On the problem of unboundedness from below of the spinor QED Hamiltonian

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Abstract

We show that the Hamiltonian $h = H_{QED} + H_2$, where $H_{QED}$ is the spinor QED Hamiltonian and $H_2$ is the positive transversal photon mass term, is unbounded from below if the electromagnetic coupling constant $\varepsilon^2$ is small enough, $\varepsilon^2 < \varepsilon_0^2$, and the transversal photon squared mass parameter $M^2$ is not too large: $0 \leq M^2 < e^2 l^2 c$, here, $l$ is the cut-off parameter, and $c$ and $\varepsilon_0^2$, positive constants which do not depend on any parameters.

0. INTRODUCTION

F.Palumbo [1] showed that the spinor QED Hamiltonian $H_{QED}$ is unbounded from below. He got this interesting result considering, in fact, the operator

$$\mathcal{H} = \frac{\int dS(0)\Omega^*_{pr} H_{QED} \Omega_{pr}}{\int dS(0)\Omega^*_{pr} \Omega_{pr}} \quad (0.1)$$

here $dS(0) \equiv \prod_{k \neq 0, \lambda = 1, 2} dq(k, \lambda)$, and $\Omega_{pr}$ is a trial function depending on the transversal photon variables $q(k, \lambda), k \neq 0, \ |k| < l, \lambda = 1, 2$, on electron
and positron degrees of freedom and on the zero momentum mode vector potential variable $q(0)$ [1],[2]. Thus, $\mathcal{H}$ depends on $q(0)$ and $\partial / \partial q(0)$. The simplest choice of the trial function enabled F.Palumbo to get the operator $\mathcal{H}$ of the form

$$H_{QED} \rightarrow \mathcal{H}_1 = -\frac{1}{2} \left( \frac{\partial}{\partial q(0)} \right)^2 + q(0) \cdot a.$$  \hspace{1cm} (0.2)

Here, $a$ does not depend on $q(0)$. The operator $\mathcal{H}_1$ is obviously unbounded from below. This result is obliged entirely to the new term $H_p$, see eq. (1.8). F.Palumbo has shown that this term has to be introduced into the QED Hamiltonian that is given in text books—see, e.g., [3]. In this work I substitute

$$dS(r) \equiv dq(0) \prod_{k \neq 0, k \neq \pm r, \lambda = 1, 2} dq(k, \lambda)$$  \hspace{1cm} (0.3a)

for $dS(0)$ in eq.(0.1) (here $r$ is a fixed value of the photon momentum) and use a more sophisticated choice of the trial function (see eqs. (2.1), (2.2), (2.7), (2.8), (2.19) and (2.23)). The result is the formula

$$H_{QED} \rightarrow \mathcal{H}_2 = - \sum_{\lambda=1,2} \left[ \frac{\partial}{\partial q(r, \lambda)} \frac{\partial}{\partial q(-r, \lambda)} + q(r, \lambda)q(-r, \lambda)\gamma^2 \right]$$  

$$+ \text{const} + O(1/V),$$  \hspace{1cm} (0.3)

$$\gamma^2 \equiv [e^2c(m/l, r/l) + e^4d(m/l, r/l)]l^2.$$  \hspace{1cm} (0.4)

Here, $e^2$ is the nonrenormalized coupling constant, $m$ is the electron mass parameter in the Lagrangian of the spinor QED, $l$ is the momentum cut-off parameter, $V$ is a large periodicity volume, $l \rightarrow \infty , V \rightarrow \infty$, and $c(x, y)$ and $d(x, y)$, some functions, $c(x, 0)$ being positive, $c(0, 0) > 0$, $d(x, y)$ is bounded if $x \geq 0, y \geq 0$. Equations (0.3) and (0.4), if $e^2 \ll 1$, indicate the existence of the negative photon squared mass of the order of magnitude $\sim e^2l^2$ and unboundedness from below of the operators $\mathcal{H}_2$ and $H_{QED}$, as well as the operator $h$ of the abstract.

