Generalized oscillator representations
for generalized Calogero Hamiltonians

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Abstract
This paper is a natural continuation of the previous paper [1] where generalized
oscillator representations for Calogero Hamiltonians with potential
\( V(x) = \alpha/x^2, \alpha \geq -\frac{1}{4} \), were constructed. In this paper, we present generalized oscillator representations
for all generalized Calogero Hamiltonians with potential \( V(x) = g_1/x^2 + g_2x^2, g_1 \geq -\frac{1}{4}, g_2 > 0 \). These representations are generally highly nonunique, but there exists an optimum representation for each Hamiltonian, representation that explicitly determines
the ground state and the ground-state energy. For generalized Calogero Hamiltonians
with coupling constants \( g_1 < -\frac{1}{4} \) or \( g_2 < 0 \), generalized oscillator representations do
not exist in agreement with the fact that the respective Hamiltonians are not bounded
from below.

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1 Introduction. Formulation of problem

Let \( \hat{H} \) be the self-adjoint (s.a. in what follows) generalized Calogero differential operation
on the positive real semiaxis,
\[
\hat{H} = -d_x^2 + g_1x^{-2} + g_2x^2, \quad d_x = \frac{d}{dx}, \quad x \in \mathbb{R}_+ ,
\]
the arbitrary real parameters \( g_1 \) and \( g_2 \) are called the coupling constants, \( g_1 \) is dimensionless
and \( g_2 \) is of dimension of the fourth degree of inverse length. By definition, the generalized Calogero Hamiltonians,\( ^1 \) commonly symbolized by \( \hat{H}_\epsilon \) are s.a. differential operators in

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\(^1\) A remark about our terminology. In the paper, we use the term “generalized” in different senses, a specific sense is clear from a context.

The term “generalized” in this context is conventionally used in the case of \( g_2 \neq 0 \); in the case of \( g_2 = 0 \), this term is omitted, and we conventionally speak about the “Calogero differential operation” and “Calogero Hamiltonians” respectively, the coupling constant \( g_1 \) is then usually denoted by \( \alpha \).

\(^2\) For specific Hamiltonians, the index \( \epsilon \) is replaced by more informative indices.
All generalized Calogero Hamiltonians with arbitrary coupling constants were constructed in [2], sec. 8.4, and their spectra and (generalized) eigenfunctions were evaluated including inversion formulas. By construction, each \( \hat{H}_c \) with fixed coupling constants \( g_1 \) and \( g_2 \) is a s.a. extension of the initial symmetric operator \( \hat{H} \) associated with \( \hat{H} \) and defined on the subspace \( D(\mathbb{R}_+) \) of smooth functions with a compact support in \((0, \infty)\). If the coupling constant \( g_1 \geq 3/4 \), the extension is unique, so that for each \( g_1 \geq 3/4 \) and any fixed \( g_2 \), \(-\infty < g_2 < \infty \), there is a unique generalized Calogero Hamiltonian \( \hat{H}_1 \) (in notation of \([2]\)). If the coupling constant \( g_1 < 3/4 \), the extension is defined nonuniquely, so that for each \( g_1 < 3/4 \) and any fixed \( g_2 \), there is a one-parameter family \( \{ \hat{H}_\nu, \nu \in [-\pi/2, \pi/2], -\pi/2 \sim \pi/2 \} \) of generalized Calogero Hamiltonians which are specified by asymptotic s.a. boundary conditions at the origin. If \( g_1 \geq -1/4 \) and \( g_2 \geq 0 \), each \( \hat{H}_c \) is bounded from below. From the general standpoint, this is a consequence of that the initial symmetric operator \( \hat{H} \) with \( g_1 \geq -1/4 \), \( g_2 \geq 0 \) is nonnegative and therefore, all its s.a. extensions are bounded from below \([3,4]\). If \( g_1 < -1/4 \), each \( \hat{H}_\nu \) is unbounded from below (which is known as “a fall to the center” in the case of \( g_2 \geq 0 \)); if \( g_2 < 0 \), all \( \hat{H}_c \) are unbounded from below (“a fall to infinity”).

In this paper, our interest is with the possibility of representing generalized Calogero Hamiltonians in the generalized oscillator form

\[
\hat{H}_c = \hat{c}^+ \hat{c} - u \hat{I},
\]

where \( \hat{c} \) and \( \hat{c}^+ \) is a pair of closed mutually adjoint first-order differential operators, \( \hat{c}^+ = (\hat{c})^+, \hat{c} = \overline{\hat{c}} = (\hat{c}^+)^+ \), \( \hat{I} \) is the identity operator and \( u \) is a real parameter, \( \text{Im} \ u = 0 \), of dimension of inverse length squared. This problem was comprehensively considered for Calogero Hamiltonians, \( g_2 = 0 \), in \([1]\). In what follows, we therefore consider the case of \( g_2 \neq 0 \), the generalized Calogero Hamiltonians proper.

The parameter \( u \) in representation \([2]\) is not unique and is not arbitrary. Each Hamiltonian \( \hat{H}_c \) allowing representation \([2]\) allows its own region of admissible values of the parameter \( u \) determined by the spectrum of the Hamiltonian, we make this region more precise just below. Accordingly, the symbols \( \hat{c}, \hat{c}^+ \) of the operator pairs in \([2]\) implicitly contain \( u \) as an argument. Moreover, generalized oscillator representations with given \( u \) for a given \( \hat{H}_c \) may allow different mutually adjoint pairs \( \hat{c}, \hat{c}^+ \) parametrized by an additional parameter, which then appears as an additional argument in the symbols \( \hat{c}, \hat{c}^+ \).

The representation \([2]\) is equivalent to the representation

\[
\hat{H}_\nu = \hat{d} \hat{d}^+ - u \hat{I},
\]

where \( \hat{d} \) and \( \hat{d}^+ \) is a pair of closed mutually adjoint first-order differential operators, it is sufficient to make the identifications \( \hat{c} = \hat{d}^+, \hat{c}^+ = \hat{d} \).

The problem we are interested in really concerns the cases of \( g_1 \geq -1/4 \) together with \( g_2 > 0 \) because representations \([2]\), or \([3]\), for a Hamiltonian \( \hat{H}_c \) imply that \( \hat{H}_c \) is bounded

\[3\] By definition, a differential operator \( \hat{f} \) is called associated with a differential operation \( f(x, d_x) \) if the operator \( \hat{f} \) acts on its domain \( D_f \) by \( \hat{f} \): \( \hat{f} \psi(x) = f \psi(x), \forall \psi(x) \in D_f \).

\[4\] The term “generalized” in this context means distributions.

\[5\] For brevity, we here use a single uniform index \( \nu \) for labelling generalized Calogero Hamiltonians with \( g_1 < 3/4 \), this notation is a condensed one in comparison with \([2]\) where more specific double indices distinguishing different regions of \( g_1 \) were used. We introduce a more detailed indexing of generalized Calogero Hamiltonians with \( g_1 < 3/4 \) in sec. 3 below as needed.
from below, such that its spectrum is bounded from below by \(-u\) (for brevity, we will say that Hamiltonians \(\hat{H}_e\) (2), or (3), are bounded from below by \(-u\)). More specifically, if \(E_0 = E_0(g_1, g_2, \nu)\) is the lower boundary of the spectrum of the Hamiltonian \(\hat{H}_e\) with given coupling constants \(g_1, g_2\) and extension parameter \(\nu\) (if \(g_1 < 3/4\), then \(E_0 \geq -u\), so that the region of admissible \(u\) for a given \(\hat{H}_e\) is determined by the condition \(u \geq -E_0\). And if the kernel of the operator \(\hat{c}\) in representation (2), or the operator \(\hat{d}^+\) in representation (3), with \(u = -E_0\) is nontrivial, \(\ker \hat{c} \neq \{0\}\), or \(\ker \hat{d}^+ \neq \{0\}\), then \(\ker \hat{c}\), or \(\ker \hat{d}^+\), is the ground space (ground state) of the Hamiltonian and \(E_0\) is its ground-state energy. Conversely, if a given Hamiltonian \(\hat{H}_e\) allows representation (2), or representation (3), with a certain \(u\) and \(\ker \hat{c} \neq \{0\}\), or \(\ker \hat{d}^+ \neq \{0\}\), then \(\ker \hat{c}\), or \(\ker \hat{d}^+\), is the ground space (ground state) of the Hamiltonian and \(u = -E_0\), the ground-state energy with opposite sign. In such a case, we will say that the representation (2), or (3), is an optimum one. We recall that all the generalized Calogero Hamiltonians with \(g_2 > 0\) have a discrete spectrum, so that we expect that if a generalized Calogero Hamiltonian with given coupling constants \(g_1 \geq -1/4\), \(g_2 > 0\) allows generalized oscillator representations, then there exists an optimum representation.

It follows from the aforesaid that when examining the possibility of generalized oscillator representations (2) or (3) for generalized Calogero Hamiltonians with given coupling constants \(g_1 \geq -1/4\), \(g_2 > 0\), we can predetermine the range of the parameter \(u\) by the condition

\[ u \geq u_0 = u_0(g_1, g_2), \]

where \(u_0\) is a quantity opposite in sign to the lower-boundary value \(E_0(g_1, g_2)\) of the spectrum of a unique Hamiltonian \(\hat{H}_1\) with given coupling constants \(g_1 \geq 3/4, g_2 > 0\) or to the maximum in \(\nu\) of the lower boundaries \(E_0(g_1, g_2, \nu)\) of the spectrum of all the Hamiltonians \(\hat{H}_\nu\) with given coupling constants \(g_1 \in [-1/4, 3/4], g_2 > 0\),

\[ \begin{align*}
  u_0 = u_0(g_1, g_2) &= \left\{ \begin{array}{ll}
  -E_0(g_1, g_2), & g_1 \geq 3/4, g_2 > 0 \\
  -\max_\nu \{E_0(g_1, g_2, \nu)\}, & g_1 \in [-1/4, 3/4], g_2 > 0
  \end{array} \right. 
\end{align*} \]

These boundaries are written down in full in (2), according to which we have

\[ u_0 = u_0(g_1, g_2) = -2\sqrt{g_2(1 + \sqrt{1/4 + g_1})}. \] (4)

The range of admissible \(u\) in the generalized oscillator representation for a specific generalized Calogero Hamiltonian \(\hat{H}_e\) belongs to the semiaxis \([u_0, \infty)\). Because the generalized oscillator representations for a given \(\hat{H}_e\) is generally highly nonunique, the question arises about an optimum representation.

As to generalized oscillator representations for generalized Calogero Hamiltonians with coupling constants \(g_1 < -1/4\) or \(g_2 < 0\), there are no such representations and can not be because these Hamiltonians are not bounded from below.

In solving the problem of generalized oscillator representations for generalized Calogero Hamiltonians proper with \(g_1 \geq -1/4\) and \(g_2 > 0\), we follow ideas and methods in (1) where a similar problem was completely solved in positive for Calogero Hamiltonians, \(g_2 = 0\). We outline them here to give an insight into a content and a structure of the paper.

The basic idea is independently constructing generalized Calogero Hamiltonians as s.a. extensions of the initial symmetric operator \(\hat{H}\) directly in the generalized oscillator forms (2) or (3) and then comparing these constructions with the known Hamiltonians in (2).
A starting point is constructing a generalized oscillator representation for the respective differential operation $\hat{H}$ that is a representation of the form
\[ \hat{H} = \hat{b}\hat{a} - u, \quad \hat{a} = \hat{b}^*, \quad \hat{b} = \hat{a}^*, \quad \text{Im} \ u = 0, \]
where $\hat{a}$ and $\hat{b}$ are finite-order differential operations mutually adjoint by Lagrange (the superscript $*$ denotes the Lagrange adjoint), see [3], $u$ is a real numerical parameter.

In principle, the parameter $u$ is a variable parameter restricted by the condition $u \geq u_0 = u_0(g_1, g_2)$ [4], so that we actually deal with a family $\{\hat{a}(u), \hat{b}(u)\}$ of differential operations mutually adjoint by Lagrange. Moreover, a given $\hat{H}$ may allow a family of different pairs $\hat{a}, \hat{b}$ parametrized by an additional parameter, let it be $\mu$, in representation (5) with given $u$. In such a situation we would actually have a two-parameter family $\{\hat{a}(\mu, u), \hat{b}(\mu, u)\}$ of different mutually adjoint pairs $\hat{a}, \hat{b}$ providing the desired representation (5) for a given $\hat{H}$.

We note that in the case of Calogero Hamiltonians, the coupling constant $g_2 = 0$, the lower boundary $u_0$ of admissible values of the parameter $u$ in representation (2), or (3), is zero. $u \geq u_0 = 0$. The same holds for the parameter $u$ in representation (5) for $\hat{H}$ with $g_2 = 0$. A specific feature of generalized Calogero Hamiltonians with coupling constant $g_2 > 0$ is that the lower boundary of admissible values of the parameter $u$ is $u_0 < 0$ and $|u_0|$ indefinitely grows with $g_2$. The same holds for the parameter $u$ in representation (5) for $\hat{H}$ with $g_2 > 0$.

Let differential operation $\hat{H}$ (1) with given coupling constants $g_1, g_2 \in \mathbb{R}^2$ allow generalized oscillator representation (5) with some $\hat{a} = \hat{a}(u)$ and $\hat{b} = \hat{b}(u)$,
\[ \hat{H} = \hat{b}(u)\hat{a}(u) - u, \quad \hat{a}(u) = \hat{b}^*(u), \quad \hat{b}(u) = \hat{a}^*(u), \]
such that $\hat{a}(u) : \mathcal{D}(\mathbb{R}_+) \to \mathcal{D}(\mathbb{R}_+)$, $\hat{b}(u) : \mathcal{D}(\mathbb{R}_+) \to \mathcal{D}(\mathbb{R}_+)$; the pair $\hat{a}(u), \hat{b}(u)$ of mutually adjoint by Lagrange differential operations may be defined nonuniquely for given $u \in \mathbb{R}$. If we introduce a pair of initial differential operators $\hat{a}(u)$ and $\hat{b}(u)$ in $L^2(\mathbb{R}_+)$ defined on $\mathcal{D}(\mathbb{R}_+)$ and associated with the respective differential operations $\hat{a}(u)$ and $\hat{b}(u)$, then the initial symmetric operator $\hat{H}$ is evidently represented as
\[ \hat{H} = \hat{b}(u)\hat{a}(u) - u\hat{I}. \]
Let $\hat{c}(u)$ and $\hat{c}^+(u)$ be a pair of closed mutually adjoint operators that are closed extentions of the respective initial operators $\hat{a}(u)$ and $\hat{b}(u)$, $\hat{a}(u) \subset \hat{c}(u)$, $\hat{b}(u) \subset \hat{c}^+(u)$, the first candidate for $\hat{c}(u)$ is the closure of $\hat{a}(u)$, $\hat{c}(u) = \overline{\hat{a}(u)}$, then $\hat{c}^+(u) = \hat{a}^+(u)$. The operator
\[ \hat{H}_{\epsilon,c} = \hat{c}^+(u)\hat{c}(u) - u\hat{I} \]
is an evident extension of $\hat{H}$, $\hat{H} \subset \hat{H}_{\epsilon,c}$. By the von Neumann theorem [5] (for a proof, see also [3]), operator $\hat{H}_{\epsilon,c}$ (7) is s.a., which means that $\hat{H}_{\epsilon,c}$ is a certain generalized Calogero Hamiltonian represented in the generalized oscillator form (2) and bounded from below by $-u$, and if the kernel of the corresponding operator $\hat{c}(u)$ is nontrivial, $\ker \hat{c}(u) \neq \{0\}$, then $\ker \hat{c}(u)$ is the ground space (ground state) of $\hat{H}_{\epsilon,c}$ and $E_0 = -u$ is its least eigenvalue (ground-state energy).

6That is why the parameter $u$ was denoted in [1] as $u^2 = (sk_0)^2$, the variable dimensionless parameter $s \geq 0$, the fixed parameter $k_0 > 0$ is of dimension of inverse length.
Similarly, let $\hat{d}(u)$ and $\hat{d}^+(u)$ be a pair of closed mutually adjoint operators that are closed extensions of the respective initial operators $\hat{b}(u)$ and $\hat{a}(u)$, $\hat{b}(u) \subset \hat{d}(u)$, $\hat{a}(u) \subset \hat{d}^+(u)$, the first candidate for $\hat{d}(u)$ is the closure of $\hat{b}(u)$, $\hat{d}(u) = \overline{\hat{b}(u)}$, then $\hat{d}^+(u) = \hat{b}^+(u)$. The operator

$$\hat{H}_{e,d} = \hat{d}(u) \hat{d}^+(u) - u \hat{I} \tag{8}$$

is a certain generalized Calogero Hamiltonian represented in the generalized oscillator form $\hat{H}$ and having the properties similar to those of the operator $\hat{H}$ with $\hat{d}^+(u)$ in place of $\hat{d}(u)$. Constructing a pair $\hat{c}(u) \supset \hat{a}(u)$, $\hat{c}^+(u) \supset \hat{b}(u)$ or a pair $\hat{d}(u) \supset \hat{b}(u)$, $\hat{d}^+(u) \supset \hat{a}(u)$ is a matter of convenience: we can start with extending $\hat{a}(u)$ to its closure or with extending $\hat{b}(u)$ to its closure thus obtaining generally different s.a. extensions of $\hat{H}$.

Varying the parameter $u$ in $\hat{H}$ in the admissible region, finding total families of admissible differential operation pairs $\hat{a}(u)$, $\hat{b}(u)$ in this representation and involving all possible mutually-adjoint extensions of the initial differential operators $\hat{a}(u)$ and $\hat{b}(u)$, we can hope to construct generalized oscillator representations (2) or (3) for all generalized Calogero Hamiltonians with $g_1 \geq -1/4$ and $g_2 > 0$ in the form (7) or (8). We show below that this hope is justified. An identification of the Hamiltonians $\hat{H}_{c,e}$ and $\hat{H}_{c,d}$ with the known generalized Calogero Hamiltonians $\hat{H}_e$ in [2] is straightforward for $g_1 \geq 3/4$ because the Hamiltonian $\hat{H}_e = \hat{H}_1$ with given $g_1 \geq 3/4$, $g_2 > 0$ is unique, while for $g_1 \in [-1/4, 3/4)$, $g_2 > 0$, an identification is achieved by evaluating the asymptotic behavior of functions belonging to the domains of $\hat{H}_{c,e}$ and $\hat{H}_{c,d}$ at the origin and comparing it with the asymptotic s.a. boundary conditions specifying different generalized Calogero Hamiltonians $\hat{H}_e = \hat{H}_r$ with given $g_1 \in [-1/4, 3/4)$, $g_2 > 0$.

We say in advance that generalized oscillator representation (2), or (3), for a given generalized Calogero Hamiltonian is generally highly nonunique; in fact, there exists a one-, or even two-, parameter family of generalized oscillator representations for each Hamiltonian, among which there exists an optimum representation.

In conclusion, we note that the ideas and methods used in this paper, as well as in [1], can be applied to constructing generalized oscillator representations for other Hamiltonians associated with s.a. second order differential operations.

## 2 Generalized Calogero Hamiltonians in generalized oscillator form

We proceed to solving the problem of constructing generalized Calogero Hamiltonians in generalized oscillator form in accordance with the program presented above.

### 2.1 Basics of constructing generalized oscillator representations for differential operation $\hat{H}$

We begin with looking into the possibility of representing generalized Calogero differential operation $\hat{H}$ [1] with coupling constants $g_1 \geq -1/4$ and $g_2 > 0$ in generalized oscillator form [3] with $u \geq u_0$ given by (4).

