Self-avoiding walks on the body-centered-cubic (BCC) and face-centered-cubic (FCC) lattices are enumerated up to lengths 28 and 24, respectively, using the length-doubling method. Analysis of the enumeration results yields values for the exponents $\gamma$ and $\nu$ which are in agreement with, but less accurate than those obtained earlier from enumeration results on the simple cubic lattice. The non-universal growth constant and amplitudes are accurately determined, yielding for the BCC lattice $\mu = 6.530520(20)$, $A = 1.1785(40)$, and $D = 1.0864(50)$, and for the FCC lattice $\mu = 10.037075(20)$, $A = 1.1736(24)$, and $D = 1.0460(50)$.

I. INTRODUCTION

The enumeration of self-avoiding walks (SAWs) on regular lattices is a classical combinatorial problem in statistical physics, with a long history, see e.g. [1, 2]. Of the three-dimensional lattices, the simple cubic (SC) lattice has drawn the most effort, starting with a paper by Orr [3] from 1947, where the number of SAWs $Z_N$ was given for all $N$ up to $N_{\text{max}} = 6$; these results were obtained by hand. In 1959, Fisher and Sykes [4] used a computer to enumerate all SAWs up to $N_{\text{max}} = 9$; Sykes and collaborators extended this to 11 terms in 1961 [5], 16 terms in 1963 [6], and 19 terms in 1972 [7]. In the following decade Guttmann [8] enumerated SAWs up to $N_{\text{max}} = 20$ in 1987, and extended this by one step in 1989 [9]. In 1992, MacDonald et al. [10] reached $N_{\text{max}} = 23$, and in 2000 MacDonald et al. [11] reached $N_{\text{max}} = 26$. In 2007, a combination of the lace expansion and the two-step method allowed for the enumeration of SAWs up to $N_{\text{max}} = 30$ steps [12]. Recently, the length-doubling method [13] was presented which allowed enumerations to be extended up to $N_{\text{max}} = 36$. To date, this is the record series for the SC lattice.

The body-centered-cubic (BCC) and face-centered-cubic (FCC) lattices are in principle equally as physically relevant as the SC lattice, but enumeration is hampered by the larger lattice coordination numbers, which detrains most enumeration methods severely. It is also very slightly more cumbersome to write computer programs to perform enumerations for these lattices. Consequentially, the SC lattice has served as the test-bed problem for new enumeration algorithms, and the literature on enumerations for the BCC and FCC lattices is far more sparse. For the BCC lattice, $Z_N$ was determined up to $N_{\text{max}} = 15$ in 1972 [7], and to $N_{\text{max}} = 16$ in 1989 [9]. The current record of $N_{\text{max}} = 21$ was obtained in 1997 by Butera and Comi [14] as the $N \to 0$ limit of the high temperature series for the susceptibility of the $N$-vector model. For the FCC lattice, enumerations up to $N_{\text{max}} = 12$ were performed in 1967 [15], and the record of $N_{\text{max}} = 14$ was achieved way back in 1979 [16].

Enumeration results derive their relevance from the ability to determine critical exponents, which, according to renormalization group theory, are believed to be shared between SAWs on various lattices and real-life polymers in solution [17]. Two such exponents are the entropic exponent $\gamma$ and the size exponent $\nu$. Given the number $Z_N$ of SAWs of all lengths up to $N_{\text{max}}$ and the sum $P_N$ of their squared end-to-end extensions, these two exponents can be extracted using the relations

$$Z_N = A \mu^N N^{\gamma-1} \left( 1 + \frac{a}{N^{\Delta_1}} + O \left( \frac{1}{N} \right) \right);$$

$$P_N = \sigma D N^{2\nu} \left( 1 + \frac{b}{N^{\Delta_1}} + O \left( \frac{1}{N} \right) \right).$$

In these expressions, the growth constant $\mu$ and the amplitudes $A$ and $D$ are non-universal (model-dependent) quantities, while the leading correction-to-scaling exponent is a universal quantity with
value $\Delta_1 = 0.528(8)$ [18]. Sub-leading corrections-to-scaling are absorbed into the $O(1/N)$ term. 

