Weak $n$-categories: opetopic and multitopic foundations

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Abstract

We generalise the concepts introduced by Baez and Dolan to define opetopes constructed from symmetric operads with a category, rather than a set, of objects. We describe the category of 1-level generalised multicategories, a special case of the concept introduced by Hermida, Makkai and Power, and exhibit a full embedding of this category in the category of symmetric operads with a category of objects. As an analogy to the Baez-Dolan slice construction, we exhibit a certain multicategory of function replacement as a slice construction in the multitopic setting, and use it to construct multitopes. We give an explicit description of the relationship between opetopes and multitopes.

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Introduction

The problem of defining a weak $n$-category has been approached in various different ways ([3], [12], [17], [20], [4], [23], [22], [19], [18]), but so far the relationship between these approaches has not been fully understood. The subject of the present paper is the relationship between the approaches given in [3] and [12].

In [3], John Baez and James Dolan give a definition of weak $n$-categories based on opetopes and opetopic sets. In [12], Claudio Hermida, Michael Makkai and John Power begin a related definition, based on multitopes and multitopic sets. In each case the definition has two components. First, the language for describing $k$-cells is set up. Then, a concept of universality is introduced, to deal with composition and coherence. Any comparison of the two approaches must therefore begin at the construction of $k$-cells, and in this paper we restrict our attention to this process. This, in the terminology of Baez and Dolan, is the theory of opetopes.

In [3], the underlying shapes of $k$-cells are shapes called ‘opetopes’ by Baez and Dolan. The starting point is the theory of (symmetric) operads. A ‘slicing’ process on operads is defined, which is the means of ‘climbing up’ through dimensions; it is eventually used to construct $(k+1)$-cells from $k$-cells. Opetopes are constructed from the slicing process iterated, and presheaves on the category of opetopes are called opetopic sets. A weak $n$-category is defined as an opetopic set with certain properties.

In [12], an analogous process is presented, with shapes called ‘multitopes’. The construction is based on multicategories in a generalised form defined in the paper. Instead of a slicing process, the construction of a ‘multicategory of function replacement’ is given. This is a more general concept, and multitopic sets are defined directly from the iteration of this process. Multitopes are then defined to arise from the terminal multitopic set, and multitopic sets are shown to arise as presheaves on the category of multitopes.

Although the multitopic approach was developed explicitly as an analogy to the opetopic approach, the exact relationship between the notions
has not previously been clear. The conspicuous difference between the two approaches is the presence in the opetopic version, and absence in the multitopic, of symmetric actions. In this paper we make explicit the relationship between opetopes and multitopes, showing that they are ‘the same up to isomorphism’.

In fact, we do not use the definition of opetopes precisely as given in [3], but rather, we develop a generalisation of the notion along lines which Baez and Dolan began but chose to abandon, for reasons unknown to the present author. Baez and Dolan work with operads having an arbitrary set of types (objects), but at the beginning of the paper they use operads having an arbitrary category of types, before restricting to the case where the category of types is small and discrete. However, the construction gives many copies of each opetope, and we need to regard these as isomorphic. So we need a category of objects in order to preserve this vital information. Without it, the isomorphisms are lost and such objects are considered to be different and in this manner the relationship between the two approaches is destroyed. We discuss this in more detail in Section 1.

Thus motivated, we study the approach presented by Baez and Dolan, but using operads with a category of objects; we refer to these as symmetric multicategories (with a category of objects), in accordance with the terminology of [12] and [17].

The approach presented by Hermida, Makkai and Power uses generalised 2-level multicategories, which have ‘upper level’ and ‘lower level’ objects. As far as the construction of multitopes, however, we have found only 1-level versions to be involved, so we consider only these, which we refer to simply as generalised multicategories.

The constructions of multitopes and opetopes are explicitly analogous, so we compare them step by step as follows.

We begin, in Section 1 with an informal overview of the whole theory. We include for completeness the theory proposed by Leinster although the formal treatment is given in a further work.

In Section 2 we define the categories \( \text{SymMulticat} \) and \( \text{GenMulticat} \), of symmetric and generalised multicategories respectively. These are the underlying theories of the two approaches.

In Section 2.3 we construct a functor

\[ \xi : \text{GenMulticat} \to \text{SymMulticat} \]

and show that it is full and faithful. Given a generalised multicategory \( M \), \( \xi \) acts by leaving the objects unchanged, but adding a symmetric action freely on the arrows. (By ‘free’ here we mean that the orbit of an arrow with \( n \) source elements is the size of the whole permutation group \( S_n \).)

Clearly not all symmetric multicategories are in the image of \( \xi \). To be in the image, a symmetric multicategory certainly must have a discrete category of objects (we call this object-discrete) and a free symmetric action.
(we call this freely symmetric). We show that these conditions are in fact necessary and sufficient. Eventually we will see that every symmetric multicategory used in the construction is equivalent to one with these properties.

In Section 3, we examine the construction of opetopes. We first define and compare the slicing processes. Our method is as follows. Given a morphism of symmetric multicategories

\[ \phi : Q \longrightarrow \xi(M) \]

we construct a morphism

\[ \phi^+ : Q^+ \longrightarrow \xi(M^+) \]

does not necessarily invertible. In particular we deduce that the functor \( \xi \) and the slicing process ‘commute’ up to equivalence, that is, for any generalised multicategory \( M \)

\[ \xi(M)^+ \simeq \xi(M^+) \]

In Section 3.3, we apply the above constructions to opetopes and multitopes. Writing \( I \) for the symmetric multicategory with one object and one arrow, a \( k \)-dimensional opetope is defined to be an object of \( I^{k+} \), the \( k \)th iterated slice of \( I \). Similarly, writing \( J \) for the generalised multicategory with one object and one arrow, a \( k \)-dimensional multitope is defined to be an object of \( J_{k+} \), the \( k \)th iterated slice of \( J \). By the above constructions, we have for each \( k \)

\[ \xi(J_{k+}) \simeq I^{k+} \]

giving a correspondence between opetopes and multitopes.

Hermida, Makkai and Power suggest that where their concept is “concrete and geometric” the Baez-Dolan concept is “abstract and conceptual”. In uniting the two approaches the reward is a concept which enjoys the elegance of being abstract and conceptual while at the same time providing a concrete, geometric description of the objects in question.

**Terminology**

i) Since we are concerned chiefly with weak \( n \)-categories, we follow Baez and Dolan ([3]) and omit the word ‘weak’ unless emphasis is required; we refer to strict \( n \)-categories as ‘strict \( n \)-categories’.

ii) We use the term ‘weak \( n \)-functor’ for an \( n \)-functor where functoriality holds up to coherent isomorphisms, and ‘lax’ functor when the constraints are not necessarily invertible.

iii) In [3] Baez and Dolan use the terms ‘operad’ and ‘types’ where we use ‘multicategory’ and ‘objects’; the latter terminology is more consistent with Leinster’s use of ‘operad’ to describe a multicategory whose ‘objects-object’ is 1.
iv) In [12] Hermida, Makkai and Power use the term ‘multitope’ for the objects constructed in analogy with the ‘opetopes’ of [3]. This is intended to reflect the fact that opetopes are constructed using operads but multitopes using multicategories, a distinction that we have removed by using the term ‘multicategory’ in both cases. However, we continue to use the term ‘opetope’ and furthermore, use it in general to refer to the analogous objects constructed in each of the theories.

v) We regard sets as sets or discrete categories with no notational distinction.

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1 Overview
In this Section we give an informal overview of the opetopic foundations for theory of \( n \)-categories. This is not intended to be a rigorous treatment, but rather, to give the reader an idea of the ‘spirit’ of the definition, the issues involved in making it, and the reason (as opposed to the proof) that the different approaches in question turn out to be equivalent. For completeness we include here discussion of Leinster’s construction ([17]) although the formal account is given in a further work ([11]).

1.1 What are opetopes?
The defining feature of the opetopic theory of \( n \)-categories is, superficially, that the underlying shapes of cells are opetopes. Below are some examples of opetopes at low dimensions.

- 0-opetope
- 1-opetope
- 2-opetopes
• a 3-opetope

• a 4-opetope

Remarks

1) Note that all edges and faces are directed, but we tend to omit the arrows as at low dimensions directions are understood.

2) The number of bars on an arrow indicate its dimension.

3) The curved arrows indicate ‘pasting’ which is otherwise difficult to represent in higher-dimensions on a 2-dimensional sheet of paper.

Compared with ordinary ‘globular’ cell shapes such as

opetopes have the following important feature: the domain of a $k$-opetope is not a single $(k - 1)$-opetope but a ‘pasting diagram’ of $(k - 1)$-opetopes.
Note that a pasting diagram can be degenerate, giving ‘nullary’ opetopes whose domain consists of an ‘empty’ pasting diagram. For example, the following is a nullary 2-opetope:

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\end{array}
\]

Cells in an opetopic \( n \)-category may thus be thought of as ‘labelled opetopes’ where the sources and targets of the constituent cells must match up where they coincide in the opetope. For example the following is a 2-cell

\[
\begin{array}{c}
a_2 \\
\downarrow \alpha \\
a_3 \\
f_1 \\
\downarrow \\
f_2 \\
\downarrow \\
a_4 \\
g \\
\end{array}
\]

and the following is a 3-cell in which some of the lower-dimensional labels have been omitted

\[
\begin{array}{c}
\alpha_1 \\
\downarrow \alpha_3 \\
\downarrow \alpha_2 \\
\end{array} \Rightarrow \begin{array}{c}
\theta \\
\beta \\
\end{array}
\]

There now arise a philosophical question and a technical question, namely: why and how do we do this?

### 1.2 Why opetopes?

Opetopes arise from the need, in a weak \( n \)-category, to record the precise way in which a composition has been performed. For example, consider the following chain of composable 1-cells:

\[
a \overset{f}{\rightarrow} b \overset{g}{\rightarrow} c \overset{h}{\rightarrow} d.
\]

This gives a unique composite in an ordinary category (or any strict \( n \)-category). However, in a bicategory (or any weak \( n \)-category) we should be wary of drawing such a diagram at all, as there is more than one composite that could be produced, for example \((hg)f\) or \(h(gf)\), which may in general be distinct.

We might record the way in which the composition has occurred by a diagram such as
indicating that first $f$ is composed with $g$, and then the result is composed with $h$. So this diagram represents the forming of the composite $h(gf)$.

