Logarithmic limits of minimal models

Jørgen Rasmussen

Centre de recherches mathématiques, Université de Montréal
Case postale 6128, succursale centre-ville, Montréal, Qc, Canada H3C 3J7
rasmusse@crm.umontreal.ca

Abstract

It is discussed how a limiting procedure of (super)conformal field theories may result in logarithmic (super)conformal field theories. The construction is illustrated by logarithmic limits of (unitary) minimal models in conformal field theory and in $N = 1$ superconformal field theory.

Keywords: Logarithmic conformal field theory, $N = 1$ superconformal field theory, minimal models.
1 Introduction

It has long been speculated that certain limits of minimal models in conformal field theory (CFT) may correspond to logarithmic CFTs (LCFTs). We refer to [1] for a survey on CFT and to [2, 3, 4] for recent reviews of LCFT. The first systematic study of LCFT appeared in [5]. Flohr, in particular, has discussed [6] how LCFTs appear at the ‘boundary’ of the set of minimal models $M(p, p')$ by considering $p$ and $p'$ not coprime, where the minimal models may be characterized by a pair of coprime integers $p > p' > 1$.

The set of models $M(p, p')$ with $p, p' \geq 1$ is obviously discrete. We suggest to say that any such model, which is not a minimal model, belongs to the boundary of the set of minimal models.

The objective here is to discuss how LCFT may be obtained by a limiting procedure different from the one used in [6], to which it does not seem to be directly related. Our approach is quite general in its own right, and is illustrated by logarithmic limits of unitary or non-unitary minimal models in CFT as well as in $N = 1$ superconformal field theory (SCFT) [7, 8, 9, 10]. We shall present a general prescription for constructing (super)conformal Jordan cells, thereby rendering the associated models logarithmic (super)conformal field theories.

The idea of our construction is to consider a sequence of conformal models labeled by an integer $n$, with focus on a pair of primary fields in each conformal model appearing in the sequence. To get a firmer grip on this, we introduce sequences of primary fields and organize the former in equivalence classes. For finite $n$, the two fields must have different conformal weights, while the weights of the associated sequences converge to the same (finite) conformal weight, $\Delta$, as $n$ approaches infinity. A Jordan-cell structure emerges if one considers a particular linear and (for finite $n$) invertible map of the two fields (or of the associated sequences) into two new fields. Since the original fields have different conformal weights, the new fields do not both have well-defined conformal weights. In the limit $n \to \infty$, the linear map is singular and thus not invertible (thereby mimicking the İnönü-Wigner or Saletan contractions known from the theory of Lie algebras), while the new set of fields make up a Jordan cell of conformal weight $\Delta$. The two-point functions of the new fields are also discussed.

A naive study of the representations of the fields in two sequences of primary fields as $n \to \infty$ would in general not allow one to distinguish between the two representations of the resulting pair of fields. The singular map mentioned above ensures such a distinction. A merit of our construction is thus that it makes manifest that certain representations remain different in the limit instead of potentially producing multiple copies of a single representation.

The replica approach to systems with disorder is based on techniques resembling the ones employed in the present work. In [11, 12], for example, logarithmic divergencies of correlators were obtained in the so-called replica limit where a certain parameter vanishes. The paradigm differs from ours, though, as we are considering infinite sequences of conformal models.
The general construction of a LCFT as a limiting procedure of CFTs is outlined in Section 2 and subsequently illustrated by logarithmic limits of minimal models. This is extended to SCFT and the $N = 1$ superconformal minimal models in Section 3. Section 4 contains some concluding remarks.

2 Logarithmic limits of CFT

2.1 General construction

A Jordan cell (of rank two) consists of two fields: a primary field, $\Phi$, of conformal weight $\Delta$, and its non-primary partner, $\Psi$. With a conventional relative normalization of the fields, we have

$$
T(z)\Phi(w) = \frac{\Delta \Phi(w)}{(z-w)^2} + \frac{\partial \Phi(w)}{z-w},
$$

$$
T(z)\Psi(w) = \frac{\Delta \Psi(w) + \Phi(w)}{(z-w)^2} + \frac{\partial \Psi(w)}{z-w}
$$

(1)

where $T$ is the Virasoro generator. The associated two-point functions are known to be of the form

$$
\langle \Phi(z)\Phi(w) \rangle = 0, \quad \langle \Phi(z)\Psi(w) \rangle = \frac{A}{(z-w)^2\Delta}, \quad \langle \Psi(z)\Psi(w) \rangle = \frac{B - 2A \ln(z-w)}{(z-w)^2\Delta}
$$

(2)

with structure constants $A$ and $B$. Our goal is to construct such a system in the limit of a sequence of ordinary CFTs.