0. The article is organized as follows. Section 1. contains rather voluminous preliminary explanations concerning: a) the spinor QED Hamiltonian and gauge-transformational properties of the variables, on which this Hamiltonian depends, b) my method of the cut-off, c) the problem of compatibility of the realistic cut-off with the Lorentz invariance and d) the idea of my proof. Section 2. contains the proof of the statement of the abstract, i.e. the
derivation of eqs. (0.3) and (0.4). Appendix A contains derivation of some
formulas which are necessary for the proof of sec. 2.

1. SOME PRELIMINARY EXPLANATIONS

Here, I shall consider the Hamiltonian $h$,

$$h = H_{QED} + H_2$$  \hspace{1cm} (1.1)

which is the sum of the QED cut-off Hamiltonian $H_{QED}$ \cite{1},\cite{4},

$$H_{QED} = H_{0ph} + H_{0f} + H_1 + H_c + H_p,$$  \hspace{1cm} (1.2)

and the positive term $H_2$:

$$H_{0ph} = \int [(\hat{\mathbf{B}}^{tr})^2 + (\text{rot}\mathbf{B}^{tr})^2]d\mathbf{x}/2$$

$$= \sum_{k \neq 0, \lambda = 1, 2} \left[ -\frac{\partial}{\partial q(k, \lambda)} \frac{\partial}{\partial q(-k, \lambda)} + k^2 q(k, \lambda) q(-k, \lambda) \right]/2, \hspace{1cm} (1.3)$$

$$H_{0f} = \sum E(p) [a^*(p, \sigma) a(p, \sigma) + b^*(p, \sigma) b(p, \sigma)]$$

$$= \int \psi_1^*(\mathbf{x}) [-i\hat{\alpha} \nabla + \beta m] \psi_1(\mathbf{x}) d\mathbf{x}, \hspace{1cm} (1.4)$$

here $E(p) = \sqrt{p^2 + m^2}$,

$$H_1 = e \int \psi_1^*(\mathbf{x}) \hat{\alpha} \psi_1(\mathbf{x}) \mathbf{B}^{tr}(\mathbf{x}) d\mathbf{x}, \hspace{1cm} (1.5)$$

$$\mathbf{B}^{tr}(\mathbf{x}) = \sum_{k \neq 0, \lambda = 1, 2} \mathbf{e}(k, \lambda) q(k, \lambda) e^{ik\mathbf{x}}/\sqrt{V}, \hspace{1cm} (1.6)$$

$$H_c = e^2 \sum_{k \neq 0} \frac{\rho(k) \rho(-k)}{2V k^2}, \hspace{1cm} (1.7)$$

here $\rho(k) = \int \psi_1^*(\mathbf{x}) \psi_1(\mathbf{x}) e^{ik\mathbf{x}} d\mathbf{x}$,

$$H_p = -\frac{1}{2} \left( \frac{\partial}{\partial q(0)} \right)^2 + e q(0) \int \psi_1^*(\mathbf{x}) \hat{\alpha} \psi_1(\mathbf{x}) d\mathbf{x}/\sqrt{V}, \hspace{1cm} (1.8)$$
\[ H_2 = M^2/2 \int \left[ \mathbf{q}(0)/\sqrt{V} + \mathbf{B}^{tr}(\mathbf{x}) \right]^2 d\mathbf{x}, \quad (1.9) \]

