ON THE DIAGONAL HOOKS OF A SYMMETRIC PARTITION

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Abstract. Using only a symmetric $p$-core and $p$-quotient, we give an explicit formula for the set of diagonal hook lengths of the associated symmetric partition.

1. Introduction

Suppose $\mathbb{N} = \{0, 1, \ldots\}$ let $n \in \mathbb{N}$ and $p$ be a prime. For standard definitions of a partition $\lambda$ of $n$, its dual $\lambda^*$ and Young diagram $[\lambda]$, a hook $h_{ij}$ of $[\lambda]$ with corner $(i, j)$, the hook length $|h_{ij}|$, the arm length and leg length of $h_{ij}$, and a $\beta$-set $X$ corresponding to $\lambda$, we refer readers to [1], [2], [3], [7].

A $\beta$-set $X$ associated to a partition $\lambda$ can be seen as a finite set of non-negative integers, represented by beads at integral points of the $x$-axis, i.e. a bead at position $x$ for each $x$ in $X$. Then $X$ is a $\beta$-set to $\lambda$ in the extended sense if we extend $X$ infinitely in both directions with beads at all negative positions and spaces at all positions to the right of the position of the largest integer $x_k \in X$. In this interpretation, $\beta$-sets equivalent to $X$ are the same infinite string of beads and spaces with the origin shifted a finite number of positions to the left. A minimal $\beta$-set $X$ is an extended set where the first space is counted as 0. If $X$ is a minimal $\beta$-set of $\lambda$, we define $|X|$ as the number of beads occurring to the right of the leftmost space.

Given a fixed integer $p$, we can arrange the nonnegative integers in an array of columns and consider the columns as runners of an abacus in order to represent $X$.

\[
\begin{array}{cccc}
0 & 1 & \cdots & p - 1 \\
p & p + 1 & 2p - 1 \\
\vdots & \ddots \\
mp & \cdots & mp + p - 1
\end{array}
\]

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The column containing $\gamma$ for $0 \leq \gamma \leq p - 1$ will be called the $\gamma$th runner of the abacus. The integers $\gamma, \gamma + p, \gamma + 2p, \cdots$ label corresponding positions $0, 1, 2, \cdots$ on the $\gamma$th runner. Placing a bead at position $x_j$ for each $x_j \in X$ gives the abacus diagram of $X$.

We deviate from standard notation by denoting by $h = (y, x]$ a hook arising from the $\beta$-set $X$ of $\lambda$ where $y \not\in X$ and $x \in X$. We define the hook length of $(y, x]$ as $x - y$. Lemma describes a bijection between the set of hooks $h_{ij}$ of the Young diagram $[\lambda]$ and the set of hooks $(y, x]$ of a $\beta$-set $X$ of $\lambda$.

**Lemma 1.1.** Let $\lambda$ be a partition of $n$ and $X$ a $\beta$-set of $\lambda$. A hook $h = (y, x]$ of $X$ corresponds to the hook $h_{ij}$ with corner node $(i, j)$ in the Young diagram $[\lambda]$ where

$$i = |z \in \mathbb{N} : z \in X, z \geq x|$$

and

$$j = |z \in \mathbb{N} : z \not\in X, z \leq y|.$$  

Additionally, the leg length and arm length of $h$ are $|z \in \mathbb{N} : z \in X, y < z < x|$ and $|z \in \mathbb{N} : z \not\in X : y < z < x|$ respectively.

**Proof.** See pg. 180 in [1].

If $h = (y, x]$ is a hook of length $p$ (henceforth a $p$-hook) of $X$ then $\{y\} \cup X - \{x\}$ is a $\beta$-set for a partition $\lambda_1$ of $n - p$. We say that $\lambda_1$ and $X_{\lambda_1}$ are achieved from $\lambda$ and $X$ respectively by removing a $p$-hook. In the opposite manner, we see that $\lambda$ and $X$ are gotten from $\lambda_1$ and $X_{\lambda_1}$ respectively by adding a $p$-hook. Subsequently, the abacus diagram of $X_{\lambda_1}$ is related to that of $X$ by moving the bead at $x \in X$ up one position on the runner. Let $X^0$ be the unique $\beta$-set obtained from $X$ by successively removing $p$-hooks until none are left. Thus $X^0$ will have no $p$-hooks. The partition $\lambda^0$ represented by $X^0$ is called the $p$-core of $\lambda$ and is uniquely determined by $\lambda$. The abacus of the $p$-core $\lambda^0$ is obtained from the abacus of $\lambda$ by pushing up the beads in each runner as high up as they can go (Theorem 2.7.16, [2]).

A hook $h = (y, x]$ of length divisible by $p$ is said to be on the $\gamma$th runner if $x$ is on the $\gamma$th runner. Then $y$ is also on the $\gamma$th runner. In particular, hooks of length divisible by $p$ are on the same runner if and only if they have the same residue modulo $p$. For $0 \leq \gamma \leq p - 1$, let $X_\gamma = \{j : \gamma + jp \in X\}$ and let $\lambda_\gamma$ be the partition represented by the $\beta$-set $X_\gamma$. Notice that this is the partition whose beads appear on the $\gamma$th runner of the abacus diagram of $\lambda$. Our convention will be that the $p$-quotient of $\lambda$ is the sequence $(\lambda_0, \cdots, \lambda_{p-1})$ obtained from $X$ where $|X| \equiv 0 \pmod{p}$. We call $X_\gamma$ the $\beta$-set of $\lambda_\gamma$ induced by $X$. 


A partition is symmetric if \( \lambda = \lambda^* \). A \( p \)-quotient \((\lambda_0, \cdots, \lambda_{p-1})\) is symmetric if \( \lambda_i = \lambda^*_{p-i-1} \) where \( 0 \leq i \leq p - 1 \). The \( p \)-quotient and \( p \)-core of a partition \( \lambda \) and its dual \( \lambda^* \) are related in the following manner.

**Lemma 1.2.** Let \( X \) be a \( \beta \)-set for \( \lambda \) such that \(|X| \equiv 0 \pmod{p}\). Let \( \lambda^* \) be the dual of \( \lambda \), let \((\lambda^*)^0\) be the \( p \)-core of \( \lambda^* \) and let \( \{\lambda^*\} = \{\lambda_0^*, \cdots, \lambda_{p-1}^*\} \) be the \( p \)-quotient of \( \lambda^* \). Then \((\lambda^*)^0 = (\lambda^0)^*\) and \((\lambda^*_{p-1})^* = \lambda^*_{p-1-\gamma}\) for \( 0 \leq \gamma \leq p - 1 \). Hence \( \lambda = \lambda^* \) if and only if \( \lambda^0 = (\lambda^0)^* \) and \((\lambda^*)^* = \lambda^*_{\gamma^*} \).

**Proof.** See Proposition 3.5 in [7]. \( \square \)

Given a symmetric partition \( \lambda \), we let \( \delta(\lambda) = \{\delta_{ii}(\lambda)\} \) be the set of diagonal hooks where \( h_{ii} = \delta_{ii}(\lambda) \). When there is no ambiguity we will set \( \delta_{ii}(\lambda) = \delta_{ii} \). By abuse of notation \( \delta_{ii} \) will also stand for the size of \( \delta_{ii}(\lambda) \), and \( \delta(\lambda) \) for the set of diagonal hook lengths of \( \lambda \).

In this paper we give an explicit formula for \( \delta(\lambda) = \{\delta_{ii}\} \) in terms of only the \( p \)-quotient and the \( p \)-core. These results are motivated by an ongoing study of the irrationalities of the character table of the alternating groups \( A(n) \), which, by a classical result of Frobenius, arise from the diagonal hook lengths of symmetric partitions of \( n \). In particular they assist in verifying that a recent refinement of McKay’s conjecture by Navarro [6] involving Galois automorphisms holds for \( A(n) \) in special cases [5].