Directly extending the arguments in [1] concerning the Calogero differential operation, $g_2 = 0$, to the generalized Calogero differential operation proper, we can assert that the
differential operation $\tilde{H}$ allows generalized oscillator representation \( \hat{H} \) iff the homogeneous differential equation

$$-\phi''(x) + (g_1 x^{-2} + g_2 x^2 + u)\phi(x) = 0,$$

or the eigenvalue problem

$$\tilde{H}\phi(x) = -\phi''(x) + (g_1 x^{-2} + g_2 x^2)\phi(x) = -u \phi(x)$$

(10)

(which can be considered a stationary Schrödinger equation with “energy” $E = -u$), has a real-valued positive solution $\phi(u; x)$,

$$\text{Im} \phi(u; x) = 0, \quad \phi(u; x) > 0, \quad x > 0,$$

and in this case, $\tilde{a}(u)$ and $\tilde{b}(u)$ are first-order differential operations of the form

$$\begin{align*}
\tilde{a}(u) &= d_x - h(u; x) = \phi(u; x) \frac{d_x}{\phi(u; x)} = \tilde{b}^*(u), \\
\tilde{b}(u) &= -d_x - h(u; x) = -\frac{1}{\phi(u; x)} d_x \phi(u; x) = \tilde{a}^*(u), \\
h(u; x) &= \phi'(u; x)/\phi(u; x) = -\phi(u; x) \left( \frac{1}{\phi(u; x)} \right)'.
\end{align*}$$

(11)

It is evident that the real-valued $h(u; x)$ is smooth in $(0, \infty)$ as a function of $x$ because the real-valued $\phi(u; x)$ is smooth and positive, so that $\tilde{a}(u) : \mathcal{D}(\mathbb{R}_+) \to \mathcal{D}(\mathbb{R}_+)$, $\tilde{b}(u) : \mathcal{D}(\mathbb{R}_+) \to \mathcal{D}(\mathbb{R}_+)$. It is also evident that the function $\phi(u; x)$ in (11) is defined up to a positive constant factor.

If eq. (9), or (11), with given coupling constants $g_1$, $g_2$ and external real parameter $u$ has no real-valued positive solution, there exists no generalized oscillator representation with given $u$ for $\tilde{H}$ with given $g_1$ and $g_2$.

If eq. (9) with given $u$ has a unique, up to a positive constant factor, real-valued positive solution $\phi(u; x)$, there exists a unique generalized oscillator representation with given $u$ for given $\tilde{H}$. It may happen that eq. (9) with different $u$ has desired solutions $\phi(u; x)$, then we get a one-parameter family $\{\tilde{a}(u), \tilde{b}(u)\}$ of different pairs $\tilde{a}(u), \tilde{b}(u)$ of differential operations (11) with the respective different functions $\phi(u; x)$ providing the desired generalized oscillator representation for given $\tilde{H}$.

It may happen that eq. (9) with given $u$ has two linearly independent real-valued positive solutions $\phi_1(u; x)$ and $\phi_2(u; x)$, then the general real-valued positive solution $\phi(u; x)$ of this equation, defined modulo a positive constant factor, is of the form $\phi(\mu, u; x) = \sin \mu \phi_1(u; x) + \cos \mu \phi_2(u; x)$ with $\mu$ belonging to a certain interval in $[0, 2\pi]$ such that the positive functions $\phi(\mu, u; x)$ with different $\mu$ are pairwise linearly independent; of course, this interval contains the segment $[0, \pi/2]$. In such a case, we get a two-parameter family $\{\tilde{a}(\mu, u), \tilde{b}(\mu, u)\}$ of different pairs $\tilde{a}(\mu, u), \tilde{b}(\mu, u)$ of differential operations (11) with the respective different functions $\phi(\mu, u; x)$ providing the desired generalized oscillator representation for given $\tilde{H}$.

For completeness, we first show that generalized Calogero differential operation $\tilde{H}$ (11) with the coupling constant $g_1 < -1/4$ or with the coupling constant $g_2 < 0$ does not allow generalized oscillator representation with whatever $u$, which is in complete agreement with the fact that the respective generalized Calogero Hamiltonians $\tilde{H}_\epsilon$ with such coupling

\footnote{The differential operations in (11) are evidently defined up to arbitrary phase factors, $\tilde{a} \to e^{i\theta(x)}\tilde{a}$, $\tilde{b} \to \tilde{b} e^{-i\theta(x)}$. These factors are irrelevant because they trivially cancel in the product $\tilde{b} \tilde{a}$; their choosing is a matter of convenience, we here choose $\theta(x) = 0$.}
constants are not bounded from below and therefore cannot be represented in generalized
oscillator form.

It is sufficient to prove that eq. (9) with \( g_1 < -1/4 \) or with \( g_2 < 0 \) and any real \( u \) has no
real-valued positive solution.

Let \( g_1 = -1/4 - \sigma^2, \sigma > 0 \). In this case, eq. (9) with any \( g_2 \) and \( u \) has a solution \( \phi_1(x) \)
whose asymptotic behavior at the origin is given by

\[
\phi_1(x) = (k_0 x)^{1/2 + i\sigma} [1 + O(x^2)], \quad x \to 0,
\]

where \( k_0 \) is an arbitrary, but fixed, parameter of dimension of inverse length. The linearly
independent solution \( \phi_2(x) \) is the complex conjugate of \( \phi_1(x) \), \( \phi_2(x) = \overline{\phi_1(x)} \). The general
real-valued solution of eq. (9) is of the form \( \phi(x) = \phi_1(x) + A \phi_1(x), \quad A = |A| e^{i\varphi} \) is an
arbitrary constant, its asymptotic behavior at the origin is given by

\[
\phi(x) = 2|A|(k_0 x)^{1/2} [1 + O(x^2)] \cos[\sigma \ln(k_0 x) + \varphi + O(x^2)], \quad x \to 0,
\]

which demonstrates an infinite number of zeroes of \( \phi(x) \) accumulated at the origin.

Let \( g_2 = -\omega^2, \omega > 0 \). In this case, eq. (9) with any \( g_1 \) and \( u \) has a solution \( \phi_1(x) \) whose
asymptotic behavior at infinity is given by

\[
\phi_1(x) = (\omega x^2)^{-1/4} e^{-i\omega x^2} [1 + O(1/x^2)], \quad x \to \infty.
\]

Again, the linearly independent solution \( \phi_2(x) \) is the complex conjugate of \( \phi_1(x) \), \( \phi_2(x) = \overline{\phi_1(x)} \), and the general real-valued solution of eq. (9) is of the form \( \phi(x) = A \phi_1(x) + \overline{A} \phi_1(x), \quad A = |A| e^{i\varphi} \). The asymptotic behavior of \( \phi(x) \) at infinity is given by

\[
2|A|/(\omega x^2)^{-1/4} [1 + O(x^2)] \cos[\frac{1}{2} \omega x^2 - \frac{u}{4\omega} \ln(\omega x^2) - \varphi + O(x^2)], \quad x \to \infty,
\]

which demonstrates an infinite number of zeroes of \( \phi(x) \) accumulated at infinity.

We are now coming to generalized oscillator representations for generalized Calogero differ-
ential operations \( \hat{H} \) (10) with coupling constants \( g_1 > -1/4 \) and \( g_2 > 0 \), which are of our
main interest.

In finding the general solution of eq. (9), it is convenient to go from the old parameters
\( g_1, g_2, u, w_0 \) to new parameters \( \kappa, v, w, w_0 \) defined by

\[
\begin{align*}
  g_1 &= -1/4 + \kappa^2, \quad \kappa = \sqrt{g_1 + 1/4} \geq 0, \\
  g_2 &= v^4, \quad v = \sqrt{g_2} > 0, \\
  u &= 4 \sqrt{g_2} w, \quad w = u/4v^2 \geq w_0, \\
  w_0 &= \frac{w_0}{4v^2} = -\frac{1}{2}(1 + \kappa) < 0,
\end{align*}
\]

(12)

\( v \) is of dimension of inverse length, \( \kappa, w \) and \( w_0 \) are dimensionless, and from the old space
variable \( x \) to a new dimensionless variable

\[
\rho = (v x)^2 \geq 0.
\]

(13)

The Ansatz

\[
\phi(x) = e^{-\rho/2} \rho^{1/4+\kappa/2} \tilde{\phi}(\rho)
\]

(14)
reduces eq. (9) to the equation
\[
\rho \frac{d^2 \tilde{\phi}(\rho)}{d\rho^2} + (\beta - \rho) \frac{d\tilde{\phi}(\rho)}{d\rho} - \alpha \tilde{\phi}(\rho) = 0,
\]
where
\[
\alpha = \frac{1}{2}(1 + \kappa) + w = \beta/2 + w \geq 0, \quad \beta = 1 + \kappa \geq 1,
\]
which is the so-called confluent hypergeometric equation. A great body of information on its solutions, which are called confluent hypergeometric functions, can be found in [6], Ch. 6 and in [7], Ch. 9.2.

When considering the fundamental systems of solutions and the respective representations for the general solution of confluent hypergeometric equation (15), we have to distinguish two cases: the case of \( \alpha > 0 \) and the case of \( \alpha = 0 \).

As the fundamental system of solutions of eq. (15) with \( \alpha > 0 \), we can take the two standard confluent hypergeometric functions, the function
\[
\Phi(\alpha, \beta; \rho) = \sum_{k=0}^{\infty} \frac{(\alpha)_k \rho^k}{(\beta)_k k!} = 1 + \frac{\alpha}{\beta} \rho + \frac{\alpha(\alpha + 1)}{\beta(\beta + 1)} \frac{\rho^2}{2!} + ..., \tag{16}
\]
where
\[
(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} = \begin{cases} 1, & k = 0, \\ \alpha(\alpha + 1)(\alpha + k - 1), & k = 1, 2, 3, ... \end{cases}
\]
is the Pochhammer symbol, and the function
\[
\Psi(\alpha, \beta; \rho) = \frac{\Gamma(1 - \beta)}{\Gamma(\alpha - \beta + 1)} \Phi(\alpha, \beta; \rho) + \frac{\Gamma(\beta - 1)}{\Gamma(\alpha)} \rho^{1-\beta} \Phi(\alpha - \beta + 1, 2 - \beta; \rho). \tag{17}
\]
The indeterminacy of the r.h.s in (17) at integers \( \beta = n + 1, n = 0, 1, 2, ... \), is resolved by the passage to the limit \( \beta \to n + 1 \), which produces the term with logarithmic factor, in particular,
\[
\Psi(\alpha, 1; \rho) = \frac{1}{\Gamma(\alpha)} \left\{ \Phi(\alpha, n + 1; \rho) \ln \frac{1}{\rho} + \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(n + 1)_r} [2\psi(1 + r) - \psi(\alpha + r)] \frac{\rho^r}{r!} \right\}, \tag{18}
\]
where \( \psi \) is the logarithmic derivative of the Euler \( \Gamma \) function, \( \psi(z) = \Gamma'(z)/\Gamma(z) \). The function \( \Psi(\alpha, \beta; \rho) \) also allows the representation
\[
\Psi(\alpha, \beta; \rho) = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt \, t^{\alpha-1} (1 + t)^{\beta-\alpha-1} e^{-t} = \rho^{1-\beta} \frac{1}{\Gamma(\alpha)} \int_0^\infty dt \, t^{\alpha-1} (\rho + t)^{\beta-\alpha-1} e^{-t}. \tag{19}
\]

The general solution of eq. (15) with \( \alpha > 0 \) is then given by
\[
\tilde{\phi}(\rho) = A \Phi(\alpha, \beta; \rho) + B \Psi(\alpha, \beta; \rho), \quad \alpha > 0, \tag{20}
\]
where \( A \) and \( B \) are arbitrary complex coefficients.
In what follows, we need the asymptotic behavior of the functions $\Phi(\alpha, \beta; \rho)$ and $\Psi(\alpha, \beta; \rho)$ as a functions of $\rho$ at the origin and at infinity, which we present in the form sufficient for our purposes. The asymptotic behavior of these functions at the origin is respectively given by

$$\Phi(\alpha, \beta; \rho) = 1 + O(\rho), \ \rho \to 0,$$

and

$$\Psi(\alpha, \beta; \rho) = \begin{cases} 
O(\rho^{1-\beta}), \ \beta \geq 2 \\
\frac{\Gamma(\beta-1)}{\Gamma(\alpha)} \rho^{1-\beta}(1 + O(\rho))+ \\
+ \frac{\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)}(1 + O(\rho)), \ \beta \in (1, 2) \ 	o \infty, \ \rho \to 0, \\
-\frac{1}{\Gamma(\alpha)} \ln \rho (1 + O(\rho))+ \\
+ [2\psi(1) - \psi(\alpha)](1 + O(\rho)), \ \beta = 1 
\end{cases}$$

while their asymptotic behavior at infinity is respectively given by

$$\Phi(\alpha, \beta; \rho) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \rho^{\alpha-\beta} e^\rho (1 + O(1/\rho)) \to \infty, \ \rho \to \infty,$$

and

$$\Psi(\alpha, \beta; \rho) = \rho^{-\alpha}(1 + O(1/\rho)) \to 0, \ \rho \to \infty.$$  

The case of $\alpha = 0$ is the exceptional case because $\Phi(0, \beta; \rho) = \Psi(0, \beta; \rho) = 1$. As the fundamental system of solutions of eq. (15) with $\alpha = 0$, we can take the functions

$$\Phi(\beta; \rho) = \Phi(0, \beta; \rho) = 1$$

and

$$\Psi(\beta; \rho) = \int_0^\rho d\tau \tau^{-\beta} e^{\tau},$$

where $a > 0$ is a certain fixed number.

The general solution of eq. (15) with $\alpha = 0$ is then given by

$$\tilde{\phi}(\rho) = A + B \Psi(\beta; \rho), \ \alpha = 0.$$  

Returning to eq. (9), we use the notation introduced in (12), (13) and (15), where $u = 4v^2w$, and have to distinguish the region $w > w_0 = -\frac{1}{2}(1 + \kappa)$ ($\alpha > 0$) and the point $w = w_0$ ($\alpha = 0$).

The general solution of eq. (9) with $g_1 \geq -1/4$, $g_2 > 0$, and $w > w_0$ is obtained by combining (14) and (20). But to get a suitable form of the asymptotic behavior of the solution at the origin, we renormalize the coefficient $B$ in (20) as follows.

$$B \to \begin{cases} 
B \frac{\Gamma(\alpha)}{\Gamma(\beta-1)}, \ g_1 > -1/4 (\kappa > 0) \\
B \frac{1}{\Gamma(\alpha)}, \ g_1 = -1/4 (\kappa = 0) 
\end{cases}.$$  

Under this convention, the general solution of eq. (21) with $w > w_0$ is given by

$$\phi(w; x) = A\phi_1(w; x) + B\phi_2(w; x), \ w > w_0 = -\frac{1}{2}(1 + \kappa),$$

$$\phi_1 (w; x) = e^{-\rho/2} \rho^{1/4 + \kappa/2} \Phi(\alpha, \beta; \rho),$$

$$\phi_2 (w; x) = \begin{cases} 
e^{-\rho/2} \rho^{1/4 + \kappa/2} \frac{\Gamma(\alpha)}{\Gamma(\beta)} \Psi(\alpha, \beta; \rho), \ g_1 > -1/4 (\kappa > 0) \\
e^{-\rho/2} \rho^{1/4} \frac{1}{\Gamma(\alpha)} \Psi(\alpha, 1; \rho), \ g_1 = -1/4 (\kappa = 0) 
\end{cases}.$$  

\footnote{This is equivalent to an evident change of the fundamental system of solutions of eq. (15).}
where the functions $\Phi(\alpha, \beta; \rho)$ and $\Psi(\alpha, \beta; \rho)$ are given by respective (16) and (17), (18) or (19). $A$ and $B$ are arbitrary complex numbers. It is remarkable that the both functions $\phi_1$ and $\phi_2$ are real-valued and positive.

According to (14) and (25), the general solution of eq. (9) with $w = w_0$ is given by

$$\phi(w_0; x) = e^{-\rho/2} \rho^{1/4+\kappa/2} (A + B \Psi(\beta; \rho)), w = w_0 = -\frac{1}{2}(1 + \kappa),$$  \hspace{1cm} (28)

where the function $\Psi(\beta; \rho)$ is given by (25).

Accordingly, we have to consider separately the region $w > w_0 = -1/2(1 + \kappa)$ and the point $w = w_0$ in future treatment. In addition, as it follows from (2), in the both cases $w > w_0$ and $w = w_0$, the point $g_1 = -1/4 (\kappa = 0)$ is naturally distinguished by a specific behavior of the functions involved at the origin, which is an essential point in the analysis. We begin with the region $g_1 > -1/4 (\kappa > 0)$ and $w > w_0$.

### 2.2 Region $g_1 > -1/4 (\kappa > 0)$, $w > w_0 = -1/2(1 + \kappa)$ ($\alpha > 0$)

#### 2.2.1 Generalized oscillator representations for $\hat{H}$, differential operations $\hat{a}$ and $\hat{b}$

In this region of parameters, the general solution of eq. (9) is given by (27) with real-valued and positive linearly-independent functions $\phi_1$ and $\phi_2$. In addition, the function $\Phi(\alpha, \beta; \rho)$ increases monotonically from 1 to $\infty$ as $\rho = (\nu \kappa)^2$ together with $x$ ranges from 0 to $\infty$, see (16), while the function $\Psi(\alpha, \beta; \rho)$ decreases monotonically from $\infty$ to 0 because the integrand of the first integral in the r.h.s. of (19) is a decreasing function of $\rho$, is nonintegrable at $\rho = 0$ and vanishes as $\rho \to \infty$. It follows by the arguments in the previous subsection, see the text after (11), that the general real-valued positive solution of eq. (2) defined modulo a positive constant factor is given by (27) with $A = \sin \mu$, $B = \cos \mu$, $\mu \in [0, \pi/2],

\begin{align*}
\phi(\mu; w; x) &= e^{-\rho/2} \rho^{1/4+\kappa/2} \left[ \Phi(\alpha, \beta; \rho) \sin \mu + \frac{\Gamma(\alpha)}{\Gamma(\kappa)} \Psi(\alpha, \beta; \rho) \cos \mu \right], \\
\mu &\in [0, \pi/2], \alpha = 1/2(1 + \kappa) + w > 0, \beta = 1 + \kappa, \kappa > 0, \rho = (\nu \kappa)^2,
\end{align*}

(29)

which implies that we have the two-parameter family $\{\hat{a}(\mu, w), \hat{b}(\mu, w); \mu \in [0, \pi/2], w \in (w_0, \infty), \kappa > 0\}$ of different pairs of mutually adjoint first-order differential operations $\hat{a}(\mu, w)$ and $\hat{b}(\mu, w)$ given by (11) with the evident substitutions $\hat{a}(u) \to \hat{a}(\mu, w)$, $\hat{b}(u) \to \hat{b}(\mu, w)$, and $\phi(u; x) \to \phi(\mu, w; x)$:

\begin{align*}
\hat{a}(\mu, w) &= \phi(\mu, w; x) \frac{1}{x} \frac{\partial}{\partial \phi(\mu, w; x)} = \hat{b}^*(\mu, w), \\
\hat{b}(\mu, w) &= -\frac{1}{\phi(\mu, w; x)} \frac{\partial}{\partial \phi(\mu, w; x)} = \hat{a}^*(\mu, w), \\
\mu &\in [0, \pi/2], w \in (w_0, \infty), \kappa \in (0, \infty),
\end{align*}

(30)
and providing a two-parameter family\(^9\) of different generalized oscillator representations (6) for generalized Calogero differential operation \(\hat{H}\) (11) with \(g_1 > -1/4\) and \(g_2 > 0\),

\[
\hat{H} = -d_x^2 + g_1 x^{-2} + g_2 x^2 = \hat{b}(\mu, w) \hat{a}(\mu, w) - 4v^2 w, \tag{31}
\]

\(\mu \in [0, \pi/2], w \in (w_0, \infty)\).

