Bounds for the concentration functions of random sums under relaxed moment conditions

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Abstract. Estimates are constructed for the deviation of the concentration functions of sums of independent random variables with finite variances from the folded normal distribution function without any assumptions concerning the existence of the moments of summands of higher orders. The obtained results are extended to Poisson-binomial, binomial and Poisson random sums. Under the same assumptions, the bounds are obtained for the approximation of the concentration functions of mixed Poisson random sums by the corresponding limit distributions. In particular, bounds are obtained for the accuracy of approximation of the concentration functions of geometric, negative binomial and Sichel random sums by the exponential, the folded variance gamma and the folded Student distribution. Numerical estimates of all the constants involved are written out explicitly.

Key words: distribution function, central limit theorem, normal distribution, folded normal distribution, uniform metric, Poisson-binomial distribution, Poisson-binomial random sum, binomial random sum, Poisson random sum, mixed Poisson random sum, geometric random sum, gamma distribution, negative binomial random sum, inverse gamma distribution, Sichel distribution, Laplace distribution, exponential distribution, folded variance gamma distribution, folded Student distribution, absolute constant.

Introduction

Assume that all the random variables considered in this paper are defined on one and the same probability space \((\Omega, \mathfrak{A}, \mathbb{P})\).

The concentration function \(Q_\xi(z)\) of a random variable \(\xi\) is defined as

\[
Q_\xi(z) = \sup_{x \in \mathbb{R}} \mathbb{P}(x \leq \xi \leq x + z), \quad z \geq 0.
\]

This notion was introduced by P. Lévy in 1937 [19], for details see [10, 27]. Concentration functions are convenient and informative characteristics of the dispersion or scatter of random variables. It is conventional and convenient to characterize the dispersion of a random variable by its variance which is very simple to understand since it is a single number. However, the cost of the simplicity of the variance is the absence of the information concerning what deviations of the random variable from its expected value are more probable than the others. This information is contained in the concentration functions.

The estimates of the rate of decrease of the concentration functions of sums of independent random variables as the number of summands grows are well known, see, e. g., [27]. However, these estimates are rather rough and do not take into consideration the corresponding change of the shape of the concentration function. Perhaps, for the first time the estimates of the concentration functions that describe the

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asymptotic change of their shapes were obtained in the paper [5]. In these papers the estimates were obtained under the assumption of existence of the third moments.

In the present paper we relax the moment conditions and construct the estimates of the concentration functions only under the condition of existence of the variances of summands. The resulting estimates make it possible to directly compare the informativeness of the concentration function with that of the variance as the measure of dispersion.

Along with purely theoretical motivation, there is a somewhat practical interest in the problems considered below. Poisson-binomial, binomial and mixed Poisson (first of all, geometric) random sums are widely used as stopped-random-walk models in many fields such as financial mathematics (Cox–Ross–Rubinstein binomial random walk model for option pricing [7]), insurance (Poisson random sums as total claim size in dynamic collective risk models [8], binomial random sums as total claim size in static portfolio risk models, geometric sums in the Pollaczek–Klínčín–Beekman representation of the ruin probability within the framework of the classical risk process [11]), reliability theory for modeling rare events [11]. It is now a tradition to admit that the distributions of elementary jumps of these random walks may have very heavy tails. The problems considered in the present paper correspond to the situation where the tails may be as heavy as possible for the normal approximation to be still adequate. A very important (if not crucial) argument in favor of consideration of approximations to the concentration functions and the corresponding bounds for their accuracy when the variance exists is that in financial mathematics the variance (or the square root of it) is often used as a synonym of the volatility. At the same time, from the Chebyshev inequality it is easy to obtain the inequality

$$D_\xi \geq \frac{1}{4} \sup_{x \geq 0} x^2 [1 - Q_\xi(z)]$$

which relates the concentration function $Q_\xi(z)$ of a random variable $\xi$ with its variance $D_\xi$. Therefore, in financial applications, the concentration function of (logarithmic) increments of a stock price or some other financial index can be used as a considerably more informative characteristic of the volatility.

The paper is organized as follows. Section 1 contains main definitions, preliminary information and auxiliary results. In Section 2 we present some estimates for the deviation of the concentration functions of sums of independent random variables with finite variances from the folded normal distribution function without any assumptions concerning the existence of the moments of summands of higher orders. The obtained results are extended to Poisson-binomial and binomial random sums in Section 3. The case of the Poisson random sums is considered in Section 4. Under the same assumptions, the bounds are obtained for the approximation of the concentration functions of general mixed Poisson random sums by the corresponding limit distributions in Section 5. As corollaries, bounds are obtained for the accuracy of approximation of the concentration functions of geometric, negative binomial and Sichel random sums by the exponential, the folded variance gamma and the folded Student distribution in Sections 6, 7 and 8, respectively. Numerical estimates of all the constants involved are written out explicitly.

1 Preliminary information and auxiliary results

Lemma 1. Let $\eta$ and $\xi$ be two random variables such that

$$\sup_x |P(\eta < x) - P(\xi < x)| \leq \delta,$$

where $\delta > 0$. Then

$$\sup_{z \geq 0} |Q_\xi(z) - Q_\eta(z)| \leq 4\delta.$$
Remark 1. If instead of (1) the “concentration function” \( \tilde{Q}_\xi(z) \) of a random variable \( \xi \) is defined as
\[
\tilde{Q}_\xi(z) = \sup_{x \in \mathbb{R}} P(x \leq \xi < x + z), \quad z > 0,
\]
then the corresponding analog of Lemma 1 can be proved with the twice less constant. Namely, if \( \eta \) and \( \xi \) are two random variables such that (2) holds, then
\[
\sup_{z \geq 0} |\tilde{Q}_\xi(z) - \tilde{Q}_\eta(z)| \leq 2\delta,
\]
see [4].

