Sum rules for leading and subleading form factors in Heavy Quark Effective Theory using the non-forward amplitude

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Abstract

Within the OPE, we formulate new sum rules in Heavy Quark Effective Theory in the heavy quark limit and at order $1/m_Q$, using the non-forward amplitude. In the heavy quark limit, these sum rules imply that the elastic Isgur-Wise function $\xi(w)$ is an alternate series in powers of $(w - 1)$. Moreover, one gets that the $n$-th derivative of $\xi(w)$ at $w = 1$ can be bounded by the $(n - 1)$-th one, and the absolute lower bound for the $n$-th derivative $(-1)^n \xi^{(n)}(1) \geq \frac{(2n+1)!!}{2^n}$. Moreover, for the curvature we find $\xi''(1) \geq \frac{1}{2} \left[ 4\rho^2 + 3(\rho^2)^2 \right]$ where $\rho^2 = -\xi'(1)$. These results are consistent with the dispersive bounds, and they strongly reduce the allowed region of the latter for $\xi(w)$. The method is extended to the subleading quantities in $1/m_Q$.

Concerning the perturbations of the Current, we derive new simple relations between the functions $\xi_3(w)$ and $\Delta \xi(w)$ and the sums $\sum_n \Delta E_j^{(n)} \tau_j^{(n)}(1) \chi_j^{(n)}(w)$ ($j = \frac{1}{2}, \frac{3}{2}$), that involve leading quantities, Isgur-Wise functions $\tau_j^{(n)}(w)$ and level spacings $\Delta E_j^{(n)}$. Our results follow because the non-forward amplitude depends on three variables $(w_i, w_f, w_{if}) = (v_i \cdot v', v_f \cdot v', v_i \cdot v_f)$, and we consider the zero recoil frontier $(w, 1, w)$ where only a finite number of $j^P$ states contribute $\left( \frac{1}{2}^+, \frac{3}{2}^+ \right)$. We also obtain new sum rules involving the elastic subleading form factors $\chi_i(w)$ ($i = 1, 2, 3$) at order $1/m_Q$ that originate from the $L_{\text{kin}}$ and $L_{\text{mag}}$ perturbations of the Lagrangian. To the sum rules contribute only the same intermediate states $(j^P, J^P) = \left( \frac{1}{2}^-, 1^- \right), \left( \frac{3}{2}^-, 1^- \right)$ that enter in the $1/m_Q^2$ corrections of the axial form factor $h_{A_i}(w)$ at zero recoil. This allows to obtain a lower bound on $-\delta^{(A)}_{1/m^2}$ in terms of the $\chi_i(w)$ and the shape of the elastic IW function $\xi(w)$. An important theoretical implication is that $\chi_1'(1), \chi_2(1)$ and $\chi_3'(1)$ ($\chi_1(1) = \chi_3(1) = 0$ from Luke theorem) must vanish when the slope and the curvature attain their lowest values $\rho^2 \to \frac{3}{2}, \sigma^2 \to \frac{15}{16}$. These constraints should be taken into account in the exclusive determination of $|V_{cb}|$.

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We will expose the main results that we have obtained in Heavy Quark Effect Theory using the non-forward amplitude and the Operator Product Expansion. We will first examine results in the heavy quark limit on the shape of the Isgur-Wise function. The method is then generalized to the study of the $1/m_Q$ perturbations, that are of two types, namely perturbations of the Current, and perturbations of the Lagrangian. This is a somewhat longer version of the talk than the one that will appear in the Proceedings.

1 Heavy quark limit

In the leading order of the heavy quark expansion of QCD, Bjorken sum rule (SR) \([1]\) relates the slope of the elastic Isgur-Wise (IW) function $\xi(w)$, to the IW functions of the transitions between the ground state and the $j^P = \frac{1}{2}^+, \frac{3}{2}^+$ excited states, $\tau_{1/2}^{(n)}(w)$, $\tau_{3/2}^{(n)}(w)$, at zero recoil $w = 1$ ($n$ is a radial quantum number). This SR leads to the lower bound $-\xi'(1) = \rho^2 \geq \frac{1}{4}$. Recently, a new SR was formulated by Uraltsev in the heavy quark limit \([2]\) involving also $\tau_{1/2}^{(n)}(1)$, $\tau_{3/2}^{(n)}(1)$, that implies, combined with Bjorken SR, the much stronger lower bound $\rho^2 \geq \frac{3}{4}$, a result that came as a big surprise. In ref. \([3]\), in order to make a systematic study in the heavy quark limit of QCD, we have developed a manifestly covariant formalism within the Operator Product Expansion (OPE). We did recover Uraltsev SR plus a new class of SR. Making a natural physical assumption, this new class of SR implies the bound $\sigma^2 \geq \frac{5}{4} \rho^2$ where $\sigma^2$ is the curvature of the IW function. Using this formalism including the whole tower of excited states $j^P$, we have recovered rigorously the bound $\sigma^2 \geq \frac{5}{4} \rho^2$ plus generalizations that extend it to all the derivatives of the IW function $\xi(w)$ at zero recoil, that is shown to be an alternate series in powers of $(w - 1)$.

Using the OPE and the trace formalism in the heavy quark limit, different initial and final four-velocities $v_i$ and $v_f$, and heavy quark currents, where $\Gamma_1$ and $\Gamma_2$ are arbitrary Dirac matrices $J_1 = \bar{h}^{(c)}_{v_i} \Gamma_1 h^{(b)}_{v_i}$, $J_2 = \bar{h}^{(b)}_{v_f} \Gamma_2 h^{(c)}_{v_f}$, the following sum rule can be written \([5]\):

$$\left\{ \sum_{D=P,V} \sum_n Tr \left[ B_f(v_f) \bar{B}_i(v_i) D^{(n)}(v') \right] Tr \left[ D^{(n)}(v') \Gamma_1 B_i(v_i) \right] \xi^{(n)}(w_i) \xi^{(n)}(w_f) \right\} = -2\xi(w_{if}) Tr \left[ B_f(v_f) \bar{B}_i(v_i) \Gamma_2 P_+ \Gamma_1 B_i(v_i) \right]. \tag{1}$$
In this formula \(v'\) is the intermediate meson four-velocity, \(P'_+ = \frac{1}{2}(1 + \not{v}')\) comes from the residue of the positive energy part of the \(c\)-quark propagator, \(\xi(w_{if})\) is the elastic Isgur-Wise function that appears because one assumes \(v_i \neq v_f\). \(\mathcal{B}_i\) and \(\mathcal{B}_f\) are the \(4 \times 4\) matrices of the ground state \(B\) or \(B^*\) mesons and \(\mathcal{D}^{(n)}\) those of all possible ground state or excited state \(D\) mesons coupled to \(\mathcal{B}_i\) and \(\mathcal{B}_f\) through the currents. In (1) we have made explicit the \(j = \frac{1}{2}^- \) \(D\) and \(D^*\) mesons and their radial excitations of quantum number \(n\). The explicit contribution of the other excited states is written below.

The variables \(w_i, w_f\) and \(w_{if}\) are defined as \(w_i = v_i \cdot v', w_f = v_f \cdot v', w_{if} = v_i \cdot v_f\). The domain of \((w_i, w_f, w_{if})\) is [3]

\[
\begin{align*}
& w_i, w_f \geq 1 \\
& w_i w_f - \sqrt{(w_i^2 - 1)(w_f^2 - 1)} \leq w_{if} \\
& \leq w_i w_f + \sqrt{(w_i^2 - 1)(w_f^2 - 1)} .
\end{align*}
\]

(2)

The SR (1) writes \(L(w_i, w_f, w_{if}) = R(w_i, w_f, w_{if})\), where \(L(w_i, w_f, w_{if})\) is the sum over the intermediate charmed states and \(R(w_i, w_f, w_{if})\) is the OPE side. Within the domain (2) one can derive relatively to any of the variables \(w_i, w_f\) and \(w_{if}\) and obtain different SR taking different limits to the frontiers of the domain.

