An Algorithm to Study of the Influence of Partial-Coherent Lights through Three-Dimensional Periodic Microstructures

I-Lin Ho*, Yia-Chung Chang, Wang-Yang Li1, Ming-Tsung Lee, and Chun-Yi Yin

Research Center for Applied Sciences, Academia Sinica, Taipei, Taiwan 115, R.O.C.
1Institute of Electro-Optical Science and Engineering, National Cheng Kung University, Tainan, Taiwan 701, R.O.C.

Received April 9, 2011; revised June 27, 2011; accepted July 28, 2011; published online October 20, 2011

A transfer-matrix algorithm is presented herein as a beginning in the study of the transmission characteristics of coherent light through three-dimensional periodic microstructures, in which the structures are treated as two-dimensional layer stacks and multiple reflections are considered negligible. The spatially correlated noise is further introduced layer by layer to realize the actual partial coherence of light and allows for statistical investigation of the spatial partially coherent optics in transparent/small-birefringence media. Numerical analyses show results comparable to those obtained using the Gaussian Schell model for free-space cases, indicating the validity of the algorithms.

1. Introduction

Nanoscale structures have achieved novel functions in electro-optic devices such as optical filters, optical modulators, phase-conjugated systems, optical attenuators, beam amplifiers, tunable lasers, holographic data storage, and even as parts of optical logic systems1–14) over the last few decades. In contrast to the assumption of the ideal coherent (plane-wave) light in most theoretical studies15–22) the associated partial-coherent characteristics and intensity distribution of the light, e.g., from light-emitting diodes (LEDs),23–25) otherwise introduce another critical issue for both fundamental research and potential industrial applications. Motivated by these concerns and on the basis of previous studies on rigorous coupled-wave algorithms20–22) (RCWA), we present a transfer-matrix algorithm as a beginning in the study of the propagation of coherent light through three-dimensional (3D) periodic microstructures. In this work, the structures are discretized into two-dimensional (2D) layer stacks composed of isotropic or small-birefringence materials, and multiple reflections are considered negligible. Referring to the Langevin equation26) for the motion of fluctuations, i.e., a random force conditioned by specified correlations is added at each time step to mimic the Brownian motion, this work similarly includes a noise phase (characterized using specified spatial correlations) layer by layer to simulate the actual partial coherence of light and allows for statistical investigation of the spatial partially coherent optics. Here, the spatial correlation length \( \tau \) of noise is assumed to be larger than the wavelength of the incidence \( \lambda \), because the intense incoherence (small \( \tau \)) could cause strong scattering, and multiple reflections thereby must be a concern. Besides, diffraction components with imaginary wave vectors along the \( \xi_{zh} \in Z \) as below), which decay exponentially through media, are omitted as well to simplify the numerical processes. The generation of the noise phase is realized using a standard numerical technique based on the Fourier transform27,28) and is formulated in the appendices for the 2D and 3D conditions. Numerical examples for the propagation of the free-space Gaussian beam and the diffractions of the beam incident through a 2D liquid-crystal grating show results comparable to those obtained using the Gaussian Schell model29–32) and could validate this study.

2. Theoretical Formulæ

In this section, we mainly derive the transfer-matrix algorithms for propagations of waves through 3D periodic microstructures, in which the multiple reflections are neglected and are hence much easier to manipulate algebraically, yet account for the effects of the Fresnel refraction and the single reflection at the interfaces of the medium. Here, the assumption of no multiple reflections is legitimate for most practical transparent materials. In what follows, a reformulation, including the noise phase of electromagnetic fields, is carried out to demonstrate the partially coherent behaviors of light.

2.1 Transfer matrix method by RCWA

As shown in Fig. 1, a 3D structure is generally stratified along the medium normal to \( N \) layers, and these layers (with a thickness of \( dz_l \)) contribute to the total thickness of the stacks \( Z_N = \sum_{l=1}^{N} dz_l \). Each layer is considered as an arrangement with homogeneous or small-birefringence materials, and the dielectric coefficients are generally described by a matrix as a function of position \( (x, y, z) \) as follows:

\[
\mathbf{e} = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{bmatrix}
\]

with

\[
\begin{align*}
\varepsilon_{xx} &= n_x^2 + (n_z^2 - n_y^2) \sin^2 \theta_o \cos^2 \phi_o, \\
\varepsilon_{xy} &= \varepsilon_{yx} = (n_z^2 - n_y^2) \sin^2 \theta_o \sin \phi_o \cos \phi_o, \\
\varepsilon_{xz} &= \varepsilon_{zx} = (n_z^2 - n_y^2) \sin \theta_o \cos \phi_o, \\
\varepsilon_{yy} &= n_y^2 + (n_z^2 - n_x^2) \sin^2 \theta_o \sin^2 \phi_o, \\
\varepsilon_{yz} &= \varepsilon_{zy} = (n_z^2 - n_x^2) \sin \theta_o \sin \phi_o, \\
\varepsilon_{zz} &= n_z^2 + (n_x^2 - n_y^2) \cos^2 \theta_o.
\end{align*}
\]

Here, \( n_x \) and \( n_y \) are extraordinary and ordinary indices of refraction of a uniaxially birefringent medium, respectively, \( \theta_o \) is the angle between the optic axes and the \( z \)-axis, and \( \phi_o \) is the angle between the projection of the optic axes on the \( xy \)-plane and \( x \)-axis.

