On Interval Non-Edge-Colorable Eulerian Multigraphs

Petros A. Petrosyan

Department of Informatics and Applied Mathematics,
Yerevan State University, 0025, Armenia
Institute for Informatics and Automation Problems,
National Academy of Sciences, 0014, Armenia
E-mail: pet_spetros@ipia.sci.am

November 19, 2013

Abstract

An edge-coloring of a multigraph $G$ with colors $1, \ldots, t$ is called an interval $t$-coloring if all colors are used, and the colors of edges incident to any vertex of $G$ are distinct and form an interval of integers. In this note, we show that all Eulerian multigraphs with an odd number of edges have no interval coloring. We also give some methods for constructing of interval non-edge-colorable Eulerian multigraphs.

1 Introduction

In this note we consider graphs which are finite, undirected, and have no loops or multiple edges and multigraphs which may contain multiple edges but no loops. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a multigraph $G$, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of $G$ by $\Delta(G)$, and the chromatic index of $G$ by $\chi'(G)$. A multigraph $G$ is Eulerian if it has a closed trail containing every edge of $G$. The terms and concepts that we do not define can be found in [7].

A proper edge-coloring of a multigraph $G$ is a coloring of the edges of $G$ such that no two adjacent edges receive the same color. If $\alpha$ is a proper edge-coloring of $G$ and $v \in V(G)$, then $S(v, \alpha)$ denotes the set of colors of edges incident to $v$. A proper edge-coloring of a multigraph $G$ with colors $1, \ldots, t$ is called an interval $t$-coloring if all colors are used, and for any vertex $v$ of $G$, the set $S(v, \alpha)$ is an interval of integers. A multigraph $G$ is interval colorable if it has an interval $t$-coloring for some positive integer $t$. The set of all interval colorable multigraphs is denoted by $\mathfrak{N}$. 
The concept of interval edge-coloring of multigraphs was introduced by Asratian and Kamalian [1]. In [1, 2], they proved the following result.

**Theorem 1.** If $G$ is a multigraph and $G \in \mathcal{N}$, then $\chi'(G) = \Delta(G)$. Moreover, if $G$ is a regular multigraph, then $G \in \mathcal{N}$ if and only if $\chi'(G) = \Delta(G)$.

Some results on interval edge-colorings of multigraphs were obtained in [5]. In [6], the authors described some methods for constructing of interval non-edge-colorable bipartite graphs and multigraphs.

In this note we show that all Eulerian multigraphs with an odd number of edges have no interval coloring. We also give some methods for constructing of interval non-edge-colorable Eulerian multigraphs.

## 2 Results

Let $G$ be a multigraph. For any $e \in E(G)$, by $G_e$ we denote the multigraph obtained from $G$ by subdividing the edge $e$. For a multigraph $G$, we define a multigraph $G^*$ as follows:

$$
V(G^*) = V(G) \cup \{u\}, \ u \notin V(G),
$$

$$
V(G^*) = E(G) \cup \{uv : v \in V(G) \text{ and } d_G(v) \text{ is odd}\}.
$$

For a graph $G$, by $L(G)$ we denote the line graph of the graph $G$.

We also need a classical result on Eulerian multigraphs.

**Euler’s Theorem.** ([3]) A connected multigraph $G$ is Eulerian if and only if every vertex of $G$ has an even degree.

Now we can prove our result.

**Theorem 2.** If $G$ is an Eulerian multigraph and $|E(G)|$ is odd, then $G \notin \mathcal{N}$.

**Proof** Suppose, to the contrary, that $G$ has an interval $t$-coloring $\alpha$ for some $t$. Since $G$ is an Eulerian multigraph, $G$ is connected and $d_G(v)$ is even for any $v \in V(G)$, by Euler’s Theorem. Since $\alpha$ is an interval coloring and all degrees of vertices of $G$ are even, we have that for any $v \in V(G)$, the set $S(v, \alpha)$ contains exactly $\frac{d_G(v)}{2}$ even colors and $\frac{d_G(v)}{2}$ odd colors. Now let $m_{odd}$ be the number of odd colors in the coloring $\alpha$. By Handshaking lemma, we obtain $m_{odd} = \frac{1}{2} \sum_{v \in V(G)} \frac{d_G(v)}{2} = \frac{|E(G)|}{2}$. Thus $|E(G)|$ is even, which is a contradiction.
Corollary 1. If $G$ is an Eulerian multigraph and $G \in \mathcal{M}$, then $|E(G)|$ is even.

Let us note that there are Eulerian graphs with an even number of edges that have no interval coloring. For example, the complete graph $K_5$ has no interval coloring. On the other hand, there are many Eulerian graphs with an even number of edges that have an interval coloring. In [2], Jaeger proved the following result.

Theorem 3. If $G$ is a connected $r$-regular graph ($r \geq 2$), $\chi'(G) = r$ and $|E(G)|$ is even, then $\chi'(L(G)) = 2r - 2$.

Since $G$ is a connected $r$-regular graph ($r \geq 2$) and $|E(G)|$ is even, we have that $L(G)$ is a connected $(2r - 2)$-regular graph with an even number of edges. Moreover, by Theorems 1 and 3 and Euler’s Theorem, we obtain the following

Corollary 2. If $G$ is a connected $r$-regular ($r \geq 2$) graph with an even number of edges and $G \in \mathcal{M}$, then $L(G)$ is an Eulerian graph with an even number of edges and $L(G) \in \mathcal{M}$.

Let us note that Theorem 2 also gives some methods for constructing of interval non-edge-colorable Eulerian multigraphs from interval colorable multigraphs.

Corollary 3. If $G$ is an Eulerian multigraph and $G \in \mathcal{M}$, then for each $e \in E(G)$, $G_e \notin \mathcal{M}$.

Corollary 4. If $G$ is a connected multigraph with an odd number of edges and $G \in \mathcal{M}$, then $G^* \notin \mathcal{M}$.

References

[1] A.S. Asratian, R.R. Kamalian, Interval colorings of edges of a multigraph, Appl. Math. 5 (1987) 25-34 (in Russian).

[2] A.S. Asratian, R.R. Kamalian, Investigation on interval edge-colorings of graphs, J. Combin. Theory Ser. B 62 (1994) 34-43.

[3] L. Euler, Solutio problematis ad geometriam situs pertinentis, Commentarii Academiae Sci. I. Petropolitanae 8 (1736) 128-140.
[4] F. Jaeger, Sur l’indice chromatique du graphe representatif des aretes d’un graphe regulier, Discrete Math. 9 (1974) 161-172.

[5] R.R. Kamalian, Interval edge-colorings of graphs, Doctoral Thesis, Novosibirsk, 1990.

[6] P.A. Petrosyan, H.H. Khachatrian, Interval non-edge-colorable bipartite graphs and multigraphs, Journal of Graph Theory, 2013, http://onlinelibrary.wiley.com/doi/10.1002/jgt.21759/pdf

[7] D.B. West, Introduction to Graph Theory, Prentice-Hall, New Jersey, 1996.