Regular bi-interpretability of Chevalley groups over local rings

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Abstract
We prove that if $G(R) = G_\pi(\Phi, R)$ ($E(R) = E_\pi(\Phi, R)$) is an (elementary) Chevalley group of rank $> 1$, $R$ is a local ring (with $\frac{1}{2}$ for the root systems $A_2, B_1, C_1, F_4, G_2$ and with $\frac{1}{3}$ for $G_2$), then the group $G(R)$ (or $(E(R))$ is regularly bi-interpretable with the ring $R$. As a consequence of this theorem, we show that the class of all Chevalley groups over local rings (with the listed restrictions) is elementarily definable, i.e., if for an arbitrary group $H$ we have $H \equiv G_\pi(\Phi, R)$, then there exists a ring $R' \equiv R$ such that $H \cong G_\pi(\Phi, R')$.

Keywords Chevalley groups · Local rings · Regular bi-interpretability · Elementary definability

Mathematics Subject Classification 03C60 · 20G35

1 Introduction, history and definitions

1.1 Elementary equivalence

Two models $\mathcal{U}$ and $\mathcal{U}'$ of the same first order language $\mathcal{L}$ (for example, two groups or two rings) are called *elementarily equivalent*, if every sentence $\varphi$ of the language $\mathcal{L}$ holds in $\mathcal{U}$ if and only if it holds in $\mathcal{U}'$. Any two finite models of the same language are elementarily equivalent if and only if they are isomorphic. Any two isomorphic models are elementarily equivalent, but for infinite models the converse fact is not true. For example, the field $\mathbb{C}$ of complex numbers and the field $\mathbb{Q}$ of algebraic numbers are...
elementarily equivalent, but not isomorphic as they have different cardinalities (for more detailed examples see, for instance, [32]).

Tarski and Maltsev pushed forward a problem of describing groups and rings (in some natural classes) that are elementarily equivalent. There were obtained several complete results for some classes of groups and rings: for example, two algebraically closed fields are elementarily equivalent if and only if they have the same characteristics (classical result); two abelian groups are elementarily equivalent if and only if they have the same special “characteristic numbers” (like $\text{Exp} A$, $\dim p^n A[p]$, etc.), which must be either the same finite numbers or infinity (Szmielew [48]); similar results with invariants were obtained for Boolean rings (Ershov–Tarski [19]), also there were obtained several other classification results. An outstanding result was the answer for old questions that were raised by Tarski around 1945: all non-abelian free groups are elementarily equivalent (Kharlampovich, Myasnikov [25]; Sela [41]). Certainly, any new such “pure classification result” is a rather rare mathematical phenomenon.

1.2 Maltsev-type theorems for linear groups

Another way of studying elementary equivalence of algebraic models is to establish a connection between derivative models (linear groups over rings and fields, automorphism groups and endomorphism rigs of different structures, etc.) and initial models (and “parameters”) used for the construction. First results in this field were obtained by Maltsev in 1961 in [31]. He proved that the groups $\mathbb{G}_n(K_1)$ and $\mathbb{G}_m(K_2)$ (where $G = \text{GL}, \text{SL}, \text{PGL}, \text{PSL}$, $n, m \geq 3$, $K_1, K_2$ are fields of characteristics 0) are elementarily equivalent if and only if $m = n$ and the fields $K_1$ and $K_2$ are elementarily equivalent.

In 1961–1971, Keisler [24] and Shelah [42] proved the next Isomorphism Theorem:

**Theorem 1.1** Two models $\mathcal{U}_1$ and $\mathcal{U}_2$ are elementarily equivalent if and only if there exists an ultrafilter $\mathcal{F}$ such that their ultrapowers are isomorphic:

$$\prod_{\mathcal{F}} \mathcal{U}_1 \cong \prod_{\mathcal{F}} \mathcal{U}_2.$$ 

Later Beidar and Mikhalev introduced another general approach to elementary equivalence of classical matrix groups [7]. Their proof was based on the Keisler–Shelah theorem and the description of the abstract isomorphisms of the groups of the type $\mathbb{G}_n(F)$, and they generalized Maltsev’s theorem for the case when $K_1$ and $K_2$ are skewfields and prime associative rings. This approach for the groups $\text{GL}_n$ was extended in [9] to the following result:

**Theorem 1.2** Let $R_1$ and $R_2$ be associative rings with 1 ($\frac{1}{2}$) with finite number of central idempotents and $m, n \geq 4$ ($m, n \geq 3$). Then $\text{GL}_m(R_1) \equiv \text{GL}_n(R_2)$ if and only if there exist central idempotents $e \in R$ and $f \in S$ such that $eM_m(R) \equiv fM_n(S)$ and $(1 - e)M_m(R) \equiv (1 - f)M_n(S)^\text{op}$.

Continuation of investigations in this field were the papers of Bunina 1998–2010 (see [10–12, 15]), where the results of Maltsev were extended for unitary linear groups.
over skewfields and associative rings with involution, and also for Chevalley groups over fields and local rings. Important for this paper is the following theorem (see [12]):

**Theorem 1.3** Let \( G = G_\pi(\Phi, R) \) and \( G' = G_\pi'(\Phi', R') \) (or \( E_\pi(\Phi, R) \) and \( E_\pi'(\Phi', R') \)) be two (elementary) Chevalley groups over infinite local rings \( R \) and \( R' \) with two invertible (in the case of the root system \( G_2 \) with three invertible), with indecomposable root systems \( \Phi, \Phi' \) of ranks > 1, with weight lattices \( \Lambda \) and \( \Lambda' \), respectively. Then the groups \( G \) and \( G' \) are elementarily equivalent if and only if the root systems \( \Phi \) and \( \Phi' \) are isomorphic, the rings \( R \) and \( R' \) are elementarily equivalent, and the lattices \( \Lambda \) and \( \Lambda' \) coincide.

Theorem 1.3 is proved partially using explicit first order formulas, partially with the usage of the Keisler–Shelah Isomorphism Theorem. In 2019 this result was extended to Chevalley groups over arbitrary commutative rings (using the algebraic results from [13] on classification of all automorphisms of Chevalley groups over commutative rings and the Keisler–Shelah Isomorphism theorem):

**Theorem 1.4** (Bunina [14]) Let \( G = G_{ad}(\Phi, R) \) and \( G' = G_{ad}(\Phi', R') \) (or \( E_{ad}(\Phi, R) \) and \( E_{ad}(\Phi', R') \)) be two adjoint (elementary) Chevalley groups with indecomposable root systems \( \Phi, \Phi' \) of ranks > 1 over infinite commutative rings \( R \) and \( R' \) (for the cases of the roots systems \( A_2, B_1, C_1, F_4 \) with \( \frac{1}{2} \) and for the system \( G_2 \) with \( \frac{1}{2} \) and \( \frac{1}{3} \)).

Then the groups \( G \) and \( G' \) are elementarily equivalent if and only if the root systems \( \Phi \) and \( \Phi' \) are isomorphic and the rings \( R \) and \( R' \) are elementarily equivalent.

### 1.3 Elementary definability

Along with Maltsev-type theorems, the problems of so-called **elementary definability** of classes of groups are quite important. These problems are formulated as follows: Suppose that we have a class \( \mathcal{G} \) of groups (rings, etc.) and an arbitrary group (rings, etc.) \( H \) which is elementarily equivalent to some \( G \in \mathcal{G} \). Is it true that \( H \in \mathcal{G} \)?

For example the class of all abelian groups is clearly elementarily definable. More generally any variety of groups is elementarily definable: if \( \mathcal{G} \) is some variety of groups defined by a system of identity relations \( \Lambda \) and a group \( H \) is elementarily equivalent to some \( G \in \mathcal{G} \), then \( H \) satisfies the same system \( \Lambda \), therefore, \( H \in \mathcal{G} \).

The next interesting example is that the class of all linear groups over fields is elementarily definable:

**Theorem 1.5** (Maltsev [30]) If \( H \) is a group that is elementarily equivalent to a linear group, then \( H \) is linear.

For the sake of completeness we include a proof.

**Proof** Let \( G \subseteq GL_n(K) \) be a linear group which is elementarily equivalent to \( H \). Enumerate the elements of \( H \) as \( H = \{h_\alpha \mid \alpha \in \pi \} \) and enumerate all relations that hold between \( h_\alpha \) by \( \{r_\beta(h_\alpha) \mid \beta \in \mu \} \). Let \( \mathcal{L} \) be the first-order language of rings together with constants \( c_{j}^{i} \) for \( \alpha \in \pi \) and \( 1 \leq i, j \leq n \). Consider the theory \( T \) consisting of the following statements: (1) The axioms of fields. (2) The statements
\[ \det(c_{i,j}^{\alpha}) \neq 0 \text{ for all } \alpha \in \kappa. \]

(3) The statements \( (c_{i,j}^{\alpha}) \neq (c_{i,j}^{\beta}) \) for all distinct \( \alpha, \beta \in \kappa \).

(4) The statements \( r_{\beta}((c_{i,j}^{\alpha})) = 1 \) for all \( \beta \in \mu \).

If \( S \) is a finite subset of \( T \), then there is a finite set \( I_0 \subset \kappa \) such that only \( c_{i,j}^{\alpha} \) for \( \alpha \in I_0 \) are involved in \( S \). Since the elements \( h_{\alpha}, \alpha \in I_0 \), satisfy all relations \( r_{\beta} \) that involve only them, we get that there are elements \( (g_{i,j}^{\alpha}) \in G \subset \text{GL}_n(K) \) that satisfy \( S \). In particular, \( S \) is consistent. By the Compactness Theorem, there is a model \( \mathbb{K} \) of \( T \). This \( \mathbb{K} \) must be a field, and the map \( h_{\alpha} \mapsto (c_{i,j}^{\alpha}) \) is an embedding \( H \to \text{GL}_n(\mathbb{K}) \).

