Stationary states of an electron in periodic structures in a constant uniform electrical field

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Abstract

On the basis of the transfer matrix technique an analytical method to investigate the stationary states, for an electron in one-dimensional periodic structures in an external electrical field, displaying the symmetry of the problem is developed. These solutions are shown to be current-carrying. It is also shown that the electron spectrum for infinite structures is continuous, and the corresponding wave functions do not satisfy the symmetry condition of the problem.

Keywords: Wannier-Stark problem, Zener tunneling, Bloch oscillates
1 Introduction

The question of how an external (constant, uniform) electrical field influences the electron motion in periodic structures has been of great interest for decades [1, 2, 3, 4]. Nevertheless, the disagreements on the nature of the energy spectrum still persist at present. Some analytical investigations [3, 5, 6] show that the energy spectrum should be discrete irrespective of the potential form and consist of the so-called Wannier-Stark ladders with uniformly spaced levels. But other works (see [4, 7, 8, 9] and references therein, including rigorous mathematical results for the smooth potentials) point to the fact that under certain restrictions on the potential the spectrum is continuous, and a discrete spectrum may exist only for the periodic structures consisting of the $\delta$ — potentials (under certain conditions) and $\delta'$ — potentials (always).

In the simplest model the problem is reduced to solving the one-dimensional Schrödinger equation whose Hamiltonian includes the periodic potential and the potential of an electric field. It is known [5] that the properties of the equation depend strongly on the choice of a gauge for the field. When a scalar potential is used, the Hamiltonian is time independent, as in the absence of the field (the problem with the zero field will be referred to as a zero-field problem (ZFP)). But its symmetry is different from the translational one. In this case it is important to reveal the changes, in the band-gap energy spectrum of the ZFP, caused by the field and to find the wave functions satisfying the new conditions of symmetry (these functions will play the role which is similar to that of the Bloch functions in the ZFP). A directly opposite situation arises for a vector representation. Now, switching on the field does not break the translational symmetry, and the Hamiltonian becomes time dependent. As a result, the electron energy is no longer a quantum number and the initial problem can be treated as the one on the Bloch electron accelerated by the field.

The mathematical difficulties associated with making use of the scalar potential are well known beginning with the famous paper [3] by Wannier. To overcome them, the author had to treat a finite number of the Bloch bands. This approximation was rightly disputed later [4, 7]. Recent investigations (see for example [8, 9]) show that the solution of the problem essentially depends on the alternation order of Bloch bands and gaps in the high energy region in the spectrum of the ZFP. It is known that for periodic finite-value potentials the band width increases to infinity with increasing energy, while the gap width vanishes. Taking into account a finite number of Bloch bands is equivalent to the fact that the whole high-energy region is a gap. Such an approximation is sure to result in a discrete spectrum.

As far as we know, the rigorous analytical solution to the problem with the scalar potential of the general form have not been found. Besides, the stationary electron states, displaying the symmetry of the problem, remain to be investigated. In this work we propose an exact analytical method to find such stationary states. The connection between them and the Bloch states is discussed here. The energy spectrum of an electron and the Zener tunneling are also considered.

2 Symmetry of wave functions

The basis for our approach is the transfer matrix method (TMM) [10], that we have used [11] for solving the ZFP. We shall remind that one of the main points of that formalism is the notion of out-of-barrier regions (OBR), where the total potential is equal to zero. Here we shall use this notion as well, having made the necessary generalizations with reference to the problem at hand. It can be made in two ways. Firstly, one may consider that the total potential in the OBR coincides with the Stark potential which is a linear function of $x$ (in this case the treatment should be based on the Airy-functions formalism). Secondly, one may consider that the potential in these regions is a constant which depends linearly on the cell number. The proportionality coefficient depends on the electrical-field strength. Both the variants can be used in our approach. However in this work we dwell on the latter because there is a more evident association with the ZFP in this case.

The stationary Schrödinger equation for a structure of $N$ periods (unit cells) may be written as
\[
\frac{d^2 \Psi}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))\Psi = 0, \tag{1}
\]

where \(E\) is the electron energy; \(m\) is its mass; \(V(x)\) is defined by expressions: \(V(x) = v(x) - n\Delta\), if \(x \in (a_n, b_{n+1})\) \((n = 0, \ldots, N - 1)\); \(V(x) = -n\Delta\), if \(x \in (b_n, a_n)\); \(b_n = nD\); \(a_n = l + nD\) \((n = 0, \ldots, N)\); \(\Delta = eE D\); \(e\) is the electron charge (by modulus); \(l\) is the OBR width; \(D\) is the structure period; \(E\) is the electric-field strength; \(v(x)\) is a bounded \(D\)-periodical function.

