GLOBAL WELL-POSEDNESS FOR THE DEFOCUSING, CUBIC NONLINEAR
SCHRÖDINGER EQUATION WITH INITIAL DATA IN A CRITICAL SPACE

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Abstract. In this note we prove global well-posedness for the defocusing, cubic nonlinear Schrödinger equation with initial data lying in a critical Sobolev space.

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1. Introduction

In this note, we discuss the defocusing, cubic, nonlinear Schrödinger equation in three dimensions,

\[ iu_t + \Delta u = F(u) = |u|^2 u, \quad u(0, x) = u_0 \in \dot{H}^{1/2}(\mathbb{R}^3). \]

Equation (1.1) has a scaling symmetry. For any \( \lambda > 0 \), if \( u \) solves (1.1), then

\[ u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x), \]

also solves (1.1). The initial data \( \lambda u_0(\lambda x) \) has \( \dot{H}^{1/2}(\mathbb{R}^3) \) norm that is invariant under the scaling (1.2).

The local theory for initial data lying in \( \dot{H}^{1/2}(\mathbb{R}^3) \) has been completely worked out, and the scaling symmetry has been shown to control the local well-posedness theory.

**Theorem 1.** Assume \( u_0 \in \dot{H}^{1/2}(\mathbb{R}^3) \), \( \|u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \leq A \). Then there exists \( \delta = \delta(A) \) such that if \( \|e^{it\Delta}u_0\|_{L^5_{t,x}(I \times \mathbb{R}^3)} < \delta \), then there exists a unique solution to (1.1) on \( I \times \mathbb{R}^3 \) with \( u \in C(I; \dot{H}^{1/2}(\mathbb{R}^3)) \), and

\[ \|u\|_{L^5_{t,x}(I \times \mathbb{R}^3)} \leq 2\delta. \]

Moreover, if \( u_{0,k} \to u_0 \) in \( \dot{H}^{1/2}(\mathbb{R}^3) \), the corresponding solutions \( u_k \to u \) in \( C(I; \dot{H}^{1/2}(\mathbb{R}^3)) \).

This theorem was proved in [CW90].
From this, it is straightforward to show that local well-posedness holds for (1.1) for any initial data \( u_0 \in \dot{H}^{1/2}(\mathbb{R}^3) \). Indeed, by the dominated convergence principle combined with Strichartz estimates, for any \( u_0 \in \dot{H}^{1/2}(\mathbb{R}^3) \),

\[
\lim_{T \to 0} \| e^{it\Delta} u_0 \|_{L^6_t([-T,T] \times \mathbb{R}^3)} = 0.
\]

Since \( \delta(A) \) is decreasing as \( A \searrow +\infty \), Strichartz estimates imply that there exists \( \delta_0 > 0 \) such that if \( \| u_0 \|_{\dot{H}^{1/2}(\mathbb{R}^3)} < \delta_0 \), (1.1) has a global solution that scatters. By scattering, we mean that there exist \( u^+_0, u^-_0 \) so that

\[
\lim_{t \to +\infty} \| u(t) - e^{it\Delta} u^+_0 \|_{\dot{H}^{1/2}} = 0,
\]

and

\[
\lim_{t \to -\infty} \| u(t) - e^{it\Delta} u^-_0 \|_{\dot{H}^{1/2}} = 0.
\]

However, it is important to note that while (1.4) holds for any fixed \( u_0 \in \dot{H}^{1/2}(\mathbb{R}^3) \), the convergence is not uniform, even for \( \| u_0 \|_{\dot{H}^{1/2}(\mathbb{R}^3)} \leq A < \infty \). Thus, one cannot conclude directly from [CW90] that a uniform bound for \( \| u(t) \|_{\dot{H}^{1/2}(\mathbb{R}^3)} \) on the entire time of the existence of the solution to (1.1) implies that the solution is global. This result was instead proved in [KM10], using concentration compactness methods.

**Theorem 2.** Suppose that \( u \) is a solution of (1.1) with initial data \( u_0 \in \dot{H}^{1/2}(\mathbb{R}^3) \) and a maximal interval of existence \( I = (T_-, T_+) \). Also assume that \( \sup_{t \in (T_-, T_+)} \| u(t) \|_{\dot{H}^{1/2}(\mathbb{R}^3)} = A < \infty \). Then \( T_+(u_0) = +\infty \), \( T_-(u_0) = -\infty \), and the solution \( u \) scatters.

It is conjectured that (1.1) is globally well-posed and scattering for any \( u_0 \in \dot{H}^{1/2}(\mathbb{R}^3) \), without the a priori assumption of a universal bound on the \( \dot{H}^{1/2} \) norm of the solution \( u(t) \). Partial progress has been made in this direction.

A solution to (1.1) has the conserved quantities mass,

\[
M(u(t)) = \int |u(t,x)|^2 \, dx = M(u(0)),
\]

and energy,

\[
E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 \, dx + \frac{1}{4} \int |u(t,x)|^4 \, dx.
\]

This fact implies global well-posedness for (1.1) with \( u_0 \in H^1_x(\mathbb{R}^3) \), where \( H^1_x(\mathbb{R}^3) \) is the inhomogeneous Sobolev space of order one. In this case, one could also prove bounds on the scattering size directly, using the interaction Morawetz estimate of [CKS04].