In an analysis to follow, we need the asymptotic behavior of the functions \(\phi(\mu, w; x)\) and \(1/\phi(\mu, w; x)\) at the origin and at infinity. According to (21), (22) and (23), (24), the asymptotic behavior of these functions at the origin is respectively given by

\[
\phi(\mu, w; x) = \begin{cases} 
O(x^{1/2-\kappa}), & \kappa \geq 1 \\
\tilde{A}(v x)^{1/2+\kappa} + \tilde{B}(v x)^{1/2-\kappa} + O(v x^{5/2-\kappa}), & \kappa \in (0, 1), \quad \mu \in [0, \pi/2), \quad x \to 0,
\end{cases}
\tag{32}
\]

\[
\tilde{A} = \tilde{A}(\mu, w) = \sin \mu - \cos \mu \frac{\Gamma(1-\kappa)\Gamma\left(\frac{1}{2}(1+\kappa) + w\right)}{\Gamma(1+\kappa)\Gamma\left(\frac{1}{2}(1-\kappa) + w\right)},
\]

\[
\tilde{B} = \tilde{B}(\mu, w) = \cos \mu.
\]

and

\[
\frac{1}{\phi(\mu, w; x)} = \begin{cases} 
O(x^{\kappa-1/2}), & \kappa \geq 1 \\
\frac{1}{B(\mu, w) 1+(A(\mu, w)/B(\mu, w))v x^{2 \kappa}} + O(x^{3/2+\kappa}), & \kappa \in (0, 1), \quad \mu \in [0, \pi/2), \quad x \to 0,
\end{cases}
\tag{33}
\]

while their asymptotic behavior at infinity is respectively given by

\[
\phi(\mu, w; x) = \begin{cases} 
\sin \mu \frac{\Gamma(1+\kappa)}{\Gamma\left(\frac{1}{2}(1+\kappa) + w\right)} (v x)^{-1/2+2 w} e^{\frac{1}{2}(v x)^2} (1 + O(x^{-2})), & \mu \in (0, \pi/2], \quad \forall \kappa > 0, \\
\mu = 0, \quad \forall \kappa > 0
\end{cases}, \quad x \to \infty, \tag{34}
\]

and

\[
\frac{1}{\phi(\mu, w; x)} = \begin{cases} 
\frac{1}{\sin \mu} \frac{\Gamma\left(\frac{1}{2}(1+\kappa) + w\right)}{\Gamma(1+\kappa)} (v x)^{1/2-2 w} e^{-\frac{1}{2}(v x)^2} (1 + O(x^{-2})), & \mu \in (0, \pi/2], \quad \forall \kappa > 0, \\
\mu = 0, \quad \forall \kappa > 0
\end{cases}, \quad x \to \infty. \tag{35}
\]

\(^9\)As a rule, we indicate the ranges of parameters \(\mu\) and \(w\) in formulas to follow only if they differ from the whole ranges, here these are \([0, \pi/2]\) for \(\mu\) and \((w_0, \infty)\) for \(w\), the range of \(\kappa\) is clear from the title of section, subsection or subsubsection, here this is \((0, \infty)\). As to the main resulting formulas, we indicate the ranges of all the parameters including \(\kappa\).
2.2.2 Initial operators \(\hat{a}\) and \(\hat{b}\)

We introduce the pairs of initial differential operators \(\hat{a}(\mu, w)\) and \(\hat{b}(\mu, w)\) in \(L^2(\mathbb{R}_+)\) defined on the subspace \(\mathcal{D}(\mathbb{R}_+)\) of smooth compactly supported functions, \(\mathcal{D}_{\hat{a}(\mu, w)} = \mathcal{D}_{\hat{b}(\mu, w)} = \mathcal{D}(\mathbb{R}_+)\), and associated with each pair of the respective differential operations \(\hat{a}(\mu, w)\) and \(\hat{b}(\mu, w)\) [30]. These operators have the property

\[
(\psi, \hat{a}(\mu, w)\xi) = \left(\hat{b}(\mu, w)\psi, \xi\right), \quad \forall \psi(x), \xi(x) \in \mathcal{D}(\mathbb{R}_+),
\]

which is easily verified by integration by parts. According to [31], the initial symmetric operator \(\hat{H}\) associated with \(\hat{H}\) and defined on \(\mathcal{D}(\mathbb{R}_+)\) allows the representation

\[
\hat{H} = \hat{a}(\mu, w)\hat{b}(\mu, w) - 4v^2w \hat{I}, \quad \forall \mu \in [0, \pi/2], \quad \forall w > w_0, \forall \omega > 0,
\]

which, in particular, implies that \(\hat{H}\) with \(g_1 > -1/4, g_2 > 0\) is bounded from below by \(-4v^2w, \)

\[
(\xi, \hat{H}\xi) = \left(\xi, (\hat{b}(\mu, w)\hat{a}(\mu, w) - 4v^2w \hat{I})\xi\right) = \]

\[
= (\hat{a}(\mu, w)\xi, \hat{a}(\mu, w)\xi) - 4v^2w (\xi, \xi) \geq -4v^2w (\xi, \xi), \quad \forall \xi(x) \in \mathcal{D}(\mathbb{R}_+),
\]

and taking the infimum of \(w\), which is \(w_0\), we obtain that \(\hat{H}\) with \(g_1 > -1/4, g_2 > 0\) is bounded from below by \(-4v^2w_0 = 2v^2(1 + \omega)\). These representations provide a basis for constructing s.a. generalized Calogero Hamiltonians \(\hat{H}_\epsilon\) in generalized oscillator form as s.a. extensions of \(\hat{H}\) [37] in accordance with the program formulated in sec.1. Namely, we should construct all possible extensions of each pair \(\hat{a}(\mu, w), \hat{b}(\mu, w)\) of initial operators with given \(\mu\) and \(w\) to a pair of closed mutually adjoint operators \(\hat{c}(\mu, w), \hat{c}^+(\mu, w), \hat{\alpha}(\mu, w) \subset \hat{c}(\mu, w), \hat{b}(\mu, w) \subset \hat{c}^+(\mu, w)\), beginning from the closure \(\hat{a}(\mu, w)\) of \(\hat{a}(\mu, w)\) or to a pair of closed mutually adjoint operators \(\hat{d}(\mu, w), \hat{d}^+(\mu, w), \hat{\alpha}(\mu, w) \subset \hat{d}(\mu, w), \hat{b}(\mu, w) \subset \hat{d}^+(\mu, w)\), beginning from the closure \(\hat{b}(\mu, w)\) of \(\hat{b}(\mu, w)\). These extensions produce the respective s.a. operators

\[
\hat{H}_{\epsilon\hat{c}(\mu, w)} = \hat{c}^+(\mu, w)\hat{c}(\mu, w) - 4v^2w \hat{I},
\]

and

\[
\hat{H}_{\epsilon\hat{d}(\mu, w)} = \hat{d}^+(\mu, w)\hat{d}(\mu, w) - 4v^2w \hat{I},
\]

which are certain generalized Calogero Hamiltonians in generalized oscillator form. It then remains to identify \(\hat{H}_{\epsilon\hat{c}(\mu, w)}\) and \(\hat{H}_{\epsilon\hat{d}(\mu, w)}\) with the known generalized Calogero Hamiltonians. It should be noted that the operators \(\hat{H}_{\epsilon\hat{c}(\mu, w)}\) and \(\hat{H}_{\epsilon\hat{d}(\mu, w)}\) are not necessarily different even if \(c(\mu, w) \neq d^+(\mu, w)\), and what is more, the l.h.s in (38) or in (39) may not depend on \(\mu\) and \(w\).

We proceed to constructing all possible extensions of an arbitrary pair of initial operators \(\hat{a}(\mu, w), \hat{b}(\mu, w)\) to a pair of closed mutually adjoint operators.

2.2.3 Adjoint operators \(\hat{a}^+\) and \(\hat{b}^+\), closed operators \(\overline{\hat{a}}\) and \(\overline{\hat{b}}\)

Because all the operators \(\hat{a}(\mu, w)\) and \(\hat{b}(\mu, w)\) are densely defined, they have the adjoints, the respective \(\hat{a}^+(\mu, w)\) and \(\hat{b}^+(\mu, w)\). The defining equation for \(\hat{a}^+(\mu, w)\), i.e., the equation for
pairs \( \psi(x) \in D_{a^+(\mu, w)} \) and \( \eta(x) = a^+(\mu, w)\psi(x) \) forming the graph of the operator \( a^+(\mu, w) \), see \[2\], sec. 2.6, reads

\[
(\psi, a^+(\mu, w)\xi) = (\eta, \xi), \quad \forall \xi(x) \in \mathcal{D}(\mathbb{R}_+).
\] (40)

The equality (36) then implies that \( b^+(\mu, w) \subset a^+(\mu, w) \): eq. (10) has solutions

\[
\psi(x) = \zeta(x), \quad \eta(x) = b^+(\mu, w)\zeta(x), \quad \forall \zeta(x) \in \mathcal{D}(\mathbb{R}_+).
\]

It follows that \( a^+(\mu, w) \) is densely defined and in turn has the adjoint \( (a^+(\mu, w))^+ \), while the operator \( a(\mu, w) \) has a closure \( \overline{a}(\mu, w) = (a^+(\mu, w))^+ \subset b^+(\mu, w) \) and \( \overline{(a^+(\mu, w))} = \overline{a}(\mu, w) \). Similarly, we obtain that \( a(\mu, w) \subset b^+(\mu, w) \), and therefore, there exists the adjoint \( (\overline{b}^+(\mu, w))^+ \) of \( \overline{b}^+(\mu, w) \), the operator \( \overline{b}(\mu, w) \) has a closure \( \overline{b}(\mu, w) = (\overline{b}^+(\mu, w))^+ \subset \overline{a}(\mu, w) \) and \( \overline{(\overline{b}^+(\mu, w))} = \overline{b}(\mu, w) \). We thus obtain the chains of inclusions

\[
\begin{align*}
\hat{a}(\mu, w) &\subset \overline{a}(\mu, w) = (a^+(\mu, w))^+ \subset b^+(\mu, w), \\
\hat{b}(\mu, w) &\subset \overline{b}(\mu, w) = (\overline{b}^+(\mu, w))^+ \subset \overline{a}(\mu, w).
\end{align*}
\] (41)

2.2.4 Domains of operators \( \hat{a}^+, \hat{b}^+, \overline{a} \) and \( \overline{b} \)

In evaluating the operators \( \hat{a}^+(\mu, w), \hat{b}^+(\mu, w), \overline{a}(\mu, w), \) and \( \overline{b}(\mu, w) \), we follow \[1\] where the case of \( g_2 = 0 \) was considered. The operators \( \hat{a}^+(\mu, w) \) and \( \hat{b}(\mu, w) \) are associated with the differential operation \( \hat{b}(\mu, w) = \hat{a}^+(\mu, w) \), while the operators \( \hat{b}^+(\mu, w) \) and \( \overline{a}(\mu, w) \) are associated with the differential operation \( \hat{a}(\mu, w) = \hat{b}^+(\mu, w) \). It is therefore sufficient to evaluate the domains of the operators involved, which either coincide with or belong to the natural domains for the respective differential operations.\[10\]

i) The domain \( D_{a^+(\mu, w)} \) of the operator \( a^+(\mu, w) \) is the natural domain for \( b^+(\mu, w) \):

\[
D_{a^+(\mu, w)} = \{ \psi(x) : \psi(x) \text{ is a.c. in } \mathbb{R}_+; \psi(x), \hat{b}(\mu, w)\psi(x) = \frac{1}{\phi(\mu, w; x)} \frac{d}{dx} (\phi(\mu, w; x)\psi(x)) = \eta(x) \in L^2(\mathbb{R}_+) \},
\] (42)

the symbol “a.c.” is a contraction of “absolutely continuous”.

According to \[12\], a generic function \( \psi(x) \) belonging to \( D_{a^+(\mu, w)} \) can be considered as the general solution of the inhomogeneous differential equation \( b(\mu, w)\psi(x) = \eta(x) \) under the additional conditions that the both \( \psi(x) \) and \( \eta(x) \) are square integrable on \( \mathbb{R}_+ \). It follows with taking estimates (32), (33) and (34), (35) into account that a generic \( \psi(x) \in D_{a^+(\mu, w)} \) allows the representation

\[
\psi(x) = \frac{1}{\phi(\mu, w; x)} \left[ C - \int_{x_0}^{x} dy \phi(\mu, w; y)\eta(y) \right], \quad \eta(x) = \hat{b}(\mu, w)\psi(x) \in L^2(\mathbb{R}_+),
\] (43)

\[10\] We recall that the natural domain \( D_{\hat{f}}^n \subset L^2(\mathbb{R}_+) \) for a given differential operation \( \hat{f} \) is the maximum possible domain for operators associated with \( \hat{f} \), see \[2\].
where the point \( x_0 \) and constant \( C \) depend on the values of \( \mu \) and \( \kappa \) as follows:

\[
\begin{align*}
\mu &= 0 : \begin{cases} 
  x_0 \in (0, \infty) \text{ for } \kappa > 1 \\
  x_0 = 0, \text{ for } \kappa \in (0, 1)
\end{cases} \quad \text{ and } \quad C = \int_{x_0}^{\infty} d\psi(0, w; y)\eta(y), \\
\mu &\in (0, \pi/2), \begin{cases} 
  x_0 \in (0, \infty) \text{ for } \kappa > 1 \\
  x_0 = 0, \text{ for } \kappa \in (0, 1)
\end{cases} \quad \text{ and } \quad C \text{ is an arbitrary constant,}
\end{align*}
\]

(44)

A subtlety is that for \( \mu \in [0, \pi/2) \), the constant \( C \) in (43) can take arbitrary values, but for \( \mu \in (0, \pi/2) \), the constant \( C \) is independent of the function \( \eta(x) \), while for \( \mu = 0 \), it is uniquely related to \( \eta \), so that the representation (43), (44) with \( \mu = 0 \) for \( \psi(x) \in D_{a^+(\mu, w)} \) is equivalent to

\[
\psi(x) = \begin{cases} 
  \frac{1}{\phi(0, w; x)} \int_{x}^{\infty} d\psi(0, w; y)\eta(y), \quad \eta(x) = \tilde{b}_{(0, w)}\psi(x) \in L^2(\mathbb{R}_+), & x \to 0, \\
  \frac{1}{\phi(0, w; x)} \int_{0}^{x} d\psi(0, w; y)\eta(y), \quad \eta(x) = \tilde{b}_{(0, w)}\psi(x) \in L^2(\mathbb{R}_+), & x \to \infty
\end{cases}
\]

(45)

while at infinity, \( \psi(x) \in D_{a^+(\mu, w)} \) vanishes,

\[
\psi(x) \to 0, \quad x \to \infty, \quad \forall \mu, \forall \kappa > 0.
\]

(46)

We note that for \( \kappa \in (0, 1) \), the domain \( D_{a^+(\mu, w)} \) of the operator \( \hat{a}^+(\mu, w) \) with \( \mu \in [0, \pi/2) \) can be represented as a direct sum of the form

\[
D_{a^+(\mu, w)} = \{ C\psi_0(\mu, w; x) \} \oplus \tilde{D}_{a^+(\mu, w)}, \quad \mu \in [0, \pi/2), \quad \kappa \in (0, 1),
\]

where the function \( \psi_0(\mu, w; x) \) belonging to \( D_{a^+(\mu, w)} \) is given by

\[
\psi_0(\mu, w; x) = \frac{1}{\phi(\mu, w; x)} \zeta(x), \quad \text{so that } \tilde{b}(\mu, w)\psi_0(\mu, w; x) = -\frac{1}{\phi(\mu, w; x)} \zeta'(x),
\]

(47)

\( \zeta(x) \) is a fixed smooth function with a compact support and equal to 1 in a neighborhood of the origin, and \( \tilde{D}_{a^+(\mu, w)} \) is the subspace of functions belonging to \( D_{a^+(\mu, w)} \) and vanishing at the origin:

\[
\tilde{D}_{a^+(\mu, w)} = \{ \psi(x) \in D_{a^+(\mu, w)} : \psi(x) = O(x^{1/2}), \quad x \to 0 \} , \quad \mu \in [0, \pi/2), \quad \kappa \in (0, 1).
\]

(48)

ii) The domain \( D_{b^+(\mu, w)} \) of the operator \( \hat{b}^+(\mu, w) \) is described quite similarly. It is the natural domain for \( \hat{a}(\mu, w) = \bar{b}^*(\mu, w) \):  

\[
D_{b^+(\mu, w)} = D_{\tilde{a}^+(\mu, w)} = \{ \chi(x) : \chi(x) \text{ is a.c. in } \mathbb{R}_+; \\
\chi(x), \alpha(\mu, w)\chi(x) = \phi(\mu, w; x)dx \left( \frac{1}{\phi(\mu, w; x)} \chi(x) \right) = \eta(x) \in L^2(\mathbb{R}_+) \}.
\]

(49)
Using arguments similar to those in the previous item i) for a function \( \psi(x) \) belonging to \( D_{b^+}(\mu, w) \) with the natural interchange \( \phi(\mu, w) \leftrightarrow 1/\phi(\mu, w) \), we establish that a generic function \( \chi(x) \) belonging to \( D_{b^+}(\mu, w) \) allows the representation

\[
\chi(x) = \phi(\mu, w; x) \left[ D + \int_{x_0}^{x} dy \frac{1}{\phi(\mu, w; y)} \eta(y) \right]
\]

where the point \( x_0 \) and constant \( D \) depend on the values of \( \mu \) and \( \kappa \) as follows:

\[
\begin{align*}
\mu &= 0 : x_0 = 0 \text{ and } \left\{ \begin{array}{l}
D = 0 \text{ for } \kappa \geq 1 \\
D \text{ is an arbitrary constant for } \kappa \in (0, 1) \\
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\mu &\in (0, \pi/2), x_0 = 0 \text{ and } \left\{ \begin{array}{l}
D = \int_0^\infty dy \frac{1}{\phi(\mu, w; y)} \eta(y) = 0 \text{ for } \kappa \geq 1 \\
D = -\int_0^\infty dy \frac{1}{\phi(\mu, w; y)} \eta(y), \ \kappa \in (0, 1) \\
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\mu &= \pi/2 : 0 < x_0 < \infty \text{ and } D = -\int_{x_0}^\infty dy \frac{1}{\phi(\mu, w; y)} \eta(y), \forall \kappa > 0.
\end{align*}
\]

A subtlety is that for \( \mu \in [0, \pi/2), \kappa \in (0, 1) \) and for \( \mu = \pi/2, \forall \kappa > 0 \), the constant \( D \) in (50) can take arbitrary values, but for \( \mu = 0 \), the constant \( D \) is independent of the function \( \eta(x) \), while for \( \mu \in (0, \pi/2), \kappa \in (0, 1) \) and for \( \mu = \pi/2, \forall \kappa > 0 \), it is uniquely related to \( \eta \).

All this implies that representation (50), (51) for \( \chi(x) \in D_{b^+}(\mu, w) \) with \( \mu \in (0, \pi/2], \forall \kappa > 0 \), is equivalent to

\[
\begin{align*}
\chi(x) &= -\phi(\mu, w; x) \int_x^\infty dy \frac{1}{\phi(\mu, w; y)} \eta(y), \ \eta(x) = \tilde{a}(\mu, w) \chi(x) \in L^2(\mathbb{R}_+),
\end{align*}
\]

where \( \mu \in (0, \pi/2], \forall \kappa > 0 \).

The asymptotic behavior of functions \( \chi(x) \in D_{b^+}(\mu, w) \) at the origin and at infinity estimated using (32)-(35) and the Cauchy–Bunyakovskii inequality for integral terms in (50) and (52) is respectively given by

\[
\chi(x) = \begin{cases} 
O(x^{1/2}), & \forall \mu, \kappa \geq 1 \text{ or } \mu = \pi/2, \ \kappa \in (0, 1) \\
D\tilde{B}(\mu, w)(\psi x)^{1/2-\kappa}[1 + O(x^{2\kappa})] + O(x^{1/2}), & \mu \in [0, \pi/2), \ \kappa \in (0, 1)
\end{cases}
\]

and by

\[
\chi(x) \to 0, x \to \infty, \ \forall \mu, \forall \kappa > 0.
\]

As it follows from (54), (51) with \( \eta(x) = 0 \), the kernel of \( \hat{b}^+(\mu, w) \) is nontrivial only for \( \mu = 0, \kappa \in (0, 1) \):

\[
\text{ker} \hat{b}^+(\mu, w) = \begin{cases} 
\{0\}, & \forall \mu, \kappa \geq 1 \text{ or } \mu \in (0, \pi/2], \kappa \in (0, 1) \\
\left\{ c \left( \phi(0, w; x) = e^{-\psi x^2/2}(\psi x)^{1/2+\kappa} \frac{\Gamma(\alpha)}{\Gamma(\beta)} \Psi(\alpha, \beta; (\psi x)^2) \right) \right\}, & \mu = 0, \kappa \in (0, 1)
\end{cases}
\]

A simple reason is that \( \hat{b}^+(\mu, w)\chi(x) = \tilde{a}(\mu, w)\chi(x) = 0 \Rightarrow \chi(x) = c\phi(\mu, w; x) \) and \( \phi(\mu, w; x) \) is square integrable on \( \mathbb{R}_+ \) only for \( \mu = 0 \) and \( \kappa \in (0, 1) \), see (32) and (34).
For \( \nu \in (0, 1) \), the domain \( D_{b^+(\mu, w)} \) of the operator \( \hat{b}^+(\mu, w) \) with \( \mu \in [0, \pi/2] \) can be represented as a direct sum of the form

\[
D_{b^+(\mu, w)} = \{ D\chi(0, \mu, w; x) \} + \hat{D}_{b^+(\mu, w)}, \mu \in [0, \pi/2], \nu \in (0, 1),
\]

where the function \( \chi(0, \mu, w; x) \) belonging to \( D_{b^+(\mu, w)} \) is given by

\[
\chi(0, \mu, w; x) = \phi(\mu, w; x)\zeta(x), \text{ so that } \hat{a}(\mu, w)\chi(0, \mu, w; x) = \phi(\mu, w; x)\zeta'(x),
\]

\( \zeta(x) \) is a fixed smooth function with a compact support equal to 1 in a neighborhood of the origin, and \( \hat{D}_{b^+(\mu, w)} \) is the subspace of functions belonging to \( D_{b^+(\mu, w)} \) and vanishing at the origin:

\[
\hat{D}_{b^+(\mu, w)} = \{ \chi(x) \in D_{b^+(\mu, w)} : \chi(x) = O(x^{1/2}), \, x \to 0 \}, \mu \in [0, \pi/2], \nu \in (0, 1).
\]  

iii) The operator \( \overline{a}(\mu, w) \) is evaluated in accordance with (41): as a restriction of \( \hat{b}^+(\mu, w) \), this operator is associated with \( \hat{a}(\mu, w) \) and its domain belongs to or coincides with \( D_{b^+(\mu, w)} \), while the defining equation for \( \overline{a}(\mu, w) \) as \( (\hat{a}^+(\mu, w))^+ \),

\[
(\chi, \hat{a}^+(\mu, w)\psi) - (\overline{a}(\mu, w)\chi, \psi) = 0, \chi(x) \in D_{\overline{a}(\mu, w)}, \forall \psi(x) \in D_{a^+(\mu, w)},
\]

is reduced to the equation for \( D_{\overline{a}(\mu, w)} \), i.e., for the functions \( \chi(x) \in D_{\overline{a}(\mu, w)} \subseteq D_{b^+(\mu, w)} \), of the form

\[
(\chi, \hat{b}(\mu, w)\psi) - (\hat{a}(\mu, w)\chi, \psi) = 0, \chi(x) \in D_{\overline{a}(\mu, w)} \subseteq D_{b^+(\mu, w)}, \forall \psi(x) \in D_{a^+(\mu, w)}.
\]  

Integrating by parts in \( (\hat{a}(\mu, w)\chi, \psi) \) and taking estimates (40), (47) and (53), (54) into account, we establish that for \( \mu \in [0, \pi/2], \nu \geq 1 \) and for \( \mu = \pi/2, \forall \nu > 0 \), eq. (58) holds identically for all \( \chi(x) \in D_{b^+(\mu, w)} \), while for \( \mu \in [0, \pi/2], \nu \in (0, 1) \), eq. (58) is reduced to

\[
\overline{D}C = 0, \forall C,
\]

which requires that \( D = 0 \).

