QMA(2) with postselection equals to NEXP

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Abstract
We study the power of QMA(2) with postselection and show that the power is equal to NEXP. Our method for showing this equality can be also used to prove that other classes with exponentially small completeness-soundness gap equals to the corresponding postselection versions.

1 Introduction
Verifying a proof sent from an unlimitedly powerful prover is one of the main issues in complexity theory. One of such complexity classes is MA, introduced by Babai [5]. This class is defined as the class of languages decided by a Merlin-Arthur system, that consists of a probabilistic polynomial time verifier called Arthur and an infinitely powerful prover called Merlin. Arthur computes with a proof sent from Merlin, and each yes instances have at least one proof that Arthur accepts with high probability. This class is important in classical complexity theory.

In quantum complexity theory, there exists a similar class called Quantum Merlin-Arthur (QMA), and this has been intensively studied since introduced by Knill [14], Kitaev [16], and Watrous [24]. In the most common setting, Merlin provides a quantum proof and Arthur is allowed to use polynomial time quantum computations.

Kobayashi et al. [17] posed a question that a concatenation of many quantum proofs can be simulated by one quantum proof. In classical settings, it is obvious that many proofs can be simulated by a concatenated one proof, but one quantum proof may not be a concatenation of non-entangled many proofs. Hence a direct simulation of the original proof system with concatenated one proof may be cheated by an entangled proof. This problem has a relation to the property of entanglement and fundamentals of quantum complexity, and it is natural to introduce the complexity class QMA(2), which is decided by a Merlin-Arthur system which uses two Merlins. QMA(2) has a natural complete problem arising from quantum chemistry, called Pure State N-Representability Problem.

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It is conjectured that witnesses without entanglements are hard to simulate by one witness, and QMA and QMA(2) are different complexity classes. For example, Blier and Tapp [9] showed that QMA(2) can solve SAT with $O(\log n)$ size witnesses and with polynomial inverse completeness-soundness gap, while such an algorithm is not known for QMA. The gap parameter of this result was improved by Aaronson et al. [4], who used $O(\sqrt{n\log n})$ length witnesses and got a constant completeness-soundness gap. Whether QMA($k$)($k > 2$), which is an analogue of QMA(2) with $k$ proofs, is equal to QMA(2) was an important problem. This problem was resolved by Harrow and Montanaro [11]. They showed QMA(2) = QMA($k$) (for any $k > 2$).

In quantum complexity theory, complexity classes with postselection has been used to show quantum supremacy of sub-universal models such as Boson Sampling [2], IQP [7, 8], and DQC1 [10, 20]. These results mean that the difficulty of the simulation of probabilistic distribution of quantum circuits relates to separation of Polynomial Hierarchy. Recently postselection beyond BQP was investigated in [21, 22]. Morimae and Nishimura [21] showed postQMA = PSPACE, where postQMA is the postselection version of QMA, using the result that PSPACE is equal to QMA with exponentially small gap [9]. They also showed several results on complexity classes with postselection.

In this article, we define postQMA(2) and show postQMA(2) = NEXP. The class postQMA($k$) is easily computed by NEXP, and hence the power of quantum computation with non-entangled witnesses and postselection is characterized exactly. Here we describe the main technical difficulty briefly, and more details are in the appendix. The previous techniques [1, 21] is preparing $\alpha|0\rangle + \beta|1\rangle$ by using output qubit and detecting $\beta \leq 0$. It is necessary to erase the garbage, that is, natural quantum computing makes a state in the form of $\alpha|0\rangle\phi_0 + \beta|1\rangle\phi_1$, but what we needs is $\alpha|0\rangle + \beta|1\rangle$. To erase the garbage, Aaronson [1] uses reversible computation of classical circuit and applying Hadamard transformation to computational basis, and Morimae and Nishimura [21] use distillation [15]. Both techniques cannot be applied to postQMA(2). The garbage of general quantum computation cannot be erased in contrast to superposition of computations of classical reversible circuit, and the witnesses cannot be restricted to eigenvectors since the witness space is not linear, therefore distillation cannot be applied to postQMA(2). These difficulties are linked to the difficulty of computation with non-entangled proofs, and it seems difficult to analyze postQMA(2) by the previous protocol. Our technique is restricting completeness to 1 or bounding completeness error, and bounding the acceptance probability even with garbage superposition. Our technique is useful for other complexity classes to prove that the corresponding classes with exponentially small completeness/soundness gaps are equal to the postselection versions. For example we define postQIP and sketch the proof that postQIP equals to QIP with exponentially small gap.

