VARIANCE OF THE EXPONENTS OF ORBIFOLD LANDAU-GINZBURG MODELS

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Abstract. We prove a formula for the variance of the set of exponents of a non-degenerate weighted homogeneous polynomial with an action of a diagonal subgroup of SL_n(C).

Introduction

Let X be a smooth compact Kähler manifold of dimension n. The Hodge numbers h^{p,q}(X) := dim_{\mathbb{C}} H^q(X, \Omega_X^p), p, q \in \mathbb{Z}, are some of the most important numerical invariants of X. They satisfy

\[ h^{p,q}(X) = h^{q,p}(X), \quad p, q \in \mathbb{Z}, \]

and the Serre duality

\[ h^{p,q}(X) = h^{n-p,n-q}(X), \quad p, q \in \mathbb{Z}. \]

The Euler number \( \chi(X) \) can also be written in terms of the Hodge numbers as

\[ \chi(X) = \sum_{p, q \in \mathbb{Z}} (-1)^{p+q} h^{p,q}(X). \]

One can easily calculate the expectation value of the distribution \( \{ q \in \mathbb{Z} \mid h^{p,q}(X) \neq 0 \} \), which is given by the formula

\[
\sum_{p, q \in \mathbb{Z}} (-1)^{p+q} q \cdot h^{p,q}(X) = \frac{1}{2} n \cdot \chi(X).
\]

Equivalently, this can be rewritten as

\[
\sum_{p, q \in \mathbb{Z}} (-1)^{p+q} \left(q - \frac{n}{2}\right) h^{p,q}(X) = 0.
\]

This means nothing else but that the mean of the distribution \( \{ q \in \mathbb{Z} \mid h^{p,q}(X) \neq 0 \} \) is \( n/2 \). It is then natural to ask what is the variance of this distribution. A formula for this variance was given by A. Libgober and J. Wood [LW] and L. Borisov [B]:

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Theorem 1 (Libgober-Wood, Borisov). One has
\[
\sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \left( q - \frac{n}{2} \right)^2 h^{p,q}(X) = \frac{1}{12} n \cdot \chi(X) + \frac{1}{6} \int_X c_1(X) \cup c_{n-1}(X),
\]
(0.1)
where \(c_i(X)\) denotes the \(i\)-th Chern class of \(X\).

If the first Chern class \(c_1(X)\) is numerically zero, then the above formula becomes
\[
\sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \left( q - \frac{n}{2} \right)^2 h^{p,q}(X) = \frac{1}{12} n \cdot \chi(X).
\]
(0.2)

Similar phenomena were discovered in singularity theory. Let us consider a polynomial \(f(x_1, \ldots, x_n)\) with an isolated singularity at the origin. There, the analogue of the set \(\{q \in \mathbb{Z} \mid h^{p,q}(X) \neq 0\}\) above will be the set of the exponents of \(f(x_1, \ldots, x_n)\), which is a set of rational numbers and is also one of the most important numerical invariants defined by the mixed Hodge structure associated to \(f(x_1, \ldots, x_n)\). Let us give two important examples.

First, suppose that \(f(x_1, \ldots, x_n)\) is a non-degenerate weighted homogeneous polynomial, namely, a polynomial with an isolated singularity at the origin with the property that there are positive rational numbers \(w_i, i = 1, \ldots, n\), such that \(f(\lambda^{w_i} x_1, \ldots, \lambda^{w_n} x_n) = \lambda f(x_1, \ldots, x_n), \lambda \in \mathbb{C}\setminus\{0\}\). We have the following properties of the exponents of \(f\):

Theorem 2 (cf. [St]). Let \(q_1 \leq q_2 \leq \cdots \leq q_\mu\) be the exponents of \(f\), where \(\mu\) is the Milnor number of \(f\) defined by
\[
\mu := \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n] \left/ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right..
\]
Then one has
\[
\mu = (-1)^n \prod_{i=1}^{n} \left( 1 - \frac{1}{w_i} \right)
\]
and
\[
\sum_{i=1}^{\mu} y^{q_i - \frac{n}{2}} = (-1)^n \prod_{i=1}^{n} \frac{y^{\frac{1}{2}} - y^{w_i - \frac{1}{2}}}{1 - y^{w_i}}.
\]
In particular, one has a duality of exponents \(q_i + q_{\mu-i+1} = n, i = 1, \ldots, \mu\), and hence
\[
\sum_{i=1}^{\mu} q_i = \frac{1}{2} n \cdot \mu.
\]

The following formula was proven by C. Hertling [H] in the context of Frobenius manifolds and an elementary proof was given by A. Dimca [D].
Theorem 3 (Hertling, Dimca). Let \( q_1 \leq q_2 \leq \cdots \leq q_\mu \) be the exponents of \( f \). One has
\[
\sum_{i=1}^{\mu} \left( q_i - \frac{n}{2} \right)^2 = \frac{1}{12} \hat{c} \cdot \mu, \quad \hat{c} := n - 2 \sum_{i=1}^{n} w_i.
\]

Next, consider the polynomial \( f(x_1, x_2, x_3) := x_1^{\alpha_1} + x_2^{\alpha_2} + x_3^{\alpha_3} - x_1 x_2 x_3 \) such that \( 1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 < 1 \). We have the following properties of the exponents of \( f \):

