RANDOM LAG SINGULAR CROSS-SPECTRUM ANALYSIS

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ABSTRACT

In a previous paper (Varadi et al. 1999), Random Lag Singular Spectrum Analysis was offered as a tool to find oscillations in very noisy and long time series. This work presents a generalization of the technique to search for common oscillations in two or more time series.

Subject headings: Sun: oscillations — methods: data analysis

1. INTRODUCTION

This paper follows up on our previous one (Varadi et al. 1999) in which Random Lag Singular Spectrum Analysis was described at length. The technique is being used to search for low-frequency solar acoustic and gravity oscillations in the Global Oscillations at Low Frequency (GOLF; Gabriel 1998) and Michelson Doppler Imager (MDI; Scherrer 1998) data. It has become apparent, however, that identifying common, simultaneous oscillatory components in these data is a more promising approach. Hence the need arose to generalize the technique for two or more time series which is described here briefly.

Singular Spectrum Analysis (SSA; Vautard & Ghil 1989; Vautard et al. 1992; Dettinger et al. 1995) was originally developed to search for oscillation in short and noisy time series that one typically encounters in geophysics. The technique computes the eigenvectors of autocorrelation matrices. The sizes of the latter are usually a third of the length of the time series which limits the feasibility of the method to time series no longer than a few thousand. It has been employed in the analysis of GOLF and MDI data by extracting the

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signal in narrow frequency bands which reduces the length of the time series to which SSA is applied (Ulrich et al. 1998; Varadi et al. 1998). This, however, made it difficult to assess the importance of candidate modes in the signal as a whole. Our previous paper on Random Lag Singular Spectrum Analysis (Varadi et al. 1999) explored connections between linear dynamical systems and SSA. It was shown that one can work with sequences of random lags when dealing with matrices of autocovariances, which are the mainstay of SSA and numerous other techniques such as autoregressive modeling (AR) (e.g., Percival & Walden 1993; Proakis & Manolakis 1996). This approach makes it possible to carry out SSA in wide frequency bands, on time series having tens of thousands of points. Here a generalization of the technique, Random Lag Singular Cross-Spectrum Analysis (RLSCSA), is introduced through a few equations, without exploring deep mathematical issues. Preliminary results obtained by RLSCSA are quite encouraging and it seems appropriate to make the technique accessible to the community expeditiously.

2. RANDOM LAG SINGULAR CROSS-SPECTRUM ANALYSIS

The technique searches for coincident patterns of oscillations in two given time series, \( x = x_1, x_2, \ldots, x_N \) and \( y = y_1, y_2, \ldots, y_N \) with the same uniform sampling in time. Random Lag Singular Spectrum Analysis (Varadi et al. 1999) does so in the case of a single time series, i.e., \( x = y \), by looking at possible linear relationships between its lagged copies. The reasoning relies on straightforward but lengthy linear algebra. In short, a linear system produces time series in which consecutive values are linear related, at least approximately. This is analogous to the point of view of AR models (Percival & Walden 1993) but it is more general. The most important difference, however, is that SSA and its generalizations do not try to fit an AR model to very noisy data but rather try to extract what appears to be signal from a noisy background.

Both time series are assumed to have zero mean. First two matrices are formed, one of which is

\[
D_x = \begin{pmatrix}
    : & : & : \\
    x_{j-1+k_1} & x_{j-1+k_2} & \cdots & x_{j-1+k_M} \\
    x_{j+k_1} & x_{j+k_2} & \cdots & x_{j+k_M} \\
    x_{j+1+k_1} & x_{j+1+k_2} & \cdots & x_{j+1+k_M} \\
    : & : & : 
\end{pmatrix}, \quad (2-1)
\]

whose columns are index-shifted (lagged) copies of the time series \( x \). Here the lags \( k_1, k_2, \ldots, k_M \) are all different, random positive integers uniformly distributed between 1 and a maximum lag \( K \). With some other set of random lags, the analogous matrix \( D_y \) is
formed for $y$. The two sets of lags, for $x$ and $y$, may or may not contain common elements. For the sake of exposition, the data matrices $D_x$ and $D_y$ are padded with zeros for those indeces of $x$ and $y$ for which no value is available, i.e., when the index is smaller than one or larger than $N$. In practice, these data matrices are not used directly. Then the $M \times M$ matrix

$$C = D_x^T D_y$$

(2-2)

is formed which consists of covariances at various lags, except for normalization factors ($^T$ denotes transpose).

