Parametrized Nash Equilibria in Atomic Splittable Congestion Games via Weighted Block Laplacians*

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Abstract. We consider atomic splittable congestion games with affine cost functions and develop an algorithm that computes all Nash equilibria of the game parametrized by the players’ demands. That is, given a game where the players’ demand rates are piece-wise linear functions of some parameter $\lambda \geq 0$, we compute a family of multi-commodity flows $x(\lambda)$ parametrized in $\lambda$ such that for all $\lambda \geq 0$ the flow $x(\lambda)$ is a Nash equilibrium for the corresponding demand rate vector $r(\lambda)$. Our algorithm is based on a novel weighted block Laplacian matrix concept for atomic splittable games. We show that the weighted block Laplacians have similar properties as ordinary weighted graph Laplacians which allows to compute the parametrized Nash equilibria by matrix pivot operations. Our algorithm is output-polynomial on all instances, and each pivot step needs only $O((nk)^{2.4})$ where $k$ is the number of players and $n$ is the number of vertices.

Keywords: Equilibrium Computation · Congestion Game · Laplacian.

1 Introduction

Congestion games are a central topic in algorithmic game theory with applications in traffic, telecommunication, and logistics. In Wardrop’s basic model [27], we are given a network with flow-dependent cost functions and a set of commodities, each specified by a source node, a target node and a fixed flow demand. In this setting, a multi-commodity flow is a Wardrop equilibrium if for each commodity all paths that carry flow have the same costs, and all unused paths do not have smaller costs. Wardrop equilibria naturally model situations where each commodity consists of a continuum of infinitesimal small players that are interested in minimizing their travel costs, e.g., travel times in the context of traffic networks, latencies in the context of telecommunication networks, and monetary costs in the context of logistics networks.

With the rise of navigation systems such as Waze and TomTom and ride sharing platforms such as Lyft and Uber, and in view of the anticipated market

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penetration of autonomous cars, it is sensible to assume that in the near future several competing companies will control significant portions of the road traffic. For these future traffic scenarios Wardrop equilibria are not sufficient models for road traffic anymore since operators of fleets of cars may be interested in minimizing the overall performance of their fleet and may, thus, be willing to sacrifice the travel time of some of the traffic controlled by them in order to improve the overall performance of their fleet, see also Catoni and Pallottino [5]. These effects, however, can be studied within the class of atomic splittable congestion games. Here, each player is associated with a source node, a target node and a fixed demand rate. A strategy of the player is to distribute her demand on the paths from her source to her target. The player is interested in minimizing the overall travel time of the flow controlled by her. As shown by Haurie and Marcotte [13], ordinary non-atomic games can be obtained as the limit of a series of atomic splittable games; in that sense atomic splittable games generalize the class of non-atomic games.

While the existence of equilibria in atomic splittable games can be established by standard fixed point arguments, much less is known regarding the computation of equilibria in atomic splittable congestion games. For affine cost functions, Cominetti et al. [6] showed that an equilibrium can be found by computing the minimum of a convex potential function, see also Huang [16] for a combinatorial algorithm for special graph topologies. Bhaskar and Lolakapuri [4] proposed two algorithms with exponential worst-case complexity that compute $\varepsilon$-approximate Nash equilibria in singleton games with convex costs. Harks and Timmermans [10] developed a polynomial time algorithm that computes an equilibrium in singleton games with player-specific affine cost functions.

All these approaches above yield a single equilibrium for a fixed vector of player demands. Moreover, the algorithms of Bhaskar and Lolakapuri [4] and Harks and Timmermans [10] work only for singleton games played on a network with two nodes. In actual traffic scenarios, the assumption that the players’ demand vector is fully known and fixed is unrealistic since demands often fluctuate. In this paper, we are interested in understanding how the equilibria in atomic splittable games change as a function of the players’ demand vectors.

1.1 Our results and techniques

We propose an algorithm that computes all equilibria of an atomic splittable congestion game as a function of the players’ demands. We assume that all cost functions are affine. More formally, consider an atomic splittable congestion game played on a graph $G = (V, E)$ with cost functions $l_e(x) = a_e x + b_e$. Further, let $r : \mathbb{R}_+ \to \mathbb{R}_+^k$ be a piece-wise linear function assigning a demand rate vector $r(\lambda)$ to each value $\lambda \geq 0$. Then, we compute functions $x : \mathbb{R}_+ \to \mathbb{R}^{mk}$ such that for all $\lambda \geq 0$ the multi-commodity flow $x(\lambda) = (x_e^*(\lambda))_{i \in [k], e \in E}$ is a pure Nash equilibrium of the atomic splittable congestion game with demand rate vector $r(\lambda)$.

Our algorithm is based on the observation that for given strategies of the other players, each player plays a best-reply. Using the well-known correspon-
dence between optimal flows and Wardrop equilibria (cf. Beckmann et al. [1]) we can view the equilibrium strategies of each player as a Wardrop equilibrium in a game where costs are replaced by marginal costs. Wardrop equilibria, in turn, have a close relation to electrical flows and, thus, graph Laplacians. Using this approach, we can reformulate the Nash equilibrium conditions as a system of linear equations of the form \( y = L \pi - b \), where \( y \in \mathbb{R}^{nk} \) is a multi-commodity excess vector containing the excess \( y_{iv}^e \) for each player \( i \) and each vertex \( v \), \( \pi \in \mathbb{R}^{nk} \) is a multi-commodity potential vector containing a (shortest-path) potential \( \pi_{iv}^e \) for each player \( i \) and each vertex \( v \), and \( b \) is a vector of offsets. The matrix \( L \in \mathbb{R}^{nk \times nk} \) is a novel structure that we call the total Laplacian of the graph. Roughly speaking, it has the same structure as a graph Laplacian except that all entries are graph Laplacians rather than scalars. We show that the total Laplacian shares many properties with regular Laplacians, i.e., it is symmetric, positive semi-definite, and has a very easy nullspace.

Using these properties, we can show that the parametrized Nash equilibrium problem can be solved by pivot operations on a generalized inverse \( L^+ \) of \( L \). In the non-degenerate case, the overall time complexity of the algorithm is \( \mathcal{O}((k n)^{2.4} + K (k n)^2) \) where \( K \) is the number of breakpoints of the output function. In case of degeneracy, we can replace the pivoting operation by solving a quadratic program, so that also in this case we have an output-polynomial algorithm.

### 1.2 Further related work

Haurie and Marcotte [13] showed that non-atomic congestion games can be obtained as a sequence of atomic splittable congestion games. Existence of pure Nash equilibria in atomic splittable congestion games follows from standard fixed point arguments (cf. Kakutani [17] and Rosen [24]). Bhaskar et al. [3] showed that the equilibria are not necessarily unique and gave several topological conditions on the network for the uniqueness of equilibria. Richman and Shimkin [23] characterized two-terminal network topologies that are necessary and sufficient for uniqueness. Harks and Timmermans [11] characterized the uniqueness of equilibria in terms of the combinatorial structure of the strategy set. The price of anarchy of atomic splittable congestion games price of anarchy has been studied by Cominetti et al. [6]. Harks [9], and Roughgarden and Schoppmann [25]. Catoni and Pallottino [5] provide a paradox of a non-atomic game where replacing the non-atomic players of one commodity by an atomic player with the same demand decreases the overall performance of that commodity. In previous work [20], we developed an algorithm that computes all Wardrop equilibria parametrized by the flow demand. In this work, we generalize this approach towards atomic splittable games. From a mathematical point of view, our algorithm is a homotopy method, for further homotopy methods for computing equilibria, see [8,14,15,18,22].
2 Preliminaries

An atomic splittable routing game is a tuple \((G, K, l)\) where \(G = (V, E)\) is a directed graph with \(n\) vertices \(V\) and \(m\) edges \(E\), the family \(K = \{(s_1, t_1, r_1), \ldots, (s_k, t_k, r_k)\}\) contains \(k\) triples each of which consisting of a source node \(s_i \in V\), a sink node \(t_i \in V\), and a demand rate \(r_i\) for each of the \(k\) players, and \(l\) is a family of strictly increasing affine linear latency functions \(l = (l_e)_{e \in E}\) with \(l_e(x) = a_e x + b_e\) for some \(a_e > 0\) and \(b_e \geq 0\).

