On Quasi-Modular Forms, Almost Holomorphic Modular Forms, and the Vector-Valued Modular Forms of Shimura

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May 7, 2014

Introduction

Modular forms have been the subject of extensive research for a very long time. Throughout this time, many generalizations of the classical notion were defined. Two such notions which we consider in this paper are vector-valued modular forms with representations, and quasi-modular forms, in which the transformation under the action of the Fuchsian group involves more functions. Holomorphic quasi-modular forms were defined in [KZ] and their relation to classical holomorphic modular forms was also considered in that reference, as well as in [MR], [A], and other references. The first aim of this paper is to extend this relation to more general modular forms and quasi-modular forms.

One type of vector-valued modular forms with representations arises from the symmetric powers of the natural action of $SL_2(\mathbb{R})$ on $\mathbb{C}^2$. The $m$th symmetric power of this representation is denoted $V_m$. Modular forms with representation $V_m$ were defined by Shimura in [Sh] following an idea of Eichler (see also [E]), and a structure theorem for cusp forms of weights 0 and 2 and representation $V_m$ for even $m$ was established by Kuga and Shimura in [KS]. The main aim of this paper is to relate (holomorphic) quasi-modular forms to holomorphic modular forms with the representation $V_m$. In fact, we obtain this result in a more general setting. Using the tools developed in order to achieve this task, we also prove a general structure theorem for modular forms with representations (or multiplier systems) involving $V_m$.

In Section 1 we present the notions of modular forms and quasi-modular forms, and study the relation between them under very general assumptions. In Section 2 we introduce modular forms with representation $V_m$, obtain the connection between all three notions, and prove the structure theorem.

*The initial stage of this research has been carried out as part of my Ph.D. thesis work at the Hebrew University of Jerusalem, Israel. The final stage of this work was carried out at TU Darmstadt and supported by the Minerva Fellowship (Max-Planck-Gesellschaft).
# 1 Modular Forms and Quasi-Modular Forms

In this Section we present the notions of modular and quasi-modular forms, and give the relations between them in a general setting.

## 1.1 Definitions and Notation

Let $H = \{ \tau \in \mathbb{C} | \Im \tau > 0 \}$ be the Poincaré upper half-plane. We shall always write $\tau = x + iy$, hence $x = \Re \tau$ and $y = \Im \tau$. The group $SL_2(\mathbb{R})$ acts on $H = \{ \tau \in \mathbb{C} | \Im \tau > 0 \}$ by Möbius transformations:

$$\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) : \tau \mapsto \frac{a \tau + b}{c \tau + d} \quad (1)$$

The measure $d\mu = \frac{dx dy}{y^2}$ is invariant under the action of $SL_2(\mathbb{R})$. A discrete subgroup $\Gamma \leq SL_2(\mathbb{R})$ is called Fuchsian if the volume of a fundamental domain (with respect to $d\mu$) is finite. A subgroup of trace $\pm 2$ elements of $\Gamma$ is called parabolic and corresponds to a cusp of $\Gamma$, i.e., to the element of $\mathbb{P}^1(\mathbb{R})$ which is stabilized by this subgroup. $\Gamma$ acts on its cusps via (extended) Möbius transformations (which correspond to conjugation on the parabolic subgroups), and the number of orbits of cusps is always finite. The quotient space $Y(\Gamma) = \Gamma \backslash H$ is a Riemann surface, which becomes a compact Riemann surface (or algebraic curve) denoted $X(\Gamma)$ after adding the (finitely many) equivalence classes of cusps of $\Gamma$.

The action of $SL_2(\mathbb{R})$ on $H$ admits a factor of automorphy, which is defined, for any elements $\gamma \in SL_2(\mathbb{R})$ and $\tau \in H$ as in Equation (1), by

$$j(\gamma, \tau) = c \tau + d \quad (2)$$

The factor of automorphy is a cocycle: The equality

$$j(\gamma \delta, \tau) = j(\gamma, \delta \tau) j(\delta, \tau) \quad (3)$$

holds for any $\tau \in H$ and $\gamma$ and $\delta$ in $SL_2(\mathbb{R})$. We shall sometimes use the alternative notation $j_{\gamma}(\tau)$ for $j(\gamma, \tau)$. The entry $c$ of the matrix $\gamma$ from Equation (1) can be written as the derivative $j'_{\gamma}(\tau)$ of the linear function $j_{\gamma}$ from Equation (2). Since it is a constant (independent of $\tau$), we omit the variable and write simply $j'_{\gamma}$.

Let $k$ and $l$ be integers, let $\Gamma \leq SL_2(\mathbb{R})$ be a Fuchsian group, and let $\rho$ be a representation of $\Gamma$ on some (finite-dimensional) complex vector space $V_{\rho}$. Throughout this paper, when a representation $\rho$ of some group $\Gamma$ is considered, $V_{\rho}$ denotes the representation space of $\rho$. A **modular form of weight** $(k, l)$ and **representation** $\rho$ with respect to $\Gamma$ is a real-analytic function $f : H \to \mathbb{C}$ satisfying the functional equation

$$f(\gamma \tau) = j(\gamma, \tau)^k j(\gamma, \tau)^l \rho(\gamma) f(\tau) \quad (4)$$

$$f(\gamma \tau) = j(\gamma, \tau)^k j(\gamma, \tau)^l \rho(\gamma) f(\tau) \quad (4)$$
for any $\gamma \in \Gamma$. A modular form of weight $(k,0)$ is said to have weight $k$. By considering subgroups of the metaplectic double cover $Mp_2(\mathbb{R})$ of $SL_2(\mathbb{R})$ (one realization of which consists of pairs of an element $\gamma \in SL_2(\mathbb{R})$ and a choice of a square root $\sqrt{j(\gamma, \tau)}$ of the automorphy factor from Equation (2)), one can consider modular forms in which the weights $k$ and $l$ are half-integral. Furthermore, we can consider the case in which the weights $k$ and $l$ are arbitrary real (and even complex) by allowing $\rho$ to be a multiplier system of weight $(k,l)$. We recall that such a multiplier system is a function $\rho: \Gamma \to \mathbb{C}$ (or, more generally, $\rho: \Gamma \to GL(V_\rho)$), which satisfies the condition

$$j^k(\gamma \delta)(\tau) \rho(\gamma) = j^k(\gamma)(\delta \tau) j^k(\delta)(\tau) \rho(\delta)$$

for any $\gamma$ and $\delta$ in $\Gamma$ and $\tau \in \mathcal{H}$ (with the appropriate choice of powers, or equivalently logarithms, of the automorphy factor functions $j_\gamma$ for $\gamma \in \Gamma$). Note that we do not require the image of $\rho$ to be unitary, hence the 1-dimensional case of our definition covers also the case of “generalized modular forms” in the sense of [KM] (but extended to arbitrary Fuchsian groups). Multiplication by $y^t$ takes a modular form of weight $(k + t, l + t)$ (and any representation or multiplier system) to a modular form of weight $(k, l)$ (and the same representation or multiplier system). It follows that using this operation we can always consider only modular forms of “holomorphic” weights. We denote the space of (real-analytic) modular forms of weight $(k, l)$ and representation (or multiplier system) $\rho$ by $M^\text{an}_{k,l}(\rho)$, and write $M^\text{an}_{k}(\rho)$ for $M^\text{an}_{k,0}(\rho)$. In some cases it will be useful to allow singularities on $\mathcal{H}$, so that the space $M^\text{sing}_{k,l}(\rho)$ (and $M^\text{sing}_{k}(\rho)$) consists of those modular forms which are real-analytic on $\mathcal{H}$ except in a discrete (\Gamma-invariant) set of points. The notation $M^\text{hol}_{k}(\rho)$ and $M^\text{morr}_{k}(\rho)$ stands for the spaces consisting of modular forms which are holomorphic (resp. meromorphic) on $\mathcal{H}$. If $\Gamma$ has cusps, then elements of $M^\text{hol}_{k}(\rho)$ are required to be holomorphic also at the cusps. In this case we denote the space of cusp forms (i.e., those elements of $M^\text{hol}_{k}(\rho)$ which vanish at the cusp) by $M^\text{cusp}_{k}(\rho)$. The space of meromorphic modular forms having all their poles at the cusps (these forms are called weakly holomorphic) is denoted $M^\text{wh}_{k}(\rho)$. For integral $k$ (and $l$), replacing $\rho$ by $\Gamma$ in all these notations denotes the corresponding spaces of modular forms with trivial (1-dimensional) representation.