It should be noted that boundary conditions at the points \(x = 0\) and \(x = a_N\) are not needed here, for we do not solve the boundary-value problem. Semi-infinite and infinite structures will be considered below. Notice also that the parameter \(l\) may be equal to zero. For the OBR always may be included into the initial potential (as the point with an infinitesimal vicinity) without changing the solution of Eq. (1) (the new and initial potentials are equivalent functions). Thus the present method can be used for any initial potential.

It is known \([3]\) that if the function \(\Psi_E(x)\) is a solution of Eq. (1), then \(\Psi_{E + \Delta}(x - D)\) is a solution too. On the basis of this statement one can assume that there are solutions (to be referred to as \(\Psi_E(x; E)\)) among those of Eq. (1), satisfying the condition

\[
\Psi_E(x + D; E) = const \cdot \Psi_E(x; E + \Delta), \tag{2}
\]

where \(const\) is a complex value. Our main goal is to find the functions \(\Psi_E(x; E)\) and to examine their properties.

The solutions to the Schrödinger equation, both with the scalar and with the vector potentials, are generally found in the form of an expansion in orthogonal functions (for example, in the Bloch functions). In this case the required and basis functions are supposed to belong to the same class of functions. The disadvantage of such an approach is that, in finding wave functions obeying the symmetry conditions, it is not always clear to what class of functions they are to belong. So, for example, in the absence of the field the same condition of the translational symmetry (it coincides with \([2]\) when \(\Delta = 0\)) leads, in the bands, to the Bloch functions bounded everywhere, but in the gaps it provides the functions unbounded at the plus or minus infinity. As will be shown further, the functions \(\Psi_E(x; E)\) are unbounded when \(x \rightarrow -\infty\). Thus, if we attempted to find these functions in the Bloch or Wannier expansions, we could obtain an incorrect result, because both sets of functions belong to other classes. The transfer matrix method is free from this demerit, for the expansions technique is not used there.

### 3 Functional equation for wave functions in the transfer matrix method

The general solution of the Eq. (1) in the OBR’s is

\[
\Psi(x; E) = A_n^{(+)}(E) \exp[ik_n(x - b_n)] + A_n^{(-)}(E) \exp[-ik_n(x - b_n)], \tag{3}
\]

where \(k_n = \sqrt{2m(E + n\Delta)/\hbar^2}; n = 0, \ldots, N.\)

Here the main problem is to find the coefficients \(A_n^{(+)}(E)\) and \(A_n^{(-)}(E); n = 0, \ldots, N.\) Once the coefficients have been found, the determination of the \(\Psi_E(x; E)\) in the barrier regions should present no principal problems. In the general case for this purpose one can use, for example, the numerical technique \([2]\).

The connection between the coefficients of the solution in the first two OBR’s is given by

\[
A_0(E) = \alpha(E) Y(E) \Gamma(E) A_1(E); \tag{4}
\]
Here $Y$ is a transfer matrix (see \[10\]), describing the barrier at the unit cell $n = 0$ (providing that there is no step at the point $b_1$), and $\alpha \cdot \Gamma$ is a matrix matching the solutions at the step at $x = b_1$:

$$Y = \begin{pmatrix} \tilde{q} & \tilde{p} \\ \tilde{p}^* & \tilde{q}^* \end{pmatrix}, \quad \Gamma = \begin{pmatrix} q_s & p_s \\ p_s & q_s \end{pmatrix}; \quad A_n = \begin{pmatrix} A_n^{(+)} \\ A_n^{(-)} \end{pmatrix}; \quad (5)$$

$$\tilde{q} = \frac{1}{\sqrt{T}} \exp[-i(J + k_0l)]; \quad \tilde{p} = \sqrt{R/T} \exp[i(\frac{\pi}{2} + F - k_0l)];$$

$q_s = (\alpha + \alpha^{-1})/2; \quad p_s = (\alpha^{-1} - \alpha)/2; \quad \alpha(E) = \sqrt{k_1(E)/k_0(E)};

the phases $J(E), F(E)$ and the transmission coefficient $T(E)$ (see \[10\]) describing the barrier in the zero cell are supposed to be known; $R = 1 - T$.

