Spectral Form Factor for Time-dependent Matrix model

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Abstract: Time-dependent matrix model can be converted to a two matrix model with coupling via path integral formulation. Two point correlation function for such a two matrix model made of $M_1$ and $M_2$, each over Gaussian unitary ensemble has been studied. Fourier transform of it gives the Spectral Form Factor, which comes with rounding off near Heisenberg Time for this case. This rounding off has been obtained for same matrix interaction ($M_1 - M_1$ correlation) and different matrix interaction ($M_1 - M_2$ correlation function)\textsuperscript{[Eq.-(1.1),Eq.-(1.2)]} with first order correction. Rounding off behavior has a $(1/N)$ expansion behavior and can be related to instanton effect near Heisenberg time. With duality relation being considered this two matrix model correlation has been related to open intersection number which connect this to Riemann surface with boundary removed.
1 Introduction

Matrix models are proven tools for the study of random geometry, strings, two-dimensional quantum gravity, chaos, M-theory (non-perturbative string theory), spins on a random lattice, etc. The time-dependent matrix model is considered with explicit time dependence for eigenvalues of the matrix. Two matrix model from the time-dependent matrix model can be constructed following [1, 2]. Two matrix model was discussed in the context of the Ising model coupled to gravity in [3]. It has been subsequently generalized for 2-d quantum gravity coupled to (p,q) conformal fields. Time-dependent matrix model and two matrix model has been studied by [2, 4–6]. Two matrix model and 2-d quantum gravity relation has been established and extensively studied in [7–11]. With a specific potential of two matrices, critical regime for various (p,q)-rational string theories has been evaluated in [12]. Matrix
models are also an important part for describing black holes and chaos within it. This has a huge section of literature [13–15].

In recent times random tensor model has been related to the random matrix model and spectral form factor. They are extensively discussed in [16–19]. Spectral form factor for the Ising chain has been discussed in [20–22]. Ramp region of SFF in late time has a universal nature due to chaotic behavior. It has been solved numerically and analytically for the SYK model in [23–26]. The same has been discussed for the gravity model in [27, 28]. Two-point correlation function and spectral form factor for two Matrix model has been studied in [1, 29, 30] previously. Here we have followed the model of Gaussian matrix with an external source, previously considered in [1].

We have computed two types of two-point correlation function:-

1. Between the same matrices (kind of auto correlation function or diagonal correlation function) defined as -

   \[ U^{M_1}_{0}(z_1, z_2) = \left\langle \frac{1}{N} \text{Tr} e^{iz_1 M_1} \frac{1}{N} \text{Tr} e^{iz_2 M_1} \right\rangle \]  

   We call it same matrix interaction

2. In other case two different matrix gives the two point correlation function (off-diagonal correlation function)

   \[ U_0(z_1, z_2) = \left\langle \frac{1}{N} \text{Tr} e^{iz_1 M_1} \frac{1}{N} \text{Tr} e^{iz_2 M_2} \right\rangle \]  

   We call it different matrix interaction

Starting with a time dependent matrix M which has a correlation function varying with time, we choose its Hamiltonian to be of type

\[ H = \frac{1}{2} \text{Tr}(p^2 + M^2) \]  

Here M is a time dependent hermitian matrix of size \( N \times N \) and \( p \) is defined by \( p = \dot{M} \) and \( M \) is \( N \times N \). Following the same procedure as in [1] with application of path integral method the time-dependent model can be readily generalized to a two matrix model with time dependent coupling in between the matrices. Reducing this time varying matrix model to two matrix model gives wide application.

Spectral Form Factor (SFF) is defined as Fourier-transform of the two point correlation function and it readily gives information about basic properties of the system like integrability or time-reversal symmetry. Two-point correlation function for a system with Hamiltonian “H” defined as in [31]

\[ \rho^{(2)}(\lambda, \mu) = \left\langle \frac{1}{N} \text{Tr} \delta(\lambda - H) \frac{1}{N} \text{Tr} \delta(\mu - H) \right\rangle \]
For a two matrix case with interaction there can be two type of correlation, one between same matrices and other one between different matrices. We have studied both of them in finite large N limit with analytical solutions.

In section 3 we have studied the two-point correlation function and its Fourier transform, Spectral form factor. Using Kernel method it can be given as a exact solution of Hermite polynomial summation. We then choose Kazakov contour-integral representation [32] of correlation function for analytical solution and use saddle point method to calculate its exact form for arbitrary order of N. Then we evaluate SFF and average it following same path as in [1, 31]. We then compare both the solutions for specific N cases.

The universality of the ramp in SFF is related to the universal Dyson kernel, which is obtained in the limit $N \rightarrow \infty$, with a fixed $N\Delta E$, where $\Delta E$ is a scaled short energy distance. To obtain this universal nature asymptotics of Hermite polynomials can be used. Hermite polynomials are the solution of the GUE kernel which in the previously described limit gives universal Dyson sine kernel as solution. Near Heisenberg time, the rounding behavior, which we will discuss may not be universal. However, the study of the rounding behavior may shed a light on important physics, such as non perturbative effect.

We have studied the rounding off behavior of SFF near Heisenberg time. This gives a smooth evolution from classical to quantum nature, and also most possibly can be connected with phase transition.

In section 4 we have studied the correlation function for same matrix interaction and again use contour integral and saddle point method to evaluate its analytical form. Its Fourier transform gives the spectral form factor and by studying both cases (Same matrix and different matrix interaction) we get complementary behavior between them.

In section 5 we looked into the $1/N$ expansion of correlation function which is important to study its connection with single/multi instantons type cases. After analytically deriving exact form of saddle point for different order we use it on different matrix interaction case. We get the perturbative nature of correction by this and compare with the solution by Hermite polynomial summation.

2 Time dependent matrix model

Consider a $N \times N$ Hermitian matrix $M$, with time dependence. [1]

Time dependent correlation function for such a matrix is defined by:-

$$
\rho(\lambda, \mu; t) = \langle \frac{1}{N} \text{Tr} \delta(\lambda - M(t_1)) \frac{1}{N} \text{Tr} \delta(\mu - M(t_2)) \rangle
$$

(2.1)
where \( t = t_1 - t_2 \) and \( t_1, t_2 \) are different times. Correlation function of this form is

\[
U(\alpha, \beta) = \frac{1}{N^2} \langle \text{Tr} e^{i \alpha M(t_1)} \text{Tr} e^{i \beta M(t_2)} \rangle
\]  

(2.2)

\( \alpha, \beta \) are the Fourier transform variables. This time dependent matrix model correlation function is equivalent to two matrix model on Gaussian ensemble correlation function. This can be shown by Path integral method and it holds for any finite \( N \).

Hamiltonian for time dependent matrix model:

\[
H = \frac{1}{2} \text{Tr}(p^2 + M^2)
\]  

(2.3)

\( M \) is dependent on \( t \) and \( p = \dot{M} \) Therefore

\[
U(\alpha, \beta) = \frac{1}{N^2} \langle \text{Tr} e^{i \alpha M(t_1)} \text{Tr} e^{i \beta M(t_2)} \rangle = \frac{1}{N^2} \langle 0 | e^{H(t_2-t_1)} (\text{Tr} e^{i \alpha M}) e^{-H t_2} | 0 \rangle
\]  

(2.4)

Now by Path integral formulation, we define

\[
\langle A | e^{-\bar{\beta} H} | B \rangle = \int_{M(\bar{\beta})=A, M(0)=B} DMe^{-\frac{1}{2} \text{Tr} \int_0^\beta (\dot{M}^2 + M^2) dt}
\]  

(2.5)

Then:

\[
U(\alpha, \beta) = \frac{1}{N^2} \int dAdB \langle 0 | e^{H t_1} | A \rangle \langle A | (\text{Tr} e^{i \alpha M}) e^{H(t_2-t_1)} (\text{Tr} e^{i \beta M}) | B \rangle \langle B | e^{-H t_2} | 0 \rangle
\]  

(2.6)

Now ground state energy of free independent \( N^2 \) fermions is \( \frac{N^2}{2} \), so

\[
\langle 0 | e^{H t_1} | A \rangle = e^{\frac{N^2}{2} t_1} e^{-\frac{1}{2} \text{Tr} A^2}
\]

\[
\langle B | e^{-H t_2} | 0 \rangle = e^{\frac{-N^2}{2} t_2} e^{-\frac{1}{2} \text{Tr} B^2}
\]  

(2.7)

Now solving

\[
\dot{M} = M
\]

\[
M(t) = P \cosh(t) + Q \sinh(t)
\]  

(2.8)

With initial condition \( M(0) = B, M(\bar{\beta}) = A; \)

\[
M(t) = B \cosh(t) + \frac{A - B \cosh(\bar{\beta})}{\sinh(\bar{\beta})} \sinh(t)
\]  

(2.9)

Now, \( d(\dot{M} M) = \dot{M}^2 + \dot{M} M - M \dot{M}^2 + M^2 \)

\[
\frac{1}{2} \text{Tr} \int_0^\beta (\dot{M}^2 + M^2) = \text{Tr} \int_0^\beta d(\dot{M} M) dt = \text{Tr}(\dot{M} M) \bigg|_0^\beta
\]

\[
= \text{Tr} \{ (BA \sinh(\bar{\beta}) + \frac{A - B \cosh(\bar{\beta})}{\sinh(\bar{\beta})} (A \cosh(\bar{\beta}) - B) \}
\]  

(2.10)

\[
= \left\{ \frac{1}{2} \frac{1}{\sinh(\bar{\beta})} \text{Tr} [(A^2 + B^2) \cosh(\bar{\beta}) - 2AB] \right\}
\]
Then $U(\alpha, \beta)$ becomes:

$$U(\alpha, \beta) = \frac{1}{N^2} \int dAdB e^{\frac{t^2}{2}(t_1-t_2)} e^{-\frac{1}{2}(TrA^2+TrB^2)} Tr e^{i\alpha A} Tr e^{i\beta B} \times \text{Exp}\{-\frac{1}{2\sinh(t)} Tr[(A^2 + B^2) \cosh(t) - 2AB]\}$$

$$U(\alpha, \beta) = \frac{1}{N^2} \left(\frac{e^t}{\sinh(t)}\right)^{\frac{N^2}{2}} \int dAdB(Tr e^{i\alpha A})(Tr e^{i\beta B}) \times e^{-\frac{1}{2\sinh(t)} Tr[(A^2 + B^2) e^t - 2AB]}$$

(2.11)

Changing variable $A \rightarrow (\sqrt{e^{-t} \sinh(t)}) \bar{A}$, $B \rightarrow (\sqrt{e^{-t} \sinh(t)}) \bar{B}$, $\alpha \rightarrow (\sqrt{\frac{e^t}{\sinh(t)}}) \bar{\alpha}$, $\beta \rightarrow (\sqrt{\frac{e^t}{\sinh(t)}}) \bar{\beta}$

$$U(\alpha, \beta) = \frac{1}{Z} \int d\bar{A}d\bar{B}(Tr e^{i\bar{\alpha} \bar{A}})(Tr e^{i\bar{\beta} \bar{B}}) \times e^{-\frac{1}{2\sinh(t)} Tr[(\bar{A}^2 + \bar{B}^2) e^t - 2\bar{A}\bar{B}]}$$

(2.12)

