Abstract Using a new implicit discretization scheme, we study in this paper the existence and uniqueness of strong solutions for a class of Lur’e dynamical systems where the set-valued feedback depends on both time and state. This work is a generalization of [3] where the time-dependent set-valued feedback is considered to acquire only weak solutions. Obviously, strong solutions and implicit discretization scheme are nice properties, especially for numerical simulation. We also provide some conditions such that the solutions are exponentially attractive. The obtained results can be used to study the time-varying Lur’e systems with errors in data. Our result is new even the set-valued feedback depends only on the time.

Keywords Lur’e dynamical systems · well-posedness · state-dependent · set-valued · normal cone.

1 Introduction

It is known that Lur’e dynamical systems have been studied intensively recently with many applications can be found in control theory, engineering and applied mathematics (see, e.g., [22] for a survey). In general, the systems consist of a smooth ordinary differential equation $\dot{x} = g(x, \lambda)$ with output $y = h(x, \lambda)$ and a static single-valued feedback $\lambda = F(y)$. In order to describe discontinuous changes of velocity more effectively, Lur’e systems with static set-valued feedback was firstly considered in [10] with a special case and then largely analyzed in [1,2,3,4,5,6,11,12,13,14]. Let us also mention that set-valued Lur’e systems can be recast into other non-smooth mathematical models [9,11,15,16,17,18] such as complementarity systems, evolution variational inequalities, projected systems, relay systems . . .

In this paper, we study the well-posedness for a class of Lur’e dynamical systems where the set-valued feedback has the form of normal cone to a moving closed, convex set which depends not only on the time but also on the state.
For more details, let be given a function $f : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, some matrices $B : \mathbb{R}^m \to \mathbb{R}^n, C : \mathbb{R}^n \to \mathbb{R}^m$, $D : \mathbb{R}^m \to \mathbb{R}^m$ and a set-valued mapping $K : [0, +\infty) \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Then we want to find an absolutely continuous function $x(\cdot) : [0, +\infty) \to \mathbb{R}^n$ with given initial point $x_0 \in \mathbb{R}^n$ such that

$$
\begin{align*}
\dot{x}(t) &= f(t, x(t)) + B\lambda(t) \text{ a.e. } t \in [0, +\infty); \\
y(t) &= Cx(t) + D\lambda(t), \\
\lambda(t) &\in -N_{K(t,x(t))}(y(t)), \; t \geq 0; \\
x(0) &= x_0,
\end{align*}
$$

where $\lambda, y : [0, +\infty) \to \mathbb{R}^m$ are two unknown connected mappings.

In [8], the authors studied the time-dependent case $K(t, x) \equiv K(t)$ with motivation for considering the viability control problem and the output regulation problem, particularly in power converters. To the best of our knowledge, it can be considered as the first work which considers non-static set-valued feedbacks with non-zero $D$ (see, e.g., [4, 5, 10, 11, 12, 13, 14] for static cases). The state-dependent case is left as an open problem. A kind of weak solution ([3, Theorem 3.1]) is obtained under the following assumptions:

(A1) The matrix $D$ is positive semidefinite, and there exists a symmetric positive definite matrix $P$ such that $\ker(D + D^T) \subseteq \ker(PB - C^T)$;

(A2) There exists a nonnegative locally essentially bounded function $\rho : [0, +\infty) \to [0, +\infty)$ such that

$$
\|f(t, x) - f(t, y)\| \leq \rho(t)\|x - y\|, \; x, y \in \mathbb{R}^n;
$$

(A3) For each $t \geq 0$, $\text{rge}(C) \cap \text{rint}(\text{rge}(N_{K(t)}^{-1} + D)) \neq \emptyset$;

(A4) For every $t \geq 0$ and each $v \in \text{rge}(C) \cap \text{rge}(N_{K(t)}^{-1} + D)$, it holds that $\text{rge}(D + D^T) \cap (N_{C(t)}^{-1} + D)^{-1}(v) \neq \emptyset$;

(A5) It holds that $\text{rge}(D) \subseteq \text{rge}(C)$ and $K : [0, \infty) \rightrightarrows \mathbb{R}^m$ has closed and convex values for each $t \geq 0$. Also, the mapping $K \cap \text{rge}(C)$ varies in an absolutely continuous manner with time; that is, there exists a locally absolutely continuous function $\mu : [0, \infty) \to \mathbb{R}^+$ such that

$$
d_H(K(t_1) \cap \text{rge}(C), K(t_2) \cap \text{rge}(C)) \leq |\mu(t_1) - \mu(t_2)|, \; t_1, t_2 \geq 0,
$$

where $d_H$ denotes the Hausdorff distance. The system $(S)$ was rewritten into a time-varying first order differential inclusion where the right-hand side can be decomposed as a maximal monotone operator and a single-valued Lipschitz function to obtain weak solutions.

The current paper generalizes [3] not only to the state-dependent moving set $K(t, x)$ but also to obtain strong solutions by using a new implicit discretization scheme. Obviously, strong solutions and the implicit discretization scheme are desired properties which are advantages for implementation in numerical simulations. In addition, we provide some conditions such that the solutions are exponentially attractive, i.e., the solutions converges to the origin with an exponential rate when
Lur’e dynamical systems with state-dependent set-valued feedback

\[ y = Cx + D\lambda \]

\[ \lambda \in -N_{K(x,y)}(y) \]

Fig. 1 Lur’e systems with state-dependent feedback.

the time is large. The obtained results can be used to study time-varying Lur’e dynamical systems with errors in data.

The paper is organized as follows. In Section 2, we recall some notation and useful fundamental results. The well-posedness and asymptotic behaviour of (S) are analyzed thoroughly in Section 3. Application for the study of time-varying Lur’e dynamical systems with errors in data is presented in Section 4. Some concluding remarks are given in Section 5.

2 Notation and Mathematical Backgrounds

Let us first introduce some notation that will be used in the sequel. Denote by \( \langle \cdot, \cdot \rangle \), \( \| \cdot \| \), \( \mathbb{B} \) the scalar product, the corresponding norm and the closed unit ball in Euclidean spaces. Let be given a closed, convex set \( K \subset \mathbb{R}^n \). The distance and the projection from a point \( s \) to \( K \) are defined respectively by

\[ d(s, K) := \inf_{x \in K} \| s - x \|, \quad \text{proj}(s; K) := x \in K \text{ such that } d(s, K) = \| s - x \|. \]

The minimal norm element of \( K \) is defined by

\[ K^0 := \text{proj}(0; K). \]

The Hausdorff distance between two closed, convex sets \( K_1, K_2 \) is given by

\[ d_H(K_1, K_2) := \max \{ \sup_{x_1 \in K_1} d(x_1, K_2), \sup_{x_2 \in K_2} d(x_2, K_1) \}. \]

We define the **indicator function** \( i_K(\cdot) \) as follows

\[ i_K(x) := \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{if } x \notin K. \end{cases} \]

The **normal cone** of a closed convex set \( K \) is given by

\[ N_K(x) := \partial i_K(x) = \{ x^* \in H : \langle x^*, y - x \rangle \leq 0, \forall y \in K \}. \]

**Definition 1** A matrix \( P \in \mathbb{R}^{n \times n} \) is called
– **positive semidefinite** if for all \( x \in \mathbb{R}^n \), we have
\[
\langle Px, x \rangle \geq 0;
\]
– **positive definite** if there exists \( \alpha > 0 \) such that for all \( x \in \mathbb{R}^n \), we have
\[
\langle Px, x \rangle \geq \alpha \|x\|^2;
\]
– **symmetric** if \( P = P^T \), i.e., for all \( x, y \in \mathbb{R}^n \), we have
\[
\langle Px, y \rangle = \langle x, Py \rangle.
\]

We have the following fundamental lemma.

**Lemma 1** Let \( D \) be a positive semidefinite matrix. Then there exists some constant \( c_1 > 0 \) such that for all \( x \in \text{rge}(D + D^T) \), we have:
\[
\langle Dx, x \rangle \geq c_1 \|x\|^2.
\]

**Remark 1** Indeed, \( c_1 \) can be chosen as the small positive eigenvalue of \( D + D^T \) if \( D + D^T \neq 0 \).

Let be given some matrices \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}, C \in \mathbb{R}^{m \times n} \) and \( D \in \mathbb{R}^{m \times m} \).

**Definition 2** The system \((A, B, C, D)\) is **passive** if there exists a symmetric positive definite matrix \( P \in \mathbb{R}^{n \times n} \) such that for all \( x \in \mathbb{R}^n, y \in \mathbb{R}^m \), we have
\[
\langle PAx, x \rangle + \langle (PB - C^T)y, x \rangle - \langle Dy, y \rangle \leq 0,
\]
or equivalently, the matrix
\[
\begin{pmatrix}
P A + A^T P & PB - C^T \\
B^T P - C & -(D + D^T)
\end{pmatrix}
\]
is positive semidefinite.