As in ref. [3] [4], we choose as initial and final states the \(B\) meson \(\mathcal{B}_i(v_i) = P_{i+}(-\gamma_5), \mathcal{B}_f(v_f) = P_{f+}(-\gamma_5)\) and vector or axial currents projected along the \(v_i\) and \(v_f\) four-velocities

\[
J_1 = \bar{h}^{(c)}_{\psi'} \not{\psi}_i h^{(b)}_{\psi_i} , \quad J_2 = \bar{h}^{(b)}_{\psi_f} \not{\psi}_f h^{(c)}_{\psi'}
\]

(3)

we obtain SR (1) with the sum of all excited states \(j^P\) in a compact form:

\[
(w_i + 1)(w_f + 1) \sum_{\ell \geq 0} \frac{\ell + 1}{2\ell + 1} S_\ell(w_i, w_f, w_{if}) \sum_n \tau^{(\ell)(n)}(w_i) \tau^{(\ell)(n)}_{\ell+1/2}(w_f) \\
+ \sum_{\ell \geq 1} S_\ell(w_i, w_f, w_{if}) \sum_n \tau^{(\ell)(n)}_{\ell-1/2}(w_i) \tau^{(\ell)(n)}_{\ell-1/2}(w_f) = (1 + w_i + w_f + w_{if})\xi(w_{if})
\]

(4)

We get, choosing instead the axial currents,

\[
J_1 = \bar{h}^{(c)}_{\psi'} \not{\psi}_i \gamma_5 h^{(b)}_{\psi_i} , \quad J_2 = \bar{h}^{(b)}_{\psi_f} \not{\psi}_f \gamma_5 h^{(c)}_{\psi'} ,
\]

(5)

\[
\sum_{\ell \geq 0} S_{\ell+1}(w_i, w_f, w_{if}) \sum_n \tau^{(\ell)(n)}_{\ell+1/2}(w_i) \tau^{(\ell)(n)}_{\ell+1/2}(w_f) + (w_i - 1)(w_f - 1)
\]

3
\[
\sum_{\ell \geq 1} \frac{\ell}{2\ell - 1} S_{\ell-1}(w_i, w_f, w_{if}) \sum_n \tau^{(\ell)(n)}_{\ell-1/2}(w_i)\tau^{(\ell)(n)}_{\ell-1/2}(w_f) = -(1-w_i-w_f+w_{if})\xi(w_{if}) \tag{6}
\]

Following the formulation of heavy-light states for arbitrary \(j^P\) given by Falk [5], we have defined in ref. [3] the IW functions \(\tau^{(\ell)(n)}_{\ell+1/2}(w)\) and \(\tau^{(\ell)(n)}_{\ell-1/2}(w)\), \(\ell\) and \(j = \ell \pm \frac{1}{2}\) being the orbital and total angular momentum of the light cloud.

In (3) and (5) \(S_n\) is given by

\[
S_n = v_{i\mu_1} \cdots v_{i\mu_n} v_{f\mu_1} \cdots v_{f\mu_n} \sum_{\lambda} \varepsilon^{(\lambda)\mu_1 \cdots \mu_n} \varepsilon^{(\lambda)\nu_1 \cdots \nu_n} \xi(w_{if}) \tag{7}
\]

One can show [3]:

\[
S_n = \sum_{0 \leq k \leq \frac{n}{2}} C_{n,k}(w_i^2 - 1)^k(w_f^2 - 1)^k(w_iw_f - w_{if})^{n-2k} \tag{8}
\]

with \(C_{n,k} = (-1)^k \frac{(n)!}{(2n-2k)! \frac{(2n)!}{(n-k)!(n-2k)!}}\).

From the sum of (4) and (6) one obtains, differentiating relatively to \(w_{if}\) [4] \((\ell \geq 0)\):

\[
\xi^{(\ell)}(1) = \frac{1}{4} (-1)^\ell \frac{1}{2\ell + 1} \left\{ \frac{\ell + 1}{2\ell + 1} \sum_{n} \left[ \tau^{(\ell)(n)}_{\ell+1/2}(1) \right]^2 + \sum_{n} \left[ \tau^{(\ell-1)(n)}_{\ell-1/2}(1) \right]^2 + \sum_{n} \left[ \tau^{(\ell)(n)}_{\ell-1/2}(1) \right]^2 \right\} . \tag{9}
\]

This relation shows that \(\xi(w)\) is an alternate series in powers of \((w - 1)\). Equation (9) reduces to Bjorken SR [1] for \(\ell = 1\). Differentiating (6) relatively to \(w_{if}\) and making \(w_i = w_f = w_{if} = 1\) one obtains:

\[
\xi^{(\ell)}(1) = \ell! \left( -1 \right)^\ell \sum_{n} \left[ \tau^{(\ell)(n)}_{\ell+1/2}(1) \right]^2 \quad (\ell \geq 0) . \tag{10}
\]

Combining (9) and (10) one obtains a SR for all \(\ell\) that reduces to Uraltsev SR [2] for \(\ell = 1\). From (9) and (10) one obtains:

\[
(-1)^\ell \xi^{(\ell)}(1) = \frac{1}{4} \frac{2\ell + 1}{\ell} \ell! \left\{ \sum_{n} \left[ \tau^{(\ell-1)(n)}_{\ell-1/2}(1) \right]^2 + \sum_{n} \left[ \tau^{(\ell)(n)}_{\ell-1/2}(1) \right]^2 \right\} . \tag{11}
\]

implying

\[
(-1)^\ell \xi^{(\ell)}(1) \geq \frac{2\ell + 1}{4} \left[ \left( -1 \right)^{\ell-1} \xi^{(\ell-1)}(1) \right] \\
\geq \frac{(2\ell + 1)!!}{2^{2\ell}} \tag{12}
\]

that gives, in particular, for the lower cases,

\[
-\xi'(1) = \rho^2 \geq \frac{3}{4}, \quad \xi''(1) \geq \frac{15}{16} \tag{13}
\]
Considering systematically the derivatives of the SR (4) and (6) relatively to \( w, w_i, w_f, w_{if} \) with the boundary conditions \( w_{if} = w_i = w_f = 1 \), one obtains a new SR:

\[
\frac{4}{3} \rho^2 + (\rho^2)^2 - \frac{5}{3} \sigma^2 + \sum_{n \neq 0} |\xi^{(n)}(1)|^2 = 0
\]  

(14)

that implies:

\[
\sigma^2 \geq \frac{1}{5} \left[ 4 \rho^2 + 3(\rho^2)^2 \right] .
\]  

(15)

There is a simple intuitive argument to understand the term \( \frac{2}{5}(\rho^2)^2 \) in the best bound (15), namely the non-relativistic quark model, i.e. a non-relativistic light quark \( q \) interacting with a heavy quark \( Q \) through a potential. The form factor has the simple form:

\[
F(k^2) = \int d\mathbf{r} \varphi_0^+(r) \exp \left( i \frac{m_q}{m_q + m_Q} k \cdot \mathbf{r} \right) \varphi_0(r)
\]  

(16)

where \( \varphi_0(r) \) is the ground state radial wave function. Identifying the non-relativistic IW function \( \xi_{NR}(w) \) with the form factor \( F(k^2) \) (16), one can prove that,

\[
\sigma_{NR}^2 \geq \frac{3}{5} \left[ \rho_{NR}^2 \right]^2 .
\]  

(17)

Thus, the non-relativistic limit is a good guide-line to study the shape of the IW function \( \xi(w) \). We have recently generalized the bound (17) to all the derivatives of \( \xi_{NR}(w) \). The method uses the positivity of matrices of moments of the ground state wave function \([6]\). We have shown that the method can be generalized to the function \( \xi(w) \) of QCD.