A feasible strategy for every player \(i \in [k]\) is to route her demand \(r_i\) between her terminal vertices \(s_i\) and \(t_i\). Thus, a strategy for player \(i\) is a \(s_i\text{-}t_i\text{-}flow\) of rate \(r_i\), i.e., a non-negative vector \(x^i = (x^i_e)_{e \in E}\) in \(\mathbb{R}^m\) satisfying

\[
\sum_{e \in \delta^+(v)} x^i_e - \sum_{e \in \delta^-(v)} x^i_e = \begin{cases} 
  r_i & \text{if } v = s_i, \\
  -r_i & \text{if } v = t_i, \\
  0 & \text{otherwise}
\end{cases}
\] (1)

for every vertex \(v \in V\). A strategy profile for all players is a vector \(x = ((x^1)^\top, \ldots, (x^k)^\top)^\top \in \mathbb{R}^{km}\) containing all flows stacked. We use the notation \((\hat{x}^i, \hat{x}^{-i})\) for the strategy profile where player \(i\) uses the flow \(\hat{x}^i\) and all other players use their flow as in the strategy profile \(x\). The latency \(l_e\) experienced by the flow of the players on some edge \(e\) depends on the total flow \(\hat{x} = (\hat{x}_e)_{e \in E}\) where \(\hat{x}_e = \sum_{i=1}^k x^i_e\). The cost paid by every player is the total latency \(C^i(x)\) experienced by the flow sent by this player, i.e., \(C^i(x) = \sum_{e \in E} x^i_e l_e(\hat{x}_e)\). We say that \(x\) is a Nash equilibrium if for every player \(i \in [k]\) there is no profitable deviation from \(x\), i.e., \(C^i(x) \leq C^i(\hat{x}^i, \hat{x}^{-i})\) for all \(s_i\text{-}t_i\text{-}flows\ \hat{x}^i\) of rate \(r_i\). We define the marginal cost of player \(i\) on edge \(e\) given the flow \(x\) by

\[
L^i_e(x) := \frac{\partial}{\partial x^i_e} x^i_e l_e(\hat{x}_e) = l_e(\hat{x}_e) + x^i_e l'_e(\hat{x}_e) = a_e \hat{x}_e + b_e + a_e x^i_e.
\]

We then obtain the following characterization of Nash equilibria, see, e.g., Bhaskar et al. [3] for a reference.

**Lemma 1.** The strategy profile \(x\) is a Nash equilibrium flow if and only if, for every player \(i\), \(x^i\) is a \(s_i\text{-}t_i\text{-}flow\) and

\[
\sum_{e \in P} L^i_e(x) \leq \sum_{e \in Q} L^i_e(x)
\] (2)

for all \(s_i\text{-}t_i\text{-}paths\ \(P, Q\) with \(x^i_e > 0\) for all \(e \in P\).

Lemma 1 states that \(x\) is a Nash equilibrium if and only if all path used by player \(i\) are also shortest path for that player with respect to the marginal costs. This enables us to give another characterization based on (shortest path) potentials.
Lemma 2. The flow $x$ is a Nash equilibrium if and only if, for every player $i$ there is a potential vector $\pi^i = (\pi^i_v)_{v \in V}$ such that

$$
\begin{align}
\pi^i_w - \pi^i_v &= L^i_e(x) \quad &\text{if } x^i_e > 0 \\
\pi^i_w - \pi^i_v &\leq L^i_e(x) \quad &\text{if } x^i_e = 0
\end{align}$$

for every edge $e = (v, w) \in E$.

As for the whole flow vector $x$ we will denote by $\pi$ the vector of all stacked potentials $\pi^i$.

We say an edge $e$ is active for some player $i$ if $\pi^i_w - \pi^i_v \leq L^i_e(x)$ is satisfied with equality. (In particular all edges used by a player (i.e. $x^i_e > 0$) are active.) Further, we define the sets $S_{e} = \{ i \in [k] \mid \pi^i_w - \pi^i_v = L^i_e(x) \}$ of players for which $e$ is active. We call the family $\mathcal{S} := (S_e)_{e \in E}$ of these sets supports. When the supports $\mathcal{S}$ are known, computing a Nash equilibrium reduces to finding a feasible flow $x$ and some potential vector $\pi$ satisfying $\pi^i_w - \pi^i_v = L^i_e(x)$ for all $i \in S_e$ and all $e \in E$. In order to find an equilibrium, we have to solve a system of equations consisting of flow conservation and potential equalities. In the special case of affine-linear cost functions these equations are linear equations and can be solved explicitly as we will show in more detail in the next section.

3 Weighted Block Laplacians

Let us fix a support $\mathcal{S}$. In this section, we work towards expressing the multi-commodity excess vector $y$ in a Nash equilibrium for support $\mathcal{S}$ as a linear system of the form $y = \mathbf{L} \pi + \mathbf{b}$, and we will derive useful properties of the matrix $\mathbf{L}$. To this end, for two players $i, j \in [k]$ and an edge $e \in E$, let

$$
\omega^i_{e} := \begin{cases} 
1 & \text{if } i \in S_e \text{ and } j \in S_e, \\
0 & \text{otherwise.}
\end{cases}
$$

We also write $\omega^i_{e}$ as a shorthand for $\omega^i_{e}$. Further, let $\kappa_e := |S_e|$ be the number of players using the edge $e$ in $\mathcal{S}$. We introduce the diagonal $m \times m$ matrix

$$
C^{ij} = \begin{cases} 
\text{diag}\left( \frac{\kappa_e}{(\kappa_e + 1)a_1} \omega_{e1}^{ij}, \ldots, \frac{\kappa_e}{(\kappa_e + 1)a_m} \omega_{en}^{ij} \right) & \text{if } i = j \\
\text{diag}\left( \frac{1}{(\kappa_e + 1)a_1} \omega_{e1}^{ij}, \ldots, \frac{1}{(\kappa_e + 1)a_m} \omega_{en}^{ij} \right) & \text{otherwise}
\end{cases}
$$

as well as the vector $\mathbf{b}^i = (\frac{1}{(\kappa_e + 1)a_1} \omega_{e1}^{i1}, \ldots, \frac{1}{(\kappa_e + 1)a_m} \omega_{en}^{i1} b_{e1}, \ldots, \frac{1}{(\kappa_e + 1)a_m} \omega_{en}^{i1} b_{en})^T$. With these definitions, we obtain the following linear equation relating Nash equilibria to their potential vectors.

Lemma 3. Let $x$ be a Nash equilibrium and $\pi$ be a corresponding potential. Then,

$$
x^i = C^{ij} \Gamma^T \pi^i - \sum_{j \neq i} C^{ij} \Gamma^T \pi^j - \mathbf{b}^i,
$$

where $\Gamma$ is the vertex-edge incidence matrix.
Proof. Equations (5a) and (5b) imply that, for every player \( i \) and every edge \( e \), either \( x^i_e = 0 \) or \( \pi^i_w - \pi^i_v = a_e \bar{x}_e + b_e + a_e x^i_e \). The latter is equivalent to
\[
x^i_e = \frac{\pi^i_w - \pi^i_v - (a_e \bar{x}_e + b_e)}{a_e}.
\]

Summing up all potential differences of players using an edge \( e \), we obtain
\[
\sum_{j \in S_e} \pi^j_w - \pi^j_v = \sum_{j \in S_e} (a_e \bar{x}_e + b_e + a_e x^j_e) = (\kappa_e + 1) a_e \bar{x}_e + \kappa_e b_e
\]
and, hence,
\[
\bar{x}_e = \frac{\sum_{j \in S_e} \pi^j_w - \pi^j_v}{(\kappa_e + 1)a_e} - \frac{\kappa_e}{(\kappa_e + 1)a_e} b_e.
\]

With (6) we finally obtain from (5) the equation
\[
x^i_e = \frac{\kappa_e}{(\kappa_e + 1)a_e} (\pi^i_w - \pi^i_v) - \sum_{j \in S_e, j \neq i} \frac{1}{(\kappa_e + 1)a_e} (\pi^j_w - \pi^j_v) - \frac{1}{(\kappa_e + 1)a_e} b_e
\]
\[
= \frac{\kappa_e}{(\kappa_e + 1)a_e} \omega^i_e (\pi^i_w - \pi^i_v) - \sum_{j \neq i} \frac{1}{(\kappa_e + 1)a_e} \omega^i_e (\pi^j_w - \pi^j_v) - \frac{1}{(\kappa_e + 1)a_e} \omega^i_e b_e.
\]

Let \( \Gamma \) be the vertex-edge incidence matrix of \( G \). Then \( \Gamma^\top \pi^i \) is a vector containing all potential differences for every edge. We then rewrite (5) in vector form
\[
x^i = C^{ij} \Gamma^\top \pi^i - \sum_{j \neq i} C^{ij} \Gamma^\top \pi^j - \tilde{b}^i,
\]

as claimed. \( \square \)

The vector \( y^i := \Gamma x^i \) contains the excess, i.e., the difference of out- and in-flow for every vertex. In order \( x^i \) to be a feasible flow, we need \( y^i_i = r_i \), \( y^i_{i'} = -r_i \), and \( y^i_{i} = 0 \) otherwise. Consider the weighted Laplacian matrices of \( G \) defined as \( L^{ij} := \Gamma C^{ij} \Gamma^\top \) for every pair of players player \( i, j \in [k] \). The matrix is a weighted Laplacian matrices of the graph \( G \) with edge weights \( \frac{\kappa_e}{(\kappa_e + 1)a_e} \omega^i_e \) if \( i = j \) or edge weight \( \frac{1}{(\kappa_e + 1)a_e} \omega^i_e \) otherwise. Since \( \omega_e^{ij} = \omega_e^{ji} \), we also have \( L^{ij} = L^{ji} \) for all \( i, j \in [k] \). Let \( y = ((y^1)^\top, \ldots, (y^k)^\top)^\top \) be the stacked excess vector. Then we can express the relation between the stacked potential vector \( \pi = ((\pi^1)^\top, \ldots, (\pi^k)^\top)^\top \) and \( y \) in the following convenient way
\[
\begin{pmatrix}
  y^1 \\
  y^2 \\
  \vdots \\
  y^k \\
\end{pmatrix}
= \begin{pmatrix}
  L^{11} & -L^{12} & \cdots & -L^{1k} \\
  -L^{21} & L^{22} & \cdots & -L^{2k} \\
  \vdots & \vdots & \ddots & \vdots \\
  -L^{k1} & -L^{k2} & \cdots & L^{kk} \\
\end{pmatrix}
\begin{pmatrix}
  \pi^1 \\
  \pi^2 \\
  \vdots \\
  \pi^k \\
\end{pmatrix}
- \begin{pmatrix}
  b^1 \\
  b^2 \\
  \vdots \\
  b^k \\
\end{pmatrix}.
\]
We call the block matrix $L \in \mathbb{R}^{kn \times kn}$ the total Laplacian matrix. It is structured as a normal Laplacian matrix, but it contains the “sub-Laplacians” $L^{ij}$ rather than real entries. We proceed to show that it has similar properties as a usual Laplacian matrix with scalar entries.

