F-NILPOTENT RINGS AND PERMANENCE PROPERTIES

JENNIFER KENKEL, KYLE MADDOX, THOMAS POLSTRA, AND AUSTYN SIMPSON

Abstract. We explore the singularity classes $F$-nilpotent, weakly $F$-nilpotent, and generalized weakly $F$-nilpotent under faithfully flat local ring maps. As an application, we show that the loci of primes in a Noetherian ring of prime characteristic which define either weakly $F$-nilpotent or $F$-nilpotent local rings are open with respect to the Zariski topology whenever $R$ is $F$-finite or essentially of finite type over an excellent local ring.

1. Introduction

Let $(R, \mathfrak{m}, k)$ denote a commutative Noetherian local ring of prime characteristic $p > 0$. For each $e \in \mathbb{N}$ let $F^e : R \rightarrow R$ denote the $e$th iterate of the Frobenius endomorphism of $R$. Iterates of the Frobenius endomorphism induce Frobenius linear maps of local cohomology

$$F^e : H^i_{\mathfrak{m}}(R) \rightarrow H^i_{\mathfrak{m}}(R)$$

and behavior of these Frobenius actions are used to classify various prime characteristic singularity classes. Among these so-called $F$-singularities include $F$-rational, $F$-injective, and $F$-nilpotent singularities, the last of which being the primary subject of study in this article.

A ring $R$ is weakly $F$-nilpotent if high enough iterates of Frobenius actions on the lower local cohomology modules with support in the maximal ideal are eventually 0. Utilizing the language of [Lyu06], a local ring $(R, \mathfrak{m}, k)$ of dimension $d$ is weakly $F$-nilpotent if and only if its $F$-depth is equal to $d$. Lyubeznik developed the notion of $F$-depth to answer a question of Grothendieck. In particular, it was shown that if $(S, \mathfrak{m}, k)$ is a regular local ring of Krull dimension $d$, prime characteristic $p > 0$, and $I \subseteq S$ an ideal then $H^i_I(S) = 0$ for all $0 \leq i < d$ if and only if $R = S/I$ is weakly $F$-nilpotent, [Lyu06, Theorem 1.1].

A local ring $(R, \mathfrak{m}, k)$ of dimension $d$ is called $F$-nilpotent if $R$ is weakly $F$-nilpotent and the nilpotent submodule of $H^d_{\mathfrak{m}}(R)$ agrees with the tight closure of the 0-submodule. Srinivas and Takagi conjectured (and proved for dimension smaller than four, [ST17, Proposition 3.8]) that a normal isolated singularity $X$ defined over $\mathbb{C}$ has $F$-nilpotent type if and only if given a log resolution $\pi : Y \rightarrow X$ one has $H^i(E, \mathcal{O}_E) = 0$ for all $i \geq 1$, where $E$ is the exceptional divisor of $\pi$.

The class of $F$-nilpotent singularities also deserves interest due to its relation to other classes of singularities. For instance, a reduced (alternatively, excellent and equidimensional) Noetherian local ring is $F$-rational if and only if it is $F$-injective and $F$-nilpotent. Further, $F$-rationality and $F$-injectivity are known to behave nicely in many aspects such as ascent and descent for sufficiently nice faithfully flat maps as well as openness of their respective

Polstra was supported by NSF Postdoctoral Research Fellowship DMS #1703856.
Simpson was supported by NSF RTG grant DMS #1246844.
loci (see e.g. [Vél95] and [DM19]). Thus, it is natural to wonder if $F$-nilpotence is similarly well-behaved. Exploring this topic is the main content of this paper, and we collect a number of results along the way concerning the more general classes of weakly $F$-nilpotent and generalized weakly $F$-nilpotent rings.

Our first result is that the $F$-nilpotent locus of prime ideals is open with respect to the Zariski topology.

**Theorem A.** Let $R$ be a ring of prime characteristic $p > 0$ which is either $F$-finite or essentially of finite type over an excellent local ring. Then the following subsets of $\text{Spec}(R)$ are open with respect to the Zariski topology:

1. $\{ p \in \text{Spec}(R) \mid R_p \text{ is weakly } F\text{-nilpotent} \}$;
2. $\{ p \in \text{Spec}(R) \mid R_p \text{ is } F\text{-nilpotent} \}$.

The proof of Theorem A in the $F$-finite scenario utilizes that Frobenius actions on local cohomology are realized as the Matlis dual of Cartier linear maps of the dualizing complex (see [ST17, Lemma 2.3]). Outside of the $F$-finite scenario we must understand the behavior of Frobenius actions on local cohomology under faithfully flat extensions. To this end, we study ascent and descent properties of local rings which are either $F$-nilpotent, weakly $F$-nilpotent, or generalized weakly $F$-nilpotent. Our results in this direction are as follows:

**Theorem B.** Let $(R, m) \to (S, n)$ be a faithfully flat map of local rings of prime characteristic $p > 0$.

1. If $S$ is weakly $F$-nilpotent then $R$ is weakly $F$-nilpotent.
2. If $S$ is $F$-nilpotent, then $R$ is $F$-nilpotent under either of the following additional assumptions:
   - $S$ is an excellent local ring or;
   - The closed fiber $S/\mathfrak{m}S$ is Cohen-Macaulay.
3. If $R$ is weakly $F$-nilpotent and $S/\mathfrak{m}S$ is Cohen-Macaulay then $S$ is weakly $F$-nilpotent.
4. If $R$ is $F$-nilpotent, $R/\sqrt{0}$ and $S \otimes_R R/\sqrt{0}$ have a test element in common, and $R \to S$ has geometrically regular fibers, then $S$ is $F$-nilpotent.

We prove additionally that the property of being (weakly) $F$-nilpotent ascends under faithfully flat purely inseparable local ring homomorphisms; see Theorem 4.5 for a precise statement. Moreover, we study the behavior of generalized weakly $F$-nilpotent rings under faithfully flat maps; see Theorem 4.7 for a precise statement.

The organization of the paper is as follows: In Section 2 we discuss relevant background material needed throughout the article. In Section 3 we utilize Cartier linear maps on dualizing complexes to begin our study of Frobenius actions on local cohomology. Of particular interest in this section is the existence of uniform bounds on the Frobenius test exponents of the localizations of a ring at prime ideals in the weakly $F$-nilpotent locus. The proofs of the ascent and descent statements in Theorem B are gathered from the results of Section 4, up to a Cohen-Macaulay assumption on the closed fiber. We also use Section 4 to further develop the theory of generalized weakly $F$-nilpotent rings as introduced by the second named author in [Mad19]. The final section of the article, Section 5, is where we record a proof of Theorem A. Finally, we use our open loci results of Section 5 to complete the proof of Theorem B.
2. Preliminary results

In this section we recall basic facts about local cohomology and tight closure which will be essential in defining the singularity classes of interest, as well as in the proofs of our main theorems. We also give an overview of the recent literature concerning these singularities.

2.1. Local cohomology and the Nagel-Schenzel isomorphism, and Frobenius actions. Let $R$ be a commutative Noetherian ring, $I \subseteq R$ an ideal, and $M$ an $R$-module. We denote by $H^i_I(M)$ the $i$th local cohomology module of $M$ with support in $I$. If $(R, m, k)$ is local of Krull dimension $d$, $x_1, \ldots, x_d$ a system parameters, and $x = x_1 \cdots x_d$ then the top local cohomology module is explicitly identified as a direct limit system:

$$H^d_m(R) \cong \lim_{\rightarrow} \left( \frac{R}{(x_1^{t_1}, \ldots, x_d^{t_d})} \xrightarrow{x} \frac{R}{(x_1^{t_1+1}, \ldots, x_d^{t_d+1})} \right)$$

Filter regular sequences allow us to identify lower local cohomologies with support in the maximal ideal as submodules of the top local cohomology module. An element $x$ is a filter regular element of $M$ if $(0 :_M x)$ has finite length. A sequence of elements $x_1, \ldots, x_i$ is a filter regular sequence of $M$ if $x_j$ is a filter regular element of $M/(x_1, \ldots, x_{j-1})M$ for each $1 \leq j \leq i$. A submodule $(0 :_M x)$ has finite length if and only if $x$ avoids all non-maximal associated primes of $M$. Therefore a simple prime avoidance argument can be used to show that $R$ admits a filter regular sequence on $M$ of length $\dim(M)$. The Nagel-Schenzel isomorphism stated below allows us to identify lower local cohomology modules with support in the maximal ideal as submodules of the top local cohomology on an ideal generated by a filter regular sequence.