To this end, let us consider a sequence of conformal models $M_n$, $n \in \mathbb{Z}_>$, with central charges, $c_n$, converging to the finite value

$$
\lim_{n \to \infty} c_n = c
$$

(3)

This appears to be a necessary condition for the limit of the sequence to exist. It is assumed that $M_n$ contains a pair of primary fields, $\varphi_n$ and $\psi_n$, of conformal weights $\Delta + a_n$ and $\Delta + b_n$, respectively, where

$$
\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0
$$

(4)

We are thus considering sequences of primary fields such as $(\varphi_1, \varphi_2, ...)$ where the element $\varphi_n$ belongs to $M_n$. Their two-point functions are assumed to be of the form

$$
\langle \varphi_n(z)\varphi_n(w) \rangle = \frac{C_\varphi}{(z-w)^{2(\Delta + a_n)}},
$$

$$
\langle \psi_n(z)\psi_n(w) \rangle = \frac{C_\psi}{(z-w)^{2(\Delta + b_n)}},
$$

$$
\langle \varphi_n(z)\psi_n(w) \rangle = 0
$$

(5)
where, for simplicity, the fields have been normalized so that the non-vanishing structure constants, $C_{\phi}$ and $C_{\psi}$, are independent of $n$. The conformal models $M_n$ could have an extended symmetry in which case the fields would be characterized by additional quantum numbers. Since our construction is essentially independent of such an eventuality, we may a priori allow $a_n = b_n$ while assuming $\langle \varphi_n \psi_n \rangle = 0$. We find, though, that consistency requires $a_n \neq b_n$ after all.

We now introduce the linear and invertible map

$$
\begin{pmatrix}
\Phi_n \\
\Psi_n
\end{pmatrix} = \begin{pmatrix}
\alpha_n & \beta_n \\
\gamma_n & \delta_n
\end{pmatrix} \begin{pmatrix}
\varphi_n \\
\psi_n
\end{pmatrix}
$$

and define

$$
\Phi := \lim_{n \to \infty} \Phi_n, \quad \Psi := \lim_{n \to \infty} \Psi_n
$$

In the picture with sequences of primary fields alluded to above, we are thereby defining two new sequences with potentially different properties as $n \to \infty$. The explicit behaviour of (7) depends, of course, on the map (6). It should be emphasized that the resulting model may not share the same standards and properties as the model obtained by considering the original sequences in the limit $n \to \infty$. A similar problem is known from Lie algebra contractions where the resulting algebra only satisfies the Jacobi identities under certain conditions.

For $\Phi$ and $\Psi$ to constitute a Jordan pair, we should consider a map which becomes singular as $n$ approaches infinity. We find that the simplest map meeting our needs is given by

$$
\begin{align*}
\alpha_n &= \sqrt{\frac{(a_n - b_n)A}{C_{\phi}}}, & \beta_n &= 0 \\
\gamma_n &= \sqrt{\frac{A}{(a_n - b_n)C_{\phi}}}, & \delta_n &= \sqrt{\frac{(a_n - b_n)B - A}{(a_n - b_n)C_{\psi}}}
\end{align*}
$$

for which it is evident that $a_n \neq b_n$. For example, we have

$$
T(z)\Psi(w) = \lim_{n \to \infty} \{T(z)\Psi_n(w)\}
$$

$$
= \lim_{n \to \infty} \left\{ \frac{(\Delta + \frac{\alpha_n \delta_n b_n - \beta_n \gamma_n a_n}{\alpha_n \delta_n - \beta_n \gamma_n}) \Psi_n(w) + \frac{\gamma_n \delta_n (a_n - b_n)}{\alpha_n \delta_n - \beta_n \gamma_n} \Phi_n(w)}{(z - w)^2} + \frac{\partial_w \Psi_n(w)}{z - w} \right\}
$$

$$
= \frac{\Delta \Psi(w) + \Phi(w)}{(z - w)^2} + \frac{\partial \Psi(w)}{z - w}
$$

where the inverse map has been used in the rewriting. The map (8) has been chosen in order to reproduce the two-point functions (2) with the structure constants given there. That this is satisfied follows from the expansion

$$
x^{\epsilon} = e^{\epsilon \ln(x)} = 1 + \epsilon \ln(x) + \frac{1}{2} \epsilon^2 \ln^2(x) + O(\epsilon^3)
$$
needed when evaluating expressions like (5) in the limit \( n \to \infty \). For example, we have

\[
\langle \Psi(z)\Psi(w) \rangle = \lim_{n \to \infty} \left\{ \langle \Psi_n(z)\Psi_n(w) \rangle \right\}
\]

\[
= \lim_{n \to \infty} \left\{ \frac{\gamma^2_n C_\phi}{(z-w)^{2(\Delta+a_n)}} + \frac{\delta^2_n C_\psi}{(z-w)^{2(\Delta+b_n)}} \right\}
\]

\[
= \frac{B - 2A \ln(z-w)}{(z-w)^{2\Delta}}
\]

(11)

It follows in this way that the fields (7) constitute a Jordan cell of conformal weight \( \Delta \).

We wish to point out that a Jordan cell also emerges in the special case when either \( a_n \) or \( b_n \) (but not both) is zero for all \( n \). An example may be found in [13] where a so-called correlated limit of the parafermionic CFT is discussed. It has been found that the construction in [13] extends to the graded parafermions as well [14].

