Entropy Anomaly in Langevin-Kramers Dynamics with a Temperature Gradient, Matrix Drag, and Magnetic Field

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Abstract We investigate entropy production in the small-mass (or overdamped) limit of Langevin-Kramers dynamics. The results generalize previous works to provide a rigorous derivation that covers systems with magnetic field as well as anisotropic (i.e. matrix-valued) drag and diffusion coefficients that satisfy a fluctuation-dissipation relation with state-dependent temperature. In particular, we derive an explicit formula for the anomalous entropy production which can be estimated from simulated paths of the overdamped system.

As a part of this work, we develop a theory for homogenizing a class of integral processes involving the position and scaled-velocity variables. This allows us to rigorously identify the limit of the entropy produced in the environment, including a bound on the convergence rate.

Keywords Langevin equation · entropy anomaly · small-mass limit · homogenization

Mathematics Subject Classification (2010) 60H10 · 82C31

1 Introduction

Langevin-Kramers equations model the motion of a noisy, damped, diffusing particle of non-zero mass, m. In the simplest case, the stochastic differential equation (SDE) has the form

\[ \frac{dq_t}{dt} = v_t dt, \quad m \frac{dv_t}{dt} = -\gamma v_t dt + \sigma dW_t, \]  

where \( \gamma \) and \( \sigma \) are the dissipation (i.e. drag) and diffusion coefficients respectively and \( W_t \) is a Wiener process. Smoluchowski [1] and Kramers [2] pioneered the study of such diffusive systems in the small-mass (or overdamped) limit; see [3] for more
on the early literature and [4,5,6,7,8,9,10,11,12,13,14] for further mathematical results in this direction.

The $m \to 0$ limit of Eq. (1) is a common and useful approximation for simulating particle paths in many realistic systems. In the simplest case, setting $m = 0$ in Eq. (1) gives the correct overdamped SDE:

$$dq_t = \gamma^{-1} \sigma dW_t.$$  \hspace{1cm} (2)

This naive derivation is known to fail when $\gamma$ is state-dependent, but a slightly more complicated SDE, incorporating an anomalous drift term, does govern the overdamped particle trajectories; see Theorem 1 below for a summary of a known convergence result. The study of the singular nature of the Langevin-Kramers system in the small-mass limit (i.e. the appearance of the anomalous drift) has a long history. See, for example, [15,16,17,18,19,20].

The overdamped SDE from Theorem 1 correctly describes the limiting behavior of the position variables in a temperature gradient and over bounded time intervals. However, naively applying it outside this domain can lead to erroneous results. For example, it is known that in systems with time-dependent diffusion and damping, the overdamped SDE can fail to capture the long-time behavior of the underdamped system; see [21] for a detailed analysis of a family of anomalous diffusion processes that exhibit this behavior. Naive approaches to the overdamped limit can also fail for systems coupled to multiple reservoirs [22].

There is also a more subtle way that use of the overdamped SDE can lead to errors; it may provide a good approximation to the statistics of particle paths, yet still fail to capture the statistics of other important observables in stochastic thermodynamics. Mathematically, this occurs when the operations of taking the small-mass limit and computing the observable do not commute. Such an occurrence is typically called an anomaly in the physics literature; see, for example, [23]. Specifically, this paper will focus on entropy production, a quantity which has attracted a great deal of study, especially in systems with a (time-dependent) temperature gradient [21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38]. For general background information on stochastic thermodynamics, see [39,40,41,42,43].

It is known that in systems with a temperature gradient, the overdamped limit of Langevin-Kramers dynamics exhibits an entropy anomaly; the entropy production associated with the limiting overdamped SDE has a deficit when compared to the small-mass limit of the entropy produced by the underdamped SDE. Derived formally in [24], the entropy anomaly can be traced to a time-reversal symmetry breaking that occurs when transitioning between the under and overdamped systems. This was generalized in [33] to allow for Brownian rotations and spatially dependent, matrix-valued drag, as well as external forces and torques. Systems in a mean flow and with Brownian rotations were studied in [29], but in a different regime than the $m \to 0$ limit considered here. A similar effect has also been studied in continuous-time Markov chains [36].

In this paper, we put one aspect of the prior formal derivations of the entropy anomaly on a rigorous footing. Specifically, we prove a convergence result as $m \to 0$, with a convergence rate bound, for the entropy produced in the environment in Langevin-Kramers dynamics. The result covers systems in a temperature gradient, allowing for magnetic field and state-dependent, matrix-valued drag. Using this, we derive an explicit formula for the anomalous contribution to the entropy.
production in two different classes of models. The formulas are expectations of functionals of the paths of the overdamped system, and so are straightforward to estimate from numerical simulations.

1.1 Background and Previous Results

The Hamiltonian of a particle of mass \( m \) in an electromagnetic field with \( C^3 \) vector potential \( \psi(t, q) \) and \( C^2 \) potential \( V(t, q) \) (setting charge \( e = 1 \)) is

\[
H(t, x) = \frac{1}{2m} \| p - \psi(t, q) \|^2 + V(t, q)
\]

where \( x \equiv (q, p) \in \mathbb{R}^n \times \mathbb{R}^n \). Allowing for an additional continuous forcing term, \( \tilde{F} \), and coupling to noise and linear drag via the \( C^2 \) matrix-valued functions \( \sigma \) and \( \gamma \) respectively, Hamilton’s equations for this system are given by the SDE

\[
dq_t = \frac{1}{m}(p_t - \psi(t, q_t))\,dt,
\]

\[
d(p_t)_i = \left( -\frac{1}{m} \tilde{\gamma}_{ij}(t, q_t)\delta^{jk}(p_t)_k - \psi_k(t, q_t) + \tilde{F}_i(t, x_t) - \partial_{x_i} V(t, q_t) \right) dt + \sigma_i(t, q_t)\,dW_t^\rho.
\]

Note that the choice of stochastic integral (Itô, Stratonovich, etc.) is not important in Eq. (5); \( q_t \) is a \( C^1 \) process and \( \sigma \) is \( C^2 \), hence all choices yield the same equation.

It is often convenient to define \( u_t \equiv p_t - \psi(t, q_t) \) and write the SDE in the equivalent form

\[
dq_t = \frac{1}{m}u_t\,dt,
\]

\[
d(u_t)_i = \left( -\frac{1}{m} \tilde{\gamma}_{ik}(t, q_t)(u_t)_k + \tilde{F}_i(t, x_t) \right) dt + \sigma_i(t, q_t)\,dW_t^\rho,
\]

where \( \tilde{\gamma} \) now includes the magnetic field,

\[
\tilde{\gamma}_{ik}(t, q) \equiv \gamma_{ik}(t, q) - H_{ik}(t, q) \equiv \gamma_{ik}(t, q) - \left( \partial_{q_i} \psi_k(t, q) - \partial_{q_k} \psi_i(t, q) \right),
\]

and

\[
F(t, x) \equiv -\partial_t \psi(t, q) - \nabla_q V(t, q) + \tilde{F}(t, x).
\]

Here and in the following we employ the summation convention for repeated indices.

In this paper we will assume the fluctuation-dissipation relation holds pointwise for a time and state-dependent effective temperature.

**Assumption 1** Define

\[
\Sigma_{ij}(t, q) = \sum_{\rho} \sigma_{i\rho}(t, q)\sigma_{j\rho}(t, q).
\]
We assume $\gamma$ and $\sigma$ are $C^2$ and
\[ \Sigma(t,q) = 2\beta^{-1}(t,q)\gamma(t,q), \tag{11} \]
where $\beta$ is a $C^2$ function that is bounded above and below by positive constants. Physically, $\beta$ is related to the time and position-dependent effective temperature by $\beta^{-1} = k_B T$, where $k_B$ is the Boltzmann constant.

The above assumption is a generalization of the classical Einstein relation, in which case $\beta$ is a constant, to a class of non-equilibrium settings. Physically, a temperature gradient can be achieved through coupling to more than one heat reservoir, as in [24]. One can also view Eq. (11) as expressing a timescale separation between fast dynamics that maintain a local approximate equilibrium, characterized by a local effective temperature, and a slow relaxation to global equilibrium. This is a commonly useful class of models, but it is still a restricted type of non-equilibrium and such a description can fail due to lack of timescale separation [44, 45]. See [46] for further examples and references related to effective temperature and fluctuation-dissipation relations.

Our derivation of the entropy anomaly relies on several previously derived estimates on the small-mass limit of Eq. (4)-Eq. (5). In [47] it was shown that, for a large class of systems generalizing Eq. (4)-Eq. (5), there exists unique global in time solutions $(q^m_t, u^m_t)$ that converge to $(q_t, 0)$ as $m \to 0$, where here $q_t$ is the solution to a certain limiting SDE. We summarize the precise mode of convergence in Theorem 1 below, which we take as the starting point for this work. See also [18] and further references therein for earlier, related results. Appendix A contains a list of assumptions that guarantee that the following theorem holds.

**Theorem 1** For any $T > 0$, $p > 0$ we have
\[ \sup_{t \in [0,T]} E \left[ \sup_{t \in [0,T]} \|u^m_t\|^p \right]^{1/p} = O(m^{1/2}), \quad \sup_{t \in [0,T]} E \left[ \|q^m_t - q_t\|^p \right]^{1/p} = O(m^{1/2}) \tag{12} \]
as $m \to 0$, where $q_t$ is the solution the Itô SDE
\[ dq_t = \tilde{\gamma}^{-1}(t,q_t)F(t,q_t,\psi(t,q_t))dt + S(t,q_t)dt + \tilde{\gamma}^{-1}(t,q_t)\sigma(t,q_t)dW_t. \tag{13} \]

$S(t,q)$, called the noise-induced drift, is an anomalous drift term that arises in the limit. It is given by
\[ S^i(t,q) \equiv \beta^{-1}(t,q)\partial_{q^i}(\tilde{\gamma}^{-1})^j(t,q). \tag{14} \]
$q_t$ also satisfies
\[ E \left[ \sup_{t \in [0,T]} \|q_t\|^p \right]^{1/p} < \infty \tag{15} \]
for all $T > 0$, $p > 0$.

The components of $\tilde{\gamma}^{-1}$ are defined such that
\[ (\tilde{\gamma}^{-1})^{ij}\tilde{\gamma}_{jk} = \delta^i_k, \tag{16} \]
and for any $v_i$ we define the contraction $(\tilde{\gamma}^{-1}v)^i = (\tilde{\gamma}^{-1})^{ij}v_j$. 

It should be emphasized that the paths of the overdamped system, $q_t$, are uniquely determined by the underdamped system, Eq. (1)-Eq. (5), without requiring a choice of stochastic integration convention as additional data. As noted above, the underdamped system is independent of the choice of convention, due to smoothness of $q_t^m$. Taking the $m \to 0$ limit then uniquely identifies the paths, $q_t$, of the overdamped system. This point, and the insight it gives into the physics of systems in a temperature gradient, was noted in [48], where it was used to compare heat generation in underdamped and overdamped SDEs.

Of course, one can write the limiting SDE for the overdamped paths, $q_t$, using whichever integration convention is desired, resulting in different formulas for the noise-induced drift. Most commonly, the Itô form (see Eq. (13)-Eq. (14)), or Stratonovich form (see Eq. (17)-Eq. (18) below) are used, though anti-Itô or anything between can be used if desired. One convention may appear most natural in a given setting, due to the vanishing of the corresponding noise-induced drift (for example, the Itô convention when $\tilde{\gamma}$ is independent of $q$), but this is not always possible to accomplish when both temperature and drag are spatially dependent. In any case, as long as the chosen integration convention is consistently paired with its noise-induced drift, one obtains a SDE that can be solved to find the overdamped paths, and all choices will result in the same solution, $q_t$.

For the purposes of computing entropy production, it will be most convenient to work with the Stratonovich convention, and its corresponding noise-induced drift. Assuming that $\sigma$ and $\gamma$ are $C^2$ and $\psi$ is $C^3$, the overdamped SDE has the Stratonovich form

$$dq_t = \tilde{\gamma}^{-1}(t, q_t) F(t, q_t, \psi(t, q_t)) dt + \tilde{S}(t, q_t) dt + \tilde{\gamma}^{-1}(t, q_t) \sigma(t, q_t) \circ dW_t,$$  \hspace{1cm} (17)

where the noise-induced drift in the Stratonovich convention is

$$\tilde{S}^i(t, q) = \beta^{-1}(t, q) \partial_q^i \left( \tilde{\gamma}^{-1} \right)^{jk}(t, q) \tilde{h}_{ik}(t, q)$$  \hspace{1cm} (18)

We will also need the convergence result from [49], concerning the joint distribution of $q_t^m$ and $z_t^m$, where

$$z_t^m \equiv u_t^m / \sqrt{m}.$$  \hspace{1cm} (19)

From Eq. (18), we see that $z_t^m$ is a scaled velocity.

The properties in Appendix A along with Assumption H imply the following.

**Theorem 2** Let $K, q > 0$, $0 < \delta < 1/2$, and $\tilde{h} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a $C^1$ function that satisfies

$$\|\nabla \tilde{h}(q, z)\| \leq K (1 + \|(q, z)\|)^n.$$  \hspace{1cm} (20)

Define

$$H_t = E \left[ \left( \frac{\beta(t, q_t)}{2\pi} \right)^{n/2} \int \tilde{h}(q_t, z) e^{-\beta(t, q_t)\|z\|^2/2} dz \right].$$  \hspace{1cm} (21)
Then
\[ E \left[ \hat{h}(q^m_t, z^m_t) \right] = H_t + O(m^\delta) \] (22)
as \( m \to 0 \).

Intuitively, this results states that, as \( m \to 0 \), the \( z \)-dependence of the joint distribution of the position, \( q^m_t \), and scaled velocity, \( z^m_t \), is well approximated by a Gibbs distribution at the local temperature, and one can approximate an observable, \( \hat{h} \), with its local equilibrium (in \( z \)) average.

1.2 Summary of Results

Our primary result is a formula for the small-mass limit of the entropy produced in the environment for the Langevin-Kramers system, Eq. (4) - Eq. (5), including the effects of a temperature gradient, magnetic field, and matrix-valued drag. The general result is found in Section 4.1, Theorem 3, including precise assumptions under which we prove convergence. Corollary 1 contains the simplified case where the vector potential, \( \psi \), vanishes. We quote it here for illustrative purposes:

Let \( S^m_{s,t} \) denote the entropy produced in the environment over a time interval \([s,t]\). For any \( 0 < \delta < 1/2 \), and assuming \( \psi = 0 \), we have
\[ E[S^m_{s,t}] = E[(\beta V)(s, q_s)] - E[(\beta V)(t, q_t)] + E \left[ \int_s^t \partial_r (\beta V)(r, q_r) dr \right] \]
\[ + \int_s^t E \left[ \beta^{-1}(r, q_r) \nabla q \cdot \left( \gamma^{-1} (V \nabla q \beta + \beta \bar{F}) \right) (r, q_r) \right] dr \]
\[ + \frac{n+2}{2} E \left[ \ln(\beta(t, q_t)/\beta(s, q_s)) \right] - E \left[ \int_s^t \left( \beta^{-1} \partial_r \beta \right) (r, q_r) dr \right] \]
\[ + \int_s^t E \left[ \left( -\nabla q V + \bar{F} \right) \cdot \gamma^{-1} (V \nabla q \beta + \beta \bar{F}) (r, q_r) \right] dr \]
\[ + \int_s^t E \left[ \beta^{-3} \nabla q \beta \cdot \left( \frac{3n+2}{6} \gamma^{-1} - \int_0^\infty Tr[\gamma e^{-2y\gamma}] \gamma^{-1} e^{-y\gamma} dy \right) \nabla q \beta \right] (r, q_r) dr \]
\[ + O(m^\delta) \]
as \( m \to 0 \). In particular, note that the \( m \to 0 \) limit can be computed from the solution paths of the overdamped SDE.