We provide a characterization for passive systems, see also [14] for another characterization.

**Lemma 2** The matrix \( D \) is positive semidefinite and \( \ker(D + D^T) \subset \text{rge}(PB - C^T) \) for some symmetric positive definite matrix \( P \) if and only if the system \((kI, B, C, D)\) is passive for some \( k \in \mathbb{R} \).

**Proof** \((\Leftarrow)\) See, e.g., [13] Proposition 3 or [4] Lemma 1.

\((\Rightarrow)\) Since \( D \) is positive semidefinite there exists some \( c_1 > 0 \) such that for all \( y \in \mathbb{R}^m \), we have
\[
\langle Dy, y \rangle = \langle Dy^{im}, y^{im} \rangle \geq c_1 \|y^{im}\|^2,
\]
where \( y^{im} \) is the orthogonal projection of \( y \) onto \( \text{rge}(D + D^T) \). Similarly, there exists some \( \alpha > 0 \) such that for all \( x \in \mathbb{R}^n \)
\[
\langle Px, x \rangle \geq \alpha \|x\|^2.
\]

We choose \( k < 0 \) satisfying the inequality
\[
2\sqrt{(-k)\alpha c_1} \geq \|PB - C^T\|.
\]
Lemma 4. Let \( x, y \in \mathbb{R}^m \), we obtain
\[
\langle (P(kI)x, x) + (PB^T - C^T)y, x \rangle - \langle Dy, y \rangle
\]
\[
k(\langle P, x \rangle) + (PB^T - C^T)\langle y, x \rangle - \langle Dy, y \rangle
\]
(since \( \ker(D + D^T) \subset \ker(PB - C^T) \))
\[
\leq k\alpha\|x\|^2 + \|PB - C^T\|\|x\||\|y\| - c_1\|y\|^2 \leq 0.
\]

Thus \((kI, B, C, D)\) is passive. □

Definition 3. A set-valued mapping \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) is called monotone if for all \( x, y \in \mathbb{R}^n, x^* \in F(x), y^* \in F(y) \), one has \( \langle x^* - y^*, x - y \rangle \geq 0 \). In addition, it is called maximal monotone if there is no monotone operator \( G \) such that the graph of \( F \) is contained strictly in the graph of \( G \).

Proposition 1. Let \( H \) be a Hilbert space, \( F : H \rightrightarrows H \) be a maximal monotone operator and let \( \lambda > 0 \). Then
1) the resolvent of \( F \) defined by \( J^\lambda_F := (I + \lambda F)^{-1} \) is a non-expansive and single-valued map from \( H \) to \( H \).
2) the Yosida approximation of \( F \) defined by \( F^\lambda := \frac{1}{\lambda}(I - J^\lambda_F) = (\lambda I + F^{-1})^{-1} \) satisfies
   i) for all \( x \in H, F^\lambda(x) \in F(J^\lambda_Fx) \),
   ii) \( F_\lambda \) is Lipschitz continuous with constant \( \frac{1}{\lambda} \) and also maximal monotone.
   iii) If \( x \in \text{dom}(F) \), then \( \|F^\lambda x\| \leq \|F^0 x\| \), where \( F^0 x \) is the element of \( Fx \) of minimal norm.

Let us recall Minty’s Theorem in the setting of Hilbert spaces (see [7,8]).

Proposition 2. Let \( H \) be a Hilbert space. Let \( F : H \rightrightarrows H \) be a monotone operator. Then \( F \) is maximal monotone if and only if \( \text{rge}(F + I) = H \).

Let be given two maximal monotone operators \( F_1 \) and \( F_2 \), we recall the definition of pseudo-distance between \( F_1 \) and \( F_2 \) introduced by Vladimirov [23] as follows
\[
\text{dis}(F_1, F_2) := \sup \left\{ \frac{\langle \eta_i - \eta_j, z_i - z_j \rangle}{\|\eta_i\| + \|\eta_j\|} : \eta_i \in F(z_i), z_i \in \text{dom}(F_i), i = 1, 2 \right\}.
\]

Lemma 3. If \( F_i = N_{A_i} \), where \( A_i \) is a closed convex set \((i = 1, 2)\) then
\[
\text{dis}(F_1, F_2) = d_H(A_1, A_2).
\]

Lemma 4. Let \( F_1, F_2 \) be two maximal monotone operators. For \( \lambda > 0, \delta > 0 \) and \( x \in \text{dom}(F_1) \), we have
\[
\|x - J^\lambda_{F_1} (x)\| \leq \lambda\|F^0_1 x\| + \text{dis}(F_1, F_2) + \sqrt{\lambda(1 + \|F^0_1 x\|)\text{dis}(F_1, F_2)}
\]
\[
\leq \lambda\|F^0_1 x\| + \text{dis}(F_1, F_2) + (\delta\text{dis}(F_1, F_2) + \lambda(1 + \|F^0_1 x\|)) \quad \frac{4\delta}{\delta}
\]
\[
\leq \frac{\lambda(1 + (4\delta + 1)\|F^0_1 x\|)}{4\delta} + (1 + \delta)\text{dis}(F_1, F_2).\]
Lemma 5 [19] Let \( F_n \) be a sequence of maximal monotone operators in a Hilbert space \( H \) such that \( \text{d} \text{is}(F_n,F) \to 0 \) as \( n \to +\infty \) for some maximal monotone operator \( F \). Suppose that \( x_n \in \text{dom}(F_n) \) with \( x_n \to x \) and that \( y_n \in F_n(x_n) \) with \( y_n \to y \) weakly for some \( x,y \in H \). Then \( x \in \text{dom}(F) \) and \( y \in F(x) \).

Let us end-up this section by recalling some versions of Gronwall’s inequality.

Lemma 6 Let \( \alpha > 0 \) and \( (u_n), (\beta_n) \) be non-negative sequences satisfying

\[
    u_n \leq \alpha + \sum_{k=0}^{n-1} \beta_k u_k \quad \forall n = 0, 1, 2, \ldots \quad (\text{with } \beta_{-1} := 0).
\]

Then, for all \( n \), we have

\[
    u_n \leq \alpha \exp\left(\sum_{k=0}^{n-1} \beta_k\right).
\]

Lemma 7 Let \( T > 0 \) be given and \( a(\cdot), b(\cdot) \in L^1([0,T];\mathbb{R}) \) with \( b(t) \geq 0 \) for almost all \( t \in [0,T] \). Let an absolutely continuous function \( w : [0,T] \to \mathbb{R}_+ \) satisfy

\[
    (1 - \alpha)w'(t) \leq a(t)w(t) + b(t)w^\alpha(t), \quad \text{a.e. } t \in [0,T]
\]

where \( 0 \leq \alpha < 1 \). Then for all \( t \in [0,T] \), we have

\[
    w^{1-\alpha}(t) \leq w^{1-\alpha}(0)\exp\left(\int_0^t a(\tau)d\tau\right) + \int_0^t \exp\left(\int_s^t a(\tau)d\tau\right)b(s)ds.
\]

3 Main results

In this section, the well-posedness and asymptotic behaviour of problem \((S)\) are studied. From (1b) and (1c) of \((S)\), it is easy to compute \( \lambda(\cdot) \) in term of \( x(\cdot) \):

\[
    \lambda(t) = -\langle N_{K(t,x(t))}^{-1} + D \rangle^{-1}(Cx(t)), \quad \text{a.e. } t \geq 0.
\]

Therefore, we can rewrite the system \((S)\) in the form of first order differential inclusion as follows

\[
    \dot{x}(t) \in f(t,x(t)) - B(N_{K(t,x(t))}^{-1} + D)^{-1}(Cx(t))
    = f(t,x(t)) - B\Phi(t,x(t),x(t)) \quad \text{a.e. } t \geq 0,
\]

where

\[
    \Phi(t,x,y) := (N_{K(t,y)}^{-1} + D)^{-1}Cx, \quad t \geq 0, x,y \in \mathbb{R}^n.
\]

Suppose that the following assumptions hold.

Assumption 1: The set-valued mapping \( K : [0, +\infty) \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) has non-empty, closed convex values such that \( K \cap \text{rge}(C) \) has non-empty values and there exist \( L_{K1} \geq 0, 0 \leq L_{K2} \leq \frac{\|C\|}{\|C\|} \) such that for all \( t,s \geq 0 \) and \( x,y \in \mathbb{R}^n \) we have

\[
    d_H(K(t,x) \cap \text{rge}(C), K(s,y) \cap \text{rge}(C)) \leq L_{K1}|t-s| + L_{K2}\|x-y\|,
\]
where \( c_2 > 0 \) is the smallest positive eigenvalue of \( CC^T \).

**Assumption 2**: The matrix \( D \) is positive semidefinite with \( \text{rge}(D) \subset \text{rge}(C) \) and

\[
\ker(D + D^T) \subset \ker(PP - C^T)
\]

for some symmetric positive definite matrix \( P \).

