PARTIAL HOLOMORPHIC SEMICONJUGACIES BETWEEN RATIONAL FUNCTIONS

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Abstract. We establish a general result on the existence of partially defined semiconjugacies between rational functions acting on the Riemann sphere. The semiconjugacies are defined on the complements to at most one-dimensional sets. They are holomorphic in a certain sense.

1. Introduction

Let \( A \subseteq \mathbb{C}P^1 \) be a G\( \delta \)-set, i.e. the intersection of countably many open sets. A map \( \Phi : A \to \mathbb{C}P^1 \) is said to be holomorphic if there is a sequence of holomorphic maps \( \Phi_n : A_n \to \mathbb{C}P^1 \) such that \( A_n \supseteq A \) are open subsets of \( \mathbb{C}P^1 \), and \( \Phi_n \) converge to \( \Phi \) uniformly on \( A \). Recall that a real semi-algebraic subset in a real algebraic variety is a set given by any boolean combination of real algebraic equations and inequalities.

The main result of this paper is the following

Main Theorem. Suppose that \( R : \mathbb{C}P^1 \to \mathbb{C}P^1 \) is a hyperbolic rational function with a finite postcritical set \( P_R \), and \( Q : \mathbb{C}P^1 \to \mathbb{C}P^1 \) is a rational function such that the diagram

\[
\begin{array}{ccc}
\mathbb{C}P^1 & \xrightarrow{R} & \mathbb{C}P^1 \\
\downarrow \tilde{\eta} & & \downarrow \eta \\
\mathbb{C}P^1 & \xrightarrow{Q} & \mathbb{C}P^1 \\
\end{array}
\]

is commutative, where \( \eta \) and \( \tilde{\eta} \) are homeomorphisms that coincide on \( R(P_R) \) and are isotopic relative to \( R(P_R) \). If \( P_R \) has at least three points, then there exists a countable union \( Z \) of real semi-algebraic sets of codimension > 0 backward invariant under \( Q \) and a holomorphic map \( \Phi : \mathbb{C}P^1 - Z \to \mathbb{C}P^1 \) such that \( R \circ \Phi = \Phi \circ Q \) on \( \mathbb{C}P^1 - Z \).

Recall that the postcritical set of a rational function \( R : \mathbb{C}P^1 \to \mathbb{C}P^1 \) is defined as the closure of the set \( \{ R^n(c) \} \), where \( c \) runs through all critical points of \( R \), and \( n \) runs through all positive integers. A rational

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function $R$ is called critically finite if its postcritical set is finite. The relation between $R$ and $Q$ resembles Thurston equivalence (in which we require that $\eta$ and $\tilde{\eta}$ coincide on $P_R$, map $P_R$ onto the postcritical set of $Q$, and be isotopic relative to $P_R$) but is in fact much weaker. If $Q$ has at least one super-attracting cycle of period $> 1$, then, as a rule, there are infinitely many different functions $R$ that satisfy the assumptions of the theorem. Note that the assumptions of the Main Theorem imply the existence of at least one super-attracting cycle of $Q$, namely, $\eta(C)$, where $C$ is a super-attracting cycle of $R$ in $R(P_R)$.

The map $\Phi$ from the Main Theorem semiconjugates the restriction of $Q$ to $\mathbb{CP}^1 - Z$ with a certain restriction of $R$. Note that the set $\mathbb{CP}^1 - Z$ is forward invariant under $Q$ since $Z$ is backward invariant. The set $Z$ can be constructed explicitly, and in many different ways. The Main Theorem is only useful in combination with the knowledge of what $Z$ is. In fact, $Z$ is very flexible and can be tailored to specific needs. Semi-algebraicity is only one possible application of this flexibility. We could have replaced semi-algebraicity with many other nice properties. The map $\Phi$ is holomorphic, in particular, continuous. Being holomorphic gives more information than continuity although many nice properties of holomorphic maps fail in our setting, e.g. the uniqueness theorem. Note, however, that the restriction of $\Phi$ to the interior of $\mathbb{CP}^1 - Z$ is holomorphic in the usual sense.

The main theorem is closely related to the regluing surgery of [T], although we will not use it explicitly. Many of the ideas used in this paper are inspired by works of M. Rees (see e.g. [R]). In Section 2 we briefly describe some particular applications of the Main Theorem, from which these relations may become clear. To prove the Main Theorem, we will use a version of Thurston’s algorithm [DH93].

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2. Supportive real semi-algebraic sets

Recall that the support of a homeomorphism $\sigma: \mathbb{CP}^1 \to \mathbb{CP}^1$ is defined as the closure of the set of all points $x \in \mathbb{CP}^1$ such that $\sigma(x) \neq x$. Let $P$ be a finite subset of $\mathbb{CP}^1$ and $\sigma: \mathbb{CP}^1 \to \mathbb{CP}^1$ a homeomorphism. We say that a closed subset $Z_0 \subset \mathbb{CP}^1$ is supportive for $(\sigma, P)$ if, for every open neighborhood $U$ of $Z_0$, there exists a homeomorphism $\tilde{\sigma}$ with the following properties:

- the homeomorphisms $\sigma$ and $\tilde{\sigma}$ coincide on $P$;
- they are isotopic relative to $P$;
Proposition 2.1. For every orientation-preserving homeomorphism \( \sigma : \mathbb{C}P^1 \to \mathbb{C}P^1 \) and every finite set \( P \subset \mathbb{C}P^1 \), there exists a closed real semi-algebraic set of positive codimension supportive for \((\sigma, P)\).

Proof. Consider a continuous one-parameter family \( \sigma_t \) of homeomorphisms connecting \( \sigma \) with the identity: \( \sigma_0 = \sigma, \sigma_1 = id \). For every point \( x \in P \) define a continuous path \( \beta_x : [0, 1] \to \mathbb{C}P^1 \) by the formula \( \beta_x(t) = \sigma_x(x) \). We can assume that the curves \( \beta_x[0, 1] \) are real semi-algebraic (e.g. by the Weierstrass approximation theorem) and the paths \( \beta_x \) are smooth.

Take any open neighborhood \( U \) of the closed real semi-algebraic set \( Z_0 = \bigcup_{x \in P} \beta_x[0, 1] \).

Define a vector field \( v_t \) on \( \mathbb{C}P^1 \) depending smoothly on \( t \) and having the following properties:

- at the point \( \beta_x(t) \), the vector field \( v_t \) is equal to \( d\beta_x(t)/dt \);
- \( v_t = 0 \) outside of a small neighborhood of \( \beta_x(t) \) contained in \( U \).

Let now \( g^t \) be the time \([0,t]\) flow of the non-autonomous differential equation \( \dot{z}(t) = v_t \). Clearly, \( g^t(\sigma(x)) = \beta_x(t) \) for every \( x \in P \), and the support of \( g^t \) is contained in \( U \).

Now define the following isotopy: \( \tilde{\sigma}_t = (g^t)^{-1} \circ \sigma_t, t \in [0,1] \). We have \( \tilde{\sigma}_0(x) = \sigma(x) \). On the other hand, the support of \( \tilde{\sigma}_1 = (g_1)^{-1} \) is contained in \( U \). Therefore, \( Z_0 \) is supportive for \((\sigma, P)\). \( \square \)

Remark 2.2. It also follows from the proof of Proposition 2.2 that there exists a continuous one-parameter family of homeomorphisms connecting \( \tilde{\sigma}_1 \) with the identity such that the supports of all these homeomorphisms are contained in \( U \). To obtain such a family, we can apply the procedure, described in the proof of Proposition 2.2, to homeomorphisms \( \sigma_t \) rather than \( \sigma \).

Proposition 2.3. Let rational functions \( Q \) and \( R \) be as in the statement of the Main Theorem. Set \( \sigma = \tilde{\eta} \circ \eta^{-1} \), so that \( \sigma \circ Q = \tilde{\eta} \circ R \circ \eta^{-1} \). There exists a closed real semi-algebraic set \( Z_0 \) of positive codimension that is supportive for \((\sigma, \eta(\tilde{R}(P_R))) \) and that is disjoint from \( \eta(\tilde{R}(P_R))) \).

