A Hyperbolic Model of Boiling Liquid

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Abstract—A model of a boiling liquid is presented that is based on the single-speed two-temperature generalized equilibrium model of a mixture with account for the forces of interaction between the fractions previously proposed by the author. The liquid fraction is assumed to be incompressible. The characteristic analysis of the model equations is carried out and their hyperbolicity is revealed. The relations for the characteristic directions and the differential relations along them are derived. The analytical formula for calculating the speed of sound in a boiling liquid is obtained. It is noted that when the phase transitions are taken into account the speed of sound in a liquid turns to be slightly lower than the speed calculated by the Wood formula. The calculation formulas are provided for the iterative nodal method of characteristics, which is used to calculate the flow during the decay of an arbitrary discontinuity in a boiling liquid without account for the interfractional heat exchange. In the calculations it is assumed that the phase transition in the boiling process occurs under the superheated state conditions when the temperature of the liquid exceeds the saturation temperature. It is demonstrated that accounting for the phase transformation leads to a significant increase in the vapor concentration in the unloading wave and to a slight increase in both the velocity of the mixture and its pressure. The concentration of the vapor fraction behind the front of the shock jump decreases.

Keywords: boiling liquid, hyperbolic model, nodal method of characteristics

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1. INTRODUCTION

The explosive boiling of a liquid following an abrupt loss in pressure is of practical and theoretical interest in view of increasingly stringent requirements to the safety of power units. The theoretical and experimental works on studying the flows of boiling liquid were outlined in [1]. The authors of [2–6] numerically investigated the phenomenon of explosive boiling of liquid at decreasing pressure using different models.

The current paper presents a new model of boiling liquid based on the single-speed generalized equilibrium (GE) hyperbolic two-temperature model from [7]. To simplify the computations, we assume that the liquid is incompressible. Earlier, the GE model was applied to calculate the flows of viscous heat conducting mixtures; however, the relaxation analogs were introduced in it instead of the original Fourier and Stokes laws [8, 9]. In the following, we show that the GE model can also be applied to studying the flows with phase transitions; in this case the type of the system of equations does not change and it remains hyperbolic. This fact allows us to exploit the well-proven numerical methods for solving the hyperbolic systems in order to integrate the equations. In the current work, we apply the nodal method of characteristics well-known in gas dynamics to integrate the equations of the mixture model. This method consists in transiting from the partial differential equations to the ordinary differential equations written along the characteristic directions. We generalize the one-dimensional variant of the method outlined in [10] to the solution of multidimensional problems by using the procedure of splitting into spatial directions.

2. MODEL OF BOILING LIQUID

The equations that describe the one-dimensional flow of a vapor–liquid mixture with phase transitions and incompressible disperse fraction ($\rho^0 = \text{const}$) can be written in divergence form

$$\frac{\partial \alpha_{sf} \rho^0}{\partial t} + \frac{\partial \alpha_{sf} \rho^0 u}{\partial x} = J, \quad \frac{\partial \alpha_{sl} \rho^0}{\partial t} + \frac{\partial \alpha_{sl} (p + \rho^0 u^2)}{\partial x} = f + J u,$$
Here, \( t \) is the time, \( u \) is the velocity of the mixture flow, \( p \) is the pressure, \( \alpha_k \) is the volume fraction, where \( k = s, t \) is the fraction indicator, where the index \( s \) denotes the vapor and the index \( t \) denotes the incompressible fraction, \( J \) is the vaporization intensity per unit volume of the mixture, \( e_k = e_v + 0.5u^2 \) is the total specific energy of the \( k \)th fraction, \( e_{pt} = e_s(p, \rho_s^0) \) and \( e_s = c_v,s \theta_v + e_o \) are the specific internal energies, where \( \theta_v \) is the temperature of the disperse fraction, \( c_v,s \) is its heat capacity, \( f \) is the density of the force of interaction between the fractions [7], which is a priori unknown and is determined during integration of system (1), and \( Q \) is the intensity of heat exchange between the fractions per unit volume of the mixture. Note that, if we ignore the forces of interaction between the fractions in Eq. (1) the system loses its hyperbolic character.

