LONG TERM BEHAVIOR OF RANDOM NAVIER-STOKES EQUATIONS DRIVEN BY COLORED NOISE

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(Communicated by Björn Schmalfuß)

Abstract. This paper is devoted to the study of long term behavior of the two-dimensional random Navier-Stokes equations driven by colored noise defined in bounded and unbounded domains. We prove the existence and uniqueness of pullback random attractors for the equations with Lipschitz diffusion terms. In the case of additive noise, we show the upper semi-continuity of these attractors when the correlation time of the colored noise approaches zero. When the equations are defined on unbounded domains, we establish the pullback asymptotic compactness of the solutions by Ball’s idea of energy equations in order to overcome the difficulty introduced by the noncompactness of Sobolev embeddings.

1. Introduction. In this paper, we study the long term behavior of the two-dimensional random Navier-Stokes equations defined in a bounded or unbounded domain $\mathcal{O} \subseteq \mathbb{R}^2$ with the only assumption that the Poincaré inequality holds: there exists a positive constant $\lambda$ such that

$$\int_{\mathcal{O}} |\nabla \phi|^2 dx \geq \lambda \int_{\mathcal{O}} |\phi|^2 dx, \quad \forall \phi \in H_0^1(\mathcal{O}). \tag{1}$$

Suppose $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{\theta_t\}_{t \in \mathbb{R}}$ is a measure-preserving transformation group on $\Omega$. 

2010 Mathematics Subject Classification. Primary: 35B40; Secondary: 35B41, 37L30.
Key words and phrases. Random attractor, colored noise, unbounded domain, Navier-Stokes equations, energy equations.
This work is supported by NSF of Chongqing grant cstc2018jcyjA0897.
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Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, consider the following non-autonomous two-dimensional random Navier-Stokes equations driven by colored noise:

\[
\begin{cases}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u = f(t, x) - \nabla p + G(t, x, u)\zeta(\theta t, \omega), & x \in \mathcal{O}, \ t > \tau, \\
\text{div} \ u = 0, & x \in \mathcal{O}, \ t > \tau,
\end{cases}
\]

along with homogeneous Dirichlet boundary condition, where $\nu$ is a positive constant, $f$ is a given function defined on $\mathbb{R} \times \mathcal{O}$, $G$ is a nonlinear function satisfying certain conditions, and $\zeta(\theta t, \omega)$ is the colored noise with correlation time $\delta > 0$.

The Navier-Stokes equations are used to model the flow of an incompressible viscous fluid of constant density enclosed in a region $\mathcal{O}$ with rigid boundary $\partial \mathcal{O}$, where $u(t, x)$ and $p(t, x)$ are, respectively, the velocity and the pressure of the fluid at point $x \in \mathcal{O}$ and time $t > \tau$, $\nu > 0$ is the kinematic viscosity of the fluid and $f = f(t, x)$ is the time dependent external force. The existence of global attractors for the Navier-Stokes equations has been extensively studied in the literature, see, e.g., [5, 7, 12, 13, 19, 34, 46, 48, 51] for the deterministic case, and [11, 22, 24, 56] for the stochastic case. More work on random attractors can be found in [8, 9, 14, 15, 16, 17, 20, 21, 22, 24, 25, 26, 27, 29, 37, 42, 47, 53, 54] for the autonomous stochastic equations; and in [18, 23, 30, 31, 55, 57, 58] for the non-autonomous stochastic systems. In this paper, we will investigate pullback random attractors of the non-autonomous random system (2) driven by colored noise.

The random system (2) can be considered as an approximation of the following Stratonovich stochastic equations:

\[
\begin{cases}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u = f(t, x) - \nabla p + G(t, x, u) \circ dW, & x \in Q, \ t > \tau, \\
\text{div} \ u = 0, & x \in Q, \ t > \tau,
\end{cases}
\]

where $W$ is a two-sided real-valued Wiener process. So far, the existence of random attractors for the stochastic equations (3) has been proved only when the diffusion term $G(t, x, u)$ is either a linear function in $u$ or has a very special structure (see [11, 22, 24]). In the present paper, we will show the existence of such attractors for the random system (2) under much less restrictions on the diffusion terms, which is in sharp contrast with the stochastic system (3) (see Remark 1, Theorem 2.8 and Theorem 2.9).

We will also investigate the limiting behavior of pullback random attractors of (2) as the correlation time $\delta \to 0$. Indeed, for additive noise, we will prove, as $\delta \to 0$, the solutions as well as the pullback random attractors of system (2) converge to that of the corresponding stochastic equations (3) (see Lemma 3.4 and Theorem 3.7).

The main difficulty of the paper is to prove the pullback asymptotic compactness of the solution operator of random system (2). When the domain is bounded, such asymptotic compactness can be obtained by the compact Sobolev embeddings, which is used in [32] to study the Wong-Zakai approximations of random Navier-Stokes equations defined in bounded domains. However, this method does not work in the case of unbounded domains due to the non-compactness of Sobolev embeddings on unbounded domains. To solve the problem, we use Ball’s idea [6, 7] of energy equations to establish the desired pullback asymptotic compactness. However, to use the method of energy equations, we must impose stronger restrictions on the diffusion term $G$ (see condition (10)) because, otherwise, the energy equation will involve the nonlinear part of $G$ and make the idea of energy equations invalid. This is why, for the existence of random attractors, we require condition (10) in the
case of unbounded domains, but only require condition (11) in the case of bounded domains. It is evident that condition (11) is much weaker than (10).

When proving the pullback asymptotic compactness of solutions of the stochastic equations (3) with additive noise, we need to convert the stochastic equations into pathwise random equations and then apply the idea of energy equations to the resulting random system. The energy equation of this converted random system does involve the nonlinear drift terms, which is different from the energy equation of system (2), where the nonlinear drift terms disappear. The same problem occurs when deriving the uniform compactness of the random attractors of system (2) with respect to $\delta$, which is needed for proving the upper semi-continuity of random attractors. In these cases, in order to apply the method of energy equations to obtain the necessary asymptotic compactness, we have to prove the convergence of the nonlinear terms contained in the corresponding energy equations. This is achieved by decomposing the nonlinear term into two parts: one is defined in a bounded domain $O_R$ with radius $R$, and the other is defined in the complement of $O_R$. We then prove the nonlinear term is convergent in $O_R$ and its tails on the complement of $O_R$ are uniformly small when $R$ is sufficiently large, from which the desired asymptotic compactness follows, see the proof of Lemma 3.6 for more details.

The colored noise (also called the Ornstein-Uhlenbeck process) was first introduced in [52, 59] to approximate the Wiener process $W$ due to the nowhere differentiability of the sample paths of $W$. On the other hand, the colored noise also has wide applications in practical systems based on the fact that stochastic fluctuations in complex systems are often correlated (see for instance [35, 36]). Indeed, colored noise has been used in many publications to study the dynamics of physical systems, see, e.g., [1, 2, 3, 28, 35, 36, 39, 40, 43, 45] and the references therein. Particularly, as pointed out in [45], one of the most crucial issues in studying stochastic dynamics is how to model the random forcing, which depends on the ratio $\tau_r/\tau_d$, where $\tau_d$ and $\tau_r$ are the time scales of the deterministic dynamical system and the random forcing, respectively. When $\tau_r/\tau_d \gg 1$, white noise should be used in general due to the fact that the deterministic dynamics is much slower than the random forcing. While when $\tau_r/\tau_d \simeq 1$, the deterministic dynamics is sensitive to the autocorrelation of the random forcing and therefore colored noise should be employed. We mention that the Wong-Zakai approximations can also be used to study the solutions and dynamics of stochastic equations. In this respect, the reader is referred to [32, 38, 41, 49, 60, 61, 44] and the references therein.

This paper is organized as follows. In the next section, we prove the existence and uniqueness of tempered random attractors for system (2) with Lipschitz diffusion terms. We then prove the convergence of solutions as well as random attractors of system (2) with additive noise as the correlation time of the colored noise vanishes.

Since the case of unbounded domains is more challenging than the bounded domains, we will only provide detailed proof of our main results for the random Navier-Stokes equations defined on unbounded domains. Our results are valid for the equations defined in bounded domains, but we will not give the details of the proof in this case. Instead, the reader is referred to [32] for the idea of proof in bounded domains.

Hereafter, we always assume the domain $O$ is unbounded unless otherwise stated.
2. Pullback attractors for Random Navier-Stokes equations. This section is devoted to the existence and uniqueness of pullback random attractors for the random two-dimensional Navier-Stokes equations driven by colored noise under certain conditions on the diffusion terms.

2.1. Cocycles for Navier-Stokes equations. In the subsection, we establish the existence of a continuous non-autonomous cocycle for the random two-dimensional Navier-Stokes equations (2).

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be the standard probability space, where \( \Omega = C_0(\mathbb{R}, \mathbb{R}) := \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \} \), with the open compact topology, \( \mathcal{F} \) is its Borel \( \sigma \)-algebra, and \( \mathbb{P} \) is the Wiener measure on \( (\Omega, \mathcal{F}) \). Consider the Wiener shift \( \{ \theta_t \}_{t \in \mathbb{R}} \) defined on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) by

\[
\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t), \quad \text{for all } \omega \in \Omega \text{ and } t \in \mathbb{R}.
\]

It is known that \( \mathbb{P} \) is an ergodic invariant measure for \( \{ \theta_t \}_{t \in \mathbb{R}} \), and the quadruple \( (\Omega, \mathcal{F}, \mathbb{P}, \{ \theta_t \}_{t \in \mathbb{R}}) \) forms a metric dynamical system, see [4]. Given \( \tau \in \mathbb{R} \), consider the following random system driven by colored noise defined on \( \mathcal{O} \times (\tau, \infty) \):

\[
\begin{cases}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u = f(t, x) - \nabla p + e^{it} G(x, u) \zeta(\theta_t \omega), & x \in \mathcal{O}, \ t > \tau, \\
\text{div } u = 0, & x \in \mathcal{O}, \ t > \tau,
\end{cases}
\]

with boundary condition

\[
u \Delta u + (u \cdot \nabla) u = f(t, x) - \nabla p + e^{it} G(x, u) \zeta(\theta_t \omega), \quad x \in \mathcal{O}, \quad t > \tau.
\]

and initial condition

\[
u \Delta u + (u \cdot \nabla) u = f(x), \quad x \in \mathcal{O},
\]

where \( \nu > 0, \mu \geq 0, f \) is a given function defined on \( \mathbb{R} \times \mathcal{O}, G \) is a nonlinear diffusion term, and \( \zeta_\delta \) is the colored noise with correlation time \( \delta > 0 \).

We consider problem (4)-(6) in the classical mathematical framework (see [50], for instance), and write \( L^2(\mathcal{O}) = [L^2(\mathcal{O})]^2 \) and \( H^1_0(\mathcal{O}) = [H^1_0(\mathcal{O})]^2 \) which are endowed, respectively, with the inner products

\[
(u, v) = \int_\mathcal{O} u \cdot v dx, \quad u, v \in L^2(\mathcal{O}),
\]

\[
((u, v)) = \int_\mathcal{O} \sum_{j=1}^2 \nabla u_j \cdot \nabla v_j dx, \quad u = (u_1, u_2), \ v = (v_1, v_2) \in H^1_0(\mathcal{O}),
\]

and norms \( \| \cdot \| = (\cdot)^{1/2}, \| \cdot \| = ((\cdot))^{1/2} \). Denote by

- \( \mathcal{V} = \{ u \in [C^\infty_0(\mathcal{O})]^2 : \text{div } u = 0 \} \),
- \( H = \text{closure of } \mathcal{V} \text{ in } L^2(\mathcal{O}) \),
- \( V = \text{closure of } \mathcal{V} \text{ in } H^1_0(\mathcal{O}) \),

with \( H \) and \( V \) endowed with the inner product and norm of, respectively, \( L^2(\mathcal{O}) \) and \( H^1_0(\mathcal{O}) \). The dual space of \( V \) is denoted by \( V^* \) with norm \( \| \cdot \|_{V^*} \), and the duality pairing between \( V \) and \( V^* \) is denoted by \( \langle \cdot, \cdot \rangle \). Let \( P \) be the orthogonal projection from \( L^2(\mathcal{O}) \) onto \( H \) (the ‘Leray projector’), and \( Au = -P \Delta u \) with domain \( D(A) = [H^2(\mathcal{O})]^2 \cap V \). Define a bilinear operator \( B : V \times V \to V^* \) such that for each \( u, v, w \in V \),

\[
\langle B(u, v), w \rangle = b(u, v, w) = \sum_{i,j=1}^2 \int_\mathcal{O} u_i \frac{\partial v_j}{\partial x_i} w_j dx.
\]
It is well known that
\[
\begin{cases}
  b(u, v, w) = -b(u, w, v) \quad \text{for } u \in H, v, w \in V; \\
  b(u, v, v) = 0 \quad \text{for } u \in H, v \in V; \\
  |b(u, v, w)| \leq \ell |u|^{\frac{1}{2}}|v|^{\frac{1}{2}}|w|^{\frac{1}{2}} \quad \text{for } u, v, w \in V,
\end{cases}
\] (7)

where \(\ell\) is a positive constant.

Throughout the paper, we assume the nonlinear diffusion term \(G(x, u) = \beta u + S(u) + h(x)\) with \(\beta \geq 0, h \in H\) and \(S : V \to H\) a continuous function satisfying
\[
|S(u) - S(v)| \leq s_1|u - v|, \quad \forall u, v \in V,
\] (8)

\[
|(S(u) - S(v), w)| \leq s_2|u - v||w|, \quad \forall u, v, w \in V,
\] (9)

where \(s_1 \geq 0, s_2 \geq 0\) are constants. In addition, if the domain \(O\) is unbounded, we assume
\[
(S(u), u) = 0, \quad \forall u \in V.
\] (10)

If \(O\) is bounded, we assume
\[
|(S(u), u)| \leq s_3 + s_4|u|^{1+\kappa_1}, \quad \forall u \in V,
\] (11)

where \(s_3\) and \(s_4\) are nonnegative numbers and \(\kappa_1 \in (0, 1)\).

**Remark 1.** Let \(S : V \to H\) be given by \(S(u) = B(g, u)\) for all \(u \in V\), where \(g \in D(A)\) is a fixed element, see e.g. [11, 10]. Then \(S\) satisfies (8)-(10). In this case, for \(G(x, u) = \beta u + B(g, u) + h(x)\), the existence of pullback random attractors for the stochastic equations (3) is unknown. But as we will prove in this paper, the random equations (4) have a tempered random attractor in \(H\).

Here, for our purpose, we recall the following known results for the Wiener process \(W(t, \omega) = \omega(t)\) in [4] and the colored noise in [33].

**Lemma 2.1.** Let the correlation time \(\delta \in (0, 1]\). There exists a \(\{\theta_t\}_{t \in \mathbb{R}}\)-invariant subset (still denoted by) \(\Omega\) of full measure, such that for \(\omega \in \Omega\),

(i) \[
\lim_{t \to \pm \infty} \frac{\omega(t)}{t} = 0; \quad (12)
\]

(ii) the mapping
\[
(t, \omega) \mapsto \zeta_{\delta}(\theta_t \omega) = -\frac{1}{\delta^2} \int_{-\infty}^{0} e^{\frac{t}{2} \theta_s \omega}(s)ds
\] (13)

is a stationary solution (also called an Ornstein-Uhlenbeck process or a colored noise) of the one-dimensional stochastic differential equation \(d\zeta_{\delta} + \frac{1}{\delta} \zeta_{\delta} dt = \frac{1}{\delta} dW\) with continuous trajectories satisfying
\[
\lim_{t \to \pm \infty} \frac{\zeta_{\delta}(\theta_t \omega)}{t} = 0 \quad \text{for every } 0 < \delta \leq 1, \quad (14)
\]

\[
\lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} \zeta_{\delta}(\theta_s \omega)ds = \mathbb{E}\zeta_{\delta} = 0 \quad \text{uniformly for } 0 < \delta \leq 1; \quad (15)
\]

and
(iii) for arbitrary $T > 0$, $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0$ such that for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau + T],$

$$\left| \int_0^t \zeta_\delta(s, \omega) ds - \omega(t) \right| < \varepsilon. \quad (16)$$

From (16), we can derive that there exist $\delta_0(\tau, \omega, T) > 0$ and $c = c(\tau, \omega, T) > 0$ such that for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau + T],$

$$\left| \int_0^t \zeta_\delta(s, \omega) ds \right| \leq c. \quad (17)$$

Let $f \in L^2_{loc}(\mathbb{R}; V^*)$. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\nu \in H$, recall that a mapping $u(\cdot, \nu) : \Omega \to H$ is called a solution of problem (4)-(6) if for every $T > 0$,

$$u(\cdot, \nu, \omega, \tau) \in C([\tau, \infty) ; H) \cap L^2(\tau, \tau + T; V)$$

and $u$ satisfies

$$(u(t), \xi) + \nu \int_\tau^t ((u, \xi)) ds + \int_\tau^t b(u, u, \xi) ds$$

$$= (u_\tau, \xi) + \int_\tau^t (f(s, \cdot, \xi)) ds + \int_\tau^t (e^{\mu s} G(x, u) \zeta_\delta(\theta s, \omega), \xi) ds,$$

for every $t > \tau$ and $\xi \in V$. Note that (18) can be rewritten as

$$\frac{du}{dt} + \nu A u + B(u, u) = f + e^{\mu t} G(x, u) \zeta_\delta(\theta t, \omega) \quad \text{in } V^*. \quad (19)$$

By the Galerkin method as in [51], one can verify that if (8)-(10) are fulfilled, then for every $\tau \in \mathbb{R}$, $u_\tau \in H$ and $\omega \in \Omega$, problem (4)-(6) has a unique solution $u$ in the sense of (19). Moreover, this solution is continuous in $u_\tau$ in $H$ and is $(\mathcal{F}, B(H))$-measurable in $\omega \in \Omega$.

