EXPLICIT STABLE MODELS OF ELLIPTIC SURFACES WITH SECTIONS

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Abstract. In this note we propose to show how to find stable models of a one-parameter family of elliptic surfaces. The strategy we use was initiated by Abramovich-Vistoli in [AvV2]: that is to say to consider a fibred surface as a map from the base curve to the moduli of stable n-pointed curves, and to consider then the Kontsevitch space of such maps that are stable. In our case we will then be describing the moduli space of stable surfaces via the moduli space of Kontsevitch stable maps to \( \overline{M}_{1,1} \), the moduli space of stable one-pointed elliptic curves. Then, following Kollár–Shepherd-Barron ([K-S]) and V. Alexeev ([A1]) we apply Mori’s Minimal Model Program in an explicit manner by means of toric geometry.

1. Introduction

1.1. Description of the problem. When speaking of a surface we will always mean a reduced, integral, normal, projective noetherian scheme of dimension 2.

For the purpose of studying the boundary of the moduli space, we need to introduce the following generalization of the well-known concept of elliptic surface (for convenience, we will maintain the name):

Definition 1.1.1. An elliptic surface with zero section is the datum of a surface \( X \) together with a proper map, \( \pi : X \to C \) to a proper curve \( C \) and a section \( \sigma : C \to X \), called the zero section, such that:

1. the generic fibre of \( \pi \) is a stable complete curve of arithmetic genus 1;
2. the zero section is not contained in the singular locus of \( \pi \).

Such an object is called relatively minimal if it is smooth and there is no \((-1)\)-curve in any fibre. Furthermore, we say it is minimal if it is smooth and contains no \((-1)\)-curve at all. Notice that if the base curve is not rational, the last two notions coincide.

Remark 1.1.2. Note that our notion of elliptic surface differs from the usual one in that in point 1 we ask the generic fibre to be stable, as opposed to the usual notion where the generic fibre is required to be smooth, and in that we do not ask the map \( \pi \) to be flat.

Whenever we are given such an object, we automatically get a rational map \( C \dashrightarrow \overline{M}_{1,1} \) to the moduli space of 1-pointed elliptic curves: \( \overline{M}_{1,1} \). If the base curve were smooth, we would then extend the map to a regular map on the whole \( C \to \overline{M}_{1,1} \). Composing this map with \( \pi \) provides us with a regular map \( X \to C \to \overline{M}_{1,1} \).

Let \( S \) be a fixed base scheme. Let \( \pi : X \to C \to S \) be a proper \( S \)-map of relative dimension 2 from a proper scheme \( X \) to a proper scheme \( C \), with \( X \to C \) generically fibred in stable curves of genus 1.

Following Alexeev and Kollár–Shepherd-Barron (cf. [A1], [A2] and [K-S]), we are naturally led to the following:

Definition 1.1.3. A pair \( (\pi : \mathcal{X} \to C \to S, \mathcal{Q}) \) consisting of an \( S \)-morphism \( X \to C \) fibred in generically stable curves of genus 1 and a section \( \mathcal{Q} \) is called stable if, for each geometric point \( s \) in \( S \):
1. \( X \rightarrow S \) and \( C \rightarrow S \) are flat morphisms of relative dimension 2 and 1 respectively;
2. the pair \((X, Q_s)\) has semi-log-canonical singularities (see section 3 for a definition);
3. the relative log-canonical sheaf \( \omega_{X/S}(Q) \) is \( \mathbb{Q} \)-Cartier;
4. the relative log-canonical sheaf \( \omega_{X/S}(Q) \) is \( S \)-ample.

If \( S = \text{Spec}(k) \) for any field \( k \), we simply say that the pair is \textit{stable}.

Similarly, a \textit{triple} \((X \rightarrow C \rightarrow S, Q, f : X_{\pi} \rightarrow C \rightarrow \mathbb{M}_{1,1})\) consisting of an \( S \) morphism \( X \rightarrow C \) fibred in generically smooth elliptic curves, a \textit{section} \( Q \) and a map \( f \) as above will be called \textit{stable} if conditions 1, 2 and 3 above hold and if \( \omega_{X/S}(Q) \otimes f^*O(3) \) is \textit{ample}.

If \( S = \text{Spec}(k) \) for any field \( k \), we simply say that the triple is \textit{stable}.

Given a relatively minimal elliptic surface \( X \) with section \( Q \), it is natural to consider its associated Weierstrass model \( Y \): this is roughly obtained by contracting all the components of the fibres that do not meet \( Q \) (see section 3 for a more precise definition). This surface \( Y \) has possibly a finite number of cuspidal fibres, with the property that near each such fibre \( Y_0 \), the surface \( Y \) has a local equation of the form \( y^2 = x^3 + ax + b \) with \( a, b \in \mathcal{O}_{C,0} \) with \( \min(\nu_0(a^3), \nu_0(b^2)) < 12 \). Here \( \nu_0(g) \) denotes the order of vanishing of \( g \in \mathcal{O}_{C,0} \) at 0. Such equations are called \textit{minimal Weierstrass equations} and \( Y \) is said to be in \textit{minimal Weierstrass form}.

Loosely speaking Weierstrass models, within the theory of elliptic surfaces, play the role that canonical models do for surfaces of general type. Indeed if the base curve \( C \) is smooth and non-rational, they constitute the log-canonical models for elliptic surfaces with sections having \( C \) as base curve (see Corollary 3.1.4).

For definition 1.1.3 to be of any use, we want the moduli functor it defines to include at least a large part of the locus of minimal Weierstrass equations. In other words, we want most pairs \((\pi : X \rightarrow C, Q)\) consisting of an elliptic surface in \textit{minimal Weierstrass equation} \( X \rightarrow C \) with section \( Q \), to be stable according definition 1.1.3. Similarly, we want most triples \((\pi : S \rightarrow C, Q, X \rightarrow C_{\pi} \rightarrow \mathbb{M}_{1,1})\) to be stable according to definition 1.1.3. In section 3 we will show that a pair \((X \rightarrow C, Q)\) with \( X \rightarrow C \) in minimal Weierstrass form is \textit{stable} if \( C \) is not rational, and that a triple \((\pi : S \rightarrow C, Q, X \rightarrow C_{\pi} \rightarrow \mathbb{M}_{1,1})\) with \( X \rightarrow C \) in minimal Weierstrass form is \textit{stable} either if \( C \) is not rational, or, if it is rational, if \( \pi : X \rightarrow \mathbb{P}^1 \) is not an isotrivial family (i.e., mapped to a point in \( \mathbb{M}_{1,1} \) via \( j \)).

This sets the course for us: the moduli problems we will be looking at are the ones defined by the functors:

\[
\mathcal{M}_{\text{pairs}} : \mathcal{S} \rightarrow \mathcal{S}
\]

where:

\[
\mathcal{M}_{\text{pairs}}(S) := \left\{ \text{pairs over } S (\pi : X \rightarrow C, Q) : \text{which are stable according to definition 1.1.3} \right\} /\text{isomorphisms}
\]

and:

\[
\mathcal{M}_{\text{triples}} : \mathcal{S} \rightarrow \mathcal{S}
\]
where:

\[ M_{\text{triples}}(S) := \left\{ \text{triples over } S (\pi: \mathcal{X} \to \mathcal{C}, Q, f : \mathcal{X} \xrightarrow{\pi} \mathcal{M}_{1,1}) : \text{which are stable according to definition 1.1.3} \right\} /\text{isomorphisms} \]

To render our functors of finite type, it is essential to fix some numerical invariants. The pair of rational numbers:

\[ A = c_1(\omega_X(Q))^2; \quad \chi = \chi(O_X) \]

for the moduli of pairs and the triple of rational numbers:

\[ A = c_1(\omega_X(Q))^2; \quad B = c_1(\omega_X(D)) \cdot c_1(f^*O_{\mathcal{M}_{1,1}}(1)); \quad C = c_1(f^*O_{\mathcal{M}_{1,1}}(1))^2, \]

for the moduli of triples. Once a moduli problem is defined, there are two crucial questions one must pose and answer (preferably positively): 1) Is the associated functor proper? Is it a Deligne-Mumford stack?

Here we will be addressing only the question of properness. Specifically we will prove that the functors \( M_{\text{pairs}} \) and \( M_{\text{triples}} \) satisfy the valuative criterion of properness. Such a criterion is usually called “stable reduction theorem” for moduli problems. In our case one has to prove that in a family of stable pairs (resp. triples) over the punctured disk, one can replace the central fibre of any compactification over the whole disk by a stable one, possibly after a base change.

1.2. Previous work. It must be said that the questions we are addressing here have been addressed and answered in much greater generality by J. Kollár and N. Shephard-Barron (cf. [K-S]) and by V. Alexeev (cf. [A1], [A2]). In [K-S] J. Kollár and N. Shephard-Barron prove a stable reduction theorem for the moduli of pairs \((X, D)\) consisting of a semi-log-canonical surface \(X\) and a \(\mathbb{Q}\)-Cartier divisor \(D\). In [A1] and [A2], V. Alexeev generalizes this to moduli of triples \((X, D, f : X \to M)\) where \(X\) is a surface with at most semi-log-canonical singularities, \(D \subset X\) is a \(\mathbb{Q}\)-Cartier divisor, \(f : X \to M\) is a proper morphism to a projective scheme \(M\) and \(\omega_X(Q) \otimes f^*A\) is \(\mathbb{Q}\)-Cartier and ample for every choice of a sufficiently ample divisor \(A\) on \(M\). Moreover he proves that the corresponding moduli functor is bounded (in particular for \(M = \text{Spec}(k)\) a point, one gets that the moduli space of pairs is bounded). Their proofs make use of the full strength of Mori’s MMP (Minimal Model Program).

Also, in the case of Weiestrass fibrations over \(\mathbb{P}^1\), in [M1], R. Miranda constructs a proper moduli space by identifying the GIT semi-stable points of the action of \(k^* \times SL(V_1)\) on a suitable subset \(T_N \subset V_{4N} \oplus V_{6N}\). Here \(V_1 = H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1))\), \(V_k = H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(k)) = \text{Sym}^k(V_1)\), and the component \(k^* \times SL(V_1)\) acts on \(T_N\) by \(\lambda(A, B) = (\lambda^{4N}, \lambda^{6N})\) and \(SL(V_1)\) acts on \(T_N\) by the action induced on it by the natural action of \(SL(V_1)\) on \(\text{Sym}^k(V_1)\).

What we set out to do here, though, is to give an explicit description of the stable reduction process (in the sense of Alexeev–Kollar–Shephard-Barron, i.e., in the sense of definition 1.1.3) in 1-parameter families and of the possible surfaces we may get at the boundary of the moduli spaces of elliptic pairs and of elliptic triples respectively. In fact, the work of Kollar–Shephard-Barron and of Alexeev proves the valuative criterion for properness abstractly by means of the MMP.

In order to explain better the significance of such an explicit description and how it compares with the construction of Kollar–Shephard-Barron and of Alexeev, we may refer to the case of the space \(\overline{M}_{g,n}\) of Deligne- Mumford (DM for short) stable curves of a given genus \(g\) with sections \(\sigma_i\) with \(i = 1, \ldots, n\), which compactifies the moduli space of smooth curves of genus \(g\) with \(n\) marked points \(M_{g,n}\) (cf. [D-M] or [H-M]). A (geometrically connected and proper)
curve $\mathcal{C} \to S$ over a scheme $S$ (resp. with sections $\sigma_i : S \to \mathcal{C}$) can be defined to be a DM stable curve in two different and equivalent ways:

the abstract description:
1. the singularities of $\mathcal{C}_s$ are at worst nodal for every closed point $s \in S$;
2. the relative dualizing sheaf $\omega_{\mathcal{C}/S}$ (resp. $\omega_{\mathcal{C}/S}(\sum_{i=1}^n \sigma_i)$) is ample.

the combinatorial description:
1. as 1 above,
2. if there is a rational irreducible component $R$ of $\mathcal{C}_s$ (for some closed point $s \in S$) then $R$ must meet the rest of $\mathcal{C}_s$ in at least 3 points (resp. $R$ must contain at least three points that are either nodes of $\mathcal{C}_s$ or marked points coming from the sections $\sigma_i$); if there is a component $E$ of arithmetic genus 1, then $E$ must meet the rest of $\mathcal{C}_S$ in at least one point.

An analogous picture holds in the case of Kontsevich stable maps.

In this work we try to transpose the work of J. Kollár and N. Shepard-Barron and of V. Alexeev into an incarnation that would correspond to the combinatorial picture of the DM-stable curves given above.

Up to now there are very few cases in which the degenerations in a 1-parameter family have been described explicitly. To name one of these, recently B. Hassett [Ha] gave a description for the boundary of the moduli of pairs $(\mathbb{P}^2, C)$ where $C$ is a plane quartic, and built an isomorphism between this space and $\overline{M}_{3}$, the moduli space of Deligne-Mumford stable genus 3 curves (this is of importance also for the understanding of the locus of limiting plane curves in $\overline{M}_g$, given that this locus coincides with the whole $\overline{M}_3$ for degree 4 curves).

The present work adds to the list of cases that have been worked out.

1.3. The strategy. The strategy for the moduli of triples is the one initiated by Abramovich and Vistoli in [R-V2]: we consider an elliptic surface as a map from the base curve to $\overline{M}_{1,1}$. By marking the points on the base curve corresponding to the cuspidal fibres, thanks to the Purity Lemma of Abramovich and Vistoli (cf. [R-V2]), we can replace these fibres by finite cyclic quotients of stable curves (we will refer to such curves as twisted curves). The idea is to now make the map from the base curve to $\overline{M}_{1,1}$ Kontsevich stable.

The result is that at the limit there are surfaces $X$ that map to curves $C$; these curves come endowed with a Kontsevich stable maps $C \to \overline{M}_{1,1}$, which correspond to the map to moduli; the general fibre of $X \to C$ is a stable curve and the components of $X$ meet along fibres that are either stable or twisted. The problem is that one had to replace the cuspidal curves in our original elliptic surfaces: the new surfaces are not in Weierstrass form anymore.

In order to deal with this problem, we prove an extension lemma (see lemma 4.2.1), that allows us to place back the cuspidal fibres, and so we can remove the extra marked points of the base.

The double curves of $X$ are either stable or the twisted fibres of Abramovich–Vistoli (cf [R-V1]). Looking at the irreducible components, we are therefore led to enlarging the class of minimal Weierstrass surfaces to what we call quasiminimal elliptic surfaces. Roughly speaking, these are elliptic surfaces $X \to C$ over smooth curves $C$, which are in Weierstrass form away from a finite number of twisted fibres and such that the local Weierstrass equation away from the twisted fibres is minimal.

The price we have to pay in removing the extra marked points, is that the map from the base curve to moduli might not be Kontsevich stable anymore: there may be isotrivial components mapping to a rational component of the base curve $C$, that meet the rest of $C$ in one or two
points. It turns out (proposition \[7.3.1\]) that the components of \(X\) dominating these isotrivial rational components are unstable. To be precise, the zero section of these components is an extremal ray if the base curve meets the rest of \(C\) in only one point, and it is contracted by the log-canonical map if it meets the rest of \(C\) in two points.

To deal with these components, we start by performing a few explicit birational transformations by means of local Weierstrass equations; in doing so we are led to enlarging the class of quasiminimal elliptic surfaces to what we call standard elliptic surfaces, which are, roughly speaking, elliptic surfaces with a finite number of twisted fibres, for which the local Weierstrass equations away from the twisted fibres are of the form \(y^2 = x^3 + ax + b\), for \(a, b \in \mathcal{O}_C, p\), with the minimum of the order of vanishing of \(a^3\) and \(b^2\) not greater than 12 at each point \(p \in C\). Note that if the above mentioned minimum does achieve 12 at some point \(p \in C\), the surface has an elliptic singularity at the point \(x = y = 0\) over \(p\) (see section \[7.2\]).

We continue by following the steps of the Minimal Model Program (MMP). By means of toric geometry we are able to explicitly perform the necessary log-flips and small log-contractions.

The forgetful functor sending, for each scheme \(S\), an \(S\)-triple \((\mathcal{X}, \mathcal{Q}, f : \mathcal{X} \to \mathcal{C} \to \overline{M}_{1,1})\) to the \(S\) pair \((\mathcal{X}, \mathcal{Q})\), is not well-defined at the level of moduli: it does not always produce a stable pair out of a stable triple. In fact, every component \(X \to \mathbb{P}^1\) of a geometric fibre \(X_s \to C_s\), for some geometric point \(s\) in \(S\), that meets \(X_s\) in less than three fibres is unstable. Therefore, we need to perform more steps of the MMP. It will turn out that the same explicit operations described in the case of triples, go through for these more general settings in which the \(j\)-invariant associated to such components \(X \to \mathbb{P}^1\) is not constant. The price we have to pay is that the map \(\mathcal{X} \to \overline{M}_{1,1}\) is no longer regular.

When the general base curve \(C\) is isomorphic to \(\mathbb{P}^1\), the log-canonical bundle for the pair \((\mathcal{X}, \mathcal{Q})\) is not nef: it is negative on \(Q\). We therefore need to contract the zero-section in the central fibre, to obtain a stable pair. In order to do so we need to perform some extra birational operations on the total space of the family \(\mathcal{X}\).

1.4. The result. We need some definitions first.

Loosely speaking a log-standard elliptic surface \(X \to C\) (see definition \[7.1.4\]) is a certain explicit locally toric blow-up (the unique such semi-logcanonical blow-up) of a pair \((Y \to C, G + F + Q)\) consisting of a standard elliptic surface \(Y \to C\) together with the marking of the zero section \(Q\) and of some (possibly twisted) fibres \(G = \bigcup_i G_i\) and \(F = \bigcup_i F_i\); the centers of the blow-ups are supported on points in \(G_i \cap Q\). We call splice the proper transform of a fibre \(G_i\) of \(Y \to C\) on which the point we blow-up is supported, and scion the proper transform of each of the \(F_i\)’s. Such a log-standard elliptic surface is called strictly stable, if \(Q^2 < -1\).

Our results are stated in Theorems \[8.2.1\], \[8.1.1\], \[8.3.1\] and \[8.4.2\].

Theorems \[8.2.1\] and \[8.1.1\] roughly speaking say that, in the case in which the base curve has genus \(g \geq 2\), at the boundary we get a union of surfaces \(X = \bigcup X_i\) attached one to the other along curves which are either stable or twisted fibers or along splices, mapping to a curve \(\pi : X \to C\), with a section \(Q\). For the moduli of pairs, \(C\) is a Deligne-Mumford stable curve and the components of \(X\) are (possibly) of two kinds:

1. strictly stable log-standard components mapping dominantly onto irreducible components of \(C\);
2. components mapped to a point of \(C\) via \(\pi\).

The components mapped to a point, which we call pseudoelliptic (see definitions \[7.1.8\] and \[7.2.4\]), are further divided in two types. For the moduli of pairs \((X, Q)\), we have:
1. **type I**: A pair \((Y, G + F)\) consisting of a surface \(Y\) endowed with a *structure morphism* \(f : Y' \to Y\), which is regular and birational, from a *log-standard* surface \((Y', Y + Q + G' + F')\), with 1 *scion* \(F'\) and a number of *splices* \(G' = \bigcup G'_i\) and \(G = \bigcup G_i\). The surface \(Y\) is attached to the rest of \(X\) along \(F\) and possibly along some (or all) of the \(G_i\)'s. Furthermore, the morphism \(Y' \to Y\) is obtained by torically (and explicitly) blowing-down the zero section \(Q\) of \(Y'\);

2. **type II**: A pair \((Y, G + F_1 + F_2)\) consisting of a surface \(Y\) endowed with a birational morphism \(f : Y' \to Y\) (the *structure morphism*) from a *log-standard* surface \((Y', Q + G' + F_1' + F_2')\) with 2 *scions* \(F_1'\) and \(F_2'\) and a number of *splices* \(G' = \bigcup G'_i\); also \(F_i = f(F'_i)\), \(G_i = f(G'_i)\) and \(G = \bigcup G_i\). Moreover \(Y\) is attached to the rest of \(X\) along the \(F_i\)'s and possibly along some (or all) of the \(G_i\)'s. Furthermore, the morphism \(Y' \to Y\) is obtained by torically (and explicitly) blowing-down the zero section \(Q\) of \(Y'\);

We still call *splice* and *scion* the images in a pseudoelliptic surface, either of type I or of type II, of the corresponding curves via the structure morphism. The attaching of a scion to a splice is étale locally described by the fan given in Theorem 7.1.2.

The attaching of a pseudoelliptic surface of type II along a marked fibre of \(X\) is described étale locally by the cone given in Theorem 7.2.1.

For the moduli of triples \((X, Q, f : X \to \overline{M}_{1,1})\) we have the same situation except for having the *isotrivial* analogues of type I and type II, and we ask that the map \(C \to \overline{M}_{1,1}\) be Kontsevich stable.

The type I surfaces arise from log-flips, and the type II ones from the log-canonical contraction of the zero section.

![Diagram](fig1.png)
In the rational base case (see Theorem 8.3.1), there is no base curve at all. At the boundary here we get a union $X = \bigcup X_i$ where $X_i$ is a pseudoelliptic surface of type $N$ with $N = 0$ or $N = 1$. Furthermore the type 1 pseudoelliptic surfaces come in pair, and they are attached one to the other (but not along as a scion).

Here, by pseudoelliptic surface of type $N$ we mean, loosely speaking, a surface $S$ and a map $g : S' \to S$ from a log-standard surface $(S \to \mathbb{P}^1, Q + \sum_{i=1}^{N} F_i + \sum_{i=1}^{N} G_i)$ with $N$ marked stable or twisted fibres $F_i$ and $n$ splices $G_i$, furthermore, $g$ is obtained by an explicit toric blow-down of the zero section $Q$ of $Y$.

In the elliptic base case (see theorem 8.4.2) we have two more types of surfaces: type $E_0$ and type $E_{1N}$.

Loosely speaking, a type $E_0$ pseudoelliptic surface (resp. type $E_{1N}$ pseudoelliptic surface) is a surface $S$ endowed with a map $g : S' \to S$ from a log-standard elliptic surface $(S \to E, Q + F)$ with one marked fibre, and with base $E$ an elliptic curve (resp. a closed chain of $\mathbb{P}^1$’s), and such that the exceptional set of $g$ is the zero-section $Q$. A type $E_0$ (resp. type $E_{1N}$) pseudoelliptic surface has an elliptic (resp. degenerate cusp) singularity at $g(Q)$.

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2. Preliminaries

In this section we will give a brief list of results we need from toric geometry and the theory of log-canonical surfaces. Varieties are always integral, reduced schemes of finite type over an algebraically closed field $k$ of characteristic zero. Let $S$ be a scheme of finite type over $k$. If $X$ is proper over $S$ we set:

$$N_1(X/S) = \{1\text{-cycles of } X/S \text{ modulo numerical equivalence}\}.$$  

We then have a bilinear pairing:

$$\text{Pic}(X) \times N_1(X/S) \to \mathbb{Q}$$

defined by extending by linearity the map that associates $\deg_C(\mathcal{L})$ to the pair $(\mathcal{L}, C)$, with $\mathcal{L}$ a line bundle on $X$ and $C$ an effective irreducible curve. We also set:

$$NE(X/S) = \{B \in N_1(X/S) : \exists a \sum a_i C_i, a_1 \in \mathbb{Q}^+ \cup \{0\}\}$$

and we denote by $\overline{NE}(X/S)$ its closure with respect to the euclidean topology in $H^2(X, \mathbb{R})$.

