Shotgun assembly threshold for lattice labeling model

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Abstract
We study the shotgun assembly problem for the lattice labeling model, where i.i.d. uniform labels are assigned to each vertex in a $d$-dimensional box of side length $n$. We wish to recover the labeling configuration on the whole box given empirical profile of labeling configurations on all boxes of side length $r$. We determine the threshold around which there is a sharp transition from impossible to recover with probability tending to 1, to possible to recover with an efficient algorithm with probability tending to 1. Our result sharpens a constant factor in a previous work of Mossel and Ross (IEEE Trans Netw Sci Eng 6(2):145–157, 2019) and thus solves a question therein.

Keywords Shotgun assembly · Lattice labeling model · Phase transition

Mathematics Subject Classification Primary 60C05; Secondary 05C78 · 62B10

1 Introduction

For shotgun assembly problems of labeled graphs, the general goal is to recover a global structure from local observations. This set of problems has substantial interests in applications such as DNA sequencing [2, 5, 13] and recovery of neural networks [9]. We learned the precise formulation and the general mathematical framework for shotgun assembly problems from the inspiring paper [11]. Inspired by [11] (which has been circulated since 2015), there has been extensive study on shotgun assembly problems for various models including random jigsaw problems [3, 4, 8, 10], random graph models [1, 6, 7, 12], random coloring model [15] and some extension of DNA sequencing model [16].

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In this paper we study the shotgun assembly problem for the lattice labeling model, whose precise mathematical formulation was proposed in [11]. For $d \geq 1$, let $\Lambda_n = \{v \in \mathbb{Z}^d : |v|_\infty \leq n - 1, v_i \geq 0 \text{ for all } 1 \leq i \leq d \}$ be the box of $n^d$ vertices with the origin $o \in \mathbb{Z}^d$ being its smallest corner (here $| \cdot |_\infty$ denotes the $\ell_\infty$-norm of a vector, and we say $x \leq y$ if $x_i \leq y_i$ for $1 \leq i \leq d$). For $q \geq 1$, let $\sigma_\tau$ be i.i.d. labels uniformly sampled from $\{1, \ldots, q\}$. For $r \geq 1$, let $\mathcal{B}_r = \mathcal{B}_{n,r}$ be the collection of all $r$-boxes (an $r$-box is a box with $r^d$ vertices) contained in $\Lambda_n$. For $B \in \mathcal{B}_r$, let $\tau_B$ be the translation which maps $B$ to $\Lambda_r$. As in [11], we wish to recover $\sigma_v$ for all $v \in \Lambda_n$ from the empirical profile $\{\sigma|_B : B \in \mathcal{B}_{n,r}\}$ where $\sigma|_B = \{\sigma_{\tau_B^{-1}(v)} : v \in \Lambda_r\}$. In words, our observations are labeling configurations in all the $r$-boxes without information on locations for these $r$-boxes. (Note that in our formulation, we choose to assume that the orientation of the $r$-box is observable to us, and the similar cases when the labeling configuration is only known up to rotation/reflection symmetry can be treated by our method similarly; see Sect. 4). We say that the labeling configuration is non-identifiable if there exist two different labeling configurations on $\Lambda_n$ with the same empirical profile $\{\sigma|_B : B \in \mathcal{B}_{n,r}\}$; otherwise we say that the labeling configuration is identifiable. Previously, the best result was due to [11] which provided upper and lower bounds on the identifiability threshold up to a multiplicative constant factor. Our main contribution determines the sharp identifiability threshold, which solves [11, Question 1.3] (in fact, we also find the explicit formula for the threshold which was mentioned as a challenging problem in [11]).

**Theorem 1.1** The following hold for any fixed $\epsilon > 0$.

For $d = 1$, with probability tending to 1 as $n \to \infty$ the labeling configuration is identifiable when $r \geq \frac{2(1+\epsilon) \log n}{\log q}$ and non-identifiable when $r \leq \frac{2(1-\epsilon) \log n}{\log q}$.

For $d \geq 2$, with probability tending to 1 as $n \to \infty$ the labeling configuration is identifiable when $r^d \geq \frac{d(1+\epsilon) \log n}{\log q}$ and non-identifiable when $r^d \leq \frac{d(1-\epsilon) \log n}{\log q}$.

Furthermore, in the aforementioned identifiable regimes, the labeling can be recovered by a polynomial-time algorithm.

**Remark 1.2** Note that the identifiability threshold for $d = 1$ was known in a much more precise manner from previous works [2, 5, 11] (see [5, Theorem 1] and [11, Proposition 3.2]): [5] follows from an observation of [14] that the identifiability is equivalent to the existence of a unique Eulerian path on graphs defined on vertices formed by sub-strings, and this also motivated some considerations in [11]. We record the result for $d = 1$ here only for completeness, and in fact we also provide a proof for non-identifiability for $d = 1$ as it seems to provide some intuition that can perhaps be grasped more easily than the proof of [5]. An interesting question is whether some extension of Eulerian path consideration would apply in higher dimensions and how that is related to our proof method. We are not quite sure about this for the following two reasons: (1) there seem to be multiple choices in building the analogous graph in higher dimensions and it is not immediately clear to us that one version of this gives a necessary and sufficient condition for identifiability; (2) perhaps more importantly it is unclear to us that it is the most productive way to try to first formulate a sufficient and necessary condition for identifiability and then to prove whether this condition occurs or not using probabilistic arguments. As one will see in our approach, we used two
related structural properties to prove identifiability and non-identifiability and *a priori*

it is unclear whether these structural properties provide a necessary and sufficient

condition for identifiability but it just turns out that both structural properties deeply

depend on whether a typical $r$-box is unique or not and as a result this allows us to

establish the sharp threshold.

**Remark 1.3** We see that there is a conceptual difference between $d = 1$ and $d \geq 2$,

which is essentially rooted in the fact that there is a phase transition for percolation

when $d \geq 2$ but not when $d = 1$. This point will be further manifested in our proof

strategy (see discussions at the beginning of Sect. 3).

**Remark 1.4** When proving non-identifiability for $d \geq 2$, we show that there exist two

subsets whose original labels are 1 and 2 such that after swapping their labels the

empirical profile for local labeling configurations on $r$-boxes remains the same (see

Proposition 2.1). We feel the second moment method we employed for the proof of

Proposition 2.1 is somewhat novel and may be useful in other contexts (see discussions

that follow Proposition 2.1 for more details).

**Remark 1.5** Theorem 1.1 holds in the case where two labeling configurations on a box

$\Lambda_n$ are viewed the same if one can be mapped to the other by rotation and/or reflection.

We discuss briefly the minor modifications required for the proof in Sect. 4.

**Remark 1.6** We expect that our method should also be useful for i.i.d. labels with a

non-uniform distribution. A natural guess is that the threshold should be determined

by $r$ at which the configuration in a typical $r$-box is unique. That being said, we do

expect some complications to arise due to the fact that some configurations in an $r$-box

is much more likely than some others, and in particular it is possible that higher order

norms for the distribution vector may play a role (see also [11, Eq. (3.3)]). Further, our

method may shed some light on models without independence, but the dependence

seems to incur substantial challenge which our current method fails to address. We

think it would be an interesting direction to consider random labels with spatial mixing

such as the Ising model in high temperatures. We expect that our method would be

helpful but even in this setting the challenge seems to be substantial enough for us to

make any convincing guess. Another interesting future direction is to investigate the

situation when observations are noisy.

**Notations:** throughout the paper, we write $a_n = O(b_n)$ if there is a constant $0 < c < 

\infty$ such that $a_n \leq cb_n$ for all $n \in \mathbb{Z}_+$. We write $a_n = o(b_n)$, $a_n \ll b_n$ or $b_n \gg a_n$

if $a_n/b_n \to 0$ as $n \to \infty$. For $u, v \in \mathbb{Z}^d$, we write $|u|_\infty = \max_{1 \leq i \leq d} |u_i|$ for the

$l_\infty$-norm of $u$ and we write $|u - v|_\infty = \max_{1 \leq i \leq d} |u_i - v_i|$ for the $l_\infty$-distance

between $u$ and $v$. The diameter of a set $A \subset \Lambda_n$ is defined as $\max_{u,v \in A} |u - v|_\infty$, and the $l_\infty$-distance between two sets $A, B \subset \mathbb{Z}^d$ is defined as $\min_{u \in A, v \in B} |u - v|_\infty$.