**Theorem 3.** If \( u \) is a solution to (1.1), on an interval \( I \), then

\[
\| u \|_{L^4_t L^4_x(I \times \mathbb{R}^3)} \lesssim \| u \|_{L^\infty_t L^2_x(I \times \mathbb{R}^3)}^2 \| u \|_{L^\infty_t \dot{H}^{1/2}(I \times \mathbb{R}^3)}^2 \lesssim E(u)^{1/2} M(u)^{3/2}.
\]

Interpolating (1.8) and (1.9) then implies

\[
\| u \|_{L^4_t L^4_x(I \times \mathbb{R}^3)}^4 \lesssim M(u)^{3/4} E(u)^{3/4},
\]

with bounds independent of \( I \subset \mathbb{R} \). Combining Strichartz estimates and local well-posedness theory, a uniform bound on (1.10) for any \( I \subset \mathbb{R} \) directly implies a uniform bound on

\[
\| u \|_{L^4_x(I \times \mathbb{R}^3)}.
\]
The argument from [CW90] implies that proving scattering is equivalent to proving
\begin{equation}
\|u\|_{L^5_t L^{20/3}_x(\mathbb{R} \times \mathbb{R}^3)} < \infty.
\end{equation}
Indeed, assuming (1.12) is true, the interval \( \mathbb{R} \) may be partitioned into finitely many pieces \( J_k \) such that
\begin{equation}
\|u\|_{L^5_t L^{20/3}_x(J_k \times \mathbb{R}^3)} \leq \delta.
\end{equation}
Then iterate the argument over the intervals \( J_k \), which proves scattering.

This argument also shows that a solution to (1.1) blowing up at a finite time \( T_0 < \infty \) is equivalent to
\begin{equation}
\|u\|_{L^5_t L^{20/3}_x([0,T_0] \times \mathbb{R}^3)} = \infty.
\end{equation}

\textbf{Remark:} Prior to [CKS+04], [Bou98a] proved scattering using the standard Morawetz estimate.

\textbf{Remark:} See [Tao06] for more details on Strichartz estimates.

Many have attempted to lower the regularity needed in order to prove global well-posedness. For any \( s > \frac{1}{2} \), the inhomogeneous Sobolev space \( H^s_x(\mathbb{R}^3) \subset \dot{H}^{1/2}(\mathbb{R}^3) \). Therefore, if \( u_0 \in H^s_x(\mathbb{R}^3) \), then it would be conjectured that the solution to (1.1) with initial data \( u_0 \) is global and scatters.

Proving a uniform bound on the \( H^s_x(\mathbb{R}^3) \) norm would be enough, since by interpolation this would guarantee a uniform bound on the \( \dot{H}^{1/2}(\mathbb{R}^3) \) norm. The difficulty is that there does not exist a conserved quantity at regularity \( s \) that controls the \( \dot{H}^s \) norm for \( \frac{1}{2} < s < 1 \).

Instead, [Bou98b] used the Fourier truncation method. Decompose the initial data
\begin{equation}
u_0 = P_{\leq N}u_0 + P_{> N}u_0 = v_0 + w_0.
\end{equation}
Then \( v_0 \in H^1(\mathbb{R}^3) \), and \( \|v_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \) is small. Thus, (1.1) has a global solution for initial data \( v_0 \) or \( w_0 \), call them \( v \) and \( w \). Since (1.1) is a nonlinear equation, it is necessary to also estimate the interaction between \( v \) and \( w \) in the nonlinearity of (1.1). Then, [Bou98b] proved global well-posedness for (1.1) with initial data \( u_0 \in H^s_x(\mathbb{R}^3) \) when \( s > \frac{11}{13} \). Moreover, [Bou98b] proved that the solution is of the form
\begin{equation}
e^{it\Delta}u_0 + v(t),
\end{equation}
where \( v(t) \in H^s_x(\mathbb{R}^3) \).

The results of [Bou98b] for (1.1) were improved using the I-method. First, [CKS+04] improved the regularity necessary for global well-posedness to \( s > \frac{5}{7} \). Then, [CKS+04] improved the necessary regularity to \( s > \frac{1}{4} \). To the author’s best knowledge, the best known regularity result is the result of [Su12], proving global well-posedness and scattering for regularity \( s > \frac{5}{7} \). For radial initial data, [Dod18] proved global well-posedness and scattering for any \( s > \frac{1}{2} \). This result is almost sharp.

In this paper, we study the cubic nonlinear Schrödinger equation (1.1) with initial data lying in the Sobolev space \( W^{\frac{2}{3}, \frac{44}{7}}_x(\mathbb{R}^3) \). That is,
\begin{equation}
\|\nabla^{\frac{1}{3}} u_0\|_{L^{\frac{44}{7}}(\mathbb{R}^3)} < \infty.
\end{equation}

\textbf{Remark:} This norm is well-defined using the Littlewood–Paley decomposition. See for example [Tay10].
This norm is preserved under the scaling (1.2), and is therefore a critical Sobolev norm. Moreover, $W^\frac{7}{6} \times H^\frac{11}{7} (\mathbb{R}^3) \subset \dot{H}^{1/2} (\mathbb{R}^3)$, so (1.1) has a local solution for this initial data. We prove global well-posedness for (1.1) with this initial data.

**Theorem 4.** The cubic nonlinear Schrödinger equation is globally well-posed for initial data $u_0 \in W^\frac{7}{6} \times H^\frac{11}{7} (\mathbb{R}^3)$.

The proof of this theorem will heavily utilize dispersive estimates. Interpolating between the fact that $e^{it\Delta}$ is a unitary operator,

$$\|e^{it\Delta} u_0\|_{L^2(\mathbb{R}^3)} = \|u_0\|_{L^2(\mathbb{R}^3)},$$

and the dispersive estimate,

$$\|e^{it\Delta} u_0\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{t^{3/2}} \|u_0\|_{L^1(\mathbb{R}^3)},$$

gives the estimate

$$\|e^{it\Delta} u_0\|_{L^7(\mathbb{R}^3)} \lesssim \frac{1}{t^{\frac{15}{14}}} \|u_0\|_{L^\frac{7}{6}(\mathbb{R}^3)}.$$

This implies that the linear solution $e^{it\Delta} u_0$ has very good behavior when $t > 1$, in fact it is integrable if true. We then rescale so that $u_0$ has a local solution on an interval $[-1, 1]$. We prove that this solution may be decomposed into

$$u(t) = e^{it\Delta} u_0 + v(t) + w(t).$$

In particular,

$$u(1) = e^{i\Delta} u_0 + v(1) + w(1).$$

The term

$$e^{i(t-1)\Delta} e^{i\Delta} u_0 = e^{it\Delta} u_0$$

has good properties when $t > 1$. We can also show that

$$\|\nabla e^{i(t-1)\Delta} v(1)\|_{L^\infty} \lesssim \frac{1}{t^{3/2}},$$

which also has good properties when $t > 1$. Finally, $w(1) \in H^1_x$ and has finite energy. Making a Gronwall argument shows that

$$\|u(t) - e^{it\Delta} u_0 - e^{i(t-1)\Delta} v(1)\|_{H^1},$$

is uniformly bounded on $[1, \infty)$. This is enough to give global well-posedness, but not scattering.

