SOLUTIONS OF THE LAPLACIAN FLOW AND COFLOW OF A LOCALLY CONFORMAL PARALLEL G$_2$-STRUCTURE

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Abstract. We study the Laplacian flow of a G$_2$-structure where this latter structure is claimed to be Locally Conformal Parallel. The first examples of long time solutions of this flow with the Locally Conformal Parallel condition are given. These examples are one-parameter families of Locally Conformal Parallel G$_2$-structures on solvable Lie groups. The found solutions are used to construct long time solutions to the Laplacian coflow starting from a Locally Conformal Parallel structure. We also study the behavior of the curvature of the solutions obtaining that for one of the examples the induced metric is Einstein along all the flow (resp. coflow).

Introduction

A G$_2$-structure on a 7-dimensional manifold $M$ can be characterized by the existence of a globally defined 3-form $\sigma$, which is called the G$_2$ form or the fundamental 3-form and it can be described locally as

$$\sigma = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

with respect to some local basis $\{e^1, \ldots, e^7\}$ of the 1-forms on $M$, which we call the adapted basis. The notation $e^{i_1 \cdots i_k}$ stands for $e^{i_1} \wedge \cdots \wedge e^{i_k}$. In the following we call G$_2$-manifold a 7-dimensional manifold endowed with a G$_2$-structure.

The existence of a G$_2$ form $\sigma$ on a manifold $M$ induces a Riemannian metric $g_\sigma$ on $M$ given by

$$g_\sigma(X,Y) \text{vol} = \frac{1}{6} \iota_X \sigma \wedge \iota_Y \sigma \wedge \sigma,$$

for any vector fields $X, Y$ on $M$, where vol is the volume form on $M$.

There are many different G$_2$-structures attending to the behavior of the exterior derivative of the G$_2$ form [2, 7], like for example, parallel, nearly parallel, closed, coclosed, Locally Conformal Parallel, and so on. In this work, we will focus our attention on Locally Conformal Parallel G$_2$-structures, (LCP for short) which are characterized by the conditions

$$d\sigma = 3 \tau \wedge \sigma,$$

$$d \ast \sigma = 4 \tau \wedge \ast \sigma,$$

with $\tau$ the Lee 1-form and $\ast$ the Hodge star operator induced by the metric (2). Analogously, these G$_2$-structures can be characterized by saying that they are of type $X_4$ in the sense of Fernández-Gray, see [7].

The development of flows in Riemannian geometry has been mainly motivated by the study of the Ricci flow. The same techniques are also useful in the study of flows involving other geometrical structures, like for example, the Kähler Ricci flow.

Concerning flows on G$_2$-manifolds, for any closed G$_2$-structure $\sigma_0$ on a manifold $M$, Bryant in [3] introduced a natural flow, the so-called Laplacian flow, given by

$$\frac{d}{dt} \sigma(t) = \Delta_\sigma \sigma(t),$$

$$\sigma(0) = \sigma_0,$$

$$d\sigma(t) = 0,$$

where $\Delta_\sigma$ is the Hodge Laplacian operator of the metric (2) determined by $\sigma(t)$. The short time existence and uniqueness of solution for the Laplacian flow of any closed G$_2$-structure, on a compact manifold $M$, has been proved by Bryant and Xu in the unpublished paper [4]. Also, long time existence and convergence
of the Laplacian flow starting near a torsion-free G\(_2\)-structure was proved in the unpublished paper [14]. In the last years, Lotay and Wei in [11], [12] and [13] have obtained many results concerning the properties of the Laplacian flow.

In [6] the first examples of noncompact manifolds with long time existence of the solution for the Laplacian flow of a closed G\(_2\)-structure are shown. Those examples are nilpotent Lie groups admitting an invariant closed G\(_2\)-structure which determines the nilsoliton which. Recently in [8] the authors studied the Laplacian flow of a closed G\(_2\)-structure on warped products of the form \(M \times S^1\) where the base space is a 6-dimensional compact manifold endowed with an SU(3)-structure. Imposing the warping function to be constant they find sufficient conditions for the existence of solution of the Laplacian flow and present some examples where \(M\) is a six-dimensional solvmanifold.

Karigiannis, McKay and Tsui in [10] introduced the Laplacian coflow. In this case the initial G\(_2\)-form is claimed to be coclosed, i.e. \(d \ast \sigma_0 = 0\). The equations of this flow can be given by

\[
\begin{align*}
\frac{d}{dt} \psi(t) &= -\Delta t \psi(t), \\
\psi(0) &= \psi_0, \\
\omega(t) &= 0,
\end{align*}
\]

with \(\psi(t) = \ast t \sigma(t)\) the 4-form. Up to now, short time existence of solution of the coflow is not known.

Assuming short time existence and uniqueness of solution, it is shown in [10] that the condition of the initial G\(_2\)-form \(\sigma_0\) to be coclosed (equiv. \(\psi_0\) closed) is preserved along the flow. In [9] Grigorian introduced a modified version of the Laplacian coflow which is called the modified Laplacian coflow and proved short time existence and uniqueness of solution for this modified flow. Recently in [1] explicit solutions for the coflow and the modified Laplacian coflow have been described. These solutions are one-parameter families of G\(_2\)-structures defined on the 7-dimensional Heisenberg Lie group. The solutions for the coflow are always ancient for every initial cocalibrated G\(_2\)-structure. The condition of the induced metric to be Ricci soliton is preserved along the coflow.

In this paper we are concerned with studying the Laplacian flow, resp. coflow, of an LCP G\(_2\)-structure on a manifold \(M\) defined as:

\[
\begin{align*}
\frac{d}{dt} \sigma(t) &= \Delta t \sigma(t), \\
\sigma(0) &= \sigma_0, \\
\omega(t) &= 3 \tau(t) \wedge \sigma(t), \\
d \ast \sigma(t) &= 4 \tau(t) \wedge \ast t \sigma(t).
\end{align*}
\]

In order to describe the first examples of solution of these flows we will consider the class of solvmanifolds described in [5]. These manifolds are rank-one solvable extensions of nilpotent Lie groups such that admit a Locally Conformal Parallel G\(_2\)-structure. Furthermore, the underlying Lie algebras corresponding to the former solvmanifolds are described in the following list:

\[
\begin{align*}
cp_1 &= (-me^{17}, -me^{27}, -me^{37}, -me^{47}, -me^{57}, -me^{67}, 0); \\
cp_2 &= \left(-\frac{4}{3} me^{17} + \frac{2}{3} me^{36}, -me^{27}, -\frac{2}{3} me^{37}, -me^{47}, -me^{57}, -\frac{2}{3} me^{67}, 0\right); \\
cp_3 &= \left(-\frac{3}{2} me^{17} + \frac{1}{2} me^{36}, \frac{1}{2} me^{45}, -me^{27}, -\frac{3}{4} me^{37}, -\frac{3}{4} me^{47}, -\frac{3}{4} me^{57}, -\frac{3}{4} me^{67}, 0\right); \\
cp_4 &= \left(-\frac{7}{5} me^{17} + \frac{2}{5} me^{36} + \frac{2}{5} me^{45}, -me^{27} - \frac{2}{5} me^{46} - \frac{4}{5} me^{37} - \frac{3}{5} me^{57} - \frac{3}{5} me^{67}, 0\right); \\
cp_5 &= \left(-\frac{5}{4} me^{17} + \frac{1}{4} me^{45}, -\frac{5}{4} me^{27} - \frac{1}{2} me^{45}, -me^{37}, -\frac{1}{2} me^{47}, -\frac{3}{4} me^{57}, -\frac{3}{4} me^{67}, 0\right); \\
cp_6 &= \left(-\frac{4}{3} me^{17} + \frac{1}{3} me^{36} + \frac{1}{3} me^{45}, -\frac{4}{3} me^{27} + \frac{1}{3} me^{46} - \frac{2}{3} me^{37} - \frac{2}{3} me^{47} - \frac{2}{3} me^{57} - \frac{2}{3} me^{67}, 0\right); \\
cp_7 &= \left(-\frac{6}{5} me^{17} + \frac{2}{5} me^{36}, -\frac{3}{5} me^{27}, -\frac{3}{5} me^{37}, \frac{2}{5} me^{26} - \frac{6}{5} me^{47}, \frac{2}{5} me^{23} - \frac{6}{5} me^{57}, -\frac{3}{5} me^{67}, 0\right).
\end{align*}
\]
where \( \{e^1, \ldots, e^7\} \) is a basis of invariant 1-forms of every solvmanifold and \( m \in \mathbb{R}^\ast \). It turns out that the 3-form \( \sigma_0 \) given by (1) in terms of the basis \( \{e^1, \ldots, e^7\} \) defines an invariant LCP structure on every solvmanifold.

