Competitive play among pairs of individuals or teams naturally leads to attempts to construct ratings and rankings of players. No where is this more evident than in sports competition, but there are manifestations in almost every field of endeavor. The question, “Who are the best?” demands an answer, and where there is demand there is bound to be a supply.

The most common statistical model for rating and rankings based on paired comparison data is the (Ralph Bradley and Milton Terry 1952) model anticipated by (Ernst Zermelo 1929) as a proposed method for rating chess players. Given \( p + 1 \) players of unknown “abilities,” \( \alpha_0, \ldots, \alpha_p \), it is postulated that the probability of player \( i \) defeating player \( j \) is given by,

\[
\pi_{ij} = \frac{\alpha_i}{\alpha_i + \alpha_j},
\]

and with a sufficiently rich history of competition player ratings can be estimated by maximum likelihood.

It is convenient to reparameterize abilities so \( \theta_i = \log \alpha_i \) and \( \pi_{ij} \) becomes,

\[
\pi_{ij} = \frac{1}{1 + \exp(-\theta_i - \theta_j)}.
\]

The (logistic) log likelihood for \( n \) binary outcomes, \( y_1, y_2, \ldots, y_n \), can be written

\[
\ell(\theta | y) = \sum_{k=1}^n y_k \log(h_{\theta}(x_k)) + (1 - y_k) \log(1 - h_{\theta}(x_k))
\]

where for match \( k \) between \( i \) and \( j \), \( x_k \) is an \( p \) vector with \( i \)th element 1, and \( j \)th element -1, and other elements 0. Without loss of generality, we can set \( \theta_0 = 0 \). When there are multiple meetings between a given pair, \( (i,j) \), binary outcomes can be aggregated to write the likelihood in terms of the corresponding binomial outcomes. In this case the log likelihood can be expressed in terms of the \( \alpha_i \)'s as,

\[
\ell(\alpha | y) = K + \sum_{i=1}^p w_i \log \alpha_i - \sum_{i<j} n_{ij} \log(\alpha_i + \alpha_j),
\]

where \( w_i \) denotes the total number of “wins” of player \( i \) against all other competitors, and \( n_{ij} \) is the total number of matches between players \( i \) and \( j \). From this it may be concluded that the This observation has played an important role in recent literature on the analysis of paired comparisons in machine learning where A/B experiments constitute an important data structure, see for example (N.B. Shah and Martin Wainwright 2018). However, this sufficiency result holds if and only if the \( n_{ij} \) are uninformative about the ratings. This is plausible if matching is generated at random or by some balanced tournament design, but certainly cannot be expected to hold if, for example, players are more likely to be matched with players of similar ability. In such cases one must resort to the MLE which offers protection against such variability in “strength of schedule.” Just counting wins is insufficient in designs like this when the \( n_{ij} \) are informative too.
Since the number of players, \( p \), can be large relative to the number of matches \( n \), some regularization of the MLE estimates may be beneficial. We compare two such regularization schemes: one employing a variant of the familiar \( \ell_1 \) penalty that encourages grouping of the estimated abilities thereby reducing their effective dimensionality, and another empirical Bayes approach that treats the ratings as if they arise from a nonparametric Gaussian mixture model. Comparisons with vanilla MLE ranking and two variants of Borda score ranking illustrates advantages of the regularization methods.

I. Grouped Lasso for Ranking

In some ranking problems it is plausible that there are only a few equivalence classes of ability, groups of players that are indistinguishable in terms of proficiency. For the Bradley-Terry model this suggests penalization of the MLE that would shrink pairwise differences of the rating parameters toward zero. This is conveniently accomplished by solving,

\[
\min \{-\ell(\alpha | y) + \lambda \| D\alpha \|_1\},
\]

where \( \| D\alpha \|_1 = \sum_{i<j} |\alpha_i - \alpha_j| \). This penalty has been proposed by (Dylan Hocking, Armand Joulin, Francis Bach and Jean-Philippe Vert 2011) and employed by (Cristiano Varin, Manuela Cattelan and David Firth 2016) to study journal rating and ranking. As \( \lambda \) is increased the \( \alpha \) parameters are pulled together into fewer and fewer groups. This is somewhat analogous to total variation regularization for nonparametric smoothing where piecewise linear fitting selects only a few places at which the derivative of the estimated function jumps. The problem is convex and thus efficiently solved by interior point methods; our preferred solver is Mosek (Mosek ApS 2021).