In Section 5, the above formula will be compared with the entropy production in the environment for the overdamped SDE, Eq. (17). A formal calculation will then result in a formula for the total entropy production in each case, and we will find that the results differ i.e. the operations of computing the entropy production and taking the small-mass limit do not commute. Specifically, in Section 5.3 we identify following deficit in the entropy production of the overdamped SDE, as compared to the small-mass limit of the entropy production of the underdamped SDE:
\[ E[S^m_{s,t}] \]
\[ = \int_s^t E \left[ \left( \beta^{-3} \nabla q \beta \cdot \left( \frac{3n+2}{6} \gamma^{-1} - \int_0^\infty Tr[\gamma e^{-2y\gamma}] \gamma^{-1} e^{-y\gamma} dy \right) \nabla q \beta \right) (r, q_r) \right] dr. \]
Again, this formula applies to the \( \psi = 0 \) case; see Eq. (97) for the general result, as well as Section 6 for the entropy anomaly in another class of models with magnetic field. As previously noted, a version of this anomalous entropy production (without magnetic field) was first derived formally in [24] and generalized in [33]. Our treatment puts one aspect of these derivations on a rigorous footing; we prove convergence of the entropy produced in the environment as \( m \to 0 \), including an explicit convergence rate bound. We also extend the result to cover systems in a magnetic field.

The main new technical contribution of this paper is a method for computing the small-mass limit (i.e. homogenization) of certain integral processes of the form

\[
\int_{s}^{t} G(r, q^{m} \cdot z^{m}) \, dr \text{ and } m^{-1/2} \int_{s}^{t} z^{m} \cdot K(r, q^{m} \cdot z^{m}) \, dr
\]

that are multi-linear in \( z \), with \( K \) being even in \( z \). This is done in Appendix B; see Theorems B5 and B6.

2 Background: Time-Inversion and Entropy Production

In this section we present a synopsis of the theory of time-inversion and entropy production in stochastic thermodynamics, using the framework in [41,43]. See also [38] for a discussion of the relationship between entropy production and the system-bath interaction.

2.1 Time-Inversion

Consider a generic SDE in Stratonovich form

\[
dx_t = b(t, x_t) \, dt + \tilde{\sigma}(t, x_t) \circ dW_t
\]

on the time interval \([0, T]\), driven by a Wiener process, \( W_t \), and smooth drift and diffusion, \( b \) and \( \tilde{\sigma} \).

A time-inversion (or time-reversal) operation on spacetime will be given by a map \((t, x) \to (t^\ast, x^\ast)\) where \( x \to x^\ast \) (which we will also write as \( \phi(x) \) and refer to as a time-inversion) is a smooth involution and \( t^\ast = T - t \). For example, if \( x = (q, p) \) has position and momentum components \( q \) and \( p \) respectively, a common choice is \((q, p)^\ast = (q, -p)\). Note that we do not assume time-inversion leaves Eq. (25) invariant; in general, dissipative terms ensure it is not invariant.

One could define the time-reversed trajectories of the original system Eq. (25) by \( \tilde{x}_t = x_{T-t} \), however this is problematic as, for example, it leads to anti-dissipation. A more physically reasonable method of defining the time-reversed dynamics is to split the drift into two components \( b = b_+ + b_- \) (called the dissipative and conservative parts, respectively [11]) and define the time-reversed process to be the solution to the SDE

\[
dx_t' = (\phi_+ b_+)(t^\ast, x_t') \, dt - (\phi_- b_-)(t^\ast, x_t') \, dt + (\phi_+ \tilde{\sigma})(t^\ast, x_t') \circ d\tilde{W}_t,
\]

where \( \phi_\ast \) denotes the pushforward of vector fields by the smooth map \( \phi \) (i.e. the operation that takes a vector field and transforms it under the coordinate transformation \( x \to \phi(x) \)) and \( \tilde{W}_t \) is any other Wiener process. We call the solution \( x'_t \) the backward process while \( x_t \) will be called the forward process.
In Eq. (26), the noise term is kept the ‘same’ for both the forward and backward processes, only transformed by \( \phi \) to the new coordinate system. The choice of driving Wiener process doesn’t impact the distribution of the solution and so can be chosen based on convenience. We also note that defining the drift, and its splitting, via the Stratonovich form of the SDE can be motivated by the fact that only the Stratonovich integral has the correct transformation property under change of coordinates [50]. For the application to the underdamped system Eq. (4)–Eq. (5), all stochastic integral choices lead to the same drift (see the comment after Eq. (5)) and so this point is inconsequential there.

Physically, the procedure outlined above is often carried out in the opposite order; one has physically motivated forward and time-reversed SDEs for \( x_t \) and \( x'_t \) and a phase-space involution, \( \phi \), and one wants to reverse engineer a splitting of the drift so that the SDEs correspond as in Eq. (26). In any case, the pair of forward and backward equations should capture the physics of what one wants to call time-reversal in a given system.

### 2.2 Entropy Production

The entropy produced by the process Eq. (25) relative to the time-reversed process Eq. (26) is defined via the Radon-Nikodym derivative of the distribution of the backward process with respect to the forward process; see [41] for details. Intuitively, it quantifies how likely paths are for the backward process, as compared to the forward process.

Specifically, the entropy produced in the environment from time \( s \) to time \( t \), \( S_{s,t}^{env} \), can be computed via the formula

\[
S_{s,t}^{env} \equiv \int_{s}^{t} 2\hat{b}_k^j(r,x_r)(\tilde{\Sigma}^{-1})_{jk}(r,x_r) \circ dx^k_r - \int_{s}^{t} 2\hat{b}_k^j(r,x_r)(\tilde{\Sigma}^{-1})_{jk}(r,x_r)b_k^j(r,x_r) + \nabla \cdot b_r(r,x_r)dr, \tag{27}
\]

where \( \tilde{\Sigma} \equiv \tilde{\sigma} \tilde{\sigma}^T \), \( \tilde{\Sigma}^{-1} \) is the pseudoinverse, \( \circ \) denotes the Stratonovich integral, and

\[
\hat{b}_k^j \equiv b_k^j - \frac{1}{2} \sum_\xi \tilde{\sigma}_j^\xi \partial_\xi \tilde{\sigma}_k^\xi. \tag{28}
\]

Note that this only depends on the splitting of the drift, and not on the form of the time-inversion map.

### 3 Time-Inversion in Langevin Dynamics

The definition of entropy production outlined in the prior section is quite flexible and powerful, but can be physically ambiguous; given a forward equation, the framework it does not uniquely single out a time-inversion map, nor does it single out a preferred splitting of the drift. These two choices should be determined by the physics of what one would like to call the time-reversed process. In many cases
the proper choices appear obvious, but in others (such as Langevin dynamics in a magnetic field) they are less clear.

The intuition guiding our choices of inversion and splitting is that the backward equation should reflect the physics of the same environment (i.e. same experimental setup) as the forward system, but with any explicit time-dependence of the environment reversed and the time-inversion should reverse velocities. In the next two subsections, we will discuss this in detail for both the underdamped and overdamped systems.

3.1 Time-Inversion for Langevin-Kramers Dynamics

First consider a general Hamiltonian system coupled to noise and dissipation:

$$dx_t = (-\Gamma(t, x_t) \nabla H_t(x_t) + \Pi(t, x_t) \nabla H_t(x_t) + G_t(x_t)) dt + \tilde{\sigma}(t, x_t) \circ dW_t,$$

where $H_t$ is a time-dependent Hamiltonian function, $\Pi$ is an antisymmetric matrix, $\Gamma$ is a symmetric, positive semidefinite matrix that results in dissipation, $G$ is an additional non-conservative force field, and $\tilde{\sigma}$ are the noise coefficients.

A canonical splitting of the drift, generalizing the terminology in [41], is given by

$$b_+ = -\Gamma \nabla H_t, \quad b_- = \Pi \nabla H_t + G_t. \quad (30)$$

Note that $b_+$ contains the dissipative component of the dynamics (at least, when $H$ is time-independent), while $b_-$ contains the conservative and external force terms.

To see what the splitting Eq. (30) means physically, we will now specialize to underdamped Langevin-Kramers dynamics i.e. the objects in Eq. (29) have the form

$$\Gamma(t, x) = \begin{pmatrix} 0 & 0 \\ 0 & \gamma(t, \psi(t)) \end{pmatrix}, \quad \tilde{\sigma}(t, x) = \begin{pmatrix} 0 \\ 0 & \sigma(t, \psi(t)) \end{pmatrix}, \quad (31)$$

$$\Pi(t, x) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad G(t, x) = (0, \tilde{F}(t, x)), \quad (32)$$

with Hamiltonian given by Eq. (3).

For the underdamped system, the splitting Eq. (30) becomes

$$b_+(t, x) = \begin{pmatrix} 0 \\ -\frac{1}{m} \gamma(t, \psi(t)) (p - \psi(t, \psi(t))) \end{pmatrix}, \quad (33)$$

$$b_-(t, x) = \begin{pmatrix} \frac{1}{m} (p - \psi(t, \psi(t))) \\ \frac{1}{m} \delta^{ij} (p_i - \psi_i(t, \psi(t))) \nabla_q \psi_j(t, \psi(t)) - \nabla_q V(t, \psi(t)) + \tilde{F}(t, x) \end{pmatrix}. \quad (34)$$

Choosing the time-reversal to be the standard phase-space involution,

$$\phi(q, p) = (q, -p),$$

gives

$$\phi b_+(t, x) = \begin{pmatrix} 0, \frac{1}{m} \gamma(t, \psi(t)) (p - \psi(t, \psi(t))) \end{pmatrix}, \quad (35)$$

$$\phi b_-(t, x) = \begin{pmatrix} \frac{1}{m} (p - \psi(t, \psi(t))) \\ \frac{1}{m} \delta^{ij} (p_i - \psi_i(t, \psi(t))) \nabla_q \psi_j(t, \psi(t)) + \nabla_q V(t, \psi(t)) - \tilde{F}(t, \psi(t), -p) \end{pmatrix}. \quad (36)$$
The time-reversed dynamics are then
\begin{align}
 dq_t' &= \frac{1}{m} (p_t' + \psi(t^*, q_t')) dt, \\
 dp_t' &= -\frac{1}{m} \delta^{ij} (\gamma_{ij}(t^*, q_t') + \partial_{q_j} \psi_j(t^*, q_t')) (p_t' + \psi(t^*, q_t'))_k dt \nonumber \\
 &\quad + ( - \partial_{q_k} V(t^*, q_t') + \tilde{F}_i(t^*, q_t', -p_t')) dt + \sigma(t^*, q_t') \circ dW_t. \tag{36}
\end{align}

Note that these equations have the same form as the original system, Eq. (1)-Eq. (5), but the explicit time dependence is reversed, the argument \( p \) in the external forcing has an additional minus sign, and the vector potential has its sign reversed. When \( \psi = 0 \) and \( \tilde{F} \) is independent of \( p \), this corresponds to the intuitive notion of reversing explicit time dependence of the environment, but otherwise keeping it the same, and comparing the paths to those in the original system.

However, in the presence of a magnetic field, the correct definition of the time-reversed equation is less clear. There seems to be two reasonable options; either the time-reversed dynamics are the same as the original system, \( \psi \rightarrow -\psi \), \( V \rightarrow -V \), \( \tilde{F} \rightarrow -\tilde{F} \), or one can envision that the same experimental setup is used, and hence leave the sign of \( \psi \) unchanged. The former corresponds to the inversion rule Eq. (34). The latter appears difficult to treat in full generality in a physically reasonable manner, but becomes more tractable in the presence of special symmetries.

Specifically, in [51] it was noted that, for a uniform magnetic field, the equations of motion (without noise or drag) are invariant under
\begin{equation}
(q_1, q_2, q_3, p_1, p_2, p_3, t) \rightarrow (-q_1, q_2, q_3, p_1, -p_2, -p_3, -t). \tag{38}
\end{equation}
As we will see, the time-inversion on phase space obtained from Eq. (38) does lead to a pair of forward and backward equations that maintain the same background environment, including the external uniform magnetic field, only with explicit time dependence reversed. In terms of entropy production, the derivation proceeds similarly to that of the standard phase-space involution. Further discussion of this case can be found in Section 6.

### 3.2 Time-Inversion for the Overdamped Limit

For the underdamped system, we used the splitting Eq. (30) to derive the time-reversed SDEs. We don’t need to independently choose such a splitting for the overdamped system, as the small-mass limit uniquely determines the forward and time-reversed paths.

Together, Theorem 11 and Eq. (17) give SDEs for the small-mass limits of the forward and backward processes respectively:
\begin{align}
 dq_t &= \gamma^{-1}(t, q_t) \left( -\partial_t \psi(t, q_t) - \nabla_q V(t, q_t) + \tilde{F}(t, q_t, \psi(t, q_t)) \right) dt \\
 &\quad + \dot{S}(t, q_t) dt + \gamma^{-1}(t, q_t) \sigma(t, q_t) \circ dW_t, \tag{39}
\end{align}
\begin{align}
 dq_t' &= \gamma^{-1}(t^*, q_t') \left( -\partial_t \psi(t^*, q_t') - \nabla_q V(t^*, q_t') + \tilde{F}(t^*, q_t', \psi(t^*, q_t')) \right) dt \\
 &\quad + \dot{S}'(t^*, q_t') dt + \gamma^{-1}(t^*, q_t') \sigma(t^*, q_t') \circ dW_t, \tag{40}
\end{align}
where $S$ and $S'$ are computed via Eq. (13) using the vector potentials $\psi$ and $-\psi$ respectively.

The natural configuration-space involution for the overdamped dynamics, inherited from Eq. (34), is simply the identity map, $q \rightarrow q$. To compute the entropy production via the framework of Section 2 we need to show that the SDE for $q'_t$ has the form Eq. (26); specifically, we need to find a Wiener process, $\tilde{W}_t$, that satisfies

$$\gamma^{-1}(t^*, q'_t) \sigma(t^*, q'_t) d\tilde{W}_t = \gamma^{-1}(t^*, q'_t) \sigma(t^*, q'_t) d\tilde{W}_t$$ (41)

i.e. we need to show that

$$d\tilde{W}_t \equiv \left(\gamma^{-1}(t^*)\right)^{-1} (t^*, q'_t) d\tilde{W}_t$$ (42)

is a Wiener process.

