Parameter and Feature Selection in Stochastic Linear Bandits

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Abstract

We study two model selection settings in stochastic linear bandits (LB). In the first setting, which we refer to as feature selection, the expected reward of the LB problem is in the linear span of at least one of $M$ feature maps (models). In the second setting, the reward parameter of the LB problem is arbitrarily selected from $M$ models represented as (possibly) overlapping balls in $\mathbb{R}^d$. However, the agent only has access to misspecified models, i.e., estimates of the centers and radii of the balls. We refer to this setting as parameter selection. For each setting, we develop and analyze an algorithm that is based on a reduction from bandits to full-information problems. This allows us to obtain regret bounds that are not worse (up to a $\sqrt{\log M}$ factor) than the case where the true model is known. The regret of our parameter selection algorithm also scales logarithmically with model uncertainty. Finally, we empirically show the effectiveness of our algorithms using synthetic and real-world experiments.

1 Introduction

Learning under bandit feedback is a class of online learning problems in which an agent interacts with the environment through a set of actions (arms), and receives rewards only from the arms that it has pulled. The goal of the agent is to maximize its expected cumulative reward without knowledge of the reward distributions of the arms. Multi-armed bandit (MAB) is the simplest form of this problem [Lai and Robbins, 1985, Auer et al., 2002a, Lattimore and Szepesvari, 2020]. Linear bandit [Dani et al., 2008, Rusmevichientong and Tsitsiklis, 2010, Abbasi-Yadkori et al., 2011] is a generalization of MAB to (possibly) infinitely many arms, each associated with a feature vector. The mean reward of each arm is assumed to be the dot product of its feature vector and an unknown parameter vector. This setting contains contextual linear bandit in which action sets and feature vectors change at every round. The main component of bandit algorithms is to balance exploration and exploitation: to decide when to explore and learn about the arms, and when to exploit and select the action with the highest estimated reward. The most common exploration strategies are optimism in the face of uncertainty (OFU) or upper confidence bound (UCB) [Auer et al., 2002a, Dani et al., 2008, Abbasi-Yadkori et al., 2011], and Thompson sampling (TS) [Thompson, 1933, Agrawal and Goyal, 2013, Russo and Van Roy, 2014, Abeille et al., 2017].

In this paper, we study model selection in stochastic linear bandits (LB), where the LB problem at hand is selected from a set of $M$ models. The agent has information about the models but does not know the identity of the one(s) that the new LB problem has been selected from. The goal of the agent is to identify the true model(s) and transfer its (their) collected experience to speedup the learning of the task at hand. It is a common scenario in many application domains that the new task belongs to a family of models that are either known accurately or with misspecification. For example, it is reasonable to assume that the customers of an online marketing website, the users of an app, or the patients in a medical trial belong to a certain number of categories based on their shopping and browsing habits or their genetic signatures. Moreover, it is also common these days that the websites, apps, and clinics have a large amount of information from each of these categories that can be used to build a model.
Model selection is particularly challenging with bandit information. A common approach is to consider each model as a black-box that runs a bandit algorithm with its own information, and then a meta algorithm plays a form of bandit-over-bandits strategy with their outcomes. These algorithms often achieve a regret of $\tilde{O}(\sqrt{MT})$, and thus, are not desirable when the number of models $M$ is large. In this paper, we consider two bandit model selection settings and show that it is possible to improve this rate so that the regret scales as $\sqrt{\log M}$ with the number of models. The main innovation in our algorithms is utilizing reductions from bandits to full-information problems, and performing model selection in the full-information setting for which much stronger results exist. The main reason for $\tilde{O}(\sqrt{MT})$ regret in bandit-over-bandits algorithms is that no information is shared among the models, i.e., when a bandit algorithm is used to take an action in a round, the resulting feedback is not shared with the other models. On the other hand, model selection in the full-information setting allows the model to share information among each other, which makes the superior $\sqrt{\log M}$ regret bound possible.

The two model selection settings we consider in this paper are: feature selection, where the mean reward of the LB problem is in the linear span of at least one of $M$ given feature maps (models), and parameter selection, where the reward parameter of the LB problem is arbitrarily selected from $M$ models represented as (possibly) overlapping balls in $\mathbb{R}^d$. Here the models can be misspecified, i.e., only estimates of the centers and radii of the balls are given to the algorithm. We derive algorithms in these settings that use reductions from bandits to full-information. Our algorithms are computationally efficient and have regret bounds that are not worse (up to a $\sqrt{\log M}$ factor) than the case where the true model is known. We achieve this by properly instantiating existing algorithmic paradigms: SquareCB [Foster and Rakhlin, 2020] and OFUL [Abbasi-Yadkori et al., 2011]. The SquareCB algorithm in its original form uses a set of static experts, but we need adaptive (learning) experts in order to have a computational efficient algorithm with the desired regret in our feature selection setting. Working with adaptive (time-varying) experts requires appropriate and non-trivial modifications to the proof of SquareCB.

There are mainly two types of reductions from bandits to full-information problems. The first one is the classical reduction that uses importance weighted estimates. A popular algorithm in this class is EXP3 that uses Exponentially Weighted Average forecaster as the full-information algorithm. The bandit model selection strategy of [Agarwal et al., 2017], known as CORRAL, also uses this type of reduction with an online mirror descent method and a carefully selected mirror map as the full-information algorithm. Given that importance weighted estimates are fed to the full-information algorithm, a $\sqrt{M}$ term is in general unavoidable in the regret of the methods that use this type of reduction. In this work, we use a different type of full-information reduction introduced by [Foster and Rakhlin, 2020] and [Abbasi-Yadkori et al., 2012]. Here, the full-information algorithm has direct access to its losses without any importance weighted estimates, and thus, allows us to obtain regrets that scales as $\sqrt{\log M}$ with the number of model.

2 Problem Formulation

In this section, we first provide a brief overview of stochastic linear bandits. We then describe the two model selection settings that we study in the paper. We conclude this section by introducing a regression oracle used by our algorithms that is based on sequential prediction with expert advice and square loss.

2.1 Stochastic Linear Bandits

A stochastic linear bandit (LB) problem is defined by a sequence of $T$ interactions of a learning agent with a stochastic environment. At each round $t \in [T]$, the agent is given a decision set $\mathcal{A}_t \subset \mathbb{R}^d$ from which it has to select an action $a_t$. Upon taking the action $a_t \in \mathcal{A}_t$, it observes a reward $y_t = \langle \phi_t(a_t), \theta_t \rangle + \eta_t$, where $\theta_t \in \mathbb{R}^d$ is the unknown reward parameter, $\phi_t(a) \in \mathbb{R}^d$ is the feature vector of action $a$ at round $t$, and $\eta_t$ is a zero-mean $R$-sub-Gaussian noise. When the features correspond to the canonical basis, this formulation reduces to multi-armed bandit. In case the features depend on both an action $a \in \mathcal{A}$ and a context $x \in \mathcal{X}$, i.e., $\phi_t(a_t) = \phi(x_t, a_t)$, this LB formulation is called contextual linear bandit. It is also common in practice that the action set is fixed and finite, i.e., $\mathcal{A} = [K]$, in which case we are in the finite $K$-action setting. The history $H_t$ of a LB algorithm up to round $t$ consists of
all the contexts, actions, and rewards that it has observed from the beginning until the end of round \( t - 1 \), i.e., \( H_t = \{(x_s, a_s, y_s)\}_{s=1}^{t-1} \), or equivalently \( H_t = \{\phi_s(a_s), y_s\}_{s=1}^{t-1} \).

The goal of the agent in LB is to maximize its expected cumulative reward in \( T \) rounds, or equivalently to minimize its \( T \)-round (pseudo) regret, i.e.,

\[
R(T, \theta^*) = \sum_{t=1}^{T} \langle \phi_t(a^*_t), \theta^* \rangle - \langle \phi_t(a_t), \theta^* \rangle,
\]

where \( a^*_t = \arg\max_{a \in A} \langle \phi_t(a), \theta^* \rangle \) is the optimal action at round \( t \).

### 2.2 Feature Selection Setting

In this setting, the agent is given a set of \( M \) feature maps \( \{\phi^i\}_{i=1}^M \) with dimension \( d \). We assume that the expected reward of the LB problem belongs to the linear span of at least one of these \( M \) models (features), i.e., there exist an \( i \in [M] \) and a \( \theta^*_i \in \mathbb{R}^d \) such that for all rounds \( t \in [T] \), contexts \( x \in \mathcal{X} \), and actions \( a \in \mathcal{A} \), we may write the mean rewards as \( \mathbb{E}[y_t] = \langle \phi^i(x, a), \theta^*_i \rangle \).

We refer to such feature maps as true models and denote them by \( i^* \). Note that the agent does not know the identity of the true model(s) \( i^* \).

As a motivational example for this setting, we can consider a recommender system that has trained \( M \) models (e.g., \( M \) neural networks) to predict the score of customer-item pairs. Each model (neural network) corresponds to a particular mood or type of the customer, or any other latent component of the customer’s state. Each model provides an embedding for customer-item pairs and the score is linear in this embedding (think of an embedding as the layer before the last layer of a trained neural network). When a new customer arrives, the recommender system should find out as soon as possible which of the \( M \) models (embeddings) is the best match to (the current mood/type of) this customer in order to recommend her desirable items.

We make the following standard assumption on the boundedness of the reward parameters and features of the \( M \) models.

**Assumption 1.** There are constants \( L, S, G \geq 0 \), such that for all \( i \in [M] \), \( t \in [T] \), \( x \in \mathcal{X} \), and \( a \in \mathcal{A} \), we have \( \|\theta^*_i\| \leq S \), \( \|\phi^i(x, a)\| \leq L \), and \( |\langle \phi^i(x, a), \theta^*_i \rangle| \leq G \).

Our goal here is to design an algorithm that minimizes transfer regret, which in this setting we define it as

\[
R(T) = \sum_{t=1}^{T} \langle \phi^* (x_t, a^*_t), \theta^*_i \rangle - \langle \phi^* (x_t, a_t), \theta^*_i \rangle,
\]

where \( a^*_t = \arg\max_{a \in A} \langle \phi^i(x_t, a), \theta^*_i \rangle \). In the results we report for this setting in Section 3, we make two assumptions: 1) the feature maps are all known (no model misspecification), and 2) the number of actions is finite, i.e., we are in the finite \( K \)-action setting described in Section 2.1. However, we believe that our algorithm and analysis can be extended to the case of having misspecified models and convex action sets using the results in [Foster et al., 2020].

### 2.3 Parameter Selection Setting

In this setting, unlike the classical setting in Section 2.1, we no longer assume that the unknown parameter \( \theta^* \) can be any vector in \( \mathbb{R}^d \). Rather, \( \theta^* \) can be generated from \( M \) possible reward models, each defined as a ball \( B(\mu_i, b_i) = \{\theta \in \mathbb{R}^d : \|\theta - \mu_i\| \leq b_i\} \), with center \( \mu_i \in \mathbb{R}^d \) and radius \( b_i \geq 0 \). Note that the models (balls) may overlap and do not have to be disjoint. The \( M \) models can be thought of the responses of \( M \) types (or clusters) of customers to different items in a recommender system or the reactions of patients with \( M \) genotypes to a set of drugs. The radii \( \{b_i\}_{i=1}^M \) represent the variation within each cluster. The reward parameter \( \theta^* \) of the new task (LB problem) is arbitrarily selected from the \( M \) models. For example, it can be adversarially selected from the union of the models, i.e., \( \theta^* \in \bigcup_{i=1}^M B(\mu_i, b_i) \). In this case, we denote by \( \mathcal{I}_\epsilon \), the set of indices of the balls that contain \( \theta^* \). Since the models are often computed from (finite) historical data, it is reasonable to assume that

\[1\] Note that we use the contextual linear bandit notation for this setting and in the corresponding sections.
only estimates of their centers \( \{\hat{\mu}_i\}_{i=1}^M \) are available, together with upper-bounds on the error of these estimates \( \{c_i\}_{i=1}^M \), such that \( \|\mu_i - \hat{\mu}_i\| \leq c_i \), for all \( i \in [M] \).

The agent has no knowledge either about \( \theta_* \) or the process according to which it has been selected. The only information given to the agent are: 1) estimates \( \hat{\mu}_i \) of the center of the models, 2) upper-bounds \( c_i \) on the errors of these estimates, and 3) the exact radii \( b_i \) of the models, for all \( i \in [M] \). This means that although \( \theta_* \) is selected from the actual models \( B(\mu_i, b_i) \), the agent has only access to estimated models \( B(\hat{\mu}_i, b_i + c_i) \) that have more uncertainty (their corresponding balls are larger). For simplicity, we assume that the exact values of radii \( \{b_i\}_{i=1}^M \) are known. However, our results can be easily extended to the case that instead of \( b_i \)'s, their estimates \( \hat{b}_i \) and upper-bounds on their errors \( c_i' \), i.e., \( \|b_i - \hat{b}_i\| \leq c_i' \), for all \( i \in [M] \), are given to the agent. In this case, the agent has to use even more uncertain estimates of the models \( B(\hat{\mu}_i, b_i + c_i + c_i') \).

Our goal is to design an algorithm that can transfer knowledge from these estimated models and learn the new task (the LB problem with parameter \( \theta_* \)) more efficiently than when it is independently learned. This goal can be quantitatively stated as minimizing the transfer regret, defined as

\[
\mathcal{R}(T) = \sup_{\theta_* \in \bigcup_{t=1}^T B(\mu_i, b_i)} \mathcal{R}(T, \theta_*),
\]

where \( \mathcal{R}(T, \theta_*) \) is the regret defined by (1). We make the following standard assumption on the boundedness of the features and expected rewards.

**Assumption 2.** There exist constants \( L, G \geq 0 \), such that \( \forall t \in [T] \) and \( \forall a \in \bigcup_{t=1}^T A_t \), we have \( \|\phi_t(a)\| \leq L \), and \( \forall \theta \in \bigcup_{t=1}^T B(\mu_i, b_i) \), we have \( |\langle \phi_t(a) , \theta \rangle| \leq G \).

### 2.4 Regression Oracle

In both model selection settings studied in the paper, our proposed algorithms use a regression oracle that is based on sequential prediction with expert advice and square loss. Following [Foster and Rakhlin, 2020] and [Foster et al., 2020], we refer to this regression oracle as \( \text{SqAlg} \). We can consider \( \text{SqAlg} \) as a meta algorithm that consists of \( M \) learning algorithms (or experts), each corresponding to one of our \( M \) models, and returns a prediction by aggregating the predictions of its experts. More precisely, at each round \( t \in [T] \), \( \text{SqAlg} \) takes the current context \( x_t \) and action \( a_t \), or equivalently \( \phi_t(a_t) \), as input, and gives them to its \( M \) experts to predict the reward of this context-action pair, i.e., \( f_i(H_t) = f_i(\phi_t(a_t); H_t) \), \( \forall i \in [M] \), given the current history \( H_t \). Then, the meta algorithm \( \text{SqAlg} \) aggregates its experts’ predictions, \( \{f_i(H_t)\}_{i=1}^M \), given their current weights, and returns its own prediction \( \hat{y}_t = \text{SqAlg}(\phi_t(a_t); H_t) \). Upon observing the actual reward \( y_t \), \( \text{SqAlg} \) updates the weights of its experts according to the difference between their predictions \( f_i(\phi_t(a_t); H_t) \) and the actual reward \( y_t \).

The regression oracles (\( \text{SqAlg} \)) used by our model selection algorithms, FS-SCB and PS-OFUL, differ in the prediction algorithm used by their experts. However, in both cases, \( \text{SqAlg} \) aggregates its experts’ predictions using an algorithm by [Haussler et al., 1998], whose pseudo-code is reported in Algorithm 3 in Appendix A. The performance of \( \text{SqAlg} \) is evaluated in terms of its regret \( \mathcal{R}_{\text{Sq}}(T) \), which is defined as its accuracy (in terms of square loss) w.r.t. the accuracy of the best expert in the set, i.e.,

\[
\sum_{t=1}^T (\hat{y}_t - y_t)^2 - \min_{i \in [M]} \sum_{t=1}^T (f_i(H_t) - y_t)^2 \leq \mathcal{R}_{\text{Sq}}(T).
\]

At each round \( t \), we define the oracle prediction for a context \( x \) and an action \( a \) as \( \hat{y}_t(x, a) := \text{SqAlg}_t(x, a; H_t) \). As shown in [Haussler et al., 1998], in case all observations and experts’ predictions are bounded in an interval of size \( \ell \), this regret can be bounded as \( \mathcal{R}_{\text{Sq}}(T) \leq \ell^2 \log M \) (see Appendix A for more details). We use this regret bound in the analysis of our proposed algorithms.

