HODGE THEORY AND REAL ORBITS IN FLAG VARIETIES

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ABSTRACT. Period domains, the classifying spaces for (pure, polarized) Hodge structures, and more generally Mumford–Tate domains, arise as open $G_\mathbb{R}$–orbits in flag varieties $G/P$ (homogeneous with respect to a complex, semisimple Lie group $G$). We investigate Hodge–theoretic aspects of the geometry and representation theory associated with flag varieties. More precisely, we prove that smooth Schubert variations of Hodge structure are necessarily homogeneous, relate the Griffiths–Yukawa coupling to the variety of lines on $G/P \hookrightarrow \mathbb{P}V$ (under a minimal homogeneous embedding), construct a large class of polarized (or “Hodge–theoretically accessible”) $G_\mathbb{R}$–orbits in $G/P$ and compute the boundary components of the polarizing nilpotent orbits, use these boundary components to define “enhanced $SL_2$–orbits”, and initiate a project to express the homology classes of these homogeneous submanifolds of $G/P$ in terms of Schubert classes.

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1. INTRODUCTION

This is the first in a planned series of papers that studies the relationship between the Hodge theory of Mumford–Tate domains $\mathcal{D} = G_\mathbb{R}/R$ and the projective geometry and $G_\mathbb{R}$–orbit structure of their compact duals $\mathcal{D} = G/P$. It continues lines of thought initiated in the first author’s work with Pearlstein [31, 32] on boundary components and in the second

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Background. Mumford–Tate domains are classifying spaces for Hodge structures on a \(\mathbb{Q}\)-vector space \(V_{\mathbb{Q}}\) with fixed Hodge tensors, including a polarizing form \(Q \in (V^*)^\otimes 2\). Recall how these arise: a (pure) Hodge structure polarized by \(Q\), is given by a nonconstant homomorphism
\[
\varphi : S^1 \to \text{Aut}(V_{\mathbb{R}}, Q)
\]
of real algebraic groups with \(Q(v, \varphi(\sqrt{-1})v) > 0\), for all \(v \in V_{\mathbb{C}} \setminus \{0\}\). Let \(G_{\mathbb{Q}}\) denote the smallest \(\mathbb{Q}\)-algebraic subgroup of \(\text{Aut}(V_{\mathbb{Q}}, Q)\) with \(G_{\mathbb{R}} \supseteq \varphi(S^1)\). This is the Mumford–Tate group of \(\varphi\). The real form \(G_{\mathbb{R}}\) is known to be reductive [20]. This group acts on \(\varphi\) by conjugation, and the orbit
\[
D \overset{\text{dfn}}{=} G_{\mathbb{R}} \cdot \varphi \cong G_{\mathbb{R}}/R
\]
is a Mumford–Tate domain. The stabilizer \(R\) is compact, and its Lie algebra \(\mathfrak{r}\) contains a compact Cartan subalgebra \(\mathfrak{t}\) of \(\mathfrak{g}_{\mathbb{R}}\) [24]. The Mumford–Tate domain is an analytic open subset of its compact dual, the projective homogeneous variety
\[
\bar{D} \overset{\text{dfn}}{=} G \cdot F_{\varphi}^* \cong G/P.
\]
Here, \(G\) is the complexification of \(G_{\mathbb{R}}\), and the action of \(G\) (and \(G_{\mathbb{R}}\)) is by left translation on the associated Hodge flag \(F_{\varphi}^* \in \bar{D}\).

One way to construct Mumford–Tate domains is to begin with a simple adjoint group \(G_{\mathbb{Q}}\), for which \(G_{\mathbb{R}}\) contains a compact maximal torus \(T\). If \(E \in i\mathfrak{t} = i\text{Lie}(T)\) is a grading element which is odd (respectively, even) on the noncompact (respectively, compact) roots, then \(\varphi(z) = e^{2\log(z)E}\) defines a weight zero Hodge structure (HS) on \(\mathfrak{g}_{\mathbb{Q}} = \text{Lie}(G_{\mathbb{Q}})\) that is polarized by \(-\langle \cdot, \cdot \rangle\), where \(\langle \cdot, \cdot \rangle\) is the Killing form on \(\mathfrak{g}_{\mathbb{Q}}\). Moreover, \(\varphi\) may lift to a HS on a representation \(V_{\mathbb{Q}}\). In either case, a sufficiently general \(G_{\mathbb{R}}\)-conjugate of this HS has Mumford–Tate group \(G_{\mathbb{Q}}\). In case \(\varphi\) lifts, the Hodge decomposition \(V_C = \oplus V^{p,q}\) is the \(E\)-eigenspace decomposition; the grading element acts on \(V^{p,q}\) by the scalar \((p-q)/2\). Then \(\varphi\) is identified with the Hodge flag \(F^p = \oplus_{a \geq p} V^{a,b}\), the compact dual \(\bar{D} = G/P\) is identified with the \(G\)-flag variety containing \(F^*\), and the Mumford–Tate domain \(D = G_{\mathbb{R}}/R\) with the open \(G_{\mathbb{R}}\)-orbit of \(F^*\) in \(\bar{D}\). The Lie algebra of \(P\) is \(\mathfrak{p} = \mathfrak{g}^{\geq 0} = \oplus_{p \geq 0} \mathfrak{g}^p\), where \(\mathfrak{g}^p = \mathfrak{g}^{p-p}\). The Lie algebra of \(R\) is a real form of \(\mathfrak{g}^0\); we shall write
\[
G^0 \overset{\text{dfn}}{=} R_{\mathbb{C}}.
\]
For the purposes of this introduction, we shall assume that \(D\) has a base point \(o = \varphi\) defined over \(\mathbb{Q}\) (cf. Definition (6.17)).

Running Example 1. A case in point, which we shall use as a running example throughout this introduction, is the \(\mathbb{Q}\)-form of the exceptional, rank two group \(G_2\) constructed in
Section 6.3. The grading element \( S^2 \in \mathfrak{t}_Q(\mathfrak{m}) \) (dual to the second simple root) induces (i) a weight zero, \((-\cdot, \cdot)\) polarized Hodge decomposition of the complex Lie algebra

\[
\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2, \quad \mathfrak{g}^{-p} = \mathfrak{g}^{-p, p},
\]

with Hodge numbers \((1,4,4,4,1)\); and (ii) a weight two polarized Hodge decomposition

\[
V = V^{0,2} \oplus V^{1,1} \oplus V^{2,0}
\]
on the standard, 7–dimensional representation with Hodge numbers \((2,3,2)\). The compact dual is the 5–dimensional adjoint variety in \( \mathbb{P}\mathfrak{g} \).

Recall that a variation of Hodge structure \( \mathcal{V} \) over a complex manifold \( S \) (in the classical sense) consists of a \( \mathbb{Q}\)-split local system \( \mathcal{V} \), and a filtration of \( \mathcal{V} \otimes \mathcal{O}_S \) by holomorphic sub-bundles \( F^* \) whose fibre-wise restrictions are Hodge flags, and which satisfy the infinitesimal period relation \( \nabla(F^*) \subset F^{*-1} \otimes \Omega^1_S \) for the flat connection \( \nabla \) on \( V \) annihilating \( \mathcal{V} \). The asymptotics of \( \mathcal{V} \) over a punctured disk \( \Delta^* \subset S \) are captured by the (limiting) mixed Hodge structure \( (V^0_\text{lim}, F^*_\text{lim}, W_\ast) \) \( \text{dfn} \) \( \lim \nabla|_{\Delta^*} \) and \( \bar{V} = \exp(-\log(s)/2\pi i)\mathcal{V} \).

If \( G_Q \) contains the Mumford–Tate groups of the fibres of \( \mathcal{V} \), the possible limiting MHS are classified by the boundary components \( \tilde{B}(N) \) of \([31]\). Moreover, \( \mathcal{V} \) induces a period mapping \( \Phi : S \to \Gamma \setminus \tilde{D} \), with \( \Gamma \) the monodromy group. We may lift \( \Phi|_{\Delta^*} \) to the upper–half plane \( \tilde{\mathcal{H}} : \mathcal{H} \to \tilde{D} \), and take the naive limit \( \lim_{\text{naive limit}} \Phi(\tau) \to \infty \) \( \Phi(\tau) \in \tilde{D} \), which lies in the analytic boundary \( \text{bd}(D) \subset \tilde{D} \). This boundary breaks into finitely many \( G_R \)–orbits, and those accessible by such Hodge–theoretic limits were termed polarizable in \([32]\).

**Schubert varieties and variations of Hodge structure.** In this paper, the term variation of Hodge structure (VHS) is used primarily in a different sense. The infinitesimal period relation (IPR) forces the differential of a period map \( \Phi \) to lie in the horizontal distribution \( T^1 \subset TD \). A subvariety (or submanifold) \( Y \subset D \) is horizontal if its tangent space \( T_y Y \subset T^1_y \) is contained in the horizontal distribution at every smooth point \( y \in Y \). For us a VHS shall simply be a horizontal subvariety/submanifold of \( D \). In particular, we will call a horizontal Schubert variety \( X \subset D \) a Schubert VHS. The Schubert VHS encode a great deal of information about the IPR and period mappings, cf. \([45]\). For example, there exists a VHS of dimension \( d \) if and only if there exists a Schubert VHS of dimension \( d \). In particular, the rank of the differential \( d\Phi \) of a period mapping is bounded above by \( \max\{\dim X \mid X \subset D \text{ is a Schubert VHS}\} \).

**Running Example 2.** The horizontal distribution is a contact 4–plane distribution. Modulo the action of \( G_2 \), there is a unique Schubert variety \( X_d \subset D \) of dimension \( d \), for \( 0 \leq d \leq 5 \). The Schubert variety is a VHS if and only if \( d \leq 2 \). See \([45]\) for additional detail.

One of the motivating questions behind this work is: when can a Schubert VHS be (up to translation) the compact dual of a Mumford–Tate subdomain of \( \tilde{D} \)? Any such Schubert variety is necessarily homogeneous and therefore smooth. The majority of Schubert varieties

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\(^{(1)}\)This notion of a “polarized” orbit is distinct from J. Wolf’s in \([47, Definition 9.1]\) where the orbit \( G_R \cdot x \subset \tilde{D} \) is polarized if it realizes the minimal homogeneous CR–structure on the homogeneous manifold \( G_R/\text{Stab}_{G_R}(x) \), cf. \([1, Remark 5.5]\).
are singular. Moreover, a smooth Schubert variety $X \subset \mathcal{D}$ need not be homogeneous. Nevertheless, we show that a smooth Schubert VHS is necessarily a homogeneous embedded Hermitian symmetric space (Theorem 4.4); moreover, with an assumption on the “$Q$-types” of $G_Q$ and $o \in D$, these are all Mumford–Tate subdomains (Corollary 4.5). These results invite comparison with a similar (but quite distinct) theorem of Friedman and Laza [22]; see the remarks after Theorem 4.4. The proof of Theorem 4.4 makes use of a recent result of J. Hong and N. Mok [27] which asserts that the smooth Schubert varieties in $G/P$, with $P$ a maximal parabolic subgroup associated with a non–short simple root, are necessarily homogeneous. The argument underlying this beautiful result is in turn based J.-M. Hwang and Mok’s program to study projective varieties by considering the variety of minimal degree rational curves passing through a point [29].

Running Example 3. The 2–dimensional Schubert VHS $X_2$ cannot be Mumford–Tate. This can be seen at once (and was first observed by the authors) by looking at its tangent space, which cannot be that of a reductive group orbit in $\mathcal{D}$. In fact, it turns out that $X_2$ is singular, and therefore not homogeneous. (The 1–dimensional Schubert VHS $X_1$ is a $\mathbb{P}^1$, and therefore homogeneous.)

Lines on $\mathcal{D}$ and the Griffiths–Yukawa coupling. When $\mathcal{D}$ is the minimal homogeneous embedding $G/P \hookrightarrow \mathcal{P} V$, the minimal degree rational curves on $\mathcal{D}$ are lines $\mathbb{P}^1 \subset \mathcal{P} V$ (degree one). Let $\mathcal{C}_o \subset \mathcal{T}_o \mathcal{D}$ be the variety of tangent directions to lines passing though $o \in \mathcal{D}$. (We will often think of $\mathcal{C}_o$ as the set of lines $o \in \mathbb{P}^1 \subset \mathcal{D}$.) Of particular interest here is the set $\mathcal{C}_o \overset{\text{dfn}}{=} \mathcal{C}_o \cap \mathcal{P} T^1$ of directions horizontal with respect to the IPR: If $P$ is a maximal parabolic, then the variety

$$X \overset{\text{dfn}}{=} \bigcup_{\mathbb{P}^1 \in \mathcal{C}_o} \mathbb{P}^1$$

swept out by these lines (which is a cone over $\mathcal{C}_o$ with vertex $o$) is a (typically singular) Schubert variety. When $P$ is a maximal parabolic associated to a non-short root, we have $\mathcal{C}_o = \tilde{\mathcal{C}}_o$ and $X$ is horizontal, cf. [38, Proposition 2.11] and Lemma 3.3. When $\mathcal{D}$ is an adjoint variety ($V = g$), $\mathcal{C}_o$ realizes the homogeneous Legendrian varieties studied by J.M. Landsberg and L. Manivel [36, 37].

Running Example 4. In our $G_2$ example, $X = X_2$ is the famous twisted cubic cone known to É. Cartan, cf. Section 5.

In this paper we will discuss two Hodge–theoretic characterizations of $\mathcal{C}_o$. The first is in terms of the Griffiths–Yukawa coupling, a differential invariant of VHS’s. (The second is quite distinct, and will be discussed below.) When the kernel of the Griffiths–Yukawa coupling is nonempty, it necessarily contains $\mathcal{C}_o$ (Theorem 5.8). Moreover, for certain compact duals (including the $A_n$, $E_6$, $E_7$, $E_8$, $F_4$, and $G_2$ adjoint varieties, cf. Lemma 5.13 and Section 5.3) equality holds ($\mathcal{C}_o$ is the kernel of the Griffiths–Yukawa coupling) for some Hodge representation of weight two (Theorem 5.11).
Polarized orbits. The polarized orbits, and the corresponding limiting mixed Hodge structures, describe the asymptotics of a variation of Hodge structure on a Mumford–Tate domain. More generally, one may take the naïve limit of nilpotent orbit $\exp(zN)F^\bullet$ on $\mathring{D}$.

In this paper our emphasis is on representation theoretic descriptions of polarized orbits and mixed Hodge structures associated with nilpotent orbits, with the motivation that such descriptions may guide the construction of geometric/motivic examples.\(^{(2)}\)

Running Example 5. The boundary $\text{bd}(D)$ contains a unique real-codimension one $G_\mathbb{R}$–orbit, which is polarizable. The weight-graded pieces of the corresponding limiting mixed Hodge structures have $\dim Gr^W_{\text{lim}} V_3 = 2$ and $\dim Gr^W_{\text{lim}} V_2 = 3$ (with Hodge numbers $(1,1,1)$). Additional detail on this example may be found in \cite[Section 6.1.3]{32}.

Section 6 gives a systematic construction of polarizable boundary strata in $\text{bd}(D) \subset \mathring{D}$ via distinguished \textquotedblleft Q–Matsuki\textquotedblright\(^{(3)}\) points on them and sets $B = \{\beta_1, \ldots, \beta_s\} \subset h^*$ of strongly orthogonal roots\(^{(4)}\), culminating in Theorem 6.38.

Remark. In the case that $D$ is Hermitian symmetric, all the boundary orbits $O \subset \text{bd}(D)$ are polarizable. From \cite[Theorem 3.2.1]{21} it may be seen that they are all parameterized by the construction of Section 6, and the parameterization is essentially that given by the Harish–Chandra compactification of $D$.

While notationally dense, the construction is very natural, and we will briefly summarize it here. With each root $\beta_j \in B$ is associated a Cayley transform $c_{\beta_j} \in G$, and we let $c_B$ denote the composite/product. From the Q–Matsuki point $o$ and $c_B$ we obtain a Q–Matsuki point $o_B \in \mathring{D}$, with corresponding Hodge flag $F^\bullet_B$, a \textit{“weight filtration”} $W^\bullet_B$, and nilpotent cone

$$\sigma_B \overset{\text{dfn}}{=} \{t^1N^B_1 + \cdots + t^sN^B_s \mid t^i \geq 0\}$$

with $N^B_j \in g_\mathbb{R}$ and $N^B_jF^p_B \subset F^{p-1}_B$, for all $p$. These objects have the properties that the nilpotent cone $\sigma_B$ underlies a multiple variable nilpotent orbit

$$(z^1, \ldots, z^s) \mapsto \exp(z^1N^B_1 + \cdots + z^sN^B_s) \cdot F^\bullet_B,$$

$W^B_\bullet = W_\bullet(\sigma_B)$, and $(F^\bullet_B, W^B_\bullet)$ is a (Q–split) Q–MHS on $g$. In particular, the MHS $(F^\bullet_B, W^B_\bullet)$ belongs to the boundary component $\mathring{B}(\sigma_B)$, and the orbit

$$O_B \overset{\text{dfn}}{=} G_\mathbb{R} \cdot o_B$$

is polarizable.\(^{(5)}\)

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\(^{(2)}\)See \cite{26} for more on this line of thought.

\(^{(3)}\)The reason for this terminology is that the Cartan subalgebra $\mathfrak{h} \subset g$ corresponding to $o$ is assumed Q-rational (hence stable under complex conjugation) and stable under the Cartan involution; hence so are its Cayley transforms.

\(^{(4)}\)The set $B$ is constrained by an additional condition imposed by the IPR.

\(^{(5)}\)The notation of this introduction differs slightly from that of Section 6, where the terms $c_B$, $o_B$, $F^\bullet_B$, $W^\bullet_B$, $\sigma_B$, $O_B$, $g_\mathbb{R}^{p,q}$ are denoted $c_\ast$, $o_\ast$, $sF^\bullet_\ast$, $sW_\ast$, $\sigma_\ast$, $O_\ast$, $s\mathfrak{g}^{p,q}$, respectively.
Boundary components. The construction also yields a semisimple element \( Y_B \in g_{\mathbb{R}} \) which grades the weight filtration. The Mumford–Tate group of \((F_B^*, W_B^*)\) is contained in

\[ G_B \overset{\text{dfn}}{=} \text{Stab}_G \{ Y_B \} \cap \text{Stab}_G \{ N_1^B \} \cap \cdots \cap \text{Stab}_G \{ N_s^B \}. \]

Defining\(^{(6)}\)

\[ D_B \overset{\text{dfn}}{=} G_{B, \mathbb{R}} \cdot F_B^* \subset G_B \cdot F_B^* \overset{\text{dfn}}{=} \tilde{D}_B, \]

the set

\[ B(\sigma_B) \overset{\text{dfn}}{=} e^C \sigma_B \setminus \tilde{B}(\sigma_B) \]

of nilpotent orbits fibres naturally over \( D_B \). Passing to the quotient by the intersection \( \Gamma \) of an arithmetic subgroup of \( G_{\mathbb{Q}} \) with \( \text{Stab}(\sigma_B) \), the surjection \( \Gamma \setminus B(\sigma_B) \rightarrow \Gamma \setminus D_B \) has the structure of an iterated fibration by generalized intermediate Jacobians.

Part of the value of this construction is that it allows us to compute the ranks of the components \( g_{\mathbb{R}}^{p, \delta} \) in the Deligne splitting \( g_{\mathbb{C}} = \bigoplus g_{\mathbb{R}}^{p, \delta} \) and the structure of the \( \tilde{B}(\sigma_B) \). For example, when \( P \) is a maximal parabolic there is exactly one orbit \( O_1 \subset \text{bd}(D) \) of real codimension one (Proposition 6.56); it is of the form \( O_B \) with \( B = \{ \beta_1 \} \), and the corresponding boundary components \( B(\sigma_B) = B(N) \) are completely worked out in Appendix A.

**Adjoint varieties and \( C_o \simeq \tilde{D}_B \).** For the fundamental adjoint varieties, the Deligne bigrading takes a very intriguing form (Figure 7.1): an element of the Weyl group of \((g, h)\) exchanges the Hodge and weight gradings, and the homogeneous Legendrian varieties \( C_o \) above reappear in the guise of \( \tilde{D}_B \). Meanwhile, by Proposition 7.15, the \( \Gamma \setminus B(\sigma_B) \) realize the intermediate Jacobian families associated to the Friedman–Laza \([22]\) weight-three maximal Hermitian VHS’s of Calabi-Yau type!

**Running Example 6.** \( \Gamma \setminus B(N) \) is just an elliptic modular surface. (The Friedman–Laza variation is the symmetric cube of a VHS of type \((1, 1)\).)

**Enhanced horizontal \( \text{SL}_2 \)-orbits and homology classes.** Another motivation for the construction of Section 6 goes back to a basic question of Borel and Haefliger \([8]\) on smoothability of cohomology classes in \( G/P \). By \([45\), Theorem 4.1], the invariant characteristic cohomology of \( \tilde{D} \) is generated by classes of Schubert VHS. Moreover, the homology class of any horizontal cycle in \( \tilde{D} \) may be expressed as a linear combination of Schubert VHS classes \([43\), Theorem 4.7\. Applying a VHS twist to Borel and Haefliger’s question, we ask: when can the class (or a multiple thereof) of a singular Schubert VHS, such as \((1, 1)\), admit a smooth, horizontal, algebraic representative? One natural source of smooth subvarieties (in fact, Mumford–Tate subdomains) which are often horizontal, are the “enhanced multivariable \( \text{SL}_2 \)-orbits” \( X(\sigma_B) \) which arise as follows. The strongly orthogonal roots \( B \) above determine a collection \( \{ \mathfrak{s}_{2}^{\beta_j} \mid \beta_j \in B \} \) of commuting \( \mathfrak{s}_2 \)'s. Letting \( \text{SL}^B_2(\mathbb{C}) \subset G_{\mathbb{C}} \) be the connected Lie subgroup with Lie algebra \( \mathfrak{s}_{2}^{\beta_j}(\mathbb{C}) = \mathfrak{s}_{2}^{\beta_1}(\mathbb{C}) \times \cdots \times \mathfrak{s}_{2}^{\beta_s}(\mathbb{C}) \), the enhanced \( \text{SL}_2 \)-orbit is

\[ X(\sigma_B) \overset{\text{dfn}}{=} \text{SL}^B_2(\mathbb{C}) \cdot G_{B, \mathbb{C}} \cdot o_B \subset \tilde{D}, \]

\(^{(6)}\)Here one should really think of \( G_B \) as acting on the associated graded of the MHS \((F_B^*, W_B^*)\). Since this is a direct sum of Hodge structures, \( D_B \) is a Mumford–Tate domain in the usual sense.
cf. Section 7.1. In the case that $\tilde{D}$ is a fundamental adjoint variety, the enhanced $\text{SL}_2$-orbit attached to $O_1$ is a VHS and a cylinder on $C_o$; that is,

$$X(\sigma_B) = X(N) \cong C_o \times \mathbb{P}^1.$$ 

Recalling that $X$ is a cone over $C_o$, cf. (1.1), we conclude this article by testing the irresistible hypothesis that $[X(N)] = [X]$.

**Running Example 7.** We will show that $[X(N)] = 2[X]$ in Section 7.6.

The case that $G = \text{SO}(2r + 1, \mathbb{C})$ is worked out in Appendix B.

**Future work.** Of the many directions one could pursue from here, among the more salient is the systematic computation of the cohomology classes of smooth VHS (and the horizontal $X(\sigma_B)$ in particular). We will undertake this in a sequel (by a different method than those of Section 7.6 and Appendix B). It is also important to generalize the construction of Section 6 in order to parameterize all polarizable orbits; this work will appear in [44], and we expect to use it to tie together combinatorial structures related to nilpotent cones and boundary orbits.

Finally, one reason for classifying Hodge–theoretic phenomena is to predict algebro-geometric ones. Turning one last time to the running example, suppose we have a 2-parameter family of algebraic surfaces $\pi : X' \to S$ for which (a subquotient of) $R^2\pi_*\mathbb{Q}$ underlies a maximal VHS with Hodge numbers $(2, 3, 2)$ and Mumford–Tate group $G_2$. Then one can ask what sort of degeneration produces a point in the “elliptic modular surface” boundary component $\Gamma \backslash B(N)$, and what sort of geometry produces a subvariation whose period map image lies in the “cubic cone” Griffiths–Yukawa kernel. Now since one expects such 2-parameter VHS to arise from certain elliptic surfaces (with internal fibration over an elliptic curve, cf. [30]), neither question looks too difficult. With some optimism, one might imagine asking the analogous questions for all the fundamental adjoint varieties.

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2. Preliminaries

In this section we set notation and review the necessary background material from Hodge theory and representation theory.

2.1. Flag varieties. Let $G$ be a connected, complex semisimple Lie group, and let $P \subset G$ be a parabolic subgroup. The homogeneous manifold $G/P$ admits the structure of a rational homogeneous variety as follows. Fix a choice of Cartan and Borel subgroups $H \subset B \subset P$. Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$ be the associated Lie algebras. The choice of Cartan determines a set of roots $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^\ast$. Given a root $\alpha \in \Delta$, let $\mathfrak{g}^\alpha \subset \mathfrak{g}$ denote the root space. Given a subspace $\mathfrak{s} \subset \mathfrak{g}$, let

$$\Delta(\mathfrak{s}) \overset{\text{df}}{=} \{ \alpha \in \Delta \mid \mathfrak{g}^\alpha \subset \mathfrak{s} \}.$$
The choice of Borel determines positive roots $\Delta^+ = \Delta(b) = \{\alpha \in \Delta \mid g^\alpha \subset b\}$. Let $S = \{\alpha_1, \ldots, \alpha_r\}$ denote the simple roots, and set

$$(2.1) \quad I = I(p) \overset{\text{dfn}}{=} \{i \mid g^{-\alpha_i} \not\subset p\}.$$ 

Note that $I(b) = \{1, \ldots, r\}$, and $I = \{i\}$ consists of single element if and only if $p$ is a maximal parabolic.

Let $\{\omega_1, \ldots, \omega_r\}$ denote the fundamental weights of $g$ and let $V$ be the irreducible $g$-representation of highest weight

$$(2.2) \quad \mu = \mu_I \overset{\text{dfn}}{=} \sum_{i \in I} \omega_i.$$ 

Assume that the representation $g \to \text{End}(V)$ ‘integrates’ to a representation $G \to \text{Aut}(V)$ of $G$. (This is always the case if $G$ is simply connected.) Let $o \in PV$ be the highest weight line in $V$. The $G$-orbit $G \cdot o \subset PV$ is the minimal homogeneous embedding of $G/P$ as a rational homogeneous variety.

Remark 2.3 (Non-minimal embeddings). More generally, suppose that $V$ is the irreducible $G$-representation of highest weight $\tilde{\mu} = \sum_{i \in I} a_i^i \omega_i$ with $0 < a_i \in \mathbb{Z}$. Again, the $G$-orbit of the highest weight line $o \in PV$ is a homogeneous embedding of $G/P$. We write $G/P \hookrightarrow PV$. The embedding is minimal if and only if $a_i = 1$ for all $i \in I$. For example, the Veronese re-embedding $v_d(P^n) \subset \mathbb{P} \text{Sym}^d \mathbb{C}^{n+1}$ of $P^n$ is if minimal if and only if $d = 1$. (Here $V = \text{Sym}^d \mathbb{C}^{n+1}$ has highest weight $d\omega_1$.)

The rational homogeneous variety $G/P$ is sometimes indicated by circling the nodes of the Dynkin diagram (Appendix C) corresponding to the index set $I(p)$.

2.2. Flag domains and Mumford–Tate domains. Let $G_{\mathbb{R}}$ be a (connected) real form of $G$. There are only finitely many $G_{\mathbb{R}}$-orbits on $G/P$. An open $G_{\mathbb{R}}$-orbit

$$D = G_{\mathbb{R}}/R$$

is a flag domain. The stabilizer $R \subset G_{\mathbb{R}}$ is the centralizer of a torus $T' \subset G_{\mathbb{R}}$ [21, Corollary 2.2.3]. The flag variety

$$\tilde{D} \overset{\text{dfn}}{=} G/P$$

is the compact dual of the flag domain.

We will be interested in the case that $D$ admits the structure of a Mumford–Tate domain. Mumford–Tate domains are generalizations of period domains, and as such arise as the classification spaces of polarized Hodge structures (possibly with additional structure); see [24] for a thorough treatment. When $D$ admits the structure of a Mumford–Tate domain the stabilizer $R$ is compact, and as a consequence there exists a compact maximal torus $T \subset G_{\mathbb{R}}$ such that $T' \subset T \subset R$ [24]. We will assume this to be the case throughout.

In the Mumford–Tate domain case, $G$ also has an underlying $\mathbb{Q}$-algebraic group $G_{\mathbb{Q}}$, whose groups of real and complex points recover $G_{\mathbb{R}}$ and $G$, respectively. We will assume this only where relevant. In a few places (mostly limited to Section 6), we shall make the stronger assumption that $G_{\mathbb{Q}}$ is a Mumford–Tate–Chevalley group (Definition 6.9).
2.3. **Grading elements.** Let \( \{ S^1, \ldots, S^r \} \) be the basis of \( \mathfrak{h} \) dual to the simple roots. A *grading element* is any member of \( \text{span}_Z \{ S^1, \ldots, S^r \} \); these are precisely the elements \( T \in \mathfrak{h} \) of the Cartan subalgebra with the property that \( \alpha(T) \in Z \) for all roots \( \alpha \in \Delta \). Since \( T \) is semisimple (as an element of \( \mathfrak{h} \)), the Lie algebra \( \mathfrak{g} \) admits a direct sum decomposition

\[
\mathfrak{g} = \bigoplus_{\ell \in Z} \mathfrak{g}^\ell
\]

into \( T \)-eigenspaces

\[
\mathfrak{g}^\ell \overset{\text{dfn}}{=} \{ \xi \in \mathfrak{g} \mid [T, \xi] = \ell \xi \}.
\]

In terms of root spaces, we have

\[
\mathfrak{g}^\ell = \bigoplus_{\alpha(T) = \ell} \mathfrak{g}^\alpha, \quad \ell \neq 0,
\]

\[
\mathfrak{g}^0 = \mathfrak{h} \oplus \bigoplus_{\alpha(T) = 0} \mathfrak{g}^\alpha.
\]

The \( T \)-eigenspace decomposition is a *graded Lie algebra decomposition* in the sense that

\[
[\mathfrak{g}^\ell, \mathfrak{g}^m] \subset \mathfrak{g}^{\ell+m},
\]

a straightforward consequence of the Jacobi identity. It follows that \( \mathfrak{g}^0 \) is a Lie subalgebra of \( \mathfrak{g} \) (in fact, reductive), and each \( \mathfrak{g}^\ell \) is a \( \mathfrak{g}^0 \)-module. The Lie algebra

\[
\mathfrak{p} = \mathfrak{p}_T = \mathfrak{g}^0 \oplus \mathfrak{g}^+,
\]

is the *parabolic subalgebra determined by the grading element* \( T \). See [45, Section 2.2] for details.

2.3.1. **Minimal grading elements.** Two distinct grading elements may determine the same parabolic \( \mathfrak{p} \). As a trivial example of this, given a grading element \( T \) and \( 0 < d \in Z \), both \( T \) and \( dT \) determine the same parabolic. Among those grading elements \( T \) determining the same parabolic (2.6), only one will have the property that \( \mathfrak{g}^\pm + \mathfrak{g}_\mathfrak{g}_0 \) generates \( \mathfrak{g}^\pm \) as an algebra. That canonical grading element is defined as follows. Given a parabolic \( \mathfrak{p} \) subalgebra (and choice of Cartan and Borel \( \mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p} \)), the *grading element associated to \( \mathfrak{p} \) is*

\[
E \overset{\text{dfn}}{=} \sum_{i \in I(\mathfrak{p})} S^i.
\]

The reductive subalgebra \( \mathfrak{g}^0 = \mathfrak{g}^0_\text{s} \oplus \mathfrak{j} \) has center \( \mathfrak{j} = \text{span}_C \{ S^i \mid i \in I \} \) and semisimple subalgebra \( \mathfrak{g}^0_\text{s} = [\mathfrak{g}^0, \mathfrak{g}^0] \). A set of simple roots for \( \mathfrak{g}^0_\text{s} \) is given by \( S(\mathfrak{g}_0) = \{ \alpha_j \mid j \notin I \} \).

2.3.2. **E–eigenspace decompositions.** Any \( \mathfrak{g} \)-representation \( U \) admits a \( E \)-eigenspace decomposition

\[
U = \bigoplus_{q \in Q} U^q,
\]

see [45, Section 2.2]. In the case that \( U = \mathfrak{g} \), we recover (2.4c). Again, the Jacobi identity implies

\[
\mathfrak{g}^\ell(U^q) \subset U^{q+\ell}.
\]
In particular, 

*each E-eigenspace* $U^q$ *is a* $\mathfrak{g}^0$-*module.*

If $V$ is the irreducible $\mathfrak{g}$–module of highest weight $\mu = \sum \mu^i \omega_i$, then the E-eigenspace decomposition is of the form $V = V^m \oplus V^{m-1} \oplus V^{m-2} \oplus \cdots$. Moreover, if $\mu^i = 0$ when $i \notin I$, then $V^m$ is the (one-dimensional) highest weight line, cf. [45, Lemma 6.3]. If $\mu^i > 0$ if and only if $i \in I$, then the $G$–orbit of $[V^m] \in \mathbb{P}V$ is a homogeneous embedding of $G/P$; the embedding is minimal if $\mu^i = 1$ for all $i \in I$.

### 2.4. Variations of Hodge structure.

This section is a very brief review of the infinitesimal period relation on $G/P$ and the solutions of this differential system, the variations of Hodge structure. The reader interested in greater detail is encouraged to consult [45] and the references therein.

The (holomorphic) tangent bundle of $\mathcal{D} = G/P$ is the $G$–homogeneous vector bundle

$$T\mathcal{D} = G \times_P (\mathfrak{g}/\mathfrak{p}).$$

By (2.5) and (2.6), the quotient $\mathfrak{g}^{\geq -1}/\mathfrak{p}$ is a $\mathfrak{p}$–module. Therefore,

$$T^1 \overset{df}{=} G \times_P (\mathfrak{g}^{\geq -1}/\mathfrak{p})$$

defines a homogeneous, holomorphic subbundle of $T\mathcal{D}$. This subbundle is the *infinitesimal period relation* (IPR). The IPR is *trivial* if $T^1 = T\mathcal{D}$. This is the case when $G/P$ is Hermitian symmetric.

For the purposes of this article, a *variation of Hodge structure* (VHS)

(7)

is a solution (or integral manifold) of the IPR; that is, a VHS is any connected complex submanifold $Z \subset G/P$ with the property that $T_z Z \subset T^1_z$ for all $z \in Z$, or any irreducible subvariety $Y \subset \mathcal{D}$ with the property that $T_y Y \subset T^1_y$ for all smooth points $y \in Y$. Alternatively, we shall sometimes refer to such subvarieties as *horizontal*.

### 2.5. Adjoint varieties.

Throughout the paper we will illustrate the ideas and results with adjoint varieties. From the Hodge–theoretic perspective, these are the simplest varieties for which the infinitesimal period relation is nontrivial: the adjoint varieties are precisely those $G/P$ for which the IPR (2.9) is a contact distribution.

Let $G \subset \text{Aut}(\mathfrak{g})$ be the adjoint group of a complex simple Lie algebra $\mathfrak{g}$, and let $\check{\alpha} \in \Delta^+$ be the *highest root*. See Table 2.1. Recall $\mathfrak{g}^{\check{\alpha}}$ is one-dimensional, and therefore a line in $\mathfrak{g}$; let $o \in \mathbb{P}\mathfrak{g}$ be the corresponding point. Then the *adjoint variety* $\mathcal{D}$ is the $G$–orbit of $o$.

Writing $\mathcal{D} = G/P$, if $T$ is the grading element associated to $P$ then Table 2.1 yields

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2,$$

where $\mathfrak{g}^2 = \mathfrak{g}^{\check{\alpha}}$.

**Remark 2.10.** The adjoint varieties are precisely the compact, simply connected, homogeneous complex contact manifolds [6].

---

(7) We shall have use for the more standard meaning of VHS (over an arbitrary complex manifold, and allowing monodromy) only in the beginning of Section 5, and in Proposition 7.15.
Table 2.1. The highest root $\tilde{\alpha}$ of $G$.

| $G$ | $\tilde{\alpha}$ |
|-----|------------------|
| $A_r$ | $\alpha_1 + \cdots + \alpha_r = \omega_1 + \omega_r$ |
| $B_r$ | $\alpha_1 + 2(\alpha_2 + \cdots + \alpha_r) = \omega_2$ |
| $C_r$ | $2(\alpha_1 + \cdots + \alpha_{r-1}) + \alpha_r = 2\omega_1$ |
| $D_r$ | $\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{r-2}) + \alpha_{r-1} + \alpha_r = \omega_2$ |
| $E_6$ | $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \omega_2$ |
| $E_7$ | $2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \omega_1$ |
| $E_8$ | $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 = \omega_8$ |
| $F_4$ | $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \omega_1$ |
| $G_2$ | $3\alpha_1 + 2\alpha_2 = \omega_2$ |

Example 2.11. (a) If $g = so(r+1,\mathbb{C})$, then the adjoint variety is $\text{Flag}(1, r + 1, \mathbb{C}r + 1)$ is the (partial) flag variety of lines in hyperplanes.

(b) If $g = sl(r, \mathbb{C})$, then the adjoint variety is the orthogonal grassmannian $OG(2, \mathbb{C}r)$ of 2–planes that are $\nu$–isotropic for a nondegenerate symmetric bilinear form $\nu$.

(c) If $g = sp_{2r}(\mathbb{C})$, then the adjoint variety is the second Veronese re-embedding $v_2(P^{2r-1}) \subset \mathbb{P}\text{Sym}^2\mathbb{C}^{2r}$.

Assume that $G \neq A_r, C_r$ so that the adjoint representation $g$ is fundamental. Equivalently, $\tilde{\alpha} = \omega_1$ (cf. Table 2.1), $g$ is the irreducible $g$–representation of highest weight $\omega_1$ and $P$ is maximal; we have $I(p) = \{i\}$ and

$$\tilde{\alpha} - \alpha_j$$

is a root if and only if $j = i$.

Table 2.2. The dimensions $n = \dim \bar{D}$ and $N = \dim \mathbb{P}g$, and degree $d = \deg \bar{D}$ for the exceptional adjoint varieties.

| $G$ | $n$ | $d$ | $N$ |
|-----|-----|-----|-----|
| $E_6$ | 21  | 151,164 | 77 |
| $E_7$ | 33  | 141,430,680 | 132 |
| $E_8$ | 57  | 126,937,516,885,200 | 247 |
| $F_4$ | 15  | 4,992 | 51 |
| $G_2$ | 5   | 18  | 13 |

2.6. Schubert varieties. In the case that $G$ is classical (one of $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$ or $Sp_{2n}(\mathbb{C})$), the Schubert varieties of a flag variety $G/P$ may be described geometrically as degeneracy loci, cf. Example 2.17. However, these descriptions, aside from not generalizing easily to the exceptional groups, are type dependent. We will utilize a representation theoretic description of the Schubert varieties that allows a uniform treatment across all flag varieties. This section does little more than establish notation for our discussion of Schubert varieties. The reader interested in greater detail is encouraged to consult [45] and the references therein.
Given a simple root $\alpha_i \in S$, let $(i) \in \text{Aut}(h^\ast)$ denote the corresponding simple reflection. The Weyl group $W \subset \text{Aut}(h^\ast)$ of $g$ is the group generated by the simple reflections $\{(i) \mid \alpha_i \in S\}$. A composition of simple reflections $(i_1) \circ (i_2) \circ \cdots \circ (i_t)$, which are understood to act on the left, is written $(i_1i_2 \cdots i_t) \in W$. The length of a Weyl group element $w$ is the minimal number
\[
|w| \triangleq \min \{ \ell \mid w = (i_1i_2 \cdots i_\ell) \}
\]
of simple reflections necessary to represent $w$.

Let $W_p \subset W$ be the subgroup generated by the simple reflections $\{(i) \mid i \notin I\}$. Then $W_p$ is naturally identified with the Weyl group of $g^0_{ss}$. The rational homogeneous variety $G/P$ decomposes into a finite number of $B$–orbits
\[
G/P = \bigcup_{W_p w \in W_p \backslash W} Bw^{-1}o
\]
which are indexed by the right cosets $W_p \backslash W$. The $B$–Schubert varieties of $G/P$ are the Zariski closures
\[
X_w \triangleq Bw^{-1}o.
\]

**Remark 2.13.** Observe that the stabilizer $P_w$ of $X_w$ in $G$ contains $B$, and is therefore a parabolic subgroup of $G$.

More generally, any $G$–translate $gX_w$ is a Schubert variety (of type $W_p w$). Define a partial order on $W_p \backslash W$ by defining $W_p w \leq W_p v$ if $X_w \subset X_v$; then $W_p \backslash W$ is the Hasse poset.

