SIMULTANEOUS EXTENSIONS OF METRICS AND ULTRAMETRICS OF HIGH POWER

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Abstract. In this paper, generalized metrics mean metrics taking values in general linearly ordered Abelian groups. Using the Hahn fields, we first prove that for every generalized metric space, if the set of the Archimedean equivalence classes of the range group of the metric has an infinite decreasing sequence, then every non-empty closed subset of the space is a uniform retract of the ambient space. Next we construct simultaneous extensions of generalized metrics and ultrametrics. From the existence of extensors of generalized metrics, we characterize the final compactness of generalized metrizable spaces using the completeness of generalized metrics.

Contents

1. Introduction 1
2. Preliminaries 11
3. Retractions 29
4. Extensors of ultrametrics and metrics of high power 33
5. Table of symbols 42
References 43

1. INTRODUCTION

Sikorski [44] introduced the notion of $\omega_\mu$-metric spaces as spaces equipped with metrics taking values in a linearly ordered Abelian group, which is an example of topological spaces of high power ($\omega_\mu$-additive spaces). There are many developments of research on $\omega_\mu$-metric ($\omega_\mu$-metrizable) spaces concerning different or common properties between ordinary metrics and generalized metrics. For instance, Nyikos–Reichel [39] and Wang [51] provided some $\omega_\mu$-metrization theorems. Stevenson–Thorn [48] proved that a topological space is $\omega_\mu$-metrizable for some $\mu$ if and only if the space have a linearly ordered uniformity. Juhász [29], Kucia–Kulpa [34], Hayes [24], and Souppouris [45] independently

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proved that all $\omega_\mu$-metrizable spaces are paracompact. Di Concilio–Guadagni [11] investigated $\omega_\mu$-metrizable spaces on which all continuous maps are uniformly continuous. Di Concilio–Guadagni [12] and Artico–Marconi–Pelant [2] discussed hyperspaces of $\omega_\mu$-metric spaces. Hung [25] discussed the amalgamation property of generalized metric, which property is related to the construction of the Urysohn universal space. Comicheo [8] proved a generalized open mapping theorem for linear spaces equipped with norms taking values in general ordered sets.

In the present paper, using the Hahn fields, we first prove that for every generalized metric space, if the set of the Archimedean equivalence classes of the range group of the metric has an infinite decreasing sequence, then every non-empty closed subset of the space is a uniform retract of the ambient space (see Theorem 1.1). This is a generalization of van Douwen’s theorem [50, Theorem 7 in Chapter 3] and Brodskiy–Dydak–Higes–Mitra’s theorem [5, Theorem 2.9]. Next we construct simultaneous extensions of ultrametrics and generalized metrics (see Theorems 1.2 and 1.3). This is an analogue of Nguyen Van Khue and Nguyen To Nhu’s theorem [31] on simultaneous extensions of metrics, and a generalization of the author’s extension theorem [26] of ultrametrics. From the existence of extensions of generalized metrics, we characterize the final compactness of generalized metrizable spaces using the completeness of generalized metrics (see Theorem 1.4). This is an analogue of Niemytzki–Tychonoff’s theorem [37].

Before stating precisely our main results, we introduce some notions and notations.

1.1. Basic definitions. Let $L$ be a set. A binary relation $R$ on $L$ is said to be a linear order if it is reflexive, transitive, antisymmetric and it satisfies $a R b$ or $b R a$ for all $a, b \in L$ (see, for example, [52, Chapter 1]). In this case, the relation $R$ is often symbolically represented as $\leq_L$, and we write $a <_L b$ if $a \leq_L b$ and $a \neq b$. We denote by $x \geq_L y$ if $y \leq_L x$. The pair $(L, \leq_L)$ is called a linearly ordered set. By abuse of notation, we simply denote by $\leq_L$ and $\geq_L$ the orders “$\leq_L$” and “$\leq_L$”, respectively, when no confusion can arise.

In this paper, for an Abelian group $G$, we always denote by $+$ the group operation on $G$, and we denote by $0_G$ its zero element. A pair $(G, \leq_G)$ of an Abelian group $G$ and a linear order $\leq_G$ on $G$ is said to be a linearly ordered Abelian group if for all $a, b, c \in G$, the inequality $a \leq_G b$ implies that $a + c \leq_G b + c$. By abuse of notation, we simply denote by $G$ the linearly ordered Abelian group $(G, \leq_G)$. For example, the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$ are linearly ordered Abelian groups. For simplicity of notation, we often write 0 instead of $0_G$. For
a linearly ordered Abelian group $G$, and for $a \in G$, we denote by $G_{\geq a}$ (resp. $G_{> a}$) the set of all $x \in G$ satisfying $a \leq x$ (resp. $a < x$).

Let $G$ be a linearly ordered Abelian group. For $x, y \in G_{> 0}$, we denote by $x \ll y$ if for all $n \in \mathbb{Z}_{\geq 1}$ we have $n \cdot x < y$. We define a relation $\sim$ on $G_{> 0}$ by $x \sim y$ if and only if there exist integers $n, m \in \mathbb{Z}_{\geq 1}$ such that $y \leq n \cdot x$ and $x \leq m \cdot y$. Then $\sim$ is an equivalence relation on $G_{> 0}$, and we denote by $[x]_{\sim}$ the equivalence class of $x$ by $\sim$. If $x \ll y$, we say that $x$ and $y$ are Archimedean equivalent to each other. The relation $\sim$ is called the Archimedean equivalence on $G$. We denote by $A(G)$ the quotient set of $G_{> 0}$ by $\sim$. If $\text{Card}(A(G)) = 1$, the group $G$ is said to be Archimedean; otherwise, it is said to be non-Archimedean, where “Card” stands for the cardinality. For example, the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$ are Archimedean. For $\alpha, \beta \in A(G)$ with $\alpha = [x]_{\sim}$ and $\beta = [y]_{\sim}$, we write $\alpha \leq \beta$ if $x \ll y$ or $x \sim y$ (for the well-definedness of the order $\leq$, see Lemma 2.6). Then $(A(G), \leq)$ becomes a linearly ordered set. Since $A(G)$ plays an important role in this paper, we use the special symbol $\leq$ as the order on $A(G)$ rather than $\leq$. By abuse of notation, we use the same symbol the symbol $\ll$ on $G_{> 0}$ as the strict order $\ll$ on $A(G)$ meaning $\alpha \leq \beta$ and $\alpha \neq \beta$. This is equivalent to $x \ll y$, where $\alpha = [x]_{\sim}$ and $\beta = [y]_{\sim}$.

**Definition 1.1.** Let $X$ be a set, and $G$ be a linearly ordered Abelian group. We say that a map $d : X \times X \to G$ is a $G$-metric if the following conditions are satisfied:

(M1) for all $x, y \in X$, the equality $d(x, y) = 0$ implies $x = y$;
(M2) for all $x \in X$, we have $d(x, x) = 0$;
(M3) for all $x, y \in X$, we have $0 \leq d(x, y)$;
(M4) for all $x, y \in X$, we have $d(x, y) = d(y, x)$;
(M5) for all $x, y, z \in X$, we have $d(x, y) \leq d(x, z) + d(z, y)$.

The condition (M5) is called the triangle inequality. For a $G$-metric $d$ on $X$, the topology on $X$ induced from $d$ is defined as the topology generated from open balls of $d$.

For a topological space $X$, we denote by $\text{Met}(X; G)$ the set of all $G$-metrics that generate the same topology of $X$. A topological space $X$ is said to be $G$-metrizable if $\text{Met}(X; G) \neq \emptyset$. A topological space is metrizable in the ordinary sense if and only if it is $\mathbb{R}$-metrizable.

**Definition 1.2.** We say that a linearly ordered set $S$ is bottomed if it has the least element, and we denote by $\bigcirc_S$ the least element. We often simply denote by $\bigcirc$ the least element $\bigcirc_S$ of $S$ when no confusion can arise. Let $S$ be a bottomed linearly ordered set. Let $X$ be a set. We say that a map $d : X \times X \to S$ is an $S$-ultrametric if the following conditions are satisfied:

(U1) for all $x, y \in X$, the equality $d(x, y) = \bigcirc$ implies $x = y$;
(U2) for all $x \in X$, we have $d(x, x) = \bigcirc$;
(U3) for all \( x, y \in X \), we have \( \bullet \leq d(x, y) \);  
(U4) for all \( x, y \in X \), we have \( d(x, y) = d(y, x) \);  
(U5) for all \( x, y, z \in X \), we have \( d(x, y) \leq d(x, z) \lor d(z, y) \), where \( \lor \) stands for the maximal operator on \( S \), i.e., \( x \lor y = \max\{x, y\} \).

The condition (U5) is called the strong triangle inequality. The topology on \( X \) induced from \( d \) is defined in a similar way to \( G \)-metrics.

For a topological space \( X \), we denote by \( \text{UMet}(X; S) \) the set of all \( S \)-ultrametrics that generate the same topology of \( X \).

We say that a topological space \( X \) is \( S \)-ultrametrizable if we have \( \text{UMet}(X; S) \neq \emptyset \). A topological space is ultrametrizable in the ordinary sense if and only if it is \( \mathbb{R}_{\geq 0} \)-ultrametrizable.

For an ordinal \( \mu \), the symbol \( \omega_\mu \) stands for the \( \mu \)-th cardinal, namely, \( \omega_\mu \) is the least ordinal whose cardinal is \( \aleph_\mu \). Note that \( \omega_0 \) is the least infinite cardinal, namely, \( \omega_0 = \{0, 1, 2, \ldots\} \).

**Definition 1.3.** For a bottomed linearly ordered set \( S \), we define the character \( \chi(S) \) of \( S \) as the minimal cardinal of all \( \kappa > 0 \) such that there exists a map \( f: \kappa + 1 \to T \) satisfying that

1. if \( \alpha, \beta < \kappa \) satisfy \( \alpha < \beta \), then \( f(\beta) < f(\alpha) \);
2. \( f(\kappa) = \bullet_S \);
3. for every \( t \in S \), there exists \( \alpha < \kappa \) with \( f(\alpha) \leq t \).

Note that \( \chi(S) = 1 \), or \( \chi(S) \) is a regular cardinal. Remark that the character \( \chi(S) \) is equal to the cofinality of \( S \setminus \{\bullet_S\} \). For example, we have \( \chi(\mathbb{R}_{\geq 0}) = \omega_0 \).

A topological space is said to be \( \omega_\mu \)-metrizable (in the sense of Sikorski) if there exists a linearly ordered group \( G \) such that \( \chi(G_{\geq 0}) = \omega_\mu \) and \( \text{Met}(X; G) \neq \emptyset \). The notion of \( \omega_\mu \)-metrizable space was introduced by Sikorski [44] as a generalization of ordinary metric spaces.

As remarked in [39] and [3], if a topological space \( X \) is \( \omega_\mu \)-metrizable, then \( X \) is \( S \)-ultrametrizable for some linearly ordered set \( S \) with \( \chi(S) = \omega_\mu \). This statement can be proven by considering the Archimedean equivalence class of \( G \) with \( \text{Met}(X; G) \neq \emptyset \) (see Lemma 2.20 and Proposition 2.21). Focusing on this fact, and based on the Archimedean equivalence on linearly ordered Abelian groups, we introduce the notion of metrizable gauges in Definition 1.5. Before metrizable gauges, we define a one-point extension of a linearly ordered set.

**Definition 1.4.** Let \( L \) be a linearly ordered set. We denote by \( \Xi(L) \) the one-point extension ordered set of \( L \) by adding the element \( \bullet_{\Xi(L)} \notin L \) to \( L \), and by defining \( \bullet_{\Xi(L)} \leq x \) for all \( x \in L \). The set \( \Xi(L) \) contains \( L \) as a ordered subset. Note that \( \Xi(L) = \{\bullet_{\Xi(L)}\} \cup L \) and \( \bullet_{\Xi(L)} \) is the minimal element of \( \Xi(L) \).

**Definition 1.5.** Let \( X \) be a topological space. We say that a cardinal \( \kappa \) is a metrizable gauge of \( X \) if there exists a linearly ordered abelian
group $G$ such that $\chi(\Xi(A(G))) = \kappa$ and $\text{Met}(X; G) \neq \emptyset$. In this case, the cardinal $\kappa$ should be 1 or a regular cardinal. We denote by $\mathcal{M}_0(X)$ the set (class) of all metrizable gauges of $X$. We say that a topological space $X$ possesses an infinite metrizable gauge if there exists $\kappa \in \mathcal{M}_0(X)$ such that $\omega_0 \leq \kappa$.

**Definition 1.6.** For a cardinal $\kappa$, we denote by $\text{GRP}(\kappa)$ the class of all linearly ordered Abelian groups $G$ such that $\chi(\Xi(A(G))) = \kappa$. Note that for an uncountable regular cardinal $\kappa$, we have $G \in \text{GRP}(\kappa)$ if and only if $\chi(G_{\geq 0}) = \kappa$ (see Proposition 2.15). We also denote by $\text{ORD}(\kappa)$ the class of all bottomed linearly ordered sets $S$ such that $\chi(S) = \kappa$. For example, we observe that $\mathbb{R} \in \text{GRP}(1)$ and $\mathbb{R}_{\geq 0} \in \text{ORD}(\omega_0)$. Note that a linearly ordered Abelian group $G$ is Archimedean if and only if $G \in \text{GRP}(1)$.

We should note the difference between the $\omega_\mu$-metrizability in the sense of Sikorski and the metrizable gauges of the present paper. For an uncountable regular cardinal $\kappa$, there exists $\mu > 0$ such that $\kappa = \omega_\mu$, and a topological space $X$ is $\omega_\mu$-metrizable if and only if $\kappa \in \mathcal{M}_0(X)$. The difference arises only in the countable case. Since all ordinary metric and ultrametric spaces are $\omega_0$-metrizable, we can not distinguish ultrametrizable spaces and metrizable and non-ultrametrizable space using the $\omega_0$-metrizability. In main results of this paper, we shall treat common properties on ultrametric spaces and $\omega_\mu$-metrizable spaces with $\mu > 0$. Using metrizable gauges, we unify those cases as the case where a topological space possesses an infinite metrizable gauge, and we can separate ultrametrizable spaces and metrizable and non-ultrametrizable spaces (see Proposition 2.32). Due to this observation, in this paper, we mainly use the metrizable gauges rather than the $\omega_\mu$-metrizability.

One of the key points of the present paper is to utilize concepts on ordered fields such as the Hahn fields, Hahn’s embedding theorem, and the Archimedean equivalence, in the theory of metrics taking values in general linearly ordered Abelian groups.

### 1.2. Main results.

Let $X$ be a topological space. A subset $A$ of $X$ is said to be a retract if there exists a continuous map $r: X \to A$ such that $r(a) = a$ for all $a \in A$. In this case, the continuous map $r$ is said to be a retraction. A topological space is said to be retractifiable if every its non-empty closed subset is a retract of the ambient space. This concept was first introduced by van Douwen [50]. There are some results on retracts of zero-dimensional spaces.

The proofs of Theorem 9.1 in Dugundji’s paper [18] and Theorem 4 in Kodama’s paper [33] contain the statement that every non-empty closed subset of an ultrametrizable space is a retract of the ambient space (see also [9]).
Engelking [19] proved that if a closed subset $F$ of a metrizable space $X$ satisfies that $\dim T(X \setminus F) = 0$, then there exists a retraction $r : X \to F$ which is a closed map, where $\dim T$ stands for the covering dimension.

In 1975, van Douwen [50] established the following results:

1. All ultrametrizable spaces and $\omega_\mu$-metrizable spaces for $\mu > 0$ are retractifiable.
2. Every non-empty closed $G_\delta$-subset of a strongly zero-dimensional collectionwise normal space is a retract of the ambient space.
3. For every ordinal $\alpha$, the space $\alpha$ with the ordered topology is (hereditarily) retractifiable.
4. The Sorgenfrey line is (hereditarily) retractifiable.
5. Every locally compact totally disconnected orderable space is retractifiable.

Kąkol–Kubzdela–Śliwa [32] showed that every compact metrizable subspace $Y$ of an ultraregular space $X$ is a retract of $X$.

For a metric space $(X, d)$, we denote by $\exp(X)$ the space of all non-empty compact subsets of $X$ equipped with the Hausdorff distance. By Michael's zero-dimensional selection theorem [36, Theorem 2], for a complete ultrametric space $(X, d)$, Tymchatyn–Zarichnyi [49] constructed a map $R : X \times \exp(X) \to X$ such that for all $(x, A) \in X \times \exp(X)$, we have $R(x, A) \in A$, and such that if $x \in A$, then $R(x, A) = x$. Stasyuk–Tymchatyn [47] proved the existence of uniformly continuous $R : X \times \exp(X) \to X$ satisfying the conditions mentioned above.

For every (ordinary) ultrametric space $(X, d)$, for every closed subset $A$ of $X$, for all $\delta \in (1, \infty)$, Brodskiy–Dydak–Higes–Mitra [5] constructed a $\delta$-Lipschitz retraction from $X$ to $A$ with respect to $d$.

The author [26] used the Brodskiy–Dydak–Higes–Mitra theorem to prove the extension theorem of ultrametrics. Since the multiplication of the real numbers was utilized in the proof in [5], the author [26] said that an analogue of the retractifiable theorem for ultrametrics valued in general bottomed linearly ordered sets seemed to be not true; however, that is false. Indeed, in [50, Theorem 7 in Chapter 3], it was proven that all $\omega_\mu$-metrizable spaces are retractifiable for all $\mu > 0$. Note that the proof of [50, Theorem 7 in Chapter 3] seems not to imply the existence of uniformly continuous retractions. In the present paper, we show the existence of uniformly continuous retractions. Using the construction of the Hahn fields (see Section 2), every linearly ordered set can be regarded as a subset of positive numbers of a linearly ordered field (see Proposition 2.12), and we can apply the argument of Brodskiy–Dydak–Higes–Mitra [5] to generalized (ultra)metrics. The following is our first result:

**Theorem 1.1.** We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in \mathcal{M}(X)$. 
Then, for every $G \in \text{GRP}(\kappa)$, for every $d \in \text{Met}(X; G)$, and for every closed non-empty subset $A$ of $X$, there exists a uniformly continuous retraction $r : X \to A$ with respect to $d$. In particular, the space $X$ is retractifiable.

Remark 1.1. In contrast to Brodskiy–Dydak–Higes–Mitra’s theorem [5, Theorem 2.9] on the existence of $\delta$-Lipschitz retractions for all $\delta \in (1, \infty)$, it is worth noting when there exits a 1-Lipschitz retraction $r : X \to A$. Artico–Moresco [3] characterize a 1-Lipschitz retract of generalized ultrametric spaces using the proximality (see Theorem 3.3 in this paper, or [3, Theorem 4.6 and Proposition 2.6]). By their theorem, we characterize closed subsets of generalized ultrametric spaces (see Corollary 3.4) using the proximality and 1-Lipschitz maps.

We next explain a result on extensors of ultrametrics and generalized metrics. Hausdorff [22] proved that for every $\mathbb{R}$-metrizable space $X$, for every closed subset $A$ of $X$, and for every $d \in \text{Met}(A; \mathbb{R})$, there exists $D \in \text{Met}(X; \mathbb{R})$ satisfying that $D|_A = d$. The author [26] proved an ultrametric version of the Hausdorff metric extension theorem, i.e., for every $S \subseteq \mathbb{R}_{\geq 0}$ with $0 \in S$ possessing a decreasing sequence convergent to 0, and for every $\mathbb{R}_{\geq 0}$-ultrametrizable space $X$, and for every closed subset $A$ of $X$, and for every $d \in \text{UMet}(A; S)$, there exists $D \in \text{UMet}(X; S)$ satisfying that $D|_A = d$.