**Lemma 4.** The total Laplacian $L$ is symmetric and positive semi-definite.

**Proof.** Since all sub-Laplacian matrices are symmetric, $L$ is symmetric by definition.

Let $C \in \mathbb{R}^{km \times km}$ be the block matrix containing all diagonal matrices $C^{ij}$, i.e.

$$
C = \begin{pmatrix}
  C_{11} & -C_{12} & \cdots & -C_{1k} \\
  -C_{21} & C_{22} & \cdots & -C_{2k} \\
  \vdots & \vdots & \ddots & \vdots \\
  -C_{k1} & -C_{k2} & \cdots & C_{kk}
\end{pmatrix}.
$$

Every row of $C$ contains exactly $k$ (possibly) non-zero elements, namely the value $\frac{\kappa_e}{(\kappa_e+1)a_e} \omega_e^{ii}$ as diagonal element and $-\frac{1}{(\kappa_e+1)a_e} \omega_e^{ij}$ for all $i \neq j \in [k]$ as off-diagonal elements for fixed $e \in E$ and $i \in [k]$. If, for this fixed $e$ and $i$, we have $\omega_e^{ii} = 0$, then the row contains only zeros. Otherwise, the difference between diagonal element and absolute of the off-diagonal elements is

$$
\frac{\kappa_e}{(\kappa_e+1)a_e} \omega_e^{ii} - \sum_{j \neq i} \frac{1}{(\kappa_e+1)a_e} \omega_e^{ij} \geq -\frac{\kappa_e - 1}{(\kappa_e+1)a_e} > 0,
$$

where for the first inequality we used that $\omega_e^{ij} \leq 1$. Thus $C$ is (weakly) diagonally dominant, symmetric and real-valued implying that $C$ is positive semi-definite.

Further, let $G$ be a $kn \times km$-block diagonal matrix containing $k$ times the matrix $\Gamma$ as block diagonal entries. Then $L = GC\Gamma^T$ and, thus, $L$ is also positive semi-definite.

As we are interested in solving the system of linear equations (9), we want to invert the matrix $L$. Thus, we are interested in the rank of $L$.

**Theorem 1.** Let $L$ be the total Laplacian matrix. Then for every vector $x \in \mathbb{R}^{kn}$ containing the stacked vectors $x^i \in \mathbb{R}^n$ for $i \in [k]$, we have

$$
Lx = 0 \iff L^{ii}x^i = 0 \text{ for all } i \in [k].
$$

**Proof.** We can use Theorem 4 from Appendix A. We only have to ensure that $L$ satisfies Assumption 1. The parts (i) and (ii) are satisfied by definition and the properties of Laplacians. For part (iii) observe that, for every $i \in [k]$, we have $C^{ii} - \sum_{i \neq j} C^{ij} = \frac{\kappa_e}{\kappa_e+1} C^{ii}$. This yields

$$
\langle x, (L^{ii} - \sum_{i \neq j} L^{ij})x \rangle = \langle \Gamma^T x, (C^{ii} - \sum_{i \neq j} C^{ij}) \Gamma^T x \rangle
\geq \frac{1}{\kappa_e} \langle \Gamma^T x, C^{ii} \Gamma^T x \rangle \geq \frac{1}{k} \langle \Gamma^T x, C^{ii} \Gamma^T x \rangle = \frac{1}{k} \langle x, L^{ii}x \rangle.
$$
and thus we can apply Theorem 5.

Theorem 1 shows that the nullspace (and thus the rank) of the total Laplacian $L$ depends solely on the nullspace of the sub-Laplacians on the diagonal.

Assuming that for every player $i$, the graph $G$ contains a single connected component with respect to $\omega^{ii}$ (this means, we consider two vertices to be connected, if there is a path between these vertices using only edges $e$ with $\omega^{ii}_e = 1$) then by the properties the standard Laplacian $L^{ii}$, we know that $\text{rank}(L^{ii}) = n - 1$ and the nullspace of $L^{ii}$ is the linear hull of the all-one vector $1$. We obtain the following direct corollary.

**Corollary 1.** If, for every player $i$, there is only one connected component with respect to $\omega^{ii}$, then

(i) $\text{rank}(L) = k(n - 1)$.
(ii) the nullspace of $L$ contains only vectors that contain stacked multiples of the all-one vector $1$.
(iii) the solution $\pi$ of (9) is unique up to an additive constant for every player potential $\pi^i$.
(iv) given the support $S$, the Nash equilibrium flows $x^i$ are unique.

The last statement follows from the fact that the flows depend only on the potential difference as it can be seen in equation (4) and thus the additive constant of the potential $\pi^i$ is irrelevant. Note that this only proves the uniqueness if the support $S$ is given, not the uniqueness of the Nash equilibrium.

## 4 Computing parametrized Nash equilibria

### 4.1 Potential directions

Recall from equation (9), that for a fixed support $S$, we have $y = L \pi - b$ where $y$ is the stacked excess vector, $\pi$ is the stacked potential vector and $L$ is the total Laplacian for $S$. We are interested how the equilibria change as the excess vector change, i.e., we want to solve the equation for $\pi$. As shown in Corollary 1 the total Laplacian matrix $L$ is singular. For the purpose of a compact notation, we will use generalized inverses—for any matrix $A$ we denote by $A^+$ a *generalized inverse* of $A$, that is a matrix $A^+$ that satisfies $AA^+A = A$. Every matrix has at least one generalized inverse and, for every vector $b$, $x = A^+b$ is a solution of the system $Ax = b$ if the system is consistent, see also [2] for a reference on generalized inverses.

We then obtain $\pi = L^+(y + b)$. The solution $\pi$ may depends on the choice of the generalized inverse $L^+$, but by Corollary 1 we know that the solution has to be unique up to additive constants for every player potential. Changing the demands of the players changes the excess vector by $\Delta y$ which results in a change of the potentials by $\Delta \pi = L^+ \Delta y$. We call this vector $\Delta \pi$ the *potential direction*. Moving along this direction in the space of all potentials yields new equilibrium potentials for demands $y + \lambda \Delta y$ as long as the support $S$ does not change.
4.2 Potential space

The equilibrium solution changes only linearly when the excess $y$ is changed linearly as long as the support $S$ does not change. Therefore, we are interested in the sets of (possible) equilibrium solutions (i.e. potential vectors $\pi$) that induce the same support.

We consider the potential space $\mathbb{R}^{kn}$ of all possible potential vectors $\pi$. Given a support $S$, we define for every edge $e$ and player $i$ the vectors

$$ w_{e,i} = \left( -\omega_e^1 \gamma_e^T, \ldots, -\omega_e^{i-1} \gamma_e^T, \kappa_e \gamma_e, -\omega_e^{i+1} \gamma_e^T, \ldots, -\omega_e^k \gamma_e^T \right)^T \in \mathbb{R}^{kn} \quad (11) $$

where $\gamma_e$ is the column of the vertex-edge incidence matrix corresponding to the edge $e$. This vector enables us to simplify the formula for the player flow on active edges to $0 \leq x_e^i = \left( \kappa_e + 1 \right) a_e \left( w_{e,i}^T \pi - b_e \right)$. Since every equilibrium potential is a shortest path potential with respect to the marginal cost, we know that for every inactive edge we have

$$ 0 \geq \pi_{i,e}^T a_e \bar{x}_e - b_e = w_{e,i}^T \pi - b_e. \quad (12) $$

Thus, given a potential $\pi$ of an equilibrium flow and the corresponding support $S$, we have

$$ w_{e,i}^T \pi - b_e \geq 0 \quad \text{if } i \in S_e, \quad w_{e,i}^T \pi - b_e \leq 0 \quad \text{if } i \notin S_e. \quad (13) $$

Given any potential vector $\pi \in \mathbb{R}^{kn}$ and any support $S$, we say $\pi$ is consistent with $S$ if (13) is satisfied. We define the potential region corresponding to the support $S$ as the set

$$ R_S := \{ \pi \in \mathbb{R}^{kn} \mid \pi \text{ is consistent with } S \}. \quad (14) $$

Every region $R_S$ is a convex subset of the potential space and bounded by the hyperplanes with normal vectors $w_{e,i}$ and the offsets $b_e$. We also call these hyperplanes boundaries induced by an edge $e$ and a player $i$. See Figure 1 for a graphical representation of a region in the potential space.

By definition, all potentials in some region are consistent with the same support $S$. Thus, given some excess direction $\Delta y$, the potential direction as defined in the previous subsection is constant in the whole region.