Kleiman’s Criterion of ampleness states that, if $X$ is $S$-proper, a divisor $A$ is $S$-ample on $X$ if and only if $A \cdot x > 0$ for each $x \in \overline{NE}(X/S) \setminus \{0\}$. An extremal ray is a ray $R \subset \overline{NE}(X/S)$ such that if $x_1 + x_2 \in R$ then $x_1, x_2 \in R$, for each $x_1, x_2 \in \overline{NE}(X/S)$.

By the cone theorem (see [MK] for the smooth case and [KMM] for the general case), given an extremal ray $R$ of $NE(X)$ there exists an extremal contraction, namely a morphism $\phi_R : X \to Y$ such that:

1. $\phi_R_* \mathcal{O}_X \simeq \mathcal{O}_Y$, i.e., $\phi_R$ has connected fibres;
2. a curve $C \subset X$ is contracted by $\phi_R$ if and only if its class $[C]$ in $\overline{NE}(X/S)$ is such that $[C] \in R$. 

Definition 2.0.1. Let \( \phi : X \rightarrow Y \) be an extremal contraction such that the codimension of the exceptional set \( E \subset X \) is \( \geq 2 \), and let \( D \) be a \( \mathbb{Q} \)-Cartier divisor of \( X \). A variety \( X^+ \), together with a birational morphism:

\[
\phi^+ : X^+ \rightarrow Y
\]

is called a flip (resp. log-flip) of \( \phi \) if:
1. \( X^+ \) has only log-canonical singularities (see section 2.1 below)
2. \( K_{X^+} \) (resp. \( K_{X^+} + D^+ \) where \( D^+ \) is the closure of \((f^+ \circ f^{-1})(D))\) is \( f^+ \)-ample.
3. the exceptional set of \( \phi^+ \) has codimension \( \geq 2 \) in \( X^+ \).

2.1. Semi-log-canonical singularities. By definition, a reduced scheme of finite type \( X \) is said to be \( \mathbb{Q} \)-Gorenstein if \( \omega_X[1] \) is locally free for some \( n \). Here \( \omega_X[1] = ((\omega_X^\otimes n)^\vee)^\vee \). For a \( \mathbb{Q} \)-Gorenstein variety, the smallest such \( n \) is called the index. Following [K-S]

Definition 2.1.1. A surface \( X \) is semi-smooth if it has only the following singularities:
1. 2-fold normal crossings with local equation \( x^2 = y^2 \)
2. pinch points with local equation \( x^2 = zy^2 \)

and

Definition 2.1.2. A good semi-resolution of a surface \( X \) is a proper map \( g : Y \rightarrow X \) satisfying the following properties
1. \( Y \) is semi-smooth
2. \( g \) is an isomorphism in the complement of a codimension two subscheme of \( Y \)
3. No component of the double curve \( D \) of \( Y \) is exceptional for \( g \).
4. The components of \( D \) and the exceptional locus of \( X \) are smooth, and meet transversally.

Finally, given a birational map \( g : Y \rightarrow X \), we call discrepancies of \( K_X + D \) associated to \( g \) those integers \( a_i \) such that:

\[
\omega_Y^\vee(D) = g^*\omega_X^\vee[D + ka_1E_1 + ... + ka_nE_n], \quad \text{where} \quad D \text{ is the pushforward of } D \text{ via } g^{-1}.
\]

When \( D \) is the empty set, we call the \( a_i \) just discrepancies.

Definition 2.1.3. A surface \( X \) is said to have semi-log-canonical singularities if
1. \( X \) is Cohen-Macaulay and \( \mathbb{Q} \)-Gorenstein
2. \( X \) is semi-smooth in codimension one
3. The discrepancies \( a_i \) of a good semi-smooth resolution of \( g : Y \rightarrow X \) are all greater than or equal to \(-1\).

It is not difficult to show that a surface \( X \) is semi-log-canonical if and only if it is \( S2 \) and its normalization \( \nu : X' \rightarrow X \) is such that the pair \((X, K_X + D)\) where \( D \subset X' \) is the double curve of \( \nu \), is log-canonical.

2.2. Toric varieties. Toric varieties are obtained by suitably patching affine toric varieties, which are, roughly speaking, normal zero sets of binomials. Given a lattice \( N \cong \mathbb{Z}^n \) and a strictly convex rational polyhedral cone \( \sigma \subset N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R} \) we will denote by \( X_\sigma \) the affine toric variety associated with \( \sigma \) (see [F]).

We denote by \( \sigma^{(1)} \) the 1-dimensional edges of \( \sigma \). The variety \( X_\sigma \) is nonsingular if and only if the primitive points of \( \sigma \) form a part of a basis of \( N \).

The toric variety \( X_\sigma \) contains an \( n \)-dimensional algebraic torus \( T = \mathbb{G}_{m}^n \) as an open dense subset, and the action of \( T \) on itself extends to a linear action on \( X_\sigma \) (hence the alternate name
torus embedding). Thus, $X_\sigma$ is a disjoint union of orbits of this action. There is a one-to-one correspondence between the orbits and the faces of $\sigma$. In particular, 1-dimensional faces $\mathbb{R}_+v_i$ correspond to codimension 1 orbits $O_{v_i}$.

A fan $\Delta \subset N$ is a collection of strongly convex rational polyhedral cones such that: 1) each face of a cone in $\Delta$ is also a cone in $\Delta$; 2) the intersection of two cones in $\Delta$ is a common face of the two. To fans one associates toric varieties, that are obtained by suitably patching the affine toric varieties corresponding to the cones of the fan.

A toric morphism $f : X_\sigma \to X_\tau$ is a dominant equivariant morphism of toric varieties corresponding to a linear map $f_\Delta : (N_\sigma, \sigma) \to (N_\tau, \tau)$.

In this paper we will write $\langle f_1, ..., f_n \rangle$ for the cone in $N_\mathbb{R}$ generated by the lattice vectors $f_1, ..., f_n \in N$ for some lattice $N$.

For a toric variety $NE(X/S)$ is a closed cone, hence $\overline{NE}(X/S) = NE(X/S)$. Indeed M. Reid in [VII] proves:

**Lemma 2.2.1.** If $f : X \to S$ is a proper toric morphism, and if $X$ is proper, then

$$NE(X/S) = \sum \mathbb{Q}^+O_w$$

where $O_w$ runs through the 1-dimensional strata of $X$ in fibre of $f$. Furthermore, if $X$ is projective, $NE(X/S)$ is spanned by a finite number of extremal rays.

Set $\Delta^k = \{ k \text{-dimensional cones of } \Delta \}$. If $\sigma_1 = \langle e_1, ..., e_{n-1}, e_n \rangle$ and $\sigma_2 = \langle e_1, ..., e_{n-1}, e_{n+1} \rangle$ and $w$ is the face $\langle e_1, ..., e_{n-1} \rangle$, we write $\sigma(w) = \sigma_1 + \sigma_2$ for the cone: $\langle e_1, ..., e_{n-1}, e_n, e_{n+1} \rangle$.

Since $\{e_1, ..., e_{n-1}, e_n\}$ is a $\mathbb{Q}$-basis for the lattice $N$, there is a relation $\sum_{i=1}^{n+1} a_i e_i = 0$. Let $I_1 = \{a_i \text{ such that } a_i < 0\}$.

If $R$ is an extremal ray (i.e., $R = \mathbb{Q}^+O_w$ with $O_w \in NE(X/S)$, an extremal ray) in a fan $F$, write $F_R$ for the fan whose walls are $\Delta_R^{n-1} = \Delta^{n-1} \setminus R$. The corresponding toric variety $Y = X(F_R)$ is the contraction of $R$.

Define a simplicial subdivision $\Delta^+ of \Delta_R$ by defining

$$\Delta^+ = \Delta_R \setminus \left( \bigcup_{w \in R} \sigma(w) \cup \bigcup_{w \in R, i \in I_1} \sigma_i(w) \right),$$

where $\sigma_i(w) := \langle e_1, ..., \hat{e}_i, ..., e_{n-1}, e_n, e_{n+1} \rangle$. M. Reid in [VII] proves:

**Theorem 2.2.2.** The toric morphism $\phi_1 : X^+ = X(\Delta^+) \to Y$ corresponding to the simplicial subdivision $\Delta^+ of \Delta_R$ is projective and an isomorphism in codimension 1. $-R$ is identified with an extremal ray of $X^+$ and $\phi_1 = \phi_{-R}$ is the contraction of $-R$.

2.3. Toric 2-dimensional isolated singularities. Let $\sigma$ be the cone generated by the vectors $f_2$ and $v = kf_1 - nf_2$ in the lattice $N = f_1 \mathbb{Z} \oplus f_2 \mathbb{Z}$. Without loss of generality, we may assume that $k$ and $n$ are coprime. Then we can choose a unique $n' \in \mathbb{Z}$ such that $0 \leq n' < k$ and $nn' \equiv 1 (mod k)$. Therefore, if $nn' = 1 - kb$ we can map $N$ isomorphically into itself and $\sigma$ to the cone $\sigma'$ generated by $f_2$ and $v' = kf_1 - n'f_2$ by means of the matrix:

$$\begin{pmatrix} n & k \\ b & -n' \end{pmatrix} \in SL_2(\mathbb{Z}),$$
thus inducing an isomorphism of toric varieties: $X(\sigma) \simeq X(\sigma')$. The surface $X(\sigma')$ is the normalization of:

$$W = \{ (x, y, z) \in \mathbb{C}^3; \ x^k = yz^{k-n'} \}$$

and it is isomorphic to the quotient of $\mathbb{C}^2$ by $\mu_k$ via the action:

$$\epsilon(x, y) = (\epsilon^x, \epsilon y)$$

for a primitive $k$-th root of unity $\epsilon$. Its minimal desingularization has as exceptional divisor a chain of rational curves $E_i$ with self intersections $E_i^2 = -a_i \leq -2$ determined by the continued fraction:

$$\frac{k}{n'} = a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_r}}}.$$ 

Following [B-P-VdV], we call such a singularity an $A_{n',k}$-singularity or a $\frac{1}{k}(n',1)$ singularity.

### 2.4. Some toric 3-dimensional isolated singularities.

Let $n_1, n_2$ and $n_3$ be generators of a 3-dimensional lattice $N$ and let $\sigma$ be the cone $\langle n_1, n_2, an_1 + bn_2 + rn_3 \rangle$. Then the affine toric variety $X(\sigma)$ is isomorphic to the quotient of $\mathbb{C}^3$ by $\mu_r$ acting as:

$$\epsilon(x, y, z) = (\epsilon^a x, \epsilon^b y, \epsilon z),$$

where $\epsilon \in \mu_r$ is a primitive $r$-th root of unity. Following M. Reid ([R3]), we shall refer to such a 3-dimensional isolated singularity as a $\frac{1}{r}(a, b, 1)$ singularity, thus indicating the order of the cyclic group and the weights with which it acts.

### 3. Weierstrass Forms

#### 3.1. Ampleness of the log-canonical divisor.

Let $\psi : Y \to C$ be a flat family of generically smooth stable curves of genus 1 over a smooth curve $C$, with zero section $Q$, and let $\pi : X \to C$ be the surface obtained by contracting all the components of the possible singular fibres disjoint from $Q$. Then we can express $X$ in a Weierstrass form in the following way. Since $\pi$ is proper and flat and since $H^2(X_y, \mathcal{O}_{X_y}) = 0$, by the theorem of base change in cohomology ([H] theorem 12.11) $R^1\pi_*\mathcal{O}_X$ is locally free, and since its rank is one, it is an invertible sheaf on $C$. Let $\mathcal{L} := (R^1\pi_*\mathcal{O}_X)^\vee$ its inverse, $\mathcal{E} := \mathcal{O}_C \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$ and $\mathcal{P} := \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ and let:

$$x : \mathcal{E} \to \mathcal{L}^2,$$

$$y : \mathcal{E} \to \mathcal{L}^3$$

and

$$z : \mathcal{E} \to \mathcal{O}_C$$

be the canonical projections onto the given factor. We have:

**Theorem 3.1.1.** There exist two sections: $g_2 \in H^0(C, \mathcal{L}^2)$ and $g_3 \in H^0(C, \mathcal{L}^3)$ such that $X$ is isomorphic to the Cartier divisor in $\mathcal{P}$ given by the equation:

$$y^2z = x^3 - g_2xz^2 - g_3z^3.$$

Moreover:

1. $\Delta = 4g_2^3 - 27g_3^2 \in H^0(C, \mathcal{L}^{12})$ is non-zero;
2. the sections $g_2^3$ and $g_3$ of $\mathcal{L}^{12}$ do not vanish to order $\geq 12$ at any point of $C$;
3. the zero section $Q$ of $X \to C$ corresponds to the section at infinity $(x, y, z) = (0, 1, 0)$.

**Proof.** The proof is the same as the one of Theorem 1’ in [M-S], although they only state the theorem in the case the elliptic fibration has no singular fibres. \[\square\]
Note that the equation makes sense since all the monomials that appear in it are sections of the vector bundle $\text{Sym}^3 E^\ast \otimes L^6$. The fact that we assumed that there is no component of the singular fibres that is disjoint from $Q$ implies that our $X$ is of this form. Notice that by the Leray spectral sequence we get:

$$\chi(O_S) = \chi(\pi_*O_S) - \chi(R^1\pi_*O_S) = \chi(O_C) - \chi(L^\vee) = d,$$

where $d = c_1(L)$.

We have the important:

**Theorem 3.1.2.** Let $\psi : Y \to C$ be a relatively minimal elliptic surface, with zero section $Q$ (in particular it has no multiple fibres). Then

$$\omega_Y \simeq \pi^*(\omega_C \otimes L)$$

**Proof.** See [B-P-VdV] theorem 12.1. \qed

Two important consequences are the following:

**Corollary 3.1.3.** For a relatively minimal non-isotrivial elliptic surface $\pi : X \to C$ with smooth base curve $C$ and zero section $Q$ the divisor:

$$L_X = K_X + Q + c_1(\pi^*\mathcal{O}_{M_{1,1}}(3))$$

is semi-ample and positive on every curve except for those $(-2)$-curves of the fibres that do not meet $Q$. More generally, for the $\mathbb{Q}$-Cartier divisor:

$$L := aQ + \pi^*(\beta + \lambda)$$

for $a \in \mathbb{Q}_+$ a number with $0 < a \geq 1$ and $\beta$ a divisor on $C$, we have:

1. if $c_1(\beta) > 0$, then $L$ is ample;
2. (a) if $0 < a < 1$, $c_1(\beta) = 0$ and $c_1(\lambda) > 0$, then $L$ is ample;
   (b) if $a = 1$, $c_1(\beta) = 0$, and $c_1(\lambda) > 0$, the line bundle $L$ is semiample, and for any irreducible curve $D \subset X$:

$$L \cdot D = 0 \text{ if and only if } D = Q;$$

3. if $c_1(\beta) < 0$ and $c_1(\beta + \lambda) > 0$ then $Q$ is an extremal ray.

Note that this includes $K_X + aQ + \pi^*(\alpha)$ for any $\alpha \in \text{Div}(C)$ with $c_1(\alpha) + 2g - 2 \geq 0$, if $g$ is the genus of $C$.

**Proof.** In what follows we will use freely that $c(\lambda) = -Q^2$.

We will first show the general part and then we will reduce to it the statement involving the canonical bundle.

First one checks easily that:

$$(aQ + \pi^*(\beta + \lambda))^2 = 2a(\beta + \lambda) \cdot Q + a^2Q^2$$

$$= 2a[c_1(\beta) + c_1(\lambda)] - a^2c_1(\lambda)$$

$$= 2a(2c_1(\beta) + (2 - a)c_1(\lambda)) \geq 2a(c_1(\beta) + c_1(\lambda)).$$

Note that this is positive in all three cases, showing that $L$ is big ($L$ is clearly effective).

Also, if $F$ is a fibral divisor (i.e., if it is the class of a fibre), since $X$ is relatively minimal, we always have:

$$L \cdot F = a > 0.$$
If \( D \subset X \) is any irreducible curve, there are two possibilities: it either maps dominantly onto \( C \) or it is a fibral divisor. Thus, we are left to deal with the former case.

So we may assume that \( D \) is an irreducible curve dominating \( C \). If \( D \neq Q \), since \( Q \) is effective, we have: \( D \cdot Q \geq 0 \), and then:

\[
L \cdot D \geq c_1(\beta + \lambda) + aD \cdot Q \geq c_1(\beta + \lambda) > 0
\]

in all the cases.

Therefore, all the cases will be differentiated according to the behavior of \( L \) on \( Q \).

We have:

\[
L \cdot Q = aQ^2 + c_1(\lambda) + c_1(\beta) = (1 - a)c_1(\lambda) + c_1(\beta)
\]

**Case (1) and Case (2) (i):** In this case we have that: \( L \cdot Q > 0 \) and we can conclude by means of the Nakai-Mosheizon criterion.

**Case (2) (ii):** In this case \( L \cdot D \geq c_1(\lambda) > 0 \) for any irreducible (non-fibral) curve \( D \neq Q \) and \( L \cdot Q = 0 \)

**Case (3):** In this case \( Q \) is the only curve on which \( L \) is negative. This entails that \( Q \) generates an extremal ray. In fact, observe that if \( c_1 + c_2 \in [Q][\mathbb{R}] \subset \overline{NE}(X) \) then \( L \cdot (c_1 + c_2) \geq 0 \).

But if \( c_1 \) and \( c_2 \) are limits of sequences \( C_{1i} \) and \( C_{2i} \) of curves that differ from \( Q \), then \( L \cdot C_{ji} > 0 \) for each \( i \) and \( j \), and thus \( L \cdot c_j \geq 0 \) for each \( j \), which is a contradiction, since it would imply that \( L \cdot (c_1 + c_2) \geq 0 \). So at least one of the \( c_i \) is in \( [Q][\mathbb{R}] \), and again, since \( Q \) is the only irreducible curve on which \( L \) is negative, both \( c_1 \) and \( c_2 \) must be in \( [Q][\mathbb{R}] \).

Now, for the statement about \( L = K_X + Q + c_1(\pi^* j^* \mathcal{O}_{\mathcal{M}_{11}(3)}) \), note that the by the canonical bundle formula (theorem 3.1.2) we have that: \( K_X = \pi^*(K_C + \lambda) \), where \( \lambda \) is a divisor class associated to \( L \). This concludes the proof.

The following corollary is a special case of the previous one, but it is worth stating separately, for heuristic reasons.

**Corollary 3.1.4.** Same hypotheses on \( \pi : X \to C \) and \( C \). If furthermore the curve \( C \) is not rational, then the divisor:

\[
L_X = K_X + Q
\]

is semi-ample and positive on every curve, except for those \((-2)\)-curves of the fibres that do not meet \( Q \).

What this means is that, if there are such \(-2\)-curves in the fibres, in order to make the log-canonical divisor ample we have to contract them. This explains the necessity of considering **Weierstrass forms** for the purpose of studying our moduli problems.

### 3.2. Log-canonical singularities of Weierstrass equations.

In this section, given a Weierstrass equation \( y^2 = x^3 + a(t)x + b(t) \) with \( a(t), b(t) \in k[[t]] \) we will only write the low degree terms of \( a(t), b(t) \).

After having dealt with the ampleness of the log-canonical divisor, the second question one needs to address in order to understand when an elliptic surface is stable is what kind of Weierstrass equations give rise to a log-canonical singularity. The following lemma answers this question:
Lemma 3.2.1. Let $\pi : S \to \text{Spec } k[[t]]$ be an elliptic surface with zero-section $Q$, given in Weierstrass form: $y^2 = x^3 + at^n x + bt^m$, with $a, b$ units in $k[[t]]$. Assume furthermore that $j \neq \infty$. Then the pair $(S, Q)$ is log-canonical if and only if $\min(3n, 2m) \leq 12$. Furthermore, if $F$ is a smooth fibre of $\pi$, the pair $(S, F + Q)$ is log-terminal if and only if $S$ is, and if $F$ is a cuspidal fibre, $(S, Q + F)$ is never log-canonical.

Proof. The proof of this lemma makes use of the list of Alexeev’s of dual graphs of log-canonical surface singularities (cf. [3] ch.3) in the sufficient direction, and by calculating the discrepancies in the necessary direction.

If $\min(3n, 2m) < 12$ then the minimal resolution has as dual graph the Kodaira graphs: $I_0, I_1, I_n, II, III, IV, I_0^*, II^*, III^*, IV^*$ (see [3], Ch. IV pg.354).

We will refer to table (IV.3.1) pag.41 of [4] for the singularities corresponding to the various Kodaira types.

In case $I_0$, and $II$ the surface is smooth, and in case $I_n^*$ it has a rational double point singularity, so there is nothing to prove. In case $III$ there is only one exceptional curve in the dual graph, and so we are in case (1) of Alexeev’s list. In the case of Kodaira type $IV$, then graph of the resolution is of type $A_2$ and for the Kodaira type $I_n$ the dual graph is of type $A_n$ and again this is in case (1) of Alexeev’s list.

In the $*$ cases things become more interesting. $I_0^*$ corresponds to a $D_4$-diagram, and this is in case (2) again with $(\Delta_1, \Delta_2, \Delta_3) = (2, 2, 2)$; $I_n^*$ corresponds to a $D_{n+4}$ graph and this is still in case (2) with $(\Delta_1, \Delta_2, \Delta_3) = (2, 2, N)$; $IV^*$ to an $E_6$ and again this is in (2) with $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 3)$; $III^*$ corresponds to an $E_7$ and we are again in case (2) with $(\Delta_1, \Delta_2, \Delta_3) = (2, 3, 4)$; $II^*$ corresponds to $E_8$ and one gets case (2) with $(\Delta_1, \Delta_2, \Delta_3) = (2, 2, 3)$. So these are all even canonical (they are all Du Val singularities).

Now, if $\min(3n, 2m) \geq 12$, then one can write $(n, m) = (4k + n', 6k + m')$ with $n', m' \geq 0$ such that either $0 \leq n' \leq 3$ or $0 \leq m' \leq 5$. One can then consider the rational map: $S' \to S$ given by $(x, y, t) = (x't^{2k}, y't^{3k}, t)$, where $S'$: $y'^2 = x'^3 + at'^n x + bt'^m$. Now:

$$\omega = \frac{dx \wedge dt}{2y} \in \Omega^2_{k(S)}$$

is a basis for the $k(S)$-module $\Omega^2_{k(S)}$, and so is

$$\omega' = \frac{dx' \wedge dt}{2y'} \in \Omega^2_{k(S')},$$

for the $k(S')$-module $\Omega^2_{k(S')}$, moreover:

$$\pi^* \omega = \frac{dx' t^{2k} \wedge dt}{2y't^{3k}} = \frac{dx' \wedge dt}{2y't^{k}} = \frac{\omega'}{t^k} \in \Omega^2_{k(S')}(kE).$$

The surface $S'$ is canonical for what has just been showed above since $\min(3n', 2m') < 12$. Therefore $S$ is not log-canonical if $k > 1$, since the discrepancies would be at least $k$. One would be tempted to say that the $k = 1$ case is settled and thereby claiming the log-canonicity in this case, but indeed the previous argument fails in this case, since the map $S' \to S$ exhibited above is not proper; there might still be exceptional divisors on a completion of $S' \to S$ with discrepancies $> 1$.