Define a cocycle $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H$ such that for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in H$,

$$\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau).$$

Then $\Phi$ is a continuous cocycle on $H$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$.

Let $D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \}$ be a tempered family of bounded nonempty subsets of $H$, that is, for every $\gamma > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to -\infty} e^{\tau t} |D(\tau + t, \theta t \omega)| = 0, \quad (20)$$

where $|D| = \sup_{u \in D} |u|$. Throughout this paper, we will use $D$ to denote the collection of all tempered families of bounded nonempty subsets of $H$, i.e.

$$D = \{ D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} : D \text{ satisfies (20)} \}. \quad (21)$$

In the sequel, we assume that $\alpha \in (0, \nu \lambda)$ is a fixed number and

$$\int_{-\infty}^\tau e^{\alpha s} \|f(s, \cdot, \cdot)\|^2_{V^*} ds < \infty, \ \forall \tau \in \mathbb{R}, \quad (22)$$

and for every positive number $\gamma$,

$$\lim_{\tau \to -\infty} e^{\tau r} \int_0^\tau e^{\alpha s} \|f(s + r, \cdot, \cdot)\|^2_{V^*} ds = 0. \quad (23)$$

Note that conditions (22) and (23) do not require $f(t, \cdot, \cdot)$ be bounded in $V^*$ as $t \to \pm \infty$. 
2.2. Uniform estimates of solutions. In this subsection, we first derive uniform estimates on the solutions of (4)-(6) and then prove the $D$-pullback asymptotic compactness by the idea of energy equations introduced by Ball in [6].

**Lemma 2.2.** Let assumptions (1), (8)-(10) and (22) hold. The for every $\delta \in (0, 1]$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \subseteq D$, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$ and $\sigma \geq \tau - t$, the solution of problem (4)-(6) satisfies

\[
|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 \leq M \left( e^{\int_{\tau-t}^{\tau}(\nu \lambda - 2\beta e^{u(t+r)} \zeta_\delta(\theta_t, \omega))dr} + \int_{\tau-t}^{\tau} e^{\int_{\tau-t}^{\tau}(\nu \lambda - 2\beta e^{u(t+r)} \zeta_\delta(\theta_t, \omega))dr} \left( \|f(s, t, \cdot)\|^2 + e^{2\mu(s+\tau)}|\zeta_\delta(\theta_t, \omega)|^2 \right) ds \right),
\]

and

\[
\int_{\tau-t}^{\tau} e^{\int_{\tau-t}^{\tau}(\nu \lambda - 2\beta e^{u(t+r)} \zeta_\delta(\theta_t, \omega))dr} \|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 ds \leq M \left( 1 + \int_{-\infty}^{0} e^{\int_{-\infty}^{0}(\nu \lambda - 2\beta e^{u(s+r)} \zeta_\delta(\theta_t, \omega))dr} \left( \|f(s, \cdot)\|^2 + e^{2\mu(s+\tau)}|\zeta_\delta(\theta_t, \omega)|^2 \right) ds \right),
\]

where $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ and $M$ is a positive constant independent of $\tau$, $\omega$, $D$, $\sigma$ and $\delta$.

**Proof.** We derive from (4)-(5) that for each $\tau \in \mathbb{R}$, $t \geq \tau$ and $\omega \in \Omega$,

\[
\frac{d}{dt} |u|^2 + 2\nu \|u\|^2 = 2\langle f(t, \cdot), u \rangle + 2e^{\mu t} \zeta_\delta(\theta_t, \omega)(G(x, u), u).
\]

Note that

\[
2\langle f(t, \cdot), u \rangle \leq \frac{1}{4} \nu \|u\|^2 + \frac{1}{\nu} \|f(t, \cdot)\|^2_{V^*},
\]

and by (10),

\[
2e^{\mu t} \zeta_\delta(\theta_t, \omega)(G(x, u), u) = 2\beta e^{\mu t} \zeta_\delta(\theta_t, \omega)|u|^2 + 2e^{\mu t} \zeta_\delta(\theta_t, \omega)(S(u), u) + 2e^{\mu t} \zeta_\delta(\theta_t, \omega)(h, u) \leq 2e^{\mu t} \beta \zeta_\delta(\theta_t, \omega)|u|^2 + \frac{1}{4} \nu |u|^2 + M_1 e^{2\mu t} |\zeta_\delta(\theta_t, \omega)|^2,
\]

where $M_1 = \frac{4}{\nu\lambda} |h|^2$. By (1) and (26)-(28), we have

\[
\begin{align*}
\frac{d}{dt} |u|^2 + (\nu \lambda - 2\beta e^{\mu t} \zeta_\delta(\theta_t, \omega))|u|^2 + \nu \|u\|^2 & \leq \frac{4}{\nu} \|f(t, \cdot)\|^2_{V^*} + M_1 e^{2\mu t} |\zeta_\delta(\theta_t, \omega)|^2.
\end{align*}
\]

Applying the Gronwall inequality to (29) over $[\tau - t, \sigma]$ and replacing $\omega$ by $\theta_{-\tau}\omega$, we have
2502  ANHUI GU, BOLING GUO AND BIXIANG WANG

By (31), we find that there exists $s_0$ such that for all $s \leq s_0$,

$$\int_{s_0}^{s} (\nu \lambda - 2\beta \zeta_0(\theta_s, \omega)) ds < \alpha s.$$  (32)

By (32) and (22) we get

$$\int_{s_0}^{s} \int_{s_0}^{t} e^{f_{s_0}^{f_{s_0}} (\nu \lambda - 2\beta \zeta_0(\theta_s, \omega)) ds \| f(s, \tau, \cdot) \|_{V(r, \cdot)}^2 ds \leq \int_{s_0}^{s} e^{\alpha s} \| f(s, \tau, \cdot) \|_{V(r, \cdot)}^2 ds < \infty,$$

which implies

$$\int_{s_0}^{s} e^{f_{s_0}^{f_{s_0}} (\nu \lambda - 2\beta \zeta_0(\theta_s, \omega)) ds \| f(s, \tau, \cdot) \|_{V(r, \cdot)}^2 ds < \infty.$$  (33)

If $\mu > 0$, by (14) we obtain

$$\lim_{r \to \infty} \int_{s_0}^{s} (\nu \lambda - 2\beta e^{\mu(r+\tau)} \zeta_0(\theta_s, \omega)) ds = \nu \lambda,$$

and hence there exists $s_1$ such that for all $s \leq s_1$,

$$\nu \lambda - 2\beta e^{\mu(r+\tau)} \zeta_0(\theta_s, \omega) > \alpha.$$  (34)

By (34) and (22) we obtain for $\mu > 0$,

$$\int_{-\infty}^{s_1} e^{f_{s_1}^{f_{s_1}} (\nu \lambda - 2\beta e^{\mu(r+\tau)} \zeta_0(\theta_s, \omega)) ds \| f(s, \tau, \cdot) \|_{V(r, \cdot)}^2 ds \leq e^{-\alpha s_1} \int_{-\infty}^{s_1} e^{\alpha s} \| f(s, \tau, \cdot) \|_{V(r, \cdot)}^2 ds < \infty,$$

which yields that for $\mu > 0$,

$$\int_{-\infty}^{\tau_0} e^{f_{s_0}^{f_{s_0}} (\nu \lambda - 2\beta e^{\mu(r+\tau)} \zeta_0(\theta_s, \omega)) ds \| f(s, \tau, \cdot) \|_{V(r, \cdot)}^2 ds < \infty.$$  (35)
It follows from (33) and (35) that for any $\mu \geq 0$,
\[
\int_{-\infty}^{-T} e^{s \int_{s}^{-T} (\nu \lambda - 2 \beta e^{\mu(r)} \tau) \zeta_t (s, \omega) \, ds} \| f(s + \tau, \cdot) \|_{V_r}^2 \, ds < \infty.
\]
(36)

Similarly, one can verify that for any $\mu \geq 0$,
\[
\int_{-\infty}^{-T} e^{2 \mu (s + \tau) \int_{s}^{-T} (\nu \lambda - 2 \beta e^{\mu(r)} \tau) \zeta_t (s, \omega) \, ds} \zeta_t (s, \omega)^2 \, ds < \infty.
\]
(37)

On the other hand, since $u_{\tau - t} \in D(\tau - t, \theta - \omega)$ and $D \in D$, by (32) and (34) we get for $\mu \geq 0$,
\[
e^{f_{\tau - t}^{-1} (\nu \lambda - 2 \beta e^{\mu(r)} \tau) \zeta_t (s, \omega) \, ds} |u_{\tau - t}|^2 \leq e^{f_{\tau - t}^{-1} (\nu \lambda - 2 \beta e^{\mu(r)} \tau) \zeta_t (s, \omega) \, ds} |D(\tau - t, \theta - \omega)|^2 \rightarrow 0,
\]
as $t \rightarrow \infty$. Thus, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$,
\[
e^{f_{\tau - t}^{-1} (\nu \lambda - 2 \beta e^{\mu(r)} \tau) \zeta_t (s, \omega) \, ds} |u_{\tau - t}|^2 \leq 1,
\]
and hence for all $t \geq T$ and $\mu \geq 0$,
\[
e^{f_{\tau - t}^{-1} (\nu \lambda - 2 \beta e^{\mu(r)} \tau) \zeta_t (s, \omega) \, ds} |u_{\tau - t}|^2 \leq e^{f_{\tau - t}^{-1} (\nu \lambda - 2 \beta e^{\mu(r)} \tau) \zeta_t (s, \omega) \, ds},
\]
which along with (30) and (36)-(37) completes the proof. ∎

As an immediate consequence of Lemma 2.2, we have

**Lemma 2.3.** Let assumptions (1), (8)-(10) and (22) hold. Then for every $\delta \in (0,1]$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $k \geq 0$ and for all $t \geq T + k$, the solution of problem (4)-(6) satisfies
\[
|u(t - k, \tau - t, \theta - \omega, u_{\tau - t})|^2 \leq M \left( e^{\int_{k}^{T} (\nu \lambda - 2 \beta e^{\mu(r)} \tau) \zeta_t (s, \omega) \, ds} \| f(s + \tau, \cdot) \|_{V_r}^2 + e^{2 \mu (s + \tau)} \zeta_t (s, \omega)^2 \right),
\]
where $u_{\tau - t} \in D(\tau - t, \theta - \omega)$ and $M$ is the same number as in Lemma 2.2, independent of $\tau$, $\omega$, $D$, $k$ and $\delta$.

**Proof.** Given $\tau \in \mathbb{R}$ and $k \geq 0$, let $\sigma = \tau - k$. Let $T = T(\tau, \omega, D, \delta)$ be the positive number claimed in Lemma 2.2. If $t \geq T + k$, we get $t \geq T$ and $\sigma \geq \tau - t$, and then the desired conclusion follows from Lemma 2.2. ∎

We need the following weak continuity for the proof of the $D$-pullback asymptotic compactness of the solutions of (4)-(6). This result can be established by the standard methods as in [46, Lemma 2.1].

**Lemma 2.4.** Suppose (1), (8)-(10) hold and $f \in L^2_{\text{loc}}(\mathbb{R}; V^*)$. Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_{\tau, n} \in H$ for all $n \in \mathbb{N}$. If $u_{\tau, n} \rightarrow u_{\tau}$ in $H$, then the solutions of problem (4)-(6) verify the following properties:
\[
u(t, \tau, \omega, u_{\tau, n}) \rightarrow u(t, \tau, \omega, u_{\tau}) \quad \text{in} \quad H, \quad \forall r \geq \tau,
\]
and
\[
u(t, \tau, \omega, u_{\tau, n}) \rightarrow u(t, \tau, \omega, u_{\tau}) \quad \text{in} \quad L^2(\tau, \tau + T; V), \quad \forall T > 0.
\]

Next, we prove the $D$-pullback asymptotic compactness of the solutions of system (4)-(6).
Lemma 2.5. Let assumptions (1), (8)-(10) and (22) hold. Then for every \( \tau \in \mathbb{R}, \omega \in \Omega, D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D, \) and \( t_n \to \infty, u_{0,n} \in D(\tau - t_n, \theta_{-t_n}, \omega), \) the sequence \( u(\tau, \tau - t_n, \theta_{-t_n}, \omega_{0,n}) \) of solutions of problem (4)-(6) has a convergent subsequence in \( H. \)

Proof. We infer from Lemma 2.2 with \( \sigma = \tau \) in (24) that, there exists \( T = T(\tau, \omega, D, \delta) > 0 \) such that for all \( t \geq T, \)

\[
|u(\tau, \tau - t, \theta_{-t}, \omega)|^2 \leq R(\tau, \omega),
\]

where

\[
R(\tau, \omega) = M\left(1 + \int_{-\infty}^{0} e^{\int_{0}^{s} (\nu \lambda - 2\beta \rho^{(r+1)} \gamma \zeta(s, \omega))ds} \left(\|f(s + \tau, \cdot)\|_{V^*} + e^{2\mu(s + \tau)}|\zeta(s, \omega)|^2\right) ds\right)
\]

and \( u_{\tau - t} \in D(\tau - t, \theta_{-t}). \) Since \( t_n \to \infty, \) there exists \( N_1 \in \mathbb{N} \) such that \( t_n \geq T \) for all \( n \geq N_1. \) Due to \( u_{0,n} \in D(\tau - t_n, \theta_{-t_n}, \omega), \) we get from (38) that for all \( n \geq N_1, \)

\[
|u(\tau, \tau - t_n, \theta_{-t_n}, \omega_{0,n})|^2 \leq R(\tau, \omega),
\]

which implies that there exists \( \bar{u} \in H \) and a subsequence (not relabelled) such that

\[
u(\tau, \tau - t_n, \theta_{-t_n}, \omega_{0,n}) \to \bar{u} \quad \text{in} \quad H.
\]

We now need to prove that the weak convergence in (39) is actually a strong one, which will complete the proof. Note that (39) implies

\[
\lim_{n \to \infty} \inf |u(\tau, \tau - t_n, \theta_{-t_n}, \omega_{0,n})| \geq |\bar{u}|.
\]

To this end, we need to show

\[
\limsup_{n \to \infty} |u(\tau, \tau - t_n, \theta_{-t_n}, \omega_{0,n})| \leq |\bar{u}|,
\]

which can be achieved by the method of energy equations. Given \( k \in \mathbb{N} \) we have

\[
u(\tau, \tau - k, \theta_{-k}, \omega_{0,n}) = u(\tau, \tau - k, \theta_{-k}, \omega_{-k}, \omega_{0,n}).
\]

