THE ADAPTIVE PROJECTED SUBGRADIENT METHOD
CONSTRAINED BY FAMILIES OF QUASI-NONEXPANSIVE MAPPINGS
AND ITS APPLICATION TO ONLINE LEARNING

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ABSTRACT. Many online, i.e., time-adaptive, inverse problems in signal processing and machine learning fall under the wide umbrella of the asymptotic minimization of a sequence of non-negative, convex, and continuous functions. To incorporate a-priori knowledge into the design, the asymptotic minimization task is usually constrained on a fixed closed convex set, which is dictated by the available a-priori information. To increase versatility towards the usage of the available information, the present manuscript extends the Adaptive Projected Subgradient Method (APSM) by introducing an algorithmic scheme which incorporates a-priori knowledge in the design via a sequence of strongly attracting quasi-nonexpansive mappings in a real Hilbert space. In such a way, the benefits offered to online learning tasks by the proposed method unfold in two ways: 1) the rich class of quasi-nonexpansive mappings provides a plethora of ways to cast a-priori knowledge, and 2) by introducing a sequence of such mappings, the proposed scheme is able to capture the time-varying nature of a-priori information. The convergence properties of the algorithm are studied, several special cases of the method with wide applicability are shown, and the potential of the proposed scheme is demonstrated by considering an increasingly important, nowadays, online sparse system/signal recovery task.

1. Introduction

Many online, i.e., time-adaptive, inverse problems in signal processing and machine learning can be recast as follows: if the non-negative integer \(n \in \mathbb{N}\) denotes discrete time, having at our disposal a sequence of multidimensional data \((a_n, d_n)_{n \in \mathbb{N}} \subset \mathbb{R}^L \times \mathbb{R}\), the objective of an online learning method is to infer a possibly time-varying unknown mapping \(x^* : \mathbb{R}^L \rightarrow \mathbb{R}\), which relates the previous data under the following model:

\[
d_n = x^*(a_n) + \zeta_n, \quad \forall n \in \mathbb{N}.
\]

In other words, at the \(n\)-th time instant, the \(L\)-dimensional input signal \(a_n\) interacts with the signal/system which underlies \(x^*\), and our observation is the real valued \(d_n\) which is contaminated by the additive noise \(\zeta_n\).

Online learning methods show distinct differences from their batch counterparts due to the following fundamental reason: batch optimization methods are mobilized after all the necessary data are available to the designer, whereas, in the online scenario, the sequential nature of the data \((a_n, d_n)_{n \in \mathbb{N}}\) dictates that at each time instant \(n\), the newly arriving \((a_n, d_n)\) should be efficiently incorporated into the learning process, without

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the need of solving the optimization task from scratch. Such a sequential mode is not prescribed only by the need for computational efficiency and savings. The online processing of data becomes an efficient tool also in cases of dynamic scenarios, where not only the probability density function of the input data \((a_n, d_n)\) changes with time, but also where the unknown mapping \(x_n\) shows a time-varying nature. In such time dependent environments, and in order to monitor the time variations of the underlying signals and systems, the designer is compelled to gradually disregard data which are associated to the remote past, and to put emphasis on recently received \((a_n, d_n)\). It becomes clear that flexible and multifaceted online learning tools are needed in order to deal with fast emerging signal processing and machine learning applications, like sparsity-aware learning \[19,35,40\], time-adaptive sensor networks \[17,20\], etc.

The unknown mapping \(x_n\) of \((\mathbf{1})\) could be either linear or non-linear. Our assumption on the linearity or not of \(x_n\) dictates the choice of possible spaces into which we perform our search for \(x_n\). If \(x_n\) is assumed linear, then our working space becomes the classical Euclidean \(\mathbb{R}^L\) \[32,47\]. On the other hand, if \(x_n\) is assumed non-linear, a mathematical sound way to model a fairly large amount of non-linear systems is to work in a possibly infinite dimensional \textit{Reproducing Kernel Hilbert Space (RKHS)} \[3\]; a strategy which has been particularly successful in machine learning and pattern recognition tasks \[11,30,37,48,53,54,57,58\]. Since the Euclidean \(\mathbb{R}^L\) is a renowned Hilbert space, and in order to offer a unifying framework for linear and non-linear systems, the stage of the following discussion will be based on a real Hilbert space \(\mathcal{H}\).

Given an estimate \(a \in \mathcal{H}\) of the unknown \(x_n\), the most common way to validate \(a\), with respect to the model \((\mathbf{1})\), is to penalize the disagreement of the observed output \(d_n\) with \(a(a_n)\), i.e., the real-valued difference \(a(a_n) - d_n\). A classical way to quantify such a perception of loss is to use the quadratic function in order to form the penalty \((a(a_n) - d_n)^2\). The popularity of the quadratic loss function is based on its optimality in estimation tasks where the contaminating noise process \((\zeta_n)_{n \in \mathbb{N}}\) is Gaussian \[31\]. However, in order to establish a general framework for estimation problems, where the noise process is not constrained to be Gaussian, and in order to build estimators which show robustness to a wide variety of outliers, we give ourselves the freedom to employ any convex function \(\mathcal{L} : \mathbb{R} \rightarrow [0, \infty)\), and not just the quadratic one, in order to quantify our perception of loss (see for example \[49\]). Having the data \((a_n, d_n)\) as parameters in the design, the following function is naturally defined on the space \(\mathcal{H}\) of our estimates: \(\Theta_n : \mathcal{H} \rightarrow [0, \infty) : a \mapsto \mathcal{L}(a(a_n) - d_n)\). Due to the online nature of the problem, i.e., the sequential data \((a_n, d_n)_{n \in \mathbb{N}}\), we end up in a sequence of loss functions \((\Theta_n)_{n \in \mathbb{N}}\). We stress here that since \(\mathcal{L}\) can be any convex function, \(\Theta_n\) is not bound to be differentiable.

Theory, e.g., Bayesian inference \[31\], as well as everyday practice suggest that apart from the information included in the training sequence \((a_n, d_n)_{n \in \mathbb{N}}\), estimation is enhanced if one employs also the \textit{a-priori knowledge} about the unknown system \(x_n\). We will abide here by the set theoretic estimation approach \[21\] and quantify the a-priori knowledge as a closed convex set \(C\) in \(\mathcal{H}\). The first attempt to attack the task of online learning as the asymptotic minimization of a sequence \((\Theta_n)_{n \in \mathbb{N}}\), over a nonempty closed convex set \(C\), was given in \[32,63\], by means of the following simple iteration, called the \textit{Adaptive Projected Subgradient Method (APSM)}; for an arbitrary initial point \(u_0 \in \mathcal{H}\), let

\[
\forall n \in \mathbb{N}, \quad u_{n+1} := \begin{cases}
P_C \left( u_n - \lambda_n \frac{\Theta_n(u_n)}{\|\Theta_n'(u_n)\|} \Theta_n'(u_n) \right), & \text{if } \Theta_n'(u_n) \neq 0, \\
P_C(u_n), & \text{if } \Theta_n'(u_n) = 0,
\end{cases}
\]

where \(\lambda_n \in (0, 2)\), \(P_C\) stands for the metric projection mapping onto \(C\), and \(\Theta_n'(u_n)\) denotes any subgradient of \(\Theta_n\) at \(u_n\), \(\forall n \in \mathbb{N}\). The previous recursion is a time-adaptive generalization of the classical algorithm of Polyak \[44\], which deals with the minimization problem of a \textit{fixed}, non-smooth, convex and continuous function \(\Theta\) over \(C\). Besides the new directions for online learning \[58\], the previous recursion has offered also
a unification of several standard algorithms in classical adaptive filtering [32, 47]. Indeed, by letting \( H := \mathbb{R}^L \), for an appropriately chosen sequence \((\Theta_n)_{n \in \mathbb{N}}\), and by substituting \( P_C \) with the identity mapping, the previous recursion [62, 63, 66] results in the classical Normalized Least Mean Squares (NLMS) [1, 41] and the, vastly used nowadays, Affine Projection Algorithm (APA) [33, 43].

It is often the case that a single closed convex set \( C \), or even better, a single metric projection mapping \( P_C \), cannot capture the diversity of the a-priori knowledge in signal processing applications. For example, in a robust beamforming problem [55], the a-priori knowledge is usually expressed as \( C = \bigcap_{m=1}^M C_m \), where \( \{C_m\}_{m=1}^M \) is a number of closed convex sets, with associated projection mappings \( \{P_{C_m}\}_{m=1}^M \) that are usually easy to compute. However, an analytic expression for \( P_C \) might not be available [55]. Secondly, erroneous a-priori information may result into an empty \( C = \bigcap_{m=1}^M C_m = \emptyset \) [55, 67]. How is it possible to deal with multiple closed convex sets \( \{C_m\}_{m=1}^M \) where an analytical expression of \( P_C \) is not available, or the \( \{C_m\}_{m=1}^M \) share an empty intersection? Avoiding the straightforward and recently popular solution of relaxing the original constraints, the study in [56] provides with a solution to the previous problem and extends [62, 63] by using a mapping \( T \), in the place of \( P_C \), which belongs to the general class of strongly attracting nonexpansive mappings. Indeed, the method [56] demonstrated its potential in a wide variety of online learning tasks, which span from classical linear adaptive filtering [67] to non-linear classification and regression tasks [58].

It is natural to ask now whether we can add more freedom to the usage of the a-priori knowledge. Our motivation is based on a couple of elementary observations. First, given the well-known fact that a nonempty closed convex set \( C \) is the set of all minimizers of the distance function \( d(\cdot, C) \) to \( C \), one of the ways to visualize a-priori knowledge could be the set of all minimizers of a generally non-smooth convex function defined on an appropriate Hilbert space \( H \). Secondly, it is often the case in practice where a minimizer of a convex function cannot be reached either by an analytical formula or a computationally cheap process. A powerful mapping, whose recursive application is known to minimize a generally non-differentiable convex function, is the subgradient projection mapping [67, 64]. It is also known that this operator belongs to the class of quasi-nonexpansive mappings [6, 7, 64], which strictly contains all the strongly attracting nonexpansive mappings, utilized in [56]. Now, the question arises naturally: does the APSM still operate when constrained by the general class of quasi-nonexpansive mappings, and can we, thus, devise a method with more freedom in incorporating a-priori information, than in the studies of [56, 62, 63]? Given the wide applicability of the APSM in online learning tasks [58], it is anticipated that such a generalization will add further flexibility to the APSM in order to tackle more challenging online learning tasks, which have been recently emerging both in signal processing and machine learning [2, 17, 19, 20, 35, 40].

The present manuscript introduces an extension of the APSM [56, 62, 63], towards a more flexible usage of the a-priori information, in two ways: 1) by considering a strictly larger class of mappings than in [56, 62, 63], and in particular, operators taken from the rich family of quasi-nonexpansive mappings, and 2) by letting these mapping to be time-varying in order to capture the, quite often in signal processing and machine learning applications, dynamic nature of the a-priori information. Put in mathematical terms, the problem to be studied is the following.

**Problem 1** (Constrained asymptotic minimization task). Given a sequence of convex, continuous, and not necessarily differentiable functions \((\Theta_n : H \to [0, \infty))_{n \in \mathbb{N}}\), and a sequence of strongly attracting quasi-nonexpansive mappings \((T_n : H \to H)_{n \in \mathbb{N}}\), with nonempty fixed point sets \((\text{Fix}(T_n))_{n \in \mathbb{N}}\), we are looking for a sequence \((u_n)_{n \in \mathbb{N}}\) that asymptotically minimizes \((\Theta_n)_{n \in \mathbb{N}}\) over \((\text{Fix}(T_n))_{n \in \mathbb{N}}\). Strictly speaking, our objective is to generate a \((u_n)_{n \in \mathbb{N}}\) such that \( \lim_{n \to \infty} \Theta_n(u_n) = 0 \), and the set of its strong cluster points \( \mathcal{S}((u_n)_{n \in \mathbb{N}}) \) lies in \( \limsup_{n \to \infty} \text{Fix}(T_n) \), i.e., \( \mathcal{S}((u_n)_{n \in \mathbb{N}}) \subseteq \limsup_{n \to \infty} \text{Fix}(T_n) \).
Our algorithmic tool to tackle the previous optimization task is the following.

**Algorithm 1.** Given an arbitrary initial point \( u_0 \in \mathcal{H} \), generate the following sequence:

\[
\forall n \in \mathbb{N}, \quad u_{n+1} := \begin{cases} 
T_n \left( u_n - \lambda_n \frac{\Theta_n'(u_n)}{\|\Theta_n'(u_n)\|} \right), & \text{if } \Theta_n'(u_n) \neq 0, \\
T_n(u_n), & \text{if } \Theta_n'(u_n) = 0,
\end{cases}
\]

where \( \lambda_n \in (0, 2) \) and \( \Theta_n'(u_n) \) stands for any subgradient of \( \Theta_n \) at \( u_n, \forall n \in \mathbb{N} \).

The manuscript is organized as follows. A series of necessary definitions and facts are included in Section 2. The algorithm and its convergence analysis follow in Section 3. Special cases of the algorithm, with a wide application range in online learning, can be found in Section 4. The potential of the method is shown in Section 5 by introducing a low-complexity time-adaptive learning technique for the increasingly important, nowadays, sparse system/signal recovery task.

## 2. Preliminaries

We start with several notations which will be frequently used in the sequel.

The set of all non-negative integers, positive integers, and real numbers will be denoted by \( \mathbb{N}, \mathbb{N}_+, \) and \( \mathbb{R} \), respectively. The set of all subsequences of \( \mathbb{N} \) will be denoted by \( \mathbb{N}_\# \), i.e., \( \mathbb{N}_\# := \{ N \subset \mathbb{N} : N \text{ is infinite} \} \). Any \( N \in \mathbb{N}_\# \) can be also denoted by the standard way of \( N = (n_k)_{k \in \mathbb{N}} \). Define, also, \( \mathbb{N}_\infty := \{ N \subset \mathbb{N} : \mathbb{N} \setminus N \text{ is finite} \} \). In other words, \( \mathbb{N}_\infty \) contains all the “neighborhoods of \( \infty \)”, with respect to \( \mathbb{N} \), while \( \mathbb{N}_\# \) is its associated “grill” \( \# \).

