Instanton Calculus and Nonperturbative Relations in $N = 2$ Supersymmetric Gauge Theories

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ABSTRACT
Using instanton calculus we check, in the weak coupling region, the nonperturbative relation

\[ \langle \text{Tr} \phi^2 \rangle = i\pi \left( F - \frac{a}{2} \frac{\partial F}{\partial a} \right) \]

obtained for a $N = 2$ globally supersymmetric gauge theory. Our computations are performed for instantons of winding number $k$, up to $k = 2$ and turn out to agree with previous nonperturbative results.
1 Introduction

In a recent work \cite{1}, Seiberg and Witten have managed to compute the quantum moduli space and the Wilsonian effective action for the Yang–Mills theory with global $N = 2$ supersymmetry ($N = 2$ SYM from now on). This achievement has been possible by using judiciously a certain number of educated guesses for the behavior of the moduli space of vacua of the theory and by exploiting the unique properties of the $N = 2$ SYM. In fact, the Wilsonian effective action of this theory, after having used the Higgs mechanism, is completely determined once a certain prepotential $\mathcal{F}$ is known \cite{2}. In turn, this prepotential $\mathcal{F}$ is determined if its global structure is known or postulated. This global structure is given by the monodromies around the singular points of the prepotential $\mathcal{F}$: the group generated by these monodromies is a subgroup of $SL(2, \mathbb{Z})$.

In the electric (or Higgs) phase of the theory the form of the prepotential is known since, due to nonrenormalization theorems, only the one loop term contributes at the perturbative level. Moreover nonperturbative corrections due to instantons must be considered, leading to the final expression

$$\mathcal{F}(\Lambda) = \frac{1}{\pi} \left( \frac{A^2}{2} \ln \frac{2A^2}{\Lambda^2} - \sum_{k=1}^{\infty} \mathcal{F}_k \frac{\Lambda^{4k}}{A^{4k-2}} \right). \quad (1.1)$$

In (1.1) $\Lambda$ is the renormalization group invariant scale and $\Lambda$ is a chiral superfield whose lowest component squared is $a^2 \equiv -2u$, i.e. it is the gauge invariant coordinate of the moduli space of vacua, when the gauge group is $SU(2)$ (which will be our choice from now on), at least for large $u$ and $a$. The coefficients $\mathcal{F}_k$ give the nonperturbative contributions due to instantons. A formidable check of the assumptions made in \cite{1}, concerning the symmetries of the moduli space, is thus given by matching the coefficients $\mathcal{F}_k$ against those obtained by instanton calculus. This check must be performed in the weak coupling region in which instanton calculus can be reliably performed.
Some work has already been done along these lines. Nonperturbative contributions induced by instantons can in fact be seen, in the framework of perturbation theory, as effective four fermion vertices to be added to the tree level Lagrangian. The computation of these effective vertices has already been performed for the case of instantons of winding number one [3, 4, 5] and two [6].

The approach presented in this paper is somewhat different in that we will check the nonperturbative relation

$$< \text{Tr} \phi^2 > = i \pi \left( \mathcal{F} - \frac{a \partial \mathcal{F}}{2 \partial a} \right)$$  \hspace{1cm} (1.2)

found in [7]. Expanding the l.h.s. in (1.2) as

$$< \text{Tr} \phi^2 (a) > = -\frac{1}{2} a^2 - \sum_{k=1}^{\infty} G_k \frac{\Lambda^{4k}}{a^{4k-2}}$$  \hspace{1cm} (1.3)

and substituting (1.1) in (1.2) we find $G_k = 2k F_k$ for a comparison with the results of [8]. The $G_k$'s can also be straightforwardly checked against the results of the recursion relation found in [7, 8].

Supersymmetric instanton calculus was developed in two distinct ways [9, 10, 11] to study supersymmetry breaking. The main difference between these two approaches consists in giving or not an expectation value to the scalar (Higgs) field of the $N = 2$ multiplet. Given the check we want to perform the right choice is to follow [9, 10, 12] where such an expectation value for the scalar field is present.