After summation of the expressions corresponding to the conservation laws over the mixture fractions, we obtain the laws of mass, momentum, and energy conservation for the mixture as a whole

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} &= 0, \\
\frac{\partial \rho u}{\partial t} + \frac{\partial (p + \rho u^2)}{\partial x} &= 0, \\
\frac{\partial \rho e}{\partial t} + \frac{\partial (\rho e + pu)}{\partial x} &= 0,
\end{align*}
\]

where \( \rho = \alpha_s \rho_s^0 + \alpha_t \rho_t^0 \) is the mixture density and \( e = \varepsilon + u^2/2 \) and \( e = (\alpha_s \rho_s^0 e_{st} + \alpha_t \rho_t^0 e_s)/\rho \) are the specific total and internal energy of the mixture. Equations (2) in the quasi-linear form read

\[
\frac{1}{\rho} \frac{D \rho}{Dt} + \frac{\partial u}{\partial x} = 0, \quad \frac{D u}{Dt} + \frac{1}{\rho} \frac{\partial \rho}{\partial x} = 0, \quad \frac{D e}{Dt} + \rho \frac{\partial u}{\partial x} = 0,
\]

where \( \frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \). The corresponding conservation laws for the vapor and liquid are written as

\[
\begin{align*}
\frac{D \rho_s^0}{Dt} + \rho_s \frac{\partial u}{\partial x} &= J, \\
\rho_s \frac{D u}{Dt} + \alpha_s \frac{D \rho}{\partial x} &= f, \\
\frac{D \rho_s^0}{Dt} + \alpha_s \frac{D u}{\partial x} &= Q + J(e_s - e_{st}),
\end{align*}
\]

\[
\begin{align*}
\frac{D \rho_s}{Dt} + \rho_s \frac{\partial u}{\partial x} &= -J, \\
\rho_s \frac{D u}{Dt} + \alpha_s \frac{D \rho}{\partial x} &= -f, \\
\frac{D \rho_s}{Dt} + \alpha_s \frac{D u}{\partial x} &= -Q.
\end{align*}
\]

Here, \( \rho_k = \alpha_k \rho_k^0 \) is the normalized density of the \( k \)th fraction. Taking into account the equalities

\[
\begin{align*}
\frac{D e_{st}}{Dt} &= \frac{\partial e_{st}}{\partial \rho} \frac{D \rho}{Dt} + \frac{\partial e_{st}}{\partial \rho_s^0} \frac{D \rho_s^0}{Dt}, \\
\frac{D \rho_s^0}{Dt} &= \frac{\partial \rho_s^0}{\partial \rho} \frac{D \rho}{Dt}, \\
\frac{D \rho_s^0}{Dt} &= \frac{1}{\alpha_s} \left( \frac{\rho_s^0}{\rho_s} - \frac{\rho_s^0}{\rho} J + \frac{\rho_s^0}{\rho} \frac{D \rho}{Dt} \right), \\
\frac{D \rho_s}{Dt} &= \frac{J}{\rho_s^0} - \frac{\alpha_s}{\rho} \frac{D \rho}{Dt},
\end{align*}
\]

we may transform the law of energy conservation for the mixture as a whole to

\[
\frac{D e}{Dt} + \rho \frac{\partial u}{\partial x} = \Pi ,
\]
A HYPERBOLIC MODEL

where

\[ c = \sqrt{\frac{p - (\rho_s^0)^2 \frac{\partial e_s}{\partial \rho_s}}{\rho_s \frac{\partial e_s}{\partial p}}}, \quad \Pi = \frac{Q + J(c_s - e_s)}{\rho_s^0} - J \left(1 - \frac{\rho_s^0}{\rho_s^0} \right) \frac{\partial e_s}{\partial \rho_s}. \] (3)

Here, \( c \) is the speed of sound in the mixture.

