Magnetic stiffness calculation for the corresponding force between two current-carrying circular filaments arbitrarily oriented in the space

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Abstract

In this article, sets of analytical formulas for calculation of nine components of magnetic stiffness of corresponding force arising between two current-carrying circular filaments arbitrarily oriented in the space are derived by using Babic’s method and the method of mutual inductance (Kalantarov-Zeitlin’s method). Formulas are presented through integral expressions, whose kernel function is expressed in terms of the elliptic integrals of the first and second kinds. Also, we obtained an additional set of expressions for calculation of components of magnetic stiffness by means of differentiation of Grover’s formula of the mutual inductance between two circular filaments with respect to appropriate coordinates. The derived sets of formulas were mutually validated and results of calculation of components of magnetic stiffness agree well to each other.

Keywords: Magnetic stiffness, Circular filaments, Mutual inductance, Magnetic force, Magnetic torque

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1. Introduction

Analytical and semi-analytical methods in the calculation of self- and mutual-inductances of conducting elements of electrical circuits and magnetic force interactions between these elements have become powerful mathematical instruments in development of power transfer, wireless communication, sensing and actuation and have been applied in a broad fields of science, including electrical and electronic engineering, medicine, physics, nuclear magnetic resonance, mechatronics and robotics, to designate the most prominent. Although, a number of efficient numerical methodes implemented in the commercially developed software are available, analytical and semi-analytical methods allow to obtaining the result of calculations in the form of a final formula with a finite number of input parameters, which when applicable may significantly reduce computation effort. Providing the direct access to a calculational formula for a user in such methods facilitate mathematical analysis of obtained results of calculation and opens an opportunity for their further optimization.

Analytical methods applied to the calculation of mutual inductance between two circular filaments and arising magnetic force, magnetic torque and corresponding magnetic stiffness when such the filament system carries electric currents is a prime example. These methods have proved their efficiency and have been successfully employed in an increasing number of applications, including electromagnetic levitation \cite{1, 2}, superconducting levitation \cite{3}, calculation of mutual inductance between thick coils \cite{4}, magnetic force and torque calculation between circular coils \cite{5,6,7}, wireless power transfer \cite{8,9,10}, electromagnetic actuation \cite{11,12,13}, micro-machined contactless inductive suspensions \cite{14,15,16} and hybrid contactless suspensions \cite{17,18,19,20}, biomedical applications \cite{21,22}, topology optimization \cite{23}, nuclear magnetic resonance \cite{24,25}, indoor positioning systems \cite{26}, navigation sensors \cite{27}, non-contact gap measurement sensors \cite{28}, wireless power transfer systems \cite{29,30}, magneto-inductive wireless communications \cite{31} and others.

In the present article, the set of formulas for calculation of nine components
of magnetic stiffness of corresponding force arising between two current-carrying circular filaments arbitrarily oriented in the space are derived by using two methods, namely, Babic’s method and the method of mutual inductance. In the first one, the components of the magnetic field at an arbitrary point of the secondary circular filament generated by the primary coil carrying electric current are calculated and then after taking the first derivatives of these components with respect to the appropriate coordinates, the set of analytical formulas for calculation of magnetic stiffness appeared in the integral form whose kernel function is expressed in terms of the elliptic integrals of the first and second kinds is derived. In the second method, the calculation of components of magnetic stiffness of the corresponding magnetic force is performed by means of finding the second derivatives of the function of mutual inductance between two circular filaments recieved by using Kalantarov-Zeitlin’s method \cite{32} with respect to the appropriated coordinates.

The article is organized in the following way. In section 2 of the paper, Babic’s method and its basic expressions are introduced and the set of analytical formulas for calculation of nine components of magnetic stiffness is derived based on this method. In section 3, the method of mutual inductance is presented. The section includes the preliminary discussion, where the set of coordinate frames necessary for determining the position of the secondary coil with respect to the primary one by using Grover’s angles is given. Also, the relationship between constants of the inclined plane equation employing to define the angular misaliment of the secondary circular filament with respect to the primary one in Babic’s approach and Grover’s angles is shown. In section 4, sets of analytical formulas for calculation of nine components of magnetic stiffness recieved by means of the introduced two approaches are mutually verified via a number of designed examples. In section 5, conclusions about obtained results are discussed. In the appendix, in addition to Babic’s method and the method of mutual inductance, Grover’s method is introduced and a set of analytical formulas for calculation of magnetic stiffness based on Grover’s method is obtained.
2. Babic’s method (BM)

Let us take into consideration two current-carrying circular filaments as showed in Fig. 1 where the center of the larger loop (primary coil) of the radius \( R_p \) is placed at the plane \( XOY \) whose center is \( O (0, 0, 0) \). The smaller circular loop (secondary coil) of the radius \( R_s \) is placed in an inclined plane whose general equation is,

\[
\lambda \equiv ax + by + cz + d = 0, \quad (1)
\]

where \( a, b, c \) and \( d \) are the components of the normal \( \vec{N} \) on the inclined plane in the center of the secondary circular segment \( C (x_c, y_c, z_c) \).

2.1. Basic expressions

The segments are with the currents \( I_p \) and \( I_s \), respectively. For circular filaments (see Fig. 1) we define, [33, 34, 35, 36]:

1) Since, the primary circular filament is placed in the plane \( XOY \) \((Z = 0)\) with the center at \( O (0, 0, 0) \). Hence, an arbitrary point \( P (x_p, y_p, z_p) \) of this filament has the following parametric coordinates (see Fig. 1):

\[
x_p = R_p \cos \phi, \quad y_p = R_p \sin \phi, \quad z_p = 0, \quad \phi \in [0, 2\pi]. \quad (2)
\]
2) The differential of the primary circular filament is given by
\[ d\vec{l}_p = R_p \{- \sin \phi, \cos \phi, 0\} d\phi, \ \phi \in [0, 2\pi]. \] (3)

3) The secondary circular filament of radius \( R_s \) is placed in the inclined plane \[ \text{(1)} \] with the center at \( C (x_c, y_c, z_c) \). The unit vector \( \vec{N} \) (the unit vector of the \( z \)-axis) at the point \( C \), which is the center of the secondary circular filament, laying in the plane \( \lambda \) is defined by
\[ \vec{N} = \left\{ \frac{a}{L}, \frac{b}{L}, \frac{c}{L} \right\}, \ \ L = \sqrt{a^2 + b^2 + c^2}. \] (4)

4) The unit vector between two points \( C \) and \( S \) they are placed in the plane \( \text{(1)} \) is
\[ \vec{u} = \{u_x, u_y, u_z\} = \left\{ -\frac{ab}{lL}, \frac{cb}{lL}, \frac{c}{L} \right\}, \ \ l = \sqrt{a^2 + c^2}. \] (5)

5) We define the unite vector \( \vec{v} \) lying in the plane \( \text{(1)} \) and mutually perpendicular on the unit vectors \( \vec{N} \) and \( \vec{u} \) as the cross-product as follows
\[ \vec{v} = \vec{N} \times \vec{u} = \{v_x, v_y, v_z\} = \left\{ -\frac{c}{l}, 0, \frac{a}{l} \right\}. \] (6)

6) An arbitrary point \( S (x_s, y_s, z_s) \) of the secondary circular filament has parametric coordinates
\[ x_s = x_c + R_s u_x \cos \vartheta + R_s v_x \sin \vartheta; \]
\[ y_s = y_c + R_s u_y \cos \vartheta + R_s v_y \sin \vartheta; \]
\[ z_s = z_c + R_s u_z \cos \vartheta + R_s v_z \sin \vartheta, \ \ \vartheta \in [0, 2\pi]. \] (7)

This is well-known parametric equation of circle in 3D space. The filamentary circular filaments are the part of this circle.

7) The differential element of the secondary circular filament is given by,
\[ d\vec{l}_s = R_s \{l_{xs}, l_{ys}, l_{zs}\} d\vartheta, \ \ \vartheta \in [0, 2\pi], \] (8)

where
\[ l_{xs} = -u_x \sin \vartheta + v_x \cos \vartheta; \]
\[ l_{ys} = -u_y \sin \vartheta + v_y \cos \vartheta; \]
\[ l_{zs} = -u_z \sin \vartheta + v_z \cos \vartheta. \] (9)
2.2. Stiffness calculation

To calculate the stiffness between two inclined circular loops as the prime interest of this article we use the analytical formulas for calculating the magnetic field produced by the primary current carrying with the current \( I_p \) at the arbitrary point \( S (x_s, y_s, z_s) \) of the secondary inclined current carrying loop with the current \( I_s \). Hence, the components of the field can be calculated as follows,

\[
B_x = \frac{\mu_0 I_p z_s k}{16\pi p^2 \sqrt{R_p p (1 - k^2)}} I_x; \tag{10}
\]

\[
B_y = \frac{\mu_0 I_p z_s k}{16\pi p^2 \sqrt{R_p p (1 - k^2)}} I_y; \tag{11}
\]

\[
B_z = -\frac{\mu_0 I_p k}{16\pi p \sqrt{R_p p (1 - k^2)}} I_z, \tag{12}
\]

where \( p = \sqrt{x_s^2 + y_s^2} \), \( I_x = x_s A \), \( I_y = y_s A \) and \( I_z = D \) with

\[
k^2 = \frac{4R_p p}{(R_p + p)^2 + z_s^2};
\]

\[
A = -2 \left[(k^2 - 2)E(k) + (2 - 2k^2)K(k)\right]; \tag{13}
\]

\[
D = -2 \left[(k^2(R_p + p) - 2p) E(k) + p \left(2 - 2k^2\right) K(k)\right].
\]

In given expressions \( K(k) \) and \( E(k) \) are the complete elliptic integrals of the first and the second kind, respectively.