For each \( k, \) let \( N_k \) be large enough such that \( t_n \geq T + k \) for all \( n \geq N_k. \) We infer from Lemma 2.3 that for \( n \geq N_k, \)

\[
|u(\tau - k, \tau - t_n, \theta_{-t_n}, \omega_{0,n})|^2 \leq M\left(\int_{-\infty}^{0} e^{\int_{0}^{s} (\nu \lambda - 2\beta \rho^{(r+1)} \gamma \zeta(s, \omega))ds} \left(\|f(s + \tau, \cdot)\|_{V^*} + e^{2\mu(s + \tau)}|\zeta(s, \omega)|^2\right) ds\right)
\]

which shows that, for each \( k \in \mathbb{N} \), the sequence \( u(\tau - k, \tau - t_n, \theta_{-t_n}, \omega_{0,n}) \) is bounded in \( H. \) By a diagonal process, we can find a subsequence (not relabelled) and an element \( \bar{u}_k \in H \) for each \( k \in \mathbb{N} \) such that

\[
u(\tau - k, \tau - t_n, \theta_{-t_n}, \omega_{0,n}) \to \bar{u}_k \quad \text{in} \quad H.
\]

By (42)-(43) and Lemma 2.4 we obtain that for each \( k \in \mathbb{N}, \)

\[
u(\tau, \tau - t_n, \theta_{-t_n}, \omega_{0,n}) \to u(\tau, \tau - k, \theta_{-k}, \theta_{-t_n}, \omega_{0,n}) \quad \text{in} \quad H,
\]

and

\[
u(\tau, \tau - k, \theta_{-k}, u(\tau - k, \theta_{-t_n}, \theta_{-t_n}, \omega_{0,n})) \to u(\tau, \tau - k, \theta_{-k}, \theta_{-t_n}, \omega_{0,n}) \quad \text{in} \quad L^2(\tau - k, \tau; V).
\]

By (39) and (44) we get

\[
u(\tau, \tau - k, \theta_{-t}, \bar{u}_k) = \bar{u}.
\]
Note that (26) shows that
\[
\frac{d}{dt}|u|^2 + (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))|u|^2 + \psi(u) = 2(f(t, \cdot), u) + 2e^{\nu\mu} \zeta_\theta(\theta,\omega)(h, u),
\]}

where \(\psi\) is a functional on \(V\) given by
\[
\psi(u) = 2\nu\|u\|^2 - \nu \lambda |u|^2, \quad \forall u \in V.
\]

Indeed, by (1),
\[
\nu\|u\|^2 \leq \psi(u) \leq 2\nu\|u\|^2 \quad \forall u \in V,
\]
which indicates that \(\psi(\cdot)\) is an equivalent norm of \(V\). Now, multiplying (47) by \(e^{\int_0^t (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr}\) and integrating the result equality from \(\sigma\) to \(\tau\) for each \(\omega \in \Omega\) and \(\tau \geq \sigma\), we get
\[
|u(\tau, \sigma, \omega, u_\omega)|^2 = \int_\sigma^\tau e^{\int_s^\tau (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr} |u_\sigma|^2
\]
\[
- \int_\sigma^\tau \int_s^\tau e^{\int_s^\tau (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr} \psi(u(s, \sigma, \omega, u_\omega))ds
\]
\[
+ 2 \int_\sigma^\tau \int_s^\tau e^{\int_s^\tau (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr} \langle f(s, \cdot), u(s, \sigma, \omega, u_\omega) \rangle ds
\]
\[
+ 2 \int_\sigma^\tau e^{\mu(s)} e^{\int_s^\tau (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr} \zeta_\theta(h, u(s, \sigma, \omega, u_\omega))ds.
\]

Now, combining (46) with (50), we find that
\[
|\bar{u}|^2 = |u(\tau, \sigma, \omega, u_\omega)|^2 = \int_{-k}^0 e^{\int_{-k}^0 (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr} |\bar{u}_k|^2
\]
\[
- \int_{-k}^0 \int_{-k}^0 e^{\int_{-k}^0 (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr} \psi(u(s + \tau, \tau - k, \theta_{-\tau,\omega}, \bar{u}_k))ds
\]
\[
+ 2 \int_{-k}^0 \int_{-k}^0 e^{\int_{-k}^0 (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr} \langle f(s + \tau, \cdot), u(s + \tau, \tau - k, \theta_{-\tau,\omega}, \bar{u}_k) \rangle ds
\]
\[
+ 2 \int_{-k}^0 e^{\mu(s)} e^{\int_{-k}^0 (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr} \zeta_\theta(h, u(s + \tau, \tau - k, \theta_{-\tau,\omega}, \bar{u}_k))ds.
\]

Also, we get from (42) and (50) that
\[
|u(\tau, \tau - t_n, \theta_{-\tau,\omega}, u_0, n)|^2 = |u(\tau, \tau - t_n, \theta_{-\tau,\omega}, u(\tau - t_n, \tau - t_n, \theta_{-\tau,\omega}, u_0, n))|^2
\]
\[
= I_1(k, n) + I_2(k, n) + I_3(k, n) + I_4(k, n),
\]
where
\[
I_1(k, n) = \int_{-k}^0 e^{\int_{-k}^0 (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr} |u(\tau - t_n, \tau - t_n, \theta_{-\tau,\omega}, u_0, n)|^2,
\]
\[
I_2(k, n) = - \int_{-k}^0 e^{\int_{-k}^0 (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr} \psi(u(s + \tau, \tau - k, \theta_{-\tau,\omega}, u(\tau - t_n, \tau - t_n, \theta_{-\tau,\omega}, u_0, n)))ds,
\]
\[
I_3(k, n) = 2 \int_{-k}^0 e^{\int_{-k}^0 (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr} \langle f(s + \tau, \cdot), u(s + \tau, \tau - k, \theta_{-\tau,\omega}, u(\tau - k, \tau - t_n, \theta_{-\tau,\omega}, u_0, n)) \rangle ds,
\]
\[
I_4(k, n) = 2 \int_{-k}^0 e^{\mu(s)} e^{\int_{-k}^0 (\nu \lambda - 2\beta \epsilon^{\nu\mu} \zeta_\theta(\theta,\omega))dr} \zeta_\theta(h, u(s + \tau, \tau - k, \theta_{-\tau,\omega}, u_0, n)).
\]
We now examine the limits of (53)-(56) as \( n \to \infty \). First, by (30) with \( \sigma = \tau - k \) and \( t = t_n \), one can see that

\[
I_1(k, n) \leq e^{\int_{0}^{t_n} (\nu \lambda - 2 \beta e^{\mu(s+\tau)} \zeta_k(\theta, \omega)) \, dr} |u_{0,n}|^2 \\
+ \frac{4}{\nu} \int_{-\infty}^{-k} e^{\int_{0}^{t} (\nu \lambda - 2 \beta e^{\mu(s+\tau)} \zeta_k(\theta, \omega)) \, dr} \|f(s + \tau, \cdot)\|^2 \, ds \\
+ M_1 \int_{-\infty}^{-k} e^{2\mu(s+\tau)} e^{\int_{0}^{t} (\nu \lambda - 2 \beta e^{\mu(s+\tau)} \zeta_k(\theta, \omega)) \, dr} |\zeta_k(\theta, \omega)|^2 \, ds.
\]

Since \( u_{0,n} \in D(\tau - t_n, \theta - t_n, \omega) \), we have

\[
e^{\int_{0}^{t_n} (\nu \lambda - 2 \beta e^{\mu(s+\tau)} \zeta_k(\theta, \omega)) \, dr} |u_{0,n}|^2 \\
\leq e^{\int_{0}^{t_n} (\nu \lambda - 2 \beta e^{\mu(s+\tau)} \zeta_k(\theta, \omega)) \, dr} |D(\tau - t_n, \theta - t_n, \omega)|^2 \to 0 \quad \text{as} \quad n \to \infty,
\]

which together with (57) shows that

\[
\limsup_{n \to \infty} I_1(k, n) \leq \frac{4}{\nu} \int_{-\infty}^{-k} e^{\int_{0}^{t} (\nu \lambda - 2 \beta e^{\mu(s+\tau)} \zeta_k(\theta, \omega)) \, dr} \|f(s + \tau, \cdot)\|^2 \, ds \\
+ M_1 \int_{-\infty}^{-k} e^{2\mu(s+\tau)} e^{\int_{0}^{t} (\nu \lambda - 2 \beta e^{\mu(s+\tau)} \zeta_k(\theta, \omega)) \, dr} |\zeta_k(\theta, \omega)|^2 \, ds.
\]

By (45) we have

\[
\limsup_{n \to \infty} I_2(k, n) = -\liminf_{n \to \infty} \left( \int_{-k}^{0} e^{\int_{0}^{t} (\nu \lambda - 2 \beta e^{\mu(s+\tau)} \zeta_k(\theta, \omega)) \, dr} \\
\cdot \psi(u(s + \tau, \tau - k, \theta_{-\tau} \omega, u(\tau - k, \tau - t_n, \theta_{-\tau} \omega, u_{0,n}))) ds \right)
\]

\[
\leq - \int_{-k}^{0} e^{\int_{0}^{t} (\nu \lambda - 2 \beta e^{\mu(s+\tau)} \zeta_k(\theta, \omega)) \, dr} \psi(u(s + \tau, \tau - k, \theta_{-\tau} \omega, \bar{u}_k)) ds,
\]

and

\[
\lim_{n \to \infty} I_3(k, n) = 2 \int_{-k}^{0} e^{\int_{0}^{t} (\nu \lambda - 2 \beta e^{\mu(s+\tau)} \zeta_k(\theta, \omega)) \, dr} \langle f(s + \tau, \cdot), u(s + \tau, \tau - k, \theta_{-\tau} \omega, \bar{u}_k) \rangle ds.
\]

By (45) we also get

\[
\lim_{n \to \infty} I_4(k, n) \\
= 2 \int_{-k}^{0} e^{\mu(s+\tau)} e^{\int_{0}^{t} (\nu \lambda - 2 \beta e^{\mu(s+\tau)} \zeta_k(\theta, \omega)) \, dr} \zeta_k(\theta, \omega) (h, u(s + \tau, \tau - k, \theta_{-\tau} \omega, \bar{u}_k)) ds.
\]
Lemma 2.6. For given $\tau, \tau - t_n, \theta_{-\tau} \omega, u_{0,n}$, we get
\[
\limsup_{n \to \infty} |u(\tau, \tau - t_n, \theta_{-\tau} \omega, u_{0,n})|^2 
\leq 4 \int_{-\infty}^{-k} e^{\int_0^s (\nu\lambda - 2\beta e^{\mu(s+r)} \zeta_k(\theta, \omega) \right) ds} \| f(s + \tau, \cdot) \|_{V^{\tau}}^2 \, ds 
+ M_1 \int_{-\infty}^{-k} e^{\mu(s+\tau)} e^{\int_0^s (\nu\lambda - 2\beta e^{\mu(s+r)} \zeta_k(\theta, \omega) \right) ds} \| \tilde{u}_{\omega, \cdot} \|_{V^{\tau}}^2 \, ds 
+ 2 \int_{-\infty}^{0} e^{\int_0^{s+\tau} (\nu\lambda - 2\beta e^{\mu(s+r)} \zeta_k(\theta, \omega) \right) ds} \| f(s + \tau, \cdot) \|_{V^{\tau}}^2 \, ds + 2 \int_{-\infty}^{0} e^{\mu(s+\tau)} e^{\int_0^{s+\tau} (\nu\lambda - 2\beta e^{\mu(s+r)} \zeta_k(\theta, \omega) \right) ds} \| \tilde{u}_{\omega, \cdot} \|_{V^{\tau}}^2 \, ds,
\] (62)
which along with (51) implies that
\[
\limsup_{n \to \infty} |u(\tau, \tau - t_n, \theta_{-\tau} \omega, u_{0,n})|^2 
\leq |\tilde{u}|^2 + 4 \int_{-\infty}^{-k} e^{\int_0^s (\nu\lambda - 2\beta e^{\mu(s+r)} \zeta_k(\theta, \omega) \right) ds} \| f(s + \tau, \cdot) \|_{V^{\tau}}^2 \, ds 
+ M_1 \int_{-\infty}^{-k} e^{\mu(s+\tau)} e^{\int_0^s (\nu\lambda - 2\beta e^{\mu(s+r)} \zeta_k(\theta, \omega) \right) ds} \| \tilde{u}_{\omega, \cdot} \|_{V^{\tau}}^2 \, ds.
\] (63)
Letting $k \to \infty$ in (63), we get (41), which along with (40) shows that
\[
\lim_{n \to \infty} u(\tau, \tau - t_n, \theta_{-\tau} \omega, u_{0,n}) = \tilde{u} \quad \text{in} \quad H,
\]
as desired. \hfill \Box

2.3. Existence of pullback attractors. We now establish the existence of $D$-pullback attractors for the two-dimensional Navier-Stokes equations (4)-(6). We first show the existence of $D$-pullback absorbing sets in $H$.

Lemma 2.6. Let assumptions (1), (8)-(10), (22) and (23) hold. Then the continuous cocycle $\Phi$ associated with problem (4)-(6) possesses a closed measurable $D$-pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.

Proof. For given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we denote
\[
K(\tau, \omega) = \{u \in H : |u|^2 \leq R(\tau, \omega)\},
\] (64)
where $R(\tau, \omega)$ is the same as in (38). Since for every $\tau \in \mathbb{R}$, $R(\tau, \cdot) : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable, we know that $K(\tau, \cdot) : \Omega \to 2^H$ is a measurable set-valued mapping. Also, by Lemma 2.2, we know that for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$,
\[
\Phi(t, \tau - t, \theta_{-\tau} \omega, D(\tau - t, \theta_{-\tau} \omega)) = u(\tau, \tau - t, \theta_{-\tau} \omega, D(\tau - t, \theta_{-\tau} \omega)) \subseteq K(\tau, \omega).
\]
Therefore, to conclude the proof, it remains to show that $K \in \mathcal{D}$. For every $\gamma > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,
\[
\lim_{t \to -\infty} e^{\gamma t} |K(\tau + t, \theta_{\tau} \omega)| = \lim_{t \to -\infty} e^{\gamma t} R(\tau + t, \theta_{\tau} \omega)
\]
Due to (13), we have
\[ \zeta(\theta_{t+\omega}) = -\frac{1}{2\delta} \int_{-\infty}^{0} e^{\sigma} \omega(r + t + \delta \sigma) d\sigma + \frac{1}{\delta} \omega(r + t), \]
and therefore for \( \delta \mu \neq 1 \), by Fubini’s theorem and the integration by parts formula, we get
\[
\int_{0}^{s} e^{\mu(r + \tau + t)} \zeta(\theta_{t+\omega})\,d\tau = -\frac{1}{\delta} \int_{0}^{s} e^{\sigma} \omega(r + t + \delta \sigma) d\sigma + \frac{1}{\delta} \omega(r + t) \int_{-\infty}^{0} e^{\mu(r + \tau + t)}\,d\tau.
\]

For \( \beta > 0, \delta \mu \neq 1 \) and \( \tau \in \mathbb{R}, \) let \( \gamma_0 = e^{-\mu \tau} |\delta \mu - 1| \cdot \min\{\frac{\nu \lambda - \alpha}{2\delta}, \frac{\nu \lambda - \alpha}{\gamma_0}\} > 0. \) By (12), we find that there exists \( T_1 = T_1(\omega) > 0 \) such that for all \(|t| \geq T_1, \)
\[ |\omega(t)| < \gamma_0 |t|. \] (67)

By (67), we note that for \( \mu > 0, t \leq -T_1 \) and \( s \leq 0, \)
\[ \left| \int_{t}^{t+s} e^{\mu \tau} \omega(r)\,d\tau \right| \leq \int_{t}^{t+s} e^{\mu \tau} |\omega(r)|\,d\tau \leq \gamma_0 \int_{t}^{t+s} e^{\mu \tau} |r|\,d\tau \leq \gamma_0 \int_{-\infty}^{0} e^{\mu \tau} |r|\,d\tau = \frac{\gamma_0}{\mu^2}. \] (68)

Also for \( t \leq -T_1, r, s \leq 0 \) and \( \delta \in (0,1], \) we have \( s + t + \delta r \leq -T_1, \) and hence
\[ |\omega(\delta r + s + t)| < \gamma_0 |\delta r + s + t| \leq \gamma_0 (|r| + |s| + |t|). \] (69)
By (69), for \( t \leq -T_1 \) and \( s \leq 0 \), we find that
\[
\left| e^{\mu(t+s)} \int_{-\infty}^{0} e^\omega (r\delta + s + t)dr \right| \leq \gamma_0 \int_{-\infty}^{0} e^\tau (|r| + |s| + |t|)dr \\
\leq \gamma_0 (|s| + |t| + 1) = \gamma_0 s - \gamma_0 t.
\] (70)
Similarly, for \( t \leq -T_1 \) and \( r \leq 0 \), we have \( t + r\delta \leq -T_1 \), and hence \( |\omega(t + r\delta)| < \gamma_0|t + r\delta| \leq \gamma_0(|t| + |r|) \). This indicates that for all \( t \leq -T_1 \),
\[
\left| e^{\mu} \int_{-\infty}^{0} e^\omega (r\delta + t)dr \right| \leq \gamma_0 \int_{-\infty}^{0} e^\tau (|r| + |t|)dr \leq \gamma_0 - \gamma_0 t.
\] (71)
which along with (66), (68) and (70) implies that for \( \mu > 0 \), \( \beta > 0 \), \( \delta \in (0,1] \) with \( \mu \delta \neq 1 \) and \( t \leq -T_1 \),
\[
-2\beta \int_{0}^{s} e^{\mu(r+r\tau+t)} \zeta_\delta (\theta + t\omega)dr \leq \frac{2\beta \gamma_0}{\beta \mu^2} e^{\mu t} (\mu - 1)^2 + 1 - 2s - 2t.
\] (72)
Now let \( \gamma_1 = \min\{\alpha + 2\mu, \frac{\lambda}{\nu}\} \). We deduce from (65) and (72) that, for \( \mu > 0 \), \( \beta > 0 \), \( \delta \in (0,1] \) with \( \mu \delta \neq 1 \) and \( t \leq -T_1 \),
\[
\lim_{t \to -\infty} e^{\gamma t} |K(t + t, \theta, \omega)| \leq \frac{2\beta \gamma_0}{\beta \mu^2} e^{\mu t} (\mu - 1)^2 + 1 - 2s - 2t.
\]
where we have used (23) and the fact that \( \int_{-\infty}^{0} e^{\gamma t} |\zeta_\delta (\theta, \omega)|^2 ds < \infty \). Note that the above convergence also holds true if \( \mu = 0 \) or \( \beta = 0 \) or \( \delta \mu = 1 \). Actually, in these special cases, the proof is simpler. \( \square \)

As an immediate result of Lemma 2.5, we obtain the following \( \mathcal{D} \)-pullback asymptotic compactness of \( \Phi \).

