An effective fluid description of scalar-vector-tensor theories under the sub-horizon and quasi-static approximations

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Abstract. We consider scalar-vector-tensor (SVT) theories with second-order equations of motion and tensor propagation speed equivalent to the speed of light. Under the sub-horizon and the quasi-static approximations we find analytical formulae for an effective dark energy fluid, i.e., sound speed, anisotropic stress as well as energy density and pressure. We took advantage of our general, analytical fluid description and showed that it is possible to design SVT cosmological models which are degenerate with ΛCDM at the background level while having gravity strength $G_{\text{eff}} < G_N$ at late-times as well as non-vanishing dark energy perturbations. We implemented SVT designer models in the widely used Boltzmann solver CLASS thus making it possible to test SVT models against astrophysical observations. Our effective fluid approach to SVT models reveals non trivial behaviour in the sound speed and the anisotropic stress well worth an investigation in light of current discrepancies in cosmological parameters such as $H_0$ and $\sigma_8$.

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\section{Introduction}

Pretty interesting discrepancies in cosmological parameters are challenging the standard model of cosmology ΛCDM \cite{1, 2}. Low red-shift measurements of pulsating (Cepheid variables) and exploding stars (Supernovae type Ia) allow a determination of the Hubble constant $H_0$ which disagrees ($\approx 5\sigma$) with the value inferred by the Planck Collaboration in the context of the ΛCDM model \cite{3–5}. A similar situation involves the strength of matter clustering parameterized as $S_8$. The ΛCDM value obtained from Cosmic Microwave Background (CMB) anisotropies differs ($2 - 3\sigma$) when compared to $S_8$ values determined by probes such as weak lensing and galaxy clustering \cite{6–19}. Although Big Bang nucleosynthesis (BBN) is a key ingredient in the standard cosmological model, there is a factor of $\approx 3.5$ discrepancy between the theoretically expected abundance of lithium-7 and the observationally inferred abundance obtained from absorption spectroscopy of metal-poor stars in the galactic halo \cite{20–25}. By measuring the sky-averaged 21 cm brightness temperature at red-shift $z \approx 20$, the EDGES experiment indicates a value for the baryon temperature cooler than expected in ΛCDM \cite{26, 27}. However, a recent analysis bears out earlier concerns and shows no evidence for non-standard cosmology \cite{28}.

These exemplifying discordances could be due to unaccounted for systematic errors in analyses of current data sets, but thus far different analyses do not show a preference for this explanation. There exists also the very interesting possibility that the lack of agreement in some cosmological parameters measured by different experiments could also be due to a misunderstanding of the underlying physics. In other words, data would be hinting at new physics not taken into consideration within the standard cosmological model \cite{29–31}. For instance, a configuration of vector fields leading to an effective Early Dark Energy (EDE) fluid with equation of state $w_{\text{EDE}}(z)$, sound speed $c^2_{\text{EDE}}$, and non-vanishing anisotropic stress $\pi_{\text{EDE}}(z)$ could simultaneously soften $H_0$ and $S_8$ tensions \cite{32}.

The concordance cosmological model ΛCDM is not only a good fit for most astrophysical measurements. Being relatively simple (i.e., it is described by six cosmological parameters), the ΛCDM model is preferred over its alternatives in analyses performing Bayesian model comparison \cite{33}. However, we must bear in mind that, despite of its success explaining observations, ΛCDM is just a pretty good Universe’s phenomenological description resting upon elements yet to be understood, e.g., the cosmological constant \cite{34, 35}, the nature of dark matter \cite{36–43}, inflation \cite{44–48}. Unravelling the nature of dark matter as well as deciphering the reason why the Universe is speeding up from fundamental physics has motivated a considerable amount of research over the past decades.

In order to address the problem of the Universe late-time accelerating expansion two paths are usually followed in alternative models to ΛCDM. On the one hand, it is expected that new kinds of matter with the right properties (e.g., a negative equation of state $w < -1/3$) will be discovered in laboratories. For instance, new particles in more complete theories of fundamental interactions could dominate the energy content at late times, avoid fine-tuning issues and be the reason why the Universe is speeding up \cite{49–51}. These exotic matter fields are collectively known as dark energy (DE). On the other hand, General Relativity (GR) might require modifications despite its success \cite{52, 53}, making plausible to consider
modified gravity (MG) theories (for a review on DE and MG models see, for instance, [54]). Nevertheless, thus far several tests performed up to extra-galactic and cosmological scales agree very well with GR [55–57].

A number of popular DE and MG models (e.g., $f(R)$, Brans-Dicke, kinetic gravity braiding, quintessence, K-essence, etc.) can be nicely encompassed in a unified framework put forward by G. W. Horndeski in 1974 [58]. The generalisation of covariant Galileons led to the rediscovery of Horndeski’s theory which ever since got a lot of attention [59–61]. The theory constitutes the most general Lorentz-invariant extension of GR in four dimensions considering non-minimal couplings between the metric tensor and a scalar field, restricting the equations of motion to being second order in the derivatives of the field functions. Although recent measurements of the propagation speed for gravitational waves severely reduced the Horndeski Lagrangian [62–70], remaining degrees of freedom are well worth an investigation.

Construction of cosmological models is as important as model testing because in this way we can decide about plausibility of different theories for describing nature. Testing cosmological models is not a trivial task and besides the huge effort dealing with astrophysical measurements (e.g., instruments design, data pipelines), it also requires development of software to compute theoretical predictions for a given model. Boltzmann solvers are codes widely used in cosmology nowadays and they have become fundamental for model testing. These codes allow to not only compute the background evolution of a cosmological model, they also solve the involved differential equations governing the evolution of perturbations. CAMB [71] and CLASS [72] are two popular Boltzmann codes which focus on linear order perturbations and compute observables such as the CMB angular power spectrum and the matter power spectrum, and thus make possible testing cosmological models against data sets.

There are in the literature a number of works where CLASS and CAMB were modified to include cosmological models differing from ΛCDM. For instance, MG models have been implemented by using functions parameterizing deviations from GR (i.e., $\mu(a,k)$, $\gamma(a,k)$) in MGCAMB [74–76]; by solving the full system of differential equations for a given model as in FRCAMB [77, 78]; by exploiting an Effective Field Theory approach as in EFTCAMB [79, 80]; by using to good advantage a gauge invariant formalism in the framework of an equation of state (EoS) approach for perturbations in CLASS [81, 82]. A remarkable step for the investigation of alternative cosmological models was the development of Hi-CLASS [83] which implements Horndeski theories and, without using neither sub-horizon nor quasi-static approximations, solves differential equations for linear order perturbations. This was indeed not a trivial task due to the high number of degrees of freedom in the theory.

It turns out that by using an Effective Fluid Approach things can become easier when implementing $f(R)$ and Horndeski theories in Boltzmann solvers [86, 87]. Quasi-static and sub-horizon approximations can be applied to these kinds of theories so that a fluid description is achieved for the effective DE fluid. Fairly general analytical expressions for the equation of state $w_{DE}(z)$, sound speed $c_{s,DE}^2(z,k)$, and anisotropic stress $\pi_{DE}(z,k)$ enable a relatively easy implementation of these kinds of theories in CLASS. Interestingly, this effective fluid approach leads to no significant loss of accuracy when compared to the exact computation of observables in Hi-CLASS and agrees pretty well with other approaches such as the EoS [87–89].

Vector fields are also present in nature and it is plausible that they might be related to a late-time accelerating universe [90–111]. After a successful construction of consistent theories

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1 Non-linear corrections are usually taken into consideration via fitting functions such as HALOFIT [73].
2 For an effective fluid description of MG see, for instance, Refs. [84, 85].
for general scalar-tensor interactions, it was shown that a similar procedure can be worked out for vector-tensor interactions, known as generalised Proca theories [112–116]. It turns out that more general theories having second-order equations of motion and simultaneously including a scalar field and a vector field, namely Scalar-Vector-Tensor (SVT) theories, were also found [117–119]. Although SVT theories encompass both Horndeski and generalised Proca theories, and therefore might have an interesting, new, richer phenomenology, they have received little attention thus far in the literature. In fact, as far as we know, there is no public Boltzmann code where SVT theories be fully implemented so that their phenomenology can be investigated in detail [120]. This is a major disadvantage for model testing because we cannot compare theoretical predictions against astrophysical measurements, hence seriously decide about viability of alternative cosmological models. In this work we show that SVT theories can be mapped into an effective fluid so that their phenomenology can be investigated through Boltzmann solvers such as CLASS. We apply quasi-static and sub-horizon approximations to SVT theories and find fairly general analytical expressions for the quantities defining the effective fluid, that is, \( w_{\text{DE}}(z) \), \( c^2_{s,\text{DE}}(z, k) \), and \( \pi_{\text{DE}}(z, k) \).\(^3\) Our approach can be useful to carry out further investigations, for instance, on general EDE and Relativistic Modified Newtonian Dynamics (RMOND) models [32, 94].

The paper is organised as follows. Our notation is set in Section 2. We discuss SVT theories in Section 3, and in Section 4 we explain the effective fluid description. Then we show that our approach allows designing cosmological models having a behaviour in good agreement with observations (Section 5). We give our concluding remarks in Section 6 and provide details of our computations in Appendices A–E.

2 Perturbations in a general dark energy model

A popular approach to explain the current accelerated expansion of the universe, as well as other observations, is to assume the existence of yet undetected matter fields generally called as dark energy. In general, these new fields are treated as fluids having equation of state \( w \), sound speed \( c^2_s \), and anisotropic stress \( \sigma \) [122]. Since we are interested in the late-time dynamics, we consider the existence of a DE fluid aside non-relativistic matter (baryon and dark matter). In addition, we take into consideration the gravitational field through the Einstein-Hilbert action so that

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa} R + L_m + L \right],
\]

where \( g \) is the determinant of the metric \( g_{\mu\nu} \), \( R \) is the Ricci scalar, \( \kappa \equiv \frac{8\pi G_N}{c^4} \), \( L_m \) is the matter Lagrangian, and \( L \) is the Lagrangian of an arbitrary fluid, which we will later identify as dark energy.\(^4\) From Eq. (2.1), as it is well-known, we can obtain the Einstein field equations through the principle of least action

\[
G_{\mu\nu} = \kappa \left( T^{(m)}_{\mu\nu} + T_{\mu\nu} \right),
\]

\(^3\)While in this work we focus on DE, SVT theories have also been studied in inflation (see, for instance, [121]).

\(^4\)Definitions throughout the paper: speed of light \( c = 1 \); \( \kappa = \frac{8\pi G_N}{c^4} \) with \( G_N \) being the bare Newton’s constant; \((-+++)\) for the metric signature; the Riemann and Ricci tensors are denoted respectively as \( R^a_{\mu\beta\nu} \) and \( R_{\mu\nu} = R^a_{\mu\alpha\nu}. \)
where $G_{\mu\nu}$ is the Einstein tensor, $T^{(m)}_{\mu\nu}$ and $T_{\mu\nu}$ are the energy-momentum tensors for matter and arbitrary fluid, respectively. Since field equations (2.2) are rather general it is usual to make a few assumptions. Recent analyses [2, 123] show that the standard cosmological model ΛCDM is in very good agreement with observations. The model assumes a flat, linearly perturbed Friedman-Lemaître-Robertson-Walker (FLRW) metric that in the Newtonian gauge reads
\[ ds^2 = a(\eta)^2 \left[ -\{1 + 2\Psi(x, \eta)\}d\eta^2 + \{1 + 2\Phi(x, \eta)\}\delta_{ij}dx^i dx^j \right], \] (2.3)
which takes into account the existence of tiny inhomogeneities in the energy distribution of the universe. In Eq. (2.3) $a$ is the scale factor, $x$ indicates spatial coordinates, $\eta$ is the conformal time, and $\Psi$ and $\Phi$ are the gravitational potentials. Observations and simulations indeed support the assumption that on large enough scales the universe is statistically homogeneous and isotropic [124–126]. Throughout the paper we assume the metric in Eq. (2.3), unless otherwise specified.

We regard that our DE fluid is an ideal fluid also having tiny perturbations, so that its energy-momentum tensor is given by
\[ T^\mu_{\nu} = P\delta^\mu_{\nu} + (\rho + P)U^\mu U_\nu, \] (2.4)
where $P, \rho,$ and $U^\mu \equiv a^{-1}(1 - \Psi, u)$ are respectively the pressure, the energy density, the velocity four-vector, and $u = x'$. The linearised energy-momentum tensor then reads
\[ T^0_0 = -(\bar{\rho} + \delta\rho), \] (2.5)
\[ T^0_i = (\bar{\rho} + \bar{P})u_i, \] (2.6)
\[ T^i_j = (\bar{P} + \delta P)\delta^i_j + \Sigma^i_j, \] (2.7)
where $\bar{\rho}(\eta)$ and $\bar{P}(\eta)$ are the background energy density and pressure of the fluid, while $\delta\rho(x, \eta)$ and $\delta P(x, \eta)$ are their respective perturbations, and $\Sigma^i_j(x, \eta) \equiv T^i_j - \delta^i_j T^k_k/3$ is its anisotropic stress tensor.\(^5\)

2.1 Background and linear perturbations

For the FLRW metric (2.3) the unperturbed Einstein field equations (2.2) read
\[ \mathcal{H}^2 = \frac{\kappa}{3}a^2(\bar{\rho}_m + \bar{\rho}), \quad \mathcal{H}' = -\frac{\kappa}{3}a^2 \left( \bar{\rho}_m + \bar{\rho} + 3\bar{P} \right), \] (2.8)
and describe the background evolution. In Eqs. (2.8), $\mathcal{H} \equiv \frac{\dot{a}}{a}$ is the conformal Hubble parameter, $\bar{\rho}_m$ is the background density of matter, and we have assumed that matter is pressure-less.\(^6\)

Now, we regard linear perturbations to the Einstein field equations (2.2) and work on the Newtonian gauge (2.3). We find
\[ 3\mathcal{H}(\mathcal{H}\Psi - \Phi') - k^2\Phi = \frac{\kappa}{2}a^2 \left( \delta T^0_0(m) + \delta T^0_0 \right), \] (2.9)
\[ k^2(\mathcal{H}\Psi - \Phi') = \frac{\kappa}{2}a^2 \left[ \bar{\rho}_m\theta_m + (\bar{\rho} + \bar{P})\theta \right], \] (2.10)
\(^5\)In our notation, a prime denotes derivative with respect to the conformal time. In addition, Greek indices run from 0 to 3 whereas Latin indices take on values from 1 to 3.
\(^6\)Note that $\mathcal{H}$ and the Hubble parameter $H$ are related through $\mathcal{H} = aH$. 





\(-\frac{k^2}{3a^2}(\Phi + \Psi) + (2\mathcal{H}' + \mathcal{H}^2)\Psi + \mathcal{H}(\Psi' - 2\Phi') - \Phi'' = \frac{\kappa}{6}a^2\delta T^i_i,\) \hspace{1cm} (2.11)

\(-k^2(\Phi + \Psi) = \frac{3\kappa}{2}a^2(\bar{\rho} + \bar{P})\sigma,\) \hspace{1cm} (2.12)

where \(\theta \equiv ik^ju_j\) is the velocity divergence, \(\theta_m\) is the velocity divergence for matter, \(k\) is the wavenumber, and we write the anisotropic stress as \((\bar{\rho} + \bar{P})\sigma \equiv -\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)\Sigma^{ij}\). We have assumed that matter has no anisotropic stress, \(\sigma_m = 0\), and that its pressure perturbation also vanishes, \(\delta T^i_i = 0\). Conservation laws for energy and momentum lead to a couple of differential equations governing the evolution of linear perturbations. If the energy-momentum tensor for our general fluid is conserved, it satisfies \(\nabla_\mu T^\mu_\nu = 0\), specifically,

\(\delta' = -(1 + w)\left(\theta + 3\Phi'\right) - 3\mathcal{H}\left(c_s^2 - w\right)\delta,\) \hspace{1cm} (2.13)

\(\theta' = -\mathcal{H}(1 - 3w)\theta - \frac{w'}{1 + w}\theta + \frac{c_s^2}{1 + w}k^2\delta - k^2\sigma + k^2\Psi,\) \hspace{1cm} (2.14)

where the equation of state parameter is defined as \(w \equiv \frac{\bar{P}}{\bar{\rho}}\), and the sound speed as \(c_s^2 \equiv \frac{\delta P}{\delta \rho}\). From Eqs. (2.13)-(2.14), it becomes clear that when the equation of state crosses \(-1\), problems emerge because there is a singularity. The trouble can be solved by a variable transformation: we use the scalar velocity perturbation \(V \equiv ik^jT^{ij}_0/\rho = (1 + w)\theta\) instead of the velocity divergence \(\theta\). In terms of this new variable the evolution equations (2.13)-(2.14) are

\(\delta' = -3(1 + w)\Phi' - \frac{V}{a^2H} - \frac{3}{a}\left(c_s^2 - w\right)\delta,\) \hspace{1cm} (2.15)

\(V' = -(1 - 3w)V + \frac{k^2}{a^2H}c_s^2\delta + \left(1 + w\right)\frac{k^2}{a^2H}\Psi - 2\frac{k^2}{3a^2H}\pi,\) \hspace{1cm} (2.16)

where we define the anisotropic stress parameter \(\pi \equiv \frac{3}{2}(1 + w)\sigma\) and a quote ‘ denotes a derivative with respect to the scale factor.

3 Scalar-Vector-Tensor theories

Although plausible, SVT theories have not gotten too much attention over the past years. These theories can accommodate Horndeski as well as generalised Proca theories and might have interesting phenomenology for the late-time universe [117]. In this section, we introduce the most general SVT Lagrangian. Let us consider

\(\mathcal{L} \equiv \sum_{i=2}^{6} \mathcal{L}_{i}^{SVT} + \sum_{i=2}^{5} \mathcal{L}_{i}^{ST} + \mathcal{L}_m,\) \hspace{1cm} (3.1)

where terms \(\mathcal{L}_{i}^{ST}\) represent the scalar-tensor interactions, and terms \(\mathcal{L}_{i}^{SVT}\) denote the scalar-vector-tensor interactions.

A scalar field \(\varphi\), a vector field \(A_\mu\), and the gravitational field \(g_{\mu\nu}\) interact with each other through the Lagrangians \(\mathcal{L}_{i}^{SVT}\) taking into account interactions with a broken U(1)
where we have used the simplified notation $g_\xi = \frac{\partial}{\partial \xi}$, for the derivative of any free function $g$ with respect to a scalar $\xi$. Let us define the different terms involved in Eqs. (3.2)-(3.6).

