Strong law of large number of a class of super-diffusions

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Abstract

In this paper we prove that, under certain conditions, a strong law of large numbers holds for a class of super-diffusions $X$ corresponding to the evolution equation

$$\partial_t u_t = L u_t + \beta u_t - \psi(u_t)$$

on a bounded domain $D$ in $\mathbb{R}^d$, where $L$ is the generator of the underlying diffusion and the branching mechanism $\psi(x, \lambda) = \frac{1}{2} \alpha(x) \lambda^2 \int_0^\infty (e^{-\lambda r} - 1 + \lambda r)n(x, dr)$ satisfies $\sup_{x \in D} \int_0^\infty (r \wedge r^2)n(x, dr) < \infty$.

Keywords Super-diffusion, martingale, point process, principal eigenvalue, strong law of large numbers

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1 Introduction

1.1 Motivation

Recently many people (see [3 4 6 7 8 9 20] and the references therein) have studied limit theorems for branching Markov processes or super-processes using the principal eigenvalue and ground state of the linear part of the characteristic equations. All the papers above, except [8], assumed that the branching mechanisms satisfy a second moment condition. In [8], a $(1 + \theta)$-moment condition, $\theta > 0$, on the branching mechanism is assumed instead.

In [1], Asmussen and Hering established a Kesten-Stigum $L \log L$ type theorem for a class branching diffusion processes under a condition which is later called a positive regular property in [2]. In [16 17] we established Kesten-Stigum $L \log L$ type theorems for super-diffusions and branching Hunt processes respectively.

This paper is a natural continuation of [16 17]. The main purpose of this paper is to establish a strong law of large numbers for a class of super-diffusions. The main tool of this paper is the stochastic integral representation of super-diffusions.

Throughout this paper, we will use the following notations. For any positive integer $k$, $C^k_b(\mathbb{R}^d)$ denotes the family of bounded functions on $\mathbb{R}^d$ whose partial derivatives of order up to $k$ are bounded and continuous, $C^k_0(\mathbb{R}^d)$ denotes the family of functions of compact support on $\mathbb{R}^d$ whose

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partial derivatives of order up to \( k \) are continuous. For any open set \( D \subset \mathbb{R}^d \), the meanings of \( C^k_b(D) \) and \( C^0_b(D) \) are similar. We denote by \( \mathcal{M}_F(D) \) the space of finite measures on \( D \) equipped with the topology of weak convergence. We will use \( \mathcal{M}_F(D)^0 \) to denote the subspace of nontrivial measures in \( \mathcal{M}_F(D) \). The integral of a function \( \varphi \) with respect to a measure \( \mu \) will often be denoted as \( \langle \varphi, \mu \rangle \).

For convenience we use the following convention throughout this paper: For any probability measure \( P \), we also use \( P \) to denote the expectation with respect to \( P \).

1.2 Model

Suppose that \( a_{ij} \in C^1_b(\mathbb{R}^d), \ i, j = 1, \cdots, d \), and that the matrix \((a_{ij})\) is symmetric and satisfies

\[
0 < a|v|^2 \leq \sum_{i,j} a_{ij}v_i v_j, \quad \text{for all } x \in \mathbb{R}^d \text{ and } v \in \mathbb{R}^d
\]

for some positive constant \( a \). We assume that \( b_i, i = 1, \cdots, d \), are bounded Borel functions on \( \mathbb{R}^d \). We will use \((\xi, \Pi_x, x \in \mathbb{R}^d)\) to denote a diffusion process on \( \mathbb{R}^d \) corresponding to the operator

\[
L = \frac{1}{2} \nabla \cdot a \nabla + b \cdot \nabla.
\]

In this paper we will always assume that \( D \) is a bounded domain in \( \mathbb{R}^d \). We will use \((\xi^D, \Pi_x, x \in D)\) to denote the process obtained by killing \( \xi \) upon exiting from \( D \), that is,

\[
\xi^D_t = \begin{cases} 
\xi_t & \text{if } t < \tau, \\
\partial & \text{if } t \geq \tau,
\end{cases}
\]

where \( \tau = \inf\{t > 0; \xi_t \notin D\} \) is the first exit time of \( D \) and \( \partial \) is a cemetery point. Any function \( f \) on \( D \) is automatically extended to \( D \cup \{\partial\} \) by setting \( f(\partial) = 0 \).

We will always assume that \( \beta \) is a bounded Borel function on \( \mathbb{R}^d \). We will use \( \{P^D_t\}_{t \geq 0} \) to denote the following Feynman-Kac semigroup

\[
P^D_t f(x) = \Pi_x \left( \exp \left( \int_0^t \beta(\xi^D_s)ds \right) f(\xi^D_t) \right), \quad x \in D.
\]

It is well known that the semigroup \( \{P^D_t\}_{t \geq 0} \) is strongly continuous in \( L^2(D) \) and, for any \( t > 0 \), \( P^D_t \) has a bounded, continuous and strictly positive density \( p^D(t, x, y) \).

Let \( \{\widehat{P}^D_t\}_{t \geq 0} \) be the dual semigroup of \( \{P^D_t\}_{t \geq 0} \) defined by

\[
\widehat{P}^D_t f(x) = \int_D p^D(t, y, x) f(y)dy, \quad x \in D.
\]

It is well known that \( \{\widehat{P}^D_t\}_{t \geq 0} \) is also strongly continuous in \( L^2(D) \).

Let \( A \) and \( \widehat{A} \) be the generators of the semigroups \( \{P^D_t\}_{t \geq 0} \) and \( \{\widehat{P}^D_t\}_{t \geq 0} \) in \( L^2(D) \) respectively. Let \( \sigma(A) \) (\( \sigma(\widehat{A}) \) resp.) denote the spectrum of \( A \) (\( \widehat{A} \), resp.). It follows from Jentzsch’s theorem ([19] Theorem V.6.6, p. 337) and the strong continuity of \( \{P^D_t\}_{t \geq 0} \) and \( \{\widehat{P}^D_t\}_{t \geq 0} \) that the common
value $\lambda_1 := \sup \text{Re}(\sigma(\mathbf{A})) = \sup \text{Re}(\sigma(\hat{\mathbf{A}}))$ is an eigenvalue of multiplicity 1 for both \(\mathbf{A}\) and \(\hat{\mathbf{A}}\), and that an eigenfunction \(\phi\) of \(\mathbf{A}\) associated with \(\lambda_1\) can be chosen to be strictly positive a.e. on \(D\) and an eigenfunction \(\tilde{\phi}\) of \(\hat{\mathbf{A}}\) associated with \(\lambda_1\) can be chosen to be strictly positive a.e. on \(D\). By [13, Proposition 2.3] we know that \(\phi\) and \(\tilde{\phi}\) are bounded and continuous on \(D\), and they are in fact strictly positive everywhere on \(D\). We choose \(\phi\) and \(\tilde{\phi}\) so that \(\int_D \phi(x)\tilde{\phi}(x)dx = 1\).

Throughout this paper we assume the following

**Assumption 1** The semigroups \(\{P^D_t\}_{t \geq 0}\) and \(\{\hat{P}^D_t\}_{t \geq 0}\) are intrinsically ultracontractive, that is, for any \(t > 0\), there exists a constant \(c_t > 0\) such that

\[
p^D(t, x, y) \leq c_t \phi(x)\tilde{\phi}(y), \quad \text{for all } (x, y) \in D \times D.
\]

Assumption 1 is a very weak regularity assumption on \(D\). It follows from [13, 14] that Assumption 1 is satisfied when \(D\) is a bounded Lipschitz domain. For other, more general, examples of domain \(D\) for which Assumption 1 is satisfied, we refer our readers to [14] and the references therein.

Define

\[
p^{\phi}(t, x, y) = e^{-\lambda_1 t} \frac{\phi(x)}{\tilde{\phi}(x)} P^D(t, x, y) \phi(y).
\]

Then it follows from [13, Theorem 2.7] that if Assumption 1 holds, then for any \(\sigma > 0\) there are positive constants \(C(\sigma)\) and \(\nu\) such that

\[
\left| p^{\phi}(t, x, y) - \phi(y)\tilde{\phi}(y) \right| = \left| \frac{e^{-\lambda_1 t} P^D(t, x, y)\phi(y)}{\phi(x)} - \phi(y)\tilde{\phi}(y) \right| \leq C(\sigma)e^{-\nu t} \phi(y)\tilde{\phi}(y), \quad x, y \in D, t > \sigma.
\]

By the definition of \(\phi\) and \(\tilde{\phi}\), it is easy to check that, for any \(t > 0\), \(p^{\phi}(t, \cdot, \cdot)\) is a probability density and that \(\phi\tilde{\phi}\) is its unique invariant probability density. [12] shows that \(p^{\phi}(t, \cdot, x)\) converges to \(\phi(x)\tilde{\phi}(x)\) uniformly with exponential rate. Denote by \(P^{\phi}_t\) the semigroup with density \(p^{\phi}(t, \cdot, \cdot)\) and \(\Pi^\phi_b\) the probability generated by \((P^\phi_t)_{t \geq 0}\) with initial distribution \(h(x)dx\) on \(D\). Then \((\xi^D, \Pi^\phi)\) is a diffusion with initial distribution \(\phi(x)\tilde{\phi}(x)dx\).