To introduce the RCWA to the stack, all of the layers are defined as having the same periodicity: \( \Lambda_x \) along the \( x \)-direction and \( \Lambda_y \) along the \( y \)-direction. The periodic permittivity \( \varepsilon(x, y, z) \) as a function of \( xy \) for the stack-layer

*E-mail address: sunta.ho@msa.hinet.net
Fig. 1. (Color online) Geometry of 3D RCWA algorithm for a multi-layer stack with 2D periodic microstructures in arbitrary arrangement with isotropic and small-birefringent materials.

\[ z = z_i \] thereby can be expanded in the Fourier series of the spatial harmonics as follows:

\[
\tilde{\varepsilon}_{ij}(\tilde{x}, \tilde{y}, \tilde{z}_i) = \sum_{g,h} \tilde{\varepsilon}_{ij,gh}(\tilde{z}_i) \exp\left(i \frac{g \tilde{x}}{\Delta x} + i \frac{h \tilde{y}}{\Delta y}\right). \tag{3}
\]

\[
\tilde{\varepsilon}_{ij,gh}(\tilde{z}_i) = \frac{\lambda}{2 \pi \Lambda_x} \frac{\lambda}{2 \pi \Lambda_y} \int_0^{\frac{2\pi}{\Delta x}} \int_0^{\frac{2\pi}{\Delta y}} \tilde{\varepsilon}_{ij}(\tilde{x}, \tilde{y}, \tilde{z}_i) \exp\left(-i \frac{g \tilde{x}}{\Delta x} - i \frac{h \tilde{y}}{\Delta y}\right) d\tilde{x} d\tilde{y} \tag{4}
\]

for which we use the following definitions: \( k_0 = \omega \sqrt{\mu_0 \varepsilon_0} = 2\pi/\lambda \), \( Y_0 = 1/Z_0 = \sqrt{\varepsilon_0/\mu_0} \), \( \tilde{r} = k_0 r \), \( \tilde{x} = k_0 x \), \( \tilde{y} = k_0 y \), \( \tilde{z} = k_0 z \), and \( d\tilde{z} = k_0 d\tilde{z} \). \( \lambda \) is the vacuum wavelength of the incident wave. \( \varepsilon_{ij}(x,y,z) \) are defined as functions of position \((x,y,z)\) and \( \tilde{\varepsilon}_{ij} \) are related to the dimensionless coordinates \((\tilde{x}, \tilde{y}, \tilde{z})\). Note that the variables \( x, y, \) and \( z \) generally represent spatial positions, and when these appear in suffix, e.g., \( \varepsilon_{ij}(x,y,z) \) they denote the orientations along the directions \( \tilde{x}, \tilde{y}, \) and \( \tilde{z} \), respectively. Moreover, the variable \( i \) is the imaginary constant number, and that in suffix, e.g., \( d\tilde{z} \), is an integer indexing number.

A parallel transform for the electromagnetic fields through the 3D structure (treated as a layer stack) is expressed in terms of Rayleigh expansions:

\[
\sqrt{Y_0} \mathbf{E}(\tilde{x}, \tilde{y}, \tilde{z}_i) = \sum_{g,h} \mathbf{E}_{gh}(\tilde{z}_i) \exp[-i(n_{ix} \tilde{x} + n_{iy} \tilde{y})], \tag{5}
\]

\[
\sqrt{Z_0} \mathbf{H}(\tilde{x}, \tilde{y}, \tilde{z}_i) = \sum_{g,h} \mathbf{H}_{gh}(\tilde{z}_i) \exp[-i(n_{ix} \tilde{x} + n_{iy} \tilde{y})]. \tag{6}
\]

Here, \( g \) and \( h \) denote the diffraction or reflection orders of light, and indicate the propagation along the direction \( \mathbf{n}_{gh} = n_{ix} \mathbf{i} + n_{iy} \mathbf{j} + n_{iz} \mathbf{k} \) with \( \tilde{\varepsilon}_{gh} = (n_{iz}^2 \varepsilon_{x} - n_{ix} n_{iy} - n_{ix} n_{iz})^{1/2} \), in which \( n_1 \) and \( n_E \) are the refraction indexes for the incident and emitted regions, respectively. \( \theta \) and \( \phi \) are the incident angles defined by sphere coordinates, and \( z \) is the normal direction for the structures.

