Interconnections of input-output Hamiltonian systems with dissipation

Arjan van der Schaft

Abstract—Recently, negative imaginary and counter-clockwise systems have attracted attention as an interesting class of systems, which is well-motivated by applications. In this paper first the formulation and extension of negative imaginary and counter-clockwise systems as (nonlinear) input-output Hamiltonian systems with dissipation is summarized. Next it is shown how by considering the time-derivative of the outputs a port-Hamiltonian system is obtained, and how this leads to the consideration of alternate passive outputs for port-Hamiltonian systems. Furthermore, a converse result to positive feedback interconnection of input-output Hamiltonian systems with dissipation is obtained, stating that the positive feedback interconnection of two systems is only possible if the systems themselves are input-output Hamiltonian systems with dissipation. This implies that the Poisson and resistive structure matrices can be redefined in such a way that the interaction between the two systems only takes place via the coupling term in the Hamiltonian of the interconnected system. Subsequently, it is shown how network interconnection of (possibly nonlinear) input-output Hamiltonian systems with dissipation results in another input-output Hamiltonian system with dissipation, and how this leads to a stability analysis of the interconnected system in terms of the Hamiltonians and output mappings of the systems associated to the vertices, as well as the topology of the network.

I. PROPERTIES OF INPUT-OUTPUT HAMILTONIAN SYSTEMS WITH DISSIPATION

Consider a linear system

\[ \dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \]
\[ y = Cx + Du, \quad y \in \mathbb{R}^m \]

with transfer matrix \( G(s) = C(Is - A)^{-1}B + D \). In [14], [19], [30] \( G(s) \) was called negative imaginary if \( D = D^T \) and the transfer matrix \( H(s) := s(G(s) - D) \) is positive real. In [2], [3], [4] the same notion (also in a nonlinear context) was coined as counter-clockwise input-output dynamics.

In [14], [30] it was shown that a minimal system \(^1\) has negative imaginary transfer matrix if and only if \( D = D^T \) and there exists an \( n \times n \) symmetric matrix \( Q \succ 0 \) such that

\[ A^TQ + QA \leq 0, \quad B = -AQ^{-1}C^T \]

Subsequently in [25] it was shown, by decomposing \( AQ^{-1} \) into its symmetric and skew-symmetric part, that a minimal system \(^1\) has negative imaginary transfer matrix if and only if it can be written as

\[ \dot{x} = [J - R](Qx - CTu) \]
\[ y = Cx + Du, \quad D = DT \]

for \( Q, J, R \) satisfying

\[ Q = QT, \quad J = -JT, \quad R = RT \geq 0 \]

with \( Q \succ 0 \).

Any system \(^3\) satisfying \(^4\) (not necessarily \( Q \succ 0 \)) was called in [25] an input-output Hamiltonian system with dissipation (IOHD system), with Hamiltonian function \( \frac{1}{2}x^TQx \).

The skew-symmetric matrix \( J \) defines a Poisson structure matrix, while \( R \) is called the resistive structure matrix; see [15], [23], [9], [22], [17], [26] for the closely related port-Hamiltonian case (as discussed below).

The definition of an IOHD system \(^3\) was extended in [25] to the nonlinear case as follows. For clarity of exposition, we will throughout only consider the case without feedthrough terms and with affine dependence on \( u \); see [25], as well as Remark 2.3 below, for the general nonlinear case.

Definition 1.1: A system described in local coordinates \( x = (x_1, \ldots, x_n) \) for some \( n \)-dimensional state space manifold \( \mathcal{X} \) as

\[ \dot{x} = [J(x) - R(x)] \left( \frac{\partial H}{\partial x}(x) - \frac{\partial CT}{\partial x}(x)u \right), \quad u \in \mathbb{R}^m \]
\[ y = C(x), \quad y \in \mathbb{R}^m \]

where the \( n \times n \) matrices \( J(x), R(x) \), depending smoothly on \( x \), satisfy

\[ J(x) = -JT(x), \quad R(x) = RT(x) \geq 0, \]

is called a nonlinear IOHD system, with Hamiltonian \( H : \mathcal{X} \to \mathbb{R} \) and output mapping \( C : \mathcal{X} \to \mathbb{R}^m \).

This definition is a generalization of the definition of an affine input-output Hamiltonian system as originally proposed in [7] and studied in e.g. [20], [21], [8]. In fact, it reduces to this definition in case \( R = 0 \) (no dissipation) and \( J \) defines a symplectic form (in particular, has full rank).

