Critical temperature of a fully anisotropic three-dimensional Ising model

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Abstract. The critical temperature of a three-dimensional Ising model on a simple cubic lattice with different coupling strengths along all three spatial directions is calculated via the transfer matrix method and a finite size scaling for \( L \times L \times \infty \) clusters \((L = 2 \text{ and } 3)\). The results obtained are compared with available calculations. An exact analytical solution is found for the \( 2 \times 2 \times \infty \) Ising chain with fully anisotropic interactions (arbitrary \( J_x, J_y \) and \( J_z \)).
1. Introduction

Great attention is given to the critical temperature calculation of the three-dimensional Ising model, for the full time extent of existence. The most considerable advances have been attained for the fully isotropic cubic lattice \( (J_x = J_y = J_z) \). The calculations are steadily improving with time, accuracy for the critical point value is down to \( 10^{-3}\% \) error: \( K_c = 0.2216595 \pm 0.0000026 \) (Ferrenberg and Landau 1991), \( K_c = 0.2216544 \pm 0.0000010 \) (Livet 1991).

The efficiency of progress is less for the partly anisotropic model when \( J_x = J_y \neq J_z \). Two cases exist here: \( J_x = J_y \geq J_z \) (a quasi-two-dimensional model) and, inversely, \( J_x = J_y \leq J_z \) (a quasi-one-dimensional model). Using the high-temperature series, the critical temperature estimates have been obtained with \( 2 \times 10^{-2}\% \) error for the fully isotropic interactions \( (J_x = J_y = J_z) \) and about \( 10^{-3}\% \) in the two-dimensional limit \( (J_z = 0) \) of quasi-two-dimensional model (see Navarro and de Jongh 1978 and references cited there). By intermediate range of interlayer couplings, the error of a phase transition temperature determination lies between these two extreme values. Conversely, in the quasi-one-dimensional case, the estimates based on the same high-temperature series are rapidly deteriorated due to the limited number of terms available in the series. As a result, one can find the critical temperatures only up \( J_x(y)/J_z = 10^{-2} \) (with exactness in the one-two significant figure range). In the quasi-one-dimensional case, the phase transition temperature has been calculated also by phenomenological renormalisation of clusters (Yurishchev and Sterlin 1991). Inasmuch, the cluster geometry reflects the physical situation; this approach (contrary to the high-temperature series expansions) yields more precise results as the anisotropy of quasi-one-dimensional system increases. By \( J_x(y)/J_z = 10^{-3} \), the critical temperature position is determined here with an accuracy of about three significant figures.

The difficulties are largest when the interactions are different along all three directions. Here there are the calculations done by real-space renormalisation group method (da Silva et al 1984) and the calculations carried out via various versions of mean field theory and variational principle (see Faleiro Ferreira 1988, 1989 and references therein). These results we discuss in detail in the third section in comparing with our computations.

In this paper, the critical temperature of a three-dimensional Ising model, with
fully anisotropic interactions, is calculated by the transfer matrix approach in combination with a finite size scaling, i.e. by the phenomenological renormalisation group method proposed by Nightingale (1976). As the clusters, we exploit the infinitely long parallelepipeds $L \times L \times \infty$ with transverse scales $L = 2$ and $3$. We succeeded in obtaining a rigorous solution for lattice with $L = 2$. For the $3 \times 3 \times \infty$ lattice, we simplify the partial eigenvalue problem for the transfer matrix of 512th order. Using the symmetry, we reduce this problem to a determination of the largest eigenvalues of the two 18th order matrices. Final calculations have already been made numerically.

2. Calculation of the critical temperature

Conforming with the phenomenological renormalisation group theory, the critical temperature $T_c$ is a fixed point of an equation (see, for instance, reviews of Nightingale 1982 and Barber 1983):

$$L \kappa_L(T_c) = L' \kappa_{L'}(T_c)$$

(1)

where

$$\kappa_L = \ln(\lambda_1/\lambda_2)$$

(2)

is the inverse correlation length in a cluster with characteristic size $L$. The quantities $\lambda_1$ and $\lambda_2$ entering into (2) are respectively largest and next largest eigenvalues of subsystem transfer matrix. Thus, the task is reduced to find the dominant eigenvalues of transfer matrices.

