A Derivative-Hilbert operator Acting on Dirichlet spaces

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Abstract Let μ be a positive Borel measure on the interval [0,1). The Hankel matrix $H_{\mu} = (\mu_{n,k})_{n,k \geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where $\mu_n = \int_{[0,1]} t^n d\mu(t)$, induces formally the operator as

$$DH_{\mu}(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right)(n+1)z^n, z \in \mathbb{D},$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function in $\mathbb{D}$. In this paper, we characterize those positive Borel measures on $[0,1)$ for which $DH_{\mu}$ is bounded (resp. compact) from Dirichlet spaces $D_\alpha(0 < \alpha \leq 2)$ into $D_\beta(2 \leq \beta < 4)$.

Keywords Hilbert operator, Dirichlet space, Carleson measure

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1 Introduction

Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ denote the class of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the Bergman space $A^p$ consists of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_P = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where $dA$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. We refer to [8] for more information about Bergman spaces.

For $\alpha \in \mathbb{R}$, the Dirichlet space $D_\alpha$ consists of all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ for which

$$\|f\|_{D_\alpha} = \sum_{n=0}^{\infty} (n+1)^{1-\alpha}|a_n|^2 < \infty.$$

We get the classical Dirichlet space $D = D_0$ if $\alpha = 0$ (See [9]), we get the Hardy space $H^2 = D_1$ if $\alpha = 1$ (see [7] [17]), and we obtain the Bergman space $A^2 = D_2$ if $\alpha = 2$. We mention [9] for a complete information on Dirichlet spaces.

Suppose that $\mu$ is a positive Borel measure on $[0,1)$. We define $H_\mu$ to be the Hankel matrix $(\mu_{n,k})_{n,k \geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where $\mu_n = \int_{[0,1]} t^n d\mu(t)$. The matrix $H_\mu$ can be seen as an operator on $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D})$ by its action on the Taylor coefficients: $\{a_n\}_{n \geq 0} \rightarrow \{\sum_{k=0}^{\infty} \mu_{n,k} a_k\}_{n \geq 0}$. Furthermore, we can formally induce the Hankel operator $\mathcal{H}_\mu$ as

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n, z \in \mathbb{D},$$

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whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. If we take the measure to be the Lebesgue measure, $\mathcal{H}_\mu$ is just the Hilbert operator. So $\mathcal{H}_\mu$ is also called generalised Hilbert operator.

The operator $\mathcal{H}_\mu$ have been extensively studied in \cite{1,2,8,17}. Galanopoulos and Peláez \cite{10} characterized those measures $\mu$ supported on $[0,1)$ such that the generalised Hilbert operator $\mathcal{H}_\mu$ is well defined and is bounded on $\mathcal{H}^1$. Chatzifountas and Girela \cite{2} described those measures $\mu$ for which $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{H}^p$ into $\mathcal{H}^q$, where $0 < p, q < \infty$. Diamantopoulos \cite{6} gave many results about the operator induced by Hankel matrices on Dirichlet space. In 2018, Girela \cite{12} introduced the operators $\mathcal{H}_\mu$ acting on certain conformally invariant spaces.

In \cite{15,16}, the second and the third authors first used the Hankel matrix defined the Derivative-Hilbert operator $\mathcal{D}\mathcal{H}_\mu$ as

$$\mathcal{D}\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) (n+1) z^n. $$

It is closed related to the generalized Hilbert operator, that is,

$$\mathcal{D}\mathcal{H}_\mu(f)(z) = (z\mathcal{H}_\mu(f)(z))^{'}. $$

So we called $\mathcal{D}\mathcal{H}_\mu$ to be the Derivative-Hilbert operator. And the second and the third authors characterized the measure $\mu$ for which $\mathcal{D}\mathcal{H}_\mu$ is a bounded (resp. compact) operator from $\mathcal{A}^p$ into $\mathcal{A}^q$ for some $p, q$ in \cite{16}. In \cite{15}, they also characterized the measure $\mu$ for which $\mathcal{D}\mathcal{H}_\mu$ is a bounded (resp. compact) operator on the Bloch space.

Let us recall the definition of the Carleson-type measure, which is a useful tool for learning about Banach spaces of analytic functions. We refer to \cite{3,14} for some results about Carleson measures.

