Abstract. We discuss the relation between completely positive quaternionic maps and the corresponding complex maps obtained via projection operation. In order to illustrate this formalism, we reobtain the (complex) qubit subdynamics of maximally entangled Bell states, as complex projection of unitary dynamics between quaternionic pure states.

1. Introduction
The usual techniques devised to describe quantum systems rely on the concept of density matrix. The state of a quantum system whose state space has finite dimension \( n \) can be represented by a \( n \times n \) quantum density matrix \( \rho \), or equivalently, by an hermitian, positive (i.e., all its diagonal matrix elements, in any basis, must be nonnegative) operator with unit trace. Any change the state of the quantum system can be represented as a mapping of the state space into itself [1], hence, such a mapping must be positive (i.e., it must preserve positivity of operators) and trace preserving. Yet, positivity is a necessary but not sufficient condition for a given map to describe a physical process. In the case of open system dynamics, that is, whenever one considers a system embedded within its environment, one usually asks for a more stringent condition than positivity, namely complete positivity (CP). Essentially, CP guarantees that (for any \( n \)) the map \( \Lambda_S \otimes I_n \) preserve the positivity of all states of the compound system \( S \otimes S_n \), where \( \Lambda_S \) is the map acting on the states of \( S \), \( S_n \) is any \( n \)-level system, and \( I_n \) the identity map acting on the states of \( S_n \).

The physical argument in support of complete positivity is that one cannot exclude that the system \( S \) might have interacted in the past with another \( n \)-level system \( S_n \). In this case one should consider the two systems together, even though only one of them has a non-trivial evolution described by the map \( \Lambda_S \), while the other is dynamically inert.

More generally, the completely positive maps are positive maps satisfying the condition that their tensor multiplication is again positive; they can be characterized as the convex set generated by the maps of the form [2]:

\[
\rho \mapsto S\rho S^\dagger, \tag{1}
\]

where \( S \) is a linear operator and \( \dagger \) denotes the hermitian conjugation.

Many relevant contributions to this research field are due to prof. Sudarshan, since the early sixties when the most general dynamical law for a quantum mechanical system with a finite number of levels was formulated [3]; we like to mention further a celebrated paper [4] where the
general form of the generator of a CP dynamical semigroup of an $N$-level quantum system was established.

Although the relation between completely positive maps and composite states is very well understood [5], the physical interpretation of the maps which are positive but not completely positive is still under investigation [6, 7].

In the last years, there have been some observations suggesting that quaternionic quantum mechanics may be useful to classify positive maps in complex quantum mechanics [8]. Moreover, in some recent papers [9, 10, 11, 12] the authors exploited the possibility of "purifying" mixed (two dimensional) complex states, describing them as pure quaternionic states.

For these reasons, we study in this paper an entangled state of two 2-dimensional subsystems and discuss the evolution of this very particular state in standard (complex) quantum mechanics (CQM) as well as in quaternionic quantum mechanics (QQM). The evolution of this physical system was already studied in Ref. [13] by means of complex (not CP) maps which just coincide with the complex projection of the CP quaternionic maps considered here.

The plan of the paper is the following. We recall some basic notations and results on QQM in Sec. 2. Next, completely positive quaternionic maps and their complex projections are considered in Sec. 3. In the same section, we shall show further how one can modify the rank of a (complex) state by adding a suitable quaternionic term (Subsec. 3.2), so that in some cases a complex dynamical map can be interpreted as the complex projection of a unitary quaternionic evolution. Then, in Sec. 4, we consider a maximally entangled Bell state of a physical system made up by two different spin 1/2 systems, and we show that the evolution of the subsystems in QQM can be suitably described by unitary quaternionic (and hence CP) maps. Finally, some concluding remarks on the quaternionic description of the compound system are collected in Sec. 5.

2. Quaternionic density matrices

We recall here some basic notations; properties of quaternionic matrices are exhaustively discussed in Ref. [14]. A (real) quaternion is usually expressed as

$$q = q_0 + q_1i + q_2j + q_3k$$

where $q_l \in \mathbb{R}$ ($l = 0, 1, 2, 3$), $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$.

