Phase transition in the Bayesian estimation of the default portfolio

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Abstract

The probability of default (PD) estimation is an important process for financial institutions. The difficulty of the estimation depends on the correlations between borrowers. In this paper, we introduce a hierarchical Bayesian estimation method using the beta binomial distribution, and consider a multi-year case with a temporal correlation. A phase transition occurs when the temporal correlation decays by power decay. When the power index is less than one, the PD estimator does not converge. It is difficult to estimate the PD with the limited historical data. Conversely, when the power index is greater than one, the convergence is the same as that of the binomial distribution. We provide a condition for the estimation of the PD and discuss the universality class of the phase transition. We investigate the empirical default data history of rating agencies, and their Fourier transformations to confirm the the correlation decay equation. The power spectrum of the decay history seems to be 1/f of the fluctuations that correspond to long memory. But the estimated power index is much greater than one. If we collect adequate historical data, the parameters can be estimated correctly.

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I. INTRODUCTION

Anomalous diffusion is an emerging subject in many fields [1–4]. The models describing such phenomena depend on long memory. These are related to the phase transition, which have received considerable interest in the sociophysics [5, 6] and econophysics [8]. In previous papers, we investigated voting models that were similar to the Keynesian beauty contest [9–13]. The model has two kinds of phase transitions. One is the information cascade transition, which is similar to the phase transition of Ising model [11]. The other is the convergence transition of super-normal diffusion [10, 14].

Estimations of the probability of default (PD) and default correlation have been obtained from empirical studies on the historical data from credit events. These two parameters are important for pricing financial products, such as synthetic CDOs [15–17]. Moreover, they are important for financial institutions to manage portfolios and are called ”long run PDs”. As the number of defaults is minimal, it is not easy to estimate these parameters [18, 19].

In this paper, we introduce a Bayesian estimation method using the beta binomial distribution [20, 21]. For the usual cases, the Merton model, which incorporates the default correlation by the correlation of the asset price movements (asset correlation), is used to estimate the PD and correlation [22]. The Monte Carlo simulation process is necessary to estimate the parameters, except for the large homogeneous portfolio limits, when the Merton model is used [17]. In the beta binomial case, the default correlation instead of the asset correlation is used [20]. Moreover, we consider a multi-year case with temporal correlation, which refers to the correlation between the years [18, 19].

A phase transition occurs when the temporal correlation decays by power-law. A power-law decay implies that the PD has a long memory compared to that of exponential decay [8]. When the power index is less than one, the estimator distribution of the PD does not converge to the delta function. Alternatively, when the power index is greater than one, the convergence is the same as that of the normal case. When the distribution does not converge, it is difficult to estimate the PD with limited data. The required condition for estimating the PD is clarified. The critical exponents for the power-law decay of the correlation function depend on the microscopic feature of the model. The universality class of the phase transition is different from those of the nonlinear Pólya urns [23, 24].

To confirm the decay form of the temporal correlation, we investigate the empirical default
data history using Fourier transformations. We determine whether the power spectrum of the default history agrees with \(1/f\) of the fluctuations \([8, 25]\). When this condition is satisfied, it corresponds to the correlation of the PD with long memory where the phase transition of the convergence exists. However, it is difficult to accurately obtain \(1/f\) of the power spectrum. The estimation of the power index is much greater than one. This demonstrates that when there is adequate historical data, the parameters (PD, default correlation, temporal correlation) can be estimated correctly.

The remainder of this paper is organized as follows. In section 2, we introduce a hierarchical Bayesian estimation method using the beta binomial distribution. In section 3, we consider the convergence of the PD estimator. In section 4, we study the phase transition of the Pólya urn with the discount factor using an analytic method and a finite-size scaling analysis. In section 5, we apply the Bayesian estimation to the empirical data of default history. Finally, the conclusions are presented in section 6.

II. BAYESIAN ESTIMATION USING BETTA BINOMIAL DISTRIBUTION

We denote the PD estimation as \(\theta\) and default correlation as \(\rho_D\), where \(0 \leq \theta \leq 1\) and \(0 \leq \rho_D \leq 1\). The distribution of \(\theta\) and \(\rho_D\) is \(P(\theta, \rho_D)\). The number of obligors in the portfolio is \(n\). \(\theta\) and \(\rho_D\) are estimated using a Bayesian estimation. We consider the Bernoulli random variables \(X_i (i = 1, 2, \cdots, n)\) that take the values 1 or 0. When the obligor, \(i\), is the default (non default), \(X_i = 1(0)\). We define \(X = \sum_{j=1}^{n} X_j\) and consider the correlation for \(X_i\), which is a default correlation, not an asset correlation.

When the number of defaults is \(k\), the Bayes formula for the posterior distribution \(P(\theta, \rho_D|X = k)\) is

\[
P(\theta, \rho_D|X = k) = \frac{P(\theta, \rho_D, X = k)}{P(X = k)} = \frac{P(X = k|\theta, \rho_D)f(\theta, \rho_D)}{P(X = k)},
\]

where \(f(\theta, \rho_D)\) is a prior distribution.

We use the beta binomial distribution for \(P(X = k|\rho_D)\). The posterior distribution is given by

\[
P(\theta, \rho_D|X = k) \propto \frac{n!}{k!(n-k)!} \frac{B(\alpha + k, n + \beta - k)}{B(\alpha, \beta)} f(\theta, \rho_D) \\
\quad \times \frac{\Gamma(\alpha + k) \Gamma(n + \beta - k)}{\Gamma(\alpha) \Gamma(\beta)} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + n)} f(\theta, \rho_D),
\]

\[(2)\]
where \( \theta = \frac{\alpha}{\alpha + \beta} \) and \( \rho_D = \frac{1}{\alpha + \beta + 1} \). Hence, we obtain the relations \( \alpha = \theta \frac{1 - \rho_D}{\rho_D} \) and \( \beta = (1 - \theta) \frac{1 - \rho_D}{\rho_D} \). Here, we use the beta function \( B(\alpha, \beta) = \Gamma(\alpha) \Gamma(\beta) / \Gamma(\alpha + \beta) \).

