Bosonized supersymmetry of anyons and supersymmetric exotic particle on the non-commutative plane

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Abstract

A covariant set of linear differential field equations, describing an $N = 1$ supersymmetric anyon system in (2+1)D, is proposed in terms of Wigner’s deformation of the bosonic Heisenberg algebra. The non-relativistic “Jackiw-Nair” limit extracts the ordinary bosonic and fermionic degrees of freedom from the Heisenberg-Wigner algebra. It yields first-order, non-relativistic wave equations for a spinning particle on the non-commutative plane that admits a Galilean exotic planar $N = 1$ supersymmetry.

1 Introduction

Low dimensional quantum field theories may behave in a remarkable way: bosons and fermions can be equivalent [1]. Similarly, non-relativistic planar (or (2+1)D relativistic) systems may exhibit a boson-fermion (or, more generally, a boson-anyon) transmutation mechanism [2].

Yet another strange behaviour has been found recently, when it was shown that, owing to these low dimensional peculiarities, some purely bosonic quantum mechanical systems may be supersymmetric [3, 4]. It appears that bosonization is related to hidden nonlocal structure. In bosonic systems with hidden supersymmetry, in particular, a nonlocal operator, namely reflection, plays the rôle of a grading operator.

Quantum mechanics can be viewed as field theory in 0 + 1 dimensions. Then it is natural to ask whether the bosonization can be extended to genuine field theories which would be hence super-Poincaré symmetric. The present paper shows that this is indeed possible, namely in $2 + 1$ dimensions.

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We start with observing that the Fock space of the Heisenberg algebra \([a^-, a^+] = 1\) can be viewed as a direct sum of two irreducible infinite-dimensional unitary representations of the Lorentz algebra \(so(1, 2)\), generated by operators which are quadratic in \(a^\pm\). The two irreducible \(so(1, 2)\) representations are labeled by the Lorentz spin \(1/4\) and \(3/4\) (or \(-1/4\) and \(-3/4\)) and are realized in even and odd subspaces of the Fock space. The latter are distinguished by the reflection operator \(R = (-1)^N = \cos \pi N, \quad N = a^+ a^-\). Note that \(R\) is a nonlocal operator (as it follows, e.g., from its infinite-series expansion). These two \(so(1, 2)\) representations form an irreducible representation of the \(osp(1|2)\) superalgebra, generated by the even (in the sense of Lie superalgebra) \(so(1, 2)\) generators and by the odd generators \(a^+\), \(a^-\). They form an \(so(1, 2)\) spinor. \(R\) plays the rôle of a grading operator of the superalgebra.

Since the spins in the two sectors differ by \(1/2\), we have a priori all ingredients needed to extend the construction to a \((2+1)D\) super-Poincaré symmetry. This can be done by treating \(a^\pm\) as describing internal (translation-invariant) spin degrees of freedom of a supersymmetric system, which lives in configuration space with cooordinates \(x^\mu\). We introduce the fields

\[
\Psi(x) = \sum_{n=0}^{\infty} \psi_n(x) |n\rangle, \quad N|n\rangle = n|n\rangle, \tag{1.1}
\]

and then postulate, “à la Dirac”, a covariant (vector) set of linear differential equations, which imply the Klein-Gordon equation and Majorana-like equation as integrability (consistency) conditions. These latter fix the two Casimir operators of the \((2+1)D\) Poincaré superalgebra, namely the mass and the superspin, which fixes in turn the “Poincaré spins” whose values, \(1/4\) and \(3/4\), are shifted by one half.

Physical states are eigenstates with opposite \(\pm 1\) eigenvalues of the reflection operator [playing, again, the rôle of a grading operator of the Poincaré superalgebra], and will obey therefore a relative Fermi statistics.

This construction provides us with a bosonized \((2+1)D\) supersymmetry: relative Fermi statistics is obtained without introducing any fermionic degrees of freedom. Everything is realized in terms of bosonic degrees of freedom alone.

It is worth noting that Heisenberg structures leading to “quartions” (particles with \(1/4\) spin) have been considered before [5, 6]. Infinite component Majorana-type anyon equations were proposed in [7, 8].

Generalizing the ordinary Heisenberg algebra to the deformed Heisenberg-Wigner algebra \([a^-, a^+] = 1 + \nu R\), where \(\nu > -1\) is a deformation parameter cf. (2.1) below, allows us to extend the construction to a pair of two anyon fields with spins \(s = 1/4(1 + \nu) > 0\) and \(s + 1/2 > 0\) cf. (2.6) (or, \(s < 0\), \(s - 1/2 < 0\) for the alternative realization of the Lorentz generators), respectively.

For the special values \(\nu = -(2k + 1)\) the algebra has finite-dimensional non-unitary representations [9], see the Discussion, Section 6. This algebra appeared implicitly in Wigner’s (1950) work [10]; its infinite-mode generalization lead to the theoretical discovery of parastatistics [11, 12].

Below we mostly focus our attention to the unitary case \(\nu > -1\). Then the generators of the deformed Heisenberg-Wigner algebra can be reinterpreted as “entangled” bosonic and fermionic degrees of freedom. Taking the special nonrelativistic limit considered by Jackiw and Nair [13] (\(s \to \infty\), \(c \to \infty\), \(s/c^2 = \kappa = \text{const}\) “extracts” these degrees of
freedom, and produces a spinning particle on the noncommutative plane. It carries an $N = 1$ superextension of the “exotic” [i.e. two-parameter centrally extended] Galilean symmetry.

Our investigations here generalize a previous attempt [14], based on a similar construction [15]. In [14] a wave equation has indeed been proposed using a spinor operator, $Q_\alpha$ in (2.7) of [14]. When restricted to the $+1$ eigenspace of the reflection operator $R$, the Klein-Gordon and the Majorana equations were implied as consistency conditions. Our wave equation could, therefore, be considered as an infinite-component Majorana-type system, analogous to those proposed before [7, 8], which would describe a relativistic anyon.

The subtle “Jackiw-Nair” non-relativistic limit [13] has been designed by these authors to derive “exotic particles” (i.e., such that Galilean boosts don’t commute) from anyons. When applied to the $R = +1$ sector of the model in [14], non-relativistic infinite-component Dirac-Majorana-type equations are obtained with analogous properties. In particular, the suitably defined Galilean boost operators $K_i$ satisfy the “exotic” commutation relation

$$[K_i, K_j] = i\kappa \epsilon_{ij}. \quad (1.2)$$

These $K_i$ take, furthermore, a natural form [(6.5) in [14]] in terms of some Foldy-Wouthuysen-type coordinates $X_i$ [(6.2) of [14]], and the “exotic” relation (1.2) is reflected by their non-commutativity

$$[X_i, X_j] = i\theta \epsilon_{ij}, \quad \theta = \frac{\kappa}{m^2}. \quad (1.3)$$

In other words, the theory yields a noncommutative plane.