By Levy’s theorem (see p.157 in [53]), $\tilde{W}_t$ is a Wiener process if

$$B^{-1} A A^T (B^{-1})^T = I, \quad A \equiv (\gamma^{-1})^{-1}, \quad B \equiv \gamma^{-1},$$ (43)

where $I$ is the identity matrix and, for the purpose of employing matrix notation, we define $\gamma_j \equiv \delta_{jk} \gamma_{kj}$ and $\sigma_j \equiv \delta_{jk} \sigma_{kj}$. For arbitrary matrices $\gamma$ and $\sigma$, Eq. (43) will not always hold. However, given our assumption of the fluctuation-dissipation relation, Eq. (14), one finds that proving Eq. (43) is equivalent to proving

$$\tilde{\gamma}(\gamma^{-1})^{-1} \gamma^{-1} \gamma^{-1} = \gamma.$$ (44)

This identity can be derived from the relations

$$\gamma = \frac{1}{2} (\gamma + \gamma^T), \quad \gamma^{-1} + (\gamma^{-1})^{-1} = 2(\gamma^{-1})^{-1} \gamma^{-1},$$ (45)

and so $\tilde{W}_t$ is a Wiener process.

Therefore, in Stratonovich form, the backward SDE can be written

$$d\tilde{q}_t' = \gamma^{-1}(t^*, q'_t) \left(-\partial_t \psi(t^*, q'_t) - \nabla_q V(t^*, q'_t) + \tilde{F}(t^*, q'_t, \psi(t^*, q'_t))\right) dt \quad + \tilde{S}(t^*, q'_t) dt + (S'(t^*, q'_t) - S(t^*, q'_t)) dt + \gamma^{-1}(t^*, q'_t) \sigma(t^*, q'_t) \circ d\tilde{W}_t,$$ (46)

where $S$ and $S'$ are computed via Eq. (14) using the vector potentials $\psi$ and $-\psi$ respectively.

In this form, the forward and backward equations are seen to correspond, as in framework of Section 2 under the splitting

$$b_+ = \gamma^{-1} F + \tilde{S} - \frac{1}{2} \left(\gamma^{-1} - (\gamma^{-1})^{-1}\right) F - \frac{1}{2} (S - S'),$$ (47)

$$b_- = \frac{1}{2} \left(\gamma^{-1} - (\gamma^{-1})^{-1}\right) F + \frac{1}{2} (S - S'),$$ (48)

where

$$F(t, q) \equiv -\partial_t \psi(t, q) - \nabla_q V(t, q) + \tilde{F}(t, q, \psi(t, q)),$$ (49)

$$\frac{1}{2} \left(\gamma^{-1} - (\gamma^{-1})^{-1}\right)^{ij} F_j = (\gamma^{-1})^{ki} H_{k\ell} (\gamma^{-1})^{j\ell} F_j,$$ (50)

$$\frac{1}{2} (S - S')^i = \beta^{-1} \partial_{q_i} \left((\gamma^{-1})^{ki} H_{k\ell} (\gamma^{-1})^{j\ell}\right).$$ (51)
Having identified the forward and backward equations in both the underdamped and overdamped regimes, we proceed in Section 4 to investigate the entropy production in the environment for the underdamped system and derive a formula for its small-mass limit.

Then, in Section 5 we compute the entropy production in the environment for the overdamped system and compare it to the limit of the entropy production in the underdamped system. This allows us to identify the entropy anomaly.

Finally, Section 6 treats a special case of a uniform magnetic field, using the alternative time-inversion obtained from Eq. (38). Here, we are also able to compute the overdamped entropy production and isolate the entropy anomaly.

4 Entropy Production for Underdamped Langevin-Kramers Dynamics

In this section we compute the entropy production in the underdamped system, Eq. (4)–Eq. (5), that results from the splitting Eq. (30).

Using Eq. (27), along with the assumption that the noise only couples to the momentum, we find

\[ S_{\text{env}}^{s,t} = -\int_s^t 2\partial_{p_r} H(r, x_r) \gamma_{jl}(r, q_r) (\Sigma^{-1})^{jk}(r, q_r) \circ d(p_r)_k \]

\[ + \int_s^t 2\partial_{p_r} H(r, x_r) \gamma_{jl}(r, q_r) (\Sigma^{-1})^{jk}(r, q_r) (-\nabla_q H + \tilde{F})_k(r, x_r) - \nabla_p \cdot \tilde{F}(r, x_r) dr. \]

The fluctuation-dissipation relation, Eq. (11), yields

\[ S_{\text{env}}^{s,t} = -\int_s^t \beta(r, q_r) \partial_{p_r} H(r, x_r) \circ d(p_r)_k \]

\[ + \int_s^t \beta(r, x_r) \partial_{p_r} H(r, x_r) (-\nabla_q H + \tilde{F})_k(r, x_r) - \nabla_p \cdot \tilde{F}(r, x_r) dr. \]

The fact that \( H \in C^{1,1}, \) \( \nabla_p H \in C^{1,2}, \) where the first index refers to the \((t,q)\)-variables and the second to the \(p\)-variables, allows us to use Itô’s formula for the Stratonovich integral to obtain

\[ \beta(t, q_t)H(t, x_t) - \beta(s, q_s)H(s, x_s) \]

\[ = \int_s^t \partial_r (\beta H)(r, x_r) dr + \int_s^t \nabla_q (\beta H)(r, x_r) \cdot dp_r + \int_s^t \nabla_p (\beta H)(r, x_r) \circ dpr. \]
Therefore
\[ S_{s,t}^{\text{env}} = - (\beta(t, q_t)H(t, x_t) - \beta(s, q_s)H(s, x_s)) + \int_s^t \partial_r (\beta H) dr \] (53)
\[ + \int_s^t H_r \nabla_q \beta_r \cdot \nabla_p H_r dr + \int_s^t \partial_r \nabla_q H_r \cdot \nabla_p H_r dr \]
\[ + \int_s^t \beta_r \nabla_p H_r \cdot (-\nabla_q H_r + \bar{F}_r) - \nabla_p \cdot \bar{F}_r dr \]
\[ = - (\beta(t, q_t)H(t, x_t) - \beta(s, q_s)H(s, x_s)) + \int_s^t \partial_r (\beta H) dr \]
\[ + \int_s^t (H_r \nabla_q \beta_r + \beta_r \bar{F}_r) \cdot \nabla_p H_r - \nabla_p \cdot \bar{F}_r dr. \]

Next, we use the form of the Hamiltonian, Eq. (3), along with an additional assumption:

**Assumption 2** For the remainder of this work, we assume \( \bar{F} \) is independent of \( p \).

Recalling \( z_m^m = u_m^m / \sqrt{m} \), the entropy produced in the environment arising from the splitting Eq. (30) can be written
\[ S_{s,t}^{\text{env},m} \] (54)
\[ = - (\beta(t, q_t^m)H(t, q_t^m, z_t^m) - \beta(s, q_s^m)H(s, q_s^m, z_s^m)) + \int_s^t \partial_r (\beta V)(r, q_r^m) dr \]
\[ + \frac{1}{2} \int_s^t \partial_r \beta(r, q_r^m) \| z_r^m \|^2 dr + \frac{1}{2 \sqrt{m}} \int_s^t \| z_r^m \|^2 \nabla_q \beta(r, q_r^m) \cdot z_r^m dr \]
\[ + \frac{1}{\sqrt{m}} \int_s^t \left( (V \nabla_q \beta)(r, q_r^m) + (\beta \bar{F})(r, q_r^m) - (\beta \partial_r \psi)(r, q_r^m) \right) \cdot z_r^m dr, \]
where \( H(t, q, z) \equiv \frac{1}{2} \| z \|^2 + V(t, q) \).

### 4.1 Small-mass Limit of the Underdamped Entropy Production

Having obtained a formula, Eq. (54), for the entropy produced in the environment, we now investigate its small-mass limit. The terms in Eq. (54) of the form \( \beta H \) converge in distribution by Theorem 2. The term \( \int_s^t \partial_r (\beta V)(r, q_r^m) dr \) will be shown to converge by using Theorem 1 (i.e. because \( q_m^m \to q_t \)).

What remains are sums of integral processes of the forms
\[ \int_s^t G(r, q_r^m, z_r^m) dr + m^{-1/2} \int_s^t z_r^m \cdot K(r, q_r^m, z_r^m) dr, \] (55)
where \( G \) and \( K \) are multi-linear in \( z \) and \( K \) is even in \( z \). These can be homogenized by the limit formulas summarized below. The proofs, which are somewhat long and technical, have been collected in Appendix B. Specifically, see Assumption B1 and Theorems B5 and B6 for a precise listing of the required properties and the resulting modes of convergence. For the remainder of this paper, we will work under Assumption B1.
To state the convergence results, first define the Gibbs distribution, pointwise in \((t,q)\),

\[
h(t,q,z) = \left( \frac{\beta(t,q)}{2\pi} \right)^{n/2} e^{-\beta(t,q)\|z\|^2/2}. \tag{56}\]

Given functions \(B^{1 \cdots, k}(t,q)\), consider the family of processes

\[
J^m_{s,t} = \int_s^t B^{1 \cdots, k}(r,q^m)(z^m_{r,1} \cdots z^m_{r,k}) \, dr, \tag{57}\]

where \(0 \leq s \leq t \leq T\). The first homogenization formula is

\[
\lim_{m \to 0} J^m_{s,t} = \int_s^t B^{1 \cdots, k}(r,q_r) \left( \int h(r,q_r,z)z_{i_1} \cdots z_{i_k} \, dz \right) \, dr \equiv J_{s,t}. \tag{58}\]

The convergence is in a sup-\(L^p\) norm and the convergence rate is \(O(m^{1/2})\); see Theorem [35] for details. Intuitively, the limit is obtained by averaging the fast scaled-velocity degrees of freedom over the pointwise equilibrium distribution in \(z\). Also note that the limiting process involves only \(q_r\), the solution to the overdamped equation, Eq. (13).

When \(k\) is odd, the equilibrium average in Eq. (58) is zero and we obtain the following leading order result in Theorem [36]:

\[
\frac{1}{\sqrt{m}}E[J^m_{s,t}] = - \int_s^t E \left[ (\nabla q^r) V(r,q_r) - \partial_r \psi(r,q_r) + \tilde{F}(r,q_r) \right] \left( \int (\nabla q^r)(r,q_r,z)h(r,q_r,z) \, dz \right) \, dr + O(m^{1/2}), \tag{59}\]

where \(h\) is given by Eq. (55) and \(\psi\) is defined from \(B\) as in Eq. (13).

Using the above limit formulas, we derive the following result concerning the small-mass limit of the expected value of the entropy production.

**Theorem 3** Let \(\delta \in (0,1/2), 0 < s \leq t, \) and recall the formula for \(S^m_{s,t} \), Eq. (54). Under Assumption [37] we have

\[
E[S^m_{s,t}] = E \left[ (\nabla V)(s,q_s) \right] - E \left[ (\beta V)(t,q_t) \right] + \frac{n+2}{2} E \left[ \ln(\beta(t,q_t)/\beta(s,q_s)) \right]
\]

\[
+ \int_s^t E \left[ \partial_r (\beta V)(r,q_r) \right] \, dr - \int_s^t E \left[ (\beta^{-1}\partial_r \beta)(r,q_r) \right] \, dr
\]

\[
+ \int_s^t E \left[ (V\nabla q^r + \beta \tilde{F} - \beta \partial_r \psi) \cdot (\tilde{\gamma}^{-1} \tilde{F}) \right](r,q_r) \, dr
\]

\[
+ \int_s^t E \left[ (\beta^{-1}\partial_r q^r) \left( (V\partial q^r \beta + \beta \tilde{F}_j - \beta \partial_q \psi_j) (\tilde{\gamma}^{-1} \tilde{\gamma}) \right) (r,q_r) \right] \, dr
\]

\[
+ \int_s^t E \left[ (\beta^{-3}\partial^2 q^r \beta) \left( \frac{n-2}{2} + \delta^{kl} \delta^{ij} G^{ij} \delta_{j_{i_k}} \delta_{k_{j_i}} \right) \right] \, dr + O(m^{\delta}). \tag{60}\]

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as \( m \to 0 \), where

\[
G^{i_1 j_1 i_2 j_2 i_3 j_3}_{i_1 i_2 i_3} = \delta^{i_1 k_1} \delta^{i_2 k_2} \delta^{j_1 k_3} \delta^{j_2 k_4} \delta^{j_3 k_5} \int_0^\infty (e^{-y^2})_{i_1 k_1} (e^{-y^2})_{i_2 k_2} (e^{-y^2})_{i_3 k_3} dy. \quad (61)
\]

Note that, for the purposes of computing the operator exponential, \( \gamma \) is considered to be the linear map \( z_i \to \gamma_i \delta^i z_k \).

**Proof** Fix \( \delta \in (0, 1/2) \) and \( 0 < s \leq t \). From Eq. (54) we see

\[
E[S_{s,t}^{m,n,v,m}] = \frac{1}{2} E \left[ \beta(s, q^m_s) \| z_s^m \|^2 \right] - \frac{1}{2} E \left[ \beta(t, q^m_t) \| z_t^m \|^2 \right] + E \left[ \int_s^t \partial_r (\beta V)(r, q^m_r) dr \right] + \frac{1}{2} E \left[ \int_s^t \partial_r \beta(r, q^m_r) \| z_r^m \|^2 dr \right] + \frac{1}{2\sqrt{m}} E \left[ \int_s^t (V \nabla \beta)(r, q^m_r) + (\beta \nabla \psi)(r, q^m_r) \right] \cdot z_r^m dr.
\]

For fixed \( t \), \( \tilde{h}(q, z) = \beta(t, q) \| z \|^2 \) is \( C^1 \) in \( (q, z) \) and satisfies Eq. (20), therefore Theorem 2 gives

\[
E \left[ \beta(t, q^m_t) \| z_t^m \|^2 \right] = \left( \frac{\beta(t, q_t)}{2\pi} \right)^{n/2} \int \beta(t, q_t) \| z \|^2 e^{-\beta(t, q_t) \| z \|^2/2} dz + O(m^\delta)
\]

\[
= \left( \frac{1}{2\pi} \right)^{n/2} \int \| w \|^2 e^{-\| w \|^2/2} dw + O(m^\delta)
\]

as \( m \to 0 \). Note that the first term is independent of \( t \), and so the first two terms in Eq. (63) cancel up to order \( m^\delta \).

