Scaling behavior of the insulator to plateau transition in topological band model

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Scaling behavior of the quantum phase transition from an insulator to a quantum Hall plateau state has often been examined within systems realizing Landau levels. We study the topological transition in energy band model with nonzero Chern number, which has same topological property as a Landau level. We find that topological band generally realizes the same universality class as the integer quantum Hall system under uniform magnetic flux for strong enough disorder scattering. Furthermore, the symmetry of the transition characterized by relations: $\sigma_{xy}(E) = 1 - \sigma_{xy}(-E)$ for Hall conductance and $\sigma_T(E) = \sigma_T(-E)$ for longitudinal Thouless conductance is observed near the transition region. We also establish that finite temperature dependence of Hall conductance is determined by inelastic scattering relaxation time, while the localization exponent $\nu$ remains unchanged by such scattering.

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Introduction—Anderson localization theory[1–3] predicts that noninteracting electrons are generally localized in two-dimensional (2D) disordered systems at zero temperature limit without a magnetic field. In contrast to the physics of localization for such systems belonging to the orthogonal class, extended (delocalized) states[4–6] exist in systems with a magnetic field or spin-orbit coupling[7], which are capable of conducting electric current going through whole samples. The topological characterization[6, 8–11] plays a central role in understanding delocalization of these systems, where topological nontrivial states characterized by nonzero Chern numbers must exist associated with delocalization in these 2D systems. The integer quantum Hall effect (IQHE) discovered for electron systems under magnetic field demonstrates universal scaling behavior[12] in accordance with a single delocalized quantum critical point $(E_c)$[5, 6, 13–15] between a trivial insulator and a quantized Hall plateau state or between two adjacent plateau states, while localization length diverges as a power law $\xi \sim |E_f - E_c|^{-\nu}$ near the quantum critical point[13–17]. Physical quantities like conductances follow the one parameter scaling law as a function of the scaling variable $(L/\xi)^{1/\nu} \propto L^{1/\nu}\Delta E$ with $\Delta E = E_f - E_c$ and $L$ being the system length. However, experiments have been done at finite temperature where near thermodynamic sample size is being cut-off by a finite length scale $L_{in} \propto T^{-\nu/2}$ representing the dephasing effect[18–20] with $p$ known as the inelastic scattering exponent. Thus the scaling parameter for finite temperature systems appear to be $T^{-\kappa}\Delta E$ with $\kappa = p/2\nu[12]$. While extensive numerical studies establish a robust universal scaling dependence on the system length with the scaling exponent $\nu \sim 2.40$–2.60 (more recent studies suggest slightly bigger value[21] than earlier results[14–17, 20, 22]). The same exponent $\nu$ has also been established[17] for lattice models with uniform magnetic flux. On the other hand, there are no well established results for understanding the temperature scaling law[12, 20] in a microscopic model incorporating disorder and inelastic scattering effects. For noninteracting system, recent numerical studies based on noncommutative Kubo formula support that with the input of $1/\tau_{in} \propto kT$, one obtains a $\kappa = 1/2\nu \sim 0.2[23]$. However, the physical original for the finite temperature scaling behavior remains not well understood. Very interestingly, experimentally observed decay exponent $\kappa \approx 0.43$ is likely to be universal suggesting the scenario that interaction must play very important role in these systems. Since the scaling behaviors for interacting and disordered systems are much harder to be settled[20], it is highly interesting to examine precisely how the inelastic scattering controls the temperature scaling of transport in noninteracting systems, which may provide a systematic understanding to experimental observations.

On a different forefront, topological band models first proposed by Haldane[24] have attracted a lot of recent attention since energy bands of such systems can be tuned to mimic uniform magnetic field, and host nontrivial fractionalized topological states[25] despite the net zero flux in the system. Such models also realize the $Z_2$ topological quantum spin Hall effect[26–28] once we include spin degrees of freedom, which have also been extensively studied for their interesting delocalization properties[29–33]. There are still open questions regarding universal properties of the insulator to quantum Hall state transition in such systems. It is unclear if such systems will realize the same universality class as the system under uniform magnetic flux since a region of extended states may exist based on the argument of possible delocalized critical states in random flux problem[34]. One can also ask what is characteristic length scale to reach the universal scaling law for such systems if they do satisfy it in thermodynamic limit. In particular, we are interested in exploring universal behaviors of the transition for both system length dependence at zero temperature and finite temperature scaling, which have not been studied simultaneously in microscopic model simulations so far.