Where $Z = N^2$ and $\lambda, \mu$ are scaled down by a factor $\{e^{-t} \sinh(t)\}$ to compensate the previous change in variable. Where $c = e^{-t}$. Change in notation in two matrix model formulation. Let the matrix be $A \rightarrow M_1$, $B \rightarrow M_2$. There is a coupling between this matrix which time dependent and denoted by $c = e^{-t}$. The Gaussian distribution is given by

$$P_A(M_1, M_2) = \frac{1}{Z_A} e^{-H_{1,2}}$$

$$H_{1,2} = \frac{1}{2} Tr M_1^2 + \frac{1}{2} Tr M_2^2 - c Tr M_1 M_2 + Tr AM_1$$

(2.13)

3 Two Matrix model correlation function and dynamical form factor

Density of state $\rho(\lambda)$ derived by Fourier transform of $U_A(z)$

$$U_A(z) = \langle \frac{1}{N} Tr e^{izM_1} \rangle$$

(3.1)

We have considered an external matrix $A$ coupled to matrix $M_1$ acting as a source. At last step we will reduce it to zero to get our desired result. The eigenvalue of $M_1$ and $M_2$ are $r_i$ and $\xi_i$.

$$U_A(z) = \frac{1}{Z_A N} \int Tr e^{izM_1} \times e^{-\frac{1}{2} Tr M_1^2 - \frac{1}{2} Tr M_2^2 + c Tr M_1 M_2 + Tr AM_1} dM_1 dM_2$$

$$= \frac{1}{NZ} \sum_{\alpha=1}^{N} e^{i z r_{\alpha}} e^{-\frac{r_{\alpha}^2}{2} \sum_{i} r_i^2 - \frac{r_{\alpha}}{2} \sum_{i} \xi_i^2 + cN \sum_{i} r_i \xi_i - N \sum_{i} a_i r_i} \frac{\Delta^2(r) \Delta^2(\xi) \prod_i dr_i \prod_i d\xi_i}{\Delta(r) \Delta(\xi) \Delta(A) \Delta(r)}$$

(3.2)

Here we have used Harish-Chandra-Itzykson-Zuber formula to change the measure from integration over matrix to integration over eigenvalues of the matrix $\Delta(r) = \prod_{i<j} (r_i - r_j)$ is the Vandermonde determinant.

$$\int e^{Tr M_1 M_2 - Tr AM_1} dM_1 dM_2 = \int e^{N(\sum_{i} r_i \xi_i - \sum_{i} a_i r_i)} \frac{\Delta^2(r) \Delta^2(\xi) \prod_i dr_i \prod_j d\xi_j}{\Delta(r) \Delta(\xi) \Delta(A) \Delta(r)}$$

(3.3)
Now using the above expression in Eq:-(3.2) we first do the gaussian integral over \( \Pi dr_i \) and get the form as:-

\[
U_A(z) = \frac{1}{NZ_A \Delta(A)} \sum_{\alpha=1}^{N} \int \prod d\xi \Delta(\xi) e^{-\frac{c}{2} \sum \xi_i^2 + \sum (\xi_i - iz \delta_{i,\alpha} - a_i)^2} \\
= \frac{1}{NZ_A \Delta(A)} \sum_{\alpha=1}^{N} \prod_{i<j} \left\{ \frac{ic}{N(1-c^2)} \left( \delta_{i,\alpha} - \delta_{j,\alpha} \right) - c(a_j - a_i) \right\} e^{-\frac{c^2}{2N(1-c^2)} - \frac{i\alpha_0}{\sqrt{1-c^2}}} \\
= \frac{1}{NZ_A} \sum_{\alpha=1}^{N} \prod_{i<j} c \left( a_i - a_j - \frac{iz}{N(1-c^2)} \right) e^{-\frac{c^2}{2N(1-c^2)} - \frac{i\alpha_0}{\sqrt{1-c^2}}} \\
= -\frac{c\sqrt{1-c^2}}{iz} \int \frac{du}{2\pi i} \prod_{\gamma=1}^{N} \left( \frac{u - a_\gamma - \frac{iz}{N(1-c^2)}}{u - a_\gamma} \right)^N e^{-\frac{c^2}{2N(1-c^2)} - \frac{i\alpha_0}{\sqrt{1-c^2}}} \\
\]  

(3.4)

If we take the external source term to zero \((a_i \to 0)\)

\[
U_0(z) = -\frac{c\sqrt{1-c^2}}{iz} \int \frac{du}{2\pi i} \left( 1 - \frac{iz}{Nu\sqrt{1-c^2}} \right)^N e^{-\frac{i\alpha_0}{\sqrt{1-c^2}}} \\
\]  

(3.5)

Correlation function is defined as the Fourier transform of this function:-

\[
\rho(\lambda) = -\frac{c\sqrt{1-c^2}}{2\pi i} \int \frac{dz}{z} e^{-iz\lambda} \int \frac{du}{2\pi i} \left( 1 - \frac{iz}{Nu\sqrt{1-c^2}} \right)^N e^{-\frac{i\alpha_0}{\sqrt{1-c^2}}} \\
\]  

(3.6)

Two level correlation function and its Fourier transform - dynamical form factor (Spectral form factor) are important measure for a model. This two level correlation function has two parts. From two interacting matrix and between the same matrix.

For two different matrix it can be written as:-

\[
\rho^2(\lambda, \mu) = \int \int \frac{dz_1 dz_2}{(2\pi)^2} e^{-i z_1 \lambda - i z_2 \mu} U_0(z_1, z_2) \\
\]  

(3.7)

\(U_0(z_1, z_2)\) computed in the same with introduction of external source \((A)\) and then letting it to zero.

\[
U_0(z_1, z_2) = \left( \frac{1}{N} \text{Tr} e^{iz_1 M_1} \frac{1}{N} \text{Tr} e^{iz_2 M_2} \right) \\
\]  

(3.8)

So we start with external source matrix \(A\) (with eigen values \(a_i, ..., a_N\)). We introduced \(dr = \prod_{i=1}^{N} dr_i\) where \(r_i\) are the eigen values of matrix \(M_1\) and in same way \(d\xi = \prod_{i=1}^{N} d\xi_i\) for \(\xi_i\) being the eigen values of \(M_2\).

\[
U_A(z_1, z_2) = \frac{1}{N^2} \sum_{a_1, a_2}^{N} \int e^{iz_1 r_{\alpha_1} + iz_2 r_{\alpha_2}} e^{-\frac{c}{2} \sum r_i^2 - \frac{c}{2} \sum \xi_i^2 + cN \sum r_i \xi_i - N \sum a_i r_i} \frac{\Delta(\xi) dr d\xi}{\Delta(A)} \\
\]  

(3.9)
We perform the gaussian integral with linear term on \( dr \) and simplify the expression as:-

\[
U_A(z_1, z_2) = \frac{1}{N^2 \Delta(A)} \sum_{\alpha_1, \alpha_2}^N \int \Delta(\xi) d\xi e^{-\frac{N(1-c^2)}{2} \sum \xi_i^2 - \sum [cN(\frac{iz_1}{N} \delta_{i,\alpha_1} + a_j + \frac{iz_2}{c} \delta_{j,\alpha_2})] \xi_i} = \frac{1}{N^2} \sum_{\alpha_1, \alpha_2}^N \prod_{i<j}(a_i + iz_1 \delta_{i,\alpha_1}) + iz_2 \delta_{i,\alpha_2} - (a_j + iz_1 \delta_{j,\alpha_1}) - \frac{iz_2}{c} \delta_{j,\alpha_2}) \\
\times \exp \left[ - \frac{z_1^2}{2N(1-c^2)} - \frac{z_2^2}{2N(1-c^2)} - \frac{iz_1 a_{\alpha_1}}{1-c^2} - \frac{ic z_2 a_{\alpha_2}}{1-c^2} - \frac{c z_1 z_2}{N(1-c^2)} \delta_{\alpha_1,\alpha_2} \right] \tag{3.10}
\]

Changing the integral by contour integral with \( \alpha_1 = \alpha_2 \) and taking the external matrix tends to zero.

\[
U'_A(z_1, z_2) = -\frac{1}{iN(z_1 + \frac{z_2}{c})} \oint \frac{du}{2\pi i} \left[ 1 - \frac{i}{Nu}(z_1 + \frac{z_2}{c}) \right]^N \exp \left[ - \frac{z_1^2}{2N(1-c^2)} - \frac{z_2^2}{2N(1-c^2)} - \frac{iz_1}{1-c^2} + \frac{ic z_2}{c} \right] \tag{3.11}
\]

changing \( u \rightarrow \bar{u}(z_1 + \frac{z_2}{c}) \)

\[
\rho(\lambda, \mu) = \frac{i}{N} \int \frac{dz_1 dz_2}{4\pi^2} \oint \frac{d\bar{u}}{2\pi i} \left[ 1 - \frac{i}{N\bar{u}} \right]^N \exp \left[ - i(z_1 + z_2 c)(z_1 + \frac{z_2}{c}) - \frac{\bar{u}}{\sqrt{1-c^2}} \right] \tag{3.12}
\]

Using the transformation \( z_1 \rightarrow \frac{1}{\sqrt{1-c^2}}(z_1' - cz_1) \) and \( z_2 \rightarrow \frac{1}{\sqrt{1-c^2}}(z_2' - c z_2) \)

\[
\rho(\lambda, \mu) = \frac{i}{4(1-c^2)N\pi^2} \int \frac{dz_1'}{2\pi i} \frac{du}{2\pi i} \left[ 1 - \frac{i}{Nu} \right]^N \exp \left[ - (z_1')^2 + \frac{\mu - c\lambda}{\sqrt{1-c^2}} z_1' \right] \times \int \frac{dz_1'}{2\pi} \exp \left[ - \frac{(z_1')^2}{2N(1-c^2)} - cz_1' \left( \frac{z_1'}{N(1-c^2)} + iz_2' c\mu + \frac{i(\lambda - c\mu)}{\sqrt{1-c^2}} \right) \right] \tag{3.13}
\]

\[
\rho(\lambda, \mu) = \frac{iN(\lambda - c\mu)^2}{2\pi i} \int \frac{dz_2'}{2\pi i} \frac{du}{2\pi i} \left[ 1 - \frac{i}{Nu} \right]^N \exp \left[ - i\mu z_2' + Nu z_2' (\mu - \frac{\lambda}{c}) - \frac{iu z_2'}{1-c^2} - \frac{Nu^2(z_2')^2}{2c^2} - \frac{(z_2')^2}{2N(1-c^2)} \right] \tag{3.14}
\]

For \( \alpha_1 \neq \alpha_2 \)

\[
U_A(z_1, z_2) = -\frac{c}{z_1 z_2} \int \frac{du dv}{(2\pi i)^2} \left( 1 - \frac{i z_1}{Nu} \right)^N \left[ 1 - \frac{z_1 z_2}{c N^2(u - v - \frac{iz_2}{cN})} \right] \tag{3.14}
\]

\[
\times \left( 1 - \frac{i z_2}{c N v} \right)^N \exp \left[ - \frac{iz_1 u}{1-c^2} - \frac{iz_2 c v}{1-c^2} - \frac{z_1^2}{2N(1-c^2)} - \frac{z_2^2}{2N(1-c^2)} \right] \tag{3.14}
\]
3.1 Disconnected Spectral Form Factor

Disconnected correlation function from the disconnected part of Eq:(3.14) is decomposed to two parts then by Fourier transform we change to two point correlation function:

$$\rho^d(\lambda, \mu) = \int U_A^d(z_1, z_2)e^{-iz_1\lambda}e^{-iz_2\mu}d\lambda d\mu$$  \hspace{1cm} (3.15)