The super-diffusion \((X^\phi, \mathbb{P}_\mu), \mu \in \mathcal{M}(D)^\prime\), we are going to study is a \((\xi^D, \psi(\lambda) - \beta\lambda)\)-super-process, which is a measure-valued Markov process with underlying spatial motion \(\xi^D\), branching rate \(dt\) and branching mechanism \(\psi(\lambda) - \beta\lambda\), where

\[
\psi(x, \lambda) = \frac{1}{2} \alpha(x)\lambda^2 + \int_0^\infty \left( e^{-r\lambda} - 1 + \lambda r \right) n(x, dr), \quad \lambda > 0,
\]

for some nonnegative bounded measurable function \(\alpha\) on \(D\) and for some \(\sigma\)-finite kernel \(n\) from \((D, \mathcal{B}(D))\) to \((\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))\), that is, \(n(x, dr)\) is a \(\sigma\)-finite measure on \(\mathbb{R}_+\) for each fixed \(x\), and \(n(\cdot, B)\) is a measurable function for each Borel set \(B \subset \mathbb{R}_+\). The measure \(\mu\) here is the initial value of \(X\).

In this paper we will always assume that

\[
\sup_{x \in D} \int_0^\infty (r \wedge r^2)n(x, dr) < \infty.
\]
Note that this assumption implies, for any fixed $\lambda > 0$, $\psi(\cdot, \lambda)$ is bounded on $D$. Define a new kernel $n^\phi(x, dr)$ from $(D, \mathcal{B}(D))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that for any nonnegative measurable function $f$ on $\mathbb{R}_+$,

$$\int_0^\infty f(r)n^\phi(x, dr) = \int_0^\infty f(r\phi(x))n(x, dr), \quad x \in D.$$  \hfill (1.4)

Then, by \ref{1.3} and the boundedness of $\phi$, $n^\phi$ satisfies

$$\sup_{x \in D} \int_0^\infty (r \wedge r^2)n^\phi(x, dr) < \infty.$$  \hfill (1.5)

### 1.3 Stochastic Integral Representation and Main Result

Let $(\Omega, \mathcal{F}, \mathbb{P}_\mu, \mu \in \mathcal{M}_F(D)^0)$ be the underlying probability space equipped with the filtration $(\mathcal{F}_t)$, which is generated by $X$ and is completed as usual with the $\mathcal{F}_\infty-$measurable and $\mathbb{P}_\mu-$negligible sets for every $\mu \in \mathcal{M}_F(D)^0$. It is known (cf. \cite[Section 6.1]{5}) that the super-diffusion $X$ is a solution to the following martingale problem: for any $\varphi \in C_2^2(D)$ and $h \in C_0^2(\mathbb{R})$,

$$h(\langle \varphi, X_t \rangle) - h(\langle \varphi, \mu \rangle) - \frac{1}{2} \int_0^t h''(\langle \varphi, X_s \rangle)\langle \alpha \varphi^2, X_s \rangle ds - \frac{1}{2} \int_0^t (\langle \varphi, X_s \rangle + r\varphi(x) - h(\langle \varphi, X_s \rangle) - h'(\langle \varphi, X_s \rangle)r\varphi(x))n(x, dr)X_s(dx)ds$$

is a martingale. Let $J$ denote the set of all jump times of $X$ and $\delta$ denote the Dirac measure. It is easy to see from the martingale problem \ref{1.6} that the only possible jumps of $X$ are point measures $r\delta_x(\cdot)$ with $r$ being a positive real number and $x \in D$. \ref{1.6} implies that the compensator of the random measure

$$N := \sum_{s \in J} \delta(s, \Delta X_s)$$

is a random measure $\widehat{N}$ on $\mathbb{R}_+ \times \mathcal{M}_F(D)^0$ such that for any nonnegative predictable function $F$ on $\mathbb{R}_+ \times \Omega \times \mathcal{M}_F(D)^0$,

$$\int \int F(s, \omega, \nu)\widehat{N}(ds, d\nu) = \mathbb{E}_\mu \left[ \int_0^\infty ds \int_D X_s(dx) \int_0^\infty F(s, \omega, r\delta_x)n(x, dr) \right] = \mathbb{E}_\mu \left[ \int_0^\infty ds \int_D X_s(dx) \int_0^\infty F(s, \omega, r\delta_x)n(x, dr) \right].$$  \hfill (1.7)

where $n(x, dr)$ is the kernel of the branching mechanism $\psi$. Therefore we have

$$\mathbb{P}_\mu \left[ \sum_{s \in J} F(s, \omega, \Delta X_s) \right] = \mathbb{E}_\mu \left[ \int_0^\infty ds \int_D X_s(dx) \int_0^\infty F(s, \omega, r\delta_x)n(x, dr) \right].$$  \hfill (1.8)

See \cite[p. 111]{5}. Let $F$ be a predictable function on $\mathbb{R}_+ \times \Omega \times \mathcal{M}_F(D)^0$ satisfying

$$\mathbb{P}_\mu \left[ \left( \sum_{s \in [0,t], s \in J} F(s, \Delta X_s)^2 \right)^{1/2} \right] < \infty, \quad \text{for all } \mu \in \mathcal{M}_F(D)^0.$$
Then the stochastic integral of $F$ with respect to the compensated random measure $N - \hat{N}$

$$\int_0^t \int_{\mathcal{M}_F(D)^0} F(s, \nu)(N - \hat{N})(ds, d\nu),$$

can be defined (cf. [15] and the reference therein) as the unique purely discontinuous martingale
(vanishing at time 0) whose jumps are indistinguishable from $1_J(s)F(s, \Delta X_s)$.

Suppose that $\varphi$ is a measurable function on $\mathbb{R}_+ \times D$. Define

$$F_\varphi(s, \nu) := \int_D \varphi(s, x)\nu(dx), \quad \nu \in \mathcal{M}_F(D)$$

whenever the integral above makes sense. We write

$$S_t^J(\varphi) = \int_0^t \int_D \varphi(s, x)S_t^J(ds, dx) := \int_0^t \int_{\mathcal{M}_F(D)^0} F_\varphi(s, \nu)(N - \hat{N})(ds, d\nu),$$

whenever the right hand of (1.10) makes sense. If $\varphi$ is bounded on $\mathbb{R}_+ \times D$, then $S_t^J(\varphi)$ is well defined. Indeed, we only need to check that

$$\mathbb{P}_\mu \left[ \left( \sum_{s \in [0, t], s \in J} F_\varphi(s, \Delta X_s)^2 \right)^{1/2} \right] < \infty, \quad \text{for all } \mu \in \mathcal{M}_F(D)^0. \quad (1.11)$$

Note that, for any $\mu \in \mathcal{M}_F(D)^0$,

$$\mathbb{P}_\mu \left[ \left( \sum_{s \in [0, t], s \in J} F_\varphi(s, \Delta X_s)^2 \right)^{1/2} \right] = \mathbb{P}_\mu \left[ \left( \sum_{s \in [0, t], s \in J} \left( \int \varphi(s, x)(\Delta X_s)(dx) \right)^2 \right)^{1/2} \right] \leq \|\varphi\|_\infty \mathbb{P}_\mu \left[ \left( \sum_{s \in [0, t], s \in J} (1, \Delta X_s)^2 I\{1, \Delta X_s \leq 1\} \right)^{1/2} \right] \leq \|\varphi\|_\infty \mathbb{P}_\mu \left[ \left( \sum_{s \in [0, t], s \in J} (1, \Delta X_s)^2 I\{1, \Delta X_s \leq 1\} \right)^{1/2} \right] + \|\varphi\|_\infty \mathbb{P}_\mu \left[ \left( \sum_{s \in [0, t], s \in J} (1, \Delta X_s)^2 I\{1, \Delta X_s > 1\} \right)^{1/2} \right].$$

Here and throughout this paper, for any set $A$, $I_A$ stands for the indicator function of $A$. Using
the first two displays on [15] p. 203, we get (1.11). Thus for any bounded function $\varphi$ on $\mathbb{R}_+ \times D$.
($S_t^J(\varphi))_{t \geq 0}$ is a martingale.

For any $\varphi \in C^2_0(D)$ and $\mu \in \mathcal{M}_F(D)$,

$$\langle \varphi, X_t \rangle = \langle \varphi, \mu \rangle + S_t^J(\varphi) + S_t^C(\varphi) + \int_0^t \langle A\varphi, X_s \rangle ds,$$  \quad (1.12)
where $S^C_t(\varphi)$ is a continuous local martingale with quadratic variation
\[
\langle S^C(\varphi) \rangle_t = \int_0^t \langle \alpha \varphi^2, X_s \rangle ds. \tag{1.13}
\]
In fact, according to [10] [11], the above is still valid when $A$ is replaced by $L + \beta$, where $L$ is the weak generator of $\xi^D$ in the sense of [10 Section 4]. Using this, [11 Corollary 2.18] and applying a limit argument, one can show that for any bounded function $g$ on $D$,
\[
\langle g, X_t \rangle = \langle P^D_t g, \mu \rangle + \int_0^t \int_D P^D_{t-s} g(x) S^I(ds, dx) + \int_0^t \int_D P^D_{t-s} g(x) S^C(ds, dx). \tag{1.14}
\]
In particular, taking $g = \phi$ in (1.14), where $\phi$ is the positive eigenfunction of $A$ defined in Section 1.1, we get that
\[
e^{-\lambda_1 t} \langle \phi, X_t \rangle = \langle \phi, \mu \rangle + \int_0^t e^{-\lambda_1 s} \int_D \phi(x) S^I(ds, dx) + \int_0^t e^{-\lambda_1 s} \int_D \phi(x) S^C(ds, dx). \tag{1.15}
\]
Set $M_t(\phi) := e^{-\lambda_1 t} \langle \phi, X_t \rangle$. Then $M_t(\phi), t \geq 0$, is a nonnegative martingale. Denote by $M_{\infty}(\phi)$ the almost sure limit of $M_t(\phi)$ as $t \to \infty$. In [16], we studied the relationship between the degeneracy property of $M_{\infty}(\phi)$ and the function $l$:
\[
l(y) := \int_1^\infty r \ln(rn^p(y, dr). \tag{1.16}
\]
**Theorem 1.1** [16 Theorem 1.1] Suppose that Assumption [7] holds, $\lambda_1 > 0$ and that $X$ is a $(\xi^D, \psi(\lambda) - \beta \lambda)$-super-diffusion. Then the following assertions hold:

1. If $\int_D l(y) \tilde{\phi}(y) dy < \infty$, then $M_{\infty}(\phi)$ is non-degenerate under $P_\mu$ for any $\mu \in M_F(D)^0$, and $M_{\infty}(\phi)$ is also the $L^1(P_\mu)$ limit of $M_t(\phi)$.