**Assumption 3**: For all \( t \geq 0 \), if \( (N_{K(t,y)}^{-1} + D)^{-1}Cx \neq \emptyset \) for some \( x, y \in \mathbb{R}^n \), it holds that \( \text{rge}(D + D^T) \cap (N_{K(t,y)}^{-1} + D)^{-1}Cx \neq \emptyset \).

**Assumption 4**: For all \( t \in [0,T] \), \( x \in \mathbb{R}^n : \text{rge}(C) \cap \text{rint}((N_{K(t,x)}^{-1} + D)) \neq \emptyset \).

**Assumption 5**: The function \( f : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuous in the first variable and \( L_f \)-Lipschitz continuous in the second variable in the sense that there exist a continuous function \( v_f : [0, +\infty) \rightarrow \mathbb{R} \) and \( L_f > 0 \) such that for all \( s, t \geq 0 \) and \( x, y \in \mathbb{R}^n \) we have

\[
\|f(t,x) - f(s,y)\| \leq |v_f(t) - v_f(s)| + L_f\|x - y\|.
\]

**Remark 2** It is not difficult to relax that the moving set \( K \) varies in an absolutely continuous way with respect to the time. For simplicity of calculation, we suppose that \( K \) moves Lipschitz continuously in both time and state.

Let be given some arbitrary real number \( T > 0 \). The following lemmas are useful.

**Lemma 8** **There exists a constant** \( c_2 > 0 \) **such that for all** \( x \in \text{rge}(CC^T) \), **we have**

\[
\langle CC^T x, x \rangle \geq c_2\|x\|^2. \tag{10}
\]

**Proof** It is easy to see that the matrix \( CC^T \) is symmetric positive semidefinite and the conclusion follows thanks to Lemma 1. In addition, \( c_2 \) can be chosen as the smallest positive eigenvalue of \( CC^T \) if \( C \neq 0 \).

**Lemma 9** *Let Assumption 1 hold. Suppose that*

\[
a_i \in N_{K(t_i,x_i)}(b_i), \quad a_i \in \mathbb{R}^n, \quad b_i \in \text{rge}(C), \quad x_i \in \mathbb{R}^n, \quad t_i \geq 0 \quad (i = 1, 2). \tag{11}
\]

*Then*

\[
\langle a_1 - a_2, b_1 - b_2 \rangle \geq -(\|a_1\| + \|a_2\|)((L_{K_1}|t_2 - t_1| + L_{K_2}\|x_1 - x_2\|)) \tag{12}
\]

*where the constant numbers* \( L_{K_1} \) *and* \( L_{K_2} > 0 \) *are defined in Assumption 1.*

**Proof** We have

\[
\langle a_1, z - b_1 \rangle \leq 0, \quad \text{for all} \quad z \in K(t_1, x_1). \tag{13}
\]

Note that \( b_2 \in K(t_2, x_2) \cap \text{rge}(C) \subset K(t_1, x_1) \cap \text{rge}(C) + (L_{K_1}|t_2 - t_1| + L_{K_2}\|x_1 - x_2\|)B \). Combining with the last inequality, we obtain

\[
\langle a_1, b_2 - b_1 \rangle \leq (L_{K_1}|t_2 - t_1| + L_{K_2}\|x_1 - x_2\|)|a_1|. \tag{14}
\]
Similarly, one has
\[
(a_2, b_1 - b_2) \leq (L_{K_1}|t_2 - t_1| + L_{K_2}\|x_1 - x_2\|)\|a_2\|.
\] (15)

From (14) and (15), we deduce that
\[
(a_1 - a_2, b_1 - b_2) \geq -\left(||a_1|| + ||a_2||\right)\left(L_{K_1}|t_2 - t_1| + L_{K_2}\|x_1 - x_2\|\right),
\]
and the conclusion follows.

**Lemma 10** Let Assumption 5 hold. Then there exists \(a_1 > 0\) such that
\[
\|f(t, x)\| \leq a_1(1 + ||x||), \quad \text{for all } t \in [0, T], \ x \in \mathbb{R}^n.
\] (16)

**Proof** We have
\[
\|f(t, x)\| \leq L_f\|x\|,
\] (17)
and the conclusion follows with \(a_1 := \max_{t \in [0, T]} \|f(t, 0)\|, \ L_f\). □

**Lemma 11** Let Assumptions 1, 2, 3 hold. Then there exist \(a_2, a_3 > 0\) such that the single-valued minimal-norm function \(\Phi^0 : [0, T] \times \mathbb{R}^{2n} \to \text{rge}(D + D^T), (t, x, y) \mapsto \Phi^0(t, x, y)\) satisfies the following properties:

a) \(\|\Phi^0(t, x, y)\| \leq a_2(1 + \|x\| + \|y\|), \ \forall (t, x, y) \in \text{dom}(\Phi^0).
\)

b) \(\|\Phi^0(t, x_1, y_1) - \Phi^0(t, x_2, y_2)\|^2 \leq a_3\|x_1 - x_2\|^2 + a_3(||\Phi^0(t, x_1, y_1)|| + ||\Phi^0(t, x_2, y_2)||)(t_1 - t_2) + ||y_1 - y_2||\), \ \forall (t, x_1, y_1) \in \text{dom}(\Phi^0), i = 1, 2.

**Proof** a) Given \((t, x, y) \in \text{dom}(\Phi^0)\). Then \((N_{K(t, y)}^{-1} + D)^{-1}Cx \neq 0\). Using Assumption 3, we can find some \(z_0 \in \text{rge}(D + D^T) \cap (N_{K(t, y)}^{-1} + D)^{-1}(Cx) = \text{rge}(D + D^T) \cap \Phi(t, x, y)\). First, we prove that \(\Phi^0(t, x, y) = z_0 \in \text{rge}(D + D^T)\). Indeed, for each \(z_1 \in \Phi(t, x, y)\), it is sufficient to show that \(\|z_1\| \geq \|z_0\|\). We can write uniquely \(z_1 = z_1^{im} + z_1^{ker}\) where \(z_1^{im} \in \text{rge}(D + D^T)\), \(z_1^{ker} \in \text{ker}(D + D^T)\) and \(\{z_1^{im}, z_1^{ker}\} = 0\). One has
\[
z_1 \in (N_{K(t, y)}^{-1} + D)^{-1}(Cx) \Leftrightarrow z_1 \in N_{K(t, y)}(C - Dz_0), \ i = 0, 1.
\] (18)

The monotonicity of \(N_{K(t, y)}\) and \(D\) allows us to deduce that \((D(z_0 - z_1), z_0 - z_1) = 0\), or equivalently \(z_1^{im} + z_1^{ker} - z_0 \in \text{ker}(D + D^T)\). Therefore, \(z_1^{im} - z_0 \in \text{ker}(D + D^T) \cap \text{rge}(D + D^T) = \{0\}\). Consequently
\[
\|z_1\|^2 = \|z_1^{im}\|^2 + \|z_1^{ker}\|^2 = \|z_0\|^2 + \|z_1^{ker}\|^2 \geq \|z_0\|^2,
\] (19)
and thus, we have \(\Phi^0(t, x, y) = z_0 \in \text{rge}(D + D^T)\).

Now, fix \((0, x_0, 0) \in \text{dom}(\Phi^0)\), where \(x_0\) is an initial point of problem (S).

Similarly as in (18) and using Lemma 3 one obtains
\[
\begin{align*}
&\langle C(x - x_0), \Phi^0(t, x, y) - \Phi^0(0, x_0, x_0) \rangle \\
&\geq \langle D(\Phi^0(t, x, y) - \Phi^0(0, x_0, x_0)), \Phi^0(t, x, y) - \Phi^0(0, x_0, x_0) \rangle \\
&\geq \langle \|\Phi^0(t, x, y)\| + \|\Phi^0(0, x_0, x_0)\|\rangle(TL_{K_1} + L_{K_2})\|y - x_0\| \\
&\geq c_1\|\Phi^0(t, x, y) - \Phi^0(0, x_0, x_0)\|^2 \\
&\geq \langle \|\Phi^0(t, x, y)\| + \|\Phi^0(0, x_0, x_0)\|\rangle(TL_{K_1} + L_{K_2})\|y - x_0\|,
\end{align*}
\] (20)
where \( c_1 > 0 \) is defined in Lemma 1. Thus we can find some \( \beta > 0 \) such that
\[
\| \Phi^0(t, x, y) \|^2 \leq \| \Phi^0(t, x, y) \| (\beta \| x \| + \beta \| y \| + \beta) + \beta(\| x \| + \beta \| y \| + 1)
\]
and the conclusion follows with \( \alpha_2 := 2\beta + 1 \).