Theorem 4 (cf. [AGV]). The set of exponents \( \{q_i\} \) of \( f \) is given by
\[
\left\{ \frac{1}{\alpha_1} + 1, \frac{2}{\alpha_1} + 1, \ldots, \frac{\alpha_1 - 1}{\alpha_1} + 1, \frac{1}{\alpha_2} + 1, \frac{2}{\alpha_2} + 1, \ldots, \frac{\alpha_2 - 1}{\alpha_2} + 1, \frac{1}{\alpha_3} + 1, \frac{2}{\alpha_3} + 1, \ldots, \frac{\alpha_3 - 1}{\alpha_3} + 1, 2 \right\}.
\]
In particular, one has
\[
\sum_{i=1}^{\mu} \left( q_i - \frac{3}{2} \right)^2 = \frac{1}{12} \mu + \frac{1}{6} \chi, \quad \chi := 2 + \sum_{i=1}^{3} \left( \frac{1}{\alpha_i} - 1 \right).
\]

The purpose of this paper is to generalize these results to pairs \((f, G)\), where \( G \subset \text{SL}_n(\mathbb{C}) \) is a finite abelian subgroup leaving \( f \) invariant. If \( f \) is weighted homogeneous, such a pair is also called an orbifold Landau-Ginzburg model because \( f \) is the potential of such a model. Our main theorem in this paper is Theorem 19. The generalization of Theorem 4 is given as Theorem 21. The similarity between smooth compact Kähler manifolds and isolated hypersurface singularities with a group action is not an accident but a matter of course. Mirror symmetry predicts a correspondence between Landau-Ginzburg models and (non-commutative) Calabi-Yau orbifolds. For example, a mirror partner of a weighted homogeneous polynomial with a group action is a fractional Calabi–Yau manifold of dimension \( \hat{c} \), which has lead us to the statement of Theorem 19.

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1. Basic properties of \( E \)-functions

Let \( G \) be a finite abelian subgroup of \( \text{SL}_n(\mathbb{C}) \) acting diagonally on \( \mathbb{C}^n \). For \( g \in G \), we denote by \( \text{Fix} g := \{ x \in \mathbb{C}^n \mid g \cdot x = x \} \) the fixed locus of \( g \) and by \( n_g := \dim \text{Fix} g \) its dimension.

We first introduce the notion of the age of an element of a finite group as follows:
Definition (IN). Let \( g \in G \) be an element and \( r \) be the order of \( g \). Then \( g \) has a unique expression of the following form

\[
g = \text{diag}(e[a_1/r], \ldots, e[a_n/r])
\]

with \( 0 \leq a_i < r \), where \( e[-] = e^{2\pi \sqrt{-1} \cdot -} \). Such an element \( g \) is often simply denoted by \( g = \frac{1}{r}(a_1, \ldots, a_n) \). The age of \( g \) is defined as

\[
\text{age}(g) := \frac{1}{r} \sum_{i=1}^{n} a_i.
\]

Since we assume that \( G \subset \text{SL}_n(\mathbb{C}) \), the age\( (g) \) is a non-negative integer for all \( g \in G \).

Definition. An element \( g \in G \) of age 1 with \( \text{Fix} g = \{0\} \) is called a junior element. The number of junior elements is denoted by \( j_G \).

Proposition 5. The function \( f^g \) has an isolated singularity at the origin.

Proof. Since \( G \) acts diagonally on \( \mathbb{C}^n \), we may assume that \( \text{Fix} g = \{x_{n_g+1} = \cdots = x_n = 0\} \) by a suitable renumbering of indices. Since \( f \) is invariant under \( G \), \( g \cdot x_i \neq x_i \) for \( i = n_g + 1, \ldots, n \) and \( \frac{\partial f}{\partial x_{n_g+1}}, \ldots, \frac{\partial f}{\partial x_n} \) form a regular sequence, we have

\[
\left( \frac{\partial f}{\partial x_{n_g+1}}, \ldots, \frac{\partial f}{\partial x_n} \right) \subset (x_{n_g+1}, \ldots, x_n).
\]

Therefore, we have

\[
\dim \mathbb{C} [x_1, \ldots, x_{n_g}] / \left( \frac{\partial f^g}{\partial x_1}, \ldots, \frac{\partial f^g}{\partial x_{n_g}} \right) = \dim \mathbb{C} [x_1, \ldots, x_n] / \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n_g}}, x_{n_g+1}, \ldots, x_n \right) \leq \dim \mathbb{C} [x_1, \ldots, x_n] / \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) < \infty.
\]

We first associate to \( f \) a natural mixed Hodge structure with an automorphism, which gives the following bi-graded vector space:

Definition. Define the bi-graded vector space \( \mathcal{H}_f := \bigoplus_{p,q} \mathcal{H}_f^{p,q} \) as

(i) If \( p + q \neq n \), then \( \mathcal{H}_f^{p,q} := 0 \).

(ii) If \( p + q = n \) and \( p \in \mathbb{Z} \), then

\[
\mathcal{H}_f^{p,q} := \text{Gr}_p^n H^{n-1}(Y_f, \mathbb{C})_1.
\]
(iii) If \( p + q = n \) and \( p \notin \mathbb{Z} \), then
\[
\mathcal{H}_{f}^{p,q} := \text{Gr}_{k}^{[p]} H^{n-1}(Y_f, \mathbb{C})_{\mathbb{C}^{n}}^{\mathbb{Z} - \nu},
\]
where \([p]\) is the largest integer less than \( p \).

