Texture dynamics including potential sterile neutrino

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Abstract

A unified form of mass matrix proposed previously for all fundamental fermions is extended to include the sterile neutrino $\nu_s$, presumed to mix mainly with $\nu_e$, while $\nu_{\tau}$ mixes strongly with $\nu_\mu$ (and both $\nu_\mu$ and $\nu_{\tau}$ weakly with $\nu_e$ and $\nu_s$). It turns out that the former mixing can be responsible for oscillations of solar neutrinos, while the latter for those of atmospheric neutrinos. It is mentioned in a footnote that $\nu_s$ is one of two sterile neutrinos, possible from a viewpoint of an algebraic construction. The charged–lepton factor $U^{(e)}$ in the leptonic four–dimensional Cabibbo—Kobayashi—Maskawa matrix $V = U^{(\nu)} \dagger U^{(e)}$ is determined through the small deviation of our previous prediction $m_\tau = 1776.80$ MeV (valid at the level of $U^{(e)} = 1$) from the experimental value $m_\tau = 1777.00^{+0.30}_{-0.27}$ MeV. Then, it gives corrections to the four–dimensional mixing of $\nu_e, \nu_\mu, \nu_{\tau}, \nu_s$ leading, in particular, to small $\nu_e \to \nu_\mu$, $\nu_e \to \nu_{\tau}$ and $\nu_\mu \to \nu_s$ oscillations (absent at the level of $U^{(e)} = 1$). However, the LSND estimates of $\bar{\nu}_\mu \to \bar{\nu}_e$ and $\nu_\mu \to \nu_e$ oscillations are too large to be explained by these corrections, though in the case of $\nu_\mu \to \nu_e$ they seem to be at the edge of such possible explanation.

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1. Introduction

Recently, in the course of extended studies on the "texture" of charged leptons $e^{-}$, $\mu^{-}$, $\tau^{-}$, neutrinos $\nu_e$, $\nu_\mu$, $\nu_\tau$, up quarks $u$, $c$, $t$ and down quarks $d$, $s$, $b$, we came to a proposal of unified algebraic structure of their mass matrices $(M^{(f)}_{ij})$ $(f = \nu$, $e$, $u$, $d)$ in the three-dimensional family space $(i,j = 1,2,3)$ [1]. The proposed structure followed from two sources. First of all, from $(i)$ an idea about the origin of three fundamental-fermion families as a consequence of some generalized Dirac-type equations (interacting with the Standard–Model gauge bosons) whose a priori infinite series is, due to an intrinsic Pauli principle, reduced (in the case of fermions) to three such equations [2]. And further, from $(ii)$ an ansatz for the fermion mass matrix expressed in terms of the suggested family characteristics [2].

This proposal in the case of leptons reads

$$
(M^{(f)}_{ij}) = \frac{1}{29} \begin{pmatrix}
\mu^{(f)}e^{(f)2} & 2\alpha^{(f)}e^{i\varphi^{(f)}} & 0 \\
2\alpha^{(f)}e^{-i\varphi^{(f)}} & 4\mu^{(f)}(80 + e^{(f)2})/9 & 8\sqrt{3}(\alpha^{(f)} - \beta^{(f)})e^{i\varphi^{(f)}} \\
0 & 8\sqrt{3}(\alpha^{(f)} - \beta^{(f)})e^{-i\varphi^{(f)}} & 24\mu^{(f)}(624 + e^{(f)2})/25
\end{pmatrix}
$$

with $f = \nu$, $e$, while $\mu^{(f)}$, $e^{(f)2}$, $\alpha^{(f)}$, $\beta^{(f)}$ and $\varphi^{(f)}$ denote real constants to be determined from the present and future experimental data for lepton masses and mixing parameters ($\mu^{(f)}$, $\alpha^{(f)}$ and $\beta^{(f)}$ are mass-dimensional).

On the base of a numerical experience we assumed that

$$
e^{(\nu)2} = 0, \quad \alpha^{(\nu)} = 0, \quad \beta^{(e)} = 0,
$$

what leads to $M^{(\nu)}_{11} = 0$ and $M^{(\nu)}_{12} = 0 = M^{(\nu)}_{21}$. Then, we determined the parameters $\mu^{(e)}$, $e^{(e)2}$ and $\alpha^{(e)}$ from the experimental values of $m_e$, $m_\mu$ and $m_\tau$, while $\mu^{(\nu)}$ and $\beta^{(\nu)}$ —

$\dagger$These imply the existence of three Dirac bispinors $\psi^{(1)}_\alpha$, $\psi^{(2)}_\alpha = (1/4)(C^{-1}\gamma^5)_{\beta_1\beta_2\beta_3\beta_4}\psi^{(3)}_\alpha$, where $\alpha = 1,2,3,4$ and $\beta_i = 1,2,3,4$ $(i = 1,2,\ldots,N-1$ with $N-1 = 0,2,4$) are Dirac bispinor indices ($\alpha$ is correlated with the presence of a whole set of Standard–Model charges suppressed in our notation, while $\beta_1,\ldots,\beta_{N-1}$ are fully antisymmetric). We interpret these bispinors as fundamental fermions from three families, corresponding to a given Standard–Model signature, i.e., as $\nu_e$, $\nu_\mu$, $\nu_\tau$ or $e^{-}$, $\mu^{-}$, $\tau^{-}$ or $u$, $c$, $t$ or $d$, $s$, $b$. 


from the possible atmospheric–neutrino oscillations as seen by Super–Kamiokande. The phases \( \phi^{(\nu)} \) and \( \phi^{(e)} \) remained undetermined.

In fact, in the case of charged leptons, assuming that the off–diagonal elements of the mass matrix \( (M_{ij}^{(e)}) \) can be treated as a small perturbation of its diagonal terms, we calculate in the lowest (quadratic) perturbative order [1]

\[
m_\tau = \frac{6}{125} (351 m_\mu - 136 m_e)
+ \frac{216 \mu^{(e)}}{3625} \left( \frac{111550}{31696 + 29 \varepsilon^{(e)} 2} - \frac{487}{320 - 5 \varepsilon^{(e)} 2} \right) \left( \frac{\alpha^{(e)}}{\mu^{(e)}} \right)^2,
\]

\[
\varepsilon^{(e)} 2 = \frac{320 m_e}{9 m_\mu - 4 m_e} + O \left( \left( \frac{\alpha^{(e)}}{\mu^{(e)}} \right)^2 \right),
\]

\[
\mu^{(e)} = \frac{29}{320} (9 m_\mu - 4 m_e) + O \left( \left( \frac{\alpha^{(e)}}{\mu^{(e)}} \right)^2 \right) \mu^{(e)}.
\]  

When the experimental \( m_e \) and \( m_\mu \) [3] are used as inputs, Eqs. (3) give [1]

\[
m_\tau = \left[ 1776.80 + 10.2112 \left( \frac{\alpha^{(e)}}{\mu^{(e)}} \right)^2 \right] \text{MeV},
\]

\[
\varepsilon^{(e)} 2 = 0.172329 + O \left( \left( \frac{\alpha^{(e)}}{\mu^{(e)}} \right)^2 \right),
\]

\[
\mu^{(e)} = 85.9924 \text{MeV} + O \left( \left( \frac{\alpha^{(e)}}{\mu^{(e)}} \right)^2 \right) \mu^{(e)}.
\]  

We can see that the predicted value of \( m_\tau \) agrees very well with its experimental figure \( m_\tau^{\text{exp}} = 1777.00^{+0.30}_{-0.27} \text{MeV} \) [3], even in the zero–order perturbative calculation. In order to estimate \( \left( \frac{\alpha^{(e)}}{\mu^{(e)}} \right)^2 \), we can take this experimental figure as another input. Then,

\[
\left( \frac{\alpha^{(e)}}{\mu^{(e)}} \right)^2 = 0.020^{+0.029}_{-0.020},
\]

so it is consistent with zero.