Let us find the first derivatives of the components of the field with respect to coordinates \( x_s, y_s \) and \( z_s \), we can write

\[
\frac{\partial B_x}{\partial y} = \frac{\mu_0 I_p}{16\pi \sqrt{R_p p (1 - k^2)}} \frac{p^{-\frac{3}{2}} z_s}{p} \left\{ \left[ -\frac{5}{2p} k \frac{\partial p}{\partial y} + \frac{1 + k^2}{1 - k^2} \frac{\partial k}{\partial y} \right] I_x + k \frac{\partial I_x}{\partial y} \right\};
\]

\[
\frac{\partial B_y}{\partial y} = \frac{\mu_0 I_p}{16\pi \sqrt{R_p p (1 - k^2)}} \frac{p^{-\frac{3}{2}} z_s}{p} \left\{ \left[ -\frac{5}{2p} k \frac{\partial p}{\partial y} + \frac{1 + k^2}{1 - k^2} \frac{\partial k}{\partial y} \right] I_y + k \frac{\partial I_y}{\partial y} \right\}; \tag{14}
\]

\[
\frac{\partial B_z}{\partial y} = \frac{\mu_0 I_p}{16\pi \sqrt{R_p p (1 - k^2)}} \frac{p^{-\frac{3}{2}} z_s}{p} \left\{ \left[ -\frac{3}{2p} k \frac{\partial p}{\partial y} + \frac{1 + k^2}{1 - k^2} \frac{\partial k}{\partial y} \right] I_z + k \frac{\partial I_z}{\partial y} \right\};
\]

\[
g = x_s, y_s, z_s.
\]

where

\[
\frac{\partial k}{\partial x_s} = \frac{x_s k^3}{8R_p p^3} \left[ R_p^2 + z_s^2 - p^2 \right], \quad \frac{\partial k}{\partial y_s} = \frac{y_s k^3}{8R_p p^3} \left[ R_p^2 + z_s^2 - p^2 \right], \quad \frac{\partial k}{\partial z_s} = -\frac{z_s k^3}{4R_p p};
\]

\[
\frac{\partial p}{\partial x_s} = \frac{x_s}{p}, \quad \frac{\partial p}{\partial y_s} = \frac{y_s}{p}, \quad \frac{\partial p}{\partial z_s} = 0;
\]

\[
(15)
\]
\[
\frac{\partial I_x}{\partial x} = A + Cx_s \frac{\partial k}{\partial x}, \quad \frac{\partial I_y}{\partial y_s} = Cx_s \frac{\partial k}{\partial y_s}, \quad \frac{\partial I_z}{\partial z_s} = Cx_s \frac{\partial k}{\partial z_s};
\]
\[
\frac{\partial l}{\partial x_s} = D \frac{x_s}{p} = T, \quad \frac{\partial l}{\partial y_s} = D \frac{y_s}{p} + T, \quad \frac{\partial l}{\partial z_s} = T;
\]
\[
C = -6x_s k [E(k) - K(k)];
\]
\[
T = -2k \frac{\partial k}{\partial x_s} [3(R_p + p) E(k) + (R_p + 3p) K(k)].
\]

Accounting for the fact that
\[
\frac{\partial l_{xs}}{\partial g} = \frac{\partial l_{ys}}{\partial g} = \frac{\partial l_{zs}}{\partial g} = 0, \quad g = x_s, y_s, z_s,
\]

magnetic stiffness calculation for the corresponding force between two current-carrying circular filaments arbitrarily oriented in the space is given by the following formulas \[36\]:

\[
S_{xx} = -\frac{\partial F_s}{\partial x_s} = -I_s R_s \int_0^{2\pi} \left[ l_y z \frac{\partial B_z}{\partial x_s} - l_z z \frac{\partial B_y}{\partial x_s} \right] d\vartheta;
\]
\[
S_{xy} = -\frac{\partial F_s}{\partial y_s} = -I_s R_s \int_0^{2\pi} \left[ l_y z \frac{\partial B_z}{\partial y_s} - l_z z \frac{\partial B_y}{\partial y_s} \right] d\vartheta;
\]
\[
S_{xz} = -\frac{\partial F_s}{\partial z_s} = -I_s R_s \int_0^{2\pi} \left[ l_y z \frac{\partial B_z}{\partial z_s} - l_z z \frac{\partial B_y}{\partial z_s} \right] d\vartheta;
\]
\[
S_{yx} = \frac{\partial F_y}{\partial x_s} = I_s R_s \int_0^{2\pi} \left[ l_y z \frac{\partial B_z}{\partial x_s} - l_z z \frac{\partial B_y}{\partial x_s} \right] d\vartheta;
\]
\[
S_{yy} = \frac{\partial F_y}{\partial y_s} = I_s R_s \int_0^{2\pi} \left[ l_y z \frac{\partial B_z}{\partial y_s} - l_z z \frac{\partial B_y}{\partial y_s} \right] d\vartheta;
\]
\[
S_{yz} = \frac{\partial F_y}{\partial z_s} = I_s R_s \int_0^{2\pi} \left[ l_y z \frac{\partial B_z}{\partial z_s} - l_z z \frac{\partial B_y}{\partial z_s} \right] d\vartheta;
\]
\[
S_{zx} = -\frac{\partial F_z}{\partial x_s} = -I_s R_s \int_0^{2\pi} \left[ l_y z \frac{\partial B_z}{\partial x_s} - l_z z \frac{\partial B_y}{\partial x_s} \right] d\vartheta;
\]
\[
S_{zy} = -\frac{\partial F_z}{\partial y_s} = -I_s R_s \int_0^{2\pi} \left[ l_y z \frac{\partial B_z}{\partial y_s} - l_z z \frac{\partial B_y}{\partial y_s} \right] d\vartheta;
\]
\[
S_{zz} = -\frac{\partial F_z}{\partial z_s} = -I_s R_s \int_0^{2\pi} \left[ l_y z \frac{\partial B_z}{\partial z_s} - l_z z \frac{\partial B_y}{\partial z_s} \right] d\vartheta.
\]

Thus, all magnetic stiffness components \[19\] - \[27\] between two inclined current-carrying loops are given in the simple integral form, over the complete elliptic
integrals of the first and the second kind. These expressions can be used for resolving the singular cases. It is necessary to use expressions (19)-(27) with the following conditions $a = c = 0, l = 0, L = |b|$ for the unit vectors $\vec{u} = \{-1,0,0\}$ and $\vec{v} = \{0,0,1\}$. The loops are perpendicular mutually.

It is clear that $S_{xy} = S_{yx}, S_{xz} = S_{zx}, S_{yz} = S_{zy}$, so that the calculation can be simplified by finding only sixth stiffness: $S_{xx}, S_{yy}, S_{zz}, S_{xy}, S_{xz}$, and $S_{yz}$. Doing further investigation one can find that

$$S_{xx} + S_{yy} + S_{zz} = 0,$$

so that the problem of the stiffness calculation can be limited to find only five components, for instance, $S_{xx}, S_{yy}, S_{xy}, S_{xz}$, and $S_{yz}$.

3. Mutual Inductance Method (MIM)

In this section, the mutual inductance method as an alternative to Babic’s method discussed above is presented. The essence of the method is that the calculation of the stiffness of the corresponding magnetic force is performed by means of finding the second derivatives of the function of mutual inductance between two circular filaments with respect to the appropriated coordinates.

3.1. Preliminary discussion

The general scheme of arbitrarily positioning of two current-carrying circular filaments with respect to each other is considered as shown in Fig. 1. The linear misalignment of the secondary circle with respect to the primary one is defined by the coordinates of the centre $C \ (x_c, y_c, z_c)$. The angular misalignment of the secondary circle can be defined by using Grover’s angles [38, page 207]. Namely, the angle of $\theta$ and $\eta$ corresponds to the angular rotation around an axis passing through the diameter of the secondary circle, and then the rotation of this axis lying on the surface $x'y'z'$ around the vertical $z'$ axis, respectively, as it is shown in Figure 2(a). Accounting for Eq. (4), these two angles have the following relationship with constants of inclined plane (1):

$$\theta = \arccos \left( \frac{c}{L} \right), \quad \eta = \arccos \left( \frac{-b}{L \sin \theta} \right).$$

(29)
Figure 2: Two manners for determining the angular position of the secondary circle with respect to the primary one: \( x'y'z' \) is the auxiliary CF the axes of which are parallel to the axes of \( XYZ \), respectively; \( x''y''z'' \) is the auxiliary CF defined in such a way that the \( x' \) and \( x'' \) are coincide, but the \( z'' \) and \( y'' \) axis is rotated by the \( \alpha \) angle with respect to the \( z' \) and \( y' \) axis, respectively.