Here, \( M^2 > 0 \). In equations (1.2)-(1.9) \( H_{0\text{ph}} \) is the Hamiltonian of free transversal photons, \( \mathbf{e}(\mathbf{k}, \lambda) \) is the polarization vector of a photon with the momentum \( \mathbf{k} \) and polarization index \( \lambda \), \( (\mathbf{k} \cdot \mathbf{e}(\mathbf{k}, \lambda)) = 0 \), \( (\mathbf{e}^*(\mathbf{k}, \lambda_1) \cdot \mathbf{e}(\mathbf{k}, \lambda_2)) = \delta_{\lambda_1, \lambda_2}, \mathbf{e}^*(\mathbf{k}, \lambda) = \mathbf{e}(-\mathbf{k}, \lambda), \mathbf{q}(0) \) is the spatially independent zero momentum mode of the vector potential \[1\], \[2\], \[4\], \( \mathbf{q}(0) = \int \mathbf{B}(\mathbf{x}) d^3x/\sqrt{V}, H_{0f} \) is the Hamiltonian of free electrons and positrons, \( m \) is the fermion mass parameter, \( a(\mathbf{p}, \sigma) \) and \( b(\mathbf{p}, \sigma) \) are the annihilation operators of the electron and positron with the momentum \( \mathbf{p} \) and spin projection \( \sigma \), \( \psi_1(\mathbf{x}) = \sum [u(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + v(\mathbf{p}, \sigma) b^*(\mathbf{p}, \sigma)] e^{ipx}/\sqrt{V}, u(\mathbf{p}, \sigma) \) and \( v(\mathbf{p}, \sigma) \) are the solutions of the Dirac equation with the energy \( \pm E(\mathbf{p}) \). The Hamiltonian \( H_1 \) describes the interaction of photons, electrons and positrons. The Hamiltonian \( H_e \) describes the Coulomb interaction between electrons and positrons. In the Fourier representation of the functions \( \mathbf{B}^{tr}(\mathbf{x}) \) and \( \psi_1(\mathbf{x}) \) one has \( |\mathbf{k}| < l, |\mathbf{p}| < l \), \( l \) being the cut-off parameter, \( V \) is large periodicity cube, \( V \) tends to infinity.

1. The Hamiltonian \( H_{QED} \) is expressed in terms of the gauge invariant quantities \( \mathbf{B}^{tr}, \mathbf{q}(0) \) and \( \psi_1 \). If the functions \( A_\mu(x) \) and \( \psi(x) \) in the Lagrangian

\[ L = -\left( \partial_\mu A_\nu - \partial_\nu A_\mu \right)^2/4 - \psi^* \gamma_4 [\gamma_\mu (\partial_\mu - ieA_\mu) + m] \psi \]

which gives rise to the Hamiltonian \( H_{QED} \), undergo gauge transformation \( A_\mu \rightarrow A_\mu + \partial \lambda(x)/\partial x_\mu, \psi \rightarrow e^{ie\lambda(x)} \psi \), the variables \( \mathbf{B}^{tr}, \mathbf{q}(0) \) remain constant, and the function \( \psi_1(\mathbf{x}) \) acquires a spatially independent phase multiplier (see, e.g.[4]). Thus, my method of the cut-off does not break down the gauge invariance.

\( H_p \) is the zero momentum mode term of the QED Hamiltonian \[1\]. The discovery of it enabled F.Palumbo to prove the unboundedness of the Hamiltonian \( H_{QED} \) from below \[1\].

1.1. Here, I am going to show that not only the Hamiltonian \( H_{QED} \), but also the Hamiltonian \( h = H_{QED} + H_2 \), \( H_2 \) being positive, see eq.(1.9), is unbounded from below if the positive quantity \( M^2 \) is not too large so that the inequality

\[ M^2 < \gamma^2, \quad (1.10) \]
is fulfilled (see eq.(0.4)). We have $c(m/l, 0) > 0$, thus $\gamma^2 > 0$ if $e^2 \ll 1$ independently of the sign of the function $d(m/l, 0)$. Thus, for small values of $e^2$ my consideration gives stronger result than that by F. Palumbo.

1.1.1. Omitting the Coulomb term in the present consideration, one gets an essentially analogous consideration for the massless Yukawa model. For this model also there holds the statement analogous to that of item 1.1, where, however, one has to take $d=0$.

1.1.2. Let us note that one cannot prove the unboundedness from below of the scalar QED-s Hamiltonian via the method of this work: in case of the scalar QED the squared oscillator frequency in eq.(0.3) is positive if $e^2 \ll 1$ (and has the order of magnitude $\sim e^2 l^2$). Thus, combining the spinor field with several charged scalar fields, one hopes to construct the QED model whose Hamiltonian is bounded from below.

