Quantum Many–Body Problems and Perturbation Theory

Alexander V. Turbiner

Laboratoire de Physique Theorique, Université Paris-Sud, France and
Instituto de Ciencias Nucleares, UNAM, A.P. 70-543, 04510 México

Abstract

We show that the existence of algebraic forms of exactly-solvable $A - B - C - D$ and $G_2, F_4$ Olshanetsky-Perelomov Hamiltonians allow to develop the algebraic perturbation theory, where corrections are computed by pure algebraic means. A classification of perturbations leading to such a perturbation theory based on representation theory of Lie algebras is given. In particular, this scheme admits an explicit study of anharmonic many-body problems. Some examples are presented.

Based on invited talks at XXIII Conference ‘Group-Theoretical Methods in Physics’, 31.7 - 5.8.2000, Dubna, Russia
and
at Workshop ‘Calogero model: thirty years after’, 24-27.5.2001, Rome, Italy

1turbiner@lyre.th.u-psud.fr, turbiner@nuclecu.unam.mx
2On leave of absence from the Institute for Theoretical and Experimental Physics, Moscow 117259, Russia.
INTRODUCTION

Quantum integrable and exactly-solvable many-body problems originated from projection method [1] (see also [2]) and/or the Hamiltonian reduction method [3] serve as a source of inspiration for many years. The goal of this talk is to explore one more feature of these problems – they can be used as zero-approximation or non-perturbed problem in order to develop constructive perturbation theory.

We begin from some preliminary knowledge which is necessary to enter to the subject. Take an infinite set of linear functional spaces \( V_n, \quad n = 0, 1, \ldots \). If they can be ordered

\[
V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots \, \subset V,
\]

then such a construction is called \textit{infinite flag (filtration)} \( V \). A flag is \textit{classical}, if \( \dim V_{n+1} = \dim V_n + 1 \), otherwise it is \textit{non-classical}. If an operator \( T \) such that

\[
T : V_n \mapsto V_n, \quad n = 0, 1, 2, \ldots ,
\]

then it implies \( T \) preserves the flag \( V \).

**General Definition** [4]

An operator \( T \) which preserves an infinite flag of finite-dimensional spaces \( \{ V_k \}_{k \in \mathbb{N}} \) (namely, each space \( V_k \) is invariant to the action of \( T \)) is called \textit{exactly-solvable operator with flag} \( \{ V_k \}_{k \in \mathbb{N}} \).

**Equivalence**

Any two functional spaces \( V_n \) are equivalent if they can be transformed one into another by multiplication on a function and/or by a change of variables.

**Restriction:**

we study linear spaces (and flags) of polynomials only (and equivalent to polynomials).

Let us consider a linear space of polynomials in \( \mathbb{C}^d(\mathbb{R}^d) \)

\[
\mathcal{P}_n^{(f)} = \{ x_1^{p_1}x_2^{p_2}\ldots x_d^{p_d} | 0 \leq \sum \alpha_i p_i \leq n \} , \quad n = 0, 1, 2, \ldots ,
\]

where \( \alpha_i \) are positive integers. We define the vector

\[
\vec{f} = (\alpha_1, \ldots , \alpha_d) ,
\]
which is called characteristic vector. Now one can build a flag
\[ \mathcal{P}_0^{(f)} \subset \mathcal{P}_1^{(f)} \subset \cdots \subset \mathcal{P}_n^{(f)} \subset \cdots, \]
which is called \( \mathcal{P}^{(f)} \). Vector
\[ \vec{f}_0 = (1, 1, \ldots, 1), \]
defines the so-called basic flag \( \mathcal{P}^{(f_0)} \) in \( \mathbb{C}^d(\mathbb{R}^d) \).

Let us consider the \( gl_{d+1} \)-algebra realized by
\[
\begin{align*}
\mathcal{J}_-^i &= \frac{\partial}{\partial x_i}, \quad i = 1, 2 \ldots d, \\
\mathcal{J}_0^i &= x_i \frac{\partial}{\partial x_j}, \quad i, j = 1, 2 \ldots d, \\
\mathcal{J}_0 &= \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i} - n, \\
\mathcal{J}_+^i &= x_i \mathcal{J}_0 = x_i \left( \sum_{j=1}^{d} x_j \frac{\partial}{\partial x_j} - n \right), \quad i = 1, 2 \ldots d,
\end{align*}
\]
where \( n \in \mathbb{C} \). If \( n \) is non-negative integer, this algebra has finite-dimensional representation and its linear space (finite-dimensional representation space) coincides to \( \mathcal{P}_n^{(f_0)} \). Therefore, these finite-dimensional representation spaces as function of \( n \) being properly ordered form flag \( \mathcal{P}^{(f_0)} \). It is obvious that the generators \( \mathcal{J}_-^0, \mathcal{J}_0^0 \), which span maximal affine subalgebra \( b \subset gl_{d+1} \), and their non-linear combinations preserve the flag \( \mathcal{P}^{(f_0)} \).

**Definition:**
The operator \( h \) is called algebraic, if it preserves a flag of polynomials.

It is rather obvious that algebraic operator is characterized by polynomial coefficients, \( \sum \text{Pol}_n \cdot \partial^n \). It can be proven

**THEOREM**

Linear differential operator \( h \) preserves the flag \( \mathcal{P}^{(f_0)} \) iff \( h = P(\mathcal{J}(b \subset gl_{d+1}^{(s)})) \), where \( P \) is a polynomial in generators of the maximal affine subalgebra \( b \) of the algebra \( gl_{d+1} \) taken in realization \( (0.3) \).
In particular, if the second order differential operator $h$ preserves the flag $\mathcal{P}^{(f_0)}$, it should have a form

$$h = P_2^{(ij)}(x)\partial_i \partial_j + P_1^{(i)}(x)\partial_i$$

where $P_2^{(ij)}(x)$ and $P_1^{(i)}(x)$ are the second and first degree polynomials in coordinates $x$’s. It is well-known hypergeometrical operator.

