Categorification of Clifford algebra via geometric induction and restriction

Caroline Gruson and Vera Serganova

INTRODUCTION

In a book published in 1981 ([7]), Andrei Zelevinsky categorified an infinite-rank PSH-algebra in terms of representations of the collection of all $GL(n, \mathbb{F})$ where $\mathbb{F}$ is a finite field. He did this using a pair of adjoint functors, the parabolic induction and its adjoint.

We intend, in this paper, to apply the same set of ideas to the categorification of an infinite Clifford algebra acting on the Fock space of semi-infinite forms, in terms of representations of the collection of all classical supergroups $SOSP(2m + 1, 2n)$, using the geometric induction functor and its adjoint called geometric restriction.

Let us start with the preliminary example of classical groups.

Let $(G_n)_{n \geq 1}$ be a family of complex classical Lie groups, $G_n$ of rank $n$, together with inclusions $G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots$

in such a way that $G_{n-1} \times \mathbb{C}^*$ is the reductive part of a maximal parabolic subgroup denoted $P_n$ of $G_n$, and we denote the maximal unipotent subgroup of $P_n$ by $U_n$.

For instance, consider $G_n = GL(n, \mathbb{C})$. We use gothic letters for the corresponding Lie algebras. We denote $\mathcal{F}_n$ the category of finite-dimensional $G_n$-modules, it is a semi-simple category and we denote $\mathcal{K}_n$ its Grothendieck group.

We use the functors $\Gamma^a_i$ and $H^j_b$ defined as follows:

$$\Gamma^a_i : \mathcal{F}_n \rightarrow \mathcal{F}_{n+1},$$

$$\Gamma^a_i(M) := H^i(G_{n+1}/P_{n+1}, \mathcal{L}(\mathbb{C}_a^* \otimes M)^*)$$

where $\mathbb{C}_a$ is the one-dimensional representation of $\mathbb{C}^*$ with character $a \in \mathbb{Z}$; we assume that $U_n$ acts trivially and $\mathcal{L}(\mathbb{C}_a^* \otimes M)^*$ is the induced vector bundle $G_n \times P_{n+1} (\mathbb{C}_a^* \otimes M)^*$.

$$H^j_b : \mathcal{F}_n \rightarrow \mathcal{F}_{n-1},$$

$$H^j_b(M) := \operatorname{Hom}_{\mathbb{C}^*}(\mathbb{C}_b^* \otimes M^*).$$

At the level of Grothendieck groups we obtain linear maps

$$\gamma^a : \mathcal{K}_n \rightarrow \mathcal{K}_{n+1},$$

$$[M] \mapsto \sum_i (-1)^i [H^i(G_{n+1}/P_{n+1}, \mathcal{L}(\mathbb{C}_a^* \otimes M)^*)^*],$$

$$\eta_b : \mathcal{K}_n \rightarrow \mathcal{K}_{n-1}$$

1Equipe de géométrie, U.M.R. 7502 du CNRS, Institut Elie Cartan, Université de Lorraine, BP 239, 54506 Vandoeuvre-les-Nancy Cedex, France. E-mail: Caroline.Gruson@univ-lorraine.fr

2Department of Mathematics, University of California, Berkeley, CA, 94720-3840 USA. E-mail: serganov@math.berkeley.edu
\([M] \mapsto \sum_j (-1)^j [\text{Hom}_C(\mathbb{C}_{b+n}, H^j(u_n, M))]\).

We set \(\mathcal{K} := \oplus_n \mathcal{K}_n\) and extend those maps to \(\mathcal{K}\). Then, applying Borel-Weil-Bott theorem, we obtain the following relations, for all \(a\) and \(b\) in \(\mathbb{Z}\):

1. \(\gamma^a \gamma^b + \gamma^b \gamma^a = 0\),
2. \(\eta_a \eta_b + \eta_b \eta_a = 0\),
3. \(\gamma^a \eta_b + \eta_b \gamma^a = \delta_{a,b} \text{Id.}\)

We recognise those relations as the ones of the infinite dimensional Clifford algebra \(\mathbb{C}\). Furthermore, we see \(\mathcal{K}\) as an irreducible representation of \(\mathbb{C}\) which is induced by the trivial representation of the subalgebra of \(\mathbb{C}\) generated by \((\eta_b)_{b \in \mathbb{Z}}\).

This provides a categorification of the Clifford algebra \(\mathbb{C}\) by the family of classical groups \((G_n)_{n \geq 1}\).

We follow the same scheme for the family of classical Lie supergroups \(\text{SOSP}(2m+1, 2n)\) when \(m\) and \(n\) vary, in this case we categorify the representation of the infinite Clifford algebra in the Fock space of semi-infinite forms. In the last section, we explain how our previous categorification work on orthosymplectic Lie superalgebras \(\mathfrak{g}\) can be understood in this context.

We would also like to mention the work of Michael Ehrig and Catharina Stroppel \(\Pi\), who used quantized symmetric pairs in order to refine our previous results on the category of finite dimensional modules over orthosymplectic Lie superalgebras and obtain a diagrammatic description of the endomorphism algebras of projective generators.

It would be very interesting now to construct a canonical basis in the Fock space of semi-infinite forms.

Finally, we would like to emphasize that what we do here can easily be done for all series of classical Lie supergroups, with minor changes only.

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1. Basic setting

We work over the field of complex numbers in the category of \(\mathbb{Z}/2\mathbb{Z}\)-graded spaces. The reader should keep it in mind when we consider symmetric and exterior powers.