Henceforth, the symbol \( \mathcal{H} \) will stand for a real Hilbert space, equipped with an inner product \( \langle \cdot, \cdot \rangle \), and a norm \( \| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle} \). In the case where \( \mathcal{H} \) becomes the Euclidean \( \mathbb{R}^L \), \( L \in \mathbb{N}_+ \), any element of \( \mathbb{R}^L \) will be denoted by boldfaced symbols. The inner product of \( \mathbb{R}^L \) will be the classical vector dot product, i.e., \( \langle v_1, v_2 \rangle := v_1^t v_2, \forall v_1, v_2 \in \mathbb{R}^L \), where the superscript \( t \) stands for vector/matrix transposition.

Given an \( x \in \mathcal{H} \) and a \( \rho > 0 \), an open ball is defined as the set \( B(x, \rho) := \{ v \in \mathcal{H} : \| x - v \| < \rho \} \), while a closed ball \( B[x, \rho] := \{ v \in \mathcal{H} : \| x - v \| \leq \rho \} \). Given \( S, \Upsilon \subset \mathcal{H} \), the relative interior of \( S \) with respect to \( \Upsilon \) is defined as \( \text{ri}_\Upsilon S := \{ \hat{v} \in S : \exists \rho > 0, \emptyset \neq (B(\hat{v}, \rho) \cap \Upsilon) \subset S \} \). The interior of \( S \) is defined as \( \text{int } S := \text{ri}_\mathcal{H} S \).

Given \( S \subset \mathcal{H} \), define the distance function to \( S \) as follows: \( d(\cdot, S) : \mathcal{H} \to [0, \infty) : x \mapsto d(x, S) := \inf \{ \| x - v \| : v \in S \} \). Given any nonempty closed convex set \( C \subset \mathcal{H} \), the (metric) projection onto \( C \) is defined as the mapping \( P_C : \mathcal{H} \to C \) which maps to an \( x \in \mathcal{H} \) the (unique) \( P_C(x) \in C \) such that \( \| x - P_C(x) \| = d(x, C) \).

**Definition 2** (Subdifferential and subgradient). Given a convex function \( \Theta : \mathcal{H} \to \mathbb{R} \), the subdifferential of \( \Theta \) is defined as the set-valued mapping:

\[
\partial \Theta : \mathcal{H} \to 2^{\mathcal{H}} : x \mapsto \partial \Theta(x) := \{ v \in \mathcal{H} : \forall y \in \mathcal{H}, \langle v, y - x \rangle + \Theta(x) \leq \Theta(y) \}.
\]

In the case where \( \Theta \) is continuous at \( x \), then \( \partial \Theta(x) \neq \emptyset \). Any element in \( \partial \Theta(x) \) will be called a subgradient of \( \Theta \) at \( x \), and will be denoted by \( \Theta'(x) \). If \( \Theta \) is Gâteaux differentiable at \( x \), then \( \partial \Theta(x) \) becomes a singleton, and the unique element of \( \partial \Theta(x) \) is nothing but the classical Gâteaux differential of \( \Theta \) at \( x \). Notice, also, the well-known fact: \( 0 \in \partial \Theta(x) \iff x \in \text{arg min}_{v \in \mathcal{H}} \Theta(v) \).

**Example 3.** The subdifferential of the metric distance function to a closed convex set \( C \subset \mathcal{H} \) is given as follows:

\[
\partial d(x, C) = \begin{cases} 
N_C(x) \cap B[0, 1], & \text{if } x \in C, \\
\{ x - P_C(x) \} / d(x, C), & \text{if } x \in \mathcal{H} \setminus C,
\end{cases}
\]

where \( N_C(x) := \{ v \in \mathcal{H} : \forall y \in C, \langle v, y - x \rangle \leq 0 \} \). Notice that \( \forall x \in \mathcal{H}, \forall d'(x, C) \in \partial d(x, C), \| d'(x, C) \| \leq 1 \).
Definition 4 ([5][7][64]). Given a mapping \( T : \mathcal{H} \to \mathcal{H} \), the set of all fixed points of \( T \), i.e., \( \text{Fix}(T) := \{ v \in \mathcal{H} : T(v) = v \} \), is called the fixed point set of \( T \). Assume a \( T : \mathcal{H} \to \mathcal{H} \) such that \( \text{Fix}(T) \neq \emptyset \). The mapping \( T \) will be called quasi-nonexpansive if \( \forall x \in \mathcal{H}, \forall v \in \text{Fix}(T), \| T(x) - v \| \leq \| x - v \| \). It can be verified that the fixed point set of a quasi-nonexpansive mapping is closed and convex, e.g., [6, Prop. 2.3 and 2.6]. If \( \| T(x) - v \| \leq \| x - v \|^2 - \| T(x) - v \|^2 \), then \( T \) will be called \( \eta \)-attracting or strongly attracting quasi-nonexpansive.

Now, if \( \forall x, y \in \mathcal{H}, \| T(x) - T(y) \| \leq \| x - y \| \), then \( T \) will be called nonexpansive. In the case where \( T \) is both nonexpansive and strongly attracting quasi-nonexpansive, then it will be called strongly attracting nonexpansive.

In particular, an 1-attracting (quasi)-nonexpansive mapping will be called firmly (quasi)-nonexpansive.

Fact 5 (Equivalent description of strongly attracting quasi-nonexpansive mappings [60,61]). The following statements are equivalent for a mapping \( T : \mathcal{H} \to \mathcal{H} \).

1. \( T \) is \( \eta \)-attracting quasi-nonexpansive.
2. \( T \) is \( \frac{1}{1+\eta} \)-averaged quasi-nonexpansive. A mapping \( T \) is called \( \alpha \)-averaged quasi-nonexpansive, with \( \alpha \in (0,1) \), if there exists a quasi-nonexpansive mapping \( R : \mathcal{H} \to \mathcal{H} \) such that \( T = (1-\alpha)I + \alpha R \).

In particular, \( T \) is firmly quasi-nonexpansive iff \( T \) is \( \frac{1}{2} \)-averaged quasi-nonexpansive. Notice that \( \forall \alpha \in (0, 1), \text{Fix}(T) = \text{Fix}(R) \), which suggests that given a quasi-nonexpansive mapping \( R \), we can always construct a strongly attracting quasi-nonexpansive \( T \) that shares the same fixed point set with \( R \).

Example 6 (Subgradient projection mapping). Given a convex continuous function \( \Theta \), such that \( \text{lev}_{\leq 0} \Theta := \{ v \in \mathcal{H} : \Theta(v) \leq 0 \} \neq \emptyset \), define the subgradient projection mapping \( T_{\Theta} : \mathcal{H} \to \mathcal{H} \) with respect to \( \Theta \) as follows:

\[
T_{\Theta}(x) := \begin{cases} 
  x - \frac{\Theta(x)}{\| \Theta(x) \|} \Theta'(x), & \text{if } x \in \mathcal{H} \setminus \text{lev}_{\leq 0} \Theta, \\
  x, & \text{if } x \in \text{lev}_{\leq 0} \Theta,
\end{cases}
\]

where \( \Theta'(x) \) is any subgradient in \( \partial \Theta(x) \). If \( I \) stands for the identity mapping in \( \mathcal{H} \), the mapping

\[
T_{\Theta}^{(\lambda)} := I + \lambda(T_{\Theta} - I), \quad \lambda \in (0, 2),
\]

will be called the relaxed subgradient projection mapping with respect to \( \Theta \). It can be verified that \( \forall \lambda \in (0, 2), \text{Fix}(T_{\Theta}^{(\lambda)}) = \text{Fix}(T_{\Theta}) = \text{lev}_{\leq 0} \Theta \) [6]. Moreover, \( \forall \lambda \in (0, 2) \), the mapping \( T_{\Theta}^{(\lambda)} \) is \( \frac{2-\alpha}{\alpha} \)-attracting quasi-nonexpansive [6].

Example 7 (Relaxed metric projection mapping). Let a nonempty closed convex set \( C \subset \mathcal{H} \) and its associated metric projection mapping \( P_C \). Then, the relaxed (metric) projection mapping, \( T_{C}^{(\alpha)} := I + \alpha(P_C - I) \), \( \alpha \in (0, 2) \), is \( \frac{2-\alpha}{\alpha} \)-attracting nonexpansive with fixed point set \( \text{Fix}(T_{C}^{(\alpha)}) = C \) [5].

Example 8 ([5,63]). Let \( T_1, T_2 \) be \( \eta_1 \)- and \( \eta_2 \)-attracting (quasi)-nonexpansive mappings, respectively. Assume also that \( \text{Fix}(T_1) \cap \text{Fix}(T_2) \neq \emptyset \). Then, the mapping \( T_1T_2 \) is \( \frac{\eta_1\eta_2}{\eta_1+\eta_2} \)-attracting (quasi)-nonexpansive, and \( \text{Fix}(T_1T_2) = \text{Fix}(T_1) \cap \text{Fix}(T_2) \).

Definition 9 (Demiclosed mapping at 0). A mapping \( T : \mathcal{H} \to \mathcal{H} \) will be called demiclosed at 0 if the following property holds: for a sequence \( (x_n)_{n \in \mathbb{N}} \subset \mathcal{H} \), and an \( x_* \in \mathcal{H} \),

\[
\begin{cases} 
  x_n \xrightarrow{n \to \infty} x_*, \\
  T(x_n) \xrightarrow{n \to \infty} 0,
\end{cases}
\]

then \( T(x_*) = 0 \), where the symbols \( \to \) and \( \rightarrow \) denote weak and strong convergence in \( \mathcal{H} \), respectively.
Example 10 ([42 Lem. 2]). If $T : \mathcal{H} \to \mathcal{H}$ is a nonexpansive mapping, then $I - T$ is demiclosed at 0.

Example 11 ([3 Prop. 6.10], [60]). Let a continuous convex function $\Theta : \mathcal{H} \to \mathbb{R}$ such that $\text{lev}_{\leq 0} \Theta \neq \emptyset$. Then, $\forall \lambda \in (0, 2)$, the mapping $I - T_{\Theta}^{(\lambda)}$ is demiclosed at 0, where $T_{\Theta}^{(\lambda)}$ stands for the relaxed subgradient projection mapping with respect to $\Theta$.

Fact 12 ([63]). Assume a sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}$, and a closed convex set $C \subseteq \mathcal{H}$. Assume that

$$\exists \kappa > 0 : \forall v \in C, \forall n \in \mathbb{N}, \quad \kappa \|x_{n+1} - x_n\|^2 \leq \|x_n - v\|^2 - \|x_{n+1} - v\|^2.$$  

If there exists, also, a hyperplane $\Pi$ such that $\text{ri}_\Pi C \neq \emptyset$, then $\exists x_* \in \mathcal{H}$ such that $x_* = \lim_{n \to \infty} x_n$.

Definition 13 (Inner and outer limits [43][46]). Given a sequence of subsets $(S_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}$, define the inner and outer limits:

$$\liminf_{n \to \infty} S_n := \left\{ x \in \mathcal{H} : \exists N \in \mathbb{N}_\infty, \exists x_n \in S_n, \forall n \in N, \text{ such that } \lim_{n \in N} x_n = x \right\} = \left\{ x \in \mathcal{H} : \limsup_{n \to \infty} d(x, S_n) = 0 \right\} = \bigcap_{N \in \mathbb{N}_\infty} \overline{\bigcup_{n \in N} S_n}$$

$$= \bigcap_{\epsilon > 0} \bigcup_{n=1}^\infty \bigcap_{k=n} (S_k + B[0, \epsilon]) \quad (3)$$

$$\limsup_{n \to \infty} S_n := \left\{ x \in \mathcal{H} : \exists N \in \mathbb{N}_\infty, \exists x_n \in S_n, \forall n \in N, \text{ such that } \lim_{n \in N} x_n = x \right\} = \left\{ x \in \mathcal{H} : \liminf_{n \to \infty} d(x, S_n) = 0 \right\} = \bigcap_{N \in \mathbb{N}_\infty} \overline{\bigcup_{n \in N} S_n}$$

$$= \bigcap_{\epsilon > 0} \bigcap_{n=1}^\infty \bigcup_{k=n} (S_k + B[0, \epsilon]) \quad (4)$$

where $S_k + B[0, \epsilon] := \{ s + b : s \in S_k, b \in B[0, \epsilon] \}$, and the overline symbol stands for the closure of a set. In a similar fashion, given a sequence of subsets $(S_n)_{n \in \mathbb{N}}$, and a subsequence $N = (n_k)_{k \in \mathbb{N}} \in \mathbb{N}_\infty$, the notation $\liminf_{n \in N} S_n$ is defined as $\liminf_{k \to \infty} S_{n_k}$. Likewise, $\limsup_{n \in N} S_n := \limsup_{k \to \infty} S_{n_k}$.

3. The Analysis of the Algorithm

3.1. A useful theorem. Prior to the analysis of Algorithm [11 we state and prove Theorem [15 which will be repeatedly used in the sequel. The proof of Theorem [15 will be based on the following assumption.

Assumption 14. Assume a sequence of mappings $(T_n : \mathcal{H} \to \mathcal{H})_{n \in \mathbb{N}}$ with nonempty fixed point sets $(\text{Fix}(T_n))_{n \in \mathbb{N}}$. For any subsequence $N \in \mathbb{N}_\infty$, for any sequence $(x_n)_{n \in N} \subseteq \mathcal{H}$, and for any $\gamma > 0$ such that $\forall n \in N$, $d(x_n, \text{Fix}(T_n)) \geq \gamma$, there exists a $\delta > 0$ such that $\liminf_{n \in N} \| (I - T_n)(x_n) \| \geq \delta$.

Theorem 15. Assume a sequence of mappings $(T_n : \mathcal{H} \to \mathcal{H})_{n \in \mathbb{N}}$, with nonempty fixed point sets $(\text{Fix}(T_n))_{n \in \mathbb{N}}$, such that Assumption [14 is satisfied.

