Note on finite temperature sum rules for vector and axial-vector spectral functions *

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Abstract

An updated analysis of vector and axial-vector spectral functions is presented. The resonant contributions to the spectral integrals are shown to be expressible as multiples of $4\pi^2 f^2_\pi$, encoding the scale of spontaneous chiral symmetry breaking in QCD. Up to order $T^2$ this behavior carries over to the case of finite temperature.

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Quantum Chromodynamics (QCD) has an approximate chiral symmetry which is spontaneously broken. This Nambu-Goldstone realization of chiral symmetry manifests itself through the presence of a quark condensate \( \langle \bar{q}q \rangle \), pseudoscalar Goldstone bosons and a non-vanishing pion decay constant, \( f_\pi \simeq 92.4 \text{ MeV} \), which sets the characteristic chiral “gap” scale, \( 4\pi f_\pi \simeq 1 \text{ GeV} \). The symmetry breaking pattern is also evident in the difference between the spectral functions of vector \((J^{PC} = 1^{--})\) and axial-vector \((J^{PC} = 1^{++})\) channels. These spectral functions would be identical if chiral symmetry were realized in the (unbroken) Wigner-Weyl mode.

At high temperatures, \( T > T_c \simeq 0.2 \text{ GeV} \), chiral symmetry is expected to be restored. The vector and axial-vector channels should become degenerate as \( T \) approaches \( T_c \). The leading temperature dependence of the corresponding current correlators has been derived in refs. [1, 2]. The result is that vector and axial-vector modes mix at finite \( T \) as a consequence of their couplings to the pionic heat bath (see also ref. [3]), but that their masses remain unchanged to leading order, \( O(T^2) \).

The \( T = 0 \) vector \((V)\) and axial-vector \((A)\) spectral functions have recently been determined with high accuracy up to invariant masses \( \sqrt{s} = m_\tau \simeq 1.78 \text{ GeV} \) (the tau lepton mass) by the ALEPH [4] and OPAL [5] collaborations. These data serve as a reliable basis for all sorts of sum rule analysis (see e.g. ref. [6]), at least for the lowest moments of the \( V \) and \( A \) spectral distributions.

The purpose of our present note is twofold. First we follow up on a previous study [7], using finite energy sum rules (FESR), in which the chiral “gap” \( 4\pi f_\pi \) was suggested as the dominant scale governing the spectral sum rules (see also ref. [8]). We give an interpretation of how this scale is encoded in the pattern of the measured \( V \) and \( A \) spectral distributions and (as a reminder) in the Weinberg \((V - A)\) sum rules. Second we study the case of finite temperature \( T \). The change of the chiral “gap” with temperature is determined by the \( T \) dependence of the pion decay constant (in fact, the one related to the time component of the axial current), as given in ref. [12]. We investigate spectral sum rules for the \( T \) dependent \( V \) and \( A \) correlators and examine their consistency with the Eletsky-Ioffe result [1, 2] to order \( T^2 \).

We begin with the vector and axial-vector correlators at zero temperature. Let \( J^\mu \) be either one of the currents \( V^\mu = \bar{u}\gamma^\mu d \) or \( A^\mu = \bar{u}\gamma^\mu\gamma_5 d \) interpolating to the negatively charged member of the corresponding mesonic isotriplet. The correlators

\[
\Pi_{J}^{\mu\nu}(q) = i \int d^4 x e^{iqx} \langle 0 | T[J^\mu(x)J^\nu(0)]|0 \rangle
\]

(1)

can be decomposed as

\[
\Pi^{\mu\nu}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi^{(1)}(q^2) + q^\mu q^\nu \Pi^{(0)}(q^2),
\]

(2)
in terms of their spin-0 and spin-1 parts, \( \Pi^{(0)} \) and \( \Pi^{(1)} \), respectively. The spin-0
part is relevant only for the axial-vector current where it represents the induced pseudoscalar (pion pole) contribution. The $V$ and $A$ spectral functions are defined as

$$v_1(q^2) = 2\pi \text{Im} \Pi_V^{(1)}(q^2),$$  
$$a_{0,1}(q^2) = 2\pi \text{Im} \Pi_A^{(0,1)}(q^2), \quad (q^2 \equiv s \geq 0).$$