Therefore a detailed analysis of the case $k = 1$ is needed. In order to do that one has to find a partial resolution and compute discrepancies. To achieve this goal, since $y^2 = x^3 + at^n x + bt^m$, with $a$ and $b$ units in $k[t]$, can be thought of as the double cover of the affine plane ramified along the curve $R$: $x^3 + at^n x + bt^m = 0$, we can simply take an embedded resolution $D \subset M$ of
this curve, blow-up again at the points in which components of $D$ with odd multiplicity meet to obtain $D' \subset M'$, and then take the double covering of the resulting surface along the total transform of $R$. This double covering has canonical singularities.

The exceptional curve $E$ of a resolution of $y^2 = x^3 + at^4 x + bt^6$ is a stable curve of arithmetic genus 1 of self-intersection $-1$ attached to a rational curve of self-intersection $-1$. Hence the singularity is log-canonical (but not log-terminal).

For the isotrivial $j = \infty$ case we have:

**Lemma 3.2.2.** The local equation $y^2 = x^2(x - \lambda t^k)$ with $\lambda$ a unit in $k[[t]]$, is semi-log-canonical if and only if $k \leq 2$.

**Proof.** The proof is very similar to the one of lemma 3.2.1. Let us denote by $X$ the surface defined by $y^2 = x^2(x - \lambda t^k)$. Let us show first that if $k \leq 2$, then the singularity is semi-log-canonical. If $k = 0$ then the equation is locally around the $(0,0,t)$ isomorphic to $u^2 = w^2$, and it is thus semi-smooth, hence semi-log-canonical. For $k = 1$, then setting $z = x - \lambda t$ shows that this singularity is isomorphic to $y^2 = x^2z$ which is again semi-smooth. For $k = 2$, after one blow-up we obtain a surface $X'$ and as exceptional divisor a nodal curve of the same type as the general fibre of $y^2 = x^2(x - \lambda t^2)$. It is now easy to see that $X'$ is semismooth and that $X$ is semi-log-canonical. If $k > 2$, then we can conclude as in lemma 3.2.1 that $X$ is not semi-log-canonical.

**Definition 3.2.3.** We call standard Weierstrass equation an equation $y^2 = x^3 + a(t)x + b(t)$, which satisfies the condition $\min(3n,2m) \leq 12$, where $a, b \in k[[t]]$ and $n$ and $m$ are respectively the order of vanishing of $a$ and $b$ at $t = 0$, or one of the form $y^2 = x^2(x - \lambda t^k)$ with $k \leq 2$.

So, we can directly infer from lemma 3.2.1 and 3.2.2 the important:

**Corollary 3.2.4.** A generically stable elliptic surface $X \to C$ mapping to a smooth curve $C$ is semi-log-canonical if and only if its local equation around each cusp is a standard Weierstrass equation.

3.3. **Types of cuspidal fibres.** In this section we will introduce a bit of terminology and notation we will be using in the discussion of the special cases in section §. Let $y^2 = x^3 + ax + b$, with $a, b \in k[[t]]$, be a standard Weierstrass equation with $j \neq \infty$ (see below for the case $j = \infty$). Let us denote with $\nu_0(f)$ the order of vanishing at $t = 0$ of a power series $f \in k[[t]]$. In the following table we set $N := \min(3\nu_0(a), 2\nu_0(b))$ and the first row will give a condition on $N$, the second will give the Euler characteristic of the corresponding Kodaira fibre (see [12] table (IV.3.1) page 41) and the third will have a symbol we associate to the corresponding singularity.

| $N$ | 0 | 6 | 2 | 10 | 3 | 9 | 4 | 8 | 12 |
|-----|---|---|---|----|---|---|---|---|----|
| $\chi$ | $n$ | $n + 6$ | 2 | 10 | 3 | 9 | 4 | 8 | $L$ |

| type | $I$ | $I^*$ | $II$ | $II^*$ | $III$ | $III^*$ | $IV$ | $IV^*$ | $L$ |

here $n$ is related to the order of vanishing of $\Delta$ (see remark below).

**Remark 3.3.1.** Our notation differs from Kodaira’s in that we identify all the $I_n$ and $I_n^*$ to two categories, namely: $I$ and $I^*$. In Kodaira’s notation the former are characterized by the order of vanishing of the discriminant $\Delta = 4a^3 + 27b^2$ (in the $I_n$ case $\nu_0(\Delta) = n$ and in the $I_n^*$
case \( \nu_0(\Delta) = n + 6 \). The reason for our notation stems from the fact that in our analysis we need not distinguish among them. Also the \( L \) case does not appear in the classical literature, because it is not a rational double point singularity. It is indeed elliptic, and hence it has moduli (for instance the \( j \) invariant of the exceptional curve) as opposed to the classical cases that do not.

Note also that if a Weierstrass equation is isotrivial with \( j = \infty \) then it is of the form
\[
y^2 = x^3 - 3\lambda^2(t)x + 2\lambda^3(t)
\]
and it can be transformed to \( y'^2 = x'^2(x' + 2\lambda(t)) \) (and vice versa). This shows that the last equation is minimal if and only if \( \nu_0(\lambda(t)) < 2 \), which could also be argued by directly computing the discriminant.

4. Abramovich-Vistoli’s fibred surfaces and prestable reduction

4.1. Abramovich-Vistoli fibred surfaces. D. Abramovich and A. Vistoli in [AV2] define families of fibered surfaces in order to compactify the moduli space of fibered surfaces, that is to say surfaces \( X \to C \) mapping flatly and properly to a curve with stable fibers and with a number of sections \( \sigma_1, \ldots, \sigma_n \) (or equivalently, to compactify the moduli space of Kontsevich stable curves into the Deligne-Mumford stack of stable \( n \)-pointed curves with fixed genus).

We will need to borrow the following definitions and results from their work:

Let \( \Gamma \) be a finite group acting on a family of Deligne-Mumford stable curves \( Y \to V \), over some scheme \( V \). Abramovich and Vistoli (in [AV2]) give the following:

Definition 4.1.1. This action is essential if each \( \gamma \in \text{Stab}(v) \), for some geometric point \( v \in V \), acts nontrivially on the fibre \( Y_s \) over \( s \).

Definition 4.1.2. (see Def. 4.1 in [AV2]) Let \( C \to S \) be a flat (not necessarily proper) family of nodal curves, \( X \to C \) a proper morphism with one dimensional fibers, and \( \sigma_1, \ldots, \sigma_\nu : C \to X \) sections of \( \rho \). We will say that \( X \to C \to S \) is a family of generically fibered surfaces if \( X \) is flat over \( S \), and the restriction of \( \rho \) to \( C_{sm} \) is a flat family of stable pointed curves. If \( S \) is the spectrum of a field we will refer to \( X \to C \) as a generically fibered surface.

Definition 4.1.3. (see Def. 4.3 in [AV2]) A triple \((U, Y \to V \to S, \Gamma)\) is called a chart for a family of generically fibred surface \( X \to C \to S \) if there is a diagram:

\[
\begin{array}{ccc}
Y & \to & X \times_C U \to X \\
\downarrow & & \downarrow \\
V & \to & U \to C \\
\downarrow & & \downarrow \\
S & = & S
\end{array}
\]

together with a group action \( \Gamma \subset \text{Aut}_S(Y \to V) \) satisfying:

1. The morphism \( U \to C \) is étale;
2. \( V \to S \) is a flat (but not necessarily proper) family of nodal curves;
3. \( \rho : Y \to V \) is a flat family of stable \( \nu \)-pointed curves of genus \( \gamma \),
4. the action of \( \Gamma \) on \( \rho \) is essential;
5. we have isomorphisms of \( S \)-schemes \( V/\Gamma \cong U \) and \( Y/\Gamma \cong U \times_C X \) compatible with the projections \( Y/\Gamma \to V/\Gamma \) and \( U \times_C X \to U \), such that the sections \( U \to U \times_C X \) induced by the \( \sigma_i \) correspond to the sections \( V/\Gamma \to Y/\Gamma \).

The fibre above \( p \) is called a twisted fibre.
Remark 4.1.4. For our purposes, we do not need the chart $(U, Y \to V, \Gamma)$ to be minimal, i.e., we will not need it to satisfy property 4.

Let $(U, Y \to V \to S, \Gamma)$ be a chart for $X \to C \to S$, then:

**Proposition 4.1.5.** Let $\Gamma' = \text{Stab}(v)$ be the stabilizer at the nodal point $v$ of a geometric fibre $V_t$ of $V \to S$ and let $T_1$ and $T_2$ be the tangent spaces of each branch at the node. Then:

1. $\Gamma'$ is cyclic and it sends each branch of $V_t$ to itself;
2. the generator $\gamma$ of $\Gamma'$ acts on $T_1$ and $T_2$ by multiplication with a primitive root of unity (of the order of $\Gamma'$).

**Proof.** See [N-V2] Proposition 4.5. \qed

In the same situation, Abramovich and Vistoli (in [N-V2]) set the following:

**Definition 4.1.6.** A chart $(U, Y \to V, \Gamma)$ is called balanced if for any nodal point of any geometric fiber of $V$, the action of the two roots of unity describing the action of a generator of the stabilizer on the tangent spaces $T_1$ and $T_2$ of the branches are inverse to each other.

Let $X \to C \to S$ be a family of generically fibered surfaces, $\alpha_1 = (U_1, Y_1 \to V_1, \Gamma_1)$, $\alpha_2 = (U_2, Y_2 \to V_2, \Gamma_2)$ two charts; call $p_i : V_1 \times_C V_2 \to V_i$ the $i$th projection. Consider the scheme

$$I = \{ \text{Isom}_{V_1 \times C V_2} (p_1^* Y_1, p_2^* Y_2) \}$$

over $V_1 \times_C V_2$ representing the functor of isomorphisms of the two families $p_1^* Y_1$ and $p_2^* Y_2$. Abramovich and Vistoli call the transition scheme from $\alpha_1$ to $\alpha_2$ the scheme theoretic closure of the section of $I$ over the inverse image $\tilde{V}$ of $C_{\text{sm}}$ in $V_1 \times_C V_2$ corresponding to the isomorphism $p_1^* Y_1 |_{\tilde{V}} \cong p_2^* Y_2 |_{\tilde{V}}$.

**Definition 4.1.7.** Two charts $(U_1, Y_1 \to V_1, \Gamma_1)$ and $(U_2, Y_2 \to V_2, \Gamma_2)$ are compatible if their transition scheme $R$ is étale over $V_1$ and $V_2$.

**Definition 4.1.8.** A family of fibered surfaces

$$\mathcal{X} \to C \to S$$

is a family of generically fibered surfaces $X \to C \to S$ such that $C \to S$ is proper, together with a collection $\{(U_\alpha, Y_\alpha \to V_\alpha, \Gamma_\alpha)\}$ of mutually compatible charts, such that the images of the $U_\alpha$ cover $C$. Such a collection of charts is called an atlas.

A family of fibered surfaces is called balanced if each chart in its atlas is balanced.

The family of generically fibered surfaces $X \to C \to S$ supporting the family of fibered surfaces $\mathcal{X} \to C \to S$ will be called a family of coarse fibered surfaces.

We can now state the theorem of theirs that we are going to be using here:

**Theorem 4.1.9.** Let $\mathcal{X}_\eta \to C_\eta \to \eta$ be a balanced stable fibered surface, with induced map $f_\eta : C_\eta \to \overline{M}_{g,n}$, with sections $\sigma_i \subset C_\eta$ and with sections $D_{\eta_i} \subset C_\eta$. Then there is a finite extension of discrete valuation rings $R \subset R_1$ and an extension

$$\begin{align*}
\mathcal{X}_\eta \times_\Delta \Delta_1 & \subset \mathcal{X}_1 \\
\downarrow & \\
C_\eta \times_\Delta \Delta_1 & \subset C_1 \\
\downarrow & \\
\{\eta_i\} & \subset \Delta_1,
\end{align*}$$

with $\Delta = \text{Spec } R$ and $\Delta_1 = \text{Spec } R_1$, such that:

1. $\mathcal{X}_1 \rightarrow \mathcal{C}_1 \rightarrow \Delta_1$ is a balanced stable family of fibered surfaces with sections $\sigma_i$;
2. there is a regular map $f_1 : \mathcal{C}_1 \rightarrow \overline{\mathcal{M}}_{g,n}$ extending $f_\eta \circ p_1$, where $p_1 : \mathcal{C}_\eta \times_\Delta \Delta_1 \rightarrow \mathcal{C}_\eta$ is the natural projection;
3. $f_1$ is Kontsevich stable
4. $\omega_{\mathcal{X}_1/\mathcal{C}_1}(\sum \sigma_i + \sum \pi_1^* D_i) \otimes f_1^* \mathcal{A}$ is ample, for some ample line bundle $\mathcal{A}$ on $\overline{\mathcal{M}}_{g,n}$.

The extension is unique up to a unique isomorphism, and its formation commutes with further finite extensions of discrete valuation rings.

Proof. See [X-V2] pg.20. prop. 2.1. and pg.28. prop. 8.13 \hfill \Box

4.2. The extension lemma and Prestable reduction. Suppose one is given an elliptic surface $\mathcal{X}_\eta \rightarrow \mathcal{C}_\eta \rightarrow \eta$ over the generic point $\eta$ of a DVR scheme $\Delta$, and with induced $j$-map $j_\eta : \mathcal{C}_\eta \rightarrow \overline{\mathcal{M}}_{1,1}$. If $\mathcal{X}_\eta \rightarrow \mathcal{C}_\eta$ has no cuspidal fibres, then we can apply the theorem of Abramovich and Vistoli (cf. Theorem 4.1.9 above), to extend $\mathcal{X}_\eta \rightarrow \mathcal{C}_\eta$ and $j_\eta$ over the whole $\Delta$. In case there are cuspidal fibres (and our surface is in Weierstrass form), the strategy we will adopt is to temporarily replace these with twisted fibres and mark the corresponding points on the base curve $\mathcal{C}_\eta$.

Once we do that, we want to be able to go back (at least generically) to Weierstrass forms. To this aim we prove the following:

Lemma 4.2.1. Let $S$ be the spectrum of a two-dimensional complete regular local ring, $S = \text{Spec } \mathcal{k}[[s,t]]$, let $W = S \setminus p$ with $p = V(s,t)$ the closed point, set $U = S \setminus V(t)$, $\eta$ the generic point of $\text{Spec } \mathcal{k}[[s]]$ and let $\pi : X_W \rightarrow W$ be a family of curves of genus $1$ with zero section, such that:

1. there is a map $j : S \rightarrow \overline{\mathcal{M}}_{1,1}$, the $j$-invariant;
2. the family $X_W |_U$, is a family of stable elliptic curves;
3. $X_W |_{\mathcal{W}}$, is an elliptic surface in minimal Weierstrass form.

then $X_W$ extends over the whole $S$, to a family $X \rightarrow S$ whose fibre over $s = 0$ is an elliptic surface in minimal Weierstrass form.

Proof. Let us consider the line bundle $\mathcal{L}'$ on $W$ whose dual is: $R^1 \mathcal{p}_* \mathcal{O}_W$. Since $S$ is non-singular, this line bundle extends to a unique line bundle $\mathcal{L}$ over the whole $S$. By the Theorem of the base change in cohomology, $\mathcal{L}^\ast \otimes k(\eta) \simeq (R^1 \mathcal{p}_* \mathcal{O}_X) |_{\mathcal{W}}$, and thus the sections $g_2$ and $g_3$ of $H^0(W_\eta, (\mathcal{L} \otimes k(\eta))^4)$ and of $H^0(W_\eta, (\mathcal{L} \otimes k(\eta))^6)$ respectively, extend to two sections $a$ and $b$ of $H^0(S, \mathcal{L}^4)$ and of $H^0(S, \mathcal{L}^6)$ respectively. So we can consider:

$$X : \{y^2 = x^3 + ax + b\} \subset \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3).$$

Note that since $a^3$ and $b^2$ are sections of the sixth power of the Hodge bundle $\mathcal{L}^6$, $a$ and $b$ are defined up to the transformation: $(a,b) \rightarrow (\lambda^4 a, \lambda^6 b)$, where $\lambda \in \mathcal{O}_S^\times$.

Claim. Let $V := V(t)$, $A := V(a)$ and $B := V(b)$, and let $A = \bigcup A_i \cup hV$ and $B = \bigcup B_i \cup kV$ be respectively the decompositions in irreducible components of $A$ and $B$; then $V$ (if $h$ and $k$ are non-zero) is the only irreducible component that $A$ and $B$ can share.

In fact, if they had another irreducible component in common, say $H$, then, since in $k[[s,t]]$ every prime ideal of height one is principal, there would be an $h \in k[[s,t]]$ such that $H = V(h)$ and so:
\[ X : y^2 = x^3 + h^a a'x + h^m b', \]

where \(a'\) and \(b'\) are non zero on \(H\).

The assumption that the curves in the family are stable away from \(V = V(t)\), is equivalent to the fact either one of \(a'\) and \(b'\) is nonzero on \(H\) and that \(2n\) and \(3m\) are divisible by \(6\). In fact, if that were not the case, by pulling back our family \(X\) via the map:

\[ k[[s, t]] \rightarrow k[[s, t]] \otimes \kappa(h) \]

where \(\kappa(h)\) is the residue field at the prime \((h)\), we would produce a family of curves that is not stable and that cannot be reduced to a stable one by changing \(a\) and \(b\) via the transformation \((a, b) \rightarrow (g^{-2k}a, g^{-3k}b)\) for some \(g \in k[[s, t]]\) and \(k \in \mathbb{Z}\). Therefore \(2n\) and \(3m\) are divisible by \(6\) and we can apply the above mentioned transformation with \(g = h\), therefore \(H\) does not exist.

Furthermore we may assume, according to the purity lemma of Abramovich-Vistoli (see Lemma 2.1, pg.3 in \([R-V]\)), that the components \(A_i\) do not intersect the \(B_i\) away from \(V\). Indeed every point of \(U\) satisfies the hypotheses of the purity lemma. If there were a point \(p \in U\) such that \(p \in A_i \cap B_i\), since \(j_U : U \rightarrow \overline{\text{M}}_{1,1}\) is well defined and extends the map \(j_{U \setminus \{p\}} : U \setminus \{p\} \rightarrow \overline{\text{M}}_{1,1}\) defined by the family over \(U \setminus \{p\}\), according to the purity lemma there would exist a stable family \(X_U \rightarrow U\) extending \(X_{U \setminus \{p\}} \rightarrow U \setminus \{p\}\), and thus \(p\) could not be in \(A_i \cap B_i\).

So, next step is to show:

\textbf{Claim.} \(A_i \cap B_i \cap V = \emptyset\).

Let us assume that some \(A_i\) does meet some \(B_j\) along \(V\). That means that there is some point \(q \in V\) such that \(a\) and \(b\) are both zero at \(q\).

We can write \(a' = t^n a''\) and \(b' = t^m b''\) where \(a''\), \(b'' \in k[[s, t]] \setminus \{t\}\) do not vanish identically along \(V\), and the non negative integers \(n\) and \(m\) can be zero if, respectively \(A\) and \(B\) don't contain \(V\). Since we are assuming that some \(A_i = V(a_i)\) does meet some \(B_j = V(b_j)\) along \(V\), there must be some point \(q \in V\) such that \(a''\) and \(b''\) are both zero at \(q\). The \(j\)-map of the statement is now a map:

\[ j : \text{Spec} k[[s, t]] \rightarrow \overline{\text{M}}_{1,1} \]

that at this point has the form:

\[ j(s, t) = \frac{1728a^3t^{3n}}{4a^3t^{3n} + 27b^2t^{2m}}. \]

This restricts to a map:

\[ j_G : G \rightarrow \overline{\text{M}}_{1,1} \]

of the same form for any divisor \(G\) of \(\text{Spec} k[[s, t]]\). In particular we can restrict \(j\) to \(A_i\) and \(B_j\), to get two morphisms:

\[ j_{A_i} : A_i \rightarrow \overline{\text{M}}_{1,1} \]

and

\[ j_{B_j} : B_j \rightarrow \overline{\text{M}}_{1,1} \]

Since these two maps are induced via restriction by \(j\), they have to coincide at \(p\). But from this we infer a contradiction, since \(j_{A_i} \equiv 0\) and \(j_{B_j} \equiv 1720\). Therefore they cannot meet along \(V\), because by hypothesis \(j : \text{Spec}[[s, t]] \rightarrow \overline{\text{M}}_{1,1}\) is well defined.
Let $M \subset \mathbb{P}^r$ be a projective scheme.

**Definition 4.2.2.** We call Kontsevich pretable a flat family of maps over a scheme $S\ f : C \to M$ if $C \to S$ is a flat family of nodal curves.

and

**Definition 4.2.3.** We call quasiminimal an elliptic surface $X \to C$ such that $X |_{C_{\text{sm}}} \to C_{\text{sm}}$ is in minimal Weierstrass form, where $C_{\text{sm}}$ is the smooth locus of $C$ and such that, for each point $p \in C_{\text{sing}}$ there is a chart $(U, Y \to V, \Gamma)$ with $X \times_C V \simeq Y/\Gamma$ centered at $p$.

The first step towards proving the stable reduction theorem is to show the following theorem.

**Theorem 4.2.4.** Let $\Delta$ be the spectrum of a DVR, $\eta \in \Delta$ the generic point and $(\mathcal{X}_\eta \to C_\eta, Q_\eta, f_\eta : C_\eta \to \overline{M}_{1,1})$, be a triple consisting of a relatively minimal elliptic surface $\mathcal{X}_\eta \to C_\eta$ with section $Q_\eta$ and Kontsevich-stable map to moduli $f_\eta$. Then we can find, after possibly a finite base change $\Delta' \to \Delta$, a map of $\Delta'$-schemes $\mathcal{X}' \to C'$ such that:

1. the fibre over the special point $0 \in \Delta'$, $\mathcal{X}_0 \to C_0$ is the semi-log-canonical union of relatively quasiminimal elliptic surfaces with section $Q_0$;
2. the double curves of $\mathcal{X}_0$ are either stable or twisted fibres;
3. $f_0 : C_0 \to \overline{M}_{1,1}$ is Kontsevich prestable.

**Proof.** We are given $(\mathcal{X}_\eta \to C_\eta \to \eta, f_\eta : C_\eta \to \overline{M}_{1,1})$.

As mentioned in the introduction to this section, we can mark the points of $C_\eta$ corresponding to cuspidal fibres of $\mathcal{X}_\eta \to C_\eta$ : let $\Sigma_{i\eta}$ be such divisor on $C_\eta$. We can then use the theorem of Abramovich-Vistoli (cf. Theorem 4.1.9), to get a triple $(\mathcal{X}' \to C' \to \Delta', Q', f' : C' \to \overline{M}_{1,1})$ consisting of a family of fibred surfaces $\mathcal{X}' \to C' \to \Delta'$, (with sections $s_i : \Delta' \to C'$ extending $\Sigma_{i\eta}$) a $\mathbb{Q}$ – Cartier divisor $Q'$ and a regular map $f' : C' \to \overline{M}_{1,1}$ such that condition 2 and the condition (stronger than 3 above) that $f_0'$ together with the sections $s_j$ be Kontsevich stable hold.

This family coincides with our family of elliptic curves in minimal Weierstrass form $\mathcal{X}_\eta \to C_\eta \to \eta$ over $C'_\eta \setminus \cup_i \Sigma_{i\eta}$. Let $\Sigma = \cup \Sigma_{i\eta}$, the closure in $C'$ of $\Sigma_{i\eta}$, and $\Sigma = \cup \Sigma_i$.

