A note on semiorthogonal decompositions for Fano fibrations

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Abstract
Fano fibrations arise naturally in the birational classification of algebraic varieties. We show that these morphisms always induce a semiorthogonal decomposition on the derived category of the fibred space, extending classic results such as Orlov’s projective bundle formula to the non-flat and singular case.

Keywords Derived categories of sheaves · Semiorthogonal decompositions · Fano fibrations

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1 Introduction

Motivation

Let \( X \) be a smooth projective variety over an algebraically closed field \( k \) of characteristic zero. Suppose that we are interested in computing the bounded derived category of coherent sheaves on \( X \). The Minimal Model Program allows us to understand \( X \) in terms of a sequence of birational transformations and a simpler kind of varieties, which we will refer to as “building blocks”. If we knew how the derived categories of these building blocks look like and how these birational transformations affect the derived category, then we could compute the derived category of \( X \).

One of the building blocks of algebraic varieties are Fano varieties, which are the absolute case of Fano fibrations. Among them, the first example are projective spaces. Beilinson established in [2] the existence of a semiorthogonal decomposition

\[
\mathbf{D}^b(\mathbb{P}^n) = (\mathbf{D}^b(k) \otimes \mathcal{O}_{\mathbb{P}^n}, \ldots, \mathbf{D}^b(k) \otimes \mathcal{O}_{\mathbb{P}^n}(n)).
\]

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This means that the smallest triangulated subcategory of $\mathbf{D}^b(\mathbb{P}^n)$ containing all the subcategories in the decomposition is $\mathbf{D}^b(\mathbb{P}^n)$ itself and that there are no non-zero morphisms from right to left in a sense made precise in Definition 4. This result was generalised by Orlov to projective bundles in [19]. In the same paper he also proves his blow-up formula, yielding a derived categorical understanding of one of the key morphisms appearing on the Minimal Model Program.

Be˘ılinson’s semiorthogonal decomposition was later on extended to smooth Fano varieties, see for example [13]. As pointed out by several authors, this generalisation can be extended to the relative case of flat Fano fibrations between smooth projective varieties, see for example [1, Proposition 2.3.6]. Our aim is to further extend this result to the case of a Fano fibration over a normal base in which the total space is klt. This generalisation is necessary from the point of view of the Minimal Model Program, since singularities appear naturally throughout the process. But it also comes at a price, namely, replacing the bounded derived category of coherent sheaves by the unbounded derived category of quasi-coherent sheaves. Semiorthogonal decompositions on the bounded derived category of coherent sheaves or on the category of perfect complexes are more interesting and harder to produce, so we also give some conditions under which this semiorthogonal decomposition of the unbounded derived category of quasi-coherent sheaves induces semiorthogonal decompositions on the bounded derived category of coherent sheaves and on the category of perfect complexes.

In order to make the statement of the main result as self-contained as possible in this introduction, let us first give two definitions.

**Definition 1** (Fano fibration) Let $X$ be a proper variety with klt singularities, meaning that the pair $(X, 0)$ is klt [10, Definition 2.34]. Let $Y$ be a normal proper variety. We will say that a morphism $f: X \to Y$ is a Fano fibration if

1. the morphism $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism,
2. the anticanonical divisor $-K_X$ on $X$ is $f$-ample, and
3. $\dim X > \dim Y$.

**Definition 2** (Relative index) Let $f: X \to Y$ be a Fano fibration. The relative index of $f$ is defined as the largest rational number $r \in \mathbb{Q}$ such that $-K_X \equiv_f rH$ for some $f$-ample Cartier divisor $H$, where $\equiv_f$ denotes numerical $f$-equivalence [10, Notation 0.4.(5)].

A variety with klt singularities is in particular normal and $\mathbb{Q}$-Gorenstein, meaning that the canonical divisor $K_X$ is $\mathbb{Q}$-Cartier; this makes condition (2) in Definition 1 well-defined. Moreover, the relative index of a Fano fibration is always strictly positive; this will be relevant in the following statement. See the beginning of Sect. 3 for more remarks concerning these definitions.