An interesting phenomenological remark is that the simple parametrization for the IW function \([7]\)

\[
\xi(w) = \left( \frac{2}{w + 1} \right)^{2\rho^2}
\]  

(18)

satisfies the inequalities (12), (15) if \( \rho^2 \geq \frac{3}{4} \).

The result (12), that shows that all derivatives at zero recoil are large, should have important phenomenological implications for the empirical fit needed for the extraction of \( |V_{cb}| \) in \( B \to D^* \ell \nu \). The usual fits to extract \( |V_{cb}| \) using a linear or linear plus quadratic dependence of \( \xi(w) \) are not accurate enough.

A considerable effort has been developed to formulate dispersive constraints on the shape of the form factors in \( \bar{B} \to D^* \ell \nu \) [8]-[9], at finite mass.
Our approach, based on Bjorken-like SR, holds in the physical region of the semileptonic decays $\bar{B} \to D^{(*)}\ell\nu$ and in the heavy quark limit. The two approaches are quite different in spirit and in their results.

Let us consider the main results of ref. [9] summarized by the one-parameter formula

$$\xi(w) \simeq 1 - 8\rho^2 z + (51\rho^2 - 10)z^2 - (252\rho^2 - 84)z^3 \quad (19)$$

with the variable $z(w)$ defined by

$$z = \frac{\sqrt{w + 1} - \sqrt{2}}{\sqrt{w + 1} + \sqrt{2}} \quad (20)$$

and the allowed range for $\rho^2$ being $-0.17 < \rho^2 < 1.51$. This domain is considerably tightened by the lower bound on $\rho^2$: $-0.17 < \rho^2 < 1.51$. This shows that our type of bounds are complementary to the bounds obtained from dispersive methods.

### 2 1/$m_Q$ perturbations of the Current

In this Section, we follow the main lines of our paper [10]. Our starting point is the $T$-product

$$T_{fi}(q) = i \int d^4 x \ e^{-iq\cdot x} < B(p_f)|T[J_f(0)J_i(x)]|B(p_i)> \quad (21)$$

where $J_f(x)$, $J_i(y)$ are the currents (the convenient notation for the subindices $i$, $f$ will appear clear below):

$$J_f(x) = \bar{b}(x)\Gamma_f c(x) \quad J_i(y) = \bar{c}(y)\Gamma_i b(y) \quad (22)$$

and $p_i$ is in general different from $p_f$.

Inserting in this expression hadronic intermediate states, $x^0 < 0$ receives contributions from the direct channel with hadrons with a single heavy quark $c$, while $x^0 > 0$ receives contributions from hadrons with $b\bar{b}$ quarks, the $Z$ diagrams:

$$T_{fi}(q) = \sum_{X_c} (2\pi)^3 \delta^3(q + p_i - p_{X_c}) < B_f|J_f(0)|X_c> < X_c|J_i(0)|B_i> \frac{q^0 + E_i - E_{X_c} + i\varepsilon}{q^0 + E_f - E_{X_{cbb}} - i\varepsilon}$$

$$- \sum_{X_{c_{bb}}} (2\pi)^3 \delta^3(q - p_f + p_{X_{c_{bb}}}) < B_f|J_i(0)|X_{c_{bb}} > < X_{c_{bb}}|J_f(0)|B_i> \frac{q^0 - E_f + E_{X_{c_{bb}}} - i\varepsilon}{q^0 + E_f - E_{X_{c_{bb}}} + i\varepsilon}. \quad (23)$$

We consider the limit $m_c \gg m_b \gg \Lambda_{QCD}$. The difference between the two energy denominators is large $q^0 - E_f + E_{X_{c_{bb}}} - (q^0 + E_i - E_{X_c}) \sim 2m_c$ and therefore, we
can in this limit neglect the second term, and we consider the imaginary part of the
direct diagram, the first term in (23), the piece proportional to \( \delta (q^0 + E_i - E_{X_c}) \).
Our conditions are, in short, as follows:

\[
\Lambda_{QCD} \ll m_b \sim m_c \sim q^0 \ll q^0 \sim m_c.
\]

(24)

To summarize, we are considering the heavy quark limit for the \( c \) quark, but we allow for a large finite mass for the \( b \) quark.

In the conditions (24), or choosing the suitable integration contour, we can write, integrating over \( q^0 \)

\[
T_{fi}^{abs}(q) \approx \sum_{X_c} (2\pi)^3 \delta^3 (q + p_i - q_{X_c}) < B_f|J_f(0)|X_c > < X_c|J_i(0)|B_i >. 
\]

(25)

Finally, integrating over \( q_{X_c} \) and defining \( v' = \frac{q + p_i}{m_c} \) one gets

\[
T_{fi}^{abs} \approx \sum_{D_n} < B_f(v_f)|J_f(0)|D_n(v') > < D_n(v')|J_i(0)|B_i(v_i) >
\]

(26)

where we have denoted by \( D_n(v') \) the charmed intermediate states.

The \( T \)-product matrix element \( T_{fi}(q) \) (21) is given, alternatively, in terms of quarks and gluons, by the expression

\[
T_{fi}(q) = -\int d^4x \ e^{-iq\cdot x} < B(p_f)|\overline{b}(0)\Gamma_f S_c(0, x)\Gamma_i b(x)|B(p_i) >
\]

(27)

where \( S_c(0, x) \) is the \( c \) quark propagator in the background of the soft gluon field
[11].

Since we are considering the absorptive part in the \( c \) heavy quark limit of the
direct graph in (23), this quantity can be then identified with (27) where \( S_c(x, 0) \) is replaced by the following expression [12]

\[
S_c(0, x) \rightarrow e^{im_c v' \cdot x} \Phi_{v'}[0, x] D_{v'}(x)
\]

(28)

where \( D_{v'}(x) \) is the cut free propagator of a heavy quark

\[
D_{v'}(x) = P'_{+} \int \frac{d^4k}{(2\pi)^4} \delta(k \cdot v')e^{ik\cdot x} = P'_{+} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \delta^4(x - v't)
\]

(29)

with the positive energy projector defined by \( P'_{+} = \frac{1}{2}(1 + \gamma') \).
The eikonal phase $\Phi_{v'}[0,x]$ in (28) corresponds to the propagation of the $c$ quark from the point $x = v't$ to the point 0, that is given by

$$\Phi_{v'}[0, v't] = P \exp \left( -i \int_0^t ds \, v' \cdot A(v's) \right).$$

(30)

This quantity takes care of the dynamics of the soft gluons in HQET along the classical path $x = v't$.

We obtain

$$T_{\text{abs}}^{fi}(q) = \int d^4x \ e^{-i(q-mc)v' \cdot x} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \delta^4(x-v't)$$

$$< B(p_f) | \overline{b}(0) \Gamma_f P' \Phi_{v'}[0,x] \Gamma_i b(x) | B(p_i) > + O(1/m_c).$$

(31)

Integrating over $x$ in (31) and making explicit (30),

$$T_{\text{abs}}^{fi}(q) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-i(q-mc)v' \cdot v't}$$

$$< B(p_f) | \overline{b}(0) \Gamma_f P' P \exp \left( -i \int_0^t ds \, v' \cdot A(v's) \right) \Gamma_i b(v't) | B(p_i) > + O(1/m_c).$$

(32)

Performing first the integration over $q^0$ one obtains simply $\delta(v'^0t)$, and making the trivial integration over $t$ one obtains finally the OPE matrix element :

$$T_{\text{abs}}^{fi} \approx < B(p_f) | \overline{b}(0) \Gamma_f \frac{1 + \gamma'^0}{2v'^0} \Gamma_i b(0) | B(p_i) > + O(1/m_c).$$

(33)

Therefore, we end up with the sum rule

$$\sum_{D_n} < B_f(v_f) | J_f(0) | D_n(v') > < D_n(v') | J_i(0) | B_i(v_i) >$$

$$= < B(v_f) | \overline{b}(0) \Gamma_f \frac{1 + \gamma'^0}{2v'^0} \Gamma_i b(0) | B(v_i) > + O(1/m_c)$$

(34)

that is valid for all powers of an expansion in $1/m_b$, but only to leading order in $1/m_c$.