Proof. Note that \( \sigma = id \) on \( \eta(\tilde{R}(P_R))) \), and \( \sigma \) is homotopic to the identity relative to the set \( \eta(\tilde{R}(P_R))) \), by the assumptions of the Main Theorem. In the proof of Proposition 2.1 we can therefore assume that \( \sigma_t(x) = x \) for all \( x \in \eta(\tilde{R}(P_R))) \) and all \( t \in [0,1] \). For these \( x \), we do not consider the curves \( \beta_x \). The rest of the proof works as before. \( \square \)
The main theorem will follow from Theorem 2.4. Suppose that $Q$ and $R$ are rational functions as in the statement of the Main Theorem, and $Z_0$ is the set from Proposition 2.3. Define the set

$$Z = \bigcup_{n=1}^{\infty} Q^{-n}(Z_0).$$

There exists a holomorphic map $\Phi : \mathbb{C}P^1 - Z \to \mathbb{C}P^1$ such that $R \circ \Phi = \Phi \circ Q$ on $\mathbb{C}P^1 - Z$.

Clearly, $Z$ is a countable union of real semi-algebraic sets of positive codimension, namely, of the iterated preimages of $Z_0$ under $Q$.

We now describe some particular applications of Theorem 2.4. Let $\mathcal{R}_2$ be the set of Möbius conjugacy classes of quadratic rational functions with marked critical points. Following M. Rees [R] and J. Milnor [M], consider the slice $\text{Per}_k(0) \subset \mathcal{R}_2$ defined by the condition that the second critical point is periodic of period $k$. The slices $\text{Per}_k(0)$ form a natural sequence of parameter curves starting with $\text{Per}_1(0)$, the plane of quadratic polynomials. We say that a critically finite rational function $R \in \text{Per}_k(0)$ is of type $C$ if the first critical point is eventually mapped to the second (periodic) critical point but does not belong to the cycle of the second critical point. Let $R \in \text{Per}_k(0)$ be any type $C$ critically finite rational function, and $Q \in \text{Per}_k(0)$ be almost any function. There are only finitely many exceptions, and all exceptional maps are critically finite. Then, as M. Rees has shown in [R], there is a homeomorphism $\sigma_{\beta} : \mathbb{C}P^1 \to \mathbb{C}P^1$, whose support is contained in an arbitrarily small neighborhood of a simple path $\beta : [0, 1] \to \mathbb{C}P^1$ such that $\sigma_{\beta}(\beta(0)) = \beta(1)$, and $\sigma_{\beta} \circ Q$ is a critically finite branched covering Thurston equivalent to $R$. It is a simple exercise to check that $Q$ and $R$ satisfy the assumptions of the Main Theorem, and that we can take $Z_0 = \beta[0, 1]$. In this way, we obtain a partial semiconjugacy between almost any function from $\text{Per}_k(0)$ and any type $C$ critically finite function. It will be defined on the complement to all pullbacks of the simple curve $Z_0$ under $Q$. E.g. for $Q$ we can take a quadratic polynomial $z \mapsto z^2 + c$, whose critical point 0 is periodic of period $k$. Then the Main Theorem implies, in particular, the topological models for captures of $Q$ introduced in [R].

3. Thurston’s algorithm

In the proof of Theorem 2.4 we will use Thurston’s algorithm (see [DH93]). We now briefly recall how it works (in a slightly more general setting than usual). Let $X$ be a topological space and $f : X \to X$
be a continuous map. Suppose that there is a topological semi-conjugacy between $f$ and a rational function acting on the Riemann sphere. Thurston’s algorithm serves to find this semi-conjugacy. It starts with a surjective continuous map $\phi_0 : X \to \mathbb{CP}^1$. Assume that there is a rational function $R_0$ and a continuous map $\phi_1 : X \to \mathbb{CP}^1$ that make the following diagram commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow{\phi_1} & & \downarrow{\phi_0} \\
\mathbb{CP}^1 & \xrightarrow{R_0} & \mathbb{CP}^1
\end{array}
$$

This is always the case if the map $\phi_0$ is a homeomorphism (in particular, $X$ is a topological sphere) and $f$ a branched covering. Indeed, we can arrange that $f$ and $\phi_0$ be smooth by small deformations preserving the critical values of $\phi_0 \circ f$. Then we consider the pullback $\kappa$ of the complex structure on $\mathbb{CP}^1$ under the map $\phi_0 \circ f$. We can integrate $\kappa$, i.e. there is a homeomorphism $\phi_1 : X \to \mathbb{CP}^1$ taking the complex structure $\kappa$ on $X$ to the standard complex structure on $\mathbb{CP}^1$. Clearly, $R_0 = \phi_0 \circ f \circ \phi_1^{-1}$ preserves the standard complex structure, hence it is a rational function. If $f$ or $\phi_0$ were not smooth, then $R_0$ constructed for smooth deformations of $f$ and $\phi_0$ will also work for $f$ and $\phi_0$, i.e. $\phi_1$ can be defined as a branch of $R_0^{-1}(\phi_0 \circ f)$. Note that $R_0$ is only defined up to precomposition with an automorphism of $\mathbb{CP}^1$, and $\phi_1$ is only defined up to post-composition with an automorphism of $\mathbb{CP}^1$.

The transition from $\phi_0$ to $\phi_1$ is the main step of Thurston’s algorithm. Doing this step repeatedly, we obtain a sequence of maps $\phi_n$. We want that $\phi_n$ converge to a semiconjugacy between $f$ and some rational function.

We will now use notation from Proposition 2.3 and Theorem 2.4. Let us consider Thurston’s algorithm for $\tilde{\sigma} \circ Q$, where $\tilde{\sigma}$ is a homeomorphism isotopic to $\sigma$ relative to the set $\eta(P_R)$. Note that the branched covering $\tilde{\sigma} \circ Q$ is Thurston equivalent to $R$, and $P = \tilde{\eta}(P_R)$ is the postcritical set of this branched covering. Indeed, all critical values of $Q$ are contained in $\eta(P_R)$; the images of these critical values under $\tilde{\sigma}$ are contained in $\tilde{\eta}(P_R) = \tilde{\sigma} \circ \eta(P_R)$; the further images under $\tilde{\sigma} \circ Q$ are contained in $P' = \eta(R(P_R))$ because the action of $\tilde{\sigma} \circ Q$ on $\tilde{\eta}(P_R)$ coincides with the action of $Q$, and $Q(\tilde{\eta}(P_R)) = P'$. By Proposition 2.3 we can assume that the support of $\tilde{\sigma}$ is contained in an arbitrarily small neighborhood $U$ of $Z_0$. We choose this neighborhood so that it is disjoint from the set $P'$.  

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Set $\hat{f} = \sigma \circ Q$. Thurston’s algorithm for $\hat{f}$ yields an infinite commutative diagram

$$
\cdots \xrightarrow{\phi_3} \mathbb{C}P^1 \xrightarrow{f} \mathbb{C}P^1 \xrightarrow{f} \mathbb{C}P^1 \xrightarrow{f} \mathbb{C}P^1 \xrightarrow{\phi_0}
$$

We can set $\phi_0 = id$. The classes of $\hat{\phi}_n$ in the Teichmüller space of $(\mathbb{C}P^1, P)$ are well defined. They depend only on the Thurston equivalence class of $\hat{f}$ and not on a particular choice of the homeomorphism $\tilde{\sigma}$. However, the maps $\hat{\phi}_n$ are only defined up to post-composition with conformal automorphisms of $\mathbb{C}P^1$. To make a definite choice of $\hat{\phi}_n$, we introduce the following normalization. Let $P_0$ be any 3-point subset of $P$. Note that the sets $Q^n(P_0)$ are disjoint from $U$ for all $n > 0$ since they lie in $P'$. We require that the restriction of every $\hat{\phi}_n$ to $P_0$ be the identity. This normalization makes the maps $\hat{\phi}_n$ uniquely defined. However, the maps $\hat{\phi}_n$ depend on the choice of $\tilde{\sigma}$. The rational functions $R_n$ are uniquely defined by the classes of $\hat{\phi}_n$ in the Teichmüller space of $(\mathbb{C}P^1, P)$ and the normalization $\hat{\phi}_n|_{P_0} = id$. Therefore, they do not depend on the choice of $\tilde{\sigma}$.