For two sets $A, B \subset \mathbb{Z}^d$ and $s > 0$, we write $A \subset_s B$ if $A \subset B$ and $A$ has pairwise

$l_\infty$-distance at least $s$. For two integers $a \leq b$, we write $[a, b] = \{c \in \mathbb{Z} : a \leq c \leq b\}$. 
2 Proof of non-identifiability

2.1 The case for $d = 1$

We first provide the proof of non-identifiability for $d = 1$, which is the (much) easier part of our main theorem. In this subsection, we assume that

$$q^r \leq n^{2(1-\epsilon)}$$ for an arbitrary fixed small $\epsilon > 0$. \hfill (2.1)

Let $I_1, \ldots, I_6$ be 6 disjoint and consecutive intervals in $\Lambda_n$, each of which has $m = \lfloor n/6 \rfloor$ vertices. For each $I_j$, let $\Gamma_j$ be a collection of $\ell = \lfloor m/r \rfloor$ disjoint intervals of $r$ vertices in $I_j$. We will show that with probability tending to 1, there exist $B_j \in \Gamma_j$ for $j = 1, 3, 4, 6$ such that

$$\sigma|_{B_1} = \sigma|_{B_4} \text{ and } \sigma|_{B_3} = \sigma|_{B_6}$$ \hfill (2.2)

and that

$$\sigma|_{J[1,m]} \neq \sigma|_{J'[1,m]}$$ \hfill (2.3)

where $J$ is the interval strictly between $B_1$ and $B_3$ and $J'$ is the interval strictly between $B_4$ and $B_6$, and $J[1,m]$ is the initial segment of $J$ with $m$ vertices (similarly for $J'[1,m]$). We see that $J$ and $J'$ divide $\Lambda_n$ into 5 disjoint intervals, and we list them from left to right as $K_1, J, K_2, J', K_3$. Let $\tau$ be a bijection on $\Lambda_n$ so that $\tau$ maps integers $0, \ldots, n-1$ as a sequence obtained from concatenating $K_1, J', K_2, J, K_3$ (i.e., we swap $J$ with $J'$—note that this can be done even when $J, J'$ have different lengths). On the event described in our claim, we see that the labeling configuration $\sigma'$ with $\sigma'(v) = \sigma(\tau(v))$ preserves the empirical profile on $r$-boxes but $\sigma' \neq \sigma$. See Fig. 1 for an illustration of this swapping procedure.
It remains to prove the claim. Let \( Z = \sum_{B \in \Gamma_1, B' \in \Gamma_4} \mathbf{1}_{\sigma_B = \sigma_{B'}}. \) Then, from straightforward computations, we have \( EZ = \ell^2 q^{-r} \geq n^{\epsilon'} \) for some \( \epsilon' > 0. \) In addition,

\[
EZ^2 \leq EZ + \ell^4 q^{-2r} = (1 + o(1))(EZ)^2,
\]

where the inequality follows from the fact that \( P(\sigma_B = \sigma_{B'}, \sigma_B = \sigma_{B'}) = q^{-2r} \) for \( B, B' \in \Gamma_1, B', B' \in \Gamma_4 \) as long as \( B \neq B \) or \( B' \neq B'. \) By Chebyshev’s inequality, we see that \( Z \geq n^{\epsilon'}/2 \) with probability tending to 1. A similar computation applies to \( \Gamma_3 \) and \( \Gamma_6. \) So this ensures the existence of \( B_1, B_3, B_4, B_6 \) satisfying (2.2). Finally, a simple union bound yields that with probability tending to 1 for any disjoint intervals \( K \) and \( K' \) which contain \( I_2 \) and \( I_5 \) respectively, we have \( \sigma_{|K[1,m]} \neq \sigma_{|K'[1,m]} \) (this is because \( P(\sigma_{|K[1,m]} = \sigma_{|K'[1,m]} < e^{-cn} \) for some constant \( c > 0 \) and the number of choices for such \( K, K' \) is only polynomial in \( n \)). This verifies (2.3) and thus completes the proof.

### 2.2 The case for \( d \geq 2 \)

In this subsection we consider the non-identifiable regime where \( d \geq 2 \) and

\[
q^{rd} \leq n^{d(1-\epsilon)} \quad \text{for an arbitrary fixed small} \ \epsilon > 0. \tag{2.4}
\]

For \( v \in \Lambda_n' = \{u \in \Lambda_n : r \leq u_i \leq n - r \text{ for all } 1 \leq i \leq d\}, \) let \( \mathfrak{R}_v = \{R_1(v), \ldots, R_{rd}(v)\} \) be the collection of all \( r \)-boxes containing \( v. \) We also set the notation so that the relative location of \( v \) in \( R_j(v) \) is the same as the relative location of \( u \) in \( R_j(u) \) for all \( u, v \in \Lambda_n'. \) For \( U \subset \Lambda_n', \) we denote \( L(U) = (L_1(U), \ldots, L_{rd}(U)) \) where \( L_j(U) \) is the empirical distribution for \( \{\sigma_{|R_j(u)} : u \in U\}. \) We view \( L_j(U) \) as a \( q^{rd-1} \)-dimensional vector where its \( s \)-th coordinate \( L_{j,s}(U) \) counts the occurrences of the \( s \)-th configuration (the ordering of the configurations is arbitrary but prefixed); see Fig. 2 for an illustration of the definition for \( L(U) \). Denote \( V_k = \{v \in \Lambda_n' : \sigma_v = k\} \) for \( k \in \{1, \ldots, q\}. \)

**Proposition 2.1** Under the assumption (2.4) with probability tending to 1 as \( n \to \infty, \) there exist \( V_1' \subset ((2r)\mathbb{Z}^d) \cap V_1, V_2' \subset ((2r)\mathbb{Z}^d) \cap V_2 \) such that \( L(V_1') = L(V_2'). \)

Proposition 2.1 readily implies the non-identifiability since we can swap the labels of \( V_1' \) and \( V_2' \) without changing \( \{\sigma_B : B \in \mathfrak{R}_n\}. \) So our main goal in this subsection is to prove Proposition 2.1. Usually in order to prove the existence of such pair of sets one first shows that the first moment is large (as implied by Lemma 2.2 below) and then one needs to employ a second moment method. However, the implementation of a standard second moment method or even a second moment method with truncation would be quite challenging (if possible at all). This is because in our case by a first moment computation the size of desirable \( V_1' \) and \( V_2' \) has to be larger than \( n^{d-\epsilon'} \) for some \( \epsilon' > 0, \) and as a result a typical pair of sets would have significant overlap which results in significant amount of correlation. However, we manage to get around this challenge since we are flexible with the size of \( V_1' \) and \( V_2', \) that is, we only need to...
show for some $M'$ (but not a fixed $M'$) there exists a desired pair of $V'_1$ and $V'_2$ of size $M'$. The key novelty in our proof lies in the definition of $\chi_L$ and $E_k(U, L)$ (see (2.7)). We first show in Lemma 2.3 that for a typical $L$ we must have $\chi_L = 1 - o(1)$ since otherwise we would have the second moment of a random variable smaller than the square of its first moment. Once we show $\chi_L = 1 - o(1)$, it is straightforward to derive Proposition 2.1. Next, we carry out the proof details according to this outline.

For any $m \geq 1$ and $U \subset \Lambda'_n$ with $|U| = m$, we let $\mathcal{L}(U)$ be the space of all realizations for $L(U)$. In addition, we assume that $U \subset 2r/\Lambda'_n$, i.e., $U \subset \Lambda'_n$ is a set which has pairwise $\ell_\infty$-distance at least $2r$ (so $R_j(U)$ and $R'_j(U)$ are disjoint for different $u, v \in U$ and for all $1 \leq j, j' \leq r^d$). In this way, the law of $L(U)$ does not depend on the particular choice of $U$ except through $|U|$, and thus we can write $\mathcal{L}(U) = \mathcal{L}(|U|) = \mathcal{L}(m)$ for simplicity. We write $\mu_U = \mu_{|U|}$ for the probability measure of $L(U)$ on $\mathcal{L}(U)$. For any $\iota > 0$, we define

$$\mathcal{L}(U, \iota) = \mathcal{L}(|U|, \iota) = \{L \in \mathcal{L}(U) : \mu_U(L) \geq \iota\}. \quad (2.5)$$

For $\delta = \epsilon/100$, we let $M = n^{d(1-\delta)}$. The very basic intuition behind Proposition 2.1 is encapsulated in the following lemma, since it implies heuristically that most of $L \in \mathcal{L}(M)$ should appear.

