THE VARCHENKO DETERMINANT FOR ORIENTED MATROIDS

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Abstract. We generalize the Varchenko matrix of a hyperplane arrangement to oriented matroids. We show that the celebrated determinant formula for the Varchenko matrix, first proved by Varchenko, generalizes to oriented matroids. It follows that the determinant only depends on the matroid underlying the oriented matroid and analogous formulas hold for cones in oriented matroids. We follow a proof strategy for the original Varchenko formula first suggested by Denham and Hanlon. Besides several technical lemmas this strategy also requires a topological result on supertopes which is of independent interest. We show that a supertope considered as a subposet of the tope poset has a contractible order complex.

1. Introduction

Let \( \mathcal{L} \) be an oriented matroid on a finite ground set \( E \) given as a set of covectors \( X = (X_e)_{e \in E} \in \{+, -, 0\}^E \). We denote by \( \mathcal{T} = \mathcal{T}(\mathcal{L}) \) the set of topes in \( \mathcal{L} \) and call for two topes \( P = (P_e)_{e \in E} \) and \( Q = (Q_e)_{e \in E} \) the set \( \text{Sep}(P, Q) = \{e \in E \mid P_e \neq -Q_e\} \) the separator of \( P \) and \( Q \). For the oriented matroid \( \mathcal{L} \) and a field \( \mathbb{K} \) we consider the polynomial ring \( \mathbb{K}[U_e \mid e \in E] \) in the set of variables \( U_e, e \in E \). We call the following matrix \( \mathfrak{V} = \mathfrak{V}(\mathcal{L}) \) the Varchenko matrix of \( \mathcal{L} \). The matrix \( \mathfrak{V} \) is the \((\# \mathcal{T} \times \# \mathcal{T})\)-matrix over \( \mathbb{K}[U_e \mid e \in E] \) with rows and columns indexed by the topes \( \mathcal{T} \) in a fixed linear order. For \( P,Q \in \mathcal{T} \) the entry \( \mathfrak{V}_{P,Q} \) in row \( P \) and column \( Q \) is given by \( \prod_{e \in \text{Sep}(P,Q)} U_e \). In particular, all entries \( \mathfrak{V}_{P,P} \) on the diagonal are equal to 1. For \( F \in \mathcal{L} \) we set \( a(F) := \prod_{e \in E \atop F_e = 0} U_e \). In this paper we prove:

\[ \text{Theorem 1.1.} \text{ Let } \mathfrak{V} \text{ be the Varchenko matrix of the oriented matroid with covector set } \mathcal{L}. \text{ Then } \]
\[ \det(\mathfrak{V}) = \prod_{F \in \mathcal{L}} (1 - a(F)^2)^{b_F}. \]

for nonnegative integers \( b_F \).

Note, that a factor \((1 - a(F)^2)\) is zero if and only if \( F \) is a tope. In this case it turns out that \( b_F = 0 \). By the convention \( 0^0 = 1 \) it follows that \( \det(\mathfrak{V}) \neq 0 \). In Corollary 5.7 we give an alternative formulation of the product formula which will shed more light on the exponents \( b_F \). In particular, it will follow that \( \det(\mathfrak{V}) \) only depends on the matroid underlying the oriented matroid defined by \( \mathcal{L} \).
If $\mathcal{L}$ is given as the set of covectors of a hyperplane arrangement in some $\mathbb{R}^n$ then $\mathfrak{W}$ is the Varchenko matrix of the hyperplane arrangement and Theorem 1.1 is Varchenko’s result from [9].

After the original proof in [9] there were attempts in [5] and [6] to provide a cleaner proof of Varchenko’s original result. Our approach generalizes ideas from [5] and [6] to oriented matroids and replaces the problematic parts from both works by alternative arguments. See also [12] for a proof of an important special case. Recently, a new proof using a different strategy was published in [1]. We have not studied this proof thoroughly and cannot judge if it generalizes to oriented matroids as well. This paper is not the first to study oriented matroid generalizations of the Varchenko determinant formula. In the works [10, 11] an approach is sketched for proving Theorem 1.1 originally for general oriented matroids in [10] and restricted to oriented matroids that allow a representation as a pseudo point configuration, only, in the subsequent [11]. Despite several attempts we were not able to follow the argumentation of either thesis. Philosophically, our work parallels the article of Brylawski and Varchenko [4] who give a matroid generalization of a determinant formula by Schechtman and Varchenko [7] for yet another important class of matrices arising in representation theory. That paper probably also motivated [10] and [11].

Besides amendments and the generalization to oriented matroids the key new ingredient in our proof of Theorem 1.1 is the following result which we consider of independent interest.

For its formulation, let $R \in \mathcal{T}$ be a fixed base tope and consider $\mathcal{T}$ as a partially ordered set with order relation $P \preceq_R Q$ if $\text{Sep}(R, P) \subseteq \text{Sep}(R, Q)$. We write $\mathcal{T}_R$ if we consider $\mathcal{T}$ with this partial order. For disjoint subsets $S^+, S^- \subseteq E$ such that $S^+ \cup S^- \neq \emptyset$ the set of topes

$$\mathcal{T}(S^+, S^-) := \{ T \in \mathcal{T} \mid T_f = + \text{ for all } f \in S^+ \text{ and } T_f = - \text{ for all } f \in S^- \}$$

is called a supertope. By [3, Proposition 4.2.6] supertopes are exactly the $T$-convex sets, i.e. the sets of topes that contain any shortest path between any of two of its members. We call a supertope $\mathcal{T}(S^+, S^-)$ a cone, if for all supertopes $\mathcal{T}(\tilde{S}^+, \tilde{S}^-)$ such that $S^+ \subseteq \tilde{S}^+$, $S^- \subseteq \tilde{S}^-$ but $(S^+, S^-) \neq (\tilde{S}^+, \tilde{S}^-)$ necessarily $\mathcal{T}(\tilde{S}^+, \tilde{S}^-) \subseteq \mathcal{T}(S^+, S^-)$. Note that our notion of cone is more general than the one from [3, Definition 10.1.1 (iii)].

One would expect that $T$-convex sets as subsets of the tope poset $\mathcal{T}_R$ are contractible. We will show that this is indeed the case.

**Theorem 1.2.** Let $R \in \mathcal{T}$ be the base tope of the poset $\mathcal{T}_R$ and $\mathcal{T}(S^+, S^-) \neq \emptyset$ be a supertope. Then $\mathcal{T}(S^+, S^-)$ considered as subposet of $\mathcal{T}_R$ is contractible.

The paper is organized as follows. In Section 2 we recall some basic notations and results from oriented matroid theory and poset topology. We then use tools from poset topology to derive results on the topology of complexes associated to oriented matroids in Section 3. In Section 4 we provide the proof of Theorem 1.2 and exhibit why we cannot follow the argumentation from [5] and [6]. In Section 5 we prove Theorem 1.1. The key step in the proof is a factorization of the Varchenko matrix, one factor for each element of the ground set $E$ (Proposition 5.3). The key ingredient of the factorization is a result on Möbius numbers which is a direct consequence of Theorem 1.2 (Corollary 4.5). Then
the determinant of each factor is analyzed. Möbius number implications of topological results from Section 3 then show that each is block upper triangular with controllable block structure (Lemma 5.6). Now Theorem 1.1 follows via basic linear algebra. As a corollary we give a description of the numbers $b_F$ which implies that the determinant only depends on the matroid underlying the oriented matroid. As a second corollary we show that the result extends to cones and hence in particular to affine oriented matroids.

