Isolated singularities for fractional Lane-Emden equations
in the Serrin’s supercritical case

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Abstract

In this paper, we give a classification of the isolated singularities of positive solutions to
the semilinear fractional elliptic equations

\[(E) \quad (-\Delta)^s u = |x|^\theta u^p \quad \text{in} \quad B_1 \setminus \{0\}, \quad u = h \quad \text{in} \quad \mathbb{R}^N \setminus B_1,\]

where \(s \in (0, 1), \ \theta \in (-2s, 0], \ p > \frac{N + \theta}{N - 2s}, \ B_1\) is the unit ball centered at the origin of \(\mathbb{R}^N\)
with \(N > 2s\). \(h\) is a nonnegative function in \(\mathbb{R}^N \setminus B_1\). Our analysis of isolated singularities
of \((E)\) is based on an integral upper bounds and the study of the Poisson problem with the
fractional Hardy operators. It is worth noting that our classification of isolated singularity
holds in the Sobolev supercritical case \(p > \frac{N + 2s + 2\theta}{N - 2s}\) for \(s \in (0, 1]\) under suitable assumption
of \(h\).

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1 Introduction

Let $s \in (0,1)$, $N > 2s$ and $B_r(y)$ be the ball of radius $r > 0$ centered at $y$ in $\mathbb{R}^N$, $B_r := B_r(0)$. Our concern of this paper is to classify the isolated singular positive solutions of the fractional Lane-Emden equation

$$\begin{cases}
(-\Delta)^s u = |x|^\theta u^p & \text{in } B_1 \setminus \{0\}, \\
\quad u = h & \text{in } \mathbb{R}^N \setminus B_1,
\end{cases} \quad (1.1)$$

where $\theta \in (-2s,0]$, $p > \frac{N + \theta}{N - 2s}$, $h$ is a nonnegative function in $\mathbb{R}^N \setminus B_1$ and $(-\Delta)^s$ is the fractional Laplacian defined by

$$(-\Delta)^s u(x) = C_{N,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon} \frac{u(x) - u(x + z)}{|z|^{N + 2s}} dz$$

with

$$C_{N,s} = 2^{2s} \pi^{-\frac{N}{2}s} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}$$

being the normalized constant, see [25], $\Gamma$ being the usual Gamma function. It is known that $(-\Delta)^s u(x)$ is well defined if $u$ is twice continuously differentiable in a neighborhood of $x$ and contained in the space $L^1(\mathbb{R}^N) := L^1(\mathbb{R}^N, \frac{dx}{|x|^{N+2s}})$. When $s = 1$ and $N \geq 3$, the isolated singularities of the Lane-Emden equation

$$\begin{cases}
-\Delta u = |x|^\theta u^p & \text{in } B_1 \setminus \{0\}, \\
\quad u = 0 & \text{on } \partial B_1
\end{cases} \quad (1.2)$$

has been studied extensively in the last decades. When $\theta = 0$ and $p \in (1, \frac{N}{N-2})$, Lions in [36] classified the singular solution by building the connection with the weak solutions of

$$-\Delta u = u^p + k\delta_0$$

and the related positive solution $u_k$ has the asymptotic behavior

$$\lim_{|x| \to 0^+} u_k(x)|x|^{N-2} = c_N k$$

for some $k \geq 0$ and the normalized constant $c_N > 0$ for $-\Delta$. In the particular case $k = 0$, the solutions have removable singularity at the origin. In the Serrin’s critical and supercritical case $p \geq \frac{N}{N-2}$, however, since it always has $k = 0$, this method fails to classify the singularities. When

$$\theta \in (-2,2) \quad \text{and} \quad p \in \left(\frac{N + \theta}{N-2}, \frac{N + 2}{N-2}\right) \setminus \left\{\frac{N + 2 + 2\theta}{N - 2}\right\},$$

the singularity of solution for (1.2) was studied by Gidas-Spruck in [29] by using analytic techniques from [5]. In this case, the positive singular solutions have the singularity:

$$u(x) = c_{p,\theta} |x|^{\frac{2 + \theta}{p-1}} (1 + o(1)) \quad \text{as } |x| \to 0^+$$

with the coefficient

$$c_{p,\theta} = \left(\frac{2 + \theta}{p-1}(N-2) - \frac{2 + \theta}{p-1}\right) \frac{\Gamma}{\Gamma(1-s)}.$$ 

When

$$\theta \in (-2,2) \quad \text{and} \quad p = \frac{N + \theta}{N-2},$$

the author in [115] gave a beautiful classification: any positive solution $u$ of (1.2) has removable singularity at origin or it has the asymptotic behavior

$$u(x) = \left(\frac{(N-2)^2}{2 + \theta}\right)^{\frac{2 + \theta}{2s}} |x|^{2-N} (-\ln |x|)^\frac{2 + \theta}{N-2} (1 + o(1)) \quad \text{as } |x| \to 0^+. $$
Moreover, the existence of a singular solution is obtained by the phase plane analysis. By the aid of these solutions, Pacard in [37] has constructed positive solutions with the prescribed singular set. Later on, Caffarelli-Gidas-Spruck in [9] (also see [35]) classified the singular solutions of (1.2) when \( \theta = 0 \), \( p = \frac{N+2}{N-2} \) and its singular solutions satisfy

\[
u(x) = \varphi_d(-\ln |x|)|x|^{-\frac{N-2}{2}}(1 + o(|x|)) \quad \text{as} \quad |x| \to 0^+,
\]

where \( \varphi_d \) may be the constant \( \frac{c_{N+2}}{N-2} \) or be a periodic function. In the Sobolev super critical case \( \theta = 0 \) and \( p > \frac{N+2}{N-2} \), [30][41] have studied the structure of positive radial solutions of (1.2), [22] constructs some non-radially symmetric solutions. But it is still open to give a full classification of isolated singularities for singular solutions. We refer to [7,23,31,39,40,43] for more singularities for elliptic problem in various setting.

Motivated by various applications and relationships to the theory of PDEs, there has been an increasing interest in Dirichlet problems with nonlocal operators and the prototype of nonlocal operator is the fractional Laplacian. The problems with fully nonlinear nonlocal operators are studied by [12,13], [11] connecting the fractional problem with the second order degenerated problem in a half space with one dimensional higher, and basic results about regularities for the fractional problem could see [11,35][11] and blowing up analysis for nonlocal problems could see [10][14][18]. Motivated by [36], the isolated singularity of (1.1) with \( \theta = 0 \) was studied in [16] for \( p \in (1, \frac{N}{N-2}) \) via the connection with the distributional solutions of

\[(-\Delta)\psi u = u^p + k\delta_0 \quad \text{in} \quad B_1 \setminus \{0\}, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_1,
\]

for which the positive singularity could be described by fundamental solution of fractional Laplacian. As well as for the Laplacian, this method fails for the classification of isolated singularities for the Serrin’s critical and supercritical case, i.e. \( p \geq \frac{N+2}{N-2} \). When \( \theta = 0 \), \( p \in (\frac{N}{N-2}, \frac{N+2}{N-2}) \), [10][32] build the platform of the isolated singularity for positive solution to

\[
\begin{align*}
\text{div}(t^{1-2s})\nabla U(t,x) &= 0 \quad \text{in} \quad B_1 \times (0,1) \\
\lim_{t \to 0^+} t^{1-2s}\partial_t U(x,t) &= U^p(x,0) \quad \text{on} \quad B_1 \setminus \{0\}
\end{align*}
\]

and in the Sobolev’s critical case \( p = \frac{N+2s}{N-2} \), the behavior of non-removable singular solution of (1.1) can be stated as following

\[
c_2|x|^{-\frac{N+2s}{2}} \leq u(x) \leq c_1|x|^{-\frac{N-2s}{2}}
\]

for some \( c_1 > c_2 > 0 \), and then [46] gives a description of singular solution near the singularity for \( p \in (\frac{N}{N-2}, \frac{N+2s}{N-2}) \) as

\[
c_4|x|^{-\frac{2s}{2s-1}} \leq u(x) \leq c_3|x|^{-\frac{2s}{2s-1}}
\]

for some \( c_3 > c_4 > 0 \). Also its estimates singularity for \( p = \frac{N}{N-2} \) could see [35]. For the existence of isolated singular solutions, [12][2] constructed a sequence of isolated singular solution of

\[(-\Delta)^\psi u = u^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}
\]

for \( p \in (\frac{N}{N-2}, \frac{N+2s}{N-2}) \), with fast decaying at infinity. We can also see [3][24] for the fractional Yamabe problem with isolated singularities in the Sobolev’s critical case. Very recently, we provided in [15] the classification of isolated singularities of (1.1) by a direct analysis method and showed the existence of a sequence of singular solutions of (1.1) with \( \theta = 0 \) and \( p = \frac{N}{N-2} \).

Our aim in this paper is to classify the positive singularity of fractional problem (1.1) in the Serrin’s supercritical case by a direct analysis method. We first state our main results. To this end, we introduce some notations. For \( \tau \in (-N, 2s) \), we denote

\[
C_s(\tau) = 2^{2s} \frac{\Gamma(\frac{N+\tau}{2s})\Gamma(\frac{\tau}{2s})}{\Gamma(-\frac{s}{2})\Gamma\left(\frac{N-2s+\tau}{2s}\right)}
\]

More properties about \( C_s(\tau) \) can be found in the next section. In particular we notice that

\[
C_s(\tau) > 0 \quad \text{for} \quad \tau \in (2s-N, 0).
\]
Let
\[ K_{p, \theta} := C_s \left( -\frac{2s + \theta}{p - 1} \right)^{\frac{1}{p-1}}, \] (1.7)
then it’s well defined since \(-\frac{2s + \theta}{p - 1} \in (2s - N, 0)\) for \(p > \frac{N + \theta}{N - 2s}\) and \(\theta > -2s\). Here we use the notation \(K_p = K_{p, 0}\).

The classification of isolated singularities of (1.1) states as following.

**Theorem 1.1** Assume that \(s \in (0, 1)\), \(N > 2s\), \(\theta = 0\)
\[ p \in \left( \frac{N}{N - 2s}, \frac{N + 2s}{N - 2s} \right) \] (1.8)
and \(h \in L^1(\mathbb{R}^N \setminus B_1)\) is nonnegative.

Let \(u_0\) be a positive solution of (1.7), then \(u_0\) has either removable singularity at the origin or there holds
\[ \frac{K_p}{C_0} \leq \liminf_{|x| \to 0^+} u_0(x)|x|^{\frac{N+\theta}{N-2s}} \leq K_p \leq \limsup_{|x| \to 0^+} u_0(x)|x|^{\frac{N+\theta}{N-2s}} \leq C_0 K_p, \]
where \(C_0 \geq 1\) is a constant from the Harnack inequality.

**Theorem 1.2** Assume that \(h\) is radially symmetric and nonnegative in \(\mathbb{R}^N \setminus B_1\), decreasing with respect to \(|x|\) and \(h(1) \geq 0\),
\[ \theta = 0 \quad \& \quad p \geq \frac{N + 2s}{N - 2s} \] (1.9)
or
\[ \theta \in (-2s, 0) \quad \& \quad p > \frac{N + \theta}{N - 2s}. \] (1.10)

Let \(u_0\) be a positive solution of (1.7) such that
\[ u_0(x) \geq h(1) \quad \text{for any} \ x \in B_1, \] (1.11)
then \(u_0\) is radially symmetric, decreasing with respect to \(|x|\). Moreover, \(u_0\) has either removable singularity at the origin or one has
\[ \frac{K_p, \theta}{C_0} \leq \liminf_{|x| \to 0^+} u_0(x)|x|^{\frac{N+\theta}{N-2s}} \leq K_{p, \theta} \leq \limsup_{|x| \to 0^+} u_0(x)|x|^{\frac{N+\theta}{N-2s}} \leq C_0 K_{p, \theta}. \] (1.12)

Our outline of the proofs is the following. The first step is to get an bound
\[ \limsup_{|x| \to 0^+} u_0(x)|x|^{\frac{N+\theta}{N-2s}} < +\infty. \] (1.13)

To this end, we build the following important upper estimate:

**Proposition 1.1** Assume that \(\theta \in (-2s, +\infty)\), \(p \geq \frac{N + \theta}{N - 2s}\) and \(u_0\) is a nonnegative classical solution of (1.7). Then we have \(|x|^\theta u_0^p \in L^1(B_1, (1 - |x|)^s dx)\) and there exists a uniform constant \(c_5 > 0\), independent of \(r\) and \(u_0\), such that
\[ \int_{B_r} |x|^\theta u_0^p dx \leq c_5 r^{N - \frac{N + \theta}{N - 2s}}, \quad \forall \ r \in (0, \frac{1}{2}). \] (1.14)

With the help of Proposition 1.1 we can develop a direct blowing up technique for \(\theta = 0\), in which we don’t have to transform the equation into an extension problem, see [10,34,46], which doesn’t require the boundedness in \(L^1_s(\mathbb{R}^N, dx)\), thanks to the local property. In our blowing up procedure, if (1.13) fails, the essential part is to keep the scaled singular solutions uniformly bounded in \(L^1_s(\mathbb{R}^N)\) for \(p > \frac{N}{N - 2s}\) by (1.14) with \(\theta = 0\) and then we pass to the limit to get a positive bounded solution for the limit equation and a contradiction arises from the nonexistence of positive bounded solution for that limit equation.
However, the blowing up technique fails for $\theta \neq 0$, since the limit equation is $(-\Delta)^{s}u = 0$ in $\mathbb{R}^{N}$ when $\theta < 0$, which has positive solutions while for $\theta > 0$ there is no limit equation at all.

Fortunately, when $\theta \in (-2s, 0)$, we can proceed a direct method of moving planes to obtain the properties of symmetry and monotonicity in the $|x|$ direction under the assumption on $h$ of symmetry and monotonicity, motivated by [27]. As a consequence, these properties improve (1.14) to our desired upper bound (1.17).

The second step is to get the estimate: for non-removable solution, there holds
\[
\liminf_{|x| \to 0^+} u_{0}(x)|x|^{\frac{2s+\theta}{p}} \leq K_{p, \theta} \leq \limsup_{|x| \to 0^+} u_{0}(x)|x|^{\frac{2s+\theta}{p}}
\]
for $p \in (\frac{N+\theta}{2s}, +\infty)$. These inequalities are motivated by the Poisson problem involving the fractional Hardy operators, where the coefficients of the Hardy potential is very sensitive to the blowing up rate of the related solutions. Here we want to emphasize that the inequalities (1.15) does not depends on the upper bounds and Harnack inequality.

Generally, we can draw a conclusion for general $\theta > -2s$:

**Theorem 1.3** Assume that $\theta \in (-2s, +\infty)$, $p > \frac{N+\theta}{2s}$ and $u_{0}$ is a nonnegative classical solution of (1.1) verifying that there exist $C_{1} \geq 1$ and $r_{1} \in (0, 1)$ such that for $x, y \in B_{r_{1}} \setminus \{0\}$
\[
 u_{0}(x) \leq C_{1} u_{0}(y) \quad \text{for } 1 \frac{1}{2} \leq \frac{|x|}{|y|} \leq 2,
\]
then
\[
\frac{K_{p, \theta}}{C_{1}} \leq \liminf_{|x| \to 0^+} u_{0}(x)|x|^{\frac{2s+\theta}{p}} \leq K_{p, \theta} \leq \limsup_{|x| \to 0^+} u_{0}(x)|x|^{\frac{2s+\theta}{p}} \leq C_{1} K_{p, \theta}.
\]

Note that (1.16) is one type of Harnack inequality, which, together with the pointwise upper bound, to obtain the lower bound $\liminf_{|x| \to 0^+} u_{0}(x)|x|^{\frac{2s+\theta}{p}} > 0$ for the non-removable solution. To make clear of our results, we explain the relationship of integral upper bound (1.14), pointwise upper bound $\limsup_{|x| \to 0^+} u_{0}(x)|x|^{\frac{2s+\theta}{p}} < +\infty$ and Harnack inequality (1.16):

(i) for $\theta = 0$: integral upper bound + blow up analysis $\Rightarrow$ pointwise upper bound $\Rightarrow$ Harnack inequality;

(ii) for $\theta \in (-2s, 0)$: integral upper bound + radial symmetry, increasing monotonicity $\Rightarrow$ pointwise upper bound $\Rightarrow$ Harnack inequality;

(iii) for $\theta \in (-2s, +\infty)$: integral upper bound + Harnack inequality $\Rightarrow$ pointwise upper bound.