Recall the definition of a unimodal distribution due to A. Ya. Khinchin. A random variable \( \xi \) is said to have the unimodal distribution, if there exists a point \( x_0 \) such that the distribution function \( F_\xi(x) \) of the random variable \( \xi \) is convex for \( x < x_0 \) and the function \( 1 - F_\xi(x) \) is convex for \( x > x_0 \). Moreover, in this case the point \( x_0 \) is called the mode of the random variable \( \xi \). It is easy to see that any unimodal distribution function is continuous everywhere, possibly, except for its mode.

The following statement was given in the book [10] without proof.

Lemma 2. Let \( \xi \) be a random variable with the symmetric unimodal distribution. Then for \( z > 0 \) we have
\[
Q_\xi(z) = P(|\xi| < \frac{z}{2}).
\]

A rigorous proof of this statement can be found in, say, [5].

Let \( X_1, X_2, \ldots \) be independent random variables with \( \mathbb{E}X_i = 0 \) and \( 0 < \mathbb{E}X_i^2 \equiv \sigma_i^2 < \infty \), \( i = 1, 2, \ldots \)

For \( n \in \mathbb{N} \) denote
\[
S_n = X_1 + \ldots + X_n, \quad B_n^2 = \sigma_1^2 + \ldots + \sigma_n^2.
\]

Let \( \Phi(x) \) be the standard normal distribution function,
\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}dz, \quad x \in \mathbb{R}.
\]

The folded normal distribution function will be denoted as \( \Phi_0(x) \),
\[
\Phi_0(x) = \begin{cases} 
2\Phi(x) - 1, & x \geq 0, \\
0, & x < 0.
\end{cases}
\]

It is easy to see that if \( \zeta \) is a random variable with the standard normal distribution function, then \( \Phi_0(x) = P(|\zeta| < x) \).

Denote
\[
\Delta_n = \sup_x |P(S_n < xB_n) - \Phi(x)|.
\]

Let \( G \) be the class of real functions \( g(x) \) of \( x \in \mathbb{R} \) such that
- \( g(x) \) is even;
- \( g(x) \) is nonnegative for all \( x \) and \( g(x) > 0 \) for \( x > 0 \);
- \( g(x) \) and \( x/g(x) \) do not decrease for \( x > 0 \).

In 1963 M. Katz [12] proved that, whatever \( g \in G \) is, if the random variables \( X_1, X_2, \ldots \) are identically distributed and \( \mathbb{E}X_1^2g(X_1) < \infty \), then there exists a finite positive constant \( C_1 \) such that
\[
\Delta_n \leq C_1 \cdot \frac{\mathbb{E}X_1^2g(X_1)}{\sigma_1^2g(\sigma_1\sqrt{n})},
\]
(4)
In 1965 this result was generalized by V. V. Petrov \cite{petrov65} to the case of non-identically distributed summands (also see \cite{petrov65}): whatever \(g \in G\) is, if \(EX_ig(X_i) < \infty, i = 1, \ldots, n\), then there exists a finite positive constant \(C_2\) such that
\[
\Delta_n \leq C_2 \frac{B_2}{B_n} g(B_n) \sum_{i=1}^{n} EX_i^2 g(X_i).
\] (5)

Everywhere in what follows the symbol \(I(A)\) will denote the indicator function of an event \(A\). For \(\varepsilon \in (0, \infty)\) denote
\[
L_n(\varepsilon) = \frac{1}{B_n^2} \sum_{i=1}^{n} EX_i^2 I(|X_i| \geq \varepsilon B_n), \quad M_n(\varepsilon) = \frac{1}{B_n^3} \sum_{i=1}^{n} E|X_i|^3 I(|X_i| < \varepsilon B_n).
\]

In 1966 L. V. Osipov \cite{osipov66} proved that there exists a finite positive absolute constant \(C_3\) such that for any \(\varepsilon \in (0, \infty)\)
\[
\Delta_n \leq C_3[L_n(\varepsilon) + M_n(\varepsilon)]
\] (also see \cite{osipov66}, Chapt V, Sect. 3, theorem 7). This inequality is of special importance. Indeed, it is easy to see that
\[
M_n(\varepsilon) \leq \frac{\varepsilon}{B_n^2} \sum_{i=1}^{n} EX_i^2 I(|X_i| \geq \varepsilon B_n) \leq \varepsilon.
\]
Hence, from (6) it follows that for any \(\varepsilon \in (0, \infty)\)
\[
\Delta_n \leq C_3(\varepsilon + L_n(\varepsilon)).
\] (7)

But, as is well known, the Lindeberg condition
\[
\lim_{n \to \infty} L_n(\varepsilon) = 0 \text{ for any } \varepsilon \in (0, \infty)
\]
is a criterion of convergence in the central limit theorem. Therefore, in terminology proposed by V. M. Zolotarev \cite{zolotarev71}, bound (7) is natural, since it relates the convergence criterion with the convergence rate and its left-hand and right-hand sides converge to zero or diverge simultaneously.

In 1968 inequality (6) in a somewhat more general form was re-proved by W. Feller \cite{feller66}, who used the method of characteristic functions to show that \(C_3 \leq 6\).