We denote by \(\mathfrak{g}_{m,n}\) the Lie superalgebra \(\mathfrak{osp}(2m+1, 2n)\) and

\[\mathfrak{g}_{\infty,\infty} = \lim_{m,n \to \infty} \mathfrak{g}_{m,n}.\]

Further more, we fix an embedding \(\mathfrak{g}_{m,n} \subset \mathfrak{g}_{\infty,\infty}\).
We also fix a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g}_{\infty,\infty} \) and the standard basis \( \{\varepsilon_i, \delta_j\}_{i,j \in \mathbb{Z}_{>0}} \). The roots of \( \mathfrak{g}_{\infty,\infty} \) in this basis are:

\[
(\pm \varepsilon_i), \ (\pm \delta_j), \\
(\pm \varepsilon_i \pm \varepsilon_j), \ (\pm \delta_i \pm \delta_j),
\]

where \( i, j \) vary from 1 to \( \infty \), and in the last line, \( i \neq j \).

Then the roots of \( \mathfrak{g}_{m,n} \) lie in the subspace generated by \( (\varepsilon_i)_{1 \leq i \leq m} \) and \( (\delta_j)_{1 \leq j \leq n} \).

We fix a Borel subalgebra \( \mathfrak{b}_0 \) of \( (\mathfrak{g}_{\infty,\infty})_0 \) with the set of positive roots \( \{\varepsilon_i, 2\delta_j, (i,j > 0), \varepsilon_i \pm \varepsilon_j, \delta_i \pm \delta_j (i > j > 0)\} \).

Inside \( \mathfrak{g}_{m,n} \), we denote by \( \mathfrak{p}_{m,n} \) (resp \( \mathfrak{p}_{m,2} \)) the unique parabolic subalgebra containing \( \mathfrak{b}_0 \) with semi-simple part \( \mathfrak{g}_{m-1,n} \) (resp \( \mathfrak{g}_{m,n-1} \)).

We denote by \( G_{m,n} \) the supergroup \( SOSP(2m+1, 2n) \) and by \( T_{m,n} \) the maximal torus of \( G_{m,n} \) with Lie algebra \( \mathfrak{h} \cap \mathfrak{g}_{m,n} \).

For fixed \( m \) and \( n \), we denote by \( \mathcal{F}_{m,n} \) the category of finite dimensional \( G_{m,n} \)-modules and by \( \mathcal{K}_{m,n} \) its Grothendieck group.

Let:

\[
\mathcal{F} := \oplus_{m,n} \mathcal{F}_{m,n} \text{ and } \mathcal{K} := \oplus_{m,n} \mathcal{K}_{m,n}.
\]

Let \( \nu \) be a character of \( T_{m,n} \). We denote by \( \mathcal{L}_\nu \) the corresponding line bundle over \( G_{m,n}/B \).

Recall the definition of the Euler characteristic. For every \( \lambda \in \Lambda_{m,n} \) we set

\[
\mathcal{E}(\lambda) := \left[ \sum_{i \geq 0} (-1)^i H^i(G_{m,n}/B, \mathcal{L}_{\lambda-\rho_B}^*) \right] \in \mathcal{K}_{m,n}.
\]

Recall also that the character of this virtual module is easy to compute, namely

\[
\text{Ch}(\mathcal{E}(\lambda)) = \frac{D_0}{D_1} \sum_{w \in \mathcal{W}_{m,n}} \varepsilon(w)e^{w(\lambda)},
\]
where $W_{m,n}$ is the Weyl group of $SO(2m + 1) \times SP(2n)$, $D_0 = \Pi_{\alpha \in \Delta^+_0}(e^{\alpha/2} - e^{-\alpha/2})$, $D_1 = \Pi_{\alpha \in \Delta^+_1}(e^{\alpha/2} + e^{-\alpha/2})$.

(Remark: We avoided indexes $m$ and $n$ in this formula since one can easily recover them from the shape of $\lambda$).

Note that if we change our choice of $B$ containing $B_0 \cap G_{m,n}$, the character of $\mathcal{E}(\lambda)$ doesn’t change, thus the class in $\mathcal{K}_{m,n}$ remains the same, see [3].

For $w \in W_{m,n}$, notice that

\[ \mathcal{E}(w(\lambda)) = \varepsilon(w)\mathcal{E}(\lambda). \]

**Proposition 1. (see [3])** The set

\[ \{ \mathcal{E}(\lambda), \lambda \in \Lambda^+_{m,n} \} \]

gives a linearly independent family in $\mathcal{K}_{m,n}$, and we denote by $\mathcal{K}(\mathcal{E})_{m,n}$ the subgroup generated by this family. We also set $\mathcal{K}(\mathcal{E}) := \oplus_{m,n} \mathcal{K}(\mathcal{E})_{m,n}$.

2. **Fock space**

Let $V$ be a countable dimensional vector space together with a basis $(v_i)_{i \in \mathbb{Z}}$ and similarly $W$ with a basis $(w_i)_{i \in \mathbb{Z}}$ with a non-degenerate pairing such that $(v_i)$ and $(w_i)$ are dual bases.

Let $\mathcal{C}(V \oplus W)$ be the Clifford algebra of $V \oplus W$, namely if we denote by $T(V \oplus W)$ the tensor algebra of $V \oplus W$,

\[ \mathcal{C}(V \oplus W) = T(V \oplus W)/(v \otimes v' + v' \otimes v, w \otimes w' + w' \otimes w, v \otimes w + w \otimes v - (v, w), v, v' \in V, w, w' \in W). \]

The Fock space of semi-infinite forms, $\mathbf{F}$, is the vector space generated by

\[ v_{i_1} \wedge \ldots \wedge v_{i_k} \wedge \ldots, \]

for $i_1 > \ldots > i_k > \ldots$ such that, for $n$ large enough, $i_n = i_{n-1} - 1$.

