SPEED: Secure, PrivatE, and Efficient Deep learning

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Abstract

This paper addresses the issue of collaborative deep learning with privacy constraints. Building upon differentially private decentralized semi-supervised learning, we introduce homomorphically encrypted operations to extend the set of threats considered so far. While previous methods relied on the existence of an hypothetical ‘trusted’ third party, we designed specific aggregation operations in the encrypted domain that allow us to circumvent this assumption. This makes our method practical to real-life scenario where data holders do not trust any third party to process their datasets. Crucially the computational burden of the approach is maintained reasonable, making it suitable to deep learning applications. In order to illustrate the performances of our method, we carried out numerical experiments using image datasets in a classification context.

1 Introduction

Modern neural networks achieve state-of-the-art performances in a variety of fields such as natural language processing\textsuperscript{36}, image recognition\textsuperscript{22} and speech recognition\textsuperscript{24}. With the large adoption of such models in several domains, including critical ones, researchers and practitioners are observing growing concerns on the security and privacy of the tools they develop. In this paper, we are especially interested in collaborative deep learning with privacy constraints.

Motivating example. An example of scenario from the field of cybersecurity is when several actors each hold a database of cybersecurity incident signatures, that have occurred on their customer networks. Building a model from a larger set of such signatures would lead to improved detection capabilities. However, these databases are highly-sensitive and highly-valuable. As such, they cannot be disclosed (notwithstanding legal barriers to do so). In such a setting, the data owners wish to collaboratively train a global model while preserving the confidentiality of their learning sets. A gold standard definition of privacy-preserving machine learning is differential privacy\textsuperscript{15,25}. In the context of collaborative learning, several recent works\textsuperscript{5,6,18,10,34,35} focused on this definition to build privacy preserving deep learning models. However, these techniques rely on a 'trusted' aggregation server that gathers non private information before processing some sanitizing scheme. In real-life scenarios the absence of such a server will jeopardize the privacy and security of the overall learning procedure. This paper presents a new approach called SPEED which obtains differential privacy guarantees without the need for a trusted aggregation server. To do so, we use a decentralized semi-supervised learning procedure summarized in Figure\textsuperscript{1}. In a nutshell, SPEED works as follows. First, every data owner builds a local model (a.k.a. teacher model) using its own private database. Then, given a new unlabeled dataset, the teacher models output encrypted predictions and send them to the server which computes a differentially private aggregation in the encrypted domain to obtain an encrypted labeled dataset. From this new dataset, a collaborative

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model (a.k.a. student model) is learned in a semi-supervised manner. Our approach is supported by strong theoretical guarantees in terms of differential privacy and provably-secure cryptography.

Outline of the paper. Section 2 relates our work to the literature. In Section 3 we give some technical background on differential privacy and homomorphic encryption. In Section 4 we first present SPEED when the aggregation server is called honest but curious i.e. it is not trusted but not completely adversarial. We then go beyond the honest but curious model and present another version of SPEED when the aggregation server is not cooperative. Section 5 presents our experimental results. SPEED achieves state-of-the-art accuracy and privacy with a mild computational overhead w.r.t previous works. Section 6 concludes the paper by stating some open questions for further works.

2 Related Work

Differential privacy (DP). Recent works considered to use differential privacy in collaborative settings close to the one we consider [5, 6, 18, 10, 34, 35]. Among them, the most efficient technique in terms of accuracy and privacy guarantees is Private Aggregation of Teacher Ensembles (PATE) first presented in [34] and refined in [35]. PATE uses semi-supervised learning to privately transfer to the student model the knowledge of the ensemble of teachers by using a differentially private aggregation method. This approach considers a setting very close to ours with the notable difference that the aggregation server is trusted. Hence applying PATE in our scenario makes the teacher models vulnerable. To tackle this issue, our work builds upon PATE idea, and adds a layer of homomorphic encryption in order for the overall learning to be kept private.

Homomorphic Encryption (HE). HE allows to perform computations over encrypted data. In particular, this can be used so that the model can perform both training and prediction without handling cleartext data. In terms of learning, the naive approach would be to have the training sets homomorphically encrypted, sent to a server for training to be done in the encrypted domain and the resulting (encrypted) model sent back to the participants for decryption. However, putting aside many subtleties, even by deploying all the arsenal available in the HE practitioner toolbox (batching, transciphering, etc.) this would be impractical as “classical” learning is both computation and know-how intensive and HE operations are intrinsically costly. As a consequence, there are only very few works that capitalize on HE for private training [21, 23] and inference [19, 26] of machine learning tasks. Moreover, since some attacks can be performed in a black-box setting, the system is still vulnerable to attacks from the end user who has access to the decryption key. In our framework, we do not use HE directly to build the model, we use it as a mean for the aggregation to be kept private. That way, we are protected against potential threats from the aggregation server, which does not have the decryption key, and we keep a manageable computational overhead.

Private aggregation. Several approaches have been considered to limit the need for a trusted server when applying differential privacy, for example by considering local differential privacy [28, 13, 27]. In practice it often results in applying to much noise, and maintaining utility can be difficult [43, 28] especially for deep learning applications. In order to recover more accuracy while keeping privacy, some works combined decentralized noise distribution (a.k.a. distributed differential privacy [39])
and encryption schemes [37, 20, 39] in the context of aggregation of distributed time-series. Our work contributes to this line of research. However, our framework, which combines distributed DP and HE, is the first one to be efficient enough to investigate deep learning applications.

3 Preliminaries

3.1 Differential privacy

Differential privacy [14] provides a guarantee that under a reasonable privacy budget \((\epsilon, \delta)\), two adjacent databases produce statistically indistinguishable results. In the following, two databases \(d\) and \(d'\) are said adjacent if they differ by at most one example.

**Definition 1.** A randomized mechanism \(A\) with output range \(R\) satisfies \((\epsilon, \delta)\)-differential privacy if for any two adjacent databases \(d, d'\) and for any subset of outputs \(S \subset R\) one has

\[
P[A(d) \in S] \leq e^\epsilon P[A(d') \in S] + \delta. \tag{1}
\]

**Definition 2.** Let \(A\) be a randomized mechanism with output range \(R\) and \(d, d'\) a pair of adjacent databases. Let \(\text{aux}\) denote an auxiliary input. For any \(o \in R\), the privacy loss at \(o\) is defined as

\[
c(o; A, \text{aux}, d, d') := \log \left( \frac{P[A(\text{aux}, d) = o]}{P[A(\text{aux}, d') = o]} \right). \tag{2}
\]

The privacy loss random variable \(C(A, \text{aux}, d, d')\) is defined as \(c(A(d); A, \text{aux}, d, d')\), i.e. the random variable defined by evaluating the privacy loss at an outcome sampled from \(A(d)\).

In deep learning, it is not simple to keep track of the privacy loss due to the numerous calls the algorithm makes to the database. To evaluate the privacy budget, it is useful to introduce the notion of moments accountant [1].

**Definition 3.** With the same notation as above, the moments accountant is defined for any \(l \in \mathbb{R}^*_+\) as

\[
\alpha_A(l) := \max_{\text{aux}, d, d'} \alpha_A(l; \text{aux}, d, d') \tag{3}
\]

where the maximum is taken over any auxiliary input \(\text{aux}\) and any pair of adjacent databases \(d, d'\) and \(\alpha_A(l; \text{aux}, d, d') := \log (\mathbb{E} [\exp(lC(A, \text{aux}, d, d'))])\) is the moment generating function of the privacy loss random variable.

The privacy analysis of our method boils down to the following theorem first introduced in [34].

**Theorem 1 [(34)].** Let \(\epsilon, l \in \mathbb{R}^*_+\). Let \(A\) be a \((\epsilon, 0)\)-differentially private mechanism and \(q \geq P[A(d) \neq o^*]\) for some outcome \(o^*\). If \(q < \frac{e^\epsilon - 1}{e^\epsilon + 1}\), then for any \(\text{aux}\) and any pair \(d, d'\) of adjacent databases, \(A\) satisfies

\[
\alpha_A(l; \text{aux}, d, d') \leq \min \left( el, \frac{e^2 l (l + 1)}{2}, \log \left( \left(1 - q\right) \left(1 - e^\epsilon q\right)^l + q e^\epsilon l\right) \right). \tag{4}
\]

Theorem [1] coupled with some properties of the moments accountant (composability and tail bound) allows one to devise the overall privacy budget \((\epsilon, \delta)\) for the learning procedure. We refer the interested reader to [34] for more details. Throughout this paper, we present theorems that can be used as building blocks for Theorem [1] and evaluate the privacy budget accordingly.

3.2 Homomorphic encryption

Let us consider \(\Lambda\) and \(\Omega\) which respectively are the set of cleartexts (a.k.a. the clear domain) and the set of ciphertexts (a.k.a. the encrypted domain). An homomorphic encryption system first consists in two algorithms \(\text{Enc}_{\text{pk}} : \Lambda \rightarrow \Omega\) and \(\text{Dec}_{\text{sk}} : \Omega \rightarrow \Lambda\) where \(\text{pk}\) and \(\text{sk}\) are data structures which represent the public encryption key and the private decryption key of the cryptosystem.

Homomorphic encryption systems are by necessity probabilistic, meaning that some randomness has to be involved in the \(\text{Enc}\) function and that the ciphertexts set \(\Omega\) is significantly much bigger than the cleartexts set \(\Lambda\). Any (decent) homomorphic encryption scheme possesses the semantic security.
property meaning that, given $\text{Enc}(m)$ and polynomially many pairs $(m_i, \text{Enc}(m_i))$ it is hard to gain any information on $m$ with a significant advantage over guessing. Most importantly, an homomorphic encryption scheme offers two additional operators $\oplus$ and $\otimes$ such that

\begin{itemize}
    \item $\text{Enc}(m_1) \oplus \text{Enc}(m_2) = \text{Enc}(m_1 + m_2) \in \Omega$.
    \item $\text{Enc}(m_1) \otimes \text{Enc}(m_2) = \text{Enc}(m_1 m_2) \in \Omega$.
\end{itemize}

When these two operators are supported without restriction by an homomorphic scheme, it is said to be a Fully Homomorphic Encryption (FHE) scheme. A FHE with $\Lambda = \mathbb{Z}_2$ is Turing-complete and, as such, is in principle sufficient to perform any computation in the encrypted domain with a computational overhead depending on the security target. In practice, though, the $\oplus$ and $\otimes$ are much more computationally costly than their clear domain counterparts and this has led to the development of several approaches to HE schemes design each with their pros and cons.