This result could be compared to the result in [Dod19] for the nonlinear wave equation. There, the author proved global well-posedness and scattering for the cubic wave equation with initial radial data in the Besov space $B^2_{1,1} \times B^1_{1,1}$. Here, we do not require radial symmetry, however, we only prove global well-posedness. We are unable to prove scattering at this time due to the lack of a scale invariant conformal symmetry.

We prove a local well-posedness result in section two, and a global result in section three. This argument could be generalized to many intercritical, defocusing nonlinear Schrödinger equations.
2. Local well-posedness

The Sobolev embedding theorem implies that $W^{\frac{5}{2},\infty}([R^3])$ is embedded into $\dot{H}^{1/2}(R^3)$. Therefore, (1.1) is locally well-posed, and there exists some $T(u_0) > 0$ such that (1.1) has a solution on $[-T, T]$ and $\|u\|_{L_t^2([-T, T] \times [R^3])} = \epsilon_0$, for some $\epsilon_0(\|u_0\|_{\dot{H}^{1/2}})$ small. After rescaling using (1.2), suppose

\begin{equation}
(2.1) \quad \|u\|_{L_{t,x}^2([-1, 1] \times [R^3])} = \epsilon_0.
\end{equation}

Since (3.18) is an admissible pair, Strichartz estimates imply

\begin{equation}
(2.2) \quad \|\nabla^{1/2}u\|_{L_t^\infty L_x^2 \cap L_t^2 L_x^6([-1, 1] \times [R^3])} \lesssim \|\nabla^{1/2}u_0\|_{L_x^2(R^3)} + \|\nabla^{1/2}u\|_{L_t^1 L_x^6([-1, 1] \times [R^3])} \lesssim \epsilon_0.
\end{equation}

Therefore,

\begin{equation}
(2.3) \quad \|\nabla^{1/2}u\|_{L_t^\infty L_x^2 \cap L_t^2 L_x^6([-1, 1] \times [R^3])} \lesssim \|u_0\|_{\dot{H}^{1/2}}.
\end{equation}

Also, by Duhamel’s principle, for any $t \in [-1, 1],

\begin{equation}
(2.4) \quad u(t) = e^{i\Delta}u_0 - i \int_0^t e^{i(t-\tau)\Delta} F(u(\tau))d\tau = u(t) + u_{nl}(t).
\end{equation}

We begin with a technical lemma. This lemma allows us to make a Littlewood–Paley decomposition of $u_{nl}$, treat each $P_j u_{nl}$ separately, and then sum up. It also implies that $u_{nl}$ retains all the properties of a solution to the linear Schrödinger equation with initial data in a Besov space.

**Remark:** In this section, all implicit constants depend on the norm $\|u_0\|_{W^{\frac{5}{2},\infty}}$.

**Remark:** Throughout this section we rely very heavily on the bilinear Strichartz estimate

\begin{equation}
(2.5) \quad \|(e^{it\Delta}P_j u_0)(e^{is\Delta}P_k v_0)\|_{L_{t,x}^2([R \times [R^3])} \lesssim 2^{-j/2}2^k \|P_j u_0\|_{L_x^2} \|P_k v_0\|_{L_x^2}.
\end{equation}

See [Bon98b] for a proof.

**Lemma 1.** Let $P_j$ be the customary Littlewood–Paley projection operator. Also suppose that $u$ is a solution to (1.1) satisfying (2.1). Then

\begin{equation}
(2.6) \quad \sum_j 2^{j/2} \|P_j F(u)\|_{L_t^1 L_x^2([-1, 1] \times [R^3])} \lesssim 1.
\end{equation}

**Proof:** Decompose the nonlinearity,

\begin{equation}
(2.7) \quad P_j F(u) = P_j F(P_{\geq j-3} u) + 3P_j((P_{\geq j-3} u)^2 (P_{\leq j-3} u)) + 3P_j((P_{j-3} \leq j-3 u)(P_{\leq j-3} u)^2).
\end{equation}

By Bernstein’s inequality, and (2.2),

\begin{equation}
(2.8) \quad \lesssim 2^{j/2} \|P_{\geq j-3} u\|_{L_t^1 L_x^2([-1, 1] \times [R^3])} \lesssim 2^{j/2} \|P_{\geq j-3} u\|_{L_t^1 L_x^2([-1, 1] \times [R^3])} \lesssim 2^{j/2} \left( \sum_{t \geq j-3} 2^{-l/6} \|\nabla^{1/6} P_t u\|_{L_t^2 L_x^6} \right)^3.
\end{equation}