2. If $\int_D l(y) \tilde{\phi}(y) dy = \infty$, then $M_{\infty}(\phi) = 0$, $P_\mu$-a.s. for any $\mu \in M_F(D)^0$.

**Remark 1.2** In [16 Theorem 1.1], we only stated that in case (1) under the extra assumption $\alpha \equiv 0$, $M_{\infty}(\phi)$ is non-degenerate under $P_\mu$ for any $\mu \in M_F(D)^0$. But actually in this case we have $P_\mu M_{\infty}(\phi) = P_\mu M_0(\phi)$ (see [16 Lemma 3.4]), and therefore $M_t(\phi)$ converges to $M_{\infty}(\phi)$ in $L^1(P_\mu)$.

For general $\alpha \geq 0$, by the $L^2$ maximum inequality, and using the fact that $\alpha$ and $\phi$ are bounded in $D$, we have
\[
\mathbb{P}_\mu \left[ \sup_{t \geq 0} \left( \int_0^t e^{-\lambda_1 s} \int_D \phi(x) S^C(ds, dx) \right)^2 \right] \\
\leq 4 \sup_{t \geq 0} \mathbb{P}_\mu \left( \int_0^t e^{-\lambda_1 s} \int_D \phi(x) S^C(ds, dx) \right)^2 \\
= 4 \int_0^\infty e^{-2\lambda_1 s} ds \int_D \alpha(x) \phi^2(x) X_s(dx) \\
= 4 \int_0^\infty e^{-\lambda_1 s} ds \int_D \phi(y) \mu(dy) \int_D p(x, y, \alpha(x) \phi(x) dx < \infty. \tag{1.17}
\]
Thus the martingale \( \left( \int_0^t e^{-\lambda s} \int_D \phi(x) S^C(ds, dx) \right)_{t \geq 0} \) converges almost surely and in \( L^1(\mathbb{P}_\mu) \). Denote the limit by \( \int_0^\infty e^{-\lambda s} \int_D \phi(x) S^C(ds, dx) \). Furthermore, we obtain that when \( \lambda_1 > 0 \) and \( \int_D l(x) \tilde{\phi}(x) dx < \infty \), the martingale \( \int_0^t e^{-\lambda s} \int_D \phi(x) S^J(ds, dx) \) converges almost surely and in \( L^1(\mathbb{P}_\mu) \) as well. Denote the limit by \( \int_0^\infty e^{-\lambda s} \int_D \phi(x) S^J(ds, dx) \). Thus it follows from (1.15) that \( M_t(\phi) \) converges to a non-degenerate \( M_\infty(\phi) \) \( \mathbb{P}_\mu \)-almost surely and in \( L^1(\mathbb{P}_\mu) \) for every \( \mu \in \mathcal{M}_F(D^0) \).

The main goal of this paper is to establish the following almost sure convergence result.

**Theorem 1.3** Suppose that Assumption 1 holds, \( \lambda_1 > 0 \) and that \( X \) is a \((\xi^D, \psi(\lambda) - \beta \lambda)\)-superdiffusion. Then there exists \( \Omega_0 \subset \Omega \) of probability one (that is, \( \mathbb{P}_\mu(\Omega_0) = 1 \) for every \( \mu \in \mathcal{M}_F(D^0) \)) such that, for every \( \omega \in \Omega_0 \) and for every nontrivial nonnegative bounded Borel function \( f \) on \( D \) with compact support whose set of discontinuous points has zero Lebesgue measure, we have

\[
\lim_{t \to \infty} \mathbb{P}_\mu(f, X_t)(\omega) = \frac{M_\infty(\phi)(\omega)}{\langle \phi, \mu \rangle}.
\]  

As a consequence of this theorem we immediately get the following

**Corollary 1.4** Suppose that Assumption 1 holds, \( \lambda_1 > 0 \) and that \( X \) is a \((\xi^D, \psi(\lambda) - \beta \lambda)\)-superdiffusion. Then there exists \( \Omega_0 \subset \Omega \) of probability one (that is, \( \mathbb{P}_\mu(\Omega_0) = 1 \) for every \( \mu \in \mathcal{M}_F(D^0) \)) such that, for every \( \omega \in \Omega_0 \) and every relatively compact Borel subset \( B \) in \( D \) of positive Lebesgue measure whose boundary is of Lebesgue measure zero, we have

\[
\lim_{t \to \infty} \mathbb{P}_\mu(X_t(B))(\omega) = \frac{M_\infty(\phi)(\omega)}{\langle \phi, \mu \rangle}.
\]

**Remark 1.5**

(i) Although we assumed in this paper that the underlying motion \( \xi^D \) is a diffusion process in a bounded domain \( D \), the arguments of this paper can be easily extended to the case when the underlying motion is a Hunt process on a locally compact separable metric space \( E \) satisfying \([17, \text{Assumption 1.1}]\) for some measure \( m \) with full support and with \( m(E) < \infty \), and the analogue of Assumption 1 above.

(ii) In \([3, 6, 7, 9, 20]\), the branching mechanism is assumed to be binary, while in the present paper we deal with general branching mechanism. \([8]\) considers a general branching mechanism under a \((1 + \theta)\)-moment condition, \( \theta > 0 \), while in the present paper, we only assume a \( L \log L \) condition. In \([8]\) the underlying motion is assumed to be a symmetric Hunt process, while in the present paper, our underlying process needs not be symmetric.

## 2 Proof of Theorem 1.3

A main step in proving Theorem 1.3 is the following result.
Theorem 2.1 Suppose that Assumption 1 holds, \( \lambda_1 > 0 \) and that \( X \) is a \((\xi^D, \psi(\lambda) - \beta \lambda)\)-super-diffusion. Then for any \( f \in B_b(D) \) and \( \mu \in \mathcal{M}_F(D)^0 \),

\[
\lim_{t \to \infty} e^{-\lambda_1 t} \langle \phi f, X_t \rangle = M_\infty(\phi) \int_D \phi(y) \tilde{\phi}(y) f(y) dy, \quad P_\mu - a.s. \quad (2.1)
\]

We will prove this result first. According to Theorem 1.1 (2), when \( \int_D \tilde{\phi}(x) l(x) dx = \infty \), we have \( M_\infty(\phi) = \lim_{t \to \infty} e^{-\lambda_1 t} \langle \phi, X_t \rangle = 0 \), \( P_\mu - a.s. \).

For any \( f \in B_b(D) \),

\[
\limsup_{t \to \infty} e^{-\lambda_1 t} \langle \phi f, X_t \rangle \leq \|f\|_\infty \limsup_{t \to \infty} e^{-\lambda_1 t} \langle \phi, X_t \rangle = 0
\]

and (2.1) follows immediately from the nonnegativity of \( f \).

It remains to prove the case when \( \int_D \tilde{\phi}(x) l(x) dx < \infty \). In the remainder of this section, we assume that the assumptions of Theorem 2.1 hold and that \( f \in B_b^+(D) \) is fixed. Define

\[
S(ds, dx) = S^f(ds, dx) + S^C(ds, dx).
\]

Using (1.14), we get

\[
e^{-\lambda_1 t} \langle \phi f, X_t \rangle = e^{-\lambda_1 t} \langle P^D_t(\phi f), \mu \rangle + e^{-\lambda_1 t} \int_0^t \int_D (P^D_{t-s}(\phi f))(x) S(ds, dx) \quad (2.2)
\]

It follows from (1.2) that

\[
\lim_{t \to \infty} e^{-\lambda_1 t} \langle P^D_t(\phi f), \mu \rangle = \langle \phi, \mu \rangle \int_D \phi(x) \tilde{\phi}(x) f(x) dx.
\]

Therefore, to prove Theorem 2.1 it suffices to show that

\[
\lim_{t \to \infty} e^{-\lambda_1 t} \int_0^t \int_D (P^D_{t-s}(\phi f))(x) S(ds, dx) = \int_D \phi(x) \tilde{\phi}(x) f(x) dx \int_0^\infty e^{-\lambda_1 s} \int_D \phi(x) S(ds, dx), \quad P_\mu - a.s. \quad (2.3)
\]

To prove the above result, we need some lemmas first.

