A Finsler geometric model for membranes on triangulated surfaces

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Abstract. In this paper, we study a new surface model for anisotropic membranes. The model is characterized by an interaction between the surface and a three-dimensional tilt order, of which projected components on the surface are recognized as a vector filed. This vector field defines a Finsler metric on the surface, and consequently the length unit on the surface becomes dependent on the position and direction. The induced interaction dynamically alters the surface tension and the bending rigidity. We numerically find that the tilts form the Kosterlitz-Thouless and low temperature configurations, which correspond to two different anisotropic phases such as disk and tubular on both of the connection-fixed and dynamically triangulated surfaces. The internal tilt order plays an important role for those anisotropic configurations.

1. Introduction
To construct a surface model, we usually assume Helfrich-Polyakov (HP) Hamiltonian, which is composed of the bond potential $S_1$ and a curvature energy $S_2$ [1, 2]. Since the HP Hamiltonian is symmetric under three-dimensional rotations, the symmetric shape transformations including the crumpling transition have long been studied in the framework of HP theory [3, 4]. To the contrary, there is a lot of rotationally non-symmetric phases in membranes, such as tubular and disk phases. The prolate-oblate transition is a shape transformation between two different non-symmetric phases [6]. Those non-symmetric states have been studied with a model called the area-difference bilayer one [7]. An external flow field can also be an origin of non-symmetric surface states [8]. Hexatic membrane model is also for an interaction between the surface and internal tilts order [9]. However, the geometric origin of the non-symmetric states still remains to be studied. In this paper, we show that a Finsler geometric (FG) surface model has three different phases including the disk and tubular phases at high bending region. The FG model is simply obtained by replacing the Riemannian metric with a Finsler metric in the HP model [10].

2. Model
The model is defined by a mapping $X : M \to \mathbb{R}^3$ from a two-dimensional surface $M$ to $\mathbb{R}^3$ (Fig. 1(a)). A smooth triangulation of $M$ corresponds to a linear triangulation in $\mathbb{R}^3$. The triangulated surface in $\mathbb{R}^3$ is obtained by splitting the faces of icosahedron, and it is characterized by the three numbers $(N, N_B, N_T)$, which are the total number of vertices, the total number of bonds, and the total number of triangles.
Figure 1. (a) A mapping $X : M \to \mathbb{R}^3$, (b) a triangle 123 and the three neighboring triangles in $\mathbb{R}^3$, and (c) a vector $v_{ij}$ along $t_{ij}$ is defined by the tilt $\sigma_i$ such that $v_{ij} = 1/c$ (or $v_{ij} = 1$) if $|\sigma_i \cdot t_{ij}| \geq \cos \theta_0$ (or $|\sigma_i \cdot t_{ij}| < \cos \theta_0$).

The Hamiltonian is given by

$$S(X, \sigma) = \lambda S_0 + S_1 + b S_2,$$

where $S_0$ is an energy for the directors (or tilts) $\sigma_i(\in \mathbb{S}^2)$. The tilt is defined at the vertex $i$ and represents some external degrees of freedom. $S_1$ the Gaussian bond potential, and $S_2$ the bending energy. These energies are defined as follows:

$$S_0 = \sum_{ij} (1 - \sigma_i \cdot \sigma_j), \quad S_1 = \sum_\Delta S_{1\Delta}, \quad S_2 = \sum_\Delta S_{2\Delta}.$$

$$S_{1\Delta} = \frac{1}{6} \left( a_1 \ell_1^2 + a_2 \ell_2^2 + a_3 \ell_3^2 \right),$$

$$S_{2\Delta} = \frac{1}{6} \left[ b_1 (1 - n_0 \cdot n_1) + b_2 (1 - n_0 \cdot n_2) + b_3 (1 - n_0 \cdot n_3) \right],$$

where $\ell_i (i = 1, 2, 3)$ is the bond length of the triangles $\Delta$, and $n_i (i = 0, 1, 2, 3)$ denote the unit normal vectors of triangles (Fig. 1(b)). $\sum_\Delta$ denotes the sum over all $\Delta$.