The paper is structured as follows: in Section 1, we introduce the Laplacian flow, resp. coflow, of an LCP \( G_2 \)-structure and we provide some technical lemmata. Section 2 is devoted to the study of solutions of both flows on a solvmanifold with underlying Lie algebra \( \mathfrak{p}^m_i \). We obtain explicit solutions for both flows with long time existence and also notice that the corresponding induced metrics are Einstein for all \( t \) in the domain of these solutions. In Section 3 we consider the Laplacian flow starting from an LCP \( G_2 \)-structure on solvmanifolds with underlying Lie algebra \( \mathfrak{p}^m_i \) with \( i \in \{2, \ldots, 7\} \) obtaining the following result of existence of solution:

**Main result 1:** Every 7-dimensional rank-one solvable extension of a nilpotent Lie group with a Locally Conformal Parallel \( G_2 \) form, \( \sigma_0 \), admits a long time solution \( \sigma(t) \) to the Laplacian flow, preserving the LCP condition along the flow, such that \( \sigma(0) = \sigma_0 \).

In Section 4 we obtain an analogous result to main result 1 for the Laplacian coflow by means of Theorem 4.2, where an explicit relation between solutions to the Laplacian flow and coflow on 7-dimensional rank-one solvable extension of nilpotent Lie groups is given. In this way, we obtain

**Main result 2:** Every 7-dimensional rank-one solvable extension of a nilpotent Lie group with a Locally Conformal Parallel \( G_2 \) form admits a long time LCP solution to the Laplacian coflow.

In the Appendix we include the expressions of the curvature for the metric induced by the solutions of the Laplacian flow previously obtained.

1. **Laplacian flow and coflow of an LCP \( G_2 \)-structure.**

Let \( M \) be a 7-dimensional manifold with a Locally Conformal Parallel \( G_2 \) form \( \sigma_0 \). We will consider the Laplacian flow, resp. coflow, of an LCP \( G_2 \)-structure with initial value \( \sigma(0) = \sigma_0 \), that is:

\[
\begin{align*}
\frac{d}{dt} \sigma(t) &= \Delta_t \sigma(t), \\
\sigma(0) &= \sigma_0, \\
\sigma_0 d\sigma(t) &= 3 \tau(t) \wedge \sigma(t), \\
d \ast_t \sigma(t) &= 4 \tau(t) \wedge \ast \sigma(t),
\end{align*}
\]

where \( \Delta_t = -d \ast_t d \ast_t + d \ast_t d \ast_t d \), is the Hodge Laplacian of the metric \( g_\sigma \) determined by the \( G_2 \) form \( \sigma(t) \) and \( \psi(t) \) is the 4-form \( \psi(t) = \ast \sigma(t) \).

For simplicity, we will call the first two equations of (4) and (5) the *evolution equation* for the 3-form and the 4-form respectively. Observe that the last two equations of (4) and (5) are simply the LCP condition (3).

Let \( S_t \) be a solvmanifold with underlying Lie algebra \( \mathfrak{p}^m_i \). In what follows we will consider a basis \( \{x^1, \ldots, x^7\} \) of invariant 1-forms on \( S_t \) given by:

\[ x^i \equiv x^i(t) = h_i(t) e^i, \]

where \( h_i(t) \) are differentiable functions, \( h_i(t) \neq 0 \) in an open real interval and \( h_i(0) = 1 \), for \( i = 1, \ldots, 7 \). Using this, we consider the one-parameter family of \( G_2 \)-structures on the solvmanifold given by:

\[
\sigma(t) = x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245}
\]

\[ = h_{127}(t)e^{127} + h_{347}(t)e^{347} + h_{567}(t)e^{567} + h_{135}(t)e^{135} - h_{146}(t)e^{146} - h_{236}(t)e^{236} - h_{245}(t)e^{245}, \]

where \( h_{ijk}(t) \) stands for the product \( h_i(t)h_j(t)h_k(t) \). This family includes the LCP structure (1) coinciding with it for \( t = 0 \). The 4-form \( \psi(t) = \ast \sigma(t) \) is given by
\[
\psi(t) = x^{3456} + x^{1256} + x^{1234} - x^{2467} + x^{2357} + x^{1457} + x^{1367}
\]

(7)

\[h_{3456}(t)e^{3456} + h_{1256}(t)e^{1256} + h_{1234}(t)e^{1234} - h_{2467}(t)e^{2467} + h_{2357}(t)e^{2357} + h_{1457}(t)e^{1457} + h_{1367}(t)e^{1367}.\]

The following result comes straightforward:

**Lemma 1.1.** Let \( \sigma(t) \) be a family of \( G_2 \)-structures on \( S_i \) given by (6), then

\[
\frac{d}{dt} \sigma(t) = \sum_{(i,j,k) \in I} \left( \frac{h'_i(t)}{h_i(t)} + \frac{h'_j(t)}{h_j(t)} + \frac{h'_k(t)}{h_k(t)} \right) x^{ijk} - \sum_{(i,j,k) \in J} \left( \frac{h'_i(t)}{h_i(t)} + \frac{h'_j(t)}{h_j(t)} + \frac{h'_k(t)}{h_k(t)} \right) x^{ijk},
\]

where \( I = \{(127), (135), (347), (567)\} \) and \( J = \{(146), (236), (245)\} \) and

\[
\frac{d}{dt} \psi(t) = \sum_{(l,m,n,o) \in K} \left( \frac{h'_l(t)}{h_l(t)} + \frac{h'_m(t)}{h_m(t)} + \frac{h'_n(t)}{h_n(t)} + \frac{h'_o(t)}{h_o(t)} \right) x^{l_mn_o} - \sum_{(l,m,n,o) \in K} \left( \frac{h'_l(t)}{h_l(t)} + \frac{h'_m(t)}{h_m(t)} + \frac{h'_n(t)}{h_n(t)} + \frac{h'_o(t)}{h_o(t)} \right) x^{2467},
\]

where \( K = \{(1234), (1256), (1367), (1457), (2357), (3456)\} \).

Let \( \sigma(t) \) be a family of \( G_2 \)-structures given by (6) solving the evolution equation for the 3-form. Then, the Laplacian \( \Delta_i \sigma(t) \) must have the following expression:

\[
\Delta_i \sigma(t) = \sum_{(i,j,k) \in I} \Delta_{ijk} x^{ijk} - \sum_{(i,j,k) \in J} \Delta_{ijk} x^{ijk},
\]

where we denote by \( \Delta_{ijk} \) the coefficient of \( x^{ijk} \) in \( \Delta \sigma(t) \) chosen with an appropriate change of sign. The situation is similar for the 4-form:

\[
\Delta_4 \sigma(t) = \sum_{(l,m,n,o) \in K} \Delta_{l_mn_o} x^{l_mn_o} - \Delta_{2467} x^{2467}.
\]

Using the previous expressions we get the following:

**Proposition 1.2.** Let \( \sigma(t) \) be a family of \( G_2 \)-structures on \( S_i \) given by (6), resp. (7). It solves the evolution equation for the Laplacian flow, resp. coflow, if and only if

\[
\Delta_{ijk} = \frac{h'_i(t)}{h_i(t)} + \frac{h'_j(t)}{h_j(t)} + \frac{h'_k(t)}{h_k(t)}, \quad (i,j,k) \in I \cup J,
\]

resp.

\[
\Delta_{l_mn_o} = -\left( \frac{h'_l(t)}{h_l(t)} + \frac{h'_m(t)}{h_m(t)} + \frac{h'_n(t)}{h_n(t)} + \frac{h'_o(t)}{h_o(t)} \right), \quad (l,m,n,o) \in K \cup (2467).
\]

Moreover

**Lemma 1.3.** Let \( \sigma(t) \) be a family of \( G_2 \)-structures on \( S_i \) given by (6) solving the evolution equation of the Laplacian flow: If \( \Delta_{abc} = \Delta_{pqr} \), then \( h_a(t)h_b(t)h_c(t) = h_p(t)h_q(t)h_r(t) \).

