Non-universal parameters, corrections and universality in Kardar–Parisi–Zhang growth

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Received 14 February 2013
Accepted 10 April 2013
Published 14 May 2013

Online at stacks.iop.org/JSTAT/2013/P05007
doi:10.1088/1742-5468/2013/05/P05007

Abstract. We present a comprehensive numerical investigation of non-universal parameters and corrections related to interface fluctuations of models belonging to the Kardar–Parisi–Zhang (KPZ) universality class, in $d = 1 + 1$, for both flat and curved geometries. We analyzed two classes of models. In the isotropic models the non-universal parameters are uniform along the surface, whereas in anisotropic growth they vary. In the latter case, which produces curved surfaces, the statistics must be computed independently along fixed directions. The ansatz $h = v_\infty t + (\Gamma t)^{1/3} \chi + \eta$, where $\chi$ is a Tracy–Widom (geometry-dependent) distribution and $\eta$ is a time-independent correction, is probed. Our numerical analysis shows that the non-universal parameter $\Gamma$ determined through the first cumulant leads to a very good accordance with the extended KPZ ansatz for all investigated models in contrast with the estimates of $\Gamma$ obtained from higher order cumulants that indicate a violation of the generalized ansatz for some of the studied models. We associate the discrepancies to corrections of an unknown nature, which hampers an accurate estimation of $\Gamma$ at finite times. The discrepancies in $\Gamma$ via different approaches are relatively small but sufficient to modify the scaling law $t^{-1/3}$ that characterizes the finite-time corrections due to $\eta$. Among the investigated models, we have revisited an off-lattice Eden model that supposedly disobeyed the shift in the mean scaling as $t^{-1/3}$ and showed that there is a crossover to the expected regime. We have found model-dependent (non-universal) corrections for cumulants of order $n \geq 2$. All investigated models are consistent with a further term of order $t^{-1/3}$ in the KPZ ansatz.
Non-universal parameters, corrections and universality in KPZ growth

**Keywords:** critical exponents and amplitudes (theory), kinetic growth processes (theory), kinetic roughening (theory), thin film deposition (theory)

Contents

1. Introduction 2
2. Isotropic flat growth 4
3. Isotropic radial growth 9
4. Anisotropic radial growth 12
5. Droplet growth 17
6. Discussion 22
   Acknowledgments 24
   References 24

1. Introduction

Dynamics of self-affine surfaces is a fascinating topic of non-equilibrium statistical physics where pattern formation, stochasticity, scale invariance and universality are joined in a unified framework [1, 2]. The Kardar–Parisi–Zhang (KPZ) equation [3]

\[ \frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} \langle \nabla h \rangle^2 + \xi, \]

proposed by Kardar, Parisi and Zhang in 1986, emerged as one of the most prominent theoretical systems and established a universality class which encompasses plenty of models [1, 2]. This equation describes the non-conservative evolution of an interface \( h(x, t) \) subjected to a white noise \( \xi \) defined by \( \langle \xi \rangle = 0 \) and \( \langle \xi(x, t)\xi(x', t') \rangle = D\delta(x - x')\delta(t - t') \). The KPZ universality class goes beyond the surface dynamics including theoretical studies in random polymers [4] and fluid transport [5]. Our understanding of the KPZ universality class has undergone noticeable progress during the last few years with its experimental realization [6]–[9] as well as the achievement of analytical solutions [4], [10]–[14] of the KPZ equation in \( 1 + 1 \) dimensions.

Self-affine interface evolution is characterized by a dynamical regime where the interface width \( w \), defined as the standard deviation of the surface heights, \( w \equiv \sqrt{\langle h^2 \rangle - \langle h \rangle^2} \), increases in time as a power law \( w \sim t^\beta \), and a stationary regime where the interface width depends on the system size as \( w \sim L^\alpha \). Here \( \beta \) and \( \alpha \) are the growth and roughness exponents [1], respectively; they assume the exact values \( \beta = 1/3 \) and \( \alpha = 1/2 \) for the KPZ class in \( d = 1 + 1 \) [3]. The universality in KPZ systems includes other important universal quantities related to the height distributions (HDs) [7, 15]. Motivated by analytical determination of the HDs of some models of the KPZ class, namely, the
Non-universal parameters, corrections and universality in KPZ growth

asymmetric exclusion process [16] and the polynuclear growth model [17, 18], Prähofer and Spohn [17] conjectured that the heights for any KPZ system are asymptotically given by

\[ h = v_\infty t + s_\lambda (\Gamma t)^{1/3} \chi \]  

(2)

where \( s_\lambda = \text{sgn}(\lambda) \), \( v_\infty \) and \( \Gamma \) are non-universal parameters while \( \chi \) is a stochastic variable with universal properties. Moreover, the conjecture states that, in the dynamical regime, \( \chi \) depends on the growth geometry and is given by Tracy–Widom (TW) distributions from random matrix theory [19] for flat and curved growth geometries. The Gaussian orthogonal ensemble (GOE) is expected for the former while Gaussian unitary ensemble (GUE) is expected for the latter. Therefore, the KPZ universality class splits into subclasses depending on the growth geometry. This conjecture has been verified by many analytical results [4, 10, 11, 13, 14], computer simulations [20]–[23] and, most excitingly, in experiments [6]–[9].

The TW distributions are conjectured for the asymptotic regime, but many experimental [6, 8], analytical [10, 24] and numerical [21, 22] works have shown the existence of a slow convergence with apparently universal properties. More specifically, the rescaled distribution \( P(q) \), where

\[ q = \frac{h - v_\infty t}{s_\lambda (\Gamma t)^{1/3}}, \]

(3)

has a shift in relation to the TW distributions that vanishes as

\[ \langle q \rangle - \langle \chi \rangle \sim t^{-1/3}. \]

(4)

An exception to the correction \( t^{-1/3} \) was obtained in the partially asymmetric simple exclusion process (PASEP) for the asymmetry parameter \( p_c = 0.7822787862 \ldots [24] \). The PASEP result suggests that corrections different from \( t^{-1/3} \) would appear in very particular situations. However, Takeuchi [23] reported off-lattice simulations of a radial Eden model where the first cumulant of the scaled distributions converges to the asymptotic GUE value as \( t^{-2/3} \), in contrast with other Eden versions [21]. It is somewhat intriguing that a stochastic model without control parameters does not have the usual correction. Still in [24], a deterministic shift \( \eta \) was found in the polynuclear growth model (PNG) and in the totally/partially asymmetric exclusion processes (TASEP/PASEP), generalizing equation (2) to

\[ h = v_\infty t + s_\lambda (\Gamma t)^{1/3} \chi + \eta + \cdots. \]

(5)

The correction \( t^{-1/3} \) in the first cumulant \( \langle q \rangle \) derives directly from equation (5). It was also shown that higher order moments have corrections \( O(t^{-2/3}) \) [24] and, consequently, all higher order cumulants of order \( n \geq 2 \) decay as \( O(t^{-2/3}) \) or faster. Sasamoto and Spohn found a result similar to equation (5) in their solution of the KPZ equation [10, 12] with the difference that \( \eta \) is random, but still independent of \( \chi \).