Theorem 2.1 ([NS94, Proposition 3.4]). Let $(R, m, k)$ be a local ring of Krull dimension $d$. Suppose $x_1, \ldots, x_i$ is a filter regular sequence of length $i$ and $x = x_1 \cdots x_i$, then

$$H^i_m(R) \cong H^0_m(H^i_{(x_1, \ldots, x_i)}(R)) \cong \lim_{\rightarrow} \left( \frac{(x_1^{t_1}, \ldots, x_i^{t_i}) :_R m^\infty}{(x_1^{t_1}, \ldots, x_i^{t_i})} \xrightarrow{x} \frac{(x_1^{t_1+1}, \ldots, x_i^{t_i+1}) :_R m^\infty}{(x_1^{t_1+1}, \ldots, x_i^{t_i+1})} \right).$$

We will repeatedly use the following lemma whose content is that a filter regular sequence is preserved under a faithfully flat extension of local rings with $0$-dimensional closed fiber. This fact in combination with Theorem 2.1 will be useful when establishing the permanence properties of Section 4.

Lemma 2.2. Let $(R, m) \to (S, n)$ be a faithfully flat local ring map with $0$-dimensional closed fiber $S/mS$. If $x_1, \ldots, x_d$ is a filter regular sequence in $R$ then $x_1, \ldots, x_d$ is a filter regular sequence in $S$.

Proof. The sequence $x_1, \ldots, x_d$ is a filter regular sequence if and only if the colon ideals $((x_1, \ldots, x_i) :_R x_{i+1})/(x_1, \ldots, x_i)$ of $R/(x_1, \ldots, x_i)$ has finite length for each $1 \leq i \leq d - 1$. Base changing by a faithfully flat map preserves colon ideals. Because $S/mS$ is $0$-dimensional the images of $x_1, \ldots, x_d$ satisfy the same finite length criterion in $S$ and therefore $x_1, \ldots, x_d$ is a filter regular sequence of $S$. □
2.2. Frobenius actions on local cohomology, Frobenius closure, and tight closure.

Throughout this subsection suppose that \((R, \mathfrak{m}, k)\) is a \(d\)-dimensional local ring of prime characteristic \(p > 0\) and \(F^e : R \to R\) is the \(e\)th iterate of the Frobenius endomorphism. Then \(F^e\) induces \(\ell^e\)-linear maps \(F^e : H^a_m(R) \to H^a_m(R)\), see [ILL07, Lecture 21]. Equivalently, the Frobenius endomorphism induces \(R\)-linear maps \(H^a_m(R) \to F^e H^a_m(R)\) where \(F^e H^a_m(R)\) is the Frobenius pushforward of \(H^a_m(R)\). Suppose that \(x_1, \ldots, x_i\) a filter regular sequence of \(R\) and identify \(H^a_m(R)\) as in Theorem 2.1. If \(\gamma \in H^a_m(R)\) is represented by the class of \(\eta + (x_1^t, \ldots, x_i^t)\) in the above direct limit system then the \(e\)th Frobenius action of \(R\) sends \(\gamma\) to the element of \(H^a_m(R)\) represented by \((\eta^{p^e} + (x_1^{tp^e}, \ldots, x_i^{tp^e})\).

Frobenius actions on local cohomology modules are often used to classify prime characteristic singularities (see [BB05, EH08, Mad19, PQ19, Smi97]). This article concerns itself with the following singularity classes defined in terms of Frobenius actions on local cohomology:

1. \(F\)-nilpotent, [BB05];
2. weakly \(F\)-nilpotent, [PQ19];
3. generalized weakly \(F\)-nilpotent, [Mad19].

We will be interested in the behavior of the above singularity classes under flat base change in Section 4. Understanding the algebraic technicalities of local cohomology modules under the identifications of Theorem 2.1 will allow us to do just that. To define these classes of singularities, we first recall the basic notions of tight closure theory. See [HH90] for a thorough treatment.

Let \(R^e\) be the complement of the union of the minimal primes of \(R\) and let \(F^e(-) = - \otimes_R F^e R\) denote the \(e\)th Frobenius pullback functor, i.e. base change along the \(e\)th iterate of Frobenius. If \(N \subseteq M\) are \(R\)-modules then we say an element \(m \in M\) is in \(N^t_M\), the tight closure of \(N\), if there exists a \(c \in R^e\) so that \(m\) is in the kernel of the following composition of maps for all \(e \gg 0\):

\[
M \to M/N \to F^e(M/N) \xrightarrow{c} F^e(M/N).
\]

Similarly, we say \(m \in M\) is in \(N^F_M\), the Frobenius closure of \(N\), if \(m\) is in the kernel of the following maps for all \(e \gg 0\):

\[
M \to M/N \to F^e(M/N).
\]

Observe that there are inclusions \(N \subseteq N^F_M \subseteq N^t_M\).

The top local cohomology module with support in the maximal ideal enjoys the following property: \(F^e(H^d_m(R)) \cong H^d_m(R)\) and \(H^d_m(R) \to F^e(H^d_m(R))\) is identified with the \(e\)th Frobenius action of the top local cohomology module. In particular, if \(\gamma \in H^d_m(R)\) then \(\gamma \in 0^*_{H^d_m(R)}\) if and only if there exists a \(c \in R^e\) so that \(\gamma\) is in the kernel of the following composition of \(R\)-linear maps for all \(e \gg 0\):

\[
H^d_m(R) \xrightarrow{F^e} F^e H^d_m(R) \xrightarrow{F^e c} F^e H^d_m(R).
\]

Similarly, \(\gamma \in 0^F_{H^d_m(R)}\) if and only if \(\gamma\) is the kernel of the following maps for all \(e \gg 0\):

\[
H^d_m(R) \xrightarrow{F^e} F^e H^d_m(R).
\]
Definition 2.3. Let \((R, \mathfrak{m}, k)\) be a local ring of prime characteristic \(p > 0\) and dimension \(d\). We say that \(R\) is weakly F-nilpotent if \(0^F_{H^n_m(R)} = H^n_m(R)\) for all \(0 \leq i < d\). If \(R\) is weakly F-nilpotent and \(0^*_{H^d_m(R)} = 0^F_{H^d_m(R)}\) then we say that \(R\) is F-nilpotent.

An equivalent description of weakly F-nilpotent rings is as follows: suppose \(x_1, \ldots, x_i\) is a filter regular sequence of length \(i\), \(x = x_1 \cdots x_i\), and identify \(H^n_m(R)\) as in Theorem 2.1. By definition, \(R\) is weakly F-nilpotent if and only if for every \(\eta \in (x_1, \ldots, x_i) : R \mathfrak{m}^\infty\) and \(e \gg 0\) there exists \(t \in \mathbb{N}\) so that \(\eta^{p^e} x^{tp^e} \in (x_1^{(t+1)p^e}, \ldots, x_i^{(t+1)p^e})\). Moreover, one observes by chasing through the direct limit identifications of \(H^n_m(R)\) that \(R\) is F-nilpotent if and only if \(R\) is weakly F-nilpotent and satisfies the following property: for every system of parameters \(x_1, \ldots, x_d\), every \(\gamma \in R\), \(c \in R^e\), then

\[
\begin{align*}
\text{(for every } e \gg 0 \text{ there exists } s \in \mathbb{N} \text{ s.t. } c \eta^{p^e} x^{sp^e} \in (x_1^{(s+t)p^e}, \ldots, x_d^{(s+t)p^e}) \Rightarrow} \\
\text{(for every } e \gg 0 \text{ there exists } s \in \mathbb{N} \text{ s.t. } \gamma^{p^e} x^{sp^e} \in (x_1^{(s+t)p^e}, \ldots, x_d^{(s+t)p^e})\).
\end{align*}
\]

There is one further class of singularities of interest in this article. It is proven in [Quy19] that the Frobenius test exponent is finite for weakly F-nilpotent rings. The second named author refined this result by isolating the salient features of weakly F-nilpotent rings used in Quy’s proof. Indeed, it is shown in [Mad19, Theorem 3.1 and Theorem 3.6] that the following class of singularities has finite Frobenius test exponent.

Definition 2.4. A local \(d\)-dimensional ring \((R, \mathfrak{m}, k)\) of prime characteristic \(p > 0\) is called generalized weakly F-nilpotent if for each \(0 \leq i < d\) the \(R\)-module \(H^i_m(R)/0^F_{H^i_m(R)}\) has finite length.

In this article we characterize generalized weakly F-nilpotent rings as local rings which are weakly F-nilpotent on the punctured spectrum in Proposition 4.6.

2.3. Relative Frobenius Actions. Let \(R \to S\) be a homomorphism of rings of prime characteristic \(p > 0\). We denote by \(F^e_{S/R}\) the map which fills in the following cocartesian diagram:

\[
\begin{array}{ccc}
R & \xrightarrow{F^e} & F^e_S R \\
\downarrow & & \downarrow \\
S & \xrightarrow{F^e_{S/R}} & F^e_{S/R} S \\
\end{array}
\]

The Radu-André Theorem equates faithful flatness of \(F^e_{S/R}\) with the property that \(R \to S\) is faithfully flat with geometrically regular fibers, a theorem we record for future reference.