1. Assume a subsequence $N \in \mathbb{N}_\infty$, a sequence $(x_n)_{n \in N} \subseteq \mathcal{H}$ and an $x_* \in \mathcal{H}$.

$$\begin{cases}
  x_n \xrightarrow{n \in N} x_*, \\
  (I - T_n)(x_n) \xrightarrow{n \in N} 0,
\end{cases}$$

then $x_* \in \liminf_{n \in N} \text{Fix}(T_n)$. 

2. Let \( \mathcal{S}((x_n)_{n \in \mathbb{N}}) \) be the set of all strong cluster points of a sequence \((x_n)_{n \in \mathbb{N}}\).

\[
\begin{cases}
\mathcal{S}((x_n)_{n \in \mathbb{N}}) \neq \emptyset, \\
(I - T_n)(x_n) \xrightarrow{n \to \infty} 0,
\end{cases}
\]

then \( \mathcal{S}((x_n)_{n \in \mathbb{N}}) \subseteq \limsup_{n \to \infty} \text{Fix}(T_n). \)

**Proof.** 1. We will prove Theorem 15.1 by contradiction, i.e., assume that \( x_* \notin \liminf_{n \in \mathbb{N}} \text{Fix}(T_n). \)

By \( \| \cdot \| \), \( \limsup_{n \in \mathbb{N}} d(x_*, \text{Fix}(T_n)) > 0 \), i.e., there exists \( \tau > 0 \), and \( \exists N' \in \mathbb{N}_\infty \), such that \( \forall n \in N' \cap N, \ d(x_*, \text{Fix}(T_n)) > \tau. \)

Moreover, since \( \lim_{n \to \infty} x_n = x_* \), there exists an \( N_0 \in \mathbb{N}_\infty \) such that \( \forall n \in N_0 \cap N, \ |x_* - x_n| < \frac{\tau}{2}. \)

Having these in mind, the triangle inequality \( |x_* - v| \leq |x_* - x_n| + |x_n - v|, \forall v \in \text{Fix}(T_n) \), leads us to the following:

\[
\forall n \in N_0 \cap N', \ d(x_n, \text{Fix}(T_n)) \geq d(x_*, \text{Fix}(T_n)) - |x_* - x_n| > \tau - \frac{\tau}{2} =: \gamma > 0.
\]

Hence, there exists a subsequence \( N'' := N_0 \cap N' \cap N \in \mathbb{N}_\infty \) such that \( \forall n \in N'', \ d(x_n, \text{Fix}(T_n)) \geq \gamma. \)

Now, by Assumption 14, there exists a \( \delta > 0 \) such that

\[
0 < \delta \leq \liminf_{n \in N''} \| (I - T_n)(x_n) \| = \lim_{n \in N''} \| (I - T_n)(x_n) \| = 0,
\]

where the last two equalities come from the fact that \( N'' \subset N \). This contradiction establishes Theorem 15.1.

2. Choose arbitrarily an \( x_* \in \mathcal{S}((x_n)_{n \in \mathbb{N}}) \). By definition, there exists a subsequence \( N \in \mathbb{N}_\infty \) such that \( \lim_{n \in \mathbb{N}} x_n = x_* \). Hence, by Theorem 15.1 \( x_* \in \liminf_{n \in \mathbb{N}} \text{Fix}(T_n) \). By Definition 13, \( \exists N_0 \in \mathbb{N}_\infty \) and \( \exists x_n' \in \text{Fix}(T_n), \forall n \in N \cap N_0 \) such that \( \lim_{n \in N \cap N_0} x_n' = x_* \).

Clearly, \( N' := N \cap N_0 \in \mathbb{N}_\infty \). In other words, \( \exists N' \in \mathbb{N}_\infty, \exists x_n' \in \text{Fix}(T_n), \forall n \in N' \) such that \( \lim_{n \in N} x_n' = x_* \), i.e., \( x_* \in \limsup_{n \to \infty} \text{Fix}(T_n) \) by Definition 13. Since \( x_* \) was chosen arbitrarily, Theorem 15.2 is established.

Next is an example of a sequence of mappings which satisfies Assumption 14 and which will be used later on in the sequel. Another example of a family of mappings which satisfies Assumption 14 and which relates to the minimization of an \( \ell_1 \)-norm loss function, will be seen in Lemma 26.4.

**Example 16.** Assume a sequence of nonempty closed convex sets \((S_n)_{n \in \mathbb{N}}\), the associated sequence of relaxed metric projection mappings

\[
T_{S_n}^{(a_n)} := I + \alpha_n (P_{S_n} - I), \quad \alpha_n \in (0, 2), \forall n \in \mathbb{N},
\]

and the existence of a sufficiently small \( \epsilon > 0 \) such that \( \alpha_n \in [\epsilon, 2), \forall n \in \mathbb{N} \). Then, the sequence of mappings \((T_{S_n}^{(a_n)})_{n \in \mathbb{N}}\) satisfies Assumption 14.

**Proof.** First of all, by Example 7 \( \forall n \in \mathbb{N}, \text{Fix}(T_{S_n}^{(a_n)}) = S_n \). Choose, now, arbitrarily an \( N \in \mathbb{N}_\infty \), a sequence \((x_n)_{n \in \mathbb{N}} \subset \mathcal{H}, \) and a \( \gamma > 0 \), such that \( \forall n \in N, \ d(x_n, \text{Fix}(T_{S_n}^{(a_n)})) = d(x_n, S_n) \geq \gamma. \) Then, it is easy to verify by the definition of \( T_{S_n}^{(a_n)} \) that

\[
\forall n \in N, \quad \| (I - T_{S_n}^{(a_n)})(x_n) \| = \alpha_n d(x_n, S_n) \geq \epsilon \gamma > 0.
\]

Therefore, there exists a \( \delta > 0 \) such that \( \liminf_{n \in \mathbb{N}} \| (I - T_{S_n}^{(a_n)})(x_n) \| \geq \delta, \) and Assumption 14 is established. \( \square \)
3.2. The Main Analysis. Given a sequence of convex, continuous, and not necessarily differentiable functions \((\Theta_n : \mathcal{H} \to [0, \infty])_{n \in \mathbb{N}}\), and a sequence of \(\eta_n\)-attracting quasi-nonexpansive mappings \((T_n : \mathcal{H} \to \mathcal{H})_{n \in \mathbb{N}}\), with \(\eta_n > 0, \forall n \in \mathbb{N}\), and with nonempty fixed point sets \((\text{Fix}(T_n))_{n \in \mathbb{N}}\), the convergence analysis of Algorithm 1 given in Theorem 18 will be based on the following series of assumptions.

**Assumption 17.**
1. There exists an \(N \in \mathbb{N}_\infty\) such that \(\forall n \in N, \Omega_n := \text{Fix}(T_n) \cap \text{lev} \leq 0 \Theta_n \neq \emptyset\).
2. There exists an \(N \in \mathbb{N}_\infty\) such that \(\Omega := \bigcap_{n \in \mathbb{N}} \Omega_n \neq \emptyset\).
3. Choose an \(\epsilon \in (0, 1]\), and let \(\forall n \in \mathbb{N}, \lambda_n \in [\epsilon, 2 - \epsilon]\).
4. The sequence \((\Theta_n^{(\eta_n)})_{n \in \mathbb{N}}\) is bounded.
5. Define \(\hat{\eta} := \inf\{\eta_n : n \in \mathbb{N}\}, \hat{\eta} := \sup\{\eta_n : n \in \mathbb{N}\}\). Then, assume that \(\hat{\eta} > 0\) and \(\hat{\eta} < \infty\).
6. The sequence of relaxed subgradient projection mappings \((T_{\Theta_n}^{(\lambda_n)})_{n \in \mathbb{N}}\) satisfies Assumption 14.
7. The sequence of mappings \((T_n)_{n \in \mathbb{N}}\) satisfies Assumption 14.
8. Assume that \(\forall n \in \mathbb{N}, T_n := T, \) where \(T\) is a strongly attracting quasi-nonexpansive mapping with \(\text{Fix}(T) \neq \emptyset\), and \(I - T\) is demidlosed at 0.
9. The set \(\mathcal{G}((u_n)_{n \in \mathbb{N}})\) of all strong cluster points of the sequence \((u_n)_{n \in \mathbb{N}}\) is nonempty.
10. There exists a hyperplane \(\Pi\) such that \(\text{ri}_\Pi(\Omega) \neq \emptyset\).

**Theorem 18 (Properties of Algorithm 1).**
1. Let Assumption 17(1) hold true. Then, \(\forall n \in \mathbb{N}, d(u_{n+1}, \Omega_n) \leq d(u_n, \Omega_n)\).
2. Let Assumption 17(2) hold true. Then, \(\forall n \in \mathbb{N}, d(u_{n+1}, \Omega) \leq d(u_n, \Omega)\).
3. Let Assumption 17(2) hold true. Then, \(\forall v \in \Omega, \) the sequence \((\|u_n - v\|)_{n \in \mathbb{N}}\) converges.
4. Let Assumption 17(2) hold true. Then, the set of all weakly sequential cluster points of the sequence \((u_n)_{n \in \mathbb{N}}\) is nonempty, i.e., \(\mathcal{W}((u_n)_{n \in \mathbb{N}}) \neq \emptyset\).
5. Let Assumptions 17(2) and 17(3) hold true. Then, \[
\lim_{n \to \infty} \left\| (I - T_{\Theta_n}^{(\lambda_n)})(u_n) \right\| = \lim_{n \to \infty} \frac{\Theta_n(u_n)}{\|\Theta_n^{(\eta_n)}(u_n)\|} = 0,
\]
where, in order to avoid ambiguities, we let \(\frac{0}{0} := 0\).
6. Let Assumptions 17(2), 17(3) and 17(4) hold true. Then, \(\lim_{n \to \infty} \Theta_n(u_n) = 0\).
7. Let Assumptions 17(2), 17(3), 17(6) and 17(7) hold true. Then, \(\mathcal{G}((u_n)_{n \in \mathbb{N}}) \subset \limsup_{n \to \infty} \text{lev} \leq 0 \Theta_n\). If, in addition, the set \(\mathcal{G}((u_n)_{n \in \mathbb{N}})\) is a singleton, i.e., there exists a \(u_*\) such that \(\{u_*\} = \mathcal{G}((u_n)_{n \in \mathbb{N}})\), then, \(u_* \in \liminf_{n \to \infty} \text{lev} \leq 0 \Theta_n\).
8. Let Assumptions 17(2) and 17(5) hold true. Then, \(\lim_{n \to \infty} (I - T_n)(T_{\Theta_n}^{(\lambda_n)})(u_n)) = 0\).
9. Let Assumptions 17(2), 17(3), 17(5), 17(7) and 17(9) hold true. Then, \(\mathcal{G}((u_n)_{n \in \mathbb{N}}) \subset \limsup_{n \to \infty} \text{Fix}(T_n)\). If, in addition, the set \(\mathcal{G}((u_n)_{n \in \mathbb{N}})\) is a singleton, i.e., there exists a \(u_*\) such that \(\{u_*\} = \mathcal{G}((u_n)_{n \in \mathbb{N}})\), then, \(u_* \in \liminf_{n \to \infty} \text{Fix}(T_n)\).
10. Let Assumptions 17(2), 17(3), 17(8) and 17(10) hold true. Then, \(\mathcal{W}((u_n)_{n \in \mathbb{N}}) \subset \text{Fix}(T)\).
11. Let Assumptions 17(2), 17(3), 17(8) and 17(9) hold true. Then, \(\mathcal{G}((u_n)_{n \in \mathbb{N}}) \subset \text{Fix}(T)\).
12. Let Assumptions 17(2), 17(3), 17(5) and 17(10) hold true. Then, \(\exists u_* \in \mathcal{H} : \lim_{n \to \infty} u_n = u_*\), i.e., \(\mathcal{G}((u_n)_{n \in \mathbb{N}}) = \{u_*\}\).

**Proof.** 1. By assumption 17(1) \(\forall n \in N, \text{lev} \leq 0 \Theta_n \neq \emptyset\). Recall also the fundamental fact that \(0 \in \partial \Theta_n(u_n) \iff u_n \in \arg \min_{v \in \mathcal{H}} \Theta_n(v)\).
Fix any \( n \in N \). Consider the case where \( u_n \notin \text{lev}_{\leq 0} \Theta_n \iff \Theta_n(u_n) > 0 \implies \Theta'_n(u_n) \neq 0 \). Then, by (2),
\[
u_{n+1} = T_{\Theta} \left( u_n - \lambda_n \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|} \Theta'_n(u_n) \right) .
\]
Now, assume that \( u_n \in \text{lev}_{\leq 0} \Theta_n \iff \Theta_n(u_n) = 0 \). If \( \Theta'_n(u_n) = 0 \), then by (2), \( u_{n+1} = T_{\Theta}(u_n) \). On the other hand, if \( \Theta'_n(u_n) \neq 0 \), then, again, \( u_{n+1} = T_{\Theta}(u_n) \), since \( \Theta_n(u_n) = 0 \). To summarize, (2) takes the following form:
\[
\forall n \in N, \quad u_{n+1} := \begin{cases} T_{\Theta} \left( u_n - \lambda_n \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|} \Theta'_n(u_n) \right) , & \text{if } u_n \notin \text{lev}_{\leq 0} \Theta_n, \\ T_{\Theta}(u_n) , & \text{if } u_n \in \text{lev}_{\leq 0} \Theta_n . \end{cases}
\]
If we combine this result with Example 6, then it can be easily verified that the previous recursion can be equivalently viewed as follows: \( \forall n \in N, u_{n+1} = T_{\Theta}(u_n) \), where \( T_{\Theta}(\lambda_n) \) stands for the relaxed subgradient projection mapping \( \text{w.r.t.} \Theta_n \).