On the other hand, making use of the HQET equations of motion, the field $b(x)$ in (34) can be decomposed into upper and lower components as follows [13]

$$b(x) = e^{-im_bv \cdot x} \left( 1 + \frac{1}{2m_b + iv \cdot D} i \overline{\gamma} \right) h_v(x)$$

(35)

where the second term corresponds to the lower components and can be expanded in a series in powers of $D_\mu/m_b$, and $v$ is an arbitrary four-velocity.
Keeping the first order in $1/m_b$, the sum rule reads

\[
\sum_{D_n} < B_f(v_f)|J_f(0)|D_n(v') > < D_n(v')|J_i(0)|B_i(v_i) >
= < B(p_f)|\overline{\eta}_{v_f}(0)\Gamma_f \frac{1 + \gamma^\mu}{2v'^0} \Gamma_i h_{v_i}(0)|B(p_i) > \\
+ \frac{1}{2m_b} < B(p_f)|\overline{\eta}_{v_f}(0)\left[(-i\not{D})\Gamma_f \frac{1 + \gamma^\mu}{2v'^0} \Gamma_i + \Gamma_f \frac{1 + \gamma^\mu}{2v'^0} \Gamma_i (i\not{D})\right] h_{v_i}(0)|B(p_i) > \\
+ O(1/m_c) + O(1/m_b^2). \quad (36)
\]

Therefore, in the OPE side we have, besides the leading dimension 3 operator

\[
O^{(3)} = \overline{\eta}_{v_f} \Gamma_f P^i_+ \Gamma_i h_{v_i} 
\]

the dimension 4 operator

\[
O^{(4)} = \overline{\eta}_{v_f} \left[(-i\not{D})\Gamma_f P^i_+ \Gamma_i + \Gamma_f \frac{1 + \gamma^\mu}{2v'^0} \Gamma_i (i\not{D})\right] h_{v_i}. \quad (38)
\]

In the SR we have to compute the l.h.s. including terms of order $1/2m_b$. These terms have been parametrized by Falk and Neubert [14] for the $1^-_2$ doublet and by Leibovich et al. [15] for the transitions between the ground state $1^-_2$ and the $1^+_2$, $3^+_2$ excited states.

A remark is in order here, that was already made in ref. [17]. Had we taken higher moments of the form $\int dq^0(q^0)^n T^{ab}_{fi}(q^0) \ (n > 0)$, instead of the lowest one $n = 0$, the integration over $q^0$ that leads to the simple sum rules (34) or (36) would involve higher dimension operators, giving a whole tower of sum rules [18, 12], even in the leading heavy quark limit. Our point of view in this paper is different. We consider the lowest moment $n = 0$, while we expand in powers of $1/m_b$, keeping the first order in this parameter.

Concerning the OPE side in (36), the dimension 4 operator $O^{(4)}$ (38) is nothing else but the $1/m_b$ perturbation of the heavy current $O^{(3)} = \overline{\eta}_{v_f} \Gamma_f P^i_+ \Gamma_i h_{v_i}$ since this operator, containing the Dirac matrix $\Gamma_f P^i_+ \Gamma_i$ between heavy quark fields, can be considered as a heavy quark current. Indeed, following Falk and Neubert, the $1/m_b$ perturbation of any heavy quark current $\overline{\eta}_{v_f} \Gamma h_{v_i}$ is given by

\[
\overline{\eta}_{v_f} \left(-\frac{i\not{D}}{2m_b}\right) \Gamma h_{v_i} + \overline{\eta}_{v_f} \Gamma \left(-\frac{i\not{D}}{2m_b}\right) h_{v_i}. \quad (39)
\]
However, this perturbation of the current does not exhaust all perturbations in $1/m_b$. We need also to compute the perturbation of the initial and final wave functions $|B_i(v_i)>$, $|B_f(v_f)>$ due to the kinetic and magnetic perturbations of the Lagrangian. This can be done easily following also the prescriptions of Falk and Neubert to compute these corrections in $1/m_b$ for the leading matrix element $<B_f(v_f)|\tau_{v_f}\Gamma_f P'_+\Gamma_i h_{va}|B_i(v_i)>$.

The final result is the following. The subleading quantities, functions of $w$, $\bar{\Lambda}\xi(w)$ and $\xi_3(w)$ [14] can be expressed in terms of leading quantities, namely the $\frac{1}{2}^- \to \frac{1}{2}^+$, $\frac{3}{2}^+$ IW functions $\tau_j^{(n)}(w)$ and the corresponding level spacings $\Delta E_j^{(n)}$ ($j = \frac{1}{2}, \frac{3}{2}$) [10]

$$\bar{\Lambda}\xi(w) = 2(w+1) \sum_n \Delta E_{3/2}^{(n)} \tau_{3/2}^{(n)}(1) \tau_{3/2}^{(n)}(w) + 2 \sum_n \Delta E_{1/2}^{(n)} \tau_{1/2}^{(n)}(1) \tau_{1/2}^{(n)}(w)$$

$$\xi_3(w) = (w+1) \sum_n \Delta E_{3/2}^{(n)} \tau_{3/2}^{(n)}(1) \tau_{3/2}^{(n)}(w) - 2 \sum_n \Delta E_{1/2}^{(n)} \tau_{1/2}^{(n)}(1) \tau_{1/2}^{(n)}(w).$$

These quantities reduce to known SR for $w=1$, respectively Voloshin SR [16] and a SR for $\xi_3(1)$ [17, 2], and generalizes them for all $w$.

The comparison of (40), (41) with the results of the BT quark model [7] is very encouraging. Within this scheme $\xi(w)$ is given by (18) with $\rho^2 = 1.02$, while one gets, for the $n = 0$ states

$$\tau_j^{(0)}(w) = \tau_j^{(0)}(1) \left( \frac{2}{w+1} \right)^{2\sigma_{2j}}$$

with $\tau_{3/2}^{(0)}(1) = 0.54$, $\sigma_{3/2}^2 = 1.50$, $\tau_{1/2}^{(0)}(1) = 0.22$ and $\sigma_{1/2}^2 = 0.83$. Assuming the reasonable saturation of the SR with the lowest $n = 0$ states [7], one gets, from the first relation (40), a sensibly constant value for $\bar{\Lambda} = 0.513 \pm 0.015$.