**Lemma 2.7.** Let assumptions (1), (8)-(10) and (22) hold. Then the continuous cocycle \( \Phi \) associated with problem (4)-(6) is \( \mathcal{D} \)-pullback asymptotically compact in \( H \), that is, for every \( \tau \in \mathbb{R}, \omega \in \Omega, D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \), and \( t_n \to \infty \), \( u_{0,n} \in D(\tau - t_n, \theta_{-t_n} \omega) \), the sequence \( \Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, u_{0,n}) \) has a convergent subsequence in \( H \).
Now, we are in a position to state the main result of this section.

**Theorem 2.8.** Let assumptions (1), (8)-(10) and (22)-(23) hold. Then the continuous cocycle \( \Phi \) associated with problem (4)-(6) has a unique \( D \)-pullback attractor \( A = \{ A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \) in \( H \).

**Proof.** The result follows from [55, 58] immediately by combining Lemma 2.6 with Lemma 2.7.

Note that Theorem 2.8 is valid when the domain \( \mathcal{O} \) is unbounded. If \( \mathcal{O} \) is bounded, then the same result remains true when condition (10) is replaced by the much weaker condition (11). In other words, the following statement is true.

**Theorem 2.9.** Let \( \mathcal{O} \) be a bounded domain in \( \mathbb{R}^2 \). Suppose (1), (8)-(9), (11) and (22)-(23) hold. Then the continuous cocycle \( \Phi \) associated with problem (4)-(6) has a unique \( D \)-pullback attractor \( A = \{ A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \) in \( H \).

**Proof.** The proof is based on the compactness arguments rather than the method of energy equations. The details are similar to that of [32] and will not be repeated here again.

3. **Convergence of Random attractors for stochastic Equations.** In this section, we study the approximations of the solutions of the following stochastic equations driven by additive white noise:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u &= f(t,x) - \nabla p + e^{\mu t}h(x) \frac{dW}{dt}, \quad x \in \mathcal{O}, \\
\text{div } u &= 0, \quad x \in \mathcal{O},
\end{aligned}
\tag{73}
\]

together with boundary condition (5) and initial condition (6), where \( \mu > 0 \) is a constant and \( h \in D(A) \) is given. For \( \delta > 0 \), consider the pathwise random equations:

\[
\begin{aligned}
\frac{\partial u_\delta}{\partial t} - \nu \Delta u_\delta + (u_\delta \cdot \nabla)u_\delta &= f(t,x) - \nabla p + e^{\mu t}h(x)\zeta_\delta(\theta_t \omega), \quad x \in \mathcal{O}, \\
\text{div } u_\delta &= 0, \quad x \in \mathcal{O}.
\end{aligned}
\tag{74}
\]

In the sequel, we assume the function \( h \) in both (73) and (74) satisfies the condition: there exists a constant \( \kappa > 0 \) such that

\[
|b(u,h,u)| \leq \kappa |u|^2, \quad \forall u \in H.
\tag{75}
\]

We now transform the stochastic system (73) into a deterministic one by

\[
v(t, \tau, \omega) = u(t, \tau, \omega) - e^{\mu t}h(x)z(\theta_t \omega),
\tag{76}
\]

where \( z(\theta_t \omega) \) is the stationary solution of the one-dimensional Ornstein-Uhlenbeck equation

\[
dz(\theta_t \omega) = -\eta z(\theta_t \omega) dt + dW,
\tag{77}
\]

which is given by, for \( \omega \in \Omega ,

\[
z(\omega) = -\eta \int_{-\infty}^{0} e^{\eta s} \omega(s) ds.
\tag{78}
\]

Note that there exists a \( \theta_t \)-invariant set (still denoted by) \( \Omega \) with full measure such that for every \( \omega \in \Omega , z(\theta_t \omega) \) is continuous in \( t \) and

\[
\begin{aligned}
\lim_{t \to \pm \infty} \frac{|z(\theta_t \omega)|}{t} &= 0, \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} z(\theta_r \omega) dr = 0 \text{ and } \lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} |z(\theta_r \omega)| dr &= \frac{1}{\sqrt{4\pi \eta}}.
\end{aligned}
\tag{79}
\]
From (73) and (76), we get
\[
\frac{dv}{dt} + \nu Av + B(v + e^{\mu t}hz, v + e^{\mu t}hz) = f + (\eta - \mu)e^{\mu t}hz - \nu e^{\mu t}zAh. 
\] (80)

Now, for each \( \omega \in \Omega \), \( \tau \in \mathbb{R} \), and \( v_\tau \in H \), system (80) with (5)-(6) possesses a unique solution \( v(\cdot, \tau, \omega, v_\tau) \in C([\tau, \infty); H) \cap L^2_{\text{loc}}(\tau, \infty; V) \). This solution is continuous in \( v_\tau \) in \( H \) and is \((\mathcal{F}, \mathcal{B}(H))\)-measurable in \( \omega \in \Omega \). Define \( \Psi_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H \) such that for every \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega \) and \( u_\tau \in H \),
\[
\Psi_0(t, \tau, \omega, u_\tau) = v(t + \tau, \tau, \theta_{-\tau} \omega, v_\tau) + e^{\mu(t+\tau)}hz(\theta_{i\omega})
= u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau),
\] (81)
where \( v_\tau = u_\tau - e^{\mu \tau}hz(\omega) \).

We now present the existence of \( \mathcal{D} \)-pullback absorbing sets of \( \Psi_0 \).

**Lemma 3.1.** Let assumptions (1), (22)-(23) and (75) hold. Then \( \Psi_0 \) has a closed measurable \( \mathcal{D} \)-pullback absorbing set \( B_0 = \{ B_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \) in \( H \) given by
\[
B_0(\tau, \omega) = \{ u \in H : |u|^2 \leq R_0(\tau, \omega) \}, \quad \text{for every } \tau \in \mathbb{R} \text{ and } \omega \in \Omega,
\] (82)
where \( R_0(\tau, \omega) \) is defined by
\[
R_0(\tau, \omega) = L_1 \left( 1 + |e^{\mu \tau}z(\omega)|^2 + \int_{-\infty}^{\tau} e^{\int_0^t (\nu \lambda - 4\kappa |e^{\mu(r+s)}z(\theta_{i\omega})|)dr} \left( \| f(s + \tau, \cdot) \|^2_{L^2_V} + \xi(s + \tau, \theta_{s\omega}) \right) ds \right),
\] (83)
with \( \xi(r, \omega) = |e^{\mu r}z(\omega)|^2 + |e^{\mu r}z(\omega)|^3 \) and \( L_1 \) being a positive constant independent of \( \tau \) and \( \omega \).

**Proof.** It follows from (80) that
\[
\frac{d}{dt} |v|^2 + 2\nu ||v||^2 + 2(B(v + e^{\mu t}hz, v + e^{\mu t}hz), v) \\
= 2(f, v) + 2e^{\mu t}((\eta - \mu)hz - \nu Ahz, v).
\] (84)

By (75) we have
\[
2|b(v + e^{\mu t}hz, v + e^{\mu t}hz)| = 2|b(v + e^{\mu t}hz, e^{\mu t}hz, v + e^{\mu t}hz)|
= 2e^{\mu t}|z||b(v + e^{\mu t}hz, h, v + e^{\mu t}hz)|
\leq 2\kappa e^{\mu t}|z||v + e^{\mu t}hz|^2 \leq 4\kappa e^{\mu t}|z||(v|^2 + |h|^2|e^{\mu t}|^2).
\] (85)

From (3) and (3) we get
\[
\frac{d}{dt} \{ |v|^2 + (\nu \lambda - 4\kappa |e^{\mu r}z(\theta_{i\omega})|)|v|^2 + \frac{1}{4} \nu ||v||^2 \} \leq \frac{4}{\nu} ||f(t, \cdot)||^2_{L^2_V} + \xi(t, \theta_{i\omega}),
\] (86)
where
\[
\xi(t, \theta_{i\omega}) = 4\nu ||h||^2 |e^{\mu r}z(\theta_{i\omega})|^2 + 4\kappa |h|^2 |e^{\mu r}z(\theta_{i\omega})|^3 + \frac{4(\eta - \mu)^2}{\nu \lambda} |h|^2 |e^{\mu r}z(\theta_{i\omega})|^2.
\]
Applying Gronwall’s inequality to (86) over \((\tau - t, \tau)\) with \(t \geq 0\) and \(\tau \in \mathbb{R}\), and replacing \(\omega\) by \(\theta - \tau\omega\), we get
\[
|u(\tau, \tau - t, \theta - \tau \omega, v_{\tau - t})|^2 \leq e^{-\int_{\tau-t}^{\tau} (\nu \lambda - 4\kappa |e^{\mu r}z(\theta - \tau \omega)) dr} |v_{\tau - t}|^2 + \frac{4}{\nu} \int_{\tau-t}^{\tau} e^{\int_{\tau-t}^{s} (\nu \lambda - 4\kappa |e^{\mu r}z(\theta - \tau \omega)) dr} \|f(s, \cdot)\|^2_{V, s} ds + \int_{\tau-t}^{\tau} e^{\int_{\tau-t}^{s} (\nu \lambda - 4\kappa |e^{\mu r}z(\theta - \tau \omega)) dr} \bar{\xi}(s, \theta_{s-\tau} \omega) ds \tag{87}
\]
By (79), we find that
\[
\lim_{r \to -\infty} |e^{\mu r}z(\theta_{r} \omega)| = 0. \tag{88}
\]
Therefore, there exists \(r_0 = r_0(\tau, \omega) < 0\) such that for all \(r \leq r_0\),
\[
\nu \lambda - 4\kappa |e^{\mu (r+\tau)}z(\theta_{r} \omega)| > \alpha. \tag{89}
\]
By (79), (89) and (22), we see that the second and the third terms on the right-hand side of (87) are well defined. Due to (76) we have
\[
u(\tau - t, \theta - \tau \omega, u_{\tau - t}) = u(\tau, \tau - t, \theta - \tau \omega, v_{\tau - t}) + e^{\mu r}z(\omega)h(x),
\]
with \(u_{\tau - t} = u_{\tau - t} = e^{\mu (r-t)}z(\theta_{r-t} \omega)h\), which along with (87) implies
\[
|u(\tau, \tau - t, \theta - \tau \omega, u_{\tau - t})|^2 \leq 2|v(\tau, \tau - t, \theta - \tau \omega, v_{\tau - t})|^2 + 2|h|^2 |e^{\mu r}z(\omega)|^2 \leq 4e^{\int_{\tau-t}^{\tau} (\nu \lambda - 4\kappa |e^{\mu r}z(\theta_{r} \omega)) dr} \left(|u_{\tau - t}|^2 + |h|^2 |e^{\mu (r-t)}z(\theta_{r-t} \omega)|^2\right)
\]
\[
+ \frac{8}{\nu} \int_{-\infty}^{0} e^{\int_{-\infty}^{s} (\nu \lambda - 4\kappa |e^{\mu (r+\tau)}z(\theta_{r} \omega)) dr} \|f(s + \tau, \cdot)\|^2_{V, s} ds + 2e^{\mu r}z(\omega)^2. \tag{90}
\]
If \(u_{\tau - t} \in D(\tau - t, \theta_{-\tau} \omega) \text{ and } \tau \in D\), then from (89) we get
\[
e^{\int_{\tau-t}^{\tau} (\nu \lambda - 4\kappa |e^{\mu (r+\tau)}z(\theta_{r} \omega)) dr} |D(\tau - t, \theta_{-\tau} \omega)|^2 \to 0 \text{ as } t \to \infty.
\]
On the other hand, by (79), we have
\[
\lim_{t \to -\infty} e^{\int_{\tau-t}^{\tau} (\nu \lambda - 4\kappa |e^{\mu (r+\tau)}z(\theta_{r} \omega)) dr} |e^{\mu (r-t)}z(\theta_{r-t} \omega)|^2 = 0.
\]
Thus, it follows from (90) that there exist \(T_1 = T_1(\tau, \omega, D) > 0\) and a positive constant \(L_1\) (independent of \(\tau, \omega\) and \(D\)) such that for all \(t \geq T_1\),
\[
|u(\tau, \tau - t, \theta - \tau \omega, u_{\tau - t})|^2 \leq L_1 \left(1 + |e^{\mu r}z(\omega)|^2 \right)
\]
\[
+ \int_{-\infty}^{0} e^{\int_{\tau-t}^{s} (\nu \lambda - 4\kappa |e^{\mu r}z(\theta_{r} \omega)) dr} \left(\|f(s + \tau, \cdot)\|^2_{V, s} + \bar{\xi}(s + \tau, \theta_{s} \omega) ds\right). \tag{91}
\]
Note that (91) implies that for all $t \geq T_1$,
\[
\Psi_0(t, \tau-t, \theta_{\tau-t}, D(\tau-t, \theta_{\tau-t})) = u(\tau-t, \theta_{\tau-t}, D(\tau-t, \theta_{\tau-t})) \subseteq B_0(\tau),
\]
where $B_0(\tau)$ is defined by (82). By using (23) and (79), one can easily obtain that $B_0$ is tempered in $H$, which along with (92) completes the proof.

Also, we can obtain the $\mathcal{D}$-pullback asymptotic compactness of $\Psi_0$ associated with system (73) by the method of energy equations (see the proof of Lemma 2.5 and Lemma 3.6). This fact along with Lemma 3.1 indicates that $\Psi_0$ possesses a unique $\mathcal{D}$-pullback attractor $A_0$ in $H$.

We now transform (74) by using the solution of the random equation driven by colored noise
\[
dy_\delta dt = -\eta y_\delta + \xi_\delta(\theta t).
\]
By (14) we find that for all $\omega \in \Omega$, the integral
\[
y_\delta(\omega) = \int_{-\infty}^{0} e^{\eta s} \xi_\delta(\theta s)ds
\]
is convergent, and hence $y_\delta : \Omega \rightarrow \mathbb{R}$ is a well defined random variable. We establish the following properties of $y_\delta$ for later purpose:

**Lemma 3.2.** Let $y_\delta$ be the random variable as defined by (94). Then the mapping
\[
(t, \omega) \mapsto y_\delta(\theta t) = e^{-\eta t} \int_{-\infty}^{t} e^{\eta s} \xi_\delta(\theta s)ds
\]
is a stationary solution of (93) with continuous trajectories. In addition, $E(y_\delta) = 0$ and for every $\omega \in \Omega$,
\[
\lim_{\delta \to 0} y_\delta(\theta t) = z(\theta t) \text{ uniformly on } [\tau, \tau + T] \text{ with } \tau \in \mathbb{R} \text{ and } T > 0; \quad (96)
\]
\[
\lim_{t \to \pm \infty} \frac{|y_\delta(\theta t)|}{|t|} = 0 \text{ uniformly for } 0 < \delta \leq \hat{\eta}; \quad (97)
\]
\[
\lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} y_\delta(\theta r)dr = 0 \text{ uniformly for } 0 < \delta \leq \hat{\eta}; \quad (98)
\]
\[
\lim_{\delta \to 0} E(|y_\delta(\omega)|) = E(|z(\omega)|), \quad (99)
\]
where $\hat{\eta} = \min\{1, \frac{1}{2\eta}\}$.