**Somewhat HE (SHE).** Somewhat homomorphic encryption schemes, such as BGV [9] or BFV [16], provide both operators but with several constraints. Indeed, in these cryptosystems the $\otimes$ operator is much more costly than the $\oplus$ operator and the cost of the former strongly depends on the multiplicative depth of the calculation, that is the maximum number of multiplications that have to be chained (although this depth can be optimized [4]). Interestingly, most SHE schemes offer a batching capability by which multiple cleartexts can be packed in one ciphertext resulting in (quite massively) parallel homomorphic operations i.e.,

$$\text{Enc}(m_1, ..., m_n) \oplus \text{Enc}(m'_1, ..., m'_n) = \text{Enc}(m_1 + m'_1, ..., m_n + m'_n)$$

(and similarly so for $\otimes$). Typically, several hundreds such slots are available and, in some circumstances, this allows to significantly speed up encrypted-domain calculations.

**Fully HE (FHE).** Fully homomorphic encryption schemes offer both the $\oplus$ and $\otimes$ operators without restrictions on multiplicative depth. At the time of writing, only the FHE-over-the-torus approach, instantiated in the TFHE cryptosystem [11], offers practical performances. In this cryptosystem, $\oplus$ and $\otimes$ have the same constant cost. On the downside, TFHE offers no batching capabilities. To get the best of all worlds, the TFHE scheme is often hybridized with SHE by means of operators allowing to homomorphically switch among several ciphertext formats [7] in order to perform each part of calculation with the most appropriate scheme (see e.g. [48]).

## 4 SPEED: a framework for Secure, Private, and Efficient Deep Learning

Let us consider a set of $n$ owners each holding a personal sensitive database $d_i$. The personal databases are disjoint, meaning that no example can be present in two different personal databases. The whole database $d = (d_1, \ldots, d_n)$ is the union of the $n$ personal databases. We aim at building a collaborative model (a.k.a. student model) mapping an input space $X$ to an output space $\{k\} = \{1, \ldots, K\}$. To do this while keeping the process private, we follow the tripartite setting illustrated by Figure 1. SPEED can be divided in two layers, the first one ensures that the learning procedure is DP by using a proxy aggregation server, and the second one adds HE to avoid threats from this server.

### 4.1 Learning procedure with a trusted (Trusted) aggregation server

**Trusted aggregation server.** In this section, we suppose that the data holders have access to a trusted aggregation server. This means that the server takes a set of inputs from the owners, process them, and erase them immediately after the processing is completed. With this kind of server, we can build the following learning procedure (first introduced in [34]) with DP guarantees:

1. For every owner $i$, learn a teacher model $f_i$ from the personal database $d_i$.
2. Define the trusted aggregation server as follows: $\mathcal{A} : x \mapsto \arg\max_{k \in [K]} [n_k + Y_k]$, where $n_k := |\{i : f_i(x) = k\}|$ and $Y_k$ is a Laplace noise with mean 0 and scale $\frac{1}{\gamma} \in \mathbb{R}_+^*$.

2“Hard” means that it requires solving a reference (conjectured) computationally hard problem on which the security of the cryptosystem hence depends. From a practical viewpoint, given a security target $\lambda$, the concrete parameters of an homomorphic scheme are chosen such that the best known (exponential-time) algorithms for solving the underlying reference problem require an order of magnitude of $2^\lambda$ nontrivial operations.

3Polynomial in $\lambda$.  

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3. Train a student model on a public unlabeled dataset. To do so, the student model selects inputs from the unlabeled dataset and submits them to the teachers. Each teacher makes a prediction and sends it to the aggregation server which then outputs a label for the student to learn.

**Privacy guarantees.** The privacy guarantees of this method come from the aggregation mechanism \( \mathcal{A} \), also known as the report noisy max scheme in the DP literature. Thanks to Theorems 1 and 2 we can evaluate the overall privacy budget of the procedure (see Table 2) with regard to the number of queries the student model makes to the aggregation server.

**Theorem 2 ([15]).** Let \( \mathcal{A} \) be the report noisy max as above. Then \( \mathcal{A} \) is \((2\gamma,0)\)-differentially private.

This learning procedure ensures privacy only if \( \mathcal{A} \) is trusted. If not, \( \mathcal{A} \) can directly look at the votes from each model, which makes the data holders vulnerable.

4.2 Learning with an Honest But Curious (HBC) aggregation server.

**Honest But Curious aggregation server.** Let us now suppose that the aggregation server is no longer trusted, but HBC instead. An aggregation server is called HBC if given a set of inputs, it applies the processing as asked but stores the input.\(^4\) To secure the aggregation w.r.t an HBC server, we add a layer of HE on top of the procedure from Section 4.1. The aggregation server will perform encrypted-domain computations under an homomorphic encryption scheme whose owner is the student model who generates and knows both \( pk \) and \( sk \). When being submitted an unlabeled input, the teachers encrypt their predictions under \( pk \) and send these encryptions to the server. The server has the responsibility to homomorphically perform the aggregation operator in order to produce an encryption of the operator output (e.g. a label) which will be sent back to the student and used by the latter for learning, after due decryption. Without the homomorphic encryption layer, an HBC server has access to all the teachers’ predictions and can exploit this knowledge at will. This access is denied when the server works in the encrypted domain therefore avoiding information leakage.

**Technical details on the HE scheme (computing the counts).** The most appropriate way for computing an histogram by means of FHE consists in having the teachers sending encrypted one-hot encodings of their vote. That is, rather than sending \( f_i(x) \), the \( i \)-th teacher sends a \( K \)-dimensional vector, say \( z^{(i)} \), whose \( f_i(x) \)-th component is an encryption of 1 while all the others are encryptions of 0. Recall that FHE schemes are necessarily probabilistic which means that each clear domain value has (astronomically) many different encryptions and that the encryptions of two clear domain values are computationally indistinguishable. We then have

\[
n_k = \sum_{i=1}^{n} z_k^{(i)}.
\]

When the sum is performed by means of the homomorphic summation operator provided by the FHE scheme over encrypted \( z_k^{(i)} \)'s, the aggregation server obtains an encryption of the number of times class \( k \) has been voted for. The server then generates the Laplace noise component (in the clear domain), which it then adds to its encryption of \( n_k \). Although homomorphic encryption schemes usually provide a ciphertext vs cleartext homomorphic addition operator, the server uses the student’s public key to encrypt the Laplace noise component and then uses the homomorphic addition operator to apply it to \( n_k \). So far, we have only needed homomorphic addition which is a good start. Then an argmax operator must be performed after the noisy histogram calculation. However, efficiently handling the highly nonlinear argmax function by means of FHE is much more challenging.

**Technical details on the HE scheme (computing the argmax).** Most prior work on secure argmax computations use some kind of interaction between a party that holds a sensitive vector of values and the party that wants to obtain the argmax over those values. The non-linearity of the argmax operator presents unique challenges that have mostly been handled by allowing the two interested parties to exchange information. This means increased communication costs and, in some cases, information leakage. This is with the exception of [48]. They provide a fully non-interactive homomorphic argmax computation scheme based on the TFHE encryption. We implemented and parametrized their scheme to fit the specific training problems presented in Section 5. We present here the main idea behind this novel FHE argmax scheme. For more details, see the original paper. The TFHE

\(^4\)By input we mean any information sent by the data holders. Hence, the Laplace noise is not stored.
encryption scheme provides a bootstrap operation that can be applied on any scalar ciphertext. Its purpose is threefold: switch the encryption key; reduce the noise; apply a non-linear operation on the underlying plaintext value. This underlying operation can be seen as a function

\[ g_{t,a,b}(x) = \begin{cases} a & \text{if } x > t \\ b & \text{if } x < t. \end{cases} \]

One notable application is that of a "sign" bootstrap: we can extract the sign of the input with the underlying function \( g_{0,1,0}(x) \). The argmax computation in the ciphertext space is made as follows. For every \( k, k', k \neq k' \), we compare the values \( n_k + Y_k \) and \( n_{k'} + Y_{k'} \) with a subtraction \((n_k + Y_k - n_{k'} - Y_{k'})\) and application of a sign bootstrap operation. This yields \( \theta_{k,k'} \), a variable with value 1 if \( n_k + Y_k > n_{k'} + Y_{k'} \) and 0 otherwise. Therefore the complexity will be quadratic in the number of classes. For a given \( k \) we can then obtain a boolean truth value (0 or 1) for whether \( n_k + Y_k \) is the maximum value. To this end, we compute

\[ \Theta_k = \sum_{i \neq k} \theta_{k,i}, \]

\( n_k \) is the max if and only if, for all \( i \) one has \( \theta_{k,i} = 1 \) i.e. \( \Theta_k = K - 1 \). We can therefore apply another bootstrap operation with \( g_{3,1,0} \). If \( \Theta_k = K - 1 \), the bootstrap will return an encryption of 1, and return an encryption of 0 otherwise. Once decrypted, the position of the only non-zero value is the argmax. Because the underlying function \( g_{t,a,b} \) is applied homomorphically, its output is inherently probabilistic. In the FHE scheme used, an error is inserted in all of the ciphertexts at encryption time to ensure an appropriate level of security. This means that if two values are too close, then the sign bootstrap operation might return the wrong result over their difference. The exact impact of this approximation on the accuracy is evaluated in Section [5]

**Privacy guarantees.** In terms of DP, the homomorphic computation of the argmax does not change the privacy guarantees because the perturbations it causes can be viewed as post-processing. See supplementary material for more details on this question.

**Remark.** Another solution would be to send the noisy histogram \( n_k + Y_k \) of the counts for each class \( k \) to the student and let her process the argmax in the clear domain. This could indeed be performed with a plain-old additively-homomorphic cryptosystem such as Paillier or (additive-flavored) ElGamal, avoiding the machinery of the homomorphic argmax. Nevertheless, this approach was put aside because sending the histogram instead of the argmax would provide much worse DP guarantees.

Despite of the fact that this model is sufficient in many real-world scenarios, in Section [4,3] we will consider an approach for going beyond this HBC aggregation server.

### 4.3 Learning Beyond the Honest But Curious (BHBC) aggregation server

**Beyond the Honest But Curious aggregation server.** In this section, we relax our assumptions on the aggregation server and chose not to trust it to generate the Laplace noise. Indeed, we consider that the server may store the noise (that it generated in clear) and couple it with an attack on the student model (e.g. model inversion) to estimate the output of the aggregation without noise, therefore breaking the DP guarantees on the sensitive data. A possible workaround to circumvent this is to delegate the noise generation to the teachers while keeping the same overall noise and thus the same privacy guarantees and the same utility as in the HBC setting.

