Universal central extensions for groups of sections on non-compact manifolds

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Abstract

We construct a central Lie group extension for the Lie group of compactly supported sections of a Lie group bundle over a sigma-compact base manifold. This generalises a result of the paper “Central extensions of groups of sections” by Neeb and Wockel, where the base manifold is assumed to be compact. In the second part of the paper, we show that this extension is universal and obtain a generalisation of a corresponding result in the paper ”Universal central extensions of gauge algebras and groups” by Janssens and Wockel, where again (in the case of Lie group extensions) the base manifold is assumed compact.

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Introduction and notation

Central extensions play an important role in the theory of infinite-dimensional Lie groups. For example, every Banach-Lie algebra $\mathfrak{g}$ is a central extension $\mathfrak{z}(\mathfrak{g}) \to \mathfrak{g} \to \text{ad}(\mathfrak{g})$, where the centre $\mathfrak{z}(\mathfrak{g})$ and $\text{ad}(\mathfrak{g})$ are integrable to a Banach-Lie group; integrability of $\mathfrak{g}$ corresponds to the existence of a corresponding central Lie group extensions (see [16]).

Inspired by the seminal work by van Est and Korthagen, Neeb elaborated the general theory of central extensions of Lie groups that are modelled over locally convex spaces in 2002 (see [11]). In particular, Neeb showed that certain central extensions of Lie algebras can be integrated to central extensions of Lie groups: If the central extension of a locally convex Lie algebra $V \to \hat{\mathfrak{g}} \to \mathfrak{g}$ (with a sequentially complete locally convex space $V$) is represented by a continuous Lie algebra cocycle $\omega: \mathfrak{g}^2 \to V$ and $G$ is a Lie group with Lie algebra $\mathfrak{g}$, one considers the so-called period homomorphism

$$\text{per}_\omega: \pi_2(G) \to V, \ [\sigma] \mapsto \int_\sigma \omega^f.$$
where $\omega^l \in \Omega^2(G, V)$ is the canonical left invariant 2-form on $G$ with $\omega^l(v, w) = \omega(v, w)$ and $\sigma$ is a smooth representative of the homotopy class $[\sigma]$. One writes $\Pi_{\omega}$ for the image of the period homomorphism and calls it the period group of $\omega$. The important result from \cite{11} is that if $\Pi_{\omega}$ is a discrete subgroup of $V$ and the adjoined action of $\mathfrak{g}$ on $\hat{\mathfrak{g}}$ integrates to a smooth action of $G$ on $\hat{\mathfrak{g}}$, then $V \hookrightarrow \hat{\mathfrak{g}} \rightarrow \mathfrak{g}$ integrates to a central extension of Lie groups (see \cite{11} Proposition 7.6 and Theorem 7.12).

In the following, a Lie group is always assumed to be modelled over a Hausdorff locally convex space.

Given two central Lie group extensions $Z_1 \hookrightarrow \hat{G}_1 \xrightarrow{\pi_1} G$ and $Z_2 \hookrightarrow \hat{G}_2 \xrightarrow{\pi_2} G$, we call a Lie group homomorphism $\varphi : \hat{G}_1 \rightarrow \hat{G}_2$ a morphism of Lie group extensions if $q_1 = q_2 \circ \varphi$. In an analogous way one defines a morphism of Lie algebra extensions. In this way one obtains categories of Lie group extensions and Lie algebra extensions, respectively, and an object in these categories is called universal if it is the initial one. In 2002 Neeb showed that under certain conditions a central extension of a Lie group is universal in the category of Lie group extensions if its corresponding Lie algebra extension is universal in the category of Lie algebra extensions (see \cite{12} Recognition Theorem (Theorem 4.13)).

The natural next step was to apply the general theory to different types of Lie groups that are modelled over locally convex spaces. Important infinite-dimensional Lie groups are current groups. These are groups of the form $C^\infty(M, G)$ where $M$ is a compact finite-dimensional manifold and $G$ is a Lie group. In 2003 Maier and Neeb constructed a universal central extension for current groups (see \cite{13}) by reducing the problem to the case of loop groups $C^\infty(S^1, G)$.

The compactness of $M$ is a strong condition but it is not possible to equip $C^\infty(M, G)$ with a reasonable Lie group structure if $M$ is non-compact. Although one has a natural Lie group structure on the group $C^\infty_c(M, G)$ of compactly supported smooth functions from a $\sigma$-compact manifold $M$ to a Lie group $G$. In this situation, $C^\infty_c(M, G)$ is the inductive limit of the Lie groups $C^\infty_c(M, G) := \{ f \in C^\infty(M, G) : \text{supp}(f) \subseteq K \}$ where $K$ runs through a compact exhaustion of $M$. The Lie algebra of $C^\infty_c(M, G)$ is given by $C^\infty_c(M, \mathfrak{g})$. In this context, $C^\infty_c(M, \mathfrak{g})$ is equipped with the canonical direct limit topology in the category of locally convex spaces. In 2004, Neeb constructed a universal central extension for $C^\infty_c(M, G)$ in important cases (see \cite{14}).

It is possible to turn the group $\Gamma(M, G)$ of sections of a Lie group bundle $G$ over a compact base manifold $M$ into a Lie group by using the construction of the Lie group structure of the gauge group from \cite{17} (see \cite{10} Appendix A). The Lie algebra of $\Gamma(M, G)$ is the Lie algebra $\Gamma(M, \mathfrak{g})$ of sections of the Lie algebra bundle $\mathfrak{g}$ that corresponds to $G$. Hence the question arises if it is possible to construct central extensions for these groups of sections. This is indeed the case and was done in 2009 by Neeb and Wockel in \cite{10}.

As mentioned above, one way to show the universality of a Lie group extension is to show the universality of the corresponding Lie algebra extension and then use the Recognition Theorem from \cite{12}. In the recent paper \cite{13} from 2013, Janssens and Wockel constructed a universal central extension of the Lie algebra $\Gamma_c(M, \mathfrak{g})$ of compactly supported sections in a Lie algebra bundle over a $\sigma$-compact manifold. They also applied this result to the central extension constructed in \cite{10}: By assuming the base manifold $M$ to be compact they obtained a universal Lie algebra extension that corresponds to the Lie group extension described in \cite{14}; they were able to show the universality of this Lie group extension.

In 2013, Schütte generalised the construction of the Lie group structure from \cite{14} by endowing the gauge group of a principal bundle over a not necessary compact base manifold $M$ with a Lie group structure, under mild hypotheses (see \cite{15}). It is clear that we can
use an analogous construction to endow the group of compactly supported sections of a Lie group bundle over a σ-compact manifold with a Lie group structure. Similarly, Neeb and Wockel already generalised the construction of the Lie group structure on a gauge group with compact base manifold from [17] to the case of section groups over compact base manifolds.

The principal aim of this paper is to construct a central extension of the Lie group of compactly supported smooth sections on a σ-compact manifold such that its corresponding Lie algebra extension is represented by the Lie algebra cocycle described in [6].

This generalises the corresponding result from [10] to the case where the base manifold is non-compact.

The proof, which combines arguments from [13] and [10] with new ideas, is discussed in Section 1. The main result is Theorem 1.28 where we show that the canonical cocycle

\[ \omega: \Gamma_c(M, \mathfrak{g})^2 \to \Omega^1_c(M, \mathcal{V})/d\Gamma_c(M, \mathfrak{g}), \quad (\gamma, \eta) \mapsto [\kappa(\gamma, \eta)] \]

can be integrated to a cocycle of Lie groups. This result generalises [10] Theorem 4.24 to the case of a non-compact base manifold.

The crucial step is to show that the period group of \( \omega \) is a discrete subgroup of \( \Omega^1_c(M, \mathcal{V})/d\Gamma_c(M, \mathfrak{g}) \). This will be discussed in Theorem 1.28 and is the complementary result to [10] Theorem 4.14.

In the second part of the paper (Section 2) we turn to the question of universality. Once constructed, the central extension it is not hard to show its universality because we can use the arguments from the compact case (6).

In the following we fix our notation:

(a) For a fibre bundle \( q: F \to M \) with total space \( F \), finite-dimensional base manifold \( M \), projection \( q \) and typical fibre \( E \) we write \( E \to F \overset{q}{\to} M \). For the space of smooth sections of such an fibre bundle we write \( \Gamma(M, F) \) and \( \Gamma_c(M, F) \) for the space of compactly supported smooth sections.

(b) Let \( H \to P \overset{\pi}{\to} M \) be a finite-dimensional principal bundle over a σ-compact manifold \( M \) with connection \( HP \subseteq TP \) and right action \( R: P \times H \to P \). Given a linear representation \( \rho: H \to \text{GL}(V) \) and \( k \in \mathbb{N}_0 \), we write

\[ \Omega^k_b(P, V)_\rho = \{ \omega \in \Omega^k_b(P, V) : (\forall g \in H) \rho(g) \circ R_g^* \omega \} \]

for the space of \( H \)-invariant \( k \)-forms on \( P \) and \( \Omega^k_b(P, V)_\rho^{\text{hor}} \) for the space of \( H \)-invariant \( k \)-forms that are horizontal with respect to \( HP \). Moreover, given a compact set \( K \subseteq M \) we define \( \Omega^k_b(P, V)_\rho = \{ \omega \in \Omega^k_b(P, V)_\rho : \text{supp}(\omega) \subseteq q^{-1}(K) \} \) and write \( \Omega^k_b(P, V)_\rho^{\text{hor}} \) for the analogous subspace in the horizontal case. We equip these spaces with the natural Fréchet-topology and write \( \Omega^k_b(P, V)_\rho \) respectively \( \Omega^k_b(P, V)_\rho^{\text{hor}} \) for the locally convex inductive limit of the spaces \( \Omega^k_b(P, V)_\rho \) respectively \( \Omega^k_b(P, V)_\rho^{\text{hor}} \).

(c) Given a manifold \( M \), we write \( C^\infty_p(\mathbb{R}, M) \) for the set of proper smooth maps from \( \mathbb{R} \) to \( M \). However, if \( F \) is the total space of a fibre bundle \( E \to F \overset{q}{\to} M \), then we define \( C^\infty_p(\mathbb{R}, F) := \{ f \in C^\infty(\mathbb{R}, F) : q \circ f \in C^\infty_p(\mathbb{R}, M) \} \).

(d) Given a finite-dimensional vector bundle \( V \to \mathbb{V} \overset{\pi}{\to} M \) over a σ-compact manifold \( M \), a compact set \( K \subseteq M \) and \( k \in \mathbb{N}_0 \) we write \( \Omega^k_c(M, V) \) for the space of \( k \)-forms on \( M \) with values in the vector bundle \( V \) and support in \( K \). Using the identification \( \Omega^k(M, V) \cong \Gamma(M, \Lambda^k T^* M \otimes V) \) we give these spaces the locally convex vector topology described in [2] and equip \( \Omega^k_c(M, V) \) with the canonical inductive limit.
topology. In Lemma A.1 we recall that if $V$ is the vector bundle associated to a principal bundle as in [10], then the canonical isomorphism $\Omega^k_c(P, V)^{\text{hor}} \cong \Omega^k_c(M, V)$ is in fact an isomorphism of locally convex spaces.