Our assumptions imply \( \beta V \) and \( \partial_r (\beta V) \) are \( C^1 \) with polynomially bounded zeroth and first derivatives in \( q \) (uniform in \( t \in [0, T] \)). Therefore

\[
|E[\beta(t, q^m_t) V(t, q^m_t)] - E[\beta(t, q_t) V(t, q_t)]| 
\leq E[|\tilde{C}(1 + \| q_t \| + \| q_t - q^m_t \|)\| q_t - q^m_t \|] = O(m^{1/2})
\]

and similarly,

\[
|E \left[ \int_s^t \partial_r (\beta V)(r, q^m_r) dr \right] - E \left[ \int_s^t \partial_r (\beta V)(r, q_r) dr \right]| = O(m^{1/2}).
\]

The integrands involving \( z_r^m \) are all multi-linear functions of the \( z \) variables, hence they can be handled using the results in Appendix B. If we consider the first
z-dependent term we see that $\partial_q \beta$ is $C^1$ with zeroth and first derivatives that are polynomially bounded in $q$, uniformly in $t \in [0, T]$. Therefore Theorem B5 gives

$$E \left[ \int_s^t \partial_r \beta(r, q_r^m) \|z_r^m\|^2 dr \right]$$

$$= E \left[ \int_s^t \partial_r \beta(r, q_r) \int \|z\|^2 h(r, q_r, z) dr \right] + O(m^{1/2})$$

$$= E \left[ n \int_s^t \beta^{-1}(r, q_r) \partial_r \beta(r, q_r) dr \right] + O(m^{1/2}).$$

The last two terms are proportional to $1/\sqrt{m}$. The integrands are rank 3 and rank 1 tensors respectively, evaluated on $z_r^m$ and have the required differentiability and polynomial boundedness properties to apply Theorem B5. Therefore

$$\frac{1}{2\sqrt{m}} E \left[ \int_s^t \|z_r^m\|^2 \nabla_q \beta(r, q_r^m) \cdot z_r^m dr \right]$$

$$+ \frac{1}{\sqrt{m}} E \left[ \int_s^t (V \nabla_q \beta)(r, q_r^m) + (\beta \tilde{F})(r, q_r^m) - (\beta \partial_r \psi)(r, q_r^m)) \cdot z_r^m dr \right]$$

$$= - \int_s^t E \left[ -\nabla_q V(r, q_r) - \partial_r \psi(r, q_r) + \tilde{F}(r, q_r) \right] \left( \int (\nabla_z \chi)(r, q_r, z) h(r, q_r, z) dz \right) dr$$

$$- \int_s^t E \left[ (\nabla_q \chi)(r, q_r, z) \cdot z h(r, q_r, z) dz \right] dr + O(m^{1/2}).$$

$h$ was defined in Eq. (65) and $\chi = \chi_1 + \chi_2$ where $\chi_1$ and $\chi_2$ are computed from

$$B^{i_1 i_2}_1(t, q) = \frac{1}{2} \delta^{i_1 i_2} (\nabla_q \beta)^{i_1}(t, q)$$

and

$$B^{i_2}_2(t, q) = ((V \nabla_q \beta)(t, q) + (\beta \tilde{F})(t, q) - (\beta \partial_r \psi)(t, q))^i$$

respectively, as described in Eq. (137) and the surrounding text.

The $\chi_i$ can be computed explicitly by using Lemma C3

$$\chi_1(z) = - \frac{1}{2} \delta^{i_1 i_2} (\nabla_q \beta)^{i_1} G^{j_1 j_2 j_3}_{i_1 i_2 i_3}$$

$$\times \left( z_{j_1} z_{j_2} z_{j_3} + 2 \beta^{-1} \left( \gamma_{j_1 j_2} \delta_{j_3 l} (\tilde{\gamma}^{-1})^{lk} + \gamma_{j_1 j_3} \delta_{j_2 l} (\tilde{\gamma}^{-1})^{lk} + \gamma_{j_2 j_3} \delta_{j_1 l} (\tilde{\gamma}^{-1})^{lk} \right) z_k \right),$$

where we suppress the $(t, q)$ dependence and define

$$G^{j_1 j_2 j_3}_{i_1 i_2 i_3} = \delta^{j_1 k_1} \delta^{j_2 k_2} \delta^{j_3 k_3} \int_0^\infty (e^{-y^5})_{i_1 k_1} (e^{-y^5})_{i_2 k_2} (e^{-y^5})_{i_3 k_3} dy,$$
Therefore

$$\int (\partial_t \chi)(z) h(z) dz = - (V \nabla_q \beta + \beta \tilde{F} - \beta \partial_t \psi) \delta \chi (\gamma^{-1})^{it}$$

$$= - \beta^{-1} \delta_{j_1 j_2} G^{i_1 i_2 j_3}_{i_1 i_2 i_3} \left( \delta^{i_3 i_4} (\nabla_q \beta)^{i_2} + \frac{1}{2} \delta^{i_3 i_2} (\nabla_q \beta)^{i_2} \right)$$

$$- \beta^{-1} \delta^{i_4 i_2} (\nabla_q \beta)^{i_3} G^{i_1 i_2 j_3}_{i_1 i_2 i_3} \left( \delta_{j_1 j_3} \delta_{j_2 k} (\gamma^{-1})^{kl} + 2 \gamma_{j_1 j_3} \delta_{j_2 k} (\gamma^{-1})^{kl} \right),$$

and

$$\int ((\nabla_q \chi)(z) \cdot z) h(z) dz = - \frac{1}{2} \delta^{i_1 i_2} \beta^{-2} \partial_{q^i} \left( \left( \nabla_q \beta \right)^{i_3} G^{i_1 i_2 j_3}_{i_1 i_2 i_3} \right) \left( \delta_{j_1 j_2} \delta_{j_3} + 2 \delta_{j_1 j_2} \delta_{j_3} \right)$$

$$- \delta^{i_4 i_2} \beta^{-1} \beta_{q^i} \left( \beta^{-1} (\nabla_q \beta)^{i_3} G^{i_1 i_2 j_3}_{i_1 i_2 i_3} (\gamma^{-1})^{i_1 i_2} (\gamma_{j_1 j_2} \delta_{j_3 k} + 2 \gamma_{j_1 j_3} \delta_{j_2 k}) \right)$$

$$- \beta^{-1} \beta_{q^i} \left( (V \partial_q \beta + \beta \tilde{F} - \beta \partial_t \psi) (\gamma^{-1})^{i_1 i_2} \right).$$

This proves

$$E[\Sigma^{e_n} h]$$

$$= E((\beta V)(s, q_s) - E((\beta V)(t, q_t)) + \int_s^t E[\partial_r (\beta V)(r, q_r)] dr$$

$$+ \frac{n}{2} \int_s^t E[\beta^{-1} (r, q_r) \partial_r \beta(r, q_r)] dr$$

$$+ \int_s^t E [(- \nabla_q V(r, q_r) - \partial_r \psi(r, q_r) + \tilde{F}(r, q_r)) \cdot Y_2(r, q_r)] dr$$

$$+ \int_s^t E [Y_2(r, q_r)] dr + O(m^8)$$

as $m \to 0$, where

$$Y_1 = (V \nabla_q \beta + \beta \tilde{F} - \beta \partial_t \psi) (\gamma^{-1})^{i_1 i_2}$$

$$+ \beta^{-1} \delta_{j_1 j_2} G^{i_1 i_2 j_3}_{i_1 i_2 i_3} \left( \delta^{i_3 i_4} (\nabla_q \beta)^{i_2} + \frac{1}{2} \delta^{i_3 i_2} (\nabla_q \beta)^{i_2} \right)$$

$$+ \beta^{-1} \delta^{i_4 i_2} (\nabla_q \beta)^{i_3} G^{i_1 i_2 j_3}_{i_1 i_2 i_3} \left( \delta_{j_1 j_3} \delta_{j_2 k} (\gamma^{-1})^{kl} + 2 \gamma_{j_1 j_3} \delta_{j_2 k} (\gamma^{-1})^{kl} \right)$$

and

$$Y_2 = \frac{1}{2} \delta^{i_1 i_2} \beta^{-2} \beta_{q^i} \left( \left( \nabla_q \beta \right)^{i_3} G^{i_1 i_2 j_3}_{i_1 i_2 i_3} \right) \left( \delta_{j_1 j_3} \delta_{j_2} + 2 \delta_{j_1 j_2} \delta_{j_3} \right)$$

$$+ \delta^{i_4 i_2} \beta^{-1} \beta_{q^i} \left( \beta^{-1} (\nabla_q \beta)^{i_3} G^{i_1 i_2 j_3}_{i_1 i_2 i_3} (\gamma^{-1})^{i_1 i_2} (\gamma_{j_1 j_2} \delta_{j_3 k} + 2 \gamma_{j_1 j_3} \delta_{j_2 k}) \right)$$

$$+ \beta^{-1} \beta_{q^i} \left( (V \partial_q \beta + \beta \tilde{F} - \beta \partial_t \psi) (\gamma^{-1})^{i_1 i_2} \right).$$
Using Itô's formula we can compute

\[ E \left[ \ln(\beta(t, q_t)/\beta(s, q_s)) \right] \]

\[ = \int_s^t E \left[ (\beta^{-1} \partial_q \beta)(r, q_r) \right] \, dr + \int_s^t E \left[ (\beta^{-1} \partial_q \beta) (\widetilde{\gamma}^{-1} F + S)^i \right] (r, q_r) \, dr \]

\[ + \int_s^t E \left[ (\beta^{-1} \partial_q \beta) (\widetilde{\gamma}^{-1} j_{ij}) \right] (r, q_r) \, dr + \int_s^t E \left[ (\beta^{-1} \partial_q \beta) (\widetilde{\gamma}^{-1} H_{kl}(\widetilde{\gamma}^{-1} j_{ij})^l \right] (r, q_r) \, dr. \]

Note the the last term vanishes by antisymmetry of \( H \) combined with symmetry of the derivative terms.

\( Y_1 \) and \( Y_2 \) can be simplified using the identities

\[ \frac{1}{2} \delta_{j1, j2} G_{i+1, i+2, i+3} G_{i+1, i+2, i+3} \delta_{i1, i2, i3} \]

\[ = -\frac{1}{2} \int_0^\infty \frac{d}{dy} \left[ \sum_{\alpha, \eta} (e^{-y})^{\alpha} (e^{-y})^{\alpha} (e^{-y})^{\alpha} \right] \, dy = \frac{n}{2} \delta_{i1, i2, i3}, \]

\[ \delta_{j1, j2} G_{i+1, i+2, i+3} G_{i+1, i+2, i+3} \delta_{i1, i2, i3} \delta_{i1, i2, i3} \]

\[ = -\int_0^\infty \frac{d}{dy} \left[ \sum_{\alpha, \eta} (e^{-y})^{\alpha} (e^{-y})^{\alpha} (e^{-y})^{\alpha} \right] \, dy = \delta_{i1, i2, i3}. \]

and, similarly,

\[ \delta^{i1, i2, i3} G_{i+1, i+2, i+3} G_{i+1, i+2, i+3} \delta_{i1, i2, i3} \delta_{i1, i2, i3} = \delta_{i1, i2, i3}. \]

These yield

\[ (Y_1)^t = (V \nabla \beta + \beta \dot{F} - \beta \partial_q \psi_i) \right( \widetilde{\gamma}^{-1} j_{ij}^l \right) + \frac{n + 2}{2} \beta^{-1} \partial_q \beta (\widetilde{\gamma}^{-1} j_{ij}^l \right) \]

\[ = \frac{n + 2}{2} \beta^{-2} \partial_q \beta \left[ \partial_q \beta (\widetilde{\gamma}^{-1} j_{ij}^l \right) + \beta^{-1} \partial_q \beta \left( (V \partial_q \beta + \beta \dot{F} - \beta \partial_q \psi_j) (\widetilde{\gamma}^{-1} j_{ij}^l \right) \]

\[ - \beta^{-3} \partial_q \beta (\nabla \beta (\widetilde{\gamma}^{-1} j_{ij}^l \right) \left( \delta^{i1, i2, i3} G_{i+1, i+2, i+3} G_{i+1, i+2, i+3} \delta_{i1, i2, i3} \delta_{i1, i2, i3} \right). \]

The final result, Eq. (83), is obtained by combining Eq. (75), Eq. (78), Eq. (82), and Eq. (83), after some cancellation and rearrangement.

In particular, when the vector potential vanishes we have the simplified result:
Corollary 1 Suppose $\psi = 0$ (and hence $\hat{\gamma} = \gamma$). Then, for any $0 < \delta < 1/2$:

\[
E[S_{s,t,m}^{env}] = E[(\beta V)(s, q_s)] - E[(\beta V)(t, q_t)] + \frac{n + 2}{2} E[\ln(\beta(t, q_t)/\beta(s, q_s))]
\]

\[
= \int_s^t E[\partial_r (\beta V)(r, q_r)] dr - \int_s^t E\left[\left(\beta^{-1} \partial_r \beta\right)(r, q_r)\right] dr
\]

\[
+ \int_s^t E \left[\beta^{-1}(r, q_r) \nabla_q \cdot \left(\gamma^{-1} (\beta \hat{F} + V \nabla_q \beta)\right)(r, q_r)\right] dr
\]

\[
+ \int_s^t E \left[\left((\beta \hat{F} + V \nabla_q \beta) \cdot \gamma^{-1} (-\nabla_q V + \hat{F})\right)(r, q_r)\right] dr
\]

\[
+ \int_s^t E \left[\beta^{-3} \nabla_q \beta \cdot \left(3n + 2\right) - \int_0^\infty \text{Tr} [\gamma e^{-2y\gamma} e^{-y\gamma} dy] \gamma^{-1} \nabla_q \beta\right](r, q_r) dr
\]

\[
+ O(m^\delta)
\]

as $m \to 0$.

We obtain further simplification in the case of scalar $\gamma$ (i.e. real-number valued, rather than matrix-valued).

Corollary 2 Suppose $\psi = 0$ and $\gamma$ is a scalar (still depending on $(t, q)$). Then

\[
\beta^{-3} \nabla_q \beta \cdot \left(3n + 2\right) - \int_0^\infty \text{Tr} [\gamma e^{-2y\gamma} e^{-y\gamma} dy] \gamma^{-1} \nabla_q \beta\right)(r, q_r)
\]

\[
= \frac{n + 2}{6} \beta^{-3} \gamma^{-1} \|\nabla_q \beta\|^2.
\]

5 Overdamped Entropy Production and Entropy Anomaly

Now we derive a formula for the entropy produced in the environment for the overdamped system, using the forward and backward equations Eq. (39) and Eq. (46), and compare it to the small-mass limit of the underdamped result from Theorem 3 in order to identify the entropy anomaly.

5.1 Entropy Production for Overdamped Langevin-Kramers Dynamics

We need the following assumption on $\nabla_q V$ to ensure that the formula for the entropy production is well-defined:

Assumption 3 $\nabla_q V$ is $C^2$.