The coefficients $a_i$ and $b_i$ in Eq. (2) for model 2 are defined by

$$a_1 = v_{13}^{-1} + v_{23}^{-1}, \quad a_2 = v_{12}^{-1} + v_{32}^{-1}, \quad a_3 = v_{21}^{-1} + v_{31}^{-1},$$

$$b_1 = v_{12}^{-1} + v_{21}^{-1}, \quad b_2 = v_{13}^{-1} + v_{31}^{-1}, \quad b_3 = v_{23}^{-1} + v_{32}^{-1},$$

$$v_{ij}^{-1} = \begin{cases} 1 & (|\sigma_i \cdot t_{ij}| < \cos \theta_0), \\ c & (|\sigma_i \cdot t_{ij}| \geq \cos \theta_0). \end{cases}$$

This definition implies that $a_i$ and $b_i$ have values in $\{2, 2c, 1+c\}$. If all $\sigma_i$ satisfy $|\sigma_i \cdot t_{ij}| < \cos \theta_0$, then we have $a_i = b_i = 2$ and therefore $S_1$ and $S_2$ reduces to the ordinary ones such as $S_1 = \sum_{ij} \ell_{ij}^2$, $S_2 = \sum_{ij} (1 - n_i \cdot n_j)$ up to a multiplicative factor $2/3$ in both models. We should note that $a_i$ becomes an effective surface tension while $b_i$ becomes an effective bending rigidity at the bond $i$. Thus, the surface tension and bending rigidity dynamically changes depending on the position on the direction on the surface.

The partition function $Z_{\text{fix}(\text{flu})}$ for the connection-fixed (or fluid) surface model is given by

$$Z_{\text{fix}(\text{flu})}(\lambda, b) = \left( \sum_{\sigma} \right) \left( \sum_{T} \right) \int_{T'} \prod_{i=1}^{N} dX_i \exp \left[ -S(X, \sigma) \right],$$

where $\int' \prod_{i=1}^{N} dX_i$ is the multiple three-dimensional integrations, which are performed by fixing the center of mass of the surface to the origin of $\mathbb{R}^3$. The $\sum_{\sigma}$ denotes the sum over tilt variables,
and $\sum_T$ denotes the sum over all possible triangulation $T$. By eliminating $\sum_T$ from $Z_u$, we have $Z(x; b)$ for the connection-fixed surface model.

The assumed Finsler metric $g_{ab}$ is given by $g_{ab} = \begin{pmatrix} v_{12}^{-2} & 0 \\ 0 & v_{13}^{-2} \end{pmatrix}$. This metric comes from the Finsler function $L(x(t), y(t)) = \sqrt{\sum_i y_i^2}/|\mathbf{v}|$, where $x = (x_1, x_2)$ and $y = (y_1, y_2) = (dx_1/dt, dx_2/dt)$ are a point and a tangential vector at $x$ on an edge of triangle in $M$, and $\mathbf{v}$ is a vector along the edge [10]. This $\sqrt{\sum_i y_i^2}/|\mathbf{v}|$ corresponds to $v_{12}^{-1}$ or $v_{13}^{-1}$.

Both of the discrete Hamiltonians $S_1$ and $S_2$ in Eq. (2) with the coefficients in Eq. (3) are obtained from the continuous Hamiltonians

$$S_1 = \int \sqrt{g} d^2x g^{ab} \partial_a X^\mu \partial_b X^\mu, \quad S_2 = \frac{1}{2} \int \sqrt{g} d^2x g^{ab} \partial_a n^\mu \partial_b n^\mu.$$ 

