Is it Possible to Describe Economical Phenomena by Methods of Statistical Physics of Open Systems?

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The methods of statistical physics of open systems are used for describing the time dependence of economic characteristics (income, profit, cost, supply, currency etc.) and their correlations with each other. Nonlinear equations (analogies of known reaction-diffusion, kinetic, Langevin equation) describing appearance of bifurcations, self-sustained oscillational processes, self-organizations in economic phenomena are offered.

I. INTRODUCTION

One of the systems consisting of a big number of elements, which exchange information, currency, goods etc. with each other is a market economy. The number of the elements (for example, of firms characterized by income, dividend and investments size, demand and supply value, labor cost and so on) is so great that application of statistical physics methods to describe market economy becomes possible. But now (as it seems) there is no a general-algorithm for subsequent theories applying methods of statistical physics of open systems to market economy (although G.Bystrai [1] - [3], A.Johansen and D.Sernette [4], D.Meadows [5], W.Weidlich [6], I.Lubashevsky [7] used ideas of statistical physics of open systems to market economy (see [8]) in the frames of mathematical formalism developed in statistical physics and describing both self-organization processes and catastrophes. To do this a number of economic concepts $x_i(t)$ ($i = 1,...N; t$ denotes time) is introduced and used, different for different economic problems, with $x_i$ being able to depend on other economic characteristics as well. The concepts mentioned are then treated as “coordinates” of an economic system. Another set of economic variables $F_\alpha(x_1,...x_N, t, \dot{x}_1,...\dot{x}_N)$ is defined as functions of $x_i(t)$, $\dot{x}_i(t)$, $\frac{\partial x_i}{\partial x_k}$ ($i, j = 1,...N, \alpha = 1,...\beta$) etc. The set of “variables” $x_i$ and their relations with $F_\alpha$ are relative, that is the “coordinates” $x_i$ in a concrete problem can turn to be functions $F_\alpha$ in another one. If we are to regard $F_\alpha(x_i(t), \dot{x}_i(t), \frac{\partial x_i}{\partial x_k}, t)$ for a given $\alpha$ (i.e., $F_\alpha$ is a currency rate of exchange) as a set of macroscopic functions characterizing the economic system under consideration, it is possible to write down well developed in statistical physics of open systems equations of nonlinear kinetic, reaction-diffusion, oscillational types. Treatment of these equations and their solutions, if being applied to a concrete economic problem, allows to describe not only well known economic regularities, but also to define the values of and relations between economic parameters, leading to drastic violations of an equilibrium.

II. FORMULATION OF THE EQUATIONS.

1. Consider an economic media consisting of $N$ elements $x_i$ ($i = 1,...N$). Let this elements define functions $F_\alpha$ (not necessary independent ones). If while changing governing parameters the elements themselves change (an active media), then, provided the diffusion type relations presence, the time behaviour of $F_\alpha(x_i(t), \dot{x}_i(t), \frac{\partial x_i}{\partial x_k}, t)$ set can be described by a reaction diffusion equations

$$\frac{dF_\alpha(x_i,t)}{dt} = \varphi_\alpha(F) + \frac{\partial}{\partial x_i}[D^{\alpha}_{ij}(F)\frac{\partial F_\alpha}{\partial x_j} - A_i(x,t)F_\alpha] + \frac{\partial}{\partial x_i}[\hat{D}^{\alpha}_{ij}(F)\frac{\partial F_\alpha}{\partial \dot{x}_j}] + (a_i - b_i(x,t)F_\alpha) \tag{1}$$

$\alpha = 1,...\beta, i, j = 1,...N, \beta$ is the number of $F_\alpha$ functions used in the problem under investigation. The full time derivative in the left-hand side of (1) can be represented as (summing up over the repeating indexes)

$$\frac{dF_\alpha}{dt} = \frac{\partial F_\alpha}{\partial t} + \frac{\partial F_\alpha}{\partial x_\beta} \frac{dx_\beta}{dt} + \frac{\partial F_\alpha}{\partial \dot{x}_\beta} \frac{d\dot{x}_\beta}{dt} \tag{2}$$

Thus, the system of equations describing an economic situation reads

$$\frac{\partial F_\alpha}{\partial t} + \frac{\partial F_\alpha}{\partial x_\beta} \frac{dx_\beta}{dt} + \frac{\partial F_\alpha}{\partial \dot{x}_\beta} \frac{d\dot{x}_\beta}{dt} = I_\alpha \tag{3}$$

where

$$I_\alpha = \varphi(F, \dot{F}_i,...t) + \frac{\partial}{\partial x_i}[D^{\alpha}_{ij}(F)\frac{\partial F_\alpha}{\partial x_j} - A_i(x,t)F_\alpha] + \frac{\partial}{\partial x_i}[\hat{D}^{\alpha}_{ij}(F)\frac{\partial F_\alpha}{\partial \dot{x}_j}] + (a_i - b_i(x,t)F_\alpha) \tag{4}$$

Here $D^{\alpha}_{ij}(F_1...F_\beta)\, , \, D^{\alpha}_{ij}(F_1...F_\beta)$ are the diffusion coefficients in the $x_i$ and $\dot{x}_i$ spaces, depending (including nonlinear dependencies) on $F; \varphi(F_1, \dot{F}_i,...t)$ is a nonlinear function, determined by the economic problem data (as well as the other parameters dependencies). In (3) $\dot{x}_i$ and $\ddot{x}_i$ characterize the rate of and the acceleration of
where \( A \) and \( B \) are "external" influences onto the economic system. In special cases \( I_\alpha \) may equals zero \( I_\alpha = 0 \) for some or all of \( \alpha \) (it corresponds neglecting the influence of the factors promoting \( F_\alpha \) varying with time).