Each right coset $W_p \backslash W$ admits a unique representative of minimal length; let
\[
W_{\text{min}}^p \simeq W_p \backslash W
\]
be the set of minimal length representatives. Given $w \in W_{\text{min}}^p$, the Schubert variety $wX_w$ is the Zariski closure of $N_w \cdot o$, where $N_w \subset G$ is a unipotent subgroup with nilpotent Lie algebra
\[
(2.14) \quad n_w \triangleq \bigoplus_{\alpha \in \Delta(w)} g^{-\alpha} \subset g^-
\]
given by
\[
(2.15) \quad \Delta(w) \triangleq \Delta^+ \cap w(\Delta^-).
\]
Moreover, $N_w \cdot o$ is an affine cell isomorphic to $n_w$, and $\dim X_w = \dim n_w = |\Delta(w)|$. Indeed
\[
T_o(wX_w) = n_w.
\]
For any $w \in W_{\text{min}}^p$ we have
\[
(2.16) \quad |w| = |\Delta(w)| = \dim X_w.
\]
(The first equality holds for all $w \in W$, cf. [12, Proposition 3.2.14(3)].)

**Example 2.17 (Schubert varieties in $OG(2, C^m)$).** Let $\nu$ be a nondegenerate, symmetric bilinear form on $C^m$, and let
\[
OG(2, C^m) \triangleq \{ E \in \text{Gr}(2, C^m) \mid \nu|_E = 0 \}.
\]
be the orthogonal Grassmannian of \( \nu \)-isotropic 2-planes in \( \mathbb{C}^m \). Fix a basis \( \{e_1, \ldots, e_m\} \) of \( \mathbb{C}^m \) with the property that
\[
\nu(e_a, e_b) = \delta_{a+b}^{m+1}.
\]
Then
\[
\mathcal{F}_p \overset{\text{dfn}}{=} \text{span}_\mathbb{C}\{e_1, \ldots, e_p\} \subset \mathbb{C}^m
\]
defines a flag \( \mathcal{F}^\bullet \) with the property that
\[
(2.18) \quad \nu(\mathcal{F}_p, \mathcal{F}_{m-p}) = 0.
\]
Any flag \( \mathcal{F}^\bullet \) satisfying (2.18) is called \( \nu \)-isotropic.

Given \( 1 \leq a < b \leq m \) with \( a + b \neq m + 1 \) and \( a, b \neq r + 1 \), define
\[
(2.19a) \quad X_{a,b}(\mathcal{F}^\bullet) \overset{\text{dfn}}{=} \{ E \in \text{OG}(2, \mathbb{C}^m) \mid \dim E \cap \mathcal{F}_a \geq 1, E \subset \mathcal{F}_b \}.
\]
If \( m = 2r + 1 \), then every Schubert variety of \( \text{OG}(2, \mathbb{C}^{2r+1}) \) is \( G \)-congruent to one of (2.19a).
If \( m = 2r \) is even, define
\[
\tilde{\mathcal{F}}^r = \text{span}_\mathbb{C}\{e_1, \ldots, e_{r-1}, e_{r+1}\}.
\]
Note that \( \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^{r-1} \subset \tilde{\mathcal{F}}^r \subset \mathcal{F}^{r+1} \subset \cdots \) is also a \( \nu \)-isotropic flag. Set
\[
(2.19b) \quad \tilde{X}_{a,r}(\mathcal{F}^\bullet) \overset{\text{dfn}}{=} \{ E \in \text{OG}(2, \mathbb{C}^m) \mid \dim E \cap \mathcal{F}_a \geq 1, E \subset \tilde{\mathcal{F}}^r \}, \quad a \leq r - 1,
\]
\[
(2.19c) \quad \tilde{X}_{r,b}(\mathcal{F}^\bullet) \overset{\text{dfn}}{=} \{ E \in \text{OG}(2, \mathbb{C}^m) \mid \dim E \cap \tilde{\mathcal{F}}^r \geq 1, E \subset \mathcal{F}_b \}, \quad b \geq r + 2.
\]
Every Schubert variety of \( \text{OG}(2, \mathbb{C}^{2r}) \) is \( G \)-congruent to one of (2.19).

2.7. Schubert VHS. The condition that the Schubert variety \( X_w \) be a VHS is equivalent to \( \Delta(w) \subset \Delta(g_1) \), where \( \Delta(w) \) is given by (2.15), cf. [45, Theorem 3.8]. A convenient way to test for this condition is as follows. Let
\[
\varrho \overset{\text{dfn}}{=} \sum_{i=1}^r \omega_i = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha
\]
be the sum of the fundamental weights (which is also half the sum of the positive roots). Define
\[
(2.20) \quad \varrho_w \overset{\text{dfn}}{=} \varrho - w(\varrho) = \sum_{\alpha \in \Delta(w)} \alpha.
\]
(See [35, (5.10.1)] for the second equality.) Then
\[
|w| \leq \varrho_w(E) \in \mathbb{Z},
\]
and equality holds if and only if \( \Delta(w) \subset \Delta(g_1) \); equivalently, \( X_w \) is a variation of Hodge structure if and only if \( \varrho_w(E) = |w| \). See [45, Section 3] for details. Let
\[
\mathcal{W}_\text{vhs} \overset{\text{dfn}}{=} \{ w \in \mathcal{W}_\text{min} \mid \varrho_w(E) = |w| \}
\]
be the set indexing the Schubert variations of Hodge structure.
Lemma 2.22. This is a continuation of the introductory Running Example. In particular, $G = G_2$ is the exceptional Lie group of rank two, $\bar{D} \subset \mathbb{P}g$ is the adjoint variety with $I(p) = \{2\}$ and $E = S^2$. Modulo the action of $G$, there is a single Schubert variety $X_{w,d} \subset \bar{D}$ of dimension $d$, $0 \leq d \leq 5$. The variety $X_{w,d}$ is horizontal (a VHS) if and only if $d \leq 2$. We have $w_1 = (2)$ with $\Delta(w_1) = \{\sigma_2\}$, and $w_2 = (12)$ with $\Delta(w_2) = \{\sigma_2, \sigma_1 + \sigma_2\}$. As will be discussed later, $X_{w_1} = \mathbb{P}^1$ is a line on $\bar{D}$, and therefore homogeneous, while $X_{w_2}$ is a singular cone over the Veronese re-embedding $v_3(\mathbb{P}^1) \subset \mathbb{P}^3$.

Lemma 2.22 and Corollary 2.23 are continuations of Example 2.17.

Lemma 2.22 (Schubert VHS in $OG(2, \mathbb{C}^m)$). Recall the Schubert varieties (2.19) of $OG(2, \mathbb{C}^m)$.

(a) The Schubert variety $X_{a,b}(\mathcal{F}^*)$ is horizontal if and only if $\mathcal{F}^a \subset (\mathcal{F}^b)^\perp$.
(b) The Schubert varieties $\tilde{X}_{a,r}(\mathcal{F}^*)$, are all horizontal.
(c) None of the Schubert varieties $\tilde{X}_{r,b}(\mathcal{F}^*)$ are horizontal.

Corollary 2.23 (Maximal Schubert VHS in $OG(2, \mathbb{C}^m)$). The maximal (with respect to containment) Schubert VHS in $SO(2, \mathbb{C}^m)$, where $m \in \{2r, 2r + 1\}$, are: the $X_{a,m-a}(\mathcal{F}^*)$, with $1 \leq a \leq r - 1$; and $\tilde{X}_{r-1,r}(\mathcal{F}^*)$, if $m = 2r$. Each of these maximal Schubert VHS is of dimension $m - 4$.

Proof of Lemma 2.22. By definition, a Schubert variety $X_w \subset OG(2, \mathbb{C}^m)$ is horizontal (equivalently, satisfies Griffith’s transversality condition) if and only if

$$dE|_{N_w \cdot 0} \subset E^\perp \overset{\text{dfn}}{=} \{v \in \mathbb{C}^m \mid \nu(E, v) = 0\}$$

on the Schubert cell $N_w \cdot a$.

(a) It is straight–forward to see that the condition $\mathcal{F}^a \subset (\mathcal{F}^b)^\perp$ implies $X_{a,b}(\mathcal{F}^*)$ is horizontal. Suppose that $\mathcal{F}^a \not\subset (\mathcal{F}^b)^\perp$; equivalently, $a + b > m$. Then the condition $a + b \neq m + 1$ (see Example 2.17) forces $a + b \geq m + 2$. It follows that

$$E(t) \overset{\text{dfn}}{=} \text{span}_{\mathbb{C}}\{e_b + te_{m+1-a}, e_a - te_{m+1-b}\} \subset X_{a,b}(\mathcal{F}^*) .$$

(In fact, $E(t)$ lies in the Schubert cell.) As (2.24) clearly fails for $E(t)$, we see that $X_{a,b}(\mathcal{F}^*)$ is not horizontal. This establishes assertion (a) of the lemma.

(b) As noted in Example 2.17, the Schubert varieties $\tilde{X}_{a,r}(\mathcal{F}^*)$ are defined for $a < r$. Since $\mathcal{F}^a \subset \mathcal{F}^r = (\mathcal{F}^r)^\perp$, it is immediate that the $\tilde{X}_{a,r}(\mathcal{F}^*)$ are horizontal.

(c) Recall that the Schubert varieties $\tilde{X}_{r,b}(\mathcal{F}^*)$ are defined for $r + 2 \leq b$, cf. Example 2.17. It follows that

$$E(t) \overset{\text{dfn}}{=} \text{span}_{\mathbb{C}}\{e_b + te_r, e_r + te_{m+1-b}\} \subset \tilde{X}_{r,b}(\mathcal{F}^*) .$$

(As above, $E(t)$ lies in the Schubert cell.) As (2.24) clearly fails for $E(t)$, we see that $\tilde{X}_{r,b}(\mathcal{F}^*)$ is not horizontal. This establishes assertion (c) of the lemma. □
2.8. Nilpotent orbits. Let \( \mathfrak{g}_R \subseteq \text{End}(V_\mathbb{R}, Q) \) be the Lie algebra of \( G_\mathbb{R} \). A \((n\text{-variable})\) nilpotent orbit on \( D \) consists of a tuple \((F^\bullet; N_1, \ldots, N_n)\) such that \( F^\bullet \in \bar{D} \), the \( N_i \in \mathfrak{g}_\mathbb{R} \) commute and \( N_i F^p \subset F^{p-1} \), and the holomorphic map \( \psi : \mathbb{C}^n \rightarrow \bar{D} \) defined by

\[
\psi(z_1, \ldots, z^n) = \exp(z_i N_i)F^\bullet
\]

has the property that \( \psi(z) \in D \) for \( \text{Im}(z_i) \gg 0 \). We shall use the term \( \sigma\text{-nilpotent orbit} \) to refer to the submanifold \( e^{\text{C}^\sigma} F^\bullet \subset \bar{D} \). The associated (open) nilpotent cone is

\[
\sigma = \{ t^i N_i \mid t^i > 0 \}.
\]

Given a nilpotent \( N \in \mathfrak{g}_\mathbb{R} \) such that \( N^{k+1} = 0 \), there exists a unique increasing filtration \( W_0(N) \subset W_1(N) \subset \cdots \subset W_{2k}(N) \) of \( V_\mathbb{R} \) with the properties that

\[
NW_\ell(N) \subset W_{\ell-2}(N)
\]

and the induced

\[
N^\ell : \text{Gr}_{k+\ell} W_\bullet(N) \rightarrow \text{Gr}_{k-\ell} W_\bullet(N)
\]

is an isomorphism for all \( \ell \leq k \). Above, \( \text{Gr}_m W_\bullet(N) = W_m(N)/W_{m-1}(N) \). Moreover,

\[
Q_\ell(u, v) = Q(u, N^\ell v)
\]

defines a nondegenerate \((-1)^{k+\ell}\)–symmetric bilinear form on \( \text{Gr}_{k+\ell} W_\bullet(N) \).

Define

\[
\text{Gr}_{k+\ell} W_\bullet(N)_{\text{prim}} = \ker \{ N^{\ell+1} : \text{Gr}_{k+\ell} W_\bullet(N) \rightarrow \text{Gr}_{k-\ell-2} W_\bullet(N) \},
\]

for all \( \ell \geq 0 \). A limiting mixed Hodge structure (or polarized mixed Hodge structure) on \( D \) is given by a pair \((F^\bullet, N)^{(8)}\) such that \( F^\bullet \in \bar{D} \), \( N \in \mathfrak{g}_\mathbb{R} \) and \( N(F^p) \subset F^{p-1} \), the filtration \( F^\bullet \) induces a weight \( m \) Hodge structure on \( \text{Gr}_m W_\bullet(N) \) for all \( m \), and the Hodge structure on \( \text{Gr}_{k+\ell} W_\bullet(N)_{\text{prim}} \) is polarized by \( Q_\ell \) for all \( \ell \geq 0 \). The notions of nilpotent orbit and limiting mixed Hodge structure are closely related. Indeed, they are equivalent when \( n = 1 \).

Theorem 2.27 (Cattani, Kaplan, Schmid). Let \( D \subset \bar{D} \) be a Mumford–Tate domain (and compact dual) for a Hodge representation of \( G_\mathbb{R} \).

(a) A pair \((F^\bullet; N)\) forms a one–variable nilpotent orbit if and only if it forms a limiting mixed Hodge structure, [15, Corollary 3.13] and [46, Theorem 6.16].

(b) Given an \( n\text{-variable} \) nilpotent orbit \((F^\bullet; N_1, \ldots, N_n)\), the weight filtration \( W_\bullet(N) \) does not depend on the choice of \( N \in \sigma \) [14, Theorem 3.3]. Let \( W_\bullet(\sigma) \) denote this common weight filtration.

The Deligne bigrading [15, 19]

\[
(2.28a) \quad V_\mathbb{C} = \bigoplus I^{p,q}
\]

associated with the limiting mixed Hodge structure is given by

\[
(2.28b) \quad I^{p,q} = F^p \cap W_{p+q} \cap \left( F^q \cap W_{p+q} + \sum_{j \geq 1} F^{q-j} \cap W_{p+q-j-1} \right).
\]

(8) Of course, the actual (real) mixed Hodge structure is the pair \((F^\bullet, W_\bullet(N))\). If \( V \) and \( \mathfrak{g} \) admit compatible \( \mathbb{Q}\)-rational structures, and \( N \in \mathfrak{g}_\mathbb{Q} \), then \((V_\mathbb{Q}, F^\bullet, W_\bullet(N)) \) is a \( \mathbb{Q}\)-mixed Hodge structure.
It is the unique bigrading of $V_C$ with the properties that
\[(2.29)\]
\[F^p = \bigoplus_{r \geq p} I^{r,*} \quad \text{and} \quad W_{\ell}(\sigma) = \bigoplus_{p+q \leq \ell} I^{p,q},\]
and
\[\overline{F^{p,q}} = I^{q,p} \mod \bigoplus_{r < q, s < p} I^{r,s}.\]

The (real) mixed Hodge structure $(F^\bullet, W_\bullet(\sigma))$ is $\mathbb{R}$-split, i.e. isomorphic to its associated graded, if $\overline{F^{p,q}} = I^{q,p}$.

3. Lines on flag varieties

Let $\hat{D}$ be the image of the minimal homogeneous embedding $G/P \hookrightarrow \mathbb{P}V$, cf. Section 2.1. Fix a highest weight vector $0 \neq v \in V$, so that $[v] = o \in \mathbb{P}V$ is the highest weight line. By (2.6), the tangent space $T_o\hat{D}$ is naturally identified with $\mathfrak{g}/\mathfrak{p}$ as a $\mathfrak{p}$–module, and with $\mathfrak{g}^-$ as a $\mathfrak{g}^0$–module; for the most part, we will work with the latter identification. The set of embedded, linear $\mathbb{P}^1 \subset \mathbb{P}V$ containing $o$ and tangent to $\hat{D}$ at that point is in bijection with $\mathbb{P}\mathfrak{g}^- = \mathbb{P}T_o\hat{D}$. To be precise, given a tangent line $[\xi] \in \mathbb{P}\mathfrak{g}^-$, we have
\[\mathbb{P}^1 = \mathbb{P}^1(o, [\xi]) \overset{\text{dfn}}{=} \mathbb{P}\text{span}_\mathbb{C}\{v, \xi(v)\} \subset \mathbb{P}V.\]

Making use of this identification, let
\[\tilde{C}_o \overset{\text{dfn}}{=} \{[\xi] \in \mathbb{P}\mathfrak{g}^- \mid \mathbb{P}^1(o, [\xi]) \subset \hat{D}\} = \{\mathbb{P}^1 \subset \mathbb{P}V \mid o \in \mathbb{P}^1 \subset \hat{D}\}\]
be the set of lines on $\hat{D}$ passing through $o$. (For a general embedding $G/P \hookrightarrow \mathbb{P}V$, not necessarily minimal, $\tilde{C}_o$ is defined to be the variety of minimal rational tangents, cf. [29].) The subvariety of lines tangent to $\mathfrak{g}^- \subset \mathfrak{g}^\sim \simeq T_o\hat{D}$ is
\[(3.1)\]
\[C_o \overset{\text{dfn}}{=} \{[\xi] \in \tilde{C}_o \mid \xi \in \mathfrak{g}^-\} = \{\mathbb{P}^1 \subset \hat{D} \mid \mathbb{P}^1 \ni o \text{ is a VHS}\}\]

Let
\[(3.2)\]
\[X \overset{\text{dfn}}{=} \bigcup_{\mathbb{P}^1 \in C_o} \mathbb{P}^1\]
be the variety swept out by the lines that pass through $o$ and are VHS.

**Lemma 3.3.** Assume that $P$ is a maximal parabolic subgroup of $G$, and let $\hat{D}$ be the minimal homogeneous embedding of $G/P$ (Section 2.1). Then

(a) $X$ is a cone over $C_o$ with vertex $o$ and a Schubert variety.
(b) $X$ is a VHS if and only if the simple root $\alpha_1$ associated with the maximal parabolic $\mathfrak{p}$ is not short.

The lemma is proved in Section 3.7.
3.1. The case that $P$ is maximal. We now recall two properties of $C_\circ$ in the case that $P$ is maximal (equivalently, $I = \{i\}$ and $E = S^1$). The results that follow are due to [38], where $C_\circ$ and $\tilde{C}_\circ$ are discussed for arbitrary (not necessarily maximal) $P$.

(a) Since $P$ is maximal, $g^{-1}$ is an irreducible $g^0$–module with highest weight line $g^{-\alpha_i}$, cf. [48, Theorem 8.13.3]. The variety of lines $C_\circ \subset \mathbb{P}g^{-1}$ is the $G^0$–orbit of this highest weight line, cf. [38, Theorem 4.3]. In particular, $C_\circ$ is a rational homogeneous variety; indeed,

\[ C_\circ \simeq G^0/(G^0 \cap Q), \]

where $Q \supset B$ is the parabolic subgroup defined by

\[ I(q) = \{j \mid g^{-\alpha_j} \not\subset q\} \overset{\text{dfn}}{=} \{j \mid \langle \alpha_i, \alpha_j \rangle \neq 0\}. \]

That is, the simple roots indexed by $I(q)$ are those adjacent to $\alpha_i$ in the Dynkin diagram of $g$; cf. [38, Proposition 2.5]. With only a few exceptions, $C_\circ$ is a $G^0$–cominuscule variety; equivalently, $C_\circ \simeq G^0/(G^0 \cap Q)$ admits the structure of a compact Hermitian symmetric space, cf. [38, Proposition 2.11].

(b) If the simple root $\alpha_i$ associated with the maximal parabolic $P$ is not short, then $C_\circ = \tilde{C}_\circ$, cf. [38, Theorem 4.8.1]. If the simple root is short, then $\tilde{C}_\circ$ is the union of two $P$–orbits, and open orbit and its boundary $C_\circ$.

3.2. The case of a general parabolic. If we drop the assumption that $P$ is maximal, then $C_\circ$ may be described as follows. Let $I$ be the index set (2.1) corresponding to $p$. Given $i \in I$, let

\[ g_i^{-1} = \{\zeta \in g^{-1} \mid [S^i, \xi] = -\xi\}. \]

Each $g_i^{-1}$ is an irreducible $G^0$–module with highest weight line $g^{-\alpha_i}$, and

\[ g^0 = \bigoplus_{i \in I} g_i^{-1} \]

is a $G^0$–module decomposition, cf. [48, Theorem 8.13.3]. Let $C_{o,i} \subset \mathbb{P}g_i^{-1}$ be the $G^0$–orbit of the highest weight line $g^{-\alpha_i}$. Then the variety of lines $C_\circ \subset \mathbb{P}g^{-1}$ is the disjoint union

\[ C_\circ = \bigsqcup_{i \in I} C_{o,i}. \]

As in the case that $P$ is maximal (Section 3.1), the assertions here are established in [38, Section 4].

3.3. Uniruling of $\tilde{\mathcal{D}}$. For ease of exposition we continue with the assumption that $P$ is maximal. However, analogous statements follow for unirulings on general $G/P$. For the more general statements, it is convenient to use Tits correspondences, which are briefly reviewed in Section 3.4.

Given $x = go \in \tilde{\mathcal{D}}$, with $g \in G$, let $C_x = gC_\circ$ denote the corresponding set of lines through $x$. (It is an exercise to show that $C_x$ is well-defined; that is, $C_x$ does not depend on our choice of $g$ yielding $x = go$.) Then

\[ C \overset{\text{dfn}}{=} \{\mathbb{P}^1 \mid \mathbb{P}^1 \in C_x, \ x \in \tilde{\mathcal{D}}\} = \bigcup_{g \in G} gC_\circ. \]
forms a uniruling of $\tilde{D}$. (As will be shown in Corollary 3.15, this uniruling is parameterized by $G/Q$ — that is, $\mathcal{C} \simeq G/Q$.)

Remark 3.6. More generally, the set of all lines on $G/P$ is $\tilde{\mathcal{C}} = \cup_{g \in G} g \tilde{C}_o$.

(a) It follows from definition (3.1) and the homogeneity of the IPR, that

$$\mathcal{C} = \{ \mathbb{P}^1 \in \tilde{\mathcal{C}} \mid \mathbb{P}^1 \text{ is a VHS} \}$$

is precisely the set of lines on $G/P$ that are integrals of the IPR.

(b) As noted in Section 3.1(b), if the simple root associated to the maximal parabolic $P$ is not short, then $\tilde{\mathcal{C}} = \mathcal{C}$ consists of a single $G$–orbit. If the simple root is short, then $\tilde{\mathcal{C}}$ consists of two $G$–orbits, an open orbit and its boundary $\mathcal{C}$, cf. [38, Theorem 4.3].

### 3.4. Tits correspondences.

Tits correspondences describe homogeneous unirulings of a rational homogeneous variety $G/P$ by homogeneously embedded, rational homogeneous subvarieties $G'/P'$; these unirulings may be used to clarify the geometry of $G/P$. The material in Sections 3.6–3.7 is taken from [17, 18]. Given two standard parabolics $P$ and $Q = P_J$, the intersection $P \cap Q$ is also a standard parabolic. (Note that $I(p \cap q) = I(p) \cup I(q)$.)

There is a natural double fibration, called the Tits correspondence, given by the diagram in Figure 3.1; here the maps $\eta$ and $\tau$ are the natural projections. Given a subset $\Sigma \subset G/Q$,

**Figure 3.1.** Tits correspondence

$$G/(P \cap Q) \xrightarrow{\eta} G/P \xrightarrow{\tau} G/Q$$

the Tits transform is $T(\Sigma) := \eta(\tau^{-1}(\Sigma))$.

### 3.5. Tits transform of a point.

The Tits transform of a point $y \in G/Q$ will play a crucial rôle in our discussion of $G/P$ and its Schubert varieties. What follows is a brief review of [39, §2.7.1].

The Tits transform $T(Q/Q)$ of the point $Q/Q \in G/Q$ is the $G'$–orbit $G'(P/P) \subset G/P$, where $G'$ is the semisimple subgroup of $G$ whose Dynkin diagram $\mathcal{D}'$ is obtained from the diagram $\mathcal{D}$ of $G$ by removing the nodes $I(q) \setminus I(p)$. Therefore, $T(Q/Q) \simeq G'/P'$, where $P' = G' \cap P$ is the parabolic subgroup of $G'$ with index set (2.1) given by $I(p') = I(p) \setminus I(q)$.

Since any point $y \in G/Q$ is of the form $y = gQ/Q$ for some $g \in G$, and $T(gQ/Q) = gT(Q/Q)$, it follows that $T(y) \simeq G'/P'$ and

$$G/P \text{ is uniruled by subvarieties } \Sigma \text{ isomorphic to } G'/P',$$

and the uniruling is parameterized by $G/Q$.

Moreover, $G/(P \cap Q)$ is the incidence space for this uniruling. Precisely,

$$G/(P \cap Q) = \{ (x, \Sigma) \in G/P \times G/Q \mid x \in \Sigma \}.$$  

(9) There is a typo in [39, §2.7.1]; on page 87, $S$ and $S'$ should be swapped. This is corrected for in the discussion above.
Remark 3.9 (\(\mathbb{P}^1\)–unirulings of \(G/P\) for maximal \(P\)). In the case that \(P \subset G\) is a maximal parabolic (Section 2.1), there is a unique \(G\)–homogeneous variety \(G/Q\) parameterizing a uniruling of \(G/P\) by lines \(\mathbb{P}^1\); it may be identified by inspection of the Dynkin diagram \(\mathcal{D}\) of \(\mathfrak{g}\) as follows. The maximality of \(P\) is equivalent to \(I(\mathfrak{p}) = \{1\}\) for some \(i\). In order to obtain a uniruling by \(\mathbb{P}^1\)'s, we must choose \(Q\) (equivalently, the index set \(I(\mathfrak{q})\)) so that \(G'/P' \cong \mathbb{P}^1\). To that end, let \(J = \{j \neq i \mid (\alpha_i, \alpha_j) \neq 0\}\) index the nodes in the Dynkin diagram that are adjacent to the \(i\)–th node. Then \(G'/P' \sim \mathbb{P}^1\) (equivalently, \(G/P\) parameterizes a uniruling of \(G/P\) by \(\mathbb{P}^1\)'s) if and only if \(J \subset I(\mathfrak{q})\). When \(I(\mathfrak{q}) = J\) we say that \(G/Q\) is the smallest rational \(G\)–homogeneous variety parameterizing a uniruling of \(G/P\) by lines \(\mathbb{P}^1\).

Examples 3.10 and 3.11 below identify the varieties \(G/Q\) parameterizing lines on the fundamental adjoint varieties (Section 2.5).

Example 3.10 (Unirulings of the orthogonal adjoint varieties). Fix a nondegenerate symmetric bilinear form \(\nu\) on \(\mathbb{C}^n\), \(n \geq 7\). Let \(G = \text{Aut}(\mathbb{C}^n, \nu)^0 = \text{SO}(\mathbb{C}^n)\) denote the identity component of the orthogonal group. The adjoint variety (Section 2.5)

\[
G/P_2 = \text{OG}(2, \mathbb{C}^n) = \{E \in \text{Gr}(2, \mathbb{C}^n) \mid \nu|_E = 0\}
\]

is the set of \(\nu\)–isotropic 2–planes. The partial flag variety

\[
G/P_{1,3} = \text{Flag}_\nu(1,3,\mathbb{C}^n) = \{F^1 \in \text{OG}(1, \mathbb{C}^n) \times \text{OG}(3, \mathbb{C}^n) \mid F^1 \subset F^3\}
\]

parameterizes a uniruling of the adjoint variety \(G/P_2\) by \(\mathbb{P}^1\)'s. Given one such flag \(F^1 \subset F^3\) the corresponding line is \(\{E \in \text{OG}(2, \mathbb{C}^n) \mid F^1 \subset E \subset F^3\}\).

In this case \(\mathcal{D}' = \mathcal{D}\setminus\{2\}\) so that \(\mathfrak{g}' = \mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{so}_{n-4}\mathbb{C}\). Additionally, \(I(\mathfrak{q}) = I(\mathfrak{p}') = \{1, 3\}\).

Example 3.11 (Unirulings of the exceptional adjoint varieties). Let \(G/P \hookrightarrow \mathbb{P}\mathfrak{q}\) be the adjoint variety (Section 2.5) of an exceptional simple Lie group \(G\). The rational homogeneous variety \(G/Q\) parameterizing a uniruling of \(G/P\) by lines \(\mathbb{P}^1\) is given in Table 3.1. In the

| Adjoint variety \(G/P\) | \(E_6/P_2\) | \(E_7/P_1\) | \(E_6/P_1\) | \(E_7/P_3\) | \(E_8/P_8\) | \(F_4/P_1\) | \(G_2/P_2\) | \(G_2/P_1\) |
|------------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|

Table 3.1. Lines on exceptional adjoint varieties

3.6. Tits transform of a Schubert variety. The Tits transform preserves Schubert varieties; that is, if \(Y \subset G/Q\) is a Schubert variety, then so is \(X = \mathcal{T}(Y) = G/P\). Moreover, \(X\) may be determined as follows. Recall (Section 2.6) that \(\mathcal{W}_q^{\text{min}}\) is the set of (unique) minimal length representatives of the right cosets \(\mathcal{W}_q\setminus W\). Each right coset also admits a unique representative of maximal length; let \(\mathcal{W}_q^{\text{max}} \subset W\) denote this set of representatives. If \(w_0\) is the longest element of \(\mathcal{W}_q\), then \(\mathcal{W}_q^{\text{max}} = \{w_0w \mid w \in \mathcal{W}_q^{\text{min}}\}\). We have \(\mathcal{W}_q\setminus W \triangleright \mathcal{W}_q^{\text{min}} \cong \mathcal{W}_q^{\text{max}}\). A proof of the following well–known lemma is given in [17].

Lemma 3.12. Let \(w \in \mathcal{W}_q^{\text{max}}\), and let \(Y_w \subset G/Q\) denote the Schubert variety indexed by the coset \(\mathcal{W}_q w\). Then the Tits transformation \(\mathcal{T}(Y_w)\) is the Schubert variety \(X_w \subset G/P\) indexed by the coset \(\mathcal{W}_p w\).
3.7. Lines through a point. The following, first observed in [18], is an immediate consequence of (3.8).

Lemma 3.13. Suppose that $G/Q$ parameterizes a uniruling of $\tilde{D} = G/P$ by $\mathbb{P}^1$s. Let $\Sigma = \tau(\eta^{-1}(o)) \subset G/Q$ denote the image of $o \in \tilde{D}$ under the Tits transform, and let $T(\Sigma) = \eta(\tau^{-1}(\Sigma)) \subset \tilde{D}$ denote the image of $\Sigma$ under the Tits transform back to $\tilde{D}$. Then $\Sigma$ is the set of lines in $\tilde{D}$ that are parameterized by $G/Q$, and pass through $o \in \tilde{D}$. Likewise,

$$T(\Sigma) = \bigcup_{o \in \mathbb{P}^1 \cap G/Q} \mathbb{P}^1$$

is the subset of $\tilde{D}$ swept out by these lines, and is naturally identified with a cone $C(\Sigma)$ over $\Sigma$ with vertex $o$.

Remark 3.14. By Lemma 3.12, the variety $T(\Sigma)$ is a Schubert variety $X_w \subset G/P$.

Corollary 3.15 (Maximal parabolics). Let $P$ be a maximal parabolic and let $Q$ be the parabolic of Section 3.1(a). Then the uniruling $C$ of $G/P$ (Section 3.3) is precisely the uniruling $G/Q$ obtained through the Tits correspondence. In particular, the variety $\Sigma$ of Lemma 3.13 is the variety $C_o$ of (3.1), and the variety $T(\Sigma)$ of Lemma 3.13 is the variety $X$ of (3.2).

Proof. By the maximality of $P$ we have $I(p) = \{1\}$, cf. Section 2.1. Recall the Levi subgroup $G^0 \subset G$ whose Lie algebra $g^0$ is the zero–eigenspace of the grading element $E = S^1$ associated with the parabolic $p$, cf. Section 2.1. Then the simple roots of the semisimple Lie algebra $g^0_{ss} = [g^0, g^0]$ are $S \{\alpha_i\}$, the simple roots of $g$ minus the $i$–th. As discussed in Section 3.1(a), we have $C_o = G^0/(G^0 \cap Q)$. It follows that $C_o = G^0_{ss}/(G^0_{ss} \cap Q)$, where $G^0_{ss} \subset G^0$ is the semisimple subgroup with Lie algebra $g^0_{ss}$. Note the index set $I(g^0_{ss} \cap q)$ associated with the parabolic $G^0_{ss} \cap Q$ by (2.1) is $I(q)$.

As discussed in Section 3.1(a), the set $I(q)$ indexes those nodes of the Dynkin diagram that are adjacent to the $i$–th node. By Remark 3.9, $G/Q$ is the minimal rational $G$–homogeneous variety parameterizing a uniruling of $G/P$ by lines $\mathbb{P}^1$s. The Tits transform $\Sigma = T(P/P) \subset G/Q$ is of the form $G'/P'$. From the descriptions of $G'$ and $P'$ in Section 3.5 and the discussion of Remark 3.9 we see that $G' = G^0_{ss}$ and $P' = G^0_{ss} \cap Q$. Thus, $C_o = \Sigma$. □

Remark 3.16 (Adjoint varieties). In Sections 7.5-7.6, we will be interested in the variety $X = T(\Sigma)$ in the case that $G/P$ is a fundamental adjoint variety. In those cases, $P$ is a maximal parabolic, so that Corollary 3.15 applies: the $\Sigma = C_o$ are listed in Table 3.2. In

| $G/P$ | $OG(2, \mathbb{C}^n)$ | $E_6/P_2$ | $E_7/P_1$ | $E_8/P_8$ | $F_4/P_1$ | $G_2/P_2$ |
|-------|---------------------|-----------|-----------|-----------|-----------|-----------|
| $C_o$ | $\mathbb{P}^1 \times \mathbb{Q}^{n-6}$ | $\text{Gr}(3, \mathbb{C}^6)$ | $\mathcal{S}_6$ | $E_7/P_7$ | $\text{LG}(3, \mathbb{C}^6)$ | $v_3(\mathbb{P}^1)$ |

the table $\mathcal{S}_6$ is a Spinor variety, one of the two connected components of the orthogonal grassmannian $OG(6, \mathbb{C}^{12})$.

Moreover, for each of the fundamental adjoint varieties, the simple root $\alpha_1$ associated with the maximal parabolic $P = P_1$ is not short. Whence Lemma 3.3 applies and the variety $X$ swept out by lines passing through a fixed point is a VHS.
Proof of Lemma 3.3. Part (a) of the lemma follows directly from Lemma 3.13 and Corollary 3.15. The argument for part (b) is based on the Tits transform recipe given by Lemma 3.12 and the characterization of Schubert VHS discussed in Section 2.7. We will make use (without explicit mention) of observations made in the proof of Corollary 3.15.

To begin, let \( P \subset G \) be a maximal parabolic with index set \( I(p) = \{ i \} \), cf. Section 2.1. Let \( G/Q \) parameterize the uniruling of \( \tilde{D} = G/P \) by \( \mathbb{P}^1 \)'s. The right coset indexing the Schubert variety \( o = P/P \in \tilde{D} \) is \( W_pw_0 \), where \( w_0 \) is the longest word of the Weyl group \( W_p \) of \( g^0 \). By Lemma 3.12, the Schubert variety \( \Sigma = T(P/P) \) is indexed by the coset \( W_qw_0 \). Let \( w_1 \in W_{\text{max}} \) be the longest representative of \( W_qw_0 = W_qw_1 \). Again, Lemma 3.12 implies \( T(\Sigma) \) is the Schubert variety indexed by the right coset \( W_pw_1 \). By Corollary 3.15, this is the Schubert variety \( X \). Let \( w \in W_{\text{min}}^p \) be a shortest representative of \( W_pw_1 = W_pw \). As discussed in Section 2.7, the Schubert variety \( X \) is a VHS if and only if \( g_w(E) = \dim X \). So the substance of the proof is to compute the integer \( g_w(E) \). Note that \( E = S^1 \), cf. (2.7).

Let \( v \in W_{\text{min}}^q \) be the shortest Weyl element indexing the Schubert variety \( \Sigma \). Let \( w_0' \in W_q \) be the longest element of the Weyl subgroup, so that \( w_0'v = w_1 \) is the longest element indexing \( \Sigma \). Then

\[
W_qw_0 = W_qv = W_qw_0'v,
\]

and \( W_pw_0'v \) indexes \( X = T(\Sigma) \). We claim that

\[
(3.17) \quad W_pw_0' = W_p(i),
\]

where \( (i) \in W \) denotes the reflection associated with the simple root \( \alpha_1 \). It follows from (3.17) that

\[
(i)v = w \in W_{\text{min}}^p
\]

is the shortest Weyl group element indexing \( X \).

Proof of (3.17). Observe that \( W_p \) is generated by the simple reflections \( \{ (j) \mid j \neq i \} \). Likewise, \( W_q \) is generated by the simple reflections \( \{ (j) \mid j \neq i - 1, i + 1 \} \). In particular, any element \( u \) of \( W_q \) may be written as \( u'(i) \) with \( u' \in W_p \). The claim follows.

We return to the proof of Lemma 3.3(b). From the discussion of Section 2.6 (specifically (2.16)) we see that

\[
|v| = |\Delta(v)| = \dim \Sigma \quad \text{and} \quad 1 + |v| = |w| = |\Delta(w)| = \dim X.
\]

The argument establishing [12, Proposition 3.2.14(5)] yields

\[
\Delta(w) = \{ \alpha_1 \} \cup (i)\Delta(v).
\]

Since \( v \) indexes a homogeneous Schubert variety \( \Sigma = G'/P' \), we see from the discussion of Section 2.6 that

\[
\Delta(v) = \{ \alpha \in \Delta(g^0) \mid \alpha(F) > 0 \},
\]

where

\[
F = \sum_{i \in I(q)} S^i
\]

is the grading element (2.7) associated with the parabolic \( q \). Since \( g^0 \) is the zero–eigenspace of \( E = S^1 \), this may be rewritten as

\[
\Delta(v) = \{ \alpha \in \Delta \mid \alpha(E) = 0, \, \alpha(F) > 0 \}.
\]
In particular, the roots of $\Delta(v)$ are precisely those positive roots $\alpha = m^i \alpha_i$ such that $m_j = 0$ and $m^j > 0$ for some $j \in I(g)$. Informally, the root $\alpha$ does not involve the simple root $\alpha_1$, but does involve some simple root adjacent to $\alpha_4$. Whence the reflection

$$(i) \alpha = \alpha - 2(\alpha, \alpha_1)(\alpha_1) \alpha_1$$

has the property that the integer $-2(\alpha, \alpha_1)/(\alpha_1, \alpha_1) \geq 1$ for every $\alpha \in \Delta(v)$. It now follows from the second equality of (2.20) that $g_w(E) > |w|$ if and only if $\alpha_4$ is short. 

\section{4. Smooth Schubert VHS}

A homogeneously embedded, homogeneous submanifold of $G/P$ is a submanifold of the form

$$g \{aP \in G/P \mid a \in G'\} = gG'/P',$$

where $g \in G$ and $G'$ is a closed Lie subgroup of $G$. Such a submanifold is isomorphic to $G'/P'$, where $P' = G' \cap P$.

Distinguished amongst the homogeneously embedded, homogeneous submanifolds of $G/P$ are the rational homogeneous subvarieties $X(D')$ corresponding to subdiagrams $D'$ of the Dynkin diagram $D$ of $g$. The correspondence is as follows: Identifying the nodes of $D$ with the simple roots $s$ of $g$, to any subdiagram we may associate a semisimple Lie subalgebra $g' \subset g$ by defining $g'$ to be the subalgebra generated by the root spaces $\{g^\pm \alpha \mid \alpha \in D'\}$. Note that the Dynkin diagram of $g'$ is naturally identified with the subdiagram $D'$. Let $G' \subset G$ be the corresponding closed, connected semisimple Lie subgroup. For such $G'$, the subgroup $P' = G' \cap P$ is a parabolic subgroup. The $G'$–orbit of $P \in G/P$ is isomorphic to $G'/P'$ and is the homogeneously embedded, rational homogeneous subvariety $X(D') \subset G/P$ corresponding to subdiagram $D' \subset D$.

Define the index set $J = \{j \mid g^{s_j} \not\subseteq g'\}$. Let $F = \sum_{j \in J} s_j$ be the corresponding grading element (Section 2.3), and let $g = \oplus g^j$ be the $F$–eigenspace decomposition (2.4). Then $g' = [g^0, g^B]$ is the semisimple component of the reductive subalgebra $g^0$. Let $Q = P_F$ be the corresponding parabolic subgroup of $G$ (with Lie algebra $q = g^0 \oplus g^+$.). It follows from the discussion of Section 3.5 that

$$(4.1) \quad X(D') \subset G/P$$

is the Tits transformation $T(Q/Q)$ of the point in $Q/Q \in G/Q$.