1 **ALGEBRAIC FORMS OF OLSHANETSKY-PERELOMOV HAMILTONIANS**

In this Section we present the algebraic form of the $A_n, BC_n, G_2, F_4$ Olshanetsky-Pereomov Hamiltonians [1, 5]. All of them will be obtained by the same procedure: (i) gauge rotation of the Hamiltonian with ground state eigenfunction and (ii) a change of variables to new variables which code symmetries of the problem. $E_0$ is the ground state energy.

- **Calogero Model ($A_{N-1}$ Rational model)**[3]

  Hamiltonian:

  $$\mathcal{H}_{Cal} = \frac{1}{2} \sum_{i=1}^{N} \left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2\right) + g \sum_{i>j}^{N} \frac{1}{(x_i - x_j)^2}$$

  Ground state:

  $$\Psi_0^{(c)}(x) = \prod_{i<j} |x_i - x_j|^{\nu} e^{-\frac{\nu}{2} \sum x_i^2} , \quad g = \nu(\nu - 1). \quad (1.1)$$

  Here

  $$h_{Cal} = 2(\Psi_0^{(c)})^{-1} (\mathcal{H}_{Cal} - E_0) \Psi_0^{(c)}$$

  New variables:

  $$Y = \sum x_i , \quad y_i = x_i - \frac{1}{N} Y , \quad i = 1, \ldots, N \ , \quad (x_1, x_2, \ldots x_N) \rightarrow (Y, \tau_n(x) = \sigma_n(y(x)) | \ n = (2 \div N)) \ , \quad (1.2)$$
where

\[ \sigma_k(x) = \sum_{i_1<i_2<...<i_k} x_{i_1} x_{i_2} \ldots x_{i_k}, \]

are elementary symmetric polynomials. Finally, the gauge rotated Calogero Hamiltonian (after separation cms)

\[ h_{\text{Cal}} = A_{ij}(\tau) \frac{\partial^2}{\partial \tau_i \partial \tau_j} + B_i(\tau) \frac{\partial}{\partial \tau_i}, \tag{1.3} \]

with

\[ A_{ij} = \frac{(N-i+1)(j-1)}{N} \tau_{i-1} \tau_{j-1} + \sum_{l=\text{max}(1,j-i)}^{j-i-2l} (j-i-2l) \tau_{i+l-1} \tau_{j-l-1}, \]

\[ B_i = -\frac{1}{N} + \nu(N-i+2)N-i+1) \tau_{i-2} + 2\omega \tau_i. \]

- Sutherland model \((A_{N-1} \text{ Trigonometric model}) \[3\]

Hamiltonian

\[ H_{\text{Suth}} = -\frac{1}{2} \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} + \frac{g}{4} \sum_{k<l}^{N} \frac{1}{\sin^2\left(\frac{1}{2}(x_k-x_l)\right)} . \]

Ground state

\[ \Psi_0^{(s)}(x) = \prod_{i<j} \sin^\nu\left(\frac{1}{2}(x_i-x_j)\right) , \quad g = \nu(\nu-1). \tag{1.4} \]

\[ h_{\text{Suth}} = -2(\Psi_0^{(s)})^{-1} (H_{\text{Suth}} - E_0) \Psi_0^{(s)}. \]

New variables

\[ (x_1, x_2, \ldots x_N) \rightarrow (e^{iY}, \eta_n(x) = \sigma_n(e^{iy(x)})|_{n=[1/(N-1)]}) , \tag{1.5} \]

where \(y\)'s are given (1.2).

Finally, the gauge rotated Sutherland Hamiltonian (after separation cms)

\[ h_{\text{Suth}} = A_{ij}(\eta) \frac{\partial^2}{\partial \eta_i \partial \eta_j} + B_i(\eta) \frac{\partial}{\partial \eta_i}, \tag{1.6} \]
with

\[ \mathcal{A}_{ij} = \frac{(N - i)}{N} \eta_i \eta_j + \sum_{l \geq \max(1, j - i)} (j - i - 2l) \eta_{i+l} \eta_{j-l} , \]

\[ \mathcal{B}_i = \left( \frac{1}{N} + \nu \right) i (N - i) \eta_i . \]

- BC\textsubscript{N} –Rational model

Hamiltonian

\[ \mathcal{H}_{BCN}^{(r)} = -\frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial^2}{\partial x_i^2} - \omega^2 x_i^2 \right) + g \sum_{i<j} \left[ \frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right] \]

\[ + \frac{g_2}{2} \sum_{i=1}^{N} \frac{1}{x_i^2} . \]

Ground state

\[ \Psi_0 = \left[ \prod_{i<j} |x_i - x_j|^{\nu} |x_i + x_j|^{\nu} \prod_{i=1}^{N} |x_i|^{\nu_2} \right] e^{-\frac{1}{2} \sum_{i=1}^{N} x_i^2} , \]

\[ g = \nu(\nu - 1) , \quad g_2 = \nu_2(\nu_2 - 1) . \] (1.7)

Here

\[ h_{BCN}^{(r)} = -2(\Psi_0)^{-1} (\mathcal{H}_{BCN}^{(r)} - E_0) \Psi_0 . \]

New variables

\[ (x_1, x_2, \ldots, x_N) \rightarrow (\sigma_k(x^2) \mid k = (1 \div N)) . \] (1.8)

Finally, the gauge rotated BC\textsubscript{N} rational Hamiltonian

\[ \hat{h}_{BCN}^{(r)} = \mathcal{A}_{ij}(\sigma) \frac{\partial^2}{\partial \sigma_i \partial \sigma_j} + \mathcal{B}_i(\sigma) \frac{\partial}{\partial \sigma_i} , \] (1.9)

with

\[ \mathcal{A}_{ij} = 4 \sum_{l \geq 0} (2l + 1 + j - i) \sigma_{i-l-1} \sigma_{j+l} , \]

\[ \mathcal{B}_i = 2 \left[ 1 + \nu_2 + 2\nu(N - i) \right] \left[ N - i + 1 \right] \sigma_{i-1} - 4 \omega_i \sigma_i . \]
• $BC_N$ – Trigonometric model