**Remark 1.1** Under the assumption of $h$ in Theorem 1.3 the method the moving plane works, all such nonnegative solutions of (1.1) are radially symmetric and decreasing with respect to $|x|$.

If (1.1) has a positive solution, then
\[
\begin{align*}
(-\Delta)^{s}u &= |x|^{\theta}u^{p} \quad \text{in } B_{1} \setminus \{0\}, \\
u &= h \quad \text{in } \mathbb{R}^{N} \setminus B_{1}
\end{align*}
\]

admits the minimal nonnegative solution $u_{\min}$. Any solution $u \in C^{s}(\mathbb{R}^{N})$ of (1.18) verifies that $\inf_{x \in B_{1}} u(x) \leq h(1)$, than $u(1) = h(1)$. A stronger version of (1.18) is
\[
\inf_{x \in B_{1}} u_{\min}(x) = h(1).
\]

Particularly,
\[
u_{\min} \equiv 0 \quad \text{if } h \equiv 0
\]
and
\[
u_{\min} > b \quad \text{if } h \equiv b \quad \text{for some } b > 0.
\]
Under these assumptions, the isolated singularity of all nonnegative solution of (1.1) could be classified.

For the existence of minimal solution \( u_{\min} \), it could be approached by the sequence of functions \( \{v_m\}_{m \in \mathbb{N}} \), which are the solutions of

\[
\begin{cases}
(-\Delta)^s v_m = |x|^\theta v_m^{p-1} & \text{in } B_1, \\
v_m = h & \text{in } \mathbb{R}^N \setminus B_1
\end{cases}
\]

with \( v_0 \) being the s-harmonic extension of \( h \) in \( B_1 \), if (1.1) has a nonnegative solution, which is an upper bound for \( \{v_m\} \).

Our methods could be extended to classify the singularity of prototype Lane-Emden equation

\[
\begin{cases}
-\Delta u = |x|^\theta u^p & \text{in } B_1 \setminus \{0\}, \\
u = h & \text{in } \partial B_1,
\end{cases}
\]  

(1.19)

in the super critical case \( p > \frac{N+2+2 \theta}{N-2} \) and \( \theta > -2 \), where \( h \geq 0 \). We have the following results:

**Theorem 1.4** Assume that \( N \geq 3, \theta > -2, p > \frac{N+2+2 \theta}{N-2}, \) \( v_0 \) is a positive solution of (1.19) and \( c_{p,\theta} \) is given in (1.3).

(i) If (1.16) holds for \( v_0 \), then \( v_0 \) has either removable singularity at the origin or one has

\[
\frac{c_{p,\theta}}{C_2} \leq \liminf_{|x| \to 0^+} v_0(x)|x|^\frac{2 \theta p}{N-2} \leq c_{p,\theta} \leq \limsup_{|x| \to 0^+} v_0(x)|x|^\frac{2 \theta p}{N-2} \leq C_2 c_{p,\theta},
\]

(1.20)

where \( C_2 \geq 1 \) is a constant from the Harnack inequality.

(ii) If \( \theta \in (-2,0] \), \( h \) is a nonnegative constant on \( \partial B_1 \) and \( v_0(x) \geq h, \forall x \in B_1 \),

(1.21)

then \( v_0 \) is radially symmetric, decreasing with respect to \( |x| \) and it is either removable at the origin or has the singularity (1.20).

It is worth noting that the limit

\[
\lim_{|x| \to 0^+} v_0(x)|x|^\frac{2 \theta p}{N-2} = c_{p,\theta}
\]

is not always true in the Sobolev supercritical case. In fact, when \( \frac{N+1}{N-1} < p < p_c(N-1), p_c(N-1) \) being the Joseph-Lundgren exponent, \([22]\) shows that

\[
-\Delta u = u^p \text{ in } \mathbb{R}^N \setminus \{0\}
\]

has infinitely many non-radially symmetric solutions with the form that

\[
u(r, \omega) = c_p r^{-\frac{2}{p-1}} w_0(\omega), \quad (r, \omega) \in \mathbb{R} \times \partial B_1,
\]

where \( w_0 \) is non-constant solution in \( \partial B_1 \). Together with our theorem 1.4, we get a corollary that \( c_{p,\theta} \in w_0[\partial B_1] \).

The remainder of this paper is organized as follows. In Section 2, we provide preliminary estimates of the fractional Hardy Poisson problem, an essential integral upper bound and Harnack inequality. In Section 3, we obtain the upper bounds of positive solutions of (1.1). Section 4 is devoted to isolated singularities of (1.1) and proofs of Theorem 1.1 and Theorem 1.2.
2 Preliminary

Our analysis of isolated singularities for solutions with fractional laplacian is based on the fractional Hardy problem. In the sequel, we do the $C^2$ extension for $h$ in $B_1$, till denoting it by $h$, such that

$$h = 0 \quad \text{in } B_2^+.$$  

We denote

$$U_0(x) = -(-\Delta)^s h(x) \quad \text{for } x \in B_1,$$  \hspace{1cm} (2.1)

which is bounded locally in $B_1$.

2.1 Poisson problem for fractional Hardy problem

Recall that for $\tau \in (-N, 2s)$, we define

$$C_s(\tau) = 2^{2s} \frac{\Gamma\left(\frac{N+\tau}{2}\right) \Gamma\left(\frac{2s-\tau}{2}\right)}{\Gamma\left(\frac{\tau}{2}\right) \Gamma\left(\frac{N-2s+\tau}{2}\right)} = -\frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \frac{|\epsilon_1 + z|^\tau + |\epsilon_1 - z|^\tau - 2}{|z|^{N+2s}} \, dz,$$  \hspace{1cm} (2.2)

where $\epsilon_1 = (1, 0, \cdots, 0) \in \mathbb{R}^N$ and there holds

$$(-\Delta)^s |\cdot|^{\tau} = C_s(\tau) |\cdot|^{\tau-2s} \quad \text{in } \mathbb{R}^N \setminus \{0\}. $$  \hspace{1cm} (2.3)

Note that $C_s(\cdot)$ has two zeros point $\{0, 2s-N\}$, i.e.

$$C_s(0) = C_s(2s-N) = 0.$$  

By [19] Lemma 2.3], we know that the function $C_s$ is strictly concave and has a unique maximum at $\{\frac{2s-N}{2}\}$ with the maximal value $2^{2s} \frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}$. Moreover,

$$C_s(\tau) = C_s(2s-N-\tau) \quad \text{for } \tau \in (-N, 2s)$$  \hspace{1cm} (2.4)

and

$$\lim_{\tau \to -N} C_s(\tau) = \lim_{\tau \to 2s} C_s(\tau) = -\infty.$$  \hspace{1cm} (2.5)

Particularly, we have that

$$C_s(\tau) > 0 \quad \text{for } \tau \in (2s-N, 0), \quad C_s(\tau) < 0 \quad \text{for } \tau \in (-N, 2s-N) \cup (0, 2s).$$  \hspace{1cm} (2.6)

Now let

$$L^s_\mu u(x) = (-\Delta)^s u(x) + \frac{\mu}{|x|^{2s}} u(x),$$

where

$$\mu \geq \mu_0 := -2^{2s} \frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}.$$  

It is shown in [19] that the linear equation

$$L^s_\mu u = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}$$

has two distinct radial solutions

$$\Phi_\mu(x) = \begin{cases} |x|^{\tau_-} & \text{if } \mu > \mu_0 \\ |x|^{-\frac{N-2s}{2}} \ln \left( \frac{1}{|x|} \right) & \text{if } \mu = \mu_0 \end{cases} \quad \text{and} \quad \Gamma_\mu(x) = |x|^{\tau_+(\mu)},$$

where $\tau_-(\mu) \leq \tau_+(\mu)$ verify that

$$\tau_-(\mu) + \tau_+(\mu) = 2s - N \quad \text{for all } \mu \geq \mu_0,$$ \hspace{1cm} (2.7)

$$\tau_-(0) = \tau_+(0) = 2s - N, \quad \tau_+(0) = 0,$$ \hspace{1cm} (2.8)

$$\lim_{\mu \to -\infty} \tau_- + \tau_+(\mu) = -N \quad \text{and} \quad \lim_{\mu \to +\infty} \tau_+(\mu) = 2s.$$


Theorem 2.1

Let
\[
C_s(\tau) + \mu > 0 \quad \text{for} \quad \tau \in (\tau_-, \tau_+)
\]
and
\[
C_s(\tau) + \mu < 0 \quad \text{for} \quad \tau \in (-N, \tau_-) \cup (\tau_+, 2s).
\]

Our analysis of singularities of positive solutions to (1.1) is based on the study of the Poisson problem involving the fractional Hardy operator:
\[
\begin{align*}
\left\{ \begin{array}{ll}
L^\mu_\nu u = f & \text{in } B_1 \setminus \{0\}, \\
\quad u = h & \text{in } \mathbb{R}^N \setminus B_1.
\end{array} \right.
\end{align*}
\]

The dual of the operator \(L^\mu_\nu\) is a weighted fractional Laplacian \((-\Delta)^\mu_\nu\) given by
\[
(-\Delta)^\mu_\nu v(x) := C_{N, s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_1} \frac{v(x) - v(z)}{|x - z|^{N + 2s}} \Gamma_\mu(z) dz.
\]
This expression is well defined for \(x \in \mathbb{R}^N \setminus \{0\}\) if \(v \in L^1(\mathbb{R}^N, \frac{\Gamma_\mu(z)}{|x - z|^{N + 2s}} dz)\) and if \(v\) is twice continuously differentiable in a neighborhood of \(x\). From [19, Proposition 3.1] there holds
\[
|(-\Delta)^\mu_\nu(\xi(x))| \leq c_0 \min \{\Lambda_\mu(x), |x|^{-N - 2s}\} \quad \text{for} \quad \xi \in C^2_c(\mathbb{R}^N), \quad x \in \mathbb{R}^N \setminus \{0\},
\]
where \(c_0 = c_0(s, \mu, \xi) > 0\) is a constant and
\[
\Lambda_\mu(x) = \begin{cases} 
1 & \text{if } \tau_+(s, \mu) > 2s - 1, \\
|x|^{1 - 2s + \tau_+(s, \mu)} & \text{if } \tau_+(s, \mu) < 2s - 1, \\
1 + (-\ln |x|)_+ & \text{if } \tau_+(s, \mu) = 2s - 1.
\end{cases}
\]

Then from [19, Theorem 1.4] we have

**Theorem 2.1** Let \(\mu \geq \mu_0\) and \(f \in C^0_{\text{loc}}(\overline{B}_1 \setminus \{0\})\) for some \(\theta \in (0, 1)\).

(i) (Existence) If \(f \in L^1(B_1, \Gamma_\mu(x) dx)\), then for every \(k \in \mathbb{R}\) there exists a solution \(u_k \in L^1(B_1, \Lambda_\mu dx)\) of problem (2.7) satisfying the distributional identity
\[
\int_{B_1} u_k(-\Delta)^\mu_\nu \xi dx = \int_{B_1} f \xi \Gamma_\mu dx + c_{s, \mu} k \xi(0) \quad \text{for all} \quad \xi \in C^2_c(B_1).
\]

(ii) (Existence and Uniqueness) If \(f \in L^\infty(B_1, |x|^\rho dx)\) for some \(\rho < 2s - \tau_+(s, \mu)\), then for every \(k \in \mathbb{R}\) there exists a unique solution \(u_k \in L^1(B_1, \Lambda_\mu dx)\) of problem (2.7) with the asymptotics
\[
\lim_{|x| \to 0^+} \frac{u_k(x)}{\Phi_\mu(x)} = k.
\]
Moreover, \(u_k\) satisfies the distributional identity (2.11).

(iii) (Nonexistence) If \(f\) is nonnegative and satisfies
\[
\int_{B_1} f \Gamma_\mu dx = +\infty,
\]
then the problem
\[
\begin{align*}
\left\{ \begin{array}{ll}
L^\mu_\nu u = f & \text{in } B_1 \setminus \{0\}, \\
\quad u \geq 0 & \text{in } \mathbb{R}^N \setminus B_1
\end{array} \right.
\end{align*}
\]
has no nonnegative distributional solution \(u \in L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \cap L^1(\mathbb{R}^N, dx)\).
A direct consequence from Theorem 2.1 with $\mu = 0$ is the nonexistence of solutions for

$$
\begin{cases}
(-\Delta)^s u = f & \text{in } B_1 \setminus \{0\}, \\
u = h & \text{in } \mathbb{R}^N \setminus B_1.
\end{cases}
$$

(2.15)

Corollary 2.1 Assume that the nonnegative function $f \in C^b_{\text{loc}}(\overline{B}_1 \setminus \{0\})$ for some $\beta \in (0, 1)$ and nonnegative function $h \in C^\infty_{\text{loc}}(B_2 \setminus B_1) \cap L^1(\mathbb{R}^N \setminus B_1, \frac{ds}{|x|^{1+\alpha+\beta}})$ with $\alpha \in (0, 1)$. Then problem (2.15) has no positive solution, if

$$
\lim_{r \to 0^+} \int_{B_1 \setminus B_r} f(x) dx = +\infty.
$$

Particularly, the above assumption is filled if there holds

$$
\liminf_{|x| \to 0^+} f(x)|x|^N > 0.
$$

Next we introduce the comparison principle of $\mathcal{L}_\mu^s$.

Lemma 2.1 Assume that $\mu \geq \mu_0$, $O$ is a bounded $C^2$ domain containing the origin and the functions $u_i \in C(\overline{O} \setminus \{0\})$ for $i = 1, 2$ verify in the classical sense

$$
\begin{cases}
\mathcal{L}_\mu^s u_1 \leq \mathcal{L}_\mu^s u_2 & \text{in } O \setminus \{0\}, \\
u_1 \leq u_2 & \text{in } \mathbb{R}^N \setminus O
\end{cases}
$$

and satisfying

$$
\limsup_{|x| \to 0^+} u_1(x)|x|^{-\tau} \leq \liminf_{|x| \to 0^+} u_2(x)|x|^{-\tau}.
$$

Then

$$
u_1 \leq u_2 \quad \text{in } O \setminus \{0\}.
$$

Proof. Let $u = u_1 - u_2$ and then

$$
\mathcal{L}_\mu^s u \leq 0 \quad \text{in } O \setminus \{0\} \quad \text{and} \quad \limsup_{|x| \to 0^+} u(x)\Phi_\mu^{-1}(x) \leq 0,
$$

then for any $\epsilon > 0$, there exists $r_\epsilon > 0$ converging to 0 as $\epsilon \to 0$ such that

$$
u \leq \epsilon \Phi_\mu \quad \text{in } B_{r_\epsilon}(0) \setminus \{0\},
$$

where $\Phi_\mu(x) = |x|^{\tau}$ is the fundamental solution of $\mathcal{L}_\mu^s$.