The quaternion skew-field $\mathbb{Q}$ is an algebra of rank 4 over $\mathbb{R}$, non commutative and endowed with an involutive anti-automorphism (conjugation) such that

$$q \rightarrow \overline{q} = q_0 - q_1i - q_2j - q_3k.$$  

The density matrix $\rho_\psi$ associated with a pure state $|\psi\rangle$ belonging to a quaternionic $n$-dimensional right Hilbert space $\mathbb{Q}^n$ is defined by [15]

$$\rho_\psi = |\psi\rangle\langle\psi|$$  

and is the same for all normalized ray representatives. By definition, density matrices $\rho_\psi$ associated with pure states, are represented by rank one, positive definite quaternionic hermitian operators on $\mathbb{Q}^n$ with unit trace. In analogy with CQM, quaternionic mixed states are described by positive quaternionic hermitian operators (density matrices) $\rho$ on $\mathbb{Q}^n$ with unit trace and rank greater than one.

The expectation value of a quaternionic hermitian operator $A$ on a state $|\psi\rangle$ can be expressed in terms of $\rho_\psi$ as [15]

$$\langle A \rangle_\psi = \langle \psi | A | \psi \rangle = \text{Re } \text{Tr}(A |\psi\rangle \langle\psi|) = \text{Re } \text{Tr}(A \rho_\psi).$$  

(3)
Expanding \( A = A_\alpha + jA_\beta \) and \( \rho = \rho_\alpha + j\rho_\beta \) in terms of complex matrices \( A_\alpha, A_\beta, \rho_\alpha \) and \( \rho_\beta \), it follows that the expectation value \( \langle A \rangle_\psi \) may depend on \( A_\beta \) or \( \rho_\beta \) only if both \( A_\beta \) and \( \rho_\beta \) are different from zero. Indeed,

\[
\langle A \rangle_\rho = \text{Re} \, \text{Tr}(A\rho) = \text{Re} \, \text{Tr}(A_\alpha\rho_\alpha - A_\beta^* \rho_\beta),
\]

where \( \ast \) denotes complex conjugation.

Thus, the expectation value of an hermitian operator \( A \) on the state \( \rho \) depends on the quaternionic parts of \( A \) and \( \rho \), only if both the observable and the state are represented by genuine quaternionic matrices.

However, if an observable \( O \) is described by a pure complex hermitian matrix, its expectation value does not depend on the quaternionic part \( j \rho_\beta \) of the state \( \rho = \rho_\alpha + j\rho_\beta \). Moreover, the expectation value predicted in the standard (complex) Quantum Mechanics for the state \( \rho_\alpha \) coincides with the one predicted in Quaternionic Quantum Mechanics (QQM) for the state \( \rho \), since

\[
\text{Tr}(O\rho_\alpha) = \text{Re} \, \text{Tr}(O\rho_\alpha) = \text{Re} \, \text{Tr}(O\rho).
\]

This simple observation is actually very important in our approach, in that it enables us to merge CQM in the (more general) framework of QQM, without modifying any theoretical prediction (as long as complex observables are taken into account).

Let us denote by \( M(\mathbb{Q}) \) and \( M(\mathbb{C}) \) the space of \( n \times m \) quaternionic and complex matrices respectively and let \( M = M_\alpha + jM_\beta \in M(\mathbb{Q}) \). We define the complex projection

\[
P : M(\mathbb{Q}) \to M(\mathbb{C})
\]

by the relation

\[
P[M] = \frac{1}{2}[M - iMi] = M_\alpha.
\]

Moreover, the probability \( P_c^\rho \) that a quaternionic state \( \rho = \rho_\alpha + j\rho_\beta \) is complex can be defined as follows:

\[
P_c^\rho := \text{Re} \, \text{Tr} \, (P[\rho]|\rho) = \text{Tr} \, (\rho_\alpha^2).
\]

