A subjective supply–demand model: the maximum Boltzmann/Shannon entropy solution

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Abstract. The present authors have put forward a projective geometry model of rational trading. The expected (mean) value of the time that is necessary to strike a deal and the profit strongly depend on the strategies adopted. A frequent trader often prefers maximal profit intensity to the maximization of profit resulting from a separate transaction because the gross profit/income is the adopted/recommended benchmark. To investigate activities that have different periods of duration we define, following the queuing theory, the profit intensity as a measure of this economic category. The profit intensity in repeated trading has a unique property of attaining its maximum at a fixed point regardless of the shape of demand curves for a wide class of probability distributions of random reverse transactions (i.e. closing of the position). These conclusions remain valid for an analogous model based on supply analysis. This type of market game is often considered in research aiming at finding an algorithm that maximizes profit of a trader who negotiates prices with the Rest of the World (a collective opponent), possessing a definite and objective supply profile. Such idealization neglects the sometimes important influence of an individual trader on the demand/supply profile of the Rest of the World and in extreme cases questions the very idea of demand/supply profile. Therefore we put forward a trading model in which the demand/supply profile of the Rest of the World induces the (rational) trader to (subjectively) presume that he/she lacks (almost) all knowledge concerning the market but his/her average frequency of trade. This point of view introduces maximum entropy principles into the model and broadens the range of economic
A subjective supply–demand model phenomena that can be perceived as a sort of thermodynamical system. As a consequence, the profit intensity has a fixed point with an astonishing connection with Fibonacci classical works and looking for the quickest algorithm for obtaining the extremum of a convex function: the profit intensity reaches its maximum when the probability of transaction is given by the golden ratio rule \((\sqrt{5} - 1)/2\). This condition sets a sharp criterion of validity of the model and can be tested with real market data.

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of, possibly random, duration we define, following queuing theory [1], the profit intensity as a measure of this economic category [2]. An acceptable definition of the profit must provide us with an additive function. It seems to us that the notion of the interval interest rate used in this paper leads to consistent results.

Nevertheless, such models, although simple and elegant, have several drawbacks from both theoretical and practical points of view. This type of market game often forms the basis for research aiming at finding an algorithm that maximizes profit of an agent who negotiates prices with the Rest of the World (a collective opponent denoted by RW in this paper), possessing a definite and objective supply profile. But one cannot claim that there is always a unique, adversary-independent probability distribution function, pdf(q), of the agent demand or supply profile expressed as a function of the price q. Actually, the RW demand profile (i.e. the shape of its pdf(q) curve) in a play against an agent, say Alice, results from the interaction among all the agents in question. Alice can probe into the RW demand (or supply) only via past events analysis (note that stock exchange regulations often allow the possibility of (at least partial) invisibility of bids—therefore one cannot be sure of the actual volume of demand or supply). Unfortunately, even a thorough analysis can produce paradoxical or unwanted recommendations (the winner’s curse etc). If such market phenomena are analyzed from the game theory point of view, we have in hand interesting new (natural?) tools for analysis of paradoxes that follow from quantum game theory [3]4. Such a possibility would be welcome because the non-Gaussian shape of the demand (supply) curve suggests the existence of half-quantum Giffen goods [11,12]. Obstacles in ‘quantization’ of such models can be overcome by replacing the maximum Boltzmann/Shannon entropy principle with the requirement that the Fisher information achieves its minimum (a discussion on the connection between the principle of the minimum of Fisher information and equations of quantum theory can be found in [13]). In this way a simple method of quantum-like game theory models that stem from statistical considerations, as we show below, allow for the analysis of subjectivity in strategy selection.

This paper is organized as follows. First we describe the merchandising mathematician model put forward in [2] and quote the relevant definitions. Then we argue for the use of logarithmic quotations and define the logarithmic rate of return and briefly describe the advantages of a projective geometry approach and scaling invariance of the resulting models [4]. The results showing the usefulness of Boltzmann/Shannon entropy as a measure of strategy quality and the probability of making profits will be given in section 3.

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3 The winner’s curse says that in auctions of some types the winner will tend to overpay (unless the bidder takes into account the winner’s curse when bidding). The auctioned item is of more or less equal value to all bidders. The winner of an auction is the bidder who submits the highest bid and is likely to overpay. The winner’s curse does not necessarily have ill effects.

4 ‘Quantization’ often suggests ways of avoiding paradoxes in game theory due to the absence of limitations of the classical theory of probability. This approach has interesting consequences in decision sciences; cf for example papers by Aerts and Czachor [5], Haven [6], Khrennikov [7], Khrennikov and Haven [8], Piotrowski and Sładkowski [9], Sornette [10] and others. Of course, we do not claim that quantum processes play an explicit role here. The reason for the use of the probability amplitude or ‘quantum probabilistic measure’ is the nondistributive orthomodular lattice occurring in some special macroscopic situations: the logical structure is the same as that for the quantum phenomena.