Another notion, namely that of a quasi-modular form, was defined first in [KZ] and considered, among others, in [A] and Section 3 of [MR]. These references restricted attention only to holomorphic (or meromorphic) functions. Here we introduce a more general class of functions, by considering real-analytic functions and allowing representations and multiplier systems. Let $k$ be a real (or even complex) number, let $d$ be a non-negative integer, let $\Gamma$ be a Fuchsian group, and let $\rho: \Gamma \to GL(V_\rho)$ be a multiplier system of weight $k$. A real-analytic function $f: \mathcal{H} \to V_\rho$ is quasi-modular form of weight $k$, depth $d$, and multiplier system $\rho$ with respect to $\Gamma$ if there exist (real-analytic) functions
\( f_r : \mathcal{H} \to V_\rho, \ 0 \leq r \leq d \) such that

\[
f(\gamma \tau) = \sum_{r=0}^{d} j(\gamma, \tau)^{k-r}(j_\gamma')^r \rho(\gamma) f_r(\tau)
\]

(5)

for any \( \gamma \in \Gamma \). The functions \( f_r \) are independent of \( \gamma \), and we assume that \( f_d \neq 0 \) (otherwise the depth of \( f \) is smaller than \( d \)). By taking \( \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) in Equation (5) we obtain (under some simple normalization assumptions on \( \rho \) and the powers of \( j(\gamma, \tau) \) if \( k \) is not integral) that \( f_0 = f \). We denote the space of quasi-modular forms (with the various differential properties and growth conditions considered above) of weight \( k \) and representation \( \rho \) with respect to \( \Gamma \) by \( \widetilde{M}_k^*(\rho) \) (with the symbol \( * \) standing for one of the superscripts \( \text{an}, \text{sing}, \text{hol}, \text{mer}, \text{cusp}, \) or \( \text{wh} \)), and \( \widetilde{M}_k^{\ast \leq d}(\rho) \) denoting those quasi-modular forms whose depth does not exceed \( d \).

If \( \Gamma \) has cusps then certain assumptions on \( \rho \) allow one to consider Fourier expansions of modular and quasi-modular forms (as well as of the functions \( f_r \) in the latter case) around the cusps of \( \Gamma \). We shall not use these expansions in this paper.

The tensor product of \( f \in M_k^*(\rho) \) and \( g \in M_k^*(\eta) \), for \( \rho \) and \( \eta \) appropriate multiplier systems of the same Fuchsian group \( \Gamma \), lies in \( M_k^{* \oplus}(\rho \otimes \eta) \). For \( * = \text{an} \) or \( * = \text{sing} \) the same statement applies for modular forms with both holomorphic and anti-holomorphic weights. Similarly, let \( f \) and \( g \) be elements in \( M_k^{* \leq m}(\rho) \) and \( M_k^{* \leq n}(\eta) \) with corresponding functions \( f_r \) for \( 0 \leq r \leq m \) and \( g_s \) for \( 0 \leq s \leq n \) respectively. Then, the tensor product \( h = f \otimes g \) lies in \( M_k^{* \leq m+n}(\rho \otimes \eta) \), with the corresponding functions \( h_t \), \( 0 \leq t \leq m + n \) being \( \sum_{r+s=t} f_r \otimes g_s \). Furthermore, the (tensor) product of almost holomorphic functions (namely, polynomials in \( \frac{1}{2iy} \) over the ring of holomorphic functions on \( \mathcal{H} \)) of depths (i.e., degrees as such polynomials) at most \( m \) and \( n \) is almost holomorphic of depth not exceeding \( m + n \). We shall treat both the additive and multiplicative structures together in the main result of this paper.

1.2 Relations between Quasi-Modular Forms and Modular Forms

Quasi-modular forms relate to modular forms according to the property presented in the next Proposition. This result is stated in [KZ] and [A] and proved in [MR] for holomorphic quasi-modular forms of integral weights with trivial representation with respect to \( SL_2(\mathbb{Z}) \) (see Proposition 132 and Remarque 133 of [MR]). However, it holds in a more general context, an observation which will turn out useful later. We begin with a result which is equivalent to Lemma 119 of [MR]:

Lemma 1.1. Take \( f \in \widetilde{M}_k^{* \leq d}(\rho) \), with corresponding functions \( f_r \), \( 0 \leq r \leq d \). Then \( f_r \in \widetilde{M}_k^{* \leq d-r}(\rho) \), with the corresponding functions being \( \binom{d}{r} f_t \) for any \( r \leq t \leq d \).
The proof of Lemma 119 of [MR] used the quasi-modularity of $E_2$ with respect to $SL_2(\mathbb{Z})$. Since we work with a more general notion and the Fuchsian group $\Gamma$ is also arbitrary, we extend his proof, with some adjustments, to the present context.

**Proof.** Take $\tau \in \mathcal{H}$ and $\gamma$ and $\delta$ in $\Gamma$. Equation (5) with $\gamma \delta \in \Gamma$ and $\tau \in \mathcal{H}$ yields

$$f(\gamma \delta \tau) = \sum_{r=0}^{d} \rho(\gamma \delta)f_r(\tau)j_{\gamma \delta}^{k-r}(\tau)(j_{\gamma \delta}')^r.$$ 

On the other hand, we can apply Equation (5) with $\gamma \in \Gamma$ and $\delta \tau \in \mathcal{H}$, to obtain, using Equation (3) and its derivative, the equality

$$f(\gamma \delta \tau) = \sum_{r=0}^{d} \sum_{s=r}^{d} \left( \sum_{r}^{s} \rho(\gamma)f_s(\delta \tau)j_{\delta}^{s+r-k}(\tau)(-j_{\delta}')^{s-r} \right) j_{\gamma \delta}^{k-r}(\tau)(j_{\gamma \delta}')^r.$$ 

Applying $\rho(\gamma)^{-1}$, the fact that both equalities hold for every $\gamma \in \Gamma$ implies

$$\rho(\delta)f_r(\tau) = \sum_{s=r}^{d} \sum_{t=r}^{s} \left( \sum_{r}^{t} \rho(\delta)f_t(\tau)j_{\delta}^{t-s}(\tau)j_{\delta}'^{t-s} \right) j_{\gamma \delta}^{k-r}(\tau)(j_{\gamma \delta}')^r.$$ 

for every $0 \leq r \leq d$.

We now use Equation (6) in order to prove by decreasing induction on $r$ the assertion of the lemma, namely

$$f_r(\delta \tau) = \sum_{t=r}^{d} \left( \sum_{r}^{t} \rho(\delta)f_t(\tau)j_{\delta}^{t-r}(\tau)j_{\delta}'^{t-r} \right)$$

for every $0 \leq r \leq d$, $\tau \in \mathcal{H}$, and $\delta \in \Gamma$. Substituting $r = d$ in Equation (6) gives the first step of the induction. Assuming that the assertion holds for any $r < s \leq d$ we find that the left hand side of Equation (6) with the index $r$ is

$$f_r(\delta \tau)j_{\delta}^{2r-k}(\tau) + \sum_{s=r+1}^{d} \sum_{t=r}^{d} \left( \sum_{r}^{t} \rho(\delta)f_t(\tau)j_{\delta}^{k-t-s}(\tau)j_{\delta}'^{t-s}j_{\delta}^{s+r-k}(\tau)(-j_{\delta}')^{s-r} \right) =$$

$$f_r(\delta \tau)j_{\delta}^{2r-k}(\tau) + \sum_{t=r+1}^{d} \left( \sum_{r}^{t} \rho(\delta)f_t(\tau)j_{\delta}^{t-r}(\tau)(j_{\delta}')^{t-r} \sum_{s=r+1}^{t} \left( \sum_{s=r+1}^{t} \right) (-1)^{s-r} \right).$$

The sum over $s$ equals $-1$ by the binomial theorem. Comparing this expression with the left hand side of Equation (6) yields the assertion for $r$. Since the differential and growth properties of the functions $f_r$ follow directly from those of $f$, this proves the lemma.

