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Abstract. We propose a conjecture stating that for resonances, \( \lambda_j \), of a noncompactly supported potential, the series \( \sum_j \frac{\text{Im} \lambda_j}{|\lambda_j|^2} \) diverges. This series appears in the Breit-Wigner approximation for a compactly supported potential, in which case it converges. We provide heuristic motivation for this conjecture and prove it in several cases.

1. Introduction and Conjectures

In this note we propose a conjecture on the asymptotic distribution of scattering resonances of a one-dimensional Schrödinger equation with a noncompactly supported, super-exponentially decreasing potential. The conjecture is motivated by the Breit-Wigner formula for compactly supported potentials. We prove this conjecture for a large class of potentials, including any analytic potential for which a conjecture of Froese [Fro97, Conjecture 1.2] holds.

Scattering resonances are by definition the poles of the meromorphic continuation of the resolvent family \( R_V(\lambda) = (-D^2 + V - \lambda^2)^{-1} \). They also may be viewed as poles of the scattering matrix \( S(\lambda) \). We let \( \text{Res}V \) be the multiset of resonances of \( V \), counted with multiplicity.

The operator \(-iS'(\lambda)S^*(\lambda)\) is known as the Eisenbud-Wigner time-delay operator, which has physical significance [Jen81]. In the case of compactly supported potentials, the Breit-Wigner approximation relates the trace of the Eisenbud-Wigner operator of a compactly supported potential to a sum over resonances.

Theorem 1.1 (Breit-Wigner approximation for compactly supported potentials). Suppose that \( V \) is compactly supported and \( \lambda_0 \in \mathbb{R} \). Then the series

\[
\sum_{\lambda \in \text{Res}V \setminus 0} \frac{|\text{Im} \lambda|}{|\lambda - \lambda_0|^2} < \infty
\]

converges, and if \( V \) is real-valued then we have

\[
\frac{1}{2\pi i} \text{tr} S'(\lambda_0)S(\lambda_0)^* = -\frac{1}{\pi} |\text{ch supp } V| - \frac{1}{2\pi} \sum_{\lambda \in \text{Res}V \setminus 0} \frac{\text{Im} \lambda}{|\lambda - \lambda_0|^2}.
\]
Here $|\text{ch supp} V|$ is the length of the convex hull of $\text{supp} V$. For a proof, see [DZ19, Theorem 2.20] or [Bac20, Theorem 3.24]. For a higher-dimensional generalization, see [GMR89], [PZ99] and [PZ01], or [BP03].

**Definition 1.2.** The *Breit-Wigner series* of an arbitrary potential $V$ is
\[
B(V) = -\sum_{\lambda \in \text{Res} V \setminus 0} \frac{\text{Im} \lambda}{|\lambda|^2}.
\]

By (1), $B(V)$ converges if $V$ is compactly supported.

The left-hand side, $\text{tr} S'(\lambda) S^*(\lambda)$, of the Breit-Wigner formula (2) is a robust object that can be defined for a large class of decaying potentials $V$. Moreover, $\text{tr} S'S^*$ depends continuously on $V$ in any reasonable topology, and it is not really affected by the support of $V$ as such. Meanwhile, the right-hand side of (2) has a term, $|\text{ch supp} V|$, which is infinite when $V$ is not compactly supported, and an infinite series, so one can ask whether the right-hand side demonstrates a sort of “cancellation of infinities.” Thus, it is natural to ask whether the convergence of the Breit-Wigner series (1) still holds when $V$ decays but is not compactly supported.

**Definition 1.3.** The potential $V$ is *super-exponentially decreasing* if for every $N \in \mathbb{N}$,
\[
|V(x)| \lesssim_N e^{-N|x|}.
\]

If $V$ is a super-exponentially decreasing potential, then resonances may viewed as the zeroes of the determinant $\text{det}(1 + \sqrt{V} R_0 \sqrt{|V|})$ [Fro97, §3], and so depend continuously on the behavior of $V$ in compact sets. However, resonances may escape to infinity or otherwise be badly behaved globally. Therefore we cannot conclude that we can take the limit of the Breit-Wigner formula as the support becomes unbounded. Yet, heuristically, one would hope that the Breit-Wigner series of a super-exponentially decreasing potential is a limit of Breit-Wigner series of compactly supported approximations. Moreover, in view of the stability of the left-hand side, we expect that as $|\text{ch supp} V| \to \infty$, $B(V) \to \infty$ as well, to achieve the aforementioned “cancellation of infinities.” Hence, we make the following bold conjecture.

