Properties of Bose gases with the Raman-induced spin–orbit coupling

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Abstract

In this paper we investigate the properties of Bose gases with Raman-induced spin–orbit (SO) coupling. It is found that the SO coupling can greatly modify the single-particle density of state, and thus leads to the non-monotonic behaviour of the condensate depletion, the Lee–Huang–Yang correction of ground-state energy and the transition temperature of a non-interacting Bose–Einstein condensate (BEC). The presence of the SO coupling also breaks the Galilean invariance, and this gives two different critical velocities, corresponding to the movement of the condensate and the impurity respectively. Finally, we show that with the SO coupling, the interactions modify the BEC transition temperature even at the Hartree–Fock level, in contrast to the ordinary Bose gas without the SO coupling. All results presented here can be directly verified in the current cold atom experiments using the Raman laser-induced gauge field.

(Some figures may appear in colour only in the online journal)

1. Introduction

One of the recent major progresses in cold atom physics is the realization of spin–orbit (SO) coupling for neutral atoms with the idea of a light-induced synthetic gauge field [1–4]. Previously, effects of the SO coupling have only been studied in fermionic matters such as electron gases, while this progress opens up the new opportunity of studying the SO coupling in bosonic systems. This investigation is interesting because the SO coupling changes many basic properties of Bose–Einstein condensation (BEC) in both quantitative and qualitative ways.

Although the SO coupling emerges at a single-particle level, it may have a significant effect on the many-body behaviour due to the modification of the single-particle density of state (DoS). One particular example is the Rashba SO-coupled Bose gases. Because of the dramatic change of the low-energy DoS, many intriguing many-body phenomena have been predicted there, and some questions still have not been thoroughly understood yet. While a number of theoretical works have studied its low-temperature properties, including ground-state phases, quantum and thermal fluctuations, Bose condensation transition and superfluidity [5–11], the experimental realization of the Rashba SO coupling remains to be a great challenge. On the other hand, the type of equal Rashba and Dresselhaus SO coupling has been realized in the current experiment with two-photon Raman transition. Although the change of the low-energy DoS is not as dramatic as the Rashba case, such an SO coupling leads to a non-monotonic behaviour of the low-energy DoS as a function of Raman strength, which should result in the nontrivial manifestation of many-body properties. So far, a few authors have studied the ground-state phase diagram, collective modes and response function for such SO-coupled condensates [12–17], but there are still several fundamental properties which have not been explored, in particular, the depletion of the condensate, the BEC transition temperature and the superfluid critical velocities. In this work, we report our theoretical studies of these properties.

2. Single-particle Hamiltonian and mean-field phase diagram

2.1. Single-particle Hamiltonian

In current experiments, two counter-propagating Raman beams are applied to couple two hyperfine levels of an alkali
atom [1–4], which gives rise to the single-particle Hamiltonian as [18] (set \( h = 1 \))

\[
\hat{H}_0 = \left( \hat{p} - k_r e_z \right)^2 / 2m + \delta / 2 \Omega^2 / 2 \left( \hat{p} + k_r e_z \right)^2 / 2m - \delta / 2 ,
\]

where \( k_r \) is the recoil momentum, \( \Omega \) is the Raman coupling strength and \( \delta \) is the two-photon detuning. With the Pauli matrix, the Hamiltonian in equation (1) can be rewritten as

\[
\hat{H}_0 = \frac{\hat{p}^2}{2m} - \mathbf{B}_p \cdot \sigma + E_r ,
\]

where \( E_r = k_r^2 / (2m) \) is the recoil energy and \( \mathbf{B}_p = (-\Omega / 2, 0, k_r / m - \delta / 2) \) depends on momentum \( k_r \) and yields the locking between spin and momentum. This Hamiltonian preserves spatial translational symmetry, and momentum \( \mathbf{p} \) is a good quantum number. Another quantum number of this Hamiltonian is ‘helicity’ \( h = \pm \), which denotes for \( \mathbf{B}_p \) parallel or anti-parallel to spin \( \sigma \). Thus, the eigenenergies of two helicity branches are given by

\[
\epsilon_{p,\pm} = \frac{\hat{p}^2}{2m} + k_r^2 / 2m \pm \sqrt{\left( \frac{p, k_r}{m} - \frac{\delta}{2} \right)^2 + \left( \frac{\Omega}{2} \right)^2} ,
\]

and their wavefunctions are given by

\[
\phi_{p, +} (\mathbf{r}) = e^{i \mathbf{p} \cdot \mathbf{r}} \left( \sin \theta_p / \cos \theta_p \right) ; \quad \phi_{p, -} (\mathbf{r}) = e^{i \mathbf{p} \cdot \mathbf{r}} \left( \cos \theta_p / -\sin \theta_p \right)
\]

with

\[
\theta_p = \arcsin \left[ \frac{1}{2} \left( 1 + \frac{p, k_r}{m - \delta / 2} \sqrt{(p, k_r)^2 + (2\Omega^2 / 4)} \right) \right]^{1/2} .
\]

With the spectrum, the single-particle DoS can be calculated straightforwardly as

\[
D(\epsilon) = \frac{1}{V} \sum_{\mathbf{p}} [\delta_D(\epsilon - \epsilon_{p, +}) + \delta_D(\epsilon - \epsilon_{p, -})] ,
\]

where \( \delta_D \) is the Dirac delta function.