Lemma 2.2 For any \( \mu \in \mathcal{M}_F(D)^0 \), we have

\[
\sum_{s > 0} I_{\{\Delta X_s(\phi) > e^{\lambda_1 s}\}} < \infty, \quad P_\mu - a.s. \quad (2.4)
\]

Proof: First note that (1.5) implies that \( \sup_{x \in D} \int_1^\infty rn\phi(x, dr) < \infty \). It is well known that for any \( g \in B_b^+(D) \),

\[
P_\mu(g, X_t) = \langle P^D_t g, \mu \rangle. \quad (2.5)
\]
By the definition of $n^\phi$ given by (1.4), we have

$$
\mathbb{P}_\mu \left[ \sum_{s>0} I_{\{\Delta X_s(\phi) > \lambda_1 s\}} \right] = \mathbb{P}_\mu \left[ \int_0^\infty ds \int_D \int_\infty^\infty I_{\{\phi(s) > \lambda_1 s\}} X_s(dx)n(x, dr) \right] = \int_0^\infty ds \int_D \mu(dy) \int_D p^D(s, y, x)dx \int_{e^{\lambda_1 s}}^\infty n^\phi(x, dr).
$$

It follows from (1.2) that there is a constant $C > 0$ such that

$$
p^D(t, y, x) \leq Ce^{\lambda t} \phi(y)\phi(x), \quad \forall \ t > 1, \ x, y \in D. \tag{2.6}
$$

Therefore we have,

$$
\mathbb{P}_\mu \left[ \sum_{s>0} I_{\{\Delta X_s(\phi) > \lambda_1 s\}} \right] \leq C_1 + C(\phi, \mu) \int_D \tilde{\phi}(x)dx \int_{0}^{\infty} e^{\lambda_1 s}ds \int_{e^{\lambda_1 s}}^\infty n^\phi(x, dr).
$$

Using Fubini’s theorem we get,

$$
\int_{0}^{\infty} e^{\lambda_1 s}ds \int_{e^{\lambda_1 s}}^\infty n^\phi(x, dr) \leq \frac{1}{\lambda_1} \int_{1}^{\infty} r n^\phi(x, dr).
$$

Hence we have

$$
\mathbb{P}_\mu \left[ \sum_{s>0} I_{\{\Delta X_s(\phi) > \lambda_1 s\}} \right] \leq \frac{C(\phi, \mu)}{\lambda_1} \int_D \tilde{\phi}(x)dx \int_{1}^{\infty} r n^\phi(x, dr) < \infty.
$$

Consequently, (2.4) holds. \qed

Define

$$
N^{(1)}_\phi := \sum_{0 < \Delta X_s(\phi) < e^{\lambda_1 s}} \delta(s, \Delta X_s) \quad \text{and} \quad N^{(2)}_\phi := \sum_{\Delta X_s(\phi) \geq e^{\lambda_1 s}} \delta(s, \Delta X_s),
$$

and denote the compensators of $N^{(1)}_\phi$ and $N^{(2)}_\phi$ by $\tilde{N}_\phi^{(1)}$ and $\tilde{N}_\phi^{(1)}$ respectively. Then for any nonnegative predictable function $F$ on $\mathbb{R}_+ \times \Omega \times \mathcal{M}_F(D)$,

$$
\int_0^\infty \int F(s, r)\tilde{N}_\phi^{(1)}(ds, dr) = \int_0^\infty ds \int_D X_s(dx) \int_{e^{\lambda_1 s}}^\infty F(s, r\phi(x))^{-1} \delta_x n^\phi(x, dr), \tag{2.7}
$$

9
and
\[ \int_0^\infty \int F(s,\nu)\hat{N}_\phi^{(2)}(ds,d\nu) = \int_0^\infty ds \int_D X_s(dx) \int_{e^{\lambda s}}^\infty F(s,r\phi(x)^{-1}\delta_x)n^\phi(x,dr). \]  
(2.8)

Let \( J_\phi^{(1)} \) denote the set of jump times of \( N_\phi^{(1)} \), and \( J_\phi^{(2)} \) the set of jump times of \( N_\phi^{(2)} \). Then
\[ \int_0^\infty \int F(s,\nu)N_\phi^{(1)}(ds,d\nu) = \sum_{s \in J_\phi^{(1)}} F(s,\omega,\Delta X_s), \]  
(2.9)
\[ \int_0^\infty \int F(s,\nu)N_\phi^{(2)}(ds,d\nu) = \sum_{s \in J_\phi^{(2)}} F(s,\omega,\Delta X_s), \]  
(2.10)

We can construct two martingale measures \( S^{J,(1)}(ds,dx) \) and \( S^{J,(2)}(ds,dx) \) respectively from \( N_\phi^{(1)}(ds,d\nu) \) and \( N_\phi^{(2)}(ds,d\nu) \), similar to the way we constructed \( S^J(ds,dx) \) from \( N(ds,d\nu) \). Then for any bounded measurable function \( g \) on \( \mathbb{R}_+ \times D \),
\[ S_t^{J,(1)}(g) = \int_0^t \int_D g(s,x)S^{(1)}(ds,dx) = \int_0^t \int_{\mathcal{M}_F(D)} F_g(s,\nu)(N_\phi^{(1)} - \hat{N}_\phi^{(1)})(ds,d\nu), \]  
(2.13)
and
\[ S_t^{J,(2)}(g) = \int_0^t \int_D g(s,x)S^{(2)}(ds,dx) = \int_0^t \int_{\mathcal{M}_F(D)} F_g(s,\nu)(N_\phi^{(2)} - \hat{N}_\phi^{(2)})(ds,d\nu), \]  
(2.14)

where \( F_g(s,\nu) = \int g(s,x)\nu(dx) \).

For any \( m,n \in \mathbb{N} \), \( \sigma > 0 \) and \( f \in \mathcal{B}_+^+(D) \), define
\[ H_{(n+m)}(f) := e^{-\lambda_1(n+m)} \int_0^{(n+m)\sigma} \int_D P^D_{(n+m)\sigma-s}(\phi f)(x)S^{J,(1)}(ds,dx) \]  
and
\[ L_{(n+m)}(f) := e^{-\lambda_1(n+m)} \int_0^{(n+m)\sigma} \int_D P^D_{(n+m)\sigma-s}(\phi f)(x)S^{J,(2)}(ds,dx). \]
Lemma 2.3 If \( \int_D l(x)\tilde{\phi}(x)dx < \infty \), then for any \( m \in \mathbb{N}, \sigma > 0, \mu \in M_{F}(D)^{0} \) and \( f \in B_{b}^{+}(D) \),

\[
\sum_{n=1}^{\infty} \mathbb{P}_{\mu} \left[ H_{(n+m)\sigma}(f) - \mathbb{P}_{\mu}(H_{(n+m)\sigma}(f)\mid F_{n\sigma}) \right]^{2} < \infty \tag{2.15}
\]

and

\[
\lim_{n \to \infty} H_{(n+m)\sigma}(f) - \mathbb{P}_{\mu}(H_{(n+m)\sigma}(f)\mid F_{n\sigma}) = 0, \quad \mathbb{P}_{\mu}\text{-a.s.} \tag{2.16}
\]

**Proof:** Since \( P_{t}^{D}(\phi f) \) is bounded in \([0, T] \times D\) for any \( T > 0 \), the process

\[
H_{t}(f) := e^{-\lambda_{1}(n+m)\sigma} \int_{0}^{t} \int_{D} P_{(n+m)\sigma-s}(\phi f)(x) S_{J,1}^{(1)}(ds, dx), \quad t \in [n\sigma, (n+m)\sigma]
\]

is a martingale with respect to \((F_{t})_{t \leq (n+m)\sigma}\). Thus

\[
\mathbb{P}_{\mu}(H_{(n+m)\sigma}(f)\mid F_{n\sigma}) = e^{-\lambda_{1}(n+m)\sigma} \int_{n\sigma}^{(n+m)\sigma} \int_{D} P_{(n+m)\sigma-s}(\phi f)(x) S_{J,1}^{(1)}(ds, dx),
\]

and hence

\[
H_{(n+m)\sigma}(f) - \mathbb{P}_{\mu}(H_{(n+m)\sigma}(f)\mid F_{n\sigma}) = e^{-\lambda_{1}(n+m)\sigma} \int_{n\sigma}^{(n+m)\sigma} \int_{D} P_{(n+m)\sigma-s}(\phi f)(x) S_{J,1}^{(1)}(ds, dx).
\]

Since

\[
M_{t} := e^{-\lambda_{1}(n+m)\sigma} \int_{n\sigma}^{t} \int_{D} P_{(n+m)\sigma-s}(\phi f)(x) S_{J,1}^{(1)}(ds, dx)
\]

\[
= \int_{n\sigma}^{t} F_{e^{-\lambda_{1}(n+m)\sigma} P_{(n+m)\sigma-s}(\phi f)(s, \nu)} N_{\phi}^{(1)}(ds, d\nu), \quad t \in [n\sigma, (n+m)\sigma]
\]

is a martingale with quadratic variation

\[
\int_{n\sigma}^{t} F_{e^{-\lambda_{1}(n+m)\sigma} P_{(n+m)\sigma-s}(\phi f)(s, \nu)}^{2} \hat{N}_{\phi}^{(1)}(ds, d\nu),
\]

we have

\[
\mathbb{P}_{\mu} \left[ H_{(n+m)\sigma}(f) - \mathbb{P}_{\mu}(H_{(n+m)\sigma}(f)\mid F_{n\sigma}) \right]^{2}
\]

\[
= \mathbb{P}_{\mu} \left[ \int_{n\sigma}^{(n+m)\sigma} F_{e^{-\lambda_{1}(n+m)\sigma} P_{(n+m)\sigma-s}(\phi f)(s, \nu)}^{2} \hat{N}_{\phi}^{(1)}(ds, d\nu) \right]
\]

\[
= \mathbb{P}_{\mu} \left[ \sum_{s \in J_{n,m}^{(1)}} F_{e^{-\lambda_{1}(n+m)\sigma} P_{(n+m)\sigma-s}(\phi f)(s, \Delta X_{s})}^{2} \right], \tag{2.17}
\]

where \( J_{n,m}^{(1)} = J^{(1)}_{\phi} \cap [n\sigma, (n+m)\sigma] \). Note that for any \( f \in B_{b}(D) \), \( \| P_{t}^{\phi} f \|_{\infty} \leq \| f \|_{\infty} \) for all \( t \geq 0 \),