2.2 Logarithmic limits of minimal models

Here we illustrate the general construction above by considering limits of minimal models. The minimal model \( \mathcal{M}(p,p') \) is characterized by the coprime integers \( p \) and \( p' \) which may be chosen to satisfy \( p > p' > 1 \), without loss of generality. The central charge is given by

\[
c = 1 - 6 \frac{(p-p')^2}{pp'}
\]

(12)

whereas the primary fields, \( \phi_{r,s} \), have conformal weights given by

\[
\Delta_{r,s} = \frac{(rp - sp')^2 - (p-p')^2}{4pp'}, \quad 1 \leq r < p', \quad 1 \leq s < p
\]

(13)

The bounds on \( r \) and \( s \) define the Kac table of admissible primary fields. With the identification

\[
\phi_{r,s} = \phi_{p'-r,p-s}
\]

(14)

there are \( (p-1)(p'-1)/2 \) distinct primary fields in the model. These models are unitary provided \( p = p' + 1 \), in which case the further constraint \( s \leq r \) takes into consideration the identification (14).

For each positive integer \( k \) we now consider the sequence of minimal models \( \mathcal{M}(kn+1,n) \), \( n \geq 2 \), where it is easily verified that \( kn+1 \) and \( n \) are relatively prime. The central charges and conformal weights are given by

\[
c^{(k,n)} = 1 - 6 \frac{(k-1)n+1)^2}{n(kn+1)}
\]

\[
= 1 - 6 \frac{(k-1)^2}{k} - 6 \frac{(k^2-1)}{k^2n} + \mathcal{O}(1/n^2)
\]

\[
\Delta_{r,s}^{(k,n)} = \frac{((kn+1)r - ns)^2 - ((k-1)n+1)^2}{4n(kn+1)}
\]

\[
= \frac{(kr - s)^2 - (k-1)^2}{4k} + \frac{k^2(r^2-1) - (s^2-1)}{4k^2n} + \mathcal{O}(1/n^2)
\]

(15)
with limits
\[ c^{(k)} = \lim_{n \to \infty} c^{(k,n)} = 1 - 6\frac{(k - 1)^2}{k}, \]
\[ \Delta_{r,s}^{(k)} = \lim_{n \to \infty} \Delta_{r,s}^{(k,n)} = \frac{(kr - s)^2 - (k - 1)^2}{4k}, \quad r, s \in \mathbb{Z}_> \] (16)

These are seen to correspond to the similar values in the (non-minimal) model \( \mathcal{M}(k, 1) \) on the boundary of the set of minimal models. The model \( \mathcal{M}(1, 1) \) is thus related to the limit of the sequence of unitary minimal models \( \mathcal{M}(n + 1, n) \), and has central charge \( c^{(1)} = 1 \). The limit of the non-unitary minimal models \( \mathcal{M}(2n + 1, n) \) is associated to \( \mathcal{M}(2, 1) \) which has appeared in the literature in studies of \( c = -2 \) LCFT [15]. The other integer central charges obtained in this way are \( c^{(3)} = -7 \) and \( c^{(6)} = -24 \). As we shall discuss presently, Jordan-cell structures can be constructed for all \( k \geq 1 \).

There is a natural embedding of the Kac table associated to \( \mathcal{M}(kn_1 + 1, n_1) \) into the Kac table associated to \( \mathcal{M}(kn_2 + 1, n_2) \) if \( n_1 \leq n_2 \), mapping \( \phi_{r,s}^{(k,n_1)} \) to \( \phi_{r,s}^{(k,n_2)} \). Note, however, that the conformal weights and representations in general will be altered. Our point here is that if \( (r, s) \) is admissible for \( n_0 \), it will be admissible for all \( n \geq n_0 \). We thus have a natural notion of sequences of primary fields: \( (\phi_{r,s}^{(k,n_0)}, \phi_{r,s}^{(k,n_0+1)}, \ldots) \). The parameter \( n_0 \) is essentially immaterial since we are concerned with the properties of the sequences as \( n \to \infty \). We therefore choose to denote a sequence simply as \( \Upsilon^{(k)}_{r,s} \) (from whose indices the minimal \( n_0 \) can be determined anyway).

These sequences may be organized in equivalence classes, where \( \Upsilon^{(k)}_{r,s} \) and \( \Upsilon^{(k)}_{u,v} \) are said to be equivalent if they approach the same conformal weight. For \( \Upsilon^{(k)}_{r,s} \) and \( \Upsilon^{(k)}_{u,v} \) to be equivalent it is required that \( (ku - v)^2 = (kr - s)^2 \), that is,

\[ I: \quad (u, v) = (r + q, s + kq), \quad r, s, u, v \in \mathbb{Z}_>, \quad q \in \mathbb{Z} \] (17)
or
\[ II: \quad (u, v) = (-r + q, -s + kq), \quad r, s, u, v \in \mathbb{Z}_>, \quad q \in \mathbb{Z} \] (18)

In either case, the approached weight is \( \Delta_{r,s}^{(k)} \) in (16). The equivalence becomes trivial (i.e., \( \Upsilon^{(k)}_{r,s} = \Upsilon^{(k)}_{u,v} \)) if \( q = 0 \) in case \( I \) or if \( q = 2r \) and \( s = kr \) in case \( II \).