### 2.2 Mean-field phase diagram

The interaction of a two-component Bose gas is generally written as

\[
\hat{H}_I = \frac{1}{2} \int \mathrm{d}^3 \mathbf{r} \left( \hat{g}_{\uparrow \uparrow} \hat{\psi}_{\uparrow} \hat{\psi}_{\uparrow} + \hat{g}_{\downarrow \downarrow} \hat{\psi}_{\downarrow} \hat{\psi}_{\downarrow} + 2 \hat{g}_{\uparrow \downarrow} \hat{\psi}_{\uparrow} \hat{\psi}_{\downarrow} \right) .
\]

With the mean-field approximation, the interaction energy is given by

\[
\mathcal{E}_I = \frac{1}{2} \int \mathrm{d}^3 \mathbf{r} \left( \hat{g}_{\uparrow \uparrow} \hat{n}_{\uparrow}^2 (\mathbf{r}) + \hat{g}_{\downarrow \downarrow} \hat{n}_{\downarrow}^2 (\mathbf{r}) + 2 \hat{g}_{\uparrow \downarrow} \hat{n}_{\uparrow} (\mathbf{r}) \hat{n}_{\downarrow} (\mathbf{r}) \right) .
\]

When both \( \delta \) and \( \Omega \) are small, \( \epsilon_- \) has two local minima located at \( k_{\pm} \). Without loss of generality, we can assume the condensate wavefunction as

\[
\varphi(\mathbf{r}) = \sqrt{n_0} \left[ \cos \alpha \left( \frac{\cos \theta_p}{-\sin \theta_p} \right) e^{\mathbf{p}_\perp \cdot \mathbf{r}} + \sin \alpha \left( \frac{\sin \theta_p}{-\cos \theta_p} \right) e^{\mathbf{p}_\perp \cdot \mathbf{r}} \right] ,
\]

where \( 0 \leq \alpha \leq \pi / 2 \), and \( \cos \alpha \) and \( \sin \alpha \) are the superposition coefficients. It gives

\[
\begin{align*}
\hat{n}_+ &= n_0 [\cos^2 \alpha \cos^2 \theta_+ + \sin^2 \alpha \cos^2 \theta_- ] \\
\hat{n}_- &= n_0 [\cos^2 \alpha \sin^2 \theta_+ + \sin^2 \alpha \sin^2 \theta_- ] \\
\end{align*}
\]

\[
+ \sin 2\alpha \cos \theta_+ \cos \theta_- \cos (\delta \mathbf{p} \cdot \mathbf{r}) \right] ,
\]

where \( \delta \mathbf{p} = \mathbf{p}_+ - \mathbf{p}_- \). In practice, one can straightforwardly substitute the expression of density equation (10) into the interaction energy equation (8), and the interaction energy \( \mathcal{E}_I \) becomes a function of \( \alpha, \mathbf{p}_\pm \) and \( \theta_\pm \). Then by minimizing the total energy with respect to \( \alpha, \mathbf{p}_\pm \) and \( \theta_\pm \), one can finally obtain the ground-state condensate wavefunction for given \( \Omega, \delta \) and interaction parameters \( g_{\uparrow \uparrow}, g_{\downarrow \downarrow} \) and \( g_{\uparrow \downarrow} \). Thus, a phase diagram can be constructed. More details have been discussed in [1, 12, 13] and here we will just emphasize some general features.

(1) If the energy minimization gives \( \alpha = 0 \) or \( \alpha = \pi / 2 \), then there is only one momentum component in the condensate wavefunction, and the densities of both spin components are uniform, as can be easily seen by setting \( \alpha = 0 \) or \( \alpha = \pi / 2 \) in equation (10). This is the ‘plane-wave’ condensate. While if the energy minimization gives \( 0 < \alpha < \pi / 2 \), both \( n_+ (\mathbf{r}) \) and \( n_- (\mathbf{r}) \) have spatially periodic modulation. This is named a ‘stripe’ condensate [5].

(2) Because of the SO coupling, the densities of both components depend on momentum. If the interaction is \( SU (2) \) invariant, i.e. \( g_{\uparrow \uparrow} = g_{\downarrow \downarrow} = g_{\uparrow \downarrow} = g \), then the interaction energy becomes \( \mathcal{E}_I = (g / 2) \int \mathrm{d}^3 \mathbf{r} \left( \hat{n}_+ (\mathbf{r}) + \hat{n}_- (\mathbf{r}) \right)^2 \). In this case, if the condensate is in the ‘plane-wave’ phase, then the interaction energy is independent of momentum, and one can take \( \mathbf{p}_\pm \) as \( k_{\pm} \) which minimizes the single-particle energy. However, because the two pseudo-spin states are taken as two hyperfine levels of atoms, they do not have to obey the \( SU (2) \) spin rotational symmetry. Thus, for a general case, three interaction parameters \( g_{\uparrow \uparrow}, g_{\downarrow \downarrow} \) and \( g_{\uparrow \downarrow} \) are all unequal. Therefore, even for the plane-wave condensate, the interaction energy also depends on momentum \( \mathbf{p}_\pm \). That is equivalent to say, even at the mean-field level the self-energy correction has momentum dependence, which effectively modifies the single-particle dispersion and changes the location of its minimum. In contrast, without the SO coupling, the mean-field self-energy correction is just a constant shift of single-particle energy. Thus, as first pointed out by the authors of [13], \( \mathbf{p}_\pm \) is shifted away from single-particle minimum \( k_{\pm} \).

(3) The stripe phase has distinct properties from the plane-wave phase. For the stripe phase, the total density will have spatial modulation, because the spin wavefunction at \( \mathbf{p}_+ \) and \( \mathbf{p}_- \) are not orthogonal, i.e. \( \cos \theta_+ \cos \theta_- + \sin \theta_+ \sin \theta_- \neq 0 \). Thus, in this system the stripe

3 However, for some special case, such as the pure Rashba SO coupling without the Zeeman field considered in [5], the mean-field self-energy correlation is also momentum independent.

