On Feller, Pollard and the Complete Monotonicity of the Mittag-Leffler Function $E_\alpha(-x)$

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Abstract

Pollard used contour integration to show that the Mittag-Leffler function is the Laplace transform of a positive function, thereby proving that it is completely monotone. He also cited personal communication by Feller of a discovery of the result by “methods of probability theory”. In his published work, Feller used the two-dimensional Laplace transform of a bivariate distribution to derive the Pollard result. But both approaches may be described as analytic, despite the occurrence of the stable distribution in Feller’s starting point and in the Pollard result itself. We adopt a Bayesian probabilistic approach that assigns a prior distribution to the scale parameter of the stable distribution. We present Feller’s method as a particular instance of such assignment. The Bayesian framework enables generalisation of the Pollard result. This leads to a novel integral representation of the Mittag-Leffler function as well as a variant arising from polynomial tilting of the stable density.

Keywords— Bayesian reasoning; complete monotonicity; stable, gamma distributions; Mittag-Leffler function, distribution; infinite divisibility.

1 Background

An infinitely differentiable function $\varphi(x)$ on $x > 0$ is completely monotone if its derivatives $\varphi^{(n)}(x)$ satisfy $(-1)^n \varphi^{(n)}(x) \geq 0$, $n \geq 0$. Bernstein’s theorem states that $\varphi(x)$ is completely monotone iff it may be expressed as

$$\varphi(x) = \int_0^\infty e^{-xt} dF(t) = \int_0^\infty e^{-xt} f(t) dt$$  \hspace{1cm} (1)

for a non-decreasing distribution function $F(t)$ with density $f(t)$, i.e. $F(t) = \int_0^t f(u)du$. The first integral in (1) is formally called the Laplace-Stieltjes transform of $F$ and the latter the (ordinary) Laplace transform of $f$. For bounded $F(t)$, $\varphi(x)$ is defined on $x \geq 0$. Integrating (1) by parts in this case gives $\varphi(x)$ in terms of the ordinary Laplace transform of $F$:

$$\varphi(x) = x \int_0^\infty e^{-xt} F(t) dt$$  \hspace{1cm} (2)
The Mittag-Leffler function $E_\alpha(x)$ is defined by the infinite series
\[
E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)} \quad \alpha \geq 0
\] (3)

For later reference, the Laplace transform of $E_\alpha(-\lambda x^\alpha)$ ($\lambda > 0$) is
\[
\int_0^\infty e^{-sx} E_\alpha(-\lambda x^\alpha) \, dx = \frac{s^{\alpha-1}}{\lambda + s^\alpha} \quad \text{Re}(s) \geq 0
\] (4)

Pollard and Feller discussed the complete monotonicity of $E_\alpha(-x)$ from different perspectives. We summarise both before presenting a Bayesian argument.

1.1 Pollard’s Approach

In a 1948 paper, Pollard [17] led with the opening remark:

“W. Feller communicated to me his discovery – by the methods of probability theory – that if $0 \leq \alpha \leq 1$ the function $E_\alpha(-x)$ is completely monotonic for $x \geq 0$. This means that it can be written in the form
\[
E_\alpha(-x) = \int_0^\infty e^{-st}dP_\alpha(t)
\]
where $P_\alpha(t)$ is nondecreasing and bounded. In this note we shall prove this fact directly and determine the function $P_\alpha(t)$ explicitly.”

[we use $P_\alpha$ where Pollard used $F_\alpha$, which we reserve for another purpose]

Having dispensed with $E_0(-x) = 1/(1 + x)$ and $E_1(-x) = e^{-x}$ since “there is nothing to be proved in these cases”, Pollard used a contour integral representation of $E_\alpha(-x)$:
\[
E_\alpha(-x) = \frac{1}{2\pi i} \oint_C s^{\alpha-1} e^s ds = \frac{1}{2\pi i} \oint_{C'} \frac{e^{\frac{s^\alpha}{2}}}{x + z} \, dz
\] (5)

to prove that
\[
p_\alpha(t) \equiv P_\alpha'(t) = \frac{1}{\alpha} f_\alpha(t^{-1/\alpha}) t^{-1-1/\alpha} \quad 0 < \alpha < 1
\] (6)

where $f_\alpha(t)$ is defined by
\[
e^{-s^\alpha} = \int_0^\infty e^{-st} f_\alpha(t) \, dt \quad 0 < \alpha < 1
\] (7)

Pollard [16] had earlier proved that $f_\alpha(t) > 0$, so that $p_\alpha(t) \geq 0$, thereby completing his proof that $E_\alpha(-x)$ is completely monotone for $0 \leq \alpha \leq 1$. Pollard stopped at the point of deriving (6), the density $p_\alpha(t) \equiv P_\alpha'(t)$. As per initial task, we proceed to discuss $P_\alpha(t)$ explicitly. We first recognise $f_\alpha(t)$ as the density of the stable distribution $F_\alpha$ on $[0, \infty)$
\[
F_\alpha(t) = \int_0^t f_\alpha(u) \, du \quad 0 < \alpha < 1
\] (8)
with normalisation \( F_\alpha(\infty) = 1 \). In turn, \( P_\alpha \) is the distribution

\[
P_\alpha(t) = \int_0^t p_\alpha(u) \, du = \frac{1}{\alpha} \int_0^t f_\alpha(u^{-1/\alpha}) u^{-1-1/\alpha} \, du
\]

Setting \( y = u^{-1/\alpha} \) gives a simple relation between \( P_\alpha \) and \( F_\alpha \):

\[
P_\alpha(t) = \int_{1^{-1/\alpha}}^\infty f_\alpha(y) \, dy = 1 - \int_0^{1^{-1/\alpha}} f_\alpha(y) \, dy = 1 - F_\alpha(t^{-1/\alpha}) \tag{9}
\]

Setting aside Pollard’s contour integral proof, it is not clear how to show directly that

\[
E_\alpha(-x) = \int_0^\infty e^{-xt} dP_\alpha(t) = \int_0^\infty e^{-xt} (1 - F_\alpha(t^{-1/\alpha})) dt = x \int_0^\infty e^{-xt} (1 - F_\alpha(t^{-1/\alpha})) dt \tag{10}
\]

Feller followed a different route, discussed next.