The pion pole term is

$$a_0(q^2) = 4\pi^2 f_\pi^2 \delta(q^2 - m_\pi^2).$$

Given the analytical properties of $\Pi_V^{(1)}(s)$ and $\Pi_A^{(0)}(s) + \Pi_A^{(1)}(s)$ in the complex $s$-plane, we now write FESR relations for their spectral functions \[7, 13\]. Assume that the spectral continuum is described in terms of perturbation theory at $s \geq s_c$ and that only the first few mass dimensions are relevant in the operator product expansion (OPE) of the current correlators evaluated at $|q^2| = s_c$. Choose a closed path which surrounds the cut along the real $s$-axis and joins a circle of radius $s_c$. The Cauchy integral around this closed path includes the integration along the circle which can be estimated using the OPE. By means of the optical theorem the integral along the positive, real axis is evaluated using experimental cross sections directly or by considering a model motivated by observation. For details see ref. \[7\]. As far as quark mass corrections in the OPE (mass perturbation in the perturbative part and the term involving the quark condensate) are concerned, we neglect them, which is justified by their numerical smallness\[†\].

A compendium of the OPE for vector and axial-vector correlators can be found in ref. \[14\]. As a result of this procedure one finds the following sum rules for the lowest moments of $v_1$ and $a_0 + a_1$:

$$\int_0^{s_c} ds \, v_1(s) = \int_0^{s_c} ds \, [a_0(s) + a_1(s)] = \frac{s_c}{2} (1 + \delta_0),$$

with the pion pole contribution

$$\int_0^{s_c} ds \, a_0(s) = 4\pi^2 f_\pi^2,$$

and

\[\text{If } \sqrt{s_c} \gg \lambda^{-1}, \text{ where } \lambda \text{ denotes a typical correlation length characterizing gauge invariant two point functions, which generalizes the local condensates, then there are no oscillations in the Minkowski-like and no exponential suppression in the euclidean-like domains } \[3\].\]

\[\text{At } T > 0 \text{ there are to order } T^2 \text{ corrections to the gluon condensate and contributions from O(3) invariant operators proportional to } m_\pi^2 [10] \text{ which hence vanish in the chiral limit. However, if the vacuum state is allowed to be affected by the heat bath (not addressed in this work) then the gluon condensate does exhibit a } T \text{ dependence } [11].\]
The right-hand sides of eqs. (6, 8) include the radiative corrections \( \delta_{0,1} \) computed in perturbative QCD. Their explicit expressions up to order \( \alpha_s^3 \) are given in ref. [7]. The first moments (8) introduce the dimension 4 gluon condensate \( \langle (\alpha_s/\pi)G^2 \rangle_{\mu<\sqrt{s_c}} \simeq (0.36 \text{ GeV})^4 \). Higher moments involve condensates of correspondingly higher dimensions and will not be considered here. The Weinberg sum rules (WSR) [15] follow immediately:

\[
\int_0^{s_c} ds \ [v_1(s) - a_1(s)] = 4\pi^2 f_\pi^2, \tag{9}
\]

\[
\int_0^{s_c} ds \ s [v_1(s) - a_1(s)] = 0 \tag{10}
\]

where the limit \( s_c \to \infty \) can be taken since the high-energy continuum parts of \( v_1 \) and \( a_1 \) are identical. In practice this asymptotic behaviour is reached at \( s_c \simeq 5 \text{ GeV}^2 \).

Before turning to the temperature dependent spectral functions, let us introduce a model of \( v_1 \) and \( a_1 \) which explicitly involves the scale of spontaneous chiral symmetry breaking, \( 4\pi f_\pi \). We recall that in refs. [7, 8] an appropriate large-\( N_c \) representation of the vector spectrum was shown to be

\[
v_1(s) = 8\pi^2 f_\pi^2 \delta(s - m_\rho^2) + \frac{1}{2} \theta(s - s_0), \tag{11}
\]

with the \( \rho \) meson mass \( m_\rho \) and the continuum threshold \( s_0 \) both expressed in terms of the chiral scale as \( \sqrt{2m_\rho} = \sqrt{s_0} = 4\pi f_\pi \). The contribution of the \( \rho \) meson to the WSR (9) is then \( 8\pi^2 f_\pi^2 \), twice that of the pion pole (7). This is what is seen in the data [4, 5] when taking the WSR integral up to \( s \simeq 1 \text{ GeV}^2 \), covering the \( \rho \) resonance. The phenomenological bookkeeping seems to follow a pattern in which the \( n \)-pion sectors of the spectrum each contribute \( n \) units of \( 4\pi^2 f_\pi^2 \) to the spectral integral, with the \( \rho \) meson collecting the strength in the \( n = 2 \) sector, for example. This conjecture suggests a parametrization