Many models belonging to the KPZ class are analytically intractable with our current knowledge, and Monte Carlo simulations are the best accessible method for probing the universality of their interface fluctuations. The analysis of flat growth can be efficiently performed using lattice models [22, 25], the computer implementations of which are commonly simple and relatively quick and have low memory demands. The
Enon-universal parameters, corrections and universality in KPZ growth

Eden model [26] and its variations [2] are basic examples of radial growth belonging to the KPZ class. However, it is well known that radial growth in lattices is distorted by anisotropy effects [27]–[30]. Therefore, off-lattice simulations [21, 23, 31] or suitable tricks using tuning parameters [21, 30] are needed to investigate interface fluctuations in the entire surface. Although off-lattice simulations of Eden growth are quite affordable in 1 + 1 dimensions [32], the generalization to higher dimensions is cumbersome [33].

In this work, we present extensive simulations of models belonging to the KPZ universality class to address the validity of the extended KPZ conjecture including the correction \( \eta \) given in equation (5). We investigated models exhibiting either isotropic or anisotropic growth. In isotropic growth the velocity is the same for all parts of the interface such that the entire profile can be used to perform statistics. On other hand, in anisotropic growth, in which the interface velocity varies along the interface, we must analyze fixed directions independently and, consequently, a large ensemble of samples are required since one or a few points from each sample are used for statistics.

We have found that the procedure to determine the model-dependent parameter \( \Gamma \) via a second cumulant may be troublesome due to strong and/or puzzling corrections to the scaling. We adopted an alternative method using the first cumulant derivative, that exhibits a monotonic convergence to the asymptotic value with a correction \( t^{-2/3} \) in all investigated models. This method allowed the determination of the shifts in the first cumulants always consistent with a decay \( t^{-1/3} \), reinforcing the generality of the KPZ ansatz. We also investigated the convergence of higher order cumulants and identified complex and non-universal behaviors. Finally, an additional term \( t^{-1/3} \) in the KPZ ansatz was found in all models considered here.

The paper is organized as follows. Section 2 presents the analysis of flat growth models concomitantly with the description of the numerical recipes applied in the present work. Sections 3 and 4 are devoted to isotropic and anisotropic radial growth, respectively. Section 5 presents an analysis of droplet growth. We give our concluding remarks in section 6.

2. Isotropic flat growth

We consider two versions of the restricted solid-on-solid (RSOS) model [34] and the ballistic deposition (BD) model [1] on initially flat substrates. Averages were computed using \( 10^4 \) samples and system sizes \( L = 2^{18} \), resulting in more than \( 2 \times 10^9 \) data points for the statistics.

The RSOS model is defined as a random deposition obeying the height restriction \( \Delta h = |h_j - h_{j+1}| \leq m \). In the BD model, particles move normally to the substrate and permanently attach to the first nearest neighbor contact with a previously deposited particle. In both models, periodic boundary conditions are assumed and the time is increased by \( \Delta t = 1/L \) for each deposition attempt.

Accurate estimates of the non-universal parameters are imperative to a reliable characterization of the corrections. The asymptotic velocity is accurately determined taking the time derivative of the mean height in equation (5):

\[
(h)_t = v_\infty + \frac{8\lambda\Gamma^{1/3}}{3}\langle \chi \rangle t^{-2/3}.
\]
Figure 1. Numerical determination of the parameter $\Gamma$ for the RSOS model grown on flat substrates, with $\Delta h \leq 1$, using different cumulants. Solid lines are non-linear regressions. The inset shows the interface velocity against $t^{-2/3}$ (symbols) and the linear regression used to determine the asymptotic velocity.

Table 1. Non-universal parameters for distinct isotropic growth models. The number in parenthesis represents the uncertainties obtained from the non-linear regressions. The parameter $\Gamma_2$ does not have uncertainty for off-lattice Eden model due to the lack of a monotonic convergence.

| Model     | $s_\lambda$ | $v_\infty$ | $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ | $\Gamma_4$ | $\langle \eta \rangle$ |
|-----------|-------------|-------------|-------------|-------------|-------------|-------------|-------------------|
| RSOS ($m = 1$) | $-1$        | $0.419\,030(3)$ | $0.252(1)$  | $0.2532(3)$ | $0.2525(2)$ | $0.2534(3)$ | $-0.32(4)$       |
| RSOS ($m = 2$) | $-1$        | $0.603\,55(1)$ | $0.812(2)$  | $0.816(9)$  | $0.811(2)$  | $0.815(2)$  | $-0.66(6)$       |
| BD        | $1$         | $2.139\,83(1)$ | $4.94(1)$   | $4.778(2)$  | $4.799(7)$  | $4.74(2)$   | $-0.9(1)$        |
| Eden D    | $1$         | $0.513\,71(2)$ | $1.00(1)$   | $\approx 1$ | $0.99(1)$   | $0.98(2)$   | $0.50(5)$        |

Then, the velocity $v_\infty$ can be extracted by extrapolating $\langle h \rangle_t$ versus $t^{-2/3}$ in a linear regression for $t \to \infty$. The inset of figure 1 shows a typical plot used to determine $v_\infty$ for the RSOS model with $m = 1$. Interface velocities for all investigated isotropic models are shown in table 1.