For the unitary matrix \( (U^{(e)}_{ij}) \), diagonalizing the mass matrix \( (M_{ij}^{(e)}) \) according to the relation \( U^{(e)\dagger} M^{(e)} U^{(e)} = \text{diag}(m_e, m_\mu, m_\tau) \), we obtain in the lowest (linear) perturbative order in \( \alpha^{(e)}/\mu^{(e)} \)

\[
(U^{(e)}_{ij}) = \frac{1}{29} \begin{pmatrix}
29 & 2 \frac{\alpha^{(e)}}{m_\mu} e^{-i \phi^{(e)}} & 0 \\
-2 \frac{\alpha^{(e)}}{m_\mu} e^{-i \phi^{(e)}} & 29 & 8 \sqrt{3} \frac{\alpha^{(e)}}{m_\tau} e^{-i \phi^{(e)}} \\
0 & -8 \sqrt{3} \frac{\alpha^{(e)}}{m_\tau} e^{-i \phi^{(e)}} & 29
\end{pmatrix},
\]
where the small $\varepsilon^{(\nu)}_2$ is neglected.

The case of neutrinos is discussed throughout the next Sections. In particular, we will determine the parameters $\mu^{(\nu)}$ and $\beta^{(\nu)}$ in Section 4 from Super-Kamiokande results.

We shall not discuss any \textit{a priori} motivation for the proposal (1), considering it simply as a detailed conjecture (the interested Reader may look for its roots in Refs. [2]). Instead, we allow in this paper for the existence of a sterile neutrino $\nu_s$ (blind to the Standard–Model interactions), extending the neutrino mass matrix $\left(M_{ij}^{(\nu)}\right)$ $(i, j = 1, 2, 3)$, as given in Eq. (1), to a $4 \times 4$ Hermitian matrix $\left(M_{ij}^{(\nu)}\right)$ $(I, J = 1, 2, 3, 4)$, where $M_{11}^{(\nu)} = M_{11}^{(\nu)}$ \footnote{Roots for this sterile neutrino may be sought again in the generalized Dirac equations (this time, without Standard–Model interactions) whose \textit{a priori} infinite number is, due to the intrinsic Pauli principle, reduced (in the case of fermions) to two such equations for the Dirac bispinors $\psi_1^{(1)}$ and $\psi_2^{(2)} = (1/6)(\gamma^5 C)_{\beta_3} \beta_4 \epsilon_{\beta_4 \beta_1 \beta_2 \beta_3} \psi_{\beta_1 \beta_2 \beta_3}$ ($\beta_1, \beta_2, \beta_3$ are fully antisymmetric). In the present paper, $\psi_3^{(1)}$ is interpreted as a sterile neutrino of the Dirac type denoted by $\nu_s$. The existence of $\psi_3^{(2)}$ as another sterile neutrino of the Dirac type is not discussed here (such second potential sterile neutrino might mix mainly with $\nu_\mu$, producing an extra disappearance mode of $\nu_\mu$, slightly correcting the effect of its dominating mode $\nu_\mu \rightarrow \nu_\tau$; for some comments \textit{cf.} Section 6).}

To this end, we supplement the former matrix by seven \textit{a priori} unknown elements $M_{14}^{(\nu)}$, $M_{41}^{(\nu)}$ and $M_{44}^{(\nu)}$. We will assume by analogy with $M_{12}^{(\nu)} = 0 = M_{21}^{(\nu)}$ and $M_{13}^{(\nu)} = 0 = M_{31}^{(\nu)}$ that

$$M_{24}^{(\nu)} = 0 = M_{42}^{(\nu)} , \ M_{34}^{(\nu)} = 0 = M_{43}^{(\nu)} , \quad (7)$$

but allow for nonzero $M_{14}^{(\nu)}$ and $M_{41}^{(\nu)}$ (as well as $M_{11}^{(\nu)}$ and $M_{44}^{(\nu)}$) in analogy with nonzero $M_{23}^{(\nu)}$ and $M_{32}^{(\nu)}$ (as well as $M_{22}^{(\nu)}$ and $M_{33}^{(\nu)}$). Note that here $M_{11}^{(\nu)} = 0$ under our particular assumption (2). If there is no sterile neutrino $\nu_s$, then also $M_{14}^{(\nu)} = 0 = M_{41}^{(\nu)}$ and $M_{44}^{(\nu)} = 0$, and we return to the $3 \times 3$ mass matrix $\left(M_{ij}^{(\nu)}\right)$ $(i, j = 1, 2, 3)$ discussed in Ref. [1].

2. Neutrino mass states

The eigenvalues of the mass matrix $\left(M_{ij}^{(\nu)}\right)$ are the masses of four neutrino mass states $\nu_1, \nu_2, \nu_3, \nu_4$. They are given as follows

\[
m_{\nu_1, \nu_4} = \frac{M_{11}^{(\nu)} + M_{14}^{(\nu)}}{2} \pm \sqrt{\left(\frac{M_{11}^{(\nu)} - M_{14}^{(\nu)}}{2}\right)^2 + |M_{14}^{(\nu)}|^2},
\]

\[
m_{\nu_2, \nu_3} = \frac{M_{22}^{(\nu)} + M_{23}^{(\nu)}}{2} \pm \sqrt{\left(\frac{M_{22}^{(\nu)} - M_{23}^{(\nu)}}{2}\right)^2 + |M_{23}^{(\nu)}|^2}. \quad (8)
\]
The corresponding unitary matrix \( (U^{(\nu)}_{IJ}) \), diagonalizing the mass matrix \( (M^{(\nu)}_{IJ}) \) according to the equality 

\[ U^{(\nu)} \dagger M^{(\nu)} U^{(\nu)} = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}, m_{\nu_4}) \]

, takes the form

\[
(U^{(\nu)}_{IJ}) = \begin{pmatrix}
\frac{1}{\sqrt{1+Y^2}} & 0 & 0 & -Y \frac{1}{\sqrt{1+Y^2}} e^{i\varphi(\nu)} \\
0 & \frac{1}{\sqrt{1+X^2}} & \frac{X}{\sqrt{1+X^2}} e^{-i\varphi(\nu)} & 0 \\
0 & \frac{X}{\sqrt{1+X^2}} e^{-i\varphi(\nu)} & \frac{1}{\sqrt{1+X^2}} & 0 \\
Y \frac{1}{\sqrt{1+Y^2}} e^{-i\varphi(\nu)} & 0 & 0 & \frac{1}{\sqrt{1+Y^2}}
\end{pmatrix},
\]

(9)

where

\[
Y = \frac{M^{(\nu)}_{11} - M^{(\nu)}_{44}}{2|M^{(\nu)}_{14}|} + \sqrt{1 + \left(\frac{M^{(\nu)}_{11} - M^{(\nu)}_{44}}{2|M^{(\nu)}_{14}|}\right)^2},
\]

\[
X = \frac{M^{(\nu)}_{22} - M^{(\nu)}_{33}}{2|M^{(\nu)}_{23}|} + \sqrt{1 + \left(\frac{M^{(\nu)}_{22} - M^{(\nu)}_{33}}{2|M^{(\nu)}_{23}|}\right)^2},
\]

(10)

when \( M^{(\nu)}_{14} = -|M^{(\nu)}_{14}| \exp i\varphi^{(\nu)} \) and \( |M^{(\nu)}_{14}| \neq 0 \), in analogy to \( M^{(\nu)}_{23} = -|M^{(\nu)}_{23}| \exp i\varphi^{(\nu)} \) and \( |M^{(\nu)}_{23}| \neq 0 \) for \( \beta^{(\nu)} > 0 \). If there is no sterile neutrino, then \( Y \to 0 \) as seen from Eq. (9), what corresponds in the case of \( M^{(\nu)}_{11} = 0 \) to the limit \( |M^{(\nu)}_{14}| \to 0 \) and \( M^{(\nu)}_{44} \to 0 \) with \( M^{(\nu)}_{44}/|M^{(\nu)}_{14}| \to \infty \).