Applying Maxwell's equations as in our previous study\(^{20,21}\) and ignoring the multiple reflections, the transfer-matrix formulae for the propagations of waves through the 3D periodic microstructures can thus be written as follows:

\[
\begin{bmatrix}
E_{qN+1}^+ \\
M_{qN+1}^+
\end{bmatrix} = S_{1N} S_{21} \cdots S_{iN} S_{i1} \begin{bmatrix}
E_{q0}^+ \\
M_{q0}^+
\end{bmatrix}. \tag{9}
\]

Here, \( S_{iN(1,...,N)} \) is the matrix representing the propagations of waves through the \( l_{th} \) structured layer. It consists of the matrix \( T_{l}^{(o)} \), which is the (column) eigenvector matrix of the characteristic matrix \( G_l \), and the diagonal matrix \( \exp[ik_l^o d\tilde{z}_i] \) with the corresponding eigenvalue \( k_l^{(o)} \).

\[
S_i = T_i^{(o)} \exp[ik_l^o d\tilde{z}_i](T_i^{(o)})^{-1}
\]

\[
G_l = \begin{bmatrix}
\hat{n}_x^{(l)} \hat{\varepsilon}_x^{(l)} \hat{\varepsilon}_x^{(l)} - \hat{n}_y^{(l)} \hat{n}_y^{(l)} - 1 & \hat{n}_x^{(l)} \hat{\varepsilon}_x^{(l)} \hat{\varepsilon}_y^{(l)} - \hat{n}_y^{(l)} \hat{n}_y^{(l)} - 1 \\
\hat{\varepsilon}_x^{(l)} \hat{\varepsilon}_y^{(l)} \hat{\varepsilon}_y^{(l)} - \hat{n}_x^{(l)} \hat{n}_x^{(l)} - 1 & \hat{\varepsilon}_x^{(l)} \hat{\varepsilon}_y^{(l)} \hat{\varepsilon}_y^{(l)} - \hat{n}_x^{(l)} \hat{n}_x^{(l)} - 1
\end{bmatrix}
\]

\[
\begin{align}
\hat{n}_x^{(l)} & = \hat{n}_x^{(l)} \hat{\varepsilon}_x^{(l)} \hat{\varepsilon}_x^{(l)} - 1 \\
\hat{n}_y^{(l)} & = \hat{n}_y^{(l)} \hat{\varepsilon}_y^{(l)} \hat{\varepsilon}_y^{(l)} - 1 \\
\hat{n}_x^{(l)} \hat{n}_y^{(l)} & = \hat{n}_x^{(l)} \hat{n}_y^{(l)} - 1 \\
\hat{\varepsilon}_x^{(l)} \hat{\varepsilon}_y^{(l)} & = \hat{\varepsilon}_x^{(l)} \hat{\varepsilon}_y^{(l)} - 1
\end{align}
\]
The notation $E$ (or $M$, $e_{ij}(x,y)$, and $h_{ij}(x,y)$) represents an $N_{g} \times N_{g}$ vector with the components $E_{ij}(x,y)$ (or $M_{ij}(x,y)$; $e_{ij}(x,y)$, and $h_{ij}(x,y)$), which correspond to the fields in the $(g,h)$-th diffraction or reflection orders. The underlying meanings of $E$ and $M$ will be discussed later. $\hat{h}_{g}(n_{0})$ is $N_{g} \times N_{h}$ diagonal matrices with the $n_{ij}$ diagonal elements $n_{ij}(n_{0})$ having the same $(g,h)$ sequence as these of $E$ and $M$, and are calculated using eqs. (7) and (8). $\hat{r}_{ij}(x,y)$ is $N_{g} \times N_{h}$ matrices with elements $\hat{r}_{ij}(x,y)$ defined by eq. (4), in which the indexes $\alpha$ and $\beta$ are decided by the relation \cite{021,022,023} (conditioned by Maxwell’s equations) $\mathbf{M} \sim \hat{h}_{g}E$, i.e., $M_{ij}(x,y) \sim \sum_{g,h} \hat{r}_{ij}(x,y)E_{ij}(x,y)$. The above-mentioned $N_{g}(x,y)$ defines the number of considered total Fourier orders $g (h)$ in the $x (y)$-direction. $1$ represents the $N_{g} \times N_{h} \times N_{g}$ identity matrix. One may understand the eq. (10) for the $l_{th}$ layer as follows: the $(T_{l}^{(0)})^{-1}$ term represents the coordinate transformation from the spatial tangential components of the fields $f_{l} = [e_{ij} h_{ij} e_{ij} h_{ij}]^{T}$ defined by eqs. (5) and (6) at the $l_{th}$ interface to the orthogonal components of the eigenmodes in the $l_{th}$ layer; the exp[$\hat{r}_{ij}(x,y)$] term describes the eigenmode propagation over the distance $\hat{d}_{l}$ (thickness of the $l_{th}$ layer); the $T_{l}^{(0)}$ term then is the inverse coordinate transformation from the eigenmode components back to the spatial tangential components of fields at the next interface. Considering the continuum of tangential fields on interfaces, these fields emitted from the $l_{th}$ layer hence can be straightforwardly treated as the incident fields $f_{l+1}$ for the $(l+1)$th layer, and allow for following of the next transfer matrix $S_{l+1}^{(0)}$ to describe the sequential propagations of fields through the $(l+1)$th layer as in eq. (9).