5 A Giffen good is one which people consume more of as price rises, violating the law of demand. Evidence for the existence of Giffen goods is limited, but microeconomic mathematical models explain why such a thing could exist. Giffen goods are named after Sir Robert Giffen, who was attributed as the author of this idea by Marshall.
There also the astonishing emergence of the golden ratio as a characterization of the inclinations towards concluding deals of the most wealthy agents is discussed. Finally we will point out some issues that are yet to be addressed.

2. The merchandising mathematician model

2.1. The profit intensity

Most market/trading activities can be reduced to the following simple scenario: one buys a good and tries to sell it, possibly, at a profit. In this section we present a simple mathematical model of such activities (repeated many times). To proceed, let us denote by $t$, $v_t$ and $v_{t+\tau}$ the beginning of an interval of the duration $\tau$, and the value of the undertaking (asset) at the beginning and at the end of the interval in question, respectively. We will measure profits with the help of the logarithmic rate of return $r_{t,t+\tau}$ defined as

$$r_{t,t+\tau} \equiv \ln \left( \frac{v_{t+\tau}}{v_t} \right).$$

The expectation value of the random variable $\xi$ in one trading cycle (buying–selling or vice versa) is denoted by $E(\xi)$. If $E(r_{t,t+\tau})$ and $E(\tau)$ are finite, we can define the profit intensity for one cycle $\rho_t$ as

$$\rho_t \equiv \frac{E(r_{t,t+\tau})}{E(\tau)}.$$  \hspace{1cm} (2)

This formula is an immediate consequence of the Wald identity [16]:

$$E(S_{\tau'}) = E(X_1)E(\tau'),$$  \hspace{1cm} (3)

where $S_{\tau'} \equiv X_1 + \cdots + X_{\tau'}$ is the sum of $\tau'$ independent, identically distributed random variables $X_k$, $k = 1, \ldots, \tau'$, and $\tau'$ is the stopping time [1, 16]. (For simplicity, we assume that the autocorrelations of returns are negligible here.) The profit intensity that we have defined in equation (2) is just the expectation value of $X_1$ in the Wald identity equation (3). The expected profit is the left-hand side of the Wald identity. If we are interested in the profit expected in a time unit, we have to, according to Wald, divide the expected profit by the expectation value of the stopping time. This leads to the formula given in equation (2). We can also calculate the variance of the profit intensity by using the proposition 10.14.4 from Resnick’s book [16]:

$$E((S_{\tau'} - \tau' E(X_1))^2) = E(\tau') \text{Var}(X_1).$$

The expectation value of the profit during an arbitrary time interval, say $[0, T]$, is given by the formula

$$\rho_{0,T} \equiv \sum_{t \in [0,T]} \rho_t.$$  \hspace{1cm} (5)

The proposed definition of the profit intensity is a very convenient starting point for analysis of various models based on the subjectivity of demand/supply ideas; see the models discussed below. Relations to the commonly used measures of profits (returns) can be easily obtained by simple algebraic manipulations and this is omitted here.

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2.2. The merchandising mathematician model

The simplest possible market consists in exchanging two goods which we will call the asset and the money and denote by $\Theta$ and $\$, respectively. The model consists in the repetition of two simple basic moves (in principle, the process is continued endlessly):

(i) The first move consists in a rational buying (see below) of the asset $\Theta$ (exchanging $\$ for $\Theta$).

(ii) The second move consists in a random selling of the amount of the asset purchased in the first move, $\Theta$ (exchanging $\Theta$ for $\$).

By rational buying we mean a purchase bound by a fixed withdrawal price $-a$ that is a ‘logarithmic quotation’ for the asset $\Theta$, $-a$, above which the trader gives up buying. A random selling can be analogously identified with the situation when the withdrawal price is set to $-\infty$ (the trader in question is always bidding against the rest of the traders). The order of these transactions can be reversed and, in fact, is conventional. Note that the quotation method does not matter to the process discussed as long as it can be repeated many times. Let $V_\Theta(t)$ and $V_\$(t)$ denote some amounts of the asset and the money specified at the moment $t$, respectively. If at that moment these assets are exchanged in the proportion $V_\$(t):V_\Theta(t)$ then we call the number

$$p_t \equiv \ln(V_\$(t)) - \ln(V_\Theta(t))$$

(6)

the logarithmic quotation for the asset $\Theta$ (in units of $\$). If the trader buys some amount of the asset $\Theta$ at the quotation $p_{t_1}$ at the moment $t_1$ and sells it at the quotation $p_{t_2}$ at the later moment $t_2$ then the trader’s profit (or more precisely the logarithmic rate of return) will be equal to

$$r_{t_1,t_2} = p_{t_2} - p_{t_1}.$$  

(7)

