The Clebsch–Gordan Rule for $U(\mathfrak{sl}_2)$, the Krawtchouk Algebras and the Hamming Graphs

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Abstract. Let $D \geq 1$ and $q \geq 3$ be two integers. Let $H(D) = H(D,q)$ denote the $D$-dimensional Hamming graph over a $q$-element set. Let $T(D)$ denote the Terwilliger algebra of $H(D)$. Let $V(D)$ denote the standard $T(D)$-module. Let $\omega$ denote a complex scalar. We consider a unital associative algebra $K_\omega$ defined by generators and relations. The generators are $A$ and $B$. The relations are:

\begin{align*}
A^2B - 2ABA + BA^2 &= B + \omega A, \\
B^2A - 2BAB + AB^2 &= A + \omega B.
\end{align*}

The algebra $K_\omega$ is the case of the Askey–Wilson algebras corresponding to the Krawtchouk polynomials. The algebra $K_\omega$ is isomorphic to $U(\mathfrak{sl}_2)$ when $\omega^2 \neq 1$. We view $V(D)$ as a $K_{1-\frac{1}{2}}$-module. We apply the Clebsch–Gordan rule for $U(\mathfrak{sl}_2)$ to decompose $V(D)$ into a direct sum of irreducible $T(D)$-modules.

Key words: Clebsch–Gordan rule; Hamming graph; Krawtchouk algebra; Terwilliger algebra

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1 Introduction

Throughout this paper, we adopt the following conventions: Fix an integer $q \geq 3$. Let $\mathbb{C}$ denote the complex number field. An algebra is meant to be a unital associative algebra. An algebra homomorphism is meant to be a unital algebra homomorphism. A subalgebra has the same unit as the parent algebra. In an algebra the commutator $[x,y]$ of two elements $x$ and $y$ is defined as $[x,y] = xy - yx$. Note that every algebra has a Lie algebra structure with Lie bracket given by the commutator.

Recall that $\mathfrak{sl}_2(\mathbb{C})$ is a three-dimensional Lie algebra over $\mathbb{C}$ with a basis $e, f, h$ satisfying

\begin{align*}
[h,e] &= 2e, \\
[h,f] &= -2f, \\
e,f] &= h.
\end{align*}

Definition 1.1. The universal enveloping algebra $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is an algebra over $\mathbb{C}$ generated by $E, F, H$ subject to the relations

\begin{align*}
[H,E] &= 2E, \\
[H,F] &= -2F, \\
[E,F] &= H.
\end{align*}

Using Definition 1.1, it is straightforward to verify the following lemma:

Lemma 1.2. Given any integer $n \geq 0$ there exists an $(n+1)$-dimensional $U(\mathfrak{sl}_2)$-module $L_n$ that has a basis $\{v_i\}_{i=0}^n$ such that

\begin{align*}
Ev_i &= (n-i+1)v_{i-1} \quad \text{for } i = 1, 2, \ldots, n, \quad Ev_0 = 0, \\
Fv_i &= (i+1)v_{i+1} \quad \text{for } i = 0, 1, \ldots, n-1, \quad Fv_n = 0, \\
Hv_i &= (n-2i)v_i \quad \text{for } i = 0, 1, \ldots, n.
\end{align*}

Note that the $U(\mathfrak{sl}_2)$-module $L_n$ is irreducible for any integer $n \geq 0$. Furthermore, the finite-dimensional irreducible $U(\mathfrak{sl}_2)$-modules are classified as follows:
Lemma 1.3. For any integer \( n \geq 0 \), each \((n + 1)\)-dimensional irreducible \( \mathfrak{u}(\mathfrak{sl}_2)\)-module is isomorphic to \( L_n \).

Proof. See [10, Section V.4] for example.

It is well known that the universal enveloping algebra of a Lie algebra is a Hopf algebra. For example, see [12, Section 5].

Lemma 1.4. The algebra \( \mathfrak{u}(\mathfrak{sl}_2) \) is a Hopf algebra on which the counit \( \varepsilon : \mathfrak{u}(\mathfrak{sl}_2) \to \mathbb{C} \), the antipode \( S : \mathfrak{u}(\mathfrak{sl}_2) \to \mathfrak{u}(\mathfrak{sl}_2) \) and the comultiplication \( \Delta : \mathfrak{u}(\mathfrak{sl}_2) \to \mathfrak{u}(\mathfrak{sl}_2) \otimes \mathfrak{u}(\mathfrak{sl}_2) \) are given by

\[
\begin{align*}
\varepsilon(E) &= 0, & \varepsilon(F) &= 0, & \varepsilon(H) &= 0, \\
S(E) &= -E, & S(F) &= -F, & S(H) &= -H, \\
\Delta(E) &= E \otimes 1 + 1 \otimes E, & \Delta(F) &= F \otimes 1 + 1 \otimes F, & \Delta(H) &= H \otimes 1 + 1 \otimes H.
\end{align*}
\]

Every \( \mathfrak{u}(\mathfrak{sl}_2) \otimes \mathfrak{u}(\mathfrak{sl}_2) \)-module can be viewed as a \( \mathfrak{u}(\mathfrak{sl}_2) \)-module via the comultiplication of \( \mathfrak{u}(\mathfrak{sl}_2) \).

Theorem 1.5. For any integers \( m, n \geq 0 \), the \( \mathfrak{u}(\mathfrak{sl}_2) \)-module \( L_m \otimes L_n \) is isomorphic to

\[
\bigoplus_{p=0}^{\min\{m,n\}} L_{m+n-2p}.
\]

Proof. See [10, Section V.5] for example.

For the rest of this paper, let \( \omega \) denote a scalar taken from \( \mathbb{C} \).

Definition 1.6. The Krawtchouk algebra \( \mathfrak{K}_\omega \) is an algebra over \( \mathbb{C} \) generated by \( A \) and \( B \) subject to the relations

\[
\begin{align*}
A^2B - 2ABA + BA^2 &= B + \omega A, & (1.1) \\
B^2A - 2BAB + AB^2 &= A + \omega B. & (1.2)
\end{align*}
\]

The algebra \( \mathfrak{K}_\omega \) is the case of the Askey–Wilson algebra corresponding to the Krawtchouk polynomials [22, Lemma 7.2]. Define \( C \) to be the following element of \( \mathfrak{K}_\omega \):

\[
C = [A, B].
\]

Lemma 1.7. The algebra \( \mathfrak{K}_\omega \) has a presentation with the generators \( A, B, C \) and the relations

\[
\begin{align*}
[A,B] &= C, & (1.3) \\
[A,C] &= B + \omega A, & (1.4) \\
[C,B] &= A + \omega B. & (1.5)
\end{align*}
\]

Proof. The relation (1.3) is immediate from the setting of \( C \). Using (1.3), the relations (1.1) and (1.2) can be written as (1.4) and (1.5), respectively. The lemma follows.

Let \( \mathcal{K}_\omega \) denote a three-dimensional Lie algebra over \( \mathbb{C} \) with a basis \( a, b, c \) satisfying

\[
[a,b] = c, \quad [a,c] = b + \omega a, \quad [c,b] = a + \omega b.
\]

By Lemma 1.7, the algebra \( \mathfrak{K}_\omega \) is the universal enveloping algebra of \( \mathcal{K}_\omega \). There is a connection between \( \mathfrak{K}_\omega \) and \( \mathfrak{u}(\mathfrak{sl}_2) \):
Theorem 1.8. There exists a unique algebra homomorphism $\zeta : \mathfrak{R}_\omega \to U(\mathfrak{sl}_2)$ that sends

$$A \mapsto \frac{1 + \omega}{2} E + \frac{1 - \omega}{2} F - \frac{\omega}{2} H, \quad B \mapsto \frac{1}{2} H, \quad C \mapsto -\frac{1 + \omega}{2} E + \frac{1 - \omega}{2} F.$$ 

Moreover, if $\omega^2 \neq 1$, then $\zeta$ is an isomorphism and its inverse sends

$$E \mapsto \frac{1}{1 + \omega} A + \frac{\omega}{1 + \omega} B - \frac{1}{1 + \omega} C, \quad F \mapsto \frac{1}{1 - \omega} A + \frac{\omega}{1 - \omega} B + \frac{1}{1 - \omega} C, \quad H \mapsto 2B.$$

Proof. It is routine to verify the result by using Definition 1.1 and Lemma 1.7. Here we provide another proof by applying [13, Lemmas 2.12 and 2.13].

