ON INVARIANT MEASURES ASSOCIATED TO WEAKLY COUPLED SYSTEMS OF KOLMOGOROV EQUATIONS

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Abstract. In this paper, we deal with weakly coupled elliptic systems \( A \) with unbounded coefficients. We prove the existence and characterize all the systems of invariant measures for the semigroup \( (T(t))_{t \geq 0} \) associated to \( A \) in \( C_b(\mathbb{R}^d; \mathbb{R}^m) \). We also show some relevant properties of the extension of \( (T(t))_{t \geq 0} \) to the \( L^p \)-spaces related to systems of invariant measures. Finally, we study the asymptotic behaviour of \( (T(t))_{t \geq 0} \) as \( t \) tends to \( +\infty \).

1. Introduction

In the last two decades, partial differential equations with unbounded coefficients have attracted the attention of many researchers, for their remarkable applications in economy and finance and for their strong connection with the theory of stochastic differential equations. Such equations appear also in the analysis of the weighted \( \bar{\partial} \)-problem in \( C^d \), in the time-dependent Born-Openheimer theory and also in the study of Navier-Stokes equations. (We refer the interested reader to [2, 7, 9, 12, 13, 15, 16] for further details.) In particular, the Cauchy problems associated to second-order differential equations of elliptic and parabolic type have been widely studied in the classical setting of bounded and continuous functions and in \( L^p \)-spaces, related to the Lebesgue measure and to the so-called invariant measures. The literature is nowadays rather rich in the case of a single equation (we refer the interested reader to [19] for further details). On the other hand, according to our knowledge less is known about the theory of systems (we refer the interested reader to [2, 5, 10, 14]) and, in particular, invariant measures for systems seem to have not been studied so far.

In this paper, we consider weakly coupled elliptic operators \( A \) defined on smooth functions \( \zeta : \mathbb{R}^d \to \mathbb{R}^m \), \( m \geq 2 \), by

\[
(A\zeta)(x) = \sum_{i,j=1}^{d} q_{ij}(x)D_{ij}\zeta(x) + \sum_{j=1}^{d} b_{j}(x)D_{j}\zeta(x) + C(x)\zeta(x) = \text{Tr}(Q(x)D^2\zeta(x)) + \langle b(x), \nabla \zeta(x) \rangle + C(x)\zeta(x). (1.1)
\]

The results in [2, 5, 10] show, that under mild assumptions on the coefficients \( q_{ij}, b_i : \mathbb{R}^d \to \mathbb{R}^m \) \( (i, j = 1, \ldots, d) \) and \( C : \mathbb{R}^d \to \mathbb{R}^{2m} \), and assuming the existence of a so-called Lyapunov function \( \varphi \) for the scalar operator \( \mathcal{A} = \sum_{i,j=1}^{d} q_{ij}D_{ij} + \sum_{i=1}^{d} b_i D_i \) (see Hypothesis 2.1(iv)), it is possible to associate a semigroup \( (T(t))_{t \geq 0} \) to \( \mathcal{A} \) in \( C_b(\mathbb{R}^d; \mathbb{R}^m) \), the space of bounded and continuous functions \( f : \mathbb{R}^d \to \mathbb{R}^m \).

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1
The semigroup is defined in the natural way: for any \( f \in C_b(\mathbb{R}^d, \mathbb{R}^m) \) and \( t > 0 \)
\( T(t)f \) is the value at \( t \) of the unique bounded classical solution of the Cauchy problem

\[
\begin{align*}
D_t u &= \mathcal{A}u, & \text{in} & \quad (0, +\infty) \times \mathbb{R}^d, \\
\quad u(0, \cdot) &= f, & \text{in} & \quad \mathbb{R}^d. 
\end{align*}
\]  

(1.2)

A variant of the classical maximum principle, based on the existence of the function \( \varphi \) can be used to show that, for any \( t > 0, x \in \mathbb{R}^d \) and \( p \in (1, +\infty) \)
\[
|\langle T(t)f \rangle(x)|^p \leq \langle T(t)|f|^p \rangle(x), \quad f \in C_b(\mathbb{R}^d; \mathbb{R}^m),
\]  

(1.3)

(see [5, Proposition 2.8]).

Differently from the case of bounded and continuous coefficients, the analysis of Markov semigroups on \( L^p \)-spaces is much more difficult. In [5] a class of nonautonomous parabolic first-order coupled systems has been considered in the Lebesgue space \( L^p(\mathbb{R}^d; \mathbb{R}^m) \), \( p \in [1, +\infty) \). Sufficient conditions, consisting of quite strong growth assumptions on the coefficients of the elliptic operator \( \mathcal{A} \), have been supplied to guarantee that the associated evolution operator extends to \( L^p(\mathbb{R}^d; \mathbb{R}^m) \). Such growth assumptions are not merely technical conditions. Indeed, already in the scalar case, the Cauchy problem (1.2) may be not well posed in the usual \( L^p \)-spaces if the coefficients of the elliptic operator \( \mathcal{A} \) are unbounded, unless they satisfy rather restrictive growth assumptions. The scalar case also shows that a way to deal with \( L^p \)-spaces, under reasonable assumptions on the coefficients of the elliptic operator, is to replace the Lebesgue measure by another measure, possibly absolutely continuous with respect to the Lebesgue one. The best situation in the scalar case is when an invariant measure \( \mu \) exists, which is a Borel probability measure such that

\[
\int_{\mathbb{R}^d} T(t)f \, d\mu = \int_{\mathbb{R}^d} f \, d\mu, \quad t > 0, \quad f \in C_b(\mathbb{R}^d),
\]

where \( (T(t))_{t \geq 0} \) is the Markov semigroup naturally associated to the elliptic operator \( \mathcal{A} \) in \( C_b(\mathbb{R}^d) \). Under quite mild assumptions, a unique invariant measure exists, it is equivalent to the Lebesgue measure and is related to the asymptotic behaviour of \( T(t) \), since

\[
\lim_{t \to +\infty} (T(t)f)(x) = \int_{\mathbb{R}^d} f \, d\mu, \quad f \in C_b(\mathbb{R}^d), \quad x \in \mathbb{R}^d.
\]

Moreover, the operators \( T(t) \) may easily be extended to contractions in \( L^p(\mathbb{R}^d) \), the \( L^p \)-space associated with the measure \( \mu \), for every \( p \in [1, +\infty) \).

In this paper we give a consistent definition of invariant measures for the semigroup \( (T(t))_{t \geq 0} \) in \( C_b(\mathbb{R}^d; \mathbb{R}^m) \), providing sufficient conditions for the existence of such measures and proving that, as in the scalar case, the vector valued semigroup \( (T(t))_{t \geq 0} \) enjoys good properties in the \( L^p \)-spaces related to these measures. It seems quite natural to expect that the single measure \( \mu \) associated to a single equation in the scalar case is replaced by an \( m \)-dimensional vector of measures associated to the \( m \) equations of the system. We call system of invariant measures for the semigroup \( (T(t))_{t \geq 0} \), a family of positive and finite Borel measures \( \{ \mu_i : i = 1, \ldots, m \} \) over \( \mathbb{R}^d \) satisfying

\[
\sum_{i=1}^m \int_{\mathbb{R}^d} (T(t)f)_i \, d\mu_i = \sum_{i=1}^m \int_{\mathbb{R}^d} f_i \, d\mu_i, \quad f \in C_b(\mathbb{R}^d; \mathbb{R}^m).
\]

We assume that the off-diagonal entries of the matrix valued function \( C \) are nonnegative functions (see Hypothesis 2.1(v)). This additional assumption, in particular, implies that the semigroup \( (T(t))_{t \geq 0} \) is nonnegative in the sense that, if the entries of the function \( f \) are all nonnegative, then \( T(t)f \) has nonnegative components as
well, for any $t > 0$. The componentwise positiveness of the semigroup $(T(t))_{t \geq 0}$ is essential in our analysis to prove the existence of a system of invariant measures. This is the reason why we confine ourselves to weakly coupled elliptic operators $A$.

About existence and uniqueness of systems of invariant measures, Theorem 3.5 shows that, under reasonable assumptions, there exists a unique (up to a multiplicative constant) system of invariant measures for the semigroup $(T(t))_{t \geq 0}$. More precisely, if $\{\mu_j : j = 1, \ldots, m\}$ is a system of invariant measures for $(T(t))_{t \geq 0}$, then there exists a positive constant $c$ such that $\mu_j = c\xi_j \mu$ for any $j = 1, \ldots, m$, where $\mu$ is the invariant measure associated to the scalar semigroup $(T(t))_{t \geq 0}$ and $\xi = (\xi_1, \ldots, \xi_m)$ is a not trivial constant vector which belongs to $\cap_{x \in \mathbb{R}} \text{Ker}(C(x))$. This crucial assumption together with the non positivity of the quadratic form associated to $C$ (see Hypothesis 2.1(iii)) are inspired by the scalar case where the existence (and consequently the uniqueness) of an invariant measure is guaranteed when the potential term of the elliptic operator identically vanishes on $\mathbb{R}^d$. See Remark 3.7 for further details.

Formula (1.3) yields immediately that $(T(t))_{t \geq 0}$ extends to a strongly continuous semigroup in $L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m) = \bigotimes_{i=1}^m L^p_{\mu_i}(\mathbb{R}^d)$ for any $p \in [1, +\infty)$. Under additional growth assumptions on the coefficients of $A$, we prove some pointwise estimate for the first- and second-order spatial derivatives of $(T(t)f)$. More precisely, we show that

$$
|D^k \frac{d}{dt} T(t)f|^p \leq \Gamma_{p,k,h}(t) T(t) \left( \sum_{j=0}^H |D^j f|^2 \right)^{\frac{p}{2}}
$$

in $\mathbb{R}^d$ for any $t > 0$, $f \in C^k_b(\mathbb{R}^d, \mathbb{R}^m)$, $p > 1$, $k \in \{1, 2\}$ and $h \in \{0, \ldots, k\}$, where $\Gamma_{p,k,h}$ is a positive function defined in $(0, +\infty)$, whose behaviour as $t$ tends to $0^+$ is sharp. Clearly, in this case each $T(t)$ is a bounded operator from $L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m)$ into $W^{2,p}_{\mu}(\mathbb{R}^d; \mathbb{R}^m)$ (the set of all functions $f \in L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m)$ whose distributional derivatives up to the second-order are in $L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m)$) for any $p \in (1, +\infty)$. Estimate (1.4) with $k = 1$ is useful also to provide a partial characterization of the domain $D(A_p)$ of the infinitesimal generator $A_p$ of the semigroup $(T(t))_{t \geq 0}$ in $L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m)$, since it allows us to prove that $D(A_p) \subset W^{1,p}_{\mu}(\mathbb{R}^d; \mathbb{R}^m)$. The complete characterization of $D(A_p)$ is out of the scope of this paper and, as the scalar case shows, it is known only in some particular cases.

Finally, we relate the system of invariant measures to the asymptotic behaviour of the function $T(t)f$ as $t \to +\infty$. More precisely, we assume that $|q_{ij}(x)| \leq c|x|^i \varphi(x)$ ($i, j = 1, \ldots, d$) and $|b_i(x, x) - c|x|^i \varphi(x)| \leq c|x|^i \varphi(x)$ as $|x| \to +\infty$, for some positive constant $c$, and show that $T(t)f$ converges to $M_{t\xi} := \left( \sum_{j=1}^m \int_{\mathbb{R}^d} f_j \mu_j \right) \xi$, locally uniformly in $\mathbb{R}^d$ as $t \to +\infty$. As a byproduct, we deduce that, if $f \in L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m)$, then the function $T(t)f$ converges to $M_{t\xi}$ also in $L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m)$ as $t \to +\infty$.

The plan of the paper is the following. First in Section 2 we introduce some known results on equations and systems of elliptic operators and prove some basic facts which are crucial in all our analysis. Section 3 is devoted to the systems of invariant measures, the analysis of the semigroup $(T(t))_{t \geq 0}$ in $L^p$-spaces associated with systems of invariant measures and pointwise estimates for the spatial derivatives up to the second-order of the function $T(t)f$. Finally, in Section 4 we study the long time behaviour of the function $T(t)f$ when it is bounded and Borel measurable and when it belongs to $L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m)$ for some $p \in [1, +\infty)$.

**Notation.** Functions with values in $\mathbb{R}^m$ are displayed in bold style. Given a function $f$ (resp. a sequence $(f_n)$) as above, we denote by $f_i$ (resp. $f_{n,i}$) its $i$-th
component (resp. the $i$-th component of the function $f_i$). By $B_0(\mathbb{R}^d, \mathbb{R}^m)$ we denote the set of all the bounded Borel measurable functions $f : \mathbb{R}^d \to \mathbb{R}^m$, where \( \|f\|_\infty = \sum_{i=1}^m \sup_{x \in \mathbb{R}^d} |f_i(x)|^2 \). For any $k \geq 0$, $C^k_0(\mathbb{R}^d, \mathbb{R}^m)$ is the space of all the functions whose components belong to $C^k_0(\mathbb{R}^d)$, where “$0$” stays for bounded. Similarly, we use the subscript “$c$” and “$0$” for spaces of functions with compact support and spaces of functions vanishing at infinity, respectively. When $k \in (0, 1)$ we use the subscript “$\text{loc}$” to denote the space of all $f \in C(\mathbb{R}^d)$ which are Hölder continuous in any compact set of $\mathbb{R}^d$. We assume that the reader is familiar with the parabolic spaces $C^{\beta+\alpha/2,k+\alpha}(I \times \mathbb{R}^d)$ ($\alpha \in [0, 1)$, $h, k \in \mathbb{N} \cup \{0\}$), and we use the subscript “$\text{loc}$” with the same meaning as above. The symbols $D_t f$, $D_i f$ and $D_{ij} f$, respectively, denote the time derivative $\frac{\partial f}{\partial t}$ and the spatial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ for any $i, j = 1, \ldots, d$. For any $k \in \mathbb{N}$, we write $|D^k_x u|^2$ to denote the sum $\sum_{j=1}^m |D^k_x u_j|^2$. If $k = 1$ we write $D_x u$ and $J_x u$ indifferently for the Jacobian matrix of $u$ with respect to the spatial variables.

By $e_j$ and $I$ we denote, respectively, the $j$-th vector of the Euclidean basis of $\mathbb{R}^m$ and the function identically equal to 1 in $\mathbb{R}^d$. The open ball in $\mathbb{R}^d$ centered at 0 with radius $r > 0$ and its closure are denoted by $B_r$ and $\overline{B}_r$, respectively.

2. Hypotheses and preliminary results

Throughout the paper, if not otherwise specified, we assume the following assumptions on the coefficients of the operator $A$ in (1.1).

**Hypotheses 2.1.**  
(i) The coefficients $q_{ij} = q_{ji}$, $b_i$ and the entries $c_{hk}$ of the nonidentically vanishing matrix valued function $C$ belong to $C^\alpha_{\text{loc}}(\mathbb{R}^d)$ for some $\alpha \in (0, 1)$;

(ii) the infimum $\mu_0$ over $\mathbb{R}^d$ of the minimum eigenvalue $\mu_Q(x)$ of the matrix $Q(x) = (q_{ij}(x))$ is positive;

(iii) $\langle C(x)y, y \rangle \leq 0$ for any $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^m$;

(iv) there exists a positive function $\varphi \in C^2(\mathbb{R}^d)$, blowing up as $|x| \to +\infty$ such that $A_2(x) \leq a - c_2(x)$ for any $x \in \mathbb{R}^d$ and some positive constants $a, c$, where $A = \text{Tr}(QD^2) + \langle b, \nabla \rangle$;

(v) the off-diagonal entries of the matrix valued function $C$ are nonnegative;

(vi) there exists $\not= \xi \in \mathbb{R}^m$ such that $\xi \in \text{Ker}(C(x))$ for any $x \in \mathbb{R}^d$;

(vii) there does not exist a nontrivial set $K \subseteq \{1, \ldots, m\}$ such that the coefficients $c_{ij}$ identically vanish on $\mathbb{R}^d$ for any $i \in K$ and $j \not\in K$.

In the following Lemma 2.2, Theorem 2.6 and Proposition 2.7 we collect some basic consequences of the previous assumptions.

**Lemma 2.2.** Let Hypotheses 2.1(iii), (vi) be satisfied. Then, the set equality $\text{Ker}(C(x)) = \text{Ker}((C(x))^*)$ holds true for any $x \in \mathbb{R}^d$. If, in addition, Hypothesis 2.1(v) is satisfied, then for any $x \in \mathbb{R}^d$ the spectrum of the matrix $C(x)$ is contained in the left-halfplane and 0 is the unique eigenvalue on the imaginary axis.

**Proof.** Fix $x \in \mathbb{R}^d$. Since 0 is an eigenvalue of $C(x)$, it is also an eigenvalue of the adjoint matrix $(C(x))^*$. Let $\xi_0$ be any vector such that $(C(x))^*\xi_0 = 0$ and, by contradiction, let us assume that $\eta := C(x)\xi_0 \neq 0$. Let us fix $\beta > 0$ and observe that

\[
\langle C(x)(\beta\xi_0 + \eta), \beta\xi_0 + \eta \rangle = \beta^2 \langle C(x)\xi_0, \xi_0 \rangle + \beta \langle C(x)\xi_0, \eta \rangle + \beta \langle C(x)\eta, \xi_0 \rangle + \langle C(x)\eta, \eta \rangle = \beta^2 \langle \xi_0, (C(x))^* \xi_0 \rangle + \beta \langle \eta, (C(x))^* \xi_0 \rangle + \langle C(x)\eta, \eta \rangle = \beta |\eta|^2 + \langle C(x)\eta, \eta \rangle.
\]
It is clear that we can fix \( \beta > 0 \) such that \( \langle C(x)(\beta \xi_0 + \eta), \beta \xi_0 + \eta \rangle > 0 \) getting to a contradiction. The inclusion \( \text{Ker}((C(x))^*) \subset \text{Ker}(C(x)) \) follows. Since \( \langle C(x)y, y \rangle = \langle (C(x))^*y, y \rangle \) for any \( y \in \mathbb{R}^m \), the same arguments above applied to \( (C(x))^* \) yield the other inclusion \( \text{Ker}(C(x)) \subset \text{Ker}((C(x))^*) \).

Let us complete the proof by checking the last statement. To begin with, we observe that \( C(x) \) has no eigenvalues \( \lambda \) with positive real part. This is clear if \( \lambda \) is real. Indeed, denoting by \( \eta \) a corresponding unit eigenvector, we would get \( 0 < \lambda = \langle C(x)\eta, \eta \rangle \) contradicting Hypothesis 2.1(iii). If \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) and \( \eta \) is as above, then \( \eta = \eta_1 + i\eta_2 \) for some \( \eta_1, \eta_2 \in \mathbb{R} \). It is immediate to check that \( 0 < \text{Re}\lambda = \langle C(x)\eta_1, \eta_1 \rangle + \langle C(x)\eta_2, \eta_2 \rangle \), again contradicting Hypothesis 2.1(iii).

To prove that \( 0 \) is the unique eigenvalue of \( C(x) \) on the imaginary axis, we fix \( \lambda \) sufficiently large such that \( \lambda + \mu > 0 \) for any real eigenvalue of \( C(x) \) and \( \lambda + c_i(x) > 0 \) for any \( i = 1, \ldots, m \). With this choice all the elements of the matrix \( C_\lambda(x) := C(x) + \lambda \mathbf{I} \) are nonnegative. Moreover, since \( \sigma(C_\lambda(x)) = \sigma(C(x)) + \lambda \), all the real eigenvalues of \( C_\lambda(x) \) are positive and \( \lambda \) is the greatest one. A generalization of Perron-Frobenius Theorem (see [23, Theorem 2.7]) implies that the spectral radius of \( C_\lambda(x) \) (i.e., the maximum of the moduli of the eigenvalues of \( C_\lambda(x) \)) belongs to \( \sigma(C_\lambda(x)) \). From this and the above remarks it follows that \( \lambda \) is the maximum of the eigenvalues of \( C_\lambda(x) \). Coming back to \( C(x) \), we conclude that this matrix has not nontrivial eigenvalues on the imaginary axis and we are done.

\[ \square \]

**Remark 2.3.** We stress that our assumptions on \( C \) in general do not imply that \( C(x) \) is symmetric for some \( x \in \mathbb{R}^d \). Indeed, it is immediate to check that the matrix valued function \( C \) defined by

\[
C(x) = \begin{pmatrix}
-\zeta_1(x) - \zeta_2(x) & \zeta_1(x) & \zeta_2(x) \\
\zeta_2(x) & -\zeta_1(x) - \zeta_2(x) & \zeta_3(x) \\
\zeta_1(x) & \zeta_3(x) & -\zeta_1(x) - \zeta_2(x) - \zeta_3(x)
\end{pmatrix}
\]

for any \( x \in \mathbb{R}^d \), satisfies Hypotheses 2.1(iii), (v)-(vii) for any triplet of positive locally Hölder continuous functions \( \zeta_1, \zeta_2, \zeta_3 : \mathbb{R}^d \to \mathbb{R} \).