There is a natural linear action of $\mathcal{C}(V \oplus W)$ on $\mathbf{F}$ given by:

\[
\forall v \in V, \quad v \bullet v_{i_1} \wedge \ldots \wedge v_{i_k} \wedge v_{i_{k+1}} \ldots = v \wedge v_{i_1} \wedge \ldots \wedge v_{i_k} \wedge v_{i_{k+1}} \ldots
\]

\[ \forall w \in W, \quad w \bullet v_{i_1} \wedge \ldots \wedge v_{i_k} \wedge v_{i_{k+1}} \ldots = \sum_j (-1)^{j-1} (w, v_{i_j}) v_{i_1} \wedge \ldots \wedge \hat{v}_{i_j} \wedge \ldots \]

Define the vacuum vector in $\mathbf{F}$ as

\[ | > := v_{-\frac{1}{2}} \wedge v_{-\frac{3}{2}} \wedge \ldots \]

then, for $i < 0$, $v_i$ acts on $| >$ by 0 as $w_j$ for $j > 0$.

We can also see $\mathbf{F}$ as an induced module the following way. Denote by $\mathcal{C}^+(V \oplus W)$ the subalgebra generated by $\{ v_i, i < 0, w_j, j > 0 \}$, consider its trivial module and induce to the whole $\mathcal{C}(V \oplus W)$: this gives another construction of $\mathbf{F}$.

Let $\lambda = \sum_{1 \leq i \leq m, 1 \leq j \leq n} a_i \varepsilon_i + b_j \delta_j \in \Lambda^+_{m,n}$. We define a $\mathbb{Z}$-linear map $f: \mathcal{K}(\mathcal{E}) \to \mathbf{F}$ such that for any $\mathcal{E}(\lambda) \in \mathcal{K}(\mathcal{E})_{m,n}$:
\[ \mathcal{E}(\lambda) \mapsto v_{a_m} \wedge \ldots \wedge v_{a_1} \wedge \ldots \wedge \hat{v}_{-b_1} \wedge \ldots \wedge \hat{v}_{-b_n} \wedge \ldots \]

### 3. Duality between geometric induction and restriction

In this section we will consider 3 different Grothendieck groups for \( G_{m,n} \) namely \( K(P)_{m,n} \) generated by the indecomposable projective modules, \( K(\mathcal{E})_{m,n} \) which we already met and \( K(L)_{m,n} := K_{m,n} \) generated by the simple modules. After tensoring by the rational numbers \( \mathbb{Q} \), \( K(P)_{m,n} \otimes \mathbb{Q} \) and \( K(\mathcal{E})_{m,n} \otimes \mathbb{Q} \) coincide (see [3]). We consider the natural pairing between \( K(P)_{m,n} \) and \( K(L)_{m,n} \), \[ \langle [P], [L] \rangle : = \dim \text{Hom}(P, L) \]. The restriction of this pairing to \( K(P)_{m,n} \times K(P)_{m,n} \) is symmetric (and therefore it is a scalar product): indeed \( \dim \text{Hom}(P_1, P_3) = \dim \text{Hom}(P_2^*, P_1^*) \) and in this case projective modules happen to be self-dual (see [6]).

**Proposition 2.** Let us extend the scalar product from \( K(P)_{m,n} \) to \( K(P)_{m,n} \otimes \mathbb{Q} \). Then the set of \( \mathcal{E}(\lambda) \), when \( \lambda \) varies in \( \Lambda^+_{m,n} \), form an orthonormal basis of \( K(P)_{m,n} \otimes \mathbb{Q} \).

**Proof.** Let \( L(\lambda) \) denote the simple module with highest weight \( \lambda \) and \( P(\lambda) \) denote its projective cover. Consider the decompositions

\[
[P(\lambda)] = \sum_{\mu} b_{\lambda, \mu} \mathcal{E}(\mu), \quad \mathcal{E}(\mu) = \sum_{\nu} a_{\mu, \nu} [L(\nu)].
\]

By the weak BGG reciprocity, [3], we have \( b_{\lambda, \mu} = a_{\mu, \lambda} \). Now, we write

\[
\mathcal{E}(\mu) = \sum_{\lambda} c_{\mu, \lambda} [P(\lambda)].
\]

Then, clearly, we have the following relation

\[
\sum_{\lambda} c_{\mu, \lambda} b_{\lambda, \nu} = \sum_{\lambda} c_{\mu, \lambda} a_{\nu, \lambda} = \delta_{\mu, \nu}.
\]

On the other hand,

\[
\langle [P(\lambda)], [L(\kappa)] \rangle = \delta_{\lambda, \kappa}.
\]

Therefore

\[
\langle \mathcal{E}(\mu), \mathcal{E}(\nu) \rangle = \sum_{\lambda, \kappa} c_{\mu, \lambda} a_{\nu, \kappa} \langle [P(\lambda)], [L(\kappa)] \rangle = \sum_{\lambda} c_{\mu, \lambda} a_{\nu, \lambda} = \delta_{\mu, \nu}.
\]

\[\square\]

Let \( G \) be a quasireductive algebraic supergroup, which is an algebraic supergroup with reductive even part (see [6] for information on their representation theory). Let \( Q \subset G \) be a parabolic subgroup with quasireductive part \( K \). Let \( \mathfrak{g}, \mathfrak{q}, \mathfrak{k} \) denote the respective Lie superalgebras, and let \( \mathfrak{r} \) denote the nil-radical of \( \mathfrak{q} \). Consider the following derived functors \( \Gamma_i : K-\text{mod} \to G-\text{mod} \) and \( H^i : G-\text{mod} \to K-\text{mod} \) defined by

\[
\Gamma_i(M) := H^i(G/Q, \mathcal{L}(M^*))^*, \quad H^i(N) := H^i(\mathfrak{r}, N).
\]
Here we denote by $L(M^*)$ the vector bundle on $G/Q$ induced from $M^*$. The collection of functors $\Gamma_i$ is referred to as geometric induction while that of $H^i$ is referred to as geometric restriction.

The following observation is due to Penkov [5].