Firstly, note that the kinetic term of the scalar field $\varphi$, the coupling between $\varphi$ and the vector field $A_\mu$, and the quadratic term of $A_\mu$ are defined respectively by

$$X_1 \equiv -\frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi, \quad X_2 \equiv -\frac{1}{2} A_\mu \nabla^\mu \varphi, \quad X_3 \equiv -\frac{1}{2} A_\mu A^\mu.$$  

(3.7)

Secondly, the antisymmetric strength tensor $F_{\mu\nu}$ and its dual $\tilde{F}^{\mu\nu}$ are constructed from $A_\mu$ as

$$F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad \tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}.$$  

(3.8)

where $\epsilon^{\mu\nu\alpha\beta} \equiv \epsilon^{\mu\nu\alpha\beta}$, $\rho\sigma$, $\rho\sigma\alpha$ is the Levi-Civita symbol. Thirdly, using $F_{\mu\nu}$ we can construct the Lorentz invariant quantities

$$F \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

$$Y_1 \equiv \nabla_\mu \varphi \nabla^\mu \varphi F^{\mu\alpha} F_\alpha, \quad Y_2 \equiv \nabla_\mu \varphi A_\nu F^{\mu\alpha} F_\alpha, \quad Y_3 \equiv A_\nu A_\rho F^{\mu\alpha} F_\alpha,$$  

(3.9)

which vanish in the scalar limit when $A_\mu \to \nabla_\mu \pi$, $\pi$ being a scalar field. Note that quantities in Eqs. (3.9) carry the intrinsic vector modes in the Lagrangian $\mathcal{L}_{2}^{\text{SVT}}$. Fourthly, in $\mathcal{L}_{3}^{\text{SVT}}$ we find a symmetric tensor constructed from $A_\mu$ as

$$S_{\mu\nu} \equiv \nabla_\mu A_\nu + \nabla_\nu A_\mu.$$  

(3.10)

Fifthly, note that the intrinsic vector modes in Lagrangians $\mathcal{L}_{5}^{\text{SVT}}$ and $\mathcal{L}_{6}^{\text{SVT}}$ are carried by the tensors $\mathcal{M}$ and $\mathcal{N}$ given by

$$\mathcal{M}^{\mu\nu}_{5} \equiv \mathcal{G}^{h_{5}}_{\rho\sigma} \tilde{F}^{\mu\rho} \tilde{F}^{\nu\sigma}, \quad \mathcal{N}^{\mu\nu}_{5} \equiv \mathcal{G}^{h_{5}}_{\rho\sigma} \tilde{F}^{\mu\rho} \tilde{F}^{\nu\sigma},$$  

$$\mathcal{M}^{\mu\nu}_{6} \equiv 2 f_{6} X_1 (\varphi, X_1) \tilde{F}^{\mu\nu} \tilde{F}_{\alpha\beta}, \quad \mathcal{N}^{\mu\nu}_{6} \equiv \frac{1}{2} f_{6} X_1 (\varphi, X_1) \tilde{F}^{\mu\nu} \tilde{F}_{\alpha\beta},$$  

(3.11)

(3.12)

where

$$\mathcal{G}^{h_{5}}_{\rho\sigma} \equiv h_{51}(\varphi, X_1) g_{\rho\sigma} + h_{52}(\varphi, X_1) \nabla_\rho \varphi \nabla_\sigma \varphi + h_{53}(\varphi, X_1) A_\rho A_\sigma + h_{54}(\varphi, X_1) A_\rho \nabla_\sigma \varphi,$$  

$$\mathcal{G}^{h_{6}}_{\rho\sigma} \equiv h_{61}(\varphi, X_1) g_{\rho\sigma} + h_{62}(\varphi, X_1) \nabla_\rho \varphi \nabla_\sigma \varphi + h_{63}(\varphi, X_1) A_\rho A_\sigma + h_{64}(\varphi, X_1) A_\rho \nabla_\sigma \varphi,$$  

(3.13)

(3.14)
are effective metrics containing possible combinations of $g_{\rho\sigma}$, $A_\rho$, and $\nabla_\sigma \varphi$, and $X_i$ is a shorthand notation for the set \{X_1, X_2, X_3\}. Finally, the double dual Riemann tensor is defined as

$$L^{\mu\nu\alpha\beta}_{\rho\sigma\gamma\delta} = \frac{1}{4} \varepsilon^{\mu
u\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} R_{\rho\sigma\gamma\delta}. \quad (3.15)$$

Scalar-tensor interactions are taken into consideration in Eq. (3.1) through the Horndeski theory

\begin{align*}
L_{ST}^{2} &= G_2(\varphi, X_1), \quad (3.16) \\
L_{ST}^{3} &= -G_3(\varphi, X_1) \Box \varphi, \quad (3.17) \\
L_{ST}^{4} &= G_4(\varphi, X_1) R + G_{4X_1}(\varphi, X_1) \left\{ (\Box \varphi)^2 - \nabla_\mu \nabla_\nu \varphi \nabla^\nu \nabla^\mu \varphi \right\}, \quad (3.18) \\
L_{ST}^{5} &= \frac{1}{6} G_{5X_1}(\varphi, X_1) \left\{ (\Box \varphi)^3 - 3 (\Box \varphi) \nabla_\mu \nabla_\nu \varphi \nabla^\nu \nabla^\mu \varphi + 2 \nabla^\mu \nabla_\sigma \varphi \nabla^\sigma \nabla_\rho \varphi \nabla^\rho \nabla_\mu \varphi \right\}. \quad (3.19)
\end{align*}

In Eqs. (3.2)-(3.6), Eqs. (3.13)-(3.14), and Eqs. (3.16)-(3.19), all the $f_i$, $\tilde{f}_i$, $h_{5i}$, $\tilde{h}_{5i}$, and $G_i$ denote free functions.

Although in its most general form the theory (3.1) has several free functions, it got significantly constrained by the discovery of gravitational waves [62–68, 70, 127–132]. In the next subsection we explain it with more details.

### 3.1 Remaining SVT theories

An anisotropic expansion of the Universe is strongly disfavoured [133], hence in this paper we restrict ourselves to the FLRW metric in Eq. (2.3). Our matter fields will also respect constraints on homogeneity and isotropy, thus we will regard a scalar field and a vector field with the following configurations

$$\varphi \equiv \varphi(\eta) + \delta \varphi(x, \eta), \quad A_\mu \equiv (A_0(\eta) + \delta A_0(x, \eta), \delta A_\iota(x, \eta)). \quad (3.20)$$

By using definitions in Eqs. (3.7)-(3.15), it is relatively easy to show that up to first order in perturbation theory

$$F_{\mu\nu} = 0, \quad \tilde{F}^{\mu\nu} = 0, \quad (3.21)$$

which implies

$$F = Y_1 = 0, \quad M_5^{\mu\nu} = N_5^{\mu\nu} = 0, \quad M_5^{\mu\alpha\beta} = N_5^{\mu\alpha\beta} = 0, \quad L_5^{\mu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = 0. \quad (3.22)$$

As a result, Lagrangians (3.2)-(3.6) get reduced. In particular, $L_{ST}^{5}$ becomes $f_2(\varphi, X_1, X_2, X_3)$, in $L_{ST}^{5}$ the terms involving the matrices $M$ and $N$ vanish, while the Lagrangian $L_{ST}^{6}$ fully disappears.

#### 3.1.1 Speed of gravitational waves

Taking into consideration the full Lagrangian of Horndeski theory (see Eqs. (3.16)-(3.19)) and the remaining parts of the Lagrangians (3.2)-(3.6), we can obtain the propagation speed of gravitational waves through the computation of the evolution equation for tensor modes. For this calculation, the perturbed metric reads

$$ds^2 = a^2(\eta) \left[ -d\eta^2 + \left\{ \delta_{ij} + h_{ij}(x, \eta) \right\} dx^i dx^j \right], \quad (3.23)$$

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where \( h_{ij} \) are the tensor modes. Using the metric (3.23), and the configuration for the fields in Eq. (3.20), we compute the gravitational field equations for the tensor modes up to first order. We get

\[
(h_{ij})'' + (2H + \gamma_T) (h_{ij})' + c_s^2 k^2 h_{ij} = 0,
\]

where \( \gamma_T \) is a drag term due to the expansionary dynamics and

\[
c_s^2 = \frac{f_4 - \frac{A_0^2 f_5 X_3}{2a^4} + G_4 + \frac{A_0^2 f_5 X_3 H}{2a^4} - \frac{A_0 f_5 \varphi'}{2a^4} - \frac{G_5 X_3 H \varphi'^3}{2a^4}}{f_4 - \frac{A_0^2 f_5 X_3}{2a^4} + G_4 - \frac{A_0^2 f_5 X_3 H}{2a^4} + \frac{A_0 f_5 \varphi'}{2a^4} - \frac{G_5 X_3 H \varphi'^3}{2a^4} + \frac{G_5 X_1 \varphi'^2 \varphi''}{2a^4}}
\]

is the speed of gravitational waves, which agrees with the result presented in Ref. [119]. As previously mentioned, observations indicate that the propagation speed of gravitational waves is practically the speed of light. If we want general SVT theories to satisfy the constraint \( c_s^2 = 1 \) without fine-tuning,\(^7\) the following free functions in the general Lagrangian have to fulfill\(^8\)

\[
f_4 = f_4(\varphi), \quad f_5 = \text{constant}, \quad G_4 = G_4(\varphi), \quad G_5 = \text{constant}.
\]

Consequently, the remaining SVT theories are given by

\[
\begin{align*}
\mathcal{L}_2^{\text{SVT}} &= f_2(\varphi, X_1, X_2, X_3), \\
\mathcal{L}_3^{\text{SVT}} &= f_3(\varphi, X_3) g^\mu\nu S_{\mu\nu} + \tilde{f}_3(\varphi, X_3) A^\mu A^\nu S_{\mu\nu}, \\
\mathcal{L}_3^{\text{ST}} &= -G_3(\varphi, X_1) \square \varphi, \\
\mathcal{L}_4^{\text{ST}} &= G_4(\varphi) R,
\end{align*}
\]

and the complete Lagrangian reads

\[
\mathcal{L} = \mathcal{L}_2^{\text{SVT}} + \mathcal{L}_3^{\text{SVT}} + \mathcal{L}_3^{\text{ST}} + \mathcal{L}_4^{\text{ST}}.
\]

Note that the Lagrangian \( \mathcal{L}_3^{\text{ST}} \) is contained in \( \mathcal{L}_2^{\text{SVT}} \), while \( \mathcal{L}_3^{\text{SVT}} \) is taken into account in \( \mathcal{L}_4^{\text{ST}} \). Since \( G_5 \) and \( f_5 \) are constants, the Lagrangians \( \mathcal{L}_5^{\text{SVT}} \) and \( \mathcal{L}_5^{\text{ST}} \) are total derivatives. As a result \( \mathcal{L}_3^{\text{SVT}} \) and \( \mathcal{L}_4^{\text{ST}} \) are disregarded in (3.31); they do not contribute to the dynamics. In the next sections, we focus on the cosmological implications of the Lagrangian (3.31). In order to avoid very long expressions within the main text, we provide our results in terms of coefficients defined in the appendices.

### 3.2 Equations of motion

Varying the action for the Lagrangian in Eq. (3.31) with respect to the metric \( g^{\mu\nu} \), we obtain the gravitational field equations

\[
\sum_{i=2}^{3} G^{(i)}_{\mu\nu} + \sum_{i=3}^{4} \mathcal{A}^{(i)}_{\mu\nu} = \frac{1}{2} T^{(m)}_{\mu\nu},
\]

while varying with respect to the scalar field \( \varphi \), and the vector field \( A_\mu \) we get

\[
\sum_{i=2}^{3} \mathcal{J}_i + \sum_{i=3}^{4} \mathcal{K}_i = 0, \quad \sum_{i=2}^{3} A^\mu_{(i)} = 0,
\]

\(^7\)In Bayesian statistics models that have to be finely tuned to fit the data are penalised by the Occam factor [134].

\(^8\)For the choice (3.26), the drag term \( \gamma_T \) vanishes, and we get the usual wave equation for a mass-less field \( h_{ij} \) propagating at the speed of light \( c_s^2 = 1 \).
respectively. The terms \( G_{\mu\nu}^{(i)} \), \( J_i \), and \( A_\mu^{(i)} \), are associated with the SVT Lagrangians \( \mathcal{L}_{SVT}^{i} \) (\( i = 2, 3 \)), while \( \mathcal{H}_{\mu\nu}^{(i)} \) and \( K_i \) are associated to \( \mathcal{L}_{ST}^{i} \) (\( i = 3, 4 \)). The expressions for these terms can be found in the appendices A.1, A.2, and A.3.

The background equations of motion are obtained after replacing the unperturbed FLRW metric (2.3), and the zeroth-order part of the scalar field and the vector field [see Eqs. (3.20)] in Eqs. (3.32) and (3.33). For the gravitational field equations, due to rotational invariance, only the “time-time” equation and one of the diagonal “space-space” equations are needed, i.e.

\[
\sum_{i=2}^{3} g_{00}^{(i)} + \sum_{i=3}^{4} \mathcal{H}_{00}^{(i)} = \frac{1}{2} T_{00}^{(m)} = \frac{1}{2} a^2 \bar{\rho}_m, \quad \sum_{i=2}^{3} g_{11}^{(i)} + \sum_{i=3}^{4} \mathcal{H}_{11}^{(i)} = \frac{1}{2} T_{11}^{(m)} = 0, \tag{3.34}
\]

where \( T_{11}^{(m)} = 0 \) since we have assumed that matter is a pressure-less fluid. For the scalar field and the vector field we obtain

\[
\sum_{i=2}^{3} \bar{J}_i + \sum_{i=3}^{4} \bar{K}_i = 0, \quad \sum_{i=2}^{3} \bar{A}_i = 0, \tag{3.35}
\]

where \( A_0^{(i)} \equiv \bar{A}_i \), since only the time component of the vector field has dynamics at the background level. The expressions for \( g_{00}^{(i)} \) and \( \mathcal{H}_{00}^{(i)} \) are found in Appendix B.1, \( g_{11}^{(i)} \) and \( \mathcal{H}_{11}^{(i)} \) in Appendix B.2, those for \( \bar{J}_i, \bar{K}_i \) and \( \bar{A}_i \) in Appendix B.3.

Before discussing the linear perturbations of the model, we want to mention that our results differ from those in Ref. [119] due to the choice of the vector field profile. In Ref. [119], the homogeneous vector field is chosen as

\[
A_\mu = (N(t)A_0(t), 0, 0, 0), \tag{3.36}
\]

where \( N(t) \) is the lapse function, which is defined in the background metric as

\[
ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{3.37}
\]

The choice (3.37) implies that \( X_3 = A_0^2/2 \), and thus the variation of \( \mathcal{L}_3^{SVT} \) with respect to \( N \) will yield no terms with \( f_{2X_3} \). Furthermore, the Lagrangian \( \mathcal{L}_3^{SVT} \) will not contribute to the first Friedman equation. In our case, these terms do appear in the first Friedman equation, as can be seen in Appendix B.1, where \( f_{2X_3} \) can be found and \( g_{00}^{(3)} \) is not zero. Having clarified this aspect, let us discuss the first order perturbations of the theory.

Since we are only interested in scalar perturbations, we take just the scalar part of the perturbed spatial component of the vector field in Eq. (3.20), namely, \( \delta A_i(x, \eta) \equiv \partial_i \psi(x, \eta) \), where \( \psi(x, \eta) \) is a scalar field. Having this in mind, the linear perturbations of the
gravitational equations are given by
\[ 0 = A_1 \frac{\Phi'}{a} + A_2 \frac{\delta \varphi'}{a} + A_3 \frac{k^2}{a^2} \Phi + A_4 \Psi + \left( A_5 \frac{k^2}{a^2} - \mu \varphi \right) \delta \varphi + A_6 \frac{\delta A_0}{a} + A_7 \frac{k^2}{a^2} \psi - \delta \rho_m, \] (3.38)

\[ 0 = C_1 \frac{\Phi'}{a} + C_2 \frac{\delta \varphi'}{a} + C_3 \Psi + C_4 \delta \varphi + C_5 \frac{\delta A_0}{a} + C_6 \psi - \frac{a \rho_m V_m}{k^2}, \] (3.39)

\[ 0 = B_1 \frac{\Phi''}{a^2} + B_2 \frac{\delta \varphi''}{a^2} + B_3 \frac{\Phi'}{a} + B_4 \frac{\delta \varphi'}{a} + B_5 \frac{\Psi'}{a} + B_6 \frac{k^2}{a^2} \Phi + \left( B_7 \frac{k^2}{a^2} + 3 \nu \varphi \right) \delta \varphi \] (3.40)

\[ + \left( B_8 \frac{k^2}{a^2} + B_9 \right) \Psi + B_{10} \frac{\delta A_0'}{a^2} + B_{11} \frac{\delta A_0}{a}, \]

\[ 0 = G_4 \left( \Psi + \Phi \right) + G_4 \varphi \delta \varphi, \] (3.41)

corresponding to the “time-time”, longitudinal “time-space”, trace “space-space”, and longitudinal trace-less “space-space” parts of the gravitational field equations (3.32), respectively. Here, \( \delta \rho_m \) and \( V_m \) are the density perturbation and the scalar velocity of matter, respectively.

The evolution of linear perturbations for the scalar field, for the temporal component, and for the spatial component of the vector field are obtained from (3.33), respectively giving
\[ 0 = D_1 \frac{\Phi''}{a^2} + D_2 \frac{\delta \varphi''}{a^2} + D_3 \frac{\Phi'}{a} + D_4 \frac{\delta \varphi'}{a} + D_5 \frac{\Psi'}{a} + \left( D_7 \frac{k^2}{a^2} + D_8 \right) \Phi \] (3.42)

\[ + \left( D_9 \frac{k^2}{a^2} - m^2 \varphi \right) \delta \varphi + \left( D_{10} \frac{k^2}{a^2} + D_{11} \right) \Psi + D_{12} \frac{\delta A_0'}{a^2} + D_{13} \frac{\delta A_0}{a} + D_{14} \frac{k^2}{a^2} \psi, \]

\[ 0 = F_1 \frac{\Phi'}{a^2} + F_2 \frac{\delta \varphi'}{a^2} + F_3 \frac{\Psi}{a} + F_4 \frac{\delta \varphi}{a} + F_5 \frac{\delta A_0}{a^2} + F_6 \frac{k^2}{a^2} \psi, \] (3.43)

\[ 0 = H_1 \frac{\Psi}{a^2} + H_2 \frac{\delta \varphi}{a^2} + H_3 \frac{\delta A_0}{a^2} + H_4 \frac{\psi}{a^2}. \] (3.44)

All the coefficients in the first-order equations (3.38)-(3.44) can be found in Appendix C.

### 4 The effective fluid approach

In this section, we show that it is possible to rearrange the equations previously obtained, in order to define an effective dark energy fluid.\(^9\) First, note that an equation similar to the gravitational field equations
\[ G_{\mu \nu} = \kappa \left( T_{\mu \nu}^{(m)} + T_{\mu \nu}^{(DE)} \right), \] (4.1)

can be obtained in SVT theories if we define the energy-momentum tensor of dark energy as
\[ T_{\mu \nu}^{(DE)} \equiv \frac{1}{\kappa} G_{\mu \nu} - 2 \left( \sum_{i=2}^{3} \mathcal{G}^{(i)}_{\mu \nu} + \sum_{i=3}^{4} \mathcal{H}^{(i)}_{\mu \nu} \right). \] (4.2)

\(^9\)Effective quantities are also studied in Ref. [135].
Then, we can extract an effective dark energy density and pressure as

\[ \tilde{\rho}_{\text{DE}} = \frac{T^{(\text{DE})}_{00}}{a^2}, \quad \tilde{P}_{\text{DE}} = \frac{1}{3a^2} \text{tr} T^{(\text{DE})}_{ij}, \] (4.3)

\[
\tilde{\rho}_{\text{DE}} = -f_2 + \frac{\varphi'^2 f_{2X_1}}{a^2} + \frac{A_0 \varphi' f_{2X_2}}{a^2} + \frac{A_0^2 f_{2X_3}}{a^2} + \frac{2A_0 \varphi' f_{3\varphi}}{a^2} - \frac{2A_0^3 \varphi' f_{3\varphi}}{a^2} - \frac{\varphi'^2 G_{3\varphi}}{a^2} - 6A_0^3 f_{3X_1} \mathcal{H} - 6A_0^3 \mathcal{f}_3 \mathcal{H} + 3\varphi'' G_{3X_1} \mathcal{H} - 6\varphi' G_{4\varphi} \mathcal{H} - \frac{6G_4 \mathcal{H}_2}{a^2} + \frac{3\mathcal{H}^2}{\kappa a^2},
\]

(4.4)

\[
\tilde{P}_{\text{DE}} = f_2 + \frac{2A_0^2 A'_0 f_{3X_3}}{a^2} + \frac{2A_0 \varphi' f_{3\varphi}}{a^2} + \frac{2A_0^2 A'_0 f_{3\mathcal{f}}}{a^2} - \varphi'' \varphi'^2 G_{3X_1} - \varphi'^2 G_{3\varphi}
\]

\[
+ \frac{2\varphi'' G_{4\varphi}}{a^2} + \frac{2\varphi'^2 G_{4\varphi}}{a^2} - \frac{2A_0^3 f_{3X_3} \mathcal{H}}{a^4} - \frac{2A_0^3 \mathcal{f}_3 \mathcal{H}}{a^4} + \frac{\varphi'^2 G_{3X_1} \mathcal{H}}{a^2} + \frac{2\varphi' G_{4\varphi} \mathcal{H}}{a^2}
\]

\[
+ \frac{2G_4 \mathcal{H}_2}{a^2} - \frac{\mathcal{H}^2}{\kappa a^2} + \frac{4G_4 \mathcal{H}'}{a^2} - \frac{2\mathcal{H}'}{\kappa a^2},
\]

(4.5)

which allows us to characterize the effective dark energy fluid by its equation of state parameter \( w_{\text{DE}} = \tilde{P}_{\text{DE}}/\tilde{\rho}_{\text{DE}} \). The background evolution is governed by the usual Friedman equations in Eq. (2.8).