Let $\sigma : \mathfrak{sl}_2(\mathbb{C}) \to U(\mathfrak{sl}_2)$ denote the canonical Lie algebra homomorphism that sends $e, f, h$ to $E, F, H$, respectively. Let $\tau : \mathfrak{K}_\omega \to \mathfrak{R}_\omega$ denote the canonical Lie algebra homomorphism that sends $a, b, c$ to $A, B, C$, respectively. By [13, Lemma 2.12], there exists a unique Lie algebra homomorphism $\phi : \mathfrak{K}_\omega \to \mathfrak{sl}_2(\mathbb{C})$ that sends

$$a \mapsto \frac{1 + \omega}{2} e + \frac{1 - \omega}{2} f - \frac{\omega}{2} h, \quad b \mapsto \frac{1}{2} h, \quad c \mapsto -\frac{1 + \omega}{2} e + \frac{1 - \omega}{2} f.$$

Applying the universal property of $\mathfrak{R}_\omega$ to the Lie algebra homomorphism $\sigma \circ \phi$, this gives the algebra homomorphism $\zeta$. Suppose that $\omega^2 \neq 1$. Then $\phi : \mathfrak{K}_\omega \to \mathfrak{sl}_2(\mathbb{C})$ is a Lie algebra isomorphism by [13, Lemma 2.13]. Applying the universal property of $U(\mathfrak{sl}_2)$ to the Lie algebra homomorphism $\tau \circ \phi^{-1}$, this gives the inverse of $\zeta$. \hfill $\blacksquare$

In this paper, we relate the above algebraic results to the Hamming graphs. We now recall the definition of Hamming graphs. Let $X$ denote a $q$-element set and let $D$ be a positive integer. The $D$-dimensional Hamming graph $H(D) = H(D, q)$ over $X$ is a simple graph whose vertex set is $X^D$ and $x, y \in X^D$ are adjacent if and only if $x, y$ differ in exactly one coordinate. Let $\partial$ denote the path-length distance function for $H(D)$. Let $\text{Mat}_{X^D}(\mathbb{C})$ stand for the algebra consisting of the square matrices over $\mathbb{C}$ indexed by $X^D$.

The adjacency matrix $A(D) \in \text{Mat}_{X^D}(\mathbb{C})$ of $H(D)$ is the 0-1 matrix such that

$$A(D)_{xy} = 1 \quad \text{if and only if} \quad \partial(x, y) = 1$$

for all $x, y \in X^D$. Fix a vertex $x \in X^D$. The dual adjacency matrix $A^*(D) \in \text{Mat}_{X^D}(\mathbb{C})$ of $H(D)$ with respect to $x$ is a diagonal matrix given by

$$A^*(D)_{yy} = D(q - 1) - q \cdot \partial(x, y)$$

for all $y \in X^D$. The Terwilliger algebra $T(D)$ of $H(D)$ with respect to $x$ is the subalgebra of $\text{Mat}_{X^D}(\mathbb{C})$ generated by $A(D)$ and $A^*(D)$ [16, 17, 18]. Let $V(D)$ denote the vector space consisting of all column vectors over $\mathbb{C}$ indexed by $X^D$. The vector space $V(D)$ has a natural $T(D)$-module structure and it is called the standard $T(D)$-module.

In [18], Terwilliger employed the endpoints, dual endpoints, diameters and auxiliary parameters to describe the irreducible modules for the known families of thin $Q$-polynomial distance-regular graphs with unbounded diameter. In [14], Tanabe gave a recursive construction of irreducible modules for the Doob graphs and his method can be adjusted to the case of $H(D)$. In [5], Go gave a decomposition of the standard module for the hypercube. In [4], Gijswijt, Schrijver and Tanaka described a decomposition of $V(D)$ in terms of the block-diagonalization of $T(D)$. In [11], Levstein, Maldonado and Penazzi applied the representation theory of $\text{GL}_2(\mathbb{C})$ to determine the structure of $T(D)$. In [20], it was shown that $V(D)$ can be viewed as a $\mathfrak{gl}_2(\mathbb{C})$-module as well as a $\mathfrak{sl}_2(\mathbb{C})$-module. In [2], Bernard, Crampé, and Vinet found a decomposition of $V(D)$ by generalizing the result on the hypercube.

In this paper, we view $V(D)$ as a $\mathfrak{R}_1^\frac{1}{2}$-module as well as a $U(\mathfrak{sl}_2)$-module in light of Theorem 1.8. Subsequently, we apply Theorem 1.5 to prove the following results:
Proposition 1.9. Let $D$ be a positive integer. For any integers $p$ and $k$ with $0 \leq p \leq D$ and $0 \leq k \leq \left\lfloor \frac{D}{2} \right\rfloor$, there exists a $(p-2k+1)$-dimensional irreducible $T(D)$-module $L_{p,k}(D)$ satisfying the following conditions:

(i) There exists a basis for $L_{p,k}(D)$ with respect to which the matrices representing $A(D)$ and $A^*(D)$ are

\[
\begin{pmatrix}
\alpha_0 & \gamma_1 & 0 \\
\beta_0 & \alpha_1 & \gamma_2 \\
\beta_1 & \alpha_2 & \ddots \\
0 & \ddots & \ddots & \gamma_{p-2k} \\
0 & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \beta_{p-2k-1} & \alpha_{p-2k}
\end{pmatrix}
\]

respectively.

(ii) There exists a basis for $L_{p,k}(D)$ with respect to which the matrices representing $A(D)$ and $A^*(D)$ are

\[
\begin{pmatrix}
\theta_0 & 0 \\
\theta_1 & \theta_2 \\
\theta_3 & \ddots & \ddots \\
0 & \ddots & \ddots & \theta_{p-2k}
\end{pmatrix}
\]

respectively.

Here the parameters $\{\alpha_i\}_{i=0}^{p-2k}$, $\{\beta_i\}_{i=0}^{p-2k-1}$, $\{\gamma_i\}_{i=1}^{p-2k}$, $\{\theta_i\}_{i=0}^{p-2k}$ are as follows:

- $\alpha_i = (q - 2)(i + k) + p - D$ for $i = 0, 1, \ldots, p - 2k$,
- $\beta_i = i + 1$ for $i = 0, 1, \ldots, p - 2k - 1$,
- $\gamma_i = (q - 1)(p - i - 2k + 1)$ for $i = 1, 2, \ldots, p - 2k$,
- $\theta_i = q(p - i - k) - D$ for $i = 0, 1, \ldots, p - 2k$.

Given a vector space $W$ and a positive integer $p$, we let

$p \cdot W = W \oplus W \oplus \cdots \oplus W$.

Theorem 1.10. Let $D$ be a positive integer. Then the standard $T(D)$-module $V(D)$ is isomorphic to

\[
\bigoplus_{p=0}^{D} \bigoplus_{k=0}^{\left\lfloor \frac{D}{2} \right\rfloor} \frac{p-2k+1}{p-k+1} \binom{D}{p} \binom{p-2k}{k} (q-2)^{D-p} \cdot L_{p,k}(D).
\]

The algebra $T(D)$ is a finite-dimensional semisimple algebra. Following from [3, Theorem 25.10], Theorem 1.10 implies the following classification of irreducible $T(D)$-modules:

Theorem 1.11. Let $D$ be a positive integer. Let $P(D)$ denote the set consisting of all pairs $(p, k)$ of integers with $0 \leq p \leq D$ and $0 \leq k \leq \left\lfloor \frac{D}{2} \right\rfloor$. Let $M(D)$ denote the set of all isomorphism classes of irreducible $T(D)$-modules. Then there exists a bijection $E : P(D) \to M(D)$ given by

$(p, k) \mapsto$ the isomorphism class of $L_{p,k}(D)$

for all $(p, k) \in P(D)$.

The paper is organized as follows: In Section 2, we give the preliminaries on the algebra $A(D)$. In Section 3, we prove Proposition 1.9 and Theorems 1.10, 1.11 by using Theorem 1.5. In Appendix A, we give the equivalent statements of Proposition 1.9 and Theorems 1.10, 1.11.
2 The Krawtchouk algebra

2.1 Finite-dimensional irreducible $R_\omega$-modules

Recall the $U(\mathfrak{sl}_2)$-module $L_n$ from Lemma 1.2. Recall the algebra homomorphism $\zeta: R_\omega \to U(\mathfrak{sl}_2)$ form Theorem 1.8. Each $U(\mathfrak{sl}_2)$-module can be viewed as a $R_\omega$-module by pulling back via $\zeta$. We express the $U(\mathfrak{sl}_2)$-module $L_n$ as a $R_\omega$-module as follows:

Lemma 2.1. For any integer $n \geq 0$, the matrices representing $A$, $B$, $C$ with respect to the basis $\{v_i\}_{i=0}^n$ for the $R_\omega$-module $L_n$ are

$$
\begin{pmatrix}
\alpha_0 & \gamma_1 & 0 \\
\beta_0 & \alpha_1 & \gamma_2 \\
\beta_1 & \alpha_2 & \ddots \\
0 & \beta_{n-1} & \gamma_n
\end{pmatrix},
\begin{pmatrix}
\theta_0 & \theta_2 & 0 \\
\beta_1 & \theta_1 & \theta_2 \\
0 & \theta_1 & \ddots \\
0 & 0 & \beta_{n-1}
\end{pmatrix},
\begin{pmatrix}
0 & -\gamma_1 & -\gamma_2 & 0 \\
\beta_0 & 0 & -\gamma_2 & 0 \\
\beta_1 & 0 & \ddots & \ddots \\
0 & \beta_{n-1} & 0 & \gamma_n
\end{pmatrix}
$$

respectively, where

$$
\alpha_i = \frac{(2i-n)\omega}{2} \quad \text{for } i = 0, 1, \ldots, n,
$$

$$
\beta_i = \frac{(i+1)(1-\omega)}{2} \quad \text{for } i = 0, 1, \ldots, n-1,
$$

$$
\gamma_i = \frac{(n-i+1)(1+\omega)}{2} \quad \text{for } i = 1, 2, \ldots, n,
$$

$$
\theta_i = \frac{n}{2} - i \quad \text{for } i = 0, 1, \ldots, n.
$$

The finite-dimensional irreducible $R_\omega$-modules are classified as follows:

Theorem 2.2.