Next,

\begin{equation}
(2.9) \quad 2^{j/2} \|(P_{\geq j-3} u)(P_{\leq j-3} u)\|_{L_t^1 L_x^2([-1, 1] \times [R^3])} \lesssim 2^{j/2} \left( \sum_{t \geq j-3} 2^{-l/4} \|\nabla^{1/4} P_t u\|_{L_t^1 L_x^6} \right)^3 \|u\|_{L_t^3 L_x^2}.
\end{equation}
Finally, by the bilinear Strichartz estimate and the Sobolev embedding properties of Littlewood–Paley projections,

\[
2^{j/2} \| (P_{j-3} \lesssim \lesssim j+3 u) (P_{j-3} \lesssim \lesssim j+3 u)^2 \|_{L^2_t L^6_x([-1,1] \times \mathbb{R}^3)} \\
\lesssim \sum_{l_i \leq l \leq j-3} \| (P_{l_1} u) (P_{j-3} \lesssim \lesssim j+3 u) \|_{L^2_t L^6_x} \\
\sum_{l_i \leq l \leq j-3} \| P_{l_2} u \|_{L^4_t L^6_x} \\
\lesssim 2^{-j/2} \| \nabla 1^{1/2} u \|_{L^2_t L^6_x} \sum_{l_i \leq j-3} 2^{l/2} (j - l) \left( \| P_{j-3} \lesssim \lesssim j+3 u_0 \|_{L^2_x} + \| P_{j-3} \lesssim \lesssim j+3 F(u) \|_{L^1_t L^2} \right) \\
\times \left( \| P_{l_1} u_0 \|_{L^2} + \| P_{l_1} F(u) \|_{L^1_t L^2} \right).
\]

By Strichartz estimates, (2.10), Plancherel’s theorem, and the fractional product rule,

\[
\sum_j 2^j \| P_j u_0 \|_{L^2_t L^6_x}^2 + \sum_j 2^j \| P_j F(u) \|_{L^1_t L^2([-1,1] \times \mathbb{R}^3)}^2 \lesssim \| u_0 \|_{H^{1/2}}^2 + \| \nabla 1^{1/2} F(u) \|_{L^1_t L^2}^2 \\
\lesssim \| u_0 \|_{H^{1/2}}^2 + \| \nabla 1^{1/2} u \|_{L^1_t L^{6/5}}^2 \| u \|_{L^4_t L^6_x}^4 \lesssim 1.
\]

Combining (2.8)–(2.10) with the Cauchy–Schwarz inequality implies

\[
\sum_j 2^{j/2} \| P_j F(u) \|_{L^1_t L^2([-1,1] \times \mathbb{R}^3)} \lesssim 1,
\]

which proves the lemma. □

Next, decompose \( u_{nl} \) in the following manner:

\[
u_{nl}(t) = -i \int_0^{(1-\delta)t} e^{i(t-\tau)\Delta} F(u(\tau)) d\tau - i \int_{(1-\delta)t}^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau = v(t) + w(t),
\]

for some \( \delta > 0 \) sufficiently small, to be specified later.

**Lemma 2.** For any \( t \in [0,1] \),

\[
\| v(t) \|_{L^\infty} \lesssim \frac{1}{\delta t^{1/2} t^{1/2}},
\]

and

\[
\| \nabla v(t) \|_{L^\infty} \lesssim \frac{1}{\delta t}.
\]

**Proof:** By the dispersive estimate, since \( \| u \|_{L^3} \lesssim \| u \|_{H^{1/2}} \) is uniformly bounded on \([0,1]\),

\[
\| v(t) \|_{L^\infty} \lesssim \| \int_0^{(1-\delta)t} e^{i(t-\tau)\Delta} F(u(\tau)) d\tau \|_{L^\infty} \lesssim \int_0^{(1-\delta)t} \frac{1}{|t-\tau|^{3/2}} \| u \|_{L^3}^2 d\tau \lesssim \frac{1}{\delta t^{1/2} t^{1/2}}.
\]

To prove (2.15), observe that by the product rule,

\[
\nabla F(u) = 2 |u|^2 \nabla u + u^2 \nabla \bar{u}.
\]

Interpolating,

\[
\| | \nabla |^{1/2} u \|_{L^2} \lesssim \| | \nabla |^{1/2} u_0 \|_{L^2} \lesssim 1,
\]

with

\[
t^{15/14} \| | \nabla |^{11/7} u \|_{L^7} \lesssim \| | \nabla |^{11/7} u_0 \|_{L^{7/6}} \lesssim 1,
\]
we have
\begin{equation}
(2.20) \quad t^{1/2} \| \nabla u \|_L^3 \lesssim 1.
\end{equation}
Making a dispersive estimate,
\begin{equation}
(2.21) \quad \|
\int_0^{(1-\delta)t} e^{i(t-\tau)\Delta} |u|^2 \nabla u(t) \|_L^\infty \lesssim \int_0^{(1-\delta)t} \frac{1}{|t-\tau|^{3/2}} \| \nabla u(t) \|_L^3 \| u \|_{L^6}^2 d\tau
\end{equation}
\begin{equation}
\lesssim \int_0^{(1-\delta)t} \frac{1}{|t-\tau|^{3/2}} \frac{1}{|\tau|^{1/2}} d\tau \lesssim \frac{1}{\delta t}.
\end{equation}

The same computation may also be made for \( u^2 \nabla \bar{u}_l \).

Next, consider the contribution of \( |u|^2 \nabla u_{nl} \). By (2.22), we can, without loss of generality, consider only one \( P_j \) Littlewood-Paley multiplier, provided the estimate is uniform in \( 2^{j/2} \| P_j F(u) \|_{L^1_t L^2_x} \).

\begin{equation}
(2.22) \quad |u|^2 (\nabla P_j u_{nl}) = |P_{\leq j} u|^2 \nabla P_j u_{nl} + 2 \text{Re}(P_{j < j} u \nabla P_j u_{nl} + |P_{j > j} u|^2 \nabla P_j u_{nl}).
\end{equation}