Let

$$Z = Y \Gamma = \begin{pmatrix} q & p \\ p^* & q^* \end{pmatrix},$$

then the relationship \[4\] can be rewritten as

$$A_0(E) = \alpha(E)Z(E)A_1(E), \quad (6)$$

and the connection between any two adjacent OBR’s will be determined by

$$A_n(E) = \alpha(E + n\Delta)Z(E + n\Delta)A_{n+1}(E); \quad n = 0, 1, \ldots, N - 1. \quad (7)$$

Providing \[7\] the connection between the zero and $N$-th unit cells can be written in the form

$$A_0(E) = \alpha_{(1,N)}(E)Z_{(1,N)}(E)A_N(E), \quad (8)$$

where

$$Z_{(1,N)}(E) = Z(E) \cdot \ldots \cdot Z(E + (N - 1)\Delta); \quad (9)$$

$$\alpha_{(1,N)}(E) = \prod_{n=0}^{N-1} \alpha(E + n\Delta) = \sqrt{\frac{k_0(E + N\Delta)}{k_0(E)}},$$

Defining for all $n$ the vector

$$\tilde{A}_n(E) = \alpha_{(1,n)}(E)A_n(E),$$

we can rewrite Eq. \[8\] as

$$A_0(E) = \tilde{A}_0(E) = Z_{(1,N)}(E)\tilde{A}_N(E). \quad \quad (10)$$

Now, by analogy with the ZFP \[13\] we will attempt to find the wave functions whose expressions for the extreme OBR’s (i.e. for the zero and $N$-th unit cells) are connected by means of symmetry. For this purpose we will demand of that the coefficients of the zero and first OBR’s must satisfy the condition

$$\tilde{A}_1(E) = C(E) \cdot A_0(E + \Delta), \quad \quad (11)$$

where $C(E)$ is a complex function. Then, by Eq. \[10\], $A_0(E)$ must obey the functional equation

$$A_0(E) = C(E)Z(E)A_0(E + \Delta). \quad \quad (12)$$
It is easy to check that $A_0(E)$ is determined by this equation with an accuracy of a scalar periodical function, $\omega(E)$: $\omega(E + \Delta) = \omega(E)$. Namely, if the function $A_0(E)$ is a solution, so will be the one $\omega(E) \cdot A_0(E)$.

Now, taking into account (11) and (12) in the relation (8) we have

$$\tilde{A}_N(E) = G_N(E)A_0(E + N\Delta),$$

(13)

where $G_N(E) = \prod_{n=0}^{N-1} C(E + n\Delta)$.

As in the ZFP [11], Eqs. (10) and (13) provide theoretically the way of deriving, in explicit form, the expressions for the $N$-barrier transfer matrix (9) in terms of $A_0(E)$, i.e. in terms of unit-cell characteristics. However as will be seen from the following, in this case it gives no preferences in calculating the $Z(1,N)(E)$.

Considering (13) and the relation $G_{n+1}(E) = C(E)G_n(E + \Delta)$, one can show that

$$\tilde{A}_{n+1}(E) = C(E)\tilde{A}_n(E + \Delta).$$

Such a connection between the coefficients of two adjacent OBR’s provides fulfilling the symmetry condition (2). Namely,

$$\Psi_E(x + D; E) = \alpha^{-1}(E)C(E)\Psi_E(x; E + \Delta).$$

(14)

So, in the TMM the symmetry condition leads to the functional equation (12) for coefficients of the general solution of the Schrödinger equation.

### 4 Solutions of the functional equation

According to the theory of functional equations [13], in order to solve Eq. (12) one needs to define the auxiliary functions $\eta_n(E)$, where $n = 0, 1, \ldots$, with help of the relationships

$$\eta_0(E) = C(E)Z(E)\eta_0(E);$$

$$\eta_n(E) = C(E)Z(E)\eta_{n-1}(E + \Delta).$$

(15)

(16)

Then the solution of Eq. (12) can be written, it is easily checked, in the form

$$A_0(E) = \lim_{n \to \infty} \eta_n(E).$$

(17)

In fact, it means that we have to solve the auxiliary equation (15) and to prove the existence of the limit (17).