The parameter $\Gamma$, in terms of the constants of the KPZ equation, is given by $\Gamma = \frac{1}{2} A^2 |\lambda|$ with $A = D/\nu \ [15, 8]$. The parameters $\lambda$ and $A$ can be determined numerically using the tilt dependence of the growth velocity and the amplitude of a two-point correlation function, respectively [35]. Alternatively, $\Gamma$ can be obtained directly from equation (2) assuming that the cumulants of $\chi$ are those of TW distributions. The scaled second cumulant

$$g_2 = \frac{\langle h^2 \rangle_c}{t^{2/3}} \to \Gamma^{2/3} \langle \chi^2 \rangle_c,$$

(7)

\text{doi:10.1088/1742-5468/2013/05/P05007}
where $\langle X^n \rangle_c$ denotes the $n$th cumulant of $X$, has been used for this purpose [6, 23, 7, 22]. In principle, one can obtain $\Gamma$ from any higher order cumulant since equation (2) yields

$$g_n = \frac{\langle h^n \rangle_c}{s^{n/3}} \rightarrow \Gamma^{n/3}\langle \chi^n \rangle_c, \quad n \geq 2. \quad (8)$$

Alternatively, the parameter $\Gamma$ can also be obtained from the first cumulant since

$$g_1 = s\lambda^3(\langle h \rangle_t - v_\infty)^{2/3} \rightarrow \Gamma^{1/3}\langle \chi \rangle. \quad (9)$$

Although widely used in many experimental [8, 6, 7] and computer [22, 23] studies, the determination of $\Gamma$ using second order cumulants has complications when there are statistical dependences among $\chi$, $\eta$ or other unknown terms in equation (5). In this case, crossed terms appear in cumulants leading to relevant corrections that, in principle, may be puzzling. Otherwise, the analysis using $\langle h \rangle_t$ is free from crossed terms and does not depend on $\eta$. Noise in numerical derivatives counts against this last method. We propose that the first cumulant derivative yields more reliable estimates of $\Gamma$ than $\langle h^2 \rangle_c$ does, as discussed in this and in the following sections.

We denote by $\Gamma_n = (g_n/\langle \chi^n \rangle_c)^{3/n}$ the value of $\Gamma$ estimated via the $n$th cumulant. Figure 1 shows the curves used to determine the non-universal parameters of the RSOS model with $m = 1$. Independently of the used cumulant, the data have finite-time corrections. To estimate the asymptotic value as well as the scaling of the correction, numerical values of $\Gamma_n(t)$, that are recorded exponentially spaced in time, are plotted as a function of $s = \ln t$. Assuming that there is a power law correction, we performed non-linear regressions in the form

$$\Gamma_n = \Gamma_n(\infty) + c \exp(-a_n s), \quad (10)$$

discarding very short times. The extrapolated values for $\Gamma_n$ are shown in table 1. All cumulants yield essentially the same asymptotic value for $\Gamma_n$, that is in very good agreement with our previous estimate using longer growth times (and fewer samples) without extrapolations [22]. The correction in $\Gamma_1$ has an exponent $a_1 = 0.661 \approx 2/3$. Assuming that the next term in equation (5) is a power law $\zeta t^{-\gamma}$, we have that

$$g_1 = \Gamma^{1/3}\langle \chi \rangle + \langle \zeta \rangle t^{-\gamma - 1/3}. \quad (11)$$

So, a correction $a_1 = 2/3$ in $g_1$, or equivalently in $\Gamma_1$, shows that the next leading term in equation (5) decays as $t^{-1/3}$. The correction in $\Gamma_2$, that has the same exponent as $\langle q^2 \rangle_c - \langle \chi^2 \rangle_c$, is $a_2 \approx 1$, while $a_3 \approx a_4 \approx 2/3$ are found for the third and fourth order cumulants. These results are corroborated in figure 2, where the differences between $g_n$, $n = 1, \ldots, 4$, and their asymptotic values (see table 2) are plotted against time. It is worth mentioning that corrections in cumulants have already been studied for the RSOS model [22] and are consistent with the present results.

The power law correction $t^{-1/3}$ in the mean is also verified with a good precision, as shown in the inset of figure 2 where the difference between the scaled height and the GOE first cumulant has a plateau when rescaled by the expected power law. Considering the variation of the plateau inside errors of $\Gamma$ and $v_\infty$, the estimated amplitude of the correction is $\langle \eta \rangle = 0.32(4)$. The mean values of $\langle \eta \rangle$ for all isotropic models are shown in table 1.
Non-universal parameters, corrections and universality in KPZ growth

Figure 2. Corrections in $g_n$ for the RSOS model grown on flat substrates. The dashed line represents a power law $t^{-1}$ and the solid ones represent the power law $t^{-2/3}$. The inset shows the difference between the first cumulant of equation (3) and the GOE mean $\langle \chi \rangle = -0.76007$ using $v_\infty = 0.419030$ and $\Gamma = 0.252$. The shift is scaled by the theoretical power law $t^{-1/3}$.

Table 2. Asymptotic values for $g_n$ for isotropic growth models. Dimensionless cumulant ratios $R = g_2/g_1^2$, $S = g_3/g_2^{3/2}$ and $K = g_4/g_2^2$ are also included. Lack of uncertainty in $g_2$ means we could not determine errors in dimensionless ratios for the Eden model.

| Model     | $g_1$     | $g_2$     | $g_3$     | $g_4$     | $R$     | $S$     | $K$     |
|-----------|-----------|-----------|-----------|-----------|---------|---------|---------|
| RSOS      | -0.4795(5)| 0.2553(1) | 0.037725(5)| 0.01075(5)| 1.110(2)| 0.292(1)| 0.1649(5)|
| ($m = 1$) |           |           |           |           |         |         |         |
| RSOS      | -0.709(1) | 0.55474(4)| 0.12162(2)| 0.0512(1) | 1.103(4)| 0.294(2)| 0.1664(5)|
| ($m = 2$) |           |           |           |           |         |         |         |
| BD        | -1.294(1) | 1.8095(6)| 0.712(1)  | 0.536(3)  | 1.080(2)| 0.292(2)| 0.163(1)|
| Eden      | -1.770(2) | $\approx$0.81| 0.161(2)  | 0.0607(5) | 0.26   | 0.22   | 0.093   |

We also simulated the RSOS model allowing a height difference $\Delta h \leq 2$. The results are essentially the same except for a non-monotonic time-dependence for $\Gamma_2$ that initially decays and then increases towards the asymptotic value, also with a correction faster than $t^{-2/3}$. For other $g_n$, the corrections are approximately $t^{-2/3}$, the same as in the $m = 1$ case. The non-universal parameters for $m = 2$ and cumulant ratios are also shown in tables 1 and 2.

