Positive contractive projections on noncommutative $L^p$-spaces and nonassociative $L^p$-spaces

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Abstract

We continue our investigation of contractive projections on noncommutative $L^p$-spaces where $1 < p < \infty$ started in [ArR19]. We improve the results of [ArR19] and we characterize precisely the positive contractive projections on a noncommutative $L^p$-space associated with a $\sigma$-finite von Neumann algebra. We connect this topic to the theory of JW$^*$-algebras. More precisely, in large cases, we are able to show that the range of a positive contractive projection is isometric to a nonassociative $L^p$-space associated to a JW$^*$-algebra.

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1 Introduction

A classical and old topic of Banach space theory is the investigation of the structure of projections and complemented subspaces. Note that a bounded operator $P : X \to X$ on a Banach space $X$ is a projection if $P^2 = P$ and a subspace $Y$ of $X$ is contractively complemented if it is the range of a contractive linear projection. It is known that a subspace of a smooth Banach space $X$ can be the range of at most one projection of norm one, see [CoS1, Theorem 6]. We refer to the surveys [Ran01] and [Mos06] for the state of the art in this area.

Suppose $1 \leq p < \infty$. For example, the contractively complemented subspaces of a classical (commutative) $L^p$-space $L^p(\Omega)$ are all isometrically isomorphic to a $L^p$-space. This is a classical result of Ando [And66] (for a finite measure space), see also [Lac74, Theorem 3 p. 162]. See [Tza69], [BeL74] and references therein for the case of general measures. This work also highlights the link between contractive projections and weighted conditional expectations. Note that Douglas previously investigated the particular case $p = 1$ in [Dou65]. Furthermore,
a subspace \( Y \) of a \( L^p \)-space is the range of a positive contractive projection if and only if \( Y \) is order isometric to some \( L^q \)-space, see [AbA02, Problem 5.4.1] and [Ran01, Theorem 4.10].

The noncommutative analogue of the \( L^p \)-space \( \ell^p \) is the Schatten space \( S^p \overset{\text{def}}{=} \{ x : \ell^2 \to \ell^2 : \| x \|_p \overset{\text{def}}{=} (\text{Tr}(|x|^p))^{\frac{1}{p}} < \infty \} \). On this Banach space, the situation is much more complicated. It is easy to see that the range of a contractive projection \( P : S^p \to S^p \) on the Schatten space \( S^p \) is not necessarily isometric to a Schatten space if \( p \neq 2 \). For example if \( \sigma : S^p \to S^p \) denotes the transpose map then it is clear that the map \( P \overset{\text{def}}{=} \frac{1}{2}(\text{Id}_{S^p} + \sigma) : S^p \to S^p \) is a positive contractive projection on the subspace of symmetric matrices of \( S^p \), in sharp contrast with the setting of classical \( L^p \)-spaces of measure spaces.

Nevertheless, the complete description and structure of contractively complemented subspaces of \( S^p \) was achieved by Arazy and Friedman in their famous and impressive memoirs [ArF78] and [ArF92] for any \( 1 \leq p \leq \infty \). Such a subspace is isometrically isomorphic to a \( \ell^p \)-space with the preduals of \( W^* \)-ternary rings of operators. Using the precise description of contractively complemented subspaces of \( S^p \) given by Arazy and Friedman, it is showed in [NO02], the authors gives the description of completely contractively complemented subspaces of noncommutative \( L^1 \)-spaces. Theses spaces coincide with the preduals of \( W^* \)-ternary rings of operators. Using the precise description of contractively complemented subspaces of \( S^p \) given by Arazy and Friedman, it is showed in [LRR09, Theorem 1.1] that any completely 1-complemented subspaces of \( S^p \) is isometric to a direct sum of spaces of the form \( S^p(H,K) \), where \( H \) and \( K \) are Hilbert spaces.

Almost forty years after the publication of the memoir [ArF92], it is proved in [ArR19] that the range of a 2-positive contractive projection on an arbitrary noncommutative \( L^p \)-space is completely order and completely isometrically isomorphic to some noncommutative \( L^q \)-space. This result can be used with completely positive contractive projections. Furthermore, a description of 2-positive contractive projections is provided and even new for Schatten spaces. We can see a 2-positive contractive projection as some kind weighted conditional expectation, which remembers the situation of classical \( L^p \)-spaces. This entails that a 2-positive contractive projection is necessarily completely positive. The approach is unrelated to the methods of Arazy and Friedman and relies on a two-sided change of density and a lifting argument of the projection at the level \( p = \infty \). The use of non-tracial Haagerup’s noncommutative \( L^p \)-spaces is crucial even for the case of Schatten spaces.

The goal of this paper is to investigate the case of positive contractive projections, which are not necessarily 2-positive. Our first result is an improvement of the main result of [ArR19]. It characterizes and describe precisely the positive contractive projections on noncommutative \( L^p \)-spaces.

**Theorem 1.1** Let \( M \) be a \( \sigma \)-finite von Neumann algebra. Suppose \( 1 < p < \infty \). A bounded map \( P : L^p(M) \to L^p(M) \) is a positive contractive projection if and only if there exist a faithful normal state \( \varphi \) on \( M \), a positive element \( h \in L^p(M,\varphi) \) with support projection \( s(h) \) and a faithful normal Jordan conditional expectation \( Q : s(h)Ms(h) \to s(h)Ms(h) \) such that

1. for any \( y \in L^p(M) \) we have \( P(y) = P(s(h)ys(h)) \),
2. $s(h)$ belongs to the centralizer of the state $\varphi$,
3. for any $x \in s(h)M(s(h))$ we have $\text{Tr}_\varphi(h^pQ(x)) = \text{Tr}_\varphi(h^px)$,
4. for any $x \in s(h)M(s(h))$, we have

$$P(h^\frac{x}{2}h^\frac{x}{2}) = h^\frac{x}{2}Q(x)h^\frac{x}{2}.$$ 

Moreover, in this case $h$ belongs to the range $\text{Ran } P$ of the projection $P$ and $Q(s(h)M(s(h))$ is a JW*-subalgebra of the JW*-algebra $(s(h)M(s(h), \varphi))$.

Here $(x, y) \mapsto x \circ y \overset{\text{def}}{=} \frac{xy + yx}{2}$ is the Jordan product which is commutative but non-associative and $\text{Tr}_\varphi$ is the trace on the Haagerup noncommutative $L^1$-space $L^1(M, \varphi)$ defined in (2.10).

Recall that a JW*-algebra is a weak* closed Jordan*-subalgebra of a von Neumann algebra. This notion was introduced by Edwards in the paper [Edw80] (see also [You78]). Note that a von Neumann algebra equipped with the Jordan product is an example of JW*-algebra. It is known that the selfadjoint parts of JW*-algebras are precisely the JW-algebras. In the continuity of the classical work [JNW34] of Jordan, von Neumann and Wigner on Jordan algebras, the theory of JW-algebras was introduced by Topping in [Top65]. By the previous remark, the theory of JW*-algebras is essentially equivalent to the theory of JW-algebras. We refer to the books [AIS03], [ARU97], [CGRP14], [CGRP18], [HOS84] and references therein for more information on these algebras. These are related to the study of the state spaces of C*-algebras, see [AHOS80], [AIS78] and the survey [Alf79]. Furthermore, these algebras are connected to bounded symmetric domains, see [HOS84, pp. 92-93] for a brief overview of this topic. Finally, the category of $\sigma$-finite JBW-algebras is equivalent to the category of facially homogeneous self-dual cones in real Hilbert spaces, see [Ioc84]. This fact can be seen as a generalization of the well-known one-to-one correspondence between $\sigma$-finite von Neumann algebras and orientable, facially homogeneous self-dual cones in complex Hilbert spaces introduced by Connes in [Con74].

A difficulty for the identification of the range of the projection $P$ is that the Jordan conditional expectation $Q$ is not necessarily selfadjoint in the sense of [JMX06, p. 122] with respect to the restriction of the state $\psi \overset{\text{def}}{=} \frac{1}{\text{Tr}_\varphi(h^p \cdot)} \text{Tr}_\varphi(h^p \cdot)$ on the von Neumann algebra $s(h)M(s(h))$, contrary to the case of classical conditional expectations on von Neumann algebras. So at the time of writing, we have difficulties to describe the range of the induced map $Q_1 : L^1(s(h)M(s(h), \psi) \rightarrow L^1(s(h)M(s(h), \psi)$ and we do not see how use Theorem 1.1 and complex interpolation for identifying the range of $P$.

The main part of this paper consists precisely to investigate the fine structure of $P$ in order to determine the range of $P$. We will prove that the range of $P$ is isometrically isomorphic to a complex interpolation space of the form $(\mathcal{N}, \mathcal{N}_+)^\#$ for a suitable compatibility, where $\mathcal{N}$ is a JW*-algebra and where $\mathcal{N}_+$ is its predual, in large cases. These spaces are nonassociative $L^p$-spaces associated with JW*-algebra, notion which we will introduce in a companion paper [Arh23].

Finally, we give related complements in another publication [Arh22]. Indeed, we study the contractively decomposable projections where contractively decomposable maps are defined in [ArK23] and [JuR04]. Finally, we refer to [HRS03], [PiX03] and [RaX03] for more information on the structure of noncommutative $L^p$-spaces.

Structure of the paper The paper is organized as follows. Section 2 gives a presentation of Haagerup noncommutative $L^p$-spaces. We equally state and prove some results on Banach space geometry which are necessary for this paper. In Section 3, we recall some information on
Jordan algebras and we examine some contractive projections which we call Jordan conditional expectations. In Section 4, we give a proof of Theorem 1.1. We closely follow the approach of [AriR19]. Lemma 2.6 allows us to clarify the proof. Section 5 contains an analysis of the structure of the projection \( P \) of Theorem 1.1. We prove that in large cases its range is isometrically isomorphic to a nonassociative \( L^p \)-space. Finally, we raise several an open problem in Section 6 related to the content of this paper.

2 Haagerup’s noncommutative \( L^p \)-spaces and Banach space geometry

It is well-known that there are several equivalent constructions of noncommutative \( L^p \)-spaces associated with a von Neumann algebra. In this paper, we will use Haagerup’s noncommutative \( L^p \)-spaces introduced in [Haa79a] and presented in a more detailed way in [Ter81]. We denote by \( s(x) \) the support of a positive operator \( x \). If \( \mathcal{M} \) is a von Neumann algebra equipped with a normal semifinite faithful trace, then the topological \( * \)-algebra of all (unbounded) \( \tau \)-measurable operators \( x \) affiliated with \( \mathcal{M} \) is denoted by \( L^p(\mathcal{M}, \tau) \).

In the sequel, we fix a normal semifinite faithful weight \( \varphi \) on a von Neumann algebra \( \mathcal{M} \) acting on a Hilbert space \( H \). The one-parameter modular automorphisms group associated with \( \varphi \) is denoted by \( \sigma^\varphi = (\sigma^\varphi_t)_{t \in \mathbb{R}} \) [Tak03, p. 92]. We denote by \( \mathfrak{m}_p^\varphi \) the set of all positive \( x \in \mathcal{M} \) such that \( \psi(x) < \infty \) and \( \mathfrak{m}_p^\varphi \) its complex linear span.

For \( 1 \leq p < \infty \), the spaces \( L^p(\mathcal{M}) \) are constructed as spaces of measurable operators relative not to \( \mathcal{M} \) but to some semifinite bigger von Neumann algebra, namely, the crossed product \( \hat{\mathcal{M}} \) of \( \mathcal{M} \) and \( \mathbb{R} \) by one of its modular automorphisms groups, that is, the von Neumann subalgebra of \( B(L^2(\Omega)) \) generated by the operators \( \pi(x) \) and \( \lambda_s \circ \text{Id}_H \), where \( x \in \mathcal{M} \) and \( s \in \mathbb{R} \), defined by

\[
(\pi(x)\xi)(t) \overset{\text{def}}{=} \sigma^\varphi_{xt}(x)(\xi(t)) \quad \text{and} \quad (\lambda_s \circ \text{Id}_H)(\xi(t)) \overset{\text{def}}{=} \xi(t-s), \quad t \in \mathbb{R}, \ \xi \in L^2(\mathbb{R}, H).
\]

For any \( s \in \mathbb{R} \), let \( W(s) \) be the unitary operator on \( L^2(\mathbb{R}, H) \) defined by

\[
(W(s)\xi)(t) \overset{\text{def}}{=} e^{-ist}\xi(t), \quad \xi \in L^2(\mathbb{R}, H).
\]

The dual action \( \hat{\sigma} : \mathbb{R} \to B(\hat{\mathcal{M}}) \) on \( \mathcal{M} \) [Tak03, p. 260] is given by

\[
\hat{\sigma}_s(x) \overset{\text{def}}{=} W(s)xW(s)^*, \quad x \in \hat{\mathcal{M}}, \ s \in \mathbb{R}.
\]

Then, by [Haa78a, Lemma 3.6] or [Tak03, p. 259], \( \hat{\pi}(\mathcal{M}) \) is the fixed subalgebra of \( \hat{\mathcal{M}} \) under the family of automorphisms \( \hat{\sigma}_s \):

\[
\hat{\pi}(\mathcal{M}) = \{ x \in \hat{\mathcal{M}} : \hat{\sigma}_s(x) = x \quad \text{for all} \ s \in \mathbb{R} \}.
\]

We identify \( \mathcal{M} \) with the subalgebra \( \pi(\mathcal{M}) \) in \( \hat{\mathcal{M}} \). If \( \psi \) is a normal semifinite weight on \( \mathcal{M} \), we denote by \( \hat{\psi} \) its Takesaki’s dual weight on the crossed product \( \mathcal{M} \), see the introduction of [Haa78b] for a simple definition using the theory of operator valued weights. This dual weight satisfies the \( \hat{\sigma} \)-invariance relation \( \hat{\psi} \circ \hat{\sigma} = \hat{\psi} \), see [Ter81, (10) p.]. In fact, Haagerup introduces an operator valued weight \( T : \mathcal{M}^+ \to \mathcal{M}^+ \) with values in the extended positive part\(^1\) \( \hat{\mathcal{M}}^+ \) of \( \mathcal{M} \) and formally defined by

\[
T(x) = \int_{\mathbb{R}} \hat{\sigma}_s(x) \, ds
\]

\(^1\) If \( \mathcal{M} = L^\infty(\Omega) \) then \( \mathcal{M}^+ \) identifies to the set of equivalence classes of measurable functions \( \Omega \to [0, \infty] \).
and shows that for a normal semifinite weight \( \psi \) on \( \mathcal{M} \), its dual weight is

\[(2.6) \quad \hat{\psi} \defeq \tilde{\psi} \circ T\]

where \( \hat{\psi} \) denotes the natural extension of the normal weight \( \psi \) to the whole of \( \mathcal{M}^+ \).

By [Str81, p. 301] [Haa78a, Th. 3.7] [Ter81, Chap. II, Lemma 1], the map \( \psi \to \hat{\psi} \) is a bijection from the set of normal semifinite weights on \( \mathcal{M} \) onto the set of normal semifinite \( \hat{\sigma} \)-invariant weights on \( \mathcal{M} \).