**Conjecture 1.4.** Let $V$ be a super-exponentially decreasing potential. The Breit-Wigner series $B(V)$ converges if and only if $V$ is compactly supported.

The conjecture can be verified in some cases where resonances can be defined, yet the potential is not super-exponentially decreasing. An example is the Pöschl-Teller well,
\[
V(x) = -\frac{2}{\cosh^2(x)}.
\]

Its resonances are given by $-i(n + 2), n \in \mathbb{N}$ [cGKN16], and so $B(V)$ diverges, yet $V$ is not super-exponentially decreasing.
The distribution of $\text{Res} V$ is in general quite difficult to study. However, Froese made a conjecture [Fro97, Conjecture 1.2] about the growth of the counting function of $\text{Res} V$, and proved that a large class of potentials, including Gaussians, satisfy his conjecture [Fro97, Theorem 1.3].

To state Froese’s conjecture, we assume that $V$ is super-exponentially decreasing, so that its Fourier-Laplace transform $\hat{V}$ is entire, and introduce the following new entire function.

**Definition 1.5.** Given a super-exponentially decreasing potential $V$, its Froese function $F$ is given by

$$F(z) = \hat{V}(2z)\hat{V}(-2z) + 1.$$  

We also recall the following classical definitions [Lev64, p. 52, p. 139].

**Definition 1.6.** Let $f$ be an entire function of order $\rho$ and normal type (that is, nonzero finite type). The indicator function $h$ of $f$ is given by

$$h(\theta) = \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r^\rho}.$$  

**Definition 1.7.** Let $f$ be an entire function of order $\rho$ and normal type. If there is a subset $I$ of $\{r : r > 0\}$ of density one such that for every $\theta$, the lim sup appearing in (4) is actually a uniform limit as $r \to \infty$ along $I$, then $f$ is said to have completely regular growth.

Henceforth we let $A(R, \theta, \varphi)$ denote the sector

$$A(R, \theta, \varphi) = \{re^{i\alpha} \in \mathbb{C} : r \leq R \text{ and } \alpha \in [\theta, \varphi]\}.$$  

We let $n(R, \theta, \varphi)$ denote the number of resonances in $A(R, \theta, \varphi)$ and let $N(R, \theta, \varphi)$ denote the number of zeroes of the Froese function $F$ in $A(R, \theta, \varphi)$. We let $n(R) = n(R, 0, 2\pi)$ and similarly for $N(R)$. With this background in place, we may recall Froese’s conjecture.

**Conjecture 1.8 (Froese).** Suppose that $V$ is super-exponentially decreasing and $\hat{V}$ has completely regular growth. Then in the lower-half plane $\mathbb{C}_-$, one has

$$|n(R, \theta, \varphi) - N(R, \theta, \varphi)| = o(R^\rho).$$  

In view of Froese’s conjecture, we formulate a weaker form of Conjecture 1.4 as follows:

**Conjecture 1.9.** Suppose that $V$ meets the hypotheses of Froese’s conjecture and $V$ is not compactly supported. Then either $B(V)$ diverges, or $V$ is a counterexample to Froese’s conjecture.
Froese’s conjecture gives a linear lower bound on the resonance-counting function $n$ (Proposition 1.11), so either all resonances except for a zero-density set are contained in arbitrarily small sectors around $R$, or $B(V)$ diverges (Lemma 3.2). So, if $B(V)$ converges and Froese’s conjecture holds, then a positive-density set of resonances is contained in arbitrarily small sectors around $R$, a result that was already proven for compactly supported potentials by Zworski [Zwo87]. The method of complex scaling rules this possibility out if $V$ is holomorphic in a conic neighborhood of $R$ [Sjö02, Corollary 12.14]. We show that certain unnatural hypotheses on the monotone, nonnegative function

$$s(\theta, \varphi) = h'(\varphi) - h'(\theta) + \rho^2 \int_\theta^\varphi h(\alpha) \, d\alpha,$$

where $h$ is the indicator function of $F$, will also rule out this possibility (Theorem 1.10).

