Classical Dynamics of Anyons and the Quantum Spectrum

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In this paper we show that (a) all the known exact solutions of the problem of N-anyons in oscillator potential precisely arise from the collective degrees of freedom, (b) the system is pseudo-integrable ala Richens and Berry. We conclude that the exact solutions are trivial thermodynamically as well as dynamically.

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I. INTRODUCTION

Non-relativistic quantum mechanics in two space dimensions admits the possibility of fractional statistics and particles obeying fractional statistics are known as “anyons”. Anyons are objects defined in the quantum framework described by a Lagrangian of the form,

\[ L = \frac{1}{2} \sum_{i=1}^{N} \dot{\vec{r}}_i^2 + \alpha \sum_{i<j} \dot{\theta}_{ij} - V(\vec{r}_i); \quad \theta_{ij} = \tan^{-1}\frac{y_i - y_j}{x_i - x_j}, \]

where \(\vec{r}_i\) denote particle coordinates and \(V(\vec{r}_i)\) is some confining potential which we choose to be the harmonic oscillator potential \(V(\vec{r}_i) = \sum_{i=1}^{N} \vec{r}_i^2\). The \(\alpha\) dependent term is the statistical interaction. The harmonic oscillator potential is convenient since the dynamics of the system is well understood in the limit \(\alpha = 0\). We may choose to describe the system of anyons either through multivalued wavefunctions which occur naturally in the quantum mechanics on multiply connected spaces or equivalently in terms of single valued wave functions in the presence of statistical interaction. Quantum mechanically a system of \(N\)-anyons confined in a harmonic oscillator potential is an exceptional system since many exact solutions to the energy eigenvalue problem are known even when the many body Hamiltonian is nonseparable.

The known results about the spectrum of \(N\)-anyons confined in oscillator potential fall into two categories: (a) Exact eigenvalues which are linear functions of the statistical parameter \(\alpha\), \(E_{jm} = 2m + |j - \alpha \frac{N(N-1)}{2}| + N\), where \(j\) and \(m\) denote the angular and radial quantum numbers, and (b) eigenvalues that are nonlinear functions of \(\alpha\) as evidenced by numerical calculations for \(N = 3, 4\) anyons and also meanfield calculations for large \(N\). The special case in this context is \(N = 2\) where the spectrum is completely subsumed by the category (a). The nonlinear spectrum displays many level crossings, some of which are of the Landau-Zener type(avoided crossings). This has lead to the conjecture that the anyon system may be nonintegrable (or even chaotic) for \(N > 3\).

Two comments are in order here: Firstly, what is the reason for the existence of exact solutions of the type (a) in this nonseparable many body system? Infact the approach of
Basu et al. [7] shows that \( j=0 \) exact solutions form a subspace of the full Hilbert space which block diagonalises the Hamiltonian. This indicates that the system may be atleast partially separable. The origin of this has not been elaborated yet to the best of our knowledge. The second comment pertains to the “integrability” of the system. If one considers the classical Lagrangian, the statistical interaction is a total derivative. The Euler-Lagrange equations of motion are therefore the same as that of an \( 2N \)-dimensional oscillator and this of course is an integrable system. Why do then the numerical results suggest nonintegrability?

In this paper we elaborate on and answer the above questions. In section 2, we briefly mention the known exact solutions to the quantum many body problem. In section 3 we outline the classical Lagrangian formulation of \( N \)-anyons and prove the system is “integrable” in the Liouville sense, i.e., it admits \( 2N \) constants of motion in involution. In section 4 we exhibit a choice of coordinates at classical level which gives partial separability. From this we can isolate a set of collective coordinates whose semiclassical quantisation leads to the known exact solutions. These solutions correspond to the periodic orbits in the phase space of the oscillator which continue to remain periodic even in the presence of anyonic interaction. In section 5 we consider the integrability aspect of the system. We argue that even though we have \( 2N \) constants of motion in involution there exist invariant surfaces which do not have the topology of \( 2N \) dimensional torus. Following Berry-Richens [13] we conclude that the many anyon system confined in oscillator potential is pseudointegrable.
II. EXACT SOLUTIONS

