Hawking radiation and thermodynamics of dynamical black holes in a phantom-dominated universe

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Abstract
We investigate the thermodynamic properties of a phantom-energy-dominated universe in the presence of a black hole in the general case of a varying equation-of-state parameter \(w(a)\). We show that all the thermodynamic quantities are regular at the phantom divide crossing, and particularly the temperature and the entropy of a phantom fluid are always positive definite. We also study the accretion process of a phantom fluid by black holes and the conditions required for the validity of the generalized second law of thermodynamics. As a result, we obtain a strictly negative chemical potential and an equation-of-state parameter \(w < -5/3\).

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1. Introduction

The discovery that the Universe is currently undergoing a period of accelerating expansion, obtained from the observations of type Ia supernovae [1, 2], inaugurates an exciting era of intense theoretical research. A variety of possible solutions to understand the mechanism driving this accelerating expansion have been debated during this decade including the cosmological constant, exotic matter and energy, modified gravity, anthropic arguments, etc. The most favored ones are dark energy based models and modified gravity theories such \(f(R)\) gravity [3], and the Dvali–Gabadadze–Porrati model of gravity [4] with an equation-of-state (EoS) parameter \(w = P/\rho < -1\) (where \(\rho\) and \(P\) are the energy density and the pressure of the cosmic fluid, respectively). The dark energy is frequently modeled as an homogeneous scalar field, and according to the value of the EoS parameter \(w\), three different cases can be distinguished: quintessence scalar fields \((-1 < w < -1/3)\) with a positive kinetic term; cosmological constant \((w = -1)\), where only the potential term contributes to both the pressure and the energy density of the field; finally, scalar fields with a negative kinetic term dubbed phantom fields \((w < -1)\). In the latter case, the Universe will suffer a crucial fate.
where the energy density and the scale factor diverge in finite time, ripping apart all bound systems of the Universe, before the Universe approaches the so-called big-rip singularity [5, 6]. It is also well known that in phantom fluid models with big-rip singularity, quantum gravity effects become dominant in the neighboring of the big-rip time [7–10]. Actually, the cosmic microwave background experiments combined with large scale structure data, the Hubble parameter measurement and luminosity measurements of type Ia supernovae [11], favor a cosmological constant, but still admit the possibility of phantom fluid driven expansion, even the latter suffer from quantum instabilities [12] and violation of the strong and dominant classical energy conditions [13].

Another important and growing field currently under investigation is related to the thermodynamic properties of an expanding universe [14]. Recent studies on phantom thermodynamics show that the entropy of the Universe is negative [15], while the generalized second law of gravitational thermodynamics (GSL) is satisfied, $\dot{S}_f + \dot{S}_C \geq 0$, where $S_f$ is the phantom fluid entropy and $S_C$ is the entropy of the cosmological horizon [16]. Another point under debate is the influence of a non-zero chemical potential on the phantom thermodynamics and its relation with the GSL [17, 18].

In this paper, the thermodynamic properties of black holes (BHs) immersed in a dark-energy-dominated expanding universe and the accretion process of phantom energy onto BHs are investigated. The first paper dealing with the latter process is due to Babichev et al. [19], where ignoring the backreaction effect of the phantom fluid on the BH, they found that the change rate of the BH mass is negative. However, in recent scenarios where the backreaction is taken into account [20, 21], it is found that the mass of the BH is always an increasing function in an expanding Friedman–Robertson–Walker (FRW) universe.

The organization of this paper is as follows. In section 2, we review the exact solution describing a BH embedded in an expanding FRW universe, recently obtained [20, 21]. In section 3, we examine the Hawking radiation at the apparent horizons (AHs), and compare with the magnitude of the phantom-energy accretion process. We will show that the former is highly suppressed, particularly at late times. In section 4, we study in a unified and general way the thermodynamics of a cosmological BH embedded in an expanding (FRW) universe with a general EoS parameter $w(a)$, and particularly we obtain regular solutions realizing the crossing of the phantom divide line (PDL). In section 5, the stability of the solutions of section 4 under the quantum correction due to the conformal anomaly is established. In section 6, we study the conditions required for the validity of the GSL when the BH is immersed in a phantom-energy-dominated era. Particularly, in order to protect the GSL, we obtain a critical mass of the BH of the order of the solar mass for particular values of the parameter $\alpha_0 = -\mu_0 n_0/\rho_0$, where $\mu_0$, $n_0$ and $\rho_0$ are the present day values of the chemical potential, the particle density and the energy density, respectively. Finally, we discuss and summarize our results in section 7.

2. Cosmological expanding black hole

The first solution of Einstein’s theory of general relativity describing a BH-like object embedded in an expanding universe was introduced by McVittie in 1933 [22], and is given in isotropic coordinates by

$$\text{d}s^2 = -\left(1 - \frac{GM_0}{2\gamma_0 \Omega^2}\right)^2 \text{d}t^2 + a^2(t) \left(1 + \frac{GM_0}{2\gamma_0 \Omega^2 r}\right)^4 (\text{d}r^2 + r^2 \text{d}\Omega^2),$$

(1)
where $a(t)$ is the scale factor and $M_0$ is the mass of the BH in the static case. In fact, when $a(t) = 1$, it reduces to the Schwarzschild solution. When the mass parameter is zero, the McVittie reduces to a spatially flat FRW solution with the scale factor $a(t)$. The global structure of (1) has been studied and particularly it has been shown that the solution possesses a spacelike singularity on the 2-sphere $r = M_0/2$, and cannot describe an embedded BH in an expanding spatially flat FRW universe [23, 24]. On the other hand, the McVittie solution is constrained by the non-accretion condition onto the central mass, and therefore is not suitable for a study of the cosmic fluid accretion process onto BHs embedded in an expanding FRW universe.

In the following, we adopt the solution describing a BH embedded in a spatially flat FRW universe [20, 21]:

$$\mathbf{d}s^2 = -\frac{B^2(r)}{A^2(r)} \mathbf{d}t^2 + a^2(t)A^4(r)(\mathbf{d}r^2 + r^2 \mathbf{d}\Omega^2),$$

where $A(r) = (1 + \frac{GM_0}{2r})$, $B(r) = (1 - \frac{GM_0}{2r})$, $a(t)$ is the scale factor and $M_0$ is the mass of the BH in the static case. In fact, when $a(t) = 1$, solution (2) reduces to the Schwarzschild solution, while when the mass parameter is zero, it reduces to a spatially flat FRW solution with the scale factor $a(t)$.