The relation between quasi-modular and modular forms is given in the following generalization of Proposition 132 and Remarque 133 of [MR] and of Theorem 1 of [A].
Proposition 1.2. Let \( f \in \tilde{M}_k^{an, \leq d}(\rho) \) with the corresponding functions \( f_r \), \( 0 \leq r \leq d \), and define the function \( F(\tau) = \sum_{r=0}^{d} \frac{f_r(\tau)}{(2iy)^r} \). Then \( F \in M_k^{an}(\rho) \).

Conversely, let \( F_s \in M_k^{an,-2s}(\rho) \) be given for every \( 0 \leq s \leq d \). In this case the function \( f(\tau) = \sum_{s=r}^{d} \binom{s}{r} \frac{F_s(\tau)}{(2iy)^s} \), with \( 0 \leq r \leq d \) (hence the depth of \( f \) is precisely \( d \) if and only if \( F_d \neq 0 \)).

Proof. Recall that the equality \( \Im \gamma \tau = \frac{y}{|\Im(\gamma \tau)|^2} \) implies
\[
\frac{1}{2i\Im \gamma \tau} = \frac{j(\gamma, \tau)^2}{2iy} - j(\gamma, \tau)j',
\]
Lemma 11 and Equation (7) allow one to write \( F(\gamma \tau) = \sum_{r=0}^{d} \frac{f_r(\gamma \tau)}{(2iy)^r} \) as
\[
\sum_{r=0}^{d} \sum_{t=r}^{d} \binom{l}{t} \rho(\gamma)f_t(\tau)j_{k-r-l}(\tau)(j'_r)^{l-r} \sum_{h=0}^{r} \binom{r}{h} \frac{j_{2h}^{r}}{(2iy)^h} (-1)^{r-h} j_{l-h}^{r-h}(\tau)(j'_r)^{r-h} = \sum_{0 \leq h \leq l \leq d} \binom{l}{h} \rho(\gamma)f_h(\tau)j(\gamma, \tau)^{k-h} (j'_r)^{l-h} \sum_{r=h}^{l} \binom{l-h}{r} (-1)^{r-h}.
\]
Since the sum over \( r \) is 1 if \( t = h \) and 0 otherwise, this expression for \( F(\gamma \tau) \) reduces to \( \sum_{d=0}^{d} \frac{a(2iy)^r}{(2iy)^h} j(\gamma, \tau)^{k} = j(\gamma, \tau)^{k} \rho(\gamma) F(\tau) \). This proves the first assertion. For the second assertion, write \( f(\gamma \tau) = \sum_{s=0}^{d} \frac{F_s(\gamma \tau)}{(2iy)^s} \). The modularity of the functions \( F_s \) and Equation (7) now show that this expression equals
\[
\sum_{s=0}^{d} \rho(\gamma)F_s(\tau)j_{s}^{k-2s}(\tau) \sum_{r=0}^{s} \binom{s}{r} j_{r}(\tau)(j'_r)^{s-r} \frac{2(s-r)}{(2iy)^{s-r}},
\]
which reduces to the desired expression by the definition of the functions \( f_r \).

This proves the proposition. \( \square \)

The two maps in Proposition 1.2 are invertible to one another in the following sense. Given \( f \in \tilde{M}_k^{an, \leq d}(\rho) \) hence \( f_r \in M_k^{an, \leq d-r}(\rho) \) for each \( 0 \leq r \leq d \) (by Lemma 11), Proposition 1.2 constructs the functions \( F_r \in M_k^{an,-2r}(\rho) \) for every such \( r \). Applying the inverse construction from Proposition 1.2 to the functions \( F_r \) yields back the original quasi-modular form \( f \). Conversely, assume that \( f \in \tilde{M}_k^{an, \leq d}(\rho) \) is obtained by Proposition 1.2 from \( F_s \in M_k^{an,-2s}(\rho) \), \( 0 \leq s \leq d \), and let \( f_r \), \( 0 \leq r \leq d \) be the corresponding functions from Equation (7). In this case the modular form in \( M_k^{an,-2r}(\rho) \) constructed from \( f_r \in M_k^{an, \leq d-r}(\rho) \) is the original modular form \( f_r \) for each \( 0 \leq r \leq d \). Hence proposition 1.2 gives, for every weight \( k \), depth bound \( d \), group \( \Gamma \), and representation or multiplier system \( \rho \), an isomorphism
\[
\tilde{M}_k^{an, \leq d}(\rho) \cong \bigoplus_{s=0}^{d} M_k^{an,-2s}(\rho), \tag{8}
\]
which also preserves most reasonable growth conditions (boundedness, linear exponential growth, exponential decay, etc.) at the cusps of $\Gamma$ (if $\Gamma$ has cusps). Proposition 1.2 and Equation (18) hold also when replacing the superscript $an$ by $sing$ throughout. Restricting our attention to an element $f$ in the subspace $M_{k, \leq d}^{hol}(\rho)$ of the isomorphism in Equation (18), the elements $F_s$ in the right hand side of that Equation (and in particular $F = F_0$ from Proposition 1.2) are almost holomorphic functions on $H$. On the other hand, if $F \in M_{k, \leq d}^{hol}(\rho)$ is almost holomorphic, we can write $F(\tau) = \sum_{r=0}^{d} \frac{f_r(\tau)}{(2\pi i)^r}$ with $f_r$, $0 \leq r \leq d$ holomorphic functions. Modularity implies $F(\gamma \tau) = \rho(\gamma)j(\gamma, \tau)kF(\gamma)$, and comparing the polynomial expansion in $\frac{1}{2\pi iy}$ over holomorphic functions on both sides yields Equation (18) for the functions $f_r$. Then the proof of Lemma 1.1 and Proposition 1.2 imply that $F_s(\gamma \tau) = \sum_{r=0}^{d} \frac{f_r(\gamma \tau)}{(2\pi i)^r}$ is in $M_{k, \leq 2d}^{hol}(\rho)$ (and almost holomorphic) for each $0 \leq r \leq d$, $F_0 = F$, and the function $f_0 : \tau \mapsto \sum_{r=0}^{d} \frac{F_r(\tau)}{(2\pi i)^r}$ is in $\tilde{M}_{k, \leq d}^{hol}(\rho)$. Hence this construction reproduces (and generalizes) the isomorphism given in Proposition 132 and Remarque 133 of [MR] or in Theorem 1 of [A].

We remark that the action of the $\mathfrak{sl}_2$-triple presented in Section 3 of [A] is a direct consequence of the holomorphic case of Proposition 1.2 (with trivial representation) and the action of $\mathfrak{sl}_2(\mathbb{C})$ on modular forms considered as functions on $SL_2(\mathbb{R})$ with the appropriate behavior under the action of the maximal compact subgroup $O(2)$ of $SL_2(\mathbb{R})$. To see this, observe that [Ve] shows that the elements $W = \begin{pmatrix} 1 & -1 \\ i & -i \end{pmatrix}$, $Z = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$, and $Z^\ast = \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$ of $\mathfrak{sl}_2(\mathbb{C})$ form an $\mathfrak{sl}_2$-triple, and they act on modular forms of weight $k$ as multiplication by $k$, the weight raising operator $2i\delta_k = 2i\frac{\partial}{\partial \tau} + \frac{k}{2\pi iy}$, and the weight lowering operator $\frac{\partial}{\partial \tau} = -2iy^2 \frac{\partial}{\partial \tau}$, respectively. Now, if $F$ arises from $f \in \tilde{M}_k^{hol}(\Gamma)$ as in Proposition 1.2 then $\delta_k F$ has “constant coefficient” $f'$ as a polynomial in $\frac{1}{2\pi iy}$ (see, for example, Proposition 135 of [MR]). On the other hand, $LF(\tau) = \sum_{r=0}^{d-1} \frac{(r+1)f_{r+1}(\tau)}{(2\pi i)^r}$ has “constant term” $f_1$. Thus, the operators acting on $\tilde{M}_k^{hol}(\Gamma)$ which are considered in [A], namely $H$ (multiplication by the weight), $D = \frac{\partial}{\partial \tau}$, and $\delta : f \mapsto f_1$, correspond to the $\mathfrak{sl}_2$-triple $W$, $Z$, and $2iZ$, respectively.