This gives the disconnected two point correlation function:

$$\rho^d(\lambda, \mu) = -c \left\{ \int \frac{dz_1}{2\pi z_1} \int \frac{du}{(2\pi i)} \left( 1 - \frac{iz_1}{Nu} \right)^N e^{-\frac{iz_1u}{1-c^2} - \frac{z_1^2}{2N(1-c^2)}} e^{-i\lambda} \right. \right. $$

$$\times \left. \left. \int \frac{dz_2}{2\pi z_2} \int \frac{dv}{(2\pi i)} \left( 1 - \frac{iz_2}{cNv} \right)^N e^{-\frac{iz_2v}{1-c^2} - \frac{z_2^2}{2N(1-c^2)}} e^{-i\mu} \right\}$$  \hspace{1cm} (3.16)

Now we move to one matrix model $\rho^1$, the density of state for matrix model, following [1] :-

$$P_A(M) = \frac{1}{Z} e^{-\frac{N}{2} TrM^2 - NTrAM}$$

$$U_A(t) = \frac{1}{N} Tr(e^{itM})$$

$$U_0(t) = -\frac{1}{it} e^{-\frac{t^2}{2N}} \int \frac{du}{2\pi i} e^{-itN(1 - \frac{it}{Nu})}$$

$$\rho^1(\lambda) = -\int \frac{dt}{2\pi} \int \frac{du}{2\pi i} \left( \frac{1}{it} \right) e^{-\frac{t^2}{2N} - itu + it\lambda} (1 - \frac{it}{Nu})^N$$  \hspace{1cm} (3.17)

Therefore two matrix two point correlation function Eq:- (3.16) is very similar to one matrix model density of states.

$$\rho_{M_1, M_2}^d(\lambda, \mu) = \rho_{M_1}^1(\lambda)\rho_{M_2}^2(\mu)$$  \hspace{1cm} (3.18)

After a Fourier transform and setting the values $\lambda = 0$ and $\mu = \omega$ we get the spectral form factor.

$$S(\tau) = \int \rho^d(0, \omega)e^{i\omega\tau}d\omega$$  \hspace{1cm} (3.19)
Figure 1. Spectral Form Factor disconnected part for two matrix model from Eq:- (3.16)

We have averaged this dynamical form factor over an interval \([0,t]\) and plot this average value:

\[
S_{\text{Average}} = \int_{0}^{t} S(\tau) d\tau
\]  

(3.20)

3.2 Connected Spectral Form Factor

Connected part has a form:-

\[
U_A(z_1, z_2) = -\frac{1}{N^2} \int \frac{du dv}{(2\pi i)^2} \left(1 - \frac{iz_1}{Nu}\right)^N \left(1 - \frac{iz_2}{cNv}\right)^N \frac{1}{u - v - \frac{iz_1}{Nu}} \frac{1}{u - v + \frac{iz_2}{cNv}} \times \exp \left[ - \frac{iz_1 u}{1 - c^2} - \frac{iz_2 cv}{1 - c^2} - \frac{1}{2N(1 - c^2)} \right]
\]

there are poles at \(u=0, v=0\). Then there is one more pole at \(v = u - \frac{iz_1}{N}\) gives same expression as for \(\alpha_1 = \alpha_2\). If \(z_1 \rightarrow z'_1 - iuN, z_2 \rightarrow z'_2 - ivcN\)

\[
\rho^2_c(\lambda, \mu) = -\oint \frac{dv}{2\pi i} \int \frac{dz}{2\pi i} \left(\frac{z_2}{cN}\right)^N \frac{1}{\frac{1}{v + i\mu N} + \frac{1}{\frac{1}{v + i\mu N}}} \times \exp \left[ -\frac{Nv^2}{2(1 - c^2)} - \frac{2N}{(1 - c^2)} + iz_2 \mu + uN\lambda \right]
\]

\[
\times \oint \frac{du}{2\pi i} \int \frac{dz}{2\pi i} \left(\frac{z_1}{cN}\right)^N \frac{1}{\frac{1}{u + iz_1 N} + \frac{1}{\frac{1}{u + iz_1 N}}} \times \exp \left[ -\frac{Nu^2}{2(1 - c^2)} - \frac{2N}{(1 - c^2)} + iz_1 \lambda + vcN\mu \right]
\]

(3.22)

So this expression is written :

\[
\rho^2_c = -K_N(\lambda, \mu) \tilde{K}_N(\lambda, \mu)
\]

(3.23)

then the kernel equation can be solved in two ways:-

1. Hermite polynomial summation

2. Saddle point approximation for the integrals
3.3 Kernel solution from hermite polynomial summation

In [1] authors has solved this via expressing the kernel as hermite polynomial in contour integral representation:

\[ H_n(\lambda) = \oint \frac{du}{2\pi i} \frac{n!}{u^{n+1}} e^{\lambda u - \frac{1}{2} u^2} \]  \hspace{1cm} (3.24)

with an auxiliary variable introduction this can be written as:

\[ H_n(\lambda) = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} \oint \frac{du}{2\pi i} \frac{n!}{u^{n+1}} e^{\lambda u + itu - \frac{1}{2} t^2} \]  \hspace{1cm} (3.25)

The kernel \( K_N(\lambda, \mu) \) can be represented as

\[ K_N(\lambda, \mu) = \sqrt{\frac{N}{2\pi}} \sum_{l=0}^{N-1} \frac{H_l(\sqrt{N}\lambda)H_l(\sqrt{N}\mu)}{l!} e^{-\frac{N}{2} \lambda^2} \]  \hspace{1cm} (3.26)

The summation of the series \((1 - (\frac{u}{\sqrt{N}})^N)/ (1 - \frac{u}{\sqrt{N}})\), shifting \( t \rightarrow \frac{t}{\sqrt{N}}, u \rightarrow -\frac{u}{\sqrt{N}}\)

\[ K_N(\lambda, \mu) = -\int_{-\infty}^{\infty} \frac{du}{2\pi i} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \left( -\frac{it}{uN} \right)^N \frac{1}{u + \frac{it}{\sqrt{N}}} e^{-\left(\frac{\sqrt{N}u^2}{2} + \frac{t^2}{2N} + it\lambda + u\mu\right)} \]  \hspace{1cm} (3.27)

Therefore the kernel in hermite polynomial term

\[ K_N(\lambda, \mu) = e^{-\frac{N}{2}(1-c^2)\mu^2} \frac{1}{N} \sum_{n=0}^{N-1} \frac{H_n(\beta\lambda)H_n(\beta\mu)}{n!} \] \hspace{1cm} (3.28)

\[ \bar{K}_N(\lambda, \mu) = e^{-\frac{N}{2}(1-c^2)\lambda^2} \frac{1}{N} \sum_{n=0}^{N-1} (c^n) \frac{H_n(\beta\lambda)H_n(\beta\mu)}{n!} \]

\( \beta = \sqrt{\frac{N}{2}(1-c^2)} \) We solve this Hermite polynomial representation and after Fourier transform we get the SFF which is again averaged over an interval;

![Graph](a) SFF for c=0.9, N=100, at whole range (b) SFF for c=0.9, N=10, at specific range to show the predicted behavior

**Figure 2.** Spectral Form Factor for two matrix model
3.4 Kernel solution from contour integral by Saddle point approximation

3.4.1 For Large N limit

At large N limit, considering \( \frac{i z_1}{N} \to 0, \frac{i z_2}{cN} \to 0 \)

\[
U_A(z_1, z_2) = -\frac{1}{N^2} \oint \frac{du dv}{(2\pi i)^2} \frac{e^{-\frac{iz_1 u}{1-c^2} - \frac{iz_2 v}{1-c^2}}}{(u - v)^2} e^{-\frac{z_1^2}{2N(1-c^2)} - \frac{z_2^2}{2N(1-c^2)}} \tag{3.29}
\]

\[
\rho(\lambda, \mu) = -\frac{1}{N^2} \oint \frac{dz_1 dz_2}{(2\pi i)^2} \oint \frac{du dv}{(2\pi i)^2} \frac{e^{-\frac{iz_1 u}{1-c^2} - \frac{iz_2 v}{1-c^2}}}{(u - v)^2} e^{-\frac{z_1^2}{2N(1-c^2)} - \frac{z_2^2}{2N(1-c^2)} - iz_1 \lambda - iz_2 \mu} \tag{3.30}
\]

After integral over \( z_1, z_2 \)

\[
\rho(\lambda, \mu) = \frac{1 - c^2}{2\pi N} \oint \frac{du dv}{(2\pi i)^2} \frac{e^{-\frac{N}{2(1-c^2)}(u+\frac{1-c^2}{u} + (1-c^2)\lambda)^2} \times e^{-\frac{N}{2(1-c^2)}(vc + \frac{1-c^2}{vc} + (1-c^2)\mu)^2}}{(u - v)^2} \tag{3.31}
\]

then,

\[
f_1(u) = -\frac{1}{2(1-c^2)}(u + \frac{1-c^2}{u} + (1-c^2)\lambda)^2
\]

\[
f_2(v) = -\frac{1}{2(1-c^2)}(vc + \frac{1-c^2}{vc} + (1-c^2)\mu)^2 \tag{3.32}
\]

Saddle points are solution of the equations \( \frac{df_1}{du} = 0, \frac{df_2}{dv} = 0 \)

\[
u^2 + \lambda(1 - c^2)u + (1 - c^2) = 0
\]

\[
u^2 + \frac{1-c^2}{c} \mu v + \frac{1-c^2}{c^2} = 0 \tag{3.33}
\]

Saddle point approximation of the integral around this points

\[
\rho^{(2)}(\lambda, \mu) = -\frac{1-c^2}{2\pi N^2} \sum \frac{e^{-N(f(u,v))}}{u(\lambda) - v(\mu)^2} \frac{1}{\sqrt{\frac{d^2 f_1}{dU^2} \frac{d^2 f_2}{dV^2}}} \tag{3.34}
\]

With transformation; \( \lambda \to \frac{2}{\sqrt{1-c^2}} \sin(\theta), \mu \to \frac{2}{\sqrt{1-c^2}} \sin(\phi) \) This gives the saddle points

\[
u = \pm \frac{1}{c} \sqrt{1-c^2} e^{\pm i\phi}
\]

3.4.2 For Finite N

Now we consider the finite N limit.

