Maximum entropy principle approach to a non-isothermal
Maxwell-Stefan diffusion model

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Abstract
In this study we apply the maximum entropy principle to derive the properly scaled velocity distribution function of Boltzmann equations for mixtures, which leads to a non-isothermal Maxwell-Stefan diffusion model. We also analyze the entropy balance law and derive the kinetic entropy production from the scaled distribution function.

Keywords: diffusion, Maxwell-Stefan model, maximum entropy principle

2010 MSC: 35Q20, 82C40, 76R50

1. Introduction
Diffusion is a process of motion of one species relative to another. If the motion occurs relative to a background medium, the process is usually described by the Fick’s law or some of its generalizations. When cross-diffusion is more prominent, like in gaseous mixtures, the Maxwell-Stefan diffusion model is more appropriate\textsuperscript{[1]}. It turned out that cross-diffusion models of Maxwell-Stefan kind can be used as a starting point for diverse applications, ranging from engineering\textsuperscript{[2]} to medicine\textsuperscript{[3]}, and they also inspired new approaches to diffusion phenomena\textsuperscript{[4]}.

Along with applications, mathematical analysis of Maxwell-Stefan equations emerged as equally important but relatively new subject\textsuperscript{[5, 6]}. It covers the aspects of existence and uniqueness of solutions\textsuperscript{[7, 8]}, large-time asymptotics\textsuperscript{[9]} and relaxation limits\textsuperscript{[10]}. Our concern, however, will be the formal derivation of Maxwell-Stefan diffusion equations and analysis of their dissipative character.

From the macroscopic point of view, Maxwell-Stefan diffusion model consists of mass balance equations for the constituents, and a kind of momentum balance relations adjoined with a closure relation due to their linear dependence. It can be derived either in the continuum framework\textsuperscript{[11, 12]}, or starting from the Boltzmann equations (BEs) for mixtures\textsuperscript{[13, 14, 15, 16]}. In the latter approach, crucial step in derivation of Maxwell-Stefan equations is the asymptotic limit enabled through appropriate scaling of the governing equations.

In this study we shall take the kinetic approach to Maxwell-Stefan equations. It is based upon formal diffusive scaling of the BEs, and crucial assumption that velocity distribution functions have the following form:

\[ f_\alpha^i(t, x, v) = \rho_\alpha^i \left( \frac{m_i}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{m_i |v - \alpha u_\alpha^i|^2}{2 k_B T} \right), \quad i = 1, \ldots, S, \]  

(1)

\textsuperscript{*}Authors acknowledge financial support (B.A.) from the COST Action, CA18232 MAT-DYN-NET, supported by the “European Cooperation in Science and Technology” through the “Short Term Scientific Mission” at the Department of Mathematics and Informatics, University of Novi Sad, Serbia, and (S.S.) from the Ministry of Education, Science and Technological Development of the Republic of Serbia (Grant No. 451-03-9/2021-14/200125).

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2. The Boltzmann equation for inert mixtures

We consider a mixture of monatomic ideal inert gases: a gaseous mixture of non-reactive species consisted of particles endowed with translational degrees of freedom only. For each species $A_i$ with $i = 1, 2, 3, \ldots, S$, let $f_i := f_i(t, x, v)$ be its distribution function, representing the densities of particles of species $A_i$ which at time $t \in \mathbb{R}^+$ are located at position $x \in \mathbb{R}^3$ and have velocity $v \in \mathbb{R}^3$. In the absence of external forces, the evolution of the non-reactive mixture can be described by the following system of BEs:

$$
\frac{\partial f_i}{\partial t} + v \cdot \frac{\partial f_i}{\partial x} = \int_{\mathbb{R}^3} \left[ f_i(v') f_i(v^*_s) - f_i(v) f_i(v_s) \right] B_i(v, v_s, \sigma) \, d\sigma \, dv_s
$$

$$
+ \sum_{j=1, j \neq i}^{S} \int_{\mathbb{R}^3} \left[ f_i(v') f_j(v^*_s) - f_i(v) f_j(v_s) \right] B_{ij}(v, v_s, \sigma) \, d\sigma \, dv_s,
$$

for $i = 1, 2, 3, \ldots, S$, where the first and second terms on the right-hand side describe collisions between particles of the same species and between particles of different species, respectively. They are called the mono-species and bi-species collision operators with $v'$, $v^*_s$ defined as follows:

$$
v' = \frac{1}{m_i + m_j} (m_i v + m_j v_s + m_j v - v_s |\sigma), \quad v^*_s = \frac{1}{m_i + m_j} (m_i v + m_j v_s - m_i v - v_s |\sigma),$$

where $\sigma \in S^2$ is an arbitrary unit vector. The formulas given in equation (3) represent all the possible solutions $(v', v^*_s) \in \mathbb{R}^3 \times \mathbb{R}^3$ of the system of equations composed of the momentum and kinetic energy conservation law, which are respectively defined as follows:

$$m_i v + m_j v_s = m_i v' + m_j v^*_s, \quad \frac{1}{2} m_i |v|^2 + \frac{1}{2} m_j |v_s|^2 = \frac{1}{2} m_i |v'|^2 + \frac{1}{2} m_j |v^*_s|^2.$$

2.1. Scaled system of Boltzmann equations

To write the system (2) of BEs in dimensionless form, we first choose macroscopic scales for the independent variables $t, x$ and $v$ associated with the distribution function. To this end, let $\tau, L$ and $T_0$ respectively be the macroscopic time, space and reference temperature scales. These macroscopic scales define two macroscopic velocity scales: $u_0 = \frac{L}{\tau}$ and $c_0 = \sqrt{\frac{2 R T_0}{m_0}}$, where $u_0$ is the speed at which some macroscopic portion of the gas is transported over a distance $L$ in time $\tau$, while $c_0$ is the speed of sound in a monatomic gas with $m_0$ the average atomic mass of the mixture. Using the macroscopic scales: $\tau, L$ and $c_0$, one can define the dimensionless time, space, velocity, distribution function and cross-sections:

$$\hat{t} = \frac{t}{\tau}, \quad \hat{x} = \frac{x}{L}, \quad \hat{v} = \frac{v}{c_0}, \quad \hat{f}_i(\hat{t}, \hat{x}, \hat{v}) = \frac{L^3 c_0^3}{N_i} f_i(t, x, v), \quad \hat{B}_i(\hat{v}, \hat{v}_s, \hat{\sigma}) = \frac{1}{c_0^4 \pi r^2} B_i(v, v_s, \sigma),$$
respectively, and rewrite the system of BEs in dimensionless form in which only two parameters appear—Mach number Ma which multiplies $\partial f_i/\partial t$, and a reciprocal of the Knudsen number Kn which multiplies the scaled collision operators:

$$Ma = \frac{u_0}{c_0}, \quad Kn = \frac{\text{mean free path}}{\text{macroscopic length scale}} = \frac{L^3}{N \times 4\pi r^2} \times \frac{1}{L}.$$ 

In the sequel we shall assume that $Ma = Kn = \alpha \ll 1$, which amounts to diffusive scaling of the BEs [2]. Removing the hats for simplicity and adding superscript $\alpha$ to the distribution function so as to differentiate the dimensionless system of BEs from equation (2), we obtain:

$$\alpha \frac{\partial f_i^\alpha}{\partial t} + v \cdot \frac{\partial f_i^\alpha}{\partial x} = \frac{1}{\alpha} Q_i^\alpha(f_i^\alpha, f_i^\alpha) + \frac{1}{\alpha} \sum_{j \neq i}^S Q_{ij}^\alpha(f_i^\alpha, f_j^\alpha). \quad (6)$$

2.2. Moment Equations

Multiplying the scaled system of BEs (6) by an arbitrary function of the velocity variable $\psi_i(v)$, and then integrate the resulting equation over all values of the velocity $v$ gives the moment equations:

$$\alpha \frac{\partial}{\partial t} \int_{R^3} \psi_i(v) f_i^\alpha dv + \frac{\partial}{\partial x} \int_{R^3} \psi_i(v) f_i^\alpha dv = \frac{1}{\alpha} \int_{R^3} \psi_i(v) Q_i^\alpha(f_i^\alpha, f_i^\alpha) dv + \frac{1}{\alpha} \sum_{j \neq i}^S \int_{R^3} \psi_i(v) Q_{ij}^\alpha(f_i^\alpha, f_j^\alpha) dv, \quad (7)$$

where the integrals on the right-hand side are the weak formulation of the mono-species and bi-species collision operators respectively defined as follows:

$$\int_{R^3} \psi_i(v) Q_i^\alpha(f_i^\alpha, f_i^\alpha) dv = -\frac{1}{4} \int_{R^3} \int_{R^2} \left[ f_i^\alpha(v') f_i^\alpha(v_*) - f_i^\alpha(v) f_i^\alpha(v_*') \right] \times \left[ \psi_i(v') + \psi_i(v_*') - \psi_i(v) - \psi_i(v_*) \right] B_i(v, v_*, \sigma) d\sigma dv_* dv, \quad (8)$$

$$\int_{R^3} \psi_i(v) Q_{ij}^\alpha(f_i^\alpha, f_j^\alpha) dv = \int_{R^3} \int_{R^2} f_i^\alpha(v) f_j^\alpha(v_*) \left[ \psi_i(v') - \psi_i(v) \right] B_{ij}(v, v_*, \sigma) d\sigma dv_* dv. \quad (9)$$

3. Maximum entropy principle

The maximum entropy principle (MEP) determines the approximate velocity distribution function, compatible with selected moments, through maximization of the kinetic entropy [17 18 19 20 21]. Our aim is to apply MEP and derive the properly scaled velocity distribution function which leads to a non-isothermal Maxwell-Stefan model. To that end we introduce the following

**Definition 1.** Kinetic entropy $H(t, x)$ is defined as:

$$H(t, x) := \sum_{i=1}^S H_i(t, x), \quad H_i(t, x) := -k_B \int_{R^3} f_i \log(b_i f_i)dv, \quad (10)$$

where $b_i$ is the dimensional constant which makes the argument of the log function dimensionless. Moments of the distribution functions, representing partial mass, momentum and energy densities, are defined as:

$$\rho_i(t, x) := \int_{R^3} m_i f_i dv, \quad \rho_i(t, x) u_i(t, x) := \int_{R^3} m_i u_i f_i dv,$$

$$\rho_i(t, x)|u_i(t, x)|^2 + 3 \rho_i(t, x) \frac{k_B}{m_i} T(t, x) := \int_{R^3} m_i |v|^2 f_i dv, \quad (11)$$

where $\rho_i$ is the mass density and $u_i$ is the mean velocity of the specie $\mathcal{A}_i$, while $T$ is the common kinetic temperature.
It will be assumed that all the species have the same kinetic temperature $T$. This simplifying assumption is appropriate for non-isothermal diffusion problems, see [22].

In the sequel we shall use the dimensionless macroscopic variables:

$$
\hat{\rho}_i = \frac{L^3}{m_i N} \rho_i, \quad \hat{u}_i = \frac{u_i}{u_0}, \quad \hat{T} = \frac{T}{T_0}, \quad \hat{H}_i = \frac{L^3}{k_B N} H_i, \quad \hat{b}_i = \frac{N}{L^3 c_0^3} b_i.
$$

and drop the hat notation for convenience. The variational problem (MEP) may now be formulated in dimensionless form as follows.

**Maximum entropy principle.** Find the velocity distribution functions $f_i^\alpha(t, x, v)$ which maximize the kinetic entropy:

$$
H^\alpha(t, x) = - \int_{\mathbb{R}^3} \sum_{i=1}^n f_i^\alpha \log(b_i f_i^\alpha) dv \to \text{max},
$$

subject to the following constraints:

$$
\rho_i^\alpha(t, x) = \int_{\mathbb{R}^3} f_i^\alpha dv, \quad \alpha \rho_i^\alpha(t, x) u_i^\alpha(t, x) = \int_{\mathbb{R}^3} v f_i^\alpha dv,
$$

$$
\alpha^2 \rho_i^\alpha(t, x) |u_i^\alpha(t, x)|^2 + 3 \rho_i^\alpha(t, x) \frac{3 m_0}{S m_i} T(t, x) = \int_{\mathbb{R}^3} |v|^2 f_i^\alpha dv.
$$

We would like to note that small parameter $\alpha$ naturally appears in dimensionless constraints (14) due to scaling introduced in previous Section.

