Linear stochastic stability analysis of nonlinear systems. Parametric destabilization of the wave propagation.

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Abstract

Straightforward method for the derivation of linearized version of stochastic stability analysis of the nonlinear differential equations is presented. Methods for the study of large time behavior of the moments are exposed. These general methods are applied to the study of the stochastic destabilization of the Langmuir waves in plasma.

1 INTRODUCTION

The influence of the external noise on the stability of the linearized dynamical systems was the subject of numerous studies on finite dimensional models, studied both in physical or mathematical literature [1]-[17]. In the following we extend part of the formalism to dynamical systems described by partial differential or integro-differential stochastic equations.

The main motivation of this study is related to the sensitivity of the onset to unstable regime, under the effect of random noise, observed in hydrodynamic turbulence [1], the effect of stochastic destabilization of the stable regime of simplest Hamiltonian systems, like linear undamped LC system with random capacitance [2], [3]. The main dynamical system of interest for the controlled thermonuclear fusion studies with magnetic confinement are described the kinetic equations. Before the study of these more complete models, we expose the main physical and mathematical principles on a more simple model: of Langmuir waves in turbulent background.
The large part of the mathematical aspects exposed here, at least those applied to finite dimensional systems, can be found in the mathematical literature. An example of application to large discrete systems, perturbed by space and temporal white noise, can be found in the works [10]-[13].

The derivation of the equations for the stochastic linearized system from the stochastic nonlinear one as well as the application to Langmuir waves, is new.

1.1 Physical model

The main class of problems studied in this article are related to the influence of the external noise on the linear stability of steady state of general, non linear physical systems. We suppose that the system is described by a set of general nonlinear partial differential, or integro-differential equations, that contains a stochastic perturbation, modelled by a random field.

In all of this part of the study the random field will be modelled by temporal white noise with general spatial correlations.

Instead of establishing a general abstract framework, in this article we illustrate the method by an example. The first interesting physical model is the linear hyperbolic evolution equation, that describe the propagation of nonlinear waves in plasma, under the effects of the external noise $n_\omega(x,t)$.

$$\frac{\partial^2 \Phi_\omega(x,t)}{\partial t^2} = \Delta \Phi_\omega - F(\Phi_\omega(x,t), n_\omega(x,t)) \quad (1)$$

Here $F(u,v)$ is a nonlinear function, having near $u = v = 0$ the linearized form $F(u,v) = \alpha u + \beta v + O(u^2 + v^2)$, with $u > 0$. We are interested in the linear stability of the solution in the deterministic case $\Phi(x,t) = 0$, under the effect of the noise $n(x,t)$, represented in fact by a random field whose statistical properties are supposed to be known. In particular we suppose that the random field is homogenous and stationary.

We illustrate now the general recipe to reduce the linear stochastic stability analysis of nonlinear systems to standard form. Similar to the deterministic case we expand $F(\Phi, n)$ in the variable $\Phi$ up to order 2 term to obtain

$$F(\Phi, n_\omega) = \alpha \Phi + M_\omega(x,t)\Phi + a_\omega(x,t) + O(\Phi^2) \quad (2)$$

where $\alpha > 0$ is constant, and the multiplicative and additive noise term are given exactly by

$$M_\omega(x,t) = \left[ \frac{\partial [F(\Phi, n_\omega(x,t)) - F(\Phi, 0)]}{\partial \Phi} \right]_{\Phi = 0} \quad (3)$$

$$a_\omega(x,t) = F(0, n_\omega(x,t)) \quad (4)$$

Neglecting the $O(\Phi^2)$ term and introducing the variable $P_\omega(x,t) = \frac{\partial \Phi_\omega(x,t)}{\partial t}$, we obtain the linear system of stochastic differential equation of the first order
\begin{align*}
    d\Phi_\omega(x, t) &= P_\omega dt \\
    dP_\omega(x, t) &= (\Delta + \alpha)\Phi dt + dW_\omega(x, t)\Phi + a_\omega(x, t)dt \\
    dW_\omega(x, t) &= M_\omega(x, t)dt
\end{align*}

Without loss of generality we can consider the case

$$E_\omega[M_\omega(x, t)] = 0$$

In the following we will approximate $M_\omega(x, t)$ by a temporal white noise, consequently we have

$$E_\omega[W_\omega(x, t)] = 0$$

$$E_\omega[W_\omega(x, t)W_\omega(x', t')] = \min(t, t')C(x, x')$$

Here $C(x, x')$ is the spatial correlation function of the random field, related to the random multiplicative terms. By using $L_p$ estimates like in the work [5], it can be proven a result similar to [6], that under general conditions the additive term does not provide exponential destabilization of the moments of the random fields $\Phi_\omega, P_\omega$. Consequently we consider in the continuation only the homogenous system, while we are interested only in the large time behavior of the linearized system Eqs. (5-7).

The previous system of stochastic differential equation can be formally rewritten as general linear stochastic differential equation in a suitable Banach space as follows

$$dY_\omega(t) = \left[\hat{A}dt + d\hat{B}_\omega(t)\right]Y_\omega(t)$$

where

$$Y_\omega = \begin{pmatrix} \Phi_\omega \\ P_\omega \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0 & 1 \\ \Delta + \alpha & 0 \end{pmatrix}$$

$$d\hat{B}_\omega(t) = \begin{pmatrix} 0 \\ W_\omega(x, t) \end{pmatrix}$$

Similar linearization procedure can be performed for the system of drift kinetic equations, describing various species of charged particles $(q_\alpha, m_\alpha), \ 1 \leq \alpha \leq A$ in a strong constant magnetic field $B$. The starting nonlinear system of equations for the functions $f_\alpha(x, v, t)$ can be written as follows

$$\frac{\partial f_\alpha(x, v, t)}{\partial t} = -v\frac{\partial f_\alpha}{\partial v} - \frac{q_\alpha}{m_\alpha}E\|v\|\frac{\partial f_\alpha}{\partial v} - \frac{E \times B}{B^2} \cdot \nabla_{\perp}f_\alpha$$

$$\varepsilon_0 \Delta_x f_\alpha = \sum_{\alpha=1}^A q_\alpha \int f_\alpha(x, v, t)dv$$
The linearization of the equations starts with the ansatz

\[ f_a(x, v, t) = \delta f_{\omega,a} + [n_0(x) + \delta n_\omega(x, t)] \exp \left[ -v^2 (\beta_0(x) + \delta \beta_\omega(x, t)) \right] \]

around the equilibrium density and inverse temperature profiles \( n_0(x) \), \( \beta_0(x) \). The stochastic effects are induced by the temperature and density fluctuations \( \delta \beta_\omega(x, t), \delta n_\omega(x, t) \). By performing the same linearization procedure we obtain a system of linear stochastic differential equations of the form Eq. (8) where

\[ Y_\omega = \begin{pmatrix} \delta f_{\omega,1}(x, v, t) \\ \delta f_{\omega,2}(x, v, t) \\ \vdots \\ \delta f_{\omega,A}(x, v, t) \end{pmatrix} \]

1.2 Mathematical background

We will study now the correct formulation of the stability problem of the solutions of Eq. (8), and describe some general methods to solve these problems.

