On a topology and limits for inductive systems of \( C^* \)-algebras over partially ordered sets

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Abstract Motivated by algebraic quantum field theory and our previous work we study properties of inductive systems of \( C^* \)-algebras over arbitrary partially ordered sets. A partially ordered set can be represented as the union of the family of its maximal upward directed subsets indexed by elements of a certain set. We consider a topology on the set of indices generated by a base of neighbourhoods. Examples of those topologies with different properties are given. An inductive system of \( C^* \)-algebras and its inductive limit arise naturally over each maximal upward directed subset. Using those inductive limits, we construct different types of \( C^* \)-algebras. In particular, for neighbourhoods of the topology on the set of indices we deal with the \( C^* \)-algebras which are the direct products of those inductive limits. The present paper is concerned with the above-mentioned topology and the algebras arising from an inductive system of \( C^* \)-algebras over a partially ordered set. We show that there exists a connection between properties of that topology and those \( C^* \)-algebras.

Keywords \( C^* \)-algebra · Inductive limit · Inductive system · Partially ordered set · Topology

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1 Introduction

The motivation for the present paper comes from algebraic quantum field theory [1]–[9] and our previous work on inductive systems of C*-algebras [10]–[12]. The general framework of algebraic quantum field theory is given by a covariant functor. Usually that functor acts from a category whose objects are topological spaces with additional structures and its morphisms are structure preserving embeddings into a category describing the algebraic structure of observables. The standard assumption in quantum physics is that the second category consists of unital C*-algebras and unital embeddings of C*-algebras. The basic tool of the algebraic approach to quantum fields over a spacetime is a net of C*-algebras defined over a partially ordered set defined as a suitable set of regions of the spacetime ordered under inclusion [1]–[3].

In the papers [7]–[9] the authors consider nets containing C*-algebras of quantum observables for the case of curved spacetimes. The net that is constructed by means of semigroup C*-algebra generated by the path semigroup for partially ordered set is studied in [10]. The paper [11] contains results on limit automorphisms for inductive sequences of Toeplitz algebras which are closely related to the facts on the mappings of topological groups [13], [14]. In [12] the authors deal with a net of C*-algebras associated to a net over a partially ordered set consisting of Hilbert spaces.

In this paper we consider a covariant functor from a category associated with an arbitrary partially ordered set K into the category of unital C*-algebras and their unital *-homomorphisms. That functor is also called an inductive system over K. Using Zorn’s lemma, the set K can be represented as the union of the family \( \{ K_i \} \) of its maximal upward directed subsets indexed by elements of a set I. We consider a topology on the set I generated by a base of neighbourhoods. For every set \( K_i, i \in I \), the original inductive system over K yields naturally the inductive system of C*-algebras over \( K_i \) and its inductive limit. Using those inductive limits, we construct different types of C*-algebras. In particular, for neighbourhoods of the topology on the set of indices we deal with the C*-algebras which are the direct products of limits for inductive systems over the sets \( K_i \).

The present paper is devoted to the study of properties of the above-mentioned topology and the C*-algebras. We show that there exists a connection between topological and algebraic structures.

The paper consists of Introduction, three sections and Appendix. The first section contains preliminaries. In the second section we consider a topology on the index set I and study its properties. Examples of those topologies with different properties are given. The third section deals with inductive limits. In this section a connection between topological and algebraic constructions is studied. Finally, Appendix contains the figures for the examples in the second section.

A part of the results in this paper was announced without proofs in [15].

2 Preliminaries

In what follows, we shall consider an arbitrary partially ordered set \( (K, \leq) \) that is not necessarily directed. The category associated to this set is denoted by the same letter \( K \). We recall that the objects of this category are the elements of the
set $K$, and, for any pair $a, b \in K$, the set of morphisms from $a$ to $b$ consists of the single element $(a, b)$ provided that $a \leq b$, and is the void set otherwise.

Further, we consider a covariant functor $F$ from the category $K$ into the category of unital $C^*$-algebras and their unital $*$-homomorphisms. As was mentioned in Introduction, such a functor is called an inductive system in the category of $C^*$-algebras over the set $(K, \leq)$. It may be given by a collection $(K, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})$ satisfying the properties from the definition of a functor. We shall write $F = (K, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})$. Here, $\{\mathfrak{A}_a \mid a \in K\}$ is a family of unital $C^*$-algebras. We also suppose that all morphisms $\sigma_{ba} : \mathfrak{A}_a \rightarrow \mathfrak{A}_b$, where $a \leq b$, are embeddings of $C^*$-algebras, i.e., unital injective $*$-homomorphisms.

Recall that the equations $\sigma_{ca} = \sigma_{cb} \circ \sigma_{ba}$ hold for all elements $a, b, c \in K$ satisfying the condition $a \leq b \leq c$. Furthermore, for each element $a \in K$ the morphism $\sigma_{aa}$ is the identity mapping.

Considering the family of all upward directed subsets of the set $(K, \leq)$ and making use of Zorn’s lemma, one can easily prove

**Proposition 1** Let $(K, \leq)$ be a partially ordered set. Then the following equality holds:

$$K = \bigcup_{i \in I} K_i, \quad (1)$$

where $\{K_i \mid i \in I\}$ is the family of all maximal upward directed subsets of $(K, \leq)$.