### 3. Quaternionic versus complex maps
We summarize in this section some previous results about quaternionic maps.

#### 3.1. Completely positive quaternionic maps and their complex projections
Let us denote by \( M_n(\mathbb{Q}) \) the space of \( n \times n \) quaternionic matrices and by \( H_n(\mathbb{Q}) \) the space of \( n \times n \) quaternionic Hermitian matrices \( \rho = \rho_\alpha + j\rho_\beta \). In analogy with the Kraus-Stinespring form of completely positive complex maps [16, 17], we introduce the following definition [8]:

A map \( \Phi : H_n(\mathbb{Q}) \to H_n(\mathbb{Q}) \) is said to be completely positive provided it can be written in the form

\[
\Phi(\rho) = \sum_s U_s \rho U_s^\dagger
\]

where \( U_s \in M_n(\mathbb{Q}) \).

Maps (7) have some interesting properties. First of all, we recall that [11]:

Let \( \rho \) be any quaternionic density matrix. Then, the real linear combination

\[
\sum_s \lambda_s V_s \rho V_s^\dagger, \quad \lambda_s > 0,
\]

is completely positive.
where $V_s = (V_\alpha + j V_\beta)_s \in M_n(Q)$, is a positive (semi)definite quaternionic hermitian matrix.

A direct consequence of above proposition is that any map (7) is a positive map in the space of quaternionic density matrices. Moreover, it is easy to see that any map (8) can be reduced to the form (7) making $V_s = \sqrt{\Lambda} U_s$.

Notice that on quaternionic vector spaces we can also construct the Kronecker product $\Phi \otimes I_n$, because the $I_n$ entries are reals, obtaining so a quaternionic positive matrix. It turns out that complete positivity of $\Phi$ (in the sense of the above definition) guarantees the positivity of $\Phi \otimes I_n$ in the quaternionic formulation, in analogy to what happens in CQM.

Moreover, the following proposition holds [10]:

The complex projection $\rho_\alpha$ of any quaternionic density matrix $\rho = \rho_\alpha + j \rho_\beta$ is a complex density matrix.

Now, let $\rho$ be a quaternionic density matrix, and let $U$ be any $n \times n$ quaternionic matrix. Making use of the decomposition in complex and purely quaternionic parts, $U = U_\alpha + U_\beta j$, we obtain

$$U \rho U^T = U_\alpha \rho_\alpha U_\alpha^T + U_\beta \rho_\beta U_\beta^T - U_\beta \rho_\beta U_\alpha^T + U_\alpha \rho_\alpha U_\beta^T + j(U_\beta \rho_\alpha U_\beta^T - U_\alpha \rho_\alpha U_\beta^T + U_\alpha \rho_\beta U_\alpha^T + U_\beta \rho_\beta U_\beta^T) .$$

(9)

Hence, the complex map

$$\rho_\alpha \rightarrow U_\alpha \rho_\alpha U_\alpha^T + U_\beta \rho_\beta U_\beta^T - U_\beta \rho_\beta U_\alpha^T + U_\alpha \rho_\alpha U_\beta^T$$

(10)

is a positive map and the composition of two any of such maps is also a positive map.

Collecting the above results we can conclude that:

Every completely positive quaternionic map of quaternionic density matrices $\rho = \rho_\alpha + j \rho_\beta$ is associated with a complex positive map of their complex projections $\rho_\alpha$.

Notice that, making $\rho_\beta = 0$ in Eq. (10) we reobtain as a corollary a known result due to Kossakowski [8] (see also [9])

The complex projection of completely positive quaternionic maps of complex density matrices $\rho_\alpha$ is a complex decomposable map of the same complex density matrix $\rho_\alpha$. (The converse also holds.)

3.2. Quaternionic versus complex states

We shall show in this subsection that one can add a suitable quaternionic part to a complex density matrix, to obtain a quaternionic state with a different rank. This trick can be particularly useful in some cases, as we shall show in the example in Sec. 4.