**Theorem 1.** The velocity distribution functions which solve the variational problem (13)-(14) have the following form:

$$
f_i^\alpha(t, x, v) = \left( \frac{5}{3 m_0} \right)^{3/2} \rho_i^\alpha(t, x) \left( \frac{m_i}{2 \pi T(t, x)} \right)^{3/2} \exp \left( - \frac{5}{3 m_0} \frac{m_i |v - \alpha u_i^\alpha(t, x)|^2}{2 T(t, x)} \right).
$$

**Proof.** Proof of the Theorem relies on the method of multipliers. Starting from (13)-(14) we define extended functional:

$$
\mathcal{H}^\alpha(t, x) := \int_{\mathbb{R}^3} \sum_{i=1}^n \left( f_i^\alpha \log(b_i f_i^\alpha) + \lambda_i^{(0)} f_i^\alpha + \lambda_i^{(1)} \cdot v f_i^\alpha + \lambda_i^{(2)} |v|^2 f_i^\alpha \right) dv \equiv \int_{\mathbb{R}^3} \mathcal{L}(v, f_i^\alpha, \nabla_v f_i^\alpha) dv,
$$

where $\lambda_i^{(0)}(t, x) \in \mathbb{R}$, $\lambda_i^{(1)}(t, x) \in \mathbb{R}^3$ and $\lambda_i^{(2)}(t, x) \in \mathbb{R}$ are the unknown multipliers. Since the density function $\mathcal{L}$ does not depend on $\nabla_v f_i^\alpha$, necessary condition for extremum is reduced to $\partial \mathcal{L} / \partial f_i^\alpha = 0$, $i = 1, \ldots, S$, which yields $S$ uncoupled equations that can be solved for $f_i^\alpha$:

$$
f_i^\alpha = \frac{1}{b_i} \exp \left[ - \left( 1 + \lambda_i^{(0)} + \lambda_i^{(1)} \cdot v + \lambda_i^{(2)} |v|^2 \right) \right].
$$

Plugging the last expression into the constraints (14), after rather straightforward computation the following relations which determine the multipliers are obtained:

$$
\lambda_i^{(2)} = \frac{5}{3} \frac{m_i}{3 m_0} \frac{1}{2 T}, \quad \lambda_i^{(1)} = - \alpha u_i^\alpha, \quad \frac{1}{b_i} \exp \left( - 1 - \lambda_i^{(0)} \right) \exp \left( \frac{1}{4 \lambda_i^{(2)}} \right) \left( \frac{\pi}{\lambda_i^{(2)}} \right)^{3/2} = \rho_i^\alpha.
$$

The proof is completed by returning these expressions into the maximizer.
4. Non-isothermal Maxwell-Stefan diffusion equations

The Maxwell-Stefan diffusion equations for the mixture of Maxwell molecules were derived in [13], and in the non-isothermal setting in [22], while in [16] the case of reactive mixture of polyatomic gases was considered. Since our interest is on inert mixtures, we shall give the non-isothermal Maxwell-Stefan diffusion equations here for the completeness. They can be obtained from the moment equations (7) by using the scaled distribution function \( \psi \) obtained by MEP. The mass/concentration and momentum balance laws for the species are obtained taking \( \psi (v) = 1 \) and \( \psi (v) = v \), respectively. The energy conservation law for the mixture is obtained taking \( \psi (v) = |v|^2/2 \) and summing up all the energy balance laws for the species.

The resulting equations read:

\[
\begin{align*}
\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x} (\rho_i u_i^a) &= 0, \\
\alpha^2 \left[ \frac{\partial}{\partial t} (\rho_i^a u_i^a) + \frac{\partial}{\partial x} \left( \rho_i^a (u_i^a \otimes u_i^a) \right) \right] + \frac{3m_0}{5m_i} \frac{\partial}{\partial x} (\rho_i^a T) &= 2\pi \sum_{j=1}^{S} ||b_{ij}||_{L1} \frac{m_j}{m_i + m_j} \rho_i^a \rho_j^a (u_j^a - u_i^a) \\
\frac{\partial}{\partial t} \sum_{i=1}^{S} \left( \alpha^2 \frac{1}{2} \rho_i^a |u_i^a|^2 + \frac{3m_0}{5m_i} 2 \rho_i^a T \right) + \frac{\partial}{\partial x} \sum_{i=1}^{S} \left[ \left( \alpha^2 \frac{1}{2} \rho_i^a |u_i^a|^2 + \frac{3m_0}{5m_i} 2 \rho_i^a T \right) u_i^a \right] &= 0.
\end{align*}
\]