Using the areal radius $\tilde{r} = r\left(1 + \frac{GM_0}{2r}\right)^2$ and $R = a\tilde{r}$, the metric takes the following suitable Painlevé–Gullstrand form:

$$\mathbf{d}s^2 = -\left(1 - \frac{2GM_0a}{R}\right)\left(1 - \frac{R^2H^2}{2GM_0a}\right) \mathbf{d}t^2 + \frac{1}{1 - \frac{2GM_0a}{R}} \left(1 - \frac{2GM_0a}{R}\right)^{-1} \mathbf{d}R^2 - 2RH\left(1 - \frac{2GM_0a}{R}\right)^{-1} \mathbf{d}r \mathbf{d}R + R^2 \mathbf{d}\Omega^2,$$

where $H = \dot{a}/a$ is the Hubble parameter and the over dot stands for derivative with respect to the cosmic time. The term $R^2H^2$ plays the role of a variable cosmological constant. Now, we introduce the time transformation $t \to \tilde{t}$ to remove the $\mathbf{d}r \mathbf{d}R$ term

$$\mathbf{d}\tilde{t} = F^{-1}(t, R) \left[ \mathbf{d}t + \frac{HR}{1 - \frac{2GM_0a}{R}} \mathbf{d}R \right],$$

where the integrating factor $F(t, R)$ satisfies

$$\partial_R F^{-1} = \partial_t \left[ \frac{F^{-1}HR}{(1 - \frac{2GM_0a}{R})^2 - H^2R^2} \right].$$

Substituting (4) into (3) and replacing $\tilde{t} \to t$, we obtain

$$\mathbf{d}s^2 = -\left[1 - \frac{2GM_0a}{R}\right] - \frac{R^2H^2}{(1 - \frac{2GM_0a}{R})} F^2 \mathbf{d}t^2 + \left[1 - \frac{2GM_0a}{R}\right] - \frac{R^2H^2}{(1 - \frac{2GM_0a}{R})} \mathbf{d}R^2 + R^2 \mathbf{d}\Omega^2.$$

The AHs are the solutions of $h^{ab}\partial_aR\partial_bR = 0$, where the metric $h_{ab}$ is defined by $\mathbf{d}s^2 = h_{ab} \mathbf{d}x^a \mathbf{d}x^b + R^2(x) \mathbf{d}\Omega^2$ and $a, b = 0, 1$, yielding

$$\left(1 - \frac{2GM_H(t)}{R} \equiv RH\right)_{R^a} = 0,$$
where we introduced the Hawking–Hayward quasi-local mass

\[ m_H(t) = M_0 a(t) \]  

A remarkable feature of this quantity is that it is coordinate independent, and consequently is recognized as the physically relevant mass of the BH. Obviously, it is always increasing in an expanding universe [21]. Therefore, the calculation of the change rate of the BH mass will lead to opposite conclusions to that of Babichev et al [19].

Discarding the unphysical branch with the lower sign in (7), the AHs are given by

\[ R_B = \frac{1}{2H} \left( 1 - \sqrt{1 - 8Gm_H(t)} \right), \quad R_C = \frac{1}{2H} \left( 1 + \sqrt{1 - 8Gm_H(t)} \right), \]

where \( R_C \) and \( R_B \) are the cosmological and the BH AH, respectively. Note that the AHs coincide at the extreme time \( t^* \) defined by \( \dot{a}(t^*) = 1/8GM_0 \). This coincidence takes place in a future or past universe depending on the kind of fluid accretion onto the BH.

Let us now consider that the fluid is described by an imperfect fluid with the stress–energy tensor given by

\[ T_{\mu\nu} = (p + \rho)u_{\mu}u_{\nu} + pg_{\mu\nu} + q_{\mu}u_{\nu} + q_{\nu}u_{\mu}, \]

where \( u^\mu = (\frac{1}{H}, 0, 0, 0) \) is the fluid four-velocity and \( q^\mu = (0, q, 0, 0) \) is a spatial vector field describing the radial energy current. Written in terms of the comoving AH, the solutions of the Einstein equations of motion are [21]

\[ H = \frac{8\pi G}{3} R_A \rho, \quad 3H + \frac{2H}{H} = -8\pi G R_A p. \]

(11)

Assuming a radial heat inflow \( (q < 0) \), the change rate of the BH mass is then

\[ \dot{m}_H = G a B^2 A |q|, \]

(12)

where \( A = \int d\theta d\phi \sqrt{g_{\Sigma}} = 4\pi r^2 a^2 A^4 \). This relation shows that the Hawking–Hayward quasi-local mass is always increasing.

### 3. Hawking radiation of the apparent horizon

For the AH to exist in an expanding universe, we have the following condition:

\[ H(t) \leq \frac{1}{8Gm_H(t^*)}. \]

(13)

This condition requires us to consider massive objects for which \( Gm_H H \) is a small quantity. Using the present day value of the Hubble parameter, we estimate the critical BH mass as \( m_H(t^*) \lesssim 10^{23} M_\odot \). This is the condition for which the engulfing of the universe by the BH is prevented [25]. Now solving \( R = ar \left( 1 + \frac{Gm_H}{2r} \right)^2 \) for \( r \), we obtain one physical solution given by

\[ r_A = \frac{1}{2a} \left( R_A - Gm_H + \sqrt{R_A^2 - 2R_A Gm_H} \right). \]

(14)

For the solution to be real, we have to impose the condition

\[ R_A > 2Gm_H, \]

(15)

which is only true in an expanding universe, due to the relation \( R_A(1 - HR_A) = 2Gm_H \). We note that the equality sign has been discarded in (15), since it leads to \( H R_A = 0 \), which does not hold for \( m_H \neq 0 \). Relation (15) means that we are dealing with systems which are small compared to the cosmological curvature, and that distances below 2\( l_p \) are naturally excluded.
Assuming now the EoS, \( p = w \rho \), and substituting (15) into Friedmann equations, one finds the following upper bounds on the densities:

\[
\rho(t) < \frac{3m_{\text{pl}}^6}{128\pi m_H^2(t)} \quad \text{and} \quad |p| < \frac{3|w|m_{\text{pl}}^6}{128\pi m_H^2(t)}.
\]  

(16)

Therefore, in the case of a phantom-energy-driven expansion, the finite increase of densities with time will avoid the big-rip singularity. On the other hand, since the BH mass increases with time, the upper bounds (16) become very small. As an estimate of the energy density we have \( \rho(t) < 0.02 \times 10^{-10} (\text{GeV})^{4} \). For the smallest super-massive BH detected in the dwarf Seyfert 1 galaxy POX 52 with \( m_H \sim 10^{5} M_{\odot} \) [26]. On the other hand, if \( m_H(t_*) \sim M_{\odot} \), we have \( \rho(t) < 0.022 (\text{GeV})^{4} \). Even if we take \( m_H \) of the order of the Planck mass, we have \( \rho(t) < \frac{3m_{\text{pl}}^4}{128\pi} \).