The logarithmic rate of return, in contrast to $p_t$, does not depend on the choice of unit used to measure the assets in question. From the projective geometry point of view [4], $r_{t_1,t_2}$ is an invariant and $p_t$ is not; cf the discussion of demand and supply curves in section 2.3. Let us suppose now that the model describes a stationary process, that is the probability distribution $\eta(p)$ of the random variable $p$ (the logarithmic quotation) does not depend on time. Note that it is sufficient to know the logarithmic quotations up to an arbitrary constant because what matters is the profit and profit is always a difference of quotations. This is analogous to the classical physics case (e.g. Newton’s gravity) where only differences of the potential matter. Therefore we suppose that the expectation value of the random variable $p$ is equal to zero, $E(p) = 0$. In addition, we also suppose that the market is large enough not to be influenced by the activity of our trader. Let the expression $[\text{sentence}]$ take the value 0 or 1 if the sentence is false or true, respectively (the Iverson convention) [17]. The mean time of a random transaction (buying or selling) is denoted by $\theta$. The value of $\theta$ is fixed in our model due to the stationarity assumption. Besides, to eliminate paradoxes (e.g. infinite profits during finite time spreads), $\theta$ should be greater than zero. Let $x$ denote the probability that the rational buying does not occur:

$$x \equiv E_\eta([p > -a]).$$

(8)
The expectation value of the rational buying time of the asset $\Theta$ is equal to
\[ \theta((1-x) + 2x(1-2x) + 3x^2(1-2x) + 4x^3(1-x) + \cdots). \] (9)

The ratio of the expected value of the duration of the whole buying–selling cycle ($\tau$) and the expected time of a random reverse transaction ($\theta$) is given by
\[
\frac{E_\eta(\tau)}{\theta} = 1 + (1-x) \sum_{k=1}^{\infty} kx^{k-1}
= 1 + (1-x) \frac{d}{dx} \sum_{k=0}^{\infty} x^k
= 1 + (1-x) \frac{d}{dx} \frac{1}{1-x} = 1 + \frac{1}{1-x}.
\]
Therefore the mean duration of the whole cycle is given by
\[
E_\eta(\tau) = (1 + (E_\eta([p \leq -a]))^{-1}) \theta. \tag{10}
\]

The logarithmic rate of return for the whole cycle is
\[
r_{t,t+\tau} = -p_\rightarrow + p_\rightarrow, \tag{11}
\]
where the random variable $p_\rightarrow$ (quotation at the moment of purchase) has the distribution restricted to the interval $(-\infty, -a]$:
\[
\frac{[p \leq -a]}{E_\eta([p \leq -a])} \eta(p). \tag{12}
\]
The random variable $p_\rightarrow$ (quotation at the moment of selling) has the distribution $\eta$, as the selling is at random. The expectation value of the profit after the whole cycle is\footnote{\(\rho_\eta(a)\) should not be mistaken for the profit intensity, also denoted by $\rho$; cf equation (2). They differ by a factor equal to the (inverse) average cycle duration.}
\[
\rho_\eta(a) = \frac{-\int_{-a}^{0} p \eta(p) \, dp}{1 + \int_{-\infty}^{-a} \eta(p) \, dp}, \tag{13}
\]
which follows from equations (5) and (11). This function has very interesting properties (we will often drop the subscript $\eta$ in the following text). First we quote [2]:

**Theorem 1.** The maximal value of the function $\rho$, $a_{\text{max}}$, lies at a fixed point of $\rho$, that is fulfills the condition
\[
\rho(a_{\text{max}}) = a_{\text{max}}. \tag{14}
\]

Such a fixed point $a_{\text{max}}$ exists and $a_{\text{max}} > 0$.\footnote{It is tempting to claim that the function $\rho$ is a contraction but this is not the case. Simple inspection reveals that if the probability has a very narrow and high maximum then $\rho$ is not a contraction in the vicinity of the maximum. Fortunately, for any realistic probability density one can start at any value of $a$ and by iteration wind up at the fixed point (cf the Banach fixed point theorem). We skip the details because they are technical and unimportant for the conclusions of the paper.}

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Example. For the normal distribution with the variance $\sigma$ and expectation value $\hat{p}$ of a random variable $p$
\[
\eta(p, \sigma) \equiv \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(p - \hat{p})^2}{2\sigma^2}\right)
\]
the expectation value of the profit during a whole cycle $\rho(a, \sigma)_{\text{normal}}$ (we have explicitly shown the dependence on the variance $\sigma$) has a nice scaling property:
\[
\rho(a, \sigma)_{\text{normal}} = \sigma \rho(a, 1)_{\text{normal}},
\]
and it is sufficient to work out the $\sigma = 1$ case only. If this is the case we get the maximal expectation value of the profit for $a = 0.27603$. Therefore, according to theorem 1, the maximal expected profit is also equal to 0.27063. Recall that in our normalization this price is $e^0 = 1$.