Since our objective is to illustrate the limiting procedure of the previous section, we should consider two equivalent but different sequences of primary fields. We have two cases to analyze. In case \( I \), and with reference to the general construction above, we consider the two primary fields \( \varphi_n^{(k)} = \phi_{r,s}^{(k,n)} \) and \( \psi_n^{(k)} = \phi_{r+q,s+kq}^{(k,n)} \) (excluding, of course, the trivial case \( q = 0 \)) in the minimal model \( \mathcal{M}(kn + 1, n) \). We find that

\[ a_n^{(k)} - b_n^{(k)} = \frac{g(2n(s - rk) - 2r - q)}{4kn(kn + 1)} \]
\[ = \frac{g(s - rk)}{2kn} - \frac{g(2s + qk)}{4k^2n^2} + O(1/n^3) \] (19)
which for large enough \( n \) is non-vanishing. The map (6) and (8) then reads

\[
\begin{pmatrix}
\Phi_{n}^{(k)} \\
\Psi_{n}^{(k)}
\end{pmatrix} = \begin{pmatrix}
\sqrt{\frac{q(2n(rk-s) r-q)A}{4n(kn+1)C_{r,s}^{(k)}}} & 0 \\
\sqrt{\frac{4n(kn+1)A}{q(2n(s-rk)-2r-q)C_{r,s}^{(k)}}} & \sqrt{\frac{q(2n(rk-s)-2r-q)B-4n(kn+1)}{q(2n(rk-s)-2r-q)C_{r+q}s+kq}}
\end{pmatrix} \begin{pmatrix}
\phi_{r,s}^{(k,n)} \\
\phi_{r+q,s+kq}^{(k,n)}
\end{pmatrix}
\]

(20)

where \( C_{r,s}^{(k)} = C_{r,s}^{(k,n)} \) has been chosen independent of \( n \). The resulting Jordan cell given by \( \Phi^{(k)} \) and \( \Psi^{(k)} \), defined as in (7), has conformal weight \( \Delta_{r,s}^{(k,n)} \) given in (16).

Similarly, in case II we consider \( \phi_{r,s}^{(k,n)} = \phi_{r,s}^{(k,n)} \) and \( \psi_{n}^{(k,n)} = \psi_{r+q,s+kq}^{(k,n)} \) (again excluding the trivial case) and find

\[
a_{n}^{(k)} - b_{n}^{(k)} = \frac{q(2n(rk-s) r-q)}{4n(kn+1)}
\]

\[
= \frac{q(rk-s)}{2kn} + \frac{q(2s-qk)}{4k^{2}n^{2}} + O(1/n^{3})
\]

(21)

The map becomes

\[
\begin{pmatrix}
\Phi_{n}^{(k)} \\
\Psi_{n}^{(k)}
\end{pmatrix} = \begin{pmatrix}
\sqrt{\frac{q(2n(rk-s) + 2r-q)A}{4n(kn+1)C_{r,s}^{(k)}}} & 0 \\
\sqrt{\frac{4n(kn+1)A}{q(2n(rk-s) + 2r-q)C_{r,s}^{(k)}}} & \sqrt{\frac{q(2n(rk-s) + 2r-q)B-4n(kn+1)}{q(2n(rk-s) + 2r-q)C_{r+q,s+kq}}}
\end{pmatrix} \begin{pmatrix}
\phi_{r,s}^{(k,n)} \\
\phi_{r+q,s+kq}^{(k,n)}
\end{pmatrix}
\]

(22)

The formulas simplify a bit in some cases. For example, the numerator in (19) factorizes as \((2kn-1)(s^{2}-r^{2})\) when \( s = s'k + 2rk \) and \( q = -r + s' \), whereas the numerator in (21) factorizes as \((2kn+1)(r^{2}-s^{2})\) when \( s = s'k \) and \( q = r + s' \).

We conclude that Jordan cells can be constructed for all conformal weights in the spectrum of \( \mathcal{M}(k,1) \) given in (16). It has also been found that there are numerous ways of constructing a Jordan cell of a given weight in that spectrum. We shall comment more on these issues in the final section. In [15] on a LCFT with \( c = -2 \) and spectrum related to \( \mathcal{M}(2,1) \), only a subset of the conformal weights in the full spectrum are associated to Jordan cells. The remaining values correspond to ordinary primary fields. Our construction above does not a priori distinguish between these two subsets as all the weights in the spectrum may be associated to Jordan cells. It would be interesting to understand the origin of this discrepancy.

### 3 Logarithmic limits of SCFT

#### 3.1 General construction

The concept of a Jordan cell carries over to the \( N = 1 \) superconformal case [16] where it may be represented straightforwardly in the superspace formalism [17].
Let $\xi = (z, \theta)$ be an $N = 1$ superspace coordinate with associated superderivative $D = \theta \partial + \partial \theta$. Since $\theta$ is a Grassmann-odd (anti-commuting) variable, a superfield, $\hat{\phi}$, expands trivially as

$$\hat{\phi}(\xi) = \phi(z) + \theta \lambda(z)$$  \hspace{1cm} (23)