A special case of (6) is the inequality
\[
\Delta_n \leq C'_3[L_n(1) + M_n(1)].
\] (8)

In the book \cite{petrov65} it was demonstrated that \(C_3 \leq 2C'_3\).

For identically distributed summands inequality (8) takes the form
\[
\Delta_n \leq \frac{C_4}{\sigma_1^2} EX_1^2 \min \left\{1, \frac{|X_1|}{\sigma_1 \sqrt{n}}\right\}.
\] (9)

In the papers \cite{paditz82, paditz83} L. Paditz showed that the constant \(C_4\) can be bounded as \(C_4 < 4.77\). In 1986 in the paper \cite{paditz86} he noted that with the account of lemma 12.2 from \cite{feller66}, using the technique developed in \cite{paditz82, paditz83}, the upper bound for \(C_4\) can be lowered to \(C_4 < 3.51\).

In 1984 A. Barbour and P. Hall \cite{barbour84} proved inequality (8) by Stein’s method and, citing Feller’s result mentioned above, stated that the method they used gave only the bound \(C'_3 \leq 18\) (although the paper itself contains only the proof of the bound \(C'_3 \leq 22\)). In 2001 L. Chen and K. Shao published the paper \cite{chen01} containing no references to Paditz’ papers \cite{paditz82, paditz83, paditz86} in which the proved inequality (8) by Stein’s method with the absolute constant \(C'_3 = 4.1\).

In 2011 V. Yu. Korolev and S. V. Popov \cite{korolev11} showed that there exist universal constants \(C_1\) and \(C_2\) which do not depend on a particular form of \(g \in G\), such that inequalities (4), (5), (8) and (9) are valid
with \( C_1 = C_4 \leq 3.0466 \) and \( C_2 = C_3' \leq 3.1905 \). This result was later improved by the same authors in the papers [16, 17], where it was shown that \( C_1 = C_2 = C_4 = C_3' \leq 2.011 \).

Moreover, in the paper [17] lower bounds were established for the universal constants \( C_1 \) and \( C_2 \). Namely, let \( g \) be an arbitrary function from the class \( \mathcal{G} \). Denote by \( \mathcal{H}_g \) the set of all random variables \( X \) satisfying the condition \( \mathbb{E} X^2 g(X) < \infty \). Denote \( C^* = \sup_{g \in \mathcal{G}} \sup_{X_i \in \mathcal{H}_g} \sum^n_{i=1} \mathbb{E} X^2_i g(X_i) \).

It is easily seen that \( C^* \) is the least possible value of the absolute constant \( C_2 \) that provides the validity of inequality (5) for all functions \( g \in \mathcal{G} \) at once. In the paper [17] it was proved that

\[
C^* \geq \sup_{z \geq 0} \frac{1}{1 + z^2} - \Phi(-z) = 0.54093\ldots
\]

In the recent paper [18] the results mentioned above were improved and extended. First, it was shown that one can take \( C_3 = C_3' \). Second, the upper bounds of the absolute constants mentioned above were sharpened and it was shown that \( C_3 \leq 1.8627 \). Third, these results were extended to Poisson-binomial, binomial and Poisson random sums. Under the same conditions, bounds were obtained for the accuracy of the approximation of the distributions of mixed Poisson random sums by the corresponding limit law. In particular, the bounds were constructed for the accuracy of approximation of the distributions of geometric, negative binomial and Poisson-inverse gamma (Sichel) random sums by the Laplace, variance gamma and Student distributions, respectively. All absolute constants were written out explicitly. The main result of the paper [18] can be formulated as follows.

**Lemma 3.** For any \( n \in \mathbb{N} \) there holds the inequality

\[
\Delta_n \leq 1.8627[L_n(1) + M_n(1)].
\]

2 Bounds for the concentration functions of non-random sums of independent random variables with finite variances

The main result of this section is the following.

**Theorem 1.** For any \( n \in \mathbb{N} \) and any \( \varepsilon > 0 \) there holds the inequality

\[
\mathbb{E} \left[ Q_{S_n}(z) - \Phi_0 \left( \frac{z}{2B_n} \right) \right] \leq 7.4508 [L_n(\varepsilon) + M_n(\varepsilon)].
\]

**Proof.** The desired assertion follows from Lemma 1 with \( \eta = S_n \), \( \mathbb{P}(\xi < x) = \Phi(x/B_n) \) and Lemmas 2 and 3.

**Corollary 1.** For any \( n \in \mathbb{N} \) and any \( \varepsilon > 0 \) there holds the inequality

\[
\mathbb{E} \left[ Q_{S_n}(z) - \Phi_0 \left( \frac{z}{2B_n} \right) \right] \leq 7.4508 [\varepsilon + L_n(\varepsilon)].
\]

Actually Corollary 1 declares that as soon as the Lindeberg condition holds, that is, the central limit theorem holds, the concentration function of the sum of independent random variables can be approximated by the folded normal distribution function with the argument appropriately linearly transformed.