We see that

$$
u = 0 < \epsilon \Phi_\mu \quad \text{in } \mathbb{R}^N \setminus O,
$$

then we have $u \leq \epsilon \Phi_\mu$ in $O \setminus \{0\}$ for any $\epsilon > 0$, which implies that $u \leq 0$ in $O \setminus \{0\}$.

□

Lemma 2.2 Assume that $\mu > \mu_0$, the nonnegative function $g \in C^b_{\text{loc}}(B_1 \setminus \{0\})$ for some $\beta \in (0, 1)$ and there exists $\tau \in \mathbb{R}$ such that

$$
g(x) \geq |x|^{-2s} \quad \text{in } B_2 \setminus \{0\}.
$$

Let $u_g \in C(B_1 \setminus \{0\})$ verify

$$
\begin{cases}
\mathcal{L}_\mu^s u_g \geq g & \text{in } B_1 \setminus \{0\}, \\
u_g \geq 0 & \text{on } \mathbb{R}^N \setminus B_1.
\end{cases}
$$

(2.17)

We have the following (i) If $\tau \in (\tau_-, \tau_+)$, then there exists $c_6 > 0$ such that

$$
u_g(x) \geq c_6 |x|^\tau \quad \text{in } B_2 \setminus \{0\};
$$

(ii) If $\tau > \tau_+$, then there exists $c_7 > 0$ such that

$$
u_g(x) \geq c_7 |x|^\tau \quad \text{in } B_2 \setminus \{0\}.
$$
Proof. (i) For $\tau \in (2s - N, 0)$, we have that

$$(-\Delta)^s |x|^\tau = C_s(\tau)|x|^{\tau-2s} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

where $C_s(\tau) > 0$ by (2.6).

We let

$$w_1(x) = |x|^\tau - 1 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

which, direct computation, verifies that

$$\mathcal{L}_\mu^s w_1 \leq (C_s(\tau) + \mu)|x|^{\tau-2} \quad \text{in} \quad B_1(0) \setminus \{0\}, \quad w_1 \leq 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_1,$$  \hspace{1cm} (2.18)

where $C_s(\tau) + \mu > 0$ for $\tau \in (\tau_-, \tau_+)$ and $\mu > \mu_0$.

Note that by the lower bound of $g$ there exists $t_0 > 0$ such that

$$\mathcal{L}_\mu^s (t_0 u_g) \geq t_0 g(x) \geq (C_s(\tau) + \mu)|x|^{\tau-2s} = \mathcal{L}_\mu^s w_1 \quad \text{in} \quad B_1 \setminus \{0\}$$

and

$$\lim_{|x| \to 0} u_g \Phi_\mu^{-1}(x) \geq 0 = \lim_{|x| \to 0} w_1 \Phi_\mu^{-1}(x), \quad t_0 u_g \geq 0 \geq w_1 \quad \text{in} \quad \mathbb{R}^N \setminus B_1.$$

Then by Lemma 2.1 we have that

$$t_0 u_g \geq w_1 \quad \text{in} \quad B_1 \setminus \{0\}.$$

(ii) For $\tilde{\tau} \in (\tau_+, \min\{\tau, 2s\})$ we set

$$w_2(x) = |x|^{\tilde{\tau}} - |x|^\tau \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

then

$$\mathcal{L}_\mu^s w_2 \leq -(C_s(\tilde{\tau}) + \mu)|x|^{\tilde{\tau}-2s} \leq -(C_s(\tau) + \mu)|x|^{\tau-2s} \quad \text{in} \quad B_1 \setminus \{0\}$$

and

$$w_2 \leq 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_1,$$

where $-(C_s(\tilde{\tau}) + \mu) > 0$ for our choice of $\tilde{\tau}$ and $\tilde{\tau} - 2s \geq \tau - 2s$. Lemma 2.1 implies that $u_g \geq w_2$ in $B_1 \setminus \{0\}$. \hfill $\Box$

Lemma 2.3 Let $\mu > \mu_0$, $g \in C^d_{\text{loc}}(B_1 \setminus \{0\})$ with $\beta \in (0, 1)$ be a nonnegative function such that there exists $\tau > \tau_-$ such that

$$g(x) \leq |x|^{\tau-2s} \quad \text{for} \quad x \in B_1 \setminus \{0\},$$

and $h$ be a nonnegative function in $C^\alpha(B_2 \setminus B_1) \cap L^1(\mathbb{R}^N \setminus B_1)$ with $\alpha \in (0, 1)$.

Let $u_g \in C(\bar{B}_1 \setminus \{0\})$ verify

$$\begin{cases}
\mathcal{L}_\mu^s u_g \leq g & \text{in} \quad B_1 \setminus \{0\}, \\
u_g = h & \text{in} \quad \mathbb{R}^N \setminus B_1, \\
\lim_{|x| \to 0^+} u_g(x)|x|^{-\tau_+} = 0.
\end{cases}$$ \hspace{1cm} (2.19)

Then (i) if $\tau \in (\tau_-, \tau_+)$ then there exists $c_8 > 0$ such that

$$u_g(x) \leq c_8 |x|^\tau \quad \text{for} \quad x \in B_1 \setminus \{0\};$$

(ii) if $\tau > \tau_+$ then there exists $c_9 > 0$ such that

$$u_g(x) \leq c_9 |x|^{\tau_+} \quad \text{for} \quad x \in B_1 \setminus \{0\}.\]
Proof. From (2.1), a direct computation shows that for $x \in B_\frac{1}{4}$

$$0 < U_0(x) = c_{N,s} \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}} \frac{h(y)}{|x-y|^{N+2s}}dy$$

$$\leq c_{N,s} \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}} \frac{h(y)}{(|y| - \frac{1}{4})^{N+2s}}dy \leq c_{10}||h||_{L_1(\mathbb{R}^N)}$$ (2.20)

by using the fact that

$$|x - y| \geq |y| - \frac{1}{4} \text{ for } |x| \leq \frac{1}{4} \text{ and } |y| \geq \frac{1}{2}.$$ (i) For $\tau \in (\tau_-, \tau_+)$, we have that

$$\mathcal{L}_s^\mu |x|^\tau = (\mathcal{C}_s(\tau) + \mu) |x|^{\tau-2s} \text{ for } x \in \mathbb{R}^N \setminus \{0\},$$

where $\mathcal{C}_s(\tau) + \mu > 0$ and $\tau - 2s < 0$.

Note that there exists $t_3 > 0$ such that

$$\mathcal{L}_s^\mu \left( t_3(u_g - h) \right) \leq t_3(g(x) + U_0) \leq (\mathcal{C}_s(\tau) + \mu) |x|^{\tau-2s} = \mathcal{L}_s^\mu |x|^\tau \text{ for } x \in B_1 \setminus \{0\},$$

$$\lim_{|x| \to 0} \left( u_g(x) - h(x) \right) |x|^{-\tau-}(x) = 0$$

and

$$t_3(u_g - h)_+ \leq |x|^\tau \text{ in } B_1 \setminus B_{\frac{1}{2}} \text{ and } t_3(u_g - h) = 0 < |x|^\tau \text{ in } \mathbb{R}^N \setminus B_1.$$ Then by Lemma (2.1) we have that

$$t_3(u_g(x) - h(x)) \leq |x|^\tau \text{ for } x \in B_1 \setminus \{0\}.$$ (ii) Take $\tilde{\tau} \in (\tau_+, 2s) \cap (\tau_+, \tau)$ and

$$\tilde{w}_3(x) = |x|^\tilde{\tau} - |x|^\tau + 1 \text{ in } B_1 \setminus \{0\} \text{ and } \tilde{w}_3(x) = 1 \text{ in } \mathbb{R}^N \setminus B_1.$$ Direct computation shows that for $x \in B_{\frac{1}{4}} \setminus \{0\}$

$$\mathcal{L}_s^\mu \tilde{w}_3(x) = \mathcal{L}_s^\mu (|x|^\tilde{\tau} - |x|^\tau) + c_{N,s} \int_{\mathbb{R}^N \setminus B_1} \frac{|y|^{\tilde{\tau}+} - |y|^\tau}{|x-y|^{N+2s}}dy$$

$$\geq - (\mathcal{C}_s(\tilde{\tau}) + \mu) |x|^\tilde{\tau}-2s - c_{N,s} \int_{\mathbb{R}^N \setminus B_1} \frac{|y|^\tilde{\tau}}{|x-y|^{N+2s}}dy$$

$$\geq - (\mathcal{C}_s(\tilde{\tau}) + \mu) |x|^\tilde{\tau}-2s - 2^{N+2s} c_{N,s} \int_{\mathbb{R}^N \setminus B_1} \frac{|y|^\tilde{\tau}}{|y|^{N+2s}} dy,$$

where $-(\mathcal{C}_s(\tilde{\tau}) + \mu) > 0$.

Thus, there exists $r_0 \in (0, \frac{1}{2}]$ such that for $x \in B_{r_0} \setminus \{0\}$

$$\mathcal{L}_s^\mu \tilde{w}_3(x) \geq - \frac{1}{2} (\mathcal{C}_s(\tilde{\tau}) + \mu) |x|^{\tilde{\tau}-2s}.$$ As a consequence, there exists $t_4 > 0$ such that

$$\mathcal{L}_s^\mu \left( t_4(u_g - h) \right) \leq t_4(g(x) + U_0) \leq - \frac{1}{2} (\mathcal{C}_s(\tau) + \mu) |x|^{\tilde{\tau}-2s} \leq \mathcal{L}_s^\mu \tilde{w}_3(x) \text{ in } B_{r_0} \setminus \{0\}$$

and

$$t_4(u_g - h) \leq \tilde{w}_3 \text{ in } \mathbb{R}^N \setminus B_{r_0}.$$ 11
Together with

\[ \lim_{|x| \to 0^+} (t_4(u_g - h)(x))|x|^{-\tau} = 0, \]

Lemma 2.1 is applied to obtain that

\[ t_4(u_g - h) \leq \bar{w}_3 \quad \text{in} \ B_{r_0} \setminus \{0\}. \]

Therefore, we obtain that

\[ u_g(x) \leq c_9|x|^\tau \quad \text{for} \ x \in B_1 \setminus \{0\}. \]

We complete the proof. \[\Box\]

2.2 Uniformly integral upper bounds

The integral upper bound (1.14) is essential for our analysis of isolated singularities of (1.1) in the Serrin’s supercritical case.

**Proof of Proposition 1.1.** Recall \( U_0 \) in (2.1), which is uniformly bounded in \( B_{\frac{1}{2}} \). Let \( \bar{u} = u_0 - h \) in \( \mathbb{R}^N \), then we have that

\[ (-\Delta)^s \bar{u} = |x|^\theta u_0^p + U_0 \quad \text{in} \ B_{1}, \quad \bar{u} = 0 \quad \text{in} \ \mathbb{R}^N \setminus B_{1}. \]

From Theorem 2.1, we have that \( |x|^\theta u_0^p + U_0 \in L^1(B_1) \), so is \( u_0 \), thanks to the boundedness of \( U_0 \). Moreover, we have that

\[ \int_{B_1} (-\Delta)^s \bar{u} \chi dx = \int_{B_1} (|x|^\theta u_0^p + U_0) \chi dx \quad \text{for all} \ \chi \in C_0^2(B_1). \]

Let \( \xi_1 \) be the positive Dirichlet eigenfunction of \( (-\Delta)^s \) related to the first eigenvalue, subject to the zero Dirichlet boundary condition in \( \mathbb{R}^N \setminus B_{1} \), i.e.

\[ \begin{cases} (-\Delta)^s u = \lambda_1 u & \text{in} \ B_1, \\ u = 0 & \text{in} \ \mathbb{R}^N \setminus B_1. \end{cases} \]

(2.21)

Then \( \lambda_1 > 0 \), \( \xi_1 \) is positive and \( \xi_1 \in C_0^2(B_1) \cap C^s(\mathbb{R}^N) \). Moreover, by the standard argument we can obtain that \( \xi_1 \in C_0^2(B_1) \) and for some \( c_{10} > 1 \) there holds

\[ \frac{1}{c_{10}} \rho^s \leq \xi_1 \leq c_{10} \rho^s \quad \text{in} \ B_1. \]

Using \( \xi_1 \) as a test function, we have that

\[ \int_{B_1} |x|^\theta u_0^p \xi_1 dx + \int_{B_1} U_0 \xi_1 dx = \lambda_1 \int_{B_1} w \xi_1 dx \leq \left( \int_{B_1} |x|^\theta u_0^p \xi_1 dx \right)^{\frac{1}{2}} \left( \int_{B_1} \xi_1 |x|^{-\tau} dx \right)^{1-\frac{1}{2}}, \]

which implies

\[ \int_{B_1} |x|^\theta u_0^p \rho^s dx \leq c_{11}, \]

where \( c_{11} > 0 \) depends on \( h \) and we used the fact that \( -\frac{\theta}{\rho - 1} > -N \), thanks to \( p \geq \frac{N+\theta}{N-2s} \). Let

\[ \xi_r(x) = \xi_1(r^{-1}x) \quad \text{for} \ x \in \mathbb{R}^N, \]

then \( \xi_r \) is the solution of

\[ \begin{cases} (-\Delta)^s u = \lambda_1 r^{-2s} u & \text{in} \ B_r, \\ u = 0 & \text{in} \ \mathbb{R}^N \setminus B_r. \end{cases} \]

(2.22)
Let 

\[ w = u_0 - \bar{u}_0, \]

where \( \bar{u}_0 \) is the \( s \)-harmonic extension of \( u_0 \) in \( B_r \), i.e. the solution of

\[
\begin{aligned}
(-\Delta)^s u &= 0 & \text{in } B_r, \\
u &= u_0 & \text{in } \mathbb{R}^N \setminus B_r.
\end{aligned}
\]

Then we have that

\[
\begin{aligned}
(-\Delta)^s w &= |x|^\theta (w + \bar{u}_0)^p & \text{in } B_r, \\
w &= 0 & \text{in } \mathbb{R}^N \setminus B_r
\end{aligned}
\]

and

\[
\int_{B_r} w(-\Delta)^s \zeta dx = \int_{\mathbb{R}^N} |x|^\theta (w + \bar{u}_0)^p \zeta dx, \quad \zeta \in \mathcal{X}_r,
\]

which implies that

\[
\int_{B_r} u_0(-\Delta)^s \zeta dx = \int_{\mathbb{R}^N} |x|^\theta u_0^p \zeta dx + \int_{B_r} \tilde{u}_0(-\Delta)^s \zeta dx, \quad \zeta \in \mathcal{X}_r.
\]

Now we take \( \zeta = \xi_r \), we have that

\[
\lambda_1 r^{2s} \int_{B_r} u_0 \xi_r dx = \int_{\mathbb{R}^N} |x|^\theta u_0^p \xi_r dx + \int_{B_r} \tilde{u}_0(-\Delta)^s \xi_r dx.
\]

Again since \( -\frac{\theta}{p-1} > -N \), we have

\[
\int_{B_r} |x|^\theta u_0^p \xi_r dx < \int_{B_r} |x|^\theta u_0^p \xi_r dx + \int_{B_r} \tilde{u}_0(-\Delta)^s \xi_r dx
\]

\[
= \lambda_1 r^{-2s} \int_{B_r} u_0 \xi_r dx
\]

\[
\leq \lambda_1 r^{-2s} \left( \int_{B_r} |x|^\theta u_0^p \xi_r dx \right)^{p-1} \left( \int_{B_r} \xi_r |x|^{-\frac{\theta}{p-1}} dx \right)^{1-\frac{1}{p}}
\]

\[
\leq \|\xi_1\|_{L^\infty(B_1)} \lambda_1 r^{-2s} \left( \int_{B_r} |x|^\theta u_0^p \xi_r dx \right)^{p-1},
\]

which implies that

\[
\int_{B_r} |x|^\theta u_0^p \xi_r dx < \|\xi_1\|_{L^\infty(B_1)} \lambda_1 r^{-2s} \left( \int_{B_r} |x|^\theta u_0^p \xi_r dx \right)^{p-1}.
\]

(2.23)

Notice that

\[ \xi_r(x) = \xi_1(r^{-1} x) \geq \min_{z \in B_\frac{r}{2}} \xi_1(z) \text{ in } B_\frac{r}{2}, \]

then

\[
\int_{B_{\frac{r}{2}}} |x|^\theta u_0^p dx < c_{12} \int_{B_{\frac{r}{2}}} |x|^\theta u_0^p \xi_r dx < c_{12} \|\xi_1\|_{L^\infty(B_1)} \lambda_1 r^{-2s} \left( \int_{B_r} |x|^\theta u_0^p \xi_r dx \right)^{p-1}.
\]

(2.24)

Replace \( r \) by \( \frac{r}{2} \), we obtain [11.14].