We consider the maximum a posteriori (MAP) estimation of Eq. (2). When the prior function \( f(\theta, \rho_D) \) is a constant function, the maximum point is

\[
\frac{\partial P(\theta, \rho_D|X = k)}{\theta} \propto \frac{(1 - \rho_D) \Gamma(\alpha + k) \Gamma(n + \beta - k)}{\Gamma(\alpha) \Gamma(\beta)} (\psi(\alpha + k) - \psi(\alpha) - \psi(\beta + n - k) + \psi(\beta))
\]

\[
= \frac{(1 - \rho_D) \Gamma(\alpha + k) \Gamma(n + \beta - k)}{\rho_D \Gamma(\alpha) \Gamma(\beta)} \left( \sum_{i=1}^{\alpha + k - 1} \frac{1}{\beta + i - 1} - \sum_{i=1}^{\alpha + k - 1} \frac{1}{\beta + i - 1} \right) = 0,
\]

(3)

where \( \psi(x) \) is the digamma function. The first term of the last term in the brackets, is a monotonously decreasing function of \( \theta \), because \( \alpha \) increases. The second term of the last term in the brackets is a monotonously increasing function about \( \theta \), because \( \beta \) decreases. When \( \theta \sim 0 \), the last term is positive. Conversely, when \( \theta \sim 1 \), the last term becomes negative. Hence, the function \( P(\theta|X = k, \rho_D) \) has one peak in the range \( 0 < \theta < 1 \). The multi-term case is provided in Appendix A.

Next, we consider the variable \( \rho_D \). The maximum point is

\[
\frac{\partial P(\theta, z|X = k)}{\partial z} \propto \frac{\Gamma(\alpha + k) \Gamma(N + \beta - k)}{\Gamma(\alpha) \Gamma(\beta)} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + n)} \left( \sum_{i=1}^{\theta z + i - 1} \frac{\theta}{\beta + i - 1} + \sum_{i=1}^{k} \frac{1 - \theta}{\beta + i - 1} - \sum_{i=1}^{k} \frac{1}{\beta + i - 1} \right) = 0,
\]

(4)

where \( z = (1 - \rho_D) / \rho_D \).

All of the terms in the last term in brackets are monotonously decreasing functions about \( z \). When \( z \sim 0 \), the last term becomes positive. Conversely, when \( z \gg 1 \), the last term becomes 0. When \( (k - 1)/n \leq \theta \leq k/n \) or is adequately close to this condition, the last term becomes positive. \( (k - 1)/\theta \leq n - 1 \) and \( (n - k - 1)/(1 - \theta) \leq n - 1 \) become \( (k - 1)/(n - 1) \leq \theta \leq k/(n - 1) \). In this case the last term increases monotonously and the peak is \( z = \infty \) and \( \rho_D = 0 \). This implies that the optimization of \( \rho_D \) is zero for the single term model. When \( \theta \) is not adequately close to \( (k - 1)/n \leq \theta \leq k/n \), the last term changes from positive to negative as \( z \) increases. Therefore, one peak occurs in \( P(\theta, z|X = k) \).

We extend the method to the multi-year case. There are \( n_i \) obligors in year \( i \) and \( k_i \) defaults occur. The prior distribution for the second year is the posterior distribution, which is calculated from the first year data. In this way the posterior distribution is updated year
after year. We write the posterior distribution $P(\theta, \rho_D|k_1, k_2)$ as

$$P(\theta, \rho_D|k_1, k_2) = \frac{P(k_2|\theta, \rho_D, k_1)P(k_1|\theta, \rho_D)f(\theta, \rho_D)}{P(k_2)}.$$  

(5)

It is natural to assume that the number of defaults is affected by the number of defaults in the previous years. This is a temporal correlation. When the default rate is high (low), it is reasonable to assume that the default rate will be high (low) in the next year. This is similar to volatility clustering, which has a long memory [26, 27]. This is confirmed using empirical data in the following sections.

We introduce the temporal correlation by adjusting $\alpha$ and $\beta$, and consider the $j$th year. The number of obligors and defaults in the $j$th year are $n_j$ and $k_j$. In the same year the correlation is $\rho_D$. We set the temporal correlation parameters between the $i$th and $j$th years; $d_{i-j}$ and $j < i$. $\alpha$ and $\beta$ are adjusted to $\alpha + \sum_{j=1}^{i-1} d_{i-j} k_j$ and $\beta + \sum_{j=1}^{i-1} d_{i-j} (n_j - k_j)$ [28]. This implies that the previous years’ data affects the present defaults. It is easy to confirm that $d_i = 1$ indicates that all of the data is correlated to $\rho_D$. When $d_i = 0$, the data is independent each year.

### III. CORRELATION DECAY

In the previous section, $d_i$ was introduced to represent the temporal correlation. To clarify the behavior of the parameter $d_i$, where $i = 1, 2, \cdots, T$ and $d_0 = 1$, the variance of the stochastic process is considered in this section. In each year the diffusion has $n_i$ steps and $k_i$ defaults, where $i = 1, 2, \cdots, T$.

The adjustments related to parameters $\alpha$ and $\beta$ are the effects of the temporal correlation from the previous conclusions. We shrink the previous years’ conclusions and add them to the initial parameters for the purpose of the adjustment process. The shrinking ratio for the interval $i$ is $d_i$.

The two terms model is examined first. We consider the relation between the first and second years. $n_1$ and $k_1$ are the numbers of obligors and defaults, respectively, in the first year. The second year parameters become $\alpha + d_1 k_1$ and $\beta + d_1 (n_1 - k_1)$. We consider the shrinking processes from $\alpha$ to $\alpha + d_1 k_1$ and $\beta$ to $\beta + d_1 (n_1 - k_1)$. The variance of the process is $n_1 d_1 pq + d_1 n_1 (n_1 - 1) pq \rho_D$, where $q = 1 - p$; that is we approximate $d_1 B_{\alpha, \beta}(k_1, n_1 - k_1) \sim B_{\alpha, \beta}(d_1 k_1, d_1 n_1 - d_1 k_1)$ where $B_{\alpha, \beta}$ is the beta binomial distribution with parameters $\alpha$ and
\( \beta \). We approximate this variance by 
\[ n_1 d_1 pq + d_1 n_1 (d_1 n_1 - 1) pq \rho_D, \]
and the difference becomes 
\[ n_1^2 pq \rho_D d_1 (1 - d_1) \geq 0. \]
Hence, the approximation is exact when \( d_1 = 0, 1 \) or \( \rho_D = 0 \). If 
\( d_1 \sim 0, 1 \) or \( \rho_D \sim 0 \), this approximation can be used.