If $I \subset \partial\mathbb{D}$ in an arc, $|I|$ denotes the length of $I$, the Carleson square $S(I)$ is defined as

$$S(I) = \{ z = re^{it} : e^{it} \in I, 1 - \left| \frac{|I|}{2\pi} \right| \leq r \leq 1 \}. $$

Suppose $0 < p < \infty$ and $\mu$ is a positive Borel measure on $\mathbb{D}$, then we call $\mu$ to be a $s$–Carleson measure if there exists a positive constant $C$ such that

$$\sup_I \frac{\mu(S(I))}{|I|^s} < \infty, \text{ for any interval } I \subset \partial\mathbb{D}. $$

We called that $\mu$ is a vanishing $s$-Carleson measure if and only if $\mu$ satisfies

$$\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^s} = 0. $$

If $\mu$ is a Borel measure on $[0,1)$, it can be seen as a Borel measure on $\mathbb{D}$ by identifying it as $\hat{\mu}(A) = \mu(A \cap [0,1))$, for every Borel set $A \subset \mathbb{D}$. In this way, for $0 < s < \infty$, we called $\mu$ to be a $s$-Carleson measure if there exists a positive constant $C$ such that

$$\mu([t,1)) \leq C(1-t)^s, \quad t \in [0,1). $$

Also, $\mu$ is a vanishing $s$-Carleson measure on $[0,1]$ if $\mu$ satisfies

$$\lim_{t \to 1^-} \frac{\mu([t,1))}{(1-t)^s} = 0. $$
Other Carleson type measures on [0, 1) have the similar statements.

In this paper, we mainly characterize the positive Borel measure $\mu$ for which the Derivative-Hilbert operator $DH_\mu$ is bounded (resp. compact) from Dirichlet spaces $D_\alpha(0 < \alpha \leq 2)$ into $D_\beta(2 \leq \beta < 4)$.

Throughout this paper, $C$ denotes a positive constant which depends only on the displayed parameters but not necessarily the same from one occurrence to the next. In addition, we say that $A \gtrsim B$ if there exist a constant $C$ (independent of $A$ and $B$) such that $A \gtrsim CB$, and $A \lesssim B$ is the same as $A \gtrsim B$. In addition, the symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2 Main results

We shall first give a sufficient condition such that the operator $DH_\mu$ is well defined on the Dirichlet space $D_\alpha$, for $\alpha \in \mathbb{R}$. And we characterize the measure $\mu$ such that $DH_\mu$ is bounded from Dirichlet spaces $D_\alpha(0 < \alpha \leq 2)$ into $D_\beta(2 \leq \beta < 4)$.

Theorem 2.1 Suppose that $\alpha \in \mathbb{R}$ and let $\mu$ be a positive Borel measure on $[0, 1)$. If the proposition $\mu_n = O(n^{-\frac{\alpha}{2}+\epsilon})$ is hold for some $\epsilon > 0$, then $DH_\mu$ is well defined on $D_\alpha$.

Proof Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n \in D_\alpha$. By Cauchy-Schwarz inequality and [2, Proposition 1], we obtain that

$$\left| \sum_{k=0}^{\infty} \mu_{n,k} a_k \right| \leq \sum_{k=0}^{\infty} \mu_{n,k} |a_k| \lesssim \sum_{k=0}^{\infty} \frac{|a_k|}{(n+k+1)^{\frac{\alpha}{2}+\epsilon}}$$

$$\approx \sum_{k=0}^{\infty} (k+1)^{\frac{\alpha+1}{2}} \frac{1}{(n+k+1)^{\frac{\alpha}{2}+\epsilon}} (k+1)^{\frac{1-\alpha}{2}} |a_k|$$

$$\lesssim \left( \sum_{k=0}^{\infty} \frac{(k+1)^{\alpha-1}}{(n+k+1)^{\alpha+2\epsilon}} \right)^{\frac{1}{2}} \left( \sum_{k=0}^{\infty} (k+1)^{1-\alpha} |a_k|^2 \right)^{\frac{1}{2}}$$

$$\lesssim \left( \sum_{k=0}^{\infty} \frac{1}{(k+1)^{1+2\epsilon}} \right)^{\frac{1}{2}} \|f\|_{D_\alpha} < \infty.$$ 

This shows that the operator $DH_\mu$ is well defined on $D_\alpha$.