The fixed point theorem recommends the following simple market strategy that maximizes the trader’s expected profit on an efficient market: accept profits equal to or greater than the one you have formerly achieved on average during the characteristic time of transaction which is, roughly speaking, an average time spread between two opposite moves of a player (e.g. buying and selling the same asset). Unfortunately, such strategy recommendations still involve some important subjective factors that reflect personal or even instantaneous attitudes towards market state.

- Would a change of information measure for market strategies result in interesting recommendations?
- Do different measures of information content result in different trading recommendations?
- Are there more useful (convenient?) information measures than the Boltzmann/Shannon one?

2.3. The demand and supply curves

The textbooks on economics abound in graphs and diagrams presenting various demand and supply curves\textsuperscript{8}. This illustrates the importance that economists attach to them despite the fact that the whole idea of supply/demand profiles has serious drawbacks, both theoretical and practical [19]. Such an approach is also possible within the MM model. To this end let us consider the functions (the subscripts $s$ and $d$ denote supply and demand, respectively)
\[
F_s \equiv E_{\eta_s}([\xi \leq x]) = \int_{-\infty}^{x} \eta_s(p) \, dp,
\]
and
\[
F_d \equiv E_{\eta_d}([\xi \leq x]) = \int_{x}^{\infty} \eta_d(p) \, dp,
\]
where we have introduced two, in general cases different, probability distributions $\eta_s$ and $\eta_d$. They may differ due to the existence of a monopoly, specific market regulations, taxes,\textsuperscript{8} Blaug [18] quotes at least a hundred such diagrams.
cultural habits and so on. Let us recall that two methods of presenting demand/supply curves prevail in the literature. The first one (the Cournot convention) is based on the assumption that the demand is a function of prices. The Anglo-Saxon literature prefers the Marshall convention with reversed roles of the coordinates. The demand or supply profile is not always a monotonic function of prices (cf the discussion below on the turning back of demand/supply curves); therefore the Marshall convention seems to be less convenient as one cannot use the notion of a function. The MM model with the price-like parameter $x$ refers to the Cournot convention. Therefore, for a given value $x$ of the logarithm of the price of an asset $\Theta$, the value of the supply function $F_s(x)$ is given by the probability of the purchase of a unit of $\Theta$ at the price $e^x$. The asset could be provided by anyone who is willing to sell it at the price $e^x$ or lower. The function $F_d(x)$ can be defined analogously. If we neglect the sources of possible differences between $\eta_s$ and $\eta_d$ and, in addition, suppose that at any fixed price there are no indifferent traders (that is everybody wants to sell or buy\(^9\)), then we can claim that

$$E_{\eta_s}([\xi \leq \eta]) + E_{\eta_d}([\xi > \eta]) = 1.$$  \hspace{1cm} (19)

The differentiation of equation (19) leads to $\eta_s = \eta_d$. Under these conditions the price $e^x$ for which $E_{\eta_s}([\xi \leq y]) = E_{\eta_d}([\xi > y]) = \frac{1}{2}$ establishes the equilibrium price (actually, the most frequent one).

2.4. The projective geometry point of view

The model described has a natural setting in the projective geometry formalism [4]. In this approach the market is described in the $N$-dimensional real projective space, $\mathbb{R}P^N$, that is the $(N+1)$-dimensional vector space $\mathbb{R}^{N+1}$ (one real coordinate for each asset) subject to the equivalence relation $v \sim \lambda v$ for $v \in \mathbb{R}^{N+1}$ and $\lambda \neq 0$. That is, we assume infinite divisibility of assets. Actually, a finite field approximation is possible [14,15]—this problem will be discussed elsewhere. For example we identify all portfolios having assets in the same proportions. The actual values can be obtained by rescaling by $\lambda$. The details can be found in [4]. In this context separate profits gained by buying or selling are not invariant (coordinate free) but there is an invariant: the anharmonic ratio of four points. For example for the exchange ratio it gives the relative change of quotation

$$\{S, Q, Q', E\} := \frac{c_{s \rightarrow E}'}{c_{s \rightarrow E}} = \frac{q_{s}' q_{E}}{q_{s}' q_{E}} = \frac{|Q' E| |Q S|}{|Q' S| |Q E|} = \frac{P(Q'E'O)}{P(Q'S'O)} \frac{P(Q'O'S)}{P(Q'O'\Theta)},$$