4 This is also different from the pure Rashba case where the spin wavefunction is orthogonal for the opposite momentum, and the total density is uniform for the stripe phase.
phase is not favoured by the density interaction part \((g/2) \int d^3r (n_{↑}(r) + n_{↓}(r))^2\). In other words, if the interaction is SU(2) invariant, the stripe phase will not exist in this system. The difference in \(g_{↑↑}; g_{↓↓} \) and \(g_{↑↓}\) is necessary for stabilizing the stripe phase. Moreover, since the non-uniform term in the total density increases as \(Ω\) increases, the stripe phase, if it exists, should be found in the small \(Ω\) regime of the phase diagram.

(4) Consider the limit \(Ω = 0\), if \(g_{↑↑} g_{↓↓} - g_{↑↓}^2 > 0\), a homogeneous mixture of two components is stable against local density fluctuations, and there will be a mixed phase within a certain detuning window. Such a mixed phase will turn into the stripe phase once \(Ω\) becomes non-zero, for instance, for the \(^{87}\)Rb case as shown in figure 1(a). While if \(g_{↑↑} g_{↓↓} - g_{↑↓}^2 < 0\), the mixed phase is not stable against phase separation even for zero \(Ω\), and there will be no stripe phase in the phase diagram, for instance, for the \(^{23}\)Na case in figure 1(b).

Hereafter, we should focus only on the SU(2)-invariant interaction. This is relevant for the experiment with Rb or Na, because the difference in \(g_{↑↑}, g_{↓↓}\) and \(g_{↑↓}\) is smaller than 1%. The generalization to the non-SU(2) interaction is straightforward. Besides, we only consider the plane-wave phase, because in these systems, the stripe phase either occupies a very small regime of the phase diagram or does not exist.

2.3. Zero-detuning case

In this work, we will particularly focus on the case with \(δ = 0\) for the following two reasons.

(1) **Density-of-state effect.** When \(Ω < 4E_r\), \(ε_−(k_z)\) has two minima at \(k_± = \pm k_0 \sqrt{1 - (Ω/E_0)^2}\), and when \(Ω > 4E_r\), \(ε_−(k_z)\) has one single minimum at \(k_z = 0\). Expanding the dispersion around its minimum, one obtains the effective mass in the \(x\) direction as

\[
m^* = \begin{cases} m & \Omega < 4E_r \\ m \left(1 - \frac{4E_r}{Ω}\right)^{-1} & \Omega > 4E_r. \end{cases}
\]

(11)

Hence, the low-energy DoS increases with \(Ω\) when \(Ω < 4E_r\) and decreases with \(Ω\) when \(Ω > 4E_r\), as shown in figure 2. The most intriguing point is at \(Ω = 4E_r\) when the single-particle dispersion behaves as \(\sim p_x^2\) at the lowest order and the low-energy DoS reaches its maximum. As we shall see in later discussion, this has important physical effects on the superfluid critical velocity and the BEC transition temperature.

(2) **\(Z_2\) symmetry and magnetization.** When \(Ω < 4E_r\), bosons condense into one of the minima, which breaks the \(Z_2\) symmetry, and the Bose condensate will have finite magnetization, while when \(Ω > 4E_r\), bosons condense at the zero-momentum state and the Bose condensate is non-magnetic. Thus, there will be a magnetic phase transition at \(Ω = 4E_r\) associated with the \(Z_2\) symmetry breaking, and a divergent spin susceptibility has been predicted and experimentally found [4, 15]. We note that such a transition exists only for \(δ = 0\), since for non-zero \(δ\) the Hamiltonian does not possess the \(Z_2\) symmetry, and the condensate phase is always magnetic.

3. **Bogoliubov theory and superfluid critical velocity**

3.1. **Bogoliubov spectrum**

We study the fluctuations around the condensate with Bogoliubov theory. Considering a plane-wave condensate at momentum \(p_0 = p_0 \hat{e}_x\), the field operator can be expanded as

\[
\hat{ψ}(r) = ϕ(r) + δ\hat{ψ}(r),
\]

where \(ϕ(r)\) is the condensate wavefunction

\[
ϕ(r) = \sqrt{n_0} \left(\cos \theta_0 \hat{b}_0 - \sin \theta_0 \hat{c}_0\right) \exp(ip_0 x)
\]

(13)
and satisfies the Gross–Pitaevskii (GP) equation

\[ [H_0(p_0) + g_{n0}] \begin{pmatrix} \cos \theta_{p_0} \\ -\sin \theta_{p_0} \end{pmatrix} = \mu \begin{pmatrix} \cos \theta_{p_0} \\ -\sin \theta_{p_0} \end{pmatrix}. \]  

(14)

Defining \( \tilde{\psi}_q \) as \( \sum_q <q|\psi_{-q,\uparrow}| \psi_{q,\downarrow}> \), the Bogoliubov Hamiltonian for the fluctuation part can be written as

\[ \mathcal{K} = \sum_{q>0} \sum_{\alpha} \tilde{\psi}_q^\dagger \kappa_q \tilde{\psi}_q - \frac{1}{2} \sum_{q>0} \left[ \varepsilon_{p_0-q,\uparrow} + \varepsilon_{p_0-q,\downarrow} - 2\mu + 3g_{n0} \right], \]

where

\[ \kappa_q = \begin{pmatrix} \kappa_0(p_0 + q) + \Sigma_N & \Sigma_A \\ \Sigma_A & \kappa_0(p_0 - q) + \Sigma_N \end{pmatrix} \]