By the same reasoning as that used to prove Theorem 1, in which the role of Lemma 3 is played by inequality 2 with the constant sharpened in [18], a result similar to Theorem 1 can be obtained in terms of the function \( g \in \mathcal{G} \).
Theorem 2. Whatever a function \( g \in \mathcal{G} \) is such that \( \mathbb{E}X_i^2g(X_i) < \infty \), \( i \geq 1 \), for any \( n \in \mathbb{N} \) there holds the inequality
\[
\sup_{z \geq 0} \left| Q_{S_n}(z) - \Phi_0 \left( \frac{z}{2B_n} \right) \right| \leq \frac{7.4508}{B_n^2g(B_n)} \sum_{i=1}^{n} \mathbb{E}X_i^2g(X_i).
\]

3 Bounds for the concentration functions of Poisson-binomial and binomial random sums

From this point on let \( X_1, X_2, \ldots \) be independent identically distributed random variables with \( \mathbb{E}X_i = 0 \) and \( 0 < \mathbb{E}X_i^2 \equiv \sigma^2 < \infty \). Let \( p_j \in (0, 1] \) be arbitrary numbers, \( j = 1, 2, \ldots \). For \( n \in \mathbb{N} \) denote \( \theta_n = p_1 + \ldots + p_n \), \( p_n = (p_1, \ldots, p_n) \). The distribution of the random variable
\[
N_{n,p} = \xi_1 + \ldots + \xi_n,
\]
where \( \xi_1, \ldots, \xi_n \) are independent random variables such that
\[
\xi_j = \begin{cases} 1 & \text{with probability } p_j, \\ 0 & \text{with probability } 1 - p_j, \end{cases}, \quad j = 1, \ldots, n,
\]
is usually called Poisson-binomial distribution with parameters \( n; p_n \). Assume that for each \( n \in \mathbb{N} \) the random variables \( N_{n,p}, X_1, X_2, \ldots \) are jointly independent. The main objects considered in this section are Poisson-binomial random sums of the form
\[
S_{N_{n,p}} = X_1 + \ldots + X_{N_{n,p}}.
\]
As this is so, if \( N_{n,p} = 0 \), then we assume \( S_{N_{n,p}} = 0 \).

For \( j \in \mathbb{N} \) introduce the random variables \( \tilde{X}_j \) by setting
\[
\tilde{X}_j = \begin{cases} X_j & \text{with probability } p_j, \\ 0 & \text{with probability } 1 - p_j. \end{cases}
\]
If the common distribution function of the random variables \( X_j \) is denoted \( F(x) \) and the distribution function with a single unit jump at zero is denoted \( E_0(x) \), then, as is easily seen,
\[
P(\tilde{X}_j < x) = p_jF(x) + (1 - p_j)E_0(x), \quad x \in \mathbb{R}, \quad j \in \mathbb{N}.
\]
It is obvious that \( \mathbb{E}\tilde{X}_j = 0 \),
\[
\mathbb{D}\tilde{X}_j = \mathbb{E}\tilde{X}_j^2 = p_j \sigma^2.
\]
(10)
In what follows the symbol \( d \) will denote coincidence of distributions.

Lemma 4. For any \( n \in \mathbb{N} \) and \( p_j \in (0, 1] \)
\[
S_{N_{n,p}} \overset{d}{=} \tilde{X}_1 + \ldots + \tilde{X}_n,
\]
(11)
where the random variables on the right-hand side of (11) are independent.

For the proof see [18].

With the account of (10) and (11) it is easy to notice that
\[
\mathbb{D}S_{N_{n,p}} = \theta_n \sigma^2.
\]
Theorem 3. For any $n \in \mathbb{N}$ and $p_j \in (0, 1], j \in \mathbb{N}$,
\[
\sup_{z \geq 0} |Q_{S_{n,p}}(z) - \Phi_0 \left( \frac{z}{2\sigma \sqrt{\theta_n}} \right)| \leq \frac{7.4508}{\sigma^2} \cdot \min \left\{ 1, \frac{|X_1|}{\sigma \sqrt{\theta_n}} \right\}.
\]

Proof. In [18] it was proved that
\[
\Delta_{n,p} \equiv \sup_x |P(S_{n,p} < x \sigma \sqrt{\theta_n}) - \Phi(x)| \leq \frac{1.8627}{\sigma^2} \cdot \min \left\{ 1, \frac{|X_1|}{\sigma \sqrt{\theta_n}} \right\}.
\]
So, the desired assertion follows from Lemmas 1, 2 and (12).

Theorem 4. Whatever a function $g \in \mathcal{G}$ is such that $\mathbb{E}X_1^2 g(X_1) < \infty$, there holds the inequality
\[
\sup_{z \geq 0} |Q_{S_{n,p}}(z) - \Phi_0 \left( \frac{z}{2\sigma \sqrt{\theta_n}} \right)| \leq \frac{7.4508 \cdot \mathbb{E}X_1^2 g(X_1)}{\sigma^2 g(\sigma \sqrt{\theta_n})}.
\]

Proof. This assertion follows from Lemmas 1, 2 and the estimate
\[
\Delta_{n,p} \leq \frac{1.8627 \cdot \mathbb{E}X_1^2 g(X_1)}{\sigma^2 g(\sigma \sqrt{\theta_n})}
\]
proved in [18].

In particular, if $p_1 = p_2 = \ldots = p$, then the Poisson-binomial distribution with parameters $n \in \mathbb{N}$ and $p_n$ becomes the classical binomial distribution with parameters $n$ and $p$:
\[
N_{n,p} \overset{d}{=} N_{n,p}, \quad P(N_{n,p} = k) = C_n^k p^k (1 - p)^{n-k}, \quad k = 0, \ldots, n.
\]
In this case $\theta_n = np$, so that $DS_{n,p} = np \sigma^2$. Note that, as it was proved in [18], if the summands of the sums $S_n$ have identical distribution, then inequalities (4) and (9) hold with the absolute constants equal to 1.8546. So, in the same way as Theorems 3 and 4 were proved, from Lemmas 1 and 2 with the account of (4) and (9) we obtain the following statements.