Consider a general linear evolution equation in a complex Banach space \( V \) with the norm \( \| \cdot \| \).

\[ \frac{dx(t)}{dt} = \hat{A}x(t); \ x(t) \in V; \quad (9) \]

where \( \hat{A} \) is a linear operator with dense domain in \( V \) and range in \( V \). We will denote by \( V^* \) the dual space of \( V \), where the transpose \( \hat{A}^T \) of \( \hat{A} \) acts.

An equivalent form, written in components (or reducing our study to the discretized form of the partial differential of integro-differential equations) is

\[ \frac{dx^i(t)}{dt} = A^i_j x^j(t) \quad (10) \]

where summation on repeated indices is understood.

Typical examples of interest will be the linearized equations used in the deterministic linear stability studies.

In the framework of the deterministic linear stability studies, in the finite dimensional case, a clear answer to the stability problem is given by the eigenvalue of \( \hat{A} \) with largest real part. In general case we denote by \( sp(\hat{A}) \) the spectrum of \( \hat{A} \) and

\[ \lambda = \max \left\{ \Re(z) \middle| z \in sp(\hat{A}) \right\} \quad (11) \]

The real number \( \lambda \) is the largest Liapunov exponent. Then for all \( \varepsilon > 0 \) we have for any solution of Eqs. (9) (10)

\[ \lim_{t \to \infty} \exp \left[ (\lambda + \varepsilon) t \right] = 0 \quad (12) \]

If \( \lambda \) is known, then the question about the stability of null solution in Eqs. (9) (10) is solved.
Consider now the stochastic generalization of Eqs. (9, 10), by adding a random, time dependent a linear term, for the sake of simplicity modelled by an operator-valued Brownian motion. In this end, we consider \((\Omega, A, \mathcal{F}_t, P)\) a filtered probability space and denote by \(\hat{B}_\omega(t)\), with \(\omega \in \Omega\), an operator-valued stochastic process adapted to the filtration \(\mathcal{F}_t\). In the our case \(\hat{B}_\omega(t)\) is an operator valued Brownian motion, or equivalently \(\hat{B}_\omega(t)x\) is an \(V\) valued Brownian motion [18] for all \(x \in V\).

**Remark 1** We will use everywhere the Itô formalism.

We have the following stochastic evolution equation,

\[
dx_\omega(t) = \left[\hat{A}dt + d\hat{B}_\omega(t)\right]x_\omega(t) \tag{13}
\]

or in components (we use the general relativity index conventions)

\[
dx^i_\omega(t) = \left[A^i_j dt + \rho^i_{j,a} dw^a_\omega(t)\right]x^j_\omega(t) \tag{14}
\]

where \(\rho^i_{j,a}\) are constants and \(w^a_\omega(t)\) are independent standard Brownian motions

\[
E_\omega \left[w^a_\omega(t)w^b_\omega(t')\right] = \min(t, t') \delta^{a,b} \tag{16}
\]

We have the matrix-valued Brownian motions \(\hat{B}_\omega(t)\), with components \(\hat{B}_{j,\omega}^i(t)\), having the correlation tensor \(C_{j,m}^i\)

\[
E_\omega \left[B_{j,\omega}^i(t)B_{n,\omega}^m(t')\right] = \min(t, t') C_{j,m}^i \tag{17}
\]

\[
C_{j,m}^i = \sum_{a=1}^{A} \rho_{j,a}^{i} \rho_{n,a}^{m} \tag{18}
\]

To this correlation tensor we associate a linear operator \(\hat{C}\) acting in the tensor product space \(V \otimes V\), so an equivalent form of Eqs. [17][18] is

\[
E_\omega \left\{ [\hat{B}_\omega(t)x] \otimes [\hat{B}_\omega(t')y] \right\} = \min(t, t') \hat{C} (x \otimes y); x \otimes y \in V \otimes V \tag{19}
\]

An equivalent form of the Eq. (19) that will be useful is

\[
E_\omega \left\{ [d\hat{B}_\omega(t)x] \otimes [d\hat{B}_\omega(t)y] \right\} = \hat{C} (x \otimes y) dt \tag{20}
\]

**Remark 2** Observe immediately that exactly due to the Itô definition, if we denote \(Y_{1}(t) = \mathbb{E}x_{\omega}(t)\), then we obtain from Eq. (13)

\[
\frac{dY_{1}(t)}{dt} = \hat{A}Y_{1}(t) \tag{21}
\]

so apparently the effect of the noise is averaged out. For higher order moments the equations will contain also the correlation effects of the noise.
Nevertheless, we can try generalize Eq. (12) as follows:

**Problem 3** Find the minimal value of $\lambda_p$ such that for $p \geq 1$ and all $\varepsilon > 0$ we have

$$
\lim_{t \to \infty} \frac{[\mathbb{E}_\omega [\|x_\omega(t)\|^p]]^{1/p}}{\exp [(\lambda_p + \varepsilon) t]} = 0 \quad (22)
$$

Simplest soluble one-dimensional cases, as well as more general results [4], suggest that in this case the exponent $\lambda$ is dependent on the exponent $p$ in a non-trivial way: in the case of systems that in deterministic case are linearly stable, it is possible that we have Eq. (22) for small noise intensity and small values of $p$ we have $\lambda_p < 0$, hence stability, but surely for sufficiently large $p$ we have $\lambda_p > 0$.

