SOME NOTES ON CONCENTRATION FOR 
\( \alpha \)-SUBEXPONENTIAL RANDOM VARIABLES

HOLGER SAMBALE

Abstract. We prove extensions of classical concentration inequalities for random variables which have \( \alpha \)-subexponential tail decay for any \( \alpha \in (0,2] \). This includes Hanson–Wright type and convex concentration inequalities in various situations. In particular, we show uniform Hanson–Wright inequalities and convex concentration results for simple random tensors in the spirit of recent work by Klochkov–Zhivotovskiy \cite{KlochkovZhivotovskiy20} and Vershynin \cite{Vershynin42}.

1. Introduction

The aim of this note is to compile a number of smaller results which extend some classical as well as more recent concentration inequalities for bounded or sub-Gaussian random variables to random variables with heavier (but still exponential-type) tails. In detail, we shall consider random variables \( X \) which satisfy
\[
\Pr(|X - \mathbb{E}X| \geq t) \leq 2 \exp(-t^\alpha/C_{1,\alpha}^\alpha)
\]
for any \( t \geq 0 \), some \( \alpha \in (0,2] \) and a suitable constant \( C_{1,\alpha} > 0 \). Such random variables are sometimes called \( \alpha \)-subexponential (for \( \alpha = 2 \), they are subgaussian) or sub-Weibull (\( \alpha \)) (cf. \cite[Definition 2.2]{Definition}.

There are several equivalent reformulations of (1.1), e.g. in terms of \( L^p \) norms:
\[
\|X\|_{L^p} \leq C_{2,\alpha} p^{1/\alpha}
\]
for any \( p \geq 1 \). Another characterization is that these random variables have finite Orlicz norms of order \( \alpha \), i.e.
\[
C_{3,\alpha} := \|X\|_{\Psi_\alpha} := \inf\{t > 0 : \mathbb{E}\exp(|X|/t)^\alpha \leq 2\} < \infty.
\]
If \( \alpha < 1 \), \( \|\cdot\|_{\Psi_\alpha} \) is actually a quasi-norm, however many norm-like properties (like a triangle-type inequality) can nevertheless be recovered up to \( \alpha \)-dependent constants (see e.g. \cite[Appendix A]{Appendix}). In fact, \( C_{1,\alpha} \), \( C_{2,\alpha} \) and \( C_{3,\alpha} \) can be chosen such that they only differ by a constant \( \alpha \)-dependent factor.

Note that \( \alpha \)-subexponential random variables have log-convex (if \( \alpha \leq 1 \)) or log-concave (if \( \alpha \geq 1 \)) tails, i.e. \( t \mapsto -\log \Pr(|X| \geq t) \) is convex or concave, respectively.

For log-convex or log-concave measures, two-sided \( L^p \) norm estimates for polynomial chaos (and as a consequence, concentration bounds) have been established over the last 25 years. In the log-convex case, results of this type have been derived for linear forms in \cite{17} and for forms of any order in \cite{21,12}. For log-concave measures,
starting with linear forms again in [14], important contributions have been made in [24, 25, 27, 3].

In this note, we mainly present four different results for functions of $\alpha$-subexponential random variables: a Hanson–Wright type inequality in Section 2, a version of the convex concentration inequality in Section 3, a uniform Hanson–Wright inequality in Section 4, and finally a convex concentration inequality for simple random tensors in Section 5. These results are partly based on and generalize recent research, e.g. [20] and [42]. In fact, they partially build upon each other: for instance, in the proofs of Section 5 we apply results both from Section 2 and Section 3. A more detailed discussion is provided in each of the sections.

Finally, let us introduce some conventions which we will use in this paper.

**Notations.** If $X_1, \ldots, X_n$ is a sequence of random variables, we denote by $X := (X_1, \ldots, X_n)$ the corresponding random vector. Moreover, we shall need the following types of norms throughout the paper:

- the norms $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $x \in \mathbb{R}^n$,
- the $L^p$ norms $\|X\|_{L^p} := (E|X|^p)^{1/p}$ for random variables $X$ (cf. (1.2)),
- the Orlicz (quasi-) norms $\|X\|_{\Psi_\alpha}$ as introduced in (1.3),
- the Hilbert–Schmidt and operator norms $\|A\|_{HS} := (\sum_{i,j} a_{ij}^2)^{1/2}$, $\|A\|_{op} := \sup\{\|Ax\|_2 : \|x\|_2 = 1\}$ for matrices $A = (a_{ij})$.

The constants appearing in this paper (typically denoted $C$ or $c$) may vary from line to line. Without subscript they are assumed to be absolute, if they depend on $\alpha$ (only) we shall write $C_\alpha$ or $c_\alpha$.

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2. A generalized Hanson–Wright inequality

Arguably, the most famous concentration result for quadratic form is the Hanson–Wright inequality, which first appeared in [16]. We may state it as follows: assuming $X_1, \ldots, X_n$ are centered, independent random variables satisfying $\|X_i\|_{\Psi_2} \leq K$ for any $i$, and $A = (a_{ij})$ is a symmetric matrix, we have for any $t \geq 0$

$$\mathbb{P}\left( |X^TAX - \mathbb{E}X^TAX| \geq t \right) \leq 2 \exp\left( - \frac{1}{C} \min\left( \frac{t^2}{K^4\|A\|_{HS}^2}, \frac{t}{K^2\|A\|_{op}} \right) \right).$$

For a modern proof, see [33], and for various developments, cf. [18, 43, 2, 4].

In this note, we provide an extension of the Hanson–Wright inequality to random variables with bounded Orlicz norms of any order $\alpha \in (0, 2]$. This complements the results in [12], where the case of $\alpha \in (0, 1]$ was considered, while for $\alpha = 2$, we get back the actual Hanson–Wright inequality.

**Theorem 2.1.** For any $\alpha \in (0, 2]$, let $X_1, \ldots, X_n$ be independent, centered random variables such that $\|X_i\|_{\Psi_\alpha} \leq K$ for any $i$, and $A = (a_{ij})$ be a symmetric matrix. Then, for any $t \geq 0$,

$$\mathbb{P}\left( |X^TAX - \mathbb{E}X^TAX| \geq t \right) \leq 2 \exp\left( - \frac{1}{C_\alpha} \min\left( \frac{t^2}{K^4\|A\|_{HS}^2}, \left( \frac{t}{K^2\|A\|_{op}} \right)^{\frac{\alpha}{2}} \right) \right).$$
Theorem 2.1 generalizes and implies a number of inequalities for quadratic forms in \( \alpha \)-subexponential random variables (in particular for \( \alpha = 1 \)) which are spread throughout the literature. For a detailed discussion, see \([12\), Remark 1.7]\). Note that it is possible to sharpen the tail estimate given by Theorem 2.1, cf. e.g. \([12\), Corollary 1.4\] for \( \alpha \in (0, 1] \) or \([3\), Theorem 3.2\] for \( \alpha \in [1, 2] \) (in fact, the proof of Theorem 2.1 works by evaluating the family of norms used therein). The main benefit of Theorem 2.1 is that it uses norms which are easily calculable and in many situations already sufficient for applications.

Before we give the proof of Theorem 2.1, let us briefly mention that for the standard Hanson–Wright inequality, a number of selected applications can be found in \([33\). Some of them were generalized to \( \alpha \)-subexponential random variables with \( \alpha \leq 1 \) in \([12\), and it is no problem to extend these proofs to any order \( \alpha \in (0, 2] \) using Theorem 2.1. Here, we just focus on a single example which yields a concentration result for the Euclidean norm of a linear transformation of a vector \( X \) having independent components with bounded Orlicz norms around the Hilbert–Schmidt norm of the transformation matrix. This is a variant and extension of \([12\), Proposition 2.1\] and will be applied in Section 5.

**Proposition 2.2.** Let \( X_1, \ldots, X_n \) be independent, centered random variables such that \( \mathbb{E} X_i^2 = 1 \) and \( \|X_i\|_{\Phi_\alpha} \leq K \) for some \( \alpha \in (0, 2] \), and let \( B \neq 0 \) be an \( m \times n \) matrix. For any \( t \geq 0 \) we have

\[
P(\|BX\|_2 - \|B\|_{\text{HS}} \geq tK^2\|B\|_{\text{op}}) \leq 2 \exp(-t^\alpha/C_\alpha).
\]  

In particular, for any \( t \geq 0 \) it holds

\[
P(\|X\|_2 - \sqrt{n} \geq tK^2) \leq 2 \exp(-t^\alpha/C_\alpha).
\]

For the proofs, let us recall some elementary relations which we will use throughout the paper to adjust the constants in the tail bounds we derive.

**Adjusting constants.** For any two constants \( C_1 > C_2 > 1 \) we have for all \( r \geq 0 \) and \( C > 0 \)

\[
C_1 \exp(-r/C) \leq C_2 \exp\left(-\frac{\log(C_2)}{C\log(C_1)}r\right)
\]

whenever the left hand side is smaller or equal to 1 (cf. e.g. \([35\), Eq. (3.1)]). Moreover, for any \( \alpha \in (0, 2) \), any \( \gamma > 0 \) and all \( t \geq 0 \), we may always estimate

\[
\exp(-t/C^2) \leq 2 \exp(-t/C')^\alpha,
\]

using \( \exp(-s^2) \leq \exp(1 - s^\alpha) \) for any \( s > 0 \) and \([2.3\). More precisely, we may choose \( C' := C/\log^{1/\alpha}(2) \). Note that strictly speaking, the range of \( t/C \leq 1 \) is not covered by \([2.3\), however in this case (in particular, choosing \( C' \) as suggested) both sides of \([2.4\) are at least 1 anyway so that the right hand side still provides a valid upper bound for any probability.