The quantum mechanical Hamiltonian corresponding to the Lagrangian in Eq.(1) is given by,

\[ H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{1}{2} \sum_{i=1}^{N} r_i^2 - \alpha \sum_{j>i=1}^{N} \ell_{ij} + \frac{\alpha^2}{2} \sum_{i \neq j, k} \frac{r_{ij}r_{ik}}{r_{ik}^2}, \]

where \( \ell_{ij} = (\vec{r}_i - \vec{r}_j) \times (\vec{p}_i - \vec{p}_j). \)

and all distances have been expressed in units of \( 1/\sqrt{m\omega} \), where \( m \) is the mass of the particle and \( \omega \) is the oscillator frequency. Notice that the statistical interaction is independent of the centre of mass. We now briefly discuss the class of exact solutions already known. For discussing the known class of exact solutions it is convenient to use the complex coordinates \( z_i, \bar{z}_i \) in terms of which the Hamiltonian takes the form,

\[ H = -2 \sum_i \partial_i \bar{\partial}_i + \frac{1}{2} \sum_i z_i \bar{z}_i - \alpha \sum_{i<j} \left( \frac{\partial_{ij}}{z_{ij}} - \frac{\bar{\partial}_{ij}}{\bar{z}_{ij}} \right) + \frac{\alpha^2}{2} \sum_{i \neq j, k} \frac{1}{z_{ij}z_{ik}}, \]

where \( \partial_i = \partial/\partial z_i; \bar{\partial}_j = \partial - \partial_j \), etc., and the eigenvalue equation is,

\[ H\psi(z_i, \bar{z}_i) = E\psi(z_i, \bar{z}_i). \]

The conserved angular momentum \( J \) with eigenvalues denoted by \( j \), is given by,

\[ J = z_i \partial_i - \bar{z}_i \bar{\partial}_i. \]

Defining variables,

\[ X = \prod_{i<j} z_{ij}; \quad t = \sum_{i=1}^{N} z_i \bar{z}_i, \]

we may classify the known exact solutions as follows:

(a) \( j < 0 \) solutions-

\[ \psi_j = |X|^j \phi_j(\bar{z}_i)e^{-t/2}; \quad j < 0 \]

4
with the energy eigenvalues given by,

\[ E = N - j + \alpha \frac{N(N - 1)}{2}. \]  

(8)

Here and in what follows \( \phi_j \) generically denotes an eigenfunction of the angular momentum operator \( J \) with eigenvalue \( j \). For elucidating the solutions the specific form of \( \phi \) is irrelevant.

(b) \( j = 0 \) solutions-

\[ \psi_0 = |X|^\alpha \phi_0(t)e^{-t/2}; \quad j = 0. \]  

(9)

with \( \phi_0(t) = \sum_{k=1}^{m} C_k t^k \) a polynomial of degree \( m \) in \( t \). The corresponding energy eigenvalues are given by,

\[ E = N + 2m + \alpha \frac{N(N - 1)}{2}, \]  

(10)

The second solution is necessarily bosonic since \( t \) is symmetric, whereas the first solution needs explicit symmetrization and antisymmetrization of the wavefunction in terms of \( \bar{z}_i \) to obtain the bosonic and fermionic wavefunctions. Since this is always possible, the degeneracy of the first type solution is exactly the same for both bosonic and fermionic type solutions for any given angular momentum \( j \) \((< 0)\). We may also take a combination (product) of the solutions the types discussed above, to get further \( j < 0 \) solutions. This then generates a infinite tower radial excitations for each value of \( j \).

(c) \( j > 0 \) solutions-

\[ \psi_j = |X|^{-\alpha} \phi_j(z_i)e^{-t/2}; \quad j > 0 \]  

(11)

with the energy eigenvalues given by,

\[ E = N + j - \alpha \frac{N(N - 1)}{2}. \]  

(12)

Caution must be exercised in choosing the value of \( j \) for these solutions since the wave function is not square integrable for all values of \( j \). Infact the lower bound is obtained requiring that the wave function be square integrable over the whole domain of \( \alpha (0 \leq \alpha \leq 1) \).
This means that \( j > (N - 1)(N - 2)/2 \). If however this condition is not satisfied then the wave function remains regular only for some values of \( \alpha \) \( (0 \leq \alpha \leq 2j/N(N - 1)) \) but not for all \( 0 \leq \alpha \leq 1 \) which gives rise to the so called noninterpolating solutions which have also been discussed in the literature. [12]