**Example 4** Consider the one dimensional model

$$
dx_\omega(t) = (-at + \rho d\omega(t)) x_\omega(t)
$$

in the Itô formalism, where $\omega(t)$ is the standard Wiener process ($\mathbb{E}_\omega \omega(t) = 0; \mathbb{E}_\omega [\omega(t)]^2 = t$), $a, \sigma$ are constants. The general solution is

$$
x_\omega(t) = x(0) \exp[-(a + \rho^2/2)t + \sigma \omega(t)]
$$

Results

$$
[\mathbb{E}_\omega |x_\omega(t)|^p]^{1/p} = |x(0)| \exp(\lambda_p t)
$$

with $\lambda_p = -a + (p - 1)\rho^2/2$.

**Conclusion 5** In the stochastic case the linear stability is determined by a scale, $\lambda_p$ of Liapunov exponents

The function $\lambda_p$ was effectively computed only in some special cases ([4], [5]).

We will see that excepting the small noise, perturbative calculations, in the physically interesting cases (partial differential equations, Vlasov-Poisson systems) computation of $\lambda_p$ for even values of $p$ is possibly only at the expense of solving eigenvalue problems with spatial dimension increased by a factor $p$.

In this moment there are no general method to compute $\lambda_p$, for all real values of $p$, even for the case when $\hat{A}$ or $\hat{B}_\omega(t)$ are $2 \times 2$ matrices. So we reformulate the Problem (3) in a more weak sense as follows

**Problem 6** For a given degree $p$, find $\lambda_p$ such that for all $\varepsilon > 0$ and all monomial $m_p(x)$ of degree $p > 1$ (i.e. $m_p(x) = x^{i_1} x^{i_2} \cdots x^{i_n}$ with $i_1 + \cdots + i_n = p$) the case $p = 1$ is trivial, see Remark [2]). In the components $x_i$ we have

$$
\lim_{t \to \infty} \frac{[\mathbb{E}_\omega [m_p(x_\omega(t))]]}{\exp [p(\lambda_p + \varepsilon) t]} = 0 \quad (23)
$$

In this end we will obtain closed sets of deterministic equations for the moments

$$
[\mathbb{E}_\omega [x_{i_1}^i (t) x_{i_2}^j (t) \cdots x_{i_n}^k (t)]]
$$

(24)
2 The deterministic equations for the moments.

2.1 The evolution of the mean values

For simplicity consider first the finite dimensional case, denote in general \( x = (x^1, x^2, ..., x^n) \). We have the following

**Remark 7** *(Backward Kolmogorov equation)* Consider a general stochastic process \( x_\omega(t) \), described by the following SDE and initial conditions

\[
\begin{align*}
    dx_\omega^i(t) &= V^i(x_\omega(t))dt + \sigma^i_a(x_\omega(t))dw^a_\omega(t); \ 1 \leq i \leq n; \ 1 \leq a \leq A \\
    E_\omega[ w^a_\omega(t)] &= 0; \ E_\omega[ w^a_\omega(t)w^b_\omega(t')] = \delta_{a,b} \min(t, t') \\
    x_{\omega}(0) &= x_0
\end{align*}
\]  

(25)

(26)

(27)

For for any differentiable function \( F(x) \) we have (for the sake of simplicity of the notations, the subscript \( \omega \) will be omitted)

\[
\begin{align*}
    dF(x_\omega(t)) &= \frac{\partial F}{\partial x^i}(x_\omega(t))V^i(x_\omega(t))dt + \frac{1}{2} \frac{\partial^2 F}{\partial x^i \partial x^j}D^{i,j}(x)dt \\
    D^{i,j}(x) &= A \sum_{a=1}^{A} \sigma^i_a(x)\sigma^j_a(x)
\end{align*}
\]  

(28)

(29)

We denote

\[
M_F(t) = E_\omega[F(x_\omega(t))]
\]

(30)

Then we have

\[
\frac{M_F(t)}{dt} = E_\omega \left[ \frac{\partial F(x_\omega(t))}{\partial x^i}V^i(x_\omega(t)) + \frac{1}{2} \frac{\partial^2 F}{\partial x^i \partial x^j}D^{i,j}(x_\omega(t)) \right]
\]

(31)

It follows from Eqs.(31, 29) that in the case when the SDE (25) is linear, it is possible to obtain a closed set of linear differential equations when \( F(x) \) belongs to a finite dimensional subspace of homogenous polynomials.

We consider now the case when Eq.(25) is a linear SDE, like Eq.(14). So

\[
\begin{align*}
    V^i(x) &= A^i_jx^j \\
    \sigma^i_a(x) &= \rho^i_{j,a}x^j
\end{align*}
\]  

(32)

(33)

We obtain from Eqs. (29, 33, 18)

\[
D^{i,j}(x) = C_{m,n}^{i,j} x^m x^n
\]

(34)

We obtain from Eqs.(31, 32, 34) the basic equation for further development

\[
\frac{M_F(t)}{dt} = E_\omega \left[ \frac{\partial F(x_\omega(t))}{\partial x^i}A^i_jx^j(t) + \frac{1}{2} \frac{\partial^2 F}{\partial x^i \partial x^j}C_{m,n}^{i,j} x^m(t) x^n(t) \right]
\]

(35)
2.2 Linear equations for the moments of order $N$.

2.2.1 Case $N = 1$.

Consider, for more clarity reasons, in Eq. (35) $F(x) \overset{def}{=} x^k$, where $k$ is fixed integer index and denote in general $Y_j^i(t) = E_\omega \left[ x^i_j(t) \right]$, as in Remark 2. 

*Recall, $x^k$ is a contravariant vector component, not power.*

Then we obtain Eq. (21), or

$$
\frac{dY_1^i(t)}{dt} = A^i_j Y_j^j(t)
$$

(36)

$$
\frac{dY_1(t)}{dt} = \hat{A}Y_1(t)
$$

(37)

The corresponding stability problem is the classical, deterministic

**Problem 8** Compute the eigenvalues, left and right eigenvectors in the eigenvalue problems

$$
\lambda u = \hat{A}u; u \in V;
$$

$$
\lambda v = \hat{A}^Tv; v \in V^*; v \text{ is left eigenvector}
$$

2.2.2 Case $N = 2$

Consider in Eq. (35) $F(x) \overset{def}{=} x^k x^l$ and denote $Y_{k,l}^j(t) = E_\omega \left[ x^k_j(t)x^l_j(t) \right]$. Then we obtain

$$
\frac{dY_{k,l}^j(t)}{dt} = A^j_k Y_j^j(t) + A^j_l Y_j^k(t) + C_{m,n}^{k,l} Y_{m,n}^j(t)
$$