**Theorem 1.10.** Suppose that $V$ meets the hypotheses and conclusion of Froese’s conjecture. If $V$ is noncompactly supported, then the Breit-Wigner series $B(V)$ will diverge provided that any one of the following criteria are true:

1. The set of resonances of $V$ contained in arbitrarily small sectors around $R$ is of zero density.
2. $V$ is holomorphic in a conic neighborhood of $R$.
3. There are $\theta \leq \varphi$ such that $0, \pi \notin [\theta, \varphi]$ and $s(\theta, \varphi) \neq 0$.
4. There is a $k \in \{0, 1\}$ and a $\theta > k\pi$ such that $s(k\pi, \theta)$ exists.

Here $s$ is given by (6), and Case 3 includes the possibility that $s(\theta, \varphi)$ does not exist. We prove Theorem 1.10 in Section 3.

In Section 2, we recall properties of the Froese function $F$ and prove the following proposition, which will be used in Section 3 and may be of independent interest:

**Proposition 1.11.** Suppose that $V$ meets the hypotheses and conclusion of Froese’s conjecture. Let $\rho$ denote the order of $\hat{V}$. If $V$ is not identically zero, then as $r \to \infty$, $n(r) \gtrsim r^\rho \geq r$.

**Notation.** We will write $f \lesssim g$ to mean that there is a constant $C > 0$ such that for every $x$ such that $|x|$ is large enough, $f(x) \leq Cg(x)$. We write $f \sim g$ to mean $g \lesssim f \lesssim g$, and use a subscript $\lesssim_y$ to mean that $C$ is allowed to depend on $y$.

Given a fixed set $I \subseteq \{r : r > 0\}$ of density one, which will always be the set $I$ that appears in Definition 1.7, we write $f \approx g$ to mean that $f(r)/g(r) \to 1$ as $r \to \infty$ along $I$ (and uniformly in all other variables). We write $f \approx 0$ to mean that $f \to 0$, the limit taken along $I$.

We write $f'(x \pm 0)$ to mean the left (−) and right (+) derivatives of a semidifferentiable function $f$. We write $x_+$ to mean $\max(x, 0)$. 
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2. Linear growth of resonances

The following properties of the Froese function $F$ follow from its definition (3) and the assumption that $\hat{V}$ is an entire function of completely regular growth:

1. $F$ has completely regular growth.
2. The order of $F$ is $\rho$.
3. For every $z \in \mathbb{C}$, $F(z) = F(-z)$.

Let $h$ be the indicator function of $F$, and let $s$ be given by (6). We recall a characterization of $s$ [Lev64, Theorem III.3].

Theorem 2.1. If $\hat{V}$ is an entire function of completely regular growth, then there is a countable, possibly empty, exceptional set $Z$ of angles such that:

1. If $\theta, \varphi \in [0, 2\pi] \setminus Z$, then
   \[ s(\theta, \varphi) = 2\pi \rho \lim_{r \to \infty} \frac{N(r, \theta, \varphi)}{r^\rho}. \]
2. $\theta \in Z$ if and only if $h'(\theta - 0) \neq h'(\theta + 0)$.

Note that $h$ is semidifferentiable, hence continuous. Moreover, $s(\theta, \cdot)$ is increasing and nonnegative for any $\theta$, and dually, $s(\cdot, \varphi)$ is decreasing and nonnegative for any $\varphi$. If $\theta, \varphi \notin Z$, then $s(\theta, \varphi)$ must exist.

We adopt the convention that if $\theta \notin Z$ then $s(\theta, \cdot)$ is right-continuous, viz.

\[ s(\theta, \varphi) = h'(\varphi + 0) - h'(\theta) + \rho^2 \int_{\theta}^{\varphi} h(\alpha) \, d\alpha. \]

Thus $s(\theta, \cdot)$ is defined and right-continuous on all of $[0, 2\pi]$.

Lemma 2.2. If $\hat{V}$ is an entire function of completely regular growth, $\beta < \theta$, and $\beta \notin Z$, then the following are equivalent:

1. $s(\beta, \cdot)$ has a jump discontinuity at $\theta$.
2. $s(\beta, \cdot)$ is not continuous at $\theta$.
3. $\theta \in Z$.