All these solutions for the energy eigenvalues have a linear dependence on \( \alpha \) with a coefficient \( \pm N(N - 1)/2 \) while the corresponding eigenfunctions are finite order polynomials apart from the overall \( |X|^{\pm\alpha} \) and the Gaussian factors. These solutions (a)-(c) cover all the known exact solutions. However it is by now known that these exact solutions form only a subset of the full Hilbert space and existence of nonlinear solutions has been shown numerically as well as through meanfield calculations. It is our aim here to understand the reason for the existence of this dichotomy.

### III. CONSTANTS OF MOTION

We begin with the analysis of the classical Lagrangian given by Eq.(1). It is convenient to write the Lagrangian in the form,

\[
L = \frac{1}{2} \sum_{i=1}^{N} \left[ \dot{\vec{r}}_i^2 - \vec{r}_i^2 \right] + \alpha \sum_{i<j} \vec{r}_{ij} \times \vec{\dot{r}}_{ij},
\]

(13)

where the dots indicate the time derivatives. The first step is to introduce relative coordinates and separate the trivial centre of mass degree of freedom. To this end we write,

\[
\vec{\rho}_a = \left[ \frac{1}{\sqrt{a(a+1)}} \sum_{k=1}^{a} \vec{r}_k - \sqrt{\frac{a}{a+1}} \vec{r}_{a+1} \right]; \quad a = 1, \ldots, N - 1
\]

(14)

with inverse relation given by,

\[
\vec{r}_i = \left[ -\sqrt{\frac{i-1}{i}} \vec{\rho}_{i-1} + \sum_{k=i}^{N-1} \frac{\vec{\rho}_k}{\sqrt{k(k+1)}} \right] + \vec{R}_{cm} \equiv A_i^a \vec{\rho}_a + \vec{R}_{cm},
\]

(15)

where \( \vec{\rho}_a, a = 1, \ldots, N - 1 \) are dimensionless relative coordinates and \( \vec{R}_{cm} \) is the centre of mass coordinate. It follows that,

\[
\sum_i A_i^a = 0; \quad \sum_i A_i^a A_i^b = \delta^{ab}.
\]
It is straightforward to see that,

\[ L = L_{CM} + L_{rel}, \]

where

\[ L_{CM} = \frac{1}{2} \left[ \dot{\mathbf{R}}_{CM}^2 - \mathbf{R}_{CM}^2 \right], \]

\[ L_{rel} = \frac{1}{2} \left[ \dot{\rho}_a^2 - \rho_a^2 \right] + \alpha \sum_{i<j} A_{ai} A_{bj} \rho_{ai} \times \tilde{\rho}_b, \]

where \( A_{ij} = A_i^a - A_j^b \). From now on we concentrate only on the \( L_{rel} \) and drop the subscript.

It is easy to see that the Euler-Lagrange equations of motion are,

\[ \ddot{\rho}_a = -\rho_a. \]  

(16)

There is no \( \alpha \) dependence in these equations since the corresponding term is a total derivative. From these equations it follows that

\[ E_a = \frac{1}{2} \left[ \dot{\rho}_a^2 + \rho_a^2 \right]; \quad l_a = \rho_a \times \tilde{\rho}_a; \quad a = 1, \ldots, N - 1 \]  

(17)

are constants of motion. In the Hamiltonian formulation the \( \dot{\rho}_a \) are expressed in terms of conjugate momenta and coordinates which do contain the \( \alpha \) dependence,

\[ P_{ax} = \frac{\partial L}{\partial \dot{\rho}_{ax}} = \dot{\rho}_{ax} - \alpha A_{ax}; \quad A_{ax} = \sum_{i<j} A_{ai} A_{bj} \rho_{bi} \rho_{aj}, \]  