The same angular misalignment can be determined through the \( \alpha \) and \( \beta \) angle, which corresponds to the angular rotation around the \( x' \) axis and then around the \( y'' \) axis, respectively, as it is shown in Figure 2(b). This additional second manner is more convenient in a case of study dynamics and stability issues, for instance, applying to axially symmetric inductive levitation systems [14, 16] in compared with Grover’s manner. These two pairs of angles have the following relationship with respect to each other such as [32]:

\[
\begin{align*}
\sin \beta &= \sin \eta \sin \theta; \\
\cos \beta \sin \alpha &= \cos \eta \sin \theta.
\end{align*}
\] (30)

Then, the mutual inductance between two circular filaments can be calculated by the following formulas, which were derived by using Kalantarov-Zeitlin’s approach in work [32] for two cases. Introducing the following dimensionless coordinates:

\[
x = \frac{x_c}{R_s}; \quad y = \frac{y_c}{R_s}; \quad z = \frac{z_c}{R_s}; \quad s = \sqrt{x^2 + y^2},
\] (31)
for the first case when the $\theta$ angle is lying in an interval of $0 \leq \theta < \pi/2$, the formula can be written as

$$M = \frac{\mu_0}{\pi} \sqrt{R_p R_s} \int_0^{2\pi} r \cdot U \cdot \Phi(k) d\varphi,$$

(32)

where

$$r = r(\theta, \eta) = \frac{\cos \theta}{\sqrt{\sin^2(\varphi - \eta) + \cos^2 \theta \cos^2(\varphi - \eta)}},$$

(33)

$$U = U(x, y, \theta, \eta) = \frac{R}{\rho^{\frac{1}{2}}} = \frac{r + t_1 \cdot \cos \varphi + t_2 \cdot \sin \varphi}{\rho^{\frac{1}{2}}},$$

(34)

$$t_1 = t_1(x, y, \theta, \eta) = x + 0.5r^2 \tan^2 \theta \sin(2(\varphi - \eta)) \cdot y,$$

$$t_2 = t_2(x, y, \theta, \eta) = y - 0.5r^2 \tan^2 \theta \sin(2(\varphi - \eta)) \cdot x,$$

(35)

$$\rho = \rho(x, y, \theta, \eta) = r + 0.5 \cdot (x \cos(\varphi) + y \sin(\varphi)) + s^2,$$

$$\Phi(k) = \frac{1}{k} \left[ \left( \frac{k^2}{2} \right) K(k) - E(k) \right],$$

(36)

and

$$k^2 = k^2(x, y, z, \theta, \eta) = \frac{4\nu \rho}{(\nu \rho + 1)^2 + \nu^2 z^2},$$

$$\nu = \frac{R_s}{R_p}, \quad z\lambda = z + r \tan(\varphi - \eta).$$

For the second case when the $\theta$ angle is equal to $\pi/2$ and the two filament circles are mutually perpendicular to each other, the formula becomes

$$M = \frac{\mu_0}{\pi} \sqrt{R_p R_s} \left\{ \int_{-1}^{1} U \cdot \Phi(k) d\tilde{\ell} + \int_{1}^{1} U \cdot \Phi(k) d\tilde{\ell} \right\},$$

(38)

where

$$U = U(x, y, \eta) = \frac{R}{\rho^{\frac{1}{2}}} = \frac{t_1 - t_2}{\rho^{\frac{1}{2}}},$$

(39)

$$t_1 = t_1(x, \eta) = \sin \eta \cdot (x + \tilde{\ell} \cos \eta),$$

$$t_2 = t_2(y, \eta) = \cos \eta \cdot (y + \tilde{\ell} \sin \eta),$$

(40)

$$\rho = \rho(x, y, \eta) = \sqrt{s^2 + 2\tilde{\ell} \cdot (x \cos(\eta) + y \sin(\eta)) + \tilde{\ell}^2},$$

and $\tilde{\ell} = \ell/R_s$ is the dimensionless variable. The functions $\Phi(k)$ and $k = k(x, y, z, \eta)$ in formula (38) have the same structures as defined by Eq. (36).
and (37), respectively. Besides that, in the elliptic module $k = k(x, y, z, \eta)$, the $z_\lambda$ function is governed as follows

$$z_\lambda = z \pm \sqrt{1 - \bar{\ell}^2}, \quad (41)$$

Note that integrating formula (38) between $-1$ and $1$, Eq. (41) is calculated with the positive sign and for the other direction the negative sign is taken.

3.2. Stiffness calculation

Assuming that the primary and secondary circular filaments carry the currents of $I_p$ and $I_s$, respectively, hence, the magnetic stiffness corresponding to the force arising between these two current-carrying circular filaments can be calculated by taking the second derivatives of the function of the magnetic energy stored in such the system with respect to the appropriate coordinates. Hence, all nine components of the magnetic stiffness can be calculated by

$$S_{gq} = -I_p I_s \frac{\partial^2 M}{\partial g \partial q}, \quad (42)$$

where $g, q = x_c, y_c$, or $z_c$. Thus, to derive formulas for calculation of the magnetic stiffness between two arbitrarily oriented circular filaments, the second derivatives of formulas of mutual inductance, namely, represented by Eq. (32) and (38) must be taken. Similar to the calculation of magnetic force in such the filament system [39], finding the second derivatives of mutual inductance is reduced to taking the second derivatives of their kernel functions.

3.2.1. For the case of $0 \leq \theta < \pi/2$

Formula (32) for calculation of mutual inductance is considered. Its kernel is defined as

$$K_r = r \cdot U \cdot \Phi(k). \quad (43)$$

According to the definitions of functions $r, U, \Phi(k)$ and $k$ given in Eq. (33), (34), (36) and (37), respectively, the second $x_c$-and $y_c$-derivative of kernel $K_r$
can be written as

\[
\frac{\partial^2 K}{\partial q^2} = \frac{\partial^2 K \cdot 1}{R_g^2} = \frac{r}{R_g^2} \left[ \frac{\partial^2 U \cdot \Phi(k) + 2 \frac{\partial U}{\partial g} \cdot \frac{d\Phi(k)}{dk} - \frac{\partial k}{\partial g}}{\partial g} \right] + U \left( \frac{d^2 \Phi(k)}{dk^2} \cdot \left( \frac{\partial k}{\partial g} \right)^2 + \frac{d\Phi(k)}{dk} \cdot \frac{\partial^2 k}{\partial g^2} \right),
\]

where \( q = x, y, \) and \( g = x, y, \)

\[
\frac{\partial U}{\partial g} = \left( \frac{\partial R \cdot \rho - 1.5 \cdot \frac{\partial R}{\partial g}}{\partial g} \right) / \rho^{2.5},
\]

\[
\frac{\partial^2 U}{\partial g^2} = \left[ -0.5 \frac{\partial R \partial \rho}{\partial g} - 1.5 \frac{\partial^2 \rho}{\partial g^2} \right] \cdot \rho - 2.5 \cdot \left[ \frac{\partial R}{\partial g} \rho - 1.5 \frac{\partial R}{\partial g} \right] \cdot \frac{\partial \rho}{\partial g},
\]

\[
\frac{\partial G}{\partial g} = \frac{\partial k}{\partial g} \cdot \cos \varphi + \frac{\partial t_2}{\partial g} \cdot \sin \varphi,
\]

\[
\frac{\partial g}{\partial g} = \frac{G}{H^2} \cdot \frac{\partial H}{\partial g} + \frac{G}{H} \cdot \frac{\partial^2 \rho}{\partial g^2},
\]

\[
G = 2/k - k(\nu + 1), \quad H = (\nu + 1)^2 + \nu z \lambda,
\]

\[
\frac{\partial G}{\partial g} = - \left[ 2/k^2 + \nu + 1 \right] \frac{\partial k}{\partial g} - k \cdot \frac{\partial k}{\partial g}, \quad \frac{\partial^2 G}{\partial g^2} = 2(\nu + 1) \frac{\partial \rho}{\partial g},
\]

\[
\frac{d\Phi(k)}{dk} = \frac{1}{k^2} \left[ \frac{2 - k^2}{2(1 - k^2)} E(k) - K(k) \right],
\]

\[
\frac{d^2 \Phi(k)}{dk^2} = - \frac{2 - k^2}{2(1 - k^2)} \frac{E(k) + (4 + 9k^2 - 5k^4) K(k)}{2k^3(k^2 - 1)^2},
\]

and when \( g = x \) we have

\[
\frac{\partial t_1}{\partial x} = 1, \quad \frac{\partial t_2}{\partial x} = -0.5 r^2 \tan^2 \theta \sin(2(\varphi - \eta)),
\]

\[
\frac{\partial \rho}{\partial x} = (r \cdot \cos \varphi + x) / \rho, \quad \frac{\partial^2 \rho}{\partial x^2} = \left( \rho - (r \cdot \cos \varphi + x) \frac{\partial \rho}{\partial x} \right) / \rho^2,
\]

and when \( g = y \) we have

\[
\frac{\partial t_1}{\partial y} = 0.5 r^2 \tan^2 \theta \sin(2(\varphi - \eta)), \quad \frac{\partial t_2}{\partial y} = 1,
\]

\[
\frac{\partial \rho}{\partial y} = (r \cdot \sin \varphi + y) / \rho, \quad \frac{\partial^2 \rho}{\partial y^2} = \left( \rho - (r \cdot \sin \varphi + y) \frac{\partial \rho}{\partial y} \right) / \rho^2.
\]

The second \( z \cdot \) derivative of kernel \( K_r \) is

\[
\frac{\partial^2 K}{\partial z^2} = \frac{\partial^2 K \cdot 1}{R_s^2} = \frac{r}{R_s^2} \left[ \frac{d^2 \Phi(k)}{dk^2} \cdot \left( \frac{\partial k}{\partial z} \right)^2 + \frac{d\Phi(k)}{dk} \cdot \frac{\partial^2 k}{\partial z^2} \right],
\]
where
\[ \frac{\partial k}{\partial z} = -\sqrt{4
u\rho} \cdot \frac{\nu^2 z_\lambda}{((\nu^2 + 1)^2 + \nu^2 z_\lambda^2)^{3/2}}, \]
\[ \frac{\partial^2 k}{\partial z^2} = \nu^2 \sqrt{4
u\rho} \cdot \frac{2\nu^2 z_\lambda^2 - (\nu^2 + 1)^2}{((\nu^2 + 1)^2 + \nu^2 z_\lambda^2)^{5/2}}. \] (51)

Note that in Eq. (50), the first and second derivative of function \( \Phi \) with respect to \( k \) are the same as in Eqs (47), respectively.