**Proposition 3.1.** Set $U_n$ to be the union of $Q^{-i}(U)$ for $i = 1, \ldots, n$. The values of $\hat{\phi}_n$ at points $z \not\in U_n$ do not depend on a particular choice of a homeomorphism $\tilde{\sigma}$ with support in $U$.

**Proof.** Indeed, different homeomorphisms $\tilde{\sigma}_0$ and $\tilde{\sigma}_1$ are isotopic relative to $P$. Let $\tilde{\sigma}_t$, $t \in [0,1]$ be an isotopy. We can assume that the support of $\tilde{\sigma}_t$ is contained in $U$ for every $t$, see Remark 2.1. Let $\hat{\phi}_{n,t}$ be the maps that correspond to $\hat{\phi}_n$ as we replace $\tilde{\sigma}$ with $\tilde{\sigma}_t$. Take $z \not\in Q^{-n}(U)$. Suppose by induction that $\hat{\phi}_{n-1,t}(\hat{f}(z))$ does not depend on $t$ (note that $\hat{f}(z) = Q(z) \not\in U_{n-1}$). Then $\hat{\phi}_{n,t}(z)$ is a continuous path such that $R_{n-1} \circ \hat{\phi}_{n,t}(z) = \hat{\phi}_{n-1}(z) \circ f$. Hence the values of this path lie in the finite set $R_{n-1}^{-1}(\hat{\phi}_{n-1}(z))$. It follows that the path is constant.

We can include the maps $\hat{\phi}_n$ into a continuous family of homeomorphisms $\hat{\phi}_t : \mathbb{C}P^1 \to \mathbb{C}P^1$ defined for all real non-negative values of $t$. This is done in the following way. By Remark 2.2, there is a continuous one-parameter family of homeomorphisms $\tilde{\sigma}_t$, $t \in [0,1]$ connecting $id$ with $\tilde{\sigma}$ such that the supports of all $\tilde{\sigma}_t$ are contained in an arbitrarily small neighborhood $U$ of $Z_0$, and $\tilde{\sigma}_t(\eta(P_R - P(P_R))) \subset Z_0$ (we use
the notation of Proposition 2.3). Consider the first step of Thurston’s algorithm for $\hat{f}_t = \hat{\sigma}_t \circ Q$:

$$
\begin{array}{c}
\mathbb{CP}^1 \xrightarrow{\hat{f}_t} \mathbb{CP}^1 \\
\phi_t \downarrow \quad \downarrow \text{id} \\
\mathbb{CP}^1 \xrightarrow{R_{t-1}} \mathbb{CP}^1
\end{array}
$$

Note that the critical values of $\hat{f}_t$ (hence also of $R_{t-1}$) different from critical values of $Q$ lie in $Z_0$. Normalize $\hat{\phi}_t$ by requiring that their restrictions to $P_0$ be the identity. Note that, for $t = 1$, we obtain the same $\hat{\phi}_1$ and $R_0$ as before. We have $\hat{f}_1 = \hat{f}$. We can now start Thurston’s algorithm with $\hat{\phi}_t$, $t \in (0, 1)$ rather than starting it with $\hat{\phi}_0 = \text{id}$ (let me stress however that we do the same algorithm for all $\hat{\phi}_t$, namely, the algorithm associated with $\hat{f}!$). In this way, we obtain a continuous path of rational functions $R_t$ and a continuous path of homeomorphisms $\hat{\phi}_t$ defined for all real nonnegative $t$ and satisfying the identity $R_t \circ \hat{\phi}_{t+1} = \hat{\phi}_t \circ \hat{f}$ for $t \geq 0$.

**Proposition 3.2.** The rational functions $R_t$ converge to a rational function Möbius conjugate to $R$.

In the sequel, we will always assume that $R_t$ converge to $R$, since nothing changes in the statement of the Main Theorem if we replace the function $R$ by its Möbius conjugate.

**Proof.** We know that $\hat{f}$ is Thurston equivalent to $R$. From Thurston’s Characterization Theorem [DH93] (in fact, from its easy part) it follows that the classes of $\hat{\phi}_n$ converge to the class of some homeomorphism $\hat{\phi}_\infty : S^2 \to \mathbb{CP}^1$ in the Teichmüller space of $(\mathbb{CP}^1, P)$. The path $t \mapsto [\hat{\phi}_t]$, $t \in [0, \infty)$ converges in the Teichmüller space of $(\mathbb{CP}^1, P)$ as well. This follows from the convergence of $[\hat{\phi}_n]$ and the contraction property of Thurston’s pullback map.

Since the class $[\hat{\phi}_\infty]$ of $\hat{\phi}_\infty$ in the Teichmüller space coincides with the class $[M \circ \hat{\phi}_\infty]$ for every Möbius transformation $M$, we can assume that $\hat{\phi}_\infty = \text{id}$ on $P_0$. Convergence of $[\hat{\phi}_t]$ to $[\hat{\phi}_\infty]$ means that there is a family of of quasiconformal homeomorphisms $h_t : \mathbb{CP}^1 \to \mathbb{CP}^1$ such that the quasiconformal constant of $h_t$ tends to 1, and the equality $\hat{\phi}_t = h_t \circ \hat{\phi}_\infty$ holds on $P$ and holds on $\mathbb{CP}^1$ up to isotopy relative to $P$. The maps $\hat{\phi}_t$ and $\hat{\phi}_\infty$ are the identity on $P_0$, hence so is $h_t$. It follows that $h_t$ converge uniformly to the identity.
Note that the branched covering $h_t^{-1} \circ R_t \circ h_{t+1}$ is homotopic to $\hat{\phi}_\infty \circ f \circ \hat{\phi}_\infty^{-1}$ relative to the set $\hat{\phi}_\infty(P)$ through branched coverings. Since $h_t \to id$, any partial limit of $R_t$ as $t \to \infty$ is a rational function homotopic to $\hat{\phi}_\infty \circ f \circ \hat{\phi}_\infty^{-1}$ relative to the set $\hat{\phi}_\infty(P)$ through branched coverings (in particular, this rational function is critically finite, hyperbolic and Thurston equivalent to $\hat{f}$). By Thurston’s Uniqueness Theorem, such rational function is unique. □

Recall that, by our assumptions, $P \cap Z = \emptyset$. Set $Z_t$ to be the union of $Q^{-i}(Z_0)$ for $i$ running from 1 to the smallest integer that is greater than or equal to $t$. We will now define a family of holomorphic maps $\Phi_t : CP^1 - Z_t \to CP^1$ with the following properties:

$$R_t \circ \Phi_{t+1} = \Phi_t \circ Q, \quad \Phi_t|_P = \hat{\phi}_t|_P,$$

where the rational functions $R_t$ are the same as before. For $z \notin U_n$, where $n \geq t$, we set $\Phi_t(z) = \hat{\phi}_t(z)$. This value is well-defined by the construction of $\hat{\phi}_t$ and the same argument as in Proposition 3.1. On the other hand, for every $z \notin Z_t$, we can choose $U$ such that $z \notin U_n$, so that the definition applies. The holomorphy of $\Phi_t$ follows from the fact that locally near $z \notin Z_t$, the function $\Phi_t(z)$ is a branch of $R_t^{-1} \circ \Phi_t \circ Q$.

The key lemma is the following:

**Lemma 3.3.** The maps $\Phi_n : CP^1 - Z \to CP^1$, $n = 1, 2, \ldots$, converge uniformly.

*Proof of Theorem 2.4 assuming Lemma 3.3.* Every map $\Phi_n$ is a restriction of a holomorphic function defined on the complement to a real semi-algebraic set of positive codimension. By definition of a holomorphic function on $CP^1 - Z$, it follows that the uniform limit $\Phi$ of $\Phi_n$ is holomorphic on $CP^1 - Z$. □

Lemma 3.3 is hard to approach directly because $CP^1 - Z$ is a bad space. Therefore, we will consider a certain compactification of it.