2. Above there were formulated equations (2a)-(2b) describing time dependence of economic parameters (considered as variables or functions) of an open non equilibrium statistical system and including the description of crisis or self-sustain oscillation processes as well. Nevertheless, so far as variables \( x_i \) themselves can depend on the other variables \( x_j \) (\( i \neq j \)) and on functions \( F_\alpha(x_i,...) \) (stress again that in economy, because of the dependence of almost all the factors on each other, the choice of variables \( x_i \) and functions \( F_\alpha(x_i,...) \) is quite an arbitrary and may change from one problem to another), one should supplement the eqs. (1)-(3) by a system of equations describing the \( x_i \) dependence on \( x_j \) and \( F_\alpha \). These equations can be chosen basing on the same ideas at the time equations, that is can be obtained by substitution of \( \frac{\partial}{\partial x} \) for \( \frac{\partial}{\partial x_0} \) or for \( \frac{\partial}{\partial F_\alpha} \). In the case of reaction-diffusion equations one then writes:

\[
\frac{dx_i}{dt} = F_{ij}(x_j,F_{\alpha}) + \frac{\partial}{\partial x_j}(D_{ij}(x_j) \frac{\partial}{\partial x_j} \cdot x_i - A_j x_i) + \\
+ \frac{\partial}{\partial F_\alpha}[\tilde{D}_{ij}(x) \frac{\partial}{\partial F_\alpha} \cdot x_i + (\tilde{a}_i - \tilde{b}_i(x,t) x_\alpha)]
\]  

(5)

\[
\frac{dx_i}{dt} = \tilde{F}_{i\alpha} + \frac{\partial}{\partial F_{\alpha}}[(D_{ij}(x_j) \frac{\partial}{\partial x_j} \cdot x_i - A_i x_\alpha)] + \\
+ \frac{\partial}{\partial x_j}[\tilde{D}_{ij}(x) \frac{\partial}{\partial x_j} \cdot x_i + (\tilde{a}_j - \tilde{b}_j(x,t) x_\alpha)]
\]  

(6)

3. One can include in eqs. (5)-(6) terms corresponding to "diffusion" in the "velocity" space. The sums of such "diffusion" terms should be added to the right-hand part of these equations. The system of equation (5)-(6) seems to be able to give a full enough description of an economic system behaviour near bifurcation points and phase transitions.

4. Let \( I_\alpha \) has the following form (here we neglect by diffusion and nonlinear with respect to \( F \) terms),

\[
a) \frac{dF_\alpha}{dt} = \frac{\partial F_\alpha}{\partial t} + \frac{\partial F_\alpha}{\partial x_\beta} \frac{dx_\beta}{dt} + \frac{\partial F_\alpha}{\partial x_\beta} \frac{dx_\beta}{dt} = 0
\]

\[
b) \frac{dF_\alpha}{dt} = \frac{\partial F_\alpha}{\partial t} + \frac{\partial F_\alpha}{\partial x_\beta} \frac{dx_\beta}{dt} + \frac{\partial F_\alpha}{\partial x_\beta} \frac{dx_\beta}{dt} = \frac{F_\alpha(x) - F_{\alpha_0}}{\tau}
\]  

(7)

Eq. (7a) corresponds to a stationary behaviour of \( F_\alpha \) and the time dependence is entirely defined by time dependencies of \( x_\beta \) and \( x_\beta \). The case of (7b) characterize the temporal behaviour of the system deviation from the equilibrium state \( F_{\alpha_0}(x^0, t) \). The relaxation time can be a function of \( F_\alpha \) (a nonlinear case).

The case of \( \varphi = \alpha_0 F + \alpha_1 F^3 \) includes nonlinear equations of Landau-Ginzburg type.