Now, since \( T_{\Theta}(\lambda_n) \) is a \( 2-\lambda_n \)-attracting quasi-nonexpansive mapping, with \( \text{Fix}(T_{\Theta}(\lambda_n)) = \text{lev}_{\leq 0} \Theta_n \), it can be easily verified by Example 8 that the mapping \( T_{\Theta}(\lambda_n) \) is \( (2-\lambda_n)^2 \)-attracting quasi-nonexpansive, with \( \text{Fix}(T_{\Theta}(\lambda_n)) = \text{Fix}(T_{\Theta}) \cap \text{Fix}(T_{\Theta}(\lambda_n)) = \text{Fix}(T_{\Theta}) \cap \text{lev}_{\leq 0} \Theta_n = \Omega_n, \forall n \in N \). Hence, by Definition 4 we have that \( \forall n \in N, \forall v \in \Omega_n \),
\[
0 \leq \frac{(2-\lambda_n)^2}{2-\lambda_n(1-\eta_n)} \| u_n - u_{n+1} \|^2 = \frac{(2-\lambda_n)^2}{2-\lambda_n(1-\eta_n)} \| u_n - T_{\Theta}(\lambda_n)(u_n) \|^2
\leq \| u_n - v \|^2 - \| T_{\Theta}(\lambda_n)(u_n) - v \|^2 = \| u_n - v \|^2 - \| u_{n+1} - v \|^2
\Rightarrow \| u_{n+1} - v \| \leq \| u_n - v \| ,
\]
If we apply \( \inf_{v \in \Omega_n} \) on both sides of (3), then we obtain Theorem 18.1
2. Due to Assumption 17.2 to the fact that \( \Omega \) is closed and convex, to \( P_{\Omega}(u_n) \in \Omega \subset \Omega_n, \forall n \in N \), and to (6), we have:
\[
\forall n \in N, \quad d(u_n, \Omega) = \| u_n - P_{\Omega}(u_n) \| \geq \| u_{n+1} - P_{\Omega}(u_n) \|
\geq \| u_{n+1} - P_{\Omega}(u_{n+1}) \| = d(u_{n+1}, \Omega),
\]
which is nothing but Theorem 18.2
3. Fix arbitrarily \( v \in \Omega \). By (5), the sequence \( \| u_n - v \| \) is non-increasing and bounded; hence convergent. This establishes Theorem 18.3
4. Since \( (u_n)_{n \in N} \) is bounded by Theorem 18.3 \( \mathcal{W}(u_n)_{n \in N} \neq \emptyset \) [27 Thm. 9.12]. This establishes Theorem 18.4
5. There is no loss of generality if we assume that \( \forall n \in N, \Theta'_n(u_n) \neq 0 \). To see this, notice that for all \( n \in N \) such that \( \Theta'_n(u_n) = 0 \), we obtain \( \Theta_n(u_n) = 0 \implies \Theta_n(u_n) = 0 \iff \Theta'_n(u_n) = 0 \iff 0 :) = 0 \). Hence, in such a case, the claim of Theorem 18.5 holds true.

Assume, now, any \( v \in \Omega \). Recall also that the mapping \( T_{\Theta} \) is quasi-nonexpansive, with \( \Omega \subset \text{Fix}(T_{\Theta}) \), \( \forall n \in N \), and easily verify \( \forall n \in N, \forall v \in \Omega \),
\[
\| u_{n+1} - v \|^2 = \left\| T_{\Theta} \left( u_n - \lambda_n \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|} \Theta'_n(u_n) \right) - v \right\|^2 \leq \| u_n - \lambda_n \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|} \Theta'_n(u_n) - v \|^2
= \left\| (u_n - v) - \lambda_n \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|} \Theta'_n(u_n) \right\|^2
= \| u_n - v \|^2 + \lambda_n^2 \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|^2} \left( u_n - v, \Theta'_n(u_n) \right) .
\]
By the definition of the subgradient, we have that $\langle v - u_n, \Theta'_n(u_n) \rangle + \Theta_n(u_n) \leq \Theta_n(v) = 0$. If we merge this into (3), we obtain the following:

$$\|u_{n+1} - v\|^2 \leq \|u_n - v\|^2 + \lambda_n^2 \frac{\Theta_n^2(u_n)}{\|\Theta'_n(u_n)\|^2} - 2\lambda_n \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|^2} = \|u_n - v\|^2 - \lambda_n(2 - \lambda_n) \frac{\Theta_n^2(u_n)}{\|\Theta'_n(u_n)\|^2}.$$ 

This implies in turn that

$$\forall n \in N, \forall v \in \Omega, \quad 0 \leq \frac{\Theta_n^2(u_n)}{\|\Theta'_n(u_n)\|^2} \leq \lambda_n(2 - \lambda_n) \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|^2} \leq \frac{\|u_n - v\|^2 - \|u_{n+1} - v\|^2}{\epsilon^2}.$$ 

However, by Theorem 18.8, the sequence $(\|u_n - v\|^2)_{n \in N}$ is convergent, and hence Cauchy. The definition of a Cauchy sequence implies that $\lim_{n \to \infty}(\|u_n - v\|^2 - \|u_{n+1} - v\|^2) = 0$. This fact and the previous inequality establish $\lim_{n \to \infty} \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|^2} = 0$.

Now, notice that for all $n \in N$:

$$\left\| u_n - T_{\Theta_n}^\lambda(u_n) \right\| = \lambda_n \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|} \leq 2 \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|}.$$ 

Take $\lim_{n \to \infty}$ on both sides of this inequality, and recall the previous result to easily verify that

$$\lim_{n \to \infty} \left\| u_n - T_{\Theta_n}^\lambda(u_n) \right\| = 0.$$ 

In other words, Theorem 18.6 holds true.

6. Since the sequence $(\Theta'_n(u_n))_{n \in N}$ is assumed bounded, there exists a $D > 0$ such that $\forall n \in N, \|\Theta'_n(u_n)\| \leq D$. Notice, now, that for all those $n \in N$ such that $\Theta'_n(u_n) \neq 0$, we have

$$\Theta_n(u_n) = \left\| \Theta'_n(u_n) \right\| \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|} \leq D \frac{\Theta_n(u_n)}{\|\Theta'_n(u_n)\|}. (9)$$

Moreover, for all those $n \in N$ such that $\Theta'_n(u_n) = 0$, it is clear by the well-known fact $0 \in \partial \Theta_n(u_n) \iff u_n \in \text{arg min}_{v \in \mathcal{H}} \Theta_n(v)$, that $\Theta_n(u_n) = 0$. If we take $\lim_{n \to \infty}$ on both sides of (9), and if we also recall Theorem 18.5, the claim is established.

7. Notice that $\forall n \in N, \text{Fix}(T_{\Theta_n}^\lambda) = \text{lev}_{\leq 0} \Theta_n$. Hence, $\mathcal{S}(\{u_n\}_{n \in N}) \subset \lim_{n \to \infty} \text{lev}_{\leq 0} \Theta_n$ is a direct consequence of Theorems 15.5 and 18.5. The claim for the case of $\mathcal{S}(\{u_n\}_{n \in N}) = \{u_*\}$ can be easily obtained if we let $N := N$ in Theorem 15.11.

8. Here we will use Definition 4 two times; one for the mapping $T_n$, and one for $T_{\Theta_n}^\lambda$. In other words, $\forall n \in N, \forall v \in \Omega$,

$$\hat{\eta} \left\| (I - T_n)(T_{\Theta_n}^\lambda(u_n)) \right\|^2 \leq \hat{\eta} \left\| T_{\Theta_n}^\lambda(u_n) - T_n T_{\Theta_n}^\lambda(u_n) \right\|^2 \leq \eta_n \left\| T_{\Theta_n}^\lambda(u_n) - T_n T_{\Theta_n}^\lambda(u_n) \right\|^2 \leq \left\| T_{\Theta_n}^\lambda(u_n) - v \right\|^2 - \left\| T_n T_{\Theta_n}^\lambda(u_n) - v \right\|^2 \leq \left\| T_{\Theta_n}^\lambda(u_n) - v \right\|^2 - \left\| u_{n+1} - v \right\|^2 \leq \left\| u_n - v \right\|^2 - \left\| u_{n+1} - v \right\|^2.$$ 

Divide the above inequality by $\hat{\eta} > 0$, recall Theorem 18.8, and take $\lim_{n \to \infty}$ on both sides of the resulting inequality to obtain $\lim_{n \to \infty} (I - T_n)(T_{\Theta_n}^\lambda(u_n)) = 0$. This establishes Theorem 18.8.
9. First, since $\mathcal{S}((u_n)_{n \in \mathbb{N}}) \neq \emptyset$, notice that $\mathcal{S}((T_{(\lambda_n)}^{(\lambda_n)}(u_n))_{n \in \mathbb{N}}) = \mathcal{S}((u_n)_{n \in \mathbb{N}})$. To establish, for example, $\mathcal{S}((u_n)_{n \in \mathbb{N}}) \subset \mathcal{S}((T_{\Theta_n}^{(\lambda_n)}(u_n))_{n \in \mathbb{N}})$, choose arbitrarily a $u_* \in \mathcal{S}((u_n)_{n \in \mathbb{N}})$, which implies that there exists a subsequence $N' \in \mathbb{N}^\#$ such that $\lim_{n \in N'} u_n = u_*$. Then, it is easy to verify that
\[
\forall n \in N', \quad \|u_n - T_{\Theta_n}^{(\lambda_n)}(u_n)\| \leq \|u_n - u_*\| + \|(I - T_{\Theta_n}^{(\lambda_n)})(u_n)\|.
\]
Take $\lim_{n \in N'}$ on both sides of the previous inequality, so that the following result is obtained by Theorem 18.5 $u_* \in \mathcal{S}((T_{\Theta_n}^{(\lambda_n)}(u_n))_{n \in \mathbb{N}})$. Similar arguments can be used in order to derive $\mathcal{S}((T_{(\lambda_n)}^{(\lambda_n)}(u_n))_{n \in \mathbb{N}}) \subset \mathcal{S}((u_n)_{n \in \mathbb{N}})$.

Now, it becomes clear under the previous discussion, that if we define $x_n := T_{\Theta_n}^{(\lambda_n)}(u_n)$, $\forall n \in \mathbb{N}$, in Theorem 15, then Theorem 18.9 becomes a direct consequence of Theorems 15 and 18.8.

10. Theorem 18.4 guarantees that $\mathfrak{W}((u_n)_{n \in \mathbb{N}}) \neq \emptyset$. Fix arbitrarily a $u_* \in \mathfrak{W}((u_n)_{n \in \mathbb{N}})$. By definition, there exists a subsequence $N' \in \mathbb{N}^\#$ such that $u_n \xrightarrow{N'} u_*$.

Recall Theorem 18.5 and easily verify that
\[
u_n - T_{\Theta_n}^{(\lambda_n)}(u_n) \xrightarrow{n \in N'} 0.
\]
This together with $u_n \xrightarrow{n \in N'} u_*$ imply that
\[
T_{\Theta_n}^{(\lambda_n)}(u_n) \xrightarrow{n \in N'} u_* \quad (10)
\]
Recall, now, Theorem 18.5 in order to obtain $(I - T)(T_{\Theta_n}^{(\lambda_n)}(u_n)) \xrightarrow{n \in N'} 0$. This result, $I$, and Definition 9 lead us to $(I - T)(u_*) = 0 \iff u_* \in \text{Fix}(T)$. This establishes Theorem 18.10.

11. This is a direct consequence of Theorem 18.10 and the well-known fact that $\mathcal{S}((u_n)_{n \in \mathbb{N}}) \subset \mathfrak{W}((u_n)_{n \in \mathbb{N}})$.

12. It is easy to verify by Assumptions 17.3 and 17.5 that
\[
\frac{2 - \lambda_n}{2 - \lambda_n(1 - \eta_n)} = \frac{2 - \lambda_n}{2 - \lambda_n + \lambda_n \eta_n} > \frac{\epsilon \eta}{2(1 + \eta)} > 0.
\]
Using also 10, we easily verify under Assumption 17.2 that
\[
\forall n \in N, \forall v \in \Omega, \quad \frac{\epsilon \eta}{2(1 + \eta)} \|u_n - u_{n+1}\|^2 \leq \|u_n - v\|^2 - \|u_{n+1} - v\|^2 \quad (11)
\]
The claim of Theorem 18.12 is a direct consequence of (11), Assumption 17.10 and Fact 12.
\]

4. Special Cases of the General Algorithm

4.1. Exploring $(T_n)_{n \in \mathbb{N}}$. The available a-priori information about the model 11 enters Algorithm 1 through the sequence of mappings $(T_n)_{n \in \mathbb{N}}$, i.e., implicitly via the sequence of sets $(\text{Fix}(T_n))_{n \in \mathbb{N}}$. Given that $n \in \mathbb{N}$ stands for time, the sequence $(T_n)_{n \in \mathbb{N}}$ aims to capture the dynamic nature of a-priori information, which is usually met in signal processing and machine learning applications. For example, it is often the case in adaptive signal processing to face a channel whose impulse response changes slowly with time. Notice also here that the sequence $(T_n)_{n \in \mathbb{N}}$ belongs to the rich family of strongly attracting quasi-nonexpansive mappings. To demonstrate the versatility offered by this class of mappings in the usage of the available a-priori knowledge, examples of such mappings, mobilized extensively in various contexts of optimization theory [7], are demonstrated in this section. More specifically, in order to apply the proposed scheme to a real-world problem, the following Example 23 considers a non-smooth loss function which infuses sparsity information in 11. Such a loss function will be incorporated in Algorithm 29 to devise an algorithmic solution to the online sparse system/signal recovery task of Section 5.

Example 19 (Resolvent). For a set-valued mapping $A : \mathcal{H} \to 2^{\mathcal{H}}$, its graph is defined as the set $\text{gph}(A) := \{(x, y) \in \mathcal{H} \times \mathcal{H} : y \in A(x)\}$. The mapping $A$ will be called monotone if $\forall (x_1, y_1), (x_2, y_2) \in \text{gph}(A), \langle x_1 - x_2, y_2 - y_1 \rangle \geq 0$ [7, 8, 38, 46]. A monotone mapping $A$ will be called maximal if no enlargement of
its graph is possible without destroying monotonicity, i.e., $\forall (x, y) \in \mathcal{H} \times \mathcal{H} \setminus \text{gh}(A)$, there exists a pair $(x_0, y_0) \in \text{gh}(A)$ such that $\langle x - x_0, y - y_0 \rangle < 0$. For example, the linear mapping induced by any positive semi-definite matrix is maximal monotone [16, Examples 12.2 and 12.7].