### 3 $1/m_Q$ perturbations of the Lagrangian

We follow closely our recent work [19]. Instead of using the OPE, we will simply use the definition of the subleading elastic $\frac{1}{2}^- \to \frac{1}{2}^-$ functions $\chi_i(w)$ ($i = 1, 2, 3$) [14]

$$<D(v')|i \int dx T[J^{cb}(0), \mathcal{L}_{\psi}^{(b)}(x)]B(v)> = \frac{1}{2m_b} \left\{ -2\chi_1(w)Tr[\mathcal{D}(v')\Gamma B(v)] + \frac{1}{2}Tr[A_{\alpha\beta}(v, v')\mathcal{D}(v')\Gamma P_+i\sigma^{\alpha\beta}B(v)] \right\}$$

$$+ \frac{1}{2}Tr[A_{\alpha\beta}(v, v')\mathcal{D}(v')\Gamma P_+i\sigma^{\alpha\beta}B(v)]$$

$$+ \frac{1}{2}Tr[A_{\alpha\beta}(v, v')\mathcal{D}(v')\Gamma P_+i\sigma^{\alpha\beta}B(v)]$$
\[< D(v')|i \int dx T[J^b(0), \mathcal{L}^{(c)}_v(x)]|B(v)> = \frac{1}{2m_c} \left\{ -2\chi_1(w) Tr \left[ \overline{\mathcal{D}}(v') \Gamma B(v) \right] \right\} \]

\[-\frac{1}{2} Tr \left[ \overline{A}_{\alpha\beta}(v', v) \overline{\mathcal{D}}(v') i\sigma^{\alpha\beta} P_+ \Gamma B(v) \right] \}

(44)

with

\[A_{\alpha\beta}(v, v') = -2\chi_2(w) (v'_\alpha \gamma_\beta - v'_\beta \gamma_\alpha) + 4\chi_3(w) i\sigma_{\alpha\beta} \]

\[\overline{A}_{\alpha\beta}(v', v) = -2\chi_2(w) (v_\alpha \gamma_\beta - v_\beta \gamma_\alpha) - 4\chi_3(w) i\sigma_{\alpha\beta} \]

(45)

where \(A = \gamma^0 A + \gamma^0\) denotes the Dirac conjugate matrix, the current \(J^b(0)\) denotes

\[J^b = \overline{\mathcal{L}}^{(c)}_v \Gamma \mathcal{L}^{(b)}_v \]

(46)

where \(\Gamma\) is any Dirac matrix, and \(\mathcal{L}^{(Q)}_v(x)\) is given by

\[\mathcal{L}^{(Q)}_v = \frac{1}{2m_Q} \left[ O^{(Q)}_{\text{kin},v} + O^{(Q)}_{\text{mag},v} \right] \]

(47)

with

\[O^{(Q)}_{\text{kin},v} = \overline{\mathcal{L}}^{(Q)}_v (iD)^2 \mathcal{L}^{(Q)}_v \]

\[O^{(Q)}_{\text{mag},v} = \frac{g_s}{2} \overline{\mathcal{L}}^{(Q)}_v \sigma^{\alpha\beta} G^{\alpha\beta} \mathcal{L}^{(Q)}_v \]

(48)

In relations (43)-(45), the \(\chi_i(w)\) \((i = 1, 2, 3)\) have dimensions of mass, and correspond to the definition given by Luke [20].

We will now insert intermediate states in the \(T\)-products (43). We can separately consider \(\mathcal{L}^{(b)}_{\text{kin},v}\) or \(\mathcal{L}^{(b)}_{\text{mag},v}\). The possible \(Z\)-diagrams involving heavy quarks contributing to the \(T\)-products are suppressed by the heavy quark mass since they are \(b\overline{c}c\) intermediate states.

Conveniently choosing the initial and final states, we find the following results:

1. With \(\mathcal{L}^{(b)}_{\text{kin},v}\), pseudoscalar initial state \(B(v) = P_+(-\gamma_5)\) and pseudoscalar final state \(\overline{D}(v') = \gamma_5 P'_+\), one finds

\[2\chi_1(w) = \sum_{n \neq 0} \frac{1}{\Delta E_{1/2}^{(n)}} \zeta^{(n)}(w) \frac{< B^{(n)}(v)|O^{(b)}_{\text{kin},v}(0)|B(v)>}{\sqrt{4m_{B^{(n)}}m_B}}. \]

(49)

In the preceding expressions the energy denominators are \(\Delta E_{1/2}^{(n)} = E_{1/2}^{(n)} - E_{1/2}^{(0)}\)

\((n \neq 0)\).

2. Consider \(\mathcal{L}^{(b)}_{\text{mag},v}\), pseudoscalar initial state \(B(v) = P_+(-\gamma_5)\) and pseudoscalar final state \(\overline{D}(v') = \gamma_5 P'_+\). Because of parity conservation by the strong interactions, the intermediate states \(B^{(n)}\) must have the same parity than the initial state \(B\).
Moreover, $\mathcal{L}_{mag,v}^{(b)}$ being a scalar and producing transitions at zero recoil, the spin of $B$ and $B^{(n)}$ must be the same. Therefore, only pseudoscalar intermediate states $B^{(n)}(0^-)$ can contribute, only states with $j = \frac{1}{2}^-$. One finds, for any current (46)

$$-4(w - 1)\chi_2(w) + 12\chi_3(w) = \sum_{n \neq 0} \frac{1}{\Delta E_{1/2}^{(n)}} \xi_{(n)}^{(2)}(w) \frac{< B^{(n)}(v)|O_{mag,v}^{(b)}(0)|B(v) >}{\sqrt{4m_{B^{(n)}}m_B}}. \quad (50)$$

It is remarkable that this linear combination depends only on $\frac{1}{2}^-$ intermediate states.

(3) Consider $\mathcal{L}_{mag,v}^{(b)}$ and a vector initial state $B^*(v, \varepsilon) = P_+ \not{\ell}$ and pseudoscalar final state $D(v') = \gamma_5 P_+$. Now we will have vector $1^-$ intermediate states, either $B^{(n)}(\frac{1}{2}^-, 1^-)$ or $B^{(n)}(\frac{3}{2}^-, 1^-)$. For the latter, we have to compute the current matrix element

$$< D(v')|J^{cb}(0)|B^{(n)}(\frac{3}{2}^-, 1^-)(v, \varepsilon) >= \tau_{3/2}^{(2)(n)}(w) Tr \left[ D(v') \Gamma F^\sigma_\nu v_\sigma \right] \quad (51)$$

where the $(\frac{3}{2}^-, 1^-)$ operator is given by

$$F_\nu^\sigma = \frac{3}{2} P_+ \varepsilon_\nu \left[ g^{\sigma \nu} - \frac{1}{3} \gamma^{\nu} (\gamma^\sigma + v^\sigma) \right] \quad (52)$$

obtained from the $(\frac{3}{2}^+, 1^+)$ operator defined by Leibovich et al. (formula (2.5) of [15]), multiplying by $(-\gamma_5)$ on the right [5]. The Isgur-Wise functions $\tau_{3/2}^{(2)(n)}(w)$ correspond to $\frac{1}{2}^- \rightarrow \frac{3}{2}^-$ transitions, the superindex ($\ell$) meaning the orbital angular momentum [4] [5] [10]. As noticed by Leibovich et al. [15], on general grounds the IW functions $\tau_{3/2}^{(2)(n)}(w)$ do not vanish at zero recoil.

One finds, for any current (46), finally

$$-4\chi_2(w)(\varepsilon \cdot v') Tr \left[ D(v') \Gamma P_+ \right] + 4\chi_3(w) Tr \left[ D(v') \Gamma P_+ \not{\ell} \right]$$

$$= -Tr \left[ D(v') \Gamma B^*(v, \varepsilon) \right] \sum_{n \neq 0} \frac{1}{\Delta E_{1/2}^{(n)}} \xi_{(n)}^{(2)}(w) \frac{< B^{(n)}(v, \varepsilon)|O_{mag,v}^{(b)}(0)|B^*(v, \varepsilon) >}{\sqrt{4m_{B^{(n)}}m_B}}$$

$$+ \left\{ \frac{3}{2} (\varepsilon \cdot v') Tr \left[ D(v') \Gamma P_+ \right] - \frac{1}{\sqrt{6}} (w - 1) Tr \left[ D(v') \Gamma P_+ \not{\ell} \right] \right\}$$

$$\sum_{n} \frac{1}{\Delta E_{3/2}^{(n)}} \tau_{3/2}^{(2)(n)}(w) \frac{< B_{3/2}^{(n)}(v, \varepsilon)|O_{mag,v}^{(b)}(0)|B^*(v, \varepsilon) >}{\sqrt{4m_{B_{3/2}^{(n)}}m_{B*}}} \quad . \quad (53)$$

The energy denominators are $\Delta E_{1/2}^{(n)} = E_{1/2}^{(n)} - E_{1/2}^{(0)} (n \neq 0)$ and $\Delta E_{3/2}^{(n)} = E_{3/2}^{(n)} - E_{1/2}^{(0)} (n \geq 0)$. 

