Dynamics of entanglement and quantum states transitions in spin-qutrit systems under classical dephasing and the relevance of the initial state

Arthur Tsamouo Tsokeng, Martin Tchoffo and Lukong Cornelius Fai
Mesoscopic and Multilayer Structures Laboratory, Department of Physics, Dschang School of Science and Technology, University of Dschang, PO Box: 67 Dschang, Cameroon
E-mail: mtchoffo2000@yahoo.fr

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Abstract
We provide a detailed analysis of the time-evolutions of free and bound entangled states in a two-qutrit system under the effects of local/non-local classical dephasing noises. Namely the Ornstein-Uhlenbeck (OU) and the static noise are considered. The importance of the initial state is also analyzed by considering two completely different initially entangled states and their respective locally equivalent states by means of local unitary operation (LUO). The qutrits isotropic and bound entangled states are studied and we analyze their dynamical evolution to infer quantum states transition in the system. We demonstrate the absence of state transitions for isotropic-like states, unlike bound entangled-like states where free entangled states may lose distillability. Disentanglement however occurs for independents environments no matter the initial state. Meanwhile, for a common environment we show that indefinite free entanglement survival can be achieved (suppression of distillability sudden death) by converting the initial state using the LUO, hence saving free entanglement in the bipartite high dimensional quantum system, useful for quantum information and processing purposes.

1. Introduction
Entanglement, a unique and particular feature of quantum systems (compared to classical ones), is a phenomenon in which the latter systems exhibit correlations not possible for classical systems. It is a particular and very important resource for a wide number of quantum information processing including quantum computation, metrology and communication [1]. As such, it has received more extensive studies from a theoretical [2, 3], as well as experimental point of view [4–6]. Entangled states have allowed the development and improvement of a great variety of information protocols including quantum dense coding [7, 8], quantum teleportation [9, 10], quantum key distribution [11, 12], quantum repeaters [13], quantum teleporting [14], sensitive measurements [15], as well as quantum technologies and photonics [16–18], among others. Notwithstanding, a major challenge of experimental implementations of quantum computation and communication is the unwanted interactions between the real quantum system and its environment. The latter interactions induce the so-called decoherence processes [19], enhancing mixedness of quantum states acting on the Hilbert space, thus destroying quantum correlations within the quantum system. Decoherence may be particularly destructive to highly nonclassical and most potentially useful entangled states.

Moreover, both theoretical and experimental studies have revealed that entanglement does not always decay in an asymptotic way and can exhibit sudden death phenomena [20–22]. On the other hand, revival and/or preservation of quantum entanglement can also be induced by the external environment [23–25]. Both phenomena have been observed in quantum systems under environmental noise with different classical or quantum properties [26, 27]. Regarding these behaviors, environmental noises can be put into two categories according to their intrinsic characteristics, namely: Markovian (with instantaneous selfcorrelations) [20] and non-Markovian (associated to environments with memory, potentially leading to the non-monotonic dynamics
of entanglement) [28]. Thus, it is very important to analyze, characterize and optimize dynamical properties of the various kinds of environmental noise on entanglement dynamics in open quantum systems, which are of particular importance in practical quantum information processing.

Previous works mostly concentrated on quantum correlations dynamics for mixed states in 2-dimensional quantum systems (qubits) as in [29–33], just to list a few. However, as compared to qubits, maximally entangled qudits violate local realism more strongly and are less affected by noise [34, 35]. Thus, it is peculiar to characterize the dynamics of quantum correlations in systems of higher dimensions, so as to construct a useful parallel with the more extensively studied case of entanglement. In fact, higher-dimensional quantum systems can be used to improve the efficiency of quantum information processing [4, 34, 36]. In this sense, the dynamics of the high-dimensional bipartite entangled systems have been investigated [37, 38]. It has been revealed that higher dimensional systems may have advantages over the qubit ones, as they provide higher channel capacities, more secure cryptography and superior quantum gates [39–41]. In view of this fact, there have been some investigations regarding qutrit systems in the recent years, among which [42–48] just to cite a few examples.