With this, we can apply Eq. (27) to the splitting Eq. (47)-Eq. (48) to get

\[
S_{s,t}^{env,0} = \int_s^t 2b^j_+(r, q_r)(\hat{\Sigma}^{-1})_{jk}(r, q_r) \circ dq_k^j
\]

\[
- \int_s^t 2b^j_-(r, q_r)(\hat{\Sigma}^{-1})_{jk}(r, q_r)b^k_-(r, q_r) + \nabla \cdot b_-(r, q_r) dr.
\]
where
\[
\dot{b}_s = (\tilde{\gamma}^{-1})^j F_j - (\tilde{\gamma}^{-1})^k \dot{\partial}_k \beta^{-1} + (\partial_j (\beta^{-1}) - F_j)(\tilde{\gamma}^{-1})^k H_{k \ell} (\tilde{\gamma}^{-1})^\ell, \\
\dot{b}_q = (\tilde{\gamma}^{-1})^k H_{k \ell} (\tilde{\gamma}^{-1})^\ell F_j + \beta^{-1} \partial_q \left((\tilde{\gamma}^{-1})^k H_{k \ell} (\tilde{\gamma}^{-1})^\ell \right), \\
\bar{S}^{ij} = 2 \beta^{-1} (\tilde{\gamma}^{-1})^{ik} \gamma_{k \ell} (\tilde{\gamma}^{-1})^{\ell j}.
\] (87)

Using Itô’s formula and Eq. (13) to compute the first term in Eq. (86), and then taking expected values, we obtain, after substantial cancellation:

\[
E[S_{s,t}^{env,0}] = E[(\beta V)(s, q_s)] - E[(\beta V)(t, q_t)] + \int_s^t E[\partial_r (\beta V)(r, q_r)] dr \\
+ E[\ln(\beta(t, q_t)/\beta(s, q_s))] - \int_s^t E[\beta^{-1} \partial_r \beta(r, q_r)] dr \\
+ \int_s^t E \left[ \left( V \partial_{q_r} \beta + \beta \ddot{F}_j - \beta \partial_r \psi_j \right)(\tilde{\gamma}^{-1})^{jk} F_k \right] (r, q_r) dr \\
+ \int_s^t E \left[ \left( \beta^{-1} \partial_q \beta \left( V \partial_{q_r} \beta + \beta \ddot{F}_j - \beta \partial_r \psi_j \right) (\tilde{\gamma}^{-1})^{jk} \right) \right] (r, q_r) dr.
\] (88)

Combining Eq. (89) with Eq. (88) results in the following relation between the under and overdamped entropy production in the environment

\[
E[S_{s,t}^{env}] = E \left[ S_{s,t}^{env,0} \right] + \frac{n}{2} E \left[ \ln(\beta(t, q_t)/\beta(s, q_s)) \right] \\
+ \int_s^t E \left[ \left( \beta^{-3} \partial_{q_r} \beta \left( \frac{n}{2} \dot{\gamma}^k + \delta^{k i j} \delta^{i r} G_{i j k l} \delta_{j l} \gamma_{i r} \right) - \beta^{-1} \partial_q \beta \right) \right] (r, q_r) dr + O(m^\delta)
\] (90)

for any \(0 < \delta < 1/2\).

When \(\psi = 0\) we can simplify further to obtain:

\[
E[S_{s,t}^{env,m}] = E \left[ S_{s,t}^{env,0} \right] + \frac{n}{2} E \left[ \ln(\beta(t, q_t)/\beta(s, q_s)) \right] \\
+ \int_s^t E \left[ \left( \beta^{-3} \nabla_q \beta \left( \frac{3n + 2}{6} - \int_0^\infty Tr \left[ \gamma e^{-2\gamma y} e^{-y^\gamma} dy \right] \gamma^{-1} \nabla_q \beta \right) \right] (r, q_r) dr + O(m^\delta)
\] (90)

for any \(0 < \delta < 1/2\).

5.2 Definition of the Anomalous Entropy Production

In this section we perform a formal calculation that motivates the definition of the anomalous entropy production.
In addition to the entropy produced in the environment, Eq. (27), the diffusing particles also produce entropy [52, 41, 42, 43], defined by
\[
E\left[ S_{s,t}^{\text{part}} \right] = -E\left[ \ln(p(t,x)) \right] + E\left[ \ln(p(s,x)) \right] 
\]
\[
= -\int \ln(p(t,x)) p(t,x) dx + \int \ln(p(s,x)) p(s,x) dx, 
\]
where \( p(t,x) \) is the density of the distribution of \( x \) with respect to Lebesgue measure. Introduced in [52], the notion of change in stochastic entropy along a particular particle path is debated in the literature. However, seeing the second line of Eq. (91), one can also view this not as the expectation of a pathwise quantity, but rather as the change in entropy of the particle’s probability distribution from time \( s \) to time \( t \); the notion of the entropy of a probability distribution is a much more established concept than the pathwise definition.

Based on the convergence in distribution result, Theorem 2, one expects that the density, \( p^{m}(t,q,z) \), of the underdamped system in the variables \( (q,z) \) satisfies
\[
p^{m}(t,q,z) = \left( \frac{\beta(t,q)}{2\pi} \right)^{n/2} e^{-\beta(t,q)\|z\|^{2}/2} p^{0}(t,q) + o(1), 
\]
where \( p^{0}(t,q) \) is the density of the overdamped solution, \( q^{t} \).

Therefore, using Eq. (91) on both the over and underdamped systems, we formally obtain the relation
\[
E\left[ S_{s,t}^{\text{part},m} \right] = E\left[ S_{s,t}^{\text{part},0} \right] - \frac{n}{2} E\left[ \ln(\beta(t,q)/\beta(s,q)) \right] + o(1). 
\]
Physically, in passing to the overdamped limit one has lost (or averaged out) half of the original degrees of freedom. The entropy in the \( z \) degrees of freedom, which in the small-mass limit are locally in equilibrium, can be thought of as the source of the logarithm term in Eq. (89).

Using Eq. (93) to compare the total entropy production
\[
E\left[ S_{s,t}^{\text{tot},m} \right] \equiv E\left[ S_{s,t}^{\text{env},m} \right] + E\left[ S_{s,t}^{\text{part},m} \right] 
\]
with \( E\left[ S_{s,t}^{\text{tot},0} \right] \), we obtain another formal relation
\[
E\left[ S_{s,t}^{\text{tot},m} \right] - E\left[ S_{s,t}^{\text{tot},0} \right] = E\left[ S_{s,t}^{\text{env},m} \right] - E\left[ S_{s,t}^{\text{env},0} \right] - \frac{n}{2} E\left[ \ln(\beta(t,q)/\beta(s,q)) \right]. 
\]
This motivates the definition of the expected anomalous entropy production (or entropy anomaly):
\[
E\left[ S_{s,t}^{\text{anom}} \right] \equiv \lim_{m \to 0} E\left[ S_{s,t}^{\text{env},m} \right] - E\left[ S_{s,t}^{\text{env},0} \right] - \frac{n}{2} E\left[ \ln(\beta(t,q)/\beta(s,q)) \right]. 
\]
5.3 Entropy Anomaly

Taking the $m \to 0$ limit of Eq. (59) and using the definition Eq. (65) yields a formula for the entropy anomaly:

**Theorem 4** Under Assumptions B1 and 3, the entropy anomaly, as defined in Eq. (96), is given by

\[
E\left[ S_{\text{anom}}^{s,t} \right] = \int_{s}^{t} \left( \beta^{-3}\partial_{q}\beta \left( \frac{N^{2}}{2}\partial_{q} \right. + \delta^{kli}_{jkl} C_{t_1 t_2 t_3}^{j_1 j_2 j_3} \delta_{j_1 j_2} \tilde{z}_{j_3 t} 
\right. 
- \delta^{kli}_{jkl} C_{t_1 t_2 t_3}^{j_1 j_2 j_3} \delta_{j_1 j_2} \tilde{z}_{j_3 t} 
\left. \left( \tilde{z}^{-1}_{j_1} \delta_{q} \beta \right) \left( r, q_r \right) \right) dr.
\]

Recall that $G_{t_1 t_2 t_3}^{j_1 j_2 j_3}$ was defined in Eq. (61).

This is a new result when $\psi \neq 0$. The case $\psi = 0$ has been previously studied by various authors. We end this section by comparing our result with theirs.

When $\psi = 0$, Eq. (97) can be simplified to

\[
E\left[ S_{\text{anom}}^{s,t} \right] = \int_{s}^{t} \left( \beta^{-3}\nabla_{q}\beta \cdot \left( \frac{3n}{2} + \frac{2}{6} \right) - \int_{0}^{\infty} \text{Tr} \left[ \gamma e^{-2y\gamma} \right] e^{-y\gamma} dy \right) \gamma^{-1} \nabla_{q}\beta \left( r, q_r \right) dr.
\]

In particular, for scalar $\gamma$ the entropy anomaly is generated by the term Eq. (85), which matches Eq. (10) in [24].

More generally, when $\psi = 0$ but $\gamma$ is matrix-valued, one can diagonalize $\gamma = U \Lambda U^{T}$ where $\Lambda$ has diagonal entries $\lambda_i$. This lets us compute

\[
\left( U^{T} \left( \frac{3n}{2} + \frac{2}{6} \right) - \int_{0}^{\infty} \text{Tr} \left[ \gamma e^{-2y\gamma} \right] e^{-y\gamma} dy \right) U \right)^{ij} = \left( \frac{1}{3} + \frac{\lambda_i}{2\lambda_i + \lambda_j} \right) \delta^{ij}.
\]

Therefore the entropy anomaly when $\psi = 0$ can be equivalently written as

\[
E\left[ S_{\text{anom}}^{s,t} \right] = k_B \int_{s}^{t} \left( \frac{2}{2T} \nabla_{q}T \cdot \left( \frac{2}{3} \gamma^{-1} + \sum_{i} \left( \gamma + 2\lambda_i I \right)^{-1} \right) \nabla_{q}T \right) \left( r, q_r \right) dr,
\]

where $\lambda_i(t, q)$ are the eigenvalues of $\gamma(t, q)$. Note that the matrix in parentheses in Eq. (100) is positive definite, hence this formula proves that the entropy anomaly is non-negative when $\psi = 0$.

The physical domains covered by this paper and [33] do differ, as the latter considers particles with both translational and rotational degrees of freedom. However, in the absence of rotation, Eq. (100) agrees with the corresponding result in [33], Eq. (51c).
6 Uniform Magnetic Field Case

Finally, in this section, we explore the consequences of using a time-reversal operation other than the standard involution, Eq. (101), for a system in a uniform magnetic field.

Motivated by our previous discussion of the symmetry operation Eq. (55) from [51], we define the time-reversal, \( \tilde{\phi} \), on phase space \( (q,p) \in \mathbb{R}^3 \times \mathbb{R}^3 \), with the action
\[
(q,p) \rightarrow (-q^1,q^2,q^3,p_1,-p_2,-p_3).
\]

(101)

In part, the following assumption ensures that the system parameters are compatible with the reversal operation, Eq. (101).

**Assumption 4** In this section, we make the additional assumptions:
1. \( \gamma \) and \( \sigma \) are scalars,
2. \( \gamma \) is independent of \( q \),
3. \( \nabla_q V \) is \( C^2 \),
4. \( \sigma \) and \( V \) are invariant under \( q^1 \rightarrow -q^1 \),
5. \( \tilde{F}(t,-q^1,q^2,q^3) = (-\tilde{F}_1(t,q^1,q^2,q^3),\tilde{F}_2(t,q^1,q^2,q^3),\tilde{F}_3(t,q^1,q^2,q^3)) \),
6. \( \psi(t,q^1,q^2,q^3) = \frac{\delta(q_1)}{4\pi}(-q^2,q^1,0) \).

Note that the above vector potential results in a uniform magnetic field of strength \( B_0 \), pointing in the \( \hat{e}_3 \) coordinate direction.

With the time-inversion Eq. (101), the splitting Eq. (30), and also the above assumption, the time-reversed dynamics are
\[
dq_i' = \frac{1}{m} \left( p_i' - \psi(q_i') \right) dt,
\]
\[
dp_i' = \left( -\frac{1}{m} \gamma(t^*) \left( (p_i')_1 - \psi_i(q_i') - \partial_{q^1} V(t^*,q_i') + \tilde{F}(t^*,q_i') \right) + \frac{1}{m} \partial_{q^2} \psi_k(q_i') \delta_{k,j} \left( (p_i')_j - \psi_j(q_i') \right) \right) dt + \sigma(t^*,q_i') \delta_{i3} dW_t^3
\]
i.e. replace all explicit \( t \) dependence with \( t^* \). Note that using Eq. (26) to obtain Eq. (103), one needs to let \( W_t \) be the Wiener process obtained by flipping the sign of the \( p_2 \) and \( p_3 \) components of \( W_t \). Recall from the discussion in Section 2.1 that the time reversed SDE can be defined using any convenient Wiener process, as the choice doesn’t impact the distribution of the solution on path space and hence doesn’t impact the notion of entropy production.

Unlike Eq. (59)-Eq. (57), the SDE Eq. (102)-Eq. (103) does reflect the intuition of a time-reversal that maintains the same background environment, including the external uniform magnetic field, only with explicit time dependence reversed.

Next we investigate the overdamped limit. The natural configuration space involution inherited from Eq. (101), call it \( \tilde{\phi} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), is
\[
(q^1,q^2,q^3) \rightarrow (-q^1,q^2,q^3).
\]

(104)

Applying Theorem 1 gives the small-mass limit of the forward and backward processes respectively. We give both the Itô and Stratonovich forms:
\[
dq_t = \frac{1}{m} \left( p_t - \psi(q_t) \right) dt
\]
\[
+ \tilde{S}(t,q_t) dt + \frac{1}{m} \gamma^{-1}(t) \sigma(t,q_t) \circ dW_t
\]
\[
= \frac{1}{m} \left( p_t - \psi(q_t) \right) dt + \tilde{S}(t,q_t) dt + \left( \frac{1}{m} \gamma^{-1}(t) \sigma(t,q_t) \right) dW_t,
\]

(105)
\[ dq_t^i = \tilde{\gamma}^{-1}(t^*) ( - \nabla_q V(t^*, q_t^i) + \tilde{F}(t^*, q_t)) \ dt \]
\[ + \tilde{S}(t^*, q_t^i) dt + \tilde{\gamma}^{-1}(t^*) \sigma(t^*, q_t^i) \circ dW_t \]
\[ = \tilde{\gamma}^{-1}(t^*) ( - \nabla_q V(t^*, q_t^i) + \tilde{F}(t^*, q_t)) \ dt + \tilde{\gamma}^{-1}(t^*) \sigma(t^*, q_t^i) dW_t, \]

where

\[ \tilde{S}(t, q) = - \frac{1}{2} \sigma(t, q) \partial^2_q \sigma(t, q) (\tilde{\gamma}^{-1}(t)) d\delta_k (\tilde{\gamma}^{-1}(t))^k \xi. \]

Define

\[ dW_t \equiv [\tilde{\phi}_*(\tilde{\gamma}^{-1})^{-1}(\tilde{\gamma}^{-1})](t^*, q_t^i) dW_t. \]

Using the formulas

\[ \tilde{\gamma}_j^i \equiv \delta^k \tilde{\gamma}_k^j = \begin{pmatrix} \gamma & B_0 & 0 \\ -B_0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix} \]

and

\[ (\tilde{\gamma}^{-1})^i_j = (\tilde{\gamma}^{-1})^k_j \delta_k^i = \begin{pmatrix} \gamma/((\gamma^2 + B_0^2) & -B_0/((\gamma^2 + B_0^2) & 0 \\ B_0/((\gamma^2 + B_0^2) & \gamma/((\gamma^2 + B_0^2) & 0 \\ 0 & 0 & \gamma^{-1} \end{pmatrix} \]

we see that

\[ (\tilde{\phi}_*(\tilde{\gamma}^{-1})^{-1}) \delta^k (\tilde{\phi}_*(\tilde{\gamma}^{-1})^{-1})^k \eta = \delta^{ij}. \]

Therefore, Levy’s theorem (see p.157 in [53]) implies \( \tilde{W}_t \) is a Wiener process.