With the help of Proposition 11.1, we have the following sharp upper bound under some suitable restrictions of \( u_0 \).

**Corollary 2.2** Assume that \( \theta > -2s \), \( p \geq \frac{N+\theta}{N-2s} \), and \( u_0 \) is a nonnegative classical solution of (1.7) verifying the Harnack inequality: there exist \( C_0 > 1 \) and \( r_1 \in (0, 1) \) such that for \( x, y \in B_{r_1} \setminus \{0\} \)

\[
u_0(x) \leq C_0 u_0(y) \quad \text{for } 1 \leq \frac{|x|}{|y|} \leq 2,
\]

(2.25)

then there exists a uniform \( c_{13} > 0 \) independent of \( r_1 \) and \( u \) such that

\[
u_0(x) \leq c_{13} |x|^{-\frac{2+\theta}{2}} \quad \text{for } 0 < |x| < 1.
\]

(2.26)
Proof. Take \(|x| = r\), then (2.25) implies that for any \(y \in B_r \setminus B_{\frac{r}{2}}\), there holds that
\[
u_0(y) \geq \frac{1}{C_0} \nu_0(x).
\]
From Proposition 1.1, we see that for any given \(|x| = r\),
\[
\frac{C_0}{(1 - \frac{1}{2})^N} \nu_0 \left| \partial B_r \nu_0^p(x) \right| r^{N - \theta} \leq \int_{B_r \setminus B_{\frac{r}{2}}} |y|^{\theta} \nu_0^p(y) \, dy < c_1 \nu_0^p \frac{r^{\theta}}{r^{2 + \theta} - \frac{2+r}{2}},
\]
which implies that
\[
u_0(x) \leq C_2 \left| u \right|^{\frac{2+\theta}{2+r}}.
\]
We complete the proof. \(\square\)

2.3 Harnack inequality

Our aim in this subsection is to obtain the Harnack inequality for singular solution of (1.1).

Proposition 2.1 Assume that \(\theta > -2s\), \(p \geq \frac{N+\theta}{N-2s}\), and \(\nu_0\) is a nonnegative solution of (1.1) such that
\[
\limsup_{|x| \rightarrow 0^+} \nu_0(x) \left| \frac{2 + \theta}{2+r} \right| < +\infty.
\]

Then there exists \(C_0 > 0\) such that for all \(r \in (0, \frac{1}{2})\)
\[
\sup_{x \in B_{2r} \setminus B_r} \nu_0(x) \leq C_0 \left( \inf_{x \in B_{2r} \setminus B_r} \nu_0(x) + \|\nu_0\|_{L^1(R^N)} \right).
\]

We assume more that \(\nu_0\) is singular at the origin, then for all \(r \in (0, \frac{1}{2})\)
\[
\sup_{x \in B_{2r} \setminus B_r} \nu_0(x) \leq C_0 \inf_{x \in B_{2r} \setminus B_r} \nu_0(x).
\]

(2.27)

Proof. For fixed \(r_0 \in (\frac{1}{2}, \frac{1}{4})\) and \(x_0 \in \mathbb{R}^N\) verifying \(|x_0| = r_0\), from there exists \(C > 0\) independent of \(\nu_0\) such that for any \(t \in (0, \frac{1}{4}]\)
\[
\sup_{x \in B_t(x_0)} \nu_0(x) < C.
\]

Let
\[
w(x) = \nu_0(x) (1 - \nu_0(4x)) - h(x) \quad \text{in} \quad \mathbb{R}^N,
\]
where \(h_\theta\) be a smooth function such that \(h_\theta = 1\) in \(B_1(0)\) and \(h_\theta = 0\) in \(\mathbb{R}^N \setminus B_2\), we recall that \(h = 0\) in \(B_{\frac{1}{2}}\). Then
\[
(-\Delta)^s w = |x|^{\theta} \nu_0^{p-1} w + (-\Delta)^s h + (-\Delta)^s (u_0(x) \eta_0(4x)) \quad \text{for} \quad x \in B_1(x_0),
\]
where \((-\Delta)^s h > 0\) in \(B_1\). Note that \(0 < \nu_0^{p-1} \leq c_{15}\) for some \(c_{15} > 0\) independent of \(\nu_0\) and
\[
0 < (-\Delta)^s h(x) + (-\Delta)^s (u_0(x) \eta_0(4x)) \leq c_{15} \left( \|h\|_{L^1(R^N)} + \|u_0\|_{L^1(R^N)} \right) \leq 2c_{15} \|u_0\|_{L^1(R^N)}.
\]

Then (22) Theorem 1.1 (also see [13] Theorem 11.1) implies that
\[
\sup_{x \in B_{t}(x_0)} \nu_0(x) \leq C_3 \left( \inf_{x \in B_t(x_0)} \nu_0(x) + \|\nu_0\|_{L^1(R^N)} \right),
\]

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Proposition 3.1 Assume that $C$

where

$\theta = 0$

Proposition 3.1 Assume that $\theta = 0$ and

$p \in \left( \frac{N}{N - 2s}, \frac{N + 2s}{N - 2s} \right)$.

Let $u_0$ be a nonnegative solution of (1.1), then there exists $c_{16} > 0$ such that

$$u_0(x) \leq c_{16} |x|^{-2s} \quad \forall x \in B_1 \setminus \{0\}. \quad (3.1)$$

Proof. The similar upper bound could see [46], where the fractional laplacian is defined by the extension to a local setting. Here we provide the direct blow-up analysis in the nonlocal setting.

Suppose by contradiction that there exists a sequence of points $\{x_k\} \subset B_{\frac{1}{2}} \setminus \{0\}$ such that $|x_k| \rightarrow +\infty$ as $k \rightarrow +\infty$ and $|x_k|^\frac{2s}{p} u_0(x_k) \rightarrow +\infty$ as $k \rightarrow +\infty$.

We can choose $x_k$ again having the property that

$$|x_k|^\frac{2s}{p} u_0(x_k) = \max_{x \in B_{\frac{1}{2}} \setminus B_{\epsilon_k}} |x|^\frac{2s}{p} u_0(x) \rightarrow +\infty \quad \text{as} \quad k \rightarrow +\infty \quad (3.2)$$

by the fact that the mapping $r \mapsto \max_{x \in B_1 \setminus B_{r^p}} |x|^\frac{2s}{p} u_0(x)$ is decreasing.

We denote

$$\phi_k(x) := \left( \frac{|x_k|}{2} - |x - x_k| \right)^\frac{2s}{p} u_0(x) \quad \text{for} \quad |x - x_k| \leq \frac{|x_k|}{2}.$$ 

Let $\bar{x}_k$ be the maximum point of $\phi_k$ in $B_{\frac{1}{2}}(x_k)$, that is,

$$\phi_k(\bar{x}_k) = \max_{|x - x_k| \leq |x_k|/2} \phi_k(x).$$

Set

$$\nu_k = \frac{1}{2} \left( \frac{|x_k|}{2} + |\bar{x}_k - x_k| \right).$$
then $0 < 2\nu < \frac{|x_k|}{2}$ and

$$\frac{|x_k|}{2} - |x - x_k| \geq \nu_k \text{ for } |x - x_k| \leq \nu_k.$$  

By the definition of $\phi_k$, for any $|x - x_k| \leq \nu_k$,

$$(2\nu_k)^{\frac{2}{p-1}} u_0(x_k) = \phi_k(x_k) \geq \phi_k(x_k) \geq \nu_k^{\frac{2}{p-1}} u_0(x),$$

which implies that

$$2^{\frac{2}{p-1}} u_0(x_k) \geq u_0(x) \text{ for any } |x - x_k| \leq \nu_k.$$  

Moreover, we see that

$$|x_k|^{\frac{2}{p-1}} u_0(x_k) \geq (2\nu_k)^{\frac{2}{p-1}} u_0(x_k) = \phi_k(x_k) \geq \phi_k(x_k) \geq \left(\frac{|x_k|}{2}\right)^{\frac{2}{p-1}} u_0(x_k) \to +\infty \text{ as } k \to +\infty$$

by the fact that $|x_k| \geq \frac{|x_k|}{2} \geq 2\nu_k$.

Denote

$$W_k(y) = \frac{1}{m_k} u_0(m_k^{-\frac{\nu_k}{2}} y - x_k), \quad \forall y \in \Omega_k \setminus \{X_k\},$$

where

$$m_k = u_0(x_k),$$

$$\Omega_k := \left\{ y \in \mathbb{R}^N : m_k^{-\frac{\nu_k}{2}} y - x_k \in B_1 \right\}$$

and

$$X_k = m_k^{-\frac{\nu_k}{2}} x_k.$$  

Note that

$$|X_k| = (u_0(x_k)|x_k|^{\frac{2}{p-1}}) \frac{2}{p-1} \to +\infty \text{ as } k \to +\infty.$$  

Thus, we have that for $y \in \Omega_k \setminus \{X_k\}$

$$(-\Delta)^s W_k(y) = \frac{1}{m_k}(-\Delta)^s u_0(m_k^{-\frac{\nu_k}{2}} y - x_k)$$

$$= \frac{1}{m_k} u_0(m_k^{-\frac{\nu_k}{2}} y - x_k)$$

$$= W_k^p(y),$$

i.e.

$$(-\Delta)^s W_k(y) = W_k^p(y) \text{ for } x \in \Omega_k \setminus \{X_k\}. \quad (3.3)$$

We claim that there is $c_{17} > 0$ independent of $k$ such that

$$\|W_k\|_{L^1(\mathbb{R}^N)} \leq c_{17}$$

and for any $\epsilon > 0$, there exists $k_1 > 0$ and $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_R(0)} W_k(y)(1 + |y|)^{-N-2s} dy \leq \epsilon. \quad (3.4)$$

In fact, since $|x_k| \to 0$, we see that

$$0 \leq \int_{\mathbb{R}^N \setminus \Omega_k} W_k(y)(1 + |y|)^{-N-2s} dy$$

$$= \frac{1}{m_k} \int_{\mathbb{R}^N \setminus \Omega_k} u_0(m_k^{-\frac{\nu_k}{2}} y - x_k)(1 + |y|)^{-N-2s} dy$$

$$= \frac{1}{m_k} \int_{\mathbb{R}^N \setminus \Omega_k} u_0(m_k^{-\frac{\nu_k}{2}} y)(1 + |y|)^{-N-2s} dy$$

$$\to c_{17}$$

as $k \to 0$. Thus, for any $\epsilon > 0$, there exists $k_1 > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_R(0)} W_k(y)(1 + |y|)^{-N-2s} dy \leq \epsilon.$$
\[ \begin{align*}
&= m_k^{-p} \int_{\mathbb{R}^N \setminus B_1} h(z) \left( m_k^{-\frac{p-1}{p}} + |z + \bar{x}_k| \right)^{-N-2s} \, dz \\
&\leq m_k^{-p} \int_{\mathbb{R}^N} h(z)(1 + |z + \bar{x}_k|)^{-N-2s} \, dz \\
&\leq u_0(\bar{x}_k)^{-p} \int_{\mathbb{R}^N} h(z)(1 + |z|)^{-N-2s} \, dz \\
&\to 0 \quad \text{as } k \to +\infty.
\end{align*} \]

Taking
\[ r_k = |\bar{x}_k| u_0(\bar{x}_k)^{\frac{p-1}{p}} \to +\infty \quad \text{as } k \to +\infty, \]
we obtain that
\[ \begin{align*}
0 &\leq \int_{B_{r_k}(X_k)} W_k(y)(1 + |y|)^{-N-2s} \, dy \\
&\leq \int_{B_{r_k}(X_k)} u_0 \left( m_k^{-\frac{p-1}{p}} y - \bar{x}_k \right) \, dy \\
&= \left[ \frac{r_k^{-N-2s}}{m_k} \right] \int_{B_{r_k}(X_k)} u_0(z) \, dz \\
&\leq r_k^{-N-2s} \left( \int_{B_{r_k}(X_k)} u_0^p(z) \, dz \right)^{\frac{1}{p}} \left( \int_{B_{r_k}(X_k)} \, dz \right)^{1 - \frac{1}{p}} \\
&= c_{18} r_k^{-N-2s} \left( \int_{B_{r_k}(X_k)} u_0^p(z) \, dz \right)^{\frac{1}{p}} \\
&\to 0 \quad \text{as } k \to +\infty,
\end{align*} \]

where we used (1.14) and the fact that \( \frac{p}{p-1} < N \).

Moreover, \( W_k(y) \leq 1 \) in \( \Omega_k \setminus B_{r_k} \), then
\[ \begin{align*}
\int_{\Omega_k \setminus B_{r_k}(X_k)} W_k(y)(1 + |y|)^{-N-2s} \, dy &\leq \int_{\Omega_k \setminus B_{r_k}(X_k)} (1 + |y|)^{-N-2s} \, dy \\
&\leq \int_{\mathbb{R}^N} (1 + |y|)^{-N-2s} \, dy
\end{align*} \]

and
\[ \begin{align*}
\int_{\left( \Omega_k \setminus B_{r_k}(X_k) \right) \setminus B_{r_k}(0)} W_k(y)(1 + |y|)^{-N-2s} \, dy \\
&\leq \int_{\mathbb{R}^N \setminus B_{r_k}(0)} (1 + |y|)^{-N-2s} \, dy \\
&\leq c_{19} R^{-2s},
\end{align*} \]

where \( c_{19} > 0 \) is independent of \( k \). Now we conclude (3.4) and the claim is proved.

Note that \( 0 < W_k \leq 2^\frac{2s}{p-1} \) in \( B_{\tilde{r}_k} \), where
\[ \tilde{r}_k = \nu_k m_k^{\frac{1}{p-1}} = (\nu_k^{\frac{2s}{p-1}} m_k^{\frac{1}{p-1}}) \to +\infty \quad \text{as } k \to +\infty, \]

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then for any $R > 0$, there exists $k_R$ for any $k \geq k_R$
\[ \|W_k\|_{C^{2s} \cap L_1(B_R)} \leq c_{20} \left( \|W_k\|_{L^1_0(\mathbb{R}^N)} + \|W_k\|_{L^\infty(\mathbb{R}^N)} + \|W_k^p\|_{L^\infty(B_{2R})} \right) \]
\[ \leq c_{20} \left( \|W_k\|_{L^1_0(\mathbb{R}^N)} + 2^{1+\frac{N-2s}{p}} \right), \]
where $\alpha \in (0, s)$ and $c_{20} > 0$.