(i) If $\omega = -1$, then any finite-dimensional irreducible $R_\omega$-module $V$ is of dimension one and there is a scalar $\mu \in \mathbb{C}$ such that $Av = \mu v$, $Bv = \mu v$ for all $v \in V$.

(ii) If $\omega = 1$, then any finite-dimensional irreducible $R_\omega$-module $V$ is of dimension one and there is a scalar $\mu \in \mathbb{C}$ such that $Av = \mu v$, $Bv = -\mu v$ for all $v \in V$.

(iii) If $\omega^2 \neq 1$, then $L_n$ is the unique $(n+1)$-dimensional irreducible $R_\omega$-module up to isomorphism for every integer $n \geq 0$.

Proof. (i) Let $n \geq 0$ be an integer. Let $V$ denote an $(n+1)$-dimensional irreducible $R_{-1}$-module. Since the trace of the left-hand side of (1.1) on $V$ is zero, the elements $A$ and $B$ have the same trace on $V$. If $n = 0$ then there exists a scalar $\mu \in \mathbb{C}$ such that $Av = Bv = \mu v$ for all $v \in V$.

To see Theorem 2.2(i), it remains to assume that $n \geq 1$ and we seek a contradiction. Applying the method proposed in [6, 7, 8], there exists a basis $\{u_i\}_{i=0}^n$ for $V$ with respect to which the matrices representing $A$ and $B$ are of the forms

$$
\begin{pmatrix}
\theta_0 & \theta_2 & 0 \\
1 & \theta_1 & \theta_2 \\
0 & 1 & \theta_n
\end{pmatrix},
\begin{pmatrix}
\theta_0 & \varphi_2 & 0 \\
\theta_1 & \varphi_2 & \ddots \\
0 & \varphi_n & \theta_n
\end{pmatrix}
$$
respectively. Here \(\{\theta_i\}_{i=0}^n\) is an arithmetic sequence with common difference \(-1\) and the sequence \(\{\varphi_i\}_{i=1}^n\) satisfies \(\varphi_{i-1} - 2\varphi_i + \varphi_{i+1} = 0\), \(1 \leq i \leq n\), where \(\varphi_0\) and \(\varphi_{n+1}\) are interpreted as zero. Solving the above recurrence yields that \(\varphi_i = 0\) for all \(i = 1, 2, \ldots, n\). Thus the subspace of \(V\) spanned by \(\{u_i\}_{i=1}^n\) is a nonzero \(\mathfrak{K}_1\)-module, which is a contradiction to the irreducibility of \(V\).

(ii) Using Definition 1.6, it is routine to verify that there exists a unique algebra isomorphism \(\mathfrak{K}_0 \to \mathfrak{K}_1\) that sends \(A\) to \(A\) and \(B\) to \(-B\). Theorem 2.2(ii) follows from Theorem 2.2(i) and the above isomorphism.

(iii) Theorem 2.2(iii) follows immediately from Lemma 1.3 and Theorem 1.8.

**Lemma 2.3.** There exists a unique algebra automorphism of \(\mathfrak{K}_\omega\) that sends \(A \mapsto B\), \(B \mapsto A\), \(C \mapsto -C\).

**Proof.** It is routine to verify the lemma by using Lemma 1.7.

**Lemma 2.4.** Suppose that \(\omega^2 \neq 1\). For any integer \(n \geq 0\), there exists a basis for the \(\mathfrak{K}_\omega\)-module \(L_n\) with respect to which the matrices representing \(A\), \(B\), \(C\) are

\[
\begin{pmatrix}
\theta_0 & 0 & \cdots & 0 \\
\theta_1 & \theta_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \theta_n
\end{pmatrix},
\begin{pmatrix}
\alpha_0 & \gamma_1 & 0 & \cdots & 0 \\
\beta_0 & \alpha_1 & \gamma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \beta_{n-1} & \alpha_n
\end{pmatrix},
\begin{pmatrix}
0 & \gamma_1 & 0 & \cdots & 0 \\
-\beta_0 & 0 & \gamma_2 & \cdots & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & -\beta_{n-1} & \gamma_n
\end{pmatrix}
\]

respectively, where

\[
\begin{align*}
\alpha_i &= \frac{(2i - n)\omega}{2} & \text{for } i = 0, 1, \ldots, n, \\
\beta_i &= \frac{(i + 1)(1 - \omega)}{2} & \text{for } i = 0, 1, \ldots, n - 1, \\
\gamma_i &= \frac{(n - i + 1)(1 + \omega)}{2} & \text{for } i = 1, 2, \ldots, n, \\
\theta_i &= \frac{n - i}{2} & \text{for } i = 0, 1, \ldots, n.
\end{align*}
\]

**Proof.** Let \(L_n'\) denote the irreducible \(\mathfrak{K}_\omega\)-module obtained by twisting the \(\mathfrak{K}_\omega\)-module \(L_n\) via the automorphism of \(\mathfrak{K}_\omega\) given in Lemma 2.3. Recall the basis \(\{v_i\}_{i=0}^n\) for \(L_n\) from Lemma 2.1. Observe that the three matrices described in Lemma 2.4 are the matrices representing \(A\), \(B\), \(C\) with respect to the basis \(\{v_i\}_{i=0}^n\) for the \(\mathfrak{K}_\omega\)-module \(L_n'\). By Theorem 2.2(iii), the \(\mathfrak{K}_\omega\)-module \(L_n'\) is isomorphic to \(L_n\). The lemma follows.

Leonard pairs were introduced in [15, 19, 21] by P. Terwilliger. Suppose that \(\omega^2 \neq 1\). By Lemmas 2.1 and 2.4, the elements \(A\) and \(B\) act on the \(\mathfrak{K}_\omega\)-module \(L_n\) as a Leonard pair. The result was first stated in [13, Theorem 6.3].

### 2.2 The Krawtchouk algebra as a Hopf algebra

Let \(\mathcal{H}\) denote an algebra. Recall that \(\mathcal{H}\) is called a *Hopf algebra* if there are two algebra homomorphisms \(\varepsilon: \mathcal{H} \to \mathbb{C}\), \(\Delta: \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}\) and a linear map \(S: \mathcal{H} \to \mathcal{H}\) that satisfy the following properties:

\[
\begin{align*}
(\text{H1}) \quad (1 \otimes \Delta) \circ \Delta &= (\Delta \otimes 1) \circ \Delta, \\
(\text{H2}) \quad m \circ (1 \otimes (\iota \circ \varepsilon)) \circ \Delta &= m \circ ((\iota \circ \varepsilon) \otimes 1) \circ \Delta = 1, \\
(\text{H3}) \quad m \circ (1 \otimes S) \circ \Delta &= m \circ (S \otimes 1) \circ \Delta = \iota \circ \varepsilon.
\end{align*}
\]
Here \( m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \) is the multiplication map and \( \iota : \mathbb{C} \to \mathcal{H} \) is the unit map defined by \( \iota(c) = c1 \) for all \( c \in \mathbb{C} \). Note that \( m \) is a linear map and \( \iota \) is an algebra homomorphism.

Suppose that (H1)–(H3) hold. Then the maps \( \varepsilon, \Delta, S \) are called the counit, comultiplication and antipode of \( \mathcal{H} \), respectively. Let \( n \) be a positive integer. The \( n \)-fold comultiplication of \( \mathcal{H} \) is the algebra homomorphism \( \Delta_n : \mathcal{H} \to \mathcal{H}^{\otimes(n+1)} \) inductively defined by

\[
\Delta_n = (1 \otimes (n-1) \otimes \Delta) \circ \Delta_{n-1}.
\]

Here \( \Delta_0 \) is interpreted as the identity map of \( \mathcal{H} \). We may regard every \( \mathcal{H}^{\otimes(n+1)} \)-module as an \( \mathcal{H} \)-module by pulling back via \( \Delta_n \). Note that

\[
\Delta_n = (1 \otimes (n-1) \otimes \Delta \otimes 1^{\otimes(i-1)}) \circ \Delta_{n-1} \quad \text{for all } i = 1, 2, \ldots, n.
\tag{2.1}
\]

It follows from (2.1) that

\[
\Delta_n = (\Delta_{n-1} \otimes 1) \circ \Delta = (1 \otimes \Delta_{n-1}) \circ \Delta.
\tag{2.2}
\]

Recall from Section 1 that \( \mathfrak{k}_\omega \) is the universal enveloping algebra of \( K_\omega \). Hence \( \mathfrak{k}_\omega \) is a Hopf algebra. For the reader’s convenience, we give a detailed verification for the Hopf algebra structure of \( \mathfrak{k}_\omega \). By an algebra antihomomorphism, we mean a unital algebra antihomomorphism.