On the other hand, bipartite entangled states can be divided into free-entangled states (FES) and bound entangled states (BES) [2, 49]. FES can be distilled under local operations and classical communication (LOCC), whereas, BES cannot be distilled to pure state entanglement. However, it is interesting that some bound entanglement can be distilled by certain procedures [50] or interaction with auxiliary systems [38, 51, 52]. These properties provide new quantum communication schemes, including secure quantum key distribution [53], remote information concentration [54], activation of teleportation fidelity [38], superactivation [52] and convertibility of pure entangled states [55]. It has been shown that certain free-entangled states of qudit-qutrit systems become non-distillable in a finite time under the influence of classical noise. Such a behavior has been termed distillability sudden death (DSD) [56].

The aim of this paper is to investigate the dynamics of free and bound entanglements in a physical model of two independent spin-qutrit particles. The subsystems are initially entangled and then coupled to either local or collective classical dephasing noise sources. Namely the static noise, recently used for describing electron transport and photon propagation in disordered structures [37, 58] and the Gaussian Ornstein-Uhlenbeck (OU) noise, which describes a normal diffusion process [59–61]. The static and the OU noises respectively simulate a non-Markovian and a Markovian environment. Furthermore, different initial states for bipartite entanglement are examined, namely the qudits isotropic and bound entangled states. By applying a local unitary operation (LLO), we also consider their locally equivalent initial states. While a LLO does not affect the static entanglement, it may have an important effect on the entanglement future trajectories [62, 63]. We use analytical techniques to detect and quantify entangled states. Specifically, FES are detected by means of negativity [64], a measure based on the Peres-Horodecki separability criterion, while for BES, we use the realignment criterion [65, 66]. Hence we analyze the above mentioned entanglement measures time-evolutions to investigate DSD phenomena as well as quantum states transitions.

The remainder of this paper is organized as follows: in section 2 we describe our physical model and define the mathematical requirements of our study. In section 3, we present the analytical expressions obtained and discuss the corresponding graphical results of their dynamics for the various cases considered. Finally, section 4 is devoted to conclusions.

2. Physical model and mathematical tools

2.1. The physical model

In the last decades, qubits have been recognized as promising units for information processing in quantum computers. But for some problems, three level or ternary systems may be applied as well. These three level systems are called qutrits. A qutrit computer.

In this section, we describe a model consisting of two non-interacting qudits $Q_a$ and $Q_b$ initially entangled and subject to classical dephasing noisy environments. The two independent qudits are coupled to noisy environments both locally and collectively as depicted in figure 1. The local coupling refers to the situation where each qudit is coupled to an independent environment $E_{a(b)}(t)$, while the collective/common coupling means that both qudits interact with the same noisy environment $C(t)$.

The dynamics of our two-qudit system under dephasing is governed by the following Hamiltonian:

$$\mathcal{H}(t) = \sum_{j=a,b} \left[ E_0 + g_L L_j(t) + g C(t) \right] S_j^{(j)}$$

where $S_j^{(j)} = |0_j\rangle \langle 0_j| - |2_j\rangle \langle 2_j|$ is the spin–1 operator in the subspace of qudit $j = a, b$. $E_0$ is the single qudit energy in the absence of dephasing. The latter is here introduced by means of stochastic processes $L_j(t)$ and $C(t)$.
which are the random variables to describe the different conditions under which the two subparties undergo decoherence due to the surroundings. $g_i$ and $g$ are the coupling strengths of each qutrit to the local and non-local dephasing noise channels respectively. We consider two types of system-environment interactions, namely the case of independent environments ($g_i = \nu = 0$ and $g = 0$), as well as the case of a common/collective environment ($g_i = 0$ and $g = \nu 
eq 0$).

The dynamics of quantum correlations under dephasing have been investigated in higher dimensional and hybrid quantum systems with few approaches [42, 45, 70–72]. The aim of this study is to analyze its effects due to environmental noise models which can be described classically, that are present within numerous quantum systems recognized as useful resources for quantum information processing tasks.

In fact, the two different kinds of classical noise that we consider in the current investigation as external dephasing environments are the Ornstein-Uhlenbeck (OU) noise and the static noise (SN). The former [59–61], describes a normal diffusion process that is Gaussian in nature [73], stationary and Markovian with the Lorentzian spectrum $S(\omega) = 4\gamma/\omega^2 + 4\gamma^2$. The OU process is characterized by the autocorrelation function $\langle \delta L_i(t)\delta L_j(t') \rangle = \langle \delta C(t)\delta C(t') \rangle = \frac{\gamma}{\omega} \exp(-\gamma |t-t'|). \gamma$ here assumed to be the same for independent and collective noise channels, plays the role of the memory parameter and defines the inverse of the correlation time (i.e. spectral width of the process).