2 Modular Forms with the Representation $V_m$

In this Section we present modular forms with multiplier systems involving $V_m$, establish the relation between these modular forms and quasi-modular forms, and determine the space of such modular forms which are holomorphic.

2.1 Quasi-Modular Forms as Modular Forms with Multiplier Systems Involving $V_m$

The group $SL_2(\mathbb{R})$ (and, more generally, $GL_+^2(\mathbb{R})$) acts naturally on the space $\mathbb{C}^2$ of complex row vectors of length 2, by $\gamma : u \mapsto \gamma u$. We denote this representation space $V_1$, and let $V_m$ be its $m$th symmetric power. We denote the action of
\( \gamma \in SL_2(\mathbb{R}) \) on an element \( u \in V_m \) by \( \gamma^m u \) (this notation should lead to no confusion with the \( m \)th power of \( \gamma \) in \( \Gamma \)). We shall also use multiplicative notation for elements in \( V_m \): If \( v_i, 1 \leq i \leq m \), are vectors in \( \mathbb{C}^2 \), we denote the image of \( \gamma \) by \( \gamma \). Hence the map \( m \) implies that each of the terms in Equation (10) lies in \( V_m \). Since \( V_m \) is a representation of the full group \( SL_2(\mathbb{R}) \), we preserve \( \Gamma \) in the notation \( \rho \) for the space \( V \) for the space \( V \) valued modular forms. Since \( V \) is a representation of the full group \( SL_2(\mathbb{R}) \), we preserve \( \Gamma \) in the notation \( \rho \) for the space \( V \) valued modular forms. Since \( V \) is a valuation of \( m \) and \( \gamma \), the fact that the elements \( (\gamma)_{\Gamma} \) and \( (\gamma)_{\Gamma} \) define an element of \( \mathcal{M}_{m+, m-}^r(\Gamma, V_m) \), and for \( m+ = m \) and \( m- = 0 \) we can replace the superscript \( an \) by \( hol \). This simple observation allows us to characterize modular forms with representations involving \( V_m \), which will be needed later on.

We start with the classical equation, stating that for every \( \gamma \in SL_2(\mathbb{R}) \) and \( \tau \in \mathcal{H} \), the equality

\[
\gamma \left( \begin{array}{c} \tau \\ 1 \end{array} \right) = j(\gamma, \tau) \left( \begin{array}{c} \gamma \tau \\ 1 \end{array} \right)
\]

holds. Hence the map \( \tau \mapsto \left( \begin{array}{c} \tau \\ 1 \end{array} \right) \) is in \( \mathcal{M}_{m+}^{hol}(\Gamma, V_1) \) for any discrete subgroup \( \Gamma \) of \( SL_2(\mathbb{R}) \). Thus, \( \tau \mapsto \left( \begin{array}{c} \tau \\ 1 \end{array} \right) \) lies in \( \mathcal{M}_{m+}^{hol}(\Gamma, V_1) \) for any such \( \Gamma \). It follows that for any pair of numbers \( m_+ \) and \( m_- \) such that \( m_+ + m_- = m \), the map \( \tau \mapsto \left( \begin{array}{c} \tau \\ 1 \end{array} \right)^{m_+} \left( \begin{array}{c} \tau \\ 1 \end{array} \right)^{m_-} \) defines an element of \( \mathcal{M}_{m_+, m_-}^{m}(\Gamma, V_m) \), and for \( m_+ = m \) and \( m_- = 0 \) we can replace the superscript \( m_+ \) by \( hol \). This simple observation allows us to characterize modular forms with representations involving \( V_m \).

**Proposition 2.1.** Let \( \Gamma \) be a Fuchsian group, let \( k \) and \( l \) be weights, and let \( \rho \) be a multiplier system of weight \((k, l)\) for \( \Gamma \). If \( f_{m_+, m_-} \in \mathcal{M}_{m_+, m_-}^{k, l}(\Gamma, V_m) \) for every pair of non-negative integers \( m_+ \) and \( m_- \) such that \( m_+ + m_- = m \) then the \( V_m \otimes V_\rho \)-valued function

\[
F(\tau) = \sum_{m_+ + m_- = m} f_{m_+, m_-}(\tau) \left( \begin{array}{c} \tau \\ 1 \end{array} \right)^{m_+} \left( \begin{array}{c} \tau \\ 1 \end{array} \right)^{m_-}
\]

lies in \( \mathcal{M}_{k,l}^{m}(V_m \otimes \rho) \). Conversely, every element of \( \mathcal{M}_{k,l}^{m}(V_m \otimes \rho) \) is obtained in this way.

**Proof.** The properties of \( f_{m_+, m_-} \) and the weights of the vectors \( \left( \begin{array}{c} \gamma \end{array} \right)^{m_+} \left( \begin{array}{c} \gamma \end{array} \right)^{m_-} \) imply that each of the terms in Equation (10) lies in \( \mathcal{M}_{k,l}^{m}(V_m \otimes \rho) \). Hence \( F \in \mathcal{M}_{k,l}^{m}(V_m \otimes \rho) \) as well. On the other hand, for any \( V_m \otimes V_\rho \)-valued function \( F \), the fact that the elements \( \left( \begin{array}{c} \gamma \end{array} \right)^{m_+} \left( \begin{array}{c} \gamma \end{array} \right)^{m_-} \) with \( m_+ + m_- = m \) form a basis for the space \( V_m \) allows us to write \( F(\tau) \) as in Equation (10), with the function \( f_{m_+, m_-} \) taking values in \( V_\rho \) for any pair \( (m_+, m_-) \). Let now \( \tau \in \mathcal{H} \) and \( \gamma \in \Gamma \) be given, and write

\[
F(\gamma \tau) = \sum_{m_+ + m_- = m} f_{m_+, m_-}(\gamma \tau) \left( \begin{array}{c} \gamma \tau \\ 1 \end{array} \right)^{m_+} \left( \begin{array}{c} \gamma \tau \\ 1 \end{array} \right)^{m_-}.
\]
On the other hand, under the assumption $F \in \mathcal{M}_{k,l}^{an}(V_m \otimes \rho)$ we must have $F(\gamma \tau) = j(\gamma, \tau)^{ik} j(\gamma, \tau)^l \left( \gamma^m \otimes \rho(\gamma) \right) F(\tau)$, and Equation (9) implies that $F(\gamma \tau)$ equals
\[
j(\gamma, \tau)^{ik} j(\gamma, \tau)^l \sum_{m_+ + m_- = m} \rho(\gamma) f^{m_+ - m_-}(\gamma, \tau)j(\gamma, \tau)^{m_+} j(\gamma, \tau)^{m_-} \left( \gamma^m \right)^{m_+} \left( \gamma^m \right)^{m_-}.
\]

As the two expressions for $F(\gamma \tau)$ are presented in the same basis for $V_m$, comparing the coefficients in front of $\left( \gamma^m \right)^{m_+} \left( \gamma^m \right)^{m_-}$ implies that $f^{m_+ - m_-}$ is in $\mathcal{M}_{k+m_-, l+m_-}^{an}(\rho)$ for any pair $(m_+, m_-)$. This proves the proposition. \[\square\]

For any $m$ and $n$ there corresponds a natural map $V_m \otimes V_n \to V_{m+n}$, which takes $\prod_{i=1}^m u_i \otimes \prod_{j=1}^n v_j$ to $\prod_{i=1}^m u_i \cdot \prod_{j=1}^n v_j$ (in the multiplicative notation). This map is obtained as the projection from $V_m \otimes V_n \cong \bigoplus_{k=0}^{\min\{m,n\}} V_{m+n-k}$ onto the component $V_{m+n}$ of maximal dimension, or onto the $S_{m+n}$-invariant part, modulo the action of $S_{m+n}$. Therefore, we consider the (tensor) product of $f \in \mathcal{M}_{k,l}^{an}(V_m \otimes \rho)$ with $g \in \mathcal{M}_{r,s}^{an}(V_m \otimes \eta)$ as an element of $\mathcal{M}_{k+r,l+s}^{an}(V_{m+n} \otimes \rho \otimes \eta)$ (with the same convention for product of elements from $\mathcal{M}_k(V_m \otimes \rho)$ etc.). This way of treating products illustrates our way of viewing the multiplicative structure of quasi-modular forms below in a clearer manner. This multiplicative notation also allows us to obtain the following corollary of Proposition 2.1, which will turn out useful when we consider quasi-modular forms with arbitrary depth later.