The basic tools are the two following statements [11, 12]:

Let $\rho = \rho_\alpha + j \rho_\beta$ be a $n$-dimensional quaternionic density matrix, and let $\text{rank } \rho = m$. Then, $m \leq \text{rank } \rho_\alpha \leq 2m$.

Conversely,

Let $\rho_\alpha$ be a $n$-dimensional complex density matrix, and let $\text{rank } \rho_\alpha = m > 1$. Then, if $[x]$ denotes the integer part of real number $x$, for any $m'$ with $\frac{m+1}{2} \leq m' \leq m$ there exists a (skew-symmetric) complex matrix $\rho_\beta$ such that $\rho = \rho_\alpha + j \rho_\beta$ is a density matrix with $\text{rank } \rho = m'$.

Let $\rho_\alpha$ be a complex density matrix, and let $\rho = \rho_\alpha + j \rho_\beta$ be a quaternionic density matrix obtained from $\rho_\alpha$, as suggested above, such that $\text{rank } \rho \leq \text{rank } \rho_\alpha$. Then, the complex projection of a unitary quaternionic evolution acting on $\rho$

$$U : \rho \rightarrow U \rho U^T = \rho_c,$$

(11)
can represent to all purposes a complex non-unitary evolution of $\rho_a$ in a complex density matrix $\rho_c$ of smaller rank.

Moreover, a direct consequence of the previous results is the following statement:

*Any complex density matrix $\rho_a$ can be obtained as the complex projection of a quaternionic pure density matrix $\rho = \rho_a + j\rho_\beta$ if and only if $\text{rank}\rho_a = 2$."

Then, given any complex density matrix $\rho_a$ such that $\text{rank}\rho_a = 2$, we can build up a quaternionic density matrix $\rho$, by adding a suitable $j\rho_\beta$ such that $\text{rank}\rho = 1$.[11, 12].

### 3.3. The complex projection of unitary dynamics of quaternionic pure states

We focus now on some consequences of results in the previous subsection in the simplest case of quaternionic unitary dynamics and their corresponding complex projections.

Let us consider an arbitrary pair of complex density matrices, $\rho_a$ and $\rho'_a$ such that $\text{rank}\rho_a \leq 2$ and $\text{rank}\rho'_a \leq 2$, and let be $\mathcal{B}$ a complex dynamical map,

$$ \mathcal{B} : \rho_a \rightarrow \rho'_a = \mathcal{B}(\rho_a). $$

Now, we can "purify" the complex states $\rho_a$ and $\rho'_a$ by adding suitable purely quaternionic terms $j\rho_\beta$ and $j\rho'_\beta$, respectively. Moreover, since any pair of quaternionic hermitian matrices admitting the same eigenvalues are unitary equivalent, the map $\mathcal{B}$ can be described as the complex projection of a quaternionic unitary map between quaternionic pure states $\rho = \rho_a + j\rho_\beta$ and $\rho' = \rho'_a + j\rho'_\beta$:

$$ U : \rho \rightarrow \rho' = U\rho U^\dagger, \quad UU^\dagger = U^\dagger U = \mathbf{I}, $$

where $\rho'_a = \mathcal{B}(\rho_a) = P[\rho']$.

In this way, stochastic dynamics of (complex) quantum mechanical systems can be interpreted in terms of the complex projection of unitary dynamics between quaternionic pure states whenever the rank of their complex density matrices is less or equal to two.

In particular, when we consider time-independent quaternionic unitary dynamics,

$$ \rho(t) = U(t)\rho(0)U^\dagger(t), \quad (12) $$

where

$$ U(t) = e^{-Ht} \quad (13) $$

with $H = H_a + jH_\beta = -H^\dagger$, the differential equation associated with the time evolution for $\rho$ reads

$$ \frac{d}{dt}\rho(t) = -[H, \rho(t)]. \quad (14) $$

Moreover, Eq. (12) and (14) reduce to

$$ \rho_a(t) = U_a\rho_a(0)U_a^\dagger + U_\beta^a\rho_\beta(0)U_\beta^T + U_a\rho_\beta(0)U_\beta^T - U_\beta^a\rho_\beta(0)U_\beta^\dagger \quad (15) $$

and

$$ \frac{d}{dt}\rho_a = -[H_a, \rho_a] + H_\beta^a\rho_\beta - \rho_\beta H_\beta, \quad (16) $$

respectively, for the complex projection of the density matrix [10].