For the defaults of the obligors, the hypothesis; \( d_1 \sim 0 \) or \( 1 \) and \( \rho_D \sim 0 \), can be set. In other words, the temporal correlation is either a high or low case, or the low correlation case. Hereafter, we use this approximation and calculate the variance of this process.

We extend the stochastic process to the multi-year case. Let \( \{ U_t; t \geq 1 \} \) be an independent and identically distributed (i.i.d.) sequence, uniformly distributed on \([0,1]\).

The discrete dynamics of the process is described by:
\[
X(t + 1) = 1_{u_{i+1} \leq Z_d(t)},
\]
when \( n_i + 1 \leq t \leq n_{i+1} \). Here \( Z_d(t) \) is given by
\[
Z_d(t) \equiv \frac{\alpha + \sum_{s=n_i}^{t} X(s) + \sum_{j=1}^{i} d_{i-j} k_j}{\alpha + \beta + (t - n_i) + \sum_{j=1}^{i} d_{i-j} n_j}.
\]

The expectation value of \( X(t) \) is \( E(X(t)) = \alpha / (\alpha + \beta) \). When \( d_i = 1 \), the process is beta binomial.

We consider the relationship between the \( i \) th and \( i + 1 \) th years. The distribution in the \( i \) th year is a beta binomial distribution. Hence, the conditional variance of the \( i + 1 \) th year is \( V_{i+1} \) using the approximation,
\[
V_{i+1} \geq \sum_{j=1}^{i+1} n_j d_{i+1-j} pq + \left( \sum_{j=1}^{i+1} n_j d_{i+1-j} \right) \left( \sum_{j=1}^{i+1} n_j d_{i+1-j} - 1 \right) pq \rho_D
\]
\[
- \sum_{j=1}^{i} n_j d_{i+1-j} pq - \left( \sum_{j=1}^{i} n_j d_{i+1-j} \right) \left( \sum_{j=1}^{i} n_j d_{i+1-j} - 1 \right) pq \rho_D
\]
\[
= pq n_{i+1} + pq n_{i+1} (n_{i+1} - 1) \rho_D + 2 pq \rho_D n_{i+1} \sum_{j=1}^{i} n_j d_{i+1-j}.
\]

This implies that the variance of \( \sum_{j=1}^{i+1} n_j d_{i+1-j} \) steps is subtracted from that of \( \sum_{j=1}^{i+1} n_j d_{i+1-j} \) steps. This corresponds to the variance for \( n_i \) steps in the \( i \) th year due to the property of the beta binomial distribution. Therefore, the correlation between the \( i \) th and \( j \) years is approximated by \( \rho_D d_{i-j} \). \( d_{i-j} \) plays the role of a discount factor in the correlation \( \rho_D \). As time progresses, the correlation is discounted. It is reasonable to assume a monotonically decreasing function for \( d_i \) because the affects decrease when the distance between \( i \) and \( j \) increase.
The total variance for the diffusion is approximated by

\[ V \geq \sum_{i=1}^{T} pqn_i + \sum_{i=1}^{T} pqn_i(n_i - 1)\rho_D + 2pq\rho_D \sum_{i>j}^{T} n_in_jd_{i-j}. \]  

The first, second, and third terms correspond to the variance for binomial distribution, constant correlation \( \rho_D \) in the portfolio, and temporal correlation, respectively.

In summary, when \( d_i \sim 0, 1 \) or \( \rho_D \sim 0 \), the correlation between the year and the year is approximated by

\[
Corr \sim \rho_D \begin{pmatrix}
1 & d_1 & d_2 & \cdots & d_T \\
& 1 & d_1 & \cdots & \vdots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & 1
\end{pmatrix}.
\]

In conclusion, for this approximation, the average PD, correlation of the Bernoulli random variables, and temporal correlation, are \( p, \rho_D, \) and \( d_i \), respectively.

In the Bayesian estimation, if the scaled variance converges as the data increases, the parameters can be estimated correctly. Conversely, if the variance does not converge, the parameters cannot be estimated. It is also a problem whether the process is stationary, as discussed for the spectrum analysis in the following sections.

It is difficult to estimate all of the \( d_i \) values because the data is limited. We introduce a priori distribution for \( d_i \). The estimation then becomes a hierarchical Bayesian estimation. It is reasonable that the priori distribution is a monotonically decreasing function. We consider the two hyper priori distributions, the exponential and power decays that have long memory.

IV. PHASE TRANSITION IN THE ESTIMATION OF PD

In this section we determine whether the PD Bayesian estimation converges. To simplify the model we set \( n_j = 1, j \geq 1 \) in Eq. (7). This does not affect the conclusion regarding the possibility of the PD estimation. Let \( \{U_t; t \geq 1\} \) be an independent and identically distributed (i.i.d.) sequence, uniformly distributed on [0,1]. The discrete dynamics of the process is described by:

\[ X(t+1) = 1_{U_{t+1} \leq Zd(t)}. \]
Here $Z_d(t)$ is the weighted sum of $X(s), s \leq t$ with the discount factor $d_{t-s}$,

$$Z_d(t) \equiv \frac{\alpha + \sum_{s=1}^t X(s)d_{t-s}}{\alpha + \beta + \sum_{s=1}^t d_{t-s}}. \quad (10)$$

This is the Pólya urn model\cite{29} with the discount factor $\{d_i\}$.

The expectation value of $X(t)$ is $E(X(t)) = \alpha/(\alpha + \beta)$. The PD estimator is $Z(t)$,

$$Z(t) \equiv \sum_{s=1}^t X(s)/t.$$  

The success of the PD estimation depends on the behavior of the variance of $Z(t)$, namely that if it converges, PD can be estimated.

\textbf{A. Stochastic differential equation}

First, the stochastic process is rewritten using $c_1(t) = \sum_{s=1}^t X(s)$;

$$c_1(t) = k \to k + 1 : P_{k,t} = \frac{\alpha + \sum_{s=1}^t X(s)d_{t-s}}{\alpha + \beta + \sum_{s=1}^t d_{t-s}},$$

$$c_1(t) = k \to k : Q_{k,t} = 1 - P_{k,t}, \quad (11)$$

where $P_{k,t}$ and $Q_{k,t}$ are the process probabilities. The sum of $P_{k,t}$ and $Q_{k,t}$ is 1.