Now, given a maximal monotone mapping $A : \mathcal{H} \to 2^{\mathcal{H}}$, and a $\xi > 0$, its resolvent $T^{(\xi)} := (I + \xi A)^{-1} : \mathcal{H} \to \mathcal{H}$ is an 1-attracting nonexpansive mapping, where $(\cdot)^{-1}$ stands for the inverse of a mapping. The fixed point set of $T^{(\xi)}$ becomes $\text{Fix}(T^{(\xi)}) = \{ x \in \mathcal{H} : 0 \in A(x) \}$. For example, in the case of a positive semi-definite matrix, this fixed point set is nothing but the null space of the matrix.

**Example 20** (Proximity mapping). Given a lower semi-continuous function $\Phi : \mathcal{H} \to \mathbb{R}$, the Moreau envelope of index $\gamma > 0$ of $\Phi$ is the function

$$\Phi^{(\gamma)} : \mathcal{H} \to \mathbb{R} : x \mapsto \inf_{y \in \mathcal{H}} \left( \Phi(y) + \frac{1}{2\gamma} \| x - y \|^2 \right).$$

Then, the proximity mapping $T_{\gamma\Phi}$ is defined as the mapping which maps to an $x \in \mathcal{H}$ the unique minimizer of \([12, 23, 24, 39]\). It can be verified that the proximity mapping $T_{\gamma\Phi}$ is 1-attracting nonexpansive with fixed point set $\text{Fix}(T_{\gamma\Phi}) = \{ x \in \mathcal{H} : \Phi(x) = \inf_{y \in \mathcal{H}} \Phi(y) \}$ [28, 24].

**Example 21** (Inconsistent a-priori information). Assume that the available a-priori knowledge about our system is a gathering of several pieces of information which take the form of the following nonempty closed convex sets: $\Gamma, \{ C_m \}_{m=1}^M$ in $\mathcal{H}$, with $M \in \mathbb{N}$. With $\Gamma$ we denote the information that our system should surely satisfy, called the absolute or hard constraint. Ideally, our solution set is $\Gamma \cap (\bigcap_{m=1}^M C_m)$. However, it is quite often the case that the available pieces of a-priori knowledge are inconsistent, i.e., the previous intersection is the empty set, e.g., [53]. To tackle such a problem, we define the following proximity function:

$$\forall x \in \mathcal{H}, \quad p(x) := \sum_{m=1}^M \beta_m d^2(x, C_m), \quad \text{where } \{ \beta_m \}_{m=1}^M \text{ are convex weights, i.e., } \{ \beta_m \}_{m=1}^M \subset (0, 1], \text{ such that } \sum_{m=1}^M \beta_m = 1.$$ 

The proximity function is everywhere Fréchet differentiable, and its differential is the mapping $p' := 2 \sum_{m=1}^M \beta_m (I - P_{C_m}) : \mathcal{H} \to \mathcal{H}$. Define, now, as our new solution set $\Xi := \arg\min \{ p(x) : x \in \Gamma \}$. The non-emptiness of $\Xi$ is guaranteed if at least one of $\{ C_m \}_{m=1}^M$ or $\Gamma$ is bounded [61]. In words, $\Xi$ is the set of all those points in $\Gamma$ that least violate, in the sense of the previous proximity function, the rest of the constraints $\{ C_m \}_{m=1}^M$. Under the previous setting, and $\forall \lambda \in (0, 2)$, the mapping $T_p := P_{\Gamma}(I - \lambda p')$, is $(1 - \frac{\lambda}{2})$-attracting nonexpansive with fixed point set $\text{Fix}(T_p) = \Xi$ [18, 22, 61, 65, 67].

**Example 22** (The class $\mathcal{I}$ of mappings [6]). For any $x, y \in \mathcal{H}$, define the following set: $H(x, y) := \{ v \in \mathcal{H} : \langle x - y, v - y \rangle \leq 0 \}$. In words, the set $H(x, y)$ is the closed halfspace onto which $y$ is the metric projection of $x$. Now, a mapping $T : \mathcal{H} \to \mathcal{H}$ is said to belong to the class $\mathcal{I}$ of mappings, if $\forall x \in \mathcal{H}, \text{Fix}(T) \subset H(x, T(x))$ [6]. An equivalent description of the class $\mathcal{I}$ is as follows: $T \in \mathcal{I}$ iff $T$ is firmly quasi-nonexpansive [6, Proposition 2.3]. Moreover, $\forall T \in \mathcal{I}$, $\text{Fix}(T) = \bigcap_{x \in \mathcal{H}} H(x, T(x))$. For example, the subgradient projection mapping $T_\Phi$ (Example [6]) belongs to this class [6, Proposition 2.3].

**Definition 23** (Sparsity-aware loss function). Henceforth, the notation $\overline{j_1, j_2}$, for any integers $j_1 \leq j_2$, will stand for $\{j_1, j_1 + 1, \ldots, j_2\}$. Assume that $\mathcal{H} := \mathbb{R}^L$, for some $L \in \mathbb{N}_+$. We introduce, here, the following sequence of convex, continuous, non-negative functions $(\Phi_n : \mathcal{H} \to [0, \infty))_{n \in \mathbb{N}}$. Given a sequence of weight vectors $(w_n)_{n \in \mathbb{N}} \subset \mathbb{R}^L$, with positive components, i.e., $w_{n,j} > 0, \forall j \in \overline{1, L}, \forall n \in \mathbb{N}$, and a positive parameter $\rho > 0$, we define

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}^L, \quad \Phi_n(x) := \max\{0, \sum_{j=1}^L w_{n,j}|x_j| - \rho\}. \quad (13)$$
It is clear that the 0-th level set for each $\Phi_n$ is a weighted $\ell_1$-ball, i.e.,

$$\forall n \in \mathbb{N}, \quad \text{lev}_0 \Phi_n = B_{\ell_1}[w_n, \rho] := \{x \in \mathbb{R}^L : \sum_{j=1}^L w_{n,j}|x_j| \leq \rho\}.$$ 

The fixed point set of the relaxed subgradient projection mapping $T_{\Phi_n}^{(v_n)}$, $v_n \in (0, 2)$, is the weighted $\ell_1$-ball, i.e., $\text{Fix}(T_{\Phi_n}^{(v_n)}) = B_{\ell_1}[w_n, \rho]$. The sequence $B_{\ell_1}[w_n, \rho]$ has been very useful in building sparsity-aware online learning methods in \cite{35,51,52}. There, the metric projection mapping $P_{B_{\ell_1}([w_n, \rho])}$ was employed, whose computation scales to the order of $O(L \log_2 L)$.

Following a different path than \cite{35,51,52}, the information carried by $(B_{\ell_1}[w_n, \rho])_{n \in \mathbb{N}}$ is viewed from an alternative angle in this study: $\forall n \in \mathbb{N}$, $B_{\ell_1}[w_n, \rho]$ is not just a closed convex set, onto which we project, but it is also the set of minimizers of the non-smooth loss function $\Phi_n$. In order to minimize the non-smooth $\Phi_n$, the subgradient information will be used. However, the employment of such an information is not possible via \cite{56,62,63}, since the subgradient projection mapping (Definition 6) belongs to the class of strongly attracting quasi-nonexpansive mappings, which is strictly larger than the class of strongly attracting nonexpansive operators, utilized in \cite{56}.

The set $B_{\ell_1}[w_n, \rho]$ is a closed convex set, and its metric projection mapping is given as follows. To save space, we give here a short description. For the full discussion, the interested reader can refer to \cite{35}.  

**Fact 24** (Metric projection mapping onto the weighted $\ell_1$-ball \cite{33}). Given $x \in \mathbb{R}^L \setminus B_{\ell_1}[w_n, \rho]$, there exists an $l_* \in \mathbb{N}$, and a set of integers $\{l_j\}_{j \in \mathbb{N}} \subset \overline{l_* + 1, L}$, such that the metric projection $P_{B_{\ell_1}[w_n, \rho]}(x)$ is given by a permutation on the components of the following vector

$$
\begin{bmatrix}
    x_1 - \frac{\sum_{i=1}^{l_*} w_{n,i}|x_i| - \rho}{\sum_{i=1}^{l_*} w_{n,i}^2} \text{sgn}(x_1) w_{n,1}, & \ldots, & x_{l_*} - \frac{\sum_{i=1}^{l_*} w_{n,i}|x_i| - \rho}{\sum_{i=1}^{l_*} w_{n,i}^2} \text{sgn}(x_{l_*}) w_{n,l_*}, & 0, & \ldots, & 0
\end{bmatrix}^T,
$$

(14)  

where

$$
\begin{cases}
    |x_j| > \frac{\sum_{i=1}^{l_*} w_{n,i}|x_i| - \rho}{\sum_{i=1}^{l_*} w_{n,i}^2} w_{n,j}, & \forall j \in \overline{1,l_*}, \\
    |x_j| \leq \frac{\sum_{i=1}^{l_*} w_{n,i}|x_i| - \rho}{\sum_{i=1}^{l_*} w_{n,i}^2} w_{n,j}, & \forall j \in \overline{l_* + 1, L}.
\end{cases}
$$

Without any loss of generality, we assume that $P_{B_{\ell_1}[w_n, \rho]}(x)$ is given by (14) in the sequel.

Regarding Definition 23, consider the following assumptions.

**Assumption 25.**  
1. The sequence of weight vectors $(w_n)_{n \in \mathbb{N}}$ is constructed such that $\forall n \in \mathbb{N}$, $\forall j \in \overline{1,L}$, $w_{n,j} \in [\epsilon, \hat{\epsilon}]$, for some $\epsilon, \hat{\epsilon} > 0$.  
2. Given the sequence of relaxed subgradient projection mappings $(T_{\Phi_n}^{(v_n)})_{n \in \mathbb{N}}$, with respect to the sequence $(\Phi_n)_{n \in \mathbb{N}}$ in Definition 23 there exists $\epsilon' \in (0, 1]$ such that $\forall n \in \mathbb{N}$, $v_n \in [\epsilon', 2 - \epsilon']$.

**Lemma 26.** The following properties hold true.  
1. The subdifferentials of the loss functions $(\Phi_n)_{n \in \mathbb{N}}$, defined in (13), are given in Table 1  
2. Let Assumption 25 hold true. Then, $\forall x \in \mathbb{R}^L$, $(\Phi_n'(x))_{n \in \mathbb{N}}$ is bounded.  
3. Let Assumption 25 hold true. Then, $\text{int}(\bigcap_{n \in \mathbb{N}} B_{\ell_1}[w_n, \rho]) \neq \emptyset$.  
4. Let Assumptions 25 and 25 hold true. Then, the sequence of relaxed subgradient projection mappings $(T_{\Phi_n}^{(v_n)})_{n \in \mathbb{N}}$ satisfies Assumption 11.
| $\sum_{j=1}^n w_{n,j} |x_j| < \rho,$ | $\partial \Phi_n(x)$ |
|---|---|
| $\sum_{j=1}^n w_{n,j} |x_j| > \rho,$ | $\mathfrak{3}_x = \emptyset,$ |
| $\sum_{j=1}^n w_{n,j} |x_j| > \rho,$ | $\mathfrak{3}_x \neq \emptyset,$ |
| $\sum_{j=1}^n w_{n,j} |x_j| = \rho,$ | $\mathfrak{3}_x = \emptyset,$ |
| $\sum_{j=1}^n w_{n,j} |x_j| = \rho,$ | $\mathfrak{3}_x \neq \emptyset,$ |

**Table 1.** Here, $\mathfrak{3}_x := \{j \in 1, L : x_j = 0\}$, and $\tau$ stands for the cardinality of $\mathfrak{3}_x$, whenever $\mathfrak{3}_x \neq \emptyset$. The conv symbol stands for the convex hull of a set.

**Proof.** 1. To save space, the calculation of the subdifferentials in Table 1 is omitted. These results can be reproduced by using standard arguments of convex analysis, e.g., [15] Thm. 25.6.

