Method of fundamental solutions with weighted average condition and dummy points

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Abstract
The aim of this paper is to develop the method of fundamental solutions using weighted average condition and dummy points. We accomplish mathematical analysis, a unique existence of an approximate solution and an exponential decay of the approximation error, for a potential problem in disk, and show some numerical experiments, which exemplify our error estimate.

Keywords Method of fundamental solutions, weighted average, dummy points, invariant scheme

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1. Introduction
Let Ω be a Jordan region in the plane with smooth boundary ∂Ω. We consider the following potential problem to find a harmonic function u satisfying a Dirichlet boundary condition:

$$\Delta u = 0 \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial \Omega,$$

where f is a given data. The invariant scheme of the method of fundamental solutions (MFS) with weighted average condition and dummy points offers an approximate solution for (1a)–(1b) as in the following procedure [1,2].

(I) Take N points \(\{y_j\}_{j=1}^N\) and \(\{z_j\}_{j=1}^N\), which are called the singular and dummy points, from the exterior of Ω.

(II) Assume an approximate solution \(u^{(N)}(x)\) as follows:

$$u^{(N)}(x) = Q_0 + \sum_{j=1}^N Q_j E_j(x)$$

with the coefficients \(\{Q_j\}_{j=0}^N \subset \mathbb{R}\) satisfying the weighted average condition

$$\langle QH \rangle = 0,$$

where

$$E_j(x) = E(x - y_j) - E(x - z_j)$$

for \(j = 1, 2, \ldots, N\), \(E(x) = (2\pi)^{-1} \log |x|\) is the fundamental solution of the Laplace operator \(\Delta\), \(\{H_j\}_{j=1}^N\) are weights, and \(\langle F \rangle = \sum_{j=1}^N F_j\).

(III) Coefficients \(\{Q_j\}_{j=0}^N\) are determined by the collocation method, that is, take N points \(\{x_j\}_{j=1}^N\), which are called the collocation points, from the boundary of Ω, and impose the following boundary conditions:

$$u^{(N)}(x_i) = f(x_i), \quad i = 1, 2, \ldots, N.$$

In [3], simple approximate solution by MFS

$$u^{(N)}(x) = \sum_{j=1}^N Q_j E(x - y_j)$$

has been analyzed mathematically. Their results were extended in [4] to the Murota’s invariant scheme

$$u^{(N)}(x) = Q_0 + \sum_{j=1}^N Q_j E(x - y_j)$$

with the invariant condition

$$\langle Q \rangle = 0.$$

Since we have replaced the kernel functions \(\{E_c(-y_j)\}_{j=1}^N\) with \(\{E_c()\}_{j=1}^N\), an approximate solution \(u^{(N)}\) satisfies the invariant properties (see, for example, [4,5]) naturally. Therefore, we can add one more condition such as the weighted average condition (3), which is nothing but the invariant condition (6) when all weights \(H_j\) are equal to 1. Our modifications (3)–(4) enable us to construct some geometrical structure-preserving numerical scheme for the one-phase Hele-Shaw problem [1,2], which is a moving boundary problem.

The aim of this paper is to extend the results in [3,4] for our modifications (3)–(4).

2. Main result
We assume that Ω is a disk \(D_{\rho}\) with radius \(\rho\) having the origin as its center, which is the same assumption as in [3,4]. We arrange the collocation points \(\{x_j\}_{j=1}^N\), the singular points \(\{y_j\}_{j=1}^N\) and the dummy points \(\{z_j\}_{j=1}^N\)

\[\text{(I)} \quad \text{Assume an approximate solution}
\]

\[\text{(II)} \quad \text{Take } \{y_j\}_{j=1}^N \text{ and } \{z_j\}_{j=1}^N, \text{ which are called the singular and dummy points, from the exterior of } \Omega.
\]

\[\text{(III)} \quad \text{Coefficients } \{Q_j\}_{j=0}^N \text{ are determined by the collocation method, that is, take } N \text{ points } \{x_j\}_{j=1}^N, \text{ which are called the collocation points, from the boundary of } \Omega, \text{ and impose the following boundary conditions:}
\]

\[u^{(N)}(x_i) = f(x_i), \quad i = 1, 2, \ldots, N. \quad (5)
\]