Let $X_0$ be the Caratheodory compactification of $CP^1 - Q^{-1}(Z_0)$. This is a compact real 2-dimensional manifold with boundary that has a canonical projection $\pi_0$ onto $CP^1$. The space $X_0$ is not necessarily connected. Define $X$ as the set of sequences $(x_n)$ in $X_0$, $n = 1, 2, \ldots$ such that $\pi_0(x_{n+1}) = Q(\pi_0(x_n))$ for all $n$. The topology on $X$ is induced from the direct product topology on the space of all sequences. Thus $X$ is a compact Hausdorff space. Define a continuous self-map $f : X \to X$ as follows: if $x$ is a sequence $(x_1, x_2, x_3, \ldots)$, then $f(x) = y$, where $y$ is the sequence $(x_2, x_3, \ldots)$. There is also a natural projection $\pi : X \to CP^1$ given by the formula $\pi(x_1, x_2, \ldots) = \pi_0(x_1)$. This projection is a
semiconjugacy between $f$ and $Q$. Note that $X$ can be identified with the projective limit of Carathéodory compactifications of $\mathbb{C}P^1 - Z_n$.

Since $P \cap Z = \emptyset$, the map $\pi$ restricted to $\pi^{-1}(P)$ is one-to-one. Set $P_f = \pi(P)$. There exists a one-parameter family of continuous maps $\phi_t : X \to \mathbb{C}P^1$ such that $\phi_t = \Phi_t \circ \pi$ on $\pi^{-1}(\mathbb{C}P^1 - Z)$. Indeed, every $\Phi_t$ extends to the Carathéodory compactification of $\mathbb{C}P^1 - Z_t$. The maps $\phi_t$ make the following diagram commutative:

\[
\begin{array}{ccc}
(X, f^{-1}(P_f)) & \xrightarrow{f} & (X, P_f) \\
\downarrow \phi_{t+1} & & \downarrow \phi_t \\
(\mathbb{C}P^1, R^{-1}(P_t)) & \xrightarrow{R_t} & (\mathbb{C}P^1, P_t)
\end{array}
\]

where $P_t = \phi_t(P_f)$. The restrictions of the maps $\phi_t$ to $P_f$ converge. This follows from the convergence of $\hat{\phi}_t$ in the Teichmüller space of $(\mathbb{C}P^1, P)$ (more precisely, from the convergence of the projections of $\hat{\phi}_t$ to the moduli space of $\mathbb{C}P^1 - P$). Hence, the restrictions of $\phi_t$ to $P_f$ converge as well. It is clear that, for every critical value $v$ of $\hat{f}$, the limit of $\hat{\phi}_t(v)$ is a critical value of $R$. It follows that the limit of $\phi_t(z)$ is in $P_R$ for every $z \in P$. It also follows that the limit of $\phi_t(x)$ is in $P_R$ for every $x \in P_f$.

**Lemma 3.4.** There exists a map $\iota : f^{-1}(P_f) \to R^{-1}(P_R)$ such that the following diagram is commutative

\[
\begin{array}{ccc}
f^{-1}(P_f) & \xrightarrow{f} & P_f \\
\downarrow \iota & & \downarrow \iota \\
R^{-1}(P_R) & \xrightarrow{R} & P_R
\end{array}
\]

and the restrictions of $\phi_t$ to the set $f^{-1}(P_f)$ converge to $\iota$.

**Proof.** Consider a point $x \in f^{-1}(P_f)$. We have proved that the points $\phi_t \circ f(x) \in P_t$ converge to some point $a \in P_R$. Let $t_n$ be any sequence such that $\phi_{t_n}(x)$ converges; denote the limit by $b$. Passing to the limit in both sides of the equation $R_{t_{n-1}} \circ \phi_{t_n}(x) = \phi_{t_{n-1}} \circ f(x)$, we obtain that $R(b) = a$. It follows that the entire $\omega$-limit set of the family $\phi_t(x)$ is contained in the finite set $R^{-1}(a)$. As the $\omega$-limit set is connected, this implies that $\phi_t(x)$ converges to $b$ as $t \to \infty$. Set $\iota(x) = b$.

The commutative diagram in the statement of the lemma is obtained by passing to the limit as $t \to \infty$ in the diagram (\ast) and using that $\phi_t(x)$ converges to a point in $P_R$ for every $x \in P_f$. \hfill $\square$

Lemma 3.3, and hence also the Main Theorem, is now reduced to
Theorem 3.5. The maps $\phi_n : X \to \mathbb{C}P^1$ converge uniformly.

If this holds, then the maps $\Phi_n = \phi_n \circ \pi^{-1}$ on $\mathbb{C}P^1 - Z$ also converge uniformly. The remaining part of the paper contains the proof of Theorem 3.5. This is a statement about uniform convergence of Thurston’s algorithm. As such, it is perhaps not surprising, although we state it for a topological space $X$ that is not $S^2$ (actually, we need nothing from the space $X$ except that it is locally compact and that some neighborhood of $P_f$ in $X$ has a structure of a Riemann surface; however, specific properties of $\phi_n$ will be used, e.g. that the restrictions of $\phi_n$ to $P_f$ converge and that $\phi_n$ are holomorphic near $P_f$). Thurston’s algorithm is generally expected to converge uniformly, and theorems to this effect have been proved in a variety of contexts. E.g. a general theorem about uniform convergence of Thurston’s algorithm has appeared in [CT]. It deals with hyperbolic but not necessarily critically finite rational functions. I am grateful to Tan Lei for showing me a draft of this work.

The underlying ideas of the proof of Theorem 3.5 can be traced back to [DHS4]. Very roughly, it is an application of the contraction principle to a certain lifting map on a certain functional space. It is even possible to state a general theorem of this sort but many fine details would make its statement too cumbersome. The things are not complicated but they are not straightforward either.

4. The space $C$

Notation: for a topological space $X$ and a metric space $Y$, we denote by $C(X,Y)$ the set of all continuous maps from $X$ to $Y$. We will always equip this set with the partially defined uniform metric (note that the uniform distance between two elements of $C(X,Y)$ may well be infinite).

In this section, we start the proof of Theorem 3.5. We first set up a suitable function space. In the next section, we prove the convergence in this space. Consider the hyperbolic critically finite rational function $R$ from the statement of the Main Theorem. Recall that $P_R$ denotes the postcritical set of $R$.

We will need a metric on $\mathbb{C}P^1 - P_R$ with certain properties:

Lemma 4.1 (Expanding metric on $\mathbb{C}P^1 - P_R$). There exists a piecewise smooth metric on $\mathbb{C}P^1 - P_R$ equal to a constant multiple of $|d\xi|/|\xi|$ near every point $z \in P_R$ for some local holomorphic coordinate $\xi$ with $\xi(z) = 0$ and such that the map $R : \mathbb{C}P^1 - R^{-1}(P_R) \to \mathbb{C}P^1 - P_R$ is uniformly expanding with respect to this metric.
Proof. It follows from hyperbolicity that there exists a neighborhood \( V_0 \) of the Julia set \( J(R) \) of \( R \) and a Riemannian metric \( g_0 = \sigma(z)|dz| \) on \( V_0 \) such that \( R \) is uniformly expanding with respect to \( g_0 \) i.e.

\[
\sigma \circ R(z)|dR(z)| \geq E_0 \sigma(z)|dz|
\]

for some \( E_0 > 1 \) and all \( z \in V_0 \cap R^{-1}(V_0) \). We can assume that \( V_0 \) is bounded by smooth curves and that \( V_0 = R^{-1}(R(V_0)) \). Now extend the metric \( g_0 \) to the set \( R(V_0) - V_0 - P_R \) by the following formula:

\[
g_0(z) = E_0 \cdot \max_i g(S_i(z)),
\]

or, equivalently,

\[
\sigma(z) = E_0 \cdot \max_i \left\{ \sigma(S_i(z)) \left| \frac{dS_i(z)}{dz} \right| \right\},
\]

where \( S_i(z) \) are all local branches of \( R^{-1} \) near \( z \). They are well defined since all critical values of \( R \) belong to \( P_R \). With this definition, inequality (1) holds also in \( R(V_0) - P_R \). Using the same formula, we can extend the metric \( g_0 \) to \( R^{m}(V_0) - P_R \) for every \( m \), hence to the complement of an arbitrarily small neighborhood of \( P_R \). The extended metric is piecewise smooth and satisfies inequality (1) provided that \( g_0 \) is defined at both \( z \) and \( R(z) \).