By the arbitrary of $R$, up to subsequence, there exists a nonnegative function $W_\infty \in L^\infty(\mathbb{R}^N)$ such that as $k \to +\infty$
\[ W_k \to W_\infty \quad \text{in} \quad C^{2s+\alpha'}_{\text{loc}}(\mathbb{R}^N) \quad \text{and in} \quad L^1_0(\mathbb{R}^N) \]  
(3.5)
for some $\alpha' \in (0, \alpha)$.

For any $x \in \mathbb{R}^N$, take $R > 4|x|$ and $R \leq \min \{ \tau_k, \tilde{\tau}_k \}$ for $k$ large enough, note that
\[ \frac{1}{C_{N,s}} (-\Delta)^s W_k(x) = \text{p.v.} \int_{B_R} \frac{W_k(x) - W_k(y)}{|x-y|^{N+2s}} dy + W_k(x) \int_{\mathbb{R}^N \setminus B_R} \frac{1}{|x-y|^{N+2s}} dy \]
\[ = E_{1,k}(x) + E_{2,k}(x) - E_{3,k}(x), \quad \text{respectively.} \]

For any $\epsilon > 0$, by convergence (3.3), we have that
\[ \left| E_{1,k}(x) - \text{p.v.} \int_{B_R} \frac{W_\infty(x) - W_\infty(y)}{|x-y|^{N+2s}} dy \right| < \frac{\epsilon}{3} \]
and there exists $R_1 > 0$ such that $R > R_1$
\[ \left| E_{2,k}(x) - W_\infty(x) \int_{\mathbb{R}^N \setminus B_R} \frac{1}{|x-y|^{N+2s}} dy \right| \leq \frac{\epsilon}{3}. \]
Furthermore, by (3.1) we have that
\[ 0 < E_{3,k}(x) \leq \int_{\mathbb{R}^N \setminus B_R} W_k(y)(1 + |y|)^{-N-2s} dy < \frac{\epsilon}{3} \]
for $k > 0$ and $R$ large enough.

Now we conclude that
\[ \lim_{k \to +\infty} (-\Delta)^s W_k(x) = (-\Delta)^s W_\infty(x) \]

and $W_\infty$ is a bounded classical solution of
\[ (-\Delta)^s W_\infty = W_\infty^p \quad \text{in} \quad \mathbb{R}^N. \]  
(3.6)

Since $W_\infty(0) = 1$, then $W_\infty > 0$ in $\mathbb{R}^N$, thanks to the nonnegative property of $W_\infty$. By [20, Theorem 3] or [21, Theorem 4.5], problem (3.6) has no bounded positive solution for $p \in (1, \frac{N+2s}{N-2s})$. We complete the proof. \hfill \Box

### 3.2 The case: $\theta \in (-2s, 0)$

When $\theta \in (-2s, 0]$, we would derive the upper bound by the integral upper bound and increasing monotonicity. We recall (1.9) and (1.10), i.e.
\[ \theta = 0 \quad \& \quad p > \frac{N+2s}{N-2s} \]
and
\[ \theta \in (-2s, 0] \quad \& \quad p > \frac{N+\theta}{N-2s} \]
The upper bound states as following.
Proposition 3.2 Assume that (1.9) or (1.10) hold for $\theta$ and $p$, function $h$ is radially symmetric in $\mathbb{R}^N \setminus B_1$ and decreasing with respect to $|x|$ and $h(1) \geq 0$.

Let $u_0$ be a nonnegative solution of (1.7) satisfying (1.11), then there exists $c_{22} > 0$ such that

$$0 < u_0(x) \leq c_{22}|x|^{-\frac{N+p}{p}} , \quad \forall x \in B_1 \setminus \{0\}. \quad (3.7)$$

In order to prove the upper bound, we first show the radial symmetry and monotonicity of positive solutions to equation (1.1).

We let

$$u = u_0 - h(1) \quad \text{ in } \mathbb{R}^N,$$

then $u \geq 0$ in $B_1$ by (1.11) and it verifies that

$$\begin{align*}
\left\{ (-\Delta)^s u & = |x|^\theta (u + h(1))^p \quad \text{ in } B_1 \setminus \{0\}, \\
u & = h - h(1) \quad \text{ in } \mathbb{R}^N \setminus B_1.
\end{align*} \quad (3.8)$$

Proposition 3.3 Let (1.9), (1.10) hold for $\theta$, $p$ respectively, $h$ be radially symmetric in $\mathbb{R}^N \setminus B_1$ and decreasing with respect to $|x|$ and $h(1) \geq 0$.

Let $u_0$ be a nonnegative solution of (1.7) satisfying (1.11), then $u = u_0 - h(1)$ is a positive solution of (3.8), $u$ is radially symmetric and decreasing with respect to $|x|$.

Our method of moving planes is motivated by [27] to deal with the solution $u$ has possibly singular at the origin. For the methods of the moving planes for integral equations, we refer to [20, 21]. For singular solutions, we will use the direct moving plane method developing from [27] and we need the following variant maximum principle for small domain.

Lemma 3.1 [27, Corollary 2.1] Let $O$ be an open and bounded subset of $\mathbb{R}^N$. Suppose that $\varphi \in L^\infty(O)$ and $w \in L^\infty(\mathbb{R}^N) \cap C(\overline{O})$ is a classical solution of

$$\begin{align*}
\left\{ (-\Delta)^s w(x) & \geq \varphi(x) w(x), \quad x \in O, \\
w(x) & \geq 0, \quad x \in O^c.
\end{align*} \quad (3.9)$$

Then there is $\delta > 0$ such that whenever $|O^c| \leq \delta$, function $w$ has to be non-negative in $O$, where $O^c = \{ x \in O \mid w(x) < 0 \}$.

For simplicity, we denote

$$\Sigma_\lambda = \{ x = (x_1, x') \in O_1 \mid x_1 > \lambda \}, \quad (3.10)$$

$$u_\lambda(x) = u(x_\lambda) \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x), \quad (3.11)$$

where $\lambda \in (0, 1)$ and $x_\lambda = (2\lambda-x_1, x')$ for $x = (x_1, x') \in \mathbb{R}^N$ and $O_1 = B_1 \setminus \{0\}$. For any subset $A$ of $\mathbb{R}^N$, we write $A_\lambda = \{ x : x \in A \}$.

The essential estimate in the procedure is to show that for any $\lambda \in (0, 1)$

$$w_\lambda \geq 0 \quad \text{in } \Sigma_\lambda.$$ 

On the contrary, we suppose that $\Sigma^+_\lambda = \{ x \in \Sigma_\lambda \mid w_\lambda(x) < 0 \} \neq \emptyset$ for $\lambda \in (0, 1)$. Let us define

$$w^+_\lambda(x) = \begin{cases} 
w_\lambda(x), & x \in \Sigma^+_\lambda, \\
0, & x \in \mathbb{R}^N \setminus \Sigma^+_\lambda
\end{cases} \quad (3.12)$$

and

$$w^-_\lambda(x) = \begin{cases} 
0, & x \in \Sigma^+_\lambda, \\
w_\lambda(x), & x \in \mathbb{R}^N \setminus \Sigma^+_\lambda.
\end{cases} \quad (3.13)$$

Hence, $w^+_\lambda(x) = w_\lambda(x) - w^-_\lambda(x)$ for all $x \in \mathbb{R}^N$. It is obvious that $(2\lambda, 0, \cdots , 0) \notin \Sigma^-_\lambda$, since $\lim_{|x|\to 0^+} u(x) = +\infty$. 

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Lemma 3.2 Assume that $\Sigma^{-}_{\lambda} \neq \emptyset$ for $\lambda \in (0, 1)$, then
\[
(-\Delta)^{s}w_{\lambda}^{-}(x) \leq 0, \quad \forall x \in \Sigma^{-}_{\lambda}.
\] (3.14)

Proof. By direct computation, for $x \in \Sigma^{-}_{\lambda}$, we have
\[
\frac{1}{C_{N,s}} (-\Delta)^{s}w_{\lambda}^{-}(x) = \int_{\mathbb{R}^{N}} \frac{w_{\lambda}^{-}(x) - w_{\lambda}^{-}(z)}{|x-z|^{N+2s}} dz = -\int_{\Sigma_{\lambda}^{-}} \frac{w_{\lambda}(z)}{|x-z|^{N+2s}} dz
\]
\[
= -\int_{(O_{1})_{\lambda} \setminus ((O_{1})_{\lambda} \cap O_{1})} \frac{w_{\lambda}(z)}{|x-z|^{N+2s}} dz
\]
\[
- \int_{(\Sigma_{\lambda} \setminus \Sigma^{-}_{\lambda}) \cup (\Sigma_{\lambda} \setminus \Sigma^{-}_{\lambda})} \frac{w_{\lambda}(z)}{|x-z|^{N+2s}} dz
\]
\[
= -I_{1} - I_{2} - I_{3}.
\]

We estimate these integrals separately. Since $u = 0$ in $(O_{1})_{\lambda} \setminus O_{1}$ and $u_{\lambda} = 0$ in $O_{1} \setminus (O_{1})_{\lambda}$, then
\[
I_{1} = \int_{(O_{1})_{\lambda} \setminus ((O_{1})_{\lambda} \cap O_{1})} \frac{w_{\lambda}(z)}{|x-z|^{N+2s}} dz = \int_{(O_{1})_{\lambda} \setminus O_{1}} \frac{u_{\lambda}(z)}{|x-z|^{N+2s}} dz
\]
\[
- \int_{O_{1} \setminus (O_{1})_{\lambda}} \frac{u(z)}{|x-z|^{N+2s}} dz
\]
\[
= \int_{(O_{1})_{\lambda} \setminus O_{1}} \frac{u_{\lambda}(z)}{|x-z|^{N+2s}} dz
\]
\[
\geq 0,
\]
since $u_{\lambda} \geq 0$ and $|x-z_{\lambda}| > |x-z|$ for all $x \in \Sigma^{-}_{\lambda}$ and $z \in (O_{1})_{\lambda} \setminus O_{1}$.

In order to decide the sign of $I_{2}$ we observe that $w_{\lambda}(z_{\lambda}) = -w_{\lambda}(z)$ for any $z \in \mathbb{R}^{N}$. Then,
\[
I_{2} = \int_{\Sigma_{\lambda} \setminus \Sigma^{-}_{\lambda}} \frac{w_{\lambda}(z)}{|x-z|^{N+2s}} dz
\]
\[
= \int_{\Sigma_{\lambda} \setminus \Sigma^{-}_{\lambda}} \frac{w_{\lambda}(z)}{|x-z|^{N+2s}} dz + \int_{\Sigma_{\lambda} \setminus \Sigma^{-}_{\lambda}} \frac{w_{\lambda}(z_{\lambda})}{|x-z_{\lambda}|^{N+2s}} dz
\]
\[
\geq 0,
\]
since $w_{\lambda} \geq 0$ in $\Sigma_{\lambda} \setminus \Sigma^{-}_{\lambda}$ and $|x-z_{\lambda}| > |x-z|$ for all $x \in \Sigma^{-}_{\lambda}$ and $z \in \Sigma_{\lambda} \setminus \Sigma^{-}_{\lambda}$.

Finally, since $w_{\lambda}(z) < 0$ for $z \in \Sigma^{-}_{\lambda}$, we deduce
\[
I_{3} = \int_{\Sigma_{\lambda}^{-}} \frac{w_{\lambda}(z)}{|x-z|^{N+2s}} dz = \int_{\Sigma_{\lambda}^{-}} \frac{w_{\lambda}(z_{\lambda})}{|x-z_{\lambda}|^{N+2s}} dz
\]
\[
= -\int_{\Sigma_{\lambda}^{-}} \frac{w_{\lambda}(z_{\lambda})}{|x-z_{\lambda}|^{N+2s}} dz \geq 0.
\]

The proof is complete. \(\square\)

Now we are ready to prove Proposition 3.3.

Proof of Proposition 3.3. For simplicity, let $h(1) = t \geq 0$ and $u_{0}$ is nontrivial, i.e. $u_{0} \geq t \ u_{0} \neq t$ in $B_{1}$.

In order to show the radial symmetry and decreasing monotonicity in $|x|$, we divide the proof into four steps.

Step 1: We prove that if $\lambda$ is close to 1, then $w_{\lambda} > 0$ in $\Sigma_{\lambda}$. From the assumption (1.1), we see that $w_{\lambda} > 0$ in $\Sigma_{\lambda}$ for $\lambda = 1$.
First we show that \( w_\lambda \geq 0 \) in \( \Sigma_\lambda \), i.e. \( \Sigma_\lambda^- \) is empty. By contradiction, we assume that \( \Sigma_\lambda^- \neq \emptyset \). It must be in \( B_1 \) now we apply (3.14) and linearity of the fractional Laplacian to obtain that, for \( x \in \Sigma_\lambda^- \),
\[
(\Delta)^s w_\lambda^+(x) \geq (\Delta)^s w_\lambda(x) = (\Delta)^s u_\lambda(x) - (\Delta)^s u(x).
\] (3.15)

Combining with (3.15) and (3.12), for \( x \in \Sigma_\lambda^- \), we have
\[
(\Delta)^s w_\lambda^+(x) \geq -|x|^\theta(u_\lambda(x) + t)^p - |x|^\theta(u(x) + t)^p
\]
\[
\quad - \varphi(x)w_\lambda^+(x),
\]
where
\[
\varphi(x) = \frac{|x|^\theta(u_\lambda(x) + t)^p - |x|^\theta(u(x) + t)^p}{u_\lambda(x) - u(x)}
\]
\[
\quad = \frac{|x|^\theta - |x|^\theta}{u_\lambda(x) - u(x)}(u_\lambda(x) + t)^p + |x|^\theta \left( \frac{(u_\lambda(x) + t)^p - (u(x) + t)^p}{u_\lambda(x) - u(x)} \right)
\]
\[
\quad < |x|^\theta \left( \frac{(u_\lambda(x) + t)^p - (u_\lambda(x) + t)^p}{(u(x) + t) - (u_\lambda(x) + t)} \right)
\]
\[
\quad \leq 2^p |x|^\theta (u(x) + t)^{p-1}, \quad \forall x \in \Sigma_\lambda^-.
\]

thanks to \( \theta \in (-2s, 0) \).

For \( x \in \Sigma_\lambda^- \subset \Sigma_\lambda \subset \mathbb{R}^N \setminus B_\lambda \), \( u_\lambda(x) < u(x) \). Moreover, there exists \( M_\lambda > 0 \) such that
\[
\|u\|_{L^\infty(\mathbb{R}^N \setminus B_\lambda)} \leq M_\lambda,
\]
then there exists \( c_{23} > 0 \) dependent of \( \lambda \) such that
\[
\|\varphi\|_{L^\infty(\Sigma_\lambda^-)} \leq c_{23}.
\] (3.16)

Note that \( M_\lambda \to \infty \) as \( \lambda \to 0 \) if \( \lim_{|x| \to 0^+} u(x) = \infty \).