The time-evolution of the Hamiltonian of a nonlinear IOHD system \(^5\) is computed as (exploiting skew-symmetry

\[^2\]For a function \( H : \mathbb{R}^n \to \mathbb{R} \) we denote by \( \frac{\partial H}{\partial x}(x) \) the \( n \)-dimensional column vector of partial derivatives of \( H \). For a mapping \( C : \mathbb{R}^n \to \mathbb{R}^m \) we denote by \( \frac{\partial CT}{\partial x}(x) \) the \( n \times m \) matrix whose \( j \)-th column consists of the partial derivatives of the \( j \)-th component function \( C_j \).
of \( J(x) \)

\[
\frac{\partial H}{\partial x} = \left( \frac{\partial H}{\partial x} (x) \right)^T [J(x) - R(x)] \left( \frac{\partial H}{\partial x} (x) - \frac{\partial C^T}{\partial x} (x) u \right) = \\
-\left( \frac{\partial H}{\partial x} (x) \right)^T R(x) \frac{\partial H}{\partial x} (x) - \left( \frac{\partial H}{\partial x} (x) \right)^T [J(x) - R(x)] \frac{\partial C^T}{\partial x} (x) u
\]

(7)

Furthermore, the time-differentiated output of (5) is

\[
z := \dot{y} = \left( \frac{\partial C^T}{\partial x} (x) \right)^T \frac{\partial H}{\partial x} (x) - \left( \frac{\partial C^T}{\partial x} (x) \right)^T \frac{\partial C}{\partial x} (x) u
\]

Using \( u^T \left( \frac{\partial C^T}{\partial x} (x) \right)^T J(x) \frac{\partial C^T}{\partial x} (x) u = 0 \) it is verified that the expression (7) can be rewritten as (leaving out arguments \( x \))

\[
\frac{\partial H}{\partial x} = u^T z - \left( \frac{\partial H}{\partial x} - u^T \frac{\partial C^T}{\partial x} \right) R \left( \frac{\partial H}{\partial x} - u^T \frac{\partial C^T}{\partial x} \right)^T + u^T \left( \frac{\partial C^T}{\partial x} \right) \frac{\partial H}{\partial x} - \left( \frac{\partial C^T}{\partial x} \right) \frac{\partial C^T}{\partial x} u
\]

\[
\leq u^T z
\]

(9)

This immediately shows passivity with respect to the output \( z \) defined by (5) if additionally the Hamiltonian \( H \) is bounded from below. In fact, the system (5) with output \( y_{PH} = z \) for a general Hamiltonian \( H \) is an input-state-output port-Hamiltonian system [22, 23, 9, 17], of the general form [26]

\[
\begin{align*}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} (x) + [G(x) - P(x)] u, \ x \in X \\
y_{PH} &= [G(x) + P(x)]^T \frac{\partial H}{\partial x} (x) + [M(x) + S(x)] u
\end{align*}
\]

(10)

with

\[
\begin{bmatrix}
R(x) & P(x) \\
PT^T(x) & S(x)
\end{bmatrix}
\]

symmetric and \( \geq 0 \),

and \( J(x) \) and \( M(x) \) skew-symmetric. This can be seen by equating

\[
G(x) = -J(x) \frac{\partial C^T}{\partial x} (x), \ P(x) = -R(x) \frac{\partial C^T}{\partial x} (x),
\]

\[
S(x) = \left( \frac{\partial C^T}{\partial x} (x) \right)^T R(x) \frac{\partial C^T}{\partial x} (x),
\]

\[
M(x) = -\left( \frac{\partial C^T}{\partial x} (x) \right)^T J(x) \frac{\partial C^T}{\partial x} (x)
\]

(11)

This leads to

**Proposition 1.2:** Given the IOHD system (5). Then its dynamics together with differentiated output \( z = \dot{y} \) defined by (8) is an input-state-output port-Hamiltonian system of the form (10) with \( y_{PH} = z \). Conversely, given a port-Hamiltonian system (10), then there exists an IOHD system with the same dynamics and output \( y = C(x), C : X \to \mathbb{R}^m \), such that \( \dot{y} = y_{PH} \) if and only if \( C \) satisfies (11).

Note that the conditions (11) can be interpreted as (generalized) integrability conditions on \( G(x), M(x), S(x) \). Indeed, for the special case of a basic input-state-output port-Hamiltonian system

\[
\begin{align*}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} (x) + g(x) u \\
y_{PH} &= g^T (x) \frac{\partial H}{\partial x} (x)
\end{align*}
\]

(12)

corresponding to \( P = 0, S = 0, M = 0 \) and \( g = G \), the conditions (11) reduce to

\[
\begin{align*}
g(x) &= G(x) = -J(x) \frac{\partial C^T}{\partial x} (x), \\
R(x) &\frac{\partial C^T}{\partial x} (x) = 0, \ \ (\frac{\partial C^T}{\partial x} (x))^T J(x) \frac{\partial C^T}{\partial x} (x) = 0
\end{align*}
\]

(13)

The first line implies that the columns \( g_j, j = 1, \ldots, m \) of the input matrix \( g(x) \) are Hamiltonian vector fields with Hamiltonians \( -C_1, \ldots, -C_m \). For \( J \) corresponding to a symplectic structure, there exist locally such functions \( C_1, \ldots, C_m \), if and only if the vector fields \( g_j \) leave the symplectic structure invariant [1], [20], [21].