**Proof.** In particular, if \( \Delta_{abc} = \Delta_{pqr} \), the terms \( x^{abc} \) and \( x^{pqr} \) must evolve in the same way under the Laplacian flow, that is:

\[
h'_a(t) + h'_b(t) + h'_c(t) = h'_p(t) + h'_q(t) + h'_r(t).
\]

We can integrate this equation:

\[
\ln h_a(t) + \ln h_b(t) + \ln h_c(t) = \ln h_p(t) + \ln h_q(t) + \ln h_r(t) + C
\]

\[
\ln(h_a(t)h_b(t)h_c(t)) = \ln(C h_p(t)h_q(t)h_r(t))
\]

\[
h_a(t)h_b(t)h_c(t) = C h_p(t)h_q(t)h_r(t),
\]
where the previous condition should be satisfied for all $t$. In particular, since $h_i(0) = 1$, we obtain that $\dot{C} = 1$, thus $\Delta_{abc} = \Delta_{pqr}$ implies $h_a(t)h_b(t)h_c(t) = h_p(t)h_q(t)h_r(t)$.

Similarly:

**Lemma 1.4.** Let $\sigma(t)$ be a family of $G_2$-structures on $S_t$ given by (6) solving the evolution equation of the LCP coflow. If $\Delta_{lmno} = \Delta_{pqrst}$, then $h_{l}(t)h_{m}(t)h_{n}(t)h_{o}(t) = h_{p}(t)h_{q}(t)h_{r}(t)h_{s}(t)$.

2. **Einstein solutions of the Laplacian flow and coflow of an LCP $G_2$-structure**

Let $\sigma(t)$ be a family of $G_2$-structures on $S_t$ given by (6). In terms of the adapted basis $\{x^1, \ldots, x^7\}$ the structure equations of the underlying Lie algebra $\mathfrak{e}^7_{1m}$ become:

\[dx^i = -\frac{m}{h_7(t)}x^7, \quad i = 1, \ldots, 6, \quad dx^7 = 0.\]

In order to solve the Laplacian flow (4) and coflow (5), we will study first the corresponding evolution equations with $\sigma_0$ given by (1) and then we observe that the LCP condition (3) is already satisfied.

**Proposition 2.1.** Let $\sigma(t)$, resp. $\psi(t)$, be a solution for the evolution equation of the Laplacian flow, resp. coflow, for the solvmanifold $S_1$. Then, the functions $h_i(t)$ should verify $h_i(t) = h_j(t)$, for $i, j \neq 7$.

**Proof.** We prove the proposition for $\sigma(t)$, since the case for $\psi(t)$ is analogous. Using the adapted basis $\{x^i\}_{i=1}^7$ and the structure equations (14), one can easily check that the Laplacian of $\sigma(t)$ has the simple expression:

\[\Delta_{ijkl}(t) = \frac{-m^2}{h_7(t)} \left[8(x^{127} + x^{347} + x^{567}) + 9(x^{135} - x^{146} - x^{236} - x^{245})\right].\]

Observe that in particular it follows equation (10). In the expression above we have two different groups of terms $\Delta_{ijk}$, namely:

\[\Delta_{127} = \Delta_{347} = \Delta_{567} = \frac{-8m^2}{h_7(t)}, \quad \Delta_{135} = \Delta_{146} = \Delta_{236} = \Delta_{245} = \frac{-9m^2}{h_7(t)}.\]

Then, using Lemma 1.3 we obtain that

\[
\begin{align*}
    h_{13}(t) &= h_{24}(t), & h_{14}(t) &= h_{23}(t), & h_{15}(t) &= h_{26}(t), & h_{12}(t) &= h_{34}(t) = h_{56}(t), \\
    h_{16}(t) &= h_{25}(t), & h_{35}(t) &= h_{46}(t), & h_{36}(t) &= h_{45}(t),
\end{align*}
\]

leading to

\[
\begin{align*}
    h_1^2(t) &= h_2^2(t), & h_3^2(t) &= h_4^2(t), & h_5^2(t) &= h_6^2(t), & h_{12}(t) &= h_{34}(t) = h_{56}(t). 
\end{align*}
\]

Observe the following: consider $h_{12}(t) = h_{34}(t)$, multiply both sides by $h_7(t)$ and use $h_{13}(t) = h_{24}(t)$. We get $h_1^2(t) = h_2^2(t)$. Similarly, we obtain $h_3^2(t) = h_4^2(t)$. So, these conditions can be simplified as:

\[
\begin{align*}
    h_1^2(t) &= h_2^2(t) = h_3^2(t) = h_4^2(t) = h_5^2(t) = h_6^2(t), & h_{12}(t) &= h_{34}(t) = h_{56}(t). 
\end{align*}
\]

Since $h_i(t)$ are continuous functions, then $|h_i(t)| = |h_j(t)|$. Moreover, since $h_i(0) = 1$, we obtain that $h_i(t) = h_j(t)$ for $i, j = 1, \ldots, 6$.

We can now present the solutions of the Laplacian flow (4) of an LCP $G_2$-structure:

**Theorem 2.2.** Let $\sigma(t)$ be a family of $G_2$-structures on $S_t$ given by (6). Then, for each $m \in \mathbb{R}^*$ there exists a unique solution for the Laplacian flow of an LCP $G_2$-structure (4) given by:

\[
h(t) = (1 - 4m^2t^\frac{3}{2}), \quad h_7(t) = (1 - 4m^2t^{\frac{3}{2}}, \quad t \in \left(-\infty, \frac{1}{4m^2}\right),
\]

where $h_i(t) = h(t)$ for any $i \neq 7$.

The metric $g_t$ induced by $\sigma(t)$ given by (2) is Einstein along all the flow and the curvature satisfies

\[
\lim_{t \to -\infty} R(g_t) = 0.
\]
Proof. First, one can check that the family of $G_2$-structures given by (6) on the solvmanifold $S_1$ is LCP. Evenmore the torsion form $\tau$ described in (3) remains constant for all $t$ and is exactly

$$\tau = me^t.$$

Now, we solve the evolution equation. It gives rise to two differential equations, corresponding to the two groups of terms of the Laplacian:

$$\begin{cases}
-8m^2 
\frac{h^2(t)}{h^2(t)} = \frac{2h'(t)}{h(t)} + \frac{h^2(t)}{h(t)}, \\
-9m^2 
\frac{h^2(t)}{h^2(t)} = \frac{3h'(t)}{h(t)},
\end{cases}$$

which are equivalent to:

$$\begin{cases}
-2m^2 
\frac{h^2(t)}{h^2(t)} = \frac{h'(t)}{h(t)}, \\
-3m^2 
\frac{h^2(t)}{h^2(t)} = \frac{h'(t)}{h(t)},
\end{cases}$$

Observe that we can solve easily the first equation:

$$h^2(t)h'(t) = -2m^2 \iff h^2(t) = (-4m^2t + C)^{1/2}.$$ 

Using the fact that $h^2(0) = 1$, we get $C = 1$ and $h^2(t) = (1 - 4m^2t)^{1/2}$. With this value for $h^2(t)$, it is also possible to solve the second equation:

$$\frac{-3m^2}{1 - 4m^2t} = \frac{h'(t)}{h(t)} \iff \frac{3}{4} \ln(1 - 4m^2t) = \ln h(t) + C.$$ 

Imposing the initial condition, we get that $h(t) = (1 - 4m^2t)^{3/4}$.