First kernel for this case:-

\[
K_N(\lambda, \mu) = \oint \frac{du}{2\pi i} \oint \frac{dz_2}{2\pi} \left( \frac{z_2}{cu N} \right)^N \frac{1}{u + \frac{i z_2}{cN}} e^{-\frac{N u^2}{2(1-c^2)} + \frac{z_2^2}{2N(1-c^2)} + iz_2 u + u N \lambda} \tag{3.35}
\]
changing $z \to zNc$

\[
K_N(\lambda, \mu) = cN \int \frac{du}{2\pi i} \int \frac{dz_N}{2\pi i} \frac{z_N^1}{u + iz_N^1} e^{-\frac{N(u^2 - z_N^2)}{2(1-c^2)}} e^{-\frac{N^2(z_N^2)^2}{2(1-c^2)} + iz_N^2 N\mu + uN\lambda}
\]

\[
K_N(\lambda, \mu) = cN \int \frac{du}{2\pi i} \int \frac{dz_N}{2\pi} \frac{1}{u + iz_N^1} e^{-N(ln(u) - ln(z_N^1)) + \frac{N^2(z_N^2)^2}{2(1-c^2)} - iz_N^1 c\mu + u\lambda}
\]

(3.36)

Saddle point equation for first kernel:

\[
f(z', u) = \frac{u^2}{2(1-c^2)} + \frac{c^2(z')^2}{2(1-c^2)} + ic\mu z' + \lambda u + \log(u) - \log(z')
\]

(3.37)

So solving the saddle point equation $\frac{\partial f}{\partial u} = 0, \frac{\partial f}{\partial z'} = 0$ and $\lambda \to \frac{2}{\sqrt{1-c^2}} \sin(\theta), \mu \to \frac{2}{\sqrt{1-c^2}} \sin(\phi)$

These gives the saddle points as :-

\[
\begin{bmatrix}
\sqrt{1-c^2} e^{-i\phi} \\
\sqrt{1-c^2} e^{i\phi} \\
\frac{\sqrt{c}}{c} e^{-i\phi} \frac{1}{1} \\
\frac{\sqrt{c}}{c} e^{i\phi} \frac{1}{1}
\end{bmatrix}
\]

Fluctuation around saddle points to get the solution of kernel:

\[
\rho^{(2)}(\lambda, \mu) = -\frac{1}{2\pi^2 N^2} \sum \frac{e^{-N(f(z', u))}}{u + iz'} \frac{1}{\sqrt{\frac{\partial^2 f}{\partial u^2} \frac{\partial^2 f}{\partial z'^2}}} 
\]

(3.38)

\[
K_N(\theta, \phi) = -\frac{i\sqrt{1-c^2} e^{i(\theta+\phi)}}{2\sqrt{\cos(\theta) \cos(\phi)}} \left( \left[ \text{Exp} \left[ \frac{1}{2} N_1(2 \log \left( \frac{z}{c} \right) + 2(i\theta + i\phi) - e^{-2i\theta} + 1) \right] \right] \right. 
\]

\[
- \frac{\text{Exp} \left[ \frac{1}{2} N_1(2 \log \left( \frac{z}{c} \right) - 2(i\theta - i\phi) - e^{-2i\theta} + e^{-2i\phi} ) \right]}{1 + ce^{i(\theta+\phi)}} 
\]

\[
+ \frac{\text{Exp} \left[ \frac{1}{2} N_1(2 \log(c) + 2(i\theta - i\phi) + e^{2i\theta} - c^{2i\phi} + i\pi) \right]}{e^{i\phi} - ce^{i\theta}} \right)
\]

(3.39)

Second kernel

\[
K_N(\lambda, \mu) = \int \frac{dv}{2\pi} \int \frac{dz_1}{2\pi} \frac{z_1^2}{vN^2} \frac{1}{v + iz_1} e^{-\frac{N^2(\frac{z_1^2}{v} - c^2)}{2(1-c^2)} + iz_1 \lambda + cv\mu N}
\]

(3.40)
changing $z \rightarrow zN$

$$K_N(\lambda, \mu) = N \oint \frac{dv}{2\pi i} \oint \frac{dz_1'}{2\pi i} \frac{v}{v + iz_1'} \frac{1}{e^{-\frac{Nz_1'^2}{2(1-c^2)} + \frac{\lambda^2}{2(1-c^2)} + iz_1'N\lambda + cv\mu N}}$$

$$K_N(\lambda, \mu) = N \oint \frac{dv}{2\pi i} \oint \frac{dz_1'}{2\pi i} \frac{1}{v + iz_1'} e^{-\frac{Nz_1'^2}{2(1-c^2)} + \frac{\lambda^2}{2(1-c^2)} + iz_1'N\lambda + cv\mu ln(v) - ln(z_1')} \quad (3.41)$$

So solving the saddle point equation $\frac{\partial f}{\partial u} = 0$, $\frac{\partial f}{\partial z_1'} = 0$ and $\lambda \rightarrow \frac{2}{\sqrt{1-c^2}} \sin(\theta), \mu \rightarrow \frac{2}{\sqrt{1-c^2}} \sin(\phi)$

saddle point equation for second kernel:-

$$\tilde{f}(z_1', u) = \left( c^2 \frac{u^2}{2} + \frac{(z_1')^2}{2(1-c^2)} + iz_1'\lambda + cv\mu + ln(v) - ln(z_1') \right) \quad (3.42)$$

So this gives the saddle points

$$\begin{bmatrix}
\frac{z_1'}{\sqrt{1-c^2} e^{i\theta} - \frac{i\sqrt{1-c^2} e^{-i\theta}}{c}} \\
\frac{-\sqrt{1-c^2} e^{i\theta} - \frac{i\sqrt{1-c^2} e^{-i\theta}}{c}}{\sqrt{1-c^2} e^{-i\theta} - \frac{i\sqrt{1-c^2} e^{i\theta}}{c}}
\end{bmatrix}$$

Fluctuation around saddle points to get the solution of kernel:-

$$\rho^{(2)}(\lambda, \mu) = -\frac{1 - c^2}{2\pi^2 N^2} \sum \frac{e^{-N(\tilde{f}(z_1', v))}}{v + iz_1'} \frac{1}{\sqrt{\rho^{(2)}_1 \rho^{(2)}_2}} \quad (3.43)$$

$$K_{22}(\theta, \phi) = \frac{i\sqrt{1-c^2} e^{\frac{i}{2}(\theta+\phi)}}{2\sqrt{\cos(\theta) \cos(\phi)}} \left( -\frac{\text{Exp}\left(\frac{1}{2}N_1(2\text{Log}(\frac{1}{c}) + 2i(\theta - \phi) - e^{-2i\theta} + e^{-2i\phi})\right)}{e^{i\theta} - ce^{i\phi}} + \frac{\text{Exp}\left(\frac{1}{2}N_1(-2\text{Log}(ic) + 2i(\theta + \phi) - e^{-2i\theta} + e^{2i\phi})\right)}{1 + ce^{i(\theta+\phi)}} + \frac{\text{Exp}\left(-\frac{1}{2}N_1(2\text{Log}(-ic) + 2i(\theta - \phi) + e^{2i\theta} - e^{2i\phi})\right)}{e^{i\phi} - ce^{i\phi}} \right) \quad (3.44)$$

Two Point Correlation Function then represented as :

$$\rho^2_c[\theta, \phi] = -cN^2 K_{11}[\theta, \phi] \times K_{22}[\theta, \phi] \quad (3.45)$$
\[ \rho^2_{\epsilon}[\theta, \phi] = \left( \frac{ic\sqrt{1 - c^2}e^{i(\theta + \phi)}}{(ce^{i\phi} - e^{i\theta})\sqrt{c^2(1 + e^{2i\phi})(1 + e^{2i\phi})}} + \frac{ic\sqrt{1 - c^2}e^{i\theta}}{2(c + e^{i(\theta + \phi)})\sqrt{c^2}e^{i(\theta - \phi)}\cos(\theta)\cos(\phi)} \right) 
\]

\[ \times \left( - \frac{2(ce^{i\theta} - e^{i\phi})\sqrt{c^2}\cos(\phi)(\cos(\theta + \phi) - i\sin(\theta + \phi))}{ic\sqrt{1 - c^2}} \right) + \left( \frac{ic\sqrt{1 - c^2}e^{i(\theta + \phi)}}{(ce^{i\phi} - e^{i\theta})\sqrt{c^2(1 + e^{2i\phi})(1 + e^{2i\phi})}} \right) \frac{ic\sqrt{1 - c^2}e^{i\phi}}{2(c + e^{i(\theta + \phi)})\sqrt{c^2}e^{i(\theta - \phi)}\cos(\theta)\cos(\phi)} \right) \]

\[ (3.46) \]

For \( \theta = 0 \)

\[ \rho^2_{\epsilon}[0, \phi] = \frac{1}{4}(1 - c^2)e^{i\phi}\sec(\phi) \left( \frac{2\text{ Exp}(\frac{1}{2}N_1(-2\text{Log}(ic) + 2i\phi + e^{2i\phi} - 1) + i\phi)}{c^2 + e^{2i\phi}} \right) \]

\[ - \left( \frac{1 + e^{i\phi}\text{ Exp}(\frac{1}{2}N_1(-2\text{Log}(ic) - 2i\phi + e^{-2i\phi} - 1))}{c^2 + e^{2i\phi}} \right) \]

\[ \times \left( 2\text{ Exp}(\frac{1}{2}N_1(-2\text{Log}(ic) - 2i\phi + e^{-2i\phi} - 1)) \right) \]

\[ - \frac{1 + e^{2i\phi}}{c^2 + e^{2i\phi}} \]

\[ + \left( \frac{2\text{ Exp}(\frac{1}{2}N_1(-2\text{Log}(ic) + 2i\phi + e^{2i\phi} - 1) + i\phi)}{c^2 + e^{2i\phi}} \right) \]

\[ (3.47) \]

Figure 3. Correlation function behavior for two matrix model using Eq. (3.48)
Transforming the equation in terms of $\omega = \frac{2 \sin \phi}{\sqrt{1 - c^2}}$

$$\rho_c[0, \omega] = \frac{(1 - c^2)(\sqrt{4 - (1 - c^2)}\omega^2 + i\sqrt{1 - c^2}\omega)}{4\sqrt{4 - (1 - c^2)}\omega^2} \left[ \left( -\sqrt{4 - (1 - c^2)}\omega^2 + i\sqrt{1 - c^2}\omega \right) \left( -1 + \frac{1}{2} \left( \sqrt{4 - (1 - c^2)}\omega^2 + i\sqrt{1 - c^2}\omega \right) \right) \times \text{Exp} \left\{ \frac{N_1}{2} \left( \sqrt{4 - (1 - c^2)}\omega^2 - i\sqrt{1 - c^2}\omega^2 - 2\text{Log}(-ic) - 1 \right) \right\} \right] +$$

$$\frac{(\sqrt{4 - (1 - c^2)}\omega^2 + i\sqrt{1 - c^2}\omega)^2 \text{Exp} \left( \frac{N_1}{2} \left( \sqrt{4 - (1 - c^2)}\omega^2 + i\sqrt{1 - c^2}\omega \right)^2 - 2\text{Log}(ic) - 1 \right)}{2(-c^2 + \frac{1}{4} (\sqrt{4 - (1 - c^2)}\omega^2 + i\sqrt{1 - c^2}\omega)^2} \times$$

$$\left[ \left( \sqrt{4 - (1 - c^2)}\omega^2 + i\sqrt{1 - c^2}\omega \right) \text{Exp} \left( \frac{N_1}{2} \left( \sqrt{4 - (1 - c^2)}\omega^2 + i\sqrt{1 - c^2}\omega \right)^2 - 2\text{Log}(-ic) - 1 \right) \right]$$

$$+ \frac{(\sqrt{4 - (1 - c^2)}\omega^2 - i\sqrt{1 - c^2}\omega)^2 \text{Exp} \left( \frac{N_1}{2} \left( \sqrt{4 - (1 - c^2)}\omega^2 - i\sqrt{1 - c^2}\omega \right)^2 - 2\text{Log}(ic) - 1 \right)}{-1 + \frac{1}{4} c^2 (\sqrt{4 - (1 - c^2)}\omega^2 - i\sqrt{1 - c^2}\omega)^2} \text{Exp} \left\{ \frac{N_1}{2} \left( \sqrt{4 - (1 - c^2)}\omega^2 - i\sqrt{1 - c^2}\omega \right)^2 - 2\text{Log}(ic) - 1 \right\} \right]$$

This two point correlation function has a similar behavior of Bessel function but it has a exponential decay within it as shown in Fig:-3. Now we need to compute Fourier transform of this two point correlation function to get the dynamical form factor.

$$S(\tau) = \int \frac{d\omega}{2\pi} e^{i\omega \tau} \rho_c^{(2)}(E, \omega) \quad (3.49)$$

We choose the singularities of the above equation to find the integral by residue theorem. The singularities are given by the following equation:

$$-1 + \frac{1}{4} c^2 (\sqrt{4 - (1 - c^2)}\omega^2 + i\sqrt{1 - c^2}\omega)^2 = 0 \quad (3.50)$$

$$\sqrt{4 - (1 - c^2)}\omega^2 = 0$$

This gives the saddle points:

$$\begin{bmatrix} \omega \to -\frac{2}{\sqrt{1 - c^2}} \\ \omega \to \frac{1}{\sqrt{1 - c^2}} \\ \omega \to -\frac{i\sqrt{c^2 - 1}}{c} \\ \omega \to \frac{i\sqrt{c^2 - 1}}{c} \end{bmatrix}$$

First two are branch points so we choose our contour avoiding this two point and compute residue w.r.t other two points.
3.5 Average of SFF

SFF can be averaged over an interval \((0,t)\) and plotting that shows a continuous behavior instead of kink at Heisenberg time

\[
\langle S(\tau) \rangle = \int_0^t d\tau S(\tau) = S_{\text{avg}}(t) \tag{3.51}
\]

Figure 4. Spectral Form Factor for two matrix model from Eq:(3.49)

Figure 5. Spectral Form Factor connected part average for two matrix model from Eq:-(3.51)
We can compare solution of SFF from both method. Both have a continuous transition behavior at Heisenberg time.