Now let \( V_1 \) be a small neighborhood of \( P_R \) such that every component of \( V_1 \) is a Jordan domain containing exactly one point of \( P_R \). By Böttcher’s theorem, there exists a holomorphic function \( \xi : V_1 \to \mathbb{C} \) with simple zeros at all points of \( P_R \) such that \( \xi \circ R(z) = \xi(z)^{\nu(z)} \), where \( \nu \) is a locally constant function on \( V_1 \) taking its values in \( \mathbb{N} \). We can also assume that \( \xi \) is a holomorphic coordinate on every component of \( V_1 \). Note that \( R \) multiplies the metric \( |d\xi|/|\xi| \) on \( V_1 \) by \( \nu(z) \). Set

\[
g_1(z) = \lambda(z) \frac{|d\xi(z)|}{|\xi(z)|},
\]

where \( \lambda \) is a locally constant function on \( V_1 \), which we define below. It suffices to define \( \lambda \) on \( P_R \). Set

\[
E_1(z) = \lim_{n \to \infty} \left( \prod_{i=0}^{n-1} \nu(R^{\circ i}(z)) \right)^{1/n}.
\]

This number is equal to the geometric mean of \( \nu \) over the cycle, to which \( z \) eventually maps. In particular \( E_1(z) > 1 \). The function \( \lambda \) on \( P_R \) is now defined by the property

\[
\lambda(R(z)) = \frac{E_1(z) \lambda(z)}{\nu(z)}.
\]
If we fix an arbitrary positive value of $\lambda$ at an arbitrarily chosen point of each periodic cycle in $P_R$, then this condition defines $\lambda$ uniquely. The metric $g_1$ on $V_1 - P_R$ thus defined gets multiplied by $E_1(z)$ under the map $R$. Define the number $E_1 > 1$ as the minimum of $E_1(z)$ over all points in $P_R$.

We now want to combine the two metrics $g_0$ and $g_1$. We can assume that $V_0 \cup V_1 = \mathbb{C}P^1 - P_R$ and that both $V_0$ and $V_1$ are bounded by smooth curves. We can also assume that there is no point $z \in V_0 - V_1$ such that $R(z) \in V_1 - V_0$ (so that every $R$-orbit that visits both $V_0$ and $V_1$ must enter the “buffer zone” $V_0 \cap V_1$). Set $g = \varepsilon g_0$ on $V_0 - V_1$, $g = g_1$ on $V_1 - V_0$, and $g = \varepsilon g_0 + g_1$ on $V_0 \cap V_1$. As we will show, the map $R$ is uniformly expanding with respect to $g$ provided that the number $\varepsilon > 0$ is small enough so that e.g. $\varepsilon g_0 \leq (\sqrt{E_1} - 1)g_1$ everywhere on $V_0 \cap V_1$. Indeed, if $z \in V_0 - V_1$ and $R(z) \in V_0 \cap V_1$, then

$$g(R(z)) = \varepsilon g_0(R(z)) + g_1(R(z)) \geq E_0 \varepsilon g_0(z) = E_0 g(z).$$

If $z \in V_0 \cap V_1$ and $R(z) \in V_0 \cap V_1$, then

$$g(R(z)) = \varepsilon g_0(R(z)) + g_1(R(z)) \geq E_0 \varepsilon g_0(z) + E_1 g_1(z) \geq \bar{E} g(z).$$

where $\bar{E} = \min(E_0, E_1)$. Finally, if $z \in V_0 \cap V_1$ and $R(z) \in V_1 - V_0$, then

$$g(R(z)) = g_1(R(z)) \geq E_1 g_1(z) = (E_1 - \sqrt{E_1}) g_1(z) + \sqrt{E_1} g_1(z) \geq \sqrt{E_1} \varepsilon g_0(z) + \sqrt{E_1} g_1(z) = \sqrt{E_1} g(z).$$

\[\square\]

In the sequel, we will write $Y$ for the space $\mathbb{C}P^1 - P_R$ equipped with the metric $g$ from Lemma 4.4. Note that the metric $g$ is proper: every closed bounded set is compact. It follows that $g$ is complete and locally compact. Let $E > 1$ be the expansion factor of $R$ with respect to the metric $g$. In the notation of Lemma 4.4, we can set $E = \min(E_0, \sqrt{E_1})$.

We will use notation from Section 3. Note that there is an open neighborhood $O$ of the set $P_f$, on which the map $\pi$ is one-to-one and such that $f(O) \subset O$. We will assume that $O$ is sufficiently small. The map $\pi$ defines a Riemann surface structure on $O$. The maps $\phi_t$ are holomorphic on the set $O$ equipped with this structure. Let $\mathcal{C}(O)$ denote the space of continuous maps $\chi : X \to \mathbb{C}P^1$ with the following properties:

1. $\chi = \iota$ on $f^{-1}(P_f)$;
2. $\chi^{-1}(P_R) \subseteq P_f$;
3. $\chi^{-1}(R^{-1}(P_R)) \subseteq f^{-1}(P_f)$;
(4) the restriction of $\chi$ to $O$ is holomorphic, and no point of $P_f$ is a critical point of $\chi$.

We will consider the following metric on $\mathcal{C}(O)$: the distance between maps $\chi$ and $\chi^* \in \mathcal{C}(O)$ is the uniform distance between the restrictions $\chi : X - P_f \to Y$ and $\chi^* : X - P_f \to Y$ measured with respect to the metric $g$ on $Y$. We need to prove that the distance between any two elements $\chi$ and $\chi^*$ of $\mathcal{C}(O)$ is finite. It suffices to make a local estimate near each point $x \in P_f$. Let $W_x$ be a small neighborhood of $\iota(x)$, and $\xi$ a holomorphic coordinate on $W_x$ such that $\xi(\iota(x)) = 0$. Let $O_x$ be a small neighborhood of $x$ contained in $O$ and such that $\chi(O_x) \subset W_x$ and $\chi^*(O_x) \subset W_x$. Since both holomorphic functions $\xi \circ \chi$ and $\xi \circ \chi^*$ have simple zeros at $x$, their ratio extends to a holomorphic function on $O_x$ taking a nonzero value at $x$. Note that the uniform distance between the maps $\chi : O_x - \{x\} \to Y$ and $\chi^* : O_x - \{x\} \to Y$ in the metric $g$ is

$$const \cdot \sup_{x' \in O_x - \{x\}} |\log \left( \frac{\xi \circ \chi(x')}{\xi \circ \chi^*(x')} \right)|$$

for some local branch of the logarithm. Indeed, the metric $g$ is equal to $\text{const} \cdot |d \log \xi|$ on $W_x$. We see that the distance between $\chi$ and $\chi^*$ is finite. A similar argument shows that the topology on $\mathcal{C}(O)$ coincides with the topology induced from the uniform metric on $C(X, \mathbb{C}P^1)$. Define $\mathcal{C}$ as the union of $\mathcal{C}(O)$ over all sufficiently small neighborhoods $O$ of $P_f$ such that $f(O) \subset O$. As a metric space, $\mathcal{C}$ is the inductive limit of the spaces $\mathcal{C}(O)$.

We will write $\tilde{Y}$ for $Y - R^{-1}(P_f)$. Then $R : \tilde{Y} \to Y$ is a proper expansion with expansion factor $E$. Being a proper map and a local homeomorphism, this map enjoys the unique path lifting property. This is a key to the following

**Lemma 4.2.** If $\gamma : [0, 1] \to \mathcal{C}(O)$ is a continuous path and $\tilde{\chi}_0 \in \mathcal{C}(O)$ is a map such that $R \circ \tilde{\chi}_0 = \gamma(0) \circ f$, then there is a unique continuous path $\tilde{\gamma} : [0, 1] \to \mathcal{C}(O)$ with the properties $\tilde{\gamma}(0) = \tilde{\chi}_0$ and $R \circ \tilde{\gamma}(t) = \gamma(t) \circ f$ for all $t \in [0, 1]$.