### 1.2 Feller’s Approach

In an illustration of the use of the two-dimensional Laplacetransform, Feller [6] (p453) considered \( 1 - F_\alpha(xt^{-1/\alpha}) \), a bivariate generalisation of (9) over \( x > 0, t > 0 \). The Laplace transform over \( x \), followed by that over \( t \) gives

\[
\frac{1}{s} \int_0^\infty e^{-\lambda t} (1 - e^{-ts^\alpha}) \, dt = \frac{1}{s^\alpha - \lambda + \frac{1}{s^\alpha}} \tag{12}
\]

By reference to (4), the right hand side of (12) is the Laplace transform of \( E_\alpha(-\lambda x^\alpha)/\lambda \). Since the two-dimensional Laplace transform equivalently can be evaluated first over \( t \) then over \( x \), Feller concluded that

\[
E_\alpha(-\lambda x^\alpha) = \lambda \int_0^\infty e^{-\lambda t} (1 - F_\alpha(xt^{-1/\alpha})) \, dt \tag{13}
\]

\[
\Rightarrow E_\alpha(-x) = \lambda \int_0^\infty e^{-t} (1 - F_\alpha(x^{1/\alpha}t^{-1/\alpha})) \, dt \tag{14}
\]

\[
(t \rightarrow xt) = x \int_0^\infty e^{-xt} (1 - F_\alpha(t^{-1/\alpha})) \, dt \tag{15}
\]

which is the Pollard result in the form (10).

Feller’s proof is based on the interchange of the order of integration (Fubini’s theorem) and the uniqueness of Laplace transforms. It can be represented by the commutative diagram below, where \( \mathcal{L}_{s|t} \) denotes the one-dimensional Laplace transform of a bivariate source function at fixed \( t \), to give a bivariate function of \((s, t)\) where \( s \) is the Laplace variable.

\[
\begin{array}{c}
1 - F_\alpha(xt^{-1/\alpha}) \xrightarrow{\mathcal{L}_{s|t} \text{ easy}} \frac{1}{s} - \frac{e^{-ts^\alpha}}{s} \\
\mathcal{L}_{\lambda|x} \xrightarrow{\text{hard}} \frac{1}{\lambda} E_\alpha(-\lambda x^\alpha) \xrightarrow{\mathcal{L}_{s|\lambda} \text{ easy}} \frac{1}{\lambda} \frac{s^{\alpha-1}}{\lambda + s^\alpha}
\end{array}
\]

\[
\tag{16}
\]
The desired proof is the “hard” direct path, which is equivalent to the “easy” indirect path. We will return to commutative diagram representation in a different context later in the paper.

Feller’s concise proof uses “methods of probability theory”, as cited by Pollard, only to the extent of choosing the bivariate distribution as input to the two-dimensional Laplace transform. Other than that, the methods by both Pollard and Feller might be described as analytic rather than probabilistic. This naturally begs the following questions:

1. What is it that amounts to a method of probability theory, at least in the context of proving that $E_\alpha(-x)$ is completely monotone?

2. What additional or complementary insight, if any, does probabilistic reasoning offer relative to an analytic perspective?

1.3 Purpose of Paper

This paper addresses both questions above. The approach is that of strict use of the sum and product rules of probability theory. We identify this as Bayesian reasoning, although our context is not one of Bayesian inference. The latter calls for explicit use of Bayes’ rule to transition from prior to posterior distribution, with the aid of a prescribed likelihood. The assignment of appropriate distribution in our context is guided by the task of proving that $E_\alpha(-x)$ is completely monotone. We first cast Feller’s argument in such terms before proceeding to a more general discussion.

1.4 Scope of Paper

The Mittag-Leffler function is of growing interest in probability theory and physics, with a diversity of applications, notably fractional calculus. A comprehensive study of the properties and applications of the Mittag-Leffler function and its numerous generalisations is beyond the scope of this paper. We consciously restrict the scope to the theme of complete monotonicity and Mittag-Leffler functions, underpinned by Bayesian reasoning.

Other studies that explicitly discuss complete monotonicity and Mittag-Leffler functions build upon complex analytic approaches similar to Pollard’s rather than the probabilistic underpinning discussed here. For example, deOliveira et al. [4] and Mainardi and Garrappa [13] studied the complete monotonicity of $x^{\beta-1}E_{\alpha,\beta}^\gamma(-x^\alpha)$, whereas Górska et al. [9] explored the complete monotonicity of $E_{\alpha,\beta}^\gamma(-x)$. $E_{\alpha,\beta}^\gamma(x)$ is the three-parameter variant of the Mittag-Leffler function, also known as the Prabhakar function. These papers comment on the fundamental importance of the complete monotonicity of Mittag-Leffler functions used in the modelling of physical phenomena, such as anomalous dielectric relaxation and viscoelasticity.

Finally, we are keenly aware that there are other views on the interpretation of “methods of probability theory”. We comment on this before discussing the Bayesian approach in detail.