2. Background on Oriented Matroids and Poset Topology

2.1. Poset Topology. In this paper we will associate various partially ordered sets, posets for short, to oriented matroids. For our purposes it turns out to be useful to consider a poset $P$ as a topological space. We do this by identifying $P$ with its order complex, respectively the geometric realization of the order complex. Recall that the order complex of a poset $P$ is the simplicial complex whose chains are the linearly ordered subsets of $P$. Using this identification we can speak about contractible and homotopy equivalent posets. We will employ the following standard tools from poset topology (see [2] for details). For their formulation we denote for a poset $P$ and $p \in P$ by $P_{\leq p}$ the subposet $\{q \in P \mid q \leq p\}$. Analogously defined are $P_{<p}, P_{>p}$ and $P_{\geq p}$. For $p \leq q$ in $P$ we write $(p, q)_P$ for the open interval $P_{>p} \cap P_{<q}$ and analogously define closed and halfopen intervals.

**Proposition 2.1** (Quillen Fiber Lemma). Let $P$ and $Q$ be posets and $f : P \to Q$ a poset map. If for all $q \in Q$ we have that $f^{-1}(Q_{\leq q})$ is contractible, then $P$ and $Q$ are homotopy equivalent.

By simple induction on $#SS$ one derives the following corollary.

**Corollary 2.2.** Let $P$ be a poset and $SS$ a subset such that $P_{<s}$ is contractible for all $s \in SS$. Then $P$ and $P \setminus SS$ are homotopy equivalent.

We will use poset topology also to prove results on the Möbius number of a poset $P$. For that we take advantage of the following well known numerical consequence of the fact that two posets are homotopy equivalent. For a poset $P$ we denote by $\mu(P)$ the Möbius number of $P$ (see [8, Chapter 3]).

**Proposition 2.3.** For two homotopy equivalent posets $P$ and $Q$ we have $\mu(P) = \mu(Q)$. In particular, if $P$ is contractible then $\mu(P) = 0$.

2.2. Oriented Matroids. As mentioned in Section 1 we consider an oriented matroid $L$ on ground set $E$ as a set of covectors $X = (X_e)_{e \in E} \in \{+, -, 0\}^E$. In our notation we follow [3] which also contains all required background information on oriented matroids. Frequently, we will use the following definitions and notations.

We order the covectors by the product order induced by the order $0 < +, -$ and write $0 = (0)_{e \in E}$ for the unique minimal covector in this order. Following our conventions, for a covector $X$ we write $(0, X)_L$ for the open interval from $0$ to $X$ in $L$. It is well known that the poset of covectors is graded and hence one can assign each covector $X \in L$ a rank $\text{rank}_L(X)$. The rank $\text{rank}(L)$ of $L$ is defined as the maximal rank of one of its covectors.
Let $A \subseteq E$ be a nonempty set. For $F \in \mathcal{L}$ we denote by $F|_A$ the covector $(F_e)_{e \in A}$. For a set $\mathcal{K}$ of covectors over $E$ we then write $\mathcal{K}|_A$ for the set $\{F|_A \mid F \in \mathcal{K}\}$ of covectors over $A$. For an oriented matroid $\mathcal{L}$ over $E$ and a nonempty subset $A \subseteq E$ the set of covectors $\mathcal{L}|_A$ defines an oriented matroid called the restriction of $\mathcal{L}$ to $A$. The contraction of $A$ in $\mathcal{L}$ is the oriented matroid $\mathcal{L}/A$ with covector set $\{F|_{E \setminus A} \mid F \in \mathcal{L}, z(F) \subseteq A\}$. In case $A = \{f\}$ is a singleton we also write $\mathcal{L}/f$ for $\mathcal{L}/A$.

As usual for a covector $X \in \mathcal{L}$ we write $X^+$ for $\{e \in E \mid X_e = +\}$ and $X^-$ for $\{e \in E \mid X_e = -\}$. In addition, we write $z(X) = \{e \in E \mid X_e = 0\}$ for its zero-set. For two covectors $X, Y \in \mathcal{L}$ their composition $X \circ Y$ is defined by $(X \circ Y)^+ = X^+ \cup (Y^+ \setminus X^-)$ and $(X \circ Y)^- = X^- \cup (Y^- \setminus X^+)$. 

Next we repeat and extend some notation already stated in Section 1. We write $\mathcal{T}(\mathcal{L})$ for the set of topes of $\mathcal{L}$ and simply $\mathcal{T}$ in case there is no danger of ambiguity. For $P, Q \in \mathcal{T}$ we denote by $\text{Sep}(P, Q)$ the separator of $P$ and $Q$. Then for fixed $R \in \mathcal{T}$ the set of topes $\mathcal{T}$ carries a poset structure defined by $P \preceq_R Q$ if and only if $\text{Sep}(P, R) \subseteq \text{Sep}(R, Q)$ (see [3] Definition 4.2.9)). We write $\mathcal{T}_R$ to denote $\mathcal{T}$ with this poset structure. For $P \preceq_R Q$ we write $(P, Q)_R$ as a shorthand for $(P, Q)_{\mathcal{T}_R}$ to denote the interval of all $P < T < Q$ in $\mathcal{T}_R$.

For $e \in E$ and $P \in \mathcal{T}$ we say that $e$ does not define a proper face of $P$ if the only covector $F \in \mathcal{L}$ with $F \leq P$ and $F_e = 0$ is $F = 0$.

We will frequently encounter the situation where $R, P \in \mathcal{T}$ and $e \in E$ are such that $+ = R_e$ and $- = P_e$. Then after reordering and reorientation we can assume the following.

**Situation 2.4.** $R = + \cdots +$ and $P = \cdots - + \cdots +$ and $e$ is the first coordinate of our sign vectors.

In the rest of the paper, we will work in the general situation unless there is a technical simplification when assuming [Situation 2.4]. In that case we will explicitly mention the assumption.

Next we state well known facts about the topology of $\mathcal{L}$ and $\mathcal{T}_R$.

**Lemma 2.5** (Lemma 4.3.11 [3]). Let $P, R \in \mathcal{T}$ set

$$F_R(P) = \{X \in (0, P)_\mathcal{L} \mid z(X) \subseteq \text{Sep}(P, R)\} \subseteq \mathcal{L}.$$ 

Then $F_R(P)$ is a filter in $(0, P)_\mathcal{L}$. If $P \neq \pm R$ then $F_R(P)$ is contractible.

For a covector $X$ we set $\text{star}(X) := \{T \in \mathcal{T} \mid X \preceq T\}$.

**Theorem 2.6** (Theorem 4.4.2 [3]). Let $\mathcal{L}$ be an oriented matroid of rank $r$ and $R \in \mathcal{T}$. For $T_1, T_2 \in \mathcal{T}_R$ such that $T_1 \leq_R T_2$ the order complex of $(T_1, T_2)_R$ is homotopy equivalent to

(i) a sphere of dimension $r - \text{rank}_\mathcal{L}(X) - 2$ if $[T_1, T_2]_R$ equals $\text{star}(X)$ for some covector $X$,

(ii) a point, i.e. it is contractible, otherwise.

For $e \in E$ and $R \in \mathcal{T}$ we write $\mathcal{T}_{R,e}$ for the poset $\{T \in \mathcal{T} \mid T_e = -R_e\} \cup \{\hat{0}\}$ with $\hat{0}$ as its least element and the remaining poset structure induced from $\mathcal{T}_R$. For $P \in \mathcal{T}_{R,e}$ we
write \((\hat{0}, P)_{R,e}\) for the interval from \(\hat{0}\) to \(P\) in \(\mathcal{T}_{R,e}\). We set
\[
(\hat{0}, P)_{R,e}^\Delta := \{ Q \in (\hat{0}, P)_{R,e} \mid \exists X \in \mathcal{L} : [Q, P]_R = \text{star}(X) \}.
\]
The following is an immediate consequence of Theorem 2.6.

**Corollary 2.7.** Let \(R \in \mathcal{T}, e \in E\) and \(P \in \mathcal{T}_{R,e}\). Then \((\hat{0}, P)_{R,e}\) and \((\hat{0}, P)_{R,e}^\Delta\) are homotopy equivalent.