**Proposition 1 ([29]).** Let \( m \in \mathbb{N} \) and \( \gamma \in \mathbb{R}^+ \). Let \( G_p^{(i)} \), for \((i,p) \in [m] \times [2]\), be i.i.d. random variables following the Gamma distribution of shape \( \frac{1}{m} \) and scale \( \frac{1}{\gamma} \). Then \( \sum_{i=1}^{m} (G_1^{(i)} - G_2^{(i)}) \) follows the Laplace distribution of mean 0 and scale \( \frac{1}{\gamma} \).

**Distributing the noise among the teachers.** This property gives rise to a secure aggregation architecture. We charge each teacher \( i \) to generate for each class \( k \) a noise drawn from \( G_{k,1}^{(i)} - G_{k,2}^{(i)} \)

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\(^3\)Another well-known example of infinitely divisible probability distribution is the Gaussian distribution which can be seen as the sum of Gaussian distributions of well chosen scale parameter. In a possible further work, we could indeed replace the (distributed) Laplace noise by a (distributed) Gaussian noise.
where $G_{k,1}^{(i)}$ and $G_{k,2}^{(i)}$ are i.i.d. random variables following the gamma distribution of shape $\frac{1}{4}$ and scale $\frac{1}{\gamma}$, where $\gamma \in \mathbb{R}^+_\ast$. Then for any query sent by the student, the teacher $i$ sends the noisy one-hot encoded vector whose $k^{th}$ coordinate corresponds to $z_k^{(i)} + G_{k,1}^{(i)} - G_{k,2}^{(i)}$.

**Remark.** Proposition 1 tells us that, when summing up the noisy vectors, the aggregation server gets the same output as if the noise were drawn from a Laplace distribution. Hence, this scheme gives exactly the same guarantees in terms of differential privacy as the one from Section 4.2 under a more conservative threat model.

**Resilience to failure of the teachers.** As we have decided no to trust the aggregation server to generate the noise necessary to the privacy guarantees, we may also assume that some teachers might fail in generating the individual noise drawn from the difference of Gamma distributions. The following theorem quantifies the privacy cost of such failures. In a nutshell, as long as half of the teachers did actually generate their noise properly, we are able to guaranty that the aggregation server ensures some amount of differential privacy.

**Theorem 3.** Let $\tau \in (\frac{1}{2}, 1)$ be the fraction of the teachers that did send their noisy votes properly to the aggregation server. The remaining teachers are assumed to have sent their votes without any noise (but still encrypted). We denote $A$ the aggregation mechanism. Then, $A$ is $(\epsilon, 0)$-differentially private, with

$$\epsilon = 2\gamma + 2(1 - \tau) \log \left( w(\tau) \left[ \frac{\Gamma(2\tau - 1)}{\Gamma(\tau)} \times \left( 1 + \frac{1 - \tau}{w(\tau)} \right) \right]^{\frac{1}{\tau}} + 1 \right)$$

where $\Gamma: x \in \mathbb{R}^+_\ast \mapsto \int_0^{\infty} t^{x-1} e^{-tx} \, dt$ and $w(\tau) = -W_{-1}((\tau - 1)e^{\tau - 1 - 2\gamma}) + \tau - 1$, $W_{-1}$ being the lower real branch of the Lambert function.

This theorem allows us to control the privacy cost by the ratio $1 - \tau$ of the teachers who failed in generating their noise. Indeed, using the asymptotic behavior of $W_{-1}[\cdot]$ one can see that our bound approaches $2\gamma$ when $\tau$ approaches 1, hence recovering the classical bound of the centralized Laplace noise. As for the differential privacy results from Section 4.1, we combine Theorems 1 and 3 to evaluate the overall privacy budget of the procedure (see Section 5 for more results).

## 5 Experimental results

**Homomorphic argmax accuracy.** As we mentioned in Section 4.2, the homomorphic computation of the argmax is inherently probabilistic. This is due both to the noise added to any ciphertext at encryption time, and to limitations of the bootstrapping operation in terms of accuracy. Table 1 summarizes the impact of this loss of accuracy on the overall performance of our scheme. On MNIST dataset [30], we evaluate the method over 4 different settings (HBC, BHBC with $\alpha = 1/0.9/0.7$) and compare the cleartext argmax to our homomorphic argmax. Our implementation of the HE argmax has an average accuracy of 99.4%, meaning that it retrieves the cleartext argmax 99.4% of the time. To obtain a more general and conservative measure of the inherent accuracy of the HE argmax (which can be applied on any dataset), we make the teachers give uniformly random answers to the queries. In this setting, most counts $n_k$ are likely to be close to one another, which makes even a classical argmax useless. This kind of scenario can be seen as worst-case, since the teacher voting is adversarial to argmax computation. Even in this scenario, and with the same parameters as for MNIST, our implementation of the HE argmax algorithm still produces an average accuracy of 90%. Hence, an accuracy of 90% can be considered a lower bound for any adaptation of this argmax technique to other datasets. Yet in practice a tweaking of the parameters can yield a better accuracy even for this worst-case scenario, at the cost of time efficiency.

**HE time overhead.** We implemented the homomorphic argmax computation presented in section 4.2. Without parallelizing, a single argmax query requires just under 4 seconds to compute on an Intel Core i7-6600U CPU. Importantly, this does not depend on the input data. The costliest operation
is the computation of $\theta$. Any other part of the scheme is negligible in comparison. Therefore, once the parameters are set, the time performance depends solely on the number of classes (the number of bootstrap comparisons is quadratic in the number of classes). As such, 100 queries require 6.5 minutes and 1000 queries 65 minutes. Of course, the queries can be performed in parallel to decrease the latency allowing for much more challenging applications.

**Learning setup.** To evaluate the performances of our framework, we test our method on MNIST dataset \[30\]. To represent the data holders, we divide the dataset in 250 equally distributed and disjoint subsets. Then we apply the following procedures. We refer the interested reader to supplementary material for more details on the hyper-parameters and learning procedure.

- **Teacher models.** Given a dataset, a data holder builds a local model by stacking two convolutional layers with max pooling and a fully connected layer with ReLu activations.
- **Student model.** Following the idea from \[34\], we train the student in a semi-supervised fashion. Unlabeled inputs are used to estimate a good prior distribution using a GAN-based technique first introduced in \[38\]. Then we use a limited amount of queries (100 here) to obtain labeled examples which we use to fine tune the model.

As the student model can substantially vary based on the selected subset of labeled examples, the out-of-sample accuracy has been evaluated 17 times, with 100 labeled examples sampled from a set of 9000 ones. For each experiment, the remaining 1000 examples have been used to evaluate the student model accuracy.

Tables 1 & 2: Table 1 presents the accuracy of our argmax algorithm. We ran it over 4 sets of 1000 queries from 250 teachers on the MNIST problem. We give here accuracy results for the 4 frameworks, an average over the 4 frameworks, and the worst-case accuracy over uniformly random inputs. Table 2 summarizes our results for MNIST dataset with 250 teachers and 100 student queries. We used an inverse noise scale $\gamma = 3.3$. The DP guarantees, computed by composable over the 100 queries, are given for $\delta = 10^{-5}$.

| Framework | Accuracy [%] |
|-----------|-------------|
| HBC       | 99.1        |
| BHBC, $\tau = 1$ | 99.8        |
| BHBC, $\tau = 0.9$ | 99.5        |
| BHBC, $\tau = 0.7$ | 99.3        |
| **Average** | **99.4**    |
| **Worst-case** | **90.0**    |

**Performances on MNIST.** Table 2 displays our experimental results for SPEED with MNIST and compares them to a non-private baseline and to Trusted framework which does not involve HE. In spite of the variability of the accuracy, we observe a tradeoff between accuracy and DP. Indeed, while the reported average accuracy does not vary much across conditions, consistent rankings of the methods have been observed, confirming the expected average rank of the method based on the amount of added noise. Importantly, it should be noted that the variance is high in each condition. It masks the fact that the distribution is highly skewed, with a majority of results in the $0.975 - 0.985$ range, and a few samplings yielding an out-of-sample accuracy around 0.90. As expected, the best DP guarantees ($\epsilon = 2.76$) are obtained in the HBC framework or, equivalently, in the BHBC framework when all the teachers generated noise ($\tau = 1$), but these are the cases where the accuracy is the lowest. On the contrary, when some teachers failed to generate noise in the BHBC framework ($\tau = 0.9$ and $\tau = 0.7$), the counts are more precise, leading to a slightly better accuracy but less DP guarantees. We refer the reader to supplementary material for additional experiments on SVHN dataset \[32\].

**6 Open question for further works**

The next step towards collaborative deep learning with privacy would be to design new aggregation operators, more suitable to FHE performances yet still providing good DP bounds. In particular, as
emphasized by Section E in the supplementary material, a linear or quadratic aggregation operator would be amenable to almost negligible homomorphic computations overhead. This lighter homomorphic layer would enable to extend the applicability of our framework to more complex datasets. Such aggregation operators would also allow to associate homomorphic calculations with verifiable computing techniques (e.g. [17]) whereby the server would provide an encrypted aggregation result along with a formal proof that aggregation was indeed done correctly. These perspectives would then allow to address threats further beyond the honest-but-curious model.

**Broader impact**

Deep learning is becoming pervasive in our connected society and has already led to countless practical applications impacting, for better or worse, our daily lives. However, its applications ecosystem has so far developed with too limited concern for user or data privacy and is subject to an expanding body of statistical attack techniques (de-anonymization, model inversion, data extraction from model memory, etc.). With growing public awareness and new legislation such as the GDPR, there is now an urgent need to tackle a number of privacy challenges in order to pave the way for the next generation of systems. Indeed, in the case of Europe for instance, being at the forefront of online citizen privacy protection should not translate into a loss of competitiveness for developing and benefiting from new advances in deep learning.

Today, many deep neural network systems are operational and legions of others can be bootstrapped with no or very little additional training data. However, when in operation, these systems need to interact with more focused user-centric data which are in urgent need for stronger privacy in the GDPR era. Additionally, we are witnessing a commoditization of machine learning techniques in well-defined scenarios (mono-database and explicit cost function). Consequently, residual value comes from cross domain databases, or multiple mono-domain databases, carefully chosen by an analyst. Still, training a neural network from several private datasets is generally not possible due to the inability to share learning data for commercial, ethical or legal reasons. Additionally, neural network paradigms are constantly emerging with an insatiable appetite for new kinds of training data. In the longer term, due to legal and societal constraints on data localization and privacy, it will become more complex to even gather the large databases needed to bootstrap future high value applications. This will especially be so in more critical fields such as the medical or genomics fields unless a paradigm shift occurs to put privacy at the core of deep learning system design and operation.