II. Empirical Bayes Posterior Mean Ranking

Another approach to regularization is to treat the unconstrained logistic estimates, \( \hat{\theta} \), as approximately independent draws from a Gaussian sequence model. When the problem design is unbalanced so the number of matched pairs are not equal, the MLE point estimates will have different precision and off-diagonal elements of their covariance matrix are also more heterogeneous. Initially, we will ignore the latter aspect and treat the estimated maximum likelihood rating parameters as a sample from a Gaussian sequence model with heterogeneous scale parameters.

Since the pairs, \( (\hat{\theta}, \hat{\sigma}) = \{(\hat{\theta}_i, \hat{\sigma}_i), \ i = 1, \cdots, p\} \) can be relatively high dimensional compared to the number of matches, \( n \), another regularization strategy is to treat the \( \hat{\theta}_i \)'s as if they arose from the mixture density,

\[
f(\theta | \hat{\theta}_i, \hat{\sigma}_i) = \int \varphi_{\sigma}(\theta - \hat{\theta}_i) dG(\theta),
\]

where \( \varphi_{\sigma} \) denotes the Gaussian density with mean zero and variance, \( \sigma^2 \). The mixing distribution \( G \) can be estimated by maximum likelihood as suggested by (Herbert Robbins 1950), (Jack Kiefer and Jack Wolfowitz 1956), (Nan Laird 1978), (Wenhua Jiang and Cun-Hui Zhang 2009), and (Roger Koenker and Ivan Mizera 2014), by solving,

\[
\min_{G \in \mathcal{G}} \{-\frac{1}{p} \sum_{i=1}^p \log f_G(\hat{\theta}_i) | f_G(\hat{\theta}_i) = \int \varphi_{\sigma}(\hat{\theta}_i - t) dG(t)\}.
\]

This problem is again convex and can be efficiently solved with the interior point implementation of Mosek. Given a \( G \) we can compute a posterior means for each \( \hat{\theta}_i \) and from these ratings, rankings may be constructed.

III. Empirical Bayes Posterior Ranks

Alternatively, we can use estimated ratings to construct ranks based on pairwise comparisons of the ratings,

\[
R_i = \sum_{j \neq i} 1\{\alpha_i \geq \alpha_j\}.
\]
When the $\alpha$’s are Gaussian (Nan Laird and Thomas Louis 1989) have considered the quadratic loss, $\sum_{i=1}^{n} (\hat{R}_i - R_i)^2$, under which the Bayes rule is the posterior mean rank. We can approximate this by,

$$\hat{R}_i = \sum_{j \neq i} P(\alpha_i \geq \alpha_j | \hat{\alpha}_1, \ldots, \hat{\alpha}_n)$$

where $\varphi_{ij}(z)$ is a bivariate Gaussian density with mean, $\mu = (\alpha_i, \alpha_j)$ and covariance matrix, $\Sigma(i,j)$, estimated from the Hessian of the MLE. The mixing distribution, $G$ is estimated by the maximum likelihood procedure described in the previous section.

### IV. A Simulation Exercise

We now compare performance of seven rating procedures intended to rank latent abilities:

- **MLE** Logistic MLE
- **KWPM** Posterior Mean Ratings
- **KWPMs** Smoothed Posterior Means
- **KWPR** Posterior Mean Ranks
- **RMLE** Group Lasso MLE
- **B** Borda Scores
- **WB** Weighted Borda Scores

Performance is evaluated by computing the mean of the Kendall rank correlation between the estimated true abilities and the estimated ratings for each procedure for 100 replications. Comparisons based Spearman’s rank correlation yield very similar conclusions, see (Persi Diaconis and Ronald L Graham 1977). In all the experimental settings there are 100 players. There are three other design dimensions of the experiment:

- **Distribution of Latent Abilities**
  - $\alpha \sim e^Z + 2, \quad Z \sim N(0,1)$
  - $\alpha \sim 0.8\delta_4 + 0.2\delta_8 + Z/3$
- **Matching Design**
  - **RS** Random Pairing
  - **LS** Similar Ability Pairing
- **Sample Size**

True abilities are either lognormal or a mixture of two Diracs contaminated by Gaussian noise. Pairings are generated completely at random or by pairing players of similar ability. In the random matching case Borda scores can be expected to perform well, however when pairings are based on similar abilities Borda scores, which ignore “strength of schedule,” falter.

