SMALL-LARGE SUBGROUPS OF LOCALLY COMPACT
ABELIAN POLISH GROUPS

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Abstract. In [13] Rosłanowski and Shelah asked whether every locally compact non-discrete group has a null but non-meager subgroup, and conversely whether it is consistent with ZFC that in every locally compact group there are no meager but non-null subgroups. The answer is affirmative for both questions [12], however, in this paper we present a much simpler proof for the special case of Abelian Polish groups.

1. Preliminaries and notations

We state that topological groups $G$ and $H$ are isomorphic, in symbols $G \simeq H$, if there is an algebraic isomorphism, which is also a homeomorphism. Under the symbol $\leq$ we mean the subgroup relation, i.e. $H \leq G$ symbols that $H$ is a subgroup of $G$, and $H \triangleleft G$ is for normal subgroups. All topological groups are assumed to be Hausdorff. (It is known that for a locally compact Hausdorff topological group $G$, if $C \triangleleft G$ is a closed normal subgroup, then $G/C$ is locally compact and Hausdorff. Also, if $G$ is Polish then so is $G/C$).

A topological group $G$ is said to be locally compact if for each $g \in G$, there is a neighborhood $B$ of $g$, which is compact (i.e. each point has an open neighborhood which has compact closure). Since in compact Hausdorff spaces the Baire category theorem is true (i.e. no nonempty open set can be covered by countably many nowhere dense sets), it is also true in locally compact Hausdorff spaces. A topological space (resp., group) is Polish if it is separable and completely metrizable. It is folklore that in Polish spaces Baire category theorem is true.

For any topological space $X$, $\mathcal{B}(X)$ denotes the Borel sets, i.e. the $\sigma$-algebra generated by the open sets.

Recall the following definition of left Haar measure:

**Definition 1.** Let $G$ be a locally compact group, and $\mu$ be a Borel measure, i.e.

$$\mu : \mathcal{B}(G) \to [0, \infty]$$

such that

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• \( \mu \) is left-invariant, i.e. for every \( g \in G, B \in \mathcal{B}(G) \)

\[
\mu(gB) = \mu(B),
\]

• \( \mu(U) > 0 \) for each open \( U \neq \emptyset \),

• \( \mu(K) < \infty \) for each compact set \( K \),

• \( \mu \) is inner regular with respect to the compact sets, that is for every \( B \) Borel

\[
\mu(B) = \sup\{\mu(K) : K \subseteq B, K \text{ is compact}\}.
\]

Then \( \mu \) is called a left Haar measure of \( G \).

It is known that a left Haar measure always exists, and it is unique up to a positive multiplicative constant. By slight abuse of notation we will indentify \( \mu \) with its completion, i.e. every set \( H \subseteq G \) is measurable if \( H \) differs from a Borel by (at most) a null Borel set, i.e.

if \( (\exists B, B' \in \mathcal{B} : B \Delta H \subseteq B' \land \mu(B') = 0) \) then \( \mu(H) := \mu(B) \).

**Remark 2.** There is an other definition of (left) Haar measure in locally compact groups, but in locally compact Polish groups (more generally in locally compact groups which are \( \sigma \)-compact) these two coincide.

From now on \( \mathcal{N} \) will denote the null-ideal, i.e.

\[
\mathcal{N} = \{H \subseteq G : \mu(H) = 0\}.
\]

It is known that for any locally compact group the null-ideals of a left Haar measure and a right Haar measure coincide, thus we can speak about null sets ([8, 442F ]).

We are interested in null, but non-meager subgroups of locally compact Abelian (LCA) Polish groups. Only the non-discrete case is of interest, since non-empty open sets are of positive measure, thus in a discrete group \( \{e\} \) has positive measure, and obviously is of second category. It is known that under the Continuum Hypothesis one can construct null but non-meager and meager but non-null subgroups in arbitrary non-discrete locally compact groups. In [13], Rosłanowski, and Shelah constructed ZFC examples for null, but non-meager subgroups of the Cantor group and the reals, and showed that it is consistent with ZFC that in the two groups every meager subgroup is null. Then they asked two questions

**Problem 5.1** [13]

(1) Does every locally compact group (with complete Haar measure) admit a null non–meager subgroup?

(2) Is it consistent that no locally compact group has a meager non–null subgroup?

In the first part of the paper we construct null but non-meager subgroups in non-discrete locally compact Abelian Polish groups. In the second section we will show that it is consistent that in locally compact Polish groups meager subgroups are always null.
2. Null but non-meager subgroups

First, using abstract Fourier analytic methods we will show that the general (Polish LCA) case can be reduced to the case of the circle group, the group of $p$-adic integers, and the direct product of countably many finite (Abelian) groups. Provided that these groups have null but non-meager subgroups, we only have to show that an arbitrary LCA group has a quotient isomorphic to one of the aforementioned three groups, and then verify that pulling-back the appropriate subgroup of the quotient will work. The idea of first finding a nice quotient group, and constructing the desired object for that group is from [7].