This is the plan of the paper: in section 2 we shall briefly discuss the basic ingredients of the Atiyah–Hitchin–Drinfeld–Manin (ADHM) construction of instantons which will be useful later on. In section 3 we introduce

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1The reader should pay attention to different normalizations. The conventions of this letter for the $F_k$’s, which have opposite sign with respect to those of [8], are connected to [8] as: $F_k = -2^{6k-2} \mathcal{F}^{KLT}_k = -i \pi 2^{2k} \mathcal{F}^M_k$. 

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the semiclassical expansion of Green functions in SUSY gauge theories, and
check the relation (1.2) against a $k = 1$ computation. In section 4 we extend
our considerations to a background of Pontryagin index $k = 2$.

2 A Brief Review of the ADHM Formalism

Before describing the actual computation we need to briefly discuss the
ADHM construction and collect some useful formulae.

As it is well–known, self–dual $SU(2)$ connections on $S^4$, can be put into
one to one correspondence with holomorphic vector bundles of rank 2 over
$\mathbb{C}P^3$ admitting a reduction of the structure group to its compact real form.
The ADHM construction [13, 14] gives all these holomorphic bundles and
consequently all $SU(2)$ connections on $S^4$. The construction is purely al-
gebraic and we find it more convenient to use quaternionic notations. The
points, $x$, of the one–dimensional quaternionic space $\mathbb{H} \equiv \mathbb{C}^2 \equiv \mathbb{R}^4$
can be conveniently represented in the form $x = x^\mu \sigma_\mu$, with $\sigma_\mu = (1, i\sigma_c), c = 1, 2, 3$.
The $\sigma_c$’s are the usual Pauli matrices. The conjugate of $x$ is $x^\dagger = x^\mu \sigma^\dagger_\mu$. A
quaternion is said to be real if it is proportional to $1$ and imaginary if it has
vanishing real part.

The prescription to find an instanton of winding number $k$ is the following:
introduce a $(k + 1) \times k$ quaternionic matrix linear in $x$

$$\Delta = a + bx.$$  \hspace{1cm} (2.1)

The (anti–hermitean) gauge connection is then written in the form

$$A_{\mu}^q = U^\dagger \partial_\mu U;$$  \hspace{1cm} (2.2)

where $U$ is a $(k + 1) \times 1$ matrix of quaternions providing an orthonormal
frame of $\text{Ker}\Delta^\dagger$. In formulae

$$\Delta^\dagger U = 0;$$  \hspace{1cm} (2.3)
\[ U^\dagger U = \mathbb{I}_2, \] (2.4)

where \( \mathbb{I}_2 \) is the two-dimensional identity matrix. The constraint (2.4) ensures that \( A^c_\mu \) is an element of the Lie algebra of the \( SU(2) \) gauge group. The condition of self-duality on the field strength of (2.2) is imposed by restricting the matrix \( \Delta \) to obey

\[ \Delta^\dagger \Delta = f^{-1} \otimes \mathbb{I}_2, \] (2.5)

with \( f \) an invertible hermitean \( k \times k \) matrix (of real numbers). In addition to the gauge freedom (right multiplication of \( U \) by a unitary quaternion) we have the freedom to perform the transformations

\[ \Delta \to Q\Delta R, \] (2.6)

with \( Q \in Sp(k+1), R \in GL(k, \mathbb{R}) \), which leave (2.2) invariant.

These symmetries can be used to simplify the expressions of \( a \) and \( b \). Exploiting this fact, in the following we will choose the matrix \( b \) to be

\[ b = -\begin{pmatrix} 0_{1\times k} \\ \mathbb{I}_{k\times k} \end{pmatrix}. \] (2.7)

From (2.2), the field strength of the gauge field can be computed and it is

\[ F_{\mu\nu} = 2(U^\dagger bf_\mu \sigma_\mu b^\dagger U), \] (2.8)

where \( \sigma_\mu = i\eta_\mu^a \sigma^a \), with \( \eta_\mu^a \) the 't Hooft symbols. Using (2.8) we also compute

\[ \text{Tr}(F_{\mu\nu}F_{\mu\nu}) = 2\Box \text{Tr} \left[ b^\dagger (1 + P)b \right], \] (2.9)

where

\[ P = UU^\dagger = 1 - \Delta f \Delta^\dagger \] (2.10)

is the projector on the kernel of \( \Delta^\dagger \). (2.3) will turn out to be important in the following.
The bosonic zero–modes, $Z_\mu$, of the gauge–fixed second order differential operator

$$M_{\mu\nu} = -D^2 (A^c) \delta_{\mu\nu} - 2 F^c_{\mu\nu}, \quad (2.11)$$

which describes the quantum fluctuations of the gauge fields, can be found by noting that the transverse fluctuations of a self–dual configuration must satisfy the relations

$$^* (D_{[\mu} Z_{\nu]} = D_{[\mu} Z_{\nu]}, \quad D_\mu Z_\mu = 0. \quad (2.12)$$