2.1. Cluster $2 \times 2 \times \infty$

Let us write the Hamiltonian for the cluster as

$$H = - \sum_i [J_x (\sigma_{1,i}\sigma_{2,i} + \sigma_{3,i}\sigma_{4,i}) + J_y (\sigma_{1,i}\sigma_{4,i} + \sigma_{2,i}\sigma_{3,i})$$

$$+ J_z (\sigma_{1,i}\sigma_{1,i+1} + \sigma_{2,i}\sigma_{2,i+1} + \sigma_{3,i}\sigma_{3,i+1} + \sigma_{4,i}\sigma_{4,i+1})].$$

(3)

The spin variables $\sigma = \pm 1$ are located in sites of a lattice $2 \times 2 \times \infty$ which has a rectangular cross section and has the symmetry planes going through its axis and the middles of opposite sides.
The transfer matrix $U$ with elements

$$
\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 | U | \sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4 \rangle = \exp \left[ \frac{1}{2} K_x (\sigma_1 \sigma_2 + \sigma_3 \sigma_4 + \sigma'_1 \sigma'_2 + \sigma'_3 \sigma'_4) + \frac{1}{2} K_y (\sigma_1 \sigma_4 + \sigma_2 \sigma_3 + \sigma'_1 \sigma'_4 + \sigma'_2 \sigma'_3) + K_z (\sigma_1 \sigma'_1 + \sigma_2 \sigma'_2 + \sigma_3 \sigma'_3 + \sigma_4 \sigma'_4) \right]
$$

(4)

where $K_x = J_x / kT$, $K_y = J_y / kT$ and $K_z = J_z / kT$ corresponds to the Hamiltonian (3).

To solve the eigenvalue problem of the transfer matrix (4), we use first the invariance property of the appropriate Hamiltonian with respect to the transformations of the group $Z_2 \otimes C_{2v}$; where $Z_2$ is a group of global reflections in the spin space, $C_{2v}$ is the point group generated by the symmetry planes of a lattice and $\otimes$ represents the direct product. Carrying out the usual group-theoretical analysis (see, e.g., Yurishchev 1989), we come to a conclusion that the $16 \times 16$ transfer matrix (4) can be reduced due to symmetry $Z_2 \otimes C_{2v}$ to a quasi-diagonal form with one subblock $5 \times 5$, four subblocks $2 \times 2$ and three ‘subblocks’ $1 \times 1$, i.e. ready-made eigenvalues.

The subblock of size $5 \times 5$ is connected with the fully symmetrical irreducible representation of the group. Due to the Perron theorem, it contains the largest eigenvalue of $U$. Basis vectors for this irreducible representation are given as

$$
\psi_1 = \frac{e_1 + e_{16}}{\sqrt{2}} \quad \psi_2 = \frac{e_2 + e_3 + e_5 + e_8 + e_9 + e_{12} + e_{14} + e_{15}}{2\sqrt{2}}
$$

$$
\psi_3 = \frac{e_4 + e_{13}}{\sqrt{2}} \quad \psi_4 = \frac{e_6 + e_{11}}{\sqrt{2}} \quad \psi_5 = \frac{e_7 + e_{10}}{\sqrt{2}}
$$

(5)

where

$$
e_1 = |1, 1, 1, 1\rangle \quad e_2 = |1, 1, 1, -1\rangle \quad \ldots \quad e_{16} = |-1, -1, -1, -1\rangle.
$$

(6)

Using these basis functions and utilizing (4), we find the matrix elements $\psi_1^* U \psi_2$ of subblock $5 \times 5$. The secular equation of this subblock has a structure (and this is the second key circumstance allowing the solution of the eigenvalue problem):

$$
\lambda^5 - a_1 \lambda^4 + a_2 \lambda^3 - \alpha a_2 \lambda^2 + \alpha^3 a_1 \lambda - \alpha^5 = 0.
$$

(7)