This, with Lemma 3.12, yields the following

\textbf{Lemma 4.2.} The homogeneously embedded, rational homogeneous subvarieties $X(D') \subset G/P$ corresponding to subdiagrams $D' \subset D$ are smooth Schubert varieties of $G/P$.

\textbf{Remark 4.3.} Let $E$ be the grading element (2.7) associated with $p$. It follows from the discussion above, that a Schubert variety $X_w$ is of the form $X(D')$ if and only if $\Delta(w) = \Delta(g^+_w)$, where $g' = \oplus g^j$ is the $E$–eigenspace decomposition of $g'$ and $\Delta(w)$ is the set (2.15). By Section 2.7, $X(D')$ is a VHS if and only if $g' = g^+_1 \oplus g^+_0 \oplus g^+_{-1}$; equivalently, $X(D')$ is Hermitian symmetric.

The main result of this section is...
Theorem 4.4. Let $X \subset \check{D} = G/P$ be a Schubert variation of Hodge structure. If $X$ is smooth, then $X$ is a product of homogeneously embedded, rational homogeneous subvarieties $X(\mathcal{D}') \subset G/P$ corresponding to subdiagrams $\mathcal{D}' \subset \mathcal{D}$. Moreover, each $X(\mathcal{D}')$ is Hermitian symmetric.

Since a homogeneously embedded Hermitian symmetric space is necessarily smooth, Theorem 4.4 is equivalent to: A Schubert VHS is smooth if and only if it is a homogeneously embedded Hermitian symmetric space.

Theorem 4.4 is proved in Sections 4.1–4.5.

When $\check{D}$ is the compact dual of a Mumford–Tate domain $D$, Friedman and Laza have shown the following [22, Theorem 1]: if $Y \subset D$ is horizontal and semi-algebraic (that is, a connected component of the intersection with $D$ of a closed subvariety of $\check{D}$), with strongly quasi-projective image in an arithmetic-group quotient $\Gamma \backslash D$, then $Y$ is a horizontal (hence Hermitian) Mumford–Tate subdomain. Theorem 4.4 says that if $Y$ is horizontal and “semi-smooth-Schubert”, then $Y$ is Hermitian; so it is a rather different result from [22], without any reference to Mumford–Tate domains or quotients. However, we do have the

Corollary 4.5. Suppose $\check{D}$ is the compact dual of a Mumford–Tate–Chevalley domain $D$ (Definition 6.17), $X \subset \check{D}$ is a smooth Schubert VHS, and $Y$ is a nonempty connected component of $X \cap D$. Then some $G_\mathbb{R}$-translate of $Y$ is a (necessarily horizontal, Hermitian symmetric) connected Mumford–Tate subdomain of $D$.

Proof. Translate $Y$ to pass through the distinguished base point $o_0$ of $D$ (Section 6.4), with corresponding grading element $T$ and Hodge structure $\varphi_0 : S^1 \to G_\mathbb{R}$. By Theorem 4.4, we have $Y = (G'^\prime_\mathbb{R})^\circ \cdot o_0.$ (10) Since $g' = g'_1 \oplus g'_0 \oplus g'_{-1}$ decomposes into a direct sum of $\mathfrak{h} \cap g'$ and a subset of $\mathfrak{h}$–root spaces, it follows that $g'$ is stable under $T \in \mathfrak{h}$. Therefore, $\varphi_0(S^1)$ normalizes $G'_\mathbb{R}$, and we set $\check{G}'_\mathbb{R} \overset{dfn}{=} \varphi_0(S^1) \cdot G'_\mathbb{R}$. By Remarks 6.10 and 6.18, this is underlain by a $\mathbb{Q}$-algebraic group $\check{G}'_\mathbb{Q}$.

Now since $\text{ad}(T)|_{g'_{-1}} = -1_{g'_{-1}}$, the (complexified) $\mathbb{Q}$–Lie-algebra-closure of a very general $\text{ad}(g'_{-1})$–translate of $T$ contains $g'_{-1} = \check{g}'_{-1}$. Therefore, so does the $\mathbb{C}$–Lie algebra of the Mumford–Tate group $\check{G}''_\mathbb{Q}$ of a very general $G'_\mathbb{Q}$-conjugate of $\varphi_0$. It follows that $Y = (\check{G}''_\mathbb{Q})^\circ \cdot o_0$, where $\check{G}''_\mathbb{Q}$ is the Mumford–Tate group of a very general point (i.e., Hodge structure) in $Y$; that is, $Y$ is a connected Mumford–Tate subdomain.

Remark 4.6. (i) The converse of Corollary 4.5 fails dramatically: the compact dual $\check{D}' \subset \check{D}$ of a horizontal (Hermitian) Mumford–Tate subdomain $D' \subset D$ need not be a Schubert variety, even when $\check{D}'$ is a maximal integral of the IPR. A large class of examples illustrating this, called enhanced SL$_2$-orbits, is constructed in Section 6; the simplest one is probably the SL$_2 \times$ SL$_2$ subdomain in the G$_2$-adjoint variety (see Sections 7.5–7.6). (ii) We expect that Corollary 4.5 itself fails for more general (non-MTC) Mumford–Tate domains, to an extent which should be describable in terms of the action of $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ in the Dynkin diagram $\mathcal{D}$.

Example 4.7 (Symplectic Grassmannians). In this example we will (a) illustrate the construction of $X(\mathcal{D}')$ from $\mathcal{D}' \subset \mathcal{D}$, and (b) observe that not every smooth Schubert variety

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(10) Here the superscript $\circ$ denotes the connected identity component.
of a rational homogeneous variety is homogeneous; in particular, the assumption that $X$ be a VHS in Theorem 4.4 is essential.

Let $\nu$ be a non-degenerate, skew-symmetric bilinear form on $\mathbb{C}^{2r}$. Fix $i \leq r$. The symplectic grassmannian

$$SG(i, \mathbb{C}^{2r}) = \{ E \in \text{Gr}(i, \mathbb{C}^{2r}) \mid \nu|_E = 0 \}$$

is a rational homogeneous variety $\mathcal{D} = G/P$, where $G = \text{Aut}(\mathbb{C}^{2r}, Q) = \text{Sp}_{2r} \mathbb{C}$ and $P = P_i$ is the maximal parabolic subgroup associated to the $i$-th simple root. The Dynkin diagram of $G$ (containing $r$ nodes) is $\mathcal{D} = \mathcal{D}^r$.

Fix a $Q$-isotropic flag $F^1 \subset F^2 \subset \cdots \subset F^{2r} = \mathbb{C}^{2r}$; here $F^d$ is a complex linear subspace of dimension $d$ and $\nu(F^d, F^{2r-d}) = 0$. Given $0 \leq a < i < b \leq 2r - a$,

$$X_{a,b} = \{ E \in SG(i, \mathbb{C}^{2r}) \mid F^a \subset E \subset F^b \}$$

is a Schubert variety. The variety $X_{a,b}$ is the homogeneous submanifold $X(\mathcal{D}')$ associated to a Dynkin subdiagram $\mathcal{D}' \subset \mathcal{D}$ if and only if either $b \leq r$, or $b = 2r - a$. In the first case ($b \leq r$), the subdiagram $\mathcal{D}'$ corresponds to the simple roots $S' = \{ \alpha_{a+1}, \ldots, \alpha_{b-1} \}$; we have $g' \simeq \mathfrak{sp}_{b-a} \mathbb{C}$ and $X_{a,b} \simeq \text{Gr}(i - a, \mathbb{C}^{b-a})$; these Schubert varieties are VHS. In the second case ($b = 2r - a$), the subdiagram $\mathcal{D}'$ corresponds to the simple roots $S' = \{ \alpha_{a-1}, \ldots, \alpha_r \}$; we have $g' \simeq \mathfrak{sp}_{2(r-a)} \mathbb{C}$ and $X_{a,b} \simeq \text{SG}(i - a, \mathbb{C}^{2(r-a)})$; these $X_{a,b}$ are VHS if and only if $i = r$ (equivalently, the symplectic grassmannian is a Hermitian symmetric Lagrangian grassmannian).

We illustrate this for $\mathcal{D} = \text{SG}(4, \mathbb{C}^{16})$; that is, $i = 4$ and $r = 8$. The encircled subdiagram corresponds to $g' = \mathfrak{sl}_6 \mathbb{C}$ and

$$X_{1,7} = \{ E \in \text{SG}(4, \mathbb{C}^{16}) \mid F^1 \subset E \subset F^7 \} = \text{Gr}(3, F^7/F^1) \simeq \text{Gr}(3, \mathbb{C}^6).$$

Likewise, the encircled subdiagram corresponds to $g' = \mathfrak{sp}_{12} \mathbb{C}$ and

$$X_{2,14} = \{ E \in \text{SG}(4, \mathbb{C}^{16}) \mid F^2 \subset E \} = \text{SG}(2, (F^2)^\perp/F^2) \simeq \text{SG}(2, \mathbb{C}^{12}).$$

Returning to the general case, we have $i = r$ if and only if $\mathcal{D}$ is a Hermitian symmetric space. In this case, the IPR is trivial so that every Schubert variety is a Schubert VHS. Moreover, a Schubert variety is smooth if and only if it is the homogenous submanifold associated to a Dynkin subdiagram, cf. [10].

In contrast to the Hermitian symmetric case, if $i < r$ and $b = 2n - a - 1$, then $X_{a,b}$ is smooth, but not homogeneous, cf. [41]. These Schubert varieties are not VHS.

Remark 4.8 (Maximal parabolic associated to non-short root). If $\mathcal{D} = G/P$ with $P$ a maximal parabolic corresponding to a non-short simple root $\alpha_4$, then all smooth Schubert varieties of $\mathcal{D}$ are homogeneously embedded, rational homogeneous varieties $X(\mathcal{D}')$ corresponding to connected subdiagrams $\mathcal{D}' \subset \mathcal{D}$ containing the $i$-th node, cf. [27, Proposition 3.7]. For example, the smooth Schubert varieties in the Grassmannian $\text{Gr}(k, \mathbb{C}^n)$ are all homogeneous; likewise, the smooth Schubert varieties of the adjoint varieties (Section 2.5) of $G = B_r, D_r, E_r, F_4, G_2$ are all homogeneous.
4.1. **Proof of Theorem 4.4: outline.** The proposition is proved in three steps. First we reduce to the case that $P$ is maximal (Section 4.2). Then, the result [27, Proposition 3.7] of Hong and Mok establishes the proposition in the case that the associated simple root is not short (Section 4.3). Third, we address the short root case (Section 4.4).

4.2. **Reduce to the case that $P$ is maximal parabolic.** Suppose that $X$ is a Schubert VHS. Let $w \in W^P$ be the associated Weyl group element (Section 2.6), so that $X = X_w = Bw^{-1}o$. The condition that the Schubert variety be a VHS is equivalent to $\Delta(w) \subset \Delta(g_1)$ (Section 2.7). By definition $\alpha \in \Delta(g_1)$ if and only if $\alpha(E) = 1$. Equivalently, $\alpha(S^i) = 1$ for exactly one $i \in I$, and $\alpha(S^j) = 0$ for all other $j \in I$. Therefore, we have a disjoint union

$$\Delta(w) = \bigsqcup_{i \in I} \Delta_i(w),$$

where

$$\Delta_i(w) \overset{\text{dfn}}{=} \{ \alpha \in \Delta(w) \mid \alpha(S^i) = 1 \}.$$  

It is straightforward to confirm that both $\Delta_i(w)$ and $\Delta^+ \setminus \Delta_i(w)$ are closed. Therefore, $\Delta_i(w)$ determines a Schubert variety $X_i \subset X$, cf. [45, Remark 3.7(c)]. (In general, $\Delta_i(w) \subset \Delta(w)$ does not imply $X_i \subset X$. In case $\varphi$ lifts it is a consequence of the fact that $X$ is a VHS, cf. [45, Appendix D].) Whence

$$X = \prod_{i \in I} X_i.$$ 

Let $\mathcal{D}$ denote the Dynkin diagram of $G$. Given $i \in I$, let $\mathcal{D} \setminus \{I\{i\}\}$ denote the subdiagram of $\mathcal{D}$ obtained by removing all nodes corresponding to $j \in I \setminus \{i\}$ (and their adjacent edges). Let $\mathcal{D}_i$ denote the connected component of $\mathcal{D} \setminus \{I\{i\}\}$ containing the $i$-th node. Let $G_i \subset G$ denote the closed, connected semisimple Lie subgroup of $G$ corresponding to $\mathcal{D}_i$. Let $\bar{D}_i \subset \bar{D}$ be the $G_i$-orbit of $o$. Then $\bar{D}_i \simeq G_i/(G_i \cap P)$ is rational homogeneous variety containing $X_i$ as a Schubert subvariety. Moreover, $X_i$ is a VHS for the IPR on $\bar{D}_i$. Finally, note that $G_i \cap P$ is a maximal parabolic subgroup of $G_i$ – the corresponding index set is just $\{i\}$. Since $X$ is smooth if and only if each $X_i$ is smooth, to prove the proposition, it suffices to show that $X_i$ is a homogeneously embedded Hermitian symmetric space.

This reduces the proof of the proposition to the case that $P$ is a maximal parabolic, which we now assume. Let $\alpha_1$ denote the associated simple root.

4.3. **The case that $\alpha_1$ is not a short root.** Suppose that $\alpha_1$ is not a short root of $G$. By [27, Proposition 3.7], the Schubert variety $X \subset \bar{D} = G/P_1$ is the homogeneously embedded, rational homogeneous subvariety corresponding to a subdiagram of $\mathcal{D}$.

4.4. **The case that $\alpha_1$ is a short root.** In this case $\bar{D} = G/P_1$ is one of the following five rational homogeneous varieties:

- Let $\nu$ be a nondegenerate symmetric bilinear form on $\mathbb{C}^{2r+1}$. Then the *orthogonal grassmannian*

$$\text{OG}(r, \mathbb{C}^{2r+1}) = \{ E \in \text{Gr}(r, \mathbb{C}^{2r+1}) \mid \nu|_E = 0 \}$$

of maximal, $\nu$-isotropic subspaces is the rational homogeneous variety $B_r/P_r = \text{Spin}_{2r+1}\mathbb{C}/P_r$, where $P_r$ is the maximal parabolic associated to the short simple root $\alpha_r$. 
Let $\nu$ be a nondegenerate skew-symmetric bilinear form on $\mathbb{C}^{2r}$. Then the \emph{symplectic grassmannian}

$$SG(i, \mathbb{C}^{2r}) = \{ E \in Gr(r, \mathbb{C}^{2r+1}) \mid \nu|_E = 0 \}$$

of $i$-dimensional $\nu$-isotropic subspaces is the rational homogeneous variety $C_r/P_1 = Sp_{2r}\mathbb{C}/P_1$, where $P_1$ is the maximal parabolic associated to the simple root $\alpha_1$. The simple root is short if and only if $i < r$.

The exceptional $F_4/P_3$, $F_4/P_4$ or $G_2/P_1$.

Unfortunately, the Hong–Mok argument does not appear to extend to $SG(i, \mathbb{C}^{2r})$, $F_4/P_3$ or $G_2/P_1$. Instead, we will see that, for each of the five $\hat{D}$ above, Theorem 4.4 follows from either (i) the Brion–Polo classification of smooth minuscule Schubert varieties, or (ii) existing descriptions of the Schubert VHS, or a combination of both.

\subsection*{4.4.1. The case that $\hat{D} = OG(r, \mathbb{C}^{2r+1})$}

In this case, $\hat{D}$ is minuscule, cf. [5]. Brion and Polo have shown that the smooth Schubert varieties of minuscule $G/P$ are homogeneous submanifolds. More precisely, let $Q \supset B$ denote the stabilizer of $X$, cf. Remark 2.13. Then $X = Q \cdot o$ by [10, Proposition 3.3(a)].

Let $q = q^0 \oplus q^+$ be the graded decomposition (2.6) associated to associated to the parabolic $q$ (and choices $q \supset b \supset h$). If $Q^0 \subset Q$ is the closed Lie subgroup with Lie algebra $q^0$, then $Q \simeq Q^0 \times q^+ = Q^0 \times \exp(q^+)$, by [12, Theorem 3.1.3]. Therefore, $X = Q \cdot o = Q^0 \cdot o$, where $Q^0_{ss}$ is the closed semisimple Lie subgroup with Lie algebra $q^0_{ss} = [q^0, q^0]$. The semisimple $Q^0_{ss}$ is the subgroup of $G$ associated to the subdiagram $\mathcal{D}' = \{ \alpha \in \mathcal{D} \mid q^\alpha \subset q^0 \}$ (Section 2.3). Thus, $X = X(\mathcal{D}')$ is the homogeneously embedded, rational homogeneous subvariety associated to a subdiagram $\mathcal{D}' \subset \mathcal{D}$.

\subsection*{4.4.2. The case that $\hat{D} = SG(i, \mathbb{C}^{2r})$}

Recall that $i < r$. In this case, the marked Dynkin diagram associated to $SG(i, \mathbb{C}^{2r})$ is

$$\begin{array}{ccccccccccccc}
1 & 2 & \cdots & i & \cdots & \hat{i} & \cdots & r
\end{array}$$

It follows from the proof of [45, Proposition 3.11](11) that there is a unique maximal Schubert variation of Hodge structure; it is the homogeneous submanifold $Z \simeq Gr(i, \mathbb{C}^r)$ associated to the circled subdiagram

So, any smooth Schubert VHS in $SG(i, \mathbb{C}^{2r})$ is a smooth Schubert variety of $Gr(i, \mathbb{C}^r)$. These are well-known to be precisely the homogeneously embedded, rational homogeneous subvarieties corresponding to connected subdiagrams of the the marked Dynkin diagram for $Gr(i, \mathbb{C}^r)$ that contain the $i$-th node, cf. [5, Section 9.3]. (Alternatively, this follows from the Brion–Polo result, since $Gr(i, \mathbb{C}^r)$ is minuscule, or Remark 4.8.)

\subsection*{4.4.3. The case that $\hat{D} = F_4/P_3$}

By [45, Example 5.16], the only Schubert variations of Hodge structure are (the trivial $o \in \hat{D}$ and) the $\mathbb{P}^1 \subset \mathbb{P}^2$ corresponding to the two subdiagrams

Therefore, any smooth Schubert VHS in $F_4/P_3$ is a homogeneously embedded, rational homogeneous subvariety corresponding to a subdiagram of $\mathcal{D}$.

\footnote{See, in particular, Step 3 of Section 7.3 in [45]; in the case under consideration, $t = 1$ and $i_t = i$.}
4.4.4. The case that \( \hat{D} = F_4/P_4 \). By [45, Example 5.17], the only Schubert variations of Hodge structure are (the trivial \( o \in \hat{D} \) and) the \( \mathbb{P}^1 \subset \mathbb{P}^2 \) corresponding to the two subdiagrams

Therefore, any smooth Schubert VHS in \( F_4/P_4 \) is a homogeneously embedded, rational homogeneous subvariety corresponding to a subdiagram of \( \mathcal{D} \).

4.4.5. The case \( \hat{D} = G_2/P_1 \). This variety is the quadric hypersurface \( \mathcal{Q}^5 \subset \mathbb{P}^6 \). By [45, Example 5.30], the only Schubert VHS are (the trivial \( o \in \hat{D} \) and) the smooth \( \mathbb{P}^1 \subset \hat{D} \) which is the homogeneous submanifold corresponding to the circled subdiagram

Therefore, any smooth Schubert VHS in \( G_2/P_1 \) is a homogeneously embedded, rational homogeneous subvariety corresponding to a subdiagram of \( \mathcal{D} \).

4.5. Fini. We have shown, in Sections 4.2–4.4, that given any rational homogeneous variety \( \hat{D} = G/P \) (here the parabolic is arbitrary – \( P \) need not be maximal) any smooth Schubert VHS \( X \subset \hat{D} \) is a product homogeneously embedded, rational homogeneous subvarieties \( X_i = X(D_i) \) corresponding to a subdiagram of \( \mathcal{D}_i \subset \mathcal{D} \). The condition that \( X \) (and therefore each \( X_i \)) be a VHS forces \( X_i \) to be Hermitian symmetric. This completes the proof of Theorem 4.4.

Remark 4.9. It is possible that [10, Theorem 2.6] can be used to prove Theorem 4.4, as it was used to establish the homogeneity of minuscule and cominuscule Schubert varieties in [10]. The arguments of [10] are representation theoretic in nature; in the proof of Theorem 4.4, we elected for the more geometric approach of [27].

5. Griffiths–Yukawa couplings and lines on \( G/P \)

A historically important special instance of the double fibrations introduced in Section 3 arose in connection with É. Cartan’s two realizations of \( G_2 \) as a transformation group of a complex 5-manifold preserving a nontrivial distribution [11, 13]. Indeed, they are related by the correspondence:

\[
\begin{array}{ccc}
G_2/B & \rightarrow & G_2/P_1 \\
\uparrow & & \downarrow \\
G_2/P_2 & \rightarrow & G_2/P_1
\end{array}
\]

(5.1)

Each of the varieties of (5.1) may be realized as the compact dual of a Mumford–Tate domain with bracket-generating horizontal distribution [24, §IV.F]. This section is motivated by an observation about the Hodge theory associated to the adjoint variety \( \hat{D} = G_2/P_2 \) and its contact (hyperplane) distribution, which we now briefly explain.

Suppose \( \mathcal{V} = \bigoplus_{j=0}^{n} \mathcal{V}^{m-j,j} \) is a variation of Hodge structure (in the classical sense) over a complex manifold \( S \), and \( \mathcal{D} \) a holomorphic differential operator of order \( n \) in a neighborhood
Letting $\mathcal{D}$ act on $\mathcal{O}_U(V)$ via the Gauss–Manin connection, the IPR forces the composition

$$\mathcal{O}_U(V^{n,0}) \to \mathcal{O}_U(V) \xrightarrow{\mathcal{D}} \mathcal{O}_U(V) \to \mathcal{O}_U(V^{0,n})$$

to be $\mathcal{O}_S$-linear (so that only the symbol of $\mathcal{D}$ matters). One therefore obtains a well-defined linear mapping

$$(5.2) \quad \text{Sym}^n T_s S \to \text{Hom}(V_s^{n,0}, V_s^{0,n}) \cong (V_s^{0,n})^\otimes 2$$

for each $s \in S$, which we shall call the Griffiths–Yukawa coupling. When $V$ comes from a family of varieties, it may be of particular interest to study the geometry of subfamilies for which (5.2) vanishes. Recent work of Katz [30] suggests that motives with Hodge numbers

$$(5.3) \quad \text{Sym}^n T_s S \to \text{Hom}(V_s^{n,0}, V_s^{0,n})$$

which pulls back to (5.2) under the period map. If $g \subset G_2$, hence admitting a period map into a quotient of $D = G_2(\mathbb{R})/(P_2 \mathbb{F}_2(\mathbb{R}) \subset \mathcal{D}$, arise from certain families of elliptically fibered surfaces. We shall not pursue the (algebro-)geometric angle here, but instead look at what representation theory can tell us.

Fixing a point $o \in D$ with stabilizer $P \subset G$ and a compact Cartan $t \subset g \cap p$ determines the grading element $T$. The action of $g$ on $V$ defines a map

$$(5.4) \quad \text{Sym}^n g^{-1} \to \text{Hom}(V^{n,0}, V^{0,n})$$

which pulls back to (5.2) under the period map. If $g = g_2$ and $T$ is the adjoint grading element, which is defined by $\alpha_1(T) = 0$ and $\alpha_2(T) = 1$ on the simple roots, and $V$ is the 7-dimensional irreducible representation (with $n = 2$ and $V^{2,0} = \mathbb{C}(e_1, e_2)$), one may ask in which horizontal tangent directions (5.3) vanishes. Writing

$$\xi \overset{\text{def}}{=} \xi_0 x^{-\alpha_2} + \xi_1 x^{-\alpha_1 - \alpha_2} + \xi_2 x^{-2\alpha_1 - \alpha_2} + \xi_3 x^{-3\alpha_1 - \alpha_2} \in g^{-1},$$

and $\{e_1^*, e_2^*\}$ for the dual basis of $V^{0,2}$, a straightforward computation (with appropriate normalizations) gives

$$(5.4) \quad e^* [\xi^2]_e = \left( \begin{array}{cc} -2\xi_1 \xi_3 + 2\xi_2^2 & \xi_1 \xi_2 - \xi_0 \xi_3 \\ \xi_1 \xi_2 - \xi_0 \xi_3 & -2\xi_0 \xi_2 + 2\xi_1^2 \end{array} \right).$$

The common vanishing locus of the matrix entries in (5.4) yields a twisted cubic curve $v_3(\mathbb{P}^1) \subset \mathbb{P} g^{-1}$, cf. [25]. Upon varying $o \in D$ (or $\hat{D}$), this recovers the field of cubic cones in the contact planes $g_{\alpha}^{-1} \subset T_o \hat{D}$ preserved by $G_2$, cf. [11].

Furthermore, it is known that the lines $\ell \subset g^{-1} \subset T_o \hat{D}$ in each cubic curve are precisely the lines through $o$ contained in $\hat{D}$, under its minimal homogeneous embedding in $\mathbb{P} g$. To see this, we apply $(\text{ad} \xi)^2$ to the highest weight vector $v = x^{3\alpha_1 + 2\alpha_2} \in g^0$ to check vanishing of the second fundamental form:

$$(\text{ad} \xi)^2 v = (-2\xi_0 \xi_2 + 2\xi_1^2) x^{\alpha_1} + (\xi_0 \xi_3 - \xi_1 \xi_2) x^{3\alpha_1 + 2\alpha_2} + (-2\xi_1 \xi_3 + 2\xi_2^2) x^{-\alpha_1} \in g^0.$$

Hence we get an identification of the variety of horizontal lines $C_o$ (cf. (3.1)) with the “variety of Yukawa vanishing directions”, which provides a homogeneous-space description of the latter and a Hodge–theoretic interpretation of $C_o$. Moreover, the argument produces equations for both as projective varieties in $\mathbb{P} g^{-1}$. The purpose of this section is to discern the degree to which both phenomena generalize.
Remark 5.12 g the simple Lie algebra \( U(5.10) \)

G/P homogeneous embedding of 

\( m \)

Theorem 5.11. Let \( G/P \) \( \hookrightarrow \mathbb{P}V \)

(Section 5.4). The equations cutting out

(5.7a)

\( \sum \in \mathbb{P}(\mathfrak{g}^{-1}) \rightarrow \text{Hom}(U^{m/2}, U^{-m/2}) \),

with \( 0 \leq m \in \mathbb{Z}, U^{m/2} \neq 0 \) and \((U^a)^* \simeq U^{-a}\) as \( g_0 \)-modules.

Remark 5.6. In general, the \( E \)-eigenvalues of \( U \) are rational. However, if \( G \) is \( \mathbb{Q} \)-algebraic and \( E \) is the grading element \( T_\varphi \) associated to a Hodge representation \((G, U_\mathbb{Q}, \varphi)\), cf. [45, Section 2.3], then the \( E \)-eigenvalues of \( U = U_\mathbb{Q} \otimes \mathbb{C} \) is of the form (5.5).

Any element of \( \text{Sym}^m(\mathfrak{g}^{-1}) \) naturally defines a \( G^0 \)-module map \( U^{m/2} \rightarrow U^{-m/2} \). Explicitly, given \( \xi \in \mathfrak{g}^{-1} \) and \( u \in U^{m/2} \), the relation (2.8) implies \( \xi^m(u) \in U^{-m/2} \). Thus, we have a \( G^0 \)-module map

\( \Psi : \text{Sym}^m(\mathfrak{g}^{-1}) \rightarrow \text{Hom}(U^{m/2}, U^{-m/2}) \),

which is the Griffiths–Yukawa coupling from above. Define

(5.7a)

\[ \mathcal{Y}_U \overset{\text{dfn}}{=} \{[\xi] \in \mathbb{P}(\mathfrak{g}^{-1}) \mid \Psi(\xi^m) = 0\} \]

In particular, given \( 0 \neq \xi \in \mathfrak{g}^{-1} \),

(5.7b)

\[ [\xi] \in \mathcal{Y}_U \quad \text{if and only if} \quad \xi^m|_{U^{m/2}} = 0. \]

Observe that \( \mathcal{Y}_U \) is a closed, \( G^0 \)-invariant subvariety of \( \mathbb{P}\mathfrak{g}^{-1} \).

Our first main goal in this section is to understand the relationship between \( \mathcal{Y}_U \) and \( \bar{C}_o \).

Theorem 5.8. If the kernel of the Griffiths–Yukawa coupling is nonempty, then it contains the (tangent directions to) lines through \( o = P/P \) in the minimal homogeneous embedding \( G/P \hookrightarrow \mathbb{P}V \) that are VHS. That is, if \( \mathcal{Y}_U \neq \emptyset \), then

(5.9)

\[ C_o \subset \mathcal{Y}_U. \]

The theorem is proved in Section 5.4. The equations cutting out \( C_o \) and \( \mathcal{Y}_U \) are given in Section 5.2. Example 5.19 is a case in which containment is strict in (5.9). A large class of examples for which equality holds is the following. Take

\[ V = \mathfrak{g}, \]

so that \( \tilde{D} \subset \mathbb{P}\mathfrak{g} \) is the adjoint variety of \( G \) (Section 2.5). The associated grading element is \( E = \sum_{i \in I} S^i \) where the set \( I \) is defined by \( \tilde{\alpha} = \sum_{i \in I} \omega_i \), cf. Table 2.1. Further assume that \( m = 2 \); equivalently, the \( E \)-eigenspace decomposition (5.5) of \( U \) is

(5.10)

\[ U = U^1 \oplus U^0 \oplus U^{-1}. \]

Theorem 5.11. Let \( V = \mathfrak{g} \), so that \( G/P \hookrightarrow \mathbb{P}\mathfrak{g} \) is the adjoint variety of \( G \). Assume this homogeneous embedding of \( G/P \) is minimal. Suppose that \( m = 2 \), so that (5.10) holds. If \( U^1 \) is a faithful representation of the Levi factor \( \mathfrak{g}^0 \), then \( C_o = \mathcal{Y}_U \).

Remark 5.12. The adjoint variety fails to be a minimal homogeneous embedding only for the simple Lie algebra \( \mathfrak{g} = \mathfrak{sp}_{2r} \mathbb{C} \) — it is the second Veronese embedding \( v_2(\mathbb{P}^{2r-1}) \subset \mathbb{P}\text{Sym}^2 \mathbb{C}^{2r} \). As such it contains no lines.
Example 5.17 is a case in which \( m = 2 \) and \( C_o = \mathcal{Y}_U \) despite the fact that the hypotheses of Theorem 5.11 fail.

Theorem 5.11 is proved in Section 5.5. Given the theorem, it is interesting to identify those irreducible representations \( U \) of \( G \) for which (5.10) holds when \( E \) is the grading element associated to an adjoint variety. These representations are listed in

**Lemma 5.13.** Let \( G/P \hookrightarrow \mathbb{P}g \) be an adjoint variety with associated grading element \( E \). Let \( U \) be an irreducible representation of highest weight \( \lambda \). Then \( \lambda(E) = 1 \) if and only if \( \lambda \) is among those listed below.

(a) \( g = \mathfrak{sl}_{r+1}\mathbb{C} \) and \( E = S^1 + S^r \): \( \lambda = \omega_i \), for any \( 1 \leq i \leq r \); in this case \( U = \bigwedge^i \mathbb{C}^{r+1} \).

(b) \( g = \mathfrak{so}_{2r+1}\mathbb{C} \) and \( E = S^2 \): either \( \lambda = \omega_1 \), in which case \( U = \mathbb{C}^{2r+1} \) is the standard representation; or \( \lambda = \omega_r \), in which case \( U \) is the spin representation.

(c) \( g = \mathfrak{sp}_{2r}\mathbb{C} \) and \( E = S^1 \): \( \lambda = \omega_i \), for any \( 1 \leq i \leq r \).

(d) \( g = \mathfrak{so}_{2r}\mathbb{C} \) and \( E = S^2 \): either \( \lambda = \omega_1 \), in which case \( U = \mathbb{C}^{2r} \) is the standard representation; or \( \lambda = \omega_{r-1} \), \( \omega_r \), in which case \( U \) is one of the spin representations.

(e) \( g = \mathfrak{e}_6 \) and \( E = S^2 \): \( \lambda = \omega_1, \omega_6 \).

\( g = \mathfrak{e}_7 \) and \( E = S^1 \): \( \lambda = \omega_7 \).

(In the case that \( g = \mathfrak{e}_8 \) and \( E = S^8 \), we have \( \lambda(E) > 1 \) for all \( \lambda \).)

(f) \( g = \mathfrak{f}_4 \) and \( E = S^1 \): \( \lambda = \omega_4 \).

\( g = \mathfrak{g}_2 \) and \( E = S^2 \): \( \lambda = \omega_1 \) and \( U = \mathbb{C}^7 \) is the standard representation.

In each of these cases, we also have \( \lambda^*(E) = 1 \).

In Section 5.3 we determine to which cases in Lemma 5.13 Theorem 5.11 applies. (By Remark 5.12, the theorem can not be applied to the symplectic adjoint variety.)

**Proof.** As a highest weight \( \lambda \) is of the form \( \lambda = \lambda^i \omega_i \) with \( 0 \leq \lambda^i \omega_i \). The tables of [9] express the \( \omega_i \) as linear combinations of the simple roots \( \alpha_j \). The lemma follows. \( \square \)

5.2. Equations cutting out \( C_o \) and \( \mathcal{Y}_U \). We begin with \( \mathcal{Y}_U \). Let

\[
\Psi^* : \text{Hom}(U^{-m/2}, U^{m/2}) \to \text{Sym}^m(g^{-1})^* 
\]

denote the dual map. To be explicit: let \( \nu \otimes u \in U^{m/2} \otimes (U^{-m/2})^* = \text{Hom}(U^{-m/2}, U^{m/2}) \), then the polynomial \( \Phi^*(\nu \otimes u) \in \text{Sym}^m(g^{-1})^* \) is given by \( \Phi^*(\nu \otimes u)(\xi) = \nu(\xi^m(u)) \), for any \( \xi \in g^{-1} \). The image

\[
\text{im} \Psi^* \subset \text{Sym}^m(g^{-1})^* \text{ is the set of polynomials defining } \mathcal{Y}_U \subset \mathbb{P}g^{-1}.
\]

Turning to \( C_o \), the following lemma characterizes \( C_o \) as the set of lines in \( g^{-1} \) whose iterated action annihilates the highest weight line of \( V \).

**Lemma 5.15.** Let \( G/P \hookrightarrow \mathbb{P}V \) be the minimal homogeneous embedding. Let \( 0 \neq v \in V \) be a highest weight vector. Let \( \xi \in g^{-1} \subset g^{-} \simeq T_o \mathcal{D} \). Then \( [\xi] \in C_o \) if and only if \( \xi^2(v) = 0 \).

Before proving the lemma, we note the following corollary which describes the equations cutting out the lines \( C_o \). Recall the \( E \)-eigenspace decomposition

\[
V = V^h \oplus V^{h-1} \oplus V^{h-2} \oplus \ldots
\]
of Section 2.3.2, and that $V^h$ is one-dimensional. Fix a highest weight vector $0 \neq v \in V^h$. Given $\nu \in (V^h)^*$ define $p_\nu \in \text{Sym}^2(g^{-1})^*$ by $p_\nu(\xi) = \nu(\xi^2 v)$. This defines a $g_0$–module map

$$\Phi^* : (V^h)^* \otimes V^h \rightarrow \text{Sym}^2(g^{-1})^*.$$  

By Lemma 5.15, this map has the property that the image

$$(5.16) \quad \text{im} \Phi^* \subset \text{Sym}^2(g^{-1})^*$$  

is the set of polynomials defining $C_o \subset \mathbb{P}g^{-1}$.

In particular, if $G/P$ is one of the fundamental adjoint varieties, then $C_o$ is the variety of lines through a fixed point $o$ (cf. 3.16); and (5.16) gives equations for $C_o$(12) as a projective variety.

**Proof of Lemma 5.15.** Define $T = \text{span}_C \{v, (g^*)v\} \subset V$. Then $T$ is the embedded tangent space to the cone $C(D) \subset V$ over $D$ at $v$. (Note that $T$ depends only on the highest weight line $o = V^h$, not on our choice of $v \in o$.)

The unique $\mathbb{P}^1 \subset PV$ containing $o$ and satisfying $\xi \in T_0\mathbb{P}^1$ is $\mathbb{P}\text{span}_C \{v, \xi(v)\}$. Since $D$ is cut out by degree two equations [5, Section 2.10], the line $\mathbb{P}^1$ is contained in $D$ if and only if the line osculates with $D$ to second order; equivalently, the second fundamental form $F^2$ vanishes at $\xi$. That is,

$$C_o = \mathbb{P}\{\xi \in g^{-1} \mid F^2(\xi) = 0\}.$$  

Since $F^2(\xi) = \xi^2(v)/T \in V/T$, the vanishing $F^2(\xi) = 0$ is equivalent to $\xi^2(v) \in T$.

If $\xi^2(v) = 0$, then it is immediate that $F^2(\xi) = 0$. Whence, $[\xi] \in C_o$.

Conversely, suppose that $[\xi] \in C_o$. Then $[\xi]$ is necessarily contained in one of the $C_{o,i}$ of (3.7). Since $C_{o,i}$ is the $G^0$–orbit of $g^{-\alpha_i} \in \mathbb{P}g^{-1}$, we may assume without loss of generality that $\xi \in g^{-\alpha_i}$. Let $\bar{\mu}$ denote the highest weight of $V$. Then $\xi^2(v)$ is a weight vector of $V$ for the weight $\bar{\mu} - 2\alpha_i$. Since $[\xi] \in C_o$, and this variety is the zero locus of the second fundamental form, $\xi^2(v)$ is necessarily of the form $av + \zeta(v)$ for some $a \in \mathbb{C}$ and $\zeta \in g^-$. Write $\zeta = \sum \zeta^\beta$ with $\zeta^\beta \in g^{-\beta}$. Each $\zeta^\beta(v)$ is a weight vector of $V$ for the weight $\bar{\mu} - \beta$. Therefore, the condition that $\xi^2(v) = av + \zeta(v)$ be a weight vector for $\bar{\mu} - 2\alpha_i$ forces $a = 0$ and all but at most one $\zeta^\beta(v)$ to vanish. If $\zeta^\beta(v) \neq 0$, then $\bar{\mu} - 2\alpha_i = \bar{\mu} - \beta$ forcing $\beta = 2\alpha_i$.

This is not possible, since $2\alpha_i$ is not a root. Therefore, $\xi^2(v) = 0$. \qed

### 5.3. Examples.

**Example 5.17 (Lemma 5.13(a)).** Let $W = \mathbb{C}^{r+1}$ be the standard representation of $g = \mathfrak{sl}_{r+1}\mathbb{C}$. Let $E = V^1 + V^r$. Then the $E$–eigenspace decomposition of $W$ is

$$W = W^1 \oplus W^0 \oplus W^{-1}, \quad \text{with } \dim_{\mathbb{C}} W^{\pm 1} = 1.$$  

Whence the $E$–eigenspace decomposition

$$U = U^1 \oplus U^0 \oplus U^{-1}$$  

of $U = \Lambda^i W$ is given by

$$U^1 = W^1 \otimes (\Lambda^{i-1}W^0),$$

$$U^0 = (W^1 \otimes (\Lambda^{i-2}W^0) \otimes W^{-1}) \oplus \Lambda^i W^0,$$

$$U^{-1} = (\Lambda^{i-1}W^0) \otimes W^{-1}.$$  

(12) and for $\mathcal{Y}_V$, when Theorem 5.11 applies.
The Levi factor is
\[ g^0 = \mathbb{C}^2 \oplus \mathfrak{sl}_{r-1} \mathbb{C} \]
with \( \mathbb{C}^2 = \text{span}\{ S^1, S^r \} \).

We claim that \( U^1 \) is not a faithful representation of \( g^0 \). To see this, first note that the central element \( S^1 \) acts on \( W^1 \) and \( W^0 \oplus W^{-1} \) by the eigenvalues \( r/(r+1) \) and \( -1/(r+1) \), respectively; the central element \( S^r \) acts on \( W^1 \oplus W^0 \) and \( W^{-1} \) by the eigenvalues \( 1/(r+1) \) and \( -r/(r+1) \), respectively. It follows that \( S^1 \) acts on \( U^1 \) by the eigenvalue \( (r+1-i)/(r+1) \), and \( S^r \) acts by the eigenvalue \( i/(r+1) \). Therefore, \( iS^1 - (r+1-i)S^r \in \mathbb{C}^2 \) acts trivially on \( U^1 \). Whence \( g^0 \) does not act faithfully on \( U^1 \), and we may not apply Theorem 5.11.