The remainder of the paper is organized as follows. Section 2 defines a new complexity class postQMA(2). Section 3 shows the main result. In section 4 we show another result about postselection to compare with the main result and discuss about the case that completeness is less than 1. In section 5 we give several open problems.
2 Preliminaries

We assume that readers are familiar with quantum computation [16] and classical computational complexity [3]. In this section we define QMA(2)(c, s) and postQMA(2)(c, s).

Definition 1 (QMA(2)(c, s)). Let \( L \) be a language. Let \( c(n), s(n) : \mathbb{Z} \to [0,1] \) be functions that can be computed in polynomial time and \( c(n) - s(n) \) is positive and larger than the inverse of some polynomial if \( n \) is sufficiently large. \( L \) is in QMA(2)(c, s) if there exist polynomials \( w(n), m(n), \) and a uniform quantum circuit family \( \{ V_x \} \) constructed in polynomial time that satisfies follows for any \( n \) and any string \( x \) with \( |x| = n \):

if \( x \in L \), then there exist 2 \( w(n) \)-qubit states \( |\psi_1\rangle \) and \( |\psi_2\rangle \) such that

\[
\Pr_{V_x(\psi_1,\psi_2)}[o = 1] \geq c,
\]

if \( x \notin L \), then for any 2 \( w(n) \)-qubit states \( |\psi_1\rangle \) and \( |\psi_2\rangle \),

\[
\Pr_{V_x(\psi_1,\psi_2)}[o = 1] \leq s.
\]

Here, \( \Pr_{V_x(\psi_1,\psi_2)}[o = 1] \) is the probability that \( V_x \) with inputs \( |\psi_1\rangle|\psi_2\rangle \) outputs \( o = 1 \). Namely it is defined by

\[
\Pr_{V_x(\psi_1,\psi_2)}[o = 1] = \Tr[|1\rangle\langle 1| \otimes I^{\otimes 2w(n)+m(n)-1}Q_x(|\psi_1\rangle\langle \psi_1| \otimes |\psi_2\rangle\langle \psi_2| \otimes |0\rangle^{\otimes m(n)}Q_x^\dagger)].
\]

Definition 2 (postQMA(2)(c, s)). Let \( L \) be a language. Let \( c(n), s(n) : \mathbb{Z} \to [0,1] \) be functions that can be computed in polynomial time and \( c(n) - s(n) \) is positive and larger than the inverse of some polynomial if \( n \) is sufficiently large. \( L \) is in postQMA(2)(c, s) if there exist polynomials \( w(n), m(n), l(n), \) and a uniform quantum circuit family \( \{ V_x \} \) constructed in polynomial time that satisfies follows for any \( n \) and any string \( x \) with \( |x| = n \):

For all 2 \( w(n) \)-qubit states \( |\psi_1\rangle \) and \( |\psi_2\rangle \),

\[
\Pr_{V_x(\psi_1,\psi_2)}[p = 1] \geq 2^{-l(n)},
\]

if \( x \in L \), then there exist 2 \( w(n) \)-qubit states \( |\psi_1\rangle \) and \( |\psi_2\rangle \) such that

\[
\Pr_{V_x(\psi_1,\psi_2)}[o = 1|p = 1] \geq c(n),
\]

if \( x \notin L \), then for any 2 \( w(n) \)-qubit states \( |\psi_1\rangle \) and \( |\psi_2\rangle \),

\[
\Pr_{V_x(\psi_1,\psi_2)}[o = 1|p = 1] \leq s(n).
\]
Here, \( \Pr_{V_x(\psi_1|\psi_2)}[o = 1|p = 1] \) is the conditional probability that \( V_x \) with inputs \( |\psi_1\rangle|\psi_2\rangle \) outputs \( o = 1 \) with the condition \( p = 1 \). This probability is calculated as follows.