b) Similarly as in (20), for all \((t_i, x_i, y_i) \in \text{dom}(\Phi^0), i = 1, 2\) we have
\[
\langle C(x_1 - x_2), \Phi^0(t_1, x_1, y_1) - \Phi^0(t_2, x_2, y_2) \rangle \geq c_1 \| \Phi^0(t_1, x_1, y_1) - \Phi^0(t_2, x_2, y_2) \|^2
- (\| \Phi^0(t_1, x_1, y_1) \| + \| \Phi^0(t_2, x_2, y_2) \| )(L_{K1}|t_1 - t_2| + L_{K2}\| y_1 - y_2 \|).
\]
(21)
Let us note that
\[
\langle C(x_1 - x_2), \Phi^0(t_1, x_1, y_1) - \Phi^0(t_2, x_2, y_2) \rangle \\
\leq \frac{c_1}{2} \| \Phi^0(t_1, x_1, y_1) - \Phi^0(t_2, x_2, y_2) \|^2 + \| C \|^2 \| x_1 - x_2 \|^2,
\]
and hence we obtain the conclusion. \( \square \)

**Lemma 12** Suppose that \( P \equiv I \), the identity matrix. Then for all \( t, x, y \in \text{dom}(\Phi) \), we have
\[
BP(t, x, y) = (B - C^T)\Phi(t, x, y) + C^T\Phi(t, x, y)
\]
\[
= (B - C^T)\Phi^0(t, x, y) + C^T\Phi(t, x, y).
\]

**Proof** It is sufficient to prove that \((B - C^T)\Phi\) is single-valued function and \((B - C^T)\Phi(t, x, y) = (B - C^T)\Phi^0(t, x, y)\). Let \( z \in \Phi(t, x, y) \). Similarly as in the proof of Lemma 11 we can write \( z = \Phi^0(t, x, y) + z_{\text{ker}} \), where \( z_{\text{ker}} \) is the projection of \( z \) onto \( \ker(D + D^T) \). Since \( \ker(D + D^T) \subseteq (B - C^T) \), we have \( (B - C^T)z = (B - C^T)(\Phi^0(t, x, y) + z_{\text{ker}}) = (B - C^T)\Phi^0(t, x, y) \) and the proof is completed.

Let us recall the following result, which is firstly given in [21] for \( D = 0 \), see also [3].

**Lemma 13** Let be given two closed convex set \( K_1, K_2 \) such that \( K_i \cap \text{rge}(C) \neq \emptyset \), \( \text{rge}(D) \subseteq \text{rge}(C) \) and let \( G_i := C^T(N_{K_i}^{-1} + D)^{-1}C, i = 1, 2 \). Then
\[
\text{dis}(G_1, G_2) \leq \frac{\| C \|^2}{c_2} d_H(K_1 \cap \text{rge}(C), K_2 \cap \text{rge}(C)),
\]
(22)
where \( c_2 > 0 \) is defined in Lemma 3.

**Proof** We have
\[
\text{dis}(G_1, G_2)
\leq \sup \left\{ \frac{\| \eta_1 - \eta_2, z_2 - z_1 \|}{1 + \| \eta_1 \| + \| \eta_2 \|} : \eta_i \in C^T(N_{K_i}^{-1} + D)^{-1}Cz_i, i = 1, 2 \right\}
\leq \sup \left\{ \frac{\| \mu_1 - C^T \mu_2, z_2 - z_1 \|}{1 + \| C^T \mu_1 \| + \| C^T \mu_2 \|} : \mu_i \in (N_{K_i}^{-1} + D)^{-1}Cz_i, i = 1, 2 \right\}
\leq \sup \left\{ \frac{\| \mu_1 - \mu_2, Cz_2 - Cz_1 \|}{1 + \| C^T \mu_1 \| + \| C^T \mu_2 \|} : \mu_i \in N_{K_i}(Cz_i - D\mu_i), z_i, i = 1, 2 \right\}
\leq \sup \left\{ \frac{\| \mu_1 - \mu_2, (Cz_2 - D\mu_1) - (Cz_1 - D\mu_1) \|}{1 + \| C^T \mu_1 \| + \| C^T \mu_2 \|} : \mu_i \in N_{K_i}(Cz_i - D\mu_i), z_i, i = 1, 2 \right\}.
\]
since \(D\) is positive semidefinite.

Let \(w_1 := \text{proj}(Cz_2 - D\mu_2, K_1 \cap \text{rge}(C))\) and \(w_2 := \text{proj}(Cz_1 - D\mu_1, K_2 \cap \text{rge}(C))\). Then we have

\[
\begin{aligned}
\langle \mu_1, (Cz_2 - D\mu_2) - (Cz_1 - D\mu_1) \rangle \\
= \langle \mu_1, Cz_2 - D\mu_2 - w_1 \rangle + \langle \mu_1, w_1 - (Cz_1 - D\mu_1) \rangle \\
\leq \langle \mu_1, Cz_2 - D\mu_2 - w_1 \rangle \quad \text{(using the property of normal cone)} \\
= \langle \nu_1, Cz_2 - D\mu_2 - w_1 \rangle \quad \text{where } \nu_1 := \text{proj}(\mu_1, \text{rge}(CC^T)) \\
\leq \|\nu_1\| d_H(\text{rge}(C)) \\
\leq \|C\| c_2 \|C^T\nu_1\| d_H(\text{rge}(C), K_2 \cap \text{rge}(C)))
\end{aligned}
\]

where the second equality holds since \(\mu_1 - \nu_1 \in \text{ker}(CC^T) = \text{ker}(C^T) \cap \text{rge}(C) \subset \text{rge}(C)\) and the third inequality is satisfied because

\[
c_2 \|\nu_1\|^2 \leq \langle CC^T\nu_1, \nu_1 \rangle \leq \|C\| \|C^T\nu_1\| \|\nu_1\|.
\]

Similarly one has

\[
\langle \mu_2, (Cz_1 - D\mu_1) - (Cz_2 - D\mu_2) \rangle \leq \frac{\|C\| c_2 \|C^T\nu_2\| d_H(\text{rge}(C), K_2 \cap \text{rge}(C)))}{1 + \|C^T\nu_1\| + \|C^T\nu_2\|}
\]

where \(\nu_2 := \text{proj}(\mu_2, \text{rge}(CC^T))\). From (24) and (25), one has

\[
\begin{aligned}
\langle \mu_1 - \mu_2, (Cz_2 - D\mu_2) - (Cz_1 - D\mu_1) \rangle \\
\leq \frac{\|C\| \|C^T\nu_1\| + \|C^T\nu_2\|}{c_2 d_H(\text{rge}(C), K_2 \cap \text{rge}(C))} d_H(\text{rge}(C), K_2 \cap \text{rge}(C))
\end{aligned}
\]

and the conclusion follows.

Now we are ready for the first main result about the existence, uniqueness of strong solutions and the Lipschitz continuous dependence of solutions on the initial conditions. Let us define the admissible set

\[
\mathcal{A} := \{x_0 \in \mathbb{R}^n : (N_{K(0, x_0)}^{-1} + D)^{-1}Cx_0 \neq 0\}.
\]

**Theorem 1** (Existence) Let Assumptions 1, 2, 3, 4, 5 hold. Then for each \(x_0 \in \mathcal{A}\), there exists a solution \(x(\cdot; x_0)\) defined on \([0, T]\) of problem (\(S\)) which is Lipschitz continuous.

**Proof** From Assumption 2, there exists \(\kappa \in \mathbb{R}\) such that \((\kappa I, B, C, D)\) is passive by using Lemma 2. By using change of variables, without loss of generality, we can suppose that \(P = I\), the identity matrix (see, e.g., [14]). Let us use the following implicit scheme to approximate [5].
Let be given some positive integer $n$. Let $h_n = T/n$ and $t^n_i = ih$ for $0 \leq i \leq n$. For $0 \leq i \leq n - 1$, we can find the sequence $(x^n_i)_{0 \leq i \leq n}$ with $x^n_0 = x_0$ as follows:

$$
\begin{align*}
    y^n_i &= x^n_i + h_n f(t^n_i, x^n_i) - h_n \kappa x^n_i \\
    x^n_{i+1} &\in y^n_i - h_n F^n_{i+1, x^n_i}^{-1}(x^n_{i+1}),
\end{align*}
$$

(28)

where $F^n_{i+1, x^n_i} := -\kappa I + B(N^{-1}_K(t^n_{i+1}, x^n_i) + D)^{-1}C$ is a maximal monotone operator (see, e.g., [14, 45]). Then we can compute $x^n_{i+1}$ uniquely as follows

$$
x^n_{i+1} = (I + h_n F^n_{i+1, x^n_i})^{-1}(y^n_i) = J^h_{F^n_{i+1, x^n_i}}(y^n_i)
$$

where $J^h_\lambda$ denotes the resolvent of $F$ of index $\lambda$ which is non-expansive. Consequently, one can obtain the algorithm to construct the sequences $(x^n_i)_{i=0}^n$ as follows.