**Theorem 3** Let $f: X \to Y$ be a Fano fibration of proper varieties over a field $k$ of characteristic zero. Let $r$ be the relative index of this Fano fibration and let $L$ be the line bundle on $X$ corresponding to an $f$-ample Cartier divisor $H$ with $-K_X \equiv_f rH$. The unbounded derived category of quasi-coherent sheaves on $X$ admits a semiorthogonal decomposition

$$\mathbf{D}(X) = \langle A_f, \mathbf{D}(Y) \boxtimes \mathcal{O}_X, \mathbf{D}(Y) \boxtimes L, \mathbf{D}(Y) \boxtimes L^\otimes 2, \ldots, \mathbf{D}(Y) \boxtimes L^\otimes \lceil r-1 \rceil \rangle,$$

where $\mathbf{D}(Y) \boxtimes L^\otimes i$ denotes the essential image of the functor $L f^*(-) \otimes L^\otimes i$ and

$$A_f = \cap_{i=0}^{\lceil r-1 \rceil} \text{Ker} \left( R f_* \mathcal{H}om(L^\otimes i, -) \right).$$

If in addition $f$ has finite Tor-dimension, then the analogous statements hold for the full subcategories of perfect complexes and for the bounded derived categories of coherent sheaves.
If \( Y \) is smooth, then \( f \) has finite Tor-dimension \([23, \text{Tag} \ 068B]\), so we obtain the claimed generalisation of \([1, \text{Proposition} \ 2.3.6]\).

**Notation and conventions**

We will follow the notation and conventions in \([8, 10] \) and \([18] \). In particular, varieties are always assumed to be irreducible. We work over a field \( k \) of characteristic zero. If \( F : A \to B \) and \( G : B \to A \) are functors, we will use the notation \( F \dashv G \) to indicate that \( F \) is left adjoint to \( G \) and \( G \) is right adjoint to \( F \). The right (resp. left) derived functor of a functor \( F \) will be denoted by \( R F \) (resp. \( L F \)), and its \( p \)th-higher derived functor by \( R^p F \) (resp. \( L^p F \)). We denote by \( D(−) \), \( D^{\text{b}}(−) \) and \( \text{Perf}(−) \) the unbounded derived category of quasi-coherent sheaves, the bounded derived category of coherent sheaves and the full subcategory of perfect complexes respectively.

**2 Categorical preliminaries**

Before we prove Theorem 3, let us introduce the category-theoretical language and tools that we need. This section follows \([15, \S 3.1] \) closely, generalising only certain statements to the unbounded setting.

**Definition 4** (Semiorthogonal decomposition) Let \( T \) be a triangulated category. We say that an ordered collection \( A_1, \ldots, A_m \subseteq T \) of strictly full triangulated subcategories is **semiorthogonal** if for all \( i < j \) and all \( A_i \in A_i \) and \( A_j \in A_j \) we have \( \text{Hom}(A_j, A_i) = 0 \).

For such a semiorthogonal collection, let \( \langle A_1, A_2 \rangle \) denote the full subcategory of all \( A \in T \) for which we can find a distinguished triangle \( A_2 \to A \to A_1 \to A_2[1] \) with \( A_1 \in A_1 \) and \( A_2 \in A_2 \), which is a strictly full triangulated subcategory. Let then \( \langle A_1, \ldots, A_m \rangle \) denote \( \ldots \langle A_1, A_2 \rangle, A_3 \ldots \rangle, A_m \), which is the smallest strictly full triangulated subcategory of \( T \) containing all the \( A_i \). If \( T = \langle A_1, \ldots, A_m \rangle \), then we say that the \( A_1, \ldots, A_m \) form a **semiorthogonal decomposition** of \( T \).

The key idea that we will use to find semiorthogonal decompositions goes back to \([3, \text{Lemma} \ 3.1] \), but we will use it as stated in \([15, \text{Lemma} \ 2.3] \):

**Lemma 5** Let \( F : T \to S \) be a triangulated functor between triangulated categories and assume there exists a right adjoint \( G : S \to T \) such that \( \text{id}_T \cong G \circ F \). Then \( F \) is fully faithful and there is a semiorthogonal decomposition

\[
S = \langle \text{Ker}(G), \text{Im}(F) \rangle,
\]

where \( \text{Im}(F) \) denotes the essential image of \( F \) and \( \text{Ker}(G) \) denotes the full subcategory of all objects \( A \in S \) such that \( G(A) \cong 0 \).