The term $H_\beta^a\rho_\beta - \rho_\beta H_\beta$ in Eq. (16) is hermitian and traceless (as a consequence of the cyclic property of the trace), like the non-Hamiltonian term in the generator of a dynamical semigroup. Hence, when we consider time-independent quaternionic unitary dynamics we can...
state that: the complex projection of any quaternionic unitary dynamics belongs to the set of complex dynamics described by a one-parameter semigroup of positive maps [10].

4. Maximally entangled Bell states

In a recent paper [13] the dynamics of two qubits coupled by a general nonlocal interaction is studied. In particular, a suitable interaction is considered such that two initially pure qubits can be entangled in an optimal way. We briefly recall here the main results in [13] regarding this particular example.

Let be given the evolution

$$\rho^{AB}(t) = U\rho^{AB}(0)U^\dagger$$

where

$$U = \cos t I_A \otimes I_B - i \sin t \sigma^A_3 \otimes \sigma^B_3$$

and

$$\rho^{AB}(0) = \rho^A(0) \otimes \rho^B(0) = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \otimes \frac{1}{2} \left( \begin{array}{cc} 1 & -i \\ i & 1 \end{array} \right).$$

The dynamics (17) leads to the following reduced complex density matrix dynamical maps obtained via partial trace:

$$\rho_A^A(t) = B^A(\rho^A_A(0)) = \frac{1}{2} \left( \begin{array}{cc} \cos 2t & 0 \\ 0 & 1 \end{array} \right),$$

$$\rho_B^B(0) = B^B(\rho^B_B(0)) = \frac{1}{2} \left( \begin{array}{cc} 1 & -i \cos 2t \\ i \cos 2t & 1 \end{array} \right).$$

As it is evident from Eq. (20), the reduced matrices $\rho^{A, B}_A$ oscillates between pure and maximally mixed states, and one can calculate the time $t_{\text{bell}}$ at which the density matrix $\rho^{AB}(t)$ of the compound system becomes a maximally entangled state.

Let us now show that the open qubit subdynamics (20) can be obtained as complex projection of quaternionic closed dynamics between quaternionic pure states. We recall that, due to Eq. (4), the expectation value of complex observables on the complex mixed state $\rho$ or on the quaternionic pure state $\rho = \rho_A + j \rho_B$ do coincide.

By "purification", the initial and final quaternionic pure qubits respectively can be easily obtained:

$$\rho_A^A(0) = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad \rho_B^B(0) = \frac{1}{2} \left( \begin{array}{cc} 1 & -i \\ i & 1 \end{array} \right)$$

and

$$\rho_A^A(t) = \frac{1}{2} \left( \begin{array}{cc} \cos 2t & 0 \\ 0 & 1 \end{array} \right) + j \frac{e^{-i\varphi}}{2} \left( \begin{array}{cc} 0 & -\sin 2t \\ \sin 2t & 0 \end{array} \right), \ \varphi \in \mathbb{R}$$

$$\rho_B^B(t) = \frac{1}{2} \left( \begin{array}{cc} 1 & -i \cos 2t \\ i \cos 2t & 1 \end{array} \right) + j \frac{e^{-i\theta}}{2} \left( \begin{array}{cc} 0 & -\sin 2t \\ \sin 2t & 0 \end{array} \right), \ \theta \in \mathbb{R}.$$
\[ U^A(t) = e^{-H^A t} = \begin{pmatrix} \cos 2t + je^{-i\varphi} \sin 2t & 0 \\ 0 & 1 \end{pmatrix}, \] (24)

\[ U^B(t) = e^{-H^B t} = \begin{pmatrix} \cos 2t + ke^{-i\theta} \sin 2t & 0 \\ 0 & 1 \end{pmatrix}, \] (25)

and the quaternionic anti-Hermitian Hamiltonians read

\[ H_A = \begin{pmatrix} -2je^{-i\varphi} & 0 \\ 0 & 0 \end{pmatrix}, \quad H_B = \begin{pmatrix} -2ke^{-i\theta} & 0 \\ 0 & 0 \end{pmatrix}. \] (26)