In this paper, we present numerical study based on calculations of longitudinal Thouless conductance[35] $\sigma_T$ and Hall conductance $\sigma_{xy}$ using the exact diagonalization method for topological band model. Our results show that a direct insulator to plateau transition indeed belongs to the same universality class as the conventional IQHE system when disorder scattering is strong. In particular, the scaling exponent $\nu$ is universal and scaling curves for both $\sigma_T$ and $\sigma_{xy}$ respect the particle-
hole symmetry for large enough system length \((L \sim 120)\), which continuously adjust themselves towards recovering the symmetry with the increase of \(L\). The temperature scaling behavior is fully determined by the inelastic scattering relaxation time \(1/\tau_{in}\) at low temperature limit, which results in a universal scaling law \(d\sigma_{xy}/dE_f \simeq (1/\tau_{in})^{-1/2\nu}\) for finite temperature conductance. While this is consistent with the scaling theory\([5, 23]\), to our best knowledge, it is for the first time being revealed based on numerical model simulations using conventional Kubo formula. Our results are consistent with the experimental finding that the electron-electron interaction only comes into play as a temperature dependent relaxation time \((1/\tau_{in} \propto (kT)^2)\), which does not change the localization length exponent \(\nu\).

![Lattice model and method—We study the Haldane model \([24]\) on the honeycomb lattice in the following tight-binding form:](image)

\[
H = (-t \sum_{\langle ij \rangle} c_i^\dagger c_j + t_1 \sum_{\langle\langle i j \rangle\rangle} e^{i\phi_{ij}} c_i^\dagger c_j + t_2 \sum_{\langle\langle\langle i j \rangle\rangle\rangle} c_i^\dagger c_j + H.c.) + \sum_i w_i c_i^\dagger c_i
\]

where \(\langle \ldots \rangle, \langle\langle \ldots \rangle\rangle\) and \(\langle\langle\langle \ldots \rangle\rangle\rangle\) denote the nearest-neighbor (NN), the next NN and the third neighbor of pairs of sites. The \(c_i^\dagger\) is a fermionic creation operator and \(w_i\) is a disorder potential uniformly distributed between \((-W/2, W/2)\). We set \(t_1 = 1, t_2 = 0.40\), and vary \(t_2\) as a parameter to tune the band width of the energy spectrum, which can realize a flatband\([36]\) at \(t_2 = 1/3\) and wider band at \(t_2 = 0\). We study finite size system with \(N = L_x L_y\) sites, where \(L_x\) represents the number of sites along each zigzag chain and \(L_y\) is the number of chains as illustrated in Fig. 1(a).

To examine critical behavior of the system for different energy bands with different band broadening, we first show two examples of densities of states \(\rho\) in Fig. 1(b). We obtain longitudinal conductance from Thouless number calculations\([35, 37]\) as \(\sigma_T = 2\pi E_T/\Delta\), where \(E_T\) is the geometry average of the energy difference for eigenstates between periodic and antiperiodic boundary conditions and \(\Delta\) is average energy level spacing. For \(t_2 = 1/3\), which realizes a near flatband\([36]\) without disorder, we can see that two bands separated by a gap as shown in Fig. 1(a) (the spectrum is symmetric about \(E = 0\) and we only show one band), each carrying a nonzero but opposite Chern number \(C = \pm 1\)\([24, 36]\), respectively. The \(\rho\) is still well broadened by random disorder at \(W = 2\). For another case with \(t_2 = 0\) and \(W = 6\), the two bands are mixed together and \(\rho\) is near constant in whole energy range due to strong disorder effect. We note that longitudinal conductance peaks are very sharp comparing to the profile of the densities of states, indicating that most of states are localized according to Anderson localization except for the states near the conductance peaks.