\[\text{In [3], simple approximate solution by MFS}
\]

\[u^{(N)}(x) = \sum_{j=1}^N Q_j E(x - y_j)
\]

\[\text{has been analyzed mathematically. Their results were extended in [4] to the Murota’s invariant scheme}
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\[u^{(N)}(x) = Q_0 + \sum_{j=1}^N Q_j E(x - y_j)
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\[\text{with the invariant condition}
\]

\[\langle Q \rangle = 0.
\]

\[\text{Since we have replaced the kernel functions } \{E_c(-y_j)\}_{j=1}^N \text{ with } \{E_c()\}_{j=1}^N, \text{ an approximate solution } u^{(N)} \text{ satisfies the invariant properties (see, for example, [4,5]) naturally. Therefore, we can add one more condition such as the weighted average condition (3), which is nothing but the invariant condition (6) when all weights } H_j \text{ are equal to 1. Our modifications (3)–(4) enable us to construct some geometrical structure-preserving numerical scheme for the one-phase Hele-Shaw problem [1,2], which is a moving boundary problem.}
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\[\text{2. Main result}
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\[\text{We assume that } \Omega \text{ is a disk } D_{\rho} \text{ with radius } \rho \text{ having the origin as its center, which is the same assumption as in [3,4]. We arrange the collocation points } \{x_j\}_{j=1}^N, \text{ the singular points } \{y_j\}_{j=1}^N \text{ and the dummy points } \{z_j\}_{j=1}^N.
\]
as follows:
\[ x_j = \rho \omega^j, \quad y_j = R \omega^j, \quad z_j = R \sigma \omega^j \]
for \( j = 1, 2, \ldots, N \), where \( R > \rho, \sigma > 1 \) and \( \omega = \exp(2\pi i/N) \). The following two theorems are main results of this paper.

**Theorem 1 (Unique existence)** There exists an approximate solution uniquely if and only if weights \( \{H_j\}_{j=1}^N \) satisfy the condition \( \langle H \rangle \neq 0 \).

We choose weights \( \{H_j\}_{j=1}^N \) satisfying
\[ |\langle H \rangle| > \delta \]
for a positive \( \delta \) independent of \( N \).

**Theorem 2 (Error estimate)** In addition to the hypothesis in Theorem 1, suppose that there exists some constant \( b \in [0, 1] \) such that the Fourier coefficients \( \{f_n\}_{n \in \mathbb{Z}} \) of \( f \) can be estimated as
\[ |f_n| = O(b^{|n|}) \quad (n \in \mathbb{Z}). \]

Then there exists a constant \( C \), which is independent of \( N \), such that the following error estimate holds:
\[ \|u - u^{(N)}\|_{L^\infty(\Omega)} \leq \begin{cases} 
C \left( \frac{\rho}{R} \right)^N & \text{if } \frac{bR^2}{\rho^2} > 1, \\
CN \left( \frac{\rho}{R} \right)^N & \text{if } \frac{bR^2}{\rho^2} = 1, \\
Cb^{N/2} & \text{if } \frac{bR^2}{\rho^2} < 1.
\end{cases} \]

3. Proofs

The following analysis is based on the previous works [3, 4]. First we prove Theorem 1. The collocation equations (5) with the weighted average condition (3) can be rewritten in the following form:
\[ GQ = f, \]
where
\[ G = \begin{pmatrix} 0 & H^T \\ 1 & \end{pmatrix}, \quad H = (H_1, H_2, \ldots, H_N)^T \in \mathbb{R}^N, \]
\[ \tilde{G} = (E_j(x_l) \mid i,j = 1, 2, \ldots, N) \in \mathbb{R}^{N \times N}, \]
\[ Q = (Q_0, Q_1, \ldots, Q_N)^T \in \mathbb{R}^{N+1}, \]
\[ f = (0, f(x_1), \ldots, f(x_N))^T \in \mathbb{R}^{N+1}. \]

