Review properties solutions the mathematical models of transition time traffic congestion

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Abstract. Mathematical model of transition time traffic congestion is a mathematical model modified from Lorenz system that have three variable i.e. deviation distance of vehicles observed with the optimal distance of vehicles, vehicles speed deviation observed with optimum speed and acceleration/braking time vehicles so that the optimum speed reached. In this article, the behavior or properties solutions of mathematical model of transition time traffic congestion are observed and the result are: 1) this model has 1 stable equilibrium point for \( \tau_0 \leq 1 \), while for \( \tau_0 > 1 \) there are 2 stable equilibrium points and 1 unstable equilibrium point, where \( \tau_0 \) is characteristic time to reached optimum velocity. 2) there is a bifurcation pitchfork with bifurcation value when \( \tau_0 \) varied. 3) solution system in geometric symmetric with one of axis. 4) the solution is bounded.

1. Introduction
Traffic congestion is a common problem in some regions in Indonesia. Congestion occurs if the volume of flow of the vehicle exceed the capacity of the largest acceptable on a path [1]. Traffic congestion not only inhibits the activity of the mobility of society but also raises the issues of the effectiveness of the mobility of the time. Variety of ways done to address the congestion problem from a variety of disciplines, including mathematics. The congestion problem can be modeled into a mathematical equation, then solved and simulated with real data. Variables used in the model traffic congestion are the speed of the vehicle, acceleration/braking time vehicle and variation of the distance between the vehicles.

Researches on mathematical model of traffic congestion has been done by some experts in the disciplines of mathematics, for example mathematical modeling in kinetic congestion pointed out by Nagel and Schreckenberg [2]. Research on the congestion is also done by Khomenko et al [3] which analyzes phase/time transition of traffic jams by defining factors of a complex network congestion. Thermodynamic theory used to describe the phase transition in the traffic flow of traffic congestion [4] or stochastic theory approach to describe the traffic flow [5].

Mathematical model of the transition time traffic congestion in the form of Lorenz system express by Khomenko et al [6]. The model is also on the research conducted by Ganji et al [1]. On the research of Ganji et al, discussed about how to solve a mathematical model of transition time congestion using Difference Transform Method and the numerical solutions with RK4. One of the research on the application of the mathematical model about the traffic congestion with Lorenz system [7].
Lorenz system itself was originally a mathematical model relating about the condition of the Earth's atmosphere, i.e. the occurrence of change of heat and cold in the Earth's atmosphere. Lorenz system has unique properties which different from others mathematic model system, i.e. the dynamic behavior and the sensitivity towards initial value. Not a few researchers who examined the Lorenz system because of the uniquely as expressed by Shakhawat and Prayeer Ahmed [8] about chaos that occurs in Lorenz system also compares the dynamic behavior of the Lorenz system in complex variables, in addition researchers Loong Zoon and Zabidin Salleh [9] performs the modifications to the Lorenz system and discovered that the properties of the Lorenz system are still valid for modifies Lorenz system.

Furthermore, this article examined about mathematical model of transition time traffic congestion as well as traits that emerge from the solution of the system as seen from the stability of equilibrium points and the geometries form.

2. Mathematical model of transition time traffic congestion
Mathematical model of transition time traffic congestion in this paper based on the Lorenz system without fluctuation of acceleration [10]:

\[
\begin{align*}
\dot{\mu} &= -\mu + v \\
\dot{v} &= -v + \tau \mu \\
\dot{\tau} &= \frac{\tau_0 - \tau - \mu v}{\sigma}
\end{align*}
\] (1)

With:
- \(\mu\): the headway deviation
- \(v\): the velocity deviation
- \(\tau\): vehicle time to reached optimum velocity
- \(\tau_0\): characteristic time to reached optimum velocity
- \(\alpha\): ratio of relaxation time reach deviation velocity equal 0 and relaxation time reach deviation headway equal 0
- \(\sigma\): ratio of relaxation time deviation acceleration equal \(\tau_0\) and relaxation time reach deviation headway equal 0
- \(t_\nu\): relaxation time to reached the deviation velocity equal 0
- \(t_\mu\): relaxation time to reached the deviation velocity equal 0
- \(t_\tau\): relaxation time to reached the deviation acceleration equal \(\tau_0\).