The adjunction that we will use to obtain semiorthogonal decompositions with Lemma 5 is the following:

**Lemma 6** Let \( f : X \to Y \) be a morphism of separated schemes of finite type over \( k \). For every \( \mathcal{E} \in D(X) \) there exists an adjunction

\[
L f^* (−) \otimes^L \mathcal{E} \dashv R f_* Rf_* \mathcal{E} \]

(1)

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between \( \text{D}(X) \) and \( \text{D}(Y) \). Moreover, if \( f \) is proper and has finite Tor-dimension and if \( \mathcal{E} \in \text{Perf}(X) \), then the same pair of functors induces adjunctions between \( \text{Perf}(X) \) and \( \text{Perf}(Y) \) and between \( \text{D}^b(X) \) and \( \text{D}^b(Y) \).

**Proof** Using Spaltenstein’s resolutions of unbounded complexes [22, Theorem A], we can define the derived functors involved on the whole unbounded derived categories of \( X \) and of \( Y \) respectively. The desired adjunction can be obtained then as the composition of the adjunctions \((-) \otimes^\mathbb{L}_{\mathcal{E}} \dashv \mathcal{R}\mathcal{H}\text{om}^\bullet(\mathcal{E}, -)\) and \( \mathcal{L}f^* \dashv \mathcal{R}f_* \) in [22, Theorem A].

The extra assumptions on \( f \) and on \( \mathcal{E} \) ensure that the strictly full subcategories of perfect complexes and of bounded complexes of coherent sheaves are preserved by these functors, so they induce adjunctions on these subcategories. See for example [23, Tag 09UA], [23, Tag 0B6G] or [20, §1.3].

\[ \square \]

**Remark 7** The functors appearing in Lemma 6 are \( Y \)-linear, meaning that they commute with tensor products by pull-backs of perfect complexes on \( Y \) along \( \text{id}_Y \) and \( f \) respectively. This is immediate in the case of the left adjoint; in the case of the right adjoint, it follows from the same argument as in [12, Lemma 2.34]. Moreover, if \( \mathcal{E} \) is a perfect complex, then the two functors commute with arbitrary direct sums. This is automatic for the left adjoint; in the case of the right adjoint, it follows from the fact that both \( \mathcal{R}f_* \) and \( \mathcal{R}\mathcal{H}\text{om}^\bullet(\mathcal{E}, -) \) commute with arbitrary direct sums, see [17, Lemma 1.4] and [17, p. 213] respectively.

**Definition 8** (Relative exceptional object) Let \( f : X \to Y \) be a morphism of separated schemes of finite type over \( k \). A complex \( \mathcal{E} \in \text{D}(X) \) is called a relative exceptional object or an \( f \)-exceptional object if it is a perfect complex and if the unit of the corresponding adjunction of Lemma 6 induces an isomorphism

\[ \mathcal{O}_Y \cong \mathcal{R}f_! \mathcal{R}\mathcal{H}\text{om}^\bullet(\mathcal{E}, \mathcal{E}). \]

**Remark 9** In [21, Theorem 3.1], a similar notion is considered. Let \( \pi : X \to S \) be a flat proper morphism between smooth schemes \( X \) and \( S \) over an algebraically closed field \( k \) of characteristic zero. For any closed point \( s \in S \), denote by \( i_s : \{s\} \to S \) the closed immersion of the point and by \( \tilde{i}_s : X_s \hookrightarrow X \) the closed immersion of the fibre \( X_s := \pi^{-1}(s) \). In this setting, objects \( \mathcal{E} \in \text{D}^b(X) = \text{Perf}(X) \) such that for all closed points \( s \in S \) the restriction \( \mathcal{L}\tilde{i}_s^*\mathcal{E} \) is an exceptional object in \( \text{D}^b(X_s) \) are considered. The proof of [21, Lemma 3.1] shows that such an object \( \mathcal{E} \) is \( \pi \)-exceptional. Conversely, if \( \mathcal{E} \) is a \( \pi \)-exceptional object in \( \text{Perf}(X) = \text{D}^b(X) \) and \( s \in S \) is a closed point, then the restriction \( \mathcal{L}\tilde{i}_s^*\mathcal{E} \) is an exceptional object in \( \text{D}^b(X_s) \). Indeed, since \( X \) is smooth, the closed immersion \( i_s : X_s \hookrightarrow X \) has finite Tor-dimension [23, Tag 068B], so by [16, Proposition 4.6.7] we have \( \mathcal{R}\mathcal{H}\text{om}^\bullet(\mathcal{L}\tilde{i}_s^*\mathcal{E}, \mathcal{L}\tilde{i}_s^*\mathcal{E}) \cong \mathcal{L}\tilde{i}_s^*\mathcal{R}\mathcal{H}\text{om}^\bullet(\mathcal{E}, \mathcal{E}) \). Since \( \pi : X \to S \) is flat and \( k \) is algebraically closed, the square