Under Hypotheses 2.1(i)-(iv), for any \( f \in C_\lambda(\mathbb{R}^d; \mathbb{R}^m) \) the Cauchy problem

\[
\begin{cases}
D_t u(t, x) = A u(t, x), & t \in (0, +\infty), \quad x \in \mathbb{R}^d, \\
u(0, x) = f(x), & x \in \mathbb{R}^d
\end{cases}
\]

admits a unique classical solution \( u \in C^{1,2}((0, +\infty) \times \mathbb{R}^d; \mathbb{R}^m) \cap C_0((0, +\infty) \times \mathbb{R}^d; \mathbb{R}^m) \). Function \( u \) satisfies the estimate \( \|u\|_{\infty} \leq \|f\|_{\infty} \) and can be obtained equivalently as the limit in \( C^{1,2}_{\text{loc}}((0, +\infty) \times \mathbb{R}^d; \mathbb{R}^m) \)

(i) of the sequence \( (u_n) \) of classical solutions to the Cauchy-Dirichlet problem

\[
\begin{cases}
D_t u_n(t, x) = A u_n(t, x), & t \in (0, +\infty), \quad x \in B_n, \\
u_n(t, x) = 0, & t \in (0, +\infty), \quad x \in \partial B_n, \\
u_n(0, x) = f(x), & x \in B_n;
\end{cases}
\]

(ii) of the sequence \( (v_n) \) of classical solutions to the Cauchy-Neumann problem

\[
\begin{cases}
D_t v_n(t, x) = A v_n(t, x), & t \in (0, +\infty), \quad x \in B_n, \\
\frac{\partial v_n}{\partial \nu}(t, x) = 0, & t \in (0, +\infty), \quad x \in \partial B_n, \\
v_n(0, x) = f(x), & x \in B_n,
\end{cases}
\]

where \( \nu \) denotes the unit exterior normal vector to \( \partial B_n \).

We refer the reader to [2, 5, 10] for more details.

The above result allowed the authors of [10] to associate a semigroup \( (T(t))_{t \geq 0} \) (in the sequel simply denoted by \( T(t) \)) of bounded operators in \( C_0(\mathbb{R}^d; \mathbb{R}^m) \) with
the operator $\mathcal{A}$ in (1.1): for any $f \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and $t > 0$, $T(t)f$ is the value at $t$ of the unique bounded classical solution to problem (2.1). In [2, Theorem 3.2], actually in a greater generality, it has been proved that the semigroup $T(t)$ admits an integral representation formula in terms of some finite Borel measures. More precisely,

$$(T(t)f)_i(x) = \sum_{j=1}^m \int_{\mathbb{R}^d} f_j(y)p_{ij}(t, x, dy), \quad f \in C_b(\mathbb{R}^d; \mathbb{R}^m). \quad (2.4)$$

The measures $p_{ij}(t, x, dy)$ are absolutely continuous with respect to the Lebesgue measure but, in general, differently from the scalar case, they are signed measures. Through formula (2.4) the semigroup $T(t)$ can be extended to $B_b(\mathbb{R}^d; \mathbb{R}^m)$, with a strong Feller semigroup (i.e., $T(t)f$ belongs to $C_b(\mathbb{R}^d)$ for any $t > 0$ and $f \in B_b(\mathbb{R}^d; \mathbb{R}^m)$; actually, $T(t)f$ belongs to $C^2(\mathbb{R}^d)$ as a consequence of interior Schauder estimates). Moreover,

$$|T(t)f|^p \leq T(t)|f|^p, \quad t > 0, \quad p > 1, \quad f \in C_b(\mathbb{R}^d; \mathbb{R}^m), \quad (2.5)$$

where $T(t)$ is the semigroup of contractions associated in $C_0(\mathbb{R}^d)$ to the operator $\mathcal{A}$ (see Hypothesis 2.1(iv)). More precisely (see [19, Chapters 1 & 9]),

**Theorem 2.4.** Under Hypothesis 2.1(i), (iii) and (iv) there exists a Markov contraction semigroup $T(t)$ associated to $\mathcal{A} = \text{Tr}(QD^2) + \langle b, \nabla \rangle$ in $C_0(\mathbb{R}^d)$. For any $f \in C_0(\mathbb{R}^d)$, $T(t)f$ is the unique solution in $C_0([0, +\infty) \times \mathbb{R}^d) \cap C^{1,2}([0, +\infty) \times \mathbb{R}^d)$ to the differential equation $D_t u - Au = 0$, which satisfies the condition $u(0, \cdot) = f$. For any $f \in C_b(\mathbb{R}^d)$ it holds that

$$(T(t)f)(x) = \int_{\mathbb{R}^d} f(y)p(t, x, dy), \quad t > 0, \quad x \in \mathbb{R}^d, \quad (2.6)$$

where each $p(t, x, dy)$ is a Borel probability measure which admits a strictly positive density with respect to the Lebesgue measure. As a byproduct, if $f \geq 0$ does not identically vanishes in $\mathbb{R}^d$, then $T(\cdot)f$ is strictly positive in $[0, +\infty) \times \mathbb{R}^d$ and

$$|T(t)f|^p \leq T(t)|f|^p,$$

in $\mathbb{R}^d$ for any $t > 0$, $p \in (1, +\infty)$ and $f \in C_b(\mathbb{R}^d)$. Finally, there exists a unique invariant measure $\mu$ associated with the semigroup $T(t)$, i.e., there exists a unique Borel probability measure $\mu$ such that

$$\int_{\mathbb{R}^d} T(t)f \, d\mu = \int_{\mathbb{R}^d} f \, d\mu, \quad f \in C_b(\mathbb{R}^d), \quad t > 0.$$

**Remark 2.5.** Some of the results in Theorem 2.4 can be extended also to the case of elliptic operators with a nontrivial potential term. More precisely, if $A_c = \text{Tr}(QD^2) + \langle b, \nabla \rangle + c$, where the diffusion and drift coefficients $q_{ij}$ and $b_j$ ($j = 1, \ldots, d$) satisfy Hypotheses 2.1 and $c \in C^0_{\text{loc}}(\mathbb{R}^d)$ is bounded from above, then the Cauchy problem

$$\begin{cases}
D_t u(t, x) = A_c u(t, x), & t \in [0, +\infty), \quad x \in \mathbb{R}^d, \\
u(0, x) = f(x), & x \in \mathbb{R}^d,
\end{cases} \quad (2.7)$$

admits (at least) one solution $u \in C^{1+\alpha/2,2+\alpha}_{\text{loc}}([0, +\infty) \times \mathbb{R}^d) \cap C([0, +\infty) \times \mathbb{R}^d)$ which satisfies the estimate $\|u(t, \cdot)\|_{\infty} \leq e^{c_0t} \|f\|_{\infty}$ for any $t > 0$, where $c_0$ is the supremum over $\mathbb{R}^d$ of the function $c$. Uniqueness may fail, but in any case, if $f \geq 0$, the above Cauchy problem admits a minimal solution $u$, in the sense that, if $v$ is any other solution, then $v(t, x) \geq u(t, x)$ for any $(t, x) \in [0, +\infty) \times \mathbb{R}^d$. This allows to associate a semigroup of bounded operators in $C_0(\mathbb{R}^d)$ with the operator $A_c$: for any $f \in C_b(\mathbb{R}^d)$, $T(t)f = u_+ - u_-$, where $u_+$ and $u_-$ are the minimum nonnegative solutions to the Cauchy problem (2.7) with $f$ being replaced, respectively, by the
positive and negative part of \( f \). Moreover, if \( f \geq 0 \) does not identically vanish on \( \mathbb{R}^d \), then \( T(t) \) is strictly positive on \( (0, +\infty) \times \mathbb{R}^d \). See [4] for further details.

In view of (2.5) and Theorem 2.4, we conclude that \( T(t) \) is a contraction semigroup in \( C_b(\mathbb{R}^d; \mathbb{R}^m) \), i.e.,
\[
| (T(t)f)(x) | \leq \| f \|_{\infty}, \quad t > 0, \ x \in \mathbb{R}^d. \tag{2.8}
\]

Hypothesis 2.1(v) is the key tool to prove the positivity of the semigroup \( T(t) \).

In the proof of the following Proposition 2.8 we shall make use of the following interior Schauder estimates.

**Theorem 2.6** (Proposition A.1 & Theorem A.2 of [2]). Let Hypotheses 2.1(i), (ii) hold. Let further \( u \in C^{1+\alpha/2,\alpha}_c((0,T] \times \mathbb{R}^d; \mathbb{R}^m) \) satisfy the differential equation
\[
D_t u = Au + g \in (0,T] \times \mathbb{R}^d, \text{ for some } g \in C^{1+\alpha/2,\alpha}_c((0,T] \times \mathbb{R}^d; \mathbb{R}^m) \text{ and } T > 0.
\]
Then, for any \( T \in (0,T) \) and any pair of bounded open sets \( \Omega_1 \) and \( \Omega_2 \) such that \( \Omega_1 \) is compactly supported in \( \Omega_2 \), there exists a positive constant \( K \), depending on \( \Omega_1, \Omega_2, \tau, T \), but being independent of \( u \), such that
\[
\| u \|_{C^{1+\alpha/2,\alpha}((\tau, T] \times \Omega_1; \mathbb{R}^m)} \leq K_1(\| u \|_{C^{1+\alpha/2,\alpha}(\tau/2, T] \times \Omega_2; \mathbb{R}^m)} + \| g \|_{C^{1+\alpha/2,\alpha}(\tau/2, T] \times \Omega_2; \mathbb{R}^m)}).
\]
Further, if \( g \in C^{1+\alpha/2,\alpha}_c((0,T] \times \mathbb{R}^d; \mathbb{R}^m) \) and \( u \in C^{1+\alpha/2,\alpha}_c((0,T] \times \mathbb{R}^d; \mathbb{R}^m) \cap C([0,T] \times \mathbb{R}^d), then, for any \( 0 < r_1 < r_2 \) there exists a positive constant \( K_2 \), depending on \( r_1, r_2 \) and \( T \) but being independent of \( u \), such that
\[
t\| D_t^2 u(t, \cdot) \|_{C([\tau, T] \times \mathbb{R}^m)} \leq K_2(\| u \|_{C([0,T] \times \mathbb{R}^m)} + \| g \|_{C^{1+\alpha/2,\alpha}(0,T] \times \mathbb{R}^m)}).
\]
for any \( t \in (0, T] \).

Throughout the paper we shall make also use of the following local (in space) compactness property of the semigroup \( T(t) \) in \( C_b(\mathbb{R}^d; \mathbb{R}^m) \).

**Proposition 2.7.** Under Hypotheses 2.1(i)-(iv), for any bounded sequence \( (f_n) \subset C_b(\mathbb{R}^d; \mathbb{R}^m) \) and for any \( t_0 > 0 \), there exists a subsequence \( (f_{n_k}) \) such that \( T(t_0)f_{n_k} \) converges uniformly in \( (t_0, +\infty) \times B_r \) for every \( r > 0 \).

**Proof.** To begin with, we observe that the Schauder estimates in Theorem 2.6 show that the sequence \( (T(t_0)f_n) \) is bounded in \( C^{1+\alpha}(B_r; \mathbb{R}^m) \) for every \( r > 0 \). Hence, by Arzelà-Ascoli theorem and a compactness argument, we can easily prove that there exists a subsequence \( (T(t_0)f_{n_k}) \) which converges locally uniformly in \( \mathbb{R}^d \) to a function \( g \in C_b(\mathbb{R}^d; \mathbb{R}^m) \).

Next, we observe that the arguments in the proof of [17, Lemma 5.3] show that \( T(t) \leq \varphi + c^{-1} a \) in \( \mathbb{R}^d \) for every \( t > 0 \), where \( \varphi \) is the function in Hypothesis 2.1(iv). From this estimate we easily deduce that \( \sup_{t>0} p(t,x,\mathbb{R}^d \setminus B_r) \leq 0 \) locally uniformly with respect to \( x \), as \( r \) tends to \( +\infty \). Indeed,
\[
p(t,x,\mathbb{R}^d \setminus B_r) = \int_{\mathbb{R}^d \setminus B_r} p(t,x,dy) \leq \frac{1}{\inf_{B_r \setminus B_r} \varphi} \int_{\mathbb{R}^d} \varphi(y)p(t,x,dy)
\leq \frac{1}{\inf_{B_r \setminus B_r} \varphi} (T(t)\varphi)(x) \leq \frac{1}{\inf_{\mathbb{R}^d \setminus \mathbb{R}_r} \varphi} (\varphi(x) + c^{-1} a)
\]
and \( \varphi \) blows up as \( |x| \to +\infty \). As a byproduct, we can infer that, if \( (\psi_k) \subset C_b(\mathbb{R}^d) \) is a bounded sequence, converging locally uniformly on \( \mathbb{R}^d \) to some function \( \psi \), then
$T(\cdot)\psi_n$ converges to $T(\cdot)\psi$ uniformly in $[0, +\infty) \times B_R$ for every $R > 0$. Indeed, if $t > 0$, then we can estimate

$$|(T(t)(\psi_n - \psi))(x)| \leq \int_{B_r} |\psi_k(y) - \psi(y)|p(t, x, dy) + \int_{\mathbb{R}^d \setminus B_r} |\psi_k(y) - \psi(y)|p(t, x, dy)$$

$$\leq \|\psi_k - \psi\|c_{A(B_r)} + 2 \sup_{k \in \mathbb{N}} \|\psi_k\|_\infty \sup_{t > 0} \sup_{x \in B_R} p(t, x, \mathbb{R}^d \setminus B_r)$$

for every $r, R, t > 0$, $k \in \mathbb{N}$ and $x \in \mathbb{R}^d$. Letting $k$ tend to $+\infty$ we obtain that

$$\limsup_{k \to +\infty} \|T(t)(\psi_n - \psi)\|_{C^0((0, +\infty) \times B_R)} \leq 2 \sup_{k \in \mathbb{N}} \|\psi_k\|_\infty \sup_{t > 0} \sup_{x \in B_R} p(t, x, \mathbb{R}^d \setminus B_r)$$

for every $r, R > 0$. Finally, letting $r$ tend to $+\infty$, we conclude that

$$\limsup_{k \to +\infty} \|T(t)(\psi_n - \psi)\|_{C^0((0, +\infty) \times B_R)} \leq 0$$

and we are done.

Coming back to the sequence $(T(t)f_{n_k})$, we observe that, by (2.5), we can estimate $|T(t)f_{n_k} - T(t-t_0)g|^2 \leq T(t-t_0)|(T(t_0)f_{n_k} - g)|^2$ in $\mathbb{R}^d$ for every $t > t_0$ and $k \in \mathbb{N}$. The above result, with $\psi_k = |T(t_0)f_{n_k} - g|$, yields immediately the assertion.

The same arguments can be used to prove the last part of the assertion. □

**Proposition 2.8.** Under Hypotheses 2.1(i)-(v), the semigroup $T(t)$ is positive, in the sense that, if $f \in C_0(\mathbb{R}^d; \mathbb{R}^m)$ has all nonnegative components, then the function $T(t)f$ has nonnegative components as well, for any $t > 0$. In particular, if $f_0$ does not identically vanish in $\mathbb{R}^d$ for some $k = 1, \ldots, m$, then $(T(t)f)_k > 0$ in $\mathbb{R}^d$ for any $t > 0$. Consequently, the measures $p_{ij}(t, x, dy)$ ($i \neq j = 1, \ldots, m$) in (2.4) are nonnegative and the measures $p_{ii}(t, x, dy)$ ($i = 1, \ldots, m$) are positive for any $t > 0$ and $x \in \mathbb{R}^d$.

**Proof.** To prove the first statement, we take advantage of a result in [22], which deals with the positivity of the semigroup in the case of bounded coefficients. For this purpose, for any $n \in \mathbb{N}$, we introduce a smooth function $\psi_n : \mathbb{R} \to \mathbb{R}$ such that

$$\psi_n(x) = \begin{cases} x & |x| \leq n, \\ n + 1 & x \geq n + 1, \\ -n - 1 & x \leq -n - 1, \end{cases}$$

and set $\Psi_n(x) = (\psi_n(x_1), \ldots, \psi_n(x_d))$ for any $x \in \mathbb{R}^d$. Let $\mathcal{A}_n$ be the elliptic operator defined as the operator $\mathcal{A}$ in (1.1) with $q_{ij}, b_i$ and $C$ being replaced by $q_{ij,n} = q_{ij} \circ \Psi_n, b_{i,n} = b_i \circ \Psi_n$ and $C_n$ with entries $c_{hk,n} = c_{hk} \circ \Psi_n, (i, j = 1, \ldots, d, h, k = 1, \ldots, m)$. Clearly, $\left(\mathcal{A}_n(x)y, y\right)$ is nonpositive for any $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^m$; thus, again [10] shows that, for any $f \in C_0(\mathbb{R}^d; \mathbb{R}^m)$, the Cauchy problem (2.1) with $\mathcal{A}$ being replaced by $\mathcal{A}_n$ admits a unique classical solution $u_n$ which is bounded in $[0, +\infty) \times \mathbb{R}^d$. Denote by $T_n(t)$ the semigroup of contractions associated in $C_0(\mathbb{R}^d; \mathbb{R}^m)$ to the operator $\mathcal{A}_n$. Since the off-diagonal entries of $C_n$ are nonnegative, [22, Theorem 1.2] implies that, if all the components of $f$ are nonnegative (as we assume from now on), then the components of $u_n$ are all nonnegative as well. The interior Schauder estimates in Theorem 2.6, Arzelà-Ascoli theorem and a diagonal argument imply that there exists a subsequence $(u_{n_k})$ which converges to a function $v$ in $C^{1,2}(\mathbb{R}^d; \mathbb{R}^m)$ for any $0 < \varepsilon < T$ and any compact set $K \subset \mathbb{R}^d$. Clearly, $v \in C^{1+1/2,1+1/2}_\text{loc}(\mathbb{R}^d; \mathbb{R}^m)$, satisfies the differential equation in (2.1) and its components are all nonnegative in $[0, +\infty) \times \mathbb{R}^d$. We claim that $v$ can be extended by continuity to $\{0\} \times \mathbb{R}^d$, by setting $v(0, \cdot) = f$. For this purpose, we fix $R > 0$ and let $\vartheta$ be a smooth cut-off function such that $\chi_{B_{R-1}} \leq \vartheta \leq \chi_{B_R}$.
The function \( v_k := \partial u_{n_k} \) is bounded and continuous, and for \( n_k > R \) it solves the Cauchy problem

\[
\begin{align*}
D_t v_k &= A v_k + g_k, \quad \text{in } (0, T] \times B_R, \\
v_k(t, x) &= 0, \quad \text{in } (0, T) \times \partial B_R, \\
v_k(0, \cdot) &= \partial f, \quad \text{in } B_R,
\end{align*}
\]

where \( g_k = -\text{Tr}(QD^2)u_{n_k} - 2(J_\omega u_{n_k})Q\nabla \vartheta - u_{n_k}(b, \nabla \vartheta) \). Note that \( g_k \) is continuous in \((0, T) \times B_R\) and \( \sqrt{t}\|g_k(t, \cdot)\|_\infty \leq C\|f\|_\infty \) for every \( t \in (0, 1] \) and some positive constant \( C \) independent of \( k \) (see Proposition 2.6). Moreover, by the variation-of-constants-formula, we can write

\[
v_k(t, x) = (T_R(t)(\partial f))(x) + \int_0^t (T_R(t-s)g_k(s, \cdot))(x)ds, \quad t \in (0, T], \ x \in B_R,
\]

where \( T_R(t) \) is the semigroup generated by the realization of \( A \) in \( C_b(B_R; \mathbb{R}^m) \) with homogeneous Dirichlet boundary conditions. Since \( v_k \equiv u_{n_k} \) in \([0, +\infty) \times B_{R-1} \) we deduce that

\[
|u_{n_k}(t, x) - f(x)| \leq |(T_R(t)(\partial f))(x) - f(x)| + c\sqrt{t}\|f\|_\infty, \quad t \in (0, 1], \ x \in B_{R-1}.
\]

Letting first \( k \) tend to +\( \infty \) and, then, \( t \) tend to 0+ we conclude that \( v \) can be extended by continuity on \( [0, +\infty) \times B_{R-1} \) by setting \( v(0, \cdot) = f \). By the arbitrariness of \( R \) we conclude that \( v \) can be extended by continuity to \([0, +\infty) \times \mathbb{R}^d \) by setting \( v(0, \cdot) = f \) in \( \mathbb{R}^d \). Hence, \( v \) is a bounded classical solution to the Cauchy problem (2.1). By the uniqueness of the solution to this problem, it follows that \( v \equiv T(\cdot)f \). Thus, we conclude that all the components of \( T(t)f \) are nonnegative.