$\sigma$ is a lattice specific constant to ensure that our amplitude “$D$” is the same as in earlier work. $\sigma$ corrects for the fact that with our definition each step of the walk is of length $\sqrt{2}$ for the BCC lattice (leading to $\sigma = 2$), and of length $\sqrt{3}$ for the FCC lattice (leading to $\sigma = 3$). Note that for bipartite lattices, of which the SC and BCC lattices are examples, there is an additional alternating “antiferromagnetic” singularity, that is sub-leading but which still must be treated carefully as the odd-even oscillations tend to become amplified by series analysis techniques. Because of universality, the exponents are clearly more interesting from a physics perspective. However, accurate estimates for the growth constant and the amplitudes can also be very helpful for many kinds of computer simulations on lattice polymers.

In this paper, we used the length-doubling method [13] to measure $Z_N$ and $P_N$ up to $N_{\text{max}} = 28$ and 24, on the BCC and FCC lattices, respectively. These lattices can be easily simulated as subsets of the SC lattice: the collection of sites in which $x$, $y$, and $z$ are either all even or all odd forms a BCC lattice, and the collection of sites $(x, y, z)$ constrained to even values for $x + y + z$ forms a FCC lattice. We then analysed these results to obtain estimates for the exponents $\gamma$ and $\nu$, the growth constant $\mu$, and the amplitudes $A$ and $D$. Our results for the two exponents $\gamma$ and $\nu$ agree with the values reported in literature which are obtained on the SC lattice, reinforcing the credibility of the literature values. Our results for the growth constant $\mu$ and the amplitudes $A$ and $D$ for the BCC and FCC lattices are the most accurate ones to date.

The manuscript is organized as follows. First, in Sec. II we present a short outline of the length-doubling method, and present the enumeration data. In Sec. III we describe the analysis method we use, before summarising our results and giving a brief conclusion in Sec. IV.

II. LENGTH-DOUBLING METHOD

We first present an intuitive description of the length-doubling method; a more formal description can be found in [13]. In the length-doubling method, the number $Z_{2N}$ of SAWs with a length of $2N$ steps, with the middle rooted in the origin, is obtained from the walks of length $N$, with one end rooted in the origin, and the number $Z_N(S)$ of times that a subset $S$ of sites is visited by such a walk of length $N$. The lowest-order estimate for $Z_{2N}$ is the number of combinations of two SAWs of length $N$, i.e. $Z_N^2$. This estimate is too large since it includes pairs of SAWs which overlap. The first correction to $Z_{2N}$ is the lowest-order estimate for the number of pairs of overlapping SAWs, which can be obtained from the number $Z_N(\{s\})$ of SAWs of length $N$ which pass through a single site $s$. The first correction is then to subtract $Z_N(\{s\})^2$, summed over all sites $s$. This first correction is too large, as it includes pairs of SAWs twice, if they intersect twice. The second correction corrects for this over-subtraction, by adding the number $Z_N(\{s, l\})$ of SAWs that pass through the pair of sites $\{s, l\}$. Continuing this process with groups of three sites, etc., the number $Z_{2N}$ of SAWs of length $2N$ can then be obtained by the length-doubling formula

$$Z_{2N} = Z_N^2 + \sum_{S \neq \emptyset} (-1)^{|S|} Z_N^2(S),$$

where $|S|$ denotes the number of sites in $S$.

The usefulness of this formula lies in the fact that the numbers $Z_N(S)$ can be obtained relatively efficiently:

- Generate each SAW of length $N$.
- Generate for each SAW each of the $2^N$ subsets $S$ of lattice sites, and increment the counter for each specific subset. Multiple counters for the same subset $S$ must be avoided; this can be achieved by sorting the sites within each subset in an unambiguous way.
- Finally, compute the sum of the squares of these counters, with a positive and negative sign for subsets with an even and odd number of sites, respectively, as in Eq. (3).

As there are $Z_N$ walks of length $N$, each visiting $2^N$ subsets of sites, the computational complexity is $O(2^N Z_N) \sim (2\mu)^N$ times some polynomial in $N$ which depends on implementation details. This compares favourably to generating all $Z_{2N} \sim \mu^{2N}$ walks of length $2N$, provided $\mu > 2$. This is
TABLE I: Enumeration results for the number of three-dimensional self-avoiding walks $Z_N$ and the sum of their squared end-to-end distances $P_N$ on the BCC lattice.