We now consider the Hawking radiation from the AH. According to the generalization of BH thermodynamics to a cosmological framework [27], the temperature on the AH is defined by

\[
T_A = \frac{\kappa_A}{2\pi},
\]

where \( \kappa_A \) is the surface gravity. In dynamical spacetime there is no time-like Killing vector, and the usual definition of the surface gravity may be modified. In this case, the surface gravity is related to the so-called trapping horizon. Here, we follow the work of Hayward [28], where the surface gravity is defined by \( K^b \nabla_a K_b = \kappa K_a \) (\( a, b = 0, 1 \)) with \( K^a = -\epsilon^{ab} \nabla_b R \) being the Kodama vector corresponding to the background given by equation (3). Evaluating all the quantities on the trapping horizon, we get the simplified form,

\[
\kappa_A = \frac{1}{2} \Box h_R | \frac{1}{2} - \frac{\kappa_A}{2}, \quad (17)
\]

Writing \( m_H \) is terms of the AH, the temperature at the AH becomes

\[
T_A = \frac{1}{4\pi R_A} \left| 1 - \frac{R_A}{H} (3H^2 + \dot{H}) \right|, \quad (18)
\]

Expanding (18) to first order in \( Gm_H \) at the BH AH and cosmic AH, we obtain

\[
T_C = \frac{H}{2\pi} \left| 1 + \frac{\dot{H}}{2H^2} \right| - \frac{Gm_H}{2}, \quad (19)
\]

\[
T_B = \frac{1}{8\pi Gm_H} \left( \frac{G\dot{H}^2 m_H}{2\pi} - \frac{4H^2 + \dot{H}}{4\pi H} \right), \quad (20)
\]

respectively. Using (15) in the first factor, along with \( R_C \leq 1/H \) in the second factor, and \( \frac{\dot{H}}{H} = \frac{3}{2}(1 + w) \), one obtains an instantaneous upper bound for temperature at the cosmic AH:

\[
T_C < \frac{|1 - 3w| m_{\text{pl}}^2}{16\pi m_H(t)}.
\]  

(21)

Relation (19) shows that the accretion process tends to lower the Hawking temperature at the cosmic AH. We note that in (20) we have not used the absolute value, as a consequence of the equivalence principle [29]. A similar behavior is also observed if one anticipates and uses the scale factor (60) derived in section 4, for a phantom-dominated era. In fact, one can see an unusual behavior of the temperature when approaching the time \( t_* \). As is shown in figure 1, the temperature at the cosmic AH increases with time, reaches a maximum, then begins to decrease, and stops at the end time \( t_* \). This means that the Hawking radiation at the cosmological AH will decrease rapidly in favor of a huge increasing of the accretion of the
phantom fluid. On the other hand, the temperature of the BH AH starts to fall at an early stage because of the increasing of the BH mass by accretion of phantom energy, and at a late stage begins increasing. However, the latter strange behavior is mainly due to the absolute value in the definition of temperature.

Independent derivations of (16) and (21) can be performed by using the following simple arguments. Imposing positivity of the \(-g_{tt}\) component of the metric and replacing the quasi-local mass by a density energy, we have \(0 < -g_{tt} = 1 - 2Gm_H/R - R^{2}H^2/R = 1 - (8\pi G/3)\rho R^2 - \frac{R^{2}H^2}{16\pi Gm_H R}.\) Hence, \(\rho < (1 - RH) / (8\pi G R^2 / 3).\) Using \(R > 2Gm_H,\) one obtains \(\rho < 3m^6_{\text{pl}} / (32\pi m^2_H),\) which is consistent with (16). If one uses the Stefan–Boltzmann law \(\rho = \sigma T^4\) and the quantum mechanical relation \(R \geq 1/T [30],\) one finds \(T < (3/4\pi \sigma)m_{\text{pl}},\) which is of the same order of magnitude as (21) for \(m_H = m_{\text{pl}}.\) Finally, we point that the extremal case corresponding to \(T_C = 0\) never occurs, since in this case we must have \(R_C = \frac{1}{120\pi^2 m_{\text{pl}}^2},\) which contradicts the condition \(R_C \geq 1/(2H).\)

Now, neglecting the accretion of radiation in a phantom-energy-dominated era, and taking into account only the semi-classical Hawking evaporation and the phantom-energy accretion term, the differential equation for the BH mass reads

\[
\frac{dm_H}{dt} = -4\pi R^2 H \sigma T^4_H + m_H H, \tag{22}
\]

where \(\sigma = N\pi^2 / 120\) is the Stefan–Boltzmann constant for massless fields with the effective degree of freedom \(N.\) Substituting the Hawking temperature associated with the BH AH, we obtain

\[
\frac{dm_H}{dt} = -\frac{\sigma H^2|1 - \frac{3}{2}(1 - \sqrt{1 - 8Gm_H H})(1 - w)|^4}{16\pi^3 (1 - \sqrt{1 - 8Gm_H H})^2} + m_H H. \tag{23}
\]

This is a complicated relation for \(m_H,\) whose behavior is shown in figure 2. Since the two terms are in competition, there exist a transition time, the \textit{phantom time}, after which the accretion...
process dominates and the BH mass increases. Consequently, the BH does not lose but gain the mass due to huge accretion of dark energy. Indeed, it is easy to show that the maximal rate gain mass for $m_H \neq 0$ is $(d m_H / dt)_{\text{max}} \sim m_{\text{pl}}^2$. It is important to note that the accretion term becomes predominant at earlier times for a massive BH. Next, we perform the same analysis on the variation of the mass inside the cosmic AH. Let us define the ratio between the radiation and the accretion term:

$$\eta_C(t) = \left| \frac{\dot{m}_{\text{Haw}}}{\dot{m}_{\text{ph}}} \right| = \frac{4 \pi R_C^2 \sigma T_C^4}{m_H H}. \quad (24)$$

Substituting (18) and repeating the procedure leading to (21), one finds

$$\eta_C(t) < N \left| 1 - 3 w \right|^4 \frac{m_{\text{pl}}^2}{245760 \pi} \left( \frac{m_{\text{pl}}}{m_H} \right)^2. \quad (25)$$

This result clearly shows that the Hawking radiation at the cosmic AH is insignificant, even near the Planck scale.