Here the 2-cells are seen to indicate composition of their domain 1-cells. This is one of the fundamental ideas of the opetopic theory, that composition is not given by an operation, but by certain higher-dimensional cells. The cells giving composites are those with a certain universal property, and there may be many such cells for any composable configuration of cells. For, as we have seen above, there may be many distinct ways of composing a given diagram of cells.

This is the motivation behind taking opetopes as the underlying shapes of cells.

1.3 How are opetopes constructed formally?

We have seen that the source of a cell is to be a pasting diagram of cells rather than just a single cell. This is expressed using the language of multicategories. A multicategory is like a category whose morphisms have as their domain a list of objects rather than just a single object. Thus arrows may be drawn as

and composition then looks like

and composition then looks like
So a $k$-cell is considered as a morphism from its constituent $(k-1)$-cells to its codomain $(k-1)$-cell. For example

\[ \begin{array}{c}
\begin{array}{ccc}
\text{1} & \text{2} & \text{3}
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{ccc}
\text{1} & \text{2} & \text{3}
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{ccc}
\text{1} & \text{2} & \text{3}
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{ccc}
\text{1} & \text{2} & \text{3}
\end{array}
\end{array} \]

This raises the immediate question: in what order should we list the constituent cells? Tom Leinster points out ([17]) that there is no way of ordering the cells that is stable under composition as required for a multicategory as above.

The three different approaches to this construction ([3], [12], [17]) arise from three different ways of dealing with this problem.

- **Baez and Dolan**

  Baez and Dolan say: include all possible orderings. For example

  \[ \begin{array}{c}
  \begin{array}{ccc}
  \text{1} & \text{2} & \text{3}
  \end{array}
  \end{array} , \begin{array}{c}
  \begin{array}{ccc}
  \text{1} & \text{2} & \text{3}
  \end{array}
  \end{array} , \begin{array}{c}
  \begin{array}{ccc}
  \text{1} & \text{2} & \text{3}
  \end{array}
  \end{array} \cdots \text{where the numbers indicate the order in which the source cells are listed.}

  So a symmetric action arises, giving the different orderings, and Baez and Dolan use symmetric multicategories for the construction.

  However, a peculiar situation arise in which arrows such as

  \[ \begin{array}{c}
  \begin{array}{c}
  \text{1} & \text{2}
  \end{array}
  \end{array} \equiv \rightarrow \begin{array}{c}
  \begin{array}{c}
  \text{1} & \text{2}
  \end{array}
  \end{array} \]

  and

  \[ \begin{array}{c}
  \begin{array}{c}
  \text{2} & \text{1}
  \end{array}
  \end{array} \equiv \rightarrow \begin{array}{c}
  \begin{array}{c}
  \text{2} & \text{1}
  \end{array}
  \end{array} \]
cannot be composed, as the ordering on the target of one does not match the ordering of the source of the other. The situation quickly escalates with more and more different possible manifestations of the same opetope arising from not only the orderings on the source cells, but also the orderings on their source cells, and so on. For example the following innocuous looking opetope

\[
\begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\node at (0,0) [circle, fill=black] (a) {};
\node at (1,1) [circle, fill=black] (b) {};
\node at (2,2) [circle, fill=black] (c) {};
\node at (3,3) [circle, fill=black] (d) {};
\draw (a) -- (b) -- (c) -- (d);
\end{tikzpicture}}
\end{array}
\quad \equiv \quad
\begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\node at (0,0) [circle, fill=black] (a) {};
\node at (1,1) [circle, fill=black] (b) {};
\node at (2,2) [circle, fill=black] (c) {};
\node at (3,3) [circle, fill=black] (d) {};
\draw (a) -- (b) -- (c) -- (d);
\end{tikzpicture}}
\end{array}
\]

has 576 possible manifestations, and the following one

\[
\begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\node at (0,0) [circle, fill=black] (a) {};
\node at (1,1) [circle, fill=black] (b) {};
\node at (2,2) [circle, fill=black] (c) {};
\node at (3,3) [circle, fill=black] (d) {};
\node at (4,4) [circle, fill=black] (e) {};
\draw (a) -- (b) -- (c) -- (d);
\end{tikzpicture}}
\quad \equiv \quad
\begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\node at (0,0) [circle, fill=black] (a) {};
\node at (1,1) [circle, fill=black] (b) {};
\node at (2,2) [circle, fill=black] (c) {};
\node at (3,3) [circle, fill=black] (d) {};
\node at (4,4) [circle, fill=black] (e) {};
\draw (a) -- (b) -- (c) -- (d);
\end{tikzpicture}}
\end{array}
\]

has 311040.

We need a way of saying that these objects ‘look the same’ and this is where the use of a category of objects comes in. The isomorphisms in this category tell us precisely this.

- Hermida, Makkai, Power

Hermida, Makkai and Power say: pick one ordering. We know that this cannot stable under composition; instead, the notion of multicategory is generalised so that this stability is not required. Rather, for each composite there is a specified re-ordering of the source elements, satisfying some coherence laws. This is a notion we refer to as generalised multicategory.

- Leinster

Leinster says: pick no ordering at all. The idea is that, fundamentally, squashing the constituent cells into a straight line is an unnatural (and indeed rather violent) thing to try and do. So instead, the source of an arrow such as

\[
\begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\node at (0,0) [circle, fill=black] (a) {};
\node at (1,1) [circle, fill=black] (b) {};
\node at (2,2) [circle, fill=black] (c) {};
\node at (3,3) [circle, fill=black] (d) {};
\node at (4,4) [circle, fill=black] (e) {};
\draw (a) -- (b) -- (c) -- (d);
\end{tikzpicture}}
\quad \equiv \quad
\begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\node at (0,0) [circle, fill=black] (a) {};
\node at (1,1) [circle, fill=black] (b) {};
\node at (2,2) [circle, fill=black] (c) {};
\node at (3,3) [circle, fill=black] (d) {};
\node at (4,4) [circle, fill=black] (e) {};
\draw (a) -- (b) -- (c) -- (d);
\end{tikzpicture}}
\end{array}
\]

is literally the diagram
expressed as a structure given by a cartesian monad $T$. This is the notion of $T$-multicategory.

These differences notwithstanding, the constructions proceed in a similar manner: a process of ‘slicing’ is used to construct $k$-cells from $(k - 1)$-cells in each of the respective frameworks.

1.4 Why are the different approaches equivalent?

At first sight, it might seem implausible that a construction with so much symmetry should give anything like a construction without any symmetry. In fact, the symmetric actions in the Baez-Dolan approach are a sort of trompe d’œil created by our attempt to view constituents of an opetope in a straight line when they simply are not in one. It is not the opetope itself that is symmetric, but only our presentations of it.

So, with the Baez-Dolan version, we end up with many isomorphic presentations of the same opetope, given by all the different orders in which we could list its components. In effect, with the Hermida-Makkai-Power version we pick one representative of each isomorphism class, and with the Leinster version, we take the whole isomorphism class as one opetope.

In the end there is a trade-off between naturality (in the informal sense of the word) and practicality. Consider the following analogy. If I tidy the papers on my desk into a neat pile, I have forced them into a straight line when they had natural positions as they were. However, they are thus easier to carry around. Likewise, the Leinster construction may seem less brutal in this sense, but the Hermida-Makkai-Power construction yields a framework that is more practical for calculating with cells.

This means that if we are to write down a set of domain cells on a piece of paper in a calculation, we can write them in some order. The Baez-Dolan construction mediates for us, giving us peace of mind that the order we chose is irrelevant, as the symmetric actions are quietly working in the shadows dealing with all the other possibilities.

1.5 How is the Baez-Dolan definition modified here?

The present author began studying the relationship between opetopes and multitopes as given, but began to encounter difficulties when examining the process of slicing. Essentially, slicing yields a multicategory whose objects are the morphisms of the original multicategory, and whose morphisms are its composition laws. Given a multicategory $Q$, Baez and Dolan define the slice multicategory $Q^+$ to be a multicategory whose set of objects is the set of arrows of $Q$. The effect is that some information has been abandoned,
or at least, concealed. That is, we have discarded the symmetries relating the arrows of $Q$ to one another. As the slicing process is iterated, progressively more information is abandoned in this manner, essentially a layer of symmetry at each stage of slicing. For the construction of opetopes, the crucial fact is that the symmetries arise \textit{precisely and exclusively} from the different possible orderings of source elements. So it is precisely these symmetries which give the vital information about which opetopes are merely different presentations of the same thing, and therefore should be isomorphic. Without it, the isomorphisms are lost and such objects are considered to be different. In this manner the relationship between the two approaches would destroyed.

However, pursuing Baez and Dolan’s original approach, using multicategories with an arbitrary category of objects, it is no longer necessary to force the category of objects of $Q^+$ to be discrete. This theory yields a different slice multicategory, in which the symmetric action in $Q$ is recorded in the morphisms of the category of objects of $Q^+$.

This modification can then be pursued throughout the definition of $n$-category (see \cite{7}, \cite{8}). The relationship between this definition and the original one is not currently clear. For low dimensions it appears that the existence of certain universal cells may eventually iron out the differences, but such explicit arguments are unfeasible for arbitrary higher dimensions. Moreover, such arguments cannot be applied to the structures underlying $n$-categories where the existence of such universals has not yet been asserted.

So what does seem clear is that the equivalences between theories as described above facilitates much further work in this area, for example, the study of the categories of opetopes and opetopic sets (\cite{7}, \cite{8}, \cite{9}, \cite{10}). Using the original definition and therefore without the help of these equivalences, this work would not have been possible.

\section{The theory of multicategories}

Opetopes are described using the language of multicategories. In each of the two theories of opetopes in question, a different underlying theory of multicategories is used. In this section we examine the two underlying theories, and we construct a way of relating these theories to one another; this relationship provides subsequent equivalences between the definitions. We adopt a concrete approach here; certain aspects of the definitions suggest a more abstract approach but this will require further work beyond the scope of this work.

\subsection{Symmetric multicategories}

In \cite{3} opetopes are constructed using symmetric multicategories. In this section we define $\text{SymMulticat}$, the category of symmetric multicategories
with a category of objects. The definition we give here includes one axiom which appears to have been omitted from [3].