As a vector space, \( \mathcal{H}_{f} \) is identified with \( \Omega_{f} := \Omega_{\mathbb{C}^{n},0}^{n} / df \wedge \Omega_{\mathbb{C}^{n},0}^{n-1} \). Note that we have
\[
\Omega_{f} = \mathcal{O}_{\mathbb{C}^{n},0} \left/ \left( \frac{\partial f}{\partial x_{1}}, \ldots, \frac{\partial f}{\partial x_{n}} \right) \cdot dx_{1} \wedge \cdots \wedge dx_{n}, \right.
\]
and that Theorem 2 can be shown based on the above equality by calculating the Poincaré polynomial of the right hand side. The \( G \)-action on \( \mathcal{H}_{f} \) can also be identified with the one on \( \Omega_{f} \). We shall use the fact that \( \mathcal{H}_{f,g} \) admits a natural \( G \)-action by restricting the \( G \)-action on \( \mathbb{C}^{n} \) to \( \text{Fix } g \) (which is well-defined since \( G \) acts diagonally on \( \mathbb{C}^{n} \)).

To the pair \((f,G)\) we can associate a natural mixed Hodge structure with an automorphism, which gives the following bi-graded vector space:

**Definition.** Define the bi-graded \( \mathbb{C} \)-vector space \( \mathcal{H}_{f,G} \) as
\[
\mathcal{H}_{f,G} := \bigoplus_{g \in G} (\mathcal{H}_{f,g})^{G}(-\text{age}(g), -\text{age}(g)), \tag{1.1}
\]
where \((\mathcal{H}_{f,g})^{G}\) denotes the \( G \)-invariant subspace of \( \mathcal{H}_{f,g} \).

Since the bi-graded vector space \( \mathcal{H}_{f,G} \) is the analog of \( \bigoplus_{p,q \in \mathbb{Z}} H^{p}(X, \Omega_{X}^{q}) \) for a smooth compact Kähler manifold \( X \), we introduce the following notion:

**Definition.** The **Hodge numbers** for the pair \((f,G)\) are
\[
h^{p,q}(f,G) := \dim_{\mathbb{C}} \mathcal{H}_{f,G}^{p,q}, \quad p, q \in \mathbb{Q}.
\]

**Definition.** The rational number \( q \) with \( \mathcal{H}_{f,G}^{p,q} \neq 0 \) is called an **exponent** of the pair \((f,G)\). The **set of exponents** of the pair \((f,G)\) is the multi-set of exponents
\[
\{ q * h^{p,q}(f,G) \mid p, q \in \mathbb{Q}, \ h^{p,q}(f,G) \neq 0 \},
\]
where by \( u * v \) we denote \( v \) copies of the rational number \( u \).

Note that \( p+q \in \mathbb{Z} \) for the rational number \( q \) with \( h^{p,q}(f,G) \neq 0 \) since \( G \subset \text{SL}_{n}(\mathbb{C}) \).

**Definition.** The **E-function** for the pair \((f,G)\) is
\[
E(f,G)(\bar{t}, \bar{t}) := \sum_{p,q \in \mathbb{Q}} (-1)^{(p-n)+q} h^{p,q}(f,G) \cdot \bar{t}^{p-\frac{n}{2}} \bar{t}^{-q-\frac{q}{2}}. \tag{1.2}
\]
Definition. The Milnor number for the pair \((f, G)\) is
\[
\mu(f, G) := E(f, G)(1, 1) = \sum_{p, q \in \mathbb{Q}} (-1)^{(p-n)+q} h^{p,q}(f, G).
\]

Theorem 6. Assume that \(f\) is a non-degenerate weighted homogeneous polynomial. Write \(g \in G\) in the form \((\lambda_1(g), \ldots, \lambda_n(g))\) where \(\lambda_i(g) = e[a_iw_i]\). The E-function for the pair \((f, G)\) is given by the following formula:
\[
E(f, G)(t, \bar{t}) = \sum_{g \in G} E_g(f, G)(t, \bar{t}) \tag{1.3}
\]
\[
E_g(f, G)(t, \bar{t}) := (-1)^n \left( \prod_{a_iw_i \in \mathbb{Z}} (t\bar{t})^{w_i} \right)^{-\frac{1}{2}} \frac{1}{|G|} \sum_{h \in G} \prod_{a_iw_i \in \mathbb{Z}} \frac{\left(\frac{1}{t}\right)^{\frac{1}{2}} - \lambda_i(h) \left(\frac{1}{t}\right)^{w_i} - \frac{1}{2}}{1 - \lambda_i(h) \left(\frac{1}{t}\right)^{w_i}}. \tag{1.4}
\]
Here \([a]\) for \(a \in \mathbb{Q}\) denotes the largest integer less than or equal to \(a\).

Proof. Theorem 2 enables us to obtain \(E_g(f, G)(t, \bar{t})\). In particular, the term
\[
\frac{1}{|G|} \sum_{h \in G} (-1)^n \prod_{a_iw_i \in \mathbb{Z}} \frac{\left(\frac{1}{t}\right)^{\frac{1}{2}} - \lambda_i(h) \left(\frac{1}{t}\right)^{w_i} - \frac{1}{2}}{1 - \lambda_i(h) \left(\frac{1}{t}\right)^{w_i}}
\]
calculates the \(G\)-invariant part of \(E(f^g, \{1\})(t, \bar{t})\) and the term \(\prod_{w_i \in \mathbb{Z}} (-1) (t\bar{t})^{w_i} \left(\frac{1}{t}\right)^{\frac{1}{2}} \right)^{-\frac{1}{2}}\) gives the contribution from the age shift \((-\text{age}(g), -\text{age}(g))\). \(\square\)

We have the following properties of the Hodge numbers \(h^{p,q}(f, G)\).