This allows to write $Z_\mu$ in terms of the quantities appearing in (2.1), as \[15\]

$$Z_\mu = U^\dagger C \bar{\sigma}_\mu f b^\dagger U - U^\dagger b f \sigma_\mu C^\dagger U. \quad (2.13)$$

For (2.13) to satisfy (2.12), the $(k + 1) \times k$ matrix $C$ must obey

$$\Delta^\dagger C = (\Delta^\dagger C)^T, \quad (2.14)$$

where the superscript $T$ stands for transposition of the quaternionic elements of the matrix (without transposing the quaternions themselves). In our case, the number of independent $Z_\mu$ is $8k$ (the dimension of the moduli space of the instanton). $C$ has $k(k + 1)$ quaternionic elements, which are subject to the $4k(k - 1)$ constraints (2.14). The number of $C$’s satisfying (2.14) is thus $8k$ as desired. This is the reason why in the following we will sometimes attach a subscript $r = 1, \ldots, 8k$ to the zero–modes.

Fermionic zero–modes are easily deduced from (2.13) by remarking that, due to $N = 2$ SUSY,

$$\lambda^{(r)}_{\dot{\beta}\dot{A}} = \sigma^\mu_{\dot{\beta}\dot{A}} Z_\mu^{(r)}, \quad (2.15)$$

where $\dot{A} = 1, 2$ labels the two SUSY charges. Furthermore, the superposition of the $Z_\mu^{(r)}$ was computed in \[15\] to be

$$(Z^{(r)}, Z^{(s)}) = \frac{8\pi^2}{g^2} \text{Tr} \left[ C_r^\dagger (1 + P_\infty) C_s \right] \equiv \frac{8\pi^2}{g^2} (C_r, C_s), \quad (2.16)$$
where \( P_\infty = 1 - b b^\dagger \) is the projector \( P \) evaluated in the limit \( |x| \to \infty \). The superpositions of the bosonic zero–modes are then tied to the Jacobian which yields the integration measure for the bosonic collective coordinates. This integration measure is easily written once the variations of the bosonic fields with respect to the instanton moduli are known. But the zero–modes \( (2.13) \) are also transverse to allow the factorization of the infinite volume of the gauge group via the introduction of the Faddeev–Popov determinant \( [16] \). It is thus useful to separate, in the expression of the bosonic zero–modes, the variation with respect to the instanton moduli from the gauge part needed to make it transverse \([17, 18]\). In the ADHM formalism this yields to the formula for the variation of the gauge connection \([19]\)

\[
\delta_r A_\mu = U^\dagger (\delta_r \Delta) \sigma_\mu f b^\dagger U - U^\dagger b f \sigma_\mu (\delta_r \Delta)^\dagger U + [D_\mu, \delta g] \tag{2.17}
\]

where \( \delta_r, r = 1, \ldots, 8k \) stand for the variations with respect to the instanton moduli, and \( \delta g, \) satisfying \( \delta g + \delta g^\dagger = 0 \), is an arbitrary infinitesimal gauge transformation. The unconstrained variations \( \delta_r \Delta \), which give the integration over the collective coordinates, cannot be easily traded with \( (2.16) \) since the \( C \)'s appearing in that formula are constrained by \( (2.14) \). The complete relation between the \( \delta_r \Delta \) and \( C \)'s is given by

\[
(C_r, C_s) = (\delta_r \Delta, \delta_s \Delta) + 2 K_{rji} M_{ij,lm} K_{slm}. \tag{2.18}
\]

The explicit expression of the matrices \( K, M \), which parametrize the freedom to transform the ADHM data as in \( (2.6) \), can be found in \([15]\).