For further differentiation of the kernel, it is recognized that
\[ \frac{\partial^2 K_r}{\partial q \partial z_c} = \frac{\partial^2 K_r}{\partial z_c \partial q}, \quad \frac{\partial^2 K_r}{\partial x_c \partial y_c} = \frac{\partial^2 K_r}{\partial y_c \partial x_c}, \] (52)
where \( q = x_c, y_c \). Using properties (52), the derivation of the second derivatives, where the variable \( z_c \) is involved, can be simplified by taking the first derivative of the kernel with respect to \( z_c \). Following this, we have
\[ \frac{\partial^2 K_r}{\partial q \partial z_c} = \frac{\partial^2 K_r}{\partial z_c \partial q} \frac{1}{R_s^2} = \frac{r}{R_s^2} \left[ \frac{\partial U}{\partial q} \frac{d\Phi(k)}{dk} \frac{\partial k}{\partial z_c} + \frac{\partial^2 U}{\partial x \partial y} \frac{d\Phi(k)}{dk} \frac{\partial^2 k}{\partial z_c} + U \left( \frac{d^2 \Phi(k)}{dk^2} \frac{\partial k}{\partial x} \frac{\partial k}{\partial y} + \frac{d\Phi(k)}{dk} \frac{\partial^2 k}{\partial x \partial y} \right) \right], \] (53)
where \( q = x_c, y_c \) and \( g = x, y \).

The second derivative with respect to variables \( x_c \) and \( y_c \) is
\[ \frac{\partial^2 K_r}{\partial x_c \partial y_c} = \frac{\partial^2 K_r}{\partial x \partial y} \frac{1}{R_s^2} = \frac{r}{R_s^2} \left[ \frac{\partial^2 U}{\partial x \partial y} \frac{d\Phi(k)}{dk} \frac{\partial k}{\partial x} + U \left( \frac{d^2 \Phi(k)}{dk^2} \frac{\partial k}{\partial x} \frac{\partial k}{\partial y} + \frac{d\Phi(k)}{dk} \frac{\partial^2 k}{\partial x \partial y} \right) \right] \] (54)

where
\[ \frac{\partial^2 U}{\partial y \partial x} = \left( \frac{\partial G_x}{\partial y} \rho - 2.5 G_x \frac{\partial \rho}{\partial y} \right) \rho^{3.5}, \]
\[ G_x = \frac{\partial R}{\partial x} \rho - 1.5 \cdot \frac{\partial \rho}{\partial x}, \]
\[ \frac{\partial^2 G_x}{\partial y^2} = \frac{\partial^2 R}{\partial y^2} \rho - 1.5 \cdot \frac{\partial^2 \rho}{\partial y^2} - 1.5 \cdot \frac{\partial^2 \rho}{\partial y \partial x}, \]
\[ \frac{\partial^2 \rho}{\partial y \partial x} = -\frac{\partial \rho}{\partial x} \cdot \frac{\partial \rho}{\partial y} \cdot \frac{1}{\rho}. \] (56)
\[
\frac{\partial^2 k}{\partial y \partial x} = \frac{\partial G}{\partial y} H - G \frac{\partial H}{\partial y} \cdot \nu \frac{\partial \rho}{\partial x} + G \frac{\partial^2 \rho}{\partial y \partial x}, \quad (57)
\]

\[
G = \frac{2}{k} - k (\nu \rho + 1), \quad H = (\nu + 1)^2 + \nu^2 z^2, \quad (58)
\]

\[
\frac{\partial G}{\partial y} = - \left[ \frac{2}{k} - \frac{\nu \rho}{2} \right] \frac{\partial k}{\partial y} - k \cdot \nu \frac{\partial \rho}{\partial y}, \quad \frac{\partial H}{\partial y} = 2(\nu + 1) \nu \frac{\partial \rho}{\partial y}. \quad (59)
\]

Hence, for this particular case when \(0 \leq \theta < \pi/2\), according to Eqs (44), (50), (53) and (55) all nine components of magnetic stiffness can be calculated as follows:

\[
S_{qq} = - \frac{\mu_0 I_p I_s \sqrt{R_p}}{\pi R_s^{3/2}} \int_0^{2\pi} r \cdot \left[ \frac{\partial^2 U}{\partial g^2} \cdot \Phi(k) + 2 \frac{\partial U}{\partial g} \cdot \frac{\partial \Phi(k)}{dk} \cdot \frac{\partial k}{\partial g} \right] d\varphi; \quad (58)
\]

\[
S_{zz} = - \frac{\mu_0 I_p I_s \sqrt{R_p}}{\pi R_s^{3/2}} \int_0^{2\pi} r \cdot \left[ \frac{d^2 \Phi(k)}{dk^2} \cdot \left( \frac{\partial k}{\partial z} \right)^2 + \frac{d \Phi(k)}{dk} \cdot \frac{\partial^2 k}{\partial g \partial z} \right] d\varphi; \quad (59)
\]

\[
S_{zz} = S_{qq} = - \frac{\mu_0 I_p I_s \sqrt{R_p}}{\pi R_s^{3/2}} \int_0^{2\pi} r \cdot \left[ \frac{\partial U}{\partial g} \cdot \frac{\partial \Phi(k)}{dk} \cdot \frac{\partial k}{\partial g} + \frac{d \Phi(k)}{dk} \cdot \frac{\partial^2 k}{\partial g \partial z} \right] d\varphi; \quad (60)
\]

\[
S_{xy} = S_{yx} = - \frac{\mu_0 I_p I_s \sqrt{R_p}}{\pi R_s^{3/2}} \int_0^{2\pi} r \cdot \left[ \frac{\partial^2 U}{\partial x \partial y} \cdot \Phi(k) + \frac{\partial U}{\partial x} \cdot \frac{\partial \Phi(k)}{dk} \cdot \frac{\partial k}{\partial x} + \frac{\partial \Phi(k)}{dk} \cdot \frac{\partial^2 k}{\partial x \partial y} \right] d\varphi, \quad (61)
\]

where \(q = x_c \) or \( y_c \), \( g = x \) or \( y \), respectively.

### 3.2.2. The second case: \( \theta = \pi/2 \)

For this case, formula (38) for calculation of mutual inductance is used. Its kernel is defined as

\[
Kr = U \cdot \Phi(k). \quad (62)
\]
Accounting for that in this case the function $U$ is defined by Eq. (39), then the second $x_c$- and $y_c$-derivative of kernel $K_r$ can be written as

$$\frac{\partial^2 K_r}{\partial q^2} = \frac{\partial^2 K_r}{\partial g^2} \frac{1}{R_z^2} = \frac{1}{R_z^2} \left[ \frac{\partial^2 U}{\partial g^2} \cdot \Phi (k) + 2 \frac{\partial U}{\partial g} \cdot \frac{d \Phi (k)}{dk} \cdot \frac{\partial k}{\partial g} + U \left( \frac{d^2 \Phi (k)}{dk^2} \cdot \frac{\partial k}{\partial g} \right)^2 \right] \cdot \left( \frac{\partial k}{\partial g} \right)^2,$$

where $g = x, y$. The first and second derivatives of function $U$ with respect to $g$ are taken analogically as shown in Eqs (45). Similar to Eqs (46), the derivatives of function $k$ are defined. When $g = x$ we have

$$\frac{\partial R}{\partial x} = \frac{\partial t_1}{\partial x} = \sin \eta, \quad \frac{\partial \rho}{\partial x} = \frac{x + \bar{\ell} \cos \eta}{\rho},$$

and when $g = y$ we can write

$$\frac{\partial R}{\partial y} = \frac{\partial t_2}{\partial y} = - \cos \eta, \quad \frac{\partial \rho}{\partial y} = \frac{y + \bar{\ell} \sin \eta}{\rho},$$

The second $z_c$-derivative of the kernel $K_r$ is defined similar to Eq. (50) as

$$\frac{\partial^2 K_r}{\partial z_c^2} = \frac{\partial^2 K_r}{\partial z^2} \frac{1}{R_z^2} = \frac{1}{R_z^2} \left[ \frac{\partial^2 U}{\partial g^2} \cdot \frac{\partial k}{\partial g} \cdot \frac{\partial k}{\partial z} + \frac{d \Phi (k)}{dk} \cdot \frac{\partial^2 k}{\partial z^2} \right],$$

where the derivatives of function $k$ have the same meaning as in Eqs. (51). Using properties (52) the derivation of the second derivatives with respect to the variables $z_c$ and $x_c$, and $z_c$ and $y_c$ can be written similar to Eq. (53) as follows

$$\frac{\partial^2 K_r}{\partial q \partial z_c} = \frac{\partial^2 K_r}{\partial g \partial z} \frac{1}{R_z^2} = \frac{1}{R_z^2} \left[ \frac{\partial U}{\partial g} \cdot \frac{d \Phi (k)}{dk} \cdot \frac{\partial k}{\partial z} + \frac{\partial^2 k}{\partial z^2} \cdot \frac{\partial k}{\partial g} \cdot \frac{\partial \rho}{\partial g} \cdot \frac{\partial \rho}{\partial z} \right],$$

where $q = x_c, y_c$ and $g = x, y$, taking into account Eqs. (64) and (65) the first and the second derivatives of $U, k$ are determined by Eqs. (45) and (54), respectively.
where nine components of magnetic stiffness can be calculated as follows:

\[
\frac{\partial^2 K_r}{\partial x \partial y} = \frac{1}{R_s^2} \left[ \frac{\partial^2 U}{\partial x \partial y} \Phi(k) + \frac{\partial U}{\partial x} \cdot \frac{d \Phi(k)}{d k} \cdot \frac{\partial k}{\partial y} + \frac{\partial^2 \Phi(k)}{d k^2} \left( \frac{\partial k}{\partial x} \cdot \frac{\partial k}{\partial y} \right) \right].
\]

(68)

In Eq. (68) all derivatives of functions \(U\) and \(k\) are determined in Eq. (55), but the definitions of function \(R\) and \(\rho\) and their respective derivatives given in Eqs. (64) and (65) must be taken into account.