Lemma 2.2 For $\iota_\ast = e^{-n^{d(1-\epsilon/2)}}$, we have $\mu_M(\mathcal{L}(M, \iota_\ast)) = 1 - o(1)$.

Proof For $U \subset 2r \Lambda'_n$ of cardinality $M$, we see that $0 \leq L_{j,s}(U) \leq M$ for all $1 \leq j \leq r^d, 1 \leq s \leq q'^{d-1}$ and thus $|\mathcal{L}(M)| \leq (M + 1)r^dq'^{d-1} \Lambda_1 \leq K$. Therefore,

$$\mu_M(\mathcal{L}(M) \setminus \mathcal{L}(M, \iota_\ast)) \leq \iota_\ast K.$$
and it remains to check that \( t_\ast = o(1/K) \). Simple algebraic manipulations yield that

\[
\log K = r^d q^{r^d-1} O(\log n) \leq O(1)n^{d(1-9\epsilon/10)} = o(\log t_\ast^{-1}),
\]
as required. \( \square \)

Let \( U_k = (2r)\mathbb{Z}^d \cap V_k \). Recall that the standard Chernoff bound for binomial random variables gives that \( \mathbb{P}(X < t) < \exp(-np - t^2/2np) \) for \( t < np \) if \( X \sim \text{Bin}(n, p) \). Since \( |U_k| \sim \text{Bin}(m, 1/q) \) where \( m \geq ((n-2r)/2r) > (n/4r)^d \), we have that for \( N = \frac{n^d}{q(16r)^d} \)

\[
\mathbb{P}(|U_k| < N) < \exp \left( -\frac{(n^d/q(4r)^d - N)^2}{2n^d/q(4r)^d} \right) \leq \exp \left( -\frac{1}{100}n^d/q(4r)^d \right) \ll n^{-100}
\]
for \( k = 1, \ldots, q \). Hence, by a union bound we get that

\[
\mathbb{P}(|U_k| \geq N \text{ for all } k = 1, \ldots, q) \geq 1 - n^{-4}. \tag{2.6}
\]

Without loss of generality we can assume that the event in (2.6) holds and as a result we may assume that \( |U_k| = N \) for \( k = 1, \ldots, q \) because if not, we can simply take a subset with cardinality \( N \) and name it as \( U_k \). For \( U \subset U_k \) and \( L \in \mathcal{L}(U) \), define

\[
\chi_L = \mathbb{P}(E_k(U, L) \mid L(U) = L), \tag{2.7}
\]
where

\[
E_k(U, L) = \bigcup_{W: W \subset U_k, W \neq U, W \cap U \neq \emptyset, |W| = |U|} \{L(W) = L\}.
\]

Note that on \( E_k(U, L) \) and \( L(U) = L \), there exist \( \emptyset \neq U' \subset U \) and \( W' \subset U_k \setminus U \) such that \( L(U') = L(W') \) (we can simply take \( U' = U \setminus W \) and \( W' = W \setminus U \) for an arbitrary \( W \) that certifies \( E_k(U, L) \)). Since we have assumed that \( |U_k| = N \) for all \( k \) and \( L(W') \) does not depend on \( \sigma_{W'} \) for \( W' \subset 2r \Lambda_n \), we see that the law of the empirical profiles \( \{L(W'): W' \subset U_k\} \) does not depend on \( k \). In addition, for any \( k' \neq k \), we have that conditioned on \( L(U) = L \), there is a coupling such that \( \{L(W'): W' \subset U_k\} \supset \{L(W'): W' \subset U_{k'}\} \). Therefore,

\[
\mathbb{P}\left( \bigcup_{\emptyset \neq U' \subset U, W' \subset U_k'} \{L(W') = L(U')\} \mid L(U) = L\right) \geq \chi_L. \tag{2.8}
\]

**Lemma 2.3** For each \( U \subset U_k \) with \( |U| = M \) and \( L \in \mathcal{L}(U, t_\ast) \), we have \( \chi_L \geq 1 - n^{-1} \).

**Proof** Suppose otherwise there exists \( L \in \mathcal{L}(U, t_\ast) \) with \( \chi_L \leq 1 - n^{-1} \). Define

\[
Z_L = \sum_{U \subset U_k: |U| = M} 1\{L(U) = L; (E_k(U, L))^c\},
\]
where \(1\{\cdot\}\) is an indicator function. Since \(\chi_L \leq 1 - n^{-1}\), we see that

\[
E Z_L \geq \left( \frac{N}{M} \right) \mu_M(L) n^{-1} \geq \left( \frac{N}{M} \right) \ell_n n^{-1}
\]

\[
\geq \left( \frac{N - M}{M} \right)^M e^{-n^{(1 - \epsilon/2)d} n^{-1}}
\]

\[
= \left( \frac{n^d / (q(16r)^d) - n^d(1 - \delta)}{n^d(1 - \delta)} \right)^{n^d(1 - \delta)} e^{-n^{(1 - \epsilon/2)d} n^{-1}}
\]

\[
\geq n^{(d\delta / 2) - n^d(1 - \delta)} e^{-n^{(1 - \epsilon/2)d} n^{-1}} \gg n^6,
\]

where the second inequality follows from \(L \in L(U, \iota^*_*)\). In addition, we can compute its second moment as

\[
E Z^2_L = \sum_{U, U' \subset U_k : |U| = |U'| = M} E(1\{L(U) = L; (E_k(U, L))^c\}) 1\{L(U') = L; (E_k(U', L))^c\})
\]

\[
\leq \sum_{U, U' \subset U_k : |U| = |U'| = M, U \cap U' = \emptyset} E(1\{L(U) = L\}) 1\{L(U') = L\}) + \sum_{U \subset U_k : |U| = M} E(1\{L(U) = L\})
\]

\[
\leq \left( \frac{N}{M} \right) \left( \frac{N - M}{M} \right) (\mu_M(L))^2 + \left( \frac{N}{M} \right) \mu_M(L),
\]

where the first inequality follows since on \((E_k(U, L))^c\) for any legitimate \(U'\) with \(U' \cap U \neq \emptyset\) and \(U' \neq U\) we have \(L(U') \neq L\). Since \(\left( \frac{N}{M} \right) \mu_M(L) \geq n^7\) and since

\[
\left( \frac{N}{M} \right) / \left( \frac{N - M}{M} \right) \geq \left( \frac{N}{N - M} \right) M \gg n^2,
\]

we have that

\[
\left( \frac{N - M}{M} \right) \mu_M(L) + 1 \ll \left( \frac{N}{M} \right) \mu_M(L)n^{-2}.
\]

Combined with (2.9) and (2.10), it yields that \(EZ^2_L \ll (EZ_L)^2\), arriving at a contradiction and thereby concluding the proof of the lemma.

**Proof of Proposition 2.1** Take a \(U \subset U_1\) with \(|U| = M\). By Lemma 2.2, with probability \(1 - o(1)\) we have that \(L(U) \in L(U, \iota^*_*)\). By (2.8) and Lemma 2.3, we see that with probability \(1 - o(1)\), there exist \(\emptyset \neq U' \subset U\) and \(W' \subset U_2\) such that \(L(U') = L(W')\). This completes the proof of the proposition.

\[\square\]
3 Proof of identifiability

In this section we prove identifiability for \( d \geq 2 \) (the identifiability for \( d = 1 \) was already known from [11, Theorem 1.1]). Recall that in this regime

\[
q^r \geq n^{d(1+\epsilon)} \quad \text{for an arbitrary fixed small } \epsilon > 0. \tag{3.1}
\]

We note that the threshold is chosen as in (3.1) since this ensures that for each \( B \in \mathcal{B}_r \) we have that \( \sigma_B \) is unique with probability tending to 1 (see Lemma 3.1)—this is a property we will repeatedly use in our proof.

