Searching for non-Kerr objects

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Abstract. We suggest a method that could be used to discriminate a Kerr black hole from any other supermassive axisymmetric astrophysical object by analyzing the gravitational-wave signal from an extreme mass ratio inspiral (EMRI). The method is based on the quite distinct qualitative features that characterize a slightly nonintegrable system. According to the Poincaré-Birkhoff theorem, whenever a resonance of frequencies arise in an axisymmetric perturbed Kerr metric, instead of the anticipated KAM curves of the integrable Kerr case, a Birkhoff chain of islands appears on a surface of sections. The orbits of this chain of islands have a fixed ratio of frequencies. The idea is to exploit this feature to check if the inspiraling low-mass object spends a finite interval of time to cross this resonance, while its orbit evolves adiabatically due to the radiation of gravitational waves.

1. Introduction
One of the most promising sources of gravitational waves for LISA [1] (the future space interferometric detector that is planned to be launched by NASA and ESA) are solar mass \((1-10 M_\odot)\) compact objects (black holes or neutron stars) orbiting around supermassive black holes \((\sim 10^5-10^8 M_\odot)\). The extreme-mass ratio of such binaries render the analytic approximate Post-Newtonian formulae, that have been produced by a large number of people (see [2] and the cited references) during the last couple of decades, an extremely accurate tool to describe the adiabatic change of the orbit of the small object in the Kerr background of the larger one. The analysis of the signal of LISA will give us the opportunity to check if the waveforms according to general relativity are the right ones. Unfortunately the general relativistic waveforms depend on too many parameters of the binary; thus a confusion problem may arise while analyzing the data since a slight deviation of the waveforms due to a different background than Kerr, or a different
theory of gravity than General Relativity may be mimicked by a pure relativistic waveform in a Kerr background with wrong parameters.

In order to make sure that whatever signal we monitor is exactly what we anticipate according to our conventional wisdom we have to think of “smoking guns”, that is of qualitatively new features that could reveal a deviation of theory from what we expect (see e.g. [3]). Such a new feature that could reveal the deviation of the background metric from Kerr is any orbital effect that could discriminate a slightly non-integrable and an integrable system (like the Kerr one).

The KAM theorem of Kolmogorov, Arnold and Moser [4] states that most of the KAM tori on which the phase orbits of an integrable system evolve are not destroyed when the system becomes slightly perturbed. However, according to the Poincaré-Birkhoff theorem the resonant tori (the ones that are characterized by commensurate ratios of fundamental frequencies) of the integrable system disintegrate when the systems become slightly perturbed and they form the so-called Birkhoff chains of islands on a surface of sections. These islands have a finite thickness and all their interior corresponds to orbits that are characterized by the same rational ratio of frequencies.

This is exactly the idea that we have quantitatively explored in order to check if the background metric on which an EMRI evolves is the integrable case of Kerr or something else. We have used the Manko-Novikov metric [5] as an example of a non-Kerr metric. This is a solution of vacuum Einstein’s equations that is axisymmetric and asymptotically flat. It is described by a different multipole moments from the Kerr metric starting from the quadrupole moment. Actually we have used the simplest family of the general Manko-Novikov solutions, which is characterized by a single parameter $q$, apart of its mass $M$ and spin $S$. The $q$ parameter measures the deviation of the quadrupole moment from the corresponding Kerr metric (the one with the same mass and spin).

We have shown that the geodesic orbits of this Manko-Novikov metric exhibit Birkhoff islands whenever a resonance is present, as expected for a slightly non-integrable system. The thickness of these islands has been measured as a function of the orbital energy, and the angular momentum, as well as the spin, and the quadrupole deviation of the metric. The size of the islands is expected to have a similar behavior for a generic type of a non-Kerr metric background.

Next we studied what will be the effect of the presence of these islands in the gravitational wave signal radiated by such an EMRI source. As the orbiting object loses energy and angular momentum the orbit changes adiabatically. As a consequence the phase orbit may pass through a chain of resonance islands. During the time interval that the orbit finds itself in such a chain of islands the corresponding frequencies encoded in the gravitational wave signal stay locked to a rational ratio, although both are changing. If such a signal behavior is observed we could be sure that the metric in which the small object orbits is not that of a Kerr black hole.

We end our analysis by presenting some indicative estimates of the time intervals spent by the signal with a locked ratio of frequencies.