First-order variables may also be extracted from the energy-momentum tensor in Eq. (4.2). In general, we obtain expressions with the following structure:

\[
\delta \rho_{\text{DE}} = (\cdots) \delta \varphi + (\cdots) \delta \varphi' + (\cdots) \Psi + (\cdots) \Phi + (\cdots) \Phi' + (\cdots) \delta A_0 + (\cdots) \psi,
\]

(4.6)

\[
\delta P_{\text{DE}} = (\cdots) \delta \varphi + (\cdots) \delta \varphi' + (\cdots) \Psi + (\cdots) \Phi + (\cdots) \Phi' + (\cdots) \delta A_0 + (\cdots) \Phi,' + (\cdots) \psi.
\]

(4.7)

\[
V_{\text{DE}} = (\cdots) \delta \varphi + (\cdots) \delta \varphi' + (\cdots) \Psi + (\cdots) \Phi + (\cdots) \delta A_0 + (\cdots) \psi.
\]

(4.8)

However, the expressions in (\cdots) are cumbersome and it is worthwhile to look for ways to simplify them. Firstly, the quasi-static approximation (QSA) allows us to consider the gravitational potentials \( \Phi \) and \( \Psi \) as nearly time-independent functions during the matter-dominated epoch, in such a way that any time derivative of these potentials can be neglected. Secondly, we assume the so-called sub-horizon approximation (SHA) where only modes deep inside the Hubble horizon are physically interesting, i.e., \( k^2 \gg \mathcal{H}^2 \). Under these approximations time derivatives acting on perturbations variables and terms of order \( \mathcal{H} \times \text{perturbation} \) can also be neglected.\(^\text{10}\) For example, applying the SHA in Eq. (3.42) we may simplify the coefficient \( D_9 \) as

\[
D_9 \delta \varphi = \left( -f_{2X_1} - \frac{2\varphi'' G_{3X_1}}{a^2} - \frac{\varphi'' \varphi'^2 G_{3X_1} X_1}{a^2} + 2G_{3\varphi} - \frac{\varphi'^2 G_{3\varphi} X_1}{a^2} - \frac{2\varphi' G_{3X_1} \mathcal{H}}{a^2} + \frac{\varphi'^2 G_{3X_1} \mathcal{H}}{a^2} \right) \delta \varphi
\]

\[
\approx \left( -f_{2X_1} - \frac{2\varphi'' G_{3X_1}}{a^2} - \frac{\varphi'' \varphi'^2 G_{3X_1} X_1}{a^2} + 2G_{3\varphi} - \frac{\varphi'^2 G_{3\varphi} X_1}{a^2} \right) \delta \varphi,
\]

(4.9)

\(^\text{10}\)See the appendix in Ref. [99] for a detailed explanation of QSA and SHA approximations.
where we neglected terms of order $\mathcal{H}\delta \varphi$. We apply the QSA and the SHA to the linear gravitational field and scalar field equations [Eqs. (3.38), (3.40), and (3.42)] to obtain

\begin{align}
0 &= A_3 \frac{k^2}{a^2} \Phi + A_5 \frac{k^2}{a^2} \psi + A_7 \frac{k^2}{a^2} \delta \rho_m, \\
0 &= B_6 \frac{k^2}{a^2} \Phi + B_7 \frac{k^2}{a^2} \psi + B_8 \frac{k^2}{a^2} \Psi, \\
0 &= D_7 \frac{k^2}{a^2} \Phi + \left( D_9 \frac{k^2}{a^2} \delta \Phi + D_{10} \frac{k^2}{a^2} \psi + D_{14} \frac{k^2}{a^2} \psi, \right. (4.10)
\end{align}

where we had into account that in some models the mass $m_{\varphi}$ of the scalar field may play a significant role in the past, given that $m_{\varphi} \gg H$ (e.g., quintessence). For the perturbed vector field equations of motion (3.43)-(3.44), we only apply the QSA and neglect derivatives of the fields obtaining

\begin{align}
0 &= F_3 \frac{\Psi}{a^2} + F_4 \frac{\delta \varphi}{a^2} + F_5 \frac{\delta A_0}{a^2} + F_6 \frac{k^2}{a^2} \psi, \\
0 &= H_1 \frac{\Psi}{a^2} + H_2 \frac{\delta \varphi}{a^2} + H_3 \frac{\delta A_0}{a^2} + H_4 \frac{k^2}{a^2} \psi. \quad (4.13)
\end{align}

If we also applied the SHA to Eq. (3.43), we would obtain that the only relevant term would be $\frac{k^2}{a^2} \psi$, yielding the trivial solution $\psi = 0$. In that case, the gravitational field equations and the scalar field equation will not be affected by the presence of the vector field; the system would be reduced to that of Horndeski theory which was already studied in Ref. [87]. Note however that our set of equations (4.10)-(4.12) agrees with results in Ref. [87] [see their Eqs. (91)-(93)] when the vector field vanishes.

Applying the approximations as explained above, we get five algebraic equations (4.10)-(4.14) that we solve for the five perturbation variables $\Psi$, $\Phi$, $\delta \varphi$, $\delta A_0$, and $\psi$. We obtain

\begin{align}
\delta \varphi &= \frac{k^2}{a^2} W_1 + W_2 \\
\frac{\delta A_0}{a} &= \frac{k^4}{a^4} W_6 + \frac{k^2}{a^2} W_7 + W_8 \\
\frac{k^2}{a^2} \Phi &= \frac{k^4}{a^4} W_1 + \frac{k^2}{a^2} W_2 + \frac{k^2}{a^2} W_3 + \frac{k^2}{a^2} W_4 + \frac{k^2}{a^2} W_5 \\
\frac{k^2}{a^2} \Psi &= \frac{k^4}{a^4} W_1 + \frac{k^2}{a^2} W_2 + \frac{k^2}{a^2} W_3 + \frac{k^2}{a^2} W_4 + \frac{k^2}{a^2} W_5 \\
\end{align}

where we took into account that some of the perturbations coefficients are related (see Appendix C),

\begin{align}
A_3 = B_6 = B_8, \quad D_{10} = A_5, \quad D_7 = B_7, \quad D_{14} = H_2, \quad A_7 = H_1, \quad F_6 = H_3. \quad (4.16)
\end{align}

The coefficients $W_i$ ($i = 1, \ldots, 15$) are given in Appendix D. Actually, we could have four dynamical degrees of freedom. From Eq. (4.14), we could retrieve $\delta A_0$ and insert it in the approximated linear equations, thus reducing one dynamical degree of freedom. Nonetheless, the procedure we follow might be clearer since the mentioned reduction yields results for $\Phi$, $\Psi$, $\varphi$, and $\psi$ much more complicated to handle.
From the expressions for the potentials in Eq. (4.15), we can characterize deviations from GR by defining the gravitational slip parameters

\[
\eta \equiv \frac{\Psi + \Phi}{\Phi} = \frac{k_4^4(W_{11} - W_{14}) + k_2^2(W_{12} - W_{15})}{k_4^4W_{11} + k_2^2W_{12} + W_{13}},
\]

(4.17)

\[
\gamma \equiv -\frac{\Phi}{\Psi} = \frac{k_4^4W_{11} + k_2^2W_{12} + W_{13}}{k_4^4W_{14} + k_2^2W_{15} + W_{13}},
\]

(4.18)

where the GR case corresponds to \(\eta = 0\) and \(\gamma = 1\). Note that the expressions of the gravitational potentials can be written as Poisson-like equations if we define parameters \(G_{\text{eff}}\) and \(Q_{\text{eff}}\) such that

\[
k_4^2a^4\Psi = -\frac{1}{2}G_{\text{eff}}G_N\delta \rho_m, \quad k_2^2a^2\Phi = \frac{1}{2}Q_{\text{eff}}\delta \rho_m.
\]

These parameters also characterize modifications to gravity. The GR case corresponds to \(G_{\text{eff}}/G_N = 1\), and \(Q_{\text{eff}} = G_N\), with \(\Phi = -\Psi\). In the case of SVT theories, as shown in Eq. (3.41), the presence of the scalar field prevents both potential to be opposite. Hence, anisotropic stress in SVT theories is sourced solely by the scalar field whenever the function \(G_4(\varphi)\) is not a constant. Note that, under this scheme, generalised Proca theories do not admit anisotropic stress.

As we will show below, the parameter \(G_{\text{eff}}\) largely determines the evolution of the growth of matter perturbations. Using the QSA and the SHA in the Eqs. (2.15) and (2.16) for matter \((w_m = c_{s,m}^2 = \pi_m = 0)\), we get

\[
\delta_m' \sim -\frac{V_m}{a^2H}, \quad V_m' \sim -\frac{V_m}{a} + \frac{k_2^2}{a^2}H \Psi.
\]

(4.20)

Therefore, differentiating the equation for \(\delta_m'\), inserting \(V_m'\) in that derivative, and using the Poisson equation for \(\Psi\) in Eq. (4.19), the evolution equation for \(\delta_m\) will be given by

\[
\delta_m''(a) + \left(\frac{3}{a} + \frac{H'(a)}{H(a)}\right)\delta_m'(a) - \frac{3}{a^3H(a)} \frac{G_{\text{eff}}G_N}{a^5H^2} \delta_m(a) = 0.
\]

(4.21)

Hence, by solving Eq. (4.21), we can determine the growth factor. We will work out a fully numerical solution of Eq. (4.21) in the Sec. 5.3 for a specific \(G_{\text{eff}}\) obtained from our designer SVT model, which we will describe in Sec. 5.

In what follows we will consider perturbations to the effective dark energy under QSA and SHA in two cases: i) non-vanishing anisotropic stress \(\pi_{\text{DE}}\); ii) \(G_4 = \text{constant}\) and hence vanishing anisotropic stress \(\pi_{\text{DE}} = 0\).

### 4.1 SVT theories with non-vanishing anisotropic stress

Now, we apply the QSA and the SHA to the quantities in Eqs. (4.6)-(4.8). We proceed as follows. Since the QSA breaks down due to rapid oscillations of the scalar field [87], we use the trace-less “space-space” equation (3.41) in order to solve for \(\delta \varphi\) in terms of the gravitational potentials. By differentiating (3.41), we can also solve for the derivatives of \(\delta \varphi\), also applying the QSA and the SHA at the end of the differentiation. For instance, for the first derivative we get

\[
\delta \varphi' = -\frac{G_4 \varphi'}{G_4} \varphi' \delta \varphi - \varphi' (\Psi + \Phi).
\]

(4.22)
This is the reason why we discriminate models with and without anisotropic stress: since $G_4 = \text{constant}$ when anisotropic stress vanishes, Eq. (4.22) would diverge. When we deal with models having a non-vanishing anisotropic stress we can then replace the potentials using the Poisson equations in Eq. (4.19), leaving all the expressions in terms of $\delta \rho_m$. We obtain

$$
\delta \rho_{DE} = \frac{k^6}{a^4} Z_1 + \frac{k^4}{a^4} Z_2 + \frac{k^2}{a^2} Z_3 + Z_4 \delta \rho_m, \quad \delta P_{DE} = \frac{1}{3 Z_{12}} \frac{k^6}{a^4} Z_8 + \frac{k^4}{a^4} Z_9 + \frac{k^2}{a^2} Z_{10} + Z_{11} \delta \rho_m,
$$

$$
a \tilde{\rho}_{DE} V_{DE} = \frac{k^4}{a^4} Z_1 + \frac{k^2}{a^2} Z_6 + \frac{k^2}{a^2} Z_7 \delta \rho_m, \quad \tilde{\rho}_{DE} \Pi_{DE} = \frac{k^2}{a^2} \frac{k^2}{a^2} (W_{14} - W_{11}) + (W_{15} - W_{12}) \delta \rho_m,
$$

$$
e_s^2 = \frac{1}{3 Z_{12}} \frac{k^6}{a^4} Z_8 + \frac{k^4}{a^4} Z_9 + \frac{k^2}{a^2} Z_{10} + Z_{11}.
$$

The coefficients $Z_i$ ($i = 1, \dots, 15$) are presented in Appendix E.1. Due to the presence of the anisotropic stress, the sound speed in Eq. (4.23) does not fully determine the stability of sub-horizon perturbations. This can be seen by solving Eq. (2.13) for $\theta$ and substituting the result (and its derivative) into (2.14). Doing so, we obtain the following second-order equation for $\delta$

$$
\delta'' + (1 - 6w) \mathcal{H}' + 3 \mathcal{H} \left( \frac{\delta P}{\delta \rho} \right)' + 3 \left[ (1 - 3w) \mathcal{H}^2 + \mathcal{H}' \right] \left( \frac{\delta P}{\rho} - \rho \delta \right) - 3 \mathcal{H} w' \delta
$$

$$
= -3(1 + w) \left[ \Phi'' + \left( 1 - 3w + \frac{w'}{1 + w} \right) \mathcal{H} \Phi' \right] - k^2 \left[ (1 + w) \Psi + \frac{\delta P}{\rho} - \frac{2}{3} \right],
$$

(4.24)

where we have used the relation $\pi = \frac{3}{2} (1 + w) \sigma$. For sub-horizon modes, the last term factorized by $k^2$ in Eq. (4.24) is the relevant term which determines the stability of perturbations. Since the potential scales as $\Psi \sim 1/k^2$ for these modes, therefore, the stability of sub-horizon perturbations is driven mainly by an effective sound speed defined as $[136]$

$$
e_s^{2, \text{eff}} = e_s^2 \frac{\tilde{\rho}_{DE} \Pi_{DE}}{3 \delta \rho_{DE}}.
$$

(4.25)

Thus far the discussion has been quite general, providing analytical expressions for the field perturbations $\delta \varphi$, $\delta A_0$, $\psi$, and the potentials $\Phi$ and $\Psi$ in Eqs. (4.15). Now we want to test our equations against known results in literature, namely: $f(R)$ theories, quintessence, and generalised Proca. We also indicate some possible, minimal modifications in the context of SVT theories.

- **$f(R)$ Theories**

Through a conformal transformation, $f(R)$ theories can be seen as a theory for a scalar field non-minimally coupled to $R$. This theory will be contained in SVT theories if we do the following identification

$$
f_2 = - \frac{RF - f}{2}, \quad f_{2 \varphi} = - \frac{R}{2}, \quad f_{2 \varphi \varphi} = - \frac{1}{2F_R}, \quad G_4 = \frac{F}{2}, \quad G_{4 \varphi} = \frac{1}{2},
$$

(4.26)
where $F \equiv f_R = \sqrt{\kappa} \varphi$, $F_R \equiv f_{RR}$; unspecified derivatives and free-functions are set to zero, and we have assumed $\kappa = 1$. Replacing (4.26) in Eqs. (4.4)-(4.5), we get the well-known expressions for the density and pressure of DE in $f(R)$ theories:

$$a^2 \rho_{DE} = -\frac{a^2 f}{2} + 3\mathcal{H}^2 + 3F\mathcal{H}' - 3HF', \quad (4.27)$$

$$a^2 P_{DE} = \frac{a^2 f}{2} - (1 + 2F)\mathcal{H}^2 - (2 + F)\mathcal{H}' + \mathcal{H}F' + F''. \quad (4.28)$$

Under the QSA and the SHA, replacing the above functions [Eq. (4.26)] in Eqs. (4.15) we get for the perturbation variables

$$\delta \varphi = \frac{F_R}{F + 3\frac{k^2}{a^2}F_R} \delta \rho_m, \quad \delta A_0 = 0, \quad \psi = 0,$

$$\Psi = -\frac{F + 4\frac{k^2}{a^2}F_R}{2\frac{k^2}{a^2}F^2 + 6\frac{k^4}{a^4}F_R} \delta \rho_m, \quad \Phi = \frac{F + 2\frac{k^2}{a^2}F_R}{2\frac{k^2}{a^2}F^2 + 6\frac{k^4}{a^4}F_R} \delta \rho_m. \quad (4.29)$$

Since $\Phi \neq -\Psi$, the slip parameters are not constants and Eqs. (4.17) and (4.18) become

$$\eta = -\frac{2\frac{k^2}{a^2}F_R}{F + 2\frac{k^2}{a^2}F_R}, \quad \gamma = \frac{F + 2\frac{k^2}{a^2}F_R}{F + 4\frac{k^2}{a^2}F_R}. \quad (4.30)$$

The effective DE perturbed quantities in Eq. (4.23) take on

$$\delta \rho_{DE} = \frac{(1 - F)F + (2 - 3F)\frac{k^2}{a^2}F_R}{F(F + 3\frac{k^2}{a^2}F_R)} \delta \rho_m, \quad \delta P_{DE} = \frac{1}{3F} \frac{2\frac{k^4}{a^4}F_R + 15\frac{k^2}{a^2}F_R F'' + 3FF''}{3\frac{k^4}{a^4}F_R + 2\frac{k^2}{a^2}F} \delta \rho_m,$$

$$\bar{\rho}_{DE} = \frac{1}{2F} \frac{(F + 6\frac{k^2}{a^2}F_R)F'}{F + 3\frac{k^2}{a^2}F_R} \delta \rho_m; \quad \bar{\rho}_{DE} \bar{\gamma}_{DE} = \frac{\frac{k^2}{a^2}F_R}{2F + 3\frac{k^2}{a^2}F_R} \delta \rho_m. \quad (4.31)$$

These results are in perfect agreement with those reported in Refs. [86, 87, 137].

- $f(R)$ + Cubic Vector Interactions

We can have minimal modifications to $f(R)$ theories by adding cubic interactions coming from the vector sector in SVT theories. If we assume $f_3 = \text{constant}$, we see that in order to get non trivial solutions, $\tilde{f}_3$ must depend at least on $\varphi$. Assuming Eqs. (4.26), $f_3 = 0$, $\tilde{f}_3 = \tilde{f}_3(\varphi)$, and $\kappa = 1$, from the background equation of motion for the scalar and vector fields (3.35), we obtain

$$2\frac{A_0^2}{a^4} \tilde{f}_3 \varphi (A_0' - \mathcal{H} A_0) = 0, \quad -2\frac{A_0^2}{a^4} \left( \tilde{f}_3 \varphi F' + 3\mathcal{H} \tilde{f}_3 \right) = 0, \quad (4.32)$$

and therefore,

$$3\mathcal{H} \tilde{f}_3 = -\tilde{f}_3 \varphi F', \quad A_0' = \mathcal{H} A_0. \quad (4.33)$$

Note that the right-hand side expression in the last equation can be recast as $(\frac{da}{a})' = 0$. In generalised Proca theories, a vector field fulfilling this condition characterizes de Sitter solutions [99], which is not the case in this vector $f(R)$ theory. Using Eq. (4.26) and Eq.
as well as the QSA and the SHA, we get the same density and pressure given in Eqs. (4.27) and (4.28), meaning that the background evolution of this new model is not modified by the inclusion of the vector field. However, perturbations do get a contribution from the vector field. Replacing the new configuration in Eqs. (4.26) and (4.33) in the expressions in Eq. (4.23), we get

\[
\delta \rho_{\text{DE}} = \frac{(1 - F)}{F R^2} \left[ 2 k^4 F + 2 F R F' A^6_0 \delta \rho_m \right] 
\]

\[
\delta P_{\text{DE}} = \frac{(1 - F)}{F R^2} \left[ 2 k^4 F + 2 F R F' A^6_0 \delta \rho_m \right] 
\]

\[
\delta \rho_{\text{DE}} V_{\text{DE}} = \frac{k^2}{a^2} \left[ \frac{k^4}{a^2} Y_1 + \frac{k^4}{a^2} Y_2 + \frac{k^4}{a^2} Y_3 + Y_4 \right] \delta \rho_m, \quad \delta P_{\text{DE}} V_{\text{DE}} = \frac{k^2}{a^2} \left[ \frac{k^4}{a^2} Y_5 + \frac{k^4}{a^2} Y_6 + \frac{k^4}{a^2} Y_7 \right] \delta \rho_m.
\]

The phenomenology of these kinds of models could therefore add new features or help addressing current discrepancies in cosmological parameters as discussed, for instance, in Ref. [32].