**Proposition 3.** For any $K$-module $M$ we have

$$\sum_i (-1)^i[\Gamma_i(M)] = \sum_i (-1)^i[H^i(G_0/Q_0, L(S^\bullet(r) \otimes M^*))].$$

**Proposition 4.** For every projective $G$-module $P$, every $K$-module $M$ and $i \geq 0$ there is a canonical isomorphism

$$\text{Hom}_G(\Gamma_i(M), P) \simeq \text{Hom}_K(M, H^i(P)).$$

**Proof.** This result is a slight generalization of Proposition 1 in [3]. We consider an injective resolution $0 \to R^0 \to R^1 \to \ldots$ of $M$ in the category of $Q$-modules. Since $\text{Hom}_G(P, \cdot)$ is an exact functor, $\text{Hom}_G(P, H^i(G/Q, M))$ is given by the $i$-th cohomology group of the complex

$$0 \to \text{Hom}_G(P, H^0(G/Q, R^0)) \to \text{Hom}_G(P, H^0(G/Q, R^1)) \to \ldots .$$

The Frobenius reciprocity implies

$$\text{Hom}_G(P, H^0(G/Q, R^0)) \simeq \text{Hom}_Q(P, R^0).$$

Thus, we obtain the isomorphism

$$\text{Hom}_G(P, H^0(G/Q, M)) \simeq \text{Ext}_Q^i(P, M).$$

We now need the following lemma.

**Lemma 1.** The restricted module $\text{Res}_K P$ is projective in the category $K - \text{mod}$. 

**Proof.** Note that $P$ is a direct summand of the induced module $\text{Ind}_{g_0}^g S$ for some semisimple $g_0$-module $S$. Using the isomorphism

$$\text{Res}_K \text{Ind}_{g_0}^g S \simeq \text{Ind}_{g_0}^g S \otimes S^\bullet(g_1/t_1),$$

we obtain that $P$ is a direct summand of some module induced from a semisimple $t_0$-module. Therefore $P$ is projective as a $K$-module. \hfill $\Box$

Applying the above lemma we can use the Koszul complex $\Lambda^i(r) \otimes U(r) \otimes P$ (where $U(r)$ is the universal enveloping algebra of $r$) and thus obtain an isomorphism

$$\text{Ext}_Q^i(P, M) \simeq \text{Hom}_K(H_i(r, P), M).$$

Now we use the double dualization and the fact that $P^*$ is also projective:

$$\text{Hom}_G(\Gamma_i(M), P) \simeq \text{Hom}_G(P^*, H^i(G/Q, M^*)) \simeq \text{Hom}_K(H_i(r, P^*), M^*) \simeq \text{Hom}_K(M, H^i(P)).$$

Hence the statement. \hfill $\Box$

Recall that for any quasireductive supergroup every projective module is injective and vice versa, [6].

**Corollary 1.** If $P$ is an injective (equivalently, projective) $G$-module, then $H^i(P)$ is an injective and projective $K$-module.
4. Two functors on $\mathcal{F}$

We choose a parabolic subalgebra $\mathfrak{p}$ which can be either $\mathfrak{p}_{m,n}$ or $\mathfrak{p}_{m,n}$ in $\mathfrak{g}_{m,n}$, where:

$$
\mathfrak{p}_{m,n} = \mathfrak{g}_{m-1,n} \oplus \mathbb{C}z \oplus \mathfrak{r}_{m,n}, \quad \mathfrak{g}_{m,n} = \mathfrak{p}_{m,n} \oplus \mathfrak{r}_{m,n}^-
$$

$$
\mathfrak{p}_{m,2} = \mathfrak{g}_{m,1} \oplus \mathbb{C}z \oplus \mathfrak{r}_{m,2}, \quad \text{and} \quad \mathfrak{g}_{m,n} = \mathfrak{p}_{m,2} \oplus \mathfrak{r}_{m,2}^-.
$$

Denote by $Z$ the center of the reductive part of the parabolic subgroup $P$ corresponding to the parabolic subalgebra we chose above (the Lie algebra of $Z$ is $\mathbb{C}z$).

For any $a \in Z$ we denote by $\mathbb{C}_a$ the corresponding character of $Z$. Since $Z$ is a one-parameter subgroup of $T$, we denote by $\mathbb{C}_a$ the associated $T$-module (in our case, $\mathbb{C}_a$ is either $\varepsilon_n$ or $\delta_n$).

Definition 1. We define the following functors:

$$
\Gamma^a_i : \mathcal{F} \to \mathcal{F}, \quad a \in \frac{1}{2} + \mathbb{Z}
$$

if $a > 0$, if $M \in \mathcal{F}_{m-1,n}$, $\Gamma^a_i(M) := H^i(G_{m,n}/P_{m,n}, \mathcal{L}(C_{(a-(m-n-\frac{1}{2})} \otimes M)^*)^*$.

if $a < 0$, if $M \in \mathcal{F}_{m,n-1}$, $\Gamma^a_i(M) := H^i(G_{m,n}/P_{m,2}, \mathcal{L}(C_{(a-(m-n-\frac{1}{2})} \otimes M)^*)^*$.

$$
H^a_i : \mathcal{F} \to \mathcal{F}, \quad b \in \frac{1}{2} + \mathbb{Z}
$$

if $b > 0$, if $M \in \mathcal{F}_{m,n}$, $H^a_i(M) := \text{Hom}_Z(C_{(b-(m-n-\frac{1}{2})} \otimes M^*, H^j(\mathfrak{r}_{m,n}, M)) \in \mathcal{F}_{m-1,n}$

if $b < 0$, if $M \in \mathcal{F}_{m,n}$, $H^a_i(M) := \text{Hom}_Z(C_{(b-(m-n-\frac{1}{2})} \otimes M^*, H^j(\mathfrak{r}_{m,2}, M)) \in \mathcal{F}_{m,n-1}$.