With a good timing to address these challenges, the framework of provably-secure cryptography, which provides a well-founded corpus of security properties and techniques, has dramatically expanded its functional capabilities beyond “just” encryption by developing tools to perform general computations directly over encrypted data. One of the flagships of these new cryptographic tools is Fully Homomorphic Encryption (FHE). FHE is relatively recent as it was shown to be theoretically possible only around 2010. Yet, after 10 years of very active research towards turning its theory into a practical reality, homomorphic encryption is ready to enter the practitioners’ toolbox to help addressing deep learning privacy challenges.

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Supplementary material

A  DP analysis of the learning procedure

In this section, we describe the procedure that computes the overall DP guarantees of the student model learning stage. We summarize this procedure in Section A.1 and demonstrate the theorems we use in Sections A.2 and A.4.

A.1 Analysis algorithm

Let us suppose that for every query \( Q \) of the student model, we have a privacy guarantee using Theorem 3 and that we can upperbound the probability \( \mathbb{P}[A(d; Q) \neq o^\ast] \) that \( A \) outputs some specific output \( o^\ast \) (in practice we choose \( o^\ast \) to be the unnoisy argmax). Then, Theorem 1 gives us an upperbound on the moments accountant per query. The computation of these building blocks is detailed in Sections A.2 and A.4 and the procedure is summarized in Algorithm 1.

Algorithm 1: Algorithm to determine the overall privacy guarantee of the learning procedure

Input : number of teachers \( n \), number of classes \( K \), ratio of successful teachers \( \tau \) (for BHBC), set of queries \( Q \), unnoisy teachers’ counts \( n_k \), inverse noise scale \( \gamma \), \( l_{\text{max}} \), \( \delta \)

Output : \( \epsilon \)

for \( l \) in \([l_{\text{max}}]\) do
  \( \alpha(l) \leftarrow 0 \)
  for query \( Q \) in \( Q \) do
    Compute the privacy cost of \( Q \) and an upperbound of \( \mathbb{P}[A(d; Q) \neq o^\ast] \);
    Derive the moments accountant \( \alpha_Q(l) \) with Theorem 1;
    \( \alpha(l) \leftarrow \alpha(l) + \alpha_Q(l) \);
  end
  \( \epsilon(l) \leftarrow \frac{\alpha(l) - \delta}{l} \);
end
\( \epsilon \leftarrow \min_{l \in \{l_{\text{max}}\}} \epsilon(l) \);

Using the moments accountant per query, we evaluate the overall moments accountant by compositability, applying the following theorem from [1].

Theorem 4 ([1]). Let \( p \in \mathbb{N} \). Suppose that a mechanism \( A \) consists of a sequence of adaptative mechanisms \( A_1, \ldots, A_p \) where \( A_i: \prod_{j=1}^{i-1} R_j \times D \mapsto R_i \). Then, for any \( l \in \mathbb{R}^+_\ast \),

\[
\alpha_A(l) \leq \sum_{i=1}^{k} \alpha_{A_i}(l).
\]

Finally, parameter \( \delta \) being chosen, the privacy guarantee is derived from the overall moments accountant applying the tail bound property, stated in Theorem 5 from [1].

Theorem 5 ([1]). For any \( \epsilon \in \mathbb{R}^+_\ast \), the mechanism is \((\epsilon, \delta)\)-differentially private for

\[
\delta = \min_l \exp(\alpha_A(l) - l\epsilon).
\]

A.2  DP guarantee per query in the BHBC framework

In this section, we consider the BHBC framework and we prove Theorem 3 from the main paper. In the following, we consider \( \gamma \in \mathbb{R}^+_\ast \), a fixed privacy parameter, introduce some functions and state some lemmas that will be useful to demonstrate Theorem 3.

Let \( \beta \in \left(\frac{1}{2}, 1\right) \). We denote

\[\text{Note that only the third value over which the minimum is taken in Theorem 3 is data-dependent and, as such, requires this upperbound of } \mathbb{P}[A(d; Q) \neq o^\ast].\]
\[ I_\beta : z \in \mathbb{R}_+ \mapsto \int_0^{+\infty} (t + z)^{\beta - 1} e^{-t} \, dt \]

\[ L_\beta : b \in \mathbb{R}_+^* \mapsto \left( \frac{\Gamma(2\beta - 1)}{\Gamma(\beta)(1-e^{-\beta})} \right) ^{1/\beta} \]

where \( \Gamma \) is the gamma function, defined as \( \Gamma : x \in \mathbb{R}_+^* \mapsto \int_0^{+\infty} t^{x-1} e^{-t} \, dt \).

**Lemma 1.** For any \( \beta \in \left( \frac{1}{2}, 1 \right) \), \( z \in \mathbb{R}_+^* \) and \( b \in \mathbb{R}_+ \), one has

\[ I_\beta (z) \geq (2\gamma)^{1-\beta} I_{\frac{\beta}{1+b}} (0) (b + 2\gamma z)^{\beta - 1} (1 - e^{-b}). \]

**Proof.** Let \( \beta \in \left( \frac{1}{2}, 1 \right) \), \( z \in \mathbb{R}_+^* \), \( a \in \mathbb{R}_+ \).

\[ I_\beta (z) \geq \int_0^a (t + z)^{\beta - 1} t^{\beta - 1} e^{-2\gamma t} \, dt \]

\[ \geq (a + z)^{\beta - 1} \int_0^a t^{\beta - 1} e^{-2\gamma t} \, dt \]

\[ = (a + z)^{\beta - 1} \left[ \int_0^{+\infty} t^{\beta - 1} e^{-2\gamma t} \, dt - \int_a^{+\infty} t^{\beta - 1} e^{-2\gamma t} \, dt \right] \]

\[ = (a + z)^{\beta - 1} \left[ I_{\frac{\beta}{1+b}} (0) - \int_0^{+\infty} (u + a)^{\beta - 1} e^{-2\gamma (u + a)} \, du \right] \]

(by the substitution \( u = t - a \))

\[ \geq (a + z)^{\beta - 1} \left[ I_{\frac{\beta}{1+b}} (0) - e^{-2\gamma a} \int_0^{+\infty} u^{\beta - 1} e^{-2\gamma u} \, du \right] \]

\[ = (2\gamma)^{1-\beta} I_{\frac{\beta}{1+b}} (0) (b + 2\gamma z)^{\beta - 1} (1 - e^{-b}) \]

(where \( b = 2\gamma a \)).

\[ \square \]

**Lemma 2.** For any \( \beta \in \left( \frac{1}{2}, 1 \right) \), \( I_\beta (0) = \left( \frac{1}{2\gamma} \right) ^{2\beta - 1} \Gamma (2\beta - 1) \).

**Proof.** Let \( \beta \in \left( \frac{1}{2}, 1 \right) \). One has

\[ I_\beta (0) = \int_0^{+\infty} t^{2\beta - 2} e^{-2\gamma t} \, dt \]

\[ = \left( \frac{1}{2\gamma} \right) ^{2\beta - 2} \int_0^{+\infty} u^{2\beta - 2} e^{-u} \frac{1}{2\gamma} \, du \]

(by the substitution \( u = 2\gamma t \))

\[ = \left( \frac{1}{2\gamma} \right) ^{2\beta - 1} \Gamma (2\beta - 1) . \]

\[ \square \]

**Lemma 3.** Let \( \tau \in \left( \frac{1}{2}, 1 \right] \) be the fraction of the teachers that did send their noisy votes properly to the aggregation server. The remaining teachers are assumed to have sent their votes without any noise (but still encrypted). We denote \( A \) the aggregation server. Then, \( A \) is \((\epsilon, 0)\)-differentially private, where

\[ \epsilon = 2\gamma + 2 \inf_{b \in \mathbb{R}_+} [(1 - \tau) \log (L_\tau (b) b + 2\gamma L_\tau (b) + 1)]. \]
Proof. Preliminaries on the generalized Laplace distribution. For every teacher \( j \) who did not fail, the noise sent by \( j \) is distributed as \( G_{1}^{(j)} - G_{2}^{(j)} \) where \( G_{1}^{(j)} \) and \( G_{2}^{(j)} \) are two i.i.d. random variables with gamma density \( u \mapsto \frac{1}{\tau} u^{\tau-1} e^{-\gamma u} \) and characteristic function \( t \mapsto \left( \frac{1}{1-i\tau} \right) \tau \) (see [29]). Hence, the characteristic function of \( G_{1}^{(j)} - G_{2}^{(j)} \) is \( \psi: t \mapsto \left( \frac{1}{1-i(1+\tau)} \right) \tau \). By summing over all the teachers who did send a noise, we get a total noise whose characteristic function is \( \psi^{\tau_n}: t \mapsto \left( \frac{1}{1+i(\tau)} \right)^{\tau n} \). The corresponding moment generating function is \( t \mapsto \left( \frac{1}{1-1-i(\tau)} \right)^{\tau} \). According to [31], this is the moment generating function of a generalized Laplace distribution whose density is

\[
 u \mapsto \begin{cases} 
 \frac{1}{\tau} \Gamma(\tau)^2 e^{\gamma u} \int_{0}^{\infty} t^{\tau-1} (t-u)^{\tau-1} e^{-2\gamma t} dt & \text{if } u \geq 0 \\
 \frac{1}{\tau} \Gamma(\tau)^2 e^{\gamma u} \int_{0}^{\infty} t^{\tau-1} (t-u)^{\tau-1} e^{-2\gamma t} dt & \text{if } u < 0
\end{cases}
\]

which is actually

\[
 u \mapsto \frac{1}{\frac{\tau}{2}} e^{\gamma|u|} \int_{|u|}^{\infty} t^{\tau-1} (t-|u|)^{\tau-1} e^{-2\gamma t} dt
\]

(note that we need \( \tau > \frac{1}{2} \) for the integral to be defined when \( u = 0 \))

\[
= \frac{1}{\frac{\tau}{2}} e^{\gamma|u|} \int_{0}^{\infty} (t' + |u|)^{\tau-1} (t')^{\tau-1} e^{-2\gamma(t'+|u|)} dt'
\]

(by substitution \( t' = t - |u| \))

\[
= \frac{1}{\frac{\tau}{2}} e^{-\gamma|u|} I_{\tau}(|u|).
\]

If \( \tau = 1 \), we recover a Laplace distribution and the result comes from the classical result on the Laplace mechanism (see e.g. [15]). In the following, we assume \( \tau < 1 \).