The neutrino states \( \nu_\alpha \equiv \nu_e, \nu_\mu, \nu_\tau, \nu_s \) (of which \( \nu_e, \nu_\mu, \nu_\tau \) denote the familiar observed neutrino weak–interaction states, while \( \nu_s \) stands for their unobservable sterile partner) are related to neutrino mass states \( \nu_J \equiv \nu_1, \nu_2, \nu_3, \nu_4 \) through the four–dimensional unitary transformation

\[
\nu_\alpha = \sum_J V^*_J \nu_J
\]

(11)

with \( (V^*_J) = (V_J)^\dagger \). Here,

\[
V_J \equiv \sum_K U^{(\nu)}_{KJ} U^{(e)}_{KJ} = \sum_K U^{(\nu)}_{KJ} U^{(e)}_{KJ} + U^{(\nu)}_{4J} \delta_{4J}
\]

(12)

with the charged–lepton diagonalizing matrix \( (U^{(e)}_{ij}) \) as given in Eq. (6) and

\[
U^{(e)}_{14} = 0, \quad U^{(e)}_{24} = 0, \quad U^{(e)}_{44} = 1.
\]

(13)

The latter equations follow from the fact that charged leptons get no sterile partner. Thus,
\[ V_{\alpha j} = \sum_k U_{k\alpha}^{(\nu)} U_{kj}^{(e)}, \quad V_{\alpha 4} = U_{4\alpha}^{(\nu)} \]  

(14)

Of course, the \( 4 \times 4 \) unitary matrix \( (V_{\alpha j}) \) is the four–dimensional lepton counterpart of the familiar Cabibbo–Kobayashi–Maskawa matrix for quarks. The charged leptons \( e^- , \mu^- , \tau^- \) (with diagonalized mass matrix) are here counterparts of the up quarks \( u , c , t \) (with diagonalized mass matrix).

From Eqs. (12) as well as (9) and (6) we can calculate the matrix elements \( V_{\alpha j} \) in the lowest (quadratic) perturbative order in \( \alpha^{(e)}/\mu^{(e)} \). The result reads (we write for convenience \( \alpha = I = 1, 2, 3, 4 \)):

\[
\begin{align*}
V_{11} &= \left[ 1 - \frac{2}{841} \left( \frac{\alpha^{(e)}}{m_\mu} \right)^2 \right] \frac{1}{\sqrt{1 + \tilde{Y}^2}}, \\
V_{22} &= \left[ 1 - \frac{2}{841} \left( \frac{\alpha^{(e)}}{m_\mu} \right)^2 - \frac{96}{841} \left( \frac{\alpha^{(e)}}{m_\tau} \right)^2 - \frac{8\sqrt{3} \alpha^{(e)}}{29 m_\tau} X e^{i(\varphi^{(e)} - \varphi^{(c)})} \right] \frac{1}{\sqrt{1 + X^2}}, \\
V_{33} &= \left[ 1 - \frac{96}{841} \left( \frac{\alpha^{(e)}}{m_\tau} \right)^2 - \frac{8\sqrt{3} \alpha^{(e)}}{29 m_\tau} X e^{-i(\varphi^{(e)} - \varphi^{(c)})} \right] \frac{1}{\sqrt{1 + X^2}}, \\
V_{12} &= \frac{2 \alpha^{(e)}}{29 m_\mu} \frac{1}{\sqrt{1 + Y^2}} e^{i\varphi^{(e)}}, \quad V_{21} = -\frac{2 \alpha^{(e)}}{29 m_\mu} \frac{1}{\sqrt{1 + Y^2}} e^{-i\varphi^{(e)}}, \\
V_{23} &= \left\{ \left[ 1 - \frac{96}{841} \left( \frac{\alpha^{(e)}}{m_\tau} \right)^2 \right] \frac{1}{\sqrt{1 + X^2}} \right\}, \\
V_{32} &= \left\{ \left[ -1 + \frac{2}{841} \left( \frac{\alpha^{(e)}}{m_\mu} \right)^2 + \frac{96}{841} \left( \frac{\alpha^{(e)}}{m_\tau} \right)^2 \right] \frac{1}{\sqrt{1 + X^2}} \right\}, \\
V_{13} &= 0, \quad V_{31} = \frac{2 \alpha^{(e)}}{29 m_\mu} \frac{X}{\sqrt{1 + X^2}} e^{-i(\varphi^{(e)} + \varphi^{(c)})}, \\
V_{14} &= \frac{Y}{\sqrt{1 + Y^2}} e^{i\varphi^{(e)}}, \quad V_{41} = \left[ -1 + \frac{2}{841} \left( \frac{\alpha^{(e)}}{m_\mu} \right)^2 \right] \frac{Y}{\sqrt{1 + Y^2}} e^{-i\varphi^{(e)}}, \\
V_{24} &= 0, \quad V_{42} = -\frac{2 \alpha^{(e)}}{29 m_\mu} \frac{Y}{\sqrt{1 + Y^2}} e^{-i(\varphi^{(e)} - \varphi^{(c)})}, \\
V_{34} &= 0, \quad V_{43} = 0, \quad V_{44} = \frac{1}{\sqrt{1 + Y^2}}. \\
\end{align*}
\]  

(15)
3. Neutrino oscillations

Once knowing the elements (15) of the lepton Cabibbo—Kobayashi—Maskawa matrix, we are able to calculate the probabilities of neutrino oscillations $\nu_\alpha \rightarrow \nu_\beta$ (in the vacuum), making use of the familiar formulae:

\[
P(\nu_\alpha \rightarrow \nu_\beta) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = \sum_{KL} V_{L\beta}^* V_{L\alpha} V_{K\beta}^* V_{K\alpha} \exp \left( i \frac{m_{\nu_L}^2 - m_{\nu_K}^2}{2|p|} t \right),
\]

where $\nu_\alpha(0) = \nu_\alpha$, $\langle \nu_\alpha | = \langle 0 | \nu_\alpha$ and $\langle \nu_\beta | \nu_\alpha \rangle = \delta_{\beta \alpha}$. Here, as usual, $t/|p| = L/E \ (c = 1 = \bar{h})$ and this is equal to $4 \times 1.2663 L/E$ if $m_{\nu_L}^2 - m_{\nu_K}^2$, $L$ and $E$ are measured in eV$^2$, m and MeV, respectively. Of course, $L$ is the source–detector distance (the baseline). In the following, it will be convenient to denote $x_{LK} = 1\frac{m_{\nu_L}^2 - m_{\nu_K}^2}{L/E}$ and use the identity \[ \cos 2x_{LK} = 1 - 2 \sin^2 x_{LK}. \]

From Eqs. (16) and (15) we deduce by explicit calculations the following neutrino oscillation formulae valid in the lowest (quadratic) perturbative order in $\alpha^{(e)}/\mu^{(e)}$:

\[
P(\nu_e \rightarrow \nu_\mu) = \frac{16}{841} \left( \frac{\alpha^{(e)}}{m_\mu} \right)^2 \times \left\{ \frac{1}{(1 + X^2)(1 + Y^2)} \left( \sin^2 x_{21} + X^2 \sin^2 x_{31} + Y^2 \sin^2 x_{24} + X^2 Y^2 \sin^2 x_{34} \right) - \frac{X^2}{(1 + X^2)^2} \sin^2 x_{32} - \frac{Y^2}{(1 + Y^2)^2} \sin^2 x_{41} \right\},
\]

\[
P(\nu_e \rightarrow \nu_\tau) = \frac{16}{841} \left( \frac{\alpha^{(e)}}{m_\mu} \right)^2 \frac{X^2}{(1 + X^2)^2} \sin^2 x_{32},
\]

\[
P(\nu_\mu \rightarrow \nu_\tau) = \left\{ \left[ 1 - \frac{4}{841} \left( \frac{\alpha^{(e)}}{m_\mu} \right)^2 \right] \frac{4X^2}{(1 + X^2)^2} + \frac{64 \sqrt{3} \alpha^{(e)}}{29 m_\tau} \frac{X(1 - X^2)}{(1 + X^2)^2} \cos \left( \varphi^{(e)} - \varphi^{(e)} \right) \right\} \sin^2 x_{32} + \frac{768}{841} \left( \frac{\alpha^{(e)}}{m_\tau} \right)^2 \left[ \left( 1 - X^2 \right) - 4X^2 \cos^2 \left( \varphi^{(e)} - \varphi^{(e)} \right) \right] \frac{1}{(1 + X^2)^2},
\]