Next, we import an auxiliary lemma to prove the main theorem in this paper.

Lemma 2.1 ([13, Theorem 318]) Let $K(x, y)$ be a real function of two variables and has the following properties:

(i) $K(x, y)$ is non-negative and homogeneous of degree -1;

(ii) $\int_0^{\infty} K(x, 1)x^{-\frac{1}{2}}dx = \int_0^{\infty} K(1, y)y^{-\frac{1}{2}}dy = C$;

(iii) $K(x, 1)x^{-\frac{1}{2}}$ is a strictly decreasing functions of $x$, and $K(1, y)y^{-\frac{1}{2}}$ of $y$; or, more generally:

(iii') $K(x, 1)x^{-\frac{1}{2}}$ decreases from $x = 1$ onwards, while the interval $(0, 1)$ can be divided into two parts, $(0, \xi)$ and $(\xi, 1)$, of which one may be null, in the first of which it decreases and in
the second of which it increases; and \( K(1, y)y^{-\frac{1}{2}} \) has similar properties; and \( K(x, x) = 0 \).

Then for every sequence \( \{a_n\}_{n \geq 0} \) in \( l^2 \), we get

\[
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} K(n, k)a_k \right)^2 \leq C^2 \sum_{n=1}^{\infty} a_n^2.
\]

In short, if \( f(z) = \sum_{n=0}^{\infty} a_nz^n \in \mathcal{H}^2 \), we have

\[
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} K(n, k)a_k \right)^2 \leq C^2 \|f\|_{\mathcal{H}^2}^2.
\]

**Theorem 2.2** Suppose that \( 0 < \alpha \leq 2, 2 \leq \beta < 4 \), and let \( \mu \) be a positive Borel measure on \([0, 1]\) which satisfies the condition in Theorem 2.1. Then the following conditions are equivalent:

(i) \( \mu \) is a \((2 - \frac{\beta + \alpha}{2})\)-Carleson measure.

(ii) \( \mu_n = O \left( \frac{1}{n^{2 - \frac{\beta + \alpha}{2}}} \right) \).

(iii) \( \mathcal{D}H_\mu \) is bounded operator from \( \mathcal{D}_\alpha \) into \( \mathcal{D}_\beta \).

Before giving the proof, let us recall some classical conclusions about the Beta function. Let

\[
B(s, t) = \int_0^1 x^{s-1}(1-x)^{t-1} dx,
\]

where \( s, t > 0 \), then we called it to be the Beta function. And we also know that the Beta function as

\[
B(s, t) = \int_0^\infty \frac{x^{s-1}}{(1+x)^{s+t}} dx,
\]

where \( \text{Re}(s) > 0, \text{Re}(t) > 0 \). It is known that the value of \( B(s, t) \) is closed related to the Gamma function, that is,

\[
B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.
\]

Now we continue to complete the proof of the Theorem 2.2.

**Proof** (i) \( \Rightarrow \) (ii). Let \( \mu \) be a finite positive Borel measure on \([0, 1]\), we have

\[
|\mu_n| \leq \int_0^1 |t|^n d\mu(t) = n \int_0^1 t^{n-1} \mu(t, 1) dt
\]

Since \( \mu \) is a \((2 - \frac{\beta + \alpha}{2})\)-Carleson measure, we obtain that

\[
\mu(t, 1) \lesssim (1-t)^{2-\frac{\beta + \alpha}{2}}, \quad \text{for any } t \in (0, 1).
\]

Hence,

\[
|\mu_n| \lesssim n \int_0^1 t^{n-1}(1-t)^{2-\frac{\beta + \alpha}{2}} dt \approx \frac{1}{n^{2 - \frac{\beta + \alpha}{2}}}
\]

As well as,

\[
\mu_n = O(n^{-\left(2 - \frac{\beta + \alpha}{2}\right)}).
\]
(ii) ⇒ (iii). First, we define two operators. For \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in D_{\alpha} \), define \( V_{\alpha}(f) \) by the formula

\[
V_{\alpha}(f)(z) = \sum_{n=0}^{\infty} (n + 1)^{\frac{1-\alpha}{2}} a_n z^n.
\]

Also, for \( g(z) = \sum_{n=0}^{\infty} b_n z^n \in H^2 \), define \( T_{\beta}(g) \) by the formula

\[
T_{\beta}(g)(z) = \sum_{n=0}^{\infty} (n + 1)^{\frac{\beta-1}{2}} b_n z^n.
\]

It is easy to check that \( V_{\alpha} \) is a bounded operator from \( D_{\alpha} \) into \( H^2 \), and \( T_{\beta} \) is a bounded operator from \( H^2 \) into \( D_{\beta} \).