1.5 Probabilistic Perspectives

The phrase ‘methods of probability theory’ used by Pollard may suggest an experiment with random outcomes as a fundamental metaphor. Indeed, Pollard’s $P_\alpha$, which is referred to as the
Mittag-Leffler distribution in the probabilistic literature, is derived as a limiting distribution of a Pólya urn scheme (e.g. Janson [12]).

Diversity of approach is commonplace in probability theory and mathematics more generally. For example, in a context of nonparametric Bayesian analysis, Ferguson [7] constructed the Dirichlet process based on the gamma distribution as the fundamental probabilistic concept, without invoking a random experiment. Blackwell and MacQueen [3] observed that the Ferguson approach “involves a rather deep study of the gamma process” as they proceeded to give an alternate construction based on the metaphor of a generalised Pólya urn scheme. Adopting the one approach is not to deny or diminish the other, but to bring attention to the diversity of thinking in probability theory, even when the end result is the same mathematical object. We look upon this as healthy complementarity rather than undesirable contestation.

We discuss complete monotonicity by methods of probability theory in the sense of Bayesian reasoning. For the purpose at hand, we have no need to invoke an underlying random experiment or indeed an explicit random variable, while not denying the latter as an alternative probabilistic approach. Hence, for example, we shall continue to express the Laplace transform of a distribution as an explicit integral rather than as an expectation $E[e^{-sX}]$ for a random variable $X$.

## 2 A Bayesian Approach

First, we note that the scale change $s \to t^{1/\alpha}s$ ($t > 0$) in (7) gives

$$e^{-ts^\alpha} = \int_0^\infty e^{-sx} f_\alpha(x t^{-1/\alpha}) t^{-1/\alpha} dx \equiv \int_0^\infty e^{-sx} f_\alpha(x|t) dx$$

(17)

where $f_\alpha(x|t) \equiv f_\alpha(x t^{-1/\alpha}) t^{-1/\alpha}$ is the stable density conditioned on the scale parameter $t$, with $f_\alpha(x) \equiv f_\alpha(x|1)$. Correspondingly, the stable distribution conditioned on $t$ is

$$F_\alpha(x|t) = \int_0^x f_\alpha(u|t) du = \int_0^{xt^{-1/\alpha}} f_\alpha(u) du \equiv F_\alpha(x t^{-1/\alpha})$$

(18)

with Laplace transform $e^{-ts^\alpha}/s$.

We then assign a distribution $G(t)$ to the scale parameter $t$ of $F_\alpha(x|t)$. Then, by the sum and product rules of probability theory, the unconditional or marginal distribution $M_\alpha(x)$ over $x$ is

$$M_\alpha(x) = \int_0^\infty F_\alpha(x|t)dG(t)$$

(19)

with Laplace transform

$$\int_0^\infty e^{-sx} M_\alpha(x) dx = \frac{1}{s} \int_0^\infty e^{-ts^\alpha} dG(t)$$

(20)

$M_\alpha$ is also referred to as a mixture distribution, arising from randomising or mixing the parameter $t$ in $F_\alpha(x|t)$ with $G(t)$. This has the same import as saying that we assign a prior distribution $G(t)$ on $t$ and we shall continue to use the latter language.

$G$ may depend on one or more parameters. A notable example is the gamma distribution $G(\mu, \lambda)$ with shape and scale parameters $\mu > 0, \lambda > 0$ respectively:

$$dG(t|\mu, \lambda) = \frac{\lambda^\mu}{\Gamma(\mu)} t^{\mu-1} e^{-\lambda t} dt$$

(21)
λ is not fundamental and may be set to λ = 1 by change of scale t → λt, while μ controls the shape of G(t|μ, λ). The marginal (19) becomes Mα(x|μ, λ), with Laplace transform

$$\int_0^\infty e^{-sx}M_{\alpha}(x|\mu, \lambda) \, dx = \frac{1}{s} \left( \frac{\lambda}{\lambda + s^\alpha} \right)^{\mu} = \frac{1}{s} \left( 1 - \frac{s^\alpha}{\lambda + s^\alpha} \right)^{\mu}$$  \hspace{1cm} (22)

We may now state Feller’s approach from a Bayesian perspective.

2.1 A Bayesian View of Feller’s Approach

The case μ = 1 in (21) gives the exponential distribution dG(t|λ) = λe^{-λt} dt. Then Mα(x|λ) ≡ Mα(x|μ, λ = 1), is

$$M_{\alpha}(x|\lambda) = \int_0^\infty F_{\alpha}(x|t)dG(t|\lambda) = \lambda \int_0^\infty F_{\alpha}(x|t)e^{-\lambda t} \, dt \hspace{1cm} (23)$$

The Laplace transform of Mα(x|λ), read from (22) with μ = 1, is

$$\int_0^\infty e^{-sx}M_{\alpha}(x|\lambda) \, dx = \frac{1}{s} - \frac{s^{\alpha-1}}{\lambda + s^\alpha}$$  \hspace{1cm} (24)

$$\Rightarrow M_{\alpha}(x|\lambda) = 1 - E_{\alpha}(-\lambda x^\alpha)$$  \hspace{1cm} (25)

$$\Rightarrow E_{\alpha}(-\lambda x^\alpha) = 1 - M_{\alpha}(x|\lambda) = \lambda \int_0^\infty (1 - F_{\alpha}(x|t))e^{-\lambda t} \, dt$$  \hspace{1cm} (26)

This reproduces Feller’s result (13) from a Bayesian perspective. The difference is purely a matter of conceptual outlook:

Feller: Study the two-dimensional Laplace transform of the bivariate distribution 1 - Fα(xt^{-1/α}), where Fα is the stable distribution. Deduce that Eα(-λx^α)/λ is the Laplace transform of 1 - Fα(xt^{-1/α}) over t at fixed x, where λ is the Laplace variable.