Let us now suppose that \( f_k \geq 0 \) does not identically vanish in \( \mathbb{R}^d \) for some \( k \). From Hypotheses 2.1(iv) and the first part of the proof it follows that \( D_1(T(\cdot)f)_k \geq A_k(T(\cdot)f)_k \) in \((0, +\infty) \times \mathbb{R}^d \), where \( A_k = \text{Tr}(QD^2) + (b, \nabla) + c_{kk} \). Since \( c_{kk} \leq 0 \), Hypothesis 2.1(iv) yields \( A_k \varphi \leq a - c\varphi \) in \( \mathbb{R}^d \). As a byproduct, a generalized version of the classical maximum principle implies that \( (T(\cdot)f)_k \geq S_k(\cdot)f_k \) in \((0, +\infty) \times \mathbb{R}^d \), where \( S_k(t) \) is the semigroup associated to the realization of \( A_k \) in \( C_b(\mathbb{R}^d) \) (see [19, Chapter 3]). By Remark 2.5, \( S_k(t)f_k \) is strictly positive in \( \mathbb{R}^d \) for any \( t > 0 \) so that \( (T(\cdot)f)_k \) is positive in \((0, +\infty) \times \mathbb{R}^d \).

Finally, let \( A \) be any Borel subset of \( \mathbb{R}^d \) and fix \( h \in \{1, \ldots, m\} \). Then, the function \( f = \chi_A \epsilon_h \) belongs to \( B_b(\mathbb{R}^d; \mathbb{R}^m) \) and there exists a bounded sequence \((f_n) \subset C_b(\mathbb{R}^d; \mathbb{R}^m) \) which converges to \( f \) pointwise almost everywhere with respect to the Lebesgue measure and, hence, with respect to each measure \( p_{ij}(t, x, dy) \) (\( i, j = 1, \ldots, m \)). Moreover, without loss of generality, we can assume that \( f_{n,i} \geq 0 \) for any \( n \in \mathbb{N} \) and \( i = 1, \ldots, m \). Combining the above facts we conclude that

\[
p_{ih}(t, x, A) = (T(t)f(x))_i = \lim_{n \to +\infty} (T(t)f_n(x))_i \geq 0
\]

for any \( t > 0, x \in \mathbb{R}^d, i = 1, \ldots, m \). Hence, the measure \( p_{ih}(t, x, dy) \) is nonnegative for any \( i = 1, \ldots, m \). Taking \( A = \mathbb{R}^d \), we also deduce that \( p_{ih}(t, x, \mathbb{R}^d) > 0 \) and the arbitrariness of \( h \) allows us to conclude.

\[\square\]

3. Systems of invariant measures

**Definition 3.1.** A family of positive and finite Borel measures \( \{\mu_i : i = 1, \ldots, m\} \) over \( \mathbb{R}^d \) is a system of invariant measures for the semigroup \( T(t) \), if for any \( f \in C_b(\mathbb{R}^d; \mathbb{R}^m) \) it holds that

\[
\sum_{i=1}^m \int_{\mathbb{R}^d} (T(t)f)_i d\mu_i = \sum_{i=1}^m \int_{\mathbb{R}^d} f_i d\mu_i.
\]

(3.1)
To begin with, we characterize the set of all the fixed point of the semigroup \( T(t) \), i.e., the set 
\[ E = \{ f \in C_b(\mathbb{R}^d; \mathbb{R}^m) : T(t)f \equiv f \text{ for any } t \geq 0 \}. \]

Hypothesis 2.1(iv) yields that \( E \) is not empty since it contains the function \( f_0 \equiv \xi \).

Hypothesis 2.1(vii) simply requires that \( m \) is the minimum coupling order in the sense that the system (2.1) does not contain any lower order system that decouples. Such assumption, which is not restrictive, allows us to prove some properties of \( E \) which yield to assume that \( \xi \) has all positive components and to deduce, as a consequence, that \( E \) is a one dimensional vector space spanned by the function \( f_0 \).

**Proposition 3.2.** \( E \) is a one-dimensional vector space of constant functions spanned by the vector \( \xi \).

**Proof.** We split the proof into three steps.

**Step 1.** Here, we prove that \( f \) belongs to \( E \) if and only if \( f \equiv \eta \) for some \( \eta \in \bigcap_{x \in \mathbb{R}^d} \text{Ker}(C(x)) \).

It is straightforward to check that, if \( f \equiv \eta \) for some \( \eta \in \bigcap_{x \in \mathbb{R}^d} \text{Ker}(C(x)) \), then \( f \) belongs to \( E \). Vice versa, let \( f \) belong to \( E \). From (2.5) it follows that \( |f|^2 = |T(t)f|^2 \leq T(t)|f|^2 \) in \( \mathbb{R}^d \) for any \( t \in (0, +\infty) \). The above inequality and the invariance property of \( \mu \), which yields
\[
\int_{\mathbb{R}^d} (|T(t)f|^2 - |f|^2) d\mu = 0, \quad t > 0,
\]
implies that, for any \( t > 0 \), \( T(t)|f|^2 = |f|^2 \mu \)-almost everywhere. Since \( \mu \) is equivalent to the Lebesgue measure and the functions \( |f|^2 \) and \( T(t)|f|^2 \) are continuous in \( \mathbb{R}^d \), we deduce that \( |T(t)f|^2 = T(t)|f|^2 = |f|^2 \) in \( \mathbb{R}^d \) for any \( t > 0 \). Based on this result, we can now prove that \( f \) is constant. To this aim, we observe that the equality \( T(\cdot)f = \eta \) implies that \( f \in C^2(\mathbb{R}^d; \mathbb{R}^m) \) and, consequently, \( |f|^2 \in C^2(\mathbb{R}^d) \).

Moreover, since \( T(\cdot)|f|^2 \) is independent of \( t \) and solves the equation \( D_t u = Au \) in \( (0, +\infty) \times \mathbb{R}^d \), it follows that \( 0 = AT(\cdot)|f|^2 = A|f|^2 = A|T(t)f|^2 \). Using this fact and Hypothesis 2.1(iii) we deduce that
\[
0 = D_t |f|^2 = D_t |T(\cdot)f|^2 = 2 \langle T(\cdot)f, D_t T(\cdot)f \rangle
\]
\[
= A|T(\cdot)f|^2 - 2 \sum_{i=1}^m |Q_x^i \nabla_x (T(\cdot)f)|^2 + 2 \langle C T(\cdot)f, T(\cdot)f \rangle
\]
\[
\leq -2 \sum_{i=1}^m |Q_x^i \nabla_x (T(\cdot)f)|^2.
\]
Thus, taking Hypothesis 2.1(ii) into account we immediately get \( J_t f = J_t T(t)f = 0 \) for any \( t > 0 \). Hence, \( f \equiv \eta \) for some \( \eta \in \mathbb{R}^m \). Clearly, \( C(x)\eta = 0 \) for any \( x \in \mathbb{R}^d \) so that \( \eta \in \bigcap_{x \in \mathbb{R}^d} \text{Ker}(C(x)) \).

**Step 2.** Here, we prove that, if \( \eta = (\eta_1, \ldots, \eta_m) \) belongs to \( \bigcap_{x \in \mathbb{R}^d} \text{Ker}(C(x)) \), then \( \tilde{\eta} := (|\eta_1|, \ldots, |\eta_m|) \) belongs to \( \bigcap_{x \in \mathbb{R}^d} \text{Ker}(C(x)) \) as well.

To this aim, let \( g \equiv \eta \), \( \tilde{g} \equiv \tilde{\eta} \) and assume that \( \eta \in \bigcap_{x \in \mathbb{R}^d} \text{Ker}(C(x)) \). By Step 1, \( g \) belongs to \( E \). Moreover, since \( T(\cdot) \) preserves positivity, \( |(T(\cdot)g_i)| \leq (T(\cdot)|\tilde{g}|) \), in \( [0, +\infty) \times \mathbb{R}^d \) for any \( j = 1, \ldots, m \). Hence, taking (2.5) into account, we get
\[
|\tilde{g}|^2 = |g|^2 = |T(t)|\tilde{g}|^2 \leq |T(t)\tilde{g}|^2 \leq T(t)|\tilde{g}|^2 = |\tilde{g}|^2, \quad t > 0,
\]
and, consequently, \( |T(t)|\tilde{g}|^2 = |\tilde{g}|^2 \). The same argument as in Step 1 implies that \( \tilde{g} \in E \).

**Step 3.** Now, we complete the proof. We claim that the entries of any vector \( \eta \in \mathbb{R}^m \setminus \{0\} \) belonging to \( \bigcap_{x \in \mathbb{R}^d} \text{Ker}(C(x)) \) are all positive or all negative. Fix any such vector \( \eta \). In view of Step 2, the vector \( \tilde{\eta} \) has all the components which
are nonnegative. By contradiction, assume that there exists \( K \subseteq \{1, \ldots, m\} \) with 
\( 1 \leq |K| < m \) such that \( \eta_i = 0 \) for any \( i \in K \). Since \( C(x)\hat{\eta} = 0 \) for any \( x \in \mathbb{R}^d \), it follows that 
\( \sum_{i \notin K} c_{ij}(x)\eta_j = 0 \) for any \( i = 1, \ldots, m \) and \( x \in \mathbb{R}^d \). In particular, 
choosing \( i \in K \) and recalling that the off-diagonal entries of the matrix \( C(x) \) are 
nonnegative for any \( x \in \mathbb{R}^d \), we conclude that \( c_{ij} \equiv 0 \) for any \( i \in K \) and \( j \notin K \) 
contradicting Hypothesis 2.1(vii). Now, we are almost done. Up to replacing \( \eta \) 
with \( -\eta \), we can assume that \( \eta_1 > 0 \). Then, all the other components are positive 
as well. Indeed, if this were not the case, the nontrivial vector \( \eta + \hat{\eta} \), which belongs 
to \( \bigcap_{x \in \mathbb{R}^d} \ker(C(x)) \), would have at least one trivial component, which can not be 
the case. It is straightforward to check that, if a subspace of \( \mathbb{R}^m \) consists of vectors 
whose entries are all positive or negative, then it is one-dimensional. \( \square \)

Remark 3.4. In view of Steps 1 and 2 in the proof of Proposition 3.2 in the rest 
of this paper, we assume that all the entries of the vector \( \xi \) are positive and \( |\xi| = 1 \).

Remark 3.5. In the particular case when the matrix \( C(x) \) is irreducible for some 
\( x \in \mathbb{R}^d \), the proof of Proposition 3.2 can be considerably simplified. Indeed, the 
Perron-Frobenius Theorem applied to the matrix \( C(x) + \lambda I \), where \( \lambda \) is any real 
number greater than the maximum of the moduli of the negative eigenvalues of 
\( C(x) \) and the moduli of the elements \( c_{ij}(x) \) \( (i = 1, \ldots, m) \), shows that the kernel 
of \( C(x) \) is one-dimensional and spanned by a vector \( \xi \) whose components are all positive.

However, we stress that our assumptions on \( C \) in general do not ensure that 
\( C(x) \) is irreducible for some \( x \in \mathbb{R}^d \). For instance, suppose that 
\[
C(x) = \begin{pmatrix} -f(x) - h(x) & f(x) & h(x) \\
f(x) & -f(x) - g(x) & g(x) \\
h(x) & g(x) & -g(x) - h(x) \end{pmatrix},
\]
for any \( x \in \mathbb{R}^d \) and some smooth, nonnegative and nontrivial functions \( f, g, h : \mathbb{R}^d \to \mathbb{R} \) 
compactly supported, respectively, in \( B_1, 3e_1 + B_1 \) and \( 6e_1 + B_1 \). It is easy to show that \( C \) satisfies Hypotheses 2.1(iii), (v)-(vii) but it is irreducible for 
no values of \( x \), since for any \( x \in \mathbb{R}^d \) at least one row of \( C(x) \) vanishes.

Now, we prove that our standing assumptions guarantee the existence of systems 
of invariant measures for \( T(t) \).

Theorem 3.5. There exist infinitely many systems of invariant measures for the 
semigroup \( T(t) \). More precisely, if \( \{\mu_j : j = 1, \ldots, m\} \) is a system of invariant 
measures for \( T(t) \), then there exists a positive constant \( c \) such that \( \mu_j = c\xi_j \mu \) for 
yany \( j = 1, \ldots, m \), where \( \mu \) is the invariant measure of the semigroup \( T(t) \).

In the proof of Theorem 3.5 we shall make use of the following result.

Proposition 3.6. Under Hypotheses 2.1(i)-(v), let \( \mathcal{R}_n \ (n \in \mathbb{N}) \) be the operator 
defined on \( C_b(\mathbb{R}^d; \mathbb{R}^m) \) by 
\[
\mathcal{R}_n f = \frac{1}{n} \sum_{k=0}^{n-1} T(k)f, \quad f \in C_b(\mathbb{R}^d; \mathbb{R}^m).
\]
Then, for any \( f \in C_b(\mathbb{R}^d; \mathbb{R}^m) \), \( \mathcal{R}_n f \) converges to \( Pf \) locally uniformly on \( \mathbb{R}^d \), as 
\( n \to +\infty \), where \( P \) is a projection onto the kernel of the operator \( I - T(1) \).

Proof. To begin with, we prove that, for any \( f \in C_b(\mathbb{R}^d; \mathbb{R}^m) \), there exists a 
subsequence \( (\mathcal{R}_{n_k}) \) converging locally uniformly to a function which belongs to 
\( \ker(I - T(1)) \). For this purpose, we fix any such function \( f \) and split 
\[
\mathcal{R}_n f = \frac{1}{n} f + T(1) \left( \frac{1}{n} \sum_{k=0}^{n-2} T(k)f \right) = \frac{1}{n} f + T(1) \left( \frac{n-1}{n} \mathcal{R}_{n-1} f \right).
\]
Since the sequence \((\mathcal{R}_n f)\) is bounded, by Proposition 2.7 there exists a subsequence \((\mathcal{R}_{n_k} f)\) which converges locally uniformly in \(\mathbb{R}^d\) to a function \(g \in C_0(\mathbb{R}^d; \mathbb{R}^m)\). Clearly, \(g\) belongs to the kernel of \(I - T(1)\). Indeed,

\[
\mathcal{R}_{n_k} f - T(1)\mathcal{R}_{n_k} f = \frac{1}{n_k} \sum_{j=0}^{n_k-1} (T(j)f - T(j+1)f) = \frac{1}{n_k} (f - T(n_k)f).
\]

Letting \(k\) tend to \(+\infty\) and taking (2.8) into account, we conclude that \(g - T(1)g = 0\).

Actually, we prove that all the sequence \((\mathcal{R}_n f)\) converges to \(g\) locally uniformly in \(\mathbb{R}^d\). For this purpose, we split \(f = g + (f - g)\). Since \(g \in \text{Ker}(I - T(1))\), \(\mathcal{R}_n g = g\) for any \(n \in \mathbb{N}\), so that, trivially, \(\mathcal{R}_n g\) converges uniformly in \(\mathbb{R}^d\). As far as the function \(f - g\) is concerned, we first observe that \(f - g\) is the local uniform limit in \(\mathbb{R}^d\) of a sequence of functions in \((I - T(1))(C_0(\mathbb{R}^d; \mathbb{R}^m))\). Indeed,

\[
f - g = f - \lim_{k \to +\infty} \mathcal{R}_{n_k} f = \lim_{k \to +\infty} \frac{1}{n_k} \sum_{j=1}^{n_k-1} (I - T(1))^j f = \lim_{k \to +\infty} (I - T(1))\zeta_k,
\]

where each function \(\zeta_k\) belongs to \(C_0(\mathbb{R}^d; \mathbb{R}^m)\) and all the limits appearing in the previous chain of equalities are local uniform in \(\mathbb{R}^d\). Now, we observe that

\[
\mathcal{R}_n (I - T(1))\zeta_k = \frac{1}{n} \sum_{j=0}^{n-1} (T(j)f - T(j+1)f) = 1 (\zeta_k - T(n)f) = \frac{1}{n} (\zeta_k - T(n)\zeta_k).
\]

Hence,

\[
\|\mathcal{R}_n (f - g)\|_{C_0(\mathbb{R}^d; \mathbb{R}^m)} \leq \|\mathcal{R}_n f - g - (I - T(1))\zeta_k\|_{C_0(\mathbb{R}^d; \mathbb{R}^m)} + \|\mathcal{R}_n (I - T(1))\zeta_k\|_{C_0(\mathbb{R}^d; \mathbb{R}^m)} + \frac{1}{n} \|\zeta_k - T(n)\zeta_k\|_{\mathbb{R}^m}, \tag{3.2}
\]

for any \(k, n \in \mathbb{N}\) and \(r > 0\). Since \(f - g - (I - T(1))\zeta_k\) vanishes locally uniformly in \(\mathbb{R}^d\) as \(k \to +\infty\), Proposition 2.7 shows that \(T(1)(f - g - (I - T(1))\zeta_k)\) converges uniformly in \((0, +\infty) \times B_r\) to zero as \(k \to +\infty\). Hence, letting first \(n\) and then \(k\) tend to \(+\infty\) in the first and last side of (3.2) we conclude that \(\mathcal{R}_n (f - g)\) converges to zero locally uniformly in \(\mathbb{R}^d\).

Let us denote by \(\mathcal{P} f\) the limit of \(\mathcal{R}_n f\) as \(n \to +\infty\). By the first part of the proof, we already know that the image of \(\mathcal{P}\) coincides with the kernel of \(I - T(1)\) and that \(\mathcal{R}_n(\mathcal{P} f) = \mathcal{P} f\) for any \(f \in C_0(\mathbb{R}^d; \mathbb{R}^m)\) and \(n \in \mathbb{N}\). Thus, letting \(n \to +\infty\) we deduce that \(\mathcal{P}^2 = \mathcal{P}\).

**Proof of Theorem 3.5.** We split the proof into three steps.

**Step 1.** Here, we show that the family of measures \(\{\nu_j : j = 1, \ldots, m\}\) defined by \(\nu_j = \xi_j \mu\) is a system of invariant measures for \(T(t)\). For this purpose, we fix \(f \in C_0(\mathbb{R}^d; \mathbb{R}^m)\). Taking Lemma 2.2 into account, it is easy to show that the function \(v = \langle T(\cdot)f, \xi \rangle\) is a bounded classical solution to the Cauchy problem

\[
\begin{cases}
D_t v(t, x) = Av(t, x), & t \in (0, +\infty), \ x \in \mathbb{R}^d, \\
v(0, x) = \langle f(x), \xi \rangle, & x \in \mathbb{R}^d.
\end{cases}
\]

Since this problem admits a unique bounded classical solution, it follows that \(v = T(\cdot)(f, \xi)\), i.e., \(\langle T(t)f(x), \xi \rangle = \langle T(\cdot)(f, \xi) \rangle(x)\) for any \(t \geq 0\) and \(x \in \mathbb{R}^d\).
For any $j = 1, \ldots, m$, let us set $\nu_j = \xi_j\mu$. Then, the above result and the invariance of $\mu$ imply that
\[
\sum_{j=1}^{m} \int_{\mathbb{R}^d} (T(t)f_j) d\nu_j = \int_{\mathbb{R}^d} T(t)(f, \xi) d\mu = \int_{\mathbb{R}^d} (f, \xi) d\mu = \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\nu_j.
\]
Hence, the family $\{\nu_j : j = 1, \ldots, m\}$ is a system of invariant measures for $T(t)$.