Hamiltonian:

$$\mathcal{H}_{BC_N}^{(t)} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{g}{4} \sum_{i<j} \left[ \frac{1}{\sin^2 \left( \frac{1}{2}(x_i - x_j) \right)} + \frac{1}{\sin^2 \left( \frac{1}{2}(x_i + x_j) \right)} \right]$$

$$+ \frac{g_2}{4} \sum_{i=1}^{N} \frac{1}{\sin^2 x_i} + \frac{g_3}{4} \sum_{i=1}^{N} \frac{1}{\sin^2 \frac{x_i}{2}} .$$

Ground state:

$$\Psi_0 = \left[ \prod_{i<j} \sin \left( \frac{x_i - x_j}{2} \right) \right]^\nu \sin \left( \frac{x_i + x_j}{2} \right) \prod_{i=1}^{N} \sin (x_i)^\nu_2 \sin \left( \frac{x_i}{2} \right)^\nu_3 ,$$

$$g = \nu(\nu - 1) , \quad g_2 = \nu_2(\nu_2 - 1) , \quad g_3 = \nu_3(\nu_3 + 2\nu_2 - 1) . \quad (1.10)$$

Here

$$h_{BC_N}^{(t)} = -2(\Psi_0)^{-1} (\mathcal{H}_{BC_N}^{(t)} - E_0) \Psi_0 .$$

New variables

$$(x_1, x_2, \ldots x_N) \rightarrow \left( \hat{\sigma}_k(x) = \sigma_k(\cos x) \mid k = (1 \div N) \right) . \quad (1.11)$$

Finally, the gauge rotated $BC_N$ trigonometric Hamiltonian

$$h_{BC_N}^{(t)} = \mathcal{A}_{ij}(\hat{\sigma}) \frac{\partial^2}{\partial \hat{\sigma}_i \partial \hat{\sigma}_j} + \mathcal{B}_i(\hat{\sigma}) \frac{\partial}{\partial \hat{\sigma}_i} , \quad (1.12)$$

with

$$\mathcal{A}_{ij} = N \hat{\sigma}_{i-1} \hat{\sigma}_{j-1} - \sum_{l \geq 0} \left[ (i - l) \hat{\sigma}_{i-l} \hat{\sigma}_{j+l} + (l + j - 1) \hat{\sigma}_{i-l-1} \hat{\sigma}_{j+l-1} 
- (i - 2 - l) \hat{\sigma}_{i-2-l} \hat{\sigma}_{j+l} - (l + j + 1) \hat{\sigma}_{i-l-1} \hat{\sigma}_{j+l+1} \right] ,$$

$$\mathcal{B}_i = \frac{\nu_3}{2} (i - N - 1) \hat{\sigma}_{j-1} - \left[ \nu_2 + \frac{\nu_3}{2} + 1 + \nu(2N - i - 1) \right] \hat{\sigma}_i 
- \nu(N - i + 1)(N - i + 2) \hat{\sigma}_{i-2} \right] .$$
• $G_2$ – Rational model

Hamiltonian:

$$H_{G_2}^{(r)} = -\frac{1}{2} \sum_{i=1}^{3} \left( \frac{\partial^2}{\partial x_i^2} - \omega^2 x_i^2 \right) + g \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

$$+ g_1 \sum_{i<j} \frac{1}{(x_k + x_i - 2x_m)^2} .$$

Ground state

$$\Psi_0 = \prod_{i<j} |x_i - x_j|^\nu \prod_{i<j} |x_i + x_j - 2x_k|^{\mu} e^{-\frac{1}{2} \omega \sum x_i^2} ,$$

$$g = \nu (\nu - 1) > -\frac{1}{4} , \ g_1 = 3\mu (\mu - 1) > -\frac{3}{4} . \quad (1.13)$$

Here

$$h_{G_2}^{(r)} = -2(\Psi_0)^{-1} (H_{G_2}^{(r)} - E_0) \Psi_0 .$$

New variables

$$Y = \sum x_i , \ y_i = x_i - \frac{1}{3} Y , \ i = 1, 2, 3 ,$$

and

$$\lambda_1 = -y_1^2 - y_2^2 - y_1 y_2 , \ \lambda_2 = [y_1 y_2 (y_1 + y_2)]^2 .$$

Finally, the gauge rotated $G_2$ rational Hamiltonian (after separation cms)

$$h_{G_2}^{(r)} = -2\lambda_1 \partial_{\lambda_1 \lambda_1}^2 - 12\lambda_2 \partial_{\lambda_1 \lambda_2}^2 + \frac{8}{3} \lambda_1^2 \lambda_2 \partial_{\lambda_2 \lambda_2}^2$$

$$- \{4\omega \lambda_1 + 2[1 + 3(\mu + \nu)]\} \partial_{\lambda_1} - (12\omega \lambda_2 - \frac{4}{3} \lambda_1^2) \partial_{\lambda_2} . \quad (1.15)$$
\begin{itemize}
    \item $G_2$ – Trigonometric model Hamiltonian
    \[
    \mathcal{H}^{(t)}_{G_2} = -\frac{1}{2} \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2} + \frac{g\alpha^2}{4} \sum_{k<l}^{3} \frac{1}{\sin^2 \left( \frac{\alpha}{2} (x_k - x_l) \right)} \\
    + \frac{g_1\alpha^2}{4} \sum_{k<l, k,l\neq m}^{3} \frac{1}{\sin^2 \left( \frac{\alpha}{2} (x_k + x_l - 2x_m) \right)}
    \]