Finally, let us point out another crucial feature of the model of a dynamical BH considered in this paper. If the expansion of the Universe is driven by the phantom fluid, the BH AH expands while the cosmic AH shrinks as the Universe expands until they meet at $R_{\text{crit}} = 1/(2H)$ at time $t_*$ solution of (13). At times $t > t_*$, both the AHs disappear leaving a proper singularity, well before the big-rip singularity is reached. The question if this singularity is naked or located inside the AH, and its connection with the violation of the cosmic censorship conjecture (CCH), is still under debate [31, 32]. To avoid discussing these topics, which are out of the scope of this paper, and the fact that the phantom-driven expansion of the Universe is non-singular, and that the radiation power can be neglected in comparison to the accretion of the phantom fluid onto the BH, particularly at a later stage, we limit the analysis in the remaining sections to the interval $t \leq t_*$, far from the big-rip singularity, and then a purely classical treatment will be considered.
4. Thermodynamics with varying $w$

We now consider the thermodynamic properties of the solution described in section 2, with a variable EoS parameter, $p(a) = w(a)\rho(a)$. The particle fluid and entropy fluid currents, $N^\mu$ and $S^\mu$, are given by

$$N^\mu = nu^\mu, \quad S^\mu = su^\mu,$$

where $n$ and $s$ are the densities of the particle number and entropy, respectively. The conservations laws, $T^\mu_\nu = 0$, $N^\alpha_\mu = 0$ and $S^\alpha_\mu = 0$, computed on the background given by (2), give the following set of differential equations:

$$\dot{\rho} + \frac{\dot{R}_A}{R_A} \rho + \frac{3}{2} H (\rho + p) = 0,$$

$$\dot{n} + \left( \frac{\rho}{\rho + p} \frac{\dot{R}_A}{R_A} + \frac{3}{2} H \right) n = 0,$$

$$\dot{s} + \left( \frac{\rho}{\rho + p} \frac{\dot{R}_A}{R_A} + \frac{3}{2} H \right) s = 0.$$

The solutions of the above equations are

$$\rho(a) = \rho_0 \left[ \frac{R_A(a_0)}{R_A(a)} \right] \left[ \frac{\dot{a}_0^{(1+w_0)}}{a^{(1+w(a))}} \right] \exp \left[ \frac{3}{2} \int_{a_0}^a \dot{a} w'(a) \ln a \right],$$

$$n(a) = n_0 \left( \frac{\dot{a}_0}{a} \right)^2 \exp \left[ \int_{a_0}^a \frac{R'(a)}{1+w(a)R(a)} \right],$$

$$s(a) = s_0 \left( \frac{\dot{a}_0}{a} \right)^2 \exp \left[ \int_{a_0}^a \frac{R'(a)}{1+w(a)R(a)} \right],$$

where the prime stands for derivative with respect to the scale factor, and $\rho_0$, $n_0$ and $s_0$ are the present day values of the corresponding quantities assumed to be positive definite. Here, we note the important corrections due to the presence of the BH. The corresponding relations in the pure FRW universe are easily obtained by setting $M_0 = 0$ in equations (29)–(31) [33].

Now, assuming that $\rho = \rho(T, n)$, $p = p(T, n)$ and using the Gibbs law

$$T \left( \frac{\partial p}{\partial T} \right)_n = p + \rho - n \left( \frac{\partial \rho}{\partial n} \right)_T,$$

we obtain

$$\left[ \frac{3}{2} H + \frac{1}{(1+w(a))} \frac{\dot{R}_A}{R_A} \right] T(a) w(a) + \dot{T}(a) \left( \frac{\partial \rho}{\partial a} \right)_n = - \left[ \frac{3}{2} H + \frac{1}{(1+w(a))} \frac{\dot{R}_A}{R_A} \right] T(a) \rho(a) w'(a).$$

Calculating $\left( \frac{\partial \rho}{\partial a} \right)_n$ from equation (29), we get the equation governing the evolution of temperature

$$w'(a)T(a) = \left[ \frac{3}{2a} (1+w(a)) + \frac{R'_A}{R_A} \right] T(a) w(a) = (1+w(a))T'(a).$$
Solving for \( w(a) \neq -1 \), we obtain

\[
T(a) = T_0 \left( \frac{w(a) + 1}{w_0 + 1} \right)^{a_0\sqrt{w_0}/a_0} \exp \left[ \int_{a_0}^a \frac{3}{2} w'(a) \ln a - \frac{w(a)}{1 + w(a)} \frac{R'(a)}{R_A(a)} \right].
\] (35)

Using this expression, we write the energy density as a function of temperature

\[
\rho(a) = \rho_0 \left( \frac{T(a)}{T_0} \right)^{ \frac{3}{2} \frac{a_0}{a_0 - w_0} } \exp \left[ \frac{3}{2} \frac{R'(a)}{R_A(a)} \int_{a_0}^a \frac{da}{1 + w(a)} \right].
\] (36)

Extracting the scale factor from (35), we obtain the generalized Stefan–Boltzmann law

\[
\rho(t) = \rho_0 \left( \frac{T(t)}{T_0} \right)^{ \frac{3}{2} \frac{a_0}{a_0 - w_0} } \exp \left[ \frac{3}{2} \frac{R'(a)}{R_A(a)} \int_{a_0}^t \frac{da}{1 + w(a)} \right].
\] (37)

Let us now scrutinize the behavior of the temperature when \( w(a) \) crosses \( -1 \). In this case, equation (34) becomes

\[
\left[ w' + \frac{R'(a)}{R(a)} \right]_{\text{PDL}} T(a) = 0.
\] (38)