Taking the limit as \( \alpha \to 0^+ \), one obtains the non-isothermal Maxwell-Stefan diffusion equations:

\[
\begin{align*}
\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x} (\rho_i u_i) &= 0, \\
\frac{3m_0}{5m_i} \frac{\partial}{\partial x} (\rho_i T) &= 2\pi \sum_{j=1}^{S} ||b_{ij}||_{L1} \frac{m_j}{m_i + m_j} \rho_i \rho_j (u_j - u_i) \\
\frac{3}{2} \frac{\partial}{\partial t} \sum_{i=1}^{S} (\rho_i T) + \frac{5}{2} \frac{\partial}{\partial x} \sum_{i=1}^{S} (\rho_i u_i T) &= 0,
\end{align*}
\]

where

\[
\rho_i = \lim_{\alpha \to 0^+} \rho_i^a \quad \text{and} \quad u_i = \lim_{\alpha \to 0^+} u_i^a.
\]

Equations (16) and (17) are up to scaling completely equivalent to the equations derived in [13, 22].

5. Entropy balance law

To derive the macroscopic entropy balance law from the BEs (2), we first have to define the entropy flux and the entropy production.

**Definition 2.** Entropy flux \( \Phi(t, x) \) and entropy production \( D(t, x) \) are defined as:

\[
\Phi(t, x) := \sum_{i=1}^{S} \Phi_i(t, x), \quad \Phi_i(t, x) := -k_B \int_{\mathbb{R}^3} v f_i \log(b_i f_i) dv,
\]

\[
D(t, x) := \sum_{i=1}^{S} D_i(t, x), \quad D_i(t, x) := -k_B \int_{\mathbb{R}^3} Q_i^b (f_i, f_i) \log(b_i f_i) dv - k_B \sum_{j=1}^{S} \int_{\mathbb{R}^3} Q_{ij}^b (f_i, f_j) \log(b_i f_i) dv.
\]

By introducing dimensionless quantities:

\[
\Phi_i = \frac{L^3}{k_B N c_0} \Phi_i, \quad D_i = \frac{L^6}{k_B N^2 4\pi r^2 c_0} D_i,
\]

we can write:

\[
\Phi(t, x) = \sum_{i=1}^{S} \Phi_i(t, x), \quad D(t, x) = \sum_{i=1}^{S} D_i(t, x).
\]
and dropping hat notation for convenience, one may determine the dimensionless partial entropy fluxes and partial entropy productions:

\[
\Phi_i^\alpha(t, x) = -\int_{\mathbb{R}^3} \mathbf{v} f_i^\alpha \log(b_i f_i^\alpha) d\mathbf{v},
\]

\[
D_i^\alpha(t, x) = -\int_{\mathbb{R}^3} Q_i^\alpha(f_i^\alpha, f_i^\alpha) \log(b_i f_i^\alpha) d\mathbf{v} - \sum_{j=1, j \neq i}^{S} \int_{\mathbb{R}^3} Q_{ij}^\alpha(f_i^\alpha, f_j^\alpha) \log(b_i f_i^\alpha) d\mathbf{v}.
\]

(18)

Setting \( \psi_i(\mathbf{v}) = -\log(b_i f_i^\alpha) \) in (17), one obtains the following scaled entropy balance law:

\[
\frac{\partial H_i^\alpha}{\partial t} + \frac{\partial \Phi_i^\alpha}{\partial x} = \frac{1}{\alpha} D_i^\alpha.
\]

(19)

It is a matter of straightforward computation to show that dimensionless entropy densities and fluxes read:

\[
H_i^\alpha(t, x) = \left\{ \log \left[ b_i \left( \frac{5}{3 m_0} \right) \rho_i^{\alpha/2} \left( \frac{m_i}{2 \pi T} \right)^{3/2} \right] - \frac{3}{2} \right\} \rho_i^{\alpha}, \quad \Phi_i^\alpha(t, x) = \alpha H_i^\alpha(t, x) u_i^\alpha
\]

(20)

However, computation of the entropy production is summarized in the following Theorem.

**Theorem 2.** The dimensionless partial entropy productions read:

\[
D_i^\alpha = \alpha^2 \frac{10\pi}{3T m_0} \sum_{j=1, j \neq i}^{S} ||b_{ij}|| L^1 \frac{m_i m_j \rho_i^{\alpha} \rho_j^{\alpha}}{m_i + m_j} \left[ \frac{m_i u_i^{\alpha} + m_j u_j^{\alpha}}{m_i + m_j} \cdot (u_i^{\alpha} - u_j^{\alpha}) - (u_i^{\alpha} \cdot (u_i^{\alpha} - u_j^{\alpha}) \right],
\]

while the dimensionless total entropy production have the form:

\[
D^\alpha = \sum_{i=1}^{S} D_i^\alpha = \alpha^2 \frac{5 \pi}{3T m_0} \sum_{i=1}^{S} \sum_{j=1, j \neq i}^{S} ||b_{ij}|| L^1 \frac{m_i m_j}{m_i + m_j} \rho_i^{\alpha} \rho_j^{\alpha} |u_j^{\alpha} - u_i^{\alpha}|^2.
\]