Corollary 7. Assume that \(f\) is a non-degenerate weighted homogeneous polynomial. We have
\[
h^{p,q}(f, G) = h^{q,p}(f, G), \quad p, q \in \mathbb{Q}.
\]

In other words, we have
\[
E(f, G)(t, \bar{t}) = E(f, G)(\bar{t}, t).
\]

Proof. This is shown by an elementary direct calculation. \(\square\)

Corollary 8. Assume that \(f\) is a non-degenerate weighted homogeneous polynomial. The Hodge numbers satisfy the “Serre duality”
\[
h^{p,q}(f, G) = h^{n-p,n-q}(f, G), \quad p, q \in \mathbb{Q}.
\]

In other words, we have
\[
E(f, G)(t, \bar{t}) = E(f, G)(t^{-1}, \bar{t}^{-1}).
\]
Proof. By using the formula
\[ w_i(-a_i) - [w_i(-a_i)] - \frac{1}{2} = -w_i a_i + [w_i a_i] + \frac{1}{2}, \]
an easy calculation yields the formula. \(\square\)

**Corollary 9.** Assume that \(f\) is a non-degenerate weighted homogeneous polynomial. The mean of the set of exponents of \((f, G)\) is \(n/2\). Namely, we have
\[ \sum_{p,q \in \mathbb{Q}} (-1)^{(p-n)+q} \left( q - \frac{n}{2} \right)^2 h^{p,q}(f, G) = 0. \]

Proof. This is obvious from the previous corollary. \(\square\)

**Definition.** Define the variance of the set of exponents of \((f, G)\) by
\[ \text{Var}_{(f,G)} := \sum_{p,q \in \mathbb{Q}} (-1)^{(p-n)+q} \left( q - \frac{n}{2} \right)^2 h^{p,q}(f, G). \]

In order to state our formula for the variance, we introduce the following notion of dimension for a polynomial \(f\) with an isolated singularity at the origin.

**Definition.** The non-negative rational number \(\hat{c}\) defined as the difference of the maximal exponent of the pair \((f, \{1\})\) and the minimal exponent of the pair \((f, \{1\})\) is called the dimension of \(f\).

**Proposition 10.** Assume that \(f\) is a non-degenerate weighted homogeneous polynomial. The dimension \(\hat{c}\) of \(f\) is given by
\[ \hat{c} := n - 2 \sum_{i=1}^{n} w_i. \]

Proof. It easily follows from Theorem \(\square\) that the maximal exponent and the minimal exponent are given by \(n - \sum_{i=1}^{n} w_i\) and \(\sum_{i=1}^{n} w_i\) respectively. \(\square\)

It is natural from the mirror symmetry point of view to expect that the variance of the set of exponents of \((f, G)\) should be given by
\[ \text{Var}_{(f,G)} = \frac{1}{12} \hat{c} \cdot \mu_{(f,G)}. \quad (1.5) \]

This will be proved in the next section.
2. Variance of the exponents

**Definition.** The $\chi_y$-genus for the pair $(f, G)$ is

$$\chi(f, G)(y) := E(f, G)(1, y).$$

We have

$$\chi(f, G)(y) = (-1)^n \sum_{g \in G} \left( y^{\text{age}(g)} \frac{n-n_g}{2} \right) \frac{1}{|G|} \sum_{h \in G} \prod_{\lambda_i(g)=1} \frac{y^{\frac{1}{2}} - \lambda_i(h)y^{w_i-\frac{1}{2}}}{1 - \lambda_i(h)y^{w_i}}.$$

One has

$$\mu(f, G) = \lim_{y \to 1} \chi(f, G)(y),$$

$$\text{Var}(f, G) = \lim_{y \to 1} \frac{d}{dy} \left( y \frac{d}{dy} \chi(f, G)(y) \right).$$

**Proposition 11.** Let

$$p_i(y) := \frac{y^{\frac{1}{2}} - \lambda_i(h)y^{w_i-\frac{1}{2}}}{1 - \lambda_i(h)y^{w_i}}.$$

(i) For $\lambda_i(h) = 1$ one has

$$\lim_{y \to 1} p_i(y) = 1 - \frac{1}{w_i}, \quad \lim_{y \to 1} \frac{d}{dy} p_i(y) = 0, \quad \lim_{y \to 1} \frac{d}{dy} \left( y \frac{d}{dy} p_i(y) \right) = \frac{1}{12}.$$

(ii) For $\lambda_i(h) \neq 1$ one has

$$\lim_{y \to 1} p_i(y) = 1, \quad \lim_{y \to 1} \frac{d}{dy} p_i(y) = \frac{1 + \lambda_i(h)}{2 - \lambda_i(h)}, \quad \lim_{y \to 1} \frac{d}{dy} \left( y \frac{d}{dy} p_i(y) \right) = -\frac{(1 - 2w_i)\lambda_i(h)}{(1 - \lambda_i(h))^2}.$$

**Proof.** For (i) see the proof of [D Proposition 5.2]. Statement (ii) follows from a similar elementary but tedious computation.

Let $I_0 := \{1, \ldots, n\}$ and let $H \subset G$ be a subgroup of $G$. For a subset $I \subset I_0$ ($I = \emptyset$ is admitted) let $H^I$ be the maximal subgroup of $H$ fixing the coordinates $x_i, i \in I$.