We will call the path $\tilde{\gamma}$ a lift of the path $\gamma$.

**Proof.** For every $t$, consider the restriction of $\gamma(t)$ to $X - P_f$. We obtain a path $\gamma_* : [0, 1] \to C(X - P_f, Y)$.

Consider the map $\mathcal{G} : C(X - f^{-1}(P_f), \tilde{Y}) \to C(X - f^{-1}(P_f), Y)$ given by the formula $\mathcal{G}(\chi) = R \circ \chi$. The map $R : \tilde{Y} \to Y$ is a proper expansion, and $X - f^{-1}(P_f)$ is a locally compact space (as a complement to finitely many points in a compact Hausdorff space). It is a standard fact from topology (see e.g. Spanier [S]) that in this case the map $\mathcal{G}$ has
the path lifting property: given a path \( \alpha : [0, 1] \to C(X - f^{-1}(P_f), Y) \) and an element \( \tilde{\chi}_0 \in C(X - f^{-1}(P_f), \tilde{Y}) \) such that \( R \circ \tilde{\chi}_0 = \alpha(0) \), there exists a unique path \( \tilde{\alpha} : [0, 1] \to C(X - f^{-1}(P_f), \tilde{Y}) \) such that \( R \circ \tilde{\alpha}(t) = \alpha(t) \) for all \( t \in [0, 1] \) and \( \tilde{\alpha}(0) = \tilde{\chi}_0 \). We take \( \alpha(t) = \gamma_\epsilon(t) \circ f \) and consider the corresponding lift \( \tilde{\alpha} \) (with \( \tilde{\chi}_0 \) as in the statement of the lemma).

It is obvious that every map \( \tilde{\alpha}(t) \) extends to a continuous map \( \tilde{\gamma}(t) \) from \( X \) to \( \mathbb{C}P^1 \) holomorphic on \( \tilde{O} \). It remains to show that the maps \( \tilde{\gamma}(t) \) belong to \( \mathcal{C}(O) \). The following two properties imply this:

1. \( \tilde{\gamma}(t) = \gamma(t) \) on \( f^{-1}(P_f) \);
2. \( \tilde{\gamma}(t)^{-1}(R^{-1}(P_R)) \subseteq f^{-1}(P_f) \).

Note that both \( \gamma(t) \) and \( \tilde{\gamma}(t) \) restricted to \( f^{-1}(P_f) \) take values in \( R^{-1}(P_R) \). For \( \tilde{\gamma}(t) \), this follows from the defining identity \( R \circ \tilde{\gamma}(t) = \gamma(t) \circ f \). Now property (1) holds by continuity (the two maps coincide for \( t = 0 \) and take values in finite sets). To prove property (2), take any \( x \in X \) such that \( \tilde{\gamma}(t)(x) \in R^{-1}(P_R) \). Then

\[
\gamma(t)(f(x)) = R \circ \tilde{\gamma}(t)(x) \in P_R,
\]

hence \( f(x) \in P_f \), hence \( x \in f^{-1}(P_f) \).

\[ \square \]

Remark 4.3. Note that any lift of a rectifiable path in \( \mathcal{C} \) is at least \( E \) times shorter than the path itself. This follows from the fact that the map \( \mathcal{G} \) is a local expansion with expansion factor \( E \).

**Proposition 4.4.** There exists a continuous family of homeomorphisms \( \psi_t : \mathbb{C}P^1 \to \mathbb{C}P^1 \) defined for sufficiently large \( t \) with the following properties:

- \( \psi_t \circ \phi_t \in \mathcal{C} \);
- \( \psi_t \to \text{id} \) uniformly as \( t \to \infty \);
- there is a neighborhood \( W \) of \( P_R \) such that the restrictions of \( \psi_t \) to \( W \) are holomorphic;

Note that a priori we cannot fix a neighborhood \( O \) of \( P_f \) such that \( \psi_t \circ \phi_t \in \mathcal{C}(O) \) for all \( t \). However, as can be seen from the proof, such neighborhood exists for every bounded interval of values of \( t \).

**Proof.** Suppose that \( t \) is sufficiently large so that \( \phi_t(x) \neq \phi_t(x') \) for \( x, x' \in f^{-1}(P_f) \) unless \( \nu(x) = \nu(x') \). Recall that \( P_t = \phi_t(P_f) \). For \( z \in R_{t-1}^{-1}(P_{t-1}) \), we set \( \psi_t(z) = \nu(x) \), where \( x \in f^{-1}(P_f) \) is any point such that \( \phi_t(x) = z \). By our assumption, the point \( \psi_t(z) \) thus defined does not depend on the choice of \( x \). We have defined the map \( \psi_t \) on \( R_{t-1}^{-1}(P_{t-1}) \). It is clear that the map \( \psi_t \) is injective on \( R_{t-1}^{-1}(P_{t-1}) \) (different points of \( R_{t-1}^{-1}(P_{t-1}) \) do not merge in the limit). Since the
sets $R_{t-1}(P_{t-1})$ and $R^{-1}(P_R)$ have the same cardinality, this map is actually a bijection between these two sets.

The uniform distance between the map $\psi_t$ on $R_{t-1}(P_{t-1})$ and the identity is bounded above by the uniform distance between $\phi_t$ on $f^{-1}(P_f)$ and $t$. Hence this distance tends to $0$ as $t \to \infty$. It follows that we can choose a continuous family $\psi_t$ so that $\psi_t \to \text{id}$ as $t \to \infty$ and that $\psi_t$ are holomorphic on some neighborhood of $P_R$ for all sufficiently large $t$.

It suffices to prove that $\psi_t \circ \phi_t$ belongs to $\mathcal{C}$. By definition $\psi_t \circ \phi_t = t$ on $f^{-1}(P_f)$. We need to check that:

1. $(\psi_t \circ \phi_t)^{-1}(P_R) \subseteq P_f$;
2. $(\psi_t \circ \phi_t)^{-1}(R^{-1}(P_R)) \subseteq f^{-1}(P_f)$.

To prove (1), suppose that $w = \psi_t \circ \phi_t(x) \in P_R$ for some $x \in X$. Since $\psi_t$ is a homeomorphism taking $P_t$ to $P_R$, and $P_R$ has the same cardinality as $P_t$, we have $\psi_t^{-1}(w) \in P_t$. Then $x \in \phi_t^{-1}(P_t) = P_f$. Property (2) can be proved by the same argument.

Theorem 3.5 (and hence the Main Theorem) reduces to the following:

**Theorem 4.5.** The sequence $\chi_n = \psi_n \circ \phi_n$ converges in $\mathcal{C}$ as $n \to \infty$ through positive integers, hence in $C(X - P_f, Y)$ and in $C(X, \mathbb{C}P^1)$.

Indeed, since $\psi_n$ converge to the identity, we conclude that $\phi_n$ converge uniformly, q.e.d. The convergence in $\mathcal{C}$ is perhaps a little surprising because the space $\mathcal{C}$ is not complete.

5. Contracting lifting

In this section, we prove Theorem 4.5 hence also the Main Theorem. We will repeatedly use contraction properties of the lifting as defined in Lemma 4.2. Note that the map $\chi_t = \psi_t \circ \phi_t \in \mathcal{C}$ depends continuously on $t$ with respect to the uniform metric in $C(X, \mathbb{C}P^1)$. It can also be arranged that, for every finite interval $[t_0, t_1]$, there exists a neighborhood $O$ of $P_f$ such that $\chi_t \in \mathcal{C}(O)$ for all $t \in [t_0, t_1]$. Hence $\chi_t$ form also a continuous family in $\mathcal{C}$.