for the appearance experiments (with the appearance modes of $\nu_\mu, \nu_\tau$), and

\[
P(\nu_e \rightarrow \nu_e) = 1 - P(\nu_e \rightarrow \nu_\mu) - P(\nu_e \rightarrow \nu_\tau) - \left[ 1 - \frac{4}{841} \left( \frac{\alpha^{(e)}}{m_\mu} \right)^2 \right] \frac{4Y^2}{(1 + Y^2)^2} \sin^2 x_{41},
\]

for the disappearance channel.
\[ P(\nu_\mu \to \nu_\mu) = 1 - P(\nu_\mu \to \nu_\nu) - P(\nu_\mu \to \nu_\tau) \]
\[ - \frac{16}{841} \left( \frac{\alpha^{(e)}}{m_\mu} \right)^2 \frac{Y^2}{(1 + Y^2)^2} \sin^2 x_{41}, \]  
(21)

\[ P(\nu_\tau \to \nu_\tau) = 1 - P(\nu_\tau \to \nu_\nu) - P(\nu_\tau \to \nu_\mu) \]  
(22)

for the disappearance experiments (with disappearance modes of \(\nu_e, \nu_\mu, \nu_\tau\)). Of all these formulae, only Eqs. (18) and (19) have two–family forms.

Notice that Eqs. (17)—(19) are invariant under the simultaneous substitution \(\varphi^{(\nu)} \to -\varphi^{(\nu)}\) and \(\varphi^{(e)} \to -\varphi^{(e)}\), what means their invariance under the replacement \(V_{K\alpha} \to V_{K\alpha}^*\).

This implies through Eq. (16) that \(P(\nu_\beta \to \nu_\alpha) = P(\nu_\alpha \to \nu_\beta)\). Thus, our neutrino oscillation formulae, valid in the lowest (quadratic) perturbative order in \(\alpha^{(e)}/\mu^{(e)}\), preserve \(T\) reversal and, by CPT invariance, also CP reflection, though the matrix \((V_{\alpha J})\) is here not real (the effect of nonreal \((V_{\alpha J})\) appears in higher perturbative orders and then spoils CP conservation). Note that CP conservation, when it works, and CPT invariance imply \(P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha)\) and \(P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha)\), respectively.

Obviously, Eqs. (20)—(22) tell us that

\[ P(\nu_\nu \to \nu_s) = \left[ 1 - 4 \frac{\alpha^{(e)}}{841} \left( \frac{\alpha^{(e)}}{m_\mu} \right)^2 \right] \frac{4Y^2}{(1 + Y^2)^2} \sin^2 x_{41}, \]  
(23)

\[ P(\nu_\mu \to \nu_s) = \frac{16}{841} \left( \frac{\alpha^{(e)}}{m_\mu} \right)^2 \frac{Y^2}{(1 + Y^2)^2} \sin^2 x_{41}, \]  
(24)

\[ P(\nu_\tau \to \nu_s) = 0, \]  
(25)

for the ”appearance” modes of unobservable sterile neutrino \(\nu_s\).

If there is no sterile neutrino \(\nu_s\), then \(Y = 0\) wherever it appears in Eqs. (17)—(24). On the other hand, if \((\alpha^{(e)}/m_\mu)^2 \to 0\), then

\[ P(\nu_\mu \to \nu_\tau) \to \frac{4X^2}{(1 + X^2)^2} \sin^2 x_{32} \]  
(26)

and

\[ P(\nu_e \to \nu_s) \to \frac{4Y^2}{(1 + Y^2)^2} \sin^2 x_{41}, \]  
(27)

while \(P(\nu_e \to \nu_\mu) \to 0, P(\nu_e \to \nu_\tau) \to 0\) and \(P(\nu_\mu \to \nu_s) \to 0\). Thus,
\[
P(\nu_e \to \nu_e) \to 1 - \frac{4Y^2}{(1 + Y^2)^2} \sin^2 x_{41}
\]

and

\[
P(\nu_\mu \to \nu_\mu) \to 1 - \frac{4X^2}{(1 + X^2)^2} \sin^2 x_{32}
\]

but \(P(\nu_\tau \to \nu_\tau) \to 1\).

Concluding, we can see that the practically decoupled oscillations \(\nu_\mu \to \nu_\tau\) and \(\nu_e \to \nu_s\) play in the framework of our neutrino "texture" an exceptional, dominating role. If there is no sterile neutrino \(\nu_s\) i.e., \(Y = 0\), then such a role is played only by \(\nu_\mu \to \nu_\tau\).

Note that in Eqs. (17)—(24) \(\sin^2 x_{L<K} \to 1/2\) if \(x_{L<K}/\pi \to \infty\), since the source and detector have always finite extensions over which \(\sin^2 x_{L<K}\) ought to be averaged (with respect to their distance \(L\)). If \(2x_{L<K}/\pi \to 0\), then \(\sin^2 x_{L<K} \to x_{L<K}^2 \to 0\).

4. Information from atmospheric neutrinos

The atmospheric neutrino experiments seem to indicate that there is a deficit of atmospheric \(\nu_\mu\)'s, caused by the neutrino oscillations corresponding to disappearance modes of \(\nu_\mu\). These result in the survival probability which, if analized in two–family form

\[
P(\nu_\mu \to \nu_\mu) = 1 - \sin^2 2\theta_{atm} \sin^2 \left(1.27\Delta m_{atm}^2 L/E\right)
\]

leads to

\[
\sin^2 2\theta_{atm} = O(1) \sim 0.8 \text{ to } 1
\]

and

\[
\Delta m_{atm}^2 \sim (0.03 \text{ to } 1) \times 10^{-2} \text{ eV}^2
\]

with the preferable value \(\Delta m_{atm}^2 \sim 0.5 \times 10^{-2} \text{ eV}^2 [4,5]\). It is usually suggested that, practically, the oscillations \(\nu_\mu \to \nu_\tau\) alone are responsible for such a deficit. Then,

\[
P(\nu_\mu \to \nu_\tau) = \sin^2 2\theta_{atm} \sin^2 \left(1.27\Delta m_{atm}^2 L/E\right)
\]
The last suggestion, neglecting the disappearance mode $\nu_\mu \to \nu_e$, is consistent with the negative result of CHOOZ long–baseline reactor experiment that finds no evidence for neutrino oscillations corresponding to the disappearance modes of $\bar{\nu}_e$, in particular $\bar{\nu}_e \to \bar{\nu}_\mu$ [6]. The region of $\sin^2 2\theta_{\text{atm}}$ and $\Delta m^2_{\text{atm}}$ indicated by Super–Kamiokande atmospheric–neutrino experiment [Eqs. (31) and (32)] lies, in fact, inside the region of $\sin^2 2\theta_{\text{CH}}$ and $\Delta m^2_{\text{CH}}$ excluded by CHOOZ (where $P(\bar{\nu}_e \to \bar{\nu}_e) = 1$).

This important message from CHOOZ experiment, restricting the strength of mixing $\nu_\mu$ with $\nu_e$, leaves a priori three options for mixing $\nu_\mu$ with $\nu_\tau$ and $\nu_s$: $\nu_\mu$ mixes dominantly with $\nu_\tau$ (while $\nu_e$ with $\nu_s$), or with $\nu_s$ (while $\nu_e$ with $\nu_\tau$), or with both $\nu_\tau$ and $\nu_s$ (while $\nu_e$ does not mix). Of these three options, our neutrino ”texture” chooses the first due to the assumptions $M_{12}^{(\nu)} = 0$ and $M_{24}^{(\nu)} = 0$, supplemented by $M_{13}^{(\nu)} = 0$ and $M_{34}^{(\nu)} = 0$. Note that the second option would correspond to different assumptions, $M_{12}^{(\nu)} = 0$ and $M_{23}^{(\nu)} = 0$, supplemented by $M_{14}^{(\nu)} = 0$ and $M_{34}^{(\nu)} = 0$; eventually, in the case of the third option there would be $M_{12}^{(\nu)} = 0$, supplemented by $M_{13}^{(\nu)} = 0$, $M_{14}^{(\nu)} = 0$ and $M_{34}^{(\nu)} = 0$.

