THE SKYRMION IN THE NUCLEUS

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Abstract

The energy levels of a skyrmion in nucleus are calculated in a field theory of skyrmions coupled to the dilaton field and the ω meson. The central potential fits well with expectations. The nucleon spin-orbit interaction derived from the omega meson in a rotating frame gives the correct level splittings. The same interaction originating from the Thomas precession effect is negligible. Energy levels are calculated for closed shell nuclei. The meson fields are obtained from a Thomas-Fermi mean field approximation to the nucleus.

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One of the key successes of conventional nuclear theory was the ability to describe in a simple manner the shell structure of nuclei. Harmonic oscillator, or Woods-Saxon potentials give the gross features of the single particle levels of nucleons. In the present work we address the question of the suitability of the Skyrme model in achieving the same goal.

In previous publications\cite{1, 2} a model of a fluid of skyrmions in nuclear matter and finite nuclei was developed in a dilute fluid approximation. The skyrmions were regarded as essentially free objects interacting via the dilaton field and the $\omega$ meson in a mean field approach. The model succeeds in reproducing the main features of nuclear matter as well as closed shell nuclei. It incorporates scale and chiral invariance. The former is manifestly broken by a potential inspired in the trace anomaly. In the present work we take advantage of the scalar fields derived from the mean field approximation in order to generate the central and spin-orbit interactions of a single skyrmion in the nucleus.

In ref. \cite{1, 2} a considerable effort was invested in generating the potential for the dilaton in order to reproduce the main features of nuclear matter and finite nuclei. As it is our aim now to focus on the single-particle energy levels of the nucleon in the nucleus, we will simplify the calculation by freeing ourselves from a specific parametrization for the dilaton potential. Instead, the measured nuclear densities will be used to produce the mean fields in a Thomas-Fermi approximation. A more consistent approach would be to use a Hartree (or Hartree-Fock) method. However, the goal of the present work is to see whether we can reproduce the key elements of the nucleon binding energy, especially the spin-orbit interaction and not the fine details. In the Walecka-type models\cite{3, 4}, this potential arises from relativistic effects in the Dirac equation. While the central potential is due to the subtraction of scalar and vector interactions, the spin-orbit force originates from their addition. The scalar and vector potentials being large, produce a sizeable spin-orbit force.
We here investigate various possible sources of spin-orbit interactions and find that there is a large potential due to the motion of the skyrmion in a quiescent nucleus. Viewed from the frame of the nucleon the nucleus flows past it. The $\omega$ meson in this frame develops a spatial component that is responsible for the spin-orbit coupling. This is a specific feature of the skyrmion picture, as evidenced by the presence of the skyrmion moment of inertia in the interaction. This would be senseless in the Dirac approach for which the nucleon is pointlike.

The lagrangian adopted for the model is

$$L = e^{2\sigma} \left[ \frac{1}{2} \Gamma_0 \partial_\mu \sigma \partial^\mu \sigma - \frac{F_\pi^2}{16} \text{tr}(L_\mu L^\mu) \right] + \frac{1}{32e^2} \text{tr}[L_\mu, L_\nu]^2$$

$$- V_\sigma - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2} e^{2\sigma} m_\omega^2 \omega_\mu - g_\nu \omega_\mu B^\mu$$

(1)

Here

$$L_\mu \equiv U^\dagger \partial_\mu U,$$

(2)

where $U(\mathbf{r}, t)$ is the chiral field, $F_\pi$ is the pion decay constant $e$ the Skyrme parameter, and

$$G_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$$

(3)

Following ref. the ansätze for the scalar and vector fields become

$$U(\mathbf{r}) = \exp[i \mathbf{\tau} \cdot \mathbf{r} F(r)],$$

$$\omega^\mu(\mathbf{R}) = \delta_{\mu 0} \omega(R)$$

(4)

where $\mathbf{R}$ measures the distance from the center of the nucleus at rest, and $\mathbf{r}$ is the coordinate from the center of the skyrmion.
Averaging over the coordinates $R$ and the corresponding momenta with the distribution function for zero temperature, the energy of the nucleus with spherical symmetry reads

$$ E = 4\pi \int R^2 dR \ E(R) $$

$$ E(R) = E_\sigma + E_\omega + E_{\text{int}} + E_{\text{sk}} \quad (5) $$

where

$$ E_\sigma = \frac{1}{2} \Gamma_0^2 e^{2\sigma} \sigma^2 + V_\sigma $$

$$ E_\omega = -\frac{1}{2} \omega^2 - e^{2\sigma} \frac{m_\omega^2 \omega^2}{2} $$

$$ E_{\text{int}} = g_V \omega B $$

$$ E_{\text{sk}} = \frac{2}{\pi^2} \int_{k_F}^{k_F'} k^2 dk \sqrt{k^2 + M^2} \quad (6) $$