Nonetheless, we claim that

\[ (5.18) \quad C_o = \mathcal{Y}_U. \]

To see this, fix a basis \( \{ e_0, e_1, \ldots, e_r \} \) of \( W \) so that
\[ W^1 = \text{span}\{ e_0 \}, \quad W^0 = \text{span}\{ e_1, \ldots, e_{r-1} \} \quad \text{and} \quad W^{-1} = \text{span}\{ e_r \}. \]

Let \( \{ e^0, \ldots, e^r \} \) be the dual basis of \( W^* \), and \( \{ e^i_j = e_j \otimes e^i | 0 \leq i, j \leq r \} \) the induced basis of \( \text{End}(W) \). Then
\[ g^{-1} = \text{span}\{ e^0_a, e^a_{r+1} | 1 \leq a \leq r-1 \} \]
and
\[ U^1 = \text{span}\{ e_0 \wedge e_1 \wedge \cdots \wedge e_{a_i-1} | 1 \leq a_1, \ldots, a_i-1 \leq r-1 \}. \]

The action of \( \xi = \xi^b e^b_0 + \xi^b e^b_{r+1} \in g^{-1} \) on \( u = e_0 \wedge e_1 \wedge \cdots \wedge e_{a_i-1} \in U^1 \) is given by
\[ \xi(u) = \left( \xi^b e_b \wedge e_{a_1} \wedge \cdots \wedge e_{a_{i-1}} - \sum_{s=1}^{i-1} (-1)^s \xi_{a_s} e_0 \wedge e_{r+1} \wedge (e_{a_1} \wedge \cdots \wedge e_{a_s} \cdots \wedge e_{a_{i-1}}) \right) \]
and
\[ \xi^2(u) = 2 \sum_{b \notin \{ a_i \}} \sum_{s=1}^{i-1} \xi^b_{a_s} e_{a_1} \wedge \cdots \wedge e_{a_{i-1}} \wedge e_b \wedge e_{r+1} + (-1)^i \sum_{s=1}^{i-1} \sum_{b \notin \{ a_i \}} \xi^b_{a_s} e_{a_1} \wedge \cdots \wedge e_{a_{i-1}} \wedge e_{r+1} \]
\[ -(-1)^i \sum_{b \notin \{ a_i \}} \xi^b_{a_s} e_{a_1} \wedge \cdots \wedge e_{a_{i-1}} \wedge e_{r+1}. \]

Whence
\[ \mathcal{Y}_U = \{ \xi^a = 0 | 1 \leq a \leq r-1 \} \cup \{ \xi_a = 0 | 1 \leq a \leq r-1 \}. \]

Next, we determine \( C_o \) by applying Lemma 5.15 to the highest root vector \( v = e^r_0 \). We find
\[ \xi^2(v) = \sum_{a=1}^{r-1} \xi^a \xi_a (e^0_0 + e^r_{r+1}) + \sum_{a,b=1}^{r-1} \xi^b \xi_c e^c_b. \]

Whence
\[ C_o = \{ \xi^a = 0 | 1 \leq a \leq r-1 \} \cup \{ \xi_a = 0 | 1 \leq a \leq r-1 \} \]
is a disjoint union $\mathbb{P}^{r-2} \sqcup \mathbb{P}^{r-2}$ of linear spaces, and (5.18) follows.

**Example 5.19** (Lemma 5.13(b,d) with the standard representation). Let $\mathfrak{g} = \mathfrak{so}_m \mathbb{C}$ and $U = \mathbb{C}^m$. Take the parabolic $\mathfrak{p}$ determined by the grading element $E = S^2$, cf. (2.6). Then $\tilde{D} = G/P$ is the adjoint variety. Note that

$\mathfrak{g} \simeq \wedge^2 U,$

as a $\mathfrak{g}$–module. The $E$–eigenspace decomposition of $U$ is

\begin{equation}
U = U^1 \oplus U^0 \oplus U^{-1}.
\end{equation}

Moreover, $U$ is a real Hodge representation with Hodge numbers $h = (2, -m - 4, -2)$.

In this case

\begin{equation}
\mathfrak{g}^0 \simeq \mathbb{C} \oplus \mathfrak{sl}_2 \mathbb{C} \oplus \mathfrak{so}_{m-4} \mathbb{C},
\end{equation}

with center $\mathbb{C} = \mathfrak{c} \{E\}$, and semisimple factor $\mathfrak{g}^0_{ss} = [\mathfrak{g}^0, \mathfrak{g}^0] = \mathfrak{sl}_2 \mathbb{C} \oplus \mathfrak{so}_{m-4} \mathbb{C}$. As a $\mathfrak{g}^0$–module $U^1 \simeq \mathbb{C}^2$. In particular, the factor $\mathfrak{so}_{m-4} \mathbb{C} \subset \mathfrak{g}^0_{ss}$ acts trivially on $U^1$. Therefore $\mathfrak{g}^0$–representation on $U^1$ is not faithful, and we may not apply Theorem 5.11.

We proceed to investigate the question of whether or not equality holds in (5.9) “by hand.” Fix bases $\{e_1, e_2\}$, $\{e_3, \ldots, e_{m-2}\}$ and $\{e_{m-1}, e_m\}$ of $U^1$, $U^0$ and $U^{-1}$, respectively, which satisfy $\nu(e_j, e_k) = 1$ if $j + k = m + 1$ and zero otherwise. Following the standard classification of complex simple Lie algebras, we will assume that

$m \geq 7.$

By definition $[\xi] \in \mathcal{Y}_U$ if and only if $\xi^2|_{U^1} = 0$, cf. (5.7); equivalently,

\begin{equation}
[\xi] \in \mathcal{Y}_U \text{ if and only if } 0 = \xi^2 e_a \text{ for } a = 1, 2.
\end{equation}

At this point, making use of a coarse dimension count, we see that

$\mathcal{C}_o \neq \mathcal{Y}_U.$

Note that $\dim \mathbb{P} \mathfrak{g}^{-1} = 2m - 9$. The characterization (5.22) imposes four constrains on $\mathcal{Y}_U$, implying $\dim \mathcal{Y}_U \geq 2m - 13$. On the other hand, from Table 3.2 we see that $\dim \mathcal{C}_o = m - 7$.

We finish this example by identifying the additional equations that cut $\mathcal{C}_o$ out from $\mathcal{Y}_U$. The highest root line of $V = \mathfrak{g}$ is spanned by

$v = e_1^{m-1} - e_2^m.$

Let $\xi \in \mathfrak{g}^{-1}$. By Lemma 5.15, the point $[\xi] \in \mathbb{P} \mathfrak{g}^{-1}$ lies in $\mathcal{C}_o$ if and only if

\[ 0 = \xi^2(v) = (\xi^2 e_1) \otimes e_1^{m-1} + 2(\xi e_1) \otimes (\xi e_1^{m-1}) + e_1 \otimes (\xi^2 e_1^{m-1}) \]
\[- (\xi^2 e_2) \otimes e_2^m - 2(\xi e_2) \otimes (\xi e_2^m) - e_2 \otimes (\xi^2 e_2^m). \]

Thus,

\begin{equation}
[\xi] \in \mathcal{C}_o \text{ if and only if } \begin{cases} 
0 = \xi^2 e_a, \xi^2 e_s, & a = 1, 2, s = m - 1, m, \\
0 = (\xi e_1) \otimes (\xi e_1^{m-1}) - (\xi e_2) \otimes (\xi e_2^m). 
\end{cases}
\end{equation}
We will see that the four equations \( \xi^2 e^s = 0 \) of (5.23) are redundant – they are satisfied by \( \mathcal{Y}_U \) – and \( C_o \) is cut out from \( \mathcal{Y}_U \) by the \( \frac{1}{2} (m - 4) (m - 5) \) equations \( (\xi e_1) \otimes (\xi e^{m-1}) - (\xi e_2) \otimes (\xi e^m) = 0 \). Write \( \xi \) as

\[
\xi = \xi^j_a (e^j_a - e^{m+1-j}_{m+1-a}) = \xi^{m+1-k}_{m+1-s} (e^{m+1-s}_{m+1-k} - e^k_s).
\]

The sums above are over \( 3 \leq j, k \leq m - 2 \). Then

\[
(5.24) \quad \xi e_a = \xi^j_a e_j = \xi^{m+1-j}_{m+1} e_{m+1-j},
\]

and

\[
(5.25) \quad \xi^2 e_a = - \sum b_j \xi^j_b e_{m+1-b}.
\]

Here the sum is over \( 1 \leq b \leq 2 \) and \( 3 \leq j \leq m - 2 \). Therefore,

\[
(5.26) \quad [\xi] \in \mathcal{Y}_U \quad \text{if and only if} \quad 0 = \sum_j \xi^{m+1-j}_{m+1} e_j \quad \text{for} \quad a = 1, 2.
\]

We compute

\[
(5.27) \quad \xi e^s = - \sum_j \xi^{m+1-j}_{m+1-s} e^j
\]

and

\[
\xi^2 e^s = - \sum_{j,s} \xi^{m+1-j}_{m+1-s} e^j e^b
\]

Substituting \( a \) for \( m + 1 - s \), we see from (5.25) that

\[
(5.28) \quad [\xi] \in \mathcal{Y} \quad \text{implies} \quad \xi^2 e^s = 0.
\]

Finally, from (5.24) and (5.26), we compute

\[
(x_1 e_1) \otimes (x e^{m-1}) - (x_2 e_2) \otimes (x e^m) = \sum_{j,k} (\xi^j_1 \xi^k_2 - \xi^j_2 \xi^k_1) e^{j+1-k}_{j+1-k}.
\]

Whence

\[
(5.29) \quad (x_1 e_1) \otimes (x e^{m-1}) - (x_2 e_2) \otimes (x e^m) = 0 \quad \text{if and only if} \quad \xi^j_1 \xi^k_2 - \xi^j_2 \xi^k_1 = 0
\]

for all \( 3 \leq j, k \leq m - 2 \). It follows from (5.22), (5.23) and (5.27) that (5.28) are the equations cutting \( C_o \) out from \( \mathcal{Y}_U \).

**Example 5.29** (Lemma 5.13(b,d) with a spin representation). We have (5.21). If \( U \) is a spin representation of \( g \), then \( U^1 \) is a spin representation of the factor \( \mathfrak{s}_0 \mathfrak{o} \mathfrak{m} \mathfrak{c} \subset \mathfrak{g}^0 \), while \( \mathfrak{s}_0 \mathfrak{c} \subset \mathfrak{g}^0 \) acts trivially. The latter implies \( U^1 \) is not a faithful \( \mathfrak{g}^0 \) representation – Theorem 5.11 does not apply.

**Example 5.30** (Lemma 5.13(e) with \( g = e_0 \)). Let \( \mathcal{U} \) be the irreducible \( e_0 (\mathbb{C}) \)–representation of highest weight \( \omega_1 \). Then \( \mathcal{U}^* \) is the irreducible representation of highest weight \( \omega_0 \). In particular, \( \mathcal{U} \not\cong \mathcal{U}^* \), so that \( \mathcal{U} \) is complex. Whence there exists a real form \( U_{\mathbb{R}} \) of \( U = \mathcal{U} \oplus \mathcal{U}^* \). Moreover, \( E = S^2 \) induces the structure of a Hodge representation of \( G_{\mathbb{R}} = E \Pi \) on \( U_{\mathbb{R}} \) with Hodge numbers \( h = (12, 30, 12) \).

In this case \( \mathfrak{g}^0 = \mathbb{C} \oplus \mathfrak{s}_0 \mathfrak{g} \mathfrak{c} \); here \( \mathbb{C} = \text{span}\{E\} \) is the center of \( \mathfrak{g}^0 \), and \( \mathfrak{s}_0 \mathfrak{g} \mathfrak{c} \) the semisimple factor \( [\mathfrak{g}^0, \mathfrak{g}^0] \). We have \( U^1 = \mathbb{C}^6 \oplus (\mathbb{C}^6)^* \) as a \( \mathfrak{s}_0 \mathfrak{g} \mathfrak{c} \)–module. This representation is faithful, and Theorem 5.11 applies.
Example 5.31 (Lemma 5.13(e) with \( g = e_7 \)). The irreducible \( g \)-module \( U \) of highest weight \( \omega_7 \) is self-dual and of dimension 56. As a \( g_\mathbb{R} \)-representation \( U \) is quaternionic: there is a real form \( U_\mathbb{R} \) of \( U = U \oplus U^* \) with the property that \( U_\mathbb{R} \) is an irreducible \( g_\mathbb{R} \)-representation. Moreover, the grading element \( E = S^1 \) induces a Hodge structure on \( U_\mathbb{R} \) with Hodge numbers \( h = (24, 64, 24) \).

In this case \( g^0 = \mathbb{C} \oplus \mathfrak{so}_{12} \mathbb{C} \) with center \( C = \text{span}\{E\} \). Additionally, \( U^1 \simeq \mathbb{C}^{12} \oplus \mathbb{C}^{12} \) is two copies of the standard representation of \( g_0^0 = \mathfrak{so}_{12} \mathbb{C} \), and is therefore a faithful \( g^0 \)-representation. Thus Theorem 5.11 applies.

Example 5.32 (Lemma 5.13(f)). The irreducible \( g \)-module \( U \) of highest weight \( \omega_4 \) is self-dual and of dimension 26. As a \( g_\mathbb{R} \)-representation \( U \) is real: there is a real form \( U_\mathbb{R} \) of \( U \). Moreover, the grading element \( E = S^1 \) induces a Hodge structure on \( U_\mathbb{R} \) with Hodge numbers \( h = (6, 14, 6) \).

In this case \( g^0 = \mathbb{C} \oplus \mathfrak{sp}_{6} \mathbb{C} \), and \( U^1 = \mathbb{C}^6 \) is the standard representation of the semisimple factor \( g_0^0 = \mathfrak{sp}_{6} \mathbb{C} \). Whence Theorem 5.11 applies.

Finally, we return to the example motivating this section:

Example 5.33 (Lemma 5.13(g)). Let \( g \) be the exceptional Lie algebra \( g_2 \). Take \( E = S^2 \), so that \( \bar{D} = G/P \) is the adjoint variety, and let \( U = \mathbb{C}^7 \) be the standard representation. The \( E \)-eigenspace decomposition of \( U \) is of the form (5.20), and \( U \) is a real Hodge representation with Hodge numbers \( h = (2, 3, 2) \).

In this case
\[
g^0 = \mathbb{C} \oplus \mathfrak{sl}_2 \mathbb{C},
\]
with \( C = \text{span}\{E\} \) the center and \( g^0_{ss} = [g^0, g^0] = \mathfrak{sl}_2 \mathbb{C} \) the semisimple factor. Moreover, \( U^1 \simeq \mathbb{C}^2 \) as a \( g^0_{ss} \)-representation. Since the center acts on \( U^1 \) by the eigenvalue 1, it follows that \( U^1 \) is a faithful representation of \( g^0 \). Whence Theorem 5.11 yields
\[
C_o = \mathcal{Y}_U.
\]

We finish this example by showing how (5.34) may be obtained (independently of Theorem 5.11) by working with the equations of Section 5.2, but without the explicit computations at the beginning of the section. We anticipate that this approach may be used to determine equality (or its failure) in (5.9) in cases to which Theorem 5.11 does not apply.

The second exterior power of the standard representation decomposes
\[
\wedge^2 U = U \oplus g_2
\]
as a \( g_2 \)-module. Given a basis \( \{e_1, e_2\} \) of \( U^1 \), the product
\[
v = e_1 \wedge e_2
\]
spans the highest weight (root) line of \( g_2 \subset \wedge^2 U \). Given \( \xi \in g^{-1} \), notice that
\[
0 = \xi^2(v) = (\xi^2 e_1) \wedge e_2 + 2(\xi e_1) \wedge (\xi e_2) + e_1 \wedge (\xi^2 e_2).
\]
From (5.7) and Lemma 5.15 we see that
\[
C_o = \{[\xi] \in \mathcal{Y}_U \mid (\xi e_1) \wedge (\xi e_2) = 0\}.
\]
As \( \xi e_1 \) and \( \xi e_2 \) both lie in \( U^0 \) and the later is of dimension three, we see that the condition that \( [\xi] \in \mathcal{Y}_U \) lie in \( C_o \) is at most three additional conditions on \( \xi \). It may be the case that that these conditions are empty: that is, every \( [\xi] \in \mathcal{Y}_U \) satisfies \( (\xi e_1) \wedge (\xi e_2) = 0; \)
equivalently, (5.34) holds. To see that this is indeed the case (without invoking Theorem 5.11) we proceed as follows.

Both \( \mathcal{Y}_U \) and \( \mathcal{C}_o \) are cut out by quadrics; let \( I_2(\mathcal{Y}_U) \) and \( I_2(\mathcal{C}_o) \) be the quadrics vanishing on \( \mathcal{Y}_U \) and \( \mathcal{C}_o \), respectively. Then both \( I_2(\mathcal{Y}_U) \) and \( I_2(\mathcal{C}_o) \) are \( \mathfrak{g}^0 \)-invariant subspaces of \( \text{Sym}^2(\mathfrak{g}^{-1})^* \). We have \( \mathfrak{g}^0 = \mathfrak{gl}_2 \mathbb{C}, \mathfrak{g}^0_{ss} = \mathfrak{sl}_2 \mathbb{C} \) and \( \mathfrak{g}^{-1} = \text{Sym}^3 \mathbb{C}^2 \) as a \( \mathfrak{g}^0_{ss} \)-module. Whence
\[
\text{Sym}^2(\mathfrak{g}^{-1})^* = \text{Sym}^2 \mathbb{C}^2 \oplus \text{Sym}^6 \mathbb{C}^2.
\]

If \( \mathcal{C}_o \subseteq \mathcal{Y}_U \), then \( 0 \neq I_2(\mathcal{Y}_U) \subseteq I_2(\mathcal{C}_o) \). Given (5.35) this forces \( I_2(\mathcal{C}_o) = \text{Sym}^2(\mathfrak{g}^{-1})^* \); equivalently, \( \mathcal{C}_o = \emptyset \), a contradiction. This establishes (5.34).

5.4. **Proof of Theorem 5.8.** Let \( I = \{i_1, \ldots, i_t\} \) be the index set (2.1) associated with the parabolic subalgebra \( \mathfrak{p} \). The proof will proceed in two steps. We begin with the case that \( \mathfrak{p} \) is a maximal parabolic (equivalently \( |I| = 1 \)); this forms the basis for the general argument.

**Lemma 5.36.** Suppose that \( \tilde{D} = G/P \) and that \( P \) is a maximal parabolic. If \( \mathcal{Y}_U \neq \emptyset \), then \( \mathcal{Y}_U \) is connected and
\[
\mathcal{C}_o \subseteq \mathcal{Y}_U.
\]

**Proof.** The parabolic \( P \) is maximal if and only if \( \mathfrak{g}^{-1} \) is an irreducible \( \mathfrak{g}^0 \)-module. In this case, \( \mathcal{C}_o \subset \mathbb{P}\mathfrak{g}^{-1} \) is the unique closed \( G^0 \)-orbit (Section 3.1). It is clear from the definition (5.7) that \( \mathcal{Y}_U \) is closed and preserved under the action of \( G^0 \). Therefore, each connected component of \( \mathcal{Y}_U \) contains a closed \( G^0 \)-orbit. \( \square \)

We now turn to the case that \( |I| > 1 \). First, we review the relationship between the lines on \( G/P \hookrightarrow \mathbb{P}\mathfrak{V} \) and \( G/P_i \hookrightarrow \mathbb{P}\mathfrak{V}_{\omega_i}, i \in I \). The lines \( \mathcal{C}_{o,i} \) on the former are described in Section 3.2. For the latter, given \( i \in I \), let \( P_i \subset G \) be the maximal parabolic associated with the index set \( \{i\} \). Let \( \tilde{D}_i \subset \mathbb{P}\mathfrak{V}_{\omega_i} \) be the image of the minimal homogeneous embedding \( G/P_i \hookrightarrow \mathbb{P}\mathfrak{V}_{\omega_i} \), and let \( \mathcal{C}_{o,i}(\tilde{D}_i) \) be the lines on \( \tilde{D}_i \) passing through \( o_i = P_i/P_i \). Then the \( \mathcal{C}_{o,i} \) component of (3.5) is
\[
(5.37) \quad \mathcal{C}_{o,i} = \{ [\xi] \in \mathbb{P}\mathfrak{g}^{-1} \mid [\xi] \in \mathcal{C}_{o,i}(\tilde{D}_i), \ [\xi] \in \mathfrak{p}_j \ \forall j \in I \setminus \{i\} \}.
\]

Recall that \( \mathfrak{E} = \sum_{i \in I} \mathfrak{s}_i \), and \( U^{m/2} \) is the eigenspace of the largest \( \mathfrak{E} \)-eigenvalue \( m/2 \) on \( U \). Let \( m_i/2 \) be the largest \( \mathfrak{s}_i \)-eigenvalue of \( U \), and let \( \mathcal{U}_i \subset U \) be the corresponding eigenspace. Then
\[
m = \sum_{i \in I} m_i \quad \text{and} \quad U^{m/2} = \bigcap_{i \in I} \mathcal{U}_i.
\]

Let \( [\xi] \in \mathcal{C}_{o,i}(\tilde{D}_i) \). Then (5.7) and Lemma (5.36) imply that \( \xi^{m_i}|_{\mathcal{U}_i} = 0 \). Since \( m_i \leq m \) and \( U^{m/2} \subset \mathcal{U}_i \), it follows that \( \xi^m|_{U^{m/2}} = 0 \). Theorem 5.8 now follows from (3.5), (5.7) and (5.37).

5.5. **Proof of Theorem 5.11.** Given Theorem 5.8 it suffices to show that
\[
(5.38) \quad \mathcal{Y}_U \subset \mathcal{C}_o.
\]

To that end, let \( [\xi] \in \mathcal{Y}_U \). Equivalently,
\[
(5.39) \quad \xi^2(u) = 0 \quad \text{for all} \quad u \in U^1.
\]
Fix a highest root vector $0 \neq v \in \mathfrak{g}^\alpha$. Given Lemma 5.15, to establish (5.38) it suffices to show that
\begin{equation}
\text{ad}_\xi^2(v) = 0.
\end{equation}

As an operator on $U$
\begin{equation*}
\text{ad}_\xi^2(v) = \xi v - 2v \xi v + v \xi v.
\end{equation*}
As an element of $\mathfrak{g}^{-1}$, the endomorphism $\xi$ lowers eigenvalues by one; that is $\xi$ maps $U^a$ into $U^{a-1}$. Likewise, as an element of $\mathfrak{g}^2 = \mathfrak{g}^\alpha$, the root vector $v$ maps $U^a$ into $U^{a+2}$. It then follows from (5.10) that
\begin{equation}
\text{ad}_\xi^2(v) \mid_{U^1} = \nu \xi \xi \mid_{U^1} \overset{(5.39)}{=} 0.
\end{equation}
On the other hand $\text{ad}_\xi^2(v) \in \mathfrak{g}^0$. By assumption $U^1$ is a faithful representation of $\mathfrak{g}^0$. So (5.41) holds if and only if (5.40) holds.

6. **Construction of limiting mixed Hodge structures**

6.1. **Overview.** In this section we construct a large class of polarized $G_\mathbb{R}$-orbits; each orbit will contain the image of a naive limit mapping $\Phi_\infty : B(N) \to \bar{D}$, where $N$ is an element of a nilpotent cone $\sigma$ (cf. Section 7.1).

Fix a rational homogeneous variety $\bar{D} = G/P$ (Section 2.1), and let $E$ be the associated grading element (2.7). Associated with $E$ is a rational form $\mathfrak{g}_Q$ of $\mathfrak{g}$ (Section 6.3). The rational form is equipped with a Hodge structure that realizes an open $G_\mathbb{R}$-orbit $D \subset \bar{D}$ as a Mumford–Tate domain (Section 6.4). We will construct a sequence of $G_\mathbb{R}$-orbits $\{O_j\}_{j=0}^s$ in $\bar{D}$ such that $O_0$ is the open orbit $D$ and
\begin{equation}
O_{j+1} \subset \partial O_j.
\end{equation}
We will see that these orbits are rational, polarized and (weakly) cuspidal in the sense of [32] (Remarks 6.40 and 6.41). The orbits arise as follows. We begin with the identity coset $o_0 = P/P \in \bar{D}$. By definition $O_0 = D$ is the $G_\mathbb{R}$-orbit of $o_0$, cf. Section 6.4. We will then define $o_j = g_j(o_0)$, where $g_j \in G$ and $c_j = \text{Ad} g_j$ is a composition of Cayley transforms (Section 6.6). These Cayley transforms are determined by a suitable set (6.29) of strongly orthogonal noncompact roots.

**Remark 6.2 (Closed orbits).** In order for $O_s$ to be the unique closed orbit, it is necessary (but not sufficient) that $s$ be the real rank of $\mathfrak{g}_R$.

**Remark 6.3 (Maximal parabolics).** In the case that $P$ is maximal (Section 6.9) we will see that, with a few exceptions, there exists a sequence $\{O_j\}_{j=0}^s$ with $s$ equal the real rank of $\mathfrak{g}_\mathbb{R}$ (Lemma 6.60).

The construction is given in Section 6.8. It is first necessary to review the representation theory underlying the construction.
6.2. **The elements** $H^\alpha \in h$. Given a root $\alpha$, define $H^\alpha \in h$ by $H^\alpha \in [g^\alpha, g^{-\alpha}]$ and $\alpha(H^\alpha) = 2$. Then

\begin{equation}
2 \left( \frac{\beta}{\alpha, \alpha} \right) = \beta(H^\alpha) \in \mathbb{Z} \quad \text{for all } \alpha, \beta \in \Delta,
\end{equation}

cf. [23]. (13) In particular, $H^\alpha$ is a grading element. Indeed, if $0 \leq m_\alpha, n_\alpha \in \mathbb{Z}$ are defined by the conditions that

$$ \beta - m_\alpha \alpha, \ldots, \beta + n_\alpha \alpha$$

is the $\alpha$–string through $\beta$, then

\begin{equation}
r_\alpha(\beta) = \beta - \beta(H^\alpha) \alpha.
\end{equation}

6.3. **A rational form.** Given a grading element $T$, we may define an integral structure on $g$ as follows.

Fix a Cartan subalgebra $h \subset g$.

The $\{H^\alpha_i \mid 1 \leq i \leq r\}$ span $h$. Complete this to a Chevalley basis

$$ \{x^\alpha \mid \alpha \in \Delta(h)\} \cup \{H^{\alpha_1}, \ldots, H^{\alpha_r}\}$$

of $g$, cf. [28, Section 25.2]. That is, $x^\alpha \in g^\alpha$ and the following properties hold: the Killing form satisfies $(H^\alpha, H^\beta) > 0$ and $(x^\alpha, x^{-\alpha}) > 0$ for all $\alpha \in \Delta$; and the Lie bracket satisfies

- $[x^\alpha, x^{-\alpha}] = H^\alpha \in \text{span}_\mathbb{Z}\{H^{\alpha_1}, \ldots, H^{\alpha_r}\}$, for all $\alpha \in \Delta(h)$,
- $[H^\beta, x^\alpha] \in \mathbb{Z} x^\alpha$, for all $\alpha, \beta \in \Delta(h)$,
- if $\alpha, \beta, \alpha + \beta \in \Delta$ and $[x_\alpha, x_\beta] = c_{\alpha, \beta} x^{\alpha+\beta}$, then $c_{-\alpha, -\beta} = -c_{\alpha, \beta} \in \mathbb{Z}$.

We emphasize that the structure coefficients with respect to the Chevalley basis are all integers. Fix $i \overset{\text{dfn}}{=} \sqrt{-1}$, and set

$$ h^j \overset{\text{dfn}}{=} i H^{\alpha_j}, $$

$$ u^\alpha \overset{\text{dfn}}{=} \begin{cases} x^\alpha - x^{-\alpha} & \text{if } \alpha(T) \text{ is even}, \\ i(x^\alpha - x^{-\alpha}) & \text{if } \alpha(T) \text{ is odd}; \end{cases} $$

$$ v^\alpha \overset{\text{dfn}}{=} \begin{cases} i(x^\alpha + x^{-\alpha}) & \text{if } \alpha(T) \text{ is even}, \\ x^\alpha + x^{-\alpha} & \text{if } \alpha(T) \text{ is odd}. \end{cases} $$

Define

\begin{equation}
(6.6) \quad g_Z = \ell_Z \oplus \ell_Z^\perp
\end{equation}

---

(13) The reader consulting other references should beware that the $H_\alpha$ of [16, 34] is not our $H^\alpha$. To be precise, $H^\alpha = \frac{2}{(\alpha, \alpha)} H_\alpha$ is the $Z_\alpha$ of [16] and the $H'_\alpha$ of [34]. However, our $H^\alpha$ is the $H_\alpha$ of [23], and the $h_\alpha$ of [28]. (Ha ha!)
by
\[(6.7)\]
\[
\mathfrak{t}_Z \overset{\text{dfn}}{=} \text{span}_\mathbb{Z}\{h^j \mid 1 \leq j \leq r\} \cup \text{span}_\mathbb{Z}\{u^\alpha, v^\alpha \mid \alpha(T) \text{ even}\},
\]
\[
\mathfrak{t}_Z^\perp \overset{\text{dfn}}{=} \text{span}_\mathbb{Z}\{u^\alpha, v^\alpha \mid \alpha(T) \text{ odd}\}.
\]

It is a straightforward exercise to confirm that \([\mathfrak{g}_Z, \mathfrak{g}_Z] \subset \mathfrak{g}_Z\) and \(\mathfrak{g}_Z = \mathfrak{g}_Z \otimes \mathbb{C}\); that is, \(\mathfrak{g}_Z\) is an integral form of \(\mathfrak{g}\). It follows that the Killing form \(B : \mathfrak{g}_Z \times \mathfrak{g}_Z \to \mathbb{Z}\) is defined over \(\mathbb{Z}\).

Likewise, \([\mathfrak{k}_Z, \mathfrak{k}_Z] \subset \mathfrak{k}_Z\), \([\mathfrak{k}_Z, \mathfrak{k}_Z^\perp] \subset \mathfrak{k}_Z^\perp\), and one may confirm that the Killing form is negative definite on \(\mathfrak{k}_Z\) and positive definite on \(\mathfrak{k}_Z^\perp\), so that (6.6) is a Cartan decomposition. Let \(\theta : \mathfrak{g}_Z \to \mathfrak{g}_Z\) denote the corresponding Cartan involution; that is, \((6.8)\)
\[\theta|_{\mathfrak{t}_Z} = 1 \quad \text{and} \quad \theta|_{\mathfrak{t}_Z^\perp} = -1.\]

**Definition 6.9.** Let \(G_\mathbb{Q}\) be the \(\mathbb{Q}\)-form of \(G_\mathbb{R}\) with Lie algebra \(\mathfrak{g}_\mathbb{Q} \overset{\text{dfn}}{=} \mathfrak{g}_\mathbb{Z} \otimes \mathbb{Q}\). When a \(\mathbb{Q}\)-algebraic group arises in this way (from the above construction), we will call it a Mumford–Tate–Chevalley (MTC) group.

**Remark 6.10.** In general, if \(G_\mathbb{Q}\) is a (\(\mathbb{Q}\)-algebraic) Mumford–Tate group of a \(\mathbb{Q}\)-Hodge structure underlying a semisimple real Lie group \(G_\mathbb{R}\), it need not be MTC. Indeed, the above construction implies that MTC groups are split over \(\mathbb{Q}(i)\). Unlike \(\mathbb{Q}\)-Chevalley groups, they need not be split over \(\mathbb{Q}\); for instance, apply the construction to \(\text{SU}(2,1)\). However, we do have that \(\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}\) is defined over \(\mathbb{Q}\) for each \(\alpha \in \Delta\).

A Cartan subalgebra of \(\mathfrak{g}_\mathbb{R}\) is given (over \(\mathbb{Z}\)) by
\[(6.11)\]
\[\mathfrak{h}_Z \overset{\text{dfn}}{=} \mathfrak{h} \cap \mathfrak{g}_Z = \text{span}_\mathbb{Z}\{iH^1, \ldots, iH^r\} \subset \mathfrak{t}_Z.\]

From \(\alpha_i(S^j) = \delta^j_i\) and \(\alpha_i(H^j) \in \mathbb{Z}\), we see that the \(S^i\) are \(\mathbb{Q}\)-linear combinations of the \(H^j\). Therefore, the \(iS^i\) are defined over \(\mathbb{Q}\). In particular, the rational form is
\[(6.12)\]
\[\mathfrak{h}_\mathbb{Q} = \text{span}_\mathbb{Q}\{iS^1, \ldots, iS^r\} \subset \mathfrak{t}_\mathbb{Q}.\]

Note that the real form
\[t \overset{\text{dfn}}{=} \mathfrak{h}_\mathbb{R} \quad \text{is compact.}\]

Let \(G_\mathbb{R} \subset G\) be the real form of \(G\) with Lie algebra \(\mathfrak{g}_\mathbb{R}\). For later use, we introduce the subalgebra
\[(6.13)\]
\[\mathfrak{sl}_2^\alpha(\mathbb{Z}) \overset{\text{dfn}}{=} \text{span}_\mathbb{Z}\{u^\alpha, iH^\alpha, v^\alpha\} \subset \mathfrak{g}_Z.\]

Let
\[(6.14)\]
\[\text{SL}_2^\alpha(\mathbb{R}) \subset G_\mathbb{R} \quad \text{and} \quad \text{SL}_2^\alpha(\mathbb{C}) \subset G\]
be the connected Lie subgroups with Lie algebras \(\mathfrak{sl}_2^\alpha(\mathbb{R})\) and \(\mathfrak{sl}_2^\alpha(\mathbb{C})\), respectively.

\(^{14}\)Of course, the construction could be modified to replace \(\mathbb{Q}(i)\) by any imaginary quadratic field.
Example 6.15 ($\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$). Consider the case that $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$, and let $T = S^1$. We may take the Chevalley basis to be
\[
x^\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H^\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x^{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]
Then the rational form (6.6) is spanned by
\[
u^\alpha = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad h^1 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad v^a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]
Observe that the real form defined by (6.6) is $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(1,1) = \{ X \in \mathfrak{sl}_2\mathbb{C} \mid X^TQ + QX = 0 \}$, where $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This is the Lie algebra of the group
\[G_{\mathbb{R}} = \text{SU}(1,1) = \{ A \in \text{SL}_2\mathbb{C} \mid A^*QA = Q \}.
\]

6.4. Hodge structure on $\mathfrak{g}_{\mathbb{Z}}$. If $\mathfrak{g} = \oplus \mathfrak{g}^p$ is the $T$–eigenspace decomposition (2.4), then (6.7) yields
\[
\mathfrak{t} \overset{\text{dfn}}{=} \mathfrak{t}_{\mathbb{Z}} \otimes \mathbb{C} = \mathfrak{g}^{\text{even}} \quad \text{and} \quad \mathfrak{t}^\perp \overset{\text{dfn}}{=} \mathfrak{k}_{\mathbb{Z}} \otimes \mathbb{C} = \mathfrak{g}^{\text{odd}}.
\]
Since $-(u, \theta \bar{u}) > 0$ for all $0 \neq u \in \mathfrak{g}$, it follows that $\mathfrak{g}^p - p = \mathfrak{g}^p$ defines a weight zero, $-\langle \cdot, \cdot \rangle$–polarized Hodge structure on $\mathfrak{g}$. (We regard $T$ as an infinitesimal Hodge structure; it is the (rescaled) derivative of a Hodge structure $\varphi_0 : S^1 \to \text{Ad}(G_{\mathbb{R}})$, cf. [45, Section 2.3].) The corresponding Hodge flag $F^\bullet$ is given by
\[
F^p = \bigoplus_{q \geq p} \mathfrak{g}^{q-p}.
\]
Note that $P = \text{Stab}_G(F^\bullet)$. This allows us to identify with the flag $F^\bullet$ with the point $o_0 \overset{\text{dfn}}{=} P/P \subseteq G/P$.

Define
\[
\bar{D} \overset{\text{dfn}}{=} G/P,
\]
and let
\[
D \overset{\text{dfn}}{=} G_{\mathbb{R}} \cdot o_0.
\]
Then $D$ is open in $\bar{D}$, and therefore a flag domain (Section 2.2).

Definition 6.17. A Mumford–Tate domain arising in this way is called a Mumford–Tate–Chevalley (MTC) domain.

Remark 6.18. From the Hodge–theoretic perspective, $D$ is the Mumford–Tate domain parameterizing weight zero Hodge structures on $\mathfrak{g}_{\mathbb{Q}}$ that are polarized by $-\langle \cdot, \cdot \rangle$ and have Mumford–Tate group contained in $G_{\mathbb{Q}}$. By construction, this $G_{\mathbb{Q}}$ is a MTC group. Moreover, the grading element $T$ associated with $o_0$ is defined over $\mathbb{Q}(i)$ and is purely imaginary. The Mumford–Tate group of $o_0$ is therefore a 1-torus with real points $\varphi_0(S^1)$.
In general, a Mumford–Tate domain (determined by Hodge–theoretic data) which is isomorphic to $D$ as a $G_\mathbb{R}$-homogeneous space, need not be MTC: this is a property which reflects the arithmetic of the $\mathbb{Q}$-Hodge tensors. The MTC case may be thought of as “maximizing” the density of Mumford–Tate subdomains in $D$.

Example 6.19 ($\mathfrak{g} = \mathfrak{sl}_2 \mathbb{C}$). This is a continuation of Example 6.15. We have $G = \text{SL}_2 \mathbb{C}$ and fix the parabolic subgroup
\[ P = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \bigg| a, b \in \mathbb{C}, \ a \neq 0 \right\}. \]

Let $\tilde{D}$ be the complex projective line $\mathbb{CP}^1$. Then the identity coset $P/P$ corresponds to the point $o_0 = (1 : 0) \in \mathbb{CP}^1$.

The $G_\mathbb{R} = \text{SU}(1,1)$–orbit
\[ D = \{(1 : z) \in \mathbb{CP}^1 | |z| < 1\} \]
of $o_0$ is naturally identified with the unit disc in $\mathbb{C}$.

6.5. Standard triples. Let $\mathfrak{g}_k$ be a Lie algebra defined over a field $k$. A standard triple in $\mathfrak{g}_k$ is a set of three elements \( \{N^+, Y, N\} \subset \mathfrak{g}_k \) such that
\[
[Y, N^+] = 2N^+,
[N^+, N] = Y \quad \text{and} \quad [Y, N] = -2N.
\]
Note that \( \{N^+, Y, N\} \) span a three–dimensional semisimple subalgebra (TDS), necessarily isomorphic to $\mathfrak{sl}_2$. We call $Y$ the neutral element, $N$ the nilnegative element and $N^+$ the nilpositive element, respectively, of the standard triple.

Example 6.20. The matrices
\[
N^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]
form a standard triple in $\mathfrak{sl}_2 \mathbb{R}$; while the matrices
\[
N^+ = \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{and} \quad N = \frac{1}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}
\]
form a standard triple in $\mathfrak{su}(1,1)$.

