The resonant drag instability (RDI): acoustic modes

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ABSTRACT

Recently, Squire & Hopkins showed any coupled dust–gas mixture is subject to a class of linear ‘resonant drag instabilities’ (RDI). These can drive large dust-to-gas ratio fluctuations even at arbitrarily small dust-to-gas mass ratios $\mu$. Here, we identify and study both resonant and non-resonant instabilities, in the simple case where the gas satisfies neutral hydrodynamics and supports acoustic waves ($\omega^2 = c_s^2 k^2$). The gas and dust are coupled via an arbitrary drag law and subject to external accelerations (e.g. gravity, radiation pressure). If there is any dust drift velocity, the system is unstable. The instabilities exist for all dust-to-gas ratios $\mu$ and their growth rates depend only weakly on $\mu$ around resonance, as $\sim \nu^{1/3}$ or $\sim \nu^{1/2}$ (depending on wavenumber). The behaviour changes depending on whether the drift velocity is larger or smaller than the sound speed $c_s$. In the supersonic regime, a ‘resonant’ instability appears with growth rate increasing without limit with wavenumber, even for vanishingly small $\nu$ and values of the coupling strength (stopping time). In the subsonic regime, non-resonant instabilities always exist, but their growth rates no longer increase indefinitely towards small wavelengths. The dimensional scalings and qualitative behaviour of the instability do not depend sensitively on the drag law or equation of state of the gas. The instabilities directly drive exponentially growing dust-to-gas-ratio fluctuations, which can be large even when the modes are otherwise weak. We discuss physical implications for cool-star winds, AGN-driven winds and torii, and starburst winds: the instabilities alter the character of these outflows and could drive clumping and/or turbulence in the dust and gas.

Key words: instabilities – turbulence – planets and satellites: formation – ISM: kinematics and dynamics – galaxies: formation.

1 INTRODUCTION

Astrophysical fluids are replete with dust, and the dynamics of the dust–gas mixture in these ‘dusty fluids’ are critical to astrochemistry, star and planet formation, ‘feedback’ from stars and active galactic nuclei (AGN) in galaxy formation, the origins and evolution heavy elements, cooling in the interstellar medium, stellar evolution in cool stars, and more. Dust is also ubiquitous as a source of extinction or contamination in almost all astrophysical contexts. As such, it is critical to understand how dust and gas interact, and whether these interactions produce phenomena that could segregate or produce novel dynamics or instabilities in the gas or dust.

Recently, Squire & Hopkins (2018b) (henceforth SH) showed that there exists a general class of previously unrecognized instabilities of dust–gas mixtures. The SH ‘resonant drag instability’ (RDI) generically appears whenever a gas system that supports some wave or linear perturbation mode (in the absence of dust) also contains dust moving with a finite drift velocity $v_d$, relative to the gas. This is unstable at a wide range of wavenumbers, but the fastest growing instabilities occur at a ‘resonance’ between the phase velocity ($v_p = \omega k$) of the ‘natural’ wave that would be present in the gas (absent dust), and the dust drift velocity projected along the wavevector direction ($v_d \cdot k \approx v_p$). Some previously well-studied instabilities – most notably the ‘streaming instability’ of grains in protostellar discs (Youdin & Goodman 2005), which is related to a resonance with the disc’s epicyclic oscillations (i.e. has maximal growth rates when $v_d \cdot k \approx \Omega$) – belong to the general RDI category. These instabilities directly generate fluctuations in the dust-to-gas ratio and the relative dynamics of the dust and gas, making them potentially critical for the host of phenomena above (see e.g. Chiang & Youdin 2010 for applications of the disc streaming instability).

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We will refer to this second component as ‘dust’ henceforth. For Balbus & Latter (2011), interacting via some generalized drag law. Consider a mixture of gas and a second component which can be linear. PERTURBATIONS 2 BASIC EQUATIONS AND LINEAR perturbations to those discussed in previous literature (Section 7), before considering applications to different astrophysical scales where our analysis breaks down (Section 6), and the relation of these instabilities to those discussed in previous literature (Section 7), before considering applications to different astrophysical systems including cool-star winds, starbursts, AGN obscuring torii and narrow-line regions, and proto-planetary discs (Section 8). We conclude in Section 9.

2 BASIC EQUATIONS AND LINEAR PERTURBATIONS

2.1 General case with constant streaming

Consider a mixture of gas and a second component which can be approximated as a pressure-free fluid (at least for linear perturbations; see Youdin & Goodman 2005 and appendix A of Jacquet, Balbus & Latter 2011), interacting via some generalized drag law. We will refer to this second component as ‘dust’ henceforth. For now we consider an ideal, inviscid gas, so the system is described by mass and momentum conservation for both fluids:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = -\nabla P + g + \frac{\rho_d}{\rho} \frac{(v - u)}{t_s},
\]

\[
\frac{\partial \rho_d}{\partial t} + \nabla \cdot (\rho_d \mathbf{v}) = 0,
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\frac{(v - u)}{t_s} + g + \mathbf{a},
\]

where \((\rho, \mathbf{u})\) and \((\rho_d, \mathbf{v})\) are the density and velocity of the gas and dust, respectively; \(g\) is the external acceleration of the gas while \(g + \mathbf{a}\) is the external acceleration of dust (i.e. \(\mathbf{a}\) is the difference in the dust and gas acceleration), and \(P\) is the gas pressure. We assume a barotropic equation of state with sound speed \(c_s^2 = \partial P/\partial \rho\) and polytropic index \(y\) (see Section 4.2 for further details). The dust experiences a drag acceleration \(\mathbf{a}_{\text{drag}} = -(v - u)/t_s\) with an arbitrary drag coefficient \(t_s\), known as the ‘stopping time’ (which can be a function of other properties). The term in \(t_s\) in the gas acceleration equation is the ‘back-reaction’ – its form is dictated by conservation of momentum.

The equilibrium (steady-state), spatially homogeneous solution to equation (1) is the dust and gas accelerating together at the same rate, with a constant relative drift velocity \(\mathbf{w}_s\):

\[
\rho_i = \langle \rho_i \rangle = \rho_0, \quad \rho_d = \langle \rho_d \rangle = \rho_{d,0} \equiv \rho_0, \quad \mathbf{u} = \langle \mathbf{u} \rangle = \mathbf{u}_0 + \left[ g + \mathbf{a} \left( \frac{\mu}{1 + \mu} \right) \right] t,
\]

\[
\mathbf{v} = \langle \mathbf{v} \rangle = \langle \mathbf{u} \rangle + \mathbf{w}_s, \quad \mathbf{w}_s \equiv \frac{a(t_s)}{1 + \mu} = \frac{\mathbf{a}(t_s)(\rho_0, \mathbf{w}_s, \ldots)}{1 + \mu},
\]

where we define the total mass ratio between the two fluids as \(\mu \equiv \langle \rho_d \rangle/\langle \rho_i \rangle\), and \((t_s) = t_s(\langle \rho_i \rangle, \langle \mathbf{v} \rangle, \ldots)\) is the value of \(t_s\) for the homogeneous solution.\(^2\)

\(^2\)Note that \(t_s\) can depend on \(\mathbf{w}_s\), so equation (2) is in general a non-linear equation for \(\mathbf{w}_s\). Let us also define the normalized drift speed \(\tilde{w}_s \equiv |\mathbf{w}_s|/c_s\), which is a key parameter in determining stability properties and will be used extensively below. (Note that this definition of \(\tilde{w}_s\) differs from that of SH: this dimensionless version is more convenient throughout this work because of our focus on the acoustic resonance; see Section 3.2.)

We now consider small perturbations \(\delta\): \(\rho = \rho_i + \delta \rho, \mathbf{u} = \mathbf{u}_s + \delta \mathbf{u}\), etc. and adopt a free-falling frame moving with the homogeneous gas solution \(\mathbf{u}_s\) (see Appendix B for details). Linearizing
some equation of state which can relate perturbations in $T$ and $c_s$ to $\rho$. Then the linearized form obeys,
\[ \frac{\delta t_s}{(t_s)} = -\zeta_s \frac{\delta \rho}{\rho_0} - \zeta_w \cdot \frac{\delta v - \delta u}{|w_s|^2}, \]
where $\zeta_s$ and $\zeta_w$ are the drag coefficients that depend on the form of $t_s$ (see Section 4).

## 2.2 Gas supported by pressure gradients and arbitrarily stratified systems

Above we considered a homogeneous, freely falling system. Another physically relevant case is when the gas is stationary (hydrostatic), which requires a pressure gradient (with $\nabla P_0 = \rho_0 g + \rho_d \cdot w / (t_s)$). This will generally involve stratification in other properties as well (e.g. gas and dust density), so more broadly we can consider arbitrary stratification of the background quantities $P_0, \rho_0, \rho_d, g$, and $w$.

As usual, if we allow such gradients, we must restrict our analysis to spatial scales shorter than the background gradient scale length $L_0$ (e.g. $k \gg |\nabla U_0|/|U_0| \sim 1/L_0$) for each variable $U_0$, or else a global solution (with appropriate boundary conditions, etc.) is obviously needed. Moreover, we must also require $\rho_0 / t_s \ll L_0$, or else the time-scale for the dust to ‘drift through’ the system scale length is much shorter than the stopping time (and no equilibrium can develop). So our analysis should be considered local in space and time, with these criteria imposing maximum spatial and timescales over which it is applicable (with actual values that are, of course, problem-dependent). We discuss these scales with various applications in Section 6.

In Appendix C, we re-derive our results, for the unstable modes considered in this paper, for hydrostatic systems with arbitrary stratification in $P_0, \rho_0, \rho_d, g$, and $w$. Provided we meet the conditions above required for our derivation to be valid (i.e. $k \gg 1/L_0$), we argue (at least to lowest order in a local approximation) that

1. The existence and qualitative (e.g. dimensional, leading-order) scalings of all the instabilities analysed here in the homogeneous case are not altered by stratification terms, and the leading-order corrections to both the real and imaginary parts (growth rates and phase velocities) of the relevant modes are usually expected to be fractionally small.

2. Pressure gradients (the term required to make the system hydrostatic) enter especially weakly at high-$k$ in the behaviour of the instabilities studied here. In our (simplified) analysis, the leading-order correction from stratification is from non-vanishing $\nabla \cdot w_s \sim \rho_d \cdot w / \nabla p_d, 0$, i.e. a background dust density and drift-velocity gradient along the direction of the drift. The sense of the resulting correction is simply that modes moving in the direction of the drift are stretched or compressed along with the background dust flow. This particular correction is therefore large only if the time-scale for the dust to drift through the dust–density gradient scale length is short compared to mode growth timescales.

3. The leading-order corrections from stratification are not necessarily stabilizing or de-stabilizing (they can increase or decrease the growth rates).

4. Introducing stratification introduces new instabilities. For example, even when the gas is stably stratified, stratification leads to

Note that we label the $\delta \rho \rho_0$ coefficient in equation (5) as $\zeta_s$ because it encodes the dependence of $t_s$ on density at constant entropy; see Appendix C.
new linear modes in the gas, e.g. Brunt–Väisälä buoyancy oscillations. As shown in SH, if these modes exist in the gas, there is a corresponding RDI (the Brunt–Väisälä RDI studied in SH), which has maximal growth rates when \( w_s \cdot k = \pm (k_{\parallel}/k) N_{BV} \), i.e. when \( w_s \cdot k \) matches the Brunt–Väisälä frequency \( N_{BV} \). We defer detailed study of these modes to a companion paper, Squire & Hopkins (2018a), since they are not acoustic instabilities and have fundamentally different behaviours and dimensional scalings (e.g. resonance exists for all \( \hat{w}_s \), but the growth rates are always lower than those of the acoustic RDI at high-\( k \) if \( \hat{w}_s > 1 \).)

In what follows, we will take the homogeneous (free-falling) case to be our ‘default’ reference case, for two reasons. (1) The homogeneous and stratified cases exhibit the same qualitative behaviours, instabilities, and modes in all limits we wish to study, but the mathematical expressions are considerably simpler in the homogeneous case. (2) As discussed in Section 6, the situations where the acoustic RDI is of the greatest astrophysical interest involve dust-driven winds (e.g. in cool stars, star-forming regions, AGN torii, etc.). Such systems are generally better approximated as being freely accelerating than in hydrostatic equilibrium.

Of course, even in a ‘free-accelerating’ system, there will still be gradients in fluid properties (e.g. as a wind expands and cools). So our focus on the homogeneous case is primarily for the sake of generality and mathematical simplicity, and must therefore be considered a local approximation in both space and time (see Section 6).

### 2.3 Neglected physics

#### 2.3.1 Magnetized gas and dust

In this paper, we focus for simplicity on a pure hydrodynamic fluid. If the system is sufficiently magnetized, new wave families appear (e.g. shear Alfvén, slow, and fast magnetosonic waves in MHD). SH show that slow and fast magnetosonic waves, just like the acoustic waves here, are subject to the RDI (even when there is no Lorentz force on the dust). For resonant modes, when the projected dust streaming velocity \( w_s \cdot \hat{k} \) matches either the slow or fast wave phase velocity, the qualitative behaviour is similar to the acoustic RDI studied here (Section 3.7.1). Further, like for hydrodynamic modes studied in detail below (Section 3), even modes that are not resonant can still be unstable (but, unsurprisingly, the MHD–dust system is more complicated; see Tytarenko, Williams & Falle 2002).

Another effect, which was not included in SH, is grain charge. If the gas is magnetized and the grains are sufficiently charged, then Lorentz forces may dominate over the aerodynamic drag laws we consider here. This regime is relevant to many astrophysical systems (even, e.g. cosmic ray instabilities; Kulsrud & Pearce 1969; Bell 2004). Lorentz forces will alter the equilibrium solution, and introduce additional dependence of the mode structure on the direction of \( k \) via cross-product terms (terms perpendicular to both the mean drift and magnetic field), although they do not generally suppress (and in many cases actually enhance) the RDI.

For these reasons, we defer a more detailed study of MHD to the follow-up study, Hopkins & Squire (2018).

#### 2.3.2 Multispecies dust

Astrophysical dust is distributed over a broad spectrum of sizes (and other internal properties), producing different \( t_s \), \( v \), \( a \) for different species. Consider de-composing the dust into subspecies \( i \). Since the dust is pressure free, the dust continuity and momentum equations in equation (1) simply become a pair of equations for each subspecies \( i \). Each has a continuity equation for \( \rho_{d,i} \) (where \( \rho  \) and momentum equation for \( v_i \), each with their own acceleration \( a_i \) and drag \( t_s,i \), but otherwise identical form to equation (1). The gas momentum equation is identical, and the gas momentum equation is modified by the replacement of the drag term \( \rho_d (v - u)/t_s \to \sum_i \rho_{d,i} (v_i - u)/t_{s,i} \). The homogeneous solution now features each grain species moving with \( w_s,i \), so the sum in the gas momentum equation becomes \( \sum_i \rho_{d,i} (v_i - u)/t_{s,i} \).

The most important grain property is usually size (this, to leading order, determines other properties such as charge). For a canonical spectrum of individual dust grain sizes \( (R_d) \), the total dust mass contained in a log-normal interval of size scales as \( \mu_i \propto d\mu/d\ln R_d \propto R_d^{3.5} \), i.e. most of the dust mass is concentrated in the largest grains (Mathis, Rumpl & Nordsieck 1977; Draine 2003). Further, for any physical dust law (see Section 4), \( t_{s,i} \) increases with \( R_d \). In most situations, we expect \( |a_i| \) to depend only weakly on \( R_d \). This occurs: (i) if the difference in dust–gas acceleration is sourced by gravity or pressure support for the gas, (ii) when the gas is directly accelerated by some additional force (e.g. radiative line-driving), or (iii) when the dust is radiatively accelerated by long-wavelength radiation. Therefore, in these cases, all of the relevant terms in the problem are dominated by the largest grains, which contain most of the mass. We therefore think of the derivation here as applying to ‘large grains.’ The finite width of the grain size distribution is expected to broaden the resonances discussed below (since there is not exactly one \( \hat{w}_s,i \), there will be a range of angles for resonance), but not significantly change the dynamics. Much smaller grains can effectively be considered tightly coupled to the gas (they will simply increase the average weight of the gas).

However, in some circumstances – for example acceleration of grains by high-frequency radiation – we may have \( |a_i| \propto R_d^{-1} \). In these cases, the ‘back-reaction’ term on the gas is dominated by small grains, however those also have the smallest \( \hat{w}_s,i \), so the sum in the gas momentum equation becomes \( \sum_i \rho_{d,i} (v_i - u)/t_{s,i} \).

#### 2.3.3 Viscosity

We neglect dissipative processes in the gas in equations (3)–(4) (e.g. bulk viscosity). Clearly, including this physics will create a minimum scale below which RDI modes may be damped. This is discussed more in Section 6.

4 If dust is radiatively accelerated by a total incident flux \( F_\lambda \) centred on some wavelength \( \lambda \), the acceleration is \( a \approx F_\lambda \sigma T_k/n_m a_k \propto Q_d R_d^2 \), where \( m_d \propto R_d^3 \) is the grain mass and \( Q_d \) is the absorption efficiency which scales as \( Q_d \sim 1/\lambda \) for \( \lambda \ll R_d \) and \( Q_d \sim R_d \) for \( \lambda \gg R_d \). So the acceleration scales \( \propto 1/\lambda R_d \) for \( \lambda \ll R_d \) and is independent of grain size for \( \lambda \gg R_d \). For ISM dust, the typical sizes of the largest grains are \( \sim 1\mu m \sim 1000 \AA \), so for many sources we expect to be in the long-wavelength limit (even in cases where sources peak at \( \ll 1000 \AA \), then gas, not dust, will typically be the dominant opacity source).
3 UNSTABLE MODES: GENERAL CASE

In this section, we outline, in full detail, the behaviour of the dispersion relation that results from equation (4). While the completely general case must be solved numerically, we can derive analytic expressions that highlight key scalings for all interesting physical regimes. To guide the reader, we start with a general overview of the different branches of the dispersion relation in Section 3.1, referring to the relevant subsections for detailed derivations. For those readers most interested in a basic picture of the instability, Figs 1–4 give a simple overview of the dispersion relation and its fastest growing modes.

3.1 Overview of results

In general, the coupled gas–dust dispersion relation (equation 7 below) admits at least two unstable modes, sometimes more. This leads to a plethora of different scalings, each valid in different regimes, which we study in detail throughout Sections 3.2–3.9. The purpose of this section is then to provide a ‘road map’ to help the reader to navigate the discussion.

An important concept, discussed above and in SH, is a mode ‘resonance.’ This occurs here when \( w_s \cdot \hat{k} = \pm \varepsilon \), and thus is always possible (for some \( \hat{k} \)) when \( |w_s| \geq c_v (\hat{u}_s \geq 1) \). As shown in SH, when \( \mu \ll 1 \) (and \( |\hat{k}| c_v t_i \gg \mu \)), modes at the resonant angle are the fastest growing, and will thus be the most important for dynamics (if they can exist). In the context of the analysis presented below, we will see that the dispersion relation changes character at resonance, and we must therefore analyse these specific mode angles separately. The connection to the matrix-based analysis of SH, which treated only the modes at the resonant angle, is outlined in Appendix A. A clear illustration of the importance of the resonant angle is shown in the right-hand panel of Fig. 1.