Let us now turn to the proof of Theorem 2.1. In what follows, we actually show that for any \( p \geq 2 \),

\[
\|X^TAX - \mathbb{E}X^TAX\|_{L^p} \leq C_\alpha K^2\left(p^{1/2}\|A\|_{\text{HS}} + p^{2/\alpha}\|A\|_{\text{op}}\right).
\]

From here, Theorem 2.1 follows by standard means (cf. \([34\), Proof of Theorem 3.6\]). Moreover, we may restrict ourselves to \( \alpha \in (1, 2] \), since the case of \( \alpha \in (0, 1] \) has been proven in \([12\).
Proof of Theorem 2.1. First we shall treat the off-diagonal part of the quadratic form. Let \( w_i^{(1)}, w_i^{(2)} \) be independent (of each other as well as of the \( X_i \)) symmetrized Weibull random variables with scale 1 and shape \( \alpha \), i.e. \( w_i^{(j)} \) are symmetric random variables with concave tails. It holds
\[
\| \sum_{i \neq j} a_{ij} X_i X_j \|_{L^p} \leq C_\alpha K^2 \| \sum_{i \neq j} a_{ij} w_i^{(1)} w_j^{(2)} \|_{L^p} \leq C_\alpha K^2 (\| A \|_{\{1,2\},p}^N + \| A \|_{\{1,2\},p}^N),
\]
where the norms \( \| A \|_{\{1,2\},p}^N \) are defined as in \cite{3}. Instead of repeating the general definitions, we will only focus on the case we need in our situation. Indeed, for the symmetric Weibull distribution with parameter \( \alpha \) we have (again, in the notation of \cite{3}) \( N(t) = t^\alpha \), and so for \( \alpha \in (1,2] \), it follows that \( \tilde{N}(t) = \min(t^2, |t|^\alpha) \). Hence, the norms can be written as follows:
\[
\| A \|_{\{1,2\},p}^N = 2 \sup \left\{ \sum_{i,j} a_{ij} x_{ij} : \sum_{i=1}^n \min \left( \sum_{j=1}^n x_{ij}^2, \left( \sum_{j=1}^n x_{ij}^2 \right)^{\alpha/2} \right) \leq p \right\},
\]
\[
\| A \|_{\{1,1\},p}^N = \sup \left\{ \sum_{i,j} a_{ij} x_{ij} y_{ij} : \sum_{i=1}^n \min(x_{ij}^2, |x_{ij}|^\alpha) \leq p, \sum_{j=1}^n \min(y_{ij}^2, |y_{ij}|^\alpha) \leq p \right\}.
\]
Before continuing with the proof, we next introduce a lemma which will help to rewrite the norms in a more tractable form.

**Lemma 2.3.** For any \( p \geq 2 \) define
\[
I_1(p) := \left\{ x = (x_{ij}) \in \mathbb{R}^{n \times n} : \sum_{i=1}^n \min \left( \sum_{j=1}^n x_{ij}^2, \left( \sum_{j=1}^n x_{ij}^2 \right)^{\alpha/2} \right) \leq p \right\},
\]
\[
I_2(p) := \left\{ x_{ij} = z_i y_{ij} \in \mathbb{R}^{n \times n} : \sum_{i=1}^n \min(|z_i|^\alpha, z_i^2) \leq p, \max_{i=1,\ldots,n} \sum_{j=1}^n y_{ij}^2 \leq 1 \right\}.
\]
Then \( I_1(p) = I_2(p) \).

**Proof.** The inclusion \( I_1(p) \supseteq I_2(p) \) is an easy calculation, and the inclusion \( I_1(p) \subseteq I_2(p) \) follows by defining \( z_i = \| (x_{ij})_j \| \) and \( y_{ij} = x_{ij}/\| (x_{ij})_j \| \) (or 0, if the norm is zero).

**Proof of Theorem 2.1, continued.** For brevity, for any matrix \( A = (a_{ij}) \) let us write \( \| A \|_m := \max_{i=1,\ldots,n} (\sum_{j=1}^n a_{ij}^2)^{1/2} \). Note that clearly, \( \| A \|_m \leq \| A \|_{op} \).

Now, fix some vector \( z \in \mathbb{R}^n \) such that \( \sum_{i=1}^n \min(|z_i|^\alpha, z_i^2) \leq p \). The condition also implies
\[
p \geq \sum_{i=1}^n |z_i|^\alpha 1_{\{|z_i|>1\}} + \sum_{i=1}^n z_i^2 1_{\{|z_i|\leq 1\}} \geq \max \left( \sum_{i=1}^n z_i^\alpha 1_{\{|z_i|>1\}}, \sum_{i=1}^n |z_i| 1_{\{|z_i|>1\}} \right),
\]
where in the second step we used \( \alpha \in [1,2] \) to estimate \( |z_i|^\alpha 1_{\{|z_i|>1\}} \geq |z_i| 1_{\{|z_i|>1\}} \). So, given any \( z \) and \( y \) satisfying the conditions of \( I_2(p) \), we can write
\[
\sum_{i,j} a_{ij} z_i y_{ij} \leq \sum_{i=1}^n |z_i| \left( \sum_{j=1}^n a_{ij}^2 \right)^{1/2} \left( \sum_{j=1}^n y_{ij}^2 \right)^{1/2} \leq \sum_{i=1}^n |z_i| \left( \sum_{j=1}^n a_{ij}^2 \right)^{1/2}.
\]
\[
\leq \sum_{i=1}^{n} |z_i| \mathbb{1}_{\{|z_i| \leq 1\}} \left( \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} + \sum_{i=1}^{n} |z_i| \mathbb{1}_{\{|z_i| > 1\}} \left( \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} \\
\leq \|A\|_{\text{HS}} \left( \sum_{i=1}^{n} z_i^2 \mathbb{1}_{\{|z_i| \leq 1\}} \right)^{1/2} + \|A\|_{\text{m}} \sum_{i=1}^{n} |z_i| \mathbb{1}_{\{|z_i| > 1\}}.
\]

So, this yields

\begin{equation}
(2.7) \quad \|A\|_{\mathbb{N}, \{1\}, \{2\}, p} \leq 2p^{1/2} \|A\|_{\text{HS}} + 2p\|A\|_{\text{m}} \leq 2p^{1/2} \|A\|_{\text{HS}} + 2p\|A\|_{\text{op}}.
\end{equation}

As for \(\|A\|_{\mathbb{N}, \{1\}, \{2\}, p}\), we can use the decomposition \(z = z_1 + z_2\), where \((z_1)_i = z_i \mathbb{1}_{\{|z_i| > 1\}}\) and \(z_2 = z - z_1\), and obtain

\[
\|A\|_{\mathbb{N}, \{1\}, \{2\}, p} \leq \sup \left\{ \sum_{ij} a_{ij}(x_i)_j(y_1)_j : \|x_1\|_\alpha \leq p^{1/\alpha}, \|y_1\|_\alpha \leq p^{1/\alpha} \right\} \\
+ 2 \sup \left\{ \sum_{ij} a_{ij}(x_1)_j(y_2)_j : \|x_1\|_\alpha \leq p^{1/\alpha}, \|y_2\|_2 \leq p^{1/2} \right\} \\
+ \sup \left\{ \sum_{ij} a_{ij}(x_2)_j(y_2)_j : \|x_2\|_2 \leq p^{1/2}, \|y_2\|_2 \leq p^{1/2} \right\} \\
= p^{2/\alpha} \sup\{\ldots\} + 2p^{1/\alpha+1/2} \sup\{\ldots\} + p\|A\|_{\text{op}}
\]

(in the braces, the conditions \(\|\cdot\|_\beta \leq p^{1/\beta}\) have been replaced by \(\|\cdot\|_\beta \leq 1\)). Clearly, since \(\|x_1\|_\alpha \leq 1\) implies \(\|x_1\|_2 \leq 1\) (and the same for \(y_1\)), all of the norms can be upper bounded by \(\|A\|_{\text{op}}\), i.e. we have

\begin{equation}
(2.8) \quad \|A\|_{\mathbb{N}, \{1\}, \{2\}, p} \leq (p^{2/\alpha} + 2p^{1/\alpha+1/2} + p)\|A\|_{\text{op}} \leq 4p^{2/\alpha}\|A\|_{\text{op}},
\end{equation}

where the last inequality follows from \(p \geq 2\) and \(1/2 \leq 1/\alpha \leq 1 \leq (\alpha + 2)/(2\alpha) \leq 2/\alpha\).

Combining the estimates (2.6), (2.7), and (2.8) yields

\[
\|\sum_{ij} a_{ij} X_i X_j\|_{L^p} \leq C_\alpha K^2 \left( 2p^{1/2} \|A\|_{\text{HS}} + 6p^{2/\alpha}\|A\|_{\text{op}} \right).
\]

To treat the diagonal terms, we use Corollary 6.1 in [12], as \(X_i^2\) are independent and satisfy \(\|X_i^2\|_{\Psi_{\rho/2}} \leq K^2\), so that it yields

\[
\mathbb{P} \left( \sum_{i=1}^{n} a_{ii}(X_i^2 - \mathbb{E} X_i^2) \geq t \right) \leq 2 \exp \left( -\frac{t^2}{C_\alpha K^2} \min \left( \frac{t}{\sum_{i=1}^{n} a_{ii}^2}, \left( \frac{t}{\max_{i=1,\ldots,n} |a_{ii}|} \right)^{\rho/2} \right) \right).
\]

Now it is clear that \(\max_{i=1,\ldots,n} |a_{ii}| \leq \|A\|_{\text{op}}\) and \(\sum_{i=1}^{n} a_{ii}^2 \leq \|A\|_{\text{HS}}^2\). In particular,

\[
\|\sum_{i=1}^{n} a_{ii} (X_i^2 - \mathbb{E} X_i^2)\|_{L^p} \leq C_\alpha K^2 (p^{1/2} \|A\|_{\text{HS}} + p^{2/\alpha}\|A\|_{\text{op}}).
\]

The claim (2.5) now follows from Minkowski’s inequality. \(\square\)

Finally, we prove Proposition 2.2.

\textbf{Proof of Proposition 2.2} It suffices to prove (2.1) for matrices satisfying \(\|B\|_{\text{HS}} = 1\), as otherwise we set \(\bar{B} = B\|B\|_{\text{HS}}^{-1}\) and use the equality

\[
\{ \|BX\|_2 - \|B\|_{\text{HS}} \geq \|B\|_{\text{op}} t \} = \{ \|\bar{B}X\|_2 - 1 \geq \|\bar{B}\|_{\text{op}} t \}.
\]

Now let us apply Theorem 2.1 to the matrix \(A := B^T B\). An easy calculation shows that \(\text{trace}(A) = \text{trace}(B^T B) = \|B\|_{\text{HS}}^2 = 1\), so that we have for any \(t \geq 0\)

\[
\mathbb{P} \left( \|BX\|_2 - 1 \geq t \right) \leq \mathbb{P} \left( \|BX\|_2^2 - 1 \geq \max(t, t^2) \right)
\]
\[ \leq 2 \exp \left( - \frac{1}{C_\alpha} \min \left( \frac{\max(t, t^2)^2}{K^4 \| B \|_{\text{op}}^2}, \frac{\max(t, t^2)}{K^4 \| B \|_{\text{op}}^2} \right)^{\alpha/2} \right) \]

\[ \leq 2 \exp \left( - \frac{1}{C_\alpha} \left( \frac{t^2}{K^4 \| B \|_{\text{op}}^2}, \frac{t^2}{K^4 \| B \|_{\text{op}}^2} \right)^{\alpha/2} \right) \]

\[ \leq 2 \exp \left( - \frac{1}{C_\alpha} \left( \frac{t}{K^2 \| B \|_{\text{op}}} \right)^{\alpha} \right). \]

Here, the first step follows from \(|z - 1| \leq \min(|z^2 - 1|, |z^2 - 1|^{1/2})\) for \(z \geq 0\), in the second step we have used the estimates \(\|A\|_{\text{op}}^\alpha \leq \|B\|_{\text{op}}^\alpha \|B\|_{\text{op}} = \|B\|_{\text{op}}^2\) and \(\|A\|_{\text{op}} \leq \|B\|_{\text{op}}^2\) and moreover the fact that since \(\mathbb{E}X_i^2 = 1\), \(K \geq C_\alpha > 0\) (cf. e.g. [12, Lemma A.2]), while the last step follows from (2.4) and (2.3). Setting \(t = K^2 s\|B\|_{\text{op}}\) for \(s \geq 0\) finishes the proof of (2.1). Finally, (2.2) follows by taking \(m = n\) and \(B = I\). \(\square\)

3. Convex concentration for random variables with bounded Orlicz norms

Assume \(X_1, \ldots, X_n\) are independent random variables each taking values in some bounded interval \([a, b]\). Then, by convex concentration as established in [38, 15, 29], for every convex 1-Lipschitz function \(f: [a, b]^n \to \mathbb{R}\),

\[ (3.1) \quad \mathbb{P}(\|f(X) - \mathbb{E}f(X)\| > t) \leq 2 \exp \left( - \frac{t^2}{2(b-a)^2} \right) \]

for any \(t \geq 0\) (see e.g. [36, Corollary 3]).