2. Lemma [26][2] can be easily established by Assumption [25][1] and Table 1.

3. Choose any $x \in B(0, \frac{\rho}{\tau^2})$. Then, $\forall j \in 1, L$, $|x_j| \leq \frac{\rho}{\tau^2}$. Moreover, $\sum_{j=1}^L w_{n,j} |x_j| \leq \sum_{j=1}^L \epsilon \frac{\rho}{\tau^2} = \rho$. Hence, $B(0, \frac{\rho}{\tau^2}) \subset B_{\ell_1}(w_n, \rho)$, $\forall n \in N$. This clearly suggests that $0 \in \text{int}(\bigcap_{n \in N} B_{\ell_1}(w_n, \rho))$, which establishes Lemma [26][3]

4. First, notice that $\forall n \in N$, $\text{Fix}(T^{(\nu_n)}) = B_{\ell_1}(w_n, \rho)$. Now, according to Assumption [14] fix arbitrarily a subsequence $N \subset \mathbb{N}$, a sequence $(x_n)_{n \in N} \subset \mathbb{R}^L$, and a $\gamma > 0$ such that $\forall n \in N$, $d(x_n, B_{\ell_1}(w_n, \rho)) \geq \gamma$. Notice by Fact [24] the following: $\forall n \in N$,

$$\gamma^2 \leq d^2(x_n, B_{\ell_1}(w_n, \rho)) = \left\| x_n - P_{B_{\ell_1}(w_n, \rho)}(x_n) \right\|^2$$

$$= \sum_{j=1}^L \left( \left( \sum_{i=1}^{l_\ast} w_{n,i} |x_{n,i}| - \rho \right)^2 \right) \frac{w_{n,j}^2}{\left( \sum_{i=1}^{l_\ast} w_{n,i}^2 \right)^2} + \sum_{j=1}^{L} \frac{x_n^2}{w_{n,j}^2}$$

$$\leq \sum_{j=1}^L \left( \left( \sum_{i=1}^{l_\ast} w_{n,i} |x_{n,i}| - \rho \right)^2 \right) \frac{w_{n,j}^2}{\left( \sum_{i=1}^{l_\ast} w_{n,i}^2 \right)^2} + \sum_{j=1}^{L} \frac{\left( \sum_{i=1}^{l_\ast} w_{n,i} |x_{n,i}| - \rho \right)^2}{\left( \sum_{i=1}^{l_\ast} w_{n,i}^2 \right)^2} w_{n,j}^2$$

$$\leq \sum_{j=1}^L \left( \left( \sum_{i=1}^{l_\ast} w_{n,i} |x_{n,i}| - \rho \right)^2 \right) \frac{\left( \sum_{i=1}^{l_\ast} w_{n,i}^2 \right)^2}{\left( \sum_{i=1}^{l_\ast} w_{n,i}^2 \right)^2} w_{n,j}^2$$

which, in turn, results into

$$\Phi_n^2(x_n) = \left( \sum_{i=1}^{L} w_{n,i} |x_{n,i}| - \rho \right)^2 \geq \gamma^2 \left( \sum_{i=1}^{l_\ast} w_{n,i}^2 \right)^2 \frac{\left( \sum_{j=1}^{L} w_{n,j}^2 \right)^2}{\left( \sum_{j=1}^{L} w_{n,j}^2 \right)^2} \geq \gamma^2 \frac{\epsilon^4}{L \epsilon^2} : \delta^2 > 0, \ \forall n \in N.$$
Notice, also, by Example 4 and Lemma 26.2 that \( \forall n \in \mathbb{N} \),
\[
\left\| (I - T_{\bar{\Phi}_n}(\nu_n)) (x_n) \right\| = \nu_n \frac{\Phi_n(x_n)}{\|\Phi_n(x_n)\|} \geq \frac{\delta'}{D} > 0,
\]
which clearly suggests that \( \exists \delta > 0 \) such that \( \liminf_{n \in \mathbb{N}} \left\| (I - T_{\bar{\Phi}_n}(\nu_n)) (x_n) \right\| \geq \delta \). This establishes Lemma 26.4.

4.2. Exploring \( (\Theta_n)_{n \in \mathbb{N}} \). In this section, the metric distance function to closed convex sets will be used in order to define a sequence of loss functions \( (\Theta_n)_{n \in \mathbb{N}} \). Such sequences have already found numerous applications in online signal processing and machine learning tasks [53, 54, 58, 66, 67], under the light, however, of the predecessors [56, 62, 63] of the present framework. In this section, this specific sequence \( (\Theta_n)_{n \in \mathbb{N}} \) will be blended with the more general class of strongly attracting quasi-nonexpansive mappings in order to construct Algorithm 29. Given the wide applicability of the techniques in [56, 62, 63], it is natural to anticipate an even larger span of usage for Algorithm 29. Such a potential will be demonstrated in Section 5 where Algorithm 29 is applied to the online sparse system/signal recovery task.

Definition 27. Assume a sequence of nonempty closed convex sets \( (S_n)_{n \in \mathbb{N}} \). Given a user-defined \( q \in \mathbb{N}_* \), let the following index set
\[
\mathcal{J}_n := \max\{0, n - q + 1\}, n, \quad \forall n \in \mathbb{N}.
\]
Notice that the sequence \( (\mathcal{J}_n)_{n \in \mathbb{N}} \) depicts a sliding window on the set \( \mathbb{N} \), of length at most \( q \).

Let us introduce a sequence of convex functions \( (\Theta_n : \mathcal{H} \to [0, \infty))_{n \in \mathbb{N}} \) inductively. For every \( n \in \mathbb{N} \), and given a \( u_n \in \mathcal{H} \), define the following active index set:
\[
\mathcal{I}_n := \{i \in \mathcal{J}_n : u_n \notin S_i\}.
\]
This set identifies those closed convex sets \( \{S_i\}_{i \in \mathcal{I}_n} \), out of \( \{S_j\}_{j \in \mathcal{J}_n} \), which add on new “information” to our learning process. The sets with indexes \( \{j \in \mathcal{J}_n : u_n \in S_j\} \) will not be processed at the time instant \( n \).

In the case where \( \mathcal{I}_n \neq \emptyset \), we introduce the set of weights \( \{\omega_i^{(n)}\}_{i \in \mathcal{I}_n} \subset (0, 1] \), such that \( \sum_{i \in \mathcal{I}_n} \omega_i^{(n)} = 1 \). Define, now, the convex function:
\[
\forall x \in \mathcal{H}, \quad \Theta_n(x) := \begin{cases} 
\sum_{i \in \mathcal{I}_n} \omega_i^{(n)} d(u_n, S_i) d(x, S_i), & \text{if } \mathcal{I}_n \neq \emptyset, \\
0, & \text{if } \mathcal{I}_n = \emptyset,
\end{cases}
\]
where \( L_n := \sum_{i \in \mathcal{I}_n} \omega_i^{(n)} d(u_n, S_i) \). We define \( L_n := 0 \) for all those \( n \in \mathbb{N} \) such that \( \mathcal{I}_n = \emptyset \).

Lemma 28. The following properties hold true for the sequence of functions \( (\Theta_n)_{n \in \mathbb{N}} \) given in (15):
1. For every \( n \in \mathbb{N} \), such that \( \mathcal{I}_n \neq \emptyset \), we have \( L_n > 0 \).
2. For every \( n \in \mathbb{N} \), \( \text{lev} \leq 0 \Theta_n = \bigcap_{i \in \mathcal{I}_n} S_i \), where we define \( \bigcap_{i \in \emptyset} S_i := \mathcal{H} \), to cover also the case where \( \mathcal{I}_n = \emptyset \).
3. The collection of all the subgradients of \( (\Theta_n)_{n \in \mathbb{N}} \) is bounded, i.e., \( \forall n \in \mathbb{N}, \forall x \in \mathcal{H}, \|\Theta'_n(x)\| \leq 1 \).
4. For any \( n \in \mathbb{N} \),
\[
\Theta'_n(u_n) = \begin{cases} 
\frac{1}{L_n} \sum_{i \in \mathcal{I}_n} \omega_i^{(n)} (u_n - P_{S_i}(u_n)), & \mathcal{I}_n \neq \emptyset, \\
0, & \mathcal{I}_n = \emptyset.
\end{cases}
\]

Proof. 1. Fix arbitrarily an \( n \in \mathbb{N} \) such that \( \mathcal{I}_n \neq \emptyset \). By the definition of \( \mathcal{I}_n \), \( \forall i \in \mathcal{I}_n, d(u_n, S_i) > 0 \). Since, also, \( \omega_i^{(n)} \in (0, 1], \forall i \in \mathcal{I}_n \), it is clear by the definition of \( L_n \) that Lemma 28.1 holds true.
2. Fix arbitrarily an \( n \in \mathbb{N} \). Assume, first, that \( \mathcal{I}_n = \emptyset \). By (15), it is clear that \( \operatorname{lev}_{\leq 0} \Theta_n = \mathcal{H} =: \bigcap_{i \in \emptyset} S_i \).

Assume, now, that \( \mathcal{I}_n \neq \emptyset \), and that \( \bigcap_{i \in \mathcal{I}_n} S_i \neq \emptyset \). It is clear by (15) that \( \bigcap_{i \in \mathcal{I}_n} S_i \subset \operatorname{lev}_{\leq 0} \Theta_n \). Assume, now, an \( x \notin \bigcap_{i \in \mathcal{I}_n} S_i \), or equivalently, \( \exists \omega \in \mathcal{I}_n \) such that \( d(x, S_{\omega}) > 0 \). Then, one can easily verify that \( \Theta_n(x) \geq \frac{\omega^\nu_n d(u_n, S_{\omega})}{L_n} d(x, S_{\omega}) > 0 \). In other words, \( x \notin \operatorname{lev}_{\leq 0} \Theta_n \), and finally \( \operatorname{lev}_{\leq 0} \Theta_n \subset \bigcap_{i \in \mathcal{I}_n} S_i \). Notice that the previous arguments hold true also in the case where \( \bigcap_{i \in \mathcal{I}_n} S_i = \emptyset \). This establishes Lemma 28.3.

3. Fix arbitrarily an \( n \in \mathbb{N} \). By (15), basic calculus on subdifferentials \([29, 46]\) suggests that

\[
\forall x \in \mathcal{H}, \quad \partial \Theta_n(x) = \begin{cases} \sum_{i \in \mathcal{I}_n} \frac{\omega^\nu_n (u_n, S_i)}{\lambda_n} \partial d(x, S_i), & \text{if } \mathcal{I}_n \neq \emptyset, \\ \{0\}, & \text{if } \mathcal{I}_n = \emptyset. \end{cases}
\]

From now and on, we deal only with the case where \( \mathcal{I}_n \neq \emptyset \), since the previous equation clearly suggests that Lemma 28.3 holds trivially in the case of \( \mathcal{I}_n = \emptyset \).

By Example 3, the subgradient \( \Theta'_n(x) \) takes the following form:

\[
\forall x \in \mathcal{H}, \quad \Theta'_n(x) = \sum_{i \in \mathcal{I}_n: \ x \notin S_i} \frac{\omega^\nu_n (u_n, S_i)}{\lambda_n} d'(x, S_i) + \sum_{i \in \mathcal{I}_n: \ x \in S_i} \frac{\omega^\nu_n (u_n, S_i)}{\lambda_n} d'(x, S_i)
\]

\[
= \sum_{i \in \mathcal{I}_n: \ x \notin S_i} \frac{\omega^\nu_n (u_n, S_i)}{\lambda_n} \frac{x - P_{S_i}(x)}{d(x, S_i)} + \sum_{i \in \mathcal{I}_n: \ x \in S_i} \frac{\omega^\nu_n (u_n, S_i)}{\lambda_n} d'(x, S_i).
\]

Hence,

\[
\forall x \in \mathcal{H}, \quad \left\| \Theta'_n(x) \right\| \leq \sum_{i \in \mathcal{I}_n: \ x \notin S_i} \frac{\omega^\nu_n (u_n, S_i)}{\lambda_n} \left\| x - P_{S_i}(x) \right\| + \sum_{i \in \mathcal{I}_n: \ x \in S_i} \frac{\omega^\nu_n (u_n, S_i)}{\lambda_n} \cdot 1
\]

\[
= \sum_{i \in \mathcal{I}_n: \ x \notin S_i} \frac{\omega^\nu_n (u_n, S_i)}{\lambda_n} + \sum_{i \in \mathcal{I}_n: \ x \in S_i} \frac{\omega^\nu_n (u_n, S_i)}{\lambda_n} = 1.
\]

This establishes Lemma 28.3.

4. Lemma 28.4 is an immediate consequence of (16). \( \square \)

**Algorithm 29.** Assume a sequence of nonempty closed convex sets \( (S_n)_{n \in \mathbb{N}} \subset \mathcal{H} \). Moreover, consider a sequence of convex continuous functions \( \Phi_n : \mathcal{H} \to \mathbb{R} \), such that \( \operatorname{lev}_{\leq 0} \Phi_n \neq \emptyset \), \( \forall n \in \mathbb{N} \). Associated to each \( \Phi_n \) is the relaxed subgradient projection mapping \( T_{\Phi_n}^{(\nu_n)} \) (see Definition 4), where \( \nu_n \in (0, 2), \forall n \in \mathbb{N} \).

For an arbitrarily chosen \( u_0 \in \mathcal{H} \), form the following sequence:

\[
\forall n \in \mathbb{N}, \quad u_{n+1} := \begin{cases} T_{\Phi_n}^{(\nu_n)} \left( u_n - \lambda_n \frac{\Theta_n(u_n)}{\left\| \Theta_n(u_n) \right\|} \Theta'_n(u_n) \right), & \text{if } \Theta'_n(u_n) \neq 0, \\ T_{\Phi_n}^{(\nu_n)}(u_n), & \text{if } \Theta'_n(u_n) = 0, \end{cases}
\]

where the sequence of functions \( (\Theta_n)_{n \in \mathbb{N}} \) is given in Definition 27, \( \Theta'_n(u_n) \) is any subgradient of \( \Theta_n \) at \( u_n \), and \( \lambda_n \in (0, 2), \forall n \in \mathbb{N} \).

Lemma 28.4 and some elementary algebra lead to the following equivalent formulation of the previous recursion:

\[
\forall n \in \mathbb{N}, \quad u_{n+1} = T_{\Phi_n}^{(\nu_n)} \left( u_n + \mu_n \left( \sum_{i \in \mathcal{I}_n} \omega^\nu_n P_{S_i}(u_n) - u_n \right) \right),
\]

(17)
where $\mu_n := \lambda_n M_n$, and
\[
M_n := \begin{cases}
\frac{\sum_{i \in I_n} \omega_i^{(n)} d^2(u_n, S_i)}{\sum_{i \in I_n} \omega_i^{(n)} (u_n - P_{S_i}(u_n))^2}, & \text{if } \sum_{i \in I_n} \omega_i^{(n)} (u_n - P_{S_i}(u_n)) \neq 0, \\
1, & \text{otherwise.}
\end{cases}
\tag{18}
\]

To avoid any ambiguity in the case where $I_n = \emptyset$, we define in \cite{17} and \cite{18}: $\sum_{i \in \emptyset} \omega_i^{(n)} (P_{S_i}(u_n) - u_n) := \sum_{i \notin \emptyset} \omega_i^{(n)} P_{S_i}(u_n) - u_n := 0$. Notice also by the convexity of $\| \cdot \|^2$ that $M_n \geq 1$, and that since $\lambda_n \in (0, 2)$, we obtain $\mu_n \in (0, 2M_n)$, i.e., the extrapolation parameter $\mu_n$ is able to take values greater than or equal to 2, $\forall n \in \mathbb{N}$.