### 2. Bisequences and diagonal hooks

A bisequence

\[
(\alpha_1, \alpha_2, \cdots, \alpha_t|\beta_1, \beta_2, \cdots, \beta_t)
\]

will be an ordered pair of strictly decreasing sequences of non-negative integers

\[
(\alpha_1, \alpha_2, \cdots, \alpha_t) \\
(\beta_1, \beta_2, \cdots, \beta_t)
\]

of the same length \( t \). For example let \( \alpha_i \) and \( \beta_i \) be the leg and arm lengths of \( \delta_{ii} \in \delta(\lambda) \). Then the sequences \((\alpha_1, \alpha_2, \cdots, \alpha_t)\) and \((\beta_1, \beta_2, \cdots, \beta_t)\), are strictly decreasing. Hence we may define the bisequence

\[
D(\lambda) = (\alpha_1, \alpha_2, \cdots, \alpha_t|\beta_1, \beta_2, \cdots, \beta_t)
\]

We define the components of \( D(\lambda) \) to be \( D(\lambda)_L = (\alpha_1, \alpha_2, \cdots, \alpha_t) \) and \( D(\lambda)_R = (\beta_1, \beta_2, \cdots, \beta_t) \). An element of \( D(\lambda) \) is an ordered pair \((\alpha_i|\beta_i)\) for some \( i \) and corresponds to a diagonal hook \( \delta_{ii} \). Then \(|D(\lambda)|\) is the number of such pairs, and equals \( t \). Note that \(|D(\lambda)_R|\) and
$|D(\lambda)_L|$ (the number of arm lengths and leg lengths of the diagonal hooks respectively) both equal $t$ as well. The dual of $D(\lambda)$ is

$$D(\lambda)^* = (\beta_1, \beta_2, \cdots, \beta_t | \alpha_1, \alpha_2, \cdots, \alpha_t).$$

Clearly $D(\lambda)^* = D(\lambda^*)$. If $\lambda$ is symmetric then $D(\lambda)_L = D(\lambda)_R$. We attach to $D(\lambda)$ a $p$-tuple $D'(\lambda)$ of bisequences.

**Definition 2.1.** Let $D'(\lambda) = (D_0(\lambda), \cdots, D_{p-1}(\lambda))$ where $D_\gamma(\lambda)$ is defined as follows.

1. If $\alpha = \gamma + mp \in D(\lambda)_L$, $0 \leq \gamma \leq p - 1$ and $m \geq 0$, we put $m$ in $D_{p-1-\gamma}(\lambda)_L$.

2. If $\beta = \gamma + mp \in D(\lambda)_R$, $0 \leq \gamma \leq p - 1$ and $m \geq 0$, we put $m$ in $D_\gamma(\lambda)_R$.

$D'(\lambda)$ is called the $p$-quotient of $D(\lambda)$.

From this definition, given $D'(\lambda)$, we can obtain $D(\lambda)$. At the moment, it is not clear that for each $\gamma$ the sequences in $D_\gamma(\lambda)_L$ and $D_\gamma(\lambda)_R$ have the same length. This will be shown to be true when $\lambda^0 = \emptyset$ in Theorem 5.2.

### 3. $\theta(\lambda)$ and Diagonal Hooks

The diagonal hooks $\delta_{ii}$ of $\lambda$ correspond to the following hooks of $\beta$-set $X = \{x_1, \cdots, x_k\}$. The largest hook $\delta_{11}$ corresponds to $(y_1, x_k]$, where $y_1$ is the position of the smallest space i.e. the minimal positive integer not included in $X$. By removing $\delta_{11}$ (that is, by moving the bead at position $x_k$ to the space $y_1$) then $\delta_{22}$ corresponds to the largest hook of $\lambda^\vee$, and so on. Thus the diagonal hooks correspond to the nested hooks starting with the longest hook in $X$, then the longest hook contained strictly within that longest hook, and so on.

Let $\lambda^\vee$ be the partition obtained from $\lambda$ by removing $\delta_{11}$. Then $X^\vee$ is the induced $\beta$-set of $\lambda^\vee$.

**Proposition 3.1.** Suppose $\lambda$ is a partition of $n$ and let $X$ be a $\beta$-set for $\lambda$. Then there exists a half-integer $\theta(\lambda)$ such that the number of beads to the right of $\theta(\lambda)$ equals the number of spaces to the left of $\theta(\lambda)$.

**Proof.** Let $\theta$ be the point at the half-integer just to the left of the smallest space of $X$. So there are 0 spaces to the left of $\theta$ and a finite number of beads to the right. Next move $\theta$ a unit distance to the right, that is, to the next half-integer on the right. One and only one of the following happens: Either the number of spaces to the left of $\theta$ increases by one or the number of of beads to the right decreases by
one. So by iterating this process we reach a point where the number of spaces to the left of $\theta$ is equal to the number of beads to the right of $\theta$. That point is then $\theta(\lambda)$.

Let $X$ be a $\beta$-set for a partition $\lambda$ (not necessarily symmetric), with maximal element $x_k \in X$. Let $X_+$ be the subset of beads to the right of $\theta(\lambda)$ and let $X_-$ be the subset of spaces to the left of $\theta(\lambda)$. We index the elements of $X_+ = \{y'_i : i \leq r\}$ so that $y'_1 < \cdots < y'_1$ with $y'_1$ as the largest bead. Correspondingly, we index the elements of $X_- = \{y_i : i \leq r\}$ so that $y_1 < \cdots < y_r$ with $y_1$ as the smallest space. In particular, $y'_i - y_i = \alpha_i + \beta_i + 1$ for all $1 \leq i \leq r$ since the length of the hook is one plus the sum of its leg and arm lengths. This relation holds whether or not $X$ is a minimal $\beta$-set since $y'_i, y_i$ and $\theta(\lambda)$ shift by the same amount when $X$ is shifted, whereas $\alpha_i$ and $\beta_i$ do not change.

In particular the $\beta$-set $X$ gotten from removing the largest diagonal hook of $X$ can be used so that $\theta(\lambda^\vee) = \theta(\lambda)$ holds. Hence we have the following.

**Lemma 3.2.** Suppose $\lambda'$ and $\lambda$ are partitions such that $|\lambda'| < |\lambda|$ and $\lambda'$ can be obtained from $\lambda$ by removing a sequence of hooks from $\lambda$. Then $\theta(\lambda) = \theta(\lambda')$.

**Proposition 3.3.** $X_-$ and $X_+$ correspond to $D(\lambda)_L$ and $D(\lambda)_R$ in the following manner. Let $\alpha_i \in D(\lambda)_L$ and $\beta_i \in D(\lambda)_R$. Then for each $y'_i \in X_+$ and $y_i \in X_-$ we have

1. $y_i = \theta(\lambda) - \frac{1}{2} - \alpha_i$.
2. $y'_i = \theta(\lambda) + \frac{1}{2} + \beta_i$

**Proof.** We proceed by induction on $s = |D(\lambda)|$. Suppose $s = 1$. Then $D(\lambda) = \{\alpha, \beta\}$ and $X = \{1, 2, \cdots, m-1, m, t_1\}$ so $\theta(\lambda) = m + \frac{1}{2}$. Recall $\alpha$ and $\beta$ are the number of beads and spaces respectively in the interval $(0, t_1)$, and hence $\alpha = \theta(\lambda) - \frac{1}{2}$ and $\beta = t - \theta(\lambda) - \frac{1}{2}$.

Consider $\lambda$ where $|D(\lambda)| = s$ and let $\lambda^\vee$ be the partition obtained by removing $h_{11}$ from $\lambda$. Then, by induction and Lemma 3.2 when $|D(\lambda^\vee)| = s - 1$ we have that

$$y_i = \theta(\lambda) - \frac{1}{2} - \alpha_i \quad y'_i = \theta(\lambda) + \frac{1}{2} + \beta_i$$

for $2 \leq i \leq r$. In particular, $y_2 = \theta(\lambda) - \frac{1}{2} - \alpha_2$ and $y'_2 = \theta(\lambda) + \frac{1}{2} + \beta_2$. But $y_2 - y_1$ is one plus the number of beads between $y_1$ and $y_2$, which is precisely the difference $\alpha_1 - \alpha_2$ by Lemma 3.1. These formulas imply $y_1 = \theta(\lambda) - \frac{1}{2} - \alpha_1$ and $y'_1 = \theta(\lambda) + \frac{1}{2} + \beta_1$.

Suppose $\lambda$ is symmetric. Then the diagonal hook lengths $\delta_{ii}$ are necessarily odd. Then $X$ has an axis of symmetry $\theta(\lambda)$ where beads and
spaces on one side are reflected respectively into spaces and beads on the other side.

**Corollary 3.4.** Suppose $\lambda$ is a symmetric partition and $X$ is a $\beta$-set in the extended sense for $\lambda$. Then there exists an axis of symmetry $\theta(\lambda)$ at a half-integer such that beads and spaces in $X$ to the right of $\theta(\lambda)$ are reflected respectively to spaces and beads in $X$ to the left of $\theta(\lambda)$.