| $N$ | $Z_N$  | $P_N$   |
|-----|--------|---------|
| 1   | 8      | 24      |
| 2   | 56     | 384     |
| 3   | 392    | 4 248   |
| 4   | 2 648  | 40 704  |
| 5   | 17 960 | 358 008 |
| 6   | 120 056| 2 987 232|
| 7   | 804 824| 23 999 880|
| 8   | 5 351 720| 187 661 376|
| 9   | 35 652 680| 1 436 494 872|
| 10  | 236 291 096| 10 816 140 768|
| 11  | 1 568 049 560| 80 339 567 112|
| 12  | 10 368 669 992| 590 168 152 512|
| 13  | 68 626 647 608| 4 294 543 350 696|
| 14  | 453 032 542 040| 31 003 097 851 872|
| 15  | 2 992 783 648 424| 222 268 142 537 848|
| 16  | 19 731 335 857 592| 1 583 984 756 090 544|
| 17  | 130 161 040 083 608| 11 228 345 566 400 136|
| 18  | 857 282 278 813 256| 79 223 666 339 548 320|
| 19  | 5 648 892 048 530 888| 556 634 161 952 309 400|
| 20  | 37 175 039 569 217 672| 3 896 382 415 388 139 840|
| 21  | 244 738 250 638 121 768| 27 181 650 674 871 447 672|
| 22  | 1 609 522 963 822 562 936| 189 042 890 267 974 827 744|
| 23  | 10 588 362 063 533 857 304| 1 311 064 323 033 684 408 072|
| 24  | 69 595 035 470 413 829 144| 9 069 398 712 299 296 227 648|
| 25  | 457 555 628 726 692 288 712| 62 590 336 418 536 387 660 248|
| 26  | 3 005 966 051 800 541 943 464| 431 019 462 253 450 273 360 416|
| 27  | 19 752 610 526 081 274 414 584| 2 962 188 249 772 759 155 770 280|
| 28  | 129 713 248 317 927 812 262 200| 20 319 964 852 485 237 389 626 176|

the case on the SC lattice, with $\mu = 4.684$, and even more so for the BCC and FCC lattices, as we will show. The length-doubling method can also compute the squared end-to-end distance, summed over all SAW configurations; for details we refer to [13]. Details on the efficient implementation of this algorithm are presented in [19].

The direct results of the length-doubling method, applied to SAWs on the BCC and FCC lattices, are presented in Tables 1 and 2, respectively. The BCC results for $N \leq 26$ and FCC results for $N \leq 22$ were obtained and verified by two independent computer programs: SAWdoubler 2.0, available from [http://www.staff.science.uu.nl/~bisse101/SAW/](http://www.staff.science.uu.nl/~bisse101/SAW/), and Raoul Schram's program. The BCC results presented for the largest problems $N = 27, 28$ were obtained by SAWdoubler 2.0 only, and the FCC results for $N = 23, 24$ were obtained by Raoul Schram's program only. Thus the largest two problem instances for each lattice were not independently verified since these require a very large amount of computer time and memory. Still, based on our analysis we believe that the given values are correct.
TABLE II: Enumeration results for the number of three-dimensional self-avoiding walks $Z_N$ and the sum of their squared end-to-end distances $P_N$ on the FCC lattice.

| $N$ | $Z_N$  | $P_N$  |
|-----|--------|--------|
| 1   | 12     | 24     |
| 2   | 132    | 576    |
| 3   | 1404   | 9816   |
| 4   | 14700  | 144288 |
| 5   | 152532 | 1951560|
| 6   | 1573716| 25021536|
| 7   | 16172148| 309080808|
| 8   | 165697044| 3714659040|
| 9   | 1693773924| 43714781448|
| 10  | 17281929564| 505948384608|
| 11  | 17606470412| 5777220825912|
| 12  | 1791455071068| 65234797723584|
| 13  | 18208650297396| 729724191726408|
| 14  | 18490737018612| 800763951530304|
| 15  | 1876240018679868| 89239258469121912|
| 16  | 19024942249966812| 977545487795069952|
| 17  | 192794447005403916| 10651662728070257016|
| 18  | 1952681556794601732| 115520552778504791136|
| 19  | 19767824914170229966| 1247619751507795096248|
| 20  | 200031316330580106948| 1342370509359486939216|
| 21  | 20233304019804218996| 14394237459787212970696|
| 22  | 20458835772261851432748| 1538749219442520114999744|
| 23  | 206801586042610941719148| 16403200314230418676555512|
| 24  | 2089765228215904826153292| 174411223302510038302309440|

III. ANALYSIS

We now proceed to analyse our series in order to extract estimates for various parameters. In addition to the expressions for $Z_N$ and $P_N/Z_N$ in Eqs (1) and (2), we also have

$$P_N = \sigma AD\mu^N N^{2\nu+\gamma-1} \left( 1 + \frac{c}{N^{\Delta_1}} + O\left( \frac{1}{N} \right) \right).$$