Recall that we have the following commutative diagram:

$$
\begin{array}{ccc}
(X, f^{-1}(P_f)) & \xrightarrow{f} & (X, P_f) \\
\downarrow \phi_t & & \downarrow \phi_t \\
(\mathbb{C}P^1, R_t^{-1}(P_t)) & \xrightarrow{R_t} & (\mathbb{C}P^1, P_t) \\
\downarrow \psi_t & & \downarrow \psi_t \\
(\mathbb{C}P^1, R^{-1}(P_R)) & \xrightarrow{\psi_t} & (\mathbb{C}P^1, P_R)
\end{array}
$$
The family $\psi_t$ can be thought of as a continuous path in $C(\mathbb{C}P^1, \mathbb{C}P^1)$ defined on the compact interval $[0, \infty]$: it suffices to set $\psi_\infty = \text{id}$. There exists a unique continuous path $t \mapsto \tilde{\psi}_t$, $t \in [1, \infty]$ such that $\tilde{\psi}_\infty = \text{id}$ and $R \circ \tilde{\psi}_t = \psi_{t-1} \circ R_{t-1}$ for all $t \geq 1$. We have $\tilde{\psi}_t = \psi_t$ on $R_{t-1}^{-1}(P_{t-1})$ by continuity, since both maps are equal to the identity for $t = \infty$, and both take values in $R^{-1}(P_R)$. Since $\psi_t \circ \phi_t \in \mathcal{C}$ and $\tilde{\psi}_t = \psi_t$ on $R_{t-1}^{-1}(P_{t-1})$, we also have $\tilde{\chi}_t = \psi_t \circ \phi_t \in \mathcal{C}$. Note that the family $\tilde{\chi}_t$ satisfies the following identity:

$$R \circ \tilde{\chi}_{t+1} = \chi_t \circ f.$$  

Indeed, we have

$$R \circ \tilde{\psi}_{t+1} \circ \phi_t = \psi_t \circ R_t \circ \phi_{t+1} = \psi_t \circ \phi_t \circ f.$$  

Note that both $\psi_t$ and $\tilde{\psi}_t$ converge to the identity as $t \to \infty$ uniformly with respect to the spherical metric. Therefore, the distance between $\chi_t$ and $\tilde{\chi}_t$ in $C(X, \mathbb{C}P^1)$ tends to 0 as $t \to \infty$.

**Proposition 5.1.** There is a continuous map $\Gamma : [t_0, \infty) \times [0, 1] \to \mathcal{C}$, where $t_0$ is a sufficiently large real number, such that

$$\Gamma(t, 0) = \chi_t, \quad \Gamma(t, 1) = \tilde{\chi}_t,$$

and the length of the path $s \mapsto \Gamma(t, s), s \in [0, 1]$ in $\mathcal{C}$ tends to 0 as $t \to \infty$. Moreover, for every $t \in [t_0, \infty)$, there exists a neighborhood $O_t$ of $P_f$ such that $\Gamma(t, s) \in \mathcal{C}(O_t)$ for all $s \in [0, 1]$. For every finite interval $[t_1, t_2]$, there exists a neighborhood $O$ of $P_f$ that is contained in $O_t$ for all $t \in [t_1, t_2]$.

**Proof.** It suffices to define a continuous map $\Psi : [t_0, \infty) \times [0, 1] \to C(\mathbb{C}P^1, \mathbb{C}P^1)$ such that $\Psi(t, 0) = \psi_t$, $\Psi(t, 1) = \tilde{\psi}_t$, and $s \mapsto \Psi(t, s)$ is a rectifiable path in $C(\mathbb{C}P^1 - P_f, Y)$, whose length tends to 0 as $t \to \infty$, and such that all $\Psi(t, s)$ are holomorphic on some fixed neighborhood $W$ of $P_R$. Then we set $\Gamma(t, s) = \Psi(t, s) \circ \phi_t$.

Fix $x \in P_f$, and set $z_t = \phi_t(x)$. Let $W_x$ be the component of $W$ containing $\iota(x)$, and $\xi$ a holomorphic coordinate on $W_x$ such that $\xi(\iota(x)) = 0$. We have $z_t \in W_x$ for all sufficiently large $t$, and $\xi(z_t) \to 0$ as $t \to \infty$. Since both $\xi \circ \psi_t(z)$ and $\xi \circ \tilde{\psi}_t(z)$ have simple zeros at $z_t$, the ratio $\xi \circ \tilde{\psi}_t/\xi \circ \psi_t$ extends holomorphically to $W_x$ and converges to 1 on $W_x$ as $t \to \infty$. It follows that the distance between $\psi_t$ and $\tilde{\psi}_t$ in $C(W_x - \{z_t\}, Y)$ tends to zero as $t \to \infty$. We set

$$\xi \circ \Psi(t, s) = \left(\xi \circ \psi_t\right) \cdot \exp \left(\frac{s \log \xi \circ \tilde{\psi}_t}{\xi \circ \psi_t}\right),$$

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on a neighborhood \( \tilde{W}_x \) of \( t(x) \) such that the right-hand side lies in \( \xi(W_x) \) (we can choose one neighborhood that will work for all sufficiently large \( t \)). The branch of the logarithm is chosen to be the closest to 0. This defines the maps \( \Psi(t, s) \) on \( \tilde{W}_x \). Note that the path \( s \mapsto \Psi(t, s) \) is rectifiable in \( C(\tilde{W}_x - \{z_t\}, Y) \), and the length of the path is equal to the distance between \( \psi_t \) and \( \tilde{\psi}_t \) in \( C(\tilde{W}_x - \{z_t\}, Y) \). The same formula defines the maps \( \Psi(t, s) \) on a neighborhood of any point from \( P_t \). Clearly, we can extend \( \Psi(t, s) \) to \( \mathbb{C}P^1 \) with desired properties. \( \square \)

In the proof of Theorem 4.5, we will need two more lemmas.

**Lemma 5.2.** Consider a rectifiable path \( \delta : [0, 1] \to \mathcal{C}(O) \). Then there is a continuous extension \( \tilde{\delta} : [0, \infty] \to \mathcal{C}(O) \) such that \( R \circ \delta(t + 1) = \delta(t) \circ f \) for all \( t \in [0, \infty) \). Moreover, the length of the extended path \( \tilde{\delta} \) is at most \( E/(E - 1) \) times the length of the original \( \delta \), and we have

\[
R \circ \delta(\infty) = \delta(\infty) \circ f.
\]

**Proof.** Consider the lift \( \alpha \) of the path \( \delta \) as in Lemma 4.2. By Remark 4.3, the length of \( \alpha \) is at most \( E^{-1} \) times the length of \( \delta \). Now we set \( \delta(t) = \alpha(t - 1) \) for \( t \in [1, 2] \). As we keep doing this extension process, we obtain more and more segments of \( \delta \), each segment being shorter than the preceding one by at least the factor \( E^{-1} \). It follows that \( \delta(t) \) converges in \( C(X - P_f, Y) \) as \( t \to \infty \). Denote the limit by \( \delta(\infty) \). Thus we obtain the extended path \( \delta : [0, \infty] \to C(X - P_f, Y) \). The length of this extended path can be estimated by a geometric series with the common ratio \( E^{-1} \): it does not exceed \( E/(E - 1) \).

It remains to prove that \( \delta(\infty) \in \mathcal{C}(O) \). The map \( \delta(\infty) \) is holomorphic on \( O \) as a uniform limit of holomorphic maps. The only non-obvious property is that the preimage of \( R^{-1}(P_R) \) under \( \delta(\infty) \) is contained in \( f^{-1}(P_f) \), or, equivalently, the image of \( X - f^{-1}(P_f) \) under \( \delta(\infty) \) is contained in \( \tilde{Y} \). Indeed, the lift \( \tilde{\delta} \) of the path \( t \mapsto \delta(t), t \in [0, \infty] \), such that \( \tilde{\delta}(0) = \delta(1) \) is unique. Therefore, we must have \( \tilde{\delta}(t) = \delta(t + 1) \) on \( X - f^{-1}(P_f) \) for all \( t \in [0, \infty] \), in particular, \( \tilde{\delta}(\infty) = \delta(\infty) \). On the other hand, by construction, the map \( \delta(\infty) \) takes the set \( X - f^{-1}(P_f) \) to \( \tilde{Y} \). Therefore, we have \( \delta(\infty)(X - f^{-1}(P_f)) \subseteq \tilde{Y} \), as desired. The equality \( R \circ \delta(\infty) = \delta(\infty) \circ f \) follows from the equality \( R \circ \tilde{\delta}(\infty) = \delta(\infty) \circ f \) on \( X - f^{-1}(P_f) \). \( \square \)

**Lemma 5.3.** There exists a real number \( \varepsilon > 0 \) such that the distance between two maps \( \chi^*, \chi^{**} \in \mathcal{C} \) is bigger than \( \varepsilon \) provided that

\[
R \circ \chi^* = \chi^* \circ f, \quad R \circ \chi^{**} = \chi^{**} \circ f.
\]
Proof. There exists a neighborhood $O$ of $P_f$ such that every map $\chi \in C$ such that $R \circ \chi = \chi \circ f$ is holomorphic on $O$. Indeed, $\chi$ is holomorphic on at least some small neighborhood of $P_f$ but the formula $R \circ \chi = \chi \circ f$ says that $\chi$ is also holomorphic on iterated pullbacks of this neighborhood under $f$.