For convenience, we define a new variable $\Delta_t$ such that

$$\Delta_t = 2c_1(t) - t. \quad (12)$$

We change the variables from $k$ to $\Delta_t$ and $X(s)$ to $Y_s = 2X(s) - 1$. Given $\Delta_t = u$, we obtain a random walk model:

$$\Delta_t = u \to u + 1 : P_{u,t} = \frac{\alpha + \sum_{s=1}^t d_{t-s}(Y_s + 1)/2}{\alpha + \beta + \sum_{s=1}^t d_{t-s}},$$

$$\Delta_t = u \to u - 1 : Q_{u,t} = 1 - P_{u,t}.$$

We now consider the continuous limit $\epsilon \to 0$,

$$Y_{\hat{\tau}} = \epsilon \Delta_{[\hat{\tau}]/\epsilon},$$

$$P(x, \tau) = \epsilon P(\Delta_{t/\epsilon}, t/\epsilon), \quad (13)$$

where $\hat{\tau} = t/\epsilon, \hat{r} = r/\epsilon$ and $y = \Delta_t/\epsilon$. On approaching the continuous limit, we can obtain the stochastic partial differential equation:

$$dY_{\hat{\tau}} = \frac{\alpha - \beta + \int_{s=1}^{\hat{\tau}} d(\hat{\tau} - s)dY_s}{\alpha + \beta + \int_{s=1}^{\hat{\tau}} d(t - s)ds} d\hat{\tau} + \sqrt{\epsilon}, \quad (14)$$
where $d(t)$ is the continuous function of $d_t$, the discount factor.

We are interested in the behavior of $Y_\tau$ in the limit $\tau \to \infty$. We assume that the stationary solution is

$$Y_\infty = \bar{v}\tau,$$

where $\bar{v}$ is constant. Substituting Eq.(15) into Eq.(14), we obtain

$$\bar{v} = \frac{\alpha - \beta + \bar{v}\hat{T}}{\alpha + \beta + \hat{T}},$$

where $\hat{T} = \lim_{\tau \to \infty} \int_1^\tau d(\tau - s)ds$.

Eq.(16) is a self-consistent equation. When $\hat{T} < \infty$, Eq.(16) is solved as $\bar{v} = (\alpha - \beta)/(\alpha + \beta)$. The process converges to the average point. On the other hand, when $\hat{T} \to \infty$, we can obtain the identical equation $\bar{v} = \bar{v}$, suggesting that the process does not converge to the delta function. The expected value of $Y_s$ is $(\alpha - \beta)/(\alpha + \beta)$. Hence, the phase transition at the point $\hat{T}$ diverges to infinity. When the distribution does not converge, we cannot estimate the parameters correctly, even if the amount of data increases. This is an important issue for the Bayesian estimation. In other words, $\hat{T} < \infty$ is the condition for the parameter estimations.

B. Correlation function and Finite size scaling analysis

To understand the phase transition, we investigated the correlation function, $C(t)$. $C(t)$ is defined as the correlation between $X(1)$ and $X(t)$:

$$C(t) \equiv \mathbb{E}(X(t + 1)|X(1) = 1) - \mathbb{E}(X(t + 1)|X(1) = 0) = \frac{\text{Cov}(X(1), X(t + 1))}{\text{V}(X(1))}. \quad (17)$$

$C(t)$ represents the propagation of the memory of $X(1)$, to later Variables, $X(t + 1)$.

To understand the relationship between the variances of $Z(t)$ and $C(t)$, the variance of $Z(t)$ can be written as

$$\text{V}(Z(t)) = \mathbb{E}_{X(1)}(\text{V}(Z(t)|X(1))) + \mathbb{E}_{X(1)}((\mathbb{E}(Z(t)|X(1)) - \mathbb{E}(Z(t)))^2), \quad (18)$$

where $\text{V}(Z(t)|X(1))$ is the conditional variance of $Z(t)$ on $X(1)$. $\mathbb{E}_{X(1)}(x)$ is the expectation value of $x$ with probability function $P(X(1))$. The second term on the right hand side of the equation represents the variance of $\mathbb{E}(Z(t)|X(1))$ from the dependence on $X(1)$. In Eq.(18),
the second term is related to $C(t)$ as it originates from the dependence of $E(Z(t)|X(1))$ on $X(1)$.

We write the second term of $C(t)$ as

$$E_X((E(Z(t)|X(1)) - E(Z(t)))^2) = \frac{1}{t^2} \frac{\alpha \beta}{(\alpha + \beta)^2} \left(\sum_{s=0}^{t-1} C(s)\right)^2.$$ 

If $c = \lim_{t \to \infty} C(t) > 0$, $\lim_{t \to \infty} V(Z(t)) > 0$ and $Z(t)$ does not converge.

We can derive the recursive relations for the conditional expectation value $E(X(t)|X(1) = x), x = 0, 1$, and the following can be obtained;

$$C(t) = \sum_{s=1}^{t} C(s-1) d_{t-s}. \tag{19}$$

This recursive relation contains all of the information regarding the asymptotic behavior of $C(t)$. If one assumes some functional form for $d_i$, with the initial condition $C(0) = 1$, we can estimate $C(t)$ for $t \geq 1$.

1. Exponential decay case

We consider the exponential decay case, $d_i = r^i, r \leq 1$. $\hat{T}$ is finite and there is no phase transition. Using Eq.\([19]\) we can obtain the next recursive relation for $C(t)$,

$$C(t) = \frac{1 + r(\alpha + \beta + \sum_{s=1}^{t-1} C(s) r^{t-1-s})}{\alpha + \beta + \sum_{s=1}^{t-1} r^{t-s}} C(t-1). \tag{20}$$

As we are interested in the asymptotic behavior of $C(t)$, we estimate the decay rate $r_{\text{eff}}$;

$$C(t) \sim r_{\text{eff}}^t$$

as

$$r_{\text{eff}} \equiv \lim_{t \to \infty} \frac{C(t)}{C(t-1)} = r + \frac{1 - r}{(\alpha + \beta)(1 - r) + 1} < 1, \tag{21}$$

where $r_{\text{eff}} < 1$ for $r < 1$, and $C(t)$ decays exponentially.