As discussed earlier, we expect the critical exponents $\gamma$ and $\nu$ and the leading correction-to-scaling exponent $\Delta_1$ to be the same for self-avoiding walks on the SC, BCC, and FCC lattices. The amplitudes $A$ and $D$ are non-universal quantities, i.e. they are lattice dependent, while $\sigma = 2$ for the BCC lattice and $\sigma = 3$ for the FCC lattice. In the analysis below, we include a subscript to indicate the appropriate lattice.

The BCC lattice is bipartite, which introduces an additional competing correction which has a factor of $(-1)^N$, so causing odd-even oscillations. We reduce the influence of this additional sub-leading correction by separately treating the sequences for even and odd $N$. See [12] for more detailed discussion on this point for the asymptotic behaviour of $Z_N$ on the SC lattice, which is also bipartite.

We now describe the method of analysis we used, which involved two stages: extrapolation of the series via a recently introduced method involving differential approximants [20], and then direct fitting of the extended series with the asymptotic forms in Eqs (1), (2), and (4). We report our final estimates in Table V at the end of the section.
A. Extrapolation

Perhaps the most powerful general-purpose method for the analysis of series arising from lattice models in statistical mechanics is the method of differential approximants, described in [21]. The basic idea is to approximate the unknown generating function $F$ by the solution of an ordinary differential equation with polynomial coefficients. In particular if we know $r$ coefficients $f_0, f_1, \cdots, f_{r-1}$ of our generating function $F$, then we can determine polynomials $Q_i(z)$ and $P(z)$ which satisfy the following $K$th order differential equation order by order:

$$
\sum_{i=0}^{K} Q_i(z) \left( z \frac{d}{dz} \right)^i F(z) = P(z).
$$

(5)

The function determined by the resulting differential equation is our approximant. The power of the method derives from the fact that such ordinary differential equations accommodate the kinds of critical behaviour that are typically seen for models of interest.

Differential approximants are extremely effective at extracting information about critical exponents from the long series that have been obtained for two-dimensional lattice models, such as self-avoiding polygons [22] or walks on the square lattice [23]. However, differential approximants have been far less successful for the shorter series available for three-dimensional models such as SAWs on the simple cubic lattice [12, 13]. For short series, it seems that corrections-to-scaling due to confluent corrections are too strong at the orders that can be reached to be able to reliably determine critical exponents. (In fact, it is extremely easy to be misled by apparent convergence, while in fact estimates have not settled down to their asymptotic values.) The method that has proved most reliable is direct fitting of the asymptotic form [12], which we describe in the next sub-section.

However, we can do better than the usual method of performing direct fits of the original series, and adopt a promising new approach recently invented by Guttmann [20], which is a hybrid of the differential approximant and direct fitting techniques. The underlying idea is to exploit the fact that differential approximants can be used to extrapolate series with high accuracy even in circumstances when the resulting estimates for critical exponents are not particularly accurate, or even when the asymptotic behaviour is non-standard such as being of stretched exponential form. The extrapolations can be extremely useful in cases where corrections-to-scaling are large, as the few extra terms they provide may be the only evidence of a clear trend from the direct fits.

We have 28 exact terms for the BCC series, and 24 exact terms for the FCC series. We used second order inhomogeneous approximants to extrapolate the series for $Z_N$, $P_N$, and $P_N/Z_N$, where we allowed the multiplying polynomials to differ by degree at most 3. In each case we calculated trimmed mean values, eliminating the outlying top and bottom 10% of estimates, with the standard deviation of the remaining extrapolated coefficients providing a proxy for the confidence interval. Note that this is an assumption, and relies on the extrapolation procedure working well for our problem. In practice, this approach of inferring the confidence interval from the spread of estimates appears to be quite reliable in the cases for which it has been tested. We have also confirmed the reliability of the extrapolations by using the method to “predict” known coefficients from truncated series. We report our extended series in Tables III and IV.

B. Direct fits

We then fitted sequences of consecutive terms of the extrapolated series for $Z_N$ and $P_N/Z_N$ to the asymptotic forms given in Eqs (1) and (2), respectively. We found that fits of $P_N/Z_N$ were superior to fits of $P_N$ for estimates of $\nu$ and the parameter $D$, and hence we do not report fits of $P_N$ here.