2. A toy model for a non-Kerr metric

2.1. The line element

In 1992 Manko and Novikov [5] constructed a multiparametric family of spacetimes, as a generalization of the Kerr metric, in order to describe the gravitational field of an arbitrary rotating and axially symmetric isolated object. It is actually an exact solution of the vacuum Einstein equations, built by a non-linear superposition of the Kerr spacetime with an arbitrary static vacuum Weyl field (see [6]). The Manko-Novikov solution that we have used (hereafter called MN metric) is a subfamily of the general multiparametric family of the Manko-Novikov solution. It depends on only three parameters: its mass monopole $M$, its spin $S$, and its dimensionless deviation from the corresponding Kerr’s quadrupole $q$. The rest multipole moments are defined from these three parameters. Thus the first four non-zero mass and current-
mass moments, $M_l$ and $S_l$ respectively, are

\[ M_0 = M, \quad S_1 = S, \quad M_2 = -M\left[\left(\frac{S}{M}\right)^2 + qM^2\right], \quad S_3 = -M\left[\left(\frac{S}{M}\right)^3 + 2qM^2\left(\frac{S}{M}\right)\right]. \tag{1} \]

By setting $q = 0$ the MN solution turns into an exact Kerr metric; thus all its multipole moments are then described by the formula

\[ M_l + iS_l = M\left(\frac{S}{M}\right)^l. \tag{2} \]

The particular MN metric in Weyl-Papapetrou coordinates is described by the following line element

\[ ds^2 = -f(dt - \omega d\phi)^2 + f^{-1}\left[e^{2\gamma}(dp^2 + dz^2) + \rho^2 d\phi^2\right] \tag{3} \]

where all metric functions $f$, $\omega$, $\gamma$ should be considered as functions of the prolate spheroidal coordinates $x, y$. The coordinates $\rho, z$ are the corresponding cylindrical coordinates which could be expressed as functions of $x, y$ as well. Thus

\[ \rho = k \sqrt{(x^2 - 1)(1 - y^2)}, \quad z = kxy \tag{4} \]

and

\[ f = e^{2\psi} \frac{A}{B}, \tag{5} \]

\[ \omega = 2ke^{-2\psi}C - 4k\frac{\alpha}{1 - \alpha^2}, \tag{6} \]

\[ e^{2\gamma} = e^{2\gamma'} \frac{A}{(x^2 - 1)(1 - \alpha^2)^2}, \tag{7} \]

\[ A = (x^2 - 1)(1 + a b)^2 - (1 - y^2)(b - a)^2, \tag{8} \]

\[ B = [(x + 1) + (x - 1)a b]^2 + [(1 + y)a + (1 - y)b]^2, \tag{9} \]

\[ C = (x^2 - 1)(1 + a b)[(b - a) - y(a + b)] + (1 - y^2)(b - a)[(1 + a b) + x(1 - a b)], \tag{10} \]

\[ \psi = \beta \frac{P_2}{R^2}, \tag{11} \]

\[ \gamma' = \ln \sqrt{\frac{x^2 - 1}{x^2 - y^2} + \frac{3\beta^2}{2R^6}(P_3^2 - P_2^2)} \]

\[ + \beta \left(-2 + \sum_{\ell=0}^{2} \frac{x - y + (-1)^{2-\ell}(x + y)}{R^{\ell+1}} P_{\ell}\right), \tag{12} \]

\[ a = -\alpha \exp \left[-2\beta \left(-1 + \sum_{\ell=0}^{2} \frac{(x - y)P_{\ell}}{R^{\ell+1}}\right)\right], \tag{13} \]

\[ b = \alpha \exp \left[2\beta \left(1 + \sum_{\ell=0}^{2} \frac{(-1)^{3-\ell}(x + y)P_{\ell}}{R^{\ell+1}}\right)\right], \tag{14} \]

\[ R = \sqrt{x^2 + y^2 - 1}, \tag{15} \]

\[ P_{\ell} = P_{\ell}\left(\frac{x y}{R}\right). \tag{16} \]
\( P_l(z) \) denotes the Legendre polynomial of order \( l \) given by

\[
P_l(z) = \frac{1}{2^l l!} \left( \frac{d}{dz} \right)^l (z^2 - 1)^l.
\] (17)