**Lemma 2.5.**

(i) There exists a unique algebra homomorphism \( \varepsilon : \mathfrak{k}_\omega \to \mathbb{C} \) given by

\[
\varepsilon(A) = 0, \quad \varepsilon(B) = 0, \quad \varepsilon(C) = 0.
\]

(ii) There exists a unique algebra homomorphism \( \Delta : \mathfrak{k}_\omega \to \mathfrak{k}_\omega \otimes \mathfrak{k}_\omega \) given by

\[
\Delta(A) = A \otimes 1 + 1 \otimes A, \quad \Delta(B) = B \otimes 1 + 1 \otimes B, \quad \Delta(C) = C \otimes 1 + 1 \otimes C.
\]

(iii) There exists a unique algebra antihomomorphism \( S : \mathfrak{k}_\omega \to \mathfrak{k}_\omega \) given by

\[
S(A) = -A, \quad S(B) = -B, \quad S(C) = -C.
\]

(iv) The algebra \( \mathfrak{k}_\omega \) is a Hopf algebra on which the counit, comultiplication and antipode are the above maps \( \varepsilon, \Delta, S \), respectively.

**Proof.** (i)–(iii) It is routine to verify Lemma 2.5(i)–(iii) by using Definition 1.6.

(iv) Using Lemma 2.5(ii), it yields that \( (1 \otimes \Delta) \circ \Delta \) and \( (\Delta \otimes 1) \circ \Delta \) agree at the generators \( A, B, C \) of \( \mathfrak{k}_\omega \). Since \( \Delta \) is an algebra homomorphism, the maps \( (1 \otimes \Delta) \circ \Delta \) and \( (\Delta \otimes 1) \circ \Delta \) are algebra homomorphisms. Hence (H1) holds for \( \mathfrak{k}_\omega \).

Let \( k = m \circ (1 \otimes (\iota \circ \varepsilon)) \circ \Delta \) and \( k' = m \circ ((\iota \circ \varepsilon) \otimes 1) \circ \Delta \). Evidently, \( k \) and \( k' \) are linear maps. Using Lemma 2.5(i), (ii) yields that

\[
k(1) = k'(1) = 1, \quad k(A) = k'(A) = A, \quad k(B) = k'(B) = B, \quad k(C) = k'(C) = C.
\]

Let \( x, y \) be any two elements of \( \mathfrak{k}_\omega \). To see that \( k = 1 \) it remains to check that \( k(xy) = k(x)k(y) \). We can write

\[
\Delta(x) = \sum_{i=1}^{n} x_i^{(1)} \otimes x_i^{(2)},
\tag{2.3}
\]

\[
\Delta(y) = \sum_{i=1}^{n} y_i^{(1)} \otimes y_i^{(2)},
\tag{2.4}
\]
where \( n \geq 1 \) is an integer and \( x_i^{(1)}, x_i^{(2)}, y_i^{(1)}, y_i^{(2)} \in \mathcal{R}_\omega \) for \( 1 \leq i \leq n \). Then
\[
k(xy) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^{(1)} \cdot y_j^{(1)} \cdot (\nu \circ \varepsilon)(x_i^{(2)}) \cdot (\nu \circ \varepsilon)(y_j^{(2)}).
\]

Since each of \((\nu \circ \varepsilon)(x_i^{(2)})\) and \((\nu \circ \varepsilon)(y_j^{(2)})\) is a scalar multiple of 1, it follows that
\[
k(xy) = \left( \sum_{i=1}^{n} x_i^{(1)} \cdot (\nu \circ \varepsilon)(x_i^{(2)}) \right) \left( \sum_{j=1}^{n} y_j^{(1)} \cdot (\nu \circ \varepsilon)(y_j^{(2)}) \right) = k(x)k(y).
\]

By a similar argument, one may show that \( k' = 1 \). Hence \((H2)\) holds for \( \mathcal{R}_\omega \).

Let \( h = m \circ (1 \otimes S) \circ \Delta \) and \( h' = m \circ (S \otimes 1) \circ \Delta \). Evidently, \( h \) and \( h' \) are linear maps. Using Lemma 2.5(ii), (iii) yields that
\[
h(1) = h'(1) = (\nu \circ \varepsilon)(1) = 1, \quad h(A) = h'(A) = (\nu \circ \varepsilon)(A) = 0,
\]
\[
h(B) = h'(B) = (\nu \circ \varepsilon)(B) = 0, \quad h(C) = h'(C) = (\nu \circ \varepsilon)(C) = 0.
\]

Let \( x, y \) be any two elements of \( \mathcal{R}_\omega \) and suppose that \( h(x) = (\nu \circ \varepsilon)(x) \) and \( h(y) = (\nu \circ \varepsilon)(y) \). To see that \( h = \nu \circ \varepsilon \), it suffices to check that \( h(xy) = h(x)h(y) \). Applying (2.3) and (2.4), one finds that
\[
h(xy) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^{(1)} y_j^{(1)} S(x_i^{(2)} y_j^{(2)}).
\]

Using the antihomomorphism property of \( S \), we obtain
\[
h(xy) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^{(1)} y_j^{(1)} S(y_j^{(2)}) S(x_i^{(2)}) = \sum_{i=1}^{n} x_i^{(1)} \left( \sum_{j=1}^{n} y_j^{(1)} S(y_j^{(2)}) \right) S(x_i^{(2)})
\]
\[
= \sum_{i=1}^{n} x_i^{(1)} h(y) S(x_i^{(2)}).
\]

Since \( h(y) = (\nu \circ \varepsilon)(y) \) is a scalar multiple of 1, it follows that
\[
h(xy) = \sum_{i=1}^{n} x_i^{(1)} S(x_i^{(2)}) h(y) = h(x)h(y).
\]

By a similar argument, one can show that \( h' = \nu \circ \varepsilon \). Hence \((H3)\) holds for \( \mathcal{R}_\omega \). The result follows.

**Theorem 2.6.** For any integers \( m, n \geq 0 \), the \( \mathcal{R}_\omega \)-module \( L_m \otimes L_n \) is isomorphic to
\[
\bigoplus_{p=0}^{\min\{m,n\}} L_{m+n-2p}.
\]

**Proof.** By Lemmas 1.4 and 2.5 along with Theorem 1.8 the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{R}_\omega & \xrightarrow{\zeta} & U(\mathfrak{sl}_2) \\
\Delta \downarrow & & \downarrow \Delta \\
\mathcal{R}_\omega \otimes \mathcal{R}_\omega & \xrightarrow{\zeta \otimes \zeta} & U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)
\end{array}
\]
Here $\Delta : U(\mathfrak{s}l_2) \to U(\mathfrak{s}l_2) \otimes U(\mathfrak{s}l_2)$ is the comultiplication of $U(\mathfrak{s}l_2)$ from Lemma 1.4 and $\Delta : \mathfrak{r}_\omega \to \mathfrak{r}_\omega \otimes \mathfrak{r}_\omega$ is the comultiplication of $\mathfrak{r}_\omega$ from Lemma 2.5(ii). Combined with Theorem 1.5, the result follows.

For the rest of this paper, the notation $\Delta$ will refer to the map from Lemma 2.5(ii) and $\Delta_n$ will stand for the corresponding $n$-fold comultiplication of $\mathfrak{r}_\omega$ for every positive integer $n$.

3 The Clebsch–Gordan rule for $U(\mathfrak{s}l_2)$ and the Hamming graph $H(D, q)$

3.1 Preliminaries on distance-regular graphs

Let $\Gamma$ denote a finite simple connected graph with vertex set $X \neq \emptyset$. Let $\partial$ denote the path-length distance function for $\Gamma$. Recall that the diameter $D$ of $\Gamma$ is defined by

$$D = \max_{x,y \in X} \partial(x, y).$$

Given any $x \in X$ let

$$\Gamma_i(x) = \{ y \in X \mid \partial(x, y) = i \} \quad \text{for } i = 0, 1, \ldots, D.$$ 

For short, we abbreviate $\Gamma(x) = \Gamma_1(x)$. We call $\Gamma$ distance-regular whenever for all $h, i, j \in \{0, 1, \ldots, D\}$ and all $x, y \in X$ with $\partial(x, y) = h$ the number $|\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of $x$ and $y$. If $\Gamma$ is distance-regular, the numbers $a_i, b_i, c_i$ for all $i = 0, 1, \ldots, D$ defined by

$$a_i = |\Gamma_i(x) \cap \Gamma(y)|, \quad b_i = |\Gamma_{i+1}(x) \cap \Gamma(y)|, \quad c_i = |\Gamma_{i-1}(x) \cap \Gamma(y)|$$

for any $x, y \in X$ with $\partial(x, y) = i$ are called the intersection numbers of $\Gamma$. Here $\Gamma_{-1}(x)$ and $\Gamma_{D+1}(x)$ are interpreted as the empty set.