Our recovering procedure will employ the following three steps to successively determine labels on vertices of \( \Lambda_n \) (here we say that we determine \( \sigma_v = k \) if every \( \sigma \) with given empirical profile satisfies \( \sigma_v = k \)):

**Step 1: initial labeling at the corner.** Determine labels on \( \Lambda_{2r} \).

**Step 2: percolation of unique \((r-1)\)-boxes.** Initially, all boxes in \( \mathcal{B}_{r-1} \) are unexplored. Inductively check each unexplored box in \( \mathcal{B}_{r-1} \) (denoted as \( B \)) where all labels have been previously determined. If \( \sigma|_B \) is unique over all \((r-1)\)-boxes, we can then find a few boxes \( B' \in \mathcal{B}_r \) so that \( \sigma|_{B'} \) agrees with \( \sigma|_B \) on a sub-\((r-1)\) box of \( B' \) (we can find \( 2^d \) such \( B' \)'s unless \( B \) is near the boundary of \( \Lambda_n \)). Therefore, from \( \sigma|_{B'} \) we can determine labels on neighboring vertices of \( B \) (they may have been determined already) and then we mark \( B \) as explored.

**Step 3: final step of recovery.** For each vertex \( v \) which is not determined after **Step 2**, check all boxes \( B \in \mathcal{B}_r \) containing \( v \). For each such \( B \), let \( B^+ \subset B \) be the collection of determined vertices in \( B \). Next, we will check whether \( \sigma|_{B^+} \) is unique over all translated copies of \( B^+ \). This is similar to the exploration process in **Step 2** where an \((r-1)\)-box was considered instead of \( B^+ \). To be precise, if there are \( B_1, B_2 \in \mathcal{B}_r \) with \( B_1 \neq B_2 \) such that \( \sigma|_{B_1^+} = \sigma|_{B_2^+} = \sigma|_{B^+} \), where \( B_i^+ \subset B_i \) is the image of \( B^+ \) under the translation which maps \( B_i \) to \( B \) for \( i = 1, 2 \) (i.e. the relative position of \( B_i^+ \) with respect to \( B_i \) is the same as the relative position of \( B^+ \) with respect to \( B \)), then \( \sigma|_{B^+} \) is not unique. (Note that even though we do not know the locations of \( B_1 \) and \( B_2 \), we can nevertheless determine whether such \( B_1 \neq B_2 \) exist or not from \( \{ \sigma|_B : B \in \mathcal{B}_r \} \).) If \( \sigma|_{B^+} \) is unique, we can then determine labels on \( B \) and thus in particular the label on \( v \).

In order to justify correctness for each aforementioned step we will use probabilistic arguments to prove some desirable structural properties which hold for a typical labeling configuration. These properties, altogether, will show that eventually we have determined labels on all vertices of \( \Lambda_n \). In addition, it is easy from the description of our procedure that the running time is polynomial in \( n \).

Next, we provide a proof for correctness of our 3-step procedure while omitting proofs for a few lemmas and propositions (in this way of exposition we hope that this can serve as an overview for our proof that helps a reader to grasp the proof sketch before jumping into details). To this end, we first introduce a few terminologies. For each set \( B \), we say \( B \) is **unique** if \( \sigma|_B \) is unique among \( \{ \sigma|_{B'} : B' \subset \Lambda_n \text{ is a translated copy of } B \} \). In particular, for each \( B \in \mathcal{B}_{r-1} \), we say \( B \) is **unique** if \( \sigma\mid_B \) is unique in \( \{ \sigma\mid_{B'} : B' \in \mathcal{B}_{r-1} \} \). For each \( B \in \mathcal{B}_r \), we say that \( B \) is **open** if each
For each $B \in \mathcal{B}_r$, we have $P(B \text{ is open}) \geq 1 - n^{-\epsilon/2}$.

Uniqueness is useful due to the following lemma.

Lemma 3.2 For $s \geq r$, if $B \in \mathcal{B}_s$ is open and labels on an $(r - 1)$-sub-box of $B$ are determined, then there is a polynomial time algorithm which determines labels on $B$.

We are now ready to prove the correctness of Step 1, as formulated in the next proposition.

Proposition 3.3 On the event that $\Lambda_2$ is open (which happens with probability tending to 1 by Lemma 3.1), we can determine labels on $\Lambda_2$.

We now turn to Step 2. Let $B_2$ be the collection of boxes in $\mathcal{B}_2$ where each coordinate of the largest corner vertex is either equal to $n$ or of the form $kr + (r - 1)$ for some integer $k$. Essentially, $B_2$ is a disjoint partition of $\Lambda_n$ into $(2r)$-boxes together with $(2^d - 1)$ shifts of the partition (where each coordinate is either shifted by $r$ or not shifted). Here the word “essentially” refers to the fact that $B_2$ is not exactly a partition (together with shifts) when $n$ is not a multiple of $r$.

Definition 3.4 For $B, B' \in B_2$, we say that $B$ is strongly neighboring to $B'$ if $|B \cap B'| \geq r^d$ and we say $B$ is weakly neighboring to $B'$ if $\min_{u \in B, v \in B'} |u - v|_\infty \leq 4r$ (here the number 4 in 4$r$ is somewhat arbitrarily chosen, and any constant that is not too small suffices here). For $S \subset B_2$, we say $B$ is strongly (respectively weakly) connected to $B'$ in $S$ if there is a strongly (respectively weakly) neighboring sequence of boxes in $S$ joining $B$ and $B'$ (note that $B, B'$ may not be in $S$). We drop $S$ from the above when $S = B_2$. If $\Lambda_2$ is open, let $C_2$ be the collection of open box $B \in B_2$ that is strongly connected to $\Lambda_2$ via open boxes (i.e., with $S$ being the collection of open boxes); otherwise let $C_2 = \emptyset$. In addition, for a collection of boxes $S$, we denote $V(S) = \bigcup_{B \in S} B$.

The following percolation type of result is the key input for analyzing Step 2.

Proposition 3.5 With probability tending to 1 as $n \to \infty$, each weakly connected component in $B_2 \setminus C_2$ has diameter (i.e., the maximal $\ell_\infty$-distance over all pairs of vertices in the union of the boxes in this component) at most $\kappa r$ where $\kappa = \kappa(d, \epsilon)$ does not depend on $n$. In addition, for any $\kappa' = \kappa'(d, \epsilon)$ (which in particular does not depend on $n$) with probability tending to 1 as $n \to \infty$ we have that

$$\text{Corner} \overset{\Delta}{=} \{v \in \Lambda_n : v_i \notin (\kappa' r, n - \kappa' r) \text{ for } 1 \leq i \leq d\} \subset V(C_2).$$

By Lemma 3.2, each vertex in $C_2$ was determined in Step 2 and thus by Proposition 3.5, with probability tending to 1 as $n \to \infty$

$$B_2 \setminus C_2 \text{ consists of weakly connected components with diameter } \leq \kappa r. \ (3.2)$$
(It is clear that each \( B \in B_{2r}\setminus C_{2r} \) is in a weakly connected component of \( B_{2r}\setminus C_{2r} \) and the point of (3.2) is that all these components have diameter at most \( \kappa r \).) This will be a very useful input for Step 3, as implied by the next proposition. For \( v \in \Lambda_N \) with \( v_1, v_2, \ldots, v_d \leq n - r \) and \( v_2 \geq r \), for \( s = 0, \ldots, r - 1 \) define

\[
B_s(v) = (v_1 + 1, v_2 - s, v_3, \ldots, v_d) + [0, r - 2] \times [0, r - 1]^d \quad (3.3)
\]

to be a collection of rectangles (in fact each rectangle is almost an \( r \)-box) neighboring \( v \) which are fully contained in \( \Lambda_n \). (We made this particular choice so that in our application later all vertices in \( B_k(v) \) have been determined in Step 2.)