\[
\Pr_{V_x(\psi_1|\psi_2)}[o = 1|p = 1] = \frac{\Pr_{V_x(\psi_1|\psi_2)}[o = 1, p = 1]}{\Pr_{V_x(\psi_1|\psi_2)}[p = 1]}
\]

Each term in LHS is defined by

\[
\Pr_{V_x(\psi_1|\psi_2)}[o = 1, p = 1] = \text{Tr}[|1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes I^\otimes 2w(n) + m(n) - 1 Q_x |\psi_1\rangle \langle \psi_1| \otimes |\psi_2\rangle \langle \psi_2| \otimes |0\rangle \langle 0| \otimes m(n) Q_x^\dagger],
\]

\[
\Pr_{V_x(\psi_1|\psi_2)}[p = 1] = \text{Tr}[I \otimes |1\rangle\langle 1| \otimes I^\otimes 2w(n) + m(n) - 1 Q_x |\psi_1\rangle \langle \psi_1| \otimes |\psi_2\rangle \langle \psi_2| \otimes |0\rangle \langle 0| \otimes m(n) Q_x^\dagger].
\]

### 3 Main Result

**Theorem 1.** There exists a constant \( s \) such that \( \text{postQMA}(2)(1, s) = \text{NEXP} \).

Theorem 1 is sufficient to prove that there is a polynomial \( p \) which satisfies \( \text{postQMA}(2)(1, 2^{-p}) = \text{NEXP} \), since we can amplify the gap between completeness and soundness by repetition of witnesses. This is also sufficient to prove that \( \text{postQMA}(2)(c, s) = \text{NEXP} \) for any constant \( c < 1 \), since it is enough to prepare a new 1 qubit \( \sqrt{c}|0\rangle + \sqrt{1-c}|1\rangle \) and accept if the original completeness 1 protocol accepts and the new qubit outputs 1. First, we prove \( \text{postQMA}(2) \subseteq \text{NEXP} \).

**Proposition 1.** \( \text{postQMA}(2) \subseteq \text{NEXP} \).

**Proof.** The witnesses of \( \text{postQMA}(2) \) are poly-length qubits, and hence exponential length classical bits can describe these witnesses with exponential precision, and the acceptance probability can be computed in non-deterministic exponential time. \( \square \)

Next, we prove that \( \text{postQMA}(2) \supseteq \text{NEXP} \). We use the next lemma from previous results.

**Lemma 1 ([6][23]).** For any polynomial \( r \), \( \text{NEXP} \subseteq \text{QMA}(2)(1, 1 - 2^{-r}) \) holds.

The sketch of this theorem is as follows: 3COLOR can be solved with log size witnesses, if the completeness/soundness gap is the inverse of a polynomial [6]. A similar proof is efficient for succinct 3COLOR [23].

The next lemma is our main technical result.

**Lemma 2.** There exists a constant \( s \) such that \( \text{QMA}(2)(1, 1 - 2^{-r}) \subseteq \text{postQMA}(2)(1, s) \) holds for any polynomial \( r \).
 Protocol 1

$Q_x$ is the circuit for $L \in \text{QMA}(2)(1, 1 - 2^{-r})$. Denote $|\psi\rangle = |\psi_{x,1}\rangle|\psi_{x,2}\rangle|0^n\rangle$: which is witnesses and ancillas.

We first prepare $|0\rangle|0\rangle|\psi\rangle$.
1. Apply $Q_x$ to $|\psi\rangle$.
2. Copy the output qubit to the second qubit.
3. Apply $Q_x^{-1}$. 
4. Prepare $\epsilon(1) - |0\rangle$ in the first qubit.
5. Apply the unitary operator $\frac{1}{1 + \epsilon^2}
\begin{pmatrix}
1 & \epsilon \\
-\epsilon & 1
\end{pmatrix}$ to the second qubit.
6. Measure the first and second qubits by projection onto \{\ket{00}, \ket{11}\}/\{\ket{01}, \ket{10}\} and postselect \{\ket{00}, \ket{11}\}.
7. Measure the first and second qubits by projection onto \{\ket{00} + \ket{11}\}/\{\ket{00} - \ket{11}, \ket{01} + \ket{10}, \ket{01} - \ket{10}\}.