To convert the fitting problem to a linear equation, we took the logarithm of the coefficients, which from Eqs (1) and (2) we expect to have the following asymptotic forms:

$$
\log Z_N = N \log \mu + (\gamma - 1) \log N + \log A + \frac{a}{N^\Delta} + O\left(\frac{1}{N}\right);
$$

(6)

$$
\log \frac{P_N}{Z_N} = 2\nu \log N + \log \sigma D + \frac{b}{N^\Delta} + O\left(\frac{1}{N}\right).
$$

(7)
TABLE III: Extrapolated coefficients of the various BCC series obtained from differential approximants. The confidence intervals are the standard deviations of the central 80% of estimates.

| N  | \(Z_N\)     | \(P_N\)   | \(P_N/Z_N\) |
|----|-------------|------------|-------------|
| 29 | 8.51984378150(70)×10^{24} | 1.39148952051(11)×10^{26} | 163.3236085142(42) |
| 30 | 5.5928609767(12)×10^{24}  | 9.5134610227(17)×10^{26}  | 170.0998979610(10) |
| 31 | 3.6720987864(23)×10^{25}  | 6.494301898(72)×10^{27}   | 176.858895340(40)  |
| 32 | 2.409709792(39)×10^{26}   | 4.427318727(75)×10^{28}   | 183.7185148677(77) |
| 33 | 1.5816383535(44)×10^{27}  | 3.014025691(25)×10^{29}   | 190.5611070(19)   |
| 34 | 1.037661297(10)×10^{28}   | 2.049378203(42)×10^{30}   | 197.49972213(33)  |
| 35 | 6.808628821(74)×10^{28}   | 1.391831542(69)×10^{31}   | 204.421701347(4)  |
| 36 | 4.46574383(26)×10^{29}    | 9.44216466(95)×10^{31}    | 211.43542011(1)   |
| 37 | 2.929428561(97)×10^{30}   | 6.3988380(13)×10^{32}     | 218.43294713(47)  |
| 38 | 1.9209657(36)×10^{31}     | 4.3321295(17)×10^{33}     | 225.51834632(32)  |

We used the linear fitting routine "lm" in the statistical programming language R to perform the fits.

In all of the fits, we biased the exponent \(\Delta_1\) of the leading correction-to-scaling term, performing the fits for three different choices of \(\Delta_1 = 0.520, 0.528, 0.536\) which correspond to the best Monte Carlo estimate of \(\Delta_1 = 0.528(8)\). We approximated the next-to-leading correction-to-scaling term with a term of order 1, only moderately greater than the spread arising from varying \(\Delta_1\) estimates over the remaining 8 cases. For the BCC lattice, we minimised the impact of the odd-even oscillations by fitting even and odd subsequences separately. We included the extrapolated coefficients in our fits, repeating the calculation for the central estimates and for values which are one standard deviation above and below them.

This procedure gave us up to nine estimates for each sequence of coefficients (from the three choices of \(\Delta_1\), and the three choices of extrapolated coefficient values). For the central parameter estimates we used the case where \(\Delta_1 = 0.528\) (the central value) in combination with the central value of the extrapolated coefficients. We also calculated the maximum and minimum parameter estimates over the remaining 8 cases.

For the BCC lattice, we found that 5 of the extrapolated coefficients gave a spread which was only moderately greater than the spread arising from varying \(\Delta_1\), effectively extending the series to 33 terms. For the FCC lattice, we found we could use 3 additional coefficients, extending the series to 27 terms.

For each of the parameter estimates, we plotted them against the expected relative magnitude of the first neglected correction-to-scaling term. This should result in approximately linear convergence as we approach the \(N \to \infty\) limit which corresponds to approaching the \(y\)-axis from the right in the following figures. In Eqs (6) and (7) we expect that the next term, which is not included in the fits, is of \(O(N^{-1-\Delta_1})\); given that \(\Delta_1 \approx 0.5\), we take the neglected term to be \(O(N^{-3/2})\). The value of \(N\) that is used in the plot is the maximum value of \(N\) in the sequence of fitted coefficients,

TABLE IV: Extrapolated coefficients of the various FCC series obtained from differential approximants. The confidence intervals are the standard deviations of the central 80% of estimates.