**Proposition 3.6** We have that

\[
P(\exists v \in \Lambda_n : B_s(v) \text{ is not unique for all } 0 \leq s \leq r - 1) = o(1).
\]

We now explain how Proposition 3.6 ensures typically all vertices are determined at the end of Step 3. Suppose otherwise there exists a vertex \( u \) that is not determined. By (3.2), \( u \) is in a weakly connected component of undetermined vertices whose diameter is at most \( \kappa r \) and in addition this component is disjoint from Corner (in our application, when defining Corner we take \( \kappa' = \kappa + 2 \)). Thus, for each such component there exists an undetermined vertex \( v \) so that one of its coordinates is in \((\kappa' r, n - \kappa' r)\) and we may assume without loss of generality that the second coordinate of \( v \) is in \((\kappa' r, n - \kappa' r)\). Since the component has diameter at most \( \kappa r \), by our choice of \( \kappa' \) we see that the second coordinates for all vertices in this component are in \((r, n - r)\). We further assume that in this component \( v \) has the largest first coordinate and \( v_1, v_2, \ldots, v_d \leq n - r \) and \( v_2 \geq r \) (the analysis is completely the same by symmetry in other cases). Since \( v \) has the largest first coordinate in this component and since all components have mutual \( \ell_\infty \)-distance at least \( 4r \) (by our definition of weakly neighboring), we claim that all vertices in \( \bigcup_{s=0}^{r-1} B_s(v) \) have been determined. In order to see this, by definition the rectangles \( B_s(v) \)’s are at the right side of \( v \) (i.e. have larger first coordinates than that of \( v \)). So these rectangles do not contain undetermined vertex from this weakly connected component since we assumed that \( v \) is the rightmost one. Here the \( \ell_\infty \) separation (i.e., the distance is at least \( 4r \) which is larger than the side length of rectangles) ensures that \( \bigcup_{s=0}^{r-1} B_s(v) \) cannot be close to other components.

By Proposition 3.6, we see that with probability \( 1 - o(1) \) we have that for all \( u \) there exists \( B_{s_u}(u) \) which is unique. We then assume without loss of generality that this event occurs. Since vertices in \( B_{s_u}(v) \) have all been determined, we can then check sequentially for \( s = 0, \ldots, r - 1 \) and for each such \( s \) we scan through the empirical profile \( \sigma_B : B \in \mathcal{B}_r \) until we find the first \( s_u \) satisfying the following property: there is a unique \( \tilde{\sigma} \in \{\sigma_B : B \in \mathcal{B}_r\} \) such that when viewed as a labeling configuration on \( \Lambda_r \) the labeling configuration of \( \tilde{\sigma}|_{\Lambda_r \setminus \{x \in \Lambda_r : x_1 = 0\}} \) agrees with \( \sigma|_{B_{s_u}(v)} \). At this point, from the value of \( s_u \) and \( \tilde{\sigma} \) we can determine the label on \( v \). This arrives at a contradiction and thus completes the proof of our theorem.

Next, we provide proofs for omitted lemmas and propositions, which are organized into three subsections corresponding to the three steps in our procedure.
3.1 Proofs for step 1

In this subsection, we provide proofs for Lemmas 3.1, 3.2 and Proposition 3.3 in order.

Proof of Lemma 3.1 For any \( B \in B_{r-1} \), we have

\[
P(B \text{ is not unique}) \leq \sum_{B' \in B_{r-1}, B' \neq B} P(\sigma |_{B'} = \sigma |_{B'}) \leq n^d q^{-(r-1)^d},
\]

(3.4)

where the last inequality follows from (3.1). Therefore, for \( B \in B_{2r} \),

\[
P(B \text{ is not open}) \leq \sum_{B' \subset B : B' \in B_{r-1}} P(B' \text{ is not unique}) \leq (r+2)^d n^d q^{-(r-1)^d} \leq n^{-\epsilon/2},
\]

where the factor of \((r+2)^d\) counts the number of \((r-1)\)-boxes in a \(2r\)-box. This completes the proof of the lemma. \(\square\)

Proof of Lemma 3.2 The proof is similar to and follows that of \([11, \text{Lemma 2.3, Proposition 3.2}]\). It suffices to show that on the event that \( B \) is open, we can determine the label for \( v \in B \) if \( v \) is neighboring to an \((r-1)\)-sub-box \( A \) whose vertices have all been determined (and for convenience we assume that \( v \) is neighboring to the surface of \( A \) with largest first coordinate). To this end, we scan through \( \{\sigma |_{B'} : B' \in B_r\} \) and let \( \tilde{\sigma} \) (viewed as a labeling configuration on \( \Lambda_r \)) be the first labeling configuration so that \( \tilde{\sigma} |_{\{w \in \Lambda_r : w_1, w_{d-1} \leq r-2\}} = \sigma |_{A} \). Since \( A \) is unique by our assumption, such translated copy in \( \tilde{\sigma} \) is unique. As a result, this indues a unique mapping \( \tau : \Lambda_r \mapsto B \) so that \( \tilde{\sigma}(u) = \sigma(\tau(u)) \) for all \( u \in \Lambda_r \). By our assumption that \( v \) is neighboring to the surface of \( A \) with largest first coordinate, we see that \( v \in \tau(\Lambda_r) \) and as a result we can determine the label on \( v \). \(\square\)

Proof of Proposition 3.3 By Lemma 3.2 (and since we assume that \( \Lambda_{2r} \) is open), it suffices to show that we can determine labels on \( \Lambda_{r-1} \). For convenience, for each \( B \in B_r \), we let \( B_1, \ldots, B_{2d} \) be the \((r-1)\)-sub-boxes of \( B \) that are contained in \( B \), and \( B_1 \) is the one that contains the smallest vertex in \( B \). Under the assumption of uniqueness of \( \Lambda_r \), we see that \( \sigma |_{\Lambda_{r-1}} \) is the unique labeling configuration so that there exists \( B \in B_r \) with \( \sigma |_{\Lambda_{r-1}} = \sigma |_{B_1} \) but there is no \( B \in B_r \) with \( \sigma |_{\Lambda_{r-1}} = \sigma |_{B_i} \) for any \( i = 2, \ldots, 2^d \). \(\square\)

3.2 Proofs for step 2

This subsection is devoted to the proof of Proposition 3.5, which closely resembles the now standard coarse graining method widely used in percolation theory. We first provide the underlying intuition. As implied by Lemma 3.1, the assumption (3.1) ensures that each \( B \in B_{2r} \) is open with probability tending to 1. Starting from some local regions whose labels have been determined, one can successively determine the labels of neighboring areas if they are contained in an open box thanks to Lemma 3.2. Therefore, the collection of open boxes defines a supercritical percolation (on a coarse
Lemma 3.7 If $B_{2r} \setminus C_{2r}$ has a weakly connected component with diameter $> \kappa r$ then there exists a weakly connected closed component $S \in \mathbb{E}$ such that either

$$|S| \geq \frac{\min_{u \in \Lambda_{2r}, w \in V(S)} |u - w|_{\infty}}{100r}$$

or

$$|S| \geq \kappa / 100.$$  

In addition, if Corner $\notin V(C_{2r})$, then there exists a weakly connected closed component $S$ with

$$|S| \geq \frac{\min_{u \in V(S), v \in \text{Corner}} |v - w|_{\infty}}{100r}.$$  

Proof We first prove the first part of the lemma. Let $C$ be the weakly connected component in $B_{2r} \setminus C_{2r}$ with the largest diameter, and let $C_{**}$ be the union of $C$ and the collection of open boxes that are strongly connected to a box in $C$ via open boxes. By our assumption, we see in particular $C_{**}$ has diameter at least $\kappa r$. Let $S \subseteq B_{2r} \setminus C_{**}$ be the collection of boxes on the outer boundary of $C_{**}$ which are not weakly enclosed by $C_{**}$, then we see that $S = \emptyset$ is a collection of weakly connected closed boxes such that $V(S) \cup \partial_{4r} \Lambda_n$ weakly separates $\Lambda_{2r}$ and $C_{**}$. Thus, it suffices to prove that (3.5) or (3.6) holds for this particular choice of $S$. To this end, let $\ell = \min_{u \in \Lambda_{2r}, w \in V(S)} |u - w|_{\infty}$. We may assume $\ell \geq 100r$ since otherwise the statement holds as $S = \emptyset$. If $|S| \leq \ell / 100r$, then out of all faces of $\Lambda_n$ which contains the origin there is at most one of them which has distance $\leq 8r$ from $S$ (since otherwise $|S| \geq (\ell - 8r)/4r > \ell / 100r$). Therefore, there exists at least one face of $\Lambda_n$ (which we denote as $F$) such that $\Lambda_{2r}$ is strongly connected in $B_{2r} \setminus S$ to each box in $B_{2r}$ that intersects $F$. If in addition, $|S| \leq \kappa / 100$, then the diameter of $C_{**}$ is at least $100r |S|$ and as a result $C_{**}$ is also strongly connected in $B_{2r} \setminus S$ to each box in $B_{2r}$ that intersects $F$. Therefore, if both (3.5) and (3.6) fail, we have that $\Lambda_{2r}$ is strongly connected in $B_{2r} \setminus S$ to each box in $C_{**}$, arriving at a contradiction (with the fact that $V(S) \cup \partial_{4r} \Lambda_n$ weakly separates $\Lambda_{2r}$ and $C_{**}$). This completes the proof of the first part.
For the second part, the proof is similar. If Corner $\not\subset V(C_{2r})$, then there exists a weakly connected component $C$ in $B_{2r}\setminus C_{2r}$ such that $V(C) \cap \text{Corner} \neq \emptyset$. Then we can define $S$ as above in the same manner. If (3.7) fails, then using the same argument above we see that both $C$ and $\Lambda_{2r}$ are strongly connected in $B_{2r}\setminus S$ to each box that intersects $F$ (where $F$ again is some face of $\Lambda_n$). This gives a contradiction and completes the proof. \qed

The following lemma is a key input in order to upper-bound probabilities for either of the two cases.