While convex concentration for bounded random variables is by now standard, there is less literature for unbounded random variables. In [31], a martingale-type approach is used, leading to a result for functionals with stochastically bounded increments. The special case of suprema of unbounded empirical processes was treated in [1, 40, 28]. Another branch of research, begun in [29] and continued e.g. in [36, 37, 13, 15, 14, 3], is based on functional inequalities (like Poincaré or log-Sobolev inequalities) restricted to convex functions and weak transport-entropy inequalities. In [20, Lemma 1.8], a generalization of (3.1) for subgaussian random variables \((\alpha = 2)\) was proven, which we may extend to any order \(\alpha \in (0, 2]\).

**Proposition 3.1.** Let \(X_1, \ldots, X_n\) be independent random variables, \(\alpha \in (0, 2]\) and \(f: \mathbb{R}^n \to \mathbb{R}\) convex and 1-Lipschitz. Then, for any \(t \geq 0\),

\[ \mathbb{P}(\|f(X) - \mathbb{E}f(X)\| > t) \leq 2 \exp \left( - \frac{t^\alpha}{C_\alpha \max_i |X_i|\|\psi_\alpha\|} \right). \]

In particular,

\[ (3.2) \quad \|f(X) - \mathbb{E}f(X)\|_{\psi_\alpha} \leq C_\alpha \max_i |X_i|\|\psi_\alpha\|. \]

Note that the main results of the following two sections can be regarded as applications of Proposition 3.1. If \(f\) is separately convex only (i.e. convex is every coordinate with the other coordinates being fixed), it is still possible to prove a corresponding result for the upper tails. Indeed, it is no problem to modify the proof below accordingly, replacing (3.1) by [7, Theorem 6.10]. Moreover, note that \(\max_i |X_i|\|\psi_\alpha\|\) cannot be replaced by \(\max_i |X_i|\|\psi_\alpha\|\) (a counterexample for \(\alpha = 2\) is provided in [20]). In general, the Orlicz norm of \(\max_i |X_i|\) will be of order \((\log n)^{1/\alpha}\) (cf. Lemma 5.6).
Proof of Proposition 3.1. Following the lines of the proof of [20, Lemma 3.5], the key step is a suitable truncation which goes back to [1]. Indeed, write

\[ X_i = X_i 1_{\{|X_i| \leq M\}} + X_i 1_{\{|X_i| > M\}} =: Y_i + Z_i \]

with \( M := 8\mathbb{E} \max_i |X_i| \) (in particular, \( M \leq C_\alpha \| \max_i |X_i| \|_{\psi_\alpha} \), cf. [12, Lemma A.2]), and let \( Y = (Y_1, \ldots, Y_n), Z = (Z_1, \ldots, Z_n) \). By the Lipschitz property of \( f \),

\[
\mathbb{P}(|f(X) - \mathbb{E} f(X)| > t) \leq \mathbb{P}(|f(Y) - \mathbb{E} f(Y)| + |f(X) - f(Y)| + |\mathbb{E} f(Y) - \mathbb{E} f(X)| > t)
\]

\[ \leq \mathbb{P}(|f(Y) - \mathbb{E} f(Y)| + \|Z\|_2 + \|\mathbb{E} f(X) - \mathbb{E} f(Y)| > t), \]

and hence it suffices to bound the terms in the last line.

Applying (3.1) to \( Y \) and using (2.4) and (2.3), we obtain

\[
\mathbb{P}(|f(Y) - \mathbb{E} f(Y)| > t) \leq 2 \exp \left( -\frac{t^\alpha}{C_\alpha \| \max_i |X_i| \|_{\psi_\alpha}^\alpha} \right).
\]

Furthermore, below we will show that

\[
\|Z\|_2 \leq C_\alpha \| \max_i |X_i| \|_{\psi_\alpha}.
\]

Hence, for any \( t \geq 0 \),

\[
\mathbb{P}(\|Z\|_2 \geq t) \leq 2 \exp \left( -\frac{t^\alpha}{C_\alpha \| \max_i |X_i| \|_{\psi_\alpha}^\alpha} \right),
\]

and by [12, Lemma A.2],

\[
\mathbb{E} \|Z\|_2 \leq C_\alpha \| \max_i |X_i| \|_{\psi_\alpha}.
\]

Temporarily writing \( K := C_\alpha \| \max_i |X_i| \|_{\psi_\alpha} \), where \( C_\alpha \) is large enough so that (3.5), (3.7) and (3.8) hold, (3.4) and (3.8) yield

\[
\mathbb{P}(|f(X) - \mathbb{E} f(X)| > t) \leq \mathbb{P}(|f(Y) - \mathbb{E} f(Y)| + \|Z\|_2 > t - K)
\]

if \( t \geq K \). Using subadditivity and invoking (3.5) and (3.7), we obtain

\[
\mathbb{P}(|f(X) - \mathbb{E} f(X)| > t) \leq 4 \exp \left( -\frac{(t - K)^\alpha}{(2K)^\alpha} \right) \leq 4 \exp \left( -\frac{t^\alpha}{c_\alpha (2K)^\alpha} \right),
\]

where the last step holds for \( t \geq K + \delta \) for some \( \delta > 0 \). This bound extends trivially to any \( t \geq 0 \) (if necessary, by a suitable change of constants). Finally, the constant in front of the exponential may be adjusted to 2 by (2.3), which finishes the proof.

It remains to show (3.6). To this end, recall the Hoffmann–Jørgensen inequality (cf. [30, Theorem 6.8]) in the following form: if \( W_1, \ldots, W_n \) are independent random variables, \( S_k := W_1 + \ldots + W_k \), and \( t \geq 0 \) is such that \( \mathbb{P}(\max_k |S_k| > t) \leq 1/8 \), then

\[
\mathbb{E} \max_k |S_k| \leq 3 \mathbb{E} \max_i |W_i| + 8t.
\]

In our case, we set \( W_i := Z_i^2, t = 0 \), and note that by Chebyshev’s inequality,

\[
\mathbb{P}(\max_i Z_i^2 > 0) = \mathbb{P}(\max_i |X_i| > M) \leq \mathbb{E} \max_i |X_i| / M = 1/8,
\]

and consequently, recalling that \( S_k = Z_1^2 + \ldots + Z_k^2 \),

\[
\mathbb{P}(\max_k |S_k| > 0) \leq \mathbb{P}(\max_i Z_i^2 > 0) \leq 1/8.
\]

Thus, together with [12, Lemma A.2], we obtain

\[
\mathbb{E} \|Z\|_2^2 \leq 3 \mathbb{E} \max_i Z_i^2 \leq C_\alpha \| \max_i Z_i^2 \|_{\psi_{\alpha/2}}.
\]
Now it is easy to see that \( \| \max_i Z_i^2 \| \Psi_{\alpha/2} \leq \| \max_i |X_i| \| \Psi_{\alpha}^2 \), so that altogether we arrive at
\[
\mathbb{E}\|Z\|_2^2 \leq C_\alpha \| \max_i |X_i| \| \Psi_{\alpha}^2
\]

Furthermore, by [30, Theorem 6.21], if \( W_1, \ldots, W_n \) are independent random variables with zero mean and \( \alpha \in (0, 1) \),
\[
\| \sum_{i=1}^n W_i \| \Psi_\alpha \leq C_\alpha (\| \sum_{i=1}^n W_i \| \Psi_\alpha + \max_i \| W_i \| \Psi_\alpha)
\]
In our case, we consider \( W_i = Z_i^2 - \mathbb{E}Z_i^2 \) and \( \alpha/2 \) (instead of \( \alpha \)). Together with the previous arguments (in particular (3.9)) and [12, Lemma A.3], this yields
\[
\| \sum_{i=1}^n (Z_i^2 - \mathbb{E}Z_i^2) \| \Psi_{\alpha/2} \leq C_\alpha (\mathbb{E}\|Z\|_2^2 - \mathbb{E}\|Z\|_2^2 + \| \max_i |Z_i^2 - \mathbb{E}Z_i^2| \| \Psi_{\alpha/2})
\]
\[
\leq C_\alpha (\mathbb{E}\|Z\|_2^2 + \max_i Z_i^2 \| \Psi_{\alpha/2}) \leq C_\alpha \| \max_i |X_i| \| \Psi_{\alpha}^2.
\]
Combining this with [12, Lemma A.3] and (3.3), we arrive at (3.6).

4. Uniform tail bounds for first and second order chaos

In this section, we discuss bounds for the tails of the supremum of certain chaos-type classes of functions. Even if we are particularly interested in quadratic forms, i.e. uniform Hanson–Wright inequalities, let us first consider linear forms.

Let \( X_1, \ldots, X_n \) be independent random variables, let \( \alpha \in (0, 2] \), and let \( \{a_{i,t}: i = 1, \ldots, n, t \in \mathcal{T} \} \) be a compact set of real numbers, where \( \mathcal{T} \) is some index set. Consider \( g(X) := \sup_{t \in \mathcal{T}} \sum_{i=1}^n a_{i,t} X_i \). Clearly, \( g \) is convex and has Lipschitz constant \( D := \sup_{t \in \mathcal{T}} (\sum_{i=1}^n a_{i,t}^2)^{1/2} \). Therefore, applying Proposition 3.1, we immediately obtain that for any \( t \geq 0 \),
\[
\mathbb{P}(\|g(X) - \mathbb{E}g(X)\| \geq t) \leq 2 \exp \left( - \frac{t^\alpha}{C_\alpha D^\alpha \| \max_i |X_i| \| \Psi_{\alpha}^2} \right).
\]
For bounded random variables, corresponding tail bounds can be found e.g. in [32, Eq. (14)], and choosing \( \alpha = 2 \) we get back this result up to constants.