(38)

In condensed notation, taking into account that to $Y_{k,l}^j(t)$ we can associate a tensor $Y_2(t) \in V \otimes V$ and to $C_{m,n}^{k,l}$ the operator $\hat{C}$ (see Eq. (20)) we have the equivalent form of Eq. (38)

$$
\frac{dY_2(t)}{dt} = \left[ \hat{A} \otimes \hat{1} + \hat{1} \otimes \hat{A} + \hat{C} \right] Y_2(t)
$$

(39)

2.2.3 Case $N = 3$

Similarly we consider $F(x) \overset{def}{=} x^k x^l x^i$ and denote $Y_{k,l,i}^j(t) = E_\omega \left[ x^k_j(t)x^l_j(t)x^i_j(t) \right]$ . Then we obtain

$$
\frac{dY_{k,l,i}^j(t)}{dt} = A^j_k Y_{k,l}^j(t) + A^j_i Y_{k,i}^k + A^j_i Y_{i,j}^k +
$$

$$
C_{m,n}^{k,l,i} Y_{m,n}^j + C_{m,n}^{k,i,l} Y_{m,n}^i + C_{m,n}^{l,i} Y_{k,m,n}^j
$$

(40)

In order to write in a more compact form, we introduce $Y_3(t) \in V \otimes V \otimes V$ with components $Y_{k,l,i}^j$ and the operators $\hat{C}_{1,2}$, $\hat{C}_{1,3}$, and $\hat{C}_{2,3}$, acting in $V \otimes V \otimes V$ according to the last terms in Eq. (38), and we obtain
\[
\frac{dY_3(t)}{dt} = (\hat{A} \otimes \hat{1} \otimes \hat{1} + \hat{1} \otimes \hat{A} \otimes \hat{1} + \hat{1} \otimes \hat{1} \otimes \hat{A} + \hat{C}_{1,2} + \hat{C}_{1,3} + \hat{C}_{2,3}) Y_3(t)
\]  

(41)

2.2.4 The general \(m\) case

With the choice \(F(x) \overset{\text{def}}{=} \prod_{j=1}^{m} x^{k_j}\) and notation \(Y_{m}^{k_1, \ldots, k_m}(t) = \mathbb{E}_{\omega} \left[ \prod_{j=1}^{m} x_j^{k_j}(t) \right]\), from Eq.(35) results

\[
\frac{dY_{m}^{k_1, \ldots, k_m}}{dt} = A_{j}^{k_1} Y_{m}^{j, k_2, \ldots, k_m} + A_{j}^{k_2} Y_{m}^{k_1, j, k_3, \ldots, k_m} + A_{j}^{k_m} Y_{m}^{k_1, \ldots, j} + \sum_{a<b}^{m} C_{a,b}^{k_1, \ldots, k_{a-1}, j, \ldots, k_{b-1}, l, \ldots, k_m}
\]

(42)

Similarly we have the compact form

\[
\frac{dY_{m}(t)}{dt} = [\hat{A} \otimes (\hat{1} \otimes (m-1)) + \hat{1} \otimes \hat{A} \otimes (\hat{1} \otimes (m-2)) + \cdots + (\hat{1} \otimes (m-1)) \otimes \hat{A} + \sum_{a<b}^{m} \hat{C}_{a,b}] Y_{m}(t)
\]

(43)

\[
\overset{\text{def.}}{=} \hat{U}_{m}(\hat{C}) Y_{m}(t)
\]

(44)

where the action of the operator \(\hat{C}_{a,b}\) in the tensor product space \(Y^{\otimes m}\) is defined according to the corresponding term in Eq.(42). Then the large time behavior of \(Y_{m}(t)\) is dominated by the term \(\exp(\lambda_{m} t)Z_{m}\) where \(\lambda_{m}\), in the finite dimensional case, is the eigenvalue with largest real part in

\[
\lambda Z_{m} = \hat{U}_{m}(\hat{C}) Z_{m}
\]

(45)

For application of the perturbative analysis in the limit of weak noise intensity (in Eq.(43) the contribution of \(\hat{C}_{a,b}\) is small) it is useful to remark that, Eq.(45) can be put in the form

\[
\lambda Y_{m} = (\hat{M}_{m} + \delta \hat{M}_{m}) Y_{m}
\]

(46)

where the unperturbed term is

\[
\hat{M}_{m} = \hat{A} \otimes (\hat{1} \otimes (m-1)) + \hat{1} \otimes \hat{A} \otimes (\hat{1} \otimes (m-2)) + \cdots + (\hat{1} \otimes (m-1)) \otimes \hat{A}
\]

(47)

and the perturbation is

\[
\delta \hat{M}_{m} = \sum_{a<b}^{m} \hat{C}_{a,b}
\]

(48)
Remark 9 If the deterministic stability problem is solved then the spectrum (including eigenvalues), left and right eigenvector of can be computed immediately.

2.2.5 Perturbative approach.

Perturbation of general, not necessary self adjoint operators. These results are generalizations of the perturbations methods from quantum mechanics to the case when the operators are not necessary self adjoint.

The formalism exposed above is useful to derive perturbative methods, to solve the eigenvalue problems like from Eq. Consider a general complex linear vector space . Suppose that the deterministic linear stability analysis was performed, so we know the eigenvalues, as well as the right eigenvectors and left eigenvectors of the operator . The left eigenvectors, in matrix notations, can be associated to the eigenvectors of the transposed . Denote the linear function on by . The transposed is defined by

\[
\langle x^*, \hat{M}x \rangle = \langle \hat{M}^T x^*, x \rangle
\] (49)

The spectrum of and identical in very general case. In particular in finite dimensions it results from the property of determinants. Much of our results are explained in the finite dimensional case and we do not discuss the specific problems related to continuous spectrum and unbounded operators.