We write \( F \) for the ‘free symmetric strict monoidal category’ monad on \( \textbf{Cat} \), and \( S_k \) for the group of permutations on \( k \) objects; we also write \( \iota \) for the identity permutation.

**Definition 2.1** A symmetric multicategory \( Q \) is given by the following data

1) A category \( o(Q) = \mathbb{C} \) of objects. We refer to \( \mathbb{C} \) as the object-category, the morphisms of \( \mathbb{C} \) as object-morphisms, and if \( \mathbb{C} \) is discrete, we say that \( Q \) is object-discrete.

2) For each \( p \in \mathcal{F}^\text{op} \times \mathbb{C} \), a set \( Q(p) \) of arrows. Writing

\[
p = (x_1, \ldots, x_k; x),
\]

an element \( f \in Q(p) \) is considered as an arrow with source and target given by

\[
s(f) = (x_1, \ldots, x_k)
\]

\[
t(f) = x
\]

and we say \( f \) has arity \( k \). We may also write \( a(Q) \) for the set of all arrows of \( Q \).

3) For each object-morphism \( f : x \rightarrow y \), an arrow \( \iota(f) \in Q(x; y) \). In particular we write \( 1_x = \iota(1_x) \in Q(x; x) \).

4) Composition: for any \( f \in Q(x_1, \ldots, x_k; x) \) and \( g_i \in Q(x_{i1}, \ldots, x_{im_i}; x_i) \) for \( 1 \leq i \leq k \), a composite

\[
f \circ (g_1, \ldots, g_k) \in Q(x_{11}, \ldots, x_{1m_1}, \ldots, x_{k1}, \ldots, x_{km_k}; x)
\]

5) Symmetric action: for each permutation \( \sigma \in S_k \), a map

\[
\sigma : \quad Q(x_1, \ldots, x_k; x) \quad \rightarrow \quad Q(x_{\sigma(1)}, \ldots, x_{\sigma(k)}; x)
\]

satisfying the following axioms:

1) Unit laws: for any \( f \in Q(x_1, \ldots, x_m; x) \), we have

\[
1_x \circ f = f = f \circ (1_{x_1}, \ldots, 1_{x_m})
\]

2) Associativity: whenever both sides are defined,

\[
f \circ (g_1 \circ (h_{11}, \ldots, h_{1m_1}), \ldots, g_k \circ (h_{k1}, \ldots, h_{km_k})) = (f \circ (g_1, \ldots, g_k)) \circ (h_{11}, \ldots, h_{1m_1}, \ldots, h_{k1}, \ldots, h_{km_k})
\]

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3) For any \( f \in Q(x_1, \ldots, x_m; x) \) and \( \sigma, \sigma' \in S_k \),

\[
(f \sigma) \sigma' = f(\sigma \sigma')
\]

4) For any \( f \in Q(x_1, \ldots, x_k; x) \), \( g_i \in Q(x_{i1}, \ldots, x_{im}; x_i) \) for \( 1 \leq i \leq k \), and \( \sigma \in S_k \), we have

\[
(f \sigma) \circ (g_{\sigma(1)}, \ldots, g_{\sigma(k)}) = f \circ (g_1, \ldots, g_k) \cdot \rho(\sigma)
\]

where \( \rho : S_k \rightarrow S_{m_1+\ldots+m_k} \) is the obvious homomorphism.

5) For any \( f \in Q(x_1, \ldots, x_k; x) \), \( g_i \in Q(x_{i1}, \ldots, x_{im}; x_i) \), and \( \sigma_i \in S_{m_i} \) for \( 1 \leq i \leq k \), we have

\[
f \circ (g_1 \sigma_1, \ldots, g_k \sigma_k) = (f \circ (g_1, \ldots, g_k)) \sigma
\]

where \( \sigma \in S_{m_1+\ldots+m_k} \) is the permutation obtained by juxtaposing the \( \sigma_i \).

6) \( \iota(f \circ g) = \iota(f) \circ \iota(g) \)

We may draw an arrow \( f \in Q(x_1, \ldots, x_k; x) \) as

\[
\begin{array}{ccc}
x_1 & x_2 & \cdots & x_k \\
\downarrow & & & \\
f & & & \\
\downarrow & & & \\
x & & & \\
\end{array}
\]

and a composite \( f \circ (g_1, \ldots, g_k) \) as

\[
\begin{array}{ccc}
x_{11} \cdots x_{1m_1} & x_{21} \cdots x_{2m_2} & x_{k1} \cdots x_{km_k} \\
\downarrow & \downarrow & \downarrow \\
g_1 & g_2 & g_k \\
\downarrow & \downarrow & \downarrow \\
\cdots & \cdots & \cdots \\
\downarrow & \downarrow & \downarrow \\
f & & \\
\downarrow & & \\
\end{array}
\]
A symmetric multicategory $Q$ may be thought of as a functor

$$Q : \mathcal{FC}^{\text{op}} \times \mathcal{C} \to \text{Set}$$

with some extra structure.

In a more abstract view, we would expect $\mathcal{F}$ to be a 2-monad on the 2-category $\text{Cat}$, which lifts via a generalised form of distributivity to a bimonad on $\text{Prof}$, the bicategory of profunctors. Then the Kleisli bicategory for this bimonad should have as objects small categories, and its 1-cells should be essentially profunctors of the form $\mathcal{FC} \to \mathcal{D}$ in the opposite category. However, the calculations involved in this description are intricate and require further work.

In this abstract view, a symmetric multicategory $Q$ would then be a monad in this bicategory. Arrows and symmetric action (Data 2, 5) are given by the action of $Q$, identities (Data 3) by the unit of the monad and composition (Data 4) by the multiplication for the monad.

**Definition 2.2** Let $Q$ and $R$ be symmetric multicategories with object-categories $\mathcal{C}$ and $\mathcal{D}$ respectively. A morphism of symmetric multicategories $F : Q \to R$ is given by

- A functor $F = F_0 : \mathcal{C} \to \mathcal{D}$
- For each arrow $f \in Q(x_1, \ldots, x_k; x)$ an arrow $Ff \in R(Fx_1, \ldots, Fx_k; Fx)$ satisfying
  - $F$ preserves identities: $F(\iota(f)) = \iota(Ff)$ so in particular $F(1_x) = 1_{Fx}$
  - $F$ preserves composition: whenever it is defined
    $$F(f \circ (g_1, \ldots, g_k)) = (Ff \circ (Fg_1, \ldots, Fg_k))$$
  - $F$ preserves symmetric action: for each $f \in Q(x_1, \ldots, x_k; x)$ and $\sigma \in S_k$
    $$F(f\sigma) = (Ff)\sigma$$

Composition of such morphisms is defined in the obvious way, and there is an obvious identity morphism $1_Q : Q \to Q$. Thus symmetric multicategories and their morphisms form a category $\text{SymMulticat}$.

**Definition 2.3** A morphism $F : Q \to R$ is an equivalence if and only if the functor $F_0 : \mathcal{C} \to \mathcal{D}$ is an equivalence, and $F$ is full and faithful. That is, given objects $x_1, \ldots, x_m, x$ the induced function

$$F : Q(x_1, \ldots, x_m; x) \to R(Fx_1, \ldots, Fx_m; Fx)$$

is an isomorphism.
Note that, given morphisms of symmetric multicategories

\[ Q \xrightarrow{F} R \xrightarrow{G} P \]

we have a result of the form ‘any 2 gives 3’, that is, if any two of \( F, G \) and \( GF \) are equivalences, then all three are equivalences.

Furthermore, we expect that \textbf{SymMulticat} may be given the structure of a 2-category, and that the equivalences in this 2-category would be the equivalences as above. However, we do not pursue this matter here.

2.2 Generalised multicategories

In [12] multitopes are constructed using ‘generalised multicategories’; in fact we need only a special case of the generalised multicategory defined in [12], that is, the ‘1-level’ case.

\textbf{Definition 2.4} A generalised multicategory \( M \) is given by

- A set \( o(M) \) of objects
- A set \( a(M) \) of arrows, with source and target functions

\[
\begin{align*}
  s : & \quad a(M) \rightarrow o(C)^* \\
  t : & \quad a(M) \rightarrow o(C)
\end{align*}
\]

where \( A^* \) denotes the set of lists of elements of a set \( A \). If

\[
  s(f) = (x_1, \ldots, x_k)
\]

we write \( s(f)_p = x_p \) and \( |s(f)| = \{1, \ldots, k\} \).

- Composition: for any \( f, g \in a(M) \) with \( t(g) = s(f)_p \), a composite \( f \circ_p g \in a(M) \) with

\[
\begin{align*}
  t(f \circ_p g) & = t(f) \\
  |s(f \circ_p g)| & \cong (|s(f)| \setminus \{p\}) \amalg |s(g)|
\end{align*}
\]

and amalgamating maps

\[
\begin{align*}
  \psi[f,g,p] : & \quad |s(f)| \setminus \{p\} \rightarrow |s(f \circ_p g)| \\
  \phi[f,g,p] : & \quad |s(g)| \rightarrow |s(f \circ_p g)|.
\end{align*}
\]

such that \( \psi \amalg \phi \) gives a bijection as above. Equivalently, writing

\[
\begin{align*}
  s(f) & = (x_1, \ldots, x_k) \\
  s(g) & = (y_1, \ldots, y_j)
\end{align*}
\]
and

\[(z_1, \ldots, z_{k+j-1}) = (x_1, \ldots, x_{p-1}, y_1, \ldots, y_j, x_{p+1}, \ldots, x_{k+j-1})\]

we have a permutation \(\chi = \chi[f, g, p] \in S_{k+j-1}\) such that

\[s(f \circ_p g) = (z_{\chi(1)}, \ldots, z_{\chi(k+j-1)})\].