We now assume that $\Gamma$ is distance-regular. Let $\text{Mat}_X(\mathbb{C})$ be the algebra consisting of the complex square matrices indexed by $X$. For all $i = 0, 1, \ldots, D$ the $i^{th}$ distance matrix $A_i \in \text{Mat}_X(\mathbb{C})$ is defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i \end{cases}$$

for all $x, y \in X$. The Bose–Mesner algebra $\mathcal{M}$ of $\Gamma$ is the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A_i$ for all $i = 0, 1, \ldots, D$. Note that the adjacency matrix $A = A_1$ of $\Gamma$ generates $\mathcal{M}$ and the matrices $\{A_i\}_{i=0}^{D}$ form a basis for $\mathcal{M}$.

Since $A$ is real symmetric and $\dim \mathcal{M} = D + 1$, it follows that $A$ has $D + 1$ mutually distinct real eigenvalues $\theta_0, \theta_1, \ldots, \theta_D$. Set $\theta_0 = b_0$ which is the valency of $\Gamma$. There exist unique $E_0, E_1, \ldots, E_D \in \mathcal{M}$ such that

$$\sum_{i=0}^{D} E_i = I \quad \text{(the identity matrix),} \quad A E_i = \theta_i E_i \quad \text{for all } i = 0, 1, \ldots, D.$$ 

The matrices $\{E_i\}_{i=0}^{D}$ form another basis for $\mathcal{M}$, and $E_i$ is called the primitive idempotent of $\Gamma$ associated with $\theta_i$ for $i = 0, 1, \ldots, D$.

Observe that $\mathcal{M}$ is closed under the Hadamard product $\odot$. The distance-regular graph $\Gamma$ is said to be $Q$-polynomial with respect to the ordering $\{E_i\}_{i=0}^{D}$ if there are scalars $a_i^*, b_i^*, c_i^*$ for all $i = 0, 1, \ldots, D$ such that $b_0^* = c_0^* = 0$, $b_{i-1}^* c_i^* \neq 0$ for all $i = 1, 2, \ldots, D$ and

$$(E_1 \odot E_i) = \frac{1}{|X|} \left( b_{i-1}^* E_{i-1} + a_i^* E_i + c_{i+1}^* E_{i+1} \right) \quad \text{for all } i = 0, 1, \ldots, D,$$

where we interpret $b_{-1}^*, c_{D+1}^*$ as any scalars in $\mathbb{C}$ and $E_{-1}, E_{D+1}$ as the zero matrix in $\text{Mat}_X(\mathbb{C})$. 

The Clebsch–Gordan Rule for $U(\mathfrak{s}l_2)$, the Krawtchouk Algebras and the Hamming Graphs
We now assume that $\Gamma$ is $Q$-polynomial with respect to $\{E_i\}_{i=0}^D$ and fix $x \in X$. For all $i = 0, 1, \ldots, D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ defined by

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i \end{cases}$$

(3.1)

for all $y \in X$. The matrix $E_i^*$ is called the $i^{th}$ dual primitive idempotent of $\Gamma$ with respect to $x$. The dual Bose–Mesner algebra $M^* = M^*(x)$ of $\Gamma$ with respect to $x$ is the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $E_i^*$ for all $i = 0, 1, \ldots, D$. Since $E_i^* E_j^* = \delta_{ij} E_i^*$ the matrices $\{E_i^*\}_{i=0}^D$ form a basis for $M^*$. For all $i = 0, 1, \ldots, D$ the $i^{th}$ dual distance matrix $A_i^* = A_i^*(x)$ is the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ defined by

$$(A_i^*)_{yy} = |X| \langle x \rangle_{xy} \quad \text{for all } y \in X.$$  

(3.2)

The matrices $\{A_i^*\}_{i=0}^D$ form another basis for $M^*$. Note that $A^* = A_1^*$ is called the dual adjacency matrix of $\Gamma$ with respect to $x$ and $A^*$ generates $M^*$ [16, Lemma 3.11].

The Terwilliger algebra $\mathcal{T}$ of $\Gamma$ with respect to $x$ is the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $M$ and $M^*$ [16, Definition 3.3]. The vector space consisting of all complex column vectors indexed by $X$ is a natural $\mathcal{T}$-module and it is called the standard $\mathcal{T}$-module [16, p. 368]. Since the algebra $\mathcal{T}$ is finite-dimensional, the irreducible $\mathcal{T}$-modules are finite-dimensional. Since the algebra $\mathcal{T}$ is closed under the conjugate-transpose map, it follows that $\mathcal{T}$ is semisimple. Hence the algebra $\mathcal{T}$ is isomorphic to

$$\bigoplus_{\text{irreducible } \mathcal{T}-\text{modules } W} \text{End}(W),$$

where the direct sum is over all non-isomorphic irreducible $\mathcal{T}$-modules $W$. Since the standard $\mathcal{T}$-module is faithful, all irreducible $\mathcal{T}$-modules are contained in the standard $\mathcal{T}$-module up to isomorphism.

Let $W$ denote an irreducible $\mathcal{T}$-module. The number $\min_{0 \leq i \leq D} \{i \mid E_i^* W \neq \{0\}\}$ is called the endpoint of $W$. The number $\min_{0 \leq i \leq D} \{i \mid E_i W \neq \{0\}\}$ is called the dual endpoint of $W$. The support of $W$ is defined as the set $\{i \mid 0 \leq i \leq D, E_i^* W \neq \{0\}\}$. The dual support of $W$ is defined as the set $\{i \mid 0 \leq i \leq D, E_i W \neq \{0\}\}$. The number $|\{i \mid 0 \leq i \leq D, E_i^* W \neq \{0\}\}| - 1$ is called the diameter of $W$. The number $|\{i \mid 0 \leq i \leq D, E_i W \neq \{0\}\}| - 1$ is called the dual diameter of $W$.

### 3.2 The adjacency matrix and the dual adjacency matrix of a Hamming graph

Let $X$ be a nonempty set and let $n$ be a positive integer. The notation

$$X^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in X\}$$

stands for the $n$-ary Cartesian product of $X$. For any $x \in X^n$, let $x_i$ denote the $i^{th}$ coordinate of $x$ for all $i = 1, 2, \ldots, n$.

Recall that $q$ stands for an integer greater than or equal to 3. For the rest of this paper, we set

$$X = \{0, 1, \ldots, q - 1\}$$

and let $D$ be a positive integer.

**Definition 3.1.** The $D$-dimensional Hamming graph $H(D) = H(D, q)$ over $X$ has the vertex set $X^D$ and $x, y \in X^D$ are adjacent if and only if $x$ and $y$ differ in exactly one coordinate.
Let \( \partial \) denote the path-length distance function for \( H(D) \). Observe that \( \partial(x, y) = |\{ i \mid 1 \leq i \leq D, \, x_i \neq y_i \}| \) for any \( x, y \in X^D \). It is routine to verify that \( H(D) \) is a distance-regular graph with diameter \( D \) and its intersection numbers are
\[
\begin{align*}
a_i &= i(q - 2), & b_i &= (D - i)(q - 1), & c_i &= i
\end{align*}
\]
for all \( i = 0, 1, \ldots, D \).

Let \( V(D) \) denote the vector space consisting of the complex column vectors indexed by \( X^D \). For convenience we write \( V = V(1) \). For any \( x \in X^D \), let \( \hat{x} \) denote the vector of \( V(D) \) with 1 in the \( x \)-coordinate and 0 elsewhere. We view any \( \hat{x} \in V(D) \) as the zero matrix in \( \text{Mat}_{X^D}(\mathbb{C}) \) and let \( I \in \text{Mat}_{X^D}(\mathbb{C}) \) denote the identity matrix in \( \text{Mat}_{X^D}(\mathbb{C}) \). We view any \( \hat{x} \in V(D) \) as the vector space consisting of the complex column vectors indexed by \( D \). Observe that it is equal to
\[
\hat{x} \rightarrow \hat{x}_1 \otimes \hat{x}_2 \otimes \cdots \otimes \hat{x}_D \text{ for all } x \in X^D.
\]

Let \( I(D) \) denote the identity matrix in \( \text{Mat}_{X^D}(\mathbb{C}) \) and let \( A(D) \) denote the adjacency matrix of \( H(D) \). We simply write \( I = I(1) \) and \( A = A(1) \).