Simple eigenvalue So to each eigenvalue corresponds the right eigenvectors and the left eigenvectors . We have

\[
\hat{M} u_{R,k} = \nu_k u_{R,k}
\] (50)

\[
\hat{M}^T v_{L,k} = \nu_k v_{L,k}
\] (51)

Proposition 10 The eigenvalues of the perturbed operator can be computed in first approximation order by

\[
\delta \nu_k = \frac{\langle v_{L,k}, \delta \hat{M} u_{R,k} \rangle}{\langle v_{L,k}, u_{R,k} \rangle}
\] (52)

Proof. The perturbed right eigenvector will be represented as . Results

\[
\left( \hat{M} + \delta \hat{M} \right) (u_{R,k} + \delta u_{R,k}) = (\nu_k + \delta \nu_k) (u_{R,k} + \delta u_{R,k})
\] (53)

We use now and drop the second order terms. We obtain

\[
\hat{M} \delta u_{R,k} + \delta \hat{M} u_{R,k} = \nu_k \delta u_{R,k} + \delta \nu_k u_{R,k}
\] (54)
By left multiplication with $\mathbf{u}_{L,k}$ results

$$
\langle \mathbf{v}_{L,k}, \hat{\mathbf{M}} \mathbf{u}_{R,k} \rangle + \langle \mathbf{v}_{L,k}, \delta \hat{\mathbf{M}} \mathbf{u}_{R,k} \rangle = v_k \langle \mathbf{v}_{L,k}, \delta \mathbf{u}_{R,k} \rangle + \delta v_k \langle \mathbf{v}_{L,k}, \mathbf{u}_{R,k} \rangle \tag{55}
$$

By using Eqs. (49, 51), the first term in l.h.s. in Eq. (55) can be rewritten as

$$
\langle \mathbf{v}_{L,k}, \hat{\mathbf{M}} \mathbf{u}_{R,k} \rangle = \langle \hat{\mathbf{M}}^T \mathbf{v}_{L,k}, \delta \mathbf{u}_{R,k} \rangle = v_k \langle \mathbf{v}_{L,k}, \delta \mathbf{u}_{R,k} \rangle \tag{56}
$$

Combined with Eq. (55) we obtain Eq. (52), that completes the proof.

**Multiple eigenvalue, semisimple (without Jordan blocks)** Must of physical cases from this category are related to some discrete or continuous symmetry group, and the explicit symmetry breaking due to asymmetric perturbation (Like Zeeman or Stark effects). Let $v_k$ an eigenvalue of multiplicity $m$. The subspace of left and right eigenvectors has the same dimension. Let $\mathbf{v}_{L,k,a}$ respectively $\mathbf{u}_{L,k,a}$, with $1 \leq a \leq m$ the set of left and right eigenvectors

$$
\hat{\mathbf{M}} \mathbf{u}_{R,k,a} = v_k \mathbf{u}_{R,k,a} \tag{56}
$$

$$
\hat{\mathbf{M}}^T \mathbf{v}_{L,k,a} = v_k \mathbf{v}_{L,k,a} \tag{57}
$$

Define the $m \times m$ matrices $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$ by matrix elements as follows

$$
F_{a,b} = \langle \mathbf{v}_{L,k,a}, \hat{\mathbf{M}} \mathbf{u}_{R,k,b} \rangle \tag{58}
$$

$$
G_{a,b} = \langle \mathbf{v}_{L,k,a}, \mathbf{u}_{R,k,b} \rangle \tag{59}
$$

Denote by $\delta v_{k,a} = z_k$ with $1 \leq a \leq m$, the sequence of $m$ roots $z_k$ of the secular equation in $z$

$$
\det \left[ \hat{\mathbf{F}} - z \hat{\mathbf{G}} \right] = 0 \tag{60}
$$

We have the following

**Proposition 11** The multiplicity, $m$ semisimple eigenvalue $v_k$ under the effect of perturbation splits into $m$ eigenvalues

$$
v_k \rightarrow v_k + \delta v_{k,a} \tag{61}
$$

**Proof.** Combination of the previous proof and from the degenerate case from quantum mechanical textbooks.

**2.2.6 The case of diagonal noise**

Our equations will be simplified in the particular case, when the noise in SDE has the form

$$
\rho^i_{j,a} = \delta^i_j \rho^i_{a} \tag{62}
$$

In this case

$$
C^{n,m}_{j,n} = \delta^i_j \delta^m_n V^{i,m}_{2} \tag{63}
$$

$$
V^{i,m}_{2} = \sum_{a=1}^{A} \rho^i_{j,a} \rho^m_{a} \tag{64}
$$
The Eq. (42) became

\[
\frac{dY_{k}}{dt} = A_{j}^{k} Y_{j,m}^{k} + A_{j}^{k} Y_{m,j}^{k} + A_{j}^{k} Y_{m,j}^{k} + \sum_{a<b} Y_{a,b}^{k} \]

(65)

3 Langmuir waves in turbulent background

3.1 Statement of the problem

The evolution of the electrostatic potential in Langmuir waves in homogenous finite temperature plasma, after rescaling \( x \rightarrow \sqrt{3} V_{\text{thermal}} x \) is given by the Klein-Gordon equation

\[
\frac{\partial^{2} \phi(x, t)}{\partial t^{2}} = \Delta \phi - m^{2} \phi 
\]

(66)

\[ m^{2} = \omega_{p,e}^{2} = \frac{n_{e} e^{2}}{\epsilon_{0} m_{e}} \]

In the case of perturbation of the background electron density \( n_{e} \rightarrow n_{e} + \text{const} \ \zeta(x, t) \) by a temporal white noise with

\[
\mathbb{E} [\zeta(x, t)] = 0 \\
\mathbb{E} [\zeta(x, t) \zeta(x', t')] = \delta(t - t') C(x, x')
\]

formally can be written as

\[
\frac{\partial^{2} \phi(x, t)}{\partial t^{2}} = \Delta_{x} \phi(x, t) - [m^{2} + \zeta(x, t)] \phi
\]

Or in a more rigorous form, by introducing the Brownian motion \( w(x, t) \) such that formally \( \zeta(x, t) = \frac{\partial w(x, t)}{\partial t} \), we obtain the system of first order PSDE

\[
d\phi_{\omega}(x, t) = p_{\omega}(x, t)dt \\
dp_{\omega}(x, t) = -K_{x} \phi_{\omega} dt + \phi dw_{\omega}(x, t)
\]

(67)

(68)

where \( K_{x} = m^{2} - \Delta_{x} \) and \( \mathbb{E}_{\omega} [w_{\omega}(x, t) w_{\omega}(x', t')] = \min(t, t') C(x, x') \).