**Step 2.** Here, we prove that
\[
\lim_{t \to +\infty} \mathcal{P}_t f := \lim_{t \to +\infty} \frac{1}{t} \int_0^t (T(s)f)(\cdot) ds = \left( \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\nu_j \right) \xi,
\]
locally uniformly on $\mathbb{R}^d$ for any $f \in C_b(\mathbb{R}^d; \mathbb{R}^m)$. First of all we observe that
\[
(P_t f)(x) = \frac{1}{t} \int_0^t (T(s)f)(x) ds + \frac{1}{t} \int_0^t (T(s + [t])f)(x) ds = \frac{1}{t} \left[ \int_0^{[t]} (T(s + k)f)(x) ds + \frac{1}{t} \int_0^{t} (T(s + [t])f)(x) ds \right] \rightarrow \frac{[t]}{t} (\mathcal{P}_t f)(x) + \frac{t}{t} ((T(1)[t])\mathcal{P}_t f)(x)
\]
for any $t > 1$, $x \in \mathbb{R}^d$ and $f \in C_b(\mathbb{R}^d; \mathbb{R}^m)$, where $[t]$ and $\{t\}$ denote respectively the integer and the fractional part of $t$. Taking Proposition 3.6 into account and observing that $\| (\mathcal{P}_t f) \|_{\infty} \leq \| f \|_{\infty}$ for any $t > 0$, from (3.4) we conclude that $P_t f$ converges to $\mathcal{P}\mathcal{P}_t f := \mathcal{P}_t f$ locally uniformly on $\mathbb{R}^d$, as $t \to +\infty$, for every $f \in C_b(\mathbb{R}^d; \mathbb{R}^m)$.

To show that $\mathcal{P}$ is a projection it is enough to prove that
\[
T(r) \circ \mathcal{P}_r = \mathcal{P}_r, \quad r \geq 0.
\]
Indeed, once (3.5) is proved, we get $\mathcal{P}_1 \circ \mathcal{P}_r = \mathcal{P}_r$, and $\mathcal{P} \circ \mathcal{P}_r = \mathcal{P}_r$, which clearly imply that $\mathcal{P}_r \circ \mathcal{P}_r = \mathcal{P}_r$. Fix $r > 0$ and observe that, for any $t > 0$, it holds that
\[
T(r) \mathcal{P}_r f = \frac{1}{t} \int_0^t (T(s)f)(\cdot) ds + \frac{1}{t} \int_t^{t+r} (T(s)f)(\cdot) ds = \mathcal{P}_r f + \frac{1}{t} \int_0^t ((T(t) - I)T(s) f)(\cdot) ds.
\]
Letting $t \to +\infty$ and taking Proposition 2.7 into account, we get (3.5). Finally, we prove that $\mathcal{P}$ is a projection on $E$. This is equivalent to showing that $f \in E$ if and only if $\mathcal{P}_r f = f$. So, let us fix $f \in E$. Then, $\mathcal{P}_r f = f$ for any $t > 0$ and, therefore, $\mathcal{P}_r f = f$. Conversely, let us assume that $\mathcal{P}_r f = f$; from (3.5) we deduce that $T(r)f = T(r)\mathcal{P}_r f = \mathcal{P}_r f = f$ for any $r > 0$, so that $f \in E$.

Since $E$ consists of constant functions, we conclude that $\mathcal{P}_r f$ is a constant function for any $f \in C_b(\mathbb{R}^d; \mathbb{R}^m)$. Hence, $\mathcal{P}_r f = M_r \xi$ for any $f \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and some bounded linear operator $f \to M_r \in \mathbb{R}$. By the Riesz representation theorem, there exists a family $\{\mu_i : i = 1, \ldots, m\}$ of finite and nonnegative Borel measures on $\mathbb{R}^d$ such that
\[
M_r = \sum_{k=1}^{m} \int_{\mathbb{R}^d} f_k d\mu_k, \quad f \in C_b(\mathbb{R}^d; \mathbb{R}^m).
\]
We claim that the previous formula can be extended to any function belonging to $C_b(\mathbb{R}^d; \mathbb{R}^m)$. To this aim, first of all we observe that $M_r$ is well defined for any $f \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and the operator $C_b(\mathbb{R}^d; \mathbb{R}^m) \ni f \to M_r$ is bounded. Now, fix $f \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and let $(f_n) \subset C_b(\mathbb{R}^d; \mathbb{R}^m)$ be a sequence converging to $f$ locally.

uniformly in $\mathbb{R}^d$ as $n \to +\infty$ and such that $\|f_n\|_{\infty} \leq \|f\|_{\infty}$ for any $n \in \mathbb{N}$. Splitting $\mathcal{P}_tf = \mathcal{P}_t(f_n + (f - f_n))$ for any $t > 0$ and $n \in \mathbb{N}$, we can estimate
\[
|\mathcal{P}_tf - \left(\sum_{k=1}^{m} \int_{\mathbb{R}^d} f_k d\mu_k\right)\xi| \leq |\mathcal{P}_tf_n - M \xi| + \|\xi\| \sum_{k=1}^{m} \int_{\mathbb{R}^d} |f_{n,k} - f_k| d\mu_k
\]
+ $\sup_{t>0} |\mathbf{T}(t)(f_n - f)|$

in $\mathbb{R}^d$ for any $t > 0$ and $n \in \mathbb{N}$. Letting $t$ tend to $+\infty$ yields
\[
\lim_{t \to +\infty} \left|\mathcal{P}_tf - \left(\sum_{k=1}^{m} \int_{\mathbb{R}^d} f_k d\mu_k\right)\xi\right| \leq \|\xi\| \sum_{k=1}^{m} \int_{\mathbb{R}^d} |f_{n,k} - f_k| d\mu_k + \sup_{t>0} |\mathbf{T}(t)(f_n - f)(x)|,
\]
for any $x \in \mathbb{R}^d$. Finally, letting $n \to +\infty$ in the above estimate and taking again Proposition 2.7 into account, we conclude that (3.6) holds true also for any $f \in C_{b}((\mathbb{R}^d, \mathbb{R}^m))$.

Now, we can complete the proof of (3.3). Since $\{\nu_j : j = 1, \ldots, m\}$ is a system of invariant measures for $\mathbf{T}(t)$, applying Fubini theorem, we easily deduce that
\[
\sum_{j=1}^{m} \int_{\mathbb{R}^d} (\mathcal{P}_f)_{j} d\nu_j = \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\nu_j
\]
for any $t > 0$ and $f \in C_b(\mathbb{R}^d, \mathbb{R}^m)$. Letting $t$ tend to $+\infty$ in the previous formula, by dominated convergence we can infer that
\[
\sum_{j=1}^{m} \int_{\mathbb{R}^d} (\mathcal{P}_f)_{j} d\nu_j = \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\nu_j
\]
or, equivalently, taking into account that $|\xi| = 1$,
\[
\sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\mu_j = \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\nu_j
\]
for any $f \in C_b(\mathbb{R}^d, \mathbb{R}^m)$. Taking $f = f \xi_j$ with $f \in C_b(\mathbb{R}^d)$ and $j = 1, \ldots, m$, we conclude that
\[
\int_{\mathbb{R}^d} f d\mu_j = \int_{\mathbb{R}^d} f d\nu_j, \quad f \in C_b(\mathbb{R}^d).
\]
This is enough to infer that $\mu_j = \nu_j$ for any $j = 1, \ldots, m$.

Step 3. Suppose that $\{\tilde{\mu}_j : j = 1, \ldots, m\}$ is another system of invariant measures for $\mathbf{T}(t)$. Then, (3.7) can be written with $\nu_j$ being replaced by $\tilde{\mu}_j$. Letting $t$ tend to $+\infty$, we deduce that
\[
\sum_{k=1}^{m} \xi_k \tilde{\mu}_k(\mathbb{R}^d) \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\nu_j = \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\tilde{\mu}_j
\]
for any $f \in C_b(\mathbb{R}^d, \mathbb{R}^m)$. Hence, $\tilde{\mu}_j = c \xi_j \mu$ for any $j = 1, \ldots, m$, where $c = \sum_{k=1}^{m} \xi_k \tilde{\mu}_k(\mathbb{R}^d)$. This completes the proof. $\square$

**Remark 3.7.** The sign condition on the quadratic form induced by $C$ is inspired by the scalar case where typically one assumes that the potential term of the elliptic operator identically vanishes on $\mathbb{R}^d$ to guarantee the existence of an invariant measure. Hypothesis 2.1(iii) seems the natural extension in the multidimensional case. If that condition is violated, then we can find examples of matrix-valued functions $C$ such that nontrivial systems of invariant measures for the associated
semigroup \( T(t) \) do not exist. Consider for instance the particular case when \( C \) is a symmetric constant matrix and assume that \( \langle C \xi, \xi \rangle > 0 \) for some \( \xi \in \mathbb{R}^m \). In this case \( \sigma(C) \cap \mathbb{R}^+ \neq \emptyset \) and the ordinary differential equation \( D_t u = Cu \) admits a solution \( u \), with all positive components, such that \( |u(t)| \geq e^{\lambda t} \) for any \( t > 0 \) and some \( \lambda > 0 \). Let us set \( u_0 = u(0) \). Then, clearly, \( u = T(t)u_0 \). It thus follows that for any \( t > 0 \) there exists \( j_t \in \{1, \ldots, m\} \) such that \( (T(t)u_0)_{j_t} \geq m^{-1/2}e^{\lambda t} \). Since \( T(t)u_0 \) is independent of \( x \), the invariance property (3.1) shows that

\[
\sum_{j=1}^m u_{0,j} \mu_j(\mathbb{R}^d) = \sum_{j=1}^m \int_{\mathbb{R}^d} u_{0,j} d\mu_j = \sum_{j=1}^m \int_{\mathbb{R}^d} (T(t)u_0)_j d\mu_j \geq (T(t)u_0)_{j_t} \mu_{j_t}(\mathbb{R}^d) \geq m^{-1/2} \min_{1 \leq j \leq m} \mu_j(\mathbb{R}^d)e^{\lambda t}
\]

for any \( t > 0 \). Letting \( t \) tend to \( +\infty \) we get to a contradiction.

On the other hand, if \( C \) is a matrix-valued function which satisfies the condition \( \sup_{x \in \mathbb{R}^d, |\xi|=1} \langle C(x)\xi, \xi \rangle < 0 \), then, by [10, Theorem 2.6], the sup-norm of the function \( T(t)f \) exponentially decreases to zero as \( t \to +\infty \) for any \( f \in C_0(\mathbb{R}^d; \mathbb{R}^m) \). Hence, if we take \( f = e_k \ (k = 1, \ldots, m) \), then using again (3.1) we obtain

\[
\mu_k(\mathbb{R}^d) = \sum_{j=1}^m \int_{\mathbb{R}^d} (T(t)e_k)_j d\mu_j
\]

and, letting \( t \) tend to \( +\infty \), by dominated convergence we conclude that \( \mu_k(\mathbb{R}^d) = 0 \) for any \( k \in \{1, \ldots, m\} \).

Example 3.8. Let \( \mathcal{A} \) be as in (1.1) with \( Q(x) = (1 + |x|^2)^\gamma Q_0 \) for any \( x \in \mathbb{R}^d \), \( Q_0 \) being a constant, symmetric and positive definite \( d \times d \)-matrix and \( \gamma \) being a nonnegative number. Let further \( b(x) = -b_0 x(1 + |x|^2)^\beta \), for any \( x \in \mathbb{R}^d \) and some positive constants \( b_0 \) and \( \beta \), and \( C \) be any \( m \times m \)-matrix, with entries in \( C_0^\infty(\mathbb{R}^d) \) for some \( \alpha \in (0, 1) \), and such that the elements on the main diagonal are negative, whereas the off-diagonal ones are positive and the sum of the elements of each row and column is zero. By the Gersgorin circle theorem, applied to \( C(x) + (C(x))^\ast \), (see [23, Theorem 1.11]), we can infer that \( \langle C(x)\eta, \eta \rangle \leq 0 \) for any \( x \in \mathbb{R}^d \) and \( y \in \mathbb{R}^m \). Moreover, we can take as \( \xi \) the vector with all entries equal to one (see Remark 2.3). It is easy to check that if \( \beta > (\gamma - 1)^+ \) then Hypothesis 2.1(iv) is satisfied as well, with \( \varphi(x) = 1 + |x|^2 \) for any \( x \in \mathbb{R}^d \). Indeed

\[
(A \varphi)(x) = 2\varphi(x)[(1 + |x|^2)^{\gamma-1}Tr(Q_0) - b_0|x|^2(1 + |x|^2)^{\beta-1}], \quad x \in \mathbb{R}^d, \quad (3.8)
\]

and the term in brackets in (3.8) tends to \( -\infty \) as \( |x| \to +\infty \). Therefore, we can determine two positive constants \( a \) and \( c \) such that \( A \varphi \leq a - c \varphi \) in \( \mathbb{R}^d \). In this case, all the assumptions in Theorem 3.5 are satisfied and consequently it can be applied.

Remark 3.9. Theorem 3.5 shows that, in general, a system of invariant measures \( \{\mu_i : i = 1, \ldots, m\} \) for \( T(t) \) does not consist only of probability measures. We can infer that each \( \mu_i \) is a probability measure if and only if \( \xi_i = 1 \) for any \( i = 1, \ldots, m \).

3.1. The semigroup \( T(t) \) in \( L^p \)-spaces. In this subsection, we prove that the semigroup \( T(t) \) can be extended, with a bounded strongly continuous semigroup, to the \( L^p \)-spaces related to any system of invariant measures \( \{\mu_i : i = 1, \ldots, m\} \) and we investigate on some of its smoothing effects in these spaces.

Throughout the section, \( \{\mu_i : i = 1, \ldots, m\} \) is any system of invariant measures for \( T(t) \). Moreover, for any \( p \in [1, +\infty) \), we write \( L^p_\mu(\mathbb{R}^d; \mathbb{R}^m) \) to denote the set \( \bigotimes_{i=1}^m L^p_\mu(\mathbb{R}^d) \), which we endow with the natural norm \( f \mapsto (\sum_{i=1}^m \int_{\mathbb{R}^d} |f_i|^p d\mu_i)^{1/p} \).
Similarly, by $W^{j,p}_\mu(\mathbb{R}^d;\mathbb{R}^m)$ we denote the Sobolev space of order $j$ of all the functions $f \in L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$ whose distributional derivatives up to the $j$-th order are in $L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$. It is normed by setting $\|f\|_{W^{j,p}_\mu(\mathbb{R}^d;\mathbb{R}^m)} = \sum_{|\alpha| \leq j} \|D^\alpha f\|_{L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)}$ for any $f \in W^{j,p}_\mu(\mathbb{R}^d;\mathbb{R}^m)$. To lighten the notation we write $\|\cdot\|_{p,\mu}$, resp. $\|\cdot\|_{p,\mu}$, resp. $\|\cdot\|_{p,\mu}$ in place of $\|\cdot\|_{L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)}$, resp. $\|\cdot\|_{L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)}$ and $\|\cdot\|_{W^{j,p}_\mu(\mathbb{R}^d;\mathbb{R}^m)}$.

**Remark 3.10.** Since the measures $\mu_i$ ($i = 1, \ldots, m$) are finite Borel measures, the space $C_b(\mathbb{R}^d;\mathbb{R}^m)$ is dense in $L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$ for any $p \in [1, +\infty)$. See [3, Remark 1.46] for further details.

**Proposition 3.11.** The semigroup $T(t)$ extends to a strongly continuous semigroup (still denoted by $T(t)$) on $L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$ for any $1 \leq p < +\infty$. Moreover for any $p \in [1, +\infty)$ it holds that

$$\|T(t)\|_{\mathcal{L}(L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m))} \leq 2^{\frac{1}{p-1}}, \quad t > 0. \tag{3.9}$$

Finally, the set $\mathcal{D} = \{ u \in C_b(\mathbb{R}^d;\mathbb{R}^m) \cap \bigcap_{p < +\infty} W^{1,p}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^m) : Au \in C_b(\mathbb{R}^d;\mathbb{R}^m) \}$ is a core for the infinitesimal generator $A_p$ of the semigroup $T(t)$ in $L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$, for any $p \in [1, +\infty)$.

**Proof.** Since $\|T(t)e_i\|_\infty \leq 1$, it follows that $\rho(t, x) \leq 1$ for any $i, k = 1, \ldots, m$, $t > 0$ and $x \in \mathbb{R}^d$. Thus, the Jensen inequality and formula (2.4) yield

$$|\langle (T(t)f)_i(x) \rangle|^p \leq 2^{p-1} \sum_{k=1}^m \left[ \int_{\mathbb{R}^d} |f_k(y)| \rho_k(t, x, dy) \right]^p \leq 2^{p-1} \sum_{k=1}^m \left[ \int_{\mathbb{R}^d} |f_k(y)| \rho_k(t, x, dy) \right]^{p-1} \int_{\mathbb{R}^d} |f_k(y)|^p \rho_k(t, x, dy) \leq 2^{p-1} \|T(t)\|_\infty \|f_1\|^p, \ldots, |f_m\|^p) \rangle_i(x)$$

for any $t > 0, x \in \mathbb{R}^d, i = 1, \ldots, m, f \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and $p \in [1, +\infty)$. Moreover, by the invariance property (3.1) we deduce that

$$\sum_{i=1}^m \left[ \int_{\mathbb{R}^d} |(T(t)f)_i|^p d\mu_i \leq 2^{p-1} \sum_{i=1}^m \left[ \int_{\mathbb{R}^d} (T(t)(|f_1|^p, \ldots, |f_m|^p))_i d\mu_i \right] \right]$$

for any $t > 0$ and $f \in C_b(\mathbb{R}^d;\mathbb{R}^m)$. Taking Remark 3.10 into account, from the previous chain of inequalities we easily deduce that $T(t)$ extends to a linear bounded operator in $L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$ and formula (3.9) follows. The semigroup property easily follows. Hence, $T(t)$ is a semigroup in $L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$.

To show that such a semigroup is strongly continuous, we first observe that $\|T(t)f - f\|_{p,\mu}$ vanishes as $t \to 0^+$ for any $f \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and $p \in [1, +\infty)$. Indeed, for any such function, $T(t)f$ converges locally uniformly to $f$ as $t \to 0^+$ and $\|T(t)f\|_\infty \leq \|f\|_\infty$; hence by the dominated convergence theorem we get the assertion.

Now, fix $f \in L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$ and let $(f_n)$ be a sequence in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ converging to $f$ in $L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$ as $n \to +\infty$. For any $i = 1, \ldots, m$ and $n \in \mathbb{N}$, we can estimate

$$\|\langle (T(t)f)_i - f_i \rangle\|_{p,\mu} \leq \|T(t)(f - f_n)\|_{p,\mu} + \|T(t)f_n - f_n\|_{p,\mu} + \|f_n - f\|_{p,\mu} \leq \|T(t)f_n - f_n\|_{p,\mu} + \|f_n - f\|_{p,\mu} \leq \left( 2^{\frac{1}{p-1}} + 1 \right) \|f_n - f\|_{p,\mu},$$

where in the last line we have used (3.9). Letting first $n$ tend to $+\infty$ and then $t$ tend to $0^+$, we deduce that $\|\langle (T(t)f)_i - f_i \rangle\|_{L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)}$ vanishes as $t \to 0^+$. 
To conclude the proof, let us prove that $\mathcal{D}$ is a core for the infinitesimal generator $A_p$ of $T(t)$ in $L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$ for any $p \in [1, +\infty)$. For this purpose, we observe that, by [10, Proposition 3.1], $\mathcal{D}$ coincides with the set of all $u \in C_0^0(\mathbb{R}^d;\mathbb{R}^m)$ such that $\sup_{t \in (0,1)} t^{-1-} \|T(t)u - u\|_{\infty} < +\infty$ and $\lim_{t \to 0^+} t^{-1} \|T(t)u - u\|_{\infty} = \mathcal{A}u$ pointwise in $\mathbb{R}^d$. Since all the $\mu_i$’s are finite measures, from the previous properties and dominated convergence we immediately deduce that, if $u \in \mathcal{D}$, then $t^{-1} \|T(t)u - u\|$ converges to $\mathcal{A}u$ in $L^p(\mathbb{R}^d;\mathbb{R}^m)$ as $t$ tends to $0^+$, for any $p \in [1, +\infty)$. Hence, $u \in D(A_p)$ and $A_p u = \mathcal{A}u$, so that $\mathcal{D} \subset D(A_p)$. Moreover, since it contains $C^\infty(\mathbb{R}^d;\mathbb{R}^m)$, $\mathcal{D}$ is dense in $L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$ (see also Remark 3.10). Finally, by [10, Proposition 3.2], $T(t)$ leaves $\mathcal{D}$ invariant. Hence, applying [11, Proposition II.1.7], we conclude that $\mathcal{D}$ is a core for $A_p$. 