4.2 SVT theories with vanishing anisotropic stress

If the DE anisotropic stress vanishes, then \( \Phi = -\Psi \) and from Eq. (3.41) \( G_4 \) does not depend on the scalar field. For the sake of simplicity, we assume \( G_4 = 1/2\kappa \) so that \( \mathcal{L}^\text{SVT}_4 \) in Eq. (3.30) equals the Einstein-Hilbert Lagrangian. Applying these assumptions in Eqs. (4.6)-(4.8) as well as the QSA and the SHA, we get

\[
\delta \rho_{\text{DE}} = \frac{k^4}{a^2} Y_1 + \frac{k^4}{a^2} Y_2 + \frac{k^4}{a^2} Y_3 + Y_4 \delta \rho_m, \quad \delta P_{\text{DE}} = \frac{1}{3} \frac{k^4}{a^2} Y_5 + \frac{k^4}{a^2} Y_6 + \frac{k^4}{a^2} Y_7 \delta \rho_m.
\]

The typical Lagrangian of a quintessence scalar field can be recovered by defining

\[
f_2 = X_1 - V(\varphi), \quad f_{2\varphi} = -V_\varphi, \quad f_{2\varphi\varphi} = -V_{\varphi\varphi}, \quad G_4 = \frac{1}{2},
\]

while any other function vanishes, and \( \kappa = 1 \). Using the definitions (4.36) in Eqs. (4.4) and (4.5) we get the usual density and pressure

\[
\bar{\rho}_{\text{DE}} = X_1 + V, \quad \bar{P}_{\text{DE}} = X_1 - V.
\]
The full perturbations in Eqs. (4.6)-(4.8) are simply given by
\[
\delta \rho_{\text{DE}} = \frac{\varphi' \delta \varphi}{a^2} + V_{\varphi} \delta \varphi - \frac{\varphi'^2}{a^2} \Psi, \quad \delta P_{\text{DE}} = \frac{\varphi' \delta \varphi}{a^2} - V_{\varphi} \delta \varphi - \frac{\varphi'^2}{a^2} \Psi, \quad (4.38)
\]
\[
\bar{\rho}_{\text{DE}} V_{\text{DE}} = k^2 a^{-1} \frac{\varphi' \delta \varphi}{a}, \quad c_{s,\text{DE}}^2 = \frac{\delta P_{\text{DE}}}{\delta \rho_{\text{DE}}}. \quad (4.39)
\]
Note however that, under the SHA and the QSA, Eqs. (4.35) provide simplified expressions
\[
\delta \rho_{\text{DE}} = \delta P_{\text{DE}} = \frac{\varphi'^2}{2k^2} \delta \rho_m, \quad \bar{\rho}_{\text{DE}} V_{\text{DE}} = 0, \quad c_{s,\text{DE}}^2 = 1. \quad (4.40)
\]
These results agree with those reported in Refs. [87].

**Quintessence + Cubic Vector Interactions**

The phenomenology in the previous example can become more interesting by introducing a vector field, as we did for \( f(R) \) theories in Subsection 4.1. Replacing the quintessence functions (4.36) in the background equation of motion for the scalar and vector fields in Eq. (3.35), and assuming that \( f_3 = 0 \) and \( \bar{f}_3 = \bar{f}_3(\varphi) \), we get
\[
\varphi'' + 2H \varphi' + a^2 V_{\varphi} - 2\bar{f}_3 \varphi \left[ A'_0 - \mathcal{H} A_0 \right] = 0, \quad -2 \frac{A_0^2}{a^2} \left( \bar{f}_3 \varphi' + 3\mathcal{H} \bar{f}_3 \right) = 0. \quad (4.41)
\]
We see that the usual Klein-Gordon equation for a scalar field is recovered when the vector field fulfills \( A'_0 = \mathcal{H} A_0 \). Using the expression in the right-hand side of Eq. (4.41) and the functions (4.36) in Eqs. (4.4)-(4.5), we get that the background density and pressure are
\[
\bar{\rho}_{\text{DE}} = X_1 + V, \quad \bar{P}_{\text{DE}} = X_1 - V - \frac{2}{3} \frac{A_0^2}{a^2} \bar{f}_3 \varphi' (A'_0 - \mathcal{H} A_0). \quad (4.42)
\]
Under the QSA and the SHA, the sound speed is now more involved due to non trivial contributions to the pressure perturbation in Eq. (4.35)
\[
\delta \rho_{\text{DE}} = \frac{\varphi'^2 + 2 \frac{A_0^2}{a^2} \bar{f}_3 \varphi'}{2k^2 - 4 \frac{A_0^2}{a^2} \bar{f}_3 \varphi'} \delta \rho_m, \quad \delta P_{\text{DE}} = \frac{\varphi'^2 - 4 \frac{A_0^2}{a^2} \bar{f}_3 \varphi'}{2k^2 - 4 \frac{A_0^2}{a^2} \bar{f}_3 \varphi'} \delta \rho_m. \quad (4.43)
\]
\[
\bar{\rho}_{\text{DE}} V_{\text{DE}} = - \frac{1}{a^2} \frac{2 A_0^2}{k^2} \bar{f}_3 \varphi' A'_0 \delta \rho_m, \quad c_{s,\text{DE}}^2 = \frac{\varphi'^2 - 4 \frac{A_0^2}{a^2} \bar{f}_3 \varphi'}{\varphi'^2 + 2 \frac{A_0^2}{a^2} \bar{f}_3 \varphi'}. \quad (4.44)
\]
Note that in this case the velocity perturbation does not vanish and the sound speed in general \( c_{s,\text{DE}}^2 \neq 1 \).

**Generalised Proca**

Generalised Proca theories are obtained from the SVT Lagrangian (3.31) by assuming that
\[
f_2 = f_2(X_3), \quad f_3 \rightarrow \frac{1}{2} f_3(X_3), \quad G_4 = \frac{1}{2}. \quad (4.45)
\]
while all the other unspecified functions vanish, due to the constraint coming from the speed of gravitational waves, namely \( c_{T}^2 = 1 \) [99], and we have assumed \( \kappa = 1 \). The second Friedman
equation and the vector field equation of motion obtained from Eqs. (3.34) and (3.35), using (4.45), are the following constraints

\[
f_2 - \frac{A_0^2 f_{2X_3}}{3a^2} + \frac{\mathcal{H}^2}{a^2} + 2 \frac{\mathcal{H}'}{a^2} + \frac{A_0 A_0' f_{2X_3}}{2a^2 \mathcal{H}} = 0, \quad \frac{A_0}{a^2} f_2 x_3 - 3 \frac{\mathcal{H}}{a^4} f_{3X_3} = 0. \tag{4.46}
\]

Using the equation in the right-hand side of Eqs. (4.46) to eliminate \( f_{3X_3} \), the corresponding background density and pressure for generalised Proca model [see Eqs. (4.4) and (4.5)] are given by

\[
\bar{\rho}_{\text{DE}} = -f_2, \quad \bar{P}_{\text{DE}} = f_2 - \frac{A_0 (A_0' - \mathcal{H} A_0)}{3a^2 \mathcal{H}} f_{2X_3}. \tag{4.47}
\]

The full perturbations in Eqs. (4.6)-(4.8) for the effective DE fluid read

\[
\delta \rho_{\text{DE}} = -f_2 - \frac{A_0^2 f_{2X_3}}{a^4} - \frac{2 A_0 f_{2X_3}}{a^2} + \frac{3 A_0^4 f_{3X_3} \mathcal{H}}{a^6} \Delta A_0 - \frac{A_0 f_{2X_3}}{3a^2 \mathcal{H}} \frac{k^2}{a^2} \Psi', \tag{4.48}
\]

\[
\bar{\rho}_{\text{DE}} V_{\text{DE}} = \frac{k^2}{a^2} \frac{f_{2X_3}}{3a^2 \mathcal{H}} \Delta A_0, \tag{4.49}
\]

\[
\delta P_{\text{DE}} = \left( \frac{A_0^2 f_{2X_3}}{3a^2} + \frac{A_0^3 f_{3X_3} \mathcal{H}}{a^6} \right) \Delta A_0 + \left( \frac{A_0^2 f_{2X_3}}{3a^2 \mathcal{H}} - \frac{A_0^3 f_{3X_3} \mathcal{H}}{a^6} \right) \Delta A_0 + \left( \frac{A_0^2 f_{2X_3}}{3a^2} - \frac{A_0^3 f_{3X_3} \mathcal{H}}{a^6} \right) \Psi - \frac{A_0^2 f_{2X_3}}{3a^2} \Psi'. \tag{4.50}
\]

These results are very similar to those reported in Ref. [138], but they are not equal since differences arise due to a different choice for the vector field profile. Under the QSA and the SHA, these perturbed quantities take the following simple form

\[
\delta \rho_{\text{DE}} = - \frac{A_0^2 f_{2X_3}}{A_0^2 f_{2X_3} + 2k^2 A_0} \delta \rho_m, \quad \bar{\rho}_{\text{DE}} V_{\text{DE}} = \frac{A_0 f_{2X_3} A_0'}{A_0^2 f_{2X_3} + 2k^2 \mathcal{H} \delta \rho_m} \approx 0,
\]

\[
\delta P_{\text{DE}} = \frac{2 A_0^2 f_{2X_3} - 3a^2 f_2}{3 A_0^2 f_{2X_3} + 2k^2} \delta \rho_m, \quad c_{s,\text{DE}}^2 = - \frac{2}{3} + \frac{2a^2 f_2}{A_0^2 f_{2X_3}}, \tag{4.51}
\]

where we have used the equation in the left-hand side of Eqs. (4.46) to eliminate \( A_0' \). Then, under these approximations, DE in generalised Proca theories is on its rest-frame, and the sound speed \( c_{s,\text{DE}}^2 \) is different from 1.

In Refs. [90, 99], authors investigated a particular model where the free functions are given by

\[
f_2 = b X_3^m, \quad f_3 = \frac{1}{2} c X_3^n, \quad G_4 = \frac{1}{2}, \tag{4.52}
\]

where \( b, c, m, n \), are constants. This power law Proca model has a phantom equation of state of dark energy when \( A_0^2 \propto \mathcal{H}^{-1} \), with \( p \equiv 2(n-m)+1 \). Because in the next example we want to show how the introduction of a scalar field can change the dynamics of the generalised Proca model (4.52), we will assume \( b = c = -1, m = 1, n = 5/2 \).

From the second Friedman equation and the equation of motion for the vector field in Eqs. (4.46), and using (4.52), we get

\[
\frac{A_0}{a^2} = -\frac{2 \sqrt{2} a^3}{5 A_0^3} + \mathcal{H} A_0 + \frac{4 \sqrt{2} a^3}{5 A_0^3} (\mathcal{H}^2 + 2 \mathcal{H}') - \frac{A_0}{a} = \frac{2^{5/8} a^{1/4}}{15^{1/4} \mathcal{H}^{1/4}}. \tag{4.53}
\]
Under these assumptions, the approximated perturbations (4.51) read
\[
\delta \rho_{\text{DE}} = -\frac{2^{1/4}a^{5/2}}{2^{1/4}a^{5/2} - \sqrt{15Hk}^2} \delta \rho_m, \quad \delta P_{\text{DE}} = -\frac{1}{3} \frac{2^{1/4}a^{5/2}}{2^{1/4}a^{5/2} - \sqrt{15Hk}^2} \delta \rho_m, \quad c_{s,\text{DE}}^2 = \frac{1}{3}.
\] (4.54)

- **Generalised Proca + Scalar Interactions**

We modify the generalised Proca model (4.52) in the following way
\[
f_2 = b X_3^m + X_1.
\] (4.55)

Taking the same powers, namely, \( b = c = -1 \), \( m = 1 \), \( n = 5/2 \), from Eqs. (4.35) the approximated perturbations in Eqs. (4.51) are modified as
\[
\delta \rho_{\text{DE}} = -\frac{2^{5/4}a^{5/2} + \sqrt{15H\varphi^2}}{2^{5/4}a^{5/2} - 2\sqrt{15Hk}^2} \delta \rho_m, \quad c_{s,\text{DE}}^2 = \frac{1}{3} \frac{2^{5/4}a^{5/2}}{2^{5/4}a^{5/2} + \sqrt{15H\varphi^2}},
\]
while \( \delta P_{\text{DE}} \) is the same given in Eq. (4.54). Therefore, in this case the sound speed of scalar perturbations \( c_{s,\text{DE}}^2 \) is in general time-dependent.

5 Designer SVT

For the SVT theories regarded in this work, Eqs. (4.23) and (4.35) represent analytical expressions for the effective DE perturbations under the QSA and the SHA. General SVT theories have several free functions (i.e., \( f_2(\varphi, X_1, X_2, X_3) \), \( f_3(\varphi, X_3) \), \( f_3(\varphi, X_3) \), \( G_3(\varphi, X_1) \), and \( G_4(\varphi) \)) which could be useful for unravelling conundrums in the standard cosmological model. In this section, we will show an example of how our effective fluid approach to SVT theories makes it possible to design a cosmological model matching the background evolution in the ΛCDM model while having non-vanishing DE perturbations. We will designate this model as SVTDES.

5.1 Designer procedure

To begin with, note that using the Leibniz rule, the general equation of motion for the scalar field in the left-hand side of Eq. (3.33) can be recast as
\[
\nabla^\mu J_\mu = K_\varphi,
\] (5.1)
where
\[
J_\mu = (-f_2X_1 + G_3X_1 \Box \varphi + 2G_3\varphi) \nabla_\mu \varphi + G_3X_1 \nabla_\mu X_1
+ \left(-\frac{1}{2} f_2X_2 - 2f_3\varphi + 4X_3\bar{f}_3\varphi\right) A_\mu,
\] (5.2)
\[
K_\varphi = f_2\varphi - 2A_\mu \nabla^\mu f_3\varphi - 2f_3\varphi \left(A_\nu \nabla^\nu A_\nu + A^\nu \nabla^\mu A_\nu\right) A_\mu + 4X_3A_\mu \nabla^\mu \bar{f}_3\varphi
+ \nabla^\mu G_3\varphi \nabla_\mu \varphi + G_4\varphi R,
\] (5.3)
where \( \Box \equiv \nabla^\mu \nabla_\mu \) is the usual Laplacian operator. Substituting the background configuration for the fields [see Eq. (3.20)] in the Eq. (5.2), we find that only the temporal component of the current \( J_\mu \) does not vanish
\[
J_0 \equiv J(\eta) = \left(-f_2X_1 - 3\frac{H\varphi'}{a^2} G_3X_1 + 2G_3\varphi\right) \varphi' + \left(-\frac{1}{2} f_2X_2 - 2f_3\varphi + \frac{A^2}{a^2} \bar{f}_3\varphi\right) A_0,
\] (5.4)
and satisfies the differential equation

$$J' + 2HJ + a^2 K_\varphi = 0. \quad (5.5)$$

When $K_\varphi = 0$, the solution of Eq. (5.5) is simply

$$J(\eta) = -\frac{J_c}{a^2}, \quad (5.6)$$

where $J_c$ is a constant. If $J_c = 0$, then the system is on the attractor solution. If $J_c \neq 0$, then the system is out of the attractor and interesting phenomenology might emerge. We will assume that $J_c$ is small, and as we will see later, it will serve as a parameter tracking deviations from $\Lambda\text{CDM}$.

Avoiding fine-tuning of the functions, the term $K_\varphi$ can be zero if we demand

$$f_{2\varphi} = 0, \quad f_{3\varphi} = 0, \quad \tilde{f}_3 = 0, \quad G_{3\varphi} = 0, \quad G_{4\varphi} = 0, \quad (5.7)$$

which implies that the remaining SVT free-functions must be of the form

$$f_2 = f_2(X_1, X_2, X_3), \quad f_3 = f_3(X_3), \quad \tilde{f}_3 = \tilde{f}_3(X_3), \quad G_3 = G_3(X_1), \quad G_4 = \text{constant}. \quad (5.8)$$

From now on we will assume $G_4 = 1/2$ and $k = 1$. Note that conditions (5.7)-(5.8) on the effective DE density (4.4) and pressure (4.5) yield an effective DE equation of state

$$w_{\text{DE}} = \frac{f_2 + (f_{3X_3} + \tilde{f}_3)}{-f_2 + \frac{\varphi'f_{2X_1}}{a} + \frac{\Lambda_0\varphi'f_{2X_2}}{a^2} + \frac{\Lambda_0^2f_{2X_3}}{a^3} - \frac{6\Lambda_0^2H_a}{a^4}(f_{3X_3} + \tilde{f}_3)} + \frac{3\varphi'^2G_3X_1H_a}{a^4} \quad (5.9)$$

Since $G_4$ is a constant, Eq. (3.41) implies $\Phi = -\Psi$, or in other words, we are designing a model with no anisotropic stress. Note that using the conditions (5.8), we can rewrite the Friedman equation [left-hand side of (3.34)], the equation of motion for the scalar field [Eqs. (5.4) and (5.6)], and the equation of motion for the vector field [right-hand side of (3.35)], respectively as

$$0 = -\frac{H^2}{a^2} + H^2_0 \Omega_{m0} - \frac{1}{3} f_2 + \frac{2}{3} X_1 f_{2X_1} + \frac{2}{3} X_2 f_{2X_2} + \frac{2}{3} X_3 f_{2X_3} + 2 \sqrt{2} X_1^{3/2} \frac{H}{a} G_{3X_1} - 4 \sqrt{2} X_3^{3/2} \frac{H}{a} (f_{3X_3} + \tilde{f}_3), \quad (5.10)$$

$$0 = \frac{J_c}{a^3} - 6 X_1 \frac{H}{a} G_{3X_1} - \sqrt{2} X_1^{1/2} f_{2X_1} - \frac{\sqrt{2}}{2} X_3^{1/2} f_{2X_2}, \quad (5.11)$$

$$0 = -\sqrt{2} X_3^{1/2} f_{2X_3} + 12 X_3 \frac{H}{a} (f_{3X_3} + \tilde{f}_3) - \frac{\sqrt{2}}{2} X_1^{1/2} f_{2X_2}, \quad (5.12)$$

where we have defined the density parameter of matter $\Omega_{m0} \equiv \rho_{m0}/3H_0^2 \approx 0.3$; $\rho_{m0}$ is the density of matter today and $H_0$ is the Hubble constant. In Eqs. (5.10)-(5.12) we replaced the fields $\varphi'$ and $A_0$ by the variables

$$X_1 = \frac{\varphi'^2}{2a^2}, \quad X_2 = \frac{\varphi'A_0}{2a^2}, \quad X_3 = \frac{A_0^2}{2a^2}. \quad (5.13)$$

We are interested in a particular model in SVT theories whose background evolution matches identically that of $\Lambda\text{CDM}$, where the Hubble parameter is given by

$$H^2 = a^2 H_0^2 (\Omega_{m0} a^{-3} + \Omega_0), \quad (5.14)$$
where $\Omega_{\Lambda 0} \approx 0.7$ is the density parameter of dark energy today. From Eq. (5.9) we can see that $f_{3X1} + \tilde{f}_3 = 0$, $G_{3X1} = 0$, and $f_2$ is constant imply $\bar{\rho}_{DE} = -1$ as in the standard cosmological model. In this case, the vector field equation of motion (5.12) is trivially satisfied while from Eq. (5.11) we see that the scalar field is on the attractor solution $J_c = 0$. Furthermore, the Friedman equation (5.10) and its solution for $\Lambda CDM$ (5.14) allow us to determine $f_2 = -3H_0^2 \Omega_{\Lambda 0}$. Such a model has vanishing DE perturbations. Next we will show that there exists a $SVT$ model matching $\Lambda CDM$ background while having non-vanishing DE perturbations.