- **Identities:** for each \(x \in o(M)\) an arrow \(1_x : x \rightarrow x \in a(M)\) satisfying the following laws

  - **Unit laws:** for any \(f \in a(M)\) with \(s(f)_p = x\) and \(t(f) = y\), we have
    \[
    1_y \circ_1 f = f = f \circ_p 1_x \\
    \chi[1_y, f, 1] = \iota = \chi[f, 1_x, p].
    \]

  - **Associativity:** for any \(f, g, h \in a(M)\) with \(s(f)_p = t(g)\) and \(s(g)_q = t(h)\) we have
    \[
    (f \circ_p g) \circ \bar{q} h = f \circ_p (g \circ_q h)
    \]
    where \(\bar{q} = \phi[f, g, p](q)\). Furthermore, the composite amalgamation maps must also be equal; that is, the following coherence conditions must be satisfied:

    \[
    \psi[f \circ_p g, h, \bar{q}] \circ \psi[f, g, p] = \psi[f, h \circ_q g, p] \\
    \psi[f \circ_p g, h, \bar{q}] \circ \bar{\phi}[f, g, p] = \phi[f, h \circ_q g, p] \circ \psi[g, h, q] \\
    \phi[f \circ_p g, h, \bar{q}] = \phi[f, h \circ_q g, p] \circ \phi[g, h, q]
    \]

    where \(\bar{\phi}\) indicates restriction to the appropriate domain. Note that the conditions concern the source elements of \(f, g\) and \(h\) respectively.

- **Commutativity:** for any \(f, g, h \in a(M)\) with \(s(f)_p = t(g)\), \(s(f)_q = t(h)\), \(p \neq q\) we have

    \[(f \circ_p g) \circ \bar{q} h = (f \circ_q h) \circ_p g\]

    where \(\bar{q} = \psi[f, g, p]\) and \(\bar{p} = \psi[f, h, q]\). As above, the composite amalgamation maps must also be equal; that is, the following coherence conditions must be satisfied:

    \[
    \psi[f \circ_p g, h, \bar{q}] \circ \psi[f, g, p] = \psi[f \circ_q h, g, \bar{p}] \circ \psi[f, h, q] \\
    \psi[f \circ_p g, h, \bar{q}] \circ \bar{\phi}[f, g, p] = \phi[f \circ_q h, g, \bar{p}] \\
    \phi[f \circ_p g, h, \bar{q}] = \psi[f \circ_q h, g, \bar{p}] \circ \phi[f, h, q].
    \]

The conditions concern the source elements of \(f, g\) and \(h\) respectively.
Note that the coherence conditions are necessary in case of repeated source elements.

**Definition 2.5** A morphism of generalised multicategories

\[ F = (F, \theta) : M \rightarrow N \]

is given by:

- for each object \( x \in o(M) \) an object \( Fx \in o(N) \)
- for each arrow \( f : (x_1, \ldots, x_k) \rightarrow x \in a(M) \)
  a transition map \( \theta_f = \theta^E_f \in S_k \) and an arrow
  \[ Ff : (Fx_{\theta^{-1}(1)}, \ldots, Fx_{\theta^{-1}(k)}) \rightarrow Fx \in a(N) \]

satisfying

- \( F \) preserves identities: \( F(1_x) = 1_{Fx} \)
- \( F \) preserves composition: if \( f, g \in a(M) \) and \( t(g) = s(f)p \) then
  \[ Ff \circ_{\theta_f(p)} Fg = F(f \circ_p g). \]

Furthermore, the following coherence conditions must be satisfied:

\[
\begin{align*}
\theta_{f_0 \circ_p g} \circ \phi[f, g, p] &= \phi[Ff, Fg, \theta_{f(p)}] \circ \theta_g \\
\theta_{f \circ_p g} \circ \psi[f, g, p] &= \psi[Ff, Fg, \theta_{f(p)}] \circ \theta_f
\end{align*}
\]

on the source elements of \( g \) and \( f \) respectively, where \( \bar{\theta} \) indicates the restriction of \( \theta \) as appropriate.

Given morphisms of generalised multicategories \( M \xrightarrow{F} N \xrightarrow{G} L \) we have a composite morphism \( H = G \circ F : M \rightarrow L \) where \( H \) is the usual composite on objects and arrows, and we put \( \theta^H_f = \theta^G_{F \circ f} \circ \theta^F_f \). There is an identity morphism \( 1_M : M \rightarrow M \) which is the usual identity on objects and arrows, with \( \theta_f = \iota \) for all \( f \in a(M) \).

Thus generalised multicategories and their morphisms form a category \textbf{GenMulticat}. We now compare the two theories of multicategories.
2.3 Relationship between symmetric and generalised multicategories

We compare symmetric and generalised multicategories by means of a functor

\[ \xi : \text{GenMulticat} \to \text{SymMulticat}. \]

Given a generalised multicategory \( M \), the idea is to generate a symmetric action freely by adding in symmetric copies of each morphism. The arrows of \( M \) are then representatives of symmetry classes of arrows of \( \xi(M) \).

We begin by constructing this functor, and then show that it is full and faithful.

We construct the functor \( \xi \) as follows. Given a generalised multicategory \( M \), we define an object-discrete symmetric multicategory \( \xi(M) = Q \) by

- **Objects:** \( o(Q) = C \) is the discrete category with objects \( o(M) \).
- **Arrows:** for each \( p = (x_1, \ldots, x_k; x) \in \mathcal{F}(C)^{\text{op}} \times C \) an element of \( Q(p) \) is given by \( (f, \sigma) \) where \( \sigma \in S_k \) and
  \[ f : (x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \to x \in a(M) \].
- **Composition:** by commutativity, it is sufficient to define
  \[
  \alpha \circ_p \beta = \alpha \circ (1_{x_1}, \ldots, 1_{x_{p-1}}, \beta, 1_{x_{p+1}}, \ldots, 1_{x_k})
  \]
  where
  \[
  \alpha = (f, \sigma) \in Q(x_1, \ldots, x_k; x) \quad \text{and} \quad \beta = (g, \tau) \in Q(y_1, \ldots, y_j; x_p). \]

Now given such \( \alpha \) and \( \beta \), we have in \( M \) arrows

\[
\begin{align*}
  f & : (x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \to x \\
  g & : (y_{\tau(1)}, \ldots, y_{\tau(j)}) \to x_p
\end{align*}
\]
giving a composite in \( M \)

\[ f \circ_{\bar{p}} g : (z_{\chi(1)}, \ldots, z_{\chi(k+j-1)}) \to x \]

where \( \bar{p} = \sigma^{-1}(p) \), \( \chi = \chi(f, g, \bar{p}) \) and

\[
(z_1, \ldots, z_{k+j-1}) = (x_{\sigma(1)}, \ldots, x_{\sigma(\bar{p}-1)}, y_{\tau(1)}, \ldots, y_{\tau(j)}, x_{\sigma(\bar{p}+1)}, \ldots, x_{\sigma(k)}).
\]

We seek a composite in \( Q \) with source

\[
(a_1, \ldots, a_{k+j-1}) = (x_1, \ldots, x_{p-1}, y_1, \ldots, y_j, x_{p+1}, \ldots, x_k)
\]
so the composite should be of the form \((f \circ \bar{p} g, \gamma)\), where \(f \circ \bar{p} g\) has source 
\((a_{\gamma(1)}, \ldots, a_{\gamma(k+j-1)})\)
in \(M\). So we define a permutation \(\gamma \in S_{j+k-1}\) by \(a_{\gamma(i)} = z_{\chi(i)}\) and we define the composite to be

\[(f, \sigma) \circ_p (g, \tau) = (f \circ \bar{p} g, \gamma)\.

Note that \(\gamma\) is determined by \(\sigma, \tau\) and \(\chi\).

- For each \(x \in C = o(M), 1_x \in Q(x; x)\) is given by \((1_x, \iota)\).
- For each permutation \(\sigma \in S_k\), we have a map
  \[\sigma : Q(x_1, \ldots, x_k; x) \rightarrow Q(x_{\sigma(1)}, \ldots, x_{\sigma(k)}; x)\]

Note that \(f\) has source \((x_{\tau(1)}, \ldots, x_{\tau(k)})\) in \(M\), and \((f, \sigma^{-1} \tau)\) on the right hand side exhibits the \(i\)th source of \(f\) to be \(x_{\sigma(\sigma^{-1} \tau)(i)} = x_{\tau(i)}\) as required.

We check that this definition satisfies the conditions for a symmetric multicategory:

1) Unit laws follow from unit laws of \textbf{GenMulticat}

2) Associativity follows from associativity in \textbf{GenMulticat} and the coherence conditions for amalgamating maps

3) \(((f, \tau) \sigma) \sigma' = (f, \sigma^{-1} \tau) \sigma' = (f, \sigma'^{-1} \sigma^{-1} \tau) = (f, \tau)(\sigma \sigma')\)

4) Given

\[(f, \tau) \in Q(x_1, \ldots, x_k; x),\]
\[(g, \mu) \in Q(y_1, \ldots, y_j, x_p)\]

and \(\sigma \in S_k\) we check that

\[(f, \tau) \sigma \circ_p (g, \mu) = (((f, \tau) \circ_p (g, \mu)) \cdot \rho(\sigma)\]

where \(\bar{p} = \sigma^{-1}(p)\) and \(\rho\) is the homomorphism indicated in Section 2.3.

The required result then follows by simultaneous composition. Note that it is sufficient to check that both expressions in question have the same first component and source (in \(Q\)), so we write \(\gamma, \gamma'\) for the permutations in the second component, without specifying what they are. Now

\[(f, \tau) \sigma \circ_p (g, \mu) = (f, \sigma^{-1} \tau) \circ_p (g, \mu) = (f \circ \tau^{-1}(p) g, \gamma)\]
with source
\[(x_{\sigma(1)}, \ldots, x_{\sigma(p-1)}, y_1, \ldots, y_j, x_{\sigma(p+1)}, \ldots, x_{\sigma(k)})\]
and
\[((f, \tau) \circ_p (g, \mu)) \cdot \rho(\sigma) = (f \circ_{\tau^{-1}(p)} g, \gamma')\]
with source
\[(z_{\rho\sigma(1)}, \ldots, z_{\rho\sigma(k+j-1)})\]
where
\[(z_1, \ldots, z_{k+j-1}) = (x_1, \ldots, x_{p-1}, y_1, \ldots, y_j, x_{p+1}, \ldots, x_k).\]
The action of $\rho(\sigma)$ is that of $\sigma$ on the $x_i$ but with $(y_1, \ldots, y_j)$ substituted for $x_p$. So
\[(z_{\rho\sigma(1)}, \ldots, z_{\rho\sigma(k+j-1)}) = (x_{\sigma(1)}, \ldots, x_{\sigma(p-1)}, y_1, \ldots, y_j, x_{\sigma(p+1)}, \ldots, x_{\sigma(k)})\]
as required.