Recall that by [Haa79b, Lemma 5.2 and Remark p. 343] and [Haa78b, Th. 1.1 (c)] the crossed product \( \mathcal{M} \) is semifinite and there is a unique normal semifinite faithful trace \( \tau = \tau_\varphi \) on \( \mathcal{M} \) satisfying

\[(2.7) \quad (D\hat{\varphi} : D\tau)_t = \lambda_t \otimes 1_{H}, \quad t \in \mathbb{R}\]

where \( (D\hat{\varphi} : D\tau)_t \) denotes the Connes cocycle [Str81, p. 48] [Tak03, p. 111] of the dual weight \( \hat{\varphi} \) with respect to \( \tau \). Moreover, \( \tau \) satisfies the relative invariance \( \tau \circ \hat{\sigma}_s = e^{-s\tau} \) for any \( s \in \mathbb{R} \) by [Haa79b, Lemma 5.2].

If \( \psi \) is a normal semifinite weight on \( \mathcal{M} \), we denote by \( h_\psi \) the Pedersen-Takesaki derivative of the dual weight \( \hat{\psi} \) with respect to \( \tau \) given by [Str81, Theorem 4.10]. By [Str81, Corollary 4.8], note that the relation of \( h_\psi \) with the Radon-Nikodym cocycle of \( \hat{\psi} \) is

\[(2.8) \quad (D\hat{\varphi} : D\tau)_t = h_\psi^t, \quad t \in \mathbb{R}.\]

By [Ter81, Chap. II, Prop. 4], the mapping \( \psi \to h_\psi \) gives a bijective correspondence between the set of all normal semifinite weights on \( \mathcal{M} \) and the set of positive selfadjoint operators \( h \) affiliated with \( \mathcal{M} \) satisfying

\[(2.9) \quad \hat{\sigma}_s(h) = e^{-s}h, \quad s \in \mathbb{R}.\]

Moreover, by [Ter81, Chap. II, Cor. 6], \( \omega \) belongs to \( \mathcal{M}^+_1 \) if and only if \( h_\omega \) belongs to \( L^0(\hat{\mathcal{M}}, \tau)_+ \). The Haagerup space \( L^1(\mathcal{M}, \varphi) \) is defined as the set \( \{ h_\omega : \omega \in \mathcal{M}_s \} \), i.e., the range of the previous map. This is a closed linear subspace of \( L^0(\mathcal{M}, \tau) \), characterized by the conditions (2.9).

By [Ter81, Chap. II, Th. 7], the mapping \( \omega \to h_\omega \), \( \mathcal{M}_s \to L^1(\mathcal{M}, \varphi) \) is a linear order isomorphism which preserves the conjugation, the module, and the left and right actions of \( \mathcal{M} \). Then \( L^1(\mathcal{M}, \varphi) \) may be equipped with a continuous linear functional \( \text{Tr}_\varphi : L^1(\mathcal{M}) \to \mathbb{C} \) defined by

\[(2.10) \quad \text{Tr}_\varphi(h_\omega) \defeq \omega(1), \quad \omega \in \mathcal{M}_s.\]

We also use the notation \( \text{Tr} \) instead of \( \text{Tr}_\varphi \). A norm on \( L^1(\mathcal{M}, \varphi) \) may be defined by \( \| h \|_1 \defeq \text{Tr}(|h|) \) for every \( h \in L^1(\mathcal{M}, \varphi) \). By [Ter81, Chap. II, Prop. 15], the map \( \mathcal{M}_s \to L^1(\mathcal{M}, \varphi) \), \( \omega \to h_\omega \) is a surjective isometry.

More generally for \( 1 \leq p \leq \infty \), the Haagerup \( L^p \)-space \( L^p(\mathcal{M}, \varphi) \) associated with the normal faithful semifinite weight \( \varphi \) is defined [Ter81, Chap. II, Def. 9] as the subset of the topological \( * \)-algebra \( L^0(\mathcal{M}, \tau) \) of all (unbounded) \( \tau \)-measurable operators \( x \) affiliated with \( \mathcal{M} \) satisfying for any \( s \in \mathbb{R} \) the condition

\[(2.11) \quad \hat{\sigma}_s(x) = e^{-\frac{s}{\tau}}x \quad \text{if} \quad p < \infty \quad \text{and} \quad \hat{\sigma}_s(x) = x \quad \text{if} \quad p = \infty\]

where \( \hat{\sigma}_s : L^0(\mathcal{M}, \tau) \to L^0(\mathcal{M}, \tau) \) is here the continuous \( * \)-automorphism obtained by a natural extension of the dual action (2.3) on \( \mathcal{M} \). By (2.4), the space \( L^\infty(\mathcal{M}, \varphi) \) coincides with \( \pi(\mathcal{M}) \) that
we identify with \( \mathcal{M} \). The spaces \( L^p(\mathcal{M}, \varphi) \) are closed selfadjoint linear subspaces of \( L^0(\mathcal{M}, \tau) \). They are closed under left and right multiplications by elements of \( \mathcal{M} \). If \( h = u|h| \) is the polar decomposition of \( h \in L^0(\mathcal{M}, \tau) \) then by \cite[Chap. II, Prop. 12]{Ter81} we have

\[
h \in L^p(\mathcal{M}, \varphi) \iff u \in \mathcal{M} \text{ and } |h| \in L^p(\mathcal{M}, \varphi).
\]

Suppose \( 1 \leq p < \infty \). By \cite[Chap. II, Prop. 12]{Ter81} and its proof, for any \( h \in L^0(\mathcal{M}, \tau)_+ \), we have \( h^p \in L^0(\mathcal{M}, \tau)_+ \). Moreover, an element \( h \in L^0(\mathcal{M}, \tau) \) belongs to \( L^p(\mathcal{M}, \varphi) \) if and only if \( |h|^p \) belongs to \( L^1(\mathcal{M}, \varphi) \). A norm on \( L^p(\mathcal{M}, \varphi) \) is then defined by the formula

\[
\|h\|_p \defeq (\text{Tr}|h|^p)^{\frac{1}{p}}
\]
if \( 1 \leq p < \infty \) and by \( \|h\|_\infty \defeq \|h\|_\mathcal{M} \), see \cite[Chap. II, Def. 14]{Ter81}.

**Case of a normal faithful linear form** If \( \varphi \) is a normal faithful linear form on \( \mathcal{M} \) then by \cite[(1.13)]{HJX10} \( h_\varphi \) belongs to \( L^1(\mathcal{M}, \varphi) \) and

\[
\varphi(x) = \text{Tr}(h_\varphi x) = \text{Tr}(x h_\varphi), \quad x \in \mathcal{M}.
\]

**Duality** Let \( p, p^* \in [1, \infty] \) with \( \frac{1}{p} + \frac{1}{p^*} = 1 \). By \cite[Chap. II, Prop. 21]{Ter81}, for any \( h \in L^p(\mathcal{M}, \varphi) \) and any \( k \in L^{p^*}(\mathcal{M}, \varphi) \) we have \( hk, kh \in L^1(\mathcal{M}, \varphi) \) and the tracial property \( \text{Tr}(hk) = \text{Tr}(kh) \).

If \( 1 \leq p < \infty \), by \cite[Ch. II, Th. 32]{Ter81} the bilinear form \( L^p(\mathcal{M}, \varphi) \times L^{p^*}(\mathcal{M}, \varphi) \to \mathbb{C}, (h, k) \mapsto \text{Tr}(hk) \) defines a duality bracket between \( L^p(\mathcal{M}, \varphi) \) and \( L^{p^*}(\mathcal{M}, \varphi) \), for which \( L^{p^*}(\mathcal{M}, \varphi) \) is (isometrically) the dual of \( L^p(\mathcal{M}, \varphi) \).

**Change of weight** It is essentially proved in \cite[p. 59]{Ter81} that \( L^p(\mathcal{M}, \varphi) \) is independent of \( \varphi \) up to an isometric isomorphism preserving the order and modular structure of \( L^p(\mathcal{M}, \varphi) \), as well as the external products and Mazur maps. In fact given two normal semifinite faithful weights \( \varphi_1, \varphi_2 \) on \( \mathcal{M} \) there is a *-isomorphism \( \kappa: \mathcal{M}_1 \to \mathcal{M}_2 \) between the crossed products \( \mathcal{M}_1 \defeq \mathcal{M} \rtimes_{\varphi_1} \mathbb{R} \) preserving \( \mathcal{M} \), as well as the dual actions and pushing the trace on \( \mathcal{M}_1 \) onto the trace on \( \mathcal{M}_2 \), that is

\[
\pi_2 = \kappa \circ \pi_1, \quad \tilde{\sigma}_2 \circ \kappa = \kappa \circ \tilde{\sigma}_1 \quad \text{and} \quad \tau_2 = \tau_1 \circ \kappa^{-1}.
\]
Furthermore, \( \kappa \) extends naturally to a topological *-isomorphism \( \kappa: L^0(\mathcal{M}_1, \tau_1) \to L^0(\mathcal{M}_2, \tau_2) \) between the algebras of measurable operators, which restricts to isometric *-isomorphisms between the noncommutative \( L^p \) -spaces \( L^p(\mathcal{M}, \varphi_1) \) and \( L^p(\mathcal{M}, \varphi_2) \), preserving the \( \mathcal{M} \)-bimodule structures.

Moreover it turns out also that for every normal semifinite faithful weight \( \psi \) on \( \mathcal{M} \), the dual weights \( \psi_1 \) corresponds through \( \kappa \), that is \( \psi_2 \circ \kappa = \psi_1 \). It follows that if \( \omega \in \mathcal{M}_* \), the corresponding Pedersen-Takesaki derivatives must verify \( h_{\omega, 2} = \tilde{\kappa}(h_{\omega, 1}) \). In particular if \( \omega \in \mathcal{M}_+^+ \), we have

\[
\text{Tr}_{\varphi_1} h_{\omega, 1} = (2.10) \omega(1) (2.10) \text{Tr}_{\varphi_2} h_{\omega, 2} = \text{Tr}_{\varphi_2} \kappa(h_{\omega, 1}).
\]
Hence \( \kappa: L^1(\mathcal{M}, \varphi_1) \to L^1(\mathcal{M}, \varphi_2) \) preserves the traces:

\[
\text{Tr}_{\varphi_1} = \text{Tr}_{\varphi_2} \circ \kappa.
\]
Since \( \kappa \) preserves the \( p \)-powers operations, i.e. \( \kappa(h^p) = (\kappa(h))^p \) for any \( h \in L^0(\mathcal{M}) \), it induces an isometry from \( L^p(\mathcal{M}, \varphi_1) \) onto \( L^p(\mathcal{M}, \varphi_2) \). It is not hard to see that this isometry is completely positive and completely isometric, a fact which is of first importance for our study.

This independence allows us to consider \( L^p(\mathcal{M}, \varphi) \) as a particular realization of an abstract space \( L^p(\mathcal{M}) \).

**Centralizer of a weight** Recall that the centralizer \([\text{Str}81, \text{p. 38}]\) of a normal semifinite faithful weight is the sub-von Neumann algebra \( \mathcal{M}^\varphi = \{ x \in \mathcal{M} : \sigma_t^\varphi(x) = x \text{ for all } t \in \mathbb{R} \} \). If \( x \in \mathcal{M} \), we have by \([\text{Str}81, \text{(2) p. 39}]\)

\[
(2.17) \quad x \in \mathcal{M}^\varphi \iff xm_\varphi \subset m_\varphi, m_\varphi x \subset m_\varphi \text{ and } \varphi(xy) = \varphi(yx) \text{ for any } y \in m_\varphi.
\]

**Reduced noncommutative \( L^p \)-spaces** If the projection \( e \) belongs to the centralizer of \( \varphi \), the restriction \( \varphi_e \) of \( \varphi \) on \( eMe \). It results from (2.17) that the weight \( \varphi_e \) is still semifinite and is well-known that we can identify \( L^p(eMe, \varphi_e) \) with the subspace \( eL^p(\mathcal{M}, \varphi)e \) of \( L^p(\mathcal{M}, \varphi) \) (see \([\text{Wat}98, \text{p. 508}]\)). Moreover, we have the following result.

**Lemma 2.1** The Haagerup trace \( \text{Tr}_\varphi \) restricts to \( \text{Tr}_{\varphi_e} \) on \( L^1(eMe) \).

Let \( e \in \mathcal{M} \) be a projection. Let us construct a normal semifinite weight on \( \mathcal{M} \) with centralizer containing \( e \). Consider two normal semifinite faithful weights \( \varphi_1 \) and \( \varphi_2 \) on \( eMe \) and \( e^\perp Me^\perp \). By \([\text{Ra}03, \text{p. 155}]\), we can define a normal semifinite faithful weight \( \varphi \) on \( \mathcal{M} \) by

\[
(2.18) \quad \varphi(x) \overset{\text{def}}{=} \varphi_1(eexe) + \varphi_2(e^\perp x e^\perp), \quad x \in \mathcal{M}_+.
\]

Moreover, \( e \) belongs to the centralizer of \( \varphi \) by (2.17) and we have \( \varphi_e = \varphi_1 \).

The following is an easy folklore observation.

**Lemma 2.2** Let \( \mathcal{M} \) be a von Neumann algebra and \( 1 \leq p < \infty \). Let \( h \) be a positive element of \( L^p(\mathcal{M}) \).

1. The map \( s(h)M_s(h) \to L^p(\mathcal{M}) \), \( x \mapsto h^\frac{1}{p} x h^\frac{1}{p} \) is injective.
2. Suppose \( 1 \leq p < \infty \). The subspace \( h^\frac{1}{p} M h^\frac{1}{p} \) is dense in \( s(h)L^p(\mathcal{M})s(h) \) for the topology of \( L^p(\mathcal{M}) \).

**Lifting result** Our main tool will be the following result of \([\text{ArR}19]\) which can be proved with the same ideas.

**Theorem 2.3** Let \( \mathcal{M} \) and \( \mathcal{N} \) be von Neumann algebras. Suppose \( 1 \leq p < \infty \). Let \( T : L^p(\mathcal{M}) \to L^p(\mathcal{N}) \) be a positive linear map. Let \( h \) be a positive element of \( L^p(\mathcal{M}) \). Then there exists a unique linear map \( v : \mathcal{M} \to s(T(h))\mathcal{N}s(T(h)) \) such that

\[
(2.19) \quad T(h^\frac{1}{p} x h^\frac{1}{p}) = T(h)^{\frac{1}{p}} v(x) T(h)^{\frac{1}{p}}, \quad x \in \mathcal{M}.
\]

Moreover, this map \( v \) is unital, contractive, positive and normal. Furthermore, if \( T \) is \( n \)-copositive then \( v \) is \( n \)-copositive.
Extension of maps on noncommutative $L^p$-spaces Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful weight $\varphi$. Suppose that $\mathcal{N}$ is another von Neumann algebra equipped with a normal faithful weight $\psi$. Consider a unital positive map $T: \mathcal{M} \to \mathcal{N}$ such that $\psi \circ T = \varphi$. Given $1 \leq p < \infty$, the map

$$
T_p: \frac{h_\varphi}{\varphi} m_{\psi} h_\varphi^* \longrightarrow L^p(\mathcal{N})
$$

extends to a contractive map $T_p$ from $L^p(\mathcal{M})$ into $L^p(\mathcal{N})$. See [HJX10, Remark 5.6].