4.3 The main algorithm

We want to develop an algorithm that, starting from a given equilibrium for demand vector $r$, computes all equilibria for demands $r + \lambda \Delta r, \lambda \geq 0$ where $\Delta r$ is some vector specifying a direction in the demand space. For a piece-wise linear function $\Delta r := \nabla r(\lambda)$ is piece-wise constant in $\lambda$. These demand vectors naturally induce corresponding vectors $y$ and $\Delta y$ in the excess space where $\Delta y$ is piece-wise constant as well. For ease of exposition, in the following, we describe only the basic version of the algorithm with constant $\Delta y$. The general case can
Fig. 1. A region in the space of vertex potentials with potential direction $\Delta \pi$. The solution curve (red line) hits the boundary induced by edge $e^*$ and player $i^*$ in the point $\pi^1$.

be easily obtained by concatenating different runs of this algorithm where the final solution of the last run serves as a starting solution of the next.

In order to compute all equilibrium potentials belonging to these excess vectors (or demand vectors, respectively), we construct a homotopy method inspired by a similar method introduced by Katzenelson [18] that computes electrical flows. The algorithm computes a piecewise linear function $\pi(\lambda)$ mapping $\lambda \geq 0$ to a potential vector belonging to an equilibrium flow for demand $r + \lambda \Delta r$ and a piecewise constant function $S$ that returns the associated supports. We call the parametrized curves $\lambda \mapsto \pi(\lambda)$ and $\lambda \mapsto S(\lambda)$ solution curve (in the potential space) and support curve, respectively. Note that the solution curve and the support curve are sufficient to reconstruct the equilibrium flows for all $\lambda \geq 0$.

The basic procedure can be described as follows.

1. Start with an equilibrium $x$ for the excess $y$ with support $S$ and with potentials $\pi(0) := \pi \in \mathbb{R}^S$ for the initial support $S(0) := S$.
2. For fixed $S$, compute the potential direction $\Delta \pi = L^+ \Delta y$.
3. Compute the maximal $\epsilon$ such that $\pi + \epsilon \Delta \pi \in \mathbb{R}^S$ for all $\lambda \leq \epsilon$. Find a new support $S'$ such that $\pi + \epsilon \Delta \pi \in R_{S'}$. Continue with 2. in the region $R_{S'}$.

Given a potential $\pi$ that is consistent with a given support $S$ (i.e. $\pi \in R_S$), we define the values

$$
\epsilon^i_e := \begin{cases} 
\frac{b_e - w_{e,i}^\top \pi}{w_{e,i}^\top \Delta \pi} & \text{if } i \notin S_e \text{ and } w_{e,i}^\top \Delta \pi > 0, \\
\frac{b_e - w_{e,i}^\top \pi}{w_{e,i}^\top \Delta \pi} & \text{if } i \in S_e \text{ and } w_{e,i}^\top \Delta \pi < 0, \\
\infty & \text{otherwise}
\end{cases}
$$

(15)

for every edge $e \in E$ and every player $i \in [k]$ and $\epsilon := \min_{e \in E, i \in [k]} \epsilon^i_e$. Then the potential $\pi + \lambda \Delta \pi$ is consistent with $S$ if and only if $\lambda \leq \epsilon$ and in the potential $\pi + \epsilon \Delta \pi$ at least one boundary is hit (see also Figure 1).

1 If $r(0) = 0$ one may start with an appropriate shortest path potential $\pi$ and appropriate supports. In general, a starting solution can be found with the convex program (22).
In order to proceed for \( \lambda \geq \epsilon \), there are two possible cases. Either there is a unique minimal \( \epsilon_e^* \) (i.e. the solution curve hits a single boundary) or there are multiple \( \epsilon_e^* \) inducing the minimal \( \epsilon \). We refer to the former as unique boundary crossing and to the latter as degenerate boundary crossing. Likewise, we call points in the potential space where more than one boundary hyperplane intersect degenerate points.

### 4.4 Unique boundary crossing

We introduce the following abuse of notation: We write \( T = S + (e, i) \) if \( i \notin S_e \), \( T_e' = S_e' \) for all edges \( e' \neq e \), and \( T_e = S_e \cup \{i\} \). Likewise, we write \( T = S - (e, i) \) if \( i \in S_e \), \( T_e' = S_e' \) for all edges \( e' \neq e \), and \( T_e = S_e \setminus \{i\} \). Thus, the support \( T = S + (e, i) \) \((T = S - (e, i))\) is the same support as \( S \) except that the status of edge \( e \) is changed from inactive to active (active to inactive) for player \( i \).

Assume the solution curve hits a unique boundary, i.e., there is a unique minimal \( \epsilon_e^* \) (i.e. the solution curve hits a single boundary) or there are multiple \( \epsilon_e^* \) with \( \epsilon_e^* \) is minimal. We want analyze what happens if we change the activity status of this edge for this player (either from active to inactive or vice versa). The following theorem gives us a relation between the inverses of the associated total Laplacian matrices. For ease of notation, denote by \( L_S \) the total Laplacian matrix that is induced by the support \( S \).

**Theorem 2.** Let \( S_1, S_2 \) be two supports with \( S_2 = S_1 \pm (e, i) \) for some edge \( e \) and player \( i \). Let \( \alpha := \pm \kappa_e (\kappa_e + 1) a_e \) where \( \kappa_e \) is the number of players using edge \( e \) including player \( i \). Further, let \( w := w_{e,i} \). Then, 

(i) the inverses of the total Laplacians satisfy 

\[
    L^+_{S_2} = L^+_{S_1} - \frac{1}{\alpha + w^T L^+_S w} L^+_{S_1} w w^T L^+_{S_1}.
\] (16)

(ii) the directions \( \Delta \pi^{(j)} := L^+_{S_1} \Delta y, j = 1, 2 \) satisfy 

\[
    \text{sgn} (w^T \Delta \pi^{(1)}) = \text{sgn} (w^T \Delta \pi^{(2)}).
\] (17)

**Proof.** We assume without loss of generality that \( i = 1 \). Further, we assume that \( e \) was inactive in \( S^{(1)} \) and is active in \( S^{(2)} \) for player \( i = 1 \). (For the other direction we just have to change signs.) Let \( \kappa_e \) be the number of players using \( e \) including player \( i \), i.e. \( \kappa_e := |S^{(2)}_e| \). Then we can express the sub-Laplacians for \( S^{(2)} \) as 

\[
    L^{1,1}_{S_2} = L^{1,1}_{S_1} + \frac{\kappa_e}{(\kappa_e + 1) a_e} \gamma_e \gamma_e^T,
\]

\[
    L^{1,j}_{S_2} = L^{1,j}_{S_1} + \frac{1}{(\kappa_e + 1) a_e} \gamma_e \gamma_e^T, \quad j \neq 1,
\]

\[
    L^{j,1}_{S_2} = L^{j,1}_{S_1} + \frac{1}{\kappa_e (\kappa_e + 1) a_e} \gamma_e \gamma_e^T, \quad j \neq 1,
\]

\[
    L^{i,j}_{S_2} = L^{i,j}_{S_1} - \frac{1}{(\kappa_e + 1) a_e} \gamma_e \gamma_e^T \quad i \neq j, i \neq 1, j \neq 1.
\]
With the constant $\alpha := \kappa_e (\kappa_e + 1) a_e$ and the vector $w := w_{e,1}$ as defined in (11) we can express the total Laplacian matrix for $S^{(2)}$ as

$$L_{S_2} = L_{S_1} + \frac{1}{\alpha} w w^\top.$$ 

We now claim that

$$\alpha + w^\top L_{S_1}^+ w > 0 \tag{18}$$

holds true independently of the choice of the generalized inverse $L_{S_1}^+$. Let $\hat{L}_{S_1}$ be the matrix obtained from $L_{S_1}$ by deleting the $k$ rows and columns belonging to the first vertex for every player. Then this matrix is non-singular (by Corollary 1) and strictly positive definite. Let $\hat{w}$ be the vector $w$ without the rows belonging to the first vertex for every player. Then it is easy to show that

$$\alpha + \hat{w}^\top \hat{L}_{S_1}^{-1} \hat{w} = \frac{\det(\hat{L}_{S_2})}{\det(\hat{L}_{S_1})} > 0.$$ 

Finally, we observe that the matrix $L_{S_1}^+$ defined as the matrix $\hat{L}_{S_1}^{-1}$ with additional zero rows and columns for the first vertex for every player is a generalized inverse of $L_{S_1}$. This together with the aforementioned fact that $L^+ w$ is unique up to additive constants for every player implies that $w^\top L_{S_1}^+ w = \hat{w}^\top \hat{L}_{S_1}^{-1} \hat{w} > 0$.

The claim (18) implies that $(\alpha + w^\top L_{S_1}^+ w)^+ = \frac{1}{\alpha + w^\top L_{S_1}^+ w}$. Thus, we obtain (10) with the Sherman-Morrison-Woodbury formula for generalized inverses (see, e.g., [12, Theorem 18.2.14]).

Using this identity, we get

$$w^\top \Delta \pi^{(2)} = w^\top (L_{S_1}^+ - \frac{1}{\alpha + w^\top L_{S_1}^+ w} L_{S_1}^+ w w^\top L_{S_1}^+) \Delta y = \underbrace{\frac{1}{\alpha + w^\top L_{S_1}^+ w} w^\top L_{S_1}^+ \Delta y}_{> 0} = \underbrace{\frac{1}{\alpha + w^\top L_{S_1}^+ w} w^\top \Delta \pi^{(1)}}_{> 0},$$

which proves the second statement. \qed

Theorem 2 shows that the generalized inverse of the Laplacian matrix after crossing the boundary can be obtained by a simple matrix multiplication. Further, the second statement ensures that the boundary crossing is well-defined in the following sense:

**Corollary 2.** Assume the solution curve hits a unique boundary induced by edge $e$ and player $i$ of the region $R_{S_1}$ in the point $\pi$. Let $S_2 := S_1 \pm (e, i)$. Then there is $\epsilon > 0$ such that

$$\pi + \lambda L_{S_2}^+ \Delta y \in R_{S_2} \tag{19}$$

for all $0 \leq \lambda < \epsilon$. 