**Corollary 2.2.** If $F \in \mathcal{M}_{k-m, l}^{an}(V_m \otimes \rho)$ then $i_m(F) = F \cdot \left( \gamma^m \right)^{m_+}$ is an element of $\mathcal{M}_{k-m-1,l}^{an}(V_{m+1} \otimes \rho)$. The map $i_m$ is injective. As for the image of $i_m$, an element $\tilde{F} \in \mathcal{M}_{k-m-1,l}^{an}(V_{m+1} \otimes \rho)$ can be expanded with respect to the basis $\left( \gamma^{m_+} \right)^{m_+} \left( \gamma^{m_-} \right)^{m_-}$ with $m_+ + m_- = m + 1$ (as in Equation (10)). Then $\tilde{F}$ lies in the image of $i_m$ if and only if the coefficient of $\left( \gamma^m \right)^{m_+ + 1}$ in this expansion vanishes.

**Proof.** Note that if $F$ is expanded as in Equation (10) then
\[
i_m(F)(\tau) = \sum_{m_+ + m_- = m} f^{m_+ - m_-}(\tau) \left( \gamma^m \right)^{m_+ + 1} \left( \gamma^m \right)^{m_-}.
\]
The modularity of $i_m(F)$ now follows from Proposition 2.1 (or from the fact that $F$ is multiplied by an element of $\mathcal{M}_{k-1,l}^{an}(\Gamma, V_1)$), and the injectivity of $i_m$ is clear. Now, images of $i_m$ have vanishing coefficient in front of $\left( \gamma^m \right)^{m_+ + 1}$ in this expansion. Conversely, let $\tilde{F} \in \mathcal{M}_{k-m-1,l}^{an}(V_{m+1} \otimes \rho)$, and expand $\tilde{F}$ as in Equation (10) (with coefficients $\tilde{f}^{m_+ - m_-}$ with $m_+ + m_- = m + 1$). Assuming that $\tilde{f}^{0,m+1} = 0$, we define $f^{m_+ - m_-} = \tilde{f}^{m_+ - m_-}$ for any pair $(m_+, m_-)$ with $m_+ + m_- = m$. Proposition 2.1 implies that $f^{m_+ - m_-} \in \mathcal{M}_{k-m-, l+m-}^{an}(\rho)$. Hence the function $F$ defined by Equation (10) with the same functions $f^{m_+ - m_-}$ lies in $\mathcal{M}_{k-m, l}^{an}(V_m \otimes \rho)$, and $\tilde{F} = i_m(F)$. This proves the corollary. \[\square\]
Once again, Proposition 2.1 and Corollary 2.2 extend to the case where the superscript $an$ is replaced by $sing$. Moreover, the holomorphicity of the function $\tau \mapsto \binom{\tau}{1}$ implies that for $l = 0$, Corollary 2.2 holds with $M_{k-n}^{an}(V_m \otimes \rho)$ replaced by any of the other spaces of the form $M_{k-2s}^{an}(\rho)$ defined above.

Assume $l = 0$ in Proposition 2.1. The action of powers of $y$ on modular forms imply that a modular form $F \in M_{k-2s}^{an}(\rho)$ can be written as

$$F(\tau) = \sum_{s=0}^{m} \frac{F_s(\tau)}{(-2iy)^s} \binom{\tau}{1}^{m-s} \binom{1}{1}^s,$$

where the function $F_s$ lies in $M_{k-2s}^{an}(\rho)$ for each $0 \leq s \leq m$. We can now state the first main result of this paper.

**Theorem 2.3.** If $f \in \tilde{M}_{k}^{an,\leq m}(\rho)$ with corresponding functions $f_r$, then the function

$$F(\tau) = \sum_{r} f_r(\tau) \binom{\tau}{1}^{m-r} \binom{1}{1}^r,$$

lies in $M_{k-n}^{an}(V_m \otimes \rho)$. Conversely, if $F \in M_{k-n}^{an}(V_m \otimes \rho)$, then we can expand it as in Equation (12), and the coefficient in front of $\binom{\tau}{1}$ is an element of $f \in \tilde{M}_{k}^{an,\leq m}(\rho)$. Moreover, every $F \in M_{k-n}^{an}(V_m \otimes \rho)$ admits an expansion as in Equation (11), and $F$ corresponds to the quasi-modular form $f$ if and only if the functions $F_s$ from Equation (11) and the functions $f_r$ from Equation (5) are mutually related as in Proposition 1.2.

**Proof.** By expanding the $r$th power of $\binom{\tau}{1} = \frac{1}{-2iy} \left[ \binom{\tau}{1} - \binom{1}{1} \right]$ in Equation (12), using the Binomial Theorem we obtain an expression for $F(\tau)$ as in Equation (11), with the function $F_s(\tau)$ being $\sum_{i=s}^{m} \binom{i}{s} f_i(\tau)$ for every $s$. According to Lemma 1.1 and Proposition 1.2, $F_s \in M_{k-n}^{an}(\rho)$ for every $s$, so that Proposition 2.1 and Equation (11) imply $F \in M_{k-n}^{an}(V_m \otimes \rho)$. Conversely, by writing $F$ as in Equation (11) and expanding the $s$th power of $\binom{\tau}{1} = \binom{\tau}{1} - 2iy \binom{1}{1}$ we obtain Equation (12) with $f_r(\tau) = \sum_{s=r}^{m} \binom{s}{r} F_s(\tau) (-2iy)^{r-s}$ and $f_0 \in \tilde{M}_{k}^{an,\leq m}(\rho)$ by Proposition 1.2. It also follows from the proof of the preceding assertions that the relations between the coefficients $F_s$ in Equation (11) and the functions $f_r$ in Equation (12) in the expansions of the same modular form $F$ agree with the relations stated in Proposition 1.2 as is evident from. This proves the theorem. \qed

The holomorphicity of the basis specified in Equation (12) shows that we can replace the superscript $an$ by any of the other superscripts defined above, and the assertion of Theorem 2.3 extends to each of these cases.

Note that Theorem 2.3 also respects the multiplicative structures in the following sense. For two elements $f \in \tilde{M}_{k}^{an}(\rho)$ and $g \in \tilde{M}_{l}^{an}(\eta)$ and their tensor product $h = f \otimes g \in \tilde{M}_{k+l}^{an,\leq m}(V_m \otimes \rho \otimes \eta)$, Theorem 2.3 yields the corresponding functions $F \in M_{k-n}^{an}(V_m \otimes \rho)$, $G \in M_{l-n}^{an}(V_n \otimes \eta)$, and $H$ in
\( \mathcal{M}_{k+l-m-n}(V_{m+n} \otimes \rho \otimes \eta) \). Then \( H = F \otimes G \) (under the convention of taking only the \( V_{m+n} \) part of \( V_m \otimes V_n \)), since if \( f_r, 0 \leq r \leq m, \) \( g_s, 0 \leq s \leq n, \) and \( h_t, 0 \leq t \leq m+n \) are the functions corresponding to \( f, g, \) and \( h \) in Equation (5) respectively, then \( h_t = \sum_{r+s=t} f_r \otimes g_s \). Indeed, taking the tensor product of the functions \( F \) and \( G \) using the expressions from Equation (12) yields the form of \( H \) in the same equation.

Theorem \( \ref{thm:modular_framework} \) establishes a modular framework for quasi-modular forms with bounded depth. The idea resembles slightly the \( \mathfrak{s}\mathfrak{l}_2(\mathbb{C}) \)-representations \( \mathcal{U}_k \) presented at the end of Section 3 of \( \ref{sec:representations} \), though these are not equivalent representations. We now use the maps \( i_m \) from Corollary \( \ref{cor:injections} \) in order to gather all the quasi-modular forms together, and also to obtain the multiplicative structure inside this ring. Let \( V_\infty \) be the direct limit of the \( V_m \) with respect to the maps \( i_m \).