Here

$$
a_1 = 2[1 + 4 \cosh(2K_x) \cosh(2K_y)] \cosh(4K_z) + 6
$$

(8)
\[
a_2 = 32 \cosh(2K_x) \cosh(2K_y)[\cosh(4K_z) \cosh^2(2K_z) - 1] \\
\quad + 8[1 + \cosh(4K_x) + \cosh(4K_y)] \sinh^2(4K_z)
\]

(9)

and

\[
\alpha = 4 \sinh^2(2K_z). 
\]

(10)

According to Sominskii (1967), an algebraic equation like (7) is a reciprocal one. This property makes it possible to easily find the roots of our equation. As a result, the largest eigenvalue of the transfer matrix (4) is equal to

\[
\lambda_1 = \frac{1}{2} r_1 + \left(\frac{1}{4} r_1^2 - \alpha^2\right)^{1/2}
\]

(11)

with

\[
r_1 = \frac{1}{2} (a_1 - \alpha) + \left[\frac{1}{4} (a_1 + \alpha)^2 + \alpha^2 - a_2\right]^{1/2}. 
\]

(12)

Solving secular equations of second-order subblocks causes no difficulties. In this issue, we obtain a complete set of eigenvalues. Sorting the eigenvalues, we seek out the next largest eigenvalue of \( U \):

\[
\lambda_2 = \{1 + \exp[2(K_x + K_y)]\} \sinh(4K_z) + \left\{1 - \exp[2(K_x + K_y)]\right\}^2 \sinh^2(4K_z) \\
\quad + 16 \exp[2(K_x + K_y)] \sinh^2(2K_z)]^{1/2}. 
\]

(13)

Note that it lies in the subblock built on basis functions which are symmetrical under all purely spatial transformations of the group and antisymmetrical under those including the spin inversion.

By \( J_x = J_y \), our solution is reduced to that of Kaufman (1949) for the Ising model on a cylinder, if the number of chains in the last model is equal to four.

It is also interesting to note that the above mention does not succeed in generalizing the model (3). All attempts in including in the Hamiltonian new interactions (e.g., external field, additional pair couplings or multiparticle forces) lead immutably to the destruction of the obvious symmetry of \( \mathbb{Z}_2 \otimes \mathbb{C}_{2v} \) or the hidden algebraic one (i.e., the reciprocal property of a secular equation).

2.2. Cluster \( 3 \times 3 \times \infty \)

We shall consider a subsystem \( 3 \times 3 \times \infty \) with cyclic boundary conditions in both transverse directions. This eliminates undesirable surface effects and at the same time
extends the symmetry group down to $\mathbb{Z}_2 \otimes (T \otimes C_{2v})$, where $T$ is a group of transverse translations and $\otimes$ implies a semidirect multiplication. The given symmetry allows one to reduce the transfer matrix $V$, of the size $512 \times 512$, to a block diagonal form in which the first and second largest eigenvalues of original matrix are located in different subblocks ($V^{(1)}$ and $V^{(2)}$, respectively) both having a dimension of $18 \times 18$. The open form of these subblocks is given in the appendix. The extraction of dominant eigenvalues from $V^{(1)}$ and $V^{(2)}$ has been carried out already by the computer.

We return again to the calculation of critical temperature. The estimates $kT_c/J_z$ obtained by a numerical solution of transcendental equation (1) are collected in table 1. By this, we also put the cyclic boundary conditions on the cluster $2 \times 2 \times \infty$, i.e. simply increase the interaction constants in transverse directions by two times: $J_x \rightarrow 2J_x$ and $J_y \rightarrow 2J_y$. In table 1, we have also inserted the critical temperature values for two limited cases: (i) $J_y = 0$, corresponding to the anisotropic two-dimensional Ising model for which the exact phase transition temperature equation is known (Onsager 1944)

$$\sinh \left( \frac{2J_x}{kT_c} \right) \sinh \left( \frac{2J_z}{kT_c} \right) = 1$$

(14)

and (ii) $J_x = J_y$, corresponding to the partly anisotropic three-dimensional Ising model for which there exists sufficiently accurate estimates of $T_c$ (Navarro and de Jongh 1978, Yurishchev and Sterlin 1991).