**Lemma 12.** Let $H \subset G$ be a subgroup of $G$ and $i \in I_0$. Then

$$\sum_{h \in H \setminus H^I} \frac{1 + \lambda_i(h)}{1 - \lambda_i(h)} = 0$$

**Proof.** One has

$$\sum_{h \in H \setminus H^I} \frac{1 + \lambda_i(h)}{1 - \lambda_i(h)} = \sum_{h \in H \setminus H^I} \frac{1}{1 - \lambda_i(h)} + \sum_{h \in H \setminus H^I} \frac{1}{\lambda_i(h^{-1}) - 1} = 0.$$
Proposition 13. Let \( r \in \mathbb{Z}, r \geq 2 \), and \( \zeta_r = e[1/r] \) be a primitive \( r \)-th root of unity. Then one has
\[
- \sum_{k=1}^{r-1} \frac{\zeta_r^k}{(1 - \zeta_r^k)^2} = \frac{r^2 - 1}{12}.
\]

Proof. One has
\[
- \sum_{k=1}^{r-1} \frac{\zeta_r^k}{(1 - \zeta_r^k)^2} = \lim_{t\to 1} q'(t) \text{ where } q(t) := - \sum_{k=1}^{r-1} \frac{1}{1 - \zeta_r^kt}.
\]
One can easily see that
\[
q(t) = \frac{-r \left( \sum_{k=0}^{r-2} tk \right) + \sum_{k=0}^{r-2} (k+1)tk}{\sum_{k=0}^{r-1} tk}.
\]
This implies
\[
\lim_{t\to 1} q'(t) = \frac{1}{r^2} \left[ \sum_{k=1}^{r-2} k(k-r+1)r - \left( \sum_{\ell=1}^{r-1} (\ell - r) \right) \left( \sum_{k=1}^{r-1} k \right) \right] = \frac{r^2 - 1}{12}.
\]

Corollary 14. Let \( H \subset G \) be a subgroup of \( G \) and \( i \in I_0 \). Then
\[
- \sum_{h \in H \setminus H^{(i)}} \frac{\lambda_i(h)}{(1 - \lambda_i(h))^2} = \frac{|H \cap H^{(i)}|(|H/H \cap H^{(i)}|^2 - 1)}{12}.
\]

Proof. The image of the factor group \( H/H \cap H^{(i)} \) under the induced character \( \lambda_i : H/H \cap H^{(i)} \to \mathbb{C}^* \) is a finite abelian subgroup of the unit circle \( S^1 \) and hence cyclic. Therefore the formula follows from Proposition 13.

Let
\[
((x)) := \begin{cases} 
  x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R}, x \notin \mathbb{Z}, \\
  0 & \text{if } x \in \mathbb{Z}.
\end{cases}
\]

Proposition 15. Let \( r \in \mathbb{Z}, r \geq 2, \zeta_r = e[1/r] \) be a primitive \( r \)-th root of unity, and \( a, b \) be integers satisfying \( 0 < a, b < r \). Then one has
\[
\frac{1}{4r} \sum_{\substack{k=1 \atop k \neq ak, bk}}^{r-1} \frac{1 + \zeta_r^ak}{1 - \zeta_r^ak} \frac{1 + \zeta_r^bk}{1 - \zeta_r^bk} = - \sum_{k=1}^{r-1} ((a \frac{r}{k}))((b \frac{r}{k})) = \frac{1 + e[x]}{1 - e[x]} = \sqrt{-1} \cot \pi x.
\]

Remark 16. The right hand side of the formula of Proposition 15 is a generalized Dedekind sum and Proposition 15 is a slight generalization of [HZ, 5.2 Theorem 1], since
\[
\frac{1 + e[x]}{1 - e[x]} = \sqrt{-1} \cot \pi x
\]
for any real number \( x \). The difference is that [HZ, 5.2 Theorem 1] is only formulated for integers \( a, b \) prime to \( r \).
Proof of Proposition \[15\]. We follow the proof of [HZ 5.2 Theorem 1]. For simplicity, we assume $b = 1$. By the formula [HZ 5.2 (2)] which goes back to Eisenstein [E], we have
\[
(\frac{q}{r}) = -\frac{1}{2r} \sum_{\ell=1}^{r-1} \zeta_r^{\ell q} \zeta_r^{\ell} + 1 \zeta_r^{\ell - 1}
\]
for any integers $q$ and $r$. (Note that there is a minor misprint in [HZ 5.2 (2)].) Applying this formula, we get
\[
\sum_{\ell=1}^{r-1} ((\frac{a\ell}{r})) ((\frac{\ell}{r})) = \frac{1}{4r^2} \sum_{\ell=1}^{r-1} \sum_{m=1}^{r-1} \zeta_r^{(m+ak)\ell} \zeta_r^{m + 1} \zeta_r^k + 1 \zeta_r^{m - 1} \zeta_r^{k - 1}
\]
\[
= \frac{1}{4r^2} \sum_{k=1, r \nmid ak}^{r-1} \zeta_r^{ak} + 1 \zeta_r^k + 1 \zeta_r^{ak} - 1 \zeta_r^k - 1 = -\frac{1}{4r^2} \sum_{k=1, r \nmid ak}^{r-1} 1 + \zeta_r^{ak} 1 + \zeta_r^k 1 - \zeta_r^{ak} 1 - \zeta_r^k,
\]
since
\[
\sum_{\ell=1}^{r} \zeta_r^{(m+ak)\ell} = \begin{cases} 0 & \text{if } m + ak \not\equiv 0 \mod r, \\ r & \text{if } m + ak \equiv 0 \mod r. \end{cases}
\]
Corollary 17. Let $K \subset J \subset I_0$. Then
\[
\frac{1}{4} \sum_{h \in G^K} \left( \sum_{j \in J \setminus K, \lambda_j(h) \neq 1} \frac{1 + \lambda_j(h)}{1 - \lambda_j(h)} \right)^2 = -|G^K| \sum_{h \in G^K} \left( \sum_{j \in J \setminus K} ((a_jw_j)) \right)^2,
\]
where $\lambda_j(h) = e[a_jw_j]$ for all $h \in G^K$ and $j \in J \setminus K$.
Proof. This follows from Proposition \[15\] by the same arguments as in the proof of Corollary \[14\].