The representation space of \( V_\infty \) is infinite-dimensional, with two possible bases being \( \{ (\tau)^{\infty - s} \}_{s=0}^\infty \) and \( \{ (\tau)^{\infty - r} \}_{r=0}^\infty \). However, images of this direct limit are \( V_\infty \)-valued functions \( \tau \in \mathcal{H} \) in which only finitely many coefficients in either basis are non-zero. After tensoring with \( V_\rho \), the image of each of the spaces \( \mathcal{M}_{k-m}^* (V_m \otimes \rho) \) in this direct limit will be referred to as modular forms (with the appropriate analytic properties) of weight \( k - \infty \) with representation \( V_\infty \otimes \rho \) (denoted \( \mathcal{M}_{k-\infty}^* (V_\infty \otimes \rho) \)), and any element \( \mathcal{M}_{k-\infty}^* (V_\infty \otimes \rho) \) arises from \( \mathcal{M}_{k-m}^* (V_m) \) for some \( m \). Even though \( k - \infty = -\infty \) for every finite \( k \), we keep \( k - \infty \) in the notation \( \mathcal{M}_{k-\infty}^* (V_\infty \otimes \rho) \), and the value of \( k \) is of course important.

Since the maps \( \mathcal{M}_{k-m}^* (V_m \otimes \rho) \rightarrow \mathcal{M}_{k-m}^* (V_m \otimes \rho) \) commute with the injections \( i_m \), we can consider the images of elements of \( \mathcal{M}_{k}^* (\rho) \) (with no depth restriction) as modular forms in \( \mathcal{M}_{k-\infty}^* (V_\infty \otimes \rho) \).

The representation \( V_\infty \) is also suitable for multiplicative properties. Indeed, given \( F \in \mathcal{M}_{k-m}^* (V_m \otimes \rho) \) and \( G \in \mathcal{M}_{l-n}^* (V_n \otimes \eta) \) with product \( H = F \otimes G \in \mathcal{M}_{k+l-m-n}^* (V_{m+n} \otimes \rho \otimes \eta) \), the equalities

\[
  i_m(F) \otimes G = i_{m+n}(H) = F \otimes i_n(G)
\]

hold in \( \mathcal{M}_{k+l-m-n-1}^* (V_{m+n+1} \otimes \rho \otimes \eta) \). Hence in the direct limit we obtain a well-defined (tensor) product map from \( \mathcal{M}_{k-\infty}^* (V_\infty \otimes \rho) \otimes \mathcal{M}_{l-\infty}^* (V_\infty \otimes \eta) \) to \( \mathcal{M}_{k+l-\infty}^* (V_\infty \otimes \rho \otimes \eta) \). As shown above, this product map corresponds to the usual tensor of quasi-modular forms. In particular, for integral \( k \) and \( l \) (or half-integral if \( \Gamma \) is a subgroup of \( MP_2(\mathbb{R}) \)) with trivial \( \rho \) and \( \eta \), we have a well-defined product map, under which the vector space \( \bigoplus_{\Gamma} \mathcal{M}_{k-\infty}^* (\Gamma, V_\infty) \) becomes a ring. Theorem \( \ref{thm:modular_framework} \) and the following discussion show that this ring is isomorphic to the ring \( \bigoplus_{\Gamma} \mathcal{M}_{k-\infty}^* (\Gamma) \) of quasi-modular forms (with various analytic properties) with a trivial representation with respect to \( \Gamma \).

### 2.2 Holomorphic Modular Forms with Multiplier Systems Involving \( V_m \)

We now turn our attention to holomorphic and meromorphic forms with multiplier systems involving \( V_m \). According to Theorem \( \ref{thm:modular_framework} \) these objects are analogous to holomorphic and meromorphic quasi-modular forms.
The spaces \( \mathcal{M}_{k-m}^*(V_m \otimes \rho) \) are most intrinsically described using the filtration arising from the maps \( i_m \). For any \( p \leq m \), denote \( \mathcal{M}_{k-m}^{p-1}(V_m \otimes \rho) \) the subspace of \( \mathcal{M}_{k-m}^*(V_m \otimes \rho) \) consisting of those elements whose expansion as in Equation (11) involves non-zero coefficients \( F_s \) only for \( s \leq p \) (equivalently, in the expansion of Equation (12) only non-zero terms \( f_r \) with \( r \leq p \) appear). Here we use an increasing filtration, unlike the decreasing filtration of [Ve] and others (though this is essentially the same filtration). Multiple applications of Theorem 2.3 show that \( \mathcal{M}_{k-m}^*(V_m \otimes \rho) \) is the image of \( \mathcal{M}_{k-m}^*(V_m \otimes \rho) \) under \( i_m \cdots i_1 \). The map \( i_m \) takes \( \mathcal{M}_{k-m}^*(V_m \otimes \rho) \) isomorphically onto \( \mathcal{M}_{k-m-1}^*(V_{m+1} \otimes \rho) \) (for \( p = m \) this is Corollary 2.2 again), which allows us to define \( \mathcal{M}_{k-m}^{p-1}(V_{\infty} \otimes \rho) \) in the direct limit. Now, Theorem 2.3 implies that for an element of \( \mathcal{M}_{k-m}^{p-1}(V_m \otimes \rho) \) the coefficient \( F_p \) in Equation (11) and the coefficient \( f_p \) in Equation (12) coincide, and this common function lies in \( \mathcal{M}_{k-2p}^*(\rho) \). Hence the space \( \mathcal{M}_{k-m}^{p-1}(V_m \otimes \rho) / \mathcal{M}_{k-m}^{p-2}(V_m \otimes \rho) \) injects into \( \mathcal{M}_{k-2p}(\rho) \). Moreover, \( i_m \) defines an isomorphism between the latter quotient space and \( \mathcal{M}_{k-m}^{p-1}(V_{m+1} \otimes \rho) / \mathcal{M}_{k-m-1}^{p-2}(V_{m+1} \otimes \rho) \), and the two injections commute with this isomorphism. Hence we identify these isomorphic quotients, and use the notation \( i_{k,\rho}^* \) (which does not involve \( m \)) for this injection. We can consider \( i_{k,\rho}^* \) as defined on the direct limit \( \mathcal{M}_{k-\infty}^{*}(V_\infty \otimes \rho) / \mathcal{M}_{k-\infty}^{*}(V_\infty \otimes \rho) \).

We can now state the structure theorem for \( \mathcal{M}_{k-m}^*(V_m \otimes \rho) \) as well as for the direct limit \( \mathcal{M}_{k-\infty}^*(V_\infty \otimes \rho) \):

**Theorem 2.4.** The map \( i_{k,\rho}^* \) is an isomorphism in all cases, except for the case where \( \Gamma \) has no cusps, \( \ast = \text{hol} \) (or equivalently \( \ast = \text{cusp} = \text{wh} \)), and the weight \( k \) is an integer between \( p + 1 \) and \( 2p \) (hence \( p > 0 \)). If \( \rho \) is a representation factoring though a finite quotient of \( \Gamma \), then the only case where \( i_{k,\rho}^* \) is not an isomorphism is where \( k = 2p > 0 \) and the representation \( \rho \) contains a trivial factor. In this case \( i_{k,\rho}^* = 0 \) and its range is non-zero.

The claim about cusp forms with trivial representation (for even \( k \) and \( m \)) appears in [Ve], and its proof (at least for weights 0 and 2) is given in [KS]. Here, we generalize it for all the various cases, and present another proof for the case where \( \Gamma \) has cusps. This new proof covers certain situations to which the proof from [KS] does not apply.