(18)

\[ P_{ay} = \frac{\partial L}{\partial \dot{\rho}_{ay}} = \dot{\rho}_{ay} + \alpha A_{ay}; \quad A_{ay} = \sum_{i<j} A_{ai} A_{bj} \rho_{bx} \rho_{aj}, \]  

(19)

and the constants of motion in relative coordinates are,

\[ E_a = \frac{1}{2} \left[ (\dot{P}_{ax} + \alpha A_{ax})^2 + (\dot{P}_{ay} - \alpha A_{ay})^2 + \rho_a^2 \right] \]  

(20)

\[ l_a = \rho_a \times \tilde{P}_a - \alpha (\rho_{ax} A_{ay} + \rho_{ay} A_{ax}). \]  

(21)

Clearly the Hamiltonian is given by

\[ H = \sum_{a=1}^{N-1} E_a. \]
Note that the $\alpha$-dependent term in $L$ is well defined only if $\vec{r}_{ij} \neq 0$ for all $i,j$. That is only if

$$A^a_{ij} A^b_{ij} \vec{r}_{a} \cdot \vec{r}_{b} \neq 0 \ \forall \ i,j$$

Consequently the expressions for $\mathcal{E}, l_a$ and $H$ are also valid only if $\vec{r}_{ij}$ is not zero. In effect the classical configuration space on which $L$ is well defined is the space

$$Q^{N-1} = R_2^{N-1} - \Delta; \ \Delta = \{ \vec{r}_a / A^c_{ij} A^d_{ij} \vec{r}_c \cdot \vec{r}_d = 0 \text{ for some pair(s) } i,j \}. \quad (22)$$

The space $Q^{N-1}$ is not simply connected. Its fundamental group is the same as the fundamental group of $\{ R_2 - (N-1) \text{ points} \}$ which is known to be nontrivial. For $N = 2$, $\pi_1 = Z$ while for $N \geq 3$, $\pi_1$ is nonabelian.

The corresponding phase space is the cotangent bundle of $Q$ on which $\mathcal{E}_a, l_a$ and $H$ are well defined. This phase space is topologically nontrivial and for our purposes we do not need the full machinery for handling this topologically nontrivial phase space. It is sufficient to pretend that the full space is still $R_2^{N-1}$ but simply avoid coincident points.

It is straightforward then to prove the following Poisson bracket relations,

$$\{ \mathcal{E}_a, \mathcal{E}_b \} = \{ l_a, l_b \} = \{ \mathcal{E}_a, l_a \} = 0 \ \forall \ a, b = 1, ..., N - 1, \quad (23)$$

$$\{ H, \mathcal{E}_a \} = \{ H, l_a \} = 0. \quad (24)$$

Thus for $4(N - 1)$ degrees of freedom $P_a, \rho_a$ we have $2(N - 1)$ constants of motion in involution. So a necessary condition for a system to be integrable is satisfied. We will tentatively refer to this system as being integrable (or potentially integrable) in the “Liouville sense”.

IV. COLLECTIVE MODES

The anyonic interaction term in $L$ is invariant under two sets of time independent transformations:

$$\vec{r}_a \rightarrow \vec{r}_a = R(\theta) \vec{r}_a \ \forall \ a, \quad (25)$$

$$\vec{r}_a \rightarrow \vec{r}_a = \lambda \vec{r}_a \ \forall \ a, \quad (26)$$
where $R(\theta)$ denotes a rotation of the vector by an angle $\theta$ in two dimensions. Therefore for any given configuration $\{\vec{\rho}_a\}$ at any given time $t$ we can always rotate the axes so that $\rho_{1y} = 0$ (say). Thus by making a time dependent rotation we can ensure that $\eta_{1y} = 0$ for all $t$. Therefore define,

$$
\vec{\rho}_a = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \eta_a,
$$

(27)

The following identities are easy to prove:

$$
\vec{\rho}_a \cdot \vec{\rho}_b = \vec{\eta}_a \cdot \vec{\eta}_b,
$$

(28)

$$
\vec{\rho}_a \times \vec{\rho}_b = \vec{\eta}_a \times \vec{\eta}_b,
$$

(29)

$$
\dot{\vec{\rho}}_a = \vec{\eta}^2_a + 2\dot{\eta}_a \times \dot{\vec{\eta}}_a + \dot{\theta}^2 \eta_a^2,
$$