**Lemma 3.8** For any $B_0 \in B_{2r}$, we have that

$$P(\exists S : B_0 \in S \text{ and } |S| \geq t) \leq n^{-8^{-d}t}.$$ 

**Proof** If there exists $S \in S$ with $B_0 \in S$ and $|S| \geq t$, then we claim that there exists a collection of disjoint and closed $2r$-boxes $S$ with $|S| \geq 4^{-d}t$ such that $S$ is $8r$-weakly-connected (here $8r$-weakly connected corresponds to the $8r$-weakly-neighboring for two boxes which means that the $\ell_\infty$-distance between these two boxes is at most $8r$). In order to see this, we can for instance take a maximal $S \subset S$ with $B_0 \in S$ such that all sets in $S$ are mutually disjoint. Since each box intersects with at most $4^d$ boxes in $S$, by maximality we see that $|S| \geq 4^{-d}t$. We can then view $S$ as a graph where vertices are $2r$-boxes in $S$ and edges are given by weakly-connected relations between boxes. In this way, we see that such $S$ with $|S| = t'$ induces at least one $8r$-weakly-connected tree on $B_{2r}$, of size $t'$, and each such tree can be encoded by its depth-first-search contour started with $B_0$ and of length $2t'$. Therefore, we can then upper-bound the number of choices for such $S$ with $|S| = t'$ by the number of $8r$-weakly-connected paths on $B_{2r}$ starting with $B_0$ and of length $2t'$. That is, the enumeration is bounded by $16^{2dt'}$.

We next wish to upper-bound the probability for all boxes in $S$ being closed. Due to disjointness, we may wish to bound this by $n^{-|S|/2}$ in light of Lemma 3.1. This is not completely correct since even for disjoint boxes their openness are not exactly independent, although the fix is easy as we explain next. For each $B \in S$, since $B$ is closed we see there exists an $(r-1)$-box $B' \subset B$ such that $\sigma|_{B'} = \sigma|_{B''}$ for some $B'' \in B_r$, in which case we draw an edge between $B'$ and $B''$. In this way, we can draw an edge from each $B \in S$ and we let $E_*$ be the collection of all the edges. We can then take a subset $E \subset E_*$ with $|E| = t'' = |S|/2$ so that there is no cycle among these edges (there is no cycle even when edges are viewed as edges between sets in $S$, that is, even when each edge between $B' \subset B$ and $B' \subset B$ for $B, B \in S$ is viewed as an edge between $B$ and $B$). On the one hand, the number of labeling configurations that are consistent with $E$ is at most $q^{\lfloor \Lambda_n \rfloor}q^{-(r-1)d}|E| = q^{\lfloor \Lambda_n \rfloor}q^{-(r-1)d}t''$ (note that the acyclic property here ensures that each edge in $E$ reduces the number of consistent labeling configurations by a factor of $q^{d(r-1)}$). On the other hand, the number of possible choices for $E$ is at most $2^{\lfloor S \rfloor}((r + 2)^d)^{t''}$, where $2^{\lfloor S \rfloor}$ bounds the number of ways to choose $t''$ $2r$-boxes from $S$, and $((r + 2)^d)^{t''}$ bounds the number of ways to choose an $(r - 1)$-box from each of these aforementioned $2r$-boxes (and these are the endpoints of $E$ contained in $S$), and $(n^d)^{t''}$ bounds the number of ways...
to choose \( r'' \) \((r - 1)\)-boxes in \( \Lambda_n \) (these are the other endpoints of \( E \)). Putting this together, we see that

\[
P( \text{all boxes in } \mathcal{S} \text{ are closed} ) \leq \frac{q^{\Lambda_n} q^{-(r-1)d} r''}{q^{\Lambda_n}} 2^{\mathcal{S}((r + 2)^d) r''} (n^d) r'' \leq n^{-\varepsilon |\mathcal{S}|/4}.
\]

Combined with the aforementioned upper bound on the enumeration for \( \mathcal{S} \), we get that

\[
P( \exists S \in S : B \in S \text{ and } |S| \geq t) \leq \sum_{t_1 \geq t} \sum_{t' \geq 4 - d t_1} 16^{2d't'} n^{-\varepsilon t'/4} \leq n^{-8 - d \varepsilon t},
\]

as required. \( \square \)

We are now ready to provide the proof of Proposition 3.5.

**Proof of Proposition 3.5** We first treat the case as in (3.5). In light of Lemma 3.8, it suffices to sum over all choices for the “starting” box \( B \). Let \( t \) be the right hand side of (3.5). Then the number of choices for \( B \in B_2r \) with \( \ell_{\infty} \)-distance at most \( 100d r t \) to \( \Lambda_1 \) is at most \( (100 t)^d \). Therefore,

\[
P( \exists S \in S : (3.5) \text{ holds} ) \leq \sum_{t \geq 1} \sum_{t' \geq t} n^{-8 - d t} (100 t)^d = o(1). \tag{3.8}
\]

Similarly we can bound the case for (3.6). In this case, there is a slight difference in bounding the enumeration for the “starting” box \( B \): the vertex \( v \) can be chosen arbitrarily and as a result the number of choices for \( B \) is at most \( n^d \). Therefore, another application of Lemma 3.8 gives that

\[
P( \exists S \in S : (3.6) \text{ holds} ) \leq \sum_{t \geq \kappa / 100^d} n^{-8 - d \varepsilon t} n^d = o(1)
\]

as long as \( \kappa = \kappa (d, \varepsilon) \) is a large enough constant. Combined with (3.8), this completes the proof of the first part of the proposition.

Finally, we show that Corner \( \subset V(C_{2r}) \) with probability \( 1 - o(1) \), for which we will use (3.7). Since \( |\text{Corner}| = O((\kappa' r)^d) \), a similar computation as in (3.8) completes the proof. \( \square \)

### 3.3 Proofs for step 3

In this subsection, we prove Proposition 3.6. For sets \( A, A' \subset \Lambda_n \), if \( A' \) is a translated copy of \( A \), we then define \( v(A, A') \) as the unique vector such that \( A' = \{ v + v(A, A') : v \in A \} \). Recall (3.3). Suppose that for \( v \in \Lambda_N \) with \( v_1, v_2, \ldots, v_d \leq n - r \) and \( v_2 \geq r \) we have that \( B_s (v) \) is not unique for all \( s = 0, \ldots, r - 1 \). Then, recursively for \( s = 0, \ldots, r - 1 \) we can pick a \( B'_s \neq B_s (v) \) such that \( \sigma |_{B'_s} = \sigma |_{B_s (v)} \), and further we pick \( B'_s \) such that \( v (B_{s-1} (v), B'_{s-1}) = v (B_s (v), B'_s) \) for \( s \geq 1 \) if this is possible. Note
that for $s_1 < s_2$ if there exist $B'_{s_1}$ and $B'_{s_2}$ such that $\nu(B_{s_1}(v), B'_{s_1}) = \nu(B_{s_2}(v), B'_{s_2})$, then it would have been possible by our rule to pick $B'_s$ for $s_1 < s < s_2$ such that $\nu(B_{s_1}(v), B'_{s_1}) = \nu(B_s(v), B'_s)$. Therefore, the translation vectors $(\nu(B_s(v), B'_s))_{s=0}^{r-1}$ can be chosen to be piecewise constant and distinct, so we can merge each piece into an interval. As a result, the interval $[0, \ldots, r - 1]$ can be partitioned into $\ell \geq 1$ intervals $I_1, \ldots, I_\ell$ such that for each $j = 1, \ldots, \ell$ we have $B_{I_j}(v) \triangleq \bigcup_{s \in I_j} B_s(v)$ satisfies $\sigma|_{B_{I_j}(v)} = \sigma|_{B'_{I_j}}$ for some $B'_{I_j}$ which is a translated copy of $B_{I_j}(v)$ with $\nu(B_{I_j}(v), B'_{I_j})$'s distinct from each other (and also not equal to the 0-vector).