**Algorithm**

**Initialization.** Let $x^n_0 := x_0$, $y^n_0 := x^n_0 + h_n f(t^n_0, x^n_0) - h_n \kappa x^n_0$.

**Iteration.** For the current points $x^n_i$ we can compute

$$
y^n_i := x^n_i + h_n f(t^n_i, x^n_i) - h_n \kappa x^n_i,
$$

and

$$
x^n_{i+1} := J^h_{F^n_{i+1, x^n_i}}(y^n_i).
$$

(29)

Clearly, the algorithm is well-defined and $x^n_{i+1} \in \text{dom}(F^n_{i+1, x^n_i}) = \text{dom}(\Phi(t^n_{i+1}, x^n_i))$ for $i = 0, ..., n - 1$. On the other hand, using Lemma 12 we can rewrite (28) as follows

$$
x^n_{i+1} \in x^n_i + h_n f(t^n_i, x^n_i) + h_n \kappa(x^n_{i+1} - x^n_i) - h_n(B - C^T)\Phi^n(t^n_{i+1}, x^n_{i+1}, x^n_i) - h_n G^n_{i+1, x^n_i}(x^n_{i+1}) \in z^n_i - h_n G^n_{i+1, x^n_i}(x^n_{i+1}),
$$

(30)

where

$$
z^n_i := x^n_i + h_n f(t^n_i, x^n_i) + h_n \kappa(x^n_{i+1} - x^n_i) - h_n(B - C^T)\Phi^n(t^n_{i+1}, x^n_{i+1}, x^n_i),
$$

and $G^n_{i+1, x^n_i} := C^T \Phi^n(t^n_{i+1}, x^n_i) C^T = (N^{-1}_K(t^n_{i+1}, x^n_i) + D)^{-1}C$ is a maximal monotone operator with $\text{dom}(G^n_{i+1, x^n_i}) = \text{dom}(\Phi(t^n_{i+1}, x^n_i))$. Therefore, we can also compute the $x^n_{i+1}$ as follows

$$
x^n_{i+1} = (I + h_n G^n_{i+1, x^n_i})^{-1}(x^n_i) = J^h_{G^n_{i+1, x^n_i}}(z^n_i).
$$

(31)

Let us note that

$$
\|(G_{t,x})^0(x)\| \leq \|C^T\|\|\Phi^0(t, x, y)\| \leq \alpha_2\|C^T\|(1 + \|x\| + \|y\|),
$$

(32)

where $\alpha_2$ is defined in Lemma 11. From (31), we have

$$
\|x^n_{i+1} - x^n_i\| = \|J^h_{G^n_{i+1, x^n_i}}(z^n_i) - x^n_i\|
\leq \|J^h_{G^n_{i+1, x^n_i}}(z^n_i) - J^h_{G^n_{i+1, x^n_i}}(x^n_i)\| + \|J^h_{G^n_{i+1, x^n_i}}(x^n_i) - x^n_i\|.
$$

(33)
Since $J_{G_{t_{i+1}}^{x^n}}^{b_n}$ is non-expansive, one has
\[
\|J_{G_{t_{i+1}}^{x^n}}^{b_n}(z^n_i) - J_{G_{t_{i+1}}^{x^n}}^{b_n}(x^n_i)\| \leq \|z^n_i - x^n_i\| \\
\leq h_n(\|f(t^n_i, x^n_i)\| + \kappa |x^n_{i+1} - x^n_i| + \|B - C^T\| \|\Phi(t^n_{i+1}, x^n_i, x^n_{i+1})\|) \\
\leq h_n(\alpha_1(1 + \|x^n_i\|) + \kappa (\|x^n_{i+1}\| + \|x^n_i\|) + \alpha_2 \|B - C^T\| (1 + \|x^n_i\| + \|x^n_{i+1}\|) \\
\leq h_n(\alpha_1 + \alpha_2 \|B - C^T\| + \kappa (1 + \|x^n_i\| + \|x^n_{i+1}\|)).
\] (34)

Let us chose some constant $\delta > 0$ such that
\[
\tilde{L}_K := (1 + \delta)L_{K2}\|C\|_{c_2} < 1.
\] (35)

Note that $x^n_i \in \text{dom}(G_{t^n_{i+1}}, x^n_{i+1})$ for $i = 0, \ldots, n - 1$ with $x^n_{-1} := x^n_0$, by using Lemmas 4, 13, Assumption 1 and (32) we obtain
\[
\|J_{G_{t_{i+1}}^{x^n}}^{b_n}(x^n_i) - x^n_i\| \leq h_n \frac{1 + (4\delta + 1)\|G_{t^n_{i+1}}^{x^n_{i+1}}(x^n_i)\|}{4\delta} + (1 + \delta)\text{dis}(G_{t_{i+1}}^{x^n_i}, G_{t_{i+1}}^{x^n_{i+1}}) \\
\leq h_n \frac{1 + (4\delta + 1)\alpha_2 \|C^T\| (1 + \|x^n_i\| + \|x^n_{i+1}\|)}{4\delta} \\
+ (1 + \delta)L_{K1}\|C\|_{c_2} + (1 + \delta)L_{K2}\|C\|_{c_2}\|x^n_i - x^n_{i+1}\|).
\] (36)

From (33), (34), (35) and (36), we can find some constant $\alpha_4 > 0$ such that
\[
\|x^n_{i+1} - x^n_i\| \leq h_n\alpha_4 (1 + \|x^n_{i+1}\| + \|x^n_i\| + \|x^n_{i+1}\|) \\
+ \tilde{L}_K\|x^n_i - x^n_{i+1}\|
\] (37)

where $\tilde{L}_K < 1$. Note that $x^n_{-1} := x^n_0$, therefore we have
\[
\|x^n_{i+1} - x^n_i\| \leq h_n\alpha_4 \sum_{j=0}^i \tilde{L}_K^j (1 + \|x^n_{i-j+1}\| + \|x^n_{i-j}\| + \|x^n_{i-j-1}\|) \\
\leq h_n\alpha_4 \frac{1}{1 - \tilde{L}_K} \sum_{j=0}^i \tilde{L}_K^j (\|x^n_{i-j+1}\| + \|x^n_{i-j}\| + \|x^n_{i-j-1}\|).
\] (38)

Consequently
\[
\|x^n_{i+1} - x^n_0\| \leq \sum_{j=0}^i \|x^n_{j+1} - x^n_j\| \\
\leq h_n\alpha_4 \left( \frac{i + 1}{1 - \tilde{L}_K} + \|x^n_{i+1}\| + \sum_{j=0}^i \tilde{L}_K^j \sum_{j=0}^i \|x^n_j\| \right) \\
\leq \frac{\alpha_4 T}{1 - \tilde{L}_K} + h_n\alpha_4 \|x^n_{i+1}\| + \frac{3h_n\alpha_4}{1 - \tilde{L}_K} \sum_{j=0}^i \|x^n_j\|.
\]

We can choose $n$ large enough such that $h_n\alpha_4 < 1/2$. Then we have
\[
\|x^n_{i+1}\| \leq \beta + \alpha_2 h_n \sum_{j=0}^i \|x^n_j\|,
\]
where
\[
\beta := \frac{\beta}{1 - \tilde{L}_K}.
\]
Combining with (38), we have

Consequently the sequence of functions

Using the discrete Gronwall’s inequality, one has

Lipschitz. Using Arzelà–Ascoli theorem, there exist a Lipschitz function

In particular,

Then, for all

We construct the sequences of functions

for each positive integer

and a subsequence, still denoted by

We define the operators

\[ \eta(t) \to t^n_i. \]

\[ \hat{x}_n(t) = \frac{x_{i+1}^n - x_i^n}{h_n}(t - t_i^n), \]

and

\[ \theta_n(t) = t^n_i, \quad \eta_n(t) = t^n_{i+1}. \]

Then, for all

\[ \sup_{t \in [0,T]} \{ |\theta_n(t) - t|, |\eta_n(t) - t| \} \leq h_n \to 0 \text{ as } n \to +\infty. \]

Consequently the sequence of functions \( (x_n(\cdot))_n \) is uniformly bounded and equi-Lipschitz. Using Arzelà–Ascoli theorem, there exist a Lipschitz function \( x(\cdot) : [0,T] \to \mathbb{R}^n \) and a subsequence, still denoted by \( (x_n(\cdot))_n \), such that

\( x_n(\cdot) \) converges strongly to \( x(\cdot) \) in \( C([0,T]; \mathbb{R}^n) \);
\( \hat{x}_n(\cdot) \) converges weakly to \( \hat{x}(\cdot) \) in \( L^2([0,T]; \mathbb{R}^n) \).