Proof. Since $w_{e,i}^T \pi - b_e = 0$ by assumption, it is clear that $\pi \in R_{S_1}$ and $\pi \in R_{S_2}$. Thus, the statement holds for $\lambda = 0$. (This is just the formalization of the fact that $\pi$ lies on the boundary between $R_{S_1}$ and $R_{S_2}$.)

Now we need to show that we can move away from $\pi$ in the direction $L_{S_2}^+ \Delta y$ of the second region without directly leaving region $R_{S_2}$. Since we assumed that edge $e$ and player $i$ is the only pair with $w_{e,i}^T \pi - b_e = 0$ at $\pi$ we can move at least an $\epsilon$ step in the direction $L_{S_2}^+ \Delta y$ without violating any of the inequalities (13) for $e' \neq e$ or $i' \neq i$. Using (17) from Theorem 2 we know that if the direction $L_{S_1}^+ \Delta y$ points towards the boundary in $R_{S_1}$ the direction $L_{S_2}^+ \Delta y$ must be directed away from that boundary in region $R_{S_2}$. This implies we can move at least $\epsilon L_{S_2}^+ \Delta y$ for some positive $\epsilon$ without leaving $R_{S_2}$.

Corollary 2 states that when the solution curve hits a unique boundary, the solution can proceed in the adjacent region induced by the same support except for changed activity status of the edge $e$ for player $i$ that induced the boundary.

4.5 Degeneracy

We now want to consider the case when two or more boundaries are hit at the same time. Note that this situation can occur if multiple edges change their status for one or more players or the status of one edge changes simultaneously for multiple players. In any case, the main idea for this situation is to add a small perturbation $\delta$ to the potential $\pi$ in order to avoid the degenerate point. Observing the perturbed solution we can obtain the right support by finitely many unique boundary crossings. In fact, we will show that this can be done implicitly using a lexicographic rule.

Assume that, starting from a potential vector $\pi$, the next potential $\pi^* := \pi + \epsilon \Delta \pi$ is a degenerate point. Then, for some $\delta > 0$, we define the perturbation vector $\delta := (\delta^{nk}, \delta^{nk-1}, \ldots, \delta^2, \delta)$ and consider the solution curve starting from the potential $\pi^0 := \pi + \delta$. (We will refer to this curve as the perturbed solution curve.) For almost every $\delta > 0$ the vectors $\delta$ and $\Delta \pi$ are linearly independent.
and thus the perturbed solution curve hits the point  \( \tilde{\pi}^1 := \pi^0 + \epsilon^0 \Delta \pi \neq \pi^* \) where  \( \epsilon^0 = \min_{e \in E, i \in [k]} \rho^0_{e, i} \) is the minimal distance from  \( \pi \) to all boundaries. The perturbed solution curve will move on to further points  \( \tilde{\pi}^1, \tilde{\pi}^2, \ldots \) until it does not hit any further boundary intersecting in the degenerate point  \( \pi^* \). See Figure 2 for a visualization of the perturbation and the resulting perturbed solution curve.

Note that, by continuity, for sufficiently small  \( \delta > 0 \), the perturbed solution curve will only cross boundaries that intersect in the degenerate points rather than other boundaries before reaching the region where the solution curve moves away from the degenerate point. Thus, we will only consider boundaries that intersect in the degenerate point for the remainder of this subsection.

The next lemma shows that the potentials  \( \tilde{\pi}^l \) and the epsilon values  \( \tilde{\epsilon}_{e,i}^l \) can be expressed as a linear function depending on  \( \delta \).

Lemma 5. There are matrices  \( M^l \in \mathbb{R}^{kn \times kn} \) and vectors  \( m^l_{e,i} \in \mathbb{R}^{kn} \) for every player  \( i \) and every edge  \( e \) such that

(i)  \( \tilde{\pi}^0 = \pi + M^0 \delta \) and  \( \tilde{\pi}^l = \pi^* + M^l \delta \) for  \( l \geq 1 \).

(ii) for every  \( e \in E, i \in [k] \),  \( \tilde{\epsilon}_{e,i}^0 = \epsilon_{e,i} + m^T_{0,e,i} \delta \) and  \( \tilde{\epsilon}_{e,i}^l = m^T_{l,e,i} \delta \) for  \( l \geq 1 \).

Proof. Denote by  \( e^*_l, i^*_l \) the player-edge pair inducing the boundary that is hit by the perturbed solution curve in the  \( l \)-th step. We define the matrices and vectors

\[
M^l := I_{nk} - \frac{1}{w^T_{e^*_l-1, i^*_l-1} \Delta \pi^l-1 w^T_{e^*_l-1, i^*_l-1}} \Delta \pi^l-1 \quad \text{for } l \geq 1
\]

\[
M^l := \prod_{j=0}^{l-1} M^{l-j} \quad \text{for } l \geq 0
\]

\[
m^l_{e,i} := -\frac{1}{w^T_{e,i} \Delta \pi^l} w^T_{e,i} M^l \quad \text{for } l \geq 0.
\]

Note that we admit  \( \infty \)-values in the vectors  \( m^l_{e,i} \) if the denominator  \( w^T_{e,i} \Delta \pi^l \) is zero. In this case, the perturbed solution curve moves parallel to the boundary induced by  \( w_{e,i} \) and, in particular, the whole vector  \( m^l_{e,i} \) contains only  \( \infty \)-values. We also write  \( m^l_{e,i} = \infty \) in this case.

With these definitions, we get  \( \tilde{\pi}^0 = \pi + M^0 \delta \) and

\[
\tilde{\epsilon}_{e,i}^0 = \frac{b_e - w^T_{e,i} \pi^0}{w^T_{e,i} \Delta \pi^0} = \epsilon_{e,i} + m^T_{0,e,i} \delta.
\]
Lemma 6. For minimum.

\[ \text{Lemma 6. For } \min. \]

Lemma 6. For minimum. minimal vector \( m \). These values are polynomials in 1 and 0, and for sufficiently small \( \delta \) the next boundary crossed is the boundary induced by the player-edge-pair \( e, i \). The next lemma proves that in every step the set of lexicographically positive vectors \( m_{l,e,i} \) contains a unique lexicographic minimum.

\[ \pi^l = \pi^{l-1} + \epsilon^{l-1}_{e^{-1}, i^{-1}} M^{l-1} \delta \]

For \( l > 1 \) we obtain by induction

\[ \pi^l = \pi^0 + \epsilon^0_{e, i} \Delta \pi^0 \]

For \( l = 1 \), we obtain

\[ \pi^1 = \pi^0 + \epsilon^0_{e, i} \Delta \pi^0 \]

When starting from the potential \( \tilde{\pi}^l \) the next boundary crossed is the boundary induced by the player-edge-pair \( e^*, i^* \) that minimizes \( \tilde{\epsilon}^l_{e^*, i^*} = m_{l,e,i}^T \delta \) for \( l = 0 \), respectively) under all positive and finite \( \tilde{\epsilon}^l_{e^*, i^*} \) values. These values are polynomials in \( \delta \) with coefficients \( m_{l,e,i} \) and, hence, for sufficiently small \( \delta \), finding the minimal \( \tilde{\epsilon}^l_{e^*, i^*} \) reduces to finding the lexicographic minimal vector \( m_{l,e,i} \). The next lemma proves that in every step the set of lexicographically positive vectors \( m^l_{l,e,i} \) contains a unique lexicographic minimum.

\[ \tilde{\epsilon}^l_{e^*, i^*} = b_e - w_{e^*, i^*}^T \tilde{\pi}^l = b_e - w_{e^*, i^*}^T \pi^* - w_{e^*, i^*}^T M^l \delta \]

Lemma 6. For \( l \geq 0 \) the set \( \{m^l_{l,e,i} : 0 < \tilde{\epsilon}^l_{e^*, i^*} < \infty\} \) has a unique lexicographic minimum.
Proof. If the set contains only one vector the claim is trivial. Otherwise we will to show that \( \mathbf{m}_{i,e,i} \neq \mathbf{m}_{i,e,2} \) for any two player-edge-pairs \((e_1, i_1), (e_2, i_2)\). For \( l = 0 \) every vector \( \mathbf{m}_{i,e,i} \) is just a multiple of the vector \( \mathbf{w}^{i,e} \) which are pairwise linear independent by definition.

For \( l \geq 1 \) we claim that
\[
\mathbf{v}^\top \mathbf{M}^l = 0 \iff \mathbf{v} \in \text{span}(\mathbf{w}^{i_1-1,i_{l-1}}),
\]
(20)
\[
\mathbf{M}^l \mathbf{v} = 0 \iff \mathbf{v} \in \text{span}(\Delta \mathbf{\pi}^{l-1}).
\]

The if-directions can be verified easily and the only if-directions follow since \( \check{\mathbf{M}}^l \) is the difference of an identity matrix and a rank 1-matrix and therefore has rank at least \( kn - 1 \).