It has been puzzling whether the negative subspace $R = -1$ could yield a supersymmetric partner of the bosonic model in [14]. Easy calculation shows, however, that the restriction of the wave equation of [14] to the subspace $R = -1$ is trivial. One of the sectors is hence lost.

In the present paper, this obstruction is overcome, though. Our clue is to introduce another operator, namely (3.20) below, which, on the one hand, allows a nontrivial $R = -1$ sector so that the $R = \pm 1$ sectors become superpartners; on the other hand, the restriction of the new operator to the $R = +1$ sector yields a theory equivalent to the one in [14].

The paper is organized as follows. In Section 2, on the basis of infinite-dimensional unitary representations of Heisenberg-Wigner algebra, we construct the anyon system admitting a bosonized $(2+1)$D $N = 1$ supersymmetry. In Section 3 a covariant set of linear differential equations for this system is identified and their general solution is discussed. In Section 4 we apply the Jackiw-Nair non-relativistic limit and discuss the resulting system of first order wave equations and its associated supersymmetry. In Section 5 we discuss the appearance of noncommutative coordinates in the theory. The last Section includes concluding remarks, where, in particular, we briefly discuss finite-dimensional representations of the deformed Heisenberg-Wigner algebra.

## 2 (2+1)D bosonized supersymmetry

Consider the deformed Heisenberg algebra [9]

$$[a^-, a^+] = 1 + \nu R, \quad R^2 = 1, \quad \{a^\pm, R\} = 0. \quad (2.1)$$
where $\nu$ is a real deformation parameter. The usual Heisenberg algebra $[a^-, a^+] = 1$ is included as a particular case $\nu = 0$. The operator

$$ N = \frac{1}{2} \{ a^+, a^- \} - \frac{1}{2} (\nu + 1), \quad [N, a^\pm] = \pm a^\pm, \tag{2.2} $$

is a number operator. Then, as in a non-deformed case, $R = (-1)^N = \cos \pi N$.

The $a^\pm$ satisfying the deformed Heisenberg algebra, satisfy also trilinear commutation relations $\{ [a^+, a^-], a^\pm \} = \pm 2a^\pm$, see Eq. (2.2), and vice versa [16]. For $\nu = p - 1$, $p = 1, 2, \ldots$, the $a^\pm$ are creation-annihilation operators of a single-mode paraboson of order $p$, characterized by the additional relation $a^- a^+ |0\rangle = p |0\rangle$, where $|0\rangle$ is a vacuum state, $a^- |0\rangle = 0$. In what follows we refer to (2.1) as to Heisenberg-Wigner (HW) algebra.

For any $\nu > -1$, the algebra admits an infinite-dimensional unitary representation realized on a Fock space; this is our main interest here. The Fock space is spanned by the states $|n\rangle$, $\langle n'|n\rangle = \delta_{n',n}$,

$$ N|n\rangle = n|n\rangle, \quad R|n\rangle = (-1)^n|n\rangle, \quad |n\rangle = C_n(a^+)^n|0\rangle, \quad n = 0, 1, \ldots, \tag{2.3} $$

where $C_n$ is a normalization coefficient.

The Fock space (as well as any finite-dimensional representation of the HW algebra) is decomposed into even and odd subspaces defined by $R|\psi\rangle_\pm = \pm |\psi\rangle_\pm$, which correspond to $n$ being even or odd. Both for finite and infinite-dimensional representations, these subspaces carry irreducible representations of the Lorentz algebra $so(1,2)$. The generators are realized as quadratic operators,

$$ J_0 = \frac{1}{4} \{ a^+, a^- \}, \quad J_\pm \equiv J_1 \pm i J_2 = \frac{1}{2} (a^\pm)^2, \tag{2.4} $$

$$ [J_\mu, J_\nu] = -i \epsilon_{\mu\nu\lambda} J^\lambda. \tag{2.5} $$

[The antisymmetric tensor is normalized by $\epsilon^{012} = 1$; the metric is $\eta_{\mu\nu} = \text{diag}(-1,1,1)$.] In the even and odd subspaces the $so(1,2)$ Casimir operator $J_\mu J^\mu$ takes the values $J_\mu J^\mu = -\alpha_+ (\alpha_+ - 1)$ and $J_\mu J^\mu = -\alpha_- (\alpha_- - 1)$, where

$$ \alpha_+ = \frac{1}{4} (1 + \nu), \quad \alpha_- = \alpha_+ + \frac{1}{2}. \tag{2.6} $$

In the unitary (infinite dimensional) case $\alpha_+ > 0$, and the states $|n\rangle$ satisfy the relations

$$ J_0 |2k\rangle = (\alpha_+ + k) |2k\rangle, \quad J_0 |2k + 1\rangle = (\alpha_- + k) |2k + 1\rangle, \quad k = 0, 1, \ldots, \tag{2.7} $$

Every infinite-dimensional unitary representation of the deformed Heisenberg algebra is therefore the direct sum of two, bounded-from-below, infinite-dimensional unitary representations of the $so(1,2)$, $D^+_\alpha \oplus D^-_{\alpha_-}$ [17] being (2+1)D analogs of the infinite-dimensional unitary representations of $so(1,3)$ discovered by Majorana (1932) [18]. The infinite-dimensional unitary representations $D^+_\alpha$ and $D^-_{\alpha_-}$ bounded from above (necessary to describe the states with negative spin values, see below) are obtained via a simple sign change in (2.4)