Proof. First observe that since $s(\beta, \cdot)$ is monotone, all discontinuities are jump discontinuities.
Suppose that $s(\beta, \cdot)$ is continuous at $\theta$. Thus
\[
\lim_{\delta \to 0} s(\theta - \delta, \theta + \delta) = \lim_{\delta \to 0} s(\beta, \theta + \delta) - s(\beta, \theta - \delta) = 0,
\]
the limit taken along $\delta > 0$ such that $\theta - \delta \notin Z$. Yet
\[
s(\theta - \delta, \theta + \delta) = h'(\theta - \delta) - h'(\theta + \delta) + \rho^2 \int_{\theta - \delta}^{\theta + \delta} h'(\alpha) \, d\alpha,
\]
and taking the limit of both sides as $\delta \to 0$ we see that $h'(\theta - 0) = h'(\theta + 0)$, so $\theta \notin Z$.

Conversely, if $s(\beta, \cdot)$ has a jump discontinuity at $\theta$ then
\[
0 < \lim_{\varepsilon \to 0} s(\beta, \theta) - s(\beta, \theta - \varepsilon) = h'(\theta + 0) - h'(\theta - 0) = \eta.
\]
so $\theta \in Z$. \hfill \Box

**Lemma 2.3.** If $\hat{V}$ is an entire function of completely regular growth and either $s(\theta, \varphi)$ is nonzero or $\theta \in Z$, then
\[
N(r, \theta, \varphi) \sim r^\rho.
\]

**Proof.** This follows immediately from Theorem 2.1 if $\theta \notin Z$ and $s(\theta, \cdot)$ is continuous at $\varphi$. So suppose otherwise.

If $\theta \in Z$, then let $\beta < \theta$, $\beta \notin Z$. Then by Lemma 2.2, $s(\beta, \cdot)$ has a jump discontinuity at $\theta$, say by $\eta > 0$. For every $\varepsilon > 0$ small enough,
\[
s(\theta - \varepsilon, \varphi) = s(\beta, \varphi) - s(\beta, \theta - \varepsilon) \geq s(\beta, \theta) - s(\beta, \theta - \varepsilon) \geq \eta.
\]
Thus
\[
N(r, \theta - \varepsilon, \varphi) \geq \frac{\eta}{\rho} r^\rho
\]
if $r$ is large enough, uniformly in $\varepsilon$. Since
\[
A(r, \theta, \varphi) = \bigcap_{\varepsilon > 0} A(r, \theta - \varepsilon, \varphi)
\]
is a closed sector, it follows that
\[
N(r, \theta, \varphi) \geq \frac{\eta}{\rho} r^\rho
\]
if $r$ is large enough. This proves the lemma in the case $\theta \in Z$.

Thus we may assume that $\theta \notin Z$ and $s(\theta, \cdot)$ is discontinuous at $\varphi$. If this happens, choose $\varphi' > \varphi$ such that $\varphi' - \varphi$ is small and $\varphi' \notin Z$. Then
\[
s(\theta, \varphi') \geq s(\theta, \varphi) > 0,
\]
so by Theorem 2.1, $N(r, \theta, \varphi') \sim r^\rho$ uniformly in $\varphi'$, again proving the lemma. \hfill \Box

**Lemma 2.4.** If $\hat{V}$ is an entire function of completely regular growth, then $h \geq 0$. 

Proof. Let $H$ be the indicator function of $\hat{V}$. Since $\hat{V}$ has completely regular growth,

$$\log|\hat{V}(re^{i\theta})| \approx H(\theta)r^\rho.$$ 

Moreover, if $T$ is any continuous function and $f \approx g$ then $T(f) \approx T(g)$, so $|\hat{V}(re^{i\theta})| \approx \exp(r^\rho H(\theta))$ and hence

$$|F(re^{i\theta})| \approx 1 + \exp(2^\rho r^\rho (H(\theta) + H(\pi + \theta))).$$

Therefore

$$\log|F(re^{i\theta})| \approx \begin{cases} 0 & \text{if } H(\theta) + H(\pi + \theta) < 0 \\ \log 2 & \text{if } H(\theta) + H(\pi + \theta) = 0 \\ 2^\rho r^\rho (H(\theta) + H(\pi + \theta)) & \text{else} \end{cases}$$

so

$$h(\theta) = 2^\rho (H(\theta) + H(\pi + \theta))_+ \geq 0,$$

which completes the proof. \hfill \Box

Proof of Proposition 1.11. We first remark that $\rho \geq 1$, a consequence of the Paley-Wiener-Schwartz theorem. Indeed, if $V$ is compactly supported, then $\rho = 1$; otherwise, either $\rho > 1$ or the type of $V$ is 0; the latter is excluded by Definition 1.7.