5) Given $(f, \tau)$ and $(g, \mu)$ as above, and $\sigma \in S_j$ we check that
\[(f, \tau) \circ_p (g, \mu) = ((f, \tau) \circ_p (g, \mu))\sigma'\]
where $\sigma' \in S_{k+j-1}$ is given by inserting $\sigma$ at the $p$th place.
Now, on the left hand side we have
\[(f, \tau) \circ_p (g, \mu) \sigma = (f, \tau) \circ_p (g, \sigma^{-1} \mu)\]
say, with source
\[(x_1, \ldots, x_{p-1}, y_{\sigma(1)}, \ldots, y_{\sigma(j)}, x_{p+1}, \ldots, x_k).\]
This agrees with the right hand side.

6) Since all object-morphisms are identities, this axiom is trivially satisfied.

So $\xi(M)$ is a symmetric multicategory.

Next we define $\xi$ on morphisms of generalised multicategories. Given a morphism $F : M \to N$ in $\text{GenMulticat}$ we define a morphism
\[\xi F : \xi M \to \xi N\]
in $\text{SymMulticat}$ as follows.
• On objects: given \(x \in o(\xi M) = o(M)\), put
\[
(\xi F)(x) = Fx \in o(N) = o(\xi N)
\]

• On arrows: given \((f, \sigma) \in \xi M(x_1, \ldots, x_k; x)\), put
\[
\xi F(f, \sigma) = (Ff, \sigma \theta_f^{-1})
\]
and check that
\[
(Ff, \sigma \theta_f^{-1}) \in \xi N(Fx_1, \ldots, Fx_k; Fx).
\]

First note that
\[
t(Ff, \sigma \theta_f^{-1}) = t(Ff) = F(t(f)) = Fx.
\]

Now
\[
s(f) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})
\]
in \(M\), so by the action of \((F, \theta)\) we have
\[
s(Ff) = (Fx_{\sigma \theta_f^{-1}(1)}, \ldots, Fx_{\sigma \theta_f^{-1}(k)})
\]
in \(N\), and so
\[
(Ff, \sigma \theta_f^{-1}) \in \xi N(Fx_1, \ldots, Fx_k; Fx)
\]
as required.

We check that this definition satisfies the laws for a morphism of symmetric multicategories:

• \(\xi F\) preserves identities: since \(\theta_{1_x} \in S_1 = \{\iota\}\), we have
\[
\xi F(1_x, \iota) = (F(1_x), \iota) = (1_{Fx}, \iota).
\]

• \(\xi F\) preserves composition: we check that \(\xi F(\alpha \circ \beta) = \xi F\alpha \circ \xi F\beta\), and the result then follows by simultaneous composition. Put
\[
\alpha = (f, \sigma) \in Q(x_1, \ldots, x_k; x)
\]
and \(\beta = (g, \tau) \in Q(y_1, \ldots, y_j; y)\).

Then
\[
\xi F(\alpha \circ \beta) = \xi F(f \circ_{\sigma^{-1}(p)} g, \gamma)
\]
\[
= (F(f \circ_{\sigma^{-1}(p)} g), \gamma \theta_f^{-1})
\]
\[
= (Ff \circ_{\theta_f \sigma^{-1}(p)} Fg, \gamma \theta_f^{-1})
\]

and this has source

\[ s(F\alpha \circ_p F\beta) = (Fx_1, \ldots, Fx_{p-1}, Fy_1, \ldots, Fy_j, Fx_{p+1}, \ldots, Fx_k). \]

For the right hand side, we have

\[ \xi F\alpha = (Ff, \sigma\theta_f^{-1}) \]
\[ \xi F\beta = (Fg, \tau\theta_g^{-1}) \]

and so the first component of \( \xi F\alpha \circ_p \xi F\beta \) is also \( Ff \circ_{\theta\sigma^{-1}(p)} Fg \). So since \( \xi F(\alpha \circ_p \beta) \) and \( \xi F\alpha \circ_p \xi F\beta \) agree in the first component and source, we have the result required.

- \( \xi F \) preserves symmetric action:
  \[
  \xi F( (f, \tau)\sigma ) = \xi F(f, \sigma^{-1}\tau) = (Ff, \sigma^{-1}\tau\theta_f^{-1}) = (Ff, \tau\theta_f^{-1})\sigma = (\xi F(f, \tau))\sigma
  \]

So \( \xi F \) is a morphism of symmetric multicategories.

We check that \( \xi \) is functorial. Clearly \( \xi 1_M = 1_{\xi M} \). Now consider morphisms of generalised multicategories

\[ M \xrightarrow{F} N \xrightarrow{G} L \]

so we need to show

\[ \xi(G \circ F) = \xi G \circ \xi F. \]

- On objects
  \[ \xi(G \circ F)(x) = (G \circ F)(x) = (\xi G \circ \xi F)(x) \]

- On arrows
  \[
  \xi(G \circ F)(f, \sigma) = ( (G \circ F)(f) , \sigma(\theta^{GF}_f)^{-1} ) = ( Gf , \sigma(\theta^{GF}_f \circ \theta^F_f)^{-1} ) = ( Gf , \sigma(\theta^F_f)^{-1}(\theta^{GF}_f)^{-1} ) = \xi G( Ff , \sigma(\theta^F_f)^{-1} ) = (\xi G \circ \xi F)(f, \tau)\sigma
  \]

So \( \xi \) is a functor as required.
Proposition 2.6 The functor $\xi : \text{GenMulticat} \rightarrow \text{SymMulticat}$ is full and faithful.

Proof. Given any morphism

$$G : \xi M \rightarrow \xi N$$

of symmetric multicategories, we show that there is a unique morphism

$$H = (H, \theta) : M \rightarrow N$$

of generalised multicategories such that

$$\xi H = G.$$

Suppose first that such an $H$ exists.

- On objects: for each object $x \in o(M) = o(\xi M)$ we must have

$$Hx = (\xi H)x = Gx.$$

- On arrows: given an arrow $f \in M(x_1, \ldots, x_k; x)$, we certainly have

$$(f, \iota) \in \xi M(x_1, \ldots, x_k; x)$$

and

$$G(f, \iota) = (f, \sigma) \in \xi N(Gx_1, \ldots, Gx_k; Gx),$$

say, where $\bar{f}$ is a morphism in $N$ with source

$$s(\bar{f}) = (Gx_{\sigma(1)}, \ldots, Gx_{\sigma(k)}).$$

Now $(\xi H)(f, \iota) = (Hf, \theta_f^{-1})$ but we must have

$$(\xi H)(f, \iota) = G(f, \iota) \quad = \quad (\bar{f}, \sigma)$$

so we must have $Hf = \bar{f}$ and $\theta_f = \sigma^{-1}$.

So we define $H$ as above and check that this satisfies the axioms for a morphism of generalised multicategories.

- $H$ preserves identities

We have

$$G(1_x, \iota) = (1_{Gx}, \iota)$$

so

$$H(1_x) = 1_{Gx} = 1_{Hx}.$$
• \(H\) preserves composition

We need to show
\[
Hf \circ_{\theta_f(p)} Hg = H(f \circ_p g)
\]
and that the coherence conditions are satisfied. Now, \(G\) preserves the composition of \(\xi M\) so
\[
G \circ_f G = G(\circ_f) \circ_p \beta.
\]
Now we have
\[
G \circ_f G = (f \circ_p g, \gamma') = (f \circ_p g, \gamma''),
\]
and
\[
G(\circ_f) \circ_p \beta = G(f \circ_p g, \gamma') = (f \circ_p g, \gamma''),\text{ say.}
\]
So these must be equal on both components. Comparing first components, we have
\[
\overline{f} \circ_p g = \overline{f} \circ_{\theta_f(p)} \overline{g}
\]
but by definition we have
\[
\overline{f} \circ_p g = H(f \circ_p g) = H(f \circ_p g).
\]
so
\[
Hf \circ_{\theta_f(p)} Hg = H(f \circ_p g)
\]
as required. Furthermore, equality of the second components gives precisely the coherence condition we require, since \(\gamma\) is formed from \(\theta_f, \theta_g\) and the amalgamation map \(\chi(f, g, \theta_f(p))\), and \(\gamma''\) is formed from \(\chi(f, g, p)\) and \(\theta_{f \circ_p g}\).

So \(H\) is a morphism of generalised multicategories; by construction it is unique such that \(\xi H = G\), so \(\xi\) is indeed full and faithful. \(\Box\)

We now give necessary and sufficient conditions for a symmetric multicategory to be in the image of \(\xi\).

**Definition 2.7** We say that a symmetric multicategory \(Q\) is freely symmetric if and only if for every arrow \(\alpha \in Q\) and permutation \(\sigma\)
\[
\alpha \sigma = \alpha \Rightarrow \sigma = \iota.
\]

**Proposition 2.8** Let \(Q\) be a symmetric multicategory. Then \(Q \cong \xi(M)\) for some generalised multicategory \(M\) if and only if \(Q\) is object-discrete and freely symmetric.
Proof. Suppose $Q \cong \xi(M)$. Then by the definition of $\xi$, $Q$ is object-discrete, with object-category $C \cong o(M)$. To show that $Q$ is freely symmetric, write $p = (x_1, \ldots, x_k; x)$, so

$$Q(p) = \{(f, \tau) \mid f \in a(M), \tau \in S_k, f : x_{\tau(1)}, \ldots, x_{\tau(k)} \to x \in M\}$$

and consider $\alpha = (f, \tau) \in Q(p)$. Now $(f, \tau)\sigma = (f, \sigma^{-1}\tau)$ so

$$\alpha\sigma = \alpha \Rightarrow \sigma^{-1}\tau = \tau \Rightarrow \sigma = 1$$

as required.

Conversely, suppose that $Q$ is object-discrete and freely symmetric. So, given an arrow $\alpha$ of arity $k$, we have distinct arrows $\alpha\sigma$ for each $\sigma \in S_k$. We define an equivalence relation $\sim$ on $a(Q)$, by

$$\alpha \sim \beta \iff \beta = \alpha\sigma \text{ for some permutation } \sigma$$

and we specify a representative of each equivalence class.

Now let $M$ be a generalised multicategory whose objects are those of $Q$, and whose arrows are the chosen representatives of the equivalence classes of $\sim$. Composition is inherited, with amalgamation maps re-ordering the sources as necessary. So associativity and commutativity are inherited; the coherence conditions for amalgamation maps are satisfied since $Q$ is freely symmetric. Observe that for each $x \in C$, the equivalence class of $1_x$ is $\{1_x\}$, so $M$ inherits identities.