Finally, the system of equations of the liquid model with phase transitions in the quasi-linear form becomes

\[ \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0, \]

\[ \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho \frac{c^2}{\partial x} \frac{\partial u}{\partial x} = \Pi, \quad \frac{\partial p_s^0}{\partial t} + u \frac{\partial p_s^0}{\partial x} + G_s \frac{\partial u}{\partial x} = I_s, \]

\[ \frac{\partial \alpha_s}{\partial t} + u \frac{\partial \alpha_s}{\partial x} + K_s \frac{\partial u}{\partial x} = L_s, \quad \frac{\partial \theta_s}{\partial t} + u \frac{\partial \theta_s}{\partial x} + Y_s \frac{\partial u}{\partial x} = Z_s, \] (4)

where

\[ G_s = \frac{p_s^0}{\alpha_s}, \quad I_s = J \frac{p_s^0 - p_s^0}{\alpha_s^0 \rho_s^0}, \quad K_s = -\alpha_s, \quad L_s = J \frac{\rho_s^0}{\rho_s^0}, \quad Y_s = \frac{p}{c_s \rho_s^0}, \quad Z_s = -\frac{Q}{\alpha_s c_s \rho_s^0}. \]

The roots of the characteristic equation of system (4) are determined from

\[ \begin{vmatrix} \lambda - u & -\rho & 0 & 0 & 0 & 0 \\ 0 & \lambda - u & -\frac{1}{\rho} & 0 & 0 & 0 \\ 0 & -\rho c^2 & \lambda - u & 0 & 0 & 0 \\ 0 & -G_s & 0 & \lambda - u & 0 & 0 \\ 0 & -K_s & 0 & 0 & \lambda - u & 0 \\ 0 & -Y_s & 0 & 0 & 0 & \lambda - u \end{vmatrix} = 0, \]

where \( \lambda = dx/dt \), and all these roots are real numbers: \( \lambda_{1,2} = u \pm \epsilon \), \( \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = u \). The relations along the characteristic directions \( dx/dt = u \pm \epsilon \) of system (4) may be found from the equation

\[ \begin{vmatrix} \lambda - u & -\rho & 0 & 0 & 0 & -\frac{u}{dt} - \rho \frac{du}{dt} \\ 0 & \lambda - u & -\frac{1}{\rho} & 0 & 0 & -\frac{u}{dt} - \frac{1}{\rho} \frac{dp}{dt} \\ 0 & -\rho c^2 & \lambda - u & 0 & 0 & \Pi - \frac{u}{dt} - \rho c^2 \frac{du}{dt} \\ 0 & -G_s & 0 & \lambda - u & 0 & I_s - \frac{u}{dt} \frac{dp_s^0}{dt} - G_s \frac{du}{dt} \\ 0 & -K_s & 0 & 0 & \lambda - u & L_s - \frac{u}{dt} \frac{d\alpha_s}{dt} - K_s \frac{du}{dt} \\ 0 & -Y_s & 0 & 0 & 0 & Z_s - \frac{u}{dt} \frac{d\theta_s}{dt} - Y_s \frac{du}{dt} \end{vmatrix} = 0. \]

We expand the determinant and obtain

\[ (u \pm \epsilon)(\rho c u \pm dp) = \pm \Pi dt. \] (5)
The following equations are valid along the trajectory $\lambda = u$
\[ dp - c^2 \frac{d\rho}{\rho} = \Pi dt, \quad d\rho^0_{st} - G_{st} \frac{d\rho}{\rho} = I_{st} dt, \quad d\alpha_{st} - K_{st} \frac{d\rho}{\rho} = L_{st} dt, \quad d\theta_s - Y_s \frac{d\rho}{\rho} = Z_{st} dt, \tag{6} \]
they directly follow from system (4).