For the matrices $S_{ext}$ and $S_{int}$ defined for the (isotropic) uniform incident ($l = 0$) and emitted ($l = N + 1$) regions, respectively, the eigenmodes are specially chosen (and symbolized as $E_{f}^{+}$ and $M_{f}^{+}$) \cite{22,23} representing the physical forward (backward) transverse electric (TE) and transverse magnetic (TM) waves, i.e., transverse electric and transverse magnetic fields corresponding to the planes of the diffraction or reflection waves, respectively. In which the transform matrix $T_{0}^{(0)}$ between the eigenmode components and the tangential components $f_{l_{th}} = [e_{ij} h_{ij} e_{ij} h_{ij}]^{T}$ for the incident region ($l = 0$) is obtained as follows:

$$S_{ext} = \begin{bmatrix} W_{1} & 0 \\ 0 & 0 \end{bmatrix} (T_{e_{0}}^{(0)})^{-1},$$

$$W_{1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$W_{2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = (T_{x}^{(0)})^{-1}T_{s_{ext}}^{(0)}.$$}

Put everything together as mentioned above, and the propagation of fields through 3D periodic microstructures can hence be evaluated using eq. (9).

### 2.2 RCWA for partially coherent waves

To study the influences of the spatial partial coherence of light, a noise phase is added layer by layer to the propagation fields of coherent waves $E(x, y, z)$ as

$$E_{\eta}(x, y, z) = E(x, y, z) \times \exp[i\pi \cdot \eta(x, y, z)].$$

Here, the noise phase $\eta(x, y, z)$ is included to simulate the partially coherent behaviors of fields with introduction of the finite (spatial) coherent length $\tau$ into the correlation relation of the phase $\eta$:

$$\langle \eta(r)\eta(r') \rangle = \gamma(r - r') = \exp \left( - \frac{\hat{r}^{2}}{2\theta^{2}} \right)$$

with $\langle \eta(r) \rangle = 0$. Obviously, an infinite coherent length $\tau \to \infty$ leads to a constant noise phase $\eta(r) = 0$, and the $E_{\eta}$ field in eq. (17) recovers to coherent waves $E$ as expected; whereas for a short coherent length $\tau \to 0$, $\eta(r)$ is random and thereby exhibits an incoherent wave. In this work, the correlation function $\gamma(r - r')$ has been chosen as Gaussian-type distribution and describes a strong (weak) phase coherence/correlation between fields at positions closer (farther) than $\tau$. The generation of the noise function $\eta(r)$ is
realized by using a standard numerical technique based on the Fourier transform and is formulated in the appendices. It is emphasized that the formulae in appendices can be straightforwardly extended to arbitrary correlation functions \( \gamma(\mathbf{r} - \mathbf{r}') \) for actual conditions. To implement RCWA with the effects of the partial coherence of light, the noise phase \( \eta(\mathbf{x}, \mathbf{y}, \mathbf{z}) \) is then included layer by layer to transfer the matrix in eq. (9) as shown below, and hence can approximately determine the 3D spatial partial-coherence.

\[
\begin{bmatrix}
E_{q,N+1}^+
M_{q,N+1}^+

E_{q,N+1}^-
M_{q,N+1}^-
\end{bmatrix} = S_{ext} \mathbf{RS}_n \cdots \mathbf{RS}_2 \mathbf{RS}_1 \mathbf{RS}_{ent} \begin{bmatrix}
E_{q,0}^+
M_{q,0}^+

E_{q,0}^-
M_{q,0}^-
\end{bmatrix}
\]

(19)

Here, \( \mathbf{R} \) indicates an operation consisting of three numerical processes in the spatial tangential fields \( f_{l\ell} = S_{l\ell} \mathbf{RS}_n \mathbf{RS}_2 \mathbf{RS}_1 \mathbf{RS}_{ent} \begin{bmatrix}
E_{l\ell}^+
M_{l\ell}^+

E_{l\ell}^-
M_{l\ell}^-
\end{bmatrix} \) is defined at the \( l \)th interface: (a) First, perform the inverse Fourier transform from the fields \( f_{l\ell} = [e_{l\ell}, h_{l\ell}, e_{l\ell}, h_{l\ell}] \) into \( (\sqrt{Z_0} \mathbf{E}, \sqrt{Z_0} \mathbf{H}) \) using eq. (5); (b) Next, include the noise phase \( e^{j\eta(\mathbf{x}, \mathbf{y}, \mathbf{z})} \) in our work, for the field \( \sqrt{Z_0} \mathbf{E}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \) using eq. (17) to calculate \( \sqrt{Z_0} \mathbf{F}_l(\mathbf{x}, \mathbf{y}, \mathbf{z}) \); (c) Finally, perform the Fourier transform of \( \sqrt{Z_0} \mathbf{E}_l(\mathbf{x}, \mathbf{y}, \mathbf{z}) \) to obtain the fields \( f_{l\ell} \) with the effect of the noise phase. By this method, the fields \( f_{l\ell} \) for one iteration is then obtained. Note that the diffraction/reflection orders \( (g, h) \) with imaginary \( \xi_{gh} \), which decay exponentially through mediums, are omitted to simplify the numerical treatments. Similar processes over 500 iterations then provide the convergent statistical results as shown in Fig. 2, with \( \tau = 5 \), 1, and 0.5 \( \mu \). Figure 3 shows the cross-sectional views of the intensity of the Gaussian beam at \( z = 35 \mu \) in Fig. 2. The results indicate that a stronger incoherence (smaller \( \tau \)) of the light causes a more intense scattering along the propagation and shows a comparable conclusion from the Gaussian Schell model.29