(20)

where $c_{s \rightarrow E} := q_{s}/q_{E}$ is the exchange ratio $S \rightarrow E$ (one gets $q_{s}$ dollars for $q_{E}$ euros) etc and $P(\Delta_{ABC})$ denotes the area of the triangle with vertices $A$, $B$, and $C$. In figure 1 the lengths of the segments $Q'$ and $Q'E$ are proportional to $q_{s}$ and $q_{E}$, respectively. The profit $r_{t+t+\tau}$ gained during the whole cycle is given by the logarithm of an appropriate anharmonic ratio (see below) that is invariant (e.g. its numerical value does not depend on units chosen to measure the assets). In the space $\mathbb{R}P^N$ this anharmonic ratio denoted by $[\Theta, U_{-\Theta}, U_{\Theta}, S]$ concerns the pair of points

$$U_{-\Theta} \equiv \{v, v \cdot e^{p_{\Theta}}, \ldots\} \quad \text{and} \quad U_{\Theta} \equiv \{w, w \cdot e^{p_{\Theta}}, \ldots\}$$

(21)

\(^9\) That is we consider only active agents.
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Figure 1. Graphical representation of exchange ratios given by equation (20). Lengths of the segments \( QS \) and \( Q\varepsilon \) are proportional to \( q_s \) and \( q_\varepsilon \).

and the pair \( \Theta \) and \$. The last pair results from the crossing of the hypersurfaces \( \Theta \) and \$ corresponding to the portfolios consisting of only one asset, \( \Theta \) or \$, respectively, and the line \( U_{-\Theta}U_{\Theta-} \). The dots represent other coordinates (not necessary equal for the two points). The line connecting \( U_{-\Theta} \) and \( U_{\Theta-} \) may be represented by the one-parameter family of vectors \( u(\lambda) \) with \( \mu \)-coordinates given by

\[
u(\lambda) \equiv \lambda(U_{-\Theta})_\mu + (1 - \lambda) \cdot (U_{\Theta-})_\mu.
\]

This implies that the values of \( \lambda \) corresponding to the points \( \Theta \) and \$ are given by the conditions

\[
u_0(\lambda) = \lambda_0(U_{-\Theta})_0 + (1 - \lambda) \cdot (U_{\Theta-})_0 = 0
\]

and

\[
u_1(\lambda) = \lambda_1(U_{-\Theta})_1 + (1 - \lambda) \cdot (U_{\Theta-})_1 = 0.
\]

Substitution of equation (21) leads to

\[
u = \frac{w}{w - \nu}
\]

and

\[
u = \frac{we^{p_{\Theta-}}}{we^{p_{\Theta-}} - ve^{p_{-\Theta}}}
\]

The calculation of the logarithm of the cross ratio \([\Theta, U_{-\Theta}, U_{\Theta-}, \$]\) on the line \( U_{-\Theta}U_{\Theta-} \) leads to

\[
\ln[U_{-\Theta},U_{\Theta-}] = \ln \left[ \frac{we^{p_{\Theta-}}}{we^{p_{\Theta-}} - ve^{p_{-\Theta}}}, 1, 0, \frac{w}{w - \nu} \right] \\
\ln \left[ \frac{we^{p_{\Theta-}}}{we^{p_{\Theta-}} - ve^{p_{-\Theta}}}, \frac{w}{w - \nu} \right] = p_{\Theta-} - p_{-\Theta}
\]

which corresponds to equation (7). Let us look more closely at the problem of trading in a single asset. Consider

\[
[\Theta, U_{-\Theta}, U_{\Theta-}, \$]
\]

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for

\[ U_{\Theta} := (v, v e^{p_{\Theta}}, \ldots) \quad \text{and} \quad U_{\Theta} := (w, w e^{p_{\Theta}}, \ldots) \]  

(29)

and the points \( \Theta \) and \( \$ \) given by crossing of the prime \( U_{\Theta} U_{\Theta} \) and one-asset portfolios: \( \bar{\Theta} \) and \( \bar{\$} \) corresponding to assets \( \Theta \) and \( \$ \). The logarithm of the cross ratio \([\Theta, U_{\Theta}, U_{\Theta}, \$] \)
on the straight line \( U_{\Theta} U_{\Theta} \) is equal to

\[
\ln[\Theta, U_{\Theta}, U_{\Theta}, \$] = \ln \left[ \frac{w e^{p_{\Theta}}}{w e^{p_{\Theta}} - v e^{p_{\Theta}}} , 1 , 0 , \frac{w}{w - v} \right] 
\]

(30)

\[
= \ln \left[ \frac{v w e^{p_{\Theta}}}{v e^{p_{\Theta}} - w} \right] = p_{\Theta} - p_{\Theta}. 
\]

(31)

In contrast to the classical economics case, the balance in the MM model does not result
in uniform quotations (prices) for the asset \( \Theta \) but only in a stationarity of the supply and
demand functions \( E_{\eta_1}(\zeta \leq x) \) and \( E_{\eta_2}(\zeta \leq x) \). Therefore the MM model is not valid
when the changes in the probabilities happen during periods shorter than or of the order
of the mean time transaction \( \theta \). Fortunately, the stochastic interpretation of the supply
and demand profiles of agents presented above remains valid in such situations. Moreover,
we can consider piecewise decreasing functions \( F_s \) and \( F_d \). These functions cease to be
probability distribution functions because their derivatives (probability densities) are not
positive definite. This generalization corresponds to the effect of turning back of the
supply and demand curves which often happens for work supplies and, in general, for the
so called Giffen goods [11, 12]. In the Marshall convention these curves are not diagrams
of functions at all and in the Cournot convention these curves are diagrams of multivalued
functions. Note that in this way probability densities that are not positive definite
(e.g. Wigner functions) gain an interesting economic reason for their very existence [20].