Moreover, for every $i \in I$ and $a \in K_i$ the set $\{b \in K \mid b \leq a\}$ is a subset of $K_i$.

Now, for each $i \in I$, we consider the inductive system $F_i = (K_i, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})$ over the upward directed set $K_i$.

Throughout the paper, for a unital algebra $\mathfrak{A}$ its unit will be denoted by $1_{\mathfrak{A}}$.

The simplest example of the inductive system $F_i$ is that in which $\{\mathfrak{A}_a \mid a \in K_i\}$ is a net of $C^*$-subalgebras of a given $C^*$-algebra $\mathfrak{A}$. By this, one means that each $\mathfrak{A}_a$ is a $C^*$-subalgebra containing the unit $1_{\mathfrak{A}}$, $\mathfrak{A}_a \subset \mathfrak{A}_b$ and $\sigma_{ba} : \mathfrak{A}_a \rightarrow \mathfrak{A}_b$ is the inclusion mapping whenever $a, b \in K_i$ and $a \leq b$. Given such a net $F_i$, the norm closure of the union of all $\mathfrak{A}_a$ is itself a $C^*$-subalgebra of $\mathfrak{A}$.

We recall the definition and some facts concerning the inductive limits for inductive systems of $C^*$-algebras (see, for example, [16, Section 11.4]).

The inductive limit of this system is a pair $(\mathfrak{A}_K, \{\sigma^K_a\})$ where $\mathfrak{A}_K$ is a $C^*$-algebra and $\{\sigma^K_a : \mathfrak{A}_a \rightarrow \mathfrak{A}_K \mid a \in K_i\}$ is a family of unital injective $*$-homomorphisms such that the following two properties are fulfilled [16, Proposition 11.4.1]:

1) For every pair elements $a, b \in K_i$ satisfying the condition $a \leq b$ the diagram

$$\begin{array}{ccc}
\mathfrak{A}_a & \xrightarrow{\sigma_{ba}} & \mathfrak{A}_b \\
\sigma_{K_i} & \searrow & \sigma^K_a \\
\mathfrak{A}_K & \nearrow & \sigma^K_b \\
\end{array}$$

is commutative, that is, the equality for mappings

$$\sigma^K_a = \sigma^K_b \circ \sigma_{ba}. \quad (2)$$
holds. Moreover, we have the following equality:

$$\mathfrak{A}^{K_i} = \bigcup_{a \in K_i} \sigma_a^{K_i}(\mathfrak{A}_a),$$  

(3)

where the bar means the closure of the set with respect to the norm topology in the $C^*$-algebra $\mathfrak{A}^{K_i}$.

2) The universal property. If $\mathfrak{B}$ is a $C^*$-algebra, $\psi_a : \mathfrak{A}_a \longrightarrow \mathfrak{B}$ is an injective $*$-homomorphism for each $a \in K_i$, and conditions analogous to those in (2) and (3) are satisfied, then there exists a $*$-isomorphism $\theta$ from $\mathfrak{A}^{K_i}$ onto $\mathfrak{B}$ such that the diagram

$$\begin{array}{ccc}
\mathfrak{A}^{K_i} & \xrightarrow{\sigma_a^{K_i}} & \mathfrak{A}_a \\
\downarrow{\psi_a} & & \downarrow{\varphi} \\
\mathfrak{B} & & \\
\end{array}$$

is commutative for every $a \in K_i$, that is, the equality $\psi_a = \varphi \circ \sigma_a^{K_i}$ holds.

The inductive limit $(\mathfrak{A}^{K_i}, \{\sigma_a^{K_i}\})$ is denoted as follows:

$$(\mathfrak{A}^{K_i}, \{\sigma_a^{K_i}\}) := \lim_{\rightarrow} F_i.$$  

The $C^*$-algebra $\mathfrak{A}^{K_i}$ itself is often called the inductive limit.

It is well known that the inductive limit can always be constructed for an inductive system in the category of $C^*$-algebras. For the convenience of the reader, we recall briefly the components of that construction (the details are contained in the proof of Proposition 11.4.1 in [16]). We shall use them in our proofs.

We consider the direct product of $C^*$-algebras

$$\prod_{a \in K_i} \mathfrak{A}_a := \left\{ (F_a) \mid \|F_a\| = \sup_{a} \|F_a\| < +\infty \right\}.$$  

It is a $C^*$-algebra relative to the pointwise operations and the supremum norm. It has a norm-closed two-sided ideal $\Sigma$ consisting of all elements $(F_a)$ in the direct product for which the net $\{\|F_a\| \mid a \in K_i\}$ converges to 0. When $a \in K_i$ the $*$-homomorphism

$$\theta_a^{K_i} : \mathfrak{A}_a \longrightarrow \prod_{b \in K_i} \mathfrak{A}_b$$

is defined at an element $A \in \mathfrak{A}_a$ as follows:

$$[\theta_a^{K_i}(A)](b) = \begin{cases} 
\sigma_{ba}(A) , & \text{if } a \leq b; \\
0 , & \text{otherwise}. 
\end{cases}$$  

The mapping $A \longrightarrow \theta_a^{K_i}(A) + \Sigma$ is a unital injective $*$-homomorphism $\sigma_a^{K_i}$ from $\mathfrak{A}_a$ into the quotient $C^*$-algebra $\prod_{a \in K_i} \mathfrak{A}_a/\Sigma$. The family $\{\sigma_a^{K_i}(\mathfrak{A}_a) \mid a \in K_i\}$ is a net of $C^*$-subalgebras of $\prod_{a \in K_i} \mathfrak{A}_a/\Sigma$, and the norm closure of $\bigcup_{a \in K_i} \sigma_a^{K_i}(\mathfrak{A}_a)$ is the inductive limit $\mathfrak{A}^{K_i}$, which is a $C^*$-subalgebra of $\prod_{a \in K_i} \mathfrak{A}_a/\Sigma$.