We finally obtain that

\[
\overline{a}(\mu, w) = \hat{b}^+(\mu, w), \, D_{\overline{a}(\mu, w)} = D_{b^+(\mu, w)} \] (49),

for \( \forall \mu, \nu \geq 1 \) and \( \mu = \pi/2, \nu \in (0, 1) \),

and

\[
\overline{a}(\mu, w) \subset \hat{b}^+(\mu, w), \, D_{\overline{a}(\mu, w)} = \hat{D}_{b^+(\mu, w)} \] (57), for \( \mu \in [0, \pi/2], \nu \in (0, 1) \),

which, in particular, implies that for \( \nu \in (0, 1) \), the asymptotic behavior at the origin of the functions belonging to the domain of any operator \( \overline{a}(\mu, w) \) is estimated as \( O(x^{1/2}) \),

\[
D_{\overline{a}(\mu, w)} \ni \chi(x) = O(x^{1/2}), \, x \to 0, \forall \mu, \forall w > w_0, \nu \in (0, 1),
\]

for \( \mu = \pi/2 \) this follows from (59) and (53), while for \( \mu \in [0, \pi/2) \) this follows from (60) and (57).

The kernel of any operator \( \overline{a}(\mu, w) \) is trivial,

\[
\ker \overline{a}(\mu, w) = \{ 0 \}, \forall \mu, \forall w > w_0, \forall \nu > 0,
\] (62)
for \( x \geq 1 \) and \( \mu = \pi/2, x \in (0, 1) \) this follows from (59) and (55), while for \( \mu \in [0, \pi/2), x \in (0, 1) \), this follows from that \( \tilde{a}(\mu, w) \chi(x) = \tilde{a}(\mu, w) \chi(x) = 0 \Rightarrow \chi(x) = c \phi(\mu, w; x) \), but the function \( \phi(\mu, w; x) \) is estimated at the origin as \( O(x^{1/2-x}) \), see (52), and therefore cannot belong to \( D_{\tilde{a}(\mu, w)} \) according to (61) (in addition, \( \phi(\mu, w; x) \) with \( \mu \in (0, \pi/2) \) is not square integrable on \( \mathbb{R}_+ \), see (54)).

iv) Quite similarly, we find

\[
\tilde{b}(\mu, w) = \hat{a}^+(\mu, w), \quad D_{\tilde{b}(\mu, w)} = D_{\hat{a}^+(\mu, w)} \quad \text{(63)}
\]

and

\[
\tilde{b}(\mu, w) \subset \tilde{a}^+(\mu, w), \quad D_{\tilde{b}(\mu, w)} = \tilde{D}_{\tilde{a}^+(\mu, w)} \quad \text{(64)}
\]

which, in particular, implies that for \( x \in (0, 1) \), the asymptotic behavior at the origin of the functions belonging to the domain of any operator \( \tilde{b}(\mu, w) \) with \( \mu \in [0, \pi/2) \) is estimated as \( O(x^{1/2}) \),

\[
D_{\tilde{b}(\mu, w)} \ni \psi(x) = O(x^{1/2}), \quad x \to 0, \quad \text{for} \quad \mu \in [0, \pi/2), \quad w \in (w_0, \infty), \quad x \in (0, 1).
\]

We note that equality (63) and inclusion (64) directly follow from the respective previous equality (59) and inclusion (60) by taking the adjoints, and only the domain \( D_{\tilde{b}(\mu, w)} \) in the last case has to be evaluated.

It is easy to prove that there are no other pairs of closed mutually adjoint operators that are extensions of each pair \( \hat{a}(\mu, w), \tilde{b}(\mu, w) \). Indeed, let \( \hat{g}, \hat{g}^+ \) be such a pair, then because \( \tilde{a}(\mu, w) \) and \( \tilde{b}(\mu, w) \) are minimum closed extensions of the respective \( \hat{a}(\mu, w) \) and \( \hat{b}(\mu, w) \), we have

\[
\hat{a}(\mu, w) \subset \tilde{a}(\mu, w) \subset \hat{g} = \hat{g}^+ + \hat{b}(\mu, w) \subset \tilde{b}(\mu, w) \subset \hat{g}^+.
\]

It follows by taking the adjoints of these inclusions that

\[
\hat{g}^+ \subseteq \hat{a}^+(\mu, w), \quad \hat{g} \subseteq \tilde{b}^+(\mu, w),
\]

so that we finally have

\[
\tilde{a}(\mu, w) \subseteq \hat{g} \subseteq \tilde{b}^+(\mu, w),
\]

in particular,

\[
D_{\tilde{a}(\mu, w)} \subseteq D_{\hat{g}} \subseteq D_{\tilde{b}^+(\mu, w)}.
\]

It then directly follows from (59) that \( \hat{g} = \tilde{a}(\mu, w) = \tilde{b}^+(\mu, w) \) and therefore \( \hat{g}^+ = \tilde{b}(\mu, w) = \hat{a}^+(\mu, w) \) for \( \mu \in [0, \pi/2), x \geq 1 \) and for \( \mu = \pi/2, x \geq 0 \), while for \( \mu \in [0, \pi/2), x \in (0, 1) \), it follows from (60), (56) that the domains \( D_{\tilde{b}^+(\mu, w)} \) and \( D_{\tilde{a}(\mu, w)} = D_{\tilde{b}(\mu, w)} \) differ by a one-dimensional subspace, so that either \( \hat{g} = \tilde{a}(\mu, w) \), and therefore \( \hat{g}^+ = \hat{a}^+(\mu, w) \), or \( \hat{g} = \tilde{b}^+(\mu, w) \), and therefore \( \hat{g}^+ = \tilde{b}(\mu, w) \).

We thus show that each pair \( \hat{a}(\mu, w), \tilde{b}(\mu, w), \mu \in [0, \pi/2), w \in (w_0, \infty) \), \( w_0 = -\frac{1}{2}(1+x), x > 0 \), of mutually adjoint by Lagrange differential operations (30) providing generalized oscillator representation (31) with \( w \in (w_0, \infty) \) for \( \hat{H} \) (11) with \( g_1 > -1/4 \) and \( g_2 > 0 \) generates a unique pair \( \tilde{a}(\mu, w) = \hat{b}^+(\mu, w), \hat{a}^+(\mu, w) = \tilde{b}(\mu, w) \) of closed mutually adjoint operators for \( \mu \in [0, \pi/2), x \geq 1 \) and for \( \mu = \pi/2, x > 0 \), while for \( \mu \in [0, \pi/2), x \in (0, 1) \), each pair \( \hat{a}(\mu, w), \tilde{b}(\mu, w) \) generates two different pairs \( \tilde{a}(\mu, w), \hat{a}^+(\mu, w) \) and \( \tilde{b}^+(\mu, w), \tilde{b}(\mu, w) \).
of closed mutually adjoint operators such that \( \tilde{a}(\mu, w) \subset \tilde{b}^+(\mu, w) \) and \( \tilde{b}(\mu, w) \subset \tilde{a}^+(\mu, w) \). The operators \( \tilde{a}(\mu, w) \) and \( \tilde{b}^+(\mu, w) \) are extensions of the initial operator \( \hat{a}(\mu, w) \), they are associated with \( \hat{a}(\mu, w) \), and their domains are given by the respective (59), (60) and (49). The operators \( \hat{b}(\mu, w) \) and \( \hat{a}^+(\mu, w) \) are extensions of the initial operator \( \hat{b}(\mu, w) \), they are associated with \( \hat{b}(\mu, w) \), and their domains are given by the respective (63), (64) and (42).

2.3 Region \( g_1 > -1/4 \ (\mathcal{N} > 0) \), \( w = w_0 = -\frac{1}{2}(1 + \mathcal{N}) \ (\alpha = 0) \)

A consideration for this region of parameters follows the standard scheme presented in the previous subsection 2.2 where \( w > w_0 \). The distinctive feature is that here, we encounter a unique pair of differential operations \( \hat{a}(u) \) and \( \hat{b}(u) \) providing generalized oscillator representation (6) with \( u = u_0 = 4v^2w_0 \) for \( \hat{H} \) with \( g_1 > -1/4 \) and \( g_2 > 0 \).

2.3.1 Generalized oscillator representations for \( \hat{H} \), differential operations \( \hat{a} \) and \( \hat{b} \)

In this region of parameters, the general solution of differential equation (9) with \( u = u_0 = 4v^2w_0 \) is given by (28). The function \( \Psi(\beta, \rho) \) given by (25) increases monotonically from \(-\infty \) to \( \infty \) as \( \rho = (vx)^2 \) together with \( x \) ranges from 0 to \( \infty \). It follows that eq. (9) with these values of parameters has a unique, up to a positive constant factor, real-valued positive solution

\[
\phi(w_0; x) = e^{-\rho/2} \rho^{1/4+\mathcal{N}/2}, \quad \rho = (vx)^2,
\]

which implies, see the text after (11), that in this case, we have a unique \(^{11}\) pair of mutually adjoint first-order differential operations \( \hat{a}(w_0) \) and \( \hat{b}(w_0) \) given by (11) with the evident substitutions \( \hat{a}(u) \rightarrow \hat{a}(w_0), \hat{b}(u) \rightarrow \hat{b}(w_0), h(u; x) \rightarrow h(w_0; x) \) and \( \phi(u; x) \rightarrow \phi(w_0; x) \):

\[
\hat{a}(w_0) = \phi(w_0; x)d_x \frac{1}{\phi(w_0; x)} = \hat{b}^*(w_0),
\]

\[
\hat{b}(w_0) = -\frac{1}{\phi(w_0; x)}d_x \phi(w_0; x) = \hat{a}^*(w_0), \quad \mathcal{N} > 0,
\]

and providing unique generalized oscillator representation (6) with \( u = 4v^2w_0 \) for generalized Calogero differential operation \( \hat{H} \) with \( g_1 > -1/4 \) and \( g_2 > 0 \):

\[
\hat{H} = -d_x^2 + g_1 x^{-2} + g_2 x^2 = \hat{b}(w_0)\hat{a}(w_0) - 4v^2w_0, \quad \mathcal{N} > 0.
\]

The asymptotic behavior of the functions \( \phi(w_0; x) \) and \( 1/\phi(w_0; x) \) at the origin and at infinity is evident from (66).

We note that all the results to follow in this subsection can be obtained from the corresponding results of the previous subsection for \( \mu = \pi/2 \) in the limit \( w \rightarrow w_0 \).

2.3.2 Initial operators \( \hat{a} \) and \( \hat{b} \)

Following subsubsection 2.2.2, we introduce the pair of initial differential operators \( \hat{a}(w_0) \) and \( \hat{b}(w_0) \) in \( \mathbb{L}^2(\mathbb{R}_+) \) associated with respective differential operations \( \hat{a}(w_0) \) and \( \hat{b}(w_0) \) (67) and

\(^{11}\) Up to irrelevant phase factors.
defined on the subspace \( \mathcal{D}(\mathbb{R}_+) \) of smooth compactly supported functions, \( D_{a(w_0)} = D_{b(w_0)} = \mathcal{D}(\mathbb{R}_+) \). These operators satisfy the relation
\[
(\psi, \hat{a}(w_0)\xi) = (\hat{b}(w_0)\psi, \xi), \quad \forall \psi(x), \xi(x) \in \mathcal{D}(\mathbb{R}_+).
\] (69)

The representation (68) for \( \hat{H} \) provide the representation
\[
\hat{H} = \hat{b}(w_0) \hat{a}(w_0) - 4v^2 w_0 \hat{I}, \quad \forall \kappa > 0,
\] (70)
for the initial symmetric operator \( \hat{H} \) associated with \( \hat{H} \) and defined on \( \mathcal{D}(\mathbb{R}_+) \), which, in particular, confirms that \( \hat{H} \) with \( g_1 > -1/4 \), \( g_2 > 0 \) is bounded from below by \(-4v^2w_0 = 2v^2(1 + \kappa)\),
\[
\left( \xi, \hat{H}\xi \right) = (\hat{a}(w_0)\xi, \hat{a}(w_0)\xi) - 4v^2 w_0(\xi, \xi) \geq 2v^2(1 + \kappa) (\xi, \xi), \quad \forall \xi(x) \in \mathcal{D}(\mathbb{R}_+).
\]

This representation is a basis for constructing, maybe new, s.a. generalized Calogero Hamiltonians \( \hat{H}_\kappa \) as s.a. extensions of \( \hat{H} \) in generalized oscillator form (7) or (8) via constructing all possible extensions of the pair \( \hat{a}(w_0), \hat{b}(w_0) \) of initial operators to a pair of closed mutually adjoint operators \( \hat{c}(w_0), \hat{\mu}(w_0) \), \( \hat{\mu}(w_0) \subset \hat{c}(w_0) \), \( \hat{a}(w_0) \subset \hat{c}(w_0) \), \( \hat{b}(w_0) \subset \hat{c}(w_0) \), beginning from the closure \( \hat{a}(w_0) \) of \( \hat{a}(w_0) \), or to a pair of closed mutually adjoint operators \( \hat{d}(w_0), \hat{d}^+(w_0) \), \( \hat{d}(w_0) \subset \hat{d}^+(w_0) \), \( \hat{b}(w_0) \subset \hat{d}(w_0) \), beginning from the closure \( \hat{b}(w_0) \) of \( \hat{b}(w_0) \).

### 2.3.3 Adjoint operators \( \hat{a}^+ \) and \( \hat{b}^+ \), closed operators \( \hat{\overline{a}} \) and \( \hat{\overline{b}} \)

By arguments completely similar to those in the previous subsecs. 2.2.3 and 2.2.4, we establish that the closures \( \hat{\overline{a}}(w_0) \) and \( \hat{\overline{b}}(w_0) \) of the operators \( \hat{a}(w_0) \) and \( \hat{b}(w_0) \), as well as the adjoints \( \hat{a}^+(w_0) \) and \( \hat{b}^+(w_0) \) of the latter, do exist and the chains of inclusions
\[
\hat{a}(w_0) \subset \hat{\overline{a}}(w_0) \subset (\hat{a}^+(w_0))^+ \subset \hat{b}^+(w_0), \\
\hat{b}(w_0) \subset \hat{\overline{b}}(w_0) \subset (\hat{b}^+(w_0))^+ \subset \hat{a}^+(w_0),
\] (71)
hold. The operators \( \hat{\overline{a}}(w_0) \) and \( \hat{\overline{b}}(w_0) \) are associated with \( \hat{a}(w_0) \), while the operators \( \hat{\overline{b}}(w_0) \) and \( \hat{\overline{a}}(w_0) \) are associated with \( \hat{b}(w_0) \), so that to specify these operators, it is sufficient to evaluate their domains following the method in subsec. 2.2.4.

### 2.3.4 Domains of operators \( \hat{a}^+ \), \( \hat{b}^+ \), \( \hat{\overline{a}} \) and \( \hat{\overline{b}} \)

i) The domain of the operator \( \hat{a}^+(w_0) \) is the natural domain for \( \hat{b}(w_0) \), which is given by a copy of (42) with the substitution \( \hat{b}(\mu, w) \rightarrow \hat{b}(w_0) \):
\[
D_{\hat{a}^+(w_0)} = D_{\hat{b}^+(w_0)} = \left\{ \psi(x) : \psi \text{ is a.c. in } \mathbb{R}_+, \psi(x), \hat{b}(w_0)\psi(x) = -\frac{1}{\phi(w_0; x)} \frac{d}{dx} (\phi(w_0; x)\psi(x)) = \eta(x) \in L^2(\mathbb{R}_+) \right\}.
\] (72)

By arguments similar to those in the item i), subsec. 2.2.4, it follows that a generic function \( \psi(x) \) belonging to \( D_{\hat{a}^+(w_0)} \) allows the representations
\[
\psi(x) = -\frac{1}{\phi(w_0; x)} \int_0^x dy \phi(w_0; y) \eta(y) = \frac{1}{\phi(w_0; x)} \int_x^\infty dy \phi(w_0; y) \eta(y), \\
\eta(x) = \hat{b}(w_0)\psi(x) \in L^2(\mathbb{R}_+), \text{ and } \int_0^\infty dy \phi(w_0; y) \eta(y) = 0.
\] (73)
The latter equality in (73) means that the range $R_{a^+(w_0)}$ of the operator $\hat{a}^+(w_0)$ is orthogonal to the one-dimensional subspace \( \{c\phi(w_0; x) \} \subset L^2(\mathbb{R}^+), \) $R_{a^+(w_0)} \perp \{c\phi(w_0; x) \}$, and thereby its closure $\overline{R_{a^+(w_0)}}$ cannot be the whole $L^2(\mathbb{R}^+)$, which in turn implies that the kernel of the adjoint operator $(\hat{a}^+(w_0))^* = \overline{\overline{a}}(w_0)$ is not trivial, \( \{c\phi(w_0; x) \} \subseteq \ker \overline{\overline{a}}(w_0) \neq \{0\} \), see below.