By Froese’s conjecture and Lemma 2.3, it suffices to show that either $\pi \in Z$ or there is an angle $\theta \in [\pi, 2\pi]$ such that $s(\pi, \theta)$ is nonzero.

To do this, we first show that $s(0, \cdot)$ is not identically zero. Suppose that it is. Then

$$h'(\varphi) = h'((\theta) + \rho^2 \int_{\varphi}^{\theta} h(\alpha) \, d\alpha,$$

yet $h$ is continuous and $\theta$ is fixed, so $h' \in C^1$ and so $h^{(2)} = -\rho^2 h$, so there are constants $c_\pm$ such that

$$h(\varphi) = c_+ e^{i\rho \varphi} + c_- e^{-i\rho \varphi}.$$ 

Since $F$ has completely regular growth, $F$ is of normal type, so $h$ is not identically zero. Since $h$ is real-valued, this implies that $h$ has a simple zero in $(0, 2\pi)$. Therefore $h$ is nonnegative, contradicting Lemma 2.4.

So either $0 \in Z$ or there is an angle $\theta \in [0, 2\pi]$ such that either $s(0, \theta) \neq 0$. Using the reflection symmetry $F(z) = F(-z)$, either $\pi \in Z$ or we may replace $\theta$ with a $\theta \in [\pi, 2\pi]$ such that $s(\pi, \theta) \neq 0$, if necessary. \hfill \Box
3. Divergence of $B(V)$

Assume that $V$ is noncompactly supported; we are ready to prove that $B(V)$ diverges. We recall that there were four sufficient conditions to check; any one would imply that $B(V)$ diverges. But Case 2 reduces to Case 1: if $V$ is holomorphic in a conic neighborhood of $\mathbf{R}$, then there are only finitely many resonances in a conic neighborhood of $\mathbf{R}$ [Sjö02, Corollary 12.14].

Similarly, Case 4 reduces to Case 1: if $s(k\pi, \theta)$ exists, then $h$ is differentiable at $k\pi$, so $k\pi \notin \mathbb{Z}$; then Lemma 2.2 implies that if $\beta < k\pi$ then $s(\beta, \cdot)$ is continuous at $k\pi$ and hence

$$\lim_{\varepsilon \to 0} \lim_{r \to \infty} \frac{N(r, k\pi - \varepsilon, k\pi + \varepsilon)}{r^\rho} = \frac{1}{2\pi \rho} \lim_{\varepsilon \to 0} s(k\pi - \varepsilon, k\pi + \varepsilon) = 0,$$

by Theorem 2.1. Since $N(r) \sim r^\rho$ by Proposition 1.11, this implies Case 1.

**Lemma 3.1.** All but finitely many resonances of $V$ are in the lower-half plane $\mathbb{C}_-$. 

**Proof.** This is well-known, but we sketch the proof; see [Fro97, §3] or [Bac20, Lemma 3.23] for the details. Let $B^1(\mathcal{H})$ denote the trace class of $\mathcal{H} = L^2(\text{supp } V)$. Choosing an appropriate branch of $\sqrt{\cdot}$, we may identify resonances with the zeroes of the function

$$D(\lambda) = \det(1 + \sqrt{V} R_0(\lambda) \sqrt{|V|}),$$

which is holomorphic in the upper-half plane $\mathbb{C}_+$ since $\sqrt{V} R_0 \sqrt{|V|}$ is holomorphic $\mathbb{C}_+ \to B^1(\mathcal{H})$. Moreover, $D(\lambda) \to 1$ as $\lambda \to \infty$ along any ray in $\mathbb{C}_+$, so there are only finitely many zeroes of $D$ in $\mathbb{C}_+$. \hfill \square

Therefore we may replace $B(V)$ with a sum over only the resonances in $\mathbb{C}_-$ without affecting its convergence properties, so that all summands in $B(V)$ are positive.

**Lemma 3.2.** Suppose that $\pi < \theta \leq \varphi < 2\pi$. If $n(r, \theta, \varphi) \gtrsim r$, then $B(V)$ diverges.

