Reduction Method for One-loop Tensor 5- and 6-point Integrals Revisited

Theodoros Diakonidis

Deutsches Elektronen-Synchrotron, DESY, Platanenallee 6, 15738 Zeuthen, Germany

A complete analytical reduction of general one-loop Feynman integrals with five legs for tensors up to rank $R = 3$ and six legs for tensors up to rank 4 is reviewed [1]. An elegant formalism with extensive use of signed minors was developed for the cancellation of leading inverse Gram determinants. The resulting compact formulae allow both for a study of analytical properties and for efficient numerical programming. Here some special numerical examples are presented.

1 Introduction

The Feynman integrals for reactions with up to four external particles have been systematically studied and evaluated in numerous studies. It is needed to be mentioned here the seminal papers [2] and [3] and the Fortran packages FF [4] and LoopTools [5], which evidently show the situation so far. The treatment of Feynman integrals with a higher multiplicity than four becomes quite involved if questions of efficiency and stability become vital, as it happens with the calculational problems related to high-dimensional phase space integrals over sums of thousands of Feynman diagrams with internal loops.

What is reviewed here is an approach which reduces the tensor integrals algebraically to sums over a small set of scalar two-, three- and four-point functions, which are assumed to be known. To accomplish this methods of ref. [6, 7] are used. The present goal is to provide compact analytic formulas for the complete reduction of tensor pentagons and hexagons to scalar master integrals, which are free of leading inverse Gram determinants. For a study of gauge invariance and of the ultraviolet (UV) and infrared (IR) singularity structure of a set of Feynman diagrams, it is evident that a complete reduction is advantageous, and it may also be quite useful for a tuned, analytical study of certain regions of potential numerical instabilities.

The numerics are obtained with two independent implementations, one made in Mathematica, and another one in Fortran. The Mathematica program hexagon.m with the reduction formulae is made publicly available [8], see also [9] for a short description. For numerical applications, one has to link the package with a program for the evaluation of scalar one- to four-point functions, e.g. with LoopTools [5][10][4], CutTools [11][12], QCDLoop [13].

2 Useful Notations

It is useful to introduce the notation for the loop integrals and also for certain determinants that occur in the recurrence relations and their solutions. The one-loop, $N$-point tensor integrals of rank $R$ in $d$-dimensional space-time are defined as,

$$I^{(N)}_{\mu_1 \ldots \mu_R}(d; \nu_1, \ldots, \nu_N) = \int \frac{d^d k}{i \pi^{d/2}} \frac{k_{\mu_1} \ldots k_{\mu_R}}{D_1^{\nu_1} \ldots D_N^{\nu_N}}$$ (1)
with propagator denominators
\[ D_j = (k - q_j)^2 - m_j^2 + i\epsilon. \]  

The determinant of an \((N+1)\times(N+1)\) matrix, known as the modified Cayley determinant is defined as: [14],
\[
()_N \equiv \begin{vmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & Y_{11} & Y_{12} & \cdots & Y_{1N} \\
1 & Y_{12} & Y_{22} & \cdots & Y_{2N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & Y_{1N} & Y_{2N} & \cdots & Y_{NN}
\end{vmatrix},
\]  

with coefficients
\[ Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i, j = 1 \ldots N). \]

All other determinants appearing are signed minors of \(()_N\), constructed by deleting \(m\) rows and \(m\) columns from \(()_N\), and multiplying with a sign factor. They will be denoted by
\[
\begin{vmatrix}
j_1 & j_2 & \cdots & j_m \\
k_1 & k_2 & \cdots & k_m
\end{vmatrix}_N \equiv (-1)^{\sum_i(j_i + k_i)} \text{sgn}_{(j)} \text{sgn}_{(k)} \begin{vmatrix}
\text{rows } j_1 \cdots j_m \text{ deleted} \\
\text{columns } k_1 \cdots k_m \text{ deleted}
\end{vmatrix},
\]  

where \(\text{sgn}_{(j)}\) and \(\text{sgn}_{(k)}\) are the signs of permutations that sort the deleted rows \(j_1 \cdots j_m\) and columns \(k_1 \cdots k_m\) into ascending order.