Before we use the above construction to prove Proposition 3.6, we derive some preliminary results as preparation. By a similar argument as in Lemma 3.1, we obtain that

$$\mathbb{P}(\exists B, B' \in \mathcal{B}_{r-1} : \min_{u \in B, u' \in B'} |u - u'|_\infty \leq 4r \text{ and } |\sigma|_B = |\sigma|_{B'} \leq n^{-\epsilon/4}. \quad (3.9)$$

The next lemma will also be useful.

**Lemma 3.9** With probability at least $1 - 2n^{-\epsilon/4}$ the following holds for all $A, A', B, B' \in \mathcal{B}_{r-1}$:

If $A \cap B \neq \emptyset, A' \cap B' \neq \emptyset$ and $\nu(A, B) \neq \nu(A', B')$, then either $\sigma|_A \neq \sigma|_{A'}$ or $\sigma|_B \neq \sigma|_{B'}$. \hfill (3.10)

**Proof** We first show that if $\min_{u \in A, u' \in A'} |u - u'|_\infty > 4r$ and if $A \cap B \neq \emptyset$ and $A' \cap B' \neq \emptyset$ and in addition $\nu(A, B) \neq \nu(A', B')$, then we have

$$\mathbb{P}(\sigma|_A = \sigma|_{A'} \text{ and } \sigma|_B = \sigma|_{B'}) = q^{-2(r-1)d}. \quad (3.11)$$

To see this, we count the number of labeling configurations on $A \cup A' \cup B \cup B'$ such that $\sigma|_A = \sigma|_{A'}$ and $\sigma|_B = \sigma|_{B'}$. Consider a graph with vertex set $A \cup A' \cup B \cup B'$ where the edge set is $\{(u, u + \nu(A, A')) : u \in A\} \cup \{(u, u' + \nu(B, B')) : u \in B\}$. Since we assumed $\min_{u \in A, u' \in A'} |u - u'|_\infty > 4r$, this graph is a bipartite graph between $A \cup B$ and $A' \cup B'$, and we claim that there is no cycle (or multiple edge) in this graph. Otherwise, for some $j \geq 1$ there exist distinct $u_1, \ldots, u_j \in A \cup B$ and $u'_1, \ldots, u'_j \in A' \cup B'$ such that

$$u_i + \nu(A, A') = u'_i \text{ and } u_{i+1} + \nu(B, B') = u'_i \text{ for } 1 \leq i \leq j,$$

where we used the convention that $u_{j+1} = u_1$. This implies that $\nu(A, A') = \nu(B, B')$, arriving at a contradiction. Therefore, each edge in this graph reduces the number of valid labeling configurations (i.e., those satisfy $\sigma|_A = \sigma|_{A'}$ and $\sigma|_B = \sigma|_{B'}$) by a factor of $q$. This implies (3.11) since the number of edges is $2(r-1)^d$. Since the number of choices for $A, A', B, B'$ with $A \cap B \neq \emptyset$ and $A' \cap B' \neq \emptyset$ is at most $n^{2d}(2r)^{2d}$, we can apply a union bound and obtain that with probability at least $1 - n^{-\epsilon/4}$ for all $A, A', B, B' \in \mathcal{B}_{r-1}$ with $\min_{u \in A, u' \in A'} |u - u'|_\infty > 4r$ we have (3.10). Combined with (3.9), this implies the lemma. \hfill $\square$
We now come back to the proof of Proposition 3.6. Without loss of generality we assume that the events in (3.9) and in the statement of Lemma 3.9 hold. Thus, we claim that it suffices to consider $B'_{I_j}$'s (which are possible realizations of $B'_{I_j}$'s) such that

$$B_{I_j}(v) \bigcap B'_{I_{j'}} = \emptyset \text{ for all } 1 \leq j, j' \leq \ell \text{ and } B_{I_j} \bigcap B'_{I_{j'}} = \emptyset \text{ for } 1 \leq j \neq j' \leq \ell.$$  

(3.12)

We first verify the former equality and we denote by $B'_s$ the realization of $B'_s$ for $s = 0, \ldots, r - 1$. Since $\sigma_{B'_s} = \sigma_{B_s(v)}$, on the complement of the event in (3.9) we have $B'_s$ has $\ell_\infty$-distance at least $4r$ away from $B_s(v)$. But $B_s(v) \bigcap B'_s(v) \neq \emptyset$ for $0 \leq s, s' \leq r$, so $B'_s$ must be disjoint from $B'_s(v)$ which readily implies the former equality. We now verify the latter equality. The partition rule of intervals implies that

$$\nu(B_{I_j}(v), B'_{I_{j'}}) \neq \nu(B_{I_j}(v), B'_{I_{j'}}) \text{ for } j \neq j'. \text{ Since } B_{I_j} \bigcap B'_{I_{j'}} \neq \emptyset, \text{ on the event in the statement of Lemma 3.9 we must have } B_{I_j} \bigcap B'_{I_{j'}} = \emptyset,$$

completing the verification of the claim.

We next count the number of labeling configurations on $\bigcup_{j=1}^\ell (B_{I_j}(v) \cup B'_{I_{j'}})$ such that $\sigma_{B_{I_j}(v)} = \sigma_{B'_{I_{j'}}}$ for $1 \leq j \leq \ell$. Consider a graph with vertex set $\bigcup_{j=1}^\ell (B_{I_j}(v) \cup B'_{I_{j'}})$ and edge set $E = \bigcup_{j=1}^\ell \{(u, u+\nu(B_{I_j}(v), B'_{I_{j'}})) : u \in B_{I_j}(v))\}$. By (3.12), we see that for each $u \in B'_{I_{j'}}$ there is a single edge incident to $u - \nu(B_{I_j}(v), B'_{I_{j'}})$, i.e., the edge between $u$ and $u - \nu(B_{I_j}(v), B'_{I_{j'}})$. As a result, in this graph there is no cycle (or multiple edge). That is to say, each edge in this graph reduces the number of configurations by a factor of $q$. Recall from (3.3) that each $B_s(v)$ is an $(r - 1) \times r \times r \times \cdots \times r$ rectangle, and thus $B_{I_j}(v) = \bigcup_{s \in I_j} B_s(v)$ is an $(r - 1) \times (r + |I_j| - 1) \times r \times \cdots \times r$ rectangle. As a result, we have $|B_{I_j}(v)| = (r - 1)r^{d-1}(r + |I_j| - 1)$. Since $E$ is a disjoint union of $\{(u, u+\nu(B_{I_j}(v), B'_{I_{j'}})) : u \in B_{I_j}(v))\}$ over $j = 1, \ldots, \ell$, we have

$$|E| = \sum_{i=1}^\ell (r - 1)r^{d-2}(r + |I_i| - 1) \geq (r - 1)^d(1 + \ell),$$

where the inequality holds since $\sum_{i=1}^\ell (r + |I_i| - 1) = \ell(r - 1) + r \geq (r - 1)(1 + \ell)$. Therefore,

$$P(\sigma_{B_{I_j}(v)} = \sigma_{B'_{I_{j'}}} \text{ for } 1 \leq j \leq \ell) \leq q^{-(r-1)^d(1+\ell)}.$$

Summing over $v \in \Lambda_n$, $1 \leq \ell \leq r$, all partitions of $I_1, \ldots, I_\ell$ and all choices of $B'_{I_{j'}}$ for $1 \leq j \leq \ell$, we derive that

\[\text{(3.13)}\]
\[
P(\exists v \in \Lambda_n : v \text{ is undetermined}) \leq \sum_{v \in \Lambda_n} \sum_{1 \leq \ell \leq r} \sum_{I_1, \ldots, I_\ell, B'_1, \ldots, B'_\ell} \sum_{j=1}^\ell P(\sigma_{B_{I_j}}(v) = \sigma_{B'_I})
\]

for \(1 \leq j \leq \ell\)

\[
\leq n^d \sum_{1 \leq \ell \leq r} (r - 1)^{n^d q^{-d(1+\ell)}} = o(1),
\]

completing the proof of Proposition 3.6.