### 3 Feature Selection Algorithm

In this section, we derive an algorithm for the feature selection setting described in Section 2.2 that is based on the SquareCB algorithm of [Foster and Rakhlin, 2020]. We refer to our algorithm as feature
Algorithm 1: Feature Selection Square-CB (FS-SCB)

1. **Input:** Models \( \{\phi^i\}_{i=1}^M \), Confident Parameter \( \delta \), Learning Rate \( \alpha \), Exploration Parameter \( \kappa \)
2. **for** \( t = 1, \ldots, T \) **do**
   3. Observe context \( x_t \)
   4. Oracle predicts: \( \hat{y}_t(x_t, a) = \text{SqAlg}_t(x_t, a; H_t) \), \( \forall a \in [K] \)
   5. Define a distribution \( p_t \) over the actions:
      \[
      p_t(a) = \begin{cases} 
      \frac{1}{\kappa + \alpha (\hat{y}_t(x_t, a) - \hat{y}_t(x_t, a') \mid a' \neq a_t)}, & a \neq a_t, \\
      1 - \sum_{a \neq a_t} p_t(a), & a = a_t,
      \end{cases}
      \]
      where \( a' = \arg\max_{a \in [K]} \hat{y}_t(x_t, a) \);
   6. Sample action \( a_t \sim p_t(\cdot) \) and play it;
   7. Observe reward \( y_t = \langle \phi^i(x_t, a_t), \theta^*_i \rangle + \eta_t \);
   8. Update \( \text{SqAlg} \) with \( (x_t, a_t, y_t) \);

**selection SquareCB** (FS-SCB). We prove an upper-bound on the transfer regret of FS-SCB in Section 3.1, and provide an overview of the related work and a discussion on our results in Section 3.2.

Algorithm 1 contains the pseudo-code of FS-SCB. At each round \( t \in [T] \), the algorithm observes a context \( x_t \in X \) and passes it to its regression oracle \( \text{SqAlg} \) to produce its reward predictions \( \hat{y}_t(x_t, a), \forall a \in [K] \) (Line 4). Each expert in \( \text{SqAlg} \) of FS-SCB corresponds to one of the \( M \) models and is a ridge regression algorithm with the feature map of that model. Expert \( i \in [M] \) predicts the reward of the context \( x_t \), for each action \( a \in [K] \), as \( f^i(x_t, a; H_t) = \langle \phi^i(x_t, a), \theta^*_i \rangle \), where \( \theta^*_i = \arg\min_{\theta} \|\Phi^T_i \theta - Y_t\|^2 + \lambda_i \|\theta\|^2 \). We may write \( \theta^*_i \) in closed-form as \( \theta^*_i = (V_i^\lambda)^{-1} \Phi^T_i Y_t \). In these equations, \( Y_t = (y_1, \ldots, y_{t-1})^{\top} \) is the reward vector; \( \Phi^T_i \) is the feature matrix of the \( i \)-th model, whose rows are \( \phi^i(x_1, a_1), \ldots, \phi^i(x_{t-1}, a_{t-1}) \); \( \lambda_i \) is the regularization parameter of model \( i \). Our analysis shows that it only needs to be larger than one, i.e., \( \lambda_i \geq 1 \); and finally \( V_i^\lambda = \lambda_i I + \Phi^T_i \Phi_i \). The meta algorithm \( \text{SqAlg} \) aggregates the experts’ predictions \( \{f^i(x_t, a; H_t)\}_{i=1}^M \), and produces its own predictions \( \hat{y}_t(x_t, a), \forall a \in [K] \), using Algorithm 3 in Appendix A.

The next step in FS-SCB is computing the action with the highest predicted reward, i.e., \( a'_t = \arg\max_{a \in [K]} \hat{y}_t(x_t, a) \), and using it to define a distribution \( p_t \in \Delta_K \) over the actions (see Eq. 5 in Line 5). The distribution \( p_t \) in (5) is defined similarly to the probability selection scheme of [Abe and Long, 1999], and assigns a probability to every action inversely proportional to the gap between its prediction and that of \( a'_t \). The algorithm then samples its action \( a_t \) from \( p_t \), observes reward \( y_t \), and feeds the tuple \( (x_t, a_t, y_t) \) to the oracle to update its weights over the experts (Lines 6 to 8). Our analysis in Section 3.1 and Appendix B suggest to set the exploration parameter to \( \kappa = K \) and the learning rate to \( \alpha = \sqrt{KT/D_T(\delta)} \), where we define \( D_T(\delta) \) in Lemma 2 and give its exact expression in Eq. 29 in Appendix B.2.

### 3.1 Regret Analysis of FS-SCB

In this section, we prove a regret bound for FS-SCB (Algorithm 1). We state the bound in Theorem 1 followed by a proof sketch. The detailed proofs are all reported in Appendix B.

**Theorem 1.** Let Assumption 1 hold and the regularization parameters \( \lambda_i \), exploration parameter \( \kappa \), and learning rate \( \alpha \) set to the values described above. Then, for any \( \delta \in (0, 1/4) \), with probability at least \( 1 - \delta \), the regret defined by (2) for FS-SCB is bounded as

\[
\mathcal{R}_{\text{FS-SCB}}(T) \leq O\left( \sqrt{2T \log(2/\delta)} + \text{RLG} \sqrt{KT(1 + \log(M)) \max_{i \in [M]} \left\{ \lambda_i S^2 + 4d \log \left( \frac{1 + TL^2}{\lambda_i d \delta} \right) \right\}} \right).
\]

**Proof Sketch.** The proof of Theorem 1 follows the steps below.

**Step 1.** We first need to bound the prediction error of the online regression oracle.
Lemma 2. For any $\delta \in (0, 1/4]$, with probability at least $1 - \delta$, we have the following bound on the prediction error of the regression oracle:

\[
\sum_{s=1}^{l-1} \left( \widehat{y}_s(x_s, a_s) - \langle \phi^{i_s}(x_s, a_s), \theta^{i_s}_s \rangle \right)^2 \leq D_t(\delta) := \mathcal{O}\left((1 + R^2L^2G^2\log(M)) \max_{i \in [M]} \left\{ \lambda_i s^2 + 4d \log \left( \frac{1 + TL^2}{\lambda_i d} \right) \right\} \right).
\]

The exact definition of $D_t(\delta)$ (see Eq. 29 in Appendix B.3) shows its dependence on the following two terms: 1) an upper-bound $Q_t$ on the prediction error of the true models, i.e.,

\[
\max_{i_s} \sum_{s=1}^{l-1} \left( \langle \phi^{i_s}(x_s, a_s), \widehat{\theta}^{i_s}_s \rangle - \langle \phi^{i_s}(x_s, a_s), \theta^{i_s}_s \rangle \right)^2 \leq Q_t,
\]

and 2) the regret $R_{\text{sq}}(t)$ of the regression oracle. Thus, the proof of Lemma 2 requires finding an expression for each of these two quantities, which we derive them in the following lemmas.

Lemma 3. For any $\delta \in (0, 1)$, with probability at least $1 - \delta$, we may write $Q_t$ defined in (6) as (see Eq. 19 in Appendix B.1 for the exact expression)

\[
Q_t = \mathcal{O}\left( \max_{i \in [M]} \left\{ \lambda_i s^2 + 4d \log (1 + T \frac{L^2}{\lambda_i d}) \right\} + R^2 \log(1/\delta) \right).
\]

Lemma 4. For any $\delta \in (0, 1)$, with probability at least $1 - \delta$, we may write the regret of the regression oracle as (see Eq. 24 in Appendix B.2 for the exact expression)

\[
R_{\text{sq}}(t) = \mathcal{O}\left( R^2L^2 \log(M) \left( G^2 + \max_{i \in [M]} \left\{ \lambda_i s^2 + d \log (1 + T \frac{L^2}{\lambda_i d}) \right\} + \log(1/\delta) \right) \right).
\]

Step 2. We then show how the overall regret of FS-SCB is related to the prediction error of the online regression oracle, $D_t(\delta)$, using the following lemma:

Lemma 5. Under the same assumptions as Theorem 1, for any $\delta \in (0, 1/4]$, with probability at least $1 - \delta$, the regret of the FS-SCB algorithm is bounded as

\[
R_{\text{FS-SCB}}(T) \leq \sqrt{2T \log(2/\delta)} + \frac{\alpha}{4} D_T(\delta) + \sum_{t=1}^{T} \sum_{a \in [K]} p_t(a) \left( \langle \phi^{i_s}(x_t, a), \theta^{i_s}_t \rangle - \langle \phi^{i_s}(x_t, a_t^*), \theta^{i_s}_t \rangle - \frac{\alpha}{4} \left( \widehat{y}_t(x_t, a) - \langle \phi^{i_s}(x_t, a), \theta^{i_s}_t \rangle \right)^2 \right).
\]

Finally, we conclude the proof of Theorem 1 by bounding the last term on the RHS of (7) using Lemma 12 (see Appendix B.5 for details).

3.2 Related Work (Feature Selection)

The most straightforward solution to the feature selection problem described in Section 2.2 is to concatenate all models (feature maps) and build a $(M \times d)$-dimensional feature, and then search for the sparse reward parameter $\theta_\epsilon \in \mathbb{R}^{Md}$ with only $d$ non-zero elements. We may then solve the resulting LB problem using a sparse LB algorithm (e.g., [Abbasi-Yadkori et al., 2012]). This approach would result in a regret bound of $\mathcal{O}(d\sqrt{MT})$, which may not be desirable when the number of models $M$ is large.

Another approach is to use the EXP4 (or SquareCB) algorithm [Auer et al., 2002b] to obtain a regret that scales only logarithmically with $M$. If we partition the linear space of each model into $\mathcal{O}(2^d)$ predictors, we will have the total number of $\mathcal{O}(M2^d)$ predictors. Predictor $(i, j) \in ([M], [2^d])$ is associated with a linear map $\theta^{ij} \in \mathbb{R}^d$ and recommends the action $\arg \max_{a \in A} \langle \phi^{i_j}(a), \theta^{ij} \rangle$. The regret of
EXP4 with this set of experts is of $\tilde{O}(\sqrt{dKT\log M})$. Although this solution has logarithmic dependence on $M$, it is still not desirable, since it is computationally expensive (requires handling $M 2^d$ predictors).

To have computational efficiency, we can use the approach of [Maillard and Munos, 2011], but this results in a $O(T^{2/3})$ regret. [Maillard and Munos, 2011] designed a model selection strategy using an EXP4 algorithm with a set of experts that are instances of the S-EXP3 algorithm of [Auer et al., 2002c]. The interesting fact is that each S-EXP3 expert is a learning algorithm and competes against a set of mappings. The overall regret of this algorithm is of $\tilde{O}(T^{2/3}/(|S|K \log K)^{1/3} \sqrt{\log M})$ (see [Bubeck and Cesa-Bianchi, 2012, Chapter 4.2]). If we apply this algorithm to our setting, the regret is not desirable as its dependence on $T$ is of $O(T^{2/3})$, which is not optimal.

The novelty of our results is that we propose a computationally efficient algorithm, whose regret has better dependence on $M$ and $T$, i.e., $\tilde{O}(\sqrt{KT\log M})$, than the existing methods. This is achieved in our FS-SCB algorithm by using SquareCB over a set of (adaptive/learning) least-squares experts.

More recently, [Papini et al., 2021] studied a feature selection problem where the reward function is linear in all $M$ feature maps (all models are realizable). Under this stronger assumption, they prove a regret bound that is competitive (up to a log $M$ factor) with that of a linear bandit algorithm that uses the best feature map. More specifically, if one of the feature maps is such that a constant regret is achievable, the overall model selection strategy also achieves a constant regret. Although our focus is not on constant regret results, we are able to achieve our model selection guarantees without requiring all models to be realizable.

### 4 Parameter Selection Algorithm

We first propose a UCB-style algorithm for the parameter selection setting described in Section 2.3. We refer to our algorithm as parameter selection OFUL (PS-OFUL). We then provide an upper-bound on its transfer regret. We conclude this section with a discussion on the existing results related to this setting.

Algorithm 2 contains the pseudo-code of PS-OFUL. The novel idea in PS-OFUL is the construction of its confidence set $C_t$ (Line 3). This construction is based on the predictions $\{\hat{y}_s\}_{s=1}^{t-1}$ by a regression oracle $\text{SqAlg}$. As described in Section 2.4, $\text{SqAlg}$ is a meta algorithm that consists of $M$ learning algorithms (or experts), and its predictions $\hat{y}_t$ are aggregates of its experts’ predictions $f^i(\phi_t(a_i); H_t)$, $\forall i \in [M]$. In PS-OFUL, each expert $i \in [M]$ is a biased regularized least-squares algorithm with bias $\hat{\mu}_i$, i.e., our estimate of the center of the $i$th model (ball). Expert $i$ predicts the reward of the context-action $\phi_t(a_i)$ as $f^i(\phi_t(a_i); H_t) = \langle \phi_t(a_i), \hat{\theta}_t^i \rangle$, where $\hat{\theta}_t^i = \text{argmin}_\theta \| \Phi_t^\top \theta - Y_t \|^2 + \lambda_i \| \theta - \hat{\mu}_i \|^2$. We may write $\hat{\theta}_t^i$ in closed-form as $\hat{\theta}_t^i = (\Phi_t^\top \Phi_t)^{-1}\Phi_t^\top(Y_t - \Phi_t \hat{\mu}_t + \hat{\mu}_i)$. In these equations, the reward vector $Y_t$ and $\Phi_t$ are defined as in Section 3; $\Phi_t$ is the feature matrix, whose rows are $\phi_1(a_1), \ldots, \phi_{t-1}(a_{t-1})$; and $\lambda_i$ is the regularization parameter of expert $i$. Our analysis of PS-OFUL in Section 4.1 and Appendix C suggests to set the regularization parameters to $\lambda_i = \frac{1}{\|a_i + c_i\|^2}$.

We can think of PS-OFUL as a model selection or transfer learning algorithm in which each expert $i$ is trying to transfer its information to the LB problem at hand through a bias $\hat{\mu}_i$, which is an estimate of the center of its model (ball), and the regression oracle $\text{SqAlg}$ is trying to select the right model (expert) by giving more weights to those whose predictions seem to be more accurate.

The PS-OFUL algorithm takes the feature map $\phi$ and models $\{B(\hat{\mu}_i, b_i + c_i)\}_{i=1}^M$ as input. At each round $t \in [T]$, it first constructs a confidence set $C_{t-1}$ using the predictions of the regression oracle $\{\hat{y}_s\}_{s=1}^{t-1}$ (Line 3 of Algorithm 2). The radius $\gamma_t(\delta)$ of the confidence set $C_t$ is defined by two terms: 1) the regret $R_{\text{sq}}(t)$ of the regression oracle $\text{SqAlg}$, defined by (4), and 2) an upper-bound $U_t$ on the prediction error of the true models (i.e., models that contain $\theta_*$), i.e.,

$$\max_{i \in I_*} \sum_{s=1}^{t-1} ((\phi_\ast(a_s), \hat{\theta}_t^i) - \langle \phi_\ast(a_s), \theta_\ast \rangle)^2 \leq U_t,$$

(8)

The exact values of $U_t$, $R_{\text{sq}}(t)$, and $\gamma_t(\delta)$ come from our analysis and have been stated in Eq. 58 in Appendix C.3. PS-OFUL then computes action $a_t$ as the one that attains the maximum optimistic reward w.r.t. the confidence set $C_{t-1}$ (Line 4). Using $a_t$, it calculates $\hat{y}_t = \text{SqAlg}_i(\phi_t(a_t); H_t)$ (Line 5). As described in Section 2.4, $\text{SqAlg}$ makes use of Algorithm 3 in Appendix A to return its prediction $\hat{y}_t$. 


as an aggregate of its experts’ predictions. Finally, PS-OFUL takes action $a_t$, observes reward $y_t$, and updates $\text{SqAlg}$ with the sample $(\phi_t(a_t), y_t)$ (Lines 6 and 7). This sample is then used within $\text{SqAlg}$ to evaluate its experts and to update their weights.