Since the direct product of $\omega$-many finite groups, and $\mathbb{R}$ has a null but non-meager subgroup, and so does the circle group by this (see [13]), we only have to construct such subgroups in the group of $p$-adic integers.

The following definition of the $p$-adic integers is from [10, (10.2)].

In Lemma 4 we prove that in an arbitrary non-discrete LCA group an appropriate open subgroup has a nice quotient group. After that, by Proposition 5 the case of $p$-adic integers is handled.

Finally we summarize, and state our main result, Theorem 7.

Definition 3. Let $p \in \mathbb{N}$ be a prime. Then consider the product set

$$\Delta_p = \prod_{i=0}^{\infty} \{0, 1, \ldots, p-1\}$$

with the following group operation. Let $x, y \in \Delta_p$. Define the series $t \in \prod_{i=0}^{\infty} \{0, 1\}$, $z \in \prod_{i=0}^{\infty} \{0, 1, \ldots, p-1\}$ inductively so that

$$x_0 + y_0 = t_0 + z_0,$$

and

$$t_i + x_{i+1} + y_{i+1} = t_{i+1}p + z_{i+1}$$

holds for each $i$. Then $z \in \Delta_p$ is the sum of $x$ and $y$.

The topology is defined by the following neighborhood base of 0 (i.e. the identically 0 sequence)

$$\Lambda_i = \{x \in \Delta_p \mid x_j = 0\text{ for each } j < i\} \quad (i \in \omega),$$

in other words, the product topology. Hence the sets of the form

$$y\Lambda_n = \{x \in \Delta_p \mid x_j = y_j\text{ for } j < n\}$$

is a base consisting of clopen sets.

Lemma 4. Let $G$ be a locally compact Abelian group, which is non-discrete. Then there is an open subgroup $G' \leq G$, and a closed subgroup $C \leq G'$ of $G'$, such that the quotient group $G'/C$ is isomorphic to one of the following:

- the circle group $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$,
- the product of $\omega$-many finite (Abelian) groups $\prod_{i \in \omega} G_i$ (with each $G_i$ having at least two elements),
Proof. By the Principal Structure Theorem of LCA groups [14, 2.4.1], there is an open subgroup $H$ of $G$ of the form $H = K \times \mathbb{R}^n$ for some $n \geq 0$, where $K$ is compact. If $n > 0$, then $K \times (\mathbb{Z} \times \mathbb{R}^{n-1}) \triangleleft H$ is a closed normal subgroup of $H$,

$$H/(K \times \mathbb{Z} \times \mathbb{R}^{n-1}) \simeq (K \times \mathbb{R} \times \mathbb{R}^{n-1})/(K \times \mathbb{Z} \times \mathbb{R}^{n-1}) \simeq \mathbb{R}/\mathbb{Z} \simeq \mathbb{T},$$

we are done. Hence we can assume that $n = 0$, i.e. $H = K$ is compact. Since $H \leq G$ is open, and $G$ was non-discrete, $H$ cannot be discrete.

Next we claim that the dual group $\hat{H}$ of $H$ is infinite. Assume on the contrary that $\hat{H}$ is finite. Then it is compact, thus the dual $\hat{\hat{H}}$ of $\hat{H}$ would be discrete (by [14, 1.2.5]). But by the Pontryagin Duality Theorem ([14, 1.7.2]) $\hat{\hat{H}}$ is isomorphic to $\hat{H}$ which was not discrete, a contradiction.

Now, by [14, 1.7.2], and [14, 2.1.2] groups of the form $H/C$ (where $C$ is a closed subgroup) are the same (isomorphic as topological groups) as dual groups of closed subgroups of $\hat{H}$. This means that having a subgroup $L \leq \hat{H}$ guarantees that there is a closed subgroup $C \leq H$ such that the quotient group

$$I = H/C \simeq \hat{L}$$

Now using [7, Thm. 2.16], and by the infiniteness of $\hat{H}$, we obtain that $\hat{H}$ has a subgroup (let’s say $L$) isomorphic to either

(i) $\mathbb{Z}$, or
(ii) $\oplus_{i \in \omega} G_i$ where each $G_i$ is a finite group of at least two elements, or
(iii) the quasicyclic group $C_p^\infty$ for some prime $p$.