$$ J_0 \rightarrow -\frac{1}{4} \{ a^-, a^+ \}, \quad J_\pm \rightarrow -\frac{1}{2} (a^\pm)^2. \tag{2.8} $$
The linear operators
\[ \mathcal{L}_1 = \frac{1}{\sqrt{2}}(a^+ + a^-), \quad \mathcal{L}_2 = \frac{i}{\sqrt{2}}(a^+ - a^-), \] (2.9)
extend the Lorentz algebra (2.5) to an \( osp(1|2) \) superalgebra, in which \( J_\mu \) and \( \mathcal{L}_\alpha \) are even and odd generators, respectively, and the grading operator is \( R \), \( [R, J_\mu] = 0, \{ R, \mathcal{L}_\alpha \} = 0 \), \( R^2 = 1 \). This superalgebra is characterized, in addition to (2.5), by the (anti)commutation relations
\[ \{ \mathcal{L}_\alpha, \mathcal{L}_\beta \} = 4i(\gamma^0)_{\alpha\beta}, \quad [J_\mu, \mathcal{L}_\alpha] = \frac{1}{2}(\gamma_\mu)_{\alpha\beta} \mathcal{L}_\beta. \] (2.10)
Here the gamma-matrices are in the Majorana representation,
\[ (\gamma^0)_{\alpha\beta} = -(\sigma^2)_{\alpha\beta}, \quad (\gamma^1)_{\alpha\beta} = i(\sigma^1)_{\alpha\beta}, \quad (\gamma^2)_{\alpha\beta} = i(\sigma^3)_{\alpha\beta}, \]
where they satisfy relations
\[ (\gamma_\mu)_{\alpha\beta} = -\eta_{\mu\nu}\epsilon_{\alpha\beta} + i\epsilon_{\mu\nu\lambda}(\gamma^\lambda)_{\alpha\beta}, \quad \gamma_{\alpha\beta} = \gamma_{\beta\alpha}, \quad \gamma_{\alpha\beta} = -\gamma_{\alpha\beta}. \]
The antisymmetric tensor \( \epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} (\epsilon^{12} = 1) \), provides us with a metric for the spinor indices, \( A^\alpha = \epsilon^{\alpha\beta} A_\beta, A_\alpha = A^\beta\epsilon_{\beta\alpha} \). Note that \( A^\alpha B_\alpha = -A_\alpha B^\alpha \) for any \( A_\alpha \) and \( B_\alpha \).

Every irreducible representation of the HW algebra is an irreducible representation of the \( osp(1|2) \) superalgebra, for which its Casimir operator
\[ \mathcal{C} = J_\mu J^\mu - \frac{i}{8} \mathcal{L}_\alpha \mathcal{L}_\alpha \] (2.11)
takes the fixed value \( \mathcal{C} = \frac{1}{16}(1 - \nu^2) \).

Since the \( so(1, 2) \) spin has a relative shift of one-half between the even and odd subspaces, it would be natural to try to realize a (2+1)D supersymmetry (in the sense of supersymmetric extension of the Poincaré symmetry) using the HW algebra and its irreducible representations. For the purpose, we treat \( a^\pm \) as internal, translation invariant variables, augmented with independent space-time coordinates \( x^\mu \) and conjugate momenta \( p_\mu, [x_\mu, p_\nu] = i\eta_{\mu\nu}, [x^\mu, a^\pm] = [p_\mu, a^\pm] = 0 \), and introduce a field (1.1). Translations and complete Lorentz generators are identified as \( P_\mu = p_\mu \) and
\[ J_\mu = -\epsilon_{\mu\nu\lambda} x^\nu p^\lambda + J_\mu. \] (2.12)
In accordance with Eqs. (2.12), (2.10), the \( \mathcal{L}_\alpha, \alpha = 1, 2 \), form a (2+1)D spinor. In its terms we define an operator
\[ Q_\alpha = \frac{i}{\sqrt{2m(1 + \nu)}} \left( (p_\gamma)_{\alpha\beta} - m\epsilon_{\alpha\beta} \right) \mathcal{L}_\beta. \] (2.13)
With the normalization depending on the mass parameter \( m \), the spinor operator \( Q_\alpha \) has the dimension of a square root of the space-time translation generator \( P_\mu \), and can be considered as a candidate for a supercharge operator of a superextended (2+1)D Poincaré algebra, for which the operator \( R \) will play the rôle of a grading operator,
\[ [R, P_\mu] = [R, J] = 0, \quad \{ R, Q_\alpha \} = 0. \]
The space-time translation and Lorentz generators, together with the operator $Q_\alpha$ satisfy the (anti)commutation relations
\begin{equation}
[P_\mu, P_\nu] = 0, \quad [J_\mu, P_\nu] = -i\epsilon_{\mu\nu\lambda}P^\lambda, \quad [J_\mu, J_\nu] = -i\epsilon_{\mu\nu\lambda}J^\lambda, \quad (2.14)
\end{equation}
\begin{equation}
[P_\mu, Q_\alpha] = 0, \quad [J_\mu, Q_\alpha] = \frac{1}{2}(\gamma_\mu)_{\alpha\beta}Q_\beta, \quad (2.15)
\end{equation}
\begin{equation}
\{Q_\alpha, Q_\beta\} = -2i(P\gamma)_{\alpha\beta}
+ \frac{2i}{m(1 + \nu)}\left[(J\gamma)_{\alpha\beta}(p^2 + m^2) - 2(p\gamma)_{\alpha\beta}((pJ - m\alpha_+)\Pi_+ + (pJ - m\alpha_-)\Pi_-)\right], \quad (2.16)
\end{equation}
where $\Pi_+ = \frac{1}{2}(1 + R)$ and $\Pi_- = \frac{1}{2}(1 - R)$ are projectors on the even and odd subspaces of the Fock space of the HW algebra, respectively. Decomposing our field as
\begin{equation}
\Psi = \Psi_+ + \Psi_-, \quad \Psi_\pm = \Pi_\pm \Psi, \quad (2.17)
\end{equation}
it is clear from Eq. (2.16) that if the components satisfy the Klein-Gordon and Majorana-like equations [18, 19],
\begin{equation}
(p^2 + m^2)\Psi_\pm = 0, \quad (pJ - m\alpha_+)\Psi_+ = 0, \quad (pJ - m\alpha_-)\Psi_- = 0, \quad (2.18)
\end{equation}
then, on shell, the relation (2.16) takes the form
\begin{equation}
\{Q_\alpha, Q_\beta\} = -2i(P\gamma)_{\alpha\beta}. \quad (2.19)
\end{equation}
As a result, we obtain that the $N = 1$ Poincaré superalgebra, (2.14), (2.15), (2.19), is a symmetry of the field system (2.18), realized without incorporating fermionic degrees of freedom. Since $R\Psi_\pm = \pm\Psi_\pm$ and $R$ is the grading operator of the Poincaré superalgebra, the fields $\Psi_+$ and $\Psi_-$ describe massive anyon states with spins $s_+ = \alpha_+$ and $s_- = s_+ + \frac{1}{2}$ and positive energy (see Eq. (2.7)), which carry relative Fermi statistics.