Considering Eqs. (13) and (16), we can write the limit (17) also as

$$A_0(E) = G_\infty(E)Z_{(1,\infty)}(E) \cdot \eta_0,$$

(18)

where $\eta_0 = \lim_{n \to \infty} \eta_0(E + n\Delta)$.

The finding of $A_0(E)$ is seen to be associated with calculating the matrix $Z_{(1,\infty)}(E)$ for the semi-infinite structure. That is the reason that to derive expressions for $Z_{(1,N)}(E)$ in terms of $A_0(E)$ is of no interest in this approach.

Let us begin with solving Eq. (13). It can be rewritten as

$$\frac{\eta_0(\cdot)}{\eta_0^{(-)}} = \frac{C^{-1} - q}{p} = \frac{p^*}{C^{-1} - q^*}; \quad \eta_0 = \begin{pmatrix} \eta_0(+) \\ \eta_0(-) \end{pmatrix}.$$  

(19)

This equation coincides by form with equation (8) (see the Ref. [11]) in the ZFP. The only difference is that the matrix $Z(E)$ describes the one-cell potential which involves the electric-field effect.
graduation of the energy scale into Bloch bands ("allowed" energy regions) and gaps ("forbidden" energy ones) arises as well. But such a division does not yield the energy spectrum to the given problem and is exclusive of an auxiliary significance.

Since $\text{det}Z(E) = 1$ the solutions of the characteristic equation (18) (the right equality) are two reciprocal quantities. In choosing the required root, for any energy region, we must proceed from the fact that the function $C(E)$ must have the limit when $E \to \infty$. Otherwise, the limit $\eta_0$ does not exist too, and hence Exp. (18) loses its meaning.

Let us show that the solutions of the auxiliary equation (19), having the properties needed, are expressed by

$$C_1(E) = \frac{1}{q + y}; \quad \eta_0^{(+)}|_1 = 1; \quad \eta_0^{(-)}|_1 = \frac{y}{p};$$

$$C_2(E) = \frac{1}{q^* - y}; \quad \eta_0^{(+)}|_2 = -\frac{y}{p^*}; \quad \eta_0^{(-)}|_2 = 1;$$

$$C_2 = C_1^{-1}, \quad \frac{y = -\frac{|p|^2 \cdot \text{sign}(u)}{|u| + \sqrt{u^2 - |p|^2}}}{u = \text{Im}(q)}.$$

First of all it should be noted that the limit $\tilde{\eta}_0$ is calculated on the set of the equidistant points $E_n$, where $E_n = E + n\Delta; \ n = 0, 1, \ldots$. The set will be denoted by $S_{E,\Delta}$, in doing so we emphasize its dependency on the parameters $E$ and $\Delta$. It is supposed that $E$ varies in the interval $(0, \Delta]$. The set $S_{E,\Delta}$ consists of the two subsets $S^0_{E,\Delta}$ and $S^f_{E,\Delta}$ whose points belong to the bands ($|u| > |p|$) and gaps ($|u| \leq |p|$), respectively. As will be shown below, the behavior of the vector-function $\tilde{\eta}_0(E_n)$ on the subsets $S^0_{E,\Delta}$ and $S^f_{E,\Delta}$ differs qualitatively. Therefore, there exists no limit, $\tilde{\eta}_0$, when both the subsets are infinite.

It follows from the general considerations that the number of points in $S^0_{E,\Delta}$ and $S^f_{E,\Delta}$ depends on the bands and gaps width as well as on their location on the energy scale. As will become clear from the following, both factors are sufficient to investigate for the rectangular barrier ($v(x) = v_0$). By using the explicit expressions for the tunneling parameters of the rectangular barrier (see for example [10]), one can show that for the matrix element $\tilde{p}$ in the high-energy region the inequality $|\tilde{p}| \lesssim (v_0/2)E^{-1}$ is valid. The asymptote of the phase $J(E)$ is the function $k_0(E)d$ ($d = b_1 - a_0$ is the barrier width); for the bigger is the electron energy, the more its motion similar to that of the free electron. Thus, in the high-energy region the centres of the gaps (the points which satisfy the equation $\sin(J(E) + k_0l) = 0$ are meant) for periodical structures formed of the rectangular barriers asymptotically coincide on the energy scale with the points $E_L$, where

$$E_L = L^2\epsilon; \quad \epsilon = \frac{\hbar^2}{2mD^2}; \quad L = 0, 1, \ldots$$