Hence, for this case when \(\theta = \pi/2\), according to Eqs (63), (67) and (68) all nine components of magnetic stiffness can be calculated as follows:

\[
S_{q,q} = -\frac{\mu_0 I_p I_s \sqrt{R_p}}{\pi R_s^{3/2}} \int_{-1}^{1} \frac{\partial^2 U}{\partial y^2} \cdot \Phi(k) + 2 \frac{\partial U}{\partial y} \cdot \frac{d \Phi(k)}{d k} \cdot \frac{\partial k}{\partial y} + \frac{d^2 \Phi(k)}{d k^2} \left( \frac{\partial k}{\partial y} \right)^2 \, d\ell + \int_{1}^{-1} \frac{\partial^2 U}{\partial y^2} \cdot \Phi(k) \, d\ell.
\]

(69)

\[
S_{x,y} = S_{y,x} = -\frac{\mu_0 I_p I_s \sqrt{R_p}}{\pi R_s^{3/2}} \int_{-1}^{1} \frac{\partial U}{\partial y} \cdot \frac{d \Phi(k)}{d k} \cdot \frac{\partial k}{\partial y} + \frac{d^2 \Phi(k)}{d k^2} \left( \frac{\partial k}{\partial y} \right)^2 \, d\ell + \int_{1}^{-1} \frac{\partial U}{\partial y} \cdot \frac{d \Phi(k)}{d k} \cdot \frac{\partial k}{\partial y} \, d\ell.
\]

(70)

where \(q = x_c\) or \(y_c\), \(g = x\) or \(y\), respectively.
Thus, the set of formulas (58)-(61) and (69)-(72) for calculation of all nine components of magnetic stiffness of the corresponding force arising between two current-carrying circular filaments arbitrarily oriented in the space are derived by using the mutual inductance method. The derived formulas are intuitively understandable for application, they can be easily programmed. For this purpose, the Matlab language was used. The Matlab files with the implemented formulas are available as supplementary materials to this article.

4. Numerical verification of derived formulas

Developed sets of formulas for calculation of nine components of magnetic stiffness of corresponding force between two current-carrying circular filaments derived by means of Babic’s method Eqs (19)-(27) and the method of mutual inductance (Kalantarov-Zeitlin’s method) Eqs (58)-(61) and Eqs (69)-(72) are mutually verified to each other through applying developed formulas to a number of examples designed in this section. In all examples below, it is assumed that the carrying currents in both coils are equal to one ampere \( I_p = I_s = 1 \, \text{A} \).

In addition to the calculation of components of magnetic stiffness of the considered filament system is supported by the set of expressions (A.15)-(A.17) derived from Grover’s formula for calculation of mutual inductance \[38\] page 207, Eq. (179)]. The derivations of these expressions are shown in Appendix A. All calculations for considered cases proved the robustness and efficiency of developed formulas.

4.1. Magnetic stiffness between circular filaments with parallel axes

The scheme for calculation of magnetic stiffness between circular filaments with parallel axes is shown in Fig. 3. The linear misalignment in the Grover notation can be defined by the geometrical parameter, \( d \), which is the distance between the planes of circles and the parameter, \( \rho \), is the distance between their axes. These parameters have the following relationship to the notation defined in this article, namely, \( z_c = d \) and \( \rho = \sqrt{x_c^2 + y_c^2} \). Fig. 3 shows the particular case, when \( \rho = y_c \).
Figure 3: Geometrical scheme of circular filaments with parallel axes denoted via Grover’s notation: $\rho$ is the distance between axes; $d$ is the distance between the coils’ planes $d = z_c$.

Example 1: (Example 16, page 74 in Babič’s work [36])

Two coaxial circular filaments for which the primary coil has a radius of $R_p = 2 \text{ m}$, and the secondary one $R_s = 1 \text{ m}$. The axial distance between filaments is $d = z_c = 1\text{ m}$. The results of calculation of diagonal and non-diagonal components of magnetic stiffness are as follows

|        | BM, Nm$^{-1}$ | MIM, Nm$^{-1}$ | GM, Nm$^{-1}$ |
|--------|---------------|----------------|---------------|
| $S_{xx}$ | $1.032010586220236 \times 10^{-7}$ | $1.032010586220216 \times 10^{-7}$ | $1.0320105862202111 \times 10^{-7}$ |
| $S_{yy}$ | $1.032010586220238 \times 10^{-7}$ | $1.032010586220216 \times 10^{-7}$ | $1.0320105862202111 \times 10^{-7}$ |
| $S_{zz}$ | $-2.064021172440475 \times 10^{-7}$ | $-2.064021172440485 \times 10^{-7}$ | $-2.064021172440499 \times 10^{-7}$ |

Analysis of the results of calculation shows that they are agree well to each other. The difference arises after thirteenth digit in a resulting number. In the
calculation of non-diagonal components, the order of magnitude corresponding to minus twenty three can be considered as approximately equal zero. In Babic’s method alternatively to Eqs (20), (24), (21) the other set of formulas, namely, Eqs (22), (26), (25) can be used for calculation of non-diagonal components. The results are the same and equal to zeros. Also, note that the sum of diagonal components in all methods is almost zero and the condition (28) is held with an accuracy of minus twenty one order of magnitude. Worth noting that in some cases this condition helps us also to restore the missed component of stiffness in the orthogonal direction in Grover’s method.

**Example 2**

Let us consider the coils having the same radii as in the previous example 1, but the center of the secondary coil is located at the point $x_c = 0$ m, $y_c = 0$ m and $z_c = 1$ m, which corresponds to the following Grover parameters: $\rho = 0.5$ m and $d = 1$ m. The results of calculation of diagonal and non-diagonal components of magnetic stiffness are as follows

| Component | BM, N m$^{-1}$ | MIM, N m$^{-1}$ | GM, N m$^{-1}$ |
|-----------|----------------|-----------------|----------------|
| $S_{xx}$  | $1.402100143236235 \times 10^{-7}$ | $1.402100143236235 \times 10^{-7}$ | $1.402100143236236 \times 10^{-7}$ |
| $S_{yy}$  | $2.118309158188127 \times 10^{-7}$ | $2.118309158188122 \times 10^{-7}$ | $2.118309158188126 \times 10^{-7}$ |
| $S_{zz}$  | $-3.520409301424362 \times 10^{-7}$ | $-3.520409301424361 \times 10^{-7}$ | $-3.520409301424361 \times 10^{-7}$ |

The component $S_{xx}$ in GM was restored from the condition (28).

**Example 3**

The coils having the same radii as in the previous examples and the center of the secondary coil is located at the point $x_c = 0$ m, $y_c = 0$ m and $z_c = 0$ m. Grover
parameters are zeros. The results of calculation of diagonal and non-diagonal components of magnetic stiffness are as follows

\[
\begin{array}{ccc}
S_{xx} & -6.367128613342259 \times 10^{-7} & -6.367128613342232 \times 10^{-7} & -6.3671286133422277 \times 10^{-7} \\
S_{yy} & -6.367128613342259 \times 10^{-7} & -6.367128613342232 \times 10^{-7} & -6.3671286133422218 \times 10^{-7} \\
S_{zz} & 1.273425722668452 \times 10^{-6} & 1.273425722668452 \times 10^{-6} & 1.273425722668449 \times 10^{-6} \\
\end{array}
\]

\[
\begin{array}{ccc}
S_{xy} & -1.752675194540019 \times 10^{-141} & -2.271727526158087 \times 10^{-22} & \text{NA} \\
S_{yx} & 0 & 0 & 0 \\
S_{xz} & 0 & 0 & 0 \\
\end{array}
\]

4.2. Magnetic stiffness between mutually perpendicular circular filaments

In this section, magnetic stiffness between mutually perpendicular current-carrying filaments is calculated. The general scheme is shown in Fig. 4. For the mutual inductance method, it corresponds to the second case, when \( \theta = \pi/2 \) and the angular misalignment is only characterized by the angle \( \eta \), and the set of formulas (69)–(72) is used.
Example 4

The two mutually perpendicular coils have the following radii, namely, \( R_p = 0.2\, \text{m} \), and the secondary one \( R_s = 0.1\, \text{m} \). The center of the secondary coil is located at the origin as shown in Fig. 5(a). The angle \( \eta \) is zero rad for MIM. For BM, the angular misalignment is characterized by the following components of plane equation (1): \( a = c = 0 \) and \( b = 1 \). The results of calculation of diagonal and non-diagonal components of magnetic stiffness are as follows

| \( S_{xx} \) | \( S_{yy} \) | \( S_{zz} \) |
|----------------|----------------|----------------|
| 0 | 0 | Not a Number (NaN) |

| \( S_{xy} \) | \( S_{yz} \) | \( S_{xz} \) |
|----------------|----------------|----------------|
| 0 | 0 | NA |

| \( S_{xy} \) | \( S_{yz} \) | \( S_{xz} \) |
|----------------|----------------|----------------|
| 0 | 2.706560599934499 \times 10^{-6} | 2.706560599933974 \times 10^{-6} |

Note that for Grover’s method it is the singular case.
Example 5

The same arrangement of coils as in example 4 is used, but the secondary coil is turned on the angle $\eta = \pi/2$ rad as shown in Fig. 5(b). For BM, the angular misalignment is characterized by the following components of plane equation (1): $b = c = 0$. The results of calculation are as follows

|          | BM, N m$^{-1}$          | MIM, N m$^{-1}$          | GM, N m$^{-1}$          |
|----------|-------------------------|-------------------------|-------------------------|
|          | Eqs (19), (23), (27)    | Eqs (69), (70)          | Eqs (A.15), (A.16)      |
| $S_{xx}$ | 0                       | 0                       | Not a Number (NaN)      |
| $S_{yy}$ | 0                       | 0                       | NaN                     |
| $S_{zz}$ | 0                       | 0                       | NaN                     |

|          | BM, N m$^{-1}$          | MIM, N m$^{-1}$          | GM, N m$^{-1}$          |
|----------|-------------------------|-------------------------|-------------------------|
|          | Eqs (20), (24), (21)    | Eqs (71), (72)          | Eqs (A.17)              |
| $S_{xy}$ | 0                       | 0                       | NA                      |
| $S_{yz}$ | 0                       | 1.657290387703741 $\times 10^{-22}$ | NaN                     |
| $S_{xz}$ | $-2.706560599933975 \times 10^{-6}$ | $-2.706560599933974 \times 10^{-6}$ | NA                      |

Note that for Grover’s method it is the singular case.