For SVT theories assuming Eq. (5.8), in general $\bar{\rho}_{DE}$ depends on $X_1$, $X_2$, and $X_3$, and thus from the Friedman equation (5.10) we see that $\mathcal{H}$ might also be also a function of these terms, i.e., $\mathcal{H} = \mathcal{H}(X_1, X_2, X_3)$. We can consider a further simplification, by noting that $X_2^2 = X_1 X_3$ up to first order in perturbations, therefore we can assume $f_2 = f_2(X_1, X_3)$ and $\mathcal{H} = \mathcal{H}(X_1, X_3)$. These assumptions imply $f_{2X_2} = 0$ and using Eqs. (5.10)-(5.12) we find

$$f_2(X_1, X_3) = -3H_0^2 \Omega_{\Lambda 0} + \frac{J_c \sqrt{2} X_1^{1/2}}{\Omega_m 0} \left[ \frac{H^2}{H_0^2} - \Omega_{\Lambda 0} \right], \quad (5.15)$$

$$f_{2X_1} = \frac{J_c H^2}{\sqrt{2} H_0^2 X_1^{1/2} \Omega_m 0} - \frac{J_c \Omega_{\Lambda 0}}{\sqrt{2} X_1^{1/2}} + \frac{2 \sqrt{2} H J_c X_1^{1/2} H X_1}{H_0^2 \Omega_m 0}, \quad (5.16)$$

$$f_{2X_3} = \frac{2 \sqrt{2} H J_c X_1^{1/2} H X_3}{H_0^2 \Omega_m 0}, \quad (5.17)$$

$$G_{3X1} = \frac{2}{3} \frac{J_c H X_1}{H_0^2 \Omega_m 0}, \quad (5.18)$$

$$f_{3X_3} + \tilde{f}_3 = \frac{1}{3} \frac{J_c X_1^{1/2} H X_3}{X_3^{1/2} H_0^2 \Omega_m 0}, \quad (5.19)$$

where we have used the expression for the Hubble parameter in the standard model $H^2 = H_0^2 (\Omega_m 0 a^{-3} + \Omega_{\Lambda 0})$. We can now assume that the Hubble parameter can be written in terms of $X_1$ and $X_3$ as

$$H(X_1, X_3) = H_0 \left( \frac{X_1}{H_0^2} \right)^{-n} \left( \frac{X_3}{H_0^2} \right)^{-m}, \quad (5.20)$$

where $n$ and $m$ are constants. Note that the units in the previous expression are correct, given that $[X_1] = [X_3] = H_0^2$. For the sake of simplicity, we further assume $f_3 = 0$ which in turn defines $\tilde{f}_3$ from Eq. (5.19) and keeps alive the vector interactions in the model. From Eqs. (5.15)-(5.20) it is possible to obtain expressions for $f_2$, $f_{2X_1}$, $f_{2X_3}$, $G_{3X1}$, $\tilde{f}_3$ in terms of $X_1$ and $X_3$. In order to close the system, we assume that $X_1$ and $X_3$ depend on $H$ as

$$X_1 = \frac{X_{10}}{H^p}, \quad X_3 = \frac{X_{30}}{H^q}, \quad (5.21)$$

where $[X_{10}] = H_0^{p+2}$, $[X_{30}] = H_0^{2+q}$, $p$ and $q$ are constants. Thus, the problem of finding a model in SVT theories with the same background as $\Lambda CDM$ is reduced to find an appropriate

\[\text{Under the conditions (5.7), the cubic interactions of SVT theories are present at the first-order level only through the combination } f_{3X_3} + \tilde{f}_3. \text{ See Appendix A where the perturbations coefficients in Eqs. (5.38)-(5.44) are shown.} \]

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set of parameters \( \{n, m, p, q\} \). From the effective DE density (4.4) and Eqs. (5.14)-(5.21) we obtain

\[
\rho_{\text{DE}} = 3H_0^2\Omega_0
\]

\[
+ \frac{4\sqrt{2}H_0^2(m+n)\tilde{J}}{\Omega_{m0}} \left[ \frac{\Omega_{m0}}{a^3} + \Omega_0 \right] \frac{(2n-1)p+2mq+2}{2} \left( 1 - \left[ \frac{\Omega_{m0}}{a^3} + \Omega_0 \right] \frac{np+mq-1}{2} \right),
\]

(5.22)

so that for

\[
np + mq = 1,
\]

(5.23)

the model matches the \( \Lambda \)CDM background evolution, while having non-vanishing perturbations: \( \tilde{\rho}_{\text{DE}} = 3H_0^2\Omega_0 \), \( \tilde{P}_{\text{DE}} = -\tilde{\rho}_{\text{DE}}, \ w_{\text{DE}} = -1 \). We choose

\[
n = 1, \quad m = -2, \quad p = 2, \quad q = \frac{1}{2},
\]

(5.24)

as a suitable set of parameters yielding manageable expressions for the perturbations in Eq. (4.35). Then, the choice (5.24) defines our SVTDES model

\[
f_2 = -3H_0^2\Omega_0 + \frac{\sqrt{2}H_0\tilde{J}X_1^{1/2}}{\Omega_{m0}} \left[ \left( \frac{X_1}{H_0^2} \right)^{-2} - \left( \frac{X_3}{H_0^2} \right)^4 - \Omega_0 \right],
\]

\[
f_3 = 0, \quad \tilde{f}_3 = \frac{2X_1^{1/2}\tilde{J}\left( \frac{X_1}{H_0^2} \right)^{-1} \left( \frac{X_3}{H_0^2} \right)^2}{3X_3^{3/2}\Omega_{m0}},
\]

\[
G_3X_1 = \frac{2\tilde{J} \left( \frac{X_1}{H_0^2} \right)^{-2} \left( \frac{X_3}{H_0^2} \right)^2}{3H_0^2\Omega_{m0}}, \quad G_4 = \frac{1}{2}.
\]

(5.25)

where \( \tilde{J} \equiv J_c/H_0 \) is dimensionless. The equations of motion for the scalar field (5.11) and the vector field (5.12) are trivially satisfied for the SVTDES model. Having defined the background evolution for the SVTDES model, we will focus on the evolution of perturbations which are defined by the coefficients \( Y_i \) in the Appendix E.2. The non-vanishing \( Y_i \) for the SVTDES model (5.25) are the coefficients \( Y_1, Y_2, Y_3, Y_5, Y_6, Y_8, Y_9, Y_{11}, Y_{12} \), which yield

\[
\delta \rho_{\text{DE}} \approx \frac{2\sqrt{2}\tilde{J}}{21\Omega_{m0}} \sqrt{a (a_{m0} + a^4\Omega_0)}
\]

\[
\times \left[ \frac{28a}{29\Omega_{m0} + 20a^3\Omega_0} + 3H_0^2 \left( \frac{5}{k^2} - \frac{96a}{8ak^2 + 273H_0^2\Omega_{m0} + 336a^3H_0^2\Omega_0} \right) \right] \delta \rho_m,
\]

(5.26)

\[
\delta P_{\text{DE}} \approx \frac{aH_0^2\tilde{J}\delta \rho_m}{3\sqrt{2}k^2\Omega_{m0} \sqrt{a (a_{m0} + a^4\Omega_0)} (8ak^2 + 273H_0^2\Omega_{m0} + 336a^3H_0^2\Omega_0)}
\]

\[
\times \left[ 13\Omega_{m0} (8ak^2 + 615H_0^2\Omega_{m0}) + 4a^3 (8ak^2 + 3669H_0^2\Omega_{m0})\Omega_0 + 5952a^6H_0^2\Omega_0^2 \right],
\]

\[
\tilde{\rho}_{\text{DE}V_{\text{DE}}} \approx -\frac{16\sqrt{2}aH_0\tilde{J}\delta \rho_m}{3\Omega_{m0} (29\Omega_{m0} + 20a^3\Omega_0) (8ak^2 + 273H_0^2\Omega_{m0} + 336a^3H_0^2\Omega_0)}
\]

\[
\times (\Omega_{m0} (20ak^2 + 1161H_0^2\Omega_{m0}) + a^3 (8ak^2 + 1791H_0^2\Omega_{m0})\Omega_0 + 576a^6H_0^2\Omega_0^2),
\]

(5.27)
c_{s,DE}^2 = \frac{H_0^2 (29\Omega_m^0 + 20a^3\Omega_{\Lambda0})}{4(\Omega_m^0 + a^3\Omega_{\Lambda0})}
\times \left( 13\Omega_m^0 (8ak^2 + 615H_0^2\Omega_m^0) + 4a^3 (8ak^2 + 3669H_0^2\Omega_m^0) \Omega_{\Lambda0} + 5952a^6H_0^2\Omega_{\Lambda0}^2 \right)
\times \left[ 32a^2k^4 + 396aH_0^2k^2\Omega_m^0 + 16965H_0^4\Omega_m^0 \right]
+ 36a^3H_0^2(24ak^2 + 905H_0^2\Omega_m^0)\Omega_{\Lambda0} + 14400a^6H_0^4\Omega_{\Lambda0}^2 \right]^{-1},
\text{(5.29)}

where we have assumed \( \tilde{J} \ll 1 \). We have replaced \( X_1 \) and \( X_3 \) in terms of \( H \) using Eqs. (5.21) and (5.24), then \( H \) can be written in terms of \( a \) by using Eq. (5.14). It becomes clear that \( \delta\rho_{DE}, \delta P_{DE}, \) and \( V_{DE} \), vanish for \( J = 0 \), therefore we recover \( \Lambda CD \), i.e., there are no dark energy perturbations. In the following subsections, we will explore the cosmological implications of the SVTDES model (5.25), always tracking deviations from \( \Lambda CD \) through the parameter \( \tilde{J} \).

5.2 Evolution of matter and dark energy perturbations

In this subsection, we numerically solve the differential equations in Eqs. (2.15) and (2.16) for matter perturbations, i.e., for \( \delta_m \) and \( V_m \).

To begin with, we check the stability of DE perturbations. Since \( G_4 \) is a constant for the SVTDES model, Eq. (3.41) implies that there is no anisotropic stress, and thus we do not have to consider an effective sound speed. The sound speed of DE perturbations in Eq. (4.35) is the key quantity driving the stability of perturbations, and for the SVTDES model in the Newtonian gauge it is given by the Eq. (5.29) which interestingly does not depend on \( \tilde{J} \). We show the evolution of \( c_{s,DE}^2 \) as a function of \( a \), for a few values of \( k \), in the left panel of Fig. 1. We would like to make some comments about the behaviour of \( c_{s,DE}^2 \). First, we can see that the squared sound speed is positive during the whole evolution, assuring that our SVTDES model avoids Laplacian instabilities. Second, the SHA indeed applies for modes well within the sound horizon, i.e., for modes such that [100]
\[ c_{s,DE}^2 k^2 \gg H^2. \]
\text{(5.30)}

Therefore, the SHA breaks down for \( c_{s,DE}^2 \approx 0 \). We can make a rough estimate of how small \( c_{s,DE}^2 \) can be so that the SHA be justifiable. Since co-moving wavenumbers relevant to the observations of large-scale structures lay in the range \( 30H_0 \lesssim k \lesssim 600H_0 \) [139], and it is reasonable to assume that during matter domination \( H^2 \approx H_0^2\Omega_m a^{-3} \), from (5.30) and using \( k \sim 300H_0 \) we get
\[ c_{s,DE}^2 \sim 3 \times 10^{-6} a^{-1}, \]
\text{(5.31)}

which provides a rough bound for modes inside the sound horizon. In Fig. 1, the solid black line shows the relation \( c_{s,DE}^2 = 3 \times 10^{-6} a^{-1} \). We see that the values taken by \( c_{s,DE}^2 \) for the two values considered, namely, \( k = 600H_0 \) and \( k = 300H_0 \) (blue dashed line and green dot-dashed line, respectively), are higher than those taken in the black line, therefore, the SHA can be safely applied. Third, note that earlier than the regime of validity of our treatment (i.e., matter dominance) as well as for modes \( k \approx H_0 \), DE perturbations propagate with speed greater than the speed of light.

Now, we focus on the differential equations (2.15) and (2.16) for matter perturbations. For pressure-less matter we have \( w_m = 0, \pi_m = 0, \) and \( c_{s,m}^2 = 0 \). Hence, matter perturbations equations read
\[ \delta_m' = -\frac{V_m(a)}{a^2 H(a)} - 3\Phi'(a), \quad V_m' = -\frac{V_m(a)}{a} - \frac{k^2 \Phi(a)}{a^2 H(a)}, \]
\text{(5.32)}
The evolution equations for matter perturbations in Eq. (5.32) couple to the DE perturbations through the gravitational potential $\Phi$. From Eqs. (2.9)-(2.10), we can eliminate $\Phi'$ and obtain

$$\Phi(a) = \frac{a^2}{2k^2} \left[ 3H_0^2\Omega_{m0}a^{-3} \left( \delta_m + \frac{3aH}{k^2}V_m \right) + \bar{\rho}_{DE} \left( \delta_{DE} + \frac{3aH}{k^2}V_{DE} \right) \right].$$  \hspace{1cm} (5.33)

Since the background of the SVTDES model is equivalent to that of $\Lambda$CDM, the Hubble parameter is given by Eq. (5.14), and the density of dark energy is $\bar{\rho}_{DE} = 3H_0^2\Omega_{\Lambda0}$, hence (5.33) is simplified to

$$\Phi(a) = \frac{3H_0^2}{2ak^2} \left[ \Omega_{m0} \left( \delta_m + \frac{3aH}{k^2}V_m \right) + \Omega_{\Lambda0}a^3 \left( \delta_{DE} + \frac{3aH}{k^2}V_{DE} \right) \right].$$  \hspace{1cm} (5.34)

In order to solve Eqs. (5.32), we need to determine $\delta_{DE}$ and $V_{DE}$. Our effective fluid approach allowed us to find analytical expressions for the perturbations $\delta\rho_{DE}$ and $V_{DE}$ which for our SVTDES model (5.25) are given by Eqs. (5.26)-(5.28), respectively.

The initial conditions required to solve Eqs. (5.32) are set by the following expressions

$$\delta_{m,i} = \delta_i a_i \left( 1 + 3\frac{a_i^2H^2(a_i)}{k^2} \right), \quad V_{m,i} = -\delta_i H_0 \Omega_{m0} a_i^{1/2},$$  \hspace{1cm} (5.35)

corresponding to the standard solutions of Eqs. (5.32) for $\delta_m$ and $V_m$ in matter dominance, i.e., assuming that $H^2 = H_0^2\Omega_{m0}a^{-3}$. The overall factor $\delta_i$ is set to unity, and we choose $a_i = 10^{-3}$, ensuring initial conditions well within the matter epoch, right after decoupling.

The evolution of $\delta_m$, $V_m/k^2$, $\delta_{DE}$, and $V_{DE}/k^2$ (their absolute values) are depicted on the right panel of Fig. 1. Note that the velocity perturbation is $u$, which is defined through $u_i \equiv -\partial_i u$ for scalar perturbations [see Eq. (2.6)]. The relation of the velocity perturbation to the scalar velocity and the velocity divergence is $V \propto \theta = ik^ju_j = k^2u$, and then $u \propto V/k^2$. 

---

**Figure 1.** Left: Evolution of the DE sound speed $c^2_{s,DE}$ for the modes $k = 600H_0$ and $k = 300H_0$. For the two values considered, we can see $c^2_{s,DE}$ is positive during the whole evolution, and it gets greater values than those in the black solid line, which marks a rough bound above which the SHA can be safely applied. Right: Evolution of $\delta_m$, $\delta_{DE}$, $V_m/k^2$, and $V_{DE}/k^2$ (their absolute values) for $J = 0.01$. The other parameters used in this figure are $\Omega_{m0} = 0.3$, $\Omega_{\Lambda0} = 0.7$, and $k = 300H_0$. The initial conditions are obtained from Eq. (5.35).
Figure 2. Left: Deviations of $G_{\text{eff}}$ from $G_N$ from decoupling to today for different values of the parameter $\tilde{J}$. We can see that deviations from GR are only at late-times (occurring around the dark energy transition at $z \approx 0.3$), and smaller for $\tilde{J} \approx 0$ as expected. Observe that for $\tilde{J} = 0.5$, the difference between $G_{\text{eff}}$ and $G_N$ is around 20% at the present time. Right: Evolution of $f\sigma_8(z)$ for the same values of $\tilde{J}$ shown in the left panel. In the case $\tilde{J} = 0.5$, deviations of SVTDES from $\Lambda$CDM are fairly noticeable. For the SVTDES model gravity is weaker than in the standard cosmological model leading to a less efficient late-time matter clustering. Other parameters used in the figures are $\Omega_m = 0.3$, $\Omega_{\Lambda} = 0.7$, $\sigma_8 = 0.8$, and $k = 300H_0$.

5.3 Solution for the growth factor

As explained in Sec. 4, under the SHA and the QSA, the parameter $G_{\text{eff}}$ plays an important role in the growth of structure, as can be seen in Eq. (4.21). In this subsection, we explore possible changes in the parameter $f\sigma_8$ within the SVTDES model due to variations in the strength of gravity which are encoded in the parameter $G_{\text{eff}}$.

From Eqs. (4.15) and (4.19), we obtain the following analytical expression for $G_{\text{eff}}$ under the QSA and the SHA

$$
\frac{G_{\text{eff}}}{G_N} = \frac{2}{a^4} W_{14} + \frac{k^2}{a^2} W_{15} + W_{13},
$$

where the coefficients $W_i$ are given in the Appendix D. Replacing the SVTDES model [Eqs. (5.25)] in the parameter $G_{\text{eff}}$ in Eq. (5.36), using the Hubble parameter of $\Lambda$CDM given in (5.14), and assuming some values for the parameter $\tilde{J}$, namely, $\tilde{J} = 0.01, 0.1, 0.5$, we can numerically solve the differential equation for $\delta_m$ in Eq. (4.21), where the initial conditions are set as $\delta_m(a_i) = a_i$ and $\delta_m'(a_i) = 1$ for a value of the scale factor $a_i$ deep in the matter era ($a_i \sim 10^{-3}$). Other parameters used in the numerical solutions are $\Omega_m = 0.3$, $\Omega_{\Lambda} = 0.7$, and $k = 300H_0$. In order to compare with observations, from this numerical solution we compute the $f\sigma_8$ function, which is defined as

$$
f\sigma_8(a) \equiv \sigma_8 \frac{a\delta_m'(a)}{\delta_m(a = 1)},
$$

where $\sigma_8 \sim 0.8$ is the expected RMS over-density in a sphere of co-moving radius equal to $8h^{-1}$ Mpc, $h$ being the normalized Hubble parameter. The results for the different values of $\tilde{J}$, aside the $\Lambda$CDM case $\tilde{J} = 0$, are shown in Fig. 2. In the left panel of Fig. 2 we can see that for $\tilde{J} = 0.5$, the modifications to GR, i.e., deviations of $G_{\text{eff}}$ from $G_N$, are fairly noticeable at late-times. This difference translates to a weaker gravity when DE becomes relevant in
the cosmic budget, which leads to a different evolution of the growth factor. In the right panel of Fig. 2, we plot $f \sigma_8(z)$ versus the data compilation from Ref. [140]. For $\tilde{J} = 0.5$, we can see that $f \sigma_8$ for SVTDES has a strong departure from $\Lambda$CDM (black solid curve) indicating a less efficient matter clustering in comparison with the standard model. Weaker gravity can be helpful in understanding the discrepancy in $S_8$ between low- and high-redshift probes. Gravity strength also decreases for smaller values of the parameter $\tilde{J} = 0.1, 0.01$, but differences with respect to $\Lambda$CDM are hardly significant.

5.4 CMB angular power spectrum and matter power spectrum

Having studied the evolution of matter perturbations in the previous subsections, here we present our results for the CMB power spectrum and the linear matter power spectrum. The advantage of the effective fluid approach is that it allows a relatively easy implementation of the SVTDES model in Boltzmann solvers. In its default version, Boltzmann codes usually have already a DE fluid implemented and parameterised by an equation of state $w$, sound speed in the fluid rest-frame $\hat{c}_s^2$, and vanishing anisotropic stress $\pi = 0$. In this work, we have computed the effective fluid quantities describing fairly general SVT theories.

We chose to carry out the implementation of the SVTDES model in the Boltzmann solver CLASS.\textsuperscript{12} Since our model matches the $\Lambda$CDM background evolution ($w_{DE} = -1$) and has vanishing anisotropic stress ($\pi_{DE} = 0$), we decided to perform the smallest number of modifications in the code. It turns out that only one modification in the module perturbations.c is required: i) the scalar velocity $V_{DE}$ (5.28) modifies the equation for $\Phi$ in the function perturbations_einstein.

The CMB temperature power spectrum and the linear matter power spectrum are shown in Fig. 3. Perturbation equations were solved by using the cosmological parameters from the 2018 Planck baseline result [2]: scalar spectrum power-law index $n_s = 0.9649$, Log power of the primordial curvature perturbations $\ln 10^{10} A_s = 3.044$, reduced Hubble parameter $h = 0.6736$, baryon density today $\omega_b = 0.0237$, cold dark matter density today $\omega_{cdm} = 0.1200$, Thomson scattering optical depth due to reionization $\tau = 0.0544$, sum of neutrino masses in eV $\sum m_\nu = 0.06$, and some values of the SVTDES parameter $\tilde{J}$. We also plot the standard $\Lambda$CDM results for reference. As it can be seen in the left panel of Fig. 3, the match in the TT CMB angular power spectrum between SVTDES and $\Lambda$CDM is almost perfect. In the right panel of Fig. 3 we can see that the agreement in the linear matter power spectrum is also quite good for SVTDES and $\Lambda$CDM models. Nonetheless, for modes $k \sim 10^{-4} - 10^{-3} h$ Mpc$^{-1}$ there is a departure from $\Lambda$CDM. Since we are working under the SHA and QSA, a big deviation is expected on large scales and late-times where the approximations are not valid.