Below, we separate our discussion into the following modes (i.e. regimes/branches of the dispersion relation):

(i) Decoupling instability, Section 3.3: If \( \varepsilon_w < -1 \), the drag on the dust decreases with increasing \( \hat{u}_s \), sufficiently rapidly that the dust and the dust completely decouple, causing an instability which separates the two. This instability exists for all \( \hat{k} \), but is not usually physically relevant (see Section 4.4).

(ii) Long-wavelength or ‘pressure-free’ modes, Section 3.4: At long wavelengths, the two unstable branches of the dispersion relation merge. This instability, which has a growth rate that scales as \( \mathcal{O}(\omega) \propto k^{2\beta} \), persists for all \( \mu \), any \( \hat{u}_s \) (it is non-resonant), and any \( \varepsilon_s \) and \( \varepsilon_w \) (except \( \varepsilon_w = 0 \), \( \varepsilon_s = 1 \)). This mode has a unique structure which does not resemble a modified sound wave or free dust drift, but arises because the drag forces on very large scales are larger than pressure gradient forces so the gas pressure terms become weak and the system resembles two frictionally coupled pressure-free fluids.

(iii) The ‘quasi-sound’ mode, Section 3.6: At shorter wavelengths, the two branches of the dispersion relation split in two. We term the first of these the ‘quasi-sound’ mode. The mode structure resembles a modified sound wave. When \( \hat{u}_s \gg 1 \), the quasi-sound mode is unstable for all \( k \), with \( \mathcal{O}(\omega(k) \propto k^{0} \) (i.e. the growth rate is constant). At resonance (Section 3.6.1), the quasi-sound mode is subdominant and its growth rate declines with increasing \( k \). The quasi-sound mode is stable for subsonic streaming (\( \hat{u}_s < 1 \)).

(iv) The ‘quasi-drift’ mode, Section 3.7: The second shorter wavelength branch is the ‘quasi-drift’ mode. The mode structure resembles modified free (undamped) grain drift. At the resonant mode angle (Section 3.7.1), the quasi-drift mode is the dominant mode in the system, with a growth rate that increases without bound as \( k \rightarrow \infty \). For a mid-range of wavelengths \( \mathcal{O}(\omega(k \propto k^{1/2}) \), while for sufficiently short wavelengths \( \mathcal{O}(\omega(k \propto k^{1/3}) \). At resonance, the mode structure also becomes ‘sound wave like’ in the gas, in some respects (Section 3.9). Away from resonance (e.g. if \( \hat{u}_s < 1 \)), the quasi-drift mode is either stable or its growth rate saturates at a constant value (i.e. \( \mathcal{O}(\omega(k \propto k^{0}) \), depending on \( \hat{u}_s \) and \( \varepsilon_s (1 + \varepsilon_w) \).

(v) The ‘uninteresting’ mode: For certain parameter choices a third unstable mode appears (it would be a fourth unstable mode if \( \varepsilon_w < -1 \), when the decoupling instability also exists). We do not analyse this mode further because it always has a (significantly) lower growth rate than either the quasi-sound or quasi-drift modes.

We also discuss the subsonic regime \( \hat{u}_s < 1 \) separately in more detail (Section 3.8), so as to highlight key scalings for this important physical regime. Finally, in Section 3.9, we consider the structure of the eigenmodes for the fastest growing modes (the long-wavelength mode and the resonant version of the quasi-drift mode), emphasizing how the resonant modes directly seed large dust-to-gas-ratio fluctuations in the gas.

3.2 General dispersion relation

Before continuing, let us define the problem. For brevity of notation, we will work in units of \( \rho_v, c_v \) and \( (t_i) \) (i.e. length units), viz.,

\[
\tilde{w}_s = \frac{|w_s|}{c_v} \quad \tilde{w} \rightarrow \omega \{t_i\} \quad \tilde{k} \rightarrow k \{t_i\},
\]

Inserting the general form for \( t_i \) (equation 5) into equation (4), we obtain the dispersion relation

\[
0 = A_0 \tilde{w}_s \tilde{B}_w \]

\[
A_0 = \mu + (\tilde{\omega} + \imath \mu)(\tilde{\sigma} + \imath)
\]

\[
B_w = \sigma (\tilde{k}^2 - \tilde{k}^2) [\tilde{\sigma}^2 + \nabla^2 \{k_1[i + 1 + \tilde{\varepsilon}_w(1 + \mu)] + i \nabla \tilde{\varepsilon}_w(1 + \mu)] - \nabla \tilde{\varepsilon}_w(1 + \mu)] + [\tilde{\sigma}^2 + \nabla^2 \{k_1[i + 1 + \mu] + i \nabla \tilde{\varepsilon}_w(1 + \mu)] - \nabla \tilde{\varepsilon}_w(1 + \mu)] + i \mu \{\tilde{\sigma}^2 \tilde{\varepsilon}_w + \nabla^2 \{k_1[1 + \tilde{\varepsilon}_w - \tilde{\varepsilon}_s] - i k_1^2 \tilde{\varepsilon}_s \}[1 + \tilde{\varepsilon}_w - \tilde{\varepsilon}_s] \}
\]

where

\[
\tilde{\sigma} = \tilde{\omega} - k \tilde{\varepsilon}_s \quad \tilde{\varepsilon}_s \equiv 1 + \tilde{\varepsilon}_w
\]

\[
k_1 = (w_s \cdot \hat{k}) \{t_i\} = \tilde{w}_s \tilde{k}_1 \equiv \tilde{w}_s \tilde{k} \cos \theta.
\]

(Note that \( \cos \theta \), the angle between \( \tilde{k} \) and \( \tilde{w}_s \), was denoted \( \psi_{lw} \) in SH to allow for simpler notation in the MHD case.) Appendix C gives more general expressions for stratified media.

Our task is to analyse the solutions to equation (7). Fig. 1 plots the growth rate of the fastest growing modes at each \( k \) for a range of \( \tilde{w}_s \), determined by exact numerical solution of equation (7). Figs 2–4 show additional examples.

3.2.1 General considerations

In equation (7), \( A_0 \) has the uninteresting zeros \( 2 \tilde{\omega} = k - i (1 + \mu) = \{\tilde{k}^2 - (1 + \mu)^2 \} / 2 \). These are damped longitudinal sound waves which decay (\( \mathcal{O}(\omega) \leq 0 \)) on a time-scale \( \sim \{t_i\} \)
for all $\mu$ and $k_0$; they are independent of $\zeta_s$ and $\zeta_w$. The interesting solutions therefore satisfy $B_{\omega} = 0$, a sixth-order polynomial in $\omega$.

For fully perpendicular modes ($k = k_\parallel$), $B_{\omega} = 0$ simplifies to $\omega^2 (\omega + i \zeta_w (1 + \mu)) (\omega^2 (i + \mu + \omega^2 k_\parallel^2 \omega^2)) = 0$; this has the solutions $\omega = 0, \omega = -i (1 + \mu) \zeta_w$, and the solutions to $\omega^2 (i + \mu + \omega^2 k_\parallel^2 \omega^2) = 0$ which correspond to damped perpendicular sound waves and decay ($\Im(\omega) < 0$) for all physical $\mu > 0$. For the general physical situation, with $\zeta_w > 0$, all unstable modes must thus have $k_\parallel \neq 0$.

### 3.3 Decoupling instability

Before considering the more general case with $k \neq 0$, it is worth noting that the perpendicular ($k_\parallel = 0$) mode above, $\omega = -i (1 + \mu) \zeta_w$ is unstable if $\zeta_w < 0$, i.e., $\zeta_w < -1$. Physically, $\zeta_w < 0$ is the statement that the dust–gas coupling becomes weaker at higher relative velocities, and instability can occur when dust and gas de-couple from one another (the gas accelerates and returns to its equilibrium without dust coupling, while the dust moves faster and faster as it accelerates, further increasing their velocity separation). As discussed below (Section 4.4) this could occur for Coulomb drag with $\tilde{w}_t > 1$; however, in this regime Coulomb drag will never realistically dominate over Epstein or Stokes drag, so we do not expect this instability to be physically relevant.

### 3.4 Long-wavelength (pressure-free) instability: $k_\parallel \ll \tilde{\mu}$

We now examine the case of long wavelengths ($k \ll 1$). If we consider terms in $\tilde{w}_t$ up to $O(k)$ for $k \ll \tilde{\mu}$, and expand $B_{\omega}$, we obtain $\omega^2 \zeta_\parallel (1 + \mu) = i \tilde{\mu} (\zeta_w - \zeta_\parallel) k_\parallel^2$ to leading order. For $\zeta_w - \zeta_\parallel > 0$, this has two unstable roots with the same imaginary part but oppositely signed real parts (waves propagating in opposite

Note that this mode depends only on $k_\parallel = \tilde{w}_t k \cos \theta$ at this order; the dependence on $\tilde{w}_t$ is implicit. The growth rate rises towards shorter wavelengths, but sublinearly. Most notably, instability exists at all dust abundances $\mu$ (and depends only weakly on that abundance, with the $1/3$ power), wavelengths $k_\parallel$ (for $k_\parallel \ll \tilde{\mu}$), accelerations or $\tilde{w}_t$, and drag coefficients $\zeta_s$ and $\zeta_w$.\footnote{Note that in the pathological case $\zeta_\parallel = \zeta_w = 1 + \zeta_w$, our approximation in equation (9) vanishes but an exact solution to equation (7) still exhibits low-$k$ instability, albeit with reduced growth rate. The reason is that the leading-order term on which equation (9) is based vanishes, so the growth rate scales with a higher power of $k_\parallel$. Instability only vanishes completely at low-$k$ when $\zeta_\parallel = 1$ and $\zeta_w = 0$, exactly.}

This mode is fundamentally distinct from either a modified sound wave or a modified dust drift mode. Rather, it is essentially a one-dimensional mode of a pressure-free, two-fluid system with drift between the two phases. To see this, we note that the pressure force on the gas scales as $\nabla P \sim k c_s^2 \delta \rho$, while the drift forces scale $\zeta_w \mu$. So, at sufficiently small $k \ll \mu$, the pressure force becomes small compared to the drag force of the dust on the gas. Perturbations perpendicular to the drift are damped on the stopping time, but
Figure 2. Spatial structure of the modes in Fig. 1 (see Section 3.9). Here, we take \( \mu = 0.01, \xi_w = 0, \dot{w}_s = 10, \) and \( \cos \theta \) shown, and plot the perturbed dust density \( \delta \rho_d \), gas density \( \delta \rho \) (in units of \( \rho_0 \), the mean density) and perturbed dust velocity \( \delta v \) and gas velocity \( \delta u \) (in units of \( c_s \)). The overall amplitude (y-axis normalization) is arbitrary. For the velocities we separate them into the magnitude of the component parallel to \( \hat{k} (\delta v \cdot \hat{k}) \), and perpendicular (\( \delta v \times \hat{k} \)). We show the spatial structure over one period, for a given \( \tilde{k} \equiv \frac{k c_s}{\langle \tau_s \rangle} \). In all cases, a lag between the dust and gas density perturbations arises because the dust decelerates when moving through the denser gas, which generates a ‘pileup’ and stronger dust–density peak, which in turn amplifies the gas response.

Top: The long-wavelength mode (Section 3.4) exhibits a nearly coherent dust–gas oscillation, with \( \delta \rho_d \approx \mu \delta \rho \) to leading order (the lag is higher order). This is not a modified sound wave, however: the phase/group velocities scale \( \propto k^{-1/3} \) (equation 9), the velocity and density responses are offset by a phase lag, and the gas+dust-density perturbation is weak (\(|\delta \rho|/\rho_0 \ll |\delta v|/c_s\); note we multiply \( \delta \rho \) plotted by 10, and \( \delta \rho_d \) by \( 10/\mu \)).

Middle: Resonant mode (Section 3.7.1), at intermediate wavelengths where the growth rate scales \( \propto k^{1/2} \) (equation 15). The wavespeed, gas density and velocity in \( \hat{k} \) direction now behave like a sound wave. The dust lag is larger (phase angle \( \sim \pi/6 \)) and because of the ‘resonance,’ where the dust motion along \( \hat{k} \) direction exactly matches the wavespeed, the effects above add coherently and generate a much stronger dust response with \(|\delta \rho_d|/|\delta \rho| \sim (2 \mu \tilde{k})^{1/2} \), a factor \( \sim (2 \tilde{k}/\mu)^{1/2} \sim 20 \) larger than the mean dust-to-gas ratio. Note the large perpendicular velocities also present.

Bottom: Resonant mode, at short wavelengths (where growth rates scale \( \propto k^{1/3} \); equation 16). This is similar to the intermediate-wavelength case except perpendicular velocities become negligible, the dust velocity response \( \delta v \) becomes weaker, and the dust-density response becomes stronger, with \(|\delta \rho_d|/|\delta \rho| \sim (4 \mu \tilde{k})^{1/3} \), a factor \( \sim 1000 \) larger than the mean dust-to-gas ratio \( \mu \).

Parallel perturbations can grow. As a result, one can recover all of the properties of this mode by simplifying to a pressure-free, one-dimensional system (\( k, \delta u, \delta v \) parallel to \( w_s \)).

At long wavelengths in particular, one might wonder whether the presence of gradients or inhomogeneity in the equilibrium solution might modify the mode here. In Appendix C, we consider a system in hydrostatic equilibrium supported by pressure gradients, with arbitrary stratification of the background quantities \( P_0, \rho_0, \rho_d, 0 \).
3.5 Short(er)-wavelength instabilities: $k_B \gg \mu$

At high-$k$, there are at least two different unstable solutions. If we assume a dispersion relation of the form $\omega \sim \mathcal{O}(k^1) + \mathcal{O}(k^n)$ where $n < 1$, and expand $B_{gw}$ to leading order in $\mu$, we obtain a dispersion relation $0 = \hat{\omega} (\hat{\omega} - k_1)(\hat{\omega}^2 - k_2)$ (1 + $\mathcal{O}(k^{-1})$). This is solved by $\hat{\omega} = \pm \hat{k} + \mathcal{O}(k^2)$ or $\hat{\omega} = k_1 + \mathcal{O}(k^2)$, each of which produces a high-$k$ branch of the dispersion relation.

In Sections 3.6 and 3.7, we study each of these branches in detail. We term the first branch, with $\hat{\omega} = \pm \hat{k} + \mathcal{O}(k^1)$, the ‘quasi-sound’ mode (Section 3.6); to leading order this is just a sound wave (the natural mode in the gas, absent drag: $\omega = \pm c_s k$). We term the second branch, with $\hat{\omega} = k_1 + \mathcal{O}(k^1)$, the ‘quasi-drift’ mode (Section 3.7); to leading order this is ‘free drift’ (the natural mode in the dust, absent drag: $\omega = w_s \cdot k$). In the analysis of each of these, we must treat modes with the resonant angle, $\cos \theta = \pm 1/w_s$, separately, because the dispersion relation fundamentally changes character. The quasi-drift mode at resonance (Section 3.7.1) is the fastest growing mode in the system (when $\hat{w}_s > 1$ and $\mu \ll 1$), with growth rates that increase without bound as $k \to \infty$. This is the resonance condition for the acoustic RDI case considered in SH (see also Appendix A).

3.6 Short(er)-wavelength instability: the ‘quasi-sound’ mode

To leading order, the quasi-sound mode satisfies $\hat{\omega} = \pm \hat{k}$ (the soundwave dispersion relation). Consider the next-leading-order term; i.e. assume $\hat{\omega} = \hat{\omega}_{qs} = \pm \hat{k} + \mathcal{O}(k^{-1})$ (where $\mathcal{O}$ is a term that is independent of $k$) and expand the dispersion relation to leading order in $k^{-1}$ (it will transpire that the solution here is valid for all $k \gg \hat{w}_s \mu$). This produces a simple linear leading-order dispersion relation for both the $\pm$ cases:

$$\hat{\omega}_{qs} \approx \pm \hat{k} - i \frac{\mu (1 + \zeta_w \cos^2 \theta \pm \hat{w}_s (1 - \zeta_s) \cos \theta)}{2},$$

(10)

where the ‘$+$’ mode applies the $+$ to all $\pm$, and vice versa. Because both signs of $\cos \theta$ are allowed, it follows that the modes are unstable ($\Im(\omega) > 0$) if

$$\hat{w}_s (1 - \zeta_s) \cos \theta > 1 + \zeta_w \cos^2 \theta.$$

(11)
Because $\zeta_w$ and $\zeta_s$ generally are order-unity or smaller, equation (11) implies that $\tilde{w}_i \gtrsim 1$ is required for this mode to be unstable. For $\zeta_w \ll 1$, the more common physical case (see Section 4), we also see that the condition (equation 11) is first met for parallel modes ($\cos \theta = \pm 1$) and that their growth rate (equation 10) is larger than oblique modes. Comparing the long-wavelength result in equation (9) to equation (10), we see that the growth rate grows with $k$ until it saturates at the constant value given by equation (10) above $k \gtrsim \tilde{w}_i \mu$. For $\tilde{w}_i \lesssim 1$, the mode becomes stable above $k \gtrsim \tilde{w}_i \mu$.

In Appendix C, we show that up to this order in $\tilde{k}$, the behaviour of this mode is not expected to change in hydrostatic or arbitrarily stratified media (the leading-order corrections appear at order $\sim 1/(k L_0)$, where $L_0$ is the gradient scale length of the system).

3.6.1 The quasi-sound mode at resonance

When $\tilde{w}_i \cos \theta = \pm 1$, the behaviour of the quasi-sound mode is modified (the series expansion we used is no longer valid; see Section 3.7.1). If we follow the same branch of the dispersion relation, then instead of the growth rate becoming constant at high-$k$, it peaks around $k_0 \sim \tilde{\mu}$ at a value $\Im(\tilde{\omega}) \approx \tilde{\mu}/4$, and then declines with increasing $k$. $k_0$ is therefore the less interesting branch in this limit, because the quasi-drift branch produces much larger growth rates.

3.7 Shorter-wavelength instability: the ‘quasi-drift’ mode

We now consider the quasi-drift mode branch of the high-$k$ limit of $\omega$, with leading-order $\tilde{\omega} = k_1$ (the free-drift dispersion relation).

3.7.1 The parallel quasi-drift mode

If we follow the same branch of the dispersion relation, then instead of the growth rate becoming constant at high-$k$, it peaks around $k_0 \sim \tilde{\mu}$ at a value $\Im(\tilde{\omega}) \approx \tilde{\mu}/4$, and then declines with increasing $k$. $k_0$ is therefore the less interesting branch in this limit, because the quasi-drift branch produces much larger growth rates.