After obtaining all eigenstates through exact diagonalization, we calculate \(\sigma_{xy}\) based on Kubo formula:

\[
\sigma = \frac{i\hbar e^2}{A} \sum_{m,n} (f_m - f_n)(m|v_x|n)(n|v_y|m) - h.c. \bigg/ (E_m - E_n)^2 + \eta^2 \bigg/, \quad (1)
\]

where \(A\) is the finite-size system area, \(v\) is the velocity operator, and \(f_{m(n)}\) is the Fermi-Dirac distribution function. \(|n\rangle\) and \(|m\rangle\) denote exact eigenstates of the system. \(\eta = 1/\tau_{in}\) is usually introduced to take into account of the finite relaxation time of the system due to inelastic scatterings.

**Hall and longitudinal conductances at T=0—** We first present zero-temperature results by setting \(\eta = 0\), where \(\sigma_{xy}\) is well defined associated with the topological invariant Chern numbers\([6, 10, 17, 22]\) of the states and the critical scaling behavior of the system is fully determined by disorder scattering. As we tune the Fermi energy \(E_f\), one can determine...
a quantum phase transition by following the evolution of Hall conductance \( \sigma_{xy} \), which is shown in Fig. 2(a) for \( t_2 = 1/3 \) and \( W = 2 \). We find that \( \sigma_{xy} \) continuously increases from zero (insulating state) to the quantized value \( e^2/h \), with data for different system sizes approximately crossing each other at one single energy \( E_c \). The transition from the insulating state to Hall plateau state becomes sharper with the increase of the system length \( L \).

Now we consider system size dependence of \( \sigma_{xy} \) to determine the scaling behavior of the quantum phase transition. According to the scaling theory of the quantum Hall system, localization length should satisfy a powerlaw behavior\[19\] near a quantum phase transition as \( \xi(E_f) \propto |E_f - E_c|^{-\nu} = |\Delta E|^{-\nu} \), and the conductance should be a function of the single parameter as the ratio \( L/\xi(E) \sim L|\Delta E|^{\nu} \), or equivalently the scaling variable \( (L/L_0)^{1/\nu} \Delta E \) (here \( L_0 \) is a constant for length scale) for large enough \( L \). As we replot data from different system sizes using the scaling variable as shown in Fig. 2(b), all data collapse onto one curve with \( \nu = 2.5 \) giving best results, consistent with the well accepted value for the insulator to plateau transition in a magnetic field. Interestingly, similar Hall conductance results are obtained for the wider band \( t_2 = 0 \) and \( W = 6 \). We plot all data from both cases together in Fig. 3(a) as a function of the scaling variable \( (L/L_0)^{1/\nu} \Delta E \) with constant \( L_0 = 24 \) and \( \nu = 2.5 \). The transition in the stronger \( W \) case is much slower, however all data points from both systems can also be collapsed together through rescaling the \( x \) variable to \( b(L/L_0)^{1/\nu} \Delta E \). We obtain \( b = 0.40 \) for the latter case (if we set \( b = 1 \) for \( W = 2 \) case), indicating the length scale in the latter is relatively bigger as \( L_0/b^{\nu} \sim 9.88L_0 \). Two curves merge into each other around the transition region, consistent with the univer-

![FIG. 2: (Color online) (a) The evolution of Hall conductance \( \sigma_{xy} \) with Fermi energy \( E_f \) for different system sizes from \( L = 24 \) to 120 for the model with \( t_2 = 1/3 \) and \( W = 2 \). All data points cross at a critical energy \( E_c \) and the transition to \( \sigma_{xy} = e^2/h \) Hall plateau is becoming sharper with the increase of the \( L \). (b) Using the scaling variable \( (L/L_0)^{1/\nu} \Delta E \) \( (\Delta E = E_f - E_c) \) to rescale all curves. All data for different \( L \) collapse into one smooth curve consistent with one parameter scaling law for the IQHE. We find best fitting results using \( \nu = 2.5 \). We choose the parameter \( L_0 = 24 \).](image1)