**Proof.** Let
\[
S = \{ Q \in (\hat{0}, P)_{R,e} \mid \exists X \in \mathcal{L} : [Q, P]_R = \text{star}(X) \}.
\]
Then \((\hat{0}, P)_{R,e}^\Delta = (\hat{0}, P)_{R,e} \setminus S\). For \(Q \in S\) Theorem 2.6 implies that \((\hat{0}, P)_{R,e} \succ Q = (Q, P)_R\) is contractible. Now the assertion follows from Corollary 2.2 \(\square\)

### 3. Some Oriented Matroid Topology

At the end of the last section we already recalled some known facts about the topology of posets associated to oriented matroids. This section now contains oriented matroid generalizations of topological results stated in [5] and [6] for hyperplane arrangements.

For \(P \in \mathcal{T}_{R,e}\) we set \(S = E \setminus \text{Sep}(P, R)\) and \(S' = \text{Sep}(P, R) \setminus \{e\}\). Let \(B \in \mathcal{T}(\mathcal{L}|_{S'})\) be the unique tope from \(\mathcal{T}(\mathcal{L}|_{S'})\) such that \(B_f = P_f = -R_f\) for all \(f \in S'\). Let \(G \in \mathcal{T}(\mathcal{L}|_S)\) be the unique tope from \(\mathcal{T}(\mathcal{L}|_S)\) such that \(G_f = P_f\) for all \(f \in S\).

We define
\[
W_{R,e}(P) = \left\{ F \in (0, P)_{\mathcal{L}} \mid \begin{array}{c}
F_e = -R_e, \\
F_{S'} \leq B, \\
\exists f \in S' : F_f = 0
\end{array} \right\} \subseteq \mathcal{L}.
\]
We consider \(W_{R,e}(P)\) as a poset with order relation inherited from \(\mathcal{L}\). Assuming Situation 2.4 we have:
\[
W_{R,e}(P) = \left\{ F \in (0, P)_{\mathcal{L}} \mid \begin{array}{c}
F_e = -, \\
F_{S'} \leq \{\}^{S'}, \\
\exists f \in S' : F_f = 0
\end{array} \right\} \subseteq \mathcal{L}.
\]
We consider the following map:
\[
\alpha_P : \left\{ \begin{array}{c}
(\hat{0}, P)_{R,e}^\Delta \\
C
\end{array} \right\} \rightarrow \alpha_P(C) := X, \quad \mathcal{L}_{\text{for the } x \in \mathcal{L}} \text{ such that } (C, P)_R = \text{star}(X)
\]

**Lemma 3.1.** Let \(R \in \mathcal{T}\) and \(e \in E\). For \(P \in \mathcal{T}_{R,e}\) and \(C \in (\hat{0}, P)_{R,e}^\Delta\) the following holds:

(i) \(z(\alpha_P(C)) = \text{Sep}(C, P)\) and \(\alpha_P(C) \in W_{R,e}(P)\).

(ii) \(\alpha_P\) is a poset map from \((\hat{0}, P)_{R,e}^\Delta\) to \(W_{R,e}(P)\).

(iii) For \(F \in W_{R,e}(P)\) the tope \(F \circ R\) is the unique maximal element in \(\alpha_P^{-1}(W_{R,e}(P) \leq F)\).

**Proof.** We assume Situation 2.4.

(i) By definition \([C, P]_R = \text{star}(X) = [X \circ R, X \circ (-R)]_R\). Hence, \(X \leq P\) and \(X_e = C_e = P_e = -\), implying \(z(X) = \text{Sep}(C, P)\) and \(\alpha_P(C) = X \in W_{R,e}(P)\).
(ii) Since $C \preceq_R C'$ in $(\hat{0}, P)_{R,e}^\Delta$ implies $\text{Sep}(C', P) \subseteq \text{Sep}(C, P)$ it follows from (i) that $\alpha_P(C) \leq \alpha_P(C')$. Hence $\alpha_P$ is a map of posets.

(iii) Let $F, F' \in W_{R,e}(P)$, $F' \leq F$ and $C = F \circ R$. We have $C' \in \alpha_P^{-1}(F')$ if and only if $[C', P]_R = [F' \circ R, F' \circ (-R)]_R$. As $z(F) \subseteq z(F')$ this implies $\text{Sep}(C', R) \subseteq \text{Sep}(C, R)$ and hence the assertion.

The following proposition allows us to determine the topology of the posets $(\hat{0}, P)_{R,e}$ through known results on $W_{R,e}(P)$.

**Proposition 3.2.** Let $R \in T$ and $e \in E$ such that $R_e = +$ and $P \in T_{R,e}$. Then:

(i) The order complex of the interval $(\hat{0}, P)_{R,e}$ and the order complex of $W_{R,e}(P)$ are homotopy equivalent.

(ii) For $e$ that do not define a proper face of $P$ the order complex of $W_{R,e}(P)$ is contractible if $\pm R \neq P$ and homotopy equivalent to a $(\text{rank}(L) - 2)$-sphere if $-R = P$.

**Proof.** (i) From Corollary 2.7 it follows that $(\hat{0}, P)_{R,e}$ and $(\hat{0}, P)_{R,e}^\Delta$ are homotopy equivalent.

Using Lemma 3.1 (iii) it follows that the order complex of each fiber $\alpha_P^{-1}(W_{R,e}(P)_{\leq F})$ for $F \in \bar{W}_{R,e}(P)$ is a cone and hence contractible. Now the Quillen Fiber Lemma, Proposition 2.1 shows that the order complexes of $(\hat{0}, P)_{R,e}^\Delta$ and $W_{R,e}(P)$ are homotopy equivalent.

(ii) Since $e$ does not define a proper face of $P$, we have $W_{R,e}(P) = F_R(P)$. If $P = -R$ then $W_{R,e}(P) = (0, P)_L$ and hence is homotopy equivalent to a $(\text{rank}(L) - 2)$-sphere. If $P \neq \pm -R$ then $W_{R,e}(P) = F_R(P)$ and Lemma 2.5 shows that $W_{R,e}(P)$ is contractible. □

We summarize the results in the following theorem.

**Theorem 3.3.** Let $P \in T_{R,e}$ such that $e$ does not define a proper face of $P$. Then the interval $(\hat{0}, P)_{R,e}$ is contractible if $-R \neq P$ and homotopy equivalent to a $(\text{rank}(L) - 2)$-sphere if $-R = P$.

**Proof.** The result is an immediate consequence of Proposition 3.2 (i) and (ii). □

The well known connection of homotopy type and Möbius-number from Proposition 2.3 yields.

**Corollary 3.4.** Let $P \in T_{R,e}$ such that $e$ does not define a proper face of $P$. Then the Möbius number $\mu((\hat{0}, P)_{R,e})$ is $0$ if $-R \neq P$ and $(-1)^{\text{rank}(L)}$ if $-R = P$.

The next result overlaps with Theorem 3.3 but also covers some of the cases where $e$ defines a face of $P$. Note that $e$ defines a proper face of $R$ if and only if $e$ defines a proper face of $-R$. Hence the interval $(\hat{0}, -R)_{R,e}$ is covered by Theorem 3.3 if $e$ does not define a proper face of $R$ and it is covered by Theorem 3.5 otherwise.
Theorem 3.5. Let $R \in \mathcal{T}$ let $e \in E$ define a proper face of $R$. Let $F \in \mathcal{L}$ be the maximal covector such that $F \leq R$ and $F_e = 0$ and choose $P_{\text{top}} \in \mathcal{T}_{R,e} \setminus \text{star}(F)$. Then $(0, P_{\text{top}})_{R,e}$ is contractible. In particular, $\mu((0, P_{\text{top}})_{R,e}) = 0$.