Notations. Let \( d = (d_1, \ldots, d_n) \) and \( d' = (d'_1, \ldots, d'_n) \) be two adjacent databases, and \( x \) an unlabeled data point. Let \( k \in [K] \). We denote \( n_k \) (respectively \( n'_k \)) the number of teachers that vote for the class \( k \) given the vector \( x \). Since \( d \) and \( d' \) are adjacent and the \( d_i \)'s (resp. \( d'_i \)'s) are disjoint, we know that at most one teacher will change between \( d \) and \( d' \), which means that \( |n_k - n'_k| \leq 1 \). Let us also denote \( A_k \) the mechanism that reports the \( k \)th noisy count for vector \( x \). Finally, let \( o \in \mathbb{R}_+ \) be a possible outcome.

Let us suppose that \( n_k \neq n'_k \).

For any auxiliary input aux, we denote

\[
c(o; A_k, aux, d, d') := \log \left( \frac{\mathbb{P}[A_k(aux, d) = o]}{\mathbb{P}[A_k(aux, d') = o]} \right)
\]

the privacy loss at \( o \) for class \( k \). Then we get the following:

\[\sum_{i=1}^{n} z_{k}^{(i)} G_{k,1}^{(i)} - G_{k,2}^{(i)} \text{ with the paper notations}\]

\[\text{Note that if } n_k = n'_k, \mathbb{P}(A_k(d) = o) = \mathbb{P}(A_k(d') = o) \text{ which would trivially complete the proof.}\]
Applying Lemmas 1 and 2, we deduce that, for all
because \(|o - n_k| \leq |o - n'_k| = (\lambda (o) + 1) |n_k - n'_k| \leq 1 + \lambda (o)|.
This bound diverges when \(\lambda\).

First bound. Let us first remark that
Let us denote \(\lambda (o) := \frac{|o - n_k|}{|n_k - n'_k|}.\) We will exhibit two bounds depending on \(\lambda (o)\). The first one is useful for small values of \(\lambda (o)\) whereas the second one is better for big values of \(\lambda (o)\). By merging these two bounds we get the expected result.

First bound. Let us first remark that

\[ |o - n'_k| \leq |o - n_k| + |n_k - n'_k| = (\lambda (o) + 1) |n_k - n'_k| \leq 1 + \lambda (o) \]

Since \(I_\tau\) is decreasing on \(\mathbb{R}_+\) (because \(\tau - 1 \leq 0\)), we have \(I_\tau (|o - n'_k|) \geq I_\tau (1 + \lambda (o))\) and \(I_\tau (|o - n_k|) \leq I_\tau (0)\). Thus,

\[ c(o; A_k, aux, d, d') \leq \gamma + \log \left( \frac{I_\tau (0)}{I_\tau (1 + \lambda (o))} \right). \]

Applying Lemmas 1 and 2, we deduce that, for all \(b \in \mathbb{R}_+\),

\[ c(o; A_k, aux, d, d') \leq \gamma + \log \left( \frac{I_\tau (0)}{I_\tau (1 + \lambda (o))} \right) + (1 - \tau) \log ([b + 2\gamma (1 + \lambda (o))] - \log (1 - e^{-b}) \]

\[ = \gamma + \log \left( \frac{\Gamma (2\tau - 1)}{\Gamma (\tau)} \right) + (1 - \tau) \log ([b + 2\gamma (1 + \lambda (o))] - \log (1 - e^{-b}) \]

This bound diverges when \(\lambda (o)\) approaches \(+\infty\), and this is why we will exhibit another bound, more useful for big values of \(\lambda (o)\).

Second bound. Let us assume that \(o \neq n_k\). Then \(\lambda (o) > 0\). Let us consider also \(t \in \mathbb{R}_+\). Then

\[ \frac{t + |o - n'_k|}{t + |o - n_k|} = 1 + \frac{|o - n'_k| - |o - n_k|}{t + |o - n_k|} \]

\[ \leq 1 + \frac{1}{\lambda (o)} \]

Since \(\tau - 1 \leq 0\), we deduce that

\[ (t + |o - n_k|)^{\tau - 1} \leq (t + |o - n'_k|)^{\tau - 1} \left( 1 + \frac{1}{\lambda (o)} \right)^{1 - \tau}. \]
As it is true for any $t \in \mathbb{R}_+^*$, we have
\[
I_r(|o - n_k|) \leq I_r(|o - n_k'|) \left(1 + \frac{1}{\lambda(o)}\right)^{1-\tau}
\]
and finally
\[
c(o; A_k, \text{aux}, d, d') \leq \gamma + (1 - \tau) \log \left(1 + \frac{1}{\lambda(o)}\right). \tag{7}
\]

**Merging the bounds.** Let us fix $b \in \mathbb{R}_+^*$, and denote
\[
f: \lambda \in \mathbb{R}_+^* \mapsto \min \left(1 + \frac{1}{\lambda}; L_r(b) [b + 2\gamma (1 + \lambda)] \right).
\]
We extend $f$ by continuity in 0 with $f(0) = L_r(b) (b + 2\gamma)$. Then, from Equations (6) and (7) and since $1 - \tau \geq 0$, we have
\[
c(o; A_k, \text{aux}, d, d') \leq \gamma + (1 - \tau) \log \left(f \left(\frac{|o - n_k|}{|n_k - n_k'|}\right)\right)
\]
Let $g: \lambda \in \mathbb{R}_+^* \mapsto 1 + \frac{b}{\lambda}$ and $h: \lambda \in \mathbb{R}_+^* \mapsto L_r(b) [b + 2\gamma (1 + \lambda)]$. $g - h$ is continuous and strictly decreasing on $\mathbb{R}_+^*$. Moreover, $\lim_{\lambda \to +\infty} (g - h) = +\infty$ and $\lim_{\lambda \to -\infty} (g - h) = -\infty$. Hence the equation $g(\lambda) = h(\lambda)$ has a unique solution on $\mathbb{R}_+^*$ that we denote $\lambda_{max}$. Since $g$ is decreasing and $h$ is increasing, one also have that $f(\lambda_{max}) = \max f$. Thus,
\[
c(o; A_k, \text{aux}, d, d') \leq \gamma + (1 - \tau) \log \left(f \left(\frac{|o - n_k|}{|n_k - n_k'|}\right)\right)
\]
Let $g(\lambda) = h(\lambda)$, then the equation $g = h$ has a unique solution on $\mathbb{R}_+^*$ that we denote $\lambda_{max}$. Since $g$ is decreasing and $h$ is increasing, one also have that $f(\lambda_{max}) = \max f$. Thus,
\[
c(o; A_k, \text{aux}, d, d') \leq \gamma + (1 - \tau) \log \left(f \left(\frac{|o - n_k|}{|n_k - n_k'|}\right)\right)
\]
Finally, with some calculus, one gets that
\[
\lambda_{max} = \frac{1 - (b + 2\gamma) L_r(b)}{4\gamma L_r(b)} + \frac{1}{2\gamma L_r(b)}.
\]
Analyzing the right part of the formula, we get
\[
[(b + 2\gamma) L_r(b) - 1]^2 + 8\gamma L_r(b) = [(b + 2\gamma) L_r(b)]^2 - 2 (b + 2\gamma) L_r(b) + 1 + 8\gamma L_r(b)
\]
\[
= [(b + 2\gamma) L_r(b)]^2 + 2 (2\gamma - b) L_r(b) + 1
\]
\[
\leq [(b + 2\gamma) L_r(b)]^2 + 2 (2\gamma + b) L_r(b) + 1
\]
(because $b \geq 0$ and $L_r(b) \geq 0$)
\[
= [(b + 2\gamma) L_r(b) + 1]^2.
\]
We then have
\[
\lambda_{max} \leq \frac{1}{2\gamma L_r(b)}.
\]
Since $h$ is increasing, from Equation (8) the following holds:
\[
c(o; A_k, \text{aux}, d, d') \leq \gamma + (1 - \tau) \log \left(\frac{1}{2\gamma L_r(b)}\right)
\]
\[
= \gamma + (1 - \tau) \log \left(L_r(b) \left[ b + 2\gamma \left(1 + \frac{1}{2\gamma L_r(b)}\right)\right]\right)
\]
\[
= \gamma + (1 - \tau) \log \left(L_r(b) \left[ b + 2\gamma \left(1 + \frac{1}{2\gamma L_r(b)}\right)\right]\right).
\]  
Since the clear counts of at most two classes differ between the two adjacent databases (because at most one teacher changes his vote), we get, by composition, that the mechanism that reports the histogram constituted of the $K$ noisy counts is $(\epsilon, 0)$-differentially private with
\[
\epsilon = 2\gamma + 2 \times (1 - \tau) \log \left(L_r(b) b + 2\gamma L_r(b) + 1\right).
\]
Finally, as report noisy max can be seen as the result of a postprocessing of the report noisy histogram, it has the same DP guarantee. \qed
Starting back from Lemma 3 we can get Theorem 3 with a little calculus. Let $\beta \in \left( \frac{1}{2}, 1 \right)$. We denote

$$\phi_\beta : b \in \mathbb{R}_+^* \mapsto L_\beta (b) b + 2\gamma L_\beta (b) + 1.$$  

**Lemma 4.** For any $\beta \in \left( \frac{1}{2}, 1 \right)$, $\phi_\beta$ has a minimum on $\mathbb{R}_+^*$ which is reached in $b_{\min} (\beta) = -W_{-1} ((\beta - 1) e^{\beta - 1 - 2\gamma}) + \beta - 1 - 2\gamma$, where $W_{-1}$ is the lower real branch of the Lambert function.

**Proof.** Let $\beta \in \left( \frac{1}{2}, 1 \right)$. For all $b \in \mathbb{R}_+^*$,

$$\phi'_\beta (b) = L'_\beta (b) b + L_\beta (b) + 2\gamma L'_\beta (b) = L'_\beta (b) [b + (\beta - 1) (e^b - 1) + 2\gamma].$$

because we have $L'_\beta (b) = \frac{L_\beta (b)}{(\beta - 1)(e^b - 1)} < 0$ for all $b \in \mathbb{R}_+^*$.

Hence we get

$$\phi'_\beta (b) \leq 0 \iff b + (\beta - 1) (e^b - 1) + 2\gamma \geq 0 \iff (\beta - 1) e^b + b + 1 - \beta + 2\gamma \geq 0.$$  

We obtain an equation of the form $a_1 e^z + a_2 z + a_3 = 0$, whose solutions depend on the value of the discriminant $b_2 e^z = (\beta - 1) e^{\beta - 1 - 2\gamma} \in (-\frac{1}{2}, 0)$.