The operator
\begin{equation}
C = P^\mu J_\mu + \frac{i}{8}Q^\alpha Q_\alpha \quad (2.20)
\end{equation}
is the Casimir operator of the superalgebra (2.14), (2.15), (2.19). Then we find that on states satisfying equations (2.18) the superspin $S = \frac{1}{m}C$ takes the value
\begin{equation}
S = \frac{1}{2}(\alpha_+ + \alpha_-). \quad (2.21)
\end{equation}
This means that the field system (2.18) realizes an irreducible representation of the (2+1)D $N = 1$ supersymmetry.

Note that in order to describe an $N = 1$ supermultiplet with negative spins $s_+ = -\alpha_+$ and $s_- = -s_+ + \frac{1}{2}$ and positive energies, it is sufficient to change a realization of the 'internal space' Lorentz generators $J_\mu$ according to Eq. (2.8) and change $m \rightarrow -m$ in all relations except for the normalization factor in the definition of the supercharge.
3 Linear differential supersymmetric field equations

Following Dirac’s idea [20, 19], let us find linear differential field equations for which the Klein-Gordon and Majorana-like equations (2.18) appear as integrability (consistency) conditions. This will provide us then with wave equations for a spinning particle on the non-commutative plane, which carries an irreducible representation of the $N = 1$ superextended exotic Galilean symmetry. Consider indeed the vector set of equations [21],

$$V^{(a)}_\mu \Psi^{(a)} = 0, \quad V^{(a)}_\mu = \alpha p_\mu - i \epsilon_{\mu \nu \lambda} p^\nu J^\lambda + m J_\mu,$$

(3.1)

where we assume that the field $\Psi^{(a)}$ carries an irreducible representation of the $so(1,2)$ Lorentz group generated by the translationally invariant operators $J_\mu$, and characterized by the relations

$$J_\mu J^\mu \Psi = -\alpha (\alpha - 1) \Psi^{(a)}, \quad J_0 \Psi^{(a)} = (\alpha + n) \Psi^{(a)}.$$

Our representations can be unitary infinite-dimensional half-bounded representations $D^+_\alpha$ realized on the even or odd subspaces of the deformed Heisenberg algebra with parameter $\nu > -1$, as described above. Alternatively, they can be finite-dimensional non-unitary representations, realized by the same algebra with parameter $\nu = -(2k + 1)$.

Since the vector operator $V^{(a)}_\mu$ satisfies the commutation relation

$$[V^{(a)}_\mu, V^{(a)}_\nu] = -i \epsilon_{\mu \nu \lambda} \left( \eta^{\lambda \rho} + (\alpha - 1) m^{-1} p^\lambda J^\rho \right) V^{(a)}_\mu,$$

(3.2)

the three equations (3.1) form a consistent set. On the other hand, this operator satisfies the relations

$$J^\mu V^{(a)}_\mu = (\alpha - 1) (p J - \alpha m),$$

(3.3)

$$p^\mu V^{(a)}_\mu = \alpha (p^2 + m^2) + m(p J - \alpha m),$$

(3.4)

$$i \epsilon^{\mu \nu \lambda} p_\nu J_\lambda V^{(a)}_\mu = \alpha (\alpha - 1) (p^2 + m^2) + (p J + (\alpha - 1) m) (p J - \alpha m).$$

(3.5)

Eq. (3.3) implies, in particular, that there is no singularity in (3.2) at $\alpha = 1$. From relations (3.3)-(3.5) we conclude that a field $\Psi^{(a)}$ which satisfies the system of vector equations (3.1) satisfies also the equations $(p^2 + m^2) \Psi^{(a)} = 0$, $(p J - m \alpha) \Psi^{(a)} = 0$. Hence $\Psi^{(a)}$ carries an irreducible representation of the $(2+1)$D Poincaré group, characterized by the mass $m$, spin $s = \alpha$, and has positive energy $p^0 > 0$ if the unitary representation $D^+_\alpha$ is chosen.

The operator $V^{(a)}_\mu$ satisfies the identity

$$W^\mu V^{(a)}_\mu \equiv 0, \quad W_\mu = (\alpha - 1)^2 p_\mu - i (\alpha - 1) \epsilon_{\mu \nu \lambda} p^\nu J^\lambda + (p J) J_\mu.$$

(3.6)

Only two components of the vector operator $V^{(a)}_\mu$ are therefore independent. All three components are necessary, however, to have a manifestly covariant set of equations (3.1) that guarantees the relativistic invariance of the theory [3].

In terms of $J_\pm = J_1 \pm i J_2$, the $so(1,2)$ commutation relations take a form $[J_0, J_\pm] = \pm J_\pm$, $[J_-, J_+] = 2 J_0$. For the bounded from below unitary representation $D^+_\alpha$ we have therefore

$$J_0 |n\rangle = (\alpha + n) |n\rangle, \quad J_+ |n\rangle = C^n_\alpha |n + 1\rangle, \quad J_- |n\rangle = C^n_{\alpha - 1} |n - 1\rangle,$$

(3.7)

$$C^n_\alpha = \sqrt{(2\alpha + n)(n + 1)}.$$  

(3.8)
Decomposing the field $\Psi^{(\alpha)}(x)$ into eigenstates of the operator $J_0$, $\Psi^{(\alpha)}(x) = \sum_{n=0}^{\infty} \psi_n|n\rangle$, provides us with an equivalent, component form of the system (3.1),

$$[\alpha(p^0 - m) - nm]\psi_n + \frac{1}{2} [C_n^a p_+ \psi_{n+1} - C_{n-1}^a p_- \psi_{n-1}] = 0, \quad (3.9)$$

$$- np_+\psi_n + C_{n-1}^a (m - p^0)\psi_{n-1} = 0, \quad (3.10)$$

$$(2\alpha + n)p_-\psi_n + C_n^a (m + p^0)\psi_{n+1} = 0. \quad (3.11)$$

The dependence of the equations can also be seen by noting that a suitable linear combination of (3.10) and (3.11), reproduces the first equation (3.9). In the same way, suitable linear combinations of any two equations from (3.9)–(3.11) show that every component satisfies the Klein-Gordon equation. On the other hand, it also imply the Majorana equation presented in component form,

$$[p^0(\alpha + n) - \alpha m]\psi_n + \frac{1}{2} [C_n^a p_+ \psi_{n+1} + C_{n-1}^a p_- \psi_{n-1}] = 0.$$