Proof. Let $P \in (0, P_{\text{top}})_{R,e}$. Then by the gate property [Exercise 4.10] the tope $Q = F \circ P \in \text{star}(F)$ is the unique tope in $\text{star}(F)$ such that for all $O \in \text{star}(F)$ we have

$$\text{Sep}(P, O) = \text{Sep}(P, Q) \cup \text{Sep}(Q, O)$$

Since $F_e = 0$ it also follows that $Q_e = -$. Since $F \leq R$, clearly $\text{Sep}(R, Q) = \text{Sep}(R, F \circ P) \subseteq \text{Sep}(R, P)$ and hence $Q \preceq_R P$. This shows $Q \in (0, P_{\text{top}})_{R,e}$. Now let $Q \preceq_R Q'$. Then $F \circ Q \preceq_R F \circ Q'$. Since $F \leq R$ it follows that $F \circ Q \preceq_R Q$. Obviously $F \circ (F \circ Q) = F \circ Q$. This shows that the map $\circ_F : (0, P_{\text{top}})_{R,e} \to (0, P_{\text{top}})_{R,e}$ is a closure operator. And hence $(0, P_{\text{top}})_{R,e}$ is homotopy equivalent to its image (see e.g, [2, Corollary 10.12]). Since $P_{\text{top}} \not\in \text{star}(F)$ and $F \circ P_{\text{top}} \in \text{star}(F) \cap (0, P_{\text{top}})_{R,e}$, it also follows that $F \circ Q \preceq_R F \circ P_{\text{top}}$ for all $Q \in (0, P_{\text{top}})_{R,e}$. Hence the image of $\circ_F$ has a maximal element and hence is contractible.

4. Supertopes

In this section we identify supertopes that are relevant for our purposes and provide the proof of Theorem 1.2. We also deduce Corollary 4.5, which is crucial for the derivation of Theorem 1.1 from Theorem 1.2.

For the identification of the supertopes in our context we study restriction maps.

Lemma 4.1. Let $R \in \mathcal{T}$ and $e \in E$ such that $R_e = +$. For $P \in (0, -R)_{R,e}$ let $S = E \setminus \text{Sep}(R, P)$ be a nonempty subset. Let $\pi_S : \mathcal{T} \to \mathcal{T}|_S$ be the map from $\mathcal{T}$ to the set of toposes $\mathcal{T}|_S = \mathcal{T}(\mathcal{L}|_S)$ of $\mathcal{L}|_S$ given by restriction. Then

(i) the map $\pi_S$ is a map of posets, $\mathcal{T}_R \to (\mathcal{T}|_S)_{R|_S}$ and restricts to a map of posets from $(0, -R)_{R,e}$ to $(\mathcal{T}|_S)_{R|_S}$ ordered by $\preceq_R$ and $\preceq_{R|_S}$ respectively.

(ii) $\text{Sep}(\pi_S(Q), \pi_S(R)) = \text{Sep}(Q, P) \cap \text{Sep}(Q, R)$.

(iii) $\pi_S(Q) = \pi_S(Q')$ if and only if $\text{Sep}(Q, P) \cap \text{Sep}(Q, R) = \text{Sep}(Q', P) \cap \text{Sep}(Q', R)$.

Proof. (i) This is obvious.

(ii) ‘$\subseteq$’: Let $f \in S$, such that $\pi_S(Q)_f \neq \pi_S(R)_f$. Then $Q_f \neq R_f$ and hence $f \in \text{Sep}(Q, R)$. If $f \notin \text{Sep}(Q, P)$ then by $f \in S = E \setminus \text{Sep}(P, R)$ we have $Q_f = P_f = R_f$. But this contradicts $Q_f \neq R_f$.

‘$\supseteq$’: Let $f \in \text{Sep}(Q, P) \cap \text{Sep}(Q, R)$. Then by $Q_f \neq P_f$ and $Q_f \neq R_f$ it follows that $R_f = P_f$ and hence $f \in S$ and $f \in \text{Sep}(\pi_S(Q), \pi_S(R))$.

(iii) Follows immediately from (ii).

The next lemma exhibits the supertopes whose topology will become crucial in the proof of Theorem 1.1.
Lemma 4.2. Let \( e \in E \) and \( e \in S \subseteq E \). Then for each \( Q|_S \in (\hat{0}, P|_S)_{R|_S,e} \) the preimage
\[
\pi^{-1}_S((\hat{0}, P|_S)_{R|_S,e}) \leq_{R|_S} Q|_S
\]
is a supertope. More precisely, if \( S = S^+ \cup S^- \) is such that \( \pi^{-1}_S(Q_S) \subseteq \mathcal{T}(S^+, S^-) \), then
\[
\pi^{-1}_S((\hat{0}, P|_S)_{R|_S,e}) \leq_{R|_S} Q|_S = \mathcal{T}(S^+, \{e\}).
\]

Proof. If \( T \in \mathcal{T}(S^+, \{e\}) \) then
\[
\{e\} \subseteq \text{Sep}(\pi_S(T), R|_S) \subseteq S^- = \text{Sep}(Q_S, R_S) \subseteq \text{Sep}(P_S, R_S)
\]
and hence \( \pi_S(T) \in (\hat{0}, P_S)_{R,e} \) and \( \pi_S(T) \preceq Q_S \). For the other inclusion assuming that \( T \in \pi^{-1}_S((\hat{0}, P_S)_{R_S,e}) \leq_{R_S} Q_S \), we have \( T_e = - \) and, as \( \pi_S(T) \preceq_{R_S} Q_S \preceq_{R_S} P_S \) also \( \text{Sep}(\pi_S(T), R_S) \subseteq S^- \). Hence we have \( T_f = + \) for all \( f \in S^+ \) which implies \( T \in \mathcal{T}(S^+, \{e\}) \). \( \Box \)

After we have seen the relevant supertopes we turn to the proof of Theorem 1.2. We assume throughout the rest of this section Situation 2.4.

Lemma 4.3. Let \( \mathcal{L} \) be an oriented matroid on \( E \) and \( \mathcal{T} \) be the set of its topes. Let \( E = S^+ \dot{\cup} S^- \dot{\cup} S^* \) be a partition of the ground set into nonempty sets \( S^+, S^- \) and \( S^* \). If for all \( f \in S^* \) there exists \( T_f^\prime \in \mathcal{T} \) such that
\[
T_{g}^f = \begin{cases} 
+ & \text{if } g \in S^+ \\
- & \text{if } g \in S^- \\
- & \text{if } g \in S^* \setminus \{f\} \\
+ & \text{if } g = f,
\end{cases}
\]
then either there exists a tope \( T_{\text{max}} \in \mathcal{T} \) satisfying
\[
T_{g}^\text{max} = \begin{cases} 
+ & \text{if } g \in S^+ \\
- & \text{if } g \in S^- \\
- & \text{if } g \in S^*
\end{cases}
\]
or there exists a tope \( T_{\text{min}} \in \mathcal{T} \) satisfying
\[
T_{g}^\text{min} = \begin{cases} 
+ & \text{if } g \in S^+ \\
- & \text{if } g \in S^- \\
+ & \text{if } g \in S^*
\end{cases}
\]
or a covector \( Y \in \mathcal{L} \) satisfying
\[
Y_{g} = \begin{cases} 
+ & \text{if } g \in S^+ \\
- & \text{if } g \in S^- \\
0 & \text{if } g \in S^0 \\
- & \text{if } g \in S^* \setminus S^0
\end{cases}
\]
for some set \( \emptyset \neq S^0 \subseteq S^* \).
Proof. We proceed by induction on $|S^*|$. If $|S^*| = 1$ the assertion is trivial. If $S^* = \{f, g\}$, then, either $f$ and $g$ are antiparallel and we find a $Y$ as required, or on a shortest path from $T^f$ to $T$ we must pass through $T^{\max}$ or $T^{\min}$. Assume $|S^*| \geq 3$ and let $g \in S^*$. If there exists some $f \in S^* \setminus \{g\}$ such that eliminating $g$ between $T^g$ and $T^f$ yields a covector $X^f$ such that $X^f_\pi \in \{0, -\}$, then $X \circ T^h$ for $h \in S^* \setminus \{f, g\}$ is an element $T^{\max}$. Hence we may assume that for all $f \in S^* \setminus \{g\}$ we find $X$ satisfying

$$X^f_h = \begin{cases} + & \text{if } h \in S^+ \\ - & \text{if } h \in S^- \\ - & \text{if } h \in S^* \setminus \{f, g\} \\ + & \text{if } h = f \\ 0 & \text{if } h = g. \end{cases}$$