(15)

and \( \kappa_0 = H_0 - \mu \) is the grand Hamiltonian of the non-interacting system, \( \Sigma_N \) and \( \Sigma_A \) are normal and anomalous self-energy, respectively,

\[ \Sigma_N = g_{n0} \begin{pmatrix} \sin^2 \theta_{p_0} + 2 \cos^2 \theta_{p_0} & -\sin \theta_{p_0} \cos \theta_{p_0} \\ -\sin \theta_{p_0} \cos \theta_{p_0} & \cos^2 \theta_{p_0} + 2 \sin^2 \theta_{p_0} \end{pmatrix}, \]

(16)

\[ \Sigma_A = g_{n0} \begin{pmatrix} \cos^2 \theta_{p_0} & -\sin \theta_{p_0} \cos \theta_{p_0} \\ -\sin \theta_{p_0} \cos \theta_{p_0} & \sin^2 \theta_{p_0} \end{pmatrix}. \]

(17)

Hence the Bogoliubov spectrum is determined by

\[ \text{Det} \left( \begin{pmatrix} \kappa_0(p_0 + q) + \Sigma_N & \Sigma_A \\ \Sigma_A & \kappa_0(p_0 - q) + \Sigma_N \end{pmatrix} \right) = 0 \]

(18)

Because of the emergence of the off-diagonal long-range order, the excitation energy should be gapless in the long wavelength limit \( q \to 0 \), which requires

\[ \text{Det}[\Sigma_A(k_0(p_0) + \Sigma_N)^{-1} \Sigma_A - (k_0(p_0) + \Sigma_N)] = 0. \]

(19)

This equation is indeed satisfied because the GP equation for the condensate wavefunction can be rewritten as

\[ \text{Det}[H_0(p_0) - \mu + \Sigma_N - \Sigma_A] = 0. \]

(20)

Equation (20) can also be regarded as a generalization of the Hugenholtz–Pines relation \( \mu = \Sigma_N - \Sigma_A \) to the SO-coupled Bose gases [19].

Solving equation (18) gives rise to the entire Bogoliubov spectrum \( E_{q,\pm} \). Examples are displayed in figure 3 for various \( \Omega/E_r \). For \( q \to 0 \), the low-energy excitation is the phonon mode with a linear dispersion \( E_{\pm}(q) = sq \). Along the direction of the Raman beam, the sound velocity is given by

\[ s = \sqrt{\frac{g_{n0}}{m} \left( 1 - \frac{\Omega^2}{16E_r^2} \right)} \]

(21)

when \( \Omega < 4E_r \) and

\[ s = \sqrt{\frac{g_{n0}}{m} \left( 1 - \frac{4E_r}{\Omega} \right)} \]

(22)

when \( \Omega > 4E_r \). These results are consistent with the effective mass approximation that gives \( s = \sqrt{g_{n0}/m} \). At \( \Omega = 4E_r \), the effective mass diverges, and the low-energy phonon mode is quadratic in \( q^2 \), as shown in figure 3(b). This is a significant difference compared to the conventional phonon mode, which is always linear in \( q \). When \( \Omega < 4E_r \), the Bogoliubov spectrum has another local minimum at finite \( q \), which is due to the double degeneracy in the single-particle spectrum. This is attributed as ‘roton minimum’ by the authors of [15].

### 3.2. Quantum and thermal condensate depletion

The Bogoliubov Hamiltonian can be straightforwardly written as

\[ \mathcal{K}^{(2)} = \sum_{q \neq 0} \left( E_q + \psi_{q,\uparrow}^\dagger \psi_{q,\downarrow} + E_q - \psi_{q,\downarrow}^\dagger \psi_{q,\uparrow} \right) \]

\[ + \frac{1}{2} \sum_{q \neq 0} \left[ E_{q,\uparrow} + E_{-q,\downarrow} - \varepsilon_{p_0-q,\uparrow} \right] \quad \varepsilon_{p_0-q,\downarrow} + 2\mu + 3g_{n0}], \]

(23)
Thermal depletion as a function of $T$ through the magnetization of the non-condensed part defined $M$ are quasi-particle operators, and $\hat{\tilde{\psi}}$ are readily calculated using the $\psi$ equation (18). The quantum and thermal depletion of condensate $n_{\text{ex},\sigma}$, $\langle \hat{\tilde{\psi}}^\dagger \hat{\tilde{\psi}} \rangle_{\sigma} = \frac{1}{2} \sum_{q \neq 0} |M_{q}^\dagger \hat{\tilde{\psi}}_q \rangle_{\sigma}^\dagger | \hat{\tilde{\psi}}_q \rangle_{\sigma}$ can thus be readily calculated using $M_q$ and $E_{q,\sigma}$.

The depletion fraction as a function of Raman coupling strength $\Omega$ and temperature $T$ is shown in figure 4. The contribution to depletion is mainly from two parts: the low-energy phonon part and the roton part. At zero temperature, the contribution is dominated by the phonon part because the roton part has a finite excitation gap. Thus, due to the non-monotonic behaviour of sound velocity discussed above, when $\Omega < 4E_r$, the quantum depletion $n_{\text{ex}}/n$ increases with $\Omega$, reaches a maximum at $\Omega = 4E_r$, and then decreases as $\Omega$ increases. At finite temperature, because the roton gap is quite small at small $\Omega$, the roton part will give a significant contribution. For $\Omega < 4E_r$, the contribution from the phonon part always increases with $\Omega$ because the phonon velocity decreases, while the contribution from the roton part decreases with $\Omega$ because the roton gap increases. Due to the interplay of these two contributions, for small $\Omega$, $n_{\text{ex}}/n$ first decreases as $\Omega$ increases, in contrast to the zero-temperature case. Then $n_{\text{ex}}/n$ increases again with $\Omega$ and the peak of $n_{\text{ex}}/n$ at $\Omega = 4E_r$ retains.