We can see from the experimental atmospheric–neutrino estimates (30) and (33) that they correspond exactly to our neutrino oscillation formulae (29) and (26), respectively. Hence, we can infer that

$$\frac{4X^2}{(1 + X^2)^2} = \sin^2 2\theta_{\text{atm}} \ , \ |m_{\nu_3}^2 - m_{\nu_2}^2| = \Delta m^2_{\text{atm}}$$

(34)

and so, with the Super–Kamiokande figures (31) and (32), we can take

$$\frac{4X^2}{(1 + X^2)^2} \sim 0.8 \ \text{to} \ 1$$

(35)

and

$$|m_{\nu_3}^2 - m_{\nu_2}^2| \sim 5 \times 10^{-3} \ \text{eV}^2$$

(36)

as two neutrino inputs.

From the input (35) we evaluate the limits

$$X \sim 0.618 \ \text{to} \ 1.$$  

(37)
However, it is not difficult to see that (because the difference (36) is kept fixed) the limit of \( X \to 1 \) is singular in the sense that then \( \mu^{(\nu)} \to 0 \) and \( \beta^{(\nu)} \to \infty \) with 
\[
\mu^{(\nu)} \beta^{(\nu)} \to (5/20.9) \times 10^{-3} \text{ eV}^2
\]
and so, \( M_{22}^{(\nu)} \to 0 \), \( M_{33}^{(\nu)} \to 0 \) and \( |M_{23}^{(\nu)}| \to \infty \) with 
\[
2(M_{22}^{(\nu)} + M_{33}^{(\nu)})|M_{23}^{(\nu)}| \to 5 \times 10^{-3} \text{ eV}^2.
\]
Then, \( m_{\nu_2, \nu_3} \to \mp 0.478 \beta^{(\nu)} \to \mp \infty \). We will restrict, therefore, the range in the input (35) to

\[
\frac{4X^2}{(1 + X^2)^2} \sim 0.8 \to 1 - 10^{-6} , \quad (38)
\]
where the particular upper limit is chosen as an illustration. Hence, \( X \sim 0.618 \) to 0.999 .

In such a case, the second Eq. (10) leads to

\[
\frac{M_{22}^{(\nu)} - M_{33}^{(\nu)}}{2|M_{23}^{(\nu)}|} = \frac{X^2 - 1}{2X} \sim -(0.500 \to 0.001) \quad (40)
\]
(respectively). On the other hand, the mass matrix \( (M_{ij}^{(\nu)}) \) as given in Eq. (1) implies that

\[
\frac{M_{22}^{(\nu)} - M_{33}^{(\nu)}}{2|M_{23}^{(\nu)}|} = -20.3 \frac{\mu^{(\nu)}}{\beta^{(\nu)}} . \quad (41)
\]
Thus, from Eqs. (40) and (41) we get

\[
\frac{\mu^{(\nu)}}{\beta^{(\nu)}} \sim 0.0246 \to 0.0000492 , \quad \frac{\beta^{(\nu)}}{\mu^{(\nu)}} \sim 40.7 \to 20300 \quad (42)
\]
(respectively).

Then, making use of Eq. (1) for the mass matrix \( (M_{ij}^{(\nu)}) \), we obtain from the second Eq. (8)

\[
m_{\nu_2, \nu_3} = \left[ 10.9 \mp 0.478 \frac{\beta^{(\nu)}}{\mu^{(\nu)}} \sqrt{1 + \left(20.3 \frac{\mu^{(\nu)}}{\beta^{(\nu)}}\right)^2} \mu^{(\nu)} \right] \mu^{(\nu)}
\]
\[
= \begin{cases} 
- (10.8 \to 9700) \mu^{(\nu)} \\
(32.7 \to 9730) \mu^{(\nu)}
\end{cases} \quad (43)
\]
(respectively). Therefore,
\[ m_{\nu_3}^2 - m_{\nu_2}^2 = (951 \text{ to } 425000) \mu^{(\nu)}^2 . \]  

(44)

Hence, using the input (36), we evaluate

\[ \mu^{(\nu)} \sim (0.00229 \text{ to } 0.000108) \text{ eV} \]  

(45)

(respectively). With the values (45) of \( \mu^{(\nu)} \), Eqs. (43) lead to

\[ m_{\nu_2} \sim -(0.0247 \text{ to } 1.05) \text{ eV} \text{, } m_{\nu_3} \sim (0.0749 \text{ to } 1.05) \text{ eV} \]  

(46)

(respectively). Here, the minus sign at \( m_{\nu_2} \) is phenomenologically irrelevant in relativistic dynamics (cf. Dirac equation). Note that

\[ m_{\nu_2}^2 \sim (0.000611 \text{ to } 1.11) \text{ eV}^2 \text{, } m_{\nu_3}^2 \sim (0.0561 \text{ to } 1.11) \text{ eV}^2 , \]  

(47)

where \( m_{\nu_3}^2 - m_{\nu_2}^2 \sim 0.005 \text{ eV}^2 \). Finally, from Eqs. (42) and (45) we evaluate

\[ \beta^{(\nu)} \sim (0.0933 \text{ to } 2.20) \text{ eV} . \]  

(48)

To summarize the above discussion of atmospheric neutrinos, we can say that the Super–Kamiokande experiment seems to transmit to us an important message about strong mixing of \( \nu_\mu \) and \( \nu_\tau \) and their rather weak mixing with \( \nu_\tau \). Such a situation is predicted just in the case of our neutrino "texture" with \( X = O(1) \) and small \( \left( \nu^{(e)}/\mu^{(e)} \right)^2 \to 0 \), the latter value being motivated by our excellent zero–order prediction of \( m_\tau \). This conclusion is independent of whether the sterile neutrino \( \nu_s \) exists or does not exist in our neutrino "texture". The predicted neutrino masses are \( |m_{\nu_2}| \sim (0.02 \text{ to } 1) \text{ eV} \) and \( m_{\nu_3} \sim (0.07 \text{ to } 1) \text{ eV} \) with \( m_{\nu_3}^2 - m_{\nu_2}^2 \sim 5 \times 10^{-3} \text{ eV}^2 \), where the upper limit \( X = 0.999 \) is considered for \( X < 1 \). For \( X \) still nearer to 1, \( m_{\nu_2} \) and \( m_{\nu_3} \) increase further.

5. Information from solar neutrinos

As is well known, the solar neutrino experiments demonstrate a deficit of solar \( \nu_e \)'s reaching the Earth, that seems to be caused by neutrino oscillations corresponding to disappearance modes of \( \nu_e \). These modes result in the survival probability, usually analyzed in two–family form
\[
P(\nu_e \rightarrow \nu_e) = 1 - \sin^2 2\theta_{\text{sol}} \sin^2 \left(1.27 \Delta m_{\text{sol}}^2 L/E \right).
\]

Here, the oscillation amplitude \(\sin^2 2\theta_{\text{sol}}\) is likely to be enhanced to a new \((\sin^2 2\theta_{\text{sol}})_{\text{matter}}\) by the resonant MSW mechanism [7] in the Sun matter (dependent on values of \(\sin^2 2\theta_{\text{sol}}\) and \(\Delta m_{\text{sol}}^2\)), though the vacuum mechanism is still not excluded. According to recent estimations [8], in the first case there are two solutions with

\[
\sin^2 2\theta_{\text{sol}} \sim 8 \times 10^{-3}, \quad \Delta m_{\text{sol}}^2 \sim 5 \times 10^{-6} \text{ eV}^2
\]

and

\[
\sin^2 2\theta_{\text{sol}} \sim 0.6, \quad \Delta m_{\text{sol}}^2 \sim 1.6 \times 10^{-5} \text{ eV}^2,
\]

while in the second case

\[
\sin^2 2\theta_{\text{sol}} \sim 0.65 - 1, \quad \Delta m_{\text{sol}}^2 \sim (5 - 8) \times 10^{-11} \text{ eV}^2.
\]

If the disappearance mode \(\nu_e \rightarrow \nu_s\) dominates, then from the two MSW solutions only the first survives. From the above three solutions, the first is considered as most favorable.

