Program complex for solving of some classes of aerohydroelasticity problems

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Abstract. On the basis of the developed mathematical models, a program complex was created for studying the dynamics and stability of an elastic pipeline, an elastic structural element in supersonic flow, elastic elements of wing profiles of several types with subsonic flow, elastic elements of vibration devices of several types. The complex generates three-dimensional graphs of element deformations, construction of animation graphs of element deformations, construction of flat graphs of deformations and strain rate at a given point or at a given time. On the basis of the constructed graphs, one can study the dynamics and draw conclusions about the stability of vibrations of elastic structural elements.

1. Introduction
In the design and operation of structures, devices, installations for various purposes interacting with a gas-liquid flow, the important problems are to ensure the reliability of their operation and increase the service life. Similar problems are inherent in many branches of technology. In particular, problems of this kind arise in mechanical engineering, aircraft engineering, instrumentation, etc. The study of the dynamic stability of deformable elements is of great importance in the design of structures interacting with a liquid or gas flow, since the effect of the flow can lead to its loss. The examples of loss of dynamic stability are: airplane wing flutter and panel flutter of plates and shells; stall flutter of turbine blades and propellers arising in the case of a flow with large angles of attack; vibrations of wires, chimneys, stiffening beams of suspension bridges, etc.

At the same time, for the operation of some technical devices, the phenomenon of excitation of oscillations under aerohydrodynamic action, indicated above as negative, is necessary. Examples of such devices related to vibration technology are devices used to intensify technological processes. For example, the devices for the preparation of homogeneous mixtures and emulsions, in particular, installations for supplying cutting fluid to the processing zone.

Thus, when designing the structures and devices that interact with a fluid flow, it is necessary to solve problems related to the study of the stability required for their functioning and operational reliability. And, consequently, it is necessary to study the stability of solutions of partial differential equations and their systems, obtained at mathematical modeling of the corresponding processes. A large number of theoretical and experimental works are devoted to the study of the stability of solutions. Among them are works [1–8] and many others. Among the works of the authors of this article on the study of the stability of solutions of partial differential equations, we note the articles [9-12].
2. Mathematical models
Let us give the mathematical formulations of plane problems of aerohydroelasticity on the dynamics and stability of a deformable (elastic, viscoelastic) element (or deformable elements) of several classes of thin-walled structures.

The plane $O_{xy}$, in which the joint oscillations of the elastic element and the flow of an ideal gas or liquid take place, is positioned so that the element is parallel to the $Ox$ axis and occupies the position $y = y_0$, $x \in [b, c]$. It is assumed that the incoming homogeneous flow with velocity $V$ and density $\rho$ is parallel to the axis $Ox$.

Let us introduce designations: $u(x,t), w(x,t), x \in (b,c)$ are functions of plate deformations in the direction of axes $Ox$ and $Oy$; $\phi(x,y,t)$ is the potential of the velocity of the disturbed flow.

The eight classes of aerohydroelasticity problems are considered:
1) The problems about investigation of the dynamic stability of a completely elastic pipeline – a hollow rod, inside which a liquid or gas flows (figure 1).

**Figure 1.** Fully elastic pipeline.

In this problem, for convenience $b = 0, c = l, y_0 = 0$ are accepted and the following designations introduced: $R_e, R_i, h$ are outer and inner pipeline radii and thickness, i.e. $R_e = R_i + h$. In this case, the aerohydrodynamic load is determined by the formula

$$P(x,t) = -\pi \rho R_e^3 \left( w_{tt} + 2V w_{tt} + V^2 w_{xx} \right). \quad (1)$$

Here and in what follows, the subscripts $x, y, t$ below denote partial derivatives with respect to $x, y, t$, respectively.

2) The problems about investigation of the dynamic stability of vibrations of an elastic element of a structure in a supersonic gas flow ($V > a$, where $a$ is the speed of sound in the flow, $M_0 = V / a > 1$ is the Mach number). In figure, 2 shows examples of supersonic gas flow around structures with an elastic element: a) two-way flow around the splitter with the formation of a shock wave; b) one-sided flow around the protective screen with the formation of a rarefaction wave.