(Note that $L$ is automatically closed, since according to [14, 1.2.5], the compactness of $H$ implies that $\hat{H}$ is discrete.) This means that

(i) if $L \simeq \mathbb{Z}$, then there is a closed subgroup $I \leq H$ such that

$$H/C \simeq \hat{L} = \mathbb{T}$$

(ii) if $L \simeq \oplus_{i \in \omega} G_i$ (where each $G_i$ has at least two elements), then first note that an LCA group is finite (compact and discrete at the same time) iff its dual is finite. Second, since the dual of the dual an LCA group is the group itself [14, 1.7.2], if $\hat{G}_i = \{0\}$ was the trivial group, then $G_i$ also would be the trivial group, therefore each $\hat{G}_i$ has at least two elements. From this we obtain that each $\hat{G}_i$ is finite, and is of cardinality at least two, and using that the dual of the direct sum (endowed with the discrete topology) is the product of the dual groups [14, 2.2.3] we get that

$$(\oplus_{i \in \omega} G_i) = \prod_{i \in \omega} \hat{G}_i.$$

Thus there is a closed subgroup $C \leq H$ such that

$$H/C \simeq \hat{L} = \prod_{i \in \omega} \hat{G}_i \quad (|\hat{G}_i| > 1)$$
(iii) if \( L \leq \hat{H} \) is isomorphic to the quasicyclic group \( C_{p^\infty} \), then as \( \hat{\Delta}_p = C_{p^\infty} \) [352, 25.2] that is equal to
\[
\hat{C}_{p^\infty} = \Delta_p,
\]
thus there is a closed subgroup \( C \leq H \) for which
\[
H/C \simeq \hat{L} = \Delta_p.
\]
This shows that there exists a closed subgroup \( C \leq H \) such that \( I \simeq H/C \) is either

the circle group, the \( p \)-adic integers, or the product of \( \omega \)-many finite groups, each
finite group having at least two elements.

\[\square\]

**Proposition 5.** Let \( p \in \mathbb{N} \) be a prime. Then the group of \( p \)-adic integers has a
null but non-meager subgroup.

Fix a non-principal ultrafilter \( \mathcal{U} \) on \( \omega \), and define the intervals
\[
I_j^k = \left[j^2 + k, (j + 1)^2\right) \subseteq \omega
\]
for each \( j \in \omega, 0 \leq k < (j + 1)^2 - j^2 \). Let \( \mu \) denote the probability Haar measure
(i.e. \( \mu(\Delta_p) = 1 \)), and
\[
H_k = \left\{ x \in \Delta_p \mid \{ j \in \omega \mid (j + 1)^2 - j^2 > k \wedge (x|_{I_j^k} \equiv 0 \vee x|_{I_j^k} \equiv p - 1) \} \in \mathcal{U} \right\}
\]
i.e. those which are constant 0-s or \( p - 1 \)-s on \( \mathcal{U} \)-almost every \( I_j^k \)-s (note that since
\( I_j^k \supseteq I_j^{k+1} \) for any \( j \),
\[
H_k \subseteq H_{k+1}.
\]
Define the set \( H \) as follows
\[
H = \bigcup_{k \in \omega} H_k = \left\{ x \in \Delta_p \mid \exists k : \{ j \in \omega \mid ((j + 1)^2 - j^2 > k) \wedge (x|_{I_j^k} \equiv 0 \vee x|_{I_j^k} \equiv p - 1) \} \in \mathcal{U} \right\}.
\]
We claim that \( H \) is the desired subgroup.

(i) \( H \) is a subgroup.

Let \( x, y \in H \). Fix \( n \in \omega \) such that \( x, y \in H_n \). Consider the following elements of \( \mathcal{U} \)
\[
U_x = \left\{ j \in \omega \mid x|_{I_j^n} \equiv 0 \vee x|_{I_j^n} \equiv p - 1 \right\} \in \mathcal{U},
\]
\[
U_y = \left\{ j \in \omega \mid y|_{I_j^n} \equiv 0 \vee y|_{I_j^n} \equiv p - 1 \right\} \in \mathcal{U},
\]
and let \( U = U_x \cap U_y \in \mathcal{U} \) denote their intersection. Now, if \( j \in U \), then the
following lemma will ensure that \( (x+y)|_{I_j^{n+1}} \equiv 0 \), or \( (x+y)|_{I_j^{n+1}} \equiv p - 1 \),
which means that \( x + y \in H^{n+1} \subseteq H \).
Lemma 6. Let $x, y \in \Delta_p$ and $m < l \in \omega$ such that

$$x_{[m,l]} \equiv 0, \text{ or } x_{[m,l]} \equiv p - 1$$

and

$$y_{[m,l]} \equiv 0, \text{ or } y_{[m,l]} \equiv p - 1.$$  

Then

$$(x + y)_{[m+1,l]} \equiv 0, \text{ or } (x + y)_{[m+1,l]} \equiv p - 1.$$  

Proof. Let $t \in \prod_{i=0}^{\infty} \{0,1\}$, $z = x + y \in \prod_{i=0}^{\infty} \{0,1,\ldots,p-1\}$ as in Definition \ref{def:recursion} i.e. (\ref{eq:1}) and (\ref{eq:2}) hold. Now we have three cases.