3 The \( k=1 \) Semiclassical Computation

We now briefly review the strategy to perform semiclassical computations in supersymmetric gauge theories, in the context of the constrained instanton method.
The classical potential for the complex scalar field $\phi$ of an $N = 2$ SYM gauge theory

$$V_D = \frac{1}{2} (\epsilon^{abc} \phi^b \phi^c)^2$$

(3.1)

has flat directions when $\phi$ is a $SU(2)$ gauge transform of $\phi = a^c \sigma_c / 2i$, where $a^c = a \delta^{c3}$ and $a$ is a complex number. Following 't Hooft [17], we shall then expand the action functional around a properly chosen field configuration, which is the solution of the equations

$$D_\mu (A) F_{\mu \nu} = 0 ,$$

(3.2)

$$D^2 (A) \phi_{cl} = 0 , \quad \lim_{|x| \to \infty} \phi_{cl} \equiv \phi_\infty = a \frac{\sigma_3}{2i} .$$

(3.3)

The first equation admits instantonic solutions. When $a = 0$, (3.3) admits the trivial solution $\phi = 0$ only. When $a \neq 0$, we shall decompose the fields $\phi, \phi^\dagger$ as

$$\phi = \phi_{cl} + \phi_Q , \quad \phi^\dagger = (\phi_{cl})^\dagger + \phi_Q^\dagger ,$$

(3.4)

and integrate over the quantum fluctuations $\phi_Q, \phi_Q^\dagger$.

The integration over bosonic zero–modes can be traded with an integration over collective coordinates, at the cost of introducing the corresponding jacobian. The existence of fermion zero–modes is the way by which Ward identities related with the group of chiral symmetries of the theory, come into play. When $a = 0$, the anomalous $U_R(1)$ symmetry

$$\lambda \longrightarrow e^{i\alpha} \lambda , \quad \phi \longrightarrow e^{2i\alpha} \phi$$

(3.5)

and gauge invariance allow a nonzero result for the Green functions with $n$ insertions of the gauge invariant quantity $(\phi^a \phi^a) (x)$ only when $n = 2k$. These correlators possess the right operator insertions needed to saturate the integration over the Grassmann parameters, and due to supersymmetry, they are also position independent. On the other hand when $a \neq 0$ the correlator
$\langle \phi^a \phi^a \rangle$ has a complete expansion in terms of instanton contributions, as in (1.3). The $k = 1$ term has been computed in [20] in the framework of the constrained superinstanton formalism of [10]. In the following we will use the component formalism, which we find simpler, to compute the $k = 1, 2$ coefficients of the instanton expansion.

Fermion zero–modes are found by solving the equation

$$D_\mu \bar{\sigma}_\mu \lambda_{\beta \dot{A}} = 0 .$$

(3.6)

For instantons of winding number $k = 1$ the whole set of solution of this equation is obtained via SUSY and superconformal transformation, which yield

$$\lambda_{a \dot{A}} = \frac{1}{2} F^{a}_{\mu \nu} (\sigma_{\mu \nu})_{a}^{\beta} \zeta_{\beta \dot{A}} ,$$

(3.7)

where $\zeta = \xi + (x - x_0)_\mu \sigma_\mu \bar{\eta} / \sqrt{2} \rho$, $\xi, \bar{\eta}$ being two arbitrary quaternions of Grassmann numbers. It is instructive, and useful for the computations to be performed in the next chapter, to deduce this result using the ADHM expressions (2.13), (2.14), (2.15). For $k = 1$ the constraint (2.14) is always satisfied since $\Delta^\dagger C$ is a single quaternion. Given a matrix

$$\Delta = \begin{pmatrix} v \\ (x_0 - x) \end{pmatrix} ,$$

(3.8)

and choosing $C$ to be

$$C = \begin{pmatrix} \sqrt{2} C_0 \\ 2 C_1 \end{pmatrix} ,$$

(3.9)

with $C_0, C_1$ two arbitrary quaternions, we can substitute (3.9) in (2.13) to find, using (2.8)

$$Z_\mu = 2 F_{\mu \nu} B_\nu ,$$

(3.10)

where

$$B = B_\mu \sigma^\mu = C_1 + (x - x_0) \frac{\bar{v}}{v^2} \frac{C_0}{\sqrt{2}}$$

(3.11)
Interpreting $C_0, C_1$ as Grassmann variables and using (2.15) we obtain (3.7) with $B$ replacing $\zeta$.