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One can obtain other linearly independent relations, taking $\Gamma = \gamma_{\mu} \gamma_{5}$.

Since the two four vectors $(v'_{\mu} - v_{\mu})$ and $[(w - 1)\varepsilon_{\mu} + (\varepsilon \cdot v') v_{\mu}]$ can be chosen to be independent, one obtains independent sum rules for $\chi_2(w)$ and $\chi_3(w)$.

To summarize, making explicit the $c$ flavor, we have obtained the sum rules

$$\chi_1(w) = \frac{1}{2} \sum_{n \neq 0} \frac{1}{\Delta E_{1/2}^{(n)}} \xi^{(n)}(w) \frac{D^{(n)}(v)|O_{\text{kin},v}^{(c)}(0)|D(v)}{\sqrt{4m_{D^{(n)}} m_{D}}}$$

(iii) The elastic subleading magnetic form factors $\chi_2(w)$ and $\chi_3(w)$ involve $D^{(1-)} \to D^{(1-)}$ transitions $\frac{1}{2}^- \to \frac{1}{2}^-$ and $\frac{3}{2}^- \to \frac{5}{2}^-$. 

(iv) $\chi_1(w)$ and $\chi_3(w)$ satisfy, as they should, Luke theorem [20],

$$\chi_1(1) = \chi_3(1) = 0$$

because the $\frac{1}{2}^- \to \frac{1}{2}^-$ IW functions at zero recoil satisfy

$$\xi^{(n)}(1) = \delta_{n,0}$$

(v) There is a linear combination of $\chi_2(w)$ and $\chi_3(w)$ that gets only contributions from $\frac{1}{2}^- \to \frac{1}{2}^-$ transitions, namely

$$-4(w-1)\chi_2(w) + 12\chi_3(w) = -3 \sum_{n \neq 0} \frac{1}{\Delta E_{1/2}^{(n)}} \xi^{(n)}(w) \frac{D^{(n)}(v,\varepsilon)|O_{\text{mag},v}^{(c)}(0)|D^{*}(v,\varepsilon)}{\sqrt{4m_{D^{(n)}} m_{D}}}$$

There are a number of striking features in relations (54)-(56).

(i) One should notice that elastic subleading form factors of the Lagrangian type are given in terms of leading IW functions, namely $\xi^{(n)}(w)$ and $\tau_{3/2}^{(2)(n)}(w)$, and subleading form factors at zero recoil.

(iii) The elastic subleading magnetic form factors $\chi_2(w)$ and $\chi_3(w)$ involve $D^{(1-)} \to D^{(1-)}$ transitions $\frac{1}{2}^- \to \frac{1}{2}^-$ and $\frac{3}{2}^- \to \frac{5}{2}^-$. 

(iv) $\chi_1(w)$ and $\chi_3(w)$ satisfy, as they should, Luke theorem [20],

$$\chi_1(1) = \chi_3(1) = 0$$

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(v) There is a linear combination of $\chi_2(w)$ and $\chi_3(w)$ that gets only contributions from $\frac{1}{2}^- \to \frac{1}{2}^-$ transitions, namely

$$-4(w-1)\chi_2(w) + 12\chi_3(w) = -3 \sum_{n \neq 0} \frac{1}{\Delta E_{1/2}^{(n)}} \xi^{(n)}(w) \frac{D^{(n)}(v,\varepsilon)|O_{\text{mag},v}^{(c)}(0)|D^{*}(v,\varepsilon)}{\sqrt{4m_{D^{(n)}} m_{D}}}$$
where the factor $-3$ is in consistency with (50), shifting from vector to pseudoscalar mesons.

It is well-known that the determination of $|V_{cb}|$ from the $\overline{B} \rightarrow D^* \ell \nu$ differential rate at zero recoil depends on the value of $h_{A1}(1)$. One interesting point is that precisely the subleading matrix elements of $O_{\text{kin}}$ and $O_{\text{mag}}$ at zero recoil, that enter in the SR (54)-(56), are related to the quantity $|h_{A1}(1)|$, as we will see now.

The following SR follows from the OPE [18] [15],

\[
|h_{A1}(1)|^2 + \sum_n \left| \frac{< D^{*\text{(n)}} \left( \begin{array}{c} 1^- \\ \frac{3}{2}^- \end{array} \right) (v, \varepsilon) | \vec{A} | B(v) > }{4 m_{D^{*\text{(n)}}} m_B} \right|^2 \]

\[
= \eta_A^2 - \frac{\mu_G^2}{3 m_c^2} - \frac{\mu^2 - \mu_G^2}{4} \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3 m_c m_b} \right) \]

(60)

where $D^{*\text{(n)}}$ are $1^-$ excited states, and

\[
\mu^2 = \frac{1}{2 m_B} < B(v) | \overline{\eta}_{v(b)} | (iD)^2 h_v^{(b)} | B(v) > \]

\[
\mu_G^2 = \frac{1}{2 m_B} < B(v) | \overline{\eta}_{v(b)} g_s \frac{1}{2} \sigma_{\alpha\beta} G^{\alpha\beta} h_v^{(b)} | B(v) > \]

(61)

In relation (60) one assumes the states at rest $v = (1, 0)$ and the axial current is space-like, orthogonal to $v$.

In the l.h.s. of relation (60),

\[
h_{A1}(1) = \eta_{A1} + \delta^{(A1)}_{1/m^2} \]

(62)

($\eta_{A1} = 1^+$ radiative corrections) because there are no first order $1/m_Q$ corrections due to Luke theorem. The sum over the squared matrix elements of $B \rightarrow D^{*\text{(n)}}(1^-)$ transitions contains two types of possible contributions, corresponding to $D^{*\text{(n)}} \left( \begin{array}{c} 1^- \\ 3^- \end{array} \right) (n \neq 0)$, and $D^{*\text{(n)}} \left( \frac{3}{2}^-, 1^- \right) (n \geq 0)$. The r.h.s. of (60) exhibits the OPE at the desired order. From the decomposition between radiative corrections and $1/m_Q^2$ corrections (62) one gets, from (60), neglecting higher order terms,

\[
-\delta^{(A1)}_{1/m^2} = \frac{\mu_G^2}{6 m_c^2} + \frac{\mu^2 - \mu_G^2}{8} \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3 m_c m_b} \right) \]

\[
+ \frac{1}{2} \sum_n \left| \frac{< D^{*\text{(n)}} \left( \begin{array}{c} 1^- \\ \frac{3}{2}^- \end{array} \right) (v, \varepsilon) | \vec{A} | B(v) > }{4 m_{D^{*\text{(n)}}} m_B} \right|^2 . \]

(63)