Note that the two values \(\Delta m_{\text{sol}}^2 = O(10^{-5} \text{ eV}^2)\) indicated in the first case as well as the value \(\Delta m_{\text{sol}}^2 = O(10^{-10} \text{ eV}^2)\) allowed in the second are situated — consistently — much below the region excluded for disappearance modes of \(\bar{\nu}_e\) by CHOOZ experiment, where \(\Delta m_{\text{CH}}^2 \gtrsim 0.9 \times 10^{-3} \text{ eV}^2\) (at the 90% confidence level) [6].

We can see from the experimental solar–neutrino estimate (49) that it may be related to our neutrino oscillation formula (28) (both being likely enhanced by the MSW mechanism). Then, only the disappearance mode \(\nu_e \rightarrow \nu_s\), described by the formula (27), contributes to the rhs of Eq. (20). In such a case we get with the use of figures (50) two other neutrino inputs

\[
\frac{4Y^2}{(1 + Y^2)^2} = \sin^2 2\theta_{\text{sol}} \sim 8 \times 10^{-3}
\]

and

\[
|m_{\nu_4}^2 - m_{\nu_1}^2| = \Delta m_{\text{sol}}^2 \sim 5 \times 10^{-6} \text{ eV}^2,
\]
if the first MSW solution is taken into account. The input (53) gives \( Y \sim 0.045 \) (and \( Y^2 \sim 2.0 \times 10^{-3} \)).

From the first Eq. (10) with \( M_{11}^{(\nu)} = 0 \) we obtain

\[
Y \simeq \frac{|M_{14}^{(\nu)}|}{M_{44}^{(\nu)}}
\]

(55)

under the assumption that \( M_{44}^{(\nu)^2} \gg \left(2|M_{14}^{(\nu)}|\right)^2 \). Thus, \( \left(|M_{14}^{(\nu)}/M_{44}^{(\nu)}|\right)^2 \sim 0.0020 \) with the input (53).

On the other hand, from the first Eq. (8) with \( M_{11}^{(\nu)} = 0 \)

\[
m_{\nu_1} \simeq -\frac{|M_{14}^{(\nu)}|^2}{M_{44}^{(\nu)}} \quad \text{,} \quad m_{\nu_4} \simeq M_{44}^{(\nu)} + \frac{|M_{14}^{(\nu)}|^2}{M_{44}^{(\nu)}}
\]

(56)

when \( M_{44}^{(\nu)^2} \gg \left(2|M_{14}^{(\nu)}|\right)^2 \). Thus, \( M_{44}^{(\nu)^2} \sim 5 \times 10^{-6} \text{ eV}^2 \) with the input (54), and so, \( |M_{14}^{(\nu)}|^2 \sim 1.0 \times 10^{-8} \text{ eV}^2 \) due to the input (53). Hence,

\[
m_{\nu_1} \sim -4.5 \times 10^{-6} \text{ eV} \quad \text{,} \quad m_{\nu_4} \sim 2.2 \times 10^{-3} \text{ eV}.
\]

(57)

Here again, the minus sign at \( m_{\nu_1} \) is phenomenologically irrelevant.

We can see from Eq. (20) that its approximate form (28) valid for \( \left(\frac{\alpha(\mu)}{m_\mu}\right)^2 \to 0 \), applied here, works very well even for the central value of \( \left(\frac{\alpha(\mu)}{m_\mu}\right)^2 \), because then

\[
\frac{4Y^2}{(1 + Y^2)^2} \sim 8 \times 10^{-3} \text{ large versus } \frac{16}{841} \left(\frac{\alpha(\mu)}{m_\mu}\right)^2 \sim 2.5 \times 10^{-4}.
\]

(58)

In fact, due to Eq. (5),

\[
0 \leq \frac{16}{841} \left(\frac{\alpha(\mu)}{m_\mu}\right)^2 \leq 6.2 \times 10^{-4},
\]

(59)

where the central value is \( 2.5 \times 10^{-4} \).

The small value (59) is the main reason, why in the case of no sterile neutrino \( \nu_s \) our neutrino oscillation formula (20) cannot be compared with the experimental estimate (49). Indeed, in this case it assumes the form

\[
P(\nu_e \to \nu_e) = 1 - \frac{16}{841} \left(\frac{\alpha(\mu)}{m_\mu}\right)^2 \frac{1}{1 + X^2} \left(\sin^2 x_{21} + X^2 \sin^2 x_{31}\right).
\]

(60)
To conclude the above discussion of solar neutrinos, we can claim that, in the framework of our "texture", only the disappearance mode of $\nu_e$ to the sterile neutrino $\nu_s$ (likely to be enhanced by the MSW mechanism in the Sun matter) can be responsible for the observed deficit of solar $\nu_e$'s. The predicted neutrino masses are $|m_{\nu_1}| \sim 2 \times 10^{-6}$ eV and $m_{\nu_4} \sim 2 \times 10^{-3}$ eV with $m_{\nu_4}^2 - m_{\nu_1}^2 \sim 5 \times 10^{-6}$ eV$^2$.

Our last remark concerns the LSND experiment that seems to detect $\nu_\mu \rightarrow \nu_e$ oscillations by observing the appearance of $\nu_e$ originating from $\nu_\mu$ produced in $\pi^+$ decay [9]. The observed excess of $\nu_e$'s, analyzed in terms of two–family oscillation formula, leads to $\sin^2 2\theta_{LSND} > 4 \times 10^{-4}$, in particular to $\sin^2 2\theta_{LSND} \sim 1.5 \times 10^{-3} - 1.5 \times 10^{-1}$ for $\Delta m_{LSND}^2 \sim 1$ eV$^2$ (cf. [9]). This shows that, due to the small value (59), the LSND magnitude of $\nu_\mu \rightarrow \nu_e$ oscillation amplitude can hardly be explained by means of our formula (17) for $P(\nu_\mu \rightarrow \nu_e) = P(\nu_e \rightarrow \nu_\mu)$ . However, there is possibly a narrow overlap near the upper limit of the range (59), corresponding to the values $\sin^2 2\theta_{LSND} \sim 6 \times 10^{-4}$ and $\Delta m_{LSND}^2 \gtrsim 1.2$ eV$^2$, located at the border of LSND allowed region (at the 95% confidence level) [9]. Unfortunately, another (earlier) LSND experiment on $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations estimates that an analogical allowed region is much narrower (cf. [9]), what excludes the above potential overlap.

Nevertheless, it may be interesting to remark that, in the case of values $m_{\nu_2}^2 \sim m_{\nu_3}^2 \gtrsim 1.2$ eV$^2$ corresponding to the large $\Delta m_{LSND}^2 \gtrsim 1.2$ eV$^2$, we can put $x_{21} \sim x_{31} \sim x_{24} \sim x_{34} \gg x_{32} \geq x_{41}$ in Eq. (17). This leads to the two–family oscillation formula

$$P(\nu_e \rightarrow \nu_\mu) \approx \frac{16}{841} \left( \frac{\alpha^{(e)}}{m_\mu} \right)^2 \sin^2 x_{21},$$

when short–baseline experiments with $x_{21} \lesssim \pi/2$ are considered (here, $1 + Y^2 \sim 1 + 0.0020 \simeq 1$).