Denote

\[
Y_{\phi,\phi}(x_{1}, x_{2}, t) = \mathbb{E}_{\omega} [\Phi_{\omega}(x_{1}, t) \Phi_{\omega}(x_{2}, t)] \\
Y_{p,\phi}(x_{1}, x_{2}, t) = \mathbb{E}_{\omega} [p_{\omega}(x_{1}, t) \Phi_{\omega}(x_{2}, t)] \\
Y_{\phi,p}(x_{1}, x_{2}, t) = \mathbb{E}_{\omega} [\Phi_{\omega}(x_{1}, t) p_{\omega}(x_{2}, t)] \\
Y_{p,p}(x_{1}, x_{2}, t) = \mathbb{E}_{\omega} [p_{\omega}(x_{1}, t) p_{\omega}(x_{2}, t)]
\]
By the same procedure we obtain the following system of linear equations for the second order correlation functions

\[
\frac{dY_{\phi,\phi}(x_1, x_2, t)}{dt} = Y_{p,\phi}(x_1, x_2, t) + Y_{\phi,p}(x_1, x_2, t)
\]

(69)

\[
\frac{dY_{p,\phi}(x_1, x_2, t)}{dt} = -\hat{K}_{x_1} Y_{\phi,\phi}(x_1, x_2, t) + Y_{p,p}(x_1, x_2, t)
\]

(70)

\[
\frac{dY_{\phi,p}(x_1, x_2, t)}{dt} = -\hat{K}_{x_1} Y_{\phi,\phi}(x_1, x_2, t) -\hat{K}_{x_2} Y_{p,\phi}(x_1, x_2, t)
\]

(71)

\[
\frac{dY_{\phi,p}(x_1, x_2, t)}{dt} = Y_{p,p}(x_1, x_2, t) + C(x_1, x_2) Y_{\phi,\phi}(x_1, x_2, t)
\]

(72)

In order to study the \( t \to \infty \) behavior of the solutions of Eqs. (69-72), we consider the particular class of solutions

\[
Y_{a,b}(x_1, x_2, t) = \exp(\lambda t) Z_{a,b}(x_1, x_2); a, b \in \{p, \phi\}
\]

(73)

and after simple algebra we find the following problem:

**Problem 12** Find \( \lambda \) with maximal real part from the generalized eigenvalue problem

\[
\left[ \lambda^4 + 2\lambda^2 \left( 2m^2 - \Delta_{x_1} - \Delta_{x_2} \right) + (\Delta_{x_1} - \Delta_{x_2})^2 - 2\lambda C(x_1, x_2) \right] Z_{\phi,\phi}(x_1, x_2) = 0
\]

(74)

The boundary conditions are obtained in a straightforward way from the boundary conditions of the original deterministic problem.

### 3.1.1 Alternative notations

A more transparent way is given by a compact notation. We will denote

\[
\begin{pmatrix}
\Psi_1 \\
\Psi_2
\end{pmatrix} = \begin{pmatrix}
\phi(x, t) \\
p(x, t)
\end{pmatrix}
\]

Define the **operator valued** \( 2 \times 2 \) matrices \( \hat{L}_\infty \) and \( \hat{J} \) as follows:

\[
\hat{L}_\infty = \begin{pmatrix}
0 & 1 \\
-K_x & 0
\end{pmatrix}
\]

\[
\hat{J} = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\]

or by components
Then Eqs. (67, 68) became

$$d\Psi(x, t) = \hat{L}^x \Psi dt + \hat{J} \Psi dw$$  \hspace{1cm} (75)$$

We use the notation

$$Y_{i_1,i_2}(x_1, x_2, t) = E_{\omega} \left[ \Psi_{i_1,\omega}(x_1, t) \Psi_{i_2,\omega}(x_2, t) \right] ; \ 1 \leq i_1, i_2 \leq 2$$ \hspace{1cm} (76)$$

Respectively Eqs. (69-72) became

$$\frac{dY_{i_1,i_2}(x_1, x_2, t)}{dt} = L_{x_1,i_1,j_1} Y_{j_1,i_2} + L_{x_2,i_2,j_2} Y_{i_1,j_2}$$ \hspace{1cm} (77)$$

$$+ J_{i_1,j_1} J_{i_2,j_2} C(x_1, x_2) ; 1 \leq i_1, i_2, j_1, j_2 \leq 2$$

Consider the stability problem of the moments $$Y_{i_1,i_2}(x_1, x_2, t)$$. From the representation

$$Y_{i_1,i_2}(x_1, x_2, t) = \exp \left( i \omega t \right) Z_{i_1,i_2}(x_1, x_2)$$  \hspace{1cm} (78)$$

and Eq. (77) the following eigenvalue problem results

$$i \omega Z_{i_1,i_2}(x_1, x_2) = L_{x_1,i_1,j_1} Z_{j_1,i_2} + L_{x_2,i_2,j_2} Z_{i_1,j_2}$$ \hspace{1cm} (79)$$

$$+ J_{i_1,j_1} J_{i_2,j_2} C(x_1, x_2)$$

In analogy to the Eq. (43) similar equations for the higher order correlation functions can be obtained.

### 3.2 Particular solutions of the eigenvalue equation (74, 79).

#### 3.2.1 Zero noise limit

This is useful for small noise limit perturbative calculations.

In line with the classical stability analysis we will denote

$$\varepsilon_k = \sqrt{m^2 + k^2} \hspace{1cm} (80)$$

$$\lambda = i \omega \hspace{1cm} (81)$$

The classical dispersion relation, for the Langmuir waves is

$$\omega_k = \pm \varepsilon_k \hspace{1cm} (82)$$

When $$C = 0$$ we have the unperturbed eigenvectors in Eq. (79) of the form

$$Z_{i_1,i_2}(x_1, x_2) = v_{i_1,i_2} \exp \left[ i (k_1 x_1 + k_2 x_2) \right]$$ \hspace{1cm} (83)$$

$$v_{i_1,i_2} = u_{i_1} u_{i_2} \hspace{1cm} (84)$$

By using Eqs. (79, 83, 84) after simple algebra we find that the generalization of the classical dispersion relation, for the case of two point correlation function, from Eq. (82) is,

$$\omega_{k_1,k_2} = \pm \varepsilon_{k_1} \pm \varepsilon_{k_2} \hspace{1cm} (85)$$

14
3.3

3.3.1 Complete spatial correlations \([17]\) .

This part is included only for the purpose of clarification the application of formalism. Can be considered as a limiting case, of the results exposed in subsection (3.3.2).