4.1 Regret Analysis of PS-OFUL

In this section, we prove a regret bound for PS-OFUL (Algorithm 2). We state the bound in Theorem 6 followed by a proof sketch. The detailed proofs are all reported in Appendix C.

Theorem 6. Let Assumption 2 hold and $\lambda_t = \frac{1}{(b_t + c_t)^2} \geq 1$, $\forall i \in [M]$. Then, for any $\delta \in (0, 1/4]$, with probability at least $1 - \delta$, the transfer-regret defined by (3) of PS-OFUL is bounded as

$$\mathcal{R}(T) = O \left( dRL \max\{1, G\} + \sqrt{1 + \log(M)} \sqrt{T \log \left( 1 + \frac{T}{d} \right) \log \left( 1 + \frac{TL^2 \max_{i \in [M]} (b_i + c_i)^2}{\delta d} \right)} \right).$$  \hspace{1cm} (9)

Proof Sketch. The proof of Theorem 6 follows the steps below.

Step 1. We first fully specify the confidence set $\mathcal{C}_t$ used by Algorithm 2 (Line 3), and prove its validity i.e., $P(\theta_* \in \mathcal{C}_t) \geq 1 - \delta$, $\forall t \in [T]$.

Theorem 7. Under the same assumptions as Theorem 6, the radius $\gamma_t(\delta)$ of the confidence set $\mathcal{C}_t$ is fully specified by Eq. 58 in Appendix C.3. Moreover, for any $\delta \in (0, 1/4]$, with probability at least $1 - \delta$, the true reward parameter $\theta_*$ lies in $\mathcal{C}_t$, i.e., $P(\theta_* \in \mathcal{C}_t) \geq 1 - \delta$.

The definition of $\gamma_t(\delta)$ in Eq. 58 shows its dependence on $U_t$ and $\mathcal{R}_{\text{Sq}}(t)$, defined by (8) and (4), respectively. Thus, the proof of Theorem 7 requires finding an expression for each of these two quantities, which we derive them in the following lemmas.

Lemma 8. Setting $\lambda_t = \frac{1}{(b_t + c_t)^2}$, $\forall i \in [M]$, with probability $1 - \delta$, we may write $U_t$, defined by (8), as (see Eq. 41 in Appendix C.1 for the exact expression)

$$U_t = O \left( dR^2 \log \left( 1 + \frac{TL^2 \max_{i \in [M]} (b_i + c_i)^2}{\delta d} \right) \right).$$

Lemma 9. Setting $\lambda_t = \frac{1}{(b_t + c_t)^2}$, $\forall i \in [M]$, with probability $1 - \delta$, we may write $\mathcal{R}_{\text{Sq}}(t)$, defined by (4), as (see Eq. 51 in Appendix C.2 for the exact expression)

$$\mathcal{R}_{\text{Sq}}(t) = O \left( dR^2 L^2 \log(M) \log \left( 1 + \frac{TL^2 \max_{i \in [M]} (b_i + c_i)^2}{\delta d} \right) \right).$$
Figure 1: Feature selection on MNIST dataset. The regrets are averaged over 100 LB problems.

Figure 2: Parameter selection on Synthetic LB (left) and CIFAR-10 (right). Results are averaged over 50 runs.

**Lemma 10.** Under the same assumptions as Theorem 6, for any $\delta \in (0, 1/4]$, with probability at least $1 - \delta$, the regret of PS-OFUL is bounded as

$$R_{\text{PS-OFUL}}(T) \leq 2Gd + 2\max\{1, G\} \sqrt{2dT \log \left(1 + \frac{T}{d}\right) \max_{t \leq T} \gamma_t(\delta)}.$$  \hfill (10)

We conclude the proof of Theorem 6 by plugging the confidence radius $\gamma_t(\delta)$ computed in Theorem 7 (Eq. 58 in Appendix C.3) into the regret bound (10).

**Remark 1 (Regret Bound).** The regret bound of PS-OFUL (9) has optimal dependence on $d$ and $T$, and more importantly, only logarithmic dependence on the number of models $M$ and the maximum model uncertainty $\max_{i \in [M]}(b_i + c_i)^2$.

### 4.2 Related Work (Parameter Selection)

[Cella et al., 2020] studied *meta learning* in stochastic linear bandit (LB), where the agent solves a sequence of LB problems, whose reward parameters $\theta_i$ are drawn from an unknown distribution $\rho$ of bounded support in $\mathbb{R}^d$. For each LB task, the agent is given an estimate of the mean of the distribution $\mu_i$ and an upper-bound of its error, and its goal is to minimize the transfer regret $R(T, \rho) = \mathbb{E}_{\theta_i \sim \rho}[\mathbb{E}[R(T, \theta_i)]]$. Their proposed algorithms assume knowing the variance term $\text{Var}_h = \mathbb{E}_{\theta_i \sim \rho}[||\theta_i - h||^2]$, for any $h \in \mathbb{R}^d$, in order to properly set their regularization parameter $\lambda$. Thus, the parameter selection setting studied in our paper can be seen as an extension of their transfer learning setting to multiple ($M$) models. Moreover, we allow the reward parameter of the new LB problem $\theta_i$ to be selected arbitrarily from the $M$ models, and consider a worst-case transfer regret (see Eq. 3) for our algorithm (instead of a regret in expectation w.r.t. $\rho$). Despite these differences, our setting is similar to theirs as we are also given an estimate of the center of each model $\mu_i$, together with an upper-bound on its error $c_i$, plus the radius $b_i$ of each model. Also similar to their results, our analysis clearly shows the importance of the choice of the regularization parameters, $\lambda_i = 1/(b_i + c_i)^2$, for obtaining a regret bound that only logarithmically depends on the maximum model uncertainty, i.e., $\max_{i \in [M]}(b_i + c_i)^2$.

Our parameter selection setting is also related to *latent* bandits [Maillard and Mannor, 2014, Hong et al., 2020] in which identifying the true latent variable is analogous to finding the correct
model. The latest work in this area is by [Hong et al., 2020] in which the agent faces a $K$-armed LB problem selected from a set of $M$ known $K$-dimensional reward vectors. They proposed UCB and TS algorithms for this setting and show that their regrets (Bayes regret in the case of TS) are bounded as $3M + 2T\varepsilon + 2R\sqrt{6MT\log T}$, where the reward vectors are known up to an error of $\varepsilon$. Comparing to their results, the regret of PS-OFUL in (9) has a better dependence on the number of models, $\sqrt{\log M}$ vs. $M$, and the model uncertainty, $\sqrt{\log(\max_{c\in[M]}(b_i + c)^2)}$ vs. $\varepsilon$. However, the number of actions $K$ does not appear in their bound, while the bound of PS-OFUL will have a $\sqrt{K}$ factor when applied to $K$-armed bandit problems. If the objective is to have a better scaling in $K$, we can use a different bandit model selection strategy, called regret balancing [Abbasi-Yadkori et al., 2020, Pacchiano et al., 2020], to obtain an improved regret that scales as $\min(\varepsilon T + \sqrt{MT}, \sqrt{KMT})$ (see Appendix E for details).

5 Experiments

We evaluate the performances of our FS-SCB and PS-OFUL algorithms using a synthetic LB problem and real-world image classification problems: MNIST [LeCun et al., 1998] and CIFAR-10 [Krizhevsky et al., 2009]. We report the details of our experiments and additional results in Appendix F.

**Feature Selection:** We train a convolutional neural network (CNN) with $M$ different number of epochs on MNIST data, and use their second layer to the last as our $d = 10$-dimensional feature maps $\{\phi^i\}_{i=1}^M$. These feature maps have test accuracy between 20% (worst model) and 97% (best model). We set the best one as true model $\phi^*$. For each class $s \in S = \{0, \ldots, 9\}$, we fit a linear model, given the feature map $\phi^s$, and obtain parameters $\{\theta^s_i\}_{i=0}^9$. At the beginning of each LB task, we select a class $s_* \in S$ uniformly at random and set its parameter to $\theta^s_*$. At each round $t \in [T]$, the learner is given an action set consists of 10 images, one from class $s_*$ and the rest randomly selected from the other classes. The reward of each action $a$ is defined as $\langle \phi^s(a), \theta^s_* + \eta_t \rangle + \eta_t \sim U[-0.5, 0.5]$, where $\phi^s(a)$ is the application of the feature map $\phi^s$ to the image corresponding to action $a$ and $\eta_t \sim U[-0.5, 0.5]$ is the noise.

In Figure 1, we compare the regret of our FS-SCB algorithm for different number of models $M$ with a regret balancing algorithm that uses OFUL baselines (RB-SCB), and three SquareCB algorithms that use the best (Oracle), second-best (with test accuracy 84%), and worst feature maps (experts). Each plot is averaged over 100 LB problems. Figure 1 shows that 1) FS-SCB always performs between the best and second-best experts, 2) the regret of PS-SCB that scales as $\sqrt{\log M}$ is close to RB-SCB (scales as $\sqrt{M}$) for small $M$, but gets much better as $M$ grows, and 3) RB-SCB has much higher variance than the other algorithms.

**Parameter Selection (Synthetic):** We first sample the center of $M = 10$ balls from a $d = 20$-dimensional Gaussian, i.e., $\{\mu_i\}_{i=1}^M \sim N(0, I_d)$, and set their radii to $b_i = b$, $\forall i \in [M]$. At the beginning of each LB task, we select a model $i_* \in [M]$ uniformly at random, and then sample the problem’s parameter from the $i_*$th ball, i.e., $\theta_* \sim B(\mu_{i_*}, b_{i_*})$. The action set at each round $t \in [T]$ consists of 10 vectors $\{\phi_t(a_i)\}_{i=1}^{10} \sim N(0, I_d)$, and the reward of the selected action $a_t$ is defined as $\langle \phi_t(a_t), \theta_* \rangle + \eta_t$, $\eta_t \sim U[-0.5, 0.5]$. Figure 2(left) compares the regret of our PS-OFUL algorithm with OFUL [Abbasi-Yadkori et al., 2011] for different sizes of the balls $b \in \{0, \sqrt{d}, 2\sqrt{d}\}$. We set the upper-bound on the reward parameter $\|\theta_*\|_2 \leq S = 10$ in the OFUL algorithm. Our results show that the regret of PS-OFUL is better than OFUL, and gets closer to it as we increase the size of the balls from $b = 0$ to $b = 2\sqrt{d} \approx 9.4$. This clearly shows the potential advantage of transfer (PS-OFUL) over individual (OFUL) learning.

**Parameter Selection (CIFAR-10):** We modify the EfficientNetV2-S Network [Tan and Le, 2021] by adding a layer of $d = 12$ neurons before the last layer and fine-tuning it on CIFAR-10 dataset. We then select this $d$-dimensional layer as our feature map $\phi$. To define our $M$ models (balls), we sample $100M$ datasets of size 500. For each dataset, we randomly select a class $s_* \in [M]$, assign reward 1 to images from $s_*$ and 0 to other images, and fit a linear model to it to obtain a parameter vector. Finally, we fit a Gaussian mixture model (GMM) with $M$ components to these $100M$ parameter vectors and use the means and covariances of the resulting clusters as the center and radii of our $M$ models (balls). At the beginning of each LB task, we select a class $s_* \in [M]$ uniformly at random. At each round $t \in [T]$, the learner is given an action set consists of 10 images, one from class $s_*$, and the rest randomly selected from the other classes. The learner receives a reward from Ber(0.9) if it selects the image from class $s_*$, and from Ber(0.1), otherwise.

In Figure 2(right), we compare the mean reward of PS-OFUL for different number of models $M$ with a regret balancing algorithm that uses OFUL baselines (RB-OFUL) [Abbasi-Yadkori et al., 2020],
OFUL (individual learning), and BIAS-OFUL [Cella et al., 2020] with bias being the center of the true model (Oracle). Figure 2 (right) shows 1) the good performance of PS-OFUL, 2) the performance of PS-OFUL gets better than RB-OFUL as $M$ grows ($\sqrt{\log M}$ vs. $\sqrt{M}$ scaling), 3) the large variance of RB-OFUL, especially in comparison to PS-OFUL, and finally 4) the advantage of transfer (PS-OFUL) over individual (OFUL) learning.

6 Conclusions and Future Works

We studied two model selection settings in LB, where the mean reward is linear in at least one of $M$ models (feature selection), and where the reward parameter is arbitrarily selected from $M$ misspecified models (parameter selection). We derived computationally efficient algorithms in these settings that are based on reductions from bandits to full-information problems, and proved regret bounds with desirable dependence on the horizon $T$ and number of models $M$.

An interesting future direction is to extend our results to the meta learning and learning-to-learn setting, where the agent starts with $M$ models, and instead of solving a single LB problem, has to solve $N$ of them one after another.
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A Sequential Prediction Algorithm

The sequential prediction algorithm \texttt{SqAlg} uses the following algorithm from [Haussler et al., 1998] (also see [Cesa-Bianchi and Lugosi, 2006, Chapter 3]) to aggregate its experts’ predictions. Algorithm 3 takes the observations \( y_t \) and experts’ predictions \( f^i_t(H_t) \) that are bounded in the known range \([\beta, \beta + \ell] \) as input. It first scales these input to the range \([0, 1] \) and uses its current weights for the experts to generate its own prediction \( \hat{y}_t \).

\textbf{Algorithm 3:} Sequential Prediction with Expert Advice

1. \textbf{Initialization:} Set the weight \( w_{1,i} = 1 \), for each expert \( i \in [M] \)
2. for \( t = 1, \ldots, T \) do
3. \hspace{1em} \textbf{Input:} Receive prediction \( f^i_t(H_t) \) by each expert \( i \in [M] \)
4. \hspace{2em} Scale experts’ predictions \( h_{i,t} = \frac{f^i_t(H_t) - \beta}{\ell}, \forall i \in [M] \)
5. \hspace{2em} Set \( v_{t,i} = \frac{w_t,i}{W}, \forall i \in [M] \), where \( W = \sum_{i=1}^{M} w_{t,i} \)
6. \hspace{2em} \textbf{Prediction:} Compute:
\[ \Delta(0) = -\frac{1}{2} \log \left( \sum_{i=1}^{M} v_{t,i} e^{-2h_{i,t}^2} \right), \quad \Delta(1) = -\frac{1}{2} \log \left( \sum_{i=1}^{M} v_{t,i} e^{-2(1-h_{i,t})^2} \right) \]

\[ \text{Predict a value } \hat{y}_t \text{ that satisfies the following conditions:} \]
\[ (\hat{y}_t)^2 \leq \Delta(0), \quad (1 - \hat{y}_t)^2 \leq \Delta(1). \]

\[ \text{Update:} \text{ Observing reward } y_t, \text{ scale it as } y'_t = \frac{y_t - \alpha}{\ell}, \text{ and update the experts’ weights:} \]
\[ w_{t+1,i} = w_{t,i} e^{-2(y'_t - h_{i,t})^2} \]
7. Return prediction \( \hat{y}_t = \beta + \ell \hat{y}'_t \)

The performance of \texttt{SqAlg} is evaluated as the accuracy (in terms of square loss) of its prediction w.r.t. the accuracy of the prediction by the best expert in the set, i.e.,
\[ \sum_{t=1}^{T} (\hat{y}_t - y_t)^2 - \min_{i \in [M]} \sum_{t=1}^{T} (f^i_t(H_t) - y_t)^2 \leq \mathcal{R}_{\text{Sq}}(T). \quad (11) \]

We call this the regret of \texttt{SqAlg} and denote it by \( \mathcal{R}_{\text{Sq}}(T) \). [Haussler et al., 1998] prove the following bound for \( \mathcal{R}_{\text{Sq}}(T) \), which we use in the analysis of our algorithms.