Theorem 2.5 ([Rad92, Theorem 4], [And93, Theorem 1]). Let \(R \to S\) be a homomorphism of rings of prime characteristic \(p > 0\). Then \(R \to S\) is faithfully flat with geometrically regular fibers if and only if the relative Frobenius map \(F^e_{S/R}\) is faithfully flat for all \(e \in \mathbb{N}\).
We will often employ another notion of relative Frobenius action when using the Nagel-Schenzel isomorphism. Let $I \subseteq R$ be an ideal of a local ring $(R, \mathfrak{m}, k)$ of prime characteristic $p > 0$. Then for each $e \in \mathbb{N}$ the Frobenius map $F^e : R/I \to R/I$ can be factored through a Frobenius linear map relative to $R$:

$$R/I \xrightarrow{F^e_R} R/I^{|p^e|} \to R/I.$$  

The first map is given by $F^e_R(a + I) = a^{p^e} + I^{|p^e|}$ and the second map is the natural projection. Note that we are suppressing the reference to the ideal $I$ in this notation. Relative Frobenius maps induce Frobenius linear maps of local cohomology modules:

$$H^i_m(R/I) \xrightarrow{F^e_R} F^e_*H^i_m(R/I^{|p^e|}).$$

We now discuss the notion of relative tight closure of a local cohomology module as defined in [PQ19]. Suppose $R/I$ is $d$-dimensional. An element $\gamma \in H^d_m(R/I)$ is an element of $0^{F^e_R}_{H^d_m(R/I)}$, the tight closure of 0 relative to $R$, if there exists $c \in R^e$ so that $\gamma$ is in the kernel of the composition

$$H^d_m(R/I) \xrightarrow{F^e_R} H^d_m(R/I^{|p^e|}) \xrightarrow{F^e c} H^d_m(R/I^{|p^e|})$$

for all $e \gg 0$. Similarly, $\gamma \in H^d_m(R/I)$ is an element of $0^{F^e_R}_{H^d_m(R/I)}$, the Frobenius closure of 0 relative to $R$, if $\gamma$ is in the kernel of the following maps for all $e \gg 0$:

$$H^d_m(R/I) \xrightarrow{F^e_R} H^d_m(R/I^{|p^e|}).$$

We say $R/I$ is weakly $F$-nilpotent relative to $R^1$ if for each $0 \leq i < d$ and $\gamma \in H^i_m(R/I)$ there exists $e \gg 0$ so that $F^e_R(\gamma)$ is the 0-element of $H^i_m(R/I^{|p^e|})$. We say $R/I$ is $F$-nilpotent relative to $R$ if $R/I$ is weakly $F$-nilpotent relative to $R$ and $0^{F^e_R}_{H^d_m(R/I)} = 0^{F^e_R}_{H^d_m(R/I)}$. The third named author and Pham Hung Quy utilized relative Frobenius actions and filter regular sequences to provide the following characterizations of weakly $F$-nilpotent and $F$-nilpotent rings:

**Theorem 2.6 ([PQ19, Theorem 5.9]).** Let $(R, \mathfrak{m}, k)$ be an excellent equidimensional local ring of prime characteristic $p > 0$ and dimension $d$. Let $x_1, \ldots, x_i$ be a filter regular sequence of length $i$. Then

1. $R$ is weakly $F$-nilpotent if and only if $R/(x_1^N, \ldots, x_i^N)$ is weakly $F$-nilpotent relative to $R$ for all $N \in \mathbb{N}$;
2. $R$ is $F$-nilpotent if and only if $R/(x_1^N, \ldots, x_i^N)$ is $F$-nilpotent relative to $R$ for all $N \in \mathbb{N}$.

3. Dualizing complexes and Frobenius actions

Let $R$ be a ring of prime characteristic $p > 0$. A Cartier linear map $\varphi : M \to M$ on an $R$-module $M$ is an $R$-linear map $F^e_*M \to M$. Suppose further that $(R, \mathfrak{m}, k)$ is a complete local ring of prime characteristic $p > 0$. We discuss how Frobenius actions on the local cohomology modules of $R$ can be understood through Cartier linear maps of Ext-modules induced by Matlis duality (see for example [BB11, Section 5] and [KMOVZ17, Section 2]).

$^1$We use this terminology in order to be consistent with [PQ19].
Any (not necessarily local) $F$-finite ring $R$ is the homomorphic image of a regular ring [Gab04], hence $R$ admits a dualizing complex $\omega^*_R$. Iterates of the Frobenius endomorphism induce Cartier linear maps on the chain complex $\omega^*_R$ and such maps can be localized to understand the behavior of Frobenius actions on local cohomology modules. We denote by $(F^e)^\vee : F^e_*\omega^*_R \to \omega^*_R$ the induced Cartier linear map obtained by evaluating at 1. The Cartier linear map $(F^e)^\vee$ is a degree preserving map on the chain complex $\omega^*_R$. In particular, for each $i \in \mathbb{Z}$ we let $(F^e)^\vee(i)$ denote degree $i$ piece of $(F^e)^\vee$. As our notation is suggestively indicating, in a local ring the Matlis dual of $(F^e)^\vee$ will be the Frobenius action on local cohomology.

If $(R, m, k)$ is a local ring of Krull dimension $d$ with dualizing complex $\omega^*_R$ then we say $\omega^*_R$ is normalized if $H^{-d}(\omega^*_R) = \omega_R$ is a canonical module for $R$. If $R$ is not assumed to be local, but is assumed to be locally equidimensional of Krull dimension $d$, then we say a dualizing complex $\omega^*_R$ is normalized if for every $p \in \text{Spec}(R)$ the shifted localized complex $(\omega^*_R \otimes R_p)[-d + \text{ht}(p)]$ is a normalized complex for $R_p$.

**Proposition 3.1.** Let $R$ be a locally equidimensional $F$-finite ring of Krull dimension $d$, $\omega^*_R$ a normalized dualizing complex of $R$, and for each $e \in \mathbb{N}$ let $(F^e)^\vee : F^e_*\omega^*_R \to \omega^*_R$ be the map of dualizing complexes induced by $F^e : R \to R$. Then for each $p \in \text{Spec}(R)$ and $i \in \mathbb{N}$ the $R_p$-Matlis dual of $(F^e)^\vee_p(-i)$ of the normalized dualizing complex $\omega^*_R = (\omega^*_R \otimes R_p)[-d + \text{ht}(p)]$ induces the $i$th Frobenius action on the local cohomology module $F^e : H^i_{R_p}(R_p) \to F^e_*H^i_{R_p}(R_p)$.

**Proof.** Dualizing complexes, the evaluation at 1 map $(F^e)^\vee : F^e_*\omega^*_R \to \omega^*_R$, and the Frobenius map are functorial and commute with localization and completion. Therefore the proposition is a consequence of [BB11, Lemma 5.1].

As a consequence of Proposition 3.1 we demonstrate the existence of uniform bounds of the Hartshorne-Speiser-Lyubeznik numbers in a (not necessarily local) ring $R$ which is either $F$-finite or essentially of finite type over an excellent local ring.

**Definition 3.2.** [HS77, Lyu97] Let $(R, m, k)$ be a $d$-dimensional local ring of prime characteristic $p > 0$. For each $0 \leq i \leq d$, consider the non-decreasing sequence of $R$-submodules $N_{i,e} := \{ z \in H^i_{m}(R) \mid F^e(z) = 0 \} \subseteq H^i_m(R)$.

1. The $i$th Hartshorne-Speiser-Lyubeznik of $R$ is defined to be

   $\text{HSL}_i(R) := \min\{ e \mid N_{i,e+j} = N_{i,e} \text{ for all } j \geq 1 \}$;

2. The Hartshorne-Speiser-Lyubeznik of $R$ is defined as

   $\text{HSL}(R) := \max\{ \text{HSL}_i(R) \mid 0 \leq i \leq d \}$.

**Corollary 3.3.** Let $R$ be a locally equidimensional ring of prime characteristic $p > 0$ and Krull dimension $d$ which is either $F$-finite or essentially of finite type over an excellent local ring. Then the set of numbers $\{ \text{HSL}_p(R) \mid p \in \text{Spec}(R) \}$ is bounded.

---

We refer the reader to [Har66] and [Sta18, Tag 0A7A] for basics of dualizing complexes.
Proof. Suppose first that $R$ is $F$-finite. Let $\omega^*_R$ be a normalized dualizing complex of $R$. Utilizing the language of [BB11], the graded pieces of the complex $\omega^*_R$ along with the corresponding graded pieces of $(F^e)^\vee$ are coherent Cartier modules on $X = \text{Spec}(R)$. Therefore we may apply [BB11, Proposition 2.14] in degrees $-d \leq i \leq 0$ to know there exists an integer $e_0$ so that the image of $(F^e)^\vee : F^e_\ast \omega^*_R \to \omega^*_R$ agrees with the image of $(F^{e_0})^\vee$ in degrees $-d \leq i \leq 0$ for all $c \geq e_0$. By Proposition 3.1 measuring the HSL numbers of $R_p$ is equivalent to understanding when the cokernel of $(F^e)^\vee$ stabilizes, from which the corollary is easily derived.