Hence, denoting $W_0$ and $W_{-1}$ the two branches of the Lambert function and $\zeta := \beta - 1 - 2\gamma$, the equation $(\beta - 1) e^x + b + 1 + \beta - 2\gamma = 0$ has two distinct solutions on $\mathbb{R}$, $-W_0 ((\beta - 1) e^z) + \zeta$ and $-W_{-1} ((\beta - 1) e^z) + \zeta$.

To select the appropriate solutions, let us study the sign of $-W_0 ((\beta - 1) e^z) + \zeta$ and $-W_{-1} ((\beta - 1) e^z) + \zeta$.

**First case:** $\zeta \leq -1$

In this case $W_0 ((\beta - 1) e^z) \geq -1$, then we have $-W_0 ((\beta - 1) e^z) + \zeta \leq 0$. Besides,

$$-W_{-1} ((\beta - 1) e^z) + \zeta > 0 \iff W_{-1} ((\beta - 1) e^z) < \zeta \iff (\beta - 1) e^x > \zeta e^z$$

(since both $W_{-1} ((\beta - 1) e^z)$ and $\zeta$ are smaller than $-1$, we can apply $y \mapsto ye^y$ which is strictly decreasing on $(-\infty, -1)$)

$$\iff \beta - 1 > \zeta \iff 2\gamma > 0.$$  

**Second case:** $\zeta > -1$

Since $W_{-1} ((\beta - 1) e^z) \leq -1$, $-W_{-1} ((\beta - 1) e^z) + \zeta > 0$. Besides,

$$-W_0 ((\beta - 1) e^z) + \zeta \leq 0 \iff W_0 ((\beta - 1) e^z) \geq \zeta \iff (\beta - 1) e^z \geq \zeta e^z$$

(since both $W_0 ((\beta - 1) e^z)$ and $\zeta$ are greater than $-1$, we can apply $y \mapsto ye^y$ which is increasing on $[-1, +\infty)$)

$$\iff 2\gamma \geq 0.$$  

Since $\gamma > 0$, in both cases we have $-W_0 ((\beta - 1) e^z) + \zeta \leq 0 < -W_{-1} ((\beta - 1) e^z) + \zeta$.

Hence, $\phi_\beta$ has a minimum on $\mathbb{R}_+^*$ which is reached in $b_{\min} (\beta) := -W_{-1} ((\beta - 1) e^z) + \zeta = -W_{-1} ((\beta - 1) e^{\beta - 1 - 2\gamma}) + \beta - 1 - 2\gamma$.  

\[\square\]
Theorem 3. Let $\tau \in \left( \frac{1}{2}, 1 \right)$ be the fraction of the teachers that did send their noisy votes properly to the aggregation server. The remaining teachers are assumed to have sent their votes without any noise (but still encrypted). We denote $\mathcal{A}$ the aggregation mechanism. Then, $\mathcal{A}$ is $(\epsilon, 0)$-differentially private, with

$$
eq 2\gamma + 2(1-\tau) \log \left( w(\tau) \frac{\Gamma(2\tau-1)}{\Gamma(\tau)} \times \left( 1 + \frac{1-\tau}{w(\tau)} \right) \right)^{\frac{1}{1-\gamma}} + 1 \right)$$

where $\Gamma: x \in \mathbb{R}_+^* \mapsto \int_0^{+\infty} e^{-x} e^{-x t} dt$ and $w(\tau) = -W_{-1} \left( (\tau - 1) e^{\tau-1-2\gamma} \right) + \tau - 1$, $W_{-1}$ being the lower real branch of the Lambert function.

Proof. By construction, $e^{-b_{\text{min}}(\tau)} = \frac{1}{b_{\text{min}}(\tau) + 1 - \tau + 2\gamma}$. Then

$$1 - e^{-b_{\text{min}}(\tau)} = \frac{b_{\text{min}}(\tau) + 2\gamma}{b_{\text{min}}(\tau) + 1 - \tau + 2\gamma}$$

and $L_\tau(b_{\text{min}}(\tau)) = \left[ \frac{\Gamma(2\tau-1)}{\Gamma(\tau)} \times \left( 1 + \frac{1-\tau}{b_{\text{min}}(\tau) + 2\gamma} \right) \right]^{\frac{1}{1-\gamma}} + 1$.

Let $o \in \mathbb{R}_+$ and $k \in [K]$ such that $n_k \neq n'_k$. Starting back from Lemma 3, we have $c(o; \mathcal{A}_k, aux, d, d') \leq \gamma + (1-\tau) \log (\phi_\tau(b_{\text{min}}(\tau)))$. Then, applying Lemma 4, we get

$$c(o; \mathcal{A}_k, aux, d, d') \leq \gamma + (1-\tau) \log (\phi_\tau(b_{\text{min}}(\tau))) = \gamma + (1-\tau) \log ((b_{\text{min}}(\tau) + 2\gamma)L_\tau(b_{\text{min}}(\tau)) + 1)$$

$$= \gamma + (1-\tau) \log \left( b_{\text{min}}(\tau) + 2\gamma \left[ \frac{\Gamma(2\tau-1)}{\Gamma(\tau)} \times \left( 1 + \frac{1-\tau}{b_{\text{min}}(\tau) + 2\gamma} \right) \right]^{\frac{1}{1-\gamma}} + 1 \right)$$

$$= \gamma + (1-\tau) \log \left( -W_{-1} \left( (\tau - 1) e^{\tau-1-2\gamma} \right) + \tau - 1 \right)$$

$$\times \left[ \frac{\Gamma(2\tau-1)}{\Gamma(\tau)} \times \left( 1 + \frac{1-\tau}{W_{-1}((\tau - 1) e^{\tau-1-2\gamma} + \tau - 1)} \right) \right]^{\frac{1}{1-\gamma}} + 1 \right)$$

\( \square \)

A.3 Influence of the HE layer on the DP guarantee per query

The computation of the homomorphic argmax induces some perturbations on the noisy counts and, as such, could harm the DP guarantees that we just gave. Nevertheless, we here show that the perturbations due to the HE layer have the same effect as some postprocessing applied on the clear noisy histogram, hence letting the DP guarantees unchanged\(^{[11]}\).

The three kinds of perturbations due to the HE layer are:

- the addition of (Gaussian) noise at the time of TFHE encryption which is inherently probabilistic
- the addition of a constant value $A$ on the noisy counts to ensure that all the noisy counts are positive (with high probability) (see Section\(^{[3]}\))
- a possible mistake on the argmax if two noisy counts are too close (see Section\(^{[5]}\) of the main paper)

The additions of Gaussian noise and constant $A$ at encryption have, by commutativity, the same effect as the addition of a sum of Gaussian noises and $nA$ after summation (and addition of the Laplace noise in the HBC case).

As far as the mistake on the argmax is concerned, it can be simulated via a probabilistic algorithm which would randomly modify some counts of the clear noisy histogram.

\(^{[11]}\)Indeed, the guarantees from Section\(^{[A.2]}\) apply to the noisy histogram as shown in the proof of Lemma\(^{[5]}\). As for the HBC case, we refer the reader to the well-known result for report noisy histogram with disjoint data subsets, e.g. in \([15]\).
A.4 Upper bound of the probability of a report noisy max mistake

In this subsection, we give an upper bound of the probability that the report noisy max mechanism gives a wrong argmax in the BHBC framework.

**Proposition 2.** Let us consider a query \( Q \in \mathbb{Q} \). Let \( \tau \in (\frac{1}{2}, 1] \) be the ratio of successful teachers. Let \( k^* \in [K] \) be the unnoisy argmax (for all \( k \in [K] \), \( n_{k^*} \geq n_k \)). Then,

\[
P[A(d; Q) \neq k^*] \leq 2^{2-4\tau} \frac{\Gamma(2\tau - 1)^2}{\Gamma(\tau)^4} \sum_{k \neq k^*} \frac{2 + \gamma(n_{k^*} - n_k)}{e^{\gamma(n_{k^*} - n_k)}}
\]

**Proof.** For \( k \in [K] \), let us denote \( Y_k \) the random variable following the generalized Laplace distribution generated by the sum of the \( \tau n \) individual noises. Using the expression of the density of the generalized Laplace distribution recalled in the proof of Lemma 3, and defining \( \Delta_k = n_{k^*} - n_k \), with some raw calculus, we get:

\[
P(n_k + Y_k \geq n_{k^*} + Y_{k^*}) = P(Y_k \geq Y_{k^*} + \Delta_k)
\]

\[
= \frac{\gamma^4}{\Gamma(\tau)^4} \int_{-\infty}^{+\infty} e^{-\gamma|t|} I_{\tau}(|t|) \int_{t + \Delta_k}^{+\infty} e^{-\gamma|u|} I_{\tau}(|u|) du dt
\]

Then,

\[
= \frac{\gamma^4}{\Gamma(\tau)^4} \times \int_{-\infty}^{+\infty} e^{-\gamma|t|} \int_{t + \Delta_k}^{+\infty} e^{-\gamma|u|} du dt
\]

(applying Lemma 2)

\[
= \frac{\gamma^4}{\Gamma(\tau)^4} \times \frac{\gamma^2}{2^{4\tau - 2}} \times \left[ \int_0^{+\infty} e^{-\gamma t} \int_{t + \Delta_k}^{+\infty} e^{-\gamma u} du dt + \int_{-\infty}^{-\Delta_k} e^{\gamma t} \left( \int_0^{+\infty} e^{\gamma u} du + \int_0^{+\infty} e^{-\gamma u} du \right) dt + \int_{0}^{-\Delta_k} e^{\gamma t} \int_{t + \Delta_k}^{+\infty} e^{-\gamma u} du dt \right].
\]

We have

\[
\int_0^{+\infty} e^{-\gamma t} \int_{t + \Delta_k}^{+\infty} e^{-\gamma u} du dt = \int_0^{+\infty} e^{-\gamma t} \frac{e^{-\gamma(t + \Delta_k)}}{\gamma} dt
\]

\[
= \frac{1}{\gamma} \int_0^{+\infty} e^{-\gamma(2t + \Delta_k)} dt
\]

\[
= \frac{e^{-\gamma \Delta_k}}{2 \gamma^2}
\]

and

\[
\int_{-\infty}^{-\Delta_k} e^{\gamma t} \left( \int_0^{+\infty} e^{\gamma u} du + \int_0^{+\infty} e^{-\gamma u} du \right) dt = \int_{-\infty}^{-\Delta_k} e^{\gamma t} \left( \frac{1 - e^{\gamma(t + \Delta_k)}}{\gamma} + \frac{1}{\gamma} \right) dt
\]

\[
= \frac{1}{\gamma} \int_{-\infty}^{-\Delta_k} \left( 2e^{\gamma t} - e^{-\gamma(2t + \Delta_k)} \right) dt
\]

\[
= \frac{2e^{-\gamma \Delta_k}}{\gamma^2} - \frac{e^{-\gamma \Delta_k}}{2 \gamma^2}
\]

\[
= \frac{3e^{-\gamma \Delta_k}}{2 \gamma^2}
\]
Besides,
\[
\int_{-\Delta_k}^{0} e^{\gamma t} \int_{t+\Delta_k}^{+\infty} e^{-\gamma u} du \ dt = \int_{-\Delta_k}^{0} e^{\gamma t} e^{-\gamma (t+\Delta_k)} dt
\]
\[
= \frac{1}{\gamma} \int_{-\Delta_k}^{0} e^{-\gamma \Delta_k} dt
\]
\[
= \frac{\Delta_k e^{-\gamma \Delta_k}}{\gamma} dt
\]