1 Construction of the Lie group extension

Convention 1.1. If not defined otherwise, $H \rightarrow P \xrightarrow{\rho} M$ denotes a finite-dimensional principal bundle over a $\sigma$-compact manifold $M$ and $\mathfrak{g}$ the Lie algebra of $H$. Moreover let $G$ be a finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ and $\kappa: \mathfrak{g} \times \mathfrak{g} \rightarrow V$ the universal invariant bilinear map on $\mathfrak{g}$ (described for instance in [3] Chapter 4). Let $\rho_G: H \times G \rightarrow G$ be a smooth action of $H$ on $G$ by Lie group automorphisms, $\rho_G: H \times \mathfrak{g} \rightarrow \mathfrak{g}$ be the derived action on $\mathfrak{g}$ (hence $\rho_G(h, \cdot) = L(\rho_G(h, \cdot)))$ and $\rho_V: H \times V \rightarrow V$ be the canonical representation on $V$ given by $\rho_V(h, \kappa(x, y)) = \kappa(\rho_G(h, x), \rho_G(h, y))$.

We emphasise that the assumption that $H$ and $G$ are finite-dimensional is not necessary to prove Theorem 1.26. Hence we obtain a generalisation of the Reduction Theorem [10, Theorem 4.14].

We write $G := P \times_{\rho_G} G$ for the associated Lie group bundle, $\mathfrak{g} := P \times_{\rho_G} \mathfrak{g}$ for the associated Lie algebra bundle and $V := P \times_{\rho_V} V$ for the associated vector bundle to $H \rightarrow P \rightarrow M$. Let $VP$ be the vertical bundle of $TP$. We fix a principal connection $H \subset VP$ on the principal bundle $P$.

Let $D_{\rho_G}: C_c^{\infty}(P, \mathfrak{g})_{\rho_G} \rightarrow \Omega^1_c(P, \mathfrak{g})^{\text{hor}}_{\rho_G}$ and $D_{\rho_V}: C_c^{\infty}(P, V)_{\rho_V} \rightarrow \Omega^1_c(P, V)^{\text{hor}}_{\rho_V}$ be the absolute derivatives corresponding to $H \subset VP$. Moreover let $d_G: \Gamma_c(M, \mathfrak{g}) \rightarrow \Omega^1_c(M, \mathfrak{g})$ and $d_V: \Gamma_c(M, V) \rightarrow \Omega^1_c(M, V)$ be the induced covariant derivation on the Lie algebra bundle $\mathfrak{g}$ and the vector bundle $V$ respectively.

In [10] Appendix A], where $M$ is compact, Neeb and Wockel endowed the group of sections $\Gamma(M, G)$ of a Lie group bundle $G$ that comes from a principal bundle $P$ with a Lie group structure. They used the identification $\Gamma(M, G) \cong C_c^{\infty}(P, G)_{\rho_G}$ and endowed the group $C_c^{\infty}(P, G)_{\rho_G}$ of $G$-invariant smooth maps from $P$ to $G$ with a Lie group structure by using the construction of a Lie group structure on the gauge group $\text{Gau}(P)$ described in [17]. To this end they replaced the conjugation of the structure group on itself by the Lie group action $\rho_G$. In the following Definition 1.2 we proceed analogously in the case where $M$ is non-compact but $\sigma$-compact. As the construction from [10] is based on [17], our analogous definition is based on [15] Chapter 4, because [15] Chapter 4 is the generalisation of [17] to the non-compact case.

Definition 1.2. (a) We equip the group

$$C_c^{\infty}(P, G)_{\rho_G} = \{ f \in C_c^{\infty}(P, G) : (\exists K \subseteq M \text{ compact}) \text{supp}(f) \subseteq q^{-1}(K) \}
$$

and $(\forall h \in H, P \in P) \rho_G(h) \circ f(pg) = f(p)$

with the infinite-dimensional Lie group structure described in [15] Chapter 4. We just replace the conjugation of $H$ on itself by the action $\rho_G$ of $H$ on $G$. We emphasise that the functions $f \in C_c^{\infty}(P, G)_{\rho_G}$ are not compactly supported in $P$ itself. The Lie algebra of $C_c^{\infty}(P, G)_{\rho_G}$ is given by the locally convex Lie algebra

$$C_c^{\infty}(P, \mathfrak{g})_{\rho_G} = \{ f \in C_c^{\infty}(P, \mathfrak{g})_{\rho_G} : (\exists K \subseteq M \text{ compact}) \text{supp}(f) \subseteq q^{-1}(K) \}.$$
(b) From [15 Chapter 4] we know \( \Gamma_c(M, \mathfrak{g}) \cong C^\infty_c(P, \mathfrak{g})_{\rho_\mathfrak{g}} \) in the sense of topological vector spaces. Now, we endow \( \Gamma_c(M, \mathfrak{g}) \) with the Lie group structure that turns the group isomorphism \( \Gamma_c(M, \mathfrak{g}) \cong C^\infty_c(P, G)_{\rho_G} \) into an isomorphism of Lie groups. Hence \( \Gamma_c(M, \mathfrak{g}) \) becomes an infinite-dimensional Lie group modelled over the locally convex space \( \Gamma_c(M, \mathfrak{g}) \).

In the following definition we fix our notation for the quotient principal bundle. For details on the well-known concept of quotient principal bundles see e.g. [3 Proposition 2.2.20].

**Definition 1.3.** Let \( N := \ker(\rho_\mathfrak{g}) \subseteq H \) and \( H/N \twoheadrightarrow P/N \xrightarrow{\pi} M \) be the quotient bundle with projection \( \pi : H/N \to M \), \( pN \mapsto q[p] \) and right action \( \overline{\pi} : H/N \times P/N \to P/N \), \( ([g], pN) \mapsto (pg)N \). We write \( \overline{\pi}_N : H/N \to GL(V) \).\( \overline{\pi}_N \) becomes the normal exterior derivative.

Remark 1.5. (b) Let \( \pi^* : \Omega^1_c(\overline{\pi}_N, V)^{\text{hor}} \to \Omega^1_c(P, V)^{\text{hor}} \) is an isomorphism of topological vector spaces and an isomorphism of chain complexes (see [A.2] (c)).

**Definition 1.4.** We define

\[
\begin{align*}
Z^1_{dR,c}(P, V)_{\rho_\mathfrak{g}} &:= \{ \omega \in \Omega^1_c(P, V)^{\text{hor}} : D_{\rho_\mathfrak{g}} \omega = 0 \}, \\
B^1_{dR,c}(P, V)_{\rho_\mathfrak{g}} &:= D_{\rho_\mathfrak{g}}(C^\infty_c(P, V)_{\rho_\mathfrak{g}}), \\
H^1_{dR,c}(P, V)_{\rho_\mathfrak{g}} &:= Z^1_{dR,c}(P, V)_{\rho_\mathfrak{g}} / B^1_{dR,c}(P, V)_{\rho_\mathfrak{g}} \cong H^1_{dR,c}(M, V), \\
Z^1_{dR,c}(P, V)_{\text{fix}} &:= Z^1_{dR,c}(P, V) \cap \Omega^1_c(P, V)_{\rho_\mathfrak{g}}, \\
B^1_{dR,c}(P, V)_{\text{fix}} &:= B^1_{dR,c}(P, V) \cap \Omega^1_c(P, V)_{\rho_\mathfrak{g}} \text{ and} \\
H^1_{dR,c}(P, V)_{\text{fix}} &:= \{ [\omega] \in H^1_{dR,c}(P, V) : [\omega] \text{ is } \rho_\mathfrak{g}-\text{invariant} \}.
\end{align*}
\]

It is possible to show the following Remark [15, 4] by a more abstract argument using that under certain conditions the fixed point functor is exact (see [10 Remark 4.12] in the compact case).

**Remark 1.5.**

(a) If \( G \) is discrete we have

\[
Z^1_{dR,c}(P, V)_{\rho_\mathfrak{g}} = Z^1_{dR,c}(P, V)_{\text{fix}} \text{ and } B^1_{dR,c}(P, V)_{\rho_\mathfrak{g}} \subseteq B^1_{dR,c}(P, V)_{\text{fix}}.
\]

(b) If \( G \) is finite we get \( B^1_{dR,c}(P, V)_{\rho_\mathfrak{g}} = B^1_{dR,c}(P, V)_{\text{fix}} \) and

\[
H^1_{dR,c}(P, V)_{\text{fix}} \cong Z^1_{dR,c}(P, V)_{\text{fix}} / B^1_{dR,c}(P, V)_{\text{fix}}.
\]

**Proof.** (a) If \( G \) is discrete there is only one connection on \( P \) namely \( HP = TP \). Hence \( D_{\rho_\mathfrak{g}} \) becomes the normal exterior derivative.

(b) Let \( n := \#G \) and \( \omega \in B^1_{dR,c}(P, V)_{\text{fix}} \) with \( \omega = df \) for \( f \in C^\infty_c(P, V) \). For \( \varphi \in C^\infty_c(P, V) \) and \( \psi \in G \) we write \( \rho_\mathfrak{g}(\psi) \varphi := \rho_\mathfrak{g}(\psi) \circ \rho_\mathfrak{g}(\varphi) \) and get

\[
\frac{1}{\#G} \sum_{g \in G} g.f \in C^\infty_c(P, V)_{\rho_\mathfrak{g}}.
\]

Moreover \( d \left[ \frac{1}{\#G} \sum_{g \in G} g.f \right] = 0 \). Hence \( B^1_{dR,c}(P, V)_{\rho_\mathfrak{g}} = B^1_{dR,c}(P, V)_{\text{fix}} \).

We consider the map \( \psi : Z^1_{dR,c}(P, V)_{\text{fix}} \to H^1_{dR,c}(P, V)_{\text{fix}}, \omega \mapsto [\omega] \). If \( [\omega] \) \( \in H^1_{dR,c}(P, V)_{\text{fix}} \) with \( \omega = df \) for \( f \in C^\infty_c(P, V) \), then

\[
[\omega] = \left[ d \left( \frac{1}{\#G} \sum_{g \in G} g.f \right) \right]
\]
Lemma 1.7. (a) The subspace \( dC^\infty_c(P, V)_{\rho V} \subseteq \Omega^1_c(P, V)_{\rho V} \) is closed (see [13]), hence the assertion follows from the fixed points of the natural action of \( \mathbb{P} \) on \( \Omega^1_c(P, V)_{\rho V} \). We know that \( B^1_{dR,c}(\mathbb{P}, V) \) is closed in \( \Omega^1_c(P, V)_{\rho V} \). Then \( B^1_{dR,c}(\mathbb{P}, V) \subseteq \ker(p) \subseteq \ker(\rho) \). It is left to show that \( \rho \) is surjective. If \( [\omega] \in H^1_{dR,c}(P, V) \), then \( \omega = \left[ \frac{1}{n} \sum_{g \in G} g \cdot \omega \right] \) and \( \omega \in Z^1_{dR,c}(P, V) \).