**Example 1.3:** Consider the linear mechanical system as considered in [15], consisting of an alternating mass-spring-mass-spring system, where the first input \( u_1 \) is the velocity of the right-hand side of the second spring, and the first input \( u_2 \) is the force on the left mass. Furthermore, differently from [15], there are dampers with damping coefficients \( d_1, d_2 \) parallel to the two springs with spring constants \( k_1, k_2 \). Denoting the masses by \( m_1, m_2 \), the elongations of the springs by \( q_{12}, q_{20} \), and the momenta of the masses by \( p_1, p_2 \), the dynamical equations are given as

\[
\begin{align*}
\dot{q}_{12} &= \frac{p_1}{m_1} - \frac{p_2}{m_2} \\
\dot{q}_{20} &= \frac{p_2}{m_2} - u_1 \\
\dot{p}_1 &= -k_1 q_{12} + u_2 \\
\dot{p}_2 &= k_1 q_{12} - k_2 q_{20}
\end{align*}
\]

(14)

This can be written as the IOHD system

\[
\begin{bmatrix}
\dot{q}_{12} \\
\dot{q}_{20} \\
\dot{p}_1 \\
\dot{p}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & d_1 & -d_1 \\
0 & -d_1 & d_1 + d_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{\partial H}{\partial q_{12}} \\
\frac{\partial H}{\partial q_{20}} \\
\frac{\partial C_1}{\partial p_1} \\
\frac{\partial C_2}{\partial p_2}
\end{bmatrix} -
\begin{bmatrix}
\frac{\partial C_1}{\partial q_{12}} \\
\frac{\partial C_2}{\partial q_{20}} \\
\frac{\partial C_1}{\partial p_2} \\
\frac{\partial C_2}{\partial p_2}
\end{bmatrix} u_1 -
\begin{bmatrix}
\frac{\partial C_1}{\partial q_{12}} \\
\frac{\partial C_2}{\partial q_{20}} \\
\frac{\partial C_1}{\partial p_2} \\
\frac{\partial C_2}{\partial p_2}
\end{bmatrix} u_2
\]

(15)

with Hamiltonian \( H(q_{12}, q_{20}, p_1, p_2) \) and outputs \( y_1 = C_1(q_{12}, q_{20}, p_1, p_2), y_2 = C_2(q_{12}, q_{20}, p_1, p_2) \) given by

\[
\begin{align*}
H(q_{12}, q_{20}, p_1, p_2) &= \frac{1}{2} k_1 q_{12}^2 + \frac{1}{2} k_2 q_{20}^2 + \frac{p_1^2}{2 m_1} + \frac{p_2^2}{2 m_2} \\
C_1(q_{12}, q_{20}, p_1, p_2) &= p_1 + p_2 - d_2 q_{20} \\
C_2(q_{12}, q_{20}, p_1, p_2) &= q_{12} + q_{20}
\end{align*}
\]

(16)

The resulting port-Hamiltonian system with respect to the differentiated outputs \( y_{PH1} = \dot{y}_1, y_{PH2} = \dot{y}_2 \) is given in the form (10), where \( G, P, S, M \) are the constant matrices given
as
\[
G = \begin{bmatrix}
0 & 0 \\
-1 & 0 \\
0 & 1 \\
d_2 & 0
\end{bmatrix}, \quad P = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
d_2 & 0
\end{bmatrix},
\]
\[
S = \begin{bmatrix}
d_2 & 0 \\
0 & 0
\end{bmatrix}, \quad M = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix},
\]
(Note that even for the case \(d_2 = 0\) the resulting port-Hamiltonian system is not anymore of the basic form (12), due to the presence of the feedthrough matrix \(M\), as already noticed (from a different point of view) in [15].