\textbf{Proposition 11} (Thm. 4.2 in [Haussler et al., 1998]). For any arbitrary sequence \( \{\{f^i_t(H_t)\}_{i=1}^{M}, \hat{y}_t, y_t\}_{t=1}^{T} \) in which the experts’ predictions \( \{\{f^i_t(H_t)\}_{i=1}^{M}\}_{t=1}^{T} \) and observations \( \{y_t\}_{t=1}^{T} \) are all bounded in \([\beta, \beta + \ell] \), the regret defined by (11) of Algorithm 3 is bounded as
\[ \mathcal{R}_{\text{Sq}}(T) \leq 2\ell^2 \log M. \]
B Proofs of Section 3

In this section, we first provide a brief overview for the steps of our proof. Then, we provide the proofs of lemmas used in Section 3.

The performance analysis of the FS-SCB algorithm requires two steps. First, we control the sum of the prediction error of the agent. Second, we show how the regret is related to the prediction error of the agent, and then we bound the regret.

**Step 1.** To control the sum of the prediction error of the agent $D_t$, we need to find two upper bounds: 1) an upper bound on the prediction error of the true model $i$, whose identity is unknown to the agent $Q_t$; 2) an upper bound on the regret caused by the online regression oracle $R_{sq}$.

First, in Lemma 3, we bound the sum of the prediction error of the true model as

$$
\sum_{s=1}^{t-1} \left( (\phi^*_i(x_s, a_s), \hat{\theta}^*_s) - (\phi^*_i(x_s, a_s), \theta^*_s) \right)^2 \leq Q_t
$$

where

$$
Q_t = 1 + 2 \left( \max_{i \in [M]} \left\{ \lambda_i S^2 + 4d \log \left( 1 + \frac{tL^2}{\lambda_id} \right) \right\} \right) + 32 R^2 \log \left( \frac{R\sqrt{8} + \sqrt{1 + \max_{i \in [M]} \left\{ \lambda_i S^2 + 4d \log \left( 1 + \frac{tL^2}{\lambda_id} \right) \right\}}}{\delta} \right).
$$

Next, in Lemma 4, we provide a high probability upper-bound on the regret caused by the online regression oracle as

$$
R_{sq}(t) \leq 8(\log M)R^2 L^2 \left( G^2 + \max_{i \in [M]} \{ \lambda_i S^2 + d \log \left( 1 + \frac{tL^2}{\lambda_id} \right) \} + \log(1/\delta) \right).
$$

Then, in Lemma 2, we show the following upper bound on the sum of prediction error of the agent:

$$
D_t(\delta) \leq 1 + 2R_{sq}(t) + 2Q_t + 4R \sqrt{2(1 + Q_t) \log \left( \frac{\sqrt{1 + Q_t}}{\delta} \right)}
$$

$$
+ 32 R^2 \log \left( \frac{R\sqrt{8} + \sqrt{1 + R_{sq}(t) + Q_t + 2R \sqrt{2(1 + Q_t) \log \left( \frac{\sqrt{1 + Q_t}}{\delta} \right)}}}{\delta} \right).
$$

**Step 2.** First in Lemma 5, we show how the regret is related to the prediction error of the agent using the Azuma’s inequality, i.e.,

$$
R_{FS-SCB}(T) \leq \sqrt{2T \log(2/\delta)} + \alpha D_T(\delta)
$$

$$
+ \sum_{t=1}^{T} \sum_{a \in [K]} p_t(x_t, a) \left( (\phi^*_i(x_t, a), \theta^*_t) - (\phi^*_i(x_t, a_t^*), \theta^*_s) \right) - \frac{\alpha}{4} \left( \hat{y}_t(x_t, a) - (\phi^*_i(x_t, a), \theta^*_s) \right)^2.
$$

Then in Appendix B.5, we put everything together and complete the proof.

B.1 Proof of Lemma 3

At each round $t$, each expert $i_\ast \in I_\ast$ estimates its reward parameter as

$$
\hat{\theta}^*_t = \arg \min_{\theta} \|(\Phi^i_t)^\top \theta - Y_t\|_2^2 + \lambda_{i_\ast} \|\theta\|_2^2.
$$

Let $V^\lambda_{i_\ast} = \lambda_{i_\ast}I + \sum_{s=1}^{t-1} \phi^*_i(x_s, a_s)\phi^*_i(x_s, a_s)^\top$. From the standard least-squares analysis, we have

$$
\sum_{s=1}^{t-1} (\phi^*_i(x_s, a_s), \hat{\theta}^*_s - y_s)^2 - \sum_{s=1}^{t-1} (\phi^*_i(x_s, a_s), \theta^*_s - y_s)^2 \leq \lambda_{i_\ast} \|\theta^*_s\|_2^2 + 2 \sum_{s=1}^{t-1} \langle \phi^*_i(x_s, a_s)^\top, (V^\lambda_{i_\ast})^{-1}\phi^*_i(x_s, a_s) \rangle.
$$
Therefore, we can write:
\[
\sum_{s=1}^{t-1} \left( \langle \phi_i^*(x_s, a_s), \hat{\theta}_s^* - \theta_s^* \rangle \right)^2 \leq \lambda_i \| \theta_s^* \|_2^2 + 2 \log \left( \frac{\det(V_t^{\tilde{\lambda}_t^*})}{\det(\lambda_i, I)} \right) + 2 \sum_{s=1}^{t-1} \eta_s \left( \langle \phi_i^*(x_s, a_s), \hat{\theta}_s^* - \theta_s^* \rangle \right).
\] (16)

The last term on the RHS of (16) can be bounded using Proposition 13 in Appendix D as
\[
\sum_{s=1}^{t-1} \eta_s \left( \langle \phi_i^*(x_s, a_s), \hat{\theta}_s^* - \theta_s^* \rangle \right) \leq R \sqrt{2 \left( 1 + \sum_{s=1}^{t-1} \left( \langle \phi_i^*(x_s, a_s), \hat{\theta}_s^* - \theta_s^* \rangle \right)^2 \right)} \log \left( \frac{1 + \sum_{s=1}^{t-1} \left( \langle \phi_i^*(x_s, a_s), \hat{\theta}_s^* - \theta_s^* \rangle \right)^2}{\delta} \right).
\]

Define \( u = \sqrt{1 + \sum_{k=1}^{t-1} \left( \langle \phi_i^*(x_k, a_k), \hat{\theta}_k^* - \theta_k^* \rangle \right)^2}, \) \( v = 1 + \lambda_i \left\| \theta_s^* \right\|_2^2 + 2 \log \left( \frac{\det(V_t^{\tilde{\lambda}_t^*})}{\det(\lambda_i, I)} \right), \) and \( w = 2R \sqrt{2 \log(s/\delta)}. \) Then, following the same machinery as the one in the proof of Lemma 8 in Section C.1, and with the use of Lemmas 17 and 18, for \( \delta \in (0, 1/4), \) with probability at least \( 1 - \delta, \) we have
\[
\sum_{s=1}^{t-1} \left( \langle \phi_i^*(x_s, a_s), \hat{\theta}_s^* - \theta_s^* \rangle \right)^2 \leq 1 + 2\lambda_i \left\| \theta_s^* \right\|_2^2 + 4 \log \left( \frac{\det(V_t^{\tilde{\lambda}_t^*})}{\det(\lambda_i, I)} \right) + 32R^2 \log \left( \frac{R\sqrt{8} + \sqrt{1 + \lambda_i S^2 + 4d \log \left( 1 + \frac{tL^2}{\lambda_i d} \right)}}{\delta} \right). \] (17)

We know that \( \left\| \theta_s^* \right\|_2^2 \leq S^2. \) Moreover, by Lemma 15 in Appendix D, we can bound the term \( \log \left( \frac{\det(V_t^{\tilde{\lambda}_t^*})}{\det(\lambda_i, I)} \right). \) Replacing these in (17), we may write
\[
\sum_{s=1}^{t-1} \left( \langle \phi_i^*(x_s, a_s), \hat{\theta}_s^* - \theta_s^* \rangle \right)^2 \leq 1 + 2\lambda_i S^2 + 8d \log \left( 1 + \frac{tL^2}{\lambda_i d} \right) + 32R^2 \log \left( \frac{R\sqrt{8} + \sqrt{1 + \lambda_i S^2 + 4d \log \left( 1 + \frac{tL^2}{\lambda_i d} \right)}}{\delta} \right). \] (18)

Since the algorithm does not know the identity of \( i^*_s, \) we derive an expression for \( Q_t, \) and conclude the proof by replacing \( i^*_s \) with the maximum over all \( i \in [M] \) in (18) as
\[
\sum_{s=1}^{t-1} \left( \langle \phi_i^*(x_s, a_s), \hat{\theta}_s^* - \theta_s^* \rangle \right)^2 \leq 1 + 2 \max_{i \in [M]} \left\{ \lambda_i S^2 + 4d \log \left( 1 + \frac{tL^2}{\lambda_i d} \right) \right\} + 32R^2 \log \left( \frac{R\sqrt{8} + \sqrt{1 + \max_{i \in [M]} \left\{ \lambda_i S^2 + 4d \log \left( 1 + \frac{tL^2}{\lambda_i d} \right) \right\}}}{\delta} \right) := Q_t.
\] (19)

### B.2 Proof of Lemma 4

To bound the regret \( R_{\text{Alg}}(t) \) of the regression oracle \( \text{SqAlg}, \) similar to the proof of Lemma 9 in Appendix C.2, we show the reward signals and the experts’ predictions are bounded with high probability. Then, we use Proposition 11 in Appendix A to complete the proof.
From (42), according to Assumption 1, we have $\langle \phi^*(x,a), \theta^*_s \rangle \leq LS$, for all $x \in X$, $a \in [k]$, and $i \in [M]$. Hence, with probability at least $1 - \delta$, WE HAVE

$$y_t \in \left[- \left(G + R \sqrt{2 \log(2/\delta)}\right), \left(G + R \sqrt{2 \log(2/\delta)}\right)\right].$$

(20)

Moreover, from (48) and Theorem 2 in [Abbasi-Yadkori et al., 2011], we have the following bound on the prediction of each expert $i$:

$$\left| \langle \phi^i(x_t, a_t), \theta^*_i \rangle \right| \leq G + RL \sqrt{\log \left(\frac{1 + \frac{L^2}{2\lambda_i}}{\delta}\right)} + L\sqrt{\lambda_i} S.$$  

(21)

Then, we may bound the range of the prediction of each expert $i \in [M]$ at round $t \in [T]$ as

$$\langle \phi^i(x_t, a_t), \theta^*_i \rangle \in \left[- \left(G + RL \sqrt{\log \left(\frac{1 + \frac{L^2}{2\lambda_i}}{\delta}\right)} + L\sqrt{\lambda_i} S\right), \left(G + RL \sqrt{\log \left(\frac{1 + \frac{L^2}{2\lambda_i}}{\delta}\right)} + L\sqrt{\lambda_i} S\right)\right].$$

(22)

Putting together (20) and (22), we conclude that for all rounds $t \in [T]$ and experts $i \in [M]$, with probability at least $1 - \delta$, the reward $y_t$ and the expert’s predictions $f^i_t(H_t)$ are in the range $[\beta, \beta + \ell]$ for

$$\beta = - \left(G + RL \sqrt{\log \left(\frac{1 + \frac{L^2}{2\lambda_i}}{\delta}\right)} + L\sqrt{\lambda_i} S\right), \quad \ell = 2 \left(G + RL \sqrt{\log \left(\frac{1 + \frac{L^2}{2\lambda_i}}{\delta}\right)} + L\sqrt{\lambda_i} S\right).$$

(23)

Using Proposition 11 in Appendix A with the bound on the observations and predictions in (23), with probability at least $1 - \delta$, we obtain the following regret bound for $\textbf{SqaAlg}$:

$$\mathcal{R}_{\text{sq}}(t) = 8R^2 L^2 \log(M) \left(G^2 + \max_{i \in [M]} \left\{\lambda_i S^2 + d \log \left(1 + \frac{L^2}{\lambda_i d}\right)\right\} + \log(1/\delta)\right),$$

(24)

in which we use the fact that for $a, b > 0$, $(a + b)^2 \leq 2a^2 + 2b^2$. This concludes our proof.

**B.3 Proof of Lemma 2**

Here, we bound the sum of the square loss of the oracle predictions, i.e.,

$$\sum_{s=1}^{t-1} \left(\tilde{y}_s(x_s, a_s) - \langle \phi^*(x_s, a_s), \theta^*_s \rangle\right)^2 \leq D_t(\delta).$$

(25)

We know that $y_t = \langle \phi^*(x_t, a_t), \theta^*_s \rangle + \eta_t$. Hence we can write

$$\left(\tilde{y}_t(x_t, a_t) - y_t\right)^2 - \left(\langle \phi^*(x_t, a_t), \tilde{\theta}^*_s \rangle - y_t\right)^2 =$$

$$\left(\tilde{y}_t(x_t, a_t) - \langle \phi^*(x_t, a_t), \theta^*_s \rangle - \eta_t\right)^2 - \left(\langle \phi^*(x_t, a_t), \theta^*_s \rangle - \langle \phi^*(x_t, a_t), \tilde{\theta}^*_s \rangle - \eta_t\right)^2$$

$$= \left(\tilde{y}_t(x_t, a_t) - \langle \phi^*(x_t, a_t), \theta^*_s \rangle\right)^2 - \left(\langle \phi^*(x_t, a_t), \tilde{\theta}^*_s \rangle - \langle \phi^*(x_t, a_t), \theta^*_s \rangle\right)^2 + 2\eta_t \left(\langle \phi^*(x_t, a_t), \tilde{\theta}^*_s \rangle - \tilde{y}_t(x_t, a_t)\right) - \tilde{y}_t(x_t, a_t))$$

$$= \left(\tilde{y}_t(x_t, a_t) - \langle \phi^*(x_t, a_t), \theta^*_s \rangle\right)^2 - \left(\langle \phi^*(x_t, a_t), \tilde{\theta}^*_s \rangle - \langle \phi^*(x_t, a_t), \theta^*_s \rangle\right)^2 + 2\eta_t \left(\langle \phi^*(x_t, a_t), \tilde{\theta}^*_s \rangle - \langle \phi^*(x_t, a_t), \theta^*_s \rangle\right) + 2\eta_t \left(\langle \phi^*(x_t, a_t), \theta^*_s \rangle - \tilde{y}_t(x_t, a_t)\right).$$

(26)

Then, from Proposition 13 in Appendix D, with probability at least $1 - \delta$, we have

$$\left|\sum_{s=1}^{t-1} \eta_s \left(\langle \phi^*(x_s, a_s), \tilde{\theta}^*_s - \theta^*_s \rangle\right)\right| \leq$$

$$R \sqrt{2 \left(1 + \sum_{s=1}^{t-1} \left(\langle \phi^*(x_s, a_s), \tilde{\theta}^*_s - \theta^*_s \rangle\right)^2\right)} \log \left(\frac{\sqrt{1 + \sum_{s=1}^{t-1} \left(\langle \phi^*(x_s, a_s), \tilde{\theta}^*_s - \theta^*_s \rangle\right)^2}}{\delta}\right).$$

(27)
and
\[
\sum_{s=1}^{t-1} \eta_s \left( \langle \phi_s^* (x_s, a_s), \theta_s^* \rangle - \hat{y}_s(x_s, a_s) \right) \leq \left. R \sqrt{2 \left( 1 + \sum_{s=1}^{t-1} \left( \langle \phi_s^* (x_s, a_s), \theta_s^* \rangle - \hat{y}_s(x_s, a_s) \right)^2 \right)} \log \left( \frac{\sqrt{1 + \sum_{s=1}^{t-1} \left( \langle \phi_s^* (x_s, a_s), \theta_s^* \rangle - \hat{y}_s \right)^2}}{\delta} \right). \right.
\]
\tag{28}