Since \( \tilde{G} \) is a circulant matrix, it can be diagonalized by discrete Fourier transform. Indeed, defining the discrete Fourier transform \( \tilde{W} \) as
\[ \tilde{W} = \frac{1}{\sqrt{N}} \omega^{i(j-1)} \mid i,j = 1, 2, \ldots, N \in \mathbb{C}^{N \times N}, \]
we have
\[ \tilde{W}^{-1}\tilde{G}\tilde{W} = \text{diag} \left( \varphi_0^{(N)}(\varphi), \varphi_1^{(N)}(\varphi), \ldots, \varphi_{N-1}^{(N)}(\varphi) \right), \]
where
\[ \varphi_p^{(N)}(z) = \sum_{k=1}^{N} \omega_{p}^{(k-1)} E_k(z). \]

Thus, defining
\[ W = \begin{pmatrix} 1 & 0^T \\ 0 & \tilde{W} \end{pmatrix} \in \mathbb{C}^{(N+1) \times (N+1)}, \]
we obtain
\[ W^{-1}GW = \begin{pmatrix} 0 & A_{12} \\ A_{21} & \text{diag} \left( \varphi_0^{(N)}(\varphi), \ldots, \varphi_{N-1}^{(N)}(\varphi) \right) \end{pmatrix}, \]
where
\[ A_{12} = \frac{1}{N}\sum_{j=1}^{N} \omega^{(i-1)(j-1)}H_j \]
\[ A_{21} = (N^{1/2}, \ldots, 0)^T \in \mathbb{C}^{N}. \]

In particular, \( \det G \) can be concretely computed as
\[ \det G = -\langle H \rangle \prod_{p=1}^{N} \varphi_p, \quad \varphi_p = \varphi_p^{(N)}(\varphi). \]

The following lemma offers us more precise nature of the functions \( \varphi_p^{(N)} \). See, for the proof, [3, Lemma 1].

**Lemma 3** For \( z = r e^{i\theta}, r < R \), we have
\[ \varphi_p^{(N)}(z) = -\frac{1}{2\pi} \log \left| \frac{z^N - R^N}{z^N - (R\sigma)^N} \right| \]
if \( p \equiv 0 \pmod{N} \), and
\[ \varphi_p^{(N)}(z) = \frac{N}{4\pi} \sum_{m=p}^{N} \frac{1}{m} \left[ \left( \frac{r}{R} \right)^{|m|} - \left( \frac{r}{R\sigma} \right)^{|m|} \right] e^{im\theta} \]
if \( p \not\equiv 0 \pmod{N} \).

Using this lemma, we can easily verify that \( \varphi_p^{(N)} \neq 0 \) for \( p = 0, 1, \ldots, N-1, \) that is, \( G \) is nonsingular if and only if \( \langle H \rangle \neq 0 \). Hence Theorem 1 has been proved.