It is easy to see that the same proof holds for linear groups over commutative rings (if we just use the axioms of commutative rings instead of the axioms of fields in (1)) or over any elementarily axiomatized class of commutative rings. Therefore the class of all linear groups over commutative rings is elementarily definable.

On the other hand, for general linear groups even over fields there is no elementary definability: there exists a group \( H \cong \text{GL}_n(R) \) which is not \( \text{GL} \) itself. To demonstrate it we will show an easy example.

**Example 1.6** Consider the group \( \text{GL}_3(R) \) and construct elementarily equivalent group which is not of the type \( \text{GL}_3 \).

To construct it we will use well-known properties (see, for example, [16]), which state that elementary equivalence respects direct products.

Consider now the field \( R \) of real numbers and any arbitrary countable field \( \mathbb{K} \) which is elementarily equivalent to \( R \). Then consider the group \( \text{SL}_3(R) \times \mathbb{K}^* \). We see that it is elementarily equivalent to the group \( \text{SL}_3(R) \times R^* \).

Let us now construct an isomorphism between \( \text{GL}_3(R) \) and \( \text{SL}_3(R) \times R^* \). Any matrix \( X \) from \( \text{GL}_3(R) \) can be represented as the product of a matrix with determinant 1 and some scalar matrix with \( (\det(X))^{1/3} \) on its diagonal, since cubic root is a bijective function in \( R \).

Finally let us prove that \( \text{SL}_3(R) \times \mathbb{K}^* \) is not a group of type \( \text{GL}_3(\mathbb{K}') \). Suppose that it is isomorphic to any \( \text{GL}_3(\mathbb{K}') \). Then from one side, considering centres of these groups we conclude that \( \mathbb{K}' \) has the same cardinality as \( \mathbb{K}^* \) (i.e., it is countable), from another side, looking at quotient groups by their centres we conclude that \( \mathbb{K}' \) has the same power as \( R \) (i.e., has the cardinality of continuum), contradiction.

In 1984, Zilber [53] proved that for algebraic groups over algebraically closed fields the problem of elementary definability has the positive answer:

**Theorem 1.7** The class of all simple algebraic groups over algebraically closed fields is elementarily definable. The same is true with respect to the subclass of all algebraic groups of the given type.

In fact this problem has positive answer for simple algebraic groups over a wider class of rings, see Myasnikov–Sohrabi [35]. Later on we will prove a generalization of this theorem.

### 1.4 First order rigidity

A new round of development of this topic has appeared recently in the papers of Nies ([37, 38], etc.), Avni, Lubotzky, Meiri ([5, 6], also see [23]), Myasnikov, Kharlampov-
ich and Sohrabi ([26, 35, 36], etc.), Segal and Tent [40], Kunyavskii, Plotkin, Vavilov [28] and others.

Since according to the Löwenheim–Skolem theorem for any infinite structure \( A \) there exists a structure \( B \) such that \( A \equiv B \) and they have different cardinalities, so classification up to elementary equivalence is always strictly wider than classification up to isomorphism. But taking finitely generated structures (groups or rings or anything else) there exists a possibility that for some of them elementary equivalence could imply isomorphism.

For example it is true for the ring \( \mathbb{Z} \) of integers: if any finitely generated ring \( A \) is elementarily equivalent to \( \mathbb{Z} \), then it is isomorphic to \( \mathbb{Z} \). It is not true for infinitely generated rings.

A question dominating research in this area is as follows: when elementary equivalence between finitely generated groups (rings) implies isomorphism. Recently, Avni et. al. [5] presented the term first-order rigidity:

**Definition 1.8** A finitely generated group (ring or other structure) \( A \) is first-order rigid if any other finitely generated group (ring or other structure) elementarily equivalent to \( A \) is isomorphic to \( A \).

There are examples of first-order rigid groups and algebras. For instance, Avni, Lubotzky and Meiri [5] showed that non-uniform higher dimensional lattices are first-order rigid. Profinite groups are first-order rigid as well [23]. Lasserre showed that under some natural conditions polycyclic groups [29] are also first-order rigid, etc.

**1.5 QFA-property**

Some groups satisfy a more strong QFA property, when a single first-order axiom distinguishes the group among all finitely generated groups. The corresponding concept was introduced in [37] for groups only, and then in [38] for arbitrary structures:

**Definition 1.9** Fix a finite signature. An infinite finitely generated structure \( \mathcal{U} \) is quasi finitely axiomatizable (QFA) if there is a first order sentence \( \varphi \) such that

1. \( \mathcal{U} \models \varphi \);
2. if \( H \) is a finitely generated structure in the same signature such that \( H \models \varphi \), then \( H \cong \mathcal{U} \).

The following groups are QFA (see [38]):

1. nilpotent groups \( UT_3(\mathbb{Z}) \);
2. metabelian groups \( \mathbb{Z}[1/m] \times \mathbb{Z} = \langle a, d \mid d^{-1}ad = a^m \text{ for any } m \geq 3 \rangle \) and \( \mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z} \) for any prime \( p \);
3. permutation groups: the subgroup of the group of permutations of \( \mathbb{Z} \) generated by the successor function and the transposition \((0, 1)\).

Speaking about rings the best (and the most important) example of QFA rings is the ring of integer \((\mathbb{Z}, +, \cdot)\) [39].
1.6 Interpretability and bi-interpretability

The real breakthrough in this area is related to applications of bi-interpretability property.

There is a well-known notion of interpretability (with or without parameters) of one structure in another one. Roughly speaking, $\mathcal{B}$ is interpretable in $\mathcal{A}$ if the elements of $\mathcal{B}$ can be represented by tuples in a definable relation $\mathcal{D}$ on $\mathcal{A}$, in such a way that equality of $\mathcal{B}$ becomes an $\mathcal{A}$-definable equivalence relation $\mathcal{E}$ on $\mathcal{D}$, and the other atomic relations on $\mathcal{B}$ are also definable.

Remark that all first-order formulas in these notions are allowed to contain parameters.

Example 1.10 A simple example is as follows: $(\mathbb{Z}, +)$ can be interpreted in $(\mathbb{N}, +)$, where the relation $\mathcal{D}$ is $\mathbb{N} \times \mathbb{N}$, addition is component-wise and $\mathcal{E}$ is the relation given by $(n, m)\mathcal{E}(n', m') \iff n' + m = n + m'$. Further examples include the quotient fields, which can be interpreted in the given integral domain, and the group $\text{GL}_n(R)$ for fixed $n \geq 1$, which can be first-order interpreted in the ring $R$.

Let us turn to the bi-interpretability definition.

We will consider the isomorphic copy of $\mathcal{B}$ that is defined in $\mathcal{A}$ by the relevant collection of formulas. For instance, taking in account the example of an interpretation of $(\mathbb{Z}, +)$ in $(\mathbb{N}, +)$ above; the actual copy of $(\mathbb{Z}, +)$ defined in $(\mathbb{N}, +)$ is the structure whose domain consists of pairs of natural numbers, whose addition is component-wise and where equality is the equivalence relation on pairs given there.

Suppose structures $\mathcal{A}$, $\mathcal{B}$ in finite signatures are given, as well as interpretations of $\mathcal{A}$ in $\mathcal{B}$, and vice versa. Then an isomorphic copy $\tilde{\mathcal{A}}$ of $\mathcal{A}$ can be defined in $\mathcal{A}$, by “decoding” $\mathcal{A}$ from the copy of $\mathcal{B}$ defined in $\mathcal{A}$.

Similarly, an isomorphic copy $\tilde{\mathcal{B}}$ of $\mathcal{B}$ can be defined in $\mathcal{B}$. An isomorphism $\Phi: \mathcal{A} \cong \tilde{\mathcal{A}}$ can be viewed as a relation on $\mathcal{A}$, and similarly for an isomorphism $\Psi: \mathcal{B} \cong \tilde{\mathcal{B}}$. Let us give a variant of a notion of bi-interpretability:

Definition 1.11 (see [21, Chapter 5]) We say that $\mathcal{A}$ and $\mathcal{B}$ are bi-interpretable (with parameters) if the isomorphisms $\Phi$ and $\Psi$ are first-order definable.

Along with the notion of bi-interpretability with parameters it is crucial to introduce a notion of regular bi-interpretability.

Definition 1.12 (see [34]) We say that a structure $\mathcal{A}$ is interpreted in a given structure $\mathcal{B}$ uniformly with respect to a subset $D \subseteq \mathcal{B}^k$ if there is one interpretation of $\mathcal{A}$ in $\mathcal{B}$ with any tuple of parameters $\overline{\varphi} \in D$.

If $\mathcal{A}$ is interpreted in $\mathcal{B}$ uniformly with respect to a absolutely definable subset $D \subseteq \mathcal{B}^k$ then we say that $\mathcal{A}$ is regularly interpretable in $\mathcal{B}$ and write in this case $\mathcal{A} \cong \Gamma(\mathcal{B}, \varphi)$, where $\varphi$ defines $D$ in $\mathcal{B}$.

Note that the interpretability without parameters (absolute interpretability) is a particular case of the regular interpretability where the set $D$ is empty.

An important application of regular interpretability is the following:
Proposition 1.13 If \( A_1 = \Gamma(\mathcal{B}_1, \varphi) \) and \( A_2 = \Gamma(\mathcal{B}_2, \varphi) \) are two regular interpretations, then \( \mathcal{B}_1 \equiv \mathcal{B}_2 \) implies \( A_1 \equiv A_2 \).

In fact all Mal'tsev-type theorems mentioned above were proved either by the Keisler–Shelah Isomorphism Theorem or by regular interpretability of a ground field/ring in a corresponding linear groups.