![FIG. 3: (Color online) (a) \( \sigma_{xy} \) for both \( W = 2 \) and \( W = 6 \) are shown to follow similar scaling curves with the scaling variable \( (L/L_0)^{1/\nu} \Delta E \). The same \( \nu = 2.5 \) is obtained for the best collapsing effect of all data. We find that the transition in the wider band case \( (W = 6) \) is much slower. (b) The two scaling curves can be rescaled together by adjusting the scaling variable by a constant \( b \) to the \( W = 6 \) curve. We obtain \( b = 0.40 \), indicating the length scale in the latter is relatively bigger as \( L_0/b^{\nu} \).](image2)
sional scaling curves discussed before for the IQHE system under a magnetic field[22]. Furthermore, we find that near the transition region, the reflection symmetry for Hall conductance as: $\sigma_{xy}(-\Delta E) = 1 - \sigma_{xy}(\Delta E)$ is satisfied despite there is no particle-hole symmetry about the transition point ($E_c$) in the original Hamiltonian.

Now we discuss the scaling behavior for Thouless conductance $\sigma_T$[35]. It has been established that the $\sigma_T$ is proportional to the longitudinal conductance[37] in the quantum Hall transition region and thus they satisfy the same scaling law. Shown in Fig. 4(a), we obtain $\sigma_T$ for different system sizes. The $E_c$ as the crossing point for $\sigma_{xy}$ of different sample sizes ($L = 24$ to $L = 120$) also coincides with the peak position of Thouless conductance $\sigma_T$. The $\sigma_T$ at the critical point shows a slow increasing with system length, which scales to a constant value at large $L$ limit. Away from the peak position, $\sigma_T$ decreases with the increase of $L$. This is consistent with a direct transition with the localization length of the system diverging at $E_c$. To check if the $\sigma_T$ satisfies the one parameter scaling law, we plot the renormalized conductances as a function of the scaling variable $(L/L_0)^{1/\nu}\Delta E$ with the same exponent $\nu = 2.5$. We see clearly that all data points are collapsing onto one curve, except data from the smaller size $L = 24$. We also find that the symmetry relation $\sigma_{xx}(-\Delta E) = \sigma(\Delta E)$ is only satisfied in a small region near $E_c$. More carefully examination suggests that the shape of the scaling curve is slowly changing with the increase of $L$ to make the curve more symmetric about the transition point. Thus we believe that, the particle-hole symmetry will be recovered over larger range of the transition at large $L$ limit. We further obtain the $\sigma_T$ for the wider band case with $t_2 = 0$ and $W = 6$. Our results are more or less symmetric about the critical point $E_c$. The $\sigma_T$ is near constant at the critical point, while all $\sigma_T$ values drop with the increase of $L$ away from $E_c$. Furthermore, we replot all data points using the scaling variable $(L/L_0)^{1/\nu}\Delta E$ and indeed all

![FIG. 4: (Color online) (a) The Thouless conductance $\sigma_T$ data for system length $L = 24$ to $L = 120$ for $t_2 = 1/3$ and $W = 2$. The peak appears at the same critical energy $E_c$ identified as the crossing point of the $\sigma_{xy}$ for different $L$. The width of each curve shrinks as $L$ increases. The $\sigma_T$ at the peak grows slowly with $L$, which scales to a constant at thermodynamic limit. (b) The renormalized $\sigma_{xx}$ is shown to follow the one parameter scaling law with the same $\nu = 2.5$. All data from larger sizes (except $L = 24$ case) collapse nicely into one curve.](image1)

![FIG. 5: (Color online) (a) The Thouless conductance $\sigma_T$ for system sizes $L = 48$ to $L = 120$ for $t_2 = 0$ and $W = 6$. (b) There are stronger finite size effect for this system due to much larger length scale. We find all data can only be approximated fitting into one curve. Interestingly, the larger sizes results ($L = 120$) are systematically deviating from the curves for smaller $L$, as $\sigma_T$ drops faster at $\Delta E < 0$ side and slower at the other side to adjust towards more symmetric curve.](image2)
data seem to collapse onto one curve as long as \( L > 48 \). The
deviation from the scaling at smaller sizes is not surprising
since the length scale in this wider band case is about one
order of magnitude bigger than \( t_2 = 1/3 \) and \( W = 2 \) case. We
also observe a stronger trend of adjusting of the shape of the
scaling curve for larger system sizes toward more symmetric
curve (it drops faster at \( \Delta E < 0 \) side and slower at the other
side for \( L = 120 \) system).