• Case 1. If $x_{[m,l]} = y_{[m,l]} \equiv 0$,

then $x_m + y_m + t_{m-1} = t_{m-1} \in \{0,1\}$, thus by (\ref{eq:2}), i.e. the recursive definition of $t$ and $z$, $t_m = 0$. From this it is easy to see that $z_{m+1} = z_{m+2} = \cdots = z_{l-1} = 0$, i.e. $z_{[m+1,l]} \equiv 0$, as desired.

• Case 2. If $x_{[m,l]} \equiv 0$, and $y_{[m,l]} \equiv p - 1$,

then $z_{[m+1,l]}$ depends only on $t_{m-1}$. If $t_{m-1} = 0$, then

$$x_m + y_m + t_{m-1} = p - 1 = 0 \cdot p + p - 1,$$

hence $z_m = p - 1$, and $t_m = 0$. Using (\ref{eq:2}) it is straightforward to check that

$$t_{m+1} = t_{m+2} = \cdots = t_{l-1} = 0,$$

and

$$z_m = z_{m+1} = \cdots = z_{l-1} = p - 1.$$

On the other hand, if $t_{m-1} = 1$, then

$$x_m + y_m + t_{m-1} = p - 1 + 1 = 1 \cdot p + 0,$$

$t_m = 1$, and $z_m = 0$. Similarly, applying (\ref{eq:2}) $l - m - 1$ times, one can get that

$$t_{m+1} = t_{m+2} = \cdots = t_{l-1} = 1,$$

and

$$z_m = z_{m+1} = \cdots = z_{l-1} = 0.$$

• Case 3. If $x_{[m,l]} = y_{[m,l]} \equiv p - 1$,

then

$$x_m + y_m + t_{m-1} = (p - 1) + (p - 1) + t_{m-1} = 1 \cdot p + p - 2 + t_{m-1},$$

i.e. $t_m = 1$. Therefore we got that

$$x_{m+1} + y_{m+1} + t_{m} = (p - 1) + (p - 1) + 1 = 1 \cdot p + p - 1,$$

i.e. $z_{m+1} = p - 1$, $t_{m+1} = 1$. Iterating this argument yields that

$$t_{m+1} = t_{m+2} = \cdots = t_{l-1} = 1,$$

and

$$z_{m+1} = z_{m+2} = \cdots = z_{l-1} = p - 1.$$

$\square$
It is left to show that \( H \) is closed under taking inverse, i.e. if \( x \in H_n \), then \(-x \in H\). We will show that if 
\[
j \in U_x = \{ l \in \omega \mid x|_{I^n_x} \equiv 0 \lor x|_{I^n_x} \equiv p - 1 \} \in \mathcal{U},
\]
and \( j > 0 \), then
\[
(-x)|_{I^n_x} \equiv 0, \text{ or } (-x)|_{I^n_x} \equiv 0.
\]
(Note that since \( 0 \in I^n_0 \), \( j > 0 \) implies that \( 0 \notin I^n_j \).) (6)

Now we will calculate (a candidate for) the inverse of \( x \). By induction on \( i \) we define 
\[
w \in \Delta_p.
\]
Let \( t \equiv 1 \in \prod_{i=0}^{\infty} \{ 0, 1 \} \) the constant \( 1 \) function, and let \( w_0 \) be such that 
\[
w_0 + x_0 = p.
\]
Assume that \( w_l \) is defined for each \( l \leq i \). Then \( w_{i+1} \) can be uniquely defined so that 
\[
w_{i+1} + x_{i+1} + t_i = z_{i+1} + x_{i+1} + 1 = 1 \cdot p
\]
holds. According to the rule of addition in \( \Delta_p \), (1), (2), 
\[
x + w = 0, \text{ i.e. } w = -x.
\]
But obviously if \( i \in I^n_i \), then (by (6)) 
\[
w_i = p - t_i - 1 - x_i,
\]
thus if \( x|_{I^n_i} \equiv 0 \), then \( w|_{I^n_i} \equiv p - 1 \), and conversely \( x|_{I^n_i} \equiv p - 1 \) implies 
\[
w|_{I^n_i} \equiv 0.
\]
(ii) \( H \) is null.