The first derivative of \(\mu\) with respect to \(t\) (\(\dot{\mu}\)) is the rate of change of distance between the vehicle's distance deviation observed with optimal vehicle distance against the stationer position that is at the moment \(\mu = 0\). The first derivative of \(v\) with respect to \(t\) (\(\dot{v}\)) is the rate of change of vehicle speed deviation against the stationer speed, namely at a time when \(v = 0\). The first derivative of \(\tau\) with respect to \(t\) (\(\dot{\tau}\)) is the rate of change of time acceleration/braking time against the stationer braking/acceleration time i.e. at the time \(\tau = \tau_0\).

2.1. The stability of equilibrium point and bifurcation
System (1) has three equilibrium point are:

\[
\begin{align*}
\bar{u}_1 &= (0, 0, \tau_0) \\
\bar{u}_2 &= (\sqrt{\tau_0 - 1}, \sqrt{\tau_0 - 1}, 1) \\
\bar{u}_3 &= (-\sqrt{\tau_0 - 1}, -\sqrt{\tau_0 - 1}, 1)
\end{align*}
\] (2)
It can be seen that the value of $\tau_0$ affect the amount of point equilibrium, because for $\tau_0 < 1$ then there is only one equilibrium point, i.e. $\bar{u}_1 = (0, 0, \tau_0)$ and for $\tau_0 > 1$ then there are three equilibrium points. System (1) is a nonlinear system so that upon Perko [11], the stability of the equilibrium point can be approached by the its linearization i.e. can be seen from the eigenvalues of the Jacobian matrix of each equilibrium point.

**Jacobian** matrix for the equilibrium point $\bar{u}_1 = (0, 0, \tau_0)$:

\[
J_{\bar{u}_1} = \begin{bmatrix}
-1 & 1 & 0 \\
\frac{\tau_0}{\alpha} - 1 & -\frac{1}{\alpha} & 0 \\
0 & 0 & -\frac{1}{\sigma}
\end{bmatrix}.
\]  

(3)

So the characteristic equation is obtained

\[
\left(-\frac{1}{\sigma} - \lambda\right)\left(-\frac{1}{\alpha} - \lambda\right)\left(-1 - \lambda\right) - \frac{\tau_0}{\alpha} = 0.
\]  

(4)

When $\tau_0 < 1$, all of the eigenvalues is negative so $\bar{u}_1$ is asymptotic stable whereas for $\tau_0 > 1$, can be used Routh-Hurwitz criterion to let easier. Table 1. is the table of Routh-Hurwitz from equation (4):

**Table 1.** Table Routh-Hurwitz $\bar{u}_1 = (0, 0, \tau_0)$ for $\tau_0 > 1$

|  $\frac{1}{\alpha\sigma}$ | $\frac{\alpha + \sigma + 1 + \tau_0\sigma}{\alpha\sigma}$ | $\frac{1}{\alpha\sigma(1 - \tau_0)}$ |
|---------------------------|-------------------------------------------------|----------------------------------|
|                           | $\frac{(a + \sigma + 1 + \tau_0\sigma)}{\alpha\sigma}$ | $\frac{(1 - \tau_0)}{(a\sigma + \alpha + \sigma)}$ | $0$ |
|                           | $\frac{1}{\alpha\sigma(1 - \tau_0)}$              | $0$                                      |

In the first column of Table 1. there is once change of sign i.e. in the fourth line it's mean that equation (4) has two root on the left imaginer axis and one root on the right imaginer axis. For $\tau_0 > 1$, there are positive eigenvalues and negative eigenvalues so $\bar{u}_1$ is not stable.