**Lemma 3.2.** Let \( D \geq 2 \) be an integer. Then
\[
A(D) = A(D - 1) \otimes I + I(D - 1) \otimes A. \tag{3.3}
\]

**Proof.** Let \( x \in X^D \) be given. Applying \( \hat{x} \) to the right-hand side of (3.3) a straightforward calculation yields that it is equal to
\[
\sum_{i=1}^{D} \sum_{y_i \in X \setminus \{x_i\}} \hat{x}_1 \otimes \cdots \otimes \hat{x}_{i-1} \otimes \hat{y}_i \otimes \hat{x}_{i+1} \otimes \cdots \otimes \hat{x}_D = A(D)\hat{x}.
\]

The lemma follows. \( \blacksquare \)

Using Lemma 3.2, a routine induction yields that \( A(D) \) has the eigenvalues
\[
\theta_i(D) = D(q - 1) - qi \text{ for all } i = 0, 1, \ldots, D.
\]

Let \( E_i(D) \) denote the primitive idempotent of \( H(D) \) associated with \( \theta_i(D) \) for all \( i = 0, 1, \ldots, D \). We simply write \( E_0 = E_0(1) \) and \( E_1 = E_1(1) \). For convenience, we interpret \( E_{-1}(D) \) and \( E_{D+1}(D) \) as the zero matrix in \( \text{Mat}_{X^D}(\mathbb{C}) \).

**Lemma 3.3.** Let \( D \geq 2 \) be an integer. Then
\[
E_i(D) = E_i(D - 1) \otimes E_0 + E_{i-1}(D - 1) \otimes E_1 \quad \text{for all } i = 0, 1, \ldots, D. \tag{3.4}
\]

**Proof.** We proceed by induction on \( D \). Let \( E_i(D)' \) denote the right-hand side of (3.4) for \( i = 0, 1, \ldots, D \). Applying Lemma 3.2 along with the induction hypothesis, it follows that
\[
\sum_{i=0}^{D} E_i(D)' = I(D), \quad A(D)E_i(D)' = \theta_i(D)E_i(D)' \quad \text{for all } i = 0, 1, \ldots, D.
\]

Hence \( E_i(D) = E_i(D)' \) for all \( i = 0, 1, \ldots, D \). The lemma follows. \( \blacksquare \)

Applying Lemma 3.3 yields that
\[
E_1(D) \otimes E_i(D) = q^{-D}(h_{i-1}^*E_{i-1}(D) + a_i^*E_i(D) + c_{i+1}^*E_{i+1}(D))
\]
for all $i = 0, 1, \ldots, D$, where
\[ a_i^* = i(q - 2), \quad b_i^* = (D - i)(q - 1), \quad c_i^* = i \]
for all $i = 0, 1, \ldots, D$. Here $b_{-1}^*, c_{D+1}^*$ are interpreted as any scalars in $\mathbb{C}$. Hence $H(D)$ is $Q$-polynomial with respect to the ordering $\{E_i(D)\}_{i=0}^D$.

Observe that the graph $H(D)$ is vertex-transitive. Without loss of generality, we can consider the dual adjacency matrix $A^*(D)$ of $H(D)$ with respect to $(0, 0, \ldots, 0) \in X^D$. We simply write $A^* = A^*(1)$.

**Lemma 3.4.** Let $D \geq 2$ be an integer. Then
\[
A^*(D) = A^*(D - 1) \otimes I + I(D - 1) \otimes A^*.
\]

**Proof.** Given $y \in X^D$ let $c_y$ denote the coefficient of $\hat{y}$ in $E_1(D) \cdot \hat{y}^D$ with respect to the basis $\{\hat{x}\}_{x \in X^D}$ for $V(D)$. By (3.2), we have
\[
A^*(D)\hat{y} = q^D c_y \hat{y} \quad \text{for all } y \in X^D.
\]
Suppose that $D \geq 2$. Using Lemma 3.3 yields that $c_y = q^{-1}c_{(y_1, \ldots, y_{D-1})} + q^{1-D}c_{y_D}$ for all $y \in X^D$. Hence
\[
A^*(D)\hat{y} = (q^{D-1}c_{(y_1, \ldots, y_{D-1})} + q^{D}c_{y_D}) \hat{y} = A^*(D - 1)(\hat{y}_1 \otimes \cdots \otimes \hat{y}_{D-1}) \otimes \hat{y}_D + \hat{y}_1 \otimes \cdots \otimes \hat{y}_{D-1} \otimes A^* \hat{y}_D
\]
\[
= (A^*(D - 1) \otimes I + I(D - 1) \otimes A^*)\hat{y}
\]
for all $y \in X^D$. The lemma follows. \[\square\]

Let $E_i^*(D)$ denote the $i^{th}$ dual primitive idempotent of $H(D)$ with respect to $(0, 0, \ldots, 0) \in X^D$ for all $i = 0, 1, \ldots, D$. We simply write $E_0^* = E_0^*(1)$ and $E_1^* = E_1^*(1)$. For convenience, we interpret $E_{i-1}^*(D)$ and $E_{D+1}^*(D)$ as the zero matrix in Mat$_{X^D}(\mathbb{C})$.

**Lemma 3.5.** Let $D \geq 2$ be an integer. Then
\[
E_i^*(D) = E_i^*(D - 1) \otimes E_0^* + E_{i-1}^*(D - 1) \otimes E_1^* \quad \text{for all } i = 0, 1, \ldots, D.
\]

**Proof.** It is straightforward to verify the lemma by using (3.1). \[\square\]

Using Lemmas 3.4 and 3.5, a routine induction yields that $A^*(D)E_i^*(D) = \theta_i^*(D)E_i^*(D)$ for all $i = 0, 1, \ldots, D$ where $\theta_i^*(D) = D(q - 1) - qi$.

### 3.3 Proofs of Proposition 1.9 and Theorems 1.10, 1.11

In this subsection, we set
\[
\omega = 1 - \frac{2}{q}.
\]
Let $T(D)$ denote the Terwilliger algebra of $H(D)$ with respect to $(0, 0, \ldots, 0) \in X^D$.

**Definition 3.6.** Let $V_0$ denote the subspace of $V$ consisting of all vectors $\sum_{i=1}^{q-1} c_i \hat{i}$, where $c_1, c_2, \ldots, c_{q-1} \in \mathbb{C}$ with $\sum_{i=1}^{q-1} c_i = 0$. Let $V_1$ denote the subspace of $V$ spanned by $\hat{0}$ and $\sum_{i=1}^{q-1} \hat{i}$.
Definition 3.7. For any \( s \in \{0,1\}^D \), we define the subspace \( V_s(D) \) of \( V(D) \) by
\[
V_s(D) = V_{s_1} \otimes V_{s_2} \otimes \cdots \otimes V_{s_D}.
\]
Note that \( V_0(1) = V_0 \) and \( V_1(1) = V_1 \).

Lemma 3.8. The vector space \( V(D) \) is equal to
\[
\bigoplus_{s \in \{0,1\}^D} V_s(D).
\]

Proof. By Definition 3.6, we have \( V = V_0 \oplus V_1 \). It follows that
\[
V(D) = V^\otimes D = (V_0 \oplus V_1)^\otimes D.
\]
The lemma follows by applying the distributive law of \( \otimes \) over \( \oplus \) to the right-hand side of the above equation.

Lemma 3.9.

(i) There exists a unique representation \( r_0 : \mathcal{R}_\omega \to \text{End}(V_0) \) that sends
\[
A \mapsto \frac{1}{q} A|V_0 + \frac{1}{q}, \quad B \mapsto \frac{1}{q} A^*|V_0 + \frac{1}{q}.
\]
Moreover, the \( \mathcal{R}_\omega \)-module \( V_0 \) is isomorphic to \( (q-2) \cdot L_0 \).

(ii) There exists a unique representation \( r_1 : \mathcal{R}_\omega \to \text{End}(V_1) \) that sends
\[
A \mapsto \frac{1}{q} A|V_1 + \frac{1}{q} - \frac{1}{2}, \quad B \mapsto \frac{1}{q} A^*|V_1 + \frac{1}{q} - \frac{1}{2}.
\]
Moreover, the \( \mathcal{R}_\omega \)-module \( V_1 \) is isomorphic to \( L_1 \).

Proof. (i) The subspace \( V_0 \) of \( V \) is invariant under \( A \) and \( A^* \) acting as scalar multiplication by \(-1\). By Lemma 2.1, the statement (i) follows.

(ii) The subspace \( V_1 \) of \( V \) is invariant under \( A \) and \( A^* \) and the matrices representing \( A \) and \( A^* \) with respect to the basis \( \tilde{0}, \sum_{i=1}^{q-1} \tilde{i} \) for \( V_1 \) are
\[
\begin{pmatrix} 0 & q-1 \\ 1 & q-2 \end{pmatrix}, \quad \begin{pmatrix} q-1 & 0 \\ 0 & -1 \end{pmatrix},
\]
respectively. By Lemma 2.1, the statement (ii) follows.