Corollary 2. For any $n \in \mathbb{N}$ and $p_j \in (0, 1], j \in \mathbb{N}$,
\[
\sup_{z \geq 0} |Q_{S_{n,p}}(z) - \Phi_0 \left( \frac{z}{2\sigma \sqrt{np}} \right)| \leq \frac{7.4184}{\sigma^2} \cdot \min \left\{ 1, \frac{|X_1|}{\sigma \sqrt{np}} \right\}.
\]

Corollary 3. Whatever a function $g \in \mathcal{G}$ is such that $\mathbb{E}X_1^2 g(X_1) < \infty$, there holds the inequality
\[
\sup_{z \geq 0} |Q_{S_{n,p}}(z) - \Phi_0 \left( \frac{z}{2\sigma \sqrt{np}} \right)| \leq \frac{7.4184 \cdot \mathbb{E}X_1^2 g(X_1)}{\sigma^2 g(\sigma \sqrt{np})}.
\]

4 Bounds for the concentration functions of Poisson random sums

In addition to the notation introduced above, let $\lambda > 0$ and $N_\lambda$ be the random variable with the Poisson distribution with parameter $\lambda$:
\[
P(N_\lambda = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \mathbb{N} \cup \{0\}.
\]
Assume that for each $\lambda > 0$ the random variables $N_\lambda, X_1, X_2, \ldots$ are jointly independent. Consider the Poisson random sum
\[
S_{N_\lambda} = X_1 + \ldots + X_{N_\lambda}.
\]
If $N_\lambda = 0$, then we set $S_{N_\lambda} = 0$. It is easy to see that $ES_\lambda = 0$ and $DS_\lambda = \lambda \sigma^2$. The accuracy of the normal approximation to the distributions of Poisson random sum was considered by many authors, see the historical surveys in [15, 28]. However, the analogs of the Katz–Osipov-type inequalities (4) and (9) under relaxed moment conditions were obtained only recently in [18]. Namely, the following statement was proved there.

**Denote**

$$\Delta_\lambda \equiv \sup_{x} |P(S_\lambda < x\sigma \sqrt{\lambda}) - \Phi(x)|.$$ 

**Lemma 5.** For any $\lambda > 0$ and any function $g \in \mathcal{G}$ such that $EX_1^2 g(X_1) < \infty$ we have

$$\Delta_\lambda \leq \frac{1.8546}{\sigma^2} EX_1^2 \min \left\{ 1, \frac{|X_1|}{\sigma \sqrt{\lambda}} \right\},$$  

(12)

$$\Delta_\lambda \leq \frac{1.8546 EX_1^2 g(X_1)}{\sigma^2 g(\sigma \sqrt{\lambda})}.$$  

(13)

Using (12) and Lemmas 1 and 2 we obtain a bound for the accuracy of the approximation of $Q_{S_\lambda}(z)$ by the folded normal distribution function.

**Theorem 5.** For any $\lambda > 0$

$$\sup_{z \geq 0} |Q_{S_\lambda}(z) - \Phi_0\left(\frac{z}{2\sigma \sqrt{\lambda}}\right)| \leq \frac{7.4184}{\sigma^2} EX_1^2 \min \left\{ 1, \frac{|X_1|}{\sigma \sqrt{\lambda}} \right\}.$$

Whereas Lemmas 1, 2 and inequality (13) yield

**Theorem 6.** Whatever a function $g \in \mathcal{G}$ is such that $EX_1^2 g(X_1) < \infty$, there holds the inequality

$$\sup_{z \geq 0} |Q_{S_\lambda}(z) - \Phi_0\left(\frac{z}{2\sigma \sqrt{\lambda}}\right)| \leq \frac{7.4184 EX_1^2 g(X_1)}{\sigma^2 g(\sigma \sqrt{\lambda})}.$$

The upper bound of the absolute constant used in Lemma 5 is uniform over the class $\mathcal{G}$. In specific cases this bound can be considerably sharpened. For example, it is obvious that $g(x) \equiv |x| \in \mathcal{G}$. For such a function $g$ inequality (13) takes the form of the classical Berry–Esseen inequality for Poisson random sums, the best current upper bound for the absolute constant in which is given in [29]:

$$\Delta_\lambda \leq \frac{E|X_1|^3}{\sigma^3 \sqrt{\lambda}},$$

so, if the third moment of the summands exist, then instead of Theorems 5 and 6 we can obtain the bound

$$\sup_{z \geq 0} |Q_{S_\lambda}(z) - \Phi_0\left(\frac{z}{2\sigma \sqrt{\lambda}}\right)| \leq \frac{1.2124 E|X_1|^3}{\sigma^3 \sqrt{\lambda}}.$$

**5 Bounds for the concentration functions of general mixed Poisson random sums**

In this section we extend the results of the preceding section to the case where the random number of summands has the mixed Poisson distribution. For convenience, in this case we introduce an “infinitely large” parameter $n \in \mathbb{N}$ and consider random variables $N_n^*$ such that for each $n \in \mathbb{N}$

$$P(N_n^* = k) = \int_0^\infty e^{-\lambda} \frac{\lambda^k}{k!} dP(\Lambda_n < \lambda), \quad k \in \mathbb{N} \cup \{0\},$$

(14)
for some positive random variable $\Lambda_n$. For simplicity $n$ may be assumed to be the scale parameter of the distribution of $\Lambda_n$ so that $\Lambda_n = n\Lambda$ where $\Lambda$ is some positive “standard” random variable in the sense, say, that $E\Lambda = 1$ (if the latter exists).