6.6. Cayley transforms. This is a brief review of Cayley transforms; see [34, Chapter VI.7] for details.

Let $\mathfrak{g}_\mathbb{R} = \mathfrak{g}_\mathbb{Z} \otimes \mathbb{R}$ be the real form of $\mathfrak{g}$ constructed in Section 6.3. By (6.12), all roots are imaginary. Fix a noncompact positive root $\alpha$. That is, $\mathfrak{g}^\alpha \subset \mathfrak{k}^\perp$; equivalently, $\alpha(\mathfrak{E})$ is odd. Then
\[
\overline{x^\alpha} = x^{-\alpha}.
\]
The Cayley transform associated to $\alpha$ is
\[
c_\alpha \overset{\text{df}}{=} \text{Ad} \left( \exp \frac{\alpha}{2}(x^{-\alpha} - x^\alpha) \right) \in \text{Ad}(G).
\]

Since $\text{Ad}(g)$ is a Lie algebra automorphism for any $g \in G$, it follows that $c_\alpha$ is a Lie algebra automorphism. Therefore,
\[
\mathfrak{h}' = c(\mathfrak{h})
\]
is a $\theta$–stable Cartan subalgebra. The root spaces of $\mathfrak{h}'$ are the Cayley transforms $'g^\alpha = c(g^\alpha)$ of the root spaces of $\mathfrak{h}$. Likewise, $'H^\alpha = c(H^\alpha)$. In particular,

$$'H^\alpha = c(H^\alpha) = x^\alpha + x^{-\alpha} = v^\alpha \in \mathfrak{t}^1_Z.$$  

The real form $'h^R = h' \cap g^R$ of $h'$ is

$$h^R \overset{\text{dfn}}{=} \ker \{ \alpha : t \to \mathfrak{i} \mathbb{R} \} \oplus \operatorname{span}_R \{'H^\alpha\}.$$ 

Additionally, we have

$$c^\alpha(x^\alpha - x^{-\alpha}) = x^\alpha - x^{-\alpha} \text{ and } c^\alpha(x^\alpha + x^{-\alpha}) = -H^\alpha.$$ 

Therefore, the root space $'g^{-\alpha} = c(g^{-\alpha})$ is spanned by

$$y^{-\alpha} \overset{\text{dfn}}{=} i c^\alpha(x^{-\alpha}) = \frac{i}{2}(x^{-\alpha} - x^\alpha - H^\alpha) \in g_Q.$$ 

Likewise, $'g^\alpha$ is spanned by

$$y^\alpha = -i c^\alpha(x^\alpha) = \frac{i}{2}(x^{-\alpha} - x^\alpha + H^\alpha) \in g_Q.$$ 

From $[x^\alpha, x^{-\alpha}] = H^\alpha$ and $c^\alpha(H^\alpha) = [c^\alpha(x^\alpha), c^\alpha(x^{-\alpha})]$, we see that

$$\{y^\alpha, 'H^\alpha, y^{-\alpha}\}$$ is a standard triple defined over $\mathbb{Q}$, cf. Section 6.5. Recall (6.13) and note that

$$\mathfrak{sl}_2^0(\mathbb{Q}) = \operatorname{span}_Q \{y^\alpha, 'H^\alpha, y^{-\alpha}\}.$$ 

Example 6.28 ($\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$). This is a continuation of Examples 6.15 and 6.19. We have

$$c^\alpha = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc} 1 & i & 1 \\ i & 1 & i \\ 1 & i & 1 \end{array} \right),$$

and

$$y^\alpha = \frac{i}{2} \left( \begin{array}{ccc} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{array} \right), \quad 'H^\alpha = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right), \quad y^{-\alpha} = \frac{i}{2} \left( \begin{array}{ccc} -1 & -1 & -1 \\ 1 & 1 & 1 \end{array} \right).$$

6.7. The weight zero $\alpha$–primitive subspace. Let

$$\Gamma^0_\alpha \overset{\text{dfn}}{=} \{ \zeta \in \mathfrak{g} | [\mathfrak{sl}_2^0, \zeta] = 0 \}$$

be the trivial isotypic component in the $\mathfrak{sl}_2^0$–module decomposition of $\mathfrak{g}$; this is the weight zero, $\alpha$–primitive subspace of $\mathfrak{g}$. Note that

*the Cayley transform $c_\alpha$ restricts to the identity on $\Gamma^0_\alpha$.*

The Jacobi identity implies $\Gamma^0_\alpha$ is a subalgebra of $\mathfrak{g}$. More precisely,

$$\Gamma^0_\alpha = \ker \{ \alpha |_\mathfrak{h} \} \oplus \bigoplus_{\beta \in \Delta(\Gamma^0_\alpha)} \mathfrak{g}^\beta,$$

where

$$\Delta(\Gamma^0_\alpha) = \{ \beta \in \Delta | \beta \pm \alpha \notin \Delta \}$$

is the set of roots that are strongly orthogonal to $\alpha$. In particular,

$$\Gamma^0_\alpha$$ is a reductive subalgebra defined over $\mathbb{Q}$. 

Let
\[ \Gamma_\alpha \overset{\text{dfn}}{=} [\Gamma_0^\alpha, \Gamma_0^\alpha] \]
be the semisimple component, and let \( \mathcal{G} \subset G \) be the connected Lie subgroup with Lie algebra \( \Gamma_\alpha \). Note that \( \mathcal{P} = \mathcal{G} \cap P \) is a parabolic subgroup, and let
\[ \mathcal{D} \overset{\text{dfn}}{=} \mathcal{G}/\mathcal{P} \]
be the associated rational homogeneous variety. Let \( o = \mathcal{P}/\mathcal{P} \in \mathcal{D} \) be the identity coset. The real form \( \Gamma_\alpha(\mathbb{R}) \cap h \subset \ker\{\alpha|_1\} \) of the Cartan subalgebra is contained in \( t \), and therefore is compact. Whence,

\[ \text{the } \mathcal{G}(\mathbb{R})-\text{orbit } D \text{ of } o \text{ is open in } \mathcal{D}. \]

6.8. **The construction.** Recall the Cartan subalgebra \( h \subset g \) with compact real form \( t \) of Section 6.3. Fix a set
\[ (6.29a) \quad \mathcal{B} = \{\beta_1, \ldots, \beta_s\} \subset h^* \]
of strongly orthogonal noncompact roots with the property that
\[ (6.29b) \quad \beta_j(E) = 1. \]
Given \( 1 \leq j \leq s \), let \( c_{\beta_j} \) be the Cayley transformation associated to \( \beta_j \) (Section 6.6). Set
\[ c_j \overset{\text{dfn}}{=} c_{\beta_j} \circ \cdots \circ c_{\beta_1}. \]
(Strong orthogonality implies that the roots \( \{\beta_{j+1}, \ldots, \beta_s\} \) are still imaginary and noncompact with respect to \( h_j \). Whence, \( c_j \) is well-defined.) By \( (6.23) \), the Cayley transform \( c_\alpha \) may be written as \( c_\alpha = \text{Ad}_{g_\alpha} \) where \( g_\alpha = \exp\pi^4(x^{-\alpha} - x_\alpha) \in G \). Let \( g_j = g_{\beta_j} \cdots g_{\beta_1} \), so that \( c_j = \text{Ad}_{g_j} \). It will be convenient to also set \( g_0 = 1 \) and \( c_0 = \text{Ad}_1 = 1 \).

Recall the weight zero polarized Hodge structure of Section 6.4. This Hodge structure is naturally identified with the identity coset \( o_0 = P/P \in \mathcal{D} \). Let \( O_0 = D \) be the the \( G_{\mathbb{R}} \)-orbit of \( o_0 \). Given \( 1 \leq j \leq s \), set
\[ o_j \overset{\text{dfn}}{=} g_j(o_0) \in \mathcal{D}, \]
and let \( O_j \subset \mathcal{D} \) be the \( G_{\mathbb{R}} \)-orbit of \( o_j \). The containment \( (6.1) \) follows from this construction, cf. \[32\].

**Remark 6.30** (Hermitian symmetric case). Note that \( c_j \) is the Cayley transform \( c_0 \) of \[3, \text{ Section 1.6}\]. Therefore, in the case that \( D \) is Hermitian symmetric, the \( O_j \) account for all the \( G_{\mathbb{R}} \)-orbits in the boundary of \( D \). That is,
\[ \overline{D} = \bigcup_j O_j. \]

**Example 6.31** \( (\mathfrak{g} = \mathfrak{sl}_2 \mathbb{C}) \). This is a continuation of Examples 6.15, 6.19 and 6.28. We have
\[ o_1 = (1 : i) \in \mathbb{C}P^1, \]
and
\[ O_1 = \{(1 : z) \mid |z| = 1\} \]
is naturally identified with the unit circle in \( \mathbb{C} \). If we define
\[ c_\alpha(t) \overset{\text{dfn}}{=} \text{Ad} \left( \exp \frac{\pi^4}{4} (x^{-\alpha} - x_\alpha) \right) = \begin{pmatrix} \cos \frac{\pi^4}{4} t & i \sin \frac{\pi^4}{4} t \\ i \sin \frac{\pi^4}{4} t & \cos \frac{\pi^4}{4} t \end{pmatrix}, \]
then the Cayley transform is \( c_\alpha = c_\alpha(1) \). Moreover, the curve \( c_\alpha(t) \cdot o_0 \) is contained in the unit disc \( \mathcal{O}_0 \) for all \( 0 \leq t < 1 \).

Associated with the Matsuki point \( o_j \) are two filtrations (6.37) of \( g \), which we now describe. Set \( h_0 = h \) and define

\[
   h_j \overset{\text{dfn}}{=} c_j(h).
\]

Then \( \dim \mathbb{R}(h_j \cap \mathfrak{t}_{\mathbb{R}}^{\perp}) = j \). Thus

\[
   (6.32) \quad s \leq \text{rank}_\mathbb{R} \mathfrak{g}_\mathbb{R}.
\]

Recall the elements \( H^\sigma \subset h_j \) of Section 6.2,\(^{15}\) and define

\[
   (6.33) \quad Y_j \overset{\text{dfn}}{=} H^{3j_1} + \ldots + H^{3j_s}.
\]

Let \( g = \oplus_j g^p \) be the \( E \)-eigenspace\(^{16}\) decomposition (2.4), and let \( g = \oplus_j g_\ell \) be the \( Y_j \)-eigenspace decomposition. That is,

\[
   (6.34) \quad jg^p \overset{\text{dfn}}{=} \{ \xi \in g \mid [E, \xi] = p \xi \},
\]

\[
   (6.35) \quad jg_\ell \overset{\text{dfn}}{=} \{ \xi \in g \mid [Y_j, \xi] = \ell \xi \}.
\]

By (6.4), the \( Y_j \)-eigenvalues are integers. As commuting semisimple endomorphisms, \( E \) and \( Y_j \) are simultaneously diagonalizable; therefore,

\[
   (6.36a) \quad jg^{p,q} \overset{\text{dfn}}{=} jg^p \cap jg^{p+q} = \{ \xi \in g \mid [E, \xi] = p\xi, [Y_j, \xi] = (p + q)\xi \}
\]

defines a direct sum decomposition

\[
   (6.36b) \quad g = \bigoplus_j jg^{p,q}.
\]

By the Jacobi identity, this is a bigraded Lie algebra decomposition; that is,

\[
   \left[ jg^{p,q}, jg^{r,s} \right] \subset jg^{p+r,q+s}.
\]

Define filtrations

\[
   (6.37) \quad jF^p \overset{\text{dfn}}{=} \bigoplus_{q \geq p} jg^q \quad \text{and} \quad jW_\ell \overset{\text{dfn}}{=} \bigoplus_{p+q \leq \ell} jg^{p,q} = \bigoplus_{k \leq \ell} jg_k.
\]

Note that \( \text{Stab}_G(jF^\bullet) = g_jPg_j^{-1} = \text{Stab}_G(o_j) \), so that there is a natural identification of the flag \( jF^\bullet \) with the Matsuki point \( o_j \in \tilde{D} \).

**Theorem 6.38.** For each \( 0 \leq j \leq s \), there exists a \( j \)-dimensional, rational nilpotent cone \( \sigma_j \subset g_Q \) such that \( \{ \exp(\mathbb{C} \sigma_j) \cdot o_j \} \in B(\sigma_j) \) is a nilpotent orbit.\(^{17}\) The weight filtration \( W_\bullet(\sigma_j) \) is the filtration \( jW_\bullet \) of (6.37). The cone \( \sigma_{j-1} \) is a face of \( \sigma_j \), for all \( 1 \leq j \leq s \).

**Remark 6.39.** As a corollary to Theorem 6.38, \( (g_Q, jF^\bullet, jW_\bullet) \) is a (limit) \( \mathbb{Q} \)-mixed Hodge structure.

\(^{15}\)Properly speaking, \( H^\sigma = H^\sigma_j \) depends on \( h_j \). Precisely, if \( H^\sigma_j \) is defined with respect to \( h \subset b \subset p \), then \( H^\sigma_j = c_j(h_j) \). We have elected to drop the subscript \( j \) to keep the notation clean and simply write \( H^\sigma \in h_j \).

\(^{16}\)Again, \( E = c_j(E) \), where \( E_0 \) is the grading element associated to \( h \subset b \subset p \).

\(^{17}\)See Section 7.1 for the definition of \( B(\sigma) \).
Lemma 6.44. The bigrading (6.36) satisfies the symmetries

\[ j\mathfrak{g}^{p,q} = j\mathfrak{g}^{q,p} \]

and

\[ \theta(j\mathfrak{g}^{p,q}) = j\mathfrak{g}^{-q,-p}. \]

Remark 6.40 (Rational and polarized). In the language of [32, Section 5], Theorem 6.38 implies that the orbit \( O_j \) is polarized and contained in the nilpotent closure of the open \( G_\mathbb{R} \)-orbit \( D = O_0 \). Moreover, from \( \sigma_j \subset \mathfrak{g}_\mathbb{Q} \) it follows that \( jW_* \) is defined over \( \mathbb{Q} \), so that \( O_j \) is rational, cf. [32, Definition 5.5].

Recall the \( Y_j \)-eigenspace \( j\mathfrak{g}_0 \) defined in (6.35). The Jacobi identity implies that \( j\mathfrak{g}_0 \) is a Lie subalgebra of \( \mathfrak{g} \). In fact, \( j\mathfrak{g}_0 \) is reductive. Let \( j\mathfrak{g}_0^{ss} = [j\mathfrak{g}_0, j\mathfrak{g}_0] \); then \( j\mathfrak{g}_0^{ss} \) is semisimple, and \( j\mathfrak{g}_0 = j\mathfrak{g}_0^{ss} \oplus j \), where \( j \) is the center of \( j\mathfrak{g}_0 \). Following [32, Definition 5.9], we say that the \( G_\mathbb{R} \)-orbit \( O_j \) is cuspidal if \( j\mathfrak{g}_0^{ss}(\mathbb{R}) \) contains a compact Cartan subalgebra; that is, a Cartan subalgebra on which the Killing form restricts to be negative definite.

Remark 6.41 (Weakly cuspidal). The orbit \( O_j \) is cuspidal in the following weak sense. In the course of the proof we will show that there exist semisimple subgroups \( G_s \subset \cdots \subset G_1 \subset G_0 = G \) with the properties that:

\( \bullet \) The orbit \( \tilde{D}_j = G_j \cdot o_j \) is a rational homogeneous variety containing the point \( o_{j+1} \).

\( \bullet \) The orbit \( D_j = G_{j,\mathbb{R}} \cdot o_j \) is open in \( \tilde{D}_j \) and equals \( \tilde{D}_j \cap O_j \) and \( o_{j+1} \in \partial D_j \).

\( \bullet \) The orbit \( C_{j+1} = G_{j,\mathbb{R}} \cdot o_{j+1} \subset \tilde{D}_j \cap O_{j+1} \) is contained in the boundary of \( D_j \) and is cuspidal in \( \tilde{D}_j \). (This will follow from Lemma 6.51 below.) Note that

\[ D_{j+1} \subset C_{j+1} \subset \partial D_j. \]

Remark 6.42 (Sub–Hodge structures). The subdomains \( \{ D_j \}_{j=0}^s \) may be described in greater detail. In the course of the proof we will show that:

\( \bullet \) The Lie algebra of \( G_j \) is

\[ \Gamma_j = \bigcap_{1 \leq k \leq j} \Gamma_{\beta_k}, \]

where the \( \Gamma_{\beta_k} \) are as in Section 6.7.

\( \bullet \) The set \( \{ \exp(C N_j) \cdot o_j \} \) is a nilpotent orbit in \( \tilde{D}_{j-1} \), for all \( 1 \leq j \leq s \), cf. (6.67).

These two observations, and results of [31], yield a natural identification of the open \( G_{j,\mathbb{R}} \)-orbit \( D_j \subset \tilde{D}_j \) as the Mumford–Tate domain of a Hodge structure \( \varphi_{j,*} \), which we now describe. Let \( W_0(N_j) \) denote the weight filtration on \( \Gamma_{j-1} \), with the convention that \( \Gamma_0 = \mathfrak{g} \). Let \( Gr_\bullet(W_0(N_j)) \) denote the associated graded and choose \( \vec{N}_j^+ \in \mathfrak{g}^{\beta_j} \). Then, given \( \ell \geq 0 \), the filtration \( jF_\bullet \) induces a weight \( \ell \) Hodge structure \( \varphi_{j,\ell} \) on \( \ker\{ (N_j^+)^{\ell+1} : Gr_\ell(W_0(N_j)) \to Gr_{-\ell-2}(W_0(N_j)) \} \) that is polarized by \( -\langle , N_j^\ell \rangle \)\(^{(18)}\).

Theorem 6.38 and the lemmas that follow are proved in Section 6.10. The first three lemmas codify some distinguished properties of the bigrading (6.36) and the Cartan subalgebra \( \mathfrak{h}_j \).

Lemma 6.44. The bigrading (6.36) satisfies the symmetries

\[ j\mathfrak{g}^{p,q} = j\mathfrak{g}^{q,p} \]

and

\[ \theta(j\mathfrak{g}^{p,q}) = j\mathfrak{g}^{-q,-p}. \]

\(^{(18)}\)In the notation of [31], the domain \( D_j \) is denoted \( D(N_j) \), and the Hodge structure \( \varphi_{j,*} \) by \( \varphi_{\text{split}} \).
Remark 6.46. By (6.37), (6.36) is the Deligne bigrading (2.28) on the $\mathbb{Q}$-MHS $(\mathfrak{g}_Q, jF_\bullet, jW_\bullet)$. It follows from (6.45a) that this MHS is $\mathbb{R}$-split, but in fact more is true. Since $Y_j \in \mathfrak{g}_Q$, $(\mathfrak{g}_Q, jF_\bullet, jW_\bullet)$ is $\mathbb{Q}$-split; and by (6.43), the other MHS in its $G_{j,\mathbb{R}}$-orbit remain $\mathbb{Q}$-split.

Lemma 6.47. For $E$ and $H^\beta$ defined with respect to the Cartan subalgebra $h_j$, we have

$$E + E = Y_j.$$ 

As a consequence, $j\mathfrak{g}^{p,q} = j\mathfrak{g}^p \cap j\mathfrak{g}^q$.

Lemma 6.48. The roots $\alpha \in \Delta(\mathfrak{g}, h_j)$ of the Cartan subalgebra $h_j$ satisfy

$$\overline{\alpha} = -\alpha + \sum_{i=1}^j \alpha(\mathbb{H}_i) \beta_i.$$ 

Consider Lemma 6.48 in the case that $j = 1$. Then (6.5) and (6.49) yield $\overline{\alpha} = -r_{\beta_1}(\alpha)$. More generally, if $j > 1$, then the strong orthogonality of the roots $\mathcal{B}$ implies $\overline{\alpha} = -r_{\beta_j} \cdots r_{\beta_1}(\alpha)$.

The following lemma will be invoked in the proof of Theorem 6.38, which is by induction.

Lemma 6.50. There exists $N_1 \in \mathfrak{g}_Q^{\beta_1}$ such that $\{\exp(\mathbb{C} N_1) \cdot a_1\} \in B(N_1)$ is a nilpotent orbit. (In particular, $O_1$ is rational and polarized.)

Lemma 6.51. The $G_\mathbb{R}$-orbit $O_1$ is cuspidal.

Lemma 6.52. If the first root $\beta_1 \in \mathcal{B}$ is simple, then the $G_\mathbb{R}$-orbit $O_1$ has codimension one in $\mathcal{D}$.

Appendix A presents a thorough study of these codimension one boundary orbits in the case that $P$ is a maximal parabolic.

Remark 6.53 (codim $O_1$). The codimension of $O_1$ is the number of roots in

$$\Delta(+, +) = \{\alpha \in \Delta \mid \alpha(E), \alpha(E) \geq 1\},$$

cf. [32, Proposition 4.1]. By Lemma 6.48

$$\Delta(+, +) = \{\alpha \in \Delta \mid 1 \leq \alpha(E) < \alpha(\mathbb{H}^\beta_1)\},$$

and $\beta_1 = \beta_1 \in \Delta(+, +)$.

Example 6.54 (codim $O_1 > 1$). In the case that $\beta_1$ is not simple there exist examples with $\text{codim}_\mathbb{R} O_1 > 1$. Let $P$ the maximal parabolic corresponding to a simple root $\alpha_1$; equivalently, $E = S^4$, cf. Section 2.1. In general it seems easiest to find examples in which $\text{codim} O_1 > 1$ when the root system $\Delta$ contains roots of distinct lengths. This is the case in each of the examples that follow, where $\alpha_1$ is a long root and $\beta_1$ is a short root.

(a) Suppose that $g = \mathfrak{so}_7 \mathbb{C}$ and $\mathfrak{i} = 2$. Then $\mathcal{D} = \text{IG}(2, \mathbb{C}^7)$ is the isotropic grassmannian. If $\beta_1 = \alpha_2 + \alpha_3$, then $\mathbb{H}^{\beta_1} = -2S^1 + S^2$. We have $\Delta(+, +) = \{(010), (011), (012)\}$, where $(a_1^1 a_2^2 a_3^3)$ denotes the root $\alpha = a_1^1 a_2$ (for example, $(011) = \alpha_2 + \alpha_3$). Thus codim $O_1 = 3$. (Likewise, the reader may confirm that $\beta_1 = (111)$ yields codim $O = 3$.)

(b) Suppose that $\mathfrak{g} = \mathfrak{sp}_6 \mathbb{C}$ and $\mathfrak{i} = 3$. Then $\mathcal{D} = \text{LG}(3, \mathbb{C}^6)$ is the (Hermitian symmetric) Lagrangian grassmannian. If $\beta_1 = \alpha_2 + \alpha_3$, then $\mathbb{H}^{\beta_1} = -S^1 + 2S^3$. We have $\Delta(+, +) = \{(001), (011), (021)\}$. (Likewise, $\beta_1 = (111)$ also yields codim $O_1 = 3$.)
(c) Suppose that $g$ is the exceptional Lie group $F_4$ and $i = 1$. Then $D \subset \mathbb{P}g$ is the adjoint variety of $g$ (Section 2.5). If $\beta_1 = (1110)$, then $H^{\beta_1} = 2S^1 - S^4$. So $\Delta(+) = \{(1000), (1100), (1110), (1120), (1220)\}$, and $\text{codim} \mathcal{O}_1 = 5$.

(d) Suppose that $g$ is the exceptional Lie group $G_2$ and $i = 2$. Again, $D \subset \mathbb{P}g$ is the adjoint variety of $g$. If $\beta_1 = 2\alpha_1 + \alpha_2$, then $H^{\beta_1} = S^1$. So $\Delta(+) = \{(21), (31), (32)\}$, and $\text{codim} \mathcal{O}_1 = 3$.

Remark 6.55 (Relationship to work of Kerr–Pearlstein). Theorem 6.38 is closely related to the discussion and results of [32]. Indeed, Lemma 6.47 implies that the bigrading (6.36) is that of [32, Lemma 3.2]. One may also view $o_j = jF^{\ast}$ as the image of a $Q$-split point in a boundary component under the naive limit map, as we explain briefly in Section 7.1.

6.9. The case that $P$ is a maximal parabolic. Assume that $G$ is simple and $P$ is a maximal parabolic. Then the grading element (2.7) is of the form

$$E = S^1.$$

Proposition 6.56. Let $P$ be a maximal parabolic in a simple Lie group $G$. Fix a real form $G_\mathbb{R}$ and an open $G_\mathbb{R}$–orbit $D \subset G/P$ admitting the structure of a Mumford–Tate domain. Then $\text{bd}(D) \subset \mathcal{D}$ contains a unique codimension one $G_\mathbb{R}$–orbit.

The proposition is proved in Section 6.10. A generalization of Proposition 6.56 holds for arbitrary (not necessarily maximal) parabolics: the number of codimension one orbits in $\text{bd}(D)$ is $|I(p)|$, cf. [44].

Remark 6.57. That an open $G_\mathbb{R}$–orbit admit the structure of a Mumford–Tate domain is a strong constraint. Given $o \in D$ of this type, with corresponding grading element $E \in \mathfrak{h} \subset g^0$, we have

$$E(\Delta_c) \subset 2\mathbb{Z} \quad \text{and} \quad E(\Delta_n) \subset 2\mathbb{Z} + 1,$$

where $\Delta_c$ and $\Delta_n$ denote the compact and noncompact roots, respectively [24]. Applying the complex Weyl group $W(g, \mathfrak{h})$ to $E$ yields points $w(o)$ in each open orbit; more precisely, it produces a bijection between double cosets in $W(g_\mathbb{R}, \mathfrak{h}_\mathbb{R}) \setminus W(g, \mathfrak{h})/W(p, \mathfrak{h})$ and open orbits [32, Theorem 4.5(ii)]. The orbit containing $w(o)$ is a Mumford–Tate domain if and only if $w(E)$ satisfies (6.58).

In fact, $D$ is often unique (or nearly so) and Proposition 6.56 often allows us to identify it from orbit incidence data.

Remark 6.59. Assume $D$ admits a Mumford–Tate domain structure. Extrapolating from the $G_2$ examples in [32, Section 6.1.3], one might hope that $\text{bd}(D)$ contains at most one $G_\mathbb{R}$–orbit in each codimension, and that products of the $c_\beta_j$ reach every orbit in $\text{bd}(D)$. That this is false can be seen in the case of the $F_4$-adjoint variety. Running $kgp$ and $kgporder$ in ATLAS [2] and interpreting the results via Matsuki duality [42, Section 6.6], we find 70 orbits (all with codimension $\leq 15$) in $\text{bd}(D)$.

As noted in (6.32), the number of elements $s$ in the set (6.29) is bounded above by the real rank of $g_\mathbb{R}$.

Lemma 6.60. (a) There exists a choice of $\mathcal{B}$ that both maximizes $s$ and contains the simple root $\alpha_1$. 

(b) Excepting the pairs

\[[6.61] \quad (g, 1) = (e_7, 5) , (e_8, 2, 2, 10) , (f_4, 2) , (g_2, 1), \]

there exists a set (6.29) with \( s \) equal to the real rank of \( g_\mathbb{R} \).

**Proof.** For most pairs \((g, 1)\) the sequence \( \{\beta_j\}_{j=1}^s \) is exhibited in [34, Appendix C.3-4]. The remainder may be identified from inspection of the root system. \( \square \)

**Corollary 6.62.** Excepting the pairs (6.61), the Cartan subalgebra \( \mathfrak{h}_s \) has maximally non-compact real form \( \mathfrak{h}_s \cap g_\mathbb{R} \).

**Remark 6.63 (Exceptional cases).** For each of the exceptional cases of (6.61) we may choose the set \( \{\beta_j\}_{j=1}^s \) as follows. Below we express the root \( \beta = b'\alpha_i \) as \( (b^1 \cdots b^r) \).

- \( B(e_7, 5) = \{(0^41^20^2), (0^31^30^0), (01^2210^2), (01^201^2), (001^4), (0^21^5)\}. \)
  Here \( s = 6 \), while \( \text{rank}_{\mathbb{R}} g_\mathbb{R} = 7 \).
- \( B(e_8, 2) = \{(01^21^50^3), (1^32^10^2), (1^22^31^0), (1^32^11^0), (1^32^11^0), (1^32^11^0)\}. \)
  Here \( s = 7 \), while \( \text{rank}_{\mathbb{R}} g_\mathbb{R} = 8 \).
- \( B(e_8, 3) = \{(0^41^20^2), (0^31^30^0), (01^2210^2), (01^201^2), (001^4), (0^21^5)\}. \)
  Here \( s = 6 \), while \( \text{rank}_{\mathbb{R}} g_\mathbb{R} = 8 \).
- \( B(f_4, 2) = \{(0100), (1110), (0120)\}. \)
  Here \( s = 3 \), while \( \text{rank}_{\mathbb{R}} g_\mathbb{R} = 4 \).
- \( B(g_2, 1) = \{(10)\}. \)
  Here \( s = 1 \), while \( \text{rank}_{\mathbb{R}} g_\mathbb{R} = 2 \).

6.10. **Proofs.**

**Proof of Lemma 6.50.** Let \( N_1 \) be the nilpotent element \( y^{-\beta_1} \in g_{\mathbb{R}} \) of (6.25). By [46, Theorem 6.16], it suffices to show that \( \exp(it N_1) \cdot o_1 \in D \) for all \( t > 0 \).

We may reduce to the case that \( g = su_2 \mathbb{C} \) and \( D = \mathbb{C}P^1 \) (of Examples 6.15, 6.19, 6.28 and 6.31) as follows. Set \( \beta = \beta_1 \) and recall the Lie subalgebra \( su_2^\beta \subset g \) of (6.13) and the Lie subgroup \( SL_2^\beta(\mathbb{C}) \subset G \) of (6.14). The Cayley transform \( c_\beta \) may be viewed as an element of \( \text{Ad}(SL_2^\beta(\mathbb{C})) \). The \( SL_2^\beta(\mathbb{C}) \)–orbit of \( o_0 \) is a \( \mathbb{C}P^1 \subset D \) containing \( o_1 \). We may identify \( o_0 \) with the point \((1 : 0) \in \mathbb{C}P^1 \) of Example 6.19, and \( o_1 \) with the point \((1 : i) \in \mathbb{C}P^1 \) of Example 6.31. Moreover, \( su_2^\beta(\mathbb{R}) \) is isomorphic with the Lie algebra \( su(1,1) \) of Example 6.15, and we may identify \( D \cap \mathbb{C}P^1 \) with the unit disc as in Example 6.19.

Under these identifications,

\[ i y^{-\beta} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \]

and

\[ \exp(it N_1) \cdot o_1 = \exp \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2}t + \frac{1}{2}it \\ -\frac{1}{2}t + i(1 - \frac{1}{2}t) \end{pmatrix}. \]
Whence \( \exp(itN_1) \cdot o_1 \) is contained in the unit disc \( D \cap \mathbb{CP}^1 \subset D \).

**Proof of Theorem 6.38.** Throughout the proof it will be helpful to keep in mind that strong orthogonality of the roots \( \mathcal{B} \) implies

\[(6.64) \text{ the Cayley transform } c_{\beta_j} \text{ restricts to the identity on } \mathfrak{sl}_2^{\beta_i} \text{ for all } j \neq i.\]

Whence, it follows from (6.13) and (6.27) that

\[(6.65) \text{ the } c_j(\mathfrak{sl}_2^{\beta_i}) \text{ are defined over } \mathbb{Q} \text{ for all } i,j.\]

The proof of the theorem is by induction.

**The inductive hypotheses.** Assume the theorem holds for \( j \), and that

\[(6.66) \sigma_j = \text{span}_{\mathbb{Q} > 0} \{N_1, \ldots, N_j\} \text{ with } N_i \in \mathfrak{g}_Q^{-\beta_i} \text{ for all } 1 \leq i \leq j.\]

Suppose furthermore that a semisimple group \( G_j \subset G \), with Lie algebra \( \Gamma_j \subset \mathfrak{g} \), and parabolic subgroup \( P_j = G_j(\mathbb{C}) \cap P \) are given with the following properties:

(a) The Lie algebra \( \Gamma_j \) is defined over \( \mathbb{Q} \). The roots of \( \Gamma_j \) are precisely those that are strongly orthogonal to the \( \{\beta_1, \ldots, \beta_j\} \); equivalently, (6.43) holds.

**Remark.** Observe that this hypothesis on the roots of \( \Gamma_j \) implies that the subalgebra \( c_j(\mathfrak{sl}_2^{\beta_i}) = \mathfrak{sl}_2^{\beta_i} \) is contained in \( \Gamma_j \) for all \( i > j \).

(b) The parabolic \( P_j = \text{Stab}_{G_j(\mathbb{C})}(o_j) \), so that \( G_j(\mathbb{C})/P_j \) is naturally identified with the complex orbit \( \mathcal{D}_j = G_j(\mathbb{C}) \cdot o_j \). The real orbit \( D_j = G_j(\mathbb{R}) \cdot o_j \) is open in \( \mathcal{D}_j \).

(c) For all \( i \geq j + 1 \) the Cayley transform \( c_{\beta_i} \) lies in \( \text{Ad}(G_j) \).

**Remark.** It follows from these last two hypotheses that \( o_{j+1} = g_{\beta_{j+1}}(o_j) \in \mathcal{D}_j \). By (6.1), the orbit \( C_{j+1} = G_j(\mathbb{R}) \cdot o_{j+1} \) is contained in the topological boundary \( \partial D_j \).

**The base case.** Set \( \sigma_0 = \{0\} \). It follows from Section 6.4 that the theorem holds for \( j = 0 \). The hypothesis (6.66) is trivial. Likewise, setting \( G_0 = G \), \( \Gamma_0 = \mathfrak{g} \), and \( P_0 = P \), the hypotheses (a-c) are immediate. (Here, \( \mathcal{D}_0 = \mathcal{D}, D_0 = D \) and \( C_1 = O_1 \).)

**The induction: part 1.** First note that (6.64) implies that the nilpotent cone

\[\sigma_j = c_{\beta_{j+1}}(\sigma_j)\]

is unchanged under the next Cayley transform.

By (6.65) the root space \( j+1\mathfrak{g}^{-\beta_{j+1}} \) is defined over \( \mathbb{Q} \). By Lemma 6.50, we may choose \( N_{j+1} \in j+1\mathfrak{g}_Q^{-\beta_{j+1}} \) so that \( \{\exp(iyN_{j+1}) \cdot o_{j+1} \} \in B_j(N_{j+1}) \) is a nilpotent orbit in \( \mathcal{D}_j \). Equivalently,

\[(6.67) \exp(iyN_{j+1}) \cdot o_{j+1} \in D_j \text{ for all } y > 0.\]

By the hypothesis (a), \( N_{j+1} \subset \Gamma_j \) so that \( \exp(\mathbb{C}N_{j+1}) \subset G_j(\mathbb{C}) \). Moreover, the hypotheses (6.66) and (a) imply that \( \exp(\sigma_j) \) commutes with \( G_j \). Whence, given any \( N \in \sigma_j \) and
\[ y > 0, \]
\[ \exp(i(N + yN_{j+1})) \cdot o_{j+1} = \exp(iN) \exp(iyN_{j+1}) \cdot o_{j+1} \]
\[ \subseteq \exp(iN) D_{j} \]
\[ = \exp(iN) G_{j}(\mathbb{R}) \cdot o_{j} \]
\[ = G_{j}(\mathbb{R}) \exp(iN) \cdot o_{j} \]
\[ \subseteq G_{j}(\mathbb{R}) D = D. \]

Setting \( \sigma_{j+1} = \text{span}_{Q > 0} \{ \sigma_{j}, N_{j+1} \} \), we have shown that \( \{ \exp(C \sigma_{j+1}) \cdot o_{j+1} \} \in B(\sigma_{j+1}) \) is a nilpotent orbit; this establishes the first two (of three) claims of Theorem 6.38 for \( j + 1 \). Also, we see that (6.66) holds for \( j + 1 \).

Given \( i \leq j + 1 \), complete the pair \( H_{\beta_{i}}, N_{i} \in g_{Q} \) to a standard triple (Section 6.5) \( \{ N_{i}^{+}, H_{i}, N_{i} \} \) with \( N_{i}^{+} \in g_{Q}^{\mathbb{R}} \). To see that the third claim \( W_{\sigma_{j+1}} = j+1 W_{\sigma_{j}} \) holds it suffices to observe that the strong orthogonality of the roots implies

\[ N^{+} = N_{1}^{+} + \cdots + N_{j+1}^{+}, \]
\[ Y_{j+1} = H_{\beta_{1}}^{+} + \cdots + H_{\beta_{j+1}}^{+}, \]
\[ N = N_{1} + \cdots + N_{j+1} \]

is a standard triple in \( g_{Q} \).

**The induction: part 2.** It remains to show that the additional inductive hypotheses (a-c) hold for \( j + 1 \). Let \( \Gamma_{j+1} \subset \Gamma_{j} \) be the Lie algebra \( \Gamma_{\beta_{j+1}} \) of Section 6.7. Then hypothesis (a) is immediate. Likewise, let \( G_{j+1} \subset G_{j} \) and \( P_{j+1} \subset P_{j} \) be the corresponding \( \mathcal{G} \) and \( \mathcal{P} \), respectively; and let \( D_{j+1} \subset D_{j+1} \) be the corresponding \( \mathcal{D} \subset \mathcal{D} \), respectively. Recall that \( D_{j+1} \) is open in \( \mathcal{D}_{j+1} \); see Section 6.7. Thus the hypothesis (b) holds.

Let \( i \geq j + 2 \). A priori the Cayley transformation \( c_{\beta_{i}} \) lies in \( \text{Ad}(G) \). However, strong orthogonality implies that \( \mathcal{B}\{ \beta_{1}, \ldots, \beta_{j+1} \} \subset \Delta(\Gamma_{j+1}) \) and

\[ \mathfrak{s}l_{2}^{\beta_{i}} = c_{j+1}(\mathfrak{s}l_{2}^{\beta_{i}}) \subset \Gamma_{j+1}, \]

Therefore, the Cayley transformation \( c_{\beta_{i}} \) lies in \( \text{Ad}(G_{j+1}) \). Whence hypothesis (c) holds.

**Proof of Lemma 6.44.** By construction the Cartan subalgebra \( \mathfrak{h} = \mathfrak{h}_{0} \) is preserved by conjugation and Cartan involution (Section 6.3). Since the Cayley transform preserves these properties, the Cartan subalgebra \( \mathfrak{h}_{j} \) is also preserved by conjugation and the Cartan involution. For any such Cartan subalgebra the roots satisfy \( \theta\alpha = -\bar{\alpha} \).

**Proof of Lemma 6.47.** Define

\[ \mathfrak{t}_{j} \overset{\text{dfn}}{=} \mathfrak{h}_{j} \cap \mathfrak{t}_{\mathbb{R}} \quad \text{and} \quad \mathfrak{a}_{j} \overset{\text{dfn}}{=} \mathfrak{h}_{j} \cap \mathfrak{a}_{\mathbb{R}}^{+}. \]

Then the real form \( \mathfrak{h}_{j}(\mathbb{R}) \overset{\text{dfn}}{=} \mathfrak{h}_{j} \cap \mathfrak{g}_{\mathbb{R}} \) is \( \mathfrak{t}_{j} \oplus \mathfrak{a}_{j} \). In analogy with (6.24),

\[ \mathfrak{t}_{j} = \bigcap_{i=1}^{j} \ker\{ \beta_{i} : \mathfrak{h}_{\mathbb{R}} \to \mathbb{C} \} \quad \text{and} \quad \mathfrak{a}_{j} = \text{span}_{\mathbb{R}}\{ H_{\beta_{1}}, \ldots, H_{\beta_{j}} \}. \]
Moreover,
\begin{equation}
(6.68) \quad \text{the roots of } h_j \text{ are } \mathbb{R}-\text{valued on } a_j \text{, and } i \mathbb{R}-\text{valued on } t_j.
\end{equation}
Since \( \alpha(E) \in \mathbb{R} \), we see from (6.68) that \( E \in i t_j \oplus a_j \). Therefore, \( E + E \in a_j \). Thus
\[ \mathbb{E} + \mathbb{E} \in \text{span}_R \{ H_{\beta_1}, \ldots, H_{\beta_s} \}. \]
Since the roots \( \beta_i \) are real, (6.29b) and (6.69) yield \( \beta_i(E + E) = \beta_i(E) + \beta_i(E) = 2. \quad \square \)

**Proof of Lemma 6.48.** By definition, the conjugate root \( \bar{\beta} \) is given by \( \bar{\beta}(\xi) = \beta(\bar{\xi}) \) for any \( \xi \in h_j \). Whence (6.49) holds when restricted to \( t_j \). It remains to show that the equation holds when restricted to \( a_j \). Since the latter is spanned by the \( \{ H_{\beta_i} \}_{i=1} \), it suffices to show that (6.49) holds when evaluated on any of these \( H_{\beta_i} \). Strong orthogonality of the roots \( \beta_i \in \mathcal{B} \) and (6.4) imply that
\begin{equation}
(6.69) \quad \frac{2 (\beta_i, \beta_i)}{(\beta_i, \beta_i)} = \beta_i(H_{\beta_i}) = 2 \delta_{ik}, \quad \text{for any } 1 \leq i, k \leq s.
\end{equation}
It follows that (6.49) holds when evaluated on the \( \{ H_{\beta_i} \}_{k=1} \). \quad \square

**Proof of Lemma 6.51.** From the definition (6.35) we see that \( h_1 \) is a Cartan subalgebra of \( g_0 \), and \( i H_{\beta_i} \) is contained in the center \( \mathfrak{z} \) of \( g_0 \). It then follows from (6.24) that \( h_1 \cap g_0^{ss} \) is a compact Cartan subalgebra of \( g_0^{ss} \). \quad \square

**Proof of Lemma 6.52.** For the duration of the proof, let \( \alpha \) denote the simple root \( \beta_1 \in \mathcal{B} \). Set \( g^{p,q} = g^{p,q}_0 \), and \( o = o_1 \) and \( \mathcal{O} = \mathcal{O}_1 \). By [32, Proposition 4.1], the (real) codimension of \( \mathcal{O} \) is the number of roots in
\[ \Delta(\mp, +) = \{ \gamma \in \Delta \mid \gamma(E), \bar{\gamma}(E) > 0 \}. \]
We will show that
\begin{equation}
(6.70) \quad \Delta(\mp, +) = \{ \alpha \};
\end{equation}
whence \( \text{codim}_\mathbb{R} \mathcal{O}_1 = 1 \), establishing the lemma. By Remark 6.53, \( \alpha \in \Delta(\mp, +) \).