After rewriting Eq. (106) as an Itô SDE driven by \( \tilde{W}_t \) and then converting to Stratonovich form, the backward SDE becomes

\[ dq_t^i = \tilde{\gamma}^{-1}(t^*) ( - \nabla_q V(t^*, q_t^i) + \tilde{F}(t^*, q_t)) \ dt \]
\[ + \tilde{S}(t^*, q_t^i) dt + \tilde{\phi}_*(\tilde{\gamma}^{-1}) \sigma(t^*, q_t^i) \circ d\tilde{W}_t. \]

The SDEs Eq. (105) and Eq. (112) are related by the time-inversion \( \tilde{\phi} \), as in Eq. (26), when the following splitting is used:

\[ b_+ (t, q) = \begin{pmatrix} \gamma(t) \sigma(t, q) V(t, q) + \tilde{F}_1(t, q) \\ \gamma(t) \sigma(t, q) V(t, q) + \tilde{F}_2(t, q) \\ \gamma^{-1}(t) \sigma(t, q) V(t, q) + \tilde{F}_3(t, q) \end{pmatrix} \]
\[ + \begin{pmatrix} \frac{1}{2(\gamma(t) + B_0)} \sigma(t, q) \partial^2_q \sigma(t, q) \\ -\frac{1}{2(\gamma(t) + B_0)} \sigma(t, q) \partial^2_q \sigma(t, q) \\ \frac{-1}{2\gamma(t)} \sigma(t, q) \partial^2_q \sigma(t, q) \end{pmatrix}, \]

\[ b_- (t, q) = \begin{pmatrix} -B_0 \gamma(t) + B_0^2 \gamma(t) \sigma(t, q) V(t, q) + \tilde{F}_2(t, q) \\ -B_0 \gamma(t) + B_0^2 \gamma(t) \sigma(t, q) V(t, q) + \tilde{F}_3(t, q) \end{pmatrix}. \]
Eq. (27) lets us compute the entropy produced in the environment for the overdamped system in terms of the splitting Eq. (113):

\[
E[S_{s,t}^{\text{env,0}}] = E[(\beta V)(s,q_s)] - E[\beta V(t,q_t)] + E[\ln (\beta(t,q_t)/\beta(s,q_s))] + \int_s^t E[\partial_\tau (\beta V)(r,q_r)] \, dr \tag{114}
\]

\[
+ \frac{1}{2} \int_s^t E \left[ \frac{B_0}{\gamma^2(t)} \partial_{q_r}^2 (\nabla_q V + \beta \tilde{F})(r,q_r) \right] \, dr \tag{115}
\]

Turning to the underdamped entropy production, we obtain the following by applying Theorem 3 (recall that this result doesn’t depend on the choice of time-inversion). Specifically, we start from Eq. (75):

For \( \delta \in (0,1/2) \) we have

\[
E[S_{s,t}^{\text{env,m}}] = E[(\beta V)(s,q_s)] - E[(\beta V)(t,q_t)] + \int_s^t E[\partial_\tau (\beta V)(r,q_r)] \, dr
\]

\[
+ \frac{n}{2} \int_s^t E \left[ \beta^{-1}(r,q_r) \partial_\tau \beta(r,q_r) \right] \, dr \tag{116}
\]

\[
+ \int_s^t E \left[ (-\nabla_q V(r,q_r) + \tilde{F}(r,q_r)) \cdot Y_1(r,q_r) \right] \, dr
\]

\[
+ \int_s^t E \left[ Y_2(r,q_r) \right] \, dr + O(m^4)
\]

as \( m \to 0 \). The integrals in the definitions of \( Y_1 \) and \( Y_2 \), see Eq. (61), Eq. (76), and Eq. (77), can be evaluated by the fact that \( \tilde{\gamma} \) and \( \tilde{\gamma}^T \) commute, and hence

\[
e^{-y\tilde{\gamma}}(e^{-y\tilde{\gamma}})^T = e^{-2yI} = (e^{-y\tilde{\gamma}})^T e^{-y\tilde{\gamma}}. \tag{117}
\]

Here, \( Y_1^T \) is the matrix defined by Eq. (109). After simplification, we obtain the formulas

\[
(Y_1)^T = \left( \nabla_q \beta + \beta \tilde{F} \right) (\tilde{\gamma}^{-1})^{ji} \delta^{kl} \tag{118}
\]

\[
+ (n+2)\gamma \beta^{-1} \partial_{q_l} \beta ((\tilde{\gamma} + 2\gamma I)^{-1})^{ji} (\tilde{\gamma}^{-1})^{kl}
\]

\[
+ (n+2)\gamma \beta^{-1} \partial_{q_l} \beta ((\tilde{\gamma} + 2\gamma I)^{-1})^{jl} (\tilde{\gamma}^{-1})^{kl} \tag{119}
\]

and

\[
Y_2 = \frac{n+2}{2} \beta^{-2} ((\tilde{\gamma} + 2\gamma I)^{-1})^{ji} \delta^{kl} \partial_{q_l} \beta \tag{120}
\]

\[
+ (n+2)\gamma \beta^{-1} ((\tilde{\gamma} + 2\gamma I)^{-1})^{jl} (\tilde{\gamma}^{-1})^{kl} \partial_{q_l} \beta \tag{121}
\]

\[
+ \beta^{-2} \partial_{q_l} \beta (V \partial_{q_l} \beta + \beta \tilde{F}_q) (\tilde{\gamma}^{-1})^{ji}
\]
where
\[
((\dot{\gamma} + 2\gamma I)^{-1})_j^i = \begin{pmatrix}
3\gamma/(9\gamma^2 + B_0^2) & -B_0/(9\gamma^2 + B_0^2) & 0 \\
B_0/(9\gamma^2 + B_0^2) & 3\gamma/(9\gamma^2 + B_0^2) & 0 \\
0 & 0 & 1/(3\gamma)
\end{pmatrix}.
\]

Recalling the definition of the entropy anomaly, Eq. (96), and using Itô’s formula on \(\ln(\beta(t,q))\), the results Eq. (114) and Eq. (115) combine (after a long computation) to yield the entropy anomaly
\[
E[S^\text{anom}_{s,t}] = \frac{n + 2}{2} \int_s^t E \left[ \left( \beta^{-3} \nabla q \beta \cdot (\dot{\gamma} + 2\gamma I)^{-1} \nabla q \beta \right) (r,q_r) \right] dr.
\]

As it should, this expression reduces to the \(\psi = 0\) result, Eq. (85), when \(B_0 = 0\). The off-diagonal terms from Eq. (119) cancel in Eq. (120) due to antisymmetry, but the magnetic field still makes a non-zero contribution via the diagonal terms. The diagonal terms are all positive and so the above formula proves that the entropy anomaly is non-negative.

7 Conclusion

We have investigated the entropy production in underdamped Langevin-Kramers dynamics in a temperature gradient and with matrix-valued drag and magnetic field, and compared this with the overdamped limit. Specifically, Theorem 3 provides a rigorous derivation of the small-mass limit of the entropy produced in the environment for the underdamped system, including a bound on the convergence rate.

Our procedure uses previously derived rigorous convergence results for process paths (Theorem 1) and the joint distribution of position and scaled velocity (Theorem 2), together with the method of homogenizing integral processes developed in Appendices B and C. These ideas should generalize to entropy production in other stochastic systems and with time-inversion rules other than those analyzed here, as well as to further, mathematically similar, observables.

When the magnetic field vanishes and the standard phase-space time-reversal operation is used, the entropy anomaly derived by our methods, Eq. (100), matches the formally derived results in [24] and [33]. Our results generalize this formula to cover a large class of systems with magnetic fields; see Eq. (97). In addition, we investigated a special case of a uniform magnetic field using an alternative time-reversal operation. There, we were also able to derive a formula for the entropy anomaly, Eq. (120). Both of these are new results, not covered by the prior studies [24,33].

A Material from [47,49]

In this appendix, we give a list of properties that, as shown in [47,49], are sufficient to guarantee that Theorems 1 and 2 hold for the solutions to the SDE Eq. (4)-Eq. (5).

Let \(\mathcal{F}_t^W\) be the natural filtration of \(W_t\) and \(\mathcal{C}\) be any sigma sub-algebra of \(\mathcal{F}\) that is independent of \(\mathcal{F}_t^W\). Define \(\mathcal{G}_t^W,\mathcal{C}\equiv \sigma(\mathcal{F}_t^W,\mathcal{C})\) and complete it with respect to \((\mathcal{G}_t^W,\mathcal{P})\) to
form $\mathcal{F}_t^{\Omega}$. Note that $(W_t, \mathcal{F}_t^{\Omega})$ is still a Brownian motion on $(\Omega, \mathcal{F}_t^{\Omega}, P)$ and this space satisfies the usual conditions \cite{53}.

For the result Eq. (22), we relied on the assumption that our filtered probability space is $(\Omega, \mathcal{F}, \mathcal{F}_t, P) \equiv (\Omega, \tilde{\mathcal{F}}_t^{\Omega}, \tilde{\mathcal{F}}_t^{\Omega}, P)$. (121)

We also need to assume:

1. $V$ is $C^2$, $\gamma$ is $C^2$, $\psi$ is $C^3$, and, letting $\alpha$ denote a multi-index, the following are bounded:
   (a) $\nabla_{\omega} V$,
   (b) $\partial_{\alpha} \psi$ if $1 \leq |\alpha| \leq 3$,
   (c) $\partial_{\alpha} \psi_{\omega}$ if $0 \leq |\alpha| \leq 2$,
   (d) $\partial_{\alpha} \gamma$ if $1 \leq |\alpha| \leq 2$,
   (e) $\partial_{\alpha} \gamma_{\omega}$ if $0 \leq |\alpha| \leq 1$.

2. There exists $\alpha, b \geq 0$ s.t. $\tilde{V}(t, q) \equiv a + b\|q\|^2 + V(t, q)$ is non-negative.

3. There exist $C > 0$ and $M > 0$ such that

$$|\partial_t V(t, q)| \leq M + C \left(\|q\|^2 + \tilde{V}(t, q)\right).$$

(122)

4. $\gamma$ is symmetric with eigenvalues bounded below by some $\lambda > 0$.

5. $\Sigma \equiv \sigma \sigma^T$ has eigenvalues bounded below by $\mu > 0$.

6. $\gamma, \tilde{F}, \partial_{\omega} \psi$, and $\sigma$ are continuous and bounded.

7. The initial conditions satisfy the following:
   (a) There exists $C > 0$ such that the (random) initial conditions satisfy $\|u_0^m\|^2 \leq C m$ for all $m > 0$ and all $\omega \in \Omega$.
   (b) Given any $p > 0$ we have $E[\|q_0^m\|^p] < \infty$ for all $m > 0$, $E[\|q_0\|^p] < \infty$, and $E[\|q_0^m - q_0\|^p]^{1/p} = O(m^{1/2})$.

8. $\nabla_{\omega} \psi$ and $\tilde{F}$ are Lipschitz in $z$ uniformly in $t$.

9. $\sigma$ is Lipschitz in $(t, q)$.

B Homogenization of Integral Processes

In this appendix, we develop the techniques necessary to investigate the entropy production in the underdamped system, Eq. (54), in the limit $m \to 0$.

General homogenization results about the $\epsilon \to 0^+$ limit of integral processes of the form

$$\int_0^t G(s, x_s^\epsilon, z_s^\epsilon) ds,$$

where $x_s^\epsilon$ come from solving some family of Hamiltonian system parametrized by $\epsilon > 0$ (analogous to $m$), can be found in \cite{54}. Here we summarize and expand on the previous technique to derive explicit formulas for the limit in the case where the integrand is multi-linear in $z$, as well as cover processes of the form $m^{-1/2} \int_0^t \tilde{z}^m \cdot R(t, q^m, \omega^m) dt$, an important case that was not treated previously.

As a starting point, let $\chi(t, q, z) : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be $C^{1,2}$, meaning $\chi$ is $C^1$ and, for each $t, q, \chi(t, q, z)$ is $C^2$ in $z$ with second derivatives continuous jointly in all variables.

Using the definitions from Section 1.4 define the operator $L$ and its formal adjoint, $L^*$, by

$$\begin{align*}
(L\chi)(t, q, z) &= \frac{1}{2} \sum_{k=1}^n (\partial_{x_k} \partial_{x_k} \chi)(t, q, z) - \tilde{\gamma}(t, q) \delta^m_i z_i (\partial_{x_k} \chi)(t, q, z), \\
(L^* h)(t, q, z) &= (1/2) \sum_{k=1}^n \partial_{x_k} h(t, q, z) + \tilde{\gamma}(t, q) \delta^m_i z_i h(t, q, z).
\end{align*}$$

(123)

(124)

As in \cite{53}, Itô's formula can be used to compute

$$\begin{align*}
\int_0^t (L\chi)(s, q_s^m, z_s^m) ds \\
= m^{1/2} (R^m_{\alpha, \omega})_{s, t} + m \left( \chi(t, q_s^m, z_s^m) - \chi(s, q_s^m, z_s^m) + (R^m_{\alpha, \omega})_{s, t} \right).
\end{align*}$$

(125)
where we define

\[(R_m^1)_{s,t} = -\int_s^t (\nabla q \chi)(r, q_m^m, z_m^m) \cdot z_m^m dr\]  
\[\quad - \int_s^t (\nabla z \chi)(r, q_m^m, z_m^m) \cdot \left[(-\partial_r \psi(r, q_m^m) + \tilde{F}(r, q_m^m) - \nabla q V(r, q_m^m)) dr + \sigma(r, q_m^m) dW_r\right]\]  

(126)

and

\[(R_m^2)_{s,t} = -\int_s^t \partial_r \chi(r, q_m^m, z_m^m) dr.\]  

(127)

Our strategy for homogenizing processes of the form \(\int_s^t G(r, q_m^m, z_m^m) dr\) is to find a function \(\tilde{G}(t, q)\) and a \(C^{1,2}\) function \(\chi(t, q, z)\) such that

\[L \chi = G - \tilde{G}.\]  

(128)

A problem of this type is termed a cell problem. It also appears in formal asymptotic methods for solving the backward Kolmogorov equation as a series in \(\sqrt{m}\) (see Chapter 11 in [55]), as well as in rigorous homogenization results (see Chapter 18 in [55]), and so its appearance as a tool here is not too surprising.

Assuming \(\tilde{G}\) and \(\chi\) exist and don’t grow too fast in \(z\), we will be able to use Eq. (125) to prove

\[\int_s^t G(r, q_m^m, z_m^m) dr \to \int_s^t \tilde{G}(r, q_r) dr\]  

as \(m \to 0\). A solution, \(h\), to the adjoint problem, \(L^* h = 0\), with \(\int h(t, q, z) dz = 1\), gives us a formula for \(\tilde{G}\) as follows. First multiply Eq. (128) by \(h\) and integrate by parts. Assuming the boundary terms are negligible, one obtains

\[\tilde{G}(t, q) = \int h(t, q, z) G(t, q, z) dz.\]  

(130)

We will be able to make the above formal derivation rigorous under the following assumptions.