A primary superfield of (super)conformal weight $\Delta$ may be characterized by

$$\hat{T}(\xi_1)\hat{\phi}(\xi_2) = \frac{\Delta \theta_{12} \hat{\phi}(\xi_2)}{(\xi_{12})^2} + \frac{\left(\theta_{12} \partial_2 + \frac{1}{2} D_2\right) \hat{\phi}(\xi_2)}{\xi_{12}}$$  \hspace{1cm} (24)

where $\xi_{12} = \xi_1 - \xi_2 - \theta_1 \theta_2$ and $\theta_{12} = \theta_1 - \theta_2$. The generator of superconformal transformations $\hat{T}$, $\hat{T}$, is the odd linear combination

$$\hat{T}(\xi) = \theta T(z) + \frac{1}{2} G(z)$$  \hspace{1cm} (25)

differing by the factor 1/2 from the convention used in [17]. $T$ is the ordinary Virasoro generator with central charge $c$, whereas $G$ is its primary spin-3/2 superpartner satisfying

$$G(z)G(w) = \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{z-w}$$  \hspace{1cm} (26)

In the expansion (23) of a primary superfield, the two fields $\phi$ and $\lambda$ are primary of weights $\Delta$ and $\Delta + 1/2$, respectively, with respect to the Virasoro generator $T$.

A superconformal Jordan cell consists of two superfields: a primary superfield, $\hat{\Phi}$, of (super)conformal weight $\Delta$, and its non-primary partner, $\hat{\Psi}$. They satisfy

$$\hat{T}(\xi_1)\hat{\Phi}(\xi_2) = \frac{\theta_{12} \Delta \hat{\Phi}(\xi_2)}{(\xi_{12})^2} + \frac{\left(\theta_{12} \partial_2 + \frac{1}{2} D_2\right) \hat{\Phi}(\xi_2)}{\xi_{12}}$$

$$\hat{T}(\xi_1)\hat{\Psi}(\xi_2) = \frac{\theta_{12} \Delta \hat{\Psi}(\xi_2) + \hat{\Phi}(\xi_2)}{(\xi_{12})^2} + \frac{\left(\theta_{12} \partial_2 + \frac{1}{2} D_2\right) \hat{\Psi}(\xi_2)}{\xi_{12}}$$  \hspace{1cm} (27)

The associated two-point functions are of the form

$$\langle \hat{\Phi}(\xi_1)\hat{\Phi}(\xi_2) \rangle = 0, \quad \langle \hat{\Phi}(\xi_1)\hat{\Psi}(\xi_2) \rangle = \frac{A}{(\xi_{12})^{2\Delta}}, \quad \langle \hat{\Psi}(\xi_1)\hat{\Psi}(\xi_2) \rangle = \frac{B - 2A \ln \xi_{12}}{(\xi_{12})^{2\Delta}}$$  \hspace{1cm} (28)

with structure constants $A$ and $B$. Our goal is to construct such a system in the limit of a sequence of SCFTs.

The construction is a straightforward extension of the one discussed in the previous section on ordinary CFT. We thus consider a sequence of superconformal models $SM_n$, $n \in \mathbb{Z}_>$, with central charges, $c_n$, converging to the finite value

$$\lim_{n \to \infty} c_n = c$$  \hspace{1cm} (29)
It is assumed that $SM_n$ contains a pair of primary superfields, $\hat{\phi}_n$ and $\hat{\psi}_n$, of weights $\Delta + a_n$ and $\Delta + b_n$, respectively, satisfying (4). Their two-point functions are assumed to be of the form

\[
\langle \hat{\phi}_n(\xi_1)\hat{\phi}_n(\xi_2) \rangle = \frac{C_{\hat{\phi}}}{(\xi_{12})^{2(\Delta + a_n)}}
\]
\[
\langle \hat{\psi}_n(\xi_1)\hat{\psi}_n(\xi_2) \rangle = \frac{C_{\hat{\psi}}}{(\xi_{12})^{2(\Delta + b_n)}}
\]
\[
\langle \hat{\phi}_n(\xi_1)\hat{\psi}_n(\xi_2) \rangle = 0
\]
(30)

We now introduce the linear and invertible map

\[
\left( \begin{array}{c}
\hat{\Phi}_n \\
\hat{\Psi}_n \\
\end{array} \right) = \left( \begin{array}{cc}
\alpha_n & \beta_n \\
\gamma_n & \delta_n \\
\end{array} \right) \left( \begin{array}{c}
\hat{\phi}_n \\
\hat{\psi}_n \\
\end{array} \right)
\]
(31)

and define the superfields

\[
\hat{\Phi} := \lim_{n \to \infty} \hat{\Phi}_n, \quad \hat{\Psi} := \lim_{n \to \infty} \hat{\Psi}_n
\]
(32)

It turns out that a map similar to (31) also applies in this case:

\[
\alpha_n = \sqrt{\frac{(a_n - b_n)A}{C_{\hat{\phi}}}}, \quad \beta_n = 0
\]
\[
\gamma_n = \sqrt{\frac{A}{(a_n - b_n)C_{\hat{\phi}}}}, \quad \delta_n = \sqrt{\frac{(a_n - b_n)B - A}{(a_n - b_n)C_{\hat{\psi}}}}
\]
(33)

as it is straightforward to show that the superfields defined in (32) indeed constitute a superconformal Jordan cell of weight $\Delta$.