\[
v_1(s) = 4\pi^2 f_\pi^2 [2d_\rho(s) + 4d_\rho'(s)] + \text{continuum}, \tag{12}
\]

\[
a_0(s) + a_1(s) = 4\pi^2 f_\pi^2 [d_\pi(s) + 3d_\pi(s)] + \text{continuum}, \tag{13}
\]

where the distributions \( d_n(s) \) are normalized as \( f ds d_n(s) = 1 \). In the “zero width” (large \( N_c \)) limit we have \( d_n(s) = \delta(s - m_n^2) \) with \( m_1 = m_\pi, m_2 = m_\rho = \sqrt{2} \cdot 2\pi f_\pi, m_3 = m_{a_1} = 4\pi f_\pi, m_4 = m_{a_1'} = \sqrt{2} \cdot 4\pi f_\pi \). With these masses the sum rules for the first moments, eq. (8), are satisfied. The actual, finite width resonances used in our analysis follow this scheme, but with their widths fitted to the data (see Appendix). The resulting spectral functions \( v_1(s) \) and \( a_1(s) \) are shown in Fig. 1a,b. By construction these model spectra satisfy the two WSR’s (9, 10).
The point to be emphasized is that the spectral strength in the resonance region is well described by a pattern of localized distributions, all of which integrate to even (for vector channels) or odd (for axial-vector channels) multiples of $4\pi^2 f^2_\pi$. Turning to finite temperature $T$, we will now show that this statement survives to order $T^2$.

At finite $T$ the correlators are expressed in terms of their Gibbs averages

$$\Pi^\mu_\nu(q; T) = \sum_n \frac{i \int d^4x \ e^{iqx} \langle n| T[J^\mu(x)J^\nu(0)]e^{-H/T}|n\rangle}{\sum_n \langle n|e^{-H/T}|n\rangle}.$$  \hspace{1cm} (14)

The primary effect at low $T$ is a mixing of $V$ and $A$ modes through their coupling to thermal pions. It was shown in ref. [1] that, to leading order in $T^2$,

$$\Pi^\mu_\nu_V(q; T) = (1 - \varepsilon)\Pi^\mu_\nu_V(q; T = 0) + \varepsilon \Pi^\mu_\nu_A(q; T = 0),$$  \hspace{1cm} (15)

$$\Pi^\mu_\nu_A(q; T) = (1 - \varepsilon)\Pi^\mu_\nu_A(q; T = 0) + \varepsilon \Pi^\mu_\nu_V(q; T = 0),$$  \hspace{1cm} (16)

with $\varepsilon \equiv T^2/(6f^2_\pi)$. To this order, poles in the correlators remain at their positions; only their residues change. Reducing eq. (15) to its spin-0 and spin-1 components, we derive for their thermal spectral functions $v_{0,1}(s; T) = 2\pi \text{Im} \Pi_V^{0,1}(s; T)$ and $a_{0,1}(s; T) = 2\pi \text{Im} \Pi_A^{0,1}(s; T)$:

$$v_0(s; T) = \varepsilon a_0(s; T = 0),$$

$$a_0(s; T) = (1 - \varepsilon)a_0(s; T = 0),$$

$$v_1(s; T) = v_1(s; 0) - \varepsilon[v_1(s; 0) - a_1(s; 0)],$$

$$a_1(s; T) = a_1(s; 0) + \varepsilon[v_1(s; 0) - a_1(s; 0)].$$  \hspace{1cm} (17)

Note that due to the coupling of the vector current to pions in the heat bath, the thermal $V$-correlator now receives an induced spin-0 (pion pole) contribution. The thermal $V - A$ mixing is evident in eqs. (17). Moreover, the reduction of strengths in $a_0, v_1$ and $a_1$ by the common factor $(1 - \varepsilon)$ is fully consistent with our previous discussion: to order $T^2$ the resonances again contribute to the spectral integral as multiples of $f^2_\pi(T) = f^2_\pi(1 - \frac{T^2}{6f^2_\pi})$. Reversing the argument in the case of the pion pole, the $T$ dependence of $f_\pi$ to this order was read off in Ref. [1] by defining $f^2_\pi(T)$ to be the residue at $T > 0$.