Convention 1.6. We assume that the identity-neighbourhood of \( H \) acts trivially on \( V \) by \( \rho_V \). Hence \( \mathbb{P} \) is a discrete Lie group. Moreover, we even assume \( \mathbb{P} \) to be finite.

Lemma 1.7. (a) The space \( \Omega^1_c(M, \mathbb{V})/d\gamma_c(M, \mathbb{V}) \) is sequentially complete.

(b) The space \( \Omega^1_c(M, \mathbb{V})/d\gamma_c(M, \mathbb{V}) \) is sequentially complete.

Proof. (a) The lemma simply says that \( d\Gamma_c(M, \mathbb{V}) \) is closed in \( \Omega^1_c(M, \mathbb{V}), \) Hence it is enough to show that the subspace \( dC^\infty_c(\mathbb{P}, V)_{\rho V} \subseteq \Omega^1_c(\mathbb{P}, V)_{\rho V} \) is closed in \( \Omega^1(\mathbb{P}, V)_{\rho V} \). We know that \( B^1_{dR,c}(\mathbb{P}, V) \) is closed in \( \Omega^1_c(\mathbb{P}, V) \), so the assertion follows from

\[
dC^\infty_c(\mathbb{P}, V)_{\rho V} = B^1_{dR,c}(\mathbb{P}, V)_{\rho V} = \bigcap_{g \in \mathbb{P}} \left\{ \omega \in B^1_{dR,c}(\mathbb{P}, V) : \pi_V(g) \cdot \mathbb{P}^g \omega = \omega \right\}
\]

is sequentially complete. To this end we show that \( \psi : \Omega^1_c(\mathbb{P}, V)_{\rho V} \to \Omega^1_c(\mathbb{P}, V)/(dC^\infty_c(\mathbb{P}, V))_{\rho V} \) is surjective and \( \ker(\psi) = dC^\infty_c(\mathbb{P}, V)_{\rho V} \). Hence it is sequential and \( \ker(\psi) = dC^\infty_c(\mathbb{P}, V)_{\rho V} \). In order to show that \( \psi \) is surjective let \( \omega \in \Omega^1_c(\mathbb{P}, V) \) such that \( \omega \) is \( \rho_V \)-invariant. Then \( \omega = \left[ \frac{1}{n} \sum_{g \in \mathbb{P}} g \cdot \omega \right] \) and \( \omega \in \Omega^1_c(\mathbb{P}, V)_{\rho V} \). Hence

\[
\Omega^1_c(\mathbb{P}, V)_{\rho V}/(dC^\infty_c(\mathbb{P}, V))_{\rho V} \to \Omega^1_c(\mathbb{P}, V)/(dC^\infty_c(\mathbb{P}, V))_{\rho V}, \quad [\omega] \mapsto [\omega]
\]

is a continuous vector space isomorphism. It is also an isomorphism of topological vector spaces, because \( \Omega^1_c(\mathbb{P}, V)/(dC^\infty_c(\mathbb{P}, V))_{\rho V} \to \Omega^1_c(\mathbb{P}, V)_{\rho V}, \quad [\omega] \mapsto [\frac{1}{n} \sum_{g \in \mathbb{P}} g \cdot \omega] \) is a continuous right-inverse.

Hence \( \Omega^1_c(\mathbb{P}, V)_{\rho V}/dC^\infty_c(\mathbb{P}, V)_{\rho V} \) is sequentially complete if and only if

\[
(\Omega^1_c(\mathbb{P}, V)/(dC^\infty_c(\mathbb{P}, V)))_{\rho V}
\]

is sequentially complete. We know that \( \Omega^1_c(\mathbb{P}, V)/(dC^\infty_c(\mathbb{P}, V)) \) is sequentially complete (see [13]), hence the assertion follows from

\[
(\Omega^1_c(\mathbb{P}, V)/(dC^\infty_c(\mathbb{P}, V)))_{\rho V} = \bigcap_{g \in \mathbb{P}} \left\{ \omega \in \Omega^1_c(\mathbb{P}, V)/(dC^\infty_c(\mathbb{P}, V)) : \pi_V(g) \cdot \mathbb{P}^g \omega = \omega \right\}
\]

and \( d(\frac{1}{n} \sum_{g \in G} g \cdot \omega) = B^1_{dR,c}(P, V)_{\rho V} \) so \( \ker(\psi) \subseteq B^1_{dR,c}(P, V)_{\rho V} \). Obviously \( B^1_{dR,c}(P, V)_{\rho V} \subseteq \ker(\psi) \). It is left to show that \( \psi \) is surjective. If \( [\omega] \in H^1_{dR,c}(P, V)_{\rho V} \), then \( [\omega] = \left[ \frac{1}{n} \sum_{g \in G} g \cdot \omega \right] \) and \( \frac{1}{n} \sum_{g \in G} g \cdot \omega \in Z^1_{dR,c}(P, V)_{\rho V} \). \( \square \)
Definition 1.8. (a) We define the locally convex, sequentially complete spaces
\[ \Omega^1_c(P, V)^{\text{hor}} := \Omega^1_c(P, V)/D_{\text{hor}} C_c(P, V) \]
and
\[ \Omega^1_c(M, V) := \Omega^1_c(M, V)/d\nu G_c(M, V). \]
With Lemma A.2 and Lemma A.1, we get
\[ \Omega^1_c(M, V) \cong \Omega^1_c(P, V)^{\text{hor}} \cong \Omega^1_c(P, V)^{\text{hor}} \cong \Omega^1_c(P, V)^{\text{fix}}. \]
In this context \( \Omega^1_c(P, V)^{\text{fix}} \) stands for the space of fixed points in \( \Omega^1_c(P, V) \) by the natural action of \( \Omega^1_c(P, V) \) on \( \Omega^1_c(P, V)^{\text{fix}} \).

(b) We define the map
\[ \omega: C_c^\infty(P, \mathfrak{g})_{\rho_\mathfrak{g}} \times C_c^\infty(P, \mathfrak{g})_{\rho_\mathfrak{g}} \to \Omega^1_c(P, V)^{\text{hor}}, (f, g) \mapsto [\kappa(f, D_{\text{hor}} g)], \]
which is the analogous map to the cocycle \( \omega_\mathfrak{g} \) defined in the compact case in [10].

Remark 1.9. Considering the vector bundles \( V(\mathcal{F}) \) from [6], we have a vector bundle isomorphism \( V \to V(\mathcal{F}) \) given by
\[ P \times_{\rho_\mathfrak{g}} V \to V(\mathcal{F}) = V(P \times_{\rho_\mathfrak{g}} \mathfrak{g}), [p, \kappa(x, y)] \mapsto \kappa_{\rho_\mathfrak{g}}(x, y) \text{ for } x, y \in \mathfrak{g}. \]
Hence our map \( \omega \) from Definition 1.8 corresponds to the cocycle \( \omega_{\mathfrak{g}} \) from [6], Chapter 1, (1.1). Therefore \( \omega \) is a continuous map to the cocycle \( \omega_{\mathfrak{g}} \) from [6].

Neeb and Wockel use Lie group homomorphisms that are pull-backs by horizontal lifts of smooth loops \( \alpha: S^1 \to M \) to reduce the proof of the discreteness of the period group to the case of \( M = S^3 \) (see [10], Definition 4.2 and Remark 4.3). But this approach does not work in the non-compact case. Instead we want to use the results from [13] of current groups. Hence we use pull-backs by horizontal lifts of proper maps \( \alpha: \mathbb{R} \to M \) (see the next definition).

Definition 1.10. We fix \( x_0 \in M \), \( p_0 \in P_{x_0} \) and \( \alpha \in C_c^\infty(\mathbb{R}, M) \) with \( \alpha(0) = x_0 \). Let \( \hat{\alpha} \in C_c^\infty(\mathbb{R}, P) \) be the unique horizontal Lift of \( \alpha \) with \( \hat{\alpha}(0) = p_0 \). We define the group homomorphism
\[ \hat{\alpha}^*: C_c^\infty(P, G)_{\rho_G} \to C_c^\infty(\mathbb{R}, G), f \mapsto f \circ \hat{\alpha} \]
and the Lie algebra homomorphism
\[ \hat{\alpha}^*: C_c^\infty(P, \mathfrak{g})_{\rho_\mathfrak{g}} \to C_c^\infty(\mathbb{R}, \mathfrak{g}), f \mapsto f \circ \hat{\alpha} \]
These maps make sense, because given \( f \in C_c^\infty(P, G) \) we have \( \text{supp}(f) \subseteq q^{-1}(L) \) for a compact set \( L \subseteq M \). We get \( \text{supp}(f \circ \hat{\alpha}) \subseteq \alpha^{-1}(L) \), because if \( f(\hat{\alpha}(t)) \neq 1 \) we get \( \hat{\alpha}(t) \in q^{-1}(L) \) and so \( \alpha(t) = q \circ \hat{\alpha}(t) \in L \). Hence \( t \in \alpha^{-1}(L) \). Now we take the closure.

Moreover we define the integration map
\[ I_\alpha: \Omega^1_c(P, V)^{\text{hor}} \to V, [\omega] \mapsto \int_\mathbb{R} \hat{\alpha}^* \omega. \]
This map is also well-defined: Let \( \omega \in \Omega^1_c(P, V)^{\text{hor}} \) with \( \text{supp}(\omega) \subseteq q^{-1}(L) \) for a compact set \( L \subseteq M \). We get \( \text{supp}(\hat{\alpha}^* \omega) \subseteq \alpha^{-1}(L) \), because if \( (\hat{\alpha}^* \omega)_t \neq 0 \) we get \( \hat{\alpha}(t) \in q^{-1}(L) \) and so \( \alpha(t) = q \circ \hat{\alpha}(t) \in L \). On the other hand
\[ (\hat{\alpha}^* D_{\text{hor}} f)(t) = (D_{\text{hor}} f)_{\hat{\alpha}(t)}(\hat{\alpha}'(t)) = (df)_{\hat{\alpha}(t)}(\hat{\alpha}'(t)) = (\hat{\alpha}^* df)(t). \]
With Stokes’ theorem \( \int_\mathbb{R} \hat{\alpha}^* df = \int_\mathbb{R} df \circ \hat{\alpha} = 0 \) for \( f \in C_c^\infty(P, V)^{\text{hor}} \).
Lemma 1.11. In the situation of Definition 1.10 the group homomorphism 
\[ \hat{\alpha}^* : C^\infty_c(P, G)_{\rho_0} \to C^\infty_c(\mathbb{R}, G), \quad f \mapsto f \circ \hat{\alpha} \]
is in fact a Lie group homomorphism such that the corresponding Lie algebra homomorphism is given by \( \hat{\alpha}^* : C^\infty_c(P, \mathfrak{g})_{\rho_0} \to C^\infty_c(\mathbb{R}, \mathfrak{g}), \quad f \mapsto f \circ \hat{\alpha} \).