The correct fermionic integration measure is given by the inverse of the determinant of the matrix whose entries are the scalar products of the fermionic zero–mode eigenfunctions. This scalar product is induced by the kinetic terms in the action and for arbitrary $SU(2)$ valued spinors $f, g$ is

$$ (f, g) \equiv \sum_{a=1}^{3} \int d^4x \ (f^a_\alpha)^* \sigma_0^{\alpha\beta} g^a_\beta. $$

(3.12)

The fermionic measure now reads

$$ d^4\xi d^4\bar{\eta} \left( \frac{g^2}{32\pi^2} \right)^4, $$

(3.13)

where $d^4\xi d^4\bar{\eta} \equiv d^2\xi_1 d^2\xi_2 d^2\bar{\eta}_1 d^2\bar{\eta}_2$. Once auxiliary fields are eliminated, the action is

$$ S = S_G + S_H + S_F + S_Y + S_D. $$

(3.14)

$S_G$ is the usual gauge field action, $S_F[\lambda, \bar{\lambda}, A] = \int d^4x \ \bar{\lambda}^{\dot{\alpha}a} \left[ D(A)\lambda^\dot{\alpha} \right]^a$ and $S_H[\phi, \phi^\dagger, A] = \int d^4x \ (D\phi)^{ta}(D\phi)^a$ are the kinetic terms for the Fermi and Bose fields minimally coupled to the gauge field $A_\mu$. The Yukawa interactions are given by

$$ S_Y[\phi, \phi^\dagger, \lambda, \bar{\lambda}] = \sqrt{2}g\epsilon^{abc} \int d^4x \ \phi^{\ast \dagger}(\lambda^b_1 \lambda^c_2) + \text{h.c.} $$

(3.15)

and finally $S_D = \int d^4x \ V_D$ comes from the potential term (3.1) for the complex scalar field $\phi$.

The evaluation of the correlator $< \phi^a \phi^a >$ in the semiclassical approximation around an instantonic background of winding number $k = 1$ yields

$$ < \phi^a \phi^a > = 
\int d^4x_0 d\rho \ \left( \mu^s \frac{2^{10} \pi^6 \rho^3}{g^8} \right) e^{-\frac{8\pi^2}{s^2} - 4\pi^2|a|^2 \rho^2} \int [\delta Q \delta \lambda] \delta \lambda \delta \phi^\dagger Q \delta \phi Q \delta c \bar{c}. $$
\[
\exp\left[-S_H[\phi_Q, \phi_Q^\dagger, A^{cl}] - S_F[\lambda, \bar{\lambda}, A^{cl}] + \frac{1}{2} \int d^4x \, Q_\mu M_{\mu\nu} Q^\nu - \int d^4x \, \bar{c}D^2(A^{cl})c \right] \\
\mu^{-(4+4)} \int d^4\xi d^4\eta \left(\frac{g^2}{32\pi^2}\right)^4 \exp \left[-S_Y[\phi_{cl} + \phi_Q, (\phi_{cl})^\dagger + \phi_Q^\dagger, \lambda(0), \bar{\lambda} = 0]\right] \\
(\phi_{cl} + \phi_Q)^a(\phi_{cl} + \phi_Q)^a(x) . \tag{3.16}
\]

Let us now explain where the different terms in (3.16) come from:

1. \(d^4x_0 d\rho \left(\mu^{8/3} \frac{\alpha_s^{3/2}}{g^2}\right)\) is the bosonic measure \([17, 18]\) after the integration over \(SU(2)/\mathbb{Z}_2\) global rotations in color space has been performed. \(x_0\) and \(\rho\) are the center and the size of the instanton (see (3.8), with \(\rho \equiv |v|\)).

2. \(S_H[\phi_{cl}, (\phi_{cl})^\dagger, A^{cl}] = 4\pi^2|a|^2\rho^2\), is the contribution of the classical Higgs action, and has been computed by ‘t Hooft \([17]\).

3. The second line include the quadratic approximation of the different kinetic operators for the quantum fluctuation of the fields and the symbol \([\delta \lambda \delta Q]\) denotes integration over nonzero–modes. \(\bar{c}\) and \(c\) are the usual ghost fields, \(\int d^4x \, \bar{c}D^2(A^{cl})c\) being the corresponding term in the action.