Obviously, we can consider alternate passive outputs for the port-Hamiltonian system (10). (An output \(\hat{y}_{PH}\) is called a passive output for (10) if \(\frac{\partial H}{\partial x}(x) \leq u^T \hat{y}_{PH}(x)\). The simplest choice is to define the new passive output \(\hat{y}_{PH} := g^T(x)\frac{\partial H}{\partial x}(x), \) with \(g(x) = G(x) - P(x)\), resulting in a new input-state-output port-Hamiltonian system of the basic form (12). In general, given the port-Hamiltonian system (10) all outputs
\[
\hat{y}_{PH} = [\hat{G}(x) + \hat{P}(x)]^T \frac{\partial H}{\partial x}(x) + [\hat{M}(x) + \hat{S}(x)]u,
\]
with \(\hat{M}(x) = -\hat{M}^T(x)\) and \(\hat{G}(x), \hat{P}(x), \hat{S}(x)\) satisfying
\[
\begin{bmatrix}
R(x) & \hat{P}(x) \\
\hat{P}^T(x) & \hat{S}(x)
\end{bmatrix} \geq 0
\]
define alternate passive outputs, corresponding to alternate port-Hamiltonian systems. Notice, however, that these alternate outputs cannot always be integrated to outputs of an IOHD system. Such new passive outputs were recently used in [6] for IDA-PBC control, continuing on e.g. [12], see also [27].

Still a larger class of passive outputs can be obtained by allowing for different \(J, R\) and \(H\) in (10), in such a way that the dynamics of (10) remains the same.

A. Mechanical systems with collocated sensors and actuators

IOHD systems show up naturally in many applications; see e.g. [14], [19], [4], [18], [7], [20], [21]. A clear example are mechanical systems with co-located position sensors and force actuators, which in the linear case are represented as the IOHD systems (with \(q\) denoting the position vector and \(p\) the momentum vector)
\[
\begin{bmatrix}
\dot{q} \\
p
\end{bmatrix} = \begin{bmatrix}
0_n & I_n \\
-I_n & -D
\end{bmatrix} \begin{bmatrix}
K & N \\
N^T & M^{-1}
\end{bmatrix} \begin{bmatrix}
q \\
p
\end{bmatrix} + \begin{bmatrix}
0 \\
L^T
\end{bmatrix} u
\]
\[
y = Lq,
\]
where \(D \geq 0\) is the damping matrix; defining the resistive structure matrix \(R = \text{diag}(0, D)\). Usually \(N = 0\) (no ‘gyroscopic forces’), in which case the Hamiltonian (total energy) is given as
\[
H(q, p) = \frac{1}{2}q^T K q + \frac{1}{2}p^T M^{-1} p,
\]
where the first term is the total potential energy (with \(K\) the compliance matrix), and the second term is the kinetic energy (with \(M\) the mass matrix).

For \(N = 0\) (20) can be rewritten into the equivalent second-order form
\[
M\ddot{q} + D\dot{q} + K q = L^T u, \quad y = Lq
\]

II. POSITIVE FEEDBACK INTERCONNECTION OF IOHD SYSTEMS

Just like the negative feedback interconnection of passive (or port-Hamiltonian) systems results in a passive (respectively, port-Hamiltonian) system, the positive feedback interconnection of IOHD systems results in another IOHD system; see [2], [3] for the counter-clockwise case. Indeed, the positive feedback interconnection
\[
u_1 = y_2 + e_1, \quad u_2 = y_1 + e_2,
\]
with \(e_1, e_2\) two external inputs, of two linear IOHD systems
\[
\dot{x}_i = \begin{bmatrix}
J_i & -R_i
\end{bmatrix}(Q_i x_i - C_i^T u_i)
\]
\[
y_i = C_i x_i, \quad i = 1, 2,
\]
with equal number of inputs and outputs can be seen [25] to result in the IOHD system
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
J_1 & 0 \\
0 & J_2
\end{bmatrix} \begin{bmatrix}
Q_1 & -C_1^T C_2 \\
-C_2^T C_1 & Q_2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} - \begin{bmatrix}
C_1^T & 0 \\
0 & C_2^T
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}
\]
\[
y_1 = \begin{bmatrix}
C_1 & 0 \\
0 & C_2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix},
\]
with interconnected Hamiltonian given as
\[
H_{int}(x_1, x_2) := \frac{1}{2} x_1^T Q_1 x_1 + \frac{1}{2} x_2^T Q_2 x_2 - x_1^T C_1^T C_2 x_2
\]
Hence the stability of the interconnected system can be characterized in terms of the interconnected Hamiltonian (26) as follows.

**Proposition 2.1:** [25] Consider two IOHD systems. The interconnected IOHD system (25) is stable having no eigenvalue at zero if the interconnected Hamiltonian (26) has a strict minimum at the origin \((x_1, x_2) = (0, 0)\). Conversely, if (25) is asymptotically stable, then the interconnected Hamiltonian (26) has a strict minimum at the origin \((x_1, x_2) = (0, 0)\).