Concerning the metric, the non-vanishing components of the curvature tensor (modulo its symmetry properties) expressed in the adapted basis $\{x_i\}_{i=1}^7$, i.e. $R_{ijkl} = g(R(x_i, x_j, x_k), x_l)$, are:

$$R_{ijkl} = -\frac{m^2}{1 - 4m^2t} \quad \text{for all } i \neq j = 1, \ldots, 7.$$ 

Thus $\lim_{t \to -\infty} R(g_t) = 0$. Moreover, an standard computation shows that the Ricci tensor $Ric(g_t)$ satisfies

$$Ric(g_t) = -\frac{6m^2}{1 - 4m^2t} g_t,$$

concluding the proof. \( \square \)

In a similar way, we now present the solutions of the Laplacian coflow of an LCP $G_2$-structure:

**Theorem 2.3.** Let $\psi(t)$ be a family of $G_2$-structures on $S_1$ given by (7). Then, for each $m \in \mathbb{R}^*$ there exists a unique solution for the Laplacian coflow of an LCP $G_2$-structure (5) given by:

$$h(t) = (1 + 6m^2t)^{1/2}, \quad h^2(t) = (1 + 6m^2t)^{1/2}, \quad t \in \left( -\frac{1}{6m^2}, +\infty \right),$$

where $h_i(t) = h(t)$ for any $i \neq 7$.

The metric induced by $\psi(t)$ is Einstein along all the flow and the curvature satisfies $\lim_{t \to +\infty} R(g_t) = 0$.

**Proof.** In this case, the evolution equation of (5) is equivalent to:

$$\begin{cases}
3m^2 
\frac{h^2(t)}{h^2(t)} = \frac{h'(t)}{h(t)}, \\
2m^2 
\frac{h^2(t)}{h^2(t)} = \frac{h'(t)}{h(t)}.
\end{cases}$$
Observe that from the first equation we get $h_7(t) = (1 + 6m^2 t)^{1/2}$, where we have used that $h_7(0) = 1$. With this value for $h_7(t)$, it is also possible to solve the second equation obtaining $h_4(t) = (1 + 6m^2 t)^{1/3}$. Again, the non-vanishing components of the curvature tensor are given by $R_{ijji} = -\frac{m^2}{1 + 6m^2 t}$ for all $i \neq j = 1, \ldots, 7$, and therefore $\lim_{t \to +\infty} R(\eta_t) = 0$. The metric is Einstein as $Ric(\eta_t) = -\frac{6m^2}{1 + 6m^2 t} \eta_t$. □

3. LONG TIME SOLUTIONS OF THE LAPLACIAN FLOW OF AN LCP $G_2$-STRUCTURE

In this section we obtain long time solutions for the Laplacian flow for $S_i$ where $i = 2, \ldots, 7$, using the family of invariant $G_2$-structures $\sigma(t)$ given by (6). Recall that by $S_i$ we denote a solvmanifold with underlying Lie algebra $\mathfrak{g}^\sigma$.

In order to solve the evolution equation we need the expression of the Laplacian for each case. Just direct computations give the following

**Lemma 3.1.** Let $\sigma(t)$ be a family of $G_2$-structures on $S_i$ given by (6), then the Laplacian $\Delta^i \sigma(t)$, where $i$ stands for the solvmanifold $S_i$, satisfies (10) and is given by:

\[
\Delta^2 \sigma(t) = -\frac{2m^2}{9} \left[ -33 \left( \frac{1}{h_7^3} - \frac{2h_7^2}{h_7 h_5^2} \right) (x^{127} - x^{236}) + \frac{36}{h_7^2} (x^{135} - x^{146} - x^{245}) + \frac{30}{h_7^2} (x^{347} + x^{567}) \right],
\]

\[
\Delta^3 \sigma(t) = \frac{m^2}{4} \left( \frac{-30}{h_7^2} + \frac{h_7^2}{h_7 h_5^2} + \frac{2h_7^2}{h_7 h_6^2} \right) x^{127} - \frac{15m^2}{2h_7^2} (x^{135} - x^{146}) - \frac{6m^2}{h_7^2} (x^{347} + x^{567})
\]

\[
\Delta^4 \sigma(t) = \frac{2m^2}{25} \left( \frac{-91}{h_7^2} + \frac{2h_7^2}{h_7 h_5^2} + \frac{2h_7^2}{h_7 h_6^2} + \frac{2h_7^2}{h_7 h_7^2} \right) x^{127} - \frac{36m^2}{5h_7^2} x^{135}
\]

\[
\Delta^5 \sigma(t) = \frac{m^2}{4} \left( \frac{-30}{h_7^2} + \frac{h_7^2}{h_7 h_5^2} + \frac{2h_7^2}{h_7 h_6^2} \right) x^{127} - \frac{15m^2}{2h_7^2} (x^{135} - x^{236}) - \frac{6m^2}{h_7^2} (x^{347} + x^{567})
\]

\[
\Delta^6 \sigma(t) = \frac{m^2}{9} \left( \frac{-64}{h_7^2} + \frac{h_7^2}{h_7 h_5^2} + \frac{h_7^2}{h_7 h_6^2} + \frac{h_7^2}{h_7 h_7^2} + \frac{h_7^2}{h_7 h_5^2} + \frac{h_7^2}{h_7 h_6^2} + \frac{h_7^2}{h_7 h_7^2} \right) x^{127} - \frac{16m^2}{3h_7^2} (x^{347} + x^{567})
\]
\[
\Delta^7 \sigma(t) = -\frac{2m^2}{25} \left( \frac{81}{h_t^2} - 2h_1 h_5 - \frac{2h_1^2}{h_2 h_6} - \frac{2h_1 h_4}{h_2 h_3 h_5} \right) x^{127} \left( \frac{36m^2}{5h_t^2} - 36m^2 - x^{245} \right) + \frac{2m^2}{25} \left( \frac{81}{h_t^2} - 2h_2 h_6 - \frac{2h_1^2}{h_2 h_3 h_6} - \frac{2h_1 h_5}{h_2 h_3 h_6} \right) x^{236} - \frac{2m^2}{25} \left( \frac{81}{h_t^2} - 2h_4 h_5 - \frac{2h_2^2}{h_2 h_3 h_6} - \frac{2h_2 h_5}{h_2 h_3 h_6} \right) x^{347} - \frac{2m^2}{25} \left( \frac{81}{h_t^2} - \frac{2h_1 h_5}{h_2 h_3 h_6} - \frac{2h_1 h_4}{h_2 h_3 h_6} \right) x^{567}.
\]

**Remark 3.2.** The expressions of \( \Delta_1^7 \psi(t) \) needed for the Laplacian coflow are easily obtained by means of the ones given in Lemma 3.1 as the Laplacian commutes with the Hodge star operator, i.e. \( \Delta_1^7 \psi(t) = *_1 \Delta_1 \sigma(t) \).

Next we prove main result 1 obtaining explicit long time solutions for the Laplacian flow (4) for all the solvmanifolds \( S_i \), \( i = 1, \ldots, 7 \).

**Theorem 3.3.** Every 7-dimensional rank-one solvable extension of a nilpotent Lie group with a Locally Conformal Parallel G\(_2\) form \( \sigma_0 \) admits a long time solution \( \sigma(t) \) to the Laplacian flow, preserving the LCP condition along the flow, such that \( \sigma(0) = \sigma_0 \).