where $k_F$ is the nucleon local, $R$ dependent, Fermi momentum and,

$$ M(R) = 4\pi \int_0^\infty r^2 dr M(r) $$

$$ M(r) = e^{2\sigma} \frac{F^2}{8} \left[ F^0 + 2 \frac{\sin^2 F}{r^2} \right] + \frac{1}{2e^2} \frac{\sin^2 F}{r^2} \left[ \frac{\sin^2 F}{r^2} + 2F'^2 \right] \quad (7) $$

is the nucleon mass. Using the virial theorem, or the skyrmion equations of motion, the mass scales as

$$ M(R) = e^\sigma M_0 \quad (8) $$

where $M_0$ is the skyrmion mass for $\sigma = 0$. In eq. $^3$ primes denote derivatives with respect to $R$, whereas in eq. $^7$ they represent derivatives with respect to $r$. Also

$$ B(R) = \frac{2}{3} \frac{k_F^3}{\pi^2} \quad (9) $$

The Euler-Lagrange equations for the mean fields become
\[ \Gamma_0^2 e^{2\sigma} \left( \sigma'' + 2\sigma'' + \frac{2\sigma'}{R} \right) - \frac{dV_\sigma}{d\sigma} + m_\omega^2 \omega^2 e^{2\sigma} - \frac{\partial E_{sk}}{\partial \sigma} = 0 \]

\[ \omega'' + \frac{2\omega'}{R} - m_\omega^2 \omega e^{2\sigma} + g_\nu B = 0 \]

(10)

The ground state of the nucleus for fixed number of nucleons is obtained by minimizing the energy, constrained by means of a Lagrange multiplier, with respect to the local wavenumbers \( k_F \) (no distinction is made here between protons and neutrons).

The algebraic equations for the multipliers become [3]

\[ \mu = g_\nu \omega + \sqrt{k_F^2 + M^2} \]

(11)

We solve the equation for the \( \omega \) meson of eq. (10) with \( B \) replaced by the measured densities and the dilaton determined by the chemical potential. In this manner we avoid tedious parametrizations of the dilaton potential in order to fit the densities. The assumption is that such a potential exists, and the results of ref. [1] support it.

A skyrmion in the nucleus, will be affected by the \( \sigma \) and \( \omega \) mean fields. These will produce the central potential. In order to uncover the spin-orbit interaction we consider three types of contributions. A Lorentz boost of the skyrmion, the Thomas precession induced by two successive Lorentz transformations and, dynamical \( \omega \) meson effects. The first contribution may be found by boosting rigidly the whole skyrmion by means of the transformation of the argument of the "hedgehog" of eq.( 4) with velocity \( \mathbf{v} \)

\[ \mathbf{r} \rightarrow \tilde{\mathbf{r}} = \mathbf{r} + \mathbf{r} \cdot \hat{\mathbf{v}} \hat{\mathbf{v}} \left( \gamma - 1 \right) - \gamma \mathbf{R} \]

(12)

where \( \gamma = \frac{1}{\sqrt{1 - v^2}} \), together with a rigid rotation of the skyrmion with collective coordinate matrices \( A[4] \)

\[ U(\tilde{\mathbf{r}}) \rightarrow A(t)U(\tilde{\mathbf{r}})A^\dagger(t) \]

(13)
Inserting the transformations in the skyrmion lagrangian of eq. (1) and after a lengthy calculation it is found that the spin-orbit potential from this transformation vanishes.

A second possible contribution to the spin-orbit interaction comes from the well-known Thomas precession that arises from two consecutive Lorentz transformations. A way to implement this transformation consists in considering the isospin vector matrix as time dependent

\[ \dot{\tau} = -\Omega_T \times \tau \]  

where \( \Omega_T \) is the Thomas frequency. Again inserting in the skyrmion lagrangian of eq. (1) the spin-orbit interaction to lowest order in the velocity is found to be

\[ U_{s.o.} \approx -\frac{S \cdot L}{2 M_0^2} \frac{\partial V_C}{R \partial R} \]  

where \( V_C \) is the central potential of the skyrmion in the nucleus, \( S \) is the spin and \( L \) the angular momentum. The same result as in standard textbook derivations\[6\]. In eq.(15) we have used the projection formula \[5\]

\[ \dot{A}^+ A = -i \frac{\tau \cdot S}{2 \lambda(R)} \]  

where \( \lambda(R) \) is the moment of inertia of the nucleon\[5\]

\[ \lambda(R) = \frac{2\pi}{3} \int r^2 \, dr \, \Lambda(R) \]

\[ \Lambda(R) = \sin^2(F) \left[ F^2 e^{2\sigma} + 4 e^2 \left( F^2 + \frac{\sin^2(F)}{r^2} \right) \right] \]  

where \( F \) is the skyrmion profile of eq. (4) and the \( R \) dependence enters through the dilaton field \( \sigma \). It will turn out, as expected beforehand, that the spin-orbit of eq. (15) is quite negligible, due to the \( \frac{1}{M_0^2} \) dependence.
The third and most important possible source of spin-orbit force is due to the transformation of the static scalar fields to a rotating frame. In order to find this potential, it is first desired to find the mean fields for a streaming nucleus. In this case there arises a spatial component of the $\omega$ meson field. An appropriate approximate ansatz for this component is