Now suppose that \( \gamma \in \Delta(\mp, +) \). The two inequalities \( \gamma(E), \bar{\gamma}(E) > 0 \) imply that both \( \gamma \) and \( \bar{\gamma} \) are positive roots. But from Lemma 6.48, we see that \( \bar{\gamma} = -\gamma + \ell \alpha \), where \( \ell = \gamma(H_0^\alpha) \). In particular, if \( \gamma \) is not a multiple of the simple root \( \alpha \), then \( \bar{\gamma} \) is a negative root. Whence \( \gamma = \alpha \). \quad \square

**Proof of Proposition 6.56.** By [32, Proposition 4.14], the codimension one \( G_\mathbb{R} \)-orbits in \( \text{bd}(G_\mathbb{R} \cdot o_0) \) are of the form \( G_\mathbb{R} \cdot (g_\beta \cdot o_0) \), with \( \beta \in \Delta(g^+) \). In fact, we must have \( \beta \in \Delta(g^4) \) in order not to violate [32, Corollary 4.4].

So let \( \beta \) be a (noncompact, positive) root with \( \beta(E) = 1 \). Recall \( \mathbb{E} = S^1 \). We will show that \( \beta \) belongs to the orbit of \( \alpha_1 \) under the subgroup \( W(g^0, h) \subset W(g_\mathbb{R}, h_\mathbb{R}) \) of the real Weyl group. It will follow that \( g_\beta \cdot o_0 \) and \( g_{\alpha_1} \cdot o_0 \) belong to the same \( G_\mathbb{R} \)-orbit, giving the desired uniqueness.

Acting by \( W(g^0, h) \) if necessary, we may assume \( (\beta, \alpha_j) \leq 0 \) for all \( j \neq 1 \); equivalently,
\begin{equation}
(6.71) \quad \alpha_j(H_\beta) \leq 0.
\end{equation}
(That is, use the Weyl group of \( G^0 \) to reflect the orthogonal projection of \( \beta \) to \( \ker(E) \) into the negative Weyl chamber.) Set \( \mathcal{B} = \{ \beta \} \) and \( g^{p,q} = g^{p,q}_0 \) for the bigrading associated to
\[ o_1 = g_3 \cdot o_0. \] For any root \( \alpha \in \Delta(g, h_1) \), write \( g^{p(\alpha), q(\alpha)} \supseteq g^\alpha \). By Lemma 6.47, \( p(\alpha) + q(\alpha) = \alpha(H^\beta) \), and of course \( p(\beta) = 1 = q(\beta) \).

Now for any positive root \( \alpha \), we have
\[ \alpha = p(\alpha)\alpha_1 + \sum_{j \neq 1} m_j(\alpha)\alpha_j \]
where \( m_j(\alpha) \geq 0 \). Since \( p(\beta) = 1 \) and \( \beta \) is positive,
\[ p(\alpha_1) + q(\alpha_1) = \alpha_1(H^\beta) = \beta(H^\beta) - \sum_{j \neq 1} m_j(\beta)\alpha_j(H^\beta) \geq \beta(H^\beta) = 2. \]
But \( p(\alpha_1) = \alpha_1(E) = \alpha_4(S^1) = 1 \), and so \( q(\alpha_1) \geq 1 \). By Remark 6.53, \( \text{codim}(G_\mathbb{R} \cdot o_1) \) is now \( \geq 2 \) unless \( \beta = \alpha_1 \).

7. Enhanced \( SL_2 \)-orbits and Adjoint Varieties

Following a brief review of boundary components of Mumford–Tate domains (Section 7.1), we begin in Section 7.2 by using the last section’s construction to produce a class of Mumford–Tate subdomains which are usually not Schubert varieties, hence will be interesting to compare with them in cohomology. (We only take this up in a limited way here, in Section 7.6 for the \( G_2 \) adjoint variety.) The remainder of this section is devoted to working out these subdomains and boundary components for the fundamental adjoint varieties, where they have striking relationships to both the varieties of lines studied in Sections 3 and 5 and the Calabi-Yau VHS studied in [22].

7.1. Boundary components and the naive limit map. First we review the definitions and recall some results from [31, 32]. Fix a Mumford–Tate domain \( D \subset \bar{D} \) and pairwise commuting nilpotents \( N_1, \ldots, N_m \in g_\mathbb{Q} \). Let \( \sigma \overset{\text{dfn}}{=} \mathbb{Q}_{\geq 0}(N_1, \ldots, N_m) \subset g_\mathbb{Q} \) be the rational nilpotent cone, with interior \( \sigma^0 \overset{\text{dfn}}{=} \mathbb{Q}_{> 0}(N_1, \ldots, N_m) \). By Theorem 2.27(b), \( W_*(\sigma) \overset{\text{dfn}}{=} W_*(N) \) is independent of \( N \in \sigma^0 \). Regarding (19)
\[ \bar{B}(\sigma) \overset{\text{dfn}}{=} \{ F^* \in \bar{D} \mid (F^*; N_1, \ldots, N_m) \text{ is a nilpotent orbit} \} \]
as a set of (limiting) \( \mathbb{Q} \)-mixed Hodge structures \( (V_\mathbb{Q}, F^*, W_*(\sigma)) \), it makes sense to consider its generic Mumford–Tate group (or that of some subset). The boundary component associated to \( \sigma \) is defined by
\[ B(\sigma) \overset{\text{dfn}}{=} \bar{B}(\sigma)/\exp(\mathbb{C} \sigma); \]
the points of \( B(\sigma) \) are \( \sigma \)-nilpotent orbits, or equivalently LMHS up to “reparametrization” (i.e. modulo \( e^{C \sigma} \)). When \( m = 1 \), we shall replace \( \sigma \) everywhere in our notation by \( N_1 \); e.g. \( \bar{B}(N_1) \). By [32, Remark 5.6], we have \( B(\sigma) = \cap_{\sigma \in \sigma^0} \bar{B}(N) \).

Let \( D(\sigma) \) be the Mumford–Tate domain parameterizing the split polarized Hodge structure \( \oplus_{\ell}(\text{Gr}_\ell W_*(\sigma), F^*) \). We have a natural map \( \bar{\rho} : \bar{B}(\sigma) \rightarrow D(\sigma) \) given by taking the associated graded Hodge structure. A choice of \( \mathbb{Q} \)-split base point in \( \bar{B}(\sigma) \) gives rise to a section \( D(\sigma) \hookrightarrow \bar{B}(\sigma) \) of \( \bar{\rho} \), and we shall sometimes blur the distinction between \( D(\sigma) \) and its image.

(19) See Section 2.8 for relevant definitions
Regarding \( N_1, \ldots, N_m \) as elements of \( \text{End}(\mathfrak{g}) \), let \( W_\bullet(\sigma) \mathfrak{g} \) be the corresponding weight filtration of \( \mathfrak{g} \). Let \( Z(\sigma) \subset G \) be the centralizer of \( \sigma \), and let \( \mathfrak{z}(\sigma) \) denote the Lie algebra of \( Z(\sigma) \). Note that

\[
\mathfrak{z}(\sigma) \subset W_0(\sigma) \mathfrak{g}.
\]

Let \( W_\bullet(\sigma) \mathfrak{z} = \mathfrak{z}(\sigma) \cap W_\bullet(\sigma) \mathfrak{g} \) be the induced filtration of \( \mathfrak{z}(\sigma) \). Then

\[
\mathfrak{z}(\sigma) \cong \bigoplus_{\ell \geq 0} \mathfrak{z}_{-\ell}(\sigma), \quad \text{where} \quad \mathfrak{z}_{-\ell}(\sigma) \overset{\text{dfn}}{=} W_{-\ell}(\sigma)_{\mathfrak{z}}/W_{-\ell-1}(\sigma)_{\mathfrak{z}}.
\]

There is a natural tower of fibrations (factoring \( \tilde{\rho} \))

\[
(7.1) \quad B(\sigma) \to \cdots \to B(\sigma)_{(k)} \overset{\rho^{(k)}_\sigma}{\to} B(\sigma)_{(k-1)} \to \cdots \to B(\sigma)_{(1)} \overset{\rho^{(1)}_\sigma}{\to} D(\sigma)
\]

with \( \rho^{(k)}_\sigma \)-fibre \( \mathfrak{z}^{(k)} \) through \( F^\bullet \) (mod \( C^\sigma \)) equal to

\[
(7.2) \quad \mathfrak{z}^{(k)} = \frac{\mathfrak{z}_{-k}(\sigma)}{F^0 \mathfrak{z}_{-k}(\sigma)}, \quad \text{for} \ k \neq 2, \quad \text{and} \quad \mathfrak{z}^{(2)} = \frac{\mathfrak{z}_{-2}(\sigma)}{C^\sigma \oplus F^0 \mathfrak{z}_{-2}(\sigma)}.
\]

(cf. [31, §7]).

Now let \( \sigma = \sigma_s = \text{span}_{Q \geq 0} \{N_1, \ldots, N_s\} \) be the cone of Theorem 6.38, with \( N_j \in s \mathfrak{g} \mathfrak{f} \), and regard \( F^\bullet \) (cf. (6.37)) as a \( Q \)-split base point in \( \tilde{B}(\sigma) \).\(^{(20)}\) The Lie algebra \( \mathfrak{g}_{\mathcal{B}(\sigma)} \) of the generic Mumford–Tate group \( G_{\mathcal{B}(\sigma)} \) of \( D(\sigma) \) may be described as follows (cf. [31, Sections 4–5]). Let \( E \) and \( Y = \sum_{j=1}^s H^{\beta_j} \) be the grading elements of (6.34)\(_{j=s} \) and (6.35)\(_{j=s} \). Set \( \phi = i(E - \bar{E}) \) (which is equal to \( i(2E - Y) \)) by Lemma 6.47). Then \( \mathfrak{g}_{\mathcal{B}(\sigma)} \) is the \( Q \)-Lie algebraic closure in \( \mathfrak{z}_0(\sigma) = \cap_{j=1}^s \Gamma_0^{\beta_j} \) of

\[
Z_0(\sigma)_{\mathbb{R}} \cdot C \phi.
\]

A consequence of our construction (and Remark 6.18) is that \( \phi \in i \mathfrak{g}_{\mathbb{R}} \), and the roots of \( \mathfrak{h}_s \) in \( \mathfrak{z}_0(\sigma) \) come in conjugate pairs defined over \( \mathbb{Q}(i) \). Therefore, \( \mathfrak{g}_{\mathcal{B}(\sigma)} \) is the \((C-)\)Lie algebraic closure of

\[
C \phi + \sum_{\beta \in \Delta(\Gamma_s), \beta(\phi) \neq 0} \mathfrak{g}^{\beta},
\]

and so \( \mathfrak{g}_{\mathcal{B}(\sigma)} \subset \Gamma_s + C \phi \) while \( \mathfrak{g}_{\mathcal{B}(\sigma)}^{s s} \subset \Gamma_s \). In most cases of interest we will have equality. In any case\(^{(21)}\) we have

\[
D(\sigma) = G_{s, \mathbb{R}} \cdot s F^\bullet \cong G_{s, \mathbb{R}} / G_{s, \mathbb{R}} \cap P
\]

(where \( \text{Lie}(G_s) = \Gamma_s \)), and this gives an open subset in the homogeneous variety

\[
\tilde{D}(\sigma) \overset{\text{dfn}}{=} G_s \cdot s F^\bullet \cong G_s / G_s \cap P.
\]

Now we could regard \( \tilde{D}(\sigma) \) as a subvariety of \( \tilde{D} \) by identifying it with \( \tilde{D}_s \overset{\text{dfn}}{=} G_s \cdot o_s \subset \tilde{D} \) (cf. Remark 6.41). However, for some purposes it is better to regard it as a separate variety and map it into \( \tilde{D} \) in a different way:

\(^{(20)}\) Though \( o_s \in D \) corresponds to this filtration, here we shall write \( s F^\bullet \) for the point of \( (D(\sigma) \subset \tilde{B}(\sigma) \) and \( o_s \) for the point of \( \tilde{D} \).

\(^{(21)}\) The discrepancy between \( \mathfrak{g}_{\mathcal{B}(\sigma)} \) and \( \Gamma_s + C \phi \) lies in \( \text{ker}(\text{ad} \phi) \), and therefore stabilizes \( s F^\bullet \).
Definition 7.4. Following [25, 32], (22) the naive limit map

\[ \Phi_\infty^\sigma : B(\sigma) \rightarrow \text{bd}(D) \]

sends \( F^\bullet \mapsto \lim_{\text{Im}(z) \to \infty} e^{zN}F^\bullet \), which is independent of the choice of \( N \in \sigma^0 \), cf. [32, Remark 5.6]).

Though \( \Phi_\infty^\sigma \) is not isomorphic onto its image (denoted \( \hat{B}(\sigma) \) in [32]), its restriction to \( D(\sigma) \) is a \( G_{s,\mathbb{R}} \)-equivariant isomorphism, which extends to a \( G_s \)-equivariant embedding of \( \tilde{D}(\sigma) \) into \( \tilde{D} \). However, the image of this embedding is not \( \tilde{D}_s \).

To present \( \tilde{D}_s \) as the image of a naive limit map, recall the commuting standard triples \( \{N_j^+, H^{\beta_j}, N_j\} \) from the proof of Theorem 6.38, and put \( \tilde{\sigma} \overset{\text{dfn}}{=} \mathbb{Q}_{\geq 0}(\langle -N_1^+, \ldots, -N_s^+ \rangle) \). Setting \( s \hat{g}_{p,q} \overset{\text{dfn}}{=} \hat{g}^{-q-p} \), \( s \hat{F}_n \overset{\text{dfn}}{=} \oplus_{p \geq q; q \in \mathbb{Z}} s \hat{g}_{p,q}^{-p} \hat{g}^{\hat{g}_{p,q}} \), \( s \hat{W}_b \overset{\text{dfn}}{=} \oplus_{p+q \leq b} s \hat{g}_{p,q}^{-p} \), and reasoning as in the proof of [32, Theorem 5.15], one shows that:

1. \( s \hat{W}_b = W_{\sigma}(\hat{g}) \) is graded by \( \hat{\gamma}^{-q} = -H^{\beta_1} - \cdots - H^{\beta_s} \);
2. \( (s \hat{F}^\bullet, -N_1^+, \ldots, -N_s^+) \) is a nilpotent orbit, and \( (s \hat{F}^\bullet, W_{\sigma}(\hat{g})) \) is \( \mathbb{Q} \)-split; and
3. \( \Phi_{\hat{\sigma}}^\tau \) sends \( s \hat{F}^\bullet \) to \( s F^\bullet \).

We conclude that \( D(\tilde{\sigma}) \) and \( \tilde{D}(\tilde{\sigma}) \) are the \( G_{s,\mathbb{R}} \) and \( G_s \)-orbits of \( s \hat{F}^\bullet \), respectively, and \( \Phi_{\hat{\sigma}}^\tau \) restricts to a \( G_s \)-equivariant isomorphism from \( \tilde{D}(\tilde{\sigma}) \) to \( \tilde{D}_s \).

7.2. Construction of the enhanced \( SL_2 \)-orbits. The above discussion gave rise to identifications of \( \tilde{D}(\sigma) \) with the two subsets \( \hat{D}_s \) and \( \Phi_\infty^\sigma(D(\sigma))^{\text{Zar}} \) of \( \hat{D} \); we shall now “interpolate” them. Via the section \( D(\sigma) \hookrightarrow \hat{B}(\sigma) \rightarrow B(\sigma) \); each point \( g \cdot s F^\bullet \in D(\sigma), \) with \( g \in G_{s,\mathbb{R}} \), produces a \( \sigma \)-nilpotent orbit \( e^{C^\sigma} g \cdot o_s \in \hat{D} \) with \( \mathbb{Q} \)-split LMHS. Taking the Zariski closure of the union of these, we have the

Definition 7.5. The enhanced (multivariable) \( SL_2 \)-orbit associated to \( (o_s, \sigma) \) is

\[ X(\sigma) \overset{\text{dfn}}{=} e^{C^\sigma}G_s \cdot o_s^{\text{Zar}} = \hat{G}_s \cdot o_s, \]

where

\[ \hat{G}_s \overset{\text{dfn}}{=} G_s \times (\prod_{j=1}^s \text{SL}_2^{\beta_j}). \]

Note that \( X(\sigma) \) does not identify with a subset of \( \hat{B}(\sigma) \), though \( e^{C^\sigma}G_s \cdot o_s \) does.

Proposition 7.6. The enhanced \( SL_2 \)-orbit \( X(\sigma) \) is the compact dual of a Mumford–Tate subdomain \( Y(\sigma) \subset D \); in particular, it is smooth.

(22) The term reduced limiting period mapping is used in [25, App. to §10], but we prefer to reserve the term “period mapping” for Hodge–theoretic classifying maps associated to families of motives. They are “naive” because they replace the limit MHS of a nilpotent orbit by its (much coarser) actual limit, which students in Hodge theory are trained to not take.
Proof. The homogeneous description in Definition 7.5 already implies smoothness. Since \( o_s \) is \( \mathbb{Q} \)-split (a fortiori \( \mathbb{R} \)-split) [32, Lemma 5.14] yields a diagram

\[
\begin{align*}
\check{D}(\sigma) \times (\mathbb{P}^1)^{\times s} &\to X(\sigma) \subset \check{D} \\
\cup \cup \cup \\
D(\sigma) \times \mathbb{G}_m^{\times s} &\to Y(\sigma) \subset D
\end{align*}
\]

in which the bottom row is given by

\[
(g \cdot s F^\bullet; \nu z_1, \ldots, z_s) \mapsto \exp(\sum s_j N_j) g \cdot s F^\bullet.
\]

(Note that the \( \mathbb{P}^1 \) factors need not be minimally embedded in \( \check{D} \).) Moreover, it is clear that the generic Mumford–Tate group of \( Y(\sigma) \) is \( G_{B(\sigma)} \times (\times_{j=1}^{s} SL_{\beta_j}^2) \), which acts transitively on \( Y(\sigma) \).

Lemma 7.7. The enhanced \( SL_2 \)-orbit \( X(\sigma) \) is horizontal (and Hermitian) if and only if \(-1 \leq E(\alpha) \leq 1 \) for every \( \alpha \in \Delta \) strongly orthogonal to \( \{\beta_1, \ldots, \beta_s\} \).

Proof. The following are equivalent:

- \( X(\sigma) \) is horizontal.
- \( Y(\sigma) \) is horizontal.
- \(-1 \leq E(\alpha) \leq 1 \) for all \( \alpha \in \Delta(Lie(\check{G}_s)) \).
- \(-1 \leq E(\alpha) \leq 1 \) for all \( \alpha \in \Delta(Lie(G_s)) = \Delta(\Gamma_s) = \bigcap_j \Delta(\Gamma_{\beta_j}) \), cf. Section 6.7.

\( \square \)

In Appendix A we determine the bigradings \( \{g^{s,q}\} \) for \( s = 1 \) and \( P \) a maximal parabolic. The condition of the lemma fails in many cases, but holds for a sizable subset which includes all the adjoint varieties (including type \( A \), where the parabolic is not maximal). We will elaborate on these cases in Sections 7.3-7.5.

One reason for studying the \( \check{X}(\sigma) \) is to express classes of (effective linear combinations of) Schubert varieties in terms of “simpler” objects. Of particular interest is the horizontal case, where \([\check{X}(\sigma)]\) can be written in terms of classes of Schubert VHS [45]. Here “simpler” might mean that \( X(\sigma) \) is smooth while the Schubert varieties [resp. VHS] of a given dimension are singular; or it could simply be the case that \( X(\sigma) \) (note its \( \mathbb{P}^1 \) factors) is decomposable while the \( \{X_w\} \) are not. Along these lines we have the easy

Proposition 7.8. If \( \check{D} = G/P \) with \( P \) a maximal parabolic corresponding to a non-short simple root, then the \( \{X(\sigma)\} \) of dimension at least 2 are never Schubert varieties.

Proof. By [27, Proposition 3.7], the smooth Schubert varieties are of the form \( X(\mathcal{D}') \) for \( \mathcal{D}' \subset \mathcal{D} \) a connected subdiagram of the Dynkin diagram (cf. Remark 4.8). So they do not have \( \mathbb{P}^1 \) factors.

\( \square \)

7.3. Fundamental adjoint varieties I: codimension one orbits. In the case that \( \check{D} = G/P \) is a fundamental adjoint variety (Section 2.5), there is a unique codimension-one \( G_\mathbb{R} \)-orbit \( O_1 \in bd(D) \) by Proposition 6.56. If \( \alpha_1 \) is the simple root associated to the maximal parabolic \( P \), then taking \( \mathcal{B} = \{\beta_1 = \alpha_1\} \) yields a distinguished base point \( o_1 \in O_1 \) via the construction of Section 6.8.
Figure 7.1. The bigrading \( g = \oplus_1 g^{p,q} \) for fundamental adjoint varieties.

We claim that the bigradings \( g = \oplus_1 g^{p,q} \) associated to \( o_1 \) exhibit a uniform appearance for all the adjoint varieties, as indicated in Figure 7.1. There, the nodes • indicate a \( g^{p,q} \) of dimension one, the once-circled nodes indicate dimension \( a \), and the twice-circled node indicates dimension \( b \), where the values of \( a, b \) are as given in Table 7.1.

| \( G \) | \( B_r (r \geq 3) \) | \( D_r (r \geq 4) \) | \( E_6 \) | \( E_7 \) | \( E_8 \) | \( F_4 \) | \( G_2 \) |
|-------|----------------|----------------|------|------|------|------|------|
| •     | 1              | 1              | 1    | 1    | 1    | 1    | 1    |
| ○     | \( 2r - 4 \)   | \( 2r - 5 \)   | 9    | 15   | 27   | 6    | 1    |
| ✐     | \( 2r^2 - 11r + 18 \) | \( 2r^2 - 13r + 24 \) | 18   | 37   | 80   | 10   | 2    |

Writing \( g^{p,q} \) for \( 1 g^{p,q} \) and \( h^{p,q} = \dim \mathbb{C} g^{p,q} \), we recall that \( g^{p,q} = g^p \cap g_{p+q} \), where the grading \( g = \oplus g^p \) [resp. \( g = \oplus g_\ell \)] is induced by \( E = S^4 \) [resp. \( H^{\beta_1} = H^{\alpha_1} \)]. By Lemma 6.44 we have \( h^{p,q} = h^{3-p} = h^{-n-p} = h^{-p,-q} \). To verify Figure 7.1 we shall also need Lemma 7.10 below, which gives \( \dim(\mathfrak{g}_\ell) = \dim(\mathfrak{g}^{-\ell}) \). It also implies

\[
   w(g^{p,q}) = w(g^p \cap g_{p+q}) = g^{-(p+q)} \cap g_{-p} = g^{-p-q,q},
\]

where \( w \in W \) is as in Lemma 7.10, and hence the additional symmetry \( h^{p,q} = h^{-p,-q} \) of the Hodge–Deligne numbers.

From Section 2.5, we know that \( \dim(g^{-2}) = \dim(g^2) = 1 \) and \( \dim(g^j) = 0 \) for \( |j| > 2 \). Hence \( \dim(g_{-2}) = \dim(g_2) = 1 \) (and \( \dim(g_\ell) = 0 \) for \( |\ell| > 2 \)), so that \( h^{p,q} = h^{q,p} \) implies \( g_{-2} = g^{-1,-1} \), \( g_2 = g^{1,1} \), \( h^{1,1} = h^{-1,-1} = 1 \). The “extra symmetry” now gives \( h^{-2,1} = h^{2,-1} = h^{-1,2} = h^{1,-2} = 1 \), as well as \( h^{1,0} = h^{-1,0} = h^{0,1} = h^{0,-1} = h^{-1,1} = h^{1,-1} = a \).

Finally, to obtain the dimensions in Table 7.1 we solve

\[
   2a + 3 = \dim_{\mathbb{C}} g^- = \dim_{\overline{\mathbb{C}}} D = n,
\]

\[
   6a + b + 6 = \dim_{\mathbb{C}} g = N + 1.
\]

using Table 2.2.

7.4. A curious symmetry. The main result of this section is Lemma 7.10, which asserts that (in the case of \( i \) associated to a fundamental adjoint representation) the decompositions \((6.34)_{j=1}\) and \((6.35)_{j=1}\) are congruent under the action of the Weyl group.
The grading element corresponding to $P$ is $S^1$. By Table 2.1, the largest $S^1$-eigenvalue on $\mathfrak{g}$ is $\tilde{\alpha}(S^1) = 2$. Therefore, the graded decomposition $(6.34)_{j=1}$ of $\mathfrak{g}$ into $S^1$-eigenspaces is
\[
\mathfrak{g} = \mathfrak{g}^2 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}.
\]
As discussed in Section 2.3.2, $\mathfrak{g}^2$ is one-dimensional; indeed,
\[
(7.9) \quad \mathfrak{g}^{\pm 2} = \mathfrak{g}^{\pm \tilde{\alpha}}.
\]

Let $\mathcal{W} \subset \text{Aut}(\Lambda_{\mathfrak{h}}) \subset \text{Aut}(\mathfrak{h}^*)$ denote the Weyl group of $\mathfrak{g}$. Recall the $\mathbb{H}^{\alpha_1}$-eigenspace decomposition $(6.35)_{j=1}$, and the filtrations $(6.37)_{j=1}$.

**Lemma 7.10.** There exists $w \in \mathcal{W}$ such that $w(\mathfrak{g}_\ell) = \mathfrak{g}^{-\ell}$. In particular, $w(W_\ell) = F^{-\ell}$.

**Corollary 7.11.** The filtration $F^\ast$ is the weight filtration associated to a nilpotent $0 \neq \tilde{N} \in \mathfrak{g}^{\tilde{\alpha}}$.

Before proving Lemma 7.10 we first establish Lemma 7.12.

**Lemma 7.12.** Define $\tilde{S} \in [\mathfrak{g}^{\tilde{\alpha}}, g^{-\tilde{\alpha}}] \subset \mathfrak{h}$ by $\tilde{\alpha}(\tilde{S}) = 2$. Then $\tilde{S} = S^1$.

**Proof of Lemma 7.12.** Let $0 \neq \xi \in \mathfrak{g}^{\alpha_j}$. Fix $\tilde{N} \in \mathfrak{g}^{\tilde{\alpha}}$ and $\tilde{N}^- \in \mathfrak{g}^{-\tilde{\alpha}}$ such that $[\tilde{N}, \tilde{N}^-] = \tilde{S}$. Then
\[
\alpha_j(\tilde{S}) \xi = [\tilde{S}, \xi] = [[\tilde{N}, \tilde{N}^-], \xi] = [[[\tilde{N}, \xi], \tilde{N}^-] + [[\xi, \tilde{N}^-], \tilde{N}]].
\]
Since $\tilde{\alpha}$ is a highest root, $\tilde{\alpha} + \alpha_j$ is not a root. Therefore, $[\tilde{N}, \xi] = 0$. Similarly, the bracket $[\xi, \tilde{N}^-]$ is nonzero if and only if $-\tilde{\alpha} + \alpha_j$ is a root; equivalently, $\tilde{\alpha} - \alpha_j$ is a root. The latter is equivalent to $\mathfrak{g}^{-\alpha_j} \subset \mathfrak{p}$; that is, $j = i$. Therefore, $\alpha_j(\tilde{S}) = 0$ if $j \neq i$. It follows that $\tilde{S}$ is a multiple of $S_1$. Since both $\tilde{\alpha}(S_1)$ and $\tilde{\alpha}(S)$ equal 2, it must be the case that $\tilde{S} = S^1$. \qed

**Proof of Lemma 7.10.** Note that $\alpha_1$ is not a short root. Since the highest root $\tilde{\alpha}$ of $\mathfrak{g}$ is also not a short root, it follows $\alpha_1$ and $\tilde{\alpha}$ have the same length. Therefore, there exists a Weyl group element $w \in \mathcal{W}$ mapping $w(-\alpha_1) = \tilde{\alpha}$, cf. [28, Lemma 10.4.C]. Therefore,
\[
w(\mathbb{H}^1) \in w[\mathfrak{g}^{-\alpha_1}, \mathfrak{g}^{\alpha_1}] = [w\mathfrak{g}^{-\alpha_1}, w\mathfrak{g}^{\alpha_1}] = [\mathfrak{g}^{\tilde{\alpha}}, \mathfrak{g}^{-\tilde{\alpha}}].
\]
By Lemma 7.12, $w(\mathbb{H}^1)$ is necessarily a multiple of $\tilde{S} = S^1$. Moreover,
\[
2 = \alpha_1(\mathbb{H}^1) = -(w^{-1}\tilde{\alpha})(\mathbb{H}^1) = -\tilde{\alpha}(w\mathbb{H}^1)
\]
forces $w(\mathbb{H}^1) = -S^1$. Given $\xi \in \mathfrak{g}_\ell$, we have
\[
\ell w(\xi) = w(\ell \xi) = w[\mathbb{H}^1, \xi] = -[S^1, w(\xi)].
\]
Thus $w(\xi) \in \mathfrak{g}^{-\ell}$, and we conclude $w(\mathfrak{g}_\ell) = \mathfrak{g}^{-\ell}$. \qed

Table 7.2 expresses the $\mathbb{H}^{\alpha_1}$ in terms of the grading elements $\{S^j\}$.

**Remark 7.13.** For the grading $\mathfrak{g} = \oplus \mathfrak{g}^p$ associated to a fundamental adjoint variety, we know that $\mathfrak{g}^{-1}$ is an irreducible $\mathfrak{g}^0$-module. It will be important in the sequel that the symmetry in Lemma 7.10 identifies $\mathfrak{g}^{-1}$ as a $\mathfrak{g}^0$-module with $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module.
Table 7.2. The grading element \( H \).

| \( G \) | \( H \) |
|---|---|
| \( D_4 \) | \(-S^1 + 2S^2 - S^3 - S^4\) |
| \( B_r, D_r \) | \(-S^1 + 2S^2 - S^3\) |
| \( E_6 \) | \(2S^2 - S^4\) |
| \( E_7 \) | \(2S^3 - S^3\) |
| \( E_8 \) | \(-S^7 + 2S^8\) |
| \( F_4 \) | \(2S^1 - S^2\) |
| \( G_2 \) | \(-S^1 + 2S^2\) |

7.5. **Fundamental adjoint varieties II: boundary components and enhanced \( SL_2 \)-orbits.** We continue with the notation of Section 7.3, and write \( N \in g^{-1,-1} \cap g_Q \) for the nil-negative element of the standard triple associated to \( \beta_1 = \alpha_4 \). In this section we briefly describe \( X(N) \) and \( B(N) \).

In each case it is clear from the description in Section 7.1 that \( g_{B(N)}^{ss} \) is the semisimple Lie subalgebra of \( g \) whose roots are the ones strongly orthogonal to the \( \alpha_1 \). Since \( g_1 \cong g_{-1} = \mathfrak{sl}_2(N) \) is a faithful representation of \( G_{B(N)}^{ss} \), and \( \phi \in g_{B(N)}^{ss} \), \( G_{B(N)}^{ss} \) is the generic Mumford–Tate group of the Hodge structures on \( g_1 \) parameterized by \( D(N) \). Moreover, the action of \( G_{B(N)}^{ss} \) on the line \( g_1 = g^2-1 \subset g_1 \) presents \( \bar{D}(N) \) as a subvariety of \( \mathbb{P}g_1 \). Since the image of \( \tilde{\alpha} \) under \( w \) (cf. Section 7.4) is the highest \( g^0 \)-weight of \( g^{-1} \), we obtain

**Proposition 7.14.** The compact dual \( \bar{D}(N) \subset \mathbb{P}g_1 \) is isomorphic to the variety of lines \( C_0 \subset \mathbb{P}g^{-1} \). We therefore have \( (\bar{D}) X(N) \cong \mathbb{P}^1 \times \bar{D}(N) \cong \mathbb{P}^1 \times C_0 \).

**Proof.** We only need to check that the \( \mathbb{P}^1 \) factor of \( X(N) \) is minimally embedded in \( \bar{D} \). This follows from \( \tilde{\alpha}(H^{p1}) = 1 \), since \( \tilde{\alpha} \) is also the highest weight of \( g \). \( \square \)

Turning to the boundary component \( B(N) \stackrel{\rho}{\to} D(N) \), we may think of points in \( D(N) \) as Hodge flags \( F^* \) on the Tate twist \( U_N \stackrel{\text{dfn}}{=} g_1(-1) = g_{-1}(-2) \) (weight 3 HS). From (7.3) we see at once that the fibres of \( \rho \) take the form \( U_N/F^2 U_N \).

Now the object that provides a partial compactification of the quotient of \( D \) by a neat arithmetic subgroup \( \mathfrak{G} \) of \( G_{Q} \), is the quotient \( \bar{B}(N) \) of \( B(N) \) by \( \mathfrak{G} \cap Z(N) \), cf. [31, Section 7].

**Proposition 7.15.** For each of the fundamental adjoint varieties, \( \bar{B}(N) \stackrel{\beta}{\to} \bar{D}(N) \) is a family of intermediate Jacobians associated to a variation of Hodge structure (with Hodge numbers \( (1,a,a,1) \)) over a Shimura variety. The corresponding variations \( \mathcal{U}_N \to D(N) \) recover (with one exception) the list of weight 3 maximal Hermitian VHS of Calabi-Yau type from [22].

**Proof.** Immediate from comparison with [22] and the fact that \( G_{B(N)}^{ss} \) is the Mumford–Tate group of \( \mathcal{U}_N \). \( \square \)

\footnote{\( G_{B(N)}^{ss} \) is the group \( G_s \) (\( s = 1 \)) of Section 7.2, but we will not write \( G_1 \) due to notational conflict with \( g_1 \) (which is obviously not its Lie algebra).}
Table 7.3. $D(N)$ and $X(N)$ for the fundamental adjoint varieties

| $g$         | $\mathfrak{so}(n+4)$ ($n \geq 3$) | $\mathfrak{e}_6$ | $\mathfrak{e}_7$ | $\mathfrak{e}_8$ | $\mathfrak{f}_4$ | $\mathfrak{g}_2$ |
|-------------|-----------------------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $\mathfrak{g}_8$ | $\mathfrak{so}(4, n)$            | $\mathfrak{E}_{II}$ | $\mathfrak{E}_{VI}$ | $\mathfrak{E}_{IX}$ | $\mathfrak{F}_I$ | $\mathfrak{G}$    |
| $\mathfrak{g}_8^\circ$ | $\mathfrak{so}(2, n-2) \oplus \mathfrak{su}(1, 1)$ | $\mathfrak{su}(3, 3)$ | $\mathfrak{so}^*(12)$ | $\mathfrak{E}_{VII}$ | $\mathfrak{sp}(3, \mathbb{R})$ | $\mathfrak{su}(1, 1)$ |
| $D(N)$      | $\mathcal{H} \times \mathcal{I}V_{n-2}$ | $\mathfrak{I}_{3, 3}$ | $\mathfrak{I}_6$ | $\mathfrak{E}_{VII}$ | $\mathfrak{III}_3$ | $\mathcal{H}$   |
| $X(N)$      | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{Q}^{n-2}$ | $\mathbb{P}^1 \times \mathfrak{Gr}(3, \mathbb{C}^6)$ | $\mathbb{P}^1 \times \mathfrak{S}_6$ | $\mathbb{P}^1 \times \mathfrak{E}_7/P_7$ | $\mathbb{P}^1 \times \mathfrak{LG}(3, \mathbb{C}^6)$ | $\mathbb{P}^1 \times v_3(\mathbb{P}^1)$ |
| $\dim(X(N))$ | $n$                               | 10                | 16                | 28                | 7                 | 2                 |

Remark 7.16. We refer to Corollary 2.29 and Theorem 6.7 of [22] for further description of these maximal variations. The missing case, labeled $I_{1, n}$ in [22, Corollary 2.29], arises in the same way from a boundary component of the (non-fundamental) adjoint variety for $A_{n+2}$.

With this addition, the compact duals $\hat{D}(N)$ yield the homogeneous Legendrian varieties as presented in [22, Theorem 6.7]; whereas the varieties of lines $C_o$ realize them (as in [37]) as the subadjoint varieties. The Weyl flip of Section 7.4 toggles between these realizations. To the authors, this suggests comparing the homology classes of $X(N)$ and the Schubert variety $\text{Cone}(C_o)$, cf. Section 7.6.

The data for all the fundamental adjoint varieties is summarized in Table 7.3, where we note that $\dim(X(N)) = a + 1$. The type of $D(N)$ as a Hermitian symmetric domain comes from [22]; otherwise, see Appendix A for more details.

Proposition 7.17. The $X(N)$ of Table 7.3 are all maximal horizontal subvarieties of $\hat{D}$.

Proof. By [45, Corollary 3.13], the dimension of a horizontal manifold is bounded by the maximal dimension of the Schubert VHS in $\hat{D}$. So Corollary 2.23 establishes the proposition for $g = \mathfrak{so}(n+4)$. Likewise [45, Example 5.9, and Corollaries 5.13 and 5.29] yield Proposition 7.17 for the exceptional $g = \mathfrak{e}_6, \mathfrak{f}_4, \mathfrak{g}_2$.

Computing as in [45, Section 5], one may confirm that the maximal Schubert VHS in the $E_7$–adjoint variety $\hat{D} = E_7(\mathbb{C})/P_7$ are all of dimension 16. Indeed, they are the $X_w$ with $\Delta(w) = \{ \alpha \in \Delta \mid \alpha(S^1) = 1, \alpha(T_w) \leq 0 \}$ given by (the first column of) Table 7.4. Similarly, the maximal Schubert VHS in the $E_8$–adjoint variety $\hat{D} = E_8(\mathbb{C})/P_8$ are all of dimension 28; they are the $X_w$ with $\Delta(w) = \{ \alpha \in \Delta \mid \alpha(S^8) = 1, \alpha(T_w) \leq 0 \}$ given by (the second column of) Table 7.4. \hfill $\Box$

7.6. Computing $[X(N)]$: a simple example. One purpose of introducing the $X(\sigma)$ was to produce smooth algebraic representatives of classes in $H_*(\hat{D}, \mathbb{Z})$. In particular, for the $G_2$ and $F_4$ adjoint varieties, none of the maximal Schubert VHS are smooth. So it seems natural to conclude this paper by computing $[X(N)]$ for the $G_2$-adjoint variety.

Recall the Schubert variety $X$ of (3.2). Because $\dim_{\mathbb{C}}(X(N)) = 2$ and $H_1(\hat{D}, \mathbb{Z}) = \mathbb{Z}[X]$, the class $[X(N)]$ will necessarily be a multiple of $[X]$. We will determine this multiple.

We begin with the description of $X$ as a Tits transform. Let $Q \subset G_2$ be the maximal parabolic associated to the first simple root, so that $G_2/Q$ is a five dimensional quadric, and the minimal homogeneous embedding lies in $\mathbb{P}V_{w_0}$. Then $G_2/Q$ parameterizes a uniruling of $\hat{D}$ by $\mathbb{P}^1$s, cf. Table 3.1. Let $G'$ be the connected simple subgroup of $G_2$ associated to
Table 7.4. The maximal Schubert VHS in the $E_7$ and $E_8$–adjoint varieties

| The grading element $T_w$ | $\epsilon_7$ | $\epsilon_8$ |
|---------------------------|------------|------------|
| $-S^1 + S^3$             | $S^2 - S^8$| $S^5 - 2S^8$|
| $-S^1 + S^5$             | $S^2 + S^6 - 3S^8$| $S^5 + S^7 - 5S^8$|
| $-2S^1 + S^3 + S^6$     | $S^2 + S^6 - 3S^8$| $S^4 + S^7 - 4S^8$|
| $-3S^1 + S^2 + S^5 + S^7$| $S^1 + S^7 - 2S^8$| $S^7 - S^8$|
| $-2S^1 + S^4 + S^7$     | $S^3 + S^7 - 3S^8$| $S^7 - S^8$|
| $-S^1 + S^2 + S^7$      |               |            |
| $S^7$                    |               |            |

the simple root $\alpha_1$, and let $\Sigma \subset G_2/Q$ be the $G'$–orbit of the identity coset $Q/Q$. By (3.8a) and Lemma 3.13, the variety $X$ is the Tits transform $T(\Sigma)$. If $\eta$ is the fundamental weight of $G'$, then $\omega_1$ restricts to $\eta$ on $g'$. It follows that $\Sigma$ is a minimal homogeneous embedding of $\mathbb{P}^1$.

Let $\Sigma(N)$ denote the $G_{ssB}(\mathbb{C})$–orbit of $Q/Q$ in $G_2/Q$. Then $X(N)$ is the Tits transform $T(\Sigma(N))$. Note that the simple root $2\alpha_1 + \alpha_2$ of $G_{ssB}(\mathbb{C})$ is image of $\alpha_1$ under the Weyl group element $w = (12) \in W$. It follows that $G_{ssB}(\mathbb{C}) = \text{Ad}_w(G')$. As a consequence we may identify $\eta$ with the fundamental weight of $G_{ssB}(\mathbb{C})$.

Claim. The restriction of $\omega_1$ to $G_{ssB}(\mathbb{C})$ is $2\eta$.