Then the image of $X^f$ in $L/g$ satisfies the assumptions of the lemma in the oriented matroid $L/g$. By induction we find either an appropriate $\tilde{X} \in L/g$ which clearly yields a $Y$ as required in $L$, or we find an element $X^{\max} \in L$ such that

$$X^{\max}_h = \begin{cases} + & \text{if } h \in S^+ \\ - & \text{if } h \in S^- \\ - & \text{if } h \in S^* \setminus \{f, g\} \\ 0 & \text{if } h = g \end{cases}$$

and $X^{\max} \circ T^f$ is as required for $f \in S^* \setminus g$ and similarly $X^{\min} \circ T^g$ in the remaining case. □

Corollary 4.4. Let $L$ be an oriented matroid on a set $E$ that satisfies the assumptions of Lemma 4.3. The for a fixed $R \in T$ the subposet $\mathcal{T}(S^+, S^-)$ of $T_R$ either has a unique maximal element or it has a unique minimal element. In particular, it is contractible.

Proof. Either $T^{\max}$, $T^{\min}$ of $Y \circ (-R)$ are as required. □

Now we are in position to prove Theorem 1.2.

Proof of Theorem 1.2. We proceed by induction on $|E \setminus (S^+ \cup S^-)|$. If $S^+ \cup S^- = E$, then $\mathcal{T}(S^+, S^-)$ is a singleton and thus contractible. If $S^+ \cup S^- \neq E$, then $S^* := E \setminus \{S^+ \cup S^-\} \neq \emptyset$. If for all $f \in S^*$ there exists $T^f$ as in Lemma 4.3, $\mathcal{T}(S^+, S^-)$ is contractible by Corollary 4.4. Hence we may assume that there exists $f \in S^*$ such that $T^f \notin T$. Let $\mathcal{T}^f_{R \setminus \{f\}}$ denote the tope poset in the oriented matroid $L \setminus f$ with base tope $R \setminus \{f\}$. By inductive assumption its subposet $\mathcal{T}_f(S^+, S^-)$ is contractible. Consider the poset map $\pi^f : \mathcal{T}(S^+, S^-) \to \mathcal{T}_f(S^+, S^-)$ given by restriction. Let $Q \in \mathcal{T}_f(S^+, S^-)$. As in Lemma 4.2 it is easy to see that

$$(\pi^f)^{-1}(Q_{\leq}) = \mathcal{T}(Q^+, S^-).$$

Clearly $S^+ \subseteq Q^+$. If $S^+ \not\subseteq Q^+$, then $(\pi^f)^{-1}(Q_{\leq})$ is contractible by inductive assumption. Consider the case that $S^+ \subseteq Q^+$. By the choice of $f$ the preimage $(\pi^f)^{-1}(Q)$ is a singleton $\{Z\}$ with $Z = \pi^f(f)$. Hence, this is the unique maximal element in $(\pi^f)^{-1}(Q_{\leq})$ and that fiber is also contractible. Hence by Proposition 2.1 $\mathcal{T}(S^+, S^-)$ and $\mathcal{T}_f(S^+, S^-)$ are homotopy equivalent and the claim follows. □
Corollary 4.5. Let $R \in \mathcal{T}$ be the base tope of the poset $\mathcal{T}_R$. Let $e \not\in S \subseteq E$. Then
\[
\sum_{Q \in \mathcal{T}(\emptyset, \{e\}) \cap S \subseteq \text{Sep}(P,Q) \cap \text{Sep}(Q,R)} \mu((\hat{0}, Q)_{R,e}) = \begin{cases} 
-1 & \text{if } S = \emptyset \\
0 & \text{if } S \neq \emptyset.
\end{cases}
\]

Proof. We prove the assertion by induction on $\#S$.

If $S = \emptyset$ then
\[
\sum_{Q \in \mathcal{T}(\emptyset, \{e\}) \cap S \subseteq \text{Sep}(P,Q) \cap \text{Sep}(Q,R)} \mu((\hat{0}, Q)_{R,e}) = 0.
\]

Assume $\#S > 0$. Set
\[
T^+ = \{ f \in E \setminus (S \cup \{e\}) \mid R_f = + \} \quad \text{and} \quad T^- = \{ f \in E \setminus S \mid R_f = - \} \cup \{e\}.
\]

Then
\[
(1) \quad \sum_{Q \in \mathcal{T}(\emptyset, \{e\}) \cap S \subseteq \text{Sep}(P,Q) \cap \text{Sep}(Q,R)} \mu((\hat{0}, Q)_{R,e}) = \sum_{Q \in \mathcal{T}(T^+, T^-)} \mu((\hat{0}, Q)_{R,e})
\]

The right hand side of (1) is the sum of M"obius function values from $\hat{0}$ to $P$ where $P \neq \hat{0}$ ranges by Theorem 1.2 over the elements of a contractible poset. By classical M"obius function theory (see e.g. [2, (9.14)]) this sum then is $-\mu(\hat{0}, \hat{0}) = -1$ plus the M"obius number of the poset. Since the poset is contractible its M"obius number is 0 and we have shown that:

\[
(2) \quad \sum_{Q \in \mathcal{T}(\emptyset, \{e\}) \cap S \subseteq \text{Sep}(P,Q) \cap \text{Sep}(Q,R)} \mu((\hat{0}, Q)_{R,e}) = -1
\]

Now rewrite the right hand side of (2) as:

\[
(3) \quad \sum_{Q \in \mathcal{T}(\emptyset, \{e\}) \cap S \subseteq \text{Sep}(P,Q) \cap \text{Sep}(Q,R)} \mu((\hat{0}, Q)_{R,e}) = \sum_{T \subseteq S} \sum_{Q \in \mathcal{T}(\emptyset, \{e\}) \cap \text{Sep}(P,Q) \cap \text{Sep}(Q,R) = T} \mu((\hat{0}, Q)_{R,e})
\]

By induction the summand $\sum_{Q \in \mathcal{T}(\emptyset, \{e\}) \cap S \subseteq \text{Sep}(P,Q) \cap \text{Sep}(Q,R) = T} \mu((\hat{0}, Q)_{R,e})$ is 0 for $T \neq S, \emptyset$ and $-1$ for $T = \emptyset$. Thus combining (2) and (3) we obtain:
\[-1 = \sum_{Q \in \mathcal{T}(\emptyset, \{e\}) \subseteq \text{Sep}(P,Q) \cap \text{Sep}(Q,R) \subseteq S} \mu((\hat{0}, Q)_{R,e})
\]

\[-1 = -1 + \sum_{Q \in \mathcal{T}(\emptyset, \{e\}) \subseteq \text{Sep}(P,Q) \cap \text{Sep}(Q,R) = S} \mu((\hat{0}, Q)_{R,e})
\]

\[
\text{From this we conclude }
\sum_{Q \in \mathcal{T}(\emptyset, \{e\}) \subseteq \text{Sep}(P,Q) \cap \text{Sep}(Q,R) = S} \mu((\hat{0}, Q)_{R,e}) = 0.
\]

\[\square\]

Remark 4.6. Denham and Hanlon mention in [5] that a “routine argument shows” that the poset on \(\{Q \in \mathcal{T}(\emptyset, \{e\}) \mid \text{Sep}(P,Q) \cap \text{Sep}(Q,R) = S\}\) induced by \(\mathcal{T}_R\) always contains a unique maximal element. While this can be shown to hold true for line arrangements, it fails already in 3-dimensional hyperplane arrangements. We sketch a counterexample in Figure 1. The element \(e\) is supposed to be the drawing plane. The tope \(R\) is below and \(P\) and the shaded region above \(e\). The separator \(\text{Sep}(P,R)\) without \(e\) is given by the thin lines, while the intersection of the remaining hyperplanes with \(e\) are the bold lines. \(S\) is given by the two bold lines that intersect in a vertex at \(R\). We sketched the poset induced on the shaded area. While it is contractible it has two maximal elements.