Numerical studies of the system were performed. To estimate $C(t)$, the recursive relation of Eq.\([19]\) is solved for $t \leq 2 \times 10^5$. A Monte Carlo sampling procedure is adopted for the variance of $Z(t)$. We obtained $10^4$ sample sequences for $\{X(t)\}, t = 1, \cdots, 2 \times 10^5$ and estimated the variance of $Z(t)$. Fig\([4]\) (a) shows the plot of $C(t)$ vs. $t$. It is clearly shown that $C(t)$ decays exponentially. Fig\([4]\) (b) shows the plot of $V(Z(t))$ vs. $t$. For all $r < 1 \in \{0.8, 0.9, 0.99\}$, $V(Z(t))$ decays as $1/t$. When $r = 1$, the $Z(t)$ distribution converges to the beta distribution. Hence, there is no phase transition for $r < 1$. 

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2. Power-law decay case

For the power-law decay case, \( d_i = \frac{1}{(1+i)^\gamma} \). When \( \gamma > 1 \), \( \hat{T} < \infty \) and the process converges to the delta function. On the other hand, when \( \gamma \leq 1 \), \( \hat{T} \) goes to the infinity, and the process does not converge.

The behaviors of \( C(t) \) and \( V(Z(t)) \) were investigated by the numerical method, as for the exponential decay case. Fig.2 (a) shows the double logarithmic plot of \( C(t) \) vs. \( t \). \( C(t) \) decays with the power-law for \( \gamma \in \{1.5, 2, 3\} \). For small \( \gamma \) as 0.5, 0.1, the slope is extremely small. Fig.2 (b) shows the double logarithmic plot of \( V(Z(t)) \) vs. \( t \). For \( r = 3.0, 2.0, 1.5 \), \( V(Z(t)) \) decays as \( 1/t \). At \( \gamma = 1 \), the slope of the decay is less than one. For \( r < 1 \), the curve becomes downward concave. These results suggest the validity of the self-consistent equation analysis.

To investigate the phase transition, we apply the finite-size scaling (FSS) analysis. We define the relaxation and second-moment correlation times, \( \tau(t) \) and \( \xi(t) \), respectively, using the \( n \)-th moment of \( C(t) \) as,

\[
M_n(t) \equiv \sum_{s=0}^{t-1} C(s)s^n,
\]

FIG. 1. Plots of (a) \( C(t) \) and (b) \( V(Z(t)) \) vs. \( t \), for \( r \in \{0.8, 0.9, 0.99\} \). For comparison, \( \exp(-0.03t)/3 \) and \( 1/t \) are potted in (a) and in (b), respectively.
FIG. 2. Plots of (a) $C(t)$ and (b) $V(Z(t))$ vs. $t$, for $\gamma \in \{3.0, 2.0, 1.5, 1.0, 0.5, 0.1\}$.

\[
\tau(t) = M_0(t), \\
\xi(t) = \sqrt{\frac{M_2(t)}{M_0(t)}}.
\]

(22)

For FSS, we assume that the scaling function $\lim_{t \to \infty} A(st)/A(t)$ for some observable $A(t)$ with the scale factor $s$ expressed as function of $\xi_t \equiv \lim_{t \to \infty} \xi(t)/t$;

\[
f_A(\xi_t) \equiv \lim_{t \to \infty} \frac{A(st)}{A(t)}
\]

We assume the next asymptotic forms for $C(t)$;

\[
C(t) \simeq \begin{cases} 
  c + \Delta C(t) & c > 0 \\
  c't^{-\delta} & c = 0
\end{cases}
\]

Here, $c = \lim_{t \to \infty} C(t)$ is the order parameter of the phase transition and $c'$ is a constant.

Using the asymptotic forms, we can classify the behavior of the scaling functions. We show the results for $f_r(\xi_t)$, $f_\xi(\xi_t)$ and $\xi_t$ in Table I. (In detail, see Appendix B)

Fig. 3 shows the numerical estimations of $\xi(2t)/\xi(t)$ and $\tau(2t)/\tau(t)$ vs. $\xi(t)/t$ with $t = 10^5$. The symbols show the fixed points under the renormalization transformation $t \to 2t$. There
TABLE I. Asymptotic behavior of \( C(t) \), and the scaling functions \( f_\tau(\xi_t) \), \( f_\xi(\xi_t) \), and \( \xi_t \). The assumed asymptotic form of \( C(t) \) is given in the second column. The second and the third columns provide the scaling functions. The last column contains the limit values of \( \xi(t)/t \).

| No. | Asymptotic behavior | \( f_\tau(\xi_t) = \lim_{t \to \infty} \frac{\tau(t)}{\tau(t)} \) | \( f_\xi(\xi_t) = \lim_{t \to \infty} \frac{\xi(t)}{\xi(t)} \) | \( \xi_t = \lim_{t \to \infty} \frac{\xi(t)}{t} \) |
|-----|---------------------|-----------------|-----------------|-----------------|
| 1   | \( C(t) \simeq c + \Delta C(t) \), \( c > 0 \) | \( s \) | \( s \) | \( 1/\sqrt{3} \) |
| 2   | \( C(t) \propto t^{-\delta} \), \( 0 < \delta < 1 \) | \( s^{1-\delta} = s^\frac{2(\xi(t))^2}{\xi(t)^2} \) | \( s \) | \( \frac{1-\delta}{\sqrt{3-\delta}} \) |
| 3   | \( C(t) \propto t^{-\delta} \), \( 1 < \delta < 3 \) | \( 1 \) | \( s^{(3-\delta)/2} \) | \( 0 \) |
| 4   | \( C(t) \propto t^{-\delta} \), \( \delta \geq 3 \) | \( 1 \) | \( 1 \) | \( 0 \) |

FIG. 3. Plots of (a) \( \xi(2t)/\xi(t) \) vs \( \xi(t)/t \) and (b) \( \tau(2t)/\tau(t) \) vs \( \xi(t)/t \). We adopt \( t = 10^5 \) and \( a = b = 1 \). The symbols show the fixed points under the renormalization transformation \( t \to 2t \).