    Ground state
    \[
    \Psi_0 = \prod_{i<j}^{3} |\sin \frac{\alpha}{2} (x_i - x_j)|^\nu \prod_{k<l, k,l\neq m}^{3} |\sin \frac{\alpha}{2} (x_i + x_j - 2x_k)|^\mu.
    \]
    \[
    g = \nu(\nu - 1) > -\frac{1}{4}, \quad g_1 = 3\mu(\mu - 1) > -\frac{3}{4} \quad (1.16)
    \]
    Here
    \[
    h^{(t)}_{G_2} = -2(\Psi_0)^{-1} (\mathcal{H}^{(t)}_{G_2} - E_0) \Psi_0.
    \]
    New variables
    \[
    Y = \sum x_i, \quad y_1 = x_1 - x_2, \quad y_2 = x_2 - x_3, \quad y_3 = x_3 - x_1, \\
    (x_1, x_2, x_3) \rightarrow (Y, \tilde{\sigma}_1, \tilde{\sigma}_2), \quad (1.17)
    \]
    \[
    \tilde{\sigma}_1 = \frac{1}{\alpha^2} \left[ \cos \alpha(y_1 - y_2) + \cos \alpha(y_2 - y_3) + \cos \alpha(y_3 - y_1) - 3 \right],
    \]
    \[
    \tilde{\sigma}_2 = \frac{4}{\alpha^6} \left[ \sin \alpha(y_1 - y_2) + \sin \alpha(y_2 - y_3) + \sin \alpha(y_3 - y_1) \right]^2.
    \]
    Finally, the gauge rotated $G_2$ trigonometric Hamiltonian (after separation cms)
    \[
    h^{(t)}_{G_2} = -(2\tilde{\sigma}_1 + \frac{\alpha^2}{2}\tilde{\sigma}_1^2 - \frac{\alpha^4}{24}\tilde{\sigma}_2)\tilde{\sigma}_1^2 - (12 + \frac{8\alpha^2}{3}\tilde{\sigma}_1)\tilde{\sigma}_2\tilde{\sigma}_1^2 + \ldots
    \]
\end{itemize}
\[ + \left( \frac{8}{3} \sigma_1^2 \sigma_2 - 2 \alpha^2 \sigma_2^2 \right) \partial_{\sigma_2}^2 - \left\{ 2[1 + 3(\mu + 2\nu)] + \frac{2}{3} (1 + 3\mu + 4\nu) \alpha^2 \sigma_1 \right\} \partial_{\sigma_1} + \]
\[ \left\{ \frac{4}{3} (1 + 4\nu) \sigma_1^2 - \left[ \frac{7}{3} + 4(\mu + \nu) \right] \alpha^2 \sigma_2 \right\} \partial_{\sigma_2} . \]

\[ (1.18) \]

• \( F_4 \)-Rational model \[ \]

Hamiltonian
\[ \mathcal{H}_{F_4}^{(r)} = \frac{1}{2} \sum_{i=1}^{4} \left( - \partial_{x_i}^2 + 4\omega^2 x_i^2 \right) + 2g \sum_{j>i} \left( \frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right) \]
\[ + 2g_1 \sum_{i=1}^{4} \frac{1}{x_i^2} + 8g_1 \sum_{\nu',s=0,1} \frac{1}{x_1 + (-1)^{\nu_2} x_2 + (-1)^{\nu_3} x_3 + (-1)^{\nu_4} x_4} . \]

Ground state
\[ \Psi_0^{(r)}(x) = (\Delta_- \Delta_+)^{\nu} (\Delta_0 \Delta)^{\mu} \exp \left( -\omega \sum_{i=1}^{4} x_i^2 \right) , \]
\[ g = \nu(\nu - 1)/2 , \quad g_1 = \mu(\mu - 1) , \]

with
\[ \Delta_+ = \prod_{j<i} (x_i \pm x_j) , \]
\[ \Delta_0 = \prod_{i=1}^{4} x_i , \]
\[ \Delta = \prod_{\nu',s=0,1} \left[ x_1 + (-1)^{\nu_2} x_2 + (-1)^{\nu_3} x_3 + (-1)^{\nu_4} x_4 \right] . \]

New variables
\[ (x_1, x_2, x_3, x_4) \rightarrow (t_1, t_3, t_4, t_6) , \]

\[ (1.20) \]
where

\[ t_1 = \sigma_1 , \]
\[ t_3 = \sigma_3 - \frac{1}{6} \sigma_1 \sigma_2 , \]
\[ t_4 = \sigma_4 - \frac{1}{4} \sigma_1 \sigma_3 + \frac{1}{12} \sigma_2^2 , \]
\[ t_6 = \sigma_4 \sigma_2 - \frac{1}{36} \sigma_2^3 - \frac{3}{8} \sigma_3^2 + \frac{1}{8} \sigma_1 \sigma_2 \sigma_3 - \frac{3}{8} \sigma_1^2 \sigma_4 . \]

and \( \sigma_a = \sigma_a(x^2) \).

Finally, gauge rotated \( F_4 \) rational Hamiltonian

\[ h^{(r)}_{F_4} = A_{ab} \frac{\partial^2}{\partial t_a \partial t_b} + (B_a + C_a) \frac{\partial}{\partial t_a} , \tag{1.21} \]

with

\[
A_{11} = 4 t_1 , \quad A_{13} = 12 t_3 ,
\]
\[
A_{14} = 16 t_4 , \quad A_{16} = 24 t_6 ,
\]
\[
A_{33} = -\frac{2}{3} t_1^2 t_3 + \frac{20}{3} t_1 t_4 , \quad A_{34} = -\frac{4}{3} t_1^2 t_4 + 8 t_6 ,
\]
\[
A_{36} = 16 t_4^2 - 2 t_1^2 t_6 , \quad A_{44} = -4 t_3 t_4 - 2 t_1 t_6 ,
\]
\[
A_{46} = -4 t_1 t_4^2 - 6 t_3 t_6 , \quad A_{66} = -12 t_3 t_4^2 - 6 t_1 t_4 t_6 ,
\]
\[
A_{ba} = A_{ab} ,
\]
\[
B_1 = 8 , \quad B_3 = -t_1^2 ,
\]
\[
B_4 = -4 t_3 , \quad B_6 = -8 t_1 t_4 .
\]
\[
C_1 = 48(\nu + \mu) - 4\omega t_1 , \quad C_3 = -2(2\nu + \mu) t_1^2 - 12\omega t_3 ,
\]
\[
C_4 = -12\nu t_3 - 16\omega t_4 , \quad C_6 = -12\nu t_1 t_4 - 24\omega t_6 .
\]
• *F*-Trigonometric model \([9]\) Hamiltonian

\[ \mathcal{H}_{F_4}^{(t)}(x) = -\frac{1}{2} \sum_{i=1}^{4} \partial_{x_i}^2 + 2gV_1(x, \beta) + \frac{g_1}{2} V_2(x, 2\beta), \]  