Proposition 18. One has
\[
\mu_{(f,G)} = \frac{(-1)^n}{|G|} \left\{ \sum_{I \subseteq I_0} \prod_{i \in I} \left( 1 - \frac{1}{w_i} \right) \left[ \sum_{J \subseteq J_0} (-1)^{|J|-|I|} |G^J|^2 \right] \right\}. \tag{2.1}
\]
Proof. Let $J \subseteq I_0$. Let $G_J$ be the set of elements of $g \in G$ with $\lambda_j(g) = 1$ for $j \in J$ and $\lambda_j(g) \neq 1$ for $j \not\in J$, i.e. the set of elements of $G$ which fix the coordinates $x_j, j \in J$, and only these coordinates. Then
\[
|G_J| = \sum_{K, J \subseteq K \subseteq I_0} (-1)^{|K|-|J|} |G^K|.
\]
Let $I \subset J$. Let $G_{I,J}$ be the set of elements $g$ of $G$ with $\lambda_i(g) = 1$ for $i \in I$ and $\Lambda_j(g) \neq 1$ for $j \in J \setminus I$ (and $\lambda_k(g)$ arbitrary for $k \in I_0 \setminus J$). Then

$$|G_{I,J}| = \sum_{K, I \subseteq K \subseteq J} (-1)^{|K|-|I|}|G^K|.$$ 

By Proposition 11 one has

$$\lim_{y \to 1} \chi(f, G)(y) = \frac{(-1)^n}{|G|} \sum_{J} |G_J| \left( \sum_{I \subseteq J} \prod_{i \in I} \left( 1 - \frac{1}{w_i} \right) |G_{I,J}| \right).$$

Now let $I \subset I_0$ be fixed. Then

$$\sum_{J \subseteq J \subseteq I_0} |G_J||G_{I,J}| = \sum_{J \subseteq J \subseteq I_0} \left( \sum_{K \subseteq K \subseteq I_0} (-1)^{|K|-|J|}|G^K| \right) \left( \sum_{L \subseteq L \subseteq J} (-1)^{|L|-|J|}|G^L| \right)$$

$$= \sum_{I \subseteq L \subseteq I_0} \sum_{K \subseteq K \subseteq I_0} \left( \sum_{J \subseteq J \subseteq K} (-1)^{|K|+|L|-|J|} \right) |G^K||G^L|$$

$$= \sum_{K \subseteq K \subseteq I_0} (-1)^{|K|-|L|}|G^K|^2;$$

since for fixed $L \subset I_0$ and $K \subset I_0$ with $L \subset K$

$$\sum_{J \subseteq J \subseteq K} (-1)^{|K|+|L|-|J|} = (-1)^{|K|-|L|}(1 - 1)^{|K|-|L|} = \begin{cases} (-1)^{|K|-|L|} & \text{for } L = K, \\ 0 & \text{otherwise.} \end{cases} \quad (2.2)$$

\[
\square
\]

Now we are ready to state the main result of our paper.

**Theorem 19.** One has

$$\text{Var}_{(f,G)} = \sum_{p,q \in \mathbb{Q}} (-1)^{(p-n)+q} \left( q - \frac{n}{2} \right)^2 h^{p,q}(f,G) = \frac{1}{12} \hat{c} \cdot \mu(f,G).$$

**Proof.** We use the notation introduced in the proof of Proposition 18. By Proposition 11 and Lemma 12 we have

$$\lim_{y \to 1} \frac{d}{dy} \left( y \frac{d}{dy} \chi(f, G)(y) \right) = A + B + C,$$
where

\[
A := \frac{(-1)^n}{|G|} \sum_{J, j \subset I_0} \sum_{g \in G_J} \left( \text{age}(g) - \frac{n - n_g}{2} \right)^2 \left[ \sum_{I_i, i \in I} \left( 1 - \frac{1}{w_i} \right) |G_{I_i J}| \right],
\]

\[
B := \frac{(-1)^n}{|G|} \sum_{J, j \subset I_0} |G_J| \left[ \sum_{I_i, i \in I} \left( 1 - \frac{1}{w_i} \right) \sum_{h \in G_{I_i J}} \frac{1}{4} \left( \sum_{j \in J \setminus I} \frac{1 + \lambda_j(h)}{1 - \lambda_j(h)} \right)^2 \right],
\]

\[
C := \frac{(-1)^n}{|G|} \sum_{J, j \subset I_0} |G_J| \times \left[ \sum_{I_i, i \in I} \left( 1 - \frac{1}{w_i} \right) \left( \sum_{i \in I} \frac{1 - 2w_i}{12} \right) - \sum_{h \in G_{I_i J}} \sum_{j \in J \setminus I} \frac{(1 - 2w_j)\lambda_j(h)}{(1 - \lambda_j(h))^2} \right].
\]

a) We first show that \( A + B = 0 \). We first take the sums in \( A \) and \( B \) in a different order:

\[
A = \frac{(-1)^n}{|G|} \sum_{I_i, i \in I} \prod_{i \subset J} \left( 1 - \frac{1}{w_i} \right) A_I, \quad A_I := \sum_{J, j \subset I_0} \sum_{g \in G_J} \left( \text{age}(g) - \frac{n - n_g}{2} \right)^2 |G_{I_i J}|,
\]

\[
B = \frac{(-1)^n}{|G|} \sum_{I_i, i \in I} \prod_{i \subset J} \left( 1 - \frac{1}{w_i} \right) B_I, \quad B_I := \sum_{J, j \subset I_0} |G_J| \left( \sum_{h \in G_{I_i J}} \frac{1}{4} \left( \sum_{j \in J \setminus I} \frac{1 + \lambda_j(h)}{1 - \lambda_j(h)} \right)^2 \right).
\]