2.5. Strategy selection

By a choice of stochastic process consistent with the MM model one can determine the
dynamics of such a model; cf [21]. Therefore we suspect that the departure from the
laws of supply and demand might be the first known example of a macroscopic reality
governed by quantum-like rules [22]. Such hypothetical quantum-like economics could
have started with the evidence given by Robert Giffen in the British Parliament [11] and
actually could have an earlier origin than quantum physics. It should be noted here that
from the quantum game theory point of view the Gauss distribution function is the only
supply (demand) curve that fulfills the physical correspondence principle. The authors will
devote a separate paper to this very interesting problem. Let us note that the distribution
functions allow for correct description of the famous Zeno paradoxes (When do grains form
a pile? When do you start to be bald?) because the introduction of probabilities removes
the original discontinuity; for example the problem of moral rightness of prices: if the
price is low (state 0) nobody wants to sell and if the price is high (state 1) everybody
wants to sell. Without the probability theory we are not able to describe intermediate
states which, in fact, are typical of the markets. Does this suggest that the MM model can

10 Economists typically place price on the vertical axis and quantity on the horizontal axis. Cournot derived the
first formula for the rule of supply and demand as a function of price and was the first to draw supply and demand
curves on a graph.
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also be applied to problems where there is a necessity of finding the maximum (minimum) of a profit intensity-like parameter?

3. Subjective character of market demand

The probability distribution \( \eta(p) \) that describes the Rest of the World strategy in the game against a single agent (Alice) is de facto of transcendental nature: agents finalizing a deal can only observe (measure) its results as values of execution prices in a way that harks back to Plato’s distinction between the idealized form of a thing and its imperfect realization in our world, as if by a shadow of the form in Plato’s Cave. They might erroneously think that the shadows are ‘real’ and not just projections of the outside world. Note that the problem is common [23]. Therefore, as statistical physics and Shannon’s information theory teach us, the best they can do is to approximate \( \eta(p) \) by the appropriate Gibbs distribution following from the minimum of their knowledge (information), that is the maximum of their information entropy, constrained by the known market parameters. In the analysis of the effectiveness of strategies the distribution function \( \eta(p) \) has to be replaced by its ‘shadow’. Let us denote the ‘shadow’ distribution function by pdf\((p)\). The key problem is that the concrete form of pdf\((p)\) strongly depends on the model of information gathering adopted, the data mining and the information measures being used. In this paper we propose to select Boltzmann/Shannon information entropy \( S \) to this end. Of course, there are other interesting alternatives and the conclusion concerning the shape of \( \eta(p) \) might differ in a dramatic way! We will refer to the class of assumptions regarding the market, measures of information, data mining, utility functions and so on adopted by the agent (Alice) in the process of determining pdf\((p)\) as Alice’s imaginoscope (Alice’s wall in Plato’s Cave).

3.1. The model

Consider the simplest case when the domain of pdf\((p)\) is one-side bounded. The assumption that the selling prices are bounded below (and buying prices bounded above) could by justified by, for example, the principle of rational production (one does not sell at prices below production costs), the existence of minimal salary, regulation concerning usury etc. In addition, let us assume that Alice uses the minimal number of estimates (that is one) for pdf\((p)\). The obvious one is the probability of making the deal \( P := E([p-a]) = \int_a^{\infty} \text{pdf}(p) \, dp \) (the Iverson convention). As we assumed that the support of pdf\((p)\) is bounded below, say by 0, for an effective reconstruction of the measure pdf\((p)\) dp she needs at least one unbounded observable. Therefore, instead of the variables \([q-\psi]\) used in the quantum-like approaches [24,25] we will use the observable of surpassing the minimal return rate \( a \) for Alice, \((p-a)(p-a)\). Under these assumptions we get the following Gibbs-like distribution function for the shadow random variable \( p \in [0, \infty) \):

\[
\text{pdf}(p) = \frac{e^{-(p-a)(p-a)/T}}{a + T}.
\] (32)

For this specific distribution, it follows that the Lagrange multiplier \( T^{-1} \) with the value fixed for the expectation value \( P \) takes the functional form of the logarithm of the withdrawal price \( a \) scaled by a factor equal to the ratio of the average number of
transactions and the average number of transactions that have not been finalized:

\[ T^{-1} = \frac{P}{1 - F^a}. \]  