**Figure 2.** Examples of flow around structures with an elastic element by supersonic gas flow.

Here $b = 0, c = l, y_0 = 0$. The aerodynamic load is determined by the expression

$$P(x,t) = -\gamma \rho (w_t + V w_x), \quad (2)$$

valid at sufficiently high supersonic flow velocities. Here $\gamma = \gamma_0 a$, where $a$ is – speed of sound in a homogeneous undisturbed flow, $\gamma_0 = 1$ at one-sided flow, $\gamma_0 = 2$ at double-sided flow.
3) The problems about investigation of the dynamic stability of the deformable aileron of the wing profile, in a subsonic flow of gas or liquid (figure 3). Here \( f^\pm(x) \) are the functions that determine the shape of the non-deformable parts of the profile, \( y_0 = 0 \).

![Figure 3. Cross-section of a wing with aileron.](image)

4) The problems about investigation of the dynamic stability of a deformable element connecting two wings in a subsonic flow of gas or liquid (figure 4). Here \( f_1^\pm(x), f_2^\pm(x) \) are the functions that determine the shape of the non-deformable parts of the profiles, \( y_0 = 0 \).

![Figure 4. Cross-section of a wing with a connecting element.](image)

5) The problems about investigation of the dynamic stability of a deformable aileron of a wing profile, in a subsonic flow of gas or liquid, in the wake of which another wing is located (figure 5). Here \( f_1^\pm(x), f_2^\pm(x) \) are the functions that determine the shape of the non-deformable parts of the profiles, \( y_0 = 0 \).

![Figure 5. Cross-section of a system of two tandem wings.](image)

6) The problems about investigation of the dynamic stability of an elastic element of the wall of a vibrating device, simulated by a rectilinear channel, inside which a stirred liquid or gas flows (figure 6).

![Figure 6. Channel, the wall of which contains a deformable element.](image)
7) The problems about investigation of the dynamic stability of two elastic symmetric synchronously oscillating elements of the walls of a vibrating device modeled by a rectilinear channel, inside which a flow of a stirred liquid or gas flows (figure 7).

![Figure 7. Channel whose walls contain deformable elements.](image)

8) The problems about investigation of the dynamic stability of an elastic element located inside a vibrating device simulated by a rectilinear channel through which a stirred liquid or gas flows (figure 8). Here \( y_0 = H / 2 \).

![Figure 8. Flow around an elastic element in a flow channel.](image)

To find the aerohydrodynamic load for problems 3-8, it is assumed that the elements are flown around by a subsonic flow of an ideal incompressible medium. A linear theory is used to describe the flow. In the model of an incompressible medium, the potential \( \phi \) satisfies to the Laplace equation

\[
\phi_{xx} + \phi_{yy} = 0, \quad (x,y) \in G, \; t \geq 0, \tag{3}
\]

where \( G \) is the flow area. The aerohydrodynamic load \( P(x,t) \) on an element is determined on the basis of a linear theory: for elements flown from both sides \( P(x,t) = \rho(\phi^+_x - \phi^-_x) + \rho V_0 (\phi^+_y - \phi^-_y) \), \( x \in (b,c) \); for elements that flow only from above \( P(x,t) = -P_0 + \rho (\phi^+_x + V \phi^-_x) \), \( x \in (b,c) \); for elements streamlined only from below \( P(x,t) = P_0 - \rho (\phi^+_x + V \phi^-_x) \), \( x \in (b,c) \). Here \( P_0 \) is the pressure in a homogeneous flow. \( \phi^+_x = \lim_{y \to y_0} \phi_x(x,y,t), \quad \phi^-_x = \lim_{y \to y_0} \phi_x(x,y,t). \) The linearized boundary conditions arising from the no-flow condition for elastic elements have the form:

\[
\phi_x(x,y_0,t) = w_x(x,t) + V w_y(x,t), \quad x \in (b,c). \tag{4}
\]

For the problems 3-8, using the method of conformal mappings of the theory of functions of a complex variable for the equation (3) with boundary conditions (4), we find the aerohydrodynamic loads

\[
P(x,t) = -\frac{\rho V}{\pi} \int_b^c \left( w_x(t) + V w_y(t) \right) \frac{\partial K(t,x)}{\partial x} \, d\tau - \frac{\rho}{\pi} \int_b^c \left( w_x(t) + V w_y(t) \right) K(t,x) \, d\tau + f(x). \tag{5}
\]

The kernels \( K(t,x) \) have an integrable singularity at \( \tau = x \).