Now let \( I \subset I_0 \) be fixed. Let \( \lambda_i(g) = e[a_i w_i] \). Then we have on one hand:

\[
A_I = \sum_{J, j \subset I_0} |G_{I_i J}| \sum_{g \in G_J} \left( \sum_{j \in I_0 \setminus J} (a_j w_j) \right)^2
= \sum_{J, j \subset I_0} |G_{I_i J}| \sum_{K, j \subset K \subset I_0} (-1)^{|K| - |J|} \sum_{g \in G^K} \left( \sum_{j \in I_0 \setminus K} (a_j w_j) \right)^2.
\]
On the other hand we have by Corollary 17

\[ B_I = \sum_{I \subset J \subset I_0} |G_J| \sum_{h \in G_{I,J}} \frac{1}{4} \left( \sum_{j \in J \setminus I} \frac{1 + \lambda_j(h)}{1 - \lambda_j(h)} \right)^2 \]

\[ = \sum_{I \subset J \subset I_0} |G_J| \sum_{K \subset J \subset I_0} (-1)^{|K| - |J|} \sum_{h \in G^K} \frac{1}{4} \left( \sum_{j \in J \setminus K} \frac{1 + \lambda_j(h)}{1 - \lambda_j(h)} \right)^2 \]

\[ = - \sum_{I \subset J \subset I_0} |G_J| \sum_{K \subset J \subset I_0} (-1)^{|K| - |J|} \sum_{h \in G^K} \left( \sum_{j \in J \setminus K} ((a_j w_j)) \right)^2 \]

For \( I \subset K \subset J \subset I_0 \) let

\[ s(K, J) := \sum_{g \in G^K} \left( \sum_{j \in J \setminus K} ((a_j w_j)) \right)^2 \]

Then

\[ A_I = \sum_{K \subset J \subset I_0} \sum_{I \subset K \subset I_0} (-1)^{|K| - |J|} |G_{I,J}| s(K, I_0) \]

\[ := \sum_{K \subset J \subset I_0} \sum_{I \subset K \subset I_0} (-1)^{|K| - |J|} \left( \sum_{L \subset J \subset K} (-1)^{|L| - |J|} |G^L| \right) s(K, I_0) \]

\[ = \sum_{L \subset J \subset I_0} \sum_{K \subset J \subset I_0} \left( \sum_{I \subset K \subset I_0} (-1)^{|K| + |L| - |J|} \right) |G^L| s(K, I_0) \]

\[ = \sum_{K \subset I_0} (-1)^{|K| - |I|} |G^K| s(K, I_0) \]
by Formula (2.2). On the other hand, we have

\[ B_I = - \sum_{K \subset I \subset J \subset I_0} (-1)^{|K|-|I|} |G_J||G^K|s(K, J) \]

\[ = - \sum_{K \subset I \subset J \subset I_0} \sum_{i \in K} (-1)^{|K|-|I|} \left( \sum_{J' \subset J} (-1)^{|J'|-|J|} |G^{J'}| \right) |G^K|s(K, J) \]

\[ = - \sum_{K \subset I \subset J \subset I_0} \sum_{i \in K} \left( \sum_{L \subset J} (-1)^{|K|-|L|-|J|} |G^L| \right) |G^K|s(K, J) \]

\[ = - \sum_{K \subset I \subset J \subset I_0} (-1)^{|K|-|I|} |G^K|s(K, I_0) = -A_I, \]

again by Formula (2.2) and since \(|G^{I_0}| = 1\). This shows that \(A + B = 0\).

b) We now consider the term \(C\). Let \(J \subset I_0\), \(I \subset J\) and \(j \in J\), \(j \not\in I\). Then it follows from Corollary 14 that

\[ - \sum_{h \in G_{I,J}} \frac{\lambda_j(h)}{(1 - \lambda_j(h))^2} = \frac{1}{12} m^{j}_{I,j}, \]

where

\[ m^{j}_{I,j} := \sum_{K, j \not\in K, i \in K \subset J} (-1)^{|K|-|I|} |G^{K \cup \{i\}}| \left( |G^K/G^{K \cup \{i\}}| \right)^2 - 1). \]

By a) we have

\[ \lim_{y \to 1} \frac{d}{dy} \left( y \frac{d}{dy} \chi(f, G)(y) \right) = C \]

\[ = \frac{(-1)^n}{|G|} \sum_{J, i \in I_0} |G_J| \left[ \sum_{I, j \subset J} \prod_{i \in I} \left( 1 - \frac{1}{w_i} \right) \left( |G_{I,J}| \left( \sum_{i \in I} \frac{1 - 2w_i}{12} \right) + \sum_{j \in J, j \not\in I} m^{j}_{I,j} \left( \frac{1 - 2w_j}{12} \right) \right) \right] \]