**Proof.** The result is proved for the solvmanifold \( S_1 \) in Theorem 2.2. Inspired by the solution obtained in this theorem we will consider families of G\(_2\)-structures of type (6) on the rest of solvmanifolds \( S_j \) where the functions \( h_i(t) \) are given by

\[
(16) 
\quad h_i(t) = (1 - \alpha m^2 t)^{\beta_i}, 
\]

for some \( \alpha \in \mathbb{R}^* \). To solve the evolution equation, one has from (8) and (9) that:

\[
\frac{d}{dt} \sigma(t) = -\frac{\alpha m^2}{1 - \alpha m^2 t} \left( \sum_{(i,j,k) \in \mathcal{I}} (\beta_i + \beta_j + \beta_k) x^{ijk} - \sum_{(i,j,k) \in \mathcal{J}} (\beta_i + \beta_j + \beta_k) x^{ijk} \right),
\]

\[
\frac{d}{dt} \psi(t) = -\frac{\alpha m^2}{1 - \alpha m^2 t} \left( \sum_{(l,m,n,o) \in \mathcal{K}} (\beta_l + \beta_m + \beta_n + \beta_o) x^{lmo} - (\beta_2 + \beta_4 + \beta_6 + \beta_7) x^{2467} \right).
\]

Let us see first that always \( \beta_7 = \frac{1}{2} \). Considering the expressions of \( \Delta \sigma(t) \) given in Lemma 3.1, observe that for any \( j = 2, \ldots, 7 \), there is a coefficient \( \Delta_{abc} \) (see (10) for notation) satisfying:

\[
\Delta_{abc} = \frac{k m^2}{h_t^2(t)},
\]

where \( k \in \mathbb{R}^* \). Now, since

\[
\frac{h'_i(t)}{h_i(t)} = -\frac{\alpha m^2 \beta_i}{1 - \alpha m^2 t},
\]

equation (12) is equivalent to:

\[
\frac{k m^2}{(1 - \alpha m^2 t)^{2\beta_7}} = -\frac{\alpha m^2 (\beta_a + \beta_b + \beta_c)}{1 - \alpha m^2 t},
\]

that is \( \beta_7 = \frac{1}{2} \) and \( \beta_a + \beta_b + \beta_c = -\frac{k}{\alpha} \).

On the other hand there also exist coefficients \( \Delta_{abc} \) of the form:

\[
\Delta_{abc} = \frac{k m^2}{1 - \alpha m^2 t} + \sum_{i=1}^r k_i m^2 (1 - \alpha m^2 t)^{s_i}, \quad k_i \in \mathbb{R}^*, s_i \in \mathbb{R},
\]

where \( s_i \) is a polynomial on variables \( \beta_j \) and we have used that \( \beta_7 = \frac{1}{2} \). Now, equation (12) can be read as:

\[
\frac{k}{1 - \alpha m^2 t} + \sum_{i=1}^r k_i (1 - \alpha m^2 t)^{s_i} = -\frac{\alpha (\beta_a + \beta_b + \beta_c)}{1 - \alpha m^2 t},
\]
or equivalently:
\[ \sum_{i=1}^{r} k_i (1 - \alpha m^2 t)^{s_i} = \frac{-k - \alpha (\beta_a + \beta_b + \beta_c)}{1 - \alpha m^2 t}. \]

This equation implies that:
\[ \begin{cases} s_1 = \cdots = s_r = -1, \\ \sum_{i=1}^{r} k_i = -k - \alpha (\beta_a + \beta_b + \beta_c). \end{cases} \]

For all cases, we obtain a system of equations in variables \( \alpha, \beta_i \), for \( i = 1, \ldots, 6 \), that has unique solution. The concrete values \( \alpha, \beta_i \) of the solutions are summarized in Table 1, where \( t \in (-\infty, \frac{1}{\alpha m^2}) \) and we have also included the data of the solutions obtained for the solvmanifold \( S_1 \). Moreover, by direct computations it can be checked that the obtained solutions \( \sigma(t) \) remain LCP for any value of \( t \) and also the torsion 1-form \( \tau \) described in (3) remains constant for all \( t \) and is exactly \( \tau = me^{t} \) for all \( S_i \).

| Solvmanifold | \( \alpha \) | \( (\beta_1, \ldots, \beta_7) \) |
|--------------|-------------|----------------|
| \( S_1 \)    | 4           | \( \left( \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2} \right) \) |
| \( S_2 \)    | 10/3        | \( \left( \frac{9}{10}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{7}{10}, \frac{1}{2} \right) \) |
| \( S_3 \)    | 3           | \( \left( \frac{1}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{1}{2} \right) \) |
| \( S_4 \)    | 14/7        | \( \left( \frac{11}{12}, \frac{11}{12}, \frac{5}{6}, \frac{5}{6}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2} \right) \) |
| \( S_5 \)    | 3           | \( \left( \frac{11}{12}, \frac{11}{12}, \frac{5}{6}, \frac{5}{6}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2} \right) \) |
| \( S_6 \)    | 8/3         | \( \left( \frac{1}{3}, \frac{3}{3}, \frac{3}{3}, \frac{3}{3}, \frac{3}{3}, \frac{1}{2} \right) \) |
| \( S_7 \)    | 14/7        | \( \left( \frac{13}{12}, \frac{10}{12}, \frac{10}{12}, \frac{13}{12}, \frac{10}{12}, \frac{10}{12}, \frac{1}{2} \right) \) |

Table 1. Defining parameters of the functions \( h_i(t) = (1 - \alpha m^2 t)^{\beta_i} \) that provide long time solution \( \sigma(t) \) of type (6) to the Laplacian flow of an LCP G\(_2\)-structure.

**Proposition 3.4.** Let \( \sigma(t) \) be one of the solutions with values described in Table 1 of the Laplacian flow of an LCP G\(_2\)-structure for the solvmanifold \( S_i \) with \( i = 2, \ldots, 7 \). Then the induced metric \( g_i \) is not Einstein and the curvature satisfies \( \lim_{t \to -\infty} R(g_i) = 0 \).

**Proof.** Consider the structure equations of the Lie algebra \( \mathfrak{cp}_i^{m} \) \( i = 2, \ldots, 7 \) described in page 2, and let us also consider a solution of the Laplacian flow of an LCP G\(_2\)-structure for the corresponding solvmanifold \( S_i \) with \( i = 2, \ldots, 7 \) given in Table 1. We can describe the structure equations of \( \mathfrak{cp}_i^{m} \) \( i = 2, \ldots, 7 \) in terms of an adapted basis of the corresponding solution of the Laplacian flow. Thus substituting the values of Table 1 on the latter structure equations is obtained that for every value of \( t \) all the non-vanishing Lie brackets are proportional to \( (1 - \alpha m^2 t)^{-\frac{1}{2}} \) for all \( \mathfrak{cp}_i^{m} \). Now from the expression of the Levi-Civita connection for an invariant metric:
\[ g(\nabla_x, x_j, x_k) = \frac{1}{2} g([x_i, x_j], x_k) + g([x_k, x_j], x_i) + g([x_k, x_i], x_j) \]
is obtained that the Christoffel symbols are also proportional to \( (1 - \alpha m^2 t)^{-\frac{1}{2}} \). Finally from the expression of the curvature \( R(g_i) \)
\[ R(g_i)(x_i, x_j, x_k, x_l) = g(\nabla_{x_i} \nabla_{x_j} x_k, x_l) - g(\nabla_{x_j} \nabla_{x_i} x_k, x_l) - g(\nabla_{x_i} x_k, \nabla_{x_j} x_l) + g(\nabla_{x_i} x_k, x_l) \]

\( \square \)
it can be easily checked that the non-vanishing coefficients are proportional to \((1 - \alpha m^2 t)^{-1}\). Thus,
\[
\lim_{t \to \infty} R(g_t) = 0.
\]

From the Riemannian curvature an explicit description of the Ricci curvature Ric\((g_t)\) can be obtained. These expressions are diagonal for all \(cp^m_i\) and are described in Table 2, where the coefficients \(C_1, C_2, C_3, C_4\) are given by:
\[
C_1 = -\frac{6m^2}{1 - \alpha m^2 t}, \quad C_2 = -\frac{m^2}{3} \left(\frac{1}{1 - \alpha m^2 t}\right), \quad C_3 = -\frac{m^2}{4} \left(\frac{1}{1 - \alpha m^2 t}\right), \quad C_4 = -\frac{m^2}{5} \left(\frac{1}{1 - \alpha m^2 t}\right).
\]
Hence, we can conclude that the induced metric \(g_t\) is not Einstein.

| Solvmanifold | Ric\((g_t)\) |
|--------------|-------------|
| \(S_1\)     | \(C_1 \text{ diag}(1,1,1,1,1,1)\) |
| \(S_2\)     | \(C_2 \text{ diag}(22,17,12,17,12,17)\) |
| \(S_3\)     | \(C_3 \text{ diag}(32,22,17,17,17,22)\) |
| \(S_4\)     | \(C_4 \text{ diag}(37,32,22,17,17,27)\) |
| \(S_5\)     | \(C_3 \text{ diag}(27,27,22,12,17,22)\) |
| \(S_6\)     | \(C_2 \text{ diag}(21,21,11,11,11,11)\) |
| \(S_7\)     | \(C_4 \text{ diag}(32,17,17,32,17,27)\) |

Table 2. Ricci tensors of the orthonormal metric expressed in the adapted basis \(\{x^i = h_i(t)e^i\}_{i=1}^7\) generated by the solutions of the Laplacian flow contained in Table 1.