III. STATISTICAL DESCRIPTION OF ECONOMICS PHENOMENON BY KINETIC EQUATIONS

Equations (3)-(4) govern the behaviour of functions \( F_\alpha \) which are set by a \( x_i \) values. Introduce therefore a set of values taken by every of the variables \( x_i \). Let them be distributed in a random way. Then introduce a function \( f_\alpha(x_i,...,t) \) of density of probability to find at the time of \( t \) values of the economic parameters to be equal to \( x_i (i = 1...N) \) (here we use one and the same symbol \( x_i \) for an economic parameter and its value). For the mean values we shall have

\[
\langle F_\alpha \rangle = \int F_\alpha(x_i,...) f_\alpha(x_i) dx_i
\]

and the equations for \( f_\alpha \) chose as

\[
\frac{df_\alpha}{dt} = I_\alpha
\]  

(8)

where \( I_\alpha \) and \( \frac{df_\alpha}{dt} \) are as follows:

\[
\frac{df_\alpha}{dt} = \frac{\partial f_\alpha}{\partial t} + \frac{\partial f_\alpha}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial f_\alpha}{\partial x_\alpha} \frac{dx_\alpha}{dt}
\]  

(9)

\[
I_\alpha = \varphi_\alpha(f,t) + \frac{\partial}{\partial x_i} [D_{ij}(f) \frac{\partial f_\alpha}{\partial x_j} - A_i(x,t) f_\alpha] + \\
+ \frac{\partial}{\partial x_i} [\tilde{D}_{ij}(f) \frac{\partial f_\alpha}{\partial x_j} + (\tilde{a}_i - \tilde{b}_i(x,t) f_\alpha)]
\]  

(10)

These equations corresponds to the statistical description of economic systems in terms of kinetic equations (well known in the theory of open systems (3)) for the economic parameters distribution function and allow to calculate the mean values for economic functions \( F_\alpha \). Adduce now one more method of statistical description of economic parameters \( x_i \) and \( F_\alpha \) based on using of equation like that of Brownian motion in thermostat. The role of the "thermostat" is played in economy by external influences.
derivatives (12)-(13) coincide with the Riemann-Liouville fractional derivatives. Necessary change ordinary derivatives by generalized Riemann-Liouville derivatives. The equations given above are quite general and with known description of economical phenomena. For inclusion of multifractal characteristics in equations (1)-(11) we must use generalized Riemann-Liouville derivatives defined in [3].

\[ D_{\gamma}^{d,t} f(t) = \left( \frac{d}{dt} \right)^n \int_a^t \frac{f(t') dt'}{\Gamma(n-d(t'))(t-t')^{d(t')-n+1}} \quad (12) \]

\[ D_{\gamma}^{d,t} f(t) = \left( \frac{d}{dt} \right)^n \int_a^b \frac{(-1)^n f(t') dt'}{\Gamma(n-d(t'))(t-t')^{d(t')-n+1}} \quad (13) \]

where \( \Gamma(x) \) is Euler’s gamma function, and \( a \) and \( b \) are some constants from \([0, \infty)\). In these definitions, as usually, \( n = \{d\} + 1 \), where \( \{d\} \) is the integer part of \( d \) if \( d \geq 0 \) (i.e. \( n-1 \leq d < n \)) and \( n = 0 \) for \( d < 0 \). If \( d = \text{const} \), the generalized fractional derivatives (GFD) (12)-(13) coincide with the Riemann-Liouville fractional derivatives (\( d \geq 0 \)) or fractional integrals (\( d < 0 \)). In the equations (12)-(13) for taking into account multifractal characteristics of economical phenomenon it is necessary change ordinary derivatives by generalized Riemann-Liouville derivatives.

IV. ECONOMICAL PHENOMENON IN THE MULTIFRACTAL ECONOMICAL SPACES

Many facts show on the possibility of useful describing of economical phenomenon and phenomenon of social life by using idea of fractal dimensions. Thus we point out the way of generalize of equations of above paragraphs for multifractal economical spaces. For including multifractal characteristics in equations (1)-(11) we must use generalized Riemann-Liouville derivatives defined in [3].

\[ \frac{d^2 F}{dt^2} + \gamma(F) \frac{dF}{dt} + \beta(F) = Y(t) \quad (11) \]

where \( \gamma(t) \) and \( \beta(t) \) are nonlinear with respect to \( F \) coefficients, \( Y(t) \) is a random process corresponding, e.g., to a random influence of prices, demand etc. The processes described by (11) can also be described in terms of the distribution function \( f(F,x,t) \) and equations like (8)-(10).

V. CONCLUSION

The equations given above are quite general and witness only a possibility of describing economic processes in terms of the mathematical formalism used in statistical physics of open systems in the economical (topological or fractal) spaces. Taking into account that in market economy one can observe catastrophes (e.g., financial crises, bankrupts, fast changes in the current exchange rate) in macro and microeconomic and stable processes like self-regulations, and the processes of development (stagnation, demolishing) are determined by statistical reasons, using of the methods of statistical physics given in the present paper may turn to be very fruitful. A general algorithm for economic processes description on the basis of statistical physics equations for open system (mainly, equations of reaction-diffusion type) is proposed. It contains wide opportunities to use nonlinear equations and governing parameters for economic problems, so far not developed in a systematic way. The methodologies described can be easily adjusted to study mathematical regularities in biology, medicine, geology etc. We ask apology for absence in this article of grate amount of cites at economical papers concerning the mathematical problems of economics because we can not find direct connection our description with known description of economical problems. The authors thank Ph.D. V.L.Kobeleva, the member of our research group, for useful and fruitful discussion and grate help.

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