Our main aim is to derive a second order analogue of (4.1), i.e. a uniform Hanson–Wright inequality. A pioneering result in this direction (for Rademacher variables) can be found in [12]. Later results include [2] (which requires the so-called concentration property), [22], [3] and [11] (certain classes of weakly dependent random variables). In [20], a uniform Hanson–Wright inequality for subgaussian random variables was proven. We may show a similar result for random variables with bounded Orlicz norms of any order \( \alpha \in (0, 2] \).

**Theorem 4.1.** Let \( X_1, \ldots, X_n \) be independent, centered random variables and \( K := \| \max_i |X_i| \| \Psi_{\alpha} \), where \( \alpha \in (0, 2] \). Let \( \mathcal{A} \) be a compact set of real symmetric \( n \times n \) matrices, and let \( f(X) := \sup_{A \in \mathcal{A}} (X^TAX - \mathbb{E}X^TAX) \). Then, for any \( t \geq 0 \),
\[
\mathbb{P}(f(X) - \mathbb{E}f(X) \geq t) \leq 2 \exp \left( - \frac{1}{C_\alpha K^\alpha \min \left( \frac{t^\alpha}{(\mathbb{E}\sup_{A \in \mathcal{A}} \| AX \|_2)^{\alpha/2}}, \frac{t^{\alpha/2}}{\sup_{A \in \mathcal{A}} \| A \|_{op}^{\alpha/2}} \right)} \right).
\]
For \( \alpha = 2 \), this gives back [20, Theorem 1.1] (up to constants and a different range of \( t \)). Comparing Theorem 4.1 to Theorem 2.1, we note that instead of a subgaussian term, we obtain an \( \alpha \)-subexponential term (which can be trivially transformed into
a subgaussian term for $t \leq \mathbb{E} \sup_{A \in \mathcal{A}} \|AX\|_2$, but this does not cover the complete $\alpha$-subexponential regime). Moreover, Theorem 4.1 only gives a bound for the upper tails. Therefore, if $\mathcal{A}$ just consists of a single matrix, Theorem 2.1 is stronger. These differences have technical reasons.

To prove Theorem 4.1, we shall follow the basic steps of [20] and modify those where the truncation comes in. Let us first repeat some tools and results. In the sequel, for a random vector $W = (W_1, \ldots, W_n)$, we shall denote

\begin{equation}
(4.2) \quad f(W) := \sup_{A \in \mathcal{A}} (W^T AW - g(A)),
\end{equation}

where $g: \mathbb{R}^{n \times n} \to \mathbb{R}$ is some function. Moreover, if $A$ is any matrix, we denote by $\text{Diag}(A)$ its diagonal part (regarded as a matrix with zero entries on its off-diagonal). The following lemma combines [20, Lemmas 3.2 & 3.5].

**Lemma 4.2.**

1. Assume the vector $W$ has independent components which satisfy $W_i \leq K$ a.s. Then, for any $t \geq 1$, we have

\[ f(W) - \mathbb{E} f(W) \leq C \left( K (\mathbb{E} \sup_{A \in \mathcal{A}} \|AW\|_2 + \mathbb{E} \sup_{A \in \mathcal{A}} \|\text{Diag}(A)W\|_2) \sqrt{t} + K^2 \sup_{A \in \mathcal{A}} \|A\|_{\text{op}} t \right) \]

with probability at least $1 - e^{-t}$.

2. Assuming the vector $W$ has independent (but not necessarily bounded) components with mean zero, we have

\[ \mathbb{E} \sup_{A \in \mathcal{A}} \|\text{Diag}(A)W\|_2 \leq C \mathbb{E} \sup_{A \in \mathcal{A}} \|AW\|_2. \]

From now on, let $X$ be the random vector from Theorem 4.1 and recall the truncated random vector $Y$ which we introduced in (3.3) (and the corresponding “remainder” $Z$). Then, Lemma 4.2 (1) for $f(Y)$ with $g(A) = \mathbb{E} X^T AX$ yields

\begin{equation}
(4.3) \quad f(Y) - \mathbb{E} f(Y) \leq C \left( M (\mathbb{E} \sup_{A \in \mathcal{A}} \|AY\|_2 + \mathbb{E} \sup_{A \in \mathcal{A}} \|\text{Diag}(A)Y\|_2) t^{1/2} + M^2 t^{2/\alpha} \sup_{A \in \mathcal{A}} \|A\|_{\text{op}} \right)
\end{equation}

with probability at least $1 - e^{-t}$ (actually, (4.3) even holds with $\alpha = 2$, but in the sequel we will have to use the weaker version given above anyway). Here we recall that $M \leq C_\alpha \|\max_i |X_i|\|_{\psi_\alpha}$.

To prove Theorem 4.1, it remains to replace the terms involving the truncated random vector $Y$ by the original vector $X$. First, by Proposition 3.3 and since $\sup_{A \in \mathcal{A}} \|AX\|_2$ is $\sup_{A \in \mathcal{A}} \|A\|_{\text{op}}$-Lipschitz, we obtain

\begin{equation}
(4.4) \quad \mathbb{P} \left( \sup_{A \in \mathcal{A}} \|AX\|_2 > \mathbb{E} \sup_{A \in \mathcal{A}} \|AX\|_2 + C_\alpha \|\max_i |X_i|\|_{\psi_\alpha} \sup_{A \in \mathcal{A}} \|A\|_{\text{op}} t^{1/\alpha} \right) \leq 2e^{-t}.
\end{equation}

Moreover, by (3.3),

\begin{equation}
(4.5) \quad \left| \mathbb{E} \sup_{A \in \mathcal{A}} \|AY\|_2 - \mathbb{E} \sup_{A \in \mathcal{A}} \|AX\|_2 \right| \leq C_\alpha \|\max_i |X_i|\|_{\psi_\alpha} \sup_{A \in \mathcal{A}} \|A\|_{\text{op}}.
\end{equation}

Next we estimate the difference between the expectations of $f(X)$ and $f(Y)$.

**Lemma 4.3.** We have

\[ |\mathbb{E} f(Y) - \mathbb{E} f(X)| \leq C_\alpha \left( \|\max_i |X_i|\|_{\psi_\alpha} \mathbb{E} \sup_{A \in \mathcal{A}} \|AX\|_2 + \|\max_i |X_i|\|_{\psi_\alpha}^2 \sup_{A \in \mathcal{A}} \|A\|_{\text{op}} \right). \]

**Proof.** First note that

\[ f(X) = \sup_{A \in \mathcal{A}} (Y^T AX - \mathbb{E} X^T AX + Z^T AX + Z^T Y). \]
We may estimate $(E \exp)\{X\}$ with probability at least $1 - e^{-t}$ for all $t \geq 1$. Using (3.7), it follows that
\[
\mathbb{E} \sup_{\alpha} |AX| \leq C_\alpha (\mathbb{E} \sup_{\alpha} |AX|)\sup_{\alpha} |A|\sup_{\alpha} |A|\mathbb{E} \sup_{\alpha} |A|.
\]
Arguing similarly and using (4.5), the same bound also holds for (4.8). Taking roots and plugging everything into (4.7) completes the proof. \hfill \Box

Finally, we prove the central result of this section.

**Proof of Theorem 4.1.** First, it immediately follows from Lemma 4.3 that
\[
\mathbb{E} f(Y) \leq \mathbb{E} f(X) + C_\alpha \left( \max_i |X_i| \|A\|_{\infty} \mathbb{E} \sup_{\alpha} |AX| + \max_i |X_i| \|A\|_{\infty} \mathbb{E} \sup_{\alpha} |A|\right).
\]
Moreover, by (4.5) and Lemma 3.2 (2),
\[
\mathbb{E} \sup_{\alpha} |AY| + \mathbb{E} \sup_{\alpha} \|\text{Diag}(A)Y\|_2 \leq C_\alpha (\mathbb{E} \sup_{\alpha} |AX| + \max_i |X_i| \|A\|_{\infty} \mathbb{E} \sup_{\alpha} |A|\).
\]
Finally, it follows from (4.6), (4.9) and (4.5) that
\[
|f(X) - f(Y)| \leq \|Z\|_2 \sup_{\alpha} |AX| + \|Z\|_2 \sup_{\alpha} |AY|
\]
with probability at least $1 - 4e^{-t}$ for all $t \geq 1$. Using (3.7), it follows that
\[
|f(X) - f(Y)| \leq C_\alpha (\max_i |X_i| \|A\|_{\infty} \mathbb{E} \sup_{\alpha} |AX|) + \max_i |X_i| \|A\|_{\infty} \mathbb{E} \sup_{\alpha} |A|\mathbb{E} \sup_{\alpha} |A|\mathbb{E} \sup_{\alpha} |A|.
\]
Combining (4.8), (4.10) and plugging into (4.3), this yields that with probability at least $1 - 6e^{-t}$ for all $t \geq 1$, \[
f(X) - \mathbb{E} f(X) \leq C_\alpha (\max_i |X_i| \|A\|_{\infty} \mathbb{E} \sup_{\alpha} |AX|) + \max_i |X_i| \|A\|_{\infty} \mathbb{E} \sup_{\alpha} |A|\mathbb{E} \sup_{\alpha} |A|.
\]
If $u \geq \max(a, b)$, it follows that
\[
\mathbb{P}(f(X) - \mathbb{E} f(X) \geq u) \leq 6 \exp \left( - \frac{1}{C_\alpha} \min \left( \left( \frac{u}{a} \right)^{\alpha}, \left( \frac{u}{b} \right)^{\alpha/2} \right) \right).
\]
By standard means (a suitable change of constants, using (2.3)), this bound may be extended to any $u \geq 0$ and the constant may be adjusted to 2. \hfill \Box
5. Random Tensors

By a simple random tensor, we mean a random tensor of the form

\[(5.1) \quad X := X_1 \otimes \cdots \otimes X_d = (X_{1,i_1} \cdots X_{d,i_d})_{i_1,\ldots,i_d} \in \mathbb{R}^{n_d},\]

where all \(X_k\) are independent random vectors in \(\mathbb{R}^n\) whose coordinates are independent, centered random variables with variance one. Concentration results for random tensors (typically for polynomial-type functions) have been shown in [26, 6, 12], for instance.