In the representation $D_{\alpha}^+$, Eq. (3.11) allows us to express the component $\psi_{n+1}$ in terms of $\psi_n$,

$$\psi_{n+1} = - \frac{2\alpha + n}{C_n^a} \frac{p_-}{p^0 + m} \psi_n. \quad (3.12)$$

Then substituting the relation (3.12) into Eq. (3.10) shows that every component satisfies the Klein-Gordon equation, and repeated application of relation (3.11) allows us to represent all higher field components in terms of the lowest one,

$$\psi_n(p) = (-1)^n \sqrt{\frac{\Gamma(2\alpha + n)}{\Gamma(n+1)\Gamma(2\alpha)}} \left(\frac{p_-}{p^0 + m}\right)^n \psi_0(p), \quad (3.13)$$

$$\psi_0(p) = \delta \left(p^0 - \sqrt{p_i^2 + m^2}\right) f(p_i), \quad (3.14)$$

where it is implied that we switched to momentum representation. Then some manipulations allow us to derive the equivalent set of independent equations

$$\sqrt{n + 2\alpha (m + p_0)}\psi_n - \sqrt{n + 1} p_+ \psi_{n+1} = 0, \quad (3.15)$$

$$\sqrt{n + 2\alpha p_-}\psi_n + \sqrt{n + 1} (m - p_0)\psi_{n+1} = 0, \quad (3.16)$$

which will be convenient for taking a special non-relativistic limit to the system, see below.

Returning to our supersymmetric system, let us introduce a notation for the even and odd states of the Fock space of the HW algebra,

$$|n\rangle_+ = |2n\rangle, \quad |n\rangle_- = |2n + 1\rangle, \quad R|n\rangle_\pm = \pm |n\rangle_\pm, \quad n = 0, 1, \ldots, \quad (3.17)$$

and rewrite decomposition (1.1), (2.17) in the form

$$\Psi(x) = \Psi_+(x) + \Psi_-(x) = \sum_{n=0}^{\infty} \left(\psi_n^+|n\rangle_+ + \psi_n^-|n\rangle_-ight). \quad (3.18)$$
Now, for every field $\Psi_+$ and $\Psi_-$, we postulate the vector set of linear differential equations (3.1) with the parameter $\alpha$ chosen to be $\alpha_+$ and $\alpha_-$, respectively. Taking the $so(1,2)$ generators $J_\mu$ as in Eq. (2.4), these fields carry the $so(1,2)$ irreducible representations $D_+^{\alpha_+}$ and $D_-^{\alpha_-}$, realized on even and odd subspaces of the Fock space, respectively. It is clear that, consistently with Eq. (2.18), we reproduce our supersymmetric system. Therefore, the covariant system of linear differential equations for the field (3.18) we were looking for is

$$V_\mu \Psi = 0,$$

$$V_\mu = V_\mu^{(\alpha_+)} \Pi_+ + V_\mu^{(\alpha_-)} \Pi_- = \frac{1}{4} (2 + \nu - R) p_\mu - i \epsilon_{\mu \nu \lambda} p^\nu J^\lambda + mJ_\mu .$$

(3.19)

(3.20)

As independent equations we can choose, again, the pair $V_+ \Psi = 0$, $V_- \Psi = 0$, whose component form can be reduced to Eqs. (3.15) and (3.16), in which the parameter $\alpha$ takes the values $\alpha_+$ and $\alpha_-$ for $\psi_+^n$ and $\psi_-^n$, respectively. The solutions of these equations for both fields $\psi_+^n$ and $\psi_-^n$ are given by Eqs. (3.13), (3.14) with correspondingly chosen value of $\alpha$. Note that in the rest frame the solutions $\Psi_+$ and $\Psi_-$ are proportional to the Fock states $|0\rangle_+ = |0\rangle$ and $|0\rangle_- = |1\rangle$, i.e., the physical state with $p_i = 0$ in the odd subspace is the first exited state of the Fock space.

Direct calculation shows that the spinor supercharge operator (2.13) satisfies with the vector operator (3.20) the relation

$$[V_\mu, Q_\alpha] = \left(D_\mu^{\nu}\right)_\alpha V_\nu ,$$

(3.21)

where $(D_\mu^{\nu})_\alpha$ is some operator, whose explicit form is not needed for us here. Relation (3.21) means that the supercharge (as well as the even generators of the $(2+1)$D superalgebra) is a symmetry generator: acting on a physical state satisfying Eq. (3.19), it produces another physical state.

Now the complex linear combinations

$$Q_\pm = Q_1 \mp iQ_2$$

(3.22)

are eigenstates of the rotation operator,

$$[J_0, Q_\pm] = \pm \frac{1}{2} Q_\pm .$$

(3.23)

In terms of generators of the HW algebra, their explicit form is

$$Q_- = \frac{i}{\sqrt{m(1 + \nu)}} \left[a^- (mR + p_0) - a^+ p_- \right] ,$$

$$Q_+ = \frac{i}{\sqrt{m(1 + \nu)}} \left[a^+ (mR - p_0) + a^- p_+ \right] .$$

(3.24)

(3.25)

In the rest frame system (where $p^0 = m$ for physical states), these reduce to

$$Q_- (p_i = 0) \approx -2i \sqrt{\frac{m}{1 + \nu}} a^- \Pi_- , \quad Q_+ (p_i = 0) \approx 2i \sqrt{\frac{m}{1 + \nu}} a^+ \Pi_+ .$$

(3.26)
From Eq. (3.26) it is clear that the operator \( Q^- \) transforms the rest frame physical state \( |0\>_- = |1\> \) into the the physical state \( |0\>_+ = |0\> \) and annihilates the latter, while the operator \( Q^+ \) acts in the opposite way. In other words, the action of the mutually conjugate operators \( Q^+ \) and \( Q^- \) corresponds to the action of ordinary nilpotent supercharges in the system with \( N = 1 \) supersymmetry.

In conclusion of this section we note that in \([14]\) instead of vector set of dependent equations, the spinor set of independent equations \([15]\)

\[
\tilde{Q}_\alpha \Psi = 0, \quad \tilde{Q}_\alpha = \left( R(p\gamma)_{\alpha}^\beta + m\epsilon_{\alpha}^\beta \right) \mathcal{L}_\beta
\]

was used \([in [14], the spinor operator is denoted by Q_\alpha] \). The relation between vector and spinor operators is given by

\[
\mathcal{L}_\alpha (\gamma_\mu)^{\alpha\beta} R\tilde{Q}_\beta = -4i \left( V^{(\alpha+)}_\mu \Pi_+ - \left( V^{(\alpha-)}_\mu + 2i\epsilon_{\mu\nu\lambda} p^\nu J^\lambda - p_\mu \right) \Pi_- \right).
\]

This relation shows that both systems are closely related. In the even subspace the two systems are indeed equivalent. This is readily seen using the component forms (3.9), (3.10), (3.11) and (3.15)–(3.16), respectively.