(1.22)

where \( g = \nu(\nu - 1)/2, \) \( g_1 = \mu(\mu - 1), \) and

\[ V_1(x, \beta) = \beta^2 \sum_{j>i} \left( \frac{1}{\sin^2 \beta(x_i - x_j)} + \frac{1}{\sin^2 \beta(x_i + x_j)} \right), \]

\[ V_2(x, 2\beta) = 2\beta^2 \sum_{i=1}^{4} \frac{1}{\sin^2 2\beta x_i} \]

\[ + 4\beta^2 \sum_{\nu' s=0,1} \frac{1}{\sin^2 \beta \left[ x_1 + (-1)^{\nu_s} x_2 + (-1)^{\nu_3} x_3 + (-1)^{\nu_4} x_4 \right]}. \]

Ground state

\[ \Psi_0^{(t)}(x, \beta) = (\Delta_+ (x, \beta) \Delta_-(x, \beta))^\nu (\Delta_0(x, 2\beta) \Delta(x, 2\beta))^\mu, \]  

(1.23)

where

\[ \Delta_+(x, \beta) = \beta^{-6} \prod_{j<i} \sin \beta(x_i \pm x_j), \]

\[ \Delta_0(x, 2\beta) = \beta^{-4} \prod_i \sin 2\beta x_i, \]

\[ \Delta(x, 2\beta) = \beta^{-8} \prod_{\nu' s=0,1} \sin \beta \left[ x_1 + (-1)^{\nu_s} x_2 + (-1)^{\nu_3} x_3 + (-1)^{\nu_4} x_4 \right]. \]

Here

\[ h_{F_4}^{(t)} = -2(\Psi_0^{(t)}(x))^{-1}(\mathcal{H}_{F_4}^{(t)} - E_0)(\Psi_0^{(t)}(x)). \]

New variables

\[ (x_1, x_2, x_3, x_4) \rightarrow (\tau_1, \tau_3, \tau_4, \tau_6), \]  

(1.24)

where

\[ \tau_1 = \sigma_1 - \frac{2\beta^2}{3}\sigma_2. \]
\begin{align*}
\tau_3 &= \sigma_3 - \frac{1}{6} \sigma_1 \sigma_2 - 2\beta^2 (\sigma_4 - \frac{1}{36} \sigma_2^2), \\
\tau_4 &= \sigma_4 - \frac{1}{4} \sigma_1 \sigma_3 + \frac{1}{12} \sigma_3^2, \\
\tau_6 &= \sigma_4 \sigma_2 - \frac{1}{36} \sigma_2^3 - \frac{3}{8} \sigma_3^2 + \frac{1}{8} \sigma_1 \sigma_2 \sigma_3 - \frac{3}{8} \sigma_1^2 \sigma_4. 
\end{align*}
\quad (1.25)

and \(\sigma_a = \sigma_a(y^2), \; y_i = \frac{\sin(\beta x_i)}{\beta}.\)

Finally, the gauge-rotated \(F_4\) trigonometric Hamiltonian

\[
h_{F_4}^{(t)} = A_{ab} \frac{\partial^2}{\partial \tau_a \partial \tau_b} + (B_a + C_a) \frac{\partial}{\partial \tau_a}, \quad a, b = 1, 3, 4, 6, \quad (1.26)
\]

where the coefficient functions are

\[
\begin{align*}
A_{11} &= 4 \tau_1 - 4\beta^2 \tau_1^2 - \frac{32}{3} \beta^4 \tau_3 - \frac{128}{9} \beta^6 \tau_4, \\
A_{13} &= 12 \tau_3 - \frac{8}{3} \beta^2 (4 \tau_1 \tau_3 + \tau_4) - \frac{32}{9} \beta^4 \tau_4, \\
A_{14} &= 16 \tau_4 - \frac{40}{3} \beta^2 \tau_1 \tau_4 - \frac{16}{3} \beta^4 \tau_6, \\
A_{16} &= 24 \tau_6 - 20 \beta^2 \tau_1 \tau_6 - \frac{32}{3} \beta^4 \tau_4^2, \\
A_{33} &= -\frac{2}{3} \tau_1^2 \tau_3 + \frac{20}{3} \tau_1 \tau_4 - \frac{8}{9} \beta^2 (18 \tau_3^2 + \tau_1^2 \tau_4 + 12 \tau_6), \\
A_{34} &= -\frac{4}{3} \tau_1^2 \tau_4 + 8 \tau_6 - \frac{4}{3} \beta^2 (\tau_1 \tau_6 + 12 \tau_3 \tau_4), \\
A_{36} &= 16 \tau_4^2 - 2 \tau_1^2 \tau_6 - \frac{8}{3} \beta^2 (9 \tau_3 \tau_6 + \tau_1 \tau_4^2), \\
A_{44} &= -4 \tau_3 \tau_4 - 2 \tau_1 \tau_6 - 24 \beta^2 \tau_4^2, \\
A_{46} &= -4 \tau_1 \tau_4^2 - 6 \tau_3 \tau_6 - 36 \beta^2 \tau_4 \tau_6, \\
A_{66} &= -12 \tau_3 \tau_4^2 - 6 \tau_1 \tau_4 \tau_6 - 8 \beta^2 (6 \tau_6^2 + \tau_4^3), \\
A_{a b} &= A_{a b}, \\
B_1 &= 8 - 8\beta^2 \tau_1, \quad B_3 = -\tau_1^2 - \frac{56}{3} \beta^2 \tau_3 - \frac{32}{9} \beta^4 \tau_4.
\end{align*}
\]
\[ B_4 = -4 \tau_3 - \frac{88}{3} \beta^2 \tau_4 , \quad B_6 = -8 \tau_1 \tau_4 - 56 \beta^2 \tau_6 . \]