Suppose $R$ is essentially of finite type over an excellent local ring $A$. The HSL-numbers of a local ring are easily seen to be preserved under faithfully flat local ring maps with 0-dimensional fiber, see [KZ19, Discussion following Remark 2.1]. We may base change by the completion of $A$ without affecting the hypotheses and assume $R$ is essentially of finite type over a complete local ring. We can then utilize a $\Gamma$-construction to find a faithfully flat and purely inseparable map $R \to R'$ so that $R'$ is $F$-finite.\footnote{For more on the existence of such ring maps, see either [HH94] or [Mur18].}

**Definition 3.4.** Let $I \subseteq R$ be an ideal and $c \in R^\circ$. We say that $e_0$ is test exponent for tight closure with respect to $c$ for $I$ if $I^* = \{x \in R \mid cz^{p^{e_0}} \in I^{[p^{e_0}]}\}$.

Sharp proved in [Sha06] that test exponents of parameter ideals in a local ring with respect to a test element depend on the HSL-numbers of that ring. Sharp’s result and Corollary 3.3 provide a uniform test exponent for tight closure of parameter ideals among all localizations of an $F$-finite ring $R$.

**Corollary 3.5.** Let $R$ be an $F$-finite locally equidimensional ring and suppose that $R$ has a completely stable test element $c$ (e.g., if $R$ is reduced). There exists an $e_0 \in \mathbb{N}$ with the following properties:

1. If $p \in \text{Spec}(R)$ and $q = (x_1, \ldots, x_{\text{ht}(p)})R_p$ is a parameter ideal of $R_p$ then $e_0$ is a test exponent for tight closure with respect to $c$ for $q^*$;

2. If $p \in \text{Spec}(R)$ then $e_0 + 1$ is a test exponent for tight closure with respect to $c$ for $0_{^g_{R_p}}(R_p)$.

**Proof.** Property (1) is immediate by Corollary 3.3 and [Sha06, Corollary 2.4(i)]. We will show that the proof of (2) reduces to showing that if $(R, \mathfrak{m}, k)$ is local of dimension $d$, $e_0$ a test exponent for parameter ideals with respect to $c$, then $e_0 + 1$ is a test exponent for $0^*_{_{H_{^g_{R}}}(R)}$.

Let $x_1, \ldots, x_d$ be a system of parameters, $q = (x_1, \ldots, x_d)$, and $x = x_1 \cdot x_2 \cdots x_d$. Suppose $\xi = [z + (x_1^n, \ldots, x_d^n)] \in H^d_{\mathfrak{m}}(R)$ and

$$cz^{p^{e_0} + 1} = [cz^{p^{e_0} + 1} + (x_1^{np^{e_0} + 1}, \ldots, x_d^{np^{e_0} + 1})] = 0.$$ 

We aim to show $\xi \in 0^*_{^g_{H_{^g_{R}}}(R)}$. There exists an integer $N$ such that

$$cz^{p^{e_0} + 1} x^N \in (x_1^{np^{e_0} + 1+N}, \ldots, x_d^{np^{e_0} + 1+N}).$$

Thus, by the colon-capturing property of tight closure,

$$cz^{p^{e_0} + 1} \in (x_1^{np^{e_0} + 1+N}, \ldots, x_d^{np^{e_0} + 1+N}) : x^N \subseteq (x_1^{np^{e_0} + 1}, \ldots, x_d^{np^{e_0} + 1})^*.$$
Thus by the proof of [Sha06, Corollary 2.4(ii)], \( z \in (x_1^n, \ldots, x_d^n)^* \), and so \( z + (x_1^n, \ldots, x_d^n) \in 0_H^d(R) \).

Similarly to the notion of test exponents for tight closure, there is a notion of a test exponent for Frobenius closure. Let \( I \) be an ideal in a Noetherian ring \( R \) of prime characteristic \( p > 0 \). Recall that the Frobenius closure of \( I \) is the ideal \( I^F = \{ r \in R \mid r^p \in I^{[p]} \} \) for some \( e \in \mathbb{N} \). It is not difficult to see that there exists an \( e_0 \), depending on \( I \), so that \( I^{[p^{e_0}]} = (I^F)^{[p^{e_0}]} \).

**Definition 3.6.** Let \( I \subseteq R \) be an ideal. The minimal \( e_0 \in \mathbb{N} \) so that \( I^{[p^{e_0}]} = (I^F)^{[p^{e_0}]} \) is called the Frobenius test exponent of \( I \) and is denoted \( \text{Fte}(I) \). If \( R \) is local then we say that \( e_0 \in \mathbb{N} \) is a Frobenius test exponent of \( R \) if \( e_0 \) bounds the Frobenius test exponent of every ideal of \( R \) generated by a full system of parameters. If such a bound exists we denote by \( \text{Fte}(R) \) the minimum Frobenius test exponent of \( R \).

It has been shown that the classes of Cohen-Macaulay, generalized Cohen-Macaulay, weakly \( F \)-nilpotent, and generalized weakly \( F \)-nilpotent rings all have finite Frobenius test exponent (see [KS06], [HKSY06], [Quy19], and [Mad19] respectively4). In [Quy19], Pham Hung Quy obtains bounds for the Frobenius test exponent of a local weakly \( F \)-nilpotent ring in terms of its HSL-numbers. Therefore we are able to uniformly bound the Frobenius test exponents of a ring \( R \) at its localizations at prime ideals defining weakly \( F \)-nilpotent local rings.

**Corollary 3.7.** Let \( R \) be a Noetherian ring of prime characteristic \( p > 0 \) which is either \( F \)-finite or essentially of finite type over an excellent local ring. Then the set

\[
\{ \text{Fte}(R_p) \mid p \in \text{Spec}(R) \text{ such that } R_p \text{ is weakly } F\text{-nilpotent} \}
\]

is bounded.

**Proof.** By Corollary 3.3 let \( \text{HSL}(R_q) < C \) for every \( q \in \text{Spec} R \). Let \( p \in \text{Spec} R \) and suppose that \( R_p \) is weakly \( F \)-nilpotent. By the proof of [Quy19, Main Theorem], one obtains

\[
\text{Fte}(R_p) \leq \sum_{i=0}^{\text{ht} p} \binom{\text{ht} p}{i} \text{HSL}_i(R_p) \leq \sum_{i=0}^{\text{ht} p} \binom{\text{ht} p}{i} \text{HSL}(R_p) < C \cdot \sum_{i=0}^{\text{ht} p} \binom{\text{ht} p}{i}.
\]

Every \( F \)-finite ring has finite Krull dimension by [Kun76, Proposition 1.1], so suppose \( R \) has Krull dimension \( d \). Then the right hand side of 3.1 has at most \( d + 1 \) terms which are all bounded by a constant independent of \( p \), hence so is \( \text{Fte}(R_p) \). \( \square \)

Pham Hung Quy has alerted us that Corollary 3.7 may also be recovered from the literature for a local weakly \( F \)-nilpotent ring. See [Quy19, Main Theorem] and [HQ19, Proposition 3.5].

4. Permanence properties

Theorem B is a consequence of the material in this section. We begin with a lemma.

**Lemma 4.1.** Let \( (R, \mathfrak{m}) \to (S, \mathfrak{n}) \) be a faithfully flat local ring map with Cohen-Macaulay closed fiber \( S/\mathfrak{m}S \) of Krull dimension \( \ell \). Then \( \text{depth}(S) = \text{depth}(R) + \ell \) and there exists \( T_1, \ldots, T_\ell \) a regular sequence of elements of \( S \) which is also a regular sequence of \( S/\mathfrak{m}S \). Moreover, for any such sequence of elements

\[\text{One cannot expect a uniform Frobenius test exponent for the class of all ideals of } R, \text{ see } [Brc06].\]
(1) $R \to S/(T_1, \ldots, T_\ell)$ is faithfully flat with 0-dimensional closed fiber,
(2) and if $x_1, \ldots, x_i$ is a parameter sequence of $R$ then $(x_1, \ldots, x_i, T_1, \ldots, T_\ell)S \cap R = (x_1, \ldots, x_i)R$.

Proof. Because $R \to S$ is faithfully flat we find that depth$(S) = $ depth$(R) + $ depth$(S/mS) = $ depth$(R) + \ell$ by [Mat86, Theorem 23.3]. By prime avoidance and the assumption that $S/mS$ is Cohen-Macaulay we may choose $T_1, \ldots, T_\ell$ in $S$ to be a regular sequence of $S$ and $S/mS$. Consider the short exact sequence
\[ 0 \to S \xrightarrow{T_1} S \to \frac{S}{(T_1)} \to 0. \]
We are assuming $R \to S$ is flat. Equivalently, $\text{Tor}_1^R(R/m, S) = 0$ and therefore there’s an exact sequence
\[ 0 \to \text{Tor}_1^R \left( \frac{R}{m}, \frac{S}{(T_1)} \right) \to \frac{S}{mS} \xrightarrow{T_1} \frac{S}{mS} \to \frac{S}{(m, T_1)}S \to 0. \]
But we have chosen $T_1$ to be $S/mS$-regular and therefore $\text{Tor}_1^R(R/m, S/(T_1)) = 0$, i.e. $R \to S/(T_1)$ is faithfully flat. By induction on the dimension of the closed fiber we find that $R \to S/(T_1, \ldots, T_\ell)$ is faithfully flat with 0-dimensional closed fiber.