Definition 3.10. For any \( s \in \{0,1\}^D \), we define the representation \( r_s(D) : \mathcal{R}_\omega \to \text{End}(V_s(D)) \) by
\[
r_s(D) = (r_{s_1} \otimes r_{s_2} \otimes \cdots \otimes r_{s_D}) \circ \Delta_{D-1}.
\]
Note that \( r_0(1) = r_0 \) and \( r_1(1) = r_1 \).

Proposition 3.11. For any integer \( D \geq 2 \) and any \( s \in \{0,1\}^D \), the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{R}_\omega & \xrightarrow{\Delta} & \mathcal{R}_\omega \otimes \mathcal{R}_\omega \\
\text{End}(V_s(D)) \downarrow r_s(D) & & \downarrow r_{s_1, s_2, \ldots, s_{D-1}}(D-1) \otimes r_{s_D} \\
\end{array}
\]
Applying the induction hypothesis the above element is equal to $r\circ \Delta_{D-2}$. Hence
\[ r_{(s_1, s_2, \ldots, s_{D-1})}(D-1) \otimes r_s = (r_{s_1} \otimes r_{s_2} \otimes \cdots \otimes r_{s_{D-1}}) \circ \Delta_{D-2} \otimes r_s = (r_{s_1} \otimes r_{s_2} \otimes \cdots \otimes r_s) \circ (\Delta_{D-2} \otimes 1). \]

By (2.2), the map $\Delta_{D-1} = (\Delta_{D-2} \otimes 1) \circ \Delta$. Combined with Definition 3.10, the following diagram commutes:

\[ \begin{array}{ccc}
\mathfrak{R}_\omega & \xrightarrow{\Delta} & \mathfrak{R}_\omega \otimes \mathfrak{R}_\omega \\
& \downarrow & \downarrow \\
\text{End}(V_s(D)) & \xleftarrow{\Delta_{D-1}} & \mathfrak{R}_\omega \otimes D
\end{array} \]

The proposition follows. \hfill \blacksquare

**Proposition 3.12.** For any $s \in \{0,1\}^D$, the representation $r_s(D) : \mathfrak{R}_\omega \to \text{End}(V_s(D))$ maps

\[ A \mapsto \frac{1}{q} \mathbf{A}(D)|_{V_s(D)} + \frac{D}{q} - \frac{1}{2} \sum_{i=1}^{D} s_i, \quad (3.5) \]

\[ B \mapsto \frac{1}{q} \mathbf{A}^*(D)|_{V_s(D)} + \frac{D}{q} - \frac{1}{2} \sum_{i=1}^{D} s_i. \quad (3.6) \]

**Proof.** We proceed by induction on $D$. By Lemma 3.9, the statement is true when $D = 1$. Suppose that $D \geq 2$. For convenience let $s' = (s_1, s_2, \ldots, s_{D-1}) \in \{0,1\}^{D-1}$. By Lemma 2.5 and Proposition 3.11, the map $r_s(D)$ sends $A$ to $r_{s'}(D-1)(A) \otimes 1 + 1 \otimes r_{s_D}(A)$.

Applying the induction hypothesis the above element is equal to

\[ \left( \frac{1}{q} \mathbf{A}(D-1)|_{V_{s'}(D-1)} + \frac{D-1}{q} - \frac{1}{2} \sum_{i=1}^{D-1} s_i \right) \otimes 1 + 1 \otimes \left( \frac{1}{q} \mathbf{A}|_{V_{s_D}} + \frac{1}{q} \frac{s_D}{2} \right) \]

\[ = \frac{\mathbf{A}(D-1)|_{V_{s'}(D-1)} \otimes 1 + 1 \otimes \mathbf{A}|_{V_{s_D}}}{q} + \frac{D}{q} - \frac{1}{2} \sum_{i=1}^{D} s_i. \]

By Lemma 3.2, the first term in the right-hand side of the above equation equals $\frac{1}{q} \mathbf{A}(D)|_{V_s(D)}$. Hence (3.5) holds. By a similar argument, (3.6) holds. The proposition follows. \hfill \blacksquare

In light of Proposition 3.12, the $\mathcal{T}(D)$-module $V_s(D)$ is a $\mathfrak{R}_\omega$-module for all $s \in \{0,1\}^D$. Combined with Lemma 3.8, the standard $\mathcal{T}(D)$-module $V(D)$ is a $\mathfrak{R}_\omega$-module.

**Lemma 3.13.** Let $p$ be a positive integer. Then the $\mathfrak{R}_\omega$-module $L_1^{\otimes p}$ is isomorphic to

\[ \bigoplus_{k=0}^{\left\lfloor \frac{p}{2} \right\rfloor} \frac{p - 2k + 1}{p - k + 1} \binom{p}{k} \cdot L_{p-2k}. \]
Proof. We proceed by induction on \( p \). If \( p = 1 \), then there is nothing to prove. Suppose that \( p \geq 2 \). Applying the induction hypothesis yields that the \( \mathfrak{R}_\omega \)-module \( L_1^{\otimes p} \) is isomorphic to

\[
\left( \bigoplus_{k=0}^{\lfloor \frac{p-1}{2} \rfloor} \frac{p-2k}{p-k} \binom{p-1}{k} \cdot L_{p-2k-1} \right) \otimes L_1.
\]

Applying the distributive law of \( \otimes \) over \( \oplus \) the above \( \mathfrak{R}_\omega \)-module is isomorphic to

\[
\left( \bigoplus_{k=0}^{\lfloor \frac{p-1}{2} \rfloor} \frac{p-2k}{p-k} \binom{p-1}{k} \cdot (L_{p-2k-1} \otimes L_1) \right).
\]

By Theorem 2.6, the \( \mathfrak{R}_\omega \)-module \( L_{p-2k-1} \otimes L_1 \) is isomorphic to

\[
\begin{cases}
L_{p-2k} \oplus L_{p-2k-2} & \text{if } 0 \leq k \leq \lfloor \frac{p}{2} \rfloor - 1, \\
L_1 & \text{else}
\end{cases}
\]

for all \( k = 0, 1, \ldots, \lfloor \frac{p-1}{2} \rfloor \). Hence the multiplicity of \( L_{p-2k} \) in \( L_1^{\otimes p} \) is equal to

\[
\frac{p-2k}{p-k} \binom{p-1}{k} + \frac{p-2k+2}{p-k+1} \binom{p-1}{k-1} = \frac{p-2k+1}{p-k+1} \binom{p}{k}
\]

for all \( k = 0, 1, \ldots, \lfloor \frac{p}{2} \rfloor \). Here \( \binom{p-1}{k-1} \) is interpreted as 0 when \( k = 0 \). The lemma follows.

Lemma 3.14. Let \( p \) be an integer with \( 0 \leq p \leq D \). Suppose that \( s \in \{0, 1\}^D \) with \( p = \sum_{i=1}^{D} s_i \). Then the \( \mathfrak{R}_\omega \)-module \( V_s(D) \) is isomorphic to

\[
\bigoplus_{k=0}^{\lfloor \frac{p}{2} \rfloor} \frac{p-2k+1}{p-k+1} \binom{p}{k} (q-2)^{D-p} \cdot L_{p-2k}.
\]

Proof. By Definition 3.7, the \( \mathfrak{R}_\omega \)-module \( V_s(D) \) is isomorphic to \( V_1^{\otimes p} \otimes V_0^{\otimes (D-p)} \). Applying Lemma 3.9 the above \( \mathfrak{R}_\omega \)-module is isomorphic to \( (q-2)^{D-p} \cdot L_1^{\otimes p} \). Combined with Lemma 3.13, the lemma follows.

Proof of Proposition 1.9. Let \( p \) and \( k \) be two integers with \( 0 \leq p \leq D \) and \( 0 \leq k \leq \lfloor \frac{p}{2} \rfloor \). Pick any \( s \in \{0, 1\}^D \) with \( p = \sum_{i=1}^{D} s_i \). By Lemma 3.14, the \( \mathfrak{R}_\omega \)-module \( V_s(D) \) contains the irreducible \( \mathfrak{R}_\omega \)-module \( L_{p-2k} \). Let \( \{v_i\}_{i=0}^{p-2k} \) and \( \{w_i\}_{i=0}^{p-2k} \) denote the two bases for \( L_{p-2k} \) described in Lemmas 2.1 and 2.4 with \( n = p - 2k \), respectively. In light of Proposition 3.12, we may view the \( \mathfrak{R}_\omega \)-submodule \( L_{p-2k} \) of \( V_s(D) \) as an irreducible \( T(D) \)-module and denoted by \( L_{p,k}(D) \). To see (i) and (ii), one may evaluate the matrices representing \( A(D) \) and \( A^*(D) \) with respect to the bases \( \{v_i\}_{i=0}^{p-2k} \) and \( \{w_i\}_{i=0}^{p-2k} \) for \( L_{p,k}(D) \), respectively. The proposition follows.