Since 
\[
H = \bigcup_{k \in \omega} H_k,
\]
it is enough to show that \( \mu(H_k) = 0 \) for each \( k \). But 
\[
\mu(H_k) = \mu\left( \left\{ x \in \Delta_p \mid \{ j \in \omega \mid (j + 1)^2 - j^2 > k \land (x|_{I^n_k} \equiv 0 \lor x|_{I^n_k} \equiv p - 1) \} \in \mathcal{U} \right\} \right) \leq \mu\left( \left\{ x \in \Delta_p \mid \{ j \in \omega \mid (j + 1)^2 - j^2 > k \land (x|_{I^n_k} \equiv 0 \lor x|_{I^n_k} \equiv p - 1) \} = \infty \right\} \right)
\]
i.e. it suffices to show that those sequences, for which there exist infinitely many \( j \)-s such that the sequence is constant 0 or \( p - 1 \) on \( I^n_j \) form a null set. Now, since for each \( m \) the \( p^m \)-many translates of the open set 
\[
\Lambda_m = \{ x \in \Delta_p \mid x_l = 0 \text{ for each } l < m \}
\]
is a partition of \( \Delta_p \), 
\[
\mu(\{ x \in \Delta_p \mid x|_{I^n_k} \equiv 0 \}) = \mu(\{ x \in \Delta_p \mid x|_{I^n_k} \equiv p - 1 \}) = \frac{1}{p^{2j+1-k}} = \frac{1}{p^{2j+1-k}}.
\]
Since we know that 
\[
\mu(\{ x \in \Delta_p \mid x|_{I^n_k} \equiv 0 \lor x|_{I^n_k} \equiv p - 1 \}) = \frac{2}{p^{2j+1-k}},
\]
we have 
\[
\| x \in \Delta_p \mid x|_{I^n_k} \equiv 0 \lor x|_{I^n_k} \equiv p - 1 \| = \frac{2}{p^{2j+1-k}},
\]
and therefore 
\[
\mu(\{ x \in \Delta_p \mid x|_{I^n_k} \equiv 0 \lor x|_{I^n_k} \equiv p - 1 \}) = \frac{2}{p^{2j+1-k}}.
\]
the following inequality will hold for arbitrary fixed \( l_0 \)
\[
\mu \left( \left\{ x \in \Delta_p \mid \left| (j \in \omega \mid (j + 1)^2 - j^2 > k \land (x|_{l_j} \equiv 0 \lor x|_{l_j} \equiv p - 1) \right) = \infty \right\} \right) = 
\mu \left( \bigcap_{j=0}^{\infty} \bigcup_{j=1}^{\infty} \left\{ x \in \Delta_p \mid (x|_{l_j} \equiv 0 \lor x|_{l_j} \equiv p - 1) \right\} \right) \leq 
\mu \left( \bigcup_{j=l_0}^{\infty} \left\{ x \in \Delta_p \mid (x|_{l_j} \equiv 0 \lor x|_{l_j} \equiv p - 1) \right\} \right).
\]

We obtained the following obvious upper bound
\[
\mu \left( \bigcup_{j=l_0}^{\infty} \left\{ x \in \Delta_p \mid (x|_{l_j} \equiv 0 \lor x|_{l_j} \equiv p - 1) \right\} \right) \leq \sum_{j=l_0}^{\infty} \frac{2}{2^{2j+1-k}},
\]
and as the latter tends to 0 when \( l_0 \) tends to infinite, we are done.

(iii) \( H \) is non-meager.

We will show that even \( H_0 \) is non-meager. For each \( j \in \omega \) define the mapping \( f_j \) as follows
\[ f_j : \{0,1,\ldots p-1\}^{[l_j]} \to \{0,1\} \]
\[ v \mapsto \begin{cases} 0 & \text{iff } v \equiv 0 \\ 1 & \text{otherwise} \end{cases} \]

Let \( f \) be the following mapping from \( \Delta_p \) to \( 2^\omega \) (where we identify \( \Delta_p \) with \( \{0,1,\ldots p-1\}^\omega \))
\[ f : \Delta_p \to 2^\omega 
 x \mapsto (f_j(x|_{l_j}))_{j \in \omega} \quad (7) \]

Now suppose that \( \Delta_p \setminus H \) is co-meager. As \( f \) is a continuous surjective open mapping between Polish spaces, according to \([2, \text{Lemma 2.6}]\) \( f \) maps a co-meager set onto a co-meager set, let \( R = f(\Delta_p \setminus H) \) denote this co-meager set in \( 2^\omega \). now, using \([4, \text{Thm. 2.2.4}]\), there exist a strictly growing sequence of non-negative integers
\[ 0 = n_0 < n_1 < \ldots, \]
and an element \( r \in 2^\omega \) such that
\[ \left\{ s \in 2^\omega \mid \left| \left\{ j : s|_{[n_j,n_{j+1}]} \equiv r|_{[n_j,n_{j+1}]} \right\} \right| = \infty \right\} \subseteq R. \quad (8) \]

Now the following disjoint sets cover \( \omega \) (since the sequence \( n_j \) is strictly growing), thus exactly one of the following set is in \( U \)
\[ U_0 = \bigcup_{k \in \omega} [n_{2k},n_{2k+1}), \quad U_1 = \bigcup_{k \in \omega} [n_{2k+1},n_{2k+2}) \quad (9) \]