Let $x_1, \ldots, x_i$ be a parameter sequence of $R$. Then modulo $(x_1, \ldots, x_i)$ the intersection ideal $(x_1, \ldots, x_i, T_1, \ldots, T_\ell)S \cap R$ is realized as the kernel of the faithfully flat map $R/(x_1, \ldots, x_i) \to S/(x_1, \ldots, x_i, T_1, \ldots, T_\ell)S$. Faithfully flat maps are injective and therefore $(x_1, \ldots, x_i, T_1, \ldots, T_\ell)S \cap R = (x_1, \ldots, x_i)R$. \hfill \□

Notation 4.2. We denote a sequence of ring elements $T_1, \ldots, T_\ell \in R$ by $T$. Moreover, for any $N \in \mathbb{N}$ we let $T^N = T_1^N, \ldots, T_\ell^N$.

We are now prepared to present a proof that the property of being either $F$-nilpotent or weakly $F$-nilpotent descends under faithfully flat local ring maps with Cohen-Macaulay closed fiber. The key ingredients are Theorem 2.1, Theorem 2.6, and Lemma 4.1. Under mild hypotheses we will be able to remove the assumption that the closed fiber is Cohen-Macaulay in Section 5 using our open loci results.

Theorem 4.3. Let $(R, m) \to (S, n)$ be a faithfully flat map of local rings of prime characteristic $p > 0$ with Cohen-Macaulay closed fiber.

(1) If $S$ is weakly $F$-nilpotent then $R$ is weakly $F$-nilpotent.
(2) If $S$ is $F$-nilpotent then $R$ is $F$-nilpotent.

Proof. Suppose $S/mS$ is an $\ell$-dimensional Cohen-Macaulay local ring. We utilize prime avoidance to choose $T_1, \ldots, T_\ell$ elements of $S$ which are regular on both $S$ and $S/mS$. Suppose further that $S$ is weakly $F$-nilpotent. Let $0 \leq i < d$ and $x_1, \ldots, x_i$ a filter regular sequence of length $i$ of $R$ and let $x = x_1 \cdots x_i$. By Theorem 2.1
\[ H^i_m(R) \cong H^0_m(H^i_{(x_1, \ldots, x_i)}(R)) \cong \lim_{\longrightarrow} \left( (x_1^t, \ldots, x_i^t :_R m^\infty) \cdot (x_1, \ldots, x_i) \right) \to \left( (x_1^{t+1}, \ldots, x_i^{t+1} :_R m^\infty) \cdot (x_1^{t+1}, \ldots, x_i^{t+1}) \right). \]
Suppose that $\eta + (x_1, x_2, \ldots, x_i)$ represents an element of $H^i_m(R)$ in the above direct limit system. In particular, there exists an $N \in \mathbb{N}$ so that $m^N \eta \subseteq (x_1, \ldots, x_i)$. By (1) of Lemma 4.1
the map $R \to S/(T)$ is faithfully flat with 0-dimensional closed fiber and therefore $x_1, \ldots, x_i$ is a filter regular sequence of length $i$ in $S/(T)$ by Lemma 2.2. Invoking Theorem 2.1 again we find that

$$H'_n(S/(T)) \cong \lim_{\rightarrow} \left( \frac{(x_1^{t_1}, \ldots, x_i^{t_i}, T)S ; : S n^\infty}{(x_1^{t_i}, \ldots, x_i^{t_i})S} \right).$$

Because $R \to S/(T)$ has 0-dimensional fiber we see that by increasing $N$ as necessary, $nN \eta \subseteq (x_1, \ldots, x_i, T)S$ and therefore $\eta + (x_1, \ldots, x_i, T)S$ represents an element of $H'_n(S/(T))$ in the above direct limit system. We are assuming $S$ is weakly $F$-nilpotent. Thus $S/(T)$ is weakly $F$-nilpotent relative to $S$ by Theorem 2.6. Therefore for all $e \gg 0$ there exists $t \in \mathbb{N}$ such that $x^{tp^r} \eta \in (x_1^{(t+1)p^r}, \ldots, x_i^{(t+1)p^r}, T^{p^r})S$. By Lemma 4.1 $(x_1^{(t+1)p^r}, \ldots, x_i^{(t+1)p^r}, T^{p^r})S \cap R = (x_1^{(t+1)p^r}, \ldots, x_i^{(t+1)p^r})R$. Therefore for all $e \gg 0$ there exists $t \in \mathbb{N}$ such that $x^{tp^r} \eta \in (x_1^{(t+1)p^r}, \ldots, x_i^{(t+1)p^r})R$, i.e., for all $e \gg 0$ the $e$th Frobenius action of $H'_n(R)$ maps the element represented by $\eta + (x_1, \ldots, x_i)$ to 0.

We now assume further that $S$ is $F$-nilpotent. To verify $R$ is $F$-nilpotent we may pass to the completion of $R$ and $S$ and assume $R$ is reduced, see [PQ19, Proposition 2.8]. We aim to show that $0_{H^d_m(R)}^r = 0_{H^d_m(R)}^s$. Let $x_1, \ldots, x_d$ be a system of parameters of $R$ and $\eta + (x_1, \ldots, x_d)$ represent an element of $0_{H^d_m(R)}^r$ under the direct limit identification

$$H'_m(R) \cong \lim_{\rightarrow} \left( \frac{R}{(x_1^{t_1}, \ldots, x_d^{t_d})} \xrightarrow{x} \frac{R}{(x_1^{t_1+1}, \ldots, x_d^{t_d+1})} \right).$$

Then there exists $c \in \mathbb{R}^p$ with the property that for all $e \in \mathbb{N}$ there exists $t \in \mathbb{N}$ so that $cn^{p^e} x^{tp^r} \in (x_1^{(t+1)p^r}, \ldots, x_d^{(t+1)p^r})$. By Theorem 2.6 we have that $0_{H^d_m(S/(T))}^r = 0_{H^d_m(S/(T))}^s$. The module $0_{H^d_m(S/(T))}^s$ is the tight closure of 0 in $H^d_m(S/(T))$ as an $S$-module and $0_{H^d_m(S/(T))}^r$ is the Frobenius closure of 0 in $H^d_m(S/(T))$ as an $S$-module. Moreover, observe that $c \in S^e$. Therefore we find that $\eta + (x_1, \ldots, x_d, T)S$ represents an element of the tight closure of 0 in $H^d_m(S/(T))$ as an $S$-module. Since $S$ is $F$-nilpotent we have that for all $e \gg 0$ there exists $t \in \mathbb{N}$ so that $\eta^{p^e} x^{tp^r} \in (x_1^{(t+1)p^r}, \ldots, x_d^{(t+1)p^r}, T^{p^r})S$. By (2) of Lemma 4.1 we find that for all $e \gg 0$ there exists $t \in \mathbb{N}$ so that $\eta^{p^e} x^{tp^r} \in (x_1^{(t+1)p^r}, \ldots, x_d^{(t+1)p^r})$, i.e., $\eta + (x_1, \ldots, x_d)$ represents the class of an element of $0_{H^d_m(R)}^r$. □

The ascent results of Theorem B are broken into several statements. We first show the property of being (weakly) $F$-nilpotent ascends under faithfully flat maps with Cohen-Macaulay fibers.

**Theorem 4.4.** Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a faithfully flat local ring map of prime characteristic $p > 0$ rings with Cohen-Macaulay closed fiber $S/\mathfrak{m}S$.

1. If $R$ is weakly $F$-nilpotent then $S$ is weakly $F$-nilpotent.
2. Suppose additionally that $R \to S$ has geometrically regular fibers and $R/\sqrt{p}$ and $S \otimes_R R/\sqrt{p}$ have a test element in common. If $R$ is $F$-nilpotent then $S$ is $F$-nilpotent.

**Proof.** As in Lemma 4.1 and Theorem 4.3 we begin by choosing elements $T_1, \ldots, T_\ell$ in $S$ which are regular in $S$ and $S/\mathfrak{m}S$. Suppose first that $R$ is weakly $F$-nilpotent. Utilizing the
notation of Notation 4.2 and the criteria of Theorem 2.6 the ring $S$ is weakly $F$-nilpotent if and only if $S/(\mathcal{T}^N)$ is weakly $F$-nilpotent relative to $S$ for each $N \in \mathbb{N}$.