The distance between the gaps centres in the high-energy region is, consequently, a multiple of the constant $\epsilon$. In this case the gaps width tends to zero with increasing $E$, and the bands width, on the contrary, does to infinity (see [7][8][9]). These findings are not changed by the presence of the step at the right boundary of the barrier, because the corresponding matrix $\Gamma$ is real, and, besides, $|p_\pm(E)| \sim E^{-1}$ as for the rectangular barrier. Namely, for the matrix $Z(E)$ we have

$$|p(E)| \lesssim \frac{v_0}{2E}, \quad \arg(q) \approx k_0(E)D,$$  \hspace{1cm} (20)

here it is also taken into account that $q = \tilde{q}a_\pm + \tilde{p}p_\pm, \ p = \tilde{q}p_\pm + \tilde{p}a_\pm; \ |\tilde{q}|^2 - |\tilde{p}|^2 = 1, \ q_\pm^2 - p_\pm^2 = 1$. The asymptotics in the high-energy region is not changed also when going to the general-form barrier, because in this case the inequality $|\tilde{p}(E)| \lesssim (v_{\text{max}}/2)E^{-1}$ holds, and the asymptotics of $p_\pm(E)$ remains the same; here $v_{\text{max}}$ is the maximum of $v(x)$ by modulus.
It follows from the above that the subset $S_{E,\Delta}^{\infty}$ is always infinite, and $S_{E,\Delta}^{\infty}$ is infinite in the exceptional ("resonance") cases only: $E = \Delta = re$, where $r$ is a rational number. There is no limit $\eta_0$ under these conditions, for the modulus of the functions $\eta_0^{(-)}(E)_1$ and $\eta_0^{(+)}(E)_2$ is equal to unity on $S_{E,\Delta}^{\infty}$, but on $S_{E,\Delta}^0$ it varies between the limits zero and unity.

At the given $\Delta$ the set of the energy values, where the "resonance" takes place, is no more than a countable set. It is eventually connected with the fact that the gaps width is zero in the limit $L \to \infty$. Any arbitrary small variation of $E$ removes the points $E_n$, beginning with the some number $N$, from the gaps. Since all these points belong to the subset $S_{E,\Delta}^{\infty}$, where the inequality $|u| > |p|$ holds, there is a such $\delta > 0$ that for all $n > N$ the condition $|u(E_n)| \geq |p(E_n)|^{1-\delta} \equiv 1$ is valid (note, that $|p|^2 < 1$ in the high-energy region). It follows from here that at these points we have

$$|y| = \frac{|p|^2}{|u| + \sqrt{u^2 - |p|^2}} < \frac{|p|^2}{|u|} \leq |p|^{1+\delta}.$$  

And hence

$$|\eta_0^{(-)}(E_n)|_1 = \frac{|y|}{|p|} \leq |p(E_n)|^{\delta} \approx \gamma E_n^{-\delta},$$

where $\gamma = (v_{max}/2)^\delta$. The same asymptotics takes place for $\eta_0^{(+)}(E_n)|_2$. It means that almost everywhere on the energy scale

$$\eta_0|_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \eta_0|_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (21)$$

Now substituting (21) into (18) we get the final expressions for two solutions of functional equation (12):

$$A^{(1)}_0 = \left( \frac{Q_{(1,\infty)} G_{\infty}}{P_{(1,\infty)} G_{\infty}} \right); \quad A^{(2)}_0 = \left( \frac{P_{(1,\infty)} G_{\infty}^{-1}}{Q_{(1,\infty)} G_{\infty}^{-1}} \right), \quad (22)$$

where $G_{\infty} = G^{(1)}_{\infty} = 1/G^{(2)}_{\infty}$; $Q_{(1,\infty)}$ and $P_{(1,\infty)}$ are the elements of $Z_{(1,\infty)}$. Then from (13) it follows that

$$A^{(1)}_{\infty}(E) = \left( \frac{G_{\infty}(E)}{0} \right); \quad A^{(2)}_{\infty}(E) = \left( \begin{array}{c} 0 \\ G_{\infty}(E)^{-1} \end{array} \right). \quad (23)$$