1.2 The Palumbo’s proof of the unboundedness from below of the QED Hamiltonian (see eq. (0.2)) is essentially based on using zero momentum mode $q(0)$ [1]. On the contrary, my consideration has little to do with the zero momentum mode term $H_p$. The unboundedness from below of the spinor QED Hamiltonian (if $e^2 \ll 1$) is the consequence of the fact, that in any QFT model with a trilinear interaction $g \times \text{fermion}^* \times \text{boson} \times \text{fermion}$ the $g^2$ perturbation theory correction to the boson squared mass is negative. The latter fact is common knowledge since long ago.

1.3. It is worth mentioning here that the starting point of my key construction (2.7), (2.8) was an attempt to get a variational estimate of the type of eq.(0.1) by using the trial function $\Omega_{pr}$, $\Omega_{pr} \equiv exp(\kappa_0 + \kappa_1 e)|0>$ where the function $\Omega_0$,

$$\Omega_0 \equiv exp(\sum_{n=0}^{\infty} \kappa_n e^n)|0> \equiv e^K|0> \quad (1.11)$$

is the ground state wave function of the Schrödinger equation

$$(H_{QED} - E)\Omega = 0, \quad (1.12)$$

and $|0>$ is the state of the bare fermion vacuum.

1.3.1. The exponential representation (1.11), being substituted into the Schrödinger equation, enables one to recurrently find functions-operators $\kappa_n$, $n = 0, 1, 2, ...$ (were the ground state to exist). These functions-operators
depend on the zero momentum mode variables $q(0)$, on the photon variables $q(k, \lambda), k \neq 0$ and on the electron and positron creation operators. Of course, the exponential representation is equivalent to the straightforward linear representation

$$\Omega_0 = \sum_{n \geq 0} \Omega_{0n} e^n.$$  

(1.13)

One should stress, however, that the representation of the type (1.11) is preferable to that of (1.13). This fact was first noticed by F. Coester and R. Haag [5]. I did systematically use the exponential representation to consider the boson models $g(\phi^4)_2, g(\phi^4)_3,$ and $g((\phi^*\phi)^2)_2$ [6]. Later I have generalized the formalism to enable one to consider also fermions [7]. This generalization essentially boils down to substituting expression (1.11) of the ground state wave function into the Schroedinger equation, multiplying this equation by the operator $e^{-K}$ and using eq. (2.10a) (see also comments after eq. (2.10a)).

1.4. The consideration of the present work is of any value only if one believes that the cut-off Schroedinger equation (1.12) governs the QED. Of course, the cut-off Schroedinger equation approach to QED, even if the Hamiltonian is bounded from below (item 1.1.2.), gives rise to problems with the Lorentz invariance -cf., e.g., analogous approach to the $g(\phi^4)_4$ model. I hope, these problems can be solved via a properly chosen realistic regularization (cut-off). I do mean the introduction into the Fourier representation of the function $B^{tr}(x)$, eq.(1.6), of a photon form-factor $F_{ph}(k, l)$ and introduction into the analogous representation of the fermion operator $\psi_1(x)$ of a fermion form-factor $F_{f}(p, l)$. These cut-off representations of the the vector potential and the fermion operators are to be used only in the interaction terms $H_1, H_c$ and $H_p$.

1.4.1. Let us denote by $m^2_n$ the n-th order perturbation theory contribution to the squared fermion mass in the spinor QED. Obviously, one has $m^2_0 = m^2$. Using the form-factors

$$F_{ph}(k, l) = \sum_{n \geq 0} F_{nph} e^{-n|k|/l}, \sum_{n \geq 0} F_{nph} = 1,$$

$$F_{f}(p, l) = \sum_{n \geq 0} F_{nf} e^{-nE(p)/l}, \sum_{n \geq 0} F_{nf} = 1,$$

(1.14)

I was able to exhibit the momentum dependence of the quantity $m^2_2$:

$$m^2_2 = e^2(m^2 \ln(l/m)\text{const}_1 + p^2 \text{const}_2 + o(1/l)).$$  

(1.15)
Here, \( o(x) \to 0 \) as \( x \to 0 \). The analogous momentum dependence show the second and third order perturbation theory contributions to the squared mass of the fundamental particle in the \( g\phi^4 \) model. I hope, it is possible to eliminate the momentum dependence of the squared mass by a proper choice of the constants \( F_n \) in formfactors and thus, to construct the Lorentz-invariant perturbation theory.