4. \(S_Y[\phi, \phi^\dagger, \lambda(0), \bar{\lambda} = 0]\) is the Yukawa action calculated with the complete expansion of the fermionic fields replaced by their projection over the zero–mode subspace. According to the index theorem for the Dirac operator in the background of a self–dual gauge field configuration, we have only zero–modes of one chirality, so that this term reduces to \(\sqrt{2} g c^{abc} \int \phi^{at}(\lambda_{1}^{(0)b}\lambda_{2}^{(0)c})\).

5. \(\mu^{8-\frac{1}{2}(4+4)} e^{-\frac{s_{\alpha_s}^2}{g^2}} = \Lambda^4\), where \(\Lambda\) is the (one loop) \(N = 2\) SYM renormalization group invariant scale with gauge group \(SU(2)\). \(\mu\) comes
from the Pauli–Villars regularization of the determinants and the exponent is $b_1 k = (n_B - n_F/2)$ where $n_B, n_F, b_1$ are the number of bosonic, fermionic zero–modes and the first coefficient of the $\beta$–function of the theory.

After the integration over $\phi, \phi^\dagger$ and the nonzero–modes, the $\phi_Q$ insertions get replaced by $\phi_{\text{inh}}$, where

$$
\phi_{\text{inh}}^a = \sqrt{2} g e^{bcd} [(D^2)^{-1}]_{ab}^{cd} (\lambda_1^{(0)d}) (\lambda_2^{(0)c}) \, ,
$$

and the determinants of the various kinetic operators cancel against each other \cite{21}. The r.h.s. of (3.16) now reads

$$
\Lambda^4 \int d^4 x_0 d\rho \left( 2^{10} \pi^6 \rho^3 \right) e^{-4\pi^2 |a|^2 \rho^2}

\int d^4 \xi d^4 \eta \left( \frac{g^2}{32\pi^2} \right)^4 \exp \left[ -\sqrt{2} g e^{abc} \int \phi_{\text{cl}}^a (\lambda_1^{(0)b}) (\lambda_2^{(0)c}) \right]

(\phi_{\text{cl}} + \phi_{\text{inh}})^a(x) \, .
$$

A straightforward calculation shows that \cite{22}

$$
S_Y [\phi_{\text{cl}}, \phi_{\text{cl}}^\dagger, \lambda^{(0)}, \bar{\lambda} = 0] = \left( \frac{a^c}{\sqrt{2}} \right) g (\bar{\eta}_1 \sigma^c \bar{\eta}_2) \left( \frac{g^2}{32\pi^2} \right)^{-1} \, .
$$

Moreover it is also easy to convince oneself that (3.17) is solved by

$$
\phi_{\text{inh}}^a = \sqrt{2} (\zeta_1 \lambda_2^a) \, ,
$$

as it can be checked by substituting in

$$
[D^2]_{ab}^{cd} \phi_{\text{inh}}^b = \sqrt{2} g e^{abc} (\lambda_1^{(0)b}) (\lambda_2^{(0)c}) \, .
$$

The Yukawa action does not contain the Grassmann parameters of the zero–modes coming from SUSY transformations. As a consequence the only
nonzero contributions are obtained by picking out the terms in the \( \phi_{\text{inh}} \) insertions which contain the SUSY solutions of the Dirac equation. We thus completely disregard the \( \phi_{\text{cl}} \) pieces. Since

\[
\phi_{\text{inh}}^a \phi_{\text{inh}}^a = -\zeta_1^2 (\lambda_2^a \lambda_2^a) = -\zeta_1^2 \zeta_2^2 (F_{\mu\nu}^a F_{\mu\nu}^a), \quad (3.22)
\]

this amounts to say

\[
(\phi_{\text{cl}} + \phi_{\text{inh}})^a (\phi_{\text{cl}} + \phi_{\text{inh}})^a \longrightarrow -\zeta_1^2 \zeta_2^2 (F_{\mu\nu}^a F_{\mu\nu}^a). \quad (3.23)
\]