Next is the static noise, due to its relevance in phenomena like Anderson localization or the transition from quantum to classical random walks [57, 58, 74]. Physically, this kind of noise can result from lattice disorder like in the specific case of coupled array of waveguides. To model the static noise in this study, the adimensional parameters $L_i(t)\equiv L$ and $C(t)\equiv C$ are assumed to be time-independent random variables following the flat probability distribution given by [58, 75]:

$$P(X) = \begin{cases} \frac{1}{\Delta_m} & \text{for } |X - d_0| \leq \frac{\Delta_m}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Here, $X = \{L; C\}$ represents the stochastic variable. $d_0$ denotes the mean value of the distributions, while $\Delta_m$ quantifies the degree of disorder of the local and collective environments. The autocorrelation function of the stochastic parameter $X$ is given by $\langle \delta X(t)\delta X(0) \rangle = \Delta_m^2/12$. This results in a power spectrum described by a $\delta$-function centered on zero frequency. Attributed to this is the long characteristic time of the noise, longer than the system-environment coupling. As such, the static noise bears the characteristic of non-Markovian noise with non-vanishing memory effects.

Once the choice of the noises parameters is performed, the unitary time-evolution operator corresponding to the various realizations of the selected stochastic processes is written as:

$$U(\{X\}, t) = \exp \left[ -i \int_0^t H(t') dt' \right] \quad \text{(hereafter } \hbar = 1)$$

The specific dynamics of the system is obtained by applying the latter to the initial state of the system. Its time-evolving state (density matrix) under the influence of the selected stochastic processes is evaluated by averaging the global state over the different noise realizations $\{X\}$. Namely $\{X\} = \{L_a(t), L_b(t)\}$ (for independent environments: $g_a = g_b = \nu, g = 0$) and $\{X\} = \{C(t)\}$ (for a common environment: $g_a = g_b = 0, g = \nu$). Such an average is given by:

$$\rho(t) = \langle \rho(\{X\}, t) \rangle = \langle U(\{X\}, t) \rho(0) U(\{X\}, t) \rangle_{\{X\}}.$$  \hspace{1cm} (4)

In equation (4), $\rho(0)$ stands for the initial state at which the system is being initially entangled. Specifically, we consider four initial states in the present paper. The first two are the qutrit isotropic state (IS) and the bound entangled state (BES) respectively given by:
\[ \rho_p = \rho_p(0) = p|\psi_+\rangle\langle \psi_+| + \frac{1-p}{9} I_9, \]

and

\[ \rho_\beta = \rho_\beta(0) = \frac{2}{7} |\psi_\beta\rangle\langle \psi_\beta| + \frac{\beta}{7} |\sigma_+\rangle\langle \sigma_+| + \frac{5-\beta}{7} |\sigma-\rangle\langle \sigma-|. \]

In the above equations, \(|\psi_+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)\) is the two-qutrit maximally entangled state, \(p \in [0; 1]\) is the purity of the state (acting as an entanglement parameter). \(I_9\) is the 9 × 9 identity matrix corresponding to full separable states. \(|\sigma_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\) and \(|\sigma-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\) are separable states. \(\beta \in [2; 5]\) is the entanglement parameter. Recall that \(\rho_\beta\) is separable for \(2 \leq \beta \leq 3\), bound entangled for \(3 \leq \beta \leq 4\), and free entangled for \(4 \leq \beta \leq 5\) [38].

The two other states we consider are the locally equivalent states \(\tilde{\rho}_p\) and \(\tilde{\rho}_\beta\) which are obtained from equations (5) and (6) respectively. They are given by

\[ \tilde{\rho}_p = U \rho_p U^\dagger \]

and

\[ \tilde{\rho}_\beta = U \rho_\beta U^\dagger. \]

\(U = \mathbb{I}_3 \otimes \theta\), with \(\theta = |0\rangle\langle 1| + |1\rangle\langle 0| + |2\rangle\langle 2|\), is a local unitary operation (LUO). It does not change the nature of the entangled states. Hence, maximally entangled states are converted into other maximally entangled states, while separable states are converted into other separable states. However, it is worth noting that LUOs do not affect the stationary entanglement, but can have a profound impact on the entanglement future trajectories [62, 63]. Once the time-evolved states have been derived, we now need to quantify entanglements in the composite system.