5. The Varchenko matrix

In this section we prove Theorem 1.1 and its corollaries. The proof consists of a factorization of the matrix \(\mathfrak{V}\) into matrices with controllable determinant.

Recall, that we assume \(\mathcal{T}\) to be linearly ordered. For any sign pattern \(\epsilon = (\epsilon_1, \epsilon_2) \in \{+,-\}^2\) let \(\mathfrak{V}^{e,\epsilon}\) be a \((\ell \times \ell)\)-matrix with rows indexed by \(\mathcal{T}\{(\{e\}, \emptyset)\}\) for \(\epsilon_1 = +\), \(\mathcal{T}(\emptyset, \{e\})\) for \(\epsilon_1 = -\) and columns indexed by \(\mathcal{T}(\{e\}, \emptyset)\) for \(\epsilon_2 = +\), \(\mathcal{T}(\emptyset, \{e\})\) for \(\epsilon_2 = -\). For a tope \(P\) indexing a row and a tope \(Q\) indexing a column we set \(\mathfrak{V}^{e,\epsilon}_{P,Q} = \mathfrak{V}_{P,Q}\). We set \(\mathcal{M}^e\) to be the \((\ell \times \ell)\)-matrix with rows indexed by \(\mathcal{T}(\{e\}, \emptyset)\), columns indexed by \(\mathcal{T}(\emptyset, \{e\})\) and entries

\[
\mathcal{M}^e_{Q,R} = \begin{cases} 
-\mu((\hat{0}, Q)_{R,e}) \cdot \mathfrak{V}_{Q,R} & \text{if } e \text{ is the maximal element of } \text{Sep}(Q,R) \\
0 & \text{otherwise}
\end{cases}
\]

where \(Q \in \mathcal{T}(\emptyset, \{e\})\) and \(R \in \mathcal{T}(\{e\}, \emptyset)\). We write \(\mathcal{I}_\ell\) for the \((\ell \times \ell)\)-identity matrix and define

\[
\mathcal{M}^e = \begin{pmatrix} \mathcal{I}_\ell & \mathcal{M}^e \\ \mathcal{M}^e & \mathcal{I}_\ell \end{pmatrix}.
\]

Lemma 5.1. Let \(e\) be the maximal element of \(E\). Then \(\mathfrak{V}^{e,(-,+)}\) factors as

\[
\mathfrak{V}^{e,(-,+)} = \mathfrak{V}^{e,(-,-)} \cdot \mathcal{M}^e.
\]
Proof. For $P \in \mathcal{T}(\emptyset, \{e\})$ and $R \in \mathcal{T}(\{e\}, \emptyset)$ the entry in row $P$ and column $R$ on the left hand side of (4) is $\mathcal{V}_{P,R}$. On the right hand side the corresponding entry is:

$$
\sum_{Q \in \mathcal{T}(\emptyset, \{e\})} \mathcal{V}_{P,Q} \cdot M_{Q,R}^e = - \sum_{Q \in \mathcal{T}(\emptyset, \{e\})} \mu((\hat{0}, Q)_{R,e}) \cdot \mathcal{V}_{P,Q} \cdot \mathcal{V}_{Q,R}
$$

*Figure 1.* The shaded region has two maximal elements
By definition for \( Q \in \mathcal{T}(\emptyset, \{ e \}) \) we have
\[
\mathfrak{V}_{P,Q} \cdot \mathfrak{V}_{Q,R} = \mathfrak{V}_{P,R} \cdot \prod_{f \in \text{Sep}(P,Q) \cap \text{Sep}(Q,R)} U_f^2
\]

Thus the claim of the lemma is proved once we have shown that for a fixed subset \( S \subseteq E \) and fixed \( Q, R \) we have:
\[
\sum_{Q \in \mathcal{T}(\emptyset, \{ e \})} \mu((\hat{0}, Q)_{R,e}) = \begin{cases} 
0 & \text{if } S \neq \emptyset \\
-1 & \text{otherwise.}
\end{cases}
\]

But this is the content of Corollary 4.5 and we are done. \( \square \)

Next we use the matrices \( M_e \) to factorize \( \mathfrak{V} \). The following lemma founds the inductive step in the factorization.

**Lemma 5.2.** Let \( e \) be the maximal element of \( E \) and let \( \mathfrak{V}_{U_e=0} \) be the matrix \( \mathfrak{V} \) after evaluating \( U_e \) to 0. Then
\[
\mathfrak{V} = \mathfrak{V}_{U_e=0} \cdot M_e
\]

**Proof.** Let \( \mathcal{T}(\emptyset, \{ e \}) = \{ P_1, \ldots, P_\ell \} \) and \( \mathcal{T}(\{ e \}, \emptyset) = \{ P_{\ell+1}, \ldots, P_{2\ell} \} \) be numbered such that \( -P_i = P_{i+i} \) for \( 1 \leq i \leq \ell \). Assume the rows and columns of \( \mathfrak{V} \) are ordered according to this numbering of \( \mathcal{T} \). This yields a block decomposition of \( \mathfrak{V} \) as
\[
\mathfrak{V} = \begin{pmatrix}
\mathfrak{V}_e(\cdot,-) & \mathfrak{V}_e(\cdot,+)
\end{pmatrix}.
\]

Since \( \mathfrak{V}_{P,Q} = \mathfrak{V}_{-P,-Q} \) it follows that \( \mathfrak{V}_e(\cdot,-) = \mathfrak{V}_e(\cdot,+), \mathfrak{V}_e(\cdot,-) = \mathfrak{V}_e(\cdot,-) \cdot M_e \) and hence
\[
\mathfrak{V}_e(\cdot,-) = \mathfrak{V}_e(\cdot,-) = \mathfrak{V}_e(\cdot,+), \mathfrak{V}_e(\cdot,-) = \mathfrak{V}_e(\cdot,+), M_e = \mathfrak{V}_e(\cdot,+), M_e.
\]

Thus
\[
\mathfrak{V} = \begin{pmatrix}
\mathfrak{V}_e(\cdot,-) & 0 \\
0 & \mathfrak{V}_e(\cdot,+)
\end{pmatrix} \cdot \begin{pmatrix}
\mathcal{I}_\ell & M_e \\
M_e & \mathcal{I}_\ell
\end{pmatrix}
= \begin{pmatrix}
\mathfrak{V}_e(\cdot,-) & 0 \\
0 & \mathfrak{V}_e(\cdot,+)
\end{pmatrix} \cdot M_e.
\]

Now the monomial \( \mathfrak{V}_{P,Q} \) has a factor \( U_e \) if and only if \( P \in \mathcal{T}(\emptyset, \{ e \}) \) and \( Q \in \mathcal{T}(\{ e \}, \emptyset) \) or \( P \in \mathcal{T}(\{ e \}, \emptyset) \) and \( Q \in \mathcal{T}(\emptyset, \{ e \}) \). Hence
\[
\mathfrak{V}_{U_e=0} = \begin{pmatrix}
\mathfrak{V}_e(\cdot,-) & 0 \\
0 & \mathfrak{V}_e(\cdot,+)
\end{pmatrix}
\]

Combining (6) und (7) yields the claim. \( \square \)

Now we are in position to state and prove the crucial factorization.
Proposition 5.3. Let $E = \{e_1 \prec \cdots \prec e_r\}$ be a fixed ordering. Then
\[ \mathfrak{M} = \mathcal{M}^{e_1} \cdots \mathcal{M}^{e_r}. \]

Proof. We will prove by downward induction on $i$ that
\[ \mathfrak{M} = \mathfrak{M}_{U_1=\cdots=U_r=0} \cdot \mathcal{M}^{e_i} \cdots \mathcal{M}^{e_r}. \]  
(8)