Definition 1.14 Two algebraic structures \( \mathcal{A} \) and \( \mathcal{B} \) are called \textit{regularly bi-interpretable} in each other if the following conditions hold:

1. \( \mathcal{A} \) and \( \mathcal{B} \) are regularly interpretable in each other, so \( \mathcal{A} \cong \Gamma(\mathcal{B}, \varphi) \) and \( \mathcal{B} \cong \Delta(\mathcal{A}, \psi) \) for some interpretations (also called \textit{codes}) \( \Gamma \) and \( \Delta \) and the corresponding formulas \( \varphi, \psi \) (without parameters). By transitivity \( \mathcal{A} \), as well as \( \mathcal{B} \), is regularly interpretable in itself, so we can write

\[
\mathcal{A} \cong (\Gamma \circ \Delta)(\mathcal{A}, \varphi^*) \quad \text{and} \quad \mathcal{B} \cong (\Delta \circ \Gamma)(\mathcal{B}, \psi^*),
\]

where \( \circ \) denotes composition of interpretations and \( \varphi^*, \psi^* \) are the corresponding formulas.

2. There is a formula \( \theta(\overline{y}, x, \overline{z}) \) in the language of \( \mathcal{A} \) such that for every tuple \( p^* \) satisfying \( \varphi^*(\overline{z}) \) in \( \mathcal{A} \) the formula \( \theta(\overline{y}, x, p^*) \) defines in \( \mathcal{A} \) the isomorphism \( \overline{\mu}_{\Gamma \circ \Delta}: (\Gamma \circ \Delta)(\mathcal{A}, p^*) \to \mathcal{A} \) and there is a formula \( \sigma(\overline{u}, x, \overline{v}) \) in the language of \( \mathcal{B} \) such that for every tuple \( q^* \) satisfying \( \psi^*(\overline{v}) \) in \( \mathcal{B} \) the formula \( \sigma(\overline{v}, x, q^*) \) defines in \( \mathcal{B} \) the isomorphism \( \overline{\mu}_{\Delta \circ \Gamma}: (\Delta \circ \Gamma)(\mathcal{B}, q^*) \to \mathcal{B} \).

If linear groups (or another derivative structures) of any concrete types over some classes of fields/rings are regularly bi-interpretable with the corresponding rings, then this class of groups (structures) is elementarily definable (see Proposition 3.7).

1.7 Bi-interpretable with the ring \( \mathbb{Z} \) and QFA property

The special interesting case is the one when it can be proved that a structure is bi-interpretable with the ring \( \mathbb{Z} \) of integers.

Khelif [27] (see also [26]) realised that one can use bi-interpretability of a finitely generated structure \( \mathcal{A} \) with \((\mathbb{Z}, +, \times)\) as a general method to prove that \( \mathcal{A} \) is QFA. Somewhat later, Scanlon independently used this method to show that each finitely generated field is QFA.

Theorem 1.15 (Nies [27]) Suppose the structure \( \mathcal{A} \) in a finite signature is bi-interpretable with the ring \( \mathbb{Z} \). Then \( \mathcal{A} \) is QFA.

By a \textit{number field} we mean a finite extension of \( \mathbb{Q} \). By the \textit{ring of integers} \( \mathcal{O} \) of a number field \( F \) we mean the subring of \( F \) consisting of all roots of monic polynomials with integer coefficients.

By [4] any ring of integers \( \mathcal{O} \) of a number field \( F \) is bi-interpretable with \( \mathbb{Z} \) (and therefore is QFA).

In the paper [35], Myasnikov and Sohrabi considered linear groups \((\text{GL}_n(\mathcal{O})), \text{SL}_n(\mathcal{O}) \) and triangular groups \( T_n(\mathcal{O}) \) over rings of integers of a field \( F \). They proved
that the groups $\SL_n(\mathcal{O})$, $n \geq 3$, are bi-interpretable (with parameters) with the ring $\mathcal{O}$ and therefore with $\mathbb{Z}$; the groups $\GL_n(\mathcal{O})$ and $T_n(\mathcal{O})$, $n \geq 3$, are bi-interpretable with $\mathcal{O}$ (and with $\mathbb{Z}$) only if $\mathcal{O}^*$ is finite. Consequently in all good cases these linear groups are QFA.

Moreover they proved that for all these “good cases” the problem of general elementary definability has positive solution:

**Theorem 1.16** (Myasnikov, Sohrabi [35]) If $n \geq 3$, $\mathcal{O}$ is the ring of integers of some number field $F$, $H$ is an arbitrary group, $H \equiv \SL_n(\mathcal{O})$ or $\mathcal{O}^*$ is finite and $H \equiv \GL_n(\mathcal{O})$ or $H \equiv T_n(\mathcal{O})$. Then $H \cong \SL_n(R)$ (or $\GL_n(R)$, $T_n(R)$ respectively) for some ring $R$ such that $R \cong \mathcal{O}$.

In this theorem they implicitly used the fact that the bi-interpretation is *regular*.

At the same time, Segal and Tent in [40] obtained the similar result for all Chevalley groups over integral domains. More precisely, they proved that:

**Theorem 1.17** Let $G(\cdot)$ be a simple Chevalley–Demazure group scheme of rank at least two, and let $R$ be an integral domain. Then $R$ and $G(R)$ are bi-interpretable provided either

1. $G$ is adjoint, or
2. $G(R)$ has finite elementary width,

assuming in case $G$ is of type $E_6$, $E_7$, $E_8$, or $F_4$ that $R$ has at least two units.

This theorem immediately implies:

**Corollary 1.18** Assume that $G$ and $R$ satisfy the hypotheses of the previous theorem. If $R$ is first order rigid (resp., QFA), in

1. the class of finitely generated rings,
2. the class of profinite rings,
3. the class of locally compact topological rings,

then $G(R)$ has the analogous property in (1) the class of finitely generated groups, (2) the class of profinite groups, (3) the class of locally compact topological groups.

All these recent papers illustrate that together with bi-interpretability with parameters it is unprecedentedly important to state regular bi-interpretability of linear groups and corresponding rings.

Thus it is quite important to prove regular bi-interpretability of Chevalley groups over local rings. As a corollary we will obtain elementary definability of these groups in the class of all groups.

We prove the following theorem and its corollary:

**Theorem 1.19** If $G(R) = G_\pi(\Phi, R)$ ($E(R) = E_\pi(\Phi, R)$) is an (elementary) Chevalley group of rank $> 1$, $R$ is a local ring (with $\frac{1}{2}$ for the root systems $B_1$, $C_1$, $F_4$, $G_2$ and with $\frac{1}{2}$ for $G_2$), then the group $G(R)$ (or $E(R)$) is regularly bi-interpretable with $R$. 
Corollary 1.20 (Elementary definability of Chevalley groups) The class of Chevalley groups over local rings is elementarily definable, i.e., if $G(R) = G_\pi(\Phi, R)$ is a Chevalley group of rank $> 1$, over a local ring $R$ (with $\frac{1}{2}$ for the root systems $A_2, B_1, C_1, F_4, G_2$ and with $\frac{1}{3}$ for $G_2$) and for an arbitrary group $H$ we have $H \equiv G(R)$, then $H \equiv G_\pi(\Phi, R')$ for some local ring $R'$, which is elementarily equivalent to $R$.

From the next section we will concentrate specially on Chevalley groups over local rings. We start with necessary definitions and references.

2 Chevalley groups over local rings and their properties

We refer for the facts related to root systems and semisimple Lie algebras to [8, 22]. More detailed information about elementary Chevalley groups is contained in the book [45], and about the Chevalley groups (also over rings) in [17, 50, 51] (see also later references in these papers).

We fix some arbitrary (indecomposable) root system $\Phi$ of the rank $l \geq 2$. We suppose that in this system there are $n$ positive and $n$ negative roots.

Additionally we fix some infinite commutative local ring $R$ with $1$ (for the root systems $A_2, B_1, C_1, F_4$ and $G_2$ with $\frac{1}{2}$ and for $G_2$ also with $\frac{1}{3}$).

We consider an arbitrary Chevalley group $G_\pi(\Phi, R)$, constructed by the root system $\Phi$, a ring $R$ and a representation $\pi$ of the corresponding Lie algebra. It is known, that Chevalley group is defined by the root system, the ring $R$ and the weight lattice of the representation $\pi$. We will denote this lattice by $\Lambda_\pi$. If we consider an elementary Chevalley group, we denote it by $E_\pi(\Phi, R)$.

The subgroup of all diagonal (in a standard basis of weight vectors) matrices of the Chevalley group $G_\pi(\Phi, R)$ is called the standard maximal torus of $G_\pi(\Phi, R)$ and is denoted by $T_\pi(\Phi, R)$. This group is isomorphic to $\text{Hom}(\Lambda_\pi, R^*)$.

Let us denote by $h(\chi)$ an element of $T_\pi(\Phi, R)$, corresponding to the homomorphism $\chi \in \text{Hom}(\Lambda(\pi), R^*)$.

In particular, $h_\alpha(u) = h(\chi_{\alpha, u}) (u \in R^*, \alpha \in \Phi)$, where $\chi_{\alpha, u} : \lambda \mapsto u^{(\lambda, \alpha)}$, $\lambda \in \Lambda_\pi$, $(\lambda, \alpha) = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$.

Relations between Chevalley groups and the corresponding elementary subgroups is an important problem in the theory of Chevalley groups over rings. For elementary Chevalley groups there exists a convenient system of generators $x_\alpha(\xi), \alpha \in \Phi, \xi \in R$, and all relations between these generators are well known. For general Chevalley groups it is not always true.

If $R$ is an algebraically closed field, then $G_\pi(\Phi, R) = E_\pi(\Phi, R)$ for any representation $\pi$. This equality is not true even for the case of fields, which are not algebraically closed.
However if $G$ is a universal group and the ring $R$ is semilocal (i.e., contains only finite number of maximal ideals), then we have the condition

$$G_{sc}(\Phi, R) = E_{sc}(\Phi, R).$$

[1, 3, 33, 44].