Consider the simplest soluble case \(C(x_1, x_2) = \sigma^2 = \text{const}\). In this case the ansatz from Eq.(83) can be used and from Eq.(79) results (see Appendix 5, Eq.(123))

\[
4\varepsilon_{k_1}^2 \varepsilon_{k_2}^2 + 2i\omega \sigma^2 = (\varepsilon_{k_1}^2 + \varepsilon_{k_2}^2 - \omega^2)^2
\]

or by using Eq.(81)

\[
\lambda^4 + 2\lambda^2(\varepsilon_{k_1}^2 + \varepsilon_{k_2}^2) + (\varepsilon_{k_1}^2 - \varepsilon_{k_2}^2)^2 = 2\lambda \sigma^2
\] \hspace{1cm} (86)

From Eq.(86) results that if \(|k_1| - |k_2|\) is sufficiently small then we have two real solutions with \(\lambda > 0\), so the random multiplicative perturbation of the background always produce parametric destabilization of the 2 point correlation functions \(Y_{\phi,\phi}(x_1, x_2, t)\). Due to the fact that \(Y_{\phi,\phi}(x_1, x_2, t)\) is symmetric with respect permutations \(x_1 \leftrightarrow x_2\), the modes with \(|k_1| \approx |k_2|\) has non-vanishing contributions. For \(|k_1| = |k_2|\) we have for the dominating mode

\[
\lambda^3 + 4\lambda \varepsilon_{k_1}^2 = 2\sigma^2
\]

For high \(|k|\) the the exponential growth is dominated by \(\lambda \approx \sigma^2/(2\varepsilon_{k_1}^2) + O(1/\varepsilon_{k_1}^4)\). The most dangerous modes are those with for \(|k_1| \approx |k_2| \approx 0\). For small \(\sigma\) we have \(\lambda_{\text{max}} \approx \frac{\sigma^2}{2m^2} + O(\sigma^0)\)

3.3.2 Homogenous system

Suppose that the random field is homogenous, so \(C(x_1, x_2) = C(x_1 - x_2)\), with \(C(-x) = C(x)\). We introduce the ”center of mass” and relative distance coordinates \((x_1 + x_2)/2, r = x_1 - x_2\) and use the ansatz in Eq.(74)

\[
Z_{\phi,\phi}(x_1, x_2) = \exp[ik(\frac{x_1 + x_2}{2})]\Psi(r)
\] \hspace{1cm} (87)

The resulting generalized eigenvalue problem is

\[
\left[\lambda^4 + 2\lambda^2(2m^2 + k^2/2 - 2\Delta_x) + 4(ik\nabla_x)^2 - 2\lambda C(x)\right]\Psi(r) = 0
\] \hspace{1cm} (88)

Temporal and spatial white noise. \([17]\) we study now the opposite case studied previously in subsubsection (3.3.1). Consider the case one dimensional version of the previous model, with Dirac-Delta correlation function \(C(x_1, x_2) = \sigma^2\delta(x_1 - x_2)\). It is known that the bound state problem in the Schrödinger equation in higher dimension with delta function potential give mathematically inconsistent results \([22]\) .
The one dimensional version of Eq. (88) is
\[ A(\lambda, k)\psi(x) - B(\lambda, k)\psi''(x) - 2\lambda\sigma^2\delta(x)\psi(x) = 0 \] (89)
\[ A(\lambda, k) = \lambda^4 + 4\lambda^2m^2 + \lambda^2k^2 \] (90)
\[ B(\lambda, k) = 4(\lambda^2 + k^2) \] (91)

From Eq.(89) results the continuity and jump conditions for the restriction of \( \psi_\pm(x) \) to the domains \( x > 0 \) respectively \( x < 0 \).

\[ \lim_{x \to 0^+} \psi_+(x) = \lim_{x \to 0^-} \psi_-(x) \] (92)
\[ B(\lambda, k) \left[ \lim_{x \to 0^+} \psi'_+(x) - \lim_{x \to 0^-} \psi'_-(x) \right] + 2\lambda\sigma^2\psi_+(0) = 0 \] (93)

The class of physically admissible (because \( \lambda \) is complex) solutions are of the form
\[ \psi_+(x) = \exp(-\alpha x) \] (94)
\[ \psi_-(x) = \exp(\alpha x) \] (95)

By straightforward calculations from Eqs.(92, 93) results
\[ B(\lambda, k)\alpha^2 = A(\lambda, k) \] (96)
\[ 2B(\lambda, k)\alpha = 2\lambda\sigma^2 \] (97)

And we obtain
\[ \sigma^4 = 4(\lambda^2 + k^2)(\lambda^2 + 4m^2 + k^2) \] (98)

From Eqs. (97, 98) results that for any root of Eq. (98) with \( \text{Re}(\lambda) \geq 0 \), the condition from Eq. (95) is fulfilled.

It is clear that now there is a threshold: when \( \sigma^4 < 4k^2(4m^2 + k^2) \) then \( \lambda \) is imaginary. For \( \sigma^4 > 4k^2(4m^2 + k^2) \) there is a real positive \( \lambda \), so the system is unstable. We remark an important mechanism: for low noise intensity first the large wavelength structures are destabilized. So by this mechanism there is transfer of energy from short to the long wavelength. The most sensitive to the destabilization are the mode with small \( k \).

Positive correlations. Consider the solutions of the Eq.(88) in the case \( C(x) \geq 0 \) and \( \lim_{|x| \to \infty} C(x) = 0 \).

Case \( k = 0 \) We are interested in the case \( \lambda \neq 0 \), so the generalized eigenvalue problem can be reformulated as follows.
\[ -\Delta_x \Psi(r) + \left[ -\frac{1}{2\lambda} C(x) \right] \Psi(r) = -(m^2 + \frac{\lambda^2}{4})\Psi(r) \] (99)
When $0 < \lambda < +\infty$, because $C > 0$, the lowest eigenvalue $E_\lambda$ of the eigenvalue problem

$$
- \Delta_x \Psi(r) + \left[ -\frac{1}{2\lambda} C(x) \right] \Psi_{0,\lambda}(r) = E_\lambda \Psi_{0,\lambda}(r)
$$

(100)

is a monotone increasing continuous function, from $-\infty$ to 0.