For the model 3 the function \( f(x) = \frac{V^2}{2\pi} \int_a^b \left( f^+_x(\tau) + f^-_x(\tau) \right) K_x(x,\tau) \, d\tau \), the kernel
\[ K(\tau, x) = 2\ln \left[ \frac{\sqrt{(x-a)(c-\tau)} + \sqrt{(\tau-a)(c-x)}}{\sqrt{(x-a)(c-\tau)} - \sqrt{(\tau-a)(c-x)}} \right]. \]

For the model 4: \[ f(x) = \frac{V^2}{2\pi} \int_a^b \left( f_{1r}'(\tau) + f_{1r}(\tau) \right) \frac{\partial K(\tau, x)}{\partial \tau} d\tau + \frac{V^2}{2\pi} \int_c^d \left( f_{2r}'(\tau) + f_{2r}(\tau) \right) \frac{\partial K(\tau, x)}{\partial \tau} d\tau, \]

\[ K(\tau, x) = 2\ln \left[ \frac{(x-a)(d-\tau) + \sqrt{(\tau-a)(d-x)}}{(x-a)(d-\tau) - \sqrt{(\tau-a)(d-x)}} \right]. \]

For the model 5: \[ f(x) = -\rho\nu^2 \int_a^b \frac{f_{1r}'(\tau) + f_{1r}(\tau)}{\tau-x} \sqrt{h(\tau)} d\tau + \rho\nu^2 \int_c^d \frac{f_{2r}'(\tau) + f_{2r}(\tau)}{\tau-x} \sqrt{h(\tau)} d\tau, \]

\[ K(\tau, x) = \int_a^c \frac{\sqrt{h(\tau)}}{\sqrt{h(x)(\tau-x)}} dx - \frac{1}{M} \int_a^c \frac{\sqrt{h(\tau)}}{\sqrt{h(x)(\tau-x)}} dx, \] where \( h(x) = (x-a)(c-x)(a_2-x)(b_2-x), \)

\[ \tilde{M} = \int_a^c \frac{dx}{\sqrt{h(x)}}. \]

For the model 6: \[ f(x) = 0, \quad K(\tau, x) = \ln \left[ \frac{2\text{cd} \left(2K(k)(x_0-b)y_0^1\right)}{\text{cd} \left(2K(k)(x_0-\tau)y_0^1\right) - \text{cd} \left(2K(k)(x_0-\tau)y_0^1\right)} \right], \] where \( K(k) \) is complete elliptic integral of the first kind, the modulus \( k \) is determined from the relation \( K(1-k^2)y_0 = 2K(k)x_0, \ cd x = -\text{sn} \ (x-K(k)), \ sn x \) is elliptic sine.

For the model 7: \[ f(x) = 0, \quad K(\tau, x) = \ln \left[ \frac{\text{cd} \left(2K(k)(x_0-\tau)y_0^1\right) + \text{cd} \left(2K(k)(x_0-y_0^1)\right)}{\text{cd} \left(2K(k)(x_0-y_0^1)\right) - \text{cd} \left(2K(k)(x_0-y_0^1)\right)} \right], \]

For the model 8: \[ K(\tau, x) = 2\gamma \frac{\text{c}h(y)(\tau-x)}{\sqrt{h(x)}} dx - 2\gamma \left( \int_b^c \frac{\sqrt{h(\tau)}}{\sqrt{h(x)}} d\tau - \frac{\tilde{N}}{\tilde{M}} \int_b^c \frac{d\tau}{\sqrt{h(\tau)}} \right) \int_b^c \frac{dx}{\sqrt{h(x)}}, \]

\[ f(x) = 0, \] where \( \gamma = \pi / H, \ h(x) = \text{shy}(x-b)\text{shy}(c-x), \ \tilde{M} = \int_b^c \frac{dx}{\sqrt{h(x)}}, \ \tilde{N} = \int_b^c \frac{dx}{\sqrt{h(x)}}, \]