Bayes: Assign an exponential prior distribution G(t|1, λ) to the scale factor t of Fα(x|t) ≡ Fα(xt^{-1/α}), where G(t|μ, λ) is the gamma distribution. Marginalise over t to generate the Feller result directly.

Feller himself might also have established the result by the latter reasoning. Under subordination of processes [6](p451), he discussed mixture distributions but he did not specifically discuss the Mittag-Leffer function in this context in his published work. The task fell on Pillai [14] to study Mα(x|μ) ≡ Mα(x|μ, λ = 1), including its infinite divisibility and the corresponding Mittag-Leffer stochastic process. He also proved that Mα(x|1) = 1 - Eα(-x^α) (as discussed above), which he referred to as the Mittag-Leffer distribution. There are thus two distributions bearing the name “Mittag-Leffer distribution”: Mα(x) = 1 - Eα(-x^α) and Pα(t) = 1 - Fα(t^{-1/α}). We shall use the term to refer to the latter distribution in the balance of our discussion.

2.2 A Bayesian Generalisation

The natural question arising from the Bayesian approach is whether there might be other choices of μ in G(μ, λ) (or indeed other choices of G altogether) that yield the Pollard result and, if so, what insight they might offer. At face value, there would appear to be nothing further to be
said since other choices of $\mu$ can be expected to lead to different results, beyond the study of the Mittag-Leffler function.

While (21) is not defined for $\mu = 0$, we note that (with $\mu \Gamma(\mu) = \Gamma(\mu + 1)$):

$$\frac{1}{\mu}dG(t|\mu, \lambda) = \frac{\lambda^\mu}{\Gamma(\mu + 1)} t^{\mu-1} e^{-\lambda t} dt$$  \hspace{1cm} (27)

$$\implies \lim_{\mu \to 0} \frac{1}{\mu}dG(t|\mu, \lambda) = t^{-1} e^{-\lambda t} dt$$  \hspace{1cm} (28)

Against our own expectation, we have discovered that this “$\mu = 0$” case (i.e. the unnormalised distribution with density $t^{-1} e^{-\lambda t}$) generates a novel integral representation of the Mittag-Leffler function. This is the main result of this paper, which we state next. We follow with a discussion of the Bayesian reasoning that led to the discovery and the generalisation that arises from that.

3 Main Contribution

**Proposition 1.** The Mittag-Leffler function $E_{\alpha}(-\lambda x^\alpha) \ (x \geq 0, \lambda > 0)$ has the integral representation

$$\alpha E_{\alpha}(-\lambda x^\alpha) = x \int_0^\infty f_\alpha(x|t) t^{-1} e^{-\lambda t} dt \quad 0 < \alpha < 1$$  \hspace{1cm} (29)

where $f_\alpha(x|t)$ is the stable density with Laplace transform $e^{-ts^\alpha}$. This leads to the Pollard result

$$E_{\alpha}(-x) = \frac{1}{\alpha} \int_0^\infty f_\alpha(u^{-1/\alpha}) u^{-1/\alpha-1} e^{-xu} du$$  \hspace{1cm} (30)

Thus $E_{\alpha}(-x)$ is completely monotone.

**Proof of Proposition 1.** The Laplace transform of the RHS of (29) is

$$\int_0^\infty e^{-sx} x \int_0^\infty f_\alpha(x|t) t^{-1} e^{-\lambda t} dt dx$$

$$= -\frac{d}{ds} \int_0^\infty t^{-1} e^{-\lambda t} \int_0^\infty e^{-sx} f_\alpha(x|t) dx dt$$

$$= -\frac{d}{ds} \int_0^\infty t^{-1} e^{-\lambda t} e^{-ts^\alpha} dt$$

$$= \alpha s^{\alpha-1} \int_0^\infty e^{-(\lambda + s^\alpha)t} dt$$

$$= \frac{\alpha s^{\alpha-1}}{\lambda + s^\alpha}$$

which is the Laplace transform of the LHS $\alpha E_{\alpha}(-\lambda x^\alpha)$ of (29). The expression (30) for $E_{\alpha}(-x)$ follows from simple substitution

$$\alpha E_{\alpha}(-x) = x^{1/\alpha} \int_0^\infty f_\alpha(x^{1/\alpha}|t) t^{-1} e^{-t} dt$$

$$= x^{1/\alpha} \int_0^\infty f_\alpha(x^{1/\alpha} t^{-1/\alpha}) t^{-1/\alpha-1} e^{-t} dt$$

$$= \int_0^\infty f_\alpha(u^{-1/\alpha}) u^{-1/\alpha-1} e^{-xu} du$$

which is the Pollard result. \qed
Since Pollard’s result follows almost trivially from (29), the pertinent question is where does this integral representation come from in the first place? Once again, if we were to accept it at face value as, perhaps, a fortunate guess (it does not take a particularly subtle form after all) then there would be nothing further to be said.

Pursuing further, we observe that $t^{-1}e^{-\lambda t}$ is the Lévy density of the infinitely divisible gamma distribution. There is indeed an intimate relationship between completely monotone functions and the theory of infinitely divisible distributions on the nonnegative half-line $\mathbb{R}_+ = [0, \infty)$. This is a topic well-studied by Feller [6]. In the balance of this paper, we shall turn to this topic. But first, we discuss a generalisation of the Pollard result that follows from a variant of (29).