6For the parallel case, the general dispersion relation $B_\omega$ simplifies to: $B_\omega \to A_\omega B_\omega^0$, with

\[ B_\omega^0 = k_0 \tilde{w}_i^2 \mu (\tilde{\omega}_s \tilde{\omega}_w - k_0 \zeta_s) + \sigma \left((\sigma + i \tilde{\omega}_w) (\tilde{\omega}_s^2 \tilde{w}_i^2 - k_0^2) + i \tilde{\omega}_i^2 \mu (\tilde{\omega}_s^2 \tilde{w}_w + k_0 (k_0 - i \tilde{\omega}_w) - \tilde{\omega} k_0 (\zeta_w + \zeta_s) - 1)\right) \]

Assuming $\tilde{\omega} = \tilde{\omega}_{QD} = k_1 + \sigma + O(k^{-1})$, and expanding to leading order in $k$, we obtain the leading-order cubic relation

\[ 0 = \sigma (\sigma + i) (\sigma + i \tilde{\omega}_w) (1 - \tilde{w}_s^2 \cos^2 \theta) - \mu (i (\tilde{\omega}_w - \zeta_s) \tilde{w}_w^2 \cos^2 \theta + \sigma (1 - \tilde{\omega}_w + (\tilde{\omega}_w^2 + (\tilde{w}_w^2 - \tilde{w}_w^2 \zeta_s - 1) \cos^2 \theta)). \]

(12)

Equation (12) is solvable in closed form but the expressions are tedious and unintuitive.7 For clarity of presentation, if we consider $\mu \ll 1$, the expression factors into a damped solution with $\sigma = -i$, and a quadratic that gives a damped and a growing solution which simplifies to

\[ \tilde{\omega}_{QD}(\mu \ll 1) \approx k_1 + i \left( \frac{\tilde{w}_w \cos \theta}{\tilde{\omega}_s} \right) \mu \left( 1 - \frac{\zeta_s}{\tilde{\omega}_w}\right). \]

(13)

This illustrates the general form of the full expression. In particular, we see that the expressions become invalid ($\Im(\tilde{\omega}) \to \infty$) at the resonant angle $\tilde{\omega}_w^2 \cos^2 \theta = 1$, which will be treated separately below (Section 3.7.1).

The requirement for instability (from the general version of equation 13) is

\[ (\tilde{w}_w^2 \cos^2 \theta - 1) (1 - \zeta_s/\tilde{\omega}_w) \geq 0. \]

(14)

We thus see that if $\zeta_s/\tilde{\omega}_w < 1$ (the more common physical case), this mode is unstable for $\tilde{w}_i \cos \theta > 1$; if $\zeta_s/\tilde{\omega}_w > 1$, however, the mode is stable for $\tilde{w}_i \cos \theta > 1$ but becomes unstable for $\tilde{w}_i \cos \theta < 1$.

Away from resonance (i.e. with $|\tilde{w}_i \cos \theta| \neq 1$), we see that, like the quasi-sound mode, the quasi-drift mode is described by the long-wavelength solution from Section 3.4, with a growth rate that

\[ \tilde{\omega}_{QD}(\mu \ll 1) \approx k_1 + i \left( \frac{\tilde{w}_w \cos \theta}{\tilde{\omega}_s} \right) \mu \left( 1 - \frac{\zeta_s}{\tilde{\omega}_w}\right). \]

(13)

Equation (12) does provide a simple closed-form solution if $\cos \theta = \pm 1$ (parallel modes), or $\tilde{\omega}_w = 0$; in these cases the growing mode solutions are:

\[ \tilde{\omega}_{QD}(\cos \theta = 1) \approx k_1 + i \tilde{\omega}_w \left[ -1 + \left( 1 + \frac{4 \mu (\tilde{\omega}_w - \zeta_s)}{\tilde{\omega}_w^2 (1 - \tilde{w}_w^2)} \right)^{1/2} \right]^{1/2} \]

\[ \tilde{\omega}_{QD}(\cos \theta = 0) \approx k_1 + \frac{1}{2} \left[ -1 + \left( 1 + \frac{4 \mu (1 - \zeta_s)}{1 - \tilde{w}_w^2 \cos^2 \theta} \right)^{1/2} \right] \]
increases with $k$ until it saturates at the constant value of equation (13): roughly $\tilde{w}_s\mu$ for $\tilde{w}_s < 1$ or $\sim \mu$ for $\tilde{w}_s > 1$. Comparing the growth rates (equations 13 and 9) we see this occurs at $k_\parallel \gtrsim \mu \tilde{w}_s^2/(1 + \tilde{w}_s^2)$ (i.e., $\sim \tilde{w}_s^2 \mu$ for $\tilde{w}_s < 1$, $\sim \mu/\tilde{w}_s$ for $\tilde{w}_s > 1$).

In Appendix C, we note that in an arbitrarily stratified background, a constant correction to the growth rate of this mode appears at leading order, with the form $\omega_{QD} \to \omega_{QD} - i \nabla \cdot \mathbf{w}_s$ (or $\omega_{QD} \to \omega_{QD} + i \rho_{d,0}^\parallel \mathbf{w}_s \cdot \nabla \mu_0$, since the dust density and velocity are related by continuity). Because this mode is moving with the mean dust motion ($\tilde{w}_s \equiv k_\parallel \mu$ or $w_s \sim k \mu$ to leading order), this is just the statement that, if there is a non-zero divergence of the background drift, the perturbation is correspondingly stretched or compressed along with the mean flow. The correction is important only if the time-scale for the dust to ‘drift through’ the global gradient scale length (in $\rho_{d,0}$ or $w_s$) is short compared to the growth time.

### 3.7.1 The quasi-drift mode at resonance

When $\tilde{w}_s \gtrsim 1$, then equation (13) (and its generalization, valid at all $\mu$) diverge as. In this case, the ‘saturation’ or maximum growth rate of the mode becomes infinite. What actually occurs is that the growth rate continues to increase without limit with increasing $k$.

In this limit, our previous series expansion at high-$k$ is invalid: we must return to $B_{\omega_0}$ and insert $\tilde{w}_s \cos \theta = \pm 1$, i.e. $k^2 = k_\perp^2$ or $k \cdot \mathbf{w}_s = 0$ and $\omega = \omega_{QD}$, $k = \pm c_1 k$, the resonance condition for the RDI. Note that when the resonant condition is met, the mode satisfies $\omega = \mathbf{w}_s \cdot k = \pm c_1 k$–i.e. to leading order it simultaneously satisfies the dispersion relation of gas absent drag (a sound wave) and dust absent drag (free drift). This effectively eliminates the restoring forces in the system, so the resulting dispersion relation $^8$ has growing solutions with $\Im(\omega_0) > 0$ for all $k_\parallel$, and the growth rate increases monotonically with $k_\parallel$ without limit (here and below we use $\omega_0$ to denote the resonant frequency).

There are two relevant regimes for this mode at resonance:

1. **The intermediate-wavelength (mid-$k$ or low-$\mu$) resonant mode:** If $\tilde{\mu} \ll k \ll \tilde{\mu}^{-1}$, the resonant solutions to $B_{\omega_0}$ are

   $$\tilde{\omega}_s(\tilde{\mu} \ll k \ll \tilde{\mu}^{-1}) \approx k_1 \left[ 1 - \tilde{\mu} + \frac{1}{2} \left( 1 - \frac{\tilde{\omega}_s}{\tilde{\omega}_w} \right) \frac{\tilde{k}^2}{4} \right]^{1/2} \tilde{\mu} \kappa_1$$

   where

   $$k_1 \equiv \left[ 1 - \tilde{\mu}^2 \left( 1 - \frac{\tilde{\omega}_s}{\tilde{\omega}_w} \right) \tilde{k}^2 \right]^{1/2}$$

   As expected, to $\mathcal{O}(\mu^{1/2})$, this matches the ‘acoustic RDI’ expression derived in SH, with the resonance between the dust drift velocity and the natural phase velocity of an acoustic wave without dust (the exact correspondence is explained in detail in Appendix A).

2. **The short-wavelength (high-$k$) resonant mode:** At larger $k_\parallel \gg \tilde{\mu}^{-1}$, expanding $\tilde{\omega} \sim \tilde{\omega}_s \tilde{k}$ to leading order in $k_\parallel$ shows that the leading-order term must obey $\tilde{\omega} = \pm \tilde{k}$, as before. Now expand to the next two orders in $k_\parallel$ as $\tilde{\omega}_s \approx \tilde{\omega}_s + \tilde{\omega}_s \tilde{k}_s + \tilde{\omega}_s \tilde{k}_s^2$, where again $\sigma$ denotes a $k$-independent part (it is easy to verify that with $v \geq 0$, any term $\tilde{\omega} = \pm \tilde{k}$, other than $v = 0$ and $v = 1$, must have $\tilde{\omega}_s = 0$ to satisfy the dispersion relation to next-leading order in $k$). This gives $2 \tilde{\omega}_s^3 + (1 + \tilde{\omega}_w - \tilde{\omega}_s) \mu_\parallel = 0$, and

   $\sigma = \frac{(1 + \tilde{\omega}_w - \tilde{\omega}_s) \mu_\parallel}{6}$

   The resonant condition is satisfied and $\tilde{\omega}_s = \tilde{\omega}_w = 0$, the dispersion relation has the simple form $\sigma = \tilde{\omega}_s (\tilde{\omega}_w + 1 + \mu / \tilde{\omega}_s) = -\mu_\parallel k^2$.

$^8$If the resonant condition is satisfied and $\tilde{\omega}_s = \tilde{\omega}_w = 0$, the dispersion relation has the simple form $\sigma = \tilde{\omega}_s (\tilde{\omega}_w + 1 + \mu / \tilde{\omega}_s) = -\mu_\parallel k^2$.

$^9$Note that at long wavelengths, $k_\parallel \ll \tilde{\mu}$, the series expansion in equation (9) is still accurate and we just obtain the solutions in Section 3.4, even at resonance.

### 3.8 Subsonic ($\tilde{w}_s < 1$) modes

In Section 3.7 above, we saw that when $\tilde{w}_s > 1$ (and $\tilde{\mu} \ll 1$) the fastest growing modes will be the long-wavelength mode (at low-$k$) and the acoustic RDI ‘resonant’ modes (at high-$k$). When the streaming is subsonic ($\tilde{w}_s < 1$) this resonance is no longer possible and the quasi-sound mode (Section 3.6) is also stabilized.
thus seems helpful to cover the subsonic mode structure in a self-contained manner, which is the purpose of this section. We collect some of the results derived in Sections 3.4–3.7 and derive a new limit of the subsonic quasi-drift mode.

At sufficiently low-$k$, the long-wavelength solutions from Section 3.4 continue to be unstable. Moreover, the ‘quasi-drift’ mode in equation (13) is still unstable if $\zeta_s > \zeta_w$ (see equation 14; in this case all $k$ are unstable). The mode then grows as in equation (9) until saturating at a maximum growth rate given by equation (13): approximately $\Delta(\omega) \sim \tilde{w}^2_s \mu$, for $k \gg \tilde{w}^2_s \mu$. From the form of equation (13) we can also see that for $\tilde{w}_s \ll 1$, the most rapidly growing mode has $\cos \theta = \pm 1$, i.e. the modes are parallel.

If $\zeta_w > \zeta_s$ (and $\tilde{w}_s < 1$), the quasi-drift mode is stabilized for . However, it persists for some intermediate range of $\tilde{k}$, which was not included in equation (13) due to our assumption $\tilde{k} \gg 1$. Specifically, the growth of $\Delta(\omega)$ with $k_1$ saturates at a similar point, but then $\Delta(\omega)$ turns over and vanishes at finite $k \gtrsim \tilde{w}_s$. Since we are interested in small $\tilde{w}_s$ and low-$\tilde{k}$, we assume $\Delta \sim \sigma \tilde{w}_s + \bar{\sigma}_1 \tilde{w}_s + \bar{\sigma}_2 \tilde{w}_s^2$ and $\dot{k} \sim O(\tilde{w}_s)$, and expand the dispersion relation to leading order in $\tilde{w}_s$. This gives two results: (i) that $\sigma$ must vanish and (ii) that $\bar{\sigma}_1$ must obey $\bar{\sigma}_1(\bar{\sigma}_1 + (1\mu - k_1^2/\tilde{w}_s^2)^2 = 0$. This gives the leading-order solution $\dot{\omega} = \pm k_1/\sqrt{1 + \mu}$. Plugging in either the $+$ or $-$ root (they give the same growth rate), we solve for the second-order term, to obtain the relation

$$\bar{\omega}_{\text{subsonic}} \approx \dot{k}_1 \left( \pm \frac{1}{1 + (1 + \mu)^{1/2}} \right) + \frac{\zeta_s}{2(1 + \mu)\zeta_w} \left( \frac{\bar{\sigma}_1}{1 + \mu} \right)$$

(iii) Resonant mode, intermediate wavelengths ($\mu \ll k \ll \tilde{k}$; equation 15): For $k_1 \approx 1$, the fastest growing mode has $k_1$ oriented at the resonant angle $\cos \theta = \pm 1/\tilde{w}_s$, such that the drift/drift velocity (in the direction of the wave propagation) is also equal to that wave/drift: $\tilde{w}_s \pm \tilde{k} = c_s$. In other words, the bulk is co-moving with the wave in the direction $\tilde{k}$.

For $\mu \ll 1$, the gas density response behaves like a sound wave, $\delta \rho/\rho_0 \approx -\delta u \cdot \delta c_s$, in phase with the velocity in the $\tilde{k}$ direction. However, the dust-density response now lags by a phase angle $\approx \pi/6$, and, more importantly, the resonance generates a strong dust-density response: $|\delta \rho_d| \sim (2\mu\kappa_1)^{1/2}|\delta \rho|$. We see the dust-density fluctuation is enhanced by a factor $\sim (2\mu\kappa_1)^{1/2} \gg 1$ relative to the mean ($\mu$), which is much stronger than for the long-wavelength mode ($\delta \rho_d \sim \delta \rho$). The resonant mode can thus generate very large dust-to-gas fluctuations even for otherwise weak modes, and the magnitude of the induced dust response increases at shorter wavelengths.

Effectively, as the dust moves into the gas density peak from the wave, it decelerates, producing a trailing ‘pileup’ of dust density behind the gas density peak, which can be large. This dust-density peak then accelerates the gas, amplifying the wave. Because of the resonance with both drift and sound speeds, these effects add coherently as the wave propagates, leading to the exponential growth of the mode.

One further interesting feature of this mode deserves mention: the velocities ($\delta v \approx \delta u$ here) are not fully aligned with $\tilde{k}$ but have a component in the $k_1$ direction, which leads the velocity in the $\tilde{k}$ direction by a phase angle $\approx \pi/4$. This is a response to the dust streaming in the $k_1$ direction and the amplitude of this term decreases with $k$.

(iii) Resonant mode, short wavelengths ($\kappa_1 \ll \tilde{k}$; equation 16): At high-$k$ with $\tilde{w}_s \geq 1$ the details of the resonant mode (and scaling of the growth rate) change. The resonant condition remains the same as at mid-$k$, however, the mode propagates with wavespeed $c_s \tilde{k}$ along the resonant angle $\cos \theta = \pm 1/\tilde{w}_s$, and the gas behaves like a sound wave (the velocities are now aligned $\delta u \propto \delta v \propto \tilde{k}$). This generates a strong dust response with the slightly modified scaling (scaling like the growth rate), with $\delta \rho_d$ lagging the gas mode by a phase angle $\sim \pi/6$. Importantly, $|\delta \rho_d|/|\delta \rho|$ continues to increase indefinitely with $k$, and in this regime, the dust-density perturbation becomes larger than the gas density perturbation in absolute units (even though the mean dust density is smaller than gas by a factor $\mu$). The dust velocity $\delta v$ is parallel to $\delta u$, but with a smaller amplitude $|\delta u|/|\delta u| \sim (\mu \kappa_1/2)^{1/3} \ll 1$, and $\delta v$ leads $\delta u$ by a phase angle $\sim \pi/6$.

\[^{10}\text{The phase angle } \pi/6 \text{ (the argument of } \mu^{1/3} \text{) appears repeatedly because the dominant imaginary terms in the dispersion relation are cubic.}\]

\[^{11}\text{Note that for } \tilde{w}_s \gg 1, \text{ the } k_1 \text{ direction is approximately the } \tilde{\omega}_s \text{ direction.} \]
4 DRAG PHYSICS

In this section, we consider different physical drag laws. This involves inserting specific forms of $\xi_s$ and $\xi_w$ into the dispersion relations derived in Section 3. Numerically calculated growth rates for representative cases are shown for comparison in Fig. 3. We also show as illustrative cases two arbitrary but constant, order-unity choices: $(\xi_s, \xi_w) = (0, 1)$ and $(\xi_s, \xi_w) = (2, 0)$. The former case illustrates that with $\xi_w < \xi_s$, the qualitative behaviour of the modes are largely similar to the constant-$t_s$ case in Fig. 1. The latter shows that when $\xi_w \gg \xi_s$, the dominant effect is to extend the instability of subsonic ($\tilde{\omega}_s < 1$) cases to high-$k$. For simplicity of notation, we again use the dimensionless variables of equation (6) in this section.

4.1 Constant drag coefficient

The simplest case is $t_s = \text{constant}$, so $\delta t_s = 0$ i.e. $\xi_s = \xi = 0$ (and $\xi_w = 1$). The characteristic polynomial simplifies to $B_\omega \equiv A_\omega B_w$ with $B'_\omega \equiv \sigma \left( \sigma + i \right) \left( \omega^2 - k^2 \right) + i \mu \left( \omega^2 \sigma - \kappa_1^2 \right) \left( m + 1 \right)$. Since $\xi_w = 1$, all pure-perpendicular modes are damped or stable.

The long-wavelength modes are unstable with growth rates,

$$\tilde{\omega}_s \approx \left( 1 \mp \frac{\sqrt{3} + i}{2} \tilde{\mu} \right) \kappa_3^{2/3}.$$  

(18)

For $\tilde{\omega}_s < 1$, these cut off at high-k with $\tilde{\omega} \approx (\mu/2)\left( \tilde{\omega}_s^2 - \kappa_3^2 \right)/(1 + \mu^2)$ (equation 17). For $\tilde{\omega}_s \geq 1$, at large $k$ the quasi-sound mode (equation 10) is present with growth rate $\mathcal{D}(\tilde{\omega}) = \mu (\tilde{\omega}_s | \cos \theta - 1)/2$ so the most rapidly growing mode is parallel. The quasi-drift mode (equation 13) is present with growth rate $\mathcal{D}(\tilde{\omega}) \sim \mu/[1 - (\tilde{\omega}_s \cos \theta)^{-2}]$. At resonance ($\cos \theta \rightarrow \pm 1/\tilde{\omega}_s$), the growth rate is,

$$\tilde{\omega}_s \approx \begin{cases} i \left( \frac{1}{4} - i \frac{\tilde{\mu}}{2} \left( \frac{1 + i}{2} \right) \left( \sqrt{\frac{2}{3}} \right) \tilde{\mu} \right)^{1/2} & (\tilde{\omega}_s < \tilde{\mu}^{-1}) \\ 1 - \frac{1}{3} \left( \frac{1 + i \sqrt{3}}{\tilde{\mu} \kappa_3} \right)^{1/3} & (\tilde{\omega}_s \gg \tilde{\mu}^{-1}) \end{cases}$$

(19)

Examples of this case ($\xi_s = \xi_w = 0$) are shown in Fig. 1, but they are similar to the other cases with $\xi_w < \xi_s$ in Fig. 3.