Estimating the integral terms in (73) with the Cauchy–Bunyakovskii inequality, we obtain that the asymptotic behavior of functions $\psi(x) \in D_{a^+(w_0)}$ at the origin and at infinity is respectively given by
\[
\psi(x) = O(x^{1/2}), \ x \to 0, \\
\psi(x) \to 0, \ x \to \infty, \ \forall \varkappa > 0.
\] (74)

ii) The domain $D_{b^+(w_0)}$ of the operator $\hat{b}^+(w_0)$ is the natural domain for $\hat{a}(w_0)$, which is given by a copy of (49) with the substitution $\hat{a}(\mu, w) \to \hat{a}(w_0)$:
\[
D_{b^+(w_0)} = D_{\hat{a}(w_0)}^0 = \{\chi(x) : \chi \text{ is a.c. in } \mathbb{R}^+, \chi(x) = \phi(w_0; x)d_x \left( \frac{1}{\phi(w_0; x)} \chi(x) \right) = \eta(x) \in L^2(\mathbb{R}^+) \},
\] (75)
whence it follows that a generic function $\chi(x) \in D_{b^+(w_0)}$ allows the representation
\[
\chi(x) = \phi(w_0; x)[D + \int_{x_0}^x dy \frac{1}{\phi(w_0; y)}\eta(y)], \ \eta(x) = \hat{b}(w_0)\chi(x) \in L^2(\mathbb{R}^+),
\] (76)
where $x_0 \in (0, \infty)$ and $D$ is an arbitrary constant, and its asymptotic behavior at the origin and at infinity estimated using the Cauchy–Bunyakovskii inequality is respectively given by
\[
\chi(x) = O(x^{1/2}), \ x \to 0, \\
\chi(x) \to 0, \ x \to \infty, \ \forall \varkappa > 0.
\] (77)

As it follows from (76) with $\eta(x) = 0$ and (66), the kernel of the operator $\hat{b}^+(w_0)$ is nontrivial,
\[
\ker \hat{b}^+(w_0) = \{c \left( \phi(w_0; x) = (\nu x)^{1/2+\varkappa}e^{-(\nu x)^2/2} \right) \}, \ \forall \varkappa > 0.
\]

iii) The domain $D_{\overline{a}(w_0)}$ of the operator $\overline{a}(w_0)$ is evaluated in accordance with (71) using arguments similar to those in the item iii), subsubsec. 2.2.4: the defining equation for $\overline{a}(w_0)$ as $(\hat{a}^+)^*(w_0)$, which is a restriction of $\hat{b}^+(w_0)$, is reduced to the equation for $D_{\overline{a}(w_0)} \subseteq D_{b^+(w_0)}$ of the form
\[
(\chi, \hat{b}(w_0)\psi) - (\hat{a}(w_0)\chi, \psi) = 0, \ \chi(x) \in D_{\overline{a}(w_0)} \subseteq D_{b^+(w_0)}, \ \forall \psi(x) \in D_{a^+(w_0)}.
\] (78)

Integrating by parts in $(\hat{a}(w_0)\chi, \psi)$ and taking asymptotic estimates (73) and (77) into account, we establish that eq. (78) holds identically for all $\chi(x) \in D_{a^+(w_0)}$, which implies that
\[
\overline{a}(w_0) = \hat{b}^+(w_0), \ D_{\overline{a}(w_0)} = D_{b^+(w_0)} (75), \ \forall \varkappa > 0,
\] (79)
in particular, the asymptotic behavior at the origin of the functions $\chi(x)$ belonging to the domain of the operator $\overline{a}(w_0)$ is estimated by a copy of (77),
\[
\chi(x) = O(x^{1/2}), \ x \to 0, \ \forall \varkappa > 0,
\] (80)
\[ \ker \overline{a}(w_0) = \{ c \left( \phi(w_0; x) = (\nu x)^{1/2+\kappa}e^{-(\nu x)^2/2} \right) \}, \forall \kappa > 0. \]  

(81)

iv) It directly follows from (79) by taking the adjoints that

\[ \overline{b}(w_0) = \hat{a}^+(w_0), \quad D_{\overline{b}(w_0)} = D_{\hat{a}^+(w_0)} \quad (72), \quad \forall \kappa > 0. \]  

(82)

By arguments similar to those in the end of subsubsec. 2.2.4, it is easy to prove that there is no other pair \( \hat{g} = \overline{g} \) and \( \hat{g}^+ \) of closed mutually adjoint operators that are extensions of the respective \( \hat{a}(w_0) \) and \( \hat{b}(w_0) \). Let \( \hat{a}(w_0) \subset \hat{g} \) and \( \hat{b}(w_0) \subset \hat{g}^+ \).

We thus show that the pair \( \hat{a}(w_0), \hat{b}(w_0) \) of mutually adjoint by Lagrange differential operations (67) providing unique generalized oscillator representation (68) for \( \hat{H} \) (1) with \( g_1 > -1/4, g_2 > 0 \) generates a unique pair \( \overline{a}(w_0) = \hat{b}^+(w_0), \quad \hat{a}^+(w_0) = \overline{b}(w_0) \) of closed mutually adjoint operators. The operator \( \overline{a}(w_0) = \hat{b}^+(\mu, w) \) is an extension of the initial operator \( \hat{a}(w_0) \), it is associated with \( \hat{a}(w_0) \), and its domain is given by (75). The operator \( \overline{b}(w_0) = \hat{a}^+(w_0) \) is an extension of the initial operator \( \hat{b}(w_0) \), it is associated with \( \hat{b}(w_0) \), and its domain is given by (72).

\section*{2.4 Region} \( g_1 = -1/4 \ (\kappa = 0), \ w > w_0 = -1/2 \ (\alpha > 0) \)

A consideration for this region of parameters literally follows the scheme of subsec. 2.2, we even do not change the notation having in mind that here \( \kappa = 0, \beta = 1 \).

The only distinction is in a specific asymptotic behavior of the real-valued positive functions \( \phi_1 \) and \( \phi_2 \) (27) with \( \beta = 1 \), the fundamental solutions of eq. (9) with \( g_1 = -1/4 \ (\kappa = 0) \), at the origin.

\subsection*{2.4.1 Generalized oscillator representations for} \( \hat{H} \), \ differential operations \( \hat{a} \) and \( \hat{b} \)

The general real-valued positive solution of eq. (9) with \( g_1 = -1/4 \ (\kappa = 0) \) and \( u = 4v^2w > -2v^2 \), defined modulo a positive constant factor, is given by

\[ \phi(\mu, w; x) = e^{-\rho^2/2}\rho^{1/4} \left[ \Phi(\alpha, 1; \rho) \sin \mu + \Gamma(\alpha) \Psi(\alpha, 1; \rho) \cos \mu \right], \]  

(83)

\[ \mu \in [0, \pi/2], \quad \alpha = 1/2 + w > 0, \quad \rho = (ux)^2, \]  

where \( \Phi(\alpha, 1; \rho) \) and \( \Psi(\alpha, 1; \rho) \) are given by the respective (16) with \( \beta = 1 \) and (18).

This implies that in this case, we have the two-parameter family \{ \( \hat{a}(\mu, w), \hat{b}(\mu, w); \mu \in [0, \pi/2], \ w \in (-1/2, \infty) \) \} of different pairs of mutually adjoint first-order differential operations \( \hat{a}(\mu, w) \) and \( \hat{b}(\mu, w) \) given by a copy of (30) with \( \phi(\mu, w; x) \) (83) instead of \( \phi(\mu, w; x) \) (29) and providing a two-parameter family of different generalized oscillator representations for generalized Calogero differential operation \( \hat{H} \) (1) with \( g_1 = -1/4, g_2 > 0, \)

\[ \hat{H} = -d_x^2 - \frac{1}{4}x^{-2} + g_2x^2 = \hat{b}(\mu, w)\hat{a}(\mu, w) - 4v^2w, \]  

\[ \mu \in [0, \pi/2], \ w \in (-1/2, \infty), \]  

(84)
which are copies of (31) with \( \alpha = 0 \). According to (21), (22) and (23), (24), the asymptotic behavior of functions \( \phi(\mu, w; x) \) (83) and \( 1/\phi(\mu, w; x) \) at the origin is respectively given by

\[
\phi(\mu, w; x) = \begin{cases} 
\tilde{A} (v x)^{1/2} + \tilde{B} (v x)^{1/2} \ln(v x) + O(x^{5/2} \ln x), & \mu \in [0, \pi/2), \ x \to 0, \\
(v x)^{1/2} + O(x^{5/2}), & \mu = \pi/2 
\end{cases}
\]

\( \tilde{A} = \tilde{A}(\mu, w) = \sin \mu \cos \mu (2 \psi(1) - \psi(\alpha)), \alpha = 1/2 + w > 0, \)
\( \tilde{B} = \tilde{B}(\mu, w) = -2 \cos \mu, \)

where \( \psi(z) = \Gamma'(z)/\Gamma(z), \) and

\[
\frac{1}{\phi(\mu, w; x)} = \begin{cases} 
\frac{1}{B(\mu, w)} (v x)^{-1/2} + O(x^{3/2} \ln x), & \mu \in [0, \pi/2), \ x \to 0, \\
(v x)^{-1/2} + O(x^{3/2}), & \mu = \pi/2 
\end{cases}
\]

while their asymptotic behavior at infinity is respectively given by

\[
\phi(\mu, w; x) = \begin{cases} 
\sin \mu \frac{1}{\Gamma(1/2+w)} (v x)^{-1/2+2w} e^{1/2(v x)^2} (1 + O(x^{-2})), & \mu \in (0, \pi/2] \\
\Gamma(1/2+w) (v x)^{-1/2-2w} e^{-1/2(v x)^2} (1 + O(x^{-2})), & \mu = 0 
\end{cases}, \ x \to \infty 
\]

and

\[
\frac{1}{\phi(\mu, w; x)} = \begin{cases} 
\frac{\Gamma(1/2+w)}{\sin \mu} (v x)^{1/2-2w} e^{-1/2(v x)^2} (1 + O(x^{-2})), & \mu \in (0, \pi/2] \\
\frac{1}{\Gamma(1/2+w)} (v x)^{1/2+2w} e^{1/2(v x)^2} (1 + O(x^{-2})), & \mu = 0 
\end{cases}, \ x \to \infty 
\]

### 2.4.2 Initial operators \( \hat{a} \) and \( \hat{b} \)

In perfect analogy to subsubsec. 2.2.2, we introduce the pairs \( \hat{a}(\mu, w), \hat{b}(\mu, w) \) of initial differential operators defined on \( \mathcal{D}(\mathbb{R}_+) \) and associated with the respective pairs of differential operations \( \tilde{a}(\mu, w), \tilde{b}(\mu, w) \). These operators have the property that is the copy of (36). The representations (83) imply that the initial symmetric operator \( \hat{H} \) defined on \( \mathcal{D}(\mathbb{R}_+) \) and associated with \( \hat{H} \) can be represented as

\[
\hat{H} = \hat{b}(\mu, w) \hat{a}(\mu, w) - 4v^2 w \hat{I}, \forall \mu \in [0, \pi/2], \forall w > -1/2, \tag{89}
\]

which, in particular, implies that \( \hat{H} \) with \( g = -1/4, g_2 > 0 \) is bounded from below by \( 2v^2 \).

The representations (83) is a basis for constructing s.a. generalized Calogero Hamiltonians \( \hat{H}_s \) with \( g = -1/4, g_2 > 0 \) as s.a. extensions of \( \hat{H} \) in generalized oscillator form coping (38) or (39) via constructing all possible extensions of each pair \( \hat{a}(\mu, w), \hat{b}(\mu, w) \) of initial operators respectively to a pair of closed mutually adjoint operators \( \hat{\tilde{c}}(\mu, w), \hat{\tilde{c}}^+(\mu, w) \) beginning from the pair \( \hat{\tilde{a}}(\mu, w), \hat{\tilde{a}}^+(\mu, w) \), the closure of \( \hat{\tilde{a}}(\mu, w) \) and its adjoint, or to a pair of closed mutually adjoint operators \( \hat{\tilde{d}}(\mu, w), \hat{\tilde{d}}^+(\mu, w) \) beginning from the pair \( \hat{\tilde{b}}(\mu, w), \hat{\tilde{b}}^+(\mu, w) \), the closure of \( \hat{\tilde{b}}(\mu, w) \) and its adjoint.

### 2.4.3 Adjoint operators \( \hat{a}^+ \) and \( \hat{b}^+ \), closed operators \( \tilde{a} \) and \( \tilde{b} \)

By arguments repeating those in subsubsec. 2.2.3 and 2.2.4, we establish that all these operators, the closures \( \tilde{a}(\mu, w), \tilde{b}(\mu, w) \), as well as the adjoints \( \hat{a}^+(\mu, w), \hat{b}^+(\mu, w) \), do exist
and satisfy the chains of inclusions

\[ \hat{a}(\mu, w) \subset \overline{a}(\mu, w) = (\hat{a}^+(\mu, w))^+ \subset \hat{b}^+(\mu, w), \]
\[ \check{b}(\mu, w) \subset \check{b}(\mu, w) = (\check{b}^+(\mu, w))^+ \subset \hat{a}^+(\mu, w), \]

the copies of (11).

The operators in the first chain are associated with \( \hat{a}(\mu, w) \), while the operators in the second chain are associated with \( \check{b}(\mu, w) \), and to specify these operators, it is sufficient to evaluate their domains repeating the reasoning in subsubsec. 2.2.4.

2.4.4 Domains of operators \( \hat{a}^+, \check{b}^+, \overline{a} \) and \( \overline{b} \)

In fact, the content of this subsubsection is a copy of that of subsubsec. 2.2.4 in the part related to the case of \( \kappa \in (0, 1) \) with the substitution of \( \phi(\mu, w; x) \) (33) \( \kappa = 0 \) for \( \phi(\mu, w; x) \) (22) \( \kappa \in (0, 1) \).

i) The domain \( D_{\hat{a}^+(\mu, w)} \) of the operator \( \hat{a}^+(\mu, w) \) is the natural domain for \( \check{b}(\mu, w) \), which is given by a copy of (12). A generic \( \psi(x) \) belonging to \( D_{\hat{a}^+(\mu, w)} \) allows the representation

\[ \psi(x) = \frac{1}{\phi(\mu, w; x)} \left[ C - \int_0^x dy \phi(\mu, w; y) \eta(y) \right], \quad \eta(x) = \check{b}(\mu, w) \psi(x) \in L^2(\mathbb{R}_+), \]

where the constant \( C \) depends on the values of \( \mu \) as follows:

\[ \mu = 0 : C = \int_0^\infty dy \phi(\mu, w; y) \eta(y), \]
\[ \mu \in (0, \pi/2) : C \text{ is an arbitrary constant}, \]
\[ \mu = \pi/2 : C = 0. \]

A subtlety is that for \( \mu \in [0, \pi/2) \), the constant \( C \) in (92) can take arbitrary values, but for \( \mu \in (0, \pi/2) \), the constant \( C \) is independent of the function \( \eta(x) \), while for \( \mu = 0 \), it is uniquely related to \( \eta \), so that the representation (91),(92) with \( \mu = 0 \) for \( \psi(x) \in D_{\hat{a}^+(0, w)} \) is equivalent to

\[ \psi(x) = \frac{1}{\phi(0, w; x)} \int_x^\infty dy \phi(0, w; y) \eta(y), \quad \eta(x) = \check{b}(0, w) \psi(x) \in L^2(\mathbb{R}_+), \quad \mu = 0. \]

The asymptotic behavior of functions \( \psi(x) \in D_{\hat{a}^+(\mu, w)} \) at the origin and at infinity estimated using (35)- (38) and the Cauchy–Bunyakovskii inequality for integral terms in (91) and (93) is respectively given by

\[ \psi(x) = \left\{ \begin{array}{ll}
\frac{C}{B(\mu, w)} (\nu x)^{-1/2} [1 + O(1/ \ln x)] + O(x^{1/2}), & \mu \in [0, \pi/2), \quad x \to 0, \\
O(x^{1/2}), & \mu = \pi/2
\end{array} \right. \]

and by

\[ \psi(x) \to 0, \quad x \to \infty, \quad \forall \mu. \]

The domain \( D_{\hat{a}^+(\mu, w)} \) of the operator \( \hat{a}^+(\mu, w) \) with \( \mu \in [0, \pi/2) \) can be represented as a direct sum of the form

\[ D_{\hat{a}^+(\mu, w)} = \{ C \psi_0(\mu, w; x) \} + \hat{D}_{\hat{a}^+(\mu, w)}, \quad \mu \in [0, \pi/2), \]

23
where the function $\psi_0(\mu, w; x)$ belonging to $D^{+}_{a}(\mu, w)$ is given by

$$
\psi_0(\mu, w; x) = \frac{1}{\phi(\mu, w; x)} \zeta(x),
$$

so that

$$
\bar{b}(\mu, w) \psi_0(\mu, w; x) = -\frac{1}{\phi(\mu, w; x)} \zeta'(x),
$$

$\zeta(x)$ is a fixed smooth function with a compact support and equal to 1 in a neighborhood of the origin, and $\bar{D}^{+}_{a}(\mu, w)$ is the subspace of functions belonging to $D^{+}_{a}(\mu, w)$ and vanishing at the origin:

$$
\bar{D}^{+}_{a}(\mu, w) = \{ \psi(x) \in D^{+}_{a}(\mu, w) : \psi(x) = O(x^{1/2}), x \to 0 \}, \mu \in [0, \pi/2).
$$

ii) The domain $D^{+}_{b}(\mu, w)$ of the operator $\hat{b}^{+}(\mu, w)$ is the natural domain for $\bar{a}(\mu, w)$, which is given by a copy of (49). A generic $\chi(x)$ belonging to $D^{+}_{b}(\mu, w)$ allows the representation

$$
\chi(x) = \phi(\mu, w; x) \left[ D + \int_{x_0}^{x} dy \frac{1}{\phi(\mu, w; y)} \eta(y) \right], \eta(x) = \bar{a}(\mu, w; x) \chi(x) \in L^2(\mathbb{R}_+),
$$

where the point $x_0$ and constant $D$ depend on the values of $\mu$ as follows:

$$
\begin{align*}
\mu &= 0 : x_0 = 0 \text{ and } D \text{ is an arbitrary constant}, \\
\mu &\in (0, \pi/2), x_0 = 0 \text{ and } D = -\int_{x_0}^{x} dy \frac{1}{\phi(\mu, w; y)} \eta(y), \\
\mu &= \pi/2 : x_0 \in (0, \infty) \text{ and } D = -\int_{x_0}^{x} dy \frac{1}{\phi(\pi/2, w; y)} \eta(y).
\end{align*}
$$

A subtlety is that the constant $D$ can take arbitrary values, but for $\mu = 0$, the constant $D$ is independent of the function $\eta(x)$, while for $\mu \in (0, \pi/2]$, it is uniquely related to $\eta$, so that representation (97), (98) for $\chi(x) \in D^{+}_{b}(\mu, w)$ with $\mu \in (0, \pi/2]$ is equivalent to

$$
\chi(x) = -\phi(\mu, w; x) \int_{x}^{x_0} dy \frac{1}{\phi(\mu, w; y)} \eta(y), \eta(x) = \bar{a}(\mu, w) \chi(x) \in L^2(\mathbb{R}_+), \mu \in (0, \pi/2].
$$

The asymptotic behavior of functions $\chi(x) \in D^{+}_{b}(\mu, w)$ at the origin and at infinity estimated using (85)-(88) and the Cauchy–Bunyakovskii inequality for integral terms in (97) and (99) is respectively given by

$$
\chi(x) = \left\{ \begin{array}{ll}
D \bar{B}(\mu, w)(ux)^{1/2} \ln(vx) [1 + O(1/\ln x)] + O(x^{1/2} \ln^{1/2} x), & x \to 0, \\
O(x^{1/2} \ln^{1/2} x), & \mu = \pi/2
\end{array} \right.
$$

and by

$$
\chi(x) \to 0, x \to \infty, \forall \mu.
$$

As it follows from (97), (98), the kernel of $\hat{b}^{+}(\mu, w)$ is nontrivial only for $\mu = 0:

$$
\ker \hat{b}^{+}(\mu, w) = \left\{ \begin{array}{ll}
\{ 0 \}, & \mu \in (0, \pi/2] \\
c \left( \phi(0, w; x) = e^{-(ux^2)/2} (v; x)^{1/2} \Gamma(\alpha, 1; (vx)^2) \right), & \mu = 0
\end{array} \right.,
$$

where $\Psi(\alpha, 1; (vx)^2)$ is the Gamma function. A simple reason is that $\hat{b}^{+}(\mu, w) \chi(x) = \bar{a}(\mu, w) \chi(x) = 0 \Rightarrow \chi(x) = c \phi(\mu, w; x)$ and $\phi(\mu, w; x)$ is square integrable on $\mathbb{R}_+$ only for $\mu = 0$, see (85) and (87).
The domain \( D_{b^+(\mu,w)} \) of the operator \( \hat{b}^+(\mu,w) \) with \( \mu \in [0,\pi/2) \) can be represented as a direct sum of the form

\[
D_{b^+(\mu,w)} = \{ D\chi_0(\mu,w)(x) \} + \hat{D}_{b^+(\mu,w)}, \quad \mu \in [0,\pi/2),
\]

where the function \( \chi_0(\mu,w;x) \) belonging to \( D_{b^+(\mu,w)} \) is given by

\[
\chi_0(\mu,w;x) = \phi(\mu,w;x)\zeta(x), \quad \text{so that } \hat{a}(\mu,w)\chi_0(\mu,w;x) = \phi(\mu,w;x)\zeta'(x),
\]

\( \zeta(x) \) is a fixed smooth function with a compact support and equal to 1 in a neighborhood of the origin, and \( \hat{D}_{b^+(\mu,w)} \) is the subspace of functions belonging to \( D_{b^+(\mu,w)} \) and vanishing at the origin as \( O(x^{1/2}\ln^{1/2}x) \):

\[
\hat{D}_{b^+(\mu,w)} = \left\{ \chi(x) \in D_{b^+(\mu,w)} : \chi(x) = O(x^{1/2}\ln^{1/2}x), \ x \to 0 \right\}, \quad \mu \in [0,\pi/2). \tag{103}
\]

iii) The domain \( D_{\tilde{a}(\mu,w)} \) of the operator \( \tilde{a} (\mu, w) \) is evaluated in accordance with (90): the defining equation for \( \tilde{a} (\mu, w) \) as \( (\hat{a}^+(\mu, w))^+ \), which is a restriction of \( \hat{b}^+(\mu, w) \), is reduced to the equation for \( D_{\tilde{a}(\mu,w)} \subseteq D_{b^+(\mu,w)} \) of the form

\[
(\chi, \hat{b}(\mu, w)\psi) - (\hat{a}(\mu, w)\chi, \psi) = 0, \quad \chi(x) \in D_{\tilde{a}(\mu,w)} \subseteq D_{b^+(\mu,w)}, \quad \forall \psi(x) \in D_{a^+(\mu,w)}. \tag{104}
\]

Integrating by parts in \( (\hat{a}(\mu, w)\chi, \psi) \) and taking estimates (94), (95) and (100), (101) into account, we establish that for \( \mu = \pi/2 \), eq. (104) holds identically for all \( \chi(x) \in D_{b^+(\mu,w)} \), while for \( \mu \in [0,\pi/2) \), eq. (104) is reduced to

\[
\overline{\nabla}C = 0, \quad \forall C,
\]

which requires that \( D = 0. \)

We finally obtain that

\[
\overline{\tilde{a}}(\pi/2,w) = \hat{b}^+(\pi/2,w), \quad D_{\overline{\tilde{a}}(\pi/2,w)} = D_{\hat{b}^+(\pi/2,w)}, \tag{105}
\]

a copy of (49) with \( \mu = \pi/2 \), and

\[
\overline{\tilde{a}}(\mu,w) \subseteq \hat{b}^+(\mu,w), \quad D_{\overline{\tilde{a}}(\mu,w)} = \hat{D}_{b^+(\mu,w)} \tag{103}, \quad \mu \in [0,\pi/2), \tag{106}
\]

which, in particular, implies that the asymptotic behavior at the origin of the functions belonging to the domain of any operator \( \overline{\tilde{a}}(\mu,w) \) is estimated as \( O(x^{1/2}\ln^{1/2}x) \),

\[
D_{\overline{\tilde{a}}(\mu,w)} \ni \chi(x) = O(x^{1/2}\ln^{1/2}x), \quad x \to 0, \ \text{for } \forall \mu, \forall w > -1/2, \tag{107}
\]

for \( \mu = \pi/2 \) this follows from (105) and (100), while for \( \mu \in [0,\pi/2) \) this follows from (106) and (103).