Now, we consider the following operators in $\mathcal{K}$: if $M \in \mathcal{F}_{m,n}$, denoting the sign of a half-integer $b$ by $\text{sgn}(b)$:

$$
\gamma^a([M]) := \text{sgn}(a)^m \sum_{i \geq 0} (-1)^i [\Gamma^a_i(M)]
$$

$$
\eta^a([M]) := \text{sgn}(b)^m \sum_{j \geq 0} (-1)^j [H^a_j(M)].
$$

Applying the results of the previous section, we get:

Proposition 5. Consider the pairing $\mathcal{K}(L) \times \mathcal{K}(P) \to Z$ defined by

$$
\langle [M], [P] \rangle := \dim \text{Hom}_{G_{m,n}}(M, P)
$$

for every projective $P \in \mathcal{F}_{m,n}$ and every $M \in \mathcal{F}_{m,n}$. Then for any $a \in \frac{1}{2} + \mathbb{Z}$ we have

$$
\langle M, \eta^a[P] \rangle = \langle \gamma^a[M], [P] \rangle.
$$

Let us restrict those linear operators to $\mathcal{K}(\mathcal{E})$. Then for every $a \in \frac{1}{2} + \mathbb{Z}$, the linear operators $\gamma^a$ and $\eta^a$ are mutually adjoint.
We can identify the Grothendieck ring with the ring of characters of finite dimensional modules (cf [3]) and so we will check the relations we need at the level of characters.

We recall the following formula ([2], prop. 1): for $P \subset G$ a parabolic subgroup of a quasireductive supergroup with Levi part $L$,

$$\sum_i (-1)^i \text{Ch}(H^i(G/P, \mathcal{L}(M^*))^*) = D \sum_{w \in W} \varepsilon(w)w \left( \frac{e^p \text{Ch}(M)}{\prod_{\alpha \in \Delta^{+}_{i,t}} (1 + e^{-\alpha})} \right),$$

where $D := \frac{D_0}{D_1}$, $D_0 = \Pi_{\alpha \in \Delta^+_{i,t}} (e^{\alpha/2} - e^{-\alpha/2})$, $D_1 = \Pi_{\alpha \in \Delta^+_{i,t}} (e^{\alpha/2} + e^{-\alpha/2})$, and the various $\Delta$ have the obvious composition (roots of $\mathfrak{g}$ if no other index, roots corresponding to a subalgebra if the subalgebra appears as index).

**Proposition 6.** Let $\nu = (a_m, \ldots, a_1|b_1, \ldots, b_n) \in \Lambda^{+}_{m,n}$. Then one has:

1. $a > 0$, if $\exists i$ s.t. $a_{i+1} > a > a_i$,

$$\gamma^a(\mathcal{E}(\nu)) = (-1)^{m-i} \mathcal{E}(a_m, \ldots, a_{i+1}, a, a_i, \ldots, a_1|b_1, \ldots, b_n),$$

and $\gamma^a(\mathcal{E}(\nu)) = 0$ if $\exists i$, $a = a_i$.

2. $a < 0$, if $\exists i$ s.t. $b_i < -a < b_{i+1}$,

$$\gamma^a(\mathcal{E}(\nu)) = (-1)^{n-i} \mathcal{E}(a_m, \ldots, a_1|b_1, \ldots, b_i, -a, b_{i+1}, \ldots, b_n),$$

and $\gamma^a(\mathcal{E}(\nu)) = 0$ if $\exists i$, $a = -b_i$.

3. $b > 0$, if $\exists i$ s.t. $b = a_i$

$$\eta^b(\mathcal{E}(\nu)) = (-1)^{m-i} \mathcal{E}(a_m, \ldots, a_{i+1}, a_{i-1}, \ldots, a_1|b_1, \ldots, b_n)$$

if $b \neq a_i \forall i$, $\eta^b(\mathcal{E}(\nu)) = 0$.

4. $b < 0$, if $\exists i$ s.t. $b = -b_i$

$$\eta^b(\mathcal{E}(\nu)) = (-1)^{n-i} \mathcal{E}(a_m, \ldots, a_1|b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$$

if $b \neq -b_i \forall i$, $\eta^b(\mathcal{E}(\nu)) = 0$.

**Proof** We will only prove (1), since (2) is analogous and then (3) and (4) follow by adjointness. Let us use [2], Theorem 1: one has, if $M$ is a $B$-module,

$$\sum_{i,j} (-1)^{i+j}[H^i(G_{m,n}/P_{m,n}, \mathcal{L}(H^j(P_{m,n}/B, \mathcal{L}(M^*))))^*] = \sum_k (-1)^k[H^k(G_{m,n}/B, \mathcal{L}(M^*))^*].$$

We take for $M$ the 1-dimensional representation $\mathbb{C}_\lambda$ with $\lambda + \rho_B = (a, a_m, \ldots, a_1|b_1, \ldots, b_n)$. Then, using the equation (4), and the definition of $\gamma^a$, we get

$$\gamma^a(\mathcal{E}(\nu)) = \mathcal{E}(\lambda) = (-1)^{m-i}(a_m, \ldots, a_{i+1}, a, a_i, \ldots, a_1|b_1, \ldots, b_n)$$

for the index $i$ of the statement. Hence the proposition. $\square$
5. Link with the Clifford algebra

Let us now interpret the map $f$ of section 2 in terms of the functors described in the previous section. The proposition 6 has the following immediate corollary:

**Corollary 2.** One has:

\[ f \circ \gamma^a = v_a \circ f \text{ for } a > 0, \]
\[ f \circ \eta_b = w_a \circ f \text{ for } a < 0, \]
\[ f \circ \eta_b = w_b \circ f \text{ for } b > 0, \]
\[ f \circ \eta_b = v_b \circ f \text{ for } b < 0, \]

where $v_a$, $w_b$ stand for the action on the Fock space of the corresponding elements of the Clifford algebra.

This gives us an action of the Clifford algebra on the Grothendieck group $K(\mathcal{E})$.