It is needless to say that the results presented in Theorem \cite{18} hold true also for Algorithm \cite{29}. Nevertheless, one can establish additional properties for Algorithm \cite{29} based on the following assumptions.

**Assumption 30.** Regarding Definition \cite{27} and Algorithm \cite{29} assume the following.
1. Let $\hat{\omega} := \inf \{ \omega_i^{(n)} : i \in I_n \neq \emptyset, n \in \mathbb{N} \} > 0$.
2. The sequences $(\Phi_n'(u_n))_{n \in \mathbb{N}}$ and $(\Phi_n'(T(\lambda_n)(u_n)))_{n \in \mathbb{N}}$ are bounded, i.e., there exists a $D > 0$ such that
   \[ \forall n \in \mathbb{N}, \max \left\{ \| \Phi_n'(u_n) \|, \| \Phi_n'(T(\lambda_n)(u_n)) \| \right\} \leq D. \]
3. $\forall n \in \mathbb{N}, \Phi_n : \mathcal{H} \to [0, \infty)$. See, for example, Definition \cite{23}.

**Theorem 31.** The following statements are valid for Algorithm \cite{29}.
1. Let Assumption \cite{17,12} hold true. Then, there exists a $D > 0$ such that $\forall n \in \mathbb{N}, L_n \leq D$.
2. Let Assumptions \cite{17,12,17,3} and \cite{30,11} hold true. Then, $\lim_{n \to \infty} \max \{ d(u_n, S_j) : j \in \mathcal{J}_n \} = 0$.
3. If Assumptions \cite{17,12,17,3,17,4} and \cite{30,11} hold true, then $\mathcal{G}((u_n)_{n \in \mathbb{N}}) \subset \limsup_{n \to \infty} S_n$. Moreover, if there exists a $u_\ast \in \mathcal{H}$ such that $\lim_{n \to \infty} u_n = u_\ast$, i.e., $\mathcal{G}((u_n)_{n \in \mathbb{N}}) = \{ u_\ast \}$, then $u_\ast \in \liminf_{n \to \infty} S_n$.
4. Let Assumptions \cite{17,12,17,3,25,2} and \cite{30,2} hold true. Then, $\limsup_{n \to \infty} \Phi_n(u_n) \leq 0$. If, in addition, Assumption \cite{30,3} holds true, then $\lim_{n \to \infty} \Phi_n(u_n) = 0$.
5. The following result applies to the next section where a system/signal recovery task is considered. Assume Algorithm \cite{29} for the case where $\mathcal{H} := \mathbb{R}^L, L \in \mathbb{N}_\ast$, equipped with the standard vector inner product. Assume, also, that the sequence of functions $(\Phi_n)_{n \in \mathbb{N}}$ is given by Definition \cite{23}.

Let Assumptions \cite{17,12,17,3} and \cite{25,11} hold true. Then, $\mathcal{G}((u_n)_{n \in \mathbb{N}}) \subset \limsup_{n \to \infty} B_{L_1}[w_n, \rho]$. If there exists a $u_\ast$ such that $\lim_{n \to \infty} u_n = u_\ast$, then $u_\ast \in \liminf_{n \to \infty} B_{L_1}[w_n, \rho]$. 

**Proof.**
1. Notice, that $\forall n \in \mathbb{N}, \forall i \in I_n, \forall v \in \Omega$,
\[ d(u_n, S_i) = \| u_n - P_{S_i}(u_n) \| \leq \| u_n - v \| + \| v - P_{S_i}(u_n) \| \leq 2 \| u_n - v \| \leq 2 \| u_{n_0} - v \|, \]
where $n_0 := \min N$, the second inequality follows from Example \cite{7} and the third one from \cite{7}. Now, by the definition of $L_n, \forall n \in \mathbb{N}$,
\[
L_n = \sum_{i \in I_n} \omega_i^{(n)} d(u_n, S_i) \leq 2 \sum_{i \in I_n} \omega_i^{(n)} \| u_{n_0} - v \| = 2 \| u_{n_0} - v \|.
\]

Choose, now, any $D > \max \{ 2 \| u_{n_0} - v \|, L_0, \ldots, L_{n_0 - 1} \}$, and notice that for such a $D$ the claim holds true.
2. Recall, here, by Definition \cite{27} that if $u_n$ is such that $I_n = \emptyset$, then $d(u_n, S_j) = 0, \forall j \in \mathcal{J}_n$. Obviously, this is equivalent to $\max \{ d(u_n, S_j) : j \in \mathcal{J}_n \} = 0$. 

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Hence, we deal only with the case of $\mathcal{I}_n \neq \emptyset$. For this case, we observe by (15) that

$$
\Theta_n(u_n) = \sum_{i \in \mathcal{I}_n} \frac{\omega_i \, d_i^2(u_n, S_i)}{L_n} \geq \sum_{i \in \mathcal{I}_n} \frac{\omega_i \, d_i^2(u_n, S_i)}{D} \\
\geq \frac{\tilde{\omega}}{D} \sum_{i \in \mathcal{I}_n} d_i^2(u_n, S_i) \geq \frac{\tilde{\omega}}{D} \max \{d_i^2(u_n, S_i) : i \in \mathcal{I}_n\},
$$

(19)

where the existence of $D > 0$ is guaranteed by Theorem 31.11.

In order to establish Theorem 18.6, i.e., $\lim_{n \to \infty} \Theta_n(u_n) = 0$, we have used Assumption 17.4 which imposes a bound on the sequence of subgradients $(\Theta'_n(u_n))_{n \in \mathbb{N}}$. However, for the case at hand, Lemma 28.3 clearly suggests that boundedness holds true by default, that Assumption 17.4 is not necessary here, and that Assumptions 17.2, 17.3 are sufficient for establishing $\lim_{n \to \infty} \Theta_n(u_n) = 0$. Having this result hold true, apply $\lim_{n \to \infty}$ on both sides of (19) to obtain $\lim_{n \to \infty} \max \{d_i(u_n, S_i) : i \in \mathcal{I}_n\} = 0$.

Recall, now, by the definition of $\mathcal{I}_n$, in Definition 27.1 that $\forall j \in \mathcal{J}_n \setminus \mathcal{I}_n$, $u_n \in S_j$ iff $d(u_n, S_j) = 0$. This clearly implies that $\max \{d_i(u_n, S_i) : i \in \mathcal{I}_n\} = \max \{d_i(u_n, S_j) : j \in \mathcal{J}_n\}$. This equality and the previously obtained result $\lim_{n \to \infty} \max \{d_i(u_n, S_i) : i \in \mathcal{I}_n\} = 0$ establish Theorem 31.2.

3. We have already seen in Theorem 31.2 that $\lim_{n \to \infty} \max \{d_i(u_n, S_j) : j \in \mathcal{J}_n\} = 0$. Since, by definition, $n \in \mathcal{J}_n$, $\forall n \in \mathbb{N}$, the previous result implies that $\lim_{n \to \infty} d(u_n, S_n) = \lim_{n \to \infty} \| (I - P_{S_n}) (u_n) \| = 0$. Having these in mind, Theorem 31.3 becomes a direct consequence of Theorem 15 and Example 16.

4. Here, we will utilize Theorems 18.5 and 18.8. To this end, notice that regarding the sequence of mappings $(T_{\theta_n}^{(u_n)})_{n \in \mathbb{N}}$, Assumption 17.5 is satisfied here; indeed, notice that $\forall n \in \mathbb{N}$, $\nu_n' \leq 2 - \nu_n \leq \nu_n$. Now, Definition 2 suggests that $\forall n \in \mathbb{N}$, $\{ T_{\theta_n}^{(u_n)}(u_n) - u_n, \Phi_n(u_n) \} + \Phi_n(u_n) \leq \Phi_n(T_{\theta_n}^{(u_n)}(u_n))$. Notice that for all those $n \in \mathbb{N}$ such that $\Phi_n'(T_{\theta_n}^{(u_n)}(u_n)) \neq 0$, we have

$$
\Phi_n(u_n) \leq \Phi_n(T_{\theta_n}^{(u_n)}(u_n)) + \langle u_n - T_{\theta_n}^{(u_n)}(u_n), \Phi_n'(u_n) \rangle \\
\leq \left\| \Phi_n'(T_{\theta_n}^{(u_n)}(u_n)) \right\| \nu_n \left\| \Phi_n(T_{\theta_n}^{(u_n)}(u_n)) \right\| + \left\| u_n - T_{\theta_n}^{(u_n)}(u_n) \right\| \left\| \Phi_n'(u_n) \right\| \\
\leq \frac{D}{\epsilon'} \left\| (I - T_{\Phi_n}^{(u_n)}(T_{\theta_n}^{(u_n)}(u_n))) \right\| + D \left\| u_n - T_{\theta_n}^{(u_n)}(u_n) \right\|.
$$

For all those $n \in \mathbb{N}$ where $\Phi_n'(T_{\theta_n}^{(u_n)}(u_n)) = 0$, we have by Definition 2 that $T_{\theta_n}^{(u_n)}(u_n) \in \arg \min_{v \in \mathcal{H}} \Phi_n(v)$, and since $\text{lev}_{\leq 0} \Phi_n \neq \emptyset$, we obtain $\Phi_n(T_{\theta_n}^{(u_n)}(u_n)) \leq 0$. Therefore, by similar steps as previously, we obtain the following inequality for such $n \in \mathbb{N}$: $\Phi_n(u_n) \leq D \left\| u_n - T_{\theta_n}^{(u_n)}(u_n) \right\|$. If we apply $\lim \sup_{n \to \infty}$ on both sides of the previous inequalities, and if we recall Theorems 18.5 and 18.8, then we obtain $\lim \sup_{n \to \infty} \Phi_n(u_n) \leq 0$. Notice that in the case where $\Phi_n : \mathcal{H} \to [0, \infty)$, $\forall n \in \mathbb{N}$, then the previous analysis leads to $\lim_{n \to \infty} \Phi_n(u_n) = 0$. This establishes Theorem 31.4.

5. First, notice that since we work in the Euclidean space $\mathbb{R}^L$, $\mathcal{S}((u_n)_{n \in \mathbb{N}}) = \mathcal{W}((u_n)_{n \in \mathbb{N}})$. Hence, the fact $\mathcal{S}((u_n)_{n \in \mathbb{N}}) \neq \emptyset$ is guaranteed by Theorem 18.4. Now, it can be verified that Theorem 31.5 is a direct consequence of Theorems 18.4, 18.9 and Lemma 26.11.

5. APPLICATION: ONLINE SPARSITY-AWARE SYSTEM/SIGNAL RECOVERY

The present section will demonstrate the potential of the previously introduced algorithms by devising a time-adaptive method for the important, nowadays, sparse system/signal recovery task. In particular, we will
use Algorithm 29 to derive a low-complexity and similarly effective variant of the technique introduced in 35–50.

Sparsity is the key characteristic of systems or signals whose representation, by means of some basis in some domain, consists of only a few nonzero coefficients, while the majority of them retain values of negligible size. The exploitation of sparsity has been attracting recently an interest of exponential growth under the Compressive Sensing or Sampling (CS) framework 12–15,28. In principle, CS allows the estimation of sparse signals and systems using fewer measurements than those previously thought to be necessary. More importantly, recovery is realized by mobilizing efficient constrained minimization schemes. Indeed, it has been shown that sparsity is favored by $\ell_1$ constrained solutions 15,16,25,26.

Recall, here, that given two integers $j_1 \leq j_2$, the notation $j_1,j_2$ stands for the set $\{j_1,j_1+1,\ldots,j_2\}$. Assume a vector $x_* := [x_{*,1}, \ldots, x_{*,L}]^t$ in the Euclidean space $\mathbb{R}^L$, $L \in \mathbb{N}$, where the superscript $t$ stands for vector transposition. If the support of $x_*$ is defined as $\text{supp}(x_*) := \{i \in 1,L : x_{*,i} \neq 0\}$, and the $\ell_0$ norm of $x_*$ is defined as the cardinality of its support, i.e., $\|x_*\|_{\ell_0} := \# \text{supp}(x_*)$, by the term “sparse” $x_*$, we refer to the case where $\|x_*\|_{\ell_0}$ is considerably smaller than $L$.

The majority of CS techniques deal with the problem of estimating a sparse system $x_*$, based on a number $K(< L)$ of measurements $(d_n)_{n=0}^{K-1} \subset \mathbb{R}$ that are generated by the following linear regression model (see (1)):

$$d_n = a_n^t x_* + \zeta_n, \quad \forall n \in \mathbb{N}. \quad (20)$$

Here, $(a_n)_{n \in \mathbb{N}} \subset \mathbb{R}^L$ are the input vectors, which excite the unknown $x_*$, and $(\zeta_n)_{n \in \mathbb{N}}$ is a real-valued discrete-time stochastic process which stands for the contaminating additive noise.

A well-known batch method for estimating the sparse $x_*$, based on a limited number $K < L$ of measurements, is provided by the Least-Absolute Shrinkage and Selection Operator (LASSO) 31,59:

$$\min\{\|Ax - d\|^2 : \|x\|_{\ell_1} \leq \|x_*\|_{\ell_1}, x \in \mathbb{R}^L\}, \quad (\forall x := [x_1, \ldots, x_L]^t \in \mathbb{R}^L, d := [d_0, \ldots, d_{K-1}]^t \in \mathbb{R}^K, A \in \mathbb{R}^{K \times L} \text{ the matrix whose rows are } (a_n^t)_{n=0}^{K-1}\}.$$

We stress here that the term “batch” method means that the data $(a_n, d_n)_{n=0}^{K-1}$ have to be available prior to the application of LASSO.

With only a few recent exceptions, i.e., 2,19,35,40,50, the majority of the proposed, so far, CS techniques are appropriate for batch mode operation 13,16,25,26. In other words, one has to wait until a fixed and predefined number $K$ of training data $(a_n, d_n)_{n=0}^{K-1}$ is available prior to application of CS processing methods, e.g., LASSO, in order to recover the corresponding signal/system estimate. Dynamic online operation for updating and improving estimates, as new measurements become available, is not feasible by batch processing methods. The development of efficient, time-adaptive, sparsity-aware techniques is of great importance in engineering, especially in cases where the signal or system under consideration is time-varying and/or the available storage resources are limited.