Through these simulations, we establish that for the topo-
logical band model we studied with strong disorder, the insu-
lator to plateau transition indeed demonstrates the same scal-
ing law as the IQHE under magnetic field consistent with the
scaling theory\[5, 20\] for the unitary class. It is important
to see that even at finite system sizes we can examine here
\( (L \sim 120) \), the one parameter scaling law is well established
for both \( \sigma_{xy} \) and \( \sigma_T \), while for the latter there is a slow tuning
of the shape of the scaling curve towards more symmetric one
upon the increase of the system length. We have also checked
that for weak disorder limit, there are much wider region of
the energy band where states appear to be delocalized or with
localization length much longer than our system sizes. One
needs to go to much larger \( L \) to see systematic scaling behav-
ior, which is beyond the scope of our current work.

**Hall conductance with finite relaxation time and the fi-
tnite temperature effect**—Now, we are ready to study the tem-
perature dependence of the quantum critical behavior. For
the finite temperature transport, one has to take into account
the\[23, 38\] finite \( \frac{1}{\tau_{in}} \propto (kT)^p \) due to the finite re-
laxation time \( \tau_{in} \) caused by the temperature dependent in-
elastic dephasing process. Theoretically, based on the scal-
ing argument\[20, 35\], it is known that the inelastic scattering
exponent \( p = 1 \) for noninteracting and \( p = 2 \) for electron-
electron interacting systems\[20\]. For disorder systems we
study, it is not well established how the finite relaxation time
will affect the Hall conductance, which has been studied re-
cently based on the noncommutative Kubo formula\[23\]. Here,
we take a different approach by following Kubo formula Eq.
(1) as it is accurate for large system sizes we consider, which
also provides the advantage of studying length scaling and fi-
nite \( T \) scaling on equal footing. As a start, we will take \( \frac{1}{\tau_{in}} \)
as a free parameter for Eq.(1) and calculate the corresponding
\( \sigma_{xy} \) at \( T = 0 \). In Fig. 6(a), we show the overall behavior of
the derivative \( \langle d\sigma_{xy}/dE_f \rangle_{E_c} \) at the transition point as a
function of the \( 1/\tau_{in} \), which characterizes the sharpness of the
insulator to plateau transition. As indicated by the straight
line fitting in the logscale plot, we find that \( \langle d\sigma_{xy}/dE_f \rangle_{E_c} \propto
(1/\tau_{in})^x \), with the exponent \( x = 0.22 \pm 0.03 \cong 1/(2\nu) \) for
two systems \( L = 60 \) and \( L = 120 \) in a wide range of \( 1/\tau_{in} \).
The relatively larger error bar is due to the sensibility of the
derivative on small changes of the \( \sigma_{xy} \) and we have taken
more than 1000 disorder configurations average to ensure ac-
curate results. At high \( 1/\tau_{in} \) side, the scaling fails around
\( 1/\tau_{in} > 0.20 \), where the effective length of the system \( L_{in} \) is

\[ \text{FIG. 6: (Color online) (a) The scaling behavior of the}
\langle d\sigma_{xy}/dE_f \rangle_{E_c} \text{ as a function of the inverse of the relaxation time}
\text{for two systems with } L = 120 \text{ and } 60 \text{ at } T = 0. \text{ (b) The}
\langle d\sigma_{xy}/dE_f \rangle_{E_c} \text{ for finite } T \sigma_{xy} \text{ as a function of } kT. \text{ The relaxa-
tion time has been chosen as } 1/\tau_{in} = ckT, \text{ where the contribution}
\text{from the Fermi-Dirac distribution sets in at higher } T. \]