### 2.2. Entanglement measures

We define here the criteria used in quantifying the entanglements in the system and to analyze their time-evolutions. Entanglement in bipartite qutrit systems can be divided into free and bound entanglements [2, 49] and both forms are investigated in the present paper. States are said to be bound entangled if they have a positive partial transpose (PPT), therefore being non-distillable under LOCC [49], which is the opposite of free entangled states. The latter have a negative partial transpose (NPT) and are distillable by means of LOCC. As such, it is suggested that any entanglement measure based on the Peres–Horodecki separability criterion (such as negativity) cannot be suitable for the quantification of bound entanglement, since it is essentially a NPT criterion and a free entanglement measure. However, one may use the realignment criterion [65, 66] to detect certain bound entangled states (BES).

Thus, in the present study we use the negativity [64] to measure free entanglement, while bound entanglement is quantified using the realignment criterion. For a two-particle state \(\rho \equiv \rho_{ab}(t)\), negativity and realignment criterion are respectively defined as

\[ N(\rho) = \frac{\|\rho^{T_b}\| - 1}{2}, \]

and

\[ R(\rho) = \max \left[ 0, \frac{\|\rho^R\| - 1}{2} \right]. \]

Where \(\| A \| = \text{Tr} \sqrt{AA^\dagger}\) is the trace norm, \(\rho^{T_b}\) is the partial transpose of state \(\rho\) with respect to subsystem \(j = a, b\). \((\rho^b)_{nj,m} = \rho_{mn,j}\) is the realigned density matrix. Both \(N(\rho) > 0\) and \(R(\rho) > 0\) indicate that the state is entangled, \(N(\rho) = 0\) and \(R(\rho) > 0\) indicates that the state is bound entangled and \(N(\rho) > 0\) means that the state is free entangled. Note that there are various BES and a single criterion is not capable to detect all of them [76]. Thus it is worth noting that the realignment criterion detects certain but not all BES. As such, we suggest the results of the present paper to be valid only for the states here considered.

### 3. Results

#### 3.1. Ornstein–Uhlenbeck (OU) noise

Under the influence of dephasing OU processes during a time interval \([0; t]\), each qutrit accumulates either the phase factor \(\phi(t) = -\nu \int_0^t L_j(t') dt'\) or \(\phi(t) = -\nu \int_0^t C(t') dt'\) respectively for independent environments (ie) and collective environment (ce) coupling. Thus, the average to be performed in equation (4) is realized over each of the above phase factors. In both system-environment coupling setups here considered, this is evaluated with:
\[ \rho_{ie}^a(t) = \langle \langle \rho(\phi_2(t), \phi_3(t)) \rangle \rangle_{\phi_2} \]
\[ \rho_{ce}^a(t) = \langle \rho(\phi(t)) \rangle_{\phi}. \]

The latter averages are achieved by utilizing the characteristic function of Gaussian random process with zero mean \([77, 78]\), and the results are expressions of the form:

\[
D_n(t) = \langle e^{i\mu(t)} \rangle = \exp \left[ -\frac{n^2\mu}{2\gamma} (e^{-\gamma t} + \gamma t - 1) \right]
\]

with \( n \in \mathbb{N} \). The time-evolved states of the system under independents or a collective dephasing OU channel(s) and for any of the initial states considered have the following form given by

\[
\rho_{\text{ite (ce)}}(t) = \begin{pmatrix}
\rho_{11} & 0 & 0 & 0 & \rho_{15} & 0 & 0 & 0 & \rho_{19} \\
0 & \rho_{22} & 0 & \rho_{24} & 0 & 0 & 0 & 0 & \rho_{29} \\
0 & 0 & \rho_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{24} & 0 & \rho_{44} & 0 & 0 & 0 & 0 & \rho_{29} \\
\rho_{15} & 0 & 0 & 0 & \rho_{55} & 0 & 0 & 0 & \rho_{15} \\
0 & 0 & 0 & 0 & \rho_{66} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho_{77} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \rho_{88} & 0 & 0 \\
\rho_{19} & \rho_{29} & 0 & \rho_{29} & \rho_{15} & 0 & 0 & 0 & \rho_{99}
\end{pmatrix}
\]