Hence, we can apply lemma 4.2.4 to remove the twisted fibres lying above $\Sigma$ and to replace them with the cuspidal curves induced by the original ones lying above $\Sigma_{i\eta}$, so that the generic fiber $\mathcal{X}'_\eta$ is now isomorphic to the original surface $\mathcal{X}_\eta \to \eta$. According to lemma 4.2.4, $\mathcal{X}'_\eta$ consists of quasiminimal elliptic surfaces. The map $f'_0$ may now be Kontsevich unstable (since we have removed the sections $s_i$), but it is clearly prestable. This concludes the proof. $\square$

We shall refer to theorem 4.2.4 as the **Prestable Reduction Theorem**.

**Remark 4.2.5.** Lemma 4.2.4 allows us to remove the extra sections we added along the cuspidal fibres to use the stable reduction theorem of [N-V2] (see section 4). But this comes with a price: when we do so, the family of maps $j : C \to \overline{M}_{1,1}$ may no longer be stable.

In fact, if there is a rational component $C$ of the central fibre $C_0$ on which $j$ is constant, and which meets the rest of the central fibre in only one point (e.g., if the surface $X = \mathcal{X}_0 |_{C \to C}$ contains two cuspidal curves or more and meets the rest of $\mathcal{X}_0$ transversally along one fibre), then $j$ is no longer stable. Indeed we will see in proposition 4.3.1 that the components of $\mathcal{X}$ that map onto such curves are not stable in the sense of Definition 1.1.3.
4.3. **The log-canonical divisor, Extremal rays and semiampleness.** As mentioned in the previous subsection, we have to deal with components of the central fibre $X_0$ that lie above a rational component of $C_0$ contracted by the $j$-map. These are then *isotrivial quasiminimal* elliptic surfaces, attached to at least one component in two possible ways: either along a smooth or a twisted fibre. By the very definition, for such a surface the $j$-map is constant, therefore the log-canonical divisor (for triples) does not have the contribution coming from $j^*\mathcal{O}_{\overline{M},1}(1)$. Hence the log-canonical divisor for triples equals $K_X + Q$ on such components.

The zero section of such a component is always a log-flipping extremal ray, if this component is attached to the rest of the central fibre only along one fibre, and will be contracted by the log-canonical bundle if the isotrivial component is attached along two fibres.

Indeed we have, even more generally, that this happens, even if the component $X \to C$ is *not isotrivial*, in case the log-canonical divisor is $L = K_X + Q$ (the one for pairs). This last divisor coincides with the log-canonical divisor for triples on the isotrivial components, as we remarked above. This will turn out to be useful when dealing with pairs $(X, Q)$ only.

**Lemma 4.3.1.** Let $X_0 = X \cup X'$ be a decomposition of the central fibre where $\pi : X \to C$ is a quasiminimal elliptic surface, where $C \simeq \mathbb{P}^1$ a smooth rational curve. Then:

1. If $X$ is attached to $X'$ only along one fibre $G$, then:

   $$L \cdot Q = -1$$

   where $Q$ is the zero-section and $L = (K_X + Q)|_{X} = G + Q + K_X$ is the log-canonical divisor;

2. If $X$ is attached to $X'$ along two fibres $G_1$ and $G_2$, then:

   $$L \cdot Q = 0$$

   where $Q$ is the zero-section and $L = (K_X + Q)|_{X} = G_1 + G_2 + Q + K_X$ is the log-canonical divisor.

**Proof.** Let us assume first that the attaching fibres are stable. In this case, the zero section $Q$ goes entirely through the smooth locus of the morphism $\pi : X \to C$. Therefore, if we denote by $K_{X/C}$ the relative canonical divisor:

$$K_{X/C} \cdot Q + Q^2 = c_1(\omega_Q \otimes \pi^*\omega_C) = 0$$

moreover the attaching fibres are reduced and therefore linearly equivalent to the general fibre $F$. Hence, since $\pi^*K_C = -2F$:

$$L \cdot Q = (K_{X/C} + Q + G_1 + \pi^*K_C) \cdot Q = (-2F + F) \cdot Q = -1$$

in the case of one attaching fibre, and:

$$L \cdot Q = (K_{X/C} + Q + G_1 + G_2 + \pi^*K_C) \cdot Q = (-2F + F + F) \cdot Q = 0$$

in the case of two stable attaching fibres.

If the attaching fibres are not stable, then there is a diagram:

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
C' & \xrightarrow{\phi} & C
\end{array}
$$
such that \( \phi : C' \to C \) is a finite morphism with Galois group \( \Gamma \), \( f : Y \to X \) is also finite and \( Y \to C' \) is a relatively minimal elliptic surface. Moreover \( \phi : C' \to C \) is ramified at two points, and:

1. \( \Gamma \simeq \mu_k \) if there is only one twisted attaching fibre with monodromy of order \( h \), in this case the branch points are 0 (where \( (X_0)_{red} = G_1 \)) and infinity;
2. \( \Gamma \simeq \mu_k \) with \( k = \text{lcm}(k_1, k_2) \) when there are two twisted attaching fibres of monodromy of orders \( k_1 \) and \( k_2 \), respectively.

Let \( Q' = f^*Q \) and \( F' = f^*G \). Then, in case (1), by the projection formula we have:

\[
L \cdot Q = \frac{1}{k} f^*L \cdot Q' = \frac{1}{k} ([K_{X'} - (k - 1)F' + F' + Q'] \cdot Q') = -1
\]

since \( f^*G_1 = F' \) and by Riemann-Hurwitz:

\[
f^*K_X = K_{X'} - (k - 1)F'.
\]

In case (2), the Riemann-Hurwitz formula now reads:

\[
f^*K_X = K_{X'} - (k_1 - 1)F' - (k_2 - 1)F'
\]

and ones again, by means of the projection formula, we can conclude:

\[
L \cdot Q = \frac{1}{k} f^*L \cdot Q' = \frac{1}{k} ([K_{X'} - (k_1 - 1)F' + (k_2 - 1)F' + k_1F' + k_2F' + Q'] \cdot Q') = 0.
\]

It will turn out to be useful to generalize these computations for any quasiminimal elliptic surface \( X \to C \) with no limitation on the number of twisted fibres and the genus of the base curve.

Let \( X_0 = X \cup X_R \) a decomposition of the central fibre such that \( X \) meets \( X_R \) in \( r \) attaching fibres (stable and twisted) \( \bigcup_{i=1}^r G_i \). Let \( Q \) be the zero-section of \( \pi : X \to C \) (where \( C \) is the base curve of \( X \)). Let \( k_i \) be order of the monodromy group around \( G_i \), and \( g \) the arithmetic genus of \( C \).

We have:

**Proposition 4.3.2.** With these hypotheses and notations, one has:

\[
L_X \cdot Q = 2g - 2 + r.
\]

**Proof.** The proof is very similar to the proof of the previous lemma. Define \( \overline{r} \) to be the integer that equals \( r \) if \( r \equiv 0(\text{mod } 2) \) and equals \( r + 1 \) if \( r \equiv 1(\text{mod } 2) \).

Let \( p_i := \pi(G_i), p_{r+1} \in C \) some extra point, and let \( I \) to be the set that equals \( \{p_1, p_2, \ldots, p_r\} \). Now let \( g : C' \to C \) be the finite ramified covering with Galois group \( \mu_k \), with \( k = \text{lcm}(k_i) \) totally ramified at the \( p_i \)'s, with \( i \in I \).

Because of the underlying structure of quasiminimal elliptic surface, there is an atlas \( (U_i, Y_i \to V_i, \Gamma_i \simeq \mu_{k_i}) \) such that \( Y_i \to V_i \) is stable. In particular, the normalization of the pull-back \( X' \to C' \) of \( X \to C \) to \( C' \) will be a family of stable curves, and therefore, if we denote by \( Q' \) the pull-back of the zero section \( Q \subset X, Q \) must be entirely contained in the smooth locus of \( X' \). Let \( f : X' \to X \) be the natural morphism.

If we indicate by \( g' \) the genus of \( C' \), it must then be the case that \( (K_{X'} + Q') \cdot Q' = 2g' - 2 \). But by Riemann-Hurwitz applied to \( C' \to C \) we obtain:

\[
2g' - 2 = k(2g - 2) + \overline{r}(k - 1)
\]
The Riemann-Hurwitz formula applied to $f$ reads:

$$K_{X'} = f^* K_X + \sum_{i=1}^r \left( \frac{k}{k_i} - 1 \right) G_i'$$

where $G_i' = (f^* G_i)_{\text{red}}$. Therefore, if we let $Q' = f^* Q$:

$$L_X \cdot Q = \frac{1}{k} (K_{X'} - \sum_{i=1}^r \left( \frac{k}{k_i} - 1 \right) G_i' + \sum_{i=1}^r \frac{k}{k_i} G_i' + Q') \cdot Q',$$

where $G_{\tau} = \emptyset$ if $r \equiv 0 \pmod{2}$ and $G_{\tau}$ is the fiber over the point $p_{r+1}$ otherwise.

Thus, if we let $\epsilon(N)$ be the function over the integers that equals 0 if $N \equiv 0 \pmod{2}$ and 1 otherwise we have:

$$L_X \cdot Q = \frac{1}{k} (k(2g - 2) + \tau(k - 1) - \epsilon(r)k + \tau) = 2g - 2 + r.$$ 

\[\Box\]

**Remark 4.3.3.** Note that a similar computation can be carried out to measure the failure of $X$ to satisfy adjunction, i.e., to compute the different (see [K], page 175 for a definition). Another point worth observing is that if $X$ has only one attaching fibre, in $X \mid_U$:

$$Q^2 = \frac{1}{k} f^* Q \cdot Q' = \frac{1}{k} Q'^2$$

so that this is the contribution a twisted fibre with monodromy $k$ gives to the self intersection of $Q$.

**Remark 4.3.4.** Note that we have showed, en passant, that:

$$f^* L_X = L_{X'} + (1 - k \epsilon(r)) G_{\tau}.$$ 

**Corollary 4.3.5.** Same hypotheses as in Proposition 4.3.2, then:

1. if $2g - 2 + r > 0$, then $L_X$ is ample;
2. if $2g - 2 + r = 0$, and $X \to C$ is non-isotrivial, then $L_X$ is semiample and for any irreducible curve $D$:
   $$L_X \cdot D = 0 \text{ if and only if } D = Q;$$
3. if $2g - 2 + r < 0$ then $Q$ is an extremal ray.

In fact, more generally the same holds for the $\mathbb{Q}$-Cartier divisor:

$$M := K_X + \sum G_i + aQ$$

with $0 < a < 1$ any rational number.

**Proof.** Recall from Remark 4.3.4 that:

$$f^* L_X = L_{X'} + (1 - k \epsilon(r)) G_{\tau},$$

where $f : X' \to X$ is as constructed in the proof of Proposition 4.3.2. Therefore:

$$L_X^2 = L_{X'}^2 + 2(1 - k \epsilon(r)) L_{X'} \cdot G'_{\tau}$$
since $G^2_\tau=0$. The surface $X'$ is now in minimal Weierstrass form, and therefore we can apply the canonical bundle formula (cf. theorem 3.1.2), and write:

$$f^*L_X = (\pi'^*(K_{C'}) + \lambda' + \sum_{i=1}^r p'_i + (1-k\epsilon(r))p'_\tau) + Q',$$

where $p'_i \in C'$ denotes the point whose fibre via $\pi'$ is $G_i$.

Note that $c_1(K_{C'}) + \sum_{i=1}^r p'_i + (1-k\epsilon(r))p'_\tau = k(2g-2+r)$.

We can then conclude (1), (2) and (3) by means of corollary 3.1.3.

We can now conclude also the more general statement about $M$ appealing to the same corollary, and by observing that if one defines the $\mathbb{Q}$-Cartier divisor on $X'$:

$$M' := K_{X'} + \sum G'_i + aQ'$$

one has that:

$$f^*M = M' + (1-k\epsilon(r))G_\tau = (\pi'^*(K_{C'}) + \lambda' + \sum_{i=1}^r p'_i + (1-k\epsilon(r))p'_\tau) + aQ'.$$

5. The special cases and standard elliptic surfaces

5.1. Special cases. Let $Y \to C \to S$ be an elliptic threefold in Weierstrass form

$$Y = V(y^2 - x^3 - ax - b) \subset \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}^2 \oplus \mathcal{L}^3),$$

where $\mathcal{L} = \omega_{Y|C}$. For any fibre $C_t$ of the base $C$ one can define an integer valued function on $C_t$ as:

$$N_{C_t}(q) = \min(\nu_q(a^3 |_{C_t}), \nu_q(b^2 |_{C_t})),$$

where $\nu_q(g)$ is the order of vanishing of any section $g \in \mathcal{O}_{C_t}$ at $q \in C_t$. In particular we will simply write $n(q)$ for this function on the central fibre $C_q$.

If $C$ contains a negative rational curve $E$ (not necessarily irreducible), one can contract it to obtain a morphism $\rho : C \to C'$. On $C'$ one can then construct an elliptic threefold $Y' \to C'$ in the following manner. Let $p'$ be the image of $E$ and $W' = C' \setminus p'$. Then, one naturally has a Weierstrass equation on $W'$ induced by $Y'|_{W'}$. In fact $\rho$ induces an isomorphism $\rho_W : W \to W'$ where $W = C \setminus E$, and, letting $\mathcal{L}_{W'} = \rho_*\omega_{Y/W}$ and $a'_{W'}$ and $b'_{W'}$ respectively the push-forward of the section $a$ on $\mathcal{L}^2 |_W$ and $b$ on $\mathcal{L}^3 |_W$, one can then define an elliptic threefold in Weierstrass form:

$$Y' = V(y^2 - x^3 - a'x - b') \subset \mathbb{P}(\mathcal{O}_{C'} \oplus \mathcal{L}'^2 \oplus \mathcal{L}'^3).$$

Since $C'$ is smooth at $p'$ one can then extend the line bundle $\mathcal{L}'_{W'}$ uniquely to a line bundle $\mathcal{L}'$ on $C'$, and correspondingly uniquely extend the sections $a'_{W'}$ and $b'_{W'}$ to global sections $a'$ and $b'$ of $\mathcal{L}'^2$ and $\mathcal{L}'^3$ respectively. To put it differently, we take the saturation $\mathcal{L}' = ((\rho_*\mathcal{L}^v)^v$ and the corresponding sections $a'$ and $b'$ to construct the Weierstrass equation presented above.

In general, given an elliptic threefold $X \to C$ (not necessarily in Weierstrass form) with no multiple fibres, but possibly twisted fibres, and a contraction of a negative curve of the base $\rho : C \to C'$, one can define an elliptic threefold $X''$ over $C'$ as the elliptic threefold $X''$ extension of a local Weierstrass model around the image point. That is to say we choose a Zariski neighborhood $Z$ of $E$ in $C$ such that the only possible twisted fibres are above the nodes of $E$. Let $W$ be a Zariski neighborhood of $E$ in $Z$ such that $X_W$ has a Weierstrass
representations away from the nodes of $E$. \( \rho : C' \to C \) induces a map \( \rho : W \to W' \) to a Zariski neighborhood of \( p' \in C' \). Now apply the construction above to \( W' \) and \( W \). This construction may lead to a threefold which does not have log-canonical singularities, even if \( X \) had log-canonical singularities to begin with.

Let \( X \to C \to \Delta \) be a 1-parameter family of elliptic surfaces with section \( Q \), with general fibre \( X_0 \) in minimal Weierstrass form. Assume furthermore that there is a component \( X_1 \to C_1 \) of the central fibre \( X_0 \) that maps to a rational nodal not necessarily irreducible curve \( C_1 \subset C \), that \( X_1 \) is attached (transversally) to another component \( X_2 \to C_2 \) of \( X_0 \) along only one fibre \( G \) (twisted or stable). One can then contract \( C_1 \) in \( C \) to a point \( p \in C' \) and therefore emulate the construction of \( X' \) above. The question one must ask oneself is: when does it happen that the surface \( X_2' \), image of \( X_2 \) via the contraction of \( X_1 \) has log-canonical singularities? Proposition 5.3.4 will answer just that. We call the cases when this occur, the special cases.

In this chapter we will use (mostly implicitly) the following well known result:

**Lemma 5.1.1.** The contraction of \( C_1 \) in \( C \) produces a surface \( C' \) that is smooth in a Zariski neighborhood of the image point.

### 5.2. The Idea

The idea of the construction of the log-canonical model, is based on the possibility (cf. section 4.1) of taking an étale neighborhood \( V \) of \( p \in C \) such that the pull-back of \( X' \) to \( V \) has a nodal fibre at \( p \), i.e., of finding a chart (not necessarily minimal) \( Y \to V, U, \Gamma \) at least of a Zariski neighborhood \( W \) of \( p \in C \). Therefore \( Y \to V \) has a Weierstrass model and we can naively contract the rational components of the central fibre of \( C \to S \) and write “explicit” equations for the analytic singularity we thus obtain.

### 5.3. Standard elliptic surfaces

Let us assume that \( C_1 \) is irreducible and at set \( p = C_1 \cap C_2 \).

Let moreover \( C_R = C_0 \setminus C_1 \). It might happen that the fibre of \( X' \) over \( p \) is twisted. But we can find a chart \( (V, Y \to U, \Gamma) \) such that the fibre over the point lying above \( p \) is stable. On \( U \) we can define two divisors \( A \) and \( B \) as follows. Let \( W \) and \( W' \) as in section 5.1, so that \( X_{W', p} \) is in Weierstrass form \( y^2 = x^3 + ax + b \subset \mathbb{P}(\mathcal{O}_{W', p} \oplus L_{W', p}^2 \oplus L_{W', p}) \) Since \( C \) is normal at \( p \), the two sections \( a \) and \( b \) extend to sections all over \( W \). Set \( A = V(a^3) \) and \( B = V(b^2) \).

**Lemma 5.3.1.** We have: \( p \notin A \cap B \).

**Proof.** Let \( U \) be a Zariski neighborhood of \( p \) such that the only twisted fibre of \( X' \mid_U \) is at \( p \) and that there are no cuspidal fibres. Let \( (V, Y \to V', \Gamma) \) be a chart of \( X'_U \to U \) as in section 4.1. Then the thesis holds, because if both \( A \) and \( B \) passed through \( p \), the family \( Y \to V' \) could not consist of stable curves. \( \square \)

Let \( c : X' \to X' \) be the contraction constructed in the previous section and \( X'_2 = c_*X_2 \). Let \( C_1 \) be an irreducible \((-1\)-cycle in \( C \) meeting the rest of \( C \) transversally in only one point \( p \). Let \( C_2 \) be the irreducible component of \( C \) meeting \( C_1 \).

At this point we ask ourselves what happens to the self intersection of the zero section \( Q_2 \) of the component \( X_2 \) to which \( X_1 \) was attached. The answer is given by the following, where we let \( Q'_2 \) be the zero-section of \( X'_2 \to C'_2 \):

**Proposition 5.3.2.** The self intersection \( Q'_2^2 \) of \( Q'_2 \) in \( X'_2 \) is:

\[
Q'_2^2 = Q_2^2 + Q_1^2
\]

where \( Q_2^2 \) is taken in \( X_2 \) and \( Q_1^2 \) is taken in \( X_1 \).
Proof. Let us consider how the number $Q_2^2 + Q_1^2$ changes after the contraction of $X_1$. Note that it is the same as $(Q_2 + Q_1) \cdot Q$, where $Q$ is the divisor of $\mathcal{X}$ swept out by the zero sections of the fibres of $\mathcal{C} \to S$. On the other hand, $Q_1 + Q_2 \equiv X_0 \cdot Q - X_R \cdot Q$ where $X_R = X_0 - (X_1 + X_2)$ is the rest of the central fibre $X_0$, and $\equiv$ denotes numerical equivalence. Hence

$$Q_2^2 + Q_1^2 = (X_0 \cdot Q - X_R \cdot Q) \cdot Q.$$  

This number does not change after performing the contraction, since this operation does not touch $X_R$ and $X_0 \cdot Q$ is constant as $t$ varies among the geometric points of $\Delta$ because of the flatness of $\mathcal{X} \to \Delta$. So

$$(Q'_2)^2 = Q_2^2 + Q_1^2$$

\[ \square \]

Our goal is to relate the invariant $\sum_{q \in C(p)} N_C(q)$ with the self intersection of the zero section, therefore suggesting that the latter should be treated as an invariant of the geometry near the zero section. This is taken care by:

**Proposition 5.3.3.** Let $X \to \mathbb{P}^1$ an elliptic surface whose singular fibres are all standard Weierstrass equations, except for possibly having at most one twisted fibre over $0 \in \mathbb{P}^1$ whose monodromy group is isomorphic to $\mu_n$. Assume also that away from $0 \in \mathbb{P}^1$, the surface $X$ is in Weierstrass form (not necessarily minimal). Then $12Q^2 = -[\sum_{q \in \mathbb{P}^1 \setminus 0} n_C(q) + \deg(j)]$ and in particular $\sum_{q \in \mathbb{P}^1 \setminus 0} n_C(q) + \deg(j) \leq 12$ if and only if $Q^2 \geq -1$.

**Proof.** Let us assume first that $X \to \mathbb{P}^1$ is not isotrivial. Noether’s formula reads:

$$\chi(O_X) = \frac{K_X^2 + \chi(X)}{12}$$

and therefore, since $K_X^2 = 0$ and $\chi(O_X) = c_1(L)$ (see section 3.1.1) we obtain:

$$\chi(X) = 12c_1(L) = -12Q^2.$$  

But since $X \to \mathbb{P}^1$ is locally trivial in the euclidean (resp. étale) topology away from the cuspidal fibres, the (étale) Euler characteristic $\chi(X)$ must equal:

$$\chi(X) = \chi(X_\eta)\chi(f^{-1}(C \setminus \delta)) + \sum_{p \in \mathbb{P}^1} \chi(X_p),$$

where $X_\eta$ is the generic fibre and $\delta = \{ p \in \mathbb{P}^1 : X_p \text{ is singular } \}$. Hence, since $\chi(X_\eta) = 0$,

$$Q^2 = -\frac{1}{12} \sum_{p \in \delta} \chi(X_p).$$

Performing a case by case analysis of the table in section 3.3 on sees that in all cases but for $I_n$ and $I_n^*$ one has that $\chi(X_p) = N_p$ and in the latter cases $\chi(X_p) = N_p + n$.

Therefore, since by Corollary IV.4.2 of [12] one has that $\deg(j) = \sum_{n \geq 1} n(i_n + i_n^*)$ where $i_n$ and $i_n^*$ indicate respectively the number of fibres of type $I_n$ and $I_n^*$, we conclude that

$$Q^2 = -\frac{1}{12} \sum_{p \in \delta} [\sum_{n \geq 1} N_p + \deg(j)].$$
If there is a twisted fibre $G = (X \otimes k(p))_{\text{red}}$ (for some point $p \in \mathbb{P}^1$), there is a ramified morphism of order $h$ (the monodromy of $G$) of the base $s : C \simeq \mathbb{P}^1 \to \mathbb{P}^1$, totally ramified at 0 and at infinity, and a diagram:

$$
\begin{array}{ccc}
X' & \to & X \\
\downarrow & & \downarrow \\
C & \xrightarrow{s} & \mathbb{P}^1
\end{array}
$$

such that $X' \to C$ is stable.