The kernel of any operator \( \overline{\tilde{a}}(\mu,w) \) is trivial,

\[
\ker \overline{\tilde{a}}(\mu,w) = \{0\}, \quad \forall \mu, \forall w > -1/2, \tag{108}
\]

for \( \mu = \pi/2 \) this follows from (105) and (102), while for \( \mu \in [0,\pi/2) \), this follows from that \( \overline{a}(\mu,w)\chi(x) = \hat{a}(\mu,w)\chi(x) = 0 \Rightarrow \chi(x) = c\phi(\mu,w;x) \), but the function \( \phi(\mu,w;x) \) is
estimated at the origin as $-2 \cos \mu (vx)^{1/2} \ln (vx) + O(x^{1/2})$, see (85), and therefore cannot belong to $D_{\hat{b} (\mu, w)}$ according to (107) (in addition, $\phi (\mu, w; x)$ with $\mu \in (0, \pi/2)$ is not square integrable on $\mathbb{R}_+$, see (87)).

iv) Quite similarly, we find

$$\tilde{b}(\pi/2, w) = \hat{a}^+ (\pi/2, w), \quad D_{\tilde{b} (\mu, w)} = D_{\hat{a}^+ (\mu, w)}, \quad \mu = \pi/2,$$

and

$$\tilde{b}(\mu, w) \subset \hat{a}^+ (\mu, w), \quad D_{\tilde{b} (\mu, w)} = \tilde{D}_{\hat{a}^+ (\mu, w)}, \quad |\mu| \in [0, \pi/2),$$

which, in particular, implies that the asymptotic behavior at the origin of the functions belonging to the domain of any operator $\tilde{b}(\mu, w)$ is estimated as $O(x^{1/2})$,

$$D_{\tilde{b} (\mu, w)} \ni \psi(x) = O(x^{1/2}), \quad x \to 0, \quad \forall \mu, \forall w > -1/2,$$

for $\mu = \pi/2$ this follows from (109) and (94), while for $\mu \in [0, \pi/2)$ this follows from (110) and (96). We note that equality (109) and inclusion (110) directly follow from the respective previous equality (105) and inclusion (106) by taking the adjoints, and only the domain $D_{\tilde{b} (\mu, w)}$ in the last case has to be evaluated.

By arguments similar to those in the end of subsubsec. 2.2.4, it is easy to prove that there are no other pairs $\hat{g} = \bar{g}$ and $\hat{g}^+$ of closed mutually adjoint operators that are extensions of the respective $\hat{a}(\mu, w)$ and $\tilde{b}(\mu, w)$, $\hat{a}(\mu, w) \subset \hat{g}$, $\tilde{b}(\mu, w) \subset \hat{g}^+$.

We thus show that each pair $\hat{a}(\mu, w)$, $\tilde{b}(\mu, w)$, $\mu \in [0, \pi/2]$, $w \in (-1/2, \infty)$, of mutually adjoint by Lagrange differential operations given by a copy of (83) with $\phi (\mu, w; x)$ (83) and providing a two-parameter family of different generalized oscillator representations (89) for generalized Calogero differential operation $\hat{H}$ (1) with $g_1 = -1/4, g_2 > 0$ generates a unique pair $\hat{\omega}(\pi/2, w) = \hat{b}^+ (\pi/2, w)$, $\hat{a}^+ (\pi/2, w) = \tilde{b}(\pi/2, w)$ of closed mutually adjoint operators for $\mu = \pi/2$, while for $\mu \in [0, \pi/2)$, each pair $\hat{a}(\mu, w)$, $\tilde{b}(\mu, w)$ generates two different pairs $\hat{\omega}(\mu, w)$, $\hat{a}^+ (\mu, w)$ and $\tilde{b}^+ (\mu, w)$, $\tilde{b}(\mu, w)$ of closed mutually adjoint operators such that $\hat{\omega}(\mu, w) \subset \hat{b}^+ (\mu, w)$ and $\tilde{b}(\mu, w) \subset \hat{a}^+ (\mu, w)$. The operators $\hat{\omega}(\mu, w)$ and $\hat{b}^+ (\mu, w)$ are extensions of the initial operator $\hat{a}(\mu, w)$, they are associated with $\hat{a}(\mu, w)$, and their domains are given by the respective (105), (106) and a copy of (49). The operators $\tilde{b}(\mu, w)$ and $\hat{a}^+ (\mu, w)$ are extensions of the initial operator $\tilde{b}(\mu, w)$, they are associated with $\tilde{b}(\mu, w)$, and their domains are given by the respective (109), (110) and a copy of (42).

2.5 Region $g_1 = -1/4$ ($\kappa = 0$), $w = w_0 = -1/2$ ($\alpha = 0$)

A consideration in this subsection is completely similar to that in subsec. 2.3, and all the results to follow can be obtained from the results of subsec. 2.3 in the limit $\kappa \to 0$ ($\beta \to 1$). We even do not change the notation having in mind that here $\kappa = 0$, $\beta = 1$.

On the other hand, all the results of this subsection can be obtained from the results of the previous subsec. 2.4 related to the case $\mu = \pi/2$ in the limit $w \to w_0 = -1/2$.

2.5.1 Generalized oscillator representations for $\hat{H}$, differential operations $\hat{a}$ and $\tilde{b}$

For these values of parameters, the general solution of differential equation (9) with $u = u_0 = 4v^2 w_0 = -2v^2$ is given by (28) with $\kappa = 0$, $\beta = 1$. The function $\Psi (1, \rho)$ given by (25)
with $\beta = 1$ increases monotonically from $-\infty$ to $\infty$ as $\rho = (vx)^2$ together with $x$ ranges from $0$ to $\infty$. It follows that eq. (10) with these values of parameters has a unique, up to a positive constant factor, real-valued positive solution

$$\phi(w_0; x) = e^{-\rho/2} \rho^{1/4}, \rho = (vx)^2,$$

(112)

which is a copy of (66) with $\kappa = 0$. This implies that in this case, we have a unique\footnote{Up to irrelevant phase factors.} pair of mutually adjoint first-order differential operations $\hat{a}(w_0)$ and $\hat{b}(w_0)$ given by a copy of (67) with $\phi(w_0; x)$ (112) instead of $\phi(w_0; x)$ (66) and providing unique generalized oscillator representation (9) with $u = -2v^2$ for generalized Calogero differential operation $\hat{H}$ (11) with $g_1 = -1/4, g_2 > 0$:

$$\hat{H} = -d_x^2 - \frac{1}{4}x^{-2} + g_2x^2 = \hat{b}(w_0)\hat{a}(w_0) + 2v^2.$$  

(113)

This representation is obtained from representation (68) in the limit $g_1 \to -1/4, \kappa \to 0$. Accordingly, a consideration in this subsection is a copy of that in subsec. 2.3 with the substitution $\phi(w_0; x)$ (112) for $\phi(w_0; x)$ (66).

The asymptotic behavior of the functions $\phi(w_0; x)$ and $1/\phi(w_0; x)$ at the origin and at infinity is evident from (112).

### 2.5.2 Initial operators $\hat{a}$ and $\hat{b}$

In perfect analogy to subsec. 2.3.2, we introduce the pair $\hat{a}(w_0), \hat{b}(w_0)$ of initial differential operators in $L^2(\mathbb{R}_+)$ defined on $\mathcal{D}(\mathbb{R}_+)$ and associated with the respective pair of differential operations $\hat{a}(w_0), \hat{b}(w_0)$ of initial operators to a pair of closed mutually adjoint operators $\hat{c}(w_0), \hat{c}^+(w_0), \hat{\alpha}(w_0) \subset \hat{c}(w_0)$, $\hat{b}(w_0) \subset \hat{c}^+(w_0)$, beginning from the the pair $\hat{a}(w_0), \hat{a}^+(w_0)$, or to a pair of closed mutually adjoint operators $\hat{d}(w_0), \hat{d}^+(w_0), \hat{\alpha}(w_0) \subset \hat{d}^+(w_0)$, $\hat{b}(w_0) \subset \hat{d}(w_0)$, beginning from the $\hat{b}(w_0)$, $\hat{b}^+(w_0)$.

### 2.5.3 Adjoint operators $\hat{a}^+$ and $\hat{b}^+$, closed operators $\overline{a}$ and $\overline{b}$

By arguments similar to those in subsubsecs. 2.2.3 and 2.2.4, we establish that the closures $\overline{a}(w_0)$ and $\overline{b}(w_0)$ of the respective operators $\hat{a}(w_0)$ and $\hat{b}(w_0)$, as well as the adjoints $\hat{a}^+(w_0)$ and $\hat{b}^+(w_0)$ of the latter, do exist and the chains of inclusions

$$\hat{a}(w_0) \subset \overline{a}(w_0) = (\hat{a}^+(w_0))^+ \subset \hat{b}^+(w_0),$$

$$\hat{b}(w_0) \subset \overline{b}(w_0) = (\hat{b}^+(w_0))^+ \subset \hat{a}^+(w_0),$$

(115)
which are the copies of (71), hold. The operators $\overline{a}(w_0)$ and $\hat{b}^+(w_0)$ are associated with $\hat{a}(w_0)$, while the operators $\hat{b}(w_0)$ and $\hat{a}^+(w_0)$ are associated with $\hat{b}(w_0)$, so that to specify these operators, it is sufficient to evaluate their domains actually repeating the content of subsubsec. 2.3.4 with the substitution $\phi(w_0; x)$ in (66) for $\phi(w_0; x)$ in (66).

2.5.4 Domains of operators $\hat{a}^+, \hat{b}^+, \overline{a}$ and $\overline{b}$

i) The domain of the operator $\hat{a}^+(w_0)$ is the natural domain for $\hat{b}(w_0)$, which is given by the copy of (72). A generic function $\psi(x)$ belonging to $D_{\hat{a}^+(w_0)}$ allows the representations

$$
\psi(x) = -\frac{1}{\phi(w_0, x)} \int_{0}^{x} dy \phi(w_0, y)\eta(y) = \frac{1}{\phi(w_0, x)} \int_{x}^{\infty} dy \phi(w_0, y)\eta(y),
$$

$$
\eta(x) = \hat{b}(w_0)\psi(x) \in L^2(\mathbb{R}_+), \quad \text{and} \quad \int_{0}^{\infty} dy \phi(w_0, y)\eta(y) = 0.
$$

The equality $\int_{0}^{\infty} dy \phi(w_0, y)\eta(y) = 0$ means that the range $R_{\hat{a}^+(w_0)}$ of the operator $\hat{a}^+(w_0)$ is orthogonal to the one-dimensional subspace $\{c\phi(w_0; x)\} \subset L^2(\mathbb{R}_+)$, $R_{\hat{a}^+(w_0)} \perp \{c\phi(w_0; x)\}$, and thereby its closure $\overline{R_{\hat{a}^+(w_0)}}$ can not be the whole $L^2(\mathbb{R}_+)$, which in turn implies that the kernel of the adjoint operator $(\hat{a}^+(w_0))^+ = \overline{a}(w_0)$ is not trivial, $\{c\phi(w_0; x)\} \subseteq \ker \overline{\hat{a}(w_0)} \neq \{0\}$, see below.

Estimating the integral terms in (116) with the Cauchy–Bunyakovskii inequality, we obtain that the asymptotic behavior of functions $\psi(x) \in D_{\hat{a}^+(w_0)}$ at the origin and at infinity is respectively given by

$$
\psi(x) = O(x^{1/2}), \quad x \to 0,
$$

$$
\psi(x) \to 0, \quad x \to \infty.
$$

ii) The domain $D_{\hat{b}^+(w_0)}$ of the operator $\hat{b}^+(w_0)$ is the natural domain for $\overline{a}(w_0)$, which is given by a copy of (75). A generic function $\chi(x)$ belonging to $D_{\hat{b}^+(w_0)}$ allows the representation

$$
\chi(x) = \phi(w_0, x) \left[ D + \int_{x_0}^{x} dy \frac{1}{\phi(w_0, y)}\eta(y) \right], \quad \eta(x) = \hat{b}(w_0)\chi(x) \in L^2(\mathbb{R}_+),
$$

where $x_0 \in (0, \infty)$ and $D$ is an arbitrary constant, and its asymptotic behavior at the origin and at infinity estimated using the Cauchy–Bunyakovskii inequality is respectively given by

$$
\chi(x) = O(x^{1/2} \ln^{1/2} x), \quad x \to 0,
$$

$$
\chi(x) \to 0, \quad x \to \infty.
$$

As it follows from (118) with $\eta(x) = 0$ and in (112), the kernel of the operator $\hat{b}^+(w_0)$ is nontrivial,

$$
\ker \hat{b}^+(w_0) = \{c \left( \phi(w_0; x) = (ux)^{1/2}e^{-(ux)^{1/2}} \right) \}.
$$

iii) The domain $D_{\overline{a}(w_0)}$ of the operator $\overline{a}(w_0)$ is evaluated in accordance with (115): the defining equation for $\overline{a}(w_0)$ as $(\hat{a}^+(w_0))^+$, which is a restriction of $\hat{b}^+(w_0)$, is reduced to the equation for $D_{\overline{a}(w_0)} \subseteq D_{\hat{b}^+(w_0)}$ of the form

$$
(\chi, \hat{b}(w_0)\psi) - (\overline{a}(w_0)\chi, \psi) = 0, \quad \chi(x) \in D_{\overline{a}(w_0)} \subseteq D_{\hat{b}^+(w_0)}, \forall \psi(x) \in D_{\hat{a}^+(w_0)}.
$$
Integrating by parts in \((\hat{a}(w_0)\chi, \psi)\) and taking asymptotic estimates \(117\) and \(119\) into account, we establish that eq. \(121\) holds identically for all \(\chi(x) \in D_{\hat{b}^+(w_0)}\), which implies that

\[
\overline{a}(w_0) = \hat{b}^+(w_0), \ D_{\overline{a}(w_0)} = D_{\hat{b}^+(w_0)} \text{ given by a copy of } (73), \tag{122}
\]

in particular, the asymptotic behavior at the origin of the functions \(\chi(x)\) belonging to the domain of the operator \(\overline{a}(w_0)\) is estimated by a copy of \(119\),

\[
\chi(x) = O(x^{1/2} \ln^{1/2} x), \ x \to 0, \tag{123}
\]

and

\[
\ker \overline{a}(w_0) = \{ c \left( \phi(w_0; x) = (ux)^{1/2} e^{-(ux)^2/2} \right) \}. \tag{124}
\]

iv) It directly follows from \(122\) by taking the adjoints that

\[
\overline{b}(w_0) = \hat{a}^+(w_0), \ D_{\overline{b}(w_0)} = D_{\hat{a}^+(w_0)} \text{ given by a copy of } (72). \tag{125}
\]

By arguments similar to those in the end of subsubsec. 2.2.4, it is easy to prove that there is no other pair \(\hat{g} = \overline{g}\) and \(\hat{g}^+\) of closed mutually adjoint operators that are extensions of the respective \(\hat{a}(w_0)\) and \(\overline{b}(w_0)\), \(\hat{a}(w_0) \subset \hat{g}, \overline{b}(w_0) \subset \hat{g}^+\).

We thus show that the pair \(\hat{a}(w_0), \overline{b}(w_0)\) of mutually adjoint by Lagrange differential operations given by a copy of \((67)\) with \(\phi(w_0; x)\) \((12)\) and providing unique generalized oscillator representation \(113\) for \(\hat{H}\) \((1)\) with \(g_1 = -1/4, g_2 > 0\) generates a unique pair \(\overline{a}(w_0) = \hat{b}^+(w_0), \hat{a}^+(w_0) = \overline{b}(w_0)\) of closed mutually adjoint operators. The operator \(\overline{a}(w_0) = \hat{b}^+(w_0)\) is an extension of the initial operator \(\hat{a}(w_0)\), it is associated with \(\overline{a}(w_0)\), and its domain is given by a copy of \((75)\). The operator \(\overline{b}(w_0) = \hat{a}^+(w_0)\) is an extension of the initial operator \(\overline{b}(w_0)\), it is associated with \(\overline{b}(w_0)\), and its domain is given by a copy of \((72)\).

2.6 Resume

The final conclusion of this section is that for each pair of coupling constants \(g_1 \geq -1/4, g_2 > 0\), we have two two-parameter families of generalized Calogero Hamiltonians in a generalized oscillator form, the family \(\{\hat{H}_{\text{ea}}\}\) of Hamiltonians

\[
\hat{H}_{\text{ea}} = \left\{ \begin{array}{ll}
\hat{H}_{\text{ea}(\mu,w)} = \hat{a}^+(\mu, w)\overline{a}(\mu, w) - 4u^2w \hat{I}, & \mu \in [0, \pi/2],\ w \in (w_0, \infty), \\
\hat{H}_{\text{ea}(w_0)} = \hat{a}^+(w_0)\overline{a}(w_0) - 4u^2w_0 \hat{I}, & u = \sqrt{g_2} > 0,\ w_0 = -\frac{1}{2}(1 + \varkappa),\ \varkappa = \sqrt{1/4 + g_1} \geq 0,
\end{array} \right. \tag{126}
\]

where the operators \(\hat{a}^+(\mu, w)\) and \(\overline{a}(\mu, w)\) are described in the respective items i) and iii) in subsec. 2.2.4 for the case of \(g_1 > -1/4 (\varkappa > 0)\) and in subsec. 2.4.4 for the case of \(g_1 = -1/4 (\varkappa = 0)\), while the operators \(\hat{a}^+(w_0)\) and \(\overline{a}(w_0)\) are described in the respective items i) and iii) in subsec. 2.3.4 for the case of \(g_1 > -1/4 (\varkappa > 0)\) and in subsec. 2.5.4 for the case of \(g_1 = -1/4 (\varkappa = 0)\), and the family \(\{\hat{H}_{eb}\}\) of Hamiltonians

\[
\hat{H}_{eb} = \left\{ \begin{array}{ll}
\hat{H}_{eb(\mu,w)} = \overline{b}(\mu, w)\hat{b}^+(\mu, w) - 4u^2w \hat{I}, & \mu \in [0, \pi/2],\ w \in (w_0, \infty), \\
\hat{H}_{eb(w_0)} = \overline{b}(w_0)\hat{b}^+(w_0) - 4u^2w_0 \hat{I},
\end{array} \right. \tag{127}
\]
where the operators $\hat{b}^+(\mu, w)$ and $\hat{b}(\mu, w)$ are described in the respective items ii) and iv) in subsubsec. 2.2.4 for the case of $g_1 = -1/4 (\varkappa = 0)$ and in subsubsec. 2.4.4 for the case of $g_1 = -1/4 (\varkappa > 0)$, while the the operators $\hat{b}^+(w_0)$ and $\hat{b}(w_0)$ are described in the respective items ii) and iv) in subsubsec. 2.3.4 for the case of $g_1 > -1/4 (\varkappa > 0)$ and in subsubsec. 2.5.4 for the case of $g_1 = -1/4 (\varkappa = 0)$.