Making a bilinear Strichartz estimate and the Cauchy–Schwarz inequality,
\begin{equation}
(2.23) \quad \| u_{\leq j} \|_{L^2_t L^1_x([0,1] \times \mathbb{R}^2)} \lesssim \sum_{j_1 \leq j_2 \leq j} \| (P_{j_1} u)(P_{j_2} \nabla u_{nl}) \|_{L^1_t L^6_x} \| P_{j_2} u \|_{L^\infty_t L^2_x}
\end{equation}
\begin{equation}
\lesssim \sum_{j_1 \leq j_2 \leq j} 2^{j_1/2 - j_2/2} 2^{j_1/2} \| P_{j_2} F(u) \|_{L^1_t L^6_x} \| \nabla |1/2 P_{j_1} u_{nl} \|_{L^2_x} + \| \nabla |1/2 P_{j_2} F(u) \|_{L^1_t L^2_x}
\end{equation}
\begin{equation}
\times (\| \nabla |1/2 P_{j_2} u_{nl} \|_{L^2_x} + \| \nabla |1/2 P_{j_2} F(u) \|_{L^1_t L^2_x}) \lesssim 1.
\end{equation}
Also, by Bernstein’s inequality,
\begin{equation}
(2.24) \quad \| \nabla P_{j_1} u_{nl} \|_{L^2_t L^1_x} \| \nabla |1/2 P_{j_2} u_{nl} \|_{L^2_x} \| \nabla |1/2 P_{j_2} F(u) \|_{L^1_t L^2_x} \lesssim 1.
\end{equation}
Therefore,
\begin{equation}
(2.25) \quad \| \int_0^{(1-\delta)t} e^{i(t-\tau)\Delta} |u|^2 \nabla u_{nl}(t) \|_L^\infty \lesssim \int_0^{(1-\delta)t} \frac{1}{|t-\tau|^{3/2}} \| u \|_{L^6}^2 \| \nabla u_{nl} \|_{L^1_t L^1_x} \lesssim \frac{1}{\delta t} \| u \|_{L^6}^2 \| \nabla u_{nl} \|_{L^1_t L^1_x} \lesssim \frac{1}{\delta t}.
\end{equation}

The same computation can be also be made for \( u^2 \nabla \bar{u}_{nl} \). This completes the proof of Lemma 2. \( \square \)

**Lemma 3.** For any \( t \in [0,1] \),
\begin{equation}
(2.26) \quad \| \nabla |1/2 w(t) \|_{L^3} \lesssim \frac{1}{\delta^{1/4} t^{1/4}}.
\end{equation}

**Proof:** First observe that by interpolation, Bernstein’s inequality, and (2.20),
\begin{equation}
(2.27) \quad \| \nabla |1/2 e^{it\Delta} u_0 \|_{L^3} \lesssim t^{1/4} \| \nabla e^{it\Delta} P_{\leq t^{-1/2}} u_0 \|_{L^3} + t^{-1/4} \| P_{t^{1/2} \leq \cdot \leq t} u_0 \|_{H^{1/2}} \lesssim t^{-1/4}.
\end{equation}

Also since \( e^{it\Delta} \) is unitary in \( L^2 \), by (2.22),
\begin{equation}
(2.28) \quad \| v(t) \|_{H^{1/2}} = \| u_{nl}(1 - \delta t) \|_{H^{1/2}} \lesssim \| \nabla |1/2 u_{nl} \|_{L^3} + \| u \|_{L^6}^2 \| u \|_{L^1}^2 \lesssim \frac{\epsilon_0^2}{\delta^{1/4} t^{1/4}}.
\end{equation}

so interpolating (2.13), (2.14), and (2.28),
\begin{equation}
(2.29) \quad \| \nabla |1/2 v \|_{L^3} \lesssim \| \nabla |1/2 v \|_{L^6} \| \nabla |1/2 v \|_{L^2}^{2/3} \| v \|_{L^6}^{4/3} \lesssim \frac{\epsilon_0^{4/3}}{\delta^{1/4} t^{1/4}}.
\end{equation}
Finally, making a dispersive estimate, for any $t \in [0, 1]$, by (2.27) and (2.29), if $\delta^{1/4} \ll \epsilon_0$, (2.30)
\[
\delta^{1/4} t^{1/4} \left| \int_{(1-\delta)t}^{t} e^{i(t-\tau)\Delta} |\nabla|^{1/2} F(u) d\tau \right|_{L^3} \lesssim \delta^{1/4} t^{1/4} \left| \int_{(1-\delta)t}^{t} \frac{1}{|t-\tau|^{1/2}} \left| |\nabla|^{1/2} u(\tau)\right|_{L^3} \left| u(\tau)\right|_{L^2}^2 d\tau \right|
\lesssim \left( \sup_{\tau \in [0,1]} \delta^{1/4} t^{1/4} \left| |\nabla|^{1/2} u\right|_{L^3}^3 \right) \lesssim \epsilon_0^4 + \left( \sup_{\tau \in [0,1]} \delta^{1/4} t^{1/4} \left| |\nabla|^{1/2} w\right|_{L^3}^3 \right).
\]

Therefore, absorbing the second term on the right hand side into the left hand side of (2.30) proves (2.26).

(2.31)
\[
\left\| |\nabla|^{1/2} w(t) \right\|_{L^3} \lesssim \frac{\epsilon_0^4}{\delta^{1/4} t^{1/4}}.
\]

\[
\square
\]

**Remark:** To make the proofs of Lemmas 2 and 3 completely rigorous, truncate $u_0$ in frequency. Then the bounds (2.13), (2.15), and (2.26) all hold on some open subset of $[0, 1]$ that contains 0. Making the bootstrap argument using the proof of Lemma 3 gives bounds on all of $[0, 1]$ that do not depend on the frequency truncation of $u_0$. Standard perturbation arguments then give the Lemmas.

Lemma 3 can be strengthened to an estimate on the $\dot{H}^1$ norm of $w$.