Thus we conclude by means of the same argument as above and the observations that $Q^2 = \frac{Q_2^2}{h}$ (see remark 4.3.3) and that if $q' \in C$ is a point mapping to $q$ via $s$, $N_{q'} = \frac{N_q}{h}$. If $X \to \mathbb{P}^1$ is isotrivial, there is nothing to prove if $j \neq \infty$, since in that case there are no $I$ or $I^*$ fibres.

\[ \tag{5.3.4} \]

\textbf{Proposition 5.3.4.} The singularity of $X_2'$ in a neighborhood of the fibre over the point $p'$ to which $C_1$ gets contracted is of type $y^2 = x^3 + ax + b$ with $\min(\nu_{q'}(a' | C_2'), \nu_{q'}(b' | C_2')) = \sum_{q \in C \setminus \{p\}} N_C(q) + \deg(j | C)$. In particular $X_2'$ is log-canonical if and only if $\sum_{q \in C \setminus \{p\}} N_C(q) + \deg(j | C) \leq 12$.

\[ \tag{5.3.3} \]

\textbf{Proof.} We can construct $X'$ as in section 5.1.

Let $C_2' = \rho \ast C_2$ and $p = C_1 \cap C_2$.

According to proposition 5.3.3 we have that:

$$
12Q_2'^2 = -\left[ \sum_{q \in C_2'} N_{C_2'}(q) + \deg(j_{C_2'}) \right]
$$

We remark that the fibre over $p'$ will no longer be twisted. Note also that even though $j'$ might only be a rational map on the whole $C'$, (the surface resulting from contracting $C_1$) it extends to a regular map when restricted to any smooth curve $B \subset C'$.

and:

$$
12(Q_1^2 + Q_2^2) = -\left[ \sum_{q \in C_2 \setminus \{p\}} N_{C_2}(q) + \deg(j_{C_2}) \right] - \left[ \sum_{q \in C_1 \setminus \{p\}} N_{C_1}(q) + \deg(j_{C_1}) \right]
$$

On the other end, according to proposition 5.3.2 one has: $12(Q_1^2 + Q_2^2) = 12Q_2'^2$. Hence:

$$
\sum_{q \in C_2} N_{C_2}(q) = \sum_{q \in C_2 \setminus \{p\}} N_{C_2}(q) + \sum_{q \in C_1 \setminus \{p\}} N_{C_1}(q) + \deg(j_{C_1})
$$

since away from $p \in C_2$ we have not changed $X_2 \to C_2$, and thus that $\deg(j_{C_2}) = \deg(j_{C_2'})$.

The proposition now follows from the observation that, since we have not changed $X_2 \to C_2$ away from $p$, it must be that $N_{C_2'}(q) = N_{C_2}(q)$ for every $q \neq p$.

\[ \tag{5.3.2} \]

What this proposition says is that one can contract all these components to begin with, before proceeding with the stable reduction. It is thus convenient to introduce the following generalization of the concept of minimal Weierstrass form:
**Definition 5.3.5.** Let $X \to C$ a be an elliptic surface mapping to a nodal irreducible curve $C$. Such a surface will be named **standard elliptic surface** if the local equation around each fibre above $C_{\text{sm}}$ is a **standard Weierstrass equation** (see definition 3.2.3) and if, for each point $p \in C_{\text{sing}}$ there is a chart $(U, Y \to V, \Gamma)$ centered at $p$ with $X \times_C U \simeq Y/\Gamma$.

We have:

**Theorem 5.3.6.** Same hypotheses as in theorem 4.2.4, then there is a finite base change of DVR schemes $\Delta' \to \Delta$ and a $\Delta'$-family of elliptic surfaces $(\mathcal{X}' \to C', \mathcal{Q}')$ with section $\mathcal{Q}'$, such that:

1. if $\eta'$ is the generic point of $\Delta'$, $\mathcal{X}'_{\eta'} \simeq \mathcal{X}_\eta$;
2. the central fibre $\mathcal{X}'_0$ is composed of standard elliptic surfaces;
3. if $X \to \mathbb{P}^1$ is a component of $\mathcal{X}'_0$ attached along only one fibre (twisted or stable), then $\mathcal{Q}'|_X < -1$ in $X$.

4. if we ask condition 3 only on isotrivial components, then there is a well defined regular map $j' : C' \to \overline{\mathcal{M}}_{1,1}$ and if we set $f' = \pi' \circ j'$, then $\omega_{\mathcal{X}'/\Delta'}(\mathcal{Q}) \otimes f'^*O_{\overline{\mathcal{M}}_{1,1}}(3)$ is ample away from those isotrivial components of the central fibre $\mathcal{X}'_0$ that meet transversally the rest of $\mathcal{X}_0'$ along one or two fibres of $\mathcal{X}_0'$.

**Proof.** According to theorem 4.2.4 there exists a finite morphisms of DVR schemes $\Delta \to \Delta$ and a triple $(\overline{\mathcal{X}} \to \overline{C}, \mathcal{Q}, j : \overline{\mathcal{X}} \to \overline{\mathcal{M}}_{1,1})$ satisfying conditions 1, 2 and $(3)$ is ample away from those isotrivial components of the central fibre $\mathcal{X}'_0$ that meet transversally the rest of $\mathcal{X}_0'$ along one or two fibres of $\mathcal{X}_0'$.

Let $B$ be a connected component of $\overline{C}_0$ consisting of a tree of rational curves (we call such a component $B$ a **tree-like component**), meeting the rest of $\overline{C}_0$ trasversally in one point, and such that in $\overline{\mathcal{X}}_B = \overline{\mathcal{X}}|_B \to \overline{B}$ we have $\mathcal{Q}_B^2 \geq -1$.

The proof will be by induction on the number $N$ of such components.

Let us assume first that $N = 1$. Let $\overline{C}_1$, be a “leaf” (i.e. an irreducible component furthest away from $C_0 \setminus B$) of $\overline{B}$. Let $\rho : \overline{C} \to \overline{C}$ be the contraction of $\overline{C}_1$. If $\overline{X}_1 \to \overline{C}_1$ is isotrivial, the map $\overline{j} : \overline{C} \to \overline{\mathcal{M}}_{1,1}$ descends to a map $\overline{j}' : \overline{C}' \to \overline{\mathcal{M}}_{1,1}$.

Let $\overline{C}_2$ be the component of $\overline{C}_0$ meeting $\overline{C}_1$, let $\overline{C}_2$ be its image in $\overline{C}$ via $\rho$ and $\rho' = \rho(\overline{C}_1)$.

We can apply proposition 5.3.4 to $\overline{C}_1$ and find a threefold $\overline{\mathcal{X}} \to \overline{C}$ that satisfies 1 and 2 (and with a regular map $\overline{j} : \overline{C} \to \overline{\mathcal{M}}_{1,1}$ if $\overline{C}_1$ is $j$-trivial) and such that the local equation of the fibre above $\rho'$ is in **standard Weierstrass form**. The image of the curve $\overline{C}_2$ is again a leaf itself, because there are no other tree-like components. According to Lemma 5.3.3, if $\overline{\mathcal{Q}}$ denotes the divisorial push-forward of $\overline{\mathcal{Q}}$:

$$\overline{\mathcal{Q}}|_{\overline{X}_{2}}^2 = \overline{\mathcal{Q}}|_{\overline{X}_{2}}^2 + \overline{\mathcal{Q}}|_{\overline{X}_{1}}^2.$$

If this number happens to be $\geq -1$, then we can apply the procedure to $\overline{C}'_2$, which is now attached to the rest of $\overline{C}'_0$ only at one point, since we have contracted $\overline{C}_1$. We can inductively iterate this procedure until we get to a component for which the self intersection of the zero-section is less than $-1$.

Let us now do the general case: we assume we know the result for any number $k$ of tree-like components with $k < N$. Let $\overline{B}$ be a tree-like component which has the property that it is attached to all the other tree-like components at one end only (such a component must exist, even though it may only consist of one edge). Then we can apply the previous argument to this tree-like component, and “prune” all its edges. Now the number of tree-like components is $N - 1$, and we can then conclude by induction.
If we perform these operations only for tree-like components $\overline{B}$ for which $\overline{X} \to \overline{B}$ is isotrivial, then we get the desired triple $(\mathcal{X}' \to \mathcal{C}', \mathcal{Q}', f : \mathcal{X}' \overset{j'}{\to} \overline{\mathcal{M}}_{1,1})$ on $\Delta' = \overline{\Delta}$. In this case, the claim about the ampleness of $\omega_{\mathcal{X}'/\mathcal{C}'}(Q) \otimes f^* \mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(3)$ away from those isotrivial components of the central $\mathcal{X}'$ that meet transversally the rest of $\mathcal{X}'_0$ in one or two fibres, is a consequence of theorem 4.1.9. In fact we have only perfomed birational operations to some surfaces with constant $j$-invariant (namely, those surfaces $X$ for which $\mathcal{Q} \mid_{X}^2 \geq -1$) which met the central fibre $\overline{X}_0$ along just one fibre, and so $\mathcal{C}' \to \overline{\mathcal{M}}_{1,1}$ is still Kontsevich prestable, i.e., $\omega_{\mathcal{C}'/\mathcal{C}'} \otimes f^* \mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(3)$ is semiample and ample away from those components $C \subset \mathcal{C}'_0$ that meet the rest of $\mathcal{C}'_0$ in one or two points and that are $j$-trivial; and since $\omega_{\mathcal{X}'/\mathcal{C}'}(Q)$ is relatively ample, we are done.

**Remark 5.3.8.**

A family of minimal elliptic surfaces, possibly after a finite base change, $(1)$ above hold and if $(2)$ holds only for those components $(3)$ away from those isotrivial components of the central $\mathcal{X}'$ that meet transversally the rest of $\mathcal{X}'_0$ in one or two fibres, is a consequence of theorem 4.1.9. In fact we have only performed birational operations to some surfaces with constant $j$-invariant (namely, those surfaces $X$ for which $\mathcal{Q} \mid_{X}^2 \geq -1$) which met the central fibre $\overline{X}_0$ along just one fibre, and so $\mathcal{C}' \to \overline{\mathcal{M}}_{1,1}$ is still Kontsevich prestable, i.e., $\omega_{\mathcal{C}'/\mathcal{C}'} \otimes f^* \mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(3)$ is semiample and ample away from those components $C \subset \mathcal{C}'_0$ that meet the rest of $\mathcal{C}'_0$ in one or two points and that are $j$-trivial; and since $\omega_{\mathcal{X}'/\mathcal{C}'}(Q)$ is relatively ample, we are done.

**Definition 5.3.7.** A pair $(X \xrightarrow{\pi} C, \mathcal{Q})$ will be called strictly prestable if:

1. $X \to C$ is a standard elliptic surface and if for each rational component $B$ of $C$ that meets the rest of $C$ in only one point;
2. $\mathcal{Q} \mid_B^2 < -1$.

Similarly, a triple $(X \xrightarrow{\pi} C, \mathcal{Q}: X \xrightarrow{\pi} C \overset{j}{\to} \overline{\mathcal{M}}_{1,1})$ will be called strictly prestable if condition (1) above hold and if (2) holds only for those components $B$ for which $X \mid_B \to B$ is isotrivial.

Therefore, theorem 5.3.6 says that one can perform the strictly prestable reduction of a family of minimal elliptic surfaces, possibly after a finite base change.

**Remark 5.3.8.**

From the formula $Q^2 = -\frac{1}{12} \sum_{p \in A} \chi(X_p)$ (see proposition 5.3.3) and from table IV.3.1 of [M2], we infer the following table for the contribution of a Kodaira fibre to $Q^2$:

| fibre type | cont. to $Q^2$ |
|-----------|-----------------|
| $I$       | $0$             |
| $I^*$     | $-\frac{1}{2}$ |
| $II$      | $-\frac{1}{6}$ |
| $II^*$    | $-\frac{1}{4}$ |
| $III$     | $-\frac{1}{3}$ |
| $III^*$   | $-\frac{1}{3}$ |
| IV        | $-\frac{1}{3}$ |
| IV*       | $-\frac{1}{3}$ |
| L         | $-\frac{N}{12}$ |

Note also that the contribution to the self intersection is exactly $-\frac{N}{12}$ (see the table in section [1.3]).

6. The Toric Picture

6.1. One attaching fibre. In this section we will show that we can perform the necessary log-flips and log-canonical contractions torically. The first step towards understanding the toric picture in the case of one attaching fibre, is to understand what it looks like on the base curve $C$, or equivalently on the zero-section $Q$.

Let $R$ be a discrete valuation ring (DVR), $\Delta = \text{Spec}(R)$, $\eta$ its generic point and $0$ its special point.

For a toric variety $Z$ with torus $T$ we write $D_Z$ for the complement of the torus $D_Z = Z \setminus T$.

Let $C \to \Delta$ be a family of nodal curves $C$. Assume that the central fibre $C_0$ has a rational component $C_1$ meeting the rest of $C_0$ transversally in only one point $p$, and assume that the singularity at $p$ is an $A_{k-1}$-singularity. Let $C_2$ be the rest of $C_0$, $S$ a divisor of $C$ meeting $C_1$ transversally in only one point and $\mathcal{I} \subset \mathcal{O}_C$ the ideal sheaf of $C_1 \cup C_2 \cup S$.

**Lemma 6.1.1.** Let $C \to \Delta$ as before. Then there is a Zariski open neighborhood $U$ of $C_1$ in $C$, a 2-dimensional toric variety $Z$ and an étale map $t : U \to Z$ such that:

1. the fan of $Z$ is $F = \langle f_1, f_1 + k_f f_2 \rangle \cup \langle f_1 + k_f f_2, f_2 \rangle$ in the lattice $N = f_1 \mathbb{Z} \oplus f_2 \mathbb{Z}$;
2. the pull-back via $t$ of the ideal of $D_Z$ is the ideal $\mathcal{I}(U)$. 
Proof. Let $U \subset \mathcal{C}$ be a Zariski open neighborhood of $C_1$ such that $U \cap (C_1 \cap S) = p$, and such that $S$ is not entirely contained in $U$. We can now contract $C_1$ to a smooth point $q$. Let $c : \mathcal{C} \to \mathcal{C}'$ be such contraction. Let $C'_2 = c'_* C_2$ and $S' = c'_* S$ and let $\mathcal{C}'$ be the ideal sheaf of $C'_2 \cup S'$. Therefore, we can find an étale neighborhood $U''$ of $q \in \mathcal{C}'$ and a map $t'' : U'' \to \mathbb{A}^2_k$ to the toric variety $\mathbb{A}^2_k$. We shrink $U$ and $U''$ if necessary so that $c^{-1}(U'') = U$.

Let $Z'$ be the toric variety whose fan $\mathcal{F}'$ is the union of the cones: $\sigma_1 = \langle f_1, f_1 + f_2 \rangle$, $\sigma_2 = \langle f_1 + f_2, f_1 + 2 f_2 \rangle$, ..., $\sigma_k = \langle f_1 + (k - 1) f_2, f_1 + k f_2 \rangle$, $\sigma_{k+1} = \langle f_1 + k f_2, f_2 \rangle$ in the lattice $N = f_1 \mathbb{Z} \oplus f_2 \mathbb{Z}$; and let $U'$ be normalization of $Z'$ in the function field $k(U)$ of $U$. Thus, by definition we have morphisms $b : U' \to U$ and $t' : U' \to Z'$ such that $t'' \mathcal{O}(DZ) \simeq \mathcal{O}(C'_1 + C'_2 + S')$ where $C'_i = b^* C_i$ and $S' = b^* S$. The surface $U''$ is in fact the minimal resolution of $U$.

The morphism $t'$ induces a rational map $t : U \dashrightarrow Z$. Let $W$ be its graph, then we have a commutative diagram:

$$
\begin{array}{ccc}
U' & \xrightarrow{\nu} & Z' \\
\downarrow b & & \downarrow \beta \\
W & \xrightarrow{\gamma} & Z \\
\downarrow p_1 & & \downarrow p_2 \\
U & \xrightarrow{t} & Z
\end{array}
$$

Where $\alpha$ is the morphism whose existence is ensured by the minimality of $U'$. If we show that $p_1$ and $p_2$ are finite, then $t$ is in fact a regular morphism and by construction it satisfies the properties of the thesis.

But $\beta \circ t' = p_2 \circ \alpha$, and since $t'' F = E$, if $E$ is the exceptional divisor of $b$ and $F$ the one of $\beta$, we have that $p_2$ is finite ($t'$ is étale). Similarly one concludes that $p_1$ is finite and therefore $t$ is a morphism.

Our goal is to look at a strictly prestable family $\mathcal{X} \to \mathcal{C}$ of elliptic surfaces, and in particular the base curve $\mathcal{C}$ is either a family of Kontsevich prestable curves (in the case of moduli of triples) or a surface obtained from a Kontsevich prestable family by contracting -1-curves. So it may happen that $\mathcal{C}$ is singular, but the singularities are of type $A_{k-1}$. In the particular context of lemma 6.1.1, the total space of the family of base curves $\mathcal{C}$ might have a singularity of type $A_{k-1}$ at $p$, in which case we want to be able to find a finite “toric” morphism from a smooth surface. We have:

**Lemma 6.1.2.** In the hypothesis of lemma 6.1.1, then there exist a ramified cyclic covering $f : V \to U$ of order $k$ ramified at $p$ and along some section $S$ and an étale map to a 2-dimensional toric variety $t' : V \to Z'$ such that:

1. the fan of $Z'$ is $\mathcal{F}' = \langle f_1', f_1' + f_2' \rangle \cup \langle f_1' + f_2', f_2' \rangle$;
2. the pull-back via $t'$ of the ideal of the toric divisor $D_{Z'}$ is the ideal of $C'_1 \cup C'_2 \cup S'$ where $C'_2 = f^* C_2$ and $S' = f^* S$;
3. $f$ induces a toric morphism $f : Z' \to Z$ given by $f_1' = f_1$ and $f_2' = kf_2$.

**Proof.** Choose a divisor $S$ that meets $C_1$ transversally in only one point, by shrinking $U$ if necessary, we can apply lemma 6.1.1 and find a toric variety $Z$ and an étale morphism $t : U \to Z$ such that $t^* \mathcal{O}_Z(\mathcal{O}_{f_1} + \mathcal{O}_{f_1+kf_2} + \mathcal{O}_{f_2}) \simeq \mathcal{O}_U(C_1 + C_2 + S)$. Since $t^* \mathcal{O}_Z(f_2) \simeq \mathcal{O}_U(S)$ the cyclic covering corresponds to taking the sub-lattice $N' = f_1' \mathbb{Z} \oplus f_2' \mathbb{Z}$ of $N = f_1 \mathbb{Z} \oplus f_2 \mathbb{Z}$ with $f_1' = f_1$ and $f_2' = kf_2$. $U'$ is the toric variety given by the fan $\mathcal{F}' = \sigma'_1 \cup \sigma'_2$ union of the two cones.
generated by \( \{ f'_1, f'_1 + f'_2 \} \) and \( \{ f'_2, f'_1 + f'_3 \} \) respectively in the lattice \( N' \). From the description of \( Z' \to Z \) we can easily read off the ramification.

Take \( V \) to be the normalization of \( U \) in \( k(Z') \) with the induced morphisms \( f : V \to U \) and \( t' : V \to Z' \). By construction, they satisfy the hypotheses of the lemma. \( \blacksquare \)

Let \( (X \to C \to \Delta, \mathcal{Q}) \) be a family of strictly prestable elliptic surfaces with zero section \( \mathcal{Q} \). Let \( X_1 \subset X_0 \) be component of the central fibre \( X_0 \), mapping down to a rational curve \( C_1 \subset C_0 \); also, let \( X_2 \subset X_0 \) be the rest of \( X_0 \) to which \( X_1 \) is attached along one and only one fibre, which is either stable or twisted. We have:

**Proposition 6.1.3.** Let \( Q_1 = \mathcal{Q} |_{C_1} \) and let \( S \) be a divisor of \( \mathcal{X} \) meeting \( X_1 \) transversally in only one stable fibre. Then there is a Zariski open neighborhood \( U \) of \( Q_1 \) in \( \mathcal{X} \) and a toric variety \( \mathcal{Y} \) with an étale morphism \( T : U \to \mathcal{Y} \) such that:

\[
T^* \mathcal{O}_Y(D_2) \simeq \mathcal{O}_U(X_1 + X_2 + S)
\]

**Proof.** Let \( U \) be a Zariski neighborhood of \( C_1 \) with an étale map from a toric variety \( t : Z \to U \) as in lemma 6.1.1.

Let \( f : V \to U \) be as in lemma 6.1.2. Since \( V \) is smooth (at least in a neighborhood of \( C_1 \cap C_2 \)), by the purity lemma of Abramovich and Vistoli (cf. Lemma 2.1, [N-V2]) the attaching fibre \( F := (X_1 \cap X_2)_{red} \) is a quotient of a stable curve by a cyclic group of order \( h \) dividing \( k \) and the normalization of the pull-back of \( \mathcal{X} |_U \) to \( V \) is a family of stable curves.

Therefore we have a diagram:

\[
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{s} & \mathcal{X} |_U \\
\downarrow{\pi'} & & \downarrow{\pi} \\
V & \xrightarrow{f} & U
\end{array}
\]

such that \( \pi' : \mathcal{V} \to V \) is a family of stable curves and the action of the Galois group \( \Gamma \) of \( f : V \to U \) extends to an action on \( \mathcal{V} \) such that \( \mathcal{V}/\Gamma \simeq \mathcal{X} |_U \).

According to lemma and 6.1.2 we have Zariski neighborhoods \( U \) of \( C_1 \) in \( C \) and \( U' \) of \( C'_1 \) in \( C' \) and toric varieties with étale maps \( U \to Z \) and \( U' \to Z' \) such that the induced map \( Z' \to Z \) is toric.

It is thus enough to show that the pull-back \( \mathcal{Y}' \) of \( \mathcal{Y} \) to \( Z' \) is toric and that the induced action of the Galois group \( \Gamma \) on \( \mathcal{Y}' \) is an action by a subgroup of the torus. In this case the quotient \( \mathcal{Y}/\Gamma \to \mathcal{X} |_U \) would be the étale map in the statement of the proposition.

Indeed, the total space of the normal bundle \( \mathcal{N}_{\mathcal{Q}/\mathcal{X}} |_U \) of the zero section \( \mathcal{Q} \) in \( \mathcal{X} \) is such a space, since every line bundle on a toric variety is a toric bundle, according to the proposition of page 63 in [F]. Thus if we can show that the action of \( \Gamma \) on \( C' \) lifts to an action of \( \Gamma \) on \( \mathcal{Y} \) and that \( \Gamma \) acts as a subgroup of the torus on \( \mathcal{Y} \).

But \( \Gamma \) is a finite group and for each point \((u, v) \in \mathcal{Y} \) it acts on the element \( v \) of finite dimensional vector space \( \mathcal{N}_{\mathcal{Q}/\mathcal{X}} \otimes \kappa(u) \), where \( \kappa(u) \) is the residue field of \( u \in V \). Therefore \( \Gamma \) must act linearly on \( \mathcal{N}_{\mathcal{Q}/\mathcal{X}} \), i.e., via multiplication by a character, and since \( \Gamma \) acts as a subgroup of \( \mathbb{C}^{*2} \) on \( U \) we are done. \( \blacksquare \)

In the following lemma, \( Q_1^2 \) denotes the self-intersection as \( \mathcal{Q} \)-divisor of \( Q_1 \) in \( X_1 \).

**Lemma 6.1.4.** In the same hypotheses as proposition 6.1.3, the fan \( \mathcal{F} \) of the toric variety \( \mathcal{U} \), is \( \mathcal{F} = \sigma_1 = \langle e_1, e_1 + e_2, e_3 \rangle \cup \sigma_2 = \langle e_3, e_1 + e_2, w \rangle \) in the lattice \( \mathcal{N} := e_1 \mathbb{Z} \oplus w \mathbb{Z} \oplus e_3 \mathbb{Z} \) where
\{e_1, e_2, e_3\} is the standard basis of \(\mathbb{R}^3\) and \(w = \frac{1}{k}(e_2 + ne_3)\). Here \(-n = kQ_1^2\) (see remark 4.3.3) and \(C\) has an \(A_{k-1}\) singularity around \(C_1 \cap C_2\).