So $M$ is a generalised multicategory, and $\xi(M) \cong Q$. Note that a different choice of representatives would give an equivalent generalised multicategory.

Definition 2.9 We call a symmetric multicategory tidy if it is freely symmetric with a category of objects equivalent to a discrete one. We write $\text{TidySymMulticat}$ for the full subcategory of $\text{SymMulticat}$ whose objects are tidy symmetric multicategories.

Lemma 2.10 A symmetric multicategory is tidy if and only if it is equivalent to one in the image of $\xi$.

Proof. We show that $Q$ is tidy if and only if $Q \cong R$ where $R$ is freely symmetric and object-discrete. The result then follows by Proposition 2.8.

Suppose $Q$ is tidy. We construct $R$ as follows. Let $C$ be the category of objects of $Q$, with $C$ equivalent to a discrete category $S$, say, by

$$\xymatrix{C \ar[r]^-F & S \ar[l]^-G}$$

Then $R$ is given by
• \( o(R) = S \).
• \( R(d_1, \ldots, d_n; d) = Q(Gd_1, \ldots, Gd_n; Gd) \).
• identities, composition and symmetric action induced from \( Q \).

Then certainly \( Q \simeq R \) and \( R \) is freely symmetric and object-discrete; the converse is clear. \( \square \)

We will later see (Section 3.3) that only tidy symmetric multicategories are needed for the construction of opetopes. We now include another result that will be useful in the next section.

**Lemma 2.11** If \( Q \) is a tidy symmetric multicategory then \( \text{elt} Q \) is equivalent to a discrete category.

**Proof.** This may be proved by direct calculation; it is also seen in Proposition 3.2. \( \square \)

Note that we write \( \text{elt} Q \) for the category of elements of \( Q \), where \( Q \) is here considered as a functor \( Q : \mathcal{F}^\text{op} \times \mathcal{C} \rightarrow \text{Set} \) with certain extra structure.

So \( \text{elt} Q \) has as objects pairs \((p, g)\) with \( p \in \mathcal{F}^\text{op} \times \mathcal{C} \) and \( g \in Q(p) \); a morphism \( \alpha : (p, g) \rightarrow (p', g') \) is an arrow \( \alpha : p \rightarrow p' \in \mathcal{F}^\text{op} \times \mathcal{C} \) such that
\[
Q(\alpha) : Q(p) \rightarrow Q(p')
\]
\[
g \mapsto g'.
\]

For example, an arrow
\[
(\sigma, f_1, f_2, f_3, f_4; f) : (x_1, x_2, x_3, x_4; x) \rightarrow (y_1, y_2, y_3, y_4; y) \in \mathcal{F}^\text{op} \times \mathcal{C}
\]
may be represented by the following diagram

```
\begin{tikzpicture}
  \node (x1) at (0,0) {$x_1$};
  \node (x2) at (1,0) {$x_2$};
  \node (x3) at (2,0) {$x_3$};
  \node (x4) at (3,0) {$x_4$};
  \node (y1) at (0,1) {$y_{\sigma(1)}$};
  \node (y2) at (1,1) {$y_{\sigma(2)}$};
  \node (y3) at (2,1) {$y_{\sigma(3)}$};
  \node (y4) at (3,1) {$y_{\sigma(4)}$};
  \node (f) at (2,0) {$f_4$};
  \node (f1) at (0,0) {$f_1$};
  \node (f2) at (1,0) {$f_2$};
  \node (f3) at (2,0) {$f_3$};
  \draw[->] (x1) -- (y1);
  \draw[->] (x2) -- (y2);
  \draw[->] (x3) -- (y3);
  \draw[->] (x4) -- (y4);
  \draw[->] (x1) -- (f);
  \draw[->] (x2) -- (f);
  \draw[->] (x3) -- (f);
  \draw[->] (x4) -- (f);
\end{tikzpicture}
```
Then, given any arrow \( g \in Q(x_1, \ldots, x_m; x) \), we have an arrow
\[
\alpha(g) = g' \in Q(y_1, \ldots, y_m; y)
\]
given by
\[
g' = (\iota(f) \circ g \circ (\iota(f_1), \ldots, \iota(f_m)) \sigma).
\]
So continuing the above example we may have:

\[
\begin{array}{c}
y_1 \quad y_2 \quad y_3 \quad y_4 \\
\downarrow \quad \quad \quad \quad \quad \downarrow g' \\
y \\
\end{array}
\quad =
\begin{array}{c}
x_1 \quad x_2 \quad x_3 \quad x_4 \\
\downarrow f \\
y \\
\end{array}
\]

Note that we may write an object \((p, g) \in \text{elt}(Q)\) simply as \(g\), since \(p\) is uniquely determined by \(g\).

### 3 The theory of opetopes

In this section we give the analogous constructions of opetopes in each theory, and show in what sense they are equivalent. That is, we show that the respective categories of \(k\)-opetopes are equivalent.

We first discuss the process by which \((k+1)\)-cells are constructed from \(k\)-cells. In [3], the ‘slice’ construction is used, giving for any symmetric multicategory \(Q\) the slice multicategory \(Q^+\). In [12] the ‘multicategory of function replacement’ is used but this has a more far-reaching role than that of the Baez-Dolan slice. For comparison with the Baez-Dolan theory, we construct a ‘slice’ which is analogous to the Baez-Dolan slice and is a special case of a multicategory of function replacement.

Opetopes and multitopes are then constructed by iterating the slicing process. We finally apply the results already established to show that the category of multitopes is equivalent to the category of opetopes.

#### 3.1 Slicing a symmetric multicategory

Let \(Q\) be a symmetric multicategory with a category \(\mathbb{C}\) of objects, so \(Q\) may be considered as a functor \(Q : \mathcal{F}^{\text{op}} \times \mathbb{C} \to \text{Set}\) with certain extra structure. The slice multicategory \(Q^+\) is given by:
• Objects: put $o(Q^+) = \text{elt}(Q)$

So the category $o(Q^+)$ has as objects pairs $(p, g)$ with $p \in F \mathcal{C}^{\text{op}} \times \mathcal{C}$ and $g \in Q(p)$; a morphism $\alpha : (p, g) \rightarrow (p', g')$ is an arrow $\alpha : p \rightarrow p' \in F \mathcal{C}^{\text{op}} \times \mathcal{C}$ such that

$$Q(\alpha) : Q(p) \longrightarrow Q(p')$$

$$g \longmapsto g'$$

Then, given any arrow

$$g \in Q(x_1, \ldots, x_m; x)$$

we have an arrow $\alpha(g) = g' \in Q(y_1, \ldots, y_m; y)$ given by

$$g' = (\iota(f) \circ g \circ (\iota(f_1), \ldots, \iota(f_m))\sigma)$$

(see Section 2.3).

• Arrows: $Q^+(f_1, \ldots, f_n; f)$ is given by the set of ‘configurations’ for composing $f_1, \ldots, f_n$ as arrows of $Q$, to yield $f$.

Writing $f_i \in Q(x_{i_1}, \ldots, x_{i_m}; x_i)$ for $1 \leq i \leq n$, such a configuration is given by $(T, \rho, \tau)$ where

1) $T$ is a planar tree with $n$ nodes. Each node is labelled by one of the $f_i$, and each edge is labelled by an object-morphism of $Q$ in such a way that the (unique) node labelled by $f_i$ has precisely $m_i$ edges going in from above, labelled by $a_{i_1}, \ldots, a_{i_m} \in \text{arr}(\mathcal{C})$, and the edge coming out is labelled $a_i \in a(\mathcal{C})$, where $\text{cod}(a_{ij}) = x_{ij}$ and $\text{dom}(a_i) = x_i$.

2) $\rho \in S_k$ where $k$ is the number of leaves of $T$.

3) $\tau : \{\text{nodes of } T\} \longrightarrow [n] = \{1, \ldots, n\}$ is a bijection such that the node $N$ is labelled by $f_{\tau(N)}$. (This specification is necessary to allow for the possibility $f_i = f_j$, $i \neq j$.)

Note that $(T, \rho)$ may be considered as a ‘combed tree’, that is, a planar tree with a ‘twisting’ of branches at the top given by $\rho$.

The arrow resulting from this composition is given by composing the $f_i$ according to their positions in $T$, with the $a_{ij}$ acting as arrows $\iota(a_{ij})$ of $Q$, and then applying $\rho$ according to the symmetric action on $Q$. This construction uniquely determines an arrow $(T, \rho, \tau) \in Q^+(f_1, \ldots, f_n; f)$.

• Composition

When it can be defined, $(T_1, \rho_1, \tau_1) \circ_m (T_2, \rho_2, \tau_2) = (T, \rho, \tau)$ is given by
1) \((T, \rho)\) is the combed tree obtained by replacing the node \(\tau_1^{-1}(m)\) by the tree \((T_2, \rho_2)\), composing the edge labels as morphisms of \(\mathbb{C}\), and then ‘combing’ the tree so that all twists are at the top.

2) \(\tau\) is the bijection which inserts the source of \(T_2\) into that of \(T_1\) at the \(m\)th place.

- **Identities:** given an object-morphism

  \[\alpha = (\sigma, f_1, \ldots, f_m; f) : g \to g',\]

  \(\nu(\alpha) \in Q^+(g; g')\) is given by a tree with one node, labelled by \(g\), twist \(\sigma\), and edges labelled by the \(f_i\) and \(f\) as in the example above.

- **Symmetric action:** \((T, \rho, \tau)\sigma = (T, \rho, \sigma^{-1}\tau)\)

This is easily seen to satisfy the axioms for a symmetric multicategory.