Using (27) and (28), the upper-bound \( \mathcal{R}_{S_q}(t) \) from (24) in Appendix B.2, and the upper-bound \( Q_t \) on the square error of the prediction of the true model in (19) in Appendix B.1, we may write (26) as
\[
\sum_{s=1}^{t-1} (\hat{y}_s(x_s, a_s) - \langle \phi_s^* (x_s, a_s), \theta_s^* \rangle)^2 \leq \mathcal{R}_{S_q}(t) + Q_t + 2R \sqrt{2(1 + Q_t) \log \left( \frac{\sqrt{1 + Q_t}}{\delta} \right)} + 2R \left( 1 + \sum_{s=1}^{t-1} \left( \langle \phi_s^* (x_s, a_s), \theta_s^* \rangle - \hat{y}_s(x_s, a_s) \right)^2 \right) \log \left( \frac{\sqrt{1 + \sum_{s=1}^{t-1} \left( \langle \phi_s^* (x_s, a_s), \theta_s^* \rangle - \hat{y}_s \right)^2}}{\delta} \right).
\]

Let \( u = \sqrt{1 + \sum_{k=1}^{t-1} (\hat{y}_k(x_k, a_k) - \langle \phi_s^* (x_k, a_k), \theta_s^* \rangle)^2} \),\( v = 1 + \mathcal{R}_{S_q}(t) + Q_t + 2R \sqrt{2(1 + Q_t) \log \left( \frac{\sqrt{1 + Q_t}}{\delta} \right)} \), and \( q = 2R \sqrt{2 \log(s/\delta)} \). Then, we follow similar machinery as the one in the proof of Lemma 8 in Appendix C.1, and with the use of Lemmas 17 and 18 in Appendix D, for \( \delta \in (0, 1/4] \), with probability at least \( 1 - \delta \), we have
\[
\sum_{s=1}^{t-1} (\hat{y}_s(x_s, a_s) - \langle \phi_s^* (x_s, a_s), \theta_s^* \rangle)^2 \leq 1 + 2\mathcal{R}_{S_q}(t) + 2Q_t + 4R \sqrt{2(1 + Q_t) \log \left( \frac{\sqrt{1 + Q_t}}{\delta} \right)} + 32R^2 \log \left( \frac{R\sqrt{8} + \sqrt{1 + \mathcal{R}_{S_q}(t) + Q_t + 2R \sqrt{2(1 + Q_t) \log \left( \frac{\sqrt{1 + Q_t}}{\delta} \right)}}}{\delta} \right) := D_t(\delta), \tag{29}
\]
where
\[
Q_t = 1 + 2 \left( \max_{i \in [M]} \left\{ \lambda_i S^2 + 4d \log \left( 1 + \frac{tL^2}{\lambda_i d} \right) \right\} \right) + 32R^2 \log \left( \frac{R\sqrt{8} + \sqrt{1 + \max_{i \in [M]} \left\{ \lambda_i S^2 + 4d \log \left( 1 + \frac{tL^2}{\lambda_i d} \right) \right\}}}{\delta} \right),
\]
and
\[
\mathcal{R}_{S_q}(t) \leq 8R^2 L^2 \log(M) \left( C^2 + \max_{i \in [M]} \left\{ \lambda_i S^2 + d \log \left( 1 + \frac{tL^2}{\lambda_i d} \right) \right\} + \log(1/\delta) \right).
\]

### B.4 Proof of Lemma 5

The inequality can be obtained using Azuma’s inequality and following similar steps as in Lemma 2 of [Foster and Rakhlin, 2020]. We may write the regret as
\[
\mathcal{R}_{FS-SCB}(T) = \sum_{t=1}^{T} \left( \langle \phi_s^* (x_t, a_t), \theta_s^* \rangle - \langle \phi_s^* (x_t, a_t), \theta_s^* \rangle \right) - \frac{\alpha}{4} \left( \hat{y}_t(x_t, a_t) - \langle \phi_s^* (x_t, a_t), \theta_s^* \rangle \right)^2
\]
\[
+ \frac{\alpha}{4} \sum_{t=1}^{T} \left( \hat{y}_t(x_t, a_t) - \langle \phi_s^* (x_t, a_t), \theta_s^* \rangle \right)^2.
\tag{30}
\]
The last term on the RHS of (30) is bounded with $D_t(\delta)$ in (29) from the result of Lemma 2 in Appendix B.3. Define filtration $F_{t-1} = \sigma((x_1, a_1, y_1), \ldots, (x_{t-1}, a_{t-1}, y_{t-1}))$. On the RHS of (30), the action $a_t$ is random. We can use the Azuma’s inequality and with probability at least $1 - \delta$, upper-bound the first term on the RHS of (30) with its expectation counterparts using the probability distribution $p_t$ as

$$R_{FS-SCB}(T) \leq \sqrt{2T \log(2/\delta)} + \frac{\alpha}{4} D_T$$

(31)

$$+ \sum_{t=1}^{T} \sum_{a \in [K]} p_t(a) \left( \langle \phi_i^*(x_t, a), \theta_i^* \rangle - \langle \phi_i^*(x_t, a^*_t), \theta_i^* \rangle - \frac{\alpha}{4} (\hat{y}_t(x_t, a) - \langle \phi_i^*(x_t, a_t), \theta_i^* \rangle)^2 \right).$$

**B.5  Proof of Theorem 1**

We first state the following lemma from [Foster and Rakhlin, 2020] to bound the first term on the RHS of (32).

**Lemma 12** (Lemma 3 in [Foster and Rakhlin, 2020]). Under Assumption 1, for the probability distribution $p_t \in \Delta_K$ defined in the FS-SCB algorithm, we may write

$$\sum_{a \in [K]} p_t(a) \left( \langle \phi_i^*(x_t, a), \theta_i^* \rangle - \langle \phi_i^*(x_t, a^*_t), \theta_i^* \rangle - \frac{\alpha}{4} (\hat{y}_t(x_t, a) - \langle \phi_i^*(x_t, a_t), \theta_i^* \rangle)^2 \right) \leq \frac{2K}{\alpha}.$$

Putting everything together, with the choice of $\alpha = \sqrt{KT/D_T(\delta)}$, with probability at least $1 - \delta$, we can show the following upper-bound on the regret of the FS-SCB algorithm:

$$R_{FS-SCB}(T) \leq 3\sqrt{KT D_T(\delta)} + \sqrt{2T \log(2/\delta)}$$

(32)

Here the upper-bound is of order

$$R_{FS-SCB}(T) \leq O\left(\sqrt{2T \log(2/\delta)} + RLG \sqrt{KT(1 + \log(M)) \max_{i \in [M]} \left\{ \lambda_i S^2 + 4d \log \left( \frac{1 + T \lambda_i^2}{\delta} \right) \right\}} \right).$$
C Proofs of Section 4

The regret analysis of the PS-OFUL algorithms requires two steps. First, in Theorem 7, we show that the confidence set $C_t$ is valid at each round $t$, i.e., for any $t, \delta > 0$, it includes the reward parameter $\theta_*$ with probability at least $1 - \delta$. Second, we show how the regret is related to the valid confidence set, and then using Lemmas 14 and 15 complete the proof.

**Step 1.** The key idea for showing the validity of the confidence set $C_t$ requires controlling the square prediction error of the online regression oracle $\hat{y}_t$, i.e., upper-bounding $\gamma_t$. In Appendix C.3, we show that we can relate this distance to the sum of two terms: $\gamma_t \leq O \left( U_t + R_{\text{Sq}}(t) \right)$, and then show how we can bound each of them.

1) Bounding $U_t$: Lemma 8 shows the worst-case upper-bound on the square error of the prediction of true model $i_*$, given that the agent does not know the identity of the true model:

$$\sum_{s=1}^{t} (\phi_s(a_s), \hat{\theta}_s^* - \theta_*)^2 \leq U_t$$

where

$$U_t \leq 3 + 8d \log \left( 1 + \frac{tL^2 \max_{i \in [M]} (b_i + c_i)^2}{d} \right) + 32R^2 \log(1/\delta) \quad (34)$$

*Proof.* The proof is provided in Appendix C.1. \qed

2) Bounding $R_{\text{Sq}}(t)$: In Lemma 9, we prove an upper-bound on the regret caused by the prediction oracle $\text{SqAlg}$, given our proposed expert predictions as (see Appendix C.2 for details).

$$R_{\text{Sq}}(t) \leq 8(G + L)^2 \log(M) + 8R^2L^2d \log(M) \log(1/\delta) + 8R^2L^2d \log(M) \log \left( 1 + \frac{tL^2 \max_{i \in [M]} (b_i + c_i)^2}{d} \right)$$

Putting these together, in Appendix C.3, we prove Theorem 7 that shows the validity of the confidence set $C_t$.

**Step 2.** In Appendix C.4, we first show how regret is related to the confidence set. In particular, we show that given the validity of of the confidence set $C_t$, i.e., for any $\delta \in (0, 1/4]$, with probability at least $1 - \delta$, $\theta_* \in C_t$, we can bound the regret as

$$R_{\text{PS-OFUL}}(T) \leq 2Gd + 2\max \{1, G\} \sqrt{2dT \log \left( 1 + \frac{T}{d} \right) \max_{d < t \leq T} \gamma_t(\delta)}.$$  

Then, in Appendix C.5, we set $\lambda_i = \frac{1}{(b_i + c_i)^2}$, for each $i \in [M]$, and use Lemmas 14 and 15 to complete the proof of Theorem 6. Here we prove a regret upper-bound of order

$$O \left( dR \max \{1, G\} \sqrt{T + \log(M)} \times \sqrt{T \log \left( 1 + \frac{T}{d} \right) \log \left( 1 + \frac{tL^2 \max_{i \in [M]} (b_i + c_i)^2}{\delta d} \right)} \right).$$

C.1 Proof of Lemma 8

At each round $s \in [T]$, each expert $i_* \in I_s$ estimates its reward parameter as

$$\hat{\theta}_s^* = \arg \min_{\theta} \| \Phi_s^T \theta - Y_s \|^2 + \lambda_i \| \theta - \hat{\mu}_i \|^2,$$

which is the output of a Follow-The-Regularized-Leader (FTRL) algorithm with quadratic regularizer $\| \theta - \hat{\mu}_i \|^2$. Following the standard FTRL analysis of online regression (see e.g., [Cesa-Bianchi and Lugosi, 2006, Chapter 11]), we have

$$\sum_{s=1}^{t} (\langle \phi_s(a_s), \hat{\theta}_s^* - y_s \rangle)^2 - \sum_{s=1}^{t} (\langle \phi_s(a_s), \theta_* - y_s \rangle)^2 \leq \lambda_i \| \theta_* - \hat{\mu}_i \|^2 + 2 \sum_{s=1}^{t} \langle \phi_s(a_s), (Y_s^\lambda_i)^{-1} \phi_s(a_s) \rangle, \quad (35)$$
where $V_t^{\lambda_i} = \lambda_i I + \sum_{s=1}^{t-1} \phi_s(a_s)\phi_s(a_s)^T$. We may write (35) as

$$\sum_{s=1}^{t} \langle \phi_s(a_s), \tilde{\theta}_s^* - \theta_* \rangle^2 \leq \lambda_i \|\theta_* - \hat{\mu}_i\|^2 + 2 \log \left( \frac{\det(V_t^{\lambda_i})}{\det(\lambda_i I)} \right) + 2 \sum_{s=1}^{t} \eta_s \langle \phi_s(a_s), \tilde{\theta}_s^* - \theta_* \rangle. \quad (36)$$

Using Proposition 13 in Appendix D, we may bound the last term on the RHS of (36) as

$$\left| \sum_{s=1}^{t} \eta_s \langle \phi_s(a_s), \tilde{\theta}_s^* - \theta_* \rangle \right| \leq R \sqrt{2 \left( 1 + \sum_{s=1}^{t} \langle \phi_s(a_s), \tilde{\theta}_s^* - \theta_* \rangle^2 \right)} \log \left( \frac{1 + \sum_{s=1}^{t} \langle \phi_s(a_s), \tilde{\theta}_s^* - \theta_* \rangle^2}{\delta} \right). \quad (37)$$

It is easy to see that (36) can be written in the form $u^2 \leq v + uw$, where $u = \sqrt{1 + \sum_{s=1}^{t} \langle \phi_s(a_s), \tilde{\theta}_s^* - \theta_* \rangle^2}$, $v = 1 + \lambda_i \|\theta_* - \hat{\mu}_i\|^2 + 2 \log \left( \frac{\det(V_t^{\lambda_i})}{\det(\lambda_i I)} \right)$, and $w = 2R \sqrt{2 \log(u/\delta)}$. Then, by applying Lemma 17 in Appendix D, we may write $u \leq \sqrt{v} + w$. Substituting for $w$, we can get $u \leq \sqrt{v} + 2R \sqrt{2 \log(u/\delta)}$. Then, by Lemma 18 in Appendix D, for $\delta \in (0, 1/4]$, we have

$$u \leq \sqrt{v} + 4R \sqrt{\log \left( \frac{2\sqrt{2}R + \sqrt{v}}{\delta} \right)}.$$

which using the inequality $(a + b)^2 \leq 2a^2 + 2b^2$, for any $a$ and $b$, we can write it as

$$u^2 \leq 2v + 32R^2 \log \left( \frac{2\sqrt{2}R + \sqrt{v}}{\delta} \right).$$

Finally, we substitute $u$ and $v$, and subtract 1 from both sides, and for $\delta \in (0, 1/4]$, we obtain

$$\sum_{s=1}^{t} \langle \phi_s(a_s), \tilde{\theta}_s^* - \theta_* \rangle^2 \leq 1 + 2\lambda_i \|\theta_* - \hat{\mu}_i\|^2 + 4 \log \left( \frac{\det(V_t^{\lambda_i})}{\det(\lambda_i I)} \right)$$

$$+ 32R^2 \log \left( \frac{2\sqrt{2}R + \sqrt{1 + \lambda_i \|\theta_* - \hat{\mu}_i\|^2 + 2 \log \left( \frac{\det(V_t^{\lambda_i})}{\det(\lambda_i I)} \right)}}{\delta} \right). \quad (38)$$

We know $\|\theta_* - \hat{\mu}_i\|^2 \leq (b_i + c_i)^2$. Moreover, by Lemma 15 in Appendix D, we can bound the term

$$\log \left( \frac{\det(V_t^{\lambda_i})}{\det(\lambda_i I)} \right).$$

Replacing these terms in (38), we have

$$\sum_{s=1}^{t} \langle \phi_s(a_s), \tilde{\theta}_s^* - \theta_* \rangle^2 \leq 1 + 2\lambda_i (b_i + c_i)^2 + 8d \log \left( 1 + \frac{tL^2}{d\lambda_i} \right)$$

$$+ 32R^2 \log \left( \frac{2\sqrt{2}R + \sqrt{1 + \lambda_i (b_i + c_i)^2 + 4 \log(1 + \frac{tL^2}{d\lambda_i})}}{\delta} \right). \quad (39)$$

Setting $\lambda_i = \frac{1}{(b_i + c_i)^2}$, as used by the PS-OFUL algorithm, we obtain

$$\sum_{s=1}^{t} \langle \phi_s(a_s), \tilde{\theta}_s^* - \theta_* \rangle^2 \leq 3 + 8d \log \left( 1 + \frac{tL^2(b_i + c_i)^2}{d} \right)$$

$$+ 32R^2 \log \left( \frac{2\sqrt{2}R + \sqrt{2 + 4d \log \left( 1 + \frac{tL^2(b_i + c_i)^2}{d} \right)}}{\delta} \right). \quad (40)$$
Since the algorithm does not know the identity of \( i^* \), we derive an expression for \( U_t \) and conclude the proof by replacing \( i^* \) with the maximum over all \( i \in [M] \) in (40), as

\[
\sum_{s=1}^{t} \langle \phi_s(a_s), \hat{\theta}_s^* - \theta_s \rangle^2 \leq 3 + 8d \log \left( 1 + \frac{tL^2 \max_{i \in [M]}(b_i + c_i)^2}{d} \right) + 32R^2 \log \left( \frac{2R + \sqrt{2 + 4d \log \left( \frac{tL^2 \max_{i \in [M]}(b_i + c_i)^2}{d} \right)}}{\delta} \right) := U_t. \tag{41}
\]

### C.2 Proof of Lemma 9

To obtain a high probability bound on the regret \( R_{sq}(t) \) of the regression oracle \( \text{SqAlg} \), we first show that the inputs to the regression oracle, i.e., reward signals \( y_t = \phi_t(a_t) + \eta_t \) and the experts’ predictions \( f^i_t(H_t) = \langle \phi_t(a_t), \hat{\theta}_i^* \rangle \) are all bounded with high probability. We then use Proposition 11 in Appendix A to complete the proof.