\[ \text{FIG. 7: (Color online) The finite temperature } \sigma_{xy} \text{ for a wide range of}
kT \text{ between } 0.0002 \text{ to } 0.02 \text{ (in units of } t) \text{ has been shown as the func-
tion of the scaling variable } (T/T_0)^{-\kappa} \Delta E \text{ (} kT_0 = 0.0002 \text{),}
\text{where all data collapse onto one curve, with best fitting exponent}
\kappa = 0.22 \approx 1/2\nu. \]
reduced to a couple of lattice constant determined by a comparison with the Hall conductance of small $L \sim 6$ size with $1/\tau_{in} = 0$. On the other hand, at small $1/\tau_{in}$ limit, the derivative saturates to a near constant due to the fact that the effective length of the system is being cut-off by the sample length when $L \leq L_{in}$, thus the Hall conductance becomes insensitive to the decreasing of the $1/\tau_{in}$. Indeed, we see that the small $1/\tau_{in}$ cut off is moving towards lower value with the increase of the sample length $L$.

According to the one parameter scaling law, one would expect that $d\sigma_{xy}(T)/dE = d\sigma_{xy}(T/L_0)^{1/2} f(0)$, where $f(x)$ is the scaling function for the Hall conductance. Our results suggest that indeed the finite $1/\tau_{in}$ effect is setting a finite dephasing length $L_{in} \propto \sqrt{\tau_{in}}$ as long as the $L$ is large enough so that $L_{in} << L$. The observed powerlaw behavior provides an explanation to the experimental results\cite{12} of the powerlaw scaling of $(d\sigma_{xy}/dE_f)|_{E_c} \propto T^{-\kappa}$. Because $1/\tau_{in} \propto (kT)^{\nu}$, one obtains $\kappa \approx p/2\nu \approx 0.4$ or $0.2$ if we take $p = 2$ for electron-electron scattering or $p = 1$ for noninteracting systems, respectively. However, there could be a finite temperature correction due to the contribution from the Fermi-Dirac distribution function. To explicitly address this effect, we calculate the finite temperature Kubo conductance with different strength of the inelastic scattering $1/\tau_{in} = c/kT$ by choosing $p = 1$ for our noninteracting system. As shown in Fig. 6(b), we find that the $(d\sigma_{xy}/dE_f)|_{E_c} \propto T^{-\kappa}$, with fitting $\kappa \approx 0.225 \pm 0.03$ for our system $L = 120$ independent of the strength of the inelastic scattering $c$. Furthermore, there is a visible higher temperature break down, which sets in earlier for smaller $c$ as the Fermi-Dirac distribution will contribute more significantly for such systems due to the sharper transition (or stronger dependence on $E_f$) near the transition point. The cut-off temperature at low $T$ limit depends on the parameter $c$ and the stronger inelastic scattering gives wider range of the powerlaw scaling since the dephasing length $L_{in}$ reaches $L$ at lower $T$.

We have also obtained $\sigma_{xy}$ at different $E_f$ for different temperatures as shown in Fig. 7 for parameter $c = 2$. We find that all data from a wide range of temperature with $kT$ varying from 0.0002 to 0.02 in units of hopping $t$, can be collapsed onto one curve using the scaling variable $(T/T_0)^{-\kappa} \Delta E$, with the best fitting $\kappa = 0.22 \sim 1/2\nu$ as expected from the scaling behavior of the $(d\sigma_{xy}/dE_f)|_{E_c}$. We suspect that the small difference between the obtained $\kappa$ and $1/2\nu = 0.2$ is due to the finite size effect (since $L_{in} << L = 120$).