**Theorem 1.** The operators $\gamma^a$ and $\eta_b$ ($a, b \in \frac{1}{2} + \mathbb{Z}$) in the Grothendieck group $K$ satisfy the Clifford relations:

\[ \eta_a \eta_b + \eta_b \eta_a = 0, \quad \gamma^a \gamma^b + \gamma^b \gamma^a = 0, \quad \gamma^a \eta_b + \eta_b \gamma^a = \delta_{a,b}. \]

**Proof** Let $a$ and $b$ be half-integers. We first show that

\[ \eta_a \eta_b + \eta_b \eta_a = 0. \]

The arguments involved in the proof depend on the signs of $a$ and $b$, we will take care of the cases $a, b > 0$ and $a > 0, b < 0$, leaving $a < 0, b < 0$ to the reader.

Assume first that $a > 0, b > 0$, let $M$ be a $g_{m,n}$ module, we consider the following increasing chain of Lie superalgebras:

\[ g_{m-2,n} \subset p_{m-1,n} \subset g_{m-1,n} \subset p_m,n \subset g_{m,n}. \]

Let $q$ be the parabolic subalgebra with reductive part equal to the direct sum of $g_{m-2,n}$ and the two-dimensional center $Z_q$, and the nilradical $r = r_{m,n} + r_{m-1,n}$. Then using the Hochschild–Serre spectral sequence for the pair $r_{m-1,n} \subset r$ we obtain

\[ \eta_a \eta_b [M] = \sum_i (-1)^i [\text{Hom}_{Z_q}(C(b-(n-1)/2)\varepsilon_m+(a-(n-m+1/2))\varepsilon_{m-1}, H^i(r, M))], \]

and

\[ \eta_b \eta_a [M] = \sum_i (-1)^i [\text{Hom}_{Z_q}(C(a-(n-1/2))\varepsilon_m+(b-(n-m+1/2))\varepsilon_{m-1}, H^i(r, M))]. \]

Now we consider the one-dimensional root subalgebra $s := g_{\beta} \subset r$ for the root $\beta = \varepsilon_m - \varepsilon_{m-1}$. Note that $s$ is the nilradical of a Borel subalgebra of the $sl(2)$ generated by $g_{\beta}$ and $g_{-\beta}$. Hence by the Kostant theorem we have for any $sl(2)$-module $N$

\[ [\text{Hom}_{Z_q}(C(a-(n-1/2))\varepsilon_m+(b-(n-m+1/2))\varepsilon_{m-1}, H^p(s, N))] = \]
\[ [\text{Hom}_{Z_q}(C(b-(n-1/2))\varepsilon_m+(a-(n-m+1/2))\varepsilon_{m-1}, H^q(s, N))] \]

for $(p, q) = (0, 1)$ or $(1, 0)$. 

Once again we apply the Hochschild–Serre spectral sequence for the pair $s \subset r$ to get
\[
\eta_r \eta_b [M] = \sum_i (-1)^{i+j} [\text{Hom}_Z(\mathbb{C}(b-(n-m+1/2))\varepsilon_m + (a-(n-m+1/2))\varepsilon_m, H^i(\mathbb{S}, \Lambda^j(r/s) \otimes M))],
\]
\[
\eta_b \eta_a [M] = \sum_i (-1)^{i+j} [\text{Hom}_Z(\mathbb{C}(a-(n-m+1/2))\varepsilon_m + (b-(n-m+1/2))\varepsilon_m, H^i(\mathbb{S}, \Lambda^j(r/s) \otimes M))].
\]
This implies the relation.

Now let $a > 0, b < 0$. Let $M$ be a $G_{m,n}$-module. Set
\[
r := r_{m,n} + r_{m-1,n}, \quad r' := r_{m,n} + r_{m,n-1}.
\]
Let $Z \subset T_{m,n}$ be the centralizer of $g_{m-1,n}$. Using Hochschild–Serre spectral sequence we obtain
\[
\eta_b \eta_a [M] = \sum_i (-1)^{i} [\text{Hom}_Z(\mathbb{C}(a-(m-n+1/2))\varepsilon_m - (b+n-m+1/2)\delta_n, H^i(r, M))]
\]
and
\[
\eta_a \eta_b [M] = \sum_i (-1)^{i} [\text{Hom}_Z(\mathbb{C}(a-(m-n+1/2))\varepsilon_m - (b+n-m+1/2)\delta_n, H^i(r', M))].
\]
Let $\alpha = \varepsilon_m - \delta_n$. Consider the root subalgebras $g_\alpha, g_{-\alpha} \subset g_{m,n}$. Note that $s := r \cap r'$ is an ideal of codimension 1 in both $r$ and $r'$ and that
\[
r = s + g_\alpha, \quad r' = s + g_{-\alpha}.
\]
Therefore by Hochschild–Serre spectral sequence we have
\[
\eta_b \eta_a = \sum_{i,j} (-1)^{i+j+m-1} [\text{Hom}_Z(\mathbb{C}(a-(m-n+1/2))\varepsilon_m - (b+n-m+1/2)\delta_n, \Lambda^j(g_{-\alpha}) \otimes H^i(s, M))],
\]
\[
\eta_a \eta_b [M] = \sum_{i,j} (-1)^{i+j+m} [\text{Hom}_Z(\mathbb{C}(a-(m-n+1/2))\varepsilon_m - (b+n-m+1/2)\delta_n, \Lambda^j(g_\alpha) \otimes H^i(s, M))].
\]
Taking into account that
\[
\sum_j (-1)^j Ch(\Lambda^j(g_\alpha)) = \frac{1}{1 + e^\alpha} = 1 + e^{-\alpha} = e^{-\alpha} \left( \sum_j (-1)^j Ch(\Lambda^j(g_{-\alpha})) \right)
\]
we obtain
\[
\sum_j (-1)^j Ch(\Lambda^j(g_\alpha) \otimes H^i(s, M)) = e^{-\alpha} \left( \sum_j (-1)^j Ch(\Lambda^j(g_{-\alpha}) \otimes H^i(s, M)) \right).
\]
Therefore
\[
Ch(\eta_a \eta_b [M]) = -Ch(\eta_b \eta_a [M]),
\]
which proves the relation.