In the odd subspace, the spinor set of equations (3.27) has only trivial solution \( \Psi_- = 0 \). The new, vector set of equations (3.19), (3.20), however, does have a nontrivial solution (3.13), (3.14) with shifted value of spin \( \alpha = \alpha_- = \alpha_+ + \frac{1}{2} \).

## 4 Jackiw-Nair nonrelativistic limit

Let us now consider the special non-relativistic “Jackiw-Nair” (JN) limit \([13]\) of our relativistic supersymmetric system,

\[
c \rightarrow \infty, \quad s \rightarrow \infty, \quad \frac{s}{c^2} = \kappa.
\]

For the purpose we need some details on the Fock representations of the HW algebra \([9]\). In the unitary case \( \nu > -1 \) the normalization coefficients in Eq. (2.3) are

\[
C_n = [n]!^{\nu}_{\nu} = \frac{1}{\prod_{l=1}^{n}[l]_{\nu}}, \quad [l]_{\nu} = l + 1 - (l!)^\nu \nu.
\]

Then, consistently with these relations and also with Eq. (2.1), we have

\[
a^+|n\> = \sqrt{[n+1]_{\nu}}|n+1\>, \quad a^-|n\> = \sqrt{[n]_{\nu}}|n-1\>.
\]

Eq. (4.2) and definition (3.17) yield the relations

\[
a^+|n\> _+ = \sqrt{2(n+2\alpha_+)}|n\> _, \quad a^+|n\> _- = \sqrt{2(n+1)}|n+1\> _+, \quad a^-|n\> _+ = \sqrt{2n}|n-1\> _-, \quad a^-|n\> _- = \sqrt{2(n+2\alpha_+)}|n\> _+.
\]

Let us introduce the ordinary bosonic operators \( b^\pm \),

\[
[b^-, b^+] = 1, \quad N_b |n\> _b = n |n\> _b, \quad N_b = b^+ b^-.
\]
and represent the states of the even and odd subspaces as
\[ |n\rangle_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes |n\rangle_b, \quad |n\rangle_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes |n\rangle_b. \] (4.5)

Then \( R = \tau_3 \otimes 1 \), where, in order to distinguish it from that in the gamma-matrices, the diagonal Pauli matrix is denoted by \( \tau_3 \). By (4.3) and (4.4), in the representation (4.5), the operators \( a^\pm \) are
\[ a^+ = \sqrt{2} \left( \tau_+ \otimes b^+ + \tau_- \otimes \sqrt{N_b + 2\alpha_+} \right), \quad a^- = \sqrt{2} \left( \tau_- \otimes b^- + \tau_+ \otimes \sqrt{N_b + 2\alpha_+} \right), \] (4.6)
where \( \tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2) \). Using \( \{\tau_+, \tau_-\} = 1, [\tau_+, \tau_-] = \tau_3 \), we have here \([a^-, a^+] = 1 + \nu\tau_3 \otimes 1\).

In this representation, the Lorentz generators (2.4) are
\[ J_0 = 1 \otimes (N_b + \alpha_+) + \frac{1}{4}(1 - \tau_3) \otimes 1, \] (4.7)
\[ J_+ = \frac{1}{2}(1 + \tau_3) \otimes b^+ \sqrt{N_b + 2\alpha_+} + \frac{1}{2}(1 - \tau_3) \otimes b^+ \sqrt{N_b + 2\alpha_+ + 1}, \] (4.8)
\[ J_- = \frac{1}{2}(1 + \tau_3) \otimes \sqrt{N_b + 2\alpha_+} b^- + \frac{1}{2}(1 - \tau_3) \otimes \sqrt{N_b + 2\alpha_+ + 1} b^- . \] (4.9)

Omitting the direct product symbol, (4.8) and (4.9) can be written alternatively
\[ J_+ = b^+ \sqrt{N_b + 2\alpha_+} + \frac{1}{2}(1 - \tau_3) \], \[ J_- = \sqrt{N_b + 2\alpha_+ + 1} b^- . \] (4.10)

Restoring the dependence on the velocity of light, \( c \), via the change \( m \to mc \), the linear differential equations (3.15), (3.16) for both fields \( \Psi_+ \) and \( \Psi_- \) are represented in operator form as
\[ \left( (mc + p_0) \sqrt{2J_0 - N_b - p_+ b^-} \right) \Psi = 0, \] (4.11)
\[ \left( p_- \sqrt{2J_0 - N_b + (mc - p_0) b^-} \right) \Psi = 0. \] (4.12)

These two equations are nothing else as the two independent equations \( V_+ \Psi = 0 \) and \( V_- \Psi = 0 \) from the covariant set (3.19).

Now let us consider a JN limit (4.1) with \( s = \alpha_+ = \frac{1}{4}(1 + \nu) \). Then, from (4.6), we find that
\[ \frac{a^+}{\nu^{1/2}} \to \tau_- \otimes 1, \quad \frac{a^-}{\nu^{1/2}} \to \tau_+ \otimes 1, \] (4.13)
i.e., the appropriately rescaled creation-annihilation operators are transformed, in the limit \( \nu \to \infty \), into fermion operators (see also [9]). We have also
\[ \frac{J_\pm}{c} \to \sqrt{2\kappa} b^\pm. \] (4.14)
On the other hand, in the JN limit, the energy and angular momentum diverge. The “renormalized” angular momentum $J_0 - \alpha_+$, becomes, in the JN limit, the operator $N_b + \frac{1}{4}(1 - \tau_3)$. Following [14], we define the ‘velocity’ operators

$$v_\pm = -\sqrt{\frac{2}{\kappa}} b^\pm, \quad [v_-, v_+] = 2\kappa^{-1}.$$  

The Galilei boosts are defined as $K_i = -\frac{1}{c} \epsilon_{ij} J_j$. Then, for the (total) rotation and Galilei boosts, we get, in the JN limit,

$$J = \epsilon_{ij} x_i p_j + \frac{1}{2} \kappa v_+ v_- + \frac{1}{4}(1 - \tau_3), \quad K_i = mx_i - tp_i + \kappa \epsilon_{ij} v_j. \quad (4.15)$$

These operators will span an “exotic” Galilei algebra, see (4.19), (4.20). Eqs. (4.13) and (4.14) show that the JN limit extracts from the parabosonic-like operators $a^\pm$ the ordinary bosonic and fermionic degrees of freedom.