It is worth noting that below we shall denote by the same symbols $\Sigma$ and $\theta_a^{K_i}$ the corresponding ideals and mappings for distinct inductive systems.
Finally, we construct the direct product for the inductive limits of the functors $F_i$ denoted by

$$\mathfrak{M}_F := \prod_{i \in I} A^K_i = \left\{ (a_i) \mid \|a_i\| = \sup_i \|a_i\| < \infty \right\}.$$ 

For additional results in the theory of $C^*$-algebras we refer the reader, for example, [17, Ch. 4, §7] and [18]. Necessary facts from the theory of categories and functors are contained, for example, in [17, Ch. 0, §2] and [19].

3 Topology on the index set $I$

Throughout this section, $K$ is an arbitrary partially ordered set that is not necessarily directed. By Proposition 1, we have equality (1).

Now we endow the index set $I$ with a topology. To this end, for every element $a \in K$ we define the set $U_a = \{ i \in I : a \in K_i \}$. Obviously, $U_a$ is a non-empty set.

Using the property of maximal upward directed sets $K_i$ from Proposition 1, it is straightforward to prove

**Lemma 1** If $a, b \in K$ such that $a \leq b$ then $U_b \subset U_a$.

We shall now show that the family of sets $\{ U_a \mid a \in K \}$ generates a topology on the index set $I$.

**Proposition 2** The family $\{ U_a \mid a \in K \}$ is a base for a topology on the set $I$.

**Proof** To prove the assertion we use Proposition 1.2.1 from [20]. According to it, we need to check two properties of a base.

Firstly, for any $U_a$ and $U_b$, $a, b \in K$, and every point $i \in U_a \cap U_b$ there exists $U_c, c \in K$, such that $i \in U_c \subset U_a \cap U_b$. Indeed, since $i \in U_a \cap U_b$ the elements $a$ and $b$ belong to the maximal upward directed set $K_i$. Hence, there is an element $c \in K_i$ satisfying the conditions $a \leq c$ and $b \leq c$. By Lemma 1 the set $U_c$ satisfies the required property.

Secondly, it is clear that for every $i \in I$ there exists an element $a \in K$ such that $i \in U_a$. 

We denote by $\tau$ the topology generated by the base $\{ U_a \mid a \in K \}$. Thus, $\tau$ is the family of all subsets of $I$ that are unions of subfamilies of $\{ U_a \mid a \in K \}$.

Below we prove the propositions describing properties of the topological space $(I, \tau)$ and give several examples.

**Proposition 3** The topological space $(I, \tau)$ is a $T_1$-space.

**Proof** Take any pair of distinct indices $i, j \in I$. The condition $i \neq j$ implies that $K_i \neq K_j$. Hence, we can take an element $a \in K_i \setminus K_j$. Then we have the conditions $i \in U_a$ and $j \notin U_a$.

**Corollary 1** For every index $i \in I$ the equality $\bigcap_{a \in K_i} U_a = \{ i \}$ holds.
Corollary 2 For every index \( i \in I \) the one-point set \( \{ i \} \) is closed.

The next example shows that \((I, \tau)\) may not be a Hausdorff space.

Example 1 We consider the set of points with integer coordinates
\[
K := \{(x, y) \mid x \in \{-1; 0; 1\}, y \in \mathbb{Z}\}.
\]

A partial order \( \leq \) on the set \( K \) is defined in the following way:
\[
(x_1, y_1) \leq (x_2, y_2) := \begin{cases} 
    x_1, x_2 \in \{-1; 1\}, x_1 = x_2, y_1 \leq y_2; \\
    x_1 \in \{-1; 1\}, x_2 = 0, y_1 < y_2.
\end{cases}
\]

It is straightforward to check that the pair \((K, \leq)\) is a partially ordered set, which is not upward directed.

One has the representation of \( K \) as the union of maximal upward directed sets \( K_i \) indexed by the set of all integers \( \mathbb{Z} \) together with two symbols \(-\infty\) and \( +\infty\), that is, \( I = \mathbb{Z} \cup \{-\infty; +\infty\} \):
\[
K = \bigcup_{i = -\infty}^{+\infty} K_i, \quad \text{where} \quad K_{-\infty} := \{(-1, y) \mid y \in \mathbb{Z}\}; \\
K_{+\infty} := \{(1, y) \mid y \in \mathbb{Z}\}
\]

and \( K_i := \{(0, i)\} \bigcup \{(x, y) \mid x \in \{-1; 1\}, y < i, y \in \mathbb{Z}\} \) for each \( i \in \mathbb{Z} \).