Normalized duality mappings Recall that a normed linear space $X$ is said to be strictly convex (or rotund) if for any $x, y \in X$ the equalities $\frac{\|x + y\|}{2} = \|x\| = \|y\|$ imply $x = y$.

Let $X$ be a Banach space. For each $x \in X$, we can associate [Pat18, Definition 2.12] the subset

$$
J_X(x) \overset{\text{def}}{=} \{ x^* \in X^* : \langle x, x^* \rangle_{X, X^*} = \|x\|_X^2 = \|x^*\|_{X^*}^2 \}
$$

of the dual $X^*$.

The multivalued operator $J_X: X \to X^*$ is called the normalized duality mapping of $X$. From the Hahn-Banach theorem, for every $x \in X$, there exists $y^* \in X^*$ with $\|y^*\|_{X^*} = 1$ such that $\langle x, y^* \rangle_{X, X^*} = \|x\|_X$. Using $x^* = \|x\|_X y^*$, we conclude that $J_X(x) \neq \emptyset$ for each $x \in X$. If the dual space $X^*$ is strictly convex, $J_X$ is single-valued.

When $X$ is a reflexive strictly convex Banach space with a strictly convex dual space $X^*$, $J_X$ is a singlevalued bijective map and its inverse $J_X^{-1}: X^* \to X^{**} = X$ is equal to $J_{X^*}: X \to X$.

If the Banach space $X$ is a noncommutative $L^p$-space, we have the following explicit description of the normalized duality mapping of [ArR19].

**Lemma 2.4** Suppose $1 < p < \infty$. If $h$ belongs to $L^p(\mathcal{M})$ with polar decomposition $h = u|h|$ then we have

$$
J_{L^p(\mathcal{M})}(h) = \|h\|_{L^p}^{2-p} |h|^{p-1} u^*.
$$

**Projections** The following is [Meg98, Theorem 3.2.6 p. 297] combined with [FHHMPZ01, 5.10 p. 148].

**Proposition 2.5** Let $X$ be a Banach space and consider a bounded map $P: X \to X$. Then $P$ is a projection if and only if $P^*: X^* \to X^*$ is a projection. In this case, $P(X)^* \text{ is isomorphic to } P^*(X^*)$.

The following lemma is a generalization of [And66, Lemma 1] and is fundamental for us.

**Lemma 2.6** Let $X$ be a smooth strictly convex reflexive Banach space. Let $P: X \to X$ be a contractive projection and $x$ be an element of $X$. Then $x$ belongs to $\text{Ran} \ P$ if and only if $J_X(x)$ belongs to $\text{Ran} \ P^*$.

**Proof** : Recall that by [Meg98, Theorem 3.2.6 p. 297] the adjoint map $P^*: X^* \to X^*$ is a contractive projection. Note that $X^*$ is strictly convex by [Meg98, Proposition 5.4.7 p. 481].
\[ \Rightarrow: \text{Suppose } x \in \text{Ran } P. \text{ We have} \]
\[ \|x\|_X^{(\ref{2.21})} = \langle x, J_X(x) \rangle_{X,X^*} = \langle P^2(x), J_X(x) \rangle_{X,X^*} = \langle P(x), P^*(J_X(x)) \rangle_{X,X^*} = \langle x, P^*(J_X(x)) \rangle_{X,X^*} \leq \|x\|X \|P^*(J_X(x))\|_{X^*} \leq \|x\| \|P^*(J_X(x))\|_{X^*} \leq \|x\|_X \|J_X(x)\|_{X^*}. \]

We infer that \( \|P^*(J_X(x))\|_{X^*} = \|x\|_X \) and \( \langle x, P^*(J_X(x)) \rangle_{X,X^*} = \|x\|_X^2 \). Since \( X^* \) is strictly convex, we conclude that \( P^*(J_X(x)) = J_X(x) \), i.e. \( J_X(x) \) belongs to \( \text{Ran } P^* \).

\[ \Leftarrow: \text{Suppose that } J_X(x) \text{ belongs to } \text{Ran } P^*. \text{ Since } X \text{ is strictly convex, the first part applied to } P^* \text{ instead of } P \text{ shows that } x = J_X(x) \text{ belongs to } \text{Ran } P. \]

We will use the following well-known result. See [ArK23, Lemma 4.8] for a slightly more general statement.

**Lemma 2.7** Let \((E_0, E_1)\) be an interpolation couple and let \(C\) be a contractively complemented subspace of \(E_0 + E_1\). We assume that the corresponding contractive projection \(P: E_0 + E_1 \to E_0 + E_1\) satisfies \(P(E_1) \subset E_1\) and that the restriction \(P: E_1 \to E_1\) is contractive for \(i = 0, 1\). Then \((E_0 \cap C, E_1 \cap C)\) is an interpolation couple and the canonical inclusion \(J: C \to E_0 + E_1\) induces an isometric isomorphism \(J\) from \((E_0 \cap C, E_1 \cap C)_\theta\) onto the subspace \(P((E_0, E_1)_\theta) = (E_0, E_1)_\theta \cap C\) of \((E_0, E_1)_\theta\).

### 3 Jordan algebras and Jordan conditional expectations

**Various Jordan algebras** A Jordan algebra \(A\) over a field \(\mathbb{K}\) is a vector space \(A\) over \(\mathbb{K}\) equipped with a commutative bilinear product that satisfies \((x^2 \circ y) \circ x = x^2 \circ (y \circ x)\) for any \(x, y \in A\), see e.g. [AlS03, Definition 1.1 p. 3]. A Jordan algebra \(A\) over \(\mathbb{R}\) is called formally real [HOS84, p. 69] if for any \(x_1, \ldots, x_n \in A\) the relation \(x_1^2 + \cdots + x_n^2 = 0\) implies \(x_1 = \cdots = x_n = 0\). Following [AlS03, Definition 1.5 p. 5], a JB-algebra is a Jordan algebra over \(\mathbb{R}\) with identity element \(1\) equipped with a complete norm satisfying the properties \(\|x \circ y\| \leq \|x\|\|y\|\), \(\|x^2\| = \|x\|^2\), \(\|x^2\| \leq \|x^2 + y^2\|\) for any \(x, y \in A\). A JBW-algebra is a JB-algebra which is a dual Banach space [HOS84, p. 111]. In this case, the predual is unique.

A JB*-algebra [HOS84, p. 91] [CGRP14, Definition 3.3.1 p. 345] is a complex Banach space \(A\) which is a complex Jordan algebra equipped with an involution satisfying
\[ (\ref{3.1}) \quad \|x \circ y\| \leq \|x\|\|y\|, \quad \|x^*\| = \|x\| \quad \text{and} \quad \|\{x, x^*, y\}\| = \|x\|^3 \]
for any \(x, y \in A\), where we use the Jordan triple product \(\{x, y, z\} \overset{\text{def}}{=} (x \circ y) \circ z + (y \circ z) \circ x - (x \circ z) \circ y\). A JBW*-algebra [CGRP18, p. 4] is a JB*-algebra which is a dual Banach space.

Let \(H\) be a complex Hilbert space. A JC-algebra [Sto13, Definition 2.1.1] [HOS84, p. 75] is a norm closed real linear subspace of selfadjoint operators of \(B(H)\) closed under the Jordan product \((x, y) \mapsto x \circ y \overset{\text{def}}{=} \frac{1}{2}(xy + yx)\). By [Sto13, p. 13], the selfadjoint part \(A_{sa}\) of a C*-algebra \(A\) is a JC-algebra.

A JC*-algebra (also called Jordan C*-algebra) is a norm-closed \(*\)-subalgebra of \((B(H), \circ)\) [CGRP14, p. 345]. If \(A\) is a JC*-algebra then \((A, \circ)\) is a JB*-algebra by [CGRP14, p. 345]. A C*-algebra \(A\) is of course a JC*-algebra.

An element \(p\) of JB-algebra such that \(p \circ p = p\) is called a projection.
Centers and factors  Two elements $a$ and $b$ of a Jordan algebra $A$ are said to operator commute [HOS84, p. 44] if for any $c \in A$ we have $(a \circ c) \circ b = a \circ (c \circ b)$. The centre $Z(A)$ of $A$ is the set of all elements of $A$ which operator commute with every element of $A$. By [HOS84, Lemma 2.5.3 p. 45], the centre is an associative subalgebra of $A$. Following [HOS84, p. 115], if the centre of a JBW-algebra $A$ only consists of scalar multiples of the identity, we say that $A$ is a JBW-factor.

If $p$ is a projection of a JW-algebra, the smallest central projection $q$ such that $q \geq p$ is called the central cover of $p$ and denoted by $c(p)$ [AIL83, Definition 2.38 p. 56]. We say that a projection $p$ of a JW-algebra $A$ is abelian if the JW-subalgebra $pAp$ is associative [HOS84, p. 122]. By [AIL83, Definition 4.24], this is equivalent to $pAp$ consists of mutually commuting elements.

We refer to [CGRP81, Theorem 6.1.40 p. 362] for the classification of JBW-factors.

JW-algebras  Recall that a (concrete) JW-algebra [HOS84, p. 95] [ARU97, p. 14] [Sto13, p. 20] is a weak* closed JC-algebra, that is a weak* closed Jordan subalgebra of $B(H)_w$, that is a real linear space of selfadjoint operators which is closed for the weak* topology and closed under the Jordan product $\circ$. Note that a JW-algebra is a JBW-algebra by [HOS84, p. 95]. Recall that a JBW-algebra is always unital by [HOS84, Lemma 4.1.7]. By [AIL83, Proposition 3.19 p. 28], two elements $x$ and $y$ of a JW-algebra operator commute if and only if $x$ and $y$ commute.

Recall that a JW-algebra $A$ is reversible [AIL83, Definition 4.24] [HOS84, p. 25] if it is closed under symmetric products, i.e. if $a_1, \ldots, a_k \in A$ then

$$a_1a_2\cdots a_k + a_k a_{k-1} \cdots a_1 \in A.$$  

Example 3.1  A spin system [HOS84, 6.1.2 p. 135] is a set $\mathcal{P}$ of at least two symmetries (i.e. selfadjoint unitaries) $\neq \pm \Id$ in $B(H)$ which satisfy $s \circ t = 0$, i.e. $st = -ts$, for any $s, t \in \mathcal{P}$ with $s \neq t$. If $\overline{\mathcal{P}}$ is the weak closure of the linear span of $\mathcal{P}$, then $S \overset{\text{def}}{=} \Id \oplus \overline{\mathcal{P}}$ is a JW-algebra [ARU97, pp. 14-15]. These JW-algebras are called (real) spin factors.

It is possible to give an abstract definition [AIL83, Definition 3.33 p. 91] of spin factors. Let $H$ be a real Hilbert space of dimension at least 2, and let $\mathbb{R}1$ denote a one dimensional real Hilbert space with unit vector 1. Let $A \overset{\text{def}}{=} H \oplus \mathbb{R}1$ and consider the product $\circ$ on $A$ defined by

$$(a + \lambda 1) \circ (b + \mu 1) \overset{\text{def}}{=} \mu a + \lambda b + ((a, b) + \lambda \mu)1,$$  

and the norm $\|a + \lambda 1\|_A \overset{\text{def}}{=} \|a\|_{H} + |\lambda|$. Then $A$ is a JB-algebra by [HOS84, Lemma 6.1.3 p. 136] which is isomorphic to $S$.

By [Top66, Theorem 3] [HOS84, Proposition 6.1.5 p. 137], two spin factors are isomorphic if and only if their real Hilbert space dimensions are equal. If card $\mathcal{P} = 4$ or card $\mathcal{P} \geq 6$ then the spin factor is non-reversible by [Sto13, Lemma 2.3.2] [HOS84, Theorem 6.2.5 p. 141].

Example 3.2  If $\mathbb{O}$ is the algebra of octonions, then the space $$H_3(\mathbb{O}) = \left\{ \begin{bmatrix} a & \alpha & \beta \\ \overline{\alpha} & b & \gamma \\ \overline{\beta} & \overline{\gamma} & c \end{bmatrix} : a, \beta, \gamma \in \mathbb{O}, a, b, c, \in \mathbb{R} \right\}$$

of hermitian 3x3 matrices with entries in $\mathbb{O}$ equipped with the product $(x, y) \mapsto xy = \frac{1}{2}(xy + yx)$ is a unital formally real Jordan algebra by [HOS84, Proposition 2.9.2 p. 69] of dimension 27.
By [HOS84, Corollary 3.1.7 p. 77] and its proof, we can equip $H_3(\mathbb{Q})$ with a norm that makes it a JB-algebra. With this structure, $H_3(\mathbb{Q})$ is a JBW-factor [CGRP18, Theorem 6.1.40 p. 362]. By [HOS84, p. 75], $H_3(\mathbb{Q})$ is not a JW-algebra. Note that by [AlS03, Theorem 4.5] every JBW-factor other than $H_3(\mathbb{Q})$ is a JW-algebra.

**Purely exceptional JBW-algebras** Following [HOS84, 7.2.1 p. 155], we say that a JB-algebra $A$ is purely exceptional [HOS84, Theorem 7.2.3 p. 155] if there is no nonzero homomorphism of $A$ into a JC-algebra. By [HOS84, Theorem 7.2.3 p. 155], a JBW-algebra $A$ can be uniquely decomposed as a direct sum

\begin{equation}
A = A_{\exp} \oplus A_{\exp}
\end{equation}

where $A_{\exp}$ is a JW-algebra and $A_{\exp}$ is a purely exceptional JBW-algebra. \textbf{SEE [Shu79]}

**Purely real JW-algebras** A real von Neumann algebra [ARU97, p. 15] is a real $\ast$-subalgebra $R$ of $B(H)$ which is weakly closed satisfying $R \cap iR = \{0\}$. Given a JW-algebra $A$, we denote\(^2\) by $\mathcal{R}(A)$ the closure for the weak* topology of the real algebra generated by $A$ in $B(H)$ (note that this algebra is closed under adjoints). If $A$ is a reversible JW-algebra then $A = \mathcal{R}(A)_{\sa}$ by [AlS03, Lemma 4.25 p. 113] and $A'' = \mathcal{R}(A) + i\mathcal{R}(A)$ by [ARU97, Theorem 1.5] and [Sto68, Theorem 2.4].

A JW-algebra $A$ is said purely real [ARU97, p. 15] if $A$ is reversible and if $\mathcal{R}(A) \cap i\mathcal{R}(A) = \{0\}$. In this case, by [ARU97, pp. 21-22] or [Sto68, Lemma 3.2], the map $\alpha: A'' \to A''$, $z + iy \mapsto z^* + iy^*$ is a $\ast$-antiautomorphism of order 2 and it is easy to check that

\begin{equation}
A = \{ x \in (A'')_{\sa} : \alpha(x) = x \}, \quad A_c = \{ x \in A'' : \alpha(x) = x \}, \quad \mathcal{R}(A) = \{ x \in A'' : \alpha(x) = x^* \}
\end{equation}

where $A_c \overset{\text{def}}{=} A + iA$ denotes the complexification of $A$. The map $P_{\can} \overset{\text{def}}{=} \frac{1 + \alpha}{2} : A'' \to A''$ is a positive contractive normal unital projection called canonical projection of $A''$ onto $A_c$.