Figure 3 shows the curves $\Gamma_n$ against time and the respective non-linear regressions used to obtain their asymptotic values for the BD model. Surprisingly, the extrapolation from $\Gamma_1$ is a little larger than the others. The difference of 3–4% is sufficient to affect the scaling law ruling the shift of the distribution in relation to GOE as shown in the inset of figure 3. The value $\Gamma = 4.94$ obtained from the first cumulant derivative yields an excellent agreement with a correction $t^{-1/3}$, while the values of $\Gamma$ extrapolated from higher order cumulants do not.

doi:10.1088/1742-5468/2013/05/P05007
The cause of this difference is unknown. A possibility could be crossed terms involving random variables in higher order cumulants that are not present in the first one. These terms would introduce complicated corrections leading to a non-monotonic convergence and $\Gamma$ could not be properly extrapolated using equation (10) for $n \geq 2$. Another possible explanation is that the asymptotic distribution of $q$ for the BD model does not converge to GOE but to a shifted GOE distribution with $\chi + a$, where $a$ is a deterministic and non-universal parameter. In this case, the deterministic shift $a$ cannot be obtained using higher order cumulants since $\langle a^n \rangle_c = 0$ for $n \geq 2$. If this last hypothesis is correct, we would have $\Gamma_1 = \Gamma(1 + a/\langle \chi \rangle)^3$ that, according to the parameters of table 1, provides a tiny shift $a \approx 0.01$. Such a small shift is easily unnoticed in scaled height distributions obtained in simulations or experiments. Indeed, the correction $t^{-1/3}$ shown in the inset of figure 3 is equally obtained in curves $\langle q \rangle - \langle \chi + a \rangle$ against $t$ (data not shown). It is important to emphasize that this shift was not predicted in the other KPZ systems investigated either in the present work or elsewhere. Independently of the numerical evidence, we believe that the non-monotonic convergence is the most likely scenario.

In [22], we have shown that the shift in the first cumulant is consistent with the usual $t^{-1/3}$ decay using the parameter $\Gamma_2 = 4.90(5)$ obtained from $\langle h^2 \rangle_c$ in simulations up to $t = 5 \times 10^4$ without extrapolation. We checked that the extrapolation of the data of [22] is consistent with our current estimate of $\Gamma_2$. A careful view of the double-logarithmic plot presented in [22] reveals a slightly bent curve, indicating a correction not so close to $t^{-1/3}$ as wished. Also, the uncertainties in $\Gamma$ and $v_\infty$ reported there are much larger than the current ones, such that the propagated error in the power law is sufficiently large to embrace the exponent $-1/3$.

Corrections in $g_n$ for the BD model are shown in figure 4. In $g_1$, we have found an exponent $a_1 \approx 2/3$, the same value found for the RSOS model. However, an unusual exponent $a_2 \approx 1/2$ was observed in the second order cumulant. To our knowledge, there are no analytical or experimental analogues for this correction. This exponent is at odds with
Figure 4. Corrections in $g_n$ for the BD model grown on flat substrates. The lines are power laws $t^{-2/3}$ ($n = 1$), $t^{-1/2}$ ($n = 2$), $t^{-1}$ ($n = 3$) and $t^{-1.25}$ ($n = 4$).

our previous analysis reporting a decay faster than $t^{-2/3}$ in plots of $\langle q^2 \rangle_c - \langle \chi^2 \rangle_c$ against time [22]. The discrepancy is due to the over-estimation of $\Gamma_2$ used in the rescaled height $q$, while in the current analysis the dependence on $\Gamma_2$ is implicit to $g_2$ and, therefore, does not involve error propagation. The third and fourth order cumulants decay with exponents $a_3 \approx 0.9$ and $a_4 \approx 1.25$, respectively, that are consistent with the previous report $O(t^{-2/3})$ or faster [22]. These results show that the exponents measured in the power laws related to the finite-time corrections are very sensitive to $\Gamma$ and, thus, a precise characterization of the corrections demands accurate estimates of $\Gamma$.

3. Isotropic radial growth

We simulated radial growth using an Eden model proposed in [23] that we have called Eden D [33], since A, B, and C versions are already defined in the literature [2, 30]. The model is defined as follows: in each time step, a particle of the cluster and a position tangent to it are randomly chosen. A new particle is added to the chosen position if this event does not imply an overlap with any other particle of the cluster. In the case of overlap, simulation proceeds to the next step. The time is increased by $\Delta t = 1/N$, where $N$ is the number of particles of the cluster at time $t$. Optimization strategies described in [32] were used to speed up the simulation. A cluster of diameter 8000 takes typically 8 min of simulation in an Intel Xeon 3.2 GHz CPU, whereas if no optimization is used the same simulation takes several hours of computation. We simulated the same version of the Eden model proposed in [23] using much better statistics (90 000 samples against 3000) and much longer growth times ($t \approx 8000$ against 2000). Considering the same interval used in [23] ($250 < t < 2000$), we have found $v_\infty = 0.51390$, in full agreement with $v_\infty = 0.5139(2)$ reported there. However, a more accurate estimate $v_\infty = 0.51371(1)$ was found for $t > 2000$ (bottom inset of figure 5).

Curves used to determine $\Gamma_n(\infty)$ for the Eden model are shown in figure 6. The second order cumulant depends non-monotonically on time as highlighted in the inset of
Figure 5. Correction in the first cumulant of the scaled height in relation to the corresponding GUE value $\langle \chi \rangle = -1.77109$ for the off-lattice Eden D model. The parameters used were $\Gamma = 1.00$ and interface velocities indicated in the legends. Dashed lines are power laws $t^{-2/3}$ and $t^{-1/3}$ as guides to the eyes. The top inset shows the local exponent against time, while the bottom one shows the velocity against $t^{-2/3}$.

Figure 6. Determination of the non-universal parameter $\Gamma$ for the off-lattice Eden D model using different cumulants. Solid lines are non-linear regressions used to extrapolate $\Gamma_n$. Inset: zoom to emphasize the non-monotonicity of $\Gamma_2$.