Finally,
\[
P(n_k + Y_k \geq n_{k^*} + Y_{k^*}) = \frac{\Gamma(2\tau - 1)^2}{\Gamma(\tau)^4} \times \frac{\gamma^2}{2^{4\tau-2}} \times \left[ \frac{2e^{-\gamma \Delta_k}}{\gamma^2} + \frac{\Delta_k e^{-\gamma \Delta_k}}{\gamma} \right]
\]
\[
= \frac{\Gamma(2\tau - 1)^2}{\Gamma(\tau)^4} \times \frac{2 + \gamma \Delta_k}{e^{\gamma \Delta_k}}
\]

The overall upper bound for \(P[A(d; Q) \neq k^*]\) is obtained using the fact that the event \((A(d; Q) \neq k^*)\) is the union of the events \((n_k + Y_k \geq n_{k^*} + Y_{k^*})\), for \(k \in [K] \setminus \{k^*\}\), and then \(P[A(d; Q) \neq k^*] \leq \sum_{k \neq k^*} P(n_k + Y_k \geq n_{k^*} + Y_{k^*})\). \(\square\)

Note that, when \(\tau\) approaches 1, we recover \(P[A(d; Q) \neq k^*] \leq \sum_{k \neq k^*} \frac{2^{4\tau-2} e^{-\gamma \Delta_k}}{\gamma^2} \leq \frac{2 + \gamma \Delta_k}{e^{\gamma \Delta_k}}\) which is the bound used by Papernot et al. in [34] and our bound for HBC framework.

Remark. The data-dependent bound \(\alpha_A(l; aux, d, d') \leq \log \left( \left( 1 - q \right) \left( 1 - q \right) \left( 1 - q \right) \right) + q e^{\mu} \) from Theorem 1 is non-monotonic in \(\gamma\). This may appear counter-intuitive since a smaller noise (greater \(\gamma\)) usually gives less privacy guarantees and, as one would expect, a bigger moments accountant. Nevertheless, a smaller noise means that the probability of outputting the true (unnoisy) argmax is closer to 1, leading naturally to a smaller moments accountant. Indeed, two adjacent databases will both output the true argmax with high probability, giving less chance to an adversary to distinguish them. This non-monotonicity of the data-dependent bound induces the non-monotonicity of the overall privacy cost \(\epsilon\). We took this phenomenon into account in our analysis. Accordingly, we chose a fairly small noise (high value of \(\gamma\)) to perform our experiments.

B More detailed problem setting

B.1 Use-case example and possible attacks

An example of scenario from the field of cybersecurity is when several actors each hold a database of cybersecurity incident signatures, that have occurred on their customer networks. Building a model from a larger set of such signatures would lead to improved detection capabilities. However, these databases are highly-sensitive and highly-valuable. As such, they cannot be disclosed (notwithstanding legal barriers to do so). In such a setting, the data owners wish to collaboratively train a global model while preserving the confidentiality of their learning sets. In order to build a global model, our actors may decide to rely on a third-party server embedded in a system architecture which has to be resistant to the threats with respect to both the aggregation server and the global model recipient (note that in our example, and in many real-world settings, all the training data providers may be recipients of the global model). Hence, it is clearly an issue if, given a particular data instance, and some (black-box [41], or white-box [47]) access to the global model, a membership inference attack [40] can indicate with high accuracy the probability that the instance has been used to train the model as it would potentially reveal that a given cyberattack scenario actually occurred within the networks supervised by one of the actors. Also, given a set of instances, the risk of a model inversion attack [45] which tries to infer sensitive attributes on the instances from a supposedly non-sensitive (often white-box) access to the model, is to be seriously taken into account as it would allow to infer e.g. that some of the networks supervised by an actor are more prone to certain kinds of cyberattacks. Lastly, it is also an issue if even features of the actors’ models can be inferred by others as Tramer et al. [42], followed by more recent works [46, 43], demonstrated that even with limited access to a model, an adversary can infer important features of that model such as the hyperparameters or...
its architecture. In this latter case, this would clearly leak some information on e.g. the detection capabilities of an actor giving a potential advantage to cyberattackers of the networks it supervises.

B.2 FHE deployment scenario and threat model

With respect to securing aggregation, we work in the following tripartite setting. The student model is the owner of the homomorphic encryption scheme under which encrypted-domain computations will be performed by the aggregation server, that is it generates and knows both $pk$ and $sk$. Then, when being submitted an unlabeled input, the teachers encrypt their predictions under $pk$ and send these encryptions to the server. The server then has the responsibility to homomorphically perform the aggregation in order to produce an encryption of the output (e.g. a label) which will be sent back to the student and used by the latter for learning, after due decryption. Homomorphic encryption thus provides a countermeasure to confidentiality threats on the teachers’ predictions from the aggregation server. In this setting, we do not address threats whereby the student model and the aggregation server collude (in which case the student model may e.g. share $sk$ with the server so that they both get access to the teachers’ predictions) or threats where the aggregation server behaves maliciously, e.g. to prevent the student model from effectively learning from the teachers, leading to more or less stealthy forms of denial-of-service. This is typical of scenarios in which homomorphic encryption intervenes and our setting thus covers the so-called HBC / BHBC threat models whereby the aggregation server is assumed to operate properly.

C FHE argmax implementation details

We implemented the FHE argmax algorithm using the C++ TFHE library [12]. Table 3 presents all of the parameters needed to reproduce our results and build a fully homomorphic argmax scheme using the TFHE library. The first two lines present our values for the standard TFHE parameters: the first line for initial ciphertext encryption; the second line for the two bootstrapping keys we use. Given the parameters that we use here, we achieve a security parameter of 110. We base the security of our scheme on the lwe-estimator script. The estimator is based on the work presented in [3] and is consistently kept up to date.

Table 3: Parameter for our implementation. The top line presents the overall security ($\lambda$), and the parameters for the initial encryption: $\sigma$ is the Gaussian noise parameter and $N$ is the size of polynomials. In the TFHE encryption scheme, there is a parameter $k$ (different from the one used in this paper) which, in our case, is always equal to 1. The second line presents the parameters needed to create the two bootstrapping keys we are using. For these two lines, we used the notations from [48] and [11]. The third line presents parameters specific to our implementation given the specificities of the data to process. $A$ is the value to add to the ciphertexts before subtracting $n_k + Y_k - n_k' - Y_k'$ as per the notations in Section 4.1 of the paper. $b_i$ is the modulus with which the values are rescaled at encryption time to obtain values in $[0, 1]$ and to allow for a correct result of the $\theta$ computation. $b_\theta^{(1)}$ is the output modulus of the first bootstrapping operation creating the $\theta$ values. $b_\theta^{(2)}$ is the output modulus of the second and final bootstrapping operation.

| $N$  | $\sigma$  | $b_i$ | $b_\theta^{(1)}$ | $b_\theta^{(2)}$ |
|------|-----------|------|------------------|------------------|
| 1024 | $1e-9$    | 64   | 6                | 4                |

The third line presents parameters that are specific to our implementation. Because of the use of a Laplace distribution (in the HBC framework) or a Gamma distribution (in the BHBC framework), the values sent by the teachers can be negative. This can be an important issue: if a value is negative, then it will be interpreted in the ciphertext space as a very high positive value and the resulting argmax will be wrong. Therefore, after summing the ciphertexts from the teachers, we add a constant value (we can add a clear value to a ciphertext value) $A$ to ensure that the $n_k + Y_k + A$ are all positive.
before subtraction. We evaluated that, given the Laplace distribution or Gamma distribution used, choosing \( A = 900 \) gives us less than a \( 2^{-64} \) probability of failure: with \( Y_k \) following a Laplace distribution (as seen in Section 4 of the main paper), then we have \( P(Y_k < -A) < 2^{-64} \). The \( b_i \) variable corresponds to the value by which we rescale the cleartexts before encryption. Indeed, the cleartext and ciphertext spaces of the TFHE encryption scheme are both \( T = ([0, 1], +) \). Additionally, for a correct \( \theta \) computation, we need to have \( \left| \frac{n_k + Y_k - n_{k'} - Y_{k'}}{b_i} \right| < \frac{1}{2} \), which is true if, for all \( k \in [K] \), \( \frac{n_k + Y_k + A}{b_i} \in [0, \frac{1}{2}) \). Since \( P(Y_k \geq A) < 2^{-64} \) by symmetry, \( b_i = 2(n + 2A) = 4100 \) (with \( n \) the number of teachers) is sufficient to have \( \left| \frac{n_k + Y_k - n_{k'} - Y_{k'}}{b_i} \right| < \frac{1}{2} \) with high probability. \( b_1^{(2)} \) is the output modulus of the first bootstrapping operation. It needs to be chosen so that we have \( \Theta_k > \frac{1}{2} \) for one and only one \( k \). That \( k \) will then be considered the argmax. \( b_2^{(2)} \) is the modulus for the final bootstrapping operation.

D Detailed experimental settings and extended results

In this section, we provide the reader with additional details regarding experimental settings, as well as complementary results obtained on the SVHN dataset.

D.1 Experimental settings for MNIST

Following PATE experimental conditions, we built our framework based, with some modifications, on the code repositories accompanying [34]. The teacher models are based on two convolutional layers with max-pooling and one fully connected layer with ReLUs. The execution environment consists in python 3 and tensorflow 1.15.0. The batch size, learning rate and max steps parameters have been respectively set to 128, 0.01 and 5000. As stated in [34], this yields an aggregate test-error rate of 93%. A semi-supervised technique proposed in [38] has been used in an execution environment consisting of python 3 and Theano 0.7. Besides modifications provided as complementary files, the learning rate and number of epochs have been set to 0.001 and 500, respectively.