Next we prove Theorem 2. Owing to the relation (8), the inverse matrix \( G^{-1} \) can be represented as
\[ G^{-1} = (G^{-1})_{jk} \mid j,k = 0, 1, \ldots, N, \]
\[ [G^{-1}]_{00} = -\frac{\varphi_0^{(N)}(\varphi)}{\langle H \rangle}, \]
\[ [G^{-1}]_{0j} = -\frac{1}{\langle H \rangle} \quad (j = 1, 2, \ldots, N), \]
\[ [G^{-1}]_{jk} = \frac{1}{N} \sum_{i=1}^{N} \omega^{-(k-1)(j-1)} \frac{\varphi_i^{(N)}(\varphi) \sum_{k=1}^{N} \omega^{(i-1)(j-1)}H_i}{\varphi_{i-1}^{(N)}(\varphi) \langle H \rangle} \quad (k = 1, 2, \ldots, N), \]
\[ [G^{-1}]_{jk} = -\frac{1}{N} \sum_{i=2}^{N} \frac{1}{\varphi_{i-1}^{(N)}(\varphi)} \sum_{i=1}^{N} \omega^{(i-k)(j-1)}H_i \]
\[ + \frac{1}{N} \sum_{p=2}^{N} \omega^{(j-k)(p-1)} \frac{1}{\varphi_{p-1}^{(N)}(\varphi)} \quad (j,k = 1, 2, \ldots, N), \]
which yields that
\[ Q_0 = \frac{1}{N(H)} \sum_{p=0}^{N-1} \frac{\varphi_p}{\varphi_p} \sum_{i=1}^{N} \varphi_i^{(i-1)} H_i \sum_{j=1}^{N} \varphi_j^{(j-1)} f(x_j) \]
and
\[ Q_j = -\frac{1}{N(H)} \sum_{p=1}^{N-1} \frac{1}{p} \varphi_p \sum_{i=1}^{N} \varphi_i^{(i-1)} H_i \sum_{k=1}^{N} \varphi_k^{(k-1)} f(x_k) \]
for \( j = 1, 2, \ldots, N \). Therefore, the approximate solution \( u^{(N)}(x) \) is written such as
\[ u^{(N)}(x) = \sum_{n=0}^{\infty} f_n + \frac{1}{N(H)} \sum_{n \neq 0} f_n \times \left[ \varphi_0(x) - \varphi_0(\rho) \sum_{i=1}^{N} \varphi_i^{(i-1)} H_i + (H) \varphi_i^{(i)}(x) \right] \]
The exact solution \( u \) for the problem (1a)–(1b) can be expressed by Fourier expansion as follows:
\[ u(x) = u(re^{i\theta}) = \sum_{n \in \mathbb{Z}} f_n \left( \frac{r}{\rho} \right)^{|n|} e^{in\theta}. \]
Since both \( u \) and \( u^{(N)} \) are harmonic in \( \Omega \) and continuous on \( \overline{\Omega} \), the error can be estimated by the maximum principle for harmonic functions as
\[ ||u - u^{(N)}||_{L^\infty(\Omega)} = \sup_{\theta \in [0, 2\pi]} |u(\rho e^{i\theta}) - u^{(N)}(\rho e^{i\theta})| \]
where
\[ g^{(N)}_n = \sup_{\theta \in [0, 2\pi]} |e^{in\theta} - \frac{1}{(H)} \left( \varphi_0(\rho) - \varphi_0(\rho e^{i\theta}) \right) \varphi_0^{(1)}(\rho) + \varphi_1(\rho e^{i\theta}) \varphi_1^{(1)}(\rho) \right| \]
for \( n \in \mathbb{Z}, n \neq 0 \). By the similar calculation as in [3, p. 516], the estimate on \( g^{(N)}_n \) can be obtained for a sufficiently large \( N \) as in the following two lemmas.

**Lemma 4** For sufficiently large \( N \) so that \((\rho/R)^N \leq 1/2\) is satisfied, we have
\[ g^{(N)}_n \leq C_{\sigma, \delta} \left( \frac{\rho}{R} \right)^{-2n}, \quad C_{\sigma, \delta} = \frac{8\sigma}{\sigma - 1} \left( 2 + \frac{1}{\delta} \right) \]
for all \( n \in \mathbb{Z} \) with \( 1 \leq n \leq N/2 \).

**Lemma 5** For sufficiently large \( N \) so that \((\rho/R)^N \leq 1/2\) is satisfied, we have
\[ g^{(N)}_n \leq C_{\delta}, \quad C_{\delta} = 2 + \frac{8}{\delta} \]
for all \( n \in \mathbb{Z}, n \neq 0 \). Note that \( g^{(N)}_n = g^{(N)}_{-n} \) holds. Therefore we have
\[ ||u - u^{(N)}||_{L^\infty(\Omega)} \leq \sum_{n=1}^{[N/2]} (|f_n| + |f_{-n}|) g^{(N)}_n + \sum_{n=[N/2]+1}^{\infty} (|f_n| + |f_{-n}|) g^{(N)}_n \]
\[ \leq 2C_{\sigma, \delta} \left( \frac{\rho}{R} \right)^N \sum_{n=1}^{N/2} b^n + 2C_{\delta} \sum_{n=[N/2]+1}^{\infty} b^n \]
\[ \equiv I_1 + I_2, \]
where \([N/2]\) denotes the integer part of \( N/2 \). The first term can be bounded as
\[ I_1 \leq \frac{2C_{\sigma, \delta}}{\rho^2/(bR^2) - 1} \left( \frac{\rho}{R} \right)^N \]
if \( \rho^2 < 1 \),
\[ I_1 \leq \frac{2C_{\sigma, \delta}}{1 - \rho^2/(bR^2)} b^{N/2} \]
if \( \rho^2 = 1 \),
\[ I_1 \leq \frac{2C_{\sigma, \delta}}{b} b^{N/2} \]
if \( \rho^2 > 1 \).
By straightforward computation, the second term can be estimated as
\[ I_2 = 2C_{\delta} b^{N/2} \]
Summarizing the above, we obtain Theorem 2.