![Graph](image)

(a) SFF for c=0.9, N=10, at whole range  
(b) SFF for c=0.9, N=100, at specific range to show the predicted behavior

Figure 6. Spectral Form Factor for two different method of solution after averaging over by rule Eq:-(3.51). Here the dotted line is for Hermite polynomial method solution. It has a rounding off behavior. The thick line is for SFF , calculated from saddle point approximation.

Fig: -6 show the rounding off nature of SFF near Heisenberg time. Both solutions have difference in magnitude but overall nature is preserved.

4 Two point correlation function between same matrices

$$U_{M_1}^0(z_1, z_2) = \langle \frac{1}{N} \text{Tr} e^{iz_1 M_1} \frac{1}{N} \text{Tr} e^{iz_2 M_1} \rangle$$  \hspace{1cm} (4.1)

Following the same change of measure and distribution function we solve this by external source matrix A. Here $\prod_{i=1}^{N} d\xi_i = d\xi$ and $\prod_{i=1}^{N} dr_i = dr$

$$U_{M_1}^A(z_1, z_2) = \frac{1}{N^2 Z_A \Delta(A)} \sum_{\alpha_1, \alpha_2}^{N} \int \int e^{iz_1 r_{\alpha_1}} e^{iz_2 r_{\alpha_2}} e^{-\frac{N}{2} \sum r_i^2 + \frac{N}{4} \sum \xi_i^2 + cN \sum \xi_i r_i - N \sum a_i r_i} \Delta(\xi) dr d\xi$$

$$U_{M_1}^A(z_1, z_2) = \frac{1}{N^2 Z_A \Delta(A)} \sum_{\alpha_1, \alpha_2}^{N} \int e^{-\frac{N}{2} \sum (c \xi_i - a_i + \frac{N}{4} (z_1 \delta_{i\alpha_1} + z_2 \delta_{i\alpha_2}) + \frac{N}{4} (z_1 \delta_{i\alpha_1} + z_2 \delta_{i\alpha_2}) - a_i) \xi_i} \Delta(\xi) d\xi$$

$$U_{M_1}^A(z_1, z_2) = \frac{1}{N^2 Z_A \Delta(A)} \sum_{\alpha_1, \alpha_2}^{N} \int e^{-\frac{N}{2} \sum (1 - c^2) \xi_i^2 + N \sum (\frac{N}{4} (z_1 \delta_{i\alpha_1} + z_2 \delta_{i\alpha_2}) - a_i) \xi_i} \Delta(\xi) d\xi$$

$$\times \text{Exp} \left[ \frac{N}{2} \sum (a_i^2 - \frac{1}{N^2} (z_1 \delta_{i\alpha_1} + z_2 \delta_{i\alpha_2})^2 - \frac{2a_i}{N} (z_1 \delta_{i\alpha_1} + z_2 \delta_{i\alpha_2})) \right]$$
Now we take the case for
This is same as one point correlation function or level density.

\[ U_A^{M_1}(z_1, z_2) = \frac{1}{N^2Z_A} \sum_{a_1, a_2}^N \prod_{i<j} \frac{(c(a_i - a_j) + \frac{iC}{N}(z_1(\delta_{j1a1} - \delta_{i1a1}) + z_2(\delta_{j2a2} - \delta_{i2a2}))}{\prod_{i<j}(a_i - a_j)} \times \exp \left[ \frac{N}{2(1 - c^2)} \sum_{i<j} \left( \frac{iC}{N}(z_1\delta_{i1a1} + z_2\delta_{i2a2}) - ca_i)^2 + \frac{N}{2} \sum (a_i^2 - \frac{1}{N^2}(z_1\delta_{i1a1} + z_2\delta_{i2a2})^2 \right) - \frac{2a_i}{N}(z_1\delta_{i1a1} + z_2\delta_{i2a2}) \right] \right] (4.3) \]

Now for \( \alpha_1 = \alpha_2 \)

\[ U_A^{M_1}(z_1, z_2) = \frac{1}{N^2Z_A} \sum_{a_1, a_2}^N \prod_{i<j} \frac{(c(a_i - a_j) + \frac{iC}{N(1 - c^2)} \{z_1(\delta_{j1a1} - \delta_{i1a1}) + z_2(\delta_{j2a2} - \delta_{i2a2})\}}{\prod_{i<j}(a_i - a_j)} \times \exp \left[ -\frac{z_1^2}{2N(1 - c^2)} - \frac{z_2^2}{2N(1 - c^2)} - \frac{iz_1a_{a1}}{N(1 - c^2)} - \frac{i\alpha_2a_{a2}}{1 - c^2} + \frac{N}{2(1 - c^2)} \sum a_i^2 \right] \right] (4.4) \]

After letting \( a_i \to 0 \)

\[ U_0^{M_1}(z_1, z_2) = \frac{c\sqrt{1 - c^2}}{iNZ_A(z_1 + z_2)} \int (1 + \frac{iz_1 + z_2}{u(1 - c^2)})^N \frac{i(z_1 + z_2)}{u(1 - c^2)^2} \times \exp \left[ -\frac{z_1^2}{2N(1 - c^2)} - \frac{z_2^2}{2N(1 - c^2)} - \frac{iz_1u}{N(1 - c^2)} - \frac{iz_2u}{1 - c^2} + \frac{N}{2(1 - c^2)} \sum a_i^2 \right] \right] (4.5) \]

This is same as one point correlation function or level density.

Now we take the case for \( \alpha_1 \neq \alpha_2 \)

\[ U_0^{M_1}(z_1, z_2) = -\frac{c(1 - c^2)^2}{Z_Az_1z_2} \int \frac{dudv}{(2\pi i)^2} \left[ 1 - \frac{iz_1}{N(1 - c^2)u} \right]^N \left[ 1 - \frac{iz_2}{N(1 - c^2)v} \right]^N \exp \left[ -\frac{z_1^2}{2N(1 - c^2)} \right] (4.6) \]
disconnected part of correlation function has the form:–

$$U_d^{M_1}(z_1, z_2) = -\frac{c(1-c^2)^2}{z_1 z_2} \int \frac{dv}{(2\pi i)^2} (1 - \frac{iz_1}{N(1-c^2)u}) (1 - \frac{iz_2}{N(1-c^2)v})^N \exp \left\{ - \frac{z_1^2}{2N(1-c^2)} - \frac{z_2^2}{2N(1-c^2)} - \frac{iz_1 u}{1-c^2} - \frac{iz_2 v}{1-c^2} \right\} \tag{4.7}$$

Now we use one transformation for connected part:– $z_1 = z'_1 - iN(1-c^2)u$, $z_2 = z'_2 - iN(1-c^2)v$

$$U_{conn}^{M_1}(z_1, z_2) = \frac{c}{N^2} \int \frac{dv}{(2\pi i)^2} \int \frac{dv}{(2\pi i)^2} \left( \frac{z'_1}{iN(1-c^2)u} \right)^N \left( \frac{z'_2}{iN(1-c^2)v} \right)^N \frac{1}{(v + \frac{iz_1}{N(1-c^2)})(u + \frac{iz_2}{N(1-c^2)})} \times \exp \left\{ - \frac{z_1^2}{2N(1-c^2)} - \frac{z_2^2}{2N(1-c^2)} - \frac{iz_1' u}{1-c^2} - \frac{iz_2' v}{1-c^2} \right\} \tag{4.8}$$

now we do the Fourier transform two get the two point correlation function:–

$$\rho_{conn}^{M_1}(\lambda, \mu) = \frac{c}{N^2} \int \frac{dz_1 dz_2}{(2\pi i)^2} \int \frac{dv}{(2\pi i)^2} \left( \frac{z_1}{iu} \right)^N \left( \frac{z_2}{iv} \right)^N \frac{1}{(v + iz_1)(u + iz_2)} \times \exp \left\{ - \frac{z_1^2}{2N(1-c^2)} - \frac{z_2^2}{2N(1-c^2)} - \frac{iz_1 u}{1-c^2} - \frac{iz_2 v}{1-c^2} \right\} \tag{4.9}$$

replacing $z_1 \rightarrow \frac{z_1}{N(1-c^2)}$ and $z_2 \rightarrow \frac{z_2}{N(1-c^2)}$

$$\rho_{conn}^{M_1}(\lambda, \mu) = c(1-c^2)^2 \int \frac{dz_1 dz_2}{(2\pi i)^2} \int \frac{dv}{(2\pi i)^2} \left( \frac{z_1}{iu} \right)^N \left( \frac{z_2}{iv} \right)^N \frac{1}{(v + iz_1)(u + iz_2)} \times \exp \left\{ - N\left\{ \frac{(1-c^2)^2}{4} + \frac{(1-c^2)^2}{4} + ic^2 u z_1 + ic^2 v z_2 + \frac{c^2 + 1}{2} u^2 + \frac{c^2 + 1}{2} v^2 \right\} \right\} \tag{4.10}$$

4.1 Solving this Integral by four-variable saddle point method.

Here we consider four variable saddle point solution discussed in [33, 34] Equation for saddle point evaluation:–

$$F(z_1, z_2, u, v) = \frac{1}{2} (c^2 + 1) (u^2 + v^2) + (1-c^2) (\lambda u + \mu v) + ic^2 (uz_1 + vz_2) + \frac{1}{2} (1-c^2) \frac{z_1^2}{2} + \frac{1}{2} (1-c^2) \frac{z_2^2}{2} + i \left( 1 - c^2 \right) (\lambda z_1 + \mu z_2) - \log(u) - \log(v) + \log(x) + \log(y) \tag{4.11}$$

So our saddle points are the simultaneous solution of four equations.

$$\frac{\partial F(z_1, z_2, u, v)}{\partial z_1} = 0, \quad \frac{\partial F(z_1, z_2, u, v)}{\partial z_2} = 0, \quad \frac{\partial F(z_1, z_2, u, v)}{\partial u} = 0, \quad \frac{\partial F(z_1, z_2, u, v)}{\partial v} = 0 \tag{4.12}$$
Which takes the form:

\[ i (1 - c^2) \lambda + ic^2 u + (1 - c^2) z_1 + \frac{1}{z_1} = 0 , \quad i (1 - c^2) \mu + ic^2 v + (1 - c^2) y + \frac{1}{y} = 0 \]

\[ (1 - c^2) \lambda + (c^2 + 1) u + ic^2 z_1 - \frac{1}{u} = 0 , \quad (1 - c^2) \mu + (c^2 + 1) v + ic^2 z_2 - \frac{1}{v} = 0 \quad (4.13) \]

Solving this equation gives sixteen set of solution as the saddle points. Now using saddle point method for four variables with the transformation:

\[ \lambda = \frac{(2\sqrt{c^2 + 1}) \sin(\theta)}{\sqrt{c^2 - 1}} \]
\[ \mu = \frac{(2\sqrt{c^2 + 1}) \sin(\phi)}{\sqrt{c^2 - 1}} \quad (4.14) \]

\[ \rho_{\text{conn}}^{M_1}(\theta, \phi) = \frac{c(-1)^{(N+1)}}{(2\pi)^4} \left\{ \frac{(1 - c^2)^2 e^{-N F(z_1, z_2, u, v)}}{v + iz_1} \frac{\partial^2 F(z_1, z_2, u, v)}{\partial^2 z_1} \frac{\partial^2 F(z_1, z_2, u, v)}{\partial^2 z_2} \right\} \]

Now changing the transformation to its previous form and setting \( \lambda = 0 \) by

\[ \omega = \frac{(2\sqrt{c^2 + 1}) \sin(\phi)}{\sqrt{c^2 - 1}} \quad (4.15) \]

Then this gives us the correlation function for same matrix model.