Proof. Let

\[
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\
\pi' \downarrow & & \downarrow \pi \\
\mathcal{C}' & \xrightarrow{g} & \mathcal{C}
\end{array}
\]

be the diagram as in 6.1.3. In an étale neighborhood of \(C_1\) the surface \(C\) is described by the quasi-projective toric surface defined by the fan \(\Delta\) obtained by the two cones generated by \(\{f_1, v = f_1 + kf_2\}\) and \(\{f_2, v\}\) respectively in the lattice \(L = f_1Z + f_2Z + (kf_2)Z\), where \(\{f_1, f_2\}\) is the standard basis for \(\mathbb{R}^2\). This is the surface we refer to in 6.1.3.

The fan of \(V\), i.e., the toric étale neighborhood of \(C'_1\) in \(\mathcal{C}'\) as in 6.1.3, is obtained by taking the two cones generated by \(\{f'_1, f'_1 + f'_2\}\) and \(\{f'_2, f'_1 + f'_2\}\) in the sub-lattice \(L' = L + \frac{1}{2}(f'_2) \subset L\), where \(f'_1 = f_1\) and \(f'_2 = kf_2\). Set \(\Lambda = Hom_Z(L, Z)\) and \(\Lambda' = Hom_Z(L', Z)\), and let \(\langle , \rangle : \Lambda'/\Lambda \times L/L' \rightarrow \mathbb{Q}/\mathbb{Z}\) the natural pairing.

The Galois group \(\Gamma\) as in 6.1.3 is isomorphic to \(L/L' \simeq \mathbb{Z}/k\mathbb{Z}\) and its action on the ring of functions of \(V\) is given by:

\[\gamma \cdot \chi^{n'} = e^{2\pi i(\gamma, n')} \cdot \chi^{n'},\]

for \(\gamma \in \Gamma\) and \(n' \in L'\); i.e., by:

\[\gamma u = u\quad \text{and} \quad \gamma uv = \zeta uv,\]

where \(u = \chi^{R'}, uv = \chi^{R'+1}\) and \(\zeta = e^{2\pi i k}\) is a primitive \(k\)-th root of unity. We refer to \(\mathbb{F}\) for the notation. Let \(\{e_1, e_2, e_3\}\) be the standard basis of \(\mathbb{R}^3\).

The fan of \(Y\) is the union of the two cones \(\sigma'_1\) and \(\sigma'_2\) generated by \(\{e_1, e_1 + e_2, e_3\}\) and \(\{e_1 + e_2, w'\}\) respectively, in the lattice \(N' = e_1Z + e_2Z + e_3Z + w'Z + (e_1 + e_2)Z\), for some vector \(w'\).

We want to find \(w'\). Well, we know that the projection onto \(e_2\mathbb{R} \oplus e_3\mathbb{R}\) along \(e_1 + e_2\) is the fan of a toric étale neighborhood of \(Q'_1\) in \(X'_1\), where \(X'_1 = \mathcal{X}' \times_{C_1} C'_1\) and \(Q'_1 = Q' \cap X'_1\) is the corresponding zero section. Let \(\pi_{e_1+e_2} : \mathbb{R}^3 \rightarrow e_2\mathbb{R} \oplus e_3\mathbb{R} \simeq \mathbb{R}^2\) be such projection. Then \(\pi_{e_1+e_2}(x, y, z) = (y-x, z)\) and \(\pi_{e_1+e_2}(w') = (-1, n)\). But \(w'\) must project onto the vector \(e_2\) via the projection \(\pi_{e_3} : \mathbb{R}^3 \rightarrow e_1\mathbb{R} \oplus e_2\mathbb{R} \simeq \mathbb{R}^2\), since the latter maps the cone of \(Y\) onto the cone of \(V\), as the ray \(e_1\mathbb{R}\) represents the zero section of \(Y\). Hence \(w' = e_2 + ne_3\).

In order to find the fan of \(Y/\Gamma\) as in 6.1.3 we need to identify a lifting in \(N'\) of the sublattice \(L\) of \(L'\). So \(N = e_1Z + e_2Z + e_3Z + w'Z + (e_1 + e_2)Z\), and we want to find \(w\). Let \(\pi_{e_3} : \mathbb{R}^3 \rightarrow e_1\mathbb{R} \oplus e_2\mathbb{R}\) be the projection. Since the divisor corresponding to \(e_3\) is the zero section \(Q'\), \(\pi_{e_3}(w) = \frac{1}{k}e_2\). Also, the action of \(\mathbb{Z}/k\mathbb{Z}\) on the divisor \(S\) corresponding to \(w\) (i.e., the pull-back to \(Y\) of the divisor in \(X\) corresponding to \(S \subset C\), with the notation as in lemma 6.1.3), is trivial, therefore it must be trivial on \(X'_1 \cap Y\). Thus \(\pi_{e_1+e_2}(w)\) must lie on \(w' = e_2 + ne_3\); from this and from \(\pi_{e_3}(w) = \frac{1}{k}e_2\), we infer that \(w = \frac{1}{k}(e_2 + ne_3)\).

6.2. Two attaching fibres. The analogous lemmas and proposition hold for the case of a chain of rational curves joining two curves in the central fibre:

Let \(C \rightarrow \Delta\) be a family of nodal curves \(C\). Assume that the central fibre \(C_0\) has a chain of rational components \(C = \bigcup_{i=1}^{N} C_i\) meeting the rest of \(C_0\) transversally in only two points \(q_1\) and \(q_2\). The singularities of \(C\) around \(p_i = C_i \cap C_{i+1}\) and the \(q_i\) are at worst \(A_{k_i-1}\) singularities, \(i = 1, \ldots, N+1\).

We have:
Lemma 6.2.1. Let $B$ be the rest of $C_0$ and $\mathcal{I} \subset \mathcal{O}_C$ the ideal sheaf of $C \cup B$. Then there is an étale neighborhood $U$ of $C$ in $\mathcal{C}$, a 2-dimensional toric variety $Z$ and an étale map $t : U \to Z$ such that:

1. the fan of $Z$ is $F = \bigcup_i \sigma_i$ where $\sigma_1 = \langle f_1, f_1 + k_1f_2 \rangle$, $\sigma_i = \langle f_1 + \sum_{j=1}^{i-1} k_j f_2, f_1 + \sum_{j=1}^{i} k_j f_2 \rangle$ and $\sigma_{r+1} = \langle f_1 + \sum_{j=1}^{r} k_j f_2, f_1 \rangle$.
2. the pull-back via $t$ of the ideal of $D_Z$ is the ideal $\mathcal{I} \otimes \mathcal{O}_U$.

Proof. The proof is analogous to the one of lemma 6.1.1, except for a few variations. Let $W$ be a Zariski neighborhood of $C$ such that $W \cap (C \cap B) = \{p, q\}$.

We can now contract $C$ to a rational double point of type $A_r$ where $r + 1 = \sum_i k_i$. Let $\rho : \mathcal{C} \to \mathcal{Y}$ be the contraction map and let $W' = \rho(W)$ and $\rho'(C)$. Choose $U'$ to be an étale neighborhood of $W'$ that splits the node. Hence, there is an isomorphism $\phi' : R = k[[x, y, t]]/(xy - t^{r+1}) \to \mathcal{O}_{W', \rho'}$, such that the pull-back of the maximal ideal $\mathfrak{m}_{W', \rho'} \subset \mathcal{O}_{W', \rho'}$ is the maximal ideal $(x, y, t) \subset R$.

The homomorphism of complete local rings $\phi'$ produces an étale map $\mathcal{C}' \to \text{Spec } R$ for some étale neighborhood $U'$ of $\rho'$ in $\mathcal{Y}$. Note that the maximal ideal $\mathfrak{m}_{C', \rho}$ is generated by local equations of the branches of $B \cap U$.

Let $Z'$ be the toric variety whose fan $F'$ is the union of the cones $\sigma_1, \ldots, \sigma_r, \sigma_{r+1}$ respectively generated by $\{f_1, f_1 + f_2\}, \ldots, \{f_1, f_1 + (r+1)f_2\}$, and $\{f_1 + f_2, f_1 + (r+1)f_2\}$ in the lattice $N = f_1\mathbb{Z} \oplus f_2\mathbb{Z}$.

Let $U' \to Z'$ be the normalization of $U$ in the function field of $Z'$. As in lemma 6.1.1 we construct a commutative diagram:

$$
\begin{array}{ccc}
U' & \xrightarrow{t'} & Z' \\
\downarrow \psi & & \downarrow \beta \\
Z & \xrightarrow{t} & Z \\
\end{array}
$$

where $Z$ is the toric variety whose fan is $F = \bigcup \sigma_i$ where $\sigma_i = \langle f_1 + \sum_{j=1}^{i} k_j f_2, f_1 + \sum_{j=1}^{i} k_j f_2 \rangle$ in $f_1\mathbb{Z} \oplus f_2\mathbb{Z}$, for $i \neq 1, r + 1$ and where $\sigma_1 = \langle f_1, f_1 + f_2 \rangle$, and $\sigma_{r+1} = \langle f_1 + f_2, f_1 \rangle$.

An argument similar to the one given there, shows that the rational map $U \dashrightarrow Z$ is in fact regular.

Let $B'' = c'B$ and let $\mathcal{I}'$ be the ideal sheaf of $B$. By construction the pull-back of the ideal sheaf of $D_Z$ is the ideal sheaf $\mathcal{I} \otimes \mathcal{O}_U$, by construction.

As in the case of one attaching fibre, we want to be able to find a toric finite covering that “untwists” all the fibres of $\mathcal{Y} \to \mathcal{C}$ above the $p_i$’s. In the hypothesis of lemma 6.2.1 let the possible $A$-singularity of $\mathcal{C}$ at the points $p_i$ be of type $A_{k_i-1}$. We have:

Lemma 6.2.2. There is a $\mu_n \times \mu_n$-covering $f : V \to U$ ramified along $C$ and along the two branches of $B$ coming off $C$ and an étale map to a 2-dimensional toric variety $t' : V \to Z'$ such that:

1. the fan of $Z'$ is $F' = \sigma'_1 = \langle f'_1, (f'_1 + k_1f_2) \rangle \cup \ldots \cup \sigma'_i = \langle f'_1 + \sum_{j=1}^{i-1} k_j f_2, f'_1 + \sum_{j=1}^{i} k_j f_2 \rangle \cup \ldots \cup \sigma'_{r+1} = \langle f'_2, f'_1 + \sum_{j=1}^{r+1} k_j f_2 \rangle$ in the lattice $N' = f'_1\mathbb{Z} \oplus f'_2\mathbb{Z}$.
2. the pull-back via $t'$ of the toric ideal sheaf $D_Z$ is the ideal sheaf $\mathcal{I}_{C' \cup B' \cup S'}|_V$ of $C' \cup B' \cup S'$ restricted on $V$, where $B' = f'B$;
3. $f$ induces a toric morphism $f : Z' \rightarrow Z$ given by $f'_1 = nf_1$ and $f'_2 = nf_2$.

Proof. We can apply lemma 6.1.1 and find a toric variety $Z$ and an étale morphism $t : Z \rightarrow U$ such that $t^*\mathcal{O}_Z(\emptyset_{f_1} + \sum_{i=1}^N \emptyset_{f_1 + k_i f_2} + \emptyset_{f_2}) \simeq \mathcal{O}_U(B + C)$.

Let $N'$ the sub-lattice of $N' = f'_1\mathbb{Z} \oplus f'_2\mathbb{Z}$ of $f_1\mathbb{Z} \oplus f_2\mathbb{Z}$ with $f'_1 = nf_1$ and $f'_2 = nf_2$ and let $Z'$ be the toric variety given by the fan $F' = \sigma'_1 \cup \bigcup_i \sigma'_i \cup \sigma'_{i+1}$ union of the cones $\sigma'_1 = \langle f'_1, (f'_1 + k_1 f_2) \rangle$, $\sigma'_i = \langle f'_1 + \sum_{j=1}^{i-1} k_j f'_2, f'_1 + \sum_{j=1}^i k_j f'_2 \rangle$ and $\sigma'_{i+1} = \langle f'_2, f'_1 + \sum_{j=1}^{i+1} k_j f'_2 \rangle$ in the lattice $N'$.

We obtain the desired ramified finite covering $V \rightarrow U$ by letting $V$ be the normalization of $U$ in the function field of $Z'$.

What we have in mind is to look at those families of elliptic surfaces with section $(\mathcal{X} \rightarrow \mathcal{C} \rightarrow \Delta, \mathcal{Q} \rightarrow \Delta)$ such that the central fibre contains a chain of rational components meeting the rest of the central fibre in only two fibres, and show that the picture is toric in this case too. This will allow us to perform the small contractions torically.

So, let $X = \bigcup_{i=1}^n X_i \subset \mathcal{X}_0$ be a chain of components of the central fibre, mapping down to a chain of rational curves $C = \bigcup_{i=1}^n C_i \subset \mathcal{C}_0$. Assume that $C$ meets $B$, the rest of $\mathcal{C}_0$, transversally in only two points, and correspondingly $X$ is attached to the rest of $\mathcal{X}_0$ along two fibres that we assume to be either stable or twisted. Let $Q = \bigcup_i Q_i = Q \mid_C$

We have:

Proposition 6.2.3. There is an étale neighborhood $\mathcal{U}$ of $Q$ in $\mathcal{X}$ and an étale morphism to a toric variety $T : \mathcal{U} \rightarrow \mathcal{Y}$ such that $T^*\mathcal{O}_Y(D_\mathcal{Y}) \simeq \mathcal{O}_U(X \mid_B)$.

Proof. As in proposition 6.1.3, we can find a diagram:

$\begin{array}{ccc}
\mathcal{V} & \xrightarrow{s} & U \\
\pi' \downarrow & & \downarrow \pi \\
\mathcal{V} & \xrightarrow{\pi} & U
\end{array}$

where $\pi' : \mathcal{V} \rightarrow V$ is a family of stable curves and the action of the Galois group $\Gamma \simeq \mu_n \times \mu_n$ of $f : V \rightarrow U$ extends to an action on $V$ such that $\Gamma \backslash V \simeq \mathcal{X}$. Let $k_i$ with $i = 2, ..., N$ and $k_1$ and $k_{N+1}$ be the orders of singularities around $p_i := C_i \cap C_{i+1}$ and $p_j := C \cap B_j$ respectively $(i = 2, ..., N$ and $j = 1, 2)$; and let $n_i$ and $n_j$ be such that the group acting non-trivially on the fibre above $p_i$ is $G_i \simeq \mathbb{Z}/n_i\mathbb{Z}$ and on the one above $q_j$ is $\Gamma_j \simeq \mathbb{Z}/n_j\mathbb{Z}$.

According to lemmas 6.2.1 and 6.2.2 we can find toric varieties and étale morphisms: $t : U \rightarrow Z$ and $t' : U' \rightarrow Z'$ such that $f : Z' \rightarrow Z$ is toric.

We can choose toric cyclic coverings:

$t_i : V_i \rightarrow U$

with Galois group $\mu_{k_i}$ and that desingularize the singularities around points $p_i$. The covering $t : U' \rightarrow U$ factors through the normalization $W$ of the fibre product $V_1 \times_U ... \times_U V_N \times_U V_{N+1}$. Let $\pi_i : W \rightarrow V_i$ be the standard projection.

Since $W$ is smooth around those points $p'_i$ that map to $p_i$ via $t_i \circ \pi_i$ and the fibres of $\mathcal{X}_i = t_i \circ \pi_i^* \mathcal{X}$ above those points are stable. Hence the same is true for $\mathcal{V}$.
Therefore on $\mathcal{V}$ we have an étale neighborhood $U'$ of the zero section $\mathcal{Q}'$ in $\mathcal{V}$ and an étale map $T' : U' \to \mathcal{Y}'$ to a toric variety, when we take as $\mathcal{Y}'$ the total space of the normal bundle of $\mathcal{Q}'$ in $\mathcal{X} : N = \mathcal{N}_{\mathcal{Q}'/\mathcal{X}'}|_{U'}$.

Now the as in proposition 6.1.3 the action of $\Gamma$ on $\mathcal{Y}'$ is linearizable, and therefore it acts as a subgroup of the torus. We take as $T : U \to \mathcal{Y}$ the quotient of $T'$. The statement that $T^*\mathcal{O}_D(D_\mathcal{F}) \simeq \mathcal{O}_U(X + \mathcal{X}^1|_B)$ follows from the analogous statement for $t$ in lemma 6.2.3, since we obviously have a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{Y} & \rightarrow & U \\
\downarrow & & \downarrow \\
Z & \rightarrow & U
\end{array}
$$

Let us keep the notation as in lemma 6.2.3 and set $Q_i = \mathcal{Q}|_{X_i}$. We have:

**Lemma 6.2.4.** The fan of the toric variety $U$ is the union of the cones $\sigma_1$, $\sigma_2$, ... $\sigma_N$, respectively generated by: $\{ t_1 e_1, w_1 = e_1 + k_1 e_2, e_3 \}$, ..., $\{ w_N = e_1 + (k_1 + ... + k_N) e_2 + (n_1 k_2 + (n_1 + n_2) k_3 + ... + (n_1 + ... + n_{N-1}) k_N) e_3, w_{N+1} = e_1 + (k_1 + ... + k_{N+1}) e_2 + (n_1 k_2 + (n_1 + n_2) k_3 + ... + (n_1 + ... + n_N) k_{N+1}) e_3 \}$ in the lattice $L = \frac{1}{n} e_1 \mathbb{Z} \oplus e_3 \mathbb{Z} \oplus (\frac{1}{n} e_2 + \frac{1}{k_1} e_3) \mathbb{Z}$, where the $n_i = Q_i^2$ and $g.c.m(a, h_1) = 1$ are the integers determining the actions on the first fibre (hence the action on all the other fibres is determined by this datum).

**Proof.** Let $(x_1, y_1, t_1), (x_2, y_2, t_2), ..., (x_{N+1}, y_{N+1}, t)$ be coordinates around $q_1 = B_1 \cap C_1, p_1 = B_2 \cap C_2$, ..., $q_2 = C_2 \cap B_2$, so that these a neighborhood of $p_i$ (resp. $q_j$) in $\mathcal{C}$ has equation $x_i y_i = t_i^k$ (resp. $x_j y_j = t_j^k$). As in lemma 6.2.3 we have a diagram:

$$
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\pi} & Z \\
\downarrow & & \downarrow \pi \\
V & \xrightarrow{\pi} & U
\end{array}
$$

where all the morphisms and varieties are toric and $\mathcal{Y}/\Gamma \simeq Z$, where $\Gamma$ is the Galois group.

The fan of $U$ is given by the cones generated by $\{ f_1, v_1 = f_1 + k_1 f_2 \}, \{ v_1, v_2 = f_1 + (k_1 + k_2) f_2 \}, ..., \{ v_N, v_{N+1} = f_1 + (k_1 + ... + k_{N+1}) f_2 \}$, in the lattice generated by $\{ f_1, v_1, ... v_{N+1} \}$ where $\{ f_1, f_2 \}$ is the standard basis of $\mathbb{R}^2$. Hence the fan of $V$ is the union of the cones $\{ f_1', v_1' = f_1' + k_1 f_2', v_2' = f_1' + (k_1 + k_2) f_2', ..., v_N', v_{N+1}' = f_1' + (k_1 + ... + k_{N+1}) f_2' \}$, where $f_1' = n f_1$ and $f_2' = n f_2$. The fan of $\mathcal{Y}$ is the union $\sigma_1' \cup ... \cup \sigma_{N+1}'$ of the cones: $\sigma_1' = \langle e_1, (e_1 + k_1 e_2) \rangle$, $\sigma_2' = \langle e_1 + k_1 e_2, w_1', e_3 \rangle$, ..., $\sigma_{N+1}' = \langle w_N', v_{N+1}' \rangle$, in the lattice $L'$ generated by $\{ e_1, e_1 + k_1 e_2, w_1',..., w_N', v_{N+1}' \}$. Our first goal is to find these vectors.

Let $\pi_{e_3} : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection along $e_3$ onto $e_1 \mathbb{R} \oplus e_2 \mathbb{R}$, and let $\pi_{w_i}' : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection along $w_i'$ onto $e_2 \mathbb{R} \oplus e_3 \mathbb{R}$. The zero section $\mathcal{Q}$ is the toric divisor of $Z$ corresponding to the ray $e_3$, hence all the $w_i$'s must project to $e_1 + (k_1 + ... + k_i) e_2$, since $\pi |_{\mathcal{Q}} : \mathcal{Q} \to U$ is an isomorphism (strictly speaking we should write $\mathcal{Q}|_U$, but with abuse of notation we shall simply write $\mathcal{Q}$ for the rest of the discussion).

Analogously the surface $X_1$ corresponds to the ray $e_1 + k_1 e_2$, the image of the cones $\sigma_1$ and $\sigma_2$ in the lattice $L'/\langle e_1 + k_1 e_2 \rangle$ should give the fan of a toric variety whose only complete toric curve (the zero section) has self intersection $-n_1$. Therefore, the vector $\pi_{e_1+k_1 e_2}(w_1') = (k_2, z)$ must be proportional to $(1, n_1)$, since $\sigma_1$ maps to the cone given by the second quadrant, and so $w_1' = (1, k_1 + k_2, n k_2)$. 


Assuming by inductive hypothesis that \( w'_N = (1, k_1 + \ldots + k_N, n_1 k_2 + \ldots + (n_1 + \ldots + n_{N-1}) k_N) \), we want to show that \( w'_{N+1} = (1, k_1 + \ldots + k_{N+1}, n_1 k_2 + \ldots + (n_1 + \ldots + n_N) k_{N+1}) \). Since \( \pi_{e_3}(w'_N) = (1, k_1 + \ldots + k_{N+1}) \), we just have to show that the last coordinate is the one claimed. Note that \( \pi_{e_3}(w'_{N+1}) = (k_N, z - n_1 k_1 + \ldots + (n_1 + \ldots + n_{N-1}) k_N) \) and and \( \pi_{w'_N}(w_{N-1}) = (-k_{N-1}, (n_1 + \ldots + n_{N-2}) k_{N-1}) \) so \( z = n_1 k_2 + \ldots + (n_1 + \ldots + n_{N-1}) k_N \). All is left to do is to identify the action of \( \Gamma \).