**Proof.** By direct calculations, we see that $y_\delta(\theta t)$ is a solution of (93). The stationarity of this solution follows by the invariance of the Wiener measure $\mathbb{P}$ with respect to the flow $\theta$.

(i) Proof of (96). First, given $t > 0$, integrating the following one-dimensional stochastic differential equation
\[
d\xi_\delta + \frac{1}{\delta} \xi_\delta dt = \frac{1}{\delta} dW;
\]
over $(0, t)$, we get
\[
\xi_\delta(\theta t) = \xi_\delta(\omega) + \frac{1}{\delta} \omega(t) - \frac{1}{\delta} \int_{0}^{t} \xi_\delta(\theta r)dr.
\]

\[
\frac{1}{\delta} \omega(t) - \frac{1}{\delta} \int_{0}^{t} \xi_\delta(\theta r)dr.
\]

\[
(100)
\]
By (100) and (95) we have
\[ y_\delta(\theta_t \omega) = e^{-\eta t} \int_{-\infty}^{t} e^{\eta s} \zeta_\delta(\theta_s \omega) ds \]
\[ = \frac{1}{\eta} \zeta_\delta(\omega) + \frac{1}{\delta} e^{-\eta t} \int_{-\infty}^{t} e^{\eta s} \omega(s) ds - \frac{1}{\eta \delta} \int_{0}^{t} \zeta_\delta(\theta_r \omega) dr + \frac{1}{\eta \delta} y_\delta(\theta_t \omega), \tag{101} \]
which implies that
\[ y_\delta(\theta_t \omega) = \frac{\delta}{\eta \delta - 1} \zeta_\delta(\omega) + \frac{\eta}{\eta \delta - 1} e^{-\eta t} \int_{-\infty}^{t} e^{\eta s} \omega(s) ds - \frac{1}{\eta \delta - 1} \int_{0}^{t} \zeta_\delta(\theta_r \omega) dr, \tag{102} \]
Then, it follows from (78) and (102) that for all \( t \in [\tau, \tau + T], \)
\[ y_\delta(\theta_t \omega) - z(\theta_t \omega) = \frac{\delta}{\eta \delta - 1} \zeta_\delta(\omega) + \frac{\eta^2 \delta}{1 - \eta \delta} e^{-\eta t} \int_{-\infty}^{t} e^{\eta s} \omega(s) ds \\ + \left( \frac{1}{1 - \eta \delta} \int_{0}^{t} \zeta_\delta(\theta_r \omega) dr - \omega(t) \right). \tag{103} \]
For the first term on the right-hand side of (103), by (13), we have
\[ \frac{\delta}{\eta \delta - 1} \zeta_\delta(\omega) = \frac{1}{\eta \delta - 1} \int_{-\infty}^{0} e^r \omega(\delta r) dr \rightarrow 0 \text{ as } \delta \rightarrow 0. \tag{104} \]
For the third term, by (16), we know that for \( \varepsilon > 0, \) there exists \( \delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0 \) such that for all \( \delta \in (0, \delta_0) \) and \( t \in [\tau, \tau + T], \)
\[ \left| \frac{1}{1 - \eta \delta} \int_{0}^{t} \zeta_\delta(\theta_r \omega) dr - \omega(t) \right| \leq \varepsilon, \tag{105} \]
which along with (103) and (104) shows that there exists \( \delta_1 = \delta_1(\tau, \omega, T, \varepsilon) \in (0, \delta_0) \) such that for all \( \delta \in (0, \delta_1) \) and \( t \in [\tau, \tau + T], \)
\[ |y_\delta(\theta_t \omega) - z(\theta_t \omega)| \leq 3\varepsilon. \tag{106} \]
This implies (96).

(ii) Proof of (97). By (12) we find that for every \( \omega \in \Omega, \) there exists \( T = T(\omega) > 0 \) such that
\[ |\omega(t)| \leq |t|, \quad \forall |t| \geq T. \tag{107} \]
On the other hand, by the continuity of \( \omega, \) there exists \( c_1 = c_1(\omega) \) such that
\[ |\omega(t)| \leq c_1, \quad \forall |t| \leq T, \]
which together with (107) yields that
\[ |\omega(t)| \leq c_1 + |t|, \quad \forall t \in \mathbb{R}. \tag{108} \]
By (104) and (108) we get, for all \( 0 < \delta < \delta_0, \)
\[ \left| \frac{\delta}{\eta \delta - 1} \zeta_\delta(\omega) \right| \leq \frac{1}{|1 - \eta \delta|} \int_{-\infty}^{0} e^s |\omega(\delta s)| ds \leq 2 \int_{-\infty}^{0} e^s (c_1 + |\delta s|) ds \leq 2 \int_{-\infty}^{0} e^s (c_1 + |s|) ds = 2(c_1 + 1). \tag{109} \]
For the last term on the right-hand side of (102), we have, for $0 < \delta < \tilde{\eta}$,

$$\left| \frac{1}{1 - \eta \delta} \int_0^t \zeta_\delta(\theta_r, \omega)dr \right| \leq 2 \left| \int_0^t \zeta_\delta(\theta_r, \omega)dr \right|,$$

which along with (15) implies that for given $\varepsilon > 0$, there exists $T_1 = T_1(\omega, \varepsilon) > 0$ such that for all $0 < \delta < \tilde{\eta}$ and $|t| \geq T_1$,

$$\left| \frac{1}{1 - \eta \delta} \int_0^t \zeta_\delta(\theta_r, \omega)dr \right| \leq \varepsilon |t|.$$(110)

On the other hand, since $0 < \delta < \tilde{\eta}$, we get

$$\left| \frac{\eta}{\eta \delta - 1} e^{-\eta t} \int_{-\infty}^t e^{\eta s} \omega(s)ds \right| \leq 2\eta e^{-\eta t} \int_{-\infty}^t e^{\eta s} |\omega(s)|ds. \quad (111)$$

We first consider the case when $t \to +\infty$. By (12), there exists $T_2 = T_2(\omega, \varepsilon) \geq T_1$ such that for all $|t| \geq T_2$,

$$|\omega(t)| < \varepsilon |t|. \quad (112)$$

Now, by (111) and (112), we get, for $t \geq T_2$,

$$\left| \frac{\eta}{\eta \delta - 1} e^{-\eta t} \int_{-\infty}^t e^{\eta s} \omega(s)ds \right| \leq 2\eta e^{-\eta t} \left( \int_{-\infty}^{T_2} e^{\eta s} |\omega(s)|ds + \int_{T_2}^t e^{\eta s} |\omega(s)|ds \right)$$

$$\leq 2\eta e^{-\eta t} \left( \int_{-\infty}^{T_2} e^{\eta s} |\omega(s)|ds + \varepsilon \int_{T_2}^t e^{\eta s} ds \right) \quad (113)$$

$$\leq 2\eta e^{-\eta t} \int_{-\infty}^{T_2} e^{\eta s} |\omega(s)|ds + 2\varepsilon (|t| + \frac{1}{\eta} e^{-\eta (t-T_2)}).$$

It follows from (102), (109), (110) and (113) that for all $t \geq T_2$ and $0 < \delta < \tilde{\eta}$,

$$\left| \frac{y_\delta(\theta_r\omega)}{t} \right| \leq 3\varepsilon + \frac{2(c_1 + 1)}{|t|} + \frac{2\varepsilon}{\eta |t|} e^{-\eta (t-T_2)} + \frac{2\eta}{|t|} e^{-\eta t} \int_{-\infty}^{T_2} e^{\eta s} |\omega(s)|ds,$$

which shows that

$$\lim_{t \to \infty} \frac{y_\delta(\theta_r\omega)}{t} = 0 \text{ uniformly for } 0 < \delta < \tilde{\eta}. \quad (114)$$

We now consider the case when $t \to -\infty$. By (110) and (111) we get for all $t \leq -T_2$ and $0 < \delta < \tilde{\eta}$,

$$\left| \frac{\eta}{\eta \delta - 1} e^{-\eta t} \int_{-\infty}^t e^{\eta s} \omega(s)ds \right| \leq 2\eta e^{-\eta t} \int_{-\infty}^t e^{\eta s} |\omega(s)|ds = 2\varepsilon (|t| + \frac{1}{\eta}). \quad (115)$$

By (102), (109), (110) and (115) we obtain, for all $t \leq -T_2$ and $0 < \delta < \tilde{\eta}$,

$$\left| \frac{y_\delta(\theta_r\omega)}{t} \right| \leq 3\varepsilon + \frac{2(c_1 + 1)}{|t|} + \frac{2\varepsilon}{\eta |t|},$$

which indicates that

$$\lim_{t \to -\infty} \frac{y_\delta(\theta_r\omega)}{t} = 0 \text{ uniformly for } 0 < \delta < \tilde{\eta}. \quad (116)$$

Then (97) follows from (114) and (116).

(iii) Proof of (98). Integrating (93) over $(0, t)$ for $t > 0$ and dividing each term by $t$, we have

$$\frac{1}{t} \int_0^t y_\delta(\theta_r\omega)dr = -\frac{1}{t\eta} y_\delta(\theta_1\omega) + \frac{1}{t\eta} y_\delta(\omega) + \frac{1}{t\eta} \int_0^t \zeta_\delta(\theta_r\omega)dr. \quad (117)$$
For the first and the third terms on the right-hand side of (117), by (97) and (15), we find that
\[
\lim_{t \to \pm \infty} \frac{1}{t \eta} y_\delta(\theta t \omega) = 0 \quad \text{uniformly for } 0 < \delta \leq \bar{\eta},
\] (118)
and
\[
\lim_{t \to \pm \infty} \frac{1}{t \eta} \int_0^t \zeta_\delta(\theta r \omega) dr = 0 \quad \text{uniformly for } 0 < \delta \leq \bar{\eta}.
\] (119)

It remains to show \( \lim_{t \to \pm \infty} \frac{1}{t \eta} y_\delta(\omega) = 0 \) uniformly for \( 0 < \delta \leq \bar{\eta} \). Letting \( t = 0 \) in (102), we find that
\[
y_\delta(\omega) = \frac{\delta}{\eta^\delta - 1} \zeta_\delta(\omega) - \frac{\eta}{\eta^\delta - 1} \int_{-\infty}^0 e^{\eta s} \omega(s) ds.
\] (120)

By (108) we have
\[
\left| \frac{\eta}{\eta^\delta - 1} \int_{-\infty}^0 e^{\eta s} \omega(s) ds \right| \leq 2\eta \int_{-\infty}^0 e^{\eta s} \omega(s) ds \leq 2c_1 + \frac{2}{\eta},
\]
which along with (109) implies for all \( 0 < \delta < \bar{\eta} \),
\[
\left| \frac{1}{t \eta} y_\delta(\omega) \right| \leq \frac{4c_1 + 2 + \frac{2}{\eta}}{\eta |t|}.
\] (121)

This indicates
\[
\lim_{t \to \pm \infty} \frac{1}{t \eta} y_\delta(\omega) = 0 \quad \text{uniformly for } 0 < \delta \leq \bar{\eta},
\]
which together with (118) and (119) implies (98).

(iv) Proof of (99). By (78) and (120) we have
\[
y_\delta(\omega) - z(\omega) = \frac{\delta}{\eta^\delta - 1} \zeta_\delta(\omega) - \frac{\eta \delta}{\eta^\delta - 1} z(\omega),
\] (122)
and hence
\[
\mathbb{E}(|y_\delta(\omega) - z(\omega)|) \leq \frac{\delta}{1 - \eta^\delta} \mathbb{E}(|\zeta_\delta(\omega)|) + \frac{\eta \delta}{1 - \eta^\delta} \mathbb{E}(|z(\omega)|).
\] (123)

We note from (13) and (79),
\[
\mathbb{E}(|z(\omega)|) = \frac{1}{\sqrt{\pi \eta}} \quad \text{and} \quad \mathbb{E}(|\zeta_\delta(\omega)|) = \frac{1}{\sqrt{\pi \delta}},
\]
which along with (123) implies (99). The proof is complete. \( \square \)

Now, define a new variable
\[
v_\delta(t, \tau, \omega) = u_\delta(t, \tau, \omega) - e^{\eta t} h_y \delta(\theta t \omega).
\] (124)

By (74) and (124), we get
\[
\frac{dv_\delta}{dt} + \nu A v_\delta + B(v_\delta + e^{\eta t} h_y \delta, v_\delta + e^{\eta t} h_y \delta) = f + (\eta - \mu) e^{\eta t} h_y \delta - \nu e^{\eta t} \delta Ah.
\] (125)

We also know that equation (125) with (5)-(6) possesses a unique solution
\[
v_\delta(\cdot, \tau, \omega, v_\delta, \tau) \in C([\tau, \infty); H) \cap L^2_{\text{loc}}(\tau, \infty; V).
\]
Moreover, the solution is continuous in initial data $v_{δ,τ}$ in $H$ and is $(\mathcal{F},B(H))$-measurable in $ω ∈ Ω$. Define a mapping $Ψ_δ : \mathbb{R}^+ × \mathbb{R} × Ω × H → H$ such that for every $t ∈ \mathbb{R}^+$, $τ ∈ \mathbb{R}$, $ω ∈ Ω$ and $u_{δ,τ} ∈ H$,

$$Ψ_δ(t, τ, ω, u_{δ,τ}) = v_δ(t + τ, τ, ω, v_{δ,τ}) + e^{μ(t+τ)}hy_δ(τ_ω) = u_δ(t + τ, τ, ω, u_δ),$$

(126)

where $u_{δ,τ} = v_{δ,τ} + e^{μτ}hy_δ(ω)$.

**Lemma 3.3.** Let assumptions (1), (22)-(23) and (75) hold. Then there exists $δ_0 > 0$ such that for all $0 < δ < δ_0$, equation (74) has a closed measurable $D$-pullback absorbing set $B_δ ∈ D$ in $H$ given by

$$B_δ(τ, ω) = \{ u_δ ∈ H : |u_δ|^2 ≤ R_δ(τ, ω) \}, \quad \text{for every} \ τ ∈ \mathbb{R} \text{ and} \ ω ∈ Ω ,$$

(127)

where $R_δ(τ, ω)$ is defined by

$$R_δ(τ, ω) = L_2 \left( 1 + |e^{μτ}y_δ(ω)|^2 \right) \int_{-∞}^0 e^{μt−4κ|e^{μt−4κ}y_δ(θ_t, ω)|}ds + \| f(s + τ, ·, ω) \|^2 \right).$$

(128)

with $ξ_δ(τ, ω) = |e^{μτ}y_δ(ω)|^2 + |e^{μτ}y_δ(ω)|^3$ and $L_2$ being a positive constant independent of $τ$, $ω$ and $δ$. Moreover, for every $τ ∈ \mathbb{R}$ and $ω ∈ Ω$,

$$\lim_{δ→0} R_δ(τ, ω) = R_0(τ, ω),$$

(129)

where $R_0(τ, ω)$ is given by (83) with $L_1$ being replaced by $L_2$.