Let $\xi$ be an extended Böttcher coordinate on a neighborhood $W$ of $P_R$ so that $\xi \circ R(z) = \xi(z)^{\nu(z)}$ for some locally constant function $\nu$ taking positive integer values. Define the Green function $G_R$ of $R$ on $W$ as $-\log |\xi|$. The function $G_R \circ \chi$ restricted to some neighborhood of $P_f$ is the same for all maps $\chi \in C$ with the property $R \circ \chi = \chi \circ f$ (by the uniqueness of Böttcher’s coordinate). Denote this function by $G_f$ and call it the Green function of $f$. The Green function $G_R$ can be extended to $\mathbb{C}P^1$ by setting

$$G_R(z) = \frac{1}{d^n} G_R(R^n(z))$$

if $R^n(z) \in W$ for some $n \geq 0$, and $G_R(z) = 0$ otherwise. Here $d = \deg(R)$. Similarly, $G_f$ extends to a function on $X$.

Take a sufficiently small number $\varepsilon_0 > 0$. The set $G_f \leq \varepsilon_0$ is mapped to the set $G_R \leq \varepsilon_0$ by both $\chi^*$ and $\chi^{**}$. Note that $\xi \circ \chi^*$ and $\xi \circ \chi^{**}$ can only differ on the set $G_f \leq \varepsilon_0$ by a locally constant factor that is a root of unity of bounded degree. Therefore, the distance between restrictions of $\xi \circ \chi^*$ and $\xi \circ \chi^{**}$ to the set $G_f \leq \varepsilon_0$ cannot take arbitrarily small nonzero values. It follows that if $\chi^*$ and $\chi^{**}$ are sufficiently close, then their restrictions to the set $G_f \leq \varepsilon_0$ must coincide.

Since all critical values of $R$ are poles of the Green function $G_R$, the distance between two different $R$-preimages of any point $z$ with $G_R(z) \geq \varepsilon_0/d$ is bounded below by some positive number uniform with respect to $z$. It follows that $\chi^*$ and $\chi^{**}$ must also coincide on the set $\varepsilon_0 \leq G_f \leq d\varepsilon_0$ provided that $\chi^*$ and $\chi^{**}$ are sufficiently close. By induction, the two maps coincide on the set $G_f > 0$. This set is dense in $X$ (because it is dense in $\mathbb{C}P^1 - Z$), hence $\chi^* = \chi^{**}$. \hfill \Box

Proof of Theorem 4.5: Throughout the proof, the parameters $t$ and $s$ will run through the interval $[0, 1]$. By a homotopy, we will always mean a homotopy between two paths in the metric space $C$ with fixed endpoints.

Consider any rectifiable path $\gamma : [0, 1] \to C$ connecting $\chi_{n-1}$ with $\chi_n$ and homotopic to the path $t \mapsto \chi_{n-1+t}$. Consider the lift $\tilde{\gamma} : [0, 1] \to C$ of $\gamma$ (as in Lemma 4.2) such that $\tilde{\gamma}(0) = \tilde{\chi}_n$. The path $\tilde{\gamma}$ is at least $E$ times shorter than $\gamma$. We claim that $\tilde{\gamma}(1) = \tilde{\chi}_{n+1}$. Indeed, since the path $\gamma$ is homotopic to the path $t \mapsto \chi_{n-1+t}$, the path $\tilde{\gamma}$ is homotopic to the path $t \mapsto \tilde{\chi}_{n+t}$ (the lifts of two homotopic paths are homotopic).
Set $L_n$ to be the infimum of the lengths of rectifiable paths in $C$ lying in $C(O)$ for some neighborhood $O$ of $P_f$, connecting $\chi_n$ to $\chi_{n+1}$ and homotopic to the path $t \mapsto \chi_{n+t}$. If $\varepsilon_n$ denotes the maximum of the lengths of the paths $\Gamma_n : s \mapsto \Gamma(n, s)$ and $\Gamma_{n+1} : s \mapsto \Gamma(n+1, s)$, then we have

$$L_n \leq E^{-1} \cdot L_{n-1} + 2\varepsilon_n$$

Indeed, if $\gamma$ is a rectifiable path in $C$ that connects $\chi_{n-1}$ with $\chi_n$, then $L_n$ is at most the length of the composition of the following paths:

- the path $\Gamma_n$ from $\chi_n$ to $\tilde{\chi}_n$;
- the path $\tilde{\gamma}$ from $\tilde{\chi}_n$ to $\tilde{\chi}_{n+1}$;
- the reversed path $\Gamma_{n+1}$ from $\tilde{\chi}_{n+1}$ to $\chi_{n+1}$.

The length of this composition can be made smaller than any fixed number exceeding $E^{-1}L_{n-1} + 2\varepsilon_n$ by choosing the path $\gamma$ in $C(O)$ for some $O$ to have length close to $L_{n-1}$. Note also that the composition is homotopic to $t \mapsto \chi_{n+t}$ provided that $\gamma$ is homotopic to $t \mapsto \chi_{n-1+t}$. The corresponding homotopy can be easily constructed using the homotopy $\Gamma$.

Take any $n_0$, then, applying the previous inequality several times, we obtain

$$L_n \leq E^{n_0-n}L_{n_0} + 2\varepsilon_{n_0}(q^{n-n_0} + q^{n-2} + \cdots + 1),$$

where $\tilde{\varepsilon}_{n_0}$ is the supremum of $\varepsilon_{n_0+1}, \ldots$. The second term in the right-hand side can be made arbitrarily small (uniformly with $n$) by choosing $n_0$ large enough. After $n_0$ has been chosen, we can choose sufficiently large $n$ to make the first term as small as we wish. It follows that $L_n \to 0$ (in particular, the distance between $\chi_n$ and $\chi_{n+1}$ tends to 0 in $C$).

Consider the composition $\delta_n$ of some path $\gamma$ of length at most $2L_n$ homotopic to $t \mapsto \chi_{n+t}$ and the path $\Gamma_{n+1}$. Reparameterize $\delta_n$ so that the parameter runs from 0 to 1. We can arrange that $\delta_n(t) \in C(O)$ for some open neighborhood $O$ of $P_f$ and all $t \in [0, 1]$. We have $R \circ \delta_n(1) = \delta_n(0) \circ f$ because $\delta_n(0) = \chi_n$ and $\delta_n(1) = \tilde{\chi}_{n+1}$.

Consider the extended path $\delta_n : [0, \infty] \to C(O)$ as in Lemma 5.2. Then we have

$$R \circ \delta_n(\infty) = \delta_n(\infty) \circ f.$$ 

The distance between $\delta_n(\infty)$ and $\delta_m(\infty)$ tends to 0 as $n$ and $m \to \infty$. By Lemma 5.3, the sequence $\delta_n(\infty)$ stabilizes, i.e. $\delta_n(\infty)$ is the same map $\chi_\infty$ for all sufficiently large $n$. We know that the distance between $\chi_n$ and $\chi_\infty$ in $C$ tends to 0. Therefore, $\chi_n$ converge to $\chi_\infty$. □
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