\[ = \frac{(-1)^n}{|G|} \sum_{I, i \in I_0} \prod_{i \in I} \left( 1 - \frac{1}{w_i} \right) \left[ \sum_{J, i \subset J \subset I_0} |G_J| \left( |G_{I,J}| \left( \sum_{i \in I} \frac{1 - 2w_i}{12} \right) + \sum_{j \in J, j \not\in I} m^{j}_{I,j} \left( \frac{1 - 2w_j}{12} \right) \right) \right]. \]
Now let $I \subset I_0$ and $j \notin I$ be fixed. Then
\[
\sum_{J \in J, I \subset J \subset I_0} |G_J|m^J_{I,J} = \sum_{J \in J, I \subset J \subset I_0} \left( \sum_{K \subseteq I \subset J \subset I_0} (-1)^{|K| - |J|}|G^K| \right) \left( \sum_{L \neq L_j, I \subset L \subset I_0} (-1)^{|L| - |I|}|G^{L \cup \{j\}}| \left( |G^L/G^{L \cup \{j\}}|^2 - 1 \right) \right)
\]
\[
= \sum_{L \neq L_j, I \subset L \subset I_0} \sum_{K \subseteq I \subset J \subset I_0} \left( \sum_{J \in J, I \subset J \subset I_0} (-1)^{|K| + |L| - |I| - |J|} \right) |G^K||G^{L \cup \{j\}}| \left( |G^L/G^{L \cup \{j\}}|^2 - 1 \right).
\]
Since $j \notin L$ but $j \in J$, the case $J = L$ and hence also $K = L$ is excluded in the sum
\[
\sum_{J \in J, I \subset J \subset K} (-1)^{|K| + |L| - |I| - |J|}.
\]
Therefore
\[
\sum_{J \in J, I \subset J \subset K} (-1)^{|K| + |L| - |I| - |J|} = \begin{cases} (-1)^{|L| - |I|} & \text{for } K = L \cup \{j\}, \\ 0 & \text{otherwise}. \end{cases}
\]
Hence we obtain
\[
\sum_{J \in J, I \subset J \subset I_0} |G_J|m^J_{I,J} = \sum_{L \neq L_j, I \subset L \subset I_0} (-1)^{|L| - |I|}|G^{L \cup \{j\}}|^2 \left( |G^L/G^{L \cup \{j\}}|^2 - 1 \right).
\]
\[
= \sum_{L \neq L_j, I \subset L \subset I_0} (-1)^{|L| - |I|} \left( |G^L|^2 - |G^{L \cup \{j\}}|^2 \right)
\]
\[
= \sum_{K \subseteq I \subset I_0} (-1)^{|K| - |I|}|G^K|^2.
\]
Therefore the statement follows from Proposition 18. \qed

3. Variance of the Exponents for Cusp Singularities with Group Actions

Let $f(x_1, x_2, x_3) := x_1^{\alpha_1} + x_2^{\alpha_2} + x_3^{\alpha_3} - x_1x_2x_3$ and $G$ be a finite subgroup of $SL_n(\mathbb{C})$ acting diagonally on $\mathbb{C}^n$ under which $f$ is invariant. Let $K_i \subset G$ be the maximal subgroup fixing the coordinate $x_i$, $i = 1, 2, 3$. Define numbers $\gamma_1, \ldots, \gamma_3$ by
\[
(\gamma_1, \ldots, \gamma_3) = \left( \frac{\alpha_i}{|G/K_i|} \ast |K_i|, i = 1, 2, 3 \right),
\]
where we omit numbers which are equal to one on the right-hand side. Define a number $\chi(f, G)$ by
\[
\chi(f, G) := 2 - 2j_G + \sum_{i=1}^8 \left( \frac{1}{\gamma_i} - 1 \right).
\]
Lemma 20. Let the pair \((f, G)\) be as above.

(i) The Milnor number of the pair \((f, G)\) is given by

\[
\mu_{(f,G)} = 2 - 2j_G + \sum_{i=1}^{s} (\gamma_i - 1). \tag{3.1}
\]

(ii) The set of exponents for the pair \((f, G)\) is given by

\[
\{1,2\} \prod \left\{ \frac{1}{\gamma_1} + 1, \frac{2}{\gamma_1} + 1, \ldots, \frac{\gamma_1 - 1}{\gamma_1} + 1 \right\} \\
\prod \left\{ \frac{1}{\gamma_2} + 1, \frac{2}{\gamma_2} + 1, \ldots, \frac{\gamma_2 - 1}{\gamma_2} + 1 \right\} \prod \ldots \\
\ldots \prod \left\{ \frac{1}{\gamma_s} + 1, \frac{2}{\gamma_s} + 1, \ldots, \frac{\gamma_s - 1}{\gamma_s} + 1 \right\} \tag{3.2}
\]

Proof. See Corollary 5.13 and the proof of Theorem 5.12 of [ET].

We have the following formula for the variance. Note that we have \(\hat{c} = 1\) by Theorem [4].

Theorem 21. Let the pair \((f, G)\) be as above. The variance of the set of exponents of \((f, G)\) is given by

\[
\text{Var}_{(f,G)} = \frac{1}{12} \mu_{(f,G)} + \frac{1}{6} \chi_{(f,G)} = \frac{1}{12} \hat{c} \cdot \mu_{(f,G)} + \frac{1}{6} \chi_{(f,G)}. \tag{3.3}
\]

Proof. Some elementary calculation yields the statement.

We have the following formula for the variance. Note that we have \(\hat{c} = 1\) by Theorem [4].

Note that the pair \((f, G)\) can be considered as a mirror partner of the orbifold curve (Deligne–Mumford stack) \(C\) which is a smooth projective curve of genus \(j_G\) with \(s\) isotropic points of orders \(\gamma_1, \ldots, \gamma_s\) (cf. Theorem 7.1 of [ET]). The above formula for the variance is compatible with this observation. In particular, the dimension of \(C\) is 1, \(\mu_{(f,G)}\) is the orbifold Euler number \(\chi(C)\) of \(C\) and \(\chi_{(f,G)}\) is the orbifold Euler characteristic of \(C\), which is the degree of the first Chern class \(c_1(C)\) of \(C\). Applying this to the formula in Theorem [1] we recover the equation (3.3).

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