Recently, in [42] new and interesting concentration bounds for simple random tensors were shown. In comparison to previous work, these inequalities focus on small values of \(t\), e.g. a regime where subgaussian tail decay holds. Moreover, in contrast to previous papers, [42] provides constants with optimal dependence on \(d\). One of these results is the following convex concentration inequality: assuming that \(n\) and \(d\) are positive integers, \(f:\mathbb{R}^{n_d} \rightarrow \mathbb{R}\) is convex and 1-Lipschitz and the \(X_{ij}\) are bounded a.s., then for any \(t \in [0,2n^{d/2}]\),

\[(5.2) \quad P(\|f(X) - Ef(X)\| > t) \leq 2 \exp\left(-\frac{t^2}{Cdn^{d-1}}\right),\]

where \(C > 0\) only depends on the bound of the coordinates. Using Theorem 2.1 and Proposition 3.1, we may extend this result to unbounded random variables as follows:

**Theorem 5.1.** Let \(n, d \in \mathbb{N}\) and \(f:\mathbb{R}^{n_d} \rightarrow \mathbb{R}\) be convex and 1-Lipschitz. Consider a simple random tensor \(X := X_1 \otimes \cdots \otimes X_d\) as in (5.1). Fix \(\alpha \in [1,2]\), and assume that \(\|X_{ij}\|_{\Psi_\alpha} \leq K\). Then, for any \(t \in [0,c_\alpha n^{d/2}(\log n)^{1/\alpha}/K]\),

\[P(\|f(X) - Ef(X)\| > t) \leq 2 \exp\left(-\frac{t}{1/C_\alpha (d^{1/2}n^{(d-1)/2}(\log n)^{1/\alpha}K)^{\alpha}}\right).\]

On the other hand, if \(\alpha \in (0,1)\), then, for any \(t \in [0,c_\alpha n^{d/2}(\log n)^{1/\alpha}d^{(1-1/\alpha)-1/2}/K]\),

\[P(\|f(X) - Ef(X)\| > t) \leq 2 \exp\left(-\frac{t}{1/C_\alpha (d^{1/\alpha}n^{(d-1)/2}(\log n)^{1/\alpha}K)^{\alpha}}\right).\]

The logarithmic factor stems from the Orlicz norm of \(\max_i \|X_i\|\) in Proposition 3.1. For a slightly sharper version which includes the explicit dependence on these norms (and also gives back (5.2) for bounded random variables and \(\alpha = 2\)), see (5.12) in the proof of Theorem 5.1. We believe that Theorem 5.1 is non-optimal for \(\alpha < 1\) as we would expect a bound of the same type as for \(\alpha \in [1,2]\). However, a key difference in the proofs is that in the case of \(\alpha \geq 1\) we can make use of moment-generating functions. This is clearly not possible if \(\alpha < 1\), so that less subtle estimates must be invoked instead.

For the proof of Theorem 5.1, we first adapt some preliminary steps and compile a number of auxiliary lemmas whose proofs are deferred to the appendix. As a start, we need some additional characterizations of \(\alpha\)-subexponential random variables via the behavior of the moment-generating functions:

**Proposition 5.2.** Let \(X\) be a random variable and \(\alpha \in (0,2]\). Then, the properties (1.1), (1.2) and (1.3) are equivalent to

\[(5.3) \quad \mathbb{E} \exp(\lambda^\alpha |X|^\alpha) \leq \exp(C_{4,\alpha}^\alpha \lambda^\alpha)\]
for all $0 \leq \lambda \leq 1/C_{4,\alpha}$. If $\alpha \in [1,2]$ and $EX = 0$, then the above properties are moreover equivalent to
\begin{align}
(5.4) \quad E \exp(\lambda X) \leq \begin{cases} 
\exp(C_{5,\alpha}^2 \lambda^2) & \text{if } |\lambda| \leq 1/C_{5,\alpha} \\
\exp(C_{5,\alpha}^{\alpha/(\alpha-1)} |\lambda|^{\alpha/(\alpha-1)}) & \text{if } |\lambda| \geq 1/C_{5,\alpha} \text{ and } \alpha > 1.
\end{cases}
\end{align}

The parameters $C_{i,\alpha}$, $i = 1, \ldots, 5$, can be chosen such that they only differ by constant $\alpha$-dependent factors. In particular, we can take $C_{i,\alpha} = c_{i,\alpha} \|X\|_{\psi_{\alpha}}$.

To continue, note that $\|X\|_2 = \prod_{i=1}^d \|X_i\|_2$. A key step in the proofs of \cite{42} is a maximal inequality which simultaneously controls the tails of $\prod_{i=1}^k \|X_i\|_2$, $k = 1, \ldots, d$, where the $X_i$ have independent subgaussian components, i.e. $\alpha = 2$. Generalizing these results to any order $\alpha \in (0,2]$ is not hard. The following preparatory lemma extends \cite{42, Lemma 3.1}. Note that in the proof (given in the appendix again), we apply Proposition 2.2.

**Lemma 5.3.** Let $X_1, \ldots, X_d \in \mathbb{R}^n$ be independent random vectors with independent, centered coordinates such that $EX_{i,j}^2 = 1$ and $\|X_{i,j}\|_{\psi_{\alpha}} \leq K$ for some $\alpha \in (0,2]$. Then, for any $t \in [0,2n^{d/2}]$,
\[
P\left( \prod_{i=1}^d \|X_i\|_2 > n^{d/2} + t \right) \leq 2 \exp\left( -\frac{1}{C_\alpha} \left( \frac{t}{K^2 d^{1/2} n^{(d-1)/2}} \right)^\alpha \right).
\]

To control all $k = 1, \ldots, d$ simultaneously, we need a generalized version of the maximal inequality \cite{42, Lemma 3.2} which we state next.

**Lemma 5.4.** Let $X_1, \ldots, X_d \in \mathbb{R}^n$ be independent random vectors with independent, centered coordinates such that $EX_{i,j}^2 = 1$ and $\|X_{i,j}\|_{\psi_{\alpha}} \leq K$ for some $\alpha \in (0,2]$. Then, for any $u \in [0,2]$,
\[
P\left( \max_{1 \leq k \leq d} \prod_{i=1}^k \|X_i\|_2 > 1 + u \right) \leq 2 \exp\left( -\frac{1}{C_\alpha} \left( \frac{n^{1/2} u}{K^2 d^{1/2}} \right)^\alpha \right).
\]

The following martingale-type bound is directly taken from \cite{42}:

**Lemma 5.5** (\cite{42}, Lemma 4.1). Let $X_1, \ldots, X_d$ be independent random vectors. For each $k = 1, \ldots, d$, let $f_k = f_k(X_1, \ldots, X_d)$ be an integrable real-valued function and $\mathcal{E}_k$ be an event that is uniquely determined by the vectors $X_1, \ldots, X_d$. Let $\mathcal{E}_{d+1}$ be the entire probability space. Suppose that for every $k = 1, \ldots, d$ we have
\[
E_{X_k} \exp(f_k) \leq \pi_k
\]
for every realisation of $X_{k+1}, \ldots, X_d$ in $\mathcal{E}_{k+1}$. Then, for $\mathcal{E} := \mathcal{E}_2 \cap \cdots \cap \mathcal{E}_d$, we have
\[
E \exp(f_1 + \cdots + f_d) I_{\mathcal{E}} \leq \pi_1 \cdots \pi_d.
\]

Finally, we need a bound for the Orlicz norm of $\max_i \|X_i\|_{\psi_{\alpha}}$.

**Lemma 5.6.** Let $X_1, \ldots, X_n$ be independent, centered random variables such that $\|X_i\|_{\psi_{\alpha}} \leq K$ for any $i$ and some $\alpha > 0$. Then,
\[
\| \max_i \|X_i\|_{\psi_{\alpha}} \leq C_\alpha K \max \left\{ \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)^{1/\alpha}, (\log n)^{1/\alpha} \left( \frac{2}{\log 2} \right)^{1/\alpha} \right\}.
\]

Here, we may choose $C_\alpha = \max\{2^{1/\alpha - 1}, 2^{1-1/\alpha}\}$. 

Again, we see that it is $2^n$ where $E$ for any realization $x$ for any

$$
(E \leq \sqrt{\frac{1}{K^2 d^{1/2}}} \alpha)
$$

we are also interested in the case of $\alpha < 1$ in the present note. The condition $EX_i = 0$ in Lemma 5.6 can easily be removed only at the expense of a different absolute constant.

We are now ready to prove Theorem 5.1.

**Proof of Theorem 5.1.** We shall adapt the arguments from [42]. First let

$$E_k := \left\{ \prod_{i=k}^d ||X_i||_2 \leq 2^n(d-k+1/2) \right\}, \quad k = 1, \ldots, d,$$

and let $E_{d+1}$ be the full space. It then follows from Lemma 5.4 for $u = 1$ that

$$P(E) \geq 1 - 2 \exp \left( - \frac{1}{C_\alpha} \left( \frac{n^{1/2}}{K^2 d^{1/2}} \right)^{\alpha} \right),$$

where $E := E_2 \cap \cdots \cap E_d$.

Now fix any realization $x_2, \ldots, x_d$ of the random vectors $X_2, \ldots, X_d$ in $E_2$ and apply Proposition 3.1 to the function $f_1(x_1)$ given by $x_1 \mapsto f(x_1, \ldots, x_d)$. Clearly, $f_1$ is convex, and since

$$|f(x \otimes x_2 \otimes \cdots \otimes x_d) - f(y \otimes x_2 \otimes \cdots \otimes x_d)| \leq ||x - y|| \prod_{i=2}^d ||x_i||_2 \leq ||x - y||_2 2^n(d-1)/2,$$

we see that it is $2^n(d-1)/2$-Lipschitz. Hence, it follows from (3.2) that

$$\|f - EX_1 f\|_{\psi_\alpha(X_1)} \leq c_\alpha n^{(d-1)/2} \max_j \|X_{1,j}\|_{\psi_\alpha}$$

for any $x_2, \ldots, x_d$ in $E_2$, where $EX_1$ denotes taking the expectation with respect to $X_1$ (which, by independence, is the same as conditionally on $X_2, \ldots, X_d$).

To continue, fix any realization $x_3, \ldots, x_d$ of the random vectors $X_3, \ldots, X_d$ which satisfy $E_3$ and apply Proposition 3.1 to the function $f_2(x_2)$ given by $x_2 \mapsto EX_1 f(X_1, x_2, \ldots, x_d)$. Again, $f_2$ is a convex function, and since

$$\|EX_1 f(X_1 \otimes x \otimes x_3 \otimes \cdots \otimes x_d) - EX_1 f(X_1 \otimes y \otimes x_3 \otimes \cdots \otimes x_d)\|
\leq EX_1 \|X_1 \otimes (x - y) \otimes x_3 \otimes \cdots \otimes x_d\|_2 \leq (\|EX_1 x_2\|_2)^{1/2} \|x - y\|_2 \prod_{i=3}^d \|x_i\|_2
\leq \sqrt{n} \|x - y\|_2 \cdot 2^n(d-1)/2 = \|x - y\|_2 \cdot 2^n(d-1)/2,$$

$f_2$ is $2^n(d-1)/2$-Lipschitz. Applying (3.2), we thus obtain

$$\|EX_1 f - EX_1 x_2 f\|_{\psi_\alpha(x_2)} \leq c_\alpha n^{(d-1)/2} \max_j \|X_{2,j}\|_{\psi_\alpha}$$

for any $x_3, \ldots, x_d$ in $E_3$. Iterating this procedure, we arrive at

$$\|EX_{1,\ldots,k-1} f - EX_{1,\ldots,k} f\|_{\psi_\alpha(X_k)} \leq c_\alpha n^{(d-1)/2} \max_j \|X_{k,j}\|_{\psi_\alpha}$$

for any realization $x_{k+1}, \ldots, x_d$ of $X_{k+1}, \ldots, X_d$ in $E_{k+1}$.