This non-monotonicity hampers an extrapolation to $\Gamma_2(\infty)$ using equation (10). Using an extrapolation similar to that used in [23], i.e. forcing a correction $a_2 = 2/3$ in $\Gamma_2$, we obtained $\Gamma_2 \approx 0.995$ considering the time interval $1000 < t < 8000$. This value is slightly smaller than $\Gamma_2 = 1.02(2)$ reported in [23]. The first cumulant derivative yields a monotonic convergence with an exponent $a_1 \approx 2/3$ and extrapolates to $\Gamma_1 = 1.00(1)$. Moreover, $\Gamma_3$ and $\Gamma_4$ also have corrections consistent with $t^{-2/3}$ and converge to values very close to 1 (see table 1).
The main plot of figure 5 shows the shift \( \langle q \rangle - \langle \chi \rangle \) against time for the Eden model. Using the velocity \( v_\infty = 0.51390 \) obtained by Takeuchi, his result, a decay much faster than \( t^{-1/3} \) \cite{23}, is reproduced. If the more accurate estimate \( v_\infty = 0.51371 \) is used instead, a decay close to \( t^{-2/3} \) is observed for short times, followed by a crossover to a smaller exponent. The top inset of figure 5 shows the local exponent against time. One can clearly see that the scaling exponent has not reached a stationary value even for our longest simulated times.

 Corrections in \( g_n \) are shown in figure 7. The quantity \( g_1 \) has an excellent agreement with a \( t^{-2/3} \) power law resulting again in a correction \( t^{-1/3} \) in the KPZ ansatz given by equation (5). Thus, a higher order term \( t^{-2/3} \) is expected in \( \langle q \rangle - \langle \chi \rangle \) explaining the short-time decay observed in figure 5. Performing a double power law regression \( \langle q \rangle - \langle \chi \rangle = at^{-1/3} + bt^{-2/3} \), an excellent fit is obtained with positive amplitudes \( a = 0.50(5) \) and \( b = 4.0(1) \). This result shows that the amplitude of the correction \( \eta \) is much smaller than in the next leading term, resulting in a very slow crossover to the asymptotic scaling law \( t^{-1/3} \). Due to the lack of monotonicity, scaling of the corrections in \( g_2 \) could not be determined with the present data. The correction in the third cumulant is \( t^{-2/3} \). Despite the large fluctuation, the fourth cumulant also decays consistently with \( t^{-2/3} \).

 Even though all the parameters in table 1 depend on the non-universal parameter \( \Gamma \), they can be combined in dimensionless cumulant ratios that must reflect the universality of \( \chi \). The relative variance \( R \), the skewness \( S \) and the kurtosis \( K \) defined by

\[
R = \frac{g_2}{g_1^2} = \frac{\langle \chi^2 \rangle_c}{\langle \chi \rangle^2},
\]

\[
S = \frac{g_3}{g_2^{3/2}} = \frac{\langle \chi^3 \rangle_c}{\langle \chi^2 \rangle_c^{3/2}},
\]

\[
K = \frac{g_4}{g_2^2} = \frac{\langle \chi^4 \rangle_c}{\langle \chi^2 \rangle_c^2},
\]

where

doi:10.1088/1742-5468/2013/05/P05007
and

$$K = \frac{g_4}{g_2^2} = \frac{\langle \chi^4 \rangle_c}{\langle \chi^2 \rangle_c^2},$$

(14)

must therefore be model-independent.

Table 2 shows the cumulant ratios for all isotropic models investigated. The results are in excellent agreement with the corresponding GOE ($R_{\text{goe}} = 1.1046$, $S_{\text{goe}} = 0.2935$ and $K_{\text{goe}} = 0.1652$) for flat models and GUE ($R_{\text{gue}} = 0.2592$, $S_{\text{gue}} = 0.2241$ and $K_{\text{gue}} = 0.09345$) for the radial Eden model. Scaled height distributions $P(q)$ for all isotropic growth models were reported elsewhere [22, 23] and therefore are omitted here. We present scaled HDs for anisotropic growth models in the following sections.

4. Anisotropic radial growth

Radial interfaces were simulated using Eden models on square lattices. Two definitions are necessary to describe the models. Peripheral sites are occupied sites with at least one empty nearest neighbor (NN), while growth sites are empty sites with at least one occupied NN. We investigated two versions. In the version Eden A, a growth site is chosen at random and occupied. For each choice, the time is increased by $1/N_g$, where $N_g$ is the number of growth sites. In the version Eden D on a lattice, a peripheral site and one of its NNs are randomly chosen. If the selected NN is empty, it receives a new particle, otherwise, the simulation runs to the next step. For each attempt, the time is incremented by $1/N_p$, where $N_p$ is the number of peripheral sites. In all versions, the growth starts with a single occupied site at the center of the lattice.

The start point to check the equivalence between isotropic and anisotropic growth is the interface width $w$, defined as the standard deviation of the radius measured in relation to the center of the lattice. At early times, it behaves as a power law $w \sim t^{\beta}$, where the exponent $\beta = 1/3$ is expected for the KPZ class in $d = 1 + 1$. However, it is well know that Eden clusters are affected by the lattice-imposed anisotropy [27, 30]. More precisely, the axial direction $\langle 10 \rangle$ grows slightly faster than the diagonal $\langle 11 \rangle$ implying that the dispersion around the mean radius of the border is asymptotically ruled by a diamond-like shape that produces a crossover from $w \sim t^{1/3}$ at short times to $w \sim t$ at long times [27, 30]. Figure 8 shows the evolution of an Eden cluster where the faster growth along $\langle 10 \rangle$ can be seen.

Since different directions have different growth velocities, we must focus on the fluctuations along a fixed direction. The radius along a given direction is defined as the distance from the origin of the farthest cluster particle along that direction. Equation (2) can be applied for an arbitrary direction with the parameters $v_\infty$ and $\Gamma$ depending on it. At a time $t$, we have a collection of radii $\{r_1(t), r_2(t), \ldots, r_N(t)\}$ along a given direction that are obtained from an ensemble of $N$ independent simulations. We analyzed axial and diagonal directions that correspond to fastest and slowest growth directions in square lattices, respectively. We simulated $2.5 \times 10^6$ clusters of diameter $5 \times 10^3$. Therefore, our statistics is performed with $10^7$ points for each analyzed direction. A typical simulation of Eden A and D takes about 3 s and 7 s, respectively, in an Intel Xeon 3.20 GHz CPU.

1 Here, we have borrowed the crystallographic notation where $\langle 10 \rangle$ represents the directions of unitary vectors $\hat{x}$, $-\hat{x}$, $\hat{y}$ and $-\hat{y}$.

doi:10.1088/1742-5468/2013/05/P05007 12
Non-universal parameters, corrections and universality in KPZ growth

Figure 8. (a) Successive borders of the on-lattice Eden D model for small sizes. The circle has the mean radius of the border. (b) First quarter of the border of a large on-lattice Eden D cluster. The inset shows a zoom of the top of the cluster.