3 Pentagons

In this chapter final results are provided of the reduction concerning ranks up to 3. More about these can be found in [15]. For the scalar 5-point function the recursion relation for the limit of \(d = 4\) is,
\[ E \equiv I_5 = \frac{1}{5!} \sum_{s=1}^{5} \binom{0}{s} I_s^4, \]  

Similarly, for the tensor integral of rank 1 (vector) in the limit \(d \to 4\) we obtain:
\[
I_5^\mu = \sum_{i=1}^{4} q_i^\mu I_{5,i},
\]
\[ I_{5,i} \equiv E_i = -\frac{1}{5!} \sum_{s=1}^{5} \binom{0}{s} \binom{i}{s} I_s^4, \]

LCWS/ILC 2008
The tensor integral of rank 2 can be written:

\[ I_{\mu \nu}^5 = \sum_{i,j=1}^{4} q_i^\mu q_j^\nu E_{ij} + g^{\mu\nu} E_{00}, \] (9)

\[ E_{ij} = \sum_{s=1}^{5} S_{ij}^{4,s} I_4^s + \sum_{s,t=1}^{5} S_{ij}^{3,st} I_3^s, \] (10)

\[ E_{00} = -\frac{1}{2} \sum_{s=1}^{5} \left( \left( \begin{array}{c} s \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ s \end{array} \right) I_4^s - \sum_{t=1}^{5} \left( \begin{array}{c} t \\ s \end{array} \right) I_3^t \right). \] (11)

Finally the tensor integral of rank 3

\[ I_{\mu \nu \lambda}^5 = \sum_{i,j,k=1}^{4} q_i^\mu q_j^\nu q_k^\lambda E_{ijk} + \sum_{k=1}^{4} g^{\mu\nu\lambda} q_k^\lambda E_{00k}, \] (12)

\[ E_{ijk} = \sum_{s=1}^{5} S_{ijk}^{4,s} I_4^s + \sum_{s,t=1}^{5} S_{ijk}^{3,st} I_3^s + \sum_{s,t,u=1}^{5} S_{ijk}^{2,stu} I_2^s, \] (13)

\[ E_{00k} = \sum_{s=1}^{5} S_{00k}^{4,s} I_4^s + \sum_{s,t=1}^{5} S_{00k}^{3,st} I_3^s + \sum_{s,t,u=1}^{5} S_{00k}^{2,stu} I_2^s. \] (14)

All coefficients, also those not explicitly defined here \( S_{ij}^{4,s}, S_{ijk}^{3,st}, S_{ijk}^{4,s}, S_{ijk}^{3,st}, S_{ij}^{2,stu} S_{ij}^{3,st} \) (see [15]), are free of the leading Gram determinants.

4 Hexagons

If the external momenta of a hexagon are 4-dimensional, their Gram determinant vanishes: \( \det = 0 \), and a linear relation between the propagators \( D_j \) exists:

\[ 1 = \sum_{j=1}^{6} \left( \begin{array}{c} 0 \\ j \end{array} \right) \left( \begin{array}{c} s \\ 0 \end{array} \right) D_j. \] (15)