*Proof.* Follows from Proposition 3.1 and Proposition 3.3. □

**Lemma 3.5.** Suppose $\lambda$ is symmetric with empty $p$-core. Then the number of beads to the right of $\theta(\lambda)$ on the $\gamma$th runner is the same as the number of empty positions to the left of $\theta(\lambda)$ on the $\gamma$th runner.

*Proof.* Let $X$ be a $\beta$-set for $\lambda$ and let $X^0$ be the $\beta$-set for $\lambda^0$. Then $X^0$ consists of $\{0, 1, 2, \ldots, t\}$. Then it is clear that symmetry about $\theta(\lambda)$ is possible only if $\theta(\lambda) = t + 1/2$. Thus all beads of $X^0$ are left of the axis and all spaces of $X^0$ are right of the axis. Hence, all beads on the $\gamma$-runner right of the axis must be accommodated by spaces on the $\gamma$-runner left of the axis. □

We will need the following relation in Section 8.

**Lemma 3.6.** If $\lambda$ has empty $p$-core, then $\theta(\lambda_\gamma) = \theta(\lambda_{\gamma'})$ for all $0 \leq \gamma, \gamma' \leq p - 1$. In particular, if $|X| = mp$, then $\theta(\lambda_\gamma) = \theta(\lambda_{\gamma''}) = m - \frac{1}{2}$.

*Proof.* Since $\lambda$ has empty $p$-core the $\beta$-set for $\lambda^0$ is $X = \{0, 1, \ldots, mp-1\}$ for some $m$. Then the abacus diagram will consist of (from north-to-south) $m$ rows of beads followed by rows of empty spaces. On each runner one begins counting at 0, hence $\theta(\lambda^0_\gamma) = \theta(\lambda^0_{\gamma'}) = m - \frac{1}{2}$. By Lemma 3.2, $\theta(\lambda^0_\gamma) = \theta(\lambda_\gamma)$ and $\theta(\lambda^0_{\gamma'}) = \theta(\lambda_{\gamma'})$. The result follows. □

We will need the following relation in Section 8.

**Lemma 3.7.** Suppose $\lambda$ has empty $p$-core. Then

$$p(\theta(\lambda_\gamma) + \frac{1}{2}) = \theta(\lambda) + \frac{1}{2}$$

for all $0 \leq \gamma \leq p - 1$.  

Proof. Since $\lambda$ has an empty $p$-core we have $X^0 = \{0, 1, 2, \ldots, mp - 1\}$ for some $m$. Hence $\theta(\lambda) = mp - \frac{1}{2}$. Since $mp$ is the total number of beads in $X$, we have
\[ p(m - 1 + 1) = mp - \frac{1}{2} + \frac{1}{2} \]
which implies $p(\theta(\lambda) + \frac{1}{2}) = \theta(\lambda) + \frac{1}{2}$ by Lemma 3.6. \hfill $\square$

Let $\lambda$ be such that $\lambda^0 \neq \emptyset$ and let $\bar{\lambda}$ be such that $\bar{\lambda} = \emptyset$ but $\bar{\lambda}_i = \lambda_i$ for $0 \leq i \leq p - 1$.

**Corollary 3.8.** For all $0 \leq \gamma \leq p - 1$ we have
\[ p(\theta(\bar{\lambda}) + \frac{1}{2}) = \theta(\lambda) + \frac{1}{2}. \]

Proof. By Lemma 3.2 \(\theta(\bar{\lambda}) = \theta(\lambda)\). The result then follows from Lemma 3.7. \hfill $\square$

We use Corollary 3.8 to offer an interpretation of $D'(\lambda)$. Suppose $\alpha_i \in D(\lambda)_{L, \gamma}$ so $\alpha_i = \gamma + \eta_i p$. Then $\gamma + \eta_i p = \theta(\lambda) - \frac{1}{2} - y_i$ by Proposition 3.3. By Corollary 3.8
\[ \gamma + \eta_i p = p(\theta(\bar{\lambda}) - \frac{1}{2}) + p - 1 - y_i. \]

Suppose $y_i = \gamma + \bar{y}_i$. Then $\bar{y}_i = \theta(\bar{\lambda}) - \eta_i - \frac{1}{2}$.

Now suppose $\beta_i \in D(\lambda)_{R, \gamma}$ so $\beta_i = \gamma + \eta_i p$. Then $\gamma + \eta_i p = y_i' - \theta(\lambda) - \frac{1}{2}$. Then
\[ \gamma + \eta_i p = y_i' - p(\theta(\bar{\lambda}) + \frac{1}{2}). \]

by Proposition 3.3 Hence $y_i' - \gamma = p(\theta(\lambda) + \frac{1}{2} + \eta_i)$. Suppose $y_i' = \gamma + \bar{y}_i'$. Then $\bar{y}_i = \theta(\bar{\lambda}) + \frac{1}{2} + \eta$. Hence $D'(\lambda) = \{D_\gamma(\lambda)\}_{0 \leq \gamma \leq p - 1}$ can be expressed as distances of the beads and spaces of the corresponding $X_\gamma$ from each $\theta(\lambda)$. We will use this observation in Section 8.

### 4. Pairs of Straddling or Non-Straddling $p$-hooks

If $(x', x]$ is a diagonal hook of $X$ corresponding to a symmetric $\lambda$, we call $x'$ the opposite position of $x$. Given two diagonal hooks $(x', x]$ and $(y', y]$ where $x < y$, we call the (non-diagonal) hooks $(y', x]$ and $(x', y]$ opposite hooks. Conversely, given opposite non-diagonal hooks $(y', x]$ and $(x', y]$ with $x < y$ we get diagonal hooks $(y', y]$ and $(x', x]$.

**Lemma 4.1.** Suppose $\lambda$ is symmetric and let $X$ be a $\beta$-set for $\lambda$ with $|X| \equiv 0 \pmod{p}$. Let $(x', x]$ be a diagonal hook of $X$. Then $x' \equiv p - 1 - \gamma \pmod{p}$ if and only if $x \equiv \gamma \pmod{p}$.
Proof. By symmetry around $\theta(\lambda)$, the number of beads and empty positions below the axis is $|X|$. Hence $\theta(\lambda) - \frac{1}{2} \equiv p - 1 \pmod{p}$. Since $x'$ and $x$ are equidistant from $\theta(\lambda)$, then $x' \equiv p - 1 - x \pmod{p}$. □

Suppose we want to reduce a symmetric partition $\lambda$ of $n - p$ by removing one $p$-hook. There is one way of doing so.

1. (The single hook case) Then $p$-hook $h = (y, x)$ is a diagonal hook $(x', x)$ where $x - x' = p$.

Suppose we want to reduce a symmetric partition $\lambda$ of $n$ to a symmetric partition $\lambda'$ of $n - 2p$ by removing two $p$-hooks. By removing two opposite $p$-hooks $h = (y, x]$ and $h' = (x', y',]$ where $h \neq h'$. There are two cases, the non-straddling case, in which $x' < y' < \theta(\lambda) < y < x$, and the straddling case, in which $x' < y < \theta(\lambda) < y' < x$.

1. (The non-straddling case). Suppose $h$ is completely to the right of $\theta(\lambda)$. Then removing $h$ and $h'$ is equivalent to replacing a diagonal hook $(x', x]$ with $(x' + p, x - p)$.

2. (The straddling case). Suppose that $h$ and $h'$ straddle $\theta(\lambda)$.

Then removing $h$ and $h'$ is equivalent to removing two diagonal hooks $(x', x]$ and $(y, y')$ where $x - x' + y' - y = 2p$.

Suppose $h = (y, x]$ and $h' = (x', y')$ are non-straddling opposite $p$-hooks of $\lambda$ (as in Figure 2). Without loss of generality, $x > y'$. Let $h = h_{ij}$, i.e. have corner $(i, j)$ in $[\lambda]$. Then the corner of $h$ is on the arm of some diagonal hook. Since

$$|\{z \in \mathbb{N} : z \notin X, z \leq y\}| > |\{z \in \mathbb{N} : z \in X, z \geq x\}|$$

by Lemma 1.1, we have $j > i$. Thus, if $x = \gamma + kp$, where $0 \leq \gamma \leq p - 1$ and $k \geq 0$, then $y \notin X$ such that $y = \gamma + (k - 1)p$. Consequently, $h = (y, x]$ of $\lambda$ corresponds to some hook $(k - 1, k]$ of $\gamma$ on the $\gamma$th runner of the $p$-abacus. We give the exact coordinates $(i_h, j_h)$ on the Young diagram $[\lambda_\gamma]$ corresponding to $(k - 1, k]$ when $\lambda$ has empty $p$-core. Define

$$A = \{z \equiv \gamma \pmod{p}, z \in X : z \geq x\}$$
$$B = \{z \equiv \gamma \pmod{p}, z \notin X : \theta(\lambda) < z \leq y\}$$
$$C = \{z \equiv -1 - \gamma \pmod{p}, z \in X : z > \theta(\lambda)\}.$$

Let $|A| = a$, $|B| = b$ and $|C| = c$. By construction, $i_h = a$.