We will use the following property [ARU97, Proposition 1.5.1].

**Proposition 3.3** A purely real JW-factor $A$ is not isomorphic to the selfadjoint part of a von Neumann algebra if and only if the von Neumann algebra $A''$ is a factor.

**Remark 3.4** The property of being purely real is not an invariant under isomorphisms. See [Ayu87, p. 1427].

We will use the observation of [ARU97, Proposition 1.1.11].

**Proposition 3.5** Let $A$ be a purely real JW-algebra. Then $A$ does not admit summands isomorphic to the selfadjoint part of a von Neumann algebra if and only if $Z(A'') = Z(A) + iZ(A)$.

The following is [HnS95, Theorem 3.2] (see also [Sto97, Theorem 3.6]).

**Theorem 3.6** Let $M$ be a von Neumann algebra and $A$ be a reversible JW-subalgebra such that $\mathcal{R}(A) \cap i\mathcal{R}(A) = \{0\}$, $A'' = \mathcal{R}(A) + i\mathcal{R}(A)$ and $Z(A) = Z(A'')_{\sa}$. Suppose $Q: M \to M$ is a faithful normal projection on $A$. Then there exists a unique faithful normal conditional expectation $E: M \to M$ on $A''$ such that if $P_{\can}: A'' \to A''$ is the canonical projection on $A$, then $Q = P_{\can} \circ E$.

\(^2\) We warn the reader that this algebra is sometimes denoted $\overline{\mathcal{R}(A)}$ in the literature.
Type Following [AlS03, Definition 3.21 p. 86], we say that a JW-algebra $A$ of type I is of type $I_n$, where $n$ is some cardinal number, if there exists an orthogonal family $p_i$ of $n$ abelian projections in $A$ such that $1 = \sum_i e_i$ and $c(e_i) = 1$ for any $i$. It follows from [HOS84, Proposition 5.3.5 p. 131] that any JBW-algebra of type I can be uniquely decomposed into a direct sum of JW-algebras of type $I_n$. We refer to [AlS03, Theorem 3.39 p. 95] for the classification of JW-factors of type I.

Decomposition An arbitrary JW-algebra $A$ can be uniquely reduced [Top65, Theorem 13] [Ayu87, Theorem 1.1] into a direct sum of five JW-algebras of the following types:

1. modular of type I (type $I_{\text{fin}}$),
2. properly nonmodular locally modular of type I (type $I_{\infty}$),
3. modular of type II (type $\text{II}_1$),
4. properly nonmodular of type II (type $\text{II}_\infty$),
5. purely nonmodular (type III).

We refer to [Top65] for the definitions of used notions. Moreover, each JW-factor belongs to one and only one of these types. Note that if a JW-algebra $A$ coincides with the selfadjoint part of a von Neumann algebra, then this decomposition agrees with the classical classification of von Neumann algebras.

Recall that by [AlS03, Corollary 4.30], a JW-algebra $A$ is reversible if and only if the $I_2$ summand of $A$ is reversible. For the factors, we have the following result [Sto66, Corollary 6.5].

**Theorem 3.7** A JW-factor is either reversible or totally non reversible (hence of type $I_2$).

**Example 3.8** By [ASS78, Proposition 2.3] [Sta81, p. 477], a type $I_1$ JW-algebra $A$ is isomorphic to the Jordan algebra $C(X,\mathbb{R})$ of all real-valued continuous functions on a compact Hausdorff hyperstonian space $X$. It is left to the reader use [DDLS16] that using [BGL22, Theorem 7.19] that this algebra is isomorphic to $L_\infty^\infty(\Omega)$ for some localizable measure space $\Omega$. There exists a unique factor of type $I_1$ up to isomorphism, the factor $\mathbb{R}1$, see [AlS03, p. 96].

**Example 3.9** The type $I_2$ JW-algebras were classified by Stacey in [Sta82, Theorem 2]. If $A$ is a JW-algebra with separable predual then $A$ has type $I_2$ if and only if there exist an index set $I$, a family $(\Omega_i)_{i \in I}$ of second countable locally compact spaces, a family $(\mu_i)_{i \in I}$ of Radon measures on the spaces $\Omega_i$ and a family $(S_i)_{i \in I}$ of spin factors, each of dimension strictly greater than 1 and at most countable giving an isomorphism

$$A = \bigoplus_{i \in I} L_\infty^\infty(\Omega_i, S_i).$$

By [HOS84, Theorem 6.1.8 p. 138], if $A$ is a JBW-algebra then $A$ is a JBW factor of type $I_2$ if and only if $A$ is isomorphic to a spin factor.

We will use the following result of Haagerup and Størmer [HaS95, Theorem 2.1].

**Theorem 3.10** Let $A$ be a JW-algebra of type $I_2$ and let $A''$ be the von Neumann algebra generated by $A$. Then there exists a faithful normal projection $P: A'' \to A''$ onto $A$ if and only if $A''$ is finite.
JW*-algebras A JW*-algebra is a weak* closed JC*-subalgebra of $B(H)$, that is a weak* closed *-subalgebra of $(B(H), \circ)$. If $M$ is a von Neumann algebra then $(M, \circ)$ is obviously a JW*-algebra. A JW*-algebra is a JBW*-algebra. The selfadjoint part of a JW*-algebra is a JW-algebra. Conversely, if $A$ is a JW-algebra (included in $B(H)$) then the complexification $A_C = A + iA$ is a JW*-subalgebra of $B(H)$. For useful results which can be used for transferring results from JW-algebras to JW*-algebras and vice versa, we refer to [BHK17, pp. 4-5] and [CGRP18, Corollary 5.1.29 p. 9].

We refer to [CGRP18, Proposition 6.1.41 p. 362] for the classification of JBW*-factors.

Example 3.11 A von Neumann algebra $M$ equipped with the Jordan product

$$x \circ y \overset{\text{def}}{=} \frac{1}{2}(xy + yx), \quad x, y \in M$$

is a JW*-algebra.

Example 3.12 By [Isi19, Proposition 25.2.2 p. 513], the space $\text{Sym}_n \overset{\text{def}}{=} \{x \in M_n : x^t = x\}$ of symmetric complex matrices of $M_n$ is a JW*-algebra (called Cartan factor of type $\Pi_n$) whose associated JW-algebra is reversible.

Example 3.13 If $n$ is an integer, by [Isi19, Proposition 25.2.2 p. 513] the space $\text{Asym}_{2n} \overset{\text{def}}{=} \{x \in M_{2n} : x^t = -x\}$ of skew-symmetric complex matrices of $M_{2n}$ is equipped with a structure of reversible JW*-algebra.

Example 3.14 By [CGRP18, Proposition 6.1.41 p. 362], the complexification $H_3(\mathbb{O})_C = H_3(\mathbb{O}_C)$ of the JBW-factor $H_3(\mathbb{O})$ is equipped with a structure of JBW*-factor.

Traces A trace on a JBW-algebra $A$ is a function $\tau$ on the set $A_+$ of positive elements of $A$ with values in $[0, +\infty]$ satisfying the following conditions:

1. $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in A_+$,
2. $\tau(\lambda x) = \lambda \tau(x)$ for any $x \in A_+$ and any $\lambda \geq 0$, where $0.(+\infty) = 0$,
3. $\tau(xsx) = \tau(x)$ for all $x \in A_+$ and all arbitrary symmetry $s$ of $A$.

The trace $\tau$ is said to be faithful if $\tau(x) > 0$ for all non-zero $x \in A_+$, finite if $\tau(1) < +\infty$, semifinite if given any $x \in A_+$ there is a non-zero $y \in A_+$, $y \leq x$ with $\tau(y) < +\infty$. The trace $\tau$ is normal if for every net $(x_\alpha)$ monotone increasing to $x$, $(x_\alpha, x \in A_+)$ we have $\tau(x_\alpha) \to \tau(x)$. We refer to [Ay85], [Ay82], [Ay92], [Kin83] an [Pe82] for more information on traces on JBW-algebras.

Every finite trace on a JBW-algebra $A$ can be extended by the linearity to a linear functional on $A$. Thus a finite trace on a JBW-algebra $A$ is a positive linear functional $\tau$ satisfying the condition $\tau(xsx) = \tau(x)$ for all $x \in A$ and all symmetries $s \in A$. By [AIS03, Lemma 5.18 p. 147], it is known that the last condition is equivalent to the formula $\tau(a \circ (b \circ c)) = \tau((a \circ b) \circ c)$ for all $a, b, c \in A$. By complexification, we obtain a positive linear functional on the associated JBW*-algebra $M$ satisfying

$$\tau(a \circ (b \circ c)) = \tau((a \circ b) \circ c), \quad a, b, c \in M.$$ 

It is known that the existence of a normal semifinite faithful trace on a JW*-algebra $A$ is characterized by a condition on $A$ called local modularity, see [ARU97, Theorem 1.2.6 p. 33]. The following is [ARU97, Corollary 1.2.10 p. 35].
Proposition 3.15 Let $A$ be a reversible JW-algebra with a normal trace $\tau$. Then the trace $\tau$ can be extended to a normal trace $\tau_1$ on the von Neumann algebra $A''$. If $\tau$ is faithful (respectively finite or semifinite) then $\tau_1$ is also faithful (respectively finite or semifinite).

Example 3.16 Consider a spin factor $S = \mathbb{R}Id \oplus \mathcal{P}$ as in Example 3.1. By [Al80, Lemma 5.21 p. 149] (see also [HOS84, Proposition 6.1.7 p. 137] and [Top66]), there exists a unique tracial state $\tau$. Moreover, the same reference shows that $\tau$ is normal, faithful and defined by $\tau(Id) = 1$ and $\tau(x) = 0$ for any $x \in \mathcal{P}$.

Example 3.17 The restriction of the trace of the matrix algebra $M_n(\mathbb{C})$ is a faithful normal finite trace on the JW-factor $H_3(\mathbb{C})$ by the proof of [HOS84, Proposition 2.9.2 p. 69].

**Projections on Jordan algebras** The notion of positivity in a JB*-algebra $A$ is defined in [CGRP18, p. 9]. We say that a positive map $T: A \to A$ on a JB*-algebra $A$ is faithful if $T(x) = 0$ for some $x \in A_+$ implies $x = 0$. Similarly to the case of C*-algebras [Str81, p. 116], we introduce the following definition.

**Definition 3.18** Let $N$ be a (unital) JW*-subalgebra of a JW*-algebra $M$. A linear map $Q: M \to M$ is called a Jordan conditional expectation on $N$ if it is a unital positive map of range $N$ which is $N$-modular, that is

\[(3.7) \quad Q(x \circ Q(y)) = Q(x) \circ Q(y), \quad x, y \in M.\]

With $x = 1$ and $y \in N$, we obtain $Q(y) = Q(1 \circ y) (3.7) = Q(1) \circ y = 1 \circ y = y$. It follows that $Q$ is the identity on $N$. Consequently, $Q$ is an idempotent mapping, that is a projection.

We introduce a similar definition for maps acting on JW-algebras.

**Definition 3.19** Let $B$ be a (unital) JW-subalgebra of a JW-algebra $A$. A map $\mathcal{E}: A \to A$ is called a Jordan conditional expectation on $B$ if it is a unital positive map of range $B$ which is $B$-modular, that is

\[(3.8) \quad \mathcal{E}(x \circ \mathcal{E}(y)) = \mathcal{E}(x) \circ \mathcal{E}(y), \quad x, y \in A.\]

We have the following elementary link between these two notions.

**Lemma 3.20** Let $B$ be a (unital) JW-subalgebra of a JW-algebra $A$. Let $\mathcal{E}: A \to A$ be a Jordan conditional expectation then $\mathcal{E}_C: A_C \to A_C$, $x_1 + ix_2 \mapsto \mathcal{E}(x_1) + i\mathcal{E}(x_2)$ is a Jordan conditional expectation on the JW*-subalgebra $B_C$.

**Proof**: Note that $\mathcal{E}_C(A_C) = \mathcal{E}_C(A + iA) = \mathcal{E}(A) + i\mathcal{E}(A)$. Since $\mathcal{E}(A)$ is JW-subalgebra of $A$, we conclude that $\mathcal{E}_C(A_C)$ is a JW*-subalgebra of the JW*-algebra of $A_C$. Finally, if $x = x_1 + ix_2$ and $y = y_1 + iy_2$ belongs to $A_C$, a simple computation\(^3\) gives (3.7).

---

3. For any $x, y \in A_C$, we have with obvious notations

\[
\mathcal{E}_C(x \circ \mathcal{E}_C(y)) = \mathcal{E}_C((x_1 + ix_2) \circ \mathcal{E}(y_1 + iy_2)) = \mathcal{E}(x_1 \circ \mathcal{E}(y_1) + ix_1 \circ \mathcal{E}(y_2) + ix_2 \circ \mathcal{E}(y_1) - x_2 \circ \mathcal{E}(y_2))
\]

\[
= \mathcal{E}(x_1 \circ \mathcal{E}(y_1)) + i\mathcal{E}(x_1) \circ \mathcal{E}(y_2) + i\mathcal{E}(x_2) \circ \mathcal{E}(y_1) - \mathcal{E}(x_2 \circ \mathcal{E}(y_2))
\]

\[(3.8) \quad \mathcal{E}(x_1 \circ \mathcal{E}(y_1) + i\mathcal{E}(x_1) \circ \mathcal{E}(y_2) + i\mathcal{E}(x_2) \circ \mathcal{E}(y_1) - \mathcal{E}(x_2 \circ \mathcal{E}(y_2))
\]

\[
= \mathcal{E}_C(x_1 + ix_2) \circ \mathcal{E}_C(y_1 + iy_2) = \mathcal{E}_C(x_1) \circ \mathcal{E}_C(y_1).
\]
The following is a simple consequence of [Sto13, Proposition 2.2.9] (see also [Eis79, Corollary 1.5] and [Bln21] for related results) but is fundamental for us. Recall that the definite set \( D \) [Sto13, Definition 2.1.4] of a positive map \( Q : A \to A \) on a C*-algebra \( A \) is defined by
\[
D = \{ x \in M_{sa} : Q(x^2) = Q(x)^2 \}.
\]

**Proposition 3.21** Let \( M \) be a von Neumann algebra. Let \( Q : M \to M \) be a weak* continuous faithful unital positive projection. Then \( Q \) is a Jordan conditional expectation and the range \( Q(M) \) is a JW*-subalgebra of \((M,\circ)\).