4. Long time solutions of the Laplacian coflow of an LCP G\(_2\)-structure

Theorems 2.2 and 2.3 show that a solvmanifold with underlying Lie algebra \(cp^m_1\) admits solutions to the Laplacian flow (4) and the Laplacian coflow (5). Every solution is given by a one-parameter family of G\(_2\)-structures (6) and the functions describing the family are of potential type in both cases.

In this section we show that there is a similar situation for the rest of the Lie algebras \(cp^m_i, i = 2, \ldots, 7\), as we find a sort of correspondence between solutions of the Laplacian flow and the Laplacian coflow. As in the case of \(cp^m_1\), they are constructed by means of families of G\(_2\)-structures given by (6) involving potential functions. Furthermore, for every Lie algebra the defining parameters of the solution functions \(h_i(t)\) and \(\tilde{h}_i(t)\) of the Laplacian flow and coflow are related by explicit formul\ae\ that we will provide.

First, as both the Laplacian flow and coflow of an LCP G\(_2\)-structure require that the solution \(\sigma(t)\) remains LCP, we study necessary and sufficient conditions that the functions that appear in a generic family (6) of G\(_2\)-structures must satisfy to preserve this condition.

**Lemma 4.1.** A family of G\(_2\) structures \(\sigma(t)\) of type (6) on a solvmanifold \(S_i\) remains Locally Conformal Parallel if and only if:
- \(cp^m_2\): \(h_{36}(t) = h_{17}(t)\).
- \(cp^m_3\): \(h_{36}(t) = h_{45}(t) = h_{17}(t)\).
- \(cp^m_4\): \(h_{17}(t) = h_{36}(t) = h_{45}(t), h_{27}(t) = h_{46}(t)\).
- \(cp^m_5\): \(h_{17}(t) = h_{45}(t), h_{27}(t) = h_{46}(t)\).
\bullet \mathfrak{p}_0^m: h_{36}(t) = h_{45}(t) = h_{17}(t), \quad h_{35}(t) = h_{46}(t) = h_{27}(t).
\bullet \mathfrak{p}_2^m: h_{36}(t) = h_{17}(t), \quad h_{23}(t) = h_{57}(t), \quad h_{26}(t) = h_{47}(t).

Proof. The proof is a straightforward computation. Just impose for every Lie algebra \( \mathfrak{p}_i^m \) the LCP condition (3) to the family of \( G_2 \) structures \( \sigma(t) \) given by (6).

Observe that the functions \( h_i(t) \) included in Table 1 satisfy the former conditions for each Lie algebra.

Now we give the main result that provides for every solvmanifold \( S_i \) based on the Lie algebra \( \mathfrak{p}_i^m \) solutions of the Laplacian coflow of an LCP \( G_2 \)-structure based on solutions of the Laplacian flow in each algebra.

Theorem 4.2. Let \( \sigma(t) \) and \( \tilde{\sigma}(t) \) be two different families of \( G_2 \) structures on \( \mathfrak{p}_i^m \) with \( i = 1, \ldots, 7 \), given by (6), where

\[ h_i(t) = (1 - \alpha m^2 t)^{\beta_i}, \quad \beta_7 = \frac{1}{2}, \quad \text{and} \quad \tilde{h}_i(t) = (1 - \gamma m^2 t)^{\delta_i}, \quad \delta_7 = \frac{1}{2}. \]

If the defining parameters of the functions \( h_i(t) \) and \( \tilde{h}_i(t) \) are related by:

\[ \gamma = \alpha \left( \frac{2 - \sum_{i=1}^{7} \beta_i}{2} \right), \quad \text{and} \quad \delta_j = \frac{1}{2} + \frac{1 - 2\beta_j}{-2 + \sum_{i=1}^{7} \beta_i} \quad \text{with} \quad j \in \{1, \ldots, 7\}. \]

Then:

(i) \( \sigma(t) \) is LCP if and only if \( \tilde{\sigma}(t) \) is LCP.
(ii) \( \sigma(t) \) solves the Laplacian flow (4) if and only if \( \hat{\psi}(t) = \star_{\sigma} \tilde{\sigma}(t) \) solves the Laplacian coflow (5).

Proof. We denote by \( (l, m, n, o) \in K \cup (2467) \) the set of complementary indexes to \( (i, j, k) \in \mathcal{I} \cup \mathcal{J} \) i.e. \( (l, m, n, o) = (i, j, k) = (1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, \hat{k}, \ldots, 7) \). With this notation, (17) implies

\[ \gamma (\delta_l + \delta_m + \delta_n + \delta_o) = -\alpha (\beta_i + \beta_j + \beta_k), \]

for all \( (l, m, n, o) \in K \cup (2467) \) and \( (l, m, n, o) = (i, j, k) \).

Now we prove the two statements of the theorem.

(i) Let us consider two pair of indexes \((i_1, j_1, k_1), (i_2, j_2, k_2) \in \mathcal{I} \cup \mathcal{J}\) such that they have a common index, let us say \( k_1 = k_2 = k \). Under this hypothesis and making use of (18) the following identities hold:

\[ \gamma(\delta_1 + \ldots + \delta_7) = \gamma(\delta_1 + \ldots + \delta_7) \]

\[ \gamma(\beta_{i_1} + \beta_{j_1} + \beta_k) + \gamma(\delta_{i_1} + \delta_{m_1} + \delta_{n_1} + \delta_{o_1}) = \gamma(\delta_{i_2} + \delta_{j_2} + \delta_k) + \gamma(\beta_{i_2} + \beta_{j_2} + \beta_k) \]

\[ \gamma([\delta_{i_2} + \delta_{j_2} - (\delta_{i_1} + \delta_{j_1})]) = -\alpha[(\beta_{i_1} + \beta_{j_1}) - (\beta_{i_2} + \beta_{j_2})]. \]

Lemma 4.1 states that for every Lie algebra \( \mathfrak{p}_i^m \) the family of \( G_2 \)-structures \( \sigma(t) \) remain LCP if and only if

\[ \beta_{i_1} + \beta_{j_1} = \beta_{i_2} + \beta_{j_2}, \]

with the indexes satisfying that \((i_1, j_1, k) \in \mathcal{I}, (i_2, j_2, k) \in \mathcal{J}\). The LCP conditions for \( \tilde{\sigma}(t) \) are exactly the same that the LCP conditions for \( \sigma(t) \) interchanging the parameters \( \beta_i \) for \( \delta_i \). Therefore, considering the non-zero values of the parameter \( \alpha \) of the solutions of the Laplacian flow included in Table 1 it is easy to see that \( \gamma \neq 0 \) in all the cases, hence we conclude that \( \sigma(t) \) is LCP if and only if \( \tilde{\sigma}(t) \) is so.