**Assumption B1** From this point on, we assume:

1. The fluctuation-dissipation relation, Assumption 2 holds.
2. The properties from Appendix A hold.
3. \(F\) is independent of \(q\).
4. \(\nabla q \beta\) and \(\tilde{F}\) are \(C^2\).
5. For any \(T > 0\) and multi-index \(\alpha\), the following are polynomially bounded in \(q\), uniformly in \(t \in [0, T]\):
   (a) \(\partial^\alpha \beta\) if \(1 \leq |\alpha| \leq 3\),
   (b) \(\partial \partial^\alpha \beta\) if \(0 \leq |\alpha| \leq 2\),
   (c) \(\partial q \beta\),
   (d) \(\partial \partial^\alpha \partial_t \beta\) if \(0 \leq |\alpha| \leq 1\),
   (e) \(\partial \partial^\alpha F\) if \(1 \leq |\alpha| \leq 2\),
   (f) \(\partial \partial^\alpha F\) if \(0 \leq |\alpha| \leq 1\),
   (g) \(\partial \partial^\alpha V\) if \(0 \leq |\alpha| \leq 1\),
   (h) \(\partial q V\) if \(|\alpha| = 2\),
   i.e. there exists \(C > 0, \tilde{p} > 0\) such that
   \[\sup_{t \in [0, T]} |\partial_t \beta(t, q)| \leq \tilde{C}(1 + \|q\|^{\tilde{p}})\]  

\[\text{and so on.}\]  

(131)
With this assumption, $L^* h = 0$ is solved by the Gibbs distribution (pointwise in $(t, q)$),

$$h(t, q, z) = \left( \frac{\beta(t, q)}{2\pi} \right)^{n/2} e^{-\beta(t,q)\|z\|^2/2}. \quad (132)$$

The integral processes we wish to homogenize are sums of multi-linear functions in $z$ i.e. they are sums of terms of the form

$$G(t, q, z) = B^{i_1, \ldots, i_k}(t, q)z_{i_1} \cdots z_{i_k}. \quad (133)$$

The solution to the cell problem, Eq. (128), for $G$'s of this form is detailed in Appendix C.

Using Lemma C3 we obtain the following general convergence result, which is used to derive entropy homogenization theorems in Section 4.1. As tools, we will primarily employ the Burkholder-Davis-Gundy inequalities, Hölder’s inequality, and Minkowski’s inequality for integrals (see, for example, Theorem 3.28 in [53] for the former, and Theorems 6.2 and 6.19 in [56] for the latter two). In essence, these are all generalizations of the triangle or Cauchy-Schwarz inequalities to (stochastic) integrals and are all used to decompose the norm of the difference between the $n$-dependent process and its purported limit into pieces, each of which we can show is negligible as $m \to 0$.

**Theorem B5** Let Assumption B1 hold, $T > 0$ and $B(t, q) : \mathbb{R} \times \mathbb{R}^n \to T^k(\mathbb{R}^n)$ (rank $k$ tensors) be $C^1$ and polynomially bounded in $m$ with polynomially bounded first derivatives, all uniformly in $t \in [0, T]$.

For $0 \leq s \leq t \leq T$, consider the family of processes

$$J^m_{s,t} = \int_s^t B^{i_1, \ldots, i_k}(r, q^m_r)(z^m_{i_1})_{q^m_{i_1}} \cdots (z^m_{i_k})_{q^m_{i_k}} dr. \quad (134)$$

Define

$$J_{s,t} = \int_s^t B^{i_1, \ldots, i_k}(r, q_r) \left( \int h(r, q_r, z)z_{i_1} \cdots z_{i_k} dz \right) dr, \quad (135)$$

where $h$ is given by Eq. (125). Then for any $p > 0$ we have

$$\sup_{0 \leq s \leq t \leq T} E \left[ \left| J^m_{s,t} - J_{s,t} \right|^p \right]^{1/p} = O(m^{1/2}) \quad (136)$$

as $m \to 0$.

**Proof** Lemma C6 implies that for each value of $B$, $\tilde{\gamma}$, and $\beta$ there exists $A_j \in T^{k-2}(\mathbb{R}^n)$, $j = 0, \ldots, [(k-1)/2]$ such that

$$\chi(z) = \sum_{j=0}^{[(k-1)/2]} A_j^{i_1, \ldots, i_k} z_{i_1} \cdots z_{i_k} \quad (137)$$

solves

$$(L \chi)(z) = B(z, \ldots, z) - \left( \frac{\beta}{2\pi} \right)^{n/2} \int B(\tilde{z}, \ldots, \tilde{z}) e^{-\beta \|\tilde{z}\|^2/2} d\tilde{z}, \quad (138)$$

where $L$ is given by Eq. (128). Considered as functions of $(\beta, \tilde{\gamma}, B)$, Lemma C6 also shows that the $A_j$ are $C^\infty$, linear in $B$, and every derivative with respect to any number of the $\beta$ and $\tilde{\gamma}$ variables is bounded by $\tilde{C}[|B|]$ for some $\tilde{C} > 0$ on any open set of the form

$$U_{\beta, R} = \{(\beta, \tilde{\gamma}, B) : \beta > \epsilon, \text{ the symmetric part of } \tilde{\gamma} \text{ has spectrum in } (\epsilon, R)\}, \quad (139)$$

where $R > \epsilon > 0$.

Assumptions I and B3 imply that $(\tilde{\gamma}(t, q), B(t, q))$ map $[0, T] \times \mathbb{R}^n$ into a region of the above form. Therefore

$$\chi(t, q, z) \equiv \chi(\tilde{\gamma}(t, q), B(t, q), z) \quad (140)$$
is $C^{1,2}$ and there exists $C, \tilde{p} > 0$ such that

\[
\sup_{t \in [0,T]} \max\{t(h(t,q,z)), t(\|t\|), \|t\|, \|\nabla t(t,q,z)\|, \|\nabla z(t,q,z)\|\} \leq \tilde{C}(1 + \|q\|)(1 + \|z\|). \tag{141}
\]

The fact that $\chi(t,q,z)$ is $C^{1,2}$ allows us to apply Eq. (142) to obtain

\[
J_m(t) = \int_t^T B^{1,\cdots,k}(r,q^m) \left( \int h(r,q^m, z,z_i \cdots z_k) dz \right) dr
\]

\[
= \int_t^T (L\chi)(r,q^m, z^m) dr
\]

\[
= m^{1/2} \left( - \int_s^t (\nabla z\chi)(r,q^m, z^m) \cdot z^m dr \right.
\]

\[
- \int_s^t (\nabla z\chi)(r,q^m, z^m) \cdot \left[ (\partial_q \psi(r,q^m) + \hat{F}(r,q^m) - \nabla q V(r,q^m) \right] dr + \sigma(r,q^m) dw_r 
\]

\[
+ m \left( \chi(t,q^m, z^m) - \chi(s, q^m, z^m) - \int_s^t \partial_q \chi(r,q^m, z^m) dr \right).
\]

Therefore, for any $\rho \geq 2$, using the Burkholder-Davis-Gundy inequalities, Minkowski's inequality for integrals, Hölder's inequality, and Assumption B1, and letting the constant $\tilde{C}$ vary from line to line, we obtain

\[
E \left[ \left| J_m(t) - \int_s^t B^{1,\cdots,k}(r,q^m) \left( \int h(r,q^m, z,z_i \cdots z_k) dz \right) dr \right|^\rho \right]^{1/\rho} \leq m^{1/2} \left( E \left[ \left| \int_s^t (\nabla z\chi)(r,q^m, z^m) \cdot \sigma(r,q^m) dw_r \right|^\rho \right]^{1/\rho}
\]

\[
+ \left[ E \left[ \left| \int_s^t (\nabla z\chi)(r,q^m, z^m) \cdot (\partial_q \psi + \hat{F}) \right|^\rho \right]^{1/\rho}
\]

\[
+ m \left( E \left[ \left| \chi(t,q^m, z^m) \right|^\rho \right]^{1/\rho}
\]

\[
+ E \left[ \left| \int_s^t \partial_q \chi(r,q^m, z^m) dr \right|^\rho \right]^{1/\rho}\right)
\]

\[
\leq \tilde{C} m^{1/2} \left( \int_s^t E \left[ \left( 1 + \|q^m\|^\tilde{p} \right)(1 + \|z^m\|^2)^p \right]^{1/\rho} dr
\]

\[
+ E \left[ \left( \int_s^t \left| (\nabla z\chi)(r,q^m, z^m) \cdot \sigma(r,q^m) \right|^{2 \rho} dr \right)^{1/\rho}\right]
\]

\[
+ \tilde{C} \left( (1 + \|q^m\|^\tilde{p})(1 + \|z^m\|^2)^p \right)^{1/\rho}
\]

\[
+ \tilde{C} \left( (1 + \|q^m\|^\tilde{p})(1 + \|z^m\|^2)^p \right)^{1/\rho}
\]

\[
\text{From this we can use Theorem [1] to find}
\]

\[
\sup_{0 \leq s \leq T} E \left[ \left| J_m(t) - \int_s^t B^{1,\cdots,k}(r,q^m) \left( \int h(r,q^m, z,z_i \cdots z_k) dz \right) dr \right|^\rho \right]^{1/\rho}
\]

\[
\leq \tilde{C} (m^{1/2}T + m(2 + T) + m^{1/2}T^1/2) \sup_{r \in [0,T]} E \left[ (1 + \|q^m\|^\tilde{p})^{1/2} \right] E \left[ (1 + \|z^m\|^2)^{2p} \right]
\]

\[
= O(m^{1/2}).
\]
We can now compute
\[
\sup_{0 \leq s \leq t \leq T} E \left[ \left| \int_0^t B^1_{i_1} \cdots i_k (r, q^m_r) \left( \int h(r, q^m_r, z) d\zeta_{i_1} \cdots d\zeta_i \right) dr \right|^{1/p} \right] \geq O(m^{1/2}) + \sup_{0 \leq s \leq t \leq T} E \left[ \left| \int_0^t B^1 \cdots i_k (r, q^m_r) \left( \int h(r, q^m_r, z) d\zeta_{i_1} \cdots d\zeta_i \right) dr \right|^{1/p} \right]
\]
\[
\leq O(m^{1/2}) + \int_0^T \left[ \left( B^1 \cdots i_k \beta^{-k/2} \right) (r, q^m_r) - \left( B^1 \cdots i_k \beta^{-k/2} \right) (r, q_r) \right]^{1/p} dr
\]
where
\[
C_{i_1 \cdots i_k} = \left( \frac{1}{2\pi} \right)^{n/2} \int e^{-\|w\|^2} w_{i_1} \cdots w_{i_k} dw.
\]  

The assumptions imply $B^1 \cdots i_k \beta^{-k/2}$ are $C^1$ with polynomially bounded first derivatives, and therefore the fundamental theorem of calculus can be used to show that
\[
\sup_{t \in [0, T]} \left[ \left( B^1 \cdots i_k \beta^{-k/2} \right) (t, q) - B^1 \cdots i_k \beta^{-k/2} (t, \tilde{q}) \right] \leq C(1 + \|q\|^p + \|q - \tilde{q}\|^p)\|q - \tilde{q}\|
\]
for some $\tilde{C}, \tilde{p} > 0$.

Therefore, again using Theorem 1, we find
\[
\sup_{0 \leq s \leq t \leq T} E \left[ \left| \int_0^t B^1_{i_k} \cdots i_k (r, q^m_r) \left( \int h(r, q^m_r, z) d\zeta_{i_1} \cdots d\zeta_i \right) dr \right|^{1/p} \right] \leq O(m^{1/2}) + \tilde{C} T \sup_{r \in [0, T]} E \left[ \left( 1 + \|q_r\|^p + \|q_r - \tilde{q}_r\|^p \right)^{2p} \right]^{1/2p} E \left[ \left( \|q^m - q_r\|^p \right)^{2p} \right]^{1/2p}
\]
\[
= O(m^{1/2}).
\]

The result for general $p > 0$ then follows from Hölder’s inequality.

**Corollary B3** If the tensor rank, $k$, is odd then $J_{s,t} = 0$ and hence
\[
\sup_{0 \leq s \leq t \leq T} E \left[ \left| \int_0^t B^1_{i_k} \cdots i_k (r, q^m_r) \left( \int h(r, q^m_r, z) d\zeta_{i_1} \cdots d\zeta_i \right) dr \right|^{1/p} \right] = O(m^{1/2})
\]
as $m \to 0$. Processes of the form $m^{-1/2} J_{s,t}^m$ for $k$ is odd do appear in the expression for the entropy production, Eq. [43]. The above corollary proves that they don’t explode in the $L^p$ norm as $m \to 0$. In fact, we will now prove that their expected values have a well behaved limit.

**Theorem B6** Let Assumption B1 hold, $T > 0$, $k$ be odd, and $B : \mathbb{R} \times \mathbb{R}^m \to T^k (\mathbb{R}^n)$ be $C^2$ with $B, \partial_i B, \partial_i \partial_j B, \partial_i \partial_j \partial_k B$, and $\partial_j B$ polynomially bounded in $q$, uniformly in $t \in [0, T] \times \mathbb{R}^n$ and consider the family of processes
\[
J_{s,t}^m = \int_0^t B^1_{i_k} \cdots i_k (r, q^m_r) (z^m_i)_{i_1} \cdots (z^m_i)_{i_k} dr
\]
for $0 \leq s \leq t \leq T$. Then, as $m \to 0$, we have
\[
\frac{1}{\sqrt{m}} E \left[ J_{s,t}^m \right] = - \int_0^t E \left[ \left( - \nabla_q V(r, q_r) - \partial_i \psi(r, q_r) + \tilde{F}(r, q_r) \right) \cdot \left( \int (\nabla_z \chi)(r, q_r, z) h(r, q_r, z) dz \right) + O(m^{1/2}) \right] dr
\]
where $h$ is given by Eq. [49] and $\chi$ is defined from $B$ as in Eq. [19].
Proof The hypotheses of Theorem 1 hold, so we can follow its proof up to Eq. (142) to obtain

\[
m^{-1/2} J_{s,t}^m = - \int_s^t (\nabla \chi \cdot \nabla m) - \int_s^t \left[ - \nabla q V(r, q^m) - \partial_r \psi(r, q^m) + \tilde{F}(r, q^m) \right] dr + \int_s^t \sigma(r, q^m) dW_r
\]

where \( \chi \) is defined in Eq. (158).

The following computation shows that

\[
M_{s,t} \equiv \int_s^t (\nabla \chi \cdot \nabla m - \sigma(r, q^m)) dW_r
\]

is a martingale (see [53]):

\[
\begin{align*}
E \left[ \int_s^t \| (\nabla \chi \cdot \nabla m - \sigma(r, q^m)) \|^2 dr \right] \\
\leq \bar{C} \| \sigma \|^2 E \left[ \int_s^t (1 + \| q^m \|)^2 (1 + \| z^m \|)^2 dr \right] \\
\leq \bar{C} \| \sigma \|^2 (t - s) \sup_{r \in [s, t]} E \left[ (1 + \| q^m \|)^4 \right]^{1/2} E \left[ (1 + \| z^m \|^k)^4 \right]^{1/2} < \infty,
\end{align*}
\]

where we used Eq. (156), Assumption B1, and Theorem 1

Therefore

\[
m^{-1/2} E[J_{s,t}^m] = - E \left[ \int_s^t (\nabla \chi \cdot \nabla m - \sigma(r, q^m)) dr \right] + O(m^{1/2}),
\]

where we used the same reasoning as in the proof of Eq. (149) to bound the last term.