(30)

$$
\vec{\rho}_a \times \dot{\vec{\rho}}_b = \dot{\theta} \vec{\eta}_a \cdot \vec{\eta}_b + \vec{\eta}_a \times \dot{\vec{\eta}}_b.
$$

(31)

Now the set $\{\eta_a; a = 1, ..., N - 1\}$ is effectively a $2(N - 1) - 1$ dimensional vector since $\eta_{1,y} = 0$ for all time $t$. We can therefore introduce the standard spherical coordinates by the usual procedure,

$$
\vec{\eta}_a \equiv R \vec{\xi}_a; \quad \xi_{1y} = 0; \quad \sum_a \xi^2_a = 1,
$$

(32)

where

$$
\xi_{N-a,x} = s_1 s_2 ... s_{2a-2} c_{2a-1},
$$

(33)

$$
\xi_{N-a,y} = s_1 s_2 ... s_{2a-1} c_{2a},
$$

(34)

$$
\xi_{1,x} = s_1 s_2 ... s_{2(N-4)},
$$

(35)

$$
\xi_{1,y} = 0,
$$

(36)

where $a = 2, ..., N - 1$ and $s_\mu = \sin \theta_\mu, c_\mu = \cos \theta_\mu$. In terms of these variables the Lagrangian may be rewritten as

$$
L = \frac{1}{2} [\dot{R} - R^2 + R^2 \dot{\theta}^2 + 2R^2 \dot{\theta} \sum_a \vec{\xi}_a \times \dot{\vec{\xi}}_a + R^2 \sum_a \dot{\vec{\xi}}_a^2] + \alpha \frac{N(N - 1)}{2} \dot{\theta} + \alpha \sum_{i<j} A^a_{ij} A^b_{ij} \vec{\xi}_a \times \dot{\vec{\xi}}_b,
$$

(37)
Since $\eta_a$ are obtained from $\rho_a$ by the same rotation matrix for all $a$, the angle $\theta(t)$ clearly describes a collective rotation of all the N-anyons about the centre of mass (say). The anyonic interaction term (a total time derivative) is manifestly independent of $R(t)$. This may also be regarded as a collective mode as discussed below. It is the semiclassical quantisation of these two modes that yields the exactly known energy eigenvalues. To elaborate these points let us consider the Euler-Lagrange equations of motion. Since the $\alpha$ dependent part of the Lagrangian is a total time derivative we can ignore it for analysing the classical equations of motion. Clearly these equations are identical to that of the oscillator equations of motions as given in Eq.(16). Translating this into the spherical coordinates we obtain,

$$\left[\ddot{R} + R - R\dot{\theta}^2\right]\vec{\xi}_a - \left[R\ddot{\theta} + 2R\dot{\theta}\right]V\vec{\xi}_a + 2\left[R - R\dot{\theta}V\right]\dot{\vec{\xi}}_a + R\dddot{\vec{\xi}}_a = 0,$$

(38)

where

$$V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and in the matrix equation above the $\vec{\xi}$ is a column vector with two elements. The matrix $V$ between two vectors essentially generates their cross product. Taking the dot product with $\vec{\xi}_a$ and summing over $a$ we find,

$$\left[\ddot{R} + R - R\dot{\theta}^2\right]\sum_a \vec{\xi}_a \times \dot{\vec{\xi}}_a + R \sum_a \vec{\xi}_a^2 = 0,$$

(39)

where we have made use of the identities,

$$\sum_a \vec{\xi}_a \cdot \dot{\vec{\xi}}_a = 0, \quad \sum_a \vec{\xi}_a \cdot \ddot{\vec{\xi}}_a = -\sum_a \vec{\xi}_a^2$$

. Taking the cross product with $\dot{\vec{\xi}}_a$ and summing over $a$ we find,

$$\left[R\ddot{\theta} + 2R\dot{\theta}\right] = -2R \sum_a \vec{\xi}_a \times \dot{\vec{\xi}}_a - R \sum_a \vec{\xi}_a \times \dddot{\vec{\xi}}_a.$$

