \( \inf d > 0 \) and \( k > 0 \). It suffices to show that \( r_{kd}(x) = r_d(x) \) for any \( x \in \mathbb{R} \). Since \( \langle kd \rangle_h = k \langle d \rangle_h \), we obtain
\[
r_{kd}(x) = \alpha \left( 2 - \frac{\langle kd \rangle_h}{kd(x)} \right)
= \alpha \left( 2 - \frac{k \langle d \rangle_h}{kd(x)} \right)
= \alpha \left( 2 - \frac{\langle d \rangle_h}{d(x)} \right)
= r_d(x)
\]
for any \( x \in \mathbb{R} \). This completes the proof.

**Proof of Theorem 2.10.** We first prove that (2) \( \Rightarrow \) (1). In the case where \( d(x) \) is a constant, by Theorem 2.6, we see that \( r(x) \) is a constant. It is known that \( \psi_c \) is the principal eigenfunction if \( d(x) \) and \( r(x) \) are constants.

We next prove that (1) \( \Rightarrow \) (2). We assume that the constant function \( \psi_c \) is the principal eigenvalue of the operator \(-L_{\lambda_0, d, r}\). By the assumption that \( c^*_d(r) = 2\sqrt{\langle d \rangle_h \langle r \rangle_a} \), we have \( k_{\lambda_0}(d, r) = -2\langle r \rangle_a \). The constant function \( \psi_c \) satisfies
\[
-(d(x)\psi'_c(x))' - 2\lambda_0 d(x)\psi_c'(x) - (\lambda_0^2 d(x) + \lambda_0 d'(x) + r(x))\psi_c(x) = k_{\lambda_0}(d, r)\psi_c(x).
\]
Thus we have
\[ \lambda_0^2 d(x) + \lambda_0 d'(x) + r(x) = 2 \langle r \rangle_a. \]
Dividing this equation by \( L \) and integrating it from 0 to \( L \), we get
\[ \lambda_0^2 \langle d \rangle_a + \langle r \rangle_a = 2 \langle r \rangle_a. \]
Substituting \( \lambda_0 = \sqrt{\langle r \rangle_a / \langle d \rangle_h} \) into this equation, we obtain
\[ \langle d \rangle_a = \langle d \rangle_h. \quad (3.7) \]
In general, by the Cauchy-Schwarz inequality, we have
\[ \left( \int_0^L d(x) \, dx \right) \left( \int_0^L \frac{1}{d(x)} \, dx \right) \geq \left( \int_0^L 1 \, dx \right)^2 = L^2, \quad (3.8) \]
and this equality holds if and only if \( d(x) \) is a constant function. From (3.8), we see that
\[ \langle d \rangle_a \geq \langle d \rangle_h. \]
The equation (3.7) means that the equality in (3.8) holds, which gives \( d(x) \) is a constant function.

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