3. Discussion

One of simplest ways in estimating the phase transition temperature in an Ising model is the mean field approximation (MFA):

$$(kT_c)_{\text{MFA}} = 2(J_x + J_y + J_z).$$

(15)

However, the accuracy is quite low (see table 2 where, for convenience of comparison, the critical temperature estimates found by various approximate methods have been given, as well as the true values obtaining from a solution of equation (14) and the precision numerical values).

The state of things is somewhat corrected by an improved mean field approximation (IMFA), taking into account the short range order effects (Faleiro
Inspecting table 2, one can see that there are considerable errors, especially for the strongly anisotropic systems.

The MFA can be considerably improved by placing the clusters in the mean field instead of separate spins. A linear chain approximation (LCA) considered by Stout and Chisholm (1962) when the one-dimensional Ising system is taken as a cluster leads to the equation

\[
\left( \frac{kT_c}{J_x} \right)_{LCA} = 2\eta \exp\left( \frac{2J_x}{kT_c} \right)
\]

where \( \eta = (J_x + J_y)/J_x \). De Bruijn (1958) has shown the solution of an equation like (16) is given by

\[
\left( \frac{kT_c}{J_x} \right)_{\text{asympt}} = 2/\left[ \ln \eta^{-1} - \ln \ln \eta^{-1} + O\left( \frac{\ln \ln \eta^{-1}}{\ln \eta^{-1}} \right) \right]
\]

when \( \eta \to 0 \). Fisher (1967) established that the formula (17) is asymptotically exact for the Ising model. Although it qualitatively describes the logarithmically slow drop of the critical temperature with an increase of coupling anisotropy, unfortunately, this asymptotical formula does not provide the acceptable precision, even at high anisotropies. For example, by \( J_x = J_y = 10^{-2}J_x \) a error of deviation from the high-temperature series estimate equals 21% and a error is 28% for the two-dimensional model \( (J_y = 0) \) for the same value of anisotropy \( (J_x/J_z = 10^{-2}) \).

During recent years a number of equations have been obtained for the critical temperature of a fully anisotropic three-dimensional Ising model within the various generalisations of mean field theory, as well a variational approach (one from them — the IMFA — we have mentioned already). Using an extended variational method and taking the sum of linear Ising chains as an auxiliary Hamiltonian, Faleiro Ferreira and Silva (1982) have found an equation for \( T_c \) via the so-called extended linear chain approximation (ELCA). The numerical solution of this equation shows that the ELCA perceptibly improves the LCA (see table 2), however, errors are still considerable. For instance, the critical point position for the two-dimensional isotropic case is overstated by 20%.

Another new approach named by Faleiro Ferreira (1989), the improved linear chain approximation (ILCA), is based upon the same auxiliary Hamiltonian but with other variational principle, leads to better results only for the isotropic three-dimensional case (this is observed in table 2).
Da Silva et al (1984) have given the calculation of a phase transition temperature for the fully anisotropic three-dimensional Potts lattice, a particular case of which is an Ising model, by the real space renormalisation group treatment. However, due to the finite size of spin blocks used, their estimates lose accuracy very rapidly with an increase of coupling anisotropy. Already only by $J_x = J_y = 10^{-1}J_z$, it follows the overestimate $kT_c/J_z = 1.4552$ which, even after carrying out the extrapolation on the rather artificial scheme proposed by authors, falls to $kT_c/J_z = 1.3986$. This value surpasses the high-temperature series value by 4.1%.

We now appeal to our results. The application of clusters makes it possible to take into consideration the specific features of a short range order and as a result, decrease the calculational error. Therefore, it is not surprising that the finite size scaling method with its hierarchy of clusters increasing in growth, allows us to determine the critical point of Ising model with more exactness than the approaches discussed above (see again a table 2). A uniform convergence (contrary to the ILCA) of the estimates with the growth of a lattice anisotropy is an important quality of the approximation.