For $\tau_0 = 1$, stability can not be inferred based on eigenvalues because the fourth line of Routh-Hurwitz table is all zero. Alternating the Liapunov function can be used to analyze the stability of equilibrium point system (1) i.e. point $\bar{u}_2 = (0, 0, \tau_0)$. The Liapunov function $V = \tau_0\mu^2 + \alpha v^2 + \sigma (\tau - \tau_0)^2$ are choosen. The derivative of $V$ with respect to $t$ is

\[
\dot{V} = \frac{\partial V}{\partial \mu} \dot{\mu}(t) + \frac{\partial V}{\partial v} \dot{v}(t) + \frac{\partial V}{\partial \tau} \dot{\tau}(t)
\]

\[
\dot{V} = -2[\tau_0 \mu^2 + v^2 + (\tau - \tau_0)^2 - 2\tau_0 \mu v].
\]  

(5)

Substituting $\tau_0 = 1$ into equation (5), obtained

\[
\dot{V} = -2[(\mu - v)^2 + (\tau - 1)^2]
\]  

(6)

Since $\dot{V}$ is negative for all value $\mu, v$ and $\tau$ then it can be inferred that equilibrium point $\bar{u}_1 = (0, 0, \tau_0)$ at $\tau_0 = 1$ is stable.

Furthermore, the Jacobian matrix for the equilibrium point $\bar{u}_2 = (\sqrt{\tau_0 - 1}, \sqrt{\tau_0 - 1}, 1)$ is
\[ J_{\bar{u}_2} = \begin{bmatrix} -1 & \frac{1}{\tau} & 0 \\ \frac{\tau}{\alpha} & -1 & \frac{\mu}{\alpha} \\ \frac{v}{\sigma} & \frac{\mu}{\sigma} & -1 \end{bmatrix} \]  

and the characteristic equation is: 
\[ \lambda^3 + \lambda^2 \left( 1 + \frac{1}{\alpha} \right) + \lambda \left( \frac{1}{\alpha} + \frac{\tau_0}{\alpha\sigma} \right) + 2\tau_0 - 2 = 0 \]  

Using Routh-Hurwitz criterion can be concluded that all of the eigenvalues is negative so \( \bar{u}_2 \) is stable. Furthermore, the Jacobian matrix for the equilibrium point \( \bar{u}_3 = (\sqrt{\tau_0 - 1}, -\sqrt{\tau_0 - 1}, 1) \) is 
\[ J_{\bar{u}_3} = \begin{bmatrix} -1 & \frac{1}{\tau} & 0 \\ \frac{\tau}{\alpha} & -1 & \frac{\mu}{\alpha} \\ \frac{v}{\sigma} & \frac{\mu}{\sigma} & -1 \end{bmatrix}. \]

Since the characteristic equation for Jacobian matrix (9) is same to equation (8), so all of the eigenvalues is negative then \( \bar{u}_3 \) stable.

Based on the analysis of stability and the number equilibrium point system (1) seen that when the parameter \( \tau_0 \) varied, it will affect the number of equilibrium points and their stability, for \( \tau_0 < 1 \) there is one equilibrium \( \bar{u}_1 \) which is stable, and for \( \tau_0 > 1 \) there are three equilibrium points, equilibrium point \( \bar{u}_1 \) which not stable and equilibrium points \( \bar{u}_2 \) and \( \bar{u}_3 \) are stable. This indicates that system (1) undergoes a bifurcation pitchfork with bifurcation value is \( \tau_0 = 1 \).

2.2. The properties of the solution of system

The properties of solutions of system (1) that will be discussed is the symmetry and the boundedness solution.

2.2.1. Symmetric. Will be proved, that if \((\mu(t), v(t), \tau(t))\) is a solution of system (1) then \((-\mu(t), -v(t), \tau(t))\) is also a solution.

Suppose that \((\mu(t), v(t), \tau(t))\) is a solution of system (1),
\[ \mu(t) = -\mu(t) + v(t) \]
\[ \dot{v}(t) = -\frac{v(t) + \tau(t)\mu(t)}{\alpha} \]
\[ \dot{\tau}(t) = \frac{\tau_0 - \tau(t) - \mu(t)v(t)}{\sigma}. \]

Substituting \((-\mu(t), -v(t), \tau(t))\) into system (1), yield
\[ -\dot{\mu}(t) = -(-\mu(t)) - v(t) \]
\[ -\dot{v}(t) = -(-v(t)) + (-\mu(t))\tau(t) \]
\[ -\dot{\tau}(t) = \frac{\tau_0 - \tau(t) - (-\mu(t))(\tau(t))}{\sigma} \]