(ii) Let \( \sigma(t) \) and \( \tilde{\sigma}(t) \) be two families of \( G_2 \)-structures of type (6) whose defining parameters of the functions \( h_i(t) \) and \( \tilde{h}_i(t) \) are related by (17). Let \( \sigma(t) \) be a solution of the Laplacian flow of an LCP \( G_2 \)-structure and we want to prove that \( \hat{\psi}(t) = \ast_{\sigma} \tilde{\sigma}(t) \) is a solution of the corresponding coflow. If \( \sigma(t) \)
is a solution of (4), then by (12) the evolution equation is equivalent to:
\[
\frac{d}{dt} \sigma(t) = \Delta \sigma(t) \iff \Delta_{ijk} = \frac{h_i}{h_j} + \frac{h_j}{h_k} + \frac{h_k}{h_i}
\]
\[
\iff \Delta_{ijk} = -\frac{1}{\alpha t} (\beta_i + \beta_j + \beta_k)
\]
\[
\iff \frac{h_i}{h_j} \Delta_{ijk} = -\alpha (\beta_i + \beta_j + \beta_k),
\]
where the latter expression must be valid for all the sets of indices $(i, j, k) \in I \cup J$. Since $\sigma(t)$ is LCP the expressions of $\Delta^\phi \sigma(t)$ given in Lemma 3.1 for every Lie algebra $\mathfrak{g}$ are simplified:

\[
\Delta^1_{\phi} \sigma(t) = -\frac{m^2}{h_2^2} \left[8(x^{127} + x^{347} + x^{567}) + 9(x^{135} - x^{146} - x^{236} - x^{245})\right],
\]
\[
\Delta^2_{\phi} \sigma(t) = -\frac{2m^2}{9h_2^2} \left[33(x^{127} - x^{236}) + 36(x^{135} - x^{146} - x^{245}) + 30(x^{347} + x^{567})\right],
\]
\[
\Delta^3_{\phi} \sigma(t) = -\frac{m^2}{h_3^2} \left[7(x^{127} - x^{236} - x^{245}) + \frac{15}{2} (x^{135} - x^{146} + 6(x^{347} + x^{567})\right],
\]
\[
\Delta^4_{\phi} \sigma(t) = -\frac{2m^2}{5h_3^2} \left[17(x^{127} - x^{146} - x^{236} - x^{245}) + 18(x^{135}) + 14(x^{347} + x^{567})\right],
\]
\[
\Delta^5_{\phi} \sigma(t) = -\frac{m^2}{h_4^2} \left[7(x^{127} - x^{146} - x^{245}) + \frac{15}{2} (x^{135} - x^{236} + 6(x^{347} + x^{567})\right],
\]
\[
\Delta^6_{\phi} \sigma(t) = -\frac{4m^2}{3h_4^2} \left[5(x^{127} + x^{135} - x^{146} - x^{236} - x^{245}) + 4(x^{347} + x^{567})\right],
\]
\[
\Delta^7_{\phi} \sigma(t) = -\frac{6m^2}{h_5^2} \left[(x^{127} - x^{236} + x^{347} + x^{567}) + \frac{6}{5} (x^{135} - x^{146} - x^{245})\right].
\]

By (i) we have that $\tilde{\sigma}(t)$ is also LCP. The simplified expressions of the Laplacians of the 4-forms are the following:

\[
\Delta^1_{\tilde{\phi}} \tilde{\psi}(t) = -\frac{m^2}{h_2^2} \left[8(y^{3456} + y^{1256} + y^{1234}) + 9(-y^{2467} + y^{2357} + y^{1457} + y^{1367})\right],
\]
\[
\Delta^2_{\tilde{\phi}} \tilde{\psi}(t) = -\frac{2m^2}{9h_2^2} \left[33(y^{3456} + y^{1457}) + 36(-y^{2467} + y^{2357} + y^{1367}) + 30(y^{1256} + y^{1234})\right],
\]
\[
\Delta^3_{\tilde{\phi}} \tilde{\psi}(t) = -\frac{m^2}{h_3^2} \left[7(y^{3456} + y^{1457} + y^{1367}) + \frac{15}{2} (-y^{2467} + y^{2357} + 6(y^{1256} + y^{1234})\right],
\]
\[
\Delta^4_{\tilde{\phi}} \tilde{\psi}(t) = -\frac{2m^2}{5h_3^2} \left[17(y^{3456} + y^{2357} + y^{1457} + y^{1367}) - 18(y^{2467}) + 14(y^{1256} + y^{1234})\right],
\]
\[
\Delta^5_{\tilde{\phi}} \tilde{\psi}(t) = -\frac{m^2}{h_4^2} \left[7(y^{3456} + y^{237} + y^{1367}) + \frac{15}{2} (-y^{2467} + y^{2357} + 6(y^{1256} + y^{1234})\right],
\]
\[
\Delta^6_{\tilde{\phi}} \tilde{\psi}(t) = -\frac{4m^2}{3h_4^2} \left[5(y^{3456} - y^{2467} + y^{2357} + y^{1457} + y^{1367}) + 4(y^{1256} + y^{1234})\right],
\]
\[
\Delta^7_{\tilde{\phi}} \tilde{\psi}(t) = -\frac{6m^2}{h_5^2} \left[(y^{3456} + y^{1457} + y^{1256} + y^{1234}) - \frac{6}{5} (y^{2467} - y^{2357} - y^{1367})\right],
\]
where \( \{y^1, \ldots, y^7\} \) is the adapted basis \( y^i = \tilde{h}_i(t)e^i \) for the \( G_2 \)-structure \( \tilde{\sigma}(t) \).

By a direct observation of the previous expressions we get the following relation

\[
(19) \quad h_2^2 \Delta_{ijk} = \tilde{h}_2^2 \tilde{\Delta}_{lmno}.
\]

linking the coefficients of \( \Delta \sigma(t) \) and \( \tilde{\Delta} \tilde{\psi}(t) \). Noticing that \( \sigma(t) \) is solution of the Laplacian flow we have the following sequence of identities:

\[
\tilde{h}_2^2 \tilde{\Delta}_{lmno} = h_2^2 \Delta_{ijk} = -\alpha(\beta_i + \beta_j + \beta_k) \overset{(18)}{=} \gamma(\delta_l + \delta_m + \delta_n + \delta_o),
\]

i.e. \( \tilde{h}_2^2 \tilde{\Delta}_{lmno} = \gamma(\delta_l + \delta_m + \delta_n + \delta_o) \) for every \( (l, m, n, o) \in \mathcal{K} \cup (2467) \), which is equivalent to \( \tilde{\psi}(t) \) be a solution of the coflow.

The converse of the statement is basically the same and we omit the proof. \( \square \)

As a consequence of the previous theorem, we provide in Table 3 for every solvmanifold \( S_i \) an explicit solution of the coflow based on the solutions of the flow contained in Table 1. We also include the information concerning the Ricci tensor of the induced metrics.

**Proposition 4.3.** Let \( \sigma(t) \) be a solution of the Laplacian coflow (5) for the solvmanifold \( S_i \) with \( i = 1, \ldots, 7 \), given by (16) with values described in Table 3. Then the induced metric \( g_t \) is not Einstein and the curvature satisfies \( \lim_{t \to \infty} R(g_t) = 0 \).