Figure 1: Protocol of postQMA(2) transferred from QMA(2)(1, 1 - 2^{-r}). While step 3 is not necessary, but we add it for our analysis.

Proof. Suppose a circuit family $\{Q_x\}$ solves $L \in \text{QMA}(2)(1, 1 - 2^{-r})$. We construct a circuit family of postQMA(2) from $\{Q_x\}$. The protocol is in Figure 1. The analysis is similar to [21], using Distillation [15], but the main difference is that there remains $|\perp\rangle$ orthogonal to the input state, since witnesses that maximize the acceptance probability may not be an eigenstate if $x$ is a no instance.

Suppose $|\psi_{x,1}\rangle|\psi_{x,2}\rangle$ is the witness that maximize the acceptance probability of $Q_x$. We use 2 qubits in addition to $|\psi_{x,1}\rangle|\psi_{x,2}\rangle|0^n\rangle$. There exist some states $|\phi_{x,0}\rangle, |\phi_{x,1}\rangle$ that satisfy the following equation

$$Q_x|\psi_{x1}\rangle|\psi_{x2}\rangle|0^n\rangle = \sqrt{p_x}|1\rangle|\phi_{x,1}\rangle + \sqrt{1-p_x}|0\rangle|\phi_{x,0}\rangle$$

(1)

$|\mathbf{1}\rangle$ is the state after step 1 in Figure 1. Prepare 1 qubit $|0\rangle$ in addition to $|\mathbf{1}\rangle$. Apply CNOT on the new qubit and the first qubit of $|\mathbf{1}\rangle$ in step 2, where the latter is the control qubit. After step 2 the state is (2).

$$\sqrt{p_x}|1\rangle|\phi_{x,1}\rangle + \sqrt{1-p_x}|0\rangle|\phi_{x,0}\rangle.$$ 

(2)

Apply $Q_x^{-1}$ on the qubits of (2) that originally $Q_x$ acts on. The next state is as follows.

$$Q_x^{-1}\sqrt{p_x}|1\rangle|\phi_{x,1}\rangle + Q_x^{-1}\sqrt{1-p_x}|0\rangle|\phi_{x,0}\rangle = \sqrt{p_x}|1\rangle|f_1\rangle + \sqrt{1-p_x}|0\rangle|f_0\rangle.$$ 

(3)

Denote $|\psi_{x,1}\rangle|\psi_{x,2}\rangle|0^n\rangle = |\psi\rangle$ and $\Pi_1 = |1\rangle\langle 1| \otimes I^{\otimes (n+m(n)-1)}$. Since $\langle \psi|\sqrt{p_x}|f_1\rangle = \langle \psi|Q_x^{-1}\Pi_1 Q_x|\psi\rangle = |\Pi_1 Q_x|\psi\rangle|^2 = p_x \langle \psi|f_1\rangle = \sqrt{p_x}$ and there exists $|\perp\rangle$ orthogonal to $|\psi\rangle$ such that $|f_1\rangle = \sqrt{p_x}|\psi\rangle + \sqrt{1-p_x}|\perp\rangle$. Similarly $|f_0\rangle$ can be written.
by $|\perp_2\rangle$ orthogonal to $|\psi\rangle$ as follows: $|f_0\rangle = \sqrt{1-p_x}|\psi\rangle + \sqrt{p_x}|\perp_2\rangle$. Note that

$$\sqrt{p_x}|f_1\rangle + \sqrt{1-p_x}|f_0\rangle = |\psi\rangle, \quad |\perp_2\rangle = -|\perp_2\rangle,$$ and $|f_0\rangle = \sqrt{1-p_x}|\psi\rangle - \sqrt{p_x}|\perp_2\rangle$.