5.4.1 Sound speed in the rest-frame

When implementing DE fluids in CLASS, it is important to bear in mind that the code uses the co-moving sound speed $c_s^2$, i.e., the sound speed in the rest-frame of the fluid, which, in general, is given by

$$c_s^2 = \frac{\delta P^C}{\delta \rho^C}, \quad (5.38)$$

where $\delta P^C$ and $\delta \rho^C$ are the pressure and density perturbations computed in the co-moving gauge. These quantities are related to quantities in the Newtonian gauge (the gauge where

\textsuperscript{12}Version v3.2.0
In summary, CLASS uses in their computations the sound speed in the co-moving gauge which is related to the Newtonian gauge by Eq. (5.40). Therefore, for a specific SVT model, we have to replace $\delta \rho_{\text{DE}}, \delta P_{\text{DE}}, V_{\text{DE}} = (1 + w_{\text{DE}}) \theta_{\text{DE}}$ from Eq. (4.23) or Eq. (4.35) depending our main results were derived) through the following transformation rules [141]

$$\delta \rho^C = \delta \rho^N + \rho' \frac{\ddot{\theta}^N}{k^2}, \quad \delta P^C = \delta P^N + \frac{\ddot{\rho} \theta^N}{k^2},$$  \hspace{1cm} (5.39)

where the superscript $N$ denotes a quantity computed in the Newtonian gauge. Therefore, the sound speed in the rest frame will be given by

$$c_s^2 = \frac{\delta P^N + \ddot{\rho} \theta^N / k^2}{\delta \rho^N + \rho' \ddot{\theta}^N / k^2}.$$  \hspace{1cm} (5.40)

Since the background pressure and density of our SVTDES model are constants, we see from the last equations that the sound speed in Eq. (5.29) is actually equivalent to the sound speed in the rest frame. However, this is not the case in general. Let us take the quintessence field as an example. Replacing the full perturbations (4.38)-(4.39) in Eq. (5.40), computing the derivatives of $\bar{\rho}_{\text{DE}}$ and $\bar{P}_{\text{DE}}$ in Eqs. (4.37), and using the relation $\phi'' = -2H \phi' - a^2 V_\phi$, we find that

$$c_s^2 = 1.$$  \hspace{1cm} (5.41)

In this case, the result is equal to $c_{s,\text{DE}}^2$ computed under the QSA and SHA [see Eq. (4.40)] since $V_{\text{DE}} = 0$ for this particular model, but it is substantially different to $c_{s,\text{DE}}^2$ given in Eq. (4.39) in the Newtonian gauge. Another interesting example concerns $f(R)$ theories where, in general, DE sound speed might depend on both time and scale. For instance, in Ref. [86] we can find expressions [see their Eqs. (61)–(68)] under QSA and SHA for the Hu & Sawicki model that allow us to obtain $c_s^2 \neq c_{s,\text{DE}}^2$.

In summary, CLASS uses in their computations the sound speed in the co-moving gauge which is related to the Newtonian gauge by Eq. (5.40). Therefore, for a specific SVT model, we have to replace $\delta \rho_{\text{DE}}, \delta P_{\text{DE}}, V_{\text{DE}} = (1 + w_{\text{DE}}) \theta_{\text{DE}}$ from Eq. (4.23) or Eq. (4.35) depending
whether or not the model has a vanishing DE anisotropic stress, and compute the derivatives of the background density and pressure in Eqs. (4.4) and (4.5).

6 Conclusions

Both scalar and vector fields are present in nature and it is reasonable that they might provide explanations for shortcomings in the standard cosmological model ΛCDM. In this work we investigated fairly general scalar-vector-tensor theories having second order equations of motion: SVT theories encompass both Horndeski and generalised Proca Lagrangians. Although these kinds of theories might provide new, interesting phenomenology for cosmology, they have been overlooked in the literature. SVT theories have various free functions taking in all relevant interactions, hence possibly richer phenomenology than in the standard model. Nevertheless, more degrees of freedom come along with more complicate equations of motion. Even though complexity in SVT theories is reduced thanks to the constraint in the propagation speed of gravitational waves $c^2_T = 1$, equations of motion remain intricate enough to find general analytical or numerical solutions.

Here, we applied an effective fluid approach to SVT theories satisfying $c^2_T = 1$. In order to decrease the complexity in the equations of motion, we carefully performed both sub-horizon and quasi-static approximations. As a result, we obtained analytical expressions describing the effective dark energy fluid, namely, equation of state $w(a)$, squared sound speed $c^2_s(a,k)$, and anisotropic stress $\pi(a,k)$. Equations (4.23) and (4.35) summarise our main results for the behaviour of perturbations, while from Eqs. (4.4)-(4.5) the equation of state is obtained.

Our analytical expressions allowed us to retrieve well known results (e.g., quintessence and $f(R)$). Moreover, we also proposed extensions to these popular theories which exemplify possible, new phenomenology, for instance, changes in quantities driving the perturbations such as the sound speed and anisotropic stress.

An interesting aspect of our investigation is that it makes it possible to design cosmological models satisfying certain conditions. As an example, we found a SVT model (dubbed SVTDES in the main text) exactly matching the background behaviour in the standard cosmological model ΛCDM, while having non-vanishing dark energy perturbations. Our effective fluid approach and the analytical solutions for the effective dark energy perturbations made it possible a relatively easy implementation of SVTDES in the Boltzmann solver CLASS. Having a code computing numerical solutions for perturbation equations in SVT cosmological models is relevant because it allows testing against measurements, e.g., CMB angular power spectra, matter power spectrum. There is however no public Boltzmann solver including a fully numerical implementation of SVT models, that is, using neither QSA nor SHA. Therefore, our results might be helpful as a reference for future exact computations testing the limitations of QSA and SHA. Since our SVTDES model has one additional parameter with respect to ΛCDM, in a model comparison it would be penalised by the Bayesian evidence. However, our example also shows that exploring the construction of cosmological models satisfying additional conditions might be well worth an investigation. Given the current discrepancies in cosmological parameters such as $H_0$ and $\sigma_8$, theories providing non trivial behaviour for $\pi$ and $c^2_s$ could alleviate the tensions while not being affected by the Occam’s razor [32].
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Numerical codes

Modified CLASS code reproducing results in this work can be found in the GitHub branch svt of the repository EFCLASS. A large part of the calculations in this paper were carried out using several Mathematica packages, like xPand. The notebooks showing these computations can be found in the GitHub repository SVT.

A General equations of motion

A.1 Gravitational field equations

Coefficients in Eq.(3.32):

\[ g^{(2)}_{\mu\nu} = -\frac{1}{2} f_{2x} A_\mu A_\nu - \frac{1}{2} f_{2g} g_{\mu\nu} - \frac{1}{4} f_{2x2} A_\mu \nabla_\mu \phi - \frac{1}{4} f_{2x2} A_\nu \nabla_\nu \phi - \frac{1}{4} f_{2x2} A_\mu \nabla_\mu \varphi - \frac{1}{2} f_{2x1} \nabla_\mu \varphi \nabla_\nu \varphi, \quad (A.1) \]

\[ g^{(3)}_{\mu\nu} = -f_3 A_\alpha \nabla_\alpha \phi + f_3 A_\phi \nabla_\phi \phi - f_3 A_\phi \nabla_\phi \varphi - f_3 A_\varphi \nabla_\varphi \phi - f_3 A_\varphi \nabla_\varphi \varphi \]

\[ \mathcal{H}^{(3)}_{\mu\nu} = -\frac{1}{2} G_3 g_{\mu\nu} \nabla_\alpha \phi \nabla_\alpha \varphi + \frac{1}{2} G_3 x_1 g_{\mu\nu} \nabla_\alpha \phi \nabla_\beta \phi \nabla_\nu \nu \varphi + \frac{1}{2} G_3 x_1 g_{\mu\nu} \nabla_\alpha \phi \nabla_\nu \nu \varphi - \frac{1}{2} G_3 x_1 \nabla_\alpha \phi \nabla_\alpha \nu \nu \varphi - \frac{1}{2} G_3 x_1 \nabla_\alpha \phi \nabla_\alpha \nu \nu \varphi. \quad (A.2) \]

A.2 Scalar field equation of motion

Coefficients on left-hand side of Eqs.(3.33):

\[ J_2 = f_2 \phi + \frac{1}{2} f_{2x} A_\alpha A_\phi + \frac{1}{2} f_{2x} \nabla_\alpha A_\phi + f_{2x1} \nabla_\alpha \phi + f_{2x1} \nabla_\alpha \varphi + f_{2x2} \nabla_\alpha \phi + \frac{1}{2} f_{2x2} \nabla_\alpha \varphi \nabla_\phi \varphi \nabla_\alpha \varphi \]

\[ - \frac{1}{4} f_{2x2} x_2 A_\alpha A_\phi A_\beta \phi - \frac{1}{2} f_{2x2} x_3 A_\alpha A_\beta \phi A_\phi A_\alpha - \frac{1}{4} f_{2x2} x_3 A_\alpha A_\beta \phi A_\phi A_\alpha - f_2 x_1 x_3 A_\alpha A_\beta \phi A_\phi A_\alpha - f_2 x_1 x_3 A_\phi A_\beta \phi A_\alpha A_\phi \]

\[ J_3 = 2 f_3 \phi A_\phi + \frac{1}{2} f_3 A_\alpha A_\phi, \quad (A.6) \]
\[ K_3 = -2G_3\varphi \nabla_\alpha \nabla^\alpha \varphi - G_3\varphi \nabla_\alpha \varphi \nabla^\alpha \varphi - G_{3X1}\nabla_\alpha \nabla_\beta \nabla^\alpha \varphi - G_{3X1}\nabla_\alpha \nabla^\alpha \varphi \nabla_\beta \nabla^\beta \varphi \\
- G_{3\varphi X1}\nabla_\alpha \varphi \nabla^\alpha \varphi \nabla_\beta \nabla^\beta \varphi + G_{3\varphi X1}\nabla^\alpha \varphi \nabla_\beta \nabla^\alpha \varphi + 2G_{3\varphi X1}\nabla^\alpha \varphi \nabla_\beta \nabla^\alpha \varphi \nabla_\gamma \nabla^\gamma \varphi \\
+ G_{3\varphi X1}\nabla^\alpha \varphi \nabla^\beta \nabla^\gamma \phi \nabla_\beta \nabla^\alpha \varphi, \quad (A.7) \\
K_4 = G_{4\varphi} R. \quad (A.8) \\
\]

A.3 Vector field equation of motion

Coefficients on right-hand side of Eqs.(3.33):

\[ A^{\mu}_{(2)} = -f_{2X3} A^{\mu} - \frac{1}{2} f_{2X2} \nabla^\mu \phi, \quad (A.9) \]

\[ A^{\mu}_{(3)} = -2f_{3\varphi} A^{\mu} \nabla_\alpha \varphi - 2f_{3X_3} A^{\mu} \nabla_\alpha A^{\alpha} - 2f_{3\varphi} A^{\mu} \nabla_\alpha \phi - 2f_{3X_3} A^{\mu} \nabla_\alpha A^{\alpha} - 2f_{3\varphi} \nabla^\mu \phi \\
+ 2f_{3X_3} A^{\alpha} \nabla^\mu A^{\alpha} + 2f_{3X_3} A^{\alpha} \nabla^\mu A^{\alpha}. \quad (A.10) \]

B Background equations of motion

B.1 “Time-Time” equation

Coefficients on the left-hand side of Eq. (3.34):

\[ G^{(2)}_{00} = \frac{1}{2} a^2 f_2 - \frac{1}{2} \varphi^2 f_{2X_1} - \frac{1}{2} A_0 \varphi f_{2X_2} - \frac{1}{2} A_0^2 f_{2X_3}, \quad (B.1) \]

\[ G^{(3)}_{00} = -A_0 \varphi \varphi f_{3\varphi} + \frac{A_0^3 \varphi f_{3\varphi}}{a^2} + \frac{3A_0^3 f_{3X_3} H}{a^2} + \frac{3A_0^3 f_{3X_3} H}{a^2}, \quad (B.2) \]

\[ \mathcal{K}^{(3)}_{00} = \frac{1}{2} \varphi^2 G_{3\varphi} - \frac{3\varphi^2 G_{3X_3} H}{2a^2}, \quad (B.3) \]

\[ \mathcal{K}^{(4)}_{00} = 3H(\varphi^2 G_{4\varphi} + G_4 H). \quad (B.4) \]

B.2 “Space-Space” equation

Coefficients on the right-hand side of Eq. (3.34):

\[ G^{(2)}_{11} = -\frac{1}{2} a^2 f_2, \quad (B.5) \]

\[ G^{(3)}_{11} = -A_0^2 A_0 f_{3X_3} - A_0 \varphi \varphi f_{3\varphi} - \frac{A_0^3 A_0 f_{3X_3}}{a^2} + \frac{A_0^3 f_{3X_3} H}{a^2} + \frac{A_0^3 f_{3X_3} H}{a^2}, \quad (B.6) \]

\[ \mathcal{K}^{(3)}_{11} = \frac{\varphi^2 \varphi^2 G_{3X_3}}{2a^2} + \frac{1}{2} \varphi^2 G_{3\varphi} - \frac{\varphi^2 G_{3X_3} H}{2a^2}, \quad (B.7) \]

\[ \mathcal{K}^{(4)}_{11} = -\varphi^2 G_{4\varphi} - G_{4\varphi}(\varphi'' + \varphi' H) - G_4 (H^2 + 2H'). \quad (B.8) \]
B.3 Background equation of motion for the scalar and vector fields

Coefficients on the left-hand side of Eq. (3.35):

\[
\begin{align*}
\mathcal{J}_2 &= - \frac{\varphi'' f_{2X_1}}{a^2} - \frac{\varphi'' \varphi f_{2X_1X_1}}{a^4} - \frac{A_0 \varphi'' f_{2X_1X_2}}{a^4} - \frac{A_0' \varphi' f_{2X_1X_3}}{2a^4} - \frac{A_0 A_0' \varphi f_{2X_1X_3}}{a^4} \\
&- \frac{A_0 f_{2X_2}}{2a^2} - \frac{A_0^2 \varphi'' f_{2X_2X_2}}{4a^4} - \frac{A_0 A_0' \varphi f_{2X_2X_2}}{2a^4} - \frac{A_0 A_0' f_{2X_2X_3}}{2a^4} + f_{2\varphi} - \frac{\varphi^2 f_{2X_1X_3}}{a^2} \\
&- \frac{A_0 \varphi' f_{2\varphi X_2}}{2a^2} - \frac{2 \varphi f_{2X_1} \mathcal{H}}{a^2} + \frac{\varphi^2 f_{2X_1X_1} \mathcal{H}}{a^4} + \frac{3 A_0 \varphi' f_{2X_1X_2} \mathcal{H}}{2a^4} + \frac{A_0' \varphi f_{2X_1X_3} \mathcal{H}}{a^4} \\
&- \frac{A_0 f_{2X_2} \mathcal{H}}{a^2} + \frac{A_0' \varphi f_{2X_2X_2} \mathcal{H}}{2a^4} + \frac{A_0 A_0' f_{2X_2X_3} \mathcal{H}}{2a^4},
\end{align*}
\]

(B.9)

\[
\begin{align*}
\mathcal{J}_3 &= - \frac{2 A_0' f_{3\varphi}}{a^2} + \frac{2 A_0^2 A_0' \bar{f}_{3\varphi}}{a^4} - \frac{4 A_0 f_{3\varphi} \mathcal{H}}{a^2} - \frac{2 A_0^3 \bar{f}_{3\varphi} \mathcal{H}}{a^4},
\end{align*}
\]

(B.10)

\[
\begin{align*}
\mathcal{K}_3 &= \frac{2 \varphi'' G_{3X_1}}{a^2} + \frac{\varphi'' \varphi' \varphi G_{3X_1X_1}}{a^4} + \frac{\varphi' \varphi G_{3X_1} \mathcal{H}}{a^2} + \frac{6 \varphi'' \mathcal{H} G_{3X_1} \mathcal{H}}{a^2} - \frac{3 \varphi'' \varphi^2 G_{3X_1X_1} \mathcal{H}}{a^4} \\
&+ \frac{4 \varphi' G_{3X_1} \mathcal{H}}{a^2} - \frac{4 \varphi' \varphi G_{3X_1X_1} \mathcal{H}}{a^4} + \frac{3 \varphi' \varphi G_{3X_1X_1} \mathcal{H}^2}{a^6} - \frac{3 \varphi' \varphi G_{3X_1} \mathcal{H}'}{a^4},
\end{align*}
\]

(B.11)

\[
\begin{align*}
\mathcal{K}_4 &= \frac{6 A_0' \varphi f_{3X_1} \mathcal{H}}{a^2}.
\end{align*}
\]

(B.12)

Coefficients on the right-hand side of Eq. (3.35):

\[
\begin{align*}
\bar{A}_2 &= \frac{\varphi f_{2X_1}}{2a^2} + \frac{A_0 f_{2X_3}}{a^2},
\end{align*}
\]

(B.13)

\[
\begin{align*}
\bar{A}_3 &= \frac{2 \varphi' f_{3\varphi}}{a^2} - \frac{2 A_0^2 \varphi' \bar{f}_{3\varphi}}{a^4} - \frac{6 A_0^2 f_{3X_3} \mathcal{H}}{a^4} - \frac{6 A_0^3 \bar{f}_{3\varphi} \mathcal{H}}{a^4}.
\end{align*}
\]

(B.14)

C Linear perturbations: coefficients

C.1 “Time-Time” equation

Coefficients in Eq. (3.38):

\[
\begin{align*}
A_1 &= \frac{6 A_0^2 f_{3X_3}}{a^3} + \frac{6 A_0^3 \bar{f}_{3\varphi}}{a^3} - \frac{3 \varphi' \varphi G_{3X_1}}{a^3} + \frac{6 \varphi' G_{3X_1}}{a} + \frac{12 G_4 \mathcal{H}}{a},
\end{align*}
\]

(C.1)

\[
\begin{align*}
A_2 &= - \frac{\varphi f_{2X_1}}{a} - \frac{\varphi \varphi f_{2X_1X_1}}{a^3} - \frac{3 A_0 \varphi^2 f_{2X_1X_2}}{2a^3} - \frac{A_0^2 \varphi' f_{2X_1X_3}}{2a^3} - \frac{A_0 f_{2X_2}}{2a^3} \\
&- \frac{A_0' \varphi f_{2X_2X_2}}{2a^3} - \frac{A_0^2 f_{2X_2X_3}}{2a^3} - \frac{2 A_0 f_{3X_3}}{a} + \frac{2 A_0^3 \bar{f}_{3\varphi}}{a^3} + \frac{2 \varphi' G_{3X_1}}{a} + \frac{\varphi'' G_{3X_1X_1}}{a^3},
\end{align*}
\]

(C.2)

\[
\begin{align*}
A_3 &= 4 G_4,
\end{align*}
\]