Recall that the Frobenius action $S \rightarrow F_e^* S$ factors as $S \rightarrow F_e^* R \otimes_R S \xrightarrow{F_e^{S/R}} F_e^* S$ where $F_e^{S/R}$ is the relative Frobenius map and $S \rightarrow F_e^* R \otimes_R S$ is the base change of the Frobenius map of $R$ to $S$, see Subsection 2.3. Tensoring with $S/(\mathcal{T}^N)$ we find that the relative Frobenius map $S/(\mathcal{T}^N) \rightarrow F_e^* S/(\mathcal{T}^{Np^e})$ has a factorization

$$S/(\mathcal{T}^N) \rightarrow F_e^* R \otimes_R S/(\mathcal{T}^N) \xrightarrow{F_e^{S/R}} F_e^* S/(\mathcal{T}^{Np^e})$$

In particular, Frobenius actions of local cohomology modules of $S$ with support in $\mathfrak{n}$ has a factorization

$$H^i_n(S/(\mathcal{T}^N)) \rightarrow H^i_n(F_e^* R \otimes_R S/(\mathcal{T}^N)) \xrightarrow{F_e^{S/R}} H^i_n(F_e^* S/(\mathcal{T}^{Np^e})).$$

By Lemma 4.1 the map $R \rightarrow S/(\mathcal{T}^N)$ is faithfully flat with 0-dimensional fiber. Therefore the map $H^i_n(S/(\mathcal{T}^N)) \rightarrow H^i_n(F_e^* R \otimes_R S/(\mathcal{T}^N))$ may be identified with

$$H^i_m(R) \otimes_R S/(\mathcal{T}^N) \rightarrow H^i_n(F_e^* R) \otimes_R S/(\mathcal{T}^N).$$

In other words, the relative Frobenius actions of local cohomology of $S/(\mathcal{T}^N)$ factor through the base change of the Frobenius actions of local cohomology of $R$. Therefore if $R$ is weakly $F$-nilpotent, i.e. all high enough iterates of Frobenius actions on lower local cohomology are 0, then $S$ is also weakly $F$-nilpotent.

We now move to (2). Without loss of generality we may assume $R$ is reduced and therefore by assumption $R$ and $S$ have a test element $c$ in common. Moreover, we are assuming $R \rightarrow S$ has geometrically regular fibers, equivalently the relative Frobenius map $F_e^{S/R} : F_e^* R \otimes_R S \rightarrow F_e^* S$ is faithfully flat for all $e \in \mathbb{N}$ by Theorem 2.5. The ring $S$ is $F$-nilpotent if and only if $S/(\mathcal{T}^N)$ is $F$-nilpotent relative to $S$ for all $N \in \mathbb{N}$ by Theorem 2.6. In particular, by (1) of this theorem it remains to show that $0^*_n H^d_{m}(S/(\mathcal{T}^N)) = 0^*_n H^d_{m}(S/(\mathcal{T}^N))$ for all $N \in \mathbb{N}$.

Because $c \in R$ the composition of $S$-linear maps

$$S \rightarrow F_e^* S \xrightarrow{F_e^c} F_e^* S$$

can be factored as

$$S \rightarrow F_e^* R \otimes_R S \xrightarrow{F_e^c \otimes_R S} F_e^* R \otimes_R S \xrightarrow{F_e^{S/R}} F_e^* S,$$

where $S \rightarrow F_e^* R \otimes_R S$ is the base change of the Frobenius map of $R$. Base changing by $S/(\mathcal{T}^N)$ and examining the induced map of the top local cohomology modules we find that

(4.1) $$H^d_n(S/(\mathcal{T}^N)) \xrightarrow{F_e^c} H^d_n(F_e^* S/(\mathcal{T}^{Np^e})) \xrightarrow{F_e^{S/R}} H^d_n(F_e^* S/(\mathcal{T}^{Np^e}))$$

factors as

(4.2) $$H^d_n(S/(\mathcal{T}^N)) \xrightarrow{F_e^{S/R} \otimes_R S/(\mathcal{T}^N)} H^d_n(F_e^* R) \otimes_R S/(\mathcal{T}^N) \xrightarrow{F_e^c \otimes_R S/(\mathcal{T}^N)} H^d_n(F_e^* R) \otimes_R S/(\mathcal{T}^N) \xrightarrow{F_e^{S/R} \otimes_R S/(\mathcal{T}^N)} H^d_n(F_e^* S/(\mathcal{T}^{Np^e})).$$

We aim to show that if \( \eta \in H_{n}(S/\mathcal{T}(N)) \) is an element of the kernel of the composition of maps in 4.1 for all \( e \gg 0 \) then \( \eta \) is an element of the kernel of \( F_{S}^{e} \) for all \( e \gg 0 \). Given such an element \( \eta \) we find that \( \eta \) is an element of the kernel of \( (F_{R}^{e} \otimes_{S} S/\mathcal{T}(N)) \circ (F_{S}^{e} \otimes_{R} S/\mathcal{T}(N)) \) from 4.2 for all \( e \gg 0 \) since \( F_{S}^{e}/R \) is faithfully flat. But \( R \to S/\mathcal{T}(N) \) is faithfully flat and we are assuming \( R \) is \( F \)-nilpotent, i.e. the kernel of \( F_{R}^{e} \circ F_{S}^{e} \) agrees with the kernel of \( F_{R}^{e} \) for all \( e \gg 0 \). Therefore \( \eta \) is an element of the kernel of \( F_{R}^{e} \otimes_{R} S/\mathcal{T}(N) \) for all \( N \gg 0 \). It follows that \( \eta \) is an element of the kernel of \( F_{S}^{e} \) for all \( e \gg 0 \) since \( F_{S}^{e} \) factors as \( F_{S}^{e}/R \circ (F_{S}^{e} \otimes_{R} S) \).

**Theorem 4.5.** Let \((R, \mathfrak{m}) \to (S, \mathfrak{n})\) be a purely inseparable local homomorphism of prime characteristic \( p > 0 \) rings.

1. If \( R \) is weakly \( F \)-nilpotent then \( S \) is weakly \( F \)-nilpotent.
2. If \( R \to S \) is faithfully flat and \( R \) is \( F \)-nilpotent then \( S \) is \( F \)-nilpotent.

**Proof.** Because \( R \to S \) is purely inseparable the induced map of spectra is a bijection. In particular, \( R \) and \( S \) have the same Krull dimension. Let \( d = \dim(R) = \dim(S) \), suppose \( R \) is weakly \( F \)-nilpotent, and let \( 0 \leq i < d \). Let \( x_1, \ldots, x_i \) be a filter regular sequence of \( R \). We first observe that \( x_1, \ldots, x_i \) is also a filter regular sequence of \( S \). Indeed, if \( x_{i+1} \) avoids all non-maximal associated primes of \((x_1, \ldots, x_i) \in R \) then \( x_{i+1} \) avoids all non-maximal associated primes of \((x_1, \ldots, x_i) \in S \) since every prime ideal of \( S \) is the radical of a prime ideal extended from \( R \). We may therefore identify the local cohomology modules \( H_{n}^{i}(R) \) and \( H_{n}^{i}(S) \) as in Theorem 2.1 with respect to the sequence \( x_1, \ldots, x_i \).

Suppose \( \eta + (x_1, \ldots, x_i) \) is a representative of an element of \( H_{n}^{i}(S) \). Because \( R \to S \) is purely inseparable there exists an \( e_0 \in \mathbb{N} \) so that \( \eta^{p_0} \in R \). Thus, after replacing \( x_1, \ldots, x_i \) by \( x_1^{p_0}, \ldots, x_i^{p_0} \) and \( \eta + (x_1, \ldots, x_i)S \) by its image under the \( e_0 \)-iterate of the Frobenius action on \( H_{n}^{i}(S) \), we may assume \( \eta \in R \). In particular, \( \eta + (x_1, \ldots, x_i) \) represents an element of \( H_{n}^{i}(R) \) and therefore, because \( R \) is weakly \( F \)-nilpotent, for all \( e \gg 0 \) there exists \( t \in \mathbb{N} \) so that \( \eta^{p} x^{tp} \in (x_1^{t+1}p, \ldots, x_i^{t+1}p)R \) where \( x = x_1 \cdots x_i \). Passing to \( S \) we see the same containment holds in the parameter ideal extended to \( S \), i.e., the \( \text{eth Frobenius action of } H_{n}^{i}(S) \) maps \( \eta + (x_1, \ldots, x_i) \) to 0.

Suppose even further that \( R \) is \( F \)-nilpotent and \( R \to S \) is faithfully flat. Similar to before, to check that \( S \) is \( F \)-nilpotent we may assume \( R \) and \( S \) are complete and that \( R \) is reduced. Let \( c \in R^{*} \) be a common test element of \( R \) and \( S \), \( x_1, \ldots, x_d \) a common system of parameters of \( R \) and \( S \), and \( \eta + (x_1, \ldots, x_d)S \) a representative of an element of \( 0_{H_{n}^{d}(S)}^{*} \).

Replacing \( x_1, \ldots, x_d \) by \( x_1^{p_0}, \ldots, x_d^{p_0} \) and \( \eta + (x_1, \ldots, x_d) \) by its image under the \( e_0 \)-iterate of the Frobenius action on \( H_{n}^{d}(R) \) we may assume \( \eta \in S \). Because \( \eta + (x_1, \ldots, x_d) \in 0_{H_{n}^{d}(S)}^{*} \) we have that for all \( e \in \mathbb{N} \) there exists \( t \in \mathbb{N} \) so that \( cp^{p} x^{tp} \in (x_1^{(t+1)p}, \ldots, x_d^{(t+1)p})S \). But \( R \to S \) is faithfully flat and therefore for all \( e \gg 0 \) there exists \( t \in \mathbb{N} \) so that \( cp^{p} x^{tp} \in (x_1^{(t+1)p}, \ldots, x_d^{(t+1)p})R \). Hence \( \eta + (x_1, \ldots, x_d) \) represents an element of \( 0_{H_{n}^{d}(R)}^{*} = 0_{H_{n}^{d}(R)}^{F} \).