\[
\begin{array}{ccc}
X_s & \xrightarrow{\pi_s} & \text{Spec}(k) \\
\downarrow \tilde{i}_s & & \downarrow \pi \\
X & \xrightarrow{\pi} & S
\end{array}
\]

is exact cartesian, see [12, Definition 2.18] and [12, Corollary 2.23]. Hence \( \mathcal{L}\tilde{i}_s^* \circ \mathcal{R}\pi_* \cong \mathcal{R}(\pi_s)_s \circ \mathcal{L}\tilde{i}_s^* \), so combining everything above with the \( \pi \)-exceptionality assumption we deduce that \( \mathcal{R}(\pi_s)_s \circ \mathcal{R}\mathcal{H}\text{om}^\bullet(\mathcal{L}\tilde{i}_s^*\mathcal{E}, \mathcal{L}\tilde{i}_s^*\mathcal{E}) \cong \mathcal{L}\tilde{i}_s^*\mathcal{O}_S \cong \mathcal{O}_{\text{Spec}(k)} \). Therefore \( \mathcal{L}\tilde{i}_s^*\mathcal{E} \in \text{D}^b(X_s) \) is an exceptional object.

**Lemma 10** (cf. [15, Lemma 3.1]) Let \( f : X \to Y \) be a morphism of separated schemes of finite type over \( k \) and let \( \mathcal{E} \in \text{D}(X) \) be an \( f \)-exceptional object. Then the unit of the corresponding adjunction of Lemma 6 is a natural isomorphism.
**Proof** The proof in [15, Lemma 3.1] still works in the unbounded setting using the functorial isomorphisms [16, Proposition 3.9.4] and [23, Tag 08DQ]:

\[
\mathcal{F} \cong \mathcal{F} \otimes \mathcal{O}_Y \cong \mathcal{F} \otimes R f_* \mathcal{RHom}^*(\mathcal{E}, \mathcal{E}) \\
\cong R f_*(\mathcal{RHom}^*(\mathcal{E}, \mathcal{E}) \otimes L f^*(\mathcal{F})) \\
\cong R f_*(\mathcal{RHom}^*(\mathcal{E}, L f^*(\mathcal{F}) \otimes L \mathcal{E})).
\]

The fact that the unit of the adjunction is a natural isomorphism follows now from [23, Tag 08DQ] without any further computation.

Alternatively, one can give a simpler proof of this lemma with the following argument due to Neeman. The composition \( R f_*(\mathcal{RHom}^*(\mathcal{E}, L f^*(-) \otimes L \mathcal{E})) \) is a triangulated functor \( F : \mathcal{D}(Y) \to \mathcal{D}(Y) \) which commutes with arbitrary direct sums, and the same is true of \( \text{id}_{\mathcal{D}(Y)} \). Let \( T \) be the largest (strictly full, triangulated) subcategory of \( \mathcal{D}(Y) \) on which the unit of the adjunction from Lemma 6 is a natural isomorphism \( \text{id}_T \cong F \). By assumption we have \( \mathcal{O}_Y \in T \). Using either the \( Y \)-linearity from Remark 7 or a direct computation as in [15, Lemma 3.1] we deduce that \( \text{Perf}(Y) \subseteq T \). Since \( F \) and \( \text{id}_{\mathcal{D}(Y)} \) commute with direct sums, \( T \) is closed under direct sums. And the smallest triangulated subcategory of \( \mathcal{D}(Y) \) closed under direct sums and containing \( \text{Perf}(Y) \) is \( \mathcal{D}(Y) \) itself [14, Lemma 2.19], so \( T = \mathcal{D}(Y) \). This argument was communicated to the author by an anonymous referee.