As a corollary of Theorems 3.3 and 4.2, we obtain main result 2:

**Theorem 4.4.** Every 7-dimensional rank-one solvable extension of a nilpotent Lie group admitting a Locally Conformal Parallel \( G_2 \) form has a solution to the Laplacian coflow (5) preserving the LCP condition.

| Solvmanifold | \( \gamma \) | \((\delta_1, \ldots, \delta_7)\) | \( \text{Ric}(\tilde{g}_t) \) |
|-------------|-------------|------------------|------------------|
| \( S_1 \)   | \(-6\)      | \((\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})\) | \(-\frac{6m^2}{1+6m^2} \text{diag}(1, 1, 1, 1, 1, 1, 1)\) |
| \( S_2 \)   | \(-\frac{16}{3}\) | \((\frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{5}{16})\) | \(-\frac{m^2}{3+8m^2} \text{diag}(22, 17, 17, 17, 17, 17, 17)\) |
| \( S_3 \)   | \(-5\)      | \((\frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17})\) | \(-\frac{m^2}{4+8m^2} \text{diag}(32, 22, 17, 17, 17, 17, 22)\) |
| \( S_4 \)   | \(-\frac{24}{7}\) | \((\frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4})\) | \(-\frac{m^2}{4+8m^2} \text{diag}(37, 22, 17, 17, 17, 17, 22)\) |
| \( S_5 \)   | \(-5\)      | \((\frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17})\) | \(-\frac{m^2}{4+8m^2} \text{diag}(27, 22, 17, 17, 17, 17, 22)\) |
| \( S_6 \)   | \(-\frac{14}{3}\) | \((\frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17})\) | \(-\frac{m^2}{4+8m^2} \text{diag}(21, 21, 11, 11, 11, 11, 16)\) |
| \( S_7 \)   | \(-\frac{24}{3}\) | \((\frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17}, \frac{3}{17})\) | \(-\frac{m^2}{4+8m^2} \text{diag}(32, 17, 17, 32, 32, 32, 17)\) |

**Table 3.** On the left side of the table there are the defining parameters of the functions \( \tilde{h}_i(t) = (1 - \gamma m^2 t)^{\delta_i} \) providing solutions \( \tilde{\sigma}(t) \) of type (6) to the Laplacian coflow. On the right side of the table the Ricci tensors of the corresponding metrics \( \tilde{g}_t \) expressed in the adapted basis \( \{y^i = \tilde{h}_i(t)e^i\}_{i=1}^7 \).
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### Appendix

| Solvmanifold | $R(g_t)$ |
|--------------|----------|
| $S_2$        | $R_{1367}(t) = R_{1637}(t) = \frac{-2}{3} C_2$, $R_{1717}(t) = \frac{-16}{3} C_2$, $R_{1736}(t) = \frac{4}{3} C_2$, $R_{1212}(t) = R_{1414}(t) = R_{1515}(t) = -4 C_2$, $R_{1313}(t) = R_{1616}(t) = R_{3636}(t) = \frac{-7}{3} C_2$, $R_{3737}(t) = R_{6767}(t) = \frac{-4}{3} C_2$, $R_{2323}(t) = R_{2626}(t) = R_{3434}(t) = R_{3535}(t) = R_{4646}(t) = R_{5656}(t) = -2 C_2$, $R_{2424}(t) = R_{2525}(t) = R_{2727}(t) = R_{4545}(t) = R_{4747}(t) = R_{5757}(t) = -3 C_2$ |
| $S_3$        | $R_{1212}(t) = \frac{-3}{2} C_3$, $R_{1313}(t) = R_{1414}(t) = R_{1515}(t) = R_{1616}(t) = \frac{-17}{4} C_3$, $R_{1367}(t) = R_{1457}(t) = -R_{1547}(t) = -R_{1637}(t) = -R_{3716}(t) = -R_{4715}(t) = R_{5714}(t) = R_{6713}(t) = \frac{-3}{2} C_3$, $R_{3645}(t) = R_{4536}(t) = \frac{-C_3}{2}$, $R_{1717}(t) = -9 C_3$, $R_{1316}(t) = R_{1745}(t) = R_{3617}(t) = R_{4531}(t) = \frac{3}{2} C_3$, $R_{2323}(t) = R_{2424}(t) = R_{2525}(t) = R_{2626}(t) = R_{3636}(t) = R_{4545}(t) = -3 C_3$, $R_{3456}(t) = -R_{3546}(t) = -R_{4635}(t) = R_{5634}(t) = \frac{C_3}{4}$, $R_{2727}(t) = -4 C_3$, $R_{3434}(t) = R_{3535}(t) = R_{3737}(t) = R_{4646}(t) = R_{4747}(t) = R_{5656}(t) = -3 C_3$, $R_{5757}(t) = R_{6767}(t) = \frac{-9}{4} C_3$ |
| $S_4$        | $R_{1234}(t) = R_{1256}(t) = -R_{1423}(t) = -R_{1625}(t) = -R_{2314}(t) = -R_{2516}(t) = R_{3412}(t) = R_{3456}(t) = -R_{3546}(t) = -R_{4635}(t) = R_{5612}(t) = R_{5634}(t) = \frac{C_4}{4}$, $R_{1367}(t) = -R_{1547}(t) = -R_{2467}(t) = R_{2647}(t) = -R_{4715}(t) = R_{4726}(t) = R_{6713}(t) = -R_{6724}(t) = \frac{-3}{5} C_4$, $R_{1313}(t) = R_{1515}(t) = \frac{-22}{5} C_4$, $R_{1414}(t) = R_{1616}(t) = -4 C_4$, $R_{1457}(t) = -R_{1637}(t) = \frac{-4}{5} C_4$, $R_{1736}(t) = R_{1745}(t) = R_{3617}(t) = R_{4531}(t) = \frac{7}{5} C_4$, $R_{1212}(t) = \frac{-12}{5} C_4$, $R_{2323}(t) = R_{2525}(t) = \frac{-24}{5} C_4$, $R_{2424}(t) = R_{2626}(t) = \frac{-17}{5} C_4$, $R_{2727}(t) = R_{4627}(t) = -\frac{6}{5} C_4$, $R_{3434}(t) = R_{4646}(t) = R_{5656}(t) = -\frac{12}{5} C_4$, $R_{3535}(t) = R_{3737}(t) = R_{5757}(t) = \frac{-16}{5} C_4$, $R_{3636}(t) = R_{4545}(t) = -3 C_4$, $R_{3645}(t) = R_{4536}(t) = \frac{-2}{5} C_4$, $R_{3716}(t) = -R_{5714}(t) = \frac{4}{5} C_4$, $R_{4747}(t) = R_{6767}(t) = \frac{-9}{5} C_4$, $R_{1717}(t) = \frac{-49}{5} C_4$. |

Table 4. Non-vanishing coefficients of the curvature of the metric $g_t$ induced by the solutions of the LCP flow expressed in the adapted basis $\{x_i\}_{i=1}^3$. 
\begin{table}[h!]
\centering
\begin{tabular}{|c|c|}
\hline
\textbf{S5} & \textbf{R(g)} \\
\hline
$R_{1234}(t) = R_{1256}(t) = -R_{1423}(t) = R_{1526}(t) = R_{1616}(t) = R_{2323}(t) = R_{2424}(t) = R_{2525}(t) =$ & \\
$R_{2626}(t) = \frac{3}{16} C_2$, & \\
$R_{1324}(t) = -R_{1423}(t) = R_{1526}(t) = -R_{1625}(t) = -R_{2314}(t) = R_{2413}(t) = R_{2615}(t) =$ & \\
$R_{3536}(t) = -R_{3645}(t) = -R_{4536}(t) = R_{4635}(t) = \frac{C_2}{17}$, & \\
$R_{1367}(t) = R_{1457}(t) = -R_{1547}(t) = R_{2357}(t) = -R_{2467}(t) = -R_{2537}(t) =$ & \\
$R_{2647}(t) = -R_{3716}(t) = -R_{3725}(t) = -R_{4715}(t) = R_{4726}(t) = R_{5714}(t) = R_{5723}(t) =$ & \\
$R_{6713}(t) = -R_{6724}(t) = \frac{C_2}{3}$, & \\
$R_{1736}(t) = R_{1745}(t) = R_{2735}(t) = -R_{2746}(t) = R_{3327}(t) = R_{3617}(t) = R_{4517}(t) =$ & \\
$-R_{4627}(t) = \frac{2}{3} C_2$, & \\
$R_{3434}(t) = R_{3714}(t) = R_{4747}(t) = R_{5656}(t) = R_{5757}(t) = R_{6767}(t) = \frac{4}{3} C_2,$ & \\
$\text{Table 5. Continuation of Table 4.}$ & \\
\hline
\end{tabular}
\end{table}

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