Moving along the path introduced in 2,19,35,40,50, the present section will deal with the case where $x_*$ is not only sparse but it is also allowed to be time-varying. For this reason, the number $K$ of available data is allowed to take values towards $\infty$. In this sense, the studies 2,19,35,40,50 operate in a framework that is different than the standard CS scenario. The major objective is no longer only the estimation of the sparse signal or system, based on a limited number of measurements. Letting $K \rightarrow \infty$ in the design, the additional task is the capability of the estimator to track possible variations of the unknown sparse system. Moreover, this has to take place at an affordable computational complexity, as required by most real time applications, where time-adaptive estimation is of interest. Consequently, the batch sparsity-aware techniques...
developed under the CS framework, e.g., LASSO or one of its variants, become unsuitable under time-varying scenarios. The focus, now, becomes the development of a framework that 1) exploits sparsity, 2) exhibits fast convergence to error floors that are as close as possible to those obtained by their batch counterparts, 3) offers good tracking performance, and 4) has low computational demands in order to meet the stringent time constraints that are imposed by most real time operation scenarios. Such a framework was demonstrated in [35, 36, 40, 50, 52]. Here, we focus on [35, 50]. Motivated by the previously presented Algorithm 29, we devise a variant of [35, 50], which shows similar performance to [35, 50], albeit its lower computational requirements.

The information at our disposal is the sequence of training data $(\mathbf{a}_n, d_n)_{n \in \mathbb{N}}$, the a-priori knowledge that the unknown $\mathbf{x}_*$ in (20) is sparse, as well as an estimate of the cardinality of the support of $\mathbf{x}_*$, i.e., $\|\mathbf{x}_*\|_{\ell_0}$. In the sequel, we will demonstrate a way to incorporate the a-priori knowledge of the estimate of $\|\mathbf{x}_*\|_{\ell_0}$ in the design as a series of closed convex sets.

In the spirit of Algorithm 29, we begin by introducing a sequence of closed convex sets $(S_n)_{n \in \mathbb{N}}$, which associate to the available training data $(\mathbf{a}_n, d_n)_{n \in \mathbb{N}}$, and quantify the deviation from the adopted model of (20) by the introduction of a user-defined tolerance $\xi \geq 0$.

**Definition 32** (Closed hyperslab). Given the online training data $(\mathbf{a}_n, d_n)_{n \in \mathbb{N}} \subset \mathbb{R}^L \times \mathbb{R}$, and a user-defined $\xi > 0$, we define the following sequence of closed convex sets, called *closed hyperslabs*:

$$\forall n \in \mathbb{N}, \quad S_n := \{ \mathbf{x} \in \mathbb{R}^L : |d_n - \mathbf{a}_n^T \mathbf{x}| \leq \xi \}.$$ 

The metric projection mapping $P_{S_n}$ can be analytically computed [54, 58], it breaks down to the metric projection onto a hyperplane, and its computational complexity scales linearly to the number of unknowns $L$.

In this section we mobilize Algorithm 29 where $(S_n)_{n \in \mathbb{N}}$ becomes the sequence of closed hyperslabs of Definition 32 and $(\Phi_n)_{n \in \mathbb{N}}$ is the sequence of sparsity-aware functions introduced in Definition 23. The Algorithm 29 with the metric projection mapping $P_{B_{t_1}(\mathbf{w}_n, \rho)}$ used instead of $T_{\Phi_n}^{(\nu_n)}$, was introduced in [35, 50].

The necessary complexity in order to compute the $P_{B_{t_1}(\mathbf{w}_n, \rho)}$ is of order $\mathcal{O}(L \log_2 L)$, needed for a sorting operation, and $\mathcal{O}(L)$ multiplications and additions [35, 50]. In the present study, due to the utilization of the relaxed subgradient projection mapping $T_{\Phi_n}^{(\nu_n)}$ in Algorithm 29 together with the simplicity of the subgradients of $\Phi_n$, seen in Table 11, we are able to cut down the computational complexity of the algorithm to $\mathcal{O}(L)$ operations. As it will be made clear by the subsequent numerical experiments, the Algorithm 29 results into a similar performance to its predecessor [35, 50].

The reason for introducing a series of weighted $\ell_1$-balls $B_{t_1}(\mathbf{w}_n, \rho)$, instead of the standard unweighted one $B_{t_1}(\mathbf{1}, \rho)$, is that 1) we have observed that the weighted $\ell_1$-balls, introduced in Definition 23, offer enhanced convergence speed, as also demonstrated in [16, 25] in a different context, and 2) the weighted balls help us easily incorporate the a-priori knowledge of the cardinality of the support of $\mathbf{x}_*$, i.e., $\|\mathbf{x}_*\|_{\ell_0}$, in the radius $\rho$, as the following lemma suggests.

**Lemma 33.** Assume that the sequence $(\mathbf{u}_n)_{n \in \mathbb{N}}$, generated by Algorithm 29 converges to the desirable $\mathbf{x}_*$.

Then, there exists an $N \in \mathbb{N}_\infty$ such that $\forall \rho \geq \|\mathbf{x}_*\|_{\ell_0}$, $\forall n \in N$, $\mathbf{u}_n \in B_{t_1}(\mathbf{w}_n, \rho)$.

**Proof.** By definition, $\sum_{i=1}^L w_{n,i} |u_{n,i}| = \sum_{i=1}^L \frac{|u_{n,i}|}{|u_{n,i}| + \tilde{\epsilon}}$, $\forall n \in \mathbb{N}$. Since $\lim_{n \to \infty} \mathbf{u}_n = \mathbf{x}_*$,

$$\limsup_{n \to \infty} \sum_{i=1}^L w_{n,i} |u_{n,i}| = \limsup_{n \to \infty} \sum_{i=1}^L \frac{|u_{n,i}|}{|u_{n,i}| + \tilde{\epsilon}} = \lim_{n \to \infty} \sum_{i=1}^L \frac{|u_{n,i}|}{|u_{n,i}| + \tilde{\epsilon}}$$

$$= \sum_{i \in \text{supp}(\mathbf{x}_*)} \frac{|x_{*,i}|}{|x_{*,i}| + \tilde{\epsilon}} + \sum_{i \notin \text{supp}(\mathbf{x}_*)} \frac{|x_{*,i}|}{|x_{*,i}| + \tilde{\epsilon}} < \sum_{i \in \text{supp}(\mathbf{x}_*)} \frac{|x_{*,i}|}{|x_{*,i}| + \tilde{\epsilon}} = \|\mathbf{x}_*\|_{\ell_0}.$$

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The previous strict inequality and the definition of \( \text{lim sup} \) suggest that there exists an \( N \in \mathbb{N}_\infty \) such that \( \forall n \in N \) we have \( \sum_{i=1}^{L} w_{n,i} u_{n,i} \leq \| x_* \| \ell_0 \). In other words, we obtain that \( \forall n \in N, \forall \rho \geq \| x_* \| \ell_0, u_n \in B_{\ell_1} [ w_n, \rho] \). This establishes Lemma 33. \( \square \)

In other words, Lemma 33 suggests that in order to have the sequence \( (u_n)_{n \in \mathbb{N}} \) converge to \( x_* \), a necessary condition is to set the radius \( \rho \), in the weighted balls \( (B_{\ell_1} [ w_n, \rho])_{n \in \mathbb{N}} \), to a value that over-estimates \( \| x_* \| \ell_0 \). This strategy will be followed in the subsequent numerical examples.

5.1. **Numerical examples.** In this section, the performance of the proposed algorithm is evaluated for both time-invariant and time-varying systems. To save space, only a couple of scenarios are considered. For extensive experiments on the behavior of similar in spirit algorithms, the interested reader is referred to \[35, 36\].

The proposed methodology is compared to a couple of recent time-adaptive methods \[2, 19\], which belong to the same algorithmic family; the cost function to be minimized is the sum of a quadratic loss, accounting for the regression model, together with an \( \ell_1 \)-norm regularization term, in order to induce sparsity into the design. The method RZ-LMS \[19\] is built upon the classical Least Mean Squares (LMS) algorithm, and employs re-weighting for the regularization term. Its computational complexity scales linearly with respect to the system unknowns, i.e., it is of order \( \mathcal{O}(L) \). Re-weighting of the \( \ell_1 \)-norm is also utilized in OCCD-TNWML \[2\], where the quadratic regression term follows the strategy in the celebrated Recursive Least Squares (RLS) method, scoring an overall computational complexity of order \( \mathcal{O}(4L^2) \).

Moreover, we mobilized batch methods for solving the classical LASSO \[9, 10, 59\], as well as its re-weighted variant \[68\]. In other words, for every batch method, each point in the respective curves is the outcome of a sub-process which takes into account all the available data available till the current time instant. It is clear that such an operation is infeasible in real-time implementations. Nevertheless, these performances will serve as benchmarks for the \( \ell_1 \)-norm regularized least squares solvers.

Fig. 1 refers to the case of a time-invariant system \( x_* \), whose length is \( L = 100 \) and only a number of 5 coefficients, placed in arbitrary positions, are nonzero, i.e., \( \| x_* \| \ell_0 = 5 \). The values of the nonzero coefficients were drawn from a Gaussian distribution of zero mean and variance equal to one. The input signal \( (u_n)_{n \in \mathbb{Z}} \) is defined as a discrete-time Gaussian process of zero mean and variance equal to 1. The vectors \( (a_n)_{n \in \mathbb{N}} \), in \[20\], are formed as follows: \( \forall n \in \mathbb{N}, a_n := [a_n, a_{n-1}, \ldots, a_{n-L+1}]^t \). The noise process \( (\zeta_n)_{n \in \mathbb{N}} \) is Gaussian with zero mean and variance equal to \( \sigma_n^2 := 0.1 \).

In Fig. 1 the tag “Proposed” refers to Algorithm 29. The curve “Proposed with exact projection mapping” refers to Algorithm 29 but with \( P_{B_{\ell_1} [ w_n, \rho]} \) in the place of \( T_{\Phi_n}^{(\nu_n)} \), \( \forall n \in \mathbb{N} \). This realization was introduced in \[35, 50\]. For both “Proposed” and “Proposed with exact projection mapping”, \( q \) was set equal to 25, \( \omega_n^{(n)} := 1/\| I_n \|, \forall n \in \mathbb{N} \), in the cases where \( I_n \neq \emptyset \), \( \rho := 6, \epsilon := 0.005, \) and \( \xi := 2\sigma_n \).

All of the parameters for the methods “LASSO” \[9, 10, 59\], “Weighted LASSO” \[68\], “OCCD-TNWML” \[2\], and “RZ-LMS” \[19\] were tuned for producing the best respective performance for the current setting. More specifically, the forgetting factor for “OCCD-TNWML” \[2\], which is an inherent parameter in any RLS-like scheme, was set equal to 1. Moreover, “RZ-LMS” \[19\] was tuned in such a way for producing the lowest error floor for the iteration \#450. Although different parameters for the “RZ-LMS” could result into faster convergence speed, this could only be obtained at the expense of higher error floors.

Fig. 1 demonstrates that “Proposed” and “Proposed with exact projection mapping” lead to similar performances. However, due to the mobilization of \( T_{\Phi_n}^{(\nu_n)} \) in “Proposed”, the computational complexity drops to \( \mathcal{O}(qL) \), as opposed to \( \mathcal{O}(qL + L \log L) \) in “Proposed with exact projection mapping”, with \( \mathcal{O}(L \log L) \) accounting for sorting operations which are necessary for the computation of the exact \( P_{B_{\ell_1} [ w_n, \rho]} \).
Figure 1. Time-invariant sparse system $\mathbf{x}_* \in \mathbb{R}^{100}$, with $\|\mathbf{x}_*\|_{\ell_0} = 5$. Here, the Mean Square Deviation (MSD) is defined as the following function on the number of the training data;

$$\text{MSD}(n) := \frac{1}{R} \sum_{r=1}^{R} \|\mathbf{x}_* - \mathbf{u}_n^{(r)}\|^2, \quad \forall n \in \mathbb{N},$$

where $R$ is the total number of independent runs of the experiment. Here, $R := 300$.

Fig. 2 refers to the case of a time-varying system. Both the number of nonzero elements of $\mathbf{x}_*$ and the values of the system’s coefficients are allowed to undergo sudden changes. This is a typical scenario used in adaptive filtering in order to study the tracking performance of an algorithm in practice. The system used in the experiments is of dimension 100. The system change is realized as follows: For the first 500 time instances, the first 5 coefficients are set equal to 1. Then, at time instance 501 the #2 and #4 coefficients are set equal to zero, and all the odd coefficients from #7 to #15 are set equal to 1. Note that the sparsity level changes at time instance 501, and it becomes 8 instead of 5. The results are shown in Fig. 2 with the noise variance being set equal to $\sigma_n^2 := 0.1$.

Notice also here the similarity in the performance of “Proposed” and “Proposed with exact projection mapping”. Moreover, the “RZ-LMS” shows better tracking ability than “OCCD-TNWL”, with the forgetting factor set equal to 0.999. In order to raise the tracking ability of the “OCCD-TNWL”, the method should be able to easily “forget” the remote past and concentrate on recent variations of the system. This is achieved by reducing the forgetting factor at the expense of an increased error floor. We chose the value of 0.96 for the forgetting factor of the “OCCD-TNWL” in order to achieve similar error floor to the “Proposed” method, for both the employed sparse systems.
Figure 2. Tracking performance for a time-varying sparse system $x_\ast \in \mathbb{R}^{100}$. The system $x_\ast$ changes suddenly at the #501 time instant. Here, as in Fig. 1, $\text{MSD}(n) := \frac{1}{R} \sum_{r=1}^{R} \| x_\ast - u_r^n \|^2$, $\forall n \in \mathbb{N}$, where $R$ is the total number of independent runs of the experiment. Similarly to Fig. 1, $R := 300$.

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