Let \( f : X \to Y \) be a morphism of separated schemes of finite type over \( k \). Let \( \mathcal{E}_1, \ldots, \mathcal{E}_m \in \mathcal{D}(X) \) be \( f \)-exceptional objects such that

\[ R f_*(\mathcal{RHom}^*(\mathcal{E}_j, \mathcal{E}_i)) = 0 \quad (2) \]

for all \( i < j \) and let \( \mathcal{D}(Y) \boxtimes \mathcal{E} \) denote the essential image of the functor \( L f^*(-) \otimes L \mathcal{E} \). Then there is a semiorthogonal decomposition

\[ \mathcal{D}(X) = \langle \bigcap_{i=1}^m \text{Ker} (R f_*(\mathcal{RHom}^*(\mathcal{E}_j, -))) \rangle \bigcup \mathcal{D}(Y) \boxtimes \mathcal{E}_1, \ldots, \mathcal{D}(Y) \boxtimes \mathcal{E}_m \].

If in addition \( f \) is proper and has finite Tor-dimension, then the analogous statements hold for \( \text{Perf}(X) \) and \( \text{Perf}(Y) \) as well as for \( \mathcal{D}^b(X) \) and \( \mathcal{D}^b(Y) \).

**Proof** We check that \( \mathcal{D}(Y) \boxtimes \mathcal{E}_i \subseteq \text{Ker}(R f_*(\mathcal{RHom}^*(\mathcal{E}_j, -))) \) for all \( i < j \). After establishing this, Lemma 10 allows us to apply Lemma 5 inductively to obtain the desired result. To check the desired inclusion we follow the argument in [14, Lemma 2.7]. We want to show that

\[ R f_*(\mathcal{RHom}^*(\mathcal{E}_j, L f^*(\mathcal{F}) \otimes L \mathcal{E}_i)) = 0 \]

for all \( \mathcal{F}, \mathcal{G} \in \mathcal{D}(Y) \). It follows from the adjunction \( L f^* \dashv R f_* \) that it suffices to show that

\[ \text{Hom}_{\mathcal{D}(Y)}(\mathcal{F}, R f_*(\mathcal{RHom}^*(\mathcal{E}_j, L f^*(\mathcal{F}) \otimes L \mathcal{E}_i))) = 0 \]

for all \( \mathcal{F}, \mathcal{G} \in \mathcal{D}(Y) \). Since \( \mathcal{E}_j \) is a perfect object in \( \mathcal{D}(X) \), we have

\[ \mathcal{RHom}^*(\mathcal{E}_j, L f^*(\mathcal{F}) \otimes L \mathcal{E}_i) \cong \mathcal{E}_j^* \otimes L \mathcal{E}_i \otimes L f^*(\mathcal{F}) \cong R \mathcal{RHom}^*(\mathcal{E}_j, \mathcal{E}_i) \otimes L f^*(\mathcal{F}) \]

for all \( \mathcal{F}, \mathcal{G} \in \mathcal{D}(Y) \), see [23, Tag 08DQ]. Using again the adjunction \( L f^* \dashv R f_* \) we further reduce our problem to showing that

\[ \text{Hom}_{\mathcal{D}(Y)}(\mathcal{F}, R f_*(\mathcal{RHom}^*(\mathcal{E}_j, \mathcal{E}_i) \otimes L f^*(\mathcal{F}))) = 0 \]

\[ \text{Hom}_{\mathcal{D}(Y)}(\mathcal{F}, R f_*(\mathcal{RHom}^*(\mathcal{E}_j, \mathcal{E}_i) \otimes L f^*(\mathcal{F}))) = 0 \]

\[ \text{Hom}_{\mathcal{D}(Y)}(\mathcal{F}, R f_*(\mathcal{RHom}^*(\mathcal{E}_j, \mathcal{E}_i) \otimes L f^*(\mathcal{F}))) = 0 \]
for all $\mathcal{F}, \mathcal{G} \in D(Y)$. Finally, the projection formula [16, Proposition 3.9.4] reduces our problem to showing that

$$\text{Hom}_{D(Y)}(\mathcal{G}, Rf_*(R\mathcal{H}om^+(\mathcal{E}_j, \mathcal{E}_i)) \otimes^L \mathcal{F}) = 0$$

for all $\mathcal{F}, \mathcal{G} \in D(Y)$, which follows then from Eq. (2) in the assumptions of the lemma. □

### 3 Proof of theorem 3

Recall that we are working over a field $k$ of characteristic zero but not necessarily algebraically closed. Let us make some remarks concerning Definition 1 and Definition 2 before proving Theorem 3.