In ([14], Theorem 5) it has been shown that \(Q\), with \(Q_1 > 0, Q_2 > 0\) is positive definite, if and only
\[
\lambda_{\max}(C_1 Q_1^{-1} C_1^T, C_2 Q_2^{-1} C_2^T) < 1
\]
where \(\lambda_{\max}(K)\) denotes the maximal eigenvalue of a symmetric matrix \(K\). An easy proof follows by the fact that \(Q > 0\) if and only if \(Q_1 > 0\) and \(Q_2 - C_2^T C_1 Q_1^{-1} C_1^T C_2 > 0\). This
last inequality is equivalent to \( Q_2^{-\frac{1}{2}} C_2^T C_1 Q_1^{-1} C_1^T C_2 Q_2^{-\frac{1}{2}} < I \), and thus to
\[
C_1 Q_1^{-1} C_1^T \cdot C_2 Q_2^{-\frac{1}{2}} Q_2^{-\frac{1}{2}} C_2^T < I,
\]
which is equivalent to (27).

This allows for the following interpretation. The \textit{dc-gain} of an IOHD system \(^3\) with \( D = 0 \) is given by the expression
\[
− C A^{-1} B = C Q^{-1} C^T
\]
(28)

Hence the interconnected IOHD system \((25)\) resulting from the positive feedback interconnection of two stable IOHD systems (\( Q_1 > 0, Q_2 > 0 \)) is stable having no eigenvalue at zero if and only if the \textit{dc loop gain is less than unity}. This can be regarded as a rephrasement of the fundamental result concerning the stability of the positive feedback interconnection of two systems with negative imaginary transfer matrices, as obtained in \( [14] \) for the general MIMO case and in \( [14] \) for the general MIMO case.

In the case of positive feedback interconnection of two IOHD systems in second-order form \((22)\) this amounts to the following corollary.

\textbf{Corollary 2.2:} Consider two systems \((22)\) with \( M_i > 0, K_i > 0 \), \( i = 1, 2 \). Then the positive feedback interconnection results in the second-order system

\[
\begin{bmatrix}
M_1 & 0 \\
0 & M_2
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix}
+
\begin{bmatrix}
D_1 & 0 \\
0 & D_2
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
+
\begin{bmatrix}
K_1 & -L_2^T L_1 \\
-L_1^T L_2 & K_2
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix}
=
\begin{bmatrix}
L_1^T & 0 \\
0 & L_2^T
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}
\]
(29)

\[
y_1 = \begin{bmatrix}
L_1 & 0 \\
0 & L_2
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix}
\]

which is stable without eigenvalue at zero if and only if
\[
\lambda_{\text{max}} \left( L_1 K_1^{-1} L_1^T \cdot L_2 K_2^{-1} L_2^T \right) < 1
\]
(30)

The positive feedback interconnection of a second-order \textit{‘plant’} system \((22)\) with a second-order \textit{‘controller’} system is known in the literature, see e.g. \([11], [10]\), as \textit{positive position feedback}. It has the advantage of being insensitive to spillover, and has favorable other robustness properties, see e.g. \([10]\) for a discussion. Note that by \((30)\) the stability of the closed-loop system only depends on the potential energies of the \textit{‘plant’} and \textit{‘controller’} system and on the matrices \( L_1, L_2 \). In particular the stability does depend not on the damping matrices \( D_1, D_2 \).

Similar to the linear case it can be seen \([25]\) that the positive feedback interconnection of two nonlinear IOHD systems\(^\dagger\) \((5)\), indexed by \( i = 1, 2 \), results in the nonlinear IOHD system
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix}
=
\begin{bmatrix}
J_1(x_1) & 0 \\
0 & J_2(x_2)
\end{bmatrix}
−
\begin{bmatrix}
R_1(x_1) & 0 \\
0 & R_2(x_2)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{\partial H_{\text{int}}}{\partial x_1}(x_1, x_2) \\
\frac{\partial H_{\text{int}}}{\partial x_2}(x_1, x_2)
\end{bmatrix}
−
\begin{bmatrix}
\frac{\partial C^T}{\partial x_1}(x_1) & 0 \\
0 & \frac{\partial C^T}{\partial x_2}(x_2)
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}
\]

\[
y_1 = \begin{bmatrix}
C_1(x_1) \\
C_2(x_2)
\end{bmatrix}
\]

(31)

with interconnected Hamiltonian \( H_{\text{int}} \) given by
\[
H_{\text{int}}(x_1, x_2) := H_1(x_1) + H_2(x_2) − C_1^T(x_1) C_2(x_2)
\]
(32)
(compare with \([3], Theorem 6\)). Like in the linear case, the stability properties of the interconnected system are determined by \( H_{\text{int}} \).