4.2 Epstein drag

The general expression (including physical dimensions) for the drag coefficient in the Epstein limit is

$$t_s = \sqrt{\frac{\pi \gamma}{8}} \frac{\rho \rho_s R_d}{\rho c_s} \left( 1 + a_s \frac{|v - u|^2}{c_s^2} \right)^{-1/2}, \quad \alpha_s \equiv \frac{9 \pi \gamma}{128}.$$  

(20)

where $\rho_s$ is the internal material density of the aerodynamic particle and $R_d$ is the particle (grain) radius. For astrophysical dust, $\rho_s \sim 1 - 3 \text{ g cm}^{-3}$, and $R_d \sim 0.001 - 1 \text{ \mu m}$ in the ISM, or in denser environments $R_d \sim 0.1 - 1000 \text{ \mu m}$ (e.g. proto-planetary discs, SNe ejecta, or cool-star atmospheres; Draine 2003). Note that Epstein drag depends on the isothermal sound speed, $c_{iso} \equiv \sqrt{k_B T/m_{\text{eff}}}$ (where $m_{\text{eff}}$ is the mean molecular weight). However, because we work in units of the sound speed $c_s \equiv \sqrt{T_p/p}$, we relate the two via the usual equation-of-state parameter $\gamma$,

$$\gamma \equiv \frac{c_s^2}{c_{iso}^2} = \frac{\rho}{p} \frac{\partial p}{\partial \rho},$$  

(21)

and will assume $\gamma$ is a constant under linear perturbations. We emphasize that the $\gamma$ here is the appropriate $\gamma$ describing how the temperature responds to compression or expansion on a wave-crossing time - roughly the same $\gamma$ appropriate for a sound wave. This means that external heating or cooling processes are only important for $\gamma$ if the heating/cooling time is shorter than the sound-crossing time (otherwise we typically expect adiabatic $\gamma$).

Note that because $t_s$ now depends on $|v - u| = |w_s|$, equation (2) for the drift velocity, $w_s = a (\tilde{\omega}_s/(1 + \mu))$, is implicit. Define $\tilde{w}_{s,0} \equiv |a| / t_0 (c_s (1 + \mu))$ where $t_0 \equiv (\pi \gamma/8)^{1/2} R_d (\rho_0 c_s)$ is the stopping time at zero relative velocity. Then the solution of equation (2) is

$$\tilde{w}_s \approx \frac{1}{2 a_s} \left[ (1 + 4 a_s \tilde{w}_{s,0}^2)^{1/2} - 1 \right],$$

(22)

which reduces to $\tilde{w}_s \approx \tilde{w}_{s,0}$ for $|a| \ll c_s / t_0$, or $\tilde{w}_s \approx a_s^{-1/4} \tilde{w}_{s,0}^{1/2}$ for $|a| \gg c_s / t_0$. With equation (20) for $t_s$ and equation (22) for $\tilde{w}_{s,0}$, $\delta t_s$ follows equation (5) with

$$\xi_s = \frac{\gamma + 1 + 2 a_s \tilde{w}_{s,0}^2}{2 (1 + a_s \tilde{w}_{s,0}^2)} \quad \xi_w = \frac{a_s \tilde{w}_s^2}{1 + a_s \tilde{w}_s^2}.$$  

(23)

From this we can derive the relevant instability behaviour for different $\gamma$ and $\tilde{w}_s$. Note $\xi_s > 0$ and $\xi_w > 0$, so the ‘decoupling’ instability (which requires $\xi_s < 0$) is not present.

In Fig. 3, for this case (as well as Stokes and Coulomb drag), we show values of $\mathcal{D}(\tilde{\omega})$ for two values of $\gamma = 2/3, 5/3$ (and a range of), which determine $\xi_s, \xi_w$. The two values of $\gamma$ are chosen to bracket the range where the behaviour changes ($\xi_s < \xi_w$ and $\xi_s > \xi_w$) and be qualitatively representative of cases where $\xi_s > \xi_w$.

4.2.1 Supersonic streaming ($\tilde{w}_s \gg 1$)

In the $\tilde{w}_s \gg 1$ limit, $\xi_s \rightarrow 1 + \mathcal{O}(\tilde{w}_s^{-2})$ (independent of $\gamma$) and $\xi_w \rightarrow 1$. This stabilizes the quasi-sound modes (equation 10) because high-$\tilde{w}_s$, the $\xi_s$ term dominates over $(1 - \xi_s)$, viz., the stronger coupling from at high relative velocity stabilizes the modes. The long-wavelength modes (equation 9) are present and saturate in the quasi-drift/resonant mode, with growth rate $\mathcal{D}(\tilde{\omega}) \sim \mu [1 - (\tilde{w}_s \cos \theta)^{-2}]^{-1} (1 - \tilde{\xi}_w/\tilde{\xi}_w)$, which approaches $\mathcal{D}(\tilde{\omega}) \sim \mu/2$ for $\tilde{w}_s \gg 1$ out-of-resonance.

At resonance, we insert the full expressions for $\xi_s$ and $\xi_w$ into equation (15) and equation (16). This gives

$$\tilde{\omega}_s \approx \frac{1}{\xi_w - \xi_s} \left( \frac{\tilde{w}_s - \xi_s}{\xi_w} \right) + \frac{1}{2} \left( \frac{\tilde{w}_s - \xi_s}{\xi_w} \right)^{1/2}.$$  

(24)
Again there is one critical point when $\tilde{\omega}_w - \zeta_w = 0$, or $\tilde{\omega}_d = 64(\gamma - 1)/(9\pi \gamma)$, where the standard long-wavelength instability vanishes. This occurs only for some specific $\tilde{w}_s$ at a given $\gamma$, so is unlikely to be of physical significance for most $\gamma$. Again, at this point, there is in fact still an instability, albeit with a reduced growth rate (see footnote 5, near equation (9); the instability only truly vanishes at $\zeta_w = 0$, $\zeta_s = 1$ exactly). However, one does approach this vanishing-point for $\gamma = 1$ as $\tilde{w}_s \to 0$ becomes sufficiently small.

This leads to a cautionary note: it is common in some subsonic ($\tilde{w}_s < c_s$) applications to drop the term in $|\text{e} - u|^2/c_s^2$ in equation (20) (i.e. simply taking $t_s \propto 1/\rho c_s$), for simplicity. If the gas is also isothermal ($\gamma = 1$), this would give $\zeta_w = 0$, $\zeta_s = 1$ exactly and the instabilities would vanish for $\tilde{w}_s \ll 1$. However, this can be misleading: although the term in $|\text{e} - u|^2/c_s^2$ is small, it does give rise to a non-zero (albeit small) growth rate. Moreover, if the equation of state is even slightly non-isothermal (e.g. $\gamma = 0.9$, 1.1), the instability is not suppressed strongly. Also, we caution that the appropriate equation of state here is that relevant under local, small-scale compression by dust and sound waves (not necessarily the same as the effective equation of state of e.g. a vertical atmosphere).

### 4.3 Stokes drag

The expression for drag in the Stokes limit – which is valid for an intermediate range of grain sizes, when $R_d \gtrsim (9/4) \lambda_{	ext{Stokes}}$ but $R_{\text{drag}} \equiv R_d|\mathcal{W}|/(\lambda_{\text{Stokes}}c_s) \lesssim 1$ – is given by multiplying the Epstein expression (equation 20) by $(4 R_d)/(9 \lambda_{\text{Stokes}})$. Here, $\lambda_{\text{Stokes}} \propto 1/(\rho \sigma_{\text{gas}})$ is the mean free-path, $\sigma_{\text{gas}}$ is the gas collision cross-section, and $R_{\text{drag}}$ is the Reynolds number of the streaming grain.

We can solve implicitly for the dust streaming velocity $\mathbf{w}_s$, which is the same as in the Epstein case (since $t_s$ depends on $|\mathbf{w} - \mathbf{u}|$ in the same manner). However, the absolute value of $t_s$ only determines our units, and the behaviour of interest depends only on the coefficients $\xi$, and $\zeta_w$. Since $R_d$ is a material property of the dust and $\sigma_{\text{gas}}$ an intrinsic property of the gas, the important aspect of the Stokes drag law is that it multiplies the Epstein law by one power of $\rho$. Although it is certainly possible $\sigma_{\text{gas}}$ might depend on density and/or temperature, lacking a specific physical model for this we will take it to be a constant for now. This simply gives $\zeta_s \to \zeta_s - 1$, relative to the scalings for Epstein drag.

When $\tilde{w}_s \ll 1$ (c.f. Section 4.2.2 for Epstein drag), and $\zeta_w = a_c \tilde{w}_s^2 + \mathcal{O}(\tilde{w}_s^4)$, and quasi-sound and resonant modes are stabilized (because $\tilde{w}_s < 1$). The quasi-drift (high-k) mode is stabilized for $\zeta_s \to \zeta_s - 1$, maximum growth rate $\zeta_s \approx a_c \tilde{w}_s^2 \mu (3 - \gamma)/4$. This is larger (smaller) than the Epstein drag growth rate for $\gamma < 5/3$ ($\gamma > 5/3$).

In the limit $\tilde{w}_s \gg 1$, the Stokes drag expression cannot formally apply because $R_d > \lambda_{\text{Stokes}}$ then implies $R_{\text{drag}} = R_d|\mathcal{W}|/(\lambda_{\text{Stokes}}c_s) \gtrsim 1$. When this is the case, either because $\tilde{w}_s$ is large or (more commonly) $R_d$ is large, there is no longer a simple drag law because the grain develops a turbulent wake. This will tend to increase the drag above the Stokes estimate (the turbulence increases the drag) with a stronger and stronger effect as $R_{\text{drag}}$ increases. Given some empirically determined scaling of $t_s$ with $R_d$, $\tilde{w}_s$, etc. (see e.g. Clair, Hamielec & Pruppacher 1970 for subsonic drag), one could still qualitatively consider such a turbulent drag within the framework above, with the properties of the turbulence determining...
and ζω. We do not do this here, but note that because \( r_{\text{grain}} \) increases with \( \tilde{w} \) and \( \rho \) (through \( \lambda_{\text{drag}} \)), we expect \( t_s \) to decrease with \( \tilde{w} \) and \( \rho \), viz., \( \xi_s > 0 > \xi_w > 0 \). The general scalings are thus likely similar to the Epstein case, but with a larger \( \xi_w \) for \( \tilde{w}_s \ll 1 \), because the velocity dependence of the drag will be significant, even for subsonic streaming.

Of course we can still simply calculate what the mode growth rates would be, if the usual Stokes expression applied even for \( \tilde{w}_s \gtrsim 1 \). This is shown in Fig. 3, for the sake of completeness.

### 4.4 Coulomb drag

The standard expression\(^{13}\) (in physical units) for \( t_s \) in the Coulomb drag limit is

\[
t_s = \sqrt{\frac{\pi \gamma}{2}} \frac{\rho_\theta R_d}{\rho \ln \Lambda} \left( \frac{k_B T}{\xi}, \frac{\rho \ln \Lambda}{\xi} \right)^2 \left[ 1 + a_C \frac{\lvert \mathbf{v} - \mathbf{u} \rvert^2}{c^2} \right],
\]

\[
\Lambda = \left( \frac{3 k_B T}{2 R_d \xi^2 U} \right)^{-\frac{1}{2}} \left( \frac{m_i k_B T}{\rho \pi} \right) \frac{a_C}{\sqrt{\frac{2 \gamma^3}{9 \pi}}},
\]

where \( \ln \Lambda \) is the Coulomb logarithm, \( \epsilon \) is the electron charge, \( \xi \) is the mean gas ion charge, \( m_i \) is the mean molecular weight, \( T \propto \rho^{\gamma-1} \) is the gas temperature, and \( U \) is the electrostatic potential of the grains, \( U \approx r_{\text{grain}} \epsilon / R_d \) (where \( r_{\text{grain}} \) is the grain charge). The behaviour of \( U \) is complicated and depends on a wide variety of environmental factors: in the different regimes considered in Draine & Salpeter (1979) they find regimes where \( U \propto \text{constant} \) and others where \( U \propto Z_{\text{grain}} \propto T \), we therefore parametrize the dependence by \( U \propto T^\xi \).

With this ansatz, we obtain

\[
\xi_s = 1 + \frac{3}{2} \frac{(\gamma - 1) - 1}{(1 + a_C \tilde{w}_s^2) \left[ 1 + a_C \tilde{w}_s^2 \right]} \approx \frac{1}{2 \ln \Lambda},
\]

\[
\xi_w = -\frac{3 a_C \tilde{w}_s^2}{1 + a_C \tilde{w}_s^2} < 0.
\]

For relevant astrophysical conditions, \( \ln \Lambda \approx 15-20 \), so the \( \ln \Lambda \) term in \( \xi_s \) is unimportant.

In general, Coulomb drag is subdominant to Epstein or Stokes drag under astrophysical conditions when the direct effects of magnetic fields on grains (i.e. Lorentz forces) are not important. None the less, the qualitative structure of the scaling produces similar features to the Epstein and Stokes drag laws, and we consider it here for completeness. In fact, grains influenced by Coulomb drag are significantly ‘more unstable’ than those influenced by Epstein or Stokes drag. For \( \tilde{w}_s \ll 1 \), \( \xi_s \rightarrow [3(\gamma - 4) + 5(3 - \gamma)] \log \Lambda / (2 \log \Lambda) \approx (5 - 3 \gamma) / 2 \) if \( \Gamma = 0 \), and \( \xi_s \rightarrow [((\gamma - 2) + (1 + \gamma)] \log \Lambda / (2 \log \Lambda) \approx (1 + \gamma) / 2 \) if \( \Gamma = 1 \). Since \( \xi_w \rightarrow 1 \), the ‘quasi-drift’ mode is unstable if \( \xi_s > 1 \) (for \( \Gamma = 0 \) this requires \( \gamma > -4 + 3 \log \Lambda / (3(1 + \log \Lambda)) \approx 0.98 \); for \( \Gamma = 1 \) this requires \( \gamma > 2 + 3 \log \Lambda / (1 + \log \Lambda) \approx 1.05 \). As noted above for the Epstein case (Section 4.2.2), because \( \xi_w \rightarrow 0 \) at small \( \tilde{w}_s \), the scaling of the ‘subsonic’ low-\( k \) mode is essentially reversed from the ‘quasi-drift’ high-\( k \) mode: when the ‘quasi-drift’ mode is stable at high-\( k \) (\( \xi_s < 1 \)) the ‘subsonic’ mode is unstable at low-\( k \), and when the ‘quasi-drift’ mode is unstable (\( \xi_s > 1 \)) the ‘subsonic’ mode is stable. In either case, whichever of the two is unstable has growth rate \( \sim k^2 \mu \lvert \xi_s \rvert / 2 \).

For \( \tilde{w}_s \gg 1 \), the drag force decreases rapidly for \( \lvert \mathbf{v} - \mathbf{u} \rvert > c_s \) (i.e. \( \xi_w \ll -1 \) when \( \tilde{w}_s \gg 1 \)). In this regime, one never expects Coulomb drag to dominate over Epstein drag (which becomes more tightly coupled at high-\( \tilde{w}_s \)), and in fact Coulomb drag alone does not allow self-consistent solutions for the equilibrium \( \tilde{w}_s \) in equation (2) without an additional Epstein or Stokes term when \( \tilde{w}_s \gg 1 \), but we consider the case briefly for completeness. We see that \( \xi_s \approx 1 \) for \( \Gamma = 0 \), and \( \xi_s \approx 2 \gamma - 1 \) for \( \Gamma = 1 \). More importantly, \( \xi_w \rightarrow -3 \). This produces the fast-growing ‘decoupling instability’ (Section 3.3), which affects all wavenumbers and has a growth rate. These modes arise from decoupling of the gas and dust: if the dust starts to move faster relative to the gas, \( t_s \) increases (the coupling becomes weaker), so the terminal/relative velocity increases further, and so on. If we ignore the decoupling mode, we see that each of the other modes we have discussed are still present: the high-\( k \) resonant mode (equation 16) has \( \Theta = (4 - 3 \gamma)/(2 \log \Lambda) \) for \( \Gamma = 0 \) and \( \Theta \approx 2(1 - \gamma) \) for \( \Gamma = 1 \).

### 5 Non-linear behaviour and turbulence

The non-linear behaviour of the coupled dust–gas system is and chaotic, and will be studied in future work with numerical simulations (Moseley et al. in preparation). Here, we briefly speculate on some possible saturation mechanisms of the acoustic RDI and subsonic instabilities.

For \( \tilde{w}_s \gtrsim 1 \), the resonant mode at the shortest wavelengths will grow fastest, with the dust density aligning locally to crests at the phase peaks with orientation \( \cos \theta = \pm 1 / \tilde{w}_s \). These will launch small-scale perturbations in the transverse directions in the gas. Because it is short wavelength, we do not expect the modes to be coherent on large scales, so this will drive small-scale turbulence in the gas in the transverse directions, while in the \( \tilde{w}_s \) direction, the modes will be stretched by the drift. For \( \tilde{w}_s < 1 \), the modes grow more slowly, and, depending on \( \xi_s \) and \( \xi_w \) (see Section 3.8), either saturate to a constant growth rate or turn over above a critical \( k \gtrsim \tilde{w}_s \). Thus, most of the power on large scales will be in modes of order this wavelength \( (k^{-1} \sim c_s^2 / (\mu \lvert \mathbf{a} \rvert)) \). If \( \mu \ll 1 \), dust will go strongly non-linear before the gas does, but eventually the non-linear terms will likely lead to turbulence in the gas and dust, at least for \( \mu \) not too small. Gas turbulence can then enhance dust-to-gas fluctuations (see e.g. numerical experiments with dust in supersonic turbulence in Hopkins & Lee 2016; Lee, Hopkins & Squire 2017). Eventually sharp dust filaments will form, and as the modes grow beyond this point, dust trajectories will cross and the fluid approximation for the dust will break down. Rayleigh–Taylor type secondary instabilities will likely appear, as regions with higher gas density are accelerated more rapidly, while those without dust are not dragged efficiently. It also seems possible that for \( \mu \ll 1 \) and/or \( \tilde{w}_s \) not very large, the modes saturate in a laminar way (e.g. by changing shape, or if the dust fluid approximation breaks down).