**Remark 12** (klt singularities) As mentioned briefly in the introduction, if we assume that $(X, 0)$ has klt singularities, then we are implicitly assuming that $X$ is normal and $\mathbb{Q}$-Gorenstein. Moreover, this also implies that $X$ has rational singularities, see [10, Proposition 2.41] and [10, Theorem 5.22]. Note also that $(X, 0)$ has klt singularities as soon as there exists some effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\Delta$ on $X$ such that $(X, \Delta)$ has klt singularities, see [10, Corollary 2.35].

**Remark 13** (Fano fibrations) Fano fibrations are a slight generalisation of Mori fibre spaces, in which we drop the condition on the relative Picard rank. It follows from conditions (1) to (3) in Definition 1 that $f$ is proper surjective with geometrically connected fibres of dimension at least 1 and that $r > 0$. The fibres of $f$ are proper over $k$ and $-K_X$ restricts to an ample divisor on each of them, so the fibres of $f$ are also projective over $k$. Moreover, since $k$ is a field of characteristic zero, it contains the field $\mathbb{Q}$ of rational numbers, so the $k$-schemes $X$ and $Y$ are $\mathbb{Q}$-schemes as well. We can therefore apply [4, Theorem 3.3.15] to deduce the existence of a dense open subset in $Y$ over which the fibres are normal, hence geometrically integral projective schemes over $k$ [7, Exercise 6.20].

**Remark 14** (Relative index) If $X$ is smooth, $k$ is algebraically closed and $Y = \text{Spec } k$ is a point, then the relative index is the usual index of a Fano variety, because numerical and linear equivalence agree on smooth projective Fano varieties [9, Proposition 2.1.2].

### Proof of theorem 3

We want to apply Lemma 11 to the line bundles in the statement of Theorem 3. Let us first check that any line bundle $\mathcal{M}$ on $X$ is a relative exceptional object. By Kawamata–Viehweg vanishing [11, Theorem 2.17.3] and the definition of Fano fibration we have a canonical isomorphism $\mathcal{O}_Y \cong Rf_*\mathcal{O}_X$. Hence the canonical isomorphisms $\mathcal{O}_Y \cong Rf_*\mathcal{O}_X \cong Rf_*\mathcal{H}om(\mathcal{M}, \mathcal{M})$ show that $\mathcal{M}$ is relative exceptional.

It remains to show the semiorthogonality condition expressed in Eq. (2). Let $0 \leq i < j \leq \lceil r - 1 \rceil$ and let $\mathcal{E}$ denote the line bundle $\mathcal{L}^{\otimes i-j}$. We want to show that

$$Rf_*\mathcal{H}om(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes i}) \cong Rf_*\mathcal{E} = 0$$

in $D(Y)$, or equivalently, that $R^p f_*\mathcal{E} = 0$ for all $p \in \mathbb{Z}$.

In degrees $p < 0$ this is true because $\mathcal{E}$ is a complex concentrated in degree zero. In degrees $p > 0$, we can again apply Kawamata–Viehweg vanishing [11, Theorem 2.17.3] to
deduce that $R^pf_*E = 0$, because $r + (i - j) \geq r - \lceil r - 1 \rceil > 0$, so $(i - j)H - K_X$ is $f$-ample.

To show the vanishing in degree $p = 0$, consider a dense open subset $V \subseteq Y$ over which $f$ is flat with normal fibres, which exists by [6, Theorem 6.9.1] and [4, Theorem 3.3.15]. Set $U := f^{-1}(V)$. Then $E|_U$ is flat over $V$ by flatness of $f|_U$. The fibres $U_y$ are positive dimensional geometrically integral projective schemes over $k$, and the restrictions $E|_{U_y}$ are antiample line bundles. Therefore $H^0(U_y, E|_{U_y}) = 0$ for all points $y \in V$ [8, Exercise III.7.1], because we may check this vanishing after base change to the algebraic closure of $k$. Applying Grauert’s theorem [8, Corollary III.12.9] we deduce that

$$(f|_U)_*(E|_U) = (f_*E)|_V = 0.$$ 

We find that every section of $f_*E$ is torsion, meaning that it maps to zero in the stalk at the generic point of $Y$. But $f_*E$ is torsion-free [5, Chapter I, Proposition 7.4.5], so we conclude that $f_*E = R^0f_*E = 0$.

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