Now suppose that \( 0 < \alpha \leq 2 \) and \( 2 \leq \beta < 4 \). For \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2 \), we consider a new operator \( S_{\mu}(f) \) as

\[
S_{\mu}(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} (n + 1)^{\frac{3-\beta}{2}} (k + 1)^{\frac{\alpha-1}{2}} \mu_{n,k} a_k \right) z^n.
\]

A direct calculation shows that

\[
\|S_{\mu}(f)(z)\|^2_{H^2} = \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} (n + 1)^{\frac{3-\beta}{2}} (k + 1)^{\frac{\alpha-1}{2}} \mu_{n,k} a_k \right|^2 \leq \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} (n + 1)^{\frac{3-\beta}{2}} (k + 1)^{\frac{\alpha-1}{2}} \mu_{n,k} |a_k| \right)^2 \lesssim \sum_{n=0}^{\infty} \left( \sum_{k=1}^{\infty} n^{\frac{3-\beta}{2}} k^{\frac{\alpha-1}{2}} \frac{|a_{k-1}|}{(n + k)^{2-\frac{\beta-\alpha}{2}}} \right)^2.
\]

Let

\[
K(x, y) = x^{\frac{3-\beta}{2}} y^{\frac{\alpha-1}{2}} \frac{1}{(x + y)^{2-\frac{\beta-\alpha}{2}}}, \quad x > 0, \ y > 0.
\]

Then we obtain that

\[
\int_{0}^{\infty} K(x, 1) x^{-\frac{\beta}{2}} dx = \int_{0}^{\infty} \frac{x^{1-\frac{\beta}{2}}}{(x + 1)^{2-\frac{\beta-\alpha}{2}}} dx = B(2 - \frac{\beta}{2}, \frac{\alpha}{2}),
\]

\[
\int_{0}^{\infty} K(1, y) y^{-\frac{\beta}{2}} dy = \int_{0}^{\infty} \frac{y^{\frac{\alpha-1}{2}}}{(y + 1)^{2-\frac{\beta-\alpha}{2}}} dy = B(\frac{\alpha}{2}, 2 - \frac{\beta}{2}).
\]

And it is clear that the functions \( K(x, 1) x^{-\frac{\beta}{2}}, K(1, y) y^{-\frac{\beta}{2}} \) are strictly decreasing. Applying Lemma 2.11 it follows that

\[
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} n^{\frac{3-\beta}{2}} k^{\frac{\alpha-1}{2}} \frac{|a_{k-1}|}{(n + k)^{2-\frac{\beta-\alpha}{2}}} \right)^2 \lesssim \left( B(2 - \frac{\beta}{2}, \frac{\alpha}{2}) \right)^2 \|f\|^2_{H^2}.
\]
This implies that the operator $S_\mu$ is bounded on $\mathcal{H}^2$.

For each $f \in D_\alpha$, it is easy to check that
$$T_\beta \circ S_\mu \circ V_\alpha(f)(z) = \sum_{n=0}^{\infty} \left( (n+1)^{\frac{\beta+1}{2}} \sum_{k=0}^{\infty} (n+1)^{\frac{\alpha+1}{2}} (k+1)^{\frac{1-\beta}{2}} \mu_{n,k} a_k \right) z^n$$
$$= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) (n+1)z^n = D\mathcal{H}_\mu(f)(z),$$

Hence, $D\mathcal{H}_\mu$ is bounded from $D_\alpha$ into $D_\beta$.