Obviously the vector \( \frac{1}{n} e_1 \) from the lattice of the base lifts to an element of the super-lattice \( L \) of \( L' \). In fact the kernel of the map:

\[
\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/k_1\mathbb{Z}
\]

as in lemma \[\text{[2.3]}\] must be contained in the cone generated by the two adjacent vectors \( e_1 \) and \( e_1 + k_1 e_2 \), since it acts trivially on the fibres above the corresponding point. Note that this kernel is generated by \( \frac{1}{n} e_1 \) and \( \frac{k}{n} e_2 \). We want to know more, namely we want to determine the kernel of the map:

\[
\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/h_1\mathbb{Z}.
\]

and in doing so, lift the vector \( \frac{1}{n} e_2 \) from the lattice of the base curve \( U \). Let \( \pi_{e_3} \) as before, then we are looking for a vector \( w \) such that \( \pi_{e_3}(w) = \frac{1}{n} e_2 \) and such that it represents the action of \( \mathbb{Z}/h_1\mathbb{Z} \) on the fibre. That is to say, we want: \( w = (0, \frac{1}{n}, z) \) with \( z \) such that \( h_1 z \in \mathbb{Z} \). Hence \( w = (0, \frac{1}{n}, \frac{a}{h_1}) \) for some \( a \) with \( g.c.d(a, h_1) = 1 \); the integer \( a \) is completely determined by the action of \( \mu_{h_1} \simeq \mathbb{Z}/h_1\mathbb{Z} \) on \( X' \). This proves the lemma.

\[\square\]

7. Toric log flips and toric log-canonical contractions

7.1. Log-flips. In this section we want to show that we can perform the log-flips torically.

We will need the following:

**Definition 7.1.1.** Given a pair \((X, D)\) consisting of a variety \(X\) and a divisor \(D = D_1 + \ldots + D_n\) (where the \(D_i\) are irreducible and reduced), we say that a pair \((T, D_T)\) consisting of a toric variety \(T\) and its toric divisor, is a **toric étale neighborhood** of a subvariety \(Y \subset X\) if there is an étale neighborhood \(u : U \to X\) of \(Y\) in \(X\) such that:

1. there is an étale map \(t : U \to T\);
2. \(t^* \mathcal{O}_T(D_T) \simeq \mathcal{O}_U(D)\);

where \(\mathcal{O}_U(D) := u^* \mathcal{O}_X(D)\).

Let \((\mathcal{X} \to \mathcal{C} \to \Delta, \mathcal{Q}, \ldots)\) be a family of strictly prestable elliptic surfaces over a DVR scheme \(\Delta\). Assume that the special fibre \(X_0 \to C_0\) contains a surface \(X_1 \to C_1\) over a rational curve \(C_1\) and a surface \(X_2 \to C_2\) along a fibre \(G_1\) which is either stable or a twisted curve. Let \(Q_i = X_i \cap Q\). In what follows, we will say that a triple \((\mathcal{Y} \to \mathcal{S} \to \Delta, \mathcal{D}, \mathcal{Y} \to \overline{\mathcal{M}}_{1,1})\) is **locally stable** if the naturally induced triple \((\mathcal{Y} \to \mathcal{S} \to \Delta, \mathcal{D}, \mathcal{Y} \to S \times \overline{\mathcal{M}}_{1,1})\) is stable.

We have:

**Theorem 7.1.2.** The log-flip of \(Q_1\) gives rise to a locally stable family \((\mathcal{X}^+, \mathcal{Q}^+) \to \mathcal{C}^+ \to \Delta\), or, which is the same the triple \((\mathcal{X}^+, \mathcal{Q}^+, \mathcal{X}^+ \to \mathcal{C}^+ \times \overline{\mathcal{M}}_{1,1})\) is stable. Moreover the central fibre decomposes as a union of surfaces \(X^+_1 \cup X^+_2\) where \(X^+_1\) is obtained from \(X_1\) by contracting \(Q_1\) and log-flipping produces a new rational curve \(Q^+_1\) which meets \(Q^+_2\) (the zero section of \(X^+_2\) ) within the smooth locus of \(X^+_2\). Furthermore, there exists an étale toric neighborhood \((T, D_T)\)
whose fan $F^+ \subset N$ is given by (keeping the notation as in lemma 6.1.4) the union of the two cones: $\sigma_1 = \langle e_1, e_2, e_3 \rangle$ and $\sigma_2 = \langle e_1 + e_2, w \rangle$. This singularity is canonical, and in particular $X_0^+$ is semilog-canonical.

Proof. First, we want to show that it is enough to perform the log-flip in an étale neighborhood of $Q$ in $\mathcal{X}$.

In fact, if $\mathcal{U}$ is an étale neighborhood of $Q$ in $\mathcal{X}$, let $\mathcal{U}^+$ be the log-flip of $Q$. It is clear that $\mathcal{X} \setminus \mathcal{U}$ and patch together to form an Artin algebraic (or analytic) space $\mathcal{X}^+$, so we just need to show that we can find an ample line bundle on it.

Let $S \subset \mathcal{C}$ any effective horizontal divisor meeting $C_1$ transversally. After possibly an étale base change $\Delta' \to \Delta$ we can make sure that $S$ is a section around $C_1$, so we may assume it is to begin with. So the bundle $\omega_{\mathcal{X}/\Delta} \otimes \mathcal{O}_\mathcal{X}(Q) \otimes \mathcal{O}_\mathcal{X}(\mathcal{X} | S)$ is $\mathbb{Q}$-Cartier and contracts $Q$, and is $\Delta$-ample on $\mathcal{X}^+$, since by hypothesis $Q$ is the only curve on which $\omega_{\mathcal{X}/\Delta} \otimes \mathcal{O}_\mathcal{X}(Q)$ fails to be positive. It therefore suffices to construct $\mathcal{X}^+$ as an algebraic space, because the fact that it possesses an ample divisor makes it automatically a scheme.

From lemmas 6.1.3 and 6.1.4 we know that there is an étale neighborhood of $Q_1$ in $\mathcal{X}$ that is toric and whose fan $F$ is the union of the two cones $\sigma_1 = \langle e_1 e_2, e_3 \rangle$ and $\sigma_2 = \langle e_3, e_1 + e_2, w \rangle$ in the lattice $N := \bigoplus_{i=1,3} \mathbb{Z} \oplus w \mathbb{Z}$, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$ and $w = \frac{1}{k}(e_2 + ne_3)$. The log-flip is thus constructed by taking the fan $F^+ = \sigma_1^+ \cup \sigma_2^+$ where $\sigma_1^+ = \langle e_1, w, e_3 \rangle$ and $\sigma_2^+ = \langle e_1 + e_2, w \rangle$ (see [1], Theorem 2.4), and now the curve $Q^+$ corresponds to the face generated by $\{e_1, w, w - e_1\}$ and $X_2^+$ to the vector $e_1 + e_2$.

The matrix $A$ associated to the vectors $\{e_1, e_3, w\}$ of the lattice $N$ is:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & \frac{1}{k} & \frac{n}{k}
\end{pmatrix}.
\]

Since $\det(A) = -\frac{1}{k}$, (i.e., $A \in SL_3(\frac{1}{k} \mathbb{Z})$) these vectors form a lattice basis for $N$, hence the toric variety corresponding to the cone $\sigma_1^+$ is smooth (see [F] page 29). The rest of the proposition is obvious. \hfill \Box

In particular, what this says is that if we have a chain of rational curves $C_1 \cup \ldots \cup C_n \subset C_0$ above which the zero-section is an extremal ray, we can contract them one by one and perform the log-flips above them inductively.

Remark 7.1.3. We can ”straighten up” the lattice $N$ in which the fans $F$ and $F^+$ live. Indeed, by applying the transformation:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & k & 0 \\
0 & -n & 1
\end{pmatrix},
\]

we can send $N$ to the lattice $f_1 \mathbb{Z} \oplus f_2 \mathbb{Z} \oplus f_3 \mathbb{Z}$, and the fan $F$ becomes $\langle f_1, f_3, f_1 + kf_2 - nf_3 \rangle \cup \langle (f_1 + kf_2 - nf_3), f_2, f_3 \rangle$ and the fan $F^+$ becomes $\langle f_1, f_2, f_3 \rangle \cup \langle (f_1 + kf_2 - nf_3), f_1, f_2 \rangle$. In particular the singular point of the threefold $\mathcal{X}^+$ through which $Q_1^+$ goes is an $\frac{1}{n}(1, k, 1)$ threecfold singularity.

We need to understand what the different surfaces after the log-flip look like, in particular what their singularities are. We have:
Lemma 7.1.4. With the notation as in theorem 7.1.2. Let \( n' \) and \( a \) be the unique integer such that \( nn' \equiv 1 \pmod{k} \) and \( 0 \leq ak - n' < n \). Then the singularity of \( X_2^+ \) at \( Q_1^+ \cap G^+ \) is an \( A_{k,m} \) singularity, where \( m = ak - n' \).

Proof. Straightforward, using the description of the fan \( F^+ \) given in the remark above, since the surface \( X_2^+ \) corresponds to the ray \( f_1 \mathbb{R}, 0 \geq 0 \), that is to say \( F^+ = \langle e_1, e_2 \rangle \cup \langle e_2, -ke_1 + ne_2 \rangle \).

and:

Lemma 7.1.5. Let \( k' \) be the unique integer such that \( kk' \equiv 1 \pmod{n} \) and \( 0 \leq an - k' < n \). Then the singularity of \( X_1^+ \) at the point to which \( Q_1 \) gets contracted is an \( A_{n,k'} \) singularity.

Proof. For this is more convinent to look at the description of \( F^+ \) in theorem 7.1.2. The surface \( X_1^+ \) corresponds to the ray \( e_1 + e_2 \). Let \( \pi : \mathbb{R}^3 \to \mathbb{R}^3 \) be the projection along that ray onto \( e_2 \mathbb{R} \oplus e_3 \mathbb{R} \). Then \( \pi(w) = (\frac{e_1}{e_2}, \frac{e_2}{e_2}) \) and \( \pi(e_1) = (-1, 0) \). The cone \( \langle \pi(w), \pi(e_1) \rangle \) in the lattice \( e_3 \mathbb{Z} \oplus (\pi(w)) \mathbb{Z} \) is equivalent, via the transformation \( \pi(w) \to f_1 \) and \( e_3 \to f_2 \), to the cone \( \langle f_1, (nf_2 - kf_1) \rangle \), in the fan \( f_1 \mathbb{Z} \oplus f_2 \mathbb{Z} \). The former gives rise to an affine toric variety isomorphic to the one in the statement (sending \( f_1 \) to \( f_2 \) and vice versa).

We will also need:

Lemma 7.1.6. Keeping the notations as above, one has that:

\[
Q_1^{+2} = \frac{1}{Q_1^+}.
\]

Proof. The fan of a toric neighborhood of \( Q_1 \) in \( X_1 \) is given by:

\[
F = \langle e_1, e_2 \rangle \cup \langle e_2, -ke_1 + ne_2 \rangle
\]

in the lattice \( N = e_1 \mathbb{Z} \oplus e_2 \mathbb{Z} \); the one of a toric neighborhood of \( Q_1^+ \) in \( X_2^+ \) is given by:

\[
F^+ = \langle e_1, e_2 \rangle \cup \langle e_2, ke_1 - ne_2 \rangle.
\]

Hence we have that \( Q_1^{+2} = -\frac{n}{k} \) and \( Q_1^{+2} = -\frac{k}{n} \).

When performing a log-flip according to theorem 7.1.2, even if the surface \( X_2 \) were standard to begin with, the surface \( X_2^+ \) in general is not. In fact it is some toric blow-up of one. We are therefore naturally led to the following:

Definition 7.1.7. A log-standard elliptic surface \((Y \to C, Q, G + F)\) is a triple consisting of an elliptic surface \( Y \to C \), its zero section \( Q \) and a marking of \( s \) curves \( G = \bigcup G_i \) and \( N \) of fibres \( F = \bigcup F_j \) with a regular birational map \( g : Y \to Y' \), called structure morphism, to a standard elliptic surface \( Y' \to C \) with zero section \( Q' \) such that:

1. the exceptional divisor \( E = \bigcup E_i \) is the disjoint union of smooth and irreducible rational curves \( E_i \) meeting the \( G_i \) transversally at one point, and \( g(E_i) = p_i = g(G_i) \cap Q \in Y' \);
2. for each \( i \) and \( j \) the curves \( G_i' := g(G_i) \) and \( F_j' = g(F_j) \) are either stable or a twisted fibres of \( Y' \) and \( G_i \) is the proper transform of \( G_i \);
3. for each \( i \) there exist étale neighborhoods \( U \to Y \) of \( E_i \subset Y \) and \( U' \to Y' \) of \( p_i \in Y' \) and morphisms to toric varieties \( t : U \to T \) and \( t' : U' \to T' \);
4. the fan of \( T \) is the union of the two cones \( \langle e_1, e_2 \rangle \cup \langle e_1, ke_1 - ne_2 \rangle \) in the lattice \( N = e_1 \mathbb{Z} \oplus e_2 \mathbb{Z} \), and \( t^*\mathcal{O}(D_T) \simeq \mathcal{O}_U(E_i + G_i + Q) \);
5. the fan of $T'$ is the cone $\langle e_2, ke_1 - ne_2 \rangle$ in $N$, and $t'' \mathcal{O}(D_T') \simeq \mathcal{O}_{U'}(G'_i + Q')$;
6. the morphisms $U \to U'$ is induced by the toric blow-up $T \to T'$ induced by the subdivision $\langle e_1, e_2 \rangle \cup \langle e_1, ke_1 - ne_2 \rangle$ of $\langle e_2, ke_1 - ne_2 \rangle$.

Here $D_Z$ is the toric divisor of a toric variety $Z$.

We shall call a divisor like $G_i$ a **splice** and one like $F_i$ a **scion**.

Something analogous happens to the surface $X_1^+$, except that we now lose the fibration structure. Let us keep the same notation as above. We have:

**Definition 7.1.8.** A **N-pseudoelliptic surface** is a pair $(Y, G + F)$ consisting of a surface $Y$, $N$ marked curves $F = \bigcup F_i$ and $s$ marked curves $G = \bigcup G_i$ with a regular birational map $g : Y' \to Y$, called **structure morphism**, from a log-standard elliptic surface $(\pi' : Y' \to \mathbb{P}^1, Q', G' + F')$ such that:

1. The proper transform of the $F_i$ are fibres of $Y' \to \mathbb{P}^1$;
2. the exceptional divisor of $g$ is the zero-section $Q$ of $Y'$ and $g$ maps $Q$ to a point $p$;
3. there exist a étale neighborhoods $V \to Y$ of $p \in Y$ and $V' \to Y'$ of $Q' \subset Y'$ and morphisms to toric varieties $\tau : V \to Z$ and $\tau' : V' \to Z'$ and a toric morphism $b : Z' \to Z$ such that $b \circ \tau' = \tau \circ g$;
4. the fan of $Z'$ is the union of the two cones $\langle e_1, e_2 \rangle \cup \langle e_1, ke_1 - ne_2 \rangle$ in the lattice $N$ and $\tau^* \mathcal{O}(D_Z) \simeq \mathcal{O}_U(F'_i + S' + Q')$ for some fibre $S'$ of $Y' \to \mathbb{P}^1$;
5. the fan of $Z$ is the cone $\langle e_2, ke_1 - ne_2 \rangle$ in the lattice $N$ and $\tau^* \mathcal{O}(D_{Z'}) \simeq \mathcal{O}_U(F_1 + S)$ where $S = g_* S'$;
6. the toric morphism $b : Z' \to Z$ is induced by the subdivision $\langle e_1, e_2 \rangle \cup \langle e_1, ke_1 - ne_2 \rangle$ of $\langle e_2, ke_1 - ne_2 \rangle$.

We call such a $Y$ **isotrivial** if $Y' \to \mathbb{P}^1$ is isotrivial. Furthermore, we call **pseudolleptic surface of type I** a 1-pseudoellitpic surface $(Y, G + F_1)$; the component $G_i$ is still called a **splice** and $F_i$ is still called a **scion**.

**Remark 7.1.9.** An $n$-pseudoelliptic surface is **log-canonical** if and only if $n \leq 2$. 

![fig.6](image-url)
After performing any number of flips, we want to make sure that we know what happens to $X^+_1$ and $X^+_2$ in particular that we know when the restriction of the log-canonical divisor is nef and big on them. This is taken care by the following proposition:

**Proposition 7.1.10.** 1. Let $Y$ be a log-standard surface, $g : Y \to Y'$ its structure morphism with $Y'$ standard. Let

$$L_{Y'} = K_{Y'} + \sum_{i=1}^{s} G'_i + \sum_{j=1}^{r} F'_j + Q'$$

and

$$L_Y = K_Y + \sum_{i=1}^{s} G_i + \sum_{j=1}^{r} F_j + Q;$$

then:

$$g^*L_{Y'} = L_Y + \sum_{i=1}^{s} E_i.$$

2. Let $Y$ be an $n$-pseudoelliptic surface with structure morphism $g : Y' \to Y$, with $Y'$ is log-standard. Let:

$$L_{Y'} = K_{Y'} + \sum_{i=1}^{s} G'_i + \sum_{j=1}^{r} F'_j + Q$$

and

$$L_Y = K_Y + \sum_{i=1}^{s} G_i + \sum_{j=1}^{r} F_j;$$

then:

$$g^*L_Y = L_{Y'} + \frac{2-n}{Q^2}Q.$$
Proof. We start with proving part (1).

We have that:

\[ g^*L_Y' = L_Y + \sum_{i=1}^{s} a_i E_i \]

given that \( \sum_{i=1}^{s} E_i \) is the exceptional divisor of \( g \). By an easy inductive argument (i.e., by performing the toric blow-ups one at the time), one can easily convince oneself that in fact all the \( a'_i's \) must be equal to each other; let us indicate this rational number by \( a \).

Let \( \pi : Y \to C \) be the projection to the base curve of \( Y \), and let \( g \) be the genus of \( C \). Furthermore, let \( k \) be the least common multiple of all the orders of monodromy around the fibres \( F_i \).

We can take a base change of order \( k \) as in proposition 4.3.2 to untwist the possible twisted fibres and get a diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
B & \xrightarrow{\phi} & C
\end{array}
\]

Now the zero section \( \overline{Q} \) of the surface \( S \) is entirely contained in the smooth locus of \( S \), and therefore, if we indicate respectively by \( \overline{F}_i \) and \( \overline{G}_j \) the proper transforms of \( F_i \) and \( G_j \), by a computation analogous to the one in proposition 4.3.2, we get:

\[ L_Y \cdot Q = \frac{1}{k} (K_S + \sum_i F_i + Q + k \sum_j G_j) \cdot \overline{Q} = 2g - 2 + s. \]

Since according to proposition 4.3.2, we have:

\[ L_{Y'} \cdot Q' = 2g - 2 + s + r, \]

we can infer that \( a = 1 \). This completes the proof of part (1).

As for part (2), we can again write:

\[ g^*L_Y = g^*(K_Y + \sum_{i=1}^{s} G_i + \sum_{j=1}^{r} F_j) = (K_{Y'} + \sum_{i=1}^{s} G'_i + \sum_{j=1}^{r} F'_j + Q') + aQ = L_{Y'} + aQ. \]

But \( g^*L_Y \cdot Q = 0 \) by the projection formula and \( L_{Y'} \cdot Q = n - 2 \) by lemma 4.3.2, therefore:

\[ n - 2 + aQ^2 = 0. \]

This proves part (2).

\[ \square \]

We need to say something about the positivity of the log-canonical bundle of a log-canonical surfaces and of n-pseudoelliptic ones. We have:

**Proposition 7.1.11.** 1. Let \( Y \to C \) be log-standard with base curve \( C \) of genus \( g \), with \( r \) \( F_j \)'s and \( n \) \( G_i \)'s. Moreover, assume that \( -1 < E_i^2 < 0 \). Then the following hold:

(a) if \( 2g - 2 + r > 0 \), then \( L_Y \) is ample;
(b) if $2g - 2 + r = 0$, then $L_Y$ is semiample, and for any irreducible curve $D$:

$$L_Y \cdot D = 0 \text{ iff } D = Q;$$

(c) if $2g - 2 + r < 0$ then $[Q] \in \overline{NE}(Y)$ is an extremal ray for $L_Y$.

2. let $Y$ be $n$-pseudoelliptic, then:

(a) if $n \geq 2$ then $L_Y$ is ample;
(b) if $n = 1$ and $Q^2 < -1$, then $L_Y$ is ample
(c) if $n = 0$, then $L_Y$ is ample if and only if $Q^2 < -4$.

Proof. We will start proving part (1).

**Proof of part (1)** Recall from Proposition 7.1.10, part (1), with the same notation as therein, that:

$$g^*L_Y = L_Y + \sum_{i=1}^{s} E_i.$$ 

Therefore, in order to show that $L_Y^2 > 0$, it suffices to show that $L_Y^2 > 0$ and $2L_Y \cdot \sum_i E_i + \sum_i E_i^2 > 0$. The first inequality is a consequence of corollary 4.3.5, and:

$$2L_Y \cdot \sum_{i=1}^{s} E_i + \sum_{i=1}^{s} E_i^2 = -\sum_{i=1}^{s} E_i^2 > 0,$$

where we have used that $L_Y \cdot \sum_{i=1}^{s} E_i = g^*L_Y \cdot \sum_{i=1}^{s} E_i + (\sum_{i=1}^{s} E_i) \cdot (\sum_{i=1}^{s} E_i)$.

Now, if $D$ is any irreducible curve on $Y$ other than one of the $G_i$’s or one of the $E_i$’s, then $L_Y \cdot D = L_Y \cdot g(D)$ which is positive according to corollary 4.3.5. In case the irreducible curve is $E_i$, then we have that:

$$L_Y \cdot E_i = -E_i^2 > 0;$$

and in case it is $G_i$, then, since $-G_i^2 = G_i \cdot E_i$, we have:

$$L_Y \cdot G_i = L_Y^2 \cdot G_i^2 - E_i \cdot G_i = Q^2 \cdot G_i^2 - E_i \cdot G_i,$$

and since $G_i^2 = \frac{1}{E_i^2} E_i \cdot G_i$ (this is obtained by writing $g^*Q^2 = G_i + \sum a_i E_i$ and intersecting with $E_i$ to obtain $a_i$ and then intersect $g^*Q^2$ with $G_i$), we have that:

$$L_Y \cdot G_i = (\frac{1}{E_i^2} - 1)E_i \cdot G_i > 0$$

since by lemma 7.1.6:

$$-1 < E_i^2 < 0.$$ 

We can then conclude part (1) with the aid of corollary 4.3.5.

We now prove part (2).

**Proof of part (2)** Recall from Proposition 7.1.10, part (2)

$$g^*L_Y = L_{Y'} + \frac{2-n}{Q^2} Q.$$ 

This implies that $L_Y \cdot G_i > 0$. If $n \geq 2$ $L_Y'$ is ample for the previous part, and $\frac{2-n}{Q^2} Q$ is effective (or 0 if $n = 2$), so part (a) is proved.

If $n = 1$, ...
\[ g^*L_Y = L_Y + \frac{1}{Q^2}Q. \]

Let \( g'': Y' \to Y'' \) the structure morphism of the log-standard surface \( Y' \). Then, if we set:

\[ M := K_{Y''} + \sum G''_i + F'' + (1 + \frac{1}{Q^2})Q'', \]

where \( Q'' = g''(Q'), F'' = g''(F') \) and \( G''_i = g''(G'_i) \), then:

\[ g^*L_Y = g''^*M + \sum E_i \]

and again we can conclude by means of part (1) and corollary 4.3.5, given that \( 0 < a := (1 + \frac{1}{Q^2}) < 1 \) and that \( \lambda > 1 \).

Let us finally analyze the case in which \( n = 0 \). As before, we have:

\[ g^*L_Y = g''^*M + \sum E_i. \]

where now:

\[ M := K_{Y''} + \sum G''_i + (1 + \frac{2}{Q^2})Q''. \]

Therefore one can once again conclude, with the aid of corollary 4.3.3 (given that \( Q^2 < -4 \)) that \( g^*L_Y \) is positive on every curve that is different from \( Q \) and from one of the \( E_i \)'s. Indeed, every other curve, except for the \( G_i \)'s (and for these the computation is exactly like in part (1) of this proposition), will not meet the \( E_i \)'s. Also, since:

\[ L^2_Y = M^2 + \sum E_i^2 \]

from the very same corollary, we get that \( L^2_Y \) is positive as soon as \( Q^2 < -4 \), in fact, we have:

\[ L_Y^2 = 2a(s - 2) + a(2 - a)\lambda + \sum E_i^2 > 0 \]

with \( a := (1 + \frac{2}{Q^2}) \) and \( \lambda = -Q^2 \), and therefore, since \( E_i^2 > -1 \), as soon as:

\[ 2a(s - 2) + a(2 - a)\lambda - s > 0 \]

and this occurs as soon as \( Q^2 < -4 \) as claimed.