**Figure 7.** Correlation function for same matrix interaction w.r.t \( \phi \)
Like the different matrix interaction case we can plot correlation function for same matrix case.

\[
\log(|\rho_c(\omega)|)\]

\[
\omega \rightarrow -\frac{2c^2}{c^2-1}, \quad \omega \rightarrow \frac{\sqrt{-c^6 + 3c^4 - 4}}{c^2 - 1}, \quad \omega \rightarrow \pm \frac{2ic^2\sqrt{c^4 - 1}}{(c^2 - 1)^2 (c^2 + 1)}, \quad \omega \rightarrow 0
\]

Finding residue w.r.t this poles gives us the Spectral Form Factor. We plotted its time average defined as:

\[
S^c_{\text{avg}}(t) = \int_0^t S(\tau) d\tau
\]

\[
S(\tau) = \int e^{i\omega \tau} \rho_{\text{conn}}^{c}(\omega) d\omega
\]
From Fig.-9 SFF for same matrix interaction($M_1 - M_1$ interaction) have $c$ dependent features. When we go towards $c \to 0$ SFF losses its previous nature and
become diverging function. In other cases it changes its magnitude as well as period with value of \( c \)

5 \( \frac{1}{N} \) expansion for correlation function

5.1 Saddle point approximation of different order (general derivation)

For Single variable saddle point calculation:

\[
\int_{-\infty}^{\infty} dx f(x)e^{Ag(x)}
\]

(5.1)

\( x_0 \) is the biggest maximum of \( g(x) \). So, \( g'(x_0) = 0 \) Now we change the integration variable to

\[
x = x_0 + \frac{y}{\sqrt{A}}
\]

(5.2)

So the taylor expansion of \( Ag(x) \) and \( f(x) \) around \( x_0 \) can be given as

\[
e^{Ag(y)}f(y) = e^{Ag(x_0)} + \frac{1}{2}g''(x_0)(f(x_0) + \frac{6yf'(x_0) + y^2f(x_0)g''(x_0)}{6\sqrt{A}} + \frac{\sqrt{2\pi}e^{Ag(x_0)}}{12A^{3/2}(-g''(x_0))^{7/2}}
\]

\[
\left((12f''(x_0)g''(x_0)^2 - 12g^{(3)}(x_0)f'(x_0)g''(x_0) + f(x_0)(5g^{(3)}(x_0)^2 + 523)
\]

\[
- 3g^{(4)}(x_0)g''(x_0))))
\]

So the saddle point expression gives

\[
\int_{-\infty}^{\infty} dy \sqrt{A}f(x)e^{Ag(y)} = \sqrt{2\pi}f(x_0)e^{Ag(x_0)} + \frac{\sqrt{2\pi}f(x_0)e^{Ag(x_0)}}{12A^{3/2}g''(x_0)^{7/2}}
\]

\[
(12f''(x_0)g''(x_0)^2 + 12g^{(3)}(x_0)f'(x_0)g''(x_0) + f(x_0)(5g^{(3)}(x_0)^2 + 3g^{(4)}(x_0)g''(x_0))
\]

(5.4)

For two variable saddle point method Now we repeat this same method for two variable saddle point approximation

\[
\int_{-\infty}^{\infty} dxdy f(x,y)e^{AH(x,y)}
\]

(5.5)

For saddle point approximation we choose the main contributing points of the integral and this set of point is given by:-

\[
\frac{\partial H(x,y)}{\partial x} |_{(x_0,y_0)} = 0, \frac{\partial H(x,y)}{\partial y} |_{(x_0,y_0)} = 0, \frac{\partial^2 H(x,y)}{\partial y \partial x} |_{(x_0,y_0)} = 0
\]

(5.6)

Now we change the integration variable to

\[
x = x_0 + \frac{w}{\sqrt{A}} \quad y = y_0 + \frac{z}{\sqrt{A}}
\]

(5.7)
Now we make Taylor expansions of $AH(x, y)$ and $F(x, y)$ around $x_0$ and $y_0$ and choose up to certain terms to get the $\frac{1}{A^n}$ terms completely for $n = \frac{1}{2}, 1$ in $e^{AH(w, z)}F(w, z)$

\[
e^{AH(w, z)}F(w, z) = \text{Exp}\{AH(x_0, y_0) + \frac{1}{2}w^2H^{(2,0)}(x_0, y_0) + F(x_0, y_0)(w^3H^{(3,0)}(x_0, y_0) + 3wz(wH^{(2,1)}(x_0, y_0) + zH^{(1,2)}(x_0, y_0)) + z^3H^{(0,3)}(x_0, y_0)\}
\]

\[
+ \frac{1}{A} \left( \frac{w^4F^{(1,0)}}{6}(x_0, y_0)H^{(3,0)}(x_0, y_0) \right) + \frac{w^2z^2F^{(1,0)}}{2}(x_0, y_0)H^{(1,2)}(x_0, y_0) + \frac{w^2F^{(2,0)}}{2}(x_0, y_0) + \frac{w^4z^2F^{(0,1)}}{6}(x_0, y_0)H^{(1,2)}(x_0, y_0) + \frac{w^2z^2F^{(1,0)}}{2}(x_0, y_0)H^{(1,2)}(x_0, y_0) + \frac{w^4z^2F^{(0,1)}}{6}(x_0, y_0)H^{(1,2)}(x_0, y_0)
\]

So

\[
\int_{-\infty}^{\infty} dxdyF(x, y)e^{AH(x,y)} = \frac{2\pi F(x_0, y_0)e^{AH(x_0, y_0)}}{A^{\frac{1}{2}}H^{(2,0)}(x_0, y_0)\sqrt{H^{(0,3)}(x_0, y_0)}} + \left(\frac{4A^2H^{(0,2)}(x_0, y_0)^{5/2}H^{(2,0)}(x_0, y_0)^{5/2}}{\pi e^{AH(x_0, y_0)}}\right) \times
\]

\[
\left\{H^{(0,2)}(x_0, y_0)^2 \left[ F^{(2,0)}(x_0, y_0)H^{(2,0)}(x_0, y_0) + 4F^{(1,0)}(x_0, y_0)H^{(3,0)}(x_0, y_0) \right] + H^{(0,2)}(x_0, y_0) \left[ 4F^{(0,2)}(x_0, y_0)H^{(2,0)}(x_0, y_0)^2 + 4F^{(1,0)}(x_0, y_0)H^{(2,0)}(x_0, y_0) \right] \right\}
\]

\[
+ 4F^{(1,0)}(x_0, y_0)H^{(1,2)}(x_0, y_0)H^{(2,0)}(x_0, y_0) + 4F^{(0,1)}(x_0, y_0)H^{(2,1)}(x_0, y_0)H^{(2,0)}(x_0, y_0) + 2F(x_0, y_0)H^{(2,0)}(x_0, y_0) + 3F(x_0, y_0)H^{(2,1)}(x_0, y_0) + 2F(x_0, y_0)H^{(1,2)}(x_0, y_0)H^{(3,0)}(x_0, y_0)
\]

\[
+ H^{(2,0)}(x_0, y_0) \left[ 4F^{(0,1)}(x_0, y_0)H^{(0,3)}(x_0, y_0) + 2H^{(0,3)}(x_0, y_0)H^{(2,1)}(x_0, y_0) \right]
\]

5.2 Second order contribution of SFF

Now we evaluate next order contribution for correlation function using second term of the (Eq.- (5.9)). We use this relation for Expression of both the kernels in Eq-
(3.36) and Eq:-(3.41). We follow the exactly same procedure thereafter and at first evaluate the two-point correlation function.

![2-point correlation function 1st correction for N=100](image1)

(a) behavior of correlation function of next order for c=0.9, N=100

![2-point correlation function 1st correction for N=100 & N=10](image2)

(b) comparison between c=0.9, N=10 and N=100

**Figure 10.** Correlation function w.r.t ω for different dimension of Matrix(N) and effect of correction

![2-point correlation function 1st correction for N=100](image3)

(a) corrected and previous correlation function for c=0.9, N=100

**Figure 11.** Comparison for correlation function w.r.t ω for dimension of Matrix(N=100) and effect of correction. Thick line is the old correlation function computed from zeroth order saddle point method term. Adding correction to it produces the dotted line of the figure.

Now we evaluate spectral form factor by Fourier transform exactly as Eq:-(3.49). We choose the singularities of this equation to find the integral by residue
The singularities are same as previous case. Then we compute residue w.r.t these points.

Figure 12. Correction of Spectral Form Factor w.r.t $\tau$ for different dimension of Matrix(N)

Exactly like previous case SFF can be averaged over an interval $(0,t)$ and plotting that shows a continuous behavior instead of kink at Heisenberg time

$$\langle S(\tau) \rangle = \int_0^t d\tau S(\tau) = S_{\text{avg}}(t)$$  \hspace{1cm} (5.10)
We can compare solution of SFF with and without correction term. Both has a continuous transition behavior at Heisenberg time. And the correction introduce an extra shift in saturation behavior.

Figure 14. First order Corrected Spectral Form Factor connected part average and previously derived Spectral Form factor for different dimension of matrix(N). The dotted line describe SFF with 1st order correction (adding the 1st order term in saddle point approximation). The thick line is old SFF from zeroth order saddle point term.

Fig:-14 shows that 1st order correction of SFF has same type of contribution even in various N. The solution add correction to old value and have the same type of rounding behavior near Heisenberg time.

6 Duality relation for Two Matrix model

Correlation function for characteristic polynomial of two matrix model has been defined in [30]

\[ J = \prod_{\alpha=1}^{k_1} \det(\lambda_\alpha - M_1) \prod_{\beta=1}^{k_2} \det(\mu_\beta - M_2) \quad (6.1) \]
$M_1$ and $M_2$ are $N \times N$ Hermitian matrix as seen in Eq:-(2.13), Eq:-(3.8) The average is over distribution for two matrix model [Eq:-(2.13)]

$$P(M_1, M_2) = \frac{1}{Z} e^{-\frac{1}{2} \text{tr}M_1^2 - \frac{1}{2} \text{tr}M_2^2 - \text{ctr}M_1M_2 - \text{tr}M_1A}$$ (6.2)

Now we use Grassmann variable $\psi_\alpha$ and $\chi_\beta$ for this integral. We know for Grassmann variable integration has determinant form given by:-

$$\int e^{cl.Ac} = \det(A)$$ (6.3)

Where $c$ and $c^\dagger$ are Grassmann variables.