As a next step, we prove
\[
\mathbf{v}^\top \mathbf{M}^l = 0 \iff \mathbf{v} \in \text{span}(\mathbf{w}^{i_1-1,i_{l-1}}),
\]
(21)
Since \( \mathbf{M}^l = \check{\mathbf{M}}^l \mathbf{M}^{l-1} \), the if-direction follows directly from (20). For \( l = 1 \), Equation (20) yields the only if-direction as well. For \( l > 1 \), by \( \mathbf{v}^\top \mathbf{M}^l = 0 \) implies \( \mathbf{v} \in \text{span}(\mathbf{w}^{i_{l-1},i_{l-2}}) \) or \( (\mathbf{M}^{l-1})^\top \mathbf{v} \in \text{span}(\mathbf{w}^{i_{l-2},i_{l-2}}) \) by induction. The latter implies \( \mathbf{w}^{i_{l-2},i_{l-2}} \Delta \mathbf{\pi}^{l-1} = (\mathbf{M}^{l-1})^\top \mathbf{v} \Delta \mathbf{\pi}^{l-1} = 0 \) by the second part of (20). With (17) from Theorem 2 we obtain \( \mathbf{w}^{i_{l-2},i_{l-2}} \Delta \mathbf{\pi}^{l-2} = 0 \) and, finally, \( \epsilon_{e,i}^{l-2} = \infty \) which is a contradiction since \( (e_{l-2}^*,i_{l-2}^*) \) minimized \( \epsilon_{e,i}^{l-2} \). Hence, (21) follows.

Now assume that for \( l \geq 1 \) we have \( \mathbf{m}_{i,e_1,i_1} = \mathbf{m}_{i,e_2,i_2} \). Then
\[
0 = \mathbf{m}_{i,e_1,i_1} - \mathbf{m}_{i,e_2,i_2} = -\frac{1}{(\mathbf{w}^{i_1,e_1})^\top \Delta \mathbf{\pi}^{l-1} \mathbf{w}^{i_1,e_1} - 1} (\mathbf{w}^{i_2,e_2})^\top \Delta \mathbf{\pi}^{l-1} \mathbf{w}^{i_2,e_2})^\top \mathbf{M}^l
\]
and, thus, by (21), we obtain \( \mathbf{w}^{i_{l-1},i_{l-1}} = \beta \hat{\mathbf{m}} \) for some factor \( \beta \in \mathbb{R} \). Using again (17), this yields
\[
\text{sgn} (\mathbf{w}^{i_{l-1},i_{l-1}}^\top \Delta \mathbf{\pi}^{l-1}) = \text{sgn} (\mathbf{w}^{i_{l-2},i_{l-2}}^\top \Delta \mathbf{\pi}^{l-1}) = \text{sgn} (\beta \hat{\mathbf{m}}^\top \Delta \mathbf{\pi}^{l-1}) = 0
\]
\[
\implies \epsilon_{e,i}^{l-2} = \infty \text{ which is a contradiction to the choice of } (e_{l-2}^*,i_{l-2}^*) \text{ as minimizer of } \epsilon_{e,i}^{l-2}.
\]

Lemma 6 proves that, for sufficiently small \( \delta > 0 \), the perturbed solution curve crosses always a unique boundary induced by some player-edge pair \((e^*_i, i^*_i)\).

In particular, the respective pairs can be obtained by recursively computing the matrices \( \mathbf{M}^l \) and vectors \( \mathbf{m}_l \), and by finding the lexicographic minimum of the set \( \{ \mathbf{m}_{i,e,i} : 0 < \epsilon_{e,i} < \infty \} \) in every step. This procedure can be iterated until none of the boundaries intersecting in the degenerate point \( \mathbf{\pi}^* \) is hit by the perturbed solution curve. The associated region is the region to proceed with the real solution curve.
4.6 Potential direction via quadratic programming

In addition to the methods for obtaining feasible potential direction presented above, we want to present another general approach to compute a feasible potential direction in any point \( \pi \) that works in the non-degenerate case as well as the degenerate case.

Let \( \pi \) be a potential associated with some equilibrium flow \( x \) for demand \( r \) inducing the excess \( y \). Let \( \pi \) further be consistent with the support \( S \). Then consider the quadratic program

\[
\begin{align*}
\min & \quad \frac{1}{2} \sum_{e \in E} a_e \left( \left( \sum_{i \in S_e} z^i_e \right)^2 + \sum_{i \in S_e} (z^i_e)^2 \right) \\
\text{s.t.} & \quad \sum_{e \in \delta^+(v); i \in S_e} z^i_e - \sum_{e \in \delta^-(v); i \in S_e} z^i_e = \Delta y^i_v, \quad \forall v \in V \forall i \in [k] \\
& \quad z^i_e = 0, \quad \forall i \notin S_e \forall e \in E \\
& \quad z^i_e \geq 0, \quad \forall i \in S_e : x^i_e = 0 \forall e \in E
\end{align*}
\]

with variables \( z^i_e \) for every edge \( e \in E \) and player \( i \in [k] \).

**Theorem 3.** Let \( z \) be an optimal solution of the quadratic program (22) and \( \lambda = (\lambda^i_v)_{e \in E, i \in [k]} \) be the KKT-multipliers corresponding the the flow conservation constraints. Then there is \( \epsilon > 0 \) such that \( x(\alpha) := x + \alpha z \) is an equilibrium flow for the demand \( r + \alpha \Delta r \) with associated potential \( \pi + \alpha \lambda \) for all \( 0 < \alpha < \epsilon \).

**Proof.** Let \( z \) be the optimal solution of (22). We define \( \bar{z}_e := \sum_{j \in S_e} z^j_e \) for all edges \( e \in E \). Then the Karush-Kuhn-Tucker (KKT) conditions imply the existence of multipliers \( \lambda^i_v \) for all \( v \in V \) and \( i \in [k] \) such that

\[
a_e \bar{z}_e + a_e z^i_e + \lambda^i_v - \lambda^i_w = 0
\]

for all \( i \in S_e \) with \( x^i_e > 0 \) and all \( e \in E \). Further, the KKT conditions imply the existence of multipliers \( \mu^i_e \) for all \( i \in S_e : x^i_e = 0 \) and \( e \in E \) such that

\[
a_e \bar{z}_e + a_e z^i_e + \lambda^i_v - \lambda^i_w - \mu^i_e = 0
\]

for all \( i \in S_e : x^i_e = 0 \) and \( e \in E \).

Now let

\[
\begin{align*}
\epsilon_1 & := \min_{e \in E} \min_{i \notin S_e} \frac{a_e \bar{z}_e + b_e - (\pi^i_u - \pi^i_v)}{\lambda^i_w - \lambda^i_e - a_e \bar{z}_e} \\
\epsilon_2 & := \min_{e \in E} \min_{i \in S_e: z^i_e < 0} \frac{x^i_e}{z^i_e} \\
\epsilon & := \min \{ \epsilon_1, \epsilon_2 \}
\end{align*}
\]
Now we consider the flow $\mathbf{x}(\alpha) := \mathbf{x} + \alpha \mathbf{z}$ and the potential $\mathbf{\pi}(\alpha) := \mathbf{\pi} + \alpha \mathbf{\lambda}$ and claim that, for $0 < \alpha < \epsilon$, they satisfy the conditions of Lemma 2 and, hence, $\mathbf{x}(\alpha)$ is a Nash equilibrium for all $0 < \alpha < \epsilon$.

Consider an edge $e$ and some player $i$. We then have the following cases.

1. The case $x^i_e(\alpha) > 0$. We claim that

$$
\lambda^i_w - \lambda^i_v = a_e \bar{z}_e + a_e z^i_e. \tag{25}
$$

There are two subcases that are possible:

(a) The flow positive before as well ($x^i_e > 0$). Then (25) follows from (23).

(b) The flow was zero before ($x^i_e = 0$). Then $z^i_e$ has to be positive (otherwise $x^i_e(\alpha)$ can not be positive). This implies by the KKT conditions that $\mu^i_e = 0$ and thus (24) implies (25).

Using (25), we get

$$
\pi^i_w(\alpha) - \pi^i_v(\alpha) = (\pi^i_w - \pi^i_v) + \alpha (\lambda^i_w - \lambda^i_v)
= (a_e \bar{x}_e + b_e + a_e x^i_e) + \alpha (a_e \bar{z}_e + a_e z^i_e)
= a_e (\bar{x}_e + \alpha \bar{z}_e) + b_e + a_e (x^i_e + \alpha z^i_e)
= a_e \bar{x}_e(\alpha) + b_e + a_e x^i_e(\alpha) = L^i_e(\mathbf{x}(\alpha)),
$$

i.e. $\mathbf{\pi}(\alpha)$ and $\mathbf{x}(\alpha)$ satisfy (3a).

2. The case $x^i_e(\alpha) = 0$. We claim that

$$
(\pi^i_w - \pi^i_v) + \alpha (\lambda^i_w - \lambda^i_v) \leq (a_e \bar{x}_e + b_e) + \alpha a_e \bar{z}_e. \tag{26}
$$

There are the following two subcases.

(a) The edge $e$ was not active for player $i$ ($i \notin S_e$). This means that

$$
a_e \bar{x}_e + b_e > \pi^i_w - \pi^i_v
$$

and (26) follows either from $\lambda^i_w - \lambda^i_v \leq a_e \bar{z}_e$ or from the definition of $\epsilon_1$.