For some values of coupling constants, these families overlap. In particular, the Hamiltonians $\hat{H}_{ea}(\mu, w)$ and $\hat{H}_{eb}(\mu, w)$, with $g_1 \geq 3/4$ coincide, see (59) and (63), $\hat{H}_{ea}(\mu, w) = \hat{H}_{eb}(\mu, w)$, $g_1 \geq 3/4$. The same holds for the Hamiltonians $\hat{H}_{ea}(w_0)$ and $\hat{H}_{eb}(w_0)$ with any $g_1 \geq -1/4$, see (79), (82) and (122), (125), so that $\{\hat{H}_{ea}\} = \{\hat{H}_{eb}\}$ for the case of $g_1 \geq 3/4$.

3 Oscillator representations for all Hamiltonians with $g_1 \geq -1/4, g_2 > 0$

3.1 Preliminaries

This section is a concluding one. Without being afraid of repeating ourselves, we give here a full answer to the question on generalized oscillator representations (7) or (equivalently) (8) for all generalized Calogero Hamiltonians $\hat{H}_e$ associated with generalized Calogero differential operation $\hat{H}$ (11) with any coupling constants $g_1, g_2 \in \mathbb{R}^2$.

An answer to the question is essentially different for different regions in the plane $\mathbb{R}^2$ of coupling constants, namely, for the open half-planes $\{g_1 < -1/4\}$ and $\{g_2 < 0\}$ (these regions are overlapping along the open quadrant $\{g_1 < -1/4, g_2 < 0\}$) and for the quadrant $\{g_1 \geq -1/4, g_2 \geq 0\}$.

As was already indicated in the beginning of sec. 1 and in subsec. 2.1, any generalized Calogero Hamiltonian $\hat{H}_e$ with coupling constants lying in the half-plane $\{g_1 < -1/4\}$ or in the half-plane $\{g_2 < 0\}$ does not allow generalized oscillator representation because such a representation would imply that $\hat{H}_e$ is bounded from below, whereas any generalized Calogero Hamiltonian with such values of coupling constants is not bounded from below [2]. This conclusion is in complete agreement with that according to subsec. 2.1, there is no generalized oscillator representation (6) for generalized Calogero differential operation $\hat{H}$ (11) with $g_1 < -1/4$ or with $g_2 < 0$.

As to the quadrant $\{g_1 \geq -1/4, g_2 \geq 0\}$, we recall that as was shown in [11], any Calogero Hamiltonian $\hat{H}_e$ with coupling constants lying on the horizontal boundary semiaxis $\{g_1 \geq -1/4, g_2 = 0\}$ of the quadrant, allows a generalized oscillator representation, moreover, a one- or two-parameter family of such representations. Thus, only the semiopen quadrant $\{g_1 \geq -1/4, g_2 > 0\}$ remains. Following the considerations advanced in the end of sec.1, we show in what follows that any generalized Calogero Hamiltonian $\hat{H}_e$ with coupling constants lying in this quadrant also allows a family of generalized oscillator representations, one- or two-parameter. For completeness, we repeat an extended version of these considerations.

A hypothesis is that the two families (126) and (127) of generalized Calogero Hamiltonians in generalized oscillator form cover all the set of the known generalized Calogero Hamiltonians with coupling constants lying in the semiopen quadrant $\{g_1 \geq -1/4, g_2 > 0\}$, see [2]. Namely, each generalized Calogero Hamiltonian with given $g_1 \geq -1/4, g_2 > 0$ can be

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13We recall that we conventionally omit the term “generalized” for Calogero Hamiltonians with $g_2 = 0$. 30
identified with one or more members of family \([126]\) or family \([127]\). This identification is trivial in the case of \(g_1 \in [3/4, \infty)\) (the semiopen quadrant \(\{g_1 \geq 3/4, g_2 > 0\}\) where there is a unique generalized Calogero Hamiltonian \(\hat{H}_1\) with given \(g_1, g_2\), so that in this case, the both \(\{\hat{H}_{ea}\}\) and \(\{\hat{H}_{eb}\}\) are reduced to a one-point set, \(\{\hat{H}_{ea}\} = \{\hat{H}_{eb}\} = \hat{H}_1\). In the case of \(g_1 \in [-1/4, 3/4) (\varkappa \in [0, 1])\) (the semiopen strip \(\{-1/4 \leq g_1 < 3/4, g_2 > 0\}\)), the procedure of identification is more complicated. According to \([2]\), for each pair of coupling constants \(g_1, g_2\) lying in the open strip \(\{-1/4 < g_1 < 3/4, g_2 > 0\}\), there exists a one-parameter family \(\{\hat{H}_{2, \nu}, \nu \in [-\pi/2, \pi/2], -\pi/2 \sim \pi/2\}\) of generalized Calogero Hamiltonians differing in their domains \(D_{\hat{H}_{2, \nu}}\), while for each pair of coupling constants \(g_1, g_2\) lying on the boundary open semiaxis \(\{g = -1/4, g_2 > 0\}\), there exists a one-parameter family \(\{\hat{H}_{3, \nu}, \nu \in [-\pi/2, \pi/2], -\pi/2 \sim \pi/2\}\) of generalized Calogero Hamiltonians differing in their domains \(D_{\hat{H}_{3, \nu}}\). Namely, the domains \(D_{\hat{H}_{2, \nu}}\) and \(D_{\hat{H}_{3, \nu}}\) are subspaces of the natural domains \(D^n_{\hat{H}}\) for the respective \(\hat{H}\), the subspaces that are specified by different s.a. asymptotic boundary conditions at the origin of the respective form

\[
D_{\hat{H}_{2, \nu}} \ni \chi(x) = c \left[(vx)^{1/2+\kappa} \sin \nu + (vx)^{1/2-\kappa} \cos \nu\right] + O(x^{3/2}), x \to 0, \kappa \in (0, 1), \quad (128)
\]

and

\[
D_{\hat{H}_{3, \nu}} \ni \chi(x) = c[ (vx)^{1/2} \sin \nu + 2(vx)^{1/2} \ln(vx) \cos \nu] + O(x^{3/2}), x \to 0, \kappa = 0. \quad (129)
\]

Therefore, an identification of a given \(\hat{H}_{2, \nu}\) with a certain \(\hat{H}_{ea}\) or \(\hat{H}_{eb}\) goes through evaluating the asymptotic behavior at the origin of functions belonging to the domain of \(\hat{H}_{ea}\) or \(\hat{H}_{eb}\) and its identification with the asymptotic boundary conditions for the certain \(\hat{H}_{2, \nu}\); the same holds for \(\hat{H}_{3, \nu}\). It may happen, and that really occurs, that \(\hat{H}_{ea(\mu, w)}\), or \(\hat{H}_{eb(\mu, w)}\), with different \(\mu, w\) have the same asymptotic behavior of functions belonging to their domains and define the same Hamiltonian.

The asymptotic boundary conditions for Hamiltonians with \(g_1 \in (-1/4, 3/4)\) and with \(g_1 = -1/4\) differ widely in their character sending us in separate consideration of the open strip \(\{-1/4 < g_1 < 3/4, g_2 > 0\}\) and the vertical boundary open semiaxis \(\{g_1 = -1/4, g_2 > 0\}\), as we did beforehand in sec. 2.

We begin a detailed consideration with the family of Hamiltonians \(\{\hat{H}_{ea}\}\) \([126]\).

### 3.2 Family \(\{\hat{H}_{ea}\}\)

#### 3.2.1 Quadrant \(\{g_1 \geq 3/4 (\varkappa \geq 1), g_2 > 0\}\)

According to \([2]\), for any pair of coupling constants \(g_1, g_2\) lying in this quadrant, there exists a unique s.a. generalized Calogero Hamiltonian \(\hat{H}_1\) defined on the natural domain \(D^n_{\hat{H}}\). It follows that the Hamiltonian \(\hat{H}_1\) with given \(g_1 \geq 3/4, g_2 > 0\) must be identified with the whole family \(\{\hat{H}_{ea}\}\) with the same \(g_1, g_2\), \(\{\hat{H}_{ea}\} = \hat{H}_1\), which yields the two types of generalized oscillator representations for this Hamiltonian:

\[
\hat{H}_1 = \hat{H}_{ea(\mu, w)} = \hat{a}^+(\mu, w)\hat{a}(\mu, w) - 4\nu^2 w \hat{I}, \quad (130)
\]

\[
\forall \mu \in [0, \pi/2], \forall w \in (w_0, \infty), w_0 = -\frac{1}{2}(1 + \varkappa), \varkappa \geq 1,
\]

\[\text{We slightly change the notation in comparison with [2] for uniformity: the extension parameter \(\nu\) entering the index of \(\hat{H}_{3, \nu}\) was denoted in [2] by \(\vartheta\), so that \(\hat{H}_{3, \nu}\) here coincides with \(\hat{H}_{3, \vartheta}\) in [2].}\]
and
\[ \hat{H}_1 = \hat{H}_{ea(w_0)} = \hat{a}^+(w_0)\overline{a}(w_0) + 2\nu^2(1 + \kappa), \kappa \geq 1. \] (131)

The formula (130) actually presents a two-parameter family of different generalized oscillator representations for a unique generalized Calogero Hamiltonian $\hat{H}_1$ with given coupling constants $g_1 \geq 3/4, g_2 > 0$. According to (52), the kernel of any operator $\overline{a}(\mu, w)$ is trivial. This implies that any of representations (130) is not an optimum one in the sense that it does not allow determining the ground state and the ground-state energy of $\hat{H}_1$; we can say only that the spectrum of $\hat{H}_1$ is bounded from below by $2\nu^2(1 + \kappa)$.

In contrast to (130), representation (131) is an optimum one: ker $\overline{a}(w_0)$ is nontrivial, it is given by (81), which implies that ker $\overline{a}(w_0)$ is the one-dimensional ground space of $\hat{H}_1$ and the ground-state energy $E_0$ of $\hat{H}_1$, which is a lower boundary of its spectrum, is
\[ E_0^{(1)} = 2\nu^2(1 + \kappa), \kappa \geq 1. \] (132)

The normalized ground state $U^{(1)}(x)$ of $\hat{H}_1$ is given by
\[ U^{(1)}(x) = \sqrt{\frac{2\nu}{\Gamma(1 + \kappa)}}(ux)^{1/2 + \kappa}e^{-(vx)^2/2}, \kappa \geq 1. \] (133)

### 3.2.2 Strip $\{-1/4 < g_1 < 3/4 \ (0 < \kappa < 1), \ g_2 > 0\}$

A consideration in this subsubsection appears to be similar to that in the previous subsubsection.

According to [2], for any pair of coupling constants $g_1, g_2$ lying in this strip, there exists a one-parameter family \{$\hat{H}_{2,\nu}, \nu \in [-\pi/2, \pi/2], -\pi/2 \sim \pi/2$\} of generalized Calogero Hamiltonians $\hat{H}_{2,\nu}$ specified by s.a. asymptotic boundary conditions at the origin (128).

An asymptotic behavior at the origin of the functions belonging to the domains of the Hamiltonians $\hat{H}_{ea}$ (126) with coupling constants $g_1, g_2$ lying in the same strip is estimated as follows. By definition of any operator $\hat{H}_{ea(\mu, w)}$, its domain belongs to or coincides with the domain of the operator $\overline{a}(\mu, w)$, $D_{H_{ea(\mu, w)}} \subseteq D_{\overline{a}(\mu, w)}$; the same holds for the operator $\hat{H}_{ea(w_0)}$, $D_{H_{ea(w_0)}} \subseteq D_{\overline{a}(w_0)}$. But according to (61) and (50), the asymptotic behavior of functions $\chi(x)$ belonging to the respective $D_{\overline{a}(\mu, w)}$ and $D_{\overline{a}(w_0)}$, is estimated as $\chi(x) = O(x^{1/2})$, which implies that the functions belonging to $D_{H_{ea(\mu, w)}, \forall \mu, \forall w \in (w_0, \infty)}$, and $D_{H_{ea(w_0)}}$ tend to zero not weaker than $x^{1/2}$ as $x \to 0$. A comparison of this estimate with (128) shows that there is only one s.a. generalized Calogero Hamiltonian with such an asymptotic behavior of the functions belonging to its domain, namely, $\hat{H}_{2,\pm\pi/2}$. It follows that the Hamiltonian $\hat{H}_{2,\pm\pi/2}$ with given coupling constants $g_1 \in (-1/4, 3/4)$, $g_2 > 0$ must be identified with the whole family \{$\hat{H}_{ea}$\} with the same $g_1, g_2$, \{$\hat{H}_{ea}$\} = $\hat{H}_{2,\pm\pi/2}$, which yields the two types of generalized oscillator representations for this Hamiltonian:
\[ \hat{H}_{2,\pm\pi/2} = \hat{H}_{ea(\mu, w)} = \hat{a}^+(\mu, w)\overline{a}(\mu, w) - 4\nu^2w\hat{I}, \] (134)
\[ \forall \mu \in [0, \pi/2], \forall w \in (w_0, \infty), \kappa \in (0, 1), \]

and
\[ \hat{H}_{2,\pm\pi/2} = \hat{H}_{ea(w_0)} = \hat{a}^+(w_0)\overline{a}(w_0) + 2\nu^2(1 + \kappa), \kappa \in (0, 1). \] (135)
These representations are evident extensions of the previous respective representations \((130)\) and \((131)\) to \(\kappa \in (0, 1)\), and a comment to them is an extension of the previous one: formula \((134)\) actually presents a two-parameter family of different generalized oscillator representations for a unique generalized Calogero Hamiltonian \(\hat{H}_{2,\pm\pi/2}\) with given coupling constants \(g_1 \in (-1, 4, 3/4)\), \(g_2 > 0\), any of representations \((134)\) is not an optimum representation because of \((52)\), while representation \((135)\) is an optimum one because of \((81)\), the ground-state energy \(E_0^{(2)}(\pm\pi/2)\) of \(\hat{H}_{2,\pm\pi/2}\), which is a lower boundary of its spectrum, is

\[
E_0^{(2)}(\pm\pi/2) = 2v^2(1 + \kappa), \quad \kappa \in (0, 1),
\]

which is an extension of \((132)\) to \(\kappa \in (0, 1)\), and the normalized ground state \(U^{(2)}_{\pm\pi/2}(x)\) of \(\hat{H}_{2,\pm\pi/2}\) is given by

\[
U^{(2)}_{\pm\pi/2}(x) = \sqrt{\frac{2v}{\Gamma(1 + \kappa)}} (v x)^{1/2 + \kappa} e^{-(vx)^2/2}, \quad \kappa \in (0, 1),
\]

which is an extension of \((133)\) to \(\kappa \in (0, 1)\).

### 3.2.3 Semiaxis \(\{g_1 = -1/4 \ (\kappa = 0), \ g_2 > 0\}\)

A consideration in this case, \(\kappa = 0\), is completely similar to that in the previous case of \(\kappa \in (0, 1)\).

According to \([2]\), for any pair of coupling constants \(g_1, g_2\) lying on this semiaxis, there exists a one-parameter family \(\{\hat{H}_{3,\nu}, \nu \in [-\pi/2, \pi/2], -\pi/2 \sim \pi/2\}\) of generalized Calogero Hamiltonians \(\hat{H}_{3,\nu}\) specified by s.a. asymptotic boundary conditions at the origin \((129)\).

An asymptotic behavior at the origin of the functions belonging to the domains of the Hamiltonians \(\hat{H}_{\nu a}\) \((126)\) with coupling constants \(g_1, g_2\) lying on the same semiaxis is estimated as follows. By definition of any operator \(\hat{H}_{\nu a}\), its domain belongs to or coincides with the domain of the operator \(\hat{a}, \hat{a}^\dagger\) is \(\hat{a}(\mu, w)\) or \(\bar{a}(\mu, w)\), i.e., \(D_{\hat{H}_{\nu a}} \subseteq D_{\hat{a}}\). But according to \((107)\) and \((123)\), the asymptotic behavior of functions \(\chi(x)\) belonging to any \(D_{\hat{a}}\), is estimated as \(\chi(x) = O(x^{1/2} \ln^{1/2} x)\), which implies that the functions belonging to any \(D_{\hat{H}_{\nu a}}\) tend to zero not weaker than \(x^{1/2} \ln^{1/2} x\) as \(x \to 0\). A comparison of this estimate with \((128)\) shows that there is only one s.a. generalized Calogero Hamiltonian with such an asymptotic behavior of the functions belonging to its domain, namely, \(\hat{H}_{3,\pm\pi/2}\). It follows that the Hamiltonian \(\hat{H}_{3,\pm\pi/2}\) with given \(g_1 = -1/4, g_2 > 0\) must be identified with the whole family \(\{\hat{H}_{\nu a}\}\) with the same \(g_1, g_2, \{\hat{H}_{\nu a}\} = \hat{H}_{3,\pm\pi/2}\), which yields the two types of generalized oscillator representations for this Hamiltonian:

\[
\hat{H}_{3,\pm\pi/2} = \hat{H}_{\nu a}(\mu, w) = \hat{a}^+(\mu, w)\bar{a}(\mu, w) - 4v^2 w I, \quad \forall \mu \in [0, \pi/2], \forall w \in (w_0, \infty), w_0 = \frac{1}{2},
\]

and

\[
\hat{H}_{3,\pm\pi/2} = \hat{H}_{\nu a}(w_0) = \hat{a}^+(w_0)\bar{a}(w_0) + 2v^2.
\]
formula (138) actually presents a two-parameter family of different generalized oscillator representations for a unique generalized Calogero Hamiltonian \( \hat{H}_{3, \pm \pi/2} \) with given coupling constants \( g_1 = -1/4, g_2 > 0 \), any of representations (138) is not an optimum representation because of (108), while representation (139) is an optimum one because of (124), the ground-state energy \( E_0^{(3)}(\pm \pi/2) \) of \( \hat{H}_{3, \pm \pi/2} \), which is a lower boundary of its spectrum, is

\[
E_0^{(3)}(\pm \pi/2) = 2v^2,
\]

which is an extension of (136) to \( \kappa = 0 \), and the normalized ground state \( U_{\pm \pi/2}^{(3)}(x) \) of \( \hat{H}_{3, \pm \pi/2} \) is given by

\[
U_{\pm \pi/2}^{(3)}(x) = \sqrt{2v}e^{(ux)^2/2},
\]

which is an extension of (137) to \( \kappa = 0 \).

### 3.3 Family \( \{ \hat{H}_{eb} \} \)

#### 3.3.1 Quadrant \( \{ g_1 \geq 3/4 \ (\kappa \geq 1), \ g_2 > 0 \} \)

For any pair of coupling constants \( g_1, g_2 \) lying in this quadrant, we have the identities

\[
\hat{b}(\mu, w) = \hat{a}^+(\mu, w), \hat{b}^+(\mu, w) = \overline{\hat{a}}(\mu, w),
\]

see the respective (63) and (59), and \( \hat{b}(w_0) = \hat{a}^+(w_0), \hat{b}^+(w_0) = \overline{\hat{a}}(w_0) \), see the respective (82) and (72). These identities and the result of subsubsec. 3.2.1 provide the identities \( \{ \hat{H}_{eb} \} = \{ \hat{H}_{oa} \} = \hat{H}_1 \) and yield equivalent forms of generalized oscillator representations for \( \hat{H}_1 \) presented in subsubsec. 3.2.1:

\[
\hat{H}_1 = \hat{H}_{eb(\mu, w)} = \hat{b}(\mu, w)\hat{b}^+(\mu, w) - 4v^2w \hat{I},
\]

\[
\forall \mu \in [0, \pi/2], \forall w \in (w_0, \infty), \kappa > 1,
\]

which is another, equivalent, form of the known two-parameter family of nonoptimum generalized oscillator representations (130) for \( \hat{H}_1 \), and

\[
\hat{H}_1 = \hat{H}_{eb(w_0)} = \hat{b}(w_0)\hat{b}^+(w_0) + 2v^2(1 + \kappa), \kappa > 1.
\]

which is another, equivalent, form of the known optimum generalized oscillator representation (131) for \( \hat{H}_1 \). Of course, the comment following (131) holds including formulas (132) and (133).