We consider five sample sizes \{1000, 5000, 10000, 50000, 100000\}. With 100 players, 1000 pairings implies teams play on average only 10 matches, so the MLE is quite noisy and shrinkage isn’t very effective. At the other extreme with 100,000 pairings, so teams on average have 1000 matches the MLE is quite accurate and there is also little room for improvement from shrinkage. For intermediate sample sizes we see considerable benefit from some of the shrinkage methods. The noisy Dirac ability mixture is particularly favorable for the unsmoothed Kiefer-Wolfowitz method, but it performs well when the ability distribution is lognormal as well. Performance of the group lasso method is rather disappointing, but this might be attributed to poor choice of the smoothing parameter $\lambda$. Both the smoothed posterior means and the posterior mean ranks rely on a smoothed version of the Kiefer-Wolfowitz $\hat{G}$, so better tuning of their bandwidth choices might improve their performance.

Figure 1 reports results for 100 players and 100 replications. When players of similar ability are more likely to meet, (DD:LS), the Borda methods of estimating ranks fails miserably, and there is a modest improvement from regularization over the performance of the MLE in this setting for both the noisy Dirac and lognormal ability distributions. With random matching Borda performs credibly and other procedures are almost indistinguishable. The
grouped lasso is also disappointing, but this may be attributable to poor $\lambda$ selection.

V. The Stigler Model of Journal Influence

(Stephen Stigler 1994) considers a model of journal influence based on pairwise citation counts. Citations in journal $j$ of papers appearing in journal $i$ are viewed as a measure of the influence of journal $i$ on journal $j$. Ratings of journal influence are formulated as a Bradley-Terry model and rankings can be evaluated from estimated ratings. We consider pairwise citation counts for 86 journal spanning the econometrics/statistics fields over the decade 2010-2019 as reported in Clarivate Journal Citation Reports. Self citations, that is citations by a journal to papers that appeared in the same journal, while perhaps interesting, are ignored in the subsequent analysis.
of the grouped lasso procedure for the top 10 journals according to the maximum likelihood estimates. *Econometrica* is the most highly rated journal and remains so over the whole range of \( \lambda \in [0, 10] \). The *Quarterly Journal of Economics* appears to be a strong second place finisher, but careful inspection of the plot reveals that the MLE very slightly favors the *Annals of Probability*. It may seem surprising that both probability journals are so highly rated, but a little reflection reveals that although they are rarely cited by the other journals they almost never cite these journals; this is the hallmark of an influential journal. Once shrinkage is applied the probability journal fade into obscurity and more mainstream journals acquire more prominence. This raises the inevitable question: How should one choose \( \lambda \)? This is always controversial. If we adopt the (Gideon Schwarz 1978) BIC criterion that penalizes parametric dimension, in this case the number of estimated groups, which decreases with \( \lambda \), we get the plot appearing in Figure 3 showing that only a very modest amount of shrinkage is desirable.

Since the grouped lasso doesn’t alter the ranking of the top journals, we turn our attention to our empirical Bayes procedure. We consider ranking based on the posterior mean of the rating parameters. As noted earlier this procedure involves no tuning parameter selection. If the maximum likelihood rating estimates were based on a balanced design and consequently had homogeneous standard errors, the posterior mean Bayes rule would produce the same ranking as the MLE. The Bayes rule must be monotone in such circumstances. Our citation data is far from balanced, so there may be potentially different rankings. Table 1 reports ranking for our five procedures:

As anticipated the grouped lasso regularization doesn’t alter the MLE ranking, and the Kiefer-Wolfowitz posterior mean ranking only flips the *Annals of Probability* and *JRSS(B)*. The Borda rankings deprecate both probability journals and elevate *JASA* and *JRSS(B)*.

VI. Conclusion

We have considered a very special but commonly employed model for inferring latent abilities from pairwise comparison data. Although there is a recent wave of theoretical support for simple estimators like Borda scores, we have seen that good performance of such measures depends critically on balanced design assumptions that may not be plausible in applications. In contrast, the classical logistic MLE underpinning the Bradley Terry model offers a more robust alternative approach. When the number of “players” is large there are opportunities to improve upon the performance of the MLE by various forms of shrinkage including a variant of the group lasso and several empirical Bayes methods based on the Kiefer-Wolfowitz nonparametric MLE. As we have argued elsewhere decision making for rating, ranking and selec-
tion are inextricably intertwined and rules for ranking can be adapted to rules for selection that bring into play control of false discovery rates and related issues.

There are several intriguing directions for future investigation. Optimal design of pairwise matching schemes particularly with dynamic selection seems challenging, but potentially very rewarding. Connections to network formation models and their analysis may be fruitful as exemplified by the reliance on results of (Gordon Simons and Yi-Ching Yao 1999) in the work of (Bryan Graham 2017). Finally, there is considerable scope for moving away from the stringent assumptions of the Bradley-Terry model to consider more flexible formulations.

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