Proof. Using the construction of the Lie group structure described in [15] Chapter 4, we can argue in the following way. Let \((V_i, \sigma_i)_{i \in \mathbb{N}}\) be a locally finite compact trivialising system of \(H \hookrightarrow P \xrightarrow{\rho} M\) and \(W_i := \alpha^{-1}(V_i)\) for \(i \in \mathbb{N}\). Then \((W_i, \text{id}_{W_i})_{i \in \mathbb{N}}\) is a trivialising system of \(\{1\} \hookrightarrow \mathbb{R} \xrightarrow{\text{id}} \mathbb{R}\) with the trivial action \(\{1\} \times G \to G\). We get the following diagram

\[ \begin{array}{ccc}
C^\infty_c(P, G)_{\rho_0} & \xrightarrow{\hat{\alpha}^*} & C^\infty_c(\mathbb{R}, G) \\
\downarrow \rho \circ \hat{\alpha} & & \downarrow f \circ \hat{\alpha} \\
\prod_{i \in \mathbb{N}} C^\infty_c(\overline{V}_i, G) & \xrightarrow{(\psi_i)_{i \in \mathbb{N}}} & \prod_{i \in \mathbb{N}} C^\infty_c(\overline{W}_i, G)
\end{array} \] (1)

where the group homomorphisms \(\psi_i\) are given by the diagramme

\[ \begin{array}{ccc}
C^\infty_c(\overline{V}_i, G) & \xrightarrow{\psi_i} & C^\infty_c(\overline{W}_i, G) \\
\downarrow \theta & & \downarrow f \circ \hat{\alpha}|_{\overline{W}_i} \\
C^\infty_c(\overline{V}_i \times H, G) & \xrightarrow{f \circ \hat{\alpha}|_{\overline{V}_i}} & C^\infty_c(P|_{\overline{V}_i}, G)
\end{array} \]

with \(\theta : C^\infty_c(\overline{V}_i, G) \to C^\infty_c(\overline{V}_i \times H, G), \quad f \mapsto ((x, h) \mapsto \rho_G(h \cdot f(x)))\) and \(\varphi_i\) the inverse of \(\overline{V}_i \times H \to P|_{\overline{V}_i}, \quad (x, h) \mapsto \sigma_i(x) h\).

Defining \(\tau_j : \overline{W}_i \to H, \quad \tau_j := \text{pr}_j \circ \varphi_i \circ \hat{\alpha}_{|\overline{W}_i}, \) for \(j \in \{1, 2\}\) and \(\hat{\rho}_G : H \times G \to G, \quad (h, g) \mapsto \rho_G(h)(g)\) the map \(\psi_i : C^\infty_c(\overline{V}_i, G) \to C^\infty_c(\overline{W}_i, G)\) is given by

\[ f \mapsto \hat{\rho}_G(\text{pr}_2 \circ \varphi_i \circ \hat{\alpha}_{|\overline{W}_i}(\bullet), f \circ \text{pr}_1 \circ \varphi_i \circ \hat{\alpha}_{|\overline{W}_i}(\bullet)) = \hat{\rho}_G(\tau_2(\bullet), f \circ \tau_1(\bullet)). \]

In order to show that (1) is commutative let \(f \in C^\infty_c(P, G)_{\rho_0}\). Then

\[ \rho_G(h) \cdot f \circ \sigma_i(x) = f(\sigma_i(x) h) = f(\varphi^{-1}(x, h)) \]

for all \((x, h) \in \overline{V}_i \times H\). Hence \(\psi_i(f \circ \sigma_i) = f \circ \hat{\alpha}_{|\overline{W}_i}\).

To show that \(\psi_i\) is a Lie group homomorphism it is enough to show that \(C^\infty_c(\overline{V}_i, G) \times \overline{W}_i \xrightarrow{\rho_G} \overline{V}_i, \quad (f, x) \mapsto \rho_G(\tau_2(x), f(\tau_1(x)))\) is smooth ([4] respectively [15] Theorem 2.25, \(\overline{W}_i\) is compact). This is the case because \(C^\infty_c(\overline{V}_i, G) \times \overline{W}_i = (f, y) \mapsto f(y)\) is smooth ([4] respectively [15] Theorem 2.26, \(\overline{V}_i\) is compact) and so \(C^\infty_c(\overline{V}_i, G) \times \overline{W}_i \xrightarrow{f \circ \hat{\alpha}} H \times G\) is smooth.

It is left to show that \(L(\hat{\alpha}^*)\) is given by \(C^\infty_c(P, \mathfrak{g})_{\rho_0} \to C^\infty_c(\mathbb{R}, \mathfrak{g}), \quad f \mapsto f \circ \hat{\alpha}\). To this end let \(f \in C^\infty_c(P, \mathfrak{g})_{\rho_0}\). We calculate

\[ L(\hat{\alpha}^*)(f) = \frac{\partial}{\partial t}|_{t=0} \hat{\alpha}^*(\exp(f(t))) = \frac{\partial}{\partial t}|_{t=0} \exp(f \circ \hat{\alpha}(t)) \]
\[ = \frac{\partial}{\partial t}|_{t=0} \exp(G \circ (t \circ f) \circ \hat{\alpha}) = \frac{\partial}{\partial t}|_{t=0} (t \circ f) \circ \hat{\alpha} = f \circ \hat{\alpha}. \]

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where \( * \) follows from

\[
e v_p \left( \frac{\partial}{\partial t} \big|_{t=0} (\exp_G \circ (t \cdot f) \circ \hat{\alpha})) = \frac{\partial}{\partial t} \big|_{t=0} (\exp_G (t \cdot f(\hat{\alpha}(p)))) = e v_p \left( \frac{\partial}{\partial t} \big|_{t=0} ((t \cdot f) \circ \hat{\alpha}) \right)
\]

for \( p \in P \).

**Definition 1.12.** We define the cocycle

\[
\omega: C_\ast^p(R, \mathfrak{g})^2 \to \Omega_1^\mathfrak{g}(R, V) = H^1_{dR,c}(R, V) \to V, \quad (f, g) \mapsto [\kappa(f, g)] \mapsto \int_R \kappa(t, g'(t)) dt.
\]

**Lemma 1.13.** Given \( x_0 \in M \), \( p_0 \in P_{x_0} \) and \( \alpha \in C^\infty_p(R, M) \) with \( \alpha(0) = x_0 \) we get

\[
I_\alpha \circ \omega = \omega \circ (\hat{\alpha}^* \times \hat{\alpha}^*).
\]

Hence the following diagramme commutes:

\[
\begin{array}{ccc}
C^\infty_p(P, \mathfrak{g})^2 & \overset{\omega}{\longrightarrow} & \Omega_1^\mathfrak{g}(P, V) \\
\hat{\alpha}^* \times \hat{\alpha}^* & \downarrow & \quad I_\alpha \\
C^\infty_p(R, \mathfrak{g})^2 & \overset{\omega_R}{\longrightarrow} & V.
\end{array}
\]

**Proof.** For \( g \in C^\infty_c(P, \mathfrak{g})_{\rho_g} \) we have

\[
(\hat{\alpha}^* D_{\rho_g} g)(t) = D_{\rho_g} g(\hat{\alpha}'(t)) = (g \circ \hat{\alpha})'(t),
\]

because \( \hat{\alpha} \) is a horizontal map. For \( f, g \in C^\infty_c(P, \mathfrak{g})_{\rho_g} \) we get

\[
I_\alpha(\omega(f, g)) = I_\alpha([\kappa(f, D_{\rho_g} g)]) = \int_R \hat{\alpha}^* \kappa(f, D_{\rho_g} g) = \int_R \kappa(f \circ \hat{\alpha}, \hat{\alpha}^* D_{\rho_g} g) = \int_R \kappa(f \circ \hat{\alpha}'(t), (g \circ \hat{\alpha})'(t)) dt = \omega_R \circ (\hat{\alpha}^* \times \hat{\alpha}^*)(f, g)
\]

The following Lemma 1.14 can be found in [10, Remark C.2 (a)].

**Lemma 1.14.** Let \( \varphi: C^i \to C^j \) be a Lie group homomorphism and \( \mathfrak{g}_i \) the Lie algebra of \( G_i \) for \( i \in \{1, 2\} \). Moreover let \( V \) be a trivial \( \mathfrak{g}_i \)-module and \( \omega \in Z^2(\mathfrak{g}_2, V) \). Then we get

\[
\text{per}_\omega \circ \pi_2(\varphi) = \text{per}_{L(\varphi)}^* \omega
\]

as an equation in the set of group homomorphism from \( \pi_2(G_1) \) to \( V \).

**Lemma 1.15.** Let \( x_0 \in M \) and \( p_0 \in P_{x_0} \) be base points and \( \alpha \in C^\infty_p(R, M) \). Then

\[
I_\alpha \circ \text{per}_\omega = \text{per}_{I_\alpha \omega} : \pi_2(C^\infty_c(P, G)_{\rho_G}) \to V.
\]
Proof. Let $\omega' \in \Omega^2_d(C^\infty_c(P, G)_{pc}, \overline{\mathcal{I}}_c(P, V)^{hor})$ be the corresponding left invariant 2-form of $\omega \in Z^2_d(C^\infty_c(P, G)_{pc}, \overline{\mathcal{I}}_c(P, V)^{hor})$. Then $I_\alpha \circ \omega' \in \Omega^2_d(C^\infty_c(P, K)_{pc}, V)$ is left invariant and

$$(I_\alpha \circ \omega')(f, g) = I_\alpha(\omega'(f, g)) = I_\alpha(\omega(f, g))$$

for $f, g \in C^\infty_c(P, G)_{pc}$. Hence $(I_\alpha \circ \omega)' = I_\alpha \circ \omega'$. For $[\sigma] \in \pi_2(C^\infty_c(P, G)_{pc})$ with a smooth representative $\sigma$ we get

$$\text{per}_{I_\alpha(\omega')}([\sigma]) = \int_{S^2} \sigma^*(I_\alpha \circ \omega') = \int_{S^2} I_\alpha \circ \sigma^* \omega' = I_\alpha \circ \text{per}_\omega([\sigma]).$$

\[ \square \]

Lemma 1.16. For a proper map $\alpha \in C^\infty_c(\mathbb{R}, M)$ and the base points $x_0 \in M$ and $p_0 \in P_{x_0}$ we get

$$I_\alpha \circ \text{per}_\omega = \text{per}_{I_\alpha(\omega')} = \text{per}_\omega \circ \pi_2(\hat{\alpha}^*) \quad (5)$$

Hence the following diagramme commutes:

$$\begin{array}{ccc}
\pi_2(C^\infty_c(P, G)_{pc}) & \xrightarrow{\text{per}_\omega} & \mathcal{I}_c(P, V)^{hor} \\
\pi_2(\hat{\alpha}^*) & \downarrow {\ast} & \downarrow {I_\alpha} \\
\pi_2(C^\infty_c(\mathbb{R}, G)) & \xrightarrow{\text{per}_\omega} & V.
\end{array}$$

Lemma 1.17. If we endow $H^1_d(M, V)$ with the canonical $\bar{H}$-module structure $\Phi: \bar{H} \times H^1_d(M, V) \rightarrow H^1_d(M, V)$, $(h, [\omega]) \mapsto [\bar{\mathcal{E}}(h) \circ \omega]$, the map $\bar{\mathcal{E}}: H^1_d(M, V) \rightarrow H^1_d(\bar{\mathcal{E}}, V)$ becomes an isomorphism of $\bar{H}$-modules, because of Lemma 1.3. Hence we get

$$H^1_d(M, V)_{fix} \cong H^1_d(M, V)^{fix} \cong \mathcal{I}_c(P, V)^{hor}$$

if we write $V_{fix}$ for the sup space of fixed points in $V$ by the action $\mathcal{E}$. With Lemma 1.3 we see that

$$H^1_d(M, V)_{fix} \rightarrow H^1_d(\mathcal{E}, V)^{fix} \rightarrow H^1_d(P, V)_{pc}, \quad \omega \mapsto [\mathcal{E}^* \mathcal{E} \omega] = [q^* \omega]$$

is an isomorphism of topological vector spaces.