By Proposition 3.1, $\lambda_\gamma$ has an axis $\theta(\lambda_\gamma)$ that is a half-integer such that the number of beads above $\theta(\lambda_\gamma)$ is the same as the number of spaces below.
Lemma 4.2. Suppose \( \lambda \) has empty \( p \)-core and \( Y = \{w_1, \ldots, w_j\} \) is the induced \( \beta \)-set for \( \lambda \). Let \( k \) be an integer such that \( 0 \leq k \leq w_j \). Then we have the following.

1. If \( k < \theta(\lambda) \) then \((p - \gamma - 1) + kp < \theta(\lambda)\)
2. If \( k > \theta(\lambda) \) then \( \gamma + kp > \theta(\lambda) \).

Proof. Follows by Proposition 3.7 and the definition of the \( p \)-quotient. \( \square \)

Lemma 4.3. Suppose \( \lambda \) is a symmetric partition with empty \( p \)-core. Consider the \( p \)-hook \( h = (y, x) \). Let \((i_h, j_h)\) be the coordinates of the corresponding 1-hook of \([\lambda_\gamma]\) for some fixed \( \gamma \). Then

\[ j_h = b + c \]

if and only if \( h \) is completely to the right of \( \theta(\lambda) \).

Proof. Suppose \( j_h = b + c \). It is clear that \( h \) is completely to the right of \( \theta(\lambda) \). Suppose \( h \) is completely to the right of \( \theta(\lambda) \). Since \( \lambda \) is symmetric, we know by Lemma 4.1 that \( C \) corresponds bijectively to the set \( \{y' \in \mathbb{N}, y' \notin X, y < \theta(\lambda) : y' \equiv \gamma \pmod{p}\} \). Hence \( c \) is also the number of empty positions less than \( \theta(\lambda) \) of residue \( \gamma \pmod{p} \). By Lemma 3.5, \( b \) is the number of empty positions between \( \theta(\lambda) \) (and including) \( y \) with residue \( \gamma \pmod{p} \). This follows since \( \lambda \) has empty \( p \)-core. Hence \( b + c \) is the total number of empty positions below and including \( y \) with residue \( \gamma \pmod{p} \). Then, by Lemma 1.1, we are done. \( \square \)

Lemma 4.4. Suppose \( \lambda \) is symmetric with empty \( p \)-core. Consider the \( p \)-hook \( h = (y, x) \). If \( h \) is completely to the right of \( \theta(\lambda) \) then

\[ a \leq c \]

Proof. By Lemma 4.1, we have that \( c \) is the number of empty positions less than \( \theta(\lambda) \) that have residue \( \gamma \pmod{p} \). By Lemma 3.5, since \( \lambda \) has empty \( p \)-core, \( c \) must be equal to the number of \( z \in X \) such that \( z > \theta(\lambda) \) and \( z \equiv \gamma \pmod{p} \). Since \( x > \theta(\lambda) \) and \( A = \{z = \gamma + jp, z \in X : z \geq x\} \), we have \( a \leq c \). \( \square \)

Proposition 4.5. A hook \((k - 1, k]\) of size 1 on \( \lambda_\gamma \) corresponds to the \( p \)-hook \( h = (y, x) \) on \( \lambda \) where \( y \) and \( x \) are completely to the right (resp. left) of \( \theta(\lambda) \) if and only if \((k - 1, k]\) occurs on an arm (resp. leg) of \([\lambda_\gamma]\).

Proof. Suppose \( h \) is to the right of \( \theta(\lambda) \). The coordinates of \((k - 1, k]\) on the Young diagram \([\lambda_\gamma]\) are \((i_h, j_h)\) where \( i_h = a \) (by definition) and \( j_h = b + c \) by Lemma 4.3. Since \( y \notin B \), \( |B| \neq 0 \) and we have \( a < b + c \),
since \(a \leq c\) by Lemma 4.4. It follows that \(i_h < j_h\). Hence \((k - 1, k]\) is a 1-hook on the arm of \([\lambda_\gamma]\).

Suppose \((k - 1, k]\) is a 1-hook on the arm of \([\lambda_\gamma]\). Clearly \(\theta(\lambda_\gamma) < k - 1\). Hence \(\theta(\lambda) < y = \gamma + (k - 1)p\) by Lemma 4.2. Since \(y < x\), \(h\) is completely to the right of \(\theta(\lambda)\).

Suppose \((k - 1, k]\) is a 1-hook on \(\lambda_\gamma\) corresponding to a \(p\)-hook \(h = (y, x]\) completely to the left of \(\theta(\lambda)\). Then \(h\) can be viewed as a hook completely to the right of \(\theta(\lambda^\star)\). Hence, by the argument above, it corresponds to a 1-hook on the arm of \(\lambda_\gamma\). Taking the dual again, \(h\) corresponds to a 1-hook on the leg of \(\lambda_\gamma\).

\[\text{Proposition 4.6.}\] Let \(\lambda\) be a symmetric partition with empty \(p\)-core. Let \(\gamma \in \{0, p - 1\}\). Then a hook \(h = (k - 1, k]\) of size 1 on \(\lambda_\gamma\) corresponds to the \(p\)-hook \(h = (y, x]\) of \(\lambda\) where \(y\) is to the left of \(\theta(\lambda)\) and \(x\) is to the right of \(\theta(\lambda)\) if and only if \((k - 1, k]\) is a diagonal hook of \([\lambda_\gamma]\).

\[\text{Proof.}\] Consider a pair \(h = (y, x]\) and \(h' = (x', y']\) of straddling \(p\)-hooks. Without loss of generality, we may suppose \(x' < y < y' < x\). Set \(y' = \gamma + jp\) and \(x = (p - 1 - \gamma) + kp\). So \(x' = \gamma + (j - 1)p\) and \(y = (p - 1 - \gamma) + (k - 1)p\).

Suppose \(E = \{z \in X, z = \gamma (\mod p), z \geq y']\) and \(F = \{z' \not\in X, z' = \gamma (\mod p) : z' \leq x']\). Let \(|E| = e\) and \(|F| = f\). By Lemma 4.1, the coordinates of the hook \((k - 1, k]\) on the Young diagram \([\lambda_\gamma]\) are \((i_h, j_h) = (e, f)\). The inequality \(x' < y < x\) is equivalent to \((k - 1)p < 2\gamma - p + 1 + jp < kp\). If \(\gamma = \frac{p - 1}{2}\), then \(h = h'\), which is impossible. Hence we only consider \(\gamma \neq \frac{p - 1}{2}\). If \(0 \leq \gamma < (p - 1)/2\), it follows that \(j = k\) and \(\theta(\lambda) = kp - 1/2\). If \((p - 1)/2 < \gamma \leq p - 1\), it follows \(j = k - 1\) and \(\theta(\lambda) = jp + (p - 1)/2\).

Since \(\theta(\lambda)\) is a half-integer, the second case is impossible and \(0 \leq \gamma < (p - 1)/2\). Now define \(E' = \{z \in X : z = \gamma (\mod p), z > \theta(\lambda)\}\) and \(F' = \{z' \not\in X : z' = \gamma (\mod p), z' < \theta(\lambda)\}\), so \(|E'| = |F'|\) by Lemma 4.5. Since \(y < \theta(\lambda) < x\) and \(x - y = p\), we have \(\{z \in X : z = \gamma (\mod p), \theta(\lambda) < z < x\} = \{z' \not\in X : z' = \gamma (\mod p), y < z' < \theta(\lambda)\} = \emptyset\). Thus \(E' = E\) and \(F' = F\) and we are done.

Suppose \((k - 1, k]\) is a hook of size 1 on the diagonal of \([\lambda_\gamma]\). Then \(k - 1 < \theta(\lambda_\gamma) < k\). Hence \((p - 1 - \gamma + (k - 1)p, \gamma + kp] = (y, x]\) straddles \(\theta(\lambda)\) by Lemma 4.2.