A base \( \{U_{(x, y)} \mid x \in \{-1; 0; 1\}, y \in \mathbb{Z}\} \) for the topology \( \tau \) on the index set \( I \) consists of the sets of three types, namely,
\[
U_{(-1, y)} := \{-\infty\} \cup \{i \in \mathbb{Z} \mid i > y\}; \\
U_{(1, y)} := \{+\infty\} \cup \{i \in \mathbb{Z} \mid i > y\}; \\
U_{(0, y)} := \{y\}.
\]

Since any two neighbourhoods of indices \(-\infty\) and \( +\infty \) have a non-empty intersection the topological space \((I, \tau)\) is not a Hausdorff space.

Proposition 4 For \( a \in K \), the set \( K^a \) is upward directed if and only if the neighbourhood \( U_a \) consists of a single point.

Proof Necessity. Assume that \( K^a \) is an upward directed set. By Zorn’s Lemma, there exists an index \( j \in I \) such that \( K^a \) is contained in the maximal upward directed set \( K_j \). Hence, \( a \in K_j \). Thus, by the definition of the neighbourhood \( U_a \), we get the inclusion
\[
\{j\} \subset U_a.
\]

Next, we shall show the reverse inclusion. In order to obtain a contradiction, we suppose that there exists an index \( i \in I \) distinct from the index \( j \) such that \( i \in U_a \). Then, we have \( a \in K_i \) and \( K_i \neq K_j \).

In this case we can take an element
\[
c \in K_i \setminus K_j.
\]

Since \( a, c \in K_i \) and \( K_i \) is an upward directed set there exists an element \( d \in K_i \) satisfying the conditions \( a \leq d \) as well as \( c \leq d \). The first condition implies that \( d \in K^a \subset K_j \). Obviously, the latter together with the maximality property of the
upward directed set $K_j$ yields the inclusion $c \in K_j$. This contradicts condition (6). Therefore, we obtain the inclusion that is reverse to (5). Thus, the equality

$$U_a = \{j\}$$

holds, as required.

Sufficiency. Let equality (7) be valid for some index $j \in I$. By the the definition of the neighbourhood $U_a$, the element $a$ belongs to the unique maximal upward directed set $K_j$.

We claim that the following inclusion holds:

$$K^a \subset K_j.$$  

Indeed, take any $b \in K^a$. Then $a \leq b$ and, by Lemma 1, $U_b \subset U_a$. Therefore, $U_b = \{j\}$ and $b \in K_j$. Consequently, we have inclusion (8), as claimed.

To show that $K^a$ is an upward directed set we take two elements $b,c \in K^a$. By (8), we have $b,c \in K_j$. Since $K_j$ is an upward directed set there exists $d \in K_j$ such that both the conditions $b \leq d$ and $c \leq d$ hold. Obviously, we have $d \in K^a$. Thus, the set $K^a$ is upward directed, as required. \qed

Now we give examples of locally compact and discrete topological spaces.

Example 2 As the set $K$ we consider the lower half-plane without the axis $y = 0$, that is,

$$K = \{(x,y) \mid x,y \in \mathbb{R}, y < 0\}.$$  

We define a partial order $\leq$ on $K$ as follows. Let us fix a positive number $a \in \mathbb{R}$. Then we put

$$(x_1, y_1) \leq (x_2, y_2) := \begin{cases} x_1 = x_2 \text{ and } y_1 = y_2; \\ y_2 - y_1 > a|x_2 - x_1|. \end{cases}$$

It is easily verified that the pair $(K, \leq)$ is a partially ordered set. Moreover, it is worth noting that this set is not upward directed.

We have the representation of $K$ as the union of maximal upward directed sets $K_i$ indexed by the set of all real numbers, that is, $I = \mathbb{R}$:

$$K = \bigcup_{i \in \mathbb{R}} K_i, \quad \text{where} \quad K_i := \{(x, y) \in K \mid -y > a|i - x|\}.$$  

Taking a point $(x_0, y_0) \in K$, one can easy see that

$$U_{(x_0, y_0)} = \left\{ i \in \mathbb{R} \mid x_0 + \frac{y_0}{a} < i < x_0 - \frac{y_0}{a} \right\}.$$  

Thus, in this example the topology $\tau$ coincides with the natural topology on the set $\mathbb{R}$ which is locally compact.
Example 3 Here, we take as $K$ the points with integer coordinates in the lower half-plane including the axis $y = 0$, that is, $K = \{(n, m) : n, m \in \mathbb{Z}, m \leq 0\}$.

We define a partial order $\leq$ on the set $K$ by the following rule:

$$(n_1, m_1) \leq (n_2, m_2) \text{ if and only if } m_2 - m_1 \geq n_2 - n_1 \geq 0.$$ 

The pair $(K, \leq)$ is a partially ordered set. It is not upward directed as well.

We have the representation of $K$ as the union of maximal upward directed sets $K_i$ indexed by the set of all integers, that is, $I = \mathbb{Z}$:

$$K = \bigcup_{i \in \mathbb{Z}} K_i,$$

where $K_i := \{(n, m) \in K | -m \geq i - n \geq 0\}$.