The solution of (38) is

\[
R_{\text{PDL}}(a) = C e^{-w'|_{\text{PDL}}(a-a_0)}, \quad T(a) \neq 0, \quad \text{or} \quad T(a) = 0.
\] (39)

where \( w'|_{\text{PDL}} \) is the value of \( w(a) \) at the PDL and \( C \) is the constant of integration. Here, we note that unlike the standard vacuum solution with \( T = 0 \) [33], the solution considered in this paper allows for a vacuum solution with a non-zero temperature. This was expected, since this temperature is associated with the AH of the BH in the absence of a dark field. The zero temperature vacuum state is recovered by setting \( M_0 = 0 \). On the other hand, it is important to observe that the expressions of temperature and energy density are regular everywhere including the phantom divide crossing. This can be easily verified by substituting the solution of the AH at the PDL, \( R(a) \approx \rho_{\text{PDL}} \) into equations (35), (36):

\[
T_{\text{PDL}}(a) = T_0 a_0^{a_0\sqrt{w_0}/a_0} 3^{3/2} \exp \left[ -1 - w_0 + \frac{3}{2} w'|_{\text{PDL}} \int_{a_0}^a da \ln a \right].
\] (40)

and

\[
\rho_{\text{PDL}}(a) = \rho_0 a_0^{3(w_0+1)/2} \exp \left[ -1 - w_0 + \frac{3}{2} w'|_{\text{PDL}} \int_{a_0}^a da \ln a \right].
\] (41)

Obviously, equation (40) shows that the temperature remains positive when \( w(a) \rightarrow -1^\pm \).

Now, making the following ansatz for the AH:

\[
R_A(a) = C e^{-w'|_{\text{PDL}}(a-a_0)} f_A(a),
\] (42)

with \( f_A(a) \rightarrow -1^\pm \), we rewrite the temperature as

\[
T(a) = T_{\text{PDL}} \left( \frac{w(a) + 1}{w(a) + 1}_{\text{PDL}} \right)^{a_0\sqrt{w_0}/a_0} \exp \left[ \frac{3}{2} w'(a) \int_{a_0}^a da \ln a + \int_{a_0}^a da \left( \frac{3}{2} w'(a) \ln a - \frac{w(a) f'_A(a)}{(1 + w(a)) f_A(a)} \right) \right].
\] (43)
which shows that the temperature is always positive definite regardless of the value of \( w(a) \). When \( M_0 = 0 \), the behavior of temperature changes: it is positive for \( w(a) > 1 \), negative for \( w(a) < -1 \) and zero for \( w(a) = -1 \) [33].

Now, we calculate the chemical potential defined by the Euler relation [34]

\[
\mu(a) = \frac{(w(a) + 1)\rho(a) - T(a)s(a)}{n(a)}. \tag{44}
\]

Using the expressions of \( \rho(a), n(a), s(a) \) and \( T(a) \), we obtain

\[
\mu(a) = \mu_0 \left[ \frac{w(a) + 1}{w_0 + 1} \right] \left[ \frac{a_0^{3w/2}}{a^{3w_0/2}} \right] \exp \left\{ \int_{a_0}^{a} da \left[ \frac{3}{2} w'(a) \ln a - \frac{w(a)}{(1 + w(a))} \frac{R'_A(a)}{R_A(a)} \right] \right\}, \tag{45}
\]

where \( \mu_0 \) is the present day chemical potential. We note that in general the sign of \( \mu_0 \) can be arbitrary, and consequently the sign of \( \mu(a) \).

The entropy of the Universe can be derived from (44) and is given by

\[
s(a) = \frac{(w(a) + 1)\rho(a) - \mu(a)n(a)}{T(a)}. \tag{46}
\]

Using again the relations for \( \rho(a), T(a), \mu(a) \) and \( n(a) \), we obtain

\[
s(a) = s_0 a^{3/2} \exp \left[ \int_{a_0}^{a} a \left[ \frac{3}{2} w'(a) \ln a - \frac{w(a)}{(1 + w(a))} \frac{R'_A(a)}{R_A(a)} \right] \right], \tag{47}
\]

where \( s_0 \) is the present day entropy given by

\[
s_0 = \left[ \frac{(w_0 + 1)\rho_0}{T_0} - \frac{\mu_0 n_0}{T_0} \right]. \tag{48}
\]

Now, defining the comoving volume by

\[
V(a) = a^{3/2} \exp \left[ \int_{a_0}^{a} a \left[ \frac{3}{2} w'(a) \ln a - \frac{w(a)}{(1 + w(a))} \frac{R'_A(a)}{R_A(a)} \right] \right], \tag{49}
\]

we derive from equation (47) the usual entropy conservation law

\[
s(a)V(a) = s_0 V_0. \tag{50}
\]

As we did for the energy density, we write the entropy in terms of temperature as

\[
s(T) = s_0 \left[ \frac{T(a)}{T_0} \frac{w_0 + 1}{w(a) + 1} \right]^{3/2} \left[ \frac{\frac{3}{2} \left( \frac{w(a)}{w_0} \right)}{\frac{3}{2} w'(a) \ln a} \right] \exp \left[ -\frac{3}{2} w(a) \int_{a_0}^{a} \frac{da w'(a) \ln a}{w(a)} \right] \times \exp \left[ \frac{1}{2} \int_{a_0}^{a} da \left( \frac{w(a)}{1 + w(a)} - \int_{a_0}^{a} \frac{da}{1 + w(a)} \right) \right] \frac{R'_A(a)}{R_A(a)} \right]. \tag{51}
\]

Let us now consider the expressions of the chemical potential and entropy at the phantom divide crossing. We easily show that

\[
\mu_{\text{PDL}}(a) = \mu_0 a_0^{-3w_0/2} a^{3/2} \exp \left\{ -1 - w_0 + \frac{3}{2} w'|_{\text{PDL}} \int_{a_0}^{a} da \ln a \right\}, \tag{52}
\]

and

\[
s_{\text{PDL}}(a) = s_0 a_0^{-3/2} \exp \left\{ -1 - w_0 + \frac{3}{2} w'|_{\text{PDL}} \int_{a_0}^{a} da \ln a \right\}. \tag{53}
\]

Note that using relations (40), (52) and (53), we can easily verify that the Euler relation remains valid at the PDL.
Finally, let us list the corresponding relations for the energy density, temperature, entropy and chemical potential in the case \( w(a) = w = \text{const} \):

\[
T(a) = T_0 \left( \frac{a_0}{a} \right)^{3w/2} \left( \frac{R_A(a_0)}{R_A(a)} \right)^{\frac{1}{1+w}},
\]