\textbf{Remark 2.3:} A general nonlinear IOHD system is defined in \([25]\) as a system of the form
\[
x = (J(x) − R(x)) \frac{\partial H}{\partial x}(x, u), \quad u \in \mathbb{R}^m,
\]
\[
y = − \frac{\partial H}{\partial u}(x, u), \quad y \in \mathbb{R}^m,
\]
(33)

for some function \( H(x, u) \), with \( R(x), J(x) \) satisfying \((6)\).

Furthermore, it is shown in \([25]\) how also the positive feedback interconnection of two general IOHD systems results in another general IOHD system, provided some transversality conditions are met. A particular case of \((33)\) is a \textit{static} system
\[
y = − \frac{\partial P}{\partial u}(u), \quad u, y \in \mathbb{R}^m,
\]
(34)

for some function \( P : \mathbb{R}^m \rightarrow \mathbb{R} \). The positive feedback interconnection of an IOHD system \((5)\) with such a static IOHD system \((34)\) results in the IOHD system \((5)\), with modified Hamiltonian given as
\[
H_{\text{cl}}(x) := H(x) + P(C(x))
\]
(35)

Conversely, it can be shown \([16]\) that any static output feedback applied to \((5)\) will result in an IOHD system with respect to the same \( J(x), R(x) \) if and only it corresponds to positive feedback interconnection with a static IOHD system \((34)\) for some \( P \).

We note that the Poisson and resistive structure of the interconnected system \((33)\) is the direct sum of the respective structures of the two component systems. This is opposite to the case of the negative feedback interconnection of two port-Hamiltonian systems \([22], [26]\), where the Poisson structure matrix of the interconnected contains an additional \textit{coupling term}, and where, on the other hand, the Hamiltonian of the interconnected system is just the sum of the Hamiltonians of the two component systems; see also \([25]\).

\section{A Converse Result}

In this section we will show that not only the positive feedback interconnection of two IOHD systems results in another IOHD system, but that, at least in the linear case, \textit{also the converse holds}: if the positive feedback interconnection of two arbitrary linear systems is an IOHD system (with

\(^3\)For the generalization to general nonlinear IOHD systems see \([25]\).
inputs \(e_1, e_2\) and outputs \(y_1, y_2\), then the two systems are necessarily IOHD as well.

This result can be seen as an analog to the converse result obtained for the negative feedback interconnection of passive systems in [13].

**Proposition 3.1:** Consider two linear systems \(\Sigma_1 = (A_1, B_1, C_1), \Sigma_2 = (A_2, B_2, C_2)\) with equal input and output dimensions. Suppose the positive feedback interconnection \(\Sigma_1, \Sigma_2\) results in a minimal system, with inputs \(e_1, e_2\) and outputs \(y_1, y_2\), that is IOHD. Then also \(\Sigma_1, \Sigma_2\) are IOHD systems.

**Proof:** Without loss of generality assume that \(C_1\) and \(C_2\) have full row rank. Since the positive feedback interconnection of \(\Sigma_1, \Sigma_2\) is a minimal IOHD system, there exists an invertible matrix \(P = Q^{-1}\), cf. \((2)\), such that

\[
\begin{bmatrix}
  P_{11} & P_{12} \\
  P_{21} & P_{22}
\end{bmatrix}
\begin{bmatrix}
  A_1^T & C_1^T B_1^T \\
  C_2^T A_2^T & A_2^T
\end{bmatrix}
\begin{bmatrix}
  P_{11} & P_{12} \\
  P_{21} & P_{22}
\end{bmatrix}
\leq 0
\]

and

\[
\begin{bmatrix}
  B_1 & 0 \\
  0 & B_2
\end{bmatrix}
= -\begin{bmatrix}
  A_1 & B_1 C_2 \\
  B_2 C_1 & A_2
\end{bmatrix}
\begin{bmatrix}
  P_{11} & P_{12} \\
  P_{12} & P_{22}
\end{bmatrix}
\begin{bmatrix}
  C_1^T \\
  0
\end{bmatrix}
\]

Then \((37)\) is written out as

\[
B_1 = -A_1 P_{11} C_1^T - B_1 C_2 P_{12} C_1^T
\]

\[
B_2 = -A_2 P_{12} C_2^T - B_2 C_1 P_{12} C_2^T
\]

\[
A_1 P_{12} C_2^T + B_1 C_2 P_{12} C_2^T = 0
\]

\[
A_2 P_{12} C_2^T + B_2 C_1 P_{12} C_2^T = 0
\]

Then the equations \((39)\) yield

\[
B_1 = -A_1 P_{11} (C_2 P_{22} C_2)^{-1}
\]

\[
B_2 = -A_2 P_{12} (C_1 P_{11} C_1)^{-1}
\]

Substituted in the right-hand sides of \((38)\) this yields

\[
B_1 = -A_1 (P_{11} - P_{12} C_2 (C_2 P_{22} C_2)^{-1} P_{12}) C_1^T
\]

\[
B_2 = -A_2 (P_{22} - P_{12} C_2 (C_1 P_{11} C_1)^{-1} P_{12}) C_2^T
\]