1.4.2. The textbook by W. Heitler [3] contains the calculation of the quantity \( m^2 \) which is the Feynman perturbation theory contribution to the quantity \( m^2 \) ([6], Chapter 6, sec. 29, item 1., equation (29.14')). The value of \( m^2 \) does not depend on \(|p|\).

1.4.3. The point, however, is that \( m^2 - m^2_F \equiv \delta m^2 \neq 0 \). The Hamiltonian \( H_{QED} \) without the Palumbo term \( H_p \) gives \( m^2 = m^2_{tr} + m^2_{c} \), where subscripts "tr" and "c" denote parts of the quantity \( m^2 \) which originate due to the exchange of transversal photons and due to the Coulomb interaction. As for the quantity \( m^2_F \), it can be represented as

\[
m^2_F = \sum_{k,\mu} \text{Trace}(A(k, p)\gamma_\mu B(k, p)\gamma_\mu) = \sum_1^4 m^2_{F\mu} = m^2_{Ftr} + m^2_{Flong} + m^2_{F4},
\]

where \( m^2_{Ftr} = m^2_{tr} \). Thus, one has \( \delta m^2 = m^2_{2c} - m^2_{2Flong} - m^2_{2F4} \). Straightforward calculation gives

\[
\delta m^2 = e^2 \int F_f(p + k, l)F_{ph}(k, l)F_f(p, l)p/k|k|^3dk,
\]

Using equations

\[
|p + k| = |k| + pk/|k| + \ldots, F_f(k, l) \to \Phi_f(|k|/l), F_{ph}(k, l) \to \Phi_{ph}(|k|/l)
\]

as \( l \to \infty \), here, \( \Phi_f(z) \) and \( \Phi_{ph}(z) \) are some functions, one gets

\[
\delta m^2 = e^2 p^2 / 3 \int_0^\infty \Phi_{ph}(z)((d/dz)\Phi_f(z))dz
\]

as \( l \to \infty \), cf. eq. (1.15).

1.4.4. If the integral here equals zero, the second order perturbation theory consideration is compatible with the Lorentz invariance.
1.5. In principle, the term $O(1/V)$ in eq. (2.23) is able to reverse the result of my consideration. Let it be, e.g., $O(1/V) = \text{const}(\sum_{\lambda=1,2} q(-r, \lambda)q(r, \lambda))^2/V$ and $\text{const} > 0$. Then, the operator (2.23) will be bounded from below so that my consideration cannot exclude possibility that the Hamiltonian $H_{QED}$ possesses the ground state. Let us denote it by $\Omega_0$. Let us also denote by $\Omega_{0\text{good}}$ the ground state of the spinor QED, which it would possess, were the operator (2.23) without the term $O(1/V)$ be bounded from below. The point is that these two vacua are as drastically different as for instance the ground states of the quantum mechanical Hamiltonians $H_{1,0}$ and $H_{-1,1/V}$, $V \to +\infty$, where $H_{a,b} = -(d/dz)^2 + az^2 + bz^4$.

1.6. Equation (0.3) and (0.4) (if $e^2 \ll 1$) testify to the existence of the squared photon mass of the order of magnitude $\sim l^2$. The $e^2$ perturbation theory contribution to this quantity is negative in the spinor QED and is positive in the scalar QED.

2. THE PROOF OF THE STATEMENT OF THE ABSTRACT

I shall prove this statement in several steps.

2. At first, I shall average the Hamiltonian (1.1) over the normalized photon state $\Omega_{ph}$,

$$\Omega_{ph} = \text{const}\exp(-\omega[q(0)^2 + \sum_{k \neq 0, k \neq \pm r, \lambda=1,2} q(k, \lambda)q(-k, \lambda)], r \equiv |r|, r, \omega > 0,$$

i.e., I shall consider the transformation

$$H_{QED} \to H_{QED1} = \int \Omega_{ph}^* H_{QED} \Omega_{ph} dS(r),$$

(2.2)

see eq. (0.3a), and analogous transformation $h \to h_1$. Then, one gets $H_p \to \text{const}, H_{0f} \to H_{0f}, H_c \to H_c,$

$$H_{0ph} \to H_{0ph1} \equiv \sum_{\lambda=1,2} \left[-\frac{\partial}{\partial q(r, \lambda)} \frac{\partial}{\partial q(-r, \lambda)} + r^2 q(r, \lambda)q(-r, \lambda)\right] + \text{const},$$