**Lemma 4.** For any $t \in [0, 1], \nabla v \lesssim \frac{1}{\delta^{1/4} t^{1/4}}$

**Proof:** Once again make use of the bilinear Strichartz estimate. Again by the product rule,

(2.33)
\[
\nabla F(u) = 2|u|^2 \nabla u + u^2 \nabla \bar{u}.
\]

First, by Strichartz estimates, (2.20), Lemma 3 and the Sobolev embedding theorem, (2.34)

\[
\left\| \int_{(1-\delta)t}^{t} e^{i(t-\tau)\Delta} \right| 2|u|^2 \nabla u + u^2 \nabla \bar{u} \right|_{L^2} \lesssim \left\| 2|u|^2 \nabla u + u^2 \nabla \bar{u} \right\|_{L^2 L_1^{5/4}}
\lesssim \delta^{1/4} t^{1/4} \left\| \nabla u \right\|_{L^\infty_x L_1^{5/4}((1-\delta,t] \times \mathbb{R}^3)} \left\| u \right\|_{L^\infty_x L_1^2((1-\delta,t] \times \mathbb{R}^3)} \left\| \nabla \right\|_{L^\infty_x L_1^{5/4}((1-\delta,t] \times \mathbb{R}^3)} \lesssim \frac{\delta^{1/4}}{t^{1/4}}.
\]

Next, by (2.26), bilinear Strichartz estimates and the Littlewood-Paley theorem, (2.35)

\[
\left\| 2|u|_{L_t^{1/2}}^2 \nabla P_j u \right\|_{L_1^2 L_1^{5/4}} \lesssim \sum_{k \geq 0} 2^{-k/2} \left( \sum_{j_1 \leq j} 2^{j_1+k} \left| P_{j_1+k} u \right|_{L_1^{5/4}}^2 \right)^{1/2} \left( \sum_{j_1 \leq j} 2^{-j_1} \left| P_{j_1} u \right|_{L_1^2} \right)^{1/2} \left\| P_j F(u) \right\|_{L_1^2 L_1^{5/4}} \lesssim \frac{1}{\delta^{1/4} t^{1/4}} \left\| |\nabla|^{1/2} F(u) \right\|_{L_1^1 L_1^2}.
\]

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Next, by Bernstein’s inequality,
\begin{align}
\|(\nabla P_j u_{nl})| u \geq j \|u \|_{{L}_2^2 \supset L^3_2} \lesssim \delta^{1/4} t^{1/4} \| \nabla^{1/2} u \|_{{L}_2^2(\mathbb{R}^3)} \| \nabla^{1/2} P_j u_{nl} \|_{{L}_2^2((1- \delta) t, t) \times \mathbb{R}^3)} \lesssim \frac{1}{\delta^{1/4} t^{1/4}} \| \nabla^{1/2} P_j F(u) \|_{{L}_2^2([0, 1] \times \mathbb{R}^3)}.
\end{align}

Summing up in \( j \) using Lemma 1 completes the proof. \( \square \)

**Remark:** The above arguments would work equally well in the time interval \([-1, 0]\).

### 3. Global well-posedness

We are now ready to prove Theorem 4. The proof will use conservation of the energy (3.8).

Decompose
\begin{align}
\tag{3.1}
u(1) = \tilde{v}(1) + w(1),
\end{align}
where
\begin{align}
\tag{3.2}
\tilde{v}(1) = u(1) + v(1),
\end{align}
and \( w(1) \) is the \( w \) in the previous section. Let \( T_0 > 1 \) be a time value for which we know that (1.1) has a solution on [0, \( T_0 \)). By standard local well-posedness arguments and (2.9), we know that such a \( T_0 \) exists. Then on [1, \( T_0 \)), decompose
\begin{align}
\tag{3.3}
u(t) = \tilde{v}(t) + w(t),
\end{align}
where \( \tilde{v}(t) \) is the solution to
\begin{align}
\tag{3.4}(i \partial_t + \Delta) \tilde{v}(t) = 0,
\end{align}
and \( w(t) \) is the solution to
\begin{align}
\tag{3.5}(i \partial_t + \Delta) w = |u|^2 u,
\end{align}
where \( w(1) = w(1, x) \).

Let \( E(t) \) denote the energy of \( w \),
\begin{align}
\tag{3.6}E(t) &= \frac{1}{2} \int |\nabla w|^2 + \frac{1}{4} \int |w|^4.
\end{align}
First observe that Lemma 4 and \( \| w \|_{\dot{H}^{1/2}} \lesssim 1 \) implies that \( E(1) < \infty \). To prove Theorem 4 it suffices to prove that for any \( T_0 > 1 \) such that (1.1) has a solution on [0, \( T_0 \))
\begin{align}
\tag{3.7}\sup_{t \in [1, T_0)} E(t) < \infty.
\end{align}
Indeed, by interpolation and the Sobolev embedding theorem, \( E(t) < \infty \) implies that \( \| w(t) \|_{L^5} < \infty \). Meanwhile, by (2.18)–(2.20), (2.14), and (2.28), \( \| \tilde{v}(t) \|_{L^5} \) is uniformly bounded on \( \mathbb{R} \). Therefore, (3.7) implies
\begin{align}
\tag{3.8}\| u \|_{L^5_1((0, T_0) \times \mathbb{R}^3)} < \infty.
\end{align}
To estimate the growth of \( E(t) \), compute the derivative in time of the energy. By (3.6),
\begin{align}
\tag{3.9}\frac{d}{dt} E(t) = -\langle \Delta w, w_t \rangle + \langle |w|^2 w, w_t \rangle = \langle |w|^2 w - |u|^2 u, w_t \rangle,
\end{align}
where \((\cdot, \cdot)\) is the inner product

$$
(f, g) = \Re \int f(x)\overline{g(x)}dx.
$$

By the product rule,

$$
\langle w_t, |u|^2u - |w|^2w \rangle = \frac{d}{dt} \langle |w|^2w, \tilde{v} \rangle + \frac{d}{dt} \langle |\tilde{v}|^2, |w|^2 \rangle + \frac{1}{2} \frac{d}{dt} \Re \int \overline{w}^2\tilde{v}^2 + \frac{d}{dt} \langle w, |\tilde{v}|^2\tilde{v} \rangle
$$

$$
-2\langle \tilde{v}_t\tilde{v}, |w|^2 \rangle - \langle |w|^2w, \tilde{v}_t \rangle - \Re \int \overline{w}^2\tilde{v}_t\overline{\tilde{v}} - 2\langle w, |\tilde{v}|^2\tilde{v}_t \rangle - \langle w, \overline{\tilde{v}}^2\tilde{v}_t \rangle.
$$