In order to identify the Hamiltonian and the wave equations of the JN limit of our supersymmetric system, we put in Eqs. (4.11), (4.12) $p_0 = -ic^{-1}\partial_t$ and $\Psi = e^{-imc^2t}\Phi$, and apply (4.1). This results in the equations

$$\left(i\partial_t - \frac{1}{2} p_+ v_+\right) \Phi = 0, \quad (p_- - mv_-) \Phi = 0, \quad (4.16)$$

where $\Phi$ is a two-component field on which the spin matrices $\tau$ act. Every such a component is decomposed in Fock space states of the bosonic operators $b^\pm$, $\Phi_\pm = \sum_{n=0}^\infty \phi^\pm_n |n\rangle_b$. The second equation in (4.16) is a constraint, allowing, as in a relativistic case, to present all higher components in terms of the lowest ones,

$$\phi^\pm_n = (-1)^n \frac{1}{\sqrt{n!}} \left(\sqrt{\frac{\kappa}{2}} \frac{p_-}{m}\right)^n \phi^\pm_0.$$  

The substitution of the second equation from (4.16) into the first one shows that every component $\phi^\pm_n$ satisfies the Schrödinger equation of a free non-relativistic particle. Adding the first equation to the second one multiplied from the left by $-\frac{1}{2} v_+$ allows us to identify finally the Hermitian operator

$$H = p_i v_i - \frac{1}{2} v_+ v_-, \quad (4.17)$$

$v_\pm = v_1 \pm iv_2$, as a Hamiltonian of the nonrelativistic system. Note that $H$ is linear in the momentum. The wave equations and the Hamiltonian coincide hence with those [14] corresponding to the spinless exotic particle on the non-commutative plane [23].

Now, let us identify the JN limit of the supercharge and corresponding superalgebra. Restoring the light speed in $Q_\pm$ then defining

$$Q^\pm_\pm = c^{-1/2} Q^\pm_\pm$$

\footnote{There is an arbitrariness in this procedure up to an additive constant [22]; here the constant is chosen in such a way that non-relativistic spin will take zero value for the state corresponding to relativistic state with spin $s = \alpha_+$.}
and taking into account Eq. (4.13), we find that in the J-N limit the supercharges reduce to

\[ Q_- = -2i\sqrt{m}\tau_+, \quad Q_+ = 2i\sqrt{m}\tau_- . \] (4.18)

The system has a superextended exotic Galilei symmetry, whose bosonic part,

\[ [\mathcal{K}_i, p_j] = im\delta_{ij}, \quad [\mathcal{K}_i, \mathcal{K}_j] = -i\kappa\epsilon_{ij}, \] (4.19)

\[ [\mathcal{K}_i, \mathcal{H}] = ip_i, \quad [\mathcal{J}, p_i] = i\epsilon_{ij}p_j, \quad [\mathcal{J}, \mathcal{K}_i] = i\epsilon_{ij}\mathcal{K}_j, \] (4.20)

is augmented by the (anti)commutation relations involving the supercharges,

\[ [\mathcal{J}, Q_\pm] = \pm \frac{1}{2}Q_\pm, \quad \{Q_+, Q_-\} = 4m, \] (4.21)

\[ [\mathcal{K}_i, Q_\pm] = [P_i, Q_\pm] = [\mathcal{H}, Q_\pm] = 0, \quad Q_\pm^2 = 0. \] (4.22)

The Casimir operators of the superextended exotic Galilei symmetry are

\[ C_1 = p_i^2 - 2m\mathcal{H}, \quad C_2 = \mathcal{J} - \epsilon_{ij}\mathcal{K}_i p_j + \kappa\mathcal{H} - \frac{1}{16}\{Q_+, Q_-\}, \] (4.23)

where \( C_2 \) is the JN limit of the superspin \( S = m^{-1}\mathcal{C} \) with \( \mathcal{C} \) given by Eq. (2.20). On states satisfying equations (4.16), these Casimir operators take the values \( C_1 = 0 \) and \( C_2 = \frac{1}{4} \).

The \( N = 1 \) supersymmetric extension (4.19), (4.20), (4.21), (4.22) of the exotic Galilei symmetry was discussed in [24, 25].

5 Noncommutative coordinates

Here we discuss how noncommutative coordinates appear in the theory.

In the relativistic theory, we start with the usual, commuting space-time coordinates \( x_\mu \). The physical subspace of the system is given by the system of equations (2.18). Taking into account the first, Klein-Gordon equation, the two other, Majorana type equations can be unified into a single equation, namely into

\[ \chi_s\Psi \equiv \left( \frac{p\mathcal{J}}{\sqrt{-p^2}} - (\alpha_+\Pi_+ + \alpha_-\Pi_-) \right)\Psi = 0. \] (5.1)

The initial space-time coordinates \( x_\mu \) are not observable with respect to the superspin equation (5.1): since \( [x_\mu, \chi_s] \neq 0 \), acting on a physical state satisfying Eq. (5.1), the operator \( x_\mu \) produces a state which does not belong to the physical subspace. Let us define instead [27]

\[ X_\mu = x_\mu + \frac{1}{p^2}\epsilon_{\mu\nu\lambda}p^\nu J^\lambda. \] (5.2)

These modified coordinates satisfy the relation

\[ [X_\mu, \chi_s] = 0, \] (5.3)
and so, are observable operators. \( X_\mu \) is a vector operator with respect to the (2+1)D Lorentz transformations generated by (2.12). However, unlike the initial coordinates, the coordinates \( X_\mu \) are not commuting,

\[
[X_\mu, X_\nu] \approx -i (\alpha_+ \Pi_+ + \alpha_- \Pi_-) \epsilon_{\mu\nu\lambda} \frac{p^\lambda}{(-p^2)^{3/2}},
\]

where the symbol \( \approx \) means “on the surface defined by equation (5.1)”. Therefore, the physical (observable) coordinates are non-commuting. They are analogous to the Foldy-Wouthuysen coordinates of the Dirac particle [28].