To investigate the dynamics of deformable elements, either linear equations are used that describe the transverse vibrations of elastic plates:

\[ D_{w_{xx,xx}}(x,t) + M_{w_{xx}}(x,t) + N(t)w_{xx}(x,t) + \beta_1w_{xx}(x,t) + \beta_2w_{xx}(x,t) + \beta_3w(x,t) = P(x,t), \ x \in (b,c), \]

or linear equations describing transverse vibrations of viscoelastic plates:

\[ M_{w_{xx}}(x,t) + N(t)w_{xx}(x,t) + \beta_2w_{xx}(x,t) + \beta_1w_{xx}(x,t), \]

\[ D \left[ w_{xx,xx}(x,t) - \int_0^t R_1(s,t)w_{xx}(x,s)ds \right] + \beta_0 \left[ w(x,t) - \int_0^t R_2(s,t)w(x,s)ds \right] = P(x,t), \ x \in (b,c), \]

or nonlinear equations describing the longitudinal-transverse vibrations of elastic plates:

\[ \left[ M_{u_{xx}}(x,t) - \beta_2F_{u_{xx}}(x,t) - EF \left[ \left( u_x(x,t) + 0.5w_{xx}^2(x,t) \right) \right] \right]_t = 0, \]

\[ M_{w_{xx}}(x,t) + D_{w_{xx,xx}}(x,t) + N(t)w_{xx}(x,t) + \beta_2w_{xx}(x,t) + \beta_1w_{xx}(x,t) + \beta_0w(x,t) - EF \left[ w_x(x,t) \left( u_x(x,t) + 0.5w_{xx}^2(x,t) \right) \right] = P(x,t), \ x \in (b,c). \]
In (6) – (8) the following designations are introduced: \( E \), \( h \), \( \rho_p \) are elastic modulus, thickness and density of the element; \( N(t) \) is compressive or tensile force of the element; \( D = EI \), \( M \) are flexural stiffness and linear mass of the element; \( I = 0.25\pi \left( R_4^4 - R_0^4 \right) \), \( F = \pi \left( R_t^2 - R_0^2 \right) \), \( M = \rho_p F \) for problem 1; \( I = h^3 / (12(1 - \nu^2)) \), \( F = h / (1 - \nu^2) \), \( M = h \rho_p \) for problems 2-8; \( \nu \) is Poisson's ratio; \( \beta_2, \beta_1 \) are coefficients of internal and external damping; \( \beta_0 \) is coefficient of stiffness of the compression layer; \( R_l(s, t), R_s(s, t) \) are relaxation kernels characterizing the viscoelastic properties of the element material and its compression layer.

For equations (6), (7), the following 6 types of fastening of the ends of elements are used:

1) the rigid clamping both ends
\[
w(b, t) = w_g(b, t) = w(c, t) = w_g(c, t) = 0;
\]

2) the hinged support of the left end and the rigid clamping of the right
\[
w(b, t) = w_{xx}(b, t) = w(c, t) = w_{xx}(c, t) = 0;
\]

3) the rigid clamping of the left end and the hinge support of the right
\[
w(b, t) = w_g(b, t) = w(c, t) = w_{xx}(c, t) = 0;
\]

4) the hinged support of both ends
\[
w(b, t) = w_{xx}(b, t) = w(c, t) = w_{xx}(c, t) = 0;
\]

5) the rigidly clamped of the left end and the free of the right
\[
w(b, t) = w_g(b, t) = w_{xx}(c, t) = w_{xx}(c, t) = 0;
\]

6) the elastic fixed of the left end and the free of the right
\[
w(b, t) = w_{xx}(c, t) = w_{xx}(c, t) = 0, \quad w_{xx}(b, t) = \alpha w_g(b, t),
\]

where number \( \alpha \) is the coefficient of stiffness of the elastic connection of the element to the structure.

For the system of equations (8) for the motionless end \((x = b \text{ or } x = c)\), it is necessary to add the condition \( u(x, t) = 0 \), and for the moving end: \( u_g(x, t) + 0.5w^2_x(x, t) = 0 \). For example, in the case of motionless rigid fixation of the left end and free right end, we obtain the following conditions:
\[
w(b, t) = w_g(b, t) = u(b, t) = w_{xx}(c, t) = w_{xx}(c, t) = u_g(c, t) + 0.5w^2_x(c, t) = 0.
\]

Excluding the cases of movable fastening of both ends of the element, we get 12 types of fasteners.

Let us also set the initial conditions:
\[
w(x, 0) = f_1(x), \quad w_g(x, 0) = f_2(x), \quad u(x, 0) = f_3(x), \quad u_g(x, 0) = f_4(x),
\]

which must be consistent with the boundary conditions.

Taking into account the aerohydrodynamic loads (1), (2), (5), the linear initial-boundary value problems for partial differential equations (6) or (7) for an unknown deformation function \( w(x, t) \) of an element, and the nonlinear initial-boundary value problems for system (8) for two unknown deformation functions \( u(x, t), w(x, t) \) of the element are formulated.