With this relation, any hexagon integral can trivially be reduced to pentagons. For example, for the scalar hexagon, one obtains the well-known result [14]:

\[ I_6 = \sum_{r=1}^{6} \left( \begin{array}{c} 0 \\ r \end{array} \right) I_5^r, \] (16)

where the scalar pentagon \( I_5^r \) on the right hand side is obtained by removing line \( r \) from the hexagon \( I_6 \). In the same way, tensor hexagons of rank \( R \) can be reduced to tensor pentagons.
of rank $R$. However, it was noticed in ref. [6] that a reduction directly to tensor pentagons of rank $R - 1$ is also possible:

$$I_{6}^{\mu_{1} \cdots \mu_{R}} = \sum_{r=1}^{6} v_{r}^{\mu_{1}} I_{5}^{\mu_{2} \cdots \mu_{R-1} r},$$

where

$$v_{r}^{\mu} \equiv - \frac{1}{6} \sum_{i=1}^{5} \left( \begin{array}{c} 0 \\
0
\end{array} \right)_{6} \left( \begin{array}{c} 0 \\
1
\end{array} \right)_{i} q_{r}^{\mu}.$$  

A more general proof of this property was given in ref. [16]. By substituting the reduction formulas for tensor pentagons into eq. (17), we can immediately express tensor hexagons in terms of scalar master integrals. In this way using the formulas of the previous section we can provide results for integrals up to 4th rank for the hexagons (see [15]).

5 Numerical results

In order to illustrate the numerical results which can be obtained with the described approach, a representative collection of tensor components will be evaluated, for some special cases which are not included in [9]. The kinematics are visualized in Figure 1.

For the evaluation of the scalar two-, three- and four-point functions, which appear after the complete reduction, we have implemented two numerical libraries:

- For massive internal particles: Looptools 2.2 [5] [10];
- If there are also massless internal particles: QCDLoop-1.4 [13].

the first one in the published package hexagon.m [8] and both of them in the Fortran implementation.
Table 1: The external four-momenta for the five-point functions; all internal and external particles are massless.

| $p_1$ | 5.0 | 0.0 | 0.0 | 5.0 |
|-------|-----|-----|-----|-----|
| $p_2$ | 5.0 | 0.0 | 0.0 | -5.0 |
| $p_3$ | -1.6554963633 | 1.2970338732 | -0.9062452085 | -0.4869198730 |
| $p_4$ | -3.8970139847 | 0.0528728505 | -2.5360890226 | 2.9584074987 |
| $p_5$ | -4.4474896520 | -1.3499067237 | 3.4423342311 | -2.4714876256 |

$m_1 = \cdots = m_5 = 0.0$

Table 2: Selected tensor components of five-point tensor functions with massless particles; kinematics defined in Table 1 (Cross checked with golem95 [17]).

| $E_0$ | 0.49975096E-03 + i 0.12807271E-02 | 0.33696138E-03 – i 0.64416161E-03 |
|-------|-----------------------------------|-----------------------------------|
| $E^1$ | -0.50336057E-03 – i 0.10928553E-02 | -0.34786666E-03 + i 0.54767334E-03 |
| $E^{12}$ | -0.11603164E-02 – i 0.17552616E-02 | -0.60899168E-03 + i 0.12327007E-02 |
| $E^{122}$ | -0.43997517E-02 – i 0.34454891E-02 | -0.10597882E-02 + i 0.36519758E-02 |

Table 3: The external four-momenta for the six-point functions; all external legs massless and the internal massive.

| $p_1$ | 5.0 | 0.0 | 0.0 | 5.0 |
|-------|-----|-----|-----|-----|
| $p_2$ | 5.0 | 0.0 | 0.0 | -5.0 |
| $p_3$ | -0.7623942818 | 0.5390582570 | -0.5220507689 | 0.1346262645 |
| $p_4$ | -3.3298826057 | 1.0349623069 | -1.1048040197 | 2.9658690580 |
| $p_5$ | -2.8267285956 | -1.4136906402 | 2.3189438782 | -0.783192500 |
| $p_6$ | -3.0809945169 | 1.9095946901 | -0.6920890895 | -2.3166760725 |

$m_1 = 1.0$, $m_2 = 1.2$, $m_3 = 1.4$, $m_4 = 1.6$, $m_5 = 1.8$, $m_6 = 2.0$

Table 4: Selected tensor components of six-point tensor functions produced by the phase space point of Table 3.

| $F_0$ | 0.54701021E-04 – i 0.67031213E-04 |
|-------|-----------------------------------|
| $F^3$ | -0.32082506E-04 + i 0.24545301E-03 |
| $F^{11}$ | 0.13862332E-04 – i 0.12247788E-03 |
| $F^{112}$ | -0.22452724E-04 – i 0.39826579E-04 |
| $F^{1122}$ | 0.15817785E-03 + i 0.26882173E-03 |

Table 4: Selected tensor components of six-point tensor functions produced by the phase space point of Table 3.