The integration over non SUSY zero–modes is then dealt with by performing the integral

\[
\int d^4 \tilde{\eta} \left( \frac{g^2}{32\pi^2} \right)^2 \exp \left[ \frac{(a^*)^g g}{\sqrt{2}} (\tilde{\eta}_1 \sigma \tilde{\eta}_2) \left( \frac{g^2}{32\pi^2} \right)^{-17} \right] = \frac{g^2}{2} (a^*)^2. \quad (3.24)
\]

(3.16) now becomes

\[
< \phi^a \phi^a > = \Lambda^4 \int d^4 x_0 d \rho \left( \frac{2^{10} \pi^6 \rho^3}{g^8} \right) e^{-4\pi^2 |a|^2 \rho^2} (F_{\mu\nu}^a F_{\mu\nu}^a) \frac{g^2}{2} (a^*)^2 \int d^4 \xi \left( \frac{g^2}{32\pi^2} \right)^2 \zeta_1^2 \zeta_2^2. \quad (3.25)
\]

A simple computation using the explicit form of the \( k = 1 \) gauge connection, shows that \( F_{\mu\nu}^a F_{\mu\nu}^a \) is a function of the difference \( x - x_0 \). We can then immediately integrate over \( x_0 \) remembering that \( \int d^4 x F_{\mu\nu}^a F_{\mu\nu}^a = 32\pi^2/g^2 \). The remaining integrations over \( \xi \) and \( \rho \) in (3.25) are trivial and yield

\[
< \phi^a \phi^a > = \frac{2 \Lambda^4}{g^4 a^2}. \quad (3.26)
\]

This result agrees with the coefficient \( G_1 \) found in [7, 8].
4 The k=2 Computation

We now come to the $k=2$ computation. There are several modifications to take into account, with respect to the $k=1$ case but the general strategy is unchanged. We start by giving the form of the matrix $\Delta$ of (2.1)

$$
\Delta = \begin{pmatrix}
v_1 & v_2 \\
 x_1 - x & d \\
d & x_2 - x
\end{pmatrix} = \begin{pmatrix}
v_1 & v_2 \\
e & d \\
d & -e
\end{pmatrix} + b(x - x_0).
$$

(4.1)

The constraint (2.5) is obeyed if

$$
d = \frac{1}{2} \frac{z}{z^2} (\bar{v}_2 v_1 - \bar{v}_1 v_2),
$$

(4.2)

with $z = x_1 - x_2$ [23]. We find it more convenient to expose the role of the center of the instanton, the part proportional to the matrix $b$ of (2.7), because this will be central in the integration which we will perform later. This is achieved with the substitutions $x_0 = (x_1 + x_2)/2$, $e = (x_1 - x_2)/2$ which gives the other form of the matrix $\Delta$ in (4.1).

We also need the form of the matrix $C$ appearing in (2.15) [24] which is constrained by (2.14). Since this constraint is very similar to (2.5) (to get convinced of this fact just think that two solutions of (2.14) are given by $C = a, b$) it is convenient to choose a form of $C$ which parallels (4.1)

$$
C = \begin{pmatrix}
v_1 & v_2 \\
 \xi_1 & \delta \\
 \delta & \xi_2
\end{pmatrix} = \begin{pmatrix}
v_1 & v_2 \\
 \eta & \delta \\
 \delta & -\eta
\end{pmatrix} - b\xi_0.
$$

(4.3)

The constraint (2.14) is satisfied imposing

$$
d = \frac{z}{z^2} [2\bar{d}\eta + \bar{v}_2 v_1 - \bar{v}_1 v_2].
$$

(4.4)

\footnote{The elements of this matrix must be interpreted as Grassmann numbers from the point of view of the functional integration.}
In analogy with the points 2, 4 of the previous calculation for the $k = 1$ case, we have to compute

$$S_H[\phi_d + \phi_{inh}, (\phi_d)^\dagger, A^{cl}] + S_Y[\phi = 0, (\phi_d)^\dagger, \lambda^{(0)}, \bar{\lambda} = 0].$$

(4.5)