5. \(D'(\lambda)\) concentrated at one or two places

Suppose \(\lambda\) is a symmetric partition. We say \(D(\lambda)\) is \emph{concentrated} at \(\{\gamma, \gamma^\star\}\) if \(D_i(\lambda) \neq \emptyset\) for \(i \in \{\gamma, \gamma^\star\}\) and \(D_i(\lambda) = \emptyset\) otherwise.
Lemma 5.1. Suppose \( \lambda \) and \( \lambda' \) are distinct partitions such that \( D(\lambda) \) is concentrated at \( \{\gamma, \gamma^*\} \) and \( D(\lambda') \) is concentrated at \( \{\gamma', \gamma'^*\} \) where \( \gamma \neq \gamma' \). If \( \{\gamma, \gamma^*\} \neq \{\gamma', \gamma'^*\} \) then \( D(\lambda) \cap D(\lambda') = \emptyset \), that is, no diagonal hook length of \( \lambda \) equals a diagonal hook length of \( \lambda' \).

Proof. Suppose not. Then there exists \( \alpha \in D(\lambda)_L \) and \( \alpha' \in D(\lambda')_L \) such that \( \alpha = \alpha' \). But \( \alpha = \gamma + mp \) and \( \alpha' = \gamma' + m'p \) so that \( \gamma = \gamma' \). This is impossible. \( \square \)

Suppose \( \lambda \) and \( \lambda' \) are symmetric partitions such that \( D(\lambda) \cap D(\lambda') = \emptyset \). Define \( \lambda + \lambda' \) to be the symmetric partition such that \( D(\lambda + \lambda') = D(\lambda) \cup D(\lambda') \). In particular, we can form \( \lambda + \lambda' \) whenever \( \lambda \) and \( \lambda' \) are concentrated on disjoint sets.

Theorem 5.2. Let \( \lambda \) be symmetric with empty \( p \)-core such that \( D'(\lambda) \) is concentrated at \( \{\gamma, \gamma^*\} \) where \( \gamma \neq \gamma^* \). Then

1. \( D'(\lambda) \) is a \( p \)-tuple of bisequences, that is, for each \( \gamma \), \( D_\gamma(\lambda)_L \) and \( D_\gamma(\lambda)_R \) are of equal lengths.
2. For each \( \gamma \), \( D_\gamma(\lambda) = D_\gamma(\lambda_\gamma) \) and \( D_{\gamma^*}(\lambda) = D_{\gamma^*}(\lambda_{\gamma^*}) \), where \( \lambda_\gamma \) and \( \lambda_{\gamma^*} \) are the \( \gamma \)-th and \( \gamma^* \)-th components of the \( p \)-quotient of \( \lambda \).
3. Suppose \( D(\lambda_\gamma) = (\sigma_1, \ldots, \sigma_w | \tau_1, \ldots, \tau_w) \). Then
   \[
   D(\lambda) = (\alpha_1, \ldots, \alpha_{2w} | \alpha_1, \ldots, \alpha_{2w})
   \]
   where \( \{\alpha_1, \ldots, \alpha_{2w}\} = \{\gamma^* + \sigma_i p, \gamma + \tau_i p : 1 \leq i \leq w\} \).

Proof. By induction on \( |\lambda| \). The minimal case is \( |\lambda| = 2p \) where \( D(\lambda) = (p - 1 - \gamma, \gamma | p - 1 - \gamma, \gamma) \). Then \( \lambda \) is comprised of just two opposite \( p \)-hooks. Hence \( |D_\gamma(\lambda)_R| = |D_\gamma(\lambda)_L| = 1 \) and part (1) follows. By definition, \( D_\gamma(\lambda) = (0 | 0) \) and \( D_{p-1-\gamma}(\lambda) = (0 | 0) \). Since \( \lambda_\gamma = (1) \) and \( \lambda_{\gamma^*} = (1) \), part (2) follows. Part (3) follows since \( D(\lambda) = (p - 1 - \gamma, \gamma | p - 1 - \gamma, \gamma) \). Now suppose \( |\lambda| = n > 2p \). By induction, we assume that the theorem holds for all partitions \( \lambda \) such that \( |\lambda| < n \). Consider \( |\lambda| = n \). Let \( h, h' \) be opposite \( p \)-hooks in \( \lambda \) and let \( \lambda^V \) be the symmetric partition gotten from removing \( h \) and \( h' \). Following the discussion preceding Lemma 4.3, there are two cases.

Case 1: (The non-straddling case) Here one obtains \( D(\lambda^V) \) from \( D(\lambda) \) by replacing an element \( (\alpha | \alpha) \) by \( (\alpha - p | \alpha - p) \) where \( \alpha - p \geq 0 \). Then \( D_\gamma(\lambda)_L \) and \( D_{\gamma^*}(\lambda^V)_L \) are the same except for some \( \sigma_\mu \in D_{\gamma^*}(\lambda)_L \) which is replaced by \( \sigma_\mu - 1 \). By symmetry, \( \sigma_\mu \) is replaced by \( \sigma_\mu - 1 \) resulting in \( D_{\gamma^*}(\lambda^V)_R \). We prove that parts (1), (2), and (3) hold.

1. By induction \( D_\gamma(\lambda^V) \) and \( D_{\gamma^*}(\lambda^V) \) are both bisequences with components of equal length. Hence the same is true for \( D_\gamma(\lambda) \) and \( D_{\gamma^*}(\lambda) \).
(2) By induction $D_\gamma(\lambda^\nu) = D(\lambda^\nu)$. By Proposition [4.5] $\lambda^\nu$ is obtained from $\lambda^\nu_\gamma$ by adding a hook of size 1 to both the leg length of the diagonal hook corresponding to $(\sigma_\mu - 1|\tau_\mu) \in D(\lambda^\nu_\gamma)$ and to the arm length of the diagonal hook $(\tau_\mu|\sigma_\mu - 1) \in D(\lambda^\nu_\gamma)$. Hence $D_\gamma(\lambda) = D(\lambda_\gamma)$ and $D_{\gamma^*}(\lambda) = D(\lambda_{\gamma^*})$.

(3) Given

\[
D(\lambda^\nu_\gamma) = (\sigma_1, \cdots, \sigma_{\mu - 1}, \cdots, \sigma_w|\tau_1, \cdots, \tau_i, \cdots, \tau_w)
\]

we have by induction that

\[
D(\lambda^\nu) = (\cdots \alpha_i', \alpha_i'' \cdots | \cdots \alpha_i', \alpha_i'' \cdots)
\]

where

\[
\alpha_i' = (p - 1 - \gamma) + \sigma_ip \quad \alpha_i'' = \gamma + \tau_ip
\]

1 \leq i \leq w \text{ and } i \neq \mu. \text{ When } i = \mu, \text{ then}

\[
\alpha_\mu' = (p - 1 - \gamma) + (\sigma_\mu - 1)p \quad \alpha_\mu'' = \gamma + \tau_\mu p
\]

It is clear by replacing $\sigma_\mu - 1$ by $\sigma_\mu$, that the desired formula for $D(\lambda)$ is obtained.

Case 2: (The straddling case) Here $D(\lambda^\nu)$ one obtains $D(\lambda)$ from by removing $(\alpha|\alpha)$ and $(\beta|\beta)$ where $\alpha + \beta + 1 = p$ (assume without loss of generality that $\alpha > \beta$). Then $D(\lambda^\nu)$ is also concentrated at $\{\gamma, \gamma^*\}$. The relation $\alpha + \beta + 1 = p$ implies that if $\alpha = \gamma$, then $\beta = p - 1 - \gamma$. Thus $(\alpha|\alpha)$ contributes a term 0 to $D_\gamma(\lambda)_L$ and a 0 to $D_{\gamma^*}(\lambda)_R$. Likewise $(\beta|\beta)$ contributes a term 0 to $D_\gamma(\lambda)_R$ and a 0 to $D_{\gamma^*}(\lambda)_R$. We prove that parts (1), (2), and (3) hold.

(1) By induction $D_\gamma(\lambda^\nu)$ is a bisquence with components of equal length. Hence $D_\gamma(\lambda)$.