For any point $(n_0, m_0) \in K$, in the space $I$ we have the neighbourhood $U_{(n_0, m_0)} = \{i \in \mathbb{Z} : n_0 \leq i \leq n_0 - m_0\}$.

Since the equality $U_{(n, 0)} = \{n\}$ holds we see that every point in the space $I = \mathbb{Z}$ is isolated. Thus, we conclude that the topology $\tau$ is discrete.

4 Inductive limits

Let us consider an inductive system $\mathcal{F} = (K, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})$, where $\mathfrak{A}_a$ is an arbitrary unital $C^*$-algebra. Then for each index $i \in I$ we can construct an inductive system $\mathcal{F}_i$ and its inductive limit $\mathfrak{A}^{K_i}$.

Further, we take any $a \in K$ and consider the direct product of $C^*$-algebras

$$\mathfrak{B}_a := \prod_{i \in U_a} \mathfrak{A}^{K_i}.$$ 

Recall, by Lemma $\prod$ for every pair $a, b \in K$ satisfying the condition $a \leq b$, we have the inclusion $U_b \subseteq U_a$. Hence, the $*$-homomorphism $\tau_{ba} : \mathfrak{B}_a \to \mathfrak{B}_b$ between $C^*$-algebras given by the rule

$$\tau_{ba}(f)(j) = f(j),$$

where $f \in \mathfrak{B}_a$ and $j \in U_b$, is well-defined.

Obviously, we have the equality $\tau_{ca} = \tau_{cb} \circ \tau_{ba}$ whenever $a, b, c \in K$ such that the condition $a \leq b \leq c$ holds.

Thus we have constructed the inductive system of $C^*$-algebras $(K, \{\mathfrak{B}_a\}, \{\tau_{ba}\})$. Therefore, for each index $i \in I$ we can consider the inductive system $(K_i, \{\mathfrak{B}_a\}, \{\tau_{ba}\})$ and its inductive limit

$$(\mathfrak{B}^{K_i}, \{\tau^{K_i}_{ba}\}) := \lim_{\to} (K_i, \{\mathfrak{B}_a\}, \{\tau_{ba}\}).$$

For the direct product of these inductive limits we introduce the following notation:

$$\mathfrak{M}_{\mathcal{F}} := \prod_{i \in I} \mathfrak{B}^{K_i}.$$ 

Further, we prove the theorems that show the connections between the structures of $C^*$-algebras $\mathfrak{A}^{K_i}$, $\mathfrak{B}^{K_i}$, $\mathfrak{M}_{\mathcal{F}}$, $\mathfrak{M}_{\mathcal{F}}$ and the properties of the topological space $(I, \tau)$. 

Theorem 1 For every index \( i \) in the set \( I \) the algebra \( \mathfrak{A}^{K_i} \) is isomorphic to a subalgebra of \( \mathfrak{B}^{K_i} \).

Proof To construct an injective \(*\)-homomorphism from the algebra \( \mathfrak{A}^{K_i} \) into the algebra \( \mathfrak{B}^{K_i} \) we proceed as follows.

Take an arbitrary neighbourhood \( U_a \) of the point \( i \). For every index \( j \in U_a \) we consider the inductive limit

\[
(\mathfrak{A}^{K_j}, \{\sigma^K_{j,a}\}) = \lim_{\rightarrow} (K_j, \{\mathfrak{A}_b\}, \{\sigma_{cb}\}).
\]

Using the family of injective \(*\)-homomorphisms \( \{\sigma^K_{j,a}: \mathfrak{A}_a \rightarrow \mathfrak{A}^{K_j} \mid j \in U_a\} \), we define a \(*\)-homomorphism of \( C^* \)-algebras \( \sigma^U_a: \mathfrak{A}_a \rightarrow \mathfrak{B}_a = \prod_{j \in U_a} \mathfrak{A}^{K_j} \) by means of the formula

\[
\sigma^U_a(\mathfrak{A})(j) := \sigma^K_{j,a}(\mathfrak{A}), \quad A \in \mathfrak{A}_a, j \in U_a.
\]

Note that the injectivity of the \(*\)-homomorphisms \( \sigma^K_{j,a} \) implies the injectivity of \( \sigma^U_a \).

Moreover, the following equalities hold:

\[
\|\sigma^U_a(\mathfrak{A})(j)\| = \|\sigma^K_{j,a}(\mathfrak{A})\| = \|\mathfrak{A}\|.
\]

Now we take two inductive systems \( (K_i, \{\mathfrak{A}_a\}, \{\sigma_{ba}\}) \) and \( (K_i, \{\mathfrak{B}_a\}, \{\tau_{ba}\}) \).

For every pair of elements \( a, b \in K_i \) satisfying the condition \( a \leq b \) we have the diagram

\[
\begin{array}{ccc}
\mathfrak{A}_a & \xrightarrow{\sigma_{ba}} & \mathfrak{A}_b \\
\downarrow{\sigma^U_a} & & \downarrow{\sigma^U_b} \\
\mathfrak{B}_a & \xrightarrow{\tau_{ba}} & \mathfrak{B}_b 
\end{array}
\]

It is commutative, that is, we have the equality for the compositions of morphisms

\[
\sigma^U_b \circ \sigma_{ba} = \tau_{ba} \circ \sigma^U_a.
\]

To show the validity of (11) let us take any element \( A \in \mathfrak{A}_a \). For every index \( j \in U_b \) we have the following equalities:

\[
[\sigma^U_b \circ \sigma_{ba}(A)](j) = \sigma^K_{j,b}(\sigma_{ba}(A)) = \sigma^K_{j,b}(A);
\]

\[
[\tau_{ba} \circ \sigma^U_a(A)](j) = \sigma^U_a(A)(j) = \sigma^K_{j,a}(A).
\]

Therefore, we obtain equation (11).