Expressions (1), (8) and (22) provide two independent functions $\Psi^{(1)}_\varepsilon(x; E)$ and $\Psi^{(2)}_\varepsilon(x; E)$. Both solutions are current-carrying. The corresponding probability flows, $I^{(1)}(E)$ and $I^{(2)}(E)$, are

$$I^{(1)}(E) = \hbar m^{-1} k_0(E)|G_{\infty}(E)|^2; \quad I^{(2)}(E) = \hbar m^{-1} k_0(E)|G_{\infty}(E)|^{-2}. \quad (24)$$

Now we have to prove that the limit in (17) exists. Otherwise Exp. (22)-(24) are meaningless.

5 On existence of the solutions $\Psi_\varepsilon(x; E)$

For a complex-valued matrix $H$ and vector $A$ let us define the norms

$$\|H\| = \max_j \sqrt{\sum_{i=1}^2 |h_{ij}|^2}; \quad j = 1, 2; \quad \|A\| = |A^{(+)}| + |A^{(-)}|.$$  

In particular, it means that $\|Z\|^2 = 1 + 2|p|^2$. Considering the first solution we will prove that for any given $E$ and $\varepsilon > 0$ we can find such number $N$ that
\[ \| \eta_n(E) - \eta_{n-1}(E) \| < \varepsilon \]

for \( n > N \). Since
\[ \eta_n(E) = G_n(E)Z_{(1,n)}(E)\eta_0(E_n), \]
(here \( E_n = E + n\Delta \)) we have
\[ \| \eta_n(E) - \eta_{n-1}(E) \| \leq |G_{n-1}(E)| \cdot \| Z_{(1,n-1)}(E) \| \cdot F(E), \]
where \( F(E) = \| C(E_n)Z(E_n)\eta_0(E_n) - \eta_0(E_{n-1}) \| \).

Let us show that the first two norms are bounded as \( n \to \infty \). We have
\[ |G_{\infty}(E)|^{-1} = \prod_{n=0}^{\infty} |C(E_n)|^{-1} = \prod_{n=0}^{\infty} |q(E_n) + y(E_n)| \leq \]
\[ \leq \prod_{n=0}^{\infty} |q(E_n)| \cdot \left( 1 + \frac{|y(E_n)|}{|q(E_n)|} \right). \quad (27) \]

The convergence of both norms in (27) is equivalent to that of the series \( \sum_{n=0}^{\infty} n^{-2} \), because \( |q| = \sqrt{1 + |p|^2}, y \sim |p|^2, |p| \sim n^{-1} \). Since this series converges, the infinite product \( |G_{\infty}(E)| \) does as well.

For the matrix describing the semi-infinite structures we have
\[ \| Z_{(1,\infty)}(E) \|^2 \leq \prod_{n=0}^{\infty} \| Z(E_n) \| = \prod_{n=0}^{\infty} (1 + 2|p(E_n)|^2). \]

Obviously, this product converges for the same reason as in (27). In addition, since
\[ \| Z_{(1,\infty)}(E) \|^2 \equiv 1 + 2\frac{R_{(1,\infty)}(E)}{T_{(1,\infty)}(E)}, \]
we have that \( T_{(1,\infty)}(E) \neq 0 \). So that the semi-infinite structure must not be absolutely opaque for an electron.

Now, it remains to be shown that \( F \) in (26) approaches zero with increasing \( n \). Using (15) we have
\[ F = \| \eta_0(E_n) - \eta_0(E_{n-1}) \| \lesssim 2\gamma n^{-\delta}. \quad (28) \]

Since the norms \( |G_{\infty}(E)| \) and \( \| Z_{(1,\infty)}(E) \| \) are bounded there is
\[ \max_j (|G_j(E)| \cdot \| Z_{(1,j)}(E) \|), \]
where \( j = 1, 2, \ldots \). Together with (28) this provides fulfilling the inequality (25), that proves the existence of the limit in (17). For the second solution the arguments are similar.

6 Conclusions

At first glance the functions \( \Psi_E(x; E) \) may be calculated by this method in the region located to the right of the zero cell only. However it should be noted that any unit cell of the periodical structure may be taken as a zero cell. Then by making use of the transfer matrix which connects solutions in the zero cell and in the regions to the left of it, one can calculate the functions \( \Psi_E(x; E) \) on the whole axis Ox.