D.2 Experimental settings for SVHN

For SVHN, two additional layers have been added to the teacher models which were learned using a node with 8 NVIDIA v100. The batch size, learning rate and max steps parameters have been respectively set to 64, 0.08 and 2000. The student model also uses the improved GAN semi-supervised model, relying on python 3 and theano 0.8.2. Besides modifications provided as complementary files, the learning rate and number of epochs have been set to 0.0003 and 600, respectively.

D.3 Additional results for SVHN

Table 4 presents our experimental results on SVHN dataset. The variance on the accuracy is much smaller than for MNIST dataset because the test set is constituted of 16032 samples. As expected, the accuracy increases when less noise is applied because less teachers noised their votes in the BHBC framework (i.e. when \( \tau \) is small). The DP guarantees are not as good as for MNIST, this is due to the high amount of queries (500) necessary to obtain a good accuracy because the learning task is more complex. Note also that the noise we used is quite small (\( \gamma = 1.1 \)), which is better for DP guarantees in the BHBC framework (see remark in Section A.4) with \( \tau < 1 \) but harms the DP guarantee for the HBC framework.

E Report noisy histogram

E.1 Homomorphic histogram computation with Partial HE schemes

On top of the types of FHE presented in Section 3.2 of the main paper, there also exist partial encryption schemes which, despite placing severe restrictions on the kind of computations which can be performed in the encrypted domain, are of practical interest.

\[ \text{https://github.com/tensorflow/privacy/tree/master/research/pate_2017} \]
\[ \text{https://github.com/openai/improved-gan} \]
Table 4: SVHN experimental results with noise inverse scale $\gamma = 1.1$, $\delta = 10^{-5}$, 500 queries

| Framework      | $\epsilon$ | Acc. [%] | HE overhead |
|----------------|------------|----------|-------------|
| Non-private    | -          | 84.7     | -           |
| Trusted        | 10.12      | 83.7     | -           |
| HBC            | 10.12      | 83.5     |             |
| BHBC, $\tau = 1$ | 10.12    | 83.5     |             |
| BHBC, $\tau = 0.9$ | 15.71   | 83.8     | 32.5 min    |
| BHBC, $\tau = 0.7$ | 31.38   | 84.6     |             |

**Partial HE (PHE).** In partial homomorphic encryption schemes, only one of the two $\oplus$ and $\otimes$ operators is supported. The most important such scheme is that of Paillier\cite{paillier1999public}, which provides the $\oplus$ operator along with a cleartext vs ciphertext multiplication operator i.e.,

\[ k \otimes \text{Enc}(m) = \text{Enc}(km) \in \Omega, \text{ with } k \in \Lambda. \]

When the aggregation function consists in just returning a noisy histogram of the votes i.e. (following the notations in Section 4.1 of the paper) the $K$

\[ n_k + Y_k \]

where $Y_k$ denotes a random variable drawn from the Laplace distribution of inverse scale parameter $\gamma \in \mathbb{R}_+^*$, then it can be computed with an additive-only homomorphic encryption scheme which is very interesting as we can either:

1. Use a plain-old additively-homomorphic cryptosystem such as Paillier or (additive-flavored) ElGamal (requiring $O(nK)$ homomorphic additions, where $n$ is the number of teachers).
2. Use a depth-0 optimized SHE scheme such as BFV or BGV which, with appropriate use of batching, will allow to compute the histogram and noise it in $O(n)$ rather than $O(nK)$ (parallel) homomorphic additions i.e. the $i$-th teacher generates a single ciphertexts of the form

\[ \text{Enc}(z_1^{(i)}, \ldots, z_K^{(i)}, 0, \ldots, 0), \]

(assuming $K \ll \kappa$) and these ciphertexts are summed following Equation 5 of the main paper.

Choosing between the two options does not only depend on the performances of the resulting aggregation operator. Indeed, from a security perspective, the aforementioned partial homomorphic encryption schemes are very well understood and can be used in real-world highly sensitive applications.\cite{paillier1999public} However, these cryptosystems are not postquantum, meaning that a hypothetical large-scale quantum computer may break them in some undetermined distant future. More advanced SHE or FHE cryptosystems all have their security based on euclidean lattice problems which, despite being postquantum, are (at present) less well understood from a practical security point of view with new (yet exponential-time) attacks regularly being published and requiring to revisit these cryptosystems’ parameters towards larger parameters and less efficiency. This state of affairs implies that, at present, SHE and FHE performances have not yet stabilized and that previously encrypted data (with a given public set of parameters) may become vulnerable. For these reasons, when dealing with high-value training data, it appears more appropriate to stick to plain-old additive HE if the application allows it (as it is the case in this section).

### E.2 Experimental results

We implemented Paillier cryptosystem in order to determine the computational overhead due to the homomorphic layer in the report noisy histogram model.

\[\text{For example, some governmental entities such as the ANSSI RGS (which is the French Government official framework for cryptographic parameter settings) explain how to parametrize the Paillier cryptosystem to achieve a given security target.}\]
Reducing online overhead. Note that the encryption function of the Paillier cryptosystem is one of the costliest operations (with the decryption). Fortunately, it can be split in a cleartext-dependent part and a cleartext-independent part which, as such, can be done offline and captures most of the encryption cost. In a nutshell \[8\], given \( m \in \mathbb{Z}_\nu \), we have \( \text{Enc}(m) = g^m r^\nu \mod \nu^2 \), where \( g \) and \( \nu \) are part of the public key and where \( r \) is uniformly drawn in \( \mathbb{Z}_\nu \). Since usually \( m \ll \nu \), the costly term is \( h = r^\nu \mod \nu^2 \) (modular exponentiation with large exponents costs a lot) which does not depend on \( m \) and, given \( h \), \( \text{Enc}(m) = g^m h \mod \nu^2 \). This trick greatly improves the practicability of our framework since these precomputations ensure that, despite its computational cost, the encryption step induces no latency. The remaining encryption operations made online are far less demanding.

Avoiding negative values. The addition of noise in both HBC and BHBC frameworks may lead to an encryption of negative values which would be treated by the modulo \( \nu \) operation as very high integers, requiring costly high-power exponentiations.\(^{16}\) In order to avoid these costly exponentiations we add a constant value \( A \) to all the noises so that the noisy counts are all positive with high probability.\(^{17}\) It is the same trick as for TFHE cryptosystem (Section \[C\]) but it is not used for the same reason. For Paillier cryptosystem, we do not need this trick to ensure that the output argmax is correct since we can filter the very high integers in the decrypted histogram and consider them as the trace of negative noises, and hence subtract \( \nu \) to the decrypted value. On the contrary, in TFHE case, as the argmax is performed in the secret domain, we cannot perform such a postprocessing. However, the trick is actually useful in Paillier scheme to reduce the encryption time as we discussed above.\(^{18}\)

Table 5: Detailed computational overhead of HE per query (milliseconds) - We used a 2048 bits modulus \( \nu \) (which is standard dimensioning to achieve strong medium-term security), 250 teachers, 10 classes and an inverse scale parameter of the noise \( \gamma = 3.3 \). The table displays, for HBC and BHBC with \( \tau = 1 \), the time of offline precomputation, encryption, homomorphic aggregation, decryption and the total online overhead which is the sum of all the operations’ times except the offline precomputation. Note that smaller values of \( \tau \) would give slightly smaller overheads since less noise would be added.

| Framework | Offline prec. | Encryption | Aggregation | Decryption | Total online ov. |
|-----------|---------------|------------|-------------|------------|------------------|
| HBC       | 218           | 0.3        | 31 (27+4)   | 216        | 247              |
| BHBC      | 218           | 4          | 29          | 222        | 255              |

Table 5 details the computational overhead per query on an Intel Core i3-3120M CPU. The detailed overhead is the same with HBC or BHBC frameworks except for two operations - the online encryption and the homomorphic aggregation. These two slight differences actually come from the same operation, namely the encryption of the noise augmented by the constant \( A \), which takes place at the aggregation step for HBC framework and at the encryption step for BHBC framework. Note that the computational overhead does not depend on the dataset we use except for the number of teachers and classes.

As shown in Table 5, the overall HE computational overhead is about 250 ms per query, which is more than an order of magnitude under the HE overhead in the report noisy max scheme (cf. Section \[C\] of the main paper). As announced, if the HE layer relies on an additive-only homomorphic cryptosystem, the cost of cryptographic security is much smaller.

E.3 Impracticability of the histogram scheme in terms of DP

Let us consider the DP analysis detailed in Section \[A.1\] and summarized in Algorithm \[1\] for the report noisy max scheme. The privacy guarantee mostly relies on the possibility of evaluating a tight data-dependent bound of the moments accountant per query. This bound is based on a privacy guarantee per query (using e.g. Theorem \[3\]) and an upperbound of the probability \( \mathbb{P}[A(d; Q) \neq o^*] \) that \( A \) does not output some specific output \( o^* \).

\(^{16}\) Because we would not have \( m \ll \nu \) anymore but, rather, \( m \) is almost \( \nu \).

\(^{17}\) We chose \( A = 900 \), ensuring that \( \mathbb{P}(Y_k < -A) < 2^{-64} \).

\(^{18}\) Note that in TFHE scheme, by contrast, the encryption time does not depend on the cleartext value.
Per query, reporting the noisy histogram has the same privacy guarantee as the report noisy max scheme because the personal databases $d_i$ are disjoint. This is true both for HBC (see for example [15]) and BHBC framework (see Theorem 3).

The crucial difference between releasing the noisy histogram or the noisy argmax comes from the impossibility of determining a reasonable upper bound for the probability of the server outputting a wrong histogram. This prevents us from using the data-dependent part of the bound from Theorem 1.

Hence we have to restrict our analysis to a more classical data-independent bound e.g. $\frac{e^{2l}(l+1)}{\delta}$. In practice, this would lead, for $\delta = 10^{-5}$, to an astronomical value of $\epsilon$ (more than 6000) for both HBC and BHBC frameworks if the noise inverse scale parameter is $\gamma = 3.3$. Even with a far more significant noise of $\gamma = 0.05$, we would still have very bad DP guarantees of 21, 273, 1608 respectively for HBC, BHBC with $\tau = 0.9$ and $\tau = 0.7$. This explains the impracticability of releasing the noisy histogram in terms of privacy guarantees. However, given the great improvement linear operators offer in terms of HE, we are currently studying a different way to circumvent the limitations we have on the DP analysis.

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19 The proof of Theorem 3 actually demonstrates the privacy guarantees for the histogram query and consider the evaluation of the argmax as a post-processing.

$20 \log \left( (1-q) \left( \frac{1-q}{1-e^\epsilon q} \right)^t + q e^\epsilon l \right)$