Assume that for each $n \in \mathbb{N}$ the random variable $N_n^\star$ is independent of the sequence $X_1, X_2, \ldots$. As above, let $S_{N_n^\star} = X_1 + \ldots + X_{N_n^\star}$ and if $N_n^\star = 0$, then $S_{N_n^\star} = 0$.

From (14) it is easily seen that, if $E\Lambda_n < \infty$, then $EN_n^\star = E\Lambda_n$ so that $DS_n = \sigma^2E\Lambda_n$.

Denote

$$\Delta_n^\star = \sup_x \{ \mathbb{P}(S_{N_n^\star} < x\sigma\sqrt{E\Lambda_n}) - \int_0^\infty \Phi\left(\frac{x}{\sqrt{\Lambda}}\right) d\mathbb{P}(\Lambda_n < \lambda E\Lambda_n) \}.$$

For $x \in \mathbb{R}$ introduce the function

$$G_n(x) = E\min\left\{ 1, \frac{|x|}{\sigma\sqrt{\Lambda_n}} \right\} = \mathbb{P}\left(\Lambda_n < \frac{x^2}{\sigma^2} \right) + \frac{|x|}{\sigma} E\frac{1}{\sqrt{\Lambda_n}}(\Lambda_n \geq \frac{x^2}{\sigma^2}).$$

The expectation in (15) exists since the random variable under the expectation sign is bounded by 1. Of course, the particular form of $G_n(x)$ depends on the particular form of the distribution of $\Lambda_n$. In [13] the following statement was proved.

**Lemma 6.** If $E\Lambda_n < \infty$, then

$$\Delta_n^\star \leq \frac{1.8546}{\sigma^2} EX_1^2 G_n(X_1) = \frac{1.8546}{\sigma^2} EX_1^2 \min\left\{ 1, \frac{|X_1|}{\sigma\sqrt{\Lambda_n}} \right\} =$$

$$= \frac{1.8546}{\sigma^2} \left[ EX_1^2 \mathbb{1}(|X_1| \geq \sigma\sqrt{\Lambda_n}) + E\frac{|X_1|^3}{\sigma\sqrt{\Lambda_n}} \mathbb{1}(|X_1| < \sigma\sqrt{\Lambda_n}) \right],$$

where the random variables $X_1$ and $\Lambda_n$ are assumed independent.

Taking into account that all scale mixtures of zero-mean normals are symmetric and unimodal, using Lemmas 1, 2 and 6 we obtain the following bound for the accuracy of the approximation of $Q_{S_{N_n^\star}}(z)$ by the scale mixture of the folded normal distribution function.

**Theorem 7.** If $E\Lambda_n < \infty$, then

$$\sup_{z \geq 0} \left| Q_{S_{N_n^\star}}(z) - \int_0^\infty \Phi_0\left(\frac{z}{2\sqrt{\Lambda}}\right) d\mathbb{P}(\Lambda_n < \lambda E\Lambda_n) \right| \leq$$

$$\leq \frac{7.4184}{\sigma^2} EX_1^2 G_n(X_1) = \frac{7.4184}{\sigma^2} EX_1^2 \min\left\{ 1, \frac{|X_1|}{\sigma\sqrt{\Lambda_n}} \right\} =$$

$$= \frac{7.4184}{\sigma^2} \left[ EX_1^2 \mathbb{1}(|X_1| \geq \sigma\sqrt{\Lambda_n}) + E\frac{|X_1|^3}{\sigma\sqrt{\Lambda_n}} \mathbb{1}(|X_1| < \sigma\sqrt{\Lambda_n}) \right],$$

where the random variables $X_1$ and $\Lambda_n$ are assumed independent.

In the subsequent sections we will consider special cases where $\Lambda_n$ has the exponential, gamma and inverse gamma distributions.

### 6 Bounds for the accuracy of approximation of the concentration functions of geometric random sums by the exponential law

In this section we consider sums of a random number of independent random variables in which the number of summands $N_n^\star$ has the geometric distribution with parameter $p = \frac{1}{1+n}$, $n \in \mathbb{N}$:

$$\mathbb{P}(N_n^\star = k) = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^k, \quad k \in \mathbb{N} \cup \{0\}. \quad (16)$$
As usual, we assume that for each \( n \in \mathbb{N} \) the random variables \( N^*_n, X_1, X_2, \ldots \) are independent. We again use the notation \( S_{N^*_n} = X_1 + \ldots + X_{N^*_n} \). If \( N^*_n = 0 \), then we set \( S_{N^*_n} = 0 \). It is easy to see that \( E N^*_n = n, DS_{N^*_n} = n \sigma^2 \). Note that for any \( k \in \mathbb{N} \cup \{0\} \)

\[
P(N^*_n = k) = \frac{1}{n} \int_{0}^{\infty} P(N^*_n = k) \exp \left\{ -\frac{\lambda}{n} \right\} d\lambda,
\]

where \( N_\lambda \) is the random variable with the Poisson distribution with parameter \( \lambda \). This means that for \( N^*_n \) representation (14) holds with \( \Lambda_n \) being an exponentially distributed random variable with parameter \( \frac{1}{n} \).

In what follows we will use traditional notation

\[
\Gamma(\alpha, z) = \int_{z}^{\infty} y^{\alpha-1} e^{-y} dy, \quad \gamma(\alpha, z) = \int_{0}^{z} y^{\alpha-1} e^{-y} dy, \quad \text{and} \quad \Gamma(\alpha) = \Gamma(\alpha, 0) = \gamma(\alpha, \infty)
\]

for the upper incomplete gamma-function, the lower incomplete gamma-function and the gamma-function itself, respectively, where \( \alpha > 0, z > 0 \).