| N  | \(Z_N\)     | \(P_N\)   | \(P_N/Z_N\) |
|----|-------------|------------|-------------|
| 25 | 2.1111652709103(46)×10^{25} | 1.8501044921473(82)×10^{27} | 87.6342803480(26) |
| 26 | 2.13224588773(38)×10^{26}  | 1.9582778101818(72)×10^{28} | 91.8410891952(22) |
| 27 | 2.1530362972(17)×10^{27}   | 2.068615279889(35)×10^{29}   | 96.079097491(11) |
| 28 | 2.1735525326(10)×10^{28}   | 2.18110187619(13)×10^{30}    | 100.347327420(41) |
| 29 | 2.1938240975(32)×10^{29}   | 2.29572427539(38)×10^{31}    | 104.6486552(13) |
| 30 | 2.2136677922(93)×10^{30}   | 2.41247069749(92)×10^{32}    | 108.9705672(37) |
| 31 | 2.238701285(63)×10^{31}    | 2.5313307684(21)×10^{33}     | 113.32449876(98) |
| 32 | 2.25340058(14)×10^{32}     | 2.6522953807(45)×10^{34}     | 117.7050374(24) |
| 33 | 2.2728013(51)×10^{33}      | 2.773566679(86)×10^{35}      | 122.11176223(56) |
which we denote $N_{\text{max}}$ in the plots.

We plot our fitted values in Figures 1–8. For ease of interpretation we converted estimates of $\log \mu$, $\log A$, and $\log D$ to estimates of $\mu$, $A$, and $D$. We note that the parameter estimates arising from the odd subsequence of the BCC series for $Z_N$ benefited dramatically from the extrapolated sequence. Examining estimates for $\gamma$ in Fig. 1, $\mu_{\text{bcc}}$ in Fig. 3, and $A_{\text{bcc}}$ in Fig. 5 we see in each case that the trend of the odd subsequence would be dramatically different were it not for the three additional odd terms in the extrapolated sequence. In other cases the additional coefficients are useful, and certainly make the trend for the estimates clearer, but are not as crucial.

Our final parameter estimates are plotted on the $y$-axes.

![Figure 1: Variation of fitted value of $\gamma$ with $N_{\text{max}}$. The line of best fit to the final six values is shown for the FCC lattice and to the final three values for the BCC lattice, separately for the odd and even values. Our final estimate is plotted on the $y$-axis.](image1)

![Figure 2: Variation of fitted value of $\nu$ with $N_{\text{max}}$. The line of best fit to the final six values is shown. Our final estimate is plotted on the $y$-axis.](image2)

![Figure 3: Variation of fitted value of $\mu_{\text{bcc}}$ with $N_{\text{max}}$. The line of best fit to the final three values is shown, separately for the odd and even values. Our final estimate is plotted on the $y$-axis.](image3)

![Figure 4: Variation of fitted value of $\mu_{\text{bcc}}$ with $N_{\text{max}}$. The line of best fit to the final six values is shown. Our final estimate is plotted on the $y$-axis.](image4)

### IV. SUMMARY AND CONCLUSION

We give our estimates for $\gamma$ and $\nu$ in Table V, where we also include estimates coming from the literature. We observe that our estimates are consistent with the literature values, but that the recent Monte Carlo estimates of $\gamma$ and $\nu$, using the pivot algorithm, are far more accurate than
the estimates from series. The estimates coming from our enumerations on the BCC and FCC lattices are not quite as precise as the estimates coming from the SC lattice only, but the fact that they are coming from two independent sources, with different systematic errors, makes these new estimates more robust.

In addition, our estimates of the non-universal quantities for the BCC lattice are $A_{\text{bcc}} = 1.1785(40)$, $D_{\text{bcc}} = 1.0864(50)$, and $\mu_{\text{bcc}} = 6.530520(20)$, which should be compared with earlier estimates of 6.5304(13) [9] from 1989, and unbiased and biased estimates respectively of 6.53036(9) and 6.53048(12) [14] from 1997. Our estimates of the non-universal quantities for the FCC lattice are $A_{\text{fcc}} = 1.1736(24)$, $D_{\text{fcc}} = 1.0460(50)$, and $\mu_{\text{fcc}} = 10.037075(20)$, which should be compared with earlier estimates of 10.03655 [16] from 1979, and 10.0364(6) [8] from 1987 (where these estimates come from different analyses of the same $N \leq 14$ term series).