Example 6

The two mutually perpendicular coils have the following radii, namely, $R_p = 1.0$ m, and the secondary one $R_s = 0.5$ m. The center of the secondary coil is located at the point $x_c = 0$m, $y_c = 2$m and $z_c = 0$m. The angle $\eta$ is zero rad for MIM. For BM, the angular misalignment is characterized by the following components of plane equation (1): $a = c = 0$ and $b = 1$. The results of calculation of diagonal and non-diagonal components of magnetic stiffness are
as follows

|        | BM, N m$^{-1}$ | MIM, N m$^{-1}$ | GM, N m$^{-1}$ |
|--------|----------------|-----------------|----------------|
|        | Eqs (19), (23), (27) | Eqs (69), (70)  | Eqs (A.15), (A.16) |
| $S_{xx}$ | 0              | 0               | $-3.68315244876568 \times 10^{-24}$ |
| $S_{yy}$ | $1.1985923988025 \times 10^{-143}$ | 0          | $8.191673408378001 \times 10^{-24}$ |
| $S_{zz}$ | $4.794368959521 \times 10^{-142}$ | 0           | $-4.508520959612321 \times 10^{-24}$ |

|        | BM, N m$^{-1}$ | MIM, N m$^{-1}$ | GM, N m$^{-1}$ |
|--------|----------------|-----------------|----------------|
|        | Eqs (20), (24), (21) | Eqs (71), (72)  | Eqs (A.17)     |
| $S_{xy}$ | 0              | 0               | NA             |
| $S_{yz}$ | $-1.368742764885786 \times 10^{-7}$ | $-1.368742764885786 \times 10^{-7}$ | $-1.368742764885786 \times 10^{-7}$ |
| $S_{xz}$ | 0              | $5.151865880056137 \times 10^{-24}$ | NA             |

**Example 7**

The two mutually perpendicular coils have the following radii, namely, $R_p = 1.0$ m, and the secondary one $R_s = 0.5$ m. The center of the secondary coil is located at the point $x_c = 0$ m, $y_c = 2$ m and $z_c = 3$ m. The angle $\eta$ is zero rad for MIM. For BM, the angular misalignment is characterized by the following components of plane equation (1): $a = c = 0$ and $b = 1$. The results of calculation of diagonal and non-diagonal components of magnetic stiffness are as follows

|        | BM, N m$^{-1}$ | MIM, N m$^{-1}$ | GM, N m$^{-1}$ |
|--------|----------------|-----------------|----------------|
|        | Eqs (19), (23), (27) | Eqs (69), (70)  | Eqs (A.15), (A.16) |
| $S_{xx}$ | $-2.2626829050505172 \times 10^{-9}$ | $-2.2626829050505171 \times 10^{-9}$ | $-2.2626829050505174 \times 10^{-9}$ |
| $S_{yy}$ | $-2.710919377082796 \times 10^{-9}$ | $-2.710919377082797 \times 10^{-9}$ | $-2.710919377082796 \times 10^{-9}$ |
| $S_{zz}$ | $4.973602282087952 \times 10^{-9}$ | $4.973602282087968 \times 10^{-9}$ | $4.97360228208797 \times 10^{-9}$ |

|        | BM, N m$^{-1}$ | MIM, N m$^{-1}$ | GM, N m$^{-1}$ |
|--------|----------------|-----------------|----------------|
|        | Eqs (20), (24), (21) | Eqs (71), (72)  | Eqs (A.17)     |
| $S_{xy}$ | $-2.364186955151571 \times 10^{-143}$ | $-2.347855544296417 \times 10^{-25}$ | NA             |
| $S_{yz}$ | $3.101402573517489 \times 10^{-9}$ | $3.101402573517489 \times 10^{-9}$ | $3.101402573517489 \times 10^{-9}$ |
| $S_{xz}$ | $7.984780962755199 \times 10^{-143}$ | $8.433113791758558 \times 10^{-26}$ | NA             |
Example 8

The two mutually perpendicular coils have the following radii, namely, \( R_p = 1.0 \text{ m} \), and the secondary one \( R_s = 0.5 \text{ m} \). The center of the secondary coil is located at the point \( x_c = 1\text{m} \), \( y_c = 2\text{m} \) and \( z_c = 3\text{m} \). The angle \( \eta \) is \( \pi/2 \text{ rad} \) for MIM as shown in Fig. 6. For BM, the angular misalignment is characterized by the following components of plane equation \( 1 \): \( b = c = 0 \) and \( a = 1 \). The
results of calculation are as follows

|               | BM, N m\(^{-1}\) | MIM, N m\(^{-1}\) | GM, N m\(^{-1}\) |
|---------------|------------------|------------------|------------------|
|               | Eqs (19), (23), (27) | Eqs (69), (70)   | Eqs (A.15), (A.16) |
| \(S_{xx}\)   | \(2.44411106760408 \times 10^{-9}\) | \(2.44411106760407 \times 10^{-9}\) | NA               |
| \(S_{yy}\)   | \(-6.546286516635751 \times 10^{-10}\) | \(-6.546286516635743 \times 10^{-10}\) | NA               |
| \(S_{zz}\)   | \(-1.789782455096833 \times 10^{-9}\) | \(-1.789782455096833 \times 10^{-9}\) | NA               |

|               | BM, N m\(^{-1}\) | MIM, N m\(^{-1}\) | GM, N m\(^{-1}\) |
|---------------|------------------|------------------|------------------|
|               | Eqs (20), (24), (21) | Eqs (71), (72)   | Eqs (A.17)       |
| \(S_{xy}\)   | \(1.042889962848133 \times 10^{-9}\) | \(1.042889962848133 \times 10^{-9}\) | NA               |
| \(S_{yz}\)   | \(-2.190346410345056 \times 10^{-9}\) | \(-2.190346410345057 \times 10^{-9}\) | NA               |
| \(S_{xz}\)   | \(1.067688019471112 \times 10^{-9}\) | \(1.067688019471113 \times 10^{-9}\) | NA               |

### 4.3. Magnetic stiffness between circular filaments arbitrarily positioned in the space

In this section, using the equation of inclined plane (1) for BM and its relationship with the Grover’s angles (29) for MIM to define different angular misalignments of the secondary coil with respect to the primary one, a number of examples with different arrangements of coils for calculation of magnetic stiffness are designed and considered below. The calculation is accompanied by the evaluation of stiffness by means of Grover’s formulas Eqs (A.15), (A.16) and (A.17) when they are applicable.
**Example 9**

The two coils have the following radii, namely, the primary one \( R_p = 0.4 \text{ m} \), and the secondary one \( R_s = 0.05 \text{ m} \). The center of the secondary coil is located at the point \( x_c = 0.1 \text{ m}, y_c = 0.15 \text{ m} \) and \( z_c = 0 \text{ m} \). For BM, the angular misalignment is characterized by the following plane equation, namely, \( 3x + 2y + z = 0.6 \). According to Eqs. (29), it corresponds to the angle \( \eta = 2.15879893034246 \text{ rad} \) (123.69006752598°) and \( \theta = 1.30024656381632 \text{ rad} \) (74.498640433063°) in notations of MIM. The coils' arrangement is shown in Fig. 7. The results of calculation are as follows

|          | BM, \( \text{N m}^{-1} \) | MIM, \( \text{N m}^{-1} \) | GM, \( \text{N m}^{-1} \) |
|----------|--------------------------|--------------------------|--------------------------|
|          | Eqs (19), (23), (27)     | Eqs (58), (59)           | Eqs (A.15), (A.16)       |
| \( S_{xx} \) | \(-5.327433787498592 \times 10^{-8}\) | \(-5.32743378749859 \times 10^{-8}\) | NA                       |
| \( S_{yy} \) | \(-7.00453721025121 \times 10^{-8}\) | \(-7.004537210251179 \times 10^{-8}\) | NA                       |
| \( S_{zz} \) | \(1.23319709977498 \times 10^{-7}\) | \(1.23319709977498 \times 10^{-7}\) | NA                       |

|          | BM, \( \text{N m}^{-1} \) | MIM, \( \text{N m}^{-1} \) | GM, \( \text{N m}^{-1} \) |
|----------|--------------------------|--------------------------|--------------------------|
|          | Eqs (20), (24), (21)     | Eqs (60), (61)           | Eq (A.17)                |
| \( S_{xy} \) | \(-1.500101622143787 \times 10^{-8}\) | \(-1.500101622143775 \times 10^{-8}\) | NA                       |
| \( S_{yx} \) | \(-2.121823074384375 \times 10^{-7}\) | \(-2.121823074384376 \times 10^{-7}\) | NA                       |
| \( S_{zz} \) | \(-2.167771909796546 \times 10^{-7}\) | \(-2.167771909796545 \times 10^{-7}\) | NA                       |
**Example 10**

The primary coil has a radius of $R_p = 4$ m, and the secondary one has a radius of $R_s = 2$ m. The center of the secondary coil is located at the point $x_c = 1$ m, $y_c = 1$ m and $z_c = -1$ m. For BM, the angular misalignment is characterized by the following plane equation, namely, $x + 2y + 3z = 0$. According to Eqs. (29), the angles $\eta$ and $\theta$ are $2.67794504458899 \text{ rad} \ (153.434948822922^\circ)$ and $0.64052312679424 \text{ rad} \ (36.6992252004899^\circ)$, respectively, in notations of MIM. The coils’ arrangement is shown in Fig. 8.