## 4 Generalisation of the Pollard Result

As mentioned in Section 1.5, $P_\alpha$ of (9) is known as the Mittag-Leffler distribution in probabilistic literature. There is a two-parameter generalisation known as the generalised Mittag-Leffler distribution $P_{\alpha,\theta}$ (Pitman [15], p70 (3.27)), also denoted by ML($\alpha$, $\theta$) (Goldschmidt and Haas [8], Ho et al. [10]). It may be written as

$$P_{\alpha,\theta}(t) = \frac{\Gamma(\theta + 1)}{\Gamma(\theta/\alpha + 1)} \int_0^t u^{\theta/\alpha} dP_\alpha(u) \quad \theta > -\alpha$$  \hspace{1cm} (31)

$P_\alpha$ is the $\theta = 0$ case: $P_\alpha \equiv P_{\alpha,0}$. Janson [12] showed that $P_{\alpha,\theta}$ may be constructed as a limiting distribution of a Pólya urn scheme. In a concise description of this construction, Goldschmidt and Haas [8] observed that “generalised Mittag-Leffler distributions arise naturally in the context of urn models”. We show that the generalised Mittag-Leffler distribution and its Laplace transform also arise naturally in the context of Bayesian reasoning.

Given the stable density $f_\alpha(x)$ ($0 < \alpha < 1$), the two-parameter density $f_{\alpha,\theta}(x) \propto x^{-\theta}f_\alpha(x)$ for some parameter $\theta$ (range discussed below) is said to be a ‘polynomially tilted’ variant of $f_\alpha(x)$ (e.g. Arbel et al. [1], Devroye [5], James [11]). Inspired by this, we consider the polynomially tilted density $f_{\alpha,\theta}(x|t) \propto x^{-\theta}f_\alpha(x|t)$ conditioned on a general scale factor $t > 0$, where $f_\alpha(x) \equiv f_\alpha(x|t = 1)$. The normalised tilted density conditioned on $t$ is

$$f_{\alpha,\theta}(x|t) = C_{\alpha,\theta}(t) x^{-\theta}f_\alpha(x|t) \quad \text{where} \quad C_{\alpha,\theta}(t) = \frac{\Gamma(\theta + 1)}{\Gamma(\theta/\alpha + 1)} t^{\theta/\alpha}$$ \hspace{1cm} (32)

so that $f_{\alpha,\theta}(x|t)$ is defined for $\theta/\alpha + 1 > 0$, or $\theta > -\alpha$.

The key idea here is to tilt the conditional stable density $f_\alpha(x|t)$ and assign a prior distribution to $t$ rather than merely tilt $f_\alpha(x) \equiv f_\alpha(x|t = 1)$. We consider a variant of (29) with the prior distribution $t^{-1}e^{-\lambda t}dt$ and $f_\alpha(x|t)$ replaced by $f_{\alpha,\theta}(x|t)$. This induces a corresponding two-parameter function $h_{\alpha,\theta}(x|\lambda)$ in place of $E_\alpha(-\lambda x^{\alpha})$, for which the following holds:

**Proposition 2.** Let $h_{\alpha,\theta}(x|\lambda)$ be defined by

$$\alpha h_{\alpha,\theta}(x|\lambda) = x \int_0^\infty f_{\alpha,\theta}(x|t) t^{-1}e^{-\lambda t} dt$$  \hspace{1cm} (33)

$$= \frac{\Gamma(\theta + 1)}{\Gamma(\theta/\alpha + 1)} \int_0^\infty f_\alpha(x|t) t^{\theta/\alpha - 1} e^{-\lambda t} dt$$ \hspace{1cm} (34)

Then $h_{\alpha,\theta}(x^{1/\alpha}) \equiv h_{\alpha,\theta}(x^{1/\alpha}|\lambda = 1)$ is completely monotone with

$$h_{\alpha,\theta}(x^{1/\alpha}) = \int_0^\infty e^{-xt}dP_{\alpha,\theta}(t)$$  \hspace{1cm} (35)
Proof of Proposition 2.

\[ \alpha h_{\alpha,\theta}(x^{1/\alpha}) = \frac{\Gamma(\theta + 1)}{\Gamma(\theta/\alpha + 1)} x^{1/\alpha} \frac{1}{\alpha} \int_0^\infty f_\alpha(x^{1/\alpha}|t) t^{\theta/\alpha-1} e^{-t} dt \]

The change of variable \( t \to xt \) leads to

\[ h_{\alpha,\theta}(x^{1/\alpha}) = \frac{\Gamma(\theta + 1)}{\Gamma(\theta/\alpha + 1)} \int_0^\infty e^{-xt} t^{\theta/\alpha} \left( \frac{1}{\alpha} f_\alpha(t^{1/\alpha}) t^{-1/\alpha-1} \right) dt \]

\[ = \frac{\Gamma(\theta + 1)}{\Gamma(\theta/\alpha + 1)} \int_0^\infty e^{-xt} t^{\theta/\alpha} dP_\alpha(t) \]

Hence \( h_{\alpha,\theta}(x^{1/\alpha}) \) is completely monotone. This generalises the Pollard result, which is the particular case \( \theta = 0 \): \( P_{\alpha,0}(t) = P_\alpha(t) \implies h_{\alpha,0}(x^{1/\alpha}) = E_\alpha(-x) \). \( \square \)

It would be consistent with the foregoing discussion to refer to \( h_{\alpha,\theta}(x^{1/\alpha}) \), as the generalised (two-parameter) Mittag-Leffler function. However, we note that there already exists a two-parameter generalised Mittag-Leffler function defined by

\[ E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} \]

where \( E_\alpha(x) \equiv E_{\alpha,1}(x) \).

This completes the discussion of the primary contribution of this paper – a Bayesian perspective on the complete monotonicity of the Mittag-Leffler function and its generalisation. We now turn to the topic of infinitely divisible distributions on \( \mathbb{R}_+ \), from which Proposition 1 arises.