We use the following relations as the caloric and temperature equations of state of the water vapor
\[ \varepsilon_{st} = \frac{(p + \rho_{st} \gamma_{st}) (1 - \rho^0_{st} b_{st}) + q_{st}}{(\gamma_{st} - 1) \rho_{st}}, \quad \theta_{st} = \frac{(p + \rho_{st} \gamma_{st}) (1 - \rho^0_{st} b_{st})}{(\gamma_{st} - 1) \rho_{st}}, \tag{7} \]
where $\gamma_{st} = 1.47$, $\rho_{st} = 0$, $q_{st} = 2.077616 \times 10^6$ J/kg, $c_{v,st} = 0.955 \times 10^3$ J/(kg K), and $b_{st} = 0$ [11]. The formula for the speed of sound (3) in the case of application of Eqs. (7), i.e., with account for phase transitions, becomes
\[ c = \frac{\sqrt{p (\gamma_{st} + \alpha_s / \alpha_{st})}}{\rho}. \]
The speed of sound in the vapor–water mixture ($\rho^0_{st} = 1000$ kg/m$^3$, $\theta_s = 293$ K) calculated by this formula $c(\alpha_s)$ under normal conditions is given in Fig. 1 for comparison, we provide the speed $c_w(\alpha_{st})$ computed by the Wood formula [12]
\[ c_w = c_{st} \sqrt{\frac{\rho^0_{st}}{\alpha_{st}}} \]
where $c_{st} = \frac{\gamma_{st} (p + \rho_{st})}{\sqrt{\rho^0_{st} (1 - b_{st} \rho^0_{st})}}$. In the figure we see that the speed of sound in the liquid with account to phase transitions appears to be somewhat lower than the speed of sound computed with the Wood formula.

3. THE NUMERICAL METHOD AND THE RESULTS OF CALCULATION

To integrate Eqs. (4) numerically, we apply the nodal method of characteristics. In this method it is sufficient to find the values of the sought variables in the node $(x_{k}, t_{n+1})$ by using their known values in the nodes at the $n$th time layer. We implement the following iteration procedure. We assume that the values of the sought variables at the zeroth iteration ($\nu = 0$) in the point $(x_{k}, t_{n+1})$ coincide with their values in the point $(x_{k}, t_{n})$; here, the characteristic directions $dx/dt = u$ and $dx/dt = u \pm c$ are approximated by the expressions:
\[ x_k - x_C^\nu = u^\nu \Delta t, \quad x_k - x_L^\nu = (u^\nu + c^\nu) \Delta t, \quad x_k - x_R^\nu = (u^\nu - c^\nu) \Delta t, \]

![Fig. 1. The character of the speed of sound in a vapor–water mixture computed with account for phase transition $c(\alpha_{st})$ (solid line) and by the Wood formula $c_w(\alpha_{st})$ (dashed line).](image-url)
where $\Delta t = t_{n+1} - t_n$. The points of intersection between the characteristic directions and the straight line $t = t_n$ (Fig. 2) are determined by

$$x_L^v = x_k - (u^v + c^v)\Delta t, \quad x_C^v = x_k - u^v\Delta t, \quad x_R^v = x_k - (u^v - c^v)\Delta t. \quad (8)$$

The parameters $(\rho, u, p, \rho_0^0, \alpha_{st}, \theta_s) (0)$ in the points $(x_L, x_C, x_R) (0)$ are found by interpolating their known values in the nodes $x_k, x_{k+1}$. We rewrite relations (5) and (6) in the finite difference form

$$p^{v+1}(x_k, t_{n+1}) - p^v(x_k, t_n) = \rho c_{||}^v(u^{v+1}(x_k, t_{n+1}) - u^v(x_k, t_n)) = \rho \frac{\Pi}{u + c_{||}} \Delta t,$$

$$p^{v+1}(x_k, t_{n+1}) - p^v(x_k, t_n) = \rho \frac{\Pi}{u - c_{||}} \Delta t,$$

$$(\rho_0^0)^{v+1}(x_k, t_{n+1}) - (\rho_0^0)^v(x_k, t_n) = \frac{G_{st}}{\rho} \rho^{v+1}(x_k, t_{n+1}) - \rho^v(x_k, t_n) = I^v(x_k, t_n) \Delta t,$$

$$\alpha_{st}^{v+1}(x_k, t_{n+1}) - \alpha_{st}^v(x_k, t_n) = \frac{K_{st}}{\rho} \rho^{v+1}(x_k, t_{n+1}) - \rho^v(x_k, t_n) = L^v(x_k, t_n) \Delta t,$$

$$\theta_s^{v+1}(x_k, t_{n+1}) - \theta_s^v(x_k, t_n) = \frac{Y_s}{\rho} \rho^{v+1}(x_k, t_{n+1}) - \rho^v(x_k, t_n) = Z^v(x_k, t_n) \Delta t,$$

$$p^{v+1}(x_k, t_{n+1}) - p^v(x_k, t_n) = \rho \frac{\Pi}{u + c_{||}} \Delta t.$$