\( \nabla \chi(t, q, z) \cdot (- \nabla q V(t, q) - \partial_r \psi(t, q) + \tilde{F}(t, q)) \) and \( \nabla \chi(t, q, z) \) are both finite sums of multi-linear functions of \( z \). Tracing the definition Eq. (156), one can see that each tensor in the sum is a \( O(1) \) function of \( t,q \) and has zeroth and first derivatives that are polynomially bounded in \( q \), uniformly in \( t \in [0, T] \). Therefore Theorem 3 applies to these integrals, giving

\[
E \left[ \int_s^t (\nabla \chi \cdot \nabla m - \sigma(r, q^m)) dr \right] = E \left[ \int_s^t (- \nabla q V(t, q) - \partial_r \psi(t, q) + \tilde{F}(t, q)) \cdot \left( \int (\nabla \chi)(r, q, z) b(r, q, z) dz \right) dr \right] + O(m^{1/2})
\]

and

\[
E \left[ \int_s^t (\nabla \chi \cdot \nabla m) dr \right] = E \left[ \int_s^t \int (\nabla \chi)(r, q, z) b(r, q, z) dz dr \right] + O(m^{1/2}).
\]

This completes the proof.
C The Cell Problem

This appendix details the solution to the cell problem, Eq. 125, a certain inhomogeneous linear partial differential equation that is useful for homogenizing integral processes. Specifically, we provide an explicit solution for the case where the inhomogeneity is a multi-linear function.

We will need the following lemma bounding the spectrum of a matrix. See, for example, Appendix A in [47] for a proof.

Lemma C1 Let $A$ be an $n \times n$ real or complex matrix with symmetric part $A^s = \frac{1}{2}(A + A^*)$. If the eigenvalues of $A^s$ are bounded above (resp. below) by $\alpha$ then the real parts of the eigenvalues of $A$ are bounded above (resp. below) by $\alpha$.

We will also need the following result, which solves a kind of generalized Lyapunov equation.

Lemma C2 Let $V$ be a finite dimensional vector space over $\mathbb{C}$, $C : V \to V$ be linear, and $B : V^k \to \mathbb{C}$ be multi-linear (i.e. $B \in T^k(V)$). If the eigenvalues of $C$ all have negative real parts then there exists a unique $A \in T^k(V)$ that satisfies $\sum \lambda_i C \cdot C = -B$ (i.e. for the $i$th term in the sum, the $i$th input is composed with $C$). $A$ is given by

$$A(v_1, \ldots, v_k) = \int_0^\infty B(e^{tc}v_1, \ldots, e^{tc}v_k)dt.$$  

Proof The eigenvalue bound implies the existence of $\tilde{C} > 0$, $\mu > 0$ such that $\|e^{\tilde{C}}\| \leq \tilde{C}e^{-\mu t}$, therefore the integral Eq. 158 exists. We have

$$\sum \lambda_i C \cdot C = -B(v_1, \ldots, v_k).$$  

Therefore Eq. 158 provides the desired solution.

To prove uniqueness, it suffices to show that $A = 0$ is the unique solution corresponding to $B = 0$. To this end, suppose $\sum \lambda_i C \cdot C = 0$. Let $\lambda_i$ be eigenvalues of $C$ and $e^i_j$ be a basis of generalized eigenvectors, where $\{e^i_j\}$ is a basis for the eigenspace corresponding to $\lambda_i$ and $Ce^i_j = \lambda_i e^i_j + e^i_{j-1}$ $(e^i_{-1} \equiv 0)$. Then

$$0 = \sum \lambda_i C(e^i_0, \ldots, e^i_{j-1}) = \left(\sum \lambda_i\right) A(e^i_0, \ldots, e^i_{j-1}).$$  

The coefficient is non-zero since the real parts of the $\lambda_i$ are all negative. Therefore

$$A(e^i_0, \ldots, e^i_{j-1}) = 0.$$  

We now show $A(e^i_{j_1}, \ldots, e^i_{j_k}) = 0$ for all choices of $i$'s and $j$'s. This will prove that $A = 0$ by multi-linearity and the fact that the $e^i_j$'s form a basis. We induct on $N = \sum j_i$. We showed it above for $N = 0$. Suppose it holds for $N - 1$. Given $j_i$ with $\sum j_i = N$ we have

$$0 = \sum \lambda_i A(e^i_{j_1}, \ldots, e^i_{j_k}) = \left(\sum \lambda_i\right) A(e^i_{j_1}, \ldots, e^i_{j_k}) + \sum \lambda_i A(e^i_{j_1}, \ldots, e^i_{j_k}).$$  

The last term vanishes by the induction hypothesis. As before, $\sum \lambda_i \neq 0$, hence $A(e^i_{j_1}, \ldots, e^i_{j_k}) = 0$. This proves the claim by induction.
Finally, the following lemma details the solution to the cell problem, Eq. 128.

**Lemma C3** Consider the differential operator \( L \) defined by

\[
(L\chi)(z) = \beta^{-1} \gamma z \partial_z \chi(z) - \gamma z \delta^N \partial_z \chi(z) \tag{163}
\]

where \( \gamma \), the symmetric part of \( \tilde{\gamma} \), is positive definite and \( \beta > 0 \).

Let \( k \geq 1 \) and \( B \in T^k(\mathbb{R}^n) \). For \( j = 0, \ldots, [(k-1)/2] \) define \( A_j \in T^{k-2j}(\mathbb{R}^n) \) inductively by

\[
A_0(v_1, \ldots, v_k) = - \int_0^\infty B(e^{-t\tilde{\gamma}}v_1, \ldots, e^{-t\tilde{\gamma}}v_k) dt \tag{164}
\]

and

\[
A_j(v_1, \ldots, v_{k-2j}) = \int_0^\infty 2^{j-1} \sum_{\alpha=1}^{k-2j} \sum_{\delta > \alpha} A_j^{\alpha\delta}(e^{-t\tilde{\gamma}}v_1, \ldots, e^{-t\tilde{\gamma}}v_{k-2j}) dt, \tag{165}
\]

where \( A_j^{\alpha\delta} \in T^{k-2j}(\mathbb{R}^n) \) is the multi-linear map with components \( A_j^{(1 \cdots i_k-2j)\gamma_{\alpha i_j}} \) and, for the purposes of taking the operator exponential, \( \tilde{\gamma} \) is to be thought of as the linear map with action \( z \mapsto \gamma z \delta^N z \).

Then

\[
\chi(z) = \sum_{j=0}^{[(k-1)/2]} A_j(z, \ldots, z) \tag{166}
\]

is a solution to the cell problem

\[
(L\chi)(z) = B(z, \ldots, z) - \left( \frac{\beta}{2\pi} \right)^n \int B(\tilde{z}, \ldots, \tilde{z}) e^{-\beta |\tilde{z}|^2/2} d\tilde{z}. \tag{167}
\]

Note that, if \( k \) is odd, the integral in Eq. 167 vanishes.

Consider the components \( A_j^{(1 \cdots i_k-2j)} \) to be functions of \( (\beta, \tilde{\gamma}, B) \), defined on the domain where \( \beta > 0 \), \( \tilde{\gamma} \) has positive definite symmetric part, and \( B \in T^k(\mathbb{R}^n) \). The \( A_j^{(1 \cdots i_k-2j)} \) are \( C^\infty \) jointly in all of their variables on this domain and are linear in \( B \).

Let \( U_{\epsilon, R} \) be the open set defined by \( \beta > \epsilon \) and the symmetric part of \( \tilde{\gamma} \) having eigenvalues in the interval \( (\epsilon, R) \). Given \( B \in T^k(\mathbb{R}^n) \), any order derivative (including the zeroth) of \( (\beta, \tilde{\gamma}) \to A_j^{(1 \cdots i_k-2j)}(\beta, \tilde{\gamma}, B) \) with respect to any combination of its variables is bounded by \( \hat{C}\|B\| \) on \( U_{\epsilon, R} \) for some \( \hat{C} > 0 \) (\( \hat{C} \) depends on \( \epsilon, R \), and the choice of derivatives, but not on \( B \)).

**Proof** For \( j = 0, \ldots, [(k-1)/2] \) let \( A_j \in T^{k-2j}(\mathbb{R}^n) \) be defined by Eq. 164. Eq. 165. Note that Lemma C4 implies that the real parts of the eigenvalues of \( -\tilde{\gamma} \) are negative, and hence the integrals in the definitions exist.

Define

\[
\chi(z) = \sum_{j=0}^{[(k-1)/2]} A_j(z, \ldots, z). \tag{168}
\]

We have

\[
(\partial_z A_j)(z, \ldots, z) = \sum_{\alpha=1}^{k-2j} A_j^{(1 \cdots i_k-2j)z_{\alpha-1} \cdots z_{\alpha-j} \delta^\alpha_{\alpha+1} \cdots z_{k-2j}}, \tag{169}
\]

\[
(\partial_{\tilde{z}} A_j)(z, \ldots, z) = \sum_{\alpha=1}^{k-2j-1} A_j^{(1 \cdots i_k-2j)z_{\alpha-1}} \sum_{\delta > \alpha} \left( \delta^\alpha_{\alpha+1} \delta^\delta_{\delta+1} \right) \prod_{\rho \neq \alpha, \delta} z_{\rho}, \tag{169}
\]
and so

\[(L\chi)(z) = \sum_{j=0}^{[(k-1)/2]} (\beta^{-1} \gamma \zeta_i (\partial_{x_1} \partial_{x_2} A_j)(z) - \gamma \xi \eta \zeta_i (\partial_{x_1} A_j)(z)) \quad (170)\]

\[= \sum_{j=0}^{[(k-1)/2]} \left( 2\beta^{-1} \sum_{\delta > \alpha}^{k-2(j-1)} A^{\delta \alpha}_{j-1} (z, \ldots, z) - \sum_{\alpha=1}^{k-2j} A_j(z, \ldots, z, z, z, \ldots, z) \right) , \]

where \(A^{\delta \alpha}_{j} \in T^{k-2(j+1)}(\mathbb{R}^n)\) is the multi-linear map with components \(A^{(1, \ldots, 1, 2)}_{\alpha} \) and it is the \(\alpha\)'th input of \(A_j(z, \ldots, z, \tilde{z}, z, \ldots, z)\) that equals \(\tilde{z}\) in the above sum.

Collecting terms involving tensors of the same degree, we have

\[(L\chi)(z) = B(z, \ldots, z) \quad (171)\]

\[= \sum_{j=0}^{[(k-1)/2]} \left( 2\beta^{-1} \sum_{\delta > \alpha}^{k-2(j-1)} A^{\delta \alpha}_{j-1} (z, \ldots, z) - \sum_{\alpha=1}^{k-2j} A_j(z, \ldots, z, z, z, \ldots, z) \right) \]

\[- \left( \sum_{\alpha=1}^{k} A_0(z, \ldots, z, z, z, \ldots, z) + B(z, \ldots, z) \right) \]

\[+ 2\beta^{-1} \sum_{\alpha=1}^{k-2[(k-1)/2]-1} \sum_{\delta > \alpha}^{k-2(j-1)-1} A^{\delta \alpha}_{j-1}(z, \ldots, z) . \]

Recalling the definition of \(A_0\) and \(A_j\) from Eq. (164) and Eq. (165), Lemma C2 implies that they satisfy

\[\sum_{\alpha=1}^{k} A_0(z, \ldots, z, z, z, \ldots, z) = -B(z, \ldots, z), \quad (172)\]

\[\sum_{\alpha=1}^{k-2j} A_j(z, \ldots, z, z, z, \ldots, z) = 2\beta^{-1} \sum_{\delta > \alpha}^{k-2(j-1)-1} A^{\delta \alpha}_{j-1}(z, \ldots, z), \quad j \geq 1. \quad (173)\]

Therefore

\[(L\chi)(z) = B(z, \ldots, z) = 2\beta^{-1} \sum_{\alpha=1}^{k-2[(k-1)/2]-1} \sum_{\delta > \alpha}^{k-2(j-1)-1} A^{\delta \alpha}_{j-1}(z, \ldots, z). \quad (174)\]

If \(k\) is odd then \(k - 2[(k-1)/2] = 1\) and therefore the second summation in Eq. (174) is empty. This gives \(L\chi(z) = B(z, \ldots, z)\) as claimed. If \(k\) is even then \(k - 2[(k-1)/2] = 2\) and \(A^{\delta \alpha}_{j-1}(z, \ldots, z) \in T^0(\mathbb{R}^k) = \mathbb{R}\). Therefore the right hand side of Eq. (174), call it \(\tilde{B}\), is a constant.

The value of \(\tilde{B}\) can be computed by integrating both sides against

\[h(z) = \left( \frac{\beta}{2\pi} \right)^n e^{-\beta \|z\|^2/2} . \quad (175)\]

Using the fact that \(\int h(z) dz = 1\) results in

\[\tilde{B} = \int (L\chi)(\tilde{z}) h(\tilde{z}) d\tilde{z} - \int B(\tilde{z}, \ldots, \tilde{z}) h(\tilde{z}) d\tilde{z} . \quad (176)\]

Integrating by parts, observing that the boundary terms vanish at infinity, and using \(L^* h = 0\), where \(L^*\) is the formal adjoint of \(L\) we find

\[\tilde{B} = - \int B(\tilde{z}, \ldots, \tilde{z}) h(\tilde{z}) d\tilde{z} . \quad (177)\]
as claimed.

We now prove the claimed smoothness and boundedness properties. Let $U$ be the subset of the $n \times n$ real matrices such that all of the eigenvalues of the symmetric part of the matrix are negative. This is an open set and the functions $G^{i_1 \ldots i_k}_{j_1 \ldots j_k} : U \to \mathbb{R},$

$$G^{i_1 \ldots i_k}_{j_1 \ldots j_k}(A) = \int_0^\infty (e^{tA})^{i_1}_{j_1} \ldots (e^{tA})^{i_k}_{j_k} dt,$$

(178)

are smooth and can be differentiated under the integral. Restricted to the subset where the eigenvalues of the symmetric part are less than $-\epsilon < 0,$ $G^{i_1 \ldots i_k}_{j_1 \ldots j_k}$ and its derivatives are all bounded. These facts can be proven by using the dominated convergence theorem, along with the formula for the derivative of the matrix exponential found in [57]. Therefore

$$A^{i_1 \ldots i_k}_{j_1 \ldots j_k}(\beta, \tilde{\gamma}, B) = -B^{i_1 \ldots i_k} G^{i_1 \ldots i_k}_{j_1 \ldots j_k}(-\tilde{\gamma})$$

(179)

which is linear in $B,$ smooth in $(\beta, \tilde{\gamma}),$ and it, along with its derivatives, are bounded by $\tilde{C} \|B\|,$ on the domain where the symmetric part of $\tilde{\gamma}$ has eigenvalues contained in $(\epsilon, R).$

Now, assume $A_{j=1}$ satisfies the desired properties. Then one can easily verify that

$$A^{i_1 \ldots i_k \ldots j}{j-2j}(\beta, \tilde{\gamma}, B)$$

(180)

$$= 2\beta^{-1} \sum_{\alpha=1}^{k-2(j-1)} \sum_{\delta > \alpha} A^{i_1 \ldots i_{\delta-1} \delta \alpha_{\delta-2} \ldots i_{k-2j-1} \ldots i_k}_{j_1 \ldots j_k}(\beta, \tilde{\gamma}, B) \frac{1}{2} (\tilde{\gamma}_{\delta \alpha} + \tilde{\gamma}_{\alpha \delta}) G^{i_1 \ldots i_{k-2j}}_{j_1 \ldots j_{k-2j}}(-\tilde{\gamma})$$

does as well. Therefore the claim holds for all $j$ by induction.

Acknowledgments

Many thanks to J. Wehr for bringing this problem to my attention and for numerous stimulating discussions.

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