In particular, \( x(0) = x_0 \). In addition, from (39), (41) and (42) we obtain

\[ \hat{x}_n(t) = f_n(t) + \kappa(x_n(\eta_n(t)) - x_n(\theta_n(t))) \]

\[ - (B - C^T)\Phi(\eta_n(t), x_n(\eta_n(t)), x_n(\theta_n(t))) \]

\[ - G_{\eta_n(t), \theta_n(t)} x_n(\eta_n(t)), \]

where \( f_n(t) := f(\theta_n(t), x_n(\theta_n(t))) \). We define the operators \( G, G_n : L^2([0,T]; \mathbb{R}^n) \to L^2([0,T]; \mathbb{R}^n) \) for each positive integer \( n \) as follows

\[ w^* \in G(w) \iff w^*(t) \in G_{t,x(t)}(w(t)) \text{ a.e. } t \in [0,T] \]

and

\[ w^* \in G_n(w) \iff w^*(t) \in G_{\eta_n(t), \theta_n(t)}(w(t)) \text{ a.e. } t \in [0,T]. \]
Using Minty’s theorem, we can conclude that $G_n, G$ are maximal monotone operators since for each $t \in [0, T]$, the operators $G_{t,x(t)}$ and $G_{\theta_n(t),x_n(\theta_n(t))}$ are maximal monotone. In addition, one has

\[
\text{dis}(G_n, G) = \sup \left\{ \frac{\int_0^T (z^*_n(t) - z^*(t), z_n(t) - z(t)) dt}{1 + \|z^*_n\|_2 + \|z^*\|_2} : z^*_n \in G_n(z_n), z^* \in G(z) \right\}
\]

\[
\leq \sup \left\{ \frac{\int_0^T \text{dis}(G_{\theta_n(t),x_n(\theta_n(t))}, G_{t,x(t)})(1 + \|z^*_n(t)\| + \|z^*(t)\|) dt}{1 + \|z^*_n\|_2 + \|z^*\|_2} : z^*_n \in G_n(z_n), z^* \in G(z) \right\}
\]

( using the definition of dis($G_{\theta_n(t),x_n(\theta_n(t))}, G_{t,x(t)})$)

\[
\leq \frac{\|C\|}{c_2} \sup \left\{ \frac{\int_0^T (L_{K_1}\eta_n(t) - I)_n + L_{K_2}\|x_n(\theta_n(t) - x(t))\|_2(1 + \|z^*_n(t)\| + \|z^*(t)\|) dt}{1 + \|z^*_n\|_2 + \|z^*\|_2} : z^*_n \in G_n(z_n), z^* \in G(z) \right\}
\]

( using Lemma \[13\] and Assumption 1)

\[
\leq \frac{\|C\|}{c_2} (L_{K_1}2\eta_n - I)_n + L_{K_2}\|x_n \circ \theta_n - x\|_2 \sup \left\{ \frac{1 + \|z^*_n\|_2 + \|z^*\|_2}{1 + \|z^*_n\|_2 + \|z^*\|_2} : z^*_n \in G_n(z_n), z^* \in G(z) \right\}
\]

\[
= \frac{\|C\|}{c_2} (L_{K_1}2\eta_n - I)_n + L_{K_2}\|x_n \circ \theta_n - x\|_2 \rightarrow 0,
\]

as $n \rightarrow +\infty$.

Using Assumption 5, Lemma \[14\], and the fact that $\dot{x}_n$ converges weakly to $\dot{x}$ in $L^2([0, T]; \mathbb{R}^n)$, we have

\[
\dot{x}_n - f_n - \kappa(x_n \circ \eta_n - x_n \circ \theta_n) + (B - C^T)\Phi^0(\eta_n, x_n \circ \eta_n, x_n \circ \theta_n)
\]

converges weakly in $L^2([0, T]; \mathbb{R}^n)$ to

\[
\dot{x} - f(\cdot, x) + (B - C^T)\Phi^0(\cdot, x, x).
\]

On the other hand, $x_n \circ \eta_n$ converges strongly $x$ in $L^2([0, T]; \mathbb{R}^n)$. Combing with \[14\] and using Lemma \[13\] we deduce that

\[
\dot{x} - f(\cdot, x) + (B - C^T)\Phi^0(\cdot, x, x) \in -G(x), \quad (45)
\]

or equivalently

\[
\dot{x}(t) - f(t, x(t)) + (B - C^T)\Phi^0(t, x(t), x(t)) \in -G_{t,x(t)}(x(t)), \quad a.e. \ t \in [0, T]. \quad (46)
\]

Consequently, one has

\[
\dot{x}(t) \in f(t, x(t)) - (B - C^T)\Phi^0(t, x(t), x(t)) - G_{t,x(t)}(x(t))
\]

\[
= f(t, x(t)) - (B - C^T)\Phi^0(t, x(t), x(t)) - C^T(N^{-1}_{K(t,x)} + D)^{-1}Cx(t)
\]

\[
= f(t, x(t)) - B(N^{-1}_{K(t,x)} + D)Cx(t), \quad a.e. \ t \in [0, T], \quad (47)
\]

and the conclusion follows. \[\square\]
Remark 3 (i) Our discretization method provides a feasible way to study the state-dependent Lur’e dynamical systems for the first time. In addition, it is remarkable that the obtained solutions are strong.

(ii) If \( B = C = I, D = 0 \) then problem \((S)\) becomes the well-known state-dependent sweeping process. Then \( c_2 = 1 \) and \( L_{K_2} < \frac{1}{c_2} = 1 \), which is accordant with the result developed in [20]. In addition, the authors in [20] provided some examples to show that the existence of solutions may lack if \( L_{K_2} \geq 1 \) and mentioned that we may not have the uniqueness of solutions even for \( L_{K_2} < 1 \). So the upper bound of \( L_{K_2} \) in Assumption 1 is optimal for our existence result.

(iii) However for the uniqueness and Lipschitz dependence of solutions on initial conditions, we can obtain the positive answer thanks to the positive semidefiniteness of \( D \), if the moving set \( K \) has a special form, namely it can be decomposed as a sum of a time-dependent moving set and a single-valued Lipschitz function.

Assumption 1’: Suppose that

\[
K(t, x) = K_1(t) + h(t, x), \quad t \geq 0, x \in \mathbb{R}^n,
\]

where \( K_1 : [0, +\infty) \to \mathbb{R}^m \) has non-empty, closed convex values and \( h : [0, +\infty) \times \mathbb{R}^n \to \text{rge}(D+D^T) \) is a single-valued mapping. In addition, there exist \( L_h, L_{h1}, L_{h2} \geq 0 \) such that for all \( s, t \geq 0 \) and \( x, y \in \mathbb{R}^n \), we have

\[
\text{dis}_H(K_1(t), K_1(s)) \leq L_{h1}|t - s|,
\]

\[
||h(t, x) - h(s, y)|| \leq L_{h2}|t - s| + L_h||x - y||.
\]

Lemma 14 Let Assumption 1’ hold. Suppose that

\[ a_i \in N_{K(t,x_i)}(b_i) \quad \text{for} \quad (a_i, b_i) \in \mathbb{R}^{2m}, x_i \in \mathbb{R}^n, \ t_i \geq 0 \ (i = 1, 2). \]

Then

\[
\langle a_1 - a_2, b_1 - b_2 \rangle \geq \langle a_1 - a_2, h(t, x_1) - h(t, x_2) \rangle.
\] (48)

Proof We have

\[ a_i \in N_{K(t,x_i)}(b_i) = N_{K_1(t)+f(t,x_i)}(b_i) = N_{K_1(t)}(b_i - f(t, x_i)). \]

Since the normal cone of a convex set is monotone, we deduce that

\[
\langle a_1 - a_2, b_1 - h(t, x_1) - b_2 + h(t, x_2) \rangle \geq 0,
\]

and the conclusion follows.

\[ \square \]

Theorem 2 (Uniqueness and Lipschitz dependence on initial condition) Let Assumptions 1’, 2, 3, 4, 5 hold. Then for each \( x_0 \in \mathbb{R}^n \) such that \( x_0 \in A \), problem \((S)\) has a unique solution \( x(\cdot; x_0) \) on \([0, T]\). In addition, the mapping \( x_0 \mapsto x(\cdot; x_0) \) is Lipschitz continuous.
Proof It is easy to see that Assumption 1 implies Assumption 1, so the existence of solutions is obtained. Now, let \( x_i \) be a solution of \((S)\) with the initial condition \( x_i(0) = x_{i0}, \ i = 1, 2 \). We have

\[
\begin{aligned}
\begin{cases}
\dot{x}_i(t) = f(t, x_i(t)) - B y_i(t), \\
y_i(t) \in (N^{-1}_K(t, x_i(t))) + D)^{-1}(C x_i(t)), \ a.e. \ t \in [0, T].
\end{cases}
\end{aligned}
\tag{49}
\]

The inclusion in \((49)\) is equivalent to

\[
y_i(t) \in N_K(t, x_i(t))(C x_i(t) - D y_i(t)).
\]