Note that the relation
\[
\gamma^a \gamma^b + \gamma^b \gamma^a = 0.
\]
follows from the relation for $\eta_a, \eta_b$ by Proposition 5.
Let us now show that if \( a > 0 \) and \( b < 0 \), then
\[
\gamma^a \eta_b + \eta_b \gamma^a = 0.
\]
The case \( a < 0 \) and \( b > 0 \) is similar and we leave it to the reader.
One should keep in mind the following diagram
\[
\begin{array}{ccc}
F_{m,n} & \xrightarrow{\eta_b} & F_{m,n-1} \\
\gamma^a \downarrow & & \downarrow \gamma^a \\
F_{m+1,n} & \xrightarrow{\eta_b} & F_{m+1,n-1}
\end{array}
\]
because we follow it to keep tracks of the weights.
Let us denote \( ChM_\gamma \) the character of \( \text{Hom}_\mathbb{Z}(\mathbb{C}_\gamma, M) \). Then one has:
\[
\begin{align*}
\gamma^a \eta_b Ch(M) &= \\
\eta_b \gamma^a Ch(M) &= \\
\end{align*}
\]
Next we compute the quotient \( Ch(X)/Ch(Y) \). One has
\[ Ch(X) = \frac{(1 - e^{-\delta_n}) \prod_{i=1}^{n-1} (1 - e^{\delta_n \pm \delta_i}) \prod_{j=1}^{n-1} (1 + e^{\delta_j - \epsilon_{m+1}})}{(1 + e^{-\delta_n}) \prod_{j=1}^{m} (1 + e^{-\delta_n \pm \epsilon_j})}, \]

\[ Ch(Y) = \frac{(1 - e^{-\delta_n}) \prod_{i=1}^{n-1} (1 - e^{-\delta_n \pm \delta_i}) \prod_{j=1}^{n} (1 + e^{\delta_j - \epsilon_{m+1}})}{(1 + e^{-\delta_n}) \prod_{j=1}^{m+1} (1 + e^{-\delta_n \pm \epsilon_j})}, \]

and the quotient turns out to be
\[ Ch(X)/Ch(Y) = e^{\epsilon_{m+1} - \delta_n}. \]

The result follows.

Let us show finally that for \( a, b > 0 \) one has
\[ \gamma^a \eta_b + \eta_b \gamma^a = \delta_{a,b} \]
where \( \delta_{a,b} \) stands for the Kronecker symbol. The proof we provide lacks functoriality at the moment, but we intend to improve it.

Let \( R : \mathcal{F}_{m,n} \to \mathcal{F}_{m,0} \) be the restriction functor and denote by the same letter the corresponding map of the Grothendieck groups. Then it follows from Proposition 3 that for any \( M \in \mathcal{F}_{m,n} \),
\[ R(\gamma^a[M]) = \gamma^{a+n}([S^\bullet((\mathfrak{r}_{m+1,n}^*)_\mathfrak{t})]R[M]). \]

On the other hand, for any Lie superalgebra \( \mathfrak{r} \) and \( \mathfrak{r} \)-module \( M \) we have
\[ \sum_i (-1)^i[H^i(\mathfrak{r}, M)] = \sum_{k,l} (-1)^{k+l}[\Lambda^l(\mathfrak{r}_\mathfrak{t})][H^k(\mathfrak{r}_\mathfrak{t}, M)]. \]

Therefore
\[ R(\eta_a[M]) = \sum_k (-1)^k \eta_{a+n}([\Lambda^k((\mathfrak{r}_{m,n}^*)_\mathfrak{t})]R[M]). \]

Therefore for \( M \in \mathcal{F}_{m,n} \) we have
\[ R(\gamma^a \eta_b[M]) = \sum_k (-1)^k \gamma^{n+a}([S^\bullet((\mathfrak{r}_{m,n}^*)_\mathfrak{t})] \eta_{b+n}([\Lambda^k((\mathfrak{r}_{m,n}^*)_\mathfrak{t})]R[M]), \]
\[ R(\eta_b \gamma^a[M]) = \sum_k (-1)^k \eta_{b+n}([\Lambda^k((\mathfrak{r}_{m+1,n}^*)_\mathfrak{t})] \gamma^{n+a}([S^\bullet((\mathfrak{r}_{m+1,n}^*)_\mathfrak{t})]R[M]). \]

Let us denote by \( U \) the standard representation of \( \mathfrak{sp}(2n) \subset \mathfrak{osp}(2m+1,2n) \) and consider it as purely odd superspace. Then
\[ Ch((\mathfrak{r}_{m,n}^*)_\mathfrak{t}) = e^{-\epsilon_m} Ch(U). \]

Therefore the above expressions can be rewritten in the form
\[ R(\gamma^a \eta_b[M]) = \sum_{k,l} (-1)^k \gamma^{n+a-l}([S^l(U)] \eta_{b+n+k}([\Lambda^k(U)]R[M])), \]
\[ R(\eta_b \gamma^a[M]) = \sum_k (-1)^k \eta_{b+n+k}([\Lambda^k(U)] \gamma^{n+a-l}([S^l(U)]R[M])). \]
Now we note that the action of $G_{m,0}$ on $U$ is trivial, hence multiplication with its exterior and symmetric powers commute with $\gamma^a$ and $\eta_b$. Thus, we have

$$R(\gamma^a \eta_b[M]) = \sum_{k,l} (-1)^k [S^l(U)][\Lambda^k(U)]\gamma^{n+a-l} \eta_b \eta_{b+n+k}(R[M]),$$

$$R(\eta_b \gamma^a[M]) = \sum_k (-1)^k([\Lambda^k(U)][S^l(U)]\eta_{b+n+k}\gamma^{n+a-l}(R[M]).$$