Let us show the difference between Chevalley groups and their elementary subgroups in the case when a ring $R$ is semilocal and a corresponding Chevalley group is not universal. In this case $G_{\pi}(\Phi, R) = E_{\pi}(\Phi, R)T_{\pi}(\Phi, R)$ (see [1, 3, 33]), and the elements $h(\chi)$ are connected with elementary generators by the formula

$$h(\chi)x_{\beta}(\xi)h(\chi)^{-1} = x_{\beta}(\chi(\beta)\xi).$$

If $\Phi$ is an irreducible root system of rank $l \geq 2$, then $E(\Phi, R)$ is always normal in $G(\Phi, R)$ (see [20, 49]). In the case of semilocal rings with $\frac{1}{2}$ it is easy to show that

$$[G(\Phi, R), G(\Phi, R)] = E(\Phi, R).$$

However in the case $l = 1$ the subgroup of elementary matrices $E_{2}(R) = E_{sc}(A_{1}, R)$ is not necessarily normal in the special linear group $SL_{2}(R) = G_{sc}(A_{1}, R)$ (see [18, 46, 47]).

We will use several facts proved in [12].

The next lemma is well known (an excellent evaluation of $N$ can be found in [43]):

**Lemma 2.1** Let $G = G_{\pi}(\Phi, R)$ be a Chevalley group, $E = E_{\pi}(\Phi, R)$ be its elementary subgroup, $R$ a semilocal ring (with $\frac{1}{2}$ for the root systems $A_{2}$, $B_{l}$, $C_{l}$, $F_{4}$, $G_{2}$ and with $\frac{1}{3}$ for $G_{2}$). Then $E = [G, G]$ and there exists such a number $N$, depending on $\Phi$, but not on $R$ (and not on a representation $\pi$), that every element of the group $E$ is a product of not more than $N$ commutators of the group $G$.

This lemma has a very important corollary:

**Corollary 2.2** Under assumptions of the previous lemma the elementary subgroup $E = E_{\pi}(\Phi, R)$ is interpretable in a Chevalley group $G = G_{\pi}(\Phi, R)$ without parameters with the same interpretation code for all Chevalley groups of the same type.

In particular, if two Chevalley groups of the same type are elementarily equivalent, then their elementary subgroups are elementarily equivalent too.

An elementary adjoint Chevalley group $E_{ad}(\Phi, R)$ is always the quotient group of $E_{\pi}(\Phi, R)$ by its center. Therefore the adjoint Chevalley group $E_{ad}(\Phi, R)$ is absolutely interpretable in the initial Chevalley group $G = G_{\pi}(\Phi, R)$ in our case, so if two Chevalley groups of the same type are elementarily equivalent, then the corresponding adjoint elementary Chevalley groups are elementarily equivalent too.

The next important step is to show that the subgroup $E_{J} = E_{ad}(\Phi, R, J)$, where $J$ is the radical of $R$, is absolutely definable in $E_{ad}(\Phi, R)$ (and so in the initial Chevalley group $G_{\pi}(\Phi, R)$). This group is generated by all $x_{\alpha}(t), \alpha \in \Phi, t \in J$, and is the biggest (unique maximal) proper normal subgroup of $E = E_{ad}(\Phi, R)$ (see [2]).

The following proposition was proved in [12]:
Proposition 2.3 ([12, Proposition 3]) The subgroup \( E_J = E_{\text{ad}}(\Phi, R, J) \) is absolutely definable in \( E = E_{\text{ad}}(\Phi, R) \), if \( R \) is as in Lemma 2.1.

For the sake of completeness we provide the reader with a sketch of the main ideas of our proof.

**Proof** The main idea is that some elements \( A \) of \( E_J \) are definable by the formula \( \varphi_N \), stating that all products of its conjugates of the length \( N \) form a normal subgroup in \( E \) which does not coincide with \( E \).

If for a given \( A \) and some length \( N \) the formula \( \varphi_N(A) \) is true, then the minimal normal subgroup of \( E \) which contains \( A \), is a proper subgroup of \( E \). As we know, every proper normal subgroup of \( E \) is contained in \( E_J \), therefore \( A \in E_J \).

Let us fix the minimal natural \( N \) such that if for some \( A \) the sentence \( \varphi_N(A) \) does not hold, then for this \( A \) no sentence \( \varphi_p(A) \), \( p > N \), holds.

Taking into account this \( N \) let us find natural \( M \) such that:

1. products \( X_1 \ldots X_k, k \leq M \), of elements of \( E \), satisfying \( \varphi_N(X) \), form a subgroup in \( E \);
2. this subgroup is normal;
3. this subgroup is not trivial;
4. this subgroup does not coincide with the whole group \( E \).

It is clear that all these assertions are first order definable.

This number \( M \) means that every element of the group \( E_J \) is generated by not more than \( M \) elements \( x_\alpha(u), u \in J \).

Now the formula

\[
\text{Normal}_{M,N}(X) := \bigwedge_{i=0}^{M} \exists X_1 \ldots \exists X_i (\varphi_N(X_1) \land \ldots \land \varphi_N(X_i) \land X = X_1 \ldots X_i)
\]

defines in \( E \) the subgroup \( E_J \). \( \square \)

We proved that the subgroup \( E_J \) is absolutely definable in the group \( E \), then, taking the quotient group \( E/E_J \), we obtain the Chevalley group \( \tilde{E} \cong E_{\text{ad}}(R/J) \), i.e., a Chevalley group over field, which is also absolutely interpretable in \( E = E_{\text{ad}}(\Phi, R) \) (and also in the initial \( G_{\pi}(\Phi, R) \)).

Recall that we have a root system \( \Phi \) of rank \( > 1 \). The set of simple roots is denoted by \( \Delta \), the set of positive roots is denoted by \( \Phi^+ \). The subgroup \( U = U(R) \) of the Chevalley group \( G(E) \) is generated by elements \( x_\alpha(t), \alpha \in \Phi^+, t \in R \), the subgroup \( V = V(R) \) is generated by elements \( x_{-\alpha}(t), \alpha \in \Phi^+, t \in R \).

For invertible \( t \in R^* \) by \( w_\alpha(t) \) we denote \( x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t) \), by \( h_\alpha(t) \) we denote \( w_\alpha(t)w_\alpha(1)^{-1} \).

The group \( H = H(R) \) is generated by all \( h_\alpha(t), \alpha \in \Phi, t \in R^* \).

The strict proof of the following proposition can be found in [12] as well. The first assertion is well known, but we also need assertions (2) and (3) for our purpose.
Proposition 2.4 (Gauss decomposition and first order formulas)

(1) Every element $x$ of a Chevalley group $G$ (respectively, of its elementary subgroup $E$) over a local ring $R$ can be represented in the form

$$x = u t v u'$$

(respectively, $x = u h v u'$),

where $u, u' \in U(R)$, $v \in V(R)$, $t \in T(R)$, $h \in H(R)$.

(2) There exists a first order formula $\varphi(\ldots)$ of the ring language and $6n + 2l$ arguments, such that for decompositions $x_1 = u_1 t_1 v_1 u'_1$ and $x_2 = u_2 t_2 v_2 u'_2$, where

$$u_i = x_{\alpha_1}(t^{(i)}_1) \ldots x_{\alpha_n}(t^{(i)}_n),$$
$$u'_i = x_{\alpha_1}(s^{(i)}_1) \ldots x_{\alpha_n}(s^{(i)}_n),$$
$$v_i = x_{-\alpha_1}(r^{(i)}_1) \ldots x_{-\alpha_n}(r^{(i)}_n),$$
$$t_i = h_{\alpha_1}(\xi^{(i)}_1) \ldots h_{\alpha_l}(\xi^{(i)}_l), \quad i = 1, 2,$$

this formula

$$\varphi(t^{(1)}_1, \ldots, t^{(1)}_n, t^{(2)}_1, \ldots, t^{(2)}_n, s^{(1)}_1, \ldots, s^{(1)}_n, s^{(2)}_1, \ldots, s^{(2)}_n, r^{(1)}_1, \ldots, r^{(1)}_n, r^{(2)}_1, \ldots, r^{(2)}_n, \xi^{(1)}_1, \ldots, \xi^{(1)}_n, \xi^{(2)}_1, \ldots, \xi^{(2)}_n)$$

is true if and only if

$$x_1 = x_2.$$

(3) Similarly, there exists a first order formula $\psi(\ldots)$ of the ring language and $9n + 3l$ arguments, such that for decompositions $x_1 = u_1 t_1 v_1 u'_1$, $x_2 = u_2 t_2 v_2 u'_2$ and $x_3 = u_3 t_3 v_3 u'_3$, where

$$u_i = x_{\alpha_1}(t^{(i)}_1) \ldots x_{\alpha_n}(t^{(i)}_n),$$
$$u'_i = x_{\alpha_1}(s^{(i)}_1) \ldots x_{\alpha_n}(s^{(i)}_n),$$
$$v_i = x_{-\alpha_1}(r^{(i)}_1) \ldots x_{-\alpha_n}(r^{(i)}_n),$$
$$t_i = h_{\alpha_1}(\xi^{(i)}_1) \ldots h_{\alpha_l}(\xi^{(i)}_l), \quad i = 1, 2, 3,$$

this formula $\psi(t^{(i)}_1, \ldots, t^{(i)}_n, s^{(i)}_1, \ldots, s^{(i)}_n, r^{(i)}_1, \ldots, r^{(i)}_n, \xi^{(i)}_1, \ldots, \xi^{(i)}_n)$ holds if and only if

$$x_3 = x_1 \cdot x_2.$$

3 Regular bi-interpretability of Chevalley groups over local rings

Now we will prove Theorem 1.19. Our proof consists of three steps:
(1) To prove that all elementary unipotent subgroups \( X_\alpha = \{ x_\alpha(t) \mid t \in R \} \) are definable in \( G = E_{ad}(\Phi, R) \) with parameters \( \bar{x} = \{ x_\alpha(1) \mid \alpha \in \Phi \} \).