(33)

A plot of the function of equation (32) is presented in figure 2. According to the Laplace indifference principle, for prices up to the value \( a \) the probability of closing the deal is constant (Alice does not sell) and for higher ones the probability distribution is exponential. The corresponding market demand function (according to Alice’s imaginoscope!) is presented in figure 3. Note that we can observe (see figure 3) that the localization of \( a \) becomes obscure due to the ‘continuous character’ of the line tangent to the plot. The demand decreases monotonically as prices approach \( a \) and then exponentially!—a strategy that maximizes profit. By inserting pdf(\( p \)) into equation (13) as the probability distribution \( \eta \) and then into the equation (14) for the fixed point of \( \rho \),

\[ \text{pdf}(p) \]

\[ \text{cdf}(p) \]

Figure 2. The subjective probability distribution function of market prices.

Figure 3. The market demand function as (subjectively) perceived by Alice (the supplier).
we obtain the condition under which Alice’s profit is maximal:

\[ \int_a^\infty p \text{pdf}(p) \, dp \quad \frac{1}{1 + \int_a^\infty \text{pdf}(p) \, dp} = a. \quad (34) \]

After integration and simple manipulations the condition (34) reduces to a very simple formula for the probability of making transactions:

\[ P = \varphi := \frac{\sqrt{5} - 1}{2} \approx 0.618. \quad (35) \]

In the case of a bounded support of the demand, the probability \( P \) of making an optimal transaction for the most profitable (golden) Alice strategy is about 0.62. Due to the obvious symmetry of the model with respect to the involution \( \Theta \leftrightarrow \$, the solutions and conclusions for the Rest of the World supply and Alice’s demand profiles are analogous.

### 3.2. Measures of information content of marker shadows

If we apply the (continuous form) formula for Boltzmann/Shannon entropy

\[ S = - \int_0^\infty \ln(\text{pdf}(p)) \, \text{pdf}(p) \, dp \]

to subjective projection of the market supply probability distribution (32) we obtain an expression for the entropy of the knowledge about the market \( S(P) \) gathered by the agent, say Alice, during transactions initiated by her strategy \( a \). Figure 4 presents plots of the function \( S - B \), the relative entropy \( S \) corrected by the terms \( B = -\ln a \) (blue) and \( B = -\ln E(p) \) (red). The market entropy calculated modulo the expectation value of the logarithmic return \( \ln E(p) \) varies in an unimportant way over the whole domain of \( P \). This is not accidental. We have already advocated [2,4] the advantages of using logarithms of mean rates. We see that it is a quite good approximation as a measure of information available about markets. Unfortunately, the corrected relative entropy \( S - B \) has one unpleasant feature: depending on the choice of the parameter (constraint), substantial changes could be observed; cf figure 4. Nevertheless, abiding by this measure of information, we can use Fisher information for information content valuation while selecting strategies (that is, determining the subjective form of \( \text{pdf}(p) \)):

\[ I := E\left(\left(\frac{\partial \ln \text{pdf}(p)}{\partial p}\right)^2\right) = \int_0^\infty \left(\frac{\partial \ln \text{pdf}(p)}{\partial p}\right)^2 \text{pdf}(p) \, dp. \quad (36) \]

Plots of \( I(P) \) modified by the corresponding factors (i.e. units) \( a^{-2} \) (blue line) and \( (E(p))^{-2} \) (red line) are presented in figure 5. Now, the two curves become similar. The observed decrease in \( I(P) \) while \( P \) increases is caused by the contribution of the integral dominating over the domain \([0, a]\) ignored by Alice for expression (36), where the probability distribution \( \text{pdf}(p) \) is constant. Acceptance of Fisher information (36) as the measure of information results in smaller susceptibility to subjective changes in attitudes. This situation suggests that a completely new approach based on Fisher information might be more appropriate. Such an approach is described in [26]. Note that it is possible to
modify formula (36) for information $I$ (the general $N$-dimensional case):

$$H := -\frac{1}{2} \sum_{k=1}^{N} \ln \left( E \left( \left( \frac{\partial \ln \text{pdf}(p_1, \ldots, p_N)}{\partial p_k} \right)^2 \right) \right)$$

(37)

to get a new entropy measure that is consistent with Fisher information $I$ in the sense that it generates the same extrema of pdf($p$). But, in addition, it has the property of being a sum of two terms of which the first depends only on $a$ or $E(p)$ (constraint entropy) and the second depends only on the probability $P$. This constraint entropy is identical to the Boltzmann/Shannon constraint entropy. We present in figure 6 the curves that correspond to plots given in figure 4. The model based on maximization of the Boltzmann/Shannon entropy seems to be reliable below the extremal value $P = (\sqrt{5} - 1)/2 \approx 0.618$ (cf figure 6) but for market games with higher probabilities of buying by WR we envisage that the model based on the extremum of the entropy (37) would be better.
4. Conclusions