Another interesting manifestation of roton minimum is through the magnetization of the non-condensed part defined as $(n_{\text{ex}}^{-} - n_{\text{ex}}^{+})/n_{\text{ex}}$. Assuming the condensate momentum $p_0 = 0$, the condensate has a positive magnetization $(n_{\text{ex}}^{-} - n_{\text{ex}}^{+})/n_0 > 0$. As shown in figure 5(a), at zero temperature, the depletion also has a positive magnetization $(n_{\text{ex}}^{-} - n_{\text{ex}}^{+})/n_{\text{ex}} > 0$. This is because the phonon contribution, which dominates the quantum depletion, originates from states whose momenta are nearby condensate momentum and possess the same magnetization as the condensate. However, at finite temperature, for small $\Omega$, the depletion can have opposite magnetizations as the condensate part, as shown in figure 5(b). And from figure 5(b), one can also see that, for instance at $\Omega = 0.5E_r$, the magnetization soon decreases to negative as temperature increases. This is because the roton contribution dominates in this regime. Recall that when $\Omega < 4E_r$, there is double minimum in this single-particle spectrum with the opposite momentum and magnetization. When the condensate takes place in one of the minima, the roton minimum in the excitation appears nearby the other minimum. Thus, the contribution from the roton minimum displays opposite magnetization with the condensate part. As we know, the phonon dispersion is linear, while the roton dispersion is quadratic. This implies that the DoS of the quasi-particle near the roton is larger than that near the phonon part. At finite temperatures higher than the roton gap, the thermal depletion near the roton minimum will exceed the depletion from phonon modes. This leads to an opposite magnetization of the total thermal depletion compared with the condensate.
In the further experiment, measuring the magnetization of the thermal depletion will thus provide evidence for the existence of the roton minimum in the excitation spectrum.

### 3.3. Beyond mean-field correction of ground-state energy

Bogoliubov theory also gives rise to a correction to mean-field energy usually named as the Lee–Huang–Yang (LHY) correction [20]. The LHY correction can be considered as a sum of all zero-point energies of excitation modes at different momenta. Without SO coupling, it is a function of $E_0$, which results in the minimum in the LHY correction. The softer the phonon mode, the smaller the correction [20]. The LHY correction can be considered as a field energy usually named as the Lee–Huang–Yang (LHY) correction. Bogoliubov theory also gives rise to a correction to mean-field energy.

#### 3.4. Superfluid critical velocity

There are two ways to measure critical velocity [11]. First, the condensate is at rest and an impurity moves with the finite velocity in the condensate. In this way, one can obtain a critical dragging velocity

$$V_{\text{drag}} = \min \left( \frac{E_\mathbf{q}}{\mathbf{q}} \right),$$

where $E_\mathbf{q}$ is the quasi-particle energy for momentum $\mathbf{q}$. The critical dragging velocity is shown in figure 7(a). One can see that $V_{\text{drag}}$ vanishes at $4E_r$, because the phonon spectrum becomes quadratic in $q_x$, at this point and the phonon velocity vanishes. One can also see that for $\Omega < 4E_r$, $V_{\text{drag}}$ is not symmetric for moving along $\hat{x}$ or along $-\hat{x}$. This is due to the non-symmetric structure of the Bogoliubov spectrum as shown in figure 3(a). Moving along $\hat{x}$, $V_{\text{drag}}$ is determined by the phonon part. Moving along $-\hat{x}$, $V_{\text{drag}}$ is determined by the roton minimum for small $\Omega$. Since the roton gap increases with $\Omega$, the $V_{\text{drag}}$ along $-\hat{x}$ also increases with $\Omega$ until the roton vanishes. After that the $V_{\text{drag}}$ along $-\hat{x}$ is also determined by the phonon mode, and will decrease with $\Omega$ until $\Omega = 4E_r$.

Second, one can consider a static impurity and let the condensate move with a finite velocity. This determines another critical velocity which can be called the critical flowing velocity $V_{\text{flow}}$. If the system is Galilean invariant, $V_{\text{drag}}$ and $V_{\text{flow}}$ should be identical. However, the SO coupling breaks the Galilean invariance, and thus these two velocities become unequal, as first pointed out in [11] for the Rashba-type SO coupling. In our system, the Galilean invariance is also broken. Considering a Galilean transformation in the $\hat{x}$ direction, the Hamiltonian in the moving frame becomes

$$H_0(\mathbf{v}) = \frac{1}{2m}[\hat{p}_x - k_x \sigma_z]^2 + \frac{1}{2} \Omega \sigma_z - v_x p_x$$

$$= \frac{1}{2m}[\hat{p}_x - (k_x \sigma_z + mv_x)]^2 + \frac{1}{2} \Omega \sigma_z - v_x k_x \sigma_z - \frac{1}{2} mv_x^2.$$ (29)

With a gauge transformation $U(x,t) = \exp[imv_x t - i(mv_x^2/2)t^2]$, the Hamiltonian becomes

$$H_0(\mathbf{v}) = \frac{1}{2m}(\hat{p}_x - k_x \sigma_z)^2 + \frac{1}{2} \Omega \sigma_z - v_x k_x \sigma_z.$$ (30)

_Figure 5. (a) Magnetization of depletion $(n_{\text{ex}} - n_{\text{ex}}^*)/n_{\text{ex}}$ as a function of $\Omega/E_r$ for various temperatures. (b) $(n_{\text{ex}} - n_{\text{ex}}^*)/n_{\text{ex}}$ as a function of $T/T_c$ for various $\Omega$. (Parameters are the same as in figure 4.)_
Figure 6. $\Gamma_{\text{LHV}}$ as a function of $\Omega/E_r$ for different $n^{1/3}a_s$ (fixed $n/k^3_r = 1$).