Hence for all \( e \gg 0 \) there exists \( t \in \mathbb{N} \) so that \( \eta^{p} x^{tp} \in (x_1^{(t+1)p}, \ldots, x_d^{(t+1)p})R \). Passing to \( S \) we find that \( \eta + (x_1, \ldots, x_d) \) represents an element of \( 0_{H_{n}^{d}(S)}^{F} \).

We next prove faithfully flat ascent and descent results for generalized weakly \( F \)-nilpotent rings. We accomplish this by first realizing generalized weakly \( F \)-nilpotent rings as rings which are weakly \( F \)-nilpotent on the punctured spectrum.
Proposition 4.6. Suppose \((R, m, k)\) is a local excellent ring of dimension \(d\) and of prime characteristic \(p > 0\). Then \(R\) is generalized weakly \(F\)-nilpotent if and only if \(R_p\) is weakly \(F\)-nilpotent for all \(p \in \text{Spec}(R) \setminus \{m\}\).

Proof. First suppose that \(R\) is \(F\)-finite. Let \(\omega_R^\bullet\) be a normalized dualizing complex of \(R\). The ring \(R\) is generalized weakly \(F\)-nilpotent if and only if \(H_m^i(R)/0^F_{H_m^i(R)}\) has finite length for all \(0 \leq i < d\). By Proposition 3.1, this is equivalent to the image of the evaluation at 1 map being finite for each \(0 \leq i < d\):

\[
(F^e)^\vee(-i) : F^e_*H^{-i}(\omega_R^\bullet) \to H^{-i}(\omega_R^\bullet).
\]

Therefore \(R\) is generalized weakly \(F\)-nilpotent if and only if for all \(0 \leq i < d\) and \(e \gg 0\) the image of \((F^e)^\vee(-i)\) is finite, i.e. \((F^e)^\vee(-i)\) is the 0-map on the punctured spectrum, i.e. \(R\) is weakly \(F\)-nilpotent on the punctured spectrum.

Now suppose \(R\) is an excellent local ring. Then \(R \to \hat{R}\) has geometrically regular fibers. By Theorem 4.4 we may assume \(R\) is complete. In which case, there exists a faithfully flat and purely inseparable map \(R \to R^\Gamma\) to some \(F\)-finite local domain \(R^\Gamma\) (see [HH94, Section 6]). The proposition now follows by Theorem 4.4 and the previous paragraph. □

Proposition 4.6 allows us to understand the property of being generalized weakly \(F\)-nilpotent under faithfully flat local ring maps.

Theorem 4.7. Let \((R, m) \to (S, n)\) be a faithfully flat map of local rings of prime characteristic \(p > 0\) with 0-dimensional fiber.

1. If \(S\) is generalized weakly \(F\)-nilpotent then \(R\) is generalized weakly \(F\)-nilpotent.
2. If \(R\) is generalized weakly \(F\)-nilpotent then \(S\) is generalized weakly \(F\)-nilpotent provided one of the following conditions is satisfied:
   • \(S/mS\) is a field, and \(R/\sqrt{0}\) and \(S \otimes_R R/\sqrt{0}\) have a test element in common;
   • \(R \to S\) is purely inseparable.

Proof. Without loss of generality we may assume \(R\) and \(S\) are complete. The theorem readily follows by Proposition 4.6, Theorem 4.3, Theorem 4.4, and Theorem 4.5. □

Recall that generalized weakly \(F\)-nilpotent rings were introduced by Maddox in [Mad19] for the purpose of finding a new class of local rings which have finite Frobenius test exponent. Prior to [Mad19], the following classes of rings were shown to have finite Frobenius test exponents:

1. Cohen-Macaulay rings, [KS06];
2. generalized Cohen-Macaulay rings, [HKSY06];
3. weakly \(F\)-nilpotent rings, [Quy19].

We provide an example of a generalized weakly \(F\)-nilpotent ring which is neither generalized Cohen-Macaulay nor weakly \(F\)-nilpotent. Therefore the main results of [Mad19] were not already covered by the results of [HKSY06, KS06, Quy19].

Example 4.8. Let \((A, m, k)\) be a weakly \(F\)-nilpotent local domain of dimension 2 and depth 1, essentially of finite type over a perfect field \(K\) of prime characteristic \(p\). Let \(B' = A[T]\) be a polynomial ring in one variable over \(A\) and \(n = (x, T)\). Define \((B, n) = (B', n)\). Then, there is a ring \(R\) with \(A \subset R \subset B\) such that the ideal \(\mathfrak{c} = \text{Ann}_R(B/R)\) is a maximal ideal...
of $R$. Further, $R_\ell$ is neither generalized Cohen-Macaulay nor weakly $F$-nilpotent, but is
generalized weakly $F$-nilpotent.

**Proof of Example 4.8.** Present $A$ as $A = (K[x_1, \cdots, x_n]/I)_p$ for some $p \in \text{Spec}(K[x_1, \cdots, x_n])$, and it suffices to assume $p = (x_1, \cdots, x_n) = (\underline{x})$, so $m = (\underline{x})$.

The map $A \to B$ is faithfully flat with regular closed fiber. Therefore the ring $(B, n)$ is weakly $F$-nilpotent of dimension 3 and depth 2 by Theorem 4.4. The ideal $mB' \subseteq n$ is prime (as $B'/mB' \cong K[T]$ is a domain) and $(B')_{mB'}$ is not Cohen-Macaulay as $A$ is not Cohen-Macaulay, so $B'$ is not generalized Cohen-Macaulay. Consequently, $B$ is not generalized Cohen-Macaulay and so we must have $H^2_n(B)$ is not of finite length.

We let $$R' = \{f(x_1, \cdots, x_n, T) \in B' \mid f(0, \cdots, 0, 1) = f(0, \cdots, 0, 1)\}.$$ Consider the ideal $c = \text{Ann}_{R'}(B'/R')$, an ideal common to both $R'$ and $B'$. One observes that $cB' = (\underline{x}, T(T - 1)) \subset n$ and $c$ is a maximal ideal of $R'$. Set $(R, c) = (R', c)$ so that there are local inclusions $$(A, m) \subset (R, c) \subset (B, n).$$ Note that $T - 1$ is a unit in $B$ so $cB = n$. Moreover, $R$ is of dimension 3 since $R \subset B$ is module-finite.

There is a short exact sequence of $R$-modules:

$$0 \longrightarrow R \longrightarrow B \longrightarrow B/R \longrightarrow 0$$

and therefore there is a long exact sequence

$$0 \longrightarrow B/R \longrightarrow H^1_c(R) \longrightarrow 0 \longrightarrow 0 \longrightarrow H^2_c(R) \longrightarrow H^2_n(B) \longrightarrow 0.$$  

The Frobenius action on $B/R$ is given by $b + R \mapsto b^p + R$, and $T^{p^e} + R \not\equiv R$ for all $e \in \mathbb{N}$, so $H^1_c(R) \cong B/R$ is not $F$-nilpotent but is finite length (since $\text{Ann}(B/R) = c$). Furthermore $H^2_c(R) \cong H^2_n(B)$ is nilpotent but is not finite length.

5. Open loci results

Theorem A is a result likely already understood by experts in the $F$-finite scenario, see [ST17, Lemma 2.3]. Nevertheless, we present a complete proof of Theorem A in the $F$-finite case for sake of completeness, as well as the essentially of finite type over an excellent local ring scenario. We begin with a consequence of Proposition 3.1.

**Lemma 5.1.** Let $R$ be a locally equidimensional $F$-finite ring of prime characteristic $p > 0$ and let $\omega^\bullet_R$ be a normalized dualizing complex of $R$. There exists a quotient $N$ of $\omega^\bullet_R := H^{-d}(\omega^\bullet_R)$ so that for each $p \in \text{Spec}(R)$ the $R_p$-Matlis dual of $N_p$ is $0_{H^d_{pR_p}(R_p)}^{\text{ht}(p)}$.

**Proof.** By Proposition 3.1 there exists an $e$ so that $0_{H^d_{pR_p}(R_p)}^{F^e}$ is the $R_p$-Matlis dual of the localized cokernel of the Cartier linear map

$$F^e_s H^{-d}(\omega^\bullet_R) \xrightarrow{(F^e_s)^\vee} H^{-d}(\omega^\bullet_R).$$
**Lemma 5.2.** Let $R$ be a locally equidimensional $F$-finite ring of prime characteristic $p > 0$ and let $\omega_R^\bullet$ be a normalized dualizing complex of $R$. Suppose further that $R$ admits a completely stable test element\(^6\), e.g. $R$ is reduced\(^6\). Then there exists a quotient $M$ of $\omega_R := H^{-d}(\omega_R^\bullet)$ so that for each $p \in \text{Spec}(R)$ the $R_p$-Matlis dual of $M$ is $0^*_{H_{pR_p}^{\text{ht}(p)}(R_p)}$.