The correction $\delta^{(A1)}_{1/m^2}$ is therefore negative, both terms being of the same sign.
The matrix elements \( \langle D^{(n)} \left( \frac{1}{2}, -\frac{3}{2} \right) | v, \varepsilon \rangle | \tilde{A}B \rangle \) have been expressed in terms of the matrix elements \( \langle D^{(n)} \left( \frac{1}{2} \right) | O^{(c)}_{\text{kin}, v}(0) | D^*(v, \varepsilon) \rangle \) and
\( \langle D^{(n)} \left( \frac{1}{2}, -\frac{3}{2} \right) | O^{(c)}_{\text{mag}, v}(0) | D^*(v, \varepsilon) \rangle \) by Leibovich et al. (formulas (4.1) and (4.3)) [15]. Hence, \(-\delta^{(A_1)}_{1/m^2}\) (63) can be written as
\[
-\delta^{(A_1)}_{1/m^2} = \frac{\mu_G^2}{6m_c^2} + \frac{1}{8} \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_c m_b} \right) \left( \mu_\pi - \mu_G^2 \right)
+ \frac{1}{2} \sum_{n} \left[ \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right) \frac{1}{\Delta E_{1/2}^{(n)}} \langle D^{(n)} \left( \frac{1}{2} \right) | v, \varepsilon \rangle | O^{(c)}_{\text{kin}, v}(0) | D^*(v, \varepsilon) \rangle \right. \\
+ \left. \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right) \frac{1}{\Delta E_{3/2}^{(n)}} \langle D^{(n)} \left( \frac{3}{2} \right) | v, \varepsilon \rangle | O^{(c)}_{\text{mag}, v}(0) | D^*(v, \varepsilon) \rangle \right]^2
\]
\[
\,
+ \frac{1}{2} \sum_{n} \left[ \frac{1}{2m_c} \frac{1}{\Delta E_{3/2}^{(n)}} \langle D^{(n)} \left( \frac{3}{2} \right) | v, \varepsilon \rangle | O^{(c)}_{\text{mag}, v}(0) | D^*(v, \varepsilon) \rangle \right]^2
\] (64)

The important point to emphasize here is that the matrix elements
\( \langle D^{(n)} \left( \frac{1}{2} \right) | O^{(c)}_{\text{kin}, v}(0) | D^*(v, \varepsilon) \rangle \) and \( \langle D^{(n)} \left( \frac{1}{2}, -\frac{3}{2} \right) | O^{(c)}_{\text{mag}, v}(0) | D^*(v, \varepsilon) \rangle \)
are precisely the same ones that enter in the SR (54)-(56). This allows to obtain an interesting lower bound on \(-\delta^{(A_1)}_{1/m^2}\).

Taking now the relevant linear combinations of the matrix elements suggested by the r.h.s. of (64), using (54), (55) and (59), and Schwarz inequality
\[
\left| \sum_n A_n B_n \right| \leq \sqrt{ \left( \sum_n |A_n|^2 \right) \left( \sum_n |B_n|^2 \right) } 
\] (65)
one finds
\[
\sum_{n \neq 0} \left[ \xi^{(n)}(w) \right]^2 \sum_{n \neq 0} \left[ \frac{1}{\Delta E_{1/2}^{(n)}} \left[ \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right) \langle D^{(n)} \left( \frac{1}{2} \right) | v, \varepsilon \rangle | O^{(c)}_{\text{kin}, v}(0) | D^*(v, \varepsilon) \rangle \right. \\
+ \left. \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right) \frac{3}{2m_b} \langle D^{(n)} \left( \frac{3}{2} \right) | v, \varepsilon \rangle | O^{(c)}_{\text{mag}, v}(0) | D^*(v, \varepsilon) \rangle \right]^2 \right. \\
\geq 4 \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right) \chi_1(w) - \frac{1}{3} \left( \frac{1}{2m_c} + \frac{3}{2m_b} \right) [-2(w - 1)\chi_2(w) + 6\chi_3(w)] \right)^2 
\] (66)
\[
\sum_n \left[ \tau^{(2)}_{3/2}(w) \right]^2 \sum_n \left[ \frac{1}{\Delta E_{3/2}^{(n)}} \left[ \frac{1}{2m_c} \langle D^{(n)} \left( \frac{3}{2} \right) \right. \\
\left. | v, \varepsilon \rangle | O^{(c)}_{\text{mag}, v}(0) | D^*(v, \varepsilon) \rangle \right]^2 
\geq \frac{32}{3} \left( \frac{1}{2m_c} \right)^2 \chi_2(w) \right]^2 
\] (67)
These two last equations imply, from (64), the inequality
\[
-\delta^{(A_1)}_{1/m^2} \geq \frac{\mu_c^2}{6m_c^2} + \frac{\mu_b^2 - \mu_c^2}{8} \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_c m_b} \right)
\]
\[
+ 2 \left\{ \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right) \chi_1(w) - \frac{1}{3} \left( \frac{1}{2m_c} + \frac{3}{2m_b} \right) \left[ -2(w-1) \chi_2(w) + 6 \chi_3(w) \right] \right\}^2
\]
\[
\sum_{n \neq 0} \left[ \xi^{(n)}(w) \right]^2
\]
\[
+ \frac{16}{3} \sum_n \left[ r^{(2)}_{3/2}(w) \right]^2 .
\] (68)

This inequality on \(-\delta^{(A_1)}_{1/m^2}\) involves on the r.h.s. elastic subleading functions \(\chi_i(w)\) (\(i = 1, 2, 3\)) in the numerator and sums over inelastic leading IW functions \(\sum_{n \neq 0} \left[ \xi^{(n)}(w) \right]^2\) and \(\sum_n \left[ r^{(2)}_{3/2}(w) \right]^2\) in the denominator. We must emphasize that this inequality is valid for all values of \(w\) and constitutes a rigorous constraint between these functions and the correction \(-\delta^{(A_1)}_{1/m^2}\). Let us point out that, near \(w = 1\), since \(\xi^{(n)}(w) \sim (w-1)\) (\(n \neq 0\)) and, due to Luke theorem \(\chi_1(w), \chi_3(w) \sim (w-1)\), the second term on the r.h.s. of (68) is a constant in the limit \(w \to 1\).

On the other hand, since \(\chi_2(w)\) is not protected by Luke theorem, \(\chi_2(1) \neq 0\) and in general, as pointed out by Leibovich et al. \(r^{(2)}_{3/2}(1) \neq 0\), the last term in the r.h.s. of (68) is also a constant for \(w = 1\).

The inequality (68) is valid for all values of \(w\), and in particular it holds in the \(w \to 1\) limit. Let us consider this limit, that gives
\[
-\delta^{(A_1)}_{1/m^2} \geq \frac{\mu_c^2}{6m_c^2} + \frac{\mu_b^2 - \mu_c^2}{8} \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_c m_b} \right)
\]
\[
+ 2 \left\{ \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right) \chi_1(1) - \frac{1}{3} \left( \frac{1}{2m_c} + \frac{3}{2m_b} \right) \left[ -2 \chi_2(1) + 6 \chi_3(1) \right] \right\}^2
\]
\[
\sum_{w \neq 0} \left[ \xi^{(n)}(1) \right]^2
\]
\[
+ \frac{16}{3} \sum_n \left[ r^{(2)}_{3/2}(1) \right]^2 .
\] (69)

On the other hand, using the OPE in the heavy quark limit, we have demonstrated above the following sum rules \([4]\)
\[
\sum_n \left[ r^{(2)}_{3/2}(1) \right]^2 = \frac{4}{5} \sigma^2 - \rho^2
\] (70)
\[
\sum_{n \neq 0} \left[ \xi^{(n)}(1) \right]^2 = \frac{5}{3} \sigma^2 - \frac{4}{3} \rho^2 - (\rho^2)^2
\] (71)
where \( \rho^2 \) and \( \sigma^2 \) are the slope and the curvature of the elastic Isgur-Wise function \( \xi(w) \).