In the case under consideration

\[
\frac{1}{n} \int_{0}^{\infty} \Phi_0 \left( \frac{x}{\sqrt{\frac{n}{\lambda}}} \right) \exp \left\{ -\frac{\lambda}{n} \right\} d\lambda = \int_{0}^{\infty} \Phi_0 \left( \frac{x}{\sqrt{\lambda}} \right) e^{-y} dy = 1 - e^{\frac{\sqrt{2}x}{2}}, \quad x \geq 0
\]

(see, e. g., lemma 12.7.1 in [13]), that is, the approximate distribution is exponential with parameter \( \sqrt{2} \).

At the same time, the function \( G_n(x) \) (see (15)) has the form

\[
G_n(x) = 1 - \exp \left\{ -\frac{x^2}{n \sigma^2} \right\} + \frac{|x|}{n \sigma} \int_{x^2/\sigma^2}^{\infty} e^{-y/n} \sqrt{\lambda} d\lambda = \gamma \left( \frac{1}{2}, \frac{x^2}{n \sigma^2} \right) + \frac{|x|}{\sigma \sqrt{n}} \Gamma \left( \frac{1}{2}, \frac{x^2}{n \sigma^2} \right).\]

So, from theorem 7 we obtain the following result.

**Corollary 4.** Let \( N^*_n \) have the geometric distribution (16). Then

\[
\sup_{z \geq 0} \left| Q_{S_{N^*_n}}(z) - 1 + \exp \left\{-\frac{z}{2 \sigma \sqrt{n}} \right\} \right| \leq \frac{7.4184}{\sigma^2} \left\{ E \left[ X_1^2 \gamma \left( \frac{1}{2}, \frac{X_1^2}{n \sigma^2} \right) \right] + \frac{1}{\sigma \sqrt{n}} E \left[ |X_1|^{3/2} \Gamma \left( \frac{1}{2}, \frac{X_1^2}{n \sigma^2} \right) \right] \right\}.
\]

### 7 Bounds for the accuracy of approximation of the concentration functions of negative binomial random sums by the folded variance-gamma distribution

The case more general than that considered in the preceding section is the case of negative binomial random sums.

Let \( r > 0 \) be an arbitrary number. Assume that representation (14) holds with \( \Lambda_n \) being a gamma-distributed random variable with the density

\[
p(\lambda) = \frac{\lambda^{r-1} e^{-\lambda/n}}{n^r \Gamma(r)} \quad \lambda > 0.
\]

Then the random variable \( N^*_n \) has the negative binomial distribution with parameters \( r \) and \( \frac{1}{n+1} \):

\[
P(N^*_n = k) = \frac{1}{n^r \Gamma(r)} \int_{0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \lambda^{r-1} e^{-\lambda/n} d\lambda = \frac{\Gamma(r+k)}{\Gamma(r) k!} \left( \frac{1}{1+n} \right)^r \left( \frac{n}{1+n} \right)^k, \quad k \in \mathbb{N} \cup \{0\}.
\]
Let
\[ V^+_r(x) \equiv \frac{1}{\Gamma(r)} \int_0^\infty \Phi_0\left(\frac{x}{\sqrt{\lambda}}\right) \lambda^{r-1} e^{-\lambda/n} d\lambda, \quad x \in \mathbb{R}, \]
be the folded symmetric variance-gamma distribution with shape parameter \( r \) (see, e.g., [20]).

In the case under consideration \( E_{N^*_n} = E \Lambda_n = nr \) so that \( DS_{N^*_n} = nr^2 \) and for any \( x \in \mathbb{R} \)
\[ \int_0^\infty \Phi_0\left(\frac{x\sqrt{\lambda}}{\sqrt{\lambda}}\right) \lambda^{r-1} e^{-\lambda/n} d\lambda = \frac{1}{\Gamma(r)} \int_0^\infty \Phi_0\left(\frac{x}{\sqrt{\lambda}}\right) \lambda^{r-1} e^{-\lambda/n} d\lambda \equiv V^+_r(x\sqrt{r}). \]

Here the function \( G_n(x) \) (see (15)) has the form
\[ G_n(x) = \frac{1}{n r \Gamma(r)} \int_0^{x^2/\sigma^2} \lambda^{r-1} e^{-\lambda/n} d\lambda + \frac{|x|}{\sigma n^r \Gamma(r)} \int_{x^2/\sigma^2}^\infty \lambda^{r-3/2} e^{-\lambda/n} d\lambda = \frac{1}{\Gamma(r)} \left[ \frac{\sqrt{r}}{\sigma^2} \sqrt{\frac{X^2}{\sigma^2}} + \frac{|x|}{\sigma \sqrt{n}} \Gamma\left(r - \frac{1}{2}, \frac{X^2}{\sigma^2}\right) \right]. \]

So, from theorem 7 we obtain the following result.