4 Minor modifications for rotation and reflection symmetry

In this section, we briefly discuss how to extend our proof to obtain the same result as in Theorem 1.1 when rotation and reflection symmetry are taken into account, as mentioned in Remark 1.5. More precisely, we say a configuration \(\sigma\) on \(\Lambda_n\) is isomorphic to \(\sigma'\) if there exists a map \(\tau\) that is a composition of rotation and reflection of \(\Lambda_n\) such that \(\sigma = \sigma'(\tau)\). In this case, the local observations are given up to isomorphism with respect to an \(r\)-box, and our goal is to recover \(\sigma\) on \(\Lambda_n\) up to isomorphism with respect to \(\Lambda_n\).

For the proof of non-identifiability, we must ensure that \(\sigma'\) is not isomorphic to \(\sigma\) where \(\sigma'\) is obtained from swapping some labels in \(\sigma\) as we described in our proofs. In the case of \(d = 1\), we can partition \(\Lambda_n\) into 8 disjoint and consecutive intervals \(I_1, \ldots, I_8\) with length \([n/8]\), and perform a swapping with \(I_2, \ldots, I_7\) replacing \(I_1, \ldots, I_6\) as in Sect. 2.1. We can then use \(I_1, I_8\) to guarantee that the \(\sigma'\) obtained from our swapping operation is not isomorphic to \(\sigma\) (since with probability tending to 1 we have that \(\sigma|_{I_1}\) is not isomorphic to \(\sigma|_{I_8}\)). In the case for \(d \geq 2\), the proof of Proposition 2.1 still works with the following additional property: with probability tending to 1 there is no rotation/reflection so that the labels that are not 1 nor 2 are preserved (this can be checked easily).

In the identifiable regime, the extension of our recovery procedure is a little more complicated as we next explain. For each box \(B\), we say \(B\) has an automorphism if there is a composition of rotation and reflection which is non-identical and maps \(\sigma|_B\) to itself. For each \(B \in \mathcal{B}_r\), we modify the definition of open such that \(B\) is open if each \(B' \in \mathcal{B}_{r-1}\) that is contained in \(B\) is unique and does not have an automorphism. The additional condition on automorphism ensures that Lemma 3.2 still holds since once a unique \((r - 1)\)-box \(A\) without automorphism is determined, the vertices neighboring to \(A\) can also be determined. Moreover, the probability that an \((r - 1)\)-box has an automorphism is at most \(2^d q^{-(r-1)d/2} = O(n^{-d/2})\), which is substantially smaller than the probability of being non-unique, so all of our probabilistic estimates remain valid.

We also need to modify Proposition 3.3 and we can do it by using the following fact: on the one hand, if the labeling configuration of \(A \in \mathcal{B}_{r-1}\) appears only once as \(\sigma|_{B'}\) where \(B'\) is an \((r - 1)\)-box in some \(r\)-box \(B\) (note that this event is measurable with respect to \(\{\sigma|_B : B \in \mathcal{B}_{n,r}\}\)), then \(A\) must lie on the corner of \(\Lambda_n\); on the other hand, if \(A \in \mathcal{B}_{r-1}\) is contained in a \(2r\)-box on a corner of \(\Lambda_n\) and if this \((2r)\)-box is
open, then the labeling configuration of $A$ appears only once as $\sigma|_{B'}$ where $B'$ is an $(r - 1)$-box in some $r$-box $B$. Assuming that $\Lambda_{2r}$ is open, this fact ensures that in any possible labeling of $\Lambda_n$ with $\{\sigma|_B : B \in \mathcal{B}_{n,r}\}$, the labeling configuration $\sigma|_{\Lambda_{r-1}}$ must appear on the corner of $\Lambda_n$. Since we only care about the labelings up to isomorphism, we can choose an arbitrary corner and put the labeling configuration $\sigma|_{\Lambda_{r-1}}$ there. We emphasize here that we can choose an arbitrary corner since by symmetry all corners a priori play the same role. That being said, when we put labeling configuration $\sigma|_{\Lambda_{r-1}}$ at this chosen corner of $\Lambda_n$, there is a unique way to do so. In other words, formally in the end we obtain $2^d$ labeling configurations depending on which corner we have chosen to begin with, but they are all isomorphic to each other.

The proofs of Lemma 3.8, Proposition 3.5, Lemma 3.9 and Proposition 3.6 will be roughly the same, except that we should choose the phantom box (i.e., the box that has the same labeling configuration as another box) together with its orientation when considering an $(r - 1)$-box as non-unique. In view of this, we can define $u(A, A')$ as the transformation in $\mathbb{R}^d$ that maps $A$ to $A'$, if $A, A' \subset \Lambda_n$ and $A$ is congruent to $A'$. In particular, if $A, A' \in \mathcal{B}_3$, then $u(A, A')$ can be seen as the composition of $v(A, A')$ and an reflection/rotation that preserves $A'$. The analysis of the reduction of a factor $q$ for the enumeration of valid labeling configurations would be the same by replacing $v(A, A')$ with $u(A, A')$ (where instead of adding $v(A, A')$ for the translation, we replace the transformation by the map $u(A, A')$). Finally, all possible orientations together only contribute a multiplicative factor of $2^d$ to the probability for each box being non-unique, and as a result all of our probabilistic estimates remain valid.

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References

1. Adhikari, K., Chakraborty, S.: Shotgun assembly of random geometric graphs. arXiv:2202.02968
2. Arratia, R., Martin, D., Reinert, G., Waterman, M.: Poisson process approximation for sequence repeats, and sequencing by hybridization. J. Comput. Biol. A J. Comput. Mol. Cell Biol. 3(3), 425–463 (1996)
3. Balister, P., Bollobás, B., Narayanan, B.: Reconstructing random jigsaws. arXiv:1707.04730
4. Bordenave, C., Feige, U., Mossel, E.: Shotgun assembly of random jigsaw puzzles. Random Struct. Algorithms 56(4), 998–1015 (2020)
5. Dyer, M., Frieze, A., Suen, S.: The probability of unique solutions of sequencing by hybridization. J. Comput. Biol. A J. Comput. Mol. Cell Biol. 1(2), 105–110 (1994)
6. Gaudio, J., Mossel, E.: Shotgun assembly of Erdős-Rényi random graphs. Electron. Commun. Prob. 27:Paper No. 5, 14 (2022)
7. Huang, H., Tikhomirov, K.: Shotgun assembly of unlabeled Erdős-Rényi graphs. arXiv:2108.09636
8. Huroyan, V., Lerman, G., Wu, H.-T.: Solving jigsaw puzzles by the graph connection Laplacian. SIAM J. Imag. Sci. 13(4), 1717–1753 (2020)
9. Keshri, S., Nemenjikakis, E., Pakman, A., Shababo, B., Paninski, L.: A shotgun sampling solution for the common input problem in neural connectivity inference. arXiv:1309.3724
10. Martinsson, A.: A linear threshold for uniqueness of solutions to random jigsaw puzzles. Combin. Prob. Comput. 28(2), 287–302 (2019)
11. Mossel, E., Ross, N.: Shotgun assembly of labeled graphs. IEEE Trans. Netw. Sci. Eng. 6(2), 145–157 (2019)
12. Mossel, E., Sun, N.: Shotgun assembly of random regular graphs. arXiv:1512.08473
13. Motahari, A.S., Bresler, G., Tse, D.N.C.: Information theory of dna shotgun sequencing. IEEE Trans. Inf. Theory 59(10), 6273–6289 (2013)
14. Pevzner, P.A.: l-tuple DNA sequencing: computer analysis. J. Biomol. Struct. Dyn. 7(1), 63–73 (1989)
15. Przykucki, M., Roberts, A., Scott, A.: Shotgun reconstruction in the hypercube. Random Struct. Algorithms 60(1), 117–150 (2022)
16. Raymond, J., Bland, R., McGoff, K.: Shotgun identification on groups. Involve 14(4), 631–682 (2021)

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