The commutativity of (11) yields the commutativity of the diagram

\[
\begin{array}{ccc}
\mathfrak{A}_a & \xrightarrow{\sigma_{ba}} & \mathfrak{A}_b \\
\downarrow{\tau^K_{b,a} \circ \sigma^U_a} & & \downarrow{\tau^K_{a,b} \circ \sigma^U_a} \\
\mathfrak{B}_a & \xrightarrow{\tau_{ba}} & \mathfrak{B}_b 
\end{array}
\]
Indeed, we have the following equalities:
\[ \tau^K_a \circ \sigma^U_a = (\tau^K_b \circ \tau_{ba}) \circ \sigma^U_a = (\tau^K_b \circ \sigma^U_b) \circ \sigma_{ba}. \]

We claim that for every \( a \in K_i \) the \( * \)-homomorphism \( \tau^K_a \circ \sigma^U_a \) is injective. To see that we take two arbitrary distinct elements \( A_1 \) and \( A_2 \) in the \( C^* \)-algebra \( \mathfrak{A}_a \). For \( l = 1, 2 \) we have the representations
\[ \tau^K_a \circ \sigma^U_a (A_l) = \theta^K_a(\sigma^U_a(A_l)) + \Sigma. \]

We need to show that the condition
\[ \theta^K_a(\sigma^U_a(A_1 - A_2)) \notin \Sigma \quad (12) \]
holds, that is, the net \( \{\|\theta^K_a(\sigma^U_a(A_1 - A_2))\| \mid x \in K_i\} \) does not converge to 0.

By (4) for every element \( x \in K_i \) one has
\[ [\theta^K_a(\sigma^U_a(A_l))](x) = \left\{ \begin{array}{ll} \tau_{xa}(\sigma^U_a(A_l)), & \text{if } a \leq x; \\ 0, & \text{otherwise}. \end{array} \right. \quad (13) \]

In view of the commutativity of diagram (10) with \( x \) instead of \( b \) we have in formula (13) the following:
\[ \tau_{xa}(\sigma^U_a(A_l)) = \sigma^U_x(\tau_{xa}(A_l)). \quad (14) \]

It follows from (13), (14), (9) and the injectivity of the \( * \)-homomorphism \( \sigma_{xa} \) that for every element \( x \in K_i \) we get
\[ \|\theta^K_a(\sigma^U_a(A_1 - A_2))\| = \left\{ \begin{array}{ll} \|A_1 - A_2\|, & \text{if } a \leq x; \\ 0, & \text{otherwise}. \end{array} \right. \quad (15) \]

Really, on the right-hand part of (15) the first row is valid because we have the equalities:
\[
\|\sigma^U_x(\tau_{xa}(A_1 - A_2))\| = \sup \left\{ \|\sigma^U_x(\tau_{xa}(A_1 - A_2))\| \mid j \in U_x \right\} = \\
= \sup \left\{ \|\sigma^U_x(\tau_{xa}(A_1 - A_2))\| \mid j \in U_x \right\} = \\
= \|\tau_{xa}(A_1 - A_2)\| = \|A_1 - A_2\|. 
\]

Hence, there exists \( \varepsilon > 0 \), for example, \( \varepsilon = \frac{1}{2}\|A_1 - A_2\| \), satisfying the following property: for every \( y \in K_i \) there is an element \( x \in K_i \) such that \( y \leq x \), \( a \leq x \) and
\[ \|\theta^K_a(\sigma^U_a(A_1 - A_2))\| \geq \varepsilon. \]

This means that condition (12) is satisfied, and the \( * \)-homomorphism \( \tau^K_a \circ \sigma^U_a \) is an injection, as claimed.

Thus, by the universal property of inductive limits, we get the injective \( * \)-homomorphism of \( C^* \)-algebras \( \mathfrak{A}_K \rightarrow \mathfrak{M}_x \), as required. \( \square \)

**Corollary 3** The \( C^* \)-algebra \( \mathfrak{M}_F \) is isomorphic to a subalgebra of \( \mathfrak{M}_\mathfrak{F} \).
Theorem 2 Let \( i \in I \) be an isolated point in the topological space \((I, \tau)\). Then the \( C^* \)-algebras \( \mathfrak{A}^{K_i} \) and \( \mathfrak{A}^{K_i} \) are isomorphic.

Proof Since the one-point set \( \{i\} \) is open in the topological space \((I, \tau)\) and the family of sets \( \{U_a \mid a \in K\} \) constitutes a base for the topology \( \tau \) there exists an element \( a \in K_i \) such that we have the equality \( U_a = \{i\} \). Fix that element \( a \in K_i \).