Before proving Theorem 2.4 we introduce another basis for the representation space of \( V_m \) in some cases. Let \( \varphi \) be an element of \( \mathcal{M}_{2,mer}^*(1)(\Gamma) \) whose corresponding functions in Equation (5) are \( f_0 = \varphi \) and \( f_1 = 1 \). Any normalized logarithmic derivative of a (meromorphic) modular form of non-zero weight can be taken as \( \varphi \). In fact, Theorem 9 of [A] shows that we can take \( \varphi \) with a unique simple pole on \( Y(\Gamma) \) at any pre-fixed point in \( Y(\Gamma) \). Then the function \( \varphi^*(\tau) = \varphi(\tau) + \frac{1}{2iz} \) lies in \( \mathcal{M}_{2,mer}^*(\Gamma) \) (by Proposition 1.2, for example), and Theorem 2.3 shows that

\[
w(\tau) = \varphi(\tau) \begin{pmatrix} \tau \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \varphi^*(\tau) \begin{pmatrix} \tau \\ 1 \end{pmatrix} + \frac{1}{-2iy} \begin{pmatrix} \tau \\ 1 \end{pmatrix}
\]  

(13)
is in $\mathcal{M}_1^{\text{mer}}(\Gamma, V_1)$. Using the basis $(\tilde{\tau})$ and $w$ for $V_1$, an argument similar to Proposition 2.4 proves the following

**Proposition 2.5.** An element $F \in \mathcal{M}_{k-m}^{\text{sing}}(V_m \otimes \rho)$ of weight $k$ and representation $V_m \otimes \rho$ decomposes as

$$F(\tau) = \sum_{t=0}^{m} f_t^w(\tau) \left( \frac{\tau}{1} \right)^{m-t} w^t,$$

where $f_t^w \in \mathcal{M}_{k-2t}^{\text{sing}}(\rho)$ for every $0 \leq t \leq m$. Conversely, if $f_t^w \in \mathcal{M}_{k-2t}^{\text{sing}}(\rho)$ for every such $t$ then the function $F$ defined in Equation (14) lies in $\mathcal{M}_{k-m}^{\text{sing}}(V_m \otimes \rho)$. In addition, $F \in \mathcal{M}_m^{\text{mer}}(V_m \otimes \rho)$ if and only if $f_t^w \in \mathcal{M}_m^{\text{mer}}(\rho)$ for all $t$.

**Proof.** The equivalence of the modularity of $F$ and the modularity of the functions $f_t^w$ is established as in the proof of Proposition 2.4. The equivalence of the meromorphy of $F$ and the meromorphy of the functions $f_t^w$ follows from the fact that the basis vectors $(\tilde{\tau})^{m-t} w^t$ are meromorphic functions of $\tau \in \mathcal{H}$. This proves the proposition. $\square$

Assume further that $\varphi \in \widetilde{\mathcal{M}}_2^{\text{hol}, \leq 1}(\Gamma)$, hence $w \in \mathcal{M}_1^{\text{hol}}(\Gamma, V_1)$. Then the proof of Proposition 2.4 implies the equivalence of $F \in \mathcal{M}_m^{\text{mer}}(V_m \otimes \rho)$ and $f_t^w \in \mathcal{M}_2^{\text{mer}}(\rho)$ for all $p$ also for $*$ being one of the superscripts $\text{an, hol, cusp,}$ and $\text{wh}$. Hence every choice of such $\varphi$ (and $w$) defines an isomorphism

$$\mathcal{M}_m^{\text{mer}}(V_m \otimes \rho) \cong \bigoplus_{t=0}^{m} \mathcal{M}_{k-2t}^{\text{mer}}(\rho).$$

The existence of $w \in \mathcal{M}_1^{\text{hol}}(\Gamma, V_1)$ which is of the form given in Equation (13) is equivalent to the existence of $\varphi \in \widetilde{\mathcal{M}}_2^{\text{hol}, \leq 1}(\Gamma)$ with the corresponding function $f_1 = 1$. Such a quasi-modular form always exists if $\Gamma$ has cusps. Theorem 4 of [A] proves this assertion using generalized Eisenstein series, though a more conceptual proof of this assertion can be obtained using Serre duality: The filtration on $\mathcal{M}_1^{\text{hol}}(\Gamma, V_1)$ leads to a short exact sequence

$$0 \rightarrow \mathcal{L}_2 \rightarrow V_1,1 \rightarrow \mathcal{L}_0 \rightarrow 0$$

of vector bundles on $X(\Gamma)$ (where $\mathcal{L}_k$ is the line bundle corresponding to modular forms of weight $k$ and $V_{k,m}$ is the vector bundle of rank $m+1$ corresponding to modular forms of weight $k$ and representation $V_m$). $H^1(X(\Gamma), \mathcal{L}_0)$ is the space dual to $H^0(X(\Gamma), \Omega^1 = \mathcal{L}_2 \otimes \mathcal{L}_{\text{cusp}})$, namely to $\mathcal{M}_0^{\text{cusp}}(\Gamma)$, and the latter space is 0 if $\Gamma$ has cusps. Therefore, the corresponding sequence of global sections, which is

$$0 \rightarrow \mathcal{M}_2^{\text{hol}}(\Gamma) \xrightarrow{\text{in}} \mathcal{M}_1^{\text{hol}}(V_1) \xrightarrow{\text{hol,1, triv}} \mathbb{C} \rightarrow 0$$

(16)

(where $\text{triv}$ is the trivial representation), is exact, and $w$ is any pre-image of 1 under $\text{hol,1, triv}$. In the classical case of subgroups of $SL_2(\mathbb{Z})$ we can take $\varphi$
to be $\frac{27}{4}E_2$, a multiple of the weight 2 quasi-modular Eisenstein series. Hence $\varphi^* = \frac{27}{4}E_2$ is a multiple of the almost holomorphic weight 2 Eisenstein series. On the other hand, if $\Gamma$ has no cusps then this proof fails (since $H^1(X(\Gamma), L_0)$ is no longer trivial), and indeed $\tilde{M}_2^{\text{hol}}(\Gamma) = M_2^{\text{hol}}(\Gamma)$ in this case (see Lemma 4 of [KS] or Theorem 4 of [A]).

**Proof of Theorem 2.4.** The assertion for $* = \text{an}$ and for $* = \text{sing}$ follows directly from Proposition 2.1. Hence we restrict attention to meromorphic modular forms. Let $f \in \mathcal{M}_\mu^{\text{mer}}(\rho)$. The operator $\delta_l = \frac{\partial}{\partial \tau} + \frac{1}{2iy}$ takes $f$ to a meromorphic form of weight $l + 2$, and the composition of $p$ consecutive such operators takes $f$ to the element

$$\delta_l^p f(\tau) = \delta_{l+2p-2} \circ \ldots \circ \delta_l f(\tau) = \sum_{r=0}^{p} \binom{p}{r} \prod_{j=1}^{r} (l + p - j) \frac{f^{(p-r)}(\tau)}{(2iy)^r}$$

of $\mathcal{M}_{l+2p}^{\text{sing}}(\rho)$. By Proposition 135 of [MR], the operators $\delta_l$ commute with the ordinary derivative $\frac{\partial}{\partial \tau}$ and the maps from Proposition 1.2. Thus, the $p$th derivative $f^{(p)}$ of $f$ is in $\mathcal{M}_l^{\text{mer}, \leq p}(\rho)$ with the functions $f_r$, being $\binom{p}{r} \prod_{j=1}^{r} (l + p - j) f^{(p-r)}(\tau)$ for each $0 \leq r \leq p$. Theorem 2.3 now shows that

$$F(\tau) = \sum_{r=0}^{p} \binom{p}{r} \prod_{j=1}^{r} (l + p - j) \cdot f^{(p-r)}(\tau) \left( \frac{\tau^m}{\tau^1} \right)^{m-r} \left( \frac{1}{0} \right)^r$$

is in $\mathcal{M}_{l+2p-m}(V_m \otimes \rho)$ for any $m \geq p$. Take $l = k - 2p$, and assume first that none of the numbers $k - p - j$, $1 \leq j \leq p$, vanish. Then the element $\prod_{r=1}^{p} \binom{p}{r} \prod_{j=1}^{r} (l + p - j)$ of $\mathcal{M}_k^{\text{mer}, \leq p}(V_m \otimes \rho)$ maps to $\iota_{k,\rho}^{\text{mer}, p}$ by Proposition 2.5 which immediately proves the case $* = \text{mer}$ for any group $\Gamma$. Moreover, given $f \in \mathcal{M}_k^{\text{mer}, 2p}(\rho)$ which is not holomorphic on $H$, Theorem 9 of [A] allows us to choose an element $\varphi \in \tilde{M}_2^{\text{mer}, \leq 1}(\Gamma)$ having poles only at the poles of $f$. In this case Equation 14 yields a meromorphic $\iota_{k,\rho}^{\text{mer}, p}$-pre-image of $f$ whose poles are only at the poles of $f$. If $\Gamma$ has cusps then we choose holomorphic $\varphi$ and $\psi$ in Equation 14. Then Proposition 2.5 and the additional equivalences arising from this choice of $\varphi$ complete the proof for the case of $\Gamma$ with cusps.