For $s \geq s_c \simeq 5 \text{ GeV}^2$ the continuum parts of the $V$ and $A$ spectra do not change with $T$, to order $T^2$: since $v_1(s) = a_1(s)$ at $T = 0$ for $s \geq s_c$, this is immediately evident from eq. (17). Considering a given invariant (note that there are thermally induced pion pole terms in the vector channel), the FESR’s for the lowest moments of the $V$ and $A$ spectral distributions are not affected to order $T^2$. For example, consider
\[
\left( 4q_\mu q_\nu - g_{\mu\nu} \right) \frac{3q^4}{3q^2} \Pi^{\mu\nu}(q^2; T) = \Pi^{(0)}(q^2; T) + \Pi^{(1)}(q^2; T),
\]

we have then

\[
\int_0^{s_c} ds \left[ v_0(s; T) + v_1(s; T) \right] = \int_0^{s_c} ds v_1(s; 0) - \varepsilon \int_0^{s_c} ds \left[ v_1(s; 0) - a_1(s; 0) - a_0(s; 0) \right].
\]

Using (4), the \( T \)-dependent part vanishes by virtue of the first Weinberg sum rule (9). An analogous statement holds for \( a_0 + a_1 \). Furthermore, it is easily seen that

\[
\int_0^{s_c} ds s \left[ v_0(s; T) + v_1(s; T) \right] = \int_0^{s_c} ds s v_1(s; 0),
\]

\[
\int_0^{s_c} ds s \left[ a_0(s; T) + a_1(s; T) \right] = \int_0^{s_c} ds s a_1(s; 0),
\]

when using the second WSR, eq. (11). To order \( T^2 \) the presence of the heat bath causes a mere redistribution of spectral strength in both the \( V \) and \( A \) sectors but such that the lowest moments of the spectral functions remain unchanged. We illustrate these features by an explicit calculation at \( T = 140 \text{ MeV} \) with the results shown in Fig. 2. It is amusing that already order \( \varepsilon \) suggests the coincidence of vector and axial-vector spectra at \( T \approx 160 \text{ MeV} \) which is not far from the critical temperature \( T_c \approx 170 \text{ MeV} \) of the chiral phase transition determined by two-flavour lattice QCD [16].

To summarize, based on the new experimental data we have shown that the resonant contributions to the spectral integrals in the \( V \) and \( A \) channels can be written as multiples of the square of the spontaneous chiral symmetry breaking scale. Up to order \( T^2 \) this pattern carries over to the case of finite temperature.

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Appendix:
To parametrize the resonances according to eqs. (12,13) we use the following functional forms for \( d_n \):

\[
d_1 \equiv d_\pi = \delta(s - m^2_\pi),
\]

\[
d_n = a_n \frac{\gamma^2_n(s)}{(s - m^2_n)^2 + \tilde{\gamma}^2_n(s)}
\]
where \( \gamma_n(s) = [b_n + c_n(s - e_n)^2] \theta [b_n + c_n(s - e_n)^2] \), with \( n = 2 \) for \( \rho \), \( n = 3 \) for \( a_1 \) and \( n = 4 \) for \( \rho' \). The parameter sets which reproduce the empirical spectra \[4, 5\] are:

\[
\begin{array}{cccccc}
\text{ } & m_n/4\pi f_\pi & a_n \text{ [GeV}^{-2}] & b_n \text{ [GeV}^2] & c_n \text{ [GeV}^{-2}] & e_n \text{ [GeV}^2] \\
\rho & 0.63 & 3.7 & 0.143 & -0.07 & 1.2 \\
a_1 & 0.96 & 0.95 & 0.75 & -0.32 & 2.0 \\
\rho' & 1.42 & 0.502 & 1.3 & -0.3 & 2.6 \\
\end{array}
\]

Note that the mass parameters \( m_n \) deviate only marginally (by less than 10% for \( \rho \) and 3% for \( a_1, \rho' \)) from the expected “large \( N_c \)” pattern (see text), and the normalization of the \( d_n(s) \) required to reproduce the empirical data is equal or close to one.
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Figure 1: Vector (a) and axial-vector (b) spectral functions as given by the parametrization $^{12,13}$ and Appendix, compared with ALEPH data $^4$ (the comparison with OPAL data $^5$ looks very similar).
Figure 2: Vector and axial-vector spectra calculated using eqs. (12,13), at temperature $T = 140$ MeV (solid lines) as compared to $T = 0$ (dashed lines).