(54)

\[
\rho(a) = \rho_0 \left( \frac{a_0}{a} \right)^{2(1+w)} \left( \frac{R_A(a_0)}{R_A(a)} \right),
\]

(55)

\[
\mu(a) = \mu_0 \left( \frac{a_0}{a} \right)^{3w/2} \left( \frac{R_A(a_0)}{R_A(a)} \right)^{\frac{1}{1+w}},
\]

(56)

\[
s(a) = s_0 \left( \frac{a_0}{a} \right)^{3/2} \left( \frac{R_A(a_0)}{R_A(a)} \right)^{\frac{1}{1+w}},
\]

(57)

\[
\rho(t) = \rho_0 \left( \frac{T(a)}{T_0} \right)^{\frac{1}{1+w}}, \quad \mu(t) = \mu_0 \left( \frac{T(a)}{T_0} \right),
\]

(58)

\[
s(T) = s_0 \left( \frac{T(a)}{T_0} \right)^{\frac{3}{2}} \left( \frac{\rho}{\rho_0} \right)^{\frac{1}{1+w}}.
\]

(59)

We note that the dependence of the thermodynamic parameters on the temperature is of the same form as in standard thermodynamics of a pure expanding FRW universe.

Before ending this section, let us reconsider explicitly the avoidance of the big-rip singularity in a phantom-dominated universe. Using equations (26) and (55), we obtain

\[
a(t) = a_0 \left[ \frac{1}{2} H_0(1+w)(t-t_s) \right]^{\frac{1}{1+w}},
\]

(60)

\[
H(t) = \frac{2}{3(1+w)(t-t_s)},
\]

(61)

where the big-rip time is

\[
t_s = t_0 - \frac{2}{3H_0(1+w)}.
\]

(62)

Using the constraint (13) on \( H(t) \), in a phantom-dominated era, one finds

\[
(t_s - t) \geq \frac{16Gm_H(t)}{3(1+w)}.
\]

(63)

Then using the FRW equations and (15), we find

\[
\rho < \frac{(t-t_s)^{-1}}{8\pi G^2 m_H(1+w)}, \quad |p| < \frac{w(t_s - t)^{-1}}{8\pi G^2 m_H(1+w)}.
\]

(64)

Substituting (63) into these relations, we reproduce relations (16). Hence, the Universe evolves toward a state where \( a, \rho, p \) and higher derivatives are finite, without ever reaching the big-rip singularity. This final state is reached in a finite time given by

\[
t_e = t_s + \frac{2[8G H_0 M_{(a_0)}]^{\frac{1}{1+w}}}{3H_0(1+w)}.
\]

(65)

This behavior of the Universe is similar to that observed in the framework of a generalized uncertainty principle (GUP) corrected FRW universe [36].
5. Quantum gravity effects and stability of the solution

The relation expressing the Hubble rate shows that curvature increases for times approaching the critical time \( t_* \), and as a consequence quantum effects are expected to be dominant for a small time interval near \( t_* \). It is like the models plagued by the big-rip singularity, where quantum effects dominate only for a small time interval near singularity [35].

Here, the quantum effects are taken into account by including the contributions from the conformal anomaly. In general, the conformal anomaly is given by [37]

\[
T = b \left( F + \frac{1}{3} \Box R \right) + b' G + b'' \Box R, \tag{66}
\]

where \( F = C_{\mu\nu\lambda\kappa} C^{\mu\nu\lambda\kappa} = \frac{1}{3} R^2 - 2 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa} \) is the square of the Weyl tensor and \( G \) is the Gauss–Bonnet invariant, \( G = R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa} \), and the coefficients \( b \) and \( b' \) are

\[
b = \frac{1}{120(4\pi)^2} (n_0 + 6n_{1/2} + 12n_1), \tag{67}
\]

\[
b' = \frac{1}{360(4\pi)^2} (n_0 + 11n_{1/2} + 62n_1), \tag{68}
\]

where \( n_0, n_{1/2} \) and \( n_1 \) are the number of scalar, Dirac fermion and vector fields, respectively.

A long but straightforward calculation gives

\[
F = \frac{12 G^2 m_H^2}{R_A^6}, \quad \Box R = -\frac{6}{R_A^2 H^2} (\dot{H} + 7H \ddot{H} + 12H H^2 + 4\dot{H}^2), \tag{69}
\]

\[
G = \frac{20 G^2 m_H^2}{R_A^6} + \frac{24}{R_A^2 H^2} (H^4 + H^2 \dot{H}). \tag{70}
\]

Substituting into (66), one obtains

\[
T_a = 12 \left( b + \frac{5}{3} b' \right) \frac{G^2 m_H^2}{R_A^6} + \frac{24b'}{R_A^2 H^2} (H^4 + H^2 \dot{H}) - \frac{(\frac{1}{3} b + b'')}{R_A^2 H^2} (6H + 42H \ddot{H} + 72H H^2 + 24\dot{H}^2).
\]

Now we assume that

\[
T_a = -\rho_a + 3 p_a, \tag{71}
\]

where \( \rho_a \) and \( p_a \) verify the conservation law given by equation (26):

\[
\dot{\rho}_a + \frac{R_A}{R_A} \rho_a + \frac{3}{2} H (\rho_a + p_a) = 0. \tag{72}
\]

Using (71), the solution to this equation is

\[
\rho_a = -\frac{1}{2 R_A a^2} \int dt (H R_A a^2 T_a). \tag{73}
\]

Since the exact calculation of \( \rho_a \) is impossible, we consider the situation where the Hawking–Hayward quasi-local mass is small (see condition (13)). In this case only the cosmic AH survives, \( R_C \approx 1/H - 2 G m_H H \), and the conformal anomaly to leading order in \( G m_H H \) reduces to

\[
T_A = \bar{T}_A (1 + 4 G m_H H), \tag{74}
\]
where $\tilde{T}_A$ is the conformal anomaly in the pure FRW universe \[38\]
\[
\tilde{T}_a = 24b'(H^4 + H^2\dot{H}) - \left(\frac{2}{3}b + b''\right)(6H + 42H\dot{H} + 24H^2 + 72H^2\ddot{H}).
\] (75)

In this case, we can write (73) as
\[
\rho_a = \left(1 + 2GmH\right)\left[-\frac{H}{a^2} \int dt a^2\tilde{T}_a\right] - \frac{2GmH}{a^3} \int dt a^3\tilde{T}_aH.
\] (76)