Proof. We show that $\Phi$ defines a $\bar{H}$-module structure. For $\bar{\mathcal{E}} \in \bar{H}$ we calculate

$$[\mathcal{E}(\bar{h}) \circ \omega] = [\mathcal{E}(\bar{h}) \circ \mathcal{E}^* \omega] = [\mathcal{E}(\bar{h}) \circ (\mathcal{E} \circ \mathcal{E}^*) \omega] = [\mathcal{E}(\bar{h}) \circ (\mathcal{E} \circ \mathcal{E}^*) \omega] = \bar{h} \mathcal{E}(\bar{h}) \omega.$$ 

For the first isomorphism in (5) we note

$$H^1_d(M, V)^{fix} = Z^1_d(M, V)^{fix}/B^1_d(M, V)^{fix}$$

as well as $Z^1_d(M, V)^{fix} = Z^1_d(M, V)$ and $B^1_d(M, V)^{fix} = B^1_d(M, V)$. \hfill \square

Convention 1.18. From now on we write $q^*: H^1_d(M, V)^{pc} \rightarrow H^1_d(M, V)^{hor}$ for the inverse of $q^*: H^1_d(M, V)^{fix} \rightarrow H^1_d(M, V)^{hor}$ for the inverse of $q^*: H^1_d(M, V)^{fix} \rightarrow H^1_d(M, V)^{hor}$, $[\omega] \mapsto [q^* \omega]$. 

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Lemma 1.19. Let \( x_0 \in M \) and \( p_0 \in P_{x_0} \) be base points and \( \alpha \in C^\infty_p(\mathbb{R}, M) \) with \( \alpha(0) = x_0 \). Moreover, let \( \tilde{\alpha} \in C^\infty(\mathbb{R}, P) \) be the unique horizontal lift of \( \alpha \) to \( P \) with \( \tilde{\alpha}(0) = p_0 \). We have the commutative diagramme

\[
\begin{array}{ccc}
H^1_{dR,c}(M, V_{fix}) & \xrightarrow{\int_\theta} & V \\
\downarrow q^* & & \downarrow \text{id} \\
H^1_{dR,c}(P, V)_{PV} & \xrightarrow{\int_\theta} & V.
\end{array}
\]

This implies \( \int_\mathbb{R} \tilde{\alpha}^* q^* \omega = \int_\mathbb{R} \alpha^* \omega \) for all \( \omega \in H^1_{dR,c}(M, V_{fix}) \), and

\[
\int_\mathbb{R} \tilde{\alpha}^* (q \circ \tilde{\alpha})^* \omega = \int_\mathbb{R} \alpha^* \omega.
\]

Proof. Given \( \omega \in H^1_{dR,c}(M, V_{fix}) \), we calculate

\[
\int_\mathbb{R} \tilde{\alpha}^* (q \circ \tilde{\alpha})^* \omega = \int_\mathbb{R} (q \circ \tilde{\alpha})^* \omega = \int_\mathbb{R} \alpha^* \omega.
\]

Using [13, Corollary IV.21] we get the following Lemma 1.20.

**Lemma 1.20.** If \( \Gamma \subseteq V \) a discrete subgroup then

\[
H^1_{dR,c}(M, \Gamma) := \left\{ [\omega] \in H^1_{dR,c}(M, V) : (\forall \alpha \in C^\infty_p(\mathbb{R}, M)) \int_\mathbb{R} \alpha^* \omega \in \Gamma \right\}
\]

is a discrete subgroup of \( \Omega^1_c(M, V) \).

**Lemma 1.21.** Because \( \kappa \) is universal and \( V \) is finite dimensional it is well known that \( \Pi_{\omega_n} \) is a discrete subgroup of \( \Omega^1_c(\mathbb{R}, V) = H^1_{dR,c}(\mathbb{R}, V) \cong V \).

Proof. We argue exactly like in the proof of [13, Proposition V.19], by combining [8, Theorem II.9] and [13, Lemma V.11].

**Remark 1.22.** Because \( \overline{\eta} : \overline{T} \to M \) is a finite covering, \( \overline{\eta} \) is a proper map and so a curve \( \overline{\eta} : \mathbb{R} \to \overline{T} \) is proper if and only if \( \eta \circ \overline{\eta} : \mathbb{R} \to M \) is proper.

**Lemma 1.23.** Let \( \overline{\eta} : \mathbb{R} \to \overline{T} \) be a proper map. We define \( \overline{\eta}_0 := \overline{\eta}(0), \overline{x}_0 := \overline{\eta}(\overline{\eta}_0) \) and \( \alpha := q \circ \overline{\eta} \). Moreover let \( p_0 \in P \) with \( \overline{\eta}(p_0) = \overline{x}_0 \) and \( \tilde{\alpha} : \mathbb{R} \to P \) be the unique horizontal lift of \( \alpha \) to \( P \) with \( \tilde{\alpha}(0) = p_0 \) then

\[
\begin{array}{ccc}
\Omega^1_c(\overline{T}, V)_{\overline{\eta}^*} & \xrightarrow{\int_\theta \pi^*} & V \\
\downarrow q^* & & \downarrow \text{id} \\
\Omega^1_c(P, V)_{PV} & \xrightarrow{\int_\theta \pi^*} & V
\end{array}
\]

commutes.
Proof. We have $\pi \circ \alpha = \overline{\alpha}$ because $\overline{\alpha}$ is the unique horizontal lift of $\alpha$ to $\overline{\mathcal{P}}$ with $\overline{\alpha}(0) = \underline{\pi}_0$ and $\pi \circ \alpha$ is also a horizontal lift of $\alpha$ to $\overline{\mathcal{P}}$ that maps $\theta$ to $\underline{\pi}_0$. Hence
\[
\int_{\mathbb{R}} \dot{\alpha}^*(\pi^*\omega) = \int_{\mathbb{R}} (\pi \circ \alpha)^*\omega = \int_{\mathbb{R}} \overline{\alpha}^*\omega
\]
for $\omega \in \Omega^1(\overline{\mathcal{P}}, \mathcal{V})_{\overline{\mathcal{P}}}$.

In [13] Remark IV.17] (where $M$ is non-compact) Neeb extends a smooth loop $\alpha: [0, 1] \to M$ by a smooth proper map $\gamma: [0, \infty[ \to M$ to a proper map $\tilde{\alpha}: \mathbb{R} \to M$ such that for all 1-forms $\theta$ with compact support one gets $\int_\alpha \theta = \int_{\tilde{\alpha}} \theta$. This construction is also used in the proof of the following Theorem [1.24]

**Theorem 1.24.** If $M$ is non-compact, we have
\[
\Pi_{\omega} \subseteq H^1_{dR,c}(M, \mathcal{V}) \subseteq \overline{\Pi}^1_c(M, \mathcal{V}).
\]
This means all forms in $\Pi_{\omega}$ are closed.

**Proof.** Because $\pi^*: \Omega^1_c(\overline{\mathcal{P}}, \mathcal{V})_{\overline{\mathcal{P}}} \to \Omega^1_c(\mathcal{P}, \mathcal{V})_{\mathcal{P}}^{hor}$ is an isomorphism of chain complexes, it is enough to show $\Pi_{\omega} \subseteq H^1_{dR,c}(\mathcal{P}, \mathcal{V}) \subseteq \Omega^1_c(\mathcal{P}, \mathcal{V})$. To this end let $\theta \in \Pi_{\omega}$ and $\overline{\pi}_0, \overline{\pi}_1: [0, 1] \to \overline{\mathcal{P}}$ be closed smooth curves in a point $\underline{\pi}_0 \in \overline{\mathcal{P}}$ that are homotopic relative $\{0, 1\}$ by a smooth homotopy $\overline{\mathcal{F}}: [0, 1]^2 \to \overline{\mathcal{P}}$. From Lemma A.3 we get that it is enough to show $\int_{\overline{\pi}_0} \theta = \int_{\overline{\pi}_1} \theta$.

By composing $\overline{\pi}_i$ respectively $\overline{\mathcal{F}}(s, \cdot)$ with a strictly increasing diffeomorphism $\varphi: [0, 1] \to [0, 1]$ whose derivatives vanish in 0 and 1, respectively, we can assume that the derivatives of $\overline{\pi}_i$ and $\overline{\mathcal{F}}(s, \cdot)$ vanish in 0 and 1 respectively, because $\int_{\overline{\pi}_i} \theta = \int_{\overline{\pi}_1 \circ \varphi} \theta$ (forward parametrization does not change line integrals).

Because $M$ is non-compact, we find a proper map $\overline{\eta}: [0, \infty[ \to \overline{\mathcal{P}}$ such that $\gamma(0) = \underline{\pi}_0$ and all derivatives vanish in 0 (see [13] Lemma VI. 5) and composition with $x \mapsto x^3$). For $i \in \{0, 1\}$ we define the smooth map
\[
\overline{\pi}_i^R: \mathbb{R} \to \overline{\mathcal{P}}, \ t \mapsto \begin{cases} 
\overline{\eta}(t) : t < 0 \\
\overline{\eta}(t^2) : t \in [0, 1] \\
\overline{\eta}(t^3) : t > 1.
\end{cases}
\]
Moreover, we define the smooth homotopy
\[
\overline{\mathcal{F}}^R: [0, 1] \times \mathbb{R} \to \overline{\mathcal{P}}, \ (s, t) \mapsto \begin{cases} 
\overline{\eta}(t) : t < 0 \\
\overline{\mathcal{F}}(s, t) : t \in [0, 1] \\
\overline{\eta}(t^3) : t > 1.
\end{cases}
\]
Hence we have $\overline{\pi}_i^R, \overline{\mathcal{F}}^R(\cdot, \cdot) \in C^p_{\alpha}(\mathbb{R}, \overline{\mathcal{P}})$ for $i \in \{0, 1\}$ and $s \in \mathbb{R}$. We define $\alpha_i := \overline{\eta} \circ \overline{\pi}_i, \overline{\eta} = \overline{\mathcal{F}} \circ \overline{\mathcal{F}}^R, \gamma := \overline{\eta} \circ \overline{\eta}$ and $\underline{x}_0 := \underline{\pi}_0$.