**Proof**: Note that \((M_{sa},\circ)\) is a JW-algebra. So the restriction \( Q|M_{sa} : M_{sa} \to M_{sa} \) is a weak* continuous faithful unital positive projection. By [Sto13, Proposition 2.2.8] applied with \( e = 1 \), \( Q(M_{sa}) \) is a JW-subalgebra\(^4\) of \((M_{sa},\circ)\). Moreover, by [Sto13, Theorem 2.2.2], \( Q(M_{sa}) \) is contained in the definite set of \( Q \). Using [Sto13, Proposition 2.1.7 (i)] in the first equality, we infer that
\[
Q(x \circ Q(y)) = Q(x) \circ Q^2(y) = Q(x) \circ Q(y), \quad x, y \in M_{sa}.
\]
Hence \( Q|M_{sa} : M_{sa} \to M_{sa} \) is a Jordan conditional expectation. With Lemma 3.20, we conclude that \( Q = (Q|M_{sa})_{c} : M \to M \) is a Jordan conditional expectation on \((Q(M_{sa}))_{c} = Q(M)\).

**Selfadjoint maps** Let \( M \) be a von Neumann algebra equipped with a normal semifinite faithful trace \( \tau \). Recall that a **positive** normal contraction \( T : M \to M \) is selfadjoint with respect to \( \tau \) [Jmx06, p. 49] if for any \( x, y \in M \cap L^1(M) \) we have \( \tau(T(xy)) = \tau(xT(y)) \).

We have a similar notion for a normal state \( \varphi \) on \( M \) instead of the trace \( \tau \), see [Jmx06, p. 122]. It is important to note that if \( \varphi \) is a normal state there is nothing to ensure that a \( \varphi \)-preserving normal Jordan conditional expectation \( Q : M \to M \) is selfadjoint contrary to the case of classical conditional expectations on von Neumann algebras. But we will show that such a map is Jordan-selfadjoint:
\[
\varphi(Q(x) \circ y) = \varphi(x \circ Q(y)), \quad x, y \in M.
\]
It is a difficulty for identify the range of its \( L^p \)-extension in Section 5. Nevertheless, we have the following fundamental observation which will be used in Section 5.

**Proposition 3.22** Let \( M \) be a von Neumann algebra equipped with a normal semifinite faithful trace. Let \( Q : M \to M \) be a trace preserving normal Jordan conditional expectation. Then \( Q \) is selfadjoint.

**Proof**: Using the preservation of the trace by \( Q \) in the third and the sixth equalities, for any \( x, y \in M \cap L^1(M) \), we obtain
\[
\tau(Q(x) \circ y) = \tau(Q(Q(x) \circ y)) = \tau(Q(x) \circ Q(y)) = \tau(Q(x \circ Q(y))) = \tau(x \circ Q(y)).
\]

Let \( A \) be a JBW-algebra equipped with a normal faithful state \( \varphi \). We say that a normal map \( Q : A \to A \) is Jordan-selfadjoint if
\[
\varphi(T(x) \circ y) = \varphi(x \circ T(y)), \quad x, y \in A.
\]

The following elementary observation is elementary but is crucial for us. This is a consequence of the computation \( \tau(x \circ y) = \frac{1}{2} \tau(xy + yx) = \tau(xy) \) which is true for any \( x, y \in M \).

\(^4\) Note that [Sto68, Theorem 2.2.2] says that \( Q(M_{sa}) \) is a JC-subalgebra of \( M \).
Proposition 3.23 Let \( \mathcal{M} \) be a von Neumann algebra equipped with a normal finite faithful trace \( \tau \). A normal map \( T: \mathcal{M} \to \mathcal{M} \) is Jordan-selfadjoint map\(^5\) if and only if \( T \) is selfadjoint.

Existence of Jordan conditional expectations Let \( A \) be a JBW-algebra and \( B \) a JBW-subalgebra of \( A \). Suppose that \( \tau \) is a faithful normal tracial state on \( A \). If we also denote by \( \tau \) the restriction of \( \tau \) on \( B \), it is essentially showed in [HaS95, Theorem 4.2] (combined with [HaH84, Remark 3.7]) that there exists a faithful normal Jordan conditional expectation \( Q: A \to A \) onto \( B \) such that \( \tau \circ Q = \tau \). See [Edw86, Theorem p. 78] for a previous preliminary result without proof. By complexification, we obtain the following result.

Proposition 3.24 Let \( \mathcal{M} \) and \( \mathcal{N} \) be JBW*-algebras such that \( \mathcal{N} \) is a subalgebra of \( \mathcal{M} \). Let \( \tau \) be a normalized normal finite faithful trace on \( \mathcal{M} \). Then there exists a trace preserving normal faithful Jordan conditional expectation \( Q: \mathcal{M} \to \mathcal{N} \) on \( \mathcal{N} \).

4 A lifting of contractive positive projections on noncom. \( L^p \)-spaces

In this section, we prove Theorem 1.1. We follow [ArR19] with some clarifications. The dependence with respect to \( \varphi \) of the Haagerup trace \( \text{Tr}_\varphi \) defined in (2.10) is the source of technical complications.

Suppose \( 1 < p < \infty \). Let \( \mathcal{M} \) be a \( \sigma \)-finite (= countably decomposable) von Neumann algebra and \( P: L^p(\mathcal{M}) \to L^p(\mathcal{M}) \) be a positive contractive projection. We define the support \( s_P \) of \( P \) as the supremum in \( \mathcal{M} \) of the supports of the positive elements in \( \text{Ran} \ P \):

\[
\tag{4.1}
 s_P \overset{\text{def}}{=} \bigvee_{h \in \text{Ran} \ P, h \geq 0} s(h).
\]

Lemma 4.1 For any \( y \in L^p(\mathcal{M}) \) we have \( P(y) = P(s_P y s_P) \).

Proof: Note that \( P^*: L^p^*(\mathcal{M}) \to L^p^*(\mathcal{M}) \) is also a positive contractive projection. Recall again that a noncommutative \( L^p \)-space is a smooth strictly convex reflexive Banach space (if \( 1 < p < \infty \)). By Lemma 2.6, the map \( J_{L^p(\mathcal{M})} \) of (2.22) induces a bijection from \( \text{Ran} \ P^* \) onto \( \text{Ran} \ P \) and this map and its inverse preserve the positivity. This remark and the formula (4.1) imply that \( s_{P^*} = s_P \). Thus for every \( y \in L^p(\mathcal{M}) \), we obtain that

\[
\left\| P(y) \right\|_{L^p(\mathcal{M})}^2 = \text{Tr} \left( P(y)J_{L^p(\mathcal{M})}(P(y)) \right) = \text{Tr} \left( yP^*(J_{L^p(\mathcal{M})}(P(y))) \right) = \text{Tr} \left( yJ_{L^p(\mathcal{M})}(P(y))s_{P^*} \right) = \text{Tr} \left( yJ_{L^p(\mathcal{M})}(P(y))s_P \right) = \text{Tr} \left( s_P y s_P J_{L^p(\mathcal{M})}(P(y)) \right).
\]

Hence if \( y s_P = 0 \) or \( s_P y = 0 \) we have \( P(y) = 0 \). Now, for any \( y \in L^p(\mathcal{M}) \), we obtain\(^6\)

\[
P(y) = P((1 - s_P)y(1 - s_P)) + P((1 - s_P)ys_P) + P(s_P y(1 - s_P)) + P(s_P y s_P) = P(s_P y s_P).
\]

Let \( \chi \) be a faithful normal state on \( \mathcal{M} \) which exists by [KaR97b, Exercise 7.6.46]. Here, we use the concrete realization \( L^p(\mathcal{M}) = L^p(\mathcal{M}, \chi) \). The following is [ArR19, Proposition 3.4]

\(^5\) Here \( \mathcal{M} \) is equipped with its canonical structure of JW-algebra.

\(^6\) Note that \( s_P (1 - s_P)y(1 - s_P) = 0 \) and similarly for the others.
Lemma 4.2 There exists a positive element \( k \) of \( \text{Ran} \, P \) such that \( s(k) = s(P) \).

We consider a positive element \( k \) as in Lemma 4.2. We have \( P(k) = k \) and \( k \in L^p(\mathcal{M}, \chi)_+ \). We can suppose that \( k \neq 0 \). Note that \( s(k)M(k) \to \mathbb{C} \), \( x \mapsto \text{Tr}_x(k^p x) \) is a faithful normal linear functional on \( s(k)M(k) \). Using the procedure (2.18), we can consider a normal faithful linear functional \( \varphi \) on \( \mathcal{M} \) such that \( s(k) \) belongs to the centralizer of \( \varphi \) and such that the reduced state \( \varphi_{s(k)} \) on \( s(k)M(k) \) satisfies

\[
(4.2) \quad \varphi_{s(k)}(x) = \text{Tr}_x(k^p x), \quad x \in s(k)M(k).
\]

Multiplying \( k \) by a constant, we can suppose that \( \varphi \) is a state. From (2.14), we have a canonical map \( \kappa \) which induces an order and isometric identification \( \kappa: L^p(\mathcal{M}, \chi) \to L^p(\mathcal{M}, \varphi) \) for all \( p \). We let

\[
(4.3) \quad h \overset{\text{def}}{=} \kappa(k), \quad \psi \overset{\text{def}}{=} \varphi_{s(k)}
\]

and \( M_h \overset{\text{def}}{=} s(h)Ms(h) \). By transport of structure, we have\(^7\)

\[
(4.4) \quad P(h) = h \quad \text{and} \quad s(h) = s(k).
\]

In particular, with Lemma 4.1 and Lemma 4.2, we obtain the first point of Theorem 1.1. Furthermore, for any \( x \in M_h \) we have

\[
(4.5) \quad \psi(x) = \varphi_{s(k)}(x) = \text{Tr}_x(k^p x) = \text{Tr}_{x}(\kappa(k)p^x) = \text{Tr}_{x}(\varphi(k)p^x) = \text{Tr}_{x}(h^p x).
\]

Since \( s(h) \) belongs to the centralizer of \( \varphi \), the noncommutative \( L^p \)-space \( L^p(\mathcal{M}_h) \overset{\text{def}}{=} L^p(\mathcal{M}_h, \psi) \) can be identified order and isometrically with the subspace \( s(h)L^p(\mathcal{M}, \varphi)s(h) \) of \( L^p(\mathcal{M}, \varphi) \). By applying Theorem 2.3 to the restriction \( P|_{L^p(\mathcal{M}_h)}: L^p(\mathcal{M}_h) \to L^p(\mathcal{M}) \) and to the positive element \( h \) of \( L^p(\mathcal{M}_h) \) which has support \( s(h) = 1_{\mathcal{M}_h} \), we see that there exists a unique linear map \( Q: \mathcal{M}_h \to s(P(h))Ms(P(h)) = M_h \) such that

\[
(4.6) \quad P(h^\frac{1}{p} x h^\frac{1}{p}) = h^\frac{1}{p} Q(x) h^\frac{1}{p}, \quad x \in \mathcal{M}_h.
\]

Moreover, this map \( Q \) is unital, contractive, normal and positive.

Lemma 4.3 The map \( Q: \mathcal{M}_h \to \mathcal{M}_h \) is faithful.

Proof: Recall that for \( 1 < p < \infty \), the norm of the space \( L^p(\mathcal{M}) \) is strictly monotone\(^8\). Now, we will show that if \( h_0 \in L^p(\mathcal{M}) \) satisfy \( 0 \leq h_0 \leq h \) and \( P(h_0) = 0 \) then \( h_0 = 0 \). We have

\[
h \overset{(4.4)}{=} P(h) = P(h) - P(h_0) = P(h - h_0)
\]

Since \( P \) is contractive, we deduce that \( \|h\|_p = \|P(h - h_0)\|_p \leq \|h - h_0\|_p \). Since \( 0 \leq h - h_0 \leq h \) we infer that \( \|h\|_p = \|h - h_0\|_p \) and finally \( h_0 = 0 \) by strict monotonicity of the \( L^p \)-norm.

Now, we can deduce that \( Q \) is faithful. Indeed, if \( x \in \mathcal{M}_h^+ \) and \( Q(x) = 0 \) we have

\[
P(h^\frac{1}{p} x h^\frac{1}{p}) \overset{(4.6)}{=} h^\frac{1}{p} Q(x) h^\frac{1}{p} = 0.
\]

By [Dix77, 1.6.9], we have \( 0 \leq x \leq \|x\|_\infty \) so \( 0 \leq h^\frac{1}{p} x h^\frac{1}{p} \leq \|x\|_\infty h \). We see that \( h^\frac{1}{p} x h^\frac{1}{p} = 0 \) by the first part of the proof. Since \( \mathcal{M}_h = s(h)Ms(h) \), we conclude by Lemma 2.2 that \( x = 0 \). \( \blacksquare \)

---

\(^7\) Here \( P: L^p(\mathcal{M}, \varphi) \to L^p(\mathcal{M}, \varphi) \).

\(^8\) That means that if \( 0 \leq x \leq y \) with \( x \neq y \) then we have \( \|x\|_p < \|y\|_p \).
Lemma 4.4 The map $Q: \mathcal{M}_h \to \mathcal{M}_h$ is a projection.

Proof: For any $x \in \mathcal{M}_h$, we have

$$P(h^{\frac{1}{2}} x h^{\frac{1}{2}}) = P^2 (h^{\frac{1}{2}} x h^{\frac{1}{2}}) \quad (4.6) \quad P(h^{\frac{1}{2}} Q(x) h^{\frac{1}{2}}) \quad (4.6) \quad h^{\frac{1}{2}} Q^2(x) h^{\frac{1}{2}}.$$

Using the uniqueness of $Q$ given by Theorem 2.3, we infer that $Q^2 = Q$, i.e. $Q$ is a projection.

Now, we prove that $Q$ is $\psi$-invariant i.e. the third point of Theorem 1.1.

Lemma 4.5 We have $\psi \circ Q = \psi$.

Proof: Since $h$ is positive, by Lemma 2.4, we have $J_{L^p(\mathcal{M})}(h) = \|h\|^{2-p} h^{p-1}$. By [PiX03, Corollary 5.2], the Banach space $L^p(\mathcal{M}_h)$ is smooth and strictly convex. Using the contractive dual map $P^*: L^p(\mathcal{M}) \to L^p(\mathcal{M})$ and Lemma 2.6, we see that $P^*(\|h\|^{2-p} h^{p-1}) = \|h\|^{2-p} h^{p-1}$, that is

$$P^*(h^{p-1}) = h^{p-1}. \quad (4.7)$$

For any $k \in L^p(\mathcal{M}_h)$, it follows that

$$\text{Tr}_\varphi \left( h^{p-1} P(k) \right) = \text{Tr}_\varphi \left( P^*(h^{p-1}) k \right) \quad (4.7) \quad \text{Tr}_\varphi(h^{p-1} k).$$

In particular, for any $x \in \mathcal{M}_h$, we have

$$\psi(Q(x)) \quad (4.5) \quad \text{Tr}_\varphi \left( h^{p-1} Q(x) \right) = \text{Tr}_\varphi \left( h^{p-1} (h^{\frac{1}{2}} Q(x) h^{\frac{1}{2}}) \right) \quad (4.6) \quad \text{Tr}_\varphi \left( h^{p-1} P(h^{\frac{1}{2}} x h^{\frac{1}{2}}) \right) \quad (4.8) \quad \text{Tr}_\varphi \left( h^{p-1} (h^{\frac{1}{2}} x h^{\frac{1}{2}}) \right) = \text{Tr}_\varphi(h^{p} x) \quad (4.5) \quad \psi(x).$$

So, we have proved the “only if” part of Theorem 1.1.