To further investigate the partially coherent behaviors of the light through the microstructures, a single liquid-crystal grating medium with the thickness \( d_{\text{LC}} = 2 \mu \) is included to explore the diffraction of light. In the case of the liquid-crystal orientation \( \theta_0 = \pi\lambda / \Lambda = \lambda \xi / 2 \Lambda \), \( \phi_0 = \pi / 2 \) is illustrated in Fig. 4. For this case, rather than employing the definition of the grating period \( \Lambda_{\text{LC}} = 1.2 \mu \) as in Fig. 4, we apply the SRC-RCWA scheme, which allows for the setting of an alternative characteristic lengths \( \Lambda_{\text{t}} \) and \( \Lambda_{\text{r}} \), e.g., \( \Lambda_{\text{r}} = 48 \mu \) = \( 24 \times \Lambda_{\text{2LC}} \) in this case. Consequently, the long-range propagation profile of the Gaussian beam in spatial space can be observed in the analyzed region, and simultaneously the small-angle scattering of the

![Fig. 2. (Color online) Intensity patterns of the free-space Gaussian beam with coherent length (a) \( \tau = 5 \mu \), (b) \( \tau = 1 \mu \), and (c) \( \tau = 0.5 \mu \) obtained by considering the statistic analyses over an ensemble of 500 iterations in the 48 x 40 \( \mu \) xz-plane.](image-url)
beam related to the partial coherence along the direction \( n_{\text{rad}} \sim 1/\lambda_\text{a} \) in eqs. (7) and (8) can be extracted. Figure 5 shows the diffraction patterns \(|\sqrt{\gamma_0}E_x(x, z)|^2 + |\sqrt{Z_0}H_y(x, z)|^2\) of the beam propagation through a forty-period liquid-crystal grating (enclosed by dashed lines) in the \( 48 \times 40 \mu m^2 \) \( xz\)-plane, in which the coherent lengths of the beam are set to be (a) \( \tau = 5 \mu m \), (b) \( \tau = 1 \mu m \), and (c) \( \tau = 0.5 \mu m \), and the results are obtained by statistical analyses over an ensemble of 500 iterations. Figure 6 shows the corresponding cross-sectional intensity of the beam at \( z = 35 \mu m \) in Fig. 5. The results indicate that the coherent electromagnetic propagation \( (\tau = 5 \mu m) \) through gratings leads to definite diffractions, as described in most reports, while the more incoherent light \( (\tau = 0.5 \mu m) \) exhibits an intense scattering similar to that in Figs. 2(c) and 3. The electromagnetic propagation with \( \tau = 1 \mu m \) in Figs. 5(b) and 6 show the results of simultaneous diffraction and scattering that could deviate from the results owing to the coherent optics.

### 4. Conclusions

In this paper, we presented the formulae of the transfer-matrix method to conduct partial-coherence analyses of 3D periodic microstructures. Two numerical analyses of the propagation of free-space Gaussian beams and the diffraction/scattering for the liquid-crystal grating are mainly applied taking into consideration of different spatial partial coherences to verify the validity of this work, and we obtained reasonable results. The algorithms are also devoted to being a feasible study for the future study of light through turbid mediums that will be related to interactions between the fluctuated electron/dipole motions and the decoherence/extinction behaviors of light.

### Acknowledgements

This work was supported in part by the National Science Council of the Republic of China under Contract Nos. NSC 99-2811-E-001-003 and NSC 98-2622-E-001-001-CC2.
Appendix A: Generation of Two-Dimensional Spatial Correlated Noises

In the following context, we introduce the generation of the fluctuation function $\eta(r)$ by the discrete Fourier method, and thereby realize the E field describing partial coherence $(0 < \tau < \infty)$ using eq. (17). First, we introduce the Fourier transform for the noise phase $\eta$ in 2D momentum space:

$$
\eta(k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta(r) \exp(-ik \cdot r) \, dr,
$$

(A-1)

$$
\eta(r) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta(k) \exp(ik \cdot r) \, dk,
$$

(A-2)

or alternatively in a discrete representation of $N \times N$ grid data with the spatial interval $\Delta$ for our case:

$$
\eta_{g,h} = \Delta^2 \sum_{u,v=0}^{N-1} \eta_{u,v} \exp\left[-\frac{2\pi i}{N} (g \cdot u + h \cdot v)\right],
$$

(g,h = 0,1,…,N−1),

(A-3)

$$
\eta_{u,v} = \frac{1}{N^2 \Delta^2} \sum_{g,h=0}^{N-1} \eta_{g,h} \exp\left[i \frac{2\pi}{N} (g \cdot u + h \cdot v)\right],
$$

(u,v = 0,1,…,N−1).