Define subsequently

\[
P_1 := P_{11} - P_{12} C_2 (C_2 P_{22} C_2)^{-1} P_{12}
\]

\[
P_2 := P_{22} - P_{12} C_2 (C_1 P_{11} C_1)^{-1} P_{12}
\]

Note that the first expression corresponds to a Schur complement of

\[
\begin{bmatrix}
  P_{11} & P_{12} C_2^T \\
  C_2 P_{12} & C_2 P_{22} C_2^T
\end{bmatrix}
\begin{bmatrix}
  I & 0 \\
  0 & C_2
\end{bmatrix}
\begin{bmatrix}
  P_{11} & P_{12} \\
  P_{12} & P_{22}
\end{bmatrix}
\begin{bmatrix}
  I & 0 \\
  0 & C_2^T
\end{bmatrix}
\]

Consequently, \(P_1\) is invertible, and positive definite if \(P\) is positive definite. Similarly, \(P_2\) is invertible, and positive definite if \(P\) is positive definite.

Finally, write out the \((1, 1)\) and \((2, 2)\) block of \((36)\) as

\[
P_{11} A_1^T + A_1 P_{11} + P_{12} C_2^T B_1^T + B_1 C_2 P_{12}^T \leq 0
\]

\[
P_{22} A_2^T + A_2 P_{22} + P_{12} C_2^T B_2^T + B_2 C_1 P_{12} \leq 0
\]

Substitution of the first line of \((40)\) into the first line of \((41)\) yields

\[
P_{11} A_1^T + A_1 P_{11} - P_{12} C_2^T (C_2 P_{22} C_2^T)^{-1} C_2 P_{12}^T A_1^T - A_1 P_{11} C_2^T (C_2 P_{22} C_2^T)^{-1} C_2 P_{12}^T \leq 0,
\]

which can be rewritten as

\[
[P_{11} - P_{12} C_2^T (C_2 P_{22} C_2^T)^{-1} C_2 P_{12}^T] A_1^T + A_1 [P_{11} - P_{12} C_2^T (C_2 P_{22} C_2^T)^{-1} C_2 P_{12}^T] \leq 0,
\]

that is, \(P_1 A_1^T + A_1 P_1 \leq 0\). Similar derivation holds for \(P_2\), altogether resulting in

\[
P_1 A_1^T + A_1 P_1 \leq 0, \quad P_2 A_2^T + A_2 P_2 \leq 0
\]

Furthermore, \((41)\) is identical to

\[
B_1 = -A_1 P_{11} C_1^T, \quad B_2 = -A_2 P_{12} C_2^T
\]

Thus \((A_1, B_1, C_1)\) and \((A_2, B_2, C_2)\) are IOHD systems with Hamiltonians \(\frac{1}{2} t_1^2 P_{11}^{-1} x_1, \frac{1}{2} t_2^2 P_{22}^{-1} x_2\) and (are negative imaginary if \(P > 0\), and thus \(P_1 > 0, P_2 > 0\).

By decomposing \(A_i P_i, i = 1, 2\), into their skew-symmetric and symmetric parts as \(A_i P_i = J_i - R_i, i = 1, 2\), it follows that the positive feedback interconnection of \(\Sigma_1\) and \(\Sigma_2\) is necessarily given as the IOHD system with respect to the Poisson structure and resistive structure matrix

\[
J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \quad R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix},
\]

which is in general different from the originally assumed Poisson structure and resistive structure matrix (containing in general coupling terms between the two component systems). Hence Proposition 5.1 stipulates that for an interconnected system which is IOHD the Poisson structure and resistive structure matrices can be always redefined as being block-diagonal, in which case, as a result, the coupling between the component systems \(\Sigma_1\) and \(\Sigma\) arises only from the coupling term in the interconnected Hamiltonian \((20)\).

Note finally that an essential part in the above proof is the fact that we consider a ‘full’ positive feedback interconnection \((23)\). In particular, a positive feedback interconnection of the form \(u_1 = y_2 + e_1, \quad u_2 = y_1\) (no \(e_2\) input) of two linear IOHD systems will result in an IOHD system with inputs \(e_1\) and \(y_1\), but the converse need not hold in this case.

**IV. NETWORKS OF IOHD SYSTEMS**

In this section it will be shown how the fact that the positive feedback interconnection of two IOHD systems is another IOHD system can be naturally extended to networks of IOHD systems.