(2) To prove that a Chevalley group \( G = G_\pi(\Phi, R) \) (or an elementary Chevalley group \( E_\sharp(\Phi, R) \)) is bi-interpretable with the ring \( R \) with parameters \( \bar{x} = \{ x_\alpha(1) \mid \alpha \in \Phi \} \).

(3) To prove that our parameters \( \bar{x} \) are definable, i.e., bi-interpretability is regular.

We will allocate one subsection for each step.

### 3.1 Definability of elementary unipotent subgroups

Definability of elementary unipotent subgroups \( X_\alpha \) was proved in [40] for integral domains, which include fields. So we can suppose that all groups \( X_\alpha \) are definable in the group \( E_{ad}(\Phi, R) \), if \( R \) is an arbitrary field.

Let us show that a root subgroup \( X_\alpha \) over a local ring is also definable.

We will define it as the intersection of the inverse image of a root subgroup of the Chevalley group \( E' = E_{ad}(\Phi, R/\text{Rad } R) \) over the residue field under canonical homomorphism and the set

\[
\{ AB \mid \forall x \ ( [x, x_\alpha(1)] = 1 \iff [A, x] = 1 \land [B, x] = 1) \}.
\]

To begin with, let us understand which elements lie in the inverse image of the root subgroup.

Let \( x_\alpha(i) \in E' \), we denote some of its inverse image under canonical homomorphism by \( g \). In \( E = E_{ad}(\Phi, R) \) there exists a Gauss decomposition (see Proposition 2.4 above). Using it we can represent \( g \) as the product of a bounded number of elements \( x_\alpha_i(t) \) of the special form. Ordering positive roots, without loss of generality, we can assume that \( \alpha \) is positive and it is \( \alpha_1 \). We have:

\[
g = x_{\alpha_1}(r_1) \ldots x_{\alpha_n}(r_n) h x_{-\alpha_1}(s_1) \ldots x_{-\alpha_n}(s_n) x_{\alpha_1}(t_1) \ldots x_{\alpha_n}(t_n),
\]

where \( \alpha_i \) are positive roots, \( r_i, s_i, t_i \in R, h \in T(R) \). The image of \( g \) under canonical homomorphism has the form (by the homomorphism properties):

\[
x_{\alpha_1}(\bar{r}_1) \ldots x_{\alpha_n}(\bar{r}_n) \bar{h} x_{-\alpha_1}(\bar{s}_1) \ldots x_{-\alpha_n}(\bar{s}_n) x_{\alpha_1}(\bar{t}_1) \ldots x_{\alpha_n}(\bar{t}_n) = x_{\alpha_1}(\bar{t}),
\]

where \( \bar{r}_i, \bar{s}_i, \bar{t}_i \in R/J \). Let us move elements with positive roots to the right side of equality:

\[
\bar{h} x_{-\alpha_1}(\bar{s}_1) \ldots x_{-\alpha_n}(\bar{s}_n) = x_{\alpha_1}(\bar{r}_n)^{-1} \ldots x_{\alpha_1}(\bar{r}_1)^{-1} x_{\alpha_1}(\bar{t}) x_{\alpha_n}(\bar{t}_n)^{-1} \ldots x_{\alpha_1}(\bar{t}_1)^{-1}.
\]

We obtained the equality \( tv = u \), where \( t \in T, v \in V, u \in U \), where \( V \) and \( U \) are the subgroups, generated by negative and positive root unipotents, respectively. Since \( TV \cap U = 1 \), both sides are equal to 1, that is, the Gauss decomposition of the original element consists only of the torus element, root unipotents with positive roots and root unipotents with negative roots, which arguments belong to the radical, so we
can assume that
\[ g = x_{\alpha_1}(r_1) \ldots x_{\alpha_n}(r_n) h \cdot x_{-\alpha_1}(s_1) \ldots x_{-\alpha_n}(s_n), \quad s_1, \ldots, s_n \in J, \]
and its image is
\[ \overline{h} \cdot x_{\alpha_1}^2 \ldots x_{\alpha_m}^2 = x_{\alpha_1}(\overline{t}), \]
therefore
\[ g = x_{\alpha_1}(r_1) \ldots x_{\alpha_n}(r_n) h \cdot x_{-\alpha_1}(s_1) \ldots x_{-\alpha_n}(s_n), \]
\( r_2, \ldots, r_n, s_1, \ldots, s_n \in J, \overline{h} = 1. \)
Thus we proved definability of \( X_{\alpha_1}E_J \). In order to prove finally that the subgroup \( X_{\alpha_1}(R) \) remains definable, we will prove the following important proposition:

**Proposition 3.1** Let \( G_{\alpha_1} = \{ g \in E \mid C_G(g) = C_G(x_{\alpha_1}(1)) \} \). Then
\[ X_{\alpha_1}(R) \supseteq X_{\alpha_1}(R) E_J \cap G_{\alpha_1} \supseteq X_{\alpha_1}(R^*). \]

**Proof** It is clear that if \( g \in X_{\alpha_1}(R^*) \), then \( g \in G_{\alpha_1} \), therefore we will prove only the first inclusion.

Let \( g \in X_{\alpha_1}(R) E_J \cap G_{\alpha_1} \). Let us recall once again the type of elements \( X_{\alpha_1}(R) E_J \) we have already proved:
\[ g = x_{\alpha_1}(r_1) \ldots x_{\alpha_n}(r_n) t x_{-\alpha_1}(s_1) \ldots x_{-\alpha_n}(s_n), \quad r_2, \ldots, r_n, s_1, \ldots, s_n \in J. \]

Note that, unlike the original Gauss decomposition, such a decomposition is uniquely defined. Indeed, suppose that
\[ u_1 t_1 v_1 = u_2 t_2 v_2. \]
If we move all the positive roots to one side: \( t_1 v_1 v_2^{-1} t_2^{-1} = u_1^{-1} u_2 \), then since \( TV \cap U = 1 \), so this decomposition is unique.

Let us stock up on an important formula
\[ x_\gamma(1) x_{-\gamma}(s) x_\gamma(1)^{-1} = h_\gamma \left( \frac{1}{1-s} \right) x_\gamma(s^2 - s) x_{-\gamma} \left( \frac{s}{1-s} \right) \quad \text{for all} \quad \gamma \in \Phi \]
(it is checked directly through a representation by matrices from \( SL_2 \)).

We always suppose that the roots \( \alpha_1, \ldots, \alpha_n \) are ordered by their height (first there are simple roots, then their sums, at the end there is the highest root).

Consider a root \( \beta \in \Phi \) such that \( \alpha_1 + \beta \notin \Phi \), and the element \( x_\beta(1) \). Since \( g \in G_{\alpha_1} \), then it commutes with \( x_\beta(1) \). It means that \( g x_\beta(1) = x_\beta(1) \). Let us consider,
how conjugation by this element acts on $g$ and its separate multipliers (let us take $\beta$ positive):

$$
\begin{align*}
g &= g^{x_\beta(1)} = x_{\alpha_1}(r_1)^{x_\beta(1)} \cdots x_{\alpha_n}(r_n)^{x_\beta(1)} t^{x_\beta(1)} x_{-\alpha_1}(s_1)^{x_\beta(1)} \cdots x_{-\alpha_n}(s_n)^{x_\beta(1)} \\
&= x_{\alpha_1}(r_1) \cdots (x_{\alpha_i}(r_i) x_{\alpha_i+\beta}(c_i r_i) \cdots) \cdots x_{\alpha_n}(r_n) \cdot (x_\beta(c) t) \\
&\quad \cdot (x_{-\alpha_1}(s_1) x_{-\alpha_1+\beta}(d_1 s_1) \cdots) \\
&\quad \cdots (h_\beta \left( \frac{1}{1-s_\beta} \right) x_\beta(s_\beta^2 - s_\beta) x_{-\beta} \left( \frac{S_\beta}{1-s_\beta} \right)) \\
&\quad \cdots (x_{-\alpha_n}(s_n) x_{-\alpha_n+\beta}(d_n s_n) \cdots).
\end{align*}
$$

Let us analyse the obtained equality.

Note that to the left of $t$ there are only unipotent elements with positive roots, that is, an element of $U$. To the right of $t$ we see the element of the torus $h_\beta \left( \frac{1}{1-s_\beta} \right)$, which we can rearrange directly to $t$ by changing only the arguments of the elements $x_\gamma(\cdot)$, we see unipotent elements with negative roots, as well as such $x_\gamma(\cdot)$ with $\gamma \in \Phi^+$, which, when moving to the left towards $T$ and $U$, cannot meet unipotent elements with the opposite root. This means that it is possible to rearrange all unipotents with positive roots that are to the right of $t$, to the left of $t$ so that $t$ and $h_\beta \left( \frac{1}{1-s_\beta} \right)$ will not change. After that, the conjugate element will be written in the form of $UTV$, that is

$$
t = t \cdot h_\beta \left( \frac{1}{1-s_\beta} \right),
$$

so $s_\beta = 0$.

Thus, $x_\beta(\cdot)$ cannot appear in the right of $t$ part of the expression during conjugation, while it can appear in the left part only in the place $x_\beta(r_\beta)$. So, in the expression $x_\beta(c)$, which is obtained by conjugating the element $t$ with the element $x_\beta(1)$, there must be $c = 0$, that is

$$
[t, x_\beta(1)] = 1.
$$

Let us consider the part $V$ after conjugation.

Conjugating $x_{-\alpha_i}(s_i)$, where $\alpha_i$ is higher than $\beta$, we can obtain additional unipotents from $V$; but if $\alpha_i$ is lower than $\beta$, then we can obtain additional unipotents from $U$.