We now combine (5.8) for $k = 1, \ldots, d$. To this end, we write

$$\Delta_k := \Delta_k(X_k, \ldots, X_d) := EX_{1,\ldots,k-1} f - EX_{1,\ldots,k} f,$$
and apply Proposition 5.2. Here we have to distinguish between the cases where \( \alpha \in [1, 2] \) and \( \alpha \in (0, 1) \). If \( \alpha \geq 1 \), we use (5.3) to arrive at a bound for the moment-generating function. Writing \( M_k := \| \max_j |X_{k,j}| \|_{\psi_\alpha} \), we obtain

\[
\mathbb{E} \exp(\lambda \Delta_k) \leq \begin{cases} 
\exp((c_\alpha n^{(d-1)/2} M_k)2\lambda^2) \\
\exp((c_\alpha n^{(d-1)/2} M_k)^{\alpha/(\alpha-1)}|\lambda|^{\alpha/(\alpha-1)})
\end{cases}
\]

for all \( x_{k+1}, \ldots, x_d \) in \( \mathcal{E}_{k+1} \), where the first line holds if \( |\lambda| \leq 1/(c_\alpha n^{(d-1)/2} M_k) \) and the second one if \( |\lambda| \geq 1/(c_\alpha n^{(d-1)/2} M_k) \) and \( \alpha > 1 \). For the simplicity of presentation, temporarily assume that \( c_\alpha n^{(d-1)/2} = 1 \) (alternatively, replace \( M_k \) by \( c_\alpha n^{(d-1)/2} M_k \) in the following arguments) and that \( M_1 \leq \ldots \leq M_d \). Using Lemma 5.5, we obtain

\[
\mathbb{E} \exp(\lambda (f - \mathbb{E}f))_{\mathcal{E}} = \mathbb{E} \exp(\lambda (\Delta_1 + \cdots + \Delta_d))_{\mathcal{E}}
\]

\[
\leq \exp((M_d^1 + \cdots + M_d^2)\lambda^2 + (M_d^\alpha/(\alpha-1) + \cdots + M_d^\alpha/(\alpha-1))|\lambda|^{\alpha/(\alpha-1)})
\]

for \( |\lambda| \in [1/M_k+1, 1/M_k] \), where we formally set \( M_0 := 0 \) and \( M_{d+1} := \infty \). In particular, setting \( M := (M_1^2 + \cdots + M_2^2)^{1/2} \), we have

\[
\mathbb{E} \exp(\lambda (f - \mathbb{E}f))_{\mathcal{E}} \leq \exp(M^2\lambda^2)
\]

for all \( \lambda \leq 1/M_d = 1/(\max_k M_k) \). Furthermore, for \( \alpha > 1 \) it is not hard to see that

\[
(M_1^2 + \cdots + M_2^2)\lambda^2 + (M_2^\alpha/(\alpha-1) + \cdots + M_d^\alpha/(\alpha-1))|\lambda|^{\alpha/(\alpha-1)} \leq M^\alpha/(\alpha-1)|\lambda|^{\alpha/(\alpha-1)}
\]

if \( |\lambda| \in [1/M_k+1, 1/M_k] \) for some \( k = 0, 1, \ldots, d-1 \) or |\lambda| \in [1/M_1, 1/M_d] for \( k = d \). Indeed, by monotonicity (divide by \( \lambda^2 \) and compare the coefficients) it suffices to check this for \( \lambda = 1/M_{k+1} \) or \( \lambda = 1/M \) if \( k = d \). The cases of \( k = 0 \) and \( k = d \) follow by simple calculations. In the general case, set \( x^2 = (M_1^2 + \cdots + M_k^2)/M_k^2 + 1 \) and \( y^\alpha/(\alpha-1) = (M_{k+2}^\alpha/(\alpha-1) + \cdots + M_d^\alpha/(\alpha-1))/M_k^\alpha/(\alpha-1) \). Clearly, \( x^2 + y^\alpha/(\alpha-1) \leq (x^2 + y^2)^{1/2} \) since \( x \geq 1 \) and \( \alpha/(\alpha-1) \geq 2 \). Moreover, \( y^\alpha \leq (M_{k+2}^\alpha + \cdots + M_d^\alpha)/M_k^\alpha \), which proves the inequality. Altogether, inserting the factor \( c_\alpha n^{(d-1)/2} \) again, we therefore obtain

\[
\mathbb{E} \exp(\lambda (f - \mathbb{E}f))_{\mathcal{E}} = \mathbb{E} \exp(\lambda (\Delta_1 + \cdots + \Delta_d))_{\mathcal{E}}
\]

\[
\leq \begin{cases} 
\exp((c_\alpha n^{(d-1)/2} M_k^2)\lambda^2) \\
\exp((c_\alpha n^{(d-1)/2})^{\alpha/(\alpha-1)} M_k^\alpha/(\alpha-1)|\lambda|^{\alpha/(\alpha-1)})
\end{cases}
\]

(5.9)

where the first line holds if \( |\lambda| \leq 1/(c_\alpha n^{(d-1)/2} M) \) and the second one if \( |\lambda| \geq 1/(c_\alpha n^{(d-1)/2} M) \) and \( \alpha > 1 \).

On the other hand, if \( \alpha < 1 \), we use (5.3). Together with Lemma 5.5 and the subadditivity of \( |\cdot|^\alpha \) for \( \alpha \in (0, 1) \), this yields

\[
\mathbb{E} \exp(\lambda^\alpha |f - \mathbb{E}f|^\alpha)_{\mathcal{E}} \leq \mathbb{E} \exp(\lambda^\alpha (|\Delta_1|^\alpha + \cdots + |\Delta_d|^\alpha))_{\mathcal{E}}
\]

\[
\leq \exp((c_\alpha n^{(d-1)/2})^{\alpha/(\alpha-1)} (M_1^\alpha + \cdots + M_d^\alpha)|\lambda|^{\alpha})
\]

(5.10)

for \( \lambda \in [0, 1/(c_\alpha n^{(d-1)/2} \max_k M_k)] \).

To finish the proof, first consider \( \alpha \in [1, 2] \). Then, for any \( \lambda > 0 \), we have

\[
P(f - \mathbb{E}f > t) \leq P(\{f - \mathbb{E}f > t\} \cap \mathcal{E}) + P(\mathcal{E}^c)
\]

\[
\leq P(\exp(\lambda (f - \mathbb{E}f))_{\mathcal{E}} > \exp(\lambda M)) + P(\mathcal{E}^c)
\]

\[
\leq \exp\left( -\left(\frac{t}{(c_\alpha n^{(d-1)/2})^{\alpha}}M\right)^\alpha \right) + 2 \exp\left( -\frac{1}{C_\alpha} \left(\frac{n^{1/2}}{K^2 d^{1/2}}\right)^\alpha \right).
\]
where the last step follows by standard arguments (similarly as in the proof of Proposition 5.2 given in the appendix), using (5.9) and (5.5). Now, assume that $t \leq c_\alpha n^{d/2}M/(K^2d^{1/2})$. Then, the right-hand side of (5.11) is dominated by the first term (possibly after adjusting constants), so that we arrive at

$$\mathbb{P}(f - \mathbb{E}f > t) \leq 3 \exp \left( - \frac{1}{C_\alpha} \left( \frac{t}{\eta(n^{d-1}/2M)} \right)^\alpha \right).$$

The same arguments hold if $f$ is replaced by $-f$. Adjusting constants by (2.3), we obtain that for any $t \in [0, c_\alpha n^{d/2}M/(K^2d^{1/2})]$, (5.12)

$$\mathbb{P}(|f(X) - \mathbb{E}f(X)| > t) \leq 2 \exp \left( - \frac{1}{C_\alpha} \left( \frac{t}{\eta(n^{d-1}/2M)} \right)^\alpha \right).$$

Now it remains to note that by Lemma 5.6 we have

$$\|\max_j |X_{i,j}|\|_{\psi_\alpha} \leq C_\alpha (\log n)^{1/\alpha} \max_j \|X_{i,j}\|_{\psi_\alpha} \leq C_\alpha (\log n)^{1/\alpha} K.$$

If $\alpha \in (0, 1)$, similarly to (5.11), using (5.10), (5.5) and Proposition 5.2,

$$\mathbb{P}(f - \mathbb{E}f > t) \leq \mathbb{P}(\{|f - \mathbb{E}f| > t\} \cap \mathcal{E}) + \mathbb{P}(\mathcal{E}^c) \leq 2 \exp \left( - \left( \frac{t}{c_\alpha n^{(d-1)/2}M_\alpha} \right)^\alpha \right) + 2 \exp \left( - \frac{1}{C_\alpha} \left( \frac{n^{1/2}}{K^2d^{1/2}} \right)^\alpha \right),$$

where $M_\alpha := (M^\alpha_1 + \ldots + M^\alpha_d)^{1/\alpha}$. The rest follows as above. \qed

**Appendix A.**

Proof of Proposition 5.2. The equivalence of (1.1), (1.2), (1.3) and (5.3) is easily seen by directly adapting the arguments from the proof of [41], Proposition 2.5.2. To see that these properties imply (5.4), first note that since in particular $\|X\|_{\psi_1} < \infty$, the bound for $|\lambda| \leq 1/C^\prime_{5,\alpha}$ directly follows from [41], Proposition 2.7.1 (e). To see the bound for large values of $|\lambda|$, we infer that by the weighted arithmetic-geometric mean inequality (with weights $\alpha - 1$ and 1),

$$y^{(\alpha-1)/\alpha}z^{1/\alpha} \leq \frac{\alpha - 1}{\alpha} y + \frac{1}{\alpha} z$$

for any $y, z \geq 0$. Setting $y := |\lambda|^{\alpha/(\alpha-1)}$ and $z := |x|^\alpha$, we may conclude that

$$\lambda x \leq \frac{\alpha - 1}{\alpha} |\lambda|^{\alpha/(\alpha-1)} + \frac{1}{\alpha} |x|^\alpha$$

for any $\lambda, x \in \mathbb{R}$. Consequently, using (5.3) assuming $C^\prime_{4,\alpha} = 1$, for any $|\lambda| \geq 1$

$$\mathbb{E} \exp(\lambda X) \leq \exp \left( \frac{\alpha - 1}{\alpha} |\lambda|^{\alpha/(\alpha-1)} \right) \mathbb{E} \exp(|X|^\alpha) \leq \exp \left( \frac{\alpha - 1}{\alpha} |\lambda|^{\alpha/(\alpha-1)} \right) \exp(1/\alpha) \leq \exp(|\lambda|^{\alpha/(\alpha-1)}).$$

This yields (5.4) for $|\lambda| \geq 1/C^\prime_{5,\alpha}$. The claim now follows by taking $C_{5,\alpha} := \max(C^\prime_{5,\alpha}, C^\prime_{5,\alpha})$.