Where the analytical expressions of the non-zero matrix elements are particularly related the form of the initial state and the type of system-environment interaction as follows:

3.1.1. Isotropic states:
In the case of the two isotropic-like states equations (5) and (7), we first have:

\[
\rho_{33} = \rho_{66} = \rho_{77} = \rho_{88} = \frac{1}{9}(1 - p), \quad \rho_{99} = \frac{1}{9}(1 + 2p),
\]

and specifically for \( \rho_{p} \) we additionally have:

\[
\rho_{11} = \rho_{55} = \rho_{99}, \quad \rho_{22} = \rho_{44} = \rho_{33}, \quad \rho_{24} = \rho_{29} = 0,
\]

\[
\rho_{15} = \frac{p}{3} \Omega_{ie(ce)}^+, \quad \rho_{19} = \frac{p}{3} \Pi_{ie(ce)}^+.
\]

whereas for \( \rho_{p} \), we rather obtain:

\[
\rho_{11} = \rho_{55} = \rho_{33}, \quad \rho_{22} = \rho_{44} = \rho_{99}, \quad \rho_{15} = \rho_{19} = 0,
\]

\[
\rho_{24} = \frac{p}{3} \Lambda_{ie(ce)}^+, \quad \rho_{29} = \frac{p}{3} \Pi_{ie(ce)}^+.
\]

In the above expressions:

\[
\Omega_{ie} = D_1^i(t), \quad \Omega_{ce} = D_2(t), \quad \Gamma_{ie} = D_3^i(t), \quad \Gamma_{ce} = D_4(t),
\]

\[
\Lambda_{ie} = \Omega_{ie}, \quad \Lambda_{ce} = 1, \quad \Pi_{ie} = D_1(t) D_2(t), \quad \Pi_{ce} = D_3(t).
\]

where the function \( D_n(t) \) is the function defined in equation (13).

Hence, from the shape of equation (14), taking into account equations (15)–(17), analytical expressions for the negativity are derived (respectively for \( \rho_{p} \) and \( \rho_{p}^c \)) after straightforward calculations as

\[
\mathcal{N}_{ie(ce)}(p, t) = \frac{1}{2} \left[ 3|\rho_{11}| + \sum_{i=\pm} (2A_{ie(ce)}^+ + B_{ie(ce)}^+) - 1 \right]
\]

and

\[
\mathcal{N}_{ce(ie)}(p, t) = \frac{1}{2} \left[ 3|\rho_{22}| + \sum_{i=\pm} (2X_{ie(ce)}^+ + Y_{ie(ce)}^+) - 1 \right].
\]

Wherein the above expressions

\[
A_{ie(ce)}^\pm = |\rho_{22} \pm \rho_{15}|, \quad B_{ie(ce)}^\pm = |\rho_{22} \pm \rho_{29}|,
\]

\[
X_{ie(ce)}^\pm = |\rho_{11} \pm \rho_{29}| \quad \text{and} \quad Y_{ie(ce)}^\pm = |\rho_{11} \pm \rho_{24}|
\]

given by equations (15)–(17). The corresponding dynamics are presented in figure 2. The dynamics of free entanglement as a function of the entanglement parameter \( p \) is presented in figure 2. For independent environments (upper panels), the free entanglement time-evolution is almost insensitive to the initial state under
consideration, and vanish asymptotically in a monotonic manner with time. The asymptotic profile is more pronounced with entanglement becoming more resistant to the destructive surroundings as the purity of the initial state rises. In the lower panels of the same figure (common environment), we note the occurrence of more pronounced entanglement sudden death phenomena as the entanglement parameter decreases. Note that the latter phenomenon is observed only for the state $\rho_p$. Meanwhile, for the initial state $\rho_{p\circ}$ (as $p \geq 0.25$), complete disentanglement can never occur (figure 2–(d)). Entanglement decays towards a non-zero saturation level that is a linear increasing function of the purity $p$ and formally given by:

$$\mathcal{N}_{\text{sat}}(p, \nu \to \infty) = \frac{1}{9}(4p - 1).$$

In fact, the panel (d) on figure 2 reveals that after certain amount of time (increasing with $p$), there is a sort of rearrangement in the mixed states of the quantum system. The latter rearrangement may create means for entanglement preservation thus avoiding disentanglement of the two-qutrit system. Note that when the two-qutrit system is initially entangled in the state $\rho_p$, a common environment is more fatal to free entanglement than independent ones. While the opposite is found for the state $\rho_{p\circ}$, where a common environment is so much more of saving entanglement than independent ones. Let us recall also that the states $\rho_p$ and $\rho_{p\circ}$ are related to each other through a local unitary operation (LUO). We thus suggest the indefinite entanglement survival to be a result of the latter conversion which enhances the rearrangement of the mixed states of the system so as to save some of its entanglement indefinitely.

### 3.1.2. Bound entangled states

Meanwhile, for bound entangled-like states equations (6) and (8), we rather find

$$\rho_{33} = \frac{5 - \beta}{21}, \quad \rho_{66} = \frac{\beta}{21}, \quad \rho_{99} = \frac{2}{21}.$$  

For $\rho_{3\circ}$, we further have

$$\rho_{11} = \rho_{33} = \rho_{99}, \quad \rho_{22} = \rho_{77} = \rho_{66}, \quad \rho_{44} = \rho_{88} = \rho_{33}, \quad \rho_{24} = \rho_{29} = 0,$$

$$\rho_{15} = \frac{2}{21} \Omega_{\text{set}(ce)}, \quad \rho_{19} = \frac{2}{21} \Gamma_{\text{set}(ce)}.$$  

And for $\rho_{3\circ}$, straightforward calculations yield

$$\rho_{11} = \rho_{88} = \rho_{66}, \quad \rho_{22} = \rho_{44} = \rho_{99}, \quad \rho_{33} = \rho_{77} = \rho_{33}, \quad \rho_{15} = \rho_{19} = 0,$$

$$\rho_{24} = \frac{2}{21} \Lambda_{\text{set}(ce)}, \quad \rho_{29} = \frac{2}{21} \Pi_{\text{set}(ce)}.$$  

Here, the negativities for initial states $\rho_{3\circ}$ and $\rho_{3\circ}$ respectively read

$$\mathcal{N}_{\text{set}(ce)}(\beta, t) = \frac{1}{4} \left[ 6|\rho_{11}| + \sum_{s=\pm} (2\beta_{s\text{set}(ce)} + \Omega_{s\text{set}(ce)}) - 2 \right].$$

---

**Figure 2.** Evolution of free entanglement (negativity) as a function of dimensionless time $\nu t$ and purity $p$ in the two-qutrits system under independent (upper panels) and a common (lower panels) dephasing Ornstein-Uhlenbeck (OU) environment(s) with $\gamma / \nu = 0.1$. Left panels: the initial state is $\rho_p$ and right panels: the initial state is $\rho_{p\circ}$. 

Furthermore, the respective realignment criteria read

\[ \mathcal{R}_{\text{ent}}(\beta, t) = \frac{1}{2} \left[ \text{nn} + \text{nn} + \text{nn} + \text{nn} + \text{nn} + \text{nn} - 2 \right] \]  

With

\[ P_{\text{ent}}^\pm = |\rho_{22} + \rho_{33} \pm \sqrt{4\rho_{15}^2 + (\rho_{22} - \rho_{33})^2}|, \quad Q_{\text{ent}}^\pm = |\rho_{22} + \rho_{33} \pm \sqrt{4\rho_{19}^2 + (\rho_{22} - \rho_{33})^2}| \]

\[ R_{\text{ent}}^\pm = |\rho_{11} + \rho_{33} \pm \sqrt{4\rho_{29}^2 + (\rho_{11} - \rho_{33})^2}|, \quad T_{\text{ent}}^\pm = |\rho_{11} + \rho_{33} \pm \sqrt{(\rho_{11} - \rho_{33})^2}|. \]