(2) By induction $D_\gamma(\lambda^\nu) = D(\lambda^\nu)$ and $D_{\gamma^*}(\lambda^\nu) = D(\lambda^\nu_{\gamma^*})$. Now we re-attach to $\lambda^\nu$ the diagonal hooks corresponding to $(\alpha|\alpha)$ and $(\beta|\beta)$, where $\alpha + \beta + 1 = p$. This is equivalent to adjoining $(0|0)$ to both $D_\gamma(\lambda)$ and $D_{\gamma^*}(\lambda')$. The effect on the partitions $\lambda^\nu_\gamma$ and $\lambda^\nu_{\gamma^*}$ will be adding a diagonal node of size 1 to each, by Proposition [4.6]. Hence $D_\gamma(\lambda) = D(\lambda_\gamma)$ and $D_{\gamma^*}(\lambda) = D(\lambda_{\gamma^*})$.

(3) Given

\[
D(\lambda^\nu_\gamma) = (\sigma_1, \cdots, \sigma_{w-1}|\tau_1, \cdots, \tau_{w-1})
\]

\[
D(\lambda^\nu_{\gamma^*}) = (\tau_1, \cdots, \tau_{w-1}|\sigma_1, \cdots, \sigma_{w-1})
\]

then by induction

\[
D(\lambda^\nu) = (\cdots \alpha_i', \alpha_i'' \cdots | \cdots \alpha_i', \alpha_i'' \cdots)
\]
Corollary 5.3. Suppose $\lambda$ is symmetric with empty $p$-core and $D(\lambda)$ is concentrated at $\{\gamma, \gamma^* : \gamma \neq \gamma^*\}$ and $D_\gamma(\lambda) = (\sigma_1, \cdots, \sigma_w | \tau_1, \cdots, \tau_w)$. Then

$$\delta(\lambda) = \cup_i \{2(\sigma_i + 1)p - 2\gamma - 1, 2\tau_i p + 2\gamma + 1\}$$

Proof. Follows from part 3 of Theorem 5.2 \hfill \Box

Example 5.4. Let $p = 5$. Suppose the 5-quotient is concentrated at $\{\gamma, \gamma^*\} = \{0, 4\}$, where $\lambda_0 = (6^2, 2)$ and $\lambda_4 = (3^2, 2^3)$. Then $D_0(\lambda) = (2, 1|5, 4)$ and $D_4(\lambda) = (5, 4|2, 1)$, $D(\lambda) = (25, 20, 14, 9|25, 20, 14, 9)$ and $\delta(\lambda) = (51, 41, 29, 19)$.

Similar results hold for the case when $\lambda$ is symmetric and $D(\lambda)$ is concentrated at $\gamma = \gamma^* = \frac{p-1}{2}$.

Theorem 5.5. Suppose $\lambda$ is a symmetric partition with empty $p$-core and let $D(\lambda)$ be concentrated at $\gamma = \gamma^* = \frac{p-1}{2}$. Then

1. $D'(\lambda)$ is a $p$-tuple of $p-1$ empty bisequences, with $D_{\frac{p-1}{2}}(\lambda) \neq \emptyset$ and $D_{\frac{p-1}{2}}(\lambda)_R$ and $D_{\frac{p-1}{2}}(\lambda)_L$ are of equal lengths.
2. $D_{\frac{p-1}{2}}(\lambda) = D(\lambda_{\frac{p-1}{2}})$.
3. Suppose $D(\lambda_{\frac{p-1}{2}}) = (w_1, \cdots, w_\mu|w_1, \cdots, w_\mu)$, and $D(\lambda_{\gamma}) = \emptyset$ when $\gamma \neq \frac{p-1}{2}$. Then

$$D(\lambda) = (z_1, \cdots, z_\mu|z_1, \cdots, z_\mu)$$

where $z_i = \frac{p-1}{2} + w_i p$.

Proof. By induction on $|\lambda|$. The minimal case is $|\lambda| = p$. In this case $D(\lambda) = (\frac{p-1}{2}|\frac{p-1}{2})$ and $D_{\frac{p-1}{2}}(\lambda) = (0|0)$. The remainder of the proof is similar to that of Theorem 5.2 \hfill \Box

Corollary 5.6. Suppose $\lambda$ is symmetric with empty $p$-core, such that $\lambda$ is concentrated at $\{\frac{p-1}{2}\}$. Then $\delta(\lambda) = \cup_i \{(2m_i + 1)p\}$ if $\delta(\lambda_{\frac{p-1}{2}}) = \cup_i \{2m_i + 1\}$ for every $(m_i|m_i) \in D(\lambda_{\frac{p-1}{2}})$.

Proof. This follows from Theorem 2, part 3. \hfill \Box
Example 5.7. Let $p = 5$. Suppose the 5-quotient is concentrated at $\{2\}$ and $\lambda_0 = (2^2)$. Then $D_2(\lambda) = (1, 0|1, 0)$ and $D(\lambda) = (7, 2|7, 2)$, $\delta(\lambda) = (15, 5)$.

6. Symmetric partitions with an empty $p$-core

Now suppose $\lambda$ is symmetric and has empty $p$-core. Fix a $\gamma$ between 0 and $\frac{p-1}{2}$. Suppose $D(\lambda_{[\gamma]}) \subseteq D(\lambda)$ is the bisequence whose $p$-quotient $D'(\lambda_{[\gamma]})$ has just the components $D_\gamma(\lambda)$ and $D_{\gamma^*}(\lambda)$. Let $\lambda_{[\gamma]}$ be the symmetric partition corresponding to $D(\lambda_{[\gamma]})$. By Lemma 5.1:

$$\lambda_{[0]}, \ldots, \lambda_{\frac{p-1}{2}}$$

have disjoint diagonals. Thus $\lambda_{[0]} + \lambda_{[1]} + \cdots + \lambda_{\frac{p-1}{2}}$ is defined in the sense described in the remark before Theorem 5.2.

Theorem 6.1. Suppose $\lambda$ is symmetric and has empty $p$-core. Then

$$\lambda = \lambda_{[0]} + \lambda_{[1]} + \cdots + \lambda_{\frac{p-1}{2}}$$

$$D(\lambda) = \coprod_{1 \leq \gamma \leq \frac{p-1}{2}} D(\lambda_{[\gamma]})$$

Proof. By Lemma 5.1, it is clear that $D(\lambda_{[\gamma]}) \cap D(\lambda_{[\mu]}) = \emptyset$ when $\gamma \neq \mu$. Let $k_\gamma = |D_\gamma(\lambda)_R|$. By Theorem 5.2 and Theorem 5.5, for a fixed $\gamma$, we have for all $1 \leq i \leq k_\gamma$ and $1 \leq j \leq t$, the diagonal hooks of $\lambda$ corresponding to $(\alpha'_{\gamma, i}, \alpha''_{\gamma, i})$, $(\alpha'_{\gamma, i}, \alpha''_{\gamma, i})$ and $(z_j, z_j)$ in $D(\lambda)$ have distinct lengths. Hence $\bigcap_{1 \leq \gamma \leq \frac{p-1}{2}} D(\lambda_{[\gamma]}) = \emptyset$. Since these exhaust the diagonal hooks arising from the $p$-quotient $D'(\lambda)$, and $\lambda$ has an empty $p$-core, $\bigcap_{1 \leq \gamma \leq \frac{p-1}{2}} D(\lambda_{[\gamma]})$ constitute all of the diagonal hook lengths of $\lambda$. $\square$

Example 6.2.

Suppose $p = 5$, $\lambda \vdash 190$ is symmetric with empty $p$-core and $\lambda_0 = (6^2, 2)$, $\lambda_1 = (3)$, $\lambda_2 = (2^2)$, $\lambda_3 = (1^3)$, $\lambda_4 = (3^2, 2^3)$. Then $D_0(\lambda) = \{2, 1|5, 4\}$, $D_1(\lambda) = \{0|2\}$, $D_2(\lambda) = \{1, 0|1, 0\}$, $D_3(\lambda) = \{2|0\}$, and $D_4(\lambda) = \{5, 4|2, 1\}$. Hence

$$D(\lambda) = \{25, 20, 14, 11, 9, 7, 3, 2|25, 20, 14, 11, 9, 7, 3, 2\}$$

and $\delta(\lambda) = (51, 41, 29, 23, 19, 15, 7, 5)$. 
7. Symmetric $p$-cores

For any partition $\lambda$, let
\[
D(\lambda)_{L,\gamma} = \{ \alpha \in D(\lambda)_L : \alpha \equiv \gamma \pmod p \}
\]
and
\[
D(\lambda)_{R,\gamma} = \{ \beta \in D(\lambda)_R : \beta \equiv \gamma \pmod p \}.
\]
Let $\lambda^0$ be a symmetric $p$-core partition. Let $D(\lambda)_\gamma$ be the set of $(\beta | \beta) \in D(\lambda)$ such that $\beta \equiv \gamma \pmod p$. Then, in particular, $D(\lambda^0) = \bigcup_\gamma D(\lambda^0)_\gamma$ and $D(\lambda^0)_\gamma \cap D(\lambda^0)_{\gamma'} = \emptyset$ for $\gamma \neq \gamma'$.