We can crudely guess the saturation amplitude of the non-linear turbulence by comparing the energy input (per unit mass) from the imposed acceleration (without including the bulk acceleration of the system),

\[
\frac{dE_{\text{accel}}}{dt} \sim \frac{d(m_{\text{dust}} v_{\text{dust}}^2) / dt}{m_{\text{dust}} + m_{\text{gas}}} \sim \frac{m_{\text{dust}} v_{\text{dust}}^2 \cdot a}{m_{\text{dust}} + m_{\text{gas}}} \sim \frac{\mu \lvert \mathbf{w}_s \rvert^2}{(1 + \mu) (\xi_s)}.
\]

\(^{13}\)Again, equation (27) is a polynomial approximation for more complex dependence on \( \lvert \mathbf{v} - \mathbf{u} \rvert \), given in Draine & Salpeter (1979). However, using this approximation versus the full expression makes no important difference to our results.
to the specific energy decay rate of turbulence
\[ \frac{dE_{\text{sub}}}{dt} \sim \frac{v_{\text{edd}}^2}{t_{\text{edd}}} \sim -\frac{\delta v_{\text{sat}}^3}{\lambda}, \]
where \( \lambda \) is the driving scale of the turbulence. Equations (29) and (30) gives \( \delta v_{\text{sat}} \sim (\hat{\mu} [w_s]^2 \lambda/(t_s))^{1/3} \). For each range of the RDI, we can then equate the turbulent dissipation rate \( \delta v_{\text{sat}}^3 \sim v_{\text{edd}}^2/t_{\text{edd}} \sim (\hat{\mu} [w_s]^2 / (t_s))^{1/3} \lambda^{-2/3} \) to the growth rate \( \dot{\omega} \), which should (in principle) allow for the estimation of a characteristic scale and saturation amplitude in the resulting turbulence. However, one finds that: (i) in the low-\( k \)-regime, with \( \dot{\omega} \sim (\hat{\mu} c_s / (t_s))^{1/3} \), the two are identical and there is no obvious characteristic \( \lambda \); (ii) in the mid-\( k \)-regime, with \( \dot{\omega} \sim (\hat{\mu} c_s / (t_s))^{1/3} \), the characteristic scale is \( \lambda \), which is outside of the range of validity of the mid-\( k \)-regime; and (iii) in the high-\( k \)-regime, with \( \dot{\omega} \sim (\hat{\mu} c_s / (t_s))^{1/3} \), the characteristic scale is \( \lambda / (c_s / (t_s)) \sim \tilde{\omega}_s^{-2} \), which is outside of the range of validity of the high-\( k \)-regime (if \( \hat{\mu} < 1 \)). Thus, we see that there is no obvious way for the system to choose a scale for resonant modes in any wavelength regime. What we instead have is that turbulence will begin on small scales and grow to larger and larger \( \lambda \), up to the scale of the system (if the given sufficiently long-time periods). One might also expect that this the characteristic scale would increase in time, in some way proportional to the growth rate at a given \( \lambda \). This suggests that \( \lambda \propto (\delta v / \lambda) t \) at early times (with the instability growing in the high-\( k \)-regime), \( \lambda \propto (\delta v / \lambda) t^{1/2} \) at intermediate times (in the mid-\( k \)-regime), then slowing to \( \lambda \propto t^{3/2} (\delta v / \lambda t^{1/2}) \) at longer times (in the long-wavelength regime). Qualitative behaviour – viz., turbulence that moves to larger and larger scales as a function of time – is observed in simulations of cosmic ray-driven instabilities, which have some similar characteristics to the dust–gas instabilities studied here (see e.g. Riquelme & Spikovksy 2009; Matthews et al. 2017).

6 SCALES WHERE OUR ANALYSIS BREAKS DOWN

We now briefly review the scales where our analysis breaks down.

(i) Non-linearity and orbit-crossing: If there is sufficiently sharp structure in the velocity or density fields, the dust trajectories become self-intersecting and the fluid approximation is invalid (for dust). In this limit, numerical simulations must be used to integrate particle trajectories directly. This should not occur in the linear regime (see appendix A of Jacquet et al. 2011 for more discussion).

(ii) Smallest spatial scales: At sufficiently short wavelengths (high-\( k \)) approaching the gas mean-free-path, dissipative effects will be important. For ionized gas, this scale is \( \lambda_{\text{gas}} \sim 10^{12} \text{ cm} (T/10^4 \text{ K})^2 (\eta_{\text{gas}}/\text{cm}^{-3})^{-1/3} \). If we assume Epstein drag with modest \( \tilde{\omega}_s \sim 1 \), this gives a dimensionless \( k_{\text{max}} \approx (2 \pi c_s / (t_s)) / \lambda_{\text{gas}} \sim 10^4 (R_d / \mu \text{ m}) / (T/10^4 \text{ K})^{-2} \gg 1 \).

In the dust, the fluid approximation breaks down on scales comparable to the dust-particle separation \( \lambda_{\text{dust}} \sim 10^6 \text{ cm} (R_d / \mu \text{ m}) (\eta_{\text{gas}}/\text{cm}^{-3})^{-1/3} (\mu / 0.01)^{-1/3} \), which is much smaller than \( \lambda_{\text{gas}} \) under most astrophysical conditions. Because each of these minimum scales (for the gas and the dust) are small, very small wavelengths (e.g. up to \( k \sim k_{\text{max}} c_s / (t_s) \)) are astrophysically relevant.

(iii) Largest spatial scales: At low-\( k \), we eventually hit new scale lengths (e.g. the gas pressure scale length). The physical scale where \( k \sim 1 \), i.e. where \( k \sim c_s / (t_s) \), can be large. For example, with Epstein drag at \( \tilde{\omega}_s \sim 1 \) this is \( k^{-1} \sim 10^{20} \text{ cm} (R_d / \mu \text{ m}) (\eta_{\text{gas}}/\text{cm}^{-3})^{-1} \). For relatively low-density starburst regions or GMCs affected by massive stars, this is only \( \sim 100 \) times smaller than the system scale, so the long-wavelength instability \((k c_s / (t_s) \ll \mu)\) will likely require a global analysis. However, in e.g. cool stars the densities are much higher and the scales correspondingly smaller; e.g. for \( \rho \sim \rho_{\odot} \), \( 10^{-12} \text{ g cm}^{-3} \) we obtain \( k_{\text{max}} c_s / (t_s) \sim 10^{-5} (R_{\odot}/100 R_{\text{sun}})^{-1} (R_d / \mu \text{ m}) \rho_{\odot}^{-1} \) (see section 8 for more details).

(iv) Maximum time-scales: Dust with \( \rho_c \) will drift through a system of size \( L_0 \) on a time-scale \( t_{\text{dust}} \sim L_0 / \rho_c \). An instability must grow faster than this to be astrophysically relevant. In Appendix C, we show that this is equivalent to the condition for background dust stratification terms to be subdominant. In units of the stopping time, the relevant time-scale is \( L_0 / (\rho_c / (t_s)) = \tilde{\omega}_s c_s / (t_s) \sim L_0 / (c_s / (t_s)) \sim \text{the time-scale criterion is closely related to the requirement that we consider modes smaller than the largest spatial scales. Another maximum time-scale is set by the time for the equilibrium solution (dust-gas) to be accelerated out of the system of size \( L_0 \), i.e. \( t_{\text{acc}} \sim (2 L_0 / \tilde{\mu} a)^{1/2} \) (or similarly, for e.g. a fast-accelerating wind to expand and change density). Noting \( \rho_c \sim |a| t_s / (1 + \mu) \), we have \( t_{\text{acc}} / t_s \sim \tilde{\mu}^{-1/2} (L_{\text{dust}} / t_s)^{1/2} \), so (since \( \tilde{\mu} \ll 1 \)) this is generally a less-stringent criterion.

7 RELATION TO PREVIOUS WORK

7.1 Winds from cool stars

In the context of dust-driven winds from red giants and other cool stars, there has been extensive work on other dust-related instabilities (including thermal instability, dust formation, Rayleigh–Taylor instabilities, magnetic cycles, etc.; see MacGregor & Stencel 1992; Hartquist & Havnes 1994; Soker 2000, 2002; Simis, Icke & Dominik 2001; Sandin & Höfner 2003; Woitke 2006a,b), but these are physically distinct from the instabilities studied here. Of course, simulations with the appropriate physics – namely, (1) explicit integration of a drag law with gas back-reaction (and compressible gas), (2) trans-sonic \( \tilde{\omega}_s \), (3) multidimensional (2D/3D) domains, and (4) sufficient resolution (for the high-\( k \) resonant modes) – should see the instabilities studied here. Most studies to date do not meet these conditions. Moreover, they often include other complicated physics (e.g. opacity and self-shielding, dust formation) which are certainly important, but make it difficult to identify the specific instability channel we describe here.

However, some authors have previously identified aspects of the instabilities described in this paper. Morris (1993) performed a much simpler linear stability analysis on a two-fluid mixture subject to drag (see also Mastrodemos, Morris & Castor 1996), and noted two unstable solutions whose growth rates saturated at high-\( k \); these are the ‘quasi-drift’ and ‘quasi-sound’ modes identified here. However, they assumed: (1) zero gas pressure (effectively \( \rho_c \rightarrow \infty \)), preventing identification of stability criteria; (2) a constant coupling coefficient; and (3) spherical symmetry (of the perturbations) which eliminates the resonant modes. Deguchi (1997) followed this up allowing for non-zero gas pressure, but retaining spherical symmetry and imposing the assumption that the dust always exactly follows
7.2 Starburst and AGN winds

In models of starbursts and AGN, there is a long literature discussing radiation pressure on grains as an acceleration mechanism for outflows or driver of turbulence (see e.g. Heckman, Armus & Miley 1990; Scoville et al. 2001; Thompson, Quataert & Murray 2005; Krumholz & Matzner 2009; Hopkins & Elvis 2010; Murray, Quataert & Thompson 2010; Hopkins, Quataert & Murray 2011; Kuiper et al. 2012; Wise et al. 2012). But almost all calculations to date treat dust and gas as perfectly coupled (so the instabilities here cannot appear). The instabilities in this paper are not related to the ‘radiative Rayleigh–Taylor’ instability of a radiation pressure-supported gas+dust fluid (Krumholz & Thompson 2012; Davis et al. 2014), nor to non-linear hydrodynamic instabilities generated by e.g. pressure gradients or entropy inversions ultimately sourced by dust ‘lifting’ material (e.g. Berruyer 1991), nor the dust sedimentation effects in ambipolar diffusion in molecular clouds discussed in Cochran & Ostriker (1977); Sandford, Whitaker & Klein (1984).

Each of these other classes of instability do not involve local dust-to-gas ratio fluctuations.

There recently has been more work exploring dust–gas decoupling in molecular cloud turbulence and shocks (integrating the explicit dust dynamics; see Hopkins & Lee 2016; Lee et al. 2017; Monceau-Baroux & Keppens 2017) which has shown this can have important effects on cooling, dust growth, and star formation. However, these studies did not identify instabilities, or include the necessary physics to capture the instabilities here, because they treated dust as a ‘passive’ species (did not include its back-reaction on the momentum of gas).

7.3 Proto-planetary discs

There has been extensive study of dust–gas instabilities and dynamics in proto-planetary discs (Youdin & Goodman 2005; Johansen & Youdin 2007; Carballido, Stone & Turner 2008; Bai & Stone 2010a,b; Pan et al. 2011; Dittrich, Klahr & Johansen 2013; Jalali 2013; Hopkins 2016; Lin & Youdin 2017). As mentioned in SH, the well-studied ‘streaming instability’ (Youdin & Goodman 2005) is in fact an example of an RDI (although this has not been noted before in this context), a connection that is explored in detail in Squire & Hopkins (2018a). However, in the streaming instability, the wave with which the dust drift ‘resonates’ is not a sound wave, but epicyclic oscillations of the gas. Similarly, as shown in SH (see also Appendix C), Brunt–Väisälä oscillations create an RDI, which may be of importance in proto-planetary discs (this is likely the cause for the instability seen in Lambrechts et al. 2016). The acoustic RDI has not been explored in this literature. In fact, it is common in these studies to simplify by assuming incompressible gas (enforcing δρ = 0), in which case all of the acoustic instabilities studied here vanish. Finally, it is worth noting that dust-induced instabilities that occur due to the mass loading of the gas caused by dust (see e.g. Garaud & Lin 2004; Takeuchi et al. 2012) or from changes to its thermodynamic properties (e.g. Lorén-Aguilar & Tate 2015, and some of the instabilities discussed in Lin & Youdin 2017), are not in the RDI class, because they do not rely on the finite drift velocity between the dust and gas phases.

7.4 Plasma instabilities

As noted in SH, the most general RDI is closely related to instabilities of two-fluid plasmas (see e.g. Tytarenko et al. 2002 for an in-depth analysis of a closely related coupled neutral gas–MHD instability). These include the Wardle (1990) instability and cosmic ray streaming instabilities (Kulsrud & Pearce 1969; Bell 2004). However, these are quite distinct physical systems and the instabilities have different linear behaviours.

8 ASTROPHYSICAL APPLICATIONS

There are a number of astrophysical contexts where this specific example of the SH instability may be important, which we review here. In the discussions below, we estimate the radiative acceleration of the dust from a \( a \sim F_i Q_i \rho_d / (c R_d) \), where \( |F_i| \sim L r^2 \) is the incident flux of radiation from a source of luminosity \( L \) at distance \( r \), \( c \) is the speed of light, and \( Q_i \) is the absorption efficiency (\( Q_i \sim 1 \) for very large grains, \( Q_i \propto R_d \) for smaller grains; see Section 2.3.2)

(i) AGN-driven outflows and the AGN ‘torus’:

Around a luminous AGN, gas and dust are strongly differentially accelerated by radiation pressure. There is some dust sublimation radius close to the AGN, interior to which dust is destroyed. The instabilities must occur outside this region in the dusty ‘torus,’ or further out still, in the galactic narrow-line region.

We assume the AGN has luminosity \( L \sim L_{46} 10^{46} \) erg s\(^{-1}\), and normalize the radius of the dusty torus to the dust sublimation radius, i.e. For a mid-plane column density \( n_{gas} r \sim N_{20} \) cm\(^{-2}\), and gas temperature \( K \), we find that we are in the highly supersonic regime with \( \tilde{v}_r \sim 100 L_{46}^{1/2} / N_{20}^{1/2} \) (dust is in the Epstein regime; see equation 22). For grains with size \( R_d \sim R_d \mu m \), the stopping time is \( (t_s) \sim 0.01 yr R_d^{1/4} L_{46}^{1/2} N_{20}^{-1/2} \) and the characteristic length scale is \( (l_s) \sim 6 \times 10^{10} \) cm \( R_d^{1/4} L_{46}^{1/2} N_{20}^{3/2} / (T_{100}/N_{20})^{1/2} \) (this is \( \sim 10^{-7} r \), and \( \sim 1000 \) times the viscous scale). Thus, the large-scale dynamics are in the long-wavelength regime \( (k \ll \mu) \), with growth time-scales (see equation 9) (where \( \lambda \) is the mode wavelength and we assume the dust-to-gas mass ratio scales with \( Z/Z_\odot \)). This is faster than the dynamical time, and the turbulent eddy turnover time, on essentially every scale inside the torus. Much smaller scale modes (\( \lambda \ll \mu \)) fall into the mid-\( k \) resonant regime, with the fastest growth time-scales of \( 3(\omega)^{-1} \sim 10–100 h \) for modes approaching the viscous scale (\( \lambda \sim 10^{-7} \) cm).

Thus, essentially all luminous AGN (\( L \gtrsim 10^{42} \) erg s\(^{-1}\)) should exhibit regions in the ‘clumpy torus’ surrounding the AGN, as well as radiation-pressure-driven AGN outflows, which are subject to the supersonic instabilities described above. This may provide a natural explanation for clumpiness, velocity substructure, and turbulence in the torus (see e.g. Krolik & Begelman 1988; Mason et al. 2006; Sánchez et al. 2006; Nenkova et al. 2008; Mor, Netzer & Elitzur 2009; Thompson et al. 2009; Hönig & Kishimoto 2010; Hopkins & Quataert 2010; Deo et al. 2011; Hopkins et al. 2012, 2016), as well as observed time-variability in AGN obscuration (McKernan & Yaqoob 1998; Risaliti, Elvis & Nicastro 2002). It is of course critical to understand whether this directly alters the AGN-driven winds in the torus region, a subject that will be addressed in future numerical simulations (see e.g. Murray, Quataert & Thompson 2005; Elitzur & Shlosman 2006; see Ciotti & Ostriker 2007; Miller, Turner & Reeves 2008; Wada, Papadopoulos & Spaans 2009; Roth et al. 2012)
gas in the narrow/broad line regions. In this case, the scaling of \( \tilde{w}_1 \) depends on the opacity of the gas, but for plausible values in the narrow-line region, and similar luminosities and densities to those used above, we find \( \tilde{w}_1 \geq 10^{-2} \). (ii) Starburst regions, radiation-pressure-driven winds, and dust in the ISM around massive stars: Similarly, consider dusty gas in molecular clouds and HI regions surrounding regions with massive stars. It has been widely postulated that radiation pressure on dust (either single-scattering from optical/UV light or multiple-scattering of IR photons) can drive local outflows from these regions, unbinding dense clumps and GMCs, and stirring GMC or ISM-scale turbulence.

Assuming geometric absorption of radiation by the dust \( (\mathcal{Q} \approx 1) \), a random patch of gas in a GMC (with temperature \( T \sim T_{1000} \) 100 K, density \( n \sim n_{10} \) 10 cm\(^{-3}\)) at a distance \( r \sim r_{10} \) pc from a source with luminosity \( L \sim L_{1000} \) 1000 L\( \odot \) has \( \tilde{w}_1 \sim 10 L_{1000}^{1/2} n_{10}^{1/2} r_{10}^{-1} \). Similarly, consider a GMC of some arbitrary total mass \( M_d \) and total size \( r \sim r_{10} \) 10 pc, which has converted a fraction of its mass into stars. If we assume a typical mass-to-light ratio for young stellar populations \( (\sim 100 L_{\odot}/M_{\odot}) \), we find \( \tilde{w}_1 \sim 10^{1/2} n_{10}^{1/2} \).

For smaller (typical ISM) \( R_d \sim 0.1 R_{100} \), the micrometer (Young et al. 2003; Ziyurys et al. 2007; Agúndez, Cernicharo & Guélin 2010; Cox et al. 2012). The latter would almost certainly trigger secondary non-linear instabilities by driving large dust–gas clumping; for example via radiative Rayleigh–Taylor instabilities, dust opacity/self-shielding effects, and dust collisions/growth in the wind. (iv) Proto-planetary discs: As discussed in Section 7, instabilities of the coupled dust–gas system in proto-planetary discs are particularly interesting, given their implications for planet formation and observable disc properties. In proto-planetary discs we expect drift velocities to be highly subsonic. For a disc with parameters following Chiang & Youdin (2010) at radius \( r \sim r_{10} \) au and surface density \( \Sigma \sim \Sigma_{\text{MSN}} \) 100 g cm\(^{-2}\), pebbles with size \( R_d \sim R_{d, cm} \) cm will have \( \tilde{w}_1 \sim 0.005 r_{10}^{2/14} R_{d, cm} \Sigma_{\text{MSN}}^{-1} \) (Nakagawa, Sekiya & Hayashi 1986). Since \( \tilde{w}_1 \ll 1 \) we expect the growth rate of the instabilities here to have a maximum value \( 3(\alpha) \sim \tilde{w}_1^2 \mu I_{1}^{-1} \). For plausible disc parameters this rate is much slower than the radial drift rate \( \tilde{v}_{\text{diff}}/r \) for the grains to drift through the disc.