Conversely, suppose that the conditions of Theorem 1.1 are satisfied. We can suppose $\|h\|_p = 1$. We introduce the reduced weight $\psi$ on the von Neumann algebra $\mathcal{M}_h \equiv s(h) \mathcal{M}_s(h)$ induced by the state $\text{Tr}_\varphi(h^{p-} )$.

In the sequel, we will use the density operator $h_\psi \in L^1(\mathcal{M}_h, \psi)$ associated with the weight $\psi$ on $\mathcal{M}_h$. From (2.14), we have a canonical map $\kappa$ which induces an order and isometric identification $\kappa: L^p(\mathcal{M}_h, \psi) \to L^p(\mathcal{M}_h, \psi_s(h))$ for all $p$. If $x \in \mathcal{M}_h$, using Lemma 2.1 in the third equality, we see that

$$\text{Tr}_\varphi(h_\psi x) \quad (2.13) \quad \psi(x) = \text{Tr}_\varphi(h^p x) = \text{Tr}_{\varphi, \kappa^{-1}}(h^p x) \quad (2.16) \quad \text{Tr}_\varphi(\kappa^{-1}(h^p x)) = \text{Tr}_\varphi(h_\psi \kappa^{-1}(h^p x)).$$

We conclude that

$$h_\psi = \kappa^{-1}(h^p). \quad (4.10)$$

With the condition 3 of Theorem 1.1, we can consider by (2.20) with $\mathcal{M}$ instead of $m_\varphi$ the contractive positive operator $Q_\varphi: L^p(\mathcal{M}_h, \psi) \to L^p(\mathcal{M}_h, \psi)$ induced by the map $Q: \mathcal{M}_h \to \mathcal{M}_h$ and defined by

$$Q_\varphi(h_\psi x h_\psi) \quad (2.20) \quad h_\psi Q(x) h_\psi, \quad x \in \mathcal{M}_h.$$
For any $x \in \mathcal{M}_h$, note that
\[
Q_p^2(h_x^\frac{1}{\psi} x h_x^\frac{1}{\psi}) = Q_p(h_x^\frac{1}{\psi} Q(x) h_x^\frac{1}{\psi}) = h_x^\frac{1}{\psi} Q^2(x) h_x^\frac{1}{\psi} = h_x^\frac{1}{\psi} Q(x) h_x^\frac{1}{\psi} = Q_p(h_x^\frac{1}{\psi} x h_x^\frac{1}{\psi}).
\]
We deduce that $Q_p^2 = Q_p$, i.e. that the map $Q_p$ is a projection. Since $s(h)$ belongs to the centralizer of $\varphi$, we have a positive identification of $L^p(\mathcal{M}_h, \varphi_{s(h)})$ in the space $L^p(\mathcal{M}, \varphi)$.

For any $x \in \mathcal{M}_h$, we have
\[
P_K(h_x^\frac{1}{\psi} x h_x^\frac{1}{\psi}) = P_K(h_x^\frac{1}{\psi} x h_x^\frac{1}{\psi}) = P(h_x^\frac{1}{\psi} x h_x^\frac{1}{\psi}) = h_x^\frac{1}{\psi} Q(x) h_x^\frac{1}{\psi} = \kappa(h_x^\frac{1}{\psi} Q(x) h_x^\frac{1}{\psi}) = \kappa(Q_p(h_x^\frac{1}{\psi} x h_x^\frac{1}{\psi})).
\]
Hence, by density we conclude that we have the following commutative diagram.

\[
\begin{array}{ccc}
L^p(\mathcal{M}, \varphi) & \overset{P}{\longrightarrow} & L^p(\mathcal{M}, \varphi) \\
\downarrow & & \downarrow \\
L^p(\mathcal{M}_h, \varphi_{s(h)}) & \overset{Q_p}{\longrightarrow} & L^p(\mathcal{M}_h, \varphi_{s(h)})
\end{array}
\]

In particular, we have the inclusion $P(L^p(\mathcal{M}_h, \varphi_{s(h)})) \subset L^p(\mathcal{M}_h, \varphi_{s(h)})$ and in addition the restriction $P|_{L^p(\mathcal{M}_h)}: L^p(\mathcal{M}_h) \to L^p(\mathcal{M}_h)$ is a positive contractive projection where we use the notation $L^p(\mathcal{M}_h) = L^p(\mathcal{M}_h, \varphi_{s(h)})$. Now, we consider the positive contractive map $R: L^p(\mathcal{M}, \varphi) \to s(h)L^p(\mathcal{M})s(h)$, $z \mapsto s(h)zs(h)$ and the canonical isometry $j: L^p(\mathcal{M}_h) \to L^p(\mathcal{M})$. Note that $R \circ j = Id_{L^p(\mathcal{M}_h)}$. Then the point 1 of Theorem 1.1 says that
\[
P = j \circ P|_{L^p(\mathcal{M}_h)} \circ R.
\]
Now, we deduce that
\[
P^2 = jP|_{L^p(\mathcal{M}_h)} RjP|_{L^p(\mathcal{M}_h)} R = j(P|_{L^p(\mathcal{M}_h)})^2 R = jP|_{L^p(\mathcal{M}_h)} R \overset{(4.12)}{=} P.
\]
We conclude that $P$ is a projection. The formula (4.12) shows that $P$ is positive and contractive.

The first part of the last sentence of Theorem 1.1 is a consequence of (4.6) with $x = 1$ and the second part can be deduced from Proposition 3.21 which also says that $Q$ is a Jordan conditional expectation. The proof is complete.

**Remark 4.6** If the contractive projection $P: L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is completely copositive, then by Theorem 2.3, the previous faithful positive normal unital projection $Q_h: \mathcal{M}_h \to \mathcal{M}_h$ is also completely copositive, hence decomposable within the meaning\(^9\) of [Sto13, Definition 1.2.8]. By [Sto13, Theorem 2.2.4], we conclude that the JW-algebra $Q(\mathcal{M}_h)$ is necessarily reversible.

**Remark 4.7** The paper [CNR04] (see also [BuP02] for related facts) furnish additional information on the range $Q(s(h)\mathcal{M}s(h))$ if we have additional knowledge on the von Neumann algebra $\mathcal{M}$. More precisely, we have the following properties.

---

\(^9\) A map is decomposable if it is the sum of a completely positive map and a completely copositive map. This notion is different of the one of [ArK23] and [Arh22].
i. If \( \mathcal{M} \) is of type I then by combining [Li92, Proposition 6.7.2] and [CNR04, Proposition 2.8], we see that the JW-algebra \( Q((s(h)M_0s(h)))_{sa} \) is also of type I.

ii. If \( \mathcal{M} \) is semifinite then with [Li92, Proposition 6.5.9] and [CNR04, Proposition 2.7] we infer that the JW-algebra \( Q((s(h)M_0s(h)))_{sa} \) is also semifinite.

iii. If \( \mathcal{M} \) is finite then using [Li92, Proposition 6.3.1] and [CNR04, Proposition 2.7] we deduce that the JW-algebra \( Q((s(h)M_0s(h)))_{sa} \) is finite, i.e. modular.

**Remark 4.8** The case where the von Neumann algebra \( \mathcal{M} \) is finite and equipped with a normalized normal finite faithful trace and where the projection \( P: L^p(\mathcal{M}) \to L^p(\mathcal{M}) \) satisfies \( P(1) = 1 \) is much simpler and instructive. Indeed, the previous proof shows that the map \( P \) is equal to the \( L^p \)-extension \( Q_p: L^p(\mathcal{M}) \to L^p(\mathcal{M}) \) of a trace preserving normal faithful Jordan conditional expectation \( Q: \mathcal{M} \to \mathcal{M} \) which is reminiscent of the classical result [AbA02, Corollary 5.53 p. 222] which says that a positive contractive projection \( P: L^p(\Omega) \to L^p(\Omega) \) on a classical \( L^p \)-space associated to a probability space \( \Omega \) which makes constant invariant is induced by a conditional expectation.

**Remark 4.9** Our assumption of \( \sigma \)-finiteness can be removed using weights and a gluing argument as in [ArR19].

**Remark 4.10** If \( \mathcal{M} \) is commutative, we recover essentially the structure of contractive projections of [Lac74, Theorem 1] in the positive case.

Note the following module map property twisted by the modular group \((\sigma^t_\varphi)_{t \in \mathbb{R}}\) which shows a new phenomenon. Recall that an element \( x \) of \( \mathcal{M} \) is an entire analytic vector with respect to the normal faithful state \( \varphi \) if the function \( t \mapsto \sigma^t_\varphi(x) \) extends to a entire function from \( \mathbb{C} \) into \( \mathcal{M} \). By [Str81, p. 32], the family \( \mathcal{M}_\alpha \) of entire analytic vectors is a weak* dense \( * \)-subalgebra of the von Neumann algebra \( \mathcal{M} \). A folklore result used in the proofs of [JuX03, Lemma 1.1] and [HJX10, Proposition 5.5] relying on [GoL99, Proposition 1.4] says that if \( x \in \mathcal{M}_\alpha \) we have

\[
x h_\varphi^\alpha = h_\varphi^\alpha \sigma_\varphi^\alpha(x)
\]

for any \( \alpha \in \mathbb{C} \) with \( \text{Re} \alpha > 0 \).

**Proposition 4.11** Let \( \mathcal{M} \) be a von Neumann algebra equipped with a normal faithful state \( \varphi \). Suppose \( 1 \leq p < \infty \). Let \( Q: \mathcal{M} \to \mathcal{M} \) be a normal Jordan conditional expectation onto a JW*-subalgebra \( Q(\mathcal{M}) \) of the JW*-algebra \((\mathcal{M}, \circ)\) such that \( \varphi = \varphi \circ Q \). Then for any entire analytic vector \( a \) belonging to \( Q(\mathcal{M}) \) such that \( \sigma^t_\varphi(a) \) also belongs to \( Q(\mathcal{M}) \) (any element of \( Q(\mathcal{M}) \) if \( p = \infty \)) we have

\[
Q_p(ax \sigma^t_\varphi(a)) = a Q_p(x) \sigma^t_\varphi(a), \quad x \in L^p(\mathcal{M}).
\]

**Proof** : The case \( p = \infty \) is a consequence of the formula \( aba = 2(a \circ b) \circ a - a^2 \circ b \). Suppose \( 1 \leq p < \infty \). For any \( y \in \mathcal{M} \), using the case \( p = \infty \) in the fourth equality, we have

\[
Q_p(ah_\varphi^\alpha y h_\varphi^\beta \sigma_\varphi^\gamma(a)) = Q_p(h_\varphi^\alpha \sigma_\varphi^\gamma(a)y \sigma_\varphi^\gamma \sigma_\varphi^\gamma(a)h_\varphi^\beta)
\]

\[
= Q_p(h_\varphi^\alpha \sigma_\varphi^\gamma(a)y \sigma_\varphi^\gamma \sigma_\varphi^\gamma(a)h_\varphi^\beta) \quad (2.20)
\]

\[
= h_\varphi^\alpha \sigma_\varphi^\gamma(a) Q(y) \sigma_\varphi^\gamma(a) h_\varphi^\beta = h_\varphi^\alpha Q(y) h_\varphi^\beta \sigma_\varphi^\gamma(a) \quad (2.20)
\]

\[
= a Q_p(h_\varphi^\alpha y h_\varphi^\beta) \sigma_\varphi^\gamma(a).
\]

We conclude by density.

\[\blacksquare\]
Remark 4.12 The assumption that $\sigma^\tau_\gamma(a)$ also belongs to $Q(M)$ is maybe useless. Note that the space of entire analytic vectors of $Q(M)$ is weak* dense in $Q(M)$. We sketch the argument. By [HaS95, Theorem 4.2], the JW-algebra $A$ associated to $Q(M)$ is invariant under the one-parameter cosine family $(\rho^\tau_i)_i \in \mathbb{R}$ associated with $\varphi$. Since $M$ is the von Neumann algebra, recall that by [HaH84, Proposition 3.6] the cosine family $(\rho^\tau_i)_i \in \mathbb{R}$ is given by $\rho^\tau_i(x) = \frac{1}{2} (\sigma^\tau_\gamma(x) + \sigma^{\tau\gamma}(x))$ where $x \in A$. Now, for any $\gamma > 0$ the element
\[
x_\gamma \overset{\text{def}}{=} \frac{1}{(2\pi \gamma^2)^{1/2}} \int_\mathbb{R} e^{-\frac{t^2}{2\gamma^2}} \sigma^\tau_i(x) \, dt = \frac{1}{(2\pi \gamma^2)^{1/2}} \int_0^\infty e^{-\frac{t^2}{2\gamma^2}} (\sigma^\tau_i(x) + \sigma^{\tau\gamma}(x)) \, dt
\]
belongs to $A$ and approximate $x \in A$ in the weak* topology when $\gamma \to 0$ by [Sun87, p. 72] [Str81, p. 33].

We have the following (easy) particular case.

Corollary 4.13 Let $M$ be a von Neumann algebra equipped with a normal finite faithful trace $\tau$. Suppose $1 \leq p \leq \infty$. Let $Q: M \to M$ be a normal Jordan conditional expectation onto a JW*-subalgebra $N$ of the JW*-algebra $(M, \circ)$ such that $\tau = \tau \circ Q$. Then
\[
Q_p(axa) = aQ_p(x)a, \quad a \in Q(M), x \in L^p(M).
\]

5 An analysis of the lifting contractive projection

Let $M$ be a von Neumann algebra and $\varphi$ be a normal faithful positive linear form on $M$. Let $Q: M \to M$ be a $\varphi$-preserving normal unital positive projection on a JW*-subalgebra $N$. We let $A = N_{au}$ which is a JW-subalgebra of $(M_{au}, \circ)$. In this section, we will make an analysis of the structure of the $L^p$-extension of $Q$, using results of Haagerup and Størmer. We also obtain information on the range.

We begin to use structure results on the JW-algebra $A$. By [Sto66, Theorem 6.4], [Sto68, Lemma 2.3], there exist projections $e_1, e_2, e_3$ in the center $Z(A)$ of the JW-algebra $A$ with sum 1 such that
1. $A_1 = e_1A$ is the selfadjoint part of the von Neumann algebra $A'$,
2. $A_2 = e_2A$ is purely real\(^{10}\),
3. $A_3 = e_3A$ is totally non reversible, hence of type 2 by Theorem 3.7.

We recall that $\varphi_{e_1}, \varphi_{e_2}$ and $\varphi_{e_3}$ denote the reduced weights of $\varphi$ on the von Neumann algebras $e_1Me_1, e_3Me_2$ and $e_3Me_3$. Note that theses weights are faithful normal positive functionals.

Lemma 5.1 Let $(e_i)_i \in I$ be a family of projections of the the center $Z(A)$ of the JW-algebra $A$ with sum 1.