Figure 9. Interface width against time for the on-lattice Eden A model. Inset (A): effective growth exponent obtained from the derivative of $\ln w$ versus $\ln t$. Inset (B): interface velocity against $t^{-2/3}$.

Figure 9 shows the interface width against time for Eden A computed for distinct directions. The interface width along direction $\langle 10 \rangle$ is larger than along direction $\langle 11 \rangle$ ($\Delta w \approx 6\%$), but they follow scaling laws in time with growth exponents $\beta_{10} = 0.335(7)$ and $\beta_{11} = 0.332(7)$, respectively, in excellent agreement with the KPZ universality class. Growth exponents for Eden A and D are shown in table 4. The inset B of figure 9 shows the analysis to determine the interface velocity for Eden A simulations. A higher velocity along the axial direction is evident.

Corrections in $g_1$ are shown in figure 10 and are consistent with a decay $t^{-2/3}$ implying that the next leading term in equation (5) is $t^{-1/3}$, in analogy with isotropic growth models.
Figure 10. Determination of the non-universal quantity $\Gamma^{1/3}(\chi)$ for on-lattice Eden models. Solid lines are non-linear regressions used to extrapolate $g_1$. The inset shows the correction in $g_1$ against time. The dashed line has slope $-2/3$.

Figure 11. Scaled cumulants $g_n = \langle r^m \rangle_c / (s_\lambda^m t^{m/3})$ for on-lattice Eden A along directions $\langle 10 \rangle$ (open symbols) and $\langle 11 \rangle$ (filled symbols). Lines are extrapolations to $t \to \infty$.

described in sections 2 and 3. The scaled cumulants against time for Eden A are shown in figure 11. Along the direction $\langle 10 \rangle$, we have found corrections consistent with $t^{-1}$ for the second cumulant in both Eden A and D. Along direction $\langle 11 \rangle$, $g_2$ has a slight non-monotonicity that complicates the extrapolation to long times. It was not possible to accurately resolve the scaling of the corrections for higher order cumulants but we have
found that they do not decay slower than $t^{-2/3}$, as illustrated in figure 12. This indicates that the correction $\eta$ is statistically independent of $\chi$ since, otherwise, corrections would decay as $t^{-1/3}$ for all higher order cumulants [8, 24]. Notice that many features of our current on-lattice simulations were observed for isotropic growth, where $t^{-2/3}$ was found for $g_1$, $g_3$ and $g_4$ and a non-monotonicity for $g_2$. Table 3 shows the non-universal parameters $v_\infty$ and $g_n$ obtained in our simulations.

One can see in table 3 that the estimates for $\Gamma_1$ are slightly larger than those for $\Gamma_2$ for all investigated models, as observed for ballistic deposition on flat substrates. In the rest of this work, we use $g_1$ to investigate the scaling and universality of the models since it yields reliable estimates of $\Gamma$ as discussed in section 2.

Due to the anisotropy, the surfaces studied here do not have translational invariance so that two-point local quantities are not well-defined. So, the parameter $A$ and, consequently,
Table 4. Universal quantities for several anisotropic models with curved geometry. Acronyms as in table 3. Cumulant ratios are defined by equations (12)–(14). The KPZ cumulant ratios correspond to the GUE distribution were taken from [17]. The lack of uncertainties in the second cumulant of EdA/⟨11⟩ and EdD/⟨11⟩ models did not allow error propagation in the cumulant ratios.

| Model     | β      | R    | S    | K    |
|-----------|--------|------|------|------|
| EdA/⟨10⟩ | 0.335(7) | 0.256(1) | 0.224(1) | 0.093(2) |
| EdA/⟨11⟩ | 0.332(7) | 0.25  | 0.22  | 0.090 |
| EdD/⟨10⟩ | 0.338(9) | 0.253(1) | 0.223(2) | 0.094(2) |
| EdD/⟨11⟩ | 0.334(10) | 0.25  | 0.22  | 0.091 |
| WRSOS     | 0.3326(5) | 0.260(2) | 0.227(1) | 0.091(4) |
| BDD       | 0.335(2) | 0.255(2) | 0.225(2) | 0.090(2) |
| KPZ       | 1/3    | 0.2592 | 0.2241 | 0.09345 |

Γ cannot be directly measured if cumulants of χ are unknown. We then define

\[ q' = \frac{r - v_\infty t}{s_\lambda g_1 t^{1/3}} \equiv \frac{q}{\langle \chi \rangle} \tag{15} \]

such that \( \langle q' \rangle \to 1 \) for \( t \to \infty \). We can obtain the scaling law governing the shift in the mean by plotting \( 1 - \langle q' \rangle \) against time using only the directly measurable parameters \( v_\infty \) and \( g_1 \). The results are shown in figure 13, where we have obtained corrections very consistent with \( t^{-1/3} \), already observed in many other systems [6, 7, 10, 21, 24]. The slow crossover to \( t^{-1/3} \) observed for off-lattice simulations of Eden D [23] was not observed in the on-lattice version. Moreover, our previous off-lattice simulations of Eden B² also yielded a shift consistent with \( t^{-1/3} \) [21]. An important remark is that the directly measurable amplitude in the scaling \( 1 - \langle q' \rangle \simeq -\langle \eta \rangle / g_1 t^{-1/3} \) also yields a way to obtain an estimate of \( \langle \eta \rangle \). The values of \( \langle \eta \rangle \) for anisotropic growth models are shown in table 3.

The dimensionless cumulant ratios obtained for both Eden models are shown in table 4. All ratios are universal and show remarkable agreement with GUE [17]. Notice that using dimensionless cumulant ratios as equations (12)–(14), all cumulants of \( \chi \) can be determined as functions of a single cumulant. This reasoning also applies to systems where \( \chi \) is unknown, raising a general strategy to probe universality in the interface distributions.

The top panel of figure 14 compares the distributions scaled according to equation (15) for Eden A in the direction ⟨10⟩ at different times with the GUE distribution scaled to a mean 1, \( \chi^* = \chi / \langle \chi \rangle \). The distributions converge to GUE as time evolves but the shift is evident. If the estimated shift \( \langle \eta \rangle = 1.7(1) \) is explicitly included by defining

\[ q^* = \frac{r - v_\infty t - \langle \eta \rangle}{s_\lambda g_1 t^{1/3}} \tag{16} \]

an excellent collapse between \( P(q^*) \) and \( P(\chi^*) \) is found (figure 14, bottom). Similar results are found for all investigated models.