Proof of Theorem 1.10. Let \( p \) be any integer with \( 0 \leq p \leq D \). By Lemma 3.14, for any \( s \in \{0, 1\}^D \) with \( p = \sum_{i=1}^{D} s_i \) the \( T(D) \)-submodule \( V_s(D) \) of \( V(D) \) is isomorphic to

\[
\bigoplus_{k=0}^{\lfloor \frac{p}{2} \rfloor} \frac{p-2k+1}{p-k+1} \binom{p}{k} (q-2)^{D-p} \cdot L_{p,k}(D).
\]

Combined with Lemma 3.8, the result follows.
Proof of Theorem 1.11. Since the standard $\mathcal{T}(D)$-module $V(D)$ contains all irreducible $\mathcal{T}(D)$-modules up to isomorphism, the map $\mathcal{E}$ is onto. Suppose that there are two pairs $(p, k)$ and $(p', k')$ in $\mathbf{P}(D)$ such that the irreducible $\mathcal{T}(D)$-module $L_{p,k}(D)$ is isomorphic to $L_{p',k'}(D)$. Since they have the same dimension, it follows that
\[ p - 2k = p' - 2k'. \tag{3.7} \]
Since $A^*(D)$ has the same eigenvalues in $L_{p,k}(D)$ and $L_{p',k'}(D)$, it follows from Proposition 1.9 that $p - k = p' - k'$. Combined with (3.7), this yields that $(p, k) = (p', k')$. Therefore, $\mathcal{E}$ is one-to-one. ■

Corollary 3.15 ([11, Corollary 3.7]). The algebra $\mathcal{T}(D)$ is isomorphic to
\[ \bigoplus_{p=0}^{D} \bigoplus_{k=0}^{\lfloor \frac{p}{2} \rfloor} \text{Mat}_{p-2k+1}(\mathbb{C}). \]
Moreover, $\dim \mathcal{T}(D) = \binom{D+4}{4}$.

Proof. By Theorem 1.11, the algebra $\mathcal{T}(D)$ is isomorphic to $\bigoplus_{p=0}^{D} \bigoplus_{k=0}^{\lfloor \frac{p}{2} \rfloor} \text{End}(L_{p,k}(D))$. Hence $\dim \mathcal{T}(D)$ is equal to
\[ \sum_{p=0}^{D} \sum_{k=0}^{\lfloor \frac{p}{2} \rfloor} (p - 2k + 1)^2 = \sum_{p=0}^{D} \binom{p+3}{3} = \binom{D+4}{4}. \]
The corollary follows. ■

A Restatements of Proposition 1.9 and Theorems 1.10, 1.11

Recall the irreducible $\mathcal{T}(D)$-module $L_{p,k}(D)$ from Proposition 1.9. Let $r$, $r^*$, $d$, $d^*$ denote the endpoint, dual endpoint, diameter, dual diameter of $L_{p,k}(D)$ respectively. It is known from [18, p. 197] that $\left\lfloor \frac{D - d}{2} \right\rfloor \leq r$, $r^* \leq D - d$. From the results of Section 3.2, we see that
\[ r = r^* = D + k - p, \quad d = d^* = p - 2k. \]
In terms of the parameters $r$ and $d$, the parameters $p$ and $k$ read as
\[ p = 2D - d - 2r, \quad k = D - d - r. \]
Thus we can restate Proposition 1.9 and Theorems 1.10, 1.11 as follows:

Proposition A.1. Let $D$ be a positive integer. For any integers $d$ and $r$ with $0 \leq d \leq D$ and $\left\lfloor \frac{D - d}{2} \right\rfloor \leq r \leq D - d$, there exists a $(d + 1)$-dimensional irreducible $\mathcal{T}(D)$-module $M_{d,r}(D)$ satisfying the following conditions:

(i) There exists a basis for $M_{d,r}(D)$ with respect to which the matrices representing $A(D)$ and $A^*(D)$ are
\[ \begin{pmatrix} \alpha_0 & \gamma_1 & 0 \\ \beta_0 & \alpha_1 & \gamma_2 \\ & \beta_1 & \alpha_2 \\ & & \ddots \\ & & & \beta_{d-1} & \alpha_d \end{pmatrix}, \quad \begin{pmatrix} \theta_0 & 0 \\ \theta_1 & 0 \\ \theta_2 & \ddots \\ \theta_{d-1} & \ddots \end{pmatrix}, \]
respectively.
(ii) There exists a basis for $M_{d,r}(D)$ with respect to which the matrices representing $A(D)$ and $A^*(D)$ are

$$
\begin{pmatrix}
\theta_0 & 0 \\
\theta_1 & \\
\theta_2 & \\
\vdots & \\
0 & \theta_d
\end{pmatrix},
\begin{pmatrix}
\alpha_0 & \gamma_1 & 0 \\
\beta_0 & \alpha_1 & \gamma_2 & \\
\beta_1 & \alpha_2 & & \\
\vdots & \vdots & \ddots & \\
0 & \beta_{d-1} & \gamma_d & \alpha_d
\end{pmatrix},
$$

respectively.

Here the parameters $\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^{d-1}, \{\gamma_i\}_{i=1}^d, \{\theta_i\}_{i=0}^d$ are as follows:

- $\alpha_i = (D - d + i - r)(q - 1) - i - r$ for $i = 0, 1, \ldots, d$,
- $\beta_i = i + 1$ for $i = 0, 1, \ldots, d - 1$,
- $\gamma_i = (q - 1)(d - i + 1)$ for $i = 1, 2, \ldots, d$,
- $\theta_i = D(q - 1) - q(i + r)$ for $i = 0, 1, \ldots, d$.

**Theorem A.2.** Let $D$ be a positive integer. Then the standard $T(D)$-module $V(D)$ is isomorphic to

$$
\bigoplus_{d=0}^D \bigoplus_{r=\left\lceil \frac{D-d}{2} \right\rceil}^{D-d} \frac{D + 1}{D - r + 1} \left( \frac{D}{2D - d - 2r} \right) \left( \frac{2D - d - 2r}{D - d - r} \right) (q - 2)^{d-D+2r} M_{d,r}(D).
$$

We illustrate Theorem A.2 for $D = 3$ and $D = 4$:

| $D$ | $d$ | $r$ | The support of $M_{d,r}(D)$ | The multiplicity of $M_{d,r}(D)$ in $V(D)$ |
|-----|-----|-----|-----------------------------|------------------------------------------|
| 3   | 1   | 0   | $\{0, 1, 2, 3\}$           | 1                                        |
|     | 2   | 1   | $\{1, 2, 3\}$              | $3(q - 2)$                               |
|     | 1   | 1   | $\{1, 2\}$                 | 2                                        |
|     | 2   | 2   | $\{2, 3\}$                 | $3(q - 2)^2$                             |
|     | 0   | 2   | $\{2\}$                    | $3(q - 2)$                               |
|     | 3   | 3   | $\{3\}$                    | $(q - 2)^3$                              |
| 4   | 1   | 0   | $\{0, 1, 2, 3, 4\}$        | 1                                        |
|     | 3   | 1   | $\{1, 2, 3, 4\}$           | $4(q - 2)$                               |
|     | 2   | 1   | $\{1, 2, 3\}$              | 3                                        |
|     | 2   | 2   | $\{2, 3, 4\}$              | $6(q - 2)^2$                             |
|     | 1   | 2   | $\{2, 3\}$                 | $8(q - 2)$                               |
|     | 3   | 3   | $\{3, 4\}$                 | $4(q - 2)^3$                             |
|     | 0   | 2   | $\{2\}$                    | 2                                        |
|     | 3   | 3   | $\{3\}$                    | $6(q - 2)^2$                             |
|     | 4   | 4   | $\{4\}$                    | $(q - 2)^4$                              |

**Theorem A.3.** Let $D$ be a positive integer. Let $P(D)$ denote the set consisting of all pairs $(d, r)$ of integers with $0 \leq d \leq D$ and $\left\lceil \frac{D-d}{2} \right\rceil \leq r \leq D - d$. Let $M(D)$ denote the set of all
isomorphism classes of irreducible $\mathcal{T}(D)$-modules. Then there exists a bijection $\mathbf{P}(D) \rightarrow \mathbf{M}(D)$ given by

$$(d, r) \mapsto \text{the isomorphism class of } M_{d, r}(D)$$

for all $(d, r) \in \mathbf{P}(D)$.

By Theorem A.3, the structure of an irreducible $\mathcal{T}(D)$-module is determined by its endpoint and its diameter. Also we can restate Corollary 3.15 as follows:

**Corollary A.4.** The algebra $\mathcal{T}(D)$ is isomorphic to

$$\bigoplus_{d=0}^{D} \left( \left\lfloor \frac{D - d}{2} \right\rfloor + 1 \right) \cdot \text{Mat}_{d+1}(\mathbb{C}).$$

Moreover, $\dim \mathcal{T}(D) = \binom{D+4}{4}$.

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