Furthermore, the respective realignment criteria read

\[ \mathcal{R}_{\text{ent}}(\beta, t) = \frac{1}{2} \left[ 8|\rho_{33}| + 4|\rho_{19}| + 2|\rho_{11} + \rho_{22} + \rho_{33}| + \sum_{i=\pm} V_{\text{ent}}^i - 2 \right] \]

and

\[ \mathcal{R}_{\text{ent}}(\beta, t) = \frac{1}{2} \left[ 4|\rho_{29}| + 2|\rho_{11} + \rho_{22} + \rho_{33}| + \sum_{i=\pm} W_{\text{ent}}^i - 1 \right]. \]

Where

\[ V_{\text{ent}}^\pm = \sqrt{(\rho_{11} - \rho_{22})^2 + (\rho_{11} - \rho_{33})^2} \]

and

\[ W_{\text{ent}} = \sqrt{(\rho_{11} - \rho_{22})^2 + (\rho_{11} - \rho_{33})^2}. \]

Results are presented in figures 3 and 4. Figures 3 and 4 show the time-evolutions of both free and bound entanglements (respectively quantified by negativity and the realignment criterion) as a function of the entanglement parameter \( \beta \). For both independent and common dephasing OU channel(s), bound entanglement suffers sudden death (panels (b) and (e) in both figures). Under the effects of independent environments (figure 3), both forms of the entanglement are nearly insensitive to the initial states here considered, hence leading approximately to the same qualitative behavior. Analyzing panels 3-(c) and 3-(f) clearly demonstrates quantum states transitions with regions of distillability sudden death (DSD: \( \mathcal{N} = 0 \) and \( \mathcal{R} > 0 \)) and regions of free entangled states (FES: \( \mathcal{N} > 0 \) and \( \mathcal{R} = 0 \)). A similar dynamical behavior is seen on the upper panels of figure 4. The DSD region shrinks a bit due to the type of system environment interaction. However, results also demonstrate that the two-qutrit system keeps indefinitely a finite amount of free entanglement, when prepared in the initial state \( \rho_{ij} \), and subject to a common OU environment (figure 4-(d)). It is worth mentioning that the saturation value of free entanglement is again a linear increasing function of the entanglement parameter \( \beta > 4 \) given by:

\[ \mathcal{R}_{\text{ent}}(\beta, \nu t \rightarrow \infty) = \frac{\beta - 4}{42} (\sqrt{4\beta} - 5). \]
This suggests once again, the relevance of the initial state (hence that of the LUO) and the physical setup in enhancing entanglement robustness in high-dimensional bipartite quantum systems. The non-vanishing free entanglement ensures the full escape from DSD phenomena as shown on figure 4-(f).

### 3.2. Static noise

On the other hand, when the two-qutrit system is subject to static disorder in either independent or a collective environment(s) interactions, the average giving the full system dynamics is performed with the following integrals using the same procedure presented by [75]:

\[
\bar{p}_a(t) = \int_{x^-}^{x^+} \int_{x^-}^{x^+} P(L_a)P(L_b)\rho(L_a, L_b, t) dL_a dL_b \bar{p}_a(t) = \int_{x^-}^{x^+} P(C)\rho(C, t) dC. \tag{30}
\]

Where \( x^\pm = d_0 \pm \frac{\Delta_m}{2} \), \( \rho(L_a, L_b, t) = U(L_a, L_b, t) \rho(0)U^*(L_a, L_b, t) \), \( \rho(C, t) = U(C, t) \rho(0)U^*(C, t) \) and \( P(\chi) \) is the probability distribution given by equation (2). The averages of the time-evolving states equation (30) contain in their explicit expressions, some terms that we denote with the terms \( G_n(t) \), and which the explicit forms are defined as

\[
G_n(t) = \frac{2\sin(n\eta \Delta_m/2)}{n\eta \Delta_m}, \quad (n \in \mathbb{N}). \tag{31}
\]

Interestingly, the above averages result in density matrices having exactly the same shape as given by equation (14). Furthermore, taking into account the initial states for any type of the system-environment coupling considered in the present paper, all the expressions from equation (15) to equation (28) remain formally the same, provided that one simply performs the following change of function

\[
D_n(t) \rightarrow G_n(t) \quad \text{with} \quad n = \{1; 2; 3; 4\}. \tag{32}
\]

This suggests that both classical dephasing noise models considered in the present study induce at a sufficiently long time, the same dynamical behavior for the composite quantum system.