The essence of the method is to compute the Singular Value Decomposition (SVD; Golub & Van Loan 1996) of this matrix, i.e.,

$$C = D_x^T D_y = E_x \Lambda E_y^T,$$

(2-3)

where $E_x$ and $E_y$ are orthogonal matrices. The matrix $\Lambda$ is diagonal and contains the so-called singular values. From (2-3) it follows that

$$(D_x E_x)^T (D_y E_y) = \Lambda.$$  

(2-4)

The columns of the $N \times M$ matrix $D_x E_x$ represent filtered versions of the original time series $x$, while $D_y E_y$ does the same for $y$. Clearly, the singular values can be interpreted as covariances between these filtered time series. One can also observe that the $i$th column of $D_x E_x$ has nonzero covariance only with the $i$th column of $D_y E_y$.

Next the columns of $D_y E_y$ are modeled by the columns of $D_x E_x$ using ordinary linear regression, the $i$th column in the former by the $i$th column in the latter. For this, the variances of the columns in both matrices have to be computed. In the case of $x$, they are the diagonal elements of the $M \times M$ matrix

$$(D_x E_x)^T (D_x E_x) = E_x^T (D_x^T D_x) E_x$$

(2-5)

and an analogous formula applies in the case of $y$. Since the original $x$ and $y$ have zero means, the same is true for the columns of $D_x E_x$ and $D_y E_y$, at least approximately in the case of large $N$ and no trends in the data. Hence one needs to compute only the scaling coefficients in the linear regression models between these columns. These can be collected to form a diagonal matrix $B_x$ to obtain

$$(D_y E_y) = D_x E_x B_x,$$

(2-6)

where $\hat{\text{}}$ signifies that this is a statistical model. Next one defines the model for the columns of $D_y$ as

$$\hat{D}_y = D_x E_x B_x E_y^T.$$  

(2-7)
Finally, we have to create a new time series $\hat{y}$ from $\hat{D}_y$. Each column of the latter is index-shifted relative to the original indexing of $y$ and thus one could use any column as $\hat{y}$. Alternatively, one can simply unshift each column and average them to obtain

$$\hat{y}_n = \frac{1}{M} \sum_{s=1}^{M} \sum_{i=1}^{M} \sum_{j=1}^{M} x_{n+(k_i^x-k_j^y)} (B_x^s)_{i} (E_x^s)_{i} (E_y^s)_{j},$$

(2-8)

where $k_i^x$ is the $i$th lag for $x$, $k_j^y$ is the $j$th lag for $y$, $s$ designates the $s$th column of a matrix and the subscripts $i$ and $j$ are row indices.

The last equation describes how to model oscillations in $y$ with those in $x$ by moving-average or finite impulse response filters. Very roughly, the oscillations in $x$ are represented by $E_x$ and those in $y$ by $E_y$, while $B_x$ contains scaling factors. At this point one can separate signal and noise by not taking into account all the linear regression models between $x$ and $y$. When the cross-correlation is large for some component $s$, it is more likely that the same oscillation is present in both time series. Hence one would not sum for all $s = 1, \ldots, M$ but for only some of them. Once the filter has been determined, perhaps the best way to proceed is by computing its spectral response i.e., its $z$ transform which boils down to computing its Fourier transform (Proakis & Manolakis 1996). This is analogous to the Maximum Entropy Method (Percival & Walden 1993).