The results of calculation are as follows:

|                | BM, N m$^{-1}$ | MIM, N m$^{-1}$ | GM, N m$^{-1}$ |
|----------------|---------------|----------------|---------------|
|                | Eqs (19), (23), (27) | Eqs (58), (59) | Eqs (A.15), (A.16) |
| $S_{xx}$      | $2.181662870952764 \times 10^{-8}$ | $2.181662870952587 \times 10^{-8}$ | NA            |
| $S_{yy}$      | $3.44815074198134 \times 10^{-8}$  | $3.448150741981526 \times 10^{-8}$ | NA            |
| $S_{zz}$      | $-5.629813612934105 \times 10^{-8}$ | $-5.629813612934103 \times 10^{-8}$ | NA            |

|                | BM, N m$^{-1}$ | MIM, N m$^{-1}$ | GM, N m$^{-1}$ |
|----------------|---------------|----------------|---------------|
|                | Eqs (20), (24), (21) | Eqs (60), (61) | Eqs (A.17) |
| $S_{xy}$      | $5.713524254486024 \times 10^{-8}$ | $5.713524254486125 \times 10^{-8}$ | NA            |
| $S_{yz}$      | $-1.070006627660674 \times 10^{-7}$ | $-1.07000662766067 \times 10^{-7}$ | NA            |
| $S_{xz}$      | $-1.105341440732599 \times 10^{-7}$ | $-1.1053414407326 \times 10^{-7}$ | NA            |
Example 11

The radii of the coils are the same as in the previous example 10, but the center of the secondary coil is located at the point $x_c = -1\text{m}$, $y_c = 1\text{m}$ and $z_c = 1\text{m}$. The angular misalignment for BM is characterized by the following plane equation, namely, $3x + 2y + 1z = 0$. According to Eqs. (29), the angles $\eta$ and $\theta$ are $2.15879893034246 \text{ rad} (123.69006752598^\circ)$ and $1.30024656381632 \text{ rad} (74.498640433063^\circ)$, respectively, in notations of MIM. The coils’ arrangement is shown in Fig. 9. The results of calculation are as follows

|          | BM, N m$^{-1}$ | MIM, N m$^{-1}$ | GM, N m$^{-1}$ |
|----------|----------------|-----------------|----------------|
|          | Eqs (20), (24), (21) | Eqs (60), (61) | Eqs (A.17) |
| $S_{xx}$ | $2.94440845967626 \times 10^{-8}$ | $2.94440845966999 \times 10^{-8}$ | NA |
| $S_{yy}$ | $-5.401491525386883 \times 10^{-8}$ | $-5.401491525387853 \times 10^{-8}$ | NA |
| $S_{zz}$ | $2.457050679419257 \times 10^{-8}$ | $2.457050679419274 \times 10^{-8}$ | NA |

|          | BM, N m$^{-1}$ | MIM, N m$^{-1}$ | GM, N m$^{-1}$ |
|----------|----------------|-----------------|----------------|
|          | Eqs (20), (24), (21) | Eqs (60), (61) | Eqs (A.17) |
| $S_{xy}$ | $-7.826596371018449 \times 10^{-9}$ | $-7.826596371018387 \times 10^{-9}$ | NA |
| $S_{yz}$ | $-2.426184704880834 \times 10^{-8}$ | $-2.426184704880832 \times 10^{-8}$ | NA |
| $S_{xz}$ | $-1.147221843342089 \times 10^{-7}$ | $-1.14722184334209 \times 10^{-7}$ | NA |
Example 12

The radii of each coil are the same and equal to 1 m. The center of the secondary coil is located at the point \( x_c = -1 \text{ m}, \ y_c = 1 \text{ m} \) and \( z_c = 0 \text{ m} \). The angular misalignment for BM is characterized by the following plane equation, namely, \( x + y + z = 0 \). According to Eqs. (29), the angles \( \eta \) and \( \theta \) are 2.35619449019234 rad \((135^\circ)\) and 0.955316618124509 rad \((54.7356103172453^\circ)\), respectively, in notations of MIM. The arrangement of the linked coils is shown in Fig. 10. The results of calculation are as follows:

|                  | BM, N m\(^{-1}\) | MIM, N m\(^{-1}\) | GM, N m\(^{-1}\) |
|------------------|-------------------|-------------------|------------------|
|                  | Eqs (19), (23), (27) | Eqs (58), (59) | Eqs (A.15), (A.16) |
| \( S_{xx} \)     | 1.046477792966786 \times 10^{-7} | 1.046477792966789 \times 10^{-7} | NaN |
| \( S_{yy} \)     | 1.046477792966786 \times 10^{-7} | 1.046477792966785 \times 10^{-7} | NaN |
| \( S_{zz} \)     | -2.092955585933573 \times 10^{-8} | -2.092955585933573 \times 10^{-7} | NaN |

|                  | BM, N m\(^{-1}\) | MIM, N m\(^{-1}\) | GM, N m\(^{-1}\) |
|------------------|-------------------|-------------------|------------------|
|                  | Eqs (20), (24), (21) | Eqs (60), (61) | Eqs (A.17) |
| \( S_{xy} \)     | 2.0146297091004 \times 10^{-7} | 2.0146297091005 \times 10^{-7} | NA |
| \( S_{yz} \)     | -2.577031542995681 \times 10^{-7} | -2.577031542995682 \times 10^{-7} | NaN |
| \( S_{xz} \)     | -2.577031542995681 \times 10^{-7} | -2.577031542995681 \times 10^{-7} | NA |

Example 13

The radii of coils and their angular orientation with respect to each other are the same as in Example 12. The center of the secondary coil is located at the
point $x_c = 0\text{m}$, $y_c = 1\text{m}$ and $z_c = -1\text{m}$. The results of calculation are as follows

|                  | BM, N m$^{-1}$ | MIM, N m$^{-1}$ | GM, N m$^{-1}$ |
|------------------|----------------|-----------------|----------------|
|                  | Eqs (19), (23), (27) | Eqs (58), (59) | Eqs (A.15), (A.16) |
| $S_{xx}$         | $2.9860909969216481 \times 10^{-7}$ | $2.9860909969216481 \times 10^{-7}$ | $2.9860909969216485 \times 10^{-7}$ |
| $S_{yy}$         | $-6.600031210636671 \times 10^{-8}$ | $-6.6000312106366671 \times 10^{-8}$ | $-6.600031210636627 \times 10^{-8}$ |
| $S_{zz}$         | $-2.326087848152819 \times 10^{-8}$ | $-2.326087848152818 \times 10^{-7}$ | $-2.326087848152822 \times 10^{-7}$ |

|                  | BM, N m$^{-1}$ | MIM, N m$^{-1}$ | GM, N m$^{-1}$ |
|------------------|----------------|-----------------|----------------|
|                  | Eqs (20), (24), (21) | Eqs (60), (61) | Eqs (A.17) |
| $S_{xy}$         | $-3.199237389569022 \times 10^{-8}$ | $-3.199237389569028 \times 10^{-8}$ | NA |
| $S_{yz}$         | $8.115657627384119 \times 10^{-7}$ | $8.115657627384113 \times 10^{-7}$ | $8.11565762738412 \times 10^{-8}$ |
| $S_{xz}$         | $2.749069587302089 \times 10^{-7}$ | $2.749069587302088 \times 10^{-7}$ | NA |

**Example 14**

The primary circular filament has a radius of $R_p = 0.2\text{m}$, while the secondary one has $R_s = 0.1\text{m}$. The centre of secondary coil is located at the point $C$ having the following coordinates $x_c = 0.1\text{m}$, $y_c = 0.1\text{m}$ and $z_c = 0.1\text{m}$. The angular misalignment for BM is defined by the following plane equation, namely, $x + y + z = 0.3$. According to Eqs. (29), the angles $\eta$ and $\theta$ are $2.35619449019234$ rad ($135^\circ$) and $0.955316618124509$ rad ($54.7356103172453^\circ$), respectively, in notations of MIM. The arrangement of the coils is shown in Fig. 11. The results
of calculation are as follows

|          | BM, N m⁻¹ | MIM, N m⁻¹ | GM, N m⁻¹ |
|----------|------------|------------|-----------|
|          | Eqs (19), (23), (27) | Eqs (58), (59) | Eqs (A.15), (A.16) |
| $S_{xx}$ | 2.868431152918931 × 10⁻⁵ | 2.868431152918925 × 10⁻⁵ | NA |
| $S_{yy}$ | 2.868431152918931 × 10⁻⁵ | 2.868431152918923 × 10⁻⁵ | NA |
| $S_{zz}$ | −5.736862305837862 × 10⁻⁵ | −5.736862305837814 × 10⁻⁵ | NA |

|          | BM, N m⁻¹ | MIM, N m⁻¹ | GM, N m⁻¹ |
|----------|------------|------------|-----------|
|          | Eqs (20), (24), (21) | Eqs (60), (61) | Eqs (A.17) |
| $S_{xy}$ | 2.397140500000452 × 10⁻⁵ | 2.397140500000446 × 10⁻⁵ | NA |
| $S_{yz}$ | 1.724024033513611 × 10⁻⁶ | 1.724024033513529 × 10⁻⁶ | NA |
| $S_{xz}$ | 1.724024033513611 × 10⁻⁶ | 1.724024033513543 × 10⁻⁶ | NA |