#### 3.2.1. Isotropic states

Figure 5 shows that under independent environments, the evolution of free entanglement (negativity) is the same, independently of the initial isotropic-like state considered. The entanglement evolution may display sudden death and revival phenomena, a particular feature of the non-Markovian nature of the static noise. Revival phenomena are more pronounced and more robust under a common environment and specifically when the initial state is \( \rho_p \). In particular for the other isotropic-like initial state (\( \rho_b \)), no sudden death is found and entanglement survives in the system with a finite level governed by equation (21). Unlike figure 2, a common environment is here more effective for entanglement robustness than independent ones, no matter the initial state considered. Moreover, the LUO (hence the initial state) is again shown here to play a significant role on the evolution of entanglement.

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![Figure 4](image-url) ---

**Figure 4.** Evolution of negativity (left panels), realignment criterion (center panels) and states transitions (right panels: contour plots) as a function of dimensionless time \( \nu t \) and entanglement parameter \( \beta \) in the two-qutrits system under a common dephasing Ornstein-Uhlenbeck process with \( \gamma / \nu = 0.1 \). Upper panels: the initial state is \( \rho_p \), and Lower panels: the initial state is \( \rho_b \).
3.2.2. Bound entangled states

In figures 6 and 7, except very few discrepancies (however negligible), the dynamical behavior of both free and bound entanglement is not very different from the one observed from figures 3 and 4. Hence these results will no more be discussed here in details (we refer the reader to the discussions on figures 3 and 4 in section 3.1.2) and we present only the major points. Quantum state transitions are observed as well as DSD phenomena. The latter are however be skipped when the initial state is transformed by mean of a LUO, and the bipartite high dimensional quantum system interacting with a common environment.

It is worth emphasizing that isotropic-like states are always separable in their negative partial transposed (NPT) regions. Hence they cannot exhibit quantum states transition since the entangled states are here always free entangled. In this sense, we have not presented the dynamics of the realignment criterion (bound entanglement) for the isotropic states since it is unnecessary in this case. We can simply say that the feature is not affected even with the application of the LUO and entanglement in isotropic qutrit states is always distillable under LOCC.
4. Conclusions

In summary, we have performed a detailed analysis of the dynamics of free and bound entanglements in a bipartite qutrit system, initially entangled and besieged by the effects of stochastic environments. Namely the static noise and the Ornstein-Uhlenbeck (OU) noise, both modeled to induce a dephasing dynamic in the system in either local or non-local interactions. The role of the initial state, hence that of local unitary operations (LUOs) on the disentanglement process is further analyzed. Specifically, both the two-qutrit isotropic and bound entangled states, as well as their locally equivalent states (by means of a LUO) are probed. Thus, we have analyzed the evolutions of free and bound entanglement to describe the various transitions between the qutrit-qutrit resultant quantum states.

Our results have demonstrated that free entanglement under the effect of external dephasing environments decay either monotonically or not depending on the Markovian or non-Markovian nature of the noise as expected. Meanwhile bound entanglement exclusively suffers of sudden death phenomena. For independent environments, the evolution of free entanglement is almost insensitive to any transformation of the initial state using a LUO. Hence exhibiting the same decaying profile, only related to the intrinsic nature of the dephasing environment. On the other hand, for a collective environment, the initial state is of a significant importance, specifically when the LUO is applied. The latter leads (after a short finite time) to a kind of rearrangement of the mixed quantum states, so as to save indefinitely some residual entanglement within the system.

We have further shown that in accordance with the separability features of isotropic-like states, quantum state transitions cannot be obtained even after applying a LUO to the initial state. Whereas, in bound entangled-like states, transitions from free to bound entangled states can be observed under the effects of both classical dephasing noise models considered. However we have also demonstrated that the latter transitions can be avoided with a simple manipulation of the initial state (by means of a LUO) and embedding the subparties under a common source of external noise. As such, free entanglement can be saved indefinitely in the composite quantum system, a very important and useful resource for a good number of applications in quantum information and processing tasks.

ORCID iDs

Martin Tchoffo @ https://orcid.org/0000-0003-4989-2751

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