Consider the maximal root $\alpha_i$, for which $-\alpha_i + \beta \in \Phi$. Conjugating the corresponding unipotent we have

$$
x_{-\alpha_i}(s_i)^{x_\beta(1)} = x_{-\alpha_i}(s_i) x_{-\alpha_i+\beta}(c \cdot s_i) x_{-\alpha_i+2\beta}(\cdots) \cdots,
$$

where $c$ is an invertible integer (since it could be equal only to $\pm 1$, $\pm 2$ or $\pm 3$, but in the second case we deal with root systems $B_1$, $C_1$, $F_4$, $G_2$ and $\frac{1}{2} \in R$ by our assumption, in the third case we deal with the root system $G_2$ and also $\frac{1}{2} \in R$ by our assumption). By the choice of $\alpha_i$ an element $x_{-\alpha_i+\beta}(c \cdot s_i)$ can no longer appear from any of the
conjugates, therefore $c_s = s_i = 0$. By the next step let us consider the next (by height) root $\alpha_j$ such that $-\alpha_j + \beta \in \Phi$, and we will come to the same conclusion $s_j = 0$.

Thus, we will consistently understand that for any root $\alpha_i$ for which $-\alpha_i + \beta \in \Phi$ the condition $s_i = 0$ takes place. So, the element $x_\beta(1)$ commutes with all non-unit unipotents located to the right of $t$, that is

$$g = g^{x_\beta(1)} = x_{\alpha_1}(r_1)^{x_\beta(1)} \ldots x_{\alpha_n}(r_n)^{x_\beta(1)} \cdot i x_{-\alpha_1}(s_1) \ldots x_{-\alpha_n}(s_n),$$

therefore

$$x_{\alpha_1}(r_1)^{x_\beta(1)} \ldots x_{\alpha_n}(r_n)^{x_\beta(1)} = x_{\alpha_1}(r_1) \ldots x_{\alpha_n}(r_n),$$

and we will come to the conclusion by the same arguments that $r_i = 0$ for all $i$ such that $\alpha_i + \beta \in \Phi$.

Therefore for all roots $\beta \in \Phi$ such that $\beta + \alpha_1 \notin \Phi \cup \{0\}$ if $\gamma + \beta \in \Phi \cup \{0\}$, then the corresponding unipotent $x_{\gamma}(t) = 1$, i.e., $t = 0$.

We will show that in most root systems in this way we will be able to “delete” all root unipotents except the first one. We divide all root systems into three categories: simply laced root systems, systems with double relations, and $G_2$.

Let us start with the $G_2$ root system, so that we do not have to go back to it.

**Lemma 3.2** Let $\Phi = G_2$, $B = \{\beta \in \Phi \mid \alpha_1 + \beta \notin \Phi \cup \{0\}\}$. By deleting from the list of roots of the system $\Phi$ all elements of $\gamma$ such that $\exists \beta \in B$ ($\beta + \gamma \in \Phi \cup \{0\}$), we delete all roots except $\alpha_1$.

**Proof** Since $\alpha_1$ can be any of the roots of the system, we will first prove the lemma for $\alpha_1$ a short, and then for a long one.

If $\alpha_1$ is a short simple root, then the system consists of the roots

$$\pm\alpha_1, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2).$$

In this case

$$B = \{\alpha_1, -\alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$  

The roots $-\alpha_1, \alpha_2, -3\alpha_1 - \alpha_2, \pm(3\alpha_1 + 2\alpha_2)$ are deleted, since they are opposite to the roots from $B$; the roots $\pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2)$ are deleted according to the root $\alpha_1 \in B$, the root $-\alpha_2$ is deleted according to the root $3\alpha_1 + 2\alpha_2 \in B$, $3\alpha_1 + \alpha_2$ is deleted according to $-3\alpha_1 - 2\alpha_2 \in B$. Therefore all roots except $\alpha_1$ are deleted.

In the case when $\alpha_1$ is a long simple root, the consideration is absolutely similar (and even easier).  

**Lemma 3.3** Let $\Phi$ be a simply laced root system, $B$ be the same as in the previous lemma. Again from the set of all root of $\Phi$ all $\gamma$ such that $\exists \beta \in B$ ($\beta + \gamma \in \Phi \cup \{0\}$) are all roots except $\alpha_1$.  

\[ Springer \]
Proof To begin with, we note that all the roots \(-\alpha_2, \ldots, -\alpha_l\) belong to \(B\), since for simple root \(\alpha_i, \alpha_j\) the difference \(\alpha_i - \alpha_j \notin \Phi\).

For any non-simple positive root \(\theta\) there exists a simple root \(\gamma\) such that \(\theta - \gamma \in \Phi^+\), therefore \(\theta\) can be deleted, since \(-\gamma \in B\).

The exception is the case when \(\gamma = \alpha_1\). This case means that there exists a pair of positive roots \(\{\alpha_1, \beta = \theta - \alpha_1\}\), for which their sum is a root. It is clear that in a simply laced root system these roots generate the subsystem \(A_2\). Then \(\beta + \alpha_1 \in \Phi, \beta + 2\alpha_1 \notin \Phi\), therefore \(\theta = \beta + \alpha_1, -\beta \in B\), and consequently since \(\alpha_1 = \theta + (-\beta) \in \Phi\), then both roots are deleted. Thus \(\theta\) (any non-simple positive root) is deleted.

All simple positive roots (except \(\alpha_1\)) are deleted, since they are opposite to the roots from \(B\). Therefore all positive roots except \(\alpha_1\) are deleted.

It is clear that the root \(-\alpha_1\) is deleted. Concerning other negative roots we mention that any root which is not collinear to \(\alpha_1\) can be positive under another choice of ordering (but with \(\alpha_1\) simple). Therefore all roots except \(\alpha_1\) can be deleted.

Lemma 3.4 For any root system \(\Phi\) if \(\langle \beta, \alpha_1 \rangle = 0\), \(g \in X_{\alpha_1}(R)E_J \cap G_{\alpha_1}\),

\[
g = x_{\alpha_1}(s_1) \ldots x_{\alpha_n}(s_n)tx_{-\alpha_1}(r_1) \ldots x_{-\alpha_n}(r_n),
\]

then

\[
[h_\beta(a), x_{\alpha_i}(s_i)] = [h_\beta(a), x_{-\alpha_i}(r_i)] = 1 \text{ for all } i = 1, \ldots, n \text{ and } a \in R^*.
\]

Proof Since

\[
h_\alpha(t)x_{-\gamma}(s)h_\alpha(t)^{-1} = x_{\gamma}(t^{(\alpha, \gamma)} \cdot s),
\]

then if \(\langle \beta, \alpha_1 \rangle = 0\), \(h_\beta(a)\) commutes with \(x_{\alpha_1}(1)\), consequently \(h_\beta(a)\) commutes with \(g\). Then

\[
g = g^{h_\beta(a)} = x_{\alpha_1}(a^{(\beta, \alpha_1)}s_1) \ldots x_{\alpha_n}(a^{(\beta, \alpha_n)}s_n)tx_{-\alpha_1}(a^{-(-\beta, \alpha_1)}r_1) \ldots x_{-\alpha_n}(a^{(-\beta, \alpha_n)}r_n).
\]

From the uniqueness of the decomposition for \(g\) we have

\[
[h_\beta(a), x_{\alpha_i}(s_i)] = [h_\beta(a), x_{-\alpha_i}(r_i)] = 1 \text{ for all } i = 1, \ldots, n.
\]
It is clear that if the roots \( \alpha_1 \) and \( \gamma \) are enclosed in the system \( A_2 \), then everything is good (see Lemma 3.3), but if they are enclosed only in the system \( B_2 \), then we need to have an additional consideration.

To finish all the cases, as we see, it remains simply to consider separately the root system \( B_2 \) and prove that it is possible to delete all the roots in it (perhaps not only using the set \( B \), but also using Lemma 3.4).

In the root system \( B_2 \) there are roots

\[ \pm \mu, \pm \nu, \pm (\mu + \nu), \pm (2\mu + \nu), \]

where the simple root \( \mu \) is short, and \( \nu \) is long.

If \( \mu = \alpha_1 \), then \( \mu, -\nu, 2\mu + \nu \in B \), therefore \( -\mu, \nu, \pm (\mu + \nu) \) and \( -(2\mu + \nu) \) are deleted. The roots \( -\nu \) and \( 2\mu + \nu \) cannot be deleted according to \( B \), therefore we will use \( h_{\alpha_1}(-1) \).

Since the roots \( \mu \) and \( \mu + \nu \) are orthogonal, by Lemma 3.4 we have

\[
[h_{\mu+\nu}(-1), x_{-\nu}(s)] = [h_{\mu+\nu}(-1), x_{2\mu+\nu}(r)] = 1 \quad \text{for} \quad s, r \text{ from decomposition of } g.
\]

Since \( \langle \mu + \nu, -\nu \rangle = -1 \), then \( -s = s \), therefore \( s = 0 \), what was required. Similarly for the root \( 2\mu + \nu \).

If \( \nu = \alpha_1 \), then \( \nu, -\mu, \mu + \nu, \pm (2\mu + \nu) \in B \), therefore \( \pm \mu, -\nu, \pm (\mu + \nu) \) and \( \pm (2\mu + \nu) \) are deleted, what was required. \( \Box \)

Therefore commuting with all suitable \( x_{\beta}(1) \) and \( h_{\nu}(a) \), our element comes to the form \( tx_{\alpha_1}(r_1) \), where \( t \) commutes with all \( x_{\beta}(1) \) for \( \beta \in B \), which clearly gives that \( t \) belongs to the group’s center. Since for an adjoint elementary Chevalley group its center is trivial, \( t = 1 \) and the proposition is proved.