### 5. Conclusion

In this article, sets of analytical formulas for calculation of nine components of magnetic stiffness of corresponding force arising between two current-carrying circular filaments arbitrarily oriented in the space have been derived by using Babic’s method and the method of mutual inductance (Kalantarov-Zeitlin’s method). Formulas are presented through integral expressions, whose kernel function is expressed in terms of the elliptic integrals of the first and second kinds. Also, the additional set of expressions for calculation of components of magnetic stiffness by means of differentiation of Grover’s formula with respect to appropriate coordinates has been obtained. Grover’s method provides the most simplest approach for calculation of magnetic stiffness, however the calculation is constrained by four components only, namely, two diagonal and two non-diagonal components instead of nine ones. Also, the GM suffers from singular cases shown, for instance, in Examples 4, 5 and 12, which limit the applicability of the method. In opposite to the GM, the set of formulas (19)–(27), (58)–(61) and (69)–(72) is deduced by BM and MIM, respectively, is universally applicable for calculation of the magnetic stiffness and covers all possible arrangements between two current-carrying circular filaments. The derived sets of formulas
Figure A.12: Geometrical scheme of current-carrying circular filaments arbitrarily oriented in the space: the Grover notations.

were mutually validated and results of calculation of components of magnetic stiffness agree well to each other.

The set of formulas (58)-(61) and (69)-(72) obtained by means of MIM and the set of expressions (A.15)-(A.17) obtained by means of GM were programmed by using the Matlab language. The Matlab files with the implemented formulas are available as supplementary materials to this article.

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Appendix A. Stiffness calculation. Grover’s method (GM) [38, page 207, Eq. (179)]

According to Grover’s notations, the linear misalignment of the centre of the secondary circle is characterised by two parameters, namely, \( d = z_c \) and \( \rho = \sqrt{x_c^2 + y_c^2} \) as shown in Figure A.12. Besides that the angular misalignment is defined in accordance with the first manner as shown in Fig. 2(a), but keeping the original Grover’s notation the angle, \( \eta \), is replaced by \( \psi \). In absence of the angular misalignment, the CF \( xyz \) assigned to the secondary circle is oriented in the following way. The \( z \)-axis is directed upward along the \( d \)-line, while the
\(y\)-axis is parallel to the \(\rho\)-line and directed in continuation of the \(\rho\)-line. Then adopting the above considered notations, Grover’s formula for calculation of mutual inductance between two circular filaments can be written as

\[
M = \frac{\mu_0 \sqrt{R_p R_s}}{2\pi} \int_0^{2\pi} U \cdot \Psi(k) d\varphi,
\]

where

\[
U = U(\gamma, \theta, \psi) = \frac{R(\gamma, \theta, \psi)}{V^{1.5}} = \frac{\cos \theta - \gamma (\cos \psi \cos \varphi - \sin \psi \cos \theta \sin \varphi)}{V^{1.5}},
\]

\[
V = V(\gamma, \theta, \psi) = \sqrt{1 - \cos(\varphi)^2 \sin(\theta)^2 + 2\gamma (\sin \psi \sin \varphi - \cos \varphi \cos \theta) + \gamma^2},
\]

\[
\Psi(k) = \frac{2}{k} \left[ \left(1 - \frac{k^2}{2}\right) K(k) - E(k) \right],
\]

\[
k^2 = k^2(\gamma, \Delta, \theta, \psi) = \frac{4\alpha V}{(\alpha V + 1)^2 + z^2},
\]

\[
\alpha = \frac{R_s}{R_p}, \Delta = \frac{d}{R_p}, \gamma = \frac{\rho}{R_s}, z = \Delta - \alpha \sin \theta \cos \varphi.
\]

The kernel of formula (A.1) is

\[
Kr = U \cdot \Psi(k).
\]

Accounting for (A.2), (A.3), (A.4) and (A.5), the second \(\rho\)-derivative of the kernel has a similar structure to Eq. (63) and becomes as follows

\[
\frac{\partial^2 Kr}{\partial \rho^2} = \frac{\partial^2 Kr}{\partial \gamma^2} = \frac{1}{R_s^2} \left[ \frac{\partial^2 U}{\partial \gamma^2} \cdot \Psi(k) + 2 \frac{\partial U}{\partial \gamma} \cdot d\Psi(k) \frac{dk}{d\gamma} \frac{\partial k}{\partial \gamma} + U \left( \frac{d^2 \Psi(k)}{dk^2} \cdot \frac{d\Psi(k)}{d\gamma} + \frac{d\Psi(k)}{dk} \cdot \frac{d^2 k}{d\gamma^2} \right) \right],
\]

where

\[
\frac{\partial U}{\partial \gamma} = \frac{J}{V^{2.5}} = \left( \frac{\partial R}{\partial \gamma} \cdot V - 1.5 \cdot R \cdot \frac{\partial V}{\partial \gamma} \right) / V^{2.5},
\]

\[
\frac{\partial^2 U}{\partial \gamma^2} = \frac{\partial J}{\partial \gamma} \cdot V - 2.5 \cdot J \cdot \frac{\partial V}{\partial \gamma},
\]

\[
\frac{\partial J}{\partial \gamma} = -0.5 \frac{\partial R}{\partial \gamma} \cdot \frac{\partial V}{\partial \gamma} - 1.5 \cdot R \cdot \frac{\partial^2 V}{\partial \gamma^2} + \cos \psi \cos \varphi + \sin \psi \cos \theta \sin \varphi,
\]

\[
\frac{\partial \gamma}{\partial \gamma} = \sin \psi \sin \varphi - \cos \varphi \cos \psi \cos \theta + \gamma,
\]

\[
\frac{\partial^2 V}{\partial \gamma^2} = \frac{V - (\sin \psi \sin \varphi - \cos \varphi \cos \psi \cos \theta + \gamma) \frac{\partial V}{\partial \gamma}}{V^2},
\]

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\[ \frac{\partial k}{\partial \gamma} = \frac{G}{H} \cdot \alpha \frac{\partial V}{\partial \gamma}. \]
\[ \frac{\partial^2 k}{\partial \gamma^2} = \frac{\partial G}{\partial \gamma} H - A \frac{\partial H}{\partial \gamma} \cdot \alpha \frac{\partial V}{\partial \gamma} + \frac{G}{H} \cdot \alpha \frac{\partial^2 V}{\partial \gamma^2}, \quad (A.9) \]

\[ G = 2/k - k(\alpha V + 1), \quad H = (\alpha V + 1)^2 + z^2, \]
\[ \frac{\partial G}{\partial \gamma} = - \left[ 2/k^2 + \alpha V + 1 \right] \frac{\partial k}{\partial \gamma} - k \cdot \alpha \frac{\partial V}{\partial \gamma}, \quad (A.10) \]

Accounting for the property
\[ \frac{\partial^2 K_r}{\partial d \partial \rho} = \frac{\partial^2 K_r}{\partial d \partial \gamma}, \quad (A.12) \]
the second derivative of the kernel with respect to variables \( d \) and \( \rho \) can be written as
\[ \frac{\partial^2 K_r}{\partial d \partial \rho} = \frac{\partial^2 K_r}{\partial d \partial \gamma}. \quad (A.13) \]

where
\[ \frac{\partial^2 k}{\partial \gamma \partial \Delta} = \frac{\partial G}{\partial \gamma} H - A \frac{\partial H}{\partial \gamma} \cdot \alpha \frac{\partial V}{\partial \gamma}, \quad (A.14) \]

The first derivatives of functions \( U \) and \( k \) with respect to variables \( \gamma \) and \( \Delta \) in the above equation are the same as determined in Eqs (A.8), (A.9) and (A.11).
Using the derivatives of the kernel obtained above, we can gain the second derivatives of Grover’s formula of mutual inductance with respect to the appropriate coordinates and write corresponding formulas for the calculation of magnetic stiffness. Taking into account Eqs (A.7), (A.10) and (A.13), we can write

\[ S_{\rho \rho} = -\mu_0 I_p I_s \frac{\sqrt{R_p}}{2\pi} \frac{\sqrt{R_1^3}}{R_1^3} \int_0^{2\pi} \frac{\partial^2 U}{\partial \gamma^2} \cdot \Psi(k) + 2 \frac{\partial U}{\partial \gamma} \cdot \frac{d\Psi(k)}{dk} \cdot \frac{\partial k}{\partial \gamma} \frac{\partial^2 k}{\partial \gamma^2} d\varphi, \]  

(A.15)

\[ S_{dd} = -\mu_0 I_p I_s \frac{\sqrt{R_s}}{2\pi} \frac{\sqrt{R_1^3}}{R_1^3} \int_0^{2\pi} \frac{d^2 \Psi(k)}{dk^2} \left( \frac{\partial k}{\partial \Delta} \right)^2 + \frac{d\Psi(k)}{dk} \cdot \frac{\partial^2 k}{\partial \Delta}^2 d\varphi, \]  

(A.16)

\[ S_{pd} = S_{dp} = -\mu_0 I_p I_s \frac{\sqrt{R_p}}{2\pi} \frac{\sqrt{R_1^3}}{R_1^3} \int_0^{2\pi} \frac{\partial U}{\partial \gamma} \cdot \frac{d\Psi(k)}{dk} \cdot \frac{\partial k}{\partial \Delta} + \frac{d\Psi(k)}{dk} \cdot \frac{\partial^2 k}{\partial \gamma \partial \Delta} d\varphi. \]  

(A.17)

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