(2.3a)
\[ H_1 \rightarrow H_{11} \equiv \frac{e}{\sqrt{V}} \sum_{p,s;\sigma;\tau;\lambda} q(s,\lambda)[a^*(p + s, \sigma)b^*(-p, \tau)A(p + s, p; \sigma, \tau) + b(-p - s, \sigma)a(p, \tau)D(p + s, p; \sigma, \tau) + a^*(p + s, \sigma)a(p, \tau)B(p + s, p; \sigma, \tau) + b(-p - s, \sigma)b^*(-p, \tau)C(p + s, p; \sigma, \tau)] \cdot e(s, \lambda) \equiv H_{11}(a^*b^*) + H_{11}(ba) + H_{11}(a^*a + bb^*), \]

\[ H_2 \rightarrow M^2(\nu^2 \sum_\lambda q(r, \lambda)q(-r, \lambda) + \text{const}) \equiv H_{21}, \]

\[ H_{QED1} = H_{0f} + H_{0ph1} + H_{11} + H_e + \text{const}, \]

Here

\[ A(p + s, p; \sigma, \tau) = u^*(p + s, \sigma)\hat{\alpha}v(p, \tau), \]
\[ B(p + s, p; \sigma, \tau) = u^*(p + s, \sigma)\hat{\alpha}u(p, \tau), \]
\[ C(p + s, p; \sigma, \tau) = v^*(p + s, \sigma)\hat{\alpha}v(p, \tau), \]
\[ D(p + s, p; \sigma, \tau) = v^*(p + s, \sigma)\hat{\alpha}u(p, \tau). \]

2.1 Then, let us determine the function \( \Omega_f \) and the operator \( K \),

\[ \Omega_f = e^K|0>, \]

\[ K = \sum_{p,s,\sigma,\tau:s=\pm r} K(p + s, p; \sigma, \tau)a^*(p + s, \sigma)b^*(-p, \tau) \equiv \sum K(p + s, p), \]

(here \( |0> \) is the state of the fermion bare vacuum: \( a(p, \sigma)|0> = b(p, \sigma)|0> = 0 \) for all values of \( p \) and \( \sigma \) by the equation

\[ (H_{0f} + H_{11}(a^*b^*))\Omega_f = 0. \]

One easily gets

\[ K(p + s, p; \sigma, \tau) = -\sum_\lambda \frac{eq(s, \lambda)A(p + s, p; \sigma, \tau) \cdot e(s, \lambda)}{\sqrt{V(E(p) + E(|p + s|))}}. \]

In order to derive eq. (2.10) from eq. (2.9), it is sufficient to multiply eq. (2.9) by the operator \( e^{-K} \) and apply the formula

\[ e^{-K}Ae^K = A + [A, K] + \frac{1}{2}[A, K], K + ... \]
(where square brackets denote the commutator), to the operators $A_1 \equiv H_{0f}$ and $A_2 \equiv H_{11}(a^* b^*)$. For the second operator all the commutators in eq. (2.10a) disappear, analogously for the operator $A_1$ the decomposition in the r.h.s. of eq. (2.10a) reduces to its first two terms. Thus, equation (2.9) becomes trivial.

(Note that if the operator $A$ were for instance bilinear in annihilation operators and not contain derivatives with respect to boson variables, the series (2.10a) would reduce to its first three terms.)

It follows from eqs. (2.7) and (2.8) that

$$\Omega_f = \prod_{p} (1 + \sum_{s=\pm r} K(p + s, p) + \frac{1}{2} (\sum_{s=\pm r} K(p + s, p))^2) |0 >. \quad (2.11)$$

Here, $\prod_{p}$ denotes the product over all values of $p$, $|p| < l$, terms with $|p + s| > l$ have to be omitted.