$$\omega = V \omega_1(R) = (\Omega \times R) \omega_1(R)$$

(18)

where $V$ is the tangential velocity of the nucleus at each $R$ and $\Omega$ the angular velocity. At the same time the nucleus baryon density develops a time dependent piece (in a nonrelativistic approximation) of the form

$$B(R) = V B_0(R)$$

(19)

where $B_0(R)$ is the static baryon density of the nucleus. Inserting eqs.(18,19) above in the lagrangain with a distribution function for a nucleus at zero temperature and averaging over the angular directions of $R$ we find the equation of motion for $\omega_1$ to be

$$\omega'' + \frac{4\omega'}{R} - m_{\omega}^2 \omega_1 e^{2\sigma} + gV B = 0$$

(20)

This is very similar to the equation of motion of the static $\omega$ in eq. (10). We solve equation (20) for each nucleus using the dilaton field of the static case and demanding a vanishing $\omega_1$ at infinity.

In order to find the corresponding spin-orbit interaction we consider a nucleon spinning at rest with a nucleus rotating with an velocity $-V(R)$ opposite to the direction of rotation of the nucleon. Using the collective coordinate quantization scheme of eq. (13), and the projection formula of eq. (16) we find

$$W_{s.o.} = \frac{-S \cdot L}{2M_0 \lambda(R)} \omega_1(R)$$

(21)
It is clear that $W_{s.o.}$ is more important than $U_{s.o.}$ due to the $\frac{1}{M_0}$ dependence. It is also a purely skyrmion spin-orbit as evidenced by the presence of the moment of inertia in the potential. In the usual Dirac type of Walecka models [3], the spin-orbit interaction arises from the coupling to the lower components of the Dirac wave function, whereas here it arises from the interaction of the rigid rotation of the nucleon with the flow of the mean fields.

The Hamiltonian of a single Syrmion embedded in a nucleus becomes

$$H = \sqrt{p^2 + M^2 + g_V \omega + W_{s.o.} + U_{s.o.}}$$

(22)

where $p$ is the nucleon momentum, the conjugated variable to the skyrmion center location $R$.

Quantizing the coordinate $R$ we obtain an effective Schrödinger equation for the radial wave function of the skyrmion center with total energy $E$

$$\left[ \frac{\partial^2}{\partial R^2} + \frac{2}{R} \frac{\partial}{\partial R} - \frac{l(l+1)}{R^2} - Q(R) \right] \Psi = 0$$

(23)

where

$$Q(R) = e^{2\sigma} M_0^2 - \left( - g_V \omega + E - W_{s.o.} \right)^2 - Z(R)$$

$$Z(R) \approx \frac{g_V}{2M(R)} \left[ \frac{\partial^2}{\partial R^2} + 2 \frac{\partial}{\partial R} \right] \omega + 2 M_0 U_{s.o.}$$

(24)

where we have expanded the square root in eq. (22) when operating on the potential to order $p^2$. The central potential entering the calculation of $U_{s.o.}$ of eq.(15) is given by

$$V_C = \frac{Q_1(R)}{2M_0}$$

(25)

where $Q_1$ is given by $Q(R)$ of eq.(24), but without the spin-orbit pieces.
We have solved the Schrödinger equation (23) for the ground state single particle levels of the magic nuclei $^{12}\text{C}$, $^{16}\text{O}$ and $^{40}\text{Ca}$. Table 1 shows the comparison between the predicted binding energies the experimental ones [8, 9], averaged over proton and neutron states. The results show that the skyrmion picture of both the central and the spin-orbit interaction is quite good. The spin-orbit originates solely from the $\omega$ meson as viewed by the rotationg skyrmion, in contradistinction to the Dirac case in which both scalar and vector fields act together to produce a large interaction.
Table 1: binding energies of single particle levels

| Nucleus | Shell  | calculated energy MeV | experimental energy MeV |
|---------|--------|------------------------|-------------------------|
|         | 1s\(\frac{1}{2}\) | 15.7                   | 16.9                     |
| \(C^{12}\) | 1p\(\frac{3}{2}\) | 36.3                   | 35.2                     |
|         | 1s\(\frac{1}{2}\) | 20.5                   | 20.1                     |
| \(O^{16}\) | 1p\(\frac{1}{2}\) | 15.6                   | 13.9                     |
|         | 1d\(\frac{5}{2}\) | 14.8                   | 12                       |
| \(C^{40}\) | 1s\(\frac{1}{2}\) | 48                     | 50±10                    |
|         | 1p\(\frac{3}{2}\) | 35                     | 34±6                     |
|         | 1p\(\frac{1}{2}\) | 30.7                   | 34±6                     |
|         | 1d\(\frac{3}{2}\) | 21                     | 18.5                     |
|         | 2s\(\frac{3}{2}\) | 15.7                   | 14.5                     |
|         | 1d\(\frac{5}{2}\) | 14.8                   | 12                       |
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