Since $\mathcal{F}_{m,0}$ is the category of representations of a purely even reductive group, we have $\mathcal{K}(\mathcal{E})_{m,0} = \mathcal{K}(L)_{m,0}$. Therefore Proposition 6 implies that for any $N \in \mathcal{F}_{m,0}$

$$\gamma^a \eta_b[N] + \eta_b \gamma^a[N] = \delta_{a,b}[N].$$

Hence,

$$(\gamma^a \eta_b + \eta_b \gamma^a)(R[M]) = \sum_{l,k} (-1)^k [S^l(U)][\Lambda^k(U)]\delta_{a+n-k,b+n+l}R[M],$$

and hence

$$(\gamma^a \eta_b + \eta_b \gamma^a)(R[M]) = \sum_{k+l=a+b} (-1)^k [S^l(U)][\Lambda^k(U)]R[M].$$

Since the Koszul complex is acyclic except in the zero degree we have the identity

$$\sum_{k+l=p} (-1)^k[\Lambda^k(U)][S^l(U)] = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Hence, the sum we compute has only one non-zero term, namely we get:

$$(\gamma^a \eta_b + \eta_b \gamma^a)(R[M]) = \delta_{a,b}R[M].$$

Since the map $R$ is injective this proves the result for $\gamma^a \eta_b + \eta_b \gamma^a$, $a, b > 0$. The case $a, b < 0$ is similar and we leave it to the reader. \qed

6. Translation functors

We would like to link this approach with the results on translation functors in [3].

Recall the Lie algebra $\mathfrak{gl}(\infty)$ which is embedded in $Cl(V \oplus W)$ as the span of $v_a w_b$, $a, b \in \frac{1}{2} + \mathbb{Z}$. The subalgebra $\mathfrak{gl}(\frac{\infty}{2})$ is generated by $v_a w_b + v_{-a} w_{-b}$, $a, b \in \frac{1}{2} + \mathbb{N}$.

Inside the Fock space $F$, we consider the subspace $F_{m,n}$ which is the image of $\mathcal{K}(\mathcal{E})_{m,n}$ under the map $f$, defined at the end of section 2.

**Remark 1.** The space $F_{m,n}$ is stable under the action of $\mathfrak{gl}(\frac{\infty}{2})$. Furthermore, it is not difficult to see that $F_{m,n}$ is isomorphic to $\Lambda^m(V_+) \otimes \Lambda^n(W_+)$ as an $\mathfrak{sl}(\frac{\infty}{2})$-module, where $V_+$ and $W_+$ are respectively the standard and costandard module of $\mathfrak{gl}(\frac{\infty}{2})$.

Consider the Cartan subalgebra $t$ of $\mathfrak{gl}(\frac{\infty}{2})$ with basis $t_a := v_a w_a + v_{-a} w_{-a}$ for all $a \in \frac{1}{2} + \mathbb{N}$, then $F$ is a semi-simple $t$-module. We denote by $\omega$ the $t$-weight of the vacuum vector: $\omega(t_a) = 1$ for all $a \in \frac{1}{2} + \mathbb{N}$. Let $\beta_a \in t^*$ be such that $\beta_a(t_b) = \delta_{a,b}$. If $\lambda = (a_m, \ldots, a_1 b_1, \ldots, b_n)$, then the $t$-weight of $f(\mathcal{E}(\lambda))$ equals

$$\beta(\lambda) := \omega + \beta_{a_1} + \cdots + \beta_{a_m} - \beta_{b_1} - \cdots - \beta_{b_n}.$$
Lemma 2. Let $\mathcal{E}(\lambda), \mathcal{E}(\mu) \in \mathcal{K}(\mathcal{E})_{m,n}$. Then $\mathcal{E}(\lambda)$ and $\mathcal{E}(\mu)$ are in the same block of $\mathcal{F}_{m,n}$ if and only if the $t$-weights of $f(\mathcal{E}(\lambda))$ and $f(\mathcal{E}(\mu))$ coincide.

Proof. The statement follows from the remark 1 after comparing with the weights denoted by $\gamma(\lambda)$ in [3] (we do not keep this notation here because we have introduced a $\gamma^a$ which is not related). The relation between those $t$-weights is $\beta(\lambda) = \omega + \gamma(\lambda)$.

Consider now the Chevalley generators of $\mathfrak{gl}(\frac{\infty}{2})$, $E_{a,a+1}$ and $E_{a+1,a}$ for all $a \in \frac{1}{2} + \mathbb{N}$. As it was shown in [3], the categorification of the action of these generators in $\Lambda^m(V_+) \otimes \Lambda^n(W_+)$ is given by the translation functors:

$$T_{a+1,a}(M) := (M \otimes E)_{\beta+\beta_{a+1}-\beta_a}, \quad T_{a,a+1}(M) := (M \otimes E)_{\beta+\beta_a-\beta_{a+1}},$$

where $E$ is the standard $\mathfrak{g}_{m,n}$-module, we assume that the $\mathfrak{g}_{m,n}$-module $M$ belongs to the block corresponding to the $t$-weight $\beta$, and by $(N)_{\beta'}$ we denote the projection of the $\mathfrak{g}_{m,n}$-module $N$ onto the block corresponding to the $t$-weight $\beta'$. By abuse of notations we denote also by $T_{a+1,a}$ and $T_{a,a+1}$ the corresponding linear operators in $\mathcal{K}(\mathcal{E})_{m,n}$.

The following statement is an immediate consequence of the remark 1 and Lemma 4 in [3].

Proposition 7. For all $a \in \frac{1}{2} + \mathbb{N}$ we have

$$f \circ T_{a+1,a} = E_{a+1,a} \circ f, \quad f \circ T_{a,a+1} = E_{a,a+1} \circ f.$$
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