Let \( U \) denote that set. Now define the following element \( y \) of \( 2^\omega \)
\[ y_i = \begin{cases} 0 & \text{if } i \in U \\ r_i & \text{otherwise} \end{cases} \]
Clearly (using (9) and the fact that \( U = U_i \) for some \( i \in \{0, 1\} \)) there are infinitely many \( j \)-s for which \( y_{[n_j, n_{j+1}]} \equiv r_{[n_j, n_{j+1}]} \), thus by (8)

\[ y \in R = f(\Delta_p \setminus H) \]

Hence, there is a

\[ z \in \Delta_p \setminus H, \quad (10) \]

for which

\[ f(z) = y. \quad (11) \]

But for each \( j \in \omega \), if

\[ j \in U \]
\[ \Downarrow \]
\[ y_j = 0 \]
\[ \Downarrow \quad \text{(by (7) and (11))} \]
\[ z_{[j^2,(j+1)^2)} \equiv 0. \]

This means, as \( U \in \mathcal{U} \), that we found that for \( \mathcal{U} \)-almost every \( j \), \( z_{[j^2,(j+1)^2)} \equiv 0 \), therefore \( z \in H_0 \) by the definition of \( H_0 \) (4), contradicting (10).

Now we are ready to state our result.

**Theorem 7.** Let \( G \) be a non-discrete Polish LCA group. Then there is a null, non-meager subgroup in \( G \).

**Proof.** According to Lemma 4 there is an open subgroup \( G' \leq G \) for which there exists a closed normal subgroup \( C \leq G' \) such that \( G'/C \) is isomorphic to one of the following

(i) the circle group \( T \simeq \mathbb{R}/\mathbb{Z} \),
(ii) the product of \( \omega \)-many finite (Abelian) groups \( \prod_{i \in \omega} G_i \) (with each \( G_i \) having at least two elements),
(iii) the group of \( p \)-adic integers (\( \Delta_p \)).

First we show that \( G'/C \) has a null but non-meager subgroup, say \( H/C \), where \( C \leq H \leq G' \). In case (i), using \cite{13} Theorem 2.3, there is a null non-meager subgroup \( L \leq \mathbb{R} \). Then the Haar measure is the Lebesgue measure on the real line (up to a positive multiplicative constant), and after identifying the circle group with \([0, 1)\), the Haar measure on it coincides with the Lebesgue measure restricted to the unit interval. Therefore the set

\[ L' = \{ r \in [0, 1) : \exists k \in \mathbb{Z} \ k + r \in L \} \subseteq [0, 1) = T \]

is a null set, and form a subgroup in the circle group. \( L' \subseteq [0, 1) \) is non-meager, since the meagerness of \( L' \) would imply that \( L \) was meager in \( \mathbb{R} \). Given \( L \), let \( H/C \) be the corresponding subgroup in \( G'/C \simeq T \)

In case (ii), i.e.

\[ G'/C \simeq \prod_{i \in \omega} F_i \quad (1 < |F_i| < \infty) \]
Remark 4.2 implies that there is a null but non-meager subgroup in $G'/C$, say $H/C$.

If $G'/C$ is isomorphic to the group of the $p$-adic integers for some prime $p$, then Proposition 5 guarantees that there is an appropriate subgroup in $G'/C$. Next we have to verify that from the nullness and non-meagerness of $H/C$ (in $G'/C$) follows the nullness and non-meagerness of $H$ in $G'$. Let $\pi : G' \to G'/C$ denote the canonical projection. Now since $G'/C$ is compact, and a Haar measure $\nu$ on $G'/C$ is Radon, using [8, 443T/(c)] $\nu$ is the push-forward of a Haar measure of $G'$. Let $\mu_{G'}$ denote that Haar measure, i.e. for each Borel set $B \subseteq G'$

$$\nu(B) = \mu(\pi^{-1}(B)).$$

Now if $B \supseteq H/C$ is a null Borel hull of $H/C$, then $\pi^{-1}(B)$ is a null Borel hull of $H$, i.e. $H \leq G'$ is a subgroup of measure zero.

For the non-meagerness, suppose that $G' \setminus H$ is co-meager in $G'$. Then using [2, Lemma 2.6] (and the fact $C \leq H$), the image of $G' \setminus H$ under the canonical projection $\pi : G' \to G'/C$

$$\pi(G' \setminus H) = (G'/C) \setminus (H/C)$$

would also be co-meager (in $G'/C$), contradicting that $H/C$ is non-meager.

Now, we have a null but non-meager subgroup $H$ in $G'$, but since $G' \leq G$ is open, it is straightforward to check that a Haar measure of $G$ restricted to $G'$ is a Haar measure of $G'$, therefore $H$ is null in $G$ too. Moreover, a non-meager set in an open subspace (namely $G'$) is also non-meager in the whole space, thus $H \leq G$ is a null non-meager subgroup.