When $x$ and $y$ are the same and the same lags are used, the filtering formula above is the same as in Random Lag Singular Spectrum Analysis with $B_x$ being identity matrix (Varadi et al. 1999). Therefore, RLSCSA is a direct generalization of Random Lag Singular Spectrum Analysis. Furthermore, as in the case of the latter, in RLSCSA the analysis can be carried out for several different lag sequences and the filters can be averaged for these cases. Also, a larger number of lags, $M$, always provides better results.

There can be a number of variations on the basic construction above. For instance, in the case of $x = y$ one can use lag sequences $k^x$ and $k^y$ which do not have common elements. This doubles the number of autocovariances one would include in the matrix $C$ as compared to Random Lag Singular Spectrum Analysis for the same number of lags $M$. In the case of $x \neq y$, one could use the same lag sequence ($k^x = k^y$) or different ones. When several time series are given, divided into two groups, $x^{(1)}, x^{(2)}, \ldots, x^{(d_x)}$ and $y^{(1)}, y^{(2)}, \ldots, y^{(d_y)}$, one can form the data matrix

$$[D_{x^{(1)}}, D_{x^{(2)}}, \ldots, D_{x^{(d_x)}}]$$

(2-9)

and an analogous one for the other set of time series. In these matrices the columns come from possibly different time series. The formulae above needs only straightforward modifications for this case although indexing becomes somewhat complicated. In (2-8), one would have filters operating on the $x$ signals whose sum would model oscillations in $y^{(i)}$ for $i = 1, 2, \ldots, d_y$. 

There can be phase lags between $x$ and $y$. There should be none, however, when they are simultaneous and independent measurements of the same physical quantity. In such cases, one can use the phase information in the filter (2-8) to further separate signal and noise. If $E_x$ and $E_y$ operate on the same signal, then the phases obtained should be close to each other. Hence one would consider as signal only those peaks in the spectral response of the filter which have nearly zero phase. Such a “phase weeding” seems to provide better results but further work is needed to fully develop this idea.

3. IMPLEMENTATION

For clarity, we provide a description of how the computations are done in practice. Instead of products of lagged time series, estimates of covariances are used to create the matrix $C$. First of all, we subtract from $x$ its mean to make sure that it is zero and also divide the resulting $x$ by its standard deviation to ensure that the numerical values are in a reasonable range. The same is done for $y$. Next one has to determine cross- and autocovariances which, in turn, are computed by fast convolution algorithms (e.g., Percival & Walden 1993; Proakis & Manolakis 1996; Varadi et al. 1999). For that, the quantities

$$
\hat{q}_j = \sum_k x_k y_{k+j}, \quad j = -(K-1), \ldots, K-1
$$

are computed the following way. If $\tilde{y}$ denotes the reverse of $y$, i.e.,

$$
\tilde{y}_i = y_{N-i+1},
$$

and the number $N_2$ is at least $2N - 1$, then

$$
\hat{q} = \frac{1}{N_2} \text{DFT}_{N_2}^{-1} (\text{DFT}_{N_2}(x) \text{DFT}_{N_2}(\tilde{y})).
$$

Here $\text{DFT}_{N_2}(x)$ denotes the complex Fourier transform on $N_2$ points without normalization, i.e., without dividing the transform with $N_2$, in either the direct or inverse transform. In formula,

$$
\begin{align*}
\text{DFT}_{N_2}(z)_k &= \sum_{j=0}^{N_2-1} z_j e^{-2\pi jk/N_2}, \quad \text{DFT}_{N_2}^{-1}(z)_j = \sum_{k=0}^{N_2-1} z_k e^{2\pi jk/N_2}.
\end{align*}
$$

Both $x$ and $\tilde{y}$ are first index-shifted to start with zero index and then are padded by zeros to length $N_2$. The latter ensures that the otherwise cyclic (or circular) convolution obtained by (3-3) is in fact acyclic (e.g., Proakis & Manolakis 1996). After computing the right hand side of (3-3), the result has to be index-shifted by $N$ in the negative direction to obtain
the values of $\hat{q}$ with the correct indexing since the $N$th element on the right hand side is actually $\hat{q}_0$. Next we compute

$$q_j = \frac{1}{N-j} \hat{q}_j, \quad j = -(K-1), \ldots, K-1,$$

(3-5)

which are unbiased estimates of the covariances — which can proved along the same lines as for autocorrelations (Proakis & Manolakis 1996).