RHS of (3) equals to

$$(p_x|1\rangle + (1-p_x)|0\rangle)|\psi\rangle + \sqrt{p_x(1-p_x)}(|1\rangle - |0\rangle)|\perp_2\rangle).$$

(4)

Define $\epsilon = 2^{-10r}$. Prepare another new 1 qubit in state $|\epsilon\rangle-|0\rangle$ (we omit the normaliation factor $1/(1+\epsilon^2)$ for convenience). The state of whole qubits are as follows.

$$(|\epsilon\rangle-|0\rangle)(p_x|1\rangle + (1-p_x)|0\rangle)|\psi\rangle + \sqrt{p_x(1-p_x)}(|1\rangle - |0\rangle)|\perp_2\rangle).$$

(5)

Apply a unitary operator $|0\rangle \rightarrow |0\rangle + \epsilon|1\rangle, |1\rangle \rightarrow |1\rangle - \epsilon|0\rangle$ on the second qubit (again we omit the normaliation factor). The whole state will become as follows:

$$(|\epsilon\rangle-|0\rangle)(p_x + (1-p_x)\epsilon)|1\rangle + ((1-p_x) - \epsilon p_x)|0\rangle)|\psi\rangle + \sqrt{p_x(1-p_x)}((1-\epsilon)|1\rangle - (1+\epsilon)|0\rangle)|\perp_2\rangle).$$

(6)

Measure the first 2 qubits by projection to the space spaned by $\{|00\rangle, |11\rangle\}/\{|01\rangle, |10\rangle\}$, and postslect $\{|00\rangle, |11\rangle\}$. The states becomes:

$$\{\epsilon(p_x + (1-p_x)\epsilon)|11\rangle - ((1-p_x) - \epsilon p_x)|00\rangle\}|\psi\rangle + \sqrt{p_x(1-p_x)}(\epsilon(1-\epsilon)|11\rangle + (1+\epsilon)|00\rangle)|\perp_2\rangle).$$

(7)

Measure the first 2 qubit by projection to $\{|00\rangle + |11\rangle\}$ and the complement, and accept if $|00\rangle + |11\rangle$ is measured.

Now we analyze the acceptance probability. If the instance is yes, then $p_x = 1$ and hence $|00\rangle + |11\rangle$ is measured with probability 1. Assume the instance is no. Hereinafter $|\mu\rangle, |\mu'\rangle$ mean vectors with $O(\epsilon)$ norms. (7) can be denoted as follows.

$$\{-(1-p_x)|00\rangle + |\mu\rangle\}|\psi\rangle + \sqrt{p_x(1-p_x)}(|00\rangle + |\mu'\rangle)|\perp_2\rangle).$$

(8)

Since $1-p_x \geq 2^{-r}$ and $\epsilon = 2^{-10r}$, the probability of projection to $\{|00\rangle + |11\rangle\}$ of this vector is $\frac{1}{\sqrt{2}} + O(\epsilon/(1-p_x))$.

Theorem 1 is proved from Lemmas 1 and 2.

4 Applications to other complexity classes and to the case that completeness is strictly smaller than 1

4.1 postQIP

In this subsection, to show an example of application of our techniques to other classes, we define postQIP and prove postQIP equals to QIP with exponentially
small gap. It can be proved that most classes with completeness one and with postselection are equal to themselves with exponentially small gap by our technique. We omit details since the technique is very similar to the main theorem. We remark that this relation is not obvious by techniques in previous research [1, 21].