□

3. Meager but non-null subgroups

The following can be found in [12], but for the sake of completeness we include the proofs. Friedman’s theorem can be found in [5]:

**Theorem 8.** In the Cohen model (that is adding $\omega_2$ Cohen reals to a model of ZFC + CH) the following holds:

$$\forall H \subseteq 2^\omega \times 2^\omega, \text{ if } H \text{ is } F_\sigma:\n
(\exists C \times D \subseteq H, C \times D \notin N) \Rightarrow (\exists A \times B \subseteq H \text{ measurable : } \mu(A \times B) > 0)$$

**Remark 9.** If $X$ is a Polish space, and $\mu$ is a $\sigma$-finite Borel measure, then for any rectangle $C \times D \subseteq X \times X$, it is non-null (wrt. the product measure $\mu \times \mu$) iff $C \notin N_\mu$ and $D \notin N_\mu$.

Next we prove that this property implies that every meager subgroup of a locally compact Polish group is of measure zero, this lemma is also from [5] (stating it only for $G = 2^\omega$, but the proof is the same).
Lemma 10. Let $G$ be a locally compact Polish group. Assume that the following holds:

$$\forall H \subseteq G \times G, \text{ if } H \text{ is } F_\sigma:\n
(\exists C \times D \subseteq H, \ C \times D \notin \mathcal{N}) \Rightarrow (\exists A \times B \subseteq H \text{ measurable}: \mu(A \times B) > 0)$$

(12)

Then every meager subgroup of $G$ is null.

Proof. Let $S \subseteq G$ be a meager subgroup, assume on the contrary that $S$ is non-null. Let $(H_i)_{i \in \omega}$ be nowhere dense closed subsets such that $\bigcup_{i \in \omega} H_i \supseteq S$. Now, if $m : G \times G \to G$ denotes the multiplication function, then $m^{-1}(\bigcup_{i \in \omega} H_i)$ is an $F_\sigma$ set, containing $S \times S$, which is non-null. But then there is a measurable rectangle $A \times B \subseteq m^{-1}(\bigcup_{i \in \omega} H_i)$, which is of positive measure, thus by our Remark 9 $\mu(A), \mu(B) > 0$. Then, due to a Steinhaus type theorem [7], $AB = m(A \times B)$ has nonempty interior. But $AB \subseteq \bigcup_{i \in \omega} H_i$ which is meager, a contradiction (by the Baire category theorem). \qed

First (in Lemma 12) we show that if every $F_\sigma$ set $H \subseteq 2^\omega \times 2^\omega$ containing a rectangle of positive outer measure contains a measurable rectangle of positive measure, then this holds for arbitrary locally compact Polish groups. This yields that it is consistent with ZFC that in a locally compact Polish group meager subgroups are always null.

Remark 11. If $X$ is a Polish space, $\nu$ is a $\sigma$-finite Borel measure on $X$, then for every measurable set $H$ there exists a Borel $B$ such that $H \triangle B$ is null (wrt. $\mu$).

Lemma 12. Assume that condition [12] holds in $2^\omega$ (i.e. every $F_\sigma$ subset of $2^\omega \times 2^\omega$ which contains a non-null rectangle must contain a measurable Haar-positive rectangle). Then (12) holds in every non-discrete locally compact Polish group $G$.

Corollary 13. It is consistent with ZFC (i.e. it holds in the Cohen model) that in every locally compact Polish group meager subgroups are null.

Proof. (Lemma 12) Let $\mu$ denote the left Haar measure on $G$, and let $\nu$ denote the Haar measure on $2^\omega$. Since locally compact Polish groups are $\sigma$-compact, the Haar measure is $\sigma$-finite.

Claim 14. There is a sequence of pairwise disjoint compact sets $C_i$ ($i \in \omega$) in $2^\omega$, for which $\nu(2^\omega \setminus \bigcup_{i \in \omega} C_i) = 0$, and similarly a sequence of compact sets $K_i$ ($i \in \omega$) in $G$ with $\mu(K_i) > 0$ and $\mu(G \setminus \bigcup_{i \in \omega} K_i) = 0$, and there exist homeomorphisms $f_i : C_i \to K_i$, and positive constants $r_i$ such that

$$\forall B \subseteq C_i \text{ Borel : } \nu(B) = r_i \mu(f_i(B)).$$

Proof. First, using the inner regularity, and the $\sigma$-compactness of $G$, let $E_i$ ($i \in \omega$) be a sequence of pairwise disjoint compact subsets of $2^\omega$ such that $\nu(2^\omega \setminus \bigcup_{i \in \omega} E_i) = 0$, and $\nu(F_i) > 0$ ($\forall i$).
Now, since $G$ is non-discrete, every open set is infinite in $G$, and an open set $U$ with compact closure is thus an infinite set. But compact sets have finite measure. Then, by the invariance of the measure, every point has the same measure, that must be 0 because there exists infinite sets with finite measure, thus we obtain that $\mu$ is a continuous measure. We can apply the isomorphism theorem for measures [11] Thm 17.41. Let $g_i : F_i \to E_i$ be a Borel isomorphism (a bijection which is Borel, and so is its inverse), for which there exists $r_i > 0$ such that