So we can write the correlation function in integral form as

$$J = \langle \int d\psi d\bar{\psi} d\chi d\bar{\chi} e^{N[\bar{\psi}(\lambda_\alpha - M_1)\psi + \bar{\chi}(\mu_\beta - M_2)\chi]} \rangle$$ (6.4)

So writing the averaging in integral form:

$$J = \frac{1}{Z} \int dM_1 dM_2 \int d\psi d\bar{\psi} d\chi d\bar{\chi} e^{N[\bar{\psi}(\lambda_\alpha - M_1)\psi + \bar{\chi}(\mu_\beta - M_2)\chi]} e^{-\frac{1}{2} \text{tr}M_1^2 - \frac{1}{2} \text{tr}M_2^2 - \text{ctr}M_1M_2 - \text{tr}M_1A}$$ (6.5)

Now we change the measure by Harish-Chandra-Itzykson-Zuber formula and get this matrix integral in eigen value integration format with use of Vandermonde determinant like previous case:

$$J = \frac{1}{\Delta(a)} \int d\psi d\bar{\psi} d\chi d\bar{\chi} \int \prod_{i=1}^{N} dq_i \prod_{i=1}^{N} dp_i \Delta(q) e^{-\frac{N}{2} \sum_{i=1}^{N} q_i^2 - \frac{N}{2} \sum_{i=1}^{N} \sum_{i=1}^{N} (c q_i + a_i + \psi_\alpha \psi_\alpha') \delta_{\alpha,\alpha'}}$$

$$\exp \left[ - \frac{N}{2} \sum_{i=1}^{N} q_i^2 + \bar{\chi}(\mu_\beta) \chi - N \sum_{i=1}^{N} \bar{\chi}(p_i) \chi_\beta' \delta_{\beta,\beta'} - \bar{\psi}(\lambda_\alpha) \psi \right]$$ (6.6)

$$J = \frac{1}{\Delta(a)} \int d\psi d\bar{\psi} d\chi d\bar{\chi} \int \prod_{i=1}^{N} dq_i \Delta(q) \exp \left\{ \frac{N}{2} \sum_{i=1}^{N} (c q_i + a_i + \psi_\alpha \psi_\alpha')^2 - \frac{N}{2} \sum_{i=1}^{N} q_i^2 \right\}$$

$$+ \bar{\chi}(\mu_\beta) \chi - N \sum_{i=1}^{N} \bar{\chi}(p_i) \chi_\beta' \delta_{\beta,\beta'} - \bar{\psi}(\lambda_\alpha) \psi \right\}$$ (6.7)

$$J = \frac{1}{\Delta(a)} \int d\psi d\bar{\psi} d\chi d\bar{\chi} \int \prod_{i=1}^{N} dq_i \Delta(q) \exp \left[ - \frac{N}{2} \sum_{i=1}^{N} q_i^2 + \bar{\chi}(\mu_\beta) \chi - N \sum_{i=1}^{N} \bar{\chi}(p_i) \chi_\beta' \delta_{\beta,\beta'} \right.$$

$$- \bar{\psi}(\lambda_\alpha) \psi + \frac{N}{2} \sum_{i=1}^{N} (c^2 q_i^2 + a_i^2 + \bar{\psi} \psi \bar{\psi} \psi + 2c \psi a_i + 2c \bar{\psi} q_i + 2a_i \bar{\psi}) \right]$$
on integrating over $q_i$ this gives four fermionic term which can be simplified by auxiliary matrices. We choose $B_1 B_2$ to be hermitian matrix of size $k_1 \times k_1 \ k_2 \times k_2$. $D$ is complex rectangular matrix of size $k_1 \times k_2$.

\[
\begin{align*}
\int dB_1 e^{-\frac{N}{2} Tr(B_1^2)+\frac{iN}{\sqrt{1-c^2}} TrB_1 \bar{\psi} \psi} &= e^{-\frac{N}{2(1-c^2)} \bar{\psi} \psi} \\
\int dB_2 e^{-\frac{N}{2} Tr(B_2^2)+\frac{iN}{\sqrt{1-c^2}} TrB_2 \bar{\chi} \chi} &= e^{-\frac{N}{2(1-c^2)} \bar{\chi} \chi} \\
\int dD d\bar{D} e^{-N TrD\bar{D} + \int \sqrt{1-c^2} Tr(D \bar{\psi} \chi + D \bar{\chi} \psi)} &= e^{\frac{N}{2(1-c^2)} \bar{\psi} \chi \psi} \\
\end{align*}
\]

Solving the integral we the integral over grassman variables as

\[
\int d\bar{\psi} d\bar{\chi} e^{\psi \left(\frac{i}{\sqrt{1-c^2}}(\bar{\lambda}_\alpha - \frac{a_i}{1-c^2}) \delta_{\alpha,\alpha'} + \frac{i}{\sqrt{1-c^2}} \bar{B}_1 \right) + \sqrt{1-c^2} D \bar{\psi} + \bar{\chi} \left(\mu_\beta - \frac{c a_i}{1-c^2} \right) \delta_{\beta,\beta'} + \frac{i}{\sqrt{1-c^2}} \bar{B}_2 \chi} = \int dLd\bar{L} e^{\frac{1}{2}L^\dagger L} = \det(X) \\
\]

With this simplification we solve the Eq.: (6.6) and get it in much simplified form

\[
J = \int dB_1 dB_2 dD d\bar{D} e^{-\frac{N}{2} Tr(B_1^2 + B_2^2 + 2D\bar{D})} \prod_{i=1}^{N} \det(X_i) \\
X_i = \left[ \begin{array}{ccc} \{\lambda_\alpha - \frac{a_i}{1-c^2} \delta_{\alpha,\alpha'} + \frac{i}{\sqrt{1-c^2}} B_1 \} & \sqrt{1-c^2} D \\
\sqrt{\frac{c}{1-c^2}} D^\dagger & \{\mu_\beta - \frac{c a_i}{1-c^2} \delta_{\beta,\beta'} + \frac{i}{\sqrt{1-c^2}} \bar{B}_2 \} \end{array} \right] \\
\]

Now we use a transformation

$B_1' \rightarrow B_1 + i\sqrt{1-c^2} \lambda_{\alpha,\alpha'} \delta_{\alpha,\alpha'}$ \quad $B_2' \rightarrow B_2 + i\sqrt{1-c^2} \mu_{\beta,\beta'} \delta_{\beta,\beta'}$

This simplifies the integral :-

\[
J = C \int dB_1 dB_2 dD d\bar{D} e^{-\frac{N}{2} Tr(B_1^2 + B_2^2 + 2D\bar{D}) - iN \sqrt{1-c^2} TrB_1 \Lambda_1 - iN \sqrt{1-c^2} TrB_2 \Lambda_2} e^{-\sum_{i=1}^{N} Tr(\log(1-K_i))} \\
\]

(6.11)

Here we have used $\det(A) = e^{Tr(\log(A))}$ and the matrix $K$ is reduced from $X$

\[
K_i = \left[ \begin{array}{cc} \frac{i\sqrt{1-c^2}}{a_i} B_1 & \sqrt{\frac{c}{1-c^2}} D^\dagger \\sqrt{\frac{c}{1-c^2}} D^\dagger \end{array} \right] \\
\]

We set $\Lambda = \alpha l$ with constraint $\alpha = \sqrt{1-c^2}$. Now we can expand $\log(1-K)$ in taylor series upto 3rd term,

\[
\log(1-K) = -K - \frac{K^2}{2} - \frac{K^3}{3} - \frac{K^4}{4} \\
\]

(6.12)
Considering upto $K^3$ gives the term in power of exponential [Eq-(6.11)] as:-

\[
J = \int dB_1 dB_2 dD \, \exp \left\{ -N \left[ iB_1 (1 - \sqrt{1 - c^2} \lambda_1) - \frac{i}{c} B_2 (1 - c\sqrt{1 - c^2} \lambda_2) \right. \right. \\
\left. \left. - \frac{1}{2} (1 - \frac{1}{c^2}) B_2^2 + \frac{i}{3} B_1^3 - \frac{i}{3c} B_2^3 + \frac{2i}{3} D D^\dagger B_1 - \frac{2i}{3c} B_2 D^\dagger D + \frac{i}{3c} D B_2 D^\dagger - \frac{i}{3} D^\dagger B_1 D \right] \right\} 
\]

(6.13)

\[
J = \int dB_1 dB_2 dD D^\dagger \exp \left\{ -i N \text{Tr}(B_1 A_1) - i N \text{Tr}(B_2 A_2) + \frac{i}{3} N \text{Tr}(B_1^3) \right. \\
\left. - \frac{N}{2} (1 - \frac{1}{c^2}) \text{Tr} B_2^2 + i N \text{Tr}(D D^\dagger B_1 - 1) - \frac{i N}{c} B_2 (-1 + D D^\dagger) \right\} 
\]

(6.14)

Now at the edge of the spectrum for the matrix $M_1$ edge scaling limit at large $N$ gives:-

\[
B_1 \sim O(N^{-\frac{1}{3}}) \quad B_2 \sim O(N^{-\frac{1}{2}}) \quad D \sim O(N^{-\frac{1}{3}}) 
\]

(6.15)

Dropping the negligible terms ($B_2 DD^\dagger \sim O(N^{-\frac{7}{3}})$)

\[
Z = \int dB_1 dB_2 DD^\dagger e^{-i N \text{Tr}(B_1 A_1) - i N \text{Tr}(B_2 A_2) + \frac{i}{3} N \text{Tr}(B_1^3) - \frac{N}{2} (1 - \frac{1}{c^2}) \text{Tr} B_2^2 + i N \text{Tr}(D D^\dagger B_1)} 
\]

(6.16)

$Q$ is the decoupled part generated after integration over $B_2$ Integrating out $D^\dagger$ and $D$ gives logarithmic term:-

\[
Z = \int dB_1 e^{\frac{i}{3} \text{Tr} B_1^3 - k_2 \text{Tr} \log(B_1) - i \text{Tr}(B_1 A_1)} 
\]

(6.17)

This has been related to Airy Matrix model coupled with a logarithmic potential (Kontsevich - Penner model) in [35]

**Derivation for $B_1^3$ term**

\[
K = \begin{bmatrix} iB_1 & \sqrt{c} D \\ D^\dagger & \frac{-iB_2}{c} \end{bmatrix} 
\]

Expanding upto 4th term

\[
\log(1 - K) = -K - \frac{K^2}{2} - \frac{K^3}{3} - \frac{K^4}{4} 
\]

(6.18)

So, $\text{Tr}(\log(1-K))$ has terms from four contribution, as trace is there we can consider only the diagonal terms in each of $\text{Tr}[K^n]$. So for $\text{Tr}[\frac{1}{2} K^3]$ term $\rightarrow$

\[
\text{Tr} \left( -\frac{i}{3} B_1^3 + \frac{1}{3} D D^\dagger B_1 + \frac{i}{3} B_1 D D^\dagger + \frac{i}{3c} D B_2 D^\dagger + \frac{i}{3} D^\dagger B_1 D - \frac{i}{3\sqrt{c}} B_2 D D^\dagger \right) 
\]