**Definition 3.** (postQIP)
A language $L$ is in postQIP if there exist a polynomial $r(n)$ and a verifier $\{V_x\}$ who can do quantum polynomial time computation and outputs 2 bit $o$ and $p$ such that:

For any prover $P$,
\[\Pr[\langle V, P \rangle(p = 1)] \geq 1/2^{r(|x|)},\]
if $x \in L$, then there exists a prover $P$ such that
\[\Pr[\langle V, P \rangle(o = 1|p = 1)] \geq 2/3,\]
if $x \notin L$, then for any prover $P$,
\[\Pr[\langle V, P \rangle(o = 1|p = 1)] \leq 1/3.\]

Here, $\Pr[\langle V, P \rangle(p = 1)]$ means the probability that $V$ outputs $p = 1$ at the end of the interactive protocol between $V$ and $P$, and $\Pr[\langle V, P \rangle(o = 1|p = 1)]$ means the conditional probability that $V$ outputs $o = 1$ at the end of the interactive protocol between $V$ and $P$ with $p = 1$.

**Definition 4.** (QIP$^{\text{exp}}$)
A language $L$ is in QIP$^{\text{exp}}$ if there exist a verifier $V$ and functions of $|x|$, $c(|x|), s(|x|)$ such that $c(|x|) - s(|x|) > \frac{1}{\exp}$, $0 \leq c(|x|), s(|x|) \leq 1$ satisfying followings:

if $x \in L$, then there exists a prover $P$ such that
\[\Pr[\langle V, P \rangle(o = 1)] \geq c(|x|),\]
if $x \notin L$, then for any prover $P$,
\[\Pr[\langle V, P \rangle(o = 1|p = 1)] \leq s(|x|).\]

**Proposition 2.** postQIP = QIP$^{\text{exp}}$

(Sketch). First we show QIP$^{\text{exp}}$ can be computed with completeness 1 by similar techniques to prove usual QIP can be computed with completeness 1 [10]. Next we use protocol 2, which is almost the same as protocol 1, except for the first state in the protocol 1, protocol 2 uses the final state of the interactive protocol just before measuring, instead of the witnesses $|\psi\rangle$ of postQMA(2)
Let $Q_x$ be the last verifier’s circuit for $L \in \text{QIP}(1, 1 - 2^{-r})$. Let $|\psi\rangle$ be the state just before applying $Q_x$:

1. Apply $Q_x$ to $|\psi\rangle$.
2. Copy the output qubit to the second qubit.
3. Apply $Q_x^{-1}$.
4. Prepare $\epsilon|1\rangle - |0\rangle$ in the first qubit.
5. Apply the unitary operator $\frac{1}{1+\epsilon^2} \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix}$ to the second qubit.
6. Measure the first and second qubits by projection onto $\{|00\rangle, |11\rangle\}/\{|01\rangle, |10\rangle\}$ and postselect $\{00, 11\}$.
7. Measure the first and second qubits by projection onto $\{00 + |11\rangle\}/\{00 - |11\rangle, 01 + |10\rangle, |01\rangle - |10\rangle\}$.

Figure 2: Protocol of postQIP transferred from QIP$(1, 1 - 2^{-r})$. Though this is almost same to postQMA(2), we include this to show an application of our technique to other classes explicitly.

4.2 The case that completeness is strictly less than 1

In this subsection we amplify the completeness/soundness gap of a protocol of which completeness is strictly smaller than 1 by postselection. We assume that $1 \gg \delta^2 \gg \epsilon$. The statement is as follows.

**Proposition 3.** For any QIP$_{\text{exp}}$ protocol with completeness $1 - \epsilon$ and soundness $1 - \delta$ ($\epsilon \ll \sqrt{\delta}$), there exists a postQIP protocol with completeness/soundness gap larger than the constant.

Proposition 2 is enough to prove that QIP$_{\text{exp}}$ with 3 or more rounds is equal to postQIP, since 3 rounds interactive protocols can be transformed to protocols with completeness 1 (normal QIP case is in [18], and almost same to the exponential gap case.). On the other hand, proposition 3 is also true for 2 rounds interactive proof, but it is not clear how to make completeness near to 1 for 2 round protocols.