For $i = r$ the assertion follows directly from Proposition 5.3. For the inductive step assume $i > 1$ and (8) holds for $i$. We know from Lemma 5.2 that if we choose a linear ordering on $E$ for which $e_{i-1}$ is the largest element then
\[ \mathfrak{M} = \mathfrak{M}_{U_1=\cdots=U_r=0} \cdot \mathcal{N}, \]
where $\mathcal{N} = (N_{Q,R})_{Q,R \in \mathcal{T}}$ is defined as
\[ N_{Q,R} = \begin{cases} 1 & \text{if } Q = R \\ -\mu((\hat{0}, Q)_{R,e_{i-1}}) \mathfrak{M}_{Q,R} & \text{if } R \in \mathcal{T}(\{e_{i-1}\}, \emptyset), Q \in \mathcal{T}(\emptyset, \{e_{i-1}\}) \\ -\mu((\hat{0}, -Q)_{-R,e_{i-1}}) \mathfrak{M}_{Q,R} & \text{if } -R \in \mathcal{T}(\{e_{i-1}\}, \emptyset), -Q \in \mathcal{T}(\emptyset, \{e_{i-1}\}) \\ 0 & \text{otherwise} \end{cases} \]

This shows:
\[ (N_{Q,R})_{U_1=\cdots=U_r=0} = \begin{cases} 1 & \text{if } Q = R \\ -\mu((\hat{0}, Q)_{R,e_{i-1}}) \mathfrak{M}_{Q,R} & \text{if } e_{i-1} \text{ is the largest element in } \text{Sep}(Q, R), R \in \mathcal{T}(\{e_{i-1}\}, \emptyset), Q \in \mathcal{T}(\emptyset, \{e_{i-1}\}) \\ -\mu((\hat{0}, -Q)_{-R,e_{i-1}}) \mathfrak{M}_{Q,R} & \text{if } e_{i-1} \text{ is the largest element in } \text{Sep}(Q, R), -R \in \mathcal{T}(\{e_{i-1}\}, \emptyset), -Q \in \mathcal{T}(\emptyset, \{e_{i-1}\}) \\ 0 & \text{otherwise} \end{cases} \]

But then $\mathcal{N}_{U_1=\cdots=U_r=0} = \mathcal{M}^{e_{i-1}}$.

Now (9) implies
\[ \mathfrak{M}_{U_1=\cdots=U_r=0} = \mathfrak{M}_{U_1=\cdots=U_r=0} \cdot \mathcal{N}_{U_1=\cdots=U_r=0} \]
\[ = \mathfrak{M}_{U_1=\cdots=U_r=0} \cdot \mathcal{M}^{e_{i-1}} \]

With the induction hypothesis this completes the induction step by
\[ \mathfrak{M} = \mathfrak{M}_{U_1=\cdots=U_r=0} \cdot \mathcal{M}^{e_i} \cdots \mathcal{M}^{e_r} \]
\[ = \mathfrak{M}_{U_1=\cdots=U_r=0} \cdot \mathcal{M}^{e_{i-1}} \cdots \mathcal{M}^{e_r}. \]

For $i = 1$ the matrix $\mathfrak{M}_{U_1=\cdots=U_r=0}$ is the identity matrix. Thus (8) yields:
\[ \mathfrak{M} = \mathcal{M}^{e_1} \cdots \mathcal{M}^{e_r}. \]

$\Box$

Let $F \in \mathcal{L}$ and $e \in z(F)$ be the maximal element of $z(F)$. Define $\mathcal{T}^{F,e}$ as the set of topes $P \in \mathcal{T}$ such that $F$ is the maximal element of $\mathcal{L}$ for which $F_e = 0$ and $F \leq P$.

Proposition 5.4. For any pair of topes $Q, R \in \mathcal{T}^{F,e}$ we have
\[ \mu((\hat{0}, \pm Q)_{\pm R,e}) = \begin{cases} (-1)^{\text{rank}L_{z(F)}} & \text{if } Q_{z(F)} = -R_{z(F)} \\ 0 & \text{otherwise} \end{cases} \]
Proof. By the definition of $\mathcal{T}^{F,e}$ we have $F \leq Q, R$. Thus, if we consider the poset $\mathcal{T}_{Rz(F),e}$ in the contraction $L/z(F)$ we find that the interval $(\hat{0}, \pm Q)_{\pm R,e}$ is isomorphic to $(0, \pm Q_{z(F)})_{\pm Rz(F),e}$. Furthermore, since $F$ is the maximal element satisfying $F_e = 0$ and $F \leq Q$, $e$ does not define a proper face of $Q_{z(F)}$. Hence the claim follows from Corollary 3.4.

We define $b_{F,e} = 0$ if $e$ is not the maximal element of $z(F)$ and $\frac{1}{2} \# \mathcal{T}^{F,e}$ otherwise. Since $P \mapsto F \circ (-P)$ is a perfect pairing on $\mathcal{T}^{F,e}$ it follows that $\mathcal{T}^{F,e}$ contains an even number of topes. In particular, $b_{F,e}$ is an integer. We denote by $\mathcal{M}^{F,e}$ the submatrix of $\mathcal{M}^e$ obtained by selecting rows and columns indexed by $\mathcal{T}^{F,e}$.

**Lemma 5.5.** Let $F \in L$ and $e \in z(F)$. If $\mathcal{T}^{F,e} \neq \emptyset$, then

$$\det(\mathcal{M}^{F,e}) = (1 - a(F)^2)^{b_{F,e}}.$$  

*Proof.* By definition of $\mathcal{M}^e$ we obtain that for $Q, R \in \mathcal{T}^{F,e}$ we have

$$\mathcal{M}^{e}_{Q,R} = \begin{cases} 
1 & \text{if } Q = R \\
-\mu((\hat{0}, Q)_{R,e}) \cdot \mathfrak{M}_{Q,R} & \text{if } e \text{ is the largest element of } \text{Sep}(Q, R), \; R \in \mathcal{T}(\{e\}, \emptyset), Q \in \mathcal{T}(\emptyset, \{e\}) \\
-\mu((\hat{0}, -Q)_{-R,e}) \cdot \mathfrak{M}_{Q,R} & \text{if } e \text{ is the largest element of } \text{Sep}(Q, R), \; -R \in \mathcal{T}(\{e\}, \emptyset), -Q \in \mathcal{T}(\emptyset, \{e\}) \\
0 & \text{otherwise}
\end{cases}$$

If $Q_{z(F)} = -R_{z(F)}$ then $\mathfrak{M}_{Q,R} = a(F)$. Using Proposition 5.4 we find

$$\mathcal{M}^{e}_{Q,R} = \begin{cases} 
1 & \text{if } Q = R \\
-(-1)^{\text{rank}(L|_S)} a(F) & \text{if } Q = F \circ (-R) \\
0 & \text{otherwise}
\end{cases}$$

We order rows and columns of $\mathcal{M}^{F,e}$ so that the elements $R$ and $F \circ (-R)$ are paired in consecutive rows and columns. With this ordering $\mathcal{M}^{F,e}$ is a block diagonal matrix having along its diagonal $b_{F,e}$ two by two matrices

$$\begin{pmatrix} 1 & -(-1)^{\text{rank}(L|_{z(F)})} a(F) \\
-(-1)^{\text{rank}(L|_{z(F)})} a(F) & 1 \end{pmatrix}$$

if $e$ is the maximal element of $z(F)$ and identity matrices otherwise. In any case we find

$$\det(\mathcal{M}^{F,e}) = (1 - a(F)^2)^{b_{F,e}}$$

as desired. 

*Lemma 5.6.* After suitably ordering $\mathcal{T}^{F,e}$ the matrix $\mathcal{M}^e$ is the block lower triangular matrix with the matrices $\mathcal{M}^{F,e}$ for $F \in L$ with $F_e = 0$ and $\mathcal{T}^{F,e} \neq \emptyset$ on the main diagonal.