(40)

Using the above two relations Eq.[38] can be rewritten as,

$$R\dddot{\vec{\xi}}_a + 2\left[R - R\dot{\theta}V\right]\dot{\vec{\xi}}_a + \left[2R\dot{\theta}\vec{\xi}_b \times \ddot{\vec{\xi}}_b + R\vec{\xi}_b^2\right]\vec{\xi}_a + \left[2R\vec{\xi}_b \times \ddot{\vec{\xi}}_b + R\vec{\xi}_b \times \dddot{\vec{\xi}}_b\right]V\vec{\xi}_a = 0,$$

(41)

where sum over $b$ is assumed. Here the equations motion for $R$ and $\theta$ are still coupled to all the other internal coordinates and as yet they are not collective coordinates. We therefore
need to impose a set of initial conditions which may lead to the separation of these two modes as collective modes from the internal variables \( \theta_\mu \). To realise this let at time \( t = 0 \), all velocities \( \dot{\vec{\xi}}_a(t = 0) = 0 \) for all \( a \). Then the above equations reduce to,

\[
R[\ddot{\vec{\xi}}_a + (\sum_b \vec{\xi}_b \times \ddot{\vec{\xi}}_b) V \vec{\xi}_a] = 0, \tag{42}
\]

\[
\ddot{R} + R - R\dot{\theta}^2 = 0, \tag{43}
\]

\[
R\ddot{\theta} + 2R\dot{\theta} = -R \sum_b \vec{\xi}_b \times \ddot{\vec{\xi}}_b. \tag{44}
\]

Now consider the first equation for \( R \neq 0 \) and \( a = 1 \). From the initial conditions it is obvious that,

\[
\ddot{\xi}_{1x} = 0; \quad \xi_{1x} \sum_b (\vec{\xi}_b \times \ddot{\vec{\xi}}_b) = 0
\]

because \( \xi_{1y} = 0 \) for all \( t \).

Now if \( \xi_{1x} \) is nonzero, then \( \sum_b \vec{\xi}_b \times \ddot{\vec{\xi}}_b = 0 \), and hence

\[
\ddot{\vec{\xi}}_b = 0 \quad \forall \quad b.
\]

Since the equations of motion are second order in \( t \), we have proved that: if \( \ddot{\vec{\xi}}_a(0) = 0 \) \( \forall \quad a \), and \( R(0), \xi_{1x}(0) \) are non zero then \( \forall \ t \),

\[
\vec{\xi}_a(t) = \vec{\xi}_a(0),
\]

\[
\ddot{R} + R - R\dot{\theta}^2 = 0,
\]

\[
R\ddot{\theta} + 2R\dot{\theta} = 0.
\]

For oscillator \( (\alpha = 0) \) \( R(0) \) and \( \xi_{1x}(0) \) being non zero is a special class of initial condition. Indeed in general for a second order equation system if both first and second order derivatives vanish at any \( t \) then this can hold only for some restricted class of coordinate values. For the equations of motions considered in the configuration space \( R^{N-1}_2 \) the restricted class is precisely characterised by \( R(0) \neq 0 \) and \( \xi_{1x}(0) \neq 0 \) However when \( \alpha \neq 0 \) the configurations space is \( Q^{N-1} \) which does not contain points which have \( R = 0 \) and \( \xi_{1x} = 0 \). Thus there are no restrictions on the initial conditions in \( Q^{N-1} \).
The above initial conditions amount to freezing the “internal motion” of the anyons. The remaining motion is a collective motion described by $R(t)$ and $\theta(t)$ which is just a $2(N-1)$ dimensional oscillator. $R(t)$ being nonzero implies that the angular momentum must be nonzero. This explains in what sense $R(t)$ can be interpreted as a collective mode. For describing the motion of collective mode the effective Lagrangian is

$$L_{\text{eff}} = \frac{1}{2} [\dot{R}^2 + R^2 \dot{\theta}^2 - R^2] + \alpha \frac{N(N-1)}{2} \dot{\theta}. \quad (45)$$

This is identical to the Lagrangian in the relative coordinate for a two anyon system with $\alpha \rightarrow \alpha \frac{N(N-1)}{2}$ and in $2(N-1)$ dimensions. Semiclassical quantisation will then reproduce all exactly known energy eigenvalues summarised in Sect.2. In this effective Lagrangian picture the only known memory of $N$ resides in the coefficient of $\alpha$. This can be understood by noting that when all the $N$-anyons are rotated by $2\pi$ about the centre of mass they also circle each other to pick up the extra phase.