Then define the modified energy,

$$
\mathcal{E}(t) = E(t) - \langle |w|^2w, \tilde{v} \rangle - \langle |\tilde{v}|^2, |w|^2 \rangle - \frac{1}{2} \Re \int \overline{w}^2\tilde{v}^2 - \langle w, |\tilde{v}|^2\tilde{v} \rangle.
$$

By Holder’s inequality, and the fact that \(\|\cdot\|_{L^\infty} \lesssim_\delta 1\) for all \(t \in [1, \infty)\) (again using (2.18) – (2.20), 2.14, and (2.28)),

$$
\langle |w|^2w, \tilde{v} \rangle + \langle |\tilde{v}|^2, |w|^2 \rangle + \frac{1}{2} \Re \int \overline{w}^2\tilde{v}^2 + \langle w, |\tilde{v}|^2\tilde{v} \rangle \lesssim E(t)^{3/4} + E(t)^{1/4}.
$$

Therefore, when \(E(t)\) is large, \(E(t) \sim \mathcal{E}(t)\). Since we are attempting to prove a uniform bound for \(E(t)\), it is enough to uniformly bound \(\mathcal{E}(t)\).

Also, by (3.11),

$$
\frac{d}{dt} \mathcal{E}(t) = -\langle |w|^2w, \tilde{v}_t \rangle - 2\langle \tilde{v}_t\tilde{v}, |w|^2 \rangle - \Re \int \overline{w}^2\tilde{v}_t\overline{\tilde{v}} - 2\langle w, |\tilde{v}|^2\tilde{v}_t \rangle - \langle w, \overline{\tilde{v}}^2\tilde{v}_t \rangle.
$$

Since \(\tilde{v}\) solves (3.4), \(\tilde{v}_t = i\Delta \tilde{v} = i\Delta u_t + i\Delta v\).

Lemma 2 implies that for any \(t > 1\),

$$
\|v(t)\|_{L^\infty} + \|\nabla v(t)\|_{L^\infty} = \|\nabla \int_0^{(1-\delta)} e^{i(t-\tau)\Delta} \langle \nabla \rangle F(u)d\tau\|_{L^\infty} \lesssim \frac{1}{\delta^{3/2}t^{3/2}}.
$$

Therefore,

$$
\langle |w|^2w, u_t \rangle = \langle |w|^2w, i\Delta v \rangle = -\langle \nabla(|w|^2w), i\nabla v \rangle \lesssim \|\nabla v\|_{L^\infty} \|\nabla w\|_{L^2} |w|^2 \lesssim \frac{1}{t^{3/2}} E(t).
$$

**Remark:** Since \(\delta > 0\) is fixed, we will ignore it from now on.

Also, by Holder’s inequality,

$$
\langle i(\Delta \tilde{v})\tilde{v}, |w|^2 \rangle \lesssim \|\nabla|^{11/7} u_t\|_{L^7} \|\nabla w\|_{L^2}^{3/7} |w|_{L^6}^{18/7} \lesssim \frac{1}{t^{11/14}} E(t)^{6/7}.
$$

This takes care of the contribution of \(\langle \tilde{v}_t, |w|^2w \rangle\).

Next, integrating by parts,\n
$$
2\langle i(\Delta \tilde{v})\tilde{v}, |w|^2 \rangle = -2\langle i|\nabla \tilde{v}|^2, |w|^2 \rangle - 2\langle i(\nabla \tilde{v})\tilde{v}, \nabla |w|^2 \rangle = -2\langle i(\nabla \tilde{v})\tilde{v}, \nabla |w|^2 \rangle.
$$

Then by Holder’s inequality, since \(\|\tilde{v}\|_{L^4} \lesssim 1\),

$$
\langle i(\nabla v)\tilde{v}, \nabla |w|^2 \rangle \lesssim \|\nabla v\|_{L^\infty} \|\tilde{v}\|_{L^4} |w|_{L^6} \|\nabla w\|_{L^2} \lesssim \frac{1}{t^{3/2}} E(t)^{3/4}.
$$
Also, by Holder’s inequality and interpolation,

\begin{equation}
\langle i(\nabla u)(u), \nabla |w|^2 \rangle \lesssim \|\nabla u\|_{L^\infty} \|u\|_{L^4} \|\nabla w\|_{L^2} \|w\|_{L^4} \lesssim \frac{1}{t} \frac{1}{t^{1/8}} E(t)^{3/4}.
\end{equation}

Finally,

\begin{equation}
\langle i(\nabla u)v, \nabla |w|^2 \rangle \lesssim \|\nabla u\|_{L^\infty} \|v\|_{L^3} \|v\|_{L^1} \|\nabla w\|_{L^2} \|w\|_{L^4} \lesssim \frac{1}{t} \frac{1}{t^{3/8}} E(t)^{3/4}.
\end{equation}

In \eqref{3.20} and \eqref{3.21} we used

**Lemma 5.**

\begin{equation}
\|u_t\|_{L^4} \lesssim \frac{1}{t^{1/8}},
\end{equation}

and

\begin{equation}
\|\nabla u_t\|_{L^\infty} \lesssim \frac{1}{t}.
\end{equation}

**Proof:** This is proved by interpolating \eqref{2.18}–\eqref{2.20}. By Bernstein’s inequality, \eqref{2.19}, \eqref{2.20}, and the Sobolev embedding theorem,

\begin{equation}
\|\nabla P_{\leq t^{-1/2}} u_t\|_{L^\infty} + \|\nabla P_{t^{-1/2}} u_t\|_{L^\infty} \lesssim \frac{1}{t}.
\end{equation}