(C.3)
\[ A_4 = \frac{\varphi'^2 f_{2X_1}}{a^2} + \frac{\varphi' f_{2X_1} X_1}{a^2} + 2A_0 \varphi'^3 f_{2X_1} X_1 + 2A_0^2 \varphi'^2 f_{2X_1, X_1} + \frac{A_0 \varphi' f_{2X_2}}{a^2} + \frac{A_0^2 \varphi' f_{2X_2 X_1}}{a^4} + \frac{A_0^2 \varphi' f_{2X_2 X_3}}{a^4} + \frac{A_0 \varphi' f_{3X_3}}{a^2} + 4A_0 \varphi' f_{3X_1} + \frac{2A_0^3 \varphi' f_{3X_1 X_3}}{a^4} + \frac{2A_0^2 \varphi' f_{3X_1 X_3} H}{a^2} + \frac{2A_0 \varphi' f_{3X_1 X_3} H}{a^6} - \frac{2A_0 \varphi' f_{3X_1 X_3} H}{a^6} - \frac{12A_0 \varphi' f_{3X_1 X_3} H}{a^2} + \frac{12 \varphi'^3 G_{3X_1 X_3} H}{a^4} + \frac{3 \varphi'^3 G_{3X_1 X_3} H}{a^6}, \tag{C.4} \]

\[ A_5 = \frac{-\varphi'^2 G_{3X_1}}{a^2} + 2G_4 \varphi, \tag{C.5} \]

\[ A_6 = \frac{-\varphi'^2 f_{2X_1} X_1}{2a^3} - \frac{\varphi' f_{2X_1, X_1}}{2a^3} - \frac{\varphi' f_{2X_1}}{2a} - \frac{\varphi f_{2X_1} X_1}{2a^3} - \frac{\varphi f_{2X_1}}{a^3} - \frac{A_0 f_{2X_3}}{a} + \frac{A_0^3 f_{2X_3} X_3}{a^3} - \frac{2 \varphi' f_{3X_1}}{a} - \frac{2 \varphi f_{3X_1} H}{a^3} - \frac{2A_0 \varphi' f_{3X_1 X_3}}{a^3} + \frac{6A_0^2 \varphi' f_{3X_1 X_3}}{a^3} + \frac{2A_0 \varphi' f_{3X_1 X_3} H}{a^3} + \frac{18A_0 \varphi' f_{3X_1 X_3} H}{a^3} + \frac{18A_0 \varphi' f_{3X_1 X_3} H}{a^3} + \frac{6A_0 \varphi' f_{3X_1 X_3} H}{a^3}, \tag{C.6} \]

\[ A_7 = \frac{2A_0^3 f_{3X_1}}{a^2} + \frac{2A_0^2 f_{3X_1}}{a^2}, \tag{C.7} \]

\[ \mu_\varphi = -\varphi f_{2\varphi} + \frac{\varphi'^2 f_{2\varphi} X_1}{a^2} + \frac{A_0 \varphi' f_{2\varphi} X_1}{a^2} + \frac{A_0^2 \varphi' f_{2\varphi} X_3}{a^2} + \frac{A_0 \varphi' f_{3\varphi}}{a^2} + \frac{2A_0 \varphi' f_{3\varphi} X_1}{a^2} + \frac{2A_0 \varphi' f_{3\varphi} X_3}{a^2} + \frac{2A_0 \varphi' f_{3\varphi} H}{a^3} - \frac{2A_0 \varphi' f_{3\varphi} H}{a^3} - \frac{\varphi'^2 G_{3\varphi}}{a^2} - \frac{6A_0 \varphi' f_{3\varphi} H}{a^4} - \frac{6A_0 \varphi' f_{3\varphi} H}{a^4} + \frac{3 \varphi'^3 G_{3\varphi} X_1 H}{a^4} - \frac{6 \varphi' G_{4\varphi} H}{a^2} - \frac{6 \varphi' G_{4\varphi} H}{a^2}, \tag{C.8} \]

C.2 Longitudinal “Time-Space” equation

Coefficients in Eq. (3.39):

\[ C_1 = 4G_4, \tag{C.9} \]

\[ C_2 = -\frac{\varphi'^2 G_{3X_1}}{a^2} + 2G_4 \varphi, \tag{C.10} \]

\[ C_3 = -\frac{2A_0 \varphi' f_{3X_1}}{a^3} - \frac{2A_0 \varphi' f_{3X_1}}{a^3} - \frac{\varphi'^3 G_{3X_1}}{a^3} - \frac{2 \varphi' G_{3X_1}}{a^3} - \frac{4G_4 H}{a}, \tag{C.11} \]

\[ C_4 = \frac{\varphi f_{2X_1}}{a} + \frac{A_0 f_{2X_2}}{a} + \frac{A_0 f_{2X_3}}{a} + \frac{2 \varphi' G_{3X_1}}{a} + \frac{2 \varphi' G_{3X_1}}{a} + \frac{3 \varphi'^2 G_{3X_1} H}{a^3} - \frac{2G_4 H}{a}, \tag{C.12} \]

\[ C_5 = \frac{2A_0^2 f_{3X_1}}{a^2} + \frac{2A_0^2 f_{3X_1}}{a^2}, \tag{C.13} \]

\[ C_6 = \frac{\varphi f_{2X_1}}{2a} + \frac{A_0 f_{2X_3}}{a} + \frac{2 \varphi' f_{3X_1}}{a^3} - \frac{2A_0 \varphi' f_{3X_1}}{a^3} - \frac{6A_0 \varphi' f_{3X_1} H}{a^3} - \frac{6A_0 \varphi' f_{3X_1} H}{a^3}. \tag{C.14} \]
C.3 Trace “Space-Space” equation

Coefficients in Eq. (3.40):

\[ B_1 = 12G_4, \quad (C.15) \]
\[ B_2 = -\frac{3\varphi'^2 G_3}{a^2} + 6G_4\varphi, \quad (C.16) \]
\[ B_3 = \frac{12\varphi' G_4}{a} + 24G_4\mathcal{H}, \quad (C.17) \]
\[ B_4 = \frac{3\varphi' f_2}{a} + \frac{3A_0 f_2}{2a} + \frac{6A_0 f_3}{a^3} - \frac{6\varphi'' \varphi' G_3}{a^3} + \frac{3\varphi'' \varphi G_3 X_1}{a^3} - \frac{6\varphi' G_3}{a^3}, \quad (C.18) \]
\[ B_5 = -\frac{6A_0^2 f_3}{a^3} - \frac{6A_0^3}{a^3} + \frac{6\varphi G_3}{a^3} - \frac{6\varphi' G_4}{a^3} - \frac{12G_4\mathcal{H}}{a^3}, \quad (C.19) \]
\[ B_6 = 4G_4, \quad (C.20) \]
\[ B_7 = 4G_4\varphi, \quad (C.21) \]
\[ B_8 = 4G_4, \quad (C.22) \]
\[ B_9 = -\frac{3\varphi'^2 f_2}{a^2} - \frac{3A_0 \varphi' f_2}{a^2} - \frac{3A_0^2 f_2}{a^2} - \frac{24A_0^2 A_0 f_3}{a^4} - \frac{6A_0^3 A_0 f_3 X_1}{a^6}, \quad (C.23) \]
\[ B_{10} = \frac{6A_0^2 f_3}{a^2} + \frac{6A_0^3}{a^2}, \quad (C.24) \]
\[ B_{11} = \frac{3\varphi' f_2}{2a} + \frac{3A_0 f_2}{a} + \frac{12A_0 A_0' f_3}{a^3} + \frac{6A_0^2 A_0^2 f_3 X_1}{a^5} + \frac{6\varphi' f_3}{a}, \quad (C.25) \]
\[ \nu_\varphi = f_{2\varphi} + \frac{2A_0^2 A_0 f_3 \varphi'}{a^4} + \frac{2A_0 \varphi' f_3 \varphi'}{a^2} + \frac{2A_0^2 A_0' f_3}{a^4} - \frac{\varphi'' \varphi' G_3}{a^4} - \frac{\varphi' G_3}{a^2} \]
\[ + \frac{2\varphi' G_{4\varphi}}{a^2} + \frac{2\varphi' G_{4\varphi}}{a^2} + \frac{2G_{4\varphi}}{a^2} + \frac{4G_{4\varphi}}{a^2}. \quad (C.26) \]
We also found the coefficient accompanying $\Phi$ in Eq. (3.40). However, it vanishes when using the equation in the right-hand side of Eq. (3.34) to eliminate $G_4$.

**C.4 Scalar field equation of motion**

Coefficients in Eq. (3.42)

\[
D_1 = \frac{3\varphi'^2 G_{3X_1}}{a^2} + 6G_4\varphi, \quad (C.27)
\]

\[
D_2 = -f_{2X_1} - \frac{\varphi'^2 f_{2X_1X_1}}{a^2} - A_0\varphi' f_{2X_1X_2} - \frac{A_0^2 f_{2X_2X_2}}{4a^2} + 2G_3\varphi + \frac{\varphi'^2 G_{3X_1}}{a^2}, \quad (C.28)
\]

\[
D_3 = -\frac{3\varphi' f_{2X_1}}{a^2} - \frac{2a}{3A_0 f_{2X_2}} - 6A_0 f_{3\varphi} - \frac{6\varphi'' \varphi' G_{3X_1}}{a^2} - 3\varphi'^2 G_{3X_1X_1} + \frac{6\varphi' G_3\varphi}{a}, \quad (C.29)
\]

\[
D_4 = -\frac{3\varphi'^2 f_{2X_1X_1}}{a^2} - \frac{\varphi'^2 \varphi'^2 f_{2X_1X_1X_1}}{a^2} + \frac{3A_0 \varphi'' \varphi'^2 f_{2X_1X_1X_2}}{a^2} - \frac{A_0^2 \varphi'^2 f_{2X_1X_1X_2}}{2a^3}, \quad (C.30)
\]
\[D_5 = \frac{\varphi' f_{2X_1}}{a} + \frac{\varphi'^3 f_{2X_1 X_1}}{a^3} + \frac{3A_0 \varphi'^2 f_{2X_1 X_2}}{2a^3} + \frac{A_0^2 \varphi' f_{2X_1 X_3}}{a^3} + \frac{A_0 f_{2X_2}}{2a} + \frac{A_0^2 \varphi' f_{2X_2 X_3}}{2a^3} + 2A_0 f_{3\varphi} + 2A_0^3 \varphi' G_{3\varphi} X_3 \frac{\varphi'^3}{a^3} - \frac{2\varphi' G_{3\varphi} X_3 \varphi}{a} - \frac{\varphi'^3 G_{3\varphi} X_1}{a^3} + \frac{9\varphi'^2 G_{3\varphi} X_1}{a^3}, \quad \text{(C.32)}
\]

\[D_7 = 4G_{4\varphi}, \quad \text{(C.33)}
\]

\[D_9 = -\frac{f_{2X_1}}{a^2} - \frac{2\varphi'' G_{3X_1}}{a^2} - \frac{\varphi'' \varphi'^2 G_{3X_1 X_1}}{a^4} + 2G_{3\varphi} - \frac{\varphi'^2 G_{3\varphi} X_1}{a^2} - \frac{2\varphi' G_{3X_1}}{a^2} + \frac{\varphi'^3 G_{3X_1} H}{a^5} - \frac{6G_{4\varphi} H}{a}, \quad \text{(C.34)}
\]

\[D_{10} = -\frac{\varphi'^2 G_{3X_1}}{a^2} + 2G_{4\varphi}, \quad \text{(C.35)}
\]

\[D_{12} = -\frac{\varphi'^2 f_{2X_1 X_3}}{2a^2} - \frac{A_0 \varphi' f_{2X_1 X_3}}{a^2} + \frac{\frac{1}{2} f_{2X_2}}{4a^2} - \frac{A_0 \varphi' f_{2X_2 X_3} - A_0^2 f_{2X_2 X_3}}{2a^2} - 2f_{3\varphi} + \frac{2A_0^2 f_{3\varphi}}{a^2}, \quad \text{(C.36)}
\]

\[D_{13} = -\frac{\varphi'^3 f_{2X_1 X_1 X_2}}{2a^5} - \frac{A_0 \varphi'' \varphi'^2 f_{2X_1 X_1 X_2}}{a^5} - \frac{3\varphi'' \varphi' f_{2X_1 X_2}}{2a^4} - \frac{A_0 \varphi'' \varphi'^2 f_{2X_1 X_2 X_3}}{2a^3} - \frac{A_0^2 \varphi' f_{2X_1 X_1} X_3}{a^3} - \frac{A_0^2 \varphi' f_{2X_1 X_2}}{a^3} - \frac{A_0^2 \varphi' f_{2X_1 X_2 X_3}}{a^3} - \frac{2A_0 f_{3\varphi X_3}}{a^3} - \frac{2A_0 A_0^3 \varphi' f_{2X_1 X_3}}{a^3} - \frac{A_0 A_0^3 \varphi' f_{2X_2 X_3}}{a^3} - \frac{A_0 A_0^3 \varphi' f_{2X_2 X_3}}{a^3} - \frac{3A_0 A_0^3 f_{2X_2 X_3}}{8a^5}
\]

\[D_{14} = -\frac{1}{2} f_{2X_2} - 2f_{3\varphi}, \quad \text{(C.38)}
\]
\[
D_{11} = \frac{2\varphi'' f_{2X_1}}{a^2} + \frac{5\varphi'' \varphi f_{2X_1}}{a^2} + \frac{\varphi'' \varphi^4 f_{2X_1}}{a^2} + \frac{2A_0 \varphi'' \varphi f_{2X_1}}{a^2} \\
+ \frac{A_0' \varphi^4 f_{2X_1}}{2a^4} + \frac{A_0^2 \varphi^2 f_{2X_1}}{2a^4} + \frac{A_0 A_0' \varphi^3 f_{2X_1}}{2a^4} + \frac{5A_0 \varphi'' f_{2X_1}}{2a^4} \\
+ \frac{5A_0' \varphi^2 f_{2X_1}}{a^4} + \frac{5A_0^2 \varphi^2 f_{2X_1}}{a^4} + \frac{3A_0' A_0' \varphi^3 f_{2X_1}}{a^4} + \frac{A_0' \varphi'' f_{2X_1}}{a^4} \\
+ \frac{2A_0^2 A_0' \varphi^2 f_{2X_1}}{a^4} + \frac{A_0^2 \varphi f_{2X_1}}{a^4} + \frac{4A_0 A_0' \varphi f_{2X_1}}{a^4} + \frac{A_0^2 A_0' \varphi f_{2X_1}}{a^4} \\
+ \frac{A_0' f_{2X_1}}{a^2} + \frac{A_0^2 \varphi f_{2X_1}}{a^2} + \frac{3A_0 A_0' \varphi f_{2X_1}}{a^2} + \frac{A_0' \varphi'' f_{2X_1}}{a^2} \\
+ A_0^2 A_0' \varphi^2 f_{2X_1} + \frac{4A_0 A_0' \varphi^4 f_{2X_1}}{a^2} + \frac{3A_0' A_0' \varphi^3 f_{2X_1}}{a^2} + \frac{5A_0^2 A_0' \varphi f_{2X_1}}{2a^2} \\
+ \frac{A_0' f_{2X_1}}{2a^4} + \frac{A_0^2 \varphi f_{2X_1}}{2a^4} + \frac{3A_0 A_0' \varphi f_{2X_1}}{2a^4} + \frac{A_0' \varphi'' f_{2X_1}}{2a^4} \\
+ \frac{A_0^2 A_0' \varphi^2 f_{2X_1}}{2a^4} + \frac{A_0^2 \varphi f_{2X_1}}{2a^4} + \frac{4A_0 A_0' \varphi f_{2X_1}}{2a^4} - \frac{A_0^2 A_0' \varphi f_{2X_1}}{2a^4} \\
+ \frac{A_0'^2 f_{2X_1}}{a^2} + \frac{2A_0^2 A_0' f_{2X_1}}{a^2} + \frac{8A_0^2 A_0' f_{2X_1}}{a^2} - \frac{4\varphi'' G_{3\varphi}}{a^2} \\
- \frac{6\varphi'' \varphi^2 G_{3\varphi X_1}}{a^2} - \frac{\varphi'' \varphi^4 G_{3\varphi X_1}}{a^2} - \frac{2A_0^4 A_0' f_{3\varphi X_1}}{a^2} - \frac{2\varphi''^2 G_{3\varphi X_1}}{a^2} \\
- \frac{\varphi''^4 G_{3\varphi X_1}}{a^2} - \frac{4\varphi' f_{2X_1}}{a^2} - \frac{2\varphi'^3 f_{2X_1}}{a^2} - \frac{\varphi'' f_{2X_1}}{a^2} - \frac{\varphi'' f_{2X_1}}{a^2} \\
- \frac{5A_0 \varphi f_{2X_1}}{a^2} - \frac{2A_0^2 \varphi^3 f_{2X_1}}{a^2} - \frac{3A_0' \varphi f_{2X_1}}{a^2} - \frac{2A_0^2 \varphi'' f_{2X_1}}{a^2} \\
- \frac{3A_0' \varphi f_{2X_1}}{a^2} - \frac{2A_0^2 \varphi^3 f_{2X_1}}{a^2} - \frac{A_0' \varphi f_{2X_1}}{a^2} - \frac{A_0^2 f_{2X_1}}{a^2} \\
- \frac{A_0^2 \varphi f_{2X_1}}{a^2} - \frac{2A_0^2 \varphi^3 f_{2X_1}}{a^2} - \frac{A_0' \varphi f_{2X_1}}{a^2} - \frac{A_0^2 \varphi f_{2X_1}}{a^2} \\
- \frac{A_0^2 \varphi f_{2X_1}}{a^2} - \frac{2A_0^2 \varphi^3 f_{2X_1}}{a^2} - \frac{A_0' \varphi f_{2X_1}}{a^2} - \frac{A_0^2 \varphi f_{2X_1}}{a^2} \\
- \frac{A_0^2 \varphi f_{2X_1}}{a^2} - \frac{2A_0^2 \varphi^3 f_{2X_1}}{a^2} - \frac{A_0' \varphi f_{2X_1}}{a^2} - \frac{A_0^2 \varphi f_{2X_1}}{a^2} \\
- \frac{24\varphi'' \varphi^3 G_{3\varphi X_1}}{a^2} - \frac{3\varphi'' \varphi^5 G_{3\varphi X_1}}{a^2} - \frac{8A_0^3 f_{3\varphi X_1}}{a^2} - \frac{8\varphi' G_{3\varphi}}{a^2} \\
+ \frac{12\varphi'^3 G_{3\varphi X_1}}{a^2} + \frac{4\varphi^5 G_{3\varphi X_1}}{a^2} + \frac{2A_0^5 f_{3\varphi X_1}}{a^2} - \frac{18\varphi'^4 G_{3\varphi X_1}}{a^2} \\
+ \frac{3\varphi''^6 G_{3\varphi X_1}}{a^2} - \frac{12G_{4\varphi}}{a^2} + \frac{12\varphi'^2 G_{3\varphi X_1}}{a^2} + \frac{3\varphi''^4 G_{3\varphi X_1}}{a^2} - \frac{12G_{4\varphi}}{a^2}, \quad \text{(C.39)}
\]
C.5 “Time” vector field equation of motion

Coefficients in Eq. (3.43):

\[
F_1 = -\frac{6 A_0^0 f_{3 X_3}}{a^2} - \frac{6 A_0^2 f_3}{a^2}, \tag{C.41}
\]

\[
F_2 = \frac{\varphi'' f_{2 X_1 X_2}}{2a^2} + \frac{A_0 \varphi'' f_{2 X_1 X_2}}{a^2} + \frac{1}{2} f_{2 X_2} + \frac{A_0 \varphi' f_{2 X_2 X_2}}{4a^2} + \frac{A_0^2 f_{2 X_2 X_2}}{2a^2} + 2 f_{3 \varphi} - \frac{2 A_0^2 \tilde{f}_{3 \varphi}}{a^2}, \tag{C.42}
\]

\[
F_3 = -\frac{\varphi'^3 f_{2 X_1 X_2}}{2a^3} - \frac{A_0 \varphi'^3 f_{2 X_1 X_2}}{a^3} - \frac{\varphi' f_{2 X_2}}{a} - \frac{A_0 \varphi'^2 f_{2 X_2 X_2}}{2a^3} - \frac{3 A_0^2 \varphi' f_{2 X_2 X_2}}{2a^3} - \frac{2 A_0 f_{2 X_2 X_2}}{a^3} - \frac{A_0^3 f_{2 X_2 X_2}}{a^3} - \frac{4 \varphi' f_{3 \varphi}}{a} - \frac{2 A_0^0 \varphi' f_{3 \varphi}}{a} - \frac{8 A_0^2 \varphi' f_{3 \varphi}}{a^3} - \frac{2 A_0^4 \varphi' f_{3 \varphi}}{a^3} + \frac{24 A_0^2 f_{3 X_1 X_3}}{a^5} + \frac{24 A_0^2 f_{3 X_1 X_3}}{a^5} + \frac{24 A_0^2 f_{3 X_1 X_3}}{a^5} + \frac{6 A_0^3 f_{3 X_1 X_3}}{a^5} + \frac{6 A_0^3 f_{3 X_1 X_3}}{a^5}, \tag{C.43}
\]

\[
F_4 = \frac{\varphi' f_{2 X_2 X_2}}{2a^2} + \frac{A_0 \varphi' f_{2 X_2 X_2}}{a^2} + \frac{2 \varphi' f_{3 \varphi}}{a} - \frac{2 A_0^0 \varphi' f_{3 \varphi}}{a} - \frac{8 A_0^2 \varphi' f_{3 \varphi}}{a^3} - \frac{6 A_0^2 \varphi' f_{3 \varphi}}{a^3} - \frac{6 A_0^2 \varphi' f_{3 \varphi}}{a^3} - \frac{6 A_0^2 \varphi' f_{3 \varphi}}{a^3}, \tag{C.44}
\]

\[
F_5 = \frac{\varphi'^2 f_{2 X_2 X_2}}{4a^2} + \frac{A_0 \varphi'^2 f_{2 X_2 X_2}}{a^2} + \frac{f_{2 X_2}}{a} + \frac{A_0^2 \varphi'^3 f_{3 X_1 X_3}}{a^3} + \frac{2 A_0 \varphi' f_{3 X_1 X_3}}{a^2} - \frac{4 A_0 \varphi' f_{3 X_1 X_3}}{a^2} - \frac{2 A_0^4 \varphi' f_{3 X_1 X_3}}{a^5} - \frac{12 A_0 f_{3 X_1 X_3}}{a^2} - \frac{6 A_0^3 f_{3 X_1 X_3}}{a^4} - \frac{12 A_0 f_{3 X_1 X_3}}{a^2} - \frac{6 A_0^3 f_{3 X_1 X_3}}{a^4}, \tag{C.45}
\]

\[
F_6 = -\frac{2 A_0 f_{3 X_3}}{a} - \frac{2 A_0 \tilde{f}_3}{a}. \tag{C.46}
\]
C.6 “Space” vector field equation of motion