Therefore we proved the following theorem:

**Theorem 3.5** If \( E = E_{\text{ad}}(\Phi, R) \) is an elementary adjoint Chevalley group of rank \( > 1 \), \( R \) is a local ring (with \( \frac{1}{2} \) for the root systems \( B_l, C_l, F_4, G_2 \) and with \( \frac{1}{3} \) for \( G_2 \)), then any root subgroup \( X_\alpha, \alpha \in \Phi \), is definable in \( E \) with parameters.

### 3.2 Bi-interpretability with parameters of \( G(R) \) and \( R \)

Now we will concentrate on proving that a group \( G_{\pi}(\Phi, R) \) (respectively, \( E_{\pi}(\Phi, R) \)) and the corresponding local ring \( R \) are bi-interpretable (with parameters \( \overline{\alpha} \)), or more strictly that the pair \( \langle G_{\pi}(\Phi, R), \overline{\alpha} \rangle \) (\( \langle E_{\pi}(\Phi, R), \overline{\alpha} \rangle \)) and the ring \( R \) are absolutely bi-interpretable.

Since in the paper [40] of Segal and Tent the similar theorem is proved for integral domains, we will use it and notice that similar arguments work in our case.

Following [40], a bi-interpretation between \( R \) and \( G(R) \) has several steps, some of them are already proved, some of them are similar to Segal–Tent, some of them are clear.

**Step 1.** An interpretation of \( E(R) \) in \( G(R) \) is a very simple absolute interpretation of the bounded commutant group in a group. Also if we speak about absolute interpretation
of \((E_{\pi}(\Phi, R), \bar{x})\) in \((G_{\pi}(\Phi, R), \bar{x})\), the parameters in one group coincide with the parameters in another one (since they always belong to the commutant).

**Step 2.** An interpretation of \(E_{\text{ad}}(R)\) in \(E(R)\) is also a very simple absolute interpretation of the quotient group of a group by it center. If we consider an interpretation of \((E_{\text{ad}}(\Phi, R), \bar{x})\) in \((E_{\pi}(\Phi, R), \bar{x})\), the parameters \(\bar{x}\) are the quotient classes of the initial \(\bar{x}\).

**Step 3.** An interpretation of \(R\) in \(E_{\text{ad}}(R)\) (as in \([40]\)) consists of an identification of \(R\) with a definable abelian subgroup \(R'\) of \(E_{\text{ad}}(R)\) such that addition \(\oplus\) in \(R'\) is the group operation in \(E_{\text{ad}}(R)\); and multiplication \(\otimes\) in \(R'\) is definable in \(E_{\text{ad}}(R)\). We will not repeat it here, since it is absolutely the same arguments as in \([40]\) and the papers about Matvev-type theorems.

**Step 4.** An interpretation of \(G' \cong G(R)\) (or \(G' \cong E(R)\)) in \(R\): according to the Gauss decomposition every element \(g \in G'\) can be represented as the product of four elements \(u_1 \in U, t \in T, v \in V, u_2 \in U\), where \(u_1, v, u_2\) uniquely correspond to vectors of the length \(n\) of elements from \(R\), \(t\) uniquely corresponds to a vector of the length \(l\) of elements from \(R^*\). Since equality and multiplication of two elements expressed by these four vectors is first order defined, we have absolute interpretation \(G'\) of \(G(R)\) (or \(E(R)\)) in \(R\).

For the group \(E(R)\) we always use this type of interpretation, but for the group \(G(R)\) it is possible also to repeat all arguments from \([40]\), since they do not depend on any properties of a given ring.

**Step 5.** As in \([40]\) we define an isomorphism \(G(R) \to G(R')\) or \(E(R) \to E(R')\), it is definable according to Proposition 2.4.

**Step 6.** An isomorphism \(R \to R'\) is the same as in the paper \([40]\).

Therefore we proved the following important theorem:

**Theorem 3.6** If \(G(R) = G_{\pi}(\Phi, R)\) (\(E(R) = E_{\pi}(\Phi, R)\)) is an (elementary) Chevalley group of rank \(> 1\), \(R\) is a local ring (with \(1/2\) for the root systems \(B_i, C_i, F_4, G_2\) and with \(1/4\) for \(G_2\)), \(\bar{x} = \{x_\alpha(1) \mid \alpha \in \Phi\}\), then the pair \((G(R), \bar{x})\) (or \((E(R), \bar{x})\)) is absolutely bi-interpretable with \(R\).

Theorem 3.6 also can be proved using results from \([52]\).

### 3.3 Definability of parameters

In this section we are going to show that our parameters \(\bar{x} = \{x_\alpha(1) \mid \alpha \in \Phi\}\) are definable up to an automorphism of the Chevalley group \(E_{\text{ad}}(\Phi, R)\).

What do we know about these parameters?

1. \(x_\alpha(1)\) and \(x_\beta(1)\) are conjugate, if \(|\alpha| = |\beta|\) and we know the order of some possible conjugating element.
2. We know all commutator relations between parameters.
3. We know that for every \(\alpha \in \Phi\) there exists a formula with parameters \(\bar{x}\), defining the set \(X_\alpha\) with operations \(\oplus\) and \(\otimes\), which becomes a local ring and all these rings for different \(\alpha\) (let us call them \(X_\alpha(\bar{x})\)) are isomorphic to each other.
4. Any element of the whole group is represented as the product $u t v u'$, where
$u, t, v, u'$ are defined through $X_\alpha$ as in Proposition 2.4, the equality of two such
representations is defined as in Proposition 2.4, multiplication of two such elements
is defined as in Proposition 2.4.

Altogether properties 1–4 can be expressed by a first order formula $\Phi(\bar{x})$. In fact it
means that there exists a ring $R' = (X_\alpha, \oplus, \otimes)$ such that $E_{ad}(\Phi, R) \cong E_{ad}(\Phi, R')$. By
[13] an isomorphism is the composition of a ring isomorphism (where $x_\alpha(t) \mapsto x_\alpha(t')$
for all $\alpha \in \Phi$ and $t \in R$) and some automorphism of the initial group $E_{ad}(\Phi, R)$.
This means that $R' \cong R$ and any parameters $\bar{t}$ satisfying the formula $\Phi$ up to an
automorphism of the group $E_{ad}(\Phi, R)$ have the form $\bar{x} = \{x_\alpha(1) | \alpha \in \Phi\}$, what was
required.

Therefore bi-interpretability of the Chevalley group $E_{ad}(\Phi, R)$ is regular.

Finally we proved our main theorem (Theorem 1.19):

If $G(R) = G_{\pi}(\Phi, R)$ ($E(R) = E_{\pi}(\Phi, R)$) is an (elementary) Chevalley group
of rank $> 1$, $R$ is a local ring (with $\frac{1}{2}$ for the root systems $B_l, C_l, F_4, G_2$ and with
$\frac{1}{3}$ for $G_2$), then the group $G(R)$ (or $E(R)$) is regularly bi-interpretable with $R$.

3.4 Elementary definability of Chevalley groups

In this section we will show that regular bi-interpretability implies elementary defin-
ability.

Proposition 3.7 If for some class of rings $\mathcal{R}$, closed under elementary equivalence,
and the class $\mathcal{J} = \{G(R) | R \in \mathcal{R}\}$ (where $G(R)$ is any type of derivative groups:
linear groups, Chevalley groups, automorphism groups, etc.) all the groups $G(R)$ are
regularly interpretable with the corresponding rings $R$ with the same interpretations,
then the class $\mathcal{J}$ is elementarily definable in class of all groups, i.e., for any group $H$
such that $H \equiv G(R)$ for some $R \in \mathcal{R}$ there exists a ring $R' \equiv R$ such that $H \cong G(R')$.

Proof Let $H$ be some group such that $H \equiv G(R)$, $R \in \mathcal{R}$. Let $\varphi(x_1, \ldots, x_m)$ be the
formula defining parameters $y_1, \ldots, y_m$, required for bi-interpretation of $R$ in $G(R)$
(recall that these parameters are defined up to some automorphism of $G(R)$). The
formula $\varphi(x_1, \ldots, x_m)$ defines in $H$ some parameters $h_1, \ldots, h_m$, let us fix them.

Let now $R \cong \Gamma(G(R), \varphi)$ be a code interpreting $R$ in $G(R)$. Let us apply the same
code to the group $H$, we will obtain some ring $R'$, elementarily equivalent to $R$.

Since $R' \equiv R$, then $R' \in \mathcal{R}$, so that $G(R') \in \mathcal{J}$. It means that the code $\Gamma(\cdot, \varphi)$ also
defines in $G(R')$ the ring $R'$, and then a code $\Delta(\cdot, \psi)$, which interprets $G(R)$ in $R$,
also interprets $G(R')$ in $R'$.

But there is a formula $\theta(\bar{y}, x, \bar{z})$ in the language of groups such that for every
tuple $\bar{p}$ satisfying $\varphi(\bar{z})$ in $G(R)$ (or in $G(R')$) the formula $\theta(\bar{y}, x, \bar{p})$ defines in $G(R)$
(in $G(R')$) the isomorphism

$\bar{\mu}_{\Delta \circ \Gamma} : (\Delta \circ \Gamma)(G(R), \bar{p}) \rightarrow G(R)$ (or $\bar{\mu}'_{\Delta \circ \Gamma} : (\Delta \circ \Gamma)(G(R'), \bar{p}') \rightarrow G(R')$).

Since $H \equiv G(R)$, the same formula defines an isomorphism between $(\Delta \circ \Gamma)(H, \bar{p}'')$ and $H$, where $\bar{p}''$ is defined in $H$ by the formula $\varphi$.  

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But the code $\Gamma(\cdot, \varphi)$ defines in $H$ a ring isomorphic to $R'$, therefore this isomorphism is between a group, constructed by $R'$ according to interpretation code $\Delta$, and the group $H$. We know that $\Delta(R') \cong G(R')$, consequently, $G(R') \cong H$, as required.