Let $D_X$ stand for the supremum metric on $\text{Met}(X; \mathbb{R})$, which can take the value $\infty$. For every metrizable space $X$, and for every closed subset $A$ of $X$, Nguyen Van Khue and Nguyen To Nhu [31] constructed maps $\Phi_1, \Phi_2 : \text{Met}(A; \mathbb{R}) \to \text{Met}(X; \mathbb{R})$ such that

1. the maps $\Phi_1$ and $\Phi_2$ are extensors; namely, for every $d \in \text{Met}(A)$, we have $\Phi_1(d)|_A = d$ and $\Phi_2(d)|_A = d$;
2. the map $\Phi_1$ is 20-Lipschitz with respect to the metrics $D_A$ and $D_X$;
3. the map $\Phi_2$ is continuous with respect to the topologies of pointwise convergence on $\text{Met}(A; \mathbb{R})$ and $\text{Met}(X; \mathbb{R})$;
4. the map $\Phi_2$ preserves orders; namely, if $d, e \in \text{Met}(A)$ satisfy $d(a, b) \leq e(a, b)$ for all $a, b \in A$, then $\Phi_2(d)(x, y) \leq \Phi_2(e)(x, y)$ for all $x, y \in X$;
5. if $X$ is completely metrizable, then $\Phi_1$ and $\Phi_2$ map any complete metric in $\text{Met}(A; \mathbb{R})$ into a complete metric in $\text{Met}(X; \mathbb{R})$.

Although their constructions of $\Phi_1$ and $\Phi_2$ need the Dugundji extension theorem and it is known that an analogue of Dugundji’s extension theorem for $\omega_\mu$-metric spaces is false in general (see [16]), we can prove an analogue of Nguyen Van Khue and Nguyen To Nhu’s theorem for generalized metrics. We shall construct monotone extensors of ultrametrics or generalized metrics (see Theorem 1.2).

For a compact ultrametrizable $X$, Tymchatyn–Zarichnyi [49] constructed maps from the set of all continuous ultrametrics defined on
closed subsets of $X$ into the set of all continuous ultrametrics defined on $X$, which is continuous with respect to the Vietoris topology. Tymchatyn–Zarichnyi constructed such a map for a complete ultrametrizable space. The author does not know whether we can constructed a Tymchatyn–Zarichnyi type map from $\bigcup_{A \in \exp(X)} \UMet(X; \mathbb{R}_{\geq 0})$ into $\UMet(X; \mathbb{R}_{\geq 0})$ or not.

Extension theorems of ultrametrics can be considered as special cases of extending a weight on the edge set of a given graph into an ultrametric on the vertex set of the graph. Dovgoshey–Martio–Vuorinen proved theorems extending weights into (pseudo)ultrametrics. Dovgoshey–Petrov also provided theorems on metrization of weighted graphs.

Let $X$ be a topological space. Let $G$ be a linearly ordered Abelian group and $S$ be a linearly ordered set such that $\Met(X; G) \neq \emptyset$ and $\UMet(X; S) \neq \emptyset$. Let $d \in \Met(X; G)$ or $d \in \UMet(X; S)$. We say that the space $(X, d)$ is complete if every Cauchy filter on $(X, d)$ has a limit point (for the definition of Cauchy filters, see Section 2).

Let $d, e \in \Met(X; G)$ (resp. $d, e \in \UMet(X; G)$). We define $d \lor e \in \Met(X; d)$ (resp. $d \lor e \in \UMet(X; S)$) by $(d \lor e)(x, y) = d(x, y) \lor e(x, y)$, where $\lor$ in the right hand side is the maximal operator on $G$ (resp. $S$).

We introduce the topology on $\Met(X; G)$ as follows: For all $\epsilon \in G_{>0}$, we denote by $\mathcal{V}(d; \epsilon)$ the set of all $e \in \Met(X; G)$ such that for all $x, y \in X$ we have $e(x, y) < d(x, y) \lor \epsilon$ and $d(x, y) < e(x, y) \lor \epsilon$. We consider that $\Met(X; G)$ is equipped with the topology induced from $\mathcal{V}(d; \epsilon) \mid d \in \Met(X; G), \epsilon \in G_{>0}$.

We introduce the notion of characteristic or $g$-characteristic subsets, which is a central concept in the present paper.

**Definition 1.7.** Let $S$ be a bottomed linearly ordered set. A subset $T$ of $S$ is said to be characteristic if $\textcircled{0}_S \in T$ and for all $s \in S \setminus \{\textcircled{0}_S\}$, there exists $t \in T \setminus \{\textcircled{0}_S\}$ such that $t \leq s$. Let $G$ be a linearly ordered Abelian group. A subset $Q$ of $G$ is said to be $g$-characteristic if $Q$ is a characteristic subset of the bottomed linearly ordered set $G_{\geq 0}$. The word “$g$-characteristic” means “group-characteristic”.

The following is our second result:

**Theorem 1.2.** We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in \mathcal{M}(X)$. Let $G \in \GRP(\kappa)$. Let $S$ be a $g$-characteristic subset of $G$. Let $A$ be a closed subset of $X$. Then there exists a map $\Phi: \Met(A, G) \to \Met(X, G)$ such that

(A1) the map $\Phi$ is continuous;
(A2) for all $d \in \Met(A; G)$, we have $\Phi(d)|_{A^2} = d$;
(A3) if $d_1, d_2 \in \Met(X; G)$ satisfy $d_1(a, b) \leq d_2(a, b)$ for all $a, b \in A$,
then for all $x, y \in X$, we have $\Phi(d_1)(x, y) \leq \Phi(d_2)(x, y)$;
(A4) if $d \in \UMet(A; G_{\geq 0})$, then $\Phi(d) \in \UMet(X; G_{\geq 0})$;
(A5) if \( d \in \text{UMet}(A; S) \), then \( \Phi(d) \in \text{UMet}(X; S) \);

(A6) for all \( d, e \in \text{Met}(A; G) \), we have \( \Phi(d \lor e) = \Phi(d) \lor \Phi(e) \).

Moreover, if there exists a complete \( G \)-metric in \( \text{Met}(X; G) \), then we can choose \( \Phi \) as a map satisfying all the conditions mentioned above and

(A7) if \( d \in \text{Met}(A; G) \) is complete, then so is \( \Phi(d) \).

To state our next result, for a topological space \( X \), and for \( S \subseteq \mathbb{R}_{\geq 0} \) with \( 0 \in S \), we define a function \( \mathcal{U}D^S_X : \text{UMet}(X, S)^2 \to [0, \infty] \) by assigning \( \mathcal{U}D^S_X(d, e) \) to the infimum of \( \epsilon \in S \sqcup \{\infty\} \) such that for all \( x, y \in X \) we have

\[
d(x, y) \leq e(x, y) \lor \epsilon,
\]

and

\[
e(x, y) \leq d(x, y) \lor \epsilon.
\]

Note that the function \( \mathcal{U}D^S_X \) is an ultrametric on \( \text{UMet}(X, S) \) valued in \( \text{CL}(S) \sqcup \{\infty\} \), where \( \text{CL}(S) \) stands for the closure of \( S \) in \( \mathbb{R}_{\geq 0} \). The ultrametric \( \mathcal{U}D^S_X \) was utilized in [26].

In the case of ordinary ultrametric spaces, the extensor in Theorem 1.2 becomes an isometric embedding.

**Theorem 1.3.** Let \( S \) be a characteristic subset of \( \mathbb{R}_{\geq 0} \). Let \( X \) be an \( \mathbb{R}_{\geq 0} \)-ultrametrizable space. Then there exists a map \( \Upsilon : \text{UMet}(A, S) \to \text{UMet}(X, S) \) such that

(B1) the map \( \Upsilon \) is an isometric embedding; namely, for all \( d_1, d_2 \in \text{UMet}(A; S) \), we have

\[
\mathcal{U}D^S_X(\Upsilon(d_1), \Upsilon(d_2)) = \mathcal{U}D^S_A(d_1, d_2);
\]

(B2) for all \( d \in \text{UMet}(A; S) \), we have \( \Upsilon(d)|_{A^2} = d \);

(B3) if \( d_1, d_2 \in \text{UMet}(A; G) \) satisfies \( d_1(a, b) \leq d_2(a, b) \) for all \( a, b \in A \), then for all \( x, y \in X \), we have \( \Upsilon(d_1)(x, y) \leq \Upsilon(d_2)(x, y) \);

(B4) for all \( d, e \in \text{UMet}(A; G) \), we have \( \Upsilon(d \lor e) = \Upsilon(d) \lor \Upsilon(e) \).

Moreover, if \( X \) is completely metrizable, then we can choose \( \Upsilon \) as a map satisfying all the conditions mentioned above and

(B5) if \( d \in \text{UMet}(A; G) \) is complete, then so is \( \Upsilon(d) \).

Theorems 1.2 and 1.3 are not only new extension theorems for ultrametrics and generalized metric spaces but also the improvement of the author’s extension theorem of ordinal ultrametrics in [26].

**Remark 1.2.** In Nguyen Van Khue and Nguyen To Nhu’s theorem, the 20-Lipschitz extensor \( \Phi_1 \) and the monotone pointwise continuous extensor \( \Phi_2 \) are constructed. In Theorem 1.3 (or Theorem 1.2), we unify these properties into a single extensor, namely, we construct a metric extensor which is isometric and monotone.
Remark 1.3. Let $G$ be a linearly ordered Abelian group. We assume that $G$ is non-Archimedean. Then, we can take $x, y \in G_{>0}$ with $x \ll y$. Put

(1.1) \[ E = \{ z \in G \mid \text{for all } n \in \mathbb{Z}_{\geq 1}, \text{we have } nz < y \}, \]
(1.2) \[ F = \{ z \in G \mid \text{there exists } n \in \mathbb{Z}_{\geq 1} \text{ such that } y \leq nz \}. \]

Then, we have $F = G \setminus E$, and $E \neq \emptyset$ and $F \neq \emptyset$ (in fact, $x \in E$ and $y \in F$). Moreover, we have $a < b$ for all $a \in E$ and $b \in F$, namely, the set $E$ (or the pair $(E,F)$) is a cut in the sense of Dedekind (see Subsection 2.1). Remark that the set $E$ has no supremum and $F$ has no infimum (see [25, Lemma 2.2]). This observation implies that if $G$ is non-Archimedean, then $G$ is not Dedekind complete, and $G$ does not have the Dedekind completion which is a linearly ordered Abelian group. Due to this phenomenon, when we defined the topology of $\text{Met}(X,G)$, we used $V(d; \epsilon)$, and we did not define a metric on $\text{Met}(X;G)$ such as $\text{UD}^S_X$ on $\text{UMet}(X;S)$, which is defined using the infimum.

We next explain an application of Theorem 1.2. Niemytzki–Tychonoff [37] proved that an $\mathbb{R}$-metrizable space $X$ is compact if and only if all $d \in \text{Met}(X;\mathbb{R})$ are complete. There are many analogues of this characterization for various geometric structures.

Nomizu–Ozeki [38] proved that a second countable connected differentiable manifold is compact if and only if all Riemannian metrics on the manifold are complete.

The author [26] showed that an $\mathbb{R}_{\geq 0}$-ultrametrizable space $X$ is compact if and only if all $d \in \text{UMet}(X;\mathbb{R}_{\geq 0})$ are complete. The author [27] also showed that a metrizable space $X$ is finite-dimensional and compact (resp. zero-dimensional and compact) if and only if the set of all doubling (resp. uniformly disconnected) metrics is dense $F_\sigma$ in $\text{Met}(X;\mathbb{R})$.

Dovgoshey–Shcherbak [17] proved that an $\mathbb{R}_{\geq 0}$-ultrametrizable space $X$ is separable if and only if for every $d \in \text{UMet}(X;\mathbb{R}_{\geq 0})$, the set \( \{ d(x,y) \mid x, y \in X \} \) is countable, and they also proved that an $\mathbb{R}_{\geq 0}$-ultrametrizable space $X$ is compact if and only if all $d \in \text{UMet}(X;\mathbb{R}_{\geq 0})$ are totally bounded.

Hausdorff [22] used the extension theorem of metrics explained before to give another proof of Niemytzki–Tychonoff’s characterization theorem. The author’s theorem [26] explained above is proven using Hausdorff’s argument.

Let $\kappa$ be a cardinal. A topological space is said to be finally $\kappa$- compact if every open cover of the space has a subcover with cardinal $< \kappa$. In this paper, using the existence of extensors of generalized metrics (Theorem 1.2) and Hausdorff’s argument, we characterized the final compactness of generalized metrizable spaces by the completeness...
of generalized metrics. This is an analogue of Niemytzki–Tychonoff’s theorem for generalized metrics.

**Theorem 1.4.** We assume that \( X \) is a topological space possessing an infinite metrizable gauge. Let \( \kappa \) be a regular cardinal with \( \kappa \in \mathcal{M}(X) \). Then the following statements are equivalent to each other.

1. The space \( X \) is finally \( \kappa \)-compact.
2. There exists \( G \in \text{GRP}(\kappa) \) such that for all \( d \in \text{Met}(X; G) \), the space \( (X, d) \) is complete.
3. For all \( G \in \text{GRP}(\kappa) \) and for all \( d \in \text{Met}(X; G) \), the space \( (X, d) \) is complete.
4. There exists \( S \in \text{ORD}(\kappa) \) such that for all \( d \in \text{UMet}(X; S) \), the space \( (X, d) \) is complete.
5. For all \( S \in \text{ORD}(\kappa) \) and for all \( d \in \text{UMet}(X; d) \), the space \( (X, d) \) is complete.

**Remark 1.4.** Since the final \( \omega_0 \)-compactness is identical with the ordinary compactness, Theorem 1.4 contains the author’s result [26, Corollary 1.3].

The organization of this paper is as follows: In Section 2, we prepare some concepts such as the Hahn groups, the Hahn fields, and Cauchy filters. We also discuss some basic statements on ordered sets, metrizable gauges, and generalized metrics and ultrametrics. In Section 3, we prove Theorem 1.1. As a consequence of the existence of retractions, and Artico–Moresco’s characterization (Theorem 3.3) of proximal subsets of generalized ultrametric spaces, we prove Corollary 3.4. In Section 4, we first discuss modifications of generalized metrics. We construct a metric vanishing on a given closed subset (see Proposition 4.1), and show basic properties of an extension of generalized metrics (see Lemma 4.3 and 4.4). We then prove Theorem 1.2 and Theorem 1.3. In Section 5, we exhibit the table of symbols appearing in this paper.

2. Preliminaries

In this section, we prepare some basic statements.

2.1. **Order structures.** We first discuss order structures such as the Dedekind completeness, the Hahn groups, and the Hahn fields.

2.1.1. **Basic statements on orders.** In the present paper, we use the set-theoretic representation of ordinals. For example, if \( \alpha, \kappa \) are ordinals, then the relation \( \alpha < \kappa \) means \( \alpha \in \kappa \), and \( \kappa + 1 = \kappa \cup \{\kappa\} \). For more information, we refer the readers to [28].

For a linearly ordered set \( (L, \leq_L) \), we define the *dual order* \( \leq_{\text{op}(L)} \) of \( \leq_L \) by \( a \leq_{\text{op}(L)} b \) if and only if \( b \leq_L a \). We denote by \( \text{op}(L) \) the ordered set \( (L, \leq_{\text{op}(L)}) \).
In Definition 2.1, we introduce a special linearly ordered set, which plays an important role in this paper.

**Definition 2.1.** For a cardinal $\kappa$, we define the ordered set $\nabla(\kappa)$ by $\nabla(\kappa) = \text{op}(\kappa + 1)$. Note that $\nabla(\kappa)$ is bottomed and $\square_{\nabla(\kappa)} = \kappa$, and note that the maximum of $\nabla(\kappa)$ is 0.

We define isotone and antitone maps, and isomorphisms between ordered sets.

**Definition 2.2.** Let $L, R$ be linearly ordered sets. A map $f : L \to R$ is said to be isotone (resp. antitone) if for all $x, y \in L$, the inequality $x \leq y$ implies $f(x) \leq f(y)$ (resp. $f(y) \leq f(x)$). An injective isotone map is said to be an isotone embedding. If $f$ is isotone and bijective, then $f^{-1}$ is also isotone. In this case, the map $f$ is called an isomorphism between ordered sets, and $L$ and $R$ are said to be isomorphic as ordered sets.

We define the notion of characteristic maps (see also Definition 1.7).

**Definition 2.3.** Let $S$ be a bottomed linearly ordered set. Let $G$ be a linearly ordered Abelian group. Let $E$ be a set. A map $f : E \to S$ (resp. $f : E \to G$) is said to be characteristic (resp. $g$-characteristic) if its image is a characteristic subset of $S$ (resp. a $g$-characteristic subset of $G$).

By the definitions of $\text{GRP}(\kappa)$ and $\text{ORD}(\kappa)$, and characters, we obtain:

**Lemma 2.1.** Let $\kappa$ be a regular cardinal. Let $G \in \text{GRP}(\kappa)$ and $S \in \text{ORD}(\kappa)$. Then there exist a characteristic isotone embedding $l : \nabla(\kappa) \to S$ and a $g$-characteristic isotone embedding $l' : \nabla(\kappa) \to G$.

Let $L$ be a linearly ordered set. A non-empty subset $A$ of $L$ is said to be bounded above (resp. bounded below) if there exists $a \in L$ such that $x \leq a$ (resp. $a \leq x$) for all $x \in A$. We say that $L$ is Dedekind complete if for every non-empty subset $A$ of $L$ which is bounded above, the supremum $\sup A$ of $A$ exists in $L$. Note that $L$ is Dedekind complete if and only if every non-empty set bounded below has the infimum. A subset $A$ of $L$ is said to be a cut if the following conditions hold true:

1. $A \neq \emptyset$;
2. $A$ is bounded above;
3. if $x \in A$ and $y \in L$ satisfy $y < x$, then $y \in A$;
4. if $\sup A$ exists, then $\sup A \in A$.

We define the Dedekind completion $L^\#$ of $L$ by the set of all cuts of $L$. We also define a map $\iota : L \to L^\#$ by $\iota(a) = \{ x \in S \mid x \leq a \}$. Then $L^\#$ is linearly ordered by the inclusion $\subseteq$, and the map $\iota$ is an isotone embedding. By abuse of notations, we represent the order on
as the same symbol as the order ≤ on L. Using the canonical map \( \iota : L \to L^\# \), we consider that \( L \subseteq L^\# \).

The statement (1) in the following lemma is deduced from the definition of \( L^\# \). The statement (2) is presented in [40, Proposition 1.1.4].

**Lemma 2.2.** Let \( L \) be a linearly ordered set. Then the Dedekind completion \( L^\# \) of \( L \) satisfies the following statements:

1. If \( L \) is bottomed, then so is \( L^\# \).
2. If \( s, t \in L^\# \) satisfy \( s < t \), then there exist \( x, y \in L \) with \( s \leq x < t \) and \( s < y \leq t \).

Let \( \kappa \) be a cardinal. Since \( \kappa + 1 \) is well-ordered and closed under the supremum operator, we obtain:

**Lemma 2.3.** Let \( \kappa \) be a cardinal. Then the linearly ordered set \( \nabla(\kappa) \) is Dedekind complete.

**Definition 2.4.** Let \( L \) be a linearly ordered set. A subset \( E \) of \( L \) is said to be cofinal (resp. coinitial) if for all \( l \in L \), there exists \( e \in E \) with \( l \leq e \) (reps. \( e \leq l \)). Let \( E \) be a set. A map \( f : E \to L \) is said to be cofinal (resp. coinitial) if its image is cofinal (resp. coinitial).

**Remark 2.1.** A cardinal \( \kappa \) is regular if and only if \( \kappa \) is infinite and \( \kappa \) is the least cardinal of all cardinal \( \nu \) satisfying that there exists a cofinal isotone embedding \( \nu \to \kappa \). Note that \( \omega_0 \) is regular, and note that if \( \kappa \) is regular, then \( \omega_0 \leq \kappa \).

By Lemma 2.2, we obtain the following two corollaries:

**Corollary 2.4.** Let \( L \) be a linearly ordered set. Then the set \( L \) is a cofinal and coinitial subset in its Dedekind completion \( L^\# \).