**Proposition 7.1.** Suppose $\lambda^0$ is a symmetric $p$-core partition. Then for $\gamma \neq \gamma^*$,
\[
D(\lambda^0)_{\gamma^*} \neq \emptyset \implies D(\lambda^0)_{\gamma^*} = \emptyset.
\]

**Proof.** Suppose $(\alpha_i | \alpha_i) \in D(\lambda^0)_\gamma$ and $(\beta_j | \beta_j) \in D(\lambda^0)_{\gamma^*}$. Then in the notation of Proposition 3.3 we have $y'_i \equiv \theta(\lambda^0) + \frac{1}{2} + \gamma \pmod p$ and $y_j \equiv \theta(\lambda^0) - \frac{1}{2} - \gamma^* \pmod p$. But $\gamma^* = -1 - \gamma \pmod p$. Thus $y'_i - y_j \equiv (\mod p)$, contradicting the assumption $\lambda^0$ is a $p$-core.

A symmetric $\lambda$ is $\gamma$-packed if $D(\lambda)_\gamma$ consists of the elements $(\gamma + ip | \gamma + ip)$ for $i = 0, 1, \ldots, r$. Let $X_{\gamma,+}$ be the subset of $X_+$ consisting of elements $y'$ where
\[
y' - \theta(\lambda) - \frac{1}{2} \equiv \gamma \pmod p.
\]
We define $X_{\gamma,-}$ similarly.

**Proposition 7.2.** Suppose $\lambda^0$ is a symmetric $p$-core and $D(\lambda)_\gamma \neq \emptyset$. Then $\lambda^0$ is $\gamma$-packed.

**Proof.** Clearly if $\lambda^0$ is not $\gamma$-packed, then there exist integers $y', z'$, greater than $\theta(\lambda^0)$ such that $z' \equiv y' \pmod p$, $z' \not\in X_+$, and $y' \in X_+$. In particular, $(z', y')$ is a $p$-hook of $\lambda^0$.

**Corollary 7.3.** (symmetric $p$-core criterion) Let $\lambda^0$ be a symmetric partition. Then $\lambda^0$ is a $p$-core if and only if for every $\gamma \in \{0, \ldots, p - 1\}$ $D(\lambda^0)_{\gamma^*} \neq \emptyset$ implies that $\lambda^0$ is $\gamma$-packed and $D(\lambda^0)_{\gamma^*} = \emptyset$.

**Proof.** Clearly, if $\lambda^0$ if a $p$-core the result follows by Proposition 7.1 and Proposition 7.2. Suppose that for each $\gamma \in \{0, \ldots, p - 1\}$, if $D(\lambda^0)_{\gamma^*} \neq \emptyset$ then $\lambda^0$ is $\gamma$-packed and $D(\lambda^0)_{\gamma^*} = \emptyset$, but that $\lambda^0$ is not a $p$-core. Then, for some $\gamma$ there exists a hook $h = (x, y')$ where $y' = \gamma + mp$, $x = \gamma + (m - 1)p$. By symmetry, we can assume that $y' > \theta(\lambda^0)$. If $x > \theta(\lambda)$, then $\lambda$ is not $\gamma$-packed, which is a contradiction. Now suppose $x < \theta(\lambda^0)$, then by symmetry there exists $x' > \theta(\lambda^0)$ such that...
Example 7.4.

Suppose $p = 5$ and $\lambda' \vdash 324$ such that $\lambda'$ is symmetric and

$$\delta(\lambda') = (69, 59, 49, 39, 29, 27, 19, 17, 9, 7).$$

In particular, $D(\lambda')_R = (34, 29, 24, 19, 14, 13, 9, 8, 4, 3)$. Hence $D(\lambda')_{R,4} = (34, 29, 24, 19, 14, 9, 4)$ and $D(\lambda')_{R,3} = (13, 8, 3)$. Hence $\lambda'$ is both 4-packed and 3-packed. Since $D(\lambda')_0 = \emptyset$ and $D(\lambda')_1 = \emptyset$, $\lambda'$ is a 5-core by Theorem 7.3.

8. Symmetric partitions with a non-empty $p$-core

We extend the results of Section 6 to the case of a symmetric partition with a non-empty $p$-core. Let $\bar{\lambda}$ be the symmetric partition that shares the $p$-quotient with $\lambda$, but has empty $p$-core. Hence $(\bar{\lambda})^0 = \emptyset$ and $(\bar{\lambda})_1 = \lambda_1$ for $0 \leq \gamma \leq p - 1$.

Now consider a symmetric partition $\lambda$ of $n$ with a non-empty $p$-core $\lambda^0$. Let $\bar{X}$ and $X^0$ be $\beta$-sets of $\lambda$ and $\lambda^0$ respectively. Since $\lambda^0 \neq \emptyset$, we have $(X^0)_{\gamma,0} \neq \emptyset$ for some $\gamma$. Then $|D(\lambda^0)\gamma| \neq \emptyset$. In particular, $|D(\lambda^0)\gamma| = d^0_\gamma$ by Proposition 7.1. The definition of $D'(\lambda)$ each $(\lambda)_{\gamma,0} \in D(\lambda^0)\gamma$ contributes an element to both $D_{\gamma,L}^*(\lambda)_L$ and $D_{\gamma,R}^*(\lambda)_R$. $(D(\lambda^0))$ contributes nothing to $D_{\gamma,L}^*(\lambda)_R$ and $D_{\gamma,L}^*(\lambda)_L$. The definition of $D'(\lambda)$ (and Proposition 7.1) forces $|D_{\gamma,L}^*(\lambda)_R| - |D_{\gamma,L}^*(\lambda)_L| = d^0_\gamma$. This implies $D'(\lambda)$ is not a $p$-tuple of bisequences. Specifically, $D_{\gamma,L}^*(\lambda) \neq D(\lambda^0)$.

Define $\Omega' \subset \{0, \ldots, p - 1\}$ so that $\gamma' \in \Omega'$ if $D_{\gamma'}(\lambda^0) \neq \emptyset$ (i.e. $d^{0}_{\gamma'} > 0$). Let $(\Omega')^* = \{0 - \gamma' - 1 : \gamma' \in \Omega'\}$ and $U = \Omega' \cup (\Omega')^*$. Define $\Omega'' = \{0, \ldots, p - 1\} - U$.

Lemma 8.1.

1. $\theta(\lambda_{\gamma'}) = \theta(\lambda_{\gamma'})$
2. $\theta(\lambda_{\gamma'}) = \theta(\lambda_{\gamma'}) + d^0_{\gamma'}$

Proof. When all beads on all the runners of the abacus of $\lambda$ are moved up completely one obtains the abacus diagram for $\lambda^0$ (Theorem 2.7.16, 2). Since $D(\lambda^0)_{\frac{p-1}{2}}$ is empty, $d^0_{\frac{p-1}{2}} = 0$ and the $\frac{p-1}{2}$-th runner of $\lambda^0$ is unchanged from the $\frac{p-1}{2}$-th runner of $\bar{\lambda}^0$. Let $\bar{X}^0$ and $X^0$ be the $\beta$-sets for $\lambda^0$ and $\bar{\lambda}^0$. Matching the $\frac{p-1}{2}$-th runners of $\lambda^0$ and $\bar{\lambda}^0$ one can superimpose the abacus of $\lambda^0$ onto the abacus of $\bar{\lambda}^0$. It follows...
that $|X_{\gamma'}^0| + d_\gamma^0 = |X_{\gamma''}^0|$ for $\gamma' \in \Omega'$. Also, $|X_{\gamma''}^0| = |X_{\gamma''}^0|$ since $d_{\gamma''}^0 = 0$ for $\gamma'' \in \Omega''$. Hence $\theta(\lambda_{\gamma'}) = \theta(\bar{\lambda}_{\gamma'}) + d_\gamma^0$ and $\theta(\lambda_{\gamma''}) = \theta(\bar{\lambda}_{\gamma''})$. The result follows since $\theta(\lambda_{\gamma'}) = \theta(\bar{\lambda}_{\gamma'})$ and $\theta(\bar{\lambda}_{\gamma''}) = \theta(\bar{\lambda}_{\gamma''})$ by Proposition 3.2.

