Modified hybrid combination synchronization of chaotic fractional order systems

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Abstract
The paper investigates a new hybrid synchronization called modified hybrid synchronization (MHS) via the active control technique. Using the active control technique, stable controllers which enable the realization of the coexistence of complete synchronization and anti-synchronization in four identical fractional order chaotic systems were derived. Numerical simulations were presented to confirm the effectiveness of the analytical technique.

Keywords Dynamical system · Modified hybrid synchronization · Fractional chaotic system · Active control

1 Introduction

A chaotic system is one whose motion is sensitive to initial conditions (Strogatz 2000). Since different initial conditions lead to different trajectories for the same dynamical system, it is expected that the trajectories cannot coincide. The possibility of two chaotic systems with different trajectories to follow the same trajectory by the introduction of a control function, as proposed by Pecora and Carroll (1990), has been an interesting research area for scientists in nonlinear dynamics. This is partly due to the applicability in different fields such as communication technology, security, neuroscience, atmospheric physics and electronics. Several chaotic systems such as Lorenz (1963), Rössler (1976), and Chen and Ueta (1999) have been defined and investigated in the literature.

There are several methods for the synchronization of chaotic systems. These methods include active control, Open Plus Closed Loop (OPCL), backstepping, feedback control, adaptive control, sliding mode and others. A comparison of performance of a modified active control method and backstepping control on synchronization of integer order system has been investigated (Ojo et al. 2013). The active control method was found “to be simpler with more stable synchronization time and hence more suitable for practical implementation”. The active control method was also found to have the best stability and convergence when compared with the direct method and OPCL method for fractional order systems (Ogunjo et al. 2017).

Generally, complete synchronization between a drive system $y_i$ and response system $x_i$ is said to occur if $\lim_{t \to +\infty} ||y_i - x_i|| = 0$ and anti-synchronization if $\lim_{t \to +\infty} ||y_i + x_i|| = 0$. If the error term is such that $\lim_{t \to +\infty} ||y_i - \alpha x_i|| = 0$, where $\alpha$ is a positive integer, we have projective synchronization. According to Zhou and Zhu (2017), $\delta$ synchronization is defined by the error given as $\lim_{t \to +\infty} ||y_i \pm x_i|| \leq \delta$, where $\delta$ has small value. Other forms of synchronization include phase synchronization, anticipated synchronization, lag synchronization, etc. The possibility of one or more of these synchronization scheme in a single synchronization has not been explored.

Different synchronization methods and techniques have been used to study synchronization between two similar integer order systems (Ojo and Ogunjo 2012), two dissimilar integer order systems of same dimension (Motallebzadeh et al. 2012; Femat and Solís-Peralles 2002), two similar or dissimilar systems with different dimensions (Ojo et al. 2013, 2014; Ogunjo 2013), three or more integer order system (compound, combination-combination synchronization Ojo et al. 2014b, a; Mahmoud et al. 2016, discrete systems (Liu 2008; Kloeden 2004; Ma et al. 2007), fractional order system of similar dimension (Lu 2005), fractional order synchronization of different dimension (Bhalekar 2014; Khan and Bhat 2017), circuit implementation of synchronization (Adelakun et al. 2017), impulsive anti-synchronization of fractional order (Meng et al. 2018), discrete fractional order
systems (Liu 2016), multiswitching (Ogunjo et al. 2018), and synchronization between integer order and fractional order systems (Chen et al. 2012).

The study of chaotic systems has evolved over time from integer order dynamical systems to cover partial differential equations, time delayed differential equations, fractional order differential equations and even time series data. The prevalence of integer order system was the lack of solution methods for fractional differential equations (Chen et al. 2009) and its inherent complexity (Gutiérrez et al. 2010). The Grünwald–Letnikov definition of fractional order systems, the fractional order derivative of order \( \alpha \) can be written as:

\[
D_t^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(t - jh) \quad (1)
\]

where \( 0 < \alpha < 1 \), \( t \) is the integration time, \( h \) is the time step, the binomial coefficients can be written in terms of the Gamma function as:

\[
\binom{\alpha}{j} = \frac{\Gamma(\alpha + 1)}{\Gamma(j + 1)\Gamma(\alpha - j + 1)}
\]

The Riemann–Liouville definition of fractional derivative is given as:

\[
D_t^{-\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t - \tau)^{\alpha-1}} d\tau \quad (2)
\]

where \( n = \lceil \alpha \rceil \). The Caputo fractional derivatives can be written as:

\[
D_t^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f^{(n)}(\tau)}{(t - \tau)^{\alpha-n+1}} d\tau, \quad n - 1 < \alpha < n \quad (3)
\]

where \( \tau \) is the integration variable. Fractional order systems have been found as a useful model in many engineering, physical and biological systems.