**Proof.** The protocol is in Figure 3. First, we analyze the acceptance probability $p_x$ of yes instances. Let $\epsilon' = 1 - p_x$. After the step 4, the state is as follows.

$$(\delta |1\rangle - |0\rangle)\{(1 - \epsilon')|1\rangle + \epsilon'|0\rangle\}|\psi\rangle + \sqrt{\epsilon'(1 - \epsilon')}(1 - |0\rangle)|\perp\rangle. \quad (9)$$

In the next equations, $|\mu\rangle$ denotes a vector with $O(\epsilon')$ norm, and $|\tau\rangle$ denotes a vector with $O(\delta)$ norm. After the rotation in step 5 and postselection in step 6,
Protocol 3

Let $Q_x$ be the last verifier’s circuit for $L \in \text{QIP}(1 - \epsilon, 1 - \delta)$ and $Q_x$ accepts no instances with probability at least $1 - \sqrt{\delta}$. Let $|\psi\rangle$ be the state just before applying $Q_x$.

1. Apply $Q_x$ to $|\psi\rangle$.
2. Copy the output qubit to the second qubit.
3. Apply $Q_x^{-1}$.
4. Prepare $\delta |1\rangle - |0\rangle$ in the first qubit.
5. Apply the unitary operator $\frac{1}{1+\delta^2} \left( \begin{array}{cc} 1 & \delta \\ -\delta & 1 \end{array} \right)$ to the second qubit.
6. Measure the first and second qubits by projection onto $\{|00\rangle, |11\rangle\}$ and postselect $\{|00\rangle, |11\rangle\}$.
7. Measure the first and second qubits by projection onto $\{|00\rangle + |11\rangle\}/\{|00\rangle - |11\rangle\}$.

Figure 3: Protocol of postQIP transferred from QIP$(1 - \epsilon, 1 - \delta)$. The differences between protocol 1 and protocol 2 are the rotations in step 4 and 5.

we have the following state.

\[
(\delta(|11\rangle + |00\rangle) + |\mu\rangle)|\psi\rangle \\
+ \sqrt{\epsilon'}(1 - \epsilon')(\delta|11\rangle - |00\rangle + |\tau\rangle)|\perp\rangle. \tag{10}
\]

Since $\sqrt{\epsilon'} \leq \sqrt{\epsilon} \ll \delta$, if we measure this state in $\{|00\rangle + |11\rangle\}$ and its orthogonal vectors, we accept with probability $1 - O(\epsilon/\delta^2)$.

Next, we analyze the acceptance probability of no instances. Let $\delta' = 1 - p_x$.

After the step 4, the state is as follows.

\[
(\delta|1\rangle - |0\rangle)\{|0\rangle + (1 - \delta')|1\rangle + \delta'|0\rangle\}|\psi\rangle \\
+ \sqrt{\delta'}(1 - \delta')(|1\rangle - |0\rangle)|\perp\rangle. \tag{11}
\]

In the next equations, $|\tau\rangle$ denotes a vector with $O(\delta \delta')$ norm, and $|\tau'\rangle$ denotes a vector with $O(\delta)$ norm. After the rotation (step 5) and postselection (step 6), the state is as follows.

\[
(\delta|11\rangle + (\delta + \delta')|00\rangle + |\tau\rangle)|\psi\rangle \\
+ \sqrt{\delta'}(1 - \delta')(\delta|11\rangle - |00\rangle + |\tau'\rangle)|\perp\rangle. \tag{12}
\]

Since $\delta \leq \delta' \ll 1$, if we measure this state in $\{|00\rangle + |11\rangle\}$ and orthogonal vectors, we accept with probability $1 - \Omega(1)$.

5 Open Problems

We conclude the paper by posing the following four questions.
• The upper bounds of extremely small gap QMA(2):
  Though double-exponential gap QMA(2) is contained in NEXP, the upper bounds of QMA(2) with infinitely small gap is not obvious. About QMA, QMA with infinitely small gap is bounded by EXPSPACE. Moreover, if the gates are represented by algebraic numbers, and the completeness is 1, then the corresponding class is bounded by PSPACE[12].

• Decide whether small gap has more power or not:
  Exponentially small gap will be more powerful than polynomial gap and some classes have strong evidence. Following examples have relatively strong evidences:
  - QMA: QMA is in PP, but QMA with exponentially small gap contains PSPACE.
  - BQP: BQP will not be able to compute NP, but BQP with exponentially small gap can compute PP.
  - QMIP*: QMIP* is an example that the corresponding class with exponential small gap is more powerful, unless QMIP* equals to the exponential time version, since [13] showed that QMIP* with exponentially small gap coincides with the exponential time version of QMIP*.