\[ C_1 = 48(\nu + \mu) - 8\beta^2(5\nu + 6\mu) \tau_1 , \quad C_3 = -2(2\nu + \mu) \tau_1^2 - 16\beta^2(3\nu + 5\mu) \tau_3 , \]
\[ C_4 = -12\nu \tau_3 - 24\beta^2(3\nu + 4\mu) \tau_4 , \quad C_6 = -12\nu \tau_1 \tau_4 - 48\beta^2(2\nu + 3\mu) \tau_6 . \]

Remarks and Comments

- \( A_N - \) and \( BC_N - \) rational and trigonometric models possess algebraic forms; their Hamiltonians (1.3), (1.6), (1.9), (1.12) preserve the same basic flag of polynomials \( \mathcal{P}(f) \).

- All \( A_N - \) and \( BC_N - \) rational and trigonometric Hamiltonians taken in algebraic form can be written as
  \[ h = P_2(J(b \subset gl_{N+1})) \]
  where \( P_2 \) is a polynomial of second degree in the generators \( J \) of the maximal affine subalgebra of the algebra \( gl_{N+1} \) in realization (0.3). One can state that \( gl_{N+1} \) is their hidden algebra.

- Both rational and trigonometric \( G_2 \) models possess algebraic forms; their Hamiltonians preserve the same flag of polynomials \( \mathcal{P}(f_{G_2}) \) with \( f_{G_2} = (1, 2) \); their hidden algebras coincide and it is some infinite-dimensional, finitely-generated algebra \( g^{(2)} \subset \text{diff}(\mathbb{C}^2) \) (see [8]).

- Both rational and trigonometric \( F_4 \) models possess algebraic forms; their Hamiltonians preserve the same flag of polynomials \( \mathcal{P}(f_{F_4}) \) with \( f_{F_4} = (1, 2, 2, 3) \); their hidden algebras coincide and it is some infinite-dimensional, finitely-generated algebra \( f^{(4)} \subset \text{diff}(\mathbb{C}^4) \) (see [9]).

- New variables (1.12), (1.15), (1.18), (1.11), (1.14), (1.17), (1.20), (1.24), in which the algebraic forms occur, usually absorb all external symmetries of model under investigation; they have a meaning of rational and trigonometric invariants in the corresponding root space; to the best of our knowledge they were used for the first time to find flat space metrics (denoted by \( A \) in \( A - B - C - D \) and \( F_4 \) examples) in rational case by V.I. Arnold [10], we will call these metrics \( A \) the Arnold metrics.

- Although the question about existence of the algebraic forms for rational and trigonometric \( E_{6,7,8} \) models was not constructively studied yet, there are almost no doubts that they should exist.
2

PERTURBATION THEORY

Existence of algebraic forms leads to a possibility to construct a special, algebraic perturbation theory – a type of perturbation theory where finding corrections is an algebraic procedure and furthermore any correction has a form of finite-order polynomial in coordinates.

Consider the spectral problem,

\[(T_0 + \lambda T_1)\phi = E\phi,\]

where \(\lambda\) is a formal parameter, and let us develop perturbation theory:

\[\phi = \sum \lambda^k \phi_k, \quad E = \sum \lambda^k E_k.\]

(2.2)

Then the following theorem holds:

**THEOREM**

Let \(T_0\) be an exactly-solvable operator with flag \(\{V_k\}_{k \in \mathbb{N}}\). Let the perturbation \(T_1\) is such that \(T_1\) is an element of space \(V_n\) from the flag and we look for \(\phi \in V\). Then the perturbation theory is algebraic: \(\exists p(k)\) such that \(k\)-th correction \(\phi_k \in V_{p(k)}\) and hence it can be found by algebraic means.

The proof is quite straightforward and is based on analysis of the equation for \(k\)th correction

\[(T_0 - E_0)\phi_k = \sum_{i=1}^{k} E_i \phi_{k-i} - T_1 \phi_{k-1}.\]

We can proceed to examples.

**Example 1.** One-dimensional Anharmonic Oscillator.

It is characterized by the Hamiltonian

\[\mathcal{H} = -\frac{1}{2} \frac{\partial^2}{\partial y^2} + \omega^2 y^2 + \frac{g}{y^2} + \lambda y^4.\]

(2.3)

\[\text{A}_1 - \text{Calogero model}\]
Ground state

\[ \psi_0 = y^\nu e^{-\frac{\omega}{2} y^2}, \quad g = \nu(\nu - 1) \], \quad E_0 = \omega(1 + 2\nu). \quad (2.4) \]

In new variable

\[ \tau = y^2, \]

the gauge-rotated Hamiltonian

\[ h = \frac{1}{\omega} \psi_0^{-1}(\mathcal{H} - E_0)\psi_0 = \]

\[ -2\tau \partial_\tau^2 + 2(\tau - \mu)\partial_\tau + \lambda \tau^2 \equiv T_0 + \lambda T_1, \]

where \( \mu \equiv \nu + 1/2 \). It is easy to check that

\[ T_0 : \mathcal{P}_n \mapsto \mathcal{P}_n, \quad E_0^{(n)} = 2n, \quad n = 0, 1, 2, \ldots, \]

\[ T_1 = \tau^2 \in \mathcal{P}_{2,3,...} \]

where \( \mathcal{P} \) is basic flag of polynomials in \( \mathbb{C} \) (see (0.1)).