Figure 7. Critical dragging (a) and flowing (b) velocity as a function of $\Omega/E_r$. Here $gn = 0.5E_r$, and $v_r = k_r/m$.

Compared to the Hamiltonian in the stationary frame, there is an additional velocity-dependent Zeeman term $v_xk_r\sigma_z$. This term cannot be gauged away, and that implies the breaking of the Galilean invariance for an SO-coupled particle. The physical effect of this term has already been observed in [4] in the collective dipole oscillation experiment. $v_{\text{flow}} \neq v_{\text{drag}}$ is another manifestation of the absence of Galilean invariance.

To determine $v_{\text{flow}}$ we shall first find out the ground-state wavefunction for the Hamiltonian in the comoving frame (equation (29)), say

$$\psi(x) = \sqrt{n_0} \left( \begin{array}{c} \cos \theta' \\ -\sin \theta' \end{array} \right) \exp(ip_0 x).$$

(31)

Then, following the similar procedure discussed above, one can find out the Bogoliubov excitation spectrum in the comoving frame, given by $E_q(v) = E_q'(v) + v \cdot q$. When $v$ is above certain critical value, $E_q$ will start to have a negative part, which indicates the instability of the condensate. This critical value defines $v_{\text{flow}}$. $v_{\text{flow}}$ of this system is shown in figure 7(b).

4. Hartree–Fock theory of the normal state and BEC transition temperature

4.1. Transition temperature of a non-interacting gas

Now, we study the normal state of the system and determine the transition temperature of the condensate. First, we consider a non-interacting gas. The condensation transition temperature $T_c$ of a uniform system is given by (set $k_B = 1$)

$$n = \int_{-\infty}^{\infty} d\epsilon \frac{D(\epsilon)}{e^{(\epsilon - \mu)/T_c} - 1},$$

(32)

where the chemical potential reaches the bottom of the single-particle spectrum, $\mu = \epsilon_{\text{min}}$. Due to the DoS effect discussed in section 1, $T_c$ decreases with $\Omega$ in the regime $\Omega < 4E_r$, reaches a minimum of finite value around $\Omega = 4E_r$, and then increases in the regime $\Omega > 4E_r$. This non-monotonic behaviour is shown in figure 8(a). In the $\Omega \to 0$ and $\Omega \to \infty$ limits, one can find a simple relation of transition temperature,

$$\frac{T_c(\Omega \to \infty)}{T_c(\Omega \to 0)} = 2^{2/3}$$

(33)

because the low-energy DoS for the later case shrinks to only half of the first one.

In a harmonic trap, the semi-classical energy of the single boson can be expressed as

$$\epsilon_{\text{p,}\pm}(r) = \frac{1}{2m} (p^2 + k_r^2) \pm (k_r p_x/m)^2 + \Omega^2/4 + \frac{1}{2} m \omega^2 r^2,$$

(34)
interaction strengths. Here \( n/\kappa^2_0 = 5 \).

with \( \omega \) being the trap frequency. With the semi-classical approximation, the DoS should be modified as

\[
D_{\text{trap}}(\epsilon) = \frac{1}{V} \int d^3 r \left[ \delta_D(\epsilon - E_{\uparrow}(\mathbf{r})) + \delta_D(\epsilon - E_{\downarrow}(\mathbf{r})) \right]
\]

and the transition temperature can be obtained from

\[
N = \int_{-\infty}^{\infty} d\epsilon \frac{D_{\text{trap}}(\epsilon)}{e^{(\epsilon - \mu)/T} - 1}
\]

when the chemical potential \( \mu \) reaches the minimum of single-particle energy at the trap centre. The result is shown in figure 9(a).

In contrast to the uniform case, one finds the minimum location of the \( T_c \) is shifted to the smaller \( \Omega \) regime even for a small particle number, while if the particle number is large enough, \( T_c \) for a small particle number, while if the particle number is large enough, \( T_c \) for a small particle number, while if the particle number is large enough, \( T_c \) for a small particle number, while if the particle number is large enough, \( T_c \) approximates the DoS should be modified as

\[
\omega_T \approx \omega_{\text{eff}} + \xi
\]

and the transition temperature can be obtained from

\[
T_c(\Omega \to \infty) = \frac{n}{\kappa_0^2} = 2^{1/3}.
\]

4.2. Mean-field shift of transition temperature for uniform gases

In the absence of the SO coupling, it is well known that the contact interaction between the particles does not affect \( T_c \) at the mean-field level [21]; this is because the Hartree–Fock self-energy only provides a constant shift of chemical potential, while with the SO coupling, as shown in the following, the interactions do have a non-trivial effect even at the mean-field level.

To construct a self-consistent Hartree–Fock theory, besides the average density for each spin component

\[
n_\sigma = \frac{1}{V} \sum_\mathbf{p} \langle \hat{\psi}_\sigma^\dagger \hat{\psi}_\sigma \rangle,
\]

we also need to introduce the mean-field parameter associating with the spin-flip term

\[
\xi = \frac{1}{V} \sum_\mathbf{p} \langle \hat{\psi}_{\sigma,\downarrow}^\dagger \hat{\psi}_{\sigma,\uparrow} \rangle.
\]

The spin-flip term, \( \xi \), is due to the Raman coupling between different spin states, which breaks the conservation of spin magnetization. In contrast, such a term is absent for an ordinary spinor Bose gas without the Raman coupling.