1. For any $x \in M$ and any $i \in I$, we have $Q(e_ixe_i) \in e_iN$.
2. For any $x \in M$, we have
\[
Q(x) = \sum_{i \in I} Q(e_ixe_i).
\]

\(^{10}\) In particular $A_2$ is the selfadjoint part of the real von Neumann algebra $\mathfrak{R}(A_2)$. 

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3. If \( i \in I \) and if \( y \) belongs to \( e_iN \), we have \( Q(y) = y \).

4. The projections \( e_i \) belongs to the centralizer \( M^\varphi \) of \( \varphi \).

**Proof**: 1. Since \( e_i \in A \), we have \( Q(e_i) = e_i \). By [Sto13, Theorem 2.2.2 (i)], the selfadjoint element \( e_i \) belongs to the definite set (3.9) of \( Q \). For any \( x \in \mathcal{M} \), we deduce by the well-known Broise’s observation [Sto80a, Lemma 4.1] [Sto13, Proposition 2.1.7 (ii)] that

\[
Q(e_i x e_i) = Q(e_i) Q(x) Q(e_i) = e_i Q(x) e_i = e_i Q(x)
\]

where we use in the last equality the fact that the element \( e_i \) of \( Z(A) \) commute with the elements of \( \text{Ran} Q = N = A + iA \). In particular, \( Q(e_i x e_i) \) belongs to \( e_iN \). Similarly, we have \( Q(e_i x e_i) = Q(x) e_i \).

2. Now, for any \( x \in \mathcal{M} \), we have

\[
\sum_{i \in I} Q(e_i x e_i) = \sum_{i \in I} e_i Q(x) = \left( \sum_{i \in I} e_i \right) Q(x) = Q(x).
\]

3. For any \( x \in \mathcal{M} \), replacing \( x \) by \( e_i x \) in (5.3), we deduce that

\[
Q(e_i x) = \sum_{j \in I} Q(e_j e_i x e_j) = Q(e_i x e_i)
\]

and similarly \( Q(x e_i) = Q(e_i x e_i) \). If \( y \in e_i N \), there exists \( x \in N \) such that \( y = e_i x \). So we have

\[
Q(y) = Q(e_i x) = \sum_{j \in I} Q(e_j e_i x e_j) = Q(e_i Q(x) e_i) = e_i Q(x) = e_i x = y.
\]

4. For any \( x \in \mathcal{M} \), using the preservation of \( \varphi \) in the first and the last equalities, we see that

\[
\varphi(e_i x) = \varphi(Q(e_i x)) = \varphi(Q(e_i x e_i)) = \varphi(Q(x)) = \varphi(x e_i).
\]

By (2.17), we conclude that \( e_i \) belongs to the centralizer \( M^\varphi \).

By the part 1 of this result, we deduce that we have a canonical isometric identification of each noncommutative \( L^p \)-space \( L^p(e_i, \mathcal{M} e_i) \) with the subspace \( e_i L^p(\mathcal{M}) e_i \) of \( L^p(\mathcal{M}) \). Note that each restriction \( Q|_{e_i, \mathcal{M} e_i}: e_i \mathcal{M} e_i \to e_i \mathcal{M} e_i \) is a faithful normal unital positive projection on \( N e_i \) which preserves the normal positive faithful linear form \( \varphi|_{e_i \mathcal{M} e_i} \). So to understand the projection \( Q \), it suffices by Lemma 5.1 to examine three cases separately.

We need the following result for the two last cases. Here, we denote by \( L^p, D(\mathcal{M}, \tau_0) \) the Dixmier noncommutative \( L^p \)-space associated with a von Neumann algebra \( \mathcal{M} \) equipped with a normal semifinite faithful trace \( \tau_0 \). If \( \tau_0 \) is finite, recall that \( m_{\tau_0} = \mathcal{M} \).

**Lemma 5.2** Let \( \mathcal{M} \) be a von Neumann algebra equipped with a normal semifinite faithful trace \( \tau_0 \). Let \( T: \mathcal{M} \to \mathcal{M} \) be a positive normal contraction which preserves \( \tau_0 \) and a normal linear form \( \varphi = \tau_0(\cdot \cdot) \) on \( \mathcal{M} \) where \( d \in L^1(\mathcal{M}, \tau) \). We denote by \( h_\varphi \) the density operator of the weight \( \varphi \). Suppose \( 1 \leq p < \infty \). Then the map \( m_{\tau_0} \to L^p(\mathcal{M}, \varphi) \), \( x \mapsto h_\varphi^{-\frac{p}{2}} x h_\varphi^{-\frac{p}{2}} \) extends to a a positive isometric map \( \Phi: L^p, D(\mathcal{M}, \tau_0) \to L^p(\mathcal{M}, \varphi) \) and if \( T_{p, H}: L^p(\mathcal{M}, \varphi) \to L^p(\mathcal{M}, \varphi) \) and \( T_{p, D}: L^p, D(\mathcal{M}, \tau_0) \to L^p, D(\mathcal{M}, \tau_0) \) denote the \( L^p \)-extension of \( T \) we have the following commutative diagram.

\[
\begin{array}{ccc}
L^p(\mathcal{M}, \varphi) & \xrightarrow{T_{p, H}} & L^p(\mathcal{M}, \varphi) \\
\Phi \downarrow & & \Phi \\
L^p, D(\mathcal{M}, \tau_0) & \xrightarrow{T_{p, D}} & L^p, D(\mathcal{M}, \tau_0)
\end{array}
\]
Proof : First, we need to recall some information on the crossed product \( \mathcal{M} \rtimes_{\sigma \tau_0} \mathbb{R} \) which is written in few lines in [Ter81, pp. 62-63]. Note that \( \mathcal{M} \rtimes_{\sigma \tau_0} \mathbb{R} \) is equal to the tensor product \( \mathcal{M} \otimes \mathcal{A} \) where \( \mathcal{A} \) is the von Neumann algebra generated by the translations \( \lambda_s : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) for \( s \in \mathbb{R} \). With the Fourier transform \( \mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) we can write \( \mathcal{F}^{-1} \lambda_s \mathcal{F} = L^\infty(\mathbb{R}) \) and

\[
\mathcal{F}^{-1} \lambda_s \mathcal{F} = e^{i s \cdot}, \quad s \in \mathbb{R}.
\]

So we have a \(*\)-isomorphism \( \Theta : \mathcal{M} \otimes \mathcal{A} \to \mathcal{M} \otimes L^\infty(\mathbb{R}) \), \( y \otimes z \mapsto y \otimes \mathcal{F}^{-1} z \mathcal{F} \). As indicated in [Ter81, p. 62], it is easy to check that the transformation \( \theta_s \defeq \Theta^{-1} \circ \hat{s} \tau_0 \circ \Theta \) of the dual action \( \hat{s} \tau_0 : \mathcal{M} \otimes \mathcal{A} \to \mathcal{M} \otimes \mathcal{A} \) defined in (2.3) under \( \Theta \) is determined by

\[
(5.5)
\theta_s(x \otimes f) = x \otimes \lambda_s(f), \quad s \in \mathbb{R}, x \in \mathcal{M}, f \in L^\infty(\mathbb{R}).
\]

Recall that \( \tau_0 \) denotes the dual weight on \( \mathcal{M} \otimes \mathcal{A} \) defined in (2.6). Let \( \tilde{\tau}_0 \defeq \tau_0 \circ \Theta^{-1} \) be the weight on \( \mathcal{M} \otimes L^\infty(\mathbb{R}) \) obtained by transport of structure. It is not difficult to prove that

\[
(5.6)
\tilde{\tau}_0 = \tau_0 \otimes \int_{-\infty}^{\infty}.
\]

Let \( \tau \defeq \tau_{\tau_0} \) be the canonical trace on the von Neumann algebra \( \mathcal{M} \rtimes_{\sigma \tau_0} \mathbb{R} = \mathcal{M} \otimes \mathcal{A} \). Set \( \tilde{\tau} \defeq \tau \circ \Theta^{-1} \) be the normal semifinite faithful trace on \( \mathcal{M} \otimes L^\infty(\mathbb{R}) \) obtained by transport of structure. Then we have

\[
(5.8)
(D \tilde{\tau}_0 : D \tilde{\tau})_t = \Theta(D \tau_0 : D \tau)_t \quad \text{(2.7)} \quad \Theta(1 \otimes \lambda_t) = 1 \otimes \mathcal{F}^{-1} \lambda_t \mathcal{F} \quad \text{(5.5)} \quad 1 \otimes e^{it} = (1 \otimes \exp)^t, \quad t \in \mathbb{R}.
\]

Hence, by (2.8), the Pedersen-Takesaki derivative \( \frac{d \tilde{\tau}_0}{dt} \) is equal to \( 1 \otimes \exp \). By [Str81, p. 62], it follows that

\[
(5.9)
\tilde{\tau} = \tilde{\tau}_0 \cdot (1 \otimes \exp)^{-1} \quad \text{(5.7)} \quad \tau_0 \otimes e^{-s} \, ds.
\]

Moreover, recall that \( h_{\tau_0} = \frac{d \tau_0}{dt} \). By transport of structure, we have

\[
(5.10)
1 \otimes \exp = \Theta(h_{\tau_0}).
\]

The isomorphism \( \Theta : \mathcal{M} \otimes \mathcal{A} \to \mathcal{M} \otimes L^\infty(\mathbb{R}) \) extends to an isomorphism \( \Theta : L^p(\mathcal{M} \otimes \mathcal{A}, \tau) \to L^p(\mathcal{M} \otimes L^\infty(\mathbb{R}), \tilde{\tau}) \) which allows to identify isometrically the Haagerup noncommutative \( L^p \)-space \( L^{p,\nu}(\mathcal{M}, \tau_0) \) with the range \( \Theta(L^p(\mathcal{M}, \tau_0)) \). By [Ter81, pp. 62-63], we have an order and isometric isomorphism \( \eta : L^{p,\nu}(\mathcal{M}, \tau_0) \to \Theta(L^p(\mathcal{M}, \tau_0)), \ x \mapsto x \otimes \exp^\tilde{\tau} \). From (2.14), we also have a canonical map \( \kappa \) which induces an order and isometric identification \( \kappa : L^p(\mathcal{M}, \tau_0) \to L^p(\mathcal{M}, \varphi) \) for all \( p \) such that

\[
(5.11)
\kappa(h_{\tau_0}) = h_{\varphi}.
\]

Now, for any \( x \in m_{\tau_0} \) we can compute

\[
\kappa \circ \Theta^{-1} \circ \eta(x) = \kappa \circ \Theta^{-1} \circ ((1 \otimes \exp^\tilde{\tau})(x \otimes 1)(1 \otimes \exp^\tilde{\tau}))
\]

\[= \kappa(\Theta^{-1}(1 \otimes \exp^\tilde{\tau}) \Theta^{-1}(x \otimes 1) \Theta^{-1}(1 \otimes \exp^\tilde{\tau})) \quad \text{(5.10)} \]

\[= \kappa(h_{\tau_0} x h_{\tau_0}) = \kappa(h_{\tau_0} x h_{\tau_0}) \quad \text{(5.11)} \quad h_{\tau_0} x h_{\tau_0} = \Phi(x). \]
By composition, we conclude that \( \Phi \) induces an order isometric isomorphism from \( L^p, D(M, \tau_0) \) into \( L^p(M, \varphi) \). If \( x \in \mathfrak{m}_{\tau_0} \), we have
\[
T_{p,H} \circ \Phi(x) = T_{p,H}(h_{p,D}^\varphi x h_{p,D}^\varphi)^{(2.20)} = h_{p,D}^\varphi T(x) h_{p,D}^\varphi = \Phi(T(x)) = \Phi \circ T_{p,D}(x).
\]
The proof is complete.

**Case 1** We consider a normal positive faithful linear form \( \varphi \) on a von Neumann algebra \( M \). We suppose that \( Q : M \to M \) is a \( \varphi \)-preserving normal unital faithful positive projection on a JW*-subalgebra \( N \) such that the associated JW-algebra \( A = N_{sa} \) is the selfadjoint part of the von Neumann algebra \( A'' \). Note that \( N = A + iA = (A'')_sa + i(A'')_sa = A'' \). Consequently, the range \( N \) of \( Q \) is a von Neumann algebra. By Tomiyama’s theorem \([Str81, p. 117]\), we conclude that \( Q : M \to M \) is a normal faithful conditional expectation on \( N \). Since \( \varphi \) is preserved, we can consider by (2.20) its \( L_p \)-extension \( Q_p : L^p(M) \to L^p(M) \). Its range is isometrically isomorphic to a noncommutative \( L_p \)-space.

**Case 2** We consider a normal positive faithful linear form \( \varphi \) on a von Neumann algebra \( M \). We suppose that \( Q : M \to M \) is a \( \varphi \)-preserving normal unital faithful positive projection on a JW*-subalgebra \( N \) such that the associated JW-algebra \( A = N_{sa} \) is purely real. We only look the case where \( A \) is a JW-factor which is not isomorphic to the selfadjoint part of a von Neumann algebra and where \( M \) is finite.

**Proposition 5.3** Let \( M \) be a finite von Neumann algebra equipped with a faithful normal positive linear form \( \varphi \). Let \( Q : M \to M \) be a \( \varphi \)-preserving normal unital positive projection on a JW*-subalgebra \( N \) such that the associated JW-algebra \( A \overset{def}{=} N_{sa} \) is a purely real factor which is not isomorphic to the selfadjoint part of a von Neumann algebra. Suppose \( 1 < p < \infty \). Then the range of the \( L_p \)-extension \( Q_p : L^p(M) \to L^p(M), h_{p,D}^\varphi \to h_{p,D}^\varphi \) is isometrically isomorphic to an interpolation space of the form \((N, N^\perp)_h\).

**Proof:** Note that \( A \subset N \subset A'' \). So, in this paragraph, we can consider \( Q : M \to A'' \).

We consider the normal faithful positive linear form \( \varphi'' \overset{def}{=} \varphi|A'' \). Note that by Proposition 3.5, we have \( Z(A) = (Z(A''))_{sa} \). By Theorem 3.6, there exists a (unique) faithful normal conditional expectation \( E : M \to A'' \) on the von Neumann algebra \( A'' \) such that \( Q = P_{can} \circ E \) where the canonical projection \( P_{can} : A'' \to A'' \) on \( M \) is defined in Section 3. We deduce that \( Q|A'' = P_{can} \). We infer that \( P_{can} \) preserves the linear form \( \varphi'' \). Now, we have
\[
\varphi = \varphi'' \circ Q = \varphi'' \circ P_{can} \circ E = \varphi'' \circ E.
\]

From Takesaki’s theorem \([Str81, Theorem 10.1 p. 130]\), we deduce that \( \varphi'' \) is semifinite and that the von Neumann algebra \( A'' \) is invariant under the modular group, i.e.
\[
\sigma^t_\varphi(A'') = A'', \quad t \in \mathbb{R}.
\]

We conclude that the noncommutative \( L_p \)-space \( L^p(A'', \varphi'') \) can be naturally isometrically identified with a subspace of \( L^p(M) \). So it suffices to understand the range of the \( L_p \)-extension of \( P_{can} \).