So we only need to check that \( L_Y \cdot E_i > 0 \), but this is clearly true, due to the fact that \( E_i^2 < 0 \) and \( M \cdot E_i = 0 \).

This concludes the proof, again thanks to Nakai-Mosheizon.

\[ \square \]

We also want to know how \( X_2^+ \) has changed. For one thing we know that now it must be a log-standard elliptic surface, even if it were only standard to begin with.

We want to know whether \( X_2^+ \) is strictly pre-stable when \( X_2 \) is. We have:
Proposition 7.1.12. Maintaining the notation as in proposition 7.1.11 part (1), the self intersection \( Q^2 \) of \( Q \) in \( Y \) is:

\[
Q^2 = Q'^2 + \frac{1}{E_i^2}
\]

where \( Q^2 \) is taken in \( Y' \). In particular, if \( Y' \) is a strictly prestable standard surface (i.e., \( Q^2 < -1 \)) and \(-1 < E_i^2 < 0 \) for each \( i \), then \( Q^2 < -1 \) as well.

Proof. Write \( g^*Q = Q' + \sum a_i E_i \), and intersect with \( E_i \), to see that it must be that \( a_i = \frac{1}{E_i^2} \) (since \( Q \cdot E_i = 1 \)). This yields:

\[
Q^2 = Q \cdot b^*Q' = Q'^2 + \frac{1}{E_i^2}
\]

which concludes the proof.

We can therefore make the following:

Definition 7.1.13. We call a triple \( (X \to C, Q, G + F) \) consisting of a log-standard elliptic surface with a given marking strictly prestable if:

1. \( (Q \mid_{X'_C})^2 < -1 \) for each rational component \( B \subset C \);
2. if \( g : Y := X \mid_B \to Y' \) is the structure morphism, then for each irreducible component \( E_i \) of the exceptional divisor \( E \) of \( g \) we have that \(-1 < E_i^2 < 0 \).

Analogously we call a quadruple \( (X \to C, Q + F, G, f : X \to C \to \overline{M}_{1,1}) \) consisting of a log-standard elliptic surface with marking and map to moduli strictly prestable if the same conditions are asked only of those rational components \( B \subset C \) for which \( j \mid_B : B \to \overline{M}_{1,1} \) is constant.

7.2. Small log-canonical contractions. Let \( (\pi : \mathcal{X} \to \mathcal{C} \to \Delta, Q) \) be a family of strictly prestable log-standard elliptic surfaces with section over a DVR scheme (or a polydisk, if one favour the analytic flavor)) \( \Delta \), such that the special fibre \( \mathcal{X}_0 \) contains a chain of surfaces \( X := X_1 \cup \ldots \cup X_N \) attached transversally along stable or twisted fibres, with base curve a chain or rational curves \( C := C_1 \cup \ldots \cup C_N \). Furthermore assume that \( X \) is attached transversally to \( \mathcal{X}_0 \setminus \mathcal{X} \) along one (twisted or stable) fibre each end. Let \( Z_1 \to B_1 \) and \( Z_2 \to B_2 \) be the adjacent surfaces (that is to say the irreducible components of \( \mathcal{X}_0 \setminus \mathcal{X} \) that meet \( X \) at each end).

Let \( \mathcal{C}' \) be obtained from \( \mathcal{C} \) by contracting the curve \( C \), \( \rho : \mathcal{C} \to \mathcal{C}' \) the contraction map and \( p = \rho(C) \).

Let also \( w_1 = e_1 + k_1 e_2, \ldots, w_i = e_1 + (k_1 + \ldots + k_i) e_2 + (n_1 + \ldots + n_{i-1}) k_{i+1} e_3, \ldots, w_{r+1} = e_1 + (k_1 + \ldots + k_{r+1}) e_2 + (n_1 + \ldots + n_r) k_{r+2} e_3 \), and \( L \) be the lattice \( L = \frac{1}{n} e_1 \mathbb{Z} \oplus e_2 \mathbb{Z} \oplus (\frac{1}{n} e_2 + \frac{a}{n} e_3) \mathbb{Z} \).

According to lemma 4.3.1 \( Q \mid_{\mathcal{C}} \) must be contracted by the log-canonical map, and in fact the contraction is described by the following:

Theorem 7.2.1. The stable model of \( \mathcal{X} \) is a family \( \mathcal{X}^c \cong \mathcal{C}' \to \Delta \) of surfaces such that the generic fibre and \( \mathcal{X}_0^c \setminus \mathcal{X} \) have not changed and \( X^c = \pi'^{-1}(p) \) is attached to the rest of the central fibre along marked curves. The singularity of this point in \( \mathcal{X}^c \) is toric, and a toric neighborhood of \( p \) in \( (\mathcal{X}^c, Z_1^c + Z_2^c) \) (where \( Z^c := \mathcal{X}^c \mid_{\rho(B)} \)) is given by the cone: \( \sigma = \langle \frac{1}{n} e_1, w_1, \ldots, w_r w_{N+1} e_3 \rangle \) in the lattice \( L \). This singularity is canonical, and in particular \( \mathcal{X}_0^c \) is semi-log-canonical.
Proof. As in the proof of theorem 7.1.2 we can reduce ourselves to finding the log-canonical contraction on an étale neighborhood, since by hypothesis $\omega_X/\Delta(Q)$ is $\Delta$-ample except for contracting $Q$. The rest is a direct consequence of \[\text{Proposition 6.2.3 and lemma 6.2.4.}\]

The normalization $X^{\omega}$ of $X^c = X^c_i$ consists of a union of normal surfaces $(Y^c_i, G_1^c, G_2^c)$ with marked double curves $G_1^c$ and $G_2^c$, and if $(Y^c_i, G_1^c, G_2^c)$ denote components with double curves of the normalization of $X$, we have a map $Y_i \to Y^c_i$ which contracts the zero section to a point $p' \in X^{\omega}$.

**Lemma 7.2.2.** There is an étale neighborhood $U$ of $p' \in Y^c_i$ with a map to a toric variety $g : U \to Z$ such that:

1. the fan of $Z$ is the cone $\Delta = \langle e_1, w = -e_1 + ne_2 \rangle$ in the lattice $N = (\frac{1}{n} e_1 + \frac{n}{h_1} e_2)Z \oplus (\frac{1}{n} e_1 + \frac{n-a}{h_1} e_2)Z$, where $a$ and $h_1$ are completely determined by the monodromy of the action around one of the two “marked” fibres $G_1$ and $G_2$.
2. $g^*O_Z(D_Z) \cong O_U(Q + G_1 + G_2)$, where $D_Z$ is the toric divisor.

Proof. The lemma is a consequence of the following two observations. On the one hand that the projection along $e_3$ onto $e_1$ must determine the action on $Q \cong \mathbb{P}^1$, which is given by choosing the lattice $\frac{1}{n} e_1$.

On the other hand, let $h_1$ be the integer such that $\mu_{h_1}$ is the subgroup of $\mu_n$ acting nontrivially on $G_1$. The kernel of the map

$$\mu_n \times \mu_n \to \mu_{h_1}$$

is determined by a vector $\nu = (\frac{1}{n}, z)$ such that $h_1z = a \in \mathbb{Z}$. Thus the claim.\]

What one has to prove now is that the log-canonical divisor on such components, which originally was only nef and big, has become ample after contracting the zero-section. In fact:

**Proposition 7.2.3.** Let $X^c$ be a $2$–pseudoelliptic surface, $\alpha : X \to X^c$, its structure morphism, and assume that $X \to C$ is a strictly stable log-standard surface ($C$ rational), with zero section $Q$. Then: $\alpha^*L_{X^c} = L_X$. In particular, $L_{X^c}$ is ample.

Proof. Write:

$$\alpha^*L_{X^c} = L_X + aQ,$$

and intersect with $Q$. By the projection formula, $\alpha^*L_{X^c} \cdot Q = 0$ and $L_X \cdot Q = 0$ according to lemma 4.3.1, therefore the conclusion, on accounts of proposition 7.1.11 part (2).\]

**Definition 7.2.4.** We call pseudoelliptic surface of type II a pair $(Y^c, G^c + F_1^c + F_2^c)$ consisting of a $2$-pseudoelliptic surface as in definition 7.1.8 with the extra conditions that $S$ in point 5) is $F_2^c$, and that we replace the fan of point 5) with the fan of lemma 7.2.2.

Such a triple is called isotrivial if the surface $Y \to Y^c$ as in lemma 7.2.2 is isotrivial.

8. The Stable reduction theorems

8.1. Stable reduction of triples. We are now ready to prove the stable reduction theorem in the relative case of elliptic surfaces with sections and endowed with a regular map to $\overline{M}_{1,1}$.

**Theorem 8.1.1.** Let $X_\eta \to C_\eta \to \eta$ be a stable elliptic surface over a smooth curve $C_\eta$. Then there is a finite extension of discrete valuation rings $R \subset R'$ and a triple $(X' \to C', Q', f' : X' \to \overline{M}_{1,1})$ over $S'$ such that:
1. \( X' \to C' \) gives rise to an extension:

\[
\begin{align*}
X_{\eta} \times_{\Delta'} \Delta' & \subset X' \\
\downarrow & \\
C_{\eta} \times_{\Delta'} \Delta' & \subset C'
\end{align*}
\]

compatible with the extension \( Q_{\eta} \times_{S} S' \subset Q' \);

2. \( C' \to \overline{M}_{1,1} \) is Kontsevich stable.

3. the components of \( X_0' \) that dominate the components of \( C_0' \) are log-standard strictly prestable quadruples \( (X, Q, G, f : X \to \overline{M}_{1,1}) \) where \( G \) consists of either one or two fibres, which are either stable or twisted; if \( X \to C \) in such a quadruple turns out to be isotrivial, then \( C \) is not rational.

4. the components of \( X_0' \) that are mapped to a point of \( C_0' \) are isotrivial log-pseudoelliptic surfaces either of type I or II. In the former case they are attached to the rest of the central fibre according to lemma 6.1.4 and in the latter according to lemma 6.2.4.

The extension is unique up to a unique isomorphism, and its formation commutes with further finite extensions of discrete valuation rings.

\textbf{Proof.} By the strictly prestable reduction theorem (theorem 5.3.6) we can find a finite base change \( \Delta' \to \Delta' \), \( \Delta' \)-schemes and \( \Delta' \)-morphisms unique up to unique isomorphisms \( (X' \to C', Q', f : X' \to \overline{M}_{1,1}) \) that extend \( (X_{\eta} \to C_{\eta}, Q_{\eta}, f_{\eta}) \), which is strictly prestable. The extension commutes with further base changes and the log-canonical divisor \( \omega_{X'/\Delta'}(Q') \otimes f'^*O_{\overline{M}_{1,1}}(3) \) is ample away from those isotrivial components of the central fibre \( X_0' \) the meet the rest of the central fibre in one or two fibres (twisted or stable).

Let \( X \to C \) such a component. Then \( j'^*O_{\overline{M}_{1,1}} \simeq O_C \), and therefore

\[
\omega_{X'/\Delta'}(Q') \otimes f'^*O_{\overline{M}_{1,1}}(3) \big|_X \simeq \omega_{X/\Delta'}(Q') \big|_X.
\]

According to lemma 4.3.1 we know that if an irreducible rational curve meets the rest of the central fibre in one point, we need to log flip the zero section above it. According to theorem 7.1.2 we can then perform the log flip to get \( (X^+ \to C^+, Q^+) \). In doing so, one produces an isotrivial pseudoelliptic surface of type I \( (X^+_1, G_1) \) attached to a log-standard elliptic surface \( (X^+_2, Q^+_2, G^+_2) \) according to the fan in theorem 7.1.2. According to the same theorem, the zero section \( Q^+_2 \) of \( X^+_2 \) misses the singular point where it meets \( X^+_1 \), so we can iterate the process, and prune the tree of all the \( j \)-trivial rational curves meeting the rest of the central fibre in only one point.

The log-canonical bundle is now nef and big after Proposition 7.1.11, since the family \( X \to C \) was strictly prestable to begin with (so it did not have any component whose zero section had self-intersection \( \geq -1 \)), and according to proposition 6.1.12 it stays such. Furthermore, according to theorem 7.1.2 this way we only produce semi-logcanonical singularities.

In order to make the log-canonical bundle \textit{ample}, we need to contract all the chains of rational curves that meet the rest of the central fibre in two ends, according to lemma 4.3.1. But this is taken care by theorem 7.2.1, and will produce \textit{isotrivial pseudoelliptic surfaces of type II}. The log-canonical bundle is now ample on accounts of proposition 7.2.3. On accounts of theorem 7.2.1 the singularities we thus obtain are at most semi-logcanonical. This ends the proof.

\( \square \)
8.2. Stable reduction for pairs. Here we want to deal with the absolute case. As it has been mentioned earlier, the steps of the MMP in a one parameter family are going to be similar to the ones performed in the case of triples, except that we now need to perform the flips and the small contractions also in cases in which the \( j \)-map is not constant.

**Theorem 8.2.1.** Let \( X_\eta \to C_\eta \to \eta \) be a stable elliptic surface over a smooth base curve \( C_\eta \) of genus \( g \geq 2 \). Then there is a finite extension of discrete valuation rings \( R \subset R' \) and a pair \( (X' \to C', Q') \) over \( S' \) such that:

1. \( X' \to C' \) gives rise to an extension:

\[
\begin{array}{ccc}
X_\eta \times_S S' & \to & X' \\
\downarrow & & \downarrow \\
C_\eta \times_S S' & \to & C'
\end{array}
\]

\[
\{\eta'\} \quad \subset \quad S',
\]

compatible with the extension \( Q_\eta \times_S S' \subset Q' \);

2. the components of \( X'_0 \) that dominate the components of \( C'_0 \) are log-standard strictly prestable triples \((X, Q, G, r)\) where \( G \) consists of either one or two fibres, which are either stable or twisted.

3. the components of \( X'_0 \) that are mapped to a point of \( C'_0 \) are log-pseudoelliptic surfaces either of type I or II. In the former case they are attached to the rest of the central fibre according to lemma 6.1.4 and in the latter according to lemma 6.2.4.

The extension is unique up to a unique isomorphism, and its formation commutes with further finite extensions of discrete valuation rings.

**Proof.** By the strictly prestable reduction theorem (theorem 7.3.6) we can find a finite base change \( \Delta' \to \Delta \), \( \Delta' \)-schemes and \( \Delta' \)-morphisms unique up to unique isomorphisms \((X' \to C, Q, f : X' \to \mathcal{M}_{1,1})\) that extend \((X_\eta \to C_\eta, Q_\eta, f_\eta)\), which strictly prestable. The extension commutes with further base changes.

The log-canonical divisor \( \omega_{X'/\Delta'}(Q') \) is ample away from those components fibred over a rational curve that meet the central fibre \( X'_0 \) along one or two fibres (stable or twisted).

In fact, let \( r : Z \to B \) be a component of \( X'_0 \). On the one hand, if \( B \) is not rational, it is obvious that \( \omega_{X'/\Delta'}(Q') \otimes \mathcal{O}_Z \simeq \omega_{X'}(Q' \mid Z) \otimes \mathcal{O}_Z(D) \), where \( D \) is the dual curve, is ample, since \( \omega_{Z/B}(Q' \mid Z) \) is relatively ample (Kollár semipositivity theorem) and \( \omega_B \) is ample. On the other hand, if \( B \) is rational but meets the rest of \( C'_0 \) in at least three points, then \( r^* \omega_B(D) \) is ample. The rest of the proof can be translated word by word from theorem 5.1.7.

Remark 8.2.2. It is worth noting that if the base curve \( C_\eta \) is rational or elliptic, then the log-canonical bundle is not ample: one needs to contract all the base curves. In this sense it is probably more natural, in the rational and elliptic base curve case, to consider the moduli of triples (with map to \( \mathcal{M}_{1,1} \)) at least if one wants to preserve the fibration structure.

8.3. The rational base case. Here we deal with one of the two cases left out from section 8.2: namely the case in which the base curve of the elliptic surface \( X_\eta \to C_\eta \), is rational. In this case, the log-canonical bundle is not ample even on the surface \( X_\eta \) itself: indeed we need to contract the zero section \( Q_\eta \), to make the log-canonical bundle ample.

We perform the stable reduction theorem for the triple \((X_\eta \to C_\eta, Q_\eta, j_\eta : C_\eta \to \mathcal{M}_{1,1})\). We may then assume to dispose of a stable \( \Delta \)-triple \((X \to C \to \Delta, Q \to \Delta, j : C \to \mathcal{M}_{1,1}) \) over
some DVR scheme $\Delta$ to begin with. We want to be able to contract the base curve in the general member.

We can now state and prove:

**Theorem 8.3.1.** Let $(X_\eta \to C_\eta \to \eta, Q_\eta \to \eta, X_\eta \to \overline{M}_{1,1})$ be a stable triple over a rational smooth base curve $C_\eta \simeq \mathbb{P}^1$. Furthermore, let $Y_\eta \to X_\eta$ be the log-canonical contraction of $Q_\eta$.

Then there is a finite extension of discrete valuation rings $R \subset R'$ and a pair $(X', Q')$ over $\Delta' = \text{Spec}(R')$ such that:

1. $X'$ gives rise to an extension:

$$\begin{align*}
X_\eta \times_\Delta \Delta' & \subset X' \\
\{\eta'\} & \subset \Delta',
\end{align*}$$

compatible with the extension $Q_\eta \times_\Delta \Delta' \subset Q'$;

2. the components of $X_0'$ are $n$-pseudoelliptic surfaces with either $n = 0$ or $n = 1$. In the former case they are attached to the rest of the central fibre according to lemma 6.1.4 and in the latter according to lemma 6.2.4.

The extension is unique up to a unique isomorphism, and its formation commutes with further finite extensions of discrete valuation rings.

**Proof.** We can apply Theorem 8.1.1 to obtain a stable triple $(Y \to C \to \Delta, Q \to \Delta, X \to \overline{M}_{1,1})$, after a possible base change (unique up to a unique isomorphism) $\Delta' \to \Delta$, that satisfies the analogous of property 1.

Either $C_0$ consists of a simple tree (i.e., a chain of rational curves meeting transversally each only one consecutive curve at one point), or every chain-like component $C \subset C_0$ (i.e., a component that consists of a chain) will meet another tree-like component in a leaf which is not an extremity (i.e., this leaf will meet two more leaves of tree-like component it belongs to).

In the first case there are two possibilities: the tree (i.e., $C_0$ has either an even or an odd number of leaves. If it has an odd number of components, there is a well-defined central leaf, that is to say the leaf which disconnects the tree in two components of equal length. In case there is an even number of components, then there is, analogously, a well defined concept of central pair of leaves.

So in the case of a simple tree of odd length, let $B$ be the central curve. We can prune, starting from the two ends (by log-flipping according to theorem 7.1.2) all the leaves that belong to the two chain-like components that $B$ disconnects from $C_0$. Let as call $Y' \to C' \to \Delta$ the family thus obtained from $Y \to C \to \Delta$, and $Q'$ the new zero-section. At this point the zero section $Q'|_B$ of the log-standard surface $Y := Y'|_B \to B$ has the property that $L_Y \cdot Q = -2$, which is the same as in the general fibre. We can now divisorially contract the zero section $Q'$ in $Y$ (e.g., by means of the line bundle $\omega_Y((1 + a)Q')$ where $a = \frac{2}{Q_\eta^2}$). The log-canonical bundle is now ample, according to proposition 7.1.11. Also, on accounts of theorems 7.1.2 and 7.2.1 the singularities thus produced are at most semi-logcanonical.

The result in the special fibre is a configuration of two chains of 0-pseudoelliptic surfaces attached to another 0-pseudoelliptic surface (the surface obtained from $Y$ by contracting the zero-section).

Analogously, if we a simple tree of even length, do the same operations word by word as above, barring that now $B$ is replaced by the central pair $B_1 \cup B_2$. 

The result, now consists of two chains of 0-pseudoelliptic surfaces as above, only now attached to a union of two 1-pseudoelliptic surfaces (the result of contracting the zero sections of $Y'|_{B_1} \cup Y'|_{B_2}$).

In the event that there are two chain-like components, we can now individuate a spine, namely the one component that is attached to the other at an extremity. In this case we first prune (by means of theorem 7.1.2) the other component and reduce ourselves to considering only the spine, in other words reducing the problem to the previous case.

We can now conclude by a simple induction argument. □

8.4. The elliptic base case. Here, at last, we deal with the final case.

We first need the following:

**Definition 8.4.1.** A surface $Y$ with a structure morphism $g : Y' \to Y$ is said to be a pseudoelliptic surface of type $E_0$ (resp. $E_{IN}$) if the surface $(Y' \to E, Q, F)$ is a log-standard elliptic surface, with one marked fibre $F$ and a zero section $Q$, mapping to an irreducible elliptic curve $E$ (resp. to a closed chain of rational curves $E$) as base curve, and if $g$ has the zero section $Q$ as exceptional curve. The singularity of $Y$ at $g(Q)$ is an elliptic (resp. degenerate cusp) singularity.

We have:

**Theorem 8.4.2.** Let $(\mathcal{X}_\eta \to \mathcal{C}_\eta \to \eta, \mathcal{Q}_\eta \to \eta, \mathcal{X}_\eta \to \overline{\mathcal{M}}_{1,1})$ be a stable triple over an elliptic smooth base curve $\mathcal{C}_\eta \cong E$. Furthermore, let $\mathcal{Y}_\eta \to \mathcal{X}_\eta$ be the pseudoelliptic surface of type $E_0$ obtained by contracting $\mathcal{Q}_\eta$.

Then there is a finite extension of discrete valuation rings $R \subset R'$ and a pair $(\mathcal{X}', \mathcal{Q}')$ over $\Delta' = \text{Spec}(R')$ such that:

1. $\mathcal{X}'$ gives rise to an extension:

\[
\mathcal{X}_\eta \times_\Delta \Delta' \subset \mathcal{X}'
\]

\[
\{\eta'\} \subset \Delta',
\]

compatible with the extension $\mathcal{Q}_\eta \times_\Delta \Delta' \subset \mathcal{Q}'$;

2. the components of the central fibre $\mathcal{X}'_0$ consist of a pseudoelliptic surface either of type $E_0$ or of type $E_{IN}$, attached to a configuration of 0-pseudolliptic and 1-pseudoelliptic surfaces as in theorem 8.3.1.

The extension is unique up to a unique isomorphism, and its formation commutes with further finite extensions of discrete valuation rings.

**Proof.** We may assume, after theorem 8.3.1, that we have a family $\mathcal{Y} \to \mathcal{C}' \to \Delta$ whose generic fibre is isomorphic to $\mathcal{X}_\eta \to \mathcal{C}'_\eta$, and whose central fibre is a surface $\mathcal{Y}_0$ whose components are a log-standard elliptic surface $(Y \to E, Q + F)$ with one marked fibre $F$ and having as a base curve $E$ either an elliptic curve or a closed chain of rational curves, attached along $F$ to a configuration of 0-pseudolliptic and 1-pseudoelliptic surfaces as in theorem 8.3.1.

We can now divisorially contract the zero section $\mathcal{Q}'$ of $\mathcal{Y} \to \mathcal{C}'$ to obtain the result stated. □

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