To summarize, we have systematically studied zero temperature and finite temperature scaling behavior of the insulator to plateau transition in topological band model. While we observe universal scaling behavior for zero temperature Hall and longitudinal conductances, we also find that the wider band with stronger disorder has much larger length scale for reaching the one parameter scaling regime. At low enough temperature, the Hall conductance follows one parameter scaling law: $\sigma_{xy} = f((T/T_0)^{-\kappa} \Delta E)$, with $\kappa = p/2\nu$ fully determined by temperature dependence of inelastic relaxation time $1/\tau_{in} \propto T^{\nu}$. Our results suggest that the electron-electron interaction is relevant for experimentally observed $\kappa$ through its temperature dependent relaxation time ($p = 2$)\cite{12} while the length scaling exponent $\nu$ remains unchanged by such inelastic relaxation effect.

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\[1\] E. Abrahams, P. W. Anderson, D. C. Licciardello, and V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).
\[2\] For a review, see, P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. 57, 287 (1985).
\[3\] A. MacKinnon and B. Kramer, Phys. Rev. Lett. 47, 1546 (1981); Z. Phys. B53, 1 (1983).
\[4\] R. B. Laughlin, Phys Rev. B, 23, 5632 (1981); B. I. Halperin, Phys Rev. B, 25, 2185 (1982).
\[5\] H. Levine, S. B. Libby and A. M. M. Pruisken, Phys. Rev. Lett. 51, 1915 (1983); A. M. M. Pruisken, Phys. Rev. Lett. 61, 1297 (1988).
\[6\] Y. Huo and R. N. Bhatt, Phys. Rev. Lett. 68, 1375 (1992).
\[7\] F. Wegner, Nucl. Phys. B, 316, 663 (1989).
\[8\] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982); D. J. Thouless, J. Phys. C, 17, L325 (1984); Q. Niu, D. J. Thouless, and Y. S. Wu, Phys. Rev. B 31, 3372 (1985).
\[9\] M. Kohmoto, Ann. Phys. (NY) 160, 343 (1985).
\[10\] D. P. Arovas et. al., Phys. Rev. Lett. 60, 619 (1988).
\[11\] D. N. Sheng and Z. Y. Weng, Phys. Rev. B 54, R11070 (1996).
\[12\] H. P. Wei et. al., Phys. Rev. Lett. 61, 1294 (1988); R. T. F. van Schaik et al., Phys. Rev. Lett. 84, 1567 (2000); Wanli Li, J. S. Xia, C. Vicente, N. S. Sullivan, W. Pan, D. C. Tsui, L. N. Pfeiffer, and K. W. West Phys Rev. B 81, 033305 (2010); Phys. Rev. Lett. b'f 102, 216801 (2009).
\[13\] H. Aoki abd T. Ando, Phys. Rev. Lett. 54, 831 (1985).
\[14\] B. Huckestein and B. Kramer, Phys. Rev. Lett. 64, 1437 (1990).
\[15\] J. T. Chalker and P. D. Coddington, J. Phys. C 21, 2665 (1988).
\[16\] S. A. Kivelson, D. -H. Lee, and S. -C Zhang, Phys. Rev. B, 59, 1316 (1999); S. Xia, C. Vicente, N. S. Sullivan, W. Pan, D. C. Tsui, L. N. Pfeiffer, and K. W. West Phys Rev. B 81, 033305 (2010); Phys. Rev. Lett. b'f 102, 216801 (2009).
\[17\] K. Yang and R. N. Bhatt, Phys. Rev. Lett. 76, 1316 (1996); Phys. Rev. B 59, 8144 (1999).
\[18\] D. J. Thouless, Phys. Rev. Lett. 39, 1167 (1977).
\[19\] P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B 22, 3519 (1980).
\[20\] B. Huckestein, Rev. Mod. Phys. 67, 357396 (1995).
\[21\] K. Slevin and T. Ohnuki, Phys. Rev. B 80, 041304 (2009); H. Obuse, A. R. Subramaniam, A. Furusaki, I. A. Gruzberg, and A. W. W. Ludwig, Phys. Rev. B 82, 035309 (2010). D. Semmler, K. Byczuk, and W. Hofstetter, Phys. Rev. B 84, 115113 (2011). M. Amado, A. V. Malyshew, A. Sedrakyan, and F. Domiguez-Adame, Phys. Rev. Lett. 107, 066402 (2011).
\[22\] D. N. Sheng and Z. Y. Weng Phys Rev. B 59, R7821 (1999); D. N. Sheng and Z. Y. Weng Phys. Rev. Lett. 80, 580 (1998).
\[23\] Yu Xue, Emil Prodan, Phys. Rev. B 87, 115141 (2013); Juntao
Song, Emil Prodan, arXiv:1301.5305.