(A-4)

Here, $u$ and $v$ ($g$ and $h$) are the indices for the spatial (momentum) space. The correlation function of the momentum space corresponding to that of the spatial space in eq. (18) can then be derived as

$$
\langle \eta(k) \eta(k') \rangle = 4\pi^2 \delta(k + k') \gamma(k)
$$

$$
= 4\pi^2 \delta(k + k') \cdot 2\pi \tau^2 \exp \left(- \frac{1}{2} k^2 \tau^2 \right),
$$

(A-5)

where $\gamma(k)$ is the Fourier transform of $\gamma(r)$ defined by eqs. (A-1)–(A-4). The discrete representation of eq. (A-5) is

$$
\langle \eta_{g,h} \eta_{g',h'} \rangle = N^2 \Delta^2 \gamma_{g,h} \delta_{g,g'} \delta_{h,h'}.
$$

(A-6)

Here, $\gamma_{g,h}$ is the discrete representation of $\gamma(k)$ and exhibits the symmetry of $\gamma_{g,h} = \gamma_{N-g,h} = \gamma_{g,N-h} = \gamma_{N-g,N-h}$. Accordingly, with the discrete correlation equation in eq. (A-6), the noise phase $\eta_{g,h}$ can be generated in the discrete momentum space using the equation:

$$
\eta_{g,h} = N \Delta \sqrt{\gamma_{g,h}} \alpha_{g,h}
$$

(A-7)

in which $\alpha_{g,h}$ is the Gaussian random number (complex number) with the average being zero, and it is conditioned as $\langle \alpha_{g,h} \alpha_{g',h'} \rangle = \delta_{g,g'} \delta_{h,h'}$, which corresponds to eq. (A-6). Definitely, these requirements for $\alpha_{g,h}$ can be realized by a simple process as follows:

$$
\alpha_{0,0} = a_{0,0}, \quad \alpha_{N/2,0} = a_{N/2,0}/\sqrt{2},
$$

$$\alpha_{N/2,0} = b_{N/2,0}, \quad a_{0,N/2} = b_{0,N/2}/\sqrt{2},
$$

$$\alpha_{N-0} = a_{N-0} / \sqrt{2},
$$

$$\alpha_{N-0} = b_{N-0} / \sqrt{2},
$$

$$\alpha_{0,0} = a_{0,0} + ib_{0,0},
$$

$$\alpha_{0,N} = a_{0,N} + ib_{0,N}/\sqrt{2},
$$

$$\alpha_{0,N} = b_{0,N}/\sqrt{2},
$$

$$\alpha_{0,0} = a_{0,0} - ib_{0,0},
$$

$$\alpha_{0,N} = a_{0,N} - ib_{0,N}/\sqrt{2}.
$$

Here, $a_{g,h}$ and $b_{g,h}$ ($g,h = 1,2,…,N/2−1$) are independent Gaussian random numbers (real numbers) with an average of zero and a variance of one. Note that it is necessary to generate $N \times N$ independent Gaussian random numbers for $N \times N$ fluctuations of $\eta_{g,h}$ exhibiting periodic characteristics. To obtain $N \times N$ fluctuations of $\eta_{g,h}$ without periodic characteristics, it is necessary to generate $2N \times 2N$ fluctuations of $\eta_{g,h}$ and take the average of $\eta_{g,h}$.

Finally, $\eta_{u,v}$ in the discrete spatial space can be obtained straightforwardly by the Fourier transform of $\eta_{g,h}$ in eq. (A-4), such that the electric field $E_{g,h}(x,y) = E \exp[i \pi \eta(x,y)]$ in the studied $xz$ region can be evaluated. A further treatment of $\eta_{u,v} \rightarrow \eta_{u,v} = \tilde{\eta}$, in which $\tilde{\eta}$ means the average noise over the $N \times N$ grid data, is executed.
to ensure the condition \( \langle \eta(r) \rangle = 0 \). The algorithm can be extended straightforwardly to arbitrary \( N \times M \) grid data. Numerical results of the noise phase \( \eta(x; z) \) with \( \tau = 5, 1, 0.5 \mu m \) are illustrated in Fig. A-1 for reference.

### Appendix B: Generation of Three-Dimensional Spatial Correlated Noises

Parallel to the process used for 2D cases in Appendix A, the generation of 3D noise phase \( \eta(r) \) by the discrete Fourier method is described in this section. Similarly, it is straightforward to extend the Fourier transform to the noise phase \( \eta(k) \) in a 3D momentum space:

\[
\eta(k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta(r) \exp(-i \mathbf{k} \cdot \mathbf{r}) \, d\mathbf{r}, \tag{B-1}
\]

or alternatively in discrete representations of \( N \times N \times N \) grid data with spatial interval \( \Delta \):

\[
\eta_{g,h,k} = \Delta^3 \sum_{u,v,\sigma=0}^{N-1} \eta_{u,v,\sigma} \exp \left[ -\frac{2\pi}{N} (g \cdot u + h \cdot v + \kappa \cdot \sigma) \right],
\]

\[
\eta_{u,v,\sigma} = \frac{1}{N^3 \Delta^3} \sum_{g,h,k=0}^{N-1} \eta_{g,h,k} \exp \left[ i \frac{2\pi}{N} (g \cdot u + h \cdot v + \kappa \cdot \sigma) \right],
\]

\[
\eta_{g,h,k} = \frac{1}{N^3 \Delta^3} \sum_{u,v,\sigma=0}^{N-1} \eta_{u,v,\sigma} \exp \left[ -\frac{2\pi}{N} (g \cdot u + h \cdot v + \kappa \cdot \sigma) \right].
\]