### 4. Numerical experiments

In this section, we present some results of numerical experiments in order to examine our error estimate. Throughout these experiments, we only consider the case where \( \rho = 1 \), which is the radius of the problem region \( \Omega \). Parameters are taken as follows:
- \( R = 2 \);
- \( \sigma = 2 \);
- \( H_j = 1 + \epsilon \cos(2j\pi/3) \) for \( j = 1, 2, \ldots, N \), where \( \epsilon = 0, 1 \) (we have \( \delta = 1 \) since \( \langle H \rangle = N \)), in Examples 6 and 7. In Example 8, \( R \) and \( \sigma \) take several values. Throughout the following examples, we use the same weights \( \{H_j\}_j \) as above.

**Example 6** We first adopt harmonic polynomials as the boundary data, that is,
\[ f(x) = \rho^m \cos(m\theta) \quad (x = \rho e^{i\theta} \in \partial\Omega), \quad m = 0, 1, \ldots, 5. \]
The result is depicted in Fig. 1, in which the horizontal and vertical axes represent \( N \) and common logarithms of errors, respectively, here and hereafter. We can observe that the errors decay exponentially with respect to \( N \), and their convergence rates agree well with the theoretical error estimate Theorem 2, which tells us that \( ||u - u^{(N)}||_{L^\infty(\Omega)} = O((1/2)^N) \). We can also see that differences of weights \( \{H_j\}_j \) affect very little behavior of errors. Note that lines corresponding to the case \( m = 0 \) do not appear in Fig. 1, since the numerical solution coincides with the exact solution in this case.

**Example 7** We next consider the case where the boundary data are logarithmic potentials:
\[ f(x) = \log |x - x_0| \quad (x \in \partial\Omega), \]
where \( x_0 \) is the singularity of \( f \), which is located outside \( \Omega \). In this case, the approximation error is estimated by Theorem 2 as follows:

\[
\|u - u^{(N)}\|_{L^\infty(\Omega)} = O \left( \max \left\{ \left( \frac{\rho}{R} \right)^N, \left( \frac{\rho}{|x_0|} \right)^{N/2} \right\} \right).
\]

The parameters \( R \) and \( x_0 \) are taken as \( R = (1 + 0.1m)\rho \) \((m = 1, 2, \ldots, 7)\) and \( x_0 = 2\rho \), respectively, which yields for \( m = 1, 2, \ldots, 7 \) that

\[
\|u - u^{(N)}\|_{L^\infty(\Omega)} = O \left( \max \left\{ \left( \frac{1}{1 + 0.1m} \right)^N, \left( \frac{1}{2} \right)^{N/2} \right\} \right).
\]

The results are depicted in Fig. 2, which exemplifies our error estimate.

**Example 8** We also compute condition number of coefficient matrix numerically by changing the parameters \( R, \sigma, \) and \( \epsilon \) as \( R = 2, 4, \sigma = 2, 10^5, \) and \( \epsilon = 0, 1, 1 \), respectively. Results in Fig. 3 imply that the linear system (7) would be ill-conditioned, and the condition number of coefficient matrix becomes a little bit large if \( \sigma \) is large, that is, the parameter \( \sigma \) should not be taken large in vain. Note also in this case that the weights \( \{H_j\}_j^N \) do not affect condition numbers.

5. Concluding remarks

In the present paper, we studied MFS with weighted average condition and dummy points for potential problem in disk, and established that an approximate solution actually exists uniquely if and only if \( \langle H \rangle \neq 0 \), and that the error decays exponentially with respect to \( N \) when the boundary datum has exponentially decaying Fourier coefficients. We showed numerical results with the boundary data being harmonic polynomials and logarithmic potentials, which exemplified our error estimate. We also computed the condition number of coefficient matrix numerically, which told us that the parameter \( \sigma \) for dummy points should not be taken so large.

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