3. Galerkin's method

The solution of equations (6) or (7) is found by the Galerkin’s method in the form
\[
w(x, t) = \sum_{k=1}^{m} a_k(t) g_k(x),
\]

where \( g_k(x) \) are basis functions, selected so that the specified boundary conditions (9) – (14) are fulfilled, and the functions \( a_k(t) \) are determined from the condition of orthogonality of the residual of the equation to the system of the basis functions.

As a basis we take the functions
\[
g_k(x) = A_k \cos \gamma_k(x - b) + B_k \sin \gamma_k(x - b) + C_k \cosh \gamma_k(x - b) + D_k \sinh \gamma_k(x - b), \quad k = 1, 2, 3, \ldots
\]
We choose the coefficients $A_k, B_k, C_k, D_k$ and the parameter $\gamma_k$ so that at each endpoint of the segment $[b,c]$ the conditions corresponding to (9) – (14) are satisfied. Then the function $w(x,t)$ in the form (17) will satisfy to conditions (9) – (14). Note that $\gamma_k$ and $g_k(x)$ are the eigenvalues and eigenfunctions of boundary value problems for the equation $g_k''''(x) = \gamma^4 g_k(x)$. These problems are self-adjoint and fully defined; therefore, the system of functions $\{g_k(x)\}_{k=1}^\infty$ is orthogonal on $[b,c]$. In this case, according to the expansion theorem, any function $U(x)$ that is fourfold continuously differentiable in $(b,c)$ and satisfying the corresponding boundary conditions can be expanded in a series $U(x) = \sum_{k=1}^m a_k g_k^{(1)}(x)$ that converges absolutely and uniformly in $(b,c)$.

Taking into account (17), the conditions for orthogonality of the residual of equation (6) to basis functions $\{g_j(x)\}_{j=1}^m$ of the form (18) allow us to write the system of equations

$$\left[D\gamma^4 a_j(t) + Ma_{jm}(t) + \beta_2 I\gamma^4 a_{jm}(t) + \beta_0 a_{jm}(t) + \beta_0 a_{jm}(t) \right] \delta_j + N(t) \sum_{k=1}^m a_k(t) \int_{b}^{c} g_{km}(x) g_j(x) dx = \int_{b}^{c} P(x,t) g_j(x) dx, \quad j = 1,2,\ldots,m. \tag{19}$$

For equation (7) we obtain

$$\left[D\gamma^4 a_j(t) + Ma_{jm}(t) + \beta_2 I\gamma^4 a_{jm}(t) + \beta_0 a_{jm}(t) + \beta_0 a_{jm}(t) \right] \delta_j + N(t) \sum_{k=1}^m a_k(t) \int_{b}^{c} g_{km}(x) g_j(x) dx - D\gamma^4 \delta_j \int_{0}^{t} R_1(s,t) a_k(s) ds - \beta_0 \delta_j \int_{0}^{t} R_2(s,t) a_k(s) ds = \int_{b}^{c} P(x,t) g_j(x) dx, \quad j = 1,2,\ldots,m. \tag{20}$$

The conditions for orthogonality of the residuals of the first two initial conditions (16) to the basis functions make it possible to find the initial values:

$$a_j(0) = \frac{1}{\delta_j} \int_{b}^{c} f_1(x) g_j(x) dx, \quad a_{jm}(0) = \frac{1}{\delta_j} \int_{b}^{c} f_2(x) g_j(x) dx. \tag{21}$$

Thus, we have obtained the Cauchy problems for systems of ordinary differential equations (19) or integro-differential equations (20) with initial conditions (21).

In the case of a motionless fixation of the ends of the element, according to the Galerkin’s method, the solution of the system of equations (8) is sought in the form

$$u(x,t) = \sum_{k=1}^m a_k(t) g_k^{(1)}(x), \quad w(x,t) = \sum_{k=1}^m b_k(t) g_k^{(2)}(x). \tag{22}$$

If the right end of the element is movable or free, then the solution of the system of equations (8) will be sought in the form

$$u(x,t) = -\frac{1}{2} \int_{b}^{x} \left( \sum_{k=1}^m b_k(t) g_k^{(2)}(s) \right)^2 ds + \sum_{k=1}^m a_k(t) g_k^{(1)}(x), \quad w(x,t) = \sum_{k=1}^m b_k(t) g_k^{(2)}(x). \tag{23}$$