**Corollary 2.5.** Let \( S \) be a bottomed linearly ordered set. Then the set \( S \) is a characteristic subset in its Dedekind completion \( S^\# \).

2.1.2. Linearly ordered Abelian groups. As mentioned in Section 1, we verify the well-definedness of the order \( \leq \) on \( A(X) \).

**Lemma 2.6.** Let \( G \) be a linearly ordered Abelian group. Let \( x, y, u, v \in G_{>0} \). If \( x \asymp y \) and \( u \asymp v \) and \( x \ll u \), then we have \( y \ll v \). Namely, the order \( \ll \) on \( A(G) \) is well-defined.

**Proof.** By \( x \asymp y \) and \( u \asymp v \), there exist integers \( N, M \in \mathbb{Z}_{\geq 1} \) such that \( y \leq N \cdot x \) and \( u \leq M \cdot v \). Then we obtain \( n \cdot y < nN \cdot x \) for all \( n \in \mathbb{Z}_{>1} \). From \( x \ll u \), it follows that for all \( n \in \mathbb{Z}_{>1} \), we have \( n \cdot y < u \). Hence \( y \ll u \). For the sake of contradiction, we suppose that there exists \( k \in \mathbb{Z}_{\geq 1} \) such that \( v \leq k \cdot y \). Then, we obtain \( u \leq Mk \cdot y \). This contradicts \( y \ll u \). Thus, we conclude that \( y \ll v \). \( \square \)
Definition 2.5. For a linearly ordered Abelian group $G$, we define the absolute value function $\text{abs}: G \to G_{\geq 0}$ by

$$\text{abs}(x) = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Some authors denote by $|x|$ the value $\text{abs}(x)$. To emphasize that it is a function on $G$, we use the symbol $\text{abs}(x)$ in this paper. We also define a map $\lambda_G: G \to \Xi(\mathcal{A}(G))$ by

$$\lambda_G(x) = \begin{cases} \text{abs}(x) \preceq & \text{if } x \neq 0; \\ \circ \Xi(\mathcal{A}(G)) & \text{if } x = 0. \end{cases}$$

By definitions of $\text{abs}$ and $\lambda_G$, we have:

Proposition 2.7. Let $G$ be a linearly ordered Abelian group. Then the following are satisfied:

1. For all $x \in G$, we have $\lambda_G(x) = \circ \Xi(\mathcal{A}(G))$ if and only if $x = 0_G$.
2. For all $x, y \in G$, we have $\text{abs}(x + y) \leq \text{abs}(x) + \text{abs}(y)$.
3. If $x, y \in G$ and $n \in \mathbb{Z}_{\geq 1}$ satisfy $\text{abs}(x) \leq n \cdot \text{abs}(y)$, then we have $\lambda_G(x) \leq \lambda_G(y)$.
4. For all $x, y \in G$, we have $\lambda_G(x + y) \leq \lambda_G(x) \lor \lambda_G(y)$, where $\lor$ is the maximum operator on $\Xi(\mathcal{A}(G))$.

Proof. The statement (1) follows from the definition of $\lambda_G$.

The statement (2) follows from the fact that for all $a \in G$, we have $a \leq \text{abs}(a)$ and $-a \leq \text{abs}(a)$.

We next verify (3). We divide the proof into two cases. Recall that $\preceq$ is the relation on $G_{\geq 0}$.

Case 1 ($x = 0$ or $y = 0$): If $x = 0$, then, by $\lambda_G(0) = \circ$, we obtain the conclusion. If $y = 0$, then, by the assumption, we have $x = y = 0$. Thus, we obtain $\lambda_G(x) = \lambda_G(y)$. This finishes the proof in Case 1.

Case 2 ($x \neq 0$ and $y \neq 0$): If $\text{abs}(x) \preceq \text{abs}(y)$, then we obtain the conclusion. If $\text{abs}(x) \neq \text{abs}(y)$, then, by the assumption that $\text{abs}(x) \leq n \cdot \text{abs}(y)$, we have $m \cdot \text{abs}(x) < \text{abs}(y)$ for all $m \in \mathbb{Z}_{\geq 1}$, namely, $\text{abs}(x) \ll \text{abs}(y)$. Lemma (2.6) implies that $\lambda_G(x) \leq \lambda_G(y)$. This finishes the proof of (3).

We next show the statement (4). We may assume that $\text{abs}(y) \leq \text{abs}(x)$. By the statement (2), we have $\text{abs}(x + y) \preceq 2 \cdot \text{abs}(x)$. Then, by the statement (3), we have $\lambda_G(x + y) \preceq \lambda_G(x)$. Since $\lambda_G(x) \lor \lambda_G(y) = \lambda_G(x)$, we obtain $\lambda_G(x + y) \leq \lambda_G(x) \lor \lambda_G(y)$. This completes the proof of the statement (4).

Proposition 2.8. Let $\kappa$ be a regular cardinal. Let $G \in \text{GRP}(\kappa)$. Then the following statements hold true.

1. For all $\epsilon \in G_{\geq 0}$, there exists $\rho \in \mathcal{A}(G)$ such that if $\lambda_G(x) \ll \rho$, then we have $\text{abs}(x) < \epsilon$. 


(2) For all \( \eta \in \mathbf{A}(G) \), there exists \( \theta \in G_{>0} \) such that if \( \text{abs}(x) < \theta \), then \( \lambda_G(x) \ll \eta \).

Proof. We first verify the statement (1). Take \( \epsilon \in G_{>0} \), and put \( \rho = [\epsilon]_\infty \). If \( \lambda_G(x) \ll \rho \), then we have \( \text{abs}(x) < \epsilon \) (if \( \epsilon \leq \text{abs}(x) \)), we have \( \rho \ll \lambda_G(x) \) by (3) in Proposition 2.7.

We next show the statement (2). Since \( \kappa \) is infinite, we can take \( \eta, \eta' \in \mathbf{A}(G) \) with \( \eta' < \eta \). Take \( \theta \in G_{>0} \) with \( \eta' = [\theta]_\infty \). If \( \text{abs}(x) < \theta \), then \( \lambda_G(x) \ll \eta' \), and hence \( \lambda_G(x) \ll \eta \). This complete the proof. \( \square \)

2.1.3. The Hahn groups and fields. For a linearly ordered set \( L \), we say that \( L \) is dually well-ordered if every its non-empty subset has the maximum. This concept is the dual of the well-ordered property.

We now define the Hahn groups and the Hahn fields. Let \( L \) be a linearly ordered set. We define \( \mathbb{H}(L) \) by the set of all maps \( f: L \to \mathbb{R} \) such that the set \( \{ x \in L \mid f(x) \neq 0_{\mathbb{R}} \} \) is a dually well-ordered subset of \( L \). We define \( f + g \) as the coordinate-wise addition, namely, \((f + g)(x) = f(x) + g(x)\). We define \( f \leq_{\mathbb{H}(L)} g \) by \( f(\theta) \leq_{\mathbb{R}} g(\theta) \), where \( \theta = \max \{ x \in L \mid f(x) \neq g(x) \} \).

For a linearly ordered Abelian group \( G \), we can define the product on the Hahn space \( \mathbb{H}(G) \) induced from \( G \) by

\[
fg(x) = \sum_{a + b = x} f(a)g(b).
\]

In this case, the group \( \mathbb{H}(G) \) becomes a field (see [21], or see also [20, Exercises 3.5.5 and 3.5.6]). To emphasize the field structure, we denote by \( \mathbb{F}(G) \) the Hahn group \( \mathbb{H}(G) \) equipped with the product structure. Some authors used \( \text{op}(L) \) instead of \( L \) and defined the Hahn fields by the set of all functions whose supports are well-ordered sets in \( \text{op}(L) \) instead of dually well-ordered sets in \( L \). The definition of the Hahn fields in this paper can be found in [13] and [8].

Lemma 2.9. Let \( L \) be a linearly ordered set. Then the Hahn group \( \mathbb{H}(L) \) is a linearly ordered Abelian group.

Proof. Take \( f, g, h \in \mathbb{H}(L) \), and assume that \( f \leq_{\mathbb{H}(L)} g \). Put \( \theta = \max \{ x \in L \mid f(x) \neq g(x) \} \). Then \( f(\theta) \leq_{\mathbb{R}} g(\theta) \). By the definition of the addition on \( \mathbb{H}(L) \), we have

\[
\theta = \max \{ x \in L \mid (f + h)(x) \neq (g + h)(x) \}.
\]

Thus, \((f + h)(\theta) \leq_{\mathbb{R}} (g + h)(\theta)\), and hence \( f + h \leq_{\mathbb{H}(L)} g + h \). Therefore we conclude that \( \mathbb{H}(L) \) is a linearly ordered Abelian group. \( \square \)

Definition 2.6. Let \( G \) be a linearly ordered Abelian group. A subset \( E \) of \( G \) is said to be full if for all \( x \in G \), there exists \( a \in E \) such that \( \text{abs}(x) \asymp \text{abs}(a) \). Let \( N \) be a set. A map \( f: N \to G \) is said to be full if its image is a full subset of \( G \).
We next discuss the relation between a linearly ordered set and the Hahn group induced from the linearly ordered set.

**Definition 2.7.** Let \((L, \leq)\) be a linearly ordered set. For \(s \in L\), we define \(e_{L,s} \in \mathbb{H}(L)\) by \(e_{L,s}(x) = 0\) if \(x \neq s\); otherwise, \(e_{L,s}(s) = 0\). We define a map \(E_L : L \to \mathbb{H}(L)\) by \(E_L(s) = e_{L,s}\).

**Lemma 2.10.** Let \(L\) be a linearly ordered set. Then the map \(E_L : L \to \mathbb{H}(L)\) satisfies the following statements:

1. For all \(s \in L\), we have \(0 < E_L(s)\).
2. The map \(E_L\) is an isotone embedding.
3. If \(s, t \in L\) satisfy \(s < t\), then we have \(E_L(s) \ll E_L(t)\).
4. For all \(f \in \mathbb{H}(L)\), we have
   \[
   \lambda_{\mathbb{H}(L)}(f) = [E_L(\max\{x \in L \mid f(x) \neq 0\})]_\mathbb{R}.
   \]
5. The map \(E_L\) is full.
6. The map \([E_L]_\mathbb{R} : L \to \mathbb{A}(\mathbb{H}(L))\) defined by \([E_L]_\mathbb{R}(s) = [E_L(s)]_\mathbb{R}\) is an isomorphism between \((L, \leq_L)\) and \((\mathbb{A}(\mathbb{H}(L)), \leq)\).

**Proof.** By the definitions of \(e_{L,s}\) and \(E_L\), we have \(0 < E_L(s)\) for all \(s \in L\), which is the condition \(\Box\).

We next show that the map \(E_L\) is an isotone embedding. Take \(s, t \in L\) satisfying that \(s < t\). In this case, we have
\[
t = \max\{x \in L \mid e_{L,s}(x) \neq e_{L,t}(x)\},
\]
and \(e_{L,s}(t) = 0\) and \(e_{L,t}(t) = 1\). Hence, for all \(m \in \mathbb{Z}_{\geq 1}\), we obtain \(m \cdot e_{L,s}(t) < e_{L,t}(t)\). Thus, we have \(E_L(s) \ll E_L(t)\). In particular, we also have \(E_L(s) < E_L(t)\). Then we conclude that the statements \(\Box\) and \(\Box\) hold true.

We next show the statement \(\Box\). Take \(f \in \mathbb{H}(L)\), and put
\[
\theta = \max\{x \in L \mid f(x) \neq 0\} \in L.
\]
By the definition of the order on \(\mathbb{H}(L)\), and by the fact that \(\mathbb{R}\) is Archimedean, we obtain \(E_L(\theta) \approx f\). Thus, the statement \(\Box\) is true.

We next verify that the map \(E_L\) is full. The statement \(\Box\) implies that the set \(\{E_L(s) \mid s \in L\}\) is full in \(\mathbb{H}(L)\). Then the map \(E_L\) is full, and hence the statement \(\Box\) is valid.

We now prove the statement \(\Box\). The statement \(\Box\) implies that the map \([E_L]_\mathbb{R}\) is an isotone embedding. From \(\Box\), the surjectivity of the map \([E_L]_\mathbb{R}\) follows. Therefore \([E_L]_\mathbb{R} : L \to \mathbb{A}(\mathbb{H}(L))\) is an isomorphism between the ordered sets \((L, \leq_L)\) and \((\mathbb{A}(\mathbb{H}(L)), \leq)\). \(\Box\)

**Remark 2.2.** Let \(G\) be a linearly ordered Abelian group. Instead of \(\lambda_{\mathbb{F}(G)}\), in the theory of valued fields, the Krull valuation \(\nu_{\mathbb{F}(G)} : \mathbb{F}(G) \to G \cup \{\infty\}\) is ordinarily used. The map \(\lambda_{\mathbb{F}(G)}\) is a dual of \(\nu_{\mathbb{F}(G)}\). The map \(\nu_{\mathbb{F}(G)}\) is sometimes called the natural valuation on \(\mathbb{F}(G)\) (see, for example, \(\Box\)).
**Definition 2.8.** For a bottomed linearly ordered set $S$, we define $S^\star = S \setminus \{\circ_S\}$.

**Definition 2.9.** Let $L$ be a linearly ordered set. We define a map $O_L: L \to \text{op}(L)$ by $O_L(x) = x$. Note $O_L$ is antitone, namely, if $x, y \in L$ satisfy $x \leq_L y$, then $O_L(y) \leq_{\text{op}(L)} O_L(x)$.

**Lemma 2.11.** Let $S$ be a bottomed linearly ordered set with $\omega_0 \leq \chi(S)$. Then the map $W_S: S^\star \to \mathbb{H}(\text{op}(S^\star))$ defined by

$$W_S(s) = -E_{\text{op}(S^\star)} \circ O_{S^\star}(s)$$

is a coinitial isotone embedding.

**Proof.** By the statement (2) in Lemma 2.10 the map $E_{\text{op}(S^\star)}: \text{op}(S^\star) \to \mathbb{H}(\text{op}(S^\star))$ is an isotone embedding. We define a map $m: \mathbb{H}(\text{op}(S^\star)) \to \mathbb{H}(\text{op}(S^\star))$ by $m(x) = -x$. Then, the maps $O_{S^\star}$ and $m$ are antitone and injective. In this case, we obtain $W_S = m \circ E_{\text{op}(S^\star)} \circ O_{S^\star}$. Thus, the map $W_S$ is an isotone embedding.

We next show that $W_S$ is coinitial. Take arbitrary $f \in \mathbb{H}(\text{op}(S^\star))$. Since $E_{\text{op}(S^\star)}$ is full (see the statement (5) in Lemma 2.10), there exists $t \in L$ satisfying that

$$[\text{abs}(f)]_\prec = [E_{\text{op}(S^\star)}(t)]_\prec.$$ 

By $\omega_0 \leq \chi(S)$, there exists $s \in S^\star$ with $s < t$. According to the statement (6) in Lemma 2.10 and $t <_{\text{op}(S^\star)} s$, we have

$$[E_{\text{op}(S^\star)}(s)]_\prec \ll [E_{\text{op}(S^\star)}(s)]_\prec.$$ 

From (2.1) and (2.2), it follows that

$$[\text{abs}(f)]_\prec \ll [E_{\text{op}(S^\star)}(s)]_\prec.$$ 

This inequality implies that $\text{abs}(f) < E_{\text{op}(S^\star)}(s)$. By $-f \leq \text{abs}(f)$, we obtain $-f < E_{\text{op}(S^\star)}(s)$. Then $-E_{\text{op}(S^\star)}(s) < f$, and hence $W_S(s) < f$. This means that the map $W_S$ is coinitial. \qed

**Definition 2.10.** For a bottomed linearly ordered set $S$, we define $\mathbb{P}(S) = \mathbb{F}(\mathbb{H}(\text{op}(S^\star)))$.

**Proposition 2.12.** Let $S$ be a bottomed linearly ordered set with $\omega_0 \leq \chi(S)$. We define a map $I: S \to \mathbb{P}(S)$ defined by

$$I(s) = \begin{cases} E_{\mathbb{H}(\text{op}(S^\star))} \circ W_S(s) & \text{if } \{0\}_S < s \\ 0_{\mathbb{P}(S)} & \text{if } s = \{0\}_S \end{cases}$$

Then the map $I$ is a $g$-characteristic isotone embedding.

**Proof.** Recall that $E_{\mathbb{H}(\text{op}(S^\star))}$ is a map $E_{\mathbb{H}(\text{op}(S^\star))}: \text{op}(S^\star) \to \mathbb{P}(S)$ defined in the same way to Definition 2.7. By Lemma 2.11, the map $W_S: S^\star \to \mathbb{H}(\text{op}(S^\star))$ is a coinitial isotone embedding, and hence $I: S \to \mathbb{P}(S)$ is an isotone embedding.
We now show that $I$ is $g$-characteristic. Take arbitrary $f \in \mathcal{P}(S)$ with $f > 0_{\mathcal{P}(S)}$. Put

$$\theta = \max \{ x \in \mathbb{H}(\text{op}(S*)) | f(x) \neq 0 \}.$$ 

Then we have $f(\theta) > 0$. By the statement (1) in Lemma 2.10, we verify that $f \asymp E_{\mathbb{H}(\text{op}(S*))}(\theta)$. Since $W_S$ is coinitial (see Lemma 2.11), there exists $s \in S^*$ such that $W_S(s) < \theta$. By the statement (3) in Lemma 2.10, we have

$$E_{\mathbb{H}(\text{op}(S*))}(W_S(s)) \not\asymp E_{\mathbb{H}(\text{op}(S*))}(\theta).$$

By (2.3) and by $f \asymp E_{\mathbb{H}(\text{op}(S*))}(\theta)$, we obtain the inequality

$$E_{\mathbb{H}(\text{op}(S*))}(W_S(s)) < f.$$ 

According to the statement (1) in Lemma 2.10, we have

$$0_{\mathcal{P}(S)} < E_{\mathbb{H}(\text{op}(S*))}(W_S(s)).$$

Since $I(s) = E_{\mathbb{H}(\text{op}(S*))}(W_S(s))$, by (2.4) and (2.5), we conclude that the map $I$ is $g$-characteristic.

\[\square\]

Remark 2.3. Based on Lemma 2.9 and Proposition 2.12 we can consider that every bottomed linearly ordered set is a $g$-characteristic subset of an ordered field.

2.1.4. Characters. We next discuss the characters of linearly ordered Abelian groups. By the definition of characters, we obtain:

**Lemma 2.13.** Let $S$ be a bottomed linearly ordered set. Then the character $\chi(S)$ of $S$ is equal to the minimal cardinal of all cardinals $\kappa$ such that there exits a characteristic isotone embedding $f: \nabla(\kappa) \to S$.

By the definition of characteristic subsets, we obtain:

**Lemma 2.14.** Let $S$ be a bottomed linearly ordered set. If $T$ is a characteristic subset of $S$, then we have $\chi(S) = \chi(T)$.

**Proposition 2.15.** Let $G$ be a linearly ordered Abelian group with $\omega_0 \leq \chi(G_{\geq 0})$. Then we have $\chi(G_{\geq 0}) = \chi(\Xi(A(G)))$.

**Proof.** Let $\kappa = \chi(G_{\geq 0})$. Note that $\kappa$ is regular. According to Lemma 2.14, we can take a characteristic map $f: \nabla(\kappa) \to G_{\geq 0}$. We define a map $F: \nabla(\kappa) \to \Xi(A(G))$ by $F(\alpha) = [f(\alpha)]_{\Xi}$ if $\alpha \neq \bigcirc_{\nabla(\kappa)}$; otherwise $F(\bigcirc_{\nabla(\kappa)}) = \bigcirc_{\Xi(A(G))}$. Then the map $F$ is characteristic; however, it can happen that $F$ is not isotone. Since $F$ is characteristic, we observe that for each $\alpha < \kappa$, we have Card($\{ \beta < \kappa | F(\alpha) = F(\beta) \}$) $< \kappa$. By this fact, and by the regularity of $\kappa$, we can take a characteristic map $l: \nabla(\kappa) \to \nabla(\kappa)$ such that $F \circ l: \nabla(\kappa) \to G_{\geq 0}$ is a characteristic isotone embedding. Thus, by Lemma 2.13, we obtain $\chi(G_{\geq 0}) \leq \chi(\Xi(A(G)))$.