Consider an undirected graph with \(N\) vertices and \(M\) edges, together with an \(N \times N\) weighted adjacency matrix \(A\) (symmetric since the graph is undirected). Furthermore, take the multi-agent point of view by associating to each of the vertices \(i \in \{1, \ldots, N\}\) a nonlinear IOHD system \((5)\), indexed by \(i\), with equal number of inputs and outputs \(m\).
(independent of \(i\)). These IOHD systems are interconnected by setting
\[
u = (A \otimes I_m) y + e,
\]
where \(u\) is the stacked \(Nm\) vector, with subvectors \(u_1, \cdots, u_N\), and analogously for \(y\) and the external inputs \(e\). (Here \(A \otimes I_m\) denotes the Kronecker product of \(A\) and \(I_m\), i.e., the \(Nm \times Nm\) matrix obtained by multiplying every element of \(A\) by the \(m \times m\) identity matrix \(I_m\).) Note that for the special case \(N = 2, M = 1, A_{12} = A_{12} = 1\), this reduces to the positive feedback interconnection \((23)\).

As in Section II it can be readily seen that the resulting multi-agent system is again an IOHD system with inputs \(e_1, \cdots, e_N\) and outputs \(y_1, \cdots, y_N\), and total interconnected Hamiltonian given as
\[
H_{\text{int}}(x_1, \cdots, x_N) := H_1(x_1) + \cdots + H_N(x_N) - \frac{1}{2} \sum_{i,j=1}^{N} A_{ij}(C_i(x_i))^T C_j(x_j),
\]
where \(x_i, H_i, C_i, i = 1, \cdots, N\), are respectively the state vectors, Hamiltonians and output mappings of the IOHD systems associated to the vertices. Stability of the resulting IOHD system is again determined by this interconnected Hamiltonian.

Another scenario, more similar to the one considered in \([29],[30]\), is to consider a directed graph, with \(N\) vertices and \(M\) edges and \(N \times M\) incidence matrix \(D\), and to associate not only to each of the vertices \(i \in \{1, \cdots, N\} a\) nonlinear IOHD system \([5]\) with equal number of inputs and outputs \(m\), but also to each of the edges \(k \in \{1, \cdots, M\}\). These IOHD systems are now naturally interconnected by setting
\[
\begin{bmatrix} u^v \\ u^e \end{bmatrix} = \begin{bmatrix} 0 & D \otimes I_m \\ D^T \otimes I_m & 0 \end{bmatrix} \begin{bmatrix} y^v \\ y^e \end{bmatrix} + \begin{bmatrix} e^v \\ e^e \end{bmatrix}
\]
\[(52)\]
Here \(u^v\) and \(y^v\) are the stacked \(Nm\) vectors of inputs and outputs of the IOHD systems associated to the vertices, and \(u^e\) and \(y^e\) are the stacked \(Mm\) vectors of inputs and outputs of the IOHD systems associated to the edges, and \(e^v, e^e\) are the external inputs associated to the vertices, respectively. The resulting system is again an IOHD system, with total Hamiltonian being given by
\[
H(x_1, \cdots, x_N, x_1^v, \cdots, x_M^e) :=
H_1^v(x_1^v) + \cdots + H_N^v(x_N^v) + H_1^e(x_1^e) + \cdots + H_M^e(x_M^e) - \sum_{i=1, k=1}^{N,M} D_{ik}(C_i^v(x_i^v))^T C_k^e(x_k^e)
\]
\[(53)\]
Here the superscripts \(v\) throughout refer to the vertex IOHD systems, and the superscripts \(e\) to the edge IOHD systems. In particular, \(x_i^v, H_i^v, C_i^v, i = 1, \cdots, N\), are the state vectors, Hamiltonians and output mappings of the vertex IOHD systems, and \(x_k^e, H_k^e, C_k^e, k = 1, \cdots, M\), are the state vectors, Hamiltonians and output mappings of the edge IOHD systems.

Note that this last setting is different from the network interconnection of passive systems associated to the vertices and edges of a directed graph as considered in \([5]\) (or in the port-Hamiltonian case in \([24]\)), since the interconnection \((52)\) again corresponds to positive feedback.

V. CONCLUSIONS AND OUTLOOK
The class of input-output Hamiltonian systems with dissipation has been further explored as an interesting class of nonlinear systems, which is well-motivated by applications, and closely related to port-Hamiltonian systems and passivity-based control. A striking result is the converse result obtained (for linear systems) in Section III, which has the interesting implication that the Poisson and resistive structure matrices of an interconnected system that is IOHD can be always redefined in such a way that the coupling between the subsystems is only via the coupling term in the interconnected Hamiltonian. This is a case that happens quite frequently in modeling of multi-physics systems.

Of course, an extension of this converse result to the nonlinear case is a topic for further research, as well as the exploration of applications of the network interconnection theory of IOHD systems initiated in Section IV.

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