5 Infinitely Divisible Distributions on \( \mathbb{R}_+ \)

Infinitely divisible distributions on \( \mathbb{R}_+ \) are covered in Feller [6] (XIII.4, XIII.7) as well as Steutel and van Harn (SvH) [20] (III). The topic has an intimate relationship with completely monotone functions, as discussed in both texts. Another relevant text in this context is Schilling et al. [19] on Bernstein functions. Sato [18] considers infinitely divisible distributions on \( \mathbb{R}^d \), but the deliberate restriction to \( \mathbb{R}_+ \) makes for simpler discussion and relates directly to the core concept of complete monotonicity that is of interest here. Nonetheless, we shall refer to Sato as appropriate.

A contribution of this paper is the representation of infinite divisibility as a commutative diagram, which readily leads to a limit relation enabling the direct generation of a Lévy measure from its associated infinitely divisible distribution. Although the diagrammatic motivation is new, the limit relation is mentioned in Steutel and van Harn [20] (III) and it is conceptually equivalent to that due to Sato [18]. It is the limit relation that leads to the representation (29) from which the Pollard result follows.
**Definition 1.** A probability distribution with Laplace transform $\varphi$ is infinitely divisible (ID) iff for $n > 0$, the positive $n$th root $\varphi^{1/n}$ is also the Laplace transform of a probability distribution.

**Theorem 1** (Feller [6], XIII.7, p450). The function $\varphi$ is the Laplace transform of an infinitely divisible probability distribution iff it takes the form $\varphi = e^{-\psi}$ where $\psi$ has a completely monotone derivative $\psi'$ and $\psi(0) = 0$.

Proof of Theorem 1. See Feller [6], XIII.7, p450. □

Infinite divisibility on $\mathbb{R}_+$ thus amounts to the study of the function $\psi$. We introduce a scale parameter $\mu > 0$ so that $\varphi(s) \to \varphi(s|\mu) = e^{-\mu\psi}$ is the Laplace transform of an infinitely divisible probability density $f(x|\mu)$. Correspondingly, $\varphi(s|\mu)^{1/n} = e^{-\mu\psi(s)/n} = \varphi(s|\frac{\mu}{n})$ is the Laplace transform of $f(x|\frac{\mu}{n})$. Since $\psi'$ is completely monotone, it is the Laplace transform of a density $r(x)$, say (which need not be normalisable – i.e. unlike $\psi(0)$, $\psi'(0)$ need not be finite). $\ell(x) = r(x)/x$ is known as the density of the Lévy measure $L(dx) = \ell(x)dx$, or simply the Lévy density. We shall also refer to the measures $R(dx) = r(x)dx = xL(dx)$ and $F(dx|\mu) = f(x|\mu)dx$.

We find it helpful to represent ID objects and relationships amongst them as a commutative diagram. If we seek the density $r(x)$ of a given ID density $f(x|\mu)$, we may proceed as illustrated in the upper diagram of (36) (where $\mathcal{L}$ denotes the Laplace transform):

1. take the Laplace transform $\varphi(s|\mu) = e^{-\mu\psi(s)}$ of $f(x|\mu)$
2. take (minus) the logarithmic derivative of $\varphi(s|\mu)$ to obtain $\mu\psi'(s)$
3. evaluate the inverse Laplace transform of $\psi'(s)$ to obtain $r(x) = x\ell(x)$

\[
\begin{align*}
\begin{array}{c}
\text{\quad $f(x|\mu)$} \\
\quad \mathcal{L} \\
\quad \downarrow \\
\quad \mu r(x) \\
\quad \mathcal{L}^{-1} \\
\quad \mu \psi'(s)
\end{array}
\quad \begin{array}{c}
\text{\quad $\varphi(s|\mu) = e^{-\mu\psi(s)}$} \\
\quad \downarrow \\
\quad \mu \psi'(s) \\
\quad \downarrow \\
\quad \mu \psi'(s)
\end{array}
\end{align*}
\]

(36)

The natural question suggested by the upper diagram is whether we can find an equivalent direct transition from $f(x|\mu)$ to $\mu r(x)$. The answer is affirmative, as formalised in the following:

**Corollary 1.1.** Let $F(dx|\mu) = f(x|\mu)dx$ be infinitely divisible with Lévy measure $L(dx) = R(dx)/x$ with density $\ell(x) = r(x)/x$. Then the following holds:

\[
\begin{align*}
\mu \ell(x) = \lim_{n \to \infty} n f(x|\frac{\mu}{n}) & \quad \text{or} \quad \mu L(dx) = \lim_{n \to \infty} n F(dx|\frac{\mu}{n}) \quad (37) \\
\mu r(x) = \lim_{n \to \infty} n x f(x|\frac{\mu}{n}) & \quad \text{or} \quad \mu R(dx) = \lim_{n \to \infty} n x F(dx|\frac{\mu}{n}) \quad (38)
\end{align*}
\]
Proof of Corollary 1.1. By Theorem 1, \( f(x|\mu) \) has Laplace transform \( \varphi(s|\mu) = e^{-\mu \psi(s)} \) so that \( \varphi(s|\mu = 0) = 1 \). Hence

\[
- \lim_{n \to \infty} n \varphi'(s|\frac{\mu}{n}) = \lim_{n \to \infty} \mu \psi'(s) \varphi(s|\frac{\mu}{n}) = \mu \psi'(s) \varphi(s|0) = \mu \psi'(s)
\]

(39)

Since \(-\varphi'(s|\mu)\) is the Laplace transform of \( xf(x|\mu)\), it follows that

\[
\mu r(x) = \lim_{n \to \infty} n x f(x|\frac{\mu}{n}) \implies \mu \ell(x) = \lim_{n \to \infty} n f(x|\frac{\mu}{n})
\]

(40)

This is invariant under scaling by \( C > 0: f(x|\mu) \to Cf(x|\mu) \).