Since each noise \( \eta_t \) is \( R \)-sub-Gaussian, from Lemma 16 in Appendix D, with probability at least \( 1 - \delta \), we have that \( |\eta_t| \leq R \sqrt{2 \log(2/\delta)} \). We also have from Assumption 2 that for each context and each action \( a \in \bigcup_{t=1}^{T} A_t \), their mean reward \( |\langle \phi_t(a), \theta_s \rangle| \leq G \). Thus, by the triangular inequality, with probability at least \( 1 - \delta \), we obtain

\[
y_t \in \left( - \left( G + R \sqrt{2 \log(2/\delta)} \right), \left( G + R \sqrt{2 \log(2/\delta)} \right) \right]. \tag{42}
\]

We now find a high probability bound for each expert’s prediction. For each round \( t \in [T] \) and model (or expert) \( i \in [M] \), we may write its prediction for the reward of the current context-action \( \phi_t(a_t) \) as

\[
\left| \langle \phi_t(a_t), \hat{\theta}_i^* \rangle \right| = \left| \langle \phi_t(a_t), \theta_s \rangle + \langle \phi_t(a_t), \hat{\theta}_i^* - \theta_s \rangle \right|
\leq |\langle \phi_t(a_t), \theta_s \rangle| + \left| \langle \phi_t(a_t), \hat{\theta}_i^* - \theta_s \rangle \right|
\leq G + \| \phi_t(a_t) \|_{(V_{\lambda_i}^*)^{-1}} \left( \| \Phi_t \eta_t \|_{(V_{\lambda_i}^*)^{-1}} + \sqrt{\lambda_i} \| \hat{\mu}_i - \theta_s \| \right), \tag{43}
\]

where we use the triangular inequality in the first inequality and of Theorem 2 in [Abbasi-Yadkori et al., 2011] for the second one.

Note that \( \| \hat{\mu}_i - \theta_s \| > (b_i + c_i) \) indicates that the reward parameter \( \theta_s \) of the problem does not belong to the \( i \)-th model. Thus, at each round \( t \in [T] \), a model (or expert) \( i \) is still considered a valid model by the algorithm, if its prediction can be bounded as

\[
\left| \langle \phi_t(a_t), \hat{\theta}_i^* \rangle \right| \leq G + \| \phi_t(a_t) \|_{(V_{\lambda_i}^*)^{-1}} \left( \| \Phi_t \eta_t \|_{(V_{\lambda_i}^*)^{-1}} + \sqrt{\lambda_i} \| \phi_t(a_t) \|_{(V_{\lambda_i}^*)^{-1}} \right). \tag{44}
\]

Checking if the inequality (44) is still valid for a model \( i \) can be potentially used by the algorithm to reduce the number of models (experts) during its execution.

We use Theorem 1 in [Abbasi-Yadkori et al., 2011] and standard matrix analysis together with our assumption that \( \| \phi_t(a_t) \| \leq L \), and bound the terms on the RHS of (44) with high probability as

\[
\| \Phi_t \eta_t \|_{(V_{\lambda_i}^*)^{-1}} \leq R \log \left( \frac{\sqrt{\det(V_{\lambda_i}^*)}}{\delta \sqrt{\det(\lambda_i I)}} \right), \tag{45}
\]

and

\[
\| \phi_t(a_t) \|_{(V_{\lambda_i}^*)^{-1}} \leq \frac{\| \phi_t(a_t) \|}{\sqrt{\lambda_{\min}(V_{\lambda_i}^*)}} \leq \frac{L}{\sqrt{\lambda_i}} \leq L, \tag{46}
\]

where \( \lambda_{\min}(V_{\lambda_i}^*) \) is the smallest eigenvalue of the matrix \( V_{\lambda_i}^* \). In the last step of (46), we use the fact that \( \lambda_i \geq 1, \forall i \in [M] \). Putting Eqs. 44, 45, and 46 together, with probability at least \( 1 - \delta \), we have

\[
\left| \langle \phi_t(a_t), \hat{\theta}_i^* \rangle \right| \leq G + RL \left( \frac{\sqrt{\det(V_{\lambda_i}^*)}}{\delta \sqrt{\det(\lambda_i I)}} + \sqrt{\lambda_i} \| \phi_t(a_t) \|_{(V_{\lambda_i}^*)^{-1}} \right). \tag{47}
\]
Using Lemma 15 in Appendix D, we may write (47) as
\[
\left| \langle \phi_t(a_t), \hat{\theta}_t^i \rangle \right| \leq G + RL \sqrt{\log \left( \frac{1 + \frac{tL^2}{\delta}}{d} \right)} + L \sqrt{\lambda_i (b_i + c_i)}.
\] (48)

Setting \( \lambda_i = \frac{1}{(b_i + c_i)^2} \) in (48), we can bound the prediction of each expert \( i \in [M] \) at round \( t \in [T] \) as
\[
\langle \phi_t(a_t), \hat{\theta}_t^i \rangle \in \left[ - \left( G + L + RL \sqrt{\log \left( \frac{1 + \frac{tL^2 \max_{j \in [M]} (b_j + c_j)^2}{\delta}}{d} \right)} \right), G + L + RL \sqrt{\log \left( \frac{1 + \frac{tL^2 \max_{j \in [M]} (b_j + c_j)^2}{\delta}}{d} \right)} \right].
\] (49)

Putting together (42) and (49), we conclude that for all rounds \( t \in [T] \) and experts \( i \in [M] \), with probability at least \( 1 - \delta \), the rewards \( y_t \) and the experts’ predictions \( f_t^i(H_t) \) are in the range \( [\beta, \beta + \ell] \) for
\[
\beta = - \left( G + L + RL \sqrt{\log \left( \frac{1 + \frac{tL^2 \max_{j \in [M]} (b_j + c_j)^2}{\delta}}{d} \right)} \right),
\]
\[
\ell = 2 \left( G + L + RL \sqrt{\log \left( \frac{1 + \frac{tL^2 \max_{j \in [M]} (b_j + c_j)^2}{\delta}}{d} \right)} \right).
\] (50)

Using Proposition 11 in Appendix A with the bound on the observations and predictions in (50), with probability at least \( 1 - \delta \), we obtain the following regret bound for \( \text{SqAlg} \):
\[
R_{\text{Sq}}(t) \leq 8(\log M) \left( (G + L)^2 + R^2 L^2 d \log \left( \frac{1 + \frac{tL^2 \max_{j \in [M]} (b_j + c_j)^2}{\delta}}{d} \right) \right),
\] (51)
in which we use the fact that for \( a, b > 0 \), \( (a + b)^2 \leq 2a^2 + 2b^2 \). This concludes our proof.

### C.3 Proof of Theorem 7

In order to fully specify the confidence set \( C_t \) and prove its validity, i.e., \( \theta_s \in \mathbb{P}(\theta_s \in C_t) \geq 1 - \delta \), we should find a high probability upper-bound \( \gamma_t(\delta) \) for the sum of the square loss of the oracle predictions, i.e.,
\[
\sum_{s=1}^{t} (\hat{y}_s - \langle \phi_s(a_s), \theta_s \rangle)^2 \leq \gamma_t(\delta).
\]

Let \( z_s = (\hat{y}_s - y_s)^2 - (\langle \phi_s(a_s), \hat{\theta}_s^* \rangle - y_s)^2 \), where \( i_s \in I_s \) is the index of a ball that contains \( \theta_s \). Since \( y_s = \langle \phi_s(a_s), \theta_s \rangle + \eta_s \), we may write
\[
z_s = (\hat{y}_s - \langle \phi_s(a_s), \theta_s \rangle - \eta_s)^2 - (\langle \phi_s(a_s), \hat{\theta}_s^* \rangle - \langle \phi_s(a_s), \theta_s \rangle - \eta_s)^2
\]
\[
= (\hat{y}_s - \langle \phi_s(a_s), \theta_s \rangle)^2 - (\langle \phi_s(a_s), \hat{\theta}_s^* \rangle - \langle \phi_s(a_s), \theta_s \rangle)^2 + 2\eta_s (\langle \phi_s(a_s), \hat{\theta}_s^* \rangle - \hat{y}_s).
\]

Since \( \sum_{s=1}^{t} z_s \leq R_{\text{Sq}}(t) \), where \( R_{\text{Sq}}(t) \) is the regret of the regression oracle at round \( t \), we have
\[
\sum_{s=1}^{t} (\hat{y}_s - \langle \phi_s(a_s), \theta_s \rangle)^2 \leq R_{\text{Sq}}(t) + \sum_{s=1}^{t} (\langle \phi_s(a_s), \hat{\theta}_s^* \rangle - \langle \phi_s(a_s), \theta_s \rangle)^2 + 2 \sum_{s=1}^{t} \eta_s (\langle \phi_s(a_s), \hat{\theta}_s^* \rangle - \hat{y}_s).
\] (52)

From the definition of \( U_t \) in (8), we may upper-bound \( \sum_{s=1}^{t} (\langle \phi_s(a_s), \hat{\theta}_s^* \rangle - \langle \phi_s(a_s), \theta_s \rangle)^2 \) with \( U_t \) and write (52) as
\[
\sum_{s=1}^{t} (\hat{y}_s - \langle \phi_s(a_s), \theta_s \rangle)^2 \leq R_{\text{Sq}}(t) + U_t + 2 \sum_{s=1}^{t} \eta_s (\langle \phi_s(a_s), \hat{\theta}_s^* \rangle - \hat{y}_s)
\]
\[
\leq R_{\text{Sq}}(t) + U_t + 2 \sum_{s=1}^{t} \eta_s (\langle \phi_s(a_s), \hat{\theta}_s^* \rangle - \theta_s) + 2 \sum_{s=1}^{t} \eta_s (\langle \phi_s(a_s), \theta_s \rangle - \hat{y}_s).
\] (53)
Then, from Proposition 13 in Appendix D, with probability at least $1 - \delta$, we have
\[
\left| \sum_{s=1}^{t} \eta_s \langle \phi_s(a_s), \hat{\theta}^*_s - \theta_s \rangle \right| \leq R \sqrt{2 \left( 1 + \sum_{s=1}^{t} \langle \phi_s(a_s), \hat{\theta}^*_s - \theta_s \rangle^2 \right)} \log \left( \frac{1 + \sum_{s=1}^{t} \langle \phi_s(a_s), \hat{\theta}^*_s - \theta_s \rangle^2}{\delta} \right)
\]
(54)
and
\[
\left| \sum_{s=1}^{t} \eta_s \langle \phi_s(a_s), \theta_s - \hat{\theta}_s \rangle \right| \leq R \sqrt{2 \left( 1 + \sum_{s=1}^{t} \langle \phi_s(a_s), \theta_s - \hat{\theta}_s \rangle^2 \right)} \log \left( \frac{1 + \sum_{s=1}^{t} \langle \phi_s(a_s), \theta_s - \hat{\theta}_s \rangle^2}{\delta} \right).
\]
(55)

Using (54) and (55), we may write (53) as
\[
\sum_{s=1}^{t} (\hat{\theta}_s - \langle \phi_s(a_s), \theta_s \rangle)^2 \leq R^2 \left( t + 2 R \sqrt{2(1 + U_t)} \log \left( \sqrt{1 + U_t}/\delta \right) \right)
\]
\[
+ R \sqrt{8 \left( 1 + \sum_{s=1}^{t} (\hat{\theta}_s - \langle \phi_s(a_s), \theta_s \rangle)^2 \right)} \log \left( \frac{1 + \sum_{s=1}^{t} (\hat{\theta}_s - \langle \phi_s(a_s), \theta_s \rangle)^2}{\delta} \right).
\]
(56)

It is easy to see that (56) can be written in the form $u^2 \leq v + uw$, where $u = \sqrt{1 + \sum_{s=1}^{t} (\langle \phi_s(a_s), \theta_s \rangle - \hat{\theta}_s)^2}$, $v = 1 + R^2 \left( t + 2 R \sqrt{2(1 + U_t)} \log (\sqrt{1 + U_t}/\delta) \right)$, and $w = R \sqrt{8 \log (u/\delta)}$. Then, by applying Lemma 17 in Appendix D, we may write $u \leq w + \sqrt{v}$. Substituting for $w$, we can get $u \leq \sqrt{v} + R \sqrt{8 \log (u/\delta)}$. Then, by Lemma 18 in Appendix D, for $\delta \in (0, 1/4]$, we have
\[
u \leq \sqrt{v} + 4R \sqrt{\log \left( \frac{R \sqrt{8} + \sqrt{v}}{\delta} \right)},
\]
which using the inequality $(a + b)^2 \leq 2a^2 + 2b^2$, for any $a$ and $b$, we can write it as
\[
u^2 \leq 2v^2 + 32R^2 \log \left( \frac{R \sqrt{8} + \sqrt{v}}{\delta} \right).
\]

Finally, we substitute $u$ and $v$, and subtract 1 from both sides, and for $\delta \in (0, 1/4]$, we obtain
\[
\sum_{s=1}^{t} (\hat{\theta}_s - \langle \phi_s(a_s), \theta_s \rangle)^2 \leq 1 + 2R^2 \left( t + 2 U_t + 4R \sqrt{2(1 + U_t)} \log \left( \sqrt{1 + U_t}/\delta \right) \right)
\]
\[
+ 32R^2 \log \left( \frac{R \sqrt{8} + 1 + R^2 \left( t + 2 U_t + 4R \sqrt{2(1 + U_t)} \log (\sqrt{1 + U_t}/\delta) \right)}{\delta} \right) := \gamma_t(\delta).
\]
(57)

Eq. 57 shows that for $\delta \in (0, 1/4]$, with probability at least $1 - \delta$, we have $\theta^* \in C_t$, which completes the proof of the validity of the confidence set $C_t$.

We can now fully specify $C_t$ by plugging $U_t$ from (41) (see Appendix C.1) and $R_{\text{sq}}(t)$ from (51) (see
Appendix C.2) into (57), and write $\gamma_t(\delta)$ as

$$
\gamma_t(\delta) := 1 + 2R_{sq}(t) + 2U_t + 4R\sqrt{2(1 + U_t) \log \left( \frac{\sqrt{1 + U_t}}{\delta} \right)} \\
+ 32R^2 \log \left( \frac{R\sqrt{8} + \sqrt{1 + R_{sq}(t) + U_t} + 2R\sqrt{2(1 + U_t) \log \left( \frac{\sqrt{1 + U_t}}{\delta} \right)}}{\delta} \right),
$$

where

$$
U_t = 3 + 8d \log \left( 1 + \frac{tL^2 \max_{i \in [M]} (b_i + c_i)^2}{d} \right) \\
+ 32R^2 \log \left( \frac{2\sqrt{2}R + \sqrt{2 + 4d \log \left( 1 + \frac{tL^2 \max_{i \in [M]} (b_i + c_i)^2}{d} \right)}}{\delta} \right),
$$

$$
R_{sq}(t) = 8 \log(M) \left( G^2 + L^2 + 2GL + R^2L^2d \log \left( 1 + \frac{tL^2 \max_{i \in [M]} (b_i + c_i)^2}{d} \right) \right),
$$

which concludes the proof.