Notice that, the time evolution of the complex projection density matrices \( \rho_{\alpha}^A(t) \) (see Eq. (5)) is given by continuous one parameter (complex) semigroups of positive trace preserving linear maps. Such maps represent the integrated form of the Markovian master equations for the complex projection density matrices given respectively by

\[ \frac{d}{dt} \rho_{\alpha}^A(t) = -[H_A, \rho_{\alpha}^A] + H_A^{*} \rho_{\beta}^A - \rho_{\beta}^A H_A = \sin 2t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \] (27)

and

\[ \frac{d}{dt} \rho_{\alpha}^B(t) = -[H_B, \rho_{\alpha}^B] + H_B^{*} \rho_{\beta}^B - \rho_{\beta}^B H_B = \sin 2t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \] (28)

The probabilities \( P_{\varepsilon}^{\rho^A}(t) \) and \( P_{\varepsilon}^{\rho^B}(t) \) that the quaternionic states \( \rho^A(t) \) and \( \rho^B(t) \) are complex are respectively given by (see Eq. (6))

\[ P_{\varepsilon}^{\rho^A}(t) = \text{Re} \, \text{Tr} \left( P[\rho^A(t)] \rho^A(t) \right) = \frac{1}{2} \left( 1 + (\cos 2t)^2 \right) \] (29)

\[ P_{\varepsilon}^{\rho^B}(t) = \text{Re} \, \text{Tr} \left( P[\rho^B(t)] \rho^B(t) \right) = \frac{1}{2} \left( 1 + (\cos 2t)^2 \right) \] (30)

which does coincide with the probabilities that the reduced complex density matrices \( \rho_{\alpha}^A(t) \) and \( \rho_{\alpha}^B(t) \) become pure; furthermore, \( \rho^{AB}(t) \) becomes a maximally entangled Bell state, at the times \( t_{\text{bell}} \) where \( P_{\varepsilon}^{\rho^{AB}}(t_{\text{bell}}) = \frac{1}{2} \) [13].

Finally, in the quaternionic formulation, the expectation value of the hermitian energy observables \( \langle H^{A,B}(t) \rangle \) on the states \( \rho^{A,B}(t) \) can be easily computed,

\[ \langle H^{A,B}(t) \rangle_{\rho^{A,B}(t)} = \text{Re} \, \text{Tr} \left( \rho^{A,B}(t) H^{A,B}(t) \right) = 1. \]

5. Quaternionic description of the compound system

It is well known that in the description of compound systems in quaternionic quantum mechanics the usual definition of Kronecker product of matrices does not hold, and even the concept of tensor product of Hilbert spaces fails, owing to the non-commutativity of the skew-field \( \mathbb{Q} \). (In order to overcome this difficulty, a concept of tensor product of quaternionic Hilbert modules has been proposed [18], which allows one to describe compound systems on a mathematically well-founded basis; unfortunately, the results obtained in this way do not agree in the complex limit with the ones of standard quantum mechanics [19]). However, as long as we limit ourselves to consider a compound system of two qubits, each subsystem can be described by a pure quaternionic qubit, which undergoes a unitary quaternionic time evolution. Hence, one can attribute to each subsystem "individual" properties, differently from what happens in the realm of CQM where reduced density matrices do not allow a similar interpretation. Nevertheless,
the correlations between subsystems do not disappear at all, but they are implicitly taken into account in such individual evolutions, as the example analyzed above has pointed out.

In conclusion, these facts point out a puzzling situation, in which the same state of a physical system is entangled in CQM, while it seems to be “separable” in QQM. We hope that further investigations (on quaternionic maps, and more generally, on QQM) will contribute to throw further light on this problem.

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