Using Lemma 14 we obtain that

\[
\begin{aligned}
&\langle y_1(t) - y_2(t), (C x_1(t) - D y_1(t)) - (C x_2(t) - D y_2(t)) \rangle \\
&\geq \langle y_1(t) - y_2(t), h(t, x_1(t)) - h(t, x_2(t)) \rangle \\
&= \langle g^{im}_1(t) - g^{im}_2(t), h(t, x_1(t)) - h(t, x_2(t)) \rangle \ 
(\text{since } \text{rge}(h) \subset \text{rge}(D + D^T)) \\
&\geq -L_h\|g^{im}_1(t) - g^{im}_2(t)\|\|x_1(t) - x_2(t)\|,
\end{aligned}
\]

where \( g^{im} \) denotes the projection of \( y \) onto \( \text{rge}(D + D^T) \). Therefore

\[
\begin{aligned}
&\langle y_1(t) - y_2(t), C x_1(t) - C x_2(t) \rangle \\
&\geq \langle y_1(t) - y_2(t), D y_1(t) - D y_2(t) \rangle - L_h\|g^{im}_1(t) - g^{im}_2(t)\|\|x_1(t) - x_2(t)\| \\
&\geq c_1\|g^{im}_1(t) - g^{im}_2(t)\|^2 - L_h\|g^{im}_1(t) - g^{im}_2(t)\|\|x_1(t) - x_2(t)\|,
\end{aligned}
\tag{50}
\]

where \( c_1 > 0 \) is defined in Lemma 1. Hence

\[
\begin{aligned}
&\langle B y_1(t) - B y_2(t), x_1(t) - x_2(t) \rangle \\
&= \langle y_1(t) - y_2(t), (C x_1(t) - C x_2(t)) + ((B - C^T)(y_1(t) - y_2(t), x_1(t) - x_2(t)) \\
&\geq c_1\|g^{im}_1(t) - g^{im}_2(t)\|^2 - L\|g^{im}_1(t) - g^{im}_2(t)\|\|x_1(t) - x_2(t)\| \\
&\text{(where } L := L_h + \|B - C^T\|) \\
&\geq \frac{-L^2}{4c_1}\|x_1(t) - x_2(t)\|^2 \ 
(\text{use the inequality } a^2 + b^2 \geq 2ab, \forall a, b \in \mathbb{R}).
\end{aligned}
\]

Consequently, we have

\[
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|x_1(t) - x_2(t)\|^2 &= \langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t) \rangle \\
&= \langle f(t, x_1(t)) - f(t, x_2(t)) - (B y_1(t) - B y_2(t)), x_1(t) - x_2(t) \rangle \\
&\leq (L_f + \frac{L^2}{4c_1})\|x_1(t) - x_2(t)\|^2 = \gamma\|x_1(t) - x_2(t)\|^2,
\end{aligned}
\]

where \( \gamma := L_f + \frac{L^2}{4c_1} \). Using Gronwall’s inequality, we obtain that

\[
\|x_1(t) - x_2(t)\| \leq \|x_1(0) - x_2(0)\|e^{\gamma t} \leq \|x_{10} - x_{20}\|e^{\gamma T}, \forall \ t \in [0, T],
\]

and the conclusion follows. \( \square \)

Remark 4 Since \( T > 0 \) is arbitrary, one can define the unique solution \( x(\cdot; x_0) \) of problem \((\hat{S})\) on \([0, +\infty)\). Now we are interested in the asymptotic behaviour of the problem \((\hat{S})\), i.e., the behaviour of solutions when the time is large.
\textbf{Theorem 3} (Globally exponential attractivity) Let all the assumptions of Theorem 1\textsuperscript{3} hold. In addition, suppose that

\[
(f(t, x), x) \leq -\sigma \|x\|^2, \quad 0 \in h(t, 0) + K_1(t) = K(t, 0), \quad \forall t \geq 0, x \in \mathbb{R}^n, \tag{51}
\]

for some \( \sigma > \frac{(L_h + \|B - C^T\|)^2}{4\delta_1} \). Then the unique solution \( x(\cdot) \) of (S) starting at a given point \( x_0 \) exponentially converges to the origin when the time is large, i.e.,

\[
\|x(t)\| \leq e^{-\delta t} \|x_0\| \to 0 \text{ as } t \to +\infty,
\]

where \( \delta := \sigma - \frac{(L_h + \|B - C^T\|)^2}{4\sigma_1} > 0 \).

\textbf{Proof} The unique solution \( x(\cdot) \) satisfies

\[
\dot{x}(t) = f(t, x(t)) - B y(t), \quad y(t) \in (N_{K(t,x(t))}^{-1} + D)^{-1}(C x(t)), \quad \text{a.e. } t \geq 0.
\]

Then

\[
y(t) \in N_{K(t,x(t))}(C x(t) - D y(t)) = N_{K(t+x(t))}(C x(t) - D y(t)) = N_{K_1(t)}(C x(t) - D y(t) - h(t, x(t))).
\]

Since \(-h(t,0) \in K_1(t)\), we have

\[
\langle y(t), C x(t) - D y(t) - h(t, x(t)) + h(t,0) \rangle \geq 0.
\]

Thus

\[
\langle y(t), C x(t) \rangle \geq \langle y(t), D y(t) \rangle + \langle y(t), h(t, x(t)) - h(t,0) \rangle \\
\geq c_1\|y^m(t)\|^2 + \langle y^m(t), h(t, x(t)) - h(t,0) \rangle \quad (\text{since } \text{rge}(h) \subset \text{rge}(D + D^T)) \\
\geq c_1\|y^m(t)\|^2 - L_h\|y^m(t)\|\|x(t)\|.
\]

Note that

\[
\frac{d}{dt} \left( \frac{1}{2} \|x(t)\|^2 \right) = \langle \dot{x}(t), x(t) \rangle = \langle f(t, x(t)) - B y(t), x(t) \rangle \\
\leq -\sigma \|x(t)\|^2 - \langle (B - C^T) y(t), x(t) \rangle - \langle y(t), C x(t) \rangle \\
\leq -\sigma \|x(t)\|^2 - \langle (B - C^T) y^m(t), x(t) \rangle + L_h\|y^m(t)\|\|x(t)\| - c_1\|y^m(t)\|^2 \\
\leq -\sigma \|x(t)\|^2 - \langle \|B - C^T\| + L_h \rangle \|y^m(t)\|\|x(t)\| - c_1\|y^m(t)\|^2 \\
\leq -\delta \|x(t)\|^2.
\]

Therefore

\[
\frac{d}{dt} (e^{2\delta t} \|x(t)\|^2) \leq 0,
\]

which implies that

\[
\|x(t)\| \leq e^{-\delta t} \|x_0\| \to 0, \quad \text{as } t \to +\infty.
\]

\( \Box \)

\textbf{Remark} 5 If only all the assumptions of Theorem 1\textsuperscript{1} are satisfied, we may not have the uniqueness of solutions. However if the moving set always contains the origin, then all solutions starting at a given point \( x_0 \in \mathcal{A} \) also tend to zeros when the time is large.
Theorem 4 (Globally exponential attractivity without uniqueness) Suppose that all the assumptions of Theorem 1 are satisfied. Furthermore, assume that
\[ \langle f(t, x), x \rangle \leq -\sigma \|x\|^2, \quad 0 \in K(t, x), \quad \forall t \geq 0, x \in \mathbb{R}^n, \] (52)
for some \( \sigma > \frac{\|B - CT\|^2}{4c_1} \). Then any solution \( x(t) \) of (S) starting at a given point \( x_0 \) exponentially converges to the origin when the time is large, i.e.,
\[ \|x(t)\| \leq e^{-\delta t} \|x_0\| \to 0 \text{ as } t \to +\infty, \]
where \( \delta := \sigma - \frac{\|B - CT\|^2}{4c_1} > 0 \).