Let us consider a table 1. In the two-dimensional isotropic limit ($J_x = 0$ and $J_x/J_z = 1$), our calculation fixes the critical temperature with error 4.3% (in the direction of overestimation). For $J_x/J_z \to 0$, this error decreases continiously. This can be easily checked by making a comparison with the exact transition temperature values presented in the next column. In particular, the value has a 1.8% error by $J_x/J_z = 10^{-1}$. A similar situation arises in the three-dimensional case with $J_x = J_y$. Here our estimates are again in excess of true values; the percentage error drops from the maximum value 3.8% in the fully isotropic case ($J_x = J_y = J_z$) and, for comparison, - to 1.7% for the $J_x/J_z = 10^{-1}$ case. A analogous picture seems to be preserved in the intermediate region $0 < J_y/J_x < 1$: by fixing $J_x/J_z$, the error smoothly moves between the limited values corresponding to $J_x/J_z = 0$ and $J_x/J_z = 1$.

4. Conclusions

In the present paper the more qualitative estimates of critical temperature in the fully anisotropic three-dimensional Ising lattice have been derived. These estimates yield the upper bound everywhere and by $J_y/J_x = const$ the error for $kT_c/J_z$ monotonically tends to zero when $J_x/J_z \to 0$. 
The accurate analytical solution has been obtained for an Ising model on the $2 \times 2 \times \infty$ lattice with fully anisotropic couplings.

The quasi-diagonalisation has been carried out for the transfer matrix of $3 \times 3 \times \infty$ Ising model with the rectangular cross section. The expressions for the matrix elements of subblocks containing the leading eigenvalues are given in detail. This permits one to easily reproduce the results presented in the article, as well as can be useful in considering other problems.

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Appendix. Explicit form of subblocks $V^{(1)}$ and $V^{(2)}$

The matrices $V^{(1)}$ and $V^{(2)}$ are symmetrical, therefore, it is enough to describe their upper triangular parts:

$$V^{(1)}_{ij} = 2 \sqrt{\frac{n_j}{n_i}} \left( \sum_{s=1}^{5} |g_s^{(i,j)}| \cosh[(2s-1)K_z] \right) \exp[\frac{1}{2}(m_i^a + m_j^a)K_x + \frac{1}{2}(m_i^b + m_j^b)K_y]$$

(A1)

and

$$V^{(2)}_{ij} = 2 \sqrt{\frac{n_j}{n_i}} \left( \sum_{s=1}^{5} g_s^{(i,j)} \sinh[(2s-1)K_z] \right) \exp[\frac{1}{2}(m_i^a + m_j^a)K_x + \frac{1}{2}(m_i^b + m_j^b)K_y]$$

(A2)

where $i \leq j = 1, 2, \ldots, 18$. The basis vectors are ordered in a non-decrease of their lengths

$$n_i = \{2, 6, 6, 12, 18, 18, 18, 18, 18, 18, 36, 36, 36, 36, 36, 36, 36, 72, 72\}. \quad (A3)$$

The quantities $m_i^a$ and $m_i^b$ have a sense of the reduced partial energies of spin configurations in $i$th vector. They are

$$m_i^a = \{9, 9, -3, -3, 5, 5, 1, 1, -3, 1, 1, -3, 5, -3, 1, -3, 1, -3\} \quad (A4)$$

and

$$m_i^b = \{9, -3, 9, -3, 5, 1, 5, 1, -3, 1, -3, 1, -3, 5, -3, 1, 1, -3\}. \quad (A5)$$