6. Outlook: an unconventional picture

If beside the sterile neutrino $\nu_s$ there exists also a second sterile neutrino mentioned in the footnote ‡, two opposite options seem to be attractive. The first was already outlined in this footnote: such a second sterile neutrino might mix mainly with $\nu_\mu$, leading to a new disappearance mode of $\nu_\mu$ that, if not too strong, would only slightly correct the effect of the dominating mode $\nu_\mu \rightarrow \nu_\tau$. In the second option, the roles of these two modes of $\nu_\mu$
would be interchanged: the new disappearance mode of $\nu_\mu$ would dominate the mode $\nu_\mu \to \nu_\tau$, and so would be responsible for producing a near–to–maximal oscillation amplitude for $\nu_\mu \to \nu_\mu$ as is observed in Super–Kamiokande. In such a case, both parameters $\alpha^{(\nu)}$ and $\beta^{(\nu)}$ might be small, perhaps zero.

Thus, this second option would create a uniform but unconventional picture of neutrino oscillations, where they would be caused essentially by mixing two sterile neutrinos with $\nu_e$ and $\nu_\mu$, respectively. At any rate, this picture would be true for the solar and atmospheric neutrinos.

Sterile neutrinos of both kinds might constitute an important part of relativistic dark matter, passive with respect to all Standard–Model interactions (including weak interactions). In such a case, they might even be the main constituents of matter in the Universe. Of course, sterile neutrinos would interact with each other and with Standard–Model active particles through gravitation.

Appendix

For the neutrino mass matrix $(M_{ij}^{(\nu)})$ as given in Eq. (1) we assumed that $\alpha^{(\nu)} = 0$ [Eq. (2)], what implied $M_{12}^{(\nu)} = 0 = M_{21}^{(\nu)}$. As we saw, this assumption, leading to weak mixing of $\nu_e$ with $\nu_\mu$ and $\nu_\tau$, is neatly consistent with negative results of CHOOZ experiment [6]. Now, we will relax such an extremal assumption, allowing for a nonzero but small $\alpha^{(\nu)}$, much smaller than the large $\beta^{(\nu)} = O(10^{-4} \text{ to } 1)$ eV [Eq. (48)] responsible for near–to–maximal mixing of $\nu_\mu$ with $\nu_\tau$, just as suggested by results of Super–Kamiokande experiment [4,5]. Although, in our neutrino ”texture” such a relaxation is not needed to understand solar neutrino experiments which are here reasonably explained by the mixing of $\nu_e$ with $\nu_s$, it may be applied to appearance experiments in the mode $\nu_\mu \to \nu_e$ or $\nu_e \to \nu_\mu$.

In this case, $(M_{ij}^{(\nu)})$ is perturbed by the matrix

$$
(\delta M_{ij}^{(\nu)}) = \frac{\alpha^{(\nu)}}{29} \begin{pmatrix}
0 & 2e^{i\phi^{(\nu)}} & 0 \\
2e^{-i\phi^{(\nu)}} & 0 & 8\sqrt{3}e^{i\phi^{(\nu)}} \\
0 & 8\sqrt{3}e^{-i\phi^{(\nu)}} & 0
\end{pmatrix}.
$$

Thus,
\[
\delta M_{12}^{(\nu)} = \frac{\alpha^{(\nu)}}{29} e^{i\varphi^{(\nu)}} = \delta M_{21}^{(\nu)*}, \quad \delta M_{23}^{(\nu)} = \frac{8\sqrt{3}\alpha^{(\nu)}}{29} e^{i\varphi^{(\nu)}} = \delta M_{32}^{(\nu)*}, \quad (A.2)
\]

while
\[
M_{12}^{(\nu)} = 0 = M_{21}^{(\nu)}, \quad M_{23}^{(\nu)} = -\frac{8\sqrt{3}\beta^{(\nu)}}{29} e^{i\varphi^{(\nu)}} = M_{32}^{(\nu)*}. \quad (A.3)
\]

Then, the total secular equation \( \det\left[ M^{(\nu)} + \delta M^{(\nu)} - 1(m_{\nu_3} + \delta m_{\nu_3}) \right] = 0 \) gives in the lowest (quadratic or linear) perturbative order in \( \alpha^{(\nu)}/\mu^{(\nu)} \) the following neutrino mass corrections:

\[
\delta m_{\nu_1} = -\frac{|\delta M_{12}^{(\nu)}|^2 M_{33}^{(\nu)}}{m_{\nu_2} m_{\nu_3}} = -\frac{|\delta M_{12}^{(\nu)}|^2}{m_{\nu_2}} \left( 1 - \frac{|M_{23}^{(\nu)}|}{m_{\nu_3}} X \right),
\]

\[
\delta m_{\nu_1, \nu_3} = \pm 2|\delta M_{23}^{(\nu)}| |M_{23}^{(\nu)}| m_{\nu_2} = \pm 2|\delta M_{23}^{(\nu)}| \frac{X}{1 + X^2}, \quad (A.4)
\]

where \( |\delta M_{12}^{(\nu)}| = 0.0690\alpha^{(\nu)}, \quad |\delta M_{23}^{(\nu)}| = 0.478\alpha^{(\nu)}, \quad M_{33}^{(\nu)} = 20.7\mu^{(\nu)} \) and \( |M_{23}^{(\nu)}| = 0.478\beta^{(\nu)} \).

When the mass matrix \( \left( M_{ij}^{(\nu)} \right) \) is perturbed by the matrix \( \left( \delta M_{ij}^{(\nu)} \right) \) given in Eq. (A.1), then the unitary diagonalizing matrix \( \left( U_{ij}^{(\nu)} \right) \) [Eq.(9)] undergoes the perturbation \( \left( \delta U_{ij}^{(\nu)} \right) \) which, after some calculations in the lowest perturbative order, can be written as follows:

\[
\left( \delta U_{ij}^{(\nu)} \right) = \begin{pmatrix}
0 & \frac{|\delta M_{12}^{(\nu)}|}{m_{\nu_2}} \frac{1}{1 + X^2} e^{i\varphi^{(\nu)}} & -\frac{|\delta M_{12}^{(\nu)}|}{m_{\nu_2}} \frac{1}{1 + X^2} e^{2i\varphi^{(\nu)}} \\
-\frac{|\delta M_{12}^{(\nu)}|^2 M_{33}^{(\nu)}}{m_{\nu_2} m_{\nu_3}} e^{-i\varphi^{(\nu)}} & 0 & \frac{|\delta M_{12}^{(\nu)}|}{m_{\nu_2}} \frac{1}{1 + X^2} e^{-2i\varphi^{(\nu)}} \\
-\frac{|\delta M_{12}^{(\nu)}|^2 |M_{23}^{(\nu)}|}{m_{\nu_2} m_{\nu_3}} e^{-2i\varphi^{(\nu)}} & -\frac{|\delta M_{23}^{(\nu)}|}{|M_{23}^{(\nu)}|^2} \frac{X(1-X^2)}{|M_{23}^{(\nu)}|^2 (1+X^2)^2} e^{-i\varphi^{(\nu)}} & 0
\end{pmatrix}, \quad (A.5)
\]

with \( M_{33}^{(\nu)}/m_{\nu_3} = 1 - |M_{23}^{(\nu)}|X/m_{\nu_3} \) and \( m_{\nu_2} = -|m_{\nu_2}| \). Here, \( Y \sim 0.0045 \) is put zero for simplicity, and \( i, j = 1, 2, 3 \) (if \( Y \neq 0 \), then \( \delta U_{21}^{(\nu)} \) and \( \delta U_{31}^{(\nu)} \) get extra factors \( 1/\sqrt{1+Y^2} \)).