If the left end of the element is movable, then the solution of the system of equations (8) can be represented in the form

$$u(x,t) = \frac{1}{2} \int_{x}^{c} \left( \sum_{k=1}^m b_k(t) g_k^{(2)}(s) \right)^2 ds + \sum_{k=1}^m a_k(t) g_k^{(1)}(x), \quad w(x,t) = \sum_{k=1}^m b_k(t) g_k^{(2)}(x). \tag{24}$$

In (22) – (24), we select the basis functions $g_k^{(1)}(x), g_k^{(2)}(x)$ so that the specified boundary conditions are satisfied, and the functions $a_k(t), b_k(t)$ are determined from the condition that the
residual of the first equation of the system is orthogonally to all basis functions \( g_k^{(1)}(x) \), and the residual of the second equation to \( g_k^{(2)}(x), k = 1 + m \). The functions \( g_k^{(2)}(x) \) take the form (18) and the functions \( g_k^{(1)}(x) \) take in the form

\[
g_k^{(1)}(x) = A_k \cos \gamma_k x + B_k \sin \gamma_k x, \quad k = 1, 2, 3, \ldots
\]  

(25)

We choose the coefficients \( A_k, B_k \) and the parameter \( \gamma_k \) so that at each endpoint of the segment \([b, c]\) one of the following conditions is fulfilled:

1) \( g_k^{(1)}(x) = 0, \)  2) \( g_k^{(1)}(x) = 0, \)  \( k = 1, 2, 3, \ldots \)  

(26)

If both ends are fixed motionless, then at the ends we take the first condition (26). If one end is fixed motionless, and the other is fixed movably or free, then at one end we use the first condition (26), and at the other the second condition (26). Then functions \( w(x, t), u(x, t) \) in the form (22) – (24) will satisfy the boundary conditions.

Note that \( \gamma_k \) and \( g_k^{(1)}(x) \) are the eigenvalues and eigenfunctions of the boundary value problem \( g^*(x) = -\gamma^2 g(x) \) with boundary conditions (26). These problems are self-adjoint and completely definite, therefore, the system of functions \( \{g_k^{(1)}(x)\}_{k=1}^m \) is orthogonal on \([b, c]\).

Substituting (22) into the system of equations (8), from the condition of the orthogonality of the residuals of the first equation (8) to the basis functions \( \{g_j^{(1)}(x)\}_{j=1}^m \), the second to \( \{g_j^{(2)}(x)\}_{j=1}^m \) the system of ordinary differential equations for \( a_j(t), b_j(t) : \)

\[
\begin{align*}
M \delta^{(1)}_j a_j(t) + EF \gamma_j^{(1)} \delta^{(1)}_j a_j(t) + \beta_2 F \gamma_j^{(2)} \delta^{(1)}_j a_j(t) - EF \sum_{i=1}^{m} A_{ij} b_i(t) b_j(t) &= 0, \\
-2 \sum_{i=1}^{m} B_{ij} b_i(t) a_j(t) - 3EF \sum_{i=1}^{m} \sum_{k=1}^{m} C_{ikj} b_i(t) b_k(t) b_j(t) + N(t) \sum_{k=1}^{m} b_k(t) \int_{b}^{c} g_k^{(2)}(x) g_j^{(2)}(x) dx &= 0, \\
[D \gamma_j^{(2)} b_j(t) + M b_j(t) + F \gamma_j^{(2)} b_j(t)] + \beta_1 b_j(t) + \beta_2 b_j(t) &= 0,
\end{align*}
\]

(27)

where \( \delta^{(1)}_j = \int_{b}^{c} g_j^{(1)}(x) dx, \) \( \delta^{(2)}_j = \int_{b}^{c} g_j^{(2)}(x) dx, \) \( A_{ij} = \int_{b}^{c} g_i^{(2)}(x) g_j^{(1)}(x) dx, \) \( B_{ij} = \int_{b}^{c} \left[ g_i^{(2)}(x) g_j^{(1)}(x) + g_i^{(1)}(x) g_j^{(2)}(x) \right] \int_{b}^{c} g_j^{(2)}(x) dx, \) \( C_{ikj} = \int_{b}^{c} g_i^{(2)}(x) g_j^{(2)}(x) g_k^{(1)}(x) dx. \)

Substituting (23), (24) into the system of equations (8), we obtain similar systems.