It remains to consider the image of $\iota_{k,\rho}^{\text{mer}, p}$ for $* = \text{hol} = \text{cusp} = \text{sing}$ in the case where $\Gamma$ has no cusps and $k$ is an integer between $p + 1$ and $2p$. We first
observe that \( \rho \) is a representation (not a multiplier system) since \( k \) is an integer. Assuming that \( \rho \) factors through a finite quotient of \( \Gamma \), we have \( M_{k-2p}^\text{hol}(\rho) = 0 \) for \( k < 2p \) (since the weight is negative), and for \( k = 2p \) the space \( M_{k-2p=0}^\text{hol}(\rho) \) consists of constant functions. A constant in \( V_\rho \) lies in \( M_0^\text{hol}(\rho) \) if and only if it lies in the maximal subspace \( V_\rho^{\text{triv}} \) of \( V_\rho \) on which \( \rho \) acts trivially. Therefore all we need to show is that if \( p > \iota \) then \( V_\rho \) consists of constant functions. A constant in \( V_\rho \) which does not vanish on \( V_\rho^{\text{triv}} \) in \( V_\rho \) implies that \( f \) is a non-zero constant complex number. Theorem 2.3 and Lemma 4 then imply that \( f_{p-1} \) is an element of \( \hat{M}_2^\text{hol}(\Gamma) \setminus M_2^\text{hol}(\Gamma) \) (here we use the assumption that \( p > 0 \), in contradiction to Theorem 4 of [A]). This shows that \( a^*_p = 0 \) in this case, which completes the proof of the theorem.

Note that the case \( p = 0 \) in Theorem 2.4 reduces to the assertion that \( a^*_k \) defines an isomorphism between the space \( M_{k-m}^* (V_m \otimes \rho) \) for any finite \( m \), or equivalently \( M_{k-m}^* (V_m \otimes \rho) \), and the space \( M_{k}^* (\rho) \) for every \( k, \Gamma, \rho \), and analytic type \( * \). Indeed, this is the isomorphism inverse to \( i_{m-1} \circ \ldots \circ i_0 \) (or to the direct limit map).

Consider now the case where \( \Gamma \) has cusps. The proof of Theorem 2.4 shows that a more general assertion is valid in this case: For any cuspidal divisor \( D \) on \( X(\Gamma) \), the restriction of the map \( i_{k,\rho}^\text{wh} \) to modular forms whose poles are bounded by \( D \) is an isomorphism onto the subspace of \( M_{k-m}^\text{wh} (V_m \otimes \rho) \) consisting of those modular forms whose polar divisor is bounded by \( D \). Observe that the degree of freedom in the choice of \( \varphi \in M_{2}^\text{hol}(\Gamma) \) (hence \( w \)) in this case corresponds to addition of an element \( h \in M_{2}^\text{hol}(\Gamma) \) to \( \varphi \) (hence adding \( h(\gamma) \) to \( w \))—see Equation (10), for example. In particular, if \( M_{2}^\text{hol}(\Gamma) = 0 \) (this is the situation, for example, when \( \Gamma = SL_2(\mathbb{Z}) \)), then \( \varphi \) and \( w \) are unique, and the decomposition of \( M_{k-m} (V_m \otimes \rho) \) into \( \bigoplus_{p=0}^{\infty} M_{k-2p}^* (\rho) \), given by Equations (14) and (15), is canonical. Returning to the general case with \( \varphi \in \hat{M}_2^\text{hol,\leq 1}(\Gamma) \), we remark that \( M_{k-m}^\text{sing}(V_m \otimes \rho) \) and \( M_{k-m}^\text{mer}(V_m \otimes \rho) \) consist of those elements \( F \) for which the decomposition in Equation (14) involves non-zero functions \( f_t \) only for \( t \leq p \). The same assertion holds for the other spaces \( M_{k-m}^\text{hol}(V_m \otimes \rho) \) if \( \varphi \in \hat{M}_{2}^\text{hol,\leq 1}(\Gamma) \), and extends to \( M_{k-m}^\text{hol}(V_m \otimes \rho) \).

We recall that if the dimension of \( V_\rho \) is finite then the spaces \( M_{k}^\text{hol}(\rho) \) are finite-dimensional. Indeed, if \( k \) is integral (or half-integral for \( \Gamma \subseteq Mp_{\mathbb{Z}}(\mathbb{R}) \)) and \( \rho \) is a representation, then \( M_{k}^\text{hol}(\rho) \) is the space of global sections of a vector bundles (of finite rank) over the compact Riemann surface \( X(\Gamma) \). For the case of multiplier systems see, for example, Proposition 9 of [KM]. If \( \Gamma \) has cusps then the same assertion holds for the spaces \( M_{k}^\text{cusp}(\rho) \), and more generally to every subspace of \( M_{k}^\text{hol}(\rho) \) defined by a bound on the polar divisor. Hence Theorem
Corollary 2.6. If \( k \) is not an integer between 1 and \( 2m \) then

\[
\dim \mathcal{M}_{k-m}^{hol}(V_m \otimes \rho) = \sum_{t=0}^{m} \dim \mathcal{M}_{k-2t}^{hol}(\rho).
\]  \hspace{1cm} (17)

If \( \Gamma \) has cusps then Equation (17) holds in general, as well as the corresponding assertion for the spaces \( \mathcal{M}^{cusp} \) and for subspaces of \( \mathcal{M}^{wh} \) in which the polar divisor is bounded by a fixed cuspidal divisor. If \( \Gamma \) has no cusps, \( k \) is an integer between 1 and \( 2m \), and the representation \( \rho \) factors through a finite quotient of \( \Gamma \), then Equation (17) still holds if \( k \) is odd. If \( k = 2p \) for some \( 0 < p \leq m \) then

\[
\dim \mathcal{M}_{2p-m}^{hol}(V_m \otimes \rho) = \sum_{t=0}^{p-1} \dim \mathcal{M}_{2p-2t}^{hol}(\rho),
\]

and the right hand side of Equation (17) is obtained from this common value by adding \( \dim \mathcal{M}_0^{hol}(\rho) = \dim V^{\text{triv}}_\rho \).

References

[A] Azaiez, N. O., The Ring of Quasimodular Forms for a Cocompact Group, J. Number Theory 128, Issue 7, 1966–1988 (2008).

[E] Eichler, M., Eine Verallgemeinerung der Abelschen Integrale, Math. Zeitschr., vol 67 no 1, 267–298 (1957).

[KS] Kuga, M., Shimura, G., On vector differential forms attached to automorphic forms, J. Math. Soc. Japan 12, 258–270 (1960).

[KZ] Kaneko, M., Zagier, D., A Generalized Jacobi Theta Function and Quasimodular Forms, in: The Moduli Space of Curves (Texel Island, 1994), Progr. Math. 129, Birkhäuser Boston, Boston, MA, 165-172 (1995).

[KM] Knopp, M., Mason, G., Generalized modular forms, J. Number Theory 99, 1–28 (2003).

[MR] Martin, F., Royer, E., Formes Modulaires et Périodes, Formes modulaires et transcendance, Sémin. Congr., vol. 12, Soc. Math. France, Paris, 1–117 (2005).

[Sh] Shimura, G., Sur les intégrales attachées aux formes automorphes, J. Math. Soc. Japan 11, 291–311 (1959).

[Ve] Verdier, J. L., Sur les intégrales attachées aux formes automorphes (d’après Goro SHIMURA), Sém. BOURBAKI 216, vol 13 (1961).

[Za] Zagier, D. et al., The 1-2-3 of Modular Forms, Universitext (2008).