Chen chaotic system has been the subject of many synchronization schemes. Two chaotic fractional order Chen systems were synchronized using a unidirectional linear error feedback coupling (Peng 2007). This has been extended to include the synchronization between fractional order Lorenz and fractional order Chen system using function projective synchronization (Zhou and Zhu 2011). Furthermore, synchronization involving fractional Chen systems has been investigated using active sliding mode controller (Tavazoei and Haeri 2008), active control (Bhalekar and Daftardar-Gejji 2010), coupling (Zhu et al. 2009), and new nonlinear control technique (Matouk and Elsadany 2014). In the unidirectional synchronization, two fractional systems can only communicate with themselves with one acting as the master and the other as a slave. The synchronization of three chaotic systems is an improvement over the two system synchronization. The fractional Chen system has been synchronized with fractional Lorenz and \( \dot{L}u \) systems using an adaptive function projective synchronization scheme with two drive and one response configuration (Xi et al. 2015). Active backstepping method was used to implement projective synchronization of Lorenz, Rossler, and Chen system (Feng et al. 2017; Runzi et al. 2011). However, this was an integer-based system implementation. This study is limited to one response. Very little work has been done on synchronization of four fractional chaotic systems. The multi-switching synchronization between four chaotic systems with Chen system as the driver was proposed (Prajapati et al. 2018). Slave–master synchronization scheme involving four systems (including Chen system) was successfully synchronized using function projective synchronization (Yang et al. 2016).

There exist several gaps in previous synchronization attempts of fractional Chen system. The two- and three-system synchronization has been superseded by the combination synchronization. Very little work has been done on the four-system combination synchronization. Many of the previous studies used approaches such as backstepping which have been shown to be of low performance. For improved security, it is intuitive to investigate the prospect of different synchronization schemes within the combination synchronization framework. In this present work, we aim to investigate the possibility of coexistence of different synchronization scheme in the synchronization of four chaotic systems (two drives and two response systems). Specifically, we aim to implement synchronization, anti-synchronization and projective synchronization on different dimensions in a fractional order combination synchronization using the method of active control. We believe, if implemented, it will enhance faster, robust and more secure information transmission. To the best of our knowledge, this has not been reported in the literature. The highlights of this study are to: perform combination synchronization on four fractional order system and implement different synchronization schemes within the combination synchronization.

2 System description

The integer order Chen system was introduced by Chen and Ueta (1999) as:

\[
\begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= (c - a)x + cy - xz \\
\dot{z} &= -bz + xy
\end{align*}
\]

where \( a, b, c \) are the constants of the system. The fractional order chaotic Chen system was introduced by Li and Chen (2004) as:
$D^\alpha x_1 = \sigma (x_2 - x_1)$
$D^\alpha x_2 = (c - a)x_1 - x_1x_3 + cx_2$  \hspace{1cm} (5)
$D^\alpha x_3 = x_1x_2 - bx_3$

The system was found to be chaotic when $(a, b, c) = (35, 3, 28)$ and $0.7 \leq \alpha \leq 0.9$. Chaos and stability of the fractional order Chen system has been established by other authors (Hegazi and Matouk 2011; Čermák and Nechvátal 2019). However, by varying parameter $a$ rather than parameter $c$ as in Li and Chen (2004), the system was found to be chaotic in the region $0.1 \leq \alpha \leq 0.1$ (Lu and Chen 2006). The phase space of the fractional order Chen system is shown in Fig. 1. The phase space was generated for the parameters $[a, b, c] = [35, 3, 28]$. The phase space in the $x_1 - x_2$ plane showed a two-lobed shape (Fig. 1a). The attractors became more obvious with a double scroll in the $x_1 - x_3$ (Fig. 1b) and $x_3 - x_2$ (Fig. 1c) planes. Figure 1c shows a cone shaped attractor in the 3D phase space. Various successful attempts have been made at synchronization of the integer, hyperchaotic, and fractional order Chen system (Hegazi and Matouk 2011; Long et al. 2010; Deng and Li 2005; Chen 2008).