Next the $M$-long sequences of random lags are selected, $k^x$ and $k^y$, using a random number generator. Having the latter, one can form the matrix $C$, more exactly its multiple by some factor. One has

$$C_{ij} = q_{k^y_j - k^x_i}.$$

(3-6)

The SVD of $C$ is computed the following way but there are other methods (Golub & Van Loan 1996). First one computes the symmetric matrix

$$C^T C = E_y \Lambda^2 E_y^T$$

(3-7)

whose eigenvectors are the columns of $E_y$. Any standard high-performance algorithm for the symmetric eigenvalue problem (Golub & Van Loan 1996) can be used to compute $E_y$. For the matrix $E_x$ one has

$$CE_y = E_x \Lambda,$$

(3-8)

which means that $E_x$ can be computed by normalizing the columns of the matrix on the left hand side. The normalizing factors are the singular values which are all nonnegative if these formulae are used. One has to be careful when very small singular values are present but this did not occur in the cases we dealt with so far.

The autocovariances in (2-5) are computed as above with $x$ replacing $y$. The diagonal matrix of scaling coefficients, $B_x$, is determined by the standard formula

$$B_k = \frac{\Lambda_k}{\sigma_k^2}$$

(3-9)

where the index $k$ refers to the $k$ element in the diagonals and $\sigma_k^2$ is the $k$th element in the diagonal of (2-3). Finally, the filters are computed using (2-8). In this, one has to make choices which components should be included, i.e., for which values of $s$ should the sum be restricted. Obviously, one should compute the correlation coefficients

$$r_k = \frac{\Lambda_k}{\sigma_k^2 \sigma_k^2}$$

(3-10)

where the superscript of $\sigma$ is used to distinguish between the standard deviations of the $k$th columns of $D_x E_x$ and $D_y E_y$. Then the filter coefficients in (2-8) are collected for
those $s$ for which $r_s$ are the largest. The computational steps starting with the selection of random lags are repeated for a number of different selections of them and the resulting filter coefficients are averaged, as in the case of Random Lag Singular Spectrum Analysis (Varadi et al. 1999).

4. AN EXAMPLE

It is worthwhile to present a “proof of concept” example. A second-order autoregressive model, i.e., a discrete, damped oscillator, was used to generate a signal by forcing the model with white noise. The signal has 50000 points from which two subsequences were selected, each 5000 long, separated from each other by 40000. In the top panel of Fig. 1, the Fourier spectrum of one of the subsequences is shown. Then white noise was added to both subsequences. As can be seen in the middle panel of Fig. 1, the cross-spectrum amplitudes of the two noisy subsequences reveal nothing about original signal. The bottom panel shows the spectral response of the RLSCSA filters. Here 500 lags were used, the maximum lag being 2500, and the filters in (2-8) were averaged for 100 different sets of random lags. The components with largest two correlations were included in the filter for each selection of lags. The signal is recovered as the third largest peak in the spectral response which we find quite encouraging. This example also illustrates that one should expect to see a number of noise peaks beside signal peaks. Still, one has to deal with a few peaks in the RLSCSA results, while the cross-spectrum exhibits hundreds of them.

We thank M. Dettinger, I. Fodor, M. Ghil, J. Leibacher, J. Pap, C. Tatsuoka and P. Yiou for enlightening discussions on time series analysis and statistics. SOHO is a project of international cooperation between ESA and NASA. This research is supported by a NASA subcontract to UCLA through Stanford University. The partial financial support of NSF Grant AST96-19574 and NASA Grant NAG5-6680 (F.V.) is also acknowledged.

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Fig. 1.— Top panel: Fourier spectrum of one subsequence of the signal. Middle panel: Amplitude of the cross-spectrum of two subsequences of the noisy signal. The original signal is completely hidden in noise. Bottom panel: RLSCSA filter response. The third largest peak is the original signal.