But QIP seems not to have such evidence. The problem that exponentially small gap has more power than polynomial gap or not remains open.

Another problem is to prove that some complexity class with exponential gap is strictly powerful than polynomial gap without any assumptions. The above examples need some computational assumptions, such as PP \neq PSPACE, NP \not\subseteq BQP and that QMIP* is strictly less powerful than exponential time of it. These assumptions will be extremely difficult to prove. To the best of our knowledge, there are no complexity classes of which exponentially small gap version has strictly stronger than polynomial gap version that we can prove without any assumptions.

• Direct amplification of exponentially small gap for quantum complexity classes with completeness strictly smaller than 1:
  Our proof needs some assumption of completeness and soundness. If completeness is strictly smaller than 1, the state |\perp\rangle remains even in computations of yes instances, and our protocol fails. The postselection technique for more general classes remains open.

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7 Appendix

Overview of PP = postBQP [1] and postQMA = PSPACE [21]

Here we briefly describe the studies of postselection of Aaronson [1] and Morimae and Nishimura [21] and why their protocol cannot be directly applied to postQMA(2).

Aaronson’s protocol to prove PP = postBQP is as follows.

1. Make $\sqrt{N} \sum_r |r\rangle$, where $N$ is a normalization factor. Here, $r$ corresponds to a random string of the original PP algorithm.
2. Compute $\sqrt{N} \sum_r |r\rangle|b_r\rangle|\text{garbage}_r\rangle$, where $b_r$ is the output of the original PP algorithm which uses $r$ as a random number, and $|\text{garbage}_r\rangle$ is the garbage that depends on $r$.
3. Erase $|\text{garbage}_r\rangle$ by classical reversible circuit computation.
4. Apply Hadamard gates to $|r\rangle$, and then the state becomes $\sqrt{N} \sum_{r,y} (-1)^{r \cdot y} |y\rangle |b_r\rangle$.
5. Measure the register $|y\rangle$ and postselect $|0\rangle$. The state becomes $p_{\text{accept}} |0\rangle + (1 - p_{\text{accept}}) |1\rangle$.
6. Prepare a new 1 qubit $\alpha |0\rangle + \beta |1\rangle$, for some $\alpha, \beta$, apply the controlled-Hadamard gate on $(\alpha |0\rangle + \beta |1\rangle)(p_{\text{accept}} |0\rangle + (1 - p_{\text{accept}}) |1\rangle)$, where $\alpha |0\rangle + \beta |1\rangle$ is the control qubit.
7. Measure the non-control qubits in the computational basis and postselect $|1\rangle$.
8. Measure the control-qubit in the computational basis.

It is critical in step 3, 4, and 5. that the computation is the linear sum of classical reversible computation. Hence it is difficult to apply this protocol to postQMA(2).

Morimae and Nishimura’s protocol [21] to prove postQMA = QMA exp (= PSPACE) is as follows.

0. Witness is the eigenvector $|\psi\rangle$ of $\Pi_0 Q_x \Pi_{\text{acc}} Q_x$.
1. Apply $Q_x$.
2. Prepare a new 1 qubit $|0\rangle$ and apply CNOT on the deciding qubit and the new qubit, where the control qubit is the deciding qubit.
3. Apply $Q_x^\dagger$ and measure ancillas by projection onto $\{\Pi_0, I - \Pi_0\}$, and post-select $\Pi_0$. The new qubit in step 2. becomes $p_{\text{accept}} |0\rangle + (1 - p_{\text{accept}}) |1\rangle$.
4-6. Similar to steps 6–8 of Aaronson’s protocol.

Preparing the eigenvector is critical. Otherwise, the 1 qubit after step 3 entangles to remaining qubits and the 1 qubit is in mixed state. Postselection on mixed states will output a useless state, like in our protocol for no instances.
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