### 3 Design and implementation of synchronization scheme

The co-existence of different synchronization scheme within commensurate fractional order Chen system will be studied. Suitable controllers are designed (Sect. 3.1) and numerical simulations presented in Sect. 3.2 to verify the proposed controllers (Bhalekar 2014).

#### 3.1 Design of controllers

Let the two drive system be defined as:

$D^\beta x_1 = \sigma (x_2 - x_1)$
$D^\beta x_2 = (c - a)x_1 - x_1x_3 + cx_2$  \hspace{1cm} (6)
$D^\beta x_3 = x_1x_2 - bx_3$

and

$D^\alpha y_1 = \sigma (y_2 - y_1)$
$D^\alpha y_2 = (c - a)y_1 - y_1y_3 + cy_2$  \hspace{1cm} (7)
$D^\alpha y_3 = y_1y_2 - by_3$

Define the two response systems as:

$D^\gamma z_1 = \sigma (z_2 - z_1) + u_4$
$D^\gamma z_2 = (c - a)z_1 - z_1z_3 + cz_2 + u_2$  \hspace{1cm} (8)
$D^\gamma z_3 = z_1z_2 - bz_3 + u_3$

and

$D^\gamma w_1 = \sigma (w_2 - w_1) + u_4$
$D^\gamma w_2 = (c - a)w_1 - w_1w_3 + cw_2 + u_5$  \hspace{1cm} (9)
$D^\gamma w_3 = w_1w_2 - bw_3 + u_6$
where the six active control functions $u_1$, $u_2$, $u_3$, $u_4$, $u_5$, $u_6$ introduced in equations 8 and 9 are control functions to be determined.

We define the error states $e_1$, $e_2$, $e_3$ as:

\[
e_1 = (x_1 + y_1) - (z_1 + w_1) \\
e_2 = (x_2 + y_2) + (z_2 + w_2) \\
e_3 = (x_3 + y_3) - \alpha(z_3 + w_3)
\] (10)

Substituting the drive systems (equations 6 and 7) and response systems (equations 8 and 9) in Eq. 10 and assuming a commensurate system, the error system is obtained as:

\[
D^\mu e_1 = -[ae_1 + ae_2 - 2a(x_2 + y_2) + u_1 + u_4] \\
D^\mu e_2 = (a + c)e_1 + ce_2 + 2c(z_1 + w_1) - 2a(x_1 + y_1) - x_1x_3 - y_1y_3 - z_1z_3 - w_1w_3 + u_2 + u_5 \\
D^\mu e_3 = -be_3 - x_1x_2 - y_1y_2 + \alpha z_1z_2 + \alpha w_1 w_2 + u_3 + u_6
\]

Active control inputs $u_i (i = 1, 2, 3, 4, 5, 6)$ are then defined as:

\[
u_1 + u_4 = -[V_1 + 2a(x_2 + y_2)] \\
u_2 + u_5 = V_2 - 2c(z_1 + w_1) + 2a(x_1 + y_1) + x_1x_3 + y_1y_3 + z_1z_3 + w_1w_3 \\
u_3 + u_6 = \frac{1}{\alpha}[-V_3 + x_1x_2 + y_1y_2 - \alpha z_1z_2 - \alpha w_1w_2]
\] (12)

where the functions $V_i$ are to be obtained. Substituting equation 12 in Eq. 11 yields

\[
D^\mu e_1 = -ae_1 - ae_2 + V_1 \\
D^\mu e_2 = (a + c)e_1 + ce_2 + V_2 \\
D^\mu e_3 = -be_3 + V_3
\] (13)