Finally, the weight coefficients $g_s^{(i,j)}$ are given as

|   |           |           |           |           |
|---|-----------|-----------|-----------|-----------|
| 1,1 | 0 0 0 0 1 | 1,2 | 0 1 0 0 0 | 2,2 | 0 -2 0 0 1 |
| 2,3 | 3 0 0 0 0 | 3,3 | 0 -2 0 0 1 | 1,4 | 0 1 0 0 0 |
| 3,4 | 3 0 0 0 0 | 4,4 | 3 -2 0 0 1 | 1,5 | 0 0 0 1 0 |
| 3,5 | 2 0 1 0 0 | 4,5 | 4 0 2 0 0 | 5,5 | 0 0 8 0 1 |
| 2,6 | -2 0 0 1 0 | 3,6 | -1 2 0 0 0 | 4,6 | -2 4 0 0 0 |
| 6,6 | 6 0 2 0 1 | 1,7 | 0 0 1 0 0 | 2,7 | -1 2 0 0 0 |
| 4,7 | -2 4 0 0 0 | 5,7 | 0 7 0 2 0 | 6,7 | 5 0 4 0 0 |
| 1,8 | 1 0 0 0 0 | 2,8 | 0 2 -1 0 0 | 3,8 | 0 2 -1 0 0 |
| 5,8 | -5 4 0 0 0 | 6,8 | 4 -3 2 0 0 | 7,8 | 4 -3 2 0 0 |
| 1,9 | 1 0 0 0 0 | 2,9 | -2 1 0 0 0 | 3,9 | -2 1 0 0 0 |
| 5,9 | -5 4 0 0 0 | 6,9 | 6 -2 1 0 0 | 7,9 | 6 -2 1 0 0 |
| 9,9 | 4 -4 0 0 1 | 1,10 | 0 0 1 0 0 | 2,10 | -1 2 0 0 0 |

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Using (A1)–(A5) and above values for the weight coefficients, one can easily find
the expressions for the matrix elements. For instance,

\[ V_{11}^{(1)} = 2 \cosh(9K_z) \exp[9(K_x + K_y)] \]

\[ V_{35}^{(1)} = 2\sqrt{3}[2 \cosh(K_z) + \cosh(5K_z)] \exp(K_x + 7K_y) \]

and

\[ V_{18,18}^{(2)} = 2\{13[\sinh(K_z) - \sinh(3K_z)] + 7 \sinh(5K_z) - 2 \sinh(7K_z) \]
\[ + \sinh(9K_z) \} \exp[-3(K_x + K_y)]. \]
References

Barber M N 1983 Phase Transitions and Critical Phenomena vol 8, ed C Domb and J L Lebowitz (New York: Academic) p 145

de Bruijn N G 1958 Asymptotic Methods in Analysis (Amsterdam: North–Holland)

sections 2.4 and 2.7

Faleiro Ferreira J R and Silva N P 1982 Phys. Status Solidi b 114 47

Faleiro Ferreira J R 1988 Phys. Status Solidi b 150 281

——1989 Phys. Status Solidi b 156 647

Ferrenberg A M and Landau D P 1991 Phys. Rev. B 44 5081

Fisher M E 1967 Phys. Rev. 162 480

Kaufman B 1949 Phys. Rev. 76 1232

Livet F 1991 Europhys. Lett. 16 139

Navarro R and de Jongh L J 1978 Physica B 94 67

Nightingale M P 1976 Physica A 83 561

——1982 J. Appl. Phys. 53 7927

Onsager L 1944 Phys. Rev. 65 117

da Silva L R, Tsallis C and Schwachheim G 1984 J. Phys. A: Math. Gen. 17 3209

Sominskii I S 1967 Elementarnaya Algebra. Dopolnitelny Kurs (Moskva: Nauka) chap

1

Stout J W and Chisholm R C 1962 J. Chem. Phys. 36 979

Yurishchev M A 1989 Phys. Status Solidi b 153 703

Yurishchev M and Sterlin A 1991 J. Phys.: Condens. Matter 3 2373
Tables and table captions

Table 1. Normalized critical temperature $kT_c/J_z$ for the fully anisotropic three-dimensional Ising model as a function of $J_x/J_z$ and $J_y/J_z$.