are two stable fixed points at \( \xi_t = 0 \) and \( \xi_t = 1/\sqrt{3} \), and one unstable fixed point at \( \xi_t = \sqrt{(1-\delta)/(3-\delta)} \simeq 0.4073 \equiv \xi_c^\ast \). If \( \xi_t > \xi_c^\ast \), then \( \xi(2t)/\xi(t) > 2 \) and \( \xi(t)/t \) moves to \( 1/\sqrt{3} \) under the transformation \( t \to 2^n t \) and \( n \to \infty \). \( \xi(t) \) diverges linearly with the system size, \( t \), at the fixed point, which reflects the memory of \( X(1) \) that remains. If \( \xi_t < \xi_c^\ast \), \( \xi(2t)/\xi(t) < 2 \) and \( \xi(t)/t \) moves to 0. \( \lim_{t \to \infty} \xi(t) < \infty \) and the memory of \( X(1) \) is lost for
sufficiently large \( t \). At the stable fixed points at \( \xi_t = 1/\sqrt{3} \) and at 0, \( \tau(2t)/\tau(t) \) becomes two and one, respectively. From the unstable fixed point at \( \xi_t = \xi^c_t \), we can estimate \( \delta \) using \( f_\tau(\xi^c_t) = 2^{1-\delta} \simeq 1.3174 \). The estimation is in accordance with the estimation from \( \xi^c_t = \sqrt{(1 - \delta)/(3 - \delta)} \simeq 0.4073 \). The results support the phase transition between the two phases, \( C(t) \simeq c + \Delta C(t), c > 0 \) and \( C(t) \propto t^{-\delta}, \delta > 1 \), in the limit \( t \to \infty \). At the critical point \( \gamma = 1, \xi_t = \xi^c_t \), and \( C(t) \propto t^{-\delta} \) with \( 0 < \delta < 1 \).

![Graphs](https://example.com/graphs.png)

**FIG. 4.** Plots of (a) \( C(t) \) and (b) \( \delta \) vs. \( \gamma \). We adopt \( t = 2 \times 10^3, 2 \times 10^5 \), and \( (\alpha, \beta) = (1,1) \) and \( (1,4) \). The conjecture presented in the main text is plotted in (b) with the thin solid line. \( \delta \) for \( \gamma = 1, \) and \( \alpha = \beta = 1 \) (thick solid and dotted black lines, respectively) are estimated by \( \xi^c_t = \sqrt{(1 - \delta)/(3 - \delta)} \) and \( \xi^c_t \) in Figure 3. (c) Plot of \( \delta \) at \( \gamma = 1 \) vs. \( \alpha + \beta \). We set the ratios \( \alpha : \beta = 1 : 1 \) and \( 1 : 4 \), and change \( \alpha + \beta \). The solid line shows the \( \delta \) estimation by solving Eq.(24).

We estimate \( c \) by \( C(2t) \) and \( \delta \) by \( \log_2 C(t)/C(2t) \) with \( t = 10^3 \) and \( 10^5 \). By comparing the values for \( t = 10^3 \) and \( 10^5 \), one can anticipate the limit behavior \( t \to \infty \). The results are shown in Fig.4. Fig.4 (a) shows \( C(2t) \) vs. \( \gamma \). For \( \gamma > 1 \), \( C(2t) \) is almost zero. For \( \gamma < 1 \), \( C(2t) \) is positive. The derivative of \( c \) at \( \gamma = 1 \) is seemingly continuous. Fig.4 (b) shows \( \delta \) vs \( \gamma \) with \( (\alpha, \beta) = (1,1) \) and \( (1,4) \). For \( \gamma > 1 \), one can anticipate that \( \delta = \gamma \) by observing the change from \( t = 10^3 \) to \( 10^5 \). For \( \gamma < 1 \), \( \delta = 0 \) which suggests that \( c > 0 \). At the critical point \( \gamma = 1, \delta \) depends on \( (\alpha, \beta) \).

Next we investigated \( \delta \) at the critical point \( \gamma = 1 \). We assume that \( C(t) \propto t^{-\delta} \). Eq.(19) can be approximated in the continuous limit as

\[
C(t) = t^{-\delta} = \frac{\int_t^t(s-1)^{-\delta}d(t-s)ds}{\alpha + \beta + \int_t^t d(t-s)ds}.
\] (23)
By the change of variable as \((t + 1)\mu = s\), we obtain
\[
\alpha + \beta \simeq \int_{1/(t+1)}^{t/(t+1)} \mu^{-\delta}(1 - \mu)^{-1} d\mu - \ln t. \tag{24}
\]

We see that \(\delta\) depends on \(\alpha\) and \(\beta\) through the combination \(\alpha + \beta\). In the limit \(t \to \infty\), when \(\delta = 1\) and \(0\), \(\alpha + \beta = 0\) and \(\alpha + \beta \to \infty\), respectively. The critical exponent \(\delta\) is in the range \(\delta \in (0, 1)\). Fig. 4(c) shows \(\delta\) vs. \(\alpha + \beta\) for \(\gamma = 1\). We adopt two cases \(\alpha : \beta = 1 : 1\) and \(1 : 4\). The symbols show the results of the numerical estimation, and the solid line shows the results by numerically solving Eq. (24). The results for \(\alpha : \beta = 1 : 1\) and \(1 : 4\) collapses onto the same curve vs. \(\alpha + \beta\), which confirms that \(\delta\) depends on \(\alpha\) and \(\beta\) through \(\alpha + \beta\).

V. IS THE TEMPORAL CORRELATION DECAY EXPONENTIAL OR POWER?

In this section, we use three data sets from the default data. Two sets are rating agency data, and the other is from a Japanese company.

A. Standard & Poor’s data

As discussed in the previous section, temporal correlation is an important issue for determining whether there is an exponential or a power decay. This affects whether the parameters are estimated correctly. In this section we investigate the temporal correlation using empirical data. First, the S&P default data from 1981 to 2017 \cite{30} are used. The average PD is 1.58 \% for all ratings, and 3.09 \% for speculative ratings. Speculative grade rating represents the rating under BBB-(Baa3). In Fig. 5(a) we show the historical default rate. The solid and dotted lines correspond to all of the samples and the speculative grade, respectively, below BBB+(Baa3).