We can describe $X_{\gamma'}$ using $\bar{X}_{\gamma'}$ and Lemma 8.1 in the following three steps which we call the $d_\gamma^0$-shift of $\bar{X}_{\gamma'}$.

1. $m_\sigma \in X_{\gamma',+}$ if $m_\sigma - d_\gamma^0 > \theta(\bar{\lambda}_{\gamma'})$ and $m_\sigma - d_\gamma^0 \in \bar{X}_{\gamma',+}$

2. $m_s \in X_{\gamma',+}$ if $\theta(\bar{\lambda}_{\gamma'}) < m_s < \theta(\bar{\lambda}_{\gamma'}) + d_\gamma^0$ and $m_s - d_\gamma^0 \notin \bar{X}_{\gamma',-}$

3. $m_t \in X_{\gamma',-}$ if $m_t - d_\gamma^0 < \theta(\bar{\lambda}_{\gamma'})$.

Now consider the following sets

$$S_{\gamma'}(\bar{\lambda})_L =: \{s : s \in \mathbb{N}, s \notin D_{\gamma'}(\bar{\lambda})_L, 0 \leq s \leq d_{\gamma'}^0 - 1\}$$

$$T_{\gamma'}(\bar{\lambda})_L =: \{t : t \in D_{\gamma'}(\bar{\lambda})_L, t \geq d_{\gamma'}^0\}.$$

Following the comments after Proposition 3.8, $S_{\gamma'}(\bar{\lambda})_L$ and $T_{\gamma'}(\bar{\lambda})_L$ are in bijection with the subsets of $\bar{X}_{\gamma}$ in steps (2) and (3) of the definition of the $d_\gamma^0$-shift. Now we can interpret $D_{\gamma'}(\lambda)$ via the $d_\gamma^0$-shift of $\bar{X}_{\gamma'}$.

**Proposition 8.2.** $D_{\gamma'}(\lambda)$ is obtained from $D_{\gamma'}(\bar{\lambda})$ in the following three steps.

1. Each $\sigma \in D_{\gamma'}(\bar{\lambda})_R$ is sent to $\sigma + d_\gamma^0 \in D_{\gamma'}(\lambda)_R$

2. Each $s \in S_{\gamma'}(\bar{\lambda})_L$ is sent to $d_\gamma^0 - s - 1 \in D_{\gamma'}(\lambda)_R$

3. Each $t \in T_{\gamma'}(\bar{\lambda})_L$ is sent to $t - d_\gamma^0 \in D_{\gamma'}(\lambda)_L$.

**Proof.** We prove part (2). Let $x_s = m_s - d_\gamma^0$. Each $x_s < \theta(\bar{\lambda}_{\gamma'})$ where $x_s \notin \bar{X}_{\gamma',-}$ where $x_s + d_\gamma^0 > \theta(\bar{\lambda}_{\gamma'})$ corresponds to some $s \in S_{\gamma'}(\bar{\lambda})_L$. Hence we have $\gamma + (x_s + d_\gamma^0)p \in X_{\gamma',+}$ by the usual $p$-quotient. Again, by Proposition 3.3,

$$\theta(\lambda) + \frac{1}{2} + \beta = \gamma + (x_s + d_\gamma^0)p$$

for some $\beta \in D(\lambda)_R$. By substitution,

$$\beta = \gamma - \theta(\lambda) - \frac{1}{2} + (\theta(\bar{\lambda}_{\gamma'}) - \frac{1}{2})p + (d_\gamma^0 - s)p.$$

By Lemma 3.7 we have

$$\beta = \gamma + (d_\gamma^0 - s - 1)p.$$

By definition of $D'(\lambda)$, $x_s + d_\gamma^0$ corresponds to $d_\gamma^0 - s - 1 \in D_{\gamma'}(\bar{\lambda})_R$.

The proofs of (1) and (3) are similar. □
Theorem 8.4. Given $\lambda^0$ and $\bar{\lambda}$,

$$D(\lambda)_R = O_1 \cup O_2 \cup O_3 \cup O_4$$

where

$$O_1 = \bigcup_{\gamma' \in \Omega'} \{\gamma' + (\sigma + d_{\gamma'}^0)p : \sigma \in D_{\gamma'}(\bar{\lambda})_R\}$$

$$O_2 = \bigcup_{\gamma' \in \Omega'} \{\gamma' + (d_{\gamma'}^0 - s - 1)p : s \in S_{\gamma'}(\bar{\lambda})\}$$

$$O_3 = \bigcup_{\gamma' \in \Omega'} \{(p - 1 - \gamma') + (t - d_{\gamma'}^0)p : t \in T_{\gamma'}(\bar{\lambda})\}$$

$$O_4 = \bigcup_{\gamma'' \in \Omega''} \{\gamma' + \mu p : \mu \in D_{\gamma''}(\bar{\lambda})_R\}.$$

Proof. This follows from the definition of $D'(\lambda)$ and Proposition 8.3. \hfill \Box

Theorem 8.4. Given $\lambda^0$ and $\bar{\lambda}$,

$$\delta(\lambda) = \mathbb{O}_1 \cup \mathbb{O}_2 \cup \mathbb{O}_3 \cup \mathbb{O}_4$$

where

$$\mathbb{O}_1 = \bigcup_{\gamma' \in \Omega'} \{2(\gamma' + (\sigma + d_{\gamma'}^0)p) + 1 : \sigma \in D_{\gamma'}(\bar{\lambda})_R\}$$

$$\mathbb{O}_2 = \bigcup_{\gamma' \in \Omega'} \{2(\gamma + (d_{\gamma'}^0 - s - 1)p) + 1 : s \in S_{\gamma'}(\lambda)_L\}$$

$$\mathbb{O}_3 = \bigcup_{\gamma' \in \Omega'} \{2((p - 1 - \gamma') + (t - d_{\gamma'}^0)p) + 1 : t \in T_{\gamma'}(\lambda)_L\}.$$  

$$\mathbb{O}_4 = \bigcup_{\gamma'' \in \Omega''} \{2(\gamma' + \mu p) + 1 : \mu \in D_{\gamma''}(\bar{\lambda})_R\}.$$

where $d_{\gamma'}^0 = |D(\lambda)^0_{\gamma'}|$, $\gamma' \in \Omega'$ and $D_{\gamma'}(\lambda)_R$, $S_{\gamma'}(\lambda)_L$ and $T_{\gamma'}(\lambda)_L$ are as above.

Proof. Follows from the Proposition 8.3 and the relationship between $D(\lambda)$ and $\delta(\lambda)$. \hfill \Box

[Note: In the case $d_{\gamma'}^0 = 0$ for all $0 \leq \gamma \leq p - 1$, Proposition 8.3 reverts to Theorem 6.1.]

Example 8.5.

Suppose $p = 5$ and $\eta \vdash 514$ such that $\eta$ is symmetric such that $\eta^0 = \lambda'$, where $\lambda'$ is as in Example 7.4. Furthermore, let $\eta_i = \lambda_i$ for $0 \leq i \leq p - 1$, where $\lambda_i$ is as in Example 6.2. In this case we have $d_3^0 = 3$, $d_4^0 = 7$, $D_3(\bar{\eta}) = \{2|0\}$, and $D_4(\bar{\eta}) = \{5,4|2,1\}$. Hence, by Proposition 8.2 we have

$$D_2(\eta)_R = \{1,0\}$$

$$D_2(\eta)_L = \{1,0\}$$

$$D_3(\eta)_R = \{2,1,0\}$$

$$D_3(\eta)_L = \emptyset$$

$$D_4(\eta)_R = \{9,8,6,5,4,3,0\}$$

$$D_4(\eta)_L = \emptyset.$$
Then by Proposition 8.3,

\[ \begin{align*}
D(\eta)_{R,2} &= \{6, 2\} \\
D(\eta)_{R,3} &= \{13, 8, 3\} \\
D(\eta)_{L,3} &= \emptyset \\
D(\eta)_{R,4} &= \{49, 44, 34, 29, 24, 19, 4\} \\
D(\eta)_{L,4} &= \emptyset.
\end{align*} \]

Finally, by Theorem 8.4 we have

\[ \delta(\eta) = (99, 69, 59, 49, 39, 37, 27, 17, 13, 9, 7, 5). \]

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