**Proof.** Let $c \in R^\circ$ be a completely stable test element and $e_0$ a test exponent for $0^*_{H_{pR_p}^{\text{ht}(p)}(R_p)}$ with respect to $c$ for all $p \in \text{Spec}(R)$ (see Corollary 3.5). Then $0^*_{H_{pR_p}^{\text{ht}(p)}(R_p)}$ is realized as the kernel of the following composition of maps:

$$H_{pR_p}^{\text{ht}(p)}(R_p) \xrightarrow{F^{0}\circ e} F^e\omega_{pR_p} \xrightarrow{F^{e}\circ c} H_{pR_p}^{\text{ht}(p)}(R_p).$$

By Proposition 3.1, the kernel of the above composition of maps is the $R_p$-Matlis dual of the localization at $p$ of the cokernel of the composition

$$F^e\omega_R \xrightarrow{F^{e}\circ c} H^{-d}(\omega_R^\bullet) \xrightarrow{(F^{e})^\vee} H^{-d}(\omega_R^\bullet).$$

\[\Box\]

The above lemmas and the results of Section 4 provide us with a proof of Theorem A.

**Theorem 5.3.** Let $R$ be a ring of prime characteristic $p > 0$ which is either $F$-finite or essentially of finite type over an excellent local ring. Then

1. $\{p \in \text{Spec}(R) \mid R_p \text{ is weakly } F\text{-nilpotent}\}$ is an open subset of $\text{Spec}(R)$;
2. $\{p \in \text{Spec}(R) \mid R_p \text{ is } F\text{-nilpotent}\}$ is an open subset $\text{Spec}(R)$.

**Proof.** Suppose first that $R$ is $F$-finite. We may assume $R$ is a reduced ring by [PQ19, Proposition 2.8 (2)]. Denote by $\omega_R^\bullet$ a dualizing complex of $R$. Let $p \in \text{Spec}(R)$ and suppose that $R_p$ is weakly $F$-nilpotent. By [PQ19, Proposition 2.8 (3) and Remark 2.9], $R_p$ is an equidimensional local ring. Thus, by replacing $R$ by a suitable localization preserving $p$, we may assume $R$ is locally equidimensional (see [Poll18, Proof of Corollary 5.3]). Suppose that $R$ has Krull dimension $d$. We shift the grading of $\omega_R^\bullet$ so that $\omega_R^\bullet$ is normalized. Then $R_p$ is weakly $F$-nilpotent if and only if for all $i < \text{ht}(p)$ the Cartier linear map $(F^e)^\vee(-i) = 0$ for all $e \gg 0$. Equivalently, $p$ is in the weakly $F$-nilpotent locus of $R$ if and only if for all $i > 0$ one has that $(F^e)^\vee(-d + i)$ localizes to the 0-map at $p$ for all $e \gg 0$. The vanishing locus of a map of finitely generated modules is an open set and therefore the weakly $F$-nilpotent locus of $R$ is indeed open.

Now suppose $p \in \text{Spec}(R)$ and $R_p$ is $F$-nilpotent. As before we may assume $R$ is reduced and locally equidimensional. Let $N$ be as in Lemma 5.1 and $M$ as in Lemma 5.2. Then $N \subseteq M$ and for each $p$ the $R_p$-Matlis dual of $(M/N)_p$ is $0^*_{H_{pR_p}^{\text{ht}(p)}(R_p)}$. The collection of primes $\{p \in \text{Spec}(R) \mid (M/N)_p = 0\}$ is open, the intersection of this open set with the weakly $F$-nilpotent locus is open, and this open set defines the $F$-nilpotent locus.

---

\(^6\)A completely stable test element is an element $c \in R$ which serves as a test element of $R$, every localization of $R$, and every completion of $R$ with respect to an ideal.

\(^6\)See [HH94, Theorem 5.10] for an explanation of why $F$-finite reduced rings admit a completely stable test element.
Now suppose that $R$ is essentially of finite type over an excellent local ring $A$. If $\hat{A}$ is the completion of $A$ at its maximal ideal then $R \to \hat{A} \otimes_A R$ is faithfully flat with geometrically regular fibers. The induced map $\text{Spec}(\hat{A} \otimes_A R) \to \text{Spec} R$ is open, hence if the (weakly) $F$-nilpotent locus of $\text{Spec}(R \otimes_A \hat{A})$ is open then so is the (weakly) $F$-nilpotent locus of $\text{Spec}(R)$. By Theorem 4.4 we may assume $R$ is essentially of finite type over a complete local ring. Hence there exists a faithfully flat and purely inseparable map $R \to R^F$ where $R^F$ is an $F$-finite ring (see [HH94, Section 6]). Therefore the subset of $\text{Spec}(R)$ of (weakly) $F$-nilpotent primes is indeed open by Theorem 4.5 and the $F$-finite case of the theorem. □

**Remark 5.4.** (1) One should not expect the conclusions of Theorem 5.3 to hold under more general hypotheses than those stated. Indeed, [Hoc73, Proposition 2] and faithfully flat descent for $F$-nilpotent (resp. weakly $F$-nilpotent) rings guarantees the existence of a locally excellent local Noetherian ring whose $F$-nilpotent (resp. weakly $F$-nilpotent) locus is not open. See [DM19, Example 5.10] for more details.

(2) In light of Proposition 4.6 and Example 4.8, one should not expect the generalized weakly $F$-nilpotent locus of a local ring of prime characteristic to be open in general.

We conclude this paper by showing that the property of being (weakly) $F$-nilpotent descends under arbitrary faithfully flat maps, thus concluding the proof of Theorem B.

**Theorem 5.5.** Let $(R, m) \to (S, n)$ be a faithfully flat map of local rings of prime characteristic $p > 0$.

(1) If $S$ is weakly $F$-nilpotent then so is $R$.

(2) If $S$ is excellent and $F$-nilpotent then $R$ is $F$-nilpotent.

**Proof.** The property of being weak $F$-nilpotent can be checked after completion. Therefore in statement (1) of the theorem we may assume $S$ is an excellent local ring. Let $p \in \text{Spec}(S)$ be a prime ideal minimal over $mS$. If $S$ is either weakly $F$-nilpotent or $F$-nilpotent then so is $S_p$ respectively by Theorem 5.3 with the stated assumptions on $S$ (cf [PQ19, Corollary 5.17]). Replacing $S$ by $S_p$ we may assume $R \to S$ is faithfully flat with 0-dimensional closed fiber. The result now follows by Theorem 4.3. □

6. ACKNOWLEDGMENTS

We are grateful to Ian Aberbach, Takumi Murayama and Pham Hung Quy for comments on earlier drafts of this article. We thank Rankeya Datta for insightful conversations. The fourth named author is grateful to his advisor, Kevin Tucker, for his constant encouragement. We also thank the anonymous referee for valuable feedback on a previous version of this article.

REFERENCES

[And93] Michel André. Homomorphismes réguliers en caractéristique p. C. R. Acad. Sci. Paris Sér. I Math., 316(7):643–646, 1993.

[BB05] Manuel Blickle and Raphael Bondud. Local cohomology multiplicities in terms of étale cohomology. Ann. Inst. Fourier (Grenoble), 55(7):2239–2256, 2005.

[BB11] Manuel Blickle and Gebhard Böckle. Cartier modules: finiteness results. J. Reine Angew. Math., 661:85–123, 2011.
[Quy19] Pham Hung Quy. On the uniform bound of Frobenius test exponents. *J. Algebra*, 518:119–128, 2019. 5, 9, 14

[Rad92] Nicolae Radu. Une classe d’anneaux noethériens. *Rev. Roumaine Math. Pures Appl.*, 37(1):79–82, 1992. 5

[Sha06] Rodney Y. Sharp. Tight closure test exponents for certain parameter ideals. *Michigan Math. J.*, 54(2):307–317, 2006. 8, 9

[Smi97] Karen E. Smith. $F$-rational rings have rational singularities. *Amer. J. Math.*, 119(1):159–180, 1997. 4

[ST17] Vasudevan Srinivas and Shunsuke Takagi. Nilpotence of Frobenius action and the Hodge filtration on local cohomology. *Adv. Math.*, 305:456–478, 2017. 1, 2, 15

[Sta18] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2018. 7

[Vé95] Juan D. Vélez. Openness of the $F$-rational locus and smooth base change. *Journal of Algebra*, 172(2):425–453, 1995. 2

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40508, USA
E-mail address: j.kenkel@uky.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA
E-mail address: kylemaddox@mail.missouri.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UT 84102 USA
E-mail address: polstra@math.utah.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO, IL 60607 USA
E-mail address: awsimps2@uic.edu