**Corollary 3.8** The class of Chevalley groups over local rings is elementarily definable, i.e., if $G(R) = G_{\pi}(\Phi, R)$ is a Chevalley group of rank $> 1$, over a local ring $R$ (with $\frac{1}{2}$ for the root systems $A_2$, $B_l$, $C_l$, $F_4$, $G_2$ and with $\frac{1}{3}$ for $G_2$) and for an arbitrary group $H$ we have $H \equiv G(R)$, then $H \equiv G_{\pi}(\Phi, R')$ for some local ring $R'$, which is elementarily equivalent to $R$.

**Proof** By the results of [12], Chevalley groups of different types (over infinite rings) are not elementarily equivalent, i.e., they cannot have different root systems or not isomorphic weight lattices. Therefore we can suppose that we consider the class $\mathcal{S} = \{G(R) = G_{\pi}(\Phi, R) \mid R$ is a local ring $\}$, where $\Phi$ and $\pi$ are fixed.

Since by Theorem 1.19 the class $\mathcal{S}$ satisfies the conditions of Proposition 3.7, by this proposition it is elementarily definable.

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**Conflict of interest** The author declares no competing interests.

**References**

1. Abe, E.: Chevalley groups over local rings. Tohoku Math. J. 21(3), 474–494 (1969)
2. Abe, E.: Normal subgroups of Chevalley groups over commutative rings. In: Stein, M.R., Dennis, R.K. (eds.) Algebraic $K$-Theory and Algebraic Number Theory. Contemporary Mathematics, vol. 83, pp. 1–17. American Mathematical Society, Providence (1989)
3. Abe, E., Suzuki, K.: On normal subgroups of Chevalley groups over commutative rings. Tohoku Math. J. 28(1), 185–198 (1976)
4. Aschenbrenner, M., Khélif, A., Naziazeno, E., Scanlon, T.: The logical complexity of finitely generated commutative rings. Int. Math. Res. Not. IMRN 2020(1), 112–166 (2020)
5. Avni, N., Lubotzky, A., Meiri, C.: First order rigidity of non-uniform higher rank arithmetic groups. Invent. Math. 217(1), 219–240 (2019)
6. Avni, N., Meiri, C.: On the model theory of higher rank arithmetic groups (2020). arXiv:2008.01793
7. Beidar, C.I., Mikhalev, A.V.: On Mal’cev’s theorem on elementary equivalence of linear groups. In: Bokut’, L.A., et al. (eds.) Proceedings of the International Conference on Algebra, Part I. Contemporary Mathematics, vol. 131.1, pp. 29–35. American Mathematical Society, Providence (1992)
8. Bourbaki, N.: Éléments de Mathématique. Fasc. XXXIV. Groupes et Algèbres de Lie. Hermann, Paris (1968)
9. Bragin, V., Bunina, E.: Elementary equivalence of linear groups over rings with a finite number of central idempotents and over Boolean rings. J. Math. Sci. 201, 438–445 (2014)
10. Bunina, E.I.: Elementary equivalence of unitary linear groups over rings and skew fields. Russian Math. Surv. 53(2), 374–376 (1998)
11. Bunina, E.I.: Elementary equivalence of Chevalley groups over fields. J. Math. Sci. 152(2), 155–190 (2008)
12. Bunina, E.I.: Elementary equivalence of Chevalley groups over local rings. Sb. Math. 201(3–4), 321–337 (2010)
13. Bunina, E.I.: Automorphisms of Chevalley groups of different types over commutative rings. J. Algebra 355(1), 154–170 (2012)
14. Bunina, E.I.: Isomorphisms and elementary equivalence of Chevalley groups over commutative rings. Sb. Math. 210(8), 1067–1091 (2019)
15. Bunina, E.I., Mikhalev, A.V.: Combinatorial and logical aspects of linear groups and Chevalley groups. Acta Appl. Math. 85(1–3), 57–74 (2005)
16. Chang, C.C., Keisler, H.J.: Model Theory. Studies in Logic and the Foundations of Mathematics, vol. 73, 3rd edn. North Holland, Amsterdam (1990)
17. Chevalley, C.: Certain schemas des groupes semi-simples. In: Séminaire Bourbaki, vol. 6219, pp. 219–234. Société Mathématique de France, Paris (1995)
18. Cohn, P.M.: On the structure of the GL₂ of a ring. Inst. Hautes Études Sci. Publ. Math. 30, 5–53 (1966)
19. Goncharov, S.S.: Countable Boolean Algebras and Decidability. Siberian School of Algebra and Logic. Consultants Bureau, New York (1997)
20. Hazrat, R., Vavilov, N.: $K_1$ of Chevalley groups are nilpotent. J. Pure Appl. Algebra 179(1–2), 99–116 (2003)
21. Hodges, W.: Model Theory. Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge (1993)
22. Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory. Graduate Texts in Mathematics, vol. 9, Springer, New York (1978)
23. Jarden, M., Lubotzky, A.: Elementary equivalence of profinite groups. Bull. London Math. Soc. 40(5), 887–896 (2008)
24. Keisler, H.J.: Ultraproducts and elementary classes. Indag. Math. 23, 477–495 (1961)
25. Kharlampovich, O., Myasnikov, A.: Elementary theory of free non-abelian groups. J. Algebra 302(2), 451–552 (2006)
26. Kharlampovich, O., Myasnikov, A., Sohrabi, M.: Rich groups, weak second-order logic, and applications. In: Kharlampovich, O., et al. (eds.) Groups and Model Theory: GAGTA Book 2, pp. 127–192. de Gruyter, Berlin (2021)
27. Khelif, A.: Bi-interprétabilité et structures QFA: étude des groupes résolubles et des anneaux commutatifs. C. R. Math. Acad. Sci. Paris 345(2), 59–61 (2007)
28. Kunyavskii, B., Plotkin, E., Vavilov, N.: Bounded generation and commutator width of Chevalley groups: function case (2022). arXiv:2204.10951
29. Lasserre, C.: Polycyclic-by-finite groups and first-order sentences. J. Algebra 396, 18–38 (2013)
30. Maltsev, A.: On isomorphic matrix representations of infinite groups. Rec. Math. [Mat. Sbornik] N. S. 8(50), 405–422 (1940) (in Russian)
31. Maltsev, A.I.: On elementary properties of linear groups. In: Problems of Mathematics and Mechanics, Novosibirsk 1961 (in Russian)
32. Marker, D.: Model Theory. Graduate Texts in Mathematics, vol. 217. Springer, New York (2002)
33. Matsumoto, H.: Sur les sous-groupes arithmétiques des groupes semi-simples déployés. Ann. Sci. École Norm. Sup. 2, 1–62 (1969)
34. Myasnikov, A.G.: Structure of models and criterion of decidability of complete theories of finite-dimensional algebras. Math. USSR-Izv. 34(2), 389–407 (1990)
35. Myasnikov, A., Sohrabi, M.: Bi-interpretablity with $\mathbb{Z}$ and models of the complete elementary theories of $SL_n(O)$, $T_n(O)$, $GL_n(O)$, $n \geq 3$ (2020). arXiv:2004.03585
36. Myasnikov, A., Sohrabi, M.: The Diophantine problem in the classical matrix groups. Izv. Math. 85(6), 1220–1256 (2021)
37. Nies, A.: Separating classes of groups by first-order sentences. Int. J. Algebra Comput. 13(3), 287–302 (2003)
38. Nies, A.: Describing groups. Bull. Symb. Logic 13(3), 305–339 (2007)
39. Oger, F., Sabbagh, G.: Quasi-finitely axiomatizable nilpotent groups. J. Group Theory 9(1), 95–106 (2006)
40. Segal, D., Tent, K.: Defining $R$ and $G(R)$. J. Eur. Math. Soc. (2022). https://doi.org/10.4171/JEMS/1255
41. Sela, Z.: Diophantine geometry over groups. VI. The elementary theory of a free group. Geom. Funct. Anal. 16(3), 707–730 (2016)
42. Shelah, S.: Every two elementarily equivalent models have isomorphic ultrapowers. Israel J. Math. 10, 224–233 (1971)
43. Smolensky, A.: Commutator width of Chevalley groups over rings of stable rank 1. J. Group Theory 22(1), 83–101 (2019)
44. Stein, M.R.: Surjective stability in dimension 0 for $K_2$ and related functors. Trans. Amer. Soc. 178(1), 165–191 (1973)
45. Steinberg, R.: Lectures on Chevalley Groups. Yale University, New Haven (1967)
46. Suslin, A.A.: One theorem of Cohn. J. Soviet Math. 17(2), 1801–1803 (1981)
47. Swan, R.G.: Generators and relations for certain special linear groups. Adv. Math. 6, 1–77 (1971)
48. Szmielew, W.: Elementary properties of Abelian groups. Fund. Math. 41, 203–271 (1955)
49. Taddei, G.: Normalité des groupes élémentaires dans les groupes de Chevalley sur un anneau. In: Bloch, S.J., et al. (eds.) Applications of Algebraic $K$-Theory to Algebraic Geometry and Number Theory, Part I, II. Contemporary Mathematics, vol. 55, pp. 693–710. American Mathematical Society, Providence (1986)
50. Vavilov, N.A.: Structure of Chevalley groups over commutative rings. In: Yamaguti, K., Kawamoto, N. (eds.) Nonassociative Algebras and Related Topics, pp. 219–335. World Scientific, River Edge (1991)
51. Vavilov, N., Plotkin, E.: Chevalley groups over commutative rings. I. Elementary calculations. Acta Appl. Math. 45(1), 73–113 (1996)
52. Vavilov, N.A., Smolensky, A., Sury, B.: Unitriangular factorization of Chevalley groups. J. Math. Sci. (N. Y.) 183(5), 584–599 (2012)
53. Zilber, B.I.: Some model theory of simple algebraic groups over algebraically closed fields. Colloq. Math. 48(2), 173–180 (1984)

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