Finally, starting with (5.4) assuming $C_{5,\alpha} = 1$, let us check (1.1). To this end, note that for any $\lambda > 0$,

$$\mathbb{P}(X \geq t) \leq \exp(-\lambda t) \mathbb{E} \exp(\lambda X) \leq \exp(-\lambda t + \lambda^2 \mathbb{1}_{\{\lambda \leq 1\}} + \lambda^{\alpha/(\alpha-1)} \mathbb{1}_{\{\lambda > 1\}}).$$
Now choose \( \lambda := t/2 \) if \( t \leq 2 \), \( \lambda := ((\alpha - 1)t/\alpha)^{\alpha - 1} \) if \( t \geq \alpha/(\alpha - 1) \) and \( \lambda := 1 \) if \( t \in (2, \alpha/(\alpha - 1)) \). This yields

\[
\mathbb{P}(X \geq t) \leq \begin{cases} 
\exp(-t^2/4) & \text{if } t \leq 2, \\
\exp(-(t - 1)) & \text{if } t \in (2, \alpha/(\alpha - 1)), \\
\exp\left(-\frac{(\alpha - 1)t^{\alpha - 1}}{\alpha} - t^\alpha\right) & \text{if } t \geq \alpha/(\alpha - 1).
\end{cases}
\]

Now use (2.3), (2.4) and the fact that \( \exp(-(t - 1)) \leq \exp(-t^\alpha/C_\alpha) \) for any \( t \in (2, \alpha/(\alpha - 1)) \). It follows that

\[
\mathbb{P}(X \geq t) \leq 2 \exp(-t^\alpha/C_\alpha^\alpha)
\]

for any \( t \geq 0 \). The same argument for \(-X\) completes the proof. \( \square \)

Proof of Lemma 5.3. By the arithmetic and geometric means inequality and since \( \mathbb{E}\|X_i\|_2 \leq \sqrt{n} \), for any \( s \geq 0 \),

\[
\mathbb{P}\left( \prod_{i=1}^d \|X_i\|_2 > (\sqrt{n} + s)^d \right) \leq \mathbb{P}\left( \frac{1}{d} \sum_{i=1}^d (\|X_i\|_2 - \sqrt{n}) > s \right)
\]

\[
\leq \mathbb{P}\left( \frac{1}{d} \sum_{i=1}^d (\|X_i\|_2 - \mathbb{E}\|X_i\|_2) > s \right).
\]

Moreover, by (2.2) and [12, Corollary A.5],

\[
\|\|X_i\|_2 - \mathbb{E}\|X_i\|_2\|_\Psi_\alpha = \|\|X_i\|_2 - \sqrt{n} - (\mathbb{E}\|X_i\|_2 - \sqrt{n})\|_\Psi_\alpha \leq C_\alpha K^2
\]

for any \( i = 1, \ldots, d \). On the other hand, if \( Y_1, \ldots, Y_d \) are independent centered random variables with \( \mathbb{E}|Y_i|^{\alpha} \leq M \), we have

\[
\mathbb{P}\left( \frac{1}{d} \left| \sum_{i=1}^d Y_i \right| \geq s \right) \leq 2 \exp\left(- \frac{1}{C_\alpha} \min\left(\left(\frac{s\sqrt{d}}{M}\right)^{\alpha}, \left(\frac{s\sqrt{d}}{M}\right)^{\alpha}\right)\right)
\]

\[
\leq 2 \exp\left(- \frac{1}{C_\alpha} \left(\frac{s\sqrt{d}}{M}\right)^{\alpha}\right).
\]

Here, the first estimate follows from [10] \((\alpha > 1)\) and [17] \((\alpha \leq 1)\), while the last step follows by (2.4). As a consequence, (A.1) can be bounded by \( 2 \exp(-s^\alpha d^{3/2}/(K^{2\alpha}C_\alpha)) \).

For \( u \in [0, 2] \) and \( s = u\sqrt{n}/2d \), we have \((\sqrt{n} + s)^d \leq n^{d/2}(1 + u)\). Plugging in, we arrive at

\[
\mathbb{P}\left( \prod_{i=1}^d \|X_i\|_2 > n^{d/2}(1 + u) \right) \leq 2 \exp\left(- \frac{1}{C_\alpha} \left(\frac{n^{1/2}u}{K^2d^{1/2}}\right)^{\alpha}\right).
\]

Now set \( u := t/n^{d/2} \). \( \square \)

Proof of Lemma 5.4. Let us first recall the partition into “binary sets” which appears in the proof of [42, Lemma 3.2]. Here we assume that \( d = 2^L \) for some \( L \in \mathbb{N} \) (if not, increase \( d \)). Then, for any \( \ell \in \{0, 1, \ldots, L\} \), we consider the partition \( \mathcal{I}_\ell \) of \( \{1, \ldots, d\} \) into \( 2^\ell \) successive (integer) intervals of length \( d_\ell := d/2^\ell \) which we call “binary intervals”. It is not hard to see that for any \( k = 1, \ldots, d \), we can partition \([1, k]\) into binary intervals of different lengths such that this partition contains at most one interval of each family \( \mathcal{I}_\ell \).

Now it suffices to prove that

\[
\mathbb{P}\left( \exists \ell \leq L, \exists I \in \mathcal{I}_\ell: \prod_{i \in I} \|X_i\|_2 > (1 + 2^{\ell/4}u)n^{d_\ell/2} \right) \leq 2 \exp\left(- \frac{1}{C_\alpha} \left(\frac{n^{1/2}u}{K^2d^{1/2}}\right)^{\alpha}\right)
\]
(cf. Step 3 of the proof of [42, Lemma 3.2], where the reduction to this case is explained in detail). To this end, for any \( \ell \in \{0, 1, \ldots, L\} \), any \( I \in \mathcal{I}_\ell \) and \( d_\ell := |I| = d/2^\ell \), we apply Lemma \ref{lem:scaling} for \( d_\ell \) and \( t := 2^{-\ell/4}n^{d_\ell/2}u \). This yields

\[
\mathbb{P}\left( \prod_{i \in I} \|X_i\|_2 > (1 + 2^{-\ell/4}u)n^{d_\ell/2} \right) \leq 2 \exp\left( -\frac{1}{C_\alpha} \left( \frac{n^{1/2}u}{2^{\ell/4}K^2d_\ell^{1/2}} \right)^\alpha \right)
\]

\[
= 2 \exp\left( -\frac{1}{C_\alpha} \left( \frac{2^{\ell/4}n^{1/2}u}{K^2d_\ell^{1/2}} \right)^\alpha \right).
\]

Altogether, we arrive at

\[
\mathbb{P}\left( \exists \ell \in \{0, 1, \ldots, L\}, \exists I \in \mathcal{I}_\ell: \prod_{i \in I} \|X_i\|_2 > (1 + 2^{-\ell/4}u)n^{d_\ell/2} \right)
\]

\[
\leq \sum_{\ell=0}^L 2^\ell \cdot 2 \exp\left( -\frac{1}{C_\alpha} \left( \frac{2^{\ell/4}n^{1/2}u}{K^2d_\ell^{1/2}} \right)^\alpha \right).
\]

We may now assume that \( (n^{1/2}u/(K^2d_\ell^{1/2}))^\alpha/C_\alpha \geq 1 \) (otherwise the bound in Lemma \ref{lem:scaling} gets trivial by adjusting \( C_\alpha \)). Using the elementary inequality \( ab \geq (a+b)/2 \) for all \( a, b \geq 1 \), we arrive at

\[
2^{\ell_\alpha/4} \frac{1}{C_\alpha} \left( \frac{n^{1/2}u}{K^2d_\ell^{1/2}} \right)^\alpha \geq \frac{1}{2} \left( 2^{\ell_\alpha/4} + \frac{1}{C_\alpha} \left( \frac{n^{1/2}u}{K^2d_\ell^{1/2}} \right)^\alpha \right).
\]

Using this in (A.2), we obtain the upper bound

\[
2 \exp\left( -\frac{1}{2C_\alpha} \left( \frac{n^{1/2}u}{K^2d_\ell^{1/2}} \right)^\alpha \right) \sum_{\ell=0}^L 2^\ell \exp(-2^{\ell_\alpha/4-1}) \leq c_\alpha \exp\left( -\frac{1}{2C_\alpha} \left( \frac{n^{1/2}u}{K^2d_\ell^{1/2}} \right)^\alpha \right).
\]

By (2.3), we can assume \( c_\alpha = 2 \). \( \square \)

To prove Lemma \ref{lem:tail_bound}, we first present a number of lemmas and auxiliary statements. In particular, recall that if \( \alpha \in (0, \infty) \), then for any \( x, y \in (0, \infty) \),

\[
(\text{A.3}) \quad c_\alpha (x^\alpha + y^\alpha) \leq (x+y)^\alpha \leq \bar{c}_\alpha (x^\alpha + y^\alpha),
\]

where \( c_\alpha := 2^{\alpha-1} \land 1 \) and \( \bar{c}_\alpha := 2^{\alpha-1} \lor 1 \). Indeed, if \( \alpha \leq 1 \), using the concavity of the function \( x \mapsto x^\alpha \) it follows by standard arguments that \( 2^{\alpha-1}(x^\alpha + y^\alpha) \leq (x+y)^\alpha \leq x^\alpha + y^\alpha \). Likewise, for \( \alpha \geq 1 \), using the convexity of \( x \mapsto x^\alpha \) we obtain \( x^\alpha + y^\alpha \leq (x+y)^\alpha \leq 2^{\alpha-1}(x^\alpha + y^\alpha) \).

**Lemma A.1.** Let \( X_1, \ldots, X_n \) be independent, centered random variables such that \( \|X_i\|_{\psi_\alpha} \leq 1 \) for some \( \alpha > 0 \). Then, if \( Y := \max_i |X_i| \) and \( c := (c_\alpha^{-1} \log n)^{1/\alpha} \), we have

\[
\mathbb{P}(Y \geq c + t) \leq 2 \exp(-c_\alpha t^\alpha)
\]

with \( c_\alpha \) as in (A.3).

**Proof.** We have

\[
\mathbb{P}(Y \geq c + t) \leq n\mathbb{P}(|X_i| \geq c + t) \leq 2n \exp(-t^\alpha)
\]

\[
\leq 2n \exp(-c_\alpha t^\alpha) = 2 \exp(-c_\alpha t^\alpha),
\]

where we have used (A.3) in the next-to-last step. \( \square \)
Lemma A.2. Let \( Y \geq 0 \) be a random variable which satisfies
\[
\mathbb{P}(Y \geq c + t) \leq 2 \exp(-t^\alpha)
\]
for some \( c \geq 0 \) and any \( t \geq 0 \). Then,
\[
\|Y\|_{\psi_\alpha} \leq \tilde{c}_\alpha^{1/\alpha} \max\left\{ \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)^{1/\alpha}, c \left( \frac{2}{\log 2} \right)^{1/\alpha} \right\}
\]
with \( \tilde{c}_\alpha \) as in (A.3).