The curves $\alpha_0$ and $\alpha_1$ are closed curves in $\underline{x}_0$ and are homotopic relative $\{0, 1\}$ by the homotopy $\overline{\mathcal{F}}$, because $\alpha_1(j) = \overline{\eta}(\underline{x}_0) = x_0$ and $\overline{\mathcal{F}}(i, \cdot) = \overline{\eta} \circ \overline{\mathcal{F}}(i, \cdot) = \overline{\eta} \circ \overline{\eta} = \alpha_i$ for $j, i \in \{0, 1\}$. Moreover
\[
\alpha_i(t) = \begin{cases} 
\gamma(-t) : t < 0 \\
\alpha_i(t) : t \in [0, 1] \\
\gamma(t) : t > 1
\end{cases}
\]
as well as

\[ F^\mathbb{R}(s, t) = \begin{cases} \gamma(t) & : t < 0 \\ F(s, t) & : t \in [0, 1] \\ \gamma(t - 1) & : t > 1 \end{cases} \]

and \( \alpha_t^\mathbb{R}, F^\mathbb{R}(s, \cdot) \in C^\infty_p(\mathbb{R}, M) \).

We choose \( p_0 \in \pi^{-1}((\mathbb{R}_0)) \). Now let \( \alpha_1^\mathbb{R} : \mathbb{R} \to P \) be the unique lift of \( \alpha_1 \) to \( P \) with \( \alpha_1(0) = p_0 \) and \( \tilde{F}^\mathbb{R} : [0, 1] \times \mathbb{R} \to P \) be the unique horizontal lift of \( F \) to \( P \) such that \( \tilde{F}^\mathbb{R}(s, 0) = p_0 \) for all \( s \in [0, 1] \). For \( i \in \{0, 1\} \) we have

\[
\int_{\partial I_i^2} \theta = \int_{\partial I_i^2} \theta = \int_{\partial I_i^2} \pi^* \theta, \tag{7}
\]

where the last equation follows from Lemma \[ 12 \] Because of \( 7 \) and Lemma \[ 13 \], it is enough to show \( \pi_2((\alpha_1^\mathbb{R})^*) = \pi_2((\tilde{F}^\mathbb{R})^*) \) as group homomorphisms from \( \pi_2(C^\infty_p(P, G)_{p_0}) \) to \( \pi_2(C^\infty(C, G)) \). From \[ 13 \] Theorem A.7 \( \pi_2(C^\infty_p(\mathbb{R}, G)) = \pi_2(C, G) \). We set \( I := [0, 1] \). Let \( \sigma : I^2 \to C^\infty_p(P, G)_{p_0} \) be continuous with \( \sigma|_{I^2} = c_{1G} \). Because \( \pi_2((\alpha_1^\mathbb{R})^*)[\sigma] = [\sigma(\cdot) \circ \alpha_1^\mathbb{R}] \) for \( i \in \{0, 1\} \) it is enough to show

\[
[\sigma(\cdot) \circ \alpha_1^\mathbb{R}] = [\sigma(\cdot) \circ \alpha_1^\mathbb{R}] \]

in \( \pi_2(C, G) \). Hence we have to construct a continuous map \( H : [0, 1] \times I^2 \to C, G \) with \( H(0, s) = \sigma(s) \circ \alpha_0^\mathbb{R} \), \( H(1, s) = \sigma(s) \circ \alpha_1^\mathbb{R} \) and \( H(s, x) = c_{1G} \) for all \( s \in [0, 1] \) and \( x \in \partial I^2 \).

We define \( H(s, x) = \sigma(x) \circ F^\mathbb{R}(s, \cdot) \) for \( s \in [0, 1] \) and \( x \in I^2 \) and want to show that \( H \) is continuous. Let \( K \subseteq M \) be compact such that \( \text{im}(\sigma) = \sigma(I^2) \subseteq C^\infty_p(P, G)_{p_0} \). For \( \ell \in C^\infty_p(P, G)_{p_0} \) we have \( \text{supp}(\ell \circ \alpha_1^\mathbb{R}) \subseteq \alpha_1^\mathbb{R}^{-1}(K) \) as well as \( \text{supp}(\ell \circ \tilde{F}^\mathbb{R}(s, \cdot)) \subseteq F^\mathbb{R}(s, \cdot)^{-1}(K) \) for \( s \in [0, 1] \). Hence \( \text{supp}(\sigma(x) \circ F^\mathbb{R}(s, \cdot)^{-1}(K)) \subseteq F^\mathbb{R}(s, \cdot)^{-1}(K) \) for \( s \in [0, 1] \) and \( x \in I^2 \).

We have

\[
F^\mathbb{R}^{-1}(K) = F^\mathbb{R}|_{[0, 1] \times [0, 1]}(K) \cup F^\mathbb{R}|_{[0, 1] \times [0, 1]}(K) \cup F^\mathbb{R}|_{[0, 1] \times [0, 1]}(K)
\]

\[
= F^\mathbb{R}|_{[0, 1] \times [0, 1]}(K) \cup \{ (0, 1] \times \gamma^{-1}(K) \} \cup \{ (0, 1] \times \gamma^{-1}(K) + 1 \}.
\]

Hence \( F^\mathbb{R}^{-1}(K) \subseteq [0, 1] \times \mathbb{R} \) is compact. Therefore

\[
L := \bigcup_{s \in [0, 1]} F^\mathbb{R}(s, \cdot)^{-1}(K) = \text{pr}_2(F^\mathbb{R}^{-1}(K)) \subseteq \mathbb{R}
\]

is compact. We have \( \text{supp}(\sigma(x) \circ F^\mathbb{R}(s, \cdot)) \subseteq L \) for all \( x \in I^2 \) and \( s \in [0, 1] \). Thus \( \text{im}(H) \subseteq C_{L}(\mathbb{R}, G) \). Therefore it is enough to show that \( H : [0, 1] \times I^2 \to C \subseteq C(\mathbb{R}, G), (s, x) \mapsto \sigma(x) \circ F^\mathbb{R}(s, \cdot) \) is continuous. We know that \( \tau : [0, 1] \to C(\mathbb{R}, P), s \mapsto F^\mathbb{R}(s, \cdot) \) is continuous and so the assertion follows from the following commutative diagramme

\[
\begin{array}{ccc}
[0, 1] \times I^2 & \xrightarrow{\tau \times \sigma} & C(\mathbb{R}, P) \times C^\infty_p(P, G)_{p_0} \\
\downarrow H & & \downarrow C(\mathbb{R}, P) \times C(P, G) \\
C(\mathbb{R}, P) \times C(P, G) & \xrightarrow{(\alpha, f)} & C(\mathbb{R}, G).
\end{array}
\]
Theorem 1.25. The period group \( \Pi_\omega = \text{im per}_\omega \) is discrete in \( \overline{\Pi}_\omega(M, V) \).

Proof. Because \( q^*: H^1_{dR,c}(M, V_{\text{can}}) \rightarrow H^1_{dR,c}(P, V)_{\rho V} \) is an isomorphism of topological vector spaces and \( \Pi_\omega \subseteq H^1_{dR,c}(M, V) = H^1_{dR,c}(P, V)_{\rho V} \), it is sufficient to show that \( \Pi_\omega \) is a discrete sub group of \( H^1_{dR,c}(M, V) \). With Lemma 1.20 and 1.21 it is enough to show
\[
\Pi_\omega \subseteq H^1_{dR,c}(M, \Pi_\mathbb{R}).
\] (8)

Let \( \beta \in \Pi_\omega \), \( \alpha \in C^\omega(\mathbb{R}, M) \) and \( [\sigma] \in \pi_2(C^\omega_c(P, G)_{\rho G}) \) with \( \beta = \text{per}_\omega([\sigma]) \). Using Lemma 1.19 and Lemma 1.16 we get
\[
\int_{\mathbb{R}} \alpha^* q_* \beta = \int_{\mathbb{R}} \hat{\alpha}^* \beta = I_\alpha \circ \text{per}_\omega([\sigma]) = \text{per}_\omega \circ \pi_2(\hat{\alpha}^*([\sigma])) \in \Pi_\omega_\mathbb{R}.
\]
Hence we get (8).

The following Theorem 1.26 is the analogous result to the Reduction Theorem ([10, Theorem 4.14]) by Neeb and Wockel.

Theorem 1.26. If the structure group \( H \) and the Lie group \( G \) are not finite-dimensional but locally exponential Lie groups that are modelled over locally convex spaces and \( \omega: \mathfrak{g}^2 \rightarrow V \) is not necessarily the universal continuous invariant bilinear form but just a continuous invariant bilinear form with values in a Fréchet space then \( \Pi_\omega \) is discrete if \( \Pi_\omega_\mathbb{R} \) is discrete in \( V \). We emphasise that the base manifold \( M \) still has to be \( \sigma \)-compact and finite-dimensional and \( \overline{\Pi} \) still has to be finite, but we do not need to assume \( \pi_2(G) \) to be trivial.

Proof. The only point (until now) where it was necessary to assume \( G \) to be finite-dimensional and \( \omega \) to be universal was in Lemma 1.21.

Theorem 1.27. If \( G \) is simply connected then the adjoint action of \( \Gamma_c(M, \mathfrak{g}) \) on the extension \( \Gamma_c(M, \mathfrak{g}) := \overline{\Pi}_c(M, V) \times_\omega \Gamma_c(M, \mathfrak{g}) \) represented by \( \omega \) integrates to a Lie group action of \( (\Gamma_c(M, \mathfrak{g}))_0 \) on \( \Gamma_c(M, \mathfrak{g}) \).

Proof. The proof goes analogously to the case where \( M \) is compact. Therefore one can transfer the proof of [10, Theorem 4.25] to this situation.

Theorem 1.28. If \( G \) is simply connected and \( \overline{\Pi} \) finite then we find a Lie group extension
\[
\overline{\Pi}_c(M, V)/\Pi_\omega \twoheadrightarrow \Gamma_c(M, \mathfrak{g})_0 \rightarrow \Gamma_c(M, \mathfrak{g})_0
\]
that corresponds to the central Lie algebra extension that is represented by \( \omega \).

Proof. We simply need to put Theorem 1.26, Theorem 1.27, Proposition 7.6 and Theorem 7.12 together.
2 Universality of the Lie group extension

In the first part of [6] Janssens and Wockel showed that the cocycle \( \omega: \Gamma_c(M, \mathcal{G})^2 \to \Omega^1_c(M, \mathcal{V}) \) is universal if \( G \) is semisimple and \( M \) is a \( \sigma \)-compact manifold. In the second part of the paper they assume the base manifold \( M \) to be compact and get a universal cocycle \( \Gamma(M, \mathcal{G})^2 \to \Omega^1(M, \mathcal{V}) \). Then they show that under certain conditions a given Lie group bundle \( G \to \mathcal{G} \to M \) with finite-dimensional Lie group \( G \) is associated to the principal frame bundle \( \text{Aut}(G) \to \text{Fr}(\mathcal{G}) \to M \). Hence they were able to use [10, Theorem 4.24] to integrate the universal Lie algebra cocycle \( \Gamma \) (see [6, Theorem I.2.]).