It follows that $\Sigma(N)$ is the second Veronese embedding of $\mathbb{P}^1$. Since $\dim_{\mathbb{C}}\Sigma(N) = 1$ and $H_2(G_2/Q, \mathbb{Z}) = \mathbb{Z}[\Sigma]$, we may conclude that

$$[\Sigma(N)] = 2[\Sigma].$$

It follows from [17, Lemmas 3.11 and 3.13] that

$$[X(N)] = 2[X].$$

Proof of claim. The Lie algebra $G_{ssB}(\mathbb{C})$ is $g' \oplus \mathbb{C}H' \oplus g^{-\alpha}$ where $\alpha = 2\alpha_1 + \alpha_2$. The fundamental weight $\eta$ is defined by $\eta(H') = 1$. On the other hand $H' = S^1$, as an element of the Cartan subalgebra of $g_2$, and $\omega_1 = 2\alpha_1 + \alpha_2$. Therefore $\omega_1(H') = 2$, and the claim follows.

The analogous computation of $[X(\sigma)]$ in the case of the $SO(2r + 1, \mathbb{C})$–adjoint variety $\tilde{D}$ is worked out in Appendix B. The systematic computation of $[X(\sigma)]$ will be taken up in a subsequent paper, using different methods.

Appendix A. Examples in codimension one

Let $G$ be simple and $P$ a maximal parabolic with associated grading element $E = S^4$. Assume that the first root $\alpha_1 \in B$ is the simple root $\alpha_1$ (Section 6.9). Then the $G_{ss}$–orbit
$O_1$ has codimension one (Lemma 6.52). In this section we will determine the associated bigradings

\[(A.1) \quad g^{p,q} = \frac{1}{2} g^{p,q}\]

for each pair $(g, i)$. Setting $N = N_1 \in g^{-\alpha_1} \subset g^{-1,1}$, we will also identify the boundary component $B(N) = \tilde{B}(N) / \exp(\mathbb{C}N)$, and the Mumford–Tate domain $D(N)$ for the split Hodge structure.

**Remark A.2 (The Hermitian symmetric case).** From these examples we will see that if $G/P_1$ is Hermitian symmetric, then the bigrading is of the form depicted in Figure A.1.b.

### A.1. Type A.

Suppose that $g = \mathfrak{sl}_{r+1} \mathbb{C}$. Then $g_\mathbb{R} = \mathfrak{su}(i, r + 1 - i)$, the roots of $g$ are

\[(A.3) \quad \Delta = \{ \pm \alpha_{jk} \mid 1 \leq j \leq k \leq r \},\]

where

\[\alpha_{jk} \overset{\text{dfn}}{=} \alpha_j + \cdots + \alpha_k,\]

and

\[(A.4) \quad g^{p,q} = 0 \quad \text{if} \mid p \mid > 1 \text{ or} \mid q \mid > 1.\]

If $i \in \{1, r\}$, then we also have $g^{1,-1} = g^{-1,1} = 0$. All other $g^{p,q}$ are nonzero. These observations are represented in Figure A.1, where each node in the $pq$–plane represents a non-trivial $g^{p,q}$. (Figure A.1.A represents the case $i \in \{1, r\}$, and Figure A.1.B represents the case $1 < i < r$.)

We now identify the Mumford–Tate domain $D(N)$ and boundary component $B(N)$. We will assume that $1 < i < r$, leaving the other two cases (which are symmetric) to the reader. In particular, the bigrading $g^{p,q}$ is of the form depicted in Figure A.1.B. In this example, $\phi = -\sqrt{-1}(S^{i-1} + S^{i+1})$. From Figure A.1.B and Lemma 6.52, we see that $g_0 = z_0(N) \oplus \langle H^1 \rangle$, so that

\[\Delta(z_0(N)) = \pm \{ \alpha_{jk} \mid i + 2 \leq j \leq k \leq r, \quad 1 \leq j \leq k \leq i - 2, \quad \text{or} \quad 1 \leq j < i < k \leq r \}.\]
To determine $g_{B(N)}$, observe that given $\phi$ acts on $g^a \subset J_0(N)$ by a nonzero scalar if and only if $j < i < k$ (in which case that scalar is $\pm 2\sqrt{-1}$). So, $g_{B(N)}$ is the $\mathbb{Q}$–Lie algebra closure of the subspace

$$\bigoplus_{j<i<k} g^{\pm \alpha_{jk}} = J_0(N) \cap (g^{-1,1} \oplus g^{1,-1}).$$

It follows that $\{\alpha_1, \ldots, \alpha_{i-2}, \alpha_{i-1} + \alpha_1 + \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_r\}$. The corresponding Dynkin diagram is

\[
\alpha_1 \alpha_2 \cdots \alpha_{i-2} \alpha_{i-1} + \alpha_1 + \alpha_{i+1} \cdots \alpha_r.
\]

Whence $g_{B(N)}^\text{ss}(\mathbb{C}) \simeq \mathfrak{sl}_{i-1}$ and

$$g_{B(N)}^\text{ss}(\mathbb{R}) \simeq \mathfrak{su}(i-1, r-i).$$

Also, $\omega^\text{res}_1 = \eta_{i-1}$, so that the compact dual is

$$\hat{D}(N) \simeq \text{Gr}(i-1, r-1).$$

Next, we have $J_-(N) \equiv g^{-1,0} \oplus g^{0,-1}$ modulo $g^{-1,-1} = g^{-\alpha_1}$. Observe that

$$\Delta(g^{0,-1}) = \{\alpha_{jk} \mid j = i + 1 \text{ or } k = i - 1\},$$

$$\Delta(g^{-1,0}) = \{-\alpha_{ik} \mid i < k\} \cup \{-\alpha_{jk} \mid j < i\},$$

and $\dim_{\mathbb{C}} g^{0,-1} = \dim_{\mathbb{C}} g^{-1,0} = (r-i) + (i-1) = r-1$. It follows from [31, Section 7] that $B(N)$ fibres over $D(N)$ with fibres $\mathbb{C}^{r-1}$.

**A.2. Type B.** Suppose that $g = \mathfrak{so}_{2r+1}$, with $r \geq 3$. The real form constructed in Section 6.3 is $g_{\mathbb{R}} = \mathfrak{so}(2i, 2(r-i) + 1)$. The roots of $g$ are

$$\Delta = \{\alpha_{j} \mid 1 \leq j \leq k \leq r\} \cup \{\beta_{jk} \mid 1 \leq j < k \leq r\},$$

where

$$\beta_{jk} \overset{\text{df}}{=} \alpha_j + \cdots + \alpha_{k-1} + 2(\alpha_k + \cdots + \alpha_r),$$

and

$$H^1 = \begin{cases} 2s^1 - s^2, & \text{if } i = 1, \\ -s^{i-1} + 2s^i - s^{i+1}, & \text{if } 1 < i < r, \\ -2s^{r-1} + 2s^r, & \text{if } i = r. \end{cases}$$

From the expressions for $\Delta$ and $H^{\alpha_i}$ above, we find:

- If $i = 1$, then (A.4) holds and all other $g^{p,q}$ are non-trivial; this bigrading is represented by Figure A.1.B.
- If $i = 2$, then the bigrading is represented by Figure A.2.A.
- If $2 < i < r$, then the bigrading is represented by Figure A.2.B.
- If $i = r$, then the bigrading is represented by Figure A.2.C.

We now identify the Mumford–Tate domain $D(N)$ and boundary component $B(N)$. The (Hermitian symmetric) case $i = 1$ is left to the reader.
Suppose that \( g_{\text{adj}} \). From Figure A.2.A, and the fact that \( g^{-1,-1} \) is one-dimensional (Lemma 6.52), we see that that \( g_0 = \mathfrak{g}_0 \). Therefore,

\[
\Delta(\mathfrak{g}_0) = \{ \alpha_{jk} \mid i + 2 \leq j \leq k \leq \ell, \text{ or } j = 1, 3 \leq k \leq \ell \} \\
\cup \{ \beta_{jk} \mid i + 2 \leq j \leq k \leq \ell \} \cup \{ \beta_{2,3} \}.
\]

It follows that the simple roots of \( \mathfrak{g}_{B(N)}^{ss} \) are \( \alpha_1 + \alpha_2 + \alpha_3, \alpha_4, \ldots, \alpha_r \) and \( \beta_{2,3} = \alpha_2 + 2(\alpha_3 + \cdots + \alpha_r) \). Whence the Dynkin diagram of \( \mathfrak{g}_{B(N)}^{ss} \) is

\[
\begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 & \quad \alpha_4 \quad \ldots \quad \alpha_{r-1} \quad \alpha_r \\
& \quad \alpha_2 + 2(\alpha_3 + \cdots + \alpha_r)
\end{align*}
\]

Thus

\[
\mathfrak{g}_{B(N)}^{ss}(\mathbb{R}) = \mathfrak{so}(2, 2r - 5) \oplus \mathfrak{su}(1, 1).
\]

Also, \( \omega_2^{res} = \eta_1 + \eta_{r-1} \) so that

\[
\hat{D}(N) = Q^{2r-5} \times \mathbb{P}^1,
\]

where \( Q^{2r-5} = \text{OG}(1, \mathbb{C}^{2r-3}) \subset \mathbb{P}^{2r-4} \) is the quadric hypersurface.

Turning to \( \mathfrak{z}_-(N) \), from Figure A.2.A we see that \( \mathfrak{z}_-(N) \equiv \mathfrak{g}^{-1} \) modulo \( \mathfrak{g}_- = \mathfrak{g}^{-\alpha_1} \). Therefore, \( B(N) = B(N)_{(1)} \to D(N) \) has fibres \( \mathfrak{g}^{-1,0} \oplus \mathfrak{g}^{-2,1} = \mathbb{C}^{2r-3} \).

This example is continued in Appendix B; there we present a thorough study of the Zariski closure of the orbit exp(\( \mathfrak{C}(N)D(N) \)).

A.2.2. **The case that \( 2 < i < r \).** Suppose that \( 2 < i < r \); that is, the bigrading \( \mathfrak{g}^{p,q} \) is of the form depicted in Figure A.2.B. It follows from the figure, and the fact that \( g^{-1,-1} \) is one-dimensional (Lemma 6.52), that \( g_0 = g_0(N) \oplus (H^i) \), so that

\[
\Delta(\mathfrak{g}_0) = \{ \alpha_{jk} \mid i + 2 \leq j \leq k \leq \ell, \text{ or } j = 1, 3 \leq k \leq \ell \} \\
\cup \{ \beta_{jk} \mid 1 \leq j < k \leq i - 1, \text{ or } j < i < k \leq \ell \} \\
\cup \{ \beta_{i+1} \mid j \leq i - 1 \text{ and } i + 2 \leq k \}
\]

Suppose that \( \alpha_{jk} \in \Delta(\mathfrak{g}_0) \), then \( \phi = -\sqrt{-1}(s^{i-1} + s^{i+1}) \) acts on the root space \( \mathfrak{g}^{\alpha_{jk}} \) by a nonzero scalar if and only if \( j < i \) (in which case the scalar is \( -2\sqrt{-1} \)). Similarly, if \( \beta_{jk} \in \Delta(\mathfrak{g}_0) \), then \( \phi \) acts on \( \mathfrak{g}^{\beta_{jk}} \) by a nonzero scalar if and only if \( j \leq i - 1 \) and \( i + 2 \leq k \), or \( \beta_{jk} \in \beta_{i+1} \) (in either case the scalar is \( -2\sqrt{-1} \)). It follows that the simple roots of \( \mathfrak{g}_{B(N)} \) are \( \alpha_1, \ldots, \alpha_{i-2}, \alpha_{i+1} + \alpha_i + \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_r \) and \( \alpha_1 + 2(\alpha_{i+1} + \cdots + \alpha_r) \). The corresponding Dynkin diagram is

\[
\begin{align*}
\alpha_1 & \quad \ldots \quad \alpha_{i-2} \quad \alpha_{i+1} + \alpha_i + \alpha_{i+1} \quad \alpha_{i+2} \quad \ldots \quad \alpha_{r-1} \quad \alpha_r \\
& \quad \alpha_1 + 2(\alpha_{i+1} + \cdots + \alpha_r)
\end{align*}
\]
From this we see that
\[
\mathfrak{g}^{ss}_{B(N)}(\mathbb{R}) = \mathfrak{so}(2i - 2, 2(r - i) - 1) \oplus \mathfrak{su}(1, 1).
\]

Also, \( \omega_1^{res} = \eta_{i-1} + \eta_{r-1} \), so that the compact dual is
\[
\tilde{D}(N) = \text{OG}(i - 1, \mathbb{C}^{2r-3}) \times \mathbb{P}^1.
\]

Turning to \( \mathfrak{z}^-(N) \), from Figure A.2.B we see that \( \mathfrak{z}^-(N) \equiv \mathfrak{g}^{-1} \) modulo \( \mathfrak{g}_{-2} = \mathfrak{g}^{-\alpha_1} \), and \( B(N) = B(N)_{(1)} \to D(N) \) has fibres \( \mathfrak{g}^{-2,1} \oplus \mathfrak{g}^{-1,0} \simeq \mathbb{C}^{2r-3} \).

A.2.3. The case that \( i = r \). It remains to address the case that \( i = r \) (Figure A.2.C). We leave it to the reader to confirm the following. The Dynkin diagram of \( \mathfrak{g}^{ss}_{B(N)} \) is

\[
\begin{array}{c}
\alpha_1 \cdots \alpha_{r-2} \alpha_{r-1} + 2(\alpha_{r-1} + \alpha_r) \\
\end{array}
\]

In particular, \( \mathfrak{g}^{ss}_{B(N)}(\mathbb{R}) = \mathfrak{su}(r) \) is the compact form of \( \mathfrak{sl}_r \mathbb{C} \) and \( D(N) = \tilde{D}(N) \simeq \mathbb{P}^{r-1} \).

Finally, \( B(N) \) fibres over \( D(N) \) with fibres \( \mathfrak{g}^{2,0} \simeq \mathbb{C}^{r-1} \).

A.3. Type C. Suppose that \( \mathfrak{g} = \mathfrak{sp}_{2r} \mathbb{C} \), with \( r \geq 2 \). The roots of \( \mathfrak{g} \) are
\[
\Delta = \pm\{\alpha_{jk} \mid j \leq k\} \cup \pm\{\gamma_{jk} \mid j \leq k \leq r - 1\},
\]

where
\[
\gamma_{jk} \overset{\text{df}}{=} \alpha_{jr} + \alpha_{k,r-1},
\]

and
\[
\mathfrak{h}^i = \begin{cases} 
2\mathfrak{g}^1 - 2\mathfrak{g}^2, & \text{if } i = 1, r = 2, \\
2\mathfrak{g}^1 - \mathfrak{g}^2, & \text{if } i = 1, 2 < r, \\
-\mathfrak{g}^i - 2\mathfrak{g}^{i+1}, & \text{if } 1 < i < r - 1, \\
-\mathfrak{g}^{r-2} + 2\mathfrak{g}^{r-1} - 2\mathfrak{g}^r, & \text{if } i = r - 1, \\
-\mathfrak{g}^{r-1} + 2\mathfrak{g}^r, & \text{if } i = r.
\end{cases}
\]

From the expressions for \( \Delta \) and \( \mathfrak{h}^{\alpha_1} \) above, we find:
- If \( i = 1 \) and \( r = 2 \), then the bigrading (A.1) is represented by Figure A.3.A.
- If \( i = 1 \) and \( r > 2 \), then the bigrading is of the form in Figure A.3.B.
- If \( 1 < i < r \), then the bigrading is represented by Figure A.3.C.
- If \( i = r \), then the bigrading is represented by Figure A.1.B.

**Figure A.3.** The nonzero \( \mathfrak{g}^{p,q} \).
Analysis of the (Hermitian symmetric) case that $i = r$ is left to the reader. For each of the cases that follow ($i < r$), the real form (6.6) constructed in Section 6.3 is

$$\mathfrak{g}_R = \mathfrak{sp}(i, r - i).$$

**A.3.1. The case that $i = 1$ and $r = 2$.** In this case (Figure A.3.A) the nonzero $\mathfrak{g}^{p,q}$ all have dimension one, with the exception of $\mathfrak{g}^{0,0} = \mathfrak{h}$ which has dimension two. In this case $\Delta(\mathfrak{g}_0) = \pm\{\alpha_1 + \alpha_2\}$ and $\mathfrak{g}_0(N)(\mathbb{C}) = \text{span}_\mathbb{C}\{\phi\} = \text{span}_\mathbb{C}\{S^2\}$ is one dimensional. In particular,

$$\mathfrak{g}_{B(N)}(\mathbb{C}) = \text{span}_\mathbb{C}\{S^2\},$$

so that the compact dual $\check{D}(N)$ is a point. From Figure A.3.A we see that $B(N) = B(N)(2) \to B(N)(1) = D(N)$ has fibre $\mathbb{C}$.

**A.3.2. The case that $i = 1$ and $2 < r$.** In this case $\mathfrak{g}_0(N)(\mathbb{C}) = \mathfrak{h}'$, so that $\check{D}(N)$ is a point. From Figure A.3.B (and Lemma 6.52) we see that the fibres of $B(N) = B(N)(2) \to B(N)(1) \to D(N)$ are $\mathfrak{f}^{(2)} = \mathfrak{g}^{-2,0} = \mathbb{C}$ and $\mathfrak{f}^{(1)} = \mathfrak{g}^{-1,0} \oplus \mathfrak{g}^{-2,1} = \mathbb{C}^{2r-4}$.

**A.3.3. The case that $1 < i < r - 1$.** The Dynkin diagram of $\mathfrak{g}_{B(N)}^{ss}$ is

$$\begin{array}{cccccccc}
\alpha_1 & 0 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{r-1} & \alpha_r \\
\alpha_{r-1} + \alpha_1 + \alpha_{i+1} & & & & & & & & \end{array}$$

so that

$$\mathfrak{g}_{B(N)}^{ss}(\mathbb{R}) = \mathfrak{sp}(i - 1, r - 1).$$

Also, $\omega_i^{\text{res}} = \eta_{i-1}$, so that

$$\check{D}(N) = \text{SG}(i - 1, \mathbb{C}^{2r-4}).$$

is a symplectic grassmannian. From Figure A.3.C (and Lemma 6.52) we see that the fibres of $B(N) = B(N)(2) \to B(N)(1) \to D(N)$ are $\mathfrak{f}^{(2)} = \mathfrak{g}^{-2,0} = \mathbb{C}$ and $\mathfrak{f}^{(1)} = \mathfrak{g}^{-1,0} \oplus \mathfrak{g}^{-2,1} = \mathbb{C}^{2r-4}$.

**A.3.4. The case that $i = r - 1$.** The Dynkin diagram of $\mathfrak{g}_{B(N)}^{ss}$ is

$$\begin{array}{cccccccc}
\alpha_1 & 0 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{r-4} & \alpha_{r-3} & 2\alpha_{r-2} + 2\alpha_{r-1} + \alpha_r \\
& & & & & & & & \end{array}$$

Therefore,

$$\mathfrak{g}_{B(N)}^{ss}(\mathbb{R}) = \mathfrak{sp}(r - 2)$$

is compact, so that $D(N) = \check{D}(N)$. Since $\omega_{r-1}^{\text{res}} = \eta_{r-2}$, we see that

$$\check{D}(N) = \text{LG}(r - 2, \mathbb{C}^{2r-4})$$

is the (Hermitian symmetric) Lagrangian grassmannian. From Figure A.3.C (and Lemma 6.52) we see that the fibres of $B(N) = B(N)(2) \to B(N)(1) \to D(N)$ are $\mathfrak{f}^{(2)} = \mathfrak{g}^{-2,0} = \mathbb{C}$ and $\mathfrak{f}^{(1)} = \mathfrak{g}^{-1,0} \oplus \mathfrak{g}^{-2,1} = \mathbb{C}^{2r-4}$.
A.4. Type D. Suppose that $\mathfrak{g} = \mathfrak{so}_2, \mathbb{C}$, with $r \geq 4$. The roots of $\mathfrak{g}$ are

$$
\Delta = \pm \{\alpha_j \mid j \leq k\} \cup \pm \{\alpha_j \mid j \leq r-2\} \cup \pm \{\varepsilon_{jk} \mid 1 \leq j < k \leq r-2\},
$$

where

$$
\alpha_j \overset{\text{dfn}}{=} \alpha_j + \cdots + \alpha_{r-2} + \alpha_r \quad \text{and} \quad \varepsilon_{jk} \overset{\text{dfn}}{=} \alpha_j + \cdots + \alpha_{k-1} + 2(\alpha_k + \cdots + \alpha_{r-2}) + \alpha_{r-1} + \alpha_r.
$$

and

$$
H^1 = \begin{cases} 
2S^1 - S^2, & \text{if } i = 1, \\
-S^{i-1} + 2S^i - S^{i+1}, & \text{if } 1 < i < r-2, \\
-S^{r-3} + 2S^{r-2} - S^{r-1} - S^r, & \text{if } i = r-2, \\
-S^{r-2} + 2S^1, & \text{if } i = r-1, r.
\end{cases}
$$

From the expressions for $\Delta$ and $H^{\alpha_1}$ above, we find:

- If $i = 1, r-1, r$, then the bigrading (A.1) is represented by Figure A.1.B.
- If $i = 2$, then the bigrading is of the form pictured in Figure A.2.A.
- If $2 < i < r-1$, then the bigrading is of the form pictured in Figure A.2.B.

Analysis of the (Hermitian symmetric) cases $i = 1, r-1, r$ will be left to the reader. For each of the cases that follow ($1 < i < r-1$), the real form (6.6) constructed in Section 6.3 is

$$
\mathfrak{g}_\mathbb{R} = \mathfrak{so}(2i, 2r - 2i).
$$

A.4.1. The case that $1 < i < r-2$. The Dynkin diagram of $\mathfrak{g}^{ss}_{B(N)}$ is

```
α_1 α_2 . . . α_{i-2} α_{i-1} + α_i + α_{i+1} . . . α_{r-3} α_{r-2} α_r
```

Whence

$$
\mathfrak{g}_{B(N)}(\mathbb{R}) = \mathfrak{so}(2i, 2r - 2i - 2) \oplus \mathfrak{su}(1, 1).
$$

We have $\omega_1^{\text{res}} = \eta_{i-1} + \eta_{r-1}$, so that

$$
\tilde{D}(N) = \text{OG}(i-1, \mathbb{C}^{2r-4}) \times \mathbb{P}^1.
$$

From Figure A.2.A&B we see that $\mathfrak{g}_{-2} = \mathfrak{g}^{-\alpha_1}$, and $B(N) = B(N)_{(i)} \rightarrow D(N)$ has fibres $\mathfrak{g}^{-2,1} \oplus \mathfrak{g}^{-1,0} \simeq \mathbb{C}^{2r-4}$.

A.4.2. The case that $i = r-2$. The Dynkin diagram of $\mathfrak{g}^{ss}_{B(N)}$ is

```
α_1 α_2 . . . α_{r-5} α_{r-4} α_{r-3} + α_{r-2} + α_{r-1} . . . α_{r-3} + α_{r-2} + α_r
```

Whence

$$
\mathfrak{g}_{B(N)}(\mathbb{R}) = \mathfrak{so}(2, 2r - 6) \oplus \mathfrak{su}(1, 1).
$$

We have $\omega_1^{\text{res}} = \eta_{i-3} + \eta_{r-2} + \eta_{r-1}$, so that

$$
\tilde{D}(N) \subset S'_{r} \times S_{r} \times \mathbb{P}^1.
$$

Here, $S'_{r}$ and $S_{r}$ are the two connected components of the orthogonal grassmannian $\text{OG}(r-2, \mathbb{C}^{2r-4})$.

From Figure A.2.B we see that $\mathfrak{g}_{-2} = \mathfrak{g}^{-\alpha_1}$, and $B(N) = B(N)_{(i)} \rightarrow D(N)$ has fibres $\mathfrak{g}^{-2,1} \oplus \mathfrak{g}^{-1,0} \simeq \mathbb{C}^{2r-4}$.
The exceptional cases. Less detail is presented in the exceptional types that follow (Sections A.5–A.9). The reader wishing to verify details will find the necessary descriptions of the root systems $\Delta$ and the Cartan matrices $(c^j_k)$ in standard references (such as [34]), and may also find Lie theory software (such as [40]) helpful, as we did.

A.5. Type $E_6$.

- If $i = 1, 6$, then the bigrading (A.1) is represented by Figure A.1.
- If $i = 2$, then the bigrading is of the form pictured in Figure A.2.
- If $i = 3, 5$, then the bigrading is represented by Figure A.2.
- If $i = 4$, then the bigrading is represented by Figure A.4.

Figure A.4. The nonzero $g^{p,q}$.

The real form $g_R$ of $g = e_6(\mathbb{C})$ (constructed in Section 6.3) is either $E_{II}$ or $E_{III}$; these two real forms are distinguished by their maximal compact subalgebras which are respectively $su(6) \oplus su(2)$ and $so(10) \oplus \mathbb{R}$. The (Hermitian symmetric) cases that $i = 1, 6$ are left to the reader. (In these two cases $g_R = E_{III}$ and $g_{B(N)}^{ss}(\mathbb{R}) = su(5, 1)$.) In each of the cases that follow $g_R = E_{II}$,

$$g_{B(N)}^{ss}(\mathbb{R}) = su(3, 3),$$

and the fibre of $B(N) = B(N)_{(1)} \rightarrow D(N)$ is $\mathbb{C}^{10}$.

A.5.1. The (adjoint) case that $i = 2$. The simple roots of $g_{B(N)}^{ss}$ are $\alpha_5, \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_1, \alpha_3$. We have $\omega_1^{res} = \eta_3$, so that the compact dual is

$$\hat{D}(N) = Gr(3, \mathbb{C}^6).$$

A.5.2. The case that $i = 3, 5$. These two cases are symmetric, so we will consider only $i = 3$. The simple roots of $g_{B(N)}^{ss}$ are $\alpha_2, \alpha_1 + \alpha_3 + \alpha_4, \alpha_5, \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$. We have $\omega_1^{res} = \eta_2 + \eta_5$, so that the compact dual is the partial flag variety

$$\hat{D}(N) = Flag(2, 5, \mathbb{C}^6).$$
A.5.3. The case that \( i = 4 \). The simple roots of \( \mathfrak{g}^\mathrm{ss}_{B(N)} \) are \( \alpha_2 + \alpha_4 + \alpha_5, \alpha_6, \alpha_3 + \alpha_4 + \alpha_5, \alpha_1, \alpha_2 + \alpha_3 + \alpha_4 \). We have \( \omega_4^\mathrm{res} = \eta_1 + \eta_3 + \eta_5 \), so that the compact dual is the partial flag variety
\[
\tilde{D}(N) = \text{Flag}(1, 3, 5, \mathbb{C}^6).
\]

A.6. Type \( E_7 \).

- If \( i = 1 \), then the bigrading is of the form pictured in Figure A.2.A.
- If \( i \in \{2, 6\} \), then the bigrading is of the form pictured in Figure A.2.B.
- If \( i \in \{3, 5\} \), then the bigrading is of the form pictured in Figure A.4.A.
- If \( i = 4 \), then the bigrading is of the form pictured in Figure A.4.B.
- If \( i = 7 \), then the bigrading is of the form depicted in Figure A.1.B.

The real form \( \mathfrak{g}_R \) of \( \mathfrak{g} = \mathfrak{e}_7(\mathbb{C}) \) (constructed in Section 6.3) is one of \( \text{E}_V, \text{E}_VI \) or \( \text{E}_VII \). The three real forms are distinguished by their maximal compact subalgebra; they are, respectively, \( \mathfrak{su}(8), \mathfrak{so}(12) \oplus \mathfrak{su}(2) \) and \( \mathfrak{e}_6 \oplus \mathbb{R} \). In each of the cases that follow
\[
\mathfrak{g}_{B(N)}(\mathbb{C}) = \mathfrak{so}_{12}\mathbb{C},
\]
and the fibre of \( B(N) = B(N)_{(1)} \to D(N) \) is \( \mathbb{C}^{16} \). The (Hermitian symmetric) case \( i = 7 \) will be left to the reader. (The real form \( \mathfrak{g}_R = \text{E}_VII \) occurs only in this case.)

A.6.1. The (adjoint) case that \( i = 1 \). In this case \( \mathfrak{g}_R = \text{E}_VI \). The simple roots of \( \mathfrak{g}^\mathrm{ss}_{B(N)} \) are \( \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 \). Thus
\[
\mathfrak{g}^\mathrm{ss}_{B(N)}(\mathbb{R}) = \mathfrak{so}^*(12).
\]

Since \( \omega_1^\mathrm{res} = \eta_6 \), the compact dual
\[
\tilde{D}(N) = S_6
\]
is the Spinor variety (one of the two connected components of the orthogonal grassmannian \( \text{OG}(6, \mathbb{C}^{12}) \)).

A.6.2. The case that \( i = 2 \). In this case \( \mathfrak{g}_R = \text{E}_V \). The simple roots of \( \mathfrak{g}^\mathrm{ss}_{B(N)} \) are \( \alpha_3, \alpha_1, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_6, \alpha_7, \alpha_5 \). Thus
\[
\mathfrak{g}^\mathrm{ss}_{B(N)}(\mathbb{R}) = \mathfrak{so}(6, 6).
\]

Since \( \omega_2^\mathrm{res} = \eta_3 \), the compact dual
\[
\tilde{D}(N) = \text{OG}(3, \mathbb{C}^{12})
\]
is an orthogonal grassmannian.

A.6.3. The case that \( i = 3 \). In this case \( \mathfrak{g}_R = \text{E}_VI \). The simple roots of \( \mathfrak{g}^\mathrm{ss}_{B(N)} \) are \( \alpha_2, \alpha_1 + \alpha_3 + \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \). Thus
\[
\mathfrak{g}^\mathrm{ss}_{B(N)}(\mathbb{R}) = \mathfrak{so}^*(12).
\]

Since \( \omega_3^\mathrm{res} = \eta_2 + \eta_6 \), the compact dual is the incidence variety
\[
\tilde{D}(N) = \{(E^2, E^6) \in \text{OG}(2, \mathbb{C}^{12}) \times S_6 \mid E^2 \subset E^6\}.
\]
A.6.4. The case that $i = 4$. In this case $\mathfrak{g}_R = \mathfrak{ev}_I$. The simple roots of $\mathfrak{g}_E^B(\mathbb{N})$ are $\alpha_2 + \alpha_3 + \alpha_4$, $\alpha_1$, $\alpha_3 + \alpha_4 + \alpha_5$, $\alpha_6$, $\alpha_7$, $\alpha_2 + \alpha_4 + \alpha_5$. Thus

$$\mathfrak{g}_E^B(\mathbb{N})(\mathbb{R}) = \mathfrak{so}^+(12).$$

Since $\omega_4^{res} = \eta_1 + \eta_3 + \eta_6$, the compact dual is the incidence variety

$$\hat{D}(\mathbb{N}) = \{(E^1, E^3, E^6) \in Q^{10} \times \text{OG}(3, C^{12}) \times S_6 \mid E^1 \subset E^3 \subset E^6\}.$$

A.6.5. The case that $i = 5$. In this case $\mathfrak{g}_R = \mathfrak{ev}_V$. The simple roots of $\mathfrak{g}_E^B(\mathbb{N})$ are $\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$, $\alpha_1$, $\alpha_3$, $\alpha_4 + \alpha_5 + \alpha_6$, $\alpha_7$, $\alpha_2$. Thus

$$\mathfrak{g}_E^B(\mathbb{N})(\mathbb{R}) = \mathfrak{so}(6, 6).$$

Since $\omega_5^{res} = \eta_1 + \eta_4$, the compact dual is the incidence variety

$$\hat{D}(\mathbb{N}) = \{(E^1, E^4) \in Q^{10} \times \text{OG}(4, C^{12}) \mid E^1 \subset E^4\},$$

where $Q^{10} = \text{OG}(1, C^{12}) \subset \mathbb{P}^{11}$ is the smooth quadric hypersurface.

A.6.6. The case that $i = 6$. In this case $\mathfrak{g}_R = \mathfrak{ev}_I$. The simple roots of $\mathfrak{g}_E^B(\mathbb{N})$ are $\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$, $\alpha_1$, $\alpha_3$, $\alpha_4 + \alpha_5 + \alpha_6$, $\alpha_7$, $\alpha_2$. Thus

$$\mathfrak{g}_E^B(\mathbb{N})(\mathbb{R}) = \mathfrak{so}^+(12).$$

Since $\omega_6^{res} = \eta_1 + \eta_5$, the compact dual is the incidence variety

$$\hat{D}(\mathbb{N}) = \{(E^1, E^5) \in Q^{10} \times \text{OG}(5, C^{12}) \mid E^1 \subset E^5\}.$$

A.7. Type $E_8$.

- If $i = 1$, then the bigrading is of the form pictured in Figure A.2.B.
- If $i \in \{2, 7\}$, then the bigrading is of the form pictured in Figure A.4.A.
- If $i \in \{3, 6\}$, then the bigrading is of the form pictured in Figure A.4.B.
- If $i = 4$, then the bigrading is of the form pictured in Figure A.5.A.
- If $i = 5$, then the bigrading is of the form pictured in Figure A.5.B.
- If $i = 8$, then the bigrading is as depicted in Figure A.2.A.

**Figure A.5.** The nonzero $\mathfrak{g}^{p,q}$. 

![Diagram](attachment:image.png)
The real form $g_R$ of $g = \mathfrak{f}_4(\mathbb{C})$ (constructed in Section 6.3) is either E VIII or E IX; the two real forms are distinguished by their maximal compact subalgebras, which are respectively, $\mathfrak{so}(16)$ and $\mathfrak{c}_7 \oplus \mathfrak{su}(2)$. In each of the examples that follow

$$g^s_{B(N)}(\mathbb{C}) = \mathfrak{c}_7(\mathbb{C}),$$

and the fibre of $B(N) = B(N)_{(1)} \to D(N)$ is $\mathbb{C}^{28}$. The real form $g^s_{B(N)}(\mathbb{R})$ will be one of E V or E VII; the two forms are distinguished by their maximal compact subalgebras, which are respectively $\mathfrak{su}(8)$ and $\mathfrak{e}_6 \oplus \mathbb{R}$. We are unaware of good geometric descriptions of the $G_{B(N)}(\mathbb{C})$–orbits $\mathcal{D}(N) \subset \mathbb{P} \mathfrak{e}_7(\mathbb{C})$, and so will only give the weight $\omega^\text{res}_i$.

A.7.1. The case that $1 \leq i \leq 7$. The values of $g^s_{B(N)}(\mathbb{R})$ and $\omega^\text{res}_i$ are recorded in the table below.

| $i$ | $g_R$ | $g^s_{B(N)}(\mathbb{R})$ | $\omega^\text{res}_i$ |
|-----|-------|-----------------|-----------------|
| 1   | E VIII | E V             | $\eta_2$        |
| 2   | E VIII | E V             | $\eta_5$        |
| 3   | E IX   | E VII           | $\eta_2 + \eta_6$|
| 4   | E IX   | E VII           | $\eta_2 + \eta_5 + \eta_7$|
| 5   | E VIII | E V             | $\eta_4 + \eta_7$|
| 6   | E VIII | E V             | $\eta_3 + \eta_7$|
| 7   | E IX   | E VII           | $\eta_1 + \eta_7$|

A.7.2. The (adjoint) case that $i = 8$. In this case $g_R = E IX$. We have $g^s_{B(N)}(\mathbb{R}) = E VII$, and $\omega^\text{res}_8 = \eta_7$, so that $\mathcal{D}(N) \subset \mathbb{P} \mathfrak{e}_7(\mathbb{C})$ is the (Hermitian symmetric) Freudenthal variety.

A.8. Type $F_4$.

- If $i = 1$, then the bigrading is as depicted in Figure A.2.A.
- If $i = 2$, then the bigrading is of the form pictured in Figure A.4.A.
- If $i = 3$, then the bigrading is of the form pictured in Figure A.6.A.
- If $i = 4$, then the bigrading is of the form pictured in Figure A.3.C.

Figure A.6. The nonzero $g^{p,q}$.

The real form $g_R$ of $g = \mathfrak{f}_4(\mathbb{C})$ (constructed in Section 6.3) is either F I or F II; the two real forms are distinguished by their maximal compact subalgebras, which are respectively, $\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$ and $\mathfrak{so}(9)$.
A.8.1. The (adjoint) case that $i = 1$. Here $\frak{g}_R = F I$. The Dynkin diagram of $\frak{g}^s_B(N)$ is 
\[ \alpha_3 \quad \alpha_1 + 2\alpha_2 + 2\alpha_3 \]
so that $\frak{g}^s_B(N)(\mathbb{R}) = \frak{sp}(3,\mathbb{R})$. The restriction of the fundamental weight $\omega_1$ to $\frak{g}^s_B(N)$ is $\eta_3$, so that 
\[ \hat{D}(N) = LG(3,\mathbb{C}^6) \]
is a Lagrangian grassmannian. The fibres of $B(N) = B(N)(1) \to D(N)$ are $\mathbb{C}^7$.

A.8.2. The case that $i = 2$. Here $\frak{g}_R = F I$. The Dynkin diagram of $\frak{g}_B(N)$ is 
\[ \alpha_1 + \alpha_2 + \alpha_3 \quad \alpha_2 + 2\alpha_3 \]
Whence 
\[ \frak{g}^s_B(N)(\mathbb{R}) \simeq \frak{sp}(3,\mathbb{R}) . \]
We have $\omega_1^{res} = 2\eta_1 + \eta_3$. Therefore, if $\nu$ is a nondegenerate skew-symmetric bilinear form on $\mathbb{C}^6$, then 
\[ \hat{D}(N) \text{ is a non-minimal embedding of Flag}_+(1,3,\mathbb{C}^6) , \]
the $\nu$–isotropic partial flags $E^1 \subset E^3 \subset \mathbb{C}^6$. The fibration $B(N) = B(N)(1) \to D(N)$ has fibre $\mathbb{C}^7$.

A.8.3. The case that $i = 3$. Here $\frak{g}_R = F II$. The Dynkin diagram of $\frak{g}_B(N)$ is 
\[ \alpha_1 \quad \alpha_1 + 2\alpha_2 + 2\alpha_3 \quad \alpha_2 + 2\alpha_3 + 2\alpha_4 \]
Whence 
\[ \frak{g}^s_B(N)(\mathbb{R}) \simeq \frak{su}(4) , \]
so that $D(N) = \hat{D}(N)$. The restriction of $\omega_3$ to $\frak{g}^s_B(N)$ is the weight $\eta_2 + \eta_3$, so that 
\[ \hat{D}(N) = Flag(2,3,\mathbb{C}^4) . \]
The fibres of $B(N) = B(N)(2) \to B(N)(1) \to D(N)$ are $\mathfrak{g}^{(2)} = \mathbb{C}^3$ and $\mathfrak{g}^{(1)} = \mathbb{C}^4$.

A.8.4. The case that $i = 4$. Here $\frak{g}_R = F II$. The Dynkin diagram of $\frak{g}_B(N)$ is 
\[ \alpha_1 \quad \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 \]
Whence 
\[ \frak{g}^s_B(N)(\mathbb{R}) = \frak{su}(4) , \]
and $D(N) = \hat{D}(N)$. Since the restriction of the fundamental weight $\omega_4$ to $\frak{g}^s_B(N)$ is $\eta_3$, we have 
\[ \hat{D}(N) = \mathbb{P}^3 . \]
The fibres of $B(N) = B(N)(2) \to B(N)(1) \to D(N)$ are $\mathfrak{g}^{(2)} = \mathbb{C}^3$ and $\mathfrak{g}^{(1)} = \mathbb{C}^4$.

A.9. Type $G_2$.

\begin{itemize}
  \item If $i = 1$, then the bigrading is of the form pictured in Figure A.6.B.
  \item If $i = 2$, then the bigrading is as depicted in Figure A.2.A.
\end{itemize}
In both cases that follow the real form $\frak{g}_R$ of $\frak{g} = \frak{g}_2(\mathbb{C})$ (constructed in Section 6.3) is the algebra $G$ with maximal compact subalgebra $\frak{su}(2) \oplus \frak{su}(2)$. 
A.9.1. The case that $i = 1$. In this case (Figure A.6.B) the nonzero $g^{p,q}$ all have dimension one, with the exception of $g^{0,0}$ which has dimension two. We have

$$g_{B(N)}^{ss}(\mathbb{R}) = \mathfrak{su}(1,1),$$

with simple root $3\alpha_1 + 2\alpha_2$. The restriction of the fundamental weight $\omega_1$ to $g_{B(N)}^{ss}$ is $\eta_1$, so that

$$\check{D}(N) = \mathbb{P}^1.$$ 

In the fibration

$$B(N) = B(N)_{(3)} \to B(N)_{(2)} \to B(N)_{(1)} \to \check{D}(N),$$

the maps $\rho^{(1)}_N$ and $\rho^{(2)}_N$ are the identity, and the fibre of $\rho^{(3)}_N$ is $\mathfrak{g}(3) \cong \mathbb{C}$.