Therefore, for \( x \in \Sigma_\lambda^- \) and then
\[
(\Delta)^s w_\lambda^+(x) \geq -\varphi(x)w_\lambda^+(x), \quad \forall x \in \Sigma_\lambda^-.
\]

Moreover, \( w_\lambda^+ = 0 \) in \( (\Sigma_\lambda^-)^c \). Choosing \( \lambda \in (0, 1) \) close enough to 1 we have \( |\Sigma_\lambda^-| \) is small and we apply Lemma 3.3 to obtain that
\[
w_\lambda = w_\lambda^+ \geq 0 \quad \text{in} \quad \Sigma_\lambda^-,
\]
which is impossible. Thus,
\[
w_\lambda \geq 0 \quad \text{in} \quad \Sigma_\lambda.
\]

If the function \( \lim_{|x| \to 0^+} u(x) = +\infty \), then \( w_\lambda \) is positive near the point \( (2\lambda, 0, \cdots, 0) \) and then \( w_\lambda \neq 0 \) in \( \mathbb{R}^N \). If \( u \) is bounded at the origin, then the solution \( u \) has removable singularity at the origin. In this case, \( u = 0 \) on \( \partial B_1 \) then \( w_\lambda \neq 0 \).

Now we claim that for \( 0 < \lambda < 1 \), \( w_\lambda > 0 \) in \( \Sigma_\lambda \).

Indeed, we assume on the contrary that there exists \( x_0 \in \Sigma_\lambda \) such that \( w_\lambda(x_0) = 0 \), i.e. \( u_\lambda(x_0) = u(x_0) \). Then
\[
(\Delta)^s w_\lambda(x_0) = (\Delta)^s u_\lambda(x_0) - (\Delta)^s u(x_0) = 0.
\] (3.17)

On the other hand, let \( K_\lambda = \{(x_1, x') \in \mathbb{R}^N \mid x_1 > \lambda\} \). Noting \( w_\lambda(z) = -w_\lambda(z) \) for any \( z \in \mathbb{R}^N \) and \( w_\lambda(x_0) = 0 \), we deduce
\[
(\Delta)^s w_\lambda(x_0) = -\int_{K_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2s}} dz - \int_{\mathbb{R}^N \setminus K_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2s}} dz
\]
\[
= -\int_{K_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2s}} dz - \int_{K_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2s}} dz
\]
\[
= -\int_{K_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2s}} dz - \frac{1}{|x_0 - z|^{N+2s}} dz.
\]

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The fact $|x_0 - z| > |x_0 - z|$ for $z \in K_\lambda$, $w_\lambda(z) \geq 0$ and $w_\lambda(z) \neq 0$ in $K_\lambda$ yield

$$(-\Delta)^s w_\lambda(x_0) < 0,$$

which contradicts $\text{(3.17)}$, completing the proof of the claim.

Step 2: We prove $\lambda_0 := \inf \{ \lambda \in (0, 1) \mid w_\lambda > 0 \} \in \Sigma_\lambda = 0$. If not, we set $\lambda_0 > 0$. Hence, $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$ and $w_{\lambda_0} \neq 0$ in $\Sigma_{\lambda_0}$. The claim in Step 1 implies $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$.

Letting $\epsilon \in (0, \lambda_0/4)$ small enough, we now show that $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$ for $\lambda = \lambda_0 - \epsilon$.

To this end, we set $D_\mu = \{ x \in \Sigma_{\lambda_0} \mid \text{dist}(x, \partial \Sigma_\lambda) \geq \mu \}$ for $\mu > 0$ small. Since $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$ and $D_\mu$ is compact, there exists $\mu_0 > 0$ such that $w_{\lambda_0} \geq \mu_0$ in $D_\mu$. By the continuity of $w_{\lambda_0}(x)$, for $\epsilon > 0$ small enough and $\lambda = \lambda_0 - \epsilon$, we have that $w_{\lambda_0}(x) \geq 0$ in $D_\mu$. Therefore, $\Sigma_{\lambda_0} \subset \Sigma_{\lambda_0} \setminus D_\mu$ and $|\Sigma_{\lambda_0}|$ is small if $\epsilon$ and $\mu$ are small. Using $\text{(3.14)}$ and proceeding as in Step 1, we have for all $x \in \Sigma_{\lambda_0}$ that

$$(-\Delta)^s w_{\lambda_0}(x) = (-\Delta)^s u_{\lambda_0}(x) - (-\Delta)^s u(x) - (-\Delta)^s w_{\lambda_0}^-(x) \geq (-\Delta)^s u_{\lambda_0}(x) - (-\Delta)^s u(x) \geq -\varphi(x) w_{\lambda_0}^+(x).$$

From $\text{(3.10)}$, $\varphi(x)$ is controlled by some constant dependent of $\lambda$.

Since $w_{\lambda_0}^+ = 0$ in $\Sigma_{\lambda_0}$ and $|\Sigma_{\lambda_0}|$ is small, for $\epsilon$ and $\mu$ small, Lemma $\text{(3.1)}$ implies that $w_{\lambda_0} \geq 0$ in $\Sigma_{\lambda_0}$. Combining with $\lambda_0 > 0$ and $w_{\lambda_0} \neq 0$ in $\Sigma_{\lambda_0}$, we obtain $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$. This contradiction arises from the definition of $\lambda_0$.

Therefore, we have that $\lambda_0 = 0$.

Step 3: By Step 2, we have $\lambda_0 = 0$, which implies that $u(-x_1, x') \geq u(x_1, x')$ for $x_1 \geq 0$. Using the same argument from the other side, we conclude that $u(-x_1, x') \leq u(x_1, x')$ for $x_1 \geq 0$ and then $u(-x_1, x') = u(x_1, x')$ for $x_1 \geq 0$. Repeating this procedure in all directions we see that $u$ is radially symmetric.

Finally, we prove $u(r)$ is strictly decreasing in $r \in (0, 1)$. Let us consider $0 < x_1 < \bar{x}_1 < 1$ and let $\lambda = \frac{x_1 + \bar{x}_1}{2}$. As proved above we have

$$w_\lambda(x) > 0 \text{ for } x \in \Sigma_\lambda.$$

Then

$$0 < w_\lambda(\bar{x}_1, 0, \ldots, 0) = w_\lambda(\bar{x}_1, 0, \ldots, 0) - u(\bar{x}_1, 0, \ldots, 0) = u(x_1, 0, \ldots, 0) - u(\bar{x}_1, 0, \ldots, 0),$$

i.e $u(x_1, 0, \ldots, 0) > u(\bar{x}_1, 0, \ldots, 0)$. From the radial symmetry of $u$ and decreasing in the direction $\bar{x}_1$, we can conclude the monotonicity of $u$. \hfill \Box

**Proof of Proposition 3.2** Note that

$$u = u_0 - h(1)$$

and Proposition $\text{(3.3)}$ shows that $u$ is radially symmetric and decreasing with respect to $|x|$, so is $u_0$. Then $\text{(2.25)}$ holds for $u_0$ and we apply Corollary $\text{(2.2)}$ to obtain that

$$u_0(x) \leq c_{\text{est}} |x|^\frac{\frac{N}{2s} - \frac{N + \theta}{N + 2s + \theta}}{N - 2s} \text{ for } x \in B_1 \setminus \{0\}.$$

We complete the proof. \hfill \Box

### 4 Classification of isolated singularity

#### 4.1 Some important estimates

**Proposition 4.1** Assume that $h \in L^1_\lambda(\mathbb{R}^N \setminus B_1)$, $\theta \in (-2s, +\infty)$ and

$$p \in \left( \frac{N + \theta}{N - 2s}, \frac{N + 2s + \theta}{N - 2s} \right).$$

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Let \( u_0 \) be a positive solution of (1.1) verifying
\[
\limsup_{|x|\to 0^+} u_0(x) = +\infty,
\]
then \( u_0 \) satisfies that
\[
\liminf_{|x|\to 0^+} u_0(x)|x|^{\frac{2s+\theta}{p-1}} \leq K_{p,\theta} \leq \limsup_{|x|\to 0^+} u_0(x)|x|^{\frac{2s+\theta}{p-1}},
\]
where \( K_{p,\theta} \) is given in (1.7).

**Proof.** We recall that
\[
K_{p,\theta} = C_s \left( -\frac{2s + \theta}{p - 1} \right)^{\frac{1}{p-1}}.
\]
For \( p \in \left( \frac{N + \theta}{N - 2s}, \frac{N + 2s + 2\theta}{N - 2s} \right) \), there holds that
\[
-\frac{2s + \theta}{p - 1} \in (2s - N, \frac{2s - N}{2})
\]
and
\[
\tau_-(-K_{p,\theta}^{-1}) = -\frac{2s + \theta}{p - 1} \quad \text{and} \quad \tau_+(-K_{p,\theta}^{-1}) = 2s - N + \frac{2s + \theta}{p - 1}.
\]
From Proposition 3.1 we have that
\[
\limsup_{|x|\to 0^+} u_0(x)|x|^{\frac{2s+\theta}{p-1}} < +\infty.
\]

Set
\[
k = \liminf_{|x|\to 0^+} u_0(x)|x|^{\frac{2s+\theta}{p-1}}. \quad (4.1)
\]

**Part 1:** we claim
\[
k \leq K_{p,\theta}.
\]
In fact, if
\[
k \in (K_{p,\theta}, +\infty],
\]
letting
\[
\epsilon_0 = \frac{k - K_{p,\theta}}{2K_{p,\theta}},
\]
then there exists \( r_1 \in (0, 1) \) such that
\[
u_0(x) \geq K_{p,\theta}(1 + \epsilon_0)|x|^{-\frac{2s+\theta}{p-1}} \quad \text{for} \quad 0 < |x| < r_1,
\]
then for \( 0 < |x| < r_1 \),
\[
u_0^{-1}(x) \geq K_{p,\theta}^{-1}(1 + \epsilon_0)^{-1}|x|^{-\theta - 2s} > K_{p,\theta}^{-1}(1 + (p - 1)\epsilon_0)|x|^{-\theta - 2s}.
\]
Therefore, \( u_0 \) verifies that
\[
\begin{align*}
\left\{
\begin{array}{ll}
(-\Delta)^s u_0 = \frac{K_{p,\theta}^{-1} + \frac{\theta - 1}{2}\epsilon_0}{|x|^2} u_0 + f_1 & \text{in} \; B_{r_1} \setminus \{0\}, \\
u_0 \geq 0 & \text{in} \; \mathbb{R}^N \setminus B_{r_1},
\end{array}
\right.
\end{align*}
(4.2)
\]
where
\[
\mu_p = -\left( K_{p,\theta}^{-1} + \frac{p - 1}{2}\epsilon_0 \right) < -K_{p,\theta}^{-1}.
\]
and

\[ f_1(x) = |x|^\theta \left( \frac{u_0^{p-1} - \frac{K_{p,\theta}^{p-1}}{2} \epsilon_0}{|x|^{2s+\theta}} \right) u_0(x) \]

\[ \geq \frac{p-1}{2} \epsilon_0 K_{p,\theta}^{p-1} |x|^{-\frac{(2s+\theta) \epsilon_0}{p-1}}. \]

Observe that

\[ \tau_+ (\mu_p) < \tau_+ (-K_{p,\theta}^{p-1}) = 2s - N + \frac{2s + \theta}{p-1} \]

that is

\[ \tau_+ (\mu_p) - \frac{(2s + \theta)p}{p-1} < -N, \]

which implies that

\[ f_1 \not\in L^1(B_{r_1}, \Gamma_{\mu_p}, dx) \]

and a contradiction arises by Theorem 2.1 from which problem (4.4) has no such positive solution. Therefore we obtain that \( k \leq K_{p,\theta} \).

**Part 2:** Set

\[ \kappa = \limsup_{|x| \to 0^+} u_0(x)|x|^{\frac{2s+\theta}{p-1}} \]  \hspace{1cm} (4.3)

and we prove that

\[ \kappa \geq K_{p,\theta}. \]

In fact, if

\[ \kappa < K_{p,\theta}, \]

letting

\[ \epsilon_1 = \min \left\{ \frac{K_{p,\theta} - \kappa}{2K_{p,\theta}}, \frac{1}{4} \right\}, \]

then there exists \( r_2 \in (0, 1) \) such that

\[ u_0(x) \leq K_{p,\theta} (1 - \epsilon_1)|x|^{-\frac{2s+\theta}{p-1}} \quad \text{for} \quad 0 < |x| < r_2, \]

and for \( 0 < |x| < r_2 \)

\[ u_0^{p-1}(x) \leq K_{p,\theta}^{p-1} (1 - \epsilon_1)^{p-1} |x|^{-\theta - 2s}. \]

Let

\[ \tilde{\mu}_p = -K_{p,\theta}^{p-1} (1 - \epsilon_1)^{p-1} > -K_{p,\theta}^{p-1}, \]

then

\[ \tau_- (\tilde{\mu}_p) < -\frac{2s + \theta}{p-1}. \]

Then \( u_0 \) is a positive solution of

\[ \begin{cases} 
(\Delta)^s u_0 = \kappa \tilde{\mu}_p^{p-1} \frac{1 - \epsilon_1}{|x|^{2s+\theta}} u_0 + f_2 & \text{in} \ B_{r_2} \setminus \{0\}, \\
\quad u_0 \geq 0 & \text{in} \ \mathbb{R}^N \setminus B_{r_2}, \\
\lim_{|x| \to 0^+} u_0(x)|x|^{-\tau_- (\tilde{\mu}_p)} = 0, 
\end{cases} \] \hspace{1cm} (4.4)

where

\[ f_2(x) = |x|^\theta \left( u_0^{p-1} - \frac{K_{p,\theta}^{p-1} (1 - \epsilon_1)^{p-1}}{|x|^{2s+\theta}} \right) u_0(x) \leq 0. \]
Then Lemma 2.3 with $\mu = \tilde{\mu}_p$ implies that

$$\limsup_{|x| \to 0^+} u_0(x)|x|^{-\tau_+ (\tilde{\mu}_p)} < +\infty,$$

where

$$\tau_+ (\tilde{\mu}_p) > \tau_+ (-K_p^{-p-1}) = 2s - N + 2s + \theta \over p - 1 > -{2s + \theta \over p - 1}.$$

Now we take the value $\tau_0 = \tau_+ (\tilde{\mu}_p) < 0$, then

$$(-\Delta)^s u_0(x) \leq c_{24} |x|^\tau_0 \in B_{r_2} \setminus \{0\}$$

and letting

$$\tau_1 := p\tau_0 + \theta + 2s,$$

if $\tau_1 > 0$,

$$u_0(x) \leq d_1 \in B_{r_2} \setminus \{0\},$$

which contradicts that $u_0$ is non-removable at the origin. Then we are done.

If $\tau_1 = 0$, applying Lemma 2.3 with $\mu = 0$ to obtain that

$$u_0(x) \leq d_1 \left( \ln \left( \frac{r_0}{|x|} + 1 \right) \right) \in B_{r_2} \setminus \{0\},$$

then for $\epsilon > 0$

$$u_0(x) \leq \tilde{d}_1 |x|^{-\epsilon} \in B_{r_2} \setminus \{0\},$$

which implies that

$$(-\Delta)^s u_0(x) \leq c_{24} |x|^\theta \epsilon \in B_{r_2} \setminus \{0\}$$

For $\epsilon > 0$ small enough, we apply Lemma 2.3 with $\mu = 0$ to obtain that

$$u_0(x) \leq d_1 \in B_{r_2} \setminus \{0\}$$

and we are done.

If $\tau_1 < 0$, applying Lemma 2.3 with $\mu = 0$ to obtain that

$$u_0(x) \leq d_1 |x|^\tau_1 \in B_{r_2} \setminus \{0\}$$

and

$$(-\Delta)^s u_0(x) \leq c_{24} |x|^\tau_1 p + \theta \in B_{r_2} \setminus \{0\},$$

thus, letting

$$\tau_2 := p\tau_1 + \theta + 2s,$$

if $\tau_2 \geq 0$, we can get that $u_0$ is bounded at the origin by the same proof of the case $\tau_1 \geq 0$ and we are done.