A closer look at $U_t$ and $R_{sq}(t)$, the two main terms in the definition of $\gamma_t(\delta)$, we may write them in terms of the dominant terms as

$$
U_t \approx 3 + 16R^2 \log(2) + 8d \log \left( 1 + \frac{tL^2 \max_{i \in [M]} (b_i + c_i)^2}{d} \right) + 32R^2 \log(1/\delta) \\
+ 32R^2 \log \left( 1 + 2R + d \log \left( 1 + \frac{tL^2 \max_{i \in [M]} (b_i + c_i)^2}{d} \right) \right),
$$

and

$$
R_{sq}(t) = 8(G + L)^2 \log(M) + 8R^2L^2d \log(M) \log(1/\delta) + 8R^2L^2d \log(M) \log \left( 1 + \frac{tL^2 \max_{i \in [M]} (b_i + c_i)^2}{d} \right)
$$

$$
= C_2 + 8R^2L^2d \log(M) \log(1/\delta) + 8R^2L^2d \log(M) \log \left( 1 + \frac{tL^2 \max_{i \in [M]} (b_i + c_i)^2}{d} \right). \quad (60)
$$

Using (59) and (60), we may write $\gamma_t(\delta)$ in terms of the dominant terms as

$$
\gamma_t(\delta) \approx 1 + 2C_1 + 2C_2 + 16R^2 \left( 4 + L^2d \log(M) \right) \log(1/\delta) \\
+ 16d \left( 1 + R^2L^2d \log(M) \right) \log \left( 1 + \frac{tL^2 \max_{i \in [M]} (b_i + c_i)^2}{d} \right). \quad (61)
$$

C.4 Proof of Lemma 10

In Theorem 7, we proved that at each round, with probability at least $1 - \delta$, the true reward parameter $\theta_*$ belongs to the confidence set $C_0$ of the PS-OFUL algorithm. Here, we show how the regret of PS-OFUL is related to the radius $\gamma_t(\delta)$ of this confidence set.

Here we assume that at the first $d$ rounds, the algorithm plays actions whose features are of the form $\phi_t(a_i) = Le_i$, $\forall i \in [d]$, where $e_i = [0, \ldots, 1, \ldots, 0]$ is a $d$-dimensional vector whose elements are all 0, except a 1 at the $i^{th}$ position. In this case, we can define a matrix $V_t$ as

$$
V_t = \sum_{s=1}^{t-1} \phi_s(a_s)^T \phi_s(a_s) + \sum_{s=d+1}^{t-1} \phi_t(a_i)^T \phi_t(a_i), \quad (62)
$$
and use it to rewrite the confidence set as

$$\mathcal{C}_{t-1} = \{ \theta \in \mathbb{R}^d : (\theta - \hat{\theta}_t) V_t (\theta - \hat{\theta}_t) + \sum_{s=1}^{t-1} (\tilde{g}_s - (\phi_s(a_s), \hat{\theta}_t))^2 \leq \gamma_t(\delta) \},$$

(63)

where $\hat{\theta}_t = \arg\min_{\theta \in \mathbb{R}^d} \sum_{s=1}^{t-1} (\tilde{g}_s - (\phi_s(a_s), \theta))^2$. The confidence set $\mathcal{C}_t$ in (63) is contained in a larger ellipsoid

$$\mathcal{C}_{t-1} \subseteq \{ \theta \in \mathbb{R}^d : (\theta - \hat{\theta}_t) V_t (\theta - \hat{\theta}_t) \leq \gamma_t(\delta) \} = \{ \theta \in \mathbb{R}^d : \| \theta - \hat{\theta}_t \|_{V_t}^2 \leq \gamma_t(\delta) \}. \tag{64}$$

Given $(a_t, \tilde{\theta}_t) = \arg\max_{a \in A_t} \max_{\theta \in \mathcal{C}_{t-1}} \langle \phi_t(a), \theta \rangle$ are the action and parameter resulted from solving the optimization problem at round $t$ of the PS-OFUL algorithm, we may write

$$\langle \phi_t(a^*_t), \theta_* \rangle - \langle \phi_t(a_t), \theta_* \rangle \leq \langle \phi_t(a_t), \hat{\theta}_t - \hat{\theta}_t \rangle - \langle \phi_t(a_t), \theta_* \rangle$$

$$= \| \phi_t(a_t) \|_{V_t^{-1}} \| \hat{\theta}_t - \hat{\theta}_t \|_{V_t} + \| \phi_t(a_t) \|_{V_t^{-1}} \| \hat{\theta}_t - \theta_* \|_{V_t}$$

$$\leq 2 \sqrt{\gamma_t(\delta)} \| \phi_t(a_t) \|_{V_t^{-1}} \| \hat{\theta}_t - \hat{\theta}_t \|_{V_t} \quad \text{(because $\theta_* - \hat{\theta}_t \in \mathcal{C}_{t-1}$).} \tag{65}$$

Since $\forall a \in \bigcup_{t=1}^{T} A_t$, we assume that $|\langle \phi(a), \theta_* \rangle| \leq G$, we can upper-bound the instantaneous regret in (65) as

$$\langle \phi_t(a^*_t), \theta_* \rangle - \langle \phi_t(a_t), \theta_* \rangle \leq 2 \min \{ G, \sqrt{\gamma_t(\delta)} \| \phi_t(a_t) \|_{V_t^{-1}} \}. \tag{66}$$

Using (66), we can bound the transfer-regret of PS-OFUL as

$$\mathcal{R}_{PS-OFUL}(T) = \sum_{t=1}^{T} \langle \phi_t(a^*_t) - \phi_t(a_t), \theta_* \rangle \leq 2Gd + \sum_{t=d+1}^{T} \langle \phi_t(a^*_t) - \phi_t(a_t), \theta_* \rangle$$

$$\leq 2Gd + 2 \sum_{t=d+1}^{T} \min \{ G, \sqrt{\gamma_t(\delta)} \| \phi_t(a_t) \|_{V_t^{-1}} \}$$

$$\leq 2Gd + 2 \sum_{t=d+1}^{T} \sqrt{\gamma_t(\delta)} \min \{ G, \| \phi_t(a_t) \|_{V_t^{-1}} \} \quad \text{(since $\gamma_t(\delta) \geq 1$)}$$

$$\leq 2Gd + 2 \left( \max_{d<t \leq T} \sqrt{\gamma_t(\delta)} \right) \sum_{t=d+1}^{T} \min \{ G, \| \phi_t(a_t) \|_{V_t^{-1}} \}$$

$$\leq 2Gd + 2 \left( \max_{d<t \leq T} \sqrt{\gamma_t(\delta)} \right) \left( \max \{ 1, G \} \right) \sum_{t=d+1}^{T} \min \{ 1, \| \phi_t(a_t) \|_{V_t^{-1}} \}$$

$$\leq 2Gd + 2 \left( \max_{d<t \leq T} \sqrt{\gamma_t(\delta)} \right) \left( \max \{ 1, G \} \right) \sqrt{T} \sum_{t=d+1}^{T} \min \{ 1, \| \phi_t(a_t) \|_{V_t^{-1}}^2 \}$$

$$\leq 2Gd + 2 \left( \max_{d<t \leq T} \sqrt{\gamma_t(\delta)} \right) \left( \max \{ 1, G \} \right) \sqrt{2T \log \left( \frac{\det(V_T)}{\det(V_d)} \right)} \tag{67}$$

where the last inequality follows from Lemma 14 in Appendix D. Then, using Lemma 15 in Appendix D, we can bound $\det(V_T) \leq \left( L^2 + \frac{T L^2}{d} \right)^d$ and $\det(V_d) = L^{2d}$. Hence, we may write (67) as

$$\mathcal{R}_{PS-OFUL}(T) \leq 2Gd + 2 \max \{ 1, G \} \sqrt{2dT \log \left( 1 + \frac{T}{d} \right) \max_{d<t \leq T} \gamma_t(\delta)}. \tag{68}$$
C.5 Proof of Theorem 6

If we substitute $\gamma_t(\delta)$ from (61) in the regret bound (68), we may write it (in terms of the dominant terms) as

$$R_{PS-OFUL}(T) \leq 2Gd + 2\sqrt{2} \max\{1, G\} \sqrt{dT \log \left(1 + \frac{T}{d}\right)}$$

$$\times \sqrt{C_3 + 16R^2(4 + L^2d \log(M)) \log(1/\delta) + 16d(1 + R^2L^2 \log(M)) \log \left(1 + \frac{TL^2 \max_{i \in [M]}(b_i + c_i)^2}{d}\right)}$$

$$= O \left( dRL \max\{1, G\} \sqrt{1 + \log(M)} \times \sqrt{T \log \left(1 + \frac{T}{d}\right)} \log \left(1 + \frac{\frac{TL^2 \max_{i \in [M]}(b_i + c_i)^2}{d}}{\delta}\right) \right).$$

where $C_3 = 1 + 2C_1 + 2C_2$, and hence $C_3 = 7 + 32R^2 \log(2) + 16(G + L)^2 \log(M)$. 


D Auxiliary Tools

Here we report auxiliary results that we use in our proofs in other appendices.

We start with stating Theorem 7 in [Abbasi-Yadkori et al., 2012], which is the self-normalized martingale tail inequality for the scalar random variables.

**Proposition 13** (Self-normalized bound for martingales). Let \( \{F_t\}_{t=1}^{\infty} \) be a filtration. Let \( \tau \) be a stopping time w.r.t to the filtration \( \{F_t\}_{t=1}^{\infty} \), i.e., the event \( \{\tau \leq t\} \) belongs to \( F_{t+1} \). Let \( \{Z_t\}_{t=1}^{\infty} \) be a sequence of real-valued variables such that \( Z_t \) is \( F_t \)-measurable. Let \( \{\eta_t\}_{t=1}^{\infty} \) be a sequence of real-valued random variables such that \( \eta_t \) is \( F_{t+1} \) measurable and is conditionally \( R \)-sub-Gaussian. Then, for any \( \delta > 0 \), with probability at least \( 1 - \delta \),

\[
\left\| \sum_{t=1}^{\tau} \eta_t Z_t \right\| \leq R \sqrt{2 \left( 1 + \sum_{t=1}^{\tau} Z_t^2 \right) \log \left( \frac{\sqrt{1 + \sum_{t=1}^{\tau} Z_t^2}}{\delta} \right)}.
\]

Next, we state a direct application of Lemma 11 in [Abbasi-Yadkori et al., 2011] that bounds the cumulative sum of \( \sum_{s=1}^{t-1} \|\phi_s(a_s)\|_{V_s^{-1}}^2 \) which plays an important role in most of the proofs for linear bandits problems.

**Lemma 14.** Let \( \lambda > 0 \) and \( V_t = \lambda I + \sum_{s=1}^{t-1} \phi_s(a_s)\phi_s^{\top}(a_s) \). If for all \( a \in \cup_{s=1}^{t-1} A_s \), we have \( \|\phi_s(a)\|_2 \leq L \), then we may write

\[
\sum_{s=1}^{t-1} \min\{1, \|\phi_s(a_s)\|_{V_s^{-1}}^2 \} \leq 2 \log \left( \frac{\det(V_t)}{\det(\lambda I)} \right).
\]

Next, we present a determinant-trace inequality matrix result.

**Lemma 15** (Determinant-Trace Inequality). Suppose \( X_1, \ldots, X_{t-1} \in \mathbb{R}^d \), and for any \( 1 \leq s \leq t-1 \), we have \( \|X_s\|_2 \leq L \). Let \( V_t = \lambda I + \sum_{s=1}^{t-1} X_s X_s^{\top} \), for some \( \lambda > 0 \). Then we have

\[
\det(V_t) \leq \left( \lambda + \frac{tL^2}{d} \right)^d.
\]

**Proof.** Let \( \alpha_1, \ldots, \alpha_d \) be the eigenvalues of \( V_t \). Since \( V_t \) is positive definite, its eigenvalues are positive. Also not that \( \det(V_t) = \Pi_{s=1}^d \alpha_s \) and \( \text{trace}(V_t) = \sum_{s=1}^d \alpha_s \). By arithmetic-geometric means inequality we have

\[
\sqrt[d]{\alpha_1 \cdots \alpha_d} \leq \frac{\alpha_1 + \cdots + \alpha_d}{d}.
\]

Therefore, \( \det(V_t) \leq \left( \frac{\text{trace}(V_t)}{d} \right)^d \). It suffices to upper-bound the trace of \( V_t \) as

\[
\text{trace}(V_t) = \text{trace}(\lambda I) + \sum_{s=1}^{t-1} \text{trace}(X_s X_s^{\top}) = d\lambda + \sum_{s=1}^{t-1} \|X_s\|_2^2 \leq d\lambda + tL^2,
\]

and the result follows. \( \square \)

Next, we state a bound on the absolute value of the \( R \)-sub-Gaussian random variable.

**Lemma 16.** Let \( \{F_t\}_{t=1}^{\infty} \) be a filtration. Let \( \{\eta_t\}_{t=1}^{\infty} \) be a real-valued stochastic process such that \( \eta_t \) is \( F_t \)-measurable and \( \eta_t \) is conditionally \( R \)-sub-Gaussian for some \( R > 0 \), i.e.,

\[
\forall \lambda \in \mathbb{R}, \quad \mathbb{E}[\eta_t | F_t] = 0, \quad \mathbb{E}\left[ e^{\lambda \eta_t} | F_t \right] \leq \exp \left( \frac{\lambda^2 R^2}{2} \right).
\]

Then, condition on filtration \( F_t \), with probability at least \( 1 - \delta \), we have \( |\eta_t| \leq R\sqrt{2\log(2/\delta)} \).
Proof. Let $\lambda > 0$. Then,
\[
\mathbb{P}(\eta \geq k \mid F_t) = \mathbb{P}(e^{\lambda \eta} \geq e^{\lambda k} \mid F_t) \leq e^{-\lambda k} \mathbb{E}[e^{\lambda \eta} \mid F_t] \quad \text{(by Markov’s inequality)}
\]
\[
\leq e^{-\lambda k} e^{\frac{\lambda^2 R^2}{2}} = \exp \left(-\lambda k + \frac{\lambda^2 R^2}{2}\right). \tag{70}
\]
Optimizing for $\lambda$, and thus, selecting $\lambda = \frac{k}{R}$, we conclude that
\[
\mathbb{P}(\eta \geq k \mid F_t) \leq e^{-\frac{k^2}{2R^2}}.
\]
Repeating this argument for $-\eta$, we also obtain $\mathbb{P}(\eta \leq -k \mid F_t) \leq e^{-\frac{k^2}{2R^2}}$. Combining these two bounds, we can conclude that
\[
\mathbb{P}(|\eta| \geq k \mid F_t) \leq 2e^{-\frac{k^2}{2R^2}}. \tag{71}
\]
From (71), with the choice of $\delta = 2e^{-\frac{k^2}{2R^2}}$, and thus $k = R\sqrt{2\log(2/\delta)}$, completes the proof. \hfill \square

Then, we state a square-root trick for positive numbers.

**Lemma 17.** Let $a, b > 0$. If $z^2 \leq a + bz$, then $z \leq \sqrt{a} + b$.

**Proof.** Let $q(z) = z^2 - bz - a$. We can rewrite the condition $z^2 \leq a + bz$ as $q(z) \leq 0$. Then we know that the quadratic polynomial $q(z)$ has the following two roots

\[
\begin{align*}
z_1^* &= \frac{b + \sqrt{b^2 + 4a}}{2} \\
z_2^* &= \frac{b - \sqrt{b^2 + 4a}}{2}.
\end{align*}
\]

Then, we know that the condition $q(z) \leq 0$, implies that $\min\{z_1^*, z_2^*\} \leq z \leq \max\{z_1^*, z_2^*\}$. Therefore, for positive numbers $a, b$, we get

\[
z \leq \max\{z_1^*, z_2^*\} = \frac{b + \sqrt{b^2 + 4a}}{2} \leq b + \sqrt{a},
\]

where for the last inequality, we use the fact that for $u, v > 0$, $\sqrt{u} + \sqrt{v} \leq \sqrt{u + v}$. \hfill \square

Next, we restate a simple logarithmic trick from [Abbasi-Yadkori et al., 2012].