**Proof.** Let $k_j = n(j, \theta, \varphi)$, so that $k_j \gtrsim j$. Let $\text{Res}^* V$ be the set of resonances $re^{i\xi}$ such that $\theta \leq \xi \leq \varphi$. Then

$$B(V) \geq - \sum_{\lambda \in \text{Res}^* V} \frac{\text{Im } \lambda}{|\lambda|^2} \geq \min(- \sin \theta, - \sin \varphi) \sum_{\lambda \in \text{Res}^* V} \frac{1}{|\lambda|} \gtrsim_{\theta, \varphi} \sum_{j=0}^{\infty} \sum_{\lambda \in \text{Res}^* V} \frac{1}{|\lambda|} \frac{j+1}{j+1} = \sum_{j=0}^{\infty} \frac{k_{j+1} - k_j}{j+1}.$$
Summing by parts,
\[
\sum_{j=0}^{J} k_{j+1} - k_{j} = \frac{k_{J+1}}{J+1} - \sum_{j=1}^{J} k_{j} \left( \frac{1}{j+1} - \frac{1}{j} \right)
\]
\[
= \frac{k_{J+1}}{J+1} + \sum_{j=1}^{J} \frac{k_{j}}{j+j^2} \gtrsim 1 + \sum_{j=1}^{J} \frac{1}{j}
\]
which \(\to \infty\) as \(J \to \infty\). \(\square\)

In Case 1, the resonances \(\lambda\) furnished by Proposition 1.11 will satisfy \(-\sin \arg \lambda > \delta\) for some sufficiently small \(\delta > 0\), so by Lemma 3.2, \(B(V)\) diverges.

Finally, we prove Case 3. By reflection, we can assume that \(\pi < \theta \leq \varphi < 2\pi\). By Lemma 2.3, Froese’s conjecture, and Proposition 1.11,
\[
n(r, \theta, \varphi) \sim N(r, \theta, \varphi) \gtrsim r.
\]
Thus Lemma 3.2 completes the proof of Theorem 1.10.

References

[Bac20] A. Backus. “A conjecture on the resonances of a non-compactly supported potential”. Bachelor’s thesis. UC Berkeley, 2020. url: https://ocf.io/abackus/thesis.pdf.

[BP03] V. Bruneau and V. Petkov. “Meromorphic continuation of the spectral shift function”. Duke Mathematical Journal 116.3 (2003), 389–430.

[cGKN16] D. Çevik, M. Gadella, Ş. Kuru, and J. Negro. “Resonances and antibound states for the PschlTeller potential: Ladder operators and SUSY partners”. Physics Letters A 380.18-19 (2016), 16001609. DOI: 10.1016/j.physleta.2016.03.003.

[DZ19] S. Dyatlov and M. Zworski. Mathematical Theory of Scattering Resonances. Graduate Studies in Mathematics. American Mathematical Society, 2019. url: http://math.mit.edu/~dyatlov/res/res_final.pdf.

[Fro97] R. Froese. “Asymptotic distribution of resonances in one dimension”. Journal of Differential Equations 137.2 (1997), 251–272.

[GMR89] C. Gérard, A. Martinez, and D. Robert. “Breit-Wigner formulas for the scattering phase and the total scattering cross-section in the semi-classical limit”. Communications in Mathematical Physics 121.2 (1989), 323–336.

[Jen81] A. Jensen. “Time-delay in potential scattering theory. Some “geometric” results”. Communications in Mathematical Physics 82.3 (1981), 435–456.

[Lev64] B.I.A. Levin. Distribution of Zeros of Entire Functions. Translations of Mathematical Monographs. American Mathematical Society, 1964.

[PZ01] V. Petkov and M. Zworski. “Semi-classical estimates on the scattering determinant”. Annales Henri Poincaré. Vol. 2. 4. Springer. 2001, 675–711.
[PZ99] V. Petkov and M. Zworski. “Breit–Wigner Approximation and the Distribution of Resonances”. *Communications in Mathematical Physics* 204.2 (1999), 329–351.

[Sjö02] J. Sjöstrand. *Lectures on resonances*. 2002. URL: http://sjostrand.perso.math.cnrs.fr/Coursgbg.pdf.

[Zwo87] M. Zworski. “Distribution of poles for scattering on the real line”. *Journal of Functional Analysis* 73.2 (1987), 277–296.