**Proof.** It follows from (125) that

$$\frac{d}{dt} |v_δ|^2 + 2ν|v_δ|^2 + 2(B(v_δ + e^{μt}hy_δ, v_δ + e^{μt}hy_δ), v_δ) = 2(f, v_δ) + 2e^{μt}((ν - μ)hy_δ - νAh_δ, v_δ).$$

(130)

From (75), we know

$$2|b(v_δ + e^{μt}hy_δ, v_δ + e^{μt}hy_δ, v_δ)| = 2|b(v_δ + e^{μt}hy_δ, v_δ + e^{μt}hy_δ, −e^{μt}hy_δ)|$$

$$= 2|b(v_δ + e^{μt}hy_δ, −e^{μt}hy_δ, v_δ + e^{μt}hy_δ)|$$

(131)

$$≤ 4κ|e^{μt}y_δ||(v_δ|^2 + |h|^2|e^{μt}y_δ|^2).$$

Combining (130) with (131), we obtain

$$\frac{d}{dt} |v_δ|^2 + (νλ - 4κ|e^{μt}y_δ(θ_tω)||v_δ|^2 + \frac{1}{4} |v_δ|^2 \leq 4ν|f(t, ·)|^2 + ξ_δ(t, θ_tω),$$

(132)

where

$$ξ_δ(t, θ_tω) = 4ν|h|^2|e^{μt}y_δ(θ_tω)|^2 + 4κ|h|^2|e^{μt}y_δ(θ_tω)|^3 + \frac{4(ν - μ)^2}{νλ}|h|^2|e^{μt}y_δ(θ_tω)|^2.$$

Again, by Gronwall’s inequality, for $τ ∈ \mathbb{R}$, $ω ∈ Ω$ and $t ≥ 0$, we get from (132) that

$$|v_δ(τ, τ - t, θ_{−τω}, v_{δ,τ−t})|^2 + \frac{1}{4} ν \int_{τ−t}^τ e^{4ν−4κ|e^{μ(τ−t−τ)}y_δ(θ_{τ−tω})|}ds ||v_δ(s, τ - t, θ_{−tω}, v_{δ,τ−t})||^2 \leq e^{4ν−4κ|e^{μ(τ−t−τ)}y_δ(θ_{τ−tω})|}ds ||v_δ,τ−t||^2 + \frac{4}{ν} \int_{−∞}^0 e^{4ν−4κ|e^{μ(τ−t−τ)}y_δ(θ_{τ−tω})|}ds ||f(s + τ, ·)||^2 \right).$$
Then it follows from (135) that there exist $\delta_0 > 0$ such that
\[
\lim_{r \to -\infty} |e^{\mu r} y_\delta(\theta, \omega)| = 0 \quad \text{uniformly for all } \delta \in (0, \delta_0).
\]
Therefore, there exists $r_0 = r_0(\tau, \omega) < 0$ (independent of $\delta$) such that for all $r \leq r_0$ and $0 < \delta < \delta_0$,
\[
\nu \lambda - 4\kappa |e^{\mu r + r} y_\delta(\theta, \omega)| > \alpha.
\]
By (22) and (134), we find that the second and the third terms on the right-hand side of (3) are well defined. We also have
\[
u_\delta(\tau - t, \theta, \omega, u_{\delta, \tau - t}) = v_\delta(\tau - t, \theta, \omega, v_{\delta, \tau - t}) + e^{\mu r} y_\delta(\omega)h(x),
\]
with $u_{\delta, \tau - t} = u_{\delta, \tau - t} - e^{\mu (r-t)} y_\delta(\theta, \omega)h$. Now, by (3) we obtain
\[
|u_\delta(\tau - t, \theta, \omega, u_{\delta, \tau - t})|^2 \leq 2|v_\delta(\tau - t, \theta, \omega, v_{\delta, \tau - t})|^2 + 2|h|^2 |e^{\mu r} y_\delta(\omega)|^2
\]
\[
\leq 4e^{-f_\delta(t, \mu \lambda - 4\kappa |e^{\mu r + r} y_\delta(\theta, \omega)|)} |u_{\delta, \tau - t}|^2 + 2|h|^2 |e^{\mu (r-t)} y_\delta(\theta, \omega)|^2
\]
\[
+ 8 \int_{-\infty}^{0} e^\delta(t, \mu \lambda - 4\kappa |e^{\mu r + r} y_\delta(\theta, \omega)|) dr \| f(s + \tau, \cdot) \|^2_{L^2} ds
\]
\[
+ 2 \int_{-\infty}^{0} e^\delta(t, \mu \lambda - 4\kappa |e^{\mu r + r} y_\delta(\theta, \omega)|) dr \| f(s + \tau, \cdot) \|^2_{L^2} ds
\]
\[
\leq \limsup_{t \to -\infty} e^{\delta(t, \mu \lambda - 4\kappa |e^{\mu r + r} y_\delta(\theta, \omega)|)} |u_{\delta, \tau - t}|^2 = 0.
\]

Now, by (97), along with (23) and (97), one can easily verify that $B_\delta$ is tempered in $H$. Therefore, $B_\delta$ is a closed measurable $D$-pullback absorbing set of $\Psi_{\delta}$. It remains to show (129). We first prove
\[
\lim_{\delta \to 0} \int_{-\infty}^{0} e^\delta(t, \mu \lambda - 4\kappa |e^{\mu r + r} y_\delta(\theta, \omega)|) dr \| f(s + \tau, \cdot) \|^2_{L^2} ds
\]
\[
= \int_{-\infty}^{0} e^\delta(t, \mu \lambda - 4\kappa |e^{\mu r + r} z(\theta, \omega)|) dr \| f(s + \tau, \cdot) \|^2_{L^2} ds.
\]
Note that
\[
\int_{-\infty}^{0} e^{\int_{s}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)y_4(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds \\
= \int_{-\infty}^{0} e^{\int_{s}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)y_4(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds \\
+ \int_{r_0}^{0} e^{\int_{r_0}^{r} (\nu \lambda - 4\kappa |e^{\nu(r+)y_4(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds \quad (139)
\]
\[
= e^{\int_{r_0}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)y_4(\theta,\omega)}))dr} \int_{-\infty}^{0} e^{\int_{s}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)y_4(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds \\
+ \int_{r_0}^{0} e^{\int_{r_0}^{r} (\nu \lambda - 4\kappa |e^{\nu(r+)y_4(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds.
\]
We now deal with the first term on the right-hand side of (139). By (96) we obtain
\[
\lim_{\delta \to 0} e^{\int_{r_0}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)y_4(\theta,\omega)}))dr} = e^{\int_{r_0}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)z(\theta,\omega)}))dr} \quad (140)
\]
By (134) we find that for all \( s \leq r_0 \) and \( 0 < \delta < \delta_0 \),
\[
e^{\int_{r_0}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)y_4(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} \leq e^{-\alpha r_0} e^{\alpha s} \|f(s + \tau, \cdot)\|^2_{V_s},
\]
which together with (22), (96) and the Lebesgue Dominated Convergence Theorem implies
\[
\lim_{\delta \to 0} \int_{-\infty}^{0} e^{\int_{s}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)y_4(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds \\
= \int_{-\infty}^{0} e^{\int_{s}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)z(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds. \quad (141)
\]
On the other hand, by (96) we get
\[
\lim_{\delta \to 0} \int_{r_0}^{0} e^{\int_{r_0}^{r} (\nu \lambda - 4\kappa |e^{\nu(r+)y_4(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds \\
= \int_{r_0}^{0} e^{\int_{r_0}^{r} (\nu \lambda - 4\kappa |e^{\nu(r+)z(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds. \quad (142)
\]
It follows from (139)-(142) that
\[
\lim_{\delta \to 0} \int_{-\infty}^{0} e^{\int_{s}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)y_4(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds \\
= e^{\int_{r_0}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)z(\theta,\omega)}))dr} \int_{-\infty}^{0} e^{\int_{s}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)z(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds \\
+ \int_{r_0}^{0} e^{\int_{r_0}^{r} (\nu \lambda - 4\kappa |e^{\nu(r+)z(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds \\
= \int_{-\infty}^{0} e^{\int_{s}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)z(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds \\
+ \int_{r_0}^{0} e^{\int_{r_0}^{r} (\nu \lambda - 4\kappa |e^{\nu(r+)z(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds \\
= \int_{-\infty}^{0} e^{\int_{s}^{0} (\nu \lambda - 4\kappa |e^{\nu(r+)z(\theta,\omega)}))dr} \|f(s + \tau, \cdot)\|^2_{V_s} ds.
which yields (138). Next, we prove
\[
\lim_{\delta \to 0} \int_{-\infty}^{0} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} y_s(\theta, \omega)|) dr \xi_\delta(s + \tau, \theta_s \omega) ds
\]
\[
= \int_{-\infty}^{0} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} z(\theta, \omega)|) dr \xi(s + \tau, \theta_s \omega) ds. 
\] (143)

By (97) we know that there exist \( s_1 \leq r_0 \) and \( \delta_1 \in (0, \delta_0) \) such that for all \( 0 < \delta < \delta_1 \),
\[
|y_\delta(\theta, \omega)| \leq |s|, \quad \forall s \leq s_1.
\] (144)

By (144) we obtain, for all \( 0 < \delta < \delta_1 \) and \( s \leq s_1 \),
\[
|\xi_\delta(s + \tau, \theta_s \omega)| = |e^{\mu(s+\tau)} y_\delta(\theta, \omega)|^2 + |e^{\mu(s+\tau)} y_\delta(\theta, \omega)|^3 \leq |e^{\mu(s+\tau)} s|^2 + |e^{\mu(s+\tau)} \delta|^3. 
\] (145)

Note that
\[
\int_{-\infty}^{0} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} y_s(\theta, \omega)|) dr \xi_\delta(s + \tau, \theta_s \omega) ds
\]
\[
= \int_{-\infty}^{s_1} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} y_s(\theta, \omega)|) dr \xi_\delta(s + \tau, \theta_s \omega) ds
\]
\[
+ \int_{s_1}^{0} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} y_s(\theta, \omega)|) dr \xi_\delta(s + \tau, \theta_s \omega) ds
\] (146)
\[
= \int_{s_1}^{0} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} y_s(\theta, \omega)|) dr \int_{-\infty}^{s_1} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} y_s(\theta, \omega)|) dr \xi_\delta(s + \tau, \theta_s \omega) ds
\]
\[
+ \int_{s_1}^{0} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} y_s(\theta, \omega)|) dr \xi_\delta(s + \tau, \theta_s \omega) ds.
\]

Similar to (3), by (96), (145) and the Lebesgue Dominated Convergence Theorem, we can get
\[
\lim_{\delta \to 0} \int_{-\infty}^{s_1} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} y_s(\theta, \omega)|) dr \xi_\delta(s + \tau, \theta_s \omega) ds
\]
\[
= \int_{-\infty}^{s_1} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} z(\theta, \omega)|) dr \xi(s + \tau, \theta_s \omega) ds. 
\] (147)

On the other hand, by (96), we find
\[
\lim_{\delta \to 0} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} y_s(\theta, \omega)|) dr = e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} z(\theta, \omega)|) dr, 
\] (148)

and
\[
\lim_{\delta \to 0} \int_{s_1}^{0} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} y_s(\theta, \omega)|) dr \xi_\delta(s + \tau, \theta_s \omega) ds
\]
\[
= \int_{s_1}^{0} e^{f_f^\delta}(\nu \lambda - 4\kappa |e^{\mu(s+\tau)} z(\theta, \omega)|) dr \xi(s + \tau, \theta_s \omega) ds. 
\] (149)

Thus, (143) follows from (147)-(149). Finally, by (96), (138) and (143) we get (129) immediately. The proof is complete. \(\Box\)

Now, we establish the convergence of solutions of (74) as \( \delta \to 0 \).
Lemma 3.4. Suppose (1) and (75) hold and \( f \in L^2_{\text{loc}}(\mathbb{R}; V^*) \). Let \( \{\delta_n\}_{n=1}^{\infty} \) be a sequence such that \( \delta_n \to 0 \). Let \( u_{\delta_n} \) and \( u \) be the solutions of (74) and (73) with initial data \( u_{\delta_n, \tau} \) and \( u_\tau \), respectively. If \( u_{\delta_n, \tau} \to u_\tau \) in \( H \) as \( n \to \infty \), then for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( t > \tau \),
\[
u_{\delta_n}(t, \tau, \omega, u_{\delta_n, \tau}) \to u(t, \tau, \omega, u_\tau) \quad \text{in} \quad H \quad \text{as} \quad n \to \infty.
\]

Proof. This proof is similar to [32, Lemma 4.4] and thus is omitted here. \( \square \)

In the sequel, we also need the following weak convergence of solutions:

Lemma 3.5. Suppose (1) and (75) hold and \( f \in L^2_{\text{loc}}(\mathbb{R}; V^*) \). Let \( \{\delta_n\}_{n=1}^{\infty} \) be a sequence such that \( \delta_n \to 0 \). Let \( v_{\delta_n} \) and \( v \) be the solutions of (125) and (80) with initial data \( v_{\delta_n, \tau} \) and \( v_\tau \), respectively. If \( v_{\delta_n, \tau} \to v_\tau \) in \( H \) as \( n \to \infty \), then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
u_{\delta_n}(r, \tau, \omega, v_{\delta_n, \tau}) \to v(r, \tau, \omega, v_\tau) \quad \text{in} \quad H, \quad \forall r \geq \tau, \quad (150)
\]
and
\[
u_{\delta_n}(\tau, \omega, v_{\delta_n, \tau}) \to v(\tau, \omega, v_\tau) \quad \text{in} \quad L^2(\tau, \tau + T; V), \quad \forall T > 0, \quad (151)
\]
where \( O_R = \{x \in O : |x| < R\} \).

Proof. Let \( v_n(r) = v_{\delta_n}(r, \tau, \omega, v_{\delta_n, \tau}) \) and \( v(r) = v(r, \tau, \omega, v_\tau) \), for \( r \geq \tau \) with \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). By the uniform estimates similar to (3), we find that for any \( T > 0 \),
\[\{v_n\}_{n=1}^{\infty} \text{ is bounded in } L^\infty(\tau, \tau + T; H) \cap L^2(\tau, \tau + T; V). \quad (153)\]
Since \( \delta_n \to 0 \), by (96), there exists \( N_1 = N_1(\tau, T, \omega) \geq 1 \) such that for all \( n \geq N_1 \) and \( t \in [\tau, \tau + T] \),
\[
|y_{\delta_n}(\theta t \omega) - z(\theta t \omega)| \leq 1. \quad (154)
\]
By the continuity of \( z(\theta t \omega) \), there exists \( c_1 = c_1(\tau, T, \omega) > 0 \) such that
\[
|z(\theta t \omega)| \leq c_1, \quad \forall t \in [\tau, \tau + T]. \quad (155)
\]
By (154)-(155) we get, for all \( n \geq N_1 \),
\[
|y_{\delta_n}(\theta t \omega)| \leq 1 + c_1, \quad \forall t \in [\tau, \tau + T]. \quad (156)
\]
Since \( h \in D(A) \), by (7) and (156) we find that for \( n \geq N_1 \) and \( t \in [\tau, \tau + T] \),
\[
\|B(v_n(t) + e^{\mu t} y_{\delta_n}(\theta t \omega), v_n(t) + e^{\mu t} y_{\delta_n}(\theta t \omega))\|_{V^*} \leq c_2 e^{2\mu t} + c_3 \|v_n(t)\|^2. \quad (157)
\]
It follows from (153) and (157) that
\[
B(v_n + e^{\mu t} y_{\delta_n}, v_n + e^{\mu t} y_{\delta_n}) \quad \text{is bounded in} \quad L^2(\tau, \tau + T; V^*). \quad (158)
\]
By (125) we have
\[
\frac{dv_n}{dt} = f + (\eta - \mu)e^{\mu t} y_{\delta_n} - \nu e^{\mu t} y_{\delta_n} A h - \nu A v_n - B(v_n + e^{\mu t} y_{\delta_n}, v_n + e^{\mu t} y_{\delta_n}). \quad (159)
\]
By (153), (158) and (159), we see that
\[
\left\{ \frac{dv_n}{dt} \right\}_{n=1}^{\infty} \quad \text{is bounded in} \quad L^2(\tau, \tau + T; V^*). \quad (160)
\]
Let $\mathcal{O}_R = \{ x \in \mathcal{O} : |x| < R \}$. Then by (153), (160) and the compactness arguments as in [50] we infer that
\[
\{ v_n \}_{n=1}^{\infty} \text{ is relatively compact in } L^2(\tau, \tau + T; \mathbb{L}^2(\mathcal{O}_R)), \quad \forall R > 0.
\] (161)
From (153) and (161), by a diagonal process, we find that for every $r$ such that $\xi$ there exist $N$ one in $H$.