*Proof.* We fix a linear ordering of $\mathcal{T}$ such that for fixed $e \in E$ and $F \in L$ the topes from $\mathcal{T}^{F,e}$ form an interval and such that the topes from $\mathcal{T}^{F,e}$ precede those of $\mathcal{T}^{F,e}$ if $F < F'$.

For this order the claim follows if we show that the entry $(\mathcal{M}^e)_{Q,R}$ is zero whenever $R \in \mathcal{T}^{F,e}$, $Q \in \mathcal{T}^{F',e}$ and $F' < F$. 

If $Q_e = R_e$ then by $Q \neq R$ we have $(\mathcal{M}^e)_{Q,R} = 0$. Hence it suffices to consider the case $Q_e \neq R_e$.

If $Q \notin \text{star}(F)$, $Q \in \mathcal{T}(\emptyset, \{e\})$ and $R \in \mathcal{T}(\{e\}, \emptyset)$ then it follows from Theorem 3.5 that $\mu((\hat{0}, Q)_{R,e}) = 0$ and therefore $(\mathcal{M}^e)_{Q,R} = 0$. Analogously if $Q \notin \text{star}(F)$, $-Q \in \mathcal{T}(\emptyset, \{e\})$ and $-R \in \mathcal{T}(\{e\}, \emptyset)$ then $\mu((\hat{0}, -Q)_{-R,e}) = 0$ and therefore $(\mathcal{M}^e)_{Q,R} = 0$.

On the other hand, if $Q \in \text{star}(F)$, then in particular $F \leq Q$. Since by definition of $\mathcal{T}^{F,e}$ we have that $F'$ is the maximal covector such that $F' \leq Q$ and $F'_e = 0$ it follows that $F \leq F'$. Since $F \neq F'$ we must have that $F < F'$, i.e. $(\mathcal{M}^e)_{Q,R}$ is an entry above the diagonal and we are done.

\[\square\]

**Proof of Theorem 1.1.** After fixing a linear order on $E$ it follows from Proposition 5.3 that $\det \mathfrak{B}$ is the product of the determinants of $\mathcal{M}^e$ for $e \in E$. By Lemma 5.6 the determinant of each $\mathcal{M}^e$ is a product of determinants of $\mathcal{M}_{F,e}^F$ for $e \in E$ and $F \in \mathcal{L}$ for which $\mathcal{T}^{F,e} \neq \emptyset$. Then Lemma 5.5 completes the proof. \[\square\]

As an immediate consequence of the proof we can give a refined version of Theorem 1.1 which also implies that $\det(\mathfrak{B})$ only depends on the matroid $\operatorname{Mat}(\mathcal{L})$ underlying the oriented matroid $\mathcal{L}$.

Let us first state an additional fact about an oriented matroid $\mathcal{L}$ over ground set $E$ and its underlying matroid $\operatorname{Mat}(\mathcal{L})$. Recall, that by [3, Theorem 4.6.5] the number of topes of $\mathcal{L}$ for which a fixed $e \in E$ does not define a proper face $e \in E$ is independent of $e$ and coincides with the $\beta$-invariant $\beta(\operatorname{Mat}(\mathcal{L}))$ of $\operatorname{Mat}(\mathcal{L})$.

**Corollary 5.7.** Let $\mathfrak{B}$ be the Varchenko matrix of the oriented matroid with covector set $\mathcal{L}$ and $M = \operatorname{Mat}(\mathcal{L})$ the matroid underlying $\mathcal{L}$. Then

$$
\det(\mathfrak{B}) = \prod_{A \subseteq E} \left(1 - \prod_{e \in A} U_e^2 \right)^{m_A},
$$

where $m_A$ is the product of number of topes in the contraction $\mathcal{L}/A$ and one half of $\beta(\operatorname{Mat}(\mathcal{L}|_A))$. In particular, it follows that $\det(\mathfrak{B})$ only depends on the matroid $M = \operatorname{Mat}(\mathcal{L})$.

**Proof.** Fix $A \subseteq E$. Using the notation of Theorem 1.1 it follows that $m_A = \sum_{F \in \mathcal{L}} b_F$.

The number of summands equals the number of topes in the contraction $\mathcal{L}/A$ of $A$. The latter only depends on $\operatorname{Mat}(\mathcal{L}/A)$ which only depends on $\operatorname{Mat}(\mathcal{L})$. By Lemma 5.5 we have $b_F = b_{F,e}$ where $e$ is the maximal element of $z(F)$. Now $b_{F,e}$ is half the number of elements of $\mathcal{T}^{F,e}$. The latter is the set of topes $P$ for which $F$ is the maximal element of $\mathcal{L}$ such that $F \leq P$ and $F_e = 0$. The map sending $P$ to $P_{z(F)}$ is then a bijection between the topes in $\mathcal{T}^{F,e}$ and the topes of $\mathcal{L}|_{z(F)}$ for which $e$ does not define a proper face. As mentioned above, by [3, Theorem 4.6.5] the number of topes in $\mathcal{L}|_{z(F)}$ for which $e \in z(F)$ does not define a proper face in is independent of $e$ and coincides with the beta invariant $\beta(\operatorname{Mat}(\mathcal{L}|_{z(F)}))$. Hence we find that $b_{F,e} = \frac{1}{2} \beta(\operatorname{Mat}(\mathcal{L}|_{z(F)}))$. Since for a nonempty subset the matroid $\operatorname{Mat}(\mathcal{L}|_A)$ depends on $A$ and $\operatorname{Mat}(\mathcal{L})$ only it follows that $m_A$ is an invariant of $\operatorname{Mat}(\mathcal{L})$. \[\square\]
Finally, as a second corollary we extend Theorem 1.1 to row and column selected submatrices of $\mathfrak{V}$ corresponding to topes in a cone. For oriented matroids coming from hyperplane arrangements this formula can also be found in [1] and [6].

**Corollary 5.8.** Let $\mathfrak{V}$ be the Varchenko matrix of the oriented matroid with covector set $\mathcal{L}$. For a subset $E' \subseteq E$ and signs $\epsilon = (\epsilon_e)_{e \in E'} \in \{+,-\}^{E'}$ such that $T(\epsilon^+, \epsilon^-)$ is a cone let $\mathfrak{V}_\epsilon$ be the matrix constructed from $\mathfrak{V}$ by selecting all rows and columns corresponding to topes $P \in T$ for which $P_e = \epsilon_e$ for $e \in E'$.

Then

$$\det(\mathfrak{V}_\epsilon) = \prod_{F \in \mathcal{L}} (1 - a(F))^2 b_{F,\epsilon},$$

for some numbers $b_{F,\epsilon}$.

**Proof.** Consider the case $E' = \{\epsilon\}$. Order $T = \{P_1 \prec \cdots \prec P_{2s}\}$ such that $(P_1)_\epsilon = \cdots = (P_s)_\epsilon = +$ and $P_{i+s} = -P_i$ for $1 \leq i \leq s$. Then $\mathfrak{V}_\epsilon$ is a block diagonal matrix with two blocks identical to $\mathfrak{V}_\epsilon$ on the main diagonal. By [Theorem 1.1] we have

$$\det(\mathfrak{V}_{\epsilon=0}) = \prod_{F \in \mathcal{L}, \emptyset \neq F} (1 - a(F))^{b_F} = \det(\mathfrak{V}_\epsilon)^2.$$ 

Since the main diagonal in $\mathfrak{V}_\epsilon$ is constant 1 and since this are the only constant entries it follows that $\det(\mathfrak{V}_\epsilon)$ has constant term +1. It follows that

$$\det(\mathfrak{V}_\epsilon) = \prod_{F \in \mathcal{L}} (1 - a(F))^{b_F}. $$

Now induction on the cardinality of $E'$ proves the assertion. \qed

**Remark 5.9.** If $E' = \{\epsilon\}$ in Corollary 5.8, i.e. in the case of an affine oriented matroid, then [3, Theorem 4.6.5] implies, that $\det(\mathfrak{V}_\epsilon)$ is still a matroid invariant.

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