The autocorrelation is shown in Fig. 6(a). The x-axis represents the year. The exponential decay and cyclical increase is confirmed. This represents the cyclical bubbles and their collapse in recent years. However, it is difficult to confirm whether the decay is exponential or power-law from the autocorrelation data alone. A Fourier transformation was applied to the PD data in Fig. 7(a), but it was still difficult to obtain confirmation. The reason is that the data was annual and its size was not very large.
FIG. 5. (a) S&P Default Rate 1981-2017. (b) Moody’s Default Rate 1920-2017. The solid and dotted lines correspond to all of the samples and the speculative grade, respectively, below BBB+(Baa3).

FIG. 6. (a) S&P autocorrelation of the default rate 1981-2017. (b) Moody’s autocorrelation of the default rate 1920-2017.

B. Moody’s data

Next we used the Moody’s default data from 1920 to 2017 for 98 years [31]. It includes the Great Depression in 1929 and Great Recession in 2008. It is one of the longest sets of default data [32]. The average default rate is 1.56% for all of the ratings, and 3.87% for the
speccative ratings. In Fig. 5 (b), we show the historical default rate.

The autocorrelation is shown in Fig. 6 (b). The x-axis represents the year. The exponential decay is seemingly confirmed in a short time. In this long history data we cannot confirm the cyclical trend that was observed in recent years. We applies a Fourier transformation to the default ratio data in Fig. 7 (b). It was difficult to confirm whether the decay is exponential or power-law from the auto correlation alone.

C. Risk Data Bank data

Next we apply the data to the risk data bank (RDB) data [33]. The data covers almost all of the enterprise data without individual owner managers in Japan. The data is monthly from 2001 to 2017. The seasonal effects are adjusted. The historical data and autocorrelation are shown in Fig. 8 (a), which is different from the previous two samples. The slow decay of the correlation was confirmed. In Fig. 8 (b) 1/f of the fluctuations was confirmed. This corresponds to the power decay of the Wiener-Khinchin theorem, which shows the relationship between the autocorrelation and power spectrum by a Fourier transformation. In Fig. 9 we show the power spectrum for each sector; construction, wholesale, real estate, retail sales, other services, and manufacturing. The solid line represents the trend. We can conclude that the temporal correlation may contain a long memory for this data. However,
it is difficult to confirm a strict power law.

FIG. 8. (a) Risk data bank autocorrelation of the default rate. (b) Risk data bank power spectrum for the default rate

FIG. 9. Plots of the spectrum analysis for (a) construction, (b) wholesale, (c) real estate, (d) retail sale, (e) other services, and (f) manufacturing.

VI. ESTIMATION OF PARAMETERS

We estimate the long run probability of default $\theta$ and the default correlation $\rho_D$ for S&P and Moody’s data by the MAP estimation. We use the uniform distribution for a
prior distribution $f(\theta, \rho_D)$. As discussed in the previous section, the exponential and power decays are used for the temporal correlation. The conclusions are given in Table II for the exponential and power decay models. We confirmed a small $r$ value that represents the small temporal correlation. The parameter $\gamma$ for the power decay is greater than the phase transition point, $\gamma = 1$. The PD and default correlation are almost the same as the estimations by the exponential and power decay models. The reason is that the power $\gamma$ is adequately large and there is only a small difference between the exponential and power decay models. The first and second year temporal correlations, $d_1$ and $d_2$, respectively, are important for representing the data.

The parameters depend on the data terms. In the recent past, the default and temporal correlations have become minimal. This may depend on the smooth financial operations of governments and central banks. Alternatively, the long history data of 100 years ago have long correlations that are less than the phase transitions. This depends on the old data before the 1980s. For the RDB data, we can estimate $\gamma = 2$, which is in the normal convergence phase. Hence, we can estimate the PD by the Bayesian formula, which we introduced.

| No. | Model                  | Exponential decay | Power decay |
|-----|------------------------|-------------------|-------------|
|     |                        | $\theta$ | $\rho_D$ | $r$ | $\theta$ | $\rho_D$ | $\gamma$ |
| 1   | Moody’s 1920-2017      | 0.96%   | 1.9%   | 0.044 | 0.95%   | 2.0%   | 4.7 |
| 2   | Moody’s 1920-2017 SG   | 2.37%   | 3.9%   | 0.044 | 2.35%   | 4.1%   | 4.7 |
| 3   | Moody’s 1981-2017      | 1.49%   | 0.7%   | 0.023 | 1.46%   | 0.7%   | 5.9 |
| 4   | Moody’s 1990-2017      | 1.65%   | 0.7%   | 0.006 | 1.70%   | 0.8%   | 7.0 |
| 5   | Moody’s 1981-2017 SG   | 4.25%   | 1.8%   | 0.020 | 4.29%   | 1.8%   | 6.0 |
| 6   | S&P 1981-2017          | 1.54%   | 0.8%   | 0.024 | 1.54%   | 0.8%   | 5.7 |
| 7   | S&P 1990-2017          | 1.72%   | 0.8%   | 0.006 | 1.72%   | 0.8%   | 7.5 |
| 8   | S&P 1990-2017 SG       | 4.21%   | 2.0%   | 0.024 | 4.17%   | 1.9%   | 5.7 |

TABLE II. MAP estimation of the parameters for the exponential and power decay models
VII. CONCLUDING REMARKS

In this paper, we introduced a hierarchical Bayesian estimation method using the Beta binomial distribution to estimate the parameters, probability of default (PD), and default correlation. Moreover, we considered a multi-year case with temporal correlation. We confirmed phase transitions when the temporal correlation decayed by a power curve, which means that the correlation had a long memory. Conversely, for the case of exponential decay, there was no phase transition. When the power index $\gamma$ was less than or equal to one, the estimator distribution of the PD converged. Conversely, when the power index was greater than 1, the distribution did not converge. The critical exponent $0 < \delta < 1$ depended on the microscopic feature of the model and the universality class of the phase transition differed from those of the nonlinear Pólya urn. We call this phase transition "short memory-long memory transition". In summary, the condition for the estimation of parameters is

$$\hat{T} = \lim_{\tau \to \infty} \int_1^\tau D(\tau-s) ds < \infty.$$ 

To confirm the form of the decay, we investigated the empirical default history data using a Fourier transformation. We determined that the power spectrum of the default history was seemingly $1/f$ of the fluctuations, which implies that the correlation had a long memory for the RDB monthly data. We applied this method to the historical data and estimated the parameters. The region of the power index provided normal convergence. We have demonstrated that, for adequate data collection, the parameters can be estimated correctly.