(b) The edge $e$ was active for player $i$ ($i \in S_e$). Then we can infer from the definition of $\epsilon_2$ that $x^i_e = 0$. (Note, that this also implies $z^i_e = 0$.) By the KKT conditions, we know that $\mu^i_e \geq 0$ and, hence, (26) follows from (24) and $\pi^i_w - \pi^i_v = a_e \bar{x}_e + b_e$ (which is true since $i \in S_e$).

Equation (26) implies that $\mathbf{\pi}(\alpha)$ and $\mathbf{x}(\alpha)$ satisfy (3b).

Overall, we have that $\mathbf{x}(\alpha)$ is a Nash equilibrium by Lemma 2. \qed

Theorem 3 can thus be used to obtain a potential direction (the KKT-multipliers $\mathbf{\lambda}$) and a flow direction (the optimal solution $\mathbf{z}$) by solving the quadratic program (22).
4.7 Termination and Time Complexity

The algorithm terminates if the solution curve enters a region where the potential direction does not point towards any further boundary, i.e. if $\epsilon^i_e = \infty$ for all edges $e$ and players $i$. We will now prove, that the solution curve will eventually enters such a region (if the graph and number of players is finite). Indeed, if the algorithm would not eventually terminate, the solution curve will traverse at least one region more than once (actually infinitely many times). This is, as the next Lemma proves, not possible.

Lemma 7. The algorithm considers every support $S$ at most once.

Proof. It is clear that two consecutive iterations have different supports. Now assume that there are two iterations $l_1$ and $l_2$ with the same supports that induces the total Laplacian matrix $L$. Then there are two different potentials $\pi^{(1)}$ and $\pi^{(2)}$ and factors $\lambda_1 < \lambda_2$ such that

$$\lambda_i \Delta y = L \pi^{(i)} + b$$

for $i = 1, 2$. This implies that

$$(\pi^{(2)} - \pi^{(1)}) = (\lambda_2 - \lambda_1)L^T \Delta y = (\lambda_2 - \lambda_1) \Delta \pi.$$

Since we assumed that $\pi^{(1)}$ and $\pi^{(2)}$ are visited in two different iterations there must have been a boundary crossing between $\lambda_1$ and $\lambda_2$. This means when moving in the direction $\Delta \pi$ from $\pi^{(1)}$ at some point the status of at least one edge changes for one player. But this is a contradiction, since the potentials $\pi^{(1)}$ and $\pi^{(2)}$ are in the same region $R$ and the regions are convex sets—hence, the solution curve could not have left this region before reaching $\pi^{(2)}$. $\square$

For finite graphs and finite number of players, there are only finitely many supports and Lemma 7 implies that the algorithm terminates in finite time. Furthermore, if the solution curve does not hit any degenerate point in the course of the algorithm, every iteration produces one function part of the piecewise linear output potential function $\pi(\lambda)$ (and, thus, also one function part of the equilibrium flow function $x(\lambda)$). Every iteration essentially consists of one application of Theorem 2. The new direction vector can be computed from the old one in $O((kn)^2)$ time due to the sparsity of the matrix $ww^T$. For the initialization of the algorithm the first (generalized) inverse of $L$ has to be computed. This can be done with fast matrix multiplication (e.g. with the Coppersmith-Winograd algorithm [7]) in $O((kn)^{2.4})$ time. Hence, the overall time complexity is $O((kn)^{2.4} + K(kn)^2)$ where $K$ is the number of breakpoints of the output function. Hence, the algorithm is output polynomial.

Note, that the factor $K$ can be exponential in the size of the graph, as there are families of graphs for which the number of breakpoints of the output functions are exponential. We give an example for such families in [19] for the case of Wardrop equilibria which is a special case of atomic splittable routing games for $k = 1$ player.
In the case of degenerate points, the number of steps needed for executing the lexicographic rule might not be polynomial. Thus, it is not clear if the algorithm with lexicographic rule is still output polynomial. But in this case, we can obtain the potential direction with Theorem 3 by solving the quadratic program which can be done in polynomial time with the ellipsoid method as shown by Kozlov et al. [21].

**Corollary 3.** The algorithm terminates runs in output polynomial time. If the algorithm does not encounter degenerate points then the run time is $O((kn)^{2.4} + K (kn)^2)$ where $K$ is the number of breakpoints of the output flow functions.

**References**

1. Beckmann, M.J., McGuire, C.B., Winsten, C.B.: Studies in the Economics of Transportation. Yale University Press, New Haven, CT (1956)
2. Ben-Israel, A., Greville, T.N.: Generalized inverses: theory and applications. Springer, New York, NY (2003)
3. Bhaskar, U., Fleischer, L., Hoy, D., Huang, C.C.: On the uniqueness of equilibrium in atomic splittable routing games. Math. Oper. Res. 40, 634–654 (2015)
4. Bhaskar, U., Lolakapuri, P.R.: Equilibrium computation in atomic splittable routing games. In: Azar, Y., Bast, H., Herman, G. (eds.) Proceedings 26th Annu. Europ. Symp. on Algorithms (ESA). LIPIcs, vol. 112, pp. 58:1–58:14 (2018)
5. Catoni, S., Pallottino, S.: Technical note – Traffic equilibrium paradoxes. Transp. Sci. 25, 240–244 (1991)
6. Cominetti, R., Correa, J.R., Stier-Moses, N.E.: The impact of oligopolistic competition in networks. Oper. Res. 57, 1421–1437 (2009)
7. Coppersmith, D., Winograd, S.: Matrix multiplication via arithmetic progressions. In: Proceedings 19th Annu. ACM Sympos. on Theory of Comput. (STOC). pp. 1–6 (1987)
8. Goldberg, P.W., Papadimitriou, C.H., Savani, R.: The complexity of the homotopy method, equilibrium selection, and Lemke-Howson solutions. ACM Transactions on Economics and Computation 1 (2013)
9. Harks, T.: Stackelberg strategies and collusion in network games with splittable flow. Theory Comput. Syst. 48, 781–802 (2011)
10. Harks, T., Timmermans, V.: Equilibrium computation in atomic splittable singleton congestion games. In: Eisenbrand, F., Koenemann, J. (eds.) Proceedings 19th Internat. Conf. Integer Program. and Comb. Optim. (IPCO). LNCS, vol. 10328, pp. 442–454 (2017)
11. Harks, T., Timmermans, V.: Uniqueness of equilibria in atomic splittable polymatroid congestion games. J. Comb. Optim. 36, 812–830 (2018)
12. Harville, D.A.: Matrix algebra from a statistician’s perspective. Springer, New York, NY (1997)
13. Haurie, A., Marcotte, P.: On the relationship between Nash-Cournot and Wardrop equilibria. Networks 15, 295–308 (1985)
14. Herings, P.J.J., Peeters, R.: Homotopy methods to compute equilibria in game theory. Econ. Theory 42, 119–156 (2010)
15. Herings, P.J.J., van den Elzen, A.H.: Computation of the Nash equilibrium selected by the tracing procedure in n-person games. Games Econ. Behav. 38, 89–117 (2002)
A Rank of block matrices

Consider the following well-known fact about strictly diagonally dominant matrices (see, e.g., [26] for a reference).

**Theorem 4.** Let \( A = (a_{ij})_{i,j \in [n]} \) be a \( n \times n \)-matrix with \( |a_{ii}| > \sum_{j=1}^{n} |a_{ij}| \) for every \( i \in [n] \). Then \( A \) is non singular.

In this appendix, we will prove a special variation of this theorem (Theorem 5) suitable for our setting of symmetric block matrices. To this end, we consider block matrices in \( \mathbb{R}^{kn \times kn} \) of the form

\[
M := \begin{pmatrix}
A_{11} & -A_{12} & -A_{13} & \cdots & -A_{1k} \\
-A_{21} & A_{22} & -A_{23} & \cdots & -A_{2k} \\
-A_{31} & -A_{32} & A_{33} & \cdots & -A_{3k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-A_{k1} & -A_{k2} & -A_{k3} & \cdots & A_{kk}
\end{pmatrix}
\]  

(27)

where every matrix \( A_{ij} \in \mathbb{R}^{n \times n} \) is a quadratic matrix. Accordingly, we consider vectors \( x \in \mathbb{R}^{kn} \) of the form

\[
x = \begin{pmatrix}
x^1 \\
\vdots \\
x^k
\end{pmatrix}
\]
where $x^i \in \mathbb{R}^n$ for all $i \in [k]$. For the remainder of this section, we assume that the block matrix $M$ satisfies the following assumptions.

**Assumption 1.** The matrix $M$ of the form (27) contains blocks $A_{ij} \in \mathbb{R}^{n \times n}$, $i, j \in [k]$ that such that

(i) $A_{ij}$ is symmetric and positive semi-definite for all $i, j \in [k]$.

(ii) $A_{ij} = A_{ji}$ for all $i, j \in [k]$.

(iii) for every $x \in \mathbb{R}^n$ it holds

$$
\langle x, (A_{ii} - \sum_{i \neq j} A_{ij})x \rangle \geq \frac{1}{k} \langle x, A_{ii}x \rangle
$$

for every $i \in [k]$.

At first, we state the following observation about vectors $x$ in the nullspace of $M$.