Proof Similarly as in the proof of Theorem 1 we know that for almost \( t \geq 0 \), one has
\[ \dot{x}(t) = f(t, x(t)) - By(t), \]
where
\[ y(t) = \mathcal{N}_{K(t, x(t))}(Cx(t) - Dy(t)). \]
The fact \( 0 \in K(t, x(t)) \) deduces that
\[ \langle y(t), 0 - Cx(t) + Dy(t) \rangle \leq 0. \]
Thus
\[ \langle y(t), Cx(t) \rangle \geq \langle y(t), Dy(t) \rangle \geq c_1 \|y^{im}(t)\|^2. \]

Therefore
\[ \frac{d}{dt} \frac{1}{2} \|x(t)\|^2 = \langle \dot{x}(t), x(t) \rangle = \langle f(t, x(t)) - By(t), x(t) \rangle \]
\[ \leq -\sigma \|x(t)\|^2 - \langle (B - CT)y(t), x(t) \rangle - \langle y(t), Cx(t) \rangle \]
\[ \leq -\sigma \|x(t)\|^2 - \langle (B - CT)y^{im}(t), x(t) \rangle - c_1 \|y^{im}(t)\|^2 \]
\[ \leq -\sigma \|x(t)\|^2 - \|B - CT\| \|y^{im}(t)\| \|x(t)\| - c_1 \|y^{im}(t)\|^2 \]
\[ \leq -\delta \|x(t)\|^2. \]
Consequently, we have
\[ \|x(t)\| \leq e^{-\delta t} \|x_0\|, \quad \forall t \geq 0, \]
and the conclusion follows. \( \square \)

4 Application for studying time-varying Lur’e system with errors in data

For simplicity, we consider the function \( f \) as some matrix \( A \). Suppose that the matrices \( A, B, C, D \) and the time-varying set \( K \) satisfy all the assumptions of Theorem 1. Then problem (S) has a unique solution. However, assume that there
are errors in measure for the matrices \( A \) and \( C \), i.e., we have the approximate matrices \( \bar{A}, \bar{C} \) and we want to know whether the following system

\[
\begin{align*}
\dot{x}(t) &= \bar{A}x(t) + B\lambda(t) \text{ a.e. } t \in [0, +\infty); \\
y(t) &= \bar{C}x(t) + D\lambda(t), \\
\lambda(t) &\in -N_{K(t)}(y(t)), \quad t \geq 0; \\
x(0) &= x_0,
\end{align*}
\]

(53a) (53b) (53c) (53d)

has a solution. Generally, \( (\bar{A}, B, \bar{C}, D, K) \) may not satisfy Assumptions \((A_1) - (A_5)\) so we can not apply the result in [3]. Let us show that we can use our result to answer this question. Indeed, we can rewrite \((\bar{S})\) as follows

\[
\begin{align*}
\dot{x}(t) &= \hat{A}x(t) + B\lambda(t) \text{ a.e. } t \in [0, +\infty); \\
\hat{y}(t) &= \bar{C}x(t) + D\lambda(t), \\
\lambda(t) &\in -\hat{N}_{\hat{K}(t)}(\hat{y}(t)) + (\bar{C} - C)x(t)), \quad t \geq 0; \\
x(0) &= x_0,
\end{align*}
\]

(54a) (54b) (54c) (54d)

where \( \hat{K}(t, x) = K(t) - (\bar{C} - C)x \). Then the systems \((\hat{A}, B, C, D, K)\) satisfies all the assumptions of Theorem 1. Consequently, \((\bar{S})\) has a solution defined on \([0, +\infty)\). If \( \bar{C} = C + \varepsilon(D + D^T) \) for some \( \varepsilon > 0 \) small enough, then the solution is unique by using Theorem 2. In addition if \( 0 \in K(t) \) for all \( t \geq 0 \), and \( \hat{A} \leq -\sigma I \), i.e.,

\[
\langle \hat{A}x, x \rangle \leq -\sigma\|x\|^2, \quad \forall x \in \mathbb{R}^n,
\]

for \( \sigma > 0 \) is large enough, then the unique solution of \((\bar{S})\) converges to the origin at exponential rate (Theorem 3).

**Example 1** Let us consider

\[
\begin{align*}
A &= -\sigma I_2, \quad B = D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = B + \varepsilon I_2, \quad K(t) = [f_1(t), +\infty) \times [f_2(t), +\infty)
\end{align*}
\]

for some \( \sigma, \varepsilon > 0 \) where \( f_1, f_2 : [0, +\infty) \to \mathbb{R} \) are two absolutely continuous functions. Then there no exists a positive symmetric matrix \( P \in \mathbb{R}^{n \times n} \) such that

\[
\ker(D + D^T) \subset \ker(PB - C^T).
\]

Therefore we can not apply the result in [3] but can use our result to deduce the existence of solutions for the associated dynamical system. Indeed, we can see that \((A, B, C, D, \bar{K})\) satisfies all assumptions of Theorem 1 where \( \bar{K}(t, x) = [f_1(t) - \varepsilon x_1, +\infty) \times [f_2(t) - \varepsilon x_2, +\infty) \).

**Remark 6** This application also suggests an idea to consider the time-varying Lur’e dynamical system when \((A, B, C, D, K)\) does not satisfy Assumptions \((A_1) - (A_5)\) by modifying the matrix \( C \) and reduce the time-varying system into the state-dependent one.
5 Conclusions

The paper studies the well-posedness and asymptotic behaviour for a class of Lur’e dynamical systems where the set-valued feedback depends not only on the time but also on the state. Let us emphasis that the obtained solutions are strong, comparing with the weak solutions acquired in [3]. The main tool is a new implicit discretization scheme, which is an advantage for implementation in numerical simulations. Some conditions are given to obtain the exponential attractivity of the solutions.

References

1. V. Acary, B. Brogliato, Numerical Methods for Nonsmooth Dynamical Systems. Applications in Mechanics and Electronics. Springer Verlag, LNACM 35, 2008.
2. A. Tanwani, B. Brogliato, C. Prieur Stability and observer design for Lur’e systems with multivalued , non-monotone, time-varying nonlinearities and state jumps, SIAM J. Control Opti., Vol. 52, No. 6, pp. 3639–3672, 2014.
3. A. Tanwani, B. Brogliato, C. Prieur Well-Posedness and Output Regulation for Implicit Time-Varying Evolution Variational Inequalities, SIAM J. Control Opti., Vol. 56, No. 2, pp. 751–781, 2018.
4. S. Adly, A. Hantoute, B. K. Le, Nonsmooth Lur’e Dynamical Systems in Hilbert Spaces, Set-Valued Var. Anal, vol 24, iss. 1, 13-35, 2016.
5. S. Adly, A. Hantoute, B. K. Le, Maximal Monotonicity and Cyclic-Monotonicity Arising in Nonsmooth Lur’e Dynamical Systems”, Journal of Mathematical Analysis and Applications, 448 (2017), no. 1, 691–706
6. S. Adly, B. K. Le, Stability and invariance results for a class of non-monotone set-valued Lur’e dynamical systems, Applicable Analysis, vol. 93, iss. 5 (2014), 1087–1105.
7. J. P. Aubin, A. Cellina, Differential Inclusions. Set-Valued Maps and Viability Theory, Springer-Verlag, Berlin, 1984.
8. H. Brezis, Op´erateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert, Math. Studies 5, North-Holland American Elsevier (1973).
9. B. Brogliato, A. Daniilidis, C. Lemaréchal, V. Acary, On the equivalence between complementarity systems, projected systems and differential inclusions, Systems and Control Letters, vol. 55, no 1, pp. 45–51, January 2006.
10. B. Brogliato Absolute stability and the Lagrange-Dirichlet theorem with monotone multivalued mappings, Systems and Control Letters 2004, 51 (5), 343-353.
11. B. Brogliato, D. Goeleven, Well-posedness, stability and invariance results for a class of multivalued Lur’e dynamical systems, Nonlinear Analysis: Theory, Methods and Applications, vol. 74, pp. 195–212, 2011.
12. B. Brogliato, D. Goeleven, Existence, uniqueness of solutions and stability of nonsmooth multivalued Lur’e dynamical systems, Journal of Convex Analysis, vol. 20, no. 3, pp. 881–900, 2013.
13. B. Brogliato, R. Lozano, B. Maschke, O. Egeland, Dissipative Systems Analysis and Control, Springer-verlag London, 2nd Edition, 2007.
14. M. K. Camlibel, J. M. Schumacher, Linear passive systems and maximal monotone mappings, Math. Program. 157 (2), pp 397–420, 2016.
15. M. G. Cojocaru, P. Daniele, A. Nagurney, Projected dynamical systems and evolutionary variational inequalities via Hilbert spaces and applications, JOTA 127 No. 3 (2005), 549–563.
16. P. Grabowski, F. M. Callier, Lur’e feedback systems with both unbounded control and observation: Well-posedness and stability using nonlinear semigroups, Nonlinear Analysis 74, 3065-3085, 2011.
17. J. Gwinner, On differential variational inequalities and projected dynamical systems -equivalence and a stability result, Discrete and Continuous Dynamical Systems 2007, issue special, 467–476.
18. J. Gwinner, On a new class of differential variational inequalities and a stability result, Math. Prog. 2015, vol. 139, issue 1-2, 205–221.
19. M. Kunze, M.D.P. Monteiro Marques, BV solutions to evolution problems with time-dependent domains, Set-Valued Anal. 5 (1997) 57–72.
20. M. Kunze, M.D.P. Monteiro Marques, An introduction to Moreau’s sweeping process, in “Impacts in Mechanical Systems. Analysis and Modelling”, (B. Brogliato, Ed), 1-60, Springer, Berlin, 2000.
21. B. K. Le, Existence of Solutions for Sweeping Processes with Local Conditions, submitted.
22. M. R. Liberzon, Essays on the absolute stability theory, Automation and Remote Control, vol. 67, no 10, pp. 1610–1644, October 2006.
23. A. A. Vladimirov, Nonstationary dissipative evolution equations in a Hilbert space, Nonlinear Anal. 17 (1991) 499–518.