The conditions for orthogonality of the residuals of the initial conditions (16) to the basis functions allow us to find the initial values:

\[
\begin{align*}
a_j(0) &= \frac{1}{\delta^{(1)}_j} \int_{b}^{c} f_j(x) g_j^{(1)}(x) dx, \quad a_{jt}(0) = \frac{1}{\delta^{(1)}_j} \int_{b}^{c} f_j(x) g_j^{(1)}(x) dx, \\
b_j(0) &= \frac{1}{\delta^{(2)}_j} \int_{b}^{c} f_j(x) g_j^{(2)}(x) dx, \quad b_{jt}(0) = \frac{1}{\delta^{(2)}_j} \int_{b}^{c} f_j(x) g_j^{(2)}(x) dx.
\end{align*}
\]

(28)

Thus, we have obtained the Cauchy problem for the system of ordinary differential equations (27) with initial conditions (28).
4. Program complex

To solve the obtained Cauchy problems, a program complex “Aerohydroelasticity” was developed. Depending on the choice of the structure model, the model of the deformable solid, the type of fixing of the elastic element, the program complex allows you to solve 114 problems of aerohydroelasticity.

To start the calculations, enter: type of construction 1–8 (figure 1–8); deformable solid model (6), (7) or (8); type of fastening of element ends (9)–(14); for the model (8) the method of fixing the ends of the element: “motionless-motionless”, “motionless-movable”, “movable-motionless”; initial conditions (16); depending on the type of construction the system parameters: \( m \) (order of approximation), \( T \) (estimated time), \( a, b, c, d, R_0, a_2, b_2, x_0, y_0, \alpha, \gamma, V, \rho, E, h, \rho_p, N(t), \nu, \beta_2, \beta_1, \beta_0, R_1(s,t), R_2(s,t) \).

Then the program checks:
- compliance of the type of structure and the type of fastening of the elastic element;
- correspondence of initial and boundary conditions

and produces:
- calculation of coefficients \( D, M, I, F \);
- calculation of eigenvalues and eigenfunctions (18), (25);
- calculation of integral members of systems (19), (20), (27);
- solving systems of differential equations (19), (20), (27);
- construction of 3D graphs of element deformations;
- construction of animation graphs of element deformations;
- plotting flat graphs of deformations and strain rates at a given point or at a given time.

5. Numerical experiment

Let’s consider an example of calculations using the program complex. Introduce: construction type 4 (wing with connecting element); deformable solid model (8); mechanical system parameters: \( V = 20 \), \( \rho = 1 \), \( E = 20.6 \cdot 10^6 \), \( \rho_p = 7850 \), \( a = 0 \), \( b = 1 \), \( c = 1.3 \), \( d = 2 \), \( h = 0.01 \), \( \nu = 0.25 \), \( \beta_0 = 4 \), \( \beta_1 = 0.4 \), \( \beta_2 = 0.4 \), \( N_0(t) = 1000 + 500 \sin(t/10) \); fastening type (11); profile shapes \( f_1^1(x) = 0.05x(b-x)^3 \), \( f_1^-(x) = -0.05x(b-x)^2 \), \( f_2^+(x) = 0.0125(x-c)(d-x)^3 \), \( f_2^-(x) = -0.0125(x-c)(d-x)^2 \); initial conditions \( f_3(x) = 0.001g_1^{(2)}(x), f_4(x) = -0.0005g_1^{(2)}(x), f_5(x) = 0.0001g_1^{(1)}(x), f_6(x) = 0.00005g_1^{(1)}(x) \); approximation order \( m = 4 \) and estimated time \( T = 5 \). The figure 9 shows an example of calculations of the transverse deformation \( w(x,t) \) and the rate of transverse vibrations \( w_v(x,t) \) at the point \( x_0 = 1.15 \) at \( t \in [0,0.05] \). The figure 10 shows an example of calculations of the transverse deformation \( w(x,t) \) and longitudinal \( u(x,t) \) deformation of an element at the time \( t_0 = 1 \) at \( x \in [1,1.3] \).

![Figure 9. Transverse deformation and velocity of transverse oscillations of element at point \( x_0 \).](image-url)
Figure 10. Transverse and longitudinal deformation of element at time $t_0$.

According to figure 9, we have the stability of vibrations of the elastic element.

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