Proof. By (A.3) and monotonicity, we have \( Y^\alpha \leq \tilde{c}_\alpha((Y - c)^\alpha_+ + c^\alpha) \), where \( x_+ := \max(x, 0) \). Thus,
\[
\mathbb{E}\exp\left( \frac{Y^\alpha}{s^\alpha} \right) \leq \exp\left( \frac{\tilde{c}_\alpha c^\alpha}{s^\alpha} \right) \mathbb{E}\exp\left( \frac{\tilde{c}_\alpha(Y - c)^\alpha_+}{s^\alpha} \right)
\]
\[
= \exp\left( \frac{c^\alpha}{t^\alpha} \right) \mathbb{E}\exp\left( \frac{(Y - c)^\alpha_+}{t^\alpha} \right) =: I_1 \cdot I_2,
\]
where we have set \( t := s\tilde{c}_\alpha^{-1/\alpha} \). Obviously, \( I_1 \leq \sqrt{2} \) if \( t \geq c(1/\log \sqrt{2})^{1/\alpha} \). As for \( I_2 \), we have
\[
I_2 = 1 + \int_1^\infty \mathbb{P}((Y - c)_+ \geq t \log y)^{1/\alpha} dy
\]
\[
\leq 1 + 2 \int_1^\infty \exp(-t^\alpha \log y) dy = 1 + 2 \int_1^\infty \frac{1}{y^{t^\alpha}} dy \leq \sqrt{2}
\]
if \( t \geq ((\sqrt{2} + 1)/(\sqrt{2} - 1))^{1/\alpha} \). Therefore, \( I_1 I_2 \leq 2 \) if \( t \geq \max\{((\sqrt{2} + 1)/(\sqrt{2} - 1))^{1/\alpha}, c(2/\log 2)^{1/\alpha}\} \), which finishes the proof. \( \square \)

Having these lemmas at hand, the proof of Lemma 5.6 is easily completed.

Proof of Lemma 5.6. The random variables \( \hat{X}_i := X_i/K \) obviously satisfy the assumptions of Lemma A.1. Hence, setting \( Y := \max_i |\hat{X}_i| = K^{-1} \max_i |X_i| \),
\[
\mathbb{P}(c_\alpha^{1/\alpha}Y \geq (\log n)^{1/\alpha} + t) \leq 2 \exp(-t^\alpha).
\]
Therefore, we may apply Lemma A.2 to \( \hat{Y} := c_\alpha^{1/\alpha}K^{-1} \max_i |X_i| \). This yields
\[
\|\hat{Y}\|_{\psi_\alpha} \leq \tilde{c}_\alpha^{1/\alpha} \max\left\{ \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)^{1/\alpha}, (\log n)^{1/\alpha} \left( \frac{2}{\log 2} \right)^{1/\alpha} \right\},
\]
i.e. the claim of Lemma 5.6 where we have set \( C := (\tilde{c}_\alpha c_\alpha^{-1})^{1/\alpha} \). \( \square \)

References

[1] R. Adamczak. A tail inequality for suprema of unbounded empirical processes with applications to Markov chains. \textit{Electron. J. Probab.}, 13:no. 34, 1000–1034, 2008.
[2] R. Adamczak. A note on the Hanson-Wright inequality for random vectors with dependencies. \textit{Electron. Commun. Probab.}, 20:no. 72, 13, 2015.
[3] R. Adamczak and R. Latała. Tail and moment estimates for chaos generated by symmetric random variables with logarithmically concave tails. \textit{Ann. Inst. Henri Poincaré Probab. Stat.}, 48(4):1103–1136, 2012.
[4] R. Adamczak, R. Latała, and R. Meller. Hanson-Wright inequality in Banach spaces. \textit{Ann. Inst. Henri Poincaré Probab. Stat.}, 56(4):2356–2376, 2020.
[5] R. Adamczak and M. Strzelecki. On the convex Poincaré inequality and weak transportation inequalities. \textit{Bernoulli}, 25(1):341–374, 2019.
[6] R. Adamczak and P. Wolff. Concentration inequalities for non-Lipschitz functions with bounded derivatives of higher order. Probab. Theory Related Fields, 162(3-4):531–586, 2015.
[7] S. Boucheron, G. Lugosi, and P. Massart. Concentration inequalities. Oxford University Press, Oxford, 2013. A nonasymptotic theory of independence, With a foreword by Michel Ledoux.
[8] V. H. de la Peña and E. Giné. Decoupling. Probability and its Applications (New York). Springer-Verlag, New York, 1999.
[9] L. H. Dicker and M. A. Erdogdu. Flexible results for quadratic forms with applications to variance components estimation. Ann. Statist., 45(1):386–414, 2017.
[10] E. D. Gluskin and S. Kwapień. Tail and moment estimates for sums of independent random variables with logarithmically concave tails. Studia Math., 114(3):303–309, 1995.
[11] F. Götze, H. Sambale, and A. Sinulis. Concentration inequalities for bounded functionals via log-Sobolev-type inequalities. J. Theoret. Probab., 34(3):1623–1652, 2021.
[12] F. Götze, H. Sambale, and A. Sinulis. Concentration inequalities for polynomials in α-sub-exponential random variables. Electron. J. Probab., 26: no. 48, 22, 2021.
[13] N. Gozlan, C. Roberto, Cyril, and P.-M. Samson. From dimension free concentration to the Poincaré inequality. Calc. Var. Partial Differential Equations, 52(3-4):899–925, 2015.
[14] N. Gozlan, Nathael, C. Roberto, P.-M. Samson, Paul-Marie, Y. Shu, and P. Tetal. Characterization of a class of weak transport-entropy inequalities on the line. Ann. Inst. Henri Poincaré Probab. Stat., 54(3):1667–1693, 2018.
[15] N. Gozlan, C. Roberto, P.-M. Samson, and P. Tetal. Kantorovich duality for general transport costs and applications. J. Funct. Anal., 273(11):3327–3405, 2017.
[16] D. L. Hanson and F. T. Wright. A bound on tail probabilities for quadratic forms in independent random variables. Ann. Math. Statist., 42:1079–1083, 1971.
[17] P. Hitczenko, S. J. Montgomery-Smith, and K. Oleszkiewicz. Moment inequalities for sums of certain independent symmetric random variables. Studia Math., 123(1):15–42, 1997.
[18] D. Hsu, S. M. Kakade, and T. Zhang. A tail inequality for quadratic forms of subgaussian random vectors. Electron. Comm. Probab., 17:no. 52, 6, 2012.
[19] W. B. Johnson and G. Schechtman. Remarks on Talagrand’s deviation inequality for Rademacher functions. In Functional analysis (Austin, TX, 1987/1989), volume 1470 of Lecture Notes in Math., pages 72–77. Springer, Berlin, 1991.
[20] Y. Klochkov and N. Zhivotovskiy. Uniform Hanson–Wright type concentration inequalities for unbounded entries via the entropy method. Electron. J. Probab., 25: no. 22, 30, 2020.
[21] K. Kolesko and R. Latała. Moment estimates for chaoses generated by symmetric random variables with logarithmically convex tails. Statist. Probab. Lett., 107: 210–214, 2015.
[22] F. Krahmer, S. Mendelson, and H. Rauhut. Suprema of chaos processes and the restricted isometry property. Comm. Pure Appl. Math., 67(11): 1877–1904, 2014.
[23] A. K. Kuchibhotla and A. Chakrabortty. Moving Beyond Sub-Gaussianity in High-Dimensional Statistics: Applications in Covariance Estimation and Linear Regression. arXiv preprint, 2018.
[24] R. Latała. Tail and moment estimates for sums of independent random vectors with logarithmically concave tails. Studia Math., 118(3): 301–304, 1996.
[25] R. Latała. Tail and moment estimates for some types of chaos. Studia Math., 135(1): 39–53, 1999.
[26] R. Latała. Estimates of moments and tails of Gaussian chaoses. *Ann. Probab.*, 24(6):2315–2331, 2006.

[27] R. Latała and R. Łochowski. Moment and tail estimates for multidimensional chaos generated by positive random variables with logarithmically concave tails. In *Stochastic inequalities and applications*, volume 56 of *Progr. Probab.*, pages 77-92. Birkhäuser, Basel, 2003.

[28] J. Lederer and S. van de Geer. New concentration inequalities for suprema of empirical processes. *Bernoulli*, 20(4):2020–2038, 2014.

[29] M. Ledoux. On Talagrand’s deviation inequalities for product measures. *ESAIM Probab. Statist.*, 1:63–87, 1995/97.

[30] M. Ledoux and M. Talagrand. *Probability in Banach spaces*, volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, Springer-Verlag, Berlin, 1991. Isoperimetry and processes.

[31] A. Marchina. Concentration inequalities for separately convex functions. *Bernoulli*, 24(4A):2906–2933, 2018.

[32] P. Massart. Some applications of concentration inequalities to statistics. *Ann. Fac. Sci. Toulouse Math. (6)*, 9(2):245–303, 2000.

[33] M. Rudelson and R. Vershynin. Hanson-Wright inequality and sub-Gaussian concentration. *Electron. Commun. Probab.*, 18:no. 82, 9, 2013.

[34] H. Sambale and A. Sinulis. Logarithmic Sobolev inequalities for finite spin systems and applications. *Bernoulli*, 26(3):1863–1890, 2020.

[35] H. Sambale and A. Sinulis. Modified log-Sobolev inequalities and two-level concentration. *ALEA Lat. Am. J. Probab. Math. Stat.*, 18:855–885, 2021.

[36] P.-M. Samson. Concentration of measure inequalities for Markov chains and Φ-mixing processes. *Ann. Probab.*, 28(1):416–461, 2000.

[37] P.-M. Samson. Concentration inequalities for convex functions on product spaces. In *Stochastic inequalities and applications*, volume 56 of *Progr. Probab.*, pages 33–52. Birkhäuser, Basel, 2003.

[38] M. Talagrand. An isoperimetric theorem on the cube and the Kintchine-Kahane inequalities. *Proc. Amer. Math. Soc.*, 104(3):905–909, 1988.

[39] M. Talagrand. New concentration inequalities in product spaces. *Invent. Math.*, 126(3):505–563, 1996.

[40] S. van de Geer and J. Lederer. The Bernstein-Orlicz norm and deviation inequalities. *Probab. Theory Related Fields*, 157(1-2):225–250, 2013.

[41] R. Vershynin. *High-dimensional probability*, volume 47 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2018.

[42] R. Vershynin. Concentration inequalities for random tensors. *Bernoulli*, 26(4):3139–3162, 2020.

[43] V. H. Vu and K. Wang. Random weighted projections, random quadratic forms and random eigenvectors. *Random Structures Algorithms*, 47(4):792–821, 2015.