In particular, the elements \( \delta U_{21}^{(\nu)} \) and \( \delta U_{31}^{(\nu)} \) are produced (in the lowest perturbative order) through the diagonalizing procedure of the total mass matrix \( \left( M_{ij}^{(\nu)} + \delta M_{ij}^{(\nu)} \right) \) in the following way:

\[
U_{21}^{(\nu)} + \delta U_{21}^{(\nu)} = -\frac{M_{11}^{(\nu)} - m_{\nu_3} - \delta m_{\nu_3}}{M_{12}^{(\nu)} + \delta M_{12}^{(\nu)}} = \frac{\delta m_{\nu_1}}{m_{\nu_2} m_{\nu_3}} = \frac{|\delta M_{12}^{(\nu)}| M_{33}^{(\nu)}}{m_{\nu_2} m_{\nu_3}} e^{-i\varphi^{(\nu)}},
\]

\[
U_{31}^{(\nu)} + \delta U_{31}^{(\nu)} = -\frac{(M_{11}^{(\nu)} - m_{\nu_3} - \delta m_{\nu_3})(M_{22}^{(\nu)} + \delta M_{22}^{(\nu)})}{(M_{12}^{(\nu)} + \delta M_{12}^{(\nu)})(M_{33}^{(\nu)} - m_{\nu_1} - \delta m_{\nu_1})} = \frac{\delta m_{\nu_1} M_{33}^{(\nu)}}{\delta M_{12}^{(\nu)} M_{33}^{(\nu)}} e^{-2i\varphi^{(\nu)}},
\]
where Eq. (A.4) for $\delta m_{\nu_1}$ is used, while $U_{21}^{(\nu)} = 0$ and $U_{31}^{(\nu)} = 0$. Of course, $M_{11}^{(\nu)} = 0$ and so, $m_{\nu_1} = 0$ for $Y = 0$.

Once $\left(\delta U_{ij}^{(\nu)}\right)$ is known, the perturbation $\left(\delta V_{ij}\right)$ of the leptonic Cabibbo—Kobayashi—Maskawa matrix $(V_{ij})$ [Eqs. (15)] can be evaluated from the definition:

$$(V_{ij} + \delta V_{ij}) = \left(U_{ik}^{(\nu)} + \delta U_{ik}^{(\nu)}\right)^\dagger \left(U_{kj}^{(e)}\right).$$

(A.7)

Here, $\nu_\alpha = \sum_j \left(V_{j\alpha}^{*} + \delta V_{j\alpha}^{*}\right) \nu_j$ with $\nu_\alpha = \nu_e, \nu_\mu, \nu_\tau$ and $\nu_j = \nu_1, \nu_2, \nu_3$. Thus, from Eq. (A.7) we obtain

$$(\delta V_{ij}) = \left(\delta U_{ik}^{(\nu)}\right)^\dagger \left(U_{kj}^{(e)}\right) \simeq \left(\delta U_{ij}^{(\nu)}\right)^\dagger,$$

(A.8)

where, in the second step, terms proportional to $(\alpha^{(\nu)}/m_{\nu_2}) (\alpha^{(e)}/m_\mu)$ are neglected versus terms proportional to $(\alpha^{(\nu)}/m_{\nu_2})$. Thus, in this case

$$\delta V_{ij} = \delta U_{ji}^{*}.$$  

(A.9)

Making use of the elements $\delta V_{ij}$ defined through Eqs. (A.9) and (A.5), we can calculate from Eqs. (16) the perturbations $\delta P(\nu_\alpha \rightarrow \nu_\beta)$ of the neutrino–oscillation probabilities $P(\nu_\alpha \rightarrow \nu_\beta)$ which were evaluated before in the case of $M_{12}^{(\nu)} = 0 = M_{21}^{(\nu)}$ [Eqs. (17) — (22)].

In particular for $\nu_e \rightarrow \nu_\mu$ oscillations, after some calculations up to the quadratic perturbative order in $\alpha^{(\nu)}/\mu^{(e)}$, we obtain the following formula correcting Eq. (17):

$$\delta P(\nu_e \rightarrow \nu_\mu) \simeq \frac{16}{841} \frac{\alpha^{(\nu)} \alpha^{(e)}}{m_{\nu_2} m_\mu} \left\{ \frac{M_{33}^{(\nu)}}{m_{\nu_3}} \left[ \sin^2(x_{21} + \varphi) + \sin^2(x_{21} - \varphi) - 2 \sin^2 \varphi \right] 
- \frac{X^2}{(1 + X^2)^2} \left[ \sin^2(x_{32} + \varphi) - \sin^2(x_{32} - \varphi) \right] \right\} 
+ \frac{16}{841} \left(\frac{\alpha^{(\nu)}}{m_{\nu_2}}\right)^2 \frac{X^2}{(1 + X^2)^2} \sin^2 x_{32},$$

(A.10)
where $\varphi = (1/2)(\varphi^{(\nu)} - \varphi^{(e)})$. Here, $Y = 0$ for simplicity, and the upper limit of $X \sim 0.999$ is taken into account. Owing to this, in our calculations leading to Eq. (A.10), the term proportional to $(M_{33}^{(\nu)}/m_{\nu_3})(1 - X^2|m_{\nu_2}|/m_{\nu_3}) \sim 9.04 \times 10^{-6}$ is negligible and the relation $x_{21} \sim x_{31} \gg x_{32}$ works. In fact, for such a value of $X$, we have $|m_{\nu_2}| \sim 1.05$ eV $\sim m_{\nu_3} (m_{\nu_2} = -|m_{\nu_2}|)$ with $m_{\nu_3}^2 - m_{\nu_2}^2 \sim 5 \times 10^{-3}$ eV$^2$ and $|m_{\nu_2}|/m_{\nu_3} \sim 0.998 \simeq 1$, while $\mu^{(\nu)} \sim 1.08 \times 10^{-4}$ eV and $\beta^{(\nu)} \sim 2.20$ eV with $\beta^{(\nu)}/m_{\nu_3} \sim 2.09$. Further,

$$M_{33}^{(\nu)} = 2.12 \times 10^{-3} \sim \frac{1 - X^2|m_{\nu_2}|/m_{\nu_3}}{1 + X^2}, \quad \frac{1}{1 + X^2} \sim 0.501, \quad \frac{X^2}{1 + X^2} \sim 0.499 \quad (A.11)$$

and, from Eqs. (A.4),

$$\delta m_{\nu_1} \sim -1.06 \times 10^{-5} \left(\frac{\alpha^{(\nu)}}{m_{\nu_2}}\right)^2 \text{eV}, \quad \delta m_{\nu_2, \nu_3} \sim \pm 0.504 \frac{\alpha^{(\nu)}}{m_{\nu_2}} \text{eV}. \quad (A.12)$$

Since the second term in Eq. (A.10) is not invariant under the change of phase sign, $\varphi \rightarrow -\varphi$, this term violates time reversal and so, CP reflection (as CPT is conserved).

The formula (A.10) can be rewritten in the following numerical form:

$$\delta P(\nu_e \rightarrow \nu_\mu) \simeq \frac{\alpha^{(\nu)}}{|m_{\nu_2}|} \left\{4.65 \times 10^{-6} \left[\sin^2(x_{21} + \varphi) + \sin^2(x_{21} - \varphi) - 2\sin^2 \varphi\right] - 5.47 \times 10^{-4} \left[\sin^2(x_{32} + \varphi) - \sin^2(x_{32} - \varphi)\right]\right\} + 4.76 \times 10^{-3} \left(\frac{\alpha^{(\nu)}}{m_{\nu_2}}\right)^2 \sin^2 x_{32}. \quad (A.13)$$

If $\varphi \rightarrow 0$ and $\alpha^{(\nu)}/|m_{\nu_2}| < 1$, the second and third term here can be neglected, when short–baseline experiments with $x_{21} \approx \pi/2$ are taken into account, since then $\sin^2 x_{32} \simeq x_{32}^2 \approx 2.5 \times 10^{-5}(\pi/2)^2$. In such a case, therefore, we get the two–family formula

$$\delta P(\nu_e \rightarrow \nu_\mu) \simeq 9.30 \times 10^{-6} \frac{\alpha^{(\nu)}}{|m_{\nu_2}|} \sin^2 x_{21} \quad (A.14)$$

Evidently, the oscillation amplitudes in the perturbative formula (A.13) are much too small to be able to help the unperturbed formula (61) in explaining the LSND estimate for $\nu_\mu \rightarrow \nu_e$ oscillations [9].
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