Searches for optimal solutions and fixed points are the key issues of contemporary mathematical economics and finance [27]. Such classical results as the generalized Brouwer theorem [28] and the Brown–Robinson iteration procedure [29] are widely applied and useful tools. The model presented in this paper model combines the two ideas with the information theoretical approach. The extension of the MM model to the randomized withdrawal price cases might also generalize the results of [30, 31] where thermodynamics of investors was considered and the temperature of portfolios was defined. The emergence of the golden ratio as a characterization of the inclinations towards concluding deals of the most wealthy agents is astonishing! This condition sets a sharp criterion of validity of the model and can be tested with real market data. Is this simply a coincidence or is there a deeper connection between these algorithms? There is no obvious answer to this question. The golden ratio emerges also in the most efficient algorithm for finding extrema of convex functions on a segment [32, 33]. It turns out that in the market game, the biggest of games, we can find traces of the golden ratio, so abundant in other phenomena; an excellent bibliography can be found in The Fibonacci Quarterly, the official journal of the Fibonacci Association.

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References

[1] Billingsley P, 1979 Probability and Measure (New York: Wiley)

[2] Piotrowski E W and Sladkowski J, 2003 Physica A 318 496

[3] Piotrowski E W and Sladkowski J, The next stage: quantum game theory, 2004 Mathematical Physics Frontiers ed C V Benton (New York: Nova Science Publishers)

[4] Piotrowski E W and Sladkowski J, 2007 Physica A 382 228

[5] Aerts D and Czachor M, 2004 J. Phys. A: Math. Gen. 37 L123

doi:10.1088/1742-5468/2009/03/P03035
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[6] Haven E, 2002 Physica A 304 507
[7] Khrennikov A Yu, Quantum-like probabilistic models outside physics, 2007 arXiv:physics/0702250v2 [physics.gen-ph]
[8] Khrennikov A Yu and Haven E, 2007 AIP Conf. Proc. 889 299
[9] Piotrowski E W and Sladkowski J, 2003 Int. J. Quant. Inform. 1 305
[10] Yukalov V I and Sornette D, Quantum decision theory, 2008 arXiv:0802.3597v1 [physics.soc-ph]
[11] Stigler G J, 1947 J. Polit. Econ. 55 152
[12] Sladkowski J, 2003 Physica A 324 234
[13] Frieden B R, 2004 Science from Fisher Information: A Unification (Cambridge: Cambridge University Press)
[14] Kustaanheimo P, 1950 Comment. Phys.-Math. Soc. Sci. Fenn. XV 19 1
[15] Järnefelt G, 1951 Ann. Acad. Sci. Fenn. AI 96 1
[16] Resnick S I, 1998 A Probability Path (Cambridge, MA: Birkhauser Boston)
[17] Graham R L, Knuth D E and Patashnik O, 1994 Concrete Mathematics (Reading, MA: Addison-Wesley)
[18] Blaug M, 1985 Economic Theory in Retrospect (Cambridge: Cambridge University Press)
[19] Osborn M F M, 2001 The Stock Market and Finance from the Physicist’s Point of View (Minneapolis, MN: Crossgar Press)
[20] Feynman R P, Negative probabilities in quantum mechanics, 1987 Quantum Implications, Essays in Honour of D Bohm ed B J Hiley and F D Peat (London: Routledge & Kegan Paul)
[21] Blaquiere A, Wave mechanics as a two-player game, 1980 Dynamical Systems and Microphysics (New York: Springer)
[22] Piotrowski E W and Sladkowski J, 2004 Quant. Finance 4 61
[23] Polyakov A M, The wall of the cave, 1998 arXiv:hep-th/9809057
[24] Piotrowski E W and Sladkowski J, 2002 Physica A 308 391
[25] Piotrowski E W and Sladkowski J, 2003 Physica A 312 208
[26] Piotrowski E W and Sladkowski J, A model of subjective supply–demand: the minimum Fisher of information solution, 2008 Talk Given at the SIGMAPHI 2008 Conf.; Cent. Eur. J. Phys. submitted
[27] Debreu G, 1981 Handbook of Mathematical Economics vol 2, ed K J Arrow and M D Intriligator (Amsterdam: Elsevier Science)
[28] Kakutani S, 1941 Duke Math. J. 8 457
[29] Robinson J, 1951 Ann. Math. 54 296
[30] Piotrowski E W and Sladkowski J, 2001 Acta Phys. Polon. B 32 597
[31] Piotrowski E W and Sladkowski J, 2001 Physica A 301 441
[32] Kiefer J, 1953 Proc. AMS 4 502
[33] Vorobiev N N, 2002 Fibonacci Numbers (Basel: Birkhauser)