1. Ground state:

   Now the ground state of \( T_0 \) is is given by \( \phi_0^{(0)} = 1 \), \( E_0^{(0)} = 0 \) and the

   First correction:

   Defining equation is

   \[ -2\tau \partial_\tau^2 \phi_1^{(0)} + 2(\tau - \mu)\partial_\tau \phi_1^{(0)} = E_1^{(0)} - \tau^2, \]

   with a solution

   \[ - \phi_1^{(0)} = \frac{1}{4} \tau^2 + \frac{\mu + 1}{2} \tau, \quad (2.5) \]

   \[ E_1^{(0)} = \mu(\mu + 1). \quad (2.6) \]

   Second correction:

   \[ -2\tau \partial_\tau^2 \phi_2^{(0)} + 2(\tau - \mu)\partial_\tau \phi_2^{(0)} = E_2^{(0)} + E_1^{(0)} \phi_1^{(0)} - \tau^2 \phi_1^{(0)}. \]
\[ \phi_{k}^{(0)} = a_{2k} \tau^{2k} + a_{2k-1} \tau^{2k-1} + \ldots + a_{2k-m} \tau^{2k-m} + \ldots . \]

Coefficients in front of leading terms can be found explicitly for any excited state(!) - they are generalized Catalan numbers of a form

\[ a_{2k-m} \sim \frac{(2k)!}{k!(k-m/2)!} \cdot \]

In standard Rayleigh-Schroedinger Perturbation Theory (RSPT) the first energy correction \( E_{1}^{(0)} = \langle 0 | T_{1} | 0 \rangle / \langle 0 | 0 \rangle \), hence

\[ E_{1}^{(0)} = \frac{\langle 0 | y^{4} | 0 \rangle}{\langle 0 | 0 \rangle} = \mu(\mu + 1) \]

therefore we can find the expectation value \( \langle 0 | y^{4} | 0 \rangle \) \textit{algebraically} (up to known normalization factor (see e.g. [3]). A comparison of other corrections in present perturbation theory and RSPT allows to find algebraically transition amplitudes between different states (correlation functions).

2. First Excited State: \( \phi_{1}^{(1)} = \tau - \mu, \quad E_{1}^{(1)} = 2 \)

First correction:

Defining equation

\[ -2\tau \partial_{\tau}^{2} \phi_{1}^{(1)} + 2(\tau - \mu) \partial_{\tau} \phi_{1}^{(1)} - 2\phi_{1}^{(1)} = (E_{1}^{(1)} - \tau^{2})(\tau - \mu) , \]

and the correction

\[ -\phi_{1}^{(1)} = \frac{1}{4}[\tau^{3} - (\mu - 3)\tau^{2} + 2(\mu + 1)(\mu - 3)\tau] , \quad (2.7) \]

\[ E_{1}^{(1)} = - (\mu + 1)(\mu - 3) . \quad (2.8) \]
It is worth to note that the developed perturbation theory in present example coincides to the so-called Dalgarno-Lewis form of perturbation theory \[11\]. In fact, it was namely this form of perturbation theory which was successfully applied by Bender and Wu \[12\] in their profound study of the problem (2.3) at \( g = 0 \).

**Example 2.** \((N - 1)\)-dimensional Anharmonic Oscillator.

Consider the following perturbed \( N \)-body Calogero model

\[
\mathcal{H} = \mathcal{H}_{Cal} + \lambda \tau_4(x), \quad N > 4
\]

\[
\tau_4(x) = \sigma_4(y) = \sum_{i_1,i_2,i_3,i_4} y_{i_1} y_{i_2} y_{i_3} y_{i_4}
\]

\[
h = h_{Cal} + \lambda \tau_4 \equiv T_0 + \lambda T_1
\]

\[
T_0 : \mathcal{P}_n^{(N-1)}(\tau) \mapsto \mathcal{P}_n^{(N-1)}(\tau), \quad n \in \mathbb{N},
\]

\[
T_1 = \tau_4 \in \mathcal{P}_1^{(N-1)},
\]

Ground State is given by

\[
\phi_0^{(0)} = 1, \quad E_0^{(1)} = 0 .
\]

**First correction:**

\[
-\phi_1^{(0)} = \frac{1}{8\omega} \tau_4 + \frac{1}{32\omega^2} \left( \frac{1}{N} + \nu \right) (N - 2)(N - 3) \tau_2
\]

\[
E_1^{(0)} = \frac{1}{32\omega^2} \left( \frac{1}{N} + \nu \right)^2 \frac{N!}{(N - 4)!}
\]

Again we can find expectation value algebraically (up to known normalization factor).

\[
E_1^{(0)} = \frac{\langle 0 | \tau_4(y) | 0 \rangle}{\langle 0 | 0 \rangle}.
\]
Second correction is of the form

\[ \phi_2 = \alpha_1 \tau_2^2 + \alpha_2 \tau_3^2 + \alpha_3 \tau_4^2 + \alpha_4 \tau_2 \tau_4 + \beta_1 \tau_2 + \beta_2 \tau_4 + \beta_3 \tau_6 , \]

where the coefficients \( \alpha \)'s and \( \beta \)'s can be easily computed.

**Example 3.** Perturbed 3-body Sutherland Model.

Take

\[ \mathcal{H} = \mathcal{H}^{(3)}_{\text{Suth}} + \lambda \eta_2 \]

where \( \mathcal{H}^{(3)}_{\text{Suth}} \) is the Hamiltonian of 3-body Sutherland model. Gauging away the ground state (1.4) and introducing new variables

\[ \eta_2 = \frac{1}{\alpha^2} \left[ \cos(\alpha y_1) + \cos(\alpha y_2) + \cos(\alpha (y_1 + y_2)) - 3 \right] , \]

\[ \eta_3 = \frac{2}{\alpha^3} \left[ \sin(\alpha y_1) + \sin(\alpha y_2) - \sin(\alpha (y_1 + y_2)) \right] , \]