Let $\mu = \chi(\Xi(A(G)))$. According to Lemma 2.13, we can take a characteristic isotone embedding $g: \nabla(\mu) \to \Xi(A(G))$. For each $\alpha < \mu$, we take $h(\alpha) \in G_{\geq 0}$ such that $g(\alpha) = [h(\alpha)]_{\Xi}$. We define $h(\mu)$
by $h(\mu) = 0_G$ (note that $\bullet_{\sigma(\mu)} = \mu$). Then $h: \nabla(\mu) \to G_{\geq 0}$ is a characteristic isotope embedding, and hence $\chi(\Xi(A(G))) \leq \chi(G_{\geq 0})$. Therefore we conclude that $\chi(G_{\geq 0}) = \chi(\Xi(A(G)))$.

Lemma 2.14 and Propositions 2.12 and 2.15 imply the following:

**Corollary 2.16.** Let $S$ be a bottomed linearly ordered set with $\omega_0 \leq \chi(S)$. Then we have $\chi(\mathbb{P}(S)) = \chi(S)$. Equivalently, we have $\mathbb{P}(S) \in \text{GRP}(\chi(S))$.

**Corollary 2.17.** Let $\kappa$ be a regular cardinal. Then we obtain $\chi(\nabla(\kappa)) = \kappa$ and $\chi(\mathbb{P}(\nabla(\kappa))) = \kappa$. Equivalently, we have $\nabla(\kappa) \in \text{ORD}(\kappa)$ and $\mathbb{P}(\nabla(\kappa)) \in \text{GRP}(\kappa)$.

**Proof.** The regularity of $\kappa$ indicates $\chi(\nabla(\kappa)) = \kappa$. Corollary 2.16 implies $\chi(\mathbb{P}(\nabla(\kappa))) = \chi(\nabla(\kappa))$, and hence $\chi(\mathbb{P}(\nabla(\kappa))) = \kappa$. □

2.1.5. Embedding theorems. The following is known as Hahn’s embedding theorem. The proof can be seen in [7] and [23].

**Theorem 2.18.** Let $G$ be a linearly ordered Abelian group. Then there exists a full isotope embedding $F: G \to \overline{\mathbb{H}}(A(G))$ which is a group homomorphism.

As a corollary of Hahn’s embedding theorem, we obtain the following statement known as Hölder’s embedding theorem:

**Theorem 2.19.** Let $G$ be a linearly ordered Abelian group. If we have $\text{Card}(A(G)) = 1$ (namely, $G$ is Archimedean), then there exists an isotope embedding $F: G \to \mathbb{R}$ which is a group homomorphism.

**Remark 2.4.** According to Theorem 2.19, if $G$ is Archimedean, then $G$ can be considered as the subgroup of $\mathbb{R}$. Thus, the group $G$ is dense in $\mathbb{R}$, or isomorphic to $\mathbb{Z}$ (see, for example, [1, Lemma 2.2]).

2.2. Metric structures. We next discuss metric structures.

2.2.1. Basic statements on metrics.

**Definition 2.11.** Let $X$ and $Y$ be topological spaces. Let $G$ and $H$ be linearly ordered Abelian groups, and let $S$ and $T$ be bottomed linearly ordered sets. Let $d \in \text{Met}(X; G)$ or $d \in \text{UMet}(X; S)$, and $e \in \text{Met}(Y; H)$ or $e \in \text{UMet}(Y; T)$. Put $E = G_{>0}$ if $d \in \text{Met}(X; G)$; otherwise $E = S^\star$, and put $F = H_{>0}$ if $e \in \text{Met}(Y; H)$; otherwise $F = T^\star$. A map $f: (X, d) \to (Y, e)$ is said to be uniformly continuous if for all $\epsilon \in F$, there exists $\delta \in E$ such that if $x, y \in X$ satisfy $d(x, y) < \delta$, then $e(f(x), f(y)) < \epsilon$. If there exists a bijection $f: (X, d) \to (Y, e)$ such that $f$ and $f^{-1}$ are uniformly continuous, we say that $(X, d)$ and $(Y, e)$ are uniformly equivalent to each other. In the case of $X = Y$, we say that $d$ and $e$ are uniformly equivalent to each other if the identity map $\text{id}_X: (X, d) \to (X, e)$ and its inverse are...
uniformly continuous. The definition of the uniform equivalence in this paper is nothing but a paraphrase of the uniform equivalence between uniform spaces induced from generalized (ultra)metrics. For uniform spaces, we refer the readers to [52] or [30].

Lemma 2.20. We assume that X is a topological space possessing an infinite metrizable gauge. Let \( \kappa \) be a regular cardinal with \( \kappa \in \mathfrak{M}(X) \). Let \( G \in \text{GRP}(\kappa) \). Let \( d \in \text{Met}(X;G) \). Then \( \lambda_G \circ d \in \text{UMet}(X;\Xi(A(G))) \), and \( d \) and \( \lambda_G \circ d \) are uniformly equivalent to each other.

Proof. From the statements (1), and (3) in Proposition 2.7, it follows that \( e \) satisfies the condition (U1)–(U4) in Definition 1.2.

By the statements (2) and (4) in Proposition 2.7, we verify that \( e \) satisfies the strong triangle inequality (the condition (U5) in Definition 1.2). Hence \( e \) is a \( \Xi(A(G)) \)-ultrametric on \( X \). The remaining part of the lemma follows from Proposition 2.8. \( \square \)

We next prove an analogue of [26, Lemma 2.2] concerning functions preserving ultrametrics.

Lemma 2.21. Let \( S \) and \( T \) be bottomed linearly ordered sets. We assume that \( \psi: S \rightarrow T \) satisfies that

1. the map \( \psi \) is isotone;
2. if \( \psi(x) = \emptyset_T \), then \( x = \emptyset_S \);
3. \( \psi \) is continuous at \( \emptyset_S \), namely, for all \( \epsilon \in T^* \), there exists \( \delta \in S^* \) such that if \( x \in S \) satisfies \( x < \delta \), then we have \( \psi(x) < \epsilon \).

Let \( d \in \text{UMet}(X;S) \). Then \( \psi \circ d \in \text{UMet}(X;T) \), and \( d \) and \( \psi \circ d \) generate the same topology on \( X \). Moreover, they are uniformly equivalent to each other.

Proof. Since \( \psi \) is isotone, the map \( \psi \circ d \) satisfies the strong triangle inequality (the condition (U5) in Definition 1.2). By the assumptions (1) and (2), the map \( \psi \circ d \) satisfies the conditions (U1)–(U4) in Definition 1.2 and hence we conclude that \( \psi \circ d \) is a \( T \)-ultrametrics. It suffices to show that \( d \) and \( \psi \circ d \) are uniformly equivalent to each other. We divide the proof into two cases.

Case 1 (\( T^* \) has the least element): By the assumptions (2) and (3), we verify that \( S^* \) has the least element. Put \( \alpha = \min S^* \) and \( \beta = \min T^* \). In this case, if \( d(x,y) < \alpha \) if and only if \( \psi \circ d(x,y) < \beta \). Then \( d \) and \( \psi \circ d \) are uniformly equivalent to each other.

Case 2 (\( T^* \) does not have the least element): Take arbitrary \( \epsilon \in T^* \). By the assumption (3), there exists \( \delta \) with \( \psi(\delta) < \epsilon \). Thus, if \( x, y \in X \) satisfy \( d(x,y) < \delta \), then \( (\psi \circ d)(x,y) < \epsilon \).

We next prove the converse. Take arbitrary \( \eta \in S^* \). Since \( T^* \) does not have the least element, we can take \( \theta \in T \) with \( \theta < \psi(\eta) \). By the assumption that \( \psi \) is isotone, if \( x, y \in X \) satisfy \( (\psi \circ d)(x,y) < \theta \),

20
then \(d(x, y) < \eta\). Therefore, the ultrametrics \(d\) and \(\psi \circ d\) are uniformly equivalent to each other. \(\square\)

**Remark 2.5.** As the author proved Lemma 2.2 in [26] using [42, Theorem 9] and [15, Theorem 2.9], we can show the converse of Lemma 2.21; namely, we can show that if \(\psi : S \rightarrow T\) satisfies that for all \(d \in \text{UMet}(d; S)\) we have \(\psi \circ d \in \text{UMet}(X; T)\), then \(\psi\) satisfies the conditions (1), (2) and (3) in Lemma 2.21. Since [26, Lemma 2.2], [42, Theorem 9] and [15, Theorem 2.9] are proven using only the order structure of \(\mathbb{R}\), we can obtain an analogue of these statements for general ordered sets.

**Lemma 2.22.** Let \(S\) be a bottomed linearly ordered set. Let \(\kappa\) be a cardinal. Let \(l : \nabla(\kappa) \rightarrow S\) be a characteristic isotone embedding. Then the map \(l\) satisfies the assumptions (1), (2) appearing in Lemma 2.21.

**Proof.** By the assumption, the map \(l\) satisfies the conditions (1) and (2). Since \(l\) is characteristic, it satisfies the condition (3). \(\square\)

**Definition 2.12.** Let \(\kappa\) be a regular cardinal. Let \(S \in \text{ORD}(\kappa)\). Let \(l : \nabla(\kappa) \rightarrow S\) be a characteristic isotone embedding. We define a map \(\zeta_{S,l} : S \rightarrow \nabla(\kappa)\) by

\[
\zeta_{S,l}(s) = \begin{cases} 
0 & \text{if } l(0) < s \\
\alpha + 1 & \text{if } l(\alpha + 1) < s \leq l(\alpha) \\
\bullet_{\nabla(\kappa)} & \text{if } s = \bullet_{\kappa}.
\end{cases}
\]

Note that 0 is the maximal element of \(\nabla(\kappa)\). Remark that \(\zeta_{S,l} : S \rightarrow \nabla(\kappa)\) satisfies the assumptions (1), (2), and (3) in Lemma 2.21.

By the definition of \(g\)-characteristic subsets, we obtain:

**Lemma 2.23.** Let \(X\) be a topological space with \(\mathcal{M}(X) \neq \emptyset\). Let \(G\) be a linearly ordered Abelian group with \(\text{Met}(X; G) \neq \emptyset\). If \(S\) is a \(g\)-characteristic subset of \(G\), we have \(\text{UMet}(X; S) \subseteq \text{Met}(X; G)\).

**Proof.** The lemma follows from the definition of \(g\)-characteristic subsets. Note that, to prove that all members in \(\text{UMet}(X; S)\) satisfy the triangle inequality, we use the fact that \(x \vee y \leq x + y\) for all \(x, y \in G_{\geq 0}\). \(\square\)

**Proposition 2.24.** Let \(\kappa\) be a regular cardinal. Let \(X\) be a topological space. Then the following conditions are equivalent to each other.

1. \(\kappa \in \mathcal{M}(X)\);
2. there exists \(G \in \text{GRP}(\kappa)\) such that \(\text{Met}(X; G) \neq \emptyset\);
3. for all \(G \in \text{GRP}(\kappa)\), we have \(\text{Met}(X; G) \neq \emptyset\);
4. there exists \(S \in \text{ORD}(\kappa)\) such that \(\text{UMet}(X; S) \neq \emptyset\);
5. for all \(S \in \text{ORD}(\kappa)\), we have \(\text{UMet}(X; S) \neq \emptyset\);
6. \(\text{UMet}(X; \nabla(\kappa)) \neq \emptyset\).
Proof. The equivalence \((1) \iff (2)\) is the definition of \(\kappa \in \mathcal{M}_G(X)\).

The implications \((3) \implies (2)\) and \((5) \implies (1)\) follows from the fact that \(\text{GRP}(\kappa) \neq \emptyset\) and \(\text{ORD}(\kappa) \neq \emptyset\) for all regular cardinals \(\kappa\) (see Corollary 2.17).

According to Proposition 2.15 for every \(G \in \text{GRP}(\kappa)\), we have \(\Xi(A(\kappa)) \in \text{ORD}(\kappa)\). Then, by Lemma 2.20, we obtain the implication \((2) \implies (1)\).

We assume the condition \((6)\). Take arbitrary \(G \in \text{GRP}(\kappa)\) and \(S \in \text{ORD}(\kappa)\). Take \(d \in \text{UMet}(X; \nabla(\kappa))\). Take a \(g\)-characteristic isotone embedding \(f\): \(\nabla(\kappa) \to G\) and a \(g\)-characteristic isotone embedding \(g\): \(\nabla(\kappa) \to S\). Then, by Lemmas 2.21 and 2.22 we have \(f \circ d \in \text{UMet}(X; G_{>0})\) and \(g \circ d \in \text{UMet}(X; S)\). Thus, we obtain the implication \((6) \implies (5)\). According to Lemma 2.23 we observe that \(f \circ d \in \text{Met}(X; G)\), and hence we obtain the implication \((6) \implies (3)\). Note that the implication \((6) \implies (3)\) can be reduced from Corollary 2.17.

To finish the proof, it suffices to show the implication \((4) \implies (1)\). We assume the condition \((4)\), and take \(d \in \text{UMet}(X; S)\). Take a characteristic isotone embedding \(l\): \(\nabla(\kappa) \to S\), and take a map \(\zeta_{S1}: S \to \nabla(\kappa)\) in Definition 2.12. We also define a map \(e: X^2 \to \nabla(\kappa)\) by \(e = \zeta_{S1} \circ d\). Since \(\zeta_{S1}\) satisfies the assumptions in Lemma 2.21 we obtain \(e \in \text{UMet}(X; \nabla(S))\). This finishes the proof.

We next observe that the Hahn group and the Hahn fields are universal in the theory of generalized metrics in some sense. Let \(X\) be a set. Let \(G\) be a linearly ordered group and \(S\) a bottomed linearly ordered set. Let \(d\) be a \(G\)-metric or \(S\)-ultrametric on \(X\). We denote by \(U(x, \epsilon; d)\) (resp. \(B(x, \epsilon; d)\)) the open (resp. closed) ball centered at \(x\) with radius \(\epsilon\).

Lemma 2.25. Let \(G\) be a linearly ordered Abelian group. Let \(X\) be a topological space with \(\text{Met}(X; G) \neq \emptyset\). Let \(d \in \text{Met}(X; G)\). By Hahn’s embedding theorem (Theorem 2.18), we can consider that \(G\) is a full subgroup of \(\mathbb{H}(\text{A}(G))\). Then, the topology of \((X, d)\) is generated by \(\{U(x, \epsilon; d) \mid x \in X, \epsilon \in \mathbb{H}(\text{A}(G))_{>0}\}\). In particular, we have \(\text{Met}(X; G) \subseteq \text{Met}(X; \mathbb{H}(\text{A}(G)))\).

Proof. The lemma follows from the fact that \(G\) is full in \(\mathbb{H}(\text{A}(G))\). \(\square\)

Let \(S\) be a bottomed linearly ordered set. We now define an extension of \(S\). If \(S^\star\) has the least element, we define a linearly ordered set \(\sigma(S)\) as follows: Take a countable set \(E = \{e_i\}_{i \in \mathbb{Z}_{\geq 0}}\) with \(E \cap S = \emptyset\). Put \(\sigma(S) = S \cup E\). We define an order on \(\sigma(S)\), which is an extension of the order of \(S\), by \(\bigcirc S < e_i < \min S^\star\) and \(e_{i+1} < e_i\) for all \(i \in \mathbb{Z}_{\geq 0}\). If \(S^\star\) does not have the least element, then we put \(\sigma(S) = S\). Note that \(\sigma(S)\) satisfies \(\omega_0 \leq \chi(\sigma(S))\).
Lemma 2.26. Let $S$ be a bottomed linearly ordered set. Let $X$ be a topological space with $\text{UMet}(X; S) \neq \emptyset$. Then, we have $\text{UMet}(X; S) \subseteq \text{UMet}(X; \mathcal{P}(\sigma(S))_{\geq 0})$. Moreover, $\text{UMet}(X; S) \subseteq \text{Met}(X; \mathcal{P}(\sigma(S)))$.

Proof. The lemma follows from Lemmas 2.21 and 2.23. □

Remark 2.6. Based on Lemma 2.26, we can observe that some statements on $G$-metrics are still valid for $S$-ultrametrics. For example, statements in the future of this paper such as Lemmas 2.37, 2.38, and 2.40 hold true for not only $G$-metrics but also $S$-ultrametric spaces.

2.2.2. Metrizable gauges. We next give a complete description of $\mathcal{M}_G(X)$ for a topological space $X$ (Proposition 2.32).

Proposition 2.27. Let $X$ be a topological space. If $1 \in \mathcal{M}_G(X)$, then $X$ is $R\geq 0$-metrizable (metrizable in the ordinary sense).

Proof. Take $G \in \text{GRP}(1)$ such that $\text{Met}(X; G) \neq \emptyset$. According to Theorem 2.19, the set $G$ can be considered as a subgroup of $R$. Thus, the space $X$ is $R$-metrizable. □

Proposition 2.28. A topological space $X$ is $R\geq 0$-ultrametrizable (ultrametrizable in the ordinary sense) if and only if $\omega_0 \in \mathcal{M}_G(X)$.

Proof. The proposition follows from Proposition 2.24 and the fact that $R\geq 0 \in \text{ORD}(\omega_0)$ (see also Lemma 2.25). □

Remark 2.7. We provide examples of theorems, similarly to Proposition 2.28, stating that values of metrics determine the topology of the underlying space. As mentioned in Section 1, Dovgoshey–Shechkab [17] proved that an $R\geq 0$-ultrametrizable space $X$ is separable if and only if for every $d \in \text{UMet}(X; R\geq 0)$, the set $\{d(x, y) \mid x, y \in X\}$ is countable. Broughan [6, Theorem 2] showed that for a topological space $X$, the following are equivalent to each other:

1. $X$ is $R\geq 0$-ultrametrizable;
2. there exists $d \in \text{Met}(X; R)$ such that $\{d(x, y) \mid x, y \in X\} \subseteq \{0\} \cup \{1/n \mid n \in \mathbb{Z}_{\geq 1}\}$;
3. there exist a characteristic subset $S$ of $R\geq 0$ isomorphic to $\nabla(\omega_0)$ as ordered sets, and $d \in \text{Met}(X; R)$ such that $\{d(x, y) \mid x, y \in X\} \subseteq S$.

Note that the metrics $d$ appearing in (2) or (3) is not assumed to be an ultrametric (see also [26, Proposition 2.14]).

The next lemma states that all $\kappa$-metrizable space (in the sense of Sikorski) are $\kappa$-additive (see [43, (viii), p129, ]).

Lemma 2.29. We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in \mathcal{M}_G(X)$. Then, for every family $\{U_i\}_{i \in I}$ of open subsets with $\text{Card}(I) < \kappa$, the set $\bigcap_{i \in I} U_i$ is open in $X$. 23
Proof. Let \( G \in \text{GRP}(\kappa) \), and \( d \in \text{Met}(X; G) \). Take a \( g \)-characteristic isotone embedding \( l : \nabla(\kappa) \to G \). The lemma follows from the fact that for every \( a \in X \), and for every \( \theta < \kappa \), we have \( U(a, l(\theta); d) \subseteq \bigcap_{\alpha < \theta} U(a, l(\alpha); d) \).

Proposition 2.30. Let \( X \) be a topological space. We assume that there exist two cardinals \( \kappa, \mu \in \text{MG}(X) \) with \( \kappa < \mu \), and assume that either of the following is satisfied:

1. \( \kappa = 1 \) and \( \omega_0 < \mu \);
2. \( \omega_0 \leq \kappa \).

Then the space \( X \) is discrete.

Proof. Put \( \theta = \max\{\omega_0, \kappa\} \). By the assumptions, we have \( \theta < \mu \). By \( \theta \in \text{MG}(X) \), in any case, for every point \( x \in X \), there exists an open set \( \{U_\alpha\}_{\alpha < \theta} \) such that \( \{x\} = \bigcap_{\alpha < \theta} U_\alpha \). By Lemma 2.29, and \( \theta < \mu \), the set \( \{x\} \) is open. Therefore the space \( X \) is discrete.

Proposition 2.31. If \( X \) is a discrete space, then \( \text{MG}(X) \) consists of and only of 1 and all regular cardinals.

Proof. Let \( \kappa \) be a cardinal such that \( \kappa = 1 \) or it is regular. Take \( G \in \text{GRP}(\kappa) \), and take \( \delta \in G_{\geq 0} \). We define a map \( d : X \times X \to G_{\geq 0} \) by \( d(x, y) = \delta \) if \( x \neq y \); otherwise, \( d(x, x) = 0 \). Then \( d \in \text{Met}(X; G) \), and hence \( \kappa \in \text{MG}(X) \).

In summary, combining Propositions 2.27, 2.28, 2.30, and 2.31 we obtain the following proposition:

Proposition 2.32. Let \( X \) be a topological space. The one and only one of the following holds true:

1. \( \text{MG}(X) = \emptyset \).
2. \( \text{MG}(X) = \{1\} \).
3. \( \text{MG}(X) = \{1, \omega_0\} \).
4. \( \text{MG}(X) = \{\kappa\} \) for some uncountable regular cardinal \( \kappa \).
5. \( \text{MG}(X) \) consists of and only of 1 and all regular cardinals.

Remark 2.8. Let \( X \) be a topological space. The space \( X \) satisfies the condition (C1) if and only if \( X \) is not \( G \)-metrizable for any \( G \in \text{GRP}(\kappa) \), nor \( S \)-ultrametrizable for any \( S \in \text{ORD}(\kappa) \), for any cardinal \( \kappa \). The space \( X \) satisfies the condition (C2) if and only if \( X \) is \( \mathbb{R} \)-metrizable and non-discrete and non-\( \mathbb{R}_{\geq 0} \)-ultrametrizable. The space \( X \) satisfies the condition (C3) if and only if \( X \) is \( \mathbb{R}_{\geq 0} \)-ultrametrizable and non-discrete. The space \( X \) satisfies the condition (C4) if and only if \( X \) is non-discrete and \( \kappa \)-metrizable in the sense of Sikorski. The space \( X \) satisfies the condition (C5) if and only if \( X \) is discrete. Note that \( X \) possessing an infinite metrizable gauge if and only if \( X \) satisfies any one of the conditions (C3), (C4), or (C5).
2.2.3. **Ultrametrics.** The following lemma can be proven by a similar method to the case of ordinary ultrametrics (see, for example, [41, The statement 4 in the page 3]).

**Lemma 2.33.** Let $X$ be a set and $S$ be a bottomed linearly ordered set. Let $d : X^2 \to S$ be an $S$-ultrametric on $X$. Then for all $x, y, z \in X$, the inequality $d(x, z) < d(y, z)$ implies $d(y, z) = d(x, y)$.

The following is well-known in the case of ordinary ultrametrics.

**Lemma 2.34.** Let $X$ be a set and $S$ be a bottomed linearly ordered set. Let $d : X^2 \to S$ be an $S$-ultrametric. Let $x \in X$ and $s \in S^\star$. Then, for all $y \in B(x, s; d)$, we have $B(x, s; d) = B(y, s; d)$. In particular, each $B(x, s; d)$ is clopen, and we have $d(a, b) \leq s$ for all $a, b \in B(x, s; d)$.

**Proof.** Take $z \in B(x, s; d)$, then $d(y, z) \leq d(y, x) \vee d(x, z) \leq s$. Thus, $B(x, s; d) \subseteq B(y, s; d)$. Similarly, we obtain $B(y, s; d) \subseteq B(x, s; d)$. The latter part follows from the former one. \hfill $\square$

**Remark 2.9.** The similar statement to Lemma 2.34 for open balls holds true. We omit the proof.

2.2.4. **Completeness.** Let $X$ be a non-empty set. A set $\mathcal{F}$ consisting of subsets of $X$ is said to be a filter on $X$ if the following conditions are satisfied:

1. $\emptyset \notin \mathcal{F}$ (in this paper, all filters are assumed to be proper);
2. $X \in \mathcal{F}$;
3. if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
4. if $E \subseteq X$ and $A \in \mathcal{F}$ satisfy $A \subseteq E$, then $E \in \mathcal{F}$.

Let $X$ be a topological space. We say that a filter $\mathcal{F}$ on $X$ converges to $p \in X$ if $\mathcal{F}$ contains all neighborhoods of $p$. In this case, the point $p$ is called a limit point of $\mathcal{F}$. We say that a filter $\mathcal{F}$ on $X$ has a cluster point if $\bigcap_{A \in \mathcal{F}} \text{CL}_X(A) \neq \emptyset$, where $\text{CL}_X$ is the closure operator of $X$.

Let $X$ be a set. Let $G$ be a linearly ordered Abelian group and $S$ be a linearly ordered set. Let $d$ be a $G$-metric (resp. an $S$-ultrametric). A filter $\mathcal{F}$ on $(X, d)$ is said to be Cauchy if for all $\epsilon \in G_{>0}$ (resp. $\epsilon \in S^\star$), there exists $A \in \mathcal{F}$ such that $d(x, y) < \epsilon$ for all $x, y \in A$. As defined in Section 1, we say that the space $(X, d)$ is complete if every Cauchy filter on $(X, d)$ has a limit point.

Let $X$ and $Y$ be sets, and $f : X \to Y$ be a map. Then, for a filter $\mathcal{F}$ on $X$, we define a filter $f_*\mathcal{F}$ on $Y$ by the filter generated by the set $\{ f(A) \mid A \in \mathcal{F} \}$. The filter $f_*\mathcal{F}$ is called the pushout filter of $\mathcal{F}$ by $f$.

For more discussions on filters, we refer the readers to [52].

The following lemme states that being a Cauchy filter is invariant under the uniform equivalence, which can be deduced from the definitions of Cauchy filters and the uniform equivalence.

**Lemma 2.35.** We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in \mathcal{MS}(X)$. \hfill 25
Let $d \in \text{Met}(X;G)$ or $d \in \text{UMet}(X;S)$ for some $G \in \text{GRP}(\kappa)$, or $S \in \text{ORD}(\kappa)$. Let $e \in \text{Met}(X;H)$ or $e \in \text{UMet}(X;T)$ for some $H \in \text{GRP}(\kappa)$, or $T \in \text{ORD}(\kappa)$. If $d$ and $e$ are uniformly equivalent, then every Cauchy filter $\mathcal{F}$ on $(X,d)$ is also Cauchy on $(X,e)$. Moreover, if $(X,d)$ is complete, then so is $(X,e)$.

**Lemma 2.37.** We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a cardinal with $\kappa \in \text{GR}(X)$. Let $G \in \text{GRP}(\kappa)$. If there exists a complete $G$-metric $d \in \text{Met}(X;G)$, then there exists a complete $\nabla(\kappa)$-ultrametric $h \in \text{UMet}(X;\nabla(\kappa))$.

*Proof.* Put $S = \Xi(A(G))$. Put $e = \lambda_G \circ d$. According to Lemmas 2.20 and 2.35, we observe that $e \in \text{UMet}(X;\Xi(A(G)))$ and $e$ is complete. Take a characteristic isotone embedding $l: \nabla(\kappa) \to S$, and take a map $\zeta_{S,l}: S \to \nabla(\kappa)$ in Definition 2.12. Put $h = \zeta_{S,l} \circ e$. By Lemma 2.21 we have $h \in \text{UMet}(X;\nabla(\kappa))$, and we verify that $e$ and $h$ are uniformly equivalent to each other. By Lemma 2.35 we conclude that $h$ is a complete $\nabla(\kappa)$-ultrametric.

We next show that a Cauchy filter on a generalized metric space has a linearly ordered (nested) subfamily, which plays a key role to discuss the completeness on generalized metric spaces.

**Lemma 2.37.** We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in \text{GR}(X)$. Let $G \in \text{GRP}(\kappa)$ and $d \in \text{Met}(X;G)$. Let $\mathcal{F}$ be a Cauchy filter on $(X,d)$. Let $l: \nabla(\kappa) \to G$ be a $g$-characteristic isotone embedding. Then, there exists a map $\mathcal{B}: \kappa \to \mathcal{F}$ such that

1. each $\mathcal{B}(\alpha)$ is a clopen subset of $X$, namely, the set $\mathcal{B}(\alpha)$ is closed and open in $X$;
2. for all $\alpha < \kappa$, and for all $x,y \in \mathcal{B}(\alpha)$, we have $d(x,y) \leq l(\alpha)$;
3. if $\alpha \leq \beta$, then $\mathcal{B}(\alpha) \supseteq \mathcal{B}(\beta)$, namely, the map $\mathcal{B}$ is an isotone map from $(\kappa,\leq)$ to the ordered set $(\{\mathcal{B}(\alpha) | \alpha < \kappa\},\supseteq)$.

*Proof.* Put $S = \Xi(A(G))$, and $e = \lambda_G \circ d \in \text{UMet}(X;S)$. Since $(X,d)$ and $(X,e)$ are uniformly equivalent to each other (see Lemma 2.20), by Lemma 2.35 the filter $\mathcal{F}$ is also Cauchy on $(X,e)$. By this observation, for each $s \in S^\star$, there exists $A \in \mathcal{F}$ such that $e(x,y) \leq s$ for all $x,y \in A$. Then, by the definition of filters, we observe that for all $s \in S^\star$, there exists $c(s) \in X$ such that $B(c(s),s;e) \in \mathcal{F}$. We define a map $\mathcal{B}: \kappa \to \mathcal{F}$ by $\mathcal{B}(\alpha) = B(c(l(\alpha)),l(\alpha);e)$. Then, by Lemma 2.34 the map $\mathcal{B}$ satisfies the conditions (1) and (2).

We next prove the condition (3). Take $\alpha, \beta < \kappa$ with $\alpha \leq \beta$. Since $\mathcal{F}$ is a filter, we have $\mathcal{B}(\alpha) \cap \mathcal{B}(\beta) \neq \emptyset$. Take $p \in \mathcal{B}(\alpha) \cap \mathcal{B}(\beta)$. Then, by Lemma 2.34 we have $\mathcal{B}(\alpha) = B(p,l(\alpha);e)$ and $\mathcal{B}(\beta) = B(p,l(\beta);e)$. By $l(\beta) \leq l(\alpha)$, we obtain $\mathcal{B}(\alpha) \supseteq \mathcal{B}(\beta)$. This completes the proof.

Let $\kappa$ be a regular cardinal. Let $X$ be a topological space with $\kappa \in \text{GR}(X)$. Let $G \in \text{GRP}(\kappa)$ and $d \in \text{Met}(X;G)$. A map from $\kappa$ to
Lemma 2.38. We assume that \( X \) is a topological space possessing an infinite metrizable gauge. Let \( \kappa \) be a regular cardinal with \( \kappa \in \mathcal{MS}(X) \). Let \( G \in \text{GRP}(\kappa) \) and \( d \in \text{Met}(X; G) \). Then the space \((X, d)\) is complete if and only if it is \( \kappa \)-complete.

Proof. First assume that \((X, d)\) is complete. Take a \( \kappa \)-sequence \( \{x_\alpha\}_{\alpha<\kappa} \). By considering the filter generated by \( \{\{x_\alpha \mid \beta \leq \alpha\} \mid \beta < \kappa\} \), and applying the completeness to this filter, we observe that \( \{x_\alpha\}_{\alpha<\kappa} \) is convergent. Then, the space \((X, d)\) is \( \kappa \)-complete.

We next assume that \((X, d)\) is \( \kappa \)-complete. Take an arbitrary Cauchy filter \( F \) on \( X \). We take a \( g \)-characteristic isotone embedding \( l: \nabla(\kappa) \to G \) and take a map \( \mathfrak{B}: \kappa \to F \) satisfying the conditions in Lemma 2.37 associated with the map \( l \). For each \( \alpha < \kappa \), we take \( x_\alpha \in \mathfrak{B}(\alpha) \). Then, by (2) and (3) in Lemma 2.37, the \( \kappa \)-sequence \( \{x_\alpha\}_{\alpha<\kappa} \) is Cauchy on \((X, d)\). Since \((X, d)\) is \( \kappa \)-complete, there exists a limit \( p \) of \( \{x_\alpha\}_{\alpha<\kappa} \). According to the condition (2) in Lemma 2.37, we have \( \bigcap_{\alpha<\kappa} \mathfrak{B}(\alpha) = \{p\} \). Then, the filter \( F \) converges to \( p \), and hence \((X, d)\) is complete. This finishes the proof. \( \square \)

Lemma 2.39. Let \( \kappa \) be a cardinal. Let \( X \) be a finally \( \kappa \)-compact topological space. If a family \( \{A_\alpha\}_{\alpha<\kappa} \) of closed subsets of \( X \) satisfies that for all \( \theta < \kappa \) we have \( \bigcap_{\beta<\theta} A_\beta \neq \emptyset \), then \( \bigcap_{\alpha<\kappa} A_\alpha \neq \emptyset \).

Proof. Supposing \( \bigcap_{\alpha<\kappa} A_\alpha = \emptyset \), and applying the final \( \kappa \)-compactness to \( \{X \setminus A_\alpha\}_{\alpha<\kappa} \), we obtain a contradiction. \( \square \)

Lemma 2.40. We assume that \( X \) is a topological space possessing an infinite metrizable gauge. Let \( \kappa \) be a regular cardinal with \( \kappa \in \mathcal{MS}(X) \). Let \( G \in \text{GRP}(\kappa) \). If \( X \) is finally \( \kappa \)-compact, then for all \( d \in \text{Met}(X; G) \), the space \((X, d)\) is complete.
Proof. Let \( F \) be a Cauchy filter on \((X,d)\). We take a \( g \)-characteristic isotone embedding \( l : \bigvee(\kappa) \to G \), and take a map \( \mathcal{B} : \kappa \to F \) satisfying the conditions in Lemma 2.37 associated with the map \( l \). By the condition (3) in Lemma 2.37, for every \( \theta < \kappa \), we have \( \bigcap_{\beta < \theta} \mathcal{B}(\beta) \neq \emptyset \).

Since \( X \) is finally \( \kappa \)-compact, by Lemma 2.39, we have \( \bigcap_{\alpha < \kappa} \mathcal{B}(\alpha) \neq \emptyset \). By the conditions (2) in Lemma 2.37, there exists \( p \in X \) with \( \bigcap_{\alpha < \kappa} \mathcal{B}(\alpha) = \{ p \} \). Then, \( F \) converges to \( p \). Thus, the space \((X,d)\) is complete. \( \square \)

Remark 2.10. Let \( \kappa \) be a cardinal. We say that a filter \( F \) is a \( \kappa \)-filter if the intersection of less than \( \kappa \) many members of \( F \) belongs to \( F \). Using the notion of \( \kappa \)-filters, the proof of Lemma 2.40 can be translated as follows: Lemma 2.37 states that every Cauchy filter \( F \) on \((X,d)\) contains a Cauchy \( \kappa \)-filter \( G \). Since \( X \) is finally \( \kappa \)-compact, by [4, Proposition 2.2], the filter \( G \) has a cluster point. Since \( G \) is Cauchy, it is convergent, and hence so does \( F \). This means that \((X,d)\) is complete.

2.3. Examples. We shall provide some examples of generalized metric spaces and ultrametrics spaces.

Definition 2.13. Let \( G \) be a linearly ordered Abelian group. We define maps \( D[\text{abs}], D[\lambda_G] : G \times G \to G \) by \( D[\text{abs}](x,y) = |x - y| \) and \( D[\lambda_G](x,y) = \lambda_G(x - y) \).

Proposition 2.41. Let \( G \) be a linearly ordered Abelian group. Then, we have \( D[\text{abs}] \in \text{Met}(G;G) \). Moreover, if \( G \in \text{GRP}(\kappa) \) for some infinite cardinal \( \kappa \), we have \( D[\lambda_G] \in \text{UMet}(G;\kappa) \).

Proof. Similarly to the proof that the order topology on \( \mathbb{R} \) are generated by \( |*| \), we can prove that the order topology is generated by the absolute value \( \text{abs} : G \to G_{\geq 0} \). By Proposition 2.8, we observe that \( D[\text{abs}] \) and \( D[\lambda_G] \) generate the same topology on \( G \). This finishes the proof. \( \square \)

Definition 2.14. Let \( S \) be a bottomed linearly ordered set. We define an ultrametric \( M_S \) by

\[
M_S(x,y) = \begin{cases} 
\bigcirc \, & \text{if } x = y; \\
x \lor y & \text{if } x \neq y,
\end{cases}
\]

where \( \lor \) means the maximum operator on \( S \). Then \( M_S \) is an \( S \)-ultrametric.

Remark 2.11. The metric \( M_S \) is a generalization of the Laflamme–Pouzet–Sauer’s construction of ultrametrics [10 Proposition 2], which also can be found in [26] and [13].

Lemma 2.42. Let \( S \) be a bottomed linearly ordered set. Let \( \kappa \) be a regular cardinal. Then the following statements hold true:

1. The set \( S^\star \) is a discrete subset of \((S,M_S)\).
If $S^\star$ does not have the least element, then the point $\bullet_S \in S$ is the unique accumulation point of $(S, M_S)$.

**Proof.** We first show the statement (1). Take arbitrary $x \in S^\star$. By the definition of $M_S$, we have $U(x, x; M_S) = \{x\}$. Thus, the set $S^\star$ is discrete.

We next verify the statement (2). Take arbitrary $x \in S^\star$. Since $S^\star$ does not have the least element, we can take $y \in S^\star$ with $y < x$. Since we have $y \in U(\bullet_S, x; M_S)$, the point $\bullet_S$ is an accumulation point of the space $(S, M_S)$.

□

**Remark 2.12.** Let $S$ be a bottomed linearly ordered set. The order topology on $S$ is not induced from $M_S$ in general.

Let $\kappa$ be a cardinal. Note that we have $\nabla(\kappa) = \kappa + 1$ as sets, and hence we obtain $\kappa \subseteq \nabla(\kappa)$, and $\kappa = \bullet_{\nabla(\kappa)} \in \nabla(\kappa)$. Applying Lemma 2.42 to the order set $\nabla(\kappa)$, we obtain:

**Corollary 2.43.** Let $\kappa$ be a regular cardinal. Then the following statements hold true:

1. The set $\kappa = \{\alpha \mid \alpha < \kappa\}$ is a discrete subset of $(\nabla(\kappa), M_{\nabla(\kappa)})$.
2. The point $\kappa \in \nabla(\kappa)$ is the unique accumulation point of the space $(\nabla(\kappa), M_{\nabla(\kappa)})$.
3. The $\kappa$-sequence $\{\alpha\}_{\alpha < \kappa}$ in $\nabla(\kappa)$ converges to the point $\kappa \in \nabla(\kappa)$.

**Proof.** The statements (1) and (2) are deduced from (1) and (2) in Lemma 2.42. The statement (3) follows from the statement (2) and the definition of $M_{\nabla(\kappa)}$. □

3. Retractions

3.1. **Proof of Theorem 1.1.** To show Theorem 1.1, we first prove the existence of Lipschitz retractions under certain conditions.

**Definition 3.1.** Let $X$ be a set. Let $G$ be a linearly ordered Abelian group. Let $d$ be a $G$-metric on $X$. Then, we define $\varrho_{d,A} : X \to G$ by

$$
\varrho_{d,A}(x) = \inf \{ d(x, a) \mid a \in A \}
$$

if the infimum in right hand side exists. If $d$ is an $S$-ultrametric on $X$ for a linearly ordered set $S$, similarly to the case of $G$-metrics, the function $\varrho_{d,A}$ is defined as $\varrho_{d,A}(x) = \inf \{ d(x, a) \mid a \in A \}$. Note that the infimum is taken in $S$ even if $S$ is a subset of some other linearly ordered set. Thus, we should actually denote by $\varrho_{S,d,A}(x)$ rather than $\varrho_{d,A}(x)$; however, in this paper, there are no confusions on sets where we take the infimum.

To show Theorem 3.1, we use a celebrated construction of Lipschitz retractions in the proof of [5, Theorem 2.9].
Theorem 3.1. Let $\kappa$ be a regular cardinal. Let $X$ be a topological space with $\kappa \in \mathcal{M}(X)$. Let $G \in \text{GRP}(\kappa)$, and $S$ be a Dedekind complete $g$-characteristic subset of $G$ (in this case, we have $\text{UMet}(X; S) \neq \emptyset$). Let $d \in \text{UMet}(X; S)$. Let $A$ be a closed subset of $X$. Let $\tau \in G$ with $1 < \tau$. Then there exists a retraction $r : X \rightarrow A$ satisfying that for all $x, y \in X$, we have $d(r(x), r(y)) \leq \tau^2 d(x, y)$.

Proof. Take a well-ordering $\leq$ on $X$. We write $x < y$ if $x \leq y$ and $x \neq y$. Remark that the well-ordering $\leq$ is not related to the topology of $X$ nor the order on $G$ in general. Since $S$ is Dedekind complete, the value $\varrho_{d, A}(x)$ always exists. For every $x \in X$, we define

$$A[x] = \{ a \in A \mid \tau^{-1} \cdot d(x, a) \leq \varrho_{d, A}(x) \}.$$ 

By $1 < \tau$, we have $\varrho_{d, A}(x) < \tau \cdot \varrho_{d, A}(x)$, and hence there exits $a \in A$ such that $d(x, a) \leq \tau \cdot \varrho_{d, A}$. Thus, each $A[x]$ is non-empty. Note that if $a \in A$, then $A[a] = \{ a \}$.

We define a map $r : X \rightarrow A$ by $r(x) = \min_{\leq} A[x]$, namely, $r(x)$ is the minimal element of $A[x]$ with respect to the well-ordering $\leq$. Since $A[a] = \{ a \}$ for all $a \in A$, we have $r(a) = a$ for all $a \in A$.

Thus, to finish the proof, we only need to show that for all $x, y \in X$, we have $d(r(x), r(y)) \leq \tau^2 d(x, y)$. For the sake of contradiction, we suppose that there exist $x, y \in X$ such that

$$(3.1) \quad \tau^2 d(x, y) < d(r(x), r(y)).$$

We may assume that $r(x) \prec r(y)$. Depending on whether $d(y, r(x)) \leq d(y, r(y))$ holds true or not, we divide the proof into two cases.

Case 1 ($d(y, r(x)) \leq d(y, r(y))$): In this case, we obtain $r(x) \in A[y]$. Since $r(x) \prec r(y)$, this contradicts the minimality of $r(y)$ in $A[y]$.

Case 2 ($d(y, r(y)) < d(y, r(x))$): We first prove that the three values $d(r(x), x)$, $d(r(x), y)$, and $d(r(x), d(y))$ are equal to each other. This means that the four points $\{r(x), r(y), x, y\}$ form a trigonal pyramid whose lengths of edges starting with $r(x)$ are equal to each other. We put $D = d(r(x), r(y))$. Applying Lemma 2.33 to $\{ y, r(x), r(y) \}$, and by $d(y, r(y)) < d(y, r(x))$, we obtain

$$(3.2) \quad D = d(r(x), r(y)) = d(r(x), y).$$

By $1 < \tau$ and (3.1), we have $d(x, y) < D$. Since $D = d(y, r(x))$, by applying Lemma 2.33 to $\{ x, y, r(x) \}$ and by $d(x, y) < D$, we have

$$(3.3) \quad d(r(x), x) = d(r(x), y) = D.$$ 

This completes our first purpose.

We next determine which two values in $\{ d(x, y), d(x, r(y)), d(y, r(y)) \}$ are equal to each other. By (3.1), by the definitions of $r(x)$ and $A[x]$, and by $r(y) \in A$, we have

$$\tau^{-1} \cdot D = \tau^{-1} \cdot d(x, r(x)) \leq \varrho_{d, A}(x) \leq d(x, r(y)).$$

30
This implies
\[(3.4)\quad \tau^{-1} \cdot D \leq d(x, r(y)).\]
By \((3.1), (3.4)\) and \(1 < \tau\), we have \(d(x, y) < d(x, r(y))\). By this inequality and by applying Lemma 2.33 to \(\{x, y, r(x)\}\), we obtain
\[(3.5)\quad d(r(y), x) = d(r(y), y).\]

From these observations, we shall deduce a contradiction. By \(r(x) < r(y)\), we see that \(r(x) \not\in A[y]\). Thus, \(\varrho_{d,A}(x) < \tau^{-1} \cdot d(r(x), y)\). Since \(D = d(r(x), y)\), there exists \(b \in A\) such that \(d(x, b) < \tau^{-1} \cdot D\). By \((3.1), (3.4)\), and \((3.5)\), and by \(b \in A\), we observe that
\[d(x, y) \leq \tau^{-1} \cdot d(x, r(y)) = \tau^{-1} \cdot d(y, r(y)) \leq \varrho_{d,A}(y) \leq d(y, b)\]
Namely, \(d(x, y) < d(y, b)\). Then, applying Lemma 2.33 to \(\{x, y, b\}\), we have \(d(x, b) = d(y, b)\). By \(d(y, b) < \tau^{-1} \cdot D\) and \(D = d(x, r(x))\), and by \(r(x) \in A[x]\), we have
\[d(x, b) = d(y, b) < \tau^{-1} \cdot d(x, r(x)) \leq \varrho_{d,A}(x)\]
Namely, \(d(x, b) < \varrho_{d,A}(x)\), which contradicts \(\varrho_{d,A}(x) \leq d(x, b)\) (recall that \(b \in A\).

In any case, we obtain a contradiction. Therefore, we conclude that \(d(r(x), r(y)) \leq \tau^2 d(x, y)\) for all \(x, y \in X\). This completes the proof. \(\square\)

**Proof of Theorem 1.1** We assume that \(X\) is a topological space possessing an infinite metrizable gauge. Let \(\kappa\) be a regular cardinal with \(\kappa \in \mathcal{M}(X)\). Let \(G \in \text{GRP}(\kappa)\) and \(d \in \text{Met}(X; G)\). Let \(A\) be a non-empty closed subset of \(X\). The latter part of the theorem follows from the former one. We only need to prove the former part of Theorem 1.1.

Put \(S = \Xi(A(G))\), and \(e = \lambda_G \circ d\). By Lemma 2.20, we have \(e \in \text{UMet}(X; S)\). Since \(S\) can be regarded as a characteristic subset of \(S^\#\) (see Lemma 2.2), and since \(S^\#\) can be regarded as a g-characteristic subset of \(\mathbb{P}(S^\#)\) (see Proposition 2.12), we can considered that \(e \in \text{UMet}(X; S^\#)\) and \(e \in \text{Met}(X; \mathbb{P}(S^\#))\) (see Lemma 2.23). Take \(\tau \in \mathbb{P}(S^\#)\) with \(1 < \tau\). Since \(S^\#\) is Dedekind complete, we can apply Theorem 3.1 to \(X, A, e,\) and \(\tau\). Thus, we obtain a retraction \(r: X \to A\) such that \(e(r(x), r(y)) \leq \tau^2 e(x, y)\) for all \(x, y \in X\). In particular, the map \(r\) is uniformly continuous with respect to \(e\). Since \(d\) and \(e = \lambda_G \circ d\) are uniformly equivalent to each other (see Lemma 2.20), the retraction \(r: X \to A\) is uniformly continuous with respect to \(d\). This finishes the proof of Theorem 1.1. \(\square\)

3.2. **Applications.** In this subsection, using the existence of retractions, we provide a characterization of the closedness of subsets of generalized metric spaces.
Let $X$ be a topological space and $d \in \operatorname{Met}(X; G)$ or $d \in \operatorname{UMet}(X; S)$. A map $f : X \to X$ is said to be $1$-Lipschitz if we have $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$.

**Proposition 3.2.** We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in \mathcal{M}_0(X)$. Let $A$ be a closed subset of $X$. Let $G \in \operatorname{GRP}(\kappa)$. Let $h \in \operatorname{Met}(X; G)$. Let $r : X \to A$ be a retraction. We define a $G$-metric $k$ on $X$ by

$$k(x, y) = h(x, y) \vee h(r(x), r(y)).$$

Then we have $k \in \operatorname{Met}(X; G)$ and the map $r$ is $1$-Lipschitz with respect to the $G$-metric $k$.

**Proof.** Since $r$ is continuous, we obtain $k \in \operatorname{Met}(X; G)$. Since $r$ is a retraction, for all $x, y \in X$, we have

$$k(r(x), r(y)) = h(r(x), r(y)) \vee h(r(r(x)), r(y)))$$

$$= h(r(x), r(y)) \vee h(r(x), r(y)) = h(r(x), r(y))$$

$$\leq k(x, y).$$

Namely, the map $r$ is $1$-Lipschitz with respect to $k$. \qed

Let $X$ be a topological space with $\mathcal{M}_0(X) \neq \emptyset$ and let $S$ be a linearly ordered set with $\operatorname{UMet}(X; S) \neq \emptyset$. Let $d \in \operatorname{UMet}(X; S)$. A subset $A$ of $X$ is said to be proximal if for all $x \in X$ there exists $a \in A$ such that

$$d(x, a) = \inf \{ d(x, z) \mid z \in A \}.$$

The proximality for $G$-metric spaces is defined in the same way.

The following theorem is a characterization of the proximality in general ultrametric spaces using $1$-Lipschitz maps. The proof can be seen in [3, Theorem 4.6 and Proposition 2.6].

**Theorem 3.3.** We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in \mathcal{M}_0(X)$. Let $A$ be a closed subset of $X$. Let $S \in \operatorname{ORD}(\kappa)$, and $d \in \operatorname{UMet}(X; S)$. Then the following two statements are equivalent to each other.

1. The set $A$ is a proximal subset of $(X, d)$.
2. There exists a $1$-Lipschitz retraction $r : X \to A$.

Using Proposition 3.2 and Theorems 1.1 and 3.3 we obtain a characterization of the closedness in a space possessing an infinite metrizable gauge.

**Corollary 3.4.** We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in \mathcal{M}_0(X)$. Let $A$ be a subset of $X$. Then the following statements are equivalent to each other.

1. The set $A$ is closed.
2. The set $A$ is a retract of $X$. 

32
(3) For all $S \in \text{ORD}(\kappa)$, there exists $d \in \text{UMet}(X; S)$ such that $A$ is a proximal subset of $(X, d)$.

(4) For all $S \in \text{ORD}(\kappa)$, there exists $d \in \text{UMet}(X; S)$ such that $A$ is a 1-Lipschitz retraction of $(X, d)$.

Proof. By Theorem 1.1, we obtain the implication $\text{(1)} \implies \text{(2)}$.

Since $X$ is Hausdorff, we obtain the implication $\text{(2)} \implies \text{(1)}$.

By Proposition 3.2, we obtain the implication $\text{(2)} \implies \text{(4)}$.

Theorem 3.3 means the equivalence $\text{(3)} \iff \text{(4)}$.

The condition $\text{(1)}$ literally implies the condition $\text{(2)}$. Thus we have the implication $\text{(1)} \implies \text{(2)}$. \hfill \Box

4. EXTENSORS OF ULTRAMETRICS AND METRICS OF HIGH POWER

4.1. Proof of Theorem 1.2. We now introduce a metric vanishing on a given closed subset. The construction is an analogue of [22].

Definition 4.1. Let $X$ be a topological space with $M\mathfrak{S}(X) \neq \emptyset$. Let $A$ be a closed subset of $X$. Let $G$ be a linearly ordered Abelian group with $\text{Met}(X; G) \neq \emptyset$. Let $S$ be a Dedekind complete $g$-characteristic subset of $G$. Let $h \in \text{UMet}(X, S)$. We define a symmetric function $\Theta[S, h, A] : X \times X \to S$ by

\[ \Theta[S, h, A](x, y) = h(x, y) \wedge (\varrho_{h,A}(x) \lor \varrho_{h,A}(y)) \]

Note that since $S$ is Dedekind complete, the function $\varrho_{h,A}$ always exists.

Proposition 4.1. We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in M\mathfrak{S}(X)$. Let $A$ be a closed subset of $X$. Let $G \in \text{GRP}(\kappa)$. Let $S$ be a Dedekind complete $g$-characteristic subset of $G$. Let $h \in \text{UMet}(X, S)$. Then the following properties are satisfied:

\begin{enumerate}
  \item the map $\Theta[S, h, A]$ is an $S$-pseudo-ultrametric on $X$, namely, it satisfies the condition $(\text{12})-(\text{15})$ in Definition 7.2;
  \item for all $x, y \in A$, we have $\Theta[S, h, A](x, y) = \bigcirc_S(= \varrho_G)$;
  \item the restriction $\Theta[S, h, A]|_{(X \setminus A) \times (X \setminus A)}$ is an $S$-ultrametric on the set $X \setminus A$, and it generates the same topology of $X \setminus A$.
\end{enumerate}

Proof. We first prove the statement $\text{(1)}$. The proof of the triangle inequality is similar to [22]. By the definition of $\varrho_{h,A}$, for all $x, y, z \in X$, we obtain the following:

\begin{enumerate}
  \item $h(x, y) \leq h(x, z) \lor h(z, y)$
  \item $\varrho_{h,A}(x) \lor \varrho_{h,A}(y) \leq (\varrho_{h,A}(x) \lor \varrho_{h,A}(z)) \lor h(z, y)$
  \item $\varrho_{h,A}(x) \lor \varrho_{h,A}(y) \leq h(x, z) \lor (\varrho_{h,A}(z) \lor \varrho_{h,A}(y))$
  \item $\varrho_{h,A}(x) \lor \varrho_{h,A}(y) \leq ((\varrho_{h,A}(x) \lor \varrho_{h,A}(z)) \lor (\varrho_{h,A}(z) \lor \varrho_{h,A}(y)))$
\end{enumerate}

These inequalities imply that $\Theta[S, h, A]$ satisfies the triangle inequality. The remaining part follows from the definition of $\Theta[S, h, A]$.

By the definitions of $\varrho_{h,A}$ and $\Theta[S, h, A]$, the statement $\text{(2)}$ is true.
We now show that the statement (3). Put $\Theta = \Theta(S,h,A)$. By the definition of $\Theta$, we verify that $\Theta$ is an $S$-ultrametric on $X \setminus A$. It suffices to show that $\Theta|_{X \setminus A \times (X \setminus A)}$ and $h|_{X \setminus A \times (X \setminus A)}$ generates the same topology.

Take arbitrary $a \in X \setminus A$, and $\epsilon \in G_{>0}$. By $G \in \text{GRP}(\kappa)$, the set $G_{>0}$ does not have the least element, and hence we can take $\delta \in G_{>0}$ such that $\delta < \min\{\epsilon, g_{h,A}(a)\}$. Take $x \in U(a, \delta, h)$. Take arbitrary $p \in A$. Since $\delta < g_{h,A}(a)$, we have $\delta < h(a, p)$. By $h(x, a) < \delta$, we have $h(x, a) < h(a, p)$. According to Lemma 2.33, we obtain $\Theta(a, x) = h(x, a)$. By $\delta < \epsilon$, we conclude that $U(a, \delta; h) \subseteq U(a, \epsilon; \Theta)$.

We next prove the converse inclusion. Take arbitrary $a \in X \setminus A$ and $\epsilon \in G_{>0}$. By $G \in \text{GRP}(\kappa)$, the set $G_{>0}$ does not have the least element, and hence we can take $\delta \in G_{>0}$ with $\delta < \min\{\epsilon, g_{h,A}(a)\}$. Take $x \in B(a, \delta; \Theta)$. Then, we have

\[(4.1) \quad \Theta(a, x) = h(a, x) \wedge (g_{h,A}(a) \lor g_{h,A}(x)) < \delta.\]

For the sake of contradiction, suppose that $\Theta(a, x) = (g_{h,A}(a) \lor g_{h,A}(x))$. Then, we have $(g_{h,A}(a) \lor g_{h,A}(x)) < \delta$, and hence $g_{h,A}(a) < \delta$. This contradicts $\delta < g_{h,A}(a)$ (the definition of $\delta$). Thus, we have $\Theta(a, x) = h(a, x)$. By (4.1) and $\delta < \epsilon$, we have $h(a, x) < \epsilon$, which implies that $U(a, \delta; \Theta) \subseteq U(a, \epsilon; h)$. This completes the proof. \qed

**Definition 4.2.** Let $X$, $Y$, and $Z$ be sets. Let $f : X \to Y$ and $d : Y \times Y \to Z$ be maps. Then, we define a map $f^*d : X \times X \to Z$ by $f^*d(x, y) = d(f(x), f(y))$. This map is called the pullback of $d$ by $f$.

**Definition 4.3.** Let $X$ be a topological space with $M_S(X) \neq \emptyset$. Let $A$ be a closed subset of $X$. Let $G$ be a linearly ordered Abelian group with $\text{Met}(X; G) \neq \emptyset$. Let $S$ be a Dedekind complete $g$-characteristic subset of $G$. Let $h \in \text{UMet}(X, S)$. Let $d \in \text{Met}(A; G)$. Let $r : X \to A$ be a uniformly continuous retraction with respect to $h$. We define a map $\Psi[S, h, r, A, d] : X \times X \to G$ by

\[\Psi[S, h, r, A, d](x, y) = r^*d(x, y) \lor \Theta[S, h, A](x, y),\]

where $r^*d$ is the pullback of $d$ by $r$.

**Lemma 4.2.** Let $X$ be a topological space with $M_S(X) \neq \emptyset$. Let $A$ be a closed subset of $X$. Let $G$ be a linearly ordered Abelian group with $\text{Met}(X; G) \neq \emptyset$. Let $S$ be a Dedekind complete $g$-characteristic subset of $G$. Let $h \in \text{UMet}(X, S)$. Let $d \in \text{Met}(A; G)$. Let $r : X \to A$ be a uniformly continuous retraction with respect to $h$. Then the map $\Psi[S, h, r, A, d]$ is a $G$-metric on $X$. Moreover, if $d$ is an $S$-ultrametric, then $\Psi[S, h, r, A, d]$ is an $S$-ultrametric.
Proof. Put $\Psi = \Psi[S, h, r, A, d]$. By the statements (2) and (3) in Proposition 4.1 and by $d \in \text{Met}(A; G)$, we observe that $\Psi$ satisfies the condition (M1) in Definition 1.1. By the statement (1) in Proposition 4.1, the map $\Psi$ satisfies the condition (M2)–(M4). From the fact that $(x + y) \vee (u + v) \leq (x \vee u) + (y \vee v)$ for all $x, y, u, v \in G$, we deduce the triangle inequality (the condition (M5)) for $\Psi$. The latter part can be proven in a similar way. \qed

The following lemma is proven using the ideas of Proof of (1-ii) in the proof of 31 Theorem 1.

Lemma 4.3. We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in M\mathcal{G}(X)$. Let $A$ be a closed subset of $X$. Let $G \in \text{GRP}(\kappa)$. Let $S$ be a Dedekind complete $g$-characteristic subset of $G$. Let $h \in \text{UMet}(X, S)$. Let $d \in \text{Met}(A; G)$. Then the $G$-metric $\Psi[S, h, r, A, d]$ generates the same topology of $X$.

Proof. Put $\Psi = \Psi[S, h, r, A, d]$ and $\Theta = \Theta[S, h, A]$. Since we have $h \in \text{UMet}(X, S)$, it suffices to show that $\Psi$ and $h$ generate the same topology.

Take $a \in X$ and $\epsilon \in G_{>0}$. If $a \notin A$, then we can take $\delta \in G_{>0}$ with $\delta < \min\{\epsilon, \varrho_{h, A}(a)\}$. Similarly to the proof of Proposition 4.1, we have $\Theta(a, x) = h(a, x)$. Hence $\Psi(a, x) = r^*d(a, x) \vee h(a, x)$. Since $r$ is continuous, there exists a sufficiently small $\eta \in G_{>0}$ such that $U(a, \eta; h) \subseteq U(a, \epsilon; \Psi)$. If $a \in A$, then we have $(\varrho_{h, A}(a) \vee \varrho_{h, A}(x)) = \varrho_{h, A}(x) \leq h(x, a)$. Thus, we also have $\Psi(a, x) = r^*d(a, x) \vee \varrho_{h, A}(x) \leq r^*d(a, x) \vee h(x, a)$. Since $r$ is continuous, there exists a sufficiently small $\eta \in G_{>0}$ such that $U(a, \eta; h) \subseteq U(a, \epsilon; \Psi)$.

We next prove the converse inclusion. Take arbitrary $a \in X$ and $\epsilon \in G_{>0}$.

Case 1 ($a \notin A$): This case is proven in the same way as the proof of Proposition 4.1. By $G \in \text{GRP}(\kappa)$, the set $G_{>0}$ does not have the least element, and hence we can take $\delta \in G_{>0}$ with $\delta < \min\{\epsilon, \varrho_{h, A}(a)\}$. Take $x \in U(a, \delta; \Psi)$. Then, we have

\begin{equation}
\tag{4.2}
\Theta(a, x) = h(a, x) \land (\varrho_{h, A}(a) \lor \varrho_{h, A}(x)) < \delta.
\end{equation}

For the sake of contradiction, we suppose that

$$\Theta(a, x) = (\varrho_{h, A}(a) \lor \varrho_{h, A}(x)).$$

Then, we have $(\varrho_{h, A}(a) \lor \varrho_{h, A}(x)) < \delta$. Hence $\varrho_{h, A}(a) < \delta$, which contradicts $\delta < \varrho_{h, A}(a)$. Thus, we obtain $\Theta(a, x) = h(a, x)$. By (4.2) and $\delta < \epsilon$, we have $h(a, x) < \epsilon$. This implies the inclusion $U(a, \delta; \Psi) \subseteq U(a, \epsilon; h)$.

Case 2 ($a \in A$): Take $\delta \in G_{>0}$ satisfying that

(a-1) $\delta < \epsilon$;
(a-2) $\delta < \varrho_{h, A}(a)$.  

35
(a-3) for all \( y \in X \) with \( d(a, y) < \delta \), we have \( h(a, y) < \epsilon \);
(a-4) if \( h(u, v) < \delta \), then \( h(r(u), r(v)) < \epsilon \).

Since \( d \) and \( h \) generate the same topology on \( A \), the condition (a-3) is guaranteed. Since \( r \) is uniformly continuous with respect to \( h \), the condition (a-4) is guaranteed.

Take \( x \in U(a, \delta; \Psi) \). We now verify \( h(a, x) < \epsilon \). By the definition of \( \Psi \), we have \( \epsilon(a, x) < \delta \) and \( \Theta(a, x) < \delta \). By the definitions of \( \epsilon \) and \( \Theta \), we obtain

\[
\begin{align*}
\text{(4.3)} & \quad d(a, r(x)) < \delta, \\
\text{(4.4)} & \quad h(a, x) \land (\varrho_{h, A}(a) \lor \varrho_{h, A}(x)) < \delta.
\end{align*}
\]

We first assume that \( h(a, x) \leq \delta \). Then, by (a-1), we have \( h(a, x) < \epsilon \).

We next assume that \( \delta < h(a, x) \). By (a-2) and (4.4), we have \( \varrho_{h, A}(x) < \delta < h(a, x) \). Then, there exists \( b \in A \) with \( \varrho_{h, A}(x) \leq h(b, x) < \delta \). Thus, we have

\[
\begin{align*}
\text{(4.5)} & \quad h(b, x) < \epsilon.
\end{align*}
\]

From \( h(b, x) < \delta \) and (a-1), it follows that \( h(r(b), r(x)) < \epsilon \). By \( r(b) = b \), we have

\[
\begin{align*}
\text{(4.6)} & \quad h(b, r(x)) < \epsilon.
\end{align*}
\]

By (a-3), and by (4.3), we have

\[
\begin{align*}
\text{(4.7)} & \quad h(a, r(x)) < \epsilon.
\end{align*}
\]

The inequalities (4.5), (4.6), and (4.7) imply

\[
\begin{align*}
h(x, a) & \leq h(x, b) \lor h(b, r(x)) \lor h(r(x), a) \\
& < \epsilon \lor \epsilon \lor \epsilon = \epsilon.
\end{align*}
\]

Thus, \( h(a, x) < \epsilon \).

Therefore, we conclude that \( U(a, \delta; \Psi) \subseteq U(a, \epsilon; h) \). This completes the proof that \( \Psi \) and \( h \) generate the same topology of \( X \). \( \square \)

**Definition 4.4.** Let \( X \) be a topological space with \( MG(X) \neq \emptyset \). Let \( A \) be a closed subset of \( X \). Let \( G \) be a linearly ordered Abelian group with \( Met(X; G) \neq \emptyset \). Let \( S \) be a \( g \)-characteristic subset of \( G \). Let \( h \in UMet(X, S) \). Let \( k \in UMet(A, S) \). Let \( r : X \to A \) be a uniformly continuous retraction with respect to \( h \). We define a map \( \Sigma[S, h, r, k] : X \times X \to S \) by

\[
\Sigma[S, h, r, k] = r^*k \lor h
\]

Note that \( \Sigma[S, h, r, k] \) is an \( S \)-ultrametric, and generates the same topology of \( X \) since \( h \in UMet(X, S) \).

The following lemma is proven using the ideas of Proof of (1-vi) in the proof of [31 Theorem 1].
Lemma 4.4. We assume that \( X \) is a topological space possessing an infinite metrizable gauge. Let \( \kappa \) be a regular cardinal with \( \kappa \in M_0(X) \). Let \( A \) be a closed subset of \( X \). Let \( G \in \text{GRP}(\kappa) \). Let \( S \) be a g-characteristic subset of \( G \). Let \( d \in \text{Met}(A; G) \). Let \( h \in \text{UMet}(X, S) \). Let \( r: X \to A \) be a uniformly continuous retraction with respect to \( h \). Let \( k \in \text{UMet}(A; S) \) be an \( S \)-ultrametric on \( A \) uniformly equivalent to \( d \). Then the map \( r \) is uniformly continuous with respect to \( \Sigma[S, h, r, k] \). If \( d \) and \( h \) are complete, then so is \( \Psi[S, \Sigma[S, h, r, k], r, A, d] \).

Proof. Put \( \Psi = \Psi[S, \Sigma[S, h, r, k], r, A, d] \) and \( u = \Sigma[S, h, r, k] \). Note that \( u \) is an \( S \)-ultrametric. Since \( r: X \to A \) is a retraction, for all \( x, y \in X \), we have

\[
u(r(x), r(y)) = k(r(r(x)), r(r(x))) \vee h(r(x), r(y))\]

\[= k(r(x), r(y)) \vee h(r(x), r(y)) \leq u(x, y) \vee h(r(x), r(y)).\]

Since \( r \) is uniformly continuous with respect to \( h \), by the definition of \( u \), the map \( r \) is uniformly continuous with respect to \( u \).

We next show that every Cauchy filter \( F \) on \( (X, \Psi) \) is also Cauchy in \( (X, u) \). We take g-characteristic isotone embeddings \( l, m: \nabla(\kappa) \to G \) such that

(a-1) for all \( \alpha < \kappa \), if \( x, y \in X \) satisfies \( u(x, y) \leq l(\alpha) \), then we have \( u(r(x), r(y)) \leq m(\alpha) \);

(a-2) for all \( \alpha < \kappa \), we have \( l(\alpha) < m(\alpha) \).

By the regularity of \( \kappa \) and by the uniform continuity of \( r \) with respect to \( u \), we can construct g-characteristic isotone embeddings \( l', m: \nabla(\kappa) \to G \) satisfying the condition (a-1). We define \( l: \nabla(\kappa) \to G \) by \( l(\alpha) = \min\{l'(\alpha), m(\alpha + 1)\} \) if \( \alpha < \kappa \), and by \( l(\kappa) = 0_G \). Then \( m \) is a g-characteristic isotone embedding, and the pair \( l, m: \nabla(\kappa) \to G \) satisfies the conditions (a-1) and (a-2).

For the sake of contradiction, we suppose that there exists a Cauchy filter \( F \) on \( (X, \Psi) \), which is not Cauchy on \( (X, u) \). Then, we obtain the following fact:

(Fact 1) there exists \( \epsilon \in G_{>0} \) such that for all \( F \in F \), and for all \( x, y \in F \), we have \( \epsilon \leq u(x, y) \).

Since \( l: \nabla(\kappa) \to G \) is a g-characteristic isotone embedding and \( F \) is Cauchy on \( (X, \Psi) \), we can take a map \( \mathcal{B}: \kappa \to F \) stated in Lemma 2.37 associated with the map \( l \).

According to the condition (2) in Lemma 2.37, the following statement holds true:

(Fact 2) we have \( \Psi(x, y) \leq l(\alpha) \) for all \( \alpha < \kappa \), and for all \( x, y \in \mathcal{B}(\alpha) \),

Thus, we obtain \( \varrho_{u,A}(x) \vee \varrho_{u,A}(y) \leq l(\alpha) \). Then, by (a-2), for each \( x \in \mathcal{B}(\alpha) \), there exists \( a_x \in A \) with \( u(x, a_x) < m(\alpha) \). By (a-1), we have \( u(a_x, r(x)) = u(r(x), r(a_x)) < m(\alpha) \). Then, for all \( \alpha < \kappa \), and for all
\( x \in \mathcal{B}(\alpha) \), we have \( u(r(x), x) \leq u(r(x), a_x) \vee u(a_x, x) \leq m(\alpha) \vee m(\alpha) = m(\alpha) \). By \((a.2)\), for all \( \alpha < \kappa \), and for all \( x \in \mathcal{B}(\alpha) \), we obtain
\begin{equation}
(4.8) \quad u(r(x), x) \leq m(\alpha).
\end{equation}

By \( r^*d(x, y) \leq \Psi(x, y) \), and by \((\text{Fact }2)\), we have \( d(r(x), r(y)) \leq l(\alpha) \) for all \( x, y \in \mathcal{B}(\alpha) \) and for all \( \alpha < \kappa \). Thus, the pushout filter \( r_\ast \mathcal{F} \) of \( \mathcal{F} \) by \( r \) is a Cauchy filter on \( (A, d) \).

Since \((A, d)\) is complete, there exists a limit \( p \in A \) of \( r_\ast \mathcal{F} \). Since \( d \) and \( u \) generate the same topology on \( A \), the point \( p \) is also a limit of \( r_\ast \mathcal{F} \) with respect to \( u \). Then, there exists \( \theta < \kappa \) satisfying that
\begin{equation}
(4.9) \quad m(\theta) < \epsilon;
\end{equation}
\begin{equation}
(4.10) \quad r(\mathcal{B}(\theta)) \subseteq U(p, \epsilon; u).
\end{equation}

From \((4.8)\), \((4.9)\), and \((4.10)\), it follows that for all \( x \in \mathcal{B}(\theta) \), we have
\( u(p, x) \leq u(p, r(x)) \vee u(r(x), x) < \epsilon \vee m(\alpha) \leq \epsilon. \)

Namely, we obtain \( \mathcal{B}(\theta) \subseteq U(p, \epsilon; u) \). This inclusion and the strong triangle inequality imply that \( u(x, y) < \epsilon \) for all \( x, y \in \mathcal{B}(\theta) \). This contradicts \((\text{Fact }1)\). Thus, the filter \( \mathcal{F} \) is Cauchy on \((X, u)\). This leads to the lemma.

\[ \square \]

**Remark 4.1.** The author does not know whether the metrics \( d \) and \( \Psi[S, \Sigma[S, h, r, k], r, A, d] \) in Lemma 4.4 are uniformly equivalent to each other or not.

**Proof of Theorem 1.2.** We assume that \( X \) is a topological space possessing an infinite metrizable gauge. Let \( \kappa \) be a regular cardinal with \( \kappa \in \mathcal{M}(X) \). Let \( G \in \mathcal{G}(\kappa) \). Let \( S \) be a \( g \)-characteristic subset of \( G \). Let \( A \) be a closed subset of \( X \).

Remark that since \( S \) is \( g \)-characteristic in \( G \), the set \( \lambda_G(S) \) is characteristic in the set \( \Xi(\mathcal{A}(G)) \). Fix a characteristic isotone embedding \( m: \nabla(\kappa) \to \Xi(\mathcal{A}(G)) \) such that \( m(\alpha) \in \lambda_G(S) \) for all \( \alpha \in \nabla(\kappa) \). For each \( \alpha \in \nabla(\kappa) \), we take \( l_\alpha \in S \) such that \( \lambda_G(l_\alpha) = m(\alpha) \). We define a map \( l: \nabla(\kappa) \to S \) by \( l(\alpha) = l_\alpha \). Then the map \( l \) is a characteristic isotone embedding, and the relation \( \lambda_G \circ l = m \) holds true. Let \( \xi_{\Xi(\mathcal{A}(G)), m}: \Xi(\mathcal{A}(G)) \to \nabla(\kappa) \) be a map in Definition 2.12 associated with \( m \). Put \( \xi = \xi_{\Xi(\mathcal{A}(G)), m} \). We put \( T = l(\nabla(\kappa)) \). Note that \( T \) is a Dedekind complete \( g \)-characteristic subset of \( G \) and \( T \subseteq S \).

For all \( \eta \in G_{>0} \), there exists \( \alpha \in \nabla(\kappa) \) such that \( m(\alpha + 1) < \lambda_G(\eta) \leq m(\alpha) \). In this case, we have \( \xi \circ \lambda_G(\eta) = \alpha + 1 \), and hence \( l \circ \xi \circ \lambda_G(\eta) = l(\alpha + 1) \). By the definition of \( l \), by \( \lambda_G \circ l = m \), and by \( m(\alpha + 1) < \lambda_G(\eta) \), we have \( l \circ \xi \circ \lambda_G(\eta) < \eta \). Namely, the following holds true:

\((\text{Fact }1)\) for all \( \eta \in G_{>0} \), we have \( l \circ \xi \circ \lambda_G(\eta) < \eta \).

Fix \( h \in \text{UMet}(X; T) \) (see Proposition 2.24). Using Theorem 1.1, we take a uniformly continuous retraction \( r: X \to A \) with respect to \( h \).
For each \( d \in \text{Met}(A; G) \), we put \( D[d] = \lambda_G \circ d \in \text{UMet}(A; \Xi(A(G))) \) (see Lemma 2.20). Let \( k[d] = I \circ \xi \circ D[d] \). According to Lemmas 2.21 and 2.22, we observe that \( k[d] \in \text{UMet}(X; T) \).

Put \( u[d] = \Sigma[T, h, r, k[d]] \). Then we have \( u[d] \in \text{UMet}(X; T) \). According to Lemma 2.35, the metrics \( d \) and \( D[d] \) are uniformly equivalent to each other. Thus, \( D[d] \) satisfies the assumption in Lemma 4.3. By Lemma 4.3 the map \( r \) is uniformly continuous with respect to \( u[d] \).

We define a map \( \Phi : \text{Met}(A; G) \to \text{Met}(X; G) \) by

\[
\Phi(d) = \Psi[T, u[d], r, A, d].
\]

We now verify that the map \( \Phi \) is a desired one. By Lemma 4.3, we observe that \( \Phi \) is certainly a map from \( \text{Met}(A; G) \) into \( \text{Met}(X; G) \).

We now show that \( \Phi \) satisfies (A1). Fix \( d \in \text{Met}(A; G) \) and \( \epsilon \in G_{>0} \). Take \( \eta \in G_{>0} \) with \( \eta < \epsilon \). Take arbitrary \( e \in V(d; \eta) \), which means that for all \( x, y \in A \) we have

\[
\begin{align*}
(4.11) & \quad d(x, y) < e(x, y) \lor \eta; \\
(4.12) & \quad e(x, y) < d(x, y) \lor \eta.
\end{align*}
\]

Since \( \lambda_G \) is isotone, by (4.11) and (4.12), for all \( x, y \in A \), we have

\[
\begin{align*}
(4.13) & \quad D[d](x, y) \leq D[e](x, y) \lor \lambda_G(\eta); \\
(4.14) & \quad D[e](x, y) \leq D[d](x, y) \lor \lambda_G(\eta).
\end{align*}
\]

Since \( I \) and \( \xi \) are isotone, by (Fact 1), and by (4.13), and (4.14), for all \( x, y \in A \), we obtain

\[
\begin{align*}
(4.15) & \quad k[d](x, y) \leq k[e](x, y) \lor \eta; \\
(4.16) & \quad k[e](x, y) \leq k[d](x, y) \lor \eta.
\end{align*}
\]

By (4.15) and (4.16), for all \( x, y \in X \), we have

\[
\begin{align*}
(4.17) & \quad r^*k[d](x, y) \leq r^*k[e](x, y) \lor \eta; \\
(4.18) & \quad r^*k[e](x, y) \leq r^*k[d](x, y) \lor \eta.
\end{align*}
\]

These inequalities imply that for all \( x, y \in X \), we have

\[
\begin{align*}
(4.19) & \quad u[d](x, y) \leq u[e](x, y) \lor \eta; \\
(4.20) & \quad u[e](x, y) \leq u[d](x, y) \lor \eta.
\end{align*}
\]

By (4.19) and (4.20), for all \( x, y \in X \), we obtain

\[
\begin{align*}
(4.21) & \quad \Theta[u[d], A](x) \leq \Theta[u[e], A](x) \lor \eta; \\
(4.22) & \quad \Theta[u[e], A](x) \leq \Theta[u[d], A](x) \lor \eta.
\end{align*}
\]

Thus, for all \( x, y \in X \), the following hold true:

\[
\begin{align*}
(4.23) & \quad \Theta[u[d], A](x, y) \leq \Theta[u[e], A](x, y) \lor \eta; \\
(4.24) & \quad \Theta[u[e], A](x, y) \leq \Theta[u[d], A](x, y) \lor \eta.
\end{align*}
\]
Therefore, by \( \eta < \epsilon \), we conclude that \( \Phi(e) \in \mathcal{V}(\Phi(d); \epsilon) \), and hence the map \( \Phi \) is continuous. This implies the condition (A1).

By the definition of \( \Phi \), we see that \( \Phi \) satisfies (A2).

The latter part of Lemma 1.2 implies the conditions (A1) and (A3) (note that \( S \subseteq \Theta \)).

\[
D[d \lor e] = D[d] \lor D[e] \quad \text{and} \quad k[d \lor e] = k[d] \lor k[e].
\]

By \( k[d \lor e] = k[d] \lor k[e] \), we have \( \varrho_{d[e]} \lor \varrho_{d[e]} = \varrho_{d[e]} \lor \varrho_{d[e]} \). Then \( \Theta_{k[d \lor e]} = \Theta_{k[d]} \lor \Theta_{k[e]} \), and hence \( \Phi(d \lor e) = \Phi(d) \lor \Phi(e) \). Thus, the condition (A6) is satisfied. The condition (A0) implies the condition (A3).

To show the condition (A7), we assume that there exists a complete \( G \)-metric \( D \) in \( \text{Met}(X; G) \). By Lemma [2.30], there exists a complete \( \nabla(\kappa) \)-ultrametric \( H \in \text{UMet}(X; \nabla(\kappa)) \). Put \( h' = \lim H \). By Lemmas [2.21] and [2.35], we have \( h' \in \text{UMet}(X; T) \) and \( h' \) is complete. Using Theorem 1.1, we can take a uniformly continuous retraction \( r' : X \to A \) with respect to \( h' \). According to Lemmas [2.20] and [2.35], if \( d \in \text{Met}(A; G) \) is complete, then so is \( D[d] \). Note that \( \Sigma[S, h', r', k[d]] \) is complete.

Using \( h' \) and \( r' \) instead of \( h \) and \( r \) in the argument discussed above, by Lemma 4.1, \( D[d] \) and \( d \) are uniformly equivalent to each other, we obtain a map \( \Phi : \text{Met}(A; G) \to \text{Met}(X; G) \) satisfying that if \( d \in \text{Met}(X; G) \) is complete, then so is the \( G \)-metrics \( \Phi(d) \). Namely, the map \( \Phi \) satisfies the condition (A7).

This finishes the proof of Theorem 1.2 \( \square \)

Proof of Theorem 1.3 Let \( S \) be a characteristic subset of \( \mathbb{R}_{\geq 0} \). Let \( X \) be an \( \mathbb{R}_{\geq 0} \)-ultrametrizable space.

Put \( G = \mathbb{P}(S) \). Then the set \( S \) can be considered as a g-characteristic subset of \( G \) (see Proposition 2.12). Let \( \Phi : \text{Met}(A; G) \to \text{Met}(X; G) \) be a map stated in Theorem 1.2. Note that \( \text{UMet}(X; S) \subseteq \text{Met}(X; G) \) (see Lemma 2.23). Put \( \Upsilon = \Phi|_{\text{UMet}(X; S)} \). By the condition (A5) in Theorem 1.2, we confirm that \( \Upsilon \) is a map from \( \text{UMet}(A; S) \) into \( \text{UMet}(X; S) \).

Similarly to the proof of Theorem 1.2, we observe that \( \Phi \) satisfies (H1)–(B3). In particular, from the argument verifying that \( e \in \mathcal{V}(d; \eta) \) implies \( \Phi(e) \in \mathcal{V}(\Phi(d); \epsilon) \), it follows that \( \Upsilon \) is an isometric embedding with respect to \( UD_A^S \) and \( UD_X^S \).

Note that, to prove the condition (B5), we need to confirm that there exists a complete \( S \)-ultrametric \( D \in \text{Met}(X; G) \). From [26, Proposition 2.17], there exists complete \( D \in \text{UMet}(X; S) \), and hence, we have \( D \in \text{Met}(X; G) \). Thus, the space \( X \) satisfies the assumption in Theorem 1.2 for the condition (A7). Hence the condition (B5) holds true.

This completes the proof of Theorem 1.3 \( \square \)

4.2. Proof of Theorem 1.4 We next show Theorem 1.4.

The following lemma is stated in [38, Theorem 1.3] (see also [4, Proposition 2.2]). The proof is similary to the case of ordinary metric spaces.
Lemma 4.5. We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in \mathcal{M}\mathcal{G}(X)$. Then the following are equivalent to each other:

1. Every $\kappa$-sequence in $X$ has a convergent $\kappa$-subsequence.
2. Every open cover of $X$ whose cardinal is $\kappa$ has a subcover with cardinal $< \kappa$.
3. The space $X$ is finally $\kappa$-compact.

Lemma 4.6. We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in \mathcal{M}\mathcal{G}(X)$. If $X$ is not finally $\kappa$-compact, then there exists a closed subset $A$ of $X$ such that $\text{Card}(A) = \kappa$ and $A$ is discrete.

Proof. Since $X$ is not finally $\kappa$-compact, by Lemma 4.5, there exists a $\kappa$-sequence $\{p_\alpha\}_{\alpha < \kappa}$ whose all $\kappa$-subsequence have no limits. Put $A = \{ p_\alpha \mid \alpha < \kappa \}$. Then, for all $p \in A$, there exists an open set $U_p$ such that $p \in U_p$ and $U_p \cap A = \{ p \}$. Thus, the set $A$ is discrete. Since all $\kappa$-subsequences of $\{p_\alpha\}_{\alpha < \kappa}$ have no limits, we conclude that for each $p \in A$, we have $\text{Card}(\{ \alpha < \kappa \mid p_\alpha = p \}) < \kappa$. Thus, by the regularity of $\kappa$, we see that $\text{Card}(A) = \kappa$. \hfill $\Box$

Proof of Theorem 4.4 We assume that $X$ is a topological space possessing an infinite metrizable gauge. Let $\kappa$ be a regular cardinal with $\kappa \in \mathcal{M}\mathcal{G}(X)$.

Since $\text{GRP}(\kappa) \neq \emptyset$ and $\text{ORD}(\kappa) \neq \emptyset$, we obtain the implications (3) $\implies$ (2) and (5) $\implies$ (4). According to Lemma 2.40, we have the implications (1) $\implies$ (3) and (1) $\implies$ (5).

To show (2) $\implies$ (4), we assume the statement (2). Then all $d \in \text{UMet}(X; G_{\geq 0})$ are complete. Put $S = G_{\geq 0}$. According to Proposition 2.15, we have $S \in \text{ORD}(\kappa)$. Hence the statement (4) holds true. Thus, we obtain the implication (2) $\implies$ (4).

To prove Theorem 4.4, we only need to show the implication (4) $\implies$ (1). We assume that $X$ is not finally $\kappa$-compact. Then, by Lemma 4.6, there exists a closed subset $A$ of $X$ such that $\text{Card}(A) = \kappa$ and $A$ is discrete. Take a bijection $f : A \rightarrow \kappa$, and put $g = f^{-1}$. Let $e = f^* M_\nabla(\kappa)$, where $M_\nabla(\kappa)$ is the ultrametric on $\nabla(\kappa)$ in Definition 2.14. Then $e \in \text{UMet}(A; \nabla(\kappa))$.

Note that $\kappa \subseteq \nabla(\kappa)$. Put $G = \mathcal{P}(\nabla(\kappa))$. The set $\nabla(\kappa)$ can be considered as a $g$-characteristic subset of $G$ (see Proposition 2.12), and hence $e \in \text{UMet}(X; G_{\geq 0})$. We take a map $\Phi : \text{Met}(A; G) \rightarrow \text{Met}(X; G)$ satisfying the conditions in Theorem 1.2. Put $d = \Phi(e)$. Then, by condition (A4) in Theorem 1.2, we obtain $d \in \text{UMet}(X; G_{\geq 0})$ with $d|_{A^2} = e$. Since $\{ \alpha \}_{\alpha < \kappa}$ is Cauchy in $(\nabla(\kappa), M_\nabla(\kappa))$ (see (3) in Corollary 2.43), the sequence $\{g(\alpha)\}_{\alpha < \kappa}$ is Cauchy in $(X, d)$. Since $A$ is closed, the sequence $\{g(\alpha)\}_{\alpha < \kappa}$ has no limits. Hence $(X, d)$ is not complete. Therefore we obtain the implication (4) $\implies$ (1). This finishes the proof of Theorem 1.4. \hfill $\Box$
| Symbol   | Description                                                                                                                                 |
|----------|---------------------------------------------------------------------------------------------------------------------------------------------|
| Met($X; G$) | The set of all $G$-metrics that generate the same topology on a topological space $X$.                                                        |
| $\odot S$ | The minimal element of a bottomed linearly ordered set $S$.                                                                               |
| $\Xi(L)$ | The one-point order extension of a linearly ordered set $L$. This is a ordered set by putting the minimal element $\odot \Xi(G)$ to $L$ (Definition 1.3). |
| $\chi(S)$ | The character of $S$ (Definition 1.3).                                                                                                     |
| $x \preccurlyeq y$ | For all $n \in \mathbb{Z}_{\geq 1}$, we have $n \cdot x < y$.                                                                           |
| $x \asymp y$ | $x$ and $y$ are Archimedean equivalent to each other. Namely, there exist $n, n \in \mathbb{Z}_{\geq 1}$ such that $y \leq n \cdot x$ and $x \leq m \cdot y$. |
| $[x]_\asymp$ | The Archimedean equivalence class of $x$.                                                                                                  |
| $A(G)$ | The set of all Archimedean classes of a linearly ordered Abelian group $G$.                                                               |
| $\alpha \preccurlyeq \beta$ | $x \preccurlyeq y$ or $x \asymp y$, where $\alpha = [x]_\asymp$ and $\beta = [y]_\asymp$.                                               |
| $\omega_0$ | The least infinite cardinal. $\omega_0 = \{0, 1, 2, \ldots, \}$                                                                           |
| GRP($\kappa$) | The class of all linearly ordered Abelian group $G$ with $\chi(A(G)) = \kappa$.                                                             |
| ORD($\kappa$) | The class of all bottomed linearly ordered set $S$ with $\chi(S) = \kappa$.                                                                 |
| $\forall(d; \epsilon)$ | The set of all $e \in \text{Met}(X; d)$ such that for all $x, y \in X$ we have $e(x, y) < d(x, y) \lor \epsilon$ and $d(x, y) < e(x, y) \lor \epsilon$. |
| $UD_X^{\epsilon}$ | The infimum of $\epsilon \in S \cup \{\infty\}$ such that for all $x, y \in X$ we have $d(x, y) \leq e(x, y) \lor \epsilon$, and $e(x, y) \leq d(x, y) \lor \epsilon$. |
| op($S$) | The ordered set with the dual order to $(S, \leq)$.                                                                                       |
| $\forall(\kappa)$ | $\text{op}(\kappa + 1)$ (Definition 2.1).                                                                                                 |
| $S^\#$ | The Dedekind completion of $(S, \leq)$.                                                                                                    |
| abs($x$) | The absolute value of $x$ of $G$ (Definition 2.5).                                                                                         |
| $\lambda_G(x)$ | The natural valuation (or covaluation) of $G$ (Definition 2.5).                                                                          |
| $\mathbb{H}(L)$ | The Hahn group induced from a linearly ordered set $L$.                                                                                  |
| $\mathbb{F}(G)$ | The Hahn field induced from a linearly ordered Abelian group $G$.                                                                           |
| $S^\star$ | $S^\star = S \setminus \{\odot S\}$ (Definition 2.8).                                                                                     |
| $\mathbb{P}(S)$ | $\mathbb{P}(S) = \mathbb{F}(\text{op}(S^\star))$ (Definition 2.10).                                                                      |
| Symbol | Definition |
|--------|------------|
| $\varsigma_{S,l}$ | The map defined in Definition 2.12 induced from $S \in \text{ORD}(\kappa)$ and a characteristic isotone embedding $l: \nabla(\kappa) \to S$. |
| $U(x, \epsilon; d)$ | The open ball centered at $x$ with radius $\epsilon$ by a $G$-metric or $S$-ultrametric $d$. |
| $B(x, \epsilon; d)$ | The closed ball centered at $x$ with radius $\epsilon$ by a $G$-metric or $S$-ultrametric $d$. |
| $M_S$ | The $S$-ultrametric on $S$ defined in Definition 2.14. |
| $\varrho_{d,A}(x)$ | The distance between the set $A$ and the point $x$ (Definition 3.1). |
| $\Theta[S,h,A]$ | The $S$-pseudo-ultrametric defined in Definition 4.1. |
| $\Psi[S,h,r,A,d]$ | The metric defined in Definition 4.3. |
| $f^*d$ | The pullback induced from $d$ and a map $f$ (Definition 4.2). |
| $\Sigma[S,h,r,k]$ | The metric defined in Definition 4.4. |
| $f_*\mathcal{F}$ | The pushout filter of $\mathcal{F}$ by a map $f$. |

**References**

1. H. Abels and A. Manoussos, *Topological generators of abelian lie groups and hypercyclic finitely generated abelian semigroups of matrices*, Adv. Math. **229** (2012), no. 3, 1862–1872.
2. G. Artico, U. Marconi, and J. Pelant, *On supercomplete $\omega_\mu$-metric spaces*, Bull. Polish Acad. Sci. Math. **44** (1996), no. 3, 299–310.
3. G. Artico and R. Moresco, *Some results on $\omega_\mu$-metric spaces*, Annali di Matematica Pura ed Applicata **128** (1981), no. 1, 241–252.
4. ______, *$\omega_\mu$-additive topological spaces*, Rend. Sem. Mat. Univ. Padova **67** (1982), 131–141.
5. N. Brodskiy, J. Dydak, J. Higes, and A. Mitra, *Dimension zero at all scales*, Topology Appl. **154** (2007), no. 14, 2729–2740.
6. K. A. Broughan, *A metric characterizing $\check{C}$ech dimension zero*, Proc. Amer. Math. Soc. **39** (1973), 437–440.
7. A. H. Clifford, *Note on Hahn’s theorem on ordered abelian groups*, Proc. Amer. Math. Soc. **5** (1954), 860–863.
8. A. B. Comicheo, *Generalized open mapping theorem for X-normed spaces*, p-Adic Numbers Ultrametric Anal. **11** (2019), no. 2, 135–150.
9. J. Dancis, *Each closed subset of metric space $X$ with Ind $X = 0$ is a retract*, Houston J. Math. **19** (1993), no. 4, 541–550.
10. C. Delhommé, C. Laflamme, M. Pouzet, and N. Sauer, *Indivisible ultrametric spaces*, Topology Appl. **155** (2008), no. 14, 1462–1478.
11. A. Di Concilio and C. Guadagni, *Uniform continuity in $\omega_\mu$-metric spaces and uc $\omega_\mu$-metric extendability*, Acta Math. Hungar. **150** (2016), no. 1, 153–166.
12. ______, *Hypertopologies on $\omega_\mu$-metric spaces*, Filomat **31** (2017), no. 13, 4063–4068.
13. D. Dordovskyi, O. Dovgoshey, and E. Petrov, *Diameter and diametrical pairs of points in ultrametric spaces*, p-Adic Numbers Ultrametric Anal. **3** (2011), no. 4, 253–262.
14. A. A. Dovgoshei and E. A. Petrov, *A subdominant pseudoultrametric on graphs*, Mat. Sb. **204** (2013), no. 8, 51–72.
15. O. Dovgoshey, On ultrametric-preserving functions, Math. Slovaca 70 (2020), no. 1, 173–182.
16. O. Dovgoshey, O. Martio, and M. Vuorinen, Metrization of weighted graphs, Ann. Comb. 17 (2013), no. 3, 455–476.
17. O. Dovgoshey and V. Shcherbak, The range of ultrametrics, compactness, and separability, Topology Appl. 305 (2022), Paper No. 107899, 19 pages.
18. J. Dugundji, Absolute neighborhood retracts and local connectedness in arbitrary metric spaces, Compos. Math. 13 (1958), 229–246 (1958).
19. R. Engelking, On closed images of the space of irrationals, Proc. Amer. Math. Soc. 21 (1969), 583–586.
20. A. J. Engler and A. Prestel, Valued fields, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005.
21. H. Hahn, Über die nichtarchimedischen Größensysteme, Sitz. ber. K. Akad. der Math. Nat. Kl. IIa 116 (1907), 601–655, (Hans Hahn Gesammelte Abhandlungen Band 1, Hans Hahn Collected Works Vol. 1, pp. 445–499, Springer, 1995).
22. F. Hausdorff, Erweiterung einer homöomorphie, Fund. Math. 16 (1930), 353–360.
23. M. Hausner and J. G. Wendel, Ordered vector spaces, Proc. Amer. Math. Soc. 3 (1952), 977–982.
24. A. Hayes, Uniformities with totally ordered bases have paracompact topologies, Proc. Cambridge Philos. Soc. 74 (1973), 67–68.
25. H. H. Hung, The amalgamation property for $G$-metric spaces, Proc. Amer. Math. Soc. 37 (1973), 53–58.
26. Y. Ishiki, An embedding, an extension, and an interpolation of ultrametrics, $p$-Adic Numbers Ultrametric Anal. Appl. 13 (2021), no. 2, 117–147.
27. ______, On dense subsets in spaces of metrics, (2021), preprint arXiv:2104.12450, to appear in Colloq. Math.
28. T. Jech, Set theory, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded.
29. I. Juhász, Untersuchungen über $\omega_{\mu}$-metrisierbare Räume, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 8 (1965), 129–145.
30. J. L. Kelley, General topology, Graduate Texts in Mathematics, No. 27, Springer-Verlag, New York-Berlin, 1975, Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.].
31. Nguyen Van Khue and Nguyen To Nhu, Two extensors of metrics, Bull. Acad. Polon. Sci. 29 (1981), 285–291.
32. J. Kąkol, A. Kubzdela, and W. Śliwa, A non-Archimedean Dugundji extension theorem, Czechoslov. Math. J. 63(138) (2013), no. 1, 157–164.
33. Y. Kodama, On LC"n metric spaces, Proc. Japan Acad. 33 (1957), 79–83.
34. A. Kucia and W. Kulpa, Spaces having uniformities with linearly ordered bases, Univ. Śląski w Katowicach—Prace Mat. 3 (1973), 45–50.
35. S. Kuhlmann, Ordered exponential fields, Fields Institute Monographs, vol. 12, American Mathematical Society, Providence, RI, 2000.
36. E. Michael, Selected Selection Theorems, Amer. Math. Monthly 63 (1956), no. 4, 233–238.
37. V. Niemytzki and A. Tychonoff, Beweis des satzes, dass ein metrisierbarer raum dann und nur dann kompakt ist, wenn er in jeder metrik vollständig ist, Fund. Math. 12 (1928), 118–120.
38. K. Nomizu and H. Ozeki, The existence of complete Riemannian metrics, Proc. Amer. Math. Soc. 12 (1961), 889–891.
39. P. Nyikos and H. C. Reichel, *On uniform spaces with linearly ordered bases. II. (ω₁-metric spaces)*, Fund. Math. 93 (1976), no. 1, 1–10.

40. H. Ochsenius and E. Olivos, *A comprehensive survey of non-Archimedean analysis in Banach spaces over fields with an infinite rank valuation*, Advances in ultrametric analysis, Contemp. Math., vol. 596, Amer. Math. Soc., Providence, RI, 2013, pp. 215–236.

41. C. Perez-Garcia and W. H. Schikhof, *Locally convex spaces over non-Archimedean valued fields*, Cambridge Studies in Advanced Mathematics, vol. 119, Cambridge University Press, Cambridge, 2010.

42. P. Pongsriiam and I. Tewmuwitpong, *Remarks on ultrametrics and metric-preserving functions*, Abstr. Appl. Anal. (2014), Art. ID 163258, 9.

43. S. Priess-Crampe, *Generalized Keller spaces*, Bull. Belg. Math. Soc. Simon Stevin 14 (2007), no. 5, 979–991.

44. R. Sikorski, *Remarks on some topological spaces of high power*, Fund. Math. 37 (1950), no. 1, 125–136.

45. D. J. Souppouris, *Generalized metric spaces are paracompact*, Math. Proc. Cambridge Philos. Soc. 77 (1975), 325–326.

46. I. S. Stares and J. E. Vaughan, *The Dugundji extension property can fail in ω₁-metrizable spaces*, Fund. Math. 150 (1996), no. 1, 11–16.

47. I. Stasyuk and E. D. Tymchatyn, *A continuous operator extending ultrametrics*, Comment. Math. Univ. Carolin. 50 (2009), no. 1, 141–151.

48. F. W. Stevenson and W. J. Thorn, *Results on ω₁-metric spaces*, Fund. Math. 65 (1969), no. 3, 317–324.

49. E. D. Tymchatyn and M. Zarichnyi, *A note on operators extending partial ultrametrics*, Comment. Math. Univ. Carolin. 46 (2005), no. 3, 515–524.

50. E. K. van Douwen, *Simultaneous extension of continuous functions*, Ph.D. thesis, Vrije Universiteit Amsterdam, 1975.

51. S. -T. Wang, *Remarks of ω₁-additive spaces*, Fund. Math. 55 (1964), 101–112.

52. S. Willard, *General topology*, Dover Publications, 2004; originally published by the Addison-Wesley Publishing Company in 1970.

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