Indeed, the term $-\frac{1}{2\lambda} C(x)$ is like an attractive potential well in a ground state problem in the Eq.(100), formally identical to Schrödinger equation. For $\lambda \to 0$ it is very deep potential well, can be proven rigorously that $E_\lambda \lim_{\lambda \to 0} E_\lambda = -\infty$. When $\lambda$ increases then there is a critical value $\lambda_{\text{crit}} > 0$ such that the bound state problem from Eq.(100) has no more solutions. So we have

$$
\lim_{\lambda \searrow \lambda_{\text{crit}}} E_\lambda = 0.
$$

(Previous asymptotic estimates can be verified by approximating the potential well with a rectangular one, at least in the case of fast correlation decay).

Results that there is a one and only one value of $\lambda > 0$ such that $E_\lambda = -(m^2 + \frac{\lambda^2}{4})$ (because the r.h.s. is decreasing, from $-m^2$ to $-\infty$).

**Conclusion 13** It follows that there exists at least an eigenvalue $\lambda > 0$ in the generalized eigenvalue problem from Eq.(99), so at least the mode $k = 0$ is destabilized for arbitrary small noise intensity.

**Case $|k| \to \infty$.** In general, Eq.(88), by suitable choice of the coordinate system $(k \parallel O_z)$, can be reformulated as follows.

$$
- \left[ \Delta_\perp + \left( 1 + \left( \frac{k}{\lambda} \right)^2 \right)^2 \frac{\partial^2}{\partial z^2} \right] \Psi_{0,\lambda}(r) + \left[ -\frac{1}{2\lambda} C(x) \right] \Psi_{0,\lambda}(r) = E_\lambda \Psi_{0,\lambda}(r)
$$

(102)

$$
-(m^2 + \frac{k^2}{4} + \frac{\lambda^2}{4}) = E_\lambda
$$

(103)

where $\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Because of the new term $(k/\lambda)^2 \frac{\partial^2}{\partial z^2}$, the ground state energy has a lower bound.

It denoting with $\delta$ the typical width of the maximum of $C(x)$ near $x = 0$, respectively $c = C(0)$, for large $k$ and small $\lambda$ we have the estimate $E_\lambda > \mathcal{O}(-\frac{c(0)^2 \delta^2}{k^2})$. So for fixed large values of $k$ there is no real $\lambda$ that satisfies Eq.(103).

**Conclusion 14**. For $k > 0$ there is a threshold for the intensity of the multiplicative noise such that the mode remains stable below this threshold.
4 Conclusions.

Similar to the simplest one dimensional case exposed in the works [2], [3], the Langmuir waves in a turbulent background are destabilized. The energy of the fluctuations related to the multiplicative noise is transferred with an exponential rate to the long wavelength fluctuations preferentially. In contradistinction with one dimensional case, there is a threshold in the noise intensity for each mode, such that below this threshold the mode remains stable.

5 Appendix

Denote the tensor with components $v_{i_1, i_2}$ from Eq. (83) by $v$. We use the notation Eq. (??) and define $2 \times 2$ matrices $\hat{E}(k), \hat{a}$ as follows

\[
\hat{E}(k) = \begin{pmatrix} 0 & 1 \\ -\varepsilon_k^2 & 0 \end{pmatrix} \quad (104)
\]

\[
\hat{a} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (105)
\]

\[
\hat{u}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (106)
\]

\[
\hat{u}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (107)
\]

and by $\hat{1}$ the $2 \times 2$ unit matrix. The eigenvalue equation (79) with ansatz Eq. (83) became

\[
i\omega v = \hat{B}v \quad (108)
\]

where

\[
\hat{B} = \hat{E}(k_1) \otimes \hat{1} + \hat{1} \otimes \hat{E}(k_2) + \sigma^2 \hat{a} \otimes \hat{a} \quad (109)
\]

From Eq. (108) results

\[
-\omega^2 v = \hat{B}^2 v \quad (110)
\]

We will use the following matrix identities, that can be verified easily

\[
\hat{E}(k)^2 = -\varepsilon_k^2 \hat{1}; \quad \hat{a}^2 = 0 \quad (111)
\]

\[
\hat{E}(k) \hat{a} = \hat{u}_1; \quad \hat{a} \hat{E}(k) = \hat{u}_2 \quad (112)
\]

\[
\hat{u}_1 + \hat{u}_2 = \hat{1} \quad (113)
\]

From Eqs. (109, 111, 112, 113) results

\[
\hat{B}^2 = -\left(\varepsilon_{k_1}^2 + \varepsilon_{k_2}^2\right) \hat{1} \otimes \hat{1} + 2\hat{E}(k_1) \otimes \hat{E}(k_2) + \\
\sigma^2 \left( \hat{1} \otimes \hat{a} + \hat{a} \otimes \hat{1} \right) \quad (114)
\]

\[
\sigma^2 \left( \hat{1} \otimes \hat{a} + \hat{a} \otimes \hat{1} \right) \quad (115)
\]
and with the notation

\[ y = (\varepsilon_{k_1}^2 + \varepsilon_{k_2}^2) - \omega^2 \]  

(116)

the eigenvalue equation (110) became

\[ \hat{C} \mathbf{v} = y \mathbf{v} \]  

(117)

with

\[ \hat{C} = 2\hat{E}(k_1) \otimes \hat{E}(k_2) + \sigma^2 \left( \hat{1} \otimes \hat{a} + \hat{a} \otimes \hat{1} \right) \]  

(118)

It follows

\[ \hat{C}^2 \mathbf{v} = y^2 \mathbf{v} \]  

(119)

and from Eq. (118) we obtain

\[ \hat{C}^2 = 4\varepsilon_{k_1}^2 \varepsilon_{k_2}^2 + 2\sigma^2 \left[ \hat{E}(k_1) \otimes \hat{1} + \hat{1} \otimes \hat{E}(k_2) + \sigma^2 \hat{a} \otimes \hat{a} \right] \]  

(120)

or compared with Eq. (109)

\[ \hat{C}^2 = 4\varepsilon_{k_1}^2 \varepsilon_{k_2}^2 + 2\sigma^2 \hat{B} \]  

(121)

So, from Eqs. (119, 121) we obtain

\[ \left[ 4\varepsilon_{k_1}^2 \varepsilon_{k_2}^2 + 2\sigma^2 \hat{B} \right] \mathbf{v} = y^2 \mathbf{v} \]  

(122)

or, combined with Eqs. (108, 116) we obtain the relation

\[ 4\varepsilon_{k_1}^2 \varepsilon_{k_2}^2 + 2i\omega \sigma^2 = \left( (\varepsilon_{k_1}^2 + \varepsilon_{k_2}^2) - \omega^2 \right)^2 \]  

(123)

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