In the non-relativistic Jackiw-Nair limit the space part of the coordinates (5.2) is transformed into [14]

\[
X_i = x_i + \frac{\kappa}{m} \epsilon_{ij} V_j + \frac{\theta}{2} \epsilon_{ij} p_j,
\]

where

\[
V_i = v_i - \frac{1}{m} p_i
\]

and \( \theta = \kappa/m^2 \). The operator in the second equation in (4.16) is a complex linear combination of the operators \( V_i, i = 1, 2 \), that behaves as an annihilation operator (an operator with nontrivial kernel). The coordinates (5.5), like the initial coordinates \( x_i \), form a 2D vector, which is a covariant object with respect to Galilei boosts generated by the \( \mathcal{K}_i \) from (4.15) (\( X_i \) commutes with \( \mathcal{K}_j \) in the same way as \( x_i \) does).

The initial coordinates \( x_i \) do not commute with the operator \(-mV_-\) appearing in the second equation from (4.16), and are subject to a Zitterbewegung-like motion under the evolution generated by the first order Hamiltonian (4.17) [14],

\[
\dot{x}_i = v_i, \quad \dot{v}_i = \frac{m}{\kappa} \epsilon_{ij} V_j.
\]

Unlike \( x_i \), the coordinates \( X_i \) commute with the constraint operator \(-mV_-\), and so are observable operators. They have the usual (Zitterbewegung-free) evolution of the coordinates of a free non-relativistic particle,

\[
\dot{X}_i = \frac{1}{m} p_i.
\]

However, the components of the coordinate (5.5) do not commute,

\[
[X_i, X_j] = i \theta \epsilon_{ij},
\]

and describe therefore a non-commutative plane [29, 14].

6 Concluding remarks

In the non-supersymmetric case, the system of dependent vector equations (3.1) was realized originally on the basis of an irreducible representation of the (2+1)D Lorentz algebra [21]. It can be replaced by an independent covariant spinor set of equations (3.27) constructed in terms of an irreducible Fock space representation of the deformed Heisenberg-Wigner algebra. The corresponding non-supersymmetric anyon system is described by a field with
nontrivial even part $\Psi_+$ alone, living in the even subspace of the Fock representation. In the present, supersymmetric, case, however, no similar substitution of the vector for spinor set of equations seems to exist for both, even and odd, fields $\Psi_+$ and $\Psi_-$. This may well explain a failure of a previous attempt to construct a bosonized (2+1)D supersymmetry [26], where minimal, spinor set of equations has been searched for. Here, like in [21], our basic set of equations (3.19), (3.20) is vector, but it is realized on an irreducible Fock space representation of the deformed HW algebra, which is an irreducible representation of the $osp(1|2)$ superalgebra. By using as a grading operator the reflection operator of the deformed HW algebra, we obtained a (2+1)D supersymmetric system avoiding introduction of fermionic Fock space.

The equations which correspond to finite-dimensional non-unitary representations of HW algebra behave similarly to the anyonic case. There is, however, an essential difference in the realization of supersymmetry. Finite-dimensional representations provide us with parafermion-like degrees of freedom described by the $a^\pm$, see Ref. [9]. In this case we have ordinary, not bosonized (2+1)D supersymmetry, realized on a system of ordinary integer or half-integer spin fields. For $\nu = (2k + 1)$, $k = 2, 3, \ldots$, the eigenvalues of the operator $J_0$ can take both signs, yielding two independent solutions of the system of linear differential equations (3.19), namely with either positive or negative energy. For the positive energy solutions the components $\psi_+^n$, $n = 0, 1, \ldots, k$, of a spin $s_+ = \frac{1}{2}k$ field and the components $\psi_-^{n'}$, $n' = 0, 1, k - 1$, of a spin $s_- = \frac{1}{2}(k - 1)$ field can be expressed in terms of the components $\psi_0^+$ and $\psi_0^-$, via the equation $V_\nu\Psi = 0$, by relations similar to (3.13). For negative energy solutions these are represented via $V_\nu\Psi = 0$ in terms of $\psi_k^+$ and $\psi_{k-1}^-$. For example, $\nu = -5$ yields a spin 1/2 fermion plus an $s = 1$ (topologically massive gauge) vector field. The case $\nu = -3$ (i.e. $k = 1$) is a special one. Now $s_- = \alpha_- = 0$, and the corresponding $so(1, 2)$ Lorentz generators $J_\mu$ in the odd subspace are trivial. Hence, the operators $V_\mu^{(\alpha_-)}$ as well as the spin fixing linear differential operator $(PJ_0 - \alpha_--m)$ disappear. For a scalar field $\Psi_-$ our construction breaks down. We still get a supermultiplet of an $s = 0$ scalar plus a spin 1/2 fermion field; for the spin-zero superpartner, however, the dynamics is quadratic Klein-Gordon equation, rather than a first-order one.

Note that the even generators $P_\mu$ and $J_\mu$ of the bosonized (2+1)D supersymmetry of the anyon system are local in internal space, associated with the operators $a^\pm$. The odd supercharge $Q_\alpha$ is, however, nonlocal due to the presence of the operator $R$. This is similar to what we have for purely bosonic quantum mechanical systems with hidden supersymmetry [4], where the Hamiltonian is local but odd supercharges are nonlocal operators.

The application of the Jackiw-Nair non-relativistic limit to our theory yields wave equations describing a spinning particle on the noncommutative plane, and that the (2+1)D Poincaré supersymmetry reduces, in this limit, to Galilean exotic $N = 1$ supersymmetry. The latter is characterized by the supercharges being square roots of one of the central charges, namely of the mass, $m$. One could expect that if the relativistic supersymmetric anyon proposed here is extended by adding ordinary fermion degrees of freedom (for instance, within the framework of a superfield approach [6]), the resulting system would possess $N = 2$ supersymmetry, half of which would be bosonized. Then subsequent application of the Jackiw-Nair limit should produce a system possessing a Galilean exotic $N = 2$ supersymmetry with new supercharges which would be the square roots of a Hamiltonian [24].
In conclusion, we have shown that bosonized supersymmetry can be realized in a (2+1)D system of anyons, constructed on the basis of infinite-dimensional unitary irreducible representations of the Heisenberg-Wigner algebra. Any such a representation carries two irreducible unitary representations of the $so(1,2)$ Lorentz group, whose spins are shifted relatively by one-half. These infinite-dimensional representations of the (2+1)D Lorentz group are analogs of the infinite-dimensional unitary representations of the (3+1)D Lorentz group, discovered by Majorana in his celebrated 1932 paper [18, 19]. It might be possible to generalize our construction of the bosonized supersymmetry here to (3+1)-dimensional field theory. This problem will be studied elsewhere.

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