### 3.2 Logarithmic limits of superconformal minimal models

A superconformal minimal model, $SM(p, p')$, is characterized by two integers whose difference is even, with $(p - p')/2$ and $p$ (or equivalently $p'$) coprime. Without loss of generality, we are here following the convention that $p \geq p' + 2$ and $p' \geq 2$. The central charge is given by

\[
c = \frac{3}{2} - 3\frac{(p - p')^2}{pp'}
\]
(34)

whereas the primary fields have conformal weights

\[
\Delta_{r,s} = \frac{(rp - sp')^2 - (p - p')^2}{8pp'} + \frac{1}{32}(1 - (-1)^{r+s}), \quad 1 \leq r < p', \quad 1 \leq s < p
\]
(35)
These fields are subject to the same field identification as in (14), and for $p$ and $p'$ both even, the field $\phi_{p'/2,p/2}$ is unaffected by this identification. The superconformal minimal model $\mathcal{SM}(p, p')$ is unitary provided $p = p' + 2$.

The Neveu-Schwarz (NS) sector contains the fields with $r + s$ even, while fields with $r + s$ odd belong to the Ramond sector. A primary field of weight $\Delta_{r,s}$ in the NS sector has a superpartner of weight $\Delta_{r,s} + 1/2$ together with which it makes up a primary superfield of weight $\Delta_{r,s}$. Since the general construction outlined above is based on superfields, we shall eventually treat the two sectors separately.

For each positive integer $k$ we consider the sequence of superconformal minimal models $\mathcal{SM}((2k-1)n + 2, n), \; n \geq 2$. It is easily verified that the difference $\{(2k-1)n + 2\} - n$ is even, and that half of this difference is relatively prime to $n$. The central charges and conformal weights are given by

$$c^{(k,n)} = \frac{3}{2} - \frac{3((2k-1)n + 2 - n)^2}{(2k-1)n + 2} \quad \text{and} \quad \Delta^{(k,n)}_{r,s} = \frac{r((2k-1)n + 2) - sn)^2 - ((2k-1)n + 2 - n)^2}{8((2k-1)n + 2)n}$$

$$+ \frac{1}{32}(1 - (-1)^{r+s})$$

with limits

$$c^{(k)} = \lim_{n \to \infty} c^{(k,n)} = \frac{3}{2} - \frac{12(k-1)^2}{2k-1}$$

$$\Delta^{(k)}_{r,s} = \lim_{n \to \infty} \Delta^{(k,n)}_{r,s}$$

$$= \frac{(r(2k-1) - s)^2 - 4(k-1)^2}{8(2k-1)} + \frac{1}{32}(1 - (-1)^{r+s})$$

$$+ \frac{1}{32}(1 - (-1)^{r+s})$$

$$r, s \in \mathbb{Z}^>$$

These are seen to correspond to the similar values in the non-minimal model $\mathcal{SM}(2k-1, 1)$ on the boundary of the set of minimal models. The model $\mathcal{SM}(1, 1)$ is thus related to the limit of the sequence of unitary minimal models $\mathcal{SM}(n + 2, n)$, and has central charge $c^{(1)} = 3/2$. The limit of the non-unitary minimal models $\mathcal{SM}(3n + 2, n)$, corresponding to $k = 2$, is associated to $\mathcal{SM}(3, 1)$ with central charge $c = -5/2$. These are the only two models of this kind with half-integer central charge.

Motivated by the construction of Jordan cells in conformal minimal models, we now consider when two sequences of primary (super)fields, $\Upsilon^{(k)}_{r,s}$ and $\Upsilon^{(k)}_{u,v}$, associated to the sequence $\mathcal{SM}((2k-1)n + 2, n)$ are equivalent. First, it is observed that a sequence in the NS sector ($r + s$ even) cannot approach the same conformal weight as a sequence in
the Ramond sector \((u + v \text{ odd})\). The condition for equivalence in either sector then reads 
\((2k - 1)r - s)^2 = ((2k - 1)u - v)^2\), that is,

\[
I : \quad (u, v) = (r + q, s + (2k - 1)q), \quad r, s, u, v \in \mathbb{Z}_>, \quad q \in \mathbb{Z} \quad (38)
\]

or

\[
II : \quad (u, v) = (-r + q, -s + (2k - 1)q), \quad r, s, u, v \in \mathbb{Z}_>, \quad q \in \mathbb{Z} \quad (39)
\]

with identity for \(q = 0\) in case I and for \(q = 2r\) and \(s = (2k - 1)r\) in case II.

Superconformal Jordan cells may now be constructed straightforwardly by combining \((38)\) with the prescription in the conformal case above. We thus work out \(a_n^{(k)} - b_n^{(k)}\) in both cases \((38)\) and \((39)\), and the superconformal Jordan cells emerge as the limit \((32)\).

Regarding the Ramond sector, we propose to deal with it in the same way as we dealt with primary fields in the previous section on ordinary CFT. Again, we work out \(a_n^{(k)} - b_n^{(k)}\) in both cases \((38)\) and \((39)\), and the conformal Jordan cells emerge as \(n\) approaches infinity.