By (167) we see that $\{ v_n \}_{n=1}^{\infty}$ such that
\[
v_{n'}(r) \rightharpoonup \xi \quad \text{in } H,
\] (162)
\[
v_{n'} \rightharpoonup \tilde{v} \quad \text{in } L^\infty(\tau, \tau + T; H),
\] (163)
\[
v_{n'} \rightarrow \tilde{v} \quad \text{in } L^2(\tau, \tau + T; V),
\] (164)
\[
v_{n'} \rightarrow \tilde{v} \quad \text{in } L^2(\tau, \tau + T; \mathbb{L}^2(\mathcal{O}_R)), \quad \forall R > 0.
\] (165)

Letting $n' \rightarrow \infty$, by (162)-(165), one can verify that $\tilde{v}$ is the solution of (80) with $\tilde{v}(\tau) = v_\tau$ and $\tilde{v}(r) = \xi$. By the uniqueness of solutions, we must have $\tilde{v} = v$, and the entire sequence $\{ v_n \}_{n=1}^{\infty}$ must converge to $v$ in the sense of (162)-(165). This completes the proof. \hfill \Box

Now, we prove the uniform compactness of the family of random attractors $A_{\delta}$ as stated below.

**Lemma 3.6.** Suppose (1), (23) and (75) hold. Let $\tau \in \mathbb{R}$ and $\omega \in \Omega$ be fixed. If $\delta_n \rightarrow 0$ and $u_n \in A_{\delta_n}(\tau, \omega)$, then the sequence $\{ u_n \}_{n=1}^{\infty}$ has a convergent subsequence in $H$.

**Proof.** Since $\delta_n \rightarrow 0$, by (129) we find that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $N_1 = N_1(\tau, \omega)$ such that for all $n \geq N_1$,
\[
R_{\delta_n}(\tau, \omega) \leq 2R_0(\tau, \omega).
\] (166)
Due to $u_n \in A_{\delta_n}(\tau, \omega)$ and $A_{\delta_n}(\tau, \omega) \subseteq B_{\delta_n}(\tau, \omega)$, by (127) and (166) we get, for all $n \geq N_1$,
\[
|u_n|^2 \leq 2R_0(\tau, \omega).
\] (167)
By (167) we see that $\{ u_n \}_{n=1}^{\infty}$ is bounded in $H$ and hence, there exists $u_0 \in H$ such that, up to a subsequence
\[
u_n \rightarrow u_0 \quad \text{in } H.
\] (168)
In what follows, we prove that the weak convergence in (168) is actually a strong one in $H$. Since $u_n \in A_{\delta_n}(\tau, \omega)$, by the invariance of $A_{\delta_n}$, for every $k \geq 1$, there exists $u_{n,k} \in A_{\delta_n}(\tau - k, \theta - k \omega)$ such that
\[
u_n = \Psi_{\delta_n}(k, \tau - k; \theta - k \omega, u_{n,k}) = u_{\delta_n}(\tau, \tau - k; \theta - \tau \omega, u_{n,k}).
\] (169)
Since $u_{n,k} \in A_{\delta_n}(\tau - k, \theta - k \omega)$ and $A_{\delta_n}(\tau - k, \theta - k \omega) \subseteq B_{\delta_n}(\tau - k, \theta - k \omega)$, by (127) and (166) we infer that for each $k \geq 1$ and $n \geq N_1(\tau - k, \theta - k \omega)$,
\[
|u_{n,k}|^2 \leq 2R_0(\tau - k, \theta - k \omega).
\] (170)
By (124) we have
\[
v_{\tilde{\delta}_n}(\tau, \tau - k; \theta - \tau \omega, v_n, k) = u_{\tilde{\delta}_n}(\tau, \tau - k; \theta - \tau \omega, u_{n,k}) - e^{\mu \tau} h(x) y_{\tilde{\delta}_n}(\omega),
\] (171)
where
\[
v_{n,k} = u_{n,k} - e^{\mu(\tau - k)} h(x) y_{\tilde{\delta}_n}(\theta - k \omega).
\] (172)
By (169) and (171) we get
\[ u_n = v_{\delta_n}(\tau, \tau - k, \theta_{-\tau} \omega, v_n, k) + e^{\mu \tau} h(x) y_{\delta_n}(\omega). \] (173)

By (170) and (172) we obtain, for \( n \geq N_1(\tau - k; \theta_{-k} \omega) \),
\[ |v_{n,k}|^2 \leq 4R_0(\tau - k; \theta_{-k} \omega) + 2e^{2\mu(\tau - k)} |h|^2 y_{\delta_n}^2(\theta_{-k} \omega). \] (174)

Now, by (96) and (174) we find that there exists \( N_2 = N_2(\tau, \omega, k) \geq N_1 \) such that for every \( k \geq 1 \) and \( n \geq N_2 \),
\[ |v_{n,k}|^2 \leq 4R_0(\tau - k; \theta_{-k} \omega) + 2e^{2\mu(\tau - k)} |h|^2 (1 + z^2(\theta_{-k} \omega)). \] (175)

Note that (168), (173) and (96) imply, as \( n \to \infty \),
\[ v_{\delta_n}(\tau, \tau - k, \theta_{-\tau} \omega, v_n, k) \to v_0 \text{ in } H \text{ with } v_0 = u_0 - e^{\mu \tau} h(x) z(\omega). \] (176)

Next, by the idea of energy equations, we examine the limit of the norm \( |v_{\delta_n}(\tau, \tau - k, \theta_{-\tau} \omega, v_n, k)|^2 \) as \( n \to \infty \) for each fixed \( k \).

By (175) we find for each fixed \( k \geq 1 \), the sequence \( \{v_{n,k}\}_{n=1}^\infty \) is bounded in \( H \), and hence, by a diagonal process, we can find a subsequence (not relabeled) such that for every \( k \geq 1 \), there exists \( \hat{v}_k \in H \) such that
\[ v_{n,k} \to \hat{v}_k \text{ in } H \text{ as } n \to \infty. \] (177)

By (177) and Lemma 3.5, we get, \( n \to \infty \),
\[ v_{\delta_n}(\tau, \tau - k, \theta_{-\tau} \omega, v_n, k) \to v(\tau, \tau - k, \theta_{-\tau} \omega, \hat{v}_k) \text{ in } H, \] (178)
\[ v_{\delta_n}(\cdot, \tau - k, \theta_{-\tau} \omega, v_n, k) \to v(\cdot, \tau - k, \theta_{-\tau} \omega, \hat{v}_k) \text{ in } L^2(\tau - k, \tau; V), \] (179)
and
\[ v_{\delta_n}(\cdot, \tau - k, \theta_{-\tau} \omega, v_n, k) \to v(\cdot, \tau - k, \theta_{-\tau} \omega, \hat{v}_k) \text{ in } L^2(\tau - k, \tau; \mathbb{L}^2(\mathcal{O}_R)), \] (180)
for any bounded domain \( \mathcal{O}_R \subseteq \mathcal{O} \). Now, by (176) and (178) we get
\[ v_0 = v(\tau, \tau - k, \theta_{-\tau} \omega, \hat{v}_k). \] (181)

By (7) and (125), we have the energy equation
\[
\frac{d}{dt} |v_{\delta_n}|^2 + 2\nu |v_{\delta_n}|^2 + 2\langle B(v_{\delta_n} + e^{\mu t} h y_{\delta_n}, e^{\mu t} h y_{\delta_n}), v_{\delta_n} \rangle \\
= 2\langle f(t, \cdot), v_{\delta_n} \rangle + 2e^{\mu t} (\eta - \mu) h - \nu Ah, v_{\delta_n} \rangle y_{\delta_n}(\theta_{t} \omega),
\]
i.e.
\[
\frac{d}{dt} |v_{\delta_n}|^2 + \nu \lambda |v_{\delta_n}|^2 + \psi(v_{\delta_n}) - 2b(v_{\delta_n} + e^{\mu t} h y_{\delta_n}, v_{\delta_n}, e^{\mu t} h y_{\delta_n}) \\
= 2\langle f(t, \cdot), v_{\delta_n} \rangle + 2e^{\mu t} (\eta - \mu) h - \nu Ah, v_{\delta_n} \rangle y_{\delta_n}(\theta_{t} \omega),
\] (182)
where \( \psi \) is a functional on \( V \) given by (48), and we know from (49) that \( \psi(\cdot) \) is an equivalent norm of \( V \). Multiplying (182) by \( e^{\nu \lambda t} \) and then integrating from \( \tau - k \) to \( \tau \) with \( \omega \) being replaced by \( \theta_{-\tau} \omega \), we get
\[ |v_{\delta_n}(\tau, \tau - k, \theta_{-\tau} \omega, v_n, k)|^2 = J_1(n, k) + J_2(n, k) + J_3(n, k) + J_4(n, k) + J_5(n, k), \] (183)
where
\[ J_1(n, k) = e^{-\nu \lambda k} |v_{n,k}|^2, \] (184)
\[ J_2(n, k) = -\int_{-\tau}^{0} e^{\nu \lambda s} \psi(v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau} \omega, v_n, k)) ds, \] (185)
Next, we examine the limits of (184)-(188) as $n \to \infty$. By (96) and (179), we have

\[
\lim_{n \to \infty} J_3(n, k) = 2 \int_{-k}^{0} e^{\nu \lambda s} (f(s + \tau, \cdot), v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau} \omega, v_{n,k})) ds,
\]

\[
J_4(n, k) = 2 \int_{-k}^{0} e^{\nu \lambda s} \mu(s + \tau) y_{\delta_n}(\theta_s \omega) \cdot ((\eta - \mu) h - \nu Ah, v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau} \omega, v_{n,k})) ds,
\]

and

\[
J_5(n, k) = 2 \int_{-k}^{0} e^{\nu \lambda s} b(v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau} \omega, v_{n,k}) + e^{\mu(s + \tau)} h y_{\delta_n}(\theta_s \omega), v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau} \omega, v_{n,k}), e^{\mu(s + \tau)} h y_{\delta_n}(\theta_s \omega)) ds.
\]

Similar to (183), by (80) and (181), we can also obtain

\[
|v_0|^2 = |v(\tau, \tau - k, \theta_{-\tau} \omega, \bar{v}_k)|^2 \leq e^{-\nu \lambda k} |\bar{v}_k|^2 - \int_{-k}^{0} e^{\nu \lambda s} \psi(v(s + \tau, \tau - k, \theta_{-\tau} \omega, \bar{v}_k)) ds + 2 \int_{-k}^{0} e^{\nu \lambda s} (f(s + \tau, \cdot), v(s + \tau, \tau - k, \theta_{-\tau} \omega, \bar{v}_k)) ds
\]

\[
+ 2 \int_{-k}^{0} e^{\nu \lambda s} e^{\mu(s + \tau)} ((\eta - \mu) h - \nu Ah, v(s + \tau, \tau - k, \theta_{-\tau} \omega, \bar{v}_k)) ds + 2 \int_{-k}^{0} e^{\nu \lambda s} b(v(s + \tau, \tau - k, \theta_{-\tau} \omega, \bar{v}_k) + e^{\mu(s + \tau)} h z(\theta_s \omega), v(s + \tau, \tau - k, \theta_{-\tau} \omega, \bar{v}_k), e^{\mu(s + \tau)} h z(\theta_s \omega)) ds.
\]

Next, we examine the limits of (184)-(188) as $n \to \infty$. First, by (175), we find that

\[
\lim_{n \to \infty} J_1(n, k) \leq 2e^{-\nu \lambda k} (4R_0(\tau - k, \theta_{-k} \omega) + 2e^{2\mu(\tau - k)}|h|^2(1 + z^2(\theta_{-k} \omega))).
\]

By (179), we have

\[
\lim_{n \to \infty} J_2(n, k) = -\liminf_{n \to \infty} \left( \int_{-k}^{0} e^{\nu \lambda s} \psi(v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau} \omega, v_{n,k})) ds \right)
\]

\[
\leq -\int_{-k}^{0} e^{\nu \lambda s} \psi(v(s + \tau, \tau - k, \theta_{-\tau} \omega, \bar{v}_k)) ds,
\]

and

\[
\lim_{n \to \infty} J_3(n, k) = 2 \int_{-k}^{0} e^{\nu \lambda s} (f(s + \tau, \cdot), v(s + \tau, \tau - k, \theta_{-\tau} \omega, \bar{v}_k)) ds.
\]

By (96) and (179), we have

\[
\lim_{n \to \infty} J_4(n, k) = 2 \int_{-k}^{0} e^{\nu \lambda s} e^{\mu(s + \tau)} z(\theta_s \omega) ((\eta - \mu) h - \nu Ah, v(s + \tau, \tau - k, \theta_{-\tau} \omega, \bar{v}_k)) ds.
\]

Next, we examine the limit of $J_5(n, k)$ given by (188) as $n \to \infty$. Note that

\[
\lim_{n \to \infty} J_5(n, k) = 2 \int_{-k}^{0} e^{\nu \lambda s} b(v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau} \omega, v_{n,k}) + e^{\mu(s + \tau)} h y_{\delta_n}(\theta_s \omega), v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau} \omega, v_{n,k}), e^{\mu(s + \tau)} h y_{\delta_n}(\theta_s \omega)) ds
\]
\[ \int_{-k}^{0} e^{\nu(s+\tau)} b(v(s+\tau, \tau-k, \theta_{-\omega}, \tilde{v}_k) + e^{\mu(s+\tau)} h z(\theta_s\omega), \] 
\[ v(s+\tau, \tau-k, \theta_{-\omega}, \tilde{v}_k), e^{\mu(s+\tau)} h z(\theta_s\omega))ds \]
\[ + \int_{-k}^{0} e^{\nu(s+\tau)} b(v_{\delta_n}(s+\tau, \tau-k, \theta_{-\omega}, v_{n,k}) - v(s+\tau, \tau-k, \theta_{-\omega}, \tilde{v}_k) \]
\[ + e^{\mu(s+\tau)} h(y_{\delta_n} - z(\theta_s\omega), v_{\delta_n}, e^{\mu(s+\tau)} h y_{\delta_n})ds \]
\[ + \int_{-k}^{0} e^{\nu(s+\tau)} b(v + e^{\mu(s+\tau)} h z(\theta_s\omega), \] 
\[ v_{\delta_n}(s+\tau, \tau-k, \theta_{-\omega}, v_{n,k}) - v(s+\tau, \tau-k, \theta_{-\omega}, \tilde{v}_k), e^{\mu(s+\tau)} h y_{\delta_n}(\theta_s\omega))ds \]
\[ + \int_{-k}^{0} e^{\nu(s+\tau)} b(v + e^{\mu(s+\tau)} h z(\theta_s\omega), v(s+\tau, \tau-k, \theta_{-\omega}, \tilde{v}_k), \] 
\[ e^{\mu(s+\tau)} h(y_{\delta_n}(\theta_s\omega) - z(\theta_s\omega)))ds. \]

We will prove the last three terms on the right-hand side of (194) converge to zero as \( n \to \infty \). We first deal with the second term on the right-hand side of (194).