The synchronization error system (Eq. 13) is a linear system with active control inputs $V_i$. We design an appropriate feedback control which stabilizes the system so that $e_i (i = 1, 2, 3) \rightarrow 0$ as $t \rightarrow \infty$, which implies that synchronization is achieved with the proposed feedback control. There are many possible choices for the control inputs $V_i$; for simplicity, we chose

\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix} = C \begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}
\] (14)

where $C$ is a $3 \times 3$ constant matrix. In order to make the closed-loop system stable, matrix $C$ should be selected in such a way that the feedback system has eigenvalues $\lambda_i$ that satisfies the equation:

\[
|\arg(\lambda_i)| > 0.5\pi \alpha, \ i = 1, 2, \ldots.
\] (15)

where $\lambda$ is the eigenvalue, $I$ is an identity matrix, and $A$ is the coefficient of the error state. There are varieties of choices for choosing matrix $C$. Matrix $C$ is chosen as:

\[
C = \begin{pmatrix}
(c - \lambda) & a & 0 \\
-(a + c) - (c + \lambda) & 0 & 0 \\
0 & 0 & (b - \lambda)
\end{pmatrix}
\] (16)

Using Eq. 16 in 14, we obtain our control function as:

\[
u_1 + u_4 = -V_1 + 2a(x_2 + y_2) \\
u_2 + u_5 = V_2 - 2c(z_1 + w_1) + 2a(x_1 + y_1) + x_1x_3 + y_1y_3 + z_1z_3 + w_1w_3 \\
u_3 + u_6 = \frac{1}{\alpha}[-V_3 + x_1x_2 + y_1y_2 - \alpha z_1z_2 - \alpha w_1w_2]
\] (17)

Based on the controllers obtained, two unique cases can be observed.

The control system can be defined as:

\[
u_1 = \frac{1}{2}[-V_1 + 2a(x_2 + y_2)] \\
u_2 = \frac{1}{2}[V_2 - 2c(z_1 + w_1) + 2a(x_1 + y_1) + x_1x_3 + y_1y_3 + z_1z_3 + w_1w_3] \\
u_3 = \frac{1}{2\alpha}[-V_3 + x_1x_2 + y_1y_2 - \alpha z_1z_2 - \alpha w_1w_2] \\
u_4 = u_1 \\
u_5 = u_2 \\
u_6 = u_3
\] (18)

The control system can also be defined as:

\[
u_1 = -V_1 + 2a(x_2 + y_2) \\
u_2 = V_2 - 2c(z_1 + w_1) + 2a(x_1 + y_1) + x_1x_3 + y_1y_3 + z_1z_3 + w_1w_3 \\
u_3 = \frac{1}{\alpha}[-V_3 + x_1x_2 + y_1y_2 - \alpha z_1z_2 - \alpha w_1w_2] \\
u_4 = 0 \\
u_5 = 0 \\
u_6 = 0
\] (19)

### 3.2 Numerical simulation of results

To verify the effectiveness of the synchronization scheme proposed in Sect. 3.1 using the method of active control, we used the initial conditions ($-10, 0.001, 37$), ($37, -5, 0$), ($-5, 0.5, 25$), and ($10, -5, 15$) for $x, y, w, z$, respectively. The order of the system was taken as 0.95 except otherwise
Fig. 2 Numerical simulation of synchronization using the control function defined in Eq. 18, where \( p_1 = (x_1 + y_1); \)
\( p_2 = (w_1 + z_1); \)
\( p_3 = (x_2 + y_2); \)
\( p_4 = (w_2 + z_2); \)
\( p_5 = (x_3 + y_3); p_6 = (w_3 + z_3) \)