The positivity of the l.h.s. of (70), (71) yield respectively the lower bounds on the curvature obtained (13) and (15). Relations (69)-(71) give finally the bound

\[
-\delta^{(A_1)}_{1/m} \geq \frac{\mu_G^2}{6m_c^2} + \frac{\mu_c^2 - \mu_b^2}{8} \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_cm_b} \right) \\
+ \frac{2}{3[5\sigma^2 - 4\rho^2 - 3(\rho^2)^2]} \left\{ \left[ \frac{1}{2m_c} - \frac{1}{2m_b} \right] 3\chi'_1(1) - \left( \frac{1}{2m_c} + \frac{3}{2m_b} \right) [-2\chi_2(1) + 6\chi'_3(1)] \right\}^2 \\
+ \frac{80}{3(4\sigma^2 - 5\rho^2)} \left[ \frac{1}{2m_c} \chi_2(1) \right]^2.
\]

(72)

A number of remarks are worth to be made here:

(i) The bounds contain an OPE piece, dependent on \( \mu_c^2, \mu_b^2, \pi \) and \( \mu_G^2 \), and a piece that bounds the inelastic contributions, given in terms of the \( 1/m_Q \) elastic quantities \( \chi'_1(1), \chi_2(1), \chi'_3(1) \) and of the slope \( \rho^2 \) and curvature \( \sigma^2 \) of the elastic IW function \( \xi(w) \).

(ii) Taking roughly constant values for \( \chi'_1(1), \chi_2(1), \chi'_3(1) \), independent of \( \rho^2 \) and \( \sigma^2 \), as suggested by the QCD Sum Rules calculations (QCDSR) [21] [22] [23], the bounds for the inelastic contributions diverge in the limit \( \rho^2 \rightarrow \frac{3}{4}, \sigma^2 \rightarrow \frac{15}{16} \), according to (13). This feature does not seem to us physical.

(iii) Therefore, one should expect that \( \chi'_1(1), \chi_2(1) \) and \( \chi'_3(1) \) vanish also in this limit. We give a demonstration of this interesting feature below.

(iv) Thus, the limit \( \rho^2 \rightarrow \frac{3}{4}, \sigma^2 \rightarrow \frac{15}{16} \) seems related to the behaviour of \( \chi_i(w) \) (\( i = 1, 2, 3 \)) near zero recoil.

(v) The feature (iii) does not appear explicitly in the QCDSR approach, where one gets roughly \( \rho^{ren}_{\infty} \approx 0.7 \), and where there is no dependence on \( \rho^2 \) of the functions \( \chi_i(w) \) (\( i = 1, 2, 3 \)).

Now we demonstrate that indeed \( \chi'_1(1), \chi_2(1) \) and \( \chi'_3(1) \) vanish in the limit \( \rho^2 \rightarrow \frac{3}{4}, \sigma^2 \rightarrow \frac{15}{16} \). At zero recoil \( w \rightarrow 1 \) we have

\[
\chi'_1(1) = \frac{1}{2} \sum_{n \neq 0} \frac{1}{\Delta E_{1/2}^{(n)}} \xi_2^{(n)}(1) \frac{< D^{(n)}(v)|O_{\text{kin}}^{(c)}(0)|D(v) >}{\sqrt{4m_D^{(n)}m_D}} \quad (73)
\]

\[
\chi_2(1) = \frac{1}{4\sqrt{6}} \sum_n \frac{1}{\Delta E_{3/2}^{(n)}} \xi_2^{(n)}(1) \frac{< D^{(n)}_{3/2}(v, \varepsilon)|O_{\text{mag}}^{(c)}(0)|D^{(n)}(v, \varepsilon) >}{\sqrt{4m_D^{(n)}m_D}} \quad (74)
\]
\[-4\chi'(1) + 12\chi'(1) = \sum_{n \neq 0} \frac{1}{\Delta E_{1/2}^{(n)}} \langle \zeta(n)'(1) | D^{(n)}(v) | O_{mag}^{(c)}(0) | D(v) > \rangle \] (75)

Using again Schwarz inequality, we obtain

\[
[\chi_1'(1)]^2 \leq \frac{1}{4} \sum_{n \neq 0} \left[ \zeta(n)'(1) \right]^2 \sum_{n \neq 0} \left[ \frac{1}{\Delta E_{1/2}^{(n)}} \langle \zeta(n)'(1) | D^{(n)}(v) | O_{kin}^{(c)}(0) | D(v) > \rangle \right]^2 \] (76)

\[
[\chi_2(1)]^2 \leq \frac{1}{96} \sum_n \left[ t_{3/2}^{(2)}(1) \right]^2 \sum_n \left[ \frac{1}{\Delta E_{3/2}^{(n)}} \langle D^{(n)}(v, \varepsilon) | O_{mag}^{(c)}(0) | D^{*(v, \varepsilon)} > \rangle \right]^2 \] (77)

\[
[-4\chi_2(1) + 12\chi'_3(1)]^2 \leq \sum_{n \neq 0} \left[ \zeta(n)'(1) \right]^2 \sum_{n \neq 0} \left[ \frac{1}{\Delta E_{1/2}^{(n)}} \langle D^{(n)}(v) | O_{mag}^{(c)}(0) | D(v) > \rangle \right]^2 \] (78)

and from relations (70) and (71),

\[
[\chi_1'(1)]^2 \leq \frac{1}{12} \left[ 5\sigma^2 - 4\rho^2 - 3(\rho^2)^2 \right] \sum_{n \neq 0} \left[ \frac{1}{\Delta E_{1/2}^{(n)}} \langle D^{(n)}(v) | O_{kin}^{(c)}(0) | D(v) > \rangle \right]^2 \] (79)

\[
[\chi_2(1)]^2 \leq \frac{1}{480} \left( 4\sigma^2 - 5\rho^2 \right) \sum_n \left[ \frac{1}{\Delta E_{3/2}^{(n)}} \langle D^{*(n)}(v, \varepsilon) | O_{mag}^{(c)}(0) | D^{*(v, \varepsilon)} > \rangle \right]^2 \] (80)

\[
[-4\chi_2(1) + 12\chi'_3(1)]^2 \leq \frac{1}{3} \left[ 5\sigma^2 - 4\rho^2 - 3(\rho^2)^2 \right] \sum_{n \neq 0} \left[ \frac{1}{\Delta E_{1/2}^{(n)}} \langle D^{(n)}(v) | O_{mag}^{(c)}(0) | D(v) > \rangle \right]^2 \] (81)

Therefore, in the limit $\rho^2 \to \frac{3}{1}, \sigma^2 \to \frac{15}{16}$, one obtains

\[
\chi_1'(1) = \chi_2(1) = \chi'_3(1) = 0 \] (82)

This is a very strong correlation relating the behaviour of the elastic IW function $\xi(w)$ to the elastic subleading IW functions $\chi_i(w)$ $(i = 1, 2, 3)$ near zero recoil.

To conclude, we have obtained bounds that relate the $1/m_Q^2$ correction of the form factor $h_{A_i}(w)$ to the $1/m_Q$ subleading form factors of the Lagrangian type $\chi_i(w)$ $(i = 1, 2, 3)$ and to the shape of the elastic Isgur-Wise $\xi(w)$. This bound should in principle be taken into account in the analysis of the exclusive determination of $|V_{cb}|$ in the channels $\bar{B} \to D(D^*)\ell\nu$. On the other hand, we have demonstrated an
important constraint on the behavior of the subleading form factors $\chi_i(w)$: in the limit $\rho^2 \to \frac{3}{4}$, $\sigma^2 \to \frac{15}{16}$, $\chi'_1(1)$, $\chi_2(1)$ and $\chi'_3(1)$ must vanish.

It would be very interesting to have a theoretical calculation of the functions $\chi_i(w) (i = 1, 2, 3)$ satisfying this constraint. Otherwise it seems questionable to try an exclusive determination of $|V_{cb}|$ by fitting the slope $\rho^2$ and considering uncorrelated subleading corrections.

In conclusion, using sum rules in HQET, as formulated in ref. [3, 4, 10, 19], we have found lower bounds for the moduli of the derivatives of $\xi(w)$ and non-trivial results on $1/m_Q$ form factors of the Current and Lagrangian types. We have also obtained a lower bound on $-\delta^{(A_1)}_{1/m^2}$. The determination of the CKM matrix element $|V_{cb}|$ in $B \to D^{(*)}l\nu$ should satisfy these constraints.

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