Due to the similarity between the two sectors, we may present the results in a unified way. In the case \((38)\) we find

\[
\left( \begin{array}{c}
\Phi_{n}^{(k)} \\
\Psi_{n}^{(k)}
\end{array} \right) = \frac{\sqrt{q(n(s-r(2k-1))-2r-q)A}}{2n((2k-1)n+2)C_{r,s}^{(k)}} \times \left( \begin{array}{c}
\phi_{r,s}^{(k,n)} \\
\phi_{r+q,s+(2k-1)q}^{(k,n)}
\end{array} \right) \quad (40)
\]

while in the case \((39)\) we have

\[
\left( \begin{array}{c}
\Phi_{n}^{(k)} \\
\Psi_{n}^{(k)}
\end{array} \right) = \frac{\sqrt{q(n(r(2k-1)-s)+2r-q)A}}{2n((2k-1)n+2)C_{r,s}^{(k)}} \times \left( \begin{array}{c}
\phi_{r,s}^{(k,n)} \\
\phi_{r+q,s-(2k-1)q}^{(k,n)}
\end{array} \right) \quad (41)
\]

In both cases, \(C_{r,s}^{(k)} = C_{-r,-s}^{(k)}\) has been chosen independent of \(n\). Also, in the NS sector the two maps \((40)\) and \((41)\) correspond to the even parts of the superfields but are identical to the maps for the superfields themselves. The maps for the superfields are therefore not written explicitly.
4 Conclusion

We have discussed how certain limits of sequences of CFTs and SCFTs may correspond to logarithmic CFTs and SCFTs. Particular emphasis has been put on minimal models, and we have found that certain logarithmic limits may be associated to non-minimal (super)conformal models on the boundary of the set of (super)conformal minimal models. An infinite family of such logarithmic limits has been proposed in the ordinary as well as in the superconformal case.

The map (6) (and (31) in the superconformal case) will in general only affect a small subset of the full spectrum of fields. One should therefore expect that different linear and invertible maps of the complete set of fields

\[
\begin{pmatrix}
\text{new set} \\
\text{old set}
\end{pmatrix} = \begin{pmatrix}
\text{(singular) matrix}
\end{pmatrix}
\begin{pmatrix}
\text{old set}
\end{pmatrix}
\] (42)

in general would result in inequivalent models in the limit \( n \to \infty \). In particular, maps that become singular in this limit seem to be prone to alter the spectrum. The naive limit of the unitary series \( \mathcal{M}(n+1, n) \), where the map is governed by the identity matrix, is related to the discussion in [18, 19] where it has been found to correspond to a non-rational but non-logarithmic CFT with \( c = 1 \).

Alternatives to the sequences \( \mathcal{M}(kn + 1, n) \) and \( \mathcal{S}_n \mathcal{M}((2k - 1)n + 2, n) \) can, of course, be envisaged. Following our general construction of (super)conformal Jordan cells, these would potentially yield different logarithmic models in the appropriate limits. A possible classification of the models thus obtainable is an interesting problem to pursue. It should be noted that the map (42), followed by a limiting procedure, in many cases will lead to a non-logarithmic model. The resulting model is most likely non-rational, though.

Here we confine ourselves to indicating how one may construct Jordan cells of any weight associated to the general model \( \mathcal{M}(p, p') \) where \( p \) and \( p' \) are coprime. To this end, let \( \mathbb{N}_{p'} \) denote the set of positive integers relatively prime to \( p' \). Consider

\[
\mathcal{M}(pn + p'^a, p'n)
\] (43)

for some positive integer \( a \). It follows that \( pn + p'^a \) and \( p'n \) are relatively prime for \( n \in \mathbb{N}_{p'} \). For \( p' \neq 1 \), one could replace \( p'^a \) with any non-trivial product of non-negative powers of the prime factors of \( p' \), and (43) would still correspond to a minimal model. For fixed \( a \), we now consider the sequence of models (43) where \( n \in S_{p'} \) and \( S_{p'} \) is an infinite subset of \( \mathbb{N}_{p'} \). As \( n \) increases and approaches infinity, the central charge and spectrum of conformal weights will approach the similar values in the model \( \mathcal{M}(p, p') \). Our prime example above corresponds to \( p' = 1 \) and \( S_{p'} = \mathbb{N}_{\geq 2} \), while another class of examples was pointed out to us by A. Nichols and is given by the minimal models \( \mathcal{M}(p^2n, pp'n + 1) \) which approach \( \mathcal{M}(p, p') \) in the limit \( n \to \infty \). These examples are obviously not the
only possibilities for obtaining the central charge and spectrum of conformal weights of \( \mathcal{M}(p, p') \), in a limiting procedure.

Our construction pertains to (super)conformal Jordan cells of rank two. We have recently found \([20]\), though, that it extends to Jordan cells of rank \( r \) \([21]\) where

\[
T(z) \Psi_{(j)}(w) = \frac{\Delta \Psi_{(j)}(w) + (1 - \delta_{j,0}) \Psi_{(j-1)}(w)}{(z - w)^2} + \frac{\partial_w \Psi_{(j)}(w)}{z - w}, \quad j = 0, 1, \ldots, r - 1 \tag{44}
\]

Here \( \Psi_{(0)} \) is a primary field while the other \( r - 1 \) fields are not.