Here, \( u, v, \) and \( \sigma \) (\( g, h, \) and \( k \)) are the indices for the spatial (momentum) space. The correlation function of \( \eta(k) \) corresponding to that of \( \eta(r) \) in spatial space in eq. (18) can then be shown as

\[
(\eta(k) \eta(k')) = 8\pi^3 \delta(k + k') \gamma(k) = 8\pi^3 \delta(k + k') \cdot \delta^{(3)} \exp \left( -\frac{1}{2} k^2 r^2 \right),
\]

where \( \gamma(k) \) is obtained by Fourier transform of \( \gamma(r) \) defined by eqs. (B-1)-(B-4). The discrete representation of eq. (B-5) is

\[
(\eta_{g,h,k} \eta_{g',h',k'}) = N^3 \Delta^3 \gamma_{g,h,k} \delta_{g',N-g} \delta_{h',N-h} \delta_{k',N-k}.
\]

Here, \( \gamma_{g,h,k} \) is the discrete representation of \( \gamma(k) \) and exhibits the symmetry of \( Y_{g,h,k} = Y_{-g,-h,-k} = Y_{g,-h,k} = Y_{g,n,-h,k} = Y_{g,-h,-k} = Y_{g,-h,n-k} = Y_{g,n,-h,k} = Y_{g,-h,n-k} \). Accordingly, by the discrete correlation in eq. (B-6), the noise phase \( \eta_{g,h,k} \) can be generated in the discrete momentum space by

\[
\eta_{g,h,k} = N^3 \Delta^3 \frac{1}{\sqrt{\gamma_{g,h,k}}} \alpha_{g,h,k},
\]

in which \( \alpha_{g,h,k} \) is the Gaussian random number (complex number) with the average being zero, and it is conditioned as \( (\alpha_{g,h,k} \alpha_{g',h',k'}) = \delta_{g,-g'} \delta_{h,-h'} \delta_{k,-k'} \), which corresponds to eq. (B-6). Definitely, these requirements for \( \alpha_{g,h,k} \) can be realized by a simple process as follows:

\[
\alpha_{0,0,0} = a_{0,0,0},
\]

\[
\alpha_{N/2,N/2,0} = a_{N/2,N/2,0},
\]

\[
\alpha_{N/2,0,0} = a_{N/2,0,0},
\]

\[
\alpha_{0,N/2,0} = a_{0,N/2,0},
\]

\[
\alpha_{0,0,N/2} = a_{0,0,N/2},
\]

\[
\alpha_{N/2,N/2,0} = b_{N/2,N/2,0},
\]

\[
\alpha_{N/2,0,0} = b_{N/2,0,0},
\]

\[
\alpha_{0,N/2,0} = b_{0,N/2,0},
\]

\[
\alpha_{g,0,0} = \frac{1}{\sqrt{2}} (a_{g,0,0} + ib_{g,0,0}),
\]

\[
\alpha_{N-g,0,0} = \frac{1}{\sqrt{2}} (a_{g,0,0} - ib_{g,0,0}),
\]

\[
\alpha_{g,N,0,0} = \frac{1}{\sqrt{2}} (a_{g,0,0} + ib_{g,N,0}),
\]

\[
\alpha_{N-g,N,0,0} = \frac{1}{\sqrt{2}} (a_{g,0,0} - ib_{g,N,0}),
\]

\[
\alpha_{g,0,N,0} = \frac{1}{\sqrt{2}} (a_{g,0,0} + ib_{0,g,0}),
\]

\[
\alpha_{0,N-g,0} = \frac{1}{\sqrt{2}} (a_{N-g,0,0} + ib_{0,g,0}),
\]

\[
\alpha_{0,N,0} = \frac{1}{\sqrt{2}} (a_{N,0,0} + ib_{0,g,0}),
\]

\[
\alpha_{N,0,0} = \frac{1}{\sqrt{2}} (a_{N,0,0} + ib_{0,g,0}),
\]