By solving system (9) at $v = 0$ for the variables $(\rho, u, p, \rho_0^0, \alpha_{st}, \theta_s) (1)$, we obtain the refined values of the sought functions in the point $(x_k, t_{n+1})$, then, from these data we compute the new coordinates $(x_L, x_C, x_R) (1)$ by Eqs. (8) and use them to determine $(\rho, u, p, \rho_0^0, \alpha_{st}, \theta_s) (2)$ by Eq. (9), where we must set $v = 1$. We continue the outlined iteration process until convergence.

To illustrate the application of the above proposed numerical procedure we consider the Riemann problem at the following parameter values at the time instance $t = 0$: “to the left” from the contact discontinuity ($x < 25$) $p_L = 0.7 \text{ MPa}, u_L = 0, (\alpha_{st})_L = 0.15$, and $\theta_{st} = 440 \text{ K}”$ and “to the right” from it ($x > 25$) $p_R = 0.3 \text{ MPa}, u_R = 0, (\alpha_{st})_R = 0.3$, and $\theta_{st} = 408.8 \text{ K}$. The remaining parameters have the
values $\rho_0 = 1000$ kg/m$^3$, $c_{rs} = 4.15 \times 10^3$ J/(kg K), and $\epsilon_0 = 1.2381 \times 10^6$ J/kg [13]. We ignore the interfractional heat exchange, that is, we suppose that $Q = 0$. The saturation temperature $\theta_s(p)$ is determined by the formula

$$\theta_s(p) = \theta_* - \frac{\theta_*}{\ln(p/p_*)},$$

where $p_* = 20.2 \times 10^9$ Pa, $\theta_* = 4200$ K, and $\theta_* = 31$ K. The intensity of the water–vapor phase transition is determined by the relation

$$J = \beta \alpha_s(\theta_s - \theta_s(p)).$$

In the computations we assume that the phase transition during boiling occurs under the conditions of the superheated state when the temperature of the liquid exceeds the saturation temperature: $\theta_s - \theta_s(p) > \Delta \theta$, where $\Delta \theta$ is the excess temperature. When the discontinuity decays, a flow is generated that exhibits shock flow moving to the right and the expansion wave moving to the left.

In Fig. 3 we present the results of the numerical calculations of the flow of the vapor–water mixture generated to the time instance $t = 0.35$ s performed both with account for vapor generation ($\beta = 0.5; 1.0; \Delta \theta = 1$ K) and without it ($\beta = 0$). We carried out computations on a mesh consisting of 500 nodes. We see in the figures that the intense vaporization appears in the region of the liquid located to the left from the water–vapor contact boundary, where the temperature of the liquid exceeds the saturation temperature. Accounting for the phase transition leads to a significant increase in the vapor concentration in the unloading wave and to a slight increase in both velocity of the mixture motion and its pressure. The concentration of the vapor fraction behind the front of the shock jump decreases. We note that the tempera-
ture of the liquid fraction $\theta$, is practically constant in the considered time interval, because we ignore the interfractional heat exchange.

4. CONCLUSIONS

In this paper we presented the model of a boiling liquid based on the conservation laws of mass, momentum, and energy written for each component of the mixture with account for the phase transition from the liquid fraction to the vapor one and vice versa. We demonstrated the hyperbolicity of the model equations of a boiling liquid. From the characteristic analysis of the model equations, we derived the analytical expression for the velocity of small perturbations in a boiling liquid. We described the iteration method for integrating the model equations based on the nodal method of characteristics and solved the Riemann problem in a boiling liquid. We revealed a significant increase in the vapor concentration in the unloading wave, which considerably changes the flow dynamics of the mixture.

The proposed model may be applied to modeling the processes during pressurization leakage of technical setups with an overheated liquid and, in particular, to studying the later stage of high-speed interaction between liquid drops and a rigid surface.

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