Also by the Bernstein inequality and the Sobolev embedding theorem, along with \eqref{3.20} and \(u_t \in \dot{H}^{1/2}\),

\begin{equation}
\|P_{\geq t^{-1/2}} u_t\|_{L^4} + \|P_{t^{-1/2}} u_t\|_{L^4} \lesssim \frac{1}{t^{1/8}}.
\end{equation}

This proves the Lemma. \(\square\)

The contribution of \(2\Re \int w^2 \bar{v}_t\) may be estimated in a similar manner as the contribution of \eqref{3.18}, except that there is an additional term to consider,

\begin{equation}
-2\Re \int i w^2 (\nabla \bar{v})^2.
\end{equation}

Interpolating \eqref{3.20} with \eqref{3.22},

\begin{equation}
-2\Re \int i w^2 (\nabla \bar{u}_t)^2 \lesssim \|\nabla u_t\|_{L^4} \|w\|_{L^4} ^2 \lesssim \frac{1}{t^{5/4}} E(t)^{1/2}.
\end{equation}

Meanwhile, replacing \(||\nabla |1/2 P_j u\|_{L^\infty} L^2\) with \(||\nabla |1/2 P_j u\|_{L^\infty} L^2\) in \eqref{2.23}–\eqref{2.24} implies that for \(t > 1\),

\begin{equation}
\|\nabla \int_0^{(1-\delta)} e^{i(t-\tau)\Delta} F(u)\|_{L^4} \lesssim \frac{1}{t^{1/2}}.
\end{equation}

Interpolating \eqref{3.24} with \eqref{3.25},

\begin{equation}
-2\Re \int i w^2 (\nabla \bar{v})^2 \lesssim \|\nabla \bar{v}\|_{L^4} ^2 \|w\|_{L^4} ^2 \lesssim \frac{1}{t^{3/2}} E(t)^{1/2}.
\end{equation}

Now treat

\begin{equation}
2\langle w, |\bar{v}^2 \bar{v}_t \rangle + \langle w, \bar{v}^2 \bar{u}_t \rangle = 2\langle w, |ar{v}|^2 (i\Delta \bar{v}) \rangle + \langle w, \bar{v}^2 (i\Delta \bar{v}) \rangle.
\end{equation}
After integrating by parts,\[ 3.31 \]
\[ \lesssim \| \nabla \tilde{v} \|_{L^4}^2 \| \tilde{v} \|_{L^4} + \| \nabla w \|_{L^2} \| \nabla \tilde{v} \|_{L^\infty} \lesssim \frac{1}{t^{5/4}} E(t)^{1/4} + \frac{1}{t} E(t)^{1/2} \| \tilde{v}(t) \|_{L^4}. \]

Interpolating \((3.14)\) with \(\| \tilde{v} \|_{L^3} \lesssim 1\) implies \(\| \tilde{v} \|_{L^4} \lesssim t^{-3/8}\), which combined with \((3.22)\) implies \(\| \tilde{v} \|_{L^4} \lesssim \frac{1}{t^{1/4}}\). Therefore, we have proved
\[ 3.32 \]
\[ \frac{d}{dt} E(t) \lesssim \frac{1}{t^{15/14}} (1 + E(t)). \]

By Gronwall’s inequality, \((3.32)\) implies a uniform bound on \(E(t)\). This implies a uniform bound on \(E(t)\), since \(E(t) \sim \tilde{E}(t)\) when \(E(t)\) is large, which proves Theorem 4.

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**References**

[Bou98a] J Bourgain. Scattering in the energy space and below for 3D NLS. *Journal d’Analyse Mathématique*, 75(1):267–297, 1998.

[Bou98b] Jean Bourgain. Refinements of Strichartz’s inequality and applications to 2D-NLS with critical nonlinearity. *International Mathematics Research Notices*, 1998(5):253–283, 1998.

[CKS+02] J Colliander, M Keel, Gigliola Staffilani, H Takaoka, and T Tao. Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation. *Mathematical Research Letters*, 9(5):659–682, 2002.

[CKS+04] James Colliander, Markus Keel, Gigliola Staffilani, Hideo Takaoka, and Terence Tao. Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on \(\mathbb{R}^3\). *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 57(8):987–1014, 2004.

[CW90] Thierry Cazenave and Fred B Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in \(H^s\). *Nonlinear Analysis: Theory, Methods & Applications*, 14(10):807–836, 1990.

[Dod18] Benjamin Dodson. Global well-posedness and scattering for nonlinear Schrödinger equations with algebraic nonlinearity when \(d = 2, 3, u_0\) radial. *Cambridge Journal of Mathematics*, 7(3):283–318, 2018.

[Dod19] Benjamin Dodson. Global well-posedness and scattering for the radial, defocusing, cubic wave equation with initial data in a critical Besov space. *Analysis & PDE*, 12(4):1023–1048, 2019.

[KM10] Carlos Kenig and Frank Merle. Scattering for \(H^{1/2}\) bounded solutions to the cubic, defocusing NLS in \(3\) dimensions. *Transactions of the American Mathematical Society*, 362(4):1937–1962, 2010.

[Su12] Qingtang Su. Global well-posedness and scattering for defocusing, cubic NLS in \(\mathbb{R}^3\). *Mathematical Research Letters*, 19(2):431–451, 2012.

[Tao06] Terence Tao. *Nonlinear Dispersive Equations: Local and Global Analysis*. Number 106. American Mathematical Soc., 2006.

[Tay10] Michael Taylor. *Partial Differential Equations III: Nonlinear Equations*, 2010.