which is equivalent to

\[
P_{t}^{D}(\phi f)(y) \leq \| f \|_{\infty} e^{\lambda_{1}t} \phi(y), \quad \forall \ t \geq 0, \ y \in D.
\]
Using (2.7) and (2.11), we obtain
\[ \mathbb{P}_\mu \left[ \sum_{s \in \mathcal{J}_{n,m}} F_{e^{-\lambda_1(n+m)s}P_{(n+m)s}^D} \rho_f(s, \Delta X_s)^2 \right] \]
\[ = \mathbb{P}_\mu \int_{n\sigma}^{(n+m)\sigma} ds \int_D X_s(dx) \int_0^{e^{\lambda_1s}} F_{e^{-\lambda_1(n+m)s}P_{(n+m)s}^D} \rho_f(s, r\rho(x)^{-1}\delta_x)n\phi(x, dr) \]
\[ = e^{-\lambda_1(n+m)s} \int_{n\sigma}^{(n+m)\sigma} ds \int_D \mu(dy) \int_D p^D(s, y, x)dx \int_0^{e^{\lambda_1s}} [P_{(n+m)s}^D(\rho_f)(x)\rho(x)^{-1}]^2 2^2 n\phi(x, dr) \]
\[ \leq \|f\|^2_\infty \int_{n\sigma}^{(n+m)\sigma} e^{-2\lambda_1s} ds \int_D \mu(dy) \int_D p^D(s, y, x)dx \int_0^{e^{\lambda_1s}} r^2 n\phi(x, dr), \]

where in the second equality we used the fact that
\[ F_{e^{-\lambda_1(n+m)s}P_{(n+m)s}^D} \rho_f(s, r\rho(x)^{-1}\delta_x) = r e^{-\lambda_1(n+m)s} \phi^{-1}(x)P_{(n+m)s}^D \phi(x) \]
and in the last inequality we used (2.18). It follows from (2.12) that there is a constant \( C > 0 \) such that
\[ p^D(s, y, x) \leq Ce^{\lambda_1s}\phi(y)\phi(x), \quad \forall \ s > \sigma, \ x, y \in D. \] (2.20)

Thus
\[ \mathbb{P}_\mu \left[ \sum_{s \in \mathcal{J}_{n,m}} F_{e^{-\lambda_1(n+m)s}P_{(n+m)s}^D} \rho_f(s, \Delta X_s)^2 \right] \]
\[ \leq C \|f\|^2_\infty \int_D \phi(x)dx \int_0^\infty e^{-\lambda_1s} ds \int_0^{e^{\lambda_1s}} r^2 n\phi(x, dr). \]

Summing over \( n \), we get
\[ \sum_{n=1}^\infty \mathbb{P}_\mu \left[ \sum_{s \in \mathcal{J}_{n,m}} F_{e^{-\lambda_1(n+m)s}P_{(n+m)s}^D} \rho_f(s, \Delta X_s)^2 \right] \]
\[ \leq \sum_{n=1}^\infty C \|f\|^2_\infty \int_D \phi(x)dx \int_0^\infty e^{-\lambda_1s} ds \int_0^{e^{\lambda_1s}} r^2 n\phi(x, dr) \]
\[ \leq C \|f\|^2_\infty \int_D \phi(x)dx \int_0^\infty dt \int_0^\infty e^{-\lambda_1s} ds \int_0^{e^{\lambda_1s}} r^2 n\phi(x, dr) \]
\[ = \frac{C}{\sigma} \|f\|^2_\infty \int_D \phi(x)dx \int_0^\infty s e^{-\lambda_1s} ds \int_0^{e^{\lambda_1s}} r^2 n\phi(x, dr) \]
\[ \leq \frac{C}{\sigma} \|f\|^2_\infty \int_D \phi(x)dx \int_1^\infty r^2 n\phi(x, dr) \int_0^{\lambda^{-1}\ln r} s e^{-\lambda_1s} ds \]
\[ + \frac{C}{\sigma} \|f\|^2_\infty \int_D \phi(x)dx \int_0^1 r^2 n\phi(x, dr) \int_0^{\infty} s e^{-\lambda_1s} ds \]
\[ =: I + II. \] (2.21)
Using (1.5) we immediately get that \( II < \infty \). On the other hand,

\[
I = \frac{C}{\lambda_1} \| f \|_\infty^2 \langle \phi, \mu \rangle \int_D \tilde{\phi}(x) dx \int_1^\infty r(\ln r + 1)n^\phi(x, dr).
\]

Now we can use \( \int_D l(x)\tilde{\phi}(x)dx < \infty \) and (1.3) to get that \( I < \infty \). The proof of (2.15) is now complete. For any \( \varepsilon > 0 \), using (2.15) and Chebyshev’s inequality we have

\[
\sum_{n=1}^\infty \mathbb{P}_\mu \left( |H_{(n+m)}(f) - H_{(n+m)}(f)|_F | F_{n\sigma} | > \varepsilon \right) \\
\leq \varepsilon^{-2} \sum_{n=1}^\infty \mathbb{P}_\mu \left( H_{(n+m)}(f) - H_{(n+m)}(f) | F_{n\sigma} |^2 \right) < \infty.
\]

Then (2.16) follows easily from the Borel-Cantelli Lemma.

\[\square\]

**Lemma 2.4** If \( \int_D l(x)\tilde{\phi}(x)dx < \infty \), then for any \( m \in \mathbb{N} \), \( \sigma > 0 \), \( \mu \in \mathcal{M}_F(D)^0 \) and \( f \in \mathcal{B}_0(D) \) we have

\[
\lim_{n \to \infty} L_{(n+m)}(f) - \mathbb{P}_\mu \left[ L_{(n+m)}(f) | F_{n\sigma} \right] = 0, \quad \mathbb{P}_\mu - \text{a.s.} (2.22)
\]

**Proof:** It is easy to see that

\[
\mathbb{P}_\mu \left[ L_{(n+m)}(f) | F_{n\sigma} \right] = e^{-\lambda_1(n+m)\sigma} \int_0^{n\sigma} \int_D P_{(n+m)\sigma-s}(\phi f)(x)S_t^{J,2}(ds, dx).
\]

Therefore,

\[
L_{(n+m)}(f) - \mathbb{P}_\mu \left[ L_{(n+m)}(f) | F_{n\sigma} \right] = e^{-\lambda_1(n+m)\sigma} \int_{n\sigma}^{(n+m)\sigma} \int_D P_{(n+m)\sigma-s}(\phi f)(x)S_t^{J,2}(ds, dx).
\]  

It follows from Lemma 2.2 that, almost surely, the support of the measure \( \nu^{(2)} \) consists of finitely many points. Hence almost surely there exists \( N_0 \in \mathbb{N} \) such that for any \( n > N_0 \),

\[
e^{-\lambda_1(n+m)\sigma} \int_{n\sigma}^{(n+m)\sigma} \int_D P_{(n+m)\sigma-s}(\phi f)(x)S_t^{J,2}(ds, dx)
\]

\[
= -e^{-\lambda_1(n+m)\sigma} \int_{n\sigma}^{(n+m)\sigma} \int_{M_F(D)} F_{P_{(n+m)\sigma-s}(\phi f)}(s, \nu) \tilde{\nu}_\phi^{(2)}(ds, dv)
\]

\[
= -e^{-\lambda_1(n+m)\sigma} \int_{n\sigma}^{(n+m)\sigma} ds \int_D X_s(dx) \phi(x) - P_{(n+m)\sigma-s}(\phi f)(x) \int_{e^{\lambda_1}}^{\infty} rn^\phi(x, dr). (2.24)
\]

where in the last equality we used (2.8) and (2.19). Using (2.18) we get

\[
\left| e^{-\lambda_1(n+m)\sigma} \int_{n\sigma}^{(n+m)\sigma} \int_D P_{(n+m)\sigma-s}(\phi f)(x)S_t^{J,2}(ds, dx) \right|
\[
\leq \| f \|_\infty \int_{n\sigma}^{(n+m)\sigma} e^{-\lambda_1 s} ds \int_D X_s(dx) \int_{e^{\lambda_1}}^{\infty} rn^\phi(x, dr). (2.25)
\]

13
On the other hand, by \( (2.20) \), we have

\[
    \mathbb{P}_\mu \left[ \int_0^\infty e^{-\lambda_1 t} \int_D X_s(x) \int_{e^{\lambda_1 t}} r n^\phi(x, dr) \right] = \int_0^\infty e^{-\lambda_1 t} \int_D \mu(dy) \int_D p^D(s, y, x) \int_{e^{\lambda_1 t}} r n^\phi(x, dr) \leq C(\phi, \mu) \int_D \mu(dy) \int_D \int_{e^{\lambda_1 n}} r n^\phi(x, dr) \int_0^{\lambda_1^{-1} \ln r} ds
\]

\[
    = C(\phi, \mu) \int_D \int_{e^{\lambda_1 n}} \varphi(x) \int_{e^{\lambda_1 n}} r \ln n r n^\phi(x, dr).
\]

Applying the dominated convergence theorem, we obtain that in \( L^1(\mathbb{P}_\mu) \),

\[
    \lim_{n \to \infty} \int_0^\infty e^{-\lambda_1 t} \int_D X_s(x) \int_{e^{\lambda_1 t}} r n^\phi(x, dr) = 0.
\]

Since \( \int_0^\infty e^{-\lambda_1 t} \int_D X_s(x) \int_{e^{\lambda_1 t}} r n^\phi(x, dr) \) is decreasing in \( n \), the above limit holds almost surely as well. Therefore, by \( (2.25) \), we have

\[
    \lim_{n \to \infty} \int_0^{(n+m)\sigma} \int_{e^{\lambda_1 (n+m)\sigma}} (P_{(n+m)\sigma-s} \phi f)(x) S^J(x)(ds, dx) = 0, \quad \mathbb{P}_\mu-\text{a.s.} \quad (2.26)
\]

Now \( (2.22) \) follows from \( (2.23) \) and \( (2.26) \). The proof is complete.