By Proposition 3.3, the von Neumann algebra \( A'' \) is a factor. Since \( M \) is finite, the von Neumann subalgebra \( A'' \) is finite. By \([KaR97b, Theorem 8.2.8]\), we deduce that \( A'' \) admits
a unique normalized finite trace $\tau$. Consider the $*$-antiautomorphism $\alpha: A'' \to A''$ defined in Section 3. Since $\tau \circ \alpha$ is also a normalized finite trace on $A''$, we conclude that $\alpha$ is trace preserving, that is $\tau \circ \alpha = \tau$. By linearity, it is immediate that the normal unital positive projection $P_{\text{can}}: A'' \to A''$ is also trace preserving. By Proposition 3.22 and Proposition 3.23, note that $P_{\text{can}}$ is selfadjoint with respect to $\tau$.

The (faithful) normal positive linear form $\varphi''$ can be written $\varphi'' = \tau(d \cdot)$ where $d$ is an element of the Dixmier noncommutative $L^1$-space $L_{\text{comm}}^1(A'', \tau)$. We denote by $h_{\varphi''}$ the density operator associated with $\varphi''$. Now by Lemma 5.2, the linear map $A'' \to L^p(A'', \varphi'')$, $x \mapsto h_{\varphi''}^{\frac{1}{p}} x h_{\varphi''}^{\frac{1}{p}}$, extends to a positive isometric map $\Phi: L^p(A'', \varphi'') \to L^p(A'', \varphi'')$ and we have the following commutative diagram.

\[
\begin{array}{ccc}
L^p(A'', \varphi'') & \xrightarrow{P_{\text{can}, p}} & L^p(A'', \varphi'') \\
\downarrow{\Phi} & & \downarrow{\Phi} \\
L^p(A'', \tau) & \xrightarrow{P_{\text{can}, p, D}} & L^p(A'', \tau)
\end{array}
\]

Since $P_{\text{can}}$ is selfadjoint, we know by [JMX06, p. 43] that $P_{\text{can}, 1, D}: L^1(A'') \to L^1(A'')$ identifies to the predual $(P_{\text{can}}, \cdot): (A'')^* \to (A'')^*$. By Proposition 2.5, the range of the latter map is isometric to the predual $(\text{Ran } P_{\text{can}})^* = \mathcal{N}_e$ of the range of $P_{\text{can}}$. With Lemma 2.7, we deduce that the range of $P_{\text{can}, p, D}$ is isometric to some interpolation space $(\mathcal{N}, \mathcal{N}_e)^{\frac{1}{p}}$. Using the commutative diagram, we deduce the same thing for $P_{\text{can}, p}$. The proof is complete. 

**Remark 5.4** It seems to the author that we can use direct theory to generalize to the case of a reversible purely real $JW^*$-algebra $\mathcal{N}$ such that $A$ does not have direct summands isomorphic to the selfadjoint part of a von Neumann algebra. Probably, with the first case 1, we can obtain the case of a reversible purely real $JW^*$-algebra $\mathcal{N}$.

**Case 3** We consider a normal positive faithful linear form $\varphi$ on a von Neumann algebra $\mathcal{M}$. We suppose that $Q: \mathcal{M} \to \mathcal{M}$ is a $\varphi$-preserving normal unital faithful positive projection on a $JW^*$-subalgebra $\mathcal{N}$ such that the associated JW-algebra $A = \mathcal{N}_{\varphi}$ is of type $I_2$. We only look the case where $A$ generates the von Neumann algebra $\mathcal{M}$ (and where $\mathcal{M}$ has separable predual). It seems to the author that it is possible to obtain a slightly more general statement with [HaS95, Lemma 2.2] but we do not know if this result allows to remove definitively this probably unnecessary assumption.

**Proposition 5.5** Let $\mathcal{M}$ be a von Neumann algebra with separable predual equipped with a faithful normal positive linear form $\varphi$. Let $Q: \mathcal{M} \to \mathcal{M}$ be a faithful normal unital positive projection which preserves $\varphi$ on a $JW^*$-algebra $\mathcal{N}$ such that the associated JW-algebra $A = \mathcal{N}_{\varphi}$ is of type $I_2$. Suppose that $A'' = \mathcal{M}$. Then the range of the $L^p$-extension $Q_{\varphi}: L^p(\mathcal{M}) \to L^p(\mathcal{M})$, $h_{\varphi}^{\frac{1}{p}} x h_{\varphi}^{\frac{1}{p}} \mapsto h_{\varphi}^{\frac{1}{p}} P(x) h_{\varphi}^{\frac{1}{p}}$ is isometrically isomorphic to an interpolation space of the form $(\mathcal{N}, \mathcal{N}_e)^{\frac{1}{p}}$.

**Proof**: By Example 3.9, there exist an index set $I$, a family $(\Omega_i)_{i \in I}$ of second countable locally compact spaces, a family of Radon measures $(\mu_i)_{i \in I}$ on the spaces $\Omega_i$ and a family $(S_i)_{i \in I}$ of spin factors, each of dimension at most countable (and strictly greater than 1) giving an isomorphism

\[A = \bigoplus_{i \in I} L^\infty(\Omega_i, \mu_i, S_i).\]
We need to take account of the generated von Neumann algebra $\mathcal{M} = A''$. So it is clear that we have a $*$-isomorphism $A'' = \bigoplus_{i \in I} L^{\infty}(\Omega_i, \mu_i, S''_i)$ for some concrete spin factor $S_i$. By Theorem 3.10, the von Neumann algebra $A''$ is necessarily finite, hence by [Li92, Proposition 6.3.1 3)] each summand of $A''$ is also finite. Note that by [Sak98, p. 68] we have a $*$-isomorphism $L^{\infty}(\Omega, \mu, S''_i) = L^{\infty}(\Omega, \mu_i) \otimes S''_i$. By [Sak98, Proposition 2.6.1], we conclude that each von Neumann algebra $S''_i$ is finite.

If $\mathcal{A}$ is the CAR algebra over the complex Hilbert space $\ell^2$, recall that by [HOS84, Theorem 6.2.2] the $C^*$-algebra generated by the spin factor $S_i$ is $*$-isomorphic to

$$
\begin{cases}
M_{2^n-1} \oplus M_{2^n-1} & \text{if } \dim S_i = 2n \\
M_{2^n} & \text{if } \dim S_i = 2n + 1 \\
\mathcal{A} & \text{if } \dim S_i = \infty
\end{cases}
$$

(5.12)

In the case where $\dim S_i = \infty$, by [KaR97b, Proposition 12.1.3] and since $A''$ is finite, the von Neumann algebra $S''_i$ is a factor of type $\text{II}_1$ which is of course hyperfinite by definition [KaR97b, p. 895]. By [KaR97b, Proposition 12.1.4], we can suppose that the spin factor $S_i$ is canonically embedding in the unique hyperfinite factor $\mathcal{R}$ of type $\text{II}_1$ with separable predual.

Now, it is easy to check that we have a $*$-isomorphism $A'' = \bigoplus_{i} L^{\infty}(\Omega_i, \mu_i, S''_i)$ where

$$
S''_i \overset{\text{def}}{=} \begin{cases}
M_{2^n-1} \oplus M_{2^n-1} & \text{if } \dim S_i = 2n \\
M_{2^n} & \text{if } \dim S_i = 2n + 1 \\
\mathcal{R} & \text{if } \dim S_i = \infty
\end{cases}
$$

(5.13)

and that with Lemma 5.1 we can reduce the problem to the case of a faithful normal unital positive projection map $Q_i : L^{\infty}(\Omega_i, S''_i) \rightarrow L^{\infty}(\Omega_i, S''_i)$ on $L^{\infty}(\Omega_i, (S_i)_C)$ which preserves a faithful normal positive linear form $\varphi_i$ where $(S_i)_C$ is the JW-$*$-algebra associated to $S_i$.

Note that by [Dix81, Corollary p. 178] we have a $*$-isomorphism $L^{\infty}(\Omega_i, S''_i) = \int_{\Omega_i} S''_i d\mu_i(\omega)$ (we can suppose that the support of $\mu_i$ is $\Omega_i$). With [KaR97a, Lemma 14.1.19] and [Tak02, Proposition 8.34 p. 285]), we can decompose the form $\varphi_i$ as a direct integral $\varphi_i = \int_{\Omega_i} \varphi_i,\omega d\mu_i(\omega)$ of faithful normal positive linear forms $\varphi_i,\omega$ on the von Neumann algebra $S''_i$. We denote by

$$Q_{i,sa} : L^{\infty}_{\mathcal{R}}(\Omega_i, (S''_i)_sa) \rightarrow L^{\infty}_{\mathcal{R}}(\Omega_i, (S''_i)_sa)$$

the restriction of $Q_i$ on the JW-algebra $L^{\infty}_{\mathcal{R}}(\Omega_i, S''_i)_sa = L^{\infty}_{\mathcal{R}}(\Omega_i, (S''_i)_sa)$ which is a projection on the JW-algebra $L^{\infty}_{\mathcal{R}}(\Omega_i, S_i)$. We can decompose $Q_{i,sa}$ as a direct integral $Q_{i,sa} = \int_{\Omega_i} Q_{i,sa,\omega} d\mu_i(\omega)$ of faithful normal projections $Q_{i,sa,\omega} : (S''_i)_sa \rightarrow (S''_i)_sa$ on $S_i$ preserving the restriction of $\varphi_i,\omega$ on $(S''_i)_sa$. By [HOS84, Lemma 4.4.13], these maps are Jordan conditional expectations. By complexification, we deduce that $Q_i = \int_{\Omega_i} Q_{i,sa,\omega,\mathcal{C}} d\mu_i(\omega)$ where $Q_{i,sa,\omega,\mathcal{C}} : S''_i \rightarrow S''_i$. By Lemma 3.20, note that $Q_{i,sa,\omega,\mathcal{C}}$ is a Jordan conditional expectation onto $S_i,\mathcal{C}$.

We equip the spin factor $S_i$ with its unique tracial state $\tau_{S_i}$, see Example 3.16. If $(\tau_{S_i})_C$ denote its complexification on its associated JW-$*$-algebra $(S_i)_C$ then by [HaS95, Lemma 2.2],

11. We warn the reader that the von Neumann algebra generated by an infinite dimensional spin factor is not necessarily a factor of type $\text{II}_1$. 

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the functional \( \tau_i \overset{\text{def}}{=} (\tau_S)_C \circ Q_{t, sa, \omega, C} \)\(^{12}\) is a normalized normal faithful trace on the von Neumann algebra \( S''_i \). Note that

\[
\tau_i \circ Q_{t, sa, \omega, C} = (\tau_S)_C \circ Q_{t, sa, \omega, C} \circ Q_{t, sa, \omega, C} = (\tau_S)_C \circ Q_{t, sa, \omega, C} = \tau_i,
\]

i.e. \( Q_{t, sa, \omega, C} \) preserves \( \tau_i \). We infer that \( Q_i = \int_{\Omega_i} Q_{t, sa, \omega, C} \, d\mu_i(\omega) \) preserves the trace \( \int_{\Omega_i} \otimes \tau_i \).

By Proposition 3.22 and Proposition 3.23, note that \( Q_i \) is selfadjoint with respect to the normal semifinite faithful trace \( \int_{\Omega_i} \otimes \tau_i \).

Each (faithful) normal positive linear form \( \varphi_{i, \omega} \) can be written \( \tau_i(d_{i, \omega} \cdot) \) where \( d_{i, \omega} \) is an element of the Dixmier noncommutative \( L^1 \)-space \( L^{1, D}(S''_i, \tau_i) \). Introducing \( d_i \overset{\text{def}}{=} \int_{\Omega_i} d_{i, \omega} \, d\mu_i(\omega) \), we have

\[
\varphi_i = \int_{\Omega_i} \varphi_{i, \omega} \, d\mu_i(\omega) = \int_{\Omega_i} \tau_i(d_{i, \omega} \cdot) \, d\mu_i(\omega)
\]

\[
= \left( \int_{\Omega_i} \otimes \tau_i \right) \left( \int_{\Omega_i} d_{i, \omega} \, d\mu_i(\omega) \cdot \right) = \left( \int_{\Omega_i} \otimes \tau_i \right)(d_i \cdot).
\]

We denote by \( h_{\varphi_i} \) the density operator associated with \( \varphi_i \). Now by Lemma 5.2, we have a positive isometric map \( \Phi: L^p(\Omega_i, \mu_i, L^{p, D}(S''_i, \tau_i)) \to L^p(\Omega^\infty(\Omega_i, \mu_i, S''_i), \varphi_i) \), \( x \mapsto h_{\varphi_i}^x \cdot, h_{\varphi_i}^x \cdot \), and we have the following commutative diagram.

\[
\begin{array}{ccc}
L^p(\Omega^\infty(\Omega_i, \mu_i, S''_i), \varphi_i) & \xrightarrow{Q_i, p} & L^p(\Omega^\infty(\Omega_i, \mu_i, S''_i), \varphi_i) \\
\Phi \downarrow & & \Phi \downarrow \\
L^p(\Omega_i, \mu_i, L^{p, D}(S''_i, \tau_i)) & \xrightarrow{Q_i, p, D} & L^p(\Omega_i, \mu_i, L^{p, D}(S''_i, \tau_i))
\end{array}
\]

Since \( Q_i \) is selfadjoint, we know by [JMX06, p. 43] that the map \( Q_i, 1, D : L^1(\Omega_i, \mu_i, L^{1, D}(S''_i, \tau_i)) \to L^1(\Omega_i, \mu_i, L^{1, D}(S''_i, \tau_i)) \) identifies to the preadjoint \( (Q_i)_*: (L^\infty(\Omega_i, \mu_i, S''_i))_* \to (L^\infty(\Omega_i, \mu_i, S''_i))_* \).

By Proposition 2.5, the range of the latter map is isometric to the predual \( \text{Ran} \,(Q_i)_* \) of \( \Omega_i \). With Lemma 2.7, we deduce that the range of \( Q_{t, p, D} \) is isometric to some interpolation space \( (L^\infty(\Omega_i, \mu_i, S''_i), L^\infty(\Omega_i, \mu_i, S''_i)) \) \(_\frac{1}{2} \). Using the commutative diagram, we deduce the same thing for \( Q_{t, p} \). The proof is complete. \( \blacksquare \)

6 Open questions

It is natural to state the following conjecture in view of our results. In a next version of this preprint, we will hope finish the case where \( \mathcal{M} \) is a finite von Neumann algebra.

**Conjecture 6.1** Suppose \( 1 < p < \infty \) with \( p \neq 2 \). Let \( X \) be a Banach space. Then \( X \) is isometric to a positively contractively complemented subspace of a Haagerup noncommutative \( L^p \)-space \( L^{p, H}(\mathcal{M}, \varphi) \) where \( \mathcal{M} \) is a \( \sigma \)-finite von Neumann algebra equipped with a normal faithful state \( \varphi \) if and only if \( X \) is isometric to a nonassociative \( L^p \)-space associated with a JW*-algebra equipped with a normal faithful state.

\( ^{12} \) Here, we consider \( Q_{t, sa, \omega, C} \) as a map \( Q_{t, sa, \omega, C} : S''_i \to (S_i)_C \).
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