Let us denote the quantity $\Omega'_f \Omega_f$ by $Q$. Eqs. (2.8) and (2.11) give

$$Q = (\prod_{p} (1 + \sum_{s=\pm r} D_1(p + s, p) + \sum_{s=\pm r} D_2(p + s, p) + D_3(p, r)),$$

$$D_1(p + s, p) = \langle 0 |^* K(p + s, p)^* K(p + s, p) |0 >,$$

$$D_2(p + s, p) = \langle 0 |^* (K(p + s, p)^*)^2 K(p + s, p)^2 |0 > /4,$$

$$D_3(p, r) = \langle 0 |^* K(p + r, p)^* K(p - r, p)^* K(p - r, p) K(p + r, p) |0 >,$$

$$D_1(p + s, p) = O(1/V), D_2(p + s, p) = O(1/V^2), D_3(p, r) = O(1/V^2). \quad (2.12)$$

We shall introduce the quantities $Q_1$ and $D_1$,

$$Q_1 = e^{D_1}, D_1 = \sum_{p, s; s=\pm r} D_1(p + s, p). \quad (2.13)$$

**2.1.1.** The following important formula holds:

$$Q = Q_1(1 + O(1/V)). \quad (2.14)$$

**2.1.2.** Equations (2.8) and (2.10) result in the definition

$$K(p + s, p) \equiv \sum_{\lambda} q(s, \lambda) K(p + s, p, \lambda). \quad (2.15)$$

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Here the function $K(p + s, p, \lambda)$ does not depend on the vector potential variables $q(0), q(k, \lambda)$. Eqs. (2.6), (2.9), (2.10), (2.14) and (2.15) give

$$\Omega_f^*(H_{0f} + H_{11})\Omega_f = \Omega_f^*(H_{11}(ba) + H_{11}(a^*a + bb^*))\Omega_f \equiv Z_1 + Z_2,$$

(2.16)

$$Z_1 = -Qe^2 \sum_{\lambda} q(r, \lambda)q(-r, \lambda)Z(r, m, l) + O(1/V),$$

(2.17)

$$Z(r, m, l) = \frac{2}{(2\pi)^3} \int_{|p|<\ell,|p+r|<\ell} \frac{E(p + r)E(p) + (pr)^2/r^2 + pr - m^2}{E(p)E(p + r)(E(p) + E(p + r))} dp.$$  

(2.17a)

The quantity $Z_2$ evidently equals zero:

$$Z_2 = 0.$$  

(2.18)

Let us introduce the notation $\Omega_f^1$:

$$\Omega_f^1 \equiv \Omega_f/\sqrt{Q}, \quad \Omega_f^*\Omega_f^1 = 1.$$  

(2.19)

There hold the formulas (see Appendix A)

$$\Omega_f^* \sum_\lambda \frac{\partial}{\partial q(r, \lambda)} \frac{\partial}{\partial q(-r, \lambda)} \Omega_f^1 = \sum_\lambda \left[ \frac{\partial}{\partial q(r, \lambda)} \frac{\partial}{\partial q(-r, \lambda)} \right] + \sum_{s = \pm r} X(s, \lambda) \frac{\partial}{\partial q(s, \lambda)} X(s, \lambda) = O(1/V),$$

(2.20)

$$Y = const + O(1/V).$$  

(2.20a)

2.1.3. Now let us consider the quantity $C$,

$$C = \Omega_f^2 H_c \Omega_f^1.$$  

(2.21)

It is convenient to represent $H_c$ in a normal form. We shall symbolically write down this representation as $H_c = const_1 + const_2(a^*a + b^*b) + const_3(a^*b^* + ba) + const_4(a^*a^*b^* + a^*b^*ba + ba^*a)$. Correspondingly, we shall represent $C$ as $C = C_1 + C_2 + C_3 + C_4$. Then, $C_1$ does not depend on the variables $q(s, \lambda), s = \pm r$, while $C_2$ and $C_4$ depend on these variables quadratically and $C_3 = 0$. Rotational invariance and dimensional considerations give

$$C = e^2 mf(m/l, r/l) - \sum_\lambda q(r, \lambda)q(-r, \lambda)e^4 d(m/l, r/l)l^2 + O(1/V).$$  

(2.22)
Here, \( f(x, y) \) and \( d(x, y) \) are some functions.