If on the otherhand we consider $\dot{R} = \dot{\theta} = 0$ at $t = 0$ class of initial conditions then we see that $\ddot{R}, \ddot{\theta}$ are nonzero and hence $R(t), \theta(t)$ do depend on $\theta_\mu$. Thus the collective motion is not fully decoupled from the “internal motion”. We therefore refer to this system as partially separable.

Incidentally the same conclusions can be drawn from the Hamiltonian formulation. For completeness we give the relevant expressions and arguments. The conjugate momenta for the hyperspherical variables are given by,

$$P_R = \dot{R}, \quad (46)$$

$$P_\theta = R^2 \dot{\theta} + R^2 \sum_{\mu} \dot{\theta}_\mu F_\mu + \alpha N(N-1), \quad (47)$$

$$P_\mu = R^2 r_\mu^2 \dot{\theta}_\mu + R^2 \dot{\theta} F_\mu + \alpha G_\mu, \quad (48)$$

where,

$$r_\mu = s_1 s_2 \ldots s_{\mu-1}, \quad \mu = 1, \ldots, 2N-3, \quad r_1 = 1$$

and $F_\mu$ and $G_\mu$ are defined through,

$$\sum_{\mu=1}^{2N-4} \dot{\theta}_\mu F_\mu \equiv \sum_{a=1}^{N-1} \xi_a \times \dot{\xi}_a$$
\[
\sum_{\mu=1}^{2N-4} \dot{\theta}_\mu G_\mu \equiv \sum_{i<j} A^a_{ij} A^b_{ij} \hat{\xi}_a \times \hat{\xi}_b.
\]

The Hamiltonian is then given by

\[
H = H_1 + H_2,
\]

where

\[
H_1 = \frac{1}{2}[P^2_R + R^2] + \frac{(P_\theta - \alpha \frac{N(N-1)}{2})^2}{2R^2},
\]

\[
H_2 = \frac{1}{2R^2} \left[ \sum_\mu \left( \frac{P_\mu - \alpha G_\mu}{r_\mu^2} \right)^2 + \frac{(P_\theta - \frac{\alpha N(N-1)}{2} - \sum_\mu \frac{F_\mu (P_\mu - \alpha G_\mu)}{r_\mu^2})^2}{1 - \sum_\mu F_\mu^2 r_\mu^2} - (P_\theta - \alpha \frac{N(N-1)}{2})^2 \right].
\]

It is easy to see that for the special initial conditions \( \dot{\theta}_\mu = 0, \) \( H_2 = 0. \) Also the Poisson bracket \( \{H_1, H_2\} \propto H_2 \) vanishes for these initial conditions. The corresponding quantum statement would \( \{H_1, H_2\} \propto H_2, \) and therefore if we consider the subspace \( \{\psi\} \) on which \( H_2 \psi = 0 \) then \( H_1 \) will act invariantly on this subspace. Further \( H = H_1 \) on this subspace and thus the eigenstates of \( H_1 \) will be exact eigenstates of the full system and conversely. But the problem of solving \( H_1 \) is analogous to that of two anyon problem with \( \alpha \rightarrow \alpha \frac{N(N-1)}{2}. \) Therefore these solutions are given by,

\[
E = 2m + |j - \alpha \frac{N(N-1)}{2}| + (N-1),
\]

\[
\psi = CR^{\left|j-\alpha\right|} \exp^{-R^2/2} f(R),
\]

\[
H_1 \psi = E \psi
\]

where \( f(R) \) is some polynomial of degree \( 2m \) in \( R. \) In general the normalisation constant \( C \) in \( \psi \) will have dependence on \( \theta_\mu \)'s, restricted by \( H_2 \psi = 0. \) These are precisely all the known solutions and the above argument shows that there are no more solutions satisfying \( H_1 \psi = E \psi \) and \( H_2 \psi = 0. \) Thus \( H_2 \psi = 0 \) characterises the subspace spanned by all the known exact solutions. So the quantum counterpart of classical partial separability implies the existence of these solutions.
V. INTEGRABILITY