For any \( m, n \in \mathbb{N}, \sigma > 0 \), set

\[
    C_{(n+m)\sigma}(f) := e^{-\lambda_1 (n+m)\sigma} \int_0^{(n+m)\sigma} \int_D (P_{(n+m)\sigma-s} \phi f)(x) S^C(ds, dx), \quad f \in B^+_b(D).
\]

Then \( \{C_{n\sigma}(f)\}_{n \in \mathbb{N}} \) is a martingale with respect to \( (\mathcal{F}_{n\sigma}) \) and

\[
    \mathbb{P}_\mu(C_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}) = e^{-\lambda_1 (n+m)\sigma} \int_0^{n\sigma} \int_D (P_{(n+m)\sigma-s} \phi f)(x) S^C(ds, dx).
\]

**Lemma 2.5** For any \( m \in \mathbb{N}, \sigma > 0, \mu \in \mathcal{M}_F(D)^0 \) and \( f \in B^+_b(D) \) we have

\[
    \lim_{n \to \infty} C_{(n+m)\sigma}(f) - \mathbb{P}_\mu[C_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}] = 0, \quad \mathbb{P}_\mu-\text{a.s.} \quad (2.27)
\]
Proof: From the quadratic variation formula (1.13),

\[
\mathbb{P}_\mu \left[ H_{(n+m)\sigma}(f) - \mathbb{P}_\mu(H_{(n+m)\sigma}(f)\mid \mathcal{F}_{n\sigma}) \right]^2 \\
= \mathbb{P}_\mu \left[ e^{-\lambda_1(n+m)\sigma} \int_{n\sigma}^{(n+m)\sigma} \int_D (\mathcal{P}^D_{(n+m)\sigma-s}\phi f(x))S^C(ds, dx) \right]^2 \\
= \int_{n\sigma}^{(n+m)\sigma} e^{-2\lambda_1(n+m)\sigma} ds \int_D \mu(dx) \int_D \mathcal{P}^D(s, x, y) \left( \mathcal{P}^D_{(n+m)\sigma-s}\phi f \right)^2(y) dy \\
= \int_{n\sigma}^{(n+m)\sigma} e^{-2\lambda_1 s} ds \int_D \mu(dx) \int_D \mathcal{P}^D(s, x, y)\phi^2(y) \left( \mathcal{P}^\phi_{(n+m)\sigma-s} f \right)^2(y) dy \\
\leq \|f\|^2_\infty \int_{n\sigma}^{(n+m)\sigma} e^{-\lambda_1 s} ds \int_D \phi(x) \mu(dx) \mathcal{P}^\phi_s \phi(y) dy \\
\leq \frac{1}{\lambda_1} \|\phi\|_\infty \|f\|^2_\infty \langle \phi, \mu \rangle e^{-\lambda_1 n\sigma}.
\]

Therefore, we have

\[
\sum_{n=1}^\infty \mathbb{P}_\mu \left[ C_{(n+m)\sigma}(f) - \mathbb{P}_\mu(C_{(n+m)\sigma}(f)\mid \mathcal{F}_{n\sigma}) \right]^2 < \infty. \tag{2.29}
\]

By the Borel-Cantelli lemma, we get (2.27). \qed

Combining the three lemmas above, we have the following result.

Lemma 2.6 If \( \int_D l(x) \tilde{\phi}(x) dx < \infty \), then for any \( m \in \mathbb{N}, \sigma > 0, \mu \in \mathcal{M}_F(D)^0 \) and \( f \in \mathcal{B}^+_0(D) \) we have

\[
\lim_{n \to \infty} e^{-\lambda_1(n+m)\sigma} \langle \phi f, X_{(n+m)\sigma} \rangle - \mathbb{P}_\mu \left[ e^{-\lambda_1(n+m)\sigma} \langle \phi f, X_{(n+m)\sigma} \rangle \mid \mathcal{F}_{n\sigma} \right] = 0, \quad \mathbb{P}_\mu - \text{a.s.} \quad \tag{2.30}
\]

Proof: From (1.14), we know that \( e^{-\lambda_1(n+m)\sigma} \langle \phi f, X_{(n+m)\sigma} \rangle \) can be decomposed into three parts:

\[
e^{-\lambda_1(n+m)\sigma} \langle \phi f, X_{(n+m)\sigma} \rangle = e^{-\lambda_1(n+m)\sigma} \langle \mathcal{P}^D_{(n+m)\sigma}(\phi f), \mu \rangle + e^{-\lambda_1(n+m)\sigma} \int_0^{(n+m)\sigma} \int_D \mathcal{P}^D_{(n+m)\sigma-s}(\phi f)(x)S(ds, dx) \\
= e^{-\lambda_1(n+m)\sigma} \langle \mathcal{P}^D_{(n+m)\sigma}(\phi f), \mu \rangle + H_{(n+m)\sigma}(f) + L_{(n+m)\sigma}(f) + C_{(n+m)\sigma}(f).
\]

Therefore,

\[
e^{-\lambda_1(n+m)\sigma} \langle \phi f, X_{(n+m)\sigma} \rangle - \mathbb{P}_\mu \left[ e^{-\lambda_1(n+m)\sigma} \langle \phi f, X_{(n+m)\sigma} \rangle \mid \mathcal{F}_{n\sigma} \right] \\
= H_{(n+m)\sigma}(f) - \mathbb{P}_\mu \left[ H_{(n+m)\sigma}(f) \mid \mathcal{F}_{n\sigma} \right] + L_{(n+m)\sigma}(f) - \mathbb{P}_\mu \left[ L_{(n+m)\sigma}(f) \mid \mathcal{F}_{n\sigma} \right] \\
+ C_{(n+m)\sigma}(f) - \mathbb{P}_\mu \left[ C_{(n+m)\sigma}(f) \mid \mathcal{F}_{n\sigma} \right].
\]

Now the conclusion of this lemma follows immediately from Lemma 2.3, Lemma 2.4 and Lemma 2.5. \qed
Theorem 2.7 If \( \int_{D} l(x) \tilde{\phi}(x) dx < \infty \), then for any \( \sigma > 0 \), \( \mu \in \mathcal{M}_F(D)^0 \) and \( f \in \mathcal{B}_b^+(D) \) we have

\[
\lim_{n \to \infty} e^{-\lambda_1 n \sigma} \langle \phi f, X_{n\sigma} \rangle = M_\infty(\phi) \int_{D} \tilde{\phi}(z) \phi(z) f(z) dz, \quad \mathbb{P}_\mu - \text{a.s.}
\]

**Proof:** By (2.5) and the Markov property of super-processes we have

\[
\mathbb{P}_\mu \left[ e^{-\lambda_1 (n+m) \sigma} \langle \phi f, X_{(n+m)\sigma} \rangle | \mathcal{F}_{n\sigma} \right] = e^{-\lambda_1 n \sigma} e^{-\lambda_1 m \sigma} P_{m \sigma}^D(\phi f), X_{n \sigma} \right). \tag{2.31}
\]

It follows from (1.2) that there exist constants \( c > 0 \) and \( \nu > 0 \) such that

\[
\left| \frac{e^{-\lambda_1 m \sigma} P_{m \sigma}^D(\phi f)(x)}{\phi(x)} - \int_{D} \tilde{\phi}(z) \phi(z) f(z) dz \right| \leq c e^{-\nu m \sigma} \int_{D} \tilde{\phi}(z) \phi(z) f(z) dz,
\]

which is equivalent to

\[
\left| \frac{e^{-\lambda_1 m \sigma} P_{m \sigma}^D(\phi f)(x)}{\phi(x) \int_{D} \tilde{\phi}(z) \phi(z) f(z) dz} - 1 \right| \leq c e^{-\nu m \sigma}.
\]

Thus there exist positive constants \( k_m \leq 1 \) and \( K_m \geq 1 \) such that

\[
k_m \phi(x) \int_{D} \tilde{\phi}(z) \phi(z) f(z) dz \leq e^{-\lambda_1 m \sigma} P_{m \sigma}^D(\phi f)(x) \leq K_m \phi(x) \int_{D} \tilde{\phi}(z) \phi(z) f(z) dz,
\]

and that \( \lim_{m \to \infty} k_m = \lim_{m \to \infty} K_m = 1 \). Hence,

\[
e^{-\lambda_1 n \sigma} \langle e^{-\lambda_1 m \sigma} P_{m \sigma}^D(\phi f), X_{n \sigma} \rangle \geq k_m e^{-\lambda_1 n \sigma} \langle \phi, X_{n \sigma} \rangle \int_{D} \tilde{\phi}(z) \phi(z) f(z) dz
\]

\[
= k_m M_{n \sigma}(\phi) \int_{D} \tilde{\phi}(z) \phi(z) f(z) dz, \quad \mathbb{P}_\mu - \text{a.s.}
\]

and

\[
e^{-\lambda_1 n \sigma} \langle e^{-\lambda_1 m \sigma} P_{m \sigma}^D(\phi f), X_{n \sigma} \rangle \leq K_m e^{-\lambda_1 n \sigma} \langle \phi, X_{n \sigma} \rangle \int_{D} \tilde{\phi}(z) \phi(z) f(z) dz
\]

\[
= K_m M_{n \sigma}(\phi) \int_{D} \tilde{\phi}(z) \phi(z) f(z) dz, \quad \mathbb{P}_\mu - \text{a.s.}
\]