Note that, given a labelled tree \(T\) with \(n\) nodes and \(k\) leaves, there is an arrow \((T, \rho, \tau) \in a(Q^+)\) for every permutation \(\rho \in S_k\) and every bijection \(\tau : \{\text{nodes of } T\} \to [n]\). Suppose

\[
\begin{align*}
s(T, \rho, \tau) &= (f_1, \ldots, f_n) \\
t(T, \rho, \tau) &= f.
\end{align*}
\]

Then, given any \(\rho_1 \in S_k, \tau : \{\text{nodes of } T\} \to [n]\), we have

\[
\begin{align*}
s(T, \rho_1\rho, \tau) &= (f_1, \ldots, f_n) \\
t(T, \rho_1\rho, \tau) &= f\rho_1
\end{align*}
\]

whereas

\[
\begin{align*}
s(T, \rho, \tau_1\tau) &= (f_{\tau_1^{-1}(1)}, \ldots, f_{\tau_1^{-1}(n)}) \\
t(T, \rho, \tau_1\tau) &= f.
\end{align*}
\]

We observe immediately that \(Q^+\) is freely symmetric, since

\[(T, \rho, \tau)\sigma = (T, \rho, \tau) \Rightarrow \sigma^{-1}\tau = \tau \Rightarrow \sigma = \nu.
\]

However \(Q^+\) is not in general object-discrete; we will later see (Proposition 3.2) that \(Q^+\) is tidy if \(Q\) is tidy.
3.2 Slicing a generalised multicategory

Given a generalised multicategory $M$, we define a slice multicategory $M_+$. We use the ‘multicategory of function replacement’ as defined in [12], which plays a role similar to (but more far-reaching than) that of the Baez-Dolan slice. The slice defined in this section is only a special case of a multicategory of function replacement, but it is sufficient for the construction of multitopes. Moreover, for the purpose of comparison it is later helpful to be able to use this closer analogy of the Baez-Dolan slice.

We first explain how this slice arises from the multicategory of function replacement as defined in [12], and then give an explicit construction of the slice multicategory that is analogous to the symmetric case. This latter construction is the one we continue to use in the rest of the work.

Using the terminology of [12], the slice is defined as follows. Let $L$ be the language with objects $o(M)$ and arrows $a(M)$, and let $F$ be the free generalised multicategory on $L$. So the objects of $F$ are the objects of $M$, and the arrows of $F$ are formal composites of arrows of $M$. We define a morphism of generalised multicategories $h: F \to M$ as the identity on objects, and on arrows the action of composing the formal composite to yield an arrow of $M$. Then we define $M_+$ to be the multicategory of function replacement on $(L, F, h)$.

Explicitly, the slice multicategory $M_+$ is a generalised multicategory given by:

- **Objects:** $o(M_+) = a(M)$.
- **Arrows:** $a(M_+)$ is given by configurations for composing arrows of $M$.

Such a configuration is given by $T = (T, \rho_T, \tau_T)$, where:

i) $T$ is a planar tree with $n$ nodes labelled by $f_1, \ldots, f_n \in a(M)$, and edges labelled by objects of $M$ in such a way that, writing $s(f_i) = (x_{i1}, \ldots, x_{im_i})$,

- the node labelled by $f_i$ has $m$ edges coming in, labelled by $x_{i1}, \ldots, x_{im_i}$ from left to right, and one edge going out, labelled by $t(f_i)$.

ii) $\rho_T \in S_k$, where $k$ is the number of leaves of $T$. The composition in $M$ given by $T$ has specified amalgamation maps giving information about the ordering of the source; $\rho_T$ is the permutation induced on the source.

iii) $\tau_T : \{\text{nodes of } T\} \to [n]$ is a bijection so that the node $N$ is labelled by $f_{\tau_T(N)}$. In fact, specifying $\tau_T$ corresponds to specifying amalgamation maps in the free multicategory $F$, and this defines the amalgamation maps of $M_+$. 

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Note that whereas in the symmetric case $\rho$ and $\tau$ may be chosen freely for any given $T$, in this case precisely one $\rho_T$ and $\tau_T$ is specified for each $T$. The source and target of such an arrow $T$ are given by $s(T) = (f_1, \ldots, f_n)$ and $t(T) = f \in a(M)$, the result of composing the $f_i$ according to their positions in $T$. Here, the tree $T$ may be thought of as a combed tree as in the symmetric case, but with all edges labelled by identities.

- **Composition**

When it can be defined, we have $T_1 \circ_m T_2 = T$ as follows:

i) $T$ is the combed labelled tree obtained from $(T_1, \tau_{T_1})$ by replacing the node $\tau_{T_1}^{-1}(m)$ by the combed tree $(T_2, \tau_{T_2})$, combing the tree and then forgetting the twist at the top.

ii) The amalgamation maps are defined to reorder the source as necessary according to $\tau_{T_1}$, $\tau_{T_2}$ and $\tau_T$.

- **Identities**: $1_f$ is the tree with one node, labelled by $f$.

This definition is easily seen to satisfy the axioms for a generalised multicategory. Note that a different choice of amalgamation maps for $F$ gives rise to different bijections $\tau_T$ and hence different amalgamation maps in $M_+$, resulting in an isomorphic slice multicategory.

### 3.3 Comparison of slice

In this section we compare the slice constructions and make precise the sense in which they correspond to one another. Recall (section 2.3) that we have defined a functor

$$\text{GenMulticat} \xrightarrow{\xi} \text{TidySymMulticat}.$$  

We now show that this functor ‘commutes’ with slicing, up to equivalence.

We will eventually prove (Corollary 3.3) that for any generalised multicategory $M$

$$\xi(M_+) \simeq \xi(M)^+.$$  

We prove this by constructing, for any morphism of symmetric multicategories $\phi : Q \rightarrow \xi(M)$ a morphism $\phi^+ : Q^+ \rightarrow \xi(M_+)$ such that

$\phi$ is an equivalence $\Rightarrow \phi^+$ is an equivalence.

The result then follows by considering the case $\phi = 1$.  

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We begin by constructing $\phi^+$. Recall

\[
\begin{align*}
o(Q^+) & = a(Q) \\
a(Q^+) & = \{(T, \rho, \tau) : T \text{ a labelled tree with } n \text{ nodes, } k \text{ leaves} \\
& \quad \rho \in S_k, \\
& \quad \tau : \{\text{nodes of } T\} \xrightarrow{\sim} [n] \\
& \quad \text{edges labelled by morphisms of } C\}
\end{align*}
\]

\[
\begin{align*}
o(\xi(M_+)) & = a(M) \\
a(\xi(M_+)) & = \{(T, \sigma) : T \text{ a labelled tree with } n \text{ nodes} \\
& \quad \sigma \in S_n \\
& \quad \text{edges labelled by identities}\}
\end{align*}
\]

The idea is that given a way of composing arrows $f_1, \ldots, f_n$ of $Q$ to an arrow $f$, we have a way of composing arrows $g_1, \ldots, g_n$ of $M$ to an arrow $g$, where

\[
\phi(f_i) = (g_i, \sigma_i) \quad \text{and} \quad \phi(f) = (g, \sigma).
\]

Observe that since $\xi M$ is object-discrete, we have $\phi a = 1$ for all object-morphisms $a \in C$.

So we define $\phi^+$ as follows:

- On objects: if $\phi(f) = (g, \sigma), \ g \in a(M)$ then put $\phi^+(f) = g$.
- On object-morphisms: since $\xi(M^+)$ is object-discrete, we must have $\phi^+(\alpha) = 1$ for all object-morphisms $\alpha$.
- On arrows: put $\phi^+ : (T, \rho, \tau) \mapsto (T, \tau \circ \tau^{-1})$, where $T$ is the labelled planar tree obtained as follows. Given a node with label $f$ say, and $\phi(f) = (g, \sigma)$:
  
  i) replace the label with $g$
  ii) ‘twist’ the inputs of the node according to $\sigma$
  iii) proceed similarly with all nodes, make all edge labels identities, then comb and ignore the twist at the top of the resulting tree (since the twist in $M_+$ is determined by the tree).

For example, suppose $T$ is given by

```
T_1  T_2  \ldots  T_n
  \downarrow
    f
```
where the $T_i$ are subtrees of $T$, and $\phi(f) = (g, \sigma)$. Then steps (i) and (ii) above give

$$T_{\sigma(1)} \quad T_{\sigma(2)} \quad \ldots \quad T_{\sigma(n)}$$

and $\bar{T}$ is then defined inductively on the subtrees. Node $N$ in $\bar{T}$ is considered to be the image of node $N$ in $T$ under the operation $T \rightarrow \bar{T}$.

Writing

$$s(T, \rho, \tau) = (f_1, \ldots, f_n)$$

and $t(T, \rho, \tau) = f$

we check that

$$s(\phi^+(T, \rho, \tau)) = (\phi^+(f_1), \ldots, \phi^+(f_n))$$

and $t(\phi^+(T, \rho, \tau)) = \phi^+(f)$.

Writing $s(\bar{T}, \tau \circ \tau_\bar{T}^{-1}) = (g_1, \ldots, g_n)$ in $\xi(M)$, we have, in $M_+$

$$s(\bar{T}) = (g_{\tau \circ \tau_\bar{T}^{-1}(1)}, \ldots, g_{\tau \circ \tau_\bar{T}^{-1}(n)})$$

so node $N$ is labelled in $\bar{T}$ by $g_{\tau \circ \tau_\bar{T}^{-1}(\tau_{\bar{T}}(N))} = g_{\tau(N)}$ and in $T$ by $f_{\tau(N)}$. So by definition of $\bar{T}$ we have

$$\phi^+(f_{\tau(N)}) = g_{\tau(N)}$$

so $\phi^+(f_i) = g_i$ for each $i$ and

$$s(\bar{T}, \tau \circ \tau_\bar{T}^{-1}) = (\phi^+(f_1), \ldots, \phi^+(f_n))$$

as required. Also, $t(\bar{T}, \tau \circ \tau_\bar{T}^{-1}) = \phi^+(f)$ by functoriality of $\phi$ and definition of composition in $\xi(M)$.

We have shown that $\phi^+$ is functorial on the object-category $o(Q^+)$; we need to check the remaining conditions for $\phi^+$ to be a morphism of symmetric multicategories. We may now assume that all edge labels are identities since they all become identities under the action of $\phi^+$.

- $\phi^+$ preserves identities:
1_f \in a(Q^+) is (T, \iota, \iota) where T has one node, labelled by f. So we have 
\phi^+(1_f) = T where T has one node, labelled by \phi^+(f), and \phi^+(1_f) = 1_{\phi^+}(f).

- \phi^+ preserves composition: We need to show
  \phi^+(\alpha \circ_m \beta) = \phi^+(\alpha) \circ_m \phi^+(\beta).

Now, the underlying trees are the same by functoriality of \phi, the permutation of leaves is the same by coherence for amalgamation maps of M, and the node ordering is the same by definition of \phi^+.