[24] F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (1988).

[25] E. Tang, J. W. Mei and X. G. Wen, Phys. Rev. Lett. 106, 236802 (2011); T. Neupert, L. Santos, C. Chamon and C. Mudry, Phys. Rev. Lett. 106, 236804 (2011); K. Sun, Z. C. Gu, H. Katsura and S. Das Sarma, Phys. Rev. Lett. 106, 236803 (2011); D. N. Sheng, Z. C. Gu, K. Sun and L. Sheng, Nature Commun. 2, 389 (2011); N. Regnault and B. Andrei Bernevig, Phys. Rev. X 1, 021014 (2011); X.-L. Qi, Phys. Rev. Lett. 107, 126803 (2011).

[26] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005). C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005).

[27] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science 314, 1757 (2006).

[28] D. N. Sheng, Z. Y. Weng, L. Sheng, F.D.M. Haldane, Phys. Rev. Lett. 97, 036808 (2006); L. Sheng, D. N. Sheng, C. S. Ting, and F. D. M. Haldane, Phys. Rev. Lett. 95, 136602 (2005).

[29] Masaru Onoda, Yshai Avishai, and Naoto Nagaosa, Phys. Rev. Lett. 98, 076802 (2007).

[30] Hideaki Obuse, Akira Furusaki, Shinsei Ryu, and Christopher Mudry, Phys. Rev. B 76, 075301 (2007).

[31] I. C. Fulga, F. Hassler, A. R. Akhmerov, and C. W. J. Beenakker, Phys. Rev. B 84, 245447 (2011); E. P. L. van Nieuwenburg, J. M. Edge, J. F. Dahlhaus, J. Tworzydo, and C. W. J. Beenakker, Phys. Rev. B 85, 165131 (2012).

[32] R. S. K. Mong, J. H. Bardarson, and J. E. Moore, Phys. Rev. Lett. 108, 076804 (2012); E. Rossi, J. H. Bardarson, M. S. Fuhrer, and S. D. Sarma, Phys. Rev. Lett. 109, 096801 (2012); A. Yamakage, K. Nomura, K.-I. Imura, and Y. Kuramoto, Phys. Rev. B 87, 205141 (2013).

[33] Yunyou Yang, Zhong Xu, L. Sheng, Baigeng Wang, D. Y. Xing, and D. N. Sheng, Phys. Rev. Lett. 107, 066602 (2011); Yunyou Yang, Huichao Li, L. Sheng, R. Shen, D N Sheng and D Y Xing, New J. Phys. 15, 083042 (2013).

[34] S. C. Zhang and D. Arovas, Phys. Rev. Lett. 72, 1886 (1994); D. N. Sheng and Z. Y. Weng, Phys. Rev. Lett. 75, 2388 (1995); Kun Yang and R. N. Bhatt, Phys. Rev. B 55, R1922 (1997); G. M. Gusev, E. B. Olshanetsky, Z. D. Kvon, N. N. Mikhailov, S. A. Dvoretsky, and J. C. Portal, Phys. Rev. Lett. 104, 166401 (2010).

[35] D.J. Thouless, Phys. Rep. 13, 93 (1974).

[36] Y.-F. Wang, Z.-C. Gu, C.-D. Gong and D. N. Sheng, Phys. Rev. Lett. 107, 146803 (2011).

[37] D. Braun, E. Hofstetter, A. MacKinnon, and G. Montambaux Phys. Rev. B 55, 7557 (1997).

[38] H. Schulz-Baldes and J. Bellissard, Rev. Mod. Phys. 10, 1 (1998).