The term in square brackets is exactly the quantum-induced energy density in the pure FRW universe. Then, using the results in \[38\], one obtains
\[
\rho_a = \left[-6b'H^4 - \left(\frac{2}{3}b + b''\right)(3\dot{H} - 6H - 18H^2\ddot{H})\right]\left(1 + 2GmH\right)
- \frac{2GmH}{a^3} \int dt a^3H\left[24b'(H^4 + H^2\dot{H})
- \left(\frac{2}{3}b + b''\right)(6\ddot{H} + 42\dot{H}H + 24H^2 + 72H^2\ddot{H})\right].
\] (77)

\[
p_A = b'(6H^4 + 8H^2\dot{H}^2) - \left(\frac{2}{3}b + b''\right)(2H + 12\dot{H}H + 9\ddot{H} + 18H^2\ddot{H})
+ \frac{2GmH}{3a^3} \int dt a^3H\left[24b'(H^4 + H^2\dot{H})
- \left(\frac{2}{3}b + b''\right)(6\ddot{H} + 42\dot{H}H + 24H^2 + 72H^2\ddot{H})\right].
\] (78)

In the following, we focus on the regime of cosmic dynamics where the Universe undergoes a phase of quasi-exponential expansion, such that $\dot{H}/\dot{H}^2 \ll 1$, and examine the stability of the solutions obtained at the crossing of the PDL when quantum effects are taken into account. In the quasi-exponential expansion, we can write $p_a \sim -\rho_a$, where
\[
\rho_a \sim -6b'H^4 - 28b'GmH^3.
\] (79)

On the other hand, the fluid pressure and energy density can be approximated by
\[
\rho_f \sim -p_f \sim \frac{3H^2}{8\pi G} + \frac{3mH^3}{4\pi G}.
\] (80)

Taking the present day values of the Hubble parameter, $H_0 \sim 10^{-33}$ eV, and $G \sim 10^{-56}$ (eV)$^{-2}$, it is easy to verify that the quantum correction is very small when crossing the PDL. Even we consider BH masses at the end of the expansion of the order of $\sim 10^{33}$ $M_\odot$, the correction term to the standard result is insignificant. Hence, the solutions given in section 4 are stable under the conformal anomaly quantum corrections.

Finally, let us study the validity of the energy conditions by taking into account the quantum effects induced by the conformal anomaly. The different types of energy conditions in cosmology are

Null energy condition (NEC) \[\iff\] $\rho + p \geq 0$,
Weak energy condition (WEC) \[\iff\] $\rho \geq 0$, $\rho + p \geq 0$,
Strong energy condition (SEC) \[\iff\] $\rho + p \geq 0$, $\rho + 3p \geq 0$,
Dominant energy condition (DEC) \[\iff\] $\rho \geq 0$, $\rho \pm p \geq 0$.

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Figure 3. The evolution of energy conditions for $w = -1.5$, and $M_0 = 0.1$. The quantum gravity coefficients are $b = 0.5$, $b' = -0.1$ and $b'' = 0$, and the big-rip time is $t_r \approx 1.33$.

In the pure FRW phantom cosmology, we have violation of all the energy conditions as can be easily deduced from equations (11). Using $a(t)$ and $H(t)$, given by equations (60) and (61), we expect a drastic change, in the behavior of matter, with the same geometry on replacing $\rho$ by $\rho + \rho_a$ and $p$ by $p + p_a$ in Friedmann equations. Then, the energy conditions are now replaced by

$$\rho + p = -\frac{1}{R_c H} \left( \frac{H}{4\pi G} + \rho_a + p_a \right), \quad \rho - p = -\frac{1}{R_c H} \left( \frac{H + 3H^2}{4\pi G} - \rho_a + p_a \right)$$

(82)

and

$$\rho + 3p = -\frac{1}{R_c H} \left( \frac{3}{4\pi G} \left(H + H^2\right) - \rho_a - 3p_a \right).$$

(83)

It is easy to show that the quantum induced $\rho_a$ and $p_a$ are significantly dominant compared to the classical energy density $\rho$ and pressure $p$. We have numerically integrated equations (73) and (66), and the resulting behavior of the energy conditions is shown in figure 3. It turns out that $\rho > 0$, $\rho + p > 0$, $\rho + 3p > 0$ and $\rho - p < 0$. Then, it is interesting to note that due to quantum effects the energy conditions are satisfied excepting the DEC, in the case of a BH immersed in a phantom-energy-dominated FRW universe.

6. Accretion of a phantom fluid and constraints on the GSL

We now consider the problem of the validity of the GSL when the effect of the backreaction effect of the phantom fluid on the BH is taken into account. We restrict our study to the scenario of a BH with a small quasi-local mass immersed in a phantom fluid-dominated era. In this case only the cosmological AH contributes, and the total entropy consists essentially
Figure 4. Variation of the critical Hawking–Hayward quasi-local mass with respect to $w$ for different values of the parameter $\alpha_0$.

of the sum of entropy of the cosmological AH and the entropy of the phantom fluid in thermal equilibrium with the cosmological AH. Indeed, using (59), we have

$$S = \left[ \frac{\pi R_C^2}{G} + s_0 V \left( \frac{\rho}{\rho_0} \right)^{\frac{1}{1+w}} \right].$$  

(84)

The first term is the entropy of the cosmological AH and the second term is the phantom fluid entropy inside a comoving volume $V$. Taking the derivative with respect to time and using the following approximation:

$$R_C \approx \frac{1}{H} = \frac{2Gm_H}{H},$$  

(85)

along with the relations $\dot{m}_H = m_0 \dot{a}/a$, $\ddot{a} = a(H^2 + \dot{H})$, we obtain to leading order in the Hawking–Hayward quasi-local mass

$$\dot{S} = -\frac{2\pi}{GH} \left[ \frac{H}{H^2} + 2Gm_H \left( 1 - \frac{H}{H^2} \right) \right]$$

$$+ \frac{s_0 V H}{(1+w)} \left( \frac{T}{T_0} \right)^{\frac{1}{1+w}} \left[ \frac{H}{H^2} + 2Gm_H \left( 1 + \frac{H}{H^2} \right) - \frac{3}{2} (1+w) \right],$$  

(86)

where we have used the continuity equations (26) and (59). In order to satisfy the GSL, $\dot{S} \geq 0$, the quantity inside the square brackets must be positive definite, yielding to the condition

$$2\dot{m}_H \geq -\frac{2\pi}{GH} \left( \frac{s_0 V H}{(1+w)} \right)^{\frac{1}{1+w}} \left( \frac{H}{H^2} \right) \left[ \frac{H}{H^2} + 2Gm_H \left( 1 + \frac{H}{H^2} \right) - \frac{3}{2} (1+w) \right].$$  

(87)

Now substituting $\frac{H}{H^2} = -\frac{3}{2} (1+w)$, we rewrite the constraint on the GSL as

$$Gm_H \geq -3 (1+w) \left( \frac{A - \frac{1+w}{A}}{A + \frac{3(1+w)(1+3w)}{1+3w}} \right),$$  

(88)

with $A = \frac{s_0 V H^2 G}{(T_0)^2}$. 