This computation involves only the contributions of the $\phi_d$ function and of the $\phi_{inh}$ field at the boundary of the physical space. It has been performed in [6] and it yields

$$S_H + S_Y = 4\pi^2|a|^2(|v_1|^2 + |v_2|^2) - 4\pi^2 \frac{\left[\text{Tr}(v_1 \bar{v}_2 - v_2 \bar{v}_1)\phi_\infty\right]^2}{|v_1|^2 + |v_2|^2 + 4(|d|^2 + |c|^2)}$$

$$+ 2\sqrt{2}\pi^2 \epsilon^{\hat{A}\hat{B}} \epsilon^{\alpha\gamma} \left[(\nu_1)_{\gamma A} (\phi_\infty)_{\alpha B} (v_1)^{\beta (\nu_1)}_{\beta \hat{B}} + \left(\frac{\text{Tr}(v_1 \bar{v}_2 - v_2 \bar{v}_1)\phi_\infty}{|v_1|^2 + |v_2|^2 + 4(|d|^2 + |c|^2)}\right)ight]$$

$$((\nu_1)_{\gamma A} (\nu_2)_{\alpha B} + 2\eta_{\gamma A} \delta_{\alpha B}),$$

(4.6)

where $\phi_\infty$ was defined in (3.3).

Let us comment on (4.6): in the Yukawa action the variable $\xi_0$ is missing. In fact, the expectation value of the scalar field has broken the conformal but not the translational invariance of the action. The effect is the appearance, in the action, of the collective coordinates related to these symmetries. As SUSY is still a symmetry, the collective coordinates associated to it must be missing in (4.6), which is what we observe. In complete analogy with the $k = 1$ case, the Grassmann parameters can now be divided into two sets: those which do not appear in the action (connected to SUSY) must be isolated in the $\phi_{inh}$ piece to cancel against the measure. Those which appear in the action, will not appear in the insertion of $\phi^a \bar{\phi}^a$: all fermionic zero–modes are lifted but the eight SUSY ones. There is another consequence of this observation. On the r.h.s. of (3.17) there are fermionic fields expanded in the basis of the zero–modes given by (2.15). See also the $k = 1$ case (3.11),

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3The following expression contains the prescription for computing $S_H + S_Y$ on the saddle point for which the only zero–modes are the left–handed ones.
for a comparison. The previous observation thus suggests us to solve (3.21) only for those fermionic fields containing the Grassmann parameters related to SUSY transformations. As a consequence (3.23) still holds. The term $F^a_{\mu\nu}F^a_{\mu\nu}$ is given by (2.9), and in (4.1) $\Delta$ was parametrized in order to be a function of $x-x_0$. Since (2.9) depends only on $\Delta$, $F^a_{\mu\nu}F^a_{\mu\nu}$ is a function of $x-x_0$ too.

Given all these observations we can write the correlator for the case $k=2$ as

$$<\phi^a\phi^a> = \frac{\Lambda^8}{S} \int d^4e d^4v_1 d^4v_2 d^4\eta d^4\nu_1 d^4\nu_2 \left( \frac{J_{Bose}}{J_{Fermi}} \right)^{1/2} e^{-s_{Bose}-s_{Fermi}} \int d^4\xi_0 (\xi_0)^2 \int d^4x_0 F^a_{\mu\nu}F^a_{\mu\nu},$$  \hspace{1cm} (4.7)

where $S = 16$ is a statistical weight computed in [15]. The integrations in the last line of (4.7) can be performed immediately after trading the $x_0$ with the $x$ integration by shifting variable, and give

$$\int d^4\xi_0 [(\xi_0)_1]^2 [(\xi_0)_2]^2 \int d^4x_0 F^a_{\mu\nu}F^a_{\mu\nu} = 4 \cdot \frac{64\pi^2}{g^2}.$$  \hspace{1cm} (4.8)

The fermionic Jacobian, $J_{Fermi}$, is obtained from (2.15), (2.16) while $J_{Bose}$ was computed in [15, 16]. Putting these results together we get

$$\left( \frac{J_{Bose}}{J_{Fermi}} \right)^{1/2} = \frac{2^{10}}{\pi^8} \frac{|e|^2 - |d|^2}{|v_1|^2 + |v_2|^2 + 4(|d|^2 + |c|^2)}.$$  \hspace{1cm} (4.9)

After substituting (4.9) in (4.7), the remaining integrations can be performed and give $5/(32\pi^2a^6g^6)[3].$

Collecting all these results we finally find

$$<\text{Tr}\phi^2> = -\frac{5}{4} \frac{\Lambda^8}{g^8a^6},$$  \hspace{1cm} (4.10)

which is in agreement with the results of [7, 8]
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