Since both functions \( \Psi_E(x; E) \) are current-carrying they increase infinitely by modulus in the classically inaccessible range when \( x \to -\infty \), according to the general properties of the one-dimensional Schrödinger
equation. Thus, for infinite structures, \( \Psi_E(x; E) \) are no solutions to the problem. However, for any \( E \) (excluding a countable set for certain values of \( \Delta \)), the solution (undegenerate) for the infinite structure can be obtained as a linear combination of these functions. As a result we arrive at two important conclusions. First, for the limited periodical potentials the energy spectrum of an electron in the problem for infinite structures is continuous (so that the Wannier-Stark states, by the model, may exist only as the quasi-stationary ones). Second, the stationary wave functions of an electron in the infinite structures, being the linear combinations of the functions \( \Psi_E(x; E) \), do not satisfy the symmetry condition \( \tilde{\Psi} = \Psi \). (There exist a mistaken opinion that the continuity of the spectrum in this problem is obvious. The following arguments are used in this case. Namely, the energy spectrum is continuous since

a) the range, where \( x \) is large, is classically accessible for an electron;

b) the periodical potential is negligible in comparison with the Stark potential when \( x \to \infty \), and, consequently, the electron motion in this range is of the free electron type (see, for example, [3]). However, it should be noted that the first statement is valid only if \( V(x) \) remains finite at the plus infinity. But if \( V(x) \to -\infty \) as \( x \to \infty \), then the electron spectrum may be both continuous and discrete, depending on the monotonicity and rate of decreasing \( V(x) \) at \( x \to \infty \) (see, for example, [4]). It follows also from this the erroneousness of the second argument, because on the whole axis \( OX \) the derivatives of \( V(x) \) (and, hence, its monotonicity) are determined by the periodical component of the potential.)

It is interesting also to dwell on the question of the connection of the given problem to the ZFP. We will start with the fact that the wave functions \( \Psi_E(x; E) \) are defined in terms of the solutions of auxiliary equation \([19]\) describing formally the electron motion in the periodic structures in the absence of an electric field. In addition, for finite structures the functions \( \Psi_E(x; E) \), by their properties, are close to the solutions of the ZFP, if \( N \Delta \ll E \) (\( N \) is the number of unit cells in the structure). In particular, if values of \( E \) are in the band, then \( \Psi_E(x; E) \), in the given interval, are close to the usual Bloch functions. It is the case when an electrical field has a weak effect on the electron with the energy \( E \). However there is no transition from the given problem to the ZFP when the periodical structure is considered on the whole axis \( Ox \). The wave functions \( \Psi_E(x; E) \) are unbounded, when \( x \to -\infty \), at any value of the electric-field strength.

Some comments should be further made about the role of the Zener tunneling (ZT) which have been the subject of great interest (see, for example, [3] and referers therein) since paper [2]. Strictly speaking, by this concept is meant the electron transitions between the bands, and therefore the latter relates to the non-stationary case. In the models with the vector potential the Zener tunneling is caused by the accelerating effect of the field, resulting in that a Bloch electron passes (tunnels) from the lower bands to the upper. In our approach we investigate stationary states. Nevertheless, we can draw some conclusions on the question. It is possible because symmetry condition \([4]\), governing the functions \( \Psi_E(x; E) \), links their \( E \)- and \( x \)-dependencies. In particular, for \( \Psi_E(x; E) \) relationship \([3]\) is valid. Note also that \( \tilde{A}_\infty \) (see \([23]\)) is a bounded non-zero value. It provides the asymptotics \( \mathcal{A}_n \sim n^{-1/4} \) and \( \mathcal{A}_0(E) \sim E^{-1/4} \) (as for Airy’s functions). Thus the probability that an electron is in the \( n \)-th unit cell or it has the energy \( E \) decreases with increasing of these parameters by the power law instead of the exponential one. This result makes the conclusion presented in Ref. [8] more precise.

Also it follows from the above that the well-known Bloch oscillations can exist only as the decaying ones. As for the experimental evidences of the long-lived Bloch oscillations and the Wannier-Stark ladders in superlattices, it is not the question of correctness of our approach. This implies only that one needs to find the mathematical model which would be more suitable for the experiments on superlattices. In the following paper we are going to present such a model.

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