Janssens and Wockel proved its universality by using the Recognition Theorem from [12] in order to apply [10, Theorem 4.24]. Once the Lie group extension was constructed, arguments in the case where \( \text{Theorem 1.28} \) instead of \( [10, \text{Theorem 4.24}] \). For the convenience of the reader, we recall the arguments in the case where \( M \) is non-compact but \( \sigma \)-compact and \( G \) is simply-connected.

Definition 2.1. Let \( G \) be a simply connected finite-dimensional semisimple Lie group with Lie algebra \( \mathfrak{g} \) and \( G \to \mathcal{G} \xrightarrow{\omega} M \) a Lie group bundle. Like in [5, Corollary 9.5.11] we turn \( \text{Aut}(G) \) into a finite-dimensional Lie group using the bijection \( L: \text{Aut}(G) \to \text{Aut}(\mathfrak{g}) \).

In particular \( \text{Aut}(G) \) becomes a Lie group with Lie algebra \( L(\text{Aut}(G)) = \text{der}(\mathfrak{g}) = \mathfrak{g} \) that acts smoothly on \( G \) by automorphisms.

Lemma 2.2. The Lie group bundle \( G \to \mathcal{G} \xrightarrow{\omega} M \) is isomorphic to the associated Lie group bundle of the frame principal bundle \( \text{Aut}(G) \to \text{Fr}(\mathcal{G}) \to M \).

Lemma 2.3. The identity component of \( \text{Aut}(G) \) acts trivially on \( V \) by the representation \( \rho_V: \text{Aut}(G) \times V \to V, (\varphi, \kappa(v, w)) \mapsto \kappa(L(\varphi)(v), L(\varphi)(w)). \)

Proof. Obviously it is enough to show that \( (\text{Aut}(\mathfrak{g}))_0 \) acts trivially by \( \rho: \text{Aut}(\mathfrak{g}) \times V(\mathfrak{g}) \to V(\mathfrak{g}), (\varphi, \kappa(x, y)) \mapsto \kappa(\varphi(x), \varphi(y)) \). For \( \rho: \text{Aut}(\mathfrak{g}) \to GL(V) \), \( \varphi \mapsto \rho(\varphi, \ast) \) and \( x, y \in \mathfrak{g} \) we have \( L(\rho)(f)(\kappa(x, y)) = d_{id}\rho(\ast, \kappa(x, y))(f) \). Defining \( \text{ev}_x: \text{Aut}(\mathfrak{g}) \to \mathfrak{g}, \varphi \mapsto \varphi(x) \) for \( x \in \mathfrak{g} \) we get

\[
\rho(\ast, \kappa(x, y)) = \kappa \circ (\text{ev}_x, \text{ev}_y).
\]

We have \( d_{id}\text{ev}_x(f) = \frac{d}{dt}|_{t=0} \exp(tf)(x) = f(x) \). Hence

\[
d_{id}\rho(\ast, \kappa(x, y))(f) = \kappa(\text{ev}_x(id), d_{id}\text{ev}_y(f)) + \kappa(d_{id}\text{ev}_x(f), \text{ev}_y(id))
\]

\[
= \kappa(x, f(y)) + \kappa(f(x), y).
\]

Because \( \mathfrak{g} \) is semisimple, \( \text{der}(\mathfrak{g}) = \text{im}(\mathfrak{g}) \). For \( z \in \mathfrak{g} \) we calculate

\[
L(\rho)(\text{ad}_z)(\kappa(x, y)) = \kappa(x, [y, z]) + \kappa([x, z], y) = \kappa(x, [y, z]) + \kappa(x, [z, y]) = 0.
\]

Hence \( \rho|_{\text{Aut}(\mathfrak{g})_0} = \text{id}_V \).

Convention 2.4. In the following we assume \( \widetilde{\text{Aut}(G)} := \text{Aut}(G)/\ker(\rho_V) \) to be finite.
Definition 2.5. Combining Convention \[2.4\] Lemma \[2.3\] and Theorem \[1.28\] we find a Lie group extension
\[\overline{\mathfrak{g}}_c(M,\mathbb{V})/\Pi_\omega \hookrightarrow \Gamma_c(M,\mathcal{G})_0 \rightarrow \Gamma_c(M,\mathcal{G})_0\]
that corresponds to the central Lie algebra extension that is represented by \(\omega\).

We write \(Z := \overline{\mathfrak{g}}_c(M,\mathbb{V})/\Pi_\omega\). If \(\pi: \Gamma_c(M,\mathcal{G})_0 \rightarrow \Gamma_c(M,\mathcal{G})_0\) is the universal covering homomorphism and \(Z \hookrightarrow H \rightarrow \Gamma_c(M,\mathcal{G})_0\) the pullback extension then [11] Remark 7.14. tells us that we get a central extension of Lie groups
\[E := Z \times \pi_1(\Gamma_c(M,\mathcal{G})_0) \hookrightarrow H \rightarrow \Gamma_c(M,\mathcal{G})_0.\]
Its corresponding Lie algebra extension is represented by \(\omega\).

The following theorem is the analogous statement to [6] Theorem I.2.] in the case of a non-compact base-manifold and simply connected typical fibre group.

Theorem 2.6. The central Lie group extension \(Z \times \pi_1(\Gamma_c(M,\mathcal{G})_0) \hookrightarrow H \rightarrow \Gamma_c(M,\mathcal{G})_0\) is universal for all abelian Lie groups modelled over Mackey-complete locally convex spaces.

Proof. According to [6] Theorem III.1] we just have to show that \(H\) is simply connected. Using [11] Remark 5.12 we have the long exact homotopy sequence
\[\pi_2(\Gamma_c(M,\mathcal{G})_0) \xrightarrow{\delta_2} \pi_1(Z \times \pi_1(\Gamma_c(M,\mathcal{G})_0)) \xrightarrow{i} \pi_1(H) \xrightarrow{\pi_0} \pi_1(\Gamma_c(M,\mathcal{G})_0) \xrightarrow{\delta_1} \pi_0(Z).\]

We show \(i = 0\): Calculating
\[\pi_1(Z \times \pi_1(\Gamma_c(M,\mathcal{G})_0)) = \pi_1(\overline{\mathfrak{g}}_c(M,\mathbb{V})/\Pi_\omega) = \Pi_\omega\]
and using [11] Proposition 5.11] we conclude that \(\delta_2\) is surjective. Hence \(i = 0\). From
\[\pi_0(Z) = \pi_1(\Gamma_c(M,\mathcal{G})_0),\]
we get that \(\delta_1\) is injective. Therefore \(p = 0\). Thus \(\pi_1(H) = 0\).

A Some differential topology

Lemma A.1. The canonical isomorphism of vector spaces
\[\Phi: \Omega^k_c(P,\mathbb{V})_{\rho} \rightarrow \Omega^k_c(M,\mathbb{V})\]
is in fact an isomorphism of topological vector spaces.

Proof. The analogous map from \(\Omega^k(P,\mathbb{V})_{\rho}\) to \(\Omega^k(M,\mathbb{V})\) is continuous. Hence given a compact set \(K \subseteq M\) we get that the corresponding map from \(\Omega^k_c(P,\mathbb{V})_{\rho}\) to \(\Omega^k_c(M,\mathbb{V})\) is continuous. Therefore \(\Phi\) is continuous. The same argument shows that the inverse of \(\Phi\) is continuous.

Remark A.2. Given the situation of Definition \[1.3\] the following holds.
(a) The vertical bundle of \(\overline{\Pi} \rightarrow \overline{P} \xrightarrow{T} M\) is given by \(V\overline{P} = T\pi(VP)\) and \(H\overline{P} := T\pi(HP)\) is a principal connection on \(\overline{P}\).
(b) Given $k \in \mathbb{N}_0$ the pullback $\pi^* : \Omega^k(\mathcal{T},V)^{\text{hor}} \to \Omega^k(P,V)^{\text{hor}}$, $\omega \mapsto \pi^* \omega$ is an isomorphism of topological vector spaces and an isomorphism of chain complexes.

(c) Given $k \in \mathbb{N}_0$ the pullback $\pi^* : \Omega^k(\mathcal{T},V)^{\text{hor}} \to \Omega^k(P,V)^{\text{hor}}$, $\omega \mapsto \pi^* \omega$ is an isomorphism of topological vector spaces and an isomorphism of chain complexes.

Proof. (a) First we show $T\pi(VP) \subseteq \ker(T\mathcal{F})$. For $v \in VP$ we get $T\mathcal{F}(T\pi(v)) = T\mathcal{F} \circ \pi(v) = Tq(v) = 0$. To see $\ker(T\mathcal{F}) \subseteq T\pi(VP)$ let $T\mathcal{F}(w) = 0$. We find $v \in TP$ with $T\pi(v) = w$. Hence $T\mathcal{F}(T\pi(v)) = T\mathcal{F} \circ \pi(v) = Tq(v) = 0$. Thus $v \in VP$ and so $w \in T\pi(VP)$.

We know that $\pi$ is a submersion. Hence $T\pi(HP) = 0$ is a sub bundle of $T\mathcal{F}$. For the same reason we get $T\pi(VP) + T\pi(HP) = T\mathcal{F}$. If $T_p\pi(v) = T_{p'}\pi(w)$ with $v \in V_pP$, $w \in H_{p'}P$ and $p, p' \in \pi^*$. Hence $0 = Tq(v) = Tq(w)$. Thus $w \in H_{p'}P$. Therefore $w = 0$ and so $T_p\pi(v)$.

(b) Without loss of generality we assume $k \in \{0,1\}$, because the case $k > 1$ is analogous to the case $k = 1$. First we show that $\pi^*$ is well-defined. Let $\omega \in \Omega^k(\mathcal{T},V)^{\text{hor}}$. We have $\pi \circ R_g = n_{[g]} \circ \pi$. Hence

$$\rho_V(g) \circ R_g \circ \pi^* \omega = n_{[g]} \circ \pi^* \omega = n_{[g]} \circ \pi^* \omega = n_{[g]} \circ \pi^* \omega = \pi^* \omega.$$ 

Moreover if $v \in V_pP$ we get $T_p\pi(v) \in V_{\pi(p)}P$ and so $\pi^* \omega_p(v) = \omega_{\pi(p)}(T_p\pi(v)) = 0$.