² The difference between Eden D and Eden B is that any nearest neighbor can be chosen in the former while only the empty ones are eligible in the latter.
5. Droplet growth

A paradigmatic model for the growth of a droplet morphology is the PNG model, where islands are randomly nucleated over existing ones and, once nucleated, they start to grow laterally with constant velocity. If we consider an initial island that grows indefinitely from a single nucleation at the origin, the PNG model has an exact asymptotic solution with a droplet shape, where the height at the origin is given by equation (2) with exactly known parameters [17, 18]. PNG model simulations with droplet geometry have been performed for comparison with analytic results [17, 20]. Other models and initial conditions leading to droplet interfaces have also been investigated [10, 16, 36].

Droplet geometry can also be investigated in more complex models for which analytical results are currently unavailable such as, for example, the BD model [38]. The growth of ballistic deposition droplets (BDD) starts with a single particle stuck to the origin of the system. Particles move ballistically along direction $-y$. If a particle visits a NN site of the aggregate, it irreversibly attaches to this position and becomes part of the aggregate. Let $L(t)$ be the size of the active region defined as the set of $x$-coordinates where growth can be tried. When particle deposition is tried the time is increased by $\Delta t = 1/L(t)$. Figure 15 shows a typical BDD.

The non-universal parameters depend on the position in the active region in analogy to the directional dependence in the Eden growth, whereas cumulant ratios and the growth exponent do not. An ensemble of $2 \times 10^7$ samples was used to compute statistics. Growth of a cluster for $t = 10^4$ ($\sim 10^8$ particles) takes about 2 s in an Intel Xeon 3.20 GHz CPU. For the sake of conciseness, we present only the results for the height fluctuations at the origin.

The curves used to determine the non-universal parameters (table 3) are shown in figures 16 and 17. The interface velocity has the usual behavior illustrated in inset (B).
Non-universal parameters, corrections and universality in KPZ growth

Figure 14. Top: distributions for the on-lattice Eden A model at different growth times scaled accordingly equation (15). Bottom: the same distributions rescaled using equation (16) that includes the shift $\langle \eta \rangle = 1.7$. Insets show zooms around the peaks.

of figure 16. However, BDD growth has strong corrections in the growth exponent as confirmed in inset (A) of figure 16, in analogy to its flat counterpart [22, 37]. The correction in the mean converges in the usual way ($t^{-1/3}$) as shown in figure 13. Notice the presence of a crossover from a faster initial decay to the regime $t^{-1/3}$, analogous to the crossover observed for the isotropic Eden D model. Similarly to the isotropic growth, corrections in $g_2$ decays with an unusually small exponent $t^{-0.45}$.

The quantity $g_1$ approaches its asymptotic value as $t^{-2/3}$, the same correction obtained for flat simulations of isotropic growth models and for all versions of the anisotropic Eden model presented in section 4. The cumulant of order $n = 3$ also has a correction very close to $t^{-2/3}$, while the cumulant of order $n = 4$ has a correction faster than $t^{-2/3}$. In particular, the correction observed in $g_3$ is slower than that observed for the flat case, demonstrating the intricate and non-universal scenarios involving corrections in cumulants.
Figure 15. Ballistic deposition droplet after deposition of 56000 particles. The first half of the deposited particles are depicted in black and the rest in green. The solid line represents the interface.

Figure 16. Interface width against time determined at the center of the BDD model. The solid line is a power law $t^{1/3}$. Inset (A) shows the effective growth exponent while inset (B) shows the interface velocity against $t^{-2/3}$.

of order $n \geq 2$. The inset of figure 17 shows the corresponding corrections in $g_n$. Finally, the growth exponent and dimensionless cumulant ratios converge to the values expected for the KPZ class as shown in table 4.

We also considered the restricted solid-on-solid (RSOS) growth model [34] with a wedge initial condition, the WRSOS model. At each time step, a site in the growing zone (of size $L(t)$) is randomly selected and its height increased by 1 if the constraint $\Delta h = |h(j, t) - h(j \pm 1, t)| \leq 1$ is satisfied, otherwise, the deposition attempt is refused.
Figure 17. Semi-logarithmic plot of $g_n$ against time for the BDD model. Solid lines are the non-linear regression used to extrapolate the asymptotic values. Inset: difference $g_n(t) - g_n(\infty)$ against time. The dashed line represents a decay $t^{-2/3}$.

Figure 18. Typical interfaces obtained for the WRSOS model for deposition times $t = 512$ (lower) and $t = 1024$ (upper).

The time is incremented by $\Delta t = 1/L(t)$ for each attempt. A wedge initial condition $h(j, 0) = |j|$, obeying $\Delta h = 1$, was considered. This initial condition implies a droplet growth since the radius of the growing zone increases with average velocity 1, while in the center ($\langle 10 \rangle$ direction) the interface has a smaller velocity (table 3). Typical interfaces for distinct times are shown in figure 18.

Figure 19 shows the interface velocity at $j = 0, 10$ and 1000. They converge to the same asymptotic value observed for the flat geometry [22], but differ at short times. This means a macroscopic shape, around the droplet center, that moves asymptotically with constant velocity. The macroscopic shape is almost perfectly fitted by a parabola $h(x) = h(0) + ax^2$ in analogy with the solution of the KPZ equation with a sharp wedge initial condition [10]. The surface statistics was computed using an ensemble of $5 \times 10^7$ samples. A typical run up
to $t = 3000$ takes about 0.2 s in an Intel Xeon 3.20 GHz CPU. Non-universal parameters obtained for the height fluctuations at the origin $j = 0$ are shown in table 3. The universal quantities shown in table 4 exhibit excellent agreement with the KPZ universality class. Far from the origin, accordance with the KPZ conjecture is also observed, but the farther from the origin the slower the convergence.