The characterization of the domain $D(A_p)$ is a very hard task and, as the scalar case reveals, $D(A_p)$ has been characterized only in rather particular situations. Still, the scalar case shows that, in general, we can not expect the semigroup $T(t)$ to be analytic in $L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$. We refer the interested reader to [19] for further details. Nevertheless, $T(t)$ has smoothing effects since maps $L^p_{\mu}(\mathbb{R}^d;\mathbb{R}^m)$ into $W^{2,p}(\mathbb{R}^d;\mathbb{R}^m)$. This property is a consequence of some pointwise estimates for the first- and second-order spatial derivatives of the semigroup $T(t)$. More precisely, we provide sufficient conditions for the estimates

$$|D^h \mathcal{T}(t)\mathcal{f}|^p \leq \Gamma_{p,k,h}(t) T(t) \left( \sum_{j=0}^{h} |D^j \mathcal{f}|^2 \right)^{\frac{p}{2}},$$

(3.10)

to hold in $\mathbb{R}^d$ for any $\mathcal{f} \in C_0^0(\mathbb{R}^d;\mathbb{R}^m)$, $t > 0$, $p > 1$, $k \in \{1,2\}$ and $h \in \{0,\ldots,k\}$, where $\Gamma_{p,k,h}$ is a positive function defined in $(0, +\infty)$. Estimates (3.10) also allow us to prove a partial characterization of $D(A_p)$ (see Corollary 3.17).

To ease the notation, in the rest of this section we set

$$\mathcal{B}_{1,u} = \sum_{|\alpha| = k}^{m} \sum_{i,j=1}^{d} (Jb \nabla_x D^\alpha u_k, \nabla_x D^\alpha u_k), \quad \mathcal{B}_2 = \left( \sum_{i,j=1}^{d} |D_{ij} b|^2 \right)^{\frac{1}{2}},$$
$$\mathcal{C}_1 = \left( \sum_{|\alpha| = i}^{m} |D^\alpha C|^2 \right)^{\frac{1}{2}}, \quad \mathcal{F}_{i,u} = \left( \sum_{k=1}^{m} \sum_{|\alpha| = i}^{m} |Q_{i,k} \nabla_x D^\alpha u_k|^2 \right)^{\frac{1}{2}},$$
$$\mathcal{Q}_1 = \left( \sum_{|\alpha| = i}^{m} |D^\alpha Q|^2 \right)^{\frac{1}{2}},$$

where, if $|\alpha| = 0$, then $D^\alpha u_k = u_k$. We also denote by $r$ the “best” function which bounds from above the quadratic form associated to the Jacobian matrix of the drift $b$, i.e., $(Jb(x)y, y) \leq r(x)|y|^2$ for any $x, y \in \mathbb{R}^d$.

Estimate (3.10) with $k = 1$ has been already proved in [5] when the differential operator $\mathcal{A}$ is in divergence form with first-order coupling. In our case, the assumptions considered in [5] and yielding (3.10) with $k = 1$ force the first-order derivatives of the entries of the matrix $C$ to be bounded. To enlighten this hypothesis we consider a set of different assumptions.

**Hypotheses 3.12.** (i) The coefficients $q_{ij}$, $b_i$ and the entries $c_{ijk}$ of the matrix valued function $C$, belong to $C^{1+\alpha}_{loc}(\mathbb{R}^d)$ for any $i, j = 1, \ldots, d$ and $h, k = 1, \ldots, m$;

(ii) for any $p \in (1, +\infty)$, there exists a positive constant $c_p$ such that

$$K_p := \sup_{\mathbb{R}^d} \left( r + (1-p)\mu_Q + \frac{1}{4(p-1)\mu_Q} \mathcal{B}_1^2 + c_p \mathcal{C}_1^2 \right) < +\infty,$$
The proof of estimate (3.10) is the content of the forthcoming Theorems 3.13 and 3.15. In the first theorem, we consider the easiest case \( k = 1 \), under the additional Hypotheses 3.12. Then, strengthening the assumptions on the coefficients of the operator \( A \), we prove estimate (3.10) with \( k = 2 \). Before entering into details, we preliminary observe that it suffices to prove (3.10) for \( p \). Then, for \( p > 2, h, k = 0, 1, 2 \), with \( h \leq k, t > 0 \) and \( f \in C_0^1(\mathbb{R}^d; \mathbb{R}^m) \), we can estimate

\[
|D_x^{j_k}T(t)f|^p = (|D_x^{j_k}T(t)f|^2)^{\frac{p}{2}} \leq \left( \Gamma_{2,k,h}(t)T(t)\sum_{j=0}^{h}|D^{j}f|^2 \right)^{\frac{p}{2}} \\
\leq (\Gamma_{2,k,h}(t))^{\frac{p}{2}} T(t)\left( \sum_{j=0}^{h}|D^{j}f|^2 \right)^{\frac{p}{2}}.
\]

We can also just consider functions in \( C_c^\infty(\mathbb{R}^d; \mathbb{R}^m) \). Once (3.10) is proved for such smooth functions, we can use a density argument to extend its validity to any \( f \in C_0^1(\mathbb{R}^d; \mathbb{R}^m) \). Indeed, for any \( f \in C_0^1(\mathbb{R}^d; \mathbb{R}^m) \) there exists a sequence \( (f_n) \subset C_c^\infty(\mathbb{R}^d; \mathbb{R}^m) \), bounded with respect to the \( C_0^1(\mathbb{R}^d; \mathbb{R}^m) \)-norm and converging to \( f \) in \( C_0^1(B_r; \mathbb{R}^m) \) for any \( r > 0 \). Writing (3.10) with \( f \) being replaced by \( f_n \) and using [2, Proposition 3.2], we can let \( n \) tend to \( +\infty \) and obtain (3.10) in its full generality.

Hence, in the proof of Theorems 3.13 and 3.15 we will assume that \( p \in (1,2] \) and \( f \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m) \) without further mentioning it.

We finally observe that, for any \( p \in (1,2] \), \( A, B > 0 \), it holds that

\[
\min_{x>0} \left( Ax^{\frac{2}{p+2}} + Bx^\frac{2}{p+2} \right) = \left( \frac{2-p}{p} \right)^{\frac{2}{p+2}} \frac{2}{p} A^{1-\frac{2}{p}} B^{\frac{2}{p+2}}.
\]

**Theorem 3.13.** Under Hypotheses 3.12, estimate (3.10) holds true, with \( k = 1 \) and \( \Gamma_{p,1,h}(t) = \gamma_{h,p} e^{C_{h,p}t^\frac{p}{2}(h-1)} \) for any \( t > 0 \), where \( \gamma_{h,p} \) and \( C_{h,p} \) are positive constants for any \( h = 0, 1 \).

**Proof.** We fix \( p, f \) and consider the solution \( v_n \) to problem (2.3) and the positive semigroup \( T_n(t) \) associated to the realization of \( A \) in \( C_0^1(\mathbb{B}_n) \) with homogeneous Neumann boundary conditions. We split the proof into two steps. In the first one we prove (3.10) with \( h = 1 \), in the second one we deal with the case \( h = 0 \).

**Step 1.** To prove (3.10) with \( h = k = 1 \) it suffices to show that

\[
|J \Sigma v_n(t,\cdot)|^p \leq e^{C_{1,p} T_n(t)} (|f|^2 + |Jf|^2)^{\frac{p}{2}}
\]

for any \( t > 0, n \in \mathbb{N} \) and some positive constant \( C_{1,p} \). Indeed, once (3.12) is proved, estimate (3.10) with \( h = k = 1, (\gamma_{1,p} = 1 \text{ and } C_{1,p} = C_p) \) will follow simply letting \( n \to +\infty \), recalling that, for any \( t > 0 \), \( T_n(t)f \) converges to \( T(t)f \), as \( n \to +\infty \), locally uniformly in \( \mathbb{R}^d \) (see Section 2). So, let us prove (3.12). For any \( \varepsilon > 0 \) and \( n \in \mathbb{N} \), let us consider the function \( w_n = (|v_n|^2 + |J \Sigma v_n|^2 + \varepsilon)^{\frac{p}{2}} \). From [18, Chapter 7, Section 10] it follows that \( w_n \in C^{1,2}(0, +\infty) \times \mathbb{B}_n \cap C_0([0, T] \times \mathbb{B}_n) \) for any \( T > 0 \). Moreover, \( w_n \) solves the problem

\[
\begin{cases}
D_{t}w_n - Aw_n = p w_n^{1-\frac{2}{p}} g, & \text{in } (0, +\infty) \times B_n, \\
\frac{\partial w_n}{\partial \nu} \leq 0, & \text{on } (0, +\infty) \times \partial B_n, \\
w_n(0) = (|f|^2 + |Jf|^2 + \varepsilon)^{\frac{p}{2}}, & \text{in } B_n,
\end{cases}
\]

(3.13)
where \( g = \psi_1 + \psi_2 + (2 - p)w_n^{-\frac{2}{p}}\psi_3 \), with
\[
\psi_1 = \mathcal{P}_{0,\nu} + \langle C\nu, \nu \rangle + \sum_{i=1}^{d} (C\nu_i, D_i\nu) - \mathcal{F}_{0,\nu}^2 - \mathcal{F}_{1,\nu}^2,
\]
\[
\psi_2 = \sum_{i,j,h,k=1}^{d} D_h q_{ij} D_{ij} v_{n,k} D_{h} v_{n,k} + \sum_{i,j,h,k=1}^{d} D_h C_{jk} v_{n,k} D_{h} v_{n,j},
\]
\[
\psi_3 = \sum_{i,j=1}^{d} q_{ij} \left( \langle \nu, D_i\nu \rangle + \sum_{h=1}^{d} (D_h \nu, D_h \nu) \right) \left( \langle \nu, D_j\nu \rangle + \sum_{t=1}^{m} (D_t \nu, D_t \nu) \right)
\]
and the boundary condition in (3.13) follows since the normal derivative of \(|\nabla x v_{n,k}|^2 \) is nonpositive in \((0, +\infty) \times \partial B_n\) for any \(k = 1, \ldots, m\) (see e.g., [6]).

Using Hypothesis 2.1(iii), the Cauchy-Schwarz and the Young’s inequalities, we estimate the functions \( \psi_i (i = 1, 2, 3) \) as follows:
\[
\psi_1 \leq r|J_x v_n|^2 - \mathcal{F}_{0,\nu}^2 - \mathcal{F}_{1,\nu}^2,
\]
\[
\psi_2 \leq D_1 |J_x v_n||D_x^2 v_n| + \mathcal{E}_1 |\nu||J_x v_n|
\]
\[
\leq \frac{1}{4c_p} |\nu|^2 + \left( \frac{1}{4(p-1)\mu_Q} \mathcal{B}_1^2 + c_p \mathcal{E}_1^2 \right) |J_x v_n|^2 + (p-1)\mu_Q |D_x^2 v_n|^2,
\]
\[
\psi_3 \leq \left( \sum_{k=1}^{m} |v_{n,k}| ||Q^\frac{1}{2} \nabla x v_{n,k}|| \right)^2 + 2 \sum_{h,k=1}^{m} |v_{n,h}||Q^\frac{1}{2} \nabla x v_{n,h}, v_{n,k}| \sum_{i=1}^{d} |D_i v_{n,k}||Q^\frac{1}{2} \nabla x D_i v_{n,k}|
\]
\[
+ \sum_{i,j,h,k=1}^{d} |D_i v_{n,h}||D_j v_{n,k}||Q^\frac{1}{2} \nabla x D_i v_{n,k}||Q^\frac{1}{2} \nabla x D_j v_{n,k}|
\]
\[
\leq \mathcal{F}_{0,\nu}^2 |\nu|^2 + \mathcal{F}_{1,\nu}^2 |J_x v_n|^2 + 2 \mathcal{F}_{0,\nu}^2 |\nu||J_x v_n|
\]
\[
\leq w_n^2 (\mathcal{F}_{0,\nu}^2 + \mathcal{F}_{1,\nu}^2),
\]
where \( c_p \) is the constant in Hypothesis 3.12(ii). Putting everything together, we get
\[
g \leq \frac{1}{4c_p} |\nu|^2 + \left( r + (1-p)\mu_Q + \frac{1}{4(p-1)\mu_Q} \mathcal{B}_1^2 + c_p \mathcal{E}_1^2 \right) |J_x v_n|^2.
\]

Using Hypothesis 3.12(ii) we conclude that \( D_x v_n - A v_n \leq C_{1,p} v_n \) in \( \mathbb{R}^d \) where \( C_{1,p} \) is the maximum between \((4c_p)^{-1}\) and \( K_p \). Hence, the function \( z_n(t) = w_n(t, \cdot) - e^{C_{1,p} t} T_n^N(t) ||f||^2 + |f|^2 + \varepsilon \) vanishes on \( \{0\} \times B_n \), satisfies the differential inequality \( D_t z_n - A z_n - C_{1,p} z_n \leq 0 \) in \((0, +\infty) \times B_n \) and its normal derivative is nonpositive on \((0, +\infty) \times \partial B_n \). The classical maximum principle yields that \( z_n \leq 0 \) in \((0, +\infty) \times B_n \), whence, letting \( \varepsilon \to 0^+ \), estimate (3.12) follows at once.

Step 2. Now, we prove (3.10) with \( k = 1 \) and \( h = 0 \). Fix \( t > 0 \). From (2.5), the previous step, the semigroup law and recalling that \((a+b)^{p/2} \leq a^{p/2} + b^{p/2}\) for any \(a, b \geq 0\), it follows that
\[
|J_x T(t) f|^p = |J_x T(t - \sigma) T(\sigma) f|^p
\]
\[
\leq e^{C_{1,p} (t-\sigma)} T(t-\sigma) |T(\sigma) f|^p + |J_x T(\sigma) f|^p
\]
\[
\leq e^{C_{1,p} (t-\sigma)} T(t) |f|^p + T(t-\sigma) |J_x T(\sigma) f|^p
\]
for any \( \sigma \in (0, t) \). Formula (2.6) and the Hölder inequality yield
\[
T(t-\sigma) |J_x T(\sigma) f|^p = T(t-\sigma) \frac{p-2}{p} |T(\sigma) f|^2 + \delta^\frac{p-2}{2p} |(T(\sigma) f|^2 + \delta^\frac{p-2}{2p})
\]

of both sides of (3.17) with respect to \( \partial B \) whence (3.10) with \( k \) for any \( \varepsilon, \delta > 0 \), whence
\[
e^{-C_{1, \rho}(t-\sigma)}|J_xT(t)f|^p \leq T(t)|f|^p + \left(1 - \frac{p}{2}\right) \varepsilon^{2\rho \sigma} T(t-\sigma)(|T\sigma|f|^2 + \delta)\frac{2}{\varepsilon^{2\rho}}
\]
\[
+ \frac{p}{2} \varepsilon^{2\rho} T(t-\sigma) \left(\left|J_xT(\sigma)f\right|^2 (|T(\sigma)|f|^2 + \delta)\frac{2}{\varepsilon^{2\rho}}\right).
\]
Integrating the previous estimate with respect to \( \sigma \in (0, t) \), we deduce
\[
|J_xT(t)f|^p \leq \frac{C_{1, p}}{1 - e^{-C_{1, \rho}t}} \left\{ (T(t)|f|^p + \left(1 - \frac{p}{2}\right) \varepsilon^{2\rho \sigma} T(t-\sigma)(|T\sigma|f|^2 + \delta)\frac{2}{\varepsilon^{2\rho}}
\]
\[
+ \frac{p}{2} \varepsilon^{2\rho} T(t-\sigma) \left(\left|J_xT(\sigma)f\right|^2 (|T(\sigma)|f|^2 + \delta)\frac{2}{\varepsilon^{2\rho}}\right) d\sigma \right\}.
\]
(3.15)

To prove the claim, we just need to show that there exists a positive constant \( k_p \) such that
\[
\int_0^t T(t-\sigma) \left(\left|J_xT(\sigma)f\right|^2 (|T(\sigma)|f|^2 + \delta)\frac{2}{\varepsilon^{2\rho}}\right) d\sigma \leq k_p T(t)(|f|^2 + \delta)\frac{2}{\varepsilon^{2\rho}}
\]
(3.16)
for any \( t > 0 \). Indeed, once (3.16) is proved, replacing it into (3.15), letting \( \delta \to 0^+ \) (see [19, Proposition 1.2.10]), using again (2.5) and minimizing with respect to \( \varepsilon > 0 \) (taking (3.11) into account), we deduce that
\[
|J_xT(t)f|^p \leq \frac{C_{1, p}}{1 - e^{-C_{1, \rho}t}} \left\{ \left[1 + \left(1 - \frac{p}{2}\right) \varepsilon^{2\rho \sigma} \right] T(t)|f|^p
\]
\[
+ \frac{p}{2} \varepsilon^{2\rho} \left[ t + \frac{p}{2} \varepsilon^{2\rho} k_p \right] T(t)|f|^p\right\},
\]
whence (3.10) with \( k = 1 \) and \( h = 0 \) follows.

We prove the above inequality with \( T(t) \) and \( T(t) \) being replaced by \( T_n(t) \) and \( T_n(t) \), respectively. Letting \( n \) tend to \( +\infty \), (3.16) will follow at once. We set
\[
\psi_n(\sigma, \cdot) = T_n(t-\sigma)(|v_n(\sigma, \cdot)|^2 + \delta)\frac{2}{\varepsilon^{2\rho}} =: T_n(t-\sigma)v_n(\sigma, \cdot)
\]
for any \( \sigma \in [0, t] \) and \( n \in \mathbb{N} \). Since the normal derivative of the function \( v_n(\sigma, \cdot) \) vanishes on \( \partial B_n \) for any \( \sigma \in (0, t) \), \( v_n(\sigma, \cdot) \) belongs to the domain of the generator \( G_n(t) \) in \( C_0(X_n) \) for any \( \sigma \in (0, t) \). Hence, \( \psi_n \) is differentiable in \( (0, t) \)
\[
\psi'_n = T_n(t-\sigma)(D_{\sigma}v_n - Av_n)
\]
\[
= pT_n(t-\sigma) \left[ v_n^{1-\frac{\rho}{p}} \left( (Cv_n, v_n) - \sum_{i,j=1}^d q_{ij}(D_i v_n, D_j v_n) \right)
\]
\[
+ (2 - p)v_n^{1-\frac{\rho}{p}} \sum_{i,j=1}^d q_{ij}(v_n, D_i v_n)(v_n, D_j v_n) \right].
\]
Applying the same arguments as in Step 1, we deduce that
\[
\psi_n(\sigma) \leq p(1 - p) \mu T_n(t-\sigma)(|v_n(\sigma, \cdot)|)^{1-\frac{\rho}{p}} |J_xv_n(\sigma, \cdot)|^2
\]
(3.17)
for any \( \sigma \in (0, t) \). Thus (3.16) follows with \( k_p = [p(p - 1)\mu_0]^{-1} \), simply integrating both sides of (3.17) with respect to \( \sigma \) in \( [h, t-h] \) and then letting \( n \to +\infty \) and \( h \to 0^+ \). The proof is so complete. \( \square \)
Estimate (3.10) with $k = 2$ is more involved and, as it has been already pointed out, it requires stronger assumptions on the coefficients of the operator $A$.