$$\nu(H) = r_i \mu(f_i(H)) \quad (H \subseteq F_i \text{ Borel})$$

Now, by Lusin’s theorem (see [11, Thm 17.12]) for every $H \subseteq C_i$ and $\varepsilon > 0$, the function $g_i|_H$ is continuous on a compact subset $H' \subseteq H$ with $\nu(H') > \nu(H) - \varepsilon$. Using this, there are pairwise disjoint compact subsets $(F^{j_i}_i)_{j \in \omega}$ in $F_i$, such that $g_i|_{F^{j_i}_i}$ is continuous, and $\nu(F_i \setminus \bigcup_{j \in \omega} F^{j_i}_i) = 0$. Let the sequence $C_i$ ($i \in \omega$) be the enumeration of the $F^{j_i}_i$s in type $\omega$, and let $f_i = g^j_i$, if $C_i = F^{j_i}_i$, then choosing $K_i$ to be $f_i(C_i)$ works. \hfill \Box

Let $C = \bigcup_{i \in \omega} C_i \subseteq 2^\omega$, $K = \bigcup_{i \in \omega} K_i \subseteq G$, $f = \bigcup_{i \in \omega} f_i$, then

$$f : C \to K$$

is a bijection which is almost everywhere defined in $2^\omega$, and

- for each $S \subseteq C \subseteq 2^\omega$, if $\nu(S) = 0$, then $\mu(f(S)) = 0$,
- if $S$ is measurable, $\nu(S) > 0$, then $f(S)$ is measurable, $\mu(f(S)) > 0$ (by our Remark 11 $S$ differs from a Borel by a null set, $f^{-1}$ is a Borel function, and $f$ maps a null set to a null set),
- for every $F_\sigma$ set $H \subseteq G$, $f^{-1}(H)$ is also $F_\sigma$ (in $2^\omega$).

Now let $H \subseteq G \times G$ be an $F_\sigma$ subset, and $D \times E \subseteq H$ be a rectangle of positive outer measure. Then $H \cap (K \times K)$ is still an $F_\sigma$ set, and by Remark 11 $(D \times E) \cap (K \times K) = \nu(G \setminus K) \times (E \cap K)$ is of positive outer measure, since $\mu(G \setminus K) = 0$. Thus from now on, we can assume that $H \subseteq K \times K$: it suffices to find measurable non-null rectangles in such $H$-s. But $(f \times f)^{-1}(H)$ is $F_\sigma$, since $f \times f = \bigcup_{i \in \omega} f_i \times (\bigcup_{j \in \omega} f_i) = \bigcup_{i,j \in \omega} f_i \times f_j$ where the $f_i$-s are homeomorphisms between compact sets, so are the $f_i \times f_j$-s. And $(f \times f)^{-1}(D \times E) = f^{-1}(D) \times f^{-1}(E)$ is a product of sets of positive outer measure, since $f$ maps a nullset to a nullset. But then $(f \times f)^{-1}(H) \subseteq 2^\omega \times 2^\omega$, that contains $(f \times f)^{-1}(D \times E) \text{ contains a measurable } A \times B \text{ with } \nu(A), \nu(B) > 0$. Now, using that $f$ maps a positive, measurable set to a positive, measurable set, we obtain that $\mu(f(A)), \mu(f(B)) > 0$, $f(A) \times f(B) \subseteq H$, hence $H$ contains a non-null measurable rectangle, indeed. \hfill \Box

**Remark 15.** For the consistency of the other direction, i.e. meager but non-null subgroups, if $\text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N})$ (in particular if $\text{CH}$ holds), then there exist a meager but non-null subgroup in $\mathbb{R}$ and $2^\omega$, see [3] Corollary 7.12. It is also true, that $\text{non}(\mathcal{N}) < \text{non}(\mathcal{M})$ implies that there exists a meager but non-null subgroup in an arbitrary locally compact Polish group.
Question 16. Is it true that CH implies the existence of meager but non-null subgroups in locally compact Polish groups?

Question 17. What can we say about small-large subgroups of not locally compact Polish groups, replacing 'null wrt. the Haar measure' by 'Haar-null' in the sense of Christensen? Do (always) exist meager but not Haar-null, and Haar-null but non-meager subgroups in not locally compact Polish groups? (A subset $X$ of a Polish group is Haar-null, if there exists a Borel probability measure $\mu$ on $G$, and a $B$ Borel set $B \supseteq X$, such that $\mu(gBh) = 0$ for each $g, h \in G$.)

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