**Lemma 8.** Let $M$ be a block matrix of the form (27). Then $x \in \ker(M)$ if and only if

$$
A_{ii}x^i = \sum_{j \neq i} A_{ij}x^j
$$

(28)

$$
A_{ip}x^p = A_{ii}x^i - \sum_{j \neq i, p} A_{ij}x^j
$$

(29)

for all $i, p \in [k]$.

**Proof.** For any $i \in [k]$, the rows $(i - 1) \cdot n + 1$ to $i \cdot n$ of the equation $Mx = 0$ are equivalent to (28) which is equivalent to (29).

Using Assumption 1 we get that the kernel of the matrices $A_{ii}$ on the diagonal is a subset of the kernel of the off-diagonal matrices $A_{ij}$.

**Lemma 9.** Let $M$ be a block matrix of the form (27) that satisfies Assumption 1. Then for every $i \in [k]$ and $x^i \in \mathbb{R}^n$ we have

$$
A_{ii}x^i = 0 \implies A_{ij}x^i = 0
$$

for all $j \in [k]$.

**Proof.** Since $A_{ij}$ is positive semi-definite for all $i, j \in [k]$ by definition, we obtain

$$
0 \leq \langle x^i, A_{ij}x^i \rangle
$$

$$
\leq \langle x^i, \sum_{p \neq i} A_{ip}x^p \rangle
$$

$$
= \langle x^i, \sum_{p \neq i} A_{ip}x^i - A_{ii}x^i \rangle
$$

$$
\leq -\frac{1}{k} \langle x^i, A_{ii}x^i \rangle = 0.
$$

Thus, $\langle x^i, A_{ij}x^i \rangle = 0$ which implies $A_{ij}x^i = 0$ since $A_{ij}$ is symmetric and positive-semi definite.
The following two lemmas constitute the main part of the proof of Theorem 5.

**Lemma 10.** Let $M$ be a block matrix of the form \[ \begin{pmatrix} A_{ii} & 0 & \cdots & 0 \\ 0 & A_{ii} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{ii} \end{pmatrix} \] that satisfies Assumption 7. Further, let $x \in \ker(M)$. Then for any $p \in [k]$

$$\langle x^p, A_{pp}x^p \rangle \geq \sum_{i \neq p} \langle x^i, A_{pi}x^i \rangle + \sum_{i \neq p} \sum_{j \neq i, p} A_{ij} \langle x^i, x^j \rangle \quad (30)$$

with equality holding if and only if $x^i \in \ker(A_{ii})$ for all $i \neq p$.

**Proof.** Let $p \in [k]$. Then using (28) and (29) we obtain

$$\langle x^p, A_{pp}x^p \rangle = \sum_{i \neq p} \langle x^i, A_{pi}x^i \rangle$$

$$= \sum_{i \neq p} \langle x^i, A_{ii}x^i \rangle$$

$$= \sum_{i \neq p} \langle x^i, A_{ii}x^i - \sum_{j \neq i} A_{ij}x^j \rangle$$

$$= \sum_{i \neq p} \langle x^i, A_{ii}x^i \rangle - \sum_{j \neq i, p} \sum_{i \neq p} A_{ij} \langle x^i, x^j \rangle + \sum_{i \neq p} \sum_{j \neq i, p} A_{ij} \langle x^i, x^j \rangle$$

$$\geq \sum_{i \neq p} \langle x^i, A_{pi}x^i \rangle + \sum_{i \neq p} \sum_{j \neq i, p} A_{ij} \langle x^i, x^j \rangle.$$

Equality holds if and only if $\sum_{i \neq p} \langle x^i, (A_{ii} - \sum_{j \neq i} A_{ij})x^i \rangle = 0$. Since every term of the sum is non-negative, equality holds if and only if for every $i \neq p$ we have

$$0 = \langle x^i, (A_{ii} - \sum_{i \neq j} A_{ij})x^i \rangle \geq \frac{1}{k} \langle x^i, A_{ii}x^i \rangle \geq 0.$$

This is equivalent to $\langle x^i, A_{ii}x^i \rangle = 0$ and thus to $A_{ii}x^i = 0$ for all $i \neq p$. \qed

**Lemma 11.** Let $M$ be a block matrix of the form \[ \begin{pmatrix} A_{ii} & 0 & \cdots & 0 \\ 0 & A_{ii} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{ii} \end{pmatrix} \] that satisfies Assumption 7 and let $x \in \ker(M)$. If, for some $r \in [k]$, we have that $\langle x^r, A_{rr}x^r \rangle \leq \langle x^i, A_{ii}x^i \rangle$ for all $i \in [k]$, then

$$\sum_{i \neq r} \langle x^i, A_{ri}x^i \rangle + \sum_{i \neq r} \sum_{j \neq i, r} A_{ij} \langle x^i, x^j \rangle \geq \langle x^r, A_{rr}x^r \rangle.$$

**Proof.** On the one hand, we obtain

$$\sum_{i \neq r} \langle x^i, A_{ri}x^i \rangle = \sum_{i \neq r} \langle x^i - x^r, A_{ri}(x^i - x^r) \rangle$$

$$+ \sum_{i \neq r} \langle x^r, A_{ri}(x^i - x^r) \rangle + \sum_{i \neq r} \langle x^i, A_{ri}x^r \rangle$$
\[ \geq \langle x^r, \sum_{i \neq r} A_{ri} x^i - \sum_{i \neq r} A_{ri} x^r \rangle + \langle x^r, \sum_{i \neq r} A_{ri} x^i \rangle \]

\[ = \langle x^r, (A_{rr} - \sum_{i \neq r} A_{ri}) x^r \rangle + \langle x^r, A_{rr} x^r \rangle \]

\[ \geq \left( 1 + \frac{1}{k} \right) \langle x^r, A_{rr} x^r \rangle \]

and on the other hand we get

\[ \sum_{i \neq r} \langle x^i, \sum_{j \neq i, r} A_{ij} (x^i - x^j) \rangle = \sum_{j \neq r} \sum_{i \neq j, r} \langle x^i - x^j, A_{ij} (x^i - x^j) \rangle \]

\[ = \sum_{j \neq r} \sum_{i \neq j, r} \langle x^i, x^j, A_{ij} (x^i - x^j) \rangle \]

\[ \geq \sum_{j \neq r} \langle x^i, \sum_{i \neq j} A_{ij} x^j \rangle - \sum_{i \neq j, r} A_{ij} x^j \]

\[ = \sum_{j \neq r} \langle x^i, (A_{jj} - \sum_{i \neq j} A_{ij}) x^j \rangle + \sum_{j \neq r} \langle x^i, A_{jr} x^j \rangle \]

\[ \geq \sum_{j \neq r} \frac{1}{k} \langle x^i, A_{jj} x^j \rangle - \langle x^r, A_{rr} x^r \rangle \]

\[ \geq \frac{1}{k} \langle x^r, A_{rr} x^r \rangle \]

and, thus, combining both inequalities we obtain the statement. \[\square\]

Finally, we can state our variant of the Theorem 4.

**Theorem 5.** Let \( M \) be a block matrix of the form \( (27) \) that satisfies Assumption 3. Then for every vector \( x \in \mathbb{R}^{k \times n} \) we have \( Mx = 0 \) if and only if \( x^i \in \ker(A_{ii}) \) for every \( i \in \{1, \ldots, k\} \).

**Proof.** For the only if-direction, observe that, for every \( i \in [k] \), by Lemma 9 we get

\[ A_{ij} x^j = A_{ji} x^i = 0 \]

for all \( j \in [k] \) since \( x^i \in \ker(A_{jj}) \). Hence, \( Mx = 0 \).

For the if-direction, let \( Mx = 0 \) and let \( r \in [k] \) be the index such that \( \langle x^r, A_{rr} x^r \rangle \leq \langle x^i, A_{ii} x^i \rangle \) for all \( i \in [k] \). Then, using Lemma 10 and Lemma 11.

\[ \langle x^r, A_{rr} x^r \rangle \leq \langle x^i, A_{ii} x^i \rangle \]
we obtain
\[
\langle x^*, A_{rr} x^* \rangle \geq \sum_{i \neq r} \langle x^i, A_{ri} x^i \rangle + \sum_{i \neq r} \left( \sum_{j \neq i, r} A_{ij} (x^i - x^j) \right) \geq \langle x^*, A_{rr} x^* \rangle. \tag{31}
\]

Thus, the inequalities in (31) are satisfied with equality, and hence Lemma 10 implies that \(x^i \in \ker(A_{ii})\) for all \(i \neq r\). Finally, with Lemma 9 we obtain
\[
A_{rr} x^r = \sum_{i \neq r} A_{ri} x^i = \sum_{i \neq r} A_{ir} x^i = 0,
\]
i.e. \(x^r \in \ker(A_{rr})\).

\(\square\)

Theorem 5 implies the following statements about the kernel and the rank of a block matrix \(M\) that satisfies Assumption 1.

**Corollary 4.** Let \(M\) be a block matrix of the form (27) that satisfies Assumption 1. Then

(i) \(\ker(M) = \{ x = ([x^1]^T, \ldots, [x^k]^T)^T \mid x^i \in \ker(A_{ii}) \text{ for all } i \in [k] \}\)

(ii) \(\text{rank}(M) = \sum_{i=1}^{k} \text{rank}(A_{ii})\),

(iii) \(M\) is regular, if and only if \(A_{ii}\) is regular for all \(i \in [k]\).