We now come back to the integrability aspect. In Sect.3 we exhibited a set of $2(N - 1)$ constants of motion and regarded the system as potentially integrable. However for the system to be integrable via action angle variables further conditions have to be satisfied [14]: If $M(\mathcal{E}_a, l_a)$ denote the set of points in phase space at which the constants of motion have values $\mathcal{E}_a, l_a$ then (a) $M$ is a $2(N - 1)$ dimensional submanifold iff $\mathcal{E}_a, l_a$ are all independent, (b) if $M$ is a submanifold which is compact and connected then $M$ is a $2(N - 1)$ dimensional torus. If these conditions are satisfied then one can introduce action-angle variables in the neighbourhood of $M$. The orbits on $M$ will all be conditionally periodic in general and for the periodic orbits one may use Bohr-Sommerfeld quantisation to get a subset of eigenvalues.

We also saw in the previous section that there exist $M(\mathcal{E}_a, l_a)$’s for which action-angle variables can be introduced. These correspond to the collective motion of N-anyons and their semiclassical quantisation will reproduce the exactly known eigenvalues described earlier. The Hamiltonian in Eq.(50) being identical in structure to the oscillator Hamiltonian, the orbits of this Hamiltonian will be in one-to-one correspondence with the orbits of the oscillator and hence they will all be periodic. However if we go away from these initial conditions some of the orbits of oscillator will cease to be periodic when $\alpha$ dependence is introduced. For instance the Euler-Lagrange equations

$$\ddot{\vec{r}}_i = -\vec{r}_i \Rightarrow \ddot{\vec{r}}_{ij} = -\vec{r}_{ij} \quad \text{(independent of } \alpha).$$

For any pair $ij, i \neq j$, $\vec{r}_{ij}$ in general describes an ellipse in the configuration space. However for orbital angular momentum zero, the ellipse degenerates to a straight line, hence $\vec{r}_{ij} = 0$ is on such an orbit. Since these points are not admissible when $\alpha \neq 0$ these orbits cannot lead to periodic orbits in the phase space. For all choices of $\mathcal{E}_a, l_a$ such that for some initial conditions the phase space orbits will have $r_{ij}$ approaching arbitrarily close to zero, $M(\mathcal{E}_a, l_a)$ cannot be a torus. ($M$, even if a manifold will fail to be compact and/or connected.) Therefore for $\alpha \neq 0$ although we have a potentially integrable system it is not generically
integrable via action-angle variables. The existence of such an $M$ indicates the possibility of “extreme sensitivity” to the initial conditions. This is a possible reason for the “level repulsions” seen numerically. This bears a resemblance to the billiard system considered by Richens and Berry [13] though the reasons for the failure of integrability via action-angle coordinates appears to be different. Following Richens and Berry we conclude that our system is Pseudointegrable [13].

VI. SUMMARY AND CONCLUSIONS

In this paper we have exhibited two properties of the many anyon system: (1) The partial separability of collective motion and (2) pseudointegrability. We have shown that property (1) explains the existence of the exactly known eigenvalues which is somewhat uncommon for a generic many body problem. This also shows that the exactly known spectrum incorporates only a somewhat trivial aspect of anyon dynamics. The nontrivial aspects although partially uncovered by numerical results are still elusive. Property (2) enables one to understand qualitatively the origin of the “level repulsion” seen in some numerical results. Further study of pseudointegrability, for example the topology of level sets $M$, may enable one to adapt different techniques to get a handle on the nontrivial aspect of the spectrum. We have demonstrated that the N-anyon system is a nontrivial example of a many body pseudointegrable system apart from the known problem of a billiard in polygonal enclosures with a regular obstacle inside. In the light of the pseudointegrability, it seems that the characterisation of quantum integrability must be understood with further conditions, for example self-adjointness of the constants of motion in involutions and the existence of common dense domains.

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