2.2. Equations (2.3a), (2.3b), (2.5) and (2.16-22) prove the formula

\[
\Omega_{f_1}^* H_{QED1} \Omega_{f_1} = - \sum_{\lambda=1,2} \left[ \frac{\partial}{\partial q(r,\lambda)} \frac{\partial}{\partial q(-r,\lambda)} \right] + 2q(r,\lambda)q(-r,\lambda)(e^2 c(m/l, r/l) + e^4 d(m/l, r/l)l^2) + \text{const} + O(1/V), c(x, 0) > 0, \tag{2.23}
\]

-cf. the formulas (0.3) and (0.4). Equation (2.4) gives

\[
\Omega_{f_1}^* H_{2,1} \Omega_{f_1} = M^2 \left( \sum_{\lambda=1,2} q(r,\lambda)q(-r,\lambda) + \text{const} \right).
\]

Last two formulas complete the task of this section.

2.2.1. The starting point of my consideration of the problem of unbound-
edness from below of the operators (2.23) and the like is the statement that the operator \(-\frac{d}{dz^2} - \gamma^2 z^2, \gamma^2 > 0\), is unbounded from below.

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APPENDIX A

Here I shall prove eq. (2.20a). Equations (2.19) and (2.20) give

\[
X(s, \lambda) = \Omega_f^* \sum_p K(p + s, p, \lambda)\Omega_f / Q + \sqrt{Q} \frac{\partial}{\partial q(s, \lambda)} (1/\sqrt{Q}), \tag{A1}
\]

\[
Y = \Omega_f^* \sum_{p_1, p_2, \lambda} K(p_1 + r, p_1, \lambda)K(p_2 - r, p_2, \lambda)\Omega_f / Q
\]

\[
+ \Omega_f^* \sum_{p, s, \lambda; s = \pm r} K(p - s, p, \lambda))\Omega_f / \sqrt{Q} \frac{\partial}{\partial q(s, \lambda)} (1/\sqrt{Q})
\]

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\[
+ \sum_{\lambda} \sqrt{Q} \frac{\partial}{\partial q(r, \lambda)} \frac{\partial}{\partial q(-r, \lambda)} \left( \frac{1}{\sqrt{Q}} \right) \equiv Y_1 + Y_2 + Y_3.
\] (A2)

Equations (2.11)- (2.14) give
\[
\Omega^* \sum_{p} K(p + s, p, \lambda) \Omega_f = \sum_{p} < 0|K(p + s, p, \lambda)|0 > Q(1 + O(1/V))
\]
\[
= \frac{1}{2} \frac{\partial}{\partial q(s, \lambda)} Q_1[1 + O(1/V)],
\] (A3)

\[
Y_1 = \sum_{p_1, p_2, \lambda} < 0|K(p_1 + r, p_1)|0 >
\]
\[
< 0|K(p_2 - r, p_2)|0 > (1 + O(1/V))
\]
\[
= \frac{1}{2} Q_1^{-2} \sum_{\lambda} \frac{\partial Q_1}{\partial q(r, \lambda)} \frac{\partial Q_1}{\partial q(-r, \lambda)} (1 + O(1/V)).
\] (A4)

It follows from equations (A2) and (A3) that
\[
Y_2 = \frac{1}{2} \sum_{s = \pm r, \lambda} \left( \frac{\partial}{\partial q(s, \lambda)} \left( \frac{1}{\sqrt{Q}} \right) \right) \left( \frac{\partial}{\partial q(-s, \lambda)} Q_1 \right) / Q_1 (1 + O(1/V)).
\] (A5)

Now note that eqs. (2.10)-(2.15) result in the formula
\[
Q = \exp \left( \text{const} \left( \sum_{\lambda = 1, 2} q(r, \lambda) q(-r, \lambda) \right) \right) (1 + O(1/V)).
\] (A6)

So, equations (A1)-(A6) and eq. (2.14) entail eq. (2.20a). This result completes the consideration of Appendix A.

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