(cf. (1.5)) we get an algebraic form:

\[ h = h_{\text{Suth}} + \lambda \eta_2 \equiv T_0 + \lambda T_1 , \]

where

\[ h_{\text{Suth}} = -(2 \eta_2 + \frac{\alpha^2}{2} \eta_2^2 - \frac{\alpha^4}{24} \eta_5^2) \partial^2_{\eta_2 \eta_2} - (6 + 4 \eta_2 + \frac{2}{3} \eta_3) \partial^2_{\eta_2 \eta_3} + (\frac{2}{3} \eta_2 - \alpha^2 \eta_3) \partial^2_{\eta_3 \eta_3} + 2(\nu + \frac{1}{3})(3 + \alpha^2 \eta_2) \partial \eta_2 + 2(\nu + \frac{1}{3}) \alpha^2 \eta_3 \partial \eta_3 \]

\[ T_0 : \mathcal{P}^{(2)}_n(\eta) \mapsto \mathcal{P}^{(2)}_n(\eta), \quad n \in \mathbb{N}, \]

\[ T_1 = \eta_2 \in \mathcal{P}^{(2)}_{1,2,3,\ldots} , \]

Ground State: \( \phi_0 = 1, \quad E_0 = 0 \)

First correction:

\[ -\phi_1 = \frac{3}{2(1 + 3\nu)\alpha^2} \eta_2 , \]
\[ E_1 = -\frac{3}{\alpha^2}. \]

Since
\[ E_1 = \frac{\langle 0|\eta_2(y)|0 \rangle}{\langle 0|0 \rangle} \]
we can find expectation value \( \langle 0|\eta_2(y)|0 \rangle \) \textit{algebraically} using known normalization factor \( \langle 0|0 \rangle \) \[5\].

\textit{Second correction:}
\[ -\phi_2 = \frac{3}{8\alpha^4(1+3\nu)(1+6\nu)} \left[ (1+12\nu) \eta_2^2 + \frac{1}{4} \eta_3^2 + \frac{9(2+13\nu+12\nu^2)}{(1+3\nu)} \eta_2 \right], \]
\[ E_2 = -\frac{27}{4\alpha^4} \frac{2+13\nu+12\nu^2}{(1+3\nu)(1+6\nu)}. \]

3 \textbf{CONCLUSION}

Algebraic forms of Calogero-Sutherland models give an opportunity study their \textit{perturbations} by algebraic means through developing a perturbation theory for single state.

Taking different perturbations and making comparison of present perturbation theory with standard Rayleigh-Schroedinger perturbation theory allow to calculate correlation functions for Calogero-Sutherland models algebraically.

Algebraic forms of Calogero-Sutherland models allow to build their Fock space representation (see \[13\]) and then develop algebraic perturbation theory in Fock space. It gives a chance to study isospectral discretizations of Calogero-Sutherland models (on different lattices) and their perturbations \[14\].

\textbf{References}

[1] M. A. Olshanetsky and A. M. Perelomov, “Quantum completely integrable systems connected with semi-simple Lie algebras,” \textit{Lett.Math.Phys.} 2 (1977) 7–13.
[2] A.Yu. Morozov, A.M. Perelomov, A. Rosly, M.A. Shifman and A.V. Turbiner, “Quasi-Exactly-Solvable Problems: One-Dimensional Analogue of Rational Conformal Field Theories”, Intern. Journ. Mod. Phys. A5, 803-843 (1990)

[3] D. Kazhdan, B. Kostant, and S. Sternberg, “Hamiltonian group actions and dynamical systems of Calogero type,” Commun. Pure Appl. Math. 31 (1978) 481–507.

[4] A.V. Turbiner, “Lie algebras and linear operators with invariant subspace”, in Lie algebras, cohomologies and new findings in quantum mechanics (N. Kamran and P. J. Olver, eds.), AMS Contemporary Mathematics, vol. 160, pp. 263–310, 1994; [funct-an/9301001] “Lie-algebras and Quasi-exactly-solvable Differential Equations”, in CRC Handbook of Lie Group Analysis of Differential Equations, Vol.3: New Trends in Theoretical Developments and Computational Methods, Chapter 12, CRC Press (N. Ibragimov, ed.), pp. 331-366, 1995 [hep-th/9409068]

[5] M. A. Olshanetsky and A. M. Perelomov, “Quantum integrable systems related to Lie algebras”, Phys. Reprs. 94 (1983) 313

[6] W. Rühl and A. V. Turbiner, “Exact solvability of the Calogero and Sutherland models”, Mod.Phys.Lett. A10 (1995) 2213–2222 [hep-th/9506105]

[7] L. Brink, A. Turbiner and N. Wyllard, “Hidden Algebras of the (super) Calogero and Sutherland models”, Journ.Math.Phys. 39 (1998) 1285-1315 [hep-th/9705219]

[8] M. Rosenbaum, A. Turbiner and A. Capella, “Solvability of the $G_2$ integrable system”, Intern. Journ. Mod. Phys. A13, (1998) 3885-3904 [solv-int/9707005]

[9] K.G. Boreskov, J.-C. Lopez V., A.V. Turbiner, “Solvability of $F_4$ integrable models”,
[10] V.I. Arnold, “Wave front evolution and equivariant Morse lemma”,
Comm.Pure Appl. Math. 29 (1976) 557-582

[11] A. Dalgarno and J.T. Lewis, “The exact calculation of long-range forces between atoms by perturbation theory”,
Proc. Royal Soc. A 233 (1955) 70-74

[12] C.M. Bender and T.T. Wu, “Anharmonic oscillator”,
Phys. Rev. 184 1231-1260 (1969)

[13] A. Turbiner, “Lie algebras in Fock space”,
(q-alg/9710012)
“Operator theory: Advances and Applications”, v.114,
‘Complex Analysis and Related Topics’, pp.265-284,
Birkhauser-Verlag, Basel-Boston-Berlin (1999),
E. Ramirez, M.V. Shapiro, L.M. Tovar, N. Vasilevski (Eds)

[14] A. Turbiner, “Canonical discretization. I. Discrete faces of (an)harmonic oscillator”,
Intern.Journ.Mod.Phys. A16, 1579-1605 (2001)