Up to a constant, the mean-field Hamiltonian in the Hartree–Fock approximation is given by

\[
H_{\text{MF}} = H_0 + \sum_\mathbf{p} \left[ g(2n_{\uparrow} + n_{\downarrow}) \hat{\psi}_{\uparrow,\mathbf{p}}^\dagger \hat{\psi}_{\uparrow,\mathbf{p}} + g(2n_{\downarrow} + n_{\uparrow}) \hat{\psi}_{\downarrow,\mathbf{p}}^\dagger \hat{\psi}_{\downarrow,\mathbf{p}} + g^2 \langle \hat{\psi}_{\sigma,\uparrow}^\dagger \hat{\psi}_{\sigma,\downarrow}^\dagger \hat{\psi}_{\sigma,\downarrow} \hat{\psi}_{\sigma,\uparrow} \rangle \right].
\]

Since we are treating the normal state above \( T_c \), spins are always unpolarized with \( n_{\uparrow} = n_{\downarrow} = n/2 \). The mean-field Hamiltonian in (38) has the same structure as the single-particle Hamiltonian, except that the Raman coupling is modified as \( \Omega_{\text{eff}} = \Omega + 2g^2 \xi \) and the energy zero point is shifted by a constant. In this sense, we can define two dressed helicity branches, with dispersions given by

\[
\epsilon_{\sigma,\pm} = \frac{1}{2m} (\mathbf{p}^2 + k_r^2) + \frac{3}{2} g \eta \pm \sqrt{(k_r p_z/m)^2 + \Omega_{\text{eff}}^2/4}.
\]

Here the mean-field parameter \( \xi \) should be solved self-consistently in combination with

\[
\xi = \frac{1}{V} \sum_\mathbf{p} \sin \theta_\mathbf{p} \cos \theta_\mathbf{p} (n_{\uparrow,\mathbf{p}} - n_{\downarrow,\mathbf{p}}).
\]

where \( \theta_\mathbf{p} \) is given by equation (5) with the bare Raman coupling replaced by \( \Omega_{\text{eff}} \), and \( n_{\sigma,\pm} \) are the Bose distribution functions with the dispersion \( \epsilon_{\sigma,\pm} \) given by equation (39). The transition temperature \( T_c \) is then determined by interaction-modified dispersions. Due to the non-trivial momentum dependence of
particle number conservation

the single-particle spectrum and

\[ \mu(\Omega/E_r) \]

\[ N = \frac{2}{\Omega_1}N \]

\[ \text{correction of} \]

\[ g(r) = \Theta \]

\[ 2 \pi \times 2.2 \text{kHz} \]

\[ 2.5 \times 10^5 \]

\[ R \]

\[ \text{dispersions (39), the interactions induce a shift of} \]

\[ T_c \]

\[ \text{from the non-interacting Bose gas.} \]

\[ \xi \]

\[ \text{is always larger than} \]

\[ n_{p,+} \]

\[ \Omega/E_r \]

\[ \text{is determined when} \]

\[ \mu(T) \]

\[ n(r, T, \mu) \]

\[ n(r) = \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{g([\epsilon_{\pm k}(r)-\mu]/\Omega_1 - 1] + \frac{1}{g([\epsilon_{\pm k}(r)-\mu]/\Omega_1 - 1]} \right\} \]

\[ \epsilon_{\pm k}(r) = \frac{1}{2m}(k^2 + k_r^2) \pm \sqrt{(k_r k_r/m)^2 + \Omega_1^2(r)/4 + V_{\text{eff}}(r)} \]

\[ V_{\text{eff}}(r) = V(r) + (3/2)gn(r) \]

\[ \text{Similar as above,} \]

\[ \Omega_{\text{eff}} \]

\[ = \Omega + 2g\xi(r), \]

\[ 45 \]

\[ \text{which also needs to be determined self-consistently as} \]

\[ \text{equation (40).} \]

\[ \text{The numerical solution gives the interaction shift of} \]

\[ T_c \]

\[ \text{inside a harmonic trap, as shown in figure 9. Here we note} \]

\[ \text{that the interaction effect always gives a negative shift of} \]

\[ \Delta T_c \]

\[ \text{in contrast to the uniform case where} \]

\[ \Delta T_c \]

\[ \text{can be either positive or negative. This is because the presence} \]

\[ \text{of the repulsive interaction reduces the central density in the} \]

\[ \text{trap, leading to a reduced} \]

\[ T_c \]

\[ \text{This effect dominates over} \]

\[ \Omega_{\text{eff}} \]

\[ \text{for a sufficient large particle number. In} \]

\[ \text{figure 9, we also plot} \]

\[ \Delta T_c/T_c^0 \]

\[ \text{as a function of} \]

\[ \Omega_c \]

\[ \text{and find that the relative shift of} \]

\[ T_c \]

\[ \text{reaches the maximum around} \]

\[ \Omega = 4E_r. \]

\[ \text{This can be qualitatively understood from} \]

\[ \text{the effective mass approximation. Using} \]

\[ m^* \]

\[ \text{one can define an effective scattering length} \]

\[ a^*_s = \frac{a_s(m^*/m)}{\Delta \xi} \]

\[ \text{harmonic length} \]

\[ \Delta \xi = \frac{a_{\Delta \xi}(m/m^*)^{1/4}}{\Delta \xi} \]

\[ \text{the shift of} \]

\[ T_c \]

\[ \text{in the} \]

\[ \text{harmoic trap \cite{23} is} \]

\[ \Delta T_c/T_c^0 = -1.32\frac{m^*}{m}N^{1/4} \propto (m^*/m)^{1/4} \]

\[ \text{As we have shown in equation (11),} \]

\[ m^*/m \]

\[ \text{is maximally enhanced at} \]

\[ \Omega = 4E_r, \]

\[ \text{so the interaction has the maximum effect on the shift of} \]

\[ T_c \]

\[ \text{at this point.} \]

\[ \text{4.3. Interaction shift of transition temperature for trapped} \]

\[ \text{gases} \]