**Corollary 5.** Let \( N^*_n \) have the negative binomial distribution (17). Then
\[ \sup_{z \geq 0} |Q_{S_{N^*_n}}(z) - V^+_r(x\sqrt{r})| \leq \frac{7.4184}{\sigma^2 \Gamma(r)} \left\{ \mathbb{E} \left[ X_1^2 \Gamma\left(r - \frac{1}{2}, \frac{X_1^2}{\sigma^2}\right) \right] \right\} + \frac{1}{\sigma \sqrt{n}} \mathbb{E} \left[ |X_1| \Gamma\left(r - \frac{1}{2}, \frac{X_1^2}{\sigma^2}\right) \right]. \]

**8 Bounds for the accuracy of approximation of the concentration functions of Poisson-inverse gamma random sums by the folded Student distribution**

Let \( r > 1 \) be an arbitrary number. Assume that representation (14) holds with \( \Lambda_n \) being an inverse-gamma-distributed random variable with parameters \( \frac{r}{2} \) and \( \frac{2}{2} \) having the density
\[ p(\lambda) = \frac{n^{r/2} \lambda^{r/2-1}}{2^{r/2} \Gamma\left(\frac{r}{2}\right)} \exp \left\{ - \frac{n}{2\lambda} \right\}, \quad \lambda > 0. \]

Then the random variable \( N^*_n \) has the so-called Poisson-inverse gamma distribution:
\[ P(N^*_n = k) = \frac{n^{r/2}}{2^{r/2} \Gamma\left(\frac{r}{2}\right)} \int_0^\infty \lambda^{k} \exp \left\{ - \frac{n}{2\lambda} \right\} d\lambda, \quad k \in \mathbb{N} \cup \{0\}, \]
which is a special case of the so-called Sichel distribution see, e.g., [30, 31]. In this case
\[ E\Lambda_n = \frac{n}{r - 2}, \]
so that
\[ DS^*_n = \frac{nr^2}{r - 2}. \]
Nevertheless, we will normalize random sums not by their mean square deviations, but by slightly different and asymptotically equivalent quantities $\sigma \sqrt{n/r}$.

As is known, if $\Lambda_n$ has the inverse gamma distribution with parameters $\frac{r}{2}$ and $\frac{n}{2}$, then $\Lambda_n^{-1}$ has the gamma distribution with the same parameters. Therefore, we have

$$
\frac{n^{r/2}}{\Gamma(\frac{r}{2})} \int_0^\infty \Phi_0\left(x \sqrt{\frac{n}{r\lambda}}\right) \exp\left\{ - \frac{n}{2\lambda} \right\} d\lambda = \frac{n^{r/2}}{\Gamma(\frac{r}{2})} \int_0^\infty \Phi_0\left(x \sqrt{\frac{n\lambda}{r}}\right) \exp\left\{ - \frac{n\lambda}{2} \right\} d\lambda =
$$

$$
= \frac{1}{2^{r/2}\Gamma(\frac{r}{2})} \int_0^\infty \Phi\left(x \sqrt{\frac{n}{r}}\right) \lambda^{r/2-1} e^{-\lambda/2} d\lambda = T_r^+(x), \quad x \in \mathbb{R},
$$

where $T_r^+(x)$ is the folded Student distribution function with parameter $r$ (“degrees of freedom”) corresponding to the density

$$
t_r^+(x) = \frac{2\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r\Gamma(\frac{r}{2})}} \left(1 + \frac{x^2}{r}\right)^{-(r+1)/2}, \quad x \geq 0,
$$

see, e.g., [3].

In this case the function $G_n(x)$ (see (15)) has the form

$$
G_n(x) = P\left(\Lambda_n^{-1} > \frac{\sigma^2}{x^2}\right) + \frac{|x|}{\sigma} E \sqrt{\Lambda_n^{-1}} \Pi\left(\Lambda_n^{-1} \leq \frac{\sigma^2}{x^2}\right) =
$$

$$
= \frac{n^{r/2}}{2^{r/2}\Gamma(\frac{r}{2})} \int_0^\infty \lambda^{r/2-1} e^{-n\lambda/2} d\lambda + \frac{|x|n^{r/2}}{2^{r/2}\sigma\Gamma(\frac{r}{2})} \int_0^\infty \lambda^{(r-1)/2} e^{-n\lambda/2} d\lambda =
$$

$$
= \frac{1}{\Gamma(\frac{r}{2})} \left[ \Gamma\left(\frac{r}{2}, \frac{n\sigma^2}{2x^2}\right) + \frac{|x|}{\sigma} \sqrt{\frac{n}{2}} \gamma\left(\frac{r+1}{2}, \frac{n\sigma^2}{2x^2}\right) \right],
$$

where $\gamma(\cdot, \cdot)$ and $\Gamma(\cdot, \cdot)$ are the lower and upper incomplete gamma-functions, respectively. So, from theorem 7 we obtain the following result.

**Corollary 6.** Let $N^*_n$ have the Poisson-inverse gamma distribution (35). Then

$$
\sup_{z > 0} \left| Q_{S_{N^*_n}}(z) - T_r^+\left(\frac{z\sqrt{r}}{2\sigma\sqrt{n}}\right) \right| \leq \frac{7.4184}{\sigma^2\Gamma(\frac{r}{2})} \left\{ E \left[ X^2 \Gamma\left(\frac{r}{2}, \frac{n\sigma^2}{2X^2}\right) \right] + \frac{1}{\sigma^2} \sqrt{n} E \left[ |X|^{3/2} \gamma\left(\frac{r+1}{2}, \frac{n\sigma^2}{2X^2}\right) \right] \right\}.
$$

**Remark 2.** In accordance with Remark 1, all the theorems and corollaries proved in this paper for the concentration functions $Q$ defined by relation (1), remain valid for the “concentration functions” $\tilde{Q}$ defined by (3). Moreover, the absolute constants in the corresponding inequalities for $\tilde{Q}$ are twice less than those in the theorems proved for $Q$.

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