We would like to comment on the many ways Jordan cells may be obtained according to our outline, cf. the observation following \([22]\), in particular. This suggests that there could be infinitely many fields or Jordan cells of a given weight. One attempt to circumvent this, if so desired, is to consider \([42]\) by supplementing the maps \([6]\) by scaling the 'unwanted' fields by \( n \) (or perhaps even higher-degree polynomials in \( n \) or \( \sqrt{n} \)) to prevent them from showing up in operator products of the 'wanted' fields. This could potentially result in finitely many fields of a given weight. The finer structure of the operator-product algebra may reveal, though, that it is impossible to avoid the unwanted fields, but this issue is beyond the scope of the present work. Field identifications may alternatively resolve the problem. It is emphasized that we are dealing with chiral fields only. As the issue of locality in LCFT appears much more subtle and complicated than in ordinary CFT, attempting to construct the full LCFT may also put severe and unexpected constraints on the permitted chiral structure.

Now, the possibility of infinitely many rank-two Jordan cells appearing with a given conformal weight could perhaps be understood as infinitely many ways of extracting rank-two Jordan cells from a single Jordan cell of infinite rank. Such a structure could possibly help explaining why some logarithmic models have been found (see \([6]\)) to display features similar to rational CFTs, despite the highly non-rational nature of LCFT.

It is known that the (super)conformal minimal models can be represented in terms of cosets of affine current algebras. The unitary series, in particular, can be given by coset constructions with diagonal embeddings \([22]\):

\[
\mathcal{M}(n + 1, n) \simeq \frac{\widehat{su}(2)_{n-2} \oplus \widehat{su}(2)_1}{\widehat{su}(2)_{n-1}}, \quad n \geq 2
\]

\[
\mathcal{SM}(n + 2, n) \simeq \frac{\widehat{su}(2)_{n-2} \oplus \widehat{su}(2)_2}{\widehat{su}(2)_n}, \quad n \geq 2 \tag{45}
\]

The levels of the constituent affine Lie algebras are indicated by subindices, and are all integer. The coset constructions of all other series of (super)conformal minimal models will involve non-integer, though fractional levels. The integer nature of the levels in \([45]\) allows one to describe the cosets in terms of gauged Wess-Zumino-Witten models. This in turn seems to suggest that the \( c = 1 \) logarithmic CFT and the \( c = 3/2 \) logarithmic SCFT discussed above may admit geometric interpretations obtained as limits of the geometries associated to the unitary series. This could potentially mimic the Penrose
limits known from studies of (super)gravity solutions and properties of space-time, see references therein. We hope to address elsewhere this exciting possibility along with the other open questions and speculations above.

Acknowledgements
The author thanks C. Cummins, P. Mathieu, A. Nichols, M. Walton and, in particular, M. Flohr for comments.

References

[1] P. Di Francesco, P. Mathieu, D. Sénéchal, *Conformal field theory* (Springer 1997).

[2] M. Flohr, Int. J. Mod. Phys. A **18** (2003) 4497.

[3] M.R. Gaberdiel, Int. J. Mod. Phys. A **18** (2003) 4593.

[4] A. Nichols, *SU(2)k logarithmic conformal field theory*, Ph.D. thesis (University of Oxford, 2002), [hep-th/0210070](https://arxiv.org/abs/hep-th/0210070)

[5] V. Gurarie, Nucl. Phys. B **410** (1993) 535.

[6] M. Flohr, Int. J. Mod. Phys. A **12** (1997) 1943.

[7] H. Eichenherr, Phys. Lett. B **151** (1985) 26.

[8] M. Bershadsky, V. Knizhnik, M. Teitelman, Phys. Lett. B **151** (1985) 31.

[9] D. Friedan, Z. Qiu, S. Shenker, Phys. Lett. B **151** (1985) 37.

[10] D. Friedan, *Notes on string theory and two-dimensional conformal field theory*, in Proc. of Workshop on Unified String Theories, Santa Barbara, 1985.

[11] J.-S. Caux, I.I. Kogan, A.M. Tsvelik, Nucl. Phys. B **466** (1996) 444.

[12] J. Cardy, *Logarithmic correlators in quenched random magnets and polymers*, [cond-mat/9911024](https://arxiv.org/abs/cond-mat/9911024)

[13] I. Bakas, K. Sfetsos, Nucl. Phys. B **639** (2002) 223.

[14] J. Rasmussen, *On string backgrounds and (logarithmic) CFT*, [hep-th/0404226](https://arxiv.org/abs/hep-th/0404226)

[15] H.G. Kausch, *Curiosities at c = −2*, [hep-th/9510149](https://arxiv.org/abs/hep-th/9510149)

[16] M. Khorrami, A. Aghamohammadi, A.M. Ghezelbash, Phys. Lett. B **439** (1998) 283.

[17] N.E. Mavromatos, R.J. Szabo, JHEP **0110** (2001) 027.
[18] I. Runkel, G.M.T. Watts, JHEP **0109** (2001) 006.

[19] D. Roggenkamp, K. Wendland, *Limits and degenerations of unitary conformal field theories*, hep-th/0308143.

[20] J. Rasmussen, *Jordan cells in logarithmic limits of conformal field theory*, hep-th/0406110.

[21] M.R. Rahimi Tabar, A. Aghamohammadi, M. Khorrami, Nucl. Phys. **B 497** (1997) 555.

[22] P. Goddard, A. Kent, D. Olive, Commun. Math. Phys. **103** (1986) 105.

[23] D. Sadri, M.M. Sheikh-Jabbari, *The plane-wave/super Yang-Mills duality*, hep-th/0310119.