We show that $\pi^*$ is bijective. For $k = 0$ let $f : P \to V$ be a smooth map with $\rho_V(g) \circ f(p) = f(gp)$ for all $p \in P$ and $g \in G$. We define $\tilde{f} : \mathcal{T} \to V$ by $\tilde{f}(p) = f(p)$. This map is well-defined because given $p, r \in P$ with $\pi(p) = \pi(r)$ we find $n \in N$ with $p = r \cdot n$ and hence $f(p) = f(r \cdot n) = \rho_V(n) \circ f(r) = f(r)$ because $N = \ker(\rho_V)$. Obviously $\tilde{f}$ is smooth and $\pi^*(\tilde{f}) = f$. It is clear that $\pi^*$ is injective. Now let $k = 1$. Again it is clear that $\pi^*$ is injective. To see that $\pi^*$ is surjective let $\eta \in \Omega^k(P,V)^{\text{hor}}$, we define $\omega \in \Omega^k(\mathcal{T},V)^{\text{hor}}$ by $\omega_{\pi(p)}(T_p\pi(v)) = \eta_p(v)$ for $p \in P$ and $v \in T_pP$. To see that this is well-defined we choose $p, r \in P$, $v \in T_pP$ and $w \in T_rP$ with $\pi(p) = \pi(r)$ and $T_p\pi(v) = T_r\pi(w)$. We find $n \in N$ with $p = r \cdot n$. Because $n \circ \pi = \pi$, it is enough to show $\eta_{\pi}(TR_n(\cdot)) = \eta_n$. We have $\pi \circ R_n = R_{[n]} \circ \pi = \pi$. Hence $T\pi \circ TR_n = T\pi$. Thus $T_p\pi(T_pR_n(\cdot)) = T_1\pi(w) = T_1\pi(v)$. Therefore we find $x \in \ker(T\pi)$ with $T_pR_n(w) + x = v$ in $T_pP$. Hence $Tq(x) = 0$, because $Tq = T\mathcal{F} \circ T\pi$. So $x \in V_pP$ and hence $\eta_{\pi}(x) = 0$. The form $\omega$ is $\mathcal{T}$-invariant because for $g \in H, p \in P$ and $v \in T_pP$ we get

$$\omega_{\pi(p)}(T_p\pi(v)) = \omega_{\pi(p)}(T_1\pi(v)) = \omega_{\pi(p)}(T_1\pi(v)) = \omega_{\pi(p)}(T_1\pi(v)) = \omega_{\pi(p)}(T_1\pi(v)).$$

Moreover $\omega$ is horizontal because given $u \in V_{\mathcal{T}}P$ with $\mathcal{T} \in \mathcal{T}$ we find $p \in P$ with $\pi(p) = \mathcal{T}$ and $v \in V_pP$ with $u = T_p\pi(v)$. Hence $\omega_{\pi(p)}(T_1\pi(v)) = \eta_p(v) = 0$. Obviously we have $\pi^* \omega = \eta$. 17
In order to show that $\pi^*$ is an isomorphism of chain complexes we choose $p \in P$, and $v, w \in T_pP$ and calculate

\[
(p^*D_{\pi_\omega}(v), w) = (D_{\pi_\omega(v)}(T\pi(v), T\pi(w))
\]

\[
= (dw)_{\pi(p)}(pr_H \circ T\pi(v), pr_H \circ T\pi(w)) = (dw)_{\pi(p)}(T\pi \circ pr_H(v), T\pi \circ pr_H(w))
\]

\[
= (p^*dw)(pr_H(v), pr_H(w)) = (D_{\pi^*\omega}(v), w). 
\]

It is left to show that $\pi^*$ is a homeomorphism. Because the corresponding spaces are Fréchet-spaces it is enough to show the continuity of $\pi^*$. We can embed $\Omega^K(\overline{P}, V)_{\rho}^{\hor}$ into $\Gamma(\overline{P}, \Lambda^kT^*\overline{P} \otimes (\overline{P} \times V))$ and $\Omega^K(P, V)_{\rho}^{\hor}$ into $\Gamma(P, \Lambda^kT^*P \otimes (P \times V))$. The map $\text{Alt}^k(\overline{P}, V) \rightarrow \text{Alt}^k(P, V), \varphi \mapsto \varphi \circ \pi$ is continuous. With the identifications $\text{Alt}^k(\overline{P}, V) \cong \Lambda^kT^*\overline{P} \otimes (\overline{P} \times V)$ and $\text{Alt}^k(TP, V) \cong \Lambda^kT^*P \otimes (P \times V)$ and the $\Omega$-Lemma (see, e.g. [11 Theorem 8.7] or [11 F.24]) we see that $\pi^*: \Gamma(\overline{P}, \Lambda^kT^*\overline{P} \otimes (\overline{P} \times V)) \rightarrow \Gamma(P, \Lambda^kT^*P \otimes (P \times V))$ is continuous.

(c) This follows from (b) and the fact that $\pi^*(\Omega^K(\overline{P}, V)_{\rho}^{\hor}) = \Omega^K(P, V)_{\rho}^{\hor}$ for a compact set $K \subseteq M$.

The proof of the following lemma uses techniques from the proof of [14 Proposition 6.13].

**Lemma A.3.** If $q: \hat{M} \rightarrow M$ is a smooth finite manifold covering, then $q^*: \Omega^K(M, V) \rightarrow \Omega^K(\hat{M}, V)$, $\omega \mapsto q^*\omega$ leads to a well-defined isomorphism of topological vector spaces $H^1_{dR,c}(M, V) \rightarrow H^1_{dR,c}(\hat{M}, V)$.

Therefore $\pi^*: H^1_{dR,c}(M, V) \rightarrow H^1_{dR,c}(\hat{M}, V)$, $\omega \mapsto \pi^*\omega$ is an isomorphism of topological vector spaces.

**Proof.** Let $n$ be the order of the covering.

The first step is to define a continuous linear map $q_k: \Omega^K(M, V) \rightarrow \Omega^K(\hat{M}, V)$ for $k \in \mathbb{N}_0$. Without loss of generality let $k = 1$. Let $\omega \in \Omega^1(M, V)$. Given $q \in M$ we find a $y$-neighbourhood $V_y \subseteq M$ that is evenly covered by open sets $U_{y, i} \subseteq \hat{M}$ with $i = 1, \ldots, n$.

We have the diffeomorphisms $q^k_y := q_{U_{y,i}}^V$. Then

\[
\tilde{\omega}^y := \frac{1}{n} \sum_{i=1}^n (q^k_y)_*\omega|_{U_{y,i}}.
\]

is a form on $V_y$. We define $q^\omega := \tilde{\omega} \in \Omega^1(M, V)$ by $\tilde{\omega}_x := \tilde{\omega}^y_x$ for $x \in V_y$. Now we show that this is a well-defined map. Let $x \in V_y \cap V_{y'}$ for $y' \in M$ with a $y'$-neighbourhood $V_{y'}$ that is evenly covered by $(U_{y',i})_{i=1,\ldots,n}$. After renumbering the sets $U_{y',i}$ we get

\[
q|_{U_{y,i}}^{-1} = q|_{U_{y',i}}^{-1}
\]

on $V_y \cap V_{y'}$ for $i = 1, \ldots, n$. Hence

\[
\tilde{\omega}^y_x = \frac{1}{n} \sum_{i=1}^n ((q^k_y)_*\omega|_{U_{y,i}})_x = \frac{1}{n} \sum_{i=1}^n ((q^k_{y'})_*\omega|_{U_{y',i}})_x = \tilde{\omega}^{y'}_x.
\]

We note that $q$ is a proper map, because it is a finite covering. Let $\text{supp}(\omega) \subseteq q^{-1}(K)$ for a compact set $K \subseteq M$. If $y \not\in K$ then $q^{-1}(\{y\}) \cap q^{-1}(K) = \emptyset$. Hence $q^{-1}(\{y\}) \cap \text{supp}(\omega) = \emptyset$. Therefore $\text{supp}(\tilde{\omega}) = \emptyset$. Hence $\tilde{\omega}$ is a well-defined map. Let $w \in V_y \cap V_{y'}$ for $y' \in M$ with a $y'$-neighbourhood $V_{y'}$ that is evenly covered by $(U_{y',i})_{i=1,\ldots,n}$.
If \( \omega \) is continuous linear. Moreover \( q_* \) is a homomorphism of chain complexes: Given \( y \in M, v, w \in T_y M \) we calculate

\[
(q_* d\omega)_y(v, w) = \frac{1}{n} \sum_i ((q_*^iy)_* d\omega|_{U_i})_y(v, w) = \frac{1}{n} \sum_i (d(q_*^iy)_* \omega|_{U_i})_y(v, w) = (dq_*\omega)_y(v, w).
\]

Now we show

\[
q_* \circ q^* = \text{id}_{\Omega^1(M, V)}.
\]

Given \( \omega \in \Omega^1_c(M, V), y \in M \) and \( v \in T_y M \) we calculate

\[
(q_*q^*\omega)_y(v) = \frac{1}{n} \sum_i (q_*q^*\omega|_{U_i})_y(v) = \frac{1}{n} \sum_i (q^*\omega|_{U_i})_{q^*^{-1}(y)}(Tq^{-1}_i(v))
\]

\[
= \frac{1}{n} \sum \omega|_{q^{-1}(y)}(Tq \circ q^{-1}_i(v)) = \omega_y(v).
\]

Hence \( q_* \circ q^* = \text{id}_{\Omega^1(M, V)} \). We know that \( q^* \) factorises to a continuous linear map \( q^*: H^1_{dR,c}(M, V) \to H^1_{dR,c}(M, V) \) and because \( q_* \) is a homomorphism of chain complexes we get a map \( q_*: H^1_{dR,c}(M, V) \to H^1_{dR,c}(M, V) \). With (9) we see

\[
q_* \circ q^* = \text{id}_{H^1_{dR,c}(M, V)}.
\]

Hence \( q_* \) is surjective. It remains to show that \( q_*: H^1_{dR,c}(M, V) \to H^1_{dR,c}(M, V) \) is also injective. To this end we show \( q_* (B^1_c(M, V)) = B^1_c(M, V) \). Given \( f \in C^\infty_c(M, V) \) we calculate

\[
q_* (df) = q_* q^* df = df.
\]

The proof of Lemma A.4 is similar to the proof of [13, Lemma II.10 (1)].

**Lemma A.4.** Let \( E \) be a finite-dimensional vector space and \( \Theta \in \Omega^1_c(M, E) \). If for all closed smooth curves \( \alpha_0, \alpha_1: [0, 1] \to M \) such that \( \alpha_0 \) is homotopy to \( \alpha_1 \) relative \( \{0, 1\} \) we get

\[
\int_{\alpha_0} \Theta = \int_{\alpha_1} \Theta,
\]

then \( \Theta \in Z^1_{dR,c}(M, E) \).
Proof. Let \( q: \tilde{M} \to M \) be the universal smooth covering of \( M \). First we show that \( q^* \theta \) is exact. To this end we show that \( q^* \theta \) is conservative. Let \( \gamma: [0,1] \to \tilde{M} \) be a smooth closed curve in a point \( p_0 \) that lies in the fibre of a point \( x_0 \in M \). Because \( \tilde{M} \) is simply connected, we find a homotopy \( H \) from \( \gamma \) to \( c_{p_0} \) relative \( \{0,1\} \). Hence \( q \circ \gamma \) is homotopy to \( e_{x_0} = q \circ c_{p_0} \) relative \( \{0,1\} \). Therefore we get

\[
\int_{\gamma} q^* \theta = \int_{\{0,1\}} \gamma^* q^* \theta = \int_{\{0,1\}} (q \circ \gamma)^* \theta = \int_{\{0,1\}} c_{x_0}^* \theta = 0.
\]

Equation (*) follows from the assumptions of the lemma. Because \( q^* \theta \) is exact we find \( f \in C^\infty(\tilde{M}, E) \) with \( q^* \theta = df \). Hence we get

\[
q^* d\theta = dq^* \theta = df = 0.
\]

Therefore \( d\theta = 0 \) because \( q \) is a submersion. \( \square \)

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