(6.19)
\[
\text{Tr}[\frac{1}{4}K^4] \text{ term } \\
\text{Tr} \left( \frac{1}{4} B^4 + \frac{i}{4} D D^\dagger B^2 - \frac{1}{4} B^1 D D^\dagger B^1 + \frac{1}{4c} D B^2 D^\dagger B^1 - \frac{1}{4} B^1 D B^1 D^\dagger + \frac{1}{4} D D^\dagger D D^\dagger \\
- \frac{1}{4c^2} D B^2 D^\dagger - \frac{1}{4} D^\dagger B^2 D + \frac{1}{4c^2} B^2 D^\dagger B_1 D + \frac{1}{4} D D^\dagger D^\dagger D - \frac{1}{4c^2} B^2 D^\dagger D \\
+ \frac{1}{4c} B^1 D B_2 D^\dagger + \frac{1}{4c} B_1 D B_2 D^\dagger + \frac{i}{4c^3} c^2 B_2 D^\dagger B_2 - \frac{1}{4c^3} c^2 D D^\dagger D_2 B_2 \right) (6.20)
\]

If we consider up to \(K^4\) term of Eq:- (6.12) This integral is solved in similar way. Now with existing edge scaling Eq:- (6.15), after integral over \(B_2\) and \(D, D^\dagger\)

\[
Z = \int dB_1 dB_2 dD dD^\dagger \exp \left[ -N \text{Tr} \left( B^1 \Lambda^1 + B_2 \Lambda_2 - \frac{1}{2} (c^2 - 1) B^2 + \frac{1}{3} B^3_1 - \frac{1}{3} B^3_2 \\
+ \frac{2}{3} D D^\dagger B^1 + \frac{2}{3} B^2 D D^\dagger D - \frac{1}{3} D B^2 D^\dagger - \frac{1}{3} D^\dagger B^1 D \right) \right] (6.21)
\]

Although \(\text{Tr}(B^4)\) term is absent in the edge scaling, this term can be derived as \([36–38]\). Two converging saddle points gives rise to fold singularity as in the \(B_3^3\) expression. This is related to Airy kernel discussed in \([39]\). For extended Airy Kernel Eq:- (6.21) cubic singularity becomes quartic term. This is expressed in terms of Pearcey function and showed in \([36, 37]\) on the level spacing distribution for hermitian random matrices with an external field. If \(H=H_0+V\) where \(H_0\) is a fixed matrix and \(V\) is an \(N \times N\) random GUE matrix. \(H_0\) has eigenvalues \(\pm a\) each with multiplicity \(N/2\). Spectrum of \(H_0\) is such that there is a gap in the average density of eigenvalues of \(H\) which is thus split into two pieces. With \(N \to \infty\) density of eigenvalues supported on single or double interval depending on size of \(a\). At the closing of gap the limiting eigenvalue distribution has Pearcey kernel structure. When the spectrum of \(H_0\) is tuned so that the gap closes limiting eigenvalue distribution have the same structure as Pearcey kernel.

Connecting the Two point correlation function with Open partition function

At first consider the equation Eq:- (6.13) with \(B_2 \rightarrow iB_2 c\), \(B_1 \rightarrow -iB_1\) and \(\Lambda_1 \rightarrow (1 - \sqrt{1 - c^2} A_1)\) and \(\Lambda_2 \rightarrow (1 + c\sqrt{1 - c^2} A_2)\)

\[
J = \int dB_1 dB_2 dD dD^\dagger \left[ -N \text{Tr} \left\{ B_1 A_1 + B_2 A_2 - \frac{1}{2} (c^2 - 1) B^2 - \frac{1}{3} B^3_1 - \frac{1}{3} B^3_2 \\
+ \frac{2}{3} D D^\dagger B_1 + \frac{2}{3} B^2 D D^\dagger D - \frac{1}{3} D B^2 D^\dagger - \frac{1}{3} D^\dagger B^1 D \right\} \right] (6.22)
\]
After integration over $dD$ and $dD^1$ and transformation $B_1 \rightarrow B_1 + \sqrt{\Lambda_1}$

$$J = \int \text{Exp} \left[ \frac{2N}{3} \text{Tr}(\Lambda^2) - \frac{N}{3} \text{Tr}(B^3_1) - N \text{Tr}(B^2_1 \sqrt{\Lambda_1}) + N \text{tr}(B_2 \Lambda_2) + N \left( \frac{c^2 - 1}{2} \right) B_2^2 - \frac{N}{3} B_2^3 + \frac{N}{3} \text{Tr}(\log(B_1 + \sqrt{\Lambda_1})) + \frac{N}{3} \text{Tr}(\log(B_2)) \right] dB_1 dB_2$$

$$J = \int_{H_{k_1} \times Z_{k_2}} dH dZ \text{Exp} \left[ \frac{2N}{3} \text{Tr}(\Lambda^2) - \frac{N}{3} \text{Tr}(H^3) - N \text{Tr}(H^2 \sqrt{\Lambda_1}) + N \text{Tr}(Z \Lambda_2) + N \left( \frac{c^2 - 1}{2} \right) Z^2 - \frac{N}{3} Z^3 + \{Nk\text{Tr}(\log(H + \sqrt{\Lambda_1})) + Nk'\text{Tr}(\log(Z))\} \right]$$

$$J = \text{Exp} \left[ \frac{\hat{N}}{3} \text{Tr}(\Lambda^2) \right] \text{Exp} \left[ - \frac{\hat{N}}{2} \text{Tr}(H^2 \sqrt{\Lambda_1}) + \frac{\hat{N}}{2} \left( \frac{c^2 - 1}{2} \right) Z \hat{Z}^t - \frac{\hat{N}}{6} \text{Tr}(H^3) \right] - \frac{\hat{N}}{2} \text{Tr}(Z \Lambda_2) + \frac{\hat{N}}{2} \text{Tr}(Z \Lambda_2) + \frac{\hat{N}}{6} \left\{ \text{Tr}(\log(H + \sqrt{\Lambda_1})) + \text{Tr}(\log(Z)) \right\} dH dZ$$

$$J = K \int_{H_{k_1} \times Z_{k_2}} \text{Exp} \left[ - \frac{\hat{N}}{2} \text{Tr}(H^2 \sqrt{\Lambda_1}) + \frac{\hat{N}}{2} \left( \frac{c^2 - 1}{2} \right) Z \hat{Z}^t - \frac{\hat{N}}{6} \text{Tr}(H^3) \right] - \frac{\hat{N}}{2} \text{Tr}(Z \Lambda_2) + \frac{\hat{N}}{6} \left\{ \text{Tr}(\log(H + \sqrt{\Lambda_1})) + \text{Tr}(\log(Z)) \right\} dH dZ$$

Now from Eq:-(6.1)

$$\Lambda_1|_\alpha = (1 - \sqrt{1 - c^2}) \lambda_\alpha \text{ and } \Lambda_2|_\beta = (1 - \sqrt{1 - c^2}) \mu_\beta \text{ } N = \frac{\hat{N}}{2}$$

and $Z_{k_2 \times k_2}$ is hermitian matrix so $Z \hat{Z}^t = Z^2$

$$J = K \int_{H_{k_1} \times Z_{k_2}} \text{Exp} \left[ - \frac{\hat{N}}{2} \text{Tr}(H^2 \sqrt{\Lambda_1}) + \frac{\hat{N}}{2} \left( \frac{c^2 - 1}{2} \right) Z \hat{Z}^t - \frac{\hat{N}}{6} \text{Tr}(H^3) \right] - \frac{\hat{N}}{2} \text{Tr}(Z \Lambda_2) \text{ } \text{det}(H + \sqrt{\Lambda_1}) \frac{\hat{N}}{\dim} \text{det}(Z) \frac{\hat{N}}{\dim}$$

Now we look at the very refined open partition function as derived in [40]. They have provided the matrix model for very refined open partition function as matrix integrals in the given form:-

$$\tilde{\mathcal{D}}^{\alpha}_{\text{open}}|_{t_t(\Lambda)} = \frac{c_{\Lambda,M}}{(2\pi)^{N^2}} \int_{H_M \times M_{N,N}} \text{Exp} \left[ - \frac{1}{2} \text{Tr} H^2 \Lambda - \frac{1}{2} \text{Tr} ZZ^t + \frac{1}{6} \text{Tr} H^3 + \frac{1}{6} \text{Tr} Z^3 + \frac{1}{2} \text{Tr} \hat{Z}^t \Theta \right]$$

$$\times \text{det} \frac{\Lambda \otimes I_N + \sqrt{\Lambda^2 \otimes I_N - I_M \otimes Z^t} - H \otimes I_N + I_M \otimes Z}{\Lambda \otimes I_N + \sqrt{\Lambda^2 \otimes I_N - I_M \otimes Z^t} - H \otimes I_N - I_M \otimes Z} dH dZ$$

$$\text{det}$$
For $N \geq 1$ the space of Hermitian matrices is denoted by $\mathcal{H}_M$ and the space of complex $N \times N$ matrices by $M_{N \times N}(\mathbb{C})$. Volume $dZ$ is denoted by

$$dZ := \prod_{1 \leq i,j \leq N} d(\text{Re}z_{i,j}) d(\text{Im}z_{i,j})$$

and Gaussian probability measure on space of complex matrices is given by

$$\frac{1}{(2\pi)^{N^2}} e^{-\frac{1}{2} \text{Tr} Z^2} dZ$$

$\theta_{i,j}, 1 \leq i,j \leq N$, are considered as an extra set of complex variables:

$$\Theta := (\theta_{i,j})_{1 \leq i,j \leq N} \in M_{N,N}(\mathbb{C})$$

$$q_m(\Theta) := \text{Tr} \Theta^m, \quad m \geq 0$$

(6.28)

And,

$$c_{\Lambda,M} := (2\pi)^{M^2} \prod_{i=1}^{M} \sqrt{\lambda_i} \prod_{1 \leq i < j \leq M} (\lambda_i + \lambda_j)$$

(6.29)

Now comparing Eq:-(6.26) and Eq:-(6.27) two matrix model two point correlation function and very refined open partition function are similar with $\sqrt{\Lambda_1} = \Lambda$ and $\Lambda_2 = \Theta$ and $K = \text{Exp} \left[ \frac{\beta}{2} \text{Tr}(\Lambda^{\frac{3}{2}}) \right]$ is the extra constant term multiplied in front. More detailed discussion on open partition function and refined open partition function can be found in [40–45].

In [30] two matrix model correlation function has been related to Kontsevich-Penner Matrix model near Heisenberg time. Using Replica method they have studied the intersection number discussion in this context. In our previous calculation we have obtained a rounding off behavior near Heisenberg time. The universal behavior of SFF ramp region Dyson sine kernel is now changed. It suggests that some new kind of description is needed in this region. Kontsevich[46] and Penner[47] Matrix models gives the edge behavior and open boundaries for the punctured open Riemann surfaces. This has been explained in [30, 42, 45, 48–51].

Universal Dyson sine kernel gives one important feature of underlying Gaussian Unitary Ensemble, its stationary nature under Dyson Brownian motion. But now universality of sine kernel are no more available. To explain the rounding off behavior we need to consider Brownian motion near edges. This brownian motion effect is related to time dependence of the model.

7 Discussion

In this paper starting from time dependent Gaussian Unitary Ensemble(GUE) matrix model we converted it in two matrix model and with contour integral representation for correlation function, SFF and average of SFF has been calculated and discussed. We have considered both type of correlation function and also the next
order contribution of $1/N$ expansion, for saddle point integral. SFF for different matrix correlation has been shown to have a rounding off near Heisenberg time $\tau = \tau_c$, a crossover in this point. In [1] it has been discussed as breakdown of one-matrix model and singularity at this point and also referred to the case of mesoscopic dirty metals discussed in [52]. For our same matrix correlation function and spectral form factor it gives a decaying average spectral form factor which is consistent with GUE behavior of SFF. Second order contribution calculated here from the $1/N$ expansion of saddle point integral gives same rounding off behavior and appear as correction to the first order solution. Here the rounding off behavior is different with increasing dimension of matrix $(N)$.

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