The three parameters \( k, \alpha, \beta \) that appear in the formulae above characterize the metric and are related to the mass \( M \), the spin \( S \), and the quadrupole deviation \( q \) through the following expressions:

\[
\alpha = -\frac{M + \sqrt{M^2 - (S/M)^2}}{(S/M)}, \quad k = M \frac{1 - \alpha^2}{1 + \alpha^2}, \quad \beta = q \left( \frac{1 + \alpha^2}{1 - \alpha^2} \right)^3.
\] (18)

2.2. Description of the orbits

In order to study the geodesic orbits in MN we have relied on a Lagrangian formulation. Thus the geodesics are simply the equations of motion described by the Lagrangian

\[
L = \frac{1}{2} \mu g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}
\] (19)

where \( \mu \) is the rest mass of the test body, and \( \tau \) is the proper time along the orbit. Besides \( \mu \) there are two integrals of motion \( p_t = -\mu E \) and \( p_\phi = \mu L_z \) due to stationarity and axisymmetry of the metric, respectively. From suitable combinations of \( E, L_z \) and the metric functions one can compute by simple integration the coordinates \( t \) and \( \phi \), once \( \rho(\tau) \) and \( z(\tau) \) are known. Thus the computation of the geodesic motion is constrained in the 4-dimensional phase space: \((\rho, \dot{\rho}, z, \dot{z})\) where the dots represent derivatives with respect to \( \tau \). An intuitive concise tool to obtain some basic characteristics of the orbits is the effective potential \( V_{\text{eff}}(\rho, z) \) (see Eq. (14) of [7]) which connects the Newtonian analogue of kinetic energy to the position of the test particle on the \( \rho - z \) plane. Thus

\[
\frac{1}{2}(\dot{\rho}^2 + \dot{z}^2) + V_{\text{eff}}(\rho, z) = 0.
\] (20)

The locus of \( V_{\text{eff}}(\rho, z) = 0 \) defines the boundary of any orbit, and its shape is characterized by \( E, L_z, q, S \) and \( M \).

While for \( q < 0 \) there is only a single region of non-plunging bound orbits \((E < 1)\), for \( q > 0 \) a new small region of non-plunging bound orbits appears between the previous region and the central region for a range of the rest parameters. Depending on \( E, L_z, S \) the two regions are either distinct or they could be joined in a single one.

In order to numerically integrate the orbits we have applied a sixth order Runge Kutta integration scheme with a constant step of integration \( \delta s = 10^{-1} \), except for the cases in which an orbit reaches the surface of section; then the integration step was gradually reduced to \( \delta s = 10^{-8} \) in order to get a fine approximation of the surface of section at \( z = 0 \). The relative error of the integration, measured through the relative variation of the Lagrangian value \( \Delta L/L = |(L_n - L)/L| \) after \( 10^3 \) crossings through the \( z = 0 \) surface of section, did not exceed the order of \( 10^{-10} \). \( (L_n) \) is the value of the Lagrangian due to the integration). For the more demanding integration in the inner region of allowed non-plunging orbits the integration step was chosen even smaller to keep the errors of \( L \) at the same order of magnitude.

By changing the initial conditions of an orbit we obtained a series of KAM invariant curves that are foliated with each other on a surface of section according to the KAM theorem. The Poincaré-Birkhoff chains of islands are fine structures and one should vary the initial conditions through very small steps to find them. Fortunately there is a very useful tool to find the suitable initial conditions that lead to resonances; it is the rotation number, that is the average fraction of a full circle along the KAM curve that the series of subsequent intersecting points (Poincaré consequents) advance around a point that represents the central periodic orbit. Whenever this number is close to a ratio of integer numbers then we are approaching a resonance. (For more
details about the use of rotation number see [7, 8].) Thus by using the rotation number as a tool to choose the initial conditions we managed to produce two chains of Birkhoff islands in the outer region of bound orbits: one corresponding to the resonance of 2/3 and one corresponding to the resonance of 1/2. If we choose initial conditions within a Birkhoff island the corresponding Poincaré consequents form a set of closed curves each lying inside the islands of that chain. In Fig. 1 we have plotted a series of KAM curves and the two aforementioned chains of islands on the surface of sections at $z = 0$ corresponding to orbits in the outer allowed region.