Given \( \varepsilon > 0 \), due to \( h \in H^1_\omega(\mathcal{O}) \), we find that there exists \( R = R(\varepsilon, h) > 0 \) such that
\[ \int_{\mathcal{O} \setminus \mathcal{O}_R} (|h(x)|^2 + |\nabla h(x)|^2) dx \leq \varepsilon^2, \]
where \( \mathcal{O}_R = \{ x \in \mathcal{O} : |x| < R \} \). For convenience, for \( u, v, w \in V \), we write
\[ b(u, v, w) = b_1(u, v, w) + b_2(u, v, w) \]
where
\[ b_1(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathcal{O} \setminus \mathcal{O}_R} u_i \frac{\partial v_j}{\partial x_i} w_j dx \]
and
\[ b_2(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathcal{O} \setminus \mathcal{O}_R} u_i \frac{\partial v_j}{\partial x_i} w_j dx. \]

Then we have
\[ \int_{-k}^{0} e^{\nu(s+\tau)} b(v_{\delta_n}(s+\tau, \tau-k, \theta_{-\omega}, v_{n,k}) - v(s+\tau, \tau-k, \theta_{-\omega}, \tilde{v}_k) \]
\[ + e^{\mu(s+\tau)} h(y_{\delta_n}(\theta_s\omega) - z(\theta_s\omega), v_{\delta_n}, e^{\mu(s+\tau)} h y_{\delta_n}(\theta_s\omega)))ds \]
\[ = \int_{-k}^{0} e^{\nu(s+\tau)} b_1(v_{\delta_n}(s+\tau, \tau-k, \theta_{-\omega}, v_{n,k}) - v(s+\tau, \tau-k, \theta_{-\omega}, \tilde{v}_k) \]
\[ + e^{\mu(s+\tau)} h(y_{\delta_n}(\theta_s\omega) - z(\theta_s\omega), v_{\delta_n}, e^{\mu(s+\tau)} h y_{\delta_n})ds \]
\[ + \int_{-k}^{0} e^{\nu(s+\tau)} b_2(v_{\delta_n}(s+\tau, \tau-k, \theta_{-\omega}, v_{n,k}) - v(s+\tau, \tau-k, \theta_{-\omega}, \tilde{v}_k) \]
\[ + e^{\mu(s+\tau)} h(y_{\delta_n}(\theta_s\omega) - z(\theta_s\omega), v_{\delta_n}, e^{\mu(s+\tau)} h y_{\delta_n})ds. \]
By (7) and (195) we get
\[
\begin{align*}
\int_{-k}^{0} e^{\nu \lambda s} b_2(v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau \omega}, v_{n,k}) - v(s + \tau, \tau - k, \theta_{-\tau \omega}, \tilde{v}_k) &+ e^{\mu(s+\tau)} h(y_{\delta_n} - z(\theta_{s \omega})), v_{\delta_n}, e^{\mu(s+\tau)} h y_{\delta_n}) \, ds \\
\leq c_1 \int_{-k}^{0} \|v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau \omega}, v_{n,k}) - v(s + \tau, \tau - k, \theta_{-\tau \omega}, \tilde{v}_k) &+ e^{\mu(s+\tau)} h(y_{\delta_n}(\theta_{s \omega}) - z(\theta_{s \omega})) \| v_{\delta_n} \| y_{\delta_n}(\theta_{s \omega}) \| ds \\
&\leq c_2 \int_{-k}^{0} \|v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau \omega}, v_{n,k})\|^2 \|y_{\delta_n}(\theta_{s \omega})\| ds \\
&\quad + \int_{-k}^{0} (\|v(s + \tau, \tau - k, \theta_{-\tau \omega}, \tilde{v}_k)\|^2 + \|h(y_{\delta_n}(\theta_{s \omega})\|^2 + \|h_z(\theta_{s \omega})\|^2) \|y_{\delta_n}(\theta_{s \omega})\| ds.
\end{align*}
\]

By (179) we see that there exists \(c_4 = c_4(\tau, \omega, k) > 0\) such that
\[
\int_{-k}^{0} \|v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau \omega}, v_{n,k})\|^2 ds \leq c_4. \tag{198}
\]

By (96), there exists \(N_3 = N_3(\tau, \omega, k) \geq N_2\) such that for all \(n \geq N_3,\)
\[
|y_{\delta_n}(\theta_{s \omega})| \leq c_5, \quad \forall s \in [-k, 0], \tag{199}
\]
where \(c_5 = c_5(\omega, k) > 0\). It follows from (197)-(199) that there exists \(c_6 = c_6(\tau, \omega, k) > 0\) such that for all \(n \geq N_3,\)
\[
\begin{align*}
\int_{-k}^{0} e^{\nu \lambda s} b_2(v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau \omega}, v_{n,k}) - v(s + \tau, \tau - k, \theta_{-\tau \omega}, \tilde{v}_k) &+ e^{\mu(s+\tau)} h(y_{\delta_n} - z(\theta_{s \omega})), v_{\delta_n}, e^{\mu(s+\tau)} h y_{\delta_n}) \, ds \\
\leq c_7 \int_{-k}^{0} |v_{\delta_n} - v + e^{\mu(s+\tau)} h(y_{\delta_n} - z)\|_{L^2(\Omega_R)}(\|v_{\delta_n} - v + e^{\mu(s+\tau)} h(y_{\delta_n} - z)\|_{L^2(\Omega_R)} &\|v_{\delta_n}\| h y_{\delta_n}(\theta_{s \omega})\| ds \\
\leq c_8 \int_{-k}^{0} |v_{\delta_n} - v + e^{\mu(s+\tau)} h(y_{\delta_n} - z)\|_{L^2(\Omega_R)}(\|v_{\delta_n}\| + \|v\| + \|v_{\delta_n}\| + \|v_{\delta_n}\| ds \\
\leq c_9 \int_{-k}^{0} |v_{\delta_n} - v + e^{\mu(s+\tau)} h(y_{\delta_n} - z)\|_{L^2(\Omega_R)}(1 + \|v_{\delta_n}\| + \|v\| ds
\end{align*}
\]

Next, we deal with the first term on the right-hand side of (196). By (7), (198)-(199) and the Young inequality, we get, for all \(n \geq N_3,\)
\[
\begin{align*}
\int_{-k}^{0} e^{\nu \lambda s} b_1(v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau \omega}, v_{n,k}) - v(s + \tau, \tau - k, \theta_{-\tau \omega}, \tilde{v}_k) &+ e^{\mu(s+\tau)} h(y_{\delta_n} - z(\theta_{s \omega})), v_{\delta_n}, e^{\mu(s+\tau)} h y_{\delta_n}) \, ds \\
\leq c_7 \int_{-k}^{0} |v_{\delta_n} - v + e^{\mu(s+\tau)} h(y_{\delta_n} - z)\|_{L^2(\Omega_R)}(\|v_{\delta_n} - v + e^{\mu(s+\tau)} h(y_{\delta_n} - z)\|_{L^2(\Omega_R)} &\|v_{\delta_n}\| h y_{\delta_n}(\theta_{s \omega})\| ds \\
\leq c_8 \int_{-k}^{0} |v_{\delta_n} - v + e^{\mu(s+\tau)} h(y_{\delta_n} - z)\|_{L^2(\Omega_R)}(\|v_{\delta_n}\| + \|v\| + \|v_{\delta_n}\| + \|v_{\delta_n}\| ds \\
\leq c_9 \int_{-k}^{0} |v_{\delta_n} - v + e^{\mu(s+\tau)} h(y_{\delta_n} - z)\|_{L^2(\Omega_R)}(1 + \|v_{\delta_n}\| + \|v\| ds
\end{align*}
\]
\[ \leq c_9 \int_{-k}^{0} \left| \psi_{\delta_n} - v + e^{\mu(s+\tau)} h(\psi_{\delta_n} - z) \right|_{L^2(\Omega)}^2 ds \\
+ c_9 \left( \int_{-k}^{0} \left| \psi_{\delta_n} - v + e^{\mu(s+\tau)} h(\psi_{\delta_n} - z) \right|_{L^2(\Omega)}^2 ds \right)^{\frac{1}{2}} \\
\cdot \left( \int_{-k}^{0} \| \psi_{\delta_n}(s, \tau - k, \theta, \omega, \nu_{n,k}) \|_{L^2(\Omega)}^2 ds \right)^{\frac{1}{2}} \\
+ c_9 \left( \int_{-k}^{0} \left| \psi_{\delta_n} - v + e^{\mu(s+\tau)} h(\psi_{\delta_n} - z) \right|_{L^2(\Omega)}^2 ds \right)^{\frac{1}{2}} \\
\cdot \left( \int_{-k}^{0} \| v(s, \tau - k, \theta, \omega, \tilde{v}_k) \|_{L^2(\Omega)}^2 ds \right)^{\frac{1}{2}} \]

which together with (180) and (96) implies that

\[ \int_{-k}^{0} e^{\nu t} b_1 (\psi_{\delta_n} - v + e^{\mu(s+\tau)} h(\psi_{\delta_n} - z(\theta, \omega)), \nu_{\delta_n}, e^{\mu(s+\tau)} h \psi_{\delta_n}) ds \to 0 \quad \text{as} \quad n \to \infty. \]  

(201)

It follows from (196), (200) and (201) that

\[ \int_{-k}^{0} e^{\nu t} b(v + e^{\mu(s+\tau)} h \psi_{\delta_n} - v(s + \tau, \tau - k, \theta, \omega, \tilde{v}_k) + e^{\mu(s+\tau)} h(\psi_{\delta_n} - z(\theta, \omega)), \nu_{\delta_n}, e^{\mu(s+\tau)} h \psi_{\delta_n}) ds \to 0 \quad \text{as} \quad n \to \infty. \]  

(202)

For the third term on the right-hand side of (194), by (7) we have

\[ \int_{-k}^{0} e^{\nu t} b(v + e^{\mu(s+\tau)} h \psi_{\delta_n} - v(s + \tau, \tau - k, \theta, \omega, \tilde{v}_k) + e^{\mu(s+\tau)} h \psi_{\delta_n} - z(\theta, \omega)), \nu_{\delta_n}, e^{\mu(s+\tau)} h \psi_{\delta_n}) ds \to 0 \quad \text{as} \quad n \to \infty. \]  

(203)

Then by the arguments similar to (202), one can verify that the right-hand side of (203) converges to zero as \( n \to \infty \). Therefore we get

\[ \int_{-k}^{0} e^{\nu t} b(v + e^{\mu(s+\tau)} h \psi_{\delta_n} - v(s + \tau, \tau - k, \theta, \omega, \tilde{v}_k) + e^{\mu(s+\tau)} h \psi_{\delta_n} - z(\theta, \omega)), \nu_{\delta_n}, e^{\mu(s+\tau)} h \psi_{\delta_n}) ds \to 0 \quad \text{as} \quad n \to \infty. \]  

(204)
For the last term on the right-hand side of (194), by (7) and (96), one can easily prove it converges to zero as \(n \to \infty\), that is, as \(n \to \infty\),
\[
\int_{-k}^{0} e^{\nu \lambda s} b(v + e^{\mu(\tau+k)} h z(\theta_s \omega), v(s + \tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k), e^{\mu(\tau+k)} h(y_{\delta_n}(\theta_s \omega) - z(\theta_s \omega))) ds \to 0.
\]

It follows from (194), (202) and (204)-(205) that, as \(n \to \infty\),
\[
\int_{-k}^{0} e^{\nu \lambda s} b(v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau} \omega, v_n, k) + e^{\mu(\tau+k)} p y_{\delta_n}(\theta_s \omega),
v_{\delta_n}(s + \tau, \tau - k, \theta_{-\tau} \omega, v_n, k), e^{\mu(\tau+k)} h y_{\delta_n}(\theta_s \omega)) ds
\to \int_{-k}^{0} e^{\nu \lambda s} b(v(s + \tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k) + e^{\mu(\tau+k)} h z(\theta_s \omega),
v(s + \tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k), e^{\mu(\tau+k)} h z(\theta_s \omega)) ds,
\]
which along with (188) implies
\[
\lim_{n \to \infty} J_5(n, k) = 2 \int_{-k}^{0} e^{\nu \lambda s} b(v(s + \tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k) + e^{\mu(\tau+k)} h z(\theta_s \omega),
v(s + \tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k), e^{\mu(\tau+k)} h z(\theta_s \omega)) ds.
\]

Now, combining (190)-(193), (206) with (183) we get
\[
\limsup_{n \to \infty} |v_{\delta_n}(\tau, \tau - k, \theta_{-\tau} \omega, v_n, k)|^2 \\
\leq 2 e^{-\nu \lambda k} (4R_0(\tau - k, \theta_{-k} \omega) + 2 e^{2\mu(\tau-k)} |h|^2(1 + z^2(\theta_{-k} \omega))
- \int_{-k}^{0} e^{\nu \lambda s} \psi(v(s + \tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k)) ds
+ 2 \int_{-k}^{0} e^{\nu \lambda s} (f(s + \tau, \cdot), v(s + \tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k)) ds
+ 2 \int_{-k}^{0} e^{\nu \lambda s} e^{\mu(\tau+k)} z(\theta_s \omega)((\eta - \mu)h - \nu Ah, v(s + \tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k)) ds
+ 2 \int_{-k}^{0} e^{\nu \lambda s} b(v(s + \tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k) + e^{\mu(\tau+k)} h z(\theta_s \omega),
v(s + \tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k), e^{\mu(\tau+k)} h z(\theta_s \omega)) ds.
\]

By (189) and (207) we obtain
\[
\limsup_{n \to \infty} |v_{\delta_n}(\tau, \tau - k, \theta_{-\tau} \omega, v_n, k)|^2 \\
\leq 2 e^{-\nu \lambda k} (4R_0(\tau - k, \theta_{-k} \omega) + 2 e^{2\mu(\tau-k)} |h|^2(1 + z^2(\theta_{-k} \omega))
+ |v(\tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k)|^2 - e^{-\nu \lambda k} |\tilde{v}_k|^2
+ 2 e^{-\nu \lambda k} (4R_0(\tau - k, \theta_{-k} \omega) + 2 e^{2\mu(\tau-k)} |h|^2(1 + z^2(\theta_{-k} \omega))
+ |v(\tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k)|^2.
\]

By (176) and (181) we have
\[
v(\tau, \tau - k, \theta_{-\tau} \omega, \tilde{v}_k) = v_0 = u_0 - e^{\mu \lambda} h(x) z(\omega).
\]
By (173) we see
\[ v_{\delta_n}(\tau, \tau - k, \theta^{-}\tau \omega, v_{n,k}) = u_n - e^{\mu \tau} h(x) y_{\delta_n}(\omega). \] (210)

It follows from (208)-(210) that
\[ \limsup_{n \to \infty} \left| u_n - e^{\mu \tau} h y_{\delta_n}(\omega) \right|^2 \leq 2 e^{-\nu \lambda k} \left( 4 R_0 (\tau - k, \theta^{-}\tau \omega) + 2 e^{2 \mu (\tau - k)} |h|^2 (1 + z^2 (\theta^{-}\tau \omega)) \right) \] (211)

Since \( R_0 \) and \( z \) are tempered, we have
\[ \lim_{k \to \infty} 2 e^{-\nu \lambda k} \left( 4 R_0 (\tau - k, \theta^{-}\tau \omega) + 2 e^{2 \mu (\tau - k)} |h|^2 (1 + z^2 (\theta^{-}\tau \omega)) \right) = 0. \]

Now, letting \( k \to \infty \) in (211), we get
\[ \limsup_{n \to \infty} \left| u_n - e^{\mu \tau} h y_{\delta_n}(\omega) \right|^2 \leq \left| u_0 - e^{\mu \tau} h z(\omega) \right|^2. \] (212)

By (168) and (96) we obtain
\[ u_n - e^{\mu \tau} h y_{\delta_n}(\omega) \to u_0 - e^{\mu \tau} h z(\omega) \quad \text{in} \quad H. \] (213)

By (212)-(213) we find
\[ u_n - e^{\mu \tau} h y_{\delta_n}(\omega) \to u_0 - e^{\mu \tau} h z(\omega) \quad \text{in} \quad H. \] (214)

Finally, by (96) and (214) we get
\[ u_n \to u_0 \quad \text{in} \quad H, \]
as desired. This completes the proof. \( \square \)

We are now ready to establish the upper semicontinuity of random attractors as \( \delta \to 0. \)

**Theorem 3.7.** Let assumptions (1), (23) and (75) hold. Then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega, \)
\[ \lim_{\delta \to 0} \text{dist}_H (A_{\delta}(\tau, \omega), A_0(\tau, \omega)) = 0. \] (215)

**Proof.** Given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega, \) define
\[ B_0(\tau, \omega) = \left\{ u \in H : |u|^2 \leq \bar{L} (1 + |e^{\mu \tau} z(\omega)|^2) + \int_{-\infty}^{0} e^{\bar{L}(\nu \lambda - 4 \kappa |e^{\mu \tau + s} (\theta^{-}\tau \omega))} ds \left( \|f(s + \tau, \cdot)\|_{L^p}^2 + \xi(s + \tau, \theta^{-}\tau \omega) \right) ds \right\}, \]
where \( \bar{L} = \max\{L_1, L_2\} \) with \( L_1 \) and \( L_2 \) are the same positive constants as in (83) and (128), respectively. Then by Lemma 3.1, \( B_0 = \{ B_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is also a \( D \)-pullback absorbing set for \( \Psi_0. \) Let \( B_\delta \) be the \( D \)-pullback absorbing set of \( \Psi_\delta \) given by (127), by (129) we know
\[ \lim_{\delta \to 0} |B_\delta(\tau, \omega)| \leq |B_0(\tau, \omega)| \quad \text{for all} \quad \tau \in \mathbb{R} \quad \text{and} \quad \omega \in \Omega. \] (216)

Then by (216), Lemma 3.4 and Lemma 3.6, we obtain (215) from [57]. \( \square \)

**Acknowledgments.** This work was initiated while the first author was visiting the Department of Mathematics at New Mexico Institute of Mining and Technology in 2017. He wishes to express his thanks to all people there for their kind hospitality.
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Received for publication June 2019.

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