Given this relatively low growth rate, we do not expect this particular sound-wave resonance (the acoustic RDI) to be dominant. However, we do expect other examples from the broad class of RDI resonances to be interesting. For example, as noted in SH and above, the well-studied disc ‘streaming instability’ is an RDI associated with the disc epicyclic frequency. Other wave families such as Brunt–Väisälä oscillations, slow magnetosonic, and Hall magnetosonic-cyclotron waves are also present with slow phase velocities, which can give rise to much larger growth rates (as compared to the acoustic RDI studied here) when \( \tilde{w}_1 \ll 1 \). These are explored in Squire & Hopkins (2018a).
a streaming velocity that ‘resonates with’ the wave phase velocity usually creates an instability (the RDI). In this work, we focus on the case where the gas is governed by neutral hydrodynamics and supports sound waves, studying the ‘acoustic RDI’ (resonance with sound waves) and a collection of other non-resonant unstable modes (these are important in certain regimes, e.g. at long wavelengths or high dust-to-gas ratios). Although neutral hydrodynamics is perhaps the simplest gas system possible, these instabilities have not (to our knowledge) been studied or identified in previous literature, despite their likely relevance for a wide variety of astrophysical systems.

We identify a spectrum of exponentially growing linear instabilities which directly source fluctuations in the dust-to-gas ratio. Under certain conditions all wavelengths feature unstable modes, some of which have growth rates that increase without limit with increasing wavenumber. We show that the basic qualitative behaviours (dimensional scalings and nature of the fastest growing modes) are not sensitive to the gas equation of state, the form of the drag law (constant drag coefficient, Epstein, Stokes, or Coulomb drag), the dust-to-gas ratio, or other details, although these do quantitatively alter the predictions. We derive stability conditions and simple closed analytic expressions for the growth rates of the instability (Section 3).

There is one critical dimensionless parameter that determines the system’s qualitative behaviour, viz., ratio of the mean dust drift velocity \( \langle |\mathbf{v}_{\text{dust}} - \mathbf{u}_{\text{gas}}|^{\text{drift}} \rangle \) to the gas sound speed \( c_s \):

\[
\tilde{w}_s \equiv \left( \frac{\mathbf{v}_{\text{dust}} - \mathbf{u}_{\text{gas}}^{\text{drift}}}{c_s} \right) = \left( \frac{\Delta a_{\text{dust-gas}}}{c_s} \right) \bar{t}_s \left( \alpha, \beta, \ldots \right) \left( \begin{array}{cc} \mu k \end{array} \right) \left( \begin{array}{cc} c_s \end{array} \right) \left( \begin{array}{cc} 1 + \mu \end{array} \right). \tag{31}
\]

Here, the drift velocity \( \tilde{w}_s \) is the ‘terminal’ velocity when the dust and gas experience accelerations which differ by some amount \( \Delta a_{\text{dust-gas}}, \bar{t}_s \) is the drag coefficient or ‘stopping time’ (determined by the drag law), and \( \mu \) is the dust-to-gas mass ratio.

When \( \tilde{w}_s \gtrsim 1 \), i.e. when the dust is moving supersonically relative to the gas, the system is strongly unstable at all wavelengths. There are multiple unstable modes but the acoustic RDI from SH (Section 3.7.1) is the most rapidly growing. The growth rate \( \mathcal{A}(\omega) \) increases without limit with increasing wavenumber \( k \) as \( \mathcal{A}(\omega) \sim (\mu_k k \bar{t}_s)^{1/3} \) (in a mid-range of \( k \)) or \( \mathcal{A}(\omega) \sim (\mu_k k \bar{t}_s)^{1/3} \) (at high-\( k \)), independent of \( \tilde{w}_s \). These modes propagate at a critical angle \( \cos \theta = \pm 1/\tilde{w}_s \) with respect to the drift direction; the wavevector is the normal speed of the drift, and the speed drift along the wavevector \( \tilde{k} \) exactly matches this, allowing the dust to coherently push gas, and generate density perturbations. The denser gas then accelerates the dust further, causing a pileup, which runs away. For modes at angles that do not match the resonance condition \( \cos \theta \neq \pm 1/\tilde{w}_s \), the growth rates saturate at finite values (i.e. \( \mathcal{A}(\omega) \) does not increase indefinitely with \( k \)).

When \( \tilde{w}_s < 1 \), i.e. when the dust is moving subsonically relative to the gas, the resonance above does not exist but there are still unstable, long-wavelength modes whose growth rate peaks or saturates above some wavenumber \( k \propto \tilde{w}_s/(c_s \bar{t}_s) \), with maximum growth rate \( \mathcal{A}(\omega) \propto \tilde{w}_s^3 \mu / \bar{t}_s \).

### 9.2 Implications, caveats, and future work

In all cases, the instabilities drive dust–gas segregation and local fluctuations in the dust-to-gas ratio, compressible fluctuations in the gas density and velocity, and clumping within the dust (Section 3.9). Non-linearly, we expect them to saturate by breaking up into turbulent motions (in both dust and gas) which can be subsonic or supersonic, and in both cases can give rise to large separations between dense gas-dominated and dust-dominated regions.

We provide simple estimates for the saturated turbulent amplitude (Section 5).

We discuss some astrophysical implications of these instabilities (Section 8) and argue that the ‘resonant’ instability is likely to be important in the dusty gas around AGN (in the torus or narrow-line regions), starbursts, giant molecular clouds, and other massive-star-forming regions, where \( \tilde{w}_s \gtrsim 1 \) almost everywhere. In the winds and photospheres of cool stars, simple estimates suggest \( \tilde{w}_s \sim 1 \), with a broad range depending on the local conditions and location in the atmosphere. Thus, we again expect these instabilities to be important. In each of these regimes, the instability may fundamentally alter the ability of the system to drive winds via radiation pressure (on the dust or the gas), and could source turbulence, velocity substructure, clumping, and potentially observable inhomogeneities in the winds.

More detailed conclusions will required detailed numerical simulations to study the non-linear evolution of these systems. Our analytic results here make it clear what physics must be included to study such instabilities – in particular, physical drag laws (with realistic density and velocity dependence) and back-reaction from the dust to the gas – and the range of scales that must be resolved. Most previous studies of such systems either did not include the appropriate drag physics or lacked the resolution to treat these modes properly. This is especially challenging for the resonant mode: because the growth rate increases without limit at high-\( k \), it could (in principle) become more important and grow ever-faster as the simulation resolution increases.

We have focused on a relatively simple case here, namely gas with a pure acoustic wave in the absence of dust. This ignores, for example, magnetic fields, which alter the mode structure and could influence the grain ‘drag’ directly (if the grains are charged); this case is explored in more detail in a companion paper, Hopkins & Squire (2018). As shown in SH, the RDI generically exists for systems that support undamped linear waves, so we expect a similar rich phenomenology of instabilities (both resonant and non-resonant) in other systems. However, it is outside the scope of this work to explore these in detail.

Another topic which we will explore in more detail in future work is the influence of a broad size spectrum of dust grains. This is discussed in Section 2.3.2, where we argue that under most conditions, we can think of the results of this work as being relevant for the large grains (specifically, the largest grains which contain a large fraction of the grain mass), because these dominate the mass and back-reaction on the gas. However as shown there, under some circumstances there is a complicated mix of terms dominated by small grains and others dominated by large grains, which could couple indirectly. Moreover, because the RDI can resonate with any wave family, it is possible that (for example) small, tightly coupled grains (which may be more stable if considered in isolation) generate wave families to which larger grains can couple via the RDI (or vice versa).

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APPENDIX A: RELATION TO THE MATRIX FORMALISM OF SQUIRE & HOPKINS (2018B)

Throughout the main text, our analysis was carried out through asymptotic expansions of the dispersion relation, so as to allow investigation into non-resonant modes (e.g. for \(|w_s| < c_s\), and the ‘long-wavelength’ modes). To clarify the link to the RDI derivation in SH, in this appendix, we calculate the acoustic RDI growth rates using the Jordan-form perturbation theory formalism of SH. We use the dimensionless variables of Section 3 (equation 6), and, for the sake of concreteness, set \(\bar{\omega} = \bar{z}\) and \(\bar{k}_z = \bar{x}\) (it was not necessary to choose a specific direction in derivation of the dispersion relation, equation 7). We also ignore \(u_t\) and \(v_t\), because these are decoupled from the sound-wave eigenmodes (these propagate in the \(\bar{k}\) direction).

From equation (3), the coupled dust–gas equations are

\[
\dot{\bar{\xi}} = \begin{pmatrix} \bar{\mu} T^{(1)}_{\rho_d} & \bar{\mu} T^{(1)}_s & \bar{\mu} T^{(1)}_g \end{pmatrix} \bar{\omega}_1 \bar{\xi},
\]

(A1)

where \(\bar{\xi} = (\delta p_d/\rho_0, \delta v_z/c_s, \delta v_z/c_s, \delta p_d/\rho_0, \delta u_t/c_s, \delta u_t/c_s)^T, \bar{k}^T = (\bar{k}_x, \bar{k}_z), T^{(1)}_{\rho_d} = (0, 0, i \bar{w}_s \bar{k}_x, T^{(1)}_s = (0, i \bar{w}_s \bar{k}_x, T^{(1)}_g \) and \(T^{(1)}_g \) are not needed,

\[
D_{\text{drag}} = \begin{pmatrix} 0 & 0 & -i \bar{\xi}_w \\ 0 & 0 & -i \bar{\xi}_w \end{pmatrix},
C_v = \begin{pmatrix} 0 & i & 0 \\ -i \bar{w}_s \bar{k}_x & 0 & -i \bar{\xi}_w \end{pmatrix},
\]

(A2)

and

\[
\mathcal{F} = \begin{pmatrix} 0 & \bar{k}_x & \bar{k}_z \\ \bar{k}_x & 0 & 0 \\ \bar{k}_z & 0 & 0 \end{pmatrix}.
\]

(A3)

When at resonance, i.e. (where \(\bar{\omega} = \bar{k}\)) is forward-propagating sound-wave eigenvalue of \(\mathcal{F}\), the matrix in equation (A1) is defective. This means that although \(\bar{\omega} = \bar{k}\) has multiplicity 2, it has only one associated eigenvector. This is associated with an RDI, the growth rate of which scales as \(\sim \mu^{1/2}\) because the matrix is singular (rather than \(\sim \mu\) as for standard perturbation theory). From SH (their equation 10), the perturbed eigenvalues in the ‘mid-k’ regime (before \(\bar{k}^T \gg D_{\text{drag}}\) in equation A1) are

\[
\bar{\omega} = \bar{k} + i \mu^{1/2} \left[ (\bar{k}_x T^{(1)}_{\rho_d} (\bar{k}^T D_{\text{drag}}^{-1} C_v \bar{k}_x)^{1/2} \right] + O(\mu). \tag{A4}
\]

Here

\[
\bar{k}_x = \frac{1}{\sqrt{2}} k \begin{pmatrix} k & k \end{pmatrix}, \quad \bar{k}_x = \frac{1}{\sqrt{2}} k \begin{pmatrix} k \\ k \end{pmatrix},
\]

(A5)

are the left and right eigenvectors of the (forward-propagating) sound wave. Equation (A4) is easily verified to be the same as equation (15) from the main text, up to \(O(\mu)\).

In the ‘high-k’ regime, the eigenvalue \(\bar{\omega} = \bar{k}\) is nearly triply defective (meaning it has multiplicity 3 with one associated eigenvector), because \(\bar{k}^T \gg D_{\text{drag}}\). The perturbed eigenvalue is then

\[
\bar{\omega} = \bar{k} + \mu^{1/3} \left[ (\bar{k}^T \bar{k}_x T^{(1)}_{\rho_d} (\bar{k}^T D_{\text{drag}}^{-1} C_v \bar{k}_x)^{1/3} \right] + O(\mu^{2/3}), \tag{A6}
\]

which matches equation (16) from the main text.

We cannot treat the ‘long-wavelength’ instability (Section 3.4) using this method, because \(\mu \ll \bar{k}\). In this regime, in other words, \(\mu T^{(1)}_{\rho_d}, \mu T^{(1)}_s\), and \(\mu T^{(1)}_g\) are no longer a small perturbation to the fluid, and there is no well-defined undamped sound wave with which the dust can resonate (see Section 3.9 and Fig. 2 for further discussion). The long-wavelength growth rate equation (9) can be derived from the matrix (equation A1) by treating \(\bar{k}\), and \(\mathcal{F}\) as a small perturbation to \(D_{\text{drag}}, C_v\), and \(T^{(1)}\) (i.e. assuming small \(k\)). However, the procedure is not particularly illuminating (or, for that matter, easier algebraically than using the dispersion relation), so we do not reproduce it here.

APPENDIX B: RELATION BETWEEN FREE-FALLING AND STATIONARY FRAMES

In Section 2.1, we transformed to a free-falling frame to analyse the instability. Here, we derive this transformation in greater detail, and relate the mode properties in the free-falling and stationary frames.

In the stationary frame, the fluid equations (equation 1) have homogeneous steady-state solutions given in equation (2). Consider small perturbations in this frame: \(\rho = \rho_0 + \delta \rho, \mu = \mu_0 + \delta \mu, \quad u = u_0 + \tilde{u} t + \delta u, \quad v = v_0 + \tilde{u} t + w_s + \delta v\), where \(\tilde{u} \equiv g + \mu / (1 + \mu)\). Note that both \(u\) and \(v\) contain both an arbitrary constant velocity offset \((u_0)\) and a linear acceleration \(\tilde{u} t\).

Inserting these into equation (1) and linearizing in the perturbative \((\delta)\) terms, we obtain the perturbation equations in the stationary frame:

\[
\begin{align*}
\frac{\partial}{\partial t} + \bar{u}_0(t) \cdot \nabla \delta \rho &= -\rho_0 \nabla \cdot \delta u, \\
\frac{\partial}{\partial t} + \bar{u}_0(t) \cdot \nabla \delta u &= -c_s^2 \frac{\nabla \delta p_d}{\rho_0} + \mu \frac{\delta v - \delta u}{\rho_0}, \\
\frac{\partial}{\partial t} + \bar{u}_0(t) \cdot \nabla \delta v &= \frac{\delta v - \delta u}{\rho_0} + \frac{w_s}{\rho_0} \frac{\delta t_s}{t_s}, \\
\bar{u}_0(t) \equiv u_0 + \tilde{u} t &= u_0 + \left[ g + \frac{\mu}{1 + \mu} \right] t.
\end{align*}
\]

(B1)

To see the relationship between these stationary-frame equations (where \(u = u_0 + \delta u\)) and those in the free-falling frame (equation 3, where \(u = \delta u\)), consider e.g. the gas continuity equation: \(\partial \rho/\partial t + \nabla \cdot (u \rho) = 0\). Compared to the free-falling equations (equation 3), we see that the time-derivative of \(\rho\) is unchanged, but the term \(\nabla \cdot (u \rho) = u \cdot (\nabla \rho) + \rho (\nabla \cdot u)\) gives rise to an additional term \((u_0 + \tilde{u} t) \cdot \nabla \delta \rho = (\tilde{u} \cdot \nabla) \delta \rho\). Note that the time-derivatives of \(u_0\) which appear in \(u\) and \(v\) are part of the homogenous solution, so do not appear in the linearized equations (equation B1).

In this stationary frame, if we make the usual Fourier ansatz, where the terms in \(\delta \propto \exp \{i (k \cdot x - \omega t)\},\) the fact that \(\tilde{u}_0\) is time-dependent prohibits a time-independent solution for \(\omega(k)\). However, note that the time derivatives \(\partial / \partial t\) in equation (B1) appear exclusively in the combination \(\partial / \partial t + \tilde{u} t\). Motivated by this, consider the modified Fourier ansatz of the form:

\[
\delta \propto \exp \left\{ i (k \cdot x - \omega t) - i \left[ \omega + \left( u_0 + \frac{1}{2} \tilde{u} t \right) \cdot k \right] t \right\},
\]

(B2)

Inserting this, one finds that the time and spatial derivatives behave as

\[
\frac{\partial}{\partial t} + \bar{u}_0(t) \cdot \nabla \delta = -i \omega \delta,
\]

(B3)

\[\nabla \delta = i k \delta.\]

(B4)
In terms of \(\omega\) and \(k\), we therefore obtain identical expressions for the dispersion relations as those derived in the main text in the free-falling frame (equation 3).

In other words, transforming from the free-falling frame to the stationary frame is equivalent to simply taking \(\omega \to \omega + u_0 \cdot k + (\ddot{a} t^2 / 2) \cdot k\). Along the direction of motion, the position of a wave crest is simply given by \(x = \omega / k + u_0 t + (1/2) \ddot{a} t^2\). So we immediately see that the offset in \(\omega\) simply corresponds to motion with the homogeneous solution, which has position \(u_0 t + (1/2) \ddot{a} t^2\). Physically, transforming into any linearly accelerating and/or uniformly boosted frame has no effect on the character of the solutions.

Another, simpler way of seeing this is to return to the original, fully general non-linear equations (equation 1), and boost to a free-falling (uniformly accelerating) frame with spatial and time coordinates \(t' = t, x' = x + u_0 t + (1/2) \ddot{a} t^2\). In a uniformly accelerating frame the local equations of motion are necessarily identical in these variables, up to the introduction of a fictitious force/acceleration \((\tilde{a}^2 = -a)\) felt by both the gas and dust. Alternatively see that the offset in \(\omega\) simply corresponds to motion with the homogeneous solution, which has position \(u_0 t + (1/2) \ddot{a} t^2\).

In Appendix C below, we show that the resulting instabilities are similar to those derived in the free-falling frame. Including background pressure/entropy gradients, we must also explicitly include an entropy equation, which takes the form \(D \rho_0 \dot{P}_0 \equiv 0\) or \(D P_0 / Dt = c_s^2 D P_0 / Dt\) (where \(D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla\)). Note the entropy equation was implicit in the main text (equation 1), because without background gradients it just trivially simplifies to \(\delta P = c_s^2 \delta \rho\) at linear order. Similarly, since pressure and density can vary independently, we de-compose the perturbations to \(t_i\) into separate pressure and density terms, i.e.

\[
\frac{\partial \delta t_i}{\partial t} = -\delta \rho \rho_0 - (\delta s - \delta t) \frac{\partial P}{\rho_0 c_s^2} - \frac{\frac{\partial \delta w}{\partial t} \cdot \nabla (\delta v - \delta s)}{|\nabla|^2},
\]

where \(\delta s = \delta s_i - \delta t\), represent perturbations to \(t_i\) from density or pressure fluctuations, respectively (with the other fixed). Note that we explicitly write this in this manner so that \(\delta s_i\) has the same meaning in the text: when \(\delta s_i \approx c_s^2 \delta \rho\) (as occurs without gradients in \(P_0\) or \(\rho_0\)), one finds \(\delta s_i \delta P / \rho_0 + (\delta t_i - \delta s_i) \delta P / c_s^2 \approx \delta \rho / \rho_0\).