A.9.2. The (adjoint) case that $i = 2$. In this case (Figure A.2.A), the dimensions of the nonzero $g^{p,q}$ are as given in Figure 7.1 and Table 7.1. We have

$$g_{B(N)}^{ss}(\mathbb{R}) = \mathfrak{su}(1,1),$$

with simple root $2\alpha_1 + \alpha_2$. The restriction of the fundamental weight $\omega_2$ to $g_{B(N)}^{ss}$ is $3\eta_1$, so that

$$\check{D}(N) = v_3(\mathbb{P}^1)$$

is the (non-minimal) Veronese re-embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$. The fibre of $B(N) = B(N)_{(1)} \to D(N)$ is $\mathbb{C}^2$.

Discussion of this example is continued in Section 7.6, where we determine the homology class $[X(N)]$ of the Zariski closure of the orbit $\exp(\mathbb{C}N)\check{D}(N)$.

APPENDIX B. COMPUTATION OF $[X(N)]$ IN $OG(2, \mathbb{C}^{2r+1})$

This is a continuation of Section A.2.1. What follows is a thorough analysis of the $X(N)$ defined in 7.2. We will give explicit, geometric descriptions of $\check{D}(N) = \overline{D(N)}$ and $X(N)$, cf. (B.1) and (B.5), respectively. From these descriptions, we will see that $X(N)$ is smooth, and obtain a $G_{B(N)}^{ss}(\mathbb{C})$–equivariant embedding $\phi_N : \check{D}(N) \to \text{Flag}_\nu(1,3,\mathbb{C}^{2r+1})$ with the property that $X(N)$ is the Tits transform of the image $\Sigma(N) = \phi_N(\check{D}(N))$, cf. (B.2). (Recall that the flag variety $\text{Flag}_\nu(1,3,\mathbb{C}^{2r+1})$ is the space parameterizing the lines on $\check{D}$, cf. Example 3.10.) This exhibits $X(N)$ as a smooth variety swept out by a set of lines parameterized by $\check{D}(N)$.

Using the description (B.5) we will express the homology class $[X(N)]$ represented by $X(N)$ as a linear combination of Schubert classes, cf. (B.8). Let $X$ be the Schubert variety (3.2) swept out by lines passing through the point $o \in \check{D}$. We will see that while $X(N)$ shares many striking similarities with $X$ (Section B.4), $X(N)$ is not homologous to (a multiple of) $X$. 
B.1. Preliminaries. Let $\nu$ denote a nondegenerate, symmetric bilinear form on $\mathbb{C}^{2r+1}$. Fix a basis $\{e_1, \ldots, e_{2r+1}\}$ of $\mathbb{C}^{2r+1}$ so that $\nu(e_a, e_b) = 1$ if $a + b = 2r + 2$, and zero otherwise. Without loss of generality, $\hat{D}$ is the $\text{SO}(\mathbb{C}^{2r+1})$-orbit of $o = \langle e_1, e_2 \rangle$, where $\langle \cdot \rangle$ denotes the complex linear span.

Moreover, if $\{e^1, \ldots, e^{2r+1}\}$ is the dual basis and $e^j_k \overset{\text{dfn}}{=} e_j \otimes e^k \in \text{End}(\mathbb{C}^{2r+1})$, then (again with no loss of generality) the simple root spaces of $\mathfrak{g}_{\mathcal{B}(N)}$ admit the following description

\[
\begin{align*}
\mathfrak{g}^{\alpha_1+\alpha_2+\alpha_3} & = \text{span}_\mathbb{C} \{e_1^4 - e^{2r-2}_2\}, \\
\mathfrak{g}^{\alpha_a} & = \text{span}_\mathbb{C} \{e_a^{a+1} - e^{2r+2-a}_2\}, \quad 4 \leq a \leq r, \\
\mathfrak{g}^{2(\alpha_3+\ldots+\alpha_r)} & = \text{span}_\mathbb{C} \{e_2^{2r-1} - e^3_r\}.
\end{align*}
\]

B.2. Geometric description of $\hat{D}(N)$. Recall that $\hat{D}(N) = \mathbb{P}^1 \times \mathcal{Q}^{2r-5}$ is the $G_{\mathcal{B}(N)}^{\text{ss}}(\mathbb{C})$-orbit of $o$. As a warm-up to describing this orbit, observe that the $\mathbb{P}^1 \subset \hat{D}(N)$ through $o$ is the $\text{SL}_2 \mathbb{C}$-orbit

$$\mathbb{P}^1_{N,o} = \{E \in \text{OG}(2, \mathbb{C}^{2r+1}) \mid \langle e_1 \rangle \subset E \subset \langle e_1, e_2, e_{2r-1} \rangle\},$$

and the $\mathcal{Q}^{2r-5} \subset \hat{D}(N)$ through $o$ is the $\text{SO}_{2r-3} \mathbb{C}$-orbit

$$\mathcal{Q}^{2r-5}_{N,o} = \{E \in \text{OG}(2, \mathbb{C}^{2r+1}) \mid \langle e_2 \rangle \subset E \subset \langle e_1, e_2, e_4, \ldots, e_{2r-2}, e_{2r+1} \rangle\}.$$

Now set

$$\mathbb{P}^{2r-4}_N \overset{\text{dfn}}{=} \mathbb{P}\langle e_1, e_2, \ldots, e_{2r-2}, e_{2r+1} \rangle$$

and

$$\mathcal{Q}^{2r-5}_N \overset{\text{dfn}}{=} \{\nu \in \mathbb{P}^{2r-4}_N \mid \nu(v,v) = 0\}.$$

Then

\begin{equation}
\hat{D}(N) = \mathcal{Q}^{2r-5} \times \mathbb{P}^1 = \{\langle u, v \rangle \mid \langle u \rangle \in \mathbb{P}^1_N, \langle v \rangle \in \mathcal{Q}^{2r-5}_N\}.
\end{equation}

B.3. Description of $X(N)$ as a Tits transform. Next we turn to $X(N)$. Recall that $N \in \mathfrak{g}^{-\alpha_2}$. Let $\text{SL}_N(2, \mathbb{C})$ be the subgroup of $\text{G}(\mathbb{C})$ associated to the simple root $\alpha_2$. Then $X(N)$ is the $\text{SL}_N(2, \mathbb{C})$-orbit of $\hat{D}(N)$. Note that the root $\alpha_2$ is orthogonal to the Dynkin diagram (A.5) of $\mathfrak{g}_{\mathcal{B}(N)}^{\text{ss}}$. From this we conclude that $G_{\mathcal{B}(N)}^{\text{ss}}(\mathbb{C}) \times \text{SL}_N(2, \mathbb{C})$ is a subgroup of $G(\mathbb{C})$, and that

\begin{equation}
X(N) \text{ is the } G_{\mathcal{B}(N)}^{\text{ss}}(\mathbb{C}) \times \text{SL}_N(2, \mathbb{C}) \text{-orbit of } o \in \hat{D}.
\end{equation}

In particular, $X(N)$ is smooth. An explicit description of $X(N)$ is given by (B.5).

Suppose $E \in \hat{D}(N)$. Then $\exp(\mathcal{CN})E = \text{SL}_N(2, \mathbb{C})E$ is a $\mathbb{P}^1$ in $\hat{D} = \text{OG}(2, \mathbb{C}^{2r+1})$. As such it is necessarily of the form $\{E' \in \hat{D} \mid F^1 \subset E' \subset F^3\}$ for fixed isotropic subspaces $F^1 \subset E \subset F^3$ dimensions one and three, respectively. The assignment $E \mapsto (F^1 \subset F^3)$ defines a $G_{\mathcal{B}(N)}^{\text{ss}}(\mathbb{C})$-equivariant map

$$\phi_N : \hat{D}(N) \rightarrow \text{Flag}_{\nu}(1, 3, \mathbb{C}^{2r+1}).$$
Recall (Example 3.10) that the flag variety \( \text{Flag}_p(1,3,\mathbb{C}^{2r+1}) \) parameterizes lines on \( \hat{D} = \text{OG}(2,\mathbb{C}^{2r+1}) \). In this context, the line \( \exp(\mathbb{C}N) \cdot \mathbb{C} \subset \hat{D} \) is the Tits transform of the point \( (F^1 \subset F^3) \in \text{Flag}_p(1,3,\mathbb{C}^{2r+1}) \). Whence,

\[
X(N) \text{ is the Tits transform of the image } \Sigma(N) \overset{\text{dfn}}{=} \phi_N(\hat{D}(N)).
\]

Note that \( g^{-\alpha_2} = \text{span}_\mathbb{C}\{e_3^2 - e_2^{2r-1}\} \). Therefore, in the case that \( E = o = \langle e_1, e_2 \rangle \), we have \( F^1 = \langle e_1 \rangle \) and \( F^3 = \langle e_1, e_2, e_3 \rangle \). Thus,

\[
\Sigma(N) \text{ is the } G_{ss}^{B(N)}(\mathbb{C}) \text{–orbit of the flag } \langle e_1 \rangle \subset \langle e_1, e_2, e_3 \rangle.
\]

This final observation, along with (A.5), yields a second proof that \( \Sigma(N) \simeq \mathbb{Q}^{2r-5} \times \mathbb{P}^1 = \hat{D}(N) \).

**Remark B.3.** It is interesting to note that \( \Sigma(N) \) is not a minimal homogeneous embedding of \( \mathbb{Q}^{2r-5} \times \mathbb{P}^1 \); it is a second Veronese re-embedding of the minimal homogeneous embedding. This is seen as follows. The minimal homogeneous embedding of the flag variety \( \text{Flag}_p(1,3,\mathbb{C}^{2r+1}) \) is the \( G(\mathbb{C}) \)-orbit of the highest weight line \( \ell \in \mathbb{P}V_{\omega_1 + \omega_3} \), where \( V_{\omega_1 + \omega_3} \) is the irreducible \( g \)-module of the highest weight \( \omega_1 + \omega_3 \). From this perspective, the point \( \ell \) corresponds to the flag \( \langle \langle e_1 \rangle \subset \langle e_1, e_2, e_3 \rangle \rangle \) in \( \text{Flag}_p(1,3,\mathbb{C}^{2r+1}) \). Let \( \eta_1, \ldots, \eta_{r-1} \) be the weights of \( g_{ss}^{B(N)} \) with respect to the simple roots (A.5). Then, when restricted to \( g_{ss}^{B(N)} \), \( \omega_1 + \omega_3 = 2(\eta_1 + \eta_{r-1}) \). From this it follows that \( \Sigma(N) \), the \( G_{ss}^{B(N)}(\mathbb{C}) \)-orbit of \( \ell \), is the second Veronese re-embedding of the minimal homogeneous embedding of \( \mathbb{Q}^{2r-5} \times \mathbb{P}^1 \).

**B.4. Similarities between \( X \) and \( X(N) \).** Recall the Schubert variety \( X \) swept out by lines passing through \( o = \langle e_1, e_2 \rangle \), cf. (3.2). As we will see in the discussion that follows, \( X \) and \( X(N) \) share many features in common. In the language of Section 3.4, \( X \) is the Tits transform of \( \Sigma \simeq \mathbb{Q}^{2r-5} \times \mathbb{P}^1 \), cf. Remark 3.16. This brings us to the first similarity between \( X \) and \( X(N) \):

**Both \( X \) and \( X(N) \) are Tits transforms of varieties isomorphic to \( \mathbb{Q}^{2r-5} \times \mathbb{P}^1 \).**

By Section 3.5 and Lemma 3.13 the variety

\[
\Sigma \overset{\text{dfn}}{=} \text{the } G' \text{–orbit of the flag } \langle e_1 \rangle \subset \langle e_1, e_2, e_3 \rangle,
\]

where \( G' \) is the semi-simple subgroup of \( G(\mathbb{C}) \) with simple roots

\[
S' \overset{\text{dfn}}{=} \{ \alpha_1, \alpha_3, \ldots, \alpha_r \}.
\]

Note the similarity with (B.2c). Moreover, \( G(\mathbb{C}) \) and \( G_{ss}^{B(N)}(\mathbb{C}) \) are congruent under the Weyl group element \( w = (234 \cdots r \cdots 4312) \); the interested reader may confirm this by checking that \( w \) maps \( S' \) to the simple roots (A.5) of \( g_{ss}^{B(N)} \). However, arguing as in Remark B.3, we see that the \( G' \)-orbit \( \Sigma \) is a minimal homogeneous embedding of \( \mathbb{Q}^{2r-5} \times \mathbb{P}^1 \). In contrast \( \Sigma(N) \) is the (non-minimal) second Veronese re-embedding of \( \mathbb{Q}^{2r-5} \times \mathbb{P}^1 \), as noted in that remark. By [7, Theorem 24.10]

\[
\deg \Sigma(N) = 2^{2r-4} \deg \Sigma.
\]

These observations might reasonably lead one to expect \( X(N) \) to be homologous to (a multiple of) \( X \). To determine whether or not that expectation is reasonable we will express the homology class \([X(N)]\) as linear combination of Schubert classes. To that end we will
need: (i) an explicit geometric description of $X(N)$, and (ii) to review the Schubert classes of $\tilde{D}$.

B.5. Geometric description of $X(N)$. Given $E \in \tilde{D}(N)$ we may explicitly describe the image $\phi_N(E) = (F^1 \subset F^3)$ as follows. By (B.1), the 2-plane $E$ is spanned by vectors $u, v$ with $\langle u \rangle \in \mathbb{P}_N^1$ and $\langle v \rangle \in \mathcal{Q}_N^{2r-5}$. We have $g^{-\alpha_2} = \text{span}_\mathbb{C}\{e_3^2 - e_{2r}^{2r-1}\}$ and, without loss of generality,

$$N = e_3^2 - e_{2r}^{2r-1}.$$  

First observe that $\exp(\mathbb{C}N)$ acts trivially on $\mathbb{P}_N^{2r-4} \ni \langle v \rangle$. Therefore, $\exp(\mathbb{C}N)v = v$. Next, write

$$u = se_2 + te_{2r-1},$$

with $(s : t) \in \mathbb{P}^1$. Given $z \in \mathbb{C}$, we compute

$$\exp(zN)u = u + z(se_3 - te_{2r}).$$

From these two observations we see that

$$F^1 = \langle v \rangle \quad \text{and} \quad F^3 = \langle v, u, u' \rangle,$$

where

$$u' = se_3 - te_{2r}.$$ 

It is also clear that $\phi_N$ is an embedding.

Any 2-plane contained in the line $\text{SL}_N(2, \mathbb{C})E = \{ E' | F^1 \subset E' \subset F^3 \}$ is of the form $E' = \langle v, v' \rangle$ with $v' \in \langle u, u' \rangle$. Any such $v'$ spans a null-line in $\mathbb{C}^4_N = \langle e_2, e_3, e_{2r-1}, e_{2r} \rangle$; conversely, any $\nu$–null line in $\mathbb{C}^4_N$ may be written as a linear combination of $u$ and $u'$ for some choice of $(s : t) \in \mathbb{P}^1$. Therefore, as $E$ ranges over $\tilde{D}(N)$, the vector $v'$ ranges over the quadric $\mathcal{Q}_N^2 \subset \mathbb{P}_N^4$ of $\nu$–null lines. From these observations we deduce that

$$X(N) = \mathcal{Q}_N^{2r-5} \times \mathbb{P}_N^1 \times \mathbb{P}_N^1 = \mathcal{Q}_N^{2r-5} \times \mathcal{Q}_N^2$$

$$\{ E = \langle u, v \rangle | \langle v \rangle \in \mathcal{Q}_N^{2r-5}, \langle u \rangle \in \mathcal{Q}_N^2 \}.$$ (B.5)

B.6. Schubert varieties of $\tilde{D}$. The Schubert varieties of $\tilde{D}$ are described as follows. Fix a $\nu$–isotropic flag $0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \cdots \subset \mathcal{F}^{2r+1} = \mathbb{C}^{2r+1}$; that is, $\mathcal{F}^d \subset \mathbb{C}^{2r+1}$ is a linear subspace of dimension $d$ and $\nu(\mathcal{F}^d, \mathcal{F}^{2r+1-d}) = 0$. The Schubert varieties of $\tilde{D}$ are of the form

$$X_{a,b}(\mathcal{F}^\bullet) = \{ E \in \tilde{D} | \dim(E \cap \mathcal{F}^a) \geq 1, \ E \subset \mathcal{F}^b \},$$

with $1 \leq a \leq r$ and $r + 2 \leq b \leq 2r + 1$, and $a + b \neq 2r + 2$. Let $x_{a,b} = [X_{a,b}]$ denote the corresponding homology class. Note that $x_{a,b}$ does not depend on our choice of $\mathcal{F}^\bullet$; this is because any two $\nu$–isotropic flags are congruent under $G(\mathbb{C})$, and the group is path connected. For notational convenience we will abbreviate $X_{a,b}(\mathcal{F}^\bullet)$ as $X_{a,b}$.

Recall the Schubert variety $X$ swept out by lines passing through $o \in \tilde{D}$, cf. (3.2). We claim that

$$X = X_{2,2r-1}.$$ (B.7)
B.7. Computation of $[X(N)]$. Note that $\dim X(N) = 2r - 3$. The Schubert varieties $x_{a,b}$ of dimension $2r - 3$ are precisely those with $a + b = 2r + 1$. Their Poincaré duals are $x_{a,b}^* = x_{a+1,b+1}$. Whence the homology class of $X(N) \subset \check{D}$ is the linear combination

$$[X(N)] = \sum_{a+b=2r+1} n_{a,b} x_{a,b},$$

and the coefficients are the non-negative intersection numbers

$$0 \leq n_{a,b} = [X(N)] \cdot x_{a+1,b+1}.$$

In Section B.7.2 we will assume that $r \geq 5$, and then show that

$$(B.8) \quad [X(N)] = n^{1,2r} x_{1,2r} + n^{2,2r-1} x_{2,2r-1} + n^{3,2r-2} x_{3,2r-2},$$

with coefficient $0 \neq n_{a,b} = \deg J_a$, where $J_a$ is the join of two quadrics, for $a = 1, 2, 3$. (The case that $r = 3$ is treated in Section B.8; the case that $r = 4$ is left as an exercise for the reader.)

Remark B.9 (The Schubert classes supporting $[X(N)]$). The three Schubert classes appearing in (B.8) admit the following descriptions. The Schubert variety $X_{2,2r-1}$ is described by (B.7) as the variety swept out by all lines $\mathbb{P}^1 \subset \check{D}$ passing through $o$. The Schubert variety

$$X_{1,2r} = \{ E \in OG(2, \mathbb{C}^{2r+1}) \mid \mathcal{F}^1 \subset E \subset \mathcal{F}^{2r} \} = OG(1, \mathbb{C}^{2r-1}) = Q^{2r-3}$$

is a smooth quadric.

The Schubert variety

$$X_{3,2r-2} = \{ E \in OG(2, \mathbb{C}^{2r+1}) \mid \dim (E \cap \mathcal{F}^3) \geq 1, \ E \subset \mathcal{F}^{2r-2} \}$$

may described as follows. Note that $OG(3, \mathbb{C}^{2r+1})$ parameterizes a uniruling of $\check{D}$ by $\mathbb{P}^2$s, and $OG(4, \mathbb{C}^{2r+1})$ parameterizes a uniruling of $\check{D}$ by varieties isomorphic to $Gr(2, 4) \simeq \mathcal{Q}^4$. To be precise, given $z \in OG(3, \mathbb{C}^{2r+1})$, the corresponding 2-plane is $\mathbb{P}^2_z = \{ E \in \check{D} \mid E \subset z \}$. Likewise, given $y \in OG(4, \mathbb{C}^{2r+1})$, the corresponding quadric is $\mathcal{Q}^4_y = \{ E \in \check{D} \mid E \subset y \}$. Note that every $\mathbb{P}^2_z$ is contained in some $\mathcal{Q}^4_y$. Fix $z = \mathcal{F}^3$. We claim that

$$(B.10) \quad X_{3,2r-2} = \bigcup_{\mathcal{Q}^4_y \supset \mathbb{P}^2_z} \mathcal{Q}^4_y;$$

that is, $X_{3,2r-2}$ is the variety swept out by all the quadrics $\mathcal{Q}^4_y \in OG(4, \mathbb{C}^{2r+1})$ that contain the fixed 2-plane $\mathbb{P}^2_z$. 

Proof of (B.7). First observe that the set of lines through $o$ is parameterized by

$$\Sigma = \{ F^* \in \text{Flag}_o (1, 3, \mathbb{C}^{2r+1}) \mid F^1 \subset o \subset F^3 \} \simeq \mathbb{P}^1 \times \mathcal{Q}^{2r-5},$$

cf. Example 3.10. Given $F^* \in \Sigma$, the corresponding line on $X$ is $\{ E \in \check{D} \mid F^1 \subset E \subset F^3 \}$. Since $F^3 \subset (F^3)^\perp \subset \langle e_1, e_2 \rangle^\perp = \mathbb{C}^{2r-1}$, we have $X \subset X_{2,2r-1}$.

Conversely, suppose that $E \in X_{2,2r-1}$. Then $E = \langle u, v \rangle$ for some $u = ae_1 + be_2$ and $v \in \langle e_1, e_2 \rangle^\perp$. If $E = o$, then $E \in X$. Assume $E \neq o$; equivalently, $v \notin \langle e_1, e_2 \rangle$. Let $u' = be_1 - ae_2$. Then $\mathbb{P}^{1}_{a, E} = \{ \langle u, sv + tu' \rangle \mid (s : t) \in \mathbb{P}^1 \}$ is a line in $X_{2,2r-1}$ passing through $o$ and $E$. It follows that $\mathbb{P}^{1}_{a, E}$ and $X_{2,2r-1}$ are contained in $X$. 

$\Box$
To prove the claim we first suppose that \( E \in X_{3,2r-2} \). Then the sum \( E + \mathcal{F}^3 \) has dimension three or four. The dimension is three if and only if \( E \subset \mathcal{F}^3 \). In this case \( E \subset \mathbb{P}^2 \subset Z \), where \( Z \) denotes the right-hand side of (B.10). If the dimension is four, then set \( y = E + \mathcal{F}^3 \).

By construction \( E \in \mathcal{Q}_y^1 \). Also \( z = \mathcal{F}^3 \subset y \), and this is equivalent to \( \mathbb{P}^2 \subset \mathcal{Q}_y^1 \). Therefore \( E \in Z \). We have established the containment \( \subset \) in (B.10).

Now suppose that \( E \in Z \). Then there exists a 4–plane \( y \in OG(4, \mathbb{C}^{2r+1}) \) such that \( z = \mathcal{F}^3 \subset y \) and \( E \in \mathcal{Q}_y^1 \). Since \( y \) is \( \nu \)–isotropic and contains \( \mathcal{F}^3 \), it is necessarily the case that \( y \subset (\mathcal{F}^3)^\perp = \mathcal{F}^{2r-2} \). Whence \( E \subset \mathcal{F}^{2r-2} \). Likewise, since both \( E \) and \( z \) are contained in \( y \), their intersection necessarily has dimension at least one. Thus, \( E \in X_{3,2r-2} \) establishing the containment \( \supset \) in (B.10), and the claim.

**Outline of the proof of (B.8).** From (B.5) and (B.6) we see that

\[
(B.11) \quad X(N) \cap X_{a+1,b+1} = \left\{ \langle u, v \rangle \left| \begin{array}{c}
\langle u \rangle \in \mathcal{Q}_N^2 \cap \mathbb{P}\mathcal{F}^{b+1} \\
\langle v \rangle \in \mathcal{Q}_N^{2r-5} \cap \mathbb{P}\mathcal{F}^{b+1} \\
\dim((u, v) \cap \mathcal{F}^{a+1}) \geq 1
\end{array} \right. \right\}.
\]

Using this formula, we will show that \( n^{a,b} = 0 \) for all \( a > 3 \) (Section B.7.2), and compute \( n^{1,2r}, n^{2,2r-1} \) and \( n^{3,2r-2} \) (Sections B.7.1, B.7.3 and B.7.4).

**B.7.1. Computation of \( n^{1,2r} \).** The integer \( n^{1,2r} \) is the number of points in the intersection of \( X(N) \) with a general Schubert variety \( X_{2,2r+1} \) representing the Poincaré dual \( x_{1,2r}^* \). From (B.11) we see that

\[
X(N) \cap X_{2,2r+1} = \left\{ \langle u, v \rangle \left| \begin{array}{c}
\langle u \rangle \in \mathcal{Q}_N^2, \langle v \rangle \in \mathcal{Q}_N^{2r-5} \\
\dim((u, v) \cap \mathcal{F}^2) \geq 1
\end{array} \right. \right\}.
\]

Observe that the set of all one-dimensional subspaces contained in an intersection of the form \( \langle u, v \rangle \cap \mathcal{F}^2 \) is in bijection with the in the intersection of \( \mathbb{P}\mathcal{F}^2 \) with the join

\[
\mathcal{J}_1 \overset{\text{df}}{=} J(\mathcal{Q}_N^2, \mathcal{Q}_N^{2r-5}).
\]

Therefore, the intersection (B.11) is nonempty (and \( n^{1,2r} \) is nonzero) if and only if the intersection \( \mathbb{P}(\mathcal{F}^2) \cap \mathcal{J}_1 \) is nonempty.

Note that both \( \mathbb{P}(\mathcal{F}^2) \) and \( J(\mathcal{Q}_N^2, \mathcal{Q}_N^{2r-5}) \) are contained in the quadric \( \mathcal{Q}_{2r-1} \subset \mathbb{P}^{2r} \). By Kleiman’s transversality theorem [33, Corollary 4], for a general choice of \( \mathcal{F}^2 \) this intersection is proper. Since \( \mathbb{P}^1 = \mathbb{P}(\mathcal{F}^2) \) and \( \mathcal{J}_1 \) are of complimentary dimensions in \( \mathcal{Q}_{2r-1} \), we see that the intersection is a (nonzero) finite number of points each with multiplicity one.

**B.7.2. Computation of \( n^{a,b} \).** The computation of the coefficients \( n^{a,b} \) is similar to that of \( n^{1,2r} \). We proceed as follows. Recall that the Schubert varieties of \( \mathcal{Q}_{2r-1} \) are

\[
X_a(\mathcal{F}^*) \overset{\text{df}}{=} \{ \langle u \rangle \in \mathcal{Q}_{2r-1} \mid u \in \mathcal{F}^a \} = \mathcal{Q}_{2r-1} \cap \mathbb{P}\mathcal{F}^a, \quad a \neq r + 1.
\]

Again, for notational convenience, we will abbreviate \( X_a(\mathcal{F}^*) \) as \( X_a \). If \( a \leq r \), then \( X_a = \mathbb{P}\mathcal{F}^a \) has dimension \( a - 1 \); if \( a \geq r + 2 \), then \( \dim X_a = a - 2 \). By Kleiman’s transversality theorem, we may choose the \( \mathcal{F}^* \) so that the Schubert varieties \( X_a \) intersect \( \mathcal{Q}_N^2, \mathcal{Q}_N^{2r-5} \) and \( \mathcal{J}_1 \) properly in \( \mathcal{Q}_{2r-1} \). In particular, as \( \dim X_a = a - 2 \) and \( \mathcal{Q}_N^m \cap X_a = \mathcal{Q}_N^m \cap \mathbb{P}\mathcal{F}^a \), we have the following.
(a) The intersection $Q^2_N \cap \mathbb{P}^b$ is empty if $a < 2r - 1$; consists of two points if $a = 2r - 1$, and is one-dimensional quadric if $a = 2r$.

(b) Assume $r \geq 5$. If $r + 2 \leq a$, then the intersection is a quadric of dimension $a - 6$.

(c) If $a \leq r$, then $X_a \cap J_1$ has dimension $a - 2$. (In particular, $X_1 \cap J_1$ is empty.)

Assume that $r \geq 5$.

The case that $r = 3$ is treated in Section B.8; we leave $4$ as an exercise for the reader.

Following Section B.6 we assume that $a + b = 2r + 1$ and proceed to compute $n^{a,b}$, keeping in mind that $1 \leq a \leq r$ and $r + 2 \leq b \leq 2r + 1$. As noted in (a) above, if $b + 1 < 2r - 1$, then $Q^2_N \cap J^{b+1}$ is empty. It follows from (B.11) that $X(N) \cap X_{a+1,b+1}$ is empty and

$$n^{a,b} = 0 \text{ if } b \leq 2r - 3 \text{ (equivalently, } a > 3).$$

So it remains to determine $n^{3,2r-2}$ and $n^{2,2r-1}$.

B.7.3. **Computation of** $n^{2,2r-1}$. In this case, $A_2 = Q^2_A \cap \mathbb{P}^{b+1}$ is a one-dimensional quadric by (a), and $B_2 = Q^{2r-5}_N \cap \mathbb{P}^{b+1}$ is a quadric of dimension $2r - 6$ by (b). By (B.11), the points of $X(N) \cap X_{a+1,b+1}$ are in bijection with the intersection of $\mathbb{P}^{a+1}$ with the join $J_2 = J(A_2, B_2)$.

The two quadrics $A_2$ and $B_2$ span linear spaces $C^3_A$ and $C^2_{B} - 4$ in $J^{b+1} = J^{2r}$. We claim that $J^1$ is transverse to

$$U_2 = C^3_A \oplus C^2_{B} - 4.$$  

This is a consequence of (c). In particular, suppose that $0 \neq u + v \in J^1$, and that $u \in C^3_A$ and $v \in C^2_{B} - 4$. Since $J^1$ is a $\nu$–isotropic line, $C^3_A$ and $C^2_{B} - 3$ are $\nu$–orthogonal, and $C^3_A \subset C^4_N$ and $C^2_{B} - 4 \subset C^2_{N} - 3$ we see that $\langle u \rangle \in Q^1_A$ and $\langle v \rangle \in Q^2_N - 5$. It follows that $\langle u + v \rangle = \mathbb{P}J^1 \in J_1$. This contradicts (c), and establishes the claim.

It follows from the claim that $J^{2r} = U_2 \oplus J^1$. In particular, $J^{a+1} \cap U_2$ is a $\nu$–isotropic $2$–plane. The Schubert variety $X_{2r}$ is cone over a smooth quadric $Q^{2r-3}$ with vertex the point $\mathbb{P}J^1$. Therefore, the intersection of $X_{2r}$ with $U_2$ is a smooth quadric $Q^{2r-3}$. Note that $\mathbb{P}^1 = \mathbb{P}(J^{a+1} \cap U_2)$ and $J_2$ both lie in $Q^{2r-3}$ and have complimentary dimension. By Kleiman’s transversality, the general translate of $\mathbb{P}^1$ by the symmetry group $\text{SO}(2r - 1) = \{g \in \text{SO}(2r + 1, \mathbb{C}) \mid g(J^1) = J^1 \text{ and } g(J^{2r}) = J^{2r}\}$ of $Q^{2r-3}$ is transverse to $J_2$. Thus, the intersection $\mathbb{P}^1 \cap J_2 = \mathbb{P}(J^3) \cap J_2$ is a set of points (each of multiplicity one) of cardinality equal to the degree of $J_2$.

**Remark** B.12. By Kleiman’s transversality we may choose $J^2$ so that $\mathbb{P}(U_2 \cap J^2) \cap J_2 = \mathbb{P}(J^2) \cap J_2$ is empty.

B.7.4. **Computation of** $n^{3,2r-2}$. In this case, $A_3 = Q^2_A \cap \mathbb{P}^{b+1}$ consists of two points by (a), and $B_3 = Q^{2r-5}_N \cap \mathbb{P}^{b+1}$ is a quadric of dimension $2r - 7$ by (b). By (B.11), the points of $X(N) \cap X_{a+1,b+1}$ are in bijection with the intersection of $\mathbb{P}^{a+1}$ with the join $J_3$ of $A_3$ and $B_3$.

The two quadrics $A_3$ and $B_3$ span linear spaces $C^2_A$ and $C^2_{B} - 5$ in $J^{b+1} = J^{2r-1}$. We claim that $J^2$ is transverse to

$$U_3 = C^2_A \oplus C^2_{B} - 5.$$  

This is a consequence of Remark B.12. In particular, suppose that $0 \neq u + v \in J^2$, and that $u \in C^2_A$ and $v \in C^2_{B} - 5$. Since $C^3_A$ and $C^2_{B} - 4$ are $\nu$–orthogonal, and $C^2_A \subset C^3_A$ and
$\mathbb{C}^{2r-5} \subset \mathbb{C}^{2r-4}$ we see that $u \in A_2$ and $v \in B_2$. It follows that $\langle u + v \rangle \in \mathcal{J}_2$. This implies $\mathbb{P}^2 \cap \mathcal{J}_2$ is nonempty, contradicting Remark B.12.

It follows from the claim that $\mathcal{F}^{2r-1} = U_3 \oplus \mathcal{F}^2$. In particular, $\mathcal{F}^{2r+1} \cap U_3$ is a $\nu$-isotropic 2-plane. The Schubert variety $X_{2r-1}$ is cone over a smooth quadric $\mathbb{Q}^{2r-5}$ with vertex the line $\mathbb{P}\mathcal{F}^2$. Therefore, the intersection of $X_{2r-1}$ with $U_3$ is a smooth quadric $\mathbb{Q}^{2r-5}$. Note that $\mathbb{P}^1 = \mathbb{P}(\mathcal{F}^{2r+1} \cap U_3)$ and $\mathcal{J}_3$ both lie in $\mathbb{Q}^{2r-5}$ and have complimentary dimension. By Kleiman’s transversality, the general translate of $\mathbb{P}^1$ by the symmetry group $\text{SO}(2r-3) = \{ g \in \text{SO}(2r+1, \mathbb{C}) \mid g(\mathcal{F}^2) = \mathcal{F}^2 \text{ and } g(\mathcal{F}^{2r-1}) = \mathcal{F}^{2r-1} \}$ of $\mathbb{Q}^{2r-5}$ is transverse to $\mathcal{J}_3$. Thus, the intersection $\mathbb{P}^1 \cap \mathcal{J}_3 = \mathbb{P}(\mathcal{F}^1) \cap \mathcal{J}_3$ is a set of points (each of multiplicity one) of cardinality equal to the degree of $\mathcal{J}_3$.

**B.8. The case $r = 3$.** In Section B.7.2 we restricted attention to the case that $r \geq 5$. As promised we consider here the case that $r = 3$. Specializing the work of Section B.6 and B.7 to $r = 3$ we see that

$$\dim X(N) = 2r - 3 = 3,$$

and there are two Schubert varieties $X = X_{1,6}$ and $Y = X_{2,5}$ of dimension three. The elements of $W_{\text{min}}^\mathfrak{p}$ indexing $x_{1,6}$ and $x_{2,5}$ are (231) and (232), respectively. Therefore,

$$[X(N)] = n^{1,6} x_{1,6} + n^{2,5} x_{2,5}$$

for integers $n^{1,6}, n^{2,5} \geq 0$. The computation of $n^{1,2r}$ in Section B.7.1 holds for all $r \geq 3$. However, we will compute the coefficients $n^{1,6}$ and $n^{2,5}$ by the BGG polynomials and divided difference operators of [4], rather than intersection theory approach of Section B.7. We find

$$[X(N)] = 2 x_{1,6} + 2 x_{2,5}.$$  \hspace{1cm} (B.13)

As a consistency check, we finish this section with some elementary observations on the degrees of the varieties involved. The Schubert varieties $X_{1,6}$ and $X_{2,5}$ are of degrees four and two, respectively (in the minimal homogeneous embedding $G/P \hookrightarrow \mathbb{P}\mathfrak{g}$). Therefore,

$$\deg X(N) = 4n^{1,6} + 2n^{2,5} \stackrel{(B.13)}{=} 12.$$ 

On the other hand, we may compute $\deg X(N)$ as follows. Set $\tau_1 = \alpha_2$. Recall (A.5) that the simple roots of $\mathfrak{g}_{\text{min}}^{\mathfrak{p}(N)}$ are $\tau_2 = \alpha_2 + 2\alpha_3$ and $\tau_3 = \alpha_1 + \alpha_2 + \alpha_3$. Note that the roots $\tau_1, \tau_2, \tau_3$ are all pairwise strongly orthogonal, and therefore form a root subsystem. Let $\mathfrak{g}(N)$ be the associated semisimple subalgebra of $\mathfrak{g}$; then strong orthogonality implies $\mathfrak{g}(N) \simeq \mathfrak{sl}_2 \mathbb{C} \oplus \mathfrak{sl}_2 \mathbb{C} \oplus \mathfrak{sl}_2 \mathbb{C}$. Since $N \in \mathfrak{g}^{-\alpha_2}$, it follows that $X(N)$ is a $G(N)$–orbit. If $\eta_1, \eta_2, \eta_3$ are the fundamental weights of $\mathfrak{g}(N)$, then the fundamental weight $\omega_2$ of $\mathfrak{g}$ restricts to $\eta_1 + \eta_2 + 2\eta_3$ on $\mathfrak{g}(N)$. It follows that

$$X(N) \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \nu_2(\mathbb{P}^1),$$

and by [7, Theorem 24.10]

$$\deg X(N) = 12.$$ 

This completes the consistency check.
Appendix C. Dynkin diagrams

For the readers convenience we include in Figure C.1 the Dynkin diagrams of the complex simple Lie algebras. Recall that: each node corresponds to a simple root $\alpha_i \in S$; two nodes are connected if and only if $\langle \alpha_i, \alpha_j \rangle \neq 0$ and in this case the number if edges is $|\alpha_i|^2 / |\alpha_j|^2 \geq 1$ (that is, $i, j$ are ordered so that the inequality holds). Below, if $G = B_r$, then $r \geq 3$; and if $G = D_r$, then $r \geq 4$.

Figure C.1. Dynkin diagrams for complex simple Lie algebras.

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {1}; \node (2) at (1,0) {2}; \node (3) at (2,0) {$\ldots$}; \node (4) at (4,0) {$r-1$}; \node (5) at (5,0) {$r$};
\draw (1) -- (2) -- (4); \draw (2) -- (3); \draw (4) -- (5);
\node (6) at (9,0) {$A_r = \mathfrak{sl}_{r+1}\mathbb{C}$};
\end{tikzpicture}
\begin{tikzpicture}
\node (1) at (0,0) {1}; \node (2) at (1,0) {2}; \node (3) at (2,0) {$\ldots$}; \node (4) at (3,0) {$r-1$}; \node (5) at (4,0) {$r$};
\draw (1) -- (2) -- (4); \draw (2) -- (3); \draw (4) -- (5);
\node (6) at (9,0) {$B_r = \mathfrak{so}_{2r+1}\mathbb{C}$};
\end{tikzpicture}
\begin{tikzpicture}
\node (1) at (0,0) {1}; \node (2) at (1,0) {2}; \node (3) at (2,0) {$\ldots$}; \node (4) at (3,0) {$r-1$}; \node (5) at (4,0) {$r$};
\draw (1) -- (2) -- (4); \draw (2) -- (3); \draw (4) -- (5);
\node (6) at (9,0) {$C_r = \mathfrak{sp}_{2r}\mathbb{C}$};
\end{tikzpicture}
\begin{tikzpicture}
\node (1) at (0,0) {1}; \node (2) at (1,0) {2}; \node (3) at (2,0) {$\ldots$}; \node (4) at (3,0) {$r-2$}; \node (5) at (4,0) {$r-1$};
\node (6) at (7,0) {$D_r = \mathfrak{so}_{2r}\mathbb{C}$};
\end{tikzpicture}
\begin{tikzpicture}
\node (1) at (0,0) {1}; \node (2) at (1,0) {3}; \node (3) at (2,0) {4}; \node (4) at (3,0) {2}; \node (5) at (4,0) {5}; \node (6) at (5,0) {6}; \node (7) at (6,0) {7};
\node (8) at (11,0) {$E_6$};
\end{tikzpicture}
\begin{tikzpicture}
\node (1) at (0,0) {1}; \node (2) at (1,0) {3}; \node (3) at (2,0) {4}; \node (4) at (3,0) {2}; \node (5) at (4,0) {5}; \node (6) at (5,0) {6}; \node (7) at (6,0) {7};
\node (8) at (11,0) {$E_7$};
\end{tikzpicture}
\begin{tikzpicture}
\node (1) at (0,0) {1}; \node (2) at (1,0) {3}; \node (3) at (2,0) {4}; \node (4) at (3,0) {2}; \node (5) at (4,0) {5}; \node (6) at (5,0) {6}; \node (7) at (6,0) {7}; \node (8) at (7,0) {8};
\node (9) at (11,0) {$E_8$};
\end{tikzpicture}
\begin{tikzpicture}
\node (1) at (0,0) {1}; \node (2) at (1,0) {2}; \node (3) at (2,0) {3}; \node (4) at (3,0) {4};
\node (5) at (10,0) {$F_4$};
\end{tikzpicture}
\begin{tikzpicture}
\node (1) at (0,0) {1}; \node (2) at (1,0) {2}; \node (3) at (2,0) {3}; \node (4) at (3,0) {4};
\node (5) at (10,0) {$G_2$};
\end{tikzpicture}
\end{center}

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