\[
\alpha_{N,0,0} = \frac{1}{\sqrt{2}} (a_{N,0,0} + ib_{0,g,0}).
\]
\[ \alpha_{N-g,N-h_0} = \frac{1}{\sqrt{2}} (a_{g,h_0} - ib_{g,h_0}), \]
\[ \alpha_{N-g,h_0} = \frac{1}{\sqrt{2}} (a_{N-g,h_0} + ib_{N-g,h_0}), \]
\[ \alpha_{g,N-h_0} = \frac{1}{\sqrt{2}} (a_{N-g,h_0} - ib_{N-g,h_0}), \]
\[ \alpha_{g,h_0,N/2} = \frac{1}{\sqrt{2}} (a_{g,h_0,N/2} + ib_{g,h_0,N/2}), \]
\[ \alpha_{N-g,h_0,N/2} = \frac{1}{\sqrt{2}} (a_{N-g,h_0,N/2} - ib_{N-g,h_0,N/2}), \]
\[ \alpha_{g,0,k} = \frac{1}{\sqrt{2}} (a_{g,0,k} + ib_{g,0,k}), \]
\[ \alpha_{g,0,N-k} = \frac{1}{\sqrt{2}} (a_{g,0,N-k} - ib_{g,0,N-k}), \]
\[ \alpha_{g,N,0,k} = \frac{1}{\sqrt{2}} (a_{g,N,0,k} - ib_{g,N,0,k}), \]
\[ \alpha_{g,0,k,N/2} = \frac{1}{\sqrt{2}} (a_{g,0,k,N/2} + ib_{g,0,k,N/2}), \]
\[ \alpha_{N-g,0,N-k} = \frac{1}{\sqrt{2}} (a_{N-g,0,N-k} - ib_{N-g,0,N-k}), \]
\[ \alpha_{g,N/2,k} = \frac{1}{\sqrt{2}} (a_{g,N/2,k} + ib_{g,N/2,k}), \]
\[ \alpha_{g,N/2,N-k} = \frac{1}{\sqrt{2}} (a_{g,N/2,N-k} - ib_{g,N/2,N-k}), \]
\[ \alpha_{0,k,h} = \frac{1}{\sqrt{2}} (a_{0,k,h} + ib_{0,k,h}), \]
\[ \alpha_{0,k,N-k} = \frac{1}{\sqrt{2}} (a_{0,k,N-k} - ib_{0,k,N-k}), \]
\[ \alpha_{0,N,k} = \frac{1}{\sqrt{2}} (a_{0,N,k} + ib_{0,N,k}), \]
\[ \alpha_{0,N,k,N/2} = \frac{1}{\sqrt{2}} (a_{0,N,k,N/2} - ib_{0,N,k,N/2}), \]
\[ \alpha_{N/2,k,h} = \frac{1}{\sqrt{2}} (a_{N/2,k,h} + ib_{N/2,k,h}), \]
\[ \alpha_{N/2,k,N-k} = \frac{1}{\sqrt{2}} (a_{N/2,k,N-k} - ib_{N/2,k,N-k}), \]
\[ \alpha_{N/2,h,k} = \frac{1}{\sqrt{2}} (a_{N/2,h,k} + ib_{N/2,h,k}), \]
\[ \alpha_{N/2,h,N-k} = \frac{1}{\sqrt{2}} (a_{N/2,h,N-k} - ib_{N/2,h,N-k}), \]
\[ \alpha_{g,h,k} = \frac{1}{\sqrt{2}} (a_{g,h,k} + ib_{g,h,k}), \]
\[ \alpha_{N-g,h,k} = \frac{1}{\sqrt{2}} (a_{N-g,h,k} - ib_{N-g,h,k}), \]
\[ \alpha_{g,h,N,k} = \frac{1}{\sqrt{2}} (a_{g,h,N,k} + ib_{g,h,N,k}), \]
\[ \alpha_{N-g,h,N,k} = \frac{1}{\sqrt{2}} (a_{N-g,h,N,k} - ib_{N-g,h,N,k}), \]
\[ \alpha_{g,h,N/2,k} = \frac{1}{\sqrt{2}} (a_{g,h,N/2,k} + ib_{g,h,N/2,k}), \]
\[ \alpha_{N-g,h,N/2,k} = \frac{1}{\sqrt{2}} (a_{N-g,h,N/2,k} - ib_{N-g,h,N/2,k}), \]
\[ \alpha_{g,h,N/2,N-k} = \frac{1}{\sqrt{2}} (a_{g,h,N/2,N-k} + ib_{g,h,N/2,N-k}), \]
\[ \alpha_{N-g,h,N/2,N-k} = \frac{1}{\sqrt{2}} (a_{N-g,h,N/2,N-k} - ib_{N-g,h,N/2,N-k}). \]

Here, \( a_{g,h,k} \) and \( b_{g,h,k} \) \((g,h,k = 1,2,\ldots,N/2 - 1)\) are independent Gaussian random numbers (real numbers) with an average of zero and a variance of one. Note that it is necessary to generate \( N \times N \times N \) independent Gaussian random numbers for \( N \times N \times N \) fluctuations of \( \eta_{g,h,k} \), exhibiting periodic characteristics. To obtain \( N \times N \times N \) fluctuations of \( \eta_{g,h,k} \) without periodic characteristics, it is necessary to generate \( 2N \times 2N \times 2N \) fluctuations of \( \eta_{g,h,k} \) and take that of \( \eta_{g,h,k} \in \{1,2,\ldots,N\} \).

Finally, \( \eta_{g,h,k} \) in the discrete spatial space can be obtained straightforwardly by the Fourier transform of \( \eta_{g,h,k} \) in eq. (B-4), such that the electric field \( \mathbf{E}_g(x,y,z) = \mathbf{E} \exp(i\eta(x,y,z)) \) in the studied \( x,y,z \) space can be evaluated.

A further treatment of the discrete representation of the noise phase \( \eta_{g,h,k} \rightarrow \eta_{g,h,k} - \bar{\eta} \), in which \( \bar{\eta} \) means the average noise over the \( N \times N \times N \) grid data, is executed to ensure the condition \( \langle \eta(x) \rangle = 0 \). The algorithm can be extended straightforwardly to arbitrary \( N \times M \times O \) periodic grid data. Numerical results of the noise phase \( \eta(x,y,z) \) with \( \tau = 5,1 \mu m \) are illustrated in Fig. B-1 for reference.

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