The finite-time corrections in the cumulants of WRSOS were also studied. The shift in the mean approaches the GUE one with the correction $t^{-1/3}$ (figure 13). The quantities $g_n$, for $n = 1, 3$ and 4 have a correction $t^{-2/3}$, while $g_2$ decays slightly faster than $t^{-2/3}$ (figure 20) in analogy with the flat simulations of RSOS.
Non-universal parameters, corrections and universality in KPZ growth

6. Discussion

We have performed extensive simulations of models belonging to the KPZ universality class in order to probe the generality of the KPZ ansatz given by equation (5) and to determine further corrections to this equation. We have analyzed two classes of models. In the isotropic growth models, all points of the interface are used to perform averages, in contrast with the anisotropic ones where the non-universal parameters vary along the surface, limiting the statistics to a single or a few points per sample.

During the numerical investigation, we have found that the determination of the non-universal parameter $\Gamma$ using the asymptotic value of the scaled second cumulant $g_2 = \langle h^2 \rangle_c / t^{2/3}$ has limited accuracy for some models, probably due to unknown puzzling corrections. We propose an alternative way to obtain $\Gamma$ using the first cumulant derivative by means of the quantity $g_1 = s\lambda 3(\langle h \rangle_c - v_\infty) t^{2/3} \rightarrow \Gamma^{1/3} \langle \chi \rangle$. It is important to mention that noise in the numerical derivative counts against this method. Here, we used very large statistics to smooth the curves. However, if statistics is limited, as is likely the case in experiments, one can use numerical procedures to smooth the derivative [39]. We also considered scaled cumulants $g_n = \langle h^n \rangle_c / (s^n \lambda^{n/3})$ of third and fourth order to determine $\Gamma$. In general, we have observed that small deviations in $\Gamma$ may suggest a fake violation of the generalized KPZ ansatz. Estimates taken from $g_1$ give very good agreement with the generalized KPZ conjecture for all investigated models. Dimensionless universal cumulant ratios are not very sensitive to corrections in cumulants, as one can see in tables 2 and 4, in which excellent agreements with GOE and GUE distributions were found for flat and curved models, respectively.

Comparing the non-universal parameters for the flat RSOS and BD models (table 1) with their droplet counterparts (table 3), we see that within the error bars they are equal, with exception of the shift $\langle \eta \rangle$. On the other hand, in the Eden D model all parameters are different for on- and off-lattice simulations. This is expected, since the lattice constraint should change the strength of fluctuations.

In off-lattice simulations of the Eden model, we determined the shift in the height distributions and our results support a crossover from $t^{-2/3}$ to $t^{-1/3}$ in the first cumulant in relation to GUE, clarifying an apparent exception to the correction $t^{-1/3}$ recently suggested by Takeuchi [23].

From the numerical side, we propose that the estimates of $\Gamma$ taken from the first cumulant are more reliable than those obtained from the higher order ones since, in principle, we know little about the nature of terms beyond $\chi$ in the KPZ ansatz. These terms may introduce anomalous corrections as, for example, in the case of statistical dependence among random variables. In fact, in contrast to the $t^{-2/3}$ correction in $g_1$ obtained for all investigated models, we have not found a standard in the corrections of higher order cumulants.

The first cumulant derivative analysis for all investigated models is in agreement with an additional term generalizing the KPZ ansatz to

$$h = v_\infty t + s\lambda (\Gamma t)^{1/3} \chi + \eta + \zeta t^{-1/3} + \cdots.$$  \hspace{1cm} (17)

Corrections $O(t^{-1/3})$ were reported also for some analytically tractable KPZ class models [24], suggesting that this term is a general property of KPZ systems. However, in
the solution of the KPZ equation with a sharp wedge initial condition a correction $O(t^{-1})$ was reported [10, 12], possibly because the non-universal parameter $\zeta$ is zero in this case.

The generalized ansatz given by equation (17) allows one to speculate on the origin of the complicated corrections obtained for cumulants of order $n \geq 2$. For instance, if $\eta$ and $\zeta$ are statistically dependent, a term $t^{-1}$ will appear in the second cumulant in addition to the leading term $t^{-2/3}$. If the amplitude of $t^{-1}$ is much larger than that of $t^{-2/3}$, one would observe a slow crossover to $t^{-2/3}$. This scenario explains the correction in $g_2$ observed for the RSOS model. Still in the field of speculation, if $\chi$ and $\eta$ are statistically dependent the two first leading terms of the correction in $g_2$ would be $t^{-1/3}$ and $t^{-2/3}$. Again, if the amplitude of the second leading term is much larger than the first one, we could explain the anomalously slow exponent observed in the BD model as a crossover between the two scaling regimes. However, a double power law regression, like that used to determine the shift correction in Eden D, does not support such a conjecture.

Figure 21. Rescaled distributions for all investigated models using equations (16) and (18) in the top and bottom panels, respectively.
Now we discuss how universality in height distributions can be checked without explicit knowledge of the parameter $\Gamma$. Performing a rescaling using the directly measurable parameters from tables 1–3, we can define the variable

$$q^* = \frac{h - v_\infty t - \langle \eta \rangle}{s_\lambda g_1 t^{1/3}}.$$  

The scaled distributions for this variable, $P(q^*)$, agree remarkably well with the GUE distribution scaled to unitary mean, $\chi^* = \chi/\langle \chi \rangle$, as shown in the top part of figure 21. Our entire analysis depends implicitly on $\langle \chi \rangle$, but does not provide information about it beyond its negative sign, since $g_1 < 0$. We can alternatively define a variable with asymptotic unitary variance using the measurable parameter $g_2$ rather than $g_1$ to find

$$q^{**} = \frac{h - v_\infty t - \langle \eta \rangle}{g_2^{1/2} t^{1/3}} \rightarrow \frac{\chi}{\sqrt{\langle \chi^2 \rangle_c}},$$

with a mean depending on the dimensionless and measurable cumulant ratio $R$ given by

$$\langle q^{**} \rangle \equiv \text{sgn}(\langle \chi \rangle) R^{-1/2} = -1.9641.$$  

Again, as shown in figure 21 bottom, a very good collapse is obtained. We remark that this kind of analysis could be very profitable to probe the universality of height distributions in systems where they are not known \textit{a priori}, such as high dimensional KPZ growth [40] or models in universality classes without analytical counterparts. Although the complete distribution is not determined, once $\langle \chi \rangle$ or $\sqrt{\langle \chi^2 \rangle_c}$ is fixed everything concerning the HD can be derived.

**Acknowledgments**

This work was partially supported by the Brazilian agencies CNPq and FAPEMIG. We thank K A Takeuchi for the fruitful discussions on [23] and H Spohn for the discussions on [10].

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