**Hypotheses 3.14.** (i) The coefficients $q_{ij}$, $b_i$ and the entries $c_{hk}$ of the matrix valued function $C$ belong to $C^2_{loc}(\mathbb{R}^d)$ for any $i, j = 1, \ldots, d$ and $h, k = 1, \ldots, m$.

(ii) there exist two positive constants $c_1, c_2$ such that for any $x \in \mathbb{R}^d$:

$$|Q(x)| \leq c_1(1 + |x|^2)\mu_Q(x),$$

$$|Q(x)x| \leq c_1(1 + |x|^2)\mu_Q(x),$$

(iii) for any $p \in (1, +\infty)$, there exist positive constants $c_{jp}$ ($j = 1, \ldots, 6$) such that

$$K_{1p} := \sup_{x \in \mathbb{R}^d} \left( r - c_{3p}\mu_Q + c_{1p}\epsilon_1^2 + \frac{3}{2(p-1)\mu_Q}Q_1^2 + c_{2p}Q_2 \right) < +\infty,$$

$$K_{2p} := \sup_{x \in \mathbb{R}^d} \left( 2r - c_{6p}\mu_Q + c_{4p}Q_2 + c_{5p}Q_3^2 + \frac{4}{(p-1)\mu_Q}Q_1^2 + Q_2 \right) < +\infty.$$
and $\zeta_n = \nabla_x(|u|^2 + \alpha\partial_n^2 J_2 u^2 + \beta\partial_n^4 |D_2^2 u|^2)$. Using the hypotheses, the Cauchy-Schwartz and Young's inequalities, as in the proof of Theorem 3.13, we estimate the terms $\psi_i$, $i = 0, \ldots, 4$. Clearly, $\psi_0 \leq \alpha\partial_n^2 |J_2 u|^2$. Moreover,

$$\psi_1 \leq \alpha\partial_n^2 \left(\frac{1}{2c_{1p}}|u|^2 + \alpha\partial_n^2 \left(\frac{3}{2(p-1)\mu_Q} \partial_1^2 + \frac{c_{1p}}{2} \partial_2^2\right) |J_2 u|^2 + \alpha\partial_n^2 \frac{p-1}{6} \mu_Q |D_2^2 u|^2, \right.$$

$$\psi_2 \leq \frac{p-1}{4} \beta\partial_n^4 \mu_Q |D_2^2 u|^2 + \beta\partial_n^4 \left(\frac{4}{(p-1)\mu_Q} \partial_1^2 + \partial_2^2\right) |D_2^2 u|^2,$$

$$\psi_3 \leq \frac{1}{4c_{5p}} \beta\partial_n^4 |u|^2 + \beta c_{1p} \partial_n^4 \partial_2^2 |J_2 u|^2 + \beta\partial_n^4 \left(\frac{1}{c_{1p}} + c_{5p} \partial_2^2\right) |D_2^2 u|^2,$$

$$\psi_4 \leq \beta\partial_n^4 (2r + c_{4p}\partial_2^2) |D_2^2 u|^2 + \frac{\beta}{4c_{4p}} \partial_n^4 \partial_2^2 |J_2 u|^2.$$

Now, we observe that

$$\frac{1}{4}|Q^\perp \zeta_n|^2 \leq |\mathcal{F}_{\theta_n} u| |u| + \alpha\partial_n (\partial_{\theta_n} \mathcal{F}_{1,u} + |Q^\perp \nabla \partial_{\theta_n}| |D_2^2 u|) |J_2 u|$$

$$+ \beta\partial_n^2 (\partial_n^2 \mathcal{F}_{2,u} + 2\partial_n |Q^\perp \nabla \partial_{\theta_n}| |D_2^2 u|) |D_2^2 u|^2$$

$$\leq (|u|^2 + \alpha \partial_n^2 |J_2 u|^2 + \beta\partial_n^4 |D_2^2 u|^2)$$

$$\times (|\mathcal{F}_{\theta_n} u| + \alpha (\partial_{\theta_n} \mathcal{F}_{1,u} + |J_2 u| |Q^\perp \nabla \partial_{\theta_n}|^2 + \beta (\partial_n^2 \mathcal{F}_{2,u} + 2\partial_n |Q^\perp \nabla \partial_{\theta_n}|)^2)$$

$$\leq \zeta_n^2 (1 + \varepsilon) \alpha\partial_n^2 \mathcal{F}_{1,u}^2 + (1 + \varepsilon) \beta\partial_n^4 \mathcal{F}_{2,u}^2$$

$$+ \varepsilon^{-1}(1 + \varepsilon) |Q^\perp \nabla \partial_{\theta_n}|^2 (\alpha |J_2 u|^2 + 4\beta\partial_n^2 |D_2^2 u|^2),$$

where we used the estimate $(a + b)^2 \leq (1 + \varepsilon)(a^2 + \varepsilon^{-1}b^2)$ which holds true for any $a, b, \varepsilon > 0$. Consequently, taking $\varepsilon = (p-1)|2(p-2)|^{-1}$, we get

$$\psi_5 + \frac{2-p}{4} z_n^{-\frac{2}{3}} |Q^\perp \zeta_n|^2 \leq \frac{1-p}{2} (\mu_Q |J_2 u|^2 + \alpha \partial_n^2 \mathcal{F}_{1,u}^2 + \beta\partial_n^4 \mathcal{F}_{2,u}^2)$$

$$+ \frac{(p^2 - 6p + 7)^+}{2(p-1)} |Q^\perp \nabla \partial_{\theta_n}|^2 (\alpha |J_2 u|^2 + 4\beta\partial_n^2 |D_2^2 u|^2).$$

Since $D_i \partial_{\theta_n} (x) = x_i (|x|^n)^{-1} \partial^i (n^{-1}|x|)$ for any $x \in \mathbb{R}^d$ and $i = 1, \ldots, d$, applying (3.18) and Young's inequality we conclude that $|Q^\perp \nabla \partial_{\theta_n}|^2 \leq M_1 \mu_Q$ in $\mathbb{R}^d$ for some positive constant $M_1$. Hence, we obtain

$$\psi_5 + \frac{2-p}{4} z_n^{-\frac{2}{3}} |Q^\perp \zeta_n|^2 \leq \frac{1-p}{2} (\mu_Q |J_2 u|^2 + \alpha \partial_n^2 \mathcal{F}_{1,u}^2 + \beta\partial_n^4 \mathcal{F}_{2,u}^2)$$

$$+ M_1 \frac{(p^2 - 6p + 7)^+}{p-1} |Q^\perp \nabla \partial_{\theta_n}|^2 (\alpha |J_2 u|^2 + 4\beta\partial_n^2 |D_2^2 u|^2).$$

It remains to estimate $\psi_6$. As above, taking the choice of $\theta$, (3.18) and (3.19) into account, we deduce that $-A\partial_{\theta_n} \leq M_2 \mu_Q$ for some positive constant $M_2$. It thus follows that

$$\psi_6 \leq 4\alpha \partial_n |Q^\perp \nabla \partial_{\theta_n}| \mathcal{F}_{1,u} |J_2 u| + 2M_2 \partial_{\theta_n} \mu_Q \left(\frac{\alpha}{2} |J_2 u|^2 + \beta\partial_n^2 |D_2^2 u|^2\right)$$

$$+ 8\beta\partial_n^3 |Q^\perp \nabla \partial_{\theta_n}| |D_2^2 u| \mathcal{F}_{2,u}$$

$$\leq \alpha\partial_n^2 \frac{p-1}{4} \mathcal{F}_{1,u}^2 + \alpha \left(\frac{16}{p-1} M_1 + \partial_{\theta_n} M_2\right) \mu_Q |J_2 u|^2$$

$$+ \beta\partial_n^4 \left(2M_2 \partial_{\theta_n} + \frac{64}{p-1} M_1\right) \mu_Q |D_2^2 u|^2 + \frac{p-1}{4} \beta\partial_n^4 \mathcal{F}_{2,u}^2.$$
Summing up we deduce that
\[ g \leq K_{3p}|u|^2 + v_0^2 \left[ \alpha \left( \frac{1 - p}{2} + \alpha \frac{\beta}{2} \frac{M_1 (p^2 - 6p + 7)^+}{p - 1} + \frac{16}{p - 1} M_1 + M_2 \right) \right] |J_u|^2 \]
\[ + \frac{1 - p}{2} + \alpha \frac{\beta}{2} \frac{M_1 (p^2 - 6p + 7)^+}{p - 1} + \frac{16}{p - 1} M_1 + M_2 \] \[ \leq \frac{4}{(p - 1)M_1 + M_2} |D_{x}^2 u|^2, \]
where \( K_{3p} = 2c_1 \alpha + (4c_5p)^{-1} \beta \). We choose \( \alpha = \alpha_p \) sufficiently small such that
\[ \alpha \left( \frac{1 - p}{2} + \alpha \frac{\beta}{2} \frac{M_1 (p^2 - 6p + 7)^+}{p - 1} + \frac{16}{p - 1} M_1 + M_2 \right) \leq \frac{p - 1}{4} \]
and, then, \( \beta \in (0, \alpha/2) \) such that
\[ \beta \left( 2M_2 + \frac{64}{p - 1} M_1 + \frac{4}{(p - 1)M_1 + M_2} \right) \leq \frac{p - 1}{24}. \]
With these choices of \( \alpha \) and \( \beta \), we conclude that
\[ g \leq K_{4p}|u|^2 + v_0^4 \left[ \frac{1 - p}{2} \frac{\mu_Q}{\mu} + r + \frac{1}{p - 1} \frac{3}{2} \frac{Q_1^2 + c_1 \sigma^2 + \beta}{4 \alpha c_4} \frac{\beta}{\mu_Q} |J_u|^2 \right] \]
\[ + \frac{1 - p}{2} \frac{\mu_Q}{\mu} + r + \frac{1}{c_1} \frac{c_5 \mu_Q + c_4}{\mu_Q} + \frac{4}{(p - 1) \mu_Q} |D_{x}^2 u|^2. \]
Taking \( \alpha \) and \( \beta \) smaller if needed, we can assume that \( g \leq K_{4p+2}p/2 \). Now, arguing as in the last part of the proof of Theorem 3.13 and letting \( n \) tend to \( +\infty \), estimate (3.10) follows in this case.

**Step 2.** Now, we consider the case \( h = 1 \). Fix \( t > 0 \). From Step 1 we get
\[ \left( |J_u T(t - \sigma)|^2 + |D_{x}^2 u(t, \cdot)|^2 \right)^{1/2} \]
\[ \leq \left( |J_u T(t - \sigma)|^2 + |D_{x}^2 T(t - \sigma)|^2 \right)^{1/2} \]
\[ \leq c \left[ \frac{p}{2} T(t - \sigma)(|J_u u|^2 + |D_{x}^2 u|^2) (u(t, \sigma))^1 - \frac{1}{2} \right] + \left( 1 - \frac{p}{2} \right) \epsilon \frac{1}{c_2} T(t - \sigma) u(t, \sigma) \]
for any \( \sigma \in (0, t) \) and \( \epsilon > 0 \). Since \( \Gamma_{p,2}(t - \sigma) = C_{p,2}(t - \sigma) \) for some positive constant \( C_p \), combining the previous two estimates we deduce
\[ e^{-C_p(t - \sigma)} (|J_u u|^2 + |D_{x}^2 u|^2) \]
\[ \leq T(t)^p + \left( 1 - \frac{p}{2} \right) \epsilon \frac{1}{c_2} T(t - \sigma) u(t, \sigma) \]
\[ + \frac{p}{2} \epsilon \frac{1}{c_2} T(t - \sigma) \left( |J_u u|^2 + |D_{x}^2 u|^2 \right) u(t, \sigma) \]
\[ \leq T(t)^p + \left( 1 - \frac{p}{2} \right) \epsilon \frac{1}{c_2} T(t - \sigma) u(t, \sigma) \]
\[ + \frac{p}{2} \epsilon \frac{1}{c_2} T(t - \sigma) \left( |J_u u|^2 + |D_{x}^2 u|^2 \right) u(t, \sigma) \]
\[ \leq T(t)^p + \left( 1 - \frac{p}{2} \right) \epsilon \frac{1}{c_2} T(t - \sigma) u(t, \sigma) \]
\[ + \frac{p}{2} \epsilon \frac{1}{c_2} T(t - \sigma) \left( |J_u u|^2 + |D_{x}^2 u|^2 \right) u(t, \sigma) \]
\[ T(t - \sigma)[T(\sigma)|f|^p + \Gamma_{2,1,1}(\sigma)T(\sigma)(|f|^2 + |Jf|^2 + \delta)^\frac{\gamma}{2}] \leq e^{\gamma\sigma T(t)}(|f|^2 + |Jf|^2 + \delta)^\frac{\gamma}{2} + T(t)|f|^p \]

for any \( \sigma \in (0, t) \) and some positive constant \( c_p \). Consequently, integrating (3.20) with respect to \( \sigma \in (0, t) \) we deduce that

\[
|D_x^2 u(t, \cdot)|^p \leq \frac{C_p}{1 - e^{-c_p t}} \left\{ \left( \frac{p}{2} \right) \frac{\epsilon^{p/2}}{2} \right\} \Gamma_{2,1,1}(\sigma)T(\sigma)(|f|^2 + |Jf|^2 + \delta)^\frac{\gamma}{2} |d\sigma\bigg|^{\frac{\gamma}{2}} + \left[ \frac{n}{2} \right] tT(t)|f|^p \]

Now we claim that there exists a positive constant \( K_p \) such that

\[
\int_0^t T(t - \sigma)|[J_x u(\sigma, \cdot)]^2 + |D_x^2 u(\sigma, \cdot)|^2 |\,d\sigma| \leq K_p \left(T(t)(|f|^2 + |Jf|^2 + \delta)^\frac{\gamma}{2} + \int_0^t T(t - \sigma) u(\sigma, \cdot) |d\sigma\bigg). \tag{3.21}
\]

Once (3.21) is proved, using again (3.20) we deduce

\[
|D_x^2 u(t, \cdot)|^p \leq \frac{C_p}{1 - e^{-c_p t}} \left[ t + \Gamma_{2,1,1}(t)|T(t)(|f|^2 + |Jf|^2 + \delta)^\frac{\gamma}{2} \right],
\]

where

\[
\Gamma_{2,1,1}(t) = \left[ \frac{(1 - p)^2}{2} \frac{\epsilon^{p/2}}{2} + \frac{p}{2} \frac{\epsilon^{p/2}}{2} K_p + \left( 1 - \frac{p}{2} \right) \frac{\epsilon^{p/2}}{2} t + \frac{p}{2} \frac{\epsilon^{p/2}}{2} K_p. \right]
\]

Letting \( \delta \to 0^+ \) and minimizing on \( \varepsilon > 0 \) we obtain (3.10) with \( k = 2, h = 1 \) and

\[
\Gamma_{2,1,1}(t) = \frac{C_p}{1 - e^{-c_p t}} \left[ t + K_p \left( \frac{\epsilon^{p/2} - 1}{c_p} + t \right)^{1/2} \left( \frac{\epsilon^{p/2} - 1}{c_p} + 1 + t \right)^{1/2} \right].
\]

To conclude, we prove (3.21). To this aim we introduce the same sequence of cut-off functions as in Step 1. For any \( \alpha > 0, t > 0, \delta \in (0, 1), n \in \mathbb{N}, \) and \( f \in C_b^2(\mathbb{R}^d) \) we define the function \( \psi_n : [0, t] \to C(B_n) \) by setting \( \psi_n(\sigma) = T_{n,\delta}^p(t - \sigma)(u_n(\sigma, \cdot) - \delta^{\nu/2}) \) for any \( \sigma \in [0, t] \), where \( u_n(\sigma, \cdot) = (|u_n(\sigma, \cdot)|^2 + \alpha \delta^2 |J_x u_n(\sigma, \cdot)|^2)^{\delta^{\nu/2}}, \) (\( u_n \)) is the sequence of solutions to the Cauchy-Dirichlet problems (2.2) and \( T_{n,\delta}^p(t) \) is the positive semigroup associated to the realization of \( A \) in \( C_b(B_n) \) with homogeneous Dirichlet boundary conditions.

Since \( u_n(\sigma, \cdot) - \delta^{\nu/2} \) vanishes on \( \partial B_n \) for any \( \delta > 0 \), taking [1, Theorem 2.3(ix)] into account, we can show that the function \( \psi_n \) is differentiable in \( (0, t) \) and

\[
\psi_n'(\sigma) = \mu T_n^p(t - \sigma)(u_n^{1 - \frac{\delta}{2}} g),
\]

where \( g \) is as in Step 1 with \( \beta = 0 \) and \( u \) being replaced by \( u_n \). Hence, we can estimate

\[
g \leq \frac{\alpha}{2c_{1p}} |u_n|^2 + \left\{ \frac{\alpha \delta^2}{2} \left[ r + \frac{3}{2(p - 1)\mu Q} \right] \frac{\epsilon^{p/2}}{2} \right\} \left[ \frac{1}{2} + \frac{\alpha}{2(p - 1)\mu Q} \left( \frac{M_1}{p - 2} + \frac{16}{p - 1} \frac{M_2}{2} \right) \right] |J_x u_n|^2
\]

The coefficient in front of \( |D_x^2 u_n|^2 \) is clearly negative and choosing properly \( \alpha \) we can make negative also the coefficient in front of \( |J_x u_n|^2 \). In this way we conclude
that \( pn_n^{1-2/p} g \leq -K_{0p}(|x|u_n|^{2} + |D^2_xu_n|^2)\|u_n\|^{1-2/p} + K_1 p u_n \) for some positive constants \( K_0 \) and \( K_1 \). Thus,
\[
\psi_n' \leq -K_{0p}T^D_n(t - \cdot )[|J_xu_n|^2 + |D^2_xu_n|^2]u_n^{1-\frac{2}{p}} + K_1 p T^D_n(t - \cdot )u_n
\]
in \((0, t)\), which we integrate with respect to \( \sigma \in (\varepsilon, t - \varepsilon) \), \( \varepsilon > 0 \). Letting first \( n \) tend to \( +\infty \) and then \( \varepsilon \) tend to \( 0^+ \), (3.21) follows.

Step 3. Estimate (3.10) with \( k = 2 \) and \( h = 0 \) can be obtained by the previous step, the semigroup law, (2.5) and Theorem 3.13. Indeed, we have
\[
|D^2_xT(t)f|^p = |D^2_xT(t/2)T(t/2)f|^p \leq \Gamma_{p, 2.1}(t/2)(|T(t/2)f|^2 + |J_xT(t/2)f|^2)^{\frac{p}{2}}
\]
\[
\leq \Gamma_{p, 2.1}(t/2)T(t/2)(|T(t/2)f|^p + |J_xT(t/2)f|^p)
\]
\[
\leq \Gamma_{p, 2.1}(t/2)T(t)|f|^p + \Gamma_{p, 2.1}(t/2)\Gamma_{p, 1.0}(t/2)T(t)|f|^p
\]
for any \( t > 0 \), whence the claim follows with \( \Gamma_{p, 2.0}(t) = \Gamma_{p, 2.1}(t/2)(1 + \Gamma_{p, 1.0}(t/2)) \).

Example 3.16. Let \( \mathcal{A} \) be as in Example 3.8 and assume further that for any \( k, s = 1, \ldots, m \), the function \( c_{ks} \) belongs to \( C^{1, \alpha}_{\text{loc}}(\mathbb{R}^d) \) and \( |\nabla c_{ks}(x)| = O(|x|^{\gamma}) \), as \( |x| \to +\infty \), with \( 0 < \alpha < \beta \leq \gamma \), then also Hypotheses 3.12 hold true and Theorem 3.13 can be applied. Indeed, since \( \mu \xi(x) = (1 + |x|^2)^{\frac{\alpha}{2}} \mu_0 \) for any \( x \in \mathbb{R}^d \), where \( \mu_0 > 0 \) is the minimum eigenvalue of the constant matrix \( Q^0 \), and the function \( r \), which bounds form above the quadratic form associated to the Jacobian matrix of \( b \), is given by \( r(x) = -b_0(1 + |x|^\beta)^{\alpha} \) for any \( x \in \mathbb{R}^d \), the sum of the first two terms in the definition of \( K_p \), which is the “good” part of \( K_p \) (see Hypotheses 3.12), behaves like \( |x|^{2(\beta + \gamma)} \) as \( |x| \to +\infty \). Now it is immediate to check that \( \mu_{q}^{-1} \xi^2 = O(|x|^{2\gamma - 2}) \) and \( \varphi^2 = O(x^{2\gamma}) \) as \( |x| \to +\infty \), where we use Landau’s formalism. Hence the supremum in Hypothesis 3.12(ii) is finite and estimate (3.10) with \( k = 1, h = 0, 1 \) holds true.