In conclusion, the length-doubling algorithm has resulted in significant extensions of the BCC and FCC series. The application of a recently invented series analysis technique [20], which combines series extrapolation from differential approximants with direct fitting of the extrapolated series, has given excellent estimates of the various critical parameters. In particular, estimates of the growth constants for the BCC and FCC lattices are far more accurate than the previous literature values.
TABLE V: Summary of our parameter estimates for $\gamma$ and $\nu$, with comparison to values from the literature. Except where noted, the series estimates for $\gamma$ and $\nu$ from the literature come from the simple cubic lattice.

| Source               | $\gamma$       | $\nu$       |
|----------------------|-----------------|--------------|
| This work            | 1.1565(50)      | 0.58785(40)  |
| [24] MC (2017)       | 1.15693(300)    | 0.587597(0)  |
| [18] MC (2016)       | 1.1588(25)      | 0.5877(12)   |
| [25] CB (2016)       | 1.15698(34)     | 0.58772(17)  |
| [26] MC (2010)       |                 | 0.587597(7)  |
| [12]$^b$ Series $N \leq 30$ (2007) | 1.1569(6)      | 0.58774(22)  |
| [27] MC (2004)       | 1.1573(2)       |              |
| [28] MC (2001)       |                 | 0.5874(2)    |
| [11] Series $N \leq 26$ (2000) | 1.1585        | 0.5875       |
| [29] MC (1998)       | 1.1575(6)       |              |
| [30] FT $d = 3$ (1998) | 1.1596(20)      | 0.5882(11)   |
| [30] FT $\epsilon$ bc (1998) | 1.1571(30)      | 0.5878(11)   |
| [14] Series $N \leq 21$ (1997) | 1.161(2)      | 0.592(2)     |
| [14] Series $N \leq 21$, biased (1997) | 1.1594(8)    | 0.5878(6)    |
| [14] BCC series $N \leq 21$ (1997) | 1.1612(8)    | 0.591(2)     |
| [14] BCC series $N \leq 21$, biased (1997) | 1.1582(8)    | 0.5879(6)    |
| [31] MCRG (1997)     | 1.16193(10)     | 0.592(3)     |
| [32] MC (1995)       |                 |              |
| [10] Series $N \leq 23$ (1992) | 1.161(2)      | 0.592(3)     |
| [9] Series $N \leq 21$ (1989) | 1.161(2)      | 0.592(3)     |

$^a$Abbreviations: MC $\equiv$ Monte Carlo, CB $\equiv$ conformal bootstrap, FT $\equiv$ field theory, MCRG $\equiv$ Monte Carlo renormalization group.

$^b$Using Eqs (74) and (75) with $0.516 \leq \Delta_1 \leq 0.54$.

Acknowledgements

This work was sponsored by NWO-Science for the use of supercomputer facilities under the project SH-349-15. Computations were carried out on the Cartesius supercomputer at SURFsara in Amsterdam. N.C. acknowledges support from the Australian Research Council under the Future Fellowship scheme (project number FT130100972) and Discovery scheme (project number DP140101110).