- \phi^+ preserves symmetric action:
  \[
  \phi^+((T, \rho, \tau)\sigma) = \phi^+(T, \rho, \sigma^{-1}_\tau) = (\bar{T}, \sigma^{-1}_\tau \circ \tau^{-1}_T) = (\bar{T}, \tau \circ \tau^{-1}_T)\sigma = (\phi^+(T, \rho, \tau))\sigma.
  \]

So \phi^+ is a morphism of symmetric multicategories.

**Proposition 3.1** Let Q be a symmetric multicategory, M a generalised multicategory and \phi : Q \rightarrow \xi(M) a morphism of symmetric multicategories. If \phi is an equivalence then \phi^+ is an equivalence.

This enables us to prove the following proposition:

**Proposition 3.2** If Q is tidy then Q^+ is tidy.

**Proof of Proposition 3.1** First we observe that given any such morphism \phi, Q is freely symmetric:

\[
\alpha\sigma = \alpha \Rightarrow \phi(\alpha\sigma) = \phi(\alpha)\sigma = \phi(\alpha) \in \xi(M) \Rightarrow \sigma = \iota,
\]

the second implication following from \xi(M) being freely symmetric.

Now, given that \phi is full, faithful and essentially surjective on the category of objects, and full and faithful, we prove the proposition in the following steps:

i) \phi^+ is surjective on objects

ii) \phi^+ is full on the category of objects

iii) \phi^+ is faithful on the category of objects
iv) $\phi^+$ is full

v) $\phi^+$ is faithful

Proof of (i). Recall the action of $\phi^+$ on objects: let $f \in o(Q^+) = a(Q)$ with $\phi(f) = (g,\sigma)$ then $\phi^+ : f \mapsto g$. Now, given any $g \in o(\xi(M_+)) = a(M)$, we have $(g,i) \in a(\xi(M))$. $\phi$ is full and surjective, so there exists $f \in a(Q)$ such that $\phi(f) = (g,\sigma)$ and $\phi^+(f) = g$. □

Proof of (ii). $\xi(M_+)$ is object-discrete so we only need to show that if $\phi^+(f_1) = \phi^+(f_2)$ then there is a morphism $f_1 \to f_2$ in $o(Q^+)$. Now

$$\phi^+(f_1) = \phi^+(f_2) \Rightarrow \phi(f_1) = \phi(f_2)\sigma \text{ for some permutation } \sigma$$

Suppose

$$f_1 : a_1, \ldots, a_n \to a$$

and $f_2\sigma : b_1, \ldots, b_n \to b$.

Then we must have $\phi(a_i) = \phi(b_i)$ for all $i$, and $\phi(a) = \phi(b)$. So there exist morphisms

$$g_i : b_i \to a_i$$

and $g : a \to b$

and we have

$$f_2\sigma = g \circ f_1 \circ (g_1, \ldots, g_n)$$

giving a morphism $f_1 \to f_2$ as required. □

Proof of (iii). An arrow $\alpha : f_1 \to f_2$ is uniquely of the form $(\sigma, g_1, \ldots, g_n; g)$ with

$$g_i : s(f_2)_{\sigma(i)} \to s(f_1)_i$$

and $g : t(f_1) \to t(f_2)$

as arrows of $C$. Since $\phi$ is faithful on the category of objects and $\xi(M)$ is object-discrete, there can only be one such map. □

Proof of (iv). Given $f_1, \ldots, f_n, f \in o(Q^+)$ and

$$(T, \sigma) : (\phi^+(f_1), \ldots, \phi^+(f_n)) \to \phi^+(f) \in \xi(M_+)$$

we seek

$$(T', \rho, \tau) : (f_1, \ldots, f_n) \to f \in Q^+$$

such that

$$\phi^+(T', \rho, \tau) = (T, \sigma)$$
i.e. such that $\bar{T}' = T$ and $\tau \circ \tau^{-1} = \sigma$.

Write $\phi(f) = (g, \alpha)$ and for each $i$, $\phi(f_i) = (g_i, \alpha_i)$. Then $\phi^+(f_i) = g_i$ and $\phi^+(f) = g$. $(T, \sigma)$ is a configuration for composing the $g_i$ to yield $g$, so we certainly have a configuration for composing the $(g_i, \alpha_i)$ to yield $g_i$ as follows: replace node label $g_i$ by $(g_i, \alpha_i)$ and insert a twist $\alpha_i^{-1}$ above the node, then comb and add the necessary twist at the top.

This gives a configuration for composing the $f_i$ as follows. We have

$$t(g_i, \alpha_i) = s(g_k, \alpha_k) \Rightarrow \phi(t(f_i)) = \phi(s(f_k)) = s(f_k).$$

Now $\phi$ is faithful on the category of objects, so there exists a morphism

$$t(f_i) \rightarrow s(f_k)$$

and we label the edge joining $t(f_i)$ and $s(f_k)$ with this object-morphism. So this gives a configuration for composing the $f_i$ to yield $h$, say, with $\phi(h) = \phi(f)$. That is, we have a morphism

$$(f_1, \ldots, f_n) \xrightarrow{\theta} h$$

such that $\phi^+(\theta) = (T, \sigma)$.

Now $\phi$ is full on the category of objects, so if $\phi(h) = \phi(f)$ then there is a morphism $\alpha : h \rightarrow f$ in $o(Q^+)$. So we have

$$(f_1, \ldots, f_n) \xrightarrow{\theta} h \xrightarrow{\iota(\alpha)} f$$

and $\phi^+(\iota(\alpha))$ is the identity since $\xi(M_+)$ is object-discrete. So

$$\phi^+(\iota(\alpha) \circ \theta) = \phi^+(\theta) = (T, \sigma)$$

as required. \qed

**Proof of (v).** Suppose $\phi^+(\alpha) = \phi^+(\beta)$. Then, writing

$$\alpha = (T_1, \rho_1, \tau_1) : (f_1, \ldots, f_n) \rightarrow f$$
$$\beta = (T_2, \rho_2, \tau_2) : (f_1, \ldots, f_n) \rightarrow f$$

we have $T_1 = T_2 = \bar{T}$, say, and $\tau_1 \circ \tau_1^{-1} = \tau_2 \circ \tau_2^{-1}$ so $\tau_1 = \tau_2$. So given any node $N$ in $\bar{T}$, its pre-image in $T_1$ has the same label $f_i$ as its pre-image in $T_2$. The same is true of edge labels, since $\phi$ is faithful on the category of objects.

Then the tree $T_1$ may be obtained from $\bar{T}$ as follows. Suppose $\phi(f_i) = (g_i, \sigma)$ and $\phi(f) = g$. Then for the node labelled by $g_i$, apply the twist $\sigma^{-1}$ to the edges above it, and then relabel the node with $f_i$. This process may also be applied to obtain the tree $T_2$. Since the process is the same in both cases, we have $T_1 = T_2 = T$, say.
Finally, suppose $f'$ is the arrow obtained from composing according to $T$. Then by the action of $\alpha$, $f = f'\rho_1$, and by the action of $\beta$, $f = f'\rho_2$. Then, since $Q$ is freely symmetric, $\rho_1 = \rho_2$, so $\alpha = \beta$ as required. □

**Proof of Proposition 3.2** Given a tidy symmetric multicategory $Q$ we need to show that $Q^+$ is also tidy.

Recall (Lemma 2.10) that a symmetric multicategory $Q$ is tidy if and only if it is equivalent to one in the image of $\xi$, $\xi M$ say, with equivalence given by

$$\phi : Q \longrightarrow \xi(M).$$

Then by Proposition 3.1 $\phi^+$ is an equivalence

$$\phi^+ : Q^+ \longrightarrow \xi(M^+)$$

so $Q^+$ is tidy as required. □

**Corollary 3.3** Let $M$ be a generalised multicategory. Then

$$\xi(M)^+ \simeq \xi(M^+)$$

as symmetric multicategories with a category of objects.

**Proof.** Put $Q = \xi(M)$, $\phi = 1$ in Proposition 3.1 □

We are now ready to give the construction of opetopes.

### 3.4 Opetopes

For any symmetric multicategory $Q$ we write

$$Q^{k+} = \begin{cases} Q & k = 0 \\ (Q^{(k-1)+})^+ & k \geq 1 \end{cases}$$

Let $I$ be the symmetric multicategory with precisely one object, precisely one (identity) object-morphism, and precisely one (identity) arrow. A $k$-dimensional opetope, or simply $k$-opetope, is defined in [3] to be an object of $I^{k+}$. We write $\mathbb{C}_k = o(I^{k+})$, the category of $k$-opetopes.

### 3.5 Multitopes

Multitopes are defined in [12] using the multicategory of function replacement. We give the same construction here, but state it in the language of slicing; this makes the analogy with Section 3.4 clear.
For any generalised multicategory $M$ we write

$$M_{k^+} = \begin{cases} M & k = 0 \\ (M_{(k-1)^+})_+ & k \geq 1 \end{cases}$$

Let $J$ be the generalised multicategory with precisely one object and precisely one (identity) morphism. Then a $k$-multitope is defined to be an object of $J_{k^+}$. We write $P_k = o(J_{k^+})$, the set of $k$-multitopes; we will also regard this as a discrete category.

### 3.6 Comparison of opetopes and multitopes

In this section we compare the construction of opetopes and multitopes, applying the results we have already established.

**Proposition 3.4** For each $k \geq 0$

$$\xi(J_{k^+}) \simeq I_{k^+}.$$

**Proof.** By induction. First observe that $\xi(J) \cong I$ and write $\phi$ for this isomorphism. So for each $k \geq 0$ we have

$$\phi^{k^+} : I_{k^+} \rightarrow \xi(J_{k^+}),$$

where

$$\phi^{k^+} = \begin{cases} \phi & k = 0 \\ (\phi^{(k-1)^+})_+ & k \geq 1 \end{cases}$$

Now $I$ is (trivially) tidy, so by Proposition 3.2 $I_{k^+}$ is tidy for each $k \geq 0$. So by Proposition 3.1 $\phi^{k^+}$ is an equivalence for all $k \geq 0$. \qed

Then on objects, the above equivalence gives the following result.

**Corollary 3.5** For each $k \geq 0$

$$P_k \simeq C_k.$$

This results shows that ‘opetopes and multitopes are the same up to isomorphism’.

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