\[ \text{Finally, we come to discuss the most realistic case where} \]

\[ \text{both interaction and trap effect are taken into account. We use} \]

\[ \text{the Hartree–Fock mean-field theory to include the interaction} \]

\[ \text{effect and use the semi-classical approximation to include the} \]

\[ \text{trap effect.} \]

\[ \text{Within the Hartree–Fock and semi-classical approximation,} \]

\[ T_c \]

\[ \text{is determined when} \]

\[ \mu(T) \]

\[ \text{reaches the minimum of} \]

\[ \text{the single-particle spectrum and} \]

\[ \mu(T) \]

\[ \text{is determined by the particle number conservation} \]

\[ N = \int d^3r n(r, T, \mu). \]

\[ 41 \]

\[ \text{Here the local density} \]

\[ n(r, T, \mu) \]

\[ \text{is given by} \]

\[ n(r) = \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{g([\epsilon_{\pm k}(r)-\mu]/\Omega_1 - 1] + \frac{1}{g([\epsilon_{\pm k}(r)-\mu]/\Omega_1 - 1]} \right\} \]

\[ 42 \]

\[ \text{in which} \]

\[ \epsilon_{\pm k}(r) = \frac{1}{2m}(k^2 + k_r^2) \pm \sqrt{(k_r k_r/m)^2 + \Omega_1^2(r)/4 + V_{\text{eff}}(r)}, \]

\[ 43 \]

\[ V_{\text{eff}}(r) = V(r) + (3/2)gn(r). \]

\[ 44 \]

\[ \text{Figure 9. (a)} \]

\[ T_c \]

\[ \text{of} \]

\[ ^{87}\text{Rb} \]

\[ \text{and non-interacting Bose gases in the harmonic trap. (b) Shift of} \]

\[ T_c \]

\[ \text{of} \]

\[ ^{87}\text{Rb} \]

\[ \text{due to the interaction as a function of} \]

\[ \Omega/E_r. \]

\[ \text{The non-interacting Bose gases are assumed to have the same atomic mass as} \]

\[ ^{87}\text{Rb}. \]

\[ \text{Here} N = 2.5 \times 10^5, \]

\[ \text{and the trapping frequency is} \]

\[ \omega = 2\pi \times 50 \text{Hz}. \]

\[ \text{displacements (39), the interactions induce a shift of} T_c \]

\[ \text{from the non-interacting Bose gas.} \]

\[ \text{Since the occupation of the lower helicity branch} n_{p,-} \]

\[ \text{is always larger than} n_{p,+}, \]

\[ \xi \]

\[ \text{is always negative and hence} \]

\[ \text{the effective Raman coupling strength} \]

\[ \Omega_{\text{eff}} \]

\[ \text{is decreased to a smaller value. Based on the non-monotonic behaviour of} \]

\[ \text{the non-interacting} T_c \]

\[ \text{discussed previously, the minimum of the condensation temperature is shifted by the interaction to a} \]

\[ \text{larger} \]

\[ \Omega \]

\[ \text{as shown in figure 8(a). Near the minimum, the} \]

\[ \text{correction of} T_c \]

\[ \text{changes its sign rapidly within a narrow} \]

\[ \text{region of} \]

\[ \Omega/E_r \]

\[ \text{as shown in figure 8(b), one can clearly see that} \]

\[ \text{the transition temperature is enhanced for} \]

\[ \text{a smaller value. Based on the non-monotonic behaviour of} \]

\[ \text{the effective Raman coupling strength} \]

\[ \Omega_{\text{eff}} \]

\[ \text{is decreased to} \]

\[ \text{a smaller value.} \]

\[ \quad \text{Based on the non-monotonic behaviour of} \]

\[ \text{condensate depletion, Lee–Huang–Yang correction of ground-state energy, and the} \]

\[ \text{transition temperature of a non-interacting Bose–Einstein condensate (BEC).} \]

\[ \text{5. Conclusion} \]

\[ \text{In this paper, we investigate the properties of Bose gases with} \]

\[ \text{the Raman-induced spin–orbit (SO) coupling. Our main results} \]

\[ \text{are summarized as follows.} \]

\[ \text{(1) The presence of the SO coupling modifies the single-particle spectrum and thus the single-particle state (DoS). At} \]

\[ \Omega = 4E_r, \]

\[ \text{the low-energy DoS reaches the maximum and the effective mass is} \]

\[ \text{maximally enhanced. The direct consequences include the vanishing of sound velocity at} \]

\[ \Omega = 4E_r, \]

\[ \text{and the non-monotonic behaviour of condensate depletion, Lee–Huang–Yang correction of ground-state energy, and the} \]

\[ \text{transition temperature of a non-interacting Bose–Einstein condensate (BEC).} \]
(2) The presence of the SO coupling breaks the Galilean invariance. As a result, the critical dragging and flowing velocity, respectively, defined in the rest frame of the condensate and the impurity, are no longer identical.

(3) In the presence of the SO coupling, a roton minimum will appear in the excitation spectrum in the regime of $\Omega < 4E_r$. As a result, the thermal depletion of the condensate can possess an opposite magnetization with the quantum depletion. Moreover, the critical dragging velocity exhibits asymmetry along different directions.

(4) In the presence of the SO coupling, the interactions shift BEC transition temperature $T_c$ even at a Hartree–Fock level. In both homogeneous and trapped systems, the interaction shift of $T_c$ shows maximum around $\Omega = 4E_r$, where the interactions take the largest effect due to the enhancement of the DoS and the effective mass.

In conclusion, we have shown that the Bose gases with the Raman-induced SO coupling can exhibit a number of non-trivial properties, as summarized above. The results revealed here can be directly verified in the current cold atom experiments using the laser-induced gauge field.

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