If $\tau_2 < 0$, iteratively, we can prove that

$$u_0(x) \leq d_1 (|x|^\tau_n + 1) \in B_{r_2} \setminus \{0\},$$

where

$$\tau_n := p\tau_{n-1} + \theta + 2s, \quad n = 1, 2, \cdots.$$

Note that

$$\tau_1 - \tau_0 = (p - 1)\tau_0 + \theta + 2s > 0,$$

thanks to $\tau_0 > -{2s \over p-1}$, and

$$\tau_n - \tau_{n-1} = p(\tau_{n-1} - \tau_{n-2}) = p^{n-1} (\tau_1 - \tau_0) \to +\infty \text{ as } n \to +\infty.$$
then there exists \( n_1 \in \mathbb{N} \) such that \( \tau_{n_1} \geq 0 \), by the same argument of the case \( \tau_1 \geq 0 \), we then obtain that \( u_0 \) is bounded at the origin which contradicts the assumption

\[
\limsup_{|x| \to 0^+} u_0(x) = +\infty.
\]

Thus, we conclude that

\[
\liminf_{|x| \to 0^+} u(x)|x|^{\frac{2s+\theta}{p-1}} \leq K_{p,\theta} \leq \limsup_{|x| \to 0^+} u(x)|x|^{\frac{2s+\theta}{p-1}}.
\]

We complete the proof. \( \square \)

Now we deal with the Sobolev’s critical and supercritical case.

**Proposition 4.2** Assume that \( h \in L^1_s(\mathbb{R}^N \setminus B_1) \), \( \theta \in (-2s, +\infty) \) and

\[
p \geq \frac{N + 2s + 2\theta}{N - 2s}.
\]

Let \( u_0 \) be a positive solution of (1.1) verifying

\[
\limsup_{|x| \to 0^+} u_0(x) = +\infty,
\]

then

\[
\liminf_{|x| \to 0^+} u_0(x)|x|^{\frac{2s+\theta}{p-1}} \leq K_{p,\theta} \leq \limsup_{|x| \to 0^+} u_0(x)|x|^{\frac{2s+\theta}{p-1}}.
\]

**Proof.** Part 1: we claim that

\[
\liminf_{|x| \to 0^+} u_0(x)|x|^{\frac{2s+\theta}{p-1}} \leq c^*\frac{1}{p-1} K_{p,\theta}.
\]

Note that for \( p \geq \frac{N + 2s + 2\theta}{N - 2s} \),

\[
\frac{2s + \theta}{p - 1} \geq \frac{2s - N}{2}.
\]

Let

\[
u_p(x) = K_{p,\theta}|x|^{-\frac{2s+\theta}{p-1}},
\]

where

\[
K_{p,\theta} = C_s\left(-\frac{2s + \theta}{p - 1}\right)^{\frac{1}{p-1}}.
\]

Note that

\[
(-\Delta)^s u_p = |x|^\theta u_p^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]

which could be written as

\[
\mathcal{L}_{-K_{p,\theta}^{p-1}} u_p := (-\Delta)^s u_p - K_{p,\theta}^{p-1}|x|^{-2s} u_p = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}
\]

and

\[
\tau_-(-K_{p,\theta}^{p-1}) = 2s - N + \frac{2s + \theta}{p - 1} \quad \text{and} \quad \tau_+(K_{p,\theta}^{p-1}) = \frac{2s + \theta}{p - 1}.
\]

By contradiction, we assume that \( u_0 \) is a positive solution of (1.1) such that

\[
\liminf_{|x| \to 0^+} u_0(x)|x|^{\frac{2s+\theta}{p-1}} > K_{p,\theta}.
\]

Then there exist \( \epsilon_0 > 0 \) and \( r_0 > 0 \) such that

\[
|x|^\theta u_0^{p-1}(x) \geq (K_{p,\theta}^{p-1} + 2\epsilon_0)|x|^{-2s} \quad \text{for any} \quad 0 < |x| < r_0.
\]
Therefore, \( u_0 \) is a super solution of
\[
\mathcal{L}^\mu_{\mu_1} u_0 \geq \epsilon_0 |x|^\theta u_0^p \quad \text{in} \ B_r \setminus \{0\},
\]
where \( r \in (0, r_0) \) and
\[
\mu_1 = -(\mathcal{K}_{p, \theta}^{p-1} + \epsilon_0) < -\mathcal{K}_{p, \theta}^{p-1}.
\]

By \[19\] Proposition 1.2], the mapping \( \mu \in (\mu_0, 0) \mapsto \tau_+(\mu) \) is continuous and strictly increasing, then
\[
\tau_+(\mu_1) < \tau_+(-\mathcal{K}_{p, \theta}^{p-1}) = \frac{-2s + \theta}{p - 1} \quad \text{for} \ p > \frac{N + 2s + 2\theta}{N - 2s}.
\]

By Lemma 2.2 with \( \mu = \mu_1 \), there exists \( c > 0 \) such that
\[
u_0(x) \geq c_25 |x|^{\tau_+(\mu_1)} \quad \text{in} \ B_r \setminus \{0\}.
\]

Let \( \tau_0 = \tau_+(\mu_1) \). If \( \tau_0 \leq -\frac{N + \theta}{p} \), where \( -\frac{N + \theta}{p} > \frac{2s + N}{2} \) for \( p \geq \frac{N + 2s + 2\theta}{N - 2s} \). Then \( |x|^\theta u_0^p \notin L^1(B_1) \), a contradiction arises from Theorem 2.1 part (iii) with \( \mu = 0 \). Then we are done.

If \( \tau_0 \in \left( -\frac{N + \theta}{p}, -\frac{2s + \theta}{p} \right) \), then
\[
(-\Delta)^s u_0(x) \geq d_0^p |x|^p \tau_0 + \theta = d_0^p |x|^|\tau_1| - 2s \quad \text{in} \ B_{\tau_0} \setminus \{0\},
\]
where
\[
\tau_1 := p\tau_0 + \theta + 2s.
\]

If \( \tau_1 \leq -\frac{N + \theta}{p} \) we are done.

If \( \tau_1 \in \left( -\frac{N + \theta}{p}, -\frac{2s + \theta}{p} \right) \), by Lemma 2.2 with \( \mu = 0 \), we have that
\[
u_0(x) \geq d_1 |x|^{\tau_1} \quad \text{in} \ B_{\tau_0} \setminus \{0\}.
\]

Iteratively, we recall that
\[
\tau_j := p\tau_{j-1} + \theta + 2s, \quad j = 1, 2, \ldots.
\]

If \( \tau_j \leq -\frac{N + \theta}{p} \) we are done, otherwise it follows from Lemma 2.2 that
\[
u_0(x) \geq d_{j+1} |x|^{\tau_{j+1}},
\]
where
\[
\tau_{j+1} = p\tau_j + \theta + 2s < \tau_j.
\]

Thanks to \( \tau_1 - \tau_0 = (p - 1)\tau_0 + \theta + 2s = \tau_{j+1} - \tau_j < 0 \), we have that
\[
\tau_j - \tau_{j-1} = p(\tau_{j-1} - \tau_{j-2}) = p^{j-1}(\tau_1 - \tau_0) \to -\infty \quad \text{as} \ j \to +\infty,
\]
then there exists \( j_0 \in \mathbb{N} \) such that
\[
\tau_{j_0} > -\frac{N + \theta}{p} \quad \text{and} \quad \tau_{j_0 + 1} \leq -\frac{N + \theta}{p}.
\]

For this, we have that
\[
(-\Delta)^s u_0(x) \geq d_{j_0}^p |x|^q \tau_{j_0 + 1} - 2s = d_{j_0}^p |x|^\tau_{j_0 + 1} \quad \text{in} \ B_{\tau_0} \setminus \{0\},
\]
then \( u_0(x) \geq d_{j_0+1} |x|^\tau_{j_0+1} \), and a contradiction arises.

**Part 2:** we claim
\[
\limsup_{|x| \to 0^+} |x|^{\frac{2s + \theta}{p - 1}} u_0(x) \geq \mathcal{K}_{p, \theta}.
\]

By contradiction, we assume that \( u_0 \) is a positive super solution of (1.1) in \( \Omega \setminus \{0\} \) such that
\[
\limsup_{|x| \to +\infty} |x|^{\frac{2s + \theta}{p - 1}} < \mathcal{K}_{p, \theta}.
\]

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Then there exist \( \epsilon_1 \in (0, \frac{1}{2}K_{p, \theta}^{-1}) \) and \( r > 0 \) such that
\[
|x|^{\theta} u_0^{p-1}(x) \leq (K_{p, \theta}^{-1} - \epsilon_1)|x|^{-2s} \quad \text{for any } 0 < |x| < r.
\]
Therefore, \( u_0 \) is a super solution of
\[
\mathcal{L}^{*}_{\mu_2} u_0 \leq 0 \quad \text{in } B_{r_0} \setminus \{0\},
\]
where
\[
\mu_2 := -K_{p, \theta}^{-1} + \epsilon_1 > -K_{p, \theta}^{-1}.
\]
Notice that for \( p \geq \frac{N + 2\theta + 2s}{N - 2s} \), we have that
\[
\tau_+(-K_{p, \theta}^{-1}) = \frac{2s + \theta}{p - 1} \geq \frac{2s - N}{2} > \tau_-(-K_{p, \theta}^{-1}).
\]
By the strictly increasing monotonicity, we have that \( \tau_+ (\mu_2) > -\frac{2s + \theta}{p - 1} \), thanks to \( \mu_2 > -K_{p, \theta}^{-1} \). By Lemma 2.3 with \( \mu = \mu_2 \), we see that
\[
u_0(x) \leq c_{2\theta}|x|^\tau_+ (\mu_2) \quad \text{in } B_{r_0} \setminus \{0\}.
\]
Let \( \tau_0 = \tau_+ (\mu_2) > -\frac{2s + \theta}{p - 1} \). Since \( \tau_0 \in \left( -\frac{2s + \theta}{p - 1}, 0 \right) \), then
\[
(\Delta)^s u_0(x) \leq c_{2\theta} |x|^{\tau_0 + \theta} = c_{2\theta} |x|^\tau_1 - 2s \quad \text{in } B_{r_0} \setminus \{0\},
\]
where
\[
\tau_1 := \tau_0 + \theta + 2s.
\]
if \( \tau_1 > 0 \),
\[
u_0(x) \leq d_1 \quad \text{in } B_{r_2} \setminus \{0\},
\]
which contradicts that \( u_0 \) is non-removable at the origin. Then we are done.

If \( \tau_1 = 0 \), applying Lemma 2.3 with \( \mu = 0 \) to obtain that
\[
u_0(x) \leq d_1 \left( \ln \frac{\tau_0}{|x|} + 1 \right) \quad \text{in } B_{r_2} \setminus \{0\},
\]
then for \( \epsilon > 0 \) small,
\[
u_0(x) \leq d_1 |x|^{-\epsilon p} \quad \text{in } B_{r_2} \setminus \{0\},
\]
which implies that
\[
(\Delta)^s u_0(x) \leq c_{2\theta} |x|^{-\epsilon p} \quad \text{in } B_{r_2} \setminus \{0\}.
\]
Then we apply Lemma 2.3 with \( \mu = 0 \) to obtain that
\[
u_0(x) \leq d_1 \quad \text{in } B_{r_2} \setminus \{0\}
\]
for \( \epsilon > 0 \) small enough and we are done.

If \( \tau_1 < 0 \), by Lemma 2.3 we have that
\[
u_0(x) \leq d_1 |x|^\tau_1 \quad \text{in } B_{r_0} \setminus \{0\}.
\]
Iteratively, we reset that
\[
\tau_j := \tau_{j-1} + \theta + 2s, \quad j = 1, 2, \ldots.
\]
Note that
\[
\tau_1 - \tau_0 = (p - 1)\tau_0 + \theta + 2s > 0
\]
thanks to \( \tau_0 > -\frac{2s + \theta}{p - 1} \).

If \( \tau_j p + \theta + 2s \geq 0 \) we are done by the same argument as \( \tau_1 \geq 0 \). If \( \tau_j p + \theta + 2s < 0 \) it follows by Lemma 2.2 that
\[
u_0(x) \geq d_{j+1} |x|^\tau_{j+1},
\]
where \[ \tau_{j+1} = p\tau_j + 2s + \theta < \tau_j. \]

Note that
\[ \tau_j - \tau_{j-1} = p(\tau_{j-1} - \tau_{j-2}) = p^{j-1}(\tau_1 - \tau_0) \rightarrow +\infty \text{ as } j \rightarrow +\infty, \]
thus, there exists \( j_0 \in \mathbb{N} \) such that \( \tau_{j_0} < 0 \) and \( \tau_{j_0+1} \geq 0 \),
and \( u_0 \) is bounded at the origin. Thus a contradiction comes from the assumption that
\[ \limsup_{|x| \rightarrow 0^+} u_0(x) = +\infty. \]
We complete the proof. \( \square \)

4.2 The isolated singularities

Proof of Theorem 1.1. Let \( u_0 \) be a nonnegative solution of (1.1). For \[ \theta = 0 \quad \& \quad p \in \left( \frac{N}{N - 2s}, \frac{N + 2s}{N - 2s} \right), \]
Proposition 3.1 gives an upper bound
\[ u_0(x) \leq c_{16}|x|^{-\frac{2s}{p-1}}, \quad \forall \, x \in B_1 \setminus \{0\}. \]
Then \( u_0 \) verifies Harnack inequality (2.27) by Proposition 2.1. We now apply Proposition 4.1 with \( p \in \left( \frac{N}{N - 2s}, \frac{N + 2s}{N - 2s} \right), \) we obtain that
\[ \frac{K_p}{C_0} \leq \liminf_{|x| \rightarrow 0^+} u(x)|x|^{-\frac{2s}{p-1}} \leq \limsup_{|x| \rightarrow 0^+} u(x)|x|^{-\frac{2s}{p-1}} \leq K_p C_0. \]
We complete the proof. \( \square \)

Proof of Theorem 1.2. When \[ \theta = 0 \quad \& \quad p \geq \frac{N + 2s}{N - 2s} \]
or
\[ \theta \in (-2s, 0) \quad \& \quad p > \frac{N + \theta}{N - 2s}, \]
and \( h \) is radially symmetric, decreasing with respect to \( |x| \), then the positive solution \( u_0 \) is radially symmetric and decreasing with respect to \( r = |x| \), provided that \( u_0(x) \geq h(1) \), then Proposition 3.2 provides an upper bound
\[ u_0(x) \leq c_{22}|x|^{-\frac{2s + \theta}{p-1}}, \quad \forall \, x \in B_1 \setminus \{0\}. \]
Then our conclusion of \( u_0 \) follows by Proposition 2.1, Proposition 1.1 and Proposition 4.1. \( \square \)

Proof of Theorem 1.3. For \( \theta \in (-2s, +\infty) \), Corollary 2.2 gives an upper bound
\[ u_0(x) \leq c_{22}|x|^{-\frac{2s + \theta}{p-1}}, \quad \forall \, x \in B_1 \setminus \{0\}. \]
Then our conclusion of \( u_0 \) follows by Proposition 4.1 and Proposition 4.2. \( \square \)

Proof of Theorem 1.4. Let \( v_0 \) be a positive solution of (1.19). We claim that
\[ v_0(x) \leq c_{28}|x|^{-\frac{2s + \theta}{p-1}} \quad \text{for} \quad 0 < |x| < 1/2 \quad (4.6) \]
for some \( c_{28} > 0. \)
We first show that for \( c_{29} > 0 \)
\[
\int_{B_r} |x|^{\theta} v_0^p dx \leq c_{29} r^{N - \frac{\alpha + 2\theta}{p}}, \quad \forall \, r \in (0, \frac{1}{2}),
\]
(4.7)

In fact, let \( r \in (0, 1/2) \) and \( v = v_0 - \tilde{v}_0 \),

where \( \tilde{v}_0 \) is the harmonic extension of \( v_0 \) in \( B_r \), i.e. the solution of

\[
\begin{cases}
- \Delta u = 0 & \text{in } B_r, \\
u = v_0 & \text{on } \partial B_r.
\end{cases}
\]

Then we have that

\[
\begin{cases}
- \Delta v = |x|^\theta (v + \tilde{v}_0)^p & \text{in } B_r \setminus \{0\}, \\
v = 0 & \text{on } \partial B_r
\end{cases}
\]

and (36) shows that in the Serrin’s supercritical since \( p > \frac{N + \theta}{N - 2} \),

\[
\int_{B_r} v (-\Delta) \eta dx = \int_{B_r} |x|^\theta (v + \tilde{v}_0)^p \eta dx, \quad \eta \in C_0^{1,1}(B_r),
\]

which implies that for \( \eta \in C_0^{1,1}(B_r) \)

\[
\int_{B_r} v_0 (-\Delta) \eta dx = \int_{B_r} |x|^\theta v_0^p \eta dx + \int_{B_r} \tilde{v}_0 (-\Delta) \eta dx = \int_{B_r} |x|^\theta v_0^p \eta dx - \int_{\partial B_r} v_0 \eta \cdot x d\sigma(x),
\]

since the normal vector pointing outside of \( B_1 \) is \( x \in \partial B_1 \).