A time step of 0.005 was used. The parameters of the system are taken as \((a, b, c) = (35, 3, 28)\). According to Petras (2011), the general numerical solution of the fractional differential equation
\[
aD^q y(t) = f(y(t), t) \tag{20}
\]
can be expressed as:
\[
y(t_k) = f(y(t_k), t_k)h^q - \sum_{j=0}^{k} c_j^{(q)} y(t_{k-j}) \tag{21}
\]
where \( c_j^{(q)} \) is given as:
\[
c_0^{(q)} = 1
\]
\[
c_j^{(q)} = \left(1 - \frac{1 + q}{j} c_{j-1}^{(q)} \right) \tag{22}
\]
The results for the two cases considered are shown in Figs. 2 and 3. In Fig. 2, the synchronization result for case described in Eq. 18 is presented. The trajectories for \((x_1 + y_1)\) and \((w_1 + z_1)\) were found to converge. This could also be confirmed from the error curve \((e_1)\) which exponentially reduces from around 6 to \(\approx 0\) around 0.8 seconds. The trajectories \((x_2 + y_2)\) and \((w_2 + z_2)\) were observed to be out of synchronization (middle figure). This confirms that anti-synchronization was observed in the second components of the four Chen system synchronized. Similarly, the trajectories of \((x_3 + y_3)\) and \((w_3 + z_3)\) were found to converge after the application of the control functions. The convergence of errors \(e_1\) and \(e_2\) further confirmed the efficiency of controllers as defined in Eq. 18. When the control function \(u_4, u_5, u_6 = 0\) (as in Eq. 19) was applied to the coupled system, the trajectories were also found to converge (Fig. 3). Synchronization was observed in both top and bottom figure as the different trajectories converged to the same values. This implies that the application of the control functions was effective in bringing the two system to the same path. In the middle figure, anti-synchronization was obtained. In Fig. 4, the order of the four system was changed from \((0.95, 0.95, 0.95)\) to \((0.95, 0.97, 0.92)\) and controllers implemented. The hybrid approach employed in this study will bring added security to the application of chaos to secure communication. From the results presented, the drives and responses were found to achieve synchronization as indicated by the convergence of the error terms to zero. The effectiveness of the proposed scheme is hereby confirmed.

The controller developed in this study was found to be simpler than those proposed in Ojo et al. (2014). Our results offered a six-term controller whereby three can be set to zero without loss of efficacy. Similarly, the error function of an hybrid projective synchronization (Khan and Chaudhary 2020) was observed to be more complicated than those obtained in this study. The hybrid combination-combination synchronization of non-identical complex chaotic system
was found to perform as well as the results obtained in this study based on the estimated time to synchronization (Khan and Nigar 2019). However, our study explores synchronization in the fractional sense, while (Khan and Nigar 2019) considered integer order systems. Results obtained in this study compare well, based on error dynamics, with hybrid projective difference combination synchronization (Khan and Chaudhary 2021), projective lag synchronization (Rajchakit et al. 2019), phase synchronization (Yadav et al. 2019), and fractional order system with delay (Li et al. 2019). However, our work is an improvement over other works as we considered both synchronization and anti-synchronization within the hybrid combination synchronization.

Fig. 3  Numerical simulation of synchronization using the control function defined in Eq. 19, where \( p_1 = (x_1 + y_1); \)
\( p_2 = (w_1 + z_1); \)
\( p_3 = (x_2 + y_2); \)
\( p_4 = (w_2 + z_2); \)
\( p_5 = (x_3 + y_3); \)
\( p_6 = (w_3 + z_3) \)

Fig. 4  Numerical simulation of synchronization using the control function defined in Eq. 18 but with \( \alpha = 0.95, 0.97, 0.92, \) where
\( p_1 = (x_1 + y_1); \)
\( p_2 = (w_1 + z_1); \)
\( p_3 = (x_2 + y_2); \)
\( p_4 = (w_2 + z_2); \)
\( p_5 = (x_3 + y_3); \)
\( p_6 = (w_3 + z_3) \)
4 Conclusion

In this paper, a new synchronization scheme is proposed and implemented. The modified hybrid synchronization that allows for the coexistence of different synchronization schemes was implemented in a compound synchronization of fractional order Chen system. In particular, the controllers consist of complete synchronization, anti-synchronization, and projective synchronization. We believe that this type of synchronization will offer better security and more robust. There is the need to investigate the performance of this type of synchronization using different synchronization schemes. Furthermore, it will be productive to study the behaviour of this scheme under different types and strength of noise. Practical implementation of this scheme is also proposed.

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Availability of data There were no data associated with this study.

Declarations

Conflict of Interest The authors hereby declare that there is no conflict of interest.

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