$(iii) \Rightarrow (i)$. For $0 < t < 1$, let $f_t(z) = (1 - t^2)^{1-\alpha} \sum_{n=0}^{\infty} t^n z^n$. We have that
$$\|f_t\|_{D_\alpha}^2 = (1 - t^2)^{2-\alpha} \sum_{n=0}^{\infty} (n+1)^{1-\alpha} t^{2n} \approx 1.$$

Therefore,
$$\|D\mathcal{H}_\mu(f_t)\|_{D_\beta}^2 \approx \sum_{n=0}^{\infty} (n+1)^{1-\beta} \left( \sum_{k=0}^{\infty} (n+1) \mu_{n,k} (1 - t^2)^{1-\frac{\beta}{2}} t^k \right)^2$$
$$\geq (1 - t^2)^{2-\alpha} \sum_{n=0}^{\infty} (n+1)^{3-\beta} \left( \sum_{k=0}^{\infty} t^k \int_t^1 \chi^{n+k} d\mu(\chi) \right)^2$$
$$\geq (1 - t^2)^{2-\alpha} \sum_{n=0}^{\infty} (n+1)^{3-\beta} \left( \sum_{k=0}^{n} t^{n+2k} \mu[t,1] \right)^2.$$

Since $D\mathcal{H}_\mu$ is bounded from $D_\alpha$ into $D_\beta$, we obtain that
$$\|D\mathcal{H}_\mu\|_{D_\beta}^2 \|f_t\|_{D_\beta}^2 \geq \|D\mathcal{H}_\mu(f_t)\|_{D_\beta}^2$$
$$\geq (1 - t^2)^{2-\alpha} \sum_{n=0}^{\infty} (n+1)^{3-\beta} \left( \sum_{k=0}^{n} t^{n+2k} \mu[t,1] \right)^2$$
$$\geq (1 - t^2)^{2-\alpha} \sum_{n=0}^{\infty} (n+1)^{5-\beta} t^{6n} (\mu[t,1])^2$$
$$\approx \frac{(\mu[t,1])^2}{(1 - t^2)^{4+\alpha-\beta}}.$$

This implies that
$$\mu[t,1] \lesssim (1 - t^2)^{2-\frac{\beta-\alpha}{2}},$$
which is equivalent to saying that $\mu$ is a $(2 - \frac{\beta-\alpha}{2})$-Carleson measure.

In particular, if we take $\alpha = \beta = 2$ in Theorem 2.2, we can obtain the following corollary which the second and the third authors have proved in [16].

**Corollary 2.1** The operator $D\mathcal{H}_\mu$ is bounded on $A^2$ if and only if the measure $\mu$ is a 2-Carleson measure.
Theorem 2.3 Suppose that $0 < \alpha \leq 2$, $2 \leq \beta < 4$, and let $\mu$ be a positive Borel measure on $[0,1]$ which satisfies the condition in Theorem [2.1]. Then the following conditions are equivalent:

(i) $\mu$ is a vanishing $(2 - \frac{\beta - \alpha}{2})$-Carleson measure.

(ii) $\mu_n = o\left(\frac{1}{n^{\frac{\beta - \alpha}{2}}}\right)$.

(iii) $\mathcal{D}\mathcal{H}_\mu$ is compact operator from $\mathcal{D}_\alpha$ into $\mathcal{D}_\beta$.

Proof (i) $\Rightarrow$ (ii). It is similar to the previous proof and will not be repeated. (ii) $\Rightarrow$ (iii). Take $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}_\alpha$. Let

$$S_{\mu,m}(f)(z) = \sum_{n=0}^{m} \left( \sum_{k=0}^{n} (n + 1) \frac{3-\beta}{2} (k + 1) \frac{\alpha - 1}{2} \mu_{n,k} a_k \right) z^n,$$

$$\mathcal{D}\mathcal{H}_{\mu,m}(f)(z) = \sum_{n=0}^{m} \left( \sum_{k=0}^{n} (n + 1) \mu_{n,k} a_k \right) z^n.$$