| $J_x/J_z$ | $J_y/J_z$ |
|----------|----------|
|          | 0.25     | 0.50  | 0.75  | 1.0   |
| 1.0      | 2.367    | 2.269 | 3.277 | 3.819 | 4.275 | 4.685 | 4.5106‡|
| 0.9      | 2.246    | 2.153 | 3.079 | 3.575 | 3.993 | 4.368 |
| 0.8      | 2.120    | 2.034 | 2.876 | 3.325 | 3.704 | 4.045 |
| 0.7      | 1.987    | 1.909 | 2.667 | 3.070 | 3.409 | 3.714 |
| 0.6      | 1.848    | 1.779 | 2.451 | 2.806 | 3.105 | 3.375 |
| 0.5      | 1.699    | 1.641 | 2.226 | 2.533 | 2.791 | 3.024 | 2.9286‡|
| 0.4      | 1.540    | 1.492 | 1.988 | 2.247 | 2.464 | 2.659 | 2.580‡|
| 0.3      | 1.365    | 1.328 | 1.733 | 1.943 | 2.117 | 2.273 | 2.219‡|
| 0.2      | 1.167    | 1.141 | 1.450 | 1.608 | 1.738 | 1.854 | 1.814‡|
| 0.1      | 0.921    | 0.905 | 1.109 | 1.211 | 1.293 | 1.366 | 1.343‡|
| 0.09     | 0.891    | 0.877 | 1.069 | 1.164 | 1.241 | 1.309 |
| 0.08     | 0.859    | 0.846 | 1.026 | 1.115 | 1.187 | 1.251 |
| 0.07     | 0.826    | 0.814 | 0.982 | 1.064 | 1.131 | 1.189 |
| 0.06     | 0.790    | 0.779 | 0.934 | 1.010 | 1.070 | 1.124 |
| 0.05     | 0.751    | 0.741 | 0.882 | 0.951 | 1.006 | 1.054 | 1.041‡|
| 0.04     | 0.707    | 0.698 | 0.825 | 0.886 | 0.935 | 0.978 |
| 0.03     | 0.657    | 0.650 | 0.760 | 0.813 | 0.856 | 0.892 |
| 0.02     | 0.597    | 0.590 | 0.683 | 0.727 | 0.761 | 0.791 |
| 0.01     | 0.513    | 0.508 | 0.579 | 0.611 | 0.637 | 0.658 | 0.65‡|
| 0.009    | 0.502    | 0.498 | 0.565 | 0.596 | 0.621 | 0.641 | 0.637§|
| 0.008    | 0.491    | 0.486 | 0.551 | 0.581 | 0.604 | 0.624 | 0.619§|
| 0.007    | 0.478    | 0.474 | 0.535 | 0.564 | 0.586 | 0.604 | 0.600§|
| 0.006    | 0.464    | 0.460 | 0.518 | 0.545 | 0.566 | 0.583 | 0.579§|
| 0.005    | 0.449    | 0.445 | 0.499 | 0.524 | 0.544 | 0.560 | 0.556§|

†: Above the critical point
‡: Below the critical point
§: Extrapolated value
|     | 0.004 | 0.431 | 0.428† | 0.478 | 0.501 | 0.519 | 0.534 | 0.531§ |
|-----|-------|-------|--------|-------|-------|-------|-------|--------|
| 0.003| 0.410 | 0.407† | 0.453 | 0.474 | 0.490 | 0.503 | 0.500§ |
| 0.002| 0.383 | 0.380† | 0.421 | 0.439 | 0.453 | 0.465 | 0.462§ |

† Onsager (1944).
‡ Navarro and de Jongh (1978).
§ Yurishchev and Sterlin (1991).
Table 2. Critical temperature estimates of the fully anisotropic three-dimensional Ising model versus calculational method.

| Method | $J_x/J_z = 1$ | $J_x/J_z = 10^{-2}$ |
|--------|---------------|---------------------|
|        | $J_y = 0$    | $J_y = J_x$         | $J_y = 0$    | $J_y = J_x$ |
| MFA    | 4            | 6                   | 2.02        | 2.04       |
| IMFA   | 3.230        | 4.933               | 1.465       | 1.487      |
| LCA    | 3.526        | 5.686               | 0.590       | 0.699      |
| ILCA   | 2.885        | 4.622               | 0.588       | 0.695      |
| ELCA   | 2.728        | 4.881               | 0.543       | 0.669      |
| Table 1| 2.367        | 4.685               | 0.513       | 0.658      |
| Exact  | 2.2691       | 4.5115              | 0.5089      | 0.65       |