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Appendix A: MAP estimation for Multi-year case

We extend the maximum a posteriori (MAP) estimation which we discussed in section 2 for the multi-year case. The number of obligors and defaults in the $j$th year are $n_j$ and $k_j$. When a prior function $f(\theta, \rho_D)$ is the constant function, the maximum point is

$$\frac{\partial P(\theta, \rho_D|n_1, \cdots, n_T, k_1, \cdots, k_T)}{\partial \theta} \propto \frac{(1 - \rho_D) \prod_{j=1}^T \Gamma(\alpha_j + k_j) \prod_{j=1}^T \Gamma(n_j + \beta_j - k_j)}{\rho_D \prod_{j=1}^T \Gamma(\alpha_j) \prod_{j=1}^T \Gamma(\beta_j)}$$
\[
\sum_{j=1}^{T} \left\{ \varphi(\alpha_j + k_j) - \varphi(\alpha_j) - \varphi(\beta_j + n_j - k_j) + \varphi(\beta_j) \right\} = (1 - \rho_D) \frac{\prod_{j=1}^{T} \Gamma(\alpha_j + k_j) \prod_{j=1}^{T} \Gamma(n_j + \beta_j - k_j)}{\rho_D \prod_{j=1}^{T} \Gamma(\alpha_j) \prod_{j=1}^{T} \Gamma(\beta_j)} \times \left\{ \sum_{j=1}^{T} \left( \sum_{i=1}^{k_j} \frac{1}{\alpha_j + i - 1} - \sum_{i=1}^{n_j - k_j} \frac{1}{\beta_j + i - 1} \right) \right\} = 0,
\]

where \( \varphi(x) \) is the digamma function. \( \alpha_j \) and \( \beta_j \) are the adjusted \( \alpha \) and \( \beta \); \( \alpha_j = \alpha + \sum_{l=1}^{j-1} d_{j-l} k_l \) and \( \beta_j = \beta + \sum_{l=1}^{j-1} d_{j-l} (n_l - k_l) \). The first term of the last term in Eq. (A1) is a monotonously decreasing function about \( \theta \), because \( \alpha \) increases. The second term of the last term is a monotonously increasing function about \( \theta \), because \( \beta \) decreases. When \( \theta \sim 0 \), the last term is positive because \( \alpha_1 = \alpha \). In contrast, when \( \theta \sim 1 \), the last term becomes negative because \( \beta_1 = \beta \). Hence, the function \( P(\theta | X = k, \rho_D) \) has one peak in the range \( 0 < \theta < 1 \).

**Appendix B: Scaling functions \( f_\xi(\xi_t) \) and \( f_\tau(\xi_t) \)**

We define the relaxation and second-moment correlation times \( \tau(t) \) and \( \xi(t) \), respectively, using the \( n \)-th moment of \( C(t) \) as in Eq. (22). If we assume that \( C(t) \propto t^{-\delta} \), \( M_n(t) \) behaves as

\[
M_n(t) \propto \begin{cases} 
\frac{1}{n+1-\delta} t^{n+1-\delta}, & \delta < n + 1, \\
\ln t, & \delta = n + 1, \\
\frac{1}{\delta-(n+1)}, & \delta > n + 1.
\end{cases}
\]

Using the asymptotic behavior of \( M_n(t) \), we find \( \tau(t) \) behaves as

\[
\tau(t) \propto \begin{cases} 
\frac{1}{1-\delta} t^{1-\delta}, & \delta < 1, \\
\ln t, & \delta = 1, \\
\text{constant}, & \delta > 1.
\end{cases}
\]

\( \xi(t) \) behaves as

\[
\xi(t) \propto \begin{cases} 
\sqrt{\frac{\delta}{3-\delta}} t, & \delta < 1, \\
t/\sqrt{\ln t}, & \delta = 1, \\
\sqrt{\frac{\delta-1}{3-\delta}} t^{(3-\delta)/2}, & 1 < \delta < 3, \\
\text{constant}, & \delta \geq 3.
\end{cases}
\]

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The scaling function for $\tau$ is defined as $f_\tau(\xi_t) \equiv \lim_{t \to \infty} \frac{\tau(st)}{\tau(t)}$, $s > 1$. From the asymptotic behavior of $\tau(t)$, we have

$$f_\tau(\xi_t) \equiv \lim_{t \to \infty} \frac{\tau(st)}{\tau(t)} = \begin{cases} s^{1-\delta} & 0 < \delta < 1 \\ 1 & \delta \geq 1 \end{cases}$$

For $\delta < 1$, $\xi_t \equiv \lim_{t \to \infty} \xi(t)/t = \lim \sqrt{(1-\delta)/(3-\delta)}$ and the scaling function is given in terms of $\xi_t$ as

$$\log_s f_\tau(\xi_t) = 1 - \delta = \frac{2(\xi_t)^2}{1 - (\xi_t)^2},$$

$\xi_t = 1/\sqrt{3}$ and $f_\tau(\xi_t) = 2$ in the limit $\delta \to 0$.

The scaling function for $\xi$ is defined as $f_\xi(\xi_t) \equiv \lim_{t \to \infty} \frac{\xi(st)}{\xi(t)}$. We have

$$f_\xi(\xi_t) \equiv \lim_{t \to \infty} \frac{\xi(st)}{\xi(t)} = \begin{cases} s & \delta \leq 1 \\ s^{(3-\delta)/2} & 1 < \delta < 3 \\ 1 & \delta \geq 3 \end{cases}$$

By the renormalization transformation $t \to s^n t$, $\lim_{n \to \infty} \xi(s^n t)/s^n = \xi(t)$ for $\delta \leq 1$. For $\delta > 1$, $\xi(s^n t)/s^n = 0$. The critical state of the system exists at $\delta < 1$.

We assume $C(t) \simeq c + \Delta C(t)$, $c > 0$ and $\Delta C(t)$ rapidly decays to zero. $\lim_{t \to \infty} \tau(t) = ct$ and $\xi_t = 1/\sqrt{3}$. $f_\xi(\xi_t) \equiv \lim_{t \to \infty} \xi(st)/\xi(t) = s$ and $f_\tau(\xi_t) \equiv \lim_{t \to \infty} \tau(st)/\tau(t) = s$ holds.

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