Without much effort one can realize that the functions \( \mathcal{B}_2 \) and \( \mathcal{D}_2 \) grow at infinity as \( |x|^{2\beta - 1} \) and \( |x|^{2\gamma - 2} \), respectively. Thus, if \( c_{ij} \in C^{2+\alpha}_{\text{loc}}(\mathbb{R}^d) \) and \( |D^2 c_{ks}(x)|^2 = O(|x|^{2\alpha}) \) as \( |x| \to +\infty \), for any \( k, s = 1, \ldots, m \), then Hypotheses 3.14 are satisfied too and Theorem 3.15 can be applied.

Starting from estimate (3.10), it is routine to prove the following partial characterization of \( D(A_p) \).

**Corollary 3.17.** Under Hypotheses 3.12, for any \( t > 0 \) and \( p \in (1, +\infty) \) the operator \( T(t) \) is bounded from \( L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m) \) into \( \mathcal{W}^{1, p}_{\mu}(\mathbb{R}^d; \mathbb{R}^m) \) and there exist two positive constants \( N_{0, p} \) and \( \omega_{1, p} \) such that
\[
\|T(t)f\|_{1, p, \mu} \leq N_{0, p} \max\{t^{-\frac{1}{2}}, 1\} \|f\|_{p, \mu}, \quad t > 0, \quad f \in L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m); \tag{3.22}
\]
\[
\|T(t)f\|_{1, p, \mu} \leq N_{0, p} e^{\omega_{1, p} t} \|f\|_{1, p, \mu}, \quad t > 0, \quad f \in \mathcal{W}^{1, p}_{\mu}(\mathbb{R}^d; \mathbb{R}^m). \tag{3.23}
\]
Moreover, \( D(A_p) \) is continuously embedded into \( \mathcal{W}^{1, p}_{\mu}(\mathbb{R}^d; \mathbb{R}^m) \) for any \( p \in (1, +\infty) \).

If also Hypotheses 3.14 are satisfied, then each operator \( T(t) \) is bounded from \( L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m) \) into \( \mathcal{W}^{2, p}_{\mu}(\mathbb{R}^d; \mathbb{R}^m) \), for any \( p \in (1, +\infty) \), and there exist two positive constants \( N_{2, p} \) and \( \omega_{2, p} \) such that
\[
\|T(t)f\|_{2, p, \mu} \leq N_{2, p} \max\{t^{-\frac{1}{2}}, 1\} \|f\|_{p, \mu}, \quad t > 0, \quad f \in L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m); \tag{3.24}
\]
\[
\|T(t)f\|_{2, p, \mu} \leq N_{2, p} e^{\omega_{2, p} t} \|f\|_{3, p, \mu}, \quad t > 0, \quad f \in \mathcal{W}^{j, p}_{\mu}(\mathbb{R}^d; \mathbb{R}^m), \quad j = 1, 2. \tag{3.25}
\]
Proof. To begin with, we prove estimates (3.22) and (3.23); the proofs of (3.24) and (3.25) are completely similar. Fix \( p \in (1, +\infty) \), \( f \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m) \). Integrating (3.10) with respect to the measure \( \mu \) we obtain that

\[
\int_{\mathbb{R}^d} |J_x T(t)f|^p d\mu \leq \Gamma_{p,1,h}(t) \int_{\mathbb{R}^d} T(t) \left( \sum_{k=0}^j |D^k f|^2 \right)^{\frac{p}{2}} d\mu \\
= \Gamma_{p,1,h}(t) \int_{\mathbb{R}^d} \left( \sum_{k=0}^h |D^k f|^2 \right)^{\frac{p}{2}} d\mu
\]

for any \( t > 0 \). Since \( \mu_i = \tilde{c}_i \mu \) for some positive constant \( \tilde{c}_i \) and any \( i = 1, \ldots, m \) (see Theorem 3.5), from the previous estimate, we conclude that

\[
\|T(t)f\|_{1,p,\mu} \leq \tilde{N}_0,p e^{\omega_1,t} t^{-\frac{1}{2}h}\|f\|_{j,p,\mu}, \quad t > 0, \ j = 0, 1,
\]

for some positive constants \( \tilde{N}_0,p \) and \( \omega_1,p \). Estimate (3.23) follows by a density argument, taking Remark 3.10 into account. If \( j = 0 \), then we can remove the exponential term from the right-hand side of the previous estimate. Indeed, for \( t > 1 \), using (3.9) we can estimate

\[
\|T(t)f\|_{1,p,\mu} = \|T(1)T(t-1)f\|_{1,p,\mu} \leq \tilde{N}_0,p e^{\omega_1,t} \|T(t-1)f\|_{p,\mu} \\
\leq 2^{\frac{1}{2h}} \tilde{N}_0,p e^{\omega_1,t} \|f\|_{p,\mu}
\]

and (3.22) follows, again by a density argument.

Let us now complete the proof, by showing that \( D(A_p) \hookrightarrow W^{1,p}(\mathbb{R}^d; \mathbb{R}^m) \). We fix \( p \in (1, +\infty) \) and observe that, in view of (3.22), the resolvent operator \( R(\lambda, A_p) \) is defined for any \( \lambda > 0 \) and

\[
R(\lambda, A_p)f = \int_0^{+\infty} e^{-\lambda t} T(t)f dt, \quad f \in L_p^p(\mathbb{R}^d; \mathbb{R}^m).
\]

Fix \( \lambda > 0 \), \( u \in D(A_p) \) and let \( f \in L_p^p(\mathbb{R}^d; \mathbb{R}^m) \) be such that \( u = R(\lambda, A_p)f \). Using (3.10) with \( h = 0 \) and \( k = 1 \), we can estimate

\[
\|J_x u\|_{p,\mu} \leq c_{1,p} \|f\|_{p,\mu} \int_0^{+\infty} t^{-\frac{1}{2}} e^{-\lambda t} dt \leq c_{2,p} \lambda^{-\frac{1}{2}} \|f\|_{p,\mu} \\
\leq c_{2,p}(\lambda^{\frac{1}{2}} \|u\|_{p,\mu} + \lambda^{-\frac{1}{2}} \|A_p u\|_{p,\mu})
\]

for some positive constants \( c_{1,p} \) and \( c_{2,p} \), independent of \( u \). The previous chain of inequalities shows that \( D(A_p) \) is compactly embedded into \( W^{1,p}(\mathbb{R}^d; \mathbb{R}^m) \). Moreover, minimizing with respect to \( \lambda > 0 \) we also conclude that

\[
\|J_x u\|_{p,\mu} \leq c_{3,p} \|u\|_{p,\mu}^{\frac{1}{2}} \|A_p u\|_{p,\mu}^{\frac{1}{2}},
\]

the constant \( c_{3,p} \) being independent of \( u \). \( \square \)

4. Asymptotic behaviour

To study the asymptotic behaviour of \( T(t)f \) as \( t \to +\infty \) we need some additional hypotheses.

**Hypothesis 4.1.** The coefficients of the operator \( A \) belong to \( C^{1+\alpha}_{\text{loc}}(\mathbb{R}^d) \). Moreover, there exists a positive constant \( c \) such that

\[
(i) \ |q_{ij}(x)| \leq c(1 + |x|^2)^{\alpha} \varphi(x), \quad (ii) \ |b(x), x| \leq c(1 + |x|^2) \varphi(x),
\]

for any \( x \in \mathbb{R}^d \) and \( i, j = 1, \ldots, m \).
Remark 4.2. We stress that, in general, Hypothesis 4.1 is not implied by Hypothesis 2.1(iv). Consider for instance the one-dimensional operator $A = qD_x^2 + bD_x$, where $q(x) = (1 + x^2)e^{x^2}$ and $b(x) = -3x(1 + x^2)e^{x^2}$ for any $x \in \mathbb{R}$. It is easy to check that the function $x \mapsto x^2 + 1$ satisfies Hypothesis 2.1(iv).

We claim that no function $\varphi$ satisfying both Hypothesis 2.1(iv) and Hypothesis 4.1 exists. We argue by contradiction and assume that such a function exists. To begin with, we observe that (4.1)(i) implies that

$$\varphi(x) \geq cx^4,$$

for some positive constant $c$. From this condition we can easily deduce that there exists an increasing sequence $(x_n)$, which blows up as $n \to +\infty$, such that

$$\varphi'(x_n) > cx_n^3 e^{x_n^2} - 1, \quad n \in \mathbb{N}.$$  \hfill (4.3)

Indeed, if this were not the case, there would exist $M_1 > 0$ such that $\varphi'(x) \leq cx^3 e^{x^2} - 1$ for any $x \geq M_1$. This inequality, integrated between $M_1$ and $x$, gives $\varphi(x) \leq cx^4/4 + K$ for some positive constant $K$, which, clearly, contradicts (4.2).

Since we are assuming that $\varphi$ satisfies Hypothesis 2.1(iv), we can determine a positive constant $M_2$ such that $q\varphi'' + b\varphi' \leq 0$ in $[M_2, +\infty)$, from which we deduce that $\varphi''(x) \leq 3x\varphi'(x)$ for any $x \geq M_2$. Let $n_0$ be the smallest integer such that $x_{n_0} \geq M_2$. Then, from the previous differential inequality we can infer that $\varphi'(x) \leq e^{3x^2/2}\varphi'(x_{n_0})$ for any $x \geq x_{n_0}$. This estimate combined with (4.3) leads us to a contradiction.

The following result plays a crucial role in the study of the asymptotic behaviour of the function $T(t)f$ as $t \to +\infty$.

Proposition 4.3. Under Hypothesis 4.1, for any $f \in C^{3+\alpha}_{c}(\mathbb{R}^d; \mathbb{R}^m)$ the $L^2(\mathbb{R}^d)$-norm of $|J_xT(t)f|$ vanishes as $t$ tends to $+\infty$.

Proof. To begin with, we recall that

$$\int_{\mathbb{R}^d} A\psi \, d\mu = 0,$$  \hfill (4.4)

for any bounded function $\psi \in D_{\text{max}}(A) = \{ u \in C_0(\mathbb{R}^d) \cap \cap_{\mu \in +\infty} W^{1,2}_{\text{loc}}(\mathbb{R}^d) : Au \in C_0(\mathbb{R}^d) \}$ (see [19, Proposition 9.1.1]). In particular, if $\psi \in C^2_0(\mathbb{R}^d)$, then $\psi^2 \in C^2(\mathbb{R}^d)$ and writing (4.4) with $\psi$ being replaced by $\psi^2$, we easily conclude that

$$\int_{\mathbb{R}^d} \psi A\psi \, d\mu = - \int_{\mathbb{R}^d} |Q^+ \nabla \psi|^2 \, d\mu.$$  \hfill (4.5)

We now introduce a decreasing function $\vartheta \in C^2(\mathbb{R})$ such that $\chi_{\langle -\infty, 1 \rangle} \leq \vartheta \leq \chi_{\langle -\infty, 2 \rangle}$ and, for any $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, we set $\vartheta_n(x) = \vartheta(n^{-1}|x|)$.

As it is immediately seen, for any $f \in C_0(\mathbb{R}^d; \mathbb{R}^m)$ and $t > 0$, the function $\vartheta_nT(t)f$ belongs to $C^2_{\text{c}}(\mathbb{R}^d; \mathbb{R}^m)$, so that, by (4.5), it follows that

$$\int_{\mathbb{R}^d} (\vartheta_n T(t)f) \, A(\vartheta_n T(t)f) \, d\mu = - \int_{\mathbb{R}^d} |Q^+ \nabla (\vartheta_n T(t)f)|^2 \, d\mu,$$  \hfill (4.6)

for any $j = 1, \ldots, m$ and $n \in \mathbb{N}$.

Now, we adapt to our situation the procedure in [8, Proposition 3.5] and [21, Proposition 2.15] (see also [20, Proposition 2.6]). We fix $f \in C_0(\mathbb{R}^d; \mathbb{R}^m), n \in \mathbb{N}$, and observe that

$$\frac{d}{dt} \| \vartheta_n T(t)f \|^2_{L^2(\mu)} = 2 \sum_{j=1}^m \int_{\mathbb{R}^d} \vartheta_n^2 (T(t)f) \, A(T(t)f) \, d\mu + 2 \int_{\mathbb{R}^d} \vartheta_n^2 \langle CT(t)f, T(t)f \rangle \, d\mu.$$
for any $t > 0$. A straightforward computation reveals that
\[
\vartheta_n A(T(t)f_j) = A(\vartheta_n(T(t)f_j) - (T(t)f_j)A\vartheta_n – 2(Q\nabla \vartheta_n, \nabla_x(T(t)f_j),
\]
which we replace in (4.7). Taking (4.6) into account, we get
\[
\frac{d}{dt}\|\vartheta_n(T(t)f_j)\|_{L^\infty}^2 \leq 2 \sum_{j=1}^m \int_{\mathbb{R}^d} \vartheta_n(Q\nabla \vartheta_n, \nabla_x(T(t)f_j))^2 d\mu - 4 \sum_{j=1}^m \int_{\mathbb{R}^d} \vartheta_n(T(t)f_j) (Q\nabla \vartheta_n, \nabla_x(T(t)f_j)) d\mu.
\]
Note that
\[
\left| \int_{\mathbb{R}^d} \vartheta_n(T(t)f_j) (Q\nabla \vartheta_n, \nabla_x(T(t)f_j)) d\mu \right|
\leq \int_{\mathbb{R}^d} |(T(t)f_j) (Q\nabla \vartheta_n, \nabla_x(T(t)f_j))| d\mu + \int_{\mathbb{R}^d} |(T(t)f_j) (Q\nabla \vartheta_n)|^2 d\mu
\leq \left( \int_{\mathbb{R}^d} |Q^{\frac{1}{2}} \nabla (\vartheta_n(T(t)f_j))^2 d\mu \right)^\frac{1}{2} \left( \int_{\mathbb{R}^d} |(T(t)f_j) (Q\nabla \vartheta_n)|^2 d\mu \right)^\frac{1}{2}
+ \int_{\mathbb{R}^d} |(T(t)f_j) (Q\nabla \vartheta_n)|^2 d\mu
\leq \frac{1}{4} \int_{\mathbb{R}^d} |Q^{\frac{1}{2}} \nabla (\vartheta_n(T(t)f_j))^2 d\mu + 2 \int_{\mathbb{R}^d} (T(t)f_j) (Q\nabla \vartheta_n)^2 d\mu,
\]
so that we can continue estimate (4.8) and obtain
\[
\frac{d}{dt}\|\vartheta_n(T(t)f_j)\|_{L^\infty}^2 \leq - \sum_{j=1}^m \int_{\mathbb{R}^d} \vartheta_n |Q^{\frac{1}{2}} \nabla (\vartheta_n(T(t)f_j))^2 d\mu - 2 \int_{\mathbb{R}^d} \vartheta_n (b, \nabla \vartheta_n) (T(t)f_j)^2 d\mu + 2\|f\|^2 \int_{\mathbb{R}^d} (|\text{Tr}(QD^2 \vartheta_n)| + 4|Q^{\frac{1}{2}} \nabla \vartheta_n|^2) d\mu.
\]
Using (4.1)(ii) we can estimate
\[
- \vartheta_n(x) (b(x), \nabla \vartheta_n(x)) (T(t)f)(x)^2 = \vartheta_n(x) |\vartheta'(n^{-1}|x|)| (b(x), x) \frac{1}{n|x|} |(T(t)f)(x)|^2
\leq c \vartheta_n(x) |\vartheta'(n^{-1}|x|)| \frac{(1 + |x|^2)}{n|x|} \varphi(x) |(T(t)f)(x)|^2
\leq 5c \|\vartheta'\|_{\infty} \|f\|^2 \frac{\varphi(x)}{B_{2n} \setminus B_n}(x)
\]
for any $x \in \mathbb{R}^d$. Hence,
\[
- \int_{\mathbb{R}^d} \vartheta_n (b, \nabla \vartheta_n) (T(t)f)^2 d\mu \leq 5c \|\vartheta'\|_{\infty} \|f\|^2 \int_{B_{2n} \setminus B_n} \varphi d\mu =: a_n
\]
and the sequence $(a_n)$ vanishes as $n \to +\infty$, since the function $\varphi$ belongs to $L^1_{\mu}(\mathbb{R}^d)$ (see [19, Chapter 9]). Similarly, using (4.1)(i), we can show that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^d} (|\text{Tr}(QD^2 \vartheta_n)| + 4|Q^{\frac{1}{2}} \nabla \vartheta_n|^2) d\mu = 0.
\]
Summing up, we have shown that
\[
\frac{d}{dt}\|\vartheta_n(T(t)f)\|_{L^\infty}^2 \leq - \sum_{j=1}^m \int_{\mathbb{R}^d} \vartheta_n |Q^{\frac{1}{2}} \nabla (\vartheta_n(T(t)f_j))^2 d\mu + b_n
\]
Applying Fatou lemma to the previous formula, we deduce that

$$\limsup_{t \to +\infty} \int_{\mathbb{R}^d} d\mu = b_n t.$$  

Applying Fatou lemma to the previous formula, we deduce that

$$\limsup_{t \to +\infty} \int_{\mathbb{R}^d} d\mu = b_n t.$$  

It thus follows that the function $|Q^{1/2} \nabla x(T(\cdot)\mathbf{f})_j|$ belongs to $L^2([0, +\infty); L^2_d(\mathbb{R}^d))$ for any $j = 1, \ldots, m$. In particular, if we set

$$h(t) = \sum_{j=1}^m \int_{\mathbb{R}^d} |\nabla x(T(t)\mathbf{f})_j|^2 d\mu,$$

then $h$ belongs to $L^1((0, +\infty))$ and its $L^1$-norm is bounded by $\mu_0^{-1} \|\mathbf{f}\|^2_{L^2_d}$, where $\mu_0$ denotes the infimum over $\mathbb{R}^d$ of the minimum eigenvalue $\mu_Q(x)$ of the matrix $Q(x)$.

We now assume that $\mathbf{f} \in C^{1+\alpha}_c(\mathbb{R}^d; \mathbb{R}^m)$. Since the coefficients of the operator $\mathcal{A}$ are in $C^{1+\alpha}_c(\mathbb{R}^d)$, the function $D_j(T(\cdot)\mathbf{f})_i$ is differentiable with respect to time (see (Th. A.1) and $D_{t_j}D_j(T(\cdot)\mathbf{f})_i = D_j(D_j(T(\cdot)\mathbf{f})_i) = D_j(\mathcal{A}T(\cdot)\mathbf{f})_i$, on $[0, +\infty) \times \mathbb{R}^d$. By [10, Proposition 3.2], $T(t)\mathbf{f} = \mathcal{A}T(t)\mathbf{f}$ for any $t > 0$. Thus it follows that $D_j(D_j(T(\cdot)\mathbf{f})_i) = D_j(T(t)\mathbf{A})$.