Further, we take the upward directed set
\[
K^a_i := \{ b \in K_i \mid a \leq b \},
\]
which is a cofinal subset in \( K_i \).

Then we consider the inductive system \( (K^a_i, \{B_b\}, \{\tau_{cb}\}) \) over the set \( K^a_i \). For every element \( b \in K^a_i \) we have the equality \( B_b = B_a \). Note that the \( C^* \)-algebra \( \mathfrak{A}_a \) consists of all the functions from the one-point set \( \{i\} \) into the \( C^* \)-algebra \( A_{K_i} \). Obviously, the \( C^* \)-algebras \( B_a \) and \( A_{K_i} \) are isomorphic. Moreover, in this system each bonding morphism \( \tau_{cb} \) is the identity mapping. Thus, one has the isomorphism of \( C^* \)-algebras
\[
\lim_{\rightarrow} (K^a_i, \{B_b\}, \{\tau_{cb}\}) \simeq A_{K_i}. \tag{16}
\]

Further, we claim that there exists an isomorphism between \( C^* \)-algebras
\[
\lim_{\rightarrow} (K^a_i, \{B_b\}, \{\tau_{cb}\}) \simeq \mathfrak{A}^{K_i}. \tag{17}
\]

The existence of such an isomorphism follows from the universal property for the inductive limits.

To show this, firstly, for every \( b \in K^a_i \) we consider the *-homomorphism
\[
\tau_{K_i}^b : \mathfrak{A}_b \longrightarrow \mathfrak{A}^{K_i}.
\]
It is injective. Indeed, we take an arbitrary non-zero element \( f \in \mathfrak{A}_b \). Then we have
\[
\tau_{K_i}^b(f) = \theta_{K_i}^b(f) + \Sigma,
\]
where the *-homomorphism \( \theta_{K_i}^b : \mathfrak{A}_b \longrightarrow \prod_{x \in K_i} \mathfrak{A}_x \) is given by the formula
\[
[\theta_{K_i}^b(f)](x) = \begin{cases} 
\tau_{xb}(f) = f, & \text{if } b \leq x; \\
0, & \text{otherwise.}
\end{cases}
\]

Obviously, we have \( \theta_{K_i}^b(f) \notin \Sigma \). Hence, \( \tau_{K_i}^b \) is an injective *-homomorphism.

It is clear that \( \tau_{K_i}^b = \tau_{c}^{K_i} \circ \tau_{cb} \) whenever \( b, c \in K^a_i \) and \( b \leq c \).

Secondly, we prove the following equality:
\[
\mathfrak{A}^{K_i} = \bigcup_{b \in K^a_i} \tau_{K_i}^b(\mathfrak{A}_b). \tag{18}
\]

To this end, we recall that one has the equality
\[
\mathfrak{A}^{K_i} = \bigcup_{x \in K_i} \tau_{x}^{K_i}(\mathfrak{A}_x). \tag{19}
\]
Certainly, the right-hand part of (18) is contained in the right-hand part of
(19). To obtain the reverse inclusion we fix an arbitrary element \( x \in K_i \). Since
\( K^a \) is cofinal in \( K_i \) there is \( b \in K^a \) such that \( x \leq b \). The commutativity of the
\( \tau \)-diagram

\[
\begin{array}{ccc}
\mathcal{B}_x & \xrightarrow{\tau_x} & \mathcal{B}_b \\
\tau^K_i & \xleftarrow{\tau^K_i} & \mathcal{G}_i
\end{array}
\]

yields the following equality for sets

\[
\tau^K_i(\tau_x(\mathcal{B}_x)) = \tau^K_i(\mathcal{B}_b)
\]

which implies the desired inclusion for sets, namely,

\[
\tau^K_i(\mathcal{B}_x) \subset \tau^K_i(\mathcal{B}_b).
\]

Consequently, the right-hand parts of (18) and (19) coincide. Thus, we have
proved equality (18).

It follows from Proposition 11.4.1(ii) in [16] that there exists an isomorphism
(17).

Finally, combining isomorphisms (16) and (17), we conclude that the \( C^\ast \)-algebras
\( \mathfrak{B}^K_i \) and \( \mathfrak{A}^K_i \) are isomorphic, as required.

The following statement is an immediate consequence of Theorem 2.

**Corollary 4** Let \( (I, \tau) \) be a discrete topological space. Then the \( C^\ast \)-algebras \( \mathfrak{M}_x \) and \( \check{\mathfrak{M}}_x \) are isomorphic.

**Theorem 3** Let \( i \in I \) be a non-isolated point with a countable neighbourhood base.
Then the algebra \( \mathfrak{B}^K_i \) has a non-trivial center.