Coefficients in Eq. (3.44):

\[ H_1 = \frac{2A_0^2f_3X_3}{a^2} + \frac{2A_0^2\tilde{f}_3}{a^2}, \quad \text{(C.47)} \]
\[ H_2 = \frac{1}{2}f_{2X_2} - 2f_{3\varphi}, \quad \text{(C.48)} \]
\[ H_3 = -\frac{2A_0f_3X_3}{a} - \frac{2A_0\tilde{f}_3}{a}, \quad \text{(C.49)} \]
\[ H_4 = -f_{2X_3} + \frac{2A_0'f_3X_3}{a^2} + \frac{2A_0\varphi'\tilde{f}_3}{a^2} + \frac{4A_0f_3X_3\mathcal{H}}{a^2} + \frac{4A_0\tilde{f}_3\mathcal{H}}{a^2}. \quad \text{(C.50)} \]

D Equations with QSA and SHA: coefficients

Coefficients in Eq. (4.15):

\[ W_1 = B_6(A_5 - B_7)H_3^2, \quad \text{(D.1)} \]
\[ W_2 = B_6[F_3H_1H_2 - F_3H_2H_3 + (B_7 - A_5)F_3H_4], \quad \text{(D.2)} \]
\[ W_3 = B_6(A_5^2 - 2A_5B_7 + B_6B_9)H_3^2, \quad \text{(D.3)} \]
\[ W_4 = B_7^2H_1(F_3H_1 - F_3H_3) + B_6B_7(F_3H_2H_3 + F_4H_1H_3 - 2F_5H_1H_2 + 2A_5F_5H_4) \]
\[ - B_6[A_5^2F_3H_4 - D_9F_3H_1H_3 + A_5(F_3H_2H_3 + F_4H_1H_3 - 2F_5H_1H_2) \]
\[ - B_6F_3H_2^2 + D_9F_3(H_3^2 + B_6H_4) + B_6H_3(F_4H_2 + H_3m_\varphi^2)], \quad \text{(D.4)} \]
\[ W_5 = B_6(F_3H_1^2 - F_3H_1H_3 + B_6F_3H_4)m_\varphi^2, \quad \text{(D.5)} \]
\[ W_6 = -[(B_7^2 + B_6B_9)H_1 + B_6(B_7 - A_5)H_2]H_3, \quad \text{(D.6)} \]
\[ W_7 = B_7^2F_3H_4 - B_6(F_4H_1H_2 - F_3H_2^2 + D_9F_3H_4 + (B_7 - A_5)F_4H_4 + H_1H_3m_\varphi^2), \quad \text{(D.7)} \]
\[ W_8 = B_6F_3H_3m_\varphi^2, \quad \text{(D.8)} \]
\[ W_9 = (B_7^2 - B_6D_9)F_5H_1 - B_6(B_7 - A_5)F_5H_2 - [(B_7^2 - B_6D_9)F_3 \]
\[ - B_6(B_7 - A_5)F_4]H_3, \quad \text{(D.9)} \]
\[ W_{10} = B_6(F_5H_1 - F_3H_3)m_\varphi^2, \quad \text{(D.10)} \]
\[ W_{11} = (B_6D_9 - A_5B_7)H_3^2, \quad \text{(D.11)} \]
\[ W_{12} = B_6F_5(H_2^2 - D_9H_4) + B_7(A_5F_5H_4 + F_3H_2H_3 - F_5H_1H_2) \]
\[ - B_6H_3(F_4H_2 + H_3m_\varphi^2), \quad \text{(D.12)} \]
\[ W_{13} = B_6F_5H_4m_\varphi^2, \quad \text{(D.13)} \]
\[ W_{14} = (B_6D_9 - B_7^2)H_3^2, \quad \text{(D.14)} \]
\[ W_{15} = B_6F_5(H_2^2 - D_9H_4) - B_6H_3(F_4H_2 + H_3m_\varphi^2) + B_7^2F_5H_4. \quad \text{(D.15)} \]
E Effective dark energy fluid

Here we use $\kappa = 1$.

E.1 Theories with non-vanishing anisotropic stress

Coefficients in Eq. (4.23):

$$
Z_1 = G_{4\varphi}(-A_2^2 B_7 G_4 - (B_6 - 2) B_6 D_9 G_{4\varphi} + A_5 B_7 (B_7 G_4 + (B_6 - 2) G_{4\varphi})) H_3^3, \tag{E.1}
$$

$$
Z_2 = -G_{4\varphi}(-B_6 G_{4\varphi}(A_5 (-F_5 H_1 H_2 + F_4 H_1 H_3 + A_6 H_2 H_3)
+ D_9 (H_3 (-A_6 H_1 - F_3 H_1 + A_4 H_3) + F_5 (H_1^2 + (B_6 - 2) H_4)))
- (B_6 - 2) (F_3 H_2^2 - H_3 (F_3 H_2 + H_3 m_\varphi^2))) + B_7 (A_5 F_5 G_4 H_1 H_2)
- 2 (B_6 - 1) F_5 G_{4\varphi} H_1 H_2 + G_{4\varphi} (B_6 F_3 H_1 - 2 F_3 H_2 + B_6 (A_6 + F_3) H_2) H_3
- A_2^2 F_5 G_4 H_4 + A_5 (B_6 - 2) F_3 G_{4\varphi} H_4 - A_5 G_4 H_3 (F_3 H_2 + H_3 m_\varphi^2))
+ B_7^2 (F_3 (G_{4\varphi} H_2^2 + A_5 G_4 H_2) + H_3 (A_4 G_{4\varphi} H_3 - A_6 G_{4\varphi} H_1 - F_3 G_{4\varphi} H_1 + G_4 H_3 m_\varphi^2))
- \frac{A_2 (A_5 - B_7) B_7 (G_{4\varphi}^2 - G_4 G_{4\varphi}^2) H_3^3 \varphi' + \frac{a}{a^2}}{a}, \tag{E.2}
$$

$$
Z_3 = G_{4\varphi} (B_6 F_5 G_{4\varphi} H_1 M_\varphi^2 - B_6 F_5 G_{4\varphi} H_2 m_\varphi^2 + 2 B_6 F_5 G_{4\varphi} H_3 m_\varphi^2 - B_6^2 F_5 G_{4\varphi} H_4 m_\varphi^2
+ A_4 G_{4\varphi} (B_5^2 H_3 H_1 + B_6 F_5 (H_3^2 - D_9 H_4) - B_6 H_3 (F_4 H_2 + H_3 m_\varphi^2))
+ A_5 G_{4\varphi} (-B_5^2 F_3 H_4 + B_6 (F_3 H_2 H_3 - F_3 H_4^2) + D_9 F_3 H_4 + (B_7 - A_5) F_4 H_4 + H_1 H_3 m_\varphi^2))
+ B_7 F_5 G_4 H_1 M_\varphi^2 - B_7 F_3 G_4 H_2 M_\varphi^2 - A_5 B_7 F_5 G_4 H_4 M_\varphi^2 + B_7^2 F_5 G_4 H_4 M_\varphi^2)
+ \frac{A_2 B_7 (G_{4\varphi}^2 - G_4 G_{4\varphi}^2) (F_3 H_2 H_3 - F_3 (H_1 H_2 + (B_7 - A_5) H_4)) \varphi'}{a^2}, \tag{E.3}
$$

$$
Z_4 = B_6 (A_1 F_5 - A_6 F_3) G_{4\varphi}^2 H_4 m_\varphi^2, \tag{E.4}
$$

$$
Z_5 = B_6 (A_2^2 - 2 A_1 B_7 + B_6 D_9) G_{4\varphi}^2 H_3^3, \tag{E.5}
$$

$$
Z_6 = G_{4\varphi} (B_5^2 H_1 F_5 H_1 - F_5 H_3) + B_6 B_7 (-2 F_5 H_1 H_2 + F_4 H_1 H_3 + F_3 H_2 H_3 + 2 A_5 F_3 H_4)
- B_6 (A_2^2 F_5 H_4 - B_6 F_3 H_2 - D_9 F_3 H_1 H_3 + A_5 (-2 F_5 H_1 H_2 + F_4 H_1 H_3 + F_3 H_2 H_3)
+ D_9 F_5 (H_4^2 + B_6 H_4) + B_6 H_3 (F_4 H_2 + H_3 m_\varphi^2))), \tag{E.6}
$$

$$
Z_7 = B_6 G_{4\varphi}^2 (F_5 (H_1^2 + B_6 H_4) - F_3 H_1 H_3) m_\varphi^2, \tag{E.7}
$$

$$
Z_8 = (A_5 - B_7) B_7 G_{4\varphi}^2 (B_7 G_4 - (B_6 - 2) G_{4\varphi}) H_3^3, \tag{E.8}
$$

$$
Z_9 = G_{4\varphi}^2 (-B_5^2 G_4 H_3 (A_5 B_1 H_2 - B_1 D_9 H_1 + B_6 D_9 H_3) + B_7^2 F_3 G_4 H_4
+ B_7^2 ((2 - B_6) F_5 G_{4\varphi} H_4 + F_5 G_4 (H_1 H_2 - A_5 H_4)
+ H_3 (-B_1 G_{4\varphi} H_1 - F_3 G_4 H_2 + B_6 G_{4\varphi} H_3 - 3 G_4 H_3 M_\varphi^2))
+ B_7 (2 - B_6) F_5 G_{4\varphi} (H_1 H_2 - A_5 H_4)
+ H_3 (B_1 B_6 G_{4\varphi} H_2 + (B_6 - 2) F_5 G_{4\varphi} H_2 + 3 A_5 G_{4\varphi} H_3 M_\varphi^2))
+ B_4 (A_5 - B_7) B_7 G_{4\varphi} (G_{4\varphi}^2 - G_4 G_{4\varphi}^2) H_3^3 \varphi' + \frac{a}{a^2}
- \frac{B_2 (A_5 - B_7) B_7 H_3^3 (G_{4\varphi}^2 - G_4 G_{4\varphi}^2 M_\varphi^2 - 2 G_4 G_{4\varphi}^2 + G_4 G_{4\varphi} G_{4\varphi} M_\varphi^2)) \varphi'^2}{a^2}
+ \frac{B_2 (A_5 - B_7) B_7 H_3^3 (G_{4\varphi} (G_{4\varphi}^2 - G_4 G_{4\varphi})) \varphi''')}{a^2}, \tag{E.9}
$$
\[ Z_{10} = G_{4\varphi}^{2}(B_{6}B_{9}G_{4\varphi}(-F_{3}H_{2}^{2} + F_{3}H_{2}H_{3} + D_{3}F_{3}H_{4} + H_{3}^{2}m_{\varphi}^{2}) \\
+ B_{11}G_{4\varphi}(B_{9}^{2}F_{3}H_{4} - B_{6}(F_{3}H_{1}H_{2} - F_{3}H_{2}^{2} + D_{3}F_{3}H_{4} + (B_{7} - A_{5})F_{3}H_{4} + H_{1}H_{3}m_{\varphi}^{2})) \\
- B_{7}(3G_{4}(-F_{3}H_{1}H_{2} + F_{3}H_{2}H_{3} + A_{3}F_{3}H_{4})\nu_{\varphi} + B_{7}F_{3}H_{4}(B_{9}G_{4\varphi} - 3G_{4}\nu_{\varphi}))) \\
+ \frac{B_{4}B_{7}G_{4\varphi}(G_{3\varphi}^{2} - G_{4G_{4\varphi}})(F_{3}H_{1}H_{2} - F_{3}H_{2}H_{3} + (B_{7} - A_{5})F_{3}H_{4})\varphi'}{a} \\
+ \frac{B_{2}B_{7}(F_{3}H_{1}H_{2} - F_{3}H_{2}H_{3} + (B_{7} - A_{5})F_{3}H_{4})(2G_{4}G_{4\varphi} - G_{4\varphi}G_{4\varphi})\varphi'^{2}}{a^{2}} \\
- \frac{B_{2}B_{7}(F_{3}H_{1}H_{2} - F_{3}H_{2}H_{3} + (B_{7} - A_{5})F_{3}H_{4})G_{4\varphi}G_{4\varphi}\varphi'^{2}}{a^{2}} \\
+ \frac{B_{2}B_{7}(F_{3}H_{1}H_{2} - F_{3}H_{2}H_{3} + (B_{7} - A_{5})F_{3}H_{4})G_{4\varphi}(G_{4\varphi}^{2} - G_{4G_{4\varphi}})\varphi''}{a^{2}}, \tag{E.10} \]
\[ Z_{11} = B_{6}(B_{11}F_{3} - B_{9}F_{5})G_{4\varphi}^{3}H_{4}m_{\varphi}, \tag{E.11} \]
\[ Z_{12} = G_{4\varphi}, \tag{E.12} \]
\[ Z_{13} = H_{3}\left(G_{4\varphi}(B_{6}G_{4\varphi}(C_{5}D_{9}H_{1} - A_{5}C_{5}H_{2} - C_{5}D_{9}H_{3}) + B_{7}(B_{6}C_{5}G_{4\varphi}H_{2} + A_{5}C_{4}G_{4}H_{3}) \\
- B_{7}^{2}(C_{5}G_{4\varphi}H_{1} + C_{4}G_{4}H_{3} - C_{3}G_{4\varphi}H_{3})\right) + \frac{(A_{5} - B_{7})B_{7}C_{2}(G_{4\varphi}^{2} - G_{4G_{4\varphi}})H_{3}\varphi'}{a}, \tag{E.13} \]
\[ Z_{14} = G_{4\varphi}(B_{7}^{2}(C_{6}G_{4\varphi}(F_{3}H_{1} - F_{3}H_{3}) + (C_{4}F_{3}G_{4} + C_{5}F_{3}G_{4\varphi} - C_{3}F_{3}G_{4\varphi})H_{4}) \\
- B_{7}(B_{6}G_{4\varphi}(C_{6}F_{3}H_{2} - C_{6}F_{3}H_{3} + C_{5}F_{3}H_{4}) \\
+ C_{4}G_{4}(-F_{3}H_{1}H_{2} + F_{3}H_{2}H_{3} + A_{5}F_{3}H_{4}) \\
+ B_{8}G_{4\varphi}(C_{6}(-D_{9}F_{3}H_{1} + A_{5}F_{3}H_{2} + D_{6}F_{3}H_{3} - A_{5}F_{4}H_{3}) \\
- C_{5}(F_{3}H_{1}H_{2} - F_{3}H_{2}^{2} + D_{9}F_{3}H_{4} - A_{5}F_{4}H_{4} + H_{1}H_{3}m_{\varphi}^{2}) \\
+ C_{3}(-F_{3}H_{2}^{2} + F_{3}H_{2}H_{3} + D_{9}F_{3}H_{4} + H_{3}^{2}m_{\varphi}^{2})) \\
- B_{7}C_{2}(G_{4\varphi}^{2} - G_{4G_{4\varphi}})(F_{3}H_{1}H_{2} - F_{3}H_{2}H_{3} + (B_{7} - A_{5})F_{3}H_{4})\varphi' \\
+ \frac{B_{6}G_{4\varphi}^{2}(C_{6}F_{3}H_{1} - C_{6}F_{3}H_{3} + C_{5}F_{3}H_{4} - C_{3}F_{3}H_{4})m_{\varphi}^{2}}{a}, \tag{E.14} \]
\[ Z_{15} = B_{6}G_{3\varphi}^{2}(C_{6}F_{3}H_{1} - C_{6}F_{3}H_{3} + C_{5}F_{3}H_{4} - C_{3}F_{3}H_{4})m_{\varphi}, \tag{E.15} \]
E.2 Theories with vanishing anisotropic stress

Coefficients in Eq. (4.35):

\begin{align*}
Y_1 &= - (A_3^3 + (B_6 - 2)D_9) H^2, \\
Y_2 &= A_3^2 F_5 H_4 + D_9 (H_3 (A_4 H_3 - A_6 H_1 - F_3 H_1) + F_5 (H_1^2 + (B_6 - 2) H_4)) \\
&\quad - (B_6 - 2) (F_5 H_1^2 - H_3 (F_4 H_2 + H_3 m_{\varphi}^2)) \\
&\quad + A_5 (H_3 (F_4 H_1 + A_6 H_2 + F_3 H_2 + H_3 \mu_\varphi) - 2 F_3 H_2 H_4), \\
Y_3 &= A_4 F_5 (H_2^2 - D_9 H_4) - F_5 H_1^2 m_{\varphi}^2 + F_3 H_1 H_3 m_{\varphi}^2 + 2 F_5 H_4 m_{\varphi}^2 - B_6 F_5 H_4 m_{\varphi}^2 \\
&\quad - A_4 H_3 (F_4 H_2 + H_3 m_{\varphi}^2) + A_6 (F_4 H_1 H_2 - F_3 H_2^2 + D_9 F_3 H_4 - A_5 F_4 H_4 \\
&\quad + H_1 H_3 m_{\varphi}^2) + F_5 H_1 H_2 H_3 \mu_\varphi - F_3 H_2 H_3 \mu_\varphi - A_5 F_3 H_4 \mu_\varphi, \\
Y_4 &= (A_4 F_5 - A_6 F_3) H_4 m_{\varphi}^2, \\
Y_5 &= (A_3^3 + B_6 D_9) H_3^2, \\
Y_6 &= 2 A_4 F_5 H_1 H_2 + B_6 F_5 H_2^2 + D_9 F_3 H_1 H_3 - A_5 (F_4 H_1 + F_3 H_2) H_3 - A_4^2 F_3 H_4 \\
&\quad - D_9 F_5 (H_1^2 + B_6 H_4) - B_6 H_3 (F_4 H_2 + H_3 m_{\varphi}^2), \\
Y_7 &= (F_5 (H_1^2 + B_6 H_4) - F_3 H_3 (F_4 H_2 + H_3 m_{\varphi}^2))^2, \\
Y_8 &= H_3 (B_{11} D_9 H_1 - A_5 B_{11} H_2 - B_{10} D_9 H_3 + 3 A_5 H_3 \nu_\varphi), \\
Y_9 &= - B_{11} (F_1 H_1 H_2 - F_3 H_2^2 + D_9 F_3 H_4 - A_5 F_4 H_4 + H_1 H_3 m_{\varphi}^2) \\
&\quad + B_9 (F_4 H_2 H_3 + D_9 F_3 H_4 + H_3^2 m_{\varphi}^2 - F_5 H_2^2) \\
&\quad + 3 (F_3 H_1 H_2 - F_3 H_2 H_3 - A_5 F_3 H_4) \nu_\varphi, \\
Y_{10} &= (B_{11} F_3 - B_9 F_3) H_4 m_{\varphi}^2, \\
Y_{11} &= H_3 (C_5 D_9 H_1 - A_5 C_5 H_2 + A_5 C_4 H_3 - C_3 D_9 H_3), \\
Y_{12} &= C_4 F_5 H_4 H_1 - C_5 F_3 H_2^2 - C_5 F_2^2 H_2 H_3 + C_5 F_2 H_2 H_3 \\
&\quad + C_6 (A_5 F_3 H_2 + D_9 F_3 H_3 - A_5 F_3 H_3 - D_9 F_3 H_1) - A_5 C_4 F_5 H_4 + C_5 D_9 F_5 H_4 \\
&\quad + C_3 H_2^2 m_{\varphi}^2 - C_5 (F_4 H_1 H_2 - F_3 H_2^2 + D_9 F_3 H_4 - A_5 F_4 H_4 + H_1 H_3 m_{\varphi}^2), \\
Y_{13} &= (C_6 F_5 H_1 - C_5 F_3 H_3 + C_5 F_3 H_4 - C_3 F_5 H_4) m_{\varphi}^2.
\end{align*}

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