When proceeding to \( \hat{H}_{eb} \) with \( g_1, g_2 \) lying in the open strip \( \{-1/4 < g_1 < 3/4, g_2 > 0 \} \), we have, in view of (59), (63) and (79), (82), to distinguish the cases \( \mu = \pi/2, w > w_0 \) and \( w = w_0 \) from the case \( \mu \in [0, \pi/2), w > w_0 \).

#### 3.3.2 Strip \( \{-1/4 < g_1 < 3/4 \ (0 < \kappa < 1), g_2 > 0 \} \), cases \( \mu = \pi/2, w > w_0 = -\frac{1}{2}(1 + \kappa) \) and \( w = w_0 \)

A consideration in this subsubsec. is completely similar to that in the previous subsubsec.

For any pair of coupling constants \( g_1, g_2 \) lying in this strip and these values of the parameters \( \mu, w \), we have the identities \( \hat{b}(\pi/2, w) = \hat{a}^+(\pi/2, w), \hat{b}^+(\pi/2, w) = \overline{\hat{a}}(\pi/2, w) \), see the respective (63) and (59), and \( \hat{b}(w_0) = \hat{a}^+(w_0), \hat{b}^+(w_0) = \overline{\hat{a}}(w_0) \), see the respective
These identities and the result of subsubsec. 3.2.2 provide the identities $H_{eb(\pi/2,w)} = H_{ea(\pi/2,w)} = \hat{H}_{2,\pm\pi/2} = \hat{H}_{ea(w_0)} = H_{eb(w_0)}$ and yield equivalent forms of a part of generalized oscillator representations for $\hat{H}_{2,\pm\pi/2}$ presented in subsubsec. 3.2.2:

$$\hat{H}_{2,\pm\pi/2} = \hat{H}_{eb(\pi/2,w)} = \tilde{b}(\pi/2,w)\tilde{b}^+(\pi/2,w) - 4v^2wI,$$

$$\forall w \in (w_0, \infty), \kappa \in (0, 1),$$

which is another, equivalent, form of the one-parameter family of nonoptimum generalized oscillator representations for $\hat{H}_{2,\pm\pi/2}$ that is a restriction of the known two-parameter family of representations (134) to $\mu = \pi/2$, and

$$\hat{H}_{2,\pm\pi/2} = \hat{H}_{eb(w_0)} = \tilde{b}(w_0)\tilde{b}^+(w_0) + 2v^2(1 + \kappa), \kappa \in (0, 1),$$

which is another, equivalent, form of the known optimum generalized oscillator representation (135) for $\hat{H}_{2,\pm\pi/2}$. Of course, the comment following (135) holds including formulas (136) and (137).

### 3.3.3 Strip $\{-1/4 < g_1 < 3/4 \ (0 < \kappa < 1), \ g_2 > 0\}$, case $\mu \in [0, \pi/2), \ w > w_0 = -\frac{1}{2}(1 + \kappa)$

In this case, we deal only with the operators $\hat{H}_{eb(\mu,w)}$ and have to find the asymptotic behavior at the origin of functions belonging to their domains.

We begin with functions $\phi(\mu, w; x)$ (29) with indicated values of parameters, namely, with representing their asymptotic behavior at the origin given in (32) in a new form:

$$\phi(\mu, w; x) = c(\mu, w)[(vx)^{1/2+\kappa} \sin \theta(\mu, w) + (vx)^{1/2-\kappa} \cos \theta(\mu, w)] + O(x^{5/2-\kappa}), \ x \to 0,$$

$$\tan \theta(\mu, w) = \tan \mu - \frac{\Gamma(1-\kappa)\Gamma\left(\frac{1}{2}(1+\kappa)+w\right)}{\Gamma(1+\kappa)\Gamma\left(\frac{1}{2}(1-\kappa)+w\right)}, \ \tan \mu \geq 0,$$

$$c(\mu, w) = \frac{\cos \kappa}{\cos \theta(\mu, w)}, \ \theta(\mu, w) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

$$\mu \in [0, \frac{\pi}{2}), \ w \in (w_0, \infty), \ \kappa \in (0, 1).$$

By definition of the operator $\hat{H}_{eb(\mu,w)}$, its domain $D_{H_{eb(\mu,w)}}$ consists of functions $\chi(x) \in D_{b^+(\mu,w)}$ such that $\tilde{b}^+(\mu,w)\chi(x) = \tilde{a}(\mu,w)\chi(x) = \eta(x) \in D_{\hat{H}_{0(\mu,w)}}$. The first condition implies that $\chi(x)$ allows representation (50) with $x_0 = 0$ and, in general, $D \neq 0$, see (51), while the second condition implies that $\eta(x) = O(x^{1/2})$, $x \to 0$, see (65). Estimating the integral term in (50) with such $\eta(x)$, we obtain that the asymptotic behavior of functions $\chi(x) \in D_{\hat{H}_{eb(\mu,w)}}$, $\mu \in [0, \pi/2), \ w \in (w_0, \infty), \ \kappa \in (0, 1)$, at the origin is given by

$$\chi(x) = c[(vx)^{1/2+\kappa} \sin \theta(\mu, w) + (vx)^{1/2-\kappa} \cos \theta(\mu, w)] + O(x^{3/2}), \ x \to 0,$$  (142)

where $c = Dc(\mu, w)$ is arbitrary. A comparison of (142) and (128) naturally identifies the parameter $\nu$ in (128) with the angle $\theta(\mu, w)$ in (142) and establishes that the Hamiltonian $\hat{H}_{2,\nu}$ with given $g_1 \in (-1/4, 3/4), \ g_2 > 0$ and $\nu \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ must be identified with all those $\hat{H}_{b(\mu,w)}$ with the same $g_1, g_2$, for which the parameters $\mu$ and $w$ are related by $\theta(\mu, w) = \nu,$
i.e., \( \{ \hat{H}_{b(\mu, w)}, \theta(\mu, w) = \nu \} = \hat{H}_{2, \nu} \). It is convenient to take \( \mu \) as an independent parameter, then \( w \) is determined from the relation \( \tan \theta(\mu, w) = \tan \nu \), or

\[
\frac{\Gamma(1 - \kappa) \Gamma\left(\frac{1}{2}(1 + \kappa) + w\right)}{\Gamma(1 + \kappa) \Gamma\left(\frac{1}{2}(1 - \kappa) + w\right)} = \tan \mu - \tan \nu \tag{143}
\]

\( \nu \in (-\pi/2, \pi/2), \mu \in [0, \pi/2), \kappa \in (0, 1), \)

considered as an equation with respect to \( w \) under the additional condition \( w > w_0 \). It can be shown that the l.h.s. of eq. (143) as a function of \( w > w_0 \) is a continuous monotonically increasing function\(^{15}\) ranging from \(-\infty \) to \( \infty \) as \( w \) ranges from \( w_0 + 0 \) to \( \infty \). It follows that eq. (143) always has a unique solution \( w = w(\mu, \nu) \) and \( w(\mu, \nu) \) as a function of \( \mu \) increases monotonically from \( w(0, \nu) > -\frac{1}{2}(1 + \kappa) \) to \( \infty \) as \( \mu \) ranges from \( 0 \) to \( \frac{\pi}{2} - 0 \). The previous result then can be written as \( \{ \hat{H}_{b(\mu, w(\mu, \nu))} \} = \hat{H}_{2, \nu} \), which yields the one-parameter family of generalized oscillator representations for the Hamiltonian \( \hat{H}_{2, \nu} \) with given coupling constants \( g_1 \in (-1/4, 3/4), g_2 > 0 : \)

\[
\hat{H}_{2, \nu} = \hat{H}_{b(\mu, w(\mu, \nu))} = \hat{b}(\mu, w(\mu, \nu)) \hat{b}^\dagger(\mu, w(\mu, \nu)) - 4\nu^2 w(\mu, \nu) \hat{I}, \tag{144}
\]

\( \nu \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \forall \mu \in [0, \pi/2), \kappa \in (0, 1), \)

the parameter is \( \mu \), and \( w = w(\mu, \nu) \) is a solution of eq. (143).

According to (55), any of representations (144) with \( \mu \in (0, \pi/2) \) is nonoptimum, we can only say that \( \hat{H}_{2, \nu} \) is bounded from below by \(-4\nu^2 w(0, \nu) \), while representation with \( \mu = 0 \) is an optimum one, the ground-state energy \( E_0^{(2)}(\nu) \) of \( \hat{H}_{2, \nu} \), which is a lower boundary of its spectrum, is

\[
E_0^{(2)}(\nu) = -4\nu^2 w(0, \nu), \kappa \in (0, 1),
\]

and the normalized ground state \( U_\nu^{(2)}(x) \) of \( \hat{H}_{2, \nu} \) is given by

\[
U_\nu^{(2)}(x) = Q_0(\nu)e^{-(\nu x)^2/2(\nu x)^{1/2+\kappa}}\Psi\left(\frac{1}{2}(1 + \kappa) + w(0, \nu), 1 + \kappa; (\nu x)^2\right), \kappa \in (0, 1),
\]

\( Q_0(\nu) \) is a normalization factor.

The ground-state energy \( E_0^{(2)}(\nu) \) is a unique solution of the equation

\[
\frac{\Gamma(1 - \kappa) \Gamma\left(\frac{1}{2}(1 + \kappa) - E/4\nu^2\right)}{\Gamma(1 + \kappa) \Gamma\left(\frac{1}{2}(1 - \kappa) - E/4\nu^2\right)} = -\tan \nu, \nu \in (-\pi/2, \pi/2), \tag{145}
\]

considered as an equation for \( E \) under the additional condition \( E < 2\nu^2(1 + \kappa) \). We are unable to present an explicit expression for \( E_0(\nu) \) except \( E_0(0) = 2\nu^2(1 - \kappa) \), but we can assert that \( E_0^{(2)}(\nu) \) monotonically increases from \(-\infty \) to \( 2\nu^2(1 + \kappa) - 0 \) as \( \nu \) ranges from \(-\pi/2 + 0 \) to \( \pi/2 - 0 \). We note that according to [2], eq. (145) determines the spectrum of \( \hat{H}_{2, \nu} \), which is a discrete one, spec\( \hat{H}_{2, \nu} = \{ E_n(\nu) \}, n = 0, 1, 2, ..., E_{n+1}(\nu) > E_n(\nu) \), and the condition \( E < 2\nu^2(1 + \kappa) \) separates out the minimum eigenvalue, the ground-state energy \( E_0^{(2)}(\nu) \).

It remains to consider \( \hat{H}_{b(\mu, w)} \) with \( g_1, g_2 \) lying on the open semiaxis \( \{ g_1 = -1/4, g_2 > 0 \} \). In view of (105), (109) and (122), (125), we have to distinguish the cases \( \mu = \pi/2, w > w_0 = -\frac{1}{2} \) and \( w = w_0 = -\frac{1}{2} \) from the case \( \mu \in [0, \pi/2), w > w_0 \).

\(^{15}\)This actually was shown in [2].
3.3.4 Semiaxis \( \{g_1 = -1/4 \ (\varkappa = 0), \ g_2 > 0 \} \), cases \( \mu = \pi/2, \ w > w_0 = -\frac{1}{2} \) and \( w = w_0 \)

A consideration in this subsection is completely similar to that in subsec. 3.3.2.

For any pair of coupling constants \( g_1, g_2 \) lying on this semiaxis and these values of the parameters \( \mu, w \), we have the identities \( \hat{b}(\pi/2, w) = \hat{a}^+ (\pi/2, w), \hat{b}^+ (\pi/2, w) = \hat{\alpha}(\pi/2, w) \), see the respective \( (109) \) and \( (105) \), and \( \bar{b}(w_0) = \hat{a}(w_0), \hat{b}^+(w_0) = \hat{\alpha}(w_0) \), see the respective \( (125) \) and \( (122) \). These identities and the result of subsec. 3.2.3 provide the identities \( \hat{H}_{eb(\pi/2, w)} = \hat{H}_{ea(\pi/2, w)} = \hat{H}_{b,\pm\pi/2} = \hat{H}_{ea(w_0)} = \hat{H}_{eb(w_0)} \) and yield equivalent forms of a part of generalized oscillator representations for \( \hat{H}_{b,\pm\pi/2} \) presented in subsec. 3.2.3:

\[
\hat{H}_{3,\pm\pi/2} = \hat{H}_{eb(\pi/2, w)} = \bar{b}(\pi/2, w)\hat{b}^+(\pi/2, w) - 4\nu^2 w \hat{I}, \quad \forall w \in (w_0, \infty),
\]

which is another, equivalent, form of the one-parameter family of nonoptimum generalized oscillator representations for \( \hat{H}_{b,\pm\pi/2} \), the parameter is \( w \), that is a restriction of the known two-parameter family of representations \( (138) \) to \( \mu = \pi/2 \), and

\[
\hat{H}_{3,\pm\pi/2} = \hat{H}_{eb(w_0)} = \bar{b}(w_0)\hat{b}^+(w_0) + 2\nu^2,
\]

which is another, equivalent, form of the known optimum generalized oscillator representation \( (139) \) for \( \hat{H}_{b,\pm\pi/2} \). Of course, the comment following \( (139) \) holds including formulas \( (140) \) and \( (141) \).

3.3.5 Semiaxis \( \{g_1 = -1/4 \ (\varkappa = 0), \ g_2 > 0 \} \), case \( \mu \in [0, \pi/2], \ w > w_0 = -\frac{1}{2} \)

A consideration in this subsection is completely similar to that in subsec. 3.3.3.

We begin with functions \( \phi(\mu, w; x) \) \( (83) \) with indicated values of parameters, namely, with representing their asymptotic behavior at the origin given in \( (85) \) in a new form:

\[
\phi(\mu, w; x) = c(\mu, w)[(vx)^{1/2} \sin \theta(\mu, w) + 2 (vx)^{1/2} \ln(vx) \cos \theta(\mu, w)] + O(x^{5/2} \ln x), \ x \to 0,
\]

\[
\tan \theta(\mu, w) = \psi(\frac{1}{2} + w) - 2\psi(1) - \tan \mu, \ \tan \mu \geq 0,
\]

\[
c(\mu, w) = -\frac{\cos \mu}{\cos \theta(\mu, w)}, \ \theta(\mu, w) \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\
\mu \in [0, \frac{\pi}{2}), \ w \in (w_0, \infty), \ w_0 = -\frac{1}{2}, \ \varkappa = 0.
\]

By definition of the operator \( \hat{H}_{eb(\mu, w)} \), its domain \( D_{\hat{H}_{eb(\mu, w)}} \) consists of functions \( \chi(x) \in D_{\hat{b}^+(\mu, w)} \) such that \( \hat{b}^+(\mu, w)\chi(x) = \hat{\alpha}(\mu, w)\chi(x) = \eta(x) \in D_{\hat{b}(\mu, w)} \). The first condition implies that \( \chi(x) \) allows representation \( (97) \) with \( x_0 = 0 \) and, in general, \( D \neq 0 \), see \( (98) \), while the second condition implies that \( \eta(x) = O(x^{1/2}), \ x \to 0, \) see \( (111) \). Estimating the integral term in \( (97) \) with such \( \eta(x) \), we obtain that the asymptotic behavior of functions \( \chi(x) \in D_{\hat{H}_{eb(\mu, w)}}, \mu \in [0, \pi/2), \ w \in (w_0, \infty), \ w_0 = -\frac{1}{2}, \ \varkappa = 0, \) at the origin is given by

\[
\chi(x) = c[(vx)^{1/2} \sin \theta(\mu, w) + 2 (vx)^{1/2} \ln(vx) \cos \theta(\mu, w)] + O(x^{3/2}), \ x \to 0,
\]

where \( c = Dc(\mu, w) \) is arbitrary. A comparison of \( (146) \) and \( (129) \) naturally identifies the parameter \( \nu \) in \( (129) \) with the angle \( \theta(\mu, w) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) in \( (146) \) and establishes that
the Hamiltonian $\hat{H}_{3,\nu}$ with given $g_1 = -1/4, g_2 > 0$ and $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2})$ must be identified with all those $\hat{H}_{b(\mu,w)}$ with the same $g_1, g_2$, for which the parameters $\mu$ and $w$ are related by $\theta(\mu, w) = \nu$, i.e., \{ $\hat{H}_{b(\mu,w)}$, $\theta(\mu, w) = \nu$ \} = $\hat{H}_{3,\nu}$. It is convenient to take $\mu$ as an independent parameter, then $w$ is determined from the relation $\tan \theta(\mu, w) = \tan \nu$, or

$$\psi(1/2 + w) = \tan \nu + \tan \mu + 2\psi(1), \nu \in (-\pi/2, \pi/2), \mu \in [0, \pi/2),$$

(147)

considered as an equation with respect to $w$ under the additional condition $w > w_0$. The function $\psi(1/2 + w)$ is a continuous monotonically increasing function of $w$ ranging from $-\infty$ to $\infty$ as $w$ ranges from $w_0 + 0$ to $\infty$. It follows that eq. (147) always has a unique solution $w = w(\mu, \nu)$ and $w(\mu, \nu)$ as a function of $\mu$ increases monotonically from $w(0, \nu) > \frac{-1}{2}$ to $\infty$ as $\mu$ ranges from $0$ to $\frac{\pi}{2} - 0$. The previous result then can be written as \{ $\hat{H}_{b(\mu,w(\mu,\nu))}$ \} = $\hat{H}_{3,\nu}$, which yields the one-parameter family of generalized oscillator representations for the Hamiltonian $\hat{H}_{3,\nu}$ with given coupling constants $g_1 = -1/4, g_2 > 0$:

$$\hat{H}_{3,\nu} = \hat{H}_{eb(\mu,w(\mu,\nu))} = \tilde{b}(\mu, w(\mu, \nu))\tilde{b}^+(\mu, w(\mu, \nu)) - 4\nu^2 w(\mu, \nu)\tilde{I}, \nu \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \forall \mu \in [0, \pi/2),$$

(148)

the parameter is $\mu$, and $w = w(\mu, \nu)$ is a solution of eq. (147).

According to (102), any of representations (148) with $\mu \in (0, \pi/2)$ is nonoptimum, we can only say that $\hat{H}_{3,\nu}$ is bounded from below by $-4\nu^2 w(0, \nu)$, while representation with $\mu = 0$ is an optimum one, the ground-state energy $E_0^{(3)}(\nu)$ of $\hat{H}_{3,\nu}$, which is a lower boundary of its spectrum, is

$$E_0^{(3)}(\nu) = -4\nu^2 w(0, \nu),$$

and the normalized ground state $U_{\nu}(x)$ of $\hat{H}_{3,\nu}$ is given by

$$U_{\nu}^{(3)}(x) = Q_0(\nu)e^{-(vx)^2/2}(vx)^{1/2}\Psi\left(\frac{1}{2} + w(0, \nu), 1; (vx)^2\right),$$

$Q_0(\nu)$ is a normalization factor.

The ground-state energy $E_0^{(3)}(\nu)$ is a unique solution of the equation

$$\psi(1/2 - \frac{E}{4\nu^2}) = \tan \nu + 2\psi(1), \nu \in (-\pi/2, \pi/2),$$

(149)

considered as an equation for $E$ under the additional condition $E < 2\nu^2$. We are unable to present an explicit expression for $E_0^{(3)}(\nu)$, but we can assert that $E_0^{(3)}(\nu)$ monotonically decreases from $2\nu^2 - 0$ to $-\infty$ as $\nu$ ranges from $-\pi/2 + 0$ to $\pi/2 - 0$.

We note that according to [2], eq. (149) determines the spectrum of $\hat{H}_{3,\nu}$, which is a discrete one, spec$\hat{H}_{3,\nu} = \{ E_n(\nu) \}, n = 0, 1, 2, ..., E_{n+1}(\nu) > E_n(\nu)$, and the condition $E < 2\nu^2$ separates out the minimum eigenvalue, the ground-state energy $E_0^{(3)}(\nu)$.

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