It can be seen that system (11) equal to system (10). It is evident that \((-\mu(t), -v(t), \tau(t))\) is also a solution of the system (1). Thus it can be concluded that solutions symmetric with respect to \( \tau \) axis Visually can be seen at Figure 1, 2, 3 and 4. In this cases the \( \tau \) axis is perpendicular to figure plane.
2.2.2. Boundedness solution. To prove that the solution system (1) is bounded, will be proven by showing that there is a bounded ellipsoid, \( E \) i.e.

\[
\tau_0 \mu^2 + a \nu^2 + \sigma(\tau + \tau_0)^2 \leq c,
\]

such that every trajectory is inside ellipsoid \( E \) and never came out of the ellipsoid \( E \).

Choose function

\[
V(\mu, \nu, \tau) \equiv \tau_0 \mu^2 + a \nu^2 + \sigma(\tau + \tau_0)^2.
\]

(12)

The derivative of \( V(\mu, \nu, \tau) \) is

\[
\dot{V} = -2[\tau_0 \mu^2 + \nu^2 + \tau^2 - \tau_0^2].
\]

(13)

Let \( D \) is an ellipsoid with equation

\[
D \equiv \tau_0 \mu^2 + \nu^2 + \tau^2 - \tau_0^2 \leq 0 \text{ or }
\]

\[
D \equiv \tau_0 \mu^2 + \nu^2 + \tau^2 \leq \tau_0^2.
\]
In other words $D$ is area that $ \dot{V} \geq 0$. Thus if the solution start in ellipsoid $D$ then the trajectory will go away from $D$. Next, it will be shown that $E$ contains $D$. Let
\[
E \equiv \tau_0 \mu^2 + \alpha \nu^2 + \sigma (\tau + \tau_0)^2 \leq c \quad \text{or} \quad \frac{\mu^2}{\tau_0^2} + \frac{\nu^2}{\tau_0^2} + \frac{\alpha^2}{\tau_0^2} \leq 1.
\]
So that the center of ellipsoid $E$ is $(0,0,-\tau_0)$ and the distance of each axis towards the center point is $\sqrt{\tau_0}, \sqrt{\tau_0^2}, \sqrt{\tau_0^2}$. Ellipsoid $D \equiv \tau_0 \mu^2 + \nu^2 + \tau^2 \leq \tau_0^2$ can be written as
\[
\frac{\mu^2}{\tau_0^2} + \frac{\nu^2}{\tau_0^2} + \frac{\tau^2}{\tau_0^2} \leq 1.
\]
So that the center of ellipsoid $D$ is $(0,0,0)$ and the distance of each axis towards the center point is $\sqrt{\tau_0}, \sqrt{\tau_0^2}, \sqrt{\tau_0^2}$. So if we choose $c = \max\{\tau_0^2, \alpha \tau_0^2, 4\sigma \tau_0^2\}$, then ellipsoid $E$ will contain $D$. So that for any $u_0$ outside ellipsoid $E$ then $u_0$ outside ellipsoid $D$. In other words, $\dot{V}(u_0) \leq -\omega$ with $\omega$ positive. It show that if a trajectory with the initial values $u_0$ then its trajectory will go to ellipsoid $E$. So that, it is evident that there is a bounded ellipsoid, $E$
\[
\tau_0 \mu^2 + \alpha \nu^2 + \sigma (\tau + \tau_0)^2 \leq c,
\]
such that every trajectory is inside ellipsoid $E$ and never came out of the ellipsoid $E$.

3. Conclusion

Properties solutions of mathematical model transition time traffic congestion without fluctuation of acceleration are 1) there is one stable equilibrium for $\tau_0 \leq 1$ and one unstable equilibrium and two stable equilibrium for $\tau_0 > 1$, 2) there is a bifurcation pitchfork when $\tau_0$ is varied, 3) the solution system in geometric symmetric with with axis $\tau$, 4) the solution is bounded.

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