These two inequalities and Lemma 2.6 imply that

\[
\limsup_{n \to \infty} e^{-\lambda_1 n \sigma} \langle \phi f, X_{n \sigma} \rangle = \limsup_{n \to \infty} e^{-\lambda_1 (n+m) \sigma} \langle \phi f, X_{(n+m)\sigma} \rangle
\]

\[
= \limsup_{n \to \infty} \mathbb{P}_\mu \left[ e^{-\lambda_1 (n+m) \sigma} \langle \phi f, X_{(n+m)\sigma} \rangle | \mathcal{F}_{n \sigma} \right]
\]

\[
= \limsup_{n \to \infty} e^{-\lambda_1 n \sigma} \langle e^{-\lambda_1 m \sigma} P_{m \sigma}^D(\phi f), X_{n \sigma} \rangle
\]

\[
\leq \limsup_{n \to \infty} K_m M_{n \sigma}(\phi) \int_{D} \tilde{\phi}(z) \phi(z) f(z) dz
\]

\[
= K_m M_\infty(\phi) \int_{D} \tilde{\phi}(z) \phi(z) f(z) dz, \quad \mathbb{P}_\mu - \text{a.s.}
\]
and that
\[
\lim_{n \to \infty} e^{-\lambda_1 n \sigma} \langle \phi f, X_{n \sigma} \rangle \geq k_m M_\infty(\phi) \int_D \bar{\phi}(z) \phi(z) f(z) dz, \quad \mathbb{P}_\mu \text{-a.s.}
\]
Letting \( m \to \infty \), we get
\[
\lim_{n \to \infty} e^{-\lambda_1 n \sigma} \langle \phi f, X_{n \sigma} \rangle = M_\infty(\phi) \int_D \bar{\phi}(z) \phi(z) f(z) dz, \quad \mathbb{P}_\mu \text{-a.s.}
\]
The proof is now complete. \( \square \)

We are now ready to give the proof of Theorem 2.1.

**Proof of Theorem 2.1**

Put \( \Delta_\sigma(f) := \sup_{0 \leq t \leq \sigma} \| P_t^\phi f - f \|_\infty \). Then for \( t \in [n \sigma, (n + 1) \sigma] \),
\[
e^{-\lambda_1 t} \left( \phi P_{(n+1)\sigma-t}^\phi f, X_t \right) - e^{-\lambda_1 t} \langle \phi f, X_t \rangle \leq e^{-\lambda_1 t} \left( \phi | P_{(n+1)\sigma-t}^\phi f - f |, X_t \right) \leq M_t(\phi) \Delta_\sigma(f). \quad (2.32)
\]
By the strong continuity of the semigroup \( (P_t^\phi) \) in \( L^\infty(D) \), we have \( \lim_{\sigma \to 0} \Delta_\sigma(f) = 0 \). Thus,
\[
\lim_{t \to \infty} \sup_{\sigma \to 0} \left| e^{-\lambda_1 t} \left( \phi P_{(n+1)\sigma-t}^\phi f, X_t \right) - e^{-\lambda_1 t} \langle \phi f, X_t \rangle \right| = 0, \quad \mathbb{P}_\mu \text{-a.s.} \quad (2.33)
\]
Therefore, to prove Theorem 2.1, we only need to show that
\[
\lim_{t \to \infty} \sup_{\sigma \to 0} e^{-\lambda_1 t} \left( \phi P_{(n+1)\sigma-t}^\phi f, X_t \right) = M_\infty(\phi) \int_D \phi(x) \bar{\phi}(x) f(x) dx, \quad \mathbb{P}_\mu \text{-a.s.} \quad (2.34)
\]
For any \( n \in \mathbb{N} \) and \( \sigma > 0 \), \( (X_t, t \in [n \sigma, (n + 1) \sigma], \mathbb{P}_\mu (\cdot | \mathcal{F}_{n \sigma}) \) can be regarded as a \( (\xi^D, \psi(\lambda) - \beta \lambda) \)-super-diffusion with initial value \( X_{n \sigma} \). Thus, for arbitrary \( g \in B_b^+ (D) \), we have by (1.14)
\[
e^{-\lambda_1 t} \langle \phi g, X_t \rangle = e^{-\lambda_1 t} \langle P_{t-n\sigma}^D (\phi g), X_{n \sigma} \rangle + e^{-\lambda_1 t} \int_{n \sigma}^t \int_{n \sigma}^{t-s} P_{t-s}^D (\phi g)(x) S(ds, dx), \quad t \in [n \sigma, (n + 1) \sigma].
\]
Taking \( g(x) = P_{(n+1)\sigma-t}^\phi f(x) \) in the above identity and using (1.1), we get
\[
e^{-\lambda_1 t} \left( \phi P_{(n+1)\sigma-t}^\phi f, X_t \right) = e^{-\lambda_1 n \sigma} \langle \phi P_{n \sigma}^\phi f, X_{n \sigma} \rangle + \int_{n \sigma}^t e^{-\lambda_1 s} \int_{n \sigma}^{t-s} (\phi P_{(n+1)\sigma-s}^\phi f)(x) S(ds, dx). \quad (2.35)
\]
Since \( \bar{\phi} \) is the invariant probability density of the semigroup \( (P_t^\phi) \), we have by Lemma 2.7
\[
\lim_{n \to \infty} e^{-\lambda_1 n \sigma} \langle \phi P_{n \sigma}^\phi f, X_{n \sigma} \rangle = M_\infty(\phi) \int_D \phi(x) \bar{\phi}(x) f(x) dx
\]
\[
= M_\infty(\phi) \int_D \phi(x) \bar{\phi}(x) f(x) dx. \quad (2.36)
\]
Hence, by (2.35) and (2.36), to prove (2.34) it suffices to show that
\[
\lim_{t \to \infty} \sup_{\sigma \to 0} \int_{n \sigma}^t e^{-\lambda_1 s} \int_{n \sigma}^{t-s} (\phi P_{(n+1)\sigma-s}^\phi f)(x) S(ds, dx) = 0, \quad \mathbb{P}_\mu \text{-a.s.} \quad (2.37)
\]
Since \(S(ds, dx) = S^J(ds, dx) + S^C(ds, dx) = S^{J,(1)}(ds, dx) + S^{J,(2)}(ds, dx) + S^C(ds, dx)\), we have
\[
\int_{n\sigma}^t e^{-\lambda_1 s} \int (\phi P_{(n+1)\sigma-s} f) (x) S(ds, dx)
= \int_{n\sigma}^t e^{-\lambda_1 s} \int (\phi P_{(n+1)\sigma-s} f) (x) S^{J,(1)}(ds, dx) + \int_{n\sigma}^t e^{-\lambda_1 s} \int (\phi P_{(n+1)\sigma-s} f) (x) S^{J,(2)}(ds, dx)
+ \int_{n\sigma}^t e^{-\lambda_1 s} \int (\phi P_{(n+1)\sigma-s} f) (x) S^C(ds, dx)
=: H_{n,t}(f) + L_{n,t}(f) + C_{n,t}(f).
\]

Thus we only need to prove that
\[
\lim_{\sigma \to 0} \lim_{n \to \infty} \sup_{t \in [n\sigma, (n+1)\sigma]} H_{n,t}(f) = 0, \quad \mathbb{P}_\mu \text{-a.s.} \tag{2.38}
\]
and
\[
\lim_{\sigma \to 0} \lim_{n \to \infty} \sup_{t \in [n\sigma, (n+1)\sigma]} L_{n,t}(f) = 0, \quad \mathbb{P}_\mu \text{-a.s.} \tag{2.39}
\]

It follows from Chebyshev’s inequality that, for any \(\varepsilon > 0\), we have
\[
\mathbb{P}_\mu \left( \sup_{t \in [n\sigma, (n+1)\sigma]} |H_{n,t}(f)| > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \mathbb{P}_\mu \left( \sup_{t \in [n\sigma, (n+1)\sigma]} \int_{n\sigma}^t e^{-\lambda_1 s} \int (\phi P_{(n+1)\sigma-s} f) (x) S^{J,(1)}(ds, dx) \right)^2 \tag{2.41}
\]
Since the process \((H_{n,t}(f); t \in [n\sigma, (n + 1)\sigma])\) is a martingale with respect to \((\mathcal{F}_t)_{t \in [n\sigma, (n+1)\sigma]}\), applying Burkholder-Davis-Gundy inequality to \(H_{n,t}(f)\) and using an argument similar to the one in the proof of Lemma 2.2, we obtain
\[
\mathbb{P}_\mu \left( \sup_{t \in [n\sigma, (n+1)\sigma]} \int_{n\sigma}^t e^{-\lambda_1 s} \int (\phi P_{(n+1)\sigma-s} f) (x) S^{J,(1)}(ds, dx) \right)^2
\leq C_1 \mathbb{P}_\mu \left( \int_{n\sigma}^{(n+1)\sigma} e^{-\lambda_1 s} \int \phi(x) P_{(n+1)\sigma-s} f (x) S^{J,(1)}(ds, dx) \right)^2
= C_1 \int_{n\sigma}^{(n+1)\sigma} e^{-2\lambda_1 s} ds \int_D \mu(dy) \int_D d\mathbb{P}^{P_{(n+1)\sigma-s} f} (y, x) \left( P_{(n+1)\sigma-s} f (x) \right)^2 (x) \int_0^{e^{\lambda_1 s}} r^2 n^\phi (x, dr)
= C_1 \int_{n\sigma}^{(n+1)\sigma} e^{-\lambda_1 s} ds \int_D \phi(y) \mu(dy) \int_D d\mathbb{P}^{P_{(n+1)\sigma-s} f} (y, x) \phi(x)^{-1} \left( P_{(n+1)\sigma-s} f (x) \right)^2 (x) \int_0^{e^{\lambda_1 s}} r^2 n^\phi (x, dr)
\leq C_2(\sigma) ||f||_{2\infty}^2 \langle \phi, \mu \rangle \int_{n\sigma}^{(n+1)\sigma} e^{-\lambda_1 s} ds \int_D \bar{\phi}(x) dx \int_0^{e^{\lambda_1 s}} r^2 n^\phi (x, dr)
\leq C_2(\sigma) ||f||_{2\infty}^2 \langle \phi, \mu \rangle \left[ \int_{n\sigma}^{(n+1)\sigma} e^{-\lambda_1 s} ds \int_D \bar{\phi}(x) dx \int_0^1 r^2 n^\phi (x, dr) \right. + \int_{n\sigma}^{(n+1)\sigma} e^{-\lambda_1 s} ds \int_D \bar{\phi}(x) dx \int_1^{e^{\lambda_1 s}} r^2 n^\phi (x, dr)
=: C_2(\sigma) ||f||_{2\infty}^2 \langle \phi, \mu \rangle (I_1(n) + I_2(n)), \tag{2.42}
\]
where $C_1$ and $C_2(\sigma)$ are positive constants independent of $n$. (1.3) implies that $\| \int_0^1 r^2 n^\phi(\cdot, dr) \|_\infty < \infty$. Thus