Now we take \( \eta = \xi_r := \xi_1(r^{-1} \cdot) \), where \( \xi_1 \) is the first eigenvalue of \( -\Delta \) in \( B_1 \) with the zero Dirichlet boundary condition. Note that \( \xi_1 \) is radially symmetric and decreasing with respect to \( |x| \) and then \( \xi_1(1) < 0 \).

Then we have that

\[
\lambda_1 r^{-2} \int_{B_r} v_0 \xi_r dx = \int_{B_r} |x|^\theta v_0^p \xi_r dx - r^{-1} \xi_1'(1) \int_{\partial B_r} v_0 d\sigma(x) > \int_{B_r} |x|^\theta v_0^p \xi_r dx.
\]

We observe that \(-\frac{\theta}{p-1} > -N\) by our assumption that \( p > \frac{N + \theta}{N - 2} \) and

\[
\int_{B_r} |x|^\theta v_0^p \xi_r dx < \lambda_1 r^{-2} \int_{B_r} v_0 \xi_r dx
\]

\[
\leq \lambda_1 r^{-2} \left( \int_{B_r} |x|^\theta v_0^p \xi_r dx \right)^{\frac{1}{p}} \left( \int_{B_r} \xi_r |x|^{-\frac{\theta}{p-1}} dx \right)^{1 - \frac{1}{p}}
\]

\[
\leq ||\xi_1||_{L^\infty(B_1)} \lambda_1 r^{(N - \frac{\theta}{p-1})(1 - \frac{1}{p}) - 2} \left( \int_{B_r} |x|^\theta v_0^p \xi_r dx \right)^{\frac{1}{p}},
\]

which implies that

\[
\int_{B_r} |x|^\theta v_0^p \xi_r dx < ||\xi_1||_{L^\infty(B_1)} \lambda_1 r^{N - \frac{\theta}{p-1} - \frac{2\theta}{p}}.
\]

Notice that

\[
\xi_r(x) = \xi_1(r^{-1}x) \geq \min_{z \in B_\frac{1}{2}} \xi_1(z) \quad \text{in } B_{\frac{1}{4}},
\]

then

\[
\int_{B_{\frac{1}{4}}} |x|^\theta v_0^p \xi_r dx < c_{30} \int_{B_r} |x|^\theta v_0^p \xi_r dx < c_{30} ||\xi_1||_{L^\infty(B_1)} \lambda_1 r^{N - \frac{\theta}{p-1} - \frac{2\theta}{p}},
\]

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Replacing $r$ by $\frac{r}{2}$, we obtain (4.7).

Part (i): Since $u_0$ verifies (1.16), thus, taking $|x| = r$, for any $y \in B_r \setminus B_\frac{r}{2}$, there holds that

$$u_0(y) \geq \frac{1}{C_1} u_0(x).$$

Now we deduce from (4.7) that for any given $|x| = r$,

$$C_0^{-p} \left(1 - \left(\frac{1}{2}\right)^N\right) \partial B_1 |v_0^p(x) r^{N-\theta} \leq \int_{B_r \setminus B_\frac{r}{2}} |y|^{\theta} v_0^p(y) dy < c_{14} r^{N-\frac{\theta}{2} - \frac{\theta}{2}},$$

which implies that for some $c_{31} \geq 1$

$$\frac{1}{c_{31}} |x|^{-\frac{2+\theta}{\beta}} \leq v_0(x) \leq c_{13} |x|^{-\frac{2+\theta}{\beta}} \quad \text{for} \quad 0 < |x| < \frac{1}{2}.$$

By similar argument in Proposition 4.2 (some basic estimates could see [17]), we obtain (1.20).

Part (ii): Let $u = v_0 - h \geq 0$, under our assumptions that $h$ is a nonnegative constant and $v_0 \geq h$ in $B_1$, then $u$ is a classical solution of

$$\begin{cases} 
-\Delta u &= |x|^\theta (u + h)^p \quad \text{in} \ B_1 \setminus \{0\}, \\
\quad u &= 0 \quad \text{on} \ \partial B_1.
\end{cases} \quad (4.8)$$

For $\theta < 0$, the moving plane works for the problem (4.8), see [6,28] and we get that $u$ is radially symmetric and decreasing with respect to $|x|$, so is $v_0$. Then (4.7) implies (4.10).

Under the upper bound (4.10), we have the Harnack inequalities, for all $r \in (0, \frac{1}{2})$

$$\sup_{x \in B_{2r} \setminus B_r} v_0(x) \leq C_1 \inf_{x \in B_{2r} \setminus B_r} v_0(x)$$

for some $C_1 > 0$. From the Harnack inequality in [10], we have that for some constant $c_{32} \geq 1$

$$\frac{1}{c_{32}} |x|^{-\frac{2+\theta}{\beta}} \leq v_0(x) \leq c_{32} |x|^{-\frac{2+\theta}{\beta}} \quad \text{for} \quad 0 < |x| < \frac{1}{2}.$$

Therefore, we obtain (1.20). \qed

5 Application to the decay at infinity

In this section, we apply the classification of isolated singularity to obtain the decay at infinity for the positive solution of

$$\begin{cases} 
(-\Delta)^s u &= |x|^\tilde{\theta} u^p \quad \text{in} \ \mathbb{R}^N \setminus \bar{B}_1, \\
\quad u &= h \quad \text{in} \ \bar{B}_1.
\end{cases} \quad (5.1)$$

**Theorem 5.1** Assume that

$$p \in \left(\frac{N + \tilde{\theta}}{N - 2s}, \frac{N + 2s + \tilde{\theta}}{N - 2s}\right) \quad \text{for} \quad \tilde{\theta} > -2s \quad (5.2)$$

or

$$p = \frac{N + 2s + \tilde{\theta}}{N - 2s} \quad \text{for} \quad \tilde{\theta} \geq 0, \quad (5.3)$$

$h \in L^1(\bar{B}_1)$ is a nonnegative and radially symmetric function such that $h(1) \geq 0$, $|x|^{N-2s} h(x)$ is increasing with respect to $|x|$ in $B_1 \setminus \{0\}$.

Let $u_{0}$ be a positive solution of (5.1) such that

$$u_0(x) \geq h(1)|x|^{2s-N} \quad \text{for} \ \forall x \in \mathbb{R}^N \setminus B_1, \quad (5.4)$$

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then \( u_0 \) is radially symmetric, either for some \( \kappa \ge h(1) \)

\[
\lim_{|x| \to +\infty} u_0(x)|x|^{N-2s} = \kappa
\]

or

\[
\frac{K_{p,\theta^*}}{C_0} \le \liminf_{|x| \to +\infty} u_0(x)|x|^\frac{2s+\theta^*}{p-1} \le \limsup_{|x| \to +\infty} u_0(x)|x|^\frac{2s+\theta^*}{p-1} \le C_0 K_{p,\theta^*},
\]

where

\[
\theta^* = p(N-2s) - N - 2s - \tilde{\theta}.
\]

**Proof.** We use the *Kelvin transformation*: let \( u_f \) be a solution of

\[
\begin{align*}
(-\Delta)^s u &= f & \text{in } \mathcal{O}, \\
u &= h & \text{in } \mathbb{R}^N \setminus \mathcal{O},
\end{align*}
\]

where \( \mathcal{O} \) is an open set of \( \mathbb{R}^N \) and \( f \) is locally Hölder continuous in \( \mathcal{O} \). Denote

\[
u^\sharp(x) = |x|^{2s-N} u_f \left( \frac{x}{|x|^2} \right) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}
\]

and

\[
\mathcal{O}^\sharp = \left\{ x \in \mathbb{R}^N : \frac{x}{|x|^2} \in \mathcal{O} \right\},
\]

then a direct computation shows that

\[
(-\Delta)^s \nu^\sharp(x) = |x|^{-2s-N} \left( (-\Delta)^s u_f \right) \left( \frac{x}{|x|^2} \right) \quad \text{for } x \in \mathcal{O}^\sharp.
\]

As a conclusion, \( \nu^\sharp \) is a solution of

\[
\begin{align*}
(-\Delta)^s \nu^\sharp &= |x|^{-2s-N} f^* & \text{in } \mathcal{O}^\sharp, \\
\nu^\sharp &= |x|^{2s-N} h^* & \text{in } \mathbb{R}^N \setminus \mathcal{O},
\end{align*}
\]

where \( f^*(x) = f \left( \frac{x}{|x|^2} \right) \) and \( h^*(x) = h \left( \frac{x}{|x|^2} \right) \).

Let \( u_0 \) be a positive solution of (5.1). Taking \( f(x) = |x|^p u_0^p(x) \) and \( \mathcal{O} = \mathbb{R}^N \setminus B_1 \) in (5.5), the function

\[
u^\sharp(x) := |x|^{2s-N} u_0 \left( \frac{x}{|x|^2} \right)
\]

is a positive solution of

\[
\begin{align*}
(-\Delta)^s \nu^\sharp &= |x|^\theta \left( u^\sharp \right)^p & \text{in } B_1 \setminus \{0\}, \\
\nu^\sharp &= |x|^{2s-N} h^* & \text{in } \mathbb{R}^N \setminus B_1,
\end{align*}
\]

where

\[
|x|^{2s-N} h^* \in L^1(\mathbb{R}^N \setminus B_1, |x|^{-N-2s} dx),
\]

which is equivalent to \( h \in L^1(B_1) \).

Since

\[
|x|^{2s-N} h^*(x) = |y|^{N-2s} h(y) \quad \text{with } x = \frac{y}{|y|^2},
\]

then \( x \in B_1 \setminus \{0\} \mapsto |x|^{2s-N} h^*(x) \) is radially symmetric and increasing with respect to \( |x| \). Note that

\[
0(x) \ge h(1)|x|^{2s-N} \quad \text{for } \forall x \in \mathbb{R}^N \setminus B_1 \quad \iff \quad \nu^\sharp(y) \ge h(1) \quad \text{for } y \in B_1 \setminus \{0\}.
\]
To obtain the asymptotic behaviors, we would apply Theorem 1.2 with $\theta^* = 0$ and $p \geq \frac{N + 2s}{N - 2s}$ by (5.3); or with $\theta^* \in (-2s, 0)$ and $p \geq \frac{N + \theta^*}{N - 2s}$ by (5.2). As a consequence, we have that $u^\delta$ is radially symmetric, decreasing with respect to $|y|$, and is either removable at the origin or verifies
\[
\frac{K_{p, \theta^*}}{C_0} \leq \liminf \frac{u^2(x)|x|^{\frac{2p + \theta^*}{p - 1}}}{|x| \to 0^+} \leq K_{p, \theta^*} \leq \limsup \frac{u^2(x)|x|^{\frac{2p + \theta^*}{p - 1}}}{|x| \to 0^+} \leq C_0 K_{p, \theta^*},
\]
which imply that $u_0$ is radially symmetric, either
\[
\lim_{|x| \to +\infty} u_0(x)|x|^{N - 2s} = u^\delta(0)
\]
or
\[
\frac{K_{p, \theta^*}}{C_0} \leq \liminf_{|x| \to +\infty} u_0(x)|x|^{\frac{2s - \theta^*}{p - 1}} \leq K_{p, \theta^*} \leq \limsup_{|x| \to +\infty} u_0(x)|x|^{\frac{2s - \theta^*}{p - 1}} \leq C_0 K_{p, \theta^*}.
\]
We complete the proof. \hfill \Box

**Theorem 5.2** Assume that $h \in L^1(B_1)$ and
\[
p = \frac{N + 2s + \tilde{\theta}}{N - 2s}
\]
for $\tilde{\theta} \in (-2s, 0)$. Let $u_0$ be a positive solution of (5.1) and $C_0 \geq 1$ be the constant from the Harnack inequality in Theorem 1.1 with $\theta = 0$. Then either for some $k > 0$
\[
\lim_{|x| \to +\infty} u_0(x)|x|^{N - 2s} = k
\]
or
\[
\frac{K_p}{C_0} \leq \liminf_{|x| \to +\infty} u_0(x)|x|^{\frac{2s - \theta^*}{p - 1}} \leq K_p \leq \limsup_{|x| \to +\infty} u_0(x)|x|^{\frac{2s - \theta^*}{p - 1}} \leq C_0 K_p.
\]

**Proof.** From the Kelvin transformation,
\[
u^\delta(x) := |x|^{2s - N} u_0\left(\frac{x}{|x|^2}\right)
\]
is a positive solution of
\[
\left\{ \begin{array}{ll}
(-\Delta)^s u^\delta = (u^\delta)^p & \text{in } B_1 \setminus \{0\}, \\
u^\delta = |x|^{2s - N} h^\ast & \text{in } \mathbb{R}^N \setminus B_1,
\end{array} \right.
\]
by our assumption (5.6) that $\tilde{\theta} = p(N - 2s) - N - 2s$.

From Theorem 1.1 $u^\delta$ is either removable at the origin or it verifies
\[
\frac{K_p}{C_0} \leq \liminf_{|x| \to 0^+} u^2(x)|x|^{\frac{2s - \theta^*}{p - 1}} \leq K_p \leq \limsup_{|x| \to 0^+} u^2(x)|x|^{\frac{2s - \theta^*}{p - 1}} \leq C_0 K_p,
\]
then we have that either
\[
\lim_{|x| \to +\infty} u_0(x)|x|^{N - 2s} = u^\delta(0)
\]
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or
\[
\frac{\mathcal{K}_p}{C_0} \leq \liminf_{|x|\to +\infty} u_0(x)|x|^{N-2s-\tilde{\theta}} \leq \mathcal{K}_p
\]
\[
\leq \limsup_{|x|\to +\infty} u_0(x)|x|^{N-2s-\tilde{\theta}} \leq C_0 \mathcal{K}_p,
\]
where we use \(\theta^* = p(N-2s) - N - 2s - \tilde{\theta} = 0\). We complete the proof. \(\square\)

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