LCWS/ILC 2008
6 Acknowledgments

Work supported the European Community’s Marie-Curie Research Training Networks MRTN-CT-2006-035505 “HEPTOOLS” and by Sonderforschungsbereich/Transregio SFB/TRR 9 of DFG “Computergestützte Theoretische Teilchenphysik”. I would also like to thank my collaborators J. Fleischer, J. Głuza, K. Kajda, T. Riemann, and especially J. B. Tausk for useful discussions.

References

[1] Presentation
http://ilcagenda.linearcollider.org/materialDisplay.py?contribId=77&sessionId=18&materialId=slides&confId=2628.

[2] Gerard ’t Hooft and M. Veltman. Scalar one loop integrals. Nucl. Phys., B153:365–401, 1979.

[3] G. Passarino and M. Veltman. One loop corrections for $e^+e^-$ annihilation into $\mu^+\mu^-$ in the Weinberg model. Nucl. Phys., B160:151, 1979.

[4] G. van Oldenborgh. FF: A package to evaluate one loop Feynman diagrams. Comput. Phys. Commun., 66:1–15, 1991.

[5] T. Hahn and M. Perez-Victoria. Automatized one-loop calculations in four and d dimensions. Comput. Phys. Commun., 118:153, 1999.

[6] J. Fleischer, F. Jegerlehner, and O. Tarasov. Algebraic reduction of one-loop Feynman graph amplitudes. Nucl. Phys., B566:423–440, 2000.

[7] A. Davydychev. A simple formula for reducing Feynman diagrams to scalar integrals. Phys. Lett., B263:107–111, 1991.

[8] Silesian Univ., Katowice, webpage http://www.us.edu.pl/~gluza/hexagon, DESY, webpage http://www-zeuthen.desy.de/theory/research/CAS.html.

[9] T. Diakonidis, J. Fleischer, J. Głuza, K. Kajda, T. Riemann, and J. B. Tausk. On the tensor reduction of one-loop pentagons and hexagons. Nucl. Phys. Proc. Suppl., 183:109–115, 2008.

[10] Thomas Hahn and Michael Rauch. News from FormCalc and LoopTools. Nucl. Phys. Proc. Suppl., 157:236–240, 2006.

[11] Andre van Hameren, Jens Vollinga, and Stefan Weinzierl. Automated computation of one-loop integrals in massless theories. Eur. Phys. J., C41:361–375, 2005.

[12] Giovanni Ossola, C. Papadopoulos, and Roberto Pittau. CutTools: a program implementing the OPP reduction method to compute one-loop amplitudes. JHEP, 03:042, 2008.

[13] R. K. Ellis and Giulia Zanderighi. Scalar one-loop integrals for QCD. JHEP, 02:002, 2008.

[14] D. B. Melrose. Reduction of Feynman diagrams. Nuovo Cim., 40:181–213, 1965.

[15] Th. Diakonidis et al. A complete reduction of one-loop tensor 5- and 6-point integrals. 2008.

[16] A. Denner and S. Dittmaier. Reduction schemes for one-loop tensor integrals. Nucl. Phys., B734:62–115, 2006.

[17] T. Binoth, J. Ph. Guillet, G. Heinrich, E. Pilon, and T. Reiter. Golem95: a numerical program to calculate one-loop tensor integrals with up to six external legs. 2008.

LCWS/ILC 2008