$$\sum_{n=1}^{\infty} I_1(n) < \infty. \quad (2.43)$$

Using Fubini’s theorem, we have

$$\sum_{n=1}^{\infty} I_2(n) = \sum_{n=1}^{\infty} \int_{n\sigma}^{(n+1)\sigma} e^{-\lambda_1 s} ds \int_D \tilde{\phi}(x) dx \int_1^{e^{\lambda_1 s}} r^2 n^\phi(x, dr)$$

$$= \int_\sigma^{\infty} e^{-\lambda_1 s} ds \int_D \tilde{\phi}(x) dx \int_1^{e^{\lambda_1 s}} r^2 n^\phi(x, dr)$$

$$\leq \int_D \tilde{\phi}(x) dx \int_1^{\infty} r^2 n^\phi(x, dr) \int_1^{\infty} e^{-\lambda_1 s} ds$$

$$= \lambda_1^{-1} \int_D \tilde{\phi}(x) dx \int_1^{\infty} r n^\phi(x, dr)$$

$$\leq \lambda_1^{-1} \int_D \tilde{\phi}(x) dx \left\| \int_1^{\infty} r n^\phi(x, dr) \right\|_\infty < \infty. \quad (2.44)$$

Combining (2.41), (2.42), (2.43) and (2.44), we get that, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}_\mu \left( \sup_{t \in [n\sigma, (n+1)\sigma]} |H^\sigma_{n,t}(f)| > \varepsilon \right) < \infty. \quad (2.45)$$

Thus by the Borel-Cantelli lemma we have, for any $\sigma > 0$,

$$\lim_{n \to \infty} \sup_{t \in [n\sigma, (n+1)\sigma]} H^\sigma_{n,t}(f) = 0, \quad \mathbb{P}_\mu - a.s. \quad (2.46)$$

Therefore (2.38) is valid.

Similarly, we can prove that

$$\lim_{n \to \infty} \sup_{t \in [n\sigma, (n+1)\sigma]} |C^\sigma_{n,t}(f)| = 0, \quad \mathbb{P}_\mu - a.s., \quad (2.47)$$

and then we get (2.40). The details are omitted here.

Using an argument similar to (2.24) we can see that almost surely there exists $N_0 \in \mathbb{N}$ such that when $n > N_0$,

$$|L^\sigma_{n,t}(f)| = \int_{n\sigma}^{t} e^{-\lambda_1 s} ds \int_D P^{\phi, \sigma}_{(n+1)\sigma-s} f(x) X_s(dx) \int_1^{e^{\lambda_1 s}} r n^\phi(x, dr)$$

$$\leq \|f\|_\infty \int_{n\sigma}^{(n+1)\sigma} e^{-\lambda_1 s} X_s(dx) \phi(x) \int_1^{e^{\lambda_1 s}} r n^\phi(x, dr)$$

$$\leq \sigma \|f\|_\infty \int_1^{\infty} r n^\phi(x, dr) \sup_{s \in [n\sigma, (n+1)\sigma]} M_s(\phi). \quad (2.48)$$

Therefore (2.39) holds. The proof of Theorem 2.1 is now complete.

The following result strengthens Theorem 2.1 in the sense that the exceptional does not depend on $f$ and $\mu$. 

19
Theorem 2.8 Suppose that Assumption [\( \Pi \)] holds, \( \lambda_1 > 0 \) and that \( X \) is a \( (\xi^D, \psi(\lambda) - \beta\lambda) \)-superdiffusion. Then there exists \( \Omega_0 \subset \Omega \) of probability one (that is, \( \mathbb{P}_\mu(\Omega_0) = 1 \) for every \( \mu \in \mathcal{M}_F(D) \)) such that, for every \( \omega \in \Omega_0 \) and for every bounded Borel measurable function \( f \) on \( \mathbb{R}^d \) with compact support whose set of discontinuous points has zero Lebesgue measure, we have
\[
\lim_{t \to \infty} e^{-\lambda t} \langle f, X_t \rangle = M_\infty(\phi) \int_D \tilde{\phi}(y)f(y)dy.
\] (2.49)

Proof: Note that there exists a countable base \( \mathcal{U} \) of open sets \( \{U_k, k \geq 1\} \) that is closed under finite unions. Define
\[
\Omega_0 := \left\{ \omega \in \Omega : \lim_{t \to \infty} e^{-\lambda t} \langle I_{U_k} \phi, X_t \rangle = M_\infty(\phi) \int_{U_k} \phi(y)\tilde{\phi}(y)dy \text{ for every } k \geq 1 \right\}.
\]
By Theorem 2.7 for any \( \mu \in \mathcal{M}_F(D) \), \( P_\mu(\Omega_0) = 1 \). For any open set \( U \), there exists a sequence of increasing open sets \( \{U_{nk}, k \geq 1\} \) in \( \mathcal{U} \) so that \( \bigcup_k^\infty U_{nk} = U \). Then for every \( \omega \in \Omega_0 \),
\[
\liminf_{t \to \infty} e^{-\lambda t} \langle I_U \phi, X_t \rangle \geq \limsup_{t \to \infty} e^{-\lambda t} \langle I_{U_{nk}} \phi, X_t \rangle \geq M_\infty(\phi) \int_{U_{nk}} \phi(y)\tilde{\phi}(y)dy \text{ for every } k \geq 1.
\]
Letting \( k \to \infty \) yields
\[
\liminf_{t \to \infty} e^{-\lambda t} \langle I_U \phi, X_t \rangle \geq M_\infty(\phi) \int_U \phi(y)\tilde{\phi}(y)dy, \quad \mathbb{P}_\mu-\text{a.s. for any } \mu \in \mathcal{M}_F(D) \). \quad (2.50)
\]
We now consider (2.49) on \( \{M_\infty(\phi) > 0\} \). For each \( \omega \in \Omega_0 \cap \{M_\infty(\phi) > 0\} \) and \( t \geq 0 \), we define two probability measures \( \nu_t \) and \( \nu \) on \( D \) respectively by
\[
\nu_t(A)(\omega) = \frac{e^{\lambda t} \langle I_A \phi, X_t \rangle(\omega)}{M_\infty(\phi)(\omega)}, \quad \text{and} \quad \nu(A) = \int_A \phi(y)\tilde{\phi}(y)dy, \quad A \in \mathcal{B}(D).
\]
Note that the measure \( \nu_t \) is well-defined for every \( t \geq 0 \). (2.50) tells us that \( \nu_t \) converges weakly to \( \nu \) as \( t \to \infty \). Since \( \phi \) is strictly positive and continuous on \( D \), for every function \( f \) on \( D \) with compact support on \( E \) whose discontinuity set has zero Lebesgue-measure (equivalently zero \( \nu \)-measure), \( g := f/\phi \) is a bounded function with compact support with the same set of discontinuity. We thus have
\[
\int_D g(x)\nu_t(dx) = \int_D g(x)\nu(dx),
\]
which is equivalent to say
\[
\lim_{t \to \infty} e^{-\lambda t} \langle f, X_t \rangle = M_\infty(\phi) \int_D \tilde{\phi}(y)f(y)dy, \quad \text{for every } \omega \in \Omega_0 \cap \{M_\infty(\phi) > 0\}. \quad (2.51)
\]
Since
\[
e^{-\lambda t} |\langle f, X_t \rangle| \leq e^{-\lambda t} (|f|, X_t) = e^{-\lambda t} (|g\phi|, X_t) \leq \|g\|_\infty M_\infty(\phi).
\]
(2.51) holds automatically on \( \{M_\infty(\phi) = 0\} \). This completes the proof of the theorem.
Lemma 2.9 For any $f \in \mathcal{B}_b(D)$, $\mu \in \mathcal{M}_F(D)$,

$$
\lim_{t \to \infty} e^{-\lambda t} \mathbb{P}_\mu(f, X_t) = \langle \phi, \mu \rangle \int_D f(y) \tilde{\phi}(y) dy. \quad (2.52)
$$

Proof: It follows from (2.5) that

$$
e^{-\lambda t} \mathbb{P}_\mu(f, X_t) = \int_D \mu(dx) e^{-\lambda t} P_t^D f(x) = \int_D \mu(dx) \int_D e^{-\lambda t} P_t^D(t, x, y) f(y) dy = \int_D \mu(dx) \phi(x) \int_D p^\phi(t, x, y) \frac{f(y)}{\phi(y)} dy.
$$

Using (1.2) and the dominated convergence theorem, we get (2.52).

Proof of Theorem 1.3 Theorem 1.3 is simply a combination of Theorem 2.8 and Lemma 2.9. \qed

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