![Figure 1](image.png)

**Figure 1.** The surface of sections of the outer region on the $(\rho, \dot{\rho})$ plane for the parameter set $E = 0.95, L_z = 3M, \chi = 0.9, q = 0.95$. There is one triplet of islands corresponding to the resonance of 2/3, and 2 doublets corresponding to the resonance of 1/2.

The inner region of bound orbits, as long as it is distinct from the outer one, has a more complicated surface of section (see Fig. 2): Most of the orbits are chaotic and the intersecting points are scattered irregularly on a large region of the available phase space on the $(\rho - \dot{\rho})$ plane. In contrast to previous investigations [9] there are also regular quasi-periodic orbits represented by the invariant curves and at least one chain of 3 resonant islands (Fig. 2) Thus the general picture of a surface of section through the whole range of phase space $(\rho - \dot{\rho})$ is that of a metric that is “close” to the integrable Kerr one. The further we are located from the central point of the field, the less important are the effects of the higher multipole moments on the orbits; thus the MN metric behaves like a perturbed Kerr and the only effect on the phase space is the appearance of the Birkhoff chains. In the inner region the higher multipole moments are essential for a geodesic orbit. Since all these moments differ from the corresponding Kerr, the metric does not look like a slightly perturbed Kerr. Instead the metric is quite different from Kerr in this region and chaos is dominant. For a more thorough study of the orbits in the MN metric see [7].

3. **Inspiraling orbits**

Next we explored the effect of resonances on an adiabatically changing orbit, that represents an actual EMRI orbit in a MN background. We have used the best known analytic formula that describes the radiation losses of an EMRI orbit in a Kerr metric [10], suitably adjusted to allow for the difference of the quadrupole moments between the two metrics. Since the radiation
Figure 2. The surface of sections at \( z = 0 \) of the inner region on the \( \rho, \dot{\rho} \) plane for the parameter set \( E = 0.95, \ L_z = 3M, \ \chi = 0.9, \ q = 0.95 \). The corresponding phase space is dominated by a chaotic sea, while there is also present a region of periodic orbits, and a very thin chain of resonant islands (thin dark regions).

Figure 3. These two diagrams show how the ratio \( \Omega_{\rho}/\Omega_z \) varies with time while the phase orbit crosses the resonance of 2/3 for two slightly different initial conditions. Both cases correspond to a MN metric with \( \chi = 0.9, \ q = 0.95 \), and orbital parameters \( E = 0.95, \ L_z = 3M \), while the ratio of masses is assumed to be \( \mu/M = 8 \times 10^{-5} \).
losses of $E$ and $L_z$ are expressed as functions of the orbital parameters, we have used such initial conditions for an orbit that starts very close to a particular resonance. From the corresponding geodesic orbit we computed its orbital parameters and then the corresponding losses. Then we assumed a linear evolution of $E(t)$ and $L_z(t)$ and we evolved the orbit accordingly. Depending on the location of the entrance point of the phase orbit in a Birkhoff island the subsequent evolution of the orbit was either a longer or a shorter passage of the orbit through that resonant island. In any case the two fundamental frequencies of the orbit remained locked to each other at a constant ratio while they both evolved. Thus a plateau in the evolution of the ratio of frequencies was present in the Fourier snapshots of the signal. The frequencies of the corresponding gravitational waves should have then analogous behavior.

This analysis is useful in order to check if the background of a detected EMRI signal is that of a Kerr or not. Although it seems quite difficult technically to incorporate a plateau effect in the bank of templates used to detect and analyze a gravitational wave signal, since this would increase dramatically the parameter space of templates, we could simply focus our attention at that part of the signal that its frequencies are close to a ratio of simple integers and look for a possible freezing of the ratio of frequencies while they evolve.

In Fig. 3 we have plotted the evolution of the ratio of the two frequencies $\Omega_{\rho}/\Omega_{z}$ as a function of time for two slightly different initial conditions. The two cases spend different time intervals inside the resonance of $2/3$; thus the corresponding plateaus differ from each other. For a typical EMRI this time interval is of the order of a few hours to many days.

Acknowledgments

T. A. acknowledges the research funding program “Kapodistrias” of ELKE (Grant No 70/4/7672) and the I.K.Y. (IKYDA 2010). G. L.-G. was supported by the Research Committee of the Academy of Athens.

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