[1] Neal Madras and Gordon Slade. *The Self-Avoiding Walk*. Probability and its applications. Birkhäuser, Boston, MA, 1993.
[2] E. J. Janse van Rensburg. *The Statistical Mechanics of Interacting Walks, Polygons, Animals and Vesicles*. Oxford University Press, Oxford, UK, second edition, 2015.
[3] W. J. C Orr. Statistical treatment of polymer solutions at infinite dilution. *Transactions Faraday Society*, 43:12–27, 1947.
[4] Michael E. Fisher and M. F. Sykes. Excluded-volume problem and the Ising model of ferromagnetism. *Physical Review*, 114:45–58, 1959.
[5] M. F. Sykes. Some counting theorems in the theory of the Ising model and the excluded volume problem. *J. Math. Phys.*, 2:52–62, 1961.
[6] M. F. Sykes. Self avoiding walks on the simple cubic lattice. *J. Chem. Phys.*, 39:410–412, 1963.
[7] M. F. Sykes, A. J. Guttmann, M. G. Watts, and P. D. Roberts. The asymptotic behaviour of self-avoiding walks and returns on a lattice. *J. Phys. A: Gen. Phys.*, 5:653–660, 1972.
[8] A. J. Guttmann. On the critical behaviour of self-avoiding walks. *J. Phys. A: Math. Gen.*, 20: 1839–1854, 1987.
[9] A. J. Guttmann. On the critical behaviour of self-avoiding walks: II. *J. Phys. A: Math. Gen.*, 22: 2807–2813, 1989.
[10] D. MacDonald, D. L. Hunter, K. Kelly, and N. Jan. Self-avoiding walks in two to five dimensions: exact enumerations and series study. J. Phys. A: Math. Gen., 25:1429–1440, 1992.
[11] D. MacDonald, S. Joseph, D. L. Hunter, L. L. Moseley, N. Jan, and A. J. Guttmann. Self-avoiding walks on the simple cubic lattice. J. Phys. A: Math. Gen., 33:5973–5983, 2000.
[12] Nathan Clisby, Richard Liang, and Gordon Slade. Self-avoiding walk enumeration via the lace expansion. J. Phys. A: Math. Theor., 40:10973–11017, 2007.
[13] R. D. Schram, G. T. Barkema, and R. H. Bisseling. Exact enumeration of self-avoiding walks. J. Stat. Mech., P06019, 2011.
[14] P. Butera and M. Comi. n-vector spin models on the simple-cubic and the body-centered-cubic lattices: A study of the critical behavior of the susceptibility and of the correlation length by high-temperature series extended to order $\beta^{21}$. Phys. Rev. B, 56:8212–8240, 1997.
[15] J. L. Martin, M. F. Sykes, and F. T. Hoe. Probability of initial ring closure for self avoiding walks on the face centered cubic and triangular lattices. J. Chem. Phys., 46:3478–3481, 1967.
[16] S. McKenzie. Self-avoiding walks on the face-centred cubic lattice. J. Phys. A: Math. Gen., 12:L267, 1979.
[17] Pierre-Gilles de Gennes. Scaling Concepts in Polymer Physics. Cornell University Press, Ithaca, NY, 1979.
[18] Nathan Clisby and Burkhard Dünweg. High-precision estimate of the hydrodynamic radius for self-avoiding walks. Phys. Rev. E, 94:052102, 2016.
[19] Raoul D. Schram, Gerard T. Barkema, and Rob H. Bisseling. SAWdoubler: a program for counting self-avoiding walks. Comput. Phys. Commun., 184:891–898, 2013.
[20] A. J. Guttmann. Series extension: predicting approximate series coefficients from a finite number of exact coefficients. J. Phys. A: Math. Theor., 49:415002, 2016.
[21] A. J. Guttmann. Asymptotic Analysis of Power-Series Expansions, volume 13 of Phase Transitions and Critical Phenomena. Academic Press, 1989.
[22] Nathan Clisby and Iwan Jensen. A new transfer-matrix algorithm for exact enumerations: self-avoiding polygons on the square lattice. J. Phys. A: Math. Theor., 45:115202, 2012.
[23] Iwan Jensen. Square lattice self-avoiding walks and biased differential approximants. J. Phys. A: Math. Theor., 49:424003, 2016.
[24] Nathan Clisby. Scale-free Monte Carlo method for calculating the critical exponent $\gamma$ of self-avoiding walks. January 2017. URL http://arxiv.org/abs/1701.08415.
[25] Hirohiko Shimada and Shinobu Hikami. Fractal dimensions of self-avoiding walks and ising high-temperature graphs in 3d conformal bootstrap. J. Stat. Phys., 165:1006–1035, 2016.
[26] Nathan Clisby. Accurate estimate of the critical exponent $\nu$ for self-avoiding walks via a fast implementation of the pivot algorithm. Phys. Rev. Lett., 104:055702, 2010.
[27] Hsiao-Ping Hsu and Peter Grassberger. Polymers confined between two parallel plane walls. J. Chem. Phys., 120:2034–41, 2004.
[28] T. Prellberg. Scaling of self-avoiding walks and self-avoiding trails in three dimensions. J. Phys. A: Math. Gen., 34:L599–L602, 2001.
[29] Sergio Caracciolo, Maria Serena Causo, and Andrea Pelissetto. High-precision determination of the critical exponent $\gamma$ for self-avoiding walks. Phys. Rev. E, 57:R1215–R1218, 1998.
[30] R. Guida and J. Zinn-Justin. Critical exponents of the $N$-vector model. J. Phys. A: Math. Gen., 31:8103–8121, 1998.
[31] Peter Belohorec. Renormalization group calculation of the universal critical exponents of a polymer molecule. PhD thesis, University of Guelph, 1997.
[32] Bin Li, Neal Madras, and Alan D. Sokal. Critical exponents, hyperscaling, and universal amplitude ratios for two- and three-dimensional self-avoiding walks. J. Stat. Phys., 80:661–754, 1995.