Notice that $S_{\mu,m}, \mathcal{D}\mathcal{H}_{\mu,m}$ are finite rank operators, then $S_{\mu,m}(f)(z)$ is compact on $\mathcal{H}^2$. Since $\mu_n$ satisfies $\mu_n = o(n^{-(2-\frac{\beta - \alpha}{2})})$, we obtain that for any $\varepsilon > 0$, there exists an $N > 0$ such that $|\mu_m| < \varepsilon n^{-(2-\frac{\beta - \alpha}{2})}$ when $m > N$. Then we note

$$(S_{\mu} - S_{\mu,m})(f)(z) = \sum_{n=m+1}^{\infty} \left( \sum_{k=0}^{n} (n + 1) \frac{3-\beta}{2} (k + 1) \frac{\alpha - 1}{2} \mu_{n,k} a_k \right) z^n,$$

$$(T_{\beta} \circ S_{\mu} \circ V_a - T_{\beta} \circ S_{\mu,m} \circ V_a)(f)(z) = \sum_{n=m+1}^{\infty} \left( \sum_{k=0}^{n} (n + 1) \mu_{n,k} a_k \right) z^n$$

$$= T_{\beta} \circ (S_{\mu} - S_{\mu,m}) \circ V_a(f)(z)$$

$$= (\mathcal{D}\mathcal{H}_{\mu} - \mathcal{D}\mathcal{H}_{\mu,m})(f)(z).$$

Therefore,

$$\| (S_{\mu} - S_{\mu,m})(f)(z) \|_{\mathcal{H}^2}^2 = \sum_{n=m+1}^{\infty} \left| \sum_{k=0}^{n} (n + 1) \frac{3-\beta}{2} (k + 1) \frac{\alpha - 1}{2} \mu_{n,k} a_k \right|^2.$$

Then for $m > N$, we have

$$\| (S_{\mu} - S_{\mu,m})(f)(z) \|_{\mathcal{H}^2}^2 \lesssim \varepsilon^2 \sum_{n=m+1}^{\infty} \left( \sum_{k=0}^{n} (n + 1) \frac{3-\beta}{2} (k + 1) \frac{\alpha - 1}{2} \mu_{n,k} a_k \right)^2.$$

By Lemma [2.1] and the proof of Theorem [2.2] we obtain

$$\| (S_{\mu} - S_{\mu,m})(f)(z) \|_{\mathcal{H}^2}^2 \lesssim \varepsilon^2 \| f \|_{\mathcal{H}^2}^2.$$

Thus,

$$\| S_{\mu} - S_{\mu,m} \|_{\mathcal{H}_2 \to \mathcal{H}_2} \lesssim \varepsilon.$$

It is clear that

$$\| \mathcal{D}\mathcal{H}_{\mu} - \mathcal{D}\mathcal{H}_{\mu,m} \|_{\mathcal{D}_\alpha \to \mathcal{D}_\beta} \lesssim \varepsilon.$$
Hence, $\mathcal{D}H_\mu$ is compact from $\mathcal{D}_\alpha$ into $\mathcal{D}_\beta$.

$(iii) \Rightarrow (i)$. For $0 < t < 1$, let $f_t(z) = (1 - t^2)^{1-\frac{\alpha}{2}} \sum_{n=0}^{\infty} t^n z^n$, we have

$$
\|f_t\|_{\mathcal{D}_\alpha}^2 = (1 - t^2)^{2-\alpha} \sum_{n=0}^{\infty} (n+1)^{1-\alpha} t^{2n} \approx 1,
$$

and $\lim_{t \to 1} f_t(z) = 0$ for any $z \in \mathbb{D}$. Since all Hilbert spaces are reflexive, we obtain that $f_t$ is convergent weakly to 0 in $\mathcal{D}_\alpha$ as $t \to 1$. By the assumption that $\mathcal{D}H_\mu$ is compact from $\mathcal{D}_\alpha$ into $\mathcal{D}_\beta$, we have

$$
\lim_{t \to 1} \|\mathcal{D}H_\mu(f_t)\|_{\mathcal{D}_\beta} = 0.
$$

Similar to the proof of Theorem 2.2, we obtain that

$$
\mu(t,1) \lesssim (1 - t)^{2 - \frac{\beta-\alpha}{2}} \|\mathcal{D}H_\mu(f_t)\|_{\mathcal{D}_\beta}.
$$

Therefore,

$$
\lim_{t \to 1} \frac{\mu(t,1)}{(1 - t)^{2 - \frac{\beta-\alpha}{2}}} = 0.
$$

Thus, $\mu$ is a vanishing $(2 - \frac{\beta-\alpha}{2})$-Carleson measure.

Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this paper.

Availability of data and material
The authors declare that all data and material in this paper are available.

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