Indeed, the partial derivatives in $S_1$ are replaced as $\partial_1 X^\mu \rightarrow X_1^\mu - X_2^\mu$, $\partial_2 X^\mu \rightarrow X_1^\mu - X_2^\mu$, and those in $S_2$ are $\partial_1 n^\mu \rightarrow \mathbf{n}_0 - \mathbf{n}_2$, $\partial_2 n^\mu \rightarrow \mathbf{n}_0 - \mathbf{n}_1$, where $\mathbf{n}_i (i = 0, 1, 2, 3)$ are shown in Fig. 1(a). The integration symbol $\int \sqrt{g} d^2x$ is simply replaced by $\sum_\Delta 1$, then we have $a_i$ and $b_i$ in Eq. (3). It is also possible to use $\sum_\Delta v_{12}^{-1} v_{13}^{-1}$ for a discretization of $\int \sqrt{g} d^2x$. Including two additional terms, which are cyclic under permutations of vertex positions $1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1$, and multiplying a factor $1/3$ to them, we have the discretization of $a_i$ and $b_i$ in Eq. (3).

3. Monte Carlo results

![Figure 2](image.png)

**Figure 2.** Snapshots obtained at (a) $\lambda = 0.56$ (spherical), (b) $\lambda = 0.68$ (disk), (c) $\lambda = 1$ (tubular). $N = 2562$, $b = 320$, and $(c, \theta_0) = (0.2, \pi/4)$ on connection-fixed surfaces. The view point of the lower snapshot is rotated $\pi/2$ around the vertical axis of the upper one in (a)–(c).

Snapshots of connection-fixed surfaces of size $N = 2562$ are shown in Figs. 1(a)–(c). We see that there are three different phases; spherical phase, disk phase and tubular phase. Brushes on the surface denote the tilts. The directions of tilts are at random in the spherical phase (Fig. 1(a)), and aligned in the tubular phase (Fig. 1(c)). In the disk phase in Fig. 1(b), the tilts
form a vortex-like configuration just like the Kosterlitz-Thouless phase in the two-dimensional Heisenberg spin model.

![Figure 3](image)

**Figure 3.** (a) M/N vs. $\lambda$, (b) $C_M$ vs. $\lambda$, (c) $S_0/N_B$ vs. $\lambda$, and (d) the maximal linear extension $L_{\text{max}}$ vs. $\lambda$. $N = 2562$ of connection-fixed surface, $b = 320$, and $(c, \theta_0) = (0.2, 45^\circ)$.

The magnetization $M$ defined by $M = \langle \sum_i \sigma_i \rangle = |\langle \sum_i \sigma_i^x, \sum_i \sigma_i^y, \sum_i \sigma_i^z \rangle|$ discontinuously changes against $\lambda$ at the phase boundary between the disk and tubular phases as shown in Fig. 3(a). This indicates that the tilt configuration is reflected in the surface shape. The variance $C_M = (1/N)(\langle (M - \langle M \rangle)^2 \rangle)$ has a peak at the boundary between the disk and spherical phases (Fig. 3(b)). The correlation energy $S_0/N_B$ rapidly changes at the same boundary, and this implies a phase transition (Fig. 3(c)). Indeed, the specific heat $C_{S_0} = (\lambda^2/N)(\langle S_0 - \langle S_0 \rangle \rangle^2)$ has a peak at the transition. The maximal linear extension $L_{\text{max}}$ of the surface changes discontinuously (almost discontinuously) at the boundary between the disk and tubular (spherical) phases (Fig. 3(d)). Thus, we confirm that an internal in-plane or external tilt order is a possible origin of anisotropy. We should note that the fluid surface model on dynamically triangulated surfaces has almost the same phase structure as the connection-fixed model.

4. Summary and conclusion

By performing Monte Carlo simulations, we have demonstrated that a Finsler geometric model has a variety of phases including anisotropic ones such as disk and tubular. The anisotropic surfaces emerge depending on the configurations of internal tilt order, which makes the surface tension and bending rigidity anisotropic. This allows us to understand that the internal tilt order is an origin of surface anisotropy. The interaction between the tilt order and surface should be studied more extensively. The author acknowledges Hideo Sekino for discussions, and he also acknowledges Hiroki Mizuno for computer analyses.

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