The lower diagram of (36) is the desired commutative diagram. To be clear, Corollary 1.1 is known. To aid comparison with the literature, Corollary 1.1 implies that, given a function \( h \) on \( \mathbb{R}^+ \) (and finite \( x \))

\[
\int_0^x h(u) R(du) = \lim_{n \to \infty} \frac{n}{\mu} \int_0^x u h(u) F(du|\frac{\mu}{n})
\]

(41)

\[
\int_0^x h(u) L(du) = \lim_{n \to \infty} \frac{n}{\mu} \int_0^x h(u) F(du|\frac{\mu}{n})
\]

(42)

The relation in SvH [20] (III(4.7)) is a particular case of (41). Also, Sato [18] (Corollary 8.9) proved the limit relation

\[
\int_{\mathbb{R}^d} h(x) L(dx) = \lim_{t \to 0} t^{-1} \int_{\mathbb{R}^d} h(x) F(dx|t)
\]

(43)

for suitably behaved \( h(x) \). Choosing \( \mathbb{R}^+ \) instead of \( \mathbb{R}^d \) in Sato’s relation (43) and setting \( t = \mu/n \) reproduces (42) where \( x = \infty \) is allowable. However, working in \( \mathbb{R}^+ \) from the outset makes for much simpler discussion of infinitely divisible distributions on \( \mathbb{R}^+ \) relative to working in \( \mathbb{R}^d \) and then trying to infer behaviour on \( \mathbb{R}^+ \) as a special case.

The contribution here is the intuitive manner in which the limit relation arises from a commutative diagram argument. Furthermore, the limiting rule from \( f(x|\mu) \) to \( \mu r(x) \) stated here explicitly preserves the scale factor \( \mu \), in keeping with the indirect route via the Laplace transform, which does not actually require the evaluation of a limit.

Aside from SvH and Sato, there appears to be limited discussion of inferring the Lévy measure or its properties directly from the corresponding infinitely divisible distribution. For instance, Barndorff-Nielsen and Hubalek [2] cited Sato’s relation at the start before turning to “the opposite problem, that of calculating \( F(dx|t) \) from \( L(dx) \)”.

Since the direct route of Corollary 1.1 and the indirect Laplace route both lead from \( f(x|\mu) \) to the same object \( \mu r(x) \), the natural question is whether Corollary 1.1 is of much practical value. The answer is that the two routes can lead to different representations of the same object \( r(x) \). Therein lies the practical value of Corollary 1.1. To that end, we first introduce some additional properties of complete monotonicity that we shall need.

6 More on Complete Monotonicity

We summarise additional properties of completely monotone functions, as covered in Feller [6], XIII, except for the proofs given here.
Proposition 3. If \( \varphi \) and \( \vartheta \) are completely monotone, so is their product \( \varphi \vartheta \)

Proof of Proposition 3. See Feller [6], XIII.4, p441. Alternatively, being completely monotone, \( \varphi \) and \( \vartheta \) are Laplace transforms of densities. The product \( \varphi \vartheta \) is thus the Laplace transform of the convolution of said densities, which is also a density. Therefore \( \varphi \vartheta \) is completely monotone.

Proposition 4. If \( \varphi \) is completely monotone and \( \eta \) is a positive function with a completely monotone derivative, \( \varphi(\eta) \) is completely monotone

Proof of Proposition 4. If \( \varphi(s) \) is completely monotone, so is \( -\varphi'(s) \). Consider \( \varphi(\eta) \) where \( \eta(s) > 0 \) and \( \eta'(s) \) is completely monotone. Of necessity, \( \varphi(\eta) > 0 \) and

\[
-\varphi'(\eta) = \left(-\frac{d\varphi(\eta)}{d\eta}\right) \eta'(s)
\] (44)

The RHS is a product of two completely monotone functions. Therefore, by Proposition 3, \( -\varphi'(\eta) \) is completely monotone. This, along with \( \varphi(\eta) > 0 \), completes the proof that \( \varphi(\eta) \) is completely monotone.

Proposition 5. If \( \varphi \) is the Laplace transform of an infinitely divisible distribution and \( \eta \) is a positive function with a completely monotone derivative, \( \varphi(\eta) \) is also the Laplace transform of an infinitely divisible distribution.

Proof of Proposition 5. By Theorem 1, \( \varphi \) is the Laplace transform of an infinitely divisible distribution iff \( \varphi = e^{-\psi} \) where \( \psi' \) is completely monotone. By Proposition 4, if \( \psi' \) is completely monotone and \( \eta \) is a positive function with a completely monotone derivative, \( \psi'(\eta) \) is completely monotone. Hence, for such \( \eta \), \( \varphi(\eta) = e^{-\psi(\eta)} \) is the Laplace transform of an infinitely divisible distribution.

We may now revisit the Bayesian formulation of Section 2.

7 Bayesian Approach Revisited

Theorem 2. Let \( m(x|\mu) \), \( f(x|y) \) and \( g(y|\mu) \) be densities on \([0, \infty)\) such that

\[
m(x|\mu) = \int_0^\infty f(x|y) g(y|\mu) \, dy
\] (45)

If \( f(x|y) \) and \( g(y|\mu) \) are infinitely divisible, with Laplace transforms \( e^{-\eta y(s)} \) and \( e^{-\mu \psi(s)} \) respectively, where \( \eta'(s) \) and \( \psi'(s) \) are completely monotone, then:

1. \( m(x|\mu) \) is also infinitely divisible
2. the Lévy density \( \xi(x) \), say, of \( m(x|\mu) \) is

\[
\xi(x) = \int_0^\infty f(x|y) \ell(y) \, dy
\] (46)

where \( \ell(y) \) is the Lévy density of \( g(y|\mu) \).
Feller [6] (p451) discussed the first part of this theorem in an example on subordination of processes. One may also refer to SvH [20] VI(Proposition 2.1). The additional contribution here is the integral representation of the Lévy density of \( m(x|\mu) \) in the second part of the theorem.