For any $n \in \mathbb{N}$, let us introduce the function $h_n : [0, +\infty) \to \mathbb{R}$ defined by

$$h_n(t) = \sum_{j=1}^m \int_{\mathbb{R}^d} |\nabla x(T(t)\mathbf{f})_j|^2 d\mu, \quad t > 0.$$  

As it is immediately seen, $h_n$ converges to $h$ in $L^1((0, +\infty))$ and pointwise in $[0, +\infty)$. Moreover, applying the dominated convergence and taking into account that the functions $D_j(D_j(T(\cdot)\mathbf{f})_i)$ and $D_j(T(\cdot)\mathbf{f})_i$ are continuous in $[0, +\infty) \times \mathbb{R}^d$, we can show that $h_n$ is differentiable in $(0, +\infty)$ and

$$h'_n(t) = \sum_{j=1}^m \int_{\mathbb{R}^d} \nabla x(T(t)\mathbf{f})_j, \nabla x(T(t)\mathcal{A} \mathbf{f})_j d\mu, \quad t > 0.$$  

By applying Cauchy-Schwarz inequality, we deduce that $\langle \nabla x(T(\cdot)\mathbf{f})_j, \nabla x(T(\cdot)\mathcal{A} \mathbf{f})_j \rangle$ belongs to $L^1((0, +\infty) \times \mathbb{R}^d; dt \times d\mu)$ for any $j = 1, \ldots, m$. Hence, the above results show that $h_n$ converges to the function $\sum_{j=1}^m \int_{\mathbb{R}^d} \nabla x(T(\cdot)\mathbf{f})_j, \nabla x(T(\cdot)\mathcal{A} \mathbf{f})_j d\mu$ as $n \to +\infty$. Since $W^{1,1}((0, +\infty)) \hookrightarrow C_b((0, +\infty))$ and $h_n$ converges in $W^{1,1}((0, +\infty))$, it follows that $h \in W^{1,1}((0, +\infty))$ and $h_n$ converges to $h$ uniformly in $(0, +\infty)$. In particular, $h$ vanishes as $t \to +\infty$. 

Now we study the asymptotic behaviour of $T(t)$. To this aim, using the same notation as in the proof of Th. 3.5, for any $\mathbf{f} \in L^1_d(\mathbb{R}^d; \mathbb{R}^m)$ we set

$$\mathcal{M}_\mathbf{f} = \sum_{j=1}^m \int_{\mathbb{R}^d} f_j d\mu_j,$$

where $\mu_j = \xi_j \mu$ for any $j = 1, \ldots, m$.

**Theorem 4.4.** Let Hypothesis 4.1 be satisfied. Then, $T(t)\mathbf{f}$ converges to $\mathcal{M}_\mathbf{f} \xi$ locally uniformly in $\mathbb{R}^d$ as $t \to +\infty$, for any $\mathbf{f} \in B_0(\mathbb{R}^d; \mathbb{R}^m)$. Further, if $\mathbf{f} \in L^1_d(\mathbb{R}^d; \mathbb{R}^m)$, then the function $T(t)\mathbf{f}$ converges to $\mathcal{M}_\mathbf{f} \xi$ in $L^1_d(\mathbb{R}^d; \mathbb{R}^m)$ as $t \to +\infty$. 
Proof. The last statement is a straightforward consequence of the first one. Indeed, if \( f \in C_b(\mathbb{R}^d; \mathbb{R}^m) \), then the first statement and the dominated convergence theorem immediately show that \( T(t)f \) converges to \( M_\xi f \) in \( L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m) \) as \( t \to +\infty \). In the general case when \( f \in L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m) \), we fix a sequence \( (f_n) \subset C_b(\mathbb{R}^d; \mathbb{R}^m) \) converging to \( f \) in \( L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m) \) as \( n \to +\infty \). Taking (3.9) into account, we can estimate

\[
\|T(t)f - M_\xi f\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)} \leq 2^{\frac{1}{p-1}}\|f - f_n\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)} + \|T(t)f_n - M_\xi f\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)}
\]

\[
+ |M_{\xi_n} - M_\xi| \left( \sum_{i=1}^m \xi_i^p \mu_i(\mathbb{R}^d) \right)^{\frac{1}{p}} .
\]

Letting first \( t \) and then \( n \) tend to \( +\infty \), we easily conclude that \( T(t)f \) converges to \( M_\xi f \) in \( L^p_{\mu}(\mathbb{R}^d; \mathbb{R}^m) \) also in this case.

In view of the strong Feller property of the semigroup \( (T(t)) \) (see Section 2), it suffices to prove the first statement for functions \( f \in C_b(\mathbb{R}^d; \mathbb{R}^m) \). Actually, we can limit ourselves to considering functions \( f \in C_\alpha^{3+\alpha}(\mathbb{R}^d; \mathbb{R}^m) \). Indeed, each \( f \in C_0(\mathbb{R}^d; \mathbb{R}^m) \) is the local uniform limit in \( \mathbb{R}^d \) of a sequence \( (f_n) \subset C_\alpha^{3+\alpha}(\mathbb{R}^d; \mathbb{R}^m) \). By Proposition 2.7, up to a subsequence, \( T(t)f_n \) converges to \( T(t)f \) uniformly in \([0, +\infty) \times \mathbb{R}^d \) for any \( R > 0 \). Hence, we can estimate

\[
\|T(t)f - M_\xi f\|_{C_b(\mathbb{R}^d; \mathbb{R}^m)} \leq \sup_{t \geq 0} \|T(t)f - T(t)f_n\|_{C_b(\mathbb{R}^d; \mathbb{R}^m)}
\]

\[
+ \|T(t)f_n - M_\xi f\|_{C_b(\mathbb{R}^d; \mathbb{R}^m)} + |M_{\xi_n} - M_\xi| ,
\]

for any \( t > 0 \) and \( R > 0 \) (take Remark 3.3 into account). Hence, if \( T(t)f_n \) converges to \( M_\xi f \) locally uniformly in \( \mathbb{R}^d \) for any \( n \in \mathbb{N} \) as \( t \to +\infty \), then letting \( t \) and \( n \) tend to \( +\infty \) in the previous estimate we conclude that \( T(t)f \) converges to \( M_\xi f \) as \( t \to +\infty \), locally uniformly in \( \mathbb{R}^d \).

Fix a function \( f \in C_\alpha^{3+\alpha}(\mathbb{R}^d; \mathbb{R}^m) \) and a sequence \( (t_n) \) diverging to \( +\infty \) such that \( t_n > 1 \) for any \( n \in \mathbb{N} \). Since the sequence \( (T(t_n-1)f) \) is bounded, by Proposition 2.7 it follows that there exists a subsequence \( (t_{n_k}) \) such that \( T(t_{n_k})f = T(1)T(t_{n_k} - 1)f \) converges to some function \( g \in C_\alpha^{3+\alpha}(\mathbb{R}^d; \mathbb{R}^m) \), locally uniformly on \( \mathbb{R}^d \). Clearly, \( (T(t_n)f)_{t_j} \) converges to \( g_j \) also in \( L^p_{\mu}(\mathbb{R}^d) \) for any \( j = 1, \ldots, m \). We claim that \( g \) is constant. Indeed, from Proposition 4.3, it follows that \( \|J_x(T(t_n)f)||_{L^2(\mathbb{R}^d)} \) vanishes as \( k \to +\infty \). Consequently, \( |J_x g| \equiv 0 \) and \( g \) is constant. Since for any \( t > 0 \) the function \( T(t)g \) is the \( L^p_{\mu} \)-limit of the sequence \( (T(t + t_n)f) \) as \( k \to +\infty \), taking again Proposition 4.3 into account and arguing as above, we deduce that the function \( h = T(t)g \) is independent of \( x \) and \( D_t h(t) = C(x)h(t) \) for any \( t > 0 \) and \( x \in \mathbb{R}^d \). Hence, we can fix \( x = 0 \).

Denote by \( \lambda_1 = 0, \lambda_2, \ldots, \lambda_s \) (\( s \leq m \)) the eigenvalues of the matrix \( C(0) \) and by \( a_i \) and \( q_i \) (\( i = 1, \ldots, s \)), respectively, their algebraic and geometric multiplicities. By the Jordan normal form theorem, we can write \( C(0) = PJP^{-1} \) for some invertible matrix \( P \) (with entries \( p_{ij} \)) and some block matrix \( J = \text{diag}(J_1, \ldots, J_s) \). Each matrix \( J_i \) has dimension \( a_i \) and itself splits into \( q_i \) sub-blocks \( J_{ij} = \lambda_i I + N_{ij} \) for some matrix \( N_{ij} \) such that \( (N_{ij})_{hk} = \delta_{h+1,k} \) for each \( h \) and \( k \). In particular, if \( n_{ij} \) denotes the dimension of the matrix \( N_{ij} \) then \( N_{ij}^{n_{ij}} \) is the trivial matrix. Based on this remark, we can infer that \( e^{tJ_{ij}} = e^{t\lambda_i}e^{tN_{ij}} \), where \( e^{tN_{ij}} \) is an upper triangular matrix with

\[
(e^{tN_{ij}})_{hk} = \frac{t^{h-k}}{(k-h)!}, \quad k \geq h .
\]

Taking Lemma 2.2 into account, which shows that \( \text{Re} \lambda_h < 0 \) for any \( h = 2, \ldots, s \), it thus follows that the norm of the matrix \( e^{tJ_{ij}} \) exponentially decreases to zero as \( t \to +\infty \) for any \( h = 2, \ldots, s \). As a byproduct of all the above remarks, if we set
\(P^{-1}g = \eta\), then for any \(i = 1, \ldots, m\) we can write
\[
(T(t)g)_i = (Pe^{tJ}P^{-1}g)_i = \sum_{j,h=1}^{m} p_{ij}(e^{tJ})_{jh}\eta_h = \sum_{j,h=1}^{a_1} p_{ij}(e^{tJ})_{jh}\eta_h + o(1, +\infty)
\]
for any \(t > 0\) where, following Landau’s formalism, we have denoted by \(o(1, +\infty)\) a function which vanishes as \(t\) tends to \(+\infty\).

Let us rewrite the sum in the last side of the previous formula, in a much more convenient way. For this purpose, we set \(T_n = 0\), \(T_k = \sum_{j=1}^{k} n_{ij}\) for \(k = 1, \ldots, a_1\) and observe that
\[
\sum_{j,h=1}^{a_1} p_{ij}(e^{tJ})_{jh}\eta_h = \sum_{k=1}^{q_1} \sum_{j,h=\text{N}_{k-1}}^{\text{N}_k} p_{ij}(e^{tJ_k})_{jh}\eta_h = \sum_{k=1}^{q_1} \sum_{j,h=\text{N}_{k-1}+1}^{\text{N}_k} p_{ij}(e^{tJ_k})_{jh}\eta_h
\]
\[
= \sum_{j=1}^{a_1} p_{ij}\eta_j + \sum_{k=1}^{q_1} \sum_{j=\text{N}_{k-1}+1}^{\text{N}_k} \sum_{h=\text{N}_{k-1}+1}^{\text{N}_k} p_{ij}(e^{tJ_k})_{jh}\eta_h.
\]
The last term in the previous chain of inequalities is a polynomial vanishing at zero, which we denote by \(q\). Summing up, we have shown that
\[
(T(t)g)_i = \sum_{j=1}^{a_1} p_{ij}\eta_j + q(t) + o(1, +\infty), \quad t > 0.
\]

Since, again by Proposition 2.7, up to a subsequence \(T(t_{n_k})g\) converges locally uniformly on \(\mathbb{R}^d\) as \(k \to +\infty\), from (4.10) we deduce that \(q\) is the null polynomial, so that
\[
(T(t)g)_i = \sum_{j=1}^{a_1} p_{ij}\eta_j + o(1, +\infty)
\]
for \(t > 0\). The above formula shows that \(T(t)g\) converges as \(t \to +\infty\). By the proof of Theorem 3.5, we know that the average of \(T(\cdot)g\) over the interval \([0, t]\) converges as \(t \to +\infty\) to \(Mg\xi\). Since the function \(T(\cdot)g\) is bounded and converges at infinity, we conclude that \(T(t)g\) converges to \(Mg\xi\) as \(t \to +\infty\). Actually \(Mg = M\xi\). Indeed, by invariance property of the measures \(\mu_j\), we can write
\[
\sum_{j=1}^{m} \int_{\mathbb{R}^d} (T(t_{n_k})f)_j d\mu_j = \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\mu_j
\]
and letting \(k\) tend to \(+\infty\) by dominated convergence, we obtain
\[
\sum_{j=1}^{m} \int_{\mathbb{R}^d} g_j d\mu_j = \sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j d\mu_j
\]
i.e., \(Mg = M\xi\).

Now, we can prove that \(T(\cdot)f\) converges to \(M\xi\) locally uniformly in \(\mathbb{R}^d\) as \(t \to +\infty\). For this purpose, we fix \(R > 0\) and estimate
\[
\|T(t)f - M\xi\|_{C(\mathbb{R}^d)} \\
\leq \|T(t - t_{n_k})f - g\|_{C(\mathbb{R}^d)} + \|T(t - t_{n_k})g - M\xi\|_{C(\mathbb{R}^d)} \\
\leq \sup_{s > 0} \|T(s)(T(t_{n_k})f - g)\|_{C(\mathbb{R}^d)} + \|T(t - t_{n_k})g - M\xi\|_{C(\mathbb{R}^d)}
\]
for any $t$ and $k \in \mathbb{N}$ such that $t - t_n > 0$. Fix $k \in \mathbb{N}$. Letting $t \to +\infty$ in the above estimate gives
\[
\limsup_{t \to +\infty} \|T(t)f - M_t \xi\|_{C(\mathbb{T}_R^n)} \leq \sup_{s > 0} \|T(s)(T(t_n)f - g)\|_{C(\mathbb{T}_R^n)}
\]
for any $k \in \mathbb{N}$. Finally, using Proposition 2.7, we can let $k$ tend to $+\infty$ and conclude that \( \limsup_{t \to +\infty} \|T(t)f - M_t \xi\|_{C(\mathbb{T}_R^n)} = 0 \), and we are done. \( \square \)

Example 4.5. Let $\mathcal{A}$ be as in Example 3.8. It is easy to show that, for any $\sigma > 0$, the function $\varphi_\sigma : \mathbb{R}^d \to \mathbb{R}$ (\( \sigma > 0 \)), defined by $\varphi_\sigma(x) = (1 + |x|^2)^{\sigma}$, for any $x \in \mathbb{R}^d$, satisfies Hypothesis 2.1(iv) too. Hence, if $\sigma > (\gamma - 1)^+$ then also Hypothesis 4.1 is satisfied and Theorem 4.4 can be applied.

Appendix A. Regularity of solutions to parabolic problems

Theorem A.1. Let $\Omega$ be a domain of $\mathbb{R}^d$ and let $\mathcal{A} = \text{Tr}(QD^2) + (b, \nabla) + C$ with the entries of the matrix-valued functions $C$, $Q$ and those of the vector-valued function $b$ in $C^{1+\alpha/2,2+\alpha}(\Omega)$ for some $k \in \mathbb{N}$ and $\alpha \in (0, 1)$. Let $u \in C^{1+k-\alpha/2,2+k-\alpha}(\Omega)$ such that $\Omega \subset \mathbb{T}_R^n$. We also deduce that $h,j \in C^{1+\alpha/2,2+\alpha}(\Omega)$ where $\hat{h},j$ is defined as the operator $\Delta$ of a diagonal operator, which generates an analytic semigroup in $C^b_R(x,\xi)$ and $\partial^\alpha D^\beta u \in C^b_R(x,\xi)$ for any $\beta \leq k$.

Proof. We begin by the case $k = 1$, which is the core of the proof. We fix two connected open sets $\Omega_1$ and $\Omega_2$ with $\Omega_1 \subset \Omega_2 \subset \Omega$, and a function $\vartheta \in C^\infty(\mathbb{R}^d)$ such that $\chi_{\Omega_1} \leq \vartheta \leq \chi_{\Omega_2}$.

Next, we set $g = D_\vartheta u - A u$, denote by $v$ the trivial extension of the function $\vartheta u$ to the whole $[0, T] \times \mathbb{R}^d$ and, for any $j = 1, \ldots, d$ and $h \in \mathbb{R} \setminus \{0\}$, introduce the operator $\Delta_{h,j}$ defined on smooth functions $\psi$ by $\Delta_{h,j} \psi = h^{-1}(\psi(\cdot + h e_j) - \psi)$, which is bounded from $C^1_b(\Omega)$ into $C^1_b(\mathbb{R}^d)$. The function $v_{h,j} = \Delta_{h,j} v := (\Delta_{h,j} v_1, \ldots, \Delta_{h,j} v_m)$ belongs to $C^{1+\alpha,2+\alpha}(\Omega)$ and $D_j v_{h,j} = \bar{A} v_{h,j} + g_{h,j}$ in $(0, T) \times \mathbb{R}^d$, where $\bar{A}$ is defined as the operator $\mathcal{A}$, with $Q$, $b$ and $C$ being replaced by $\bar{Q} = \eta Q + (1 - \eta)I$, $\bar{b} = \eta b$, $\bar{C} = \eta C$ for some function $\eta \in C^\infty(\Omega)$ such that $\eta \equiv 1$ in $\Omega_2$, and $g_{h,j} = (g_{h,j}^{(1)}, \ldots, g_{h,j}^{(m)})$, where
\[
g_{h,j}^{(k)} = \Delta_{h,j}(\partial g_k - u_{10} A_0 \vartheta - 2(\bar{Q} \nabla \eta, \nabla x u_k) + \text{Tr}(\Delta_{h,j} \bar{Q}) D^2_{\eta} v_k (\cdot, \cdot + h e_j))
\]
for any $k = 1, \ldots, m$, with $A_0 = \text{Tr}(QD^2) + (b, \nabla)$. Being a bounded perturbation of a diagonal operator, which generates an analytic semigroup in $C^b(\mathbb{R}^d, \mathbb{R}^m)$, the operator $\bar{A}$ itself is the generator of an analytic semigroup, which has maximal $C^\alpha$-regularity. Since, due to our assumptions, $v_{h,j}(0, \cdot)$ and $g_{h,j}$ belong to $C^{2+\alpha}(\Omega)$ and $C^{2+\alpha}(\Omega)$, respectively, this means that we can estimate
\[
\|v_{h,j}\|_{C^{1+\alpha/2,2+\alpha}(\Omega \times \mathbb{R}^d, \mathbb{R}^m)} \leq c_1 (\|v_{h,j}(0, \cdot)\|_{C^2(\mathbb{R}^d, \mathbb{R}^m)} + \|g_{h,j}\|_{C^{0+\alpha}(\Omega \times \mathbb{R}^d, \mathbb{R}^m)})
\]
\[
\leq c_2 (\|u(0, \cdot)\|_{C^1(\mathbb{R}^d, \mathbb{R}^m)} + \|g\|_{C^{0+\alpha}(\Omega \times \mathbb{R}^d, \mathbb{R}^m)})
\]
for any $|h| < h_0 := \text{dist}(\Omega, \partial \Omega)$ and some positive constants $c_1$ and $c_2$, independent of the functions involved, where $\Omega_2 = \Omega_2 + B_{h_0}$. Taking into account that $v_{h,j}$ converges to $D_j v$ pointwise on $[0, T] \times \mathbb{R}^d$ as $h \to 0$, a compactness argument shows that $D_j v$ belongs to $C^{1+\alpha/2,2+\alpha}(\Omega \times \mathbb{R}^d, \mathbb{R}^m)$. As a byproduct, $D_j u \in C^{1+\alpha/2,2+\alpha}(\Omega \times \mathbb{R}^d, \mathbb{R}^m)$. We also deduce that $D_j u$ is continuously differentiable
in \([0, T] \times \Omega\) with respect to the variable \(x_j\). This is enough to infer that
\[ D_tD_j u = D_jD_t u \text{ in } [0, T] \times \Omega. \]

Now, suppose that the claim holds for some \(k > 1\) and set \(g = D_t u - \mathcal{A} u\).

Differentiating the equation \(D_t u = \mathcal{A} u + g\) \(k\)-times with respect to the spatial variables, we conclude that, for any \(\beta\), with length \(k\), the function \(w = D^\beta u\) solves the differential equation \(D_t w = \mathcal{A} w + g_\beta\), where \(g_\beta\) is a linear combination of the spatial derivatives of \(u\) up to the order \(k + 1\) with coefficients which are the derivatives of the coefficients up to the order \(k\) of operator \(\mathcal{A}\). As a consequence, \(g_\beta\) belongs to \(C^{\alpha/2,1+\alpha}[0, T] \times \Omega; \mathbb{R}^m\) and from the first part of the proof we conclude that \(D^\beta u \in C^{\alpha/2,2+\alpha}\) \([0, T] \times \Omega; \mathbb{R}^m\). The arbitrariness of \(\beta\) implies that \(u \in C^{1,2+\alpha+k}[0, T] \times \Omega; \mathbb{R}^m\). \(\Box\)

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