(21)

(22)

**Proof.** To obtain the partial entropy, we need to evaluate the integrals given in equation (18) and we will focus on the case Maxwell molecules. This means that the collision cross-sections depend on the deviation angle \( \theta \in [0, \pi] \) between \( \mathbf{v} - \mathbf{v}_* \) and \( \mathbf{\sigma} \) only. Specifically,

\[
B_{ij}(\mathbf{v}, \mathbf{v}_*, \mathbf{\sigma}) = b_{ij}\left( \frac{\mathbf{v} - \mathbf{v}_*}{\mathbf{v} - \mathbf{v}_*} \cdot \mathbf{\sigma} \right) = b_{ij}(\cos \theta).
\]

Moreover, we assume that \( b_{ij} \) is an even function and is such that \( b_{ij} \in L^1(-1, 1) \) following Grad’s angular cut-off assumption. Therefore,

\[
\int_{S^2} B_{ij}(\mathbf{v}, \mathbf{v}_*, \mathbf{\sigma}) d\mathbf{\sigma} = 2 \pi ||b_{ij}|| L^1 \quad \text{and} \quad \int_{S^2} \sigma B_{ij}(\mathbf{v}, \mathbf{v}_*, \mathbf{\sigma}) d\mathbf{\sigma} = 0,
\]

(23)

see (13). Furthermore, using the definition of the species distribution functions (15) in \( \log(b_i f_i^\alpha) \), one can rewrite equation (18) as follows:

\[
D_i^\alpha = \frac{5 m_i}{6 T m_0} k_B \int_{\mathbb{R}^3} |\mathbf{v}|^2 Q_i^m(f_i^\alpha, f_i^\alpha) d\mathbf{v} - \frac{10 m_i k_B}{6 T m_0} u_i^{\alpha} \cdot \int_{\mathbb{R}^3} v Q_i^m(f_i^\alpha, f_i^\alpha) d\mathbf{v}
\]

\[
+ \frac{5 m_i}{6 T m_0} k_B \sum_{j=1, j \neq i}^{S} \int_{\mathbb{R}^3} |\mathbf{v}|^2 Q_{ij}^m(f_i^\alpha, f_j^\alpha) d\mathbf{v} - \frac{10 m_i k_B}{6 T m_0} u_i^{\alpha} \cdot \sum_{j=1, j \neq i}^{S} \int_{\mathbb{R}^3} v Q_{ij}^m(f_i^\alpha, f_j^\alpha) d\mathbf{v}
\]

(24)
The integrals $A$, $B$, $G$ and $H$ all vanish, that is
\[ A = 0 \quad B = 0 \quad G = 0 \quad H = 0. \] (25)

To show that the integrals $H$ and $G$ both vanish, we use the weak formulation of the mono-species and bi-species collision operators given in equations (8) and (9), respectively, with $\psi(v) = 1$. To show that the integrals $A$ and $B$ both vanish, we use the weak formulation of the mono-species collision operator (8) with $\psi(v) = |v|^2$ and $\psi(v) = v$, respectively, together with the conservation laws of kinetic energy and linear momentum for mono-species collisions: equations (3) and (23), with $m_j$ replaced by $m_i$. Furthermore, using the weak form of the bi-species collision operator (9) with $\psi(v) = |v|^2$ and $\psi(v) = v$, respectively, together with equations (3) and (23), one obtains that the values of the integrals $E$ and $F$ are as given below:
\[
E = \alpha^2 4 \pi ||b||_{L^1} \frac{m_j \rho_j^\alpha \rho_i^\alpha}{(m_i + m_j)^2} (m_i u_i^\alpha + m_j u_j^\alpha) \cdot (u_j^\alpha - u_i^\alpha), \quad F = \alpha \frac{m_j}{m_i + m_j} 2 \pi ||b||_{L^1} \rho_i^\alpha \rho_j^\alpha (u_j^\alpha - u_i^\alpha). \] (26)

Substituting equations (25) and (26) into (24) gives (21). Summing equation (21) over all species gives (22).