**Proof of Theorem 2.** Since \( f(x|y) \) and \( g(y|\mu) \) are infinitely divisible, by Theorem 1, their Laplace transforms take the form \( e^{-\eta(s)} \) and \( e^{-\mu\psi(s)} \) respectively, where \( \eta'(s) \) and \( \psi'(s) \) are completely monotone. In turn, \( e^{-\mu\psi(\eta(s))} \) is the Laplace transform of \( m(x|\mu) \) induced by (45). Hence, by Proposition 5, \( m(x|\mu) \) is infinitely divisible.

By Corollary 1.1 combined with (45), the density \( \rho(x) = x\xi(x) \), where \( \xi(x) \) is the Lévy density of \( m(x|\mu) \), is given by the limit

\[
\mu \rho(x) = \lim_{n \to \infty} nx m(x|y_n) = x \int_0^\infty f(x|y) \lim_{n \to \infty} n g(y|\mu) \, dy
\]

\[
= \mu x \int_0^\infty f(x|y) \ell(y) \, dy
\]

\[
\Rightarrow \frac{\rho(x)}{x} \equiv \xi(x) = \int_0^\infty f(x|y) \ell(y) \, dy
\]

where \( \ell(y) = r(y)/y \) is the Lévy density of \( g(y|\mu) \).

Theorem 2 holds for any pair of infinitely divisible densities \( (f, g) \). We now turn to a particular choice of \( (f, g) \).

### 7.1 Stable/Gamma Case

Let \( f(x|y) \) be the stable density \( f_\alpha(x|y) \) for \( 0 < \alpha < 1 \) and \( g(y|\mu) \) the gamma density to give \( m_\alpha(x|\mu, \lambda) \):

\[
m_\alpha(x|\mu, \lambda) = \frac{\lambda^{\mu}}{\Gamma(\mu)} \int_0^\infty f_\alpha(x|y) y^{\mu-1} e^{-\lambda y} \, dy
\]

(49)

With \( \eta(s) = s^\alpha \), \( \psi(s) = \log(1+s/\lambda) \), the commutative diagram of \( m_\alpha(x|\mu, \lambda) \) is:

\[
m_\alpha(x|\mu, \lambda) \xrightarrow{\mu\alpha E_\alpha(-\lambda x^\alpha)} \frac{\lambda^{\mu}}{\Gamma(\mu)} \int_0^\infty f_\alpha(x|y) y^{\mu-1} e^{-\lambda y} \, dy
\]

We recognise \( s^\alpha/(\lambda + s^\alpha) \) as the Laplace transform of the Mittag-Leffler function \( E_\alpha(-\lambda x^\alpha) \). Hence the entry \( \mu\alpha E_\alpha(-\lambda x^\alpha) \) in the bottom left corner, which is arrived at by following the path of the commutative diagram involving Laplace transforms. The equivalent, direct path from top left to bottom left corner is given by (46) in Theorem 2, where \( \ell(y) = y^{-1} e^{-\lambda y} \) is the Lévy density of the gamma distribution and, by inspection, \( \xi(x) = \alpha E_\alpha(-\lambda x^\alpha)/x \). Hence (46) becomes

\[
\alpha E_\alpha(-\lambda x^\alpha) = x \int_0^\infty f_\alpha(x|y) y^{-1} e^{-\lambda y} \, dy
\]

which is precisely the assertion of Proposition 1 that we sought to justify by appeal to infinite divisibility.
8 Conclusion

Pollard proved the complete monotonicity of the Mittag-Leffler function using methods of complex analysis. He also cited personal communication by Feller that he had discovered a proof based on “methods of probability theory”. In his published work, Feller derived the result using the the two-dimensional Laplace transform of a bivariate distribution involving the stable distribution on a positive variable. As published, both Pollard’s and Feller’s approaches are actually analytic rather than probabilistic, despite the stable distribution appearing in Feller’s approach and in the Pollard result itself.

In this paper, we adopted Bayesian reasoning as the fundamental probabilistic approach to the problem. We assigned a prior distribution to the scale factor of the stable distribution. In particular, we discussed the assignment of a gamma distribution. The special case of the exponential prior distribution reproduced the Feller result with ease.

Importantly, we discovered a novel integral representation of the Mittag-Leffler distribution. With the aid of the polynomially tilted stable density, we proceeded to prove the complete monotonicity of a generalised Mittag-Leffler function by establishing that it is the Laplace transform of the generalised Mittag-Leffler distribution, thereby generalising the Pollard result.

The novel integral representation arises from choosing the Lévy measure of the infinitely divisible gamma distribution as a prior distribution. Accordingly, we presented a discussion of infinite divisibility on the positive half-line, which led to the discovery. In this context, we have found it helpful to invoke a commutative diagram representation of infinite divisibility.

On a philosophical note, we have taken “methods of probability theory” to refer to Bayesian reasoning, placing an accent on distributions and the sum and product rules of probability theory. This is by no means to dismiss an alternative approach based on products and powers of specified random variables rather than direct assignment of distributions. We trust that the Bayesian view will nonetheless find appeal amongst both probabilists, to whom random variables are often the staple, and physicists, who routinely take an analytic view in the study of Mittag-Leffler functions without invoking an underlying random experiment or phenomenon.

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