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We know from equation (12) that the Hawking–Hayward quasi-local mass is an increasing function of time in an expanding universe, \( m_H \geq 0 \). Then, imposing the positiveness of the rhs of equation (88) and assuming the positiveness of the entropy, we have one solution given by

\[
- \frac{(5 + 3w)(1 + w)}{(1 + 3w)} > A \geq 0,
\]

which is satisfied for an EoS parameter in the range \( w \leq -5/3 \). Using the expression of \( s_0 \) given in (48) and defining \( \alpha_0 = -\mu_0 \rho_0 / \rho_0 \), an immediate consequence of the positiveness of the entropy is that \( w \geq -1 - \alpha_0 \). Since \( w \leq -5/3 \), we get a lower bound estimate to the present day value of the chemical potential,

\[
\alpha_0 \geq \frac{7}{3},
\]

which show that the chemical potential in a phantom-energy-dominated era is strictly negative. Now, with \( w = -2 \) and the present day parameters, we obtain the following bound on the entropy:

\[
s_0 < 3.5 \times 10^{-4} \text{GeV}^{-3}.
\]

Or in terms of \( \alpha_0 \) and using (90), we have

\[
\frac{7}{3} \leq \alpha_0 < 1 + 1.26 \times 10^{23} T_0.
\]

Now, if we take \( T_0 \approx 10^{-19} \text{GeV} \), we have \( \alpha_0 \lesssim 10^{24} \). Here, we would like to point that the bounds on the present day values of the entropy and the parameter \( \alpha_0 \) are of the same order of that obtained by Lima et al [39]. However, the calculation in this paper is performed by ignoring the backreaction of the phantom fluid on the BH. In fact, they used the Schwarzschild metric, which cannot describe the properties of BHs embedded in an expanding FRW universe. As we explicitly show, taking into account the backreaction effect, even in a low density background, leads to a drastic constraint on the EoS parameter, and allows for a thermodynamically stable phantom regime with positive entropy and temperature, and a negative chemical potential. Our analysis meet the previous analysis performed in [17] and completely rules out the approach based on the zero chemical potential [18].

Now, using the obvious relation \( m_H = m_H / H \), we obtain the quasi-local BH mass above which the accretion of the phantom fluid onto the BH is permitted:

\[
m_H \geq - \frac{3}{GH} \frac{(1 + w)}{(1 + 3w)} \left[ \frac{A - \frac{1+w}{T}}{A + \frac{5+3w}{1+3w}} \right].
\]

Using the constraint on the GSL (89), we obtain

\[
m_H \gtrsim m_{H, \text{crit}} = - \frac{6}{GH} \frac{A - \frac{1+w}{T}}{(5 + 3w)}.
\]

In terms of \( \alpha_0 \) and the present day quantities, the critical quasi-local mass is

\[
m_{H, \text{crit}} = -42.27 \times 10^{-20} \left[ \frac{1 + w + \alpha_0}{5 + 3w} \right] \frac{M_\odot}{T_0}.
\]

For large values of \( \alpha_0 \approx 10^{24} \) and \( T_0 \approx 10^{-19} \text{GeV} \), the critical quasi-local mass is approximately \( m_{H, \text{crit}} \approx 10^{23} M_\odot \). This is a huge value, allowing all BHs in the universe to accrete the phantom fluid. We note that in section 3, we have shown that \( m_{H, \text{crit}} \) is the maximal allowed mass which prevents the engulfing of the Universe by the BH. On the other hand, small values of \( \alpha_0 \) give the critical BH masses of the order of the solar mass. For instance, taking \( \alpha_0 = 2, w = -2 \) and \( T_0 \approx 10^{-19} \text{GeV} \), we obtain \( m_{H, \text{crit}} \approx 4.2 M_\odot \). Our
results are similar to that obtained in [39], where the EoS parameter is restricted to values less than $-1$ and the backreaction effect ignored. We have plotted, in figure 4, the present day critical quasi-local mass with respect to $w$ for different values of the parameter $\alpha_0$. Obviously, the value of the parameter $\alpha_0$ is very important in determining a critical quasi-local mass of the order of the solar mass. Finally, if we set the chemical potential to zero, the critical mass reduces to

$$m_{\mu,\text{crit}} = -\frac{3}{2\pi} \frac{(1 + w)}{(5 + 3w)} \frac{\rho_0 V H}{T_0} \left(\frac{T}{T_0}\right)^\frac{1}{w},$$

which is negative for $w < -5/3$. This result is expected from relation (90).

7. Conclusion

In summary, we have investigated the thermodynamic properties of BHs immersed in an expanding spatially flat FRW universe, for a general EoS parameter $w(a)$. Particularly, we found that the temperature of the dark fluid is always positive regardless of the value of the time-varying EoS parameter, and that the instantaneous vacuum state is characterized by a non-zero temperature, entropy and a chemical potential, respectively. Another important result is that all the thermodynamic parameters are regular at the phantom divide crossing. We have also analyzed the accretion process of the phantom fluid onto BHs with a small Hawking–Hayward quasi-local mass, and the constraints on the validity of the GSL. Particularly, assuming the positiveness of the phantom fluid entropy, we have shown that the phantom fluid may have a negative definite chemical potential in order to satisfy the GSL. We have also obtained, for an EoS parameter within the interval $w < -5/3$, a critical quasi-local mass of the BH, above which the GSL is always protected. The present analysis show that taking into account the backreaction effect of the phantom fluid on the BH, even in a low-density background, leads naturally to positive temperature and a negative chemical potential, and may contribute to resolve the controversy on the subject [17, 18, 39].

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