**Lemma 18** (Proposition 10 in [Abbasi-Yadkori et al., 2012]). Let $c \geq 1$, $q > 0$, $\delta \in (0, 1/4]$. If $s \geq 1$ and $s \leq c + \sqrt{\log(s/\delta)}$, then we have $s \leq c + q\sqrt{2\log(\frac{c+q}{\delta})}$. 

29
E  Relation to Latent Bandits

In this section, we informally show that if the goal in latent bandits is to have a better scaling with the number of actions \( K \) (e.g., the number of actions \( K \) is much larger than the number of latent states \( M \)), we can use a different bandit model selection strategy, called regret balancing [Abbasi-Yadkori et al., 2020, Pacchiano et al., 2020] to obtain an improved regret that scales as \( \min\{\varepsilon T + \sqrt{MT}, \sqrt{KMT}\} \). This rate is the best of the regret of PS-OFUL, which scales as \( \sqrt{KT} \), and the regret of the latent bandit algorithm of [Hong et al., 2020], which scales as \( \varepsilon T + \sqrt{MT} \).

In regret balancing, in each round, the model selection strategy chooses one of \( M \) base algorithms. We denote by \( N_{i,t} \), the number of times that the base algorithm \( i \) has been selected up to round \( t \), and by \( R_{i,t} \), the cumulative rewards of this base algorithm during these \( N_{i,t} \) rounds. Given a reference regret bound \( U : [T] \rightarrow \mathbb{R} \), in each round \( t \in [T] \), the algorithm first finds the optimistic base algorithm \( I_t \) and its value \( b_t \), i.e.,

\[
I_t = \arg\max_{i \in [M]} \frac{R_{i,t}}{N_{i,t}} + \frac{U(N_{i,t})}{N_{i,t}}, \quad b_t = \frac{R_{I_t,t}}{N_{I_t,t}} + \frac{U(N_{I_t,t})}{N_{I_t,t}},
\]

and then takes the action recommended by \( I_t \) and uses its observed reward to update the base algorithm \( I_t \).

We can apply regret balancing to the problem of latent bandits in the following way. We consider \( M + 1 \) base algorithms: one that plays UCB, and \( M \), each corresponds to a latent value and always plays the greedy action of that latent model (which is guaranteed to be \( \varepsilon \)-accurate by assumption). If the regret balancing strategy selects the UCB base algorithm in all rounds, it would suffer the regret \( \sqrt{KT} + \sqrt{T} \), and if it selects the optimal base algorithm, i.e., the base algorithm corresponding to the correct latent model, it would suffer the regret \( \varepsilon T + \sqrt{T} \). Note that by regret, we mean the actual regret (not pseudo-regret, and thus, \( \sqrt{T} \) is the consequence of noise in the reward signal. Thus, we select the reference regret bound of our regret balancing strategy as \( U(t) = \min\{\varepsilon T + \sqrt{T}, \sqrt{KMT} + \sqrt{T}\} \). We may write the regret of the resulting regret balancing strategy as follows:

\[
\mathcal{R}(T) \overset{(a)}{=} \sum_{i=1}^{M+1} N_{i,T} \mu_* - R_{i,T} \overset{(b)}{=} \sum_{i=1}^{M+1} N_{i,T} b_T - R_{i,T} \overset{(c)}{=} \sum_{i=1}^{M+1} U(N_{i,T}) \\
\leq \sum_{i=1}^{M+1} \min \left\{ \varepsilon N_{i,T} + \sqrt{N_{i,T}}, \sqrt{K N_{i,T}} + \sqrt{N_{i,T}} \right\} \\
\leq \min \left\{ \sum_{i=1}^{M+1} (\varepsilon N_{i,T} + \sqrt{N_{i,T}}), \sum_{i=1}^{M+1} (\sqrt{K N_{i,T}} + \sqrt{N_{i,T}}) \right\} \overset{(d)}{=} \min \left\{ \varepsilon T + \sum_{i=1}^{M+1} \sqrt{N_{i,T}}, \sum_{i=1}^{M+1} (\sqrt{K N_{i,T}} + \sqrt{N_{i,T}}) \right\} \\
\leq \min \left\{ \varepsilon T + \sqrt{M+1} \sum_{i=1}^{M+1} \sqrt{T/M+1}, \sum_{i=1}^{M+1} \left( \sqrt{K T} + \sqrt{M+1} \right) \right\} \\
= \min \left\{ \varepsilon T + \sqrt{(M+1)T}, \sqrt{K(M+1)T} + \sqrt{(M+1)T} \right\} \\
= O \left( \min \left\{ \varepsilon T + \sqrt{MT}, \sqrt{KMT} \right\} \right),
\]

which concludes our claim. Note that we used the following steps in our above derivations: (a) \( \mu_* \) is the mean of the optimal arm. (b) This is because with high probability we have \( \mu_* \leq b_t, \forall t \in [T] \). (c) This is from the definition \( b_t \) in (72). (d) This is due to the fact that \( \sum_{i=1}^{M+1} N_{i,T} = T \). (e) The maximizer of \( \sum_{i=1}^{M+1} \sqrt{N_{i,T}} \), subject to \( \sum_{i=1}^{M+1} N_{i,T} = T \), is when all \( \{N_{i,T}\}_{i=1}^{M+1} \) are equal.
F More on Experimental Results

We evaluate the performances of FS-SCB and PS-OFUL algorithms in a synthetic linear bandit problem and real-world image classification problems on CIFAR-10, CIFAR-100 [Krizhevsky et al., 2009], and MNIST datasets [LeCun et al., 1998].

F.1 Feature Selection

F.1.1 MNIST Dataset

MNIST dataset consists of 60000 training and 10000 test images of size $28 \times 28$, each belonging to one of 10 classes. We train a convolutional neural network (CNN) with $M$ different number of epochs on MNIST data, and use their second layer to the last as our $d = 10$-dimensional feature maps $\{\phi_i\}_{i=1}^M$. These feature maps have test accuracy between 20% (worst model) and 97% (best model). We set the best one as true model $\phi_\star$. For each class $s \in \mathcal{S} = \{0, \ldots, 9\}$, we fit a linear model, given the feature map $\phi_\star$, and obtain parameters $\{\theta_{\star s}\}_{s=0}^9$. At the beginning of each LB task, we select a class $s_\star \in \mathcal{S}$ uniformly at random and set its parameter to $\theta_{\star s_\star}^\star$. At each round $t \in [T]$, the learner is given an action set consists of 10 images, one from class $s_\star$ and the rest randomly selected from the other classes. The reward of each action $a$ is defined as $\langle \phi_{\star s_\star}(a), \theta_{\star s_\star}^\star \rangle + \eta_t \in [0, 1]$, where $\phi_{\star s_\star}(a)$ is the application of the feature map $\phi_\star$ to the image corresponding to action $a$ and $\eta_t \sim U[-0.5, 0.5]$ is the noise.

In Figure 3, we compare the regret of our FS-SCB algorithm for different number of models $M$ with a regret balancing algorithm that uses SquareCB baselines (RB-SCB), and three SquareCB algorithms that use the best (Oracle), second-best (with test accuracy 84%), and worst feature maps (experts). Each plot is averaged over 100 LB problems. Figure 1 shows that 1) FS-SCB always performs between the best and second-best experts, 2) the regret of FS-SCB that scales as $\sqrt{\log M}$ is close to RB-SCB (scales as $\sqrt{M}$) for small $M$, but gets much better as $M$ grows, and 3) RB-SCB has much higher variance than the other algorithms.

F.1.2 CIFAR-100 Dataset

CIFAR-100 dataset consists of 50000 training and 10000 test images of size $32 \times 32$, each belonging to one of 100 classes. We extracted the features of the images by fine tuning and taking the output of the second-to-last layer of the EfficientNet-B0 Network [Tan and Le, 2019] and got the feature matrix
of dimension $50000 \times 1280$. For all experts $i \in [M]$, we multiply this feature matrix with a Gaussian random matrix of dimension $1280 \times d_i$ for $d_i \in [2, 128]$ to get the $d_i$ dimensional feature maps $\phi^i$. These feature maps have accuracy between 5\% (worst model) and 78\% (best model). We set the best one as true model $\phi^{i*}$. For each class $s \in \mathcal{S} = \{0, \ldots, 99\}$, we fit a linear model, given the feature map $\phi^{i*}$ and obtain parameters $\{\theta_s^{i*}\}_{s=0}^{99}$. At the beginning of each LB task, we select a class $s_i \in \mathcal{S}$ uniformly at random and set its parameter to $\theta_{s_i}^{i*}$. At each round $t \in [T]$, the learner is given an action set consists of 10 images, one from class $s_*$ and the rest randomly selected from the other classes. The reward of each action $a$ is defined as $\langle \phi^{i*}(a), \theta_s^{i*} \rangle + \eta_t \in [0, 1]$, where $\phi^{i*}(a)$ is the application of the feature map $\phi^{i*}$ to the image corresponding to action $a$ and $\eta_t \sim \mathcal{U}[-0.5, 0.5]$ is the noise.

In Figure 4, we compare the regret of our FS-SCB algorithm for different number of models $M$ with a regret balancing algorithm that uses SquareCB baselines (RB-SCB) and aggregate them according to (72), and three SquareCB algorithms that use the best (Oracle), second-best (with test accuracy 55\%), and worst feature maps (experts). Each plot is averaged over 100 LB problems. Figure 1 shows that 1) FS-SCB always performs close to the best and second-best experts, 2) the regret of FS-SCB that scales as $\sqrt{\log M}$ is close to RB-SCB (scales as $\sqrt{M}$) for small $M$, but gets much better as $M$ grows, and 3) RB-SCB has much higher variance than the other algorithms.

### F.2 Parameter Selection

#### F.2.1 Synthetic Linear Bandit

We first sample the center of $M = 10$ balls from a $d = 20$-dimensional Gaussian, i.e., $\{\mu_i\}_{i=1}^M \sim \mathcal{N}(0, I_d)$, and set their radii to $b_i = b$, $\forall i \in [M]$. At the beginning of each LB task, we select a model $i_* \in [M]$ uniformly at random, and then sample the problem’s parameter from the $i_*$th ball, i.e., $\theta_* \sim B(\mu_{i_*}, b_{i_*})$. The action set at each round $t \in [T]$ consists of 10 vectors $\{\phi_t(a_j)\}_{j=1}^{10} \sim \mathcal{N}(0, I_d)$, and the reward of the selected action $a_t$ is defined as $\langle \phi_t(a_i), \theta_* \rangle + \eta_t$, $\eta_t \sim \mathcal{U}[-0.5, 0.5]$. Figure 5 compares the regret of our PS-OFUL algorithm with OFUL [Abbasi-Yadkori et al., 2011] for different sizes of the balls $b \in \{0, \sqrt{d}, 2\sqrt{d}\}$. We set the upper-bound on the reward parameter $\|\theta_*\|_2 \leq S = 10$ in the OFUL algorithm. Our results show that the regret of PS-OFUL is better than OFUL, and gets closer to it as we increase the size of the balls from $b = 0$ to $b = 2\sqrt{d} \approx 9$. This clearly shows the potential advantage of transfer (PS-OFUL) over individual (OFUL) learning.

![Figure 5: Parameter selection on synthetic linear bandit. The results are averaged over 50 runs.](image)

#### F.2.2 Image Classification on CIFAR-10 Dataset

CIFAR-10 dataset consists of 50000 test and 10000 training images of size $32 \times 32$, each belonging to one of 10 classes. We modify the EfficientNetV2-S Network [Tan and Le, 2021] by adding a layer of $d = 12$ neurons before the last layer and fine-tuning it on CIFAR-10 dataset. We then select this $d$-dimensional layer as our feature map $\phi$. To define our $M$ models (balls), we sample $100M$ datasets of size 500. For each dataset, we randomly select a class $s_* \in [M]$, assign reward 1 to images from $s_*$ and 0 to other images, and fit a linear model to it to obtain a parameter vector. Finally, we fit a Gaussian mixture model (GMM) with $M$ components to these $100M$ parameter vectors and use the means and covariances of the resulting clusters as the center and radii of our $M$ models (balls). At the beginning of each LB task, we select a class $s_* \in [M]$ uniformly at random. At each round $t \in [T]$, the learner is
Figure 6: Parameter selection on CIFAR-10 dataset, where 100M datasets of size 500 are used to define the balls. The results are averaged over 50 runs.

Figure 7: Parameter selection on CIFAR-10 dataset, where 10M datasets of size 50 are used to define the balls. The results are averaged over 50 runs.

given an action set consists of 10 images, one from class $s_*$ and the rest randomly selected from the other classes. The learner receives a reward from $\text{Ber}(0.9)$ if it selects the image from class $s_*$, and from $\text{Ber}(0.1)$, otherwise.

In Figure 6, we compare the mean reward of PS-OFUL for different number of models $M$ with a regret balancing algorithm that uses OFUL baselines (RB-OFUL) [Abbasi-Yadkori et al., 2020], OFUL (individual learning), and BIAS-OFUL [Cella et al., 2020] with bias being the center of the true model (Oracle). Figure 6 shows 1) the good performance of PS-OFUL, 2) the performance of PS-OFUL gets better than RB-OFUL as $M$ grows ($\sqrt{\log M}$ vs. $\sqrt{M}$ scaling), 3) the large variance of RB-OFUL, especially in comparison to PS-OFUL, and finally 4) the advantage of transfer (PS-OFUL) over individual (OFUL) learning.

Impact of the model estimates: In order to demonstrate the impact of the accuracy of the model center estimates as well as the radii of the balls, we defined a less accurate set of $M$ models (balls) using 10M datasets of size 50 (as opposed to 100M datasets of size 500). In Figure 7, we compare the mean reward of PS-OFUL for different number of models $M$ with RB-OFUL, OFUL, and BIAS-OFUL. It is shown that with decreasing in the accuracy of the model, the performance of the PS-OFUL and RB-OFUL get closer to the one of OFUL.
Figure 8: Parameter selection on MNIST dataset, where 100M datasets of size 500 are used to define the balls. The results are averaged over 50 runs.

Figure 9: Parameter selection on MNIST dataset, where 10M datasets of size 50 are used to define the balls. The results are averaged over 50 runs.

F.2.3 Image Classification on MNIST Dataset

MNIST dataset consists of 60000 test and 10000 training images of size 28 × 28, each belonging to one of 10 classes. We train a CNN with d = 12 neurons on second-to-last layer on MNIST dataset with 98% accuracy. We then select this d-dimensional layer as our feature map φ. To define our M models (balls), we sample 100M datasets of size 500. For each dataset, we randomly select a class s∗ ∈ [M], assign reward 1 to images from s∗ and 0 to other images, and fit a linear model to it to obtain a parameter vector. Finally, we fit a Gaussian mixture model (GMM) with M components to these 100M parameter vectors and use the means and covariances of the resulting clusters as the center and radii of our M models (balls). At the beginning of each LB task, we select a class s∗ ∈ [M] uniformly at random. At each round t ∈ [T], the learner is given an action set consists of 10 images, one from class s∗ and the rest randomly selected from the other classes. The learner receives a reward from Ber(0.9) if it selects the image from class s∗, and from Ber(0.1), otherwise.

In Figure 8, we compare the mean reward of PS-OFUL for different number of models M with a regret balancing algorithm that uses OFUL baselines (RB-OFUL) [Abbasi-Yadkori et al., 2020], OFUL (individual learning), and BIAS-OFUL [Cella et al., 2020] with bias being the center of the true model (Oracle). Figure 8 shows 1) the good performance of PS-OFUL, 2) the performance of PS-OFUL gets better than RB-OFUL as M grows ( √ logM vs. √M scaling), 3) the large variance of RB-OFUL, especially in comparison to PS-OFUL, and finally 4) the advantage of transfer (PS-OFUL) over individual (OFUL) learning.

Impact of the model estimates: In order to demonstrate the impact of the accuracy of the model center estimates as well as the radii of the balls, we defined a less accurate set of M models (balls) using 10M datasets of size 50 (as opposed to 100M datasets of size 500). In Figure 9, we compare the mean reward of PS-OFUL for different number of models M with RB-OFUL, OFUL, and BIAS-OFUL.