**Proof** It is clear that in the space \( (I, \tau) \) one can construct a countable neighbourhood base \( \{U_{a_n} \mid a_n \in K_i, n \in \mathbb{N}\} \) at the point \( i \in I \) satisfying the following conditions:

\[
U_{a_1} \supset U_{a_2} \supset U_{a_3} \supset \ldots;
\]

\[
a_1 \leq a_2 \leq a_3 \leq \ldots.
\]

Further, for each \( n \in \mathbb{N} \) we consider the subset \( W_n \) in \( I \) given by

\[
W_n := U_{a_n} \setminus U_{a_{n+1}}.
\]

Using these sets, we define the element \( f \) in the subalgebra \( \prod_{i \in I} \text{Cl}_i \) of the
algebra \( \mathfrak{M}_x \). Namely, for every \( i \in I \) the value of the function \( f \) at the point \( i \) is
defined as follows:

\[
f(i) = \begin{cases} 
0, & \text{if } i \notin U_{a_1}; \\
\mathfrak{B}^K_i, & \text{if } i \in W_{2k-1}, k \in \mathbb{N}; \\
0, & \text{if } i \in W_{2k}, k \in \mathbb{N}.
\end{cases}
\]
Now for every element $a_n$ we consider two elements $f_{a_n}$ and $g_{a_n}$ in the subalgebra $\prod_{i \in U_{a_n}} C_{\mathfrak{M}_{K_i}}$ of the algebra $\mathfrak{M}_{a_n}$. We put $f_{a_n} := f|_{U_{a_n}}$, that is, the function $f_{a_n}$ is the restriction of the function $f$ to the neighbourhood $U_{a_n}$, and $g_{a_n} := \mathfrak{M}_{a_n} - f_{a_n}$.

Together with the functions $f_{a_n}$ and $g_{a_n}$ we define two elements $\tilde{f}$ and $\tilde{g}$ in the inductive limit $\mathfrak{M}^K$, by

$$\tilde{f} := \tau^K_{a_n}(f_{a_n}) \quad \text{and} \quad \tilde{g} := \tau^K_{a_n}(g_{a_n}). \quad (21)$$

It is clear that these elements do not depend on the choice of the index $a_n$.

Obviously, we have the equality $\tilde{f} \cdot \tilde{g} = 0$ as well as $\tilde{f} + \tilde{g} = \mathfrak{M}_{K_i}$. Thus the elements $\tilde{f}$ and $\tilde{g}$ are non-trivial projections in the $C^*$-algebra $\mathfrak{M}^K$.

We claim that the elements $\tilde{f}$ and $\tilde{g}$ belong to the center of the algebra $\mathfrak{M}^K_i$. To this end, we take an element $A = h + \Sigma$ of the $C^*$-algebra $\mathfrak{M}^K_i$, where $h \in \prod_{x \in K_i} \mathfrak{M}_x$.

Let us show that the following equality holds:

$$\tilde{f} \cdot A = A \cdot \tilde{f}. \quad (22)$$

Indeed, for the left-hand part of (22) we have the expression:

$$\tilde{f} \cdot A = \tau^K_{a_n}(f_{a_n}) \cdot (h + \Sigma) = \tau^K_{a_n}(f_{a_n}) \cdot h + \tau^K_{a_n}(f_{a_n}) \cdot \Sigma. \quad (23)$$

Analogously, for the right-hand part of (22) we get the representation

$$A \cdot \tilde{f} = h \cdot \tau^K_{a_n}(f_{a_n}) + \Sigma. \quad (24)$$

Further, for $x \in K_i$ we have

$$[\theta^K_{a_n}(f_{a_n}) \cdot h](x) = \begin{cases} \tau^K_{a_n}(f_{a_n}) \cdot h(x), & \text{if } a_n \leq x; \\ 0, & \text{otherwise}. \end{cases} \quad (25)$$

For the function $\tau^K_{a_n}(f_{a_n}) \cdot h(x)$ from the algebra $\mathfrak{M}_x$ we take its value at a point $j \in U_x$:

$$[\tau^K_{a_n}(f_{a_n}) \cdot h(x)](j) = [\tau^K_{a_n}(f_{a_n})](j) \cdot [h(x)](j) = f_{a_n}(j) \cdot [h(x)](j). \quad (26)$$

Changing the order of the factors in (26) and (25), one gets the similar expressions for the summand $h \cdot \theta^K_{a_n}(f_{a_n})$ in the right-hand part of (24).

By (21) and the definition of the function $f_{a_n}$, we obtain the equality

$$f_{a_n}(j) \cdot [h(x)](j) = [h(x)](j) \cdot f_{a_n}(j).$$

Therefore the elements in the right-hand parts of (23) and (24) are the same. Hence, equality (22) is proved. Similarly one can prove equality (22) with $\tilde{g}$ instead of $\tilde{f}$.

It follows that the elements $\tilde{f}$ and $\tilde{g}$ belong to the center of the algebra $\mathfrak{M}^K_i$, as claimed. \(\Box\)

As a consequence of Theorem 5 and the definition of the $C^*$-algebra $\mathfrak{M}_X$ we have the following statement.

**Corollary 5** Let $(I, \tau)$ be a first-countable topological space without isolated points. Then the $C^*$-algebra $\mathfrak{M}_X$ has a non-trivial center.
Appendix: Figures for Examples

Example 1

\[ U_{(1,y)} = \{ +\infty \} \cup \{ y + 1, y + 2, \ldots \} \]
\[ U_{(-1,y)} = \{ -\infty \} \cup \{ y + 1, y + 2, \ldots \} \]

Fig. 1. Non Hausdorff space

Example 2

\[ y - y_1 = a(x - x_1) \]
\[ y - y_1 = -a(x - x_1) \]

Fig. 2. Locally compact space
Example

Fig. 3. Discrete space

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