We will show that to leading order, only the ‘total’ term \(\delta s_i\) appears. Combining this and equation (C1) with equation (1), and subtracting the steady-state solution, we obtain the linearized equations:

\[
\frac{\partial \delta \rho}{\partial t} = -\rho_0 \nabla \cdot \delta \mathbf{u} - \delta \mathbf{u} \cdot \nabla \rho_0,
\]

\[
\frac{\partial \delta \mathbf{u}}{\partial t} = -\nabla \delta P - \delta \mathbf{u} \cdot \nabla \rho_0 + \mu \left(\frac{\delta \mathbf{v} - \delta \mathbf{s}}{t_i}\right),
\]

\[
\frac{\partial \delta P}{\partial t} + \delta \mathbf{u} \cdot \nabla P_0 = c_s^2 \left(\frac{\partial \delta \rho}{\partial t} + \delta \mathbf{u} \cdot \nabla \rho_0\right),
\]

\[
\frac{\partial \delta \mathbf{v}}{\partial t} + \delta \mathbf{u} \cdot \nabla \mathbf{v} = -\delta \mathbf{u} + \frac{\delta \mathbf{w}_s}{t_i} - (\delta \mathbf{v} \cdot \nabla) \mathbf{w}_s.
\]

Note that if we take \(\mu \to 0\), the gas equations immediately reduce to the familiar standard equations for acoustic perturbations in a stratified fluid (Bray & Loughhead 1974).

C2 Degrees of freedom and validity

Locally, equation (C1) permits arbitrary 3D gradients in \(P_0, \rho_0, \mathbf{v}_d, \mathbf{w}_s\), and each component of \(\mathbf{w}_s\), with only one constraint equation.16 Moreover, if the problem has arbitrary 3D asymmetry we must consider 3D wavevectors \(\mathbf{k}\) (we cannot treat the \(k_z\) direction as symmetric in the plane perpendicular to \(\mathbf{w}_s\)). Formally therefore

16In equation (C1), \(\mathbf{w}_s \cdot \nabla \rho_0 = -\rho_0 \mathbf{v} \cdot \mathbf{w}_s\); each equation removes one degree of freedom if it is to be a true equilibrium. The equations for \(\nabla P_0\) and \((\mathbf{w}_s \cdot \nabla) \mathbf{w}_s\), only relate these quantities in equilibrium to the arbitrary input vectors \(\mathbf{g}\) and \(\mathbf{a}\); they do not reduce the number of degrees of freedom of the problem.
this introduces 18 degrees of freedom into the dispersion relation. Fortunately, as shown below, only a couple of these degrees of freedom have any influence on the modes, within the constraints required for our local derivation to be valid.

As noted above, lacking a global solution and/or boundary conditions, equation (C3) is valid only up to leading order in $\mathcal{O}(k_U/k)$ ($k \gg k_U$), where $k_U$ is the gradient scale length of some background quantity $U$. Moreover, if the velocities $v$ (e.g. $w$, or the mode phase/group velocities $v_0 \sim c_s$) are non-zero, then our derivation is also only valid on a full-scale time-scale $\Delta t \ll 1/(v k_U)$.

Over time-scales longer than this, the mode and/or incoming dust travels a distance greater than $k_U^{-1}$, outside the domain where our local gradient expansion is valid. Thus, we also require $|\omega| \gg v k_U$ (although if $\omega \sim c_s k$ to leading order, this condition is identical to $k \gg k_U$).

Another obvious requirement is that the dust stopping length $L_{\text{stop}} \sim |w|_s \langle t_s \rangle$ (the distance the dust travels in one stopping time) is small compared to the gradient scale lengths of the system ($|w|_s \langle t_s \rangle \ll k_U^{-1}$). Otherwise the dust simply drifts through a full scale length without feeling significant coupling to the gas. In that case the system could never meaningfully reach local equilibrium and a global solution is clearly required.

Considering the dust-density and drift-speed scale lengths, $k_{\rho d, 0} = |\nabla \rho|_0/|\nabla w|_0 \approx k_0 = |\nabla \cdot w|_0/|w|_0$ (related by equation C1), we see that $L_{\text{stop}} \ll k_{\rho d, 0}$ or $L_{\text{stop}} \ll k_0^{-1}$ is equivalent to $|w|_s \langle t_s \rangle \ll \rho_d/|\nabla \rho_d|_0 \sim |w|_s/|\nabla w|_0$, i.e. $|\langle t_s \rangle \nabla \cdot w_s|_0 \ll 1$.

C3 Dispersion relation and scalings (simplified case)

Above we noted the full set of gradients introduces 18 degrees of freedom. Analysing this is generally uninteresting, however, and many parameter combinations have no effect on the modes, or are formally allowed but unphysical.

The analysis is greatly simplified if we consider one of two cases: (a) either gravity or the external acceleration dominates, i.e. $|g| \gg |a|$ (e.g. dust settling through a hydrostatic, self-gravitating atmosphere) or $|g| \ll |a|$ (e.g. radiative acceleration of a dust-driven wind) or (b) $g$ and $a$ are parallel. In either of these cases, the equilibrium solution should be symmetric about this preferred axis. Then the gradient terms can be expressed as

$$\nabla \cdot w_s = \frac{|w|_s}{c_s} \Lambda_w, \quad \nabla \cdot \rho_d = - \frac{\rho_d}{c_s} \Lambda_w, $$

$$\nabla \cdot P_b = \frac{c_s^2 \rho_0}{c_s} \Lambda_{P_b}, \quad \nabla \cdot \rho_0 = \frac{\rho_0}{c_s} \Lambda_{\rho_0}, $$

$$\Lambda_w = \frac{c_s (\langle g + a \rangle \cdot \nabla \cdot w_s (\langle t_s \rangle)}{|w_s|_0} \left[ \frac{|g + a|}{|w_s|_0} \langle t_s \rangle + \mu |w|_s \right],$$

$$\Lambda_{P_b} = \frac{c_s (\langle g + a \rangle \cdot \nabla \cdot P_b (\langle t_s \rangle)}{c_s} = \frac{\langle g + a \rangle \cdot \nabla \cdot P_b (\langle t_s \rangle)}{c_s},$$

where $\nabla \cdot \equiv \nabla \cdot \nabla = \nabla |\cdot|$, the gradient along the drift direction, and the latter two equations are constraints arising from the momentum equations. These equations define three dimensionless parameters, $\Lambda_{\rho_0}, \Lambda_{P_b}$, which are proportional to the relevant gradient scale lengths in the parallel direction (e.g. $\Lambda_w = k_w c_s (\langle t_s \rangle = -k_{\rho d, 0} c_s (\langle t_s \rangle$, $\Lambda_P = k_p c_s (\langle t_s \rangle = (1/\gamma) k_{P_b} c_s (\langle t_s \rangle$. Since we have allowed for arbitrary background entropy profiles, there is no equation to determine $\nabla \cdot \rho_d$ and $\Lambda_{\rho_d}$ is an arbitrary parameter. For an adiabatic (isentropic) background pressure gradient, $\Lambda_{\rho_d} = \Lambda_P$ (this is convenient below and the reason for our particular definition here), while for a pure entropy gradient (with constant background density), $\Lambda_{\rho_d} = 0$.

Earlier we noted that $|w|_s \langle t_s \rangle |k_s| = \langle t_s \rangle |\nabla \cdot w_s| = \tilde{w}_s |\Lambda_{\rho_d}| \ll 1$ was required for our derivation to be valid. We typically expect $|w|_s \approx |g + a| (\langle t_s \rangle$ (the normal terminal velocity if the gas is in hydrostatic equilibrium), so $|\Lambda_{\rho_d}| \ll 1$, and this is satisfied (so long as $\tilde{w}_s$ is not extremely large, which is not usually expected in systems of interest). For streaming in a pressure-supported atmosphere that is only weakly perturbed by the dust (i.e. when $\mathcal{P}_b \approx \rho_d g$), we see that $|\Lambda_P| \ll 1$ is usually satisfied if the dust stopping length $L_{\text{stop}} \sim |w|_s \langle t_s \rangle$ is smaller than the pressure scale length $c_s /|\mu P|^{-1}$ (otherwise, a global solution is needed). If instead $\tilde{g}$ is weak (e.g. for highly supersonic streaming in a dust-driven wind), we see that $|\Lambda_P| \approx \mu \tilde{w}_s$.

For convenience of notation below, we define the generic inverse scale length $\Lambda \equiv \max(|\Lambda_{\rho_0}|, |\Lambda_{P_b}|, |\Lambda_P|)$, so the full set of conditions for a local derivation to be valid from Section C2 above become $|\tilde{w}_s| \Lambda \ll \min[1, |\tilde{a}|]$, and $\Lambda \ll \tilde{k}$.

As discussed in more rigorous mathematical detail in Squire & Hopkins (2018a), where we explore the Brunt–Väisälä RDI, at this point it is in principle possible to consider a fully general WKBJ analysis, assuming that the linear perturbations $\delta P$, etc. have the form $\exp [i \epsilon^{-1} \sum_{n=0}^\infty e^n Q(x)]$, keeping all terms in the background and deriving an expression for the frequencies $\omega$ to lowest order in the expansion parameter $\epsilon \ll 1$ (with $\epsilon$ some appropriate function of $\Lambda/\tilde{k}$). However, this is not enlightening: the expressions in full generality can only be expressed as complicated integro-differential functions of the background (which is unspecified), which can only be evaluated numerically (and then only if the background profiles are specified; see e.g. Bender & Orszag 1978). Moreover, the ordering of the expansion is fundamentally ambiguous, since above we note multiple independent small parameters (e.g. $|\Lambda/\tilde{k}|$ and $|\tilde{w}_s| \Lambda$) as well as other parameters which may also be small under some circumstances (e.g. $\mu$ or $\tilde{w}_s$). And there is no unique or obvious ‘preferred’ background as there is for common pure-hydrodynamic cases (e.g. an exponentially stratified vertical atmosphere), since we have introduced stratification of the dust properties. So instead we will consider a simpler local approximation in which we assume $|\Lambda/\tilde{k}| \ll 1$, $|\tilde{w}_s| \Lambda$, and that each of the background gradient terms $\Lambda_{\rho_0}, \Lambda_{P_b}, \Lambda_P$ is constant, so we can Fourier-decompose the perturbations keeping only the lowest order WKBJ term in $\Lambda/\tilde{k}$ (i.e. our usual Fourier ansatz for the perturbations), and solve them ‘locally’ in an infinitesimally small region about the ‘origin’ where the background quantities $\rho_0$, etc. and their gradients are defined.

Bear in mind, this means our solutions will only be valid to lowest order in this expansion, and should be regarded somewhat heuristically: but this still allows us to see if there are leading-order corrections which could be important when $|\Lambda/\tilde{k}| \ll 1$.

Finally, then, the full dispersion relation in this simplified case is a 9th-order polynomial, with roots given by the eigenvalues of
\[ b_0 = \hat{k}_i + i \{ \mu \hat{\xi}_i (\zeta_i - 1) + \Lambda_P \} , \]
\[ b_1 = -i \hat{\xi}_i \hat{w}_i \hat{k}_i \]
\[ b_2 = -i \hat{\xi}_i \hat{w}_i (\hat{k}_i - i \Lambda_w) , \]
\[ b_3 = \hat{w}_i (\hat{k}_i - i \Lambda_w) \]
\[ b_4 = \mu (\hat{k}_i + i \Lambda_w) , \]
\[ b_5 = \hat{k}_i + i \mu \hat{w}_i \zeta_p \]
\[ \kappa = |\hat{w}_i \times \hat{k}| = \hat{k} \sin \theta , \]

where

we use the same dimensionless units as in equation (7).

C4 Solutions without dust

Absent dust (i.e. for gas alone, \( \mu = 0 \)), the dispersion relation simplifies dramatically as one might expect. However, the presence of background gradients still modifies the dispersion relation from (sound waves in a homogeneous background, in dimensionless units) to

\[ \omega^2 = \omega_0^2 (k^2 + \Lambda_P \Lambda_w) + k^2 \Lambda_P (\Lambda_P - \Lambda_w) \]

(\( \hat{k}_i \) is the component of \( \hat{k} \) perpendicular to \( \nabla P_0 \)). This has the usual solution branches (e.g. Bray & Loughhead 1974) given by

\[ \omega^2 = \left( \frac{1}{2} [k^2 + \Lambda_P \Lambda_w] \pm \left[ (k^2 + \Lambda_P \Lambda_w)^2 + 4 k^2 \Lambda_P (\Lambda_P - \Lambda_w) \right]^{1/2} \right) / \Lambda_1 \]

where at \( k \gg |\Lambda_P| \) the ‘+’ branch corresponds to a weakly modified sound wave, with \( \omega_0 \approx k^2 + \Lambda_P (\Lambda_w + (\hat{k}_i / k^2) \Lambda_w - \Lambda_P) \), and the ‘−’ branch corresponds to buoyancy oscillations with \( \omega_0 \approx (\hat{k}_i / k^2) \Lambda_w (\Lambda_w - \Lambda_P) \). From this we see that, with our definitions, the usual Brunt–Väisälä frequency is \( N_{BV} = \Lambda_P (\Lambda_w - \Lambda_P) \).

Note that the leading-order terms in the dispersion relation (relevant for both the sound-wave and buoyancy oscillation regime) are correctly captured here by our local (leading-order) analysis. But the next-to-leading-order term in \( |\Lambda_1 / k| \) in the modified sound wave above does not match that usually derived from a more accurate WKBJ expansion for sound waves in an exponentially stratified, plane-parallel atmosphere (see e.g. Lighthill 2001; Clarke & Carstell 2007), except for special values of \( \Lambda_w \). This owes to (1) different assumptions about what is held constant (e.g. we assume here the \( \Lambda \) quantities are constant, whereas the usual pure-hydrodynamic analysis assumes \( \nabla P_0 / \rho_0 = \mu / \Lambda \) is constant) and (2) the local approximation described in Section C3 above, made for generality. We note this to remind the reader that subleading-order terms here, while given for completeness, should be regarded as heuristic and more detailed conclusions require solutions that actually specify the background gradients.

C5 Solutions with dust: numerical examples

In Fig. C1, we present numerical solutions for the full linearized equations including both dust and gas, comparing hydrostatic systems with arbitrary background gradients (equation C3) to the homogeneous (free-falling) systems analysed in the main text. For any given value of the gradients, we obtain from equation (C3) a ninth-order dispersion relation for \( \omega \), as a function of each of the gradients, as well as the independent variables studied in the hydrostatic case (\( \hat{w}_i, \mu, \zeta_i, \zeta_p, \xi_w, \hat{k}_i \), etc.). We discuss analytic approximations to the solutions for each relevant mode below.

We compare five different assumptions for the nature of the gradients in Fig. C1, and for each assumption, compare four different actual values of the gradients. These different gradient assumptions are as follows:

(i) Homogeneous: This is the homogeneous (free-falling) case from the text (all gradients in the background quantities neglected).

(ii) \( \nabla P_0 \): Here, we consider a hydrostatic system, which therefore must have a pressure gradient following equation (C1), offsetting the net acceleration. But we neglect all other gradient terms, i.e. consider only a simple pressure gradient aligned along the drift/acceleration direction, of the form in equation (C4), with value of the gradient (in our dimensionless units) of \( \Lambda_P \). We note that since we include no density gradient, the Brunt–Väisälä frequency in the gas is \( N_{BV}^2 = -\Lambda_P^2 \), i.e. the system is hydrodynamically unstable. The effects of this gradient on the growth rates, relative to the homogeneous case, are small at \( \hat{k} \gg |\Lambda_P| \), but at smaller \( \hat{k} \) the sense is always to enhance instability (but a global solution is really required in this limit).

(iii) \( \nabla P_0, \rho_0 \): We also include a gas density gradient along the same direction, of the form in equation (C4) with \( \Lambda_P = 2 \Lambda_w \). Now, the Brunt–Väisälä frequency is \( N_{BV} = \Lambda_P \), so the hydrodynamic system (in the absence of dust) is unconditionally stable. We have experimented with a range of values of \( |\Lambda_P / \Lambda_w| \), and find that our results at \( \hat{k} \gg \max(|\Lambda_P|, |\Lambda_w|) \) are very weakly sensitive to \( |\Lambda_P / \Lambda_w| \), particularly at high-\( \hat{k} \). At low-\( \hat{k} \), when \( \Lambda_w \sim \Lambda_P < 0 \), this actually produces closer agreement with the homogeneous case than (i) where we considered \( \nabla P_0 \) alone (the density and pressure gradient effects partially cancel). For \( \Lambda_w > 0 \), the growth rates are further enhanced at low-\( \hat{k} \), owing to the fact that \( \nabla \mu \) along the drift direction is non-zero.

(iv) \( \nabla P_0, \rho_0, \rho_d, \xi_w \): Here, we follow equation (C4) and impose gradients in the pressure, gas density, dust density, and drift velocity, all along the drift direction. We again take \( \Lambda_P = 2 \Lambda_w \), and for \( \Lambda_w \) take \( \Lambda_w = -\Lambda_P \) for our ‘low-\( \Lambda \)’ case (\( |\Lambda_w| = 10^{-4} \)) or for our ‘high-\( \Lambda \)’ case (\( |\Lambda_w| = 10^{-1} \)). These values of \( |\Lambda_w| \) ensure that the condition noted above for the solutions to exist, \( |\hat{w}_i| \| \Lambda_w \| < 1 \), is met (i.e. that the free-streaming scale is shorter than the gradient scale length). The sign of \( \Lambda_w \) is chosen such that gas and dust densities increase in the same direction. Adding dust-density and drift-velocity gradients appears to make a small difference, relative to solutions that already include pressure and density gradients. We will show below that the dust-density gradients dominate the leading-order corrections to the growth rates of the modes at high-\( \hat{k} \); however, in the figure these corrections are small enough so as to be essentially invisible, even though they are technically the leading-order correction.

(v) Random \( \nabla (\text{all}) \): In case (iv), we imposed gradients in the drift direction only, following equation (C4). For completeness here, we now set every component of every gradient to a different non-zero value. There are 18 gradient components: we first set the four aligned components defined above \( \Lambda_P, \Lambda_w, \Lambda_w \), and \( \Lambda_w \), above, and then set all other components. These are drawn as uniform random numbers with values between \( -|\Lambda| \) and \( +|\Lambda| \), where \( \Lambda = (\Lambda_P \Lambda_w \Lambda_w \Lambda_w) \) for each component of \( (\nabla P_0, \nabla \rho_0, \nabla \rho_d, \nabla \xi_w) \), respectively. A couple of these components are redrawn as necessary until a set is obtained which (1) ensures the hydrodynamic system (without dust) is stably stratified (\( N_{BV}^2 > 0 \)) and (2) satisfies the constraint equation (C1). We also randomly determine the orientation of \( \hat{k}_i \) in the plane perpendicular to \( \hat{w}_i \). Despite adding a large number of degrees of freedom and complexity to the dispersion relation, we see that this has generally small effects on the solutions, compared to the much simpler cases above.

For each of the gradient systems described above and shown in Fig. C1, we compare two absolute values of the gradients (labelled by \( \Lambda_P \)), one of which (\( |\Lambda_w| = 10^{-4} \)) is sufficiently small that \( \Lambda \sim |\Lambda| \) falls into the wavelength range where the ‘long-wavelength’ mode dominates, and the other (\( |\Lambda_P| = 10^{-1} \)) is much larger so that it falls around the ‘mid-\( k \)’ resonant mode. We also compare two signs of the gradients along the \( \hat{w}_i \) direction: for \( \Lambda_P > 0 \), pressure, gas,
Figure C1. Effects of stratification (background gradients) on the growth rates of the acoustic RDI. We show growth rates versus wavenumber (as in Fig. 1), calculated from the full solution to the ninth-order dispersion relation (equation C3) allowing for arbitrary gradients in $P_0$, $P_0$, and each component of $w_s$. For simplicity, we take $\zeta_\nu = 0$, and show (with thick lines) a supersonic ($w_s = 0.5$) case with $k$ oriented at the resonant angle $\cos \theta = 1/\tilde{w}_s$ and (with thin lines) a subsonic case ($w_s = 0.5$) with parallel $k$ ($\cos \theta = 1$). We consider five regimes as described in Section C5: (i) Homogeneous: The case from the main text (neglecting background gradients; $\Lambda_\nu = \Lambda_\rho = \Lambda_w = 0$). (ii) $\nabla P$: A hydrostatic system (external acceleration balanced by a pressure gradient obeying equation C1), with negligible gradients in other quantities ($\Lambda_\nu = \Lambda_\rho = \Lambda_w = 0$). (iii) $\nabla P$: We include gradients along the direction of $w_s$, so opposite signs correspond to pressure increasing (+) or decreasing (−) along the drift direction. (iii) : We include gradients in gas pressure and density with $\Lambda_\nu = 2 \Lambda_P$, so the gas system (without dust) is stably stratified. (iv) $\nabla V$: We include gradients along $\tilde{w}_s$ in all properties (gas pressure and density, dust density and streaming velocity), with $\Lambda_\nu = 2 \Lambda_P$, and $\Lambda_w = -\Lambda_P$ (for $|\Lambda_P| = 10^{-3}$ cases) or (for $|\Lambda_P| = 10^{-4}$ cases, because $|\tilde{w}_s/\Lambda_\nu| = 1$ is required for equilibria to exist). (v) Random $V$: We impose gradients as in case (iv), but also impose a gradient in every non-parallel direction (18 total gradient components), each set to a random number with value between $-|\Lambda_P|$ and $+|\Lambda_P|$ (for the appropriate $\Lambda$ of each quantity). The derivation in the main text requires $k \gg |\Lambda|$ – i.e. without a global solution our dispersion relation is only valid on scales smaller than the gradient scale length – so we indicate $k < |\Lambda_P|$ (i.e. $k c_s \langle t_s \rangle < |\Lambda_P|$, shaded) and $k = 10|\Lambda_P|$ (dashed vertical line) to show where the solutions are physical. In all cases, we see the predictions rapidly converge to the homogeneous case for $k \gg |\Lambda|$, as expected.

and dust density increase along the drift direction, while for $\Lambda_P < 0$ they decrease. For simplicity, we focus on the case with $\zeta_\nu = \zeta_w = 0$ (constant $t_s$), and consider a single value of $\mu = 0.1$ and two representative values of $\tilde{w}_s$ (a supersonic case with and a subsonic case with $\tilde{w}_s = 0.5$). For the supersonic case, we consider modes at the resonant angle $\cos \theta = 1/\tilde{w}_s$, while for the subsonic case we consider aligned modes $\cos \theta = 1$ (which are the fastest growing in the homogeneous case).

Overall, the dispersion relations shown Fig. C1 are sufficient to demonstrate the key qualitative behaviours that arise. At lower $\mu$, one does have to go to slightly higher $\tilde{k}/|\Lambda|$ before the growth rates converge to the homogeneous prediction, as we derive in more detail below. For Epstein or Stokes drag, with $\zeta_\nu$, $\zeta_\rho$, and $\zeta_w$ all non-zero and $\gamma$ in the range $\gamma \approx 0 \rightarrow 2$, the qualitative effects of gradients and magnitude of the deviations from the homogeneous case are very similar to the cases shown here. For Coulomb drag, the fact that at low-$k$ the ‘decoupling mode’ already exists with high growth rates means that the effects of gradients at low-$k$ are even less important than the cases studied here.

As in the text, for a given $k$ and mode angle, Fig. C1 only shows the most rapidly growing mode. There are new, albeit slower growing modes, which appear in the presence of stratification. At certain angles not studied here, the Brunt–Väisälä RDI can also appear. This causes subsonic streaming to be unstable at all $k$ with growth rates $\sim |\mu \tilde{w}_s| \Lambda_0^{1/2}$, at the Brunt–Väisälä resonant angle. This is discussed in Section C6.5 below, and in more detail in Squire & Hopkins (2017a).

C6 Mode structure

The full ninth-order dispersion relation with 18 degrees of freedom is not helpful to write out in full. To understand the relevant behaviour, here we consider each of the key limiting regimes as analysed in Section 3 of the main text, but including the leading-order corrections for arbitrary background gradients.
C6.1 The long-wavelength/pressure-free (low-k) mode

First consider behaviour at low-k, following Section 3.4 from the text. Expand the dispersion relation to leading order in \(k_j \ll \mu \ll 1\), bearing in mind that we require \(|\mu| \ll k\) for the validity of the derivation. The dispersion relation can then be written

\[
\left( \frac{\omega}{c} \right)^3 = i \left( 1 - \frac{\xi_j}{\xi_w} \right) + \left( \frac{\omega}{c} \right) \tilde{\mu}^{1/3} k_j^{2/3} \frac{\lambda_{\mu}}{k} + O(k_j^{2+m} \lambda^{1+m}) ,
\]

with \(n > 0, m > 0, \) and

\[
\sigma \equiv \tilde{\mu}^{1/3} k_j^{2/3} \lambda_{\mu} \approx \tilde{\mu}_j \approx \lambda_{\mu} + \lambda_w ,
\]

where the latter equality \((\lambda_{\mu} \approx \lambda_p + \lambda_w)\) arises from the general statement \((\lambda_{\mu} = -\mu^{-1} \tilde{\mu} \cdot \nabla \nu = \dot{\tilde{\mu}}_j \cdot [\rho_{\mu}^{-1} \nabla \rho_{\mu} - \rho_{\mu,0}^{-1} \nabla \rho_{\mu,0}])\) using the approximations of Section C3.

The dimensionless term \(\epsilon_{D} \equiv \tilde{\mu}^{1/3} k_j^{2/3} \lambda_{\mu} / \tilde{k} \sim O(\lambda_{\mu} / \tilde{k})\) gives the (fractional) correction to the mode growth rate. If this is small, this gives exactly the dispersion relation from the text in the homogeneous case (equation 9), with a small normalization correction \(\tilde{\omega} \approx i (1 - \xi_j / \xi_w) \sigma^{1/3} (1 + i \epsilon_{D}) / (1 - \xi_j / \xi_w)^3\) (where \(i\) is a complex argument with \(|i| = 1\), which depends on the signs of \(1 - \xi_j / \xi_w\) and \(\lambda_{\mu}\), and the solution branch chosen). The correction is therefore small so long as \(\tilde{\mu}^{1/3} k_j^{2/3} |\lambda_{\mu} / k| / 3 \ll 1\). However, for the local approximation to be valid we require \(|\lambda_{\mu} / k| \lesssim \lambda / k \ll k / \tilde{k}\), thus \(\epsilon_{D} \approx 0\), we are explicitly taking the limit \(k_j \ll \mu \ll 1\), and physically we have \(\tilde{\mu} \ll 1\). Thus, \(\epsilon_{D}\) in the leading-order correction is small. Moreover, it is worth noting that the nature of equation (C6) is such that the correction term in \(\sigma(\lambda_{\mu})\) is not stabilizing; solution branches always exist where it (weakly) increases the growth rate.

To summarize, if, in the first place, we meet the conditions required for our local derivation to be valid \((k \gg \Lambda)\) and for the long-wavelength mode to exist \((k \ll \mu)\), then we are almost always guaranteed to also meet conditions for the background gradient terms to be irrelevant for the mode.

C6.2 Non-resonant, short-wavelength (high-k) quasi-sound and quasi-drift modes

Now consider the dispersion relation in the high-k limit as in Sections 3.6 and 3.7. Off-resonance (far from \(\cos \theta \approx \pm 1 / \dot{\xi}_s\)) we obtain an identical expression to that in the main text for the ‘quasi-sound’ mode (equation 10; with leading-order real part \(\omega \approx \pm c_s \), k). More precisely, to third-from-leading order in \(k\), no terms in \(\Lambda\) appear.

For the off-resonant ‘quasi-drift’ mode (equation 13; with leading-order real part \(\omega \approx \omega_s \cdot k\)), we obtain a leading-order correction \(\omega_{QD} = \omega_{QD}(\Lambda_{P} = \Lambda_{W} = 0) + i \dot{\rho}_{\mu,0} \omega_s \cdot k\), \(\nabla \rho_{\mu,0} + O(\lambda_{\mu}^{2}, \mu_{\mu})\) (where \(\dot{\rho}_{\mu,0} \approx -\ddot{\omega}_s \cdot \rho_{\mu,0} \approx -\ddot{\omega}_s \cdot \rho_{\mu,0}\)). Because this mode (to leading order) is moving with the dust drift, the statement is simply that the mode (whose growth rate is proportional to the dust density \(\rho_{\mu,0}\) grows (decays) in strength along with the mean dust density, as the dust drifts into regions of higher (lower) density. This amounts to a constant offset in the growth rate, important only if (a) the quasi-drift mode is present at high-k and (b) the angle is sufficiently far from resonance (where the growth rates one would obtain with \(\Lambda_{W} = 0\) become small, in our dimensionless units, compared to \(\Lambda_{\mu}\), since \(\omega_{QD} \rightarrow \infty\) as \(\theta\) approaches the resonant angle. However, as noted above, we must have \(|\ddot{\omega}_s \cdot \Lambda_{W}| \sim |(i\lambda_{s}) \rho_{\mu,0}^{-1} \cdot \nu \cdot \nabla \rho_{\mu,0}| \sim |(i\lambda_{s}) \nabla \cdot \omega_s| \ll 1\) for the derivation to be valid, so the correction is necessarily small.

C6.3 The intermediate-wavelength (mid-k) resonant mode

Following Section 3.7.1, now consider the mid-k and high-k modes at the ‘resonant angle’ where \(\omega_s \cdot k = 0\) and \(o_{0\theta}\) is the natural sound-wave frequency of the system without dust. As noted in Section C4 this is modified, albeit weakly, from the pure sound-wave case by the background gradients to \(\tilde{\omega}_{s}^{2} = (1/2) (k_s^{2} + \lambda_{P} \lambda_{W} (k_s^{2} + \lambda_{P} \lambda_{W}) + 4 \tilde{k}_{s}^{2} \lambda_{W} / (\lambda_{P} + \lambda_{W})^{2})\) or \(o_{0\theta} = \tilde{k} \hat{k} (1 + (1/2) / (\lambda_{P} \lambda_{W})^{2} [\lambda_{P} / (\lambda_{W} + (\hat{k} / \hat{k})^{2})] (2 - \lambda_{P} / \lambda_{W}) + O((\hat{k} / \hat{k})^{4})\). This correspondingly shifts the resonant angle, \(\cos \theta = \tilde{k} \hat{k} / (1 + (1/2) / (\lambda_{P} \lambda_{W})^{2} (1 + (1/2) / (\lambda_{P} / (\hat{k} / \hat{k})^{2})) + O((\hat{k} / \hat{k})^{4})\).

With this \(o_{0\theta}\) and \(\hat{k}\), taking \(\mu \ll k_j \ll \mu^{-1}\) where the mid-k mode is relevant, we obtain the leading-order correction \(\epsilon\) to the growth rate,

\[
\tilde{\omega} = k_j + i \frac{1}{2} \left( \frac{1 - \xi_j}{\xi_w} \right) \mu^{1/2} \left[ 1 + \epsilon + O \left( \frac{\lambda^{1+m}}{\hat{k}^{1+m}} \right) \right],
\]

\[
\epsilon \equiv \frac{1}{2} \left( 1 + i \right) \frac{\omega_s \cdot \nabla \rho_{\mu,0}}{\mu^{1/2} \hat{k}^{1/2} \rho_{\mu,0}} \approx \frac{1}{2} \left( \frac{1 + \epsilon}{1 + \mu^{1/2} \hat{k}^{1/2} \rho_{\mu,0}} \right) \tilde{\omega}_s \cdot \tilde{\omega}_s.
\]

where \(m \geq 0, n \geq 0\). Recall, \(|\ddot{\omega}_s \cdot \Lambda_{W}| \ll 1\) is required for our derivation, so the fractional correction \(\epsilon\) should usually be small. However, unlike all the still higher order corrections from the \(\Lambda\) terms, which are un-ambiguously small at all \(k\) where our derivation is valid, the leading-order fractional correction here has a power of \(\sim \mu^{-1/2}\), so could be important at sufficiently small \(\mu\).

Equivalently, we can take the imaginary part of equation (C8) to write the growth rate as \(2 \Im(\tilde{\omega}) \approx (1 - \xi_j / \xi_w) \mu^{1/2} (1 - \dot{\xi}_s / \dot{\xi}_s)\). We see that the leading-order term in \(\Lambda\) is the same (up to a constant pre-factor) constant offset in the growth rate that we saw in the
off-resonant quasi-drift mode. Since the absolute correction to the growth rate is constant (or, equivalently, the fractional correction \( \epsilon \) scales \( \propto k^{-1/2} \), it must be negligible at high-\( k \), specifically when \( k \gg \langle \tilde{v}_d, \Lambda_w \rangle / \mu \). Now recall from Section 3.6.1 that this mid-\( k \) mode is present (and is the fastest growing mode) for \( k \) in the range \( \hat{\mu} \ll k \ll \hat{\mu}^{-1} \). Since \( |\tilde{v}_d, \Lambda_w| \ll 1 \), this means there must always exist a range of \( k \) where \( (\tilde{v}_d, \Lambda_w)^2 / \mu \ll k \ll 1/\mu \) and thus the correction term \( \epsilon \) is negligible.

However, if \( \mu \) is very small, such that \( \hat{\mu} \approx |\tilde{v}_d, \Lambda_w|^2 \ll 1 \), then at small \( k \) where \( k \ll \hat{\mu} \ll (\tilde{v}_d, \Lambda_w)^2 / \mu \), the growth rate of this mode can be modified significantly. The mode will then either grow faster or slower, depending on whether the dust is drifting into regions of higher or lower dust density on a time-scale short compared to the mode-growth time.

C6.4 The short-wavelength (high-\( k \)) resonant mode

Again taking the resonant condition and expanding the dispersion relation, now at high-\( k \) as in Section 3.7.1, we find it is identical to the homogeneous (\( \Lambda_P = \Lambda_w = 0 \)) case at leading (\( O(\kappa_L) \)) and next-to-leading (\( O(\kappa_L^2) \)) orders. The first correction term from background gradients appears at third-to-leading order, in the constant (\( O(\kappa_L^3) \)) correction to the growth rate \( \sigma \) in equation (16), where

\[
\sigma \rightarrow \sigma + \left( \frac{t_s}{3} \right) \left[ \frac{\hat{k} + (\Theta - 1) \tilde{w}_s \cdot (\nabla \otimes w_s) \cdot \hat{k}}{\Theta} - \frac{w_s \cdot \nabla \rho_d,0}{\rho_d,0} \right],
\]

where \( \Theta \equiv 1 - \zeta_w / \tilde{v}_d^2 \) and \( \otimes \) denotes the outer product. This is not surprising, since the gradients in the gas properties only enter the resonant mode in the gas at \( O(|\Lambda|/k^2) \) at high-\( k \), and (as noted for the ‘quasi-drift’ mode above) a divergence in the dust velocity/density \( \Lambda_w \) enters as a constant offset in the growth rate for modes moving with the mean dust motion.

Because \( |\tilde{v}_d, \Lambda_w| \ll 1 \), and since this correction only appears in the constant term (while the dominant term in the growth rate is increasing with \( k \)), it becomes a vanishingly small correction to the mode at high-\( k \).

C6.5 New instabilities: the Brunt–Väisälä RDI

In addition to the acoustic modes above, which we showed are not fundamentally altered by the background gradient terms, new unstable modes appear due to the stratification. As noted above, with these gradient terms, the dispersion relation for \( \omega_0 \) is modified to include two branches: both the usual sound-wave modes (\( \omega_0 \approx \pm c_s, k \)) and buoyancy modes (\( \omega_0 \approx \pm N_{BV} \), the Brunt–Väisälä frequency). As shown in SH, any mode of the gas without dust introduces a corresponding RDI when \( w_s \cdot k = \omega_0 \), and the Brunt–Väisälä RDI is one of the examples discussed there (within the Boussinesq approximation, which eliminates the sound waves). These modes have \( k \approx \pm N_{BV}/(|w_s|/t_s) \sim \Lambda/(|w_s|/t_s) \) (recall that \( N_{BV}^2 = \Lambda_P \) \( (\Lambda_P - \Lambda) \)), and growth rates \( \Im(\omega) \sim (\hat{\mu} \tilde{v}_d, \Lambda_P)^{1/2} \) in our units. However, the Brunt–Väisälä RDI is fundamentally distinct from the acoustic RDI (the resonance is with buoyancy oscillations with \( \omega_0 \approx \) constant, not sound waves), so we do not show or discuss them here, but instead explore them separately, in a more detailed analysis (which also allows for explicitly incompressible or compressible fluids) in Squire & Hopkins (2018a). We also note that they are also never the fastest growing mode when the acoustic RDI resonance is possible (\( \tilde{v}_d > 1 \) and \( k \gg \Lambda \)), although they could certainly be important and the fastest growing mode if the acoustic RDI is not present.

C7 Summary

We have considered the dispersion relation allowing every component of the gradients of \( P_0, \rho_0, \rho_d,0, \) and \( w_s \) to have arbitrary values, subject only to the constraints in Section C2 necessary for our local approximation to the equations of motion to be valid (\( k \gg |\Lambda| \), \( |\tilde{v}_d, \Lambda| \ll 1 \)). It is worth noting that at leading order in \( \Lambda/k \) (and up to third-from-leading order in the other relevant expansion parameters for each mode considered above) the pressure gradient term, which allows the system to be hydrostatic and motivated this study, does not appear. Likewise for any transverse gradient terms.

In fact, the leading-order corrections all follow from the derivative of the background dust properties (density or drift velocity) along the direction of the drift. These corrections, which appear for those modes that are (to leading order) ‘moving with’ the drift, have a simple physical interpretation. Because the relevant mode growth rates depend on the dust-to-gas ratio (and drift velocity), the physical statement is simply that as a mode moves into regions of larger (smaller) dust-to-gas ratio, the mode growth rates correspondingly increase (decrease). However, these would represent significant corrections to the growth rates (relative to the spatially homogeneous case in the main text) only if the parameter \( \tilde{v}_d \Lambda_w \sim (t_s) \nabla \cdot w_s \sim (t_s) \rho_d,0^{-1} w_s \cdot \nabla \rho_d,0 \) were large – i.e. if the dust ‘free-streaming’ length were larger than the gradient scale length of the equilibrium dust distribution. Obviously in this regime our local expansion is invalid.

Finally, we note that these corrections do not fundamentally alter the character or dimensional scalings of the relevant acoustic RDI, provided \( k \gg \Lambda \) (they only modify the growth rates by some numerical pre-factor). Most importantly